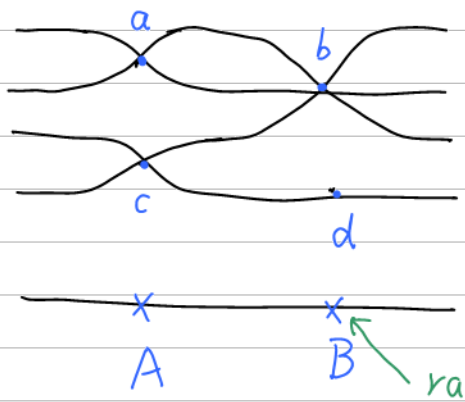
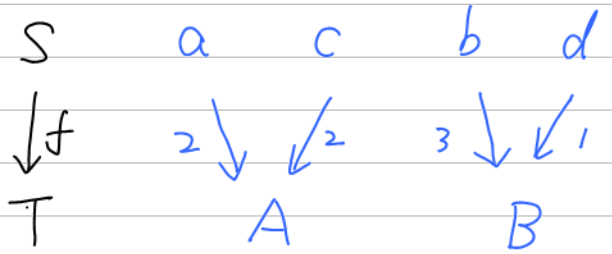
Belyi's Theorem.

1. Ramification
2. Irreducible algebraic curves
3. Statement of Belyi's theorem
4. A one-side proof: (a) ⇒ (b)

1. Ramification





ramification points

$\mathrm{Ram}(f) = \{a, b, c\}$
$\mathrm{Branch}(f) = \{A, B\}$

$f(\mathrm{Ram}(f)) = \mathrm{Branch}(f)$
$f^{-1}(\mathrm{Branch}(f)) \subseteq \mathrm{Ram}(f)$

$S \setminus \mathrm{Ram}(f) \to T \setminus \mathrm{Branch}(f)$ covering map

定义 B.3.1 设 $e \in \mathbb{Z}_{\geq 1} \sqcup \{\infty\}$.

- 若 e 有限, 按 $z \mapsto z^e$ 定义单位开圆盘到自身的连续满射 $f_e : \mathscr{D} \to \mathscr{D}$;
- 若 $e = \infty$, 按 $\tau \mapsto \exp(2\pi i \tau)$ 定义 $\mathscr{H} \sqcup \{\infty\}$ 到 $\mathscr{D}$ 的连续满射 f_e, 映 ∞ 为 0.

两种情形下都称 f_e 为 e 次**标准分歧复叠**.

定义 B.3.3 连续满射 $f : S \to T$ 具备以下性质时称为**分歧复叠**: 对每个 $t \in T$ 存在开邻域 $V \ni t$ 和 S 的一族无交开子集 $\{U_i\}_{i \in I}$ 使得 $f^{-1}(V) = \bigsqcup_{i \in I} U_i$, 而且对每个 $i \in I$, 皆存在 $e \in \mathbb{Z} \sqcup \{\infty\}$ 和从 $U_i \xrightarrow{f} V$ 到标准分歧复叠 f_e 的同胚, 使 t 对应到 $0 \in \mathscr{D}$.

定义–定理 B.3.4 设 $f : S \to T$ 是分歧复叠, 则对任意 $s \in S$ 及其邻域 U_1, 总存在开邻域 $U \ni s, U \subset U_1$ 使得 $f^{-1}(f(s)) \cap U = \{s\}$ 而 $U \smallsetminus \{s\} \xrightarrow{f} f(U \smallsetminus \{s\})$ 是复叠映射, 其次数 $e(s)$ 称为 f 在 s 处的**分歧指数**, 它只和 s 与 f 相关.

定义 B.3.5 设 $f : S \to T$ 为分歧复叠. 满足 $e(s) > 1$ 的点 $s \in S$ 称为**分歧点**. 全体分歧点构成 S 的子集 $\mathrm{Ram}(f)$.

