Eine Woche, ein Beispiel

6.12 Condensed

Main Ref: https://people.mpim-bonn.mpg.de/scholze/Course%20Summer%2022.html

That's already so well written. I collect some notations here purely for self-study, and I believe this document is useless for other people.

Condensed set

Def (pro-étale site $*_{\text{proét}}$)

Category: Prof

Cover: for $S \in \mathrm{Prof}$,

$$\mathrm{Cov}(S) = \left\{ \{S_i \xrightarrow{f_i} S\}_{i\in I} \text{ in Prof} \;\middle|\; \begin{array}{l} I \text{ finite} \\ S = \bigcup_{i\in I} f_i(S_i) \end{array} \right\}$$

Naive def. ⚠ Caveat: Prof is large. Need minor modification.

$$\begin{aligned}
\mathrm{CondSet} &= \mathrm{Sh}(*_{\text{proét}}) \\
&= \left\{ X : \mathrm{Prof}^{op} \longrightarrow \mathrm{Set} \;\middle|\; \begin{array}{l} X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \\ 0 \to X(S) \to X(T) \rightrightarrows X(T\times_S T) \quad T \twoheadrightarrow S \end{array} \right\} \\
&= \{ X : \mathrm{qcProj}^{op} \longrightarrow \mathrm{Set} \mid X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \} \\
\mathrm{CondAb} &= \mathrm{Sh}(*_{\text{proét}}, \mathrm{Ab}) \\
\mathrm{Cond}(\mathcal{C}) &= \mathrm{Sh}(*_{\text{proét}}, \mathcal{C}) \\
\mathrm{Cond}(R) &= \mathrm{Cond}(\mathrm{Mod}(R)) = \mathrm{Sh}(*_{\text{proét}}, \mathrm{Mod}(R)) \qquad R \in \mathrm{Ring}
\end{aligned}$$

when $R \in \mathrm{Cond}(\mathrm{Ring})$, require compatability.

Analytic ring and complete condensed A-module

Def. ⚠ Preliminary

An analytic ring is $\mathcal{A} = (A, \mathcal{M}_A(-), \delta)$, where

- $A \in \mathrm{Cond}(\mathrm{Ring})$
- $\mathcal{M}_A : \mathrm{Prof} \longrightarrow \mathrm{Cond}(A) \qquad S \longmapsto \mathcal{M}_A(S)$
- $\delta_S : S \longrightarrow \mathcal{M}_A(S) \qquad s \longmapsto \delta_s$
- $$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde f"] \\ & A \end{tikzcd} \qquad \tilde f : \begin{array}{c} \mu \\ \downarrow \\ \int f \mu \end{array}$$
- $M \in \mathrm{Cond}(A)$ is complete if
$$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde f"] \\ & M \end{tikzcd}$$

We require that the full subcategory

$\mathrm{Cond}_{cpl}(\mathcal{A}) := \{\text{complete condensed } \mathcal{A}\text{-modules}\} \subseteq \mathrm{Cond}(A)$

should be abelian category.

Liquid vector spaces. $S \in \mathrm{Prof}$.

$$\mathcal{M}(S) = \{ f : C(S;\mathbb{R}) \to \mathbb{R} \mid f \text{ cont} \} = \mathbb{R}[S]^{\blacksquare}$$

$$\mathcal{M}(S)_{\ell^p \le c} = \varprojlim_i \mathcal{M}(S_i)_{\ell^p \le c} \subseteq \varprojlim_i \mathbb{R}^{\oplus S_i}$$

$$\mathcal{M}_p(S) = \bigcup_{c>0} \mathcal{M}(S)_{\ell^p \le c}$$

$$\mathcal{M}_{<p}(S) = \bigcup_{0<q<p} \mathcal{M}_q(S)$$

Def. Let $V \in \mathrm{CondAb}$ and $0 < p \le 1$.
V is *p-liquid* if

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_{<p}(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd}$$

equiv: $\forall q<p$

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_q(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd}$$

equiv:

$$\bigoplus_j \mathcal{M}_{<p}(T_j) \longrightarrow \bigoplus_i \mathcal{M}_{<p}(S_i) \longrightarrow V \longrightarrow 0$$