Eine Woche, ein Beispiel
5.4. line bundles on abelian varieties

Ref: follows [2025.04.13].
Most contents in this document can be found in [BL04, Chap 2 and Appendix B].

Goal: For $A = V/\Lambda$, identify

$$\mathrm{Pic}(A) \underset{\text{hidden sheaf argument?}}{\cong} H^1(\Lambda, H^0(\mathcal{O}_V^*)) \overset{\text{def}}{=\!=} \mathcal{P}(\Lambda)$$

algebraic info — gp cohom info — analytic info

$\mathcal{L}$ — $a_{\mathcal{L}} : \Lambda \times V \to \mathbb{C}$ — (H, χ)

$$a_{\mathcal{L}}(\lambda, v) = \chi(\lambda) \exp\left(\pi H(\lambda, v) + \tfrac{\pi}{2} H(\lambda, \lambda)\right)$$

Thm (Appell-Humbert) [BL04, p32]

$$\begin{tikzcd}
 & \{\chi\} \arrow[d, phantom, "\shortparallel"] & \{(H,\chi)\} \arrow[d, phantom, "\shortparallel"] & \{H\} \arrow[d, phantom, "\shortparallel"] & \\
0 \arrow[r] & \mathrm{Hom}(\Lambda, S^1) \arrow[r] \arrow[d, "\cong"'] & \mathcal{P}(\Lambda) \arrow[r] \arrow[d, "\cong"'] & NS(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] \arrow[d, equal, "\text{def}"] & \mathrm{Pic}(A) \arrow[r] & NS(A) \arrow[r] & 0 \\
 & \hat{A} & & &
\end{tikzcd}$$

where a polarization lives

where

$$NS(A) = \left\{ H : V \times V \to \mathbb{C} \;\middle|\; \begin{array}{l} H \text{ Hermitian} \\ \mathrm{Im}\, H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$\mathcal{P}(\Lambda) = \left\{ (H, \chi) \;\middle|\; \begin{array}{l} H \in NS(A) \\ \chi : \Lambda \to S^1 \text{ semicharacter w.r.t. } H, \text{i.e.,} \\ \chi(\lambda + \mu) = \chi(\lambda)\chi(\mu)\exp(\pi i\, \mathrm{Im} H(\lambda, \mu)) \\ \qquad \forall \lambda, \mu \in \Lambda \end{array} \right\}$$

1. Cohomology of abelian varieties (Betti & Hodge)

Thm. [BL04, Thm 1.4.1 b)]
We have

$$
\begin{array}{rclcl}
\Omega & := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \overset{V^*}{=} & = H^{1,0}(A) & \cong \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) & f\,dz \\
\bar{\Omega} & := \mathrm{Hom}_{\bar{\mathbb{C}}}(V,\mathbb{C}) & = H^{0,1}(A) & \cong \mathrm{Hom}_{\mathbb{R}}(V,i\mathbb{R}) & \bar{f}\,d\bar{z} \\
\Omega\oplus\bar{\Omega} & & = H^1(A;\mathbb{C}) & = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) &
\end{array}
$$

Proof.

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \\
l & \longmapsto & \mathrm{Re}\, l \\
k(-) - ik(i-) & \longleftarrow & k
\end{array}
\qquad
\begin{array}{l}
dz = dx + i\,dy \longmapsto dx \\
i\,dz = -dy + i\,dx \longmapsto -dy
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\bar{\mathbb{C}}}(V,\mathbb{C}) & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,i\mathbb{R}) \\
l & \longmapsto & i\,\mathrm{Im}\, l \\
-k(i-) + ik(-) & \longleftarrow & ik
\end{array}
\qquad
\begin{array}{l}
d\bar{z} = dx - i\,dy \longmapsto -i\,dy \\
i\,d\bar{z} = dy + i\,dx \longmapsto i\,dx
\end{array}
$$

