Eine Woche, ein Beispiel

1.21. complex multilinear algebra

The title comes from
http://staff.ustc.edu.cn/~wangzuoq/Courses/16F-Manifolds/Notes/Lec16.pdf

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

M: cplx mfld, $p \in M$ $\qquad$ $M_{\mathbb{R}}$: M viewed as smooth mfld, not base change

better: M_{sm}

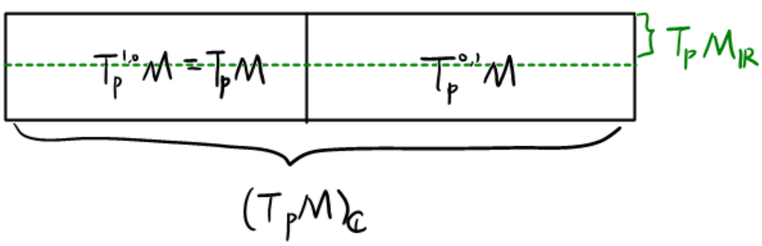
e.g. $M = \mathbb{C}^3$ $\quad p = 0$

Notation	base field	dim	basis	name	[YS20]
T_pM	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_pM_{\mathbb{R}}$	$\mathbb{R}$	6	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	real tangent vector	$T_p^{\mathbb{R}}M$
$(T_pM)_{\mathbb{C}} := T_pM_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ or $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	complexified tangent vector	$T_p^{\mathbb{C}}M$
$T_p^{1,0}M = T_pM$	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p^{0,1}M$	$\mathbb{C}$	3	$\frac{\partial}{\partial \bar{z}_i}$	anti-holomorphic tangent vector	
T_p^*M	$\mathbb{C}$	3	dz_i	holomorphic 1-form	Ω_p^1
$T_p^*M_{\mathbb{R}} \hat{=} \Omega^1_{\mathbb{R},p}$	$\mathbb{R}$	6	dx_i, dy_i	real 1-form	
$(T_p^*M)_{\mathbb{C}} := T_p^*M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$dz_i, d\bar{z}_i$ or dx, dy	complexified 1-form	$T_p^{*\mathbb{C}}M = A_p^1$
$T_p^{1,0,*}M \hat{=} \Omega_p^{1,0} = T_p^*M$	$\mathbb{C}$	3	dz_i	(1,0)-form	$T_p^{*1,0}M = A_p^{1,0}$
$T_p^{0,1,*}M \hat{=} \Omega_p^{1,0}$	$\mathbb{C}$	3	$d\bar{z}_i$	(0,1)-form	$T_p^{*0,1}M = A_p^{0,1}$

$\Omega^i, \Omega^{i,j}$: sheaves on M

Rmk. We don't have any natural identification between T_pM & $T_pM_{\mathbb{R}}$.
Notice that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $-\frac{1}{2}i$ is not real, so $\frac{\partial}{\partial z} \notin T_pM_{\mathbb{R}}$.

although our geometrical intuition of T_pM is often $T_pM_{\mathbb{R}}$



Reminder: the (induced) almost complex structure is defined as

$$J: T_pM_{\mathbb{R}} \longrightarrow T_pM_{\mathbb{R}}$$
$$\frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial y_i}$$
$$\frac{\partial}{\partial y_i} \longmapsto -\frac{\partial}{\partial x_i}$$

$$\rightsquigarrow \quad J: T_pM \longrightarrow T_pM$$

$$J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right)\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$J\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) = \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right)\begin{pmatrix} i & \\ & -i \end{pmatrix}$$