Eine Woche, ein Beispiel

4.17 preliminary facts of representions of p-adic groups

> Main reference: The Local Langlands Conjecture for GL(2) by Colin J. BushnellGuy Henniart.
> [https://link.springer.com/book/10.1007/3-540-31511-X]

Process.

1. Basic properties
   - Smoothness
   - Irreducibility and unitary
   - Reduction to smaller cardinal.
2. Examples: $\mathcal{O}$, $\mathcal{O}^\times$, F, $F^\times$
3. Construction of new reps.
   - Special sub & quotient rep
   - Duality
   - Ind and c-Ind
   - Other constructions
4. Hecke algebra
5. Intertwining properties.

1. Basic properties

1.1. Smoothness

G: loc. profinite group

V: cplx v.s.

$\rho: G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V) \qquad g \longmapsto [v \longmapsto g.v]$

Def. (ρ, V) is smooth if

$\forall\, v \in V,\ \exists\, K \leq G$ cpt open s.t. $k.v = v \quad \forall k \in K$

$\mathrm{Rep}(G) = \{\text{sm rep of } G\}$ is a full subcategory of $\{\text{rep of } G\}$.

Rmk. Any sub/quotient rep of $(\rho, V) \in \mathrm{Rep}(G)$ is smooth.

$H \leq G$ cpt, $(\rho, V) \in \mathrm{Rep}(G) \Rightarrow (\rho|_H, V) \in \mathrm{Rep}(H)$

Rmk. For fcts, smoothness has a different meaning.

Recall the definition of $C^\infty(G)$ & $C_c^\infty(G)$:

$C^\infty(G) := \{f : G \to \mathbb{C} \mid f \text{ is loc. const}\}$

$C_c^\infty(G) := \{f \in C^\infty(G) \mid \operatorname{supp} f \subset G \text{ is cpt}\}$

1.2. Irreducibility and unitary

$\mathrm{Irr}(G) = \{(\rho, V) \in \mathrm{Rep}(G) \mid \rho \text{ is a irreducible rep}\}$

$\widehat{G} = \{(\rho, V) \in \mathrm{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$

$\overset{[p13]}{=\!=} \{\chi : G \longrightarrow \mathbb{C}^\times \mid \ker \chi \text{ is open}\}$

$\overset{[(1.6)]}{=\!=} \{\chi : G \longrightarrow \mathbb{C}^\times \mid \chi \text{ is continuous}\}$

Rmk. The notation is slightly different with the original reference.

Rmk. $\widehat{G} \subseteq \mathrm{Irr}(G) \subseteq \mathrm{Rep}(G)$

When [p15] G is cpt, or

[p22] $G/Z(G)$ is cpt with G/K countable, we get $\mathrm{Ind}(G) = \mathrm{Irr}(G)$;

when [p21] G is abelian and G/K is countable, $\mathrm{Ind}(G) = \widehat{G}$.

($\exists\, K \leq G$ cpt open, countable = at most countable here)

Rmk. A more general result is as follows.

Prop. Let $(\rho, V) \in \mathrm{Rep}(G)$, G/K countable. $\exists\, K \leq G$ cpt open
Let $Z(G) \leq H \leq G \quad H \leq G$ open $\quad H/Z(G)$ is cpt.
Then $(\rho|_H, V) \in \mathrm{Rep}(H)$ is semisimple.

To prove this we need the following lemma. (when applied, it would be $K_0 Z(G) \leq H$)

Lemma. Let $H \leq G$ open, $[G : H] < +\infty$, $(\rho, V) \in \mathrm{Rep}(G)$. Then
ρ is G-semisimple $\Leftrightarrow$ $\rho|_H$ is H-semisimple.

Def (Action as character)

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $\chi \in \widehat{H}$.

We say H acts on V as χ if $\rho|_H$ decompose as follows:

$$\rho|_H : H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \mathrm{Aut}_{\mathbb{C}}(V)$$

We may denote χ by χ_ρ or χ_H. When $H = Z(G)$, χ is denoted by ω_ρ.

Def (Contain irr rep)

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $(\sigma, W) \in \mathrm{Irr}(H)$.

We say ρ contains σ, or σ occurs in ρ, if

$$\mathrm{Hom}_H(\mathrm{Res}_H^G \rho, \sigma) \neq 0$$

i.e., σ can be realized as a quotient subrep of $\mathrm{Res}_H^G \rho$.

Cor. When H acts on V as χ_ρ, ρ contains χ_ρ.

Def (Unitary operator) V: Hilbert space.

$U \in \mathrm{Aut}_{\mathbb{C}}(V)$ is called an unitary operator if

$$\langle Uv, Uw \rangle = \langle v, w \rangle \qquad \forall v, w \in V$$

Def (Unitary rep) V: Hilbert space.

$(\rho, V) \in \mathrm{Rep}(G)$ is unitary if $\rho(g)$ is an unitary operator $(\forall g \in G)$.

E.g. $\chi \in \widehat{G}$ is unitary if $\mathrm{Im}\,\chi \subseteq S^1$

Rmk. When $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$, any $\chi \in \widehat{G}$ is unitary.

1.3. Reduction to smaller cardinal

Admissibility

(ρ, V) is admissible if $\dim_{\mathbb{C}} V^K < +\infty$ for $\forall K \leq G$ cpt open.

Countable hypothesis

$\exists/\forall$ $K \leq G$ cpt open, G/K is countable.

Assuming countable hypothesis, we get

$$(\rho, V) \in \mathrm{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \mathrm{End}_G(V) = \mathbb{C} \\ \rho \text{ acts on } V \text{ as a character } \omega_\rho \end{cases}$$

$$\xRightarrow{G \text{ is abelian}} \dim_{\mathbb{C}} V = 1.$$

2. Examples: $\mathcal{O}$, $\mathcal{O}^\times$, F, $F^\times$

3. Construction of new reps

3.1. Special sub & quotient rep.