

Eine Woche, ein Beispiel

12.17 calculation of NMD

Goal: compute normal Morse data (NMD)

$$\{f \geq 0\} \xrightarrow{\sim} X \xleftarrow{\sim} \{f < 0\}$$

$$\begin{aligned} \text{NMD}(\mathcal{F}', S) &= (R\Gamma_{\{f|_{N(X)} \geq f(x)\}}(\mathcal{F}'|_{N(X)})_x \\ &\stackrel{\substack{S = \{x\} \\ X \text{ is cone} \\ f(x) = 0 \\ \text{compatible}}}{=} (R\Gamma_{\{f \geq 0\}}(\mathcal{F}'))_x \\ &= L_x^* i^* \mathcal{F}' \\ &= R\Gamma(X, \{f < 0\}, \mathcal{F}') \\ &= \text{Fiber}(R\Gamma(X, \mathcal{F}') \longrightarrow R\Gamma(\{f < 0\}, \mathcal{F}')) \\ &= \text{Fiber}(\mathcal{F}_x \longrightarrow R\Gamma(L_x, \mathcal{F}')) \end{aligned}$$

1. low dimensional case
2. quadratic hypersurface
3. du val singularity
4. other quantities

Ref:

https://bastian.riek.me/blog/posts/2019/morse_theory/

<https://oldbookstonew.blogspot.com/>

Contains the following books:

[MilnorMT]: Morse Theory by Milnor

[MilnorCC]: Characteristic Classes by Stasheff and Milnor

[MilnorSing]: singular points of complex hypersurfaces by Milnor

[Maxim20]: notes on vanishing cycles and applications

<https://people.math.wisc.edu/~lmaxim/vanishing.pdf>

1. low dimensional case

E.g. $X = \mathbb{CP}^1$ $f: \mathbb{CP}^1 \dashrightarrow \mathbb{C} \xrightarrow{\operatorname{Re} z} \mathbb{R}$ $L_X = \{*\}$ $S = \{\infty\}$
 $\infty \mapsto 0$

\mathcal{F}	$NMD(\mathcal{F}, S)$	\mathcal{F}_x	$R\Gamma(L_x, \mathcal{F})$
$i_* \mathbb{Q}_{\{\infty\}}$	\mathbb{Q}	\mathbb{Q}	0
$\mathbb{Q}_{\mathbb{CP}^1}[1]$	0	$\mathbb{Q}[1]$	$\mathbb{Q}[1]$
$Rj_* \mathbb{Q}_{\mathbb{C}}[1]$	\mathbb{Q}	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}[1]$
$j_* \mathbb{Q}_{\mathbb{C}}[1]$	\mathbb{Q}	0	$\mathbb{Q}[1]$
$P(\phi)$	\mathbb{Q}^2	\mathbb{Q}	$\mathbb{Q}[1]$

E.g. $X = \{z^2 = z^3\}$ $f: X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1} \mathbb{C} \xrightarrow{\operatorname{Re} z} \mathbb{R}$ $L_X = \{a, b\}$ $S = \{0\}$

\mathcal{F}	$NMD(\mathcal{F}, S)$	\mathcal{F}_x	$R\Gamma(L_x, \mathcal{F})$
$i_* \mathbb{Q}_Z$	\mathbb{Q}	\mathbb{Q}	0
$\mathbb{Q}_X[1]$	\mathbb{Q}	$\mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$Rj_* \mathbb{Q}_U[1]$	\mathbb{Q}^2	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$j_* \mathbb{Q}_U[1]$	\mathbb{Q}^2	0	$\mathbb{Q}^2[1]$
$P(\phi)$	\mathbb{Q}^3	\mathbb{Q}	$\mathbb{Q}^3[1]$

E.g. $X = \mathbb{C} \cup_{\{0\}} \mathbb{C} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$

$f: X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1 + z_2} \mathbb{C} \xrightarrow{\text{Re } z} \mathbb{R} \quad l_x = \{a, b\} \quad S = \{0\}$

