

Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

<https://people.math.harvard.edu/~gross/preprints/sat.pdf>

Task. Complete the following tables.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$		$\mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$	
Id	\mathbb{Z}					$\mathbb{Z}[X^*(T)] /_{I_T} \rtimes \mathbb{Z}[W_f]$
$G \times \mathbb{C}^*$	$\mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$					\mathcal{H}_{ext}
$B \times \mathbb{C}^*$	$\mathbb{Z}[X^*(T)][t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

We use the shorthand.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
G	$R(T)^W$	$R(T)$		$R(T) \otimes_{R(G)} R(T)$		$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T) \otimes_{R(G)} R(T)$		$R(T) \otimes_{R(G)} R(T) \otimes_{R(G)} R(T)$		
Id	\mathbb{Z}					$R(T) /_{I_T} \rtimes \mathbb{Z}[W_f]$
$G \times \mathbb{C}^*$	$R(G)[t^{\pm 1}]$					\mathcal{H}_{ext}
$B \times \mathbb{C}^*$	$R(T)[t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

$$\begin{aligned}
 R(B) &= \mathbb{Z}[X^*(T)] &= \mathcal{H}(\hat{\Gamma}(F), \hat{\Gamma}(\mathcal{O}_F)) \\
 R(G) &= \mathbb{Z}[X^*(T)]^W &\neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
 R(G)[q^{\pm \frac{1}{2}}] &= \mathbb{Z}[X^*(T)]^W[q^{\pm \frac{1}{2}}] &= \mathcal{H}_{sph}[q^{\pm \frac{1}{2}}] \\
 R(G \times \mathbb{C}^*) &= \mathbb{Z}[X^*(T)]^W[t^{\pm 1}] \\
 K^{G \times \mathbb{C}^*}(St) &= \mathcal{H}_{ext} &\neq \mathcal{H}(\hat{G}(F), I)
 \end{aligned}$$

Here is an initial example.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm 1}]$		$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}] / (z_1 - z_2)(z_1 - z_2^{-1})$		$\mathbb{Z}[W_{ext}] = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}]$
B	$\mathbb{Z}[y^{\pm 1}]$	$\mathbb{Z}[y^{\pm 1}, z] / (z - y)(z - y^{-1})$		$\mathbb{Z}[y_1^{\pm 1}, z_1, z_2] / ((z_1 - y_1)(z_1 - y_1^{-1}), (z_2 - y_1)(z_2 - y_1^{-1}))$		
Id	\mathbb{Z}	$\mathbb{Z}[z] / (z - 1)^2$		$\mathbb{Z}[z_1, z_2] / (z_1 - 1)^2, (z_2 - 1)^2$		$R(T) /_{I_T} \rtimes \mathbb{Z}[W_f] = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}] / (\bar{z}_w - 1)^2$
$SL_2 \times \mathbb{C}^*$	$\mathbb{Z}[x, t^{\pm 1}]$					$\mathcal{H}_{ext} = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}, t^{\pm 1}]$
$B \times \mathbb{C}^*$	$\mathbb{Z}[y^{\pm 1}, t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

This is our final task. Most of the notations are still not fixed.

$K^-(-)$	pt	$\mathcal{F}_d \quad \widetilde{Rep}_d(\mathcal{Q})$	$\mathcal{F}_d \times \mathcal{F}_{d'}$	$\mathcal{Z}_d = \prod_{d, d'} \mathcal{Z}_{d, d'}$
G_d	$R(T_d)^{w_d}$	$R(T_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$	
B_d	$R(T_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d) \otimes_{R(G_d)} R(T_d)$	
Id	\mathbb{Z}			
$G_d \times \mathbb{C}^*$	$R(G_d)[t^{\pm 1}]$			
$B_d \times \mathbb{C}^*$	$R(T_d)[t^{\pm 1}]$			
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$			

Conclusions on the summer vacation.

I guess that most part of my tasks coincide with this paper:

http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

Sadly, I only found it in the last week of the vacation.

