Eine Woche, ein Beispiel

2.4 exercise sheet for six functors

Ref: from [23.11.19]

Review: For X sm mfld, $f: X \longrightarrow pt$, compute

$Rf_* \mathcal{F} =$ $\qquad$ $f^* \mathbb{Q} =$

$Rf_! \mathcal{F} =$ $\qquad$ $f^! \mathbb{Q} =$

$f_! \dashv f^!$ $\approx$ Poincaré duality

six fctor formalism $\approx$ cohomology theory in other categories

Proper base change formula:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^* Rg_! \cong Rg'_! f'^*$$

$$g^! Rf_* \cong Rf'_* g'^!$$

[Hint: Yoneda + adjunction]

Today: six functor (between spaces, $X \to Y$) + homological algebra (over one space, over A) $\Rightarrow$ algebraic topology (classical)

six functor	homological algebra	algebraic topology
sheaf	module	cohomology
	$\Downarrow$	
$f^* \dashv f_*$, $f_! \dashv f^!$	rep theory	
$\otimes \dashv$ Hom	Tor, Ext	
	derived category	
	LES	
	spectral seq	
	acyclic resolution	

characteristic class

vector bundle

$\pi_n(X, \cdot)$

spectra

cobordism

slogan:

SES	induces	LES,
filtration	induces	spectral sequence.

To expend a little bit,

$$\begin{tikzcd}
 & \text{SES} \arrow[r] & \text{LES} & \\
 & \text{filtration} \arrow[r] & \text{spectral seq} \arrow[dr] & \\
\text{resolution} \arrow[ur, dashed] \arrow[rrr, "\text{prefer acyclic}"'] & & & \text{comparison between sheaf cohomology \& other cohomology}
\end{tikzcd}$$

Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh".
Let us leave spectral seq in other people's notes.

1. open-closed formalism
2. open cover
3. filtrations from chain complex
4. filtration by $\mathcal{H}^i(\mathcal{F}^\cdot)$
5. filtration by $\mathcal{F}^\cdot$
6. Hodge related filtration

We will assume the Poincaré lemma:

$$H^\cdot(\mathbb{R}^n;\underline{\mathbb{Q}}) \cong \mathbb{Q}$$

Since $\mathbb{R}^n$ is contractible, you know how to prove it.

Methods to construct SES: $\begin{cases} \text{check by stalks} \\ \text{filtration by } \mathcal{H}^i(\mathcal{F}) \\ \text{filtration by } \mathcal{F}^i \end{cases}$

method	spectral seq	LES	cohomology/resolution
check by stalks	… for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech (open cover)
			closed cover
	coefficient		—
filtration by $\mathcal{H}^i(\mathcal{F})$	Grothendieck		
	Leray-Serre	Gysin	Euler class
			Hodge-Tate
Hodge related			de Rham
filtration by $\mathcal{F}^i$	Hodge-de Rham		Hodge-de Rham $H^q(X,\Omega^p) \Rightarrow H^{p+q}(X)$
need resolution			Dolbeault $H^q(X,\Omega^p) = H^{p,q}(X)$
to get "another" complex	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
which	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
I don't know	Eilenberg-Moore		
	Green		for Koszul cohomology
	⋮		⋮

For more spectral sequences, see:
https://en.wikipedia.org/wiki/Spectral_sequence
https://github.com/CubicBear/SpectralSequences/blob/main/SpecralSequences.pdf

mixed Hodge theory = using two filtrations of $Rj_*\mathbb{Q}_U$ in the same time
to get two filtrations of $H^\cdot(U;\mathbb{Q})$

1. open-closed formalism

related: comparison of $j_!$ & j_*
one-point compactification.

Observe the following pictures:

$$\begin{tikzcd} Z \arrow[r, "i"] & X & \mathcal{U} \arrow[l, hook', "j"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{D}(Z) \arrow[r, "i_* = i_!"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

Black box:

0. We assume some nice conditions.
   e.g. in the category $\mathrm{Haus}^{\mathrm{loc.\,cpt}}$, and $Z \subset X$ is loc. contractable.
   Under these conditions,
1. $i_* = i_!$, $j^* = j^!$
2. $j_!, i^*, j^*, i_*$ are exact.

Ex. 1. Shows that

$$i^* i_* = i^! i_* = \mathrm{Id}_{\mathcal{D}(Z)} \qquad j^* j_! = j^* Rj_* = \mathrm{Id}_{\mathcal{D}(\mathcal{U})}$$

$$i^* j_! = 0, \qquad j^* i_* = 0, \qquad i^! Rj_* = 0$$

—— : base change
〰〰 : check stalkwise.