Ex. 1) Calculate $\mathrm{Branch}(f)$, where
$f: \mathbb{PC}^1 \to \mathbb{PC}^1 \quad z \mapsto z^3 + \frac{1}{z^3}$

2) Suppose $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are maps between RSs, prove $\mathrm{Branch}(g \circ f) = \mathrm{Branch}(g) \cup g(\mathrm{Branch}(f))$

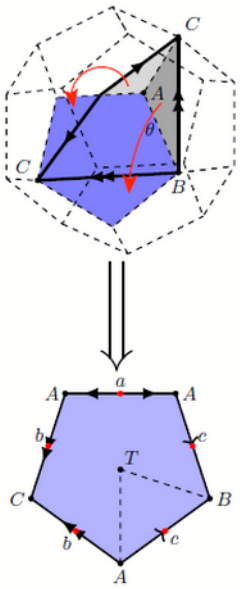


图 1.7

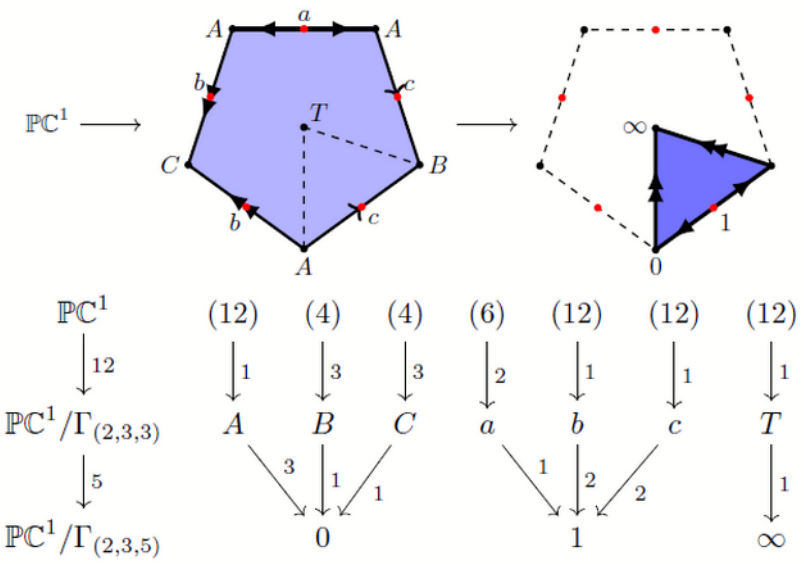


图 1.8　分歧点与分歧指标

(useless figure)

2. Irreducible algebraic curves.

Remark 1.94 We have shown the equivalence between the following classes of objects:

(1) Compact Riemann surfaces S.
(2) *Function fields in one variable* (i.e. finite extensions of $\mathbb{C}(X)$).
(3) Irreducible algebraic curves $C : F(X, Y) = 0$.

RS S ⟶ finite extensions of $\mathbb{C}(X)$
PET (primitive element theorem)
Irr alg curves $C : F(X,Y) = 0$

Theorem 1.86 *Let*

$$\begin{aligned} F(X,Y) &= p_0(X)Y^n + p_1(X)Y^{n-1} + \cdots + p_n(X) \\ &= q_0(Y)X^m + q_1(Y)X^{m-1} + \cdots + q_m(Y) \end{aligned}$$

be an irreducible polynomial. If $n \geq 1$ define

$$S_F^X = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_Y(x,y) \neq 0, p_0(x) \neq 0\}$$

and, similarly, if $m \geq 1$ set

$$S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0, F_X(x,y) \neq 0, q_0(y) \neq 0\}$$

Then:

(i) S_F^X *and* S_F^Y *are connected Riemann surfaces on which the coordinate functions* $\mathbf{x}$ *and* $\mathbf{y}$ *are holomorphic functions.*

(ii) *There exists a unique compact and connected Riemann surface* $S = S_F$ *that* contains S_F^X *and* S_F^Y.

(iii) *The coordinate functions* $\mathbf{x}$ *and* $\mathbf{y}$ *extend to meromorphic functions on* S.

