Eine Woche, ein Beispiel

3.13 dual variety

Dual variety is useful is the research of subvarieties of P^n (and symplectic geometry). We emphasize the embedding here.

Main reference:
https://arxiv.org/abs/math/0112028v1
other ref:
Discriminants, Resultants, and Multidimensional Determinants by Israel M. GelfandMikhail M. KapranovAndrei V. Zelevinsky.
https://en.wikipedia.org/wiki/Dual_curve

A vivid animation: https://www.youtube.com/watch?v=HTXpf4jDgYE
Some pictures: https://www.ima.umn.edu/materials/2006-2007/W9.18-22.06/2203/Piene_190906.pdf

Goal.

1. Definition
2. Basic properties
   - Reflexivity theorem
   - dimension and defect
   - d, g, b, f, δ, k
3. Basic examples

Let $K = \bar{K}$ be a field, V a v.s. of dim $n+1$.

1. Definition

Def (Dual variety)

Let $X \subset \mathbb{P}V$: irr proj variety

X_{sm} : smooth locus

$I_X^\circ := \{(z, H) \mid z \in X_{sm},\ H \in \mathbb{P}V^*,\ T_z X \subset H\}$

$I_X := \overline{I_X^\circ}$.

Then $X^* := pr_2(I_X)$ is called the dual variety of X.

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd}$$

$\mathbb{P}V^* = \mathbb{P}(V^*)$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\operatorname{def} X}{X^*} \end{tikzcd}$$

$\dim V = n+1$

$\dim X = m$

$\operatorname{def} X = \operatorname{codim}_{\mathbb{P}V^*} X^* - 1$

Relation with symplectic geometry

Def (Lagrangian construction)

Let M be a sm proj irr variety, $Y \subset M$ be any irr subvariety.
We define
$$\mathrm{Lag}(Y) := \overline{N^*_{Y_{sm}} M} \quad \text{(closure in } T^*M)$$

Def. Any set $S \subset T^*M$ is called conical if S is closed under scalar multiplication.

Rmk. [Thm 1.9] $\mathrm{Lag}(Y)$ is a conical Lagrangian subvariety, and every conical Lagrangian subvariety S is of this form, i.e.
$$S = \mathrm{Lag}(\pi(S)) \qquad \pi: T^*M \longrightarrow M$$

Rmk. $\mathrm{Lag}(Y)$ is an analog of I_X. see the following picture:

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd} \qquad \begin{tikzcd} & V \times V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ V & & V^* \end{tikzcd}$$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\mathrm{def}X}{X^*} \end{tikzcd} \qquad \begin{tikzcd} & \overset{n+1}{\mathrm{Lag}(Y)} \arrow[dl, "n-m"'] \arrow[dr, "\text{typically } 1"] & \\ \overset{m+1}{Y} & & \overset{n-\mathrm{def}X}{Y^*} \end{tikzcd}$$

$Y = \mathrm{Cone}(X) \quad \Rightarrow \quad Y^* = \mathrm{Cone}(X^*)$

2. Basic properties

2.1. Thm (Reflexivity thm) $X^{**} = X$

Sketch of proof.

$$X \xrightarrow{\cong} X^{**}$$
$$\Leftrightarrow \quad [\,(z,H) \in I_X^\circ \Leftrightarrow (H,z) \in I_{X^*}^\circ\,]$$
$$\Leftrightarrow \quad I_X \cong I_{X^*} \quad \text{under the iso } \mathbb{P}V \times \mathbb{P}V^* \xrightarrow{\sim} \mathbb{P}V^* \times \mathbb{P}V^{**}$$
$$\Leftrightarrow \quad \mathrm{Lag}(Y) \cong \mathrm{Lag}(Y^*) \quad \text{where } Y := \mathrm{Cone}(X) \quad Y^* := \mathrm{Cone}(X^*)$$
under the iso $\quad T^*V \cong V \times V^* \cong V^* \times V \cong T^*V^*$

Under this iso, $\mathrm{Lag}(Y)$ is a conical Lagrangian subvariety of T^*V^*, so
$$\mathrm{Lag}(Y) \cong \mathrm{Lag}(pr_2(\mathrm{Lag}(Y))) \cong \mathrm{Lag}(Y^*)$$

2.2. Dimension and defect

Def (Defect) $\operatorname{def} X := \operatorname{codim}_{\mathbb{P}V^*} X^* - 1$. $\Rightarrow \dim X^* = n-1-\operatorname{def} X$

Typically, $\operatorname{def} X = 0$.

Def (Ruled space) X is ruled in proj subspaces of dim r if

$\forall x \in X \quad \exists L$: proj subspace of dim r s.t. $x \in L \subseteq X$.

Rmk. Sufficient to check $x \in X_{sm}$.

E.g. $X = V(xw - yz)$ is ruled in proj subspaces of dim 1,

$X = V(x^3+y^3+z^3+w^3)$ is not ruled. (Strictly speaking, it's ruled in dim 0)

Prop. [Thm 1.12]

$$\operatorname{def} X = r \iff X \text{ is (maximal) ruled in proj subspaces of } \dim r.$$

[Proof. Since $X = X^{**}$, the statement is equivalent to

$\dim X = n-r-1 \Rightarrow X^*$ is ruled in proj subspaces of dim r.

For any $(z, H) \in I_X^\circ$, $pr_1^{-1}(z) \cap I_X^\circ \cong \{z\} \times \mathbb{P}^r$ is maped by pr_2 to

a proj subspace L of $\mathbb{P}V^*$, s.t. $\dim L = r$ & $H \in L \subseteq X^*$.]

Rmk. Now we know that

$$\left.\begin{array}{l} X \text{ is smooth} \overset{\text{Thm 1.10}}{\Rightarrow} I_X \text{ is smooth} \\ X \text{ is not ruled} \iff \operatorname{def} X = 0 \iff X^* \text{ hypersurface} \iff pr_2 \text{ is birational} \end{array}\right\} \iff pr_2 \text{ is a resolution}$$

E.g. When $X = V(xw - yz)$, $\dim X^* = 3-1-1 = 1$;

when $X = V(x^3+y^3+z^3+w^3)$, $\dim X^* = 3-1-0 = 2$, $pr_2 : I_X \to X^*$ is birational.

Def. When X is not ruled, Δ_X is the polynomial defining X^*, which is unique up to scaling.

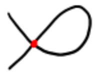
By doing so, some potential problems for the genus formula and other formula will be solved. Moreover, we don't need to do case by case analysis in those specific examples.

We now assume $k = \mathbb{C}$.

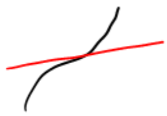
2.3. $d, g, b, f, \delta, \kappa$

Def.
- d: degrees
- g: geo genus
- b: #bitangents, δ: #double points
- f: # flexs, κ: # cusps

bitangent



ordinary double



inflection

cuspidal

Formulas:

$$\begin{cases} d^* \\ g^* \\ b^* \quad \delta^* \\ f^* \quad \kappa^* \end{cases} = \begin{cases} d(d-1) - 2\delta - 3\kappa & \text{(called Plücker-Clebsch formula)} \\ g = \frac{1}{2}(d-1)(d-2) - \delta - \kappa & \text{by genus formula} \\ \delta \quad b \\ \kappa \quad b \end{cases}$$