2. (for category fans)
   $i_*, j_*, j_!$ are fully faithful, and
   i_*, j^*, j_* preserve injectives.

   $i^!$, Rj_* are not defined for the abelian category

3. One has SES

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_! i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

You might prefer

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0$$

Well, writing every term by left adjoint fctors is better for me.

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

Define $R\Gamma(X, Z; \mathbb{Q}) := R\Gamma(X; j_! \underline{\mathbb{Q}})$

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 special cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the triangle

$$j_! \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* i^* Rj_* \mathcal{F} \xrightarrow{+1}$$

which measures the difference between $j_! \mathcal{F}$ & $j_* \mathcal{F}$.

3. Shows that

$$H_c^\cdot(X) \cong H^\cdot(\bar{X}, \{\infty\}; \mathbb{Z})$$

for one pt compactification $\iota: X \hookrightarrow \bar{X}$.

Try to compute $H_c^\cdot(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X\mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{D^b(X;\mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}})$$

$$R\pi_{X*}\mathbb{D}_X\mathcal{F}^\cdot = \mathrm{Hom}_{D^b(\{*\};\mathbb{Q})}(\pi_{X,!}\mathcal{F}^\cdot, \mathbb{Q}) = H_c^\cdot(X;\mathcal{F}^\cdot)^*$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad \mathbb{D}_X(\mathcal{F}^\cdot[n]) = (\mathbb{D}_X\mathcal{F}^\cdot)[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

"left adjoint can be pulled out" $f: Y \longrightarrow X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then

$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructable setting,

$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad (2)$$

for $\mathcal{F}^\cdot \in \mathcal{D}^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".

E.g. For a finite graph (as a topo space) X.

$$sk_0 X \xhookrightarrow{\ i\ } X \xleftarrow{\ j\ } X - sk_0 X \qquad \text{1-cells}$$

$$0 \longrightarrow j_! j^! \mathbb{Q}_X \longrightarrow \mathbb{Q}_X \longrightarrow i_* i^* \mathbb{Q}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \mathbb{Q}_{X - sk_0 X} \longrightarrow \mathbb{Q}_X \longrightarrow i_! \mathbb{Q}_{sk_0 X} \longrightarrow 0$$

Take $R\pi_{X,!}$

$$\overset{=\oplus_e \mathbb{Q}}{H^1_c(X - sk_0 X)} \longrightarrow H^1_c(X) \longrightarrow \overset{=0}{H^1_c(sk_0 X)} \xrightarrow{+1}$$

$$0 \longrightarrow \overset{=0}{H^0_c(X - sk_0 X)} \longrightarrow H^0_c(X) \longrightarrow \underset{=\oplus_v \mathbb{Q}}{H^0_c(sk_0 X)} \longrightarrow H^1_c(X - sk_0 X)$$

This calculates the sheaf cohomology as simplicial cohomology.

Ex. Shows that

$$H^i_c(\mathbb{R}) = \begin{cases} \mathbb{Q} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial/cellular cohomology, using the following stratification:

$$\emptyset \subset sk^0 X \subset sk^1 X \subset \cdots\ sk^n X = X \quad \text{} \text{filtration}$$

$$0 \subset i_* \mathbb{Q}_{sk^0 X} \Leftarrow i_* \mathbb{Q}_{sk^1 X} \Leftarrow \cdots\ \mathbb{Q}_{sk^n X} = \mathbb{Q}_X$$

Ex. derive the Wang LES for the cpt supp version. over S^1

2. open cover

Ex. For an open cover $X = \mathcal{U}_1 \cup \mathcal{U}_2$, deduce the SES

$$\begin{array}{l} 0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus j_! \underline{\mathbb{Q}}_{\mathcal{U}_2} \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \longleftarrow 0 \\ \underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \xrightarrow{+1} \end{array} \quad (3)$$

⚠ We omit the derived symbol and some subscripts in this section. $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$

(3) works for general sheaf

and, induce from (3) the MV sequence:

$$\begin{array}{l} \xleftarrow{+1} H^k_c(X) \longleftarrow H^k_c(\mathcal{U}_1) \oplus H^k_c(\mathcal{U}_2) \longleftarrow H^k_c(\mathcal{U}_1 \cap \mathcal{U}_2) \\ H^k(X) \longrightarrow H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \longrightarrow H^k(\mathcal{U}_1 \cap \mathcal{U}_2) \xrightarrow{+1} \end{array}$$

