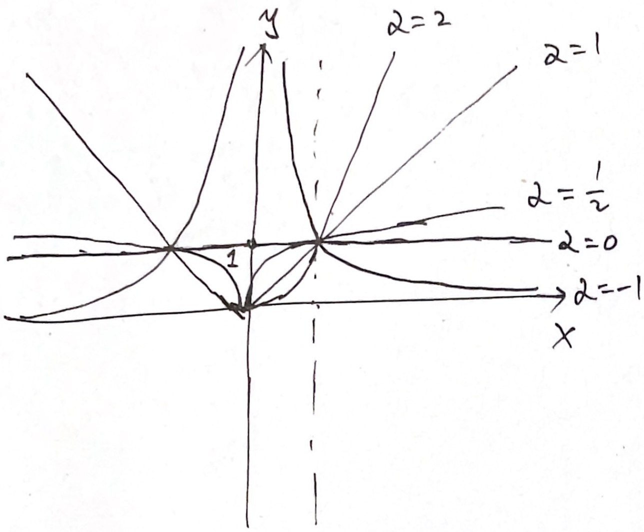
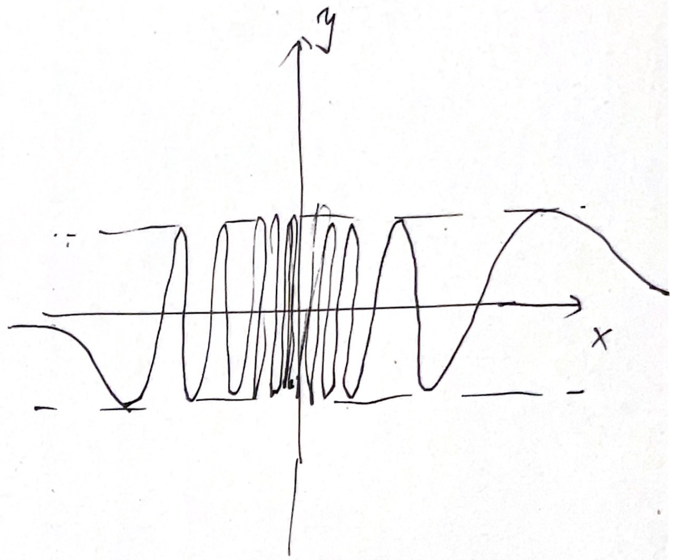
Tutorial 6 & Exercise 4.5.

Today we review and do exercises.

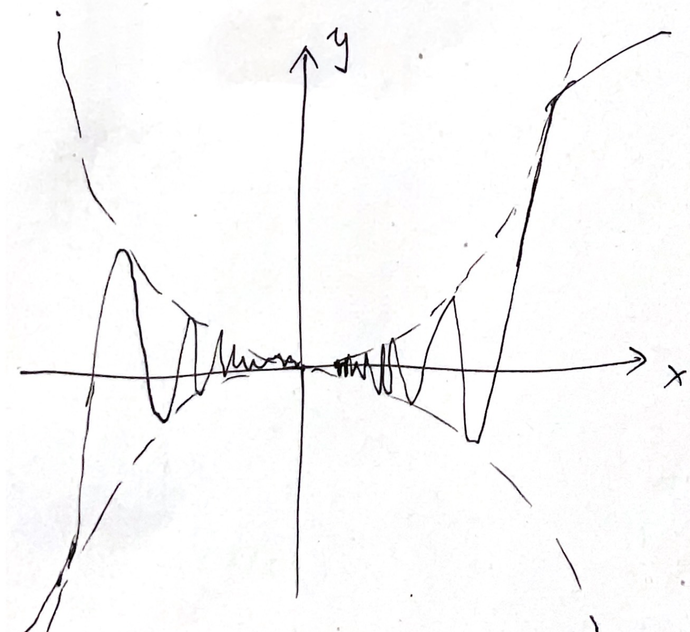
Ex 4.1. $\quad f_\alpha(x) = |x|^\alpha \sin\left(\frac{1}{x}\right)$