Some possible tasks to work on:

1. Work out what $K_0^{\text{Id}}(\mathcal{B})$ is.

ref:

In [3264], the author computes the Chow group of $G(2,4)$.

<https://pbelmans.ncag.info/blog/2018/08/22/rank-flag-varieties/>

The module structure is easy, see

[<https://math.stackexchange.com/questions/1012699/when-does-a-smooth-projective-variety-x-have-a-free-grothendieck-group>]

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.

- Bernstein presentation

- try to understand the center of $\mathcal{H}(G(F), I)$

- How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory

- How can the Hecke algebra be realized as a Hecke algebra?

ref. [Hecke, Sec 10-17], [Williamson 11.4-12.2]

3. Try to understand what the Hall algebra / Quantum group is.

ref: [Lec 1-4, Appendix 4, <https://arxiv.org/pdf/math/0611617.pdf>]

- understand $\mathcal{H}_{\text{Rep}_K^{\text{nil}}(\mathcal{Q})}$ where $\mathcal{Q} = \cdot \rightarrow \cdot \rightarrow \cdot$

[Lec 2-3]

- understand $\mathcal{H}_{\mathbb{P}^1} \cong \bigcup \mathcal{U}_\nu(\widehat{\mathfrak{sl}}_2)$

[Lec 4]

$$\mathcal{H}_{\text{Tor}(\mathbb{P}^1)} \cong \bigotimes_{x \in \mathbb{P}^1} \mathcal{H}_{\text{Tor}, x}$$

- define (Quantum) Kac-Moody / loop algs

[Appendix 4]

- Why is that

$$K_0(\text{Rep}^{\mathbb{Z}}(\mathcal{R})) = \mathcal{U}_q(\mathfrak{n}(\mathcal{Q}))$$