\mathcal{F}	$NMD(\mathcal{F}, S)$	\mathcal{F}_x	$R\Gamma(l_x, \mathcal{F})$
$i_* \underline{\mathbb{Q}}_Z$	\mathbb{Q}	\mathbb{Q}	0
$\underline{\mathbb{Q}}_X[1]$	\mathbb{Q}	$\mathbb{Q}[1]$	$\mathbb{Q}^*[1]$
$Rj_* \underline{\mathbb{Q}}_U[1]$	\mathbb{Q}^*	$\mathbb{Q}^* \oplus \mathbb{Q}^*[1]$	$\mathbb{Q}^*[1]$
$j_* \underline{\mathbb{Q}}_U[1]$	\mathbb{Q}^*	0	$\mathbb{Q}^*[1]$
$\pi^* \mathbb{Q}[-1]$	\mathbb{Q}	$\mathbb{Q} \oplus \mathbb{Q}^*[1]$	$\mathbb{Q}^*[1]$
$IC(\underline{\mathbb{Q}}_U[1])$	0	$\mathbb{Q}^*[1]$	$\mathbb{Q}^*[1]$

E.g. $X = X_3 \quad f: X \hookrightarrow \mathbb{C}^3 \xrightarrow{z_3} \mathbb{C} \xrightarrow{\text{Re } z} \mathbb{R} \quad l_x = \mathbb{C}^\times \quad S = \{0\}$

\mathcal{F}	$NMD(\mathcal{F}, S)$	\mathcal{F}_x	$R\Gamma(l_x, \mathcal{F})$
$i_* \underline{\mathbb{Q}}_Z$	\mathbb{Q}	\mathbb{Q}	0
$\underline{\mathbb{Q}}_X[2] = \pi^* \mathbb{Q}[-2]$	\mathbb{Q}	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$Rj_* \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[2] \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$j_* \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	0	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$IC(\underline{\mathbb{Q}}_U[2])$	\mathbb{Q}	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$

2. quadratic hypersurface

Cohomology list

$$X_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \dots + z_n^2 = 0\}$$

\downarrow

$$M_n := \{[z_1 : \dots : z_n] \in \mathbb{CP}^{n-1} \mid z_1^2 + z_2^2 + \dots + z_n^2 = 0\}$$

$H^i(X; \mathbb{Z})$	0	1	2	3	4	5	6	7	8	9	10	11	...
$M_2 \cong \{a, b\}$	\mathbb{Z}^2												
$M_3 \cong \mathbb{CP}^1$	\mathbb{Z}		\mathbb{Z}										
M_4	\mathbb{Z}		\mathbb{Z}^2		\mathbb{Z}								
M_5	\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}						
M_6	\mathbb{Z}		\mathbb{Z}		\mathbb{Z}^2		\mathbb{Z}		\mathbb{Z}				
M_7	\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		\mathbb{Z}		

This table is computed by Lefschetz hyperplane theorem and Chern class.

$H^i(X; \mathbb{Z})$	0	1	2	3	4	5	6	7	8	9	10	11	...
$M_2 - M_1 \cong \{a, b\}$	\mathbb{Z}^2												
$M_3 - M_2 \cong \mathbb{C}^\times$	\mathbb{Z}	\mathbb{Z}											
$M_4 - M_3$	\mathbb{Z}		\mathbb{Z}										
$M_5 - M_4$	\mathbb{Z}			\mathbb{Z}									
$M_6 - M_5$	\mathbb{Z}				\mathbb{Z}								
$M_7 - M_6$	\mathbb{Z}					\mathbb{Z}							

This table is computed by open-closed formalism. (Q-coefficient)

Using the Morse theory, one can show that (A variant of [Maxim20, Example 2.18])

$$M_n - M_{n-1} \sim S^{n-2}$$

\uparrow homotopy equivalence

$H^i(X; \mathbb{Q})$	0	1	2	3	4	5	6	7	8	9	10	11	...
$X_2 - 0 \cong \mathbb{C}^* \sqcup \mathbb{C}^*$	\mathbb{Q}^2	\mathbb{Q}^2											
$X_3 - 0$	\mathbb{Q}		$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Q}									
$X_4 - 0$	\mathbb{Q}		\mathbb{Q}	\mathbb{Q}		\mathbb{Q}							
$X_5 - 0$	\mathbb{Q}				$\mathbb{Z}/2\mathbb{Z}$			\mathbb{Q}					
$X_6 - 0$	\mathbb{Q}				\mathbb{Q}	\mathbb{Q}				\mathbb{Q}			
$X_7 - 0$	\mathbb{Q}						$\mathbb{Z}/2\mathbb{Z}$					\mathbb{Q}	

(n-3)-connected truncation

This table is computed by spectral sequence and Euler class.
Using the Morse theory, one can show that

$X_7 - 0$ is (n-3)-connected.