$y = |x|^\alpha$



$y = \sin\left(\frac{1}{x}\right)$

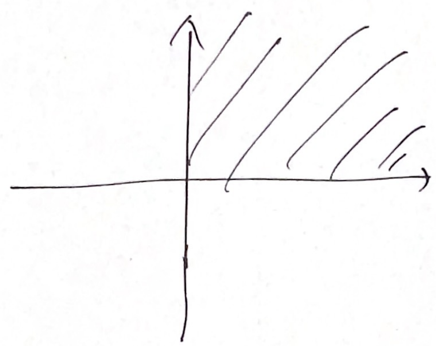


$y = |x|^2 \sin\left(\frac{1}{x}\right)$

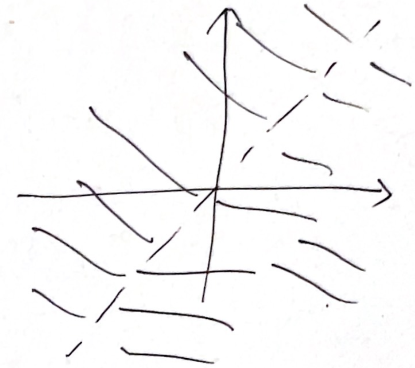
$$\overline{\lim_{x\to 0}}\, f_\alpha(x) = \begin{cases} +\infty & \alpha<0 \\ 1 & \alpha=0 \\ 0 & \alpha>0 \end{cases}$$

Ex 2. Extend fcts. (Symmetric extension is easier!)

a) $\tilde{f}(x,y) = \sqrt{|x|+|y|}$



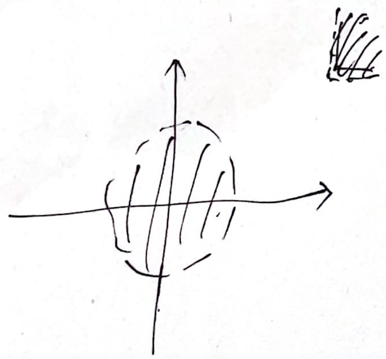
b). No way. Since $\lim\limits_{\substack{(x,y)\to(0,0) \\ x \neq y}} g(x,y) \nexists$

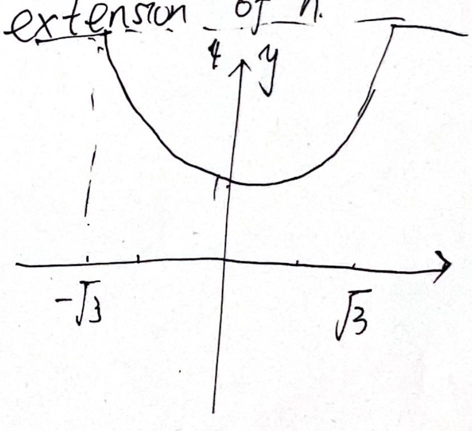


c) $\tilde{h}(x,y) = \begin{cases} 1+x^2+y^2 & |x^2+y^2| \leq 3 \\ 4 & |x^2+y^2| > 3 \end{cases}$

Verify: ① $\tilde{h}$ is cont

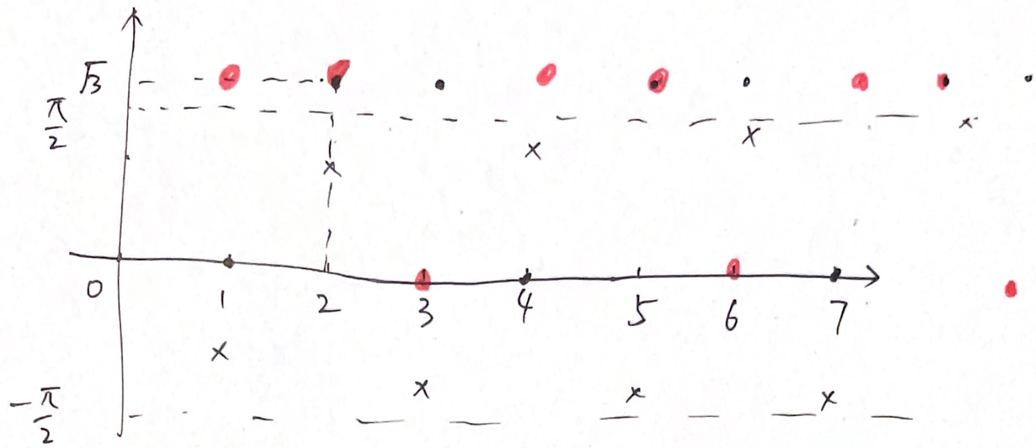
② $\tilde{h}$ is an extension of h.





$h(x,0$

Ex 3.