(iv) *The branching points of* $\mathbf{x}$ *(resp.* $\mathbf{y}$*) lie in the finite set* $S \smallsetminus S_F^X$ *(resp.* $S \smallsetminus S_F^Y$*).*

E.g. Klein quartic $F(X,Y) = X^3Y + Y^3 + X$

$$S_F^X = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0,\ x^3 + 3y^2 \neq 0,\ 1 \neq 0\}$$
$$S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0,\ 3x^2 + 1 \neq 0,\ y \neq 0\}$$
$$S_F^X \cup S_F^Y = \{(x,y) \in \mathbb{C}^2 \mid F(x,y) = 0\} - \{(0,0)\}$$

$$\begin{array}{ccc} S_F^X & & S_F \\ \downarrow & \Rightarrow & \downarrow \\ \mathbb{CP}^1 - \{\text{finite pts}\} & & \mathbb{CP}^1 \end{array}$$

Rmk. 1. S_F is generally not the corresponding proj variety (projective) ? Don't understand it now

2. If the corresponding proj variety is RS (no singularity) ⇒ it's the required one. (Riemann Surface)

Def (defined over $k \subseteq \mathbb{C}$)

We shall say that a Riemann surface S *is defined over a field* $K \subset \mathbb{C}$ (or that K is a *field of definition of* S) if $S \simeq S_F$ for some irreducible polynomial $F(X,Y) = \sum a_{ij} X^i Y^j$ with coefficients $a_{ij} \in K$.

$S \simeq S_F \qquad F(X,Y) = \sum a_{ij} X^i Y^j \quad a_{ij} \in k$

E.g. Elliptic curve over $\mathbb{Q}$

(1) S_{F_1} : $F_1(X,Y) = Y^2 - [X^3 - 1]$

(2) S_{F_2} : $F_2(X,Y) = Y^2 - [X^3 - \pi^3]$

Reason: (x,y)

$$\downarrow \qquad \downarrow$$

S_{F_1} : $F_1(W,Z) = Z^2 - [W^3 - 1]$

$$\left(\frac{y}{\pi\sqrt{\pi}}\right)^2 - \left[\left(\frac{x}{\pi}\right)^3 - 1\right] = 0 \quad \text{well-defined} \quad \left(\frac{x}{\pi}, \frac{y}{\pi\sqrt{\pi}}\right)$$

3. Claim of Belyi's theorem

Philosophy: functions reflect properties of space.
- topology
- geometry
- arithmetic

Theorem 3.1 (Belyi's Theorem) *Let S be a compact Riemann surface. The following statements are equivalent:*

(a) S is defined over $\overline{\mathbb{Q}}$.

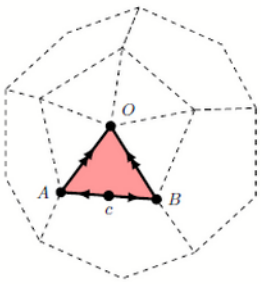
(b) S admits a morphism $f : S \longrightarrow \mathbb{P}^1$ with at most three branching values. Rmk

defined over $\bar{\mathbb{Q}}$ $\Leftrightarrow$ admits a Belyi fct.

Rmk $n := \#\,\mathrm{Branch}(f)$
- when $n = 0, 1, 2$, $S \cong \mathbb{P}^1$;
- when $n = 3$, we can assume $\mathrm{Branch}(f) = \{\infty, 0, 1\}$

Eg. of (b).

1) $\mathbb{PC}^1 \xrightarrow{\pi} \mathbb{PC}^1/\Gamma \cong \mathbb{PC}^1$ (Of course $\mathbb{PC}^1$ is defined over $\bar{\mathbb{Q}}$)



$$\begin{array}{ccccccc} \mathbb{PC}^1 & & A, B, \cdots (12) & & c, \cdots (20) & & O, \cdots (30) \\ \pi \downarrow 60 & & \downarrow 5 & & \downarrow 3 & & \downarrow 2 \\ \mathbb{PC}^1/\Gamma_{(2,3,5)} & & 0 & & 1 & & \infty \end{array}$$