To compute the stalk of IC sheaf, one truncates in the middle.
Z-coefficient cohomology need more work on Euler class.

Suspect: $X_5 - 0$:

$$\begin{array}{ccccccc} \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

$X_6 - 0$:

$$\begin{array}{ccccccccc} \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} & \xrightarrow{(\cdot, \cdot)} & \mathbb{Z}^2 & \xrightarrow{(\cdot)} & \mathbb{Z} & \xrightarrow{1} & \mathbb{Z} \\ \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}^2 & & \mathbb{Z} & & \mathbb{Z} \end{array}$$

"reason":

$$\begin{array}{ccc} \mathbb{Z} \cong H^2(\mathbb{C}P^4) & \longrightarrow & H^2(M_5) \cong \mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{Z} \cong H^6(\mathbb{C}P^4) & \longrightarrow & H^6(M_5) \cong \mathbb{Z} \end{array}$$

$$\begin{array}{ccc} 1 = ev_{\mathcal{O}(-1)} & \xrightarrow{\quad} & ev_{\mathcal{O}(-1)} = 1 \\ \downarrow & & \downarrow \\ 1 & \xrightarrow{\quad} & 2 \end{array}$$

↑

3 hyperplane intersect at 2 pts.

$H^i(X_n - 0, M_n - M_{n-1}; \mathbb{Q}) \setminus i$	0	1	2	3	4	5	6	7	8	9	10	11	...
2	0	\mathbb{Q}^2											
3			\mathbb{Q}	\mathbb{Q}									
4				\mathbb{Q}		\mathbb{Q}							
5					\mathbb{Q}			\mathbb{Q}					
6						\mathbb{Q}				\mathbb{Q}			
7							\mathbb{Q}					\mathbb{Q}	

After truncation, only the red one remains.

The $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{F}_2$ - coefficient is as follows. (Assuming $NMD(X, IC(\mathbb{Z}_u))$ support at index 0)

$H^i(X_n - 0, M_n - M_{n-1}; \mathbb{F}_2) \setminus i$	0	1	2	3	4	5	6	7	8	9	10	11	...
2		\mathbb{F}_2^2											
3			\mathbb{F}_2	\mathbb{F}_2									
4				\mathbb{F}_2		\mathbb{F}_2							
5					\mathbb{F}_2			\mathbb{F}_2					
6						\mathbb{F}_2				\mathbb{F}_2			
7							\mathbb{F}_2					\mathbb{F}_2	

After truncation, nothing remains.

Possible direct calculation:

$$\begin{aligned}
 H^i(X_n - 0, M_n - M_{n-1}; \mathbb{Z}) &\cong H_{2n-2-i}^{BM}((M_n - M_{n-1}) \times \mathbb{R}_{>0} \cup (X_{n-1} - 0) \times \{0\}; \mathbb{Z}) \\
 &\cong H_{2n-3-i}^{BM}(S^{n-2}; \mathbb{Q})
 \end{aligned}$$

↑ blow up

Conclusion: for $X_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \dots + z_n^2 = 0\}$,

$$NMD(X_n; IC(\mathbb{Q}_u[n-1])) = \begin{cases} \mathbb{Q}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

3. du val singularity

<https://math.stackexchange.com/questions/40351/what-are-the-finite-subgroups-of-su-2-c>

Name	$R(x, y, z)$	gp G	$\#G$	G/G'	det (Cartan)
A_n	$x^2 + y^2 + z^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$
D_n	$x^2 + y^2 z + z^{n-1}$	$BD_{2(n-2)} = \text{Dic}_{n-2}$	$4(n-2)$	$\begin{cases} \mathbb{Z}/4\mathbb{Z}, & n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & n \text{ even} \end{cases}$	4 dicyclic
E_6	$x^2 + y^3 + z^4$	$BT \cong SL_3(\mathbb{F}_3)$	24	$\mathbb{Z}/3\mathbb{Z}$	3
E_7	$x^2 + y^3 + yz^3$	$BO \cong 2 \cdot S_4^-$	48	$\mathbb{Z}/2\mathbb{Z}$	2 (48, 28)
E_8	$x^2 + y^3 + z^5$	$BD \cong SL_2(\mathbb{F}_5)$	120	1	1