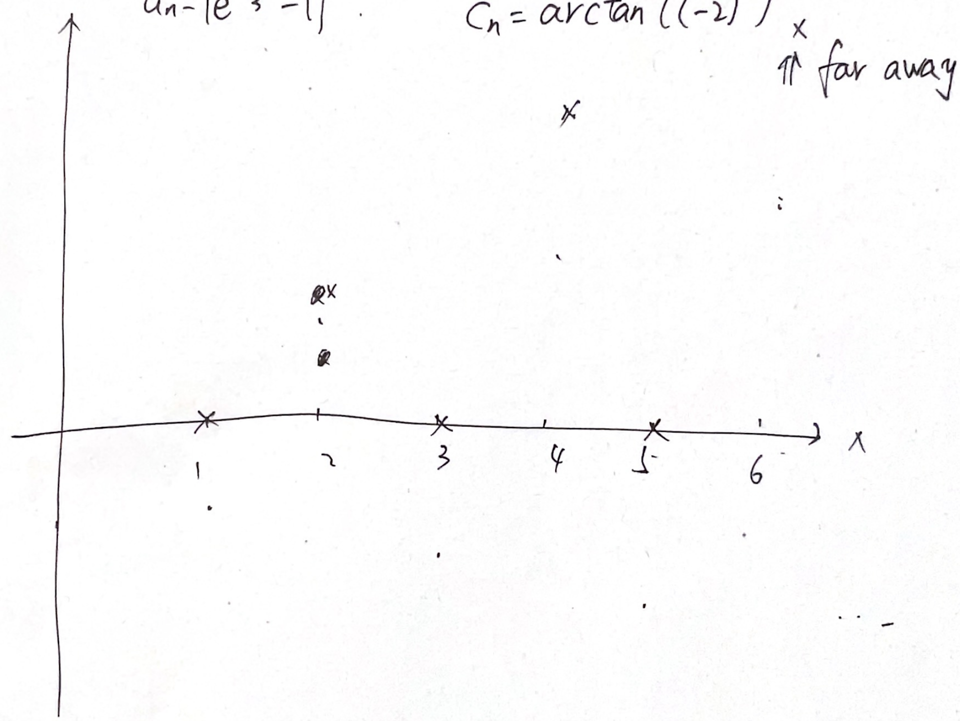


(a) ●

$a_n = |e^{\frac{2\pi i n}{3}} - 1|$

(c) ×

$c_n = \arctan((-2)^n)$



(b) .

$b_n = (-1)^n \sum_{k=1}^{n} \frac{1}{k}$

d) ×

$d_n = (-2)^n + 2^n$

Ex 4.4. Let $A = \begin{pmatrix} 0 & 2i \\ -2i & 0 \end{pmatrix}$

(a) Compute A^T, A^H. Hermitian conjugation

(b). Define

$$q_A : \mathbb{C}^2 \longrightarrow \mathbb{C} \qquad (x,y) \longmapsto (x,y)A^T \overline{\begin{pmatrix} x \\ y \end{pmatrix}}$$

Verify that $q_A(x,y) = 4\,\mathrm{Im}(x\bar{y})$, and

for fixed $y \in \mathbb{C}$, $\max\limits_{|x|=1} q_A(x,y) = 4|y|$

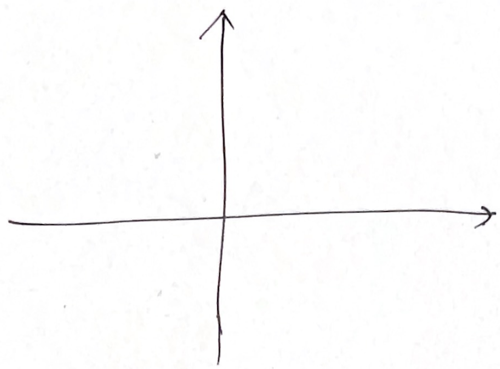
(c). Write

$$A : \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \qquad v \longmapsto Av$$

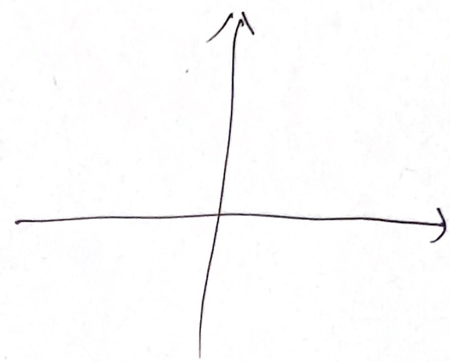
Compute $\|A\| := \sup\limits_{v \neq 0} \dfrac{\|Av\|}{\|v\|}$.

Ex 5.2.

