Eine Woche, ein Beispiel

Ref: from [23.11.19]

4.28 naive $\otimes$ - Hom adjunction

Notation: - A: associate ring allowed to be non-commutative, contains 1
- There are two systems to write category of A-modules.

$$\begin{aligned} & \mathrm{Mod}_A &= \ & A\text{-}\mathrm{Mod} & & \ni {}_A M \\ (\mathrm{Mod}_A)^{op} \neq \ & \mathrm{Mod}_{A^{op}} &= \ & \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} & & \ni M_A \\ & \mathrm{Mod}_{A\otimes B^{op}} &= \ & A\text{-}\mathrm{Mod}\text{-}B & & \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \quad \ni {}_B M_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $Ob(Ring) \cong Ob(Ring^{op})$, $A^{op} \notin Ob(Ring^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset Sh(\mathrm{Spec}\, A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

1. definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

⋮

6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-}\mathrm{Mod}\text{-}B \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$

$$\begin{array}{ccccc} \mathrm{Hom}_B^A(-,-) : & (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} & \times\ B\text{-}\mathrm{Mod}\text{-}A & \longrightarrow & \mathbb{Z}\text{-}\mathrm{Mod} \\ \parallel & \parallel & \parallel & & \parallel \\ \mathrm{Hom}_{B\otimes_{\mathbb{Z}} A^{op}}(-,-) & (\mathbb{Z}\text{-}\mathrm{Mod}\text{-}B\otimes_{\mathbb{Z}} A^{op})^{\overline{op}} & \times\ (B\otimes_{\mathbb{Z}} A^{op}\text{-}\mathrm{Mod}\text{-}\mathbb{Z})^{\overline{op}} & \longrightarrow & \mathbb{Z}\text{-}\mathrm{Mod}\text{-}\mathbb{Z} \end{array}$$

$${}_AX_B, \quad {}_BY_C, \quad {}_CZ_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{op}} X$ in $A\text{-}\mathrm{Mod}\text{-}C = C^{op}\text{-}\mathrm{Mod}\text{-}A^{op}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\mathrm{Hom}_A(A, X) \cong X$

2. adjunction $_BX_A$, $_CY_B$, $_CZ_D$. we get

$$\mathrm{Hom}_C(Y \otimes_B X, Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y,Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set

$$\{f : Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c\}$$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y, -)"] \end{tikzcd}$$