Hint: Apply $R\pi_{X,!}$ & $R\pi_{X,*}$, see [StackProject, 01E9]

Ex. Derived the Wang LES. over S^1

Ex. For an open cover $X = \bigcup_{i \in \Lambda} \mathcal{U}_i$, Λ finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{\mathcal{U}_i} \longleftarrow \bigoplus_{i<j} j_! \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longleftarrow \cdots \; j_! \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longleftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i} \longrightarrow \bigoplus_{i<j} Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow \cdots \; Rj_* \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longrightarrow 0$$

When $\{\mathcal{U}_i\}_{i \in \Lambda}$ is a good cover, $H^\cdot(\mathcal{U}_{i_1 \cdots i_k}) = H^0(\mathcal{U}_{i_1, \cdots i_k})$,

acyclic in AG

one can compute $H^\cdot(X)$ by applying $R\pi_{X,*}$:

$$0 \longrightarrow \bigoplus_i \Gamma(\mathcal{U}_i) \xrightarrow{d^1} \bigoplus_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j) \xrightarrow{d^2} \cdots \; \Gamma(\bigcap_i \mathcal{U}_i) \longrightarrow 0$$

$$\Downarrow \mathrm{Ker}/\mathrm{Im}$$

$$H^0(X) \qquad H^1(X) \qquad \cdots \qquad H^{\#\Lambda - 1}(X)$$

Rmk. When X is paracompact & Hausdorff, "limited" Čech = sheaf

e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$\mathcal{F} \longrightarrow \bigoplus_i Rj_* \mathcal{F}|_{\mathcal{U}_i} \qquad \mathcal{F} \longrightarrow \bigoplus_{x \in X} \mathcal{F}_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

$\# \Lambda = 3$ case:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & \\
 & & \mathcal{F}_0 \arrow[ur] \arrow[dr] & & & & \mathcal{F}_2 \arrow[ur] \arrow[dr] & & \\
0 \arrow[r] & \mathbb{Q}_X \arrow[ur] \arrow[rr] & & \bigoplus_i Rj_*\mathbb{Q}_{U_i} \arrow[rr, "d^1"] \arrow[dr] & & \bigoplus_i Rj_*\mathbb{Q}_{U_i\cap U_j} \arrow[ur] \arrow[rr, "d^2"] & & Rj_*\mathbb{Q}_{\bigcap_i U_i} \arrow[r, "d^3"] & 0 \\
 & & & & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & & \\
 & & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

$$\mathcal{F}_2 = Rj_*\mathbb{Q}_{\bigcap_i U_i} \quad\Rightarrow\quad H^\bullet(\mathcal{F}_2) = \ker d^3$$

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{F}_1) \arrow[r] & \bigoplus_{i<j}\Gamma(U_i\cap U_j) \arrow[r, "d^2"] & H^0(\mathcal{F}_2) \arrow[r] & H^1(\mathcal{F}_1) \arrow[r] & 0 \arrow[r] & \overset{=0}{H^1(\mathcal{F}_2)} \arrow[r, "+1"] & {}
\end{tikzcd}$$

$$\Rightarrow\quad H^i(\mathcal{F}_1) = \begin{cases} \ker d^3/\operatorname{Im} d^2, & i=1 \\ \ker d^2, & i=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{F}_0) \arrow[r] & \bigoplus_i\Gamma(U_i) \arrow[r, "d^1"] & H^0(\mathcal{F}_1) \arrow[r] & H^1(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & H^1(\mathcal{F}_1) \arrow[r] & H^2(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & \overset{=0}{H^2(\mathcal{F}_1)} \arrow[r, "+1"] & {}
\end{tikzcd}$$

$$\Rightarrow\quad H^i(X) = H^i(\mathcal{F}_0) = \begin{cases} \ker d^3/\operatorname{Im} d^2 & i=2 \\ \ker d^2/\operatorname{Im} d^1 & i=1 \\ \ker d^1 & i=0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When $\{U_i\}_{i\in\Lambda}$ is not a good cover,
one needs Čech-to-derived functor spectral seq to compute $H^\cdot(X)$.

Rmk. Closed cover can be computed in a similar way.