在 $0, 1, \infty$ 处分歧, π is a Belyi fct

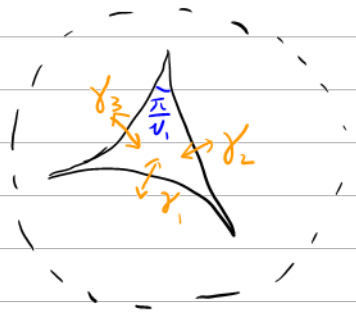
图 1.2 例: 正十二面体

2) $\mathcal{H}^*/\Gamma(N) \xrightarrow{\pi} \mathcal{H}^*/SL_2(\mathbb{Z}) \xrightarrow[\sim]{j} \mathbb{PC}^1$

The ramification points here are called elliptive points.
"Why modular form / modular space is an arithmetic object"

3) $\mathbb{D}/K \longrightarrow \mathbb{D}/\Gamma$

$\downarrow$ to make $\mathbb{D}/K$ cpt



$\Gamma_2 = \langle \gamma_1\gamma_2,\ \gamma_2\gamma_3,\ \gamma_3\gamma_1 \rangle$

4. a one-side proof: (a) $\Rightarrow$ (b)

(1) an example

e.g. for S_{F_λ} $F_\lambda: Y^2 = X(X-1)(X-\lambda)$ $\lambda \in \overline{\mathbb{Q}} \cap (0,1)$

① F_λ is defined over $\overline{\mathbb{Q}}$

② x: ramified at $\infty, 0, 1, \lambda$.

③ adjustment: $P_\lambda \circ x$ ramified at 3 points, where

$\lambda = \frac{m}{m+n}$ $\quad P_\lambda(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n$

S_{F_λ}

$x \downarrow$ $\Downarrow\Downarrow\Downarrow\Downarrow$

$\mathbb{P}^1$ 0 1 λ ∞ $\quad a$

$P_\lambda \downarrow$ $\searrow$ $\downarrow\downarrow$ $\Downarrow ?$

$\mathbb{P}^1$ 0 1 ∞ $P_\lambda(a)$

$0 = P_\lambda'(a) = \frac{(m+n)^{m+n}}{m^m n^n}[ma + n(1-a)]a^{m-1}(1-a)^{n-1} \Rightarrow a = 0, 1, \lambda$

Thm. (Belyi's Theorem) Suppose S is a cpt RS, then TFAE:

(a) S is defined over $\overline{\mathbb{Q}}$.

(b0) $\exists f: S \to \mathbb{P}^1$ with $\mathrm{Branch}(f) = \{\infty, 0, 1\}$.

(b1) $\exists f: S \to \mathbb{P}^1$ with $\#\,\mathrm{Branch}(f) \le 3$

(b2) $\exists f: S \to \mathbb{P}^1$ with $\mathrm{Branch}(f) \subseteq \overline{\mathbb{Q}} \cup \{\infty\}$.

(b3) $\exists f: S \to \mathbb{P}^1$ with $\mathrm{Branch}(f) \subseteq \overline{\mathbb{Q}} \cup \{\infty\}$

Sketch

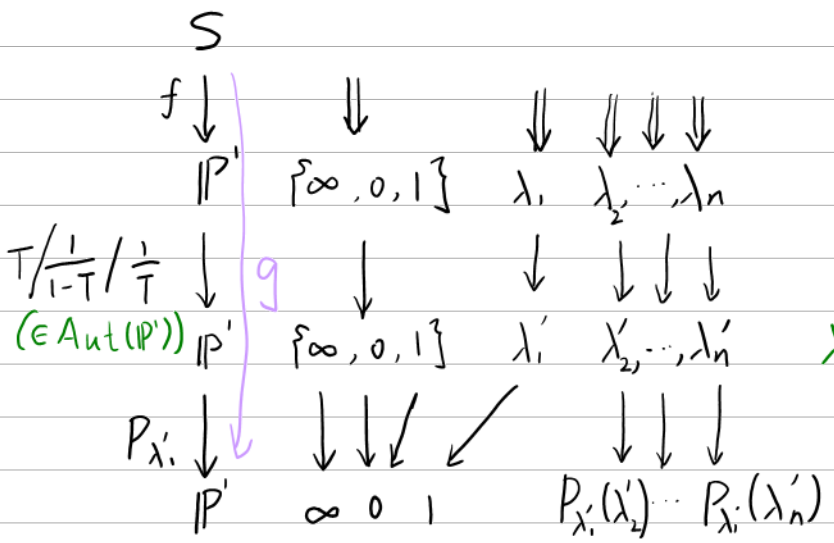
(a) $\xrightarrow{\text{Verify}}$ (b3) $\xrightarrow{(4)}$ (b2) $\xrightarrow{(2)(5)}$ (b1) $\xrightarrow{\text{Rmk}}$ (b0)