(a) $2x^4 - \sin(2\pi x) = 1$

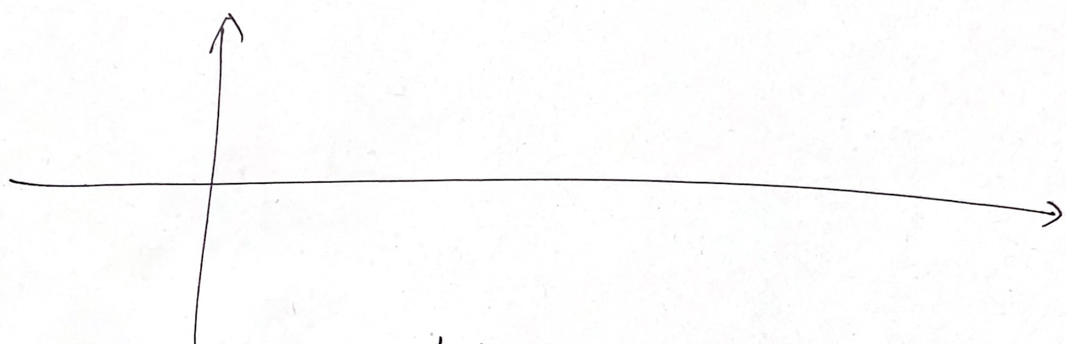


(b) $\arctan(x) = -e^x + 4$



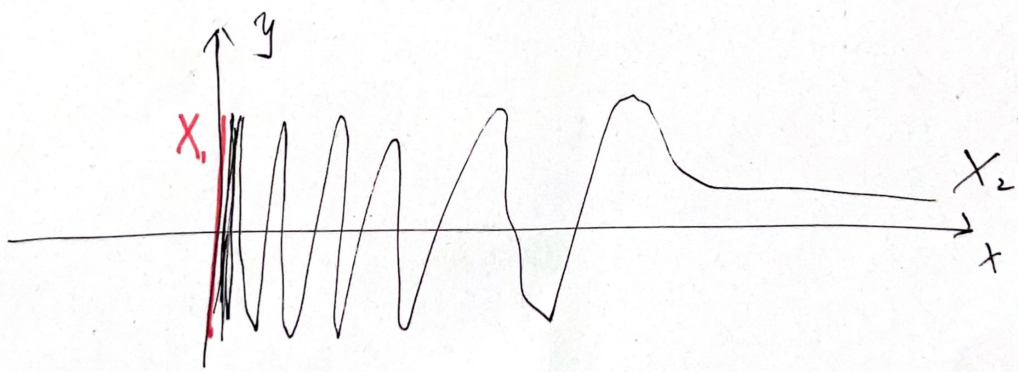
! solution.

(c) $\sin(x^2) = \sin\left(x^2 + \frac{3}{8}\right)$



infinite solution

Ex 5.3. Typo: $X_1 = \{(0,y) \in \mathbb{R}^2 \mid y \in [-1,1]\} \subseteq \mathbb{R}^2$



a). Show that X_1, X_2 are path connected & connected.

path connected for X_2: for

$$a = (x_1, \sin(\tfrac{1}{x_1})) \quad , \; b = (x_2, \sin(\tfrac{1}{x_2})) \in X_2 ,$$

we have path

$$\gamma : [0,1] \longrightarrow X_2$$
$$t \longmapsto \left(tx_1 + (1-t)x_2 , \sin\left(\frac{1}{tx_1+(1-t)x_2}\right)\right)$$

which connects a and b.

Connected for X_2: for topo spaces,

path connected $\Rightarrow$ connected

(b) Show that $X := X_1 \sqcup X_2$ is connected.

Lemma: $\overline{X_2} = X$ in $\mathbb{R}^2$, hence in X.

Proof: If $\mathcal{U} \subseteq X$ is open & closed, then

$$\left.\begin{array}{l} \mathcal{U} \cap X_2 \subseteq X_2 \text{ is open \& closed} \\ X_2 \text{ is connected} \end{array}\right\} \Rightarrow \mathcal{U} \cap X_2 = X_2 \text{ or } \emptyset$$

w.l.o.g. $\mathcal{U} \cap X_2 = X_2$. We get

$$\mathcal{U} = \overline{\mathcal{U}} \supset \overline{\mathcal{U} \cap X_2} = \overline{X_2} = X \quad \text{in } X$$

$\Rightarrow \mathcal{U} = X$ $\square$

Ex 4.4.(b)