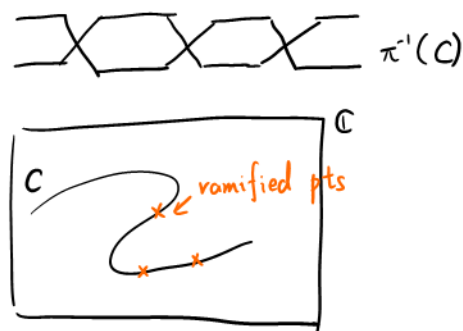
$\mathcal{U} = \text{link}$	0	1	2	3
$H^*(\mathcal{U}; \mathbb{Z})$	\mathbb{Z}	0	G/G'	\mathbb{Z}
$H_*(\mathcal{U}; \mathbb{Z})$	\mathbb{Z}	G/G'	0	\mathbb{Z}

$$\begin{aligned}
 L_X & \text{ homotopic equiv to } \begin{cases} S' \\ S'VS' \end{cases} \\
 \Rightarrow H^*(L_X; \mathbb{Z}) &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] \\ \mathbb{Z} \oplus \mathbb{Z}^2[-1] \end{cases} \\
 \Rightarrow H^*(\mathcal{U}, L_X; \mathbb{Q}) &= \begin{cases} \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] \\ \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] \end{cases} \\
 \Rightarrow \text{NMD}(X; IC(\mathbb{Q}_X[2])) &= \begin{cases} \mathbb{Q} \\ \mathbb{Q}^2 \end{cases}
 \end{aligned}$$

$A_n \& D_n$
 E_6, E_7, E_8
 $A_n \& D_n$
 E_6, E_7, E_8
 $A_n \& D_n$
 E_6, E_7, E_8
 $A_n \& D_n$
 E_6, E_7, E_8

Three different arguments for $L_X \sim S'$ or $S'VS'$:

- ① Morse index [MilnorSing, Theorem 6.5 & 5.11]
- ② Riemann surface, contract to $\pi^{-1}(C)$
- ③ Join construction, see [Maxim 20, Example 2.18]



These singularities can be used to understand 2-dimensional weighted projective spaces. For weighted projective spaces, the local charts are of form \mathbb{C}^n quotient cyclic group.

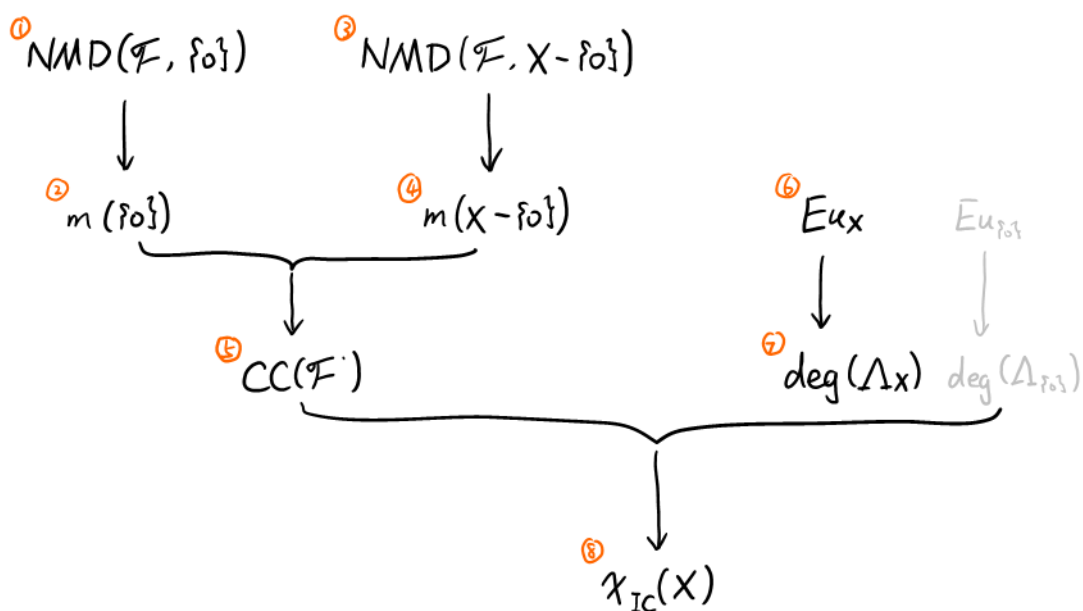
The topology of cone is still easy to compute by spectral sequence. For the result, see:
http://www.map.mpin-bonn.mpg.de/Fake_lens_spaces

However, the equations become harder to get, and we don't know the topology of the link. I just believe that there should be a answer for all these singularities.