(3)

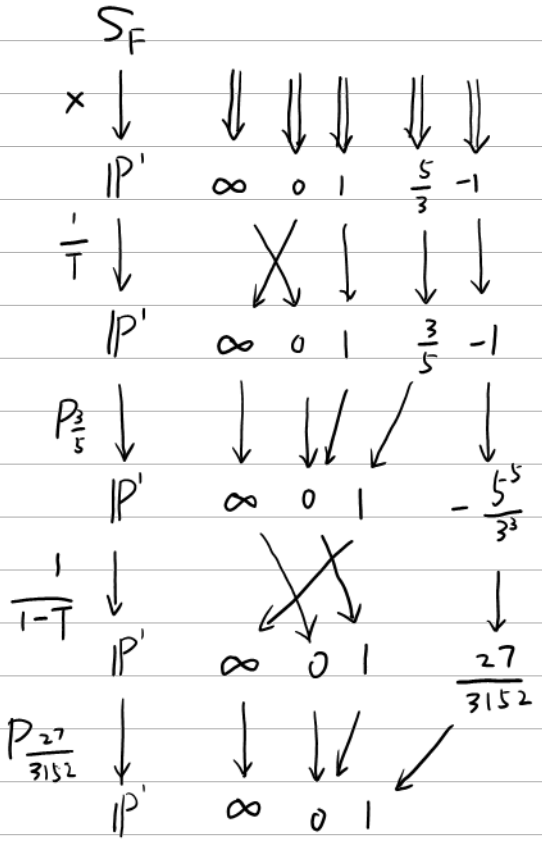
(2): (b2) $\Rightarrow$ (b1)

Use induction.

[Suppose $f: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \lambda_1, \cdots, \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$
need construct $g: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \mu_1, \cdots, \mu_{n-1}\} \subset \mathbb{Q} \cup \{\infty\}$]



Eg. of (2) $F: y^2 = x(x-1)(x+1)(x-\frac{5}{3})$



$\lambda = \frac{3}{5} = \frac{3}{3+2}$ $m=3$ $n=2$

$P_{\frac{3}{5}}(T) = \frac{5^5}{3^3 2^2} T^3 (1-T)^2$

$P_{\frac{27}{3152}}(T) = \frac{3152^{3152}}{27^{27} 3125^{3125}} T^{27} (1-T)^{3125}$

$f(x,y) = P_{\frac{27}{3152}}\left(\frac{1}{1 - P_{\frac{3}{5}}(\frac{1}{x})}\right)$

(3): (a) $\Rightarrow$ (b3)

[$S \cong S_F$ $\quad F(X,Y) = \sum a_{ij} X^i Y^j \in \bar{\mathbb{Q}}[X,Y]$ irr
Verify $\quad x: S_F \to \mathbb{P}^1$ only ramified in $\bar{\mathbb{Q}} \cup \{\infty\}$.]