Ex. Use the closed cover, try to compute

$$H_\cdot^{BM}(Z;\mathbb{Q}) \qquad Z := \{(z_1,z_2,z_3)\in\mathbb{C}^3 \mid z_1z_2z_3=0\} =$$ 

$$\text{or} \quad Z = \{z_1z_2z_3(z_1+z_2+z_3)=0\}$$

In the first case, $\mathbb{C}^3 - Z \cong \mathbb{C}^\times\times\mathbb{C}^\times\times\mathbb{C}^\times$ You can use this fact to double check results.

closed cover of Z $\approx$ "excessive cover" of $\mathbb{C}^3-Z$

$\underset{\text{no relation}}{\neq}$ open cover of $\mathbb{C}^3-Z$

Rmk. stratification & open cover are two main tools to extract topological information.
They appear with different names in different fields.
Once you realize them, you can apply the six-functor machine to analyze structures.

stratification with extra properties $\begin{cases} \text{CW cplx} \\ \text{triangulization} \\ \text{dessin d'enfant} \\ \text{affine paving} \\ \text{Whitney stratification} \\ \vdots \end{cases}$

Q. How to construct stratifications?
A. For me, there are two efficient methods: $\begin{cases} \text{orbit of gp action} \\ \text{Morse theory} \end{cases}$

That's why some geometrical problems are finally reduced to combinatorical / analytic problems.
Other fields come to geometry by providing stratifications.

In fact, there is only one method:
find a fct $f: X \longrightarrow Y$, and get stratification of X from Y.
get better stratification by analyzing f

E.g.
1. Morse theory $\quad f: X \longrightarrow \mathbb{R}$
2. tessellation $\quad f: \mathcal{H} \longrightarrow \mathcal{H}/\Gamma$
3. semi-continuous fct $\quad f: X \longrightarrow \mathbb{N}_{\geq 0}$ $\quad$ e.g. $f(p) = \dim T_pX$
4. my master thesis $\quad f: Gr(X) \longrightarrow Gr(S) \times Gr(X/S)$
5. orbit of gp action $\quad f: X \longrightarrow X/G$

3. filtrations from chain complex [Stack Project, 0118]

Lots of filtrations are obtained just from the naive complex.

"chopping pork ribs"

Consider a chain complex $C^\cdot$:

$$\cdots \xrightarrow{d^{-2}} C^{-2} \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^{0}} C^{0} \xrightarrow{d^{1}} C^{1} \xrightarrow{d^{2}} C^{2} \xrightarrow{d^{3}} \cdots$$

One can make a "stupid" truncation

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] & C^{-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \sigma_{\geq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \sigma_{\leq -1} C^\cdot \longrightarrow 0$$

One can also make a canonical truncation

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & \operatorname{Ker} d^1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^0 \arrow[r, "d^1"] & C^1 \arrow[r, "d^2"] & C^2 \arrow[r, "d^3"] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \tau_{\leq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \tau_{\geq 1} C^\cdot \longrightarrow 0$$

Using these truncations, one can easily construct filtrations:

$$0 \subset \cdots \subset \sigma_{\geq 1} C^\cdot \overset{C^\cdot[-1]}{\subset} \sigma_{\geq 0} C^\cdot \overset{C^0}{\subset} \sigma_{\geq -1} C^\cdot \subset \cdots \subset C^\cdot$$

$$0 \subset \cdots \subset \tau_{\leq -1} C^\cdot \overset{\mathcal{H}^0(C^\cdot)}{\subset} \tau_{\leq 0} C^\cdot \overset{\mathcal{H}^1(C^\cdot)[1]}{\subset} \tau_{\leq 1} C^\cdot \subset \cdots \subset C^\cdot$$

Rmk. 1. These two filtrations have opposite directions!
(a striking feature for me)

2. The "stupid" truncation extracts pieces of the chain cplx, while the canonical truncation extracts cohomology. (Ker/Im)
Therefore, only the canonical truncation can be defined on the derived category.
This information is culmulated in the standard/natural t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$.
One has adjoint fctors:

$$\begin{tikzcd} \mathcal{D}_{\leq 0} \arrow[r, bend left, "\iota_{\leq 0}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\tau_{\leq 0}"] \arrow[r, bend left, "\tau_{\geq 1}"] \arrow[r, phantom, "\perp"] & \mathcal{D}_{\geq 1} \arrow[l, bend left, "\iota_{\geq 1}"] \end{tikzcd}$$

where $\iota_{\leq 0}, \iota_{\geq 1}$ are natural inclusions.

The following notations are from: https://ncatlab.org/nlab/show/t-structure

$\mathcal{D}_{\leq 0}$: t-co-connective objects
$\mathcal{D}_{\geq 0}$: t - connective objects
$\tau_{\geq 0}$: connective cover

Let's apply these filtrations!