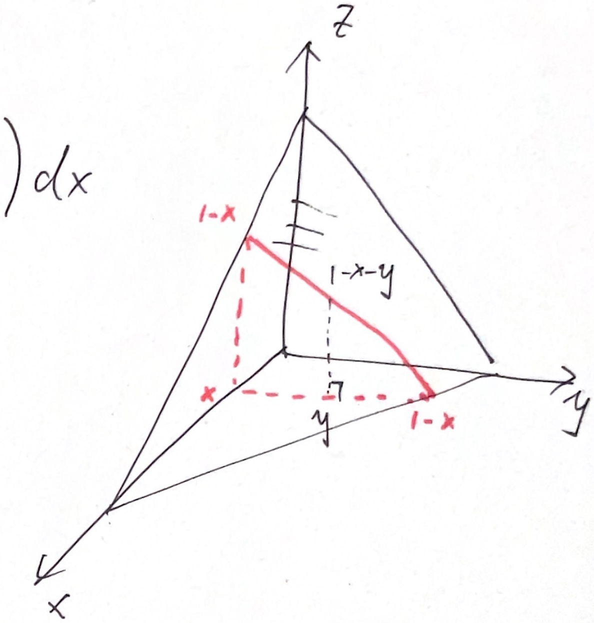
$$\int_B 1 \, dV$$

$$= \int_0^1 \left(\int_0^{1-x} \left(\int_0^{1-x-y} 1 \, dz \right) dy \right) dx$$

$$= \int_0^1 \left(\int_0^{1-x} (1-x-y) \, dy \right) dx$$

$$= \cdots$$

$$= \frac{1}{6}$$

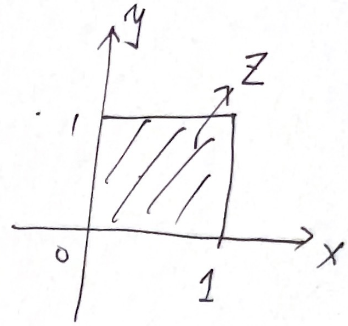


Integral.

Task 1. a). $\int_{Z} xy\,dx\,dy$

$= \int_0^1 \left(\int_0^1 xy\,dy \right) dx$

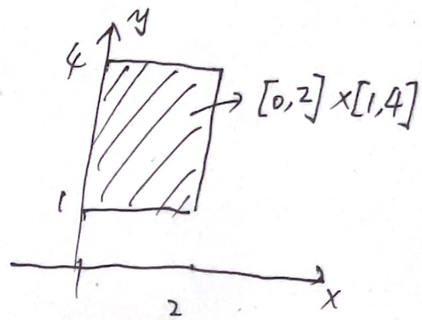
$= \cdots$



b) $\int_{[0,2]\times[1,4]} (x^3y + xy^3)\,dx\,dy$

$= \int_0^2 \left(\int_1^4 (x^3y + xy^3)\,dx \right) dy$

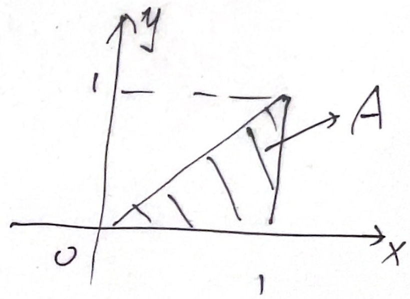
$= \cdots$



Task 2. (a) $\int_A xy\,dx\,dy$

$= \int_0^1 \left(\int_0^x xy\,dx \right) dy$

$= \cdots$

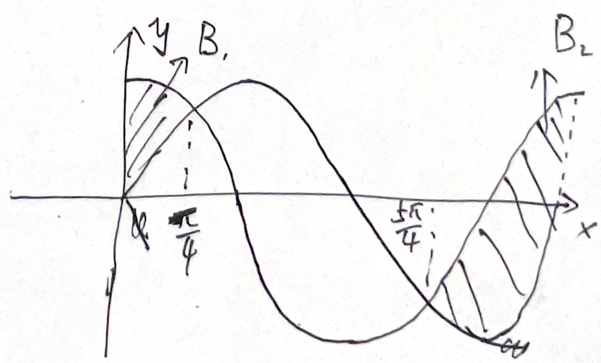


(b) $v_2(B)$

$= v_2(B_1) + v_2(B_2)$

$= \int_0^{\frac{\pi}{4}} \left(\int_{\sin x}^{\cos x} 1\,dy \right) dx + \int_{\frac{5\pi}{4}}^{2\pi} \left(\int_{\sin x}^{\cos x} 1\,dy \right) dx$

$= \cdots$



Task 3 Let $Z = [0,1] \times [0,1]$.

(a) Construct a bounded set $M \subseteq Z$ s.t.

$\mathbb{1}_M$ is not Riemannian integrable

A: $M = (\mathbb{Q} \times \mathbb{R}) \cap Z$

(b) Construct $f: Z \longrightarrow \mathbb{R}$ s.t.

f is not Riemannian integrable, but $|f|$ is.

A: $f = \mathbb{1}_M - \mathbb{1}_{Z \setminus M}$.