$$
\begin{array}{ccc}
\Omega\oplus\bar{\Omega} & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \\
(f\,dz, \bar{g}\,d\bar{z}) & \longmapsto & ? \\
 & & \\
(dz, -i\,d\bar{z}) & \longleftarrow & dz = dx + i\,dy \\
(i\,dz, i\,d\bar{z}) & \longleftarrow & i\,dz = i\,dx - dy \\
(dz, d\bar{z}) & \longleftarrow & d\bar{z} = dx - i\,dy \\
(i\,dz, -i\,d\bar{z}) & \longleftarrow & i\,d\bar{z} = i\,dx + dy
\end{array}
$$

$$
\begin{array}{ccc}
((f_1 + i f_2)\,dz, 0) & \longmapsto & f_1\,dx - f_2\,dy \\
(0, (g_1 - i g_2)\,d\bar{z}) & \longmapsto & -i(g_1\,dy + g_2\,dx)
\end{array}
\qquad f_1, f_2, g_1, g_2 \in C^\infty(A;\mathbb{R})
$$

Cor. $H^q(A;\Omega_A^p) \cong \Lambda^p\Omega \otimes \Lambda^q\overline{\Omega}$ $\qquad \Omega_A^p := \mathrm{Alt}^p\,\Omega_A$

Proof Sketch

$$
\begin{aligned}
H^q(A;\Omega_A^p) &\cong H_{\bar\partial}^{p,q}(A) \\
&= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V/\Lambda\}/\sim \\
&= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } \Lambda\}/\sim \\
&= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } V\} \\
&= \Lambda^p\Omega \otimes \Lambda^q\overline{\Omega}
\end{aligned}
$$

Another proof, though essentially the same:

<u>Step 1</u> Ω_A is a free $\mathcal{O}_A$-module with rank n, so

$$\Omega_A \cong \mathcal{O}_A \otimes_{\mathbb{C}} V^*$$

$$\Rightarrow \Omega_A^p = \Lambda^p\Omega_A = \mathcal{O}_A \otimes_{\mathbb{C}} \Lambda^p\Omega$$

<u>Step 2</u> By Dolbeault resolution,

$$H^q(A;\mathcal{O}_A) \cong H^q(\mathcal{A}^{0,\cdot}_{A\times\mathbb{C}}(A)) \cong H_{\bar\partial}^{0,q}(A) \cong \Lambda^q\overline{\Omega}$$

$\uparrow$ trivial l.b. over A

Lemma [BL04, Prop 2.1.6]

Let

$$NS(A) := \mathrm{Pic}(A)/\mathrm{Pic}^{\circ}_{\mathrm{red}}(A) \;=\!=\; H^2(A;\mathbb{Z}) \cap H^{1,1}(A)$$

$$NS'(A) := \left\{ \omega : V \times V \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \omega \ \mathbb{R}\text{-bilinear alternating form} \\ \omega(ix, iy) = \omega(x,y) \\ \omega(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$NS''(A) = \left\{ H : V \times V \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} H \ \text{Hermitian} \\ \\ \mathrm{Im}\, H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

↑imaginary part

Then

$$NS(A) \cong NS'(A) \cong NS''(A).$$

As a reminder, H Hermitian:

$$H(au, bv) = \bar{a}b\, H(u,v) \quad + \mathbb{R}\text{-linear}$$

$$H(u,v) = \overline{H(v,u)}$$

crspds to the matrix M s.t. $M^H = M$

Hint. Consider the ambient spaces:

$$\begin{array}{ccccccc} & & NS(A) & & NS'(A) & & NS''(A) \\ & & \cap & & \cap & & \cap \\ H^1(\mathcal{O}_A^*) & \xrightarrow{c_1} & H^2(A;\mathbb{Z}) & \hookrightarrow & H^2(A;\mathbb{R}) & \hookrightarrow & H^2(A;\mathbb{C}) \end{array}$$