Eine Woche, ein Beispiel
7.23 trace theorem and Sobolev embedding

This is a continuation of [23.05.28].
In the statement of propositions, all fcts are real-valued fcts.

Prop. For $0 \leq k \leq n$, $s > \frac{k}{2}$, one can construct cont linear fcts

$$H^s(\mathbb{R}^n) \longrightarrow H^{s-\frac{k}{2}}(\mathbb{R}^{n-k})$$
$$\cup \qquad\qquad \cup$$
$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^{n-k})$$
$$f \longmapsto f|_{\{0\} \times \mathbb{R}^{n-k}}$$

Proof. Denote $V := \mathbb{R}^k$, $W := \mathbb{R}^{n-k}$, then $V \times W = \mathbb{R}^n$, $W \hookrightarrow V \times W$, reduce to show:

$$\{0\} \times \mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

$$\exists C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \qquad \|f|_W\|_{H^{s-\frac{k}{2}}} \leq \|f\|_{H^s}$$

Step 1. Express $\widehat{f|_W}(\xi_2)$ in terms of $\hat{f}(\xi)$, by using Fourier transform twice.

$$\left. \begin{array}{l} f(0, x_2) = \int_W e^{i\langle x_2, \xi_2 \rangle} \widehat{f|_W}(\xi_2) \, d\xi_2 \\ f(0, x_2) = \int_{V \times W} e^{i\langle x, \xi \rangle} \hat{f}(\xi) \, d\xi \\ \qquad = \int_W e^{i\langle x_2, \xi_2 \rangle} \left(\int_V \hat{f}(\xi) \, d\xi_1 \right) d\xi_2 \end{array} \right\} \Rightarrow \widehat{f|_W}(\xi_2) = \int_V \hat{f}(\xi) \, d\xi_1$$

Step 2. Expand.
$$\|f|_W\|^2_{H^{s-\frac{k}{2}}} = \|\widehat{f|_W}\|^2_{L^2(W, (|\xi_2|^2+1)^{s-\frac{k}{2}} d\xi_2)}$$
$$= \int_W (\widehat{f|_W}(\xi_2))^2 (|\xi_2|^2+1)^{s-\frac{k}{2}} \, d\xi_2$$
$$= \int_W \left(\int_V \hat{f}(\xi) d\xi_1 \right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}} \, d\xi_2$$

$$\|f\|^2_{H^s} = \|\hat{f}\|^2_{L^2(V \times W, (|\xi|^2+1)^s d\xi)}$$
$$= \int_{V \times W} (\hat{f}(\xi))^2 (|\xi|^2+1)^s \, d\xi$$
$$= \int_W \left(\int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s \, d\xi_1 \right) d\xi_2$$

Therefore, the problem reduce to $d\xi_2 \approx d\xi_1$

$$\left(\int_V \hat{f}(\xi) \, d\xi_1 \right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}} \leq C \int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s \, d\xi_1.$$

Step 3. Use Hölder inequality to simplify. Since

$$\left(\int_V \hat{f}(\xi)\, d\xi_1\right)^2 \leq \int_V \hat{f}(\xi)^2 (|\xi|^2+1)^s\, d\xi_1 \cdot \int_V (|\xi|^2+1)^{-s}\, d\xi_1,$$

the problem reduces to

$$\int_V (|\xi|^2+1)^{-s}\, d\xi_1 \cdot \left(|\xi_2|^2+1\right)^{s-\frac{k}{2}} \leq C.$$

Step 4. Compute $\int_V (|\xi|^2+1)^{-s}\, d\xi_1$ directly.

$$\int_V (|\xi|^2+1)^{-s}\, d\xi_1$$

$$= \int_V \frac{1}{(|\xi_1|^2 + |\xi_2|^2+1)^s}\, d\xi_1$$

$$\xrightarrow[\;a>0\;]{a^2 = |\xi_2|^2+1} \int_V \frac{1}{(|\xi_1|^2 + a^2)^s}\, d\xi_1$$

$$= \int_V \frac{1}{(|\frac{\xi_1}{a}|^2 +1)^s}\, d\xi_1 \cdot a^{k-2s}$$

$$= C a^{k-2s} = C \left(|\xi_2|^2+1\right)^{\frac{k}{2}-s}$$

where $C = \int_V (|x|^2+1)^{-s}\, dx < +\infty$

$$\Rightarrow \int_V (|\xi|^2+1)^{-s}\, d\xi_1 \cdot (|\xi_2|^2+1)^{s-\frac{k}{2}} \leq C \qquad \square$$

Rmk. The original C in the proposition can be taken by

$$C = (2\pi)^k \int_{\mathbb{R}} \frac{1}{(|x|^2+1)^s}\, dx.$$

Here, C depends on the definition of the norm $\|\cdot\|_{H^s}$, $\|\cdot\|_{H^{s-\frac{n}{2}}}$. Therefore, usually we don't write down C explicitly.