4. filtration by $\mathcal{H}^i(\mathcal{F})$

(+ B simply connected to simplify the argument)

Ex. Suppose that $\pi: E \longrightarrow B$ is an oriented S^k-bundle.
Analyze $R\pi_* \mathbb{Q}_E$, and apply $R\pi_{B,*}$ to get the Gysin sequence:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[+1]{eu_\pi \wedge}$$

Hint: consider the easy case $(B = \{*\},\ E \cong S^k \times B)$ to see the shape of $R\pi_* \mathbb{Q}_E$.

Q: Why does $\pi_*: \mathcal{D}^b(E) \longrightarrow \mathcal{D}^b(B)$ takes injective objects to π_{B*}-acyclic objects?
A: [Vo02, p195] By definition, π_* takes flasque sheaves to flasque sheaves.
Similarly, $\pi_!$ takes c-soft sheaves to c-soft sheaves.

Rmk. 1. Here we can't use the "stupid" truncation, because $R\pi_* \mathbb{Q}_E$ lies in the derived category.

2. You can generalize it to fiber bundle. then you will get the Leray-Serre spectral sequence.
Think how the following conditions simplify the final results.
① π is oriented S^k-bundle
② B is simply-connected
③ (Leray-Hirsch) $H^*(E;\mathbb{Q}) \longrightarrow H^*(F;\mathbb{Q})$ is surjection
④ $\pi_1(B)$ acts on $H^*(F)$ trivially.

3. This is also a special case of Grothendieck-Serre spectral sequence.

⚠ $R^i\pi_* \mathbb{Q}_E = \mathcal{H}^i(R\pi_* \mathbb{Q}_E)$
$\neq$ the i-th term in the "chain cplx" $R\pi_* \mathbb{Q}_E$ (undefined)

5. filtration by $\mathcal{F}^\cdot$

For a single sheaf F, we have no way to produce filtrations.
Since we only care about H^i(F), we may replace F by another complex C^*, which is usually achieved by resolution:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \cdots \quad \text{exact}$$

i.e.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & C^0 \arrow[r, "d^1"] & C^1 \arrow[r, "d^2"] & C^2 \arrow[r, "d^3"] & \cdots \end{tikzcd} \quad \text{iso in } \mathcal{D}(X)$$

Then, one can use "stupid" truncation to get filtrations, and finally spectral sequences.
(The canonical truncation won't give you anything new)

E.p. When $C^\cdot$ is Γ-acyclic, the spectral seq deg, and

$$H^i(X;\mathcal{F}) \cong \mathcal{H}^i(\Gamma(C^\cdot))$$

E.g. The spectral seqs in Section 1,2 are special cases.

Coefficient spectral seq

E.g. For $r \in \mathbb{Z}_{>0}$, consider the SES

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times r} \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow 0$$

which induces the iso

$$\underline{\mathbb{Z}/r\mathbb{Z}} \simeq [\longrightarrow \underset{-1}{\underline{\mathbb{Z}}} \xrightarrow{\times r} \underset{0}{\underline{\mathbb{Z}}} \longrightarrow]$$

the triangle

$$\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}[1] \xrightarrow{+1}$$

and the LES

$$H^k(X;\mathbb{Z}) \longrightarrow H^k(X;\mathbb{Z}/r\mathbb{Z}) \longrightarrow H^{k+1}(X;\mathbb{Z}) \xrightarrow[+1]{\times r}$$

Ex. Compute $H^k(\mathbb{RP}^2;\mathbb{Z}/8\mathbb{Z})$ in this way.

A: $$H^k(\mathbb{RP}^2;\mathbb{Z}/8\mathbb{Z}) = \begin{cases} \mathbb{Z}/8\mathbb{Z}, & k=0 \\ \mathbb{Z}/2\mathbb{Z}, & k=1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Rmk. In general, one can get coefficient spectral seq [FF16, 20.5.B, p318] through the coefficient seq: G abelian

$$\cdots \xrightarrow{\partial_2} \underline{G_1} \xrightarrow{\partial_1} \underline{G_0} \xrightarrow{\varepsilon} \underline{G} \longrightarrow 0$$

Q: Can we recover the inflation-restriction seq in this way?

$$0 \longrightarrow H^1(G/H, M^H) \xrightarrow{\text{Inf}} H^1(G,M) \xrightarrow{\text{Res}} H^1(H,M)$$

A: I think no. We haven't introduce $\overset{L}{\otimes}$ & $R\,\underline{\mathrm{Hom}}$.