$x|_{S_F^X} : S_F^X \to \mathbb{C}$ is unramified, so

the possible ramified points:

① ∞

② $x : p_0(X) = 0$ $\qquad F(X,Y) = p_0(X)Y^n + \cdots p_n(X)$

③ $x \ (\{(x,y) \mid F(x,y) = F_Y(x,y) = 0\})$

(4): (b3) $\Rightarrow$ (b2)

[Suppose $f: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \lambda_1, \cdots, \lambda_n\} \subset \bar{\mathbb{Q}} \cup \{\infty\}$
need construct $g: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \mu_1, \cdots, \mu_{n'}\} \subset \mathbb{Q} \cup \{\infty\}$]

Denote $\quad B_1(f) = Branch(f) \sim \mathbb{Q} \cup \{\infty\} := \{\lambda_1, \cdots \lambda_m\}$

$B_2(f) = Branch(f) \cap (\mathbb{Q} \cup \{\infty\})$

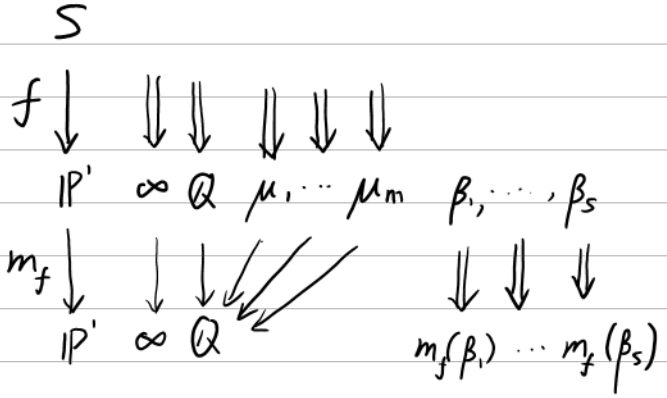
Let $m_f(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $B_1(f)$

[Induction on $\deg m_f(T)$:
construct $g: S \to \mathbb{P}^1$ s.t $\deg m_g(T) < \deg m_f(T)$
or $B_1(g) = \emptyset$
By convention $m_g(T) = 0$]

Claim: $\deg m_{m_f \circ f} < \deg m_f$

Let $\Phi := \{\beta_1, \cdots \beta_s\}$ be the roots of m_f'
then $B_1(m_f \circ f) = \{m_f(\beta_1), \cdots, m_f(\beta_s)\} \sim \mathbb{Q}$

Notation: Let $p(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $\Phi \sim \mathbb{Q}$



We have

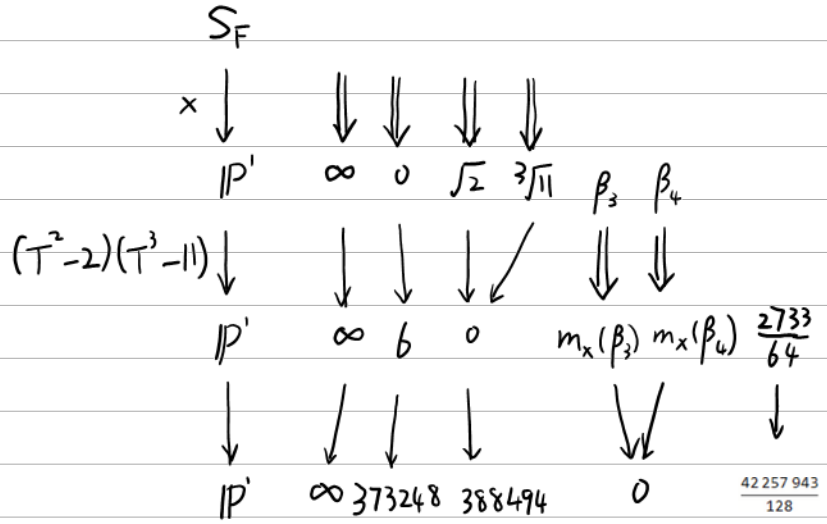
$$\deg m_{m_f \circ f} \leq \deg p \leq \deg m_f' < \deg m_f$$

($\leq \deg p$: def of min poly)

($\leq \deg m_f'$:) $\deg \min(m_f(\beta_i)) \leq \deg \min(\beta_i)$

$\beta_i = \sigma(\beta_j) \Rightarrow m_f(\beta_i) = \sigma(m_f(\beta_j))$

E.g. of (4) $\quad S_F: \; y^2 = x(x-\sqrt{2})(x-\sqrt[3]{11})$



$B_1(x) = \{\sqrt{2}, \sqrt[3]{11}\}$
$m_x(T) = (T^2-2)(T^3-11)$
$m_x'(T) = T(T-2)(2T^2+7T+14)$
$\Phi = \{\beta_1, \beta_2, \beta_3, \beta_4\}$
$= \{0, 2, \frac{1}{4}(-7 \pm 3\sqrt{7}\,i)\}$

