Eine Woche, ein Beispiel

9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

$$\boxed{\begin{array}{l} \text{— vector bundle over manifold} \\ \text{— module over ring} \end{array}}$$

There are hundreds of different versions of it:

- vector bundle over manifold 几何/几何分析
  - diffe mfld · (real) differential v.b. over (real) differential mfld
  - Riemann surface · cplx (analytic) line bundle over Riemann surface
- sheaf over space 代数几何
  - scheme theory · locally free sheaf on scheme
  - · coherent sheaf on scheme
  - geo rep theory · local system over (real/cplx) mfld
  - · perverse sheaf over Riemann surface (derived)
- module over ring 代数
  - comm alg · f.g module over Noetherian commutative ring (with 1)
  - rep of grp · group representation over group ($\rightsquigarrow$ group algebra)
  - quiver theory · quiver representation over quiver ($\rightsquigarrow$ path algebra, bound quiver algebra)
  - Lie algebra · Lie alg representation over Lie alg ($\rightsquigarrow$ universal enveloping algebra)
- Arithmetic Geometry 范数 ⇝ p进分析
  - · hermitian line bundle over projective arithmetic variety $\mathcal{X}$
  - · adelic line bundle over essentially quasi-proj scheme
  - · over Berkovich analytic space X^{an}
  - · over formal scheme $\mathrm{Spf} A$
  - · over rigid-analytic space K-affinoid space
  - · over adic space $\mathrm{Spa}(A, A^+)$

Goal
- structures & invariants
- classifications of
  special v.b, mfld, sub v.b. submfld
- symmetry & quotient
- special functors
- homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n.

(by homotopy group! $\rightsquigarrow$ generalized Picard group?)

Q: Is this group structure natural?

ref: https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres

Thm. $\{\text{rank } m \ K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $\quad K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ — rank m / n	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/12\mathbb{Z}$	$(\mathbb{Z}/12\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{RP}^\infty \simeq K(\mathbb{Z}/2\mathbb{Z}, 1)$

$\pi_{n-1}(GL_m(\mathbb{C}))$ — rank m / n	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{CP}^\infty \simeq K(\mathbb{Z}, 2)$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes$, Λ^k, Sym^k, Res, Ind

For the other spaces:
https://math.stackexchange.com/questions/383838/classifying-vector-bundles
http://www.ms.uky.edu/~guillou/F18/751Notes.pdf
It's still not so explicit.

$\{\text{rank } m \ K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_K(m, \infty)]$ $\quad K = \mathbb{R}, \mathbb{C}$

M: paracompact