Singular cohomology

We present a defective proof in [Vo02, Theorem 4.47].
The bug has been discussed in the stack exchange:
https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology

[Stackexchange]	$\mathcal{S}^k$	$\tilde{\mathcal{S}}^k$	$\mathcal{S}^k(X)_0$
[Vo02]	C^q_{sing}	$\mathcal{C}^q_{sing}$	$C^q_{sing}(\mathcal{U}, \mathbb{Z})_0$

$X \in \mathcal{T}op$.

Def The presheaf $\mathcal{S}^k \in Psh(X)$ is defined as

$$\mathcal{S}^k(\mathcal{U}) = \text{singular } k\text{-cochains on } \mathcal{U} = \mathrm{Hom}_{Abel}(\mathrm{Sing}_k(\mathcal{U}); \mathbb{Z})$$

where

$$\mathrm{Sing}_k(\mathcal{U}) = \text{singular } k\text{-chains on } \mathcal{U} = \langle \sigma : \Delta^k \longrightarrow \mathcal{U} \rangle_{\text{free. Abel}}$$

and $\tilde{\mathcal{S}}^k := (\mathcal{S}^k)^{sh}$ is called the sheaf of singular cochains.

Def

$$H^k_{sing}(X; \mathbb{Z}) := \mathcal{H}^k(\mathcal{S}^\cdot(X))$$

Thm For X loc contractible,

$$H^k(X; \mathbb{Z}) \cong H^k_{sing}(X; \mathbb{Z})$$

Sketch of the wrong proof: Still hard. I don't check details.

1. Construct a complex of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\mathcal{S}}^0 \longrightarrow \tilde{\mathcal{S}}^1 \longrightarrow \tilde{\mathcal{S}}^2 \longrightarrow \cdots \qquad (\star)$$

2. Show that $(\star)$ is exact when X is loc contractible. $\Rightarrow \mathbb{Z} \simeq \tilde{\mathcal{S}}^\cdot$
3. Show that $\tilde{\mathcal{S}}^k$ is flabby, then

$$H^k(X; \mathbb{Z}) \cong \mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X))$$

Reduced to show: $\mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X)) \cong \mathcal{H}^k(\mathcal{S}^\cdot(X))$

4. Conside the exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^\cdot(X)_0 \longrightarrow \mathcal{S}^\cdot(X) \longrightarrow \tilde{\mathcal{S}}^\cdot(X) \longrightarrow \mathcal{S}^\cdot(X)^0 \longrightarrow 0$$

shows that:

① $\mathcal{S}^\cdot(X)^0 = 0$ ⚠ This is wrong. We ignore this bug...

② $\mathcal{S}^\cdot(X)_0 = \left\{ \alpha \in \mathcal{S}^\cdot(\mathcal{U}) \;\middle|\; \begin{array}{l} \exists \text{ open covering } \{\mathcal{U}_i\}_i \text{ of } \mathcal{U} \text{ s.t.} \\ \alpha|_{\mathcal{U}_i} = 0 \quad \forall i \end{array} \right\}$

③ $\mathcal{H}^k(\mathcal{S}^\cdot(X)_0) = 0$

Finally,

$$H^k(X; \mathbb{Z}) \cong \mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X)) \cong \mathcal{H}^k(\mathcal{S}^\cdot(X)) \overset{\text{def}}{=} H^k_{sing}(X; \mathbb{Z}).$$

Q. Can one shows $\mathcal{H}^k(\mathcal{S}^\cdot(X)^0) = 0$ to replace the bug in 4.①?

6. Hodge related filtration

Smooth de Rham resolution

The classical story is the real de Rham cohomology.

Thm For X mfld, one has iso

$$H^k(X;\underline{\mathbb{R}}) \cong H^k_{dR}(X;\mathbb{R}) := \mathcal{H}^k(\Omega_X(X))$$

Here,

$$\Omega_X^k = \text{sheaf of smooth } k\text{-forms with coefficient } \mathbb{R}$$

https://en.wikipedia.org/wiki/De_Rham_cohomology#Proof

Sketch of proof

1. Construct a cplx of sheaves

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega_X^0 \xrightarrow{d^1} \Omega_X^1 \xrightarrow{d^2} \Omega_X^2 \xrightarrow{d^3} \cdots \qquad (*)$$

2. Show that $(*)$ is exact by Poincaré lemma. [Cl17, Lec1, Lemma 5]

3. Show that Ω_X^i is Γ-acyclic [Cl17, Lec1, Lemma 4], then

$$H^k(X;\underline{\mathbb{R}}) \cong \mathcal{H}^k(\Omega_X(X)) \stackrel{\text{def}}{=\!=} H^k_{dR}(X;\mathbb{R})$$