$f := m_x \circ x$
$B_1(f) = \{\beta_3, \beta_4\}$
$m_f(T) = 32T^2 - 2733T + 388494$
$m_f'(T) = 64T - 2733$
$\Phi' = \{\frac{2733}{64}\}$

(5): (b2) $\Rightarrow$ (b1)

[Suppose $f: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \lambda_1, \cdots, \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$
need construct $g: S \to \mathbb{P}^1$ ramified at $\{\infty, 0, 1\}$]

[Idea: find a "good ramified fct" $G(T) = \prod_{i=1}^{n} (x-\lambda_i)^{a_i}$ $a_i \in \mathbb{N} - \{0\}$
$\uparrow$ $\mathrm{Ram}(G) \subseteq \{\lambda_1, \cdots \lambda_n\}$]

① Suppose $\mathrm{Branch}(f) \subseteq \mathbb{Z} \cup \{\infty\}$

② Let $\quad y_i := \dfrac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)} \qquad V := \prod_{j>i} (\lambda_j - \lambda_i) \qquad a_i := V y_i$

③ Verified: i) $a_i \in \mathbb{N} - \{0\} \Rightarrow G(\lambda_i) = 0$ or ∞
ii) $\sum a_i = 0 \Rightarrow G(\infty) = 1$

$$a_n = \prod_{n>j>i} (\lambda_j - \lambda_i) = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_{n-1} & \cdots & \lambda_1^{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \qquad a_{n-1} = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_{n-2} & \cdots & \lambda_{n-2}^{n-2} & 0 \\ 0 & \cdots & & 0 & 1 \\ 1 & \lambda_n & \cdots & \lambda_n & 0 \end{vmatrix}$$

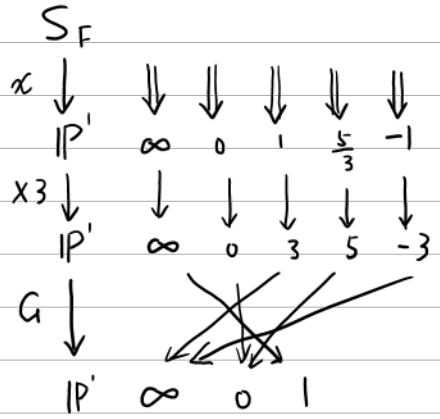
$$\Rightarrow \sum a_i = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-2} & 1 \end{vmatrix} = 0$$

iii) $\mathrm{Ram}(G) \subseteq \{\lambda_1, \cdots \lambda_n\}$

$$\frac{G'(T)}{G(T)} = (\log G(T))' = \sum_{i=1}^{n} \frac{a_i}{x-\lambda_i} = \sum_{i=1}^{n} \frac{V y_i}{x-\lambda_i} = \frac{V}{\prod_{i=1}^{n}(x-\lambda_i)}$$

$$\Rightarrow G'(T) = V \prod_{i=1}^{n} (x-\lambda_i)^{a_i - 1}$$

Eg. of (5) $F: y^2 = x(x-1)(x+1)(x-\frac{5}{3})$



$\{\lambda_1, \dots \lambda_4, \infty\} = \{-3, 0, 3, 5, \infty\}$

$V = \prod_{j>i} (\lambda_j - \lambda_i) = 4320$

$\{y_1, y_2, y_3, y_4\} = \{-\frac{1}{144}, \frac{1}{45}, -\frac{1}{36}, \frac{1}{80}\}$

$\{a_1, a_2, a_3, a_4\} = \{-30, 96, -120, 54\}$

$$G(x) = \frac{x^{96}(x-5)^{54}}{(x+3)^{30}(x-3)^{120}}$$

$$G'(x) = 4320 \frac{x^{95}(x-5)^{53}}{(x+3)^{29}(x-3)^{119}}$$