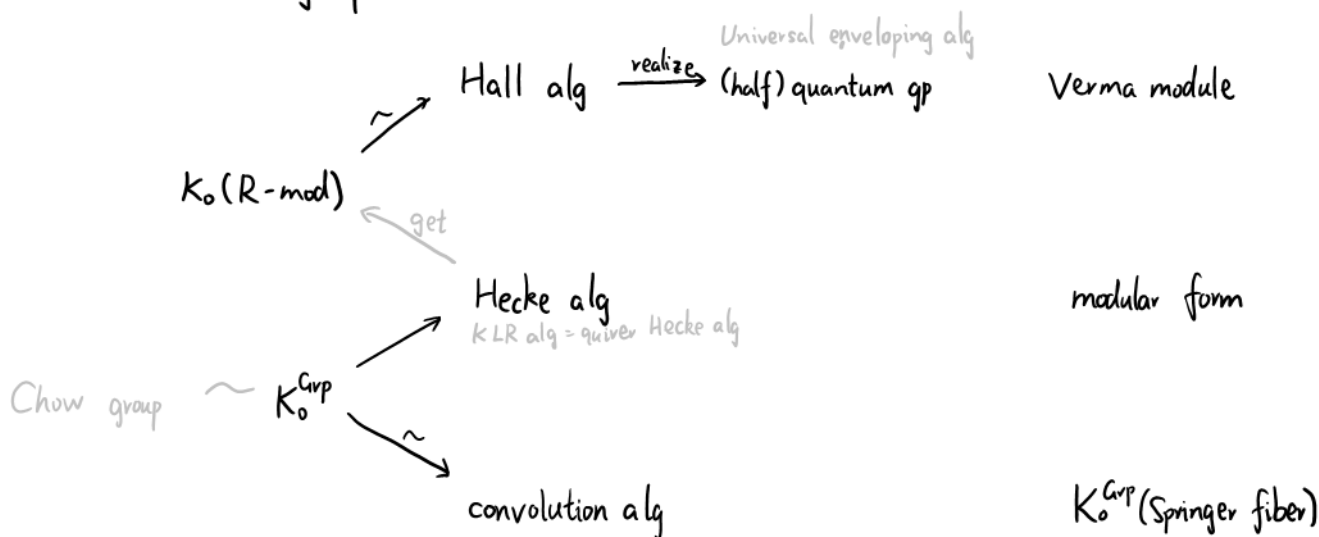
where

$$\mathcal{R} = \bigoplus_d H^{G \times \mathbb{C}^\times, BM}(\mathbb{Z}_d)$$

and what is

$$K_0(\text{Rep}^{\mathbb{Z}}(\bigoplus_d K_0^{G \times \mathbb{C}^\times}(\mathbb{Z}_d))) ?$$

4. Work out the big picture



5. A closer check of Satake iso

$$\begin{aligned}
 &K_0 \quad \text{combinations} \quad \text{Hecke alg} \\
 &R(B) = \mathbb{Z}[X^*(T)] = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F)) \\
 &R(G) = \mathbb{Z}[X^*(T)]^W \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
 &R(G)[q^{\pm \frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W [q^{\pm \frac{1}{2}}] = \mathcal{H}_{\text{sph}}[q^{\pm \frac{1}{2}}] \\
 &R(G \times \mathbb{C}^*) = \mathbb{Z}[X^*(T)]^W [t^{\pm \frac{1}{2}}] \\
 &K^{G \times \mathbb{C}^*}(St) = \mathcal{H}_{\text{ext}} \neq \mathcal{H}(\hat{G}(F), I)
 \end{aligned}$$

It's claimed by my schoolmate that

$$K_0(\text{Perv}_B(G/B)) \cong \mathcal{H}(G, B)$$

\uparrow
 sym monoidal structure
 induced from the convolution

then, what is

$$\begin{array}{ccc}
 K_0^B(\mathcal{B}) & \cong & ? \\
 K_0^{\text{Id}}(\mathcal{B}) & \cong & ? \\
 ? & \cong & \mathcal{H}(S_{m+n}, S_m \times S_n)
 \end{array}$$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces:
(e.g. in the case of \cdot , $\cdot \circ$, $\cdot \rightarrow \cdot$)

G, B, T

B, T^*B, St

$\mathfrak{g}, \widehat{\mathfrak{g}}, \mathfrak{g}^{sv}, \widehat{\mathfrak{g}}^{sv}, N, \widetilde{N}, h, n$

$\widehat{\mathfrak{g}}^h, \mathcal{O}_h, \Delta_w^h$

7. Try to understand what Kazhdan-Lusztig polynomials are [KL2], and
- Compute the transformation matrix between
 $\{[T_w^*], w \in W_f\}$ and $\{[\Delta_w^h], w \in W_f\}$? [Ka Sai]?
- understand what standard/crystal basis is
- understand the relationship between KL poly and crystal basis
- see if it is related to two basis in $\text{Rep}(G)$ (irr reps & multiplicative basis)

8. Try to understand the module part, i.e.,
- numbers of components of the Springer fiber
- how does $K_0^{Gr}(St)$ act on $K_0^{Gr}(\text{Springer fiber})$
- does that occupy "all rep" of $K_0^{Gr}(St)$

9. Ways of finding multiplication structure

1. By direct computation (with techniques)

Hecke algebra

double coset calculus

2. By formulas as alg-isos

$K_0^G(B)$

induction formula

3. By geometrical computation

cohomology

Chow group

cup product? de Rham calculus
intersection theory

4. By deformation (indirect)

$H_{top}^{BM}(St)$

$K_0^{G \times G}(St)$

10. Different views on the double coset

$$B \backslash G / B = (* / B) \times_{* / G} (* / B)$$

- as a set
- as flag variety quotient B -action
- as a stack
- groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $H(G, B)$, H_{sph} , (Haff)	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG....	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:

- 3 Seminars (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
 - L -parameters
 - moduli in AG
 - some following developments of the modular form (different type of q -ps, Hecke operators...)
 - reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.