POU $\Rightarrow$ fine $\xRightarrow{\text{paracpt}}$ acyclic

Rmk. You can change coefficient from $\mathbb{R}$ to $\mathbb{C}$, where

$$\begin{aligned}\Omega^k_{X,\mathbb{C}} &= \text{sheaf of smooth } k\text{-forms with coefficient } \mathbb{C}\\ &= \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} = \Omega_X^k \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}}\end{aligned}$$

↑ it can be viewed as $\underline{\mathbb{R}}$-module

then

$$H^k(X;\underline{\mathbb{C}}) \cong H^k_{dR}(X;\mathbb{C}) := \mathcal{H}^k(\Omega_{X,\mathbb{C}}(X)).$$

Now we step into the world of cplx mfld.

Hereafter, X: cplx mfld,

$$\Omega_X^k = \text{sheaf of } \underline{\text{holomorphic}} \text{ } k\text{-forms} \qquad = \Omega_X^{k,0}$$

$$\text{"="} \ \{ f dz_1 \wedge \cdots \wedge dz_k \mid f : X \longrightarrow \mathbb{C} \text{ holomorphic} \}$$

↑ no $d\bar{z}$

while

$$\mathcal{A}^k_{sm} := \Omega^k_{X_{sm}} = \text{sheaf of } \underline{\text{smooth}} \text{ } k\text{-forms}$$

$$\text{"="} \ \{ f dx_1 \wedge \cdots \wedge dx_k \mid f : X \longrightarrow \mathbb{R} \text{ smooth} \}$$

↑ $d\bar{z}$ is allowed

$$\mathcal{A}^k := \Omega^k_{X_{sm},\mathbb{C}} \text{"="} \{ f dx_1 \wedge \cdots \wedge dx_k \mid f : X \longrightarrow \mathbb{C} \text{ smooth} \} = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$$

Ω: holomorphic $\qquad \Omega^k = \Omega^{k,0}$

$\mathcal{A}$: C^∞ $\qquad \mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}$

This abuse of notation confuses me a lot.

For example, we still have the resolution (called the holomorphic de Rham resolution)

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \Omega_X^0 \xrightarrow[=\partial]{d^1} \Omega_X^1 \xrightarrow[=\partial]{d^2} \Omega_X^2 \xrightarrow[=\partial]{d^3} \cdots \qquad (**)$$

but $\Omega_X^\cdot$ is no longer Γ-acyclic, and

$$\dim_{\mathbb{C}} \Gamma(X; \Omega_X^k) < +\infty \qquad \text{for } X \text{ proj.}$$

$$H^q(X; \Omega_X^p) \longrightarrow H^q(X; \Omega_X^{p+1}) \text{ is 0-map} \qquad \text{for } X \text{ proj.}$$

Find reference for the 0-map. (E_1-degeneration)

Do we get the 0-map by the dim argument + Hodge decomposition?

Do we have any elegant proof to replace it?

[Vo 02, p214]

1. Let S be a compact complex surface.
   (a) Show that the Frölicher spectral sequence of S degenerates at E_3 for degree reasons. What are the possibly non-zero differentials d_2?

Holomorphic de Rham resolution & Hodge-Tate

For simplicity, let's consider a proj curve $X/\mathbb{C}$.

cpt kähler is enough

Where do we use kähler condition?

Hodge de-Rham

$$\underline{\mathbb{C}} \simeq [\underset{0}{\Omega^0} \longrightarrow \underset{1}{\Omega^1}]$$

"stupid" filtration degenerates to the triangle

$$\Omega^1[-1] \longrightarrow \underline{\mathbb{C}} \longrightarrow \Omega^0 \xrightarrow{+1}$$

$$\begin{tikzcd}
H^{-1}(X;\Omega^1) \overset{=0}{} \arrow[r] & H^0(X;\mathbb{C}) \arrow[r] & H^0(X;\Omega^0) \arrow[dll, out=-10, in=170] \\
H^0(X;\Omega^1) \arrow[r] & H^1(X;\mathbb{C}) \arrow[r] & H^1(X;\Omega^0) \arrow[dll, out=-10, in=170] \\
H^1(X;\Omega^1) \arrow[r] & H^2(X;\mathbb{C}) \arrow[r] & H^2(X;\Omega^0) \overset{=0}{}
\end{tikzcd}$$

$\rightsquigarrow$ Hodge filtration:

$$0 \longrightarrow H^0(X;\Omega^1) \longrightarrow H^1(X;\mathbb{C}) \longrightarrow H^1(X;\Omega^0) \longrightarrow 0$$