4. other quantities

Setting M : analytic mfd e.g. $M = \mathbb{C}^n$ or $\mathbb{C}P^n$
 $X \subset M$ analytic variety of $\dim_{\mathbb{C}} X = m$
 $S: \emptyset \subset \{o\} \subset X$ where o is the only singularity
 $x_0 \in X - \{o\}$
 $\mathcal{F} \in \text{Perv}_S(X)$
 $\mathcal{L} := \mathcal{F}|_{X-\{o\}}[-m]$ local system on $X - \{o\}$ with rank r
 Special case: $\mathcal{F} = IC(\mathcal{L}[m])$

Task: Compute the following quantities.



Here we use notations in <https://arxiv.org/abs/2105.13069v2>. 6-8 comes from my supervisor's notation, if needed I should find some references for the definition.

① See the examples before

② $m(\{0\}) = \chi(NMD(\mathcal{F}, \{0\}))$

③ $NMD(\mathcal{F}, X - \{0\}) \cong \mathcal{F}_{x_0} \cong \mathbb{Q}^r[m]$

④ $m(X - \{0\}) = (-1)^{\dim_{\mathbb{C}}(X - \{0\})} \chi(NMD(\mathcal{F}, X - \{0\}))$
 $= (-1)^m \cdot (-1)^m \cdot r$
 $= r$

⑤ $CC(\mathcal{F}) = m(X - \{0\}) [\overline{T_{X - \{0\}}^* M}] + m(\{0\}) [\overline{T_{\{0\}}^* M}]$
 $= r [T_X^* M] + m(\{0\}) [T_{\{0\}}^* M]$
 $= r \Delta_X + m(\{0\}) \Delta_{\{0\}}$
 recall: $[T_X^* M] = [\overline{T_{X - \{0\}}^* M}]$ $\Delta_{\overline{S}} := [T_S^* M]$

⑥ Need to check the definition.

For $X \subset \mathbb{C}^2$ cuspidal cubic, $\text{Sing}(X) = \{p_0\}$,

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X - \{p_0\} \\ 2 & p = p_0 \end{cases}$$

In general. from my memory it looks like:

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X_{sm} \\ \geq 1 & p \in X - X_{sm} \end{cases}$$

$$\begin{aligned}
 \textcircled{7} \quad \deg(\Delta_X) &:= \#(\Delta_X \cdot \Delta_M) && \text{in } T^*M \\
 &= (-1)^m \chi(X, Eu_X) \\
 &= (-1)^m (\chi(X - \{o\}) \cdot Eu_X(o) + \chi(\{o\}) \cdot Eu_X(o)) \\
 &= (-1)^m (\chi(X - \{o\}) + Eu_X(o)) && = -2 \text{ for } X=M=\mathbb{CP}^1
 \end{aligned}$$

$$\begin{aligned}
 \deg(\Delta_{\{o\}}) &:= \#(\Delta_{\{o\}} \cdot \Delta_M) \\
 &= \chi(\{o\}, Eu_{\{o\}}) \\
 &= \chi(\{o\}) \cdot Eu_{\{o\}}(o) \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \textcircled{8} \quad (-1)^m \chi_{IC}(X) &= \deg(CC(\mathcal{F})) && \text{Here, } \mathcal{F} = IC(\mathbb{Q}_{X-\{o\}}[m]), r=1 \\
 &= \deg(r\Delta_X + m(\{o\})\Delta_{\{o\}}) \\
 &= r \cdot \deg \Delta_X + m(\{o\}) \deg \Delta_o \\
 &= \deg \Delta_X + m(\{o\})
 \end{aligned}$$

$$\Rightarrow \chi_{IC}(X) = \chi(X - \{o\}) + Eu_X(o) + (-1)^m m(\{o\})$$

X	$\chi(X - \{o\})$	$Eu_X(o)$	$m(\{o\})$	$\chi(X)$
\mathbb{C}	0	1	0	1
$\{y^2 = x^3\}$	0	2	1	1