Prop. For $n, k \geq 0$, $s > k + \frac{n}{2}$, one can construct cont linear fcts.

$$H^s(\mathbb{R}^n) \longrightarrow C_0^k(\mathbb{R}^n)$$
$$\cup \qquad\qquad \cup$$
$$\mathcal{S}(\mathbb{R}^n) =\!=\!= \mathcal{S}(\mathbb{R}^n)$$

Proof. Only show the $k=0$ case, then $s > \frac{n}{2}$.

Once we show:
$$\exists C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \quad \|f\|_{L^\infty} \leq C \|f\|_{H^s},$$
by the completeness of $C_0^0(\mathbb{R}^n)$, the embedding is then well-defined.

$$\|f\|_{L^\infty}^2 \leq \|\hat{f}\|_{L^1}^2$$
$$= \left(\int_{\mathbb{R}^n} \hat{f}(\xi)\, d\xi \right)^2$$
$$\leq \int_{\mathbb{R}^n} (|\xi|^2 + 1)^{-s}\, d\xi \int_{\mathbb{R}^n} (\hat{f}(\xi))^2 (|\xi|^2 + 1)^s\, d\xi$$
$$= C^2 \|f\|_{H^s}^2$$

Here, $C = \sqrt{\int_{\mathbb{R}} \frac{1}{(|\xi|^2 + 1)^s}\, d\xi} < +\infty$ $\qquad\qquad\square$

Prop (in final exam). For $n \geq 0$, $s > \frac{n}{2}$, one can construct cont linear fcts.

$$L^1(\mathbb{R}^n) \longrightarrow H^{-s}(\mathbb{R}^n)$$
$$\cup \qquad\qquad \cup$$
$$\mathcal{S}(\mathbb{R}^n) =\!=\!= \mathcal{S}(\mathbb{R}^n)$$

Proof. Reduced to show:
$$\exists C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \quad \|f\|_{H^{-s}} \lesssim \|f\|_{L^1}.$$

$$\|f\|_{H^{-s}}^2 = \|\hat{f}\|_{L^2(\mathbb{R}^n, (|\xi|^2+1)^{-s} d\xi)}^2$$
$$= \int_{\mathbb{R}^n} (\hat{f}(\xi))^2 (|\xi|^2 + 1)^{-s}\, d\xi$$
$$\leq \int_{\mathbb{R}^n} (|\xi|^2 + 1)^{-s}\, d\xi \ \|\hat{f}\|_{L^\infty}^2$$
$$\leq \int_{\mathbb{R}^n} (|\xi|^2 + 1)^{-s}\, d\xi \ \|\hat{f}\|_{L^1}^2$$
$$= \int_{\mathbb{R}^n} (|\xi|^2 + 1)^{-s}\, d\xi \cdot (2\pi)^n \|f\|_{L^1}^2$$
$$= C^2 \|f\|_{L^1}^2$$

Here,
$$C = \sqrt{\int_{\mathbb{R}^n} \frac{1}{(|\xi|^2+1)^s}\, d\xi} \cdot (2\pi)^{\frac{n}{2}} < +\infty$$

The general case is as follows.

Exercise 6.3. Let $n = p + q$, $1 \leq p \leq n$, and write $\mathbb{R}_x^n := \mathbb{R}_{x'}^p \oplus \mathbb{R}_{x''}^q$ (i.e., for $x \in \mathbb{R}^n$ let $x' := (x_1, \ldots, x_p)$ and $x'' := (x_{p+1}, \ldots, x_n)$). Furthermore, let $s > k + p/2$ for some $k \in \mathbb{Z}_+$.

(a) Show that the assignment

$$H^s(\mathbb{R}^n) \ni f \mapsto \left(x' \mapsto f(x', \cdot) \in H^{s-p/2}(\mathbb{R}^q) \right)$$

is a well-defined continuous linear map $H^s(\mathbb{R}^n) \to C_0^k(\mathbb{R}^p, H^{s-p/2}(\mathbb{R}^q))$.

(b) Identify the Trace Theorem and the Sobolev Embedding Theorem as special cases.

Rmk. Using the same method, one can argue that

Prop. For $n \geq 0$, one can construct cont linear fcts.

$$
\begin{array}{ccc}
L^1(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & C_0(\mathbb{R}^n) \\
\cup & & \cup \\
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & \mathcal{S}(\mathbb{R}^n) \\
f & \longmapsto & \hat{f}
\end{array}
\qquad
\begin{array}{ccc}
L^2(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & L^2(\mathbb{R}^n) \\
\cup & & \cup \\
\mathcal{S}(\mathbb{R}^n) & \xrightarrow{\mathcal{F}} & \mathcal{S}(\mathbb{R}^n) \\
f & \longmapsto & \hat{f}
\end{array}
$$