Hodge Tate

similar to Euler class (I'm foolish in this field)

$$\nu : \underset{\hat{\mathcal{O}}_X}{X_{\text{proét}}} \longrightarrow X_{\text{ét}}$$

$$\begin{array}{ccccc}
R^0\nu_*\hat{\mathcal{O}}_X & \longrightarrow & R\nu_*\hat{\mathcal{O}}_X & \longrightarrow & R^1\nu_*\hat{\mathcal{O}}_X[-1] \xrightarrow{+1} \\
\wr\| & & & & \wr\| \\
\Omega^0_X & & & & \Omega^1_X[-1]
\end{array}$$

$$\begin{tikzcd}
H^0(X;\Omega^0) \arrow[r] & H^0(X;\mathbb{C}) \arrow[r] & H^{-1}(X;\Omega^1) \overset{=0}{} \arrow[dll, out=-10, in=170] \\
H^1(X;\Omega^0) \arrow[r] & H^1(X;\mathbb{C}) \arrow[r] & H^0(X;\Omega^1) \arrow[dll, out=-10, in=170] \\
H^1(X;\Omega^0) \overset{=0}{} \arrow[r] & H^2(X;\mathbb{C}) \arrow[r] & H^2(X;\Omega^1)
\end{tikzcd}$$

$$\rightsquigarrow \quad 0 \longrightarrow H^1(X;\Omega^0) \longrightarrow H^1(X;\mathbb{C}) \longrightarrow H^0(X;\Omega^1) \longrightarrow 0$$

"Euler class of ν" is trivial?

Q. Combining these two filtrations, does it give Hodge decomposition? in the p-adic sense

Hodge decomposition in the algebraic surface case.

$$\chi(\mathcal{O}_X)\ \chi(\Omega_X)\ \chi(\omega_X) \qquad\qquad e := \chi_{top}(X)$$

$$\begin{array}{ccccc}
 & & 1 & & \\
 & q & & q & \\
p_g & & h^{1,1} & & p_g \\
 & q & & q & \\
 & & 1 & &
\end{array}
\qquad
\begin{array}{c}
b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0
\end{array}
\qquad
\begin{array}{ccccc}
 & & h^{2,2} & & \\
 & h^{1,2} & & h^{2,1} & \\
h^{0,2} & & h^{1,1} & & h^{2,0} \\
 & h^{0,1} & & h^{1,0} & \\
 & & h^{0,0} & &
\end{array}$$

p_a

$c_2^2 = e$

$c_1 = K^2 = 12\chi(\mathcal{O}_X) - e$

Dolbeault resolution [Vo02, p94]

For $\pi_E : E \longrightarrow X$ holo v.b., let

$$\mathcal{A}_E^{p,q} = \text{sheaf of } \underline{\text{smooth}} \ (p,q)\text{-forms} \quad \text{with coefficient as sections in } E$$

$$\mathcal{E} = \text{sheaf of holo sections of } E \ \triangleq \Omega_E^{0,0}$$

One can construct a cplx of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{A}_E^{0,0} \xrightarrow{\bar\partial} \mathcal{A}_E^{0,1} \xrightarrow{\bar\partial} \mathcal{A}_E^{0,2} \xrightarrow{\bar\partial} \cdots \qquad (***)$$

show that $(***)$ is exact, and $\mathcal{A}_E^{p,q}$ is π_*-acyclic.

$$\Rightarrow \qquad H^k(X;\mathcal{E}) \cong \mathcal{H}^k(\mathcal{A}_E^{0,\cdot}(X))$$

E.g. When $\mathcal{E} = \Omega_X^p$, $\mathcal{A}_E^{0,q} = \mathcal{A}^{p,q}$, one gets

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar\partial} \mathcal{A}^{p,1} \xrightarrow{\bar\partial} \mathcal{A}^{p,2} \xrightarrow{\bar\partial} \cdots \qquad (***)'$$

$$\Rightarrow \qquad H^q(X;\Omega_X^p) \cong \mathcal{H}^q(\mathcal{A}_E^{0,\cdot}(X)) \stackrel{\text{def}}{=\!=} H_{\bar\partial}^{p,q}(X)$$

Rmk. $(*)$, $(**)$ & $(***)'$ are related by the following double complex:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
0 \arrow[r] & \mathcal{A}^{0,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{1,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{2,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{3,1} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \cdots \\
0 \arrow[r] & \mathcal{A}^{0,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{1,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{2,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \mathcal{A}^{3,0} \arrow[u, "\bar\partial"] \arrow[r, "\partial"] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$