Taylor's formula is the fundamental result for differential, while Stokes' formula is the fundamental result for integral.

Session 4 & Exercise 2.

You did a good job last time!

Q?

Task 3 Compute the Taylor expansion of

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad f(x,y) = e^{-x^2-2y^2}$$

at $(0,0)$ for the first two terms.

Recall the Taylor series (under technical conditions)

$$f(\vec{x}) \sim \sum_{\substack{d=(d_1,\dots,d_n) \\ \in \mathbb{Z}_{\geq 0}^{\oplus n}}} \frac{1}{d!} \frac{\partial^{|d|} f}{\partial x^d}(\vec{a}) \, (\vec{x}-\vec{a})^d$$

$$= \sum_{d_1,\dots,d_n=0}^{+\infty} \frac{1}{\prod_{k=1}^{n}(d_k!)} \frac{\partial^{\sum_k d_k} f}{\partial x_1^{d_1}\cdots\partial x_n^{d_n}}(\vec{a}) \, \cancel{(\vec{x}-\vec{a})^{\sum_k d_k}} \prod_{k=1}^{n}(x_k-a_k)^{d_k}$$

$$= \sum_{d_1=0}^{\infty} \cdots \sum_{d_n=0}^{\infty} \frac{(x_1-a_1)^{d_1}\cdots(x_n-a_n)^{d_n}}{d_1!\cdots d_n!} \frac{\partial^{d_1+\dots+d_n} f}{\partial x_1^{d_1}\cdots\partial x_n^{d_n}}(\vec{a})$$

where

$d = (d_1,\dots,d_n)$ $\qquad |d| = d_1 + \dots + d_n$

$\vec{x} = (x_1,\dots,x_n)$ $\qquad d! = d_1!\cdots d_n!$

$\vec{a} = (a_1,\dots,a_n)$ $\qquad \dfrac{\partial^{|d|}}{\partial x^d} = \dfrac{\partial^{|d|}}{\partial x_1^{d_1}\cdots\partial x_n^{d_n}}$

$$(\vec{x}-\vec{a})^d = (x_1-a_1)^{d_1}\cdots(x_n-a_n)^{d_n}.$$

In ptc, under mild conditions,

Peano form of the remainder

$$f(\vec{x}) = f(\vec{a}) + \sum_{k=1}^{n} f_k(\vec{a})(x_k - a_k) + \frac{1}{2!}\sum_{i,j=1}^{n} f_{ij}(\vec{a})(x_i - a_i)(x_j - a_j) + o(|\vec{x}-\vec{a}|^2)$$

$$= f(\vec{a}) + (f_1(\vec{a}), \dots, f_n(\vec{a}))\begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix}$$

$$+ \frac{1}{2}(x_1 - a_1, \dots, x_n - a_n)\begin{pmatrix} f_{11}(\vec{a}) & \cdots & f_{1n}(\vec{a}) \\ \vdots & & \vdots \\ f_{n1}(\vec{a}) & \cdots & f_{nn}(\vec{a}) \end{pmatrix}\begin{pmatrix} x_1 - a_1 \\ \vdots \\ x_n - a_n \end{pmatrix} + o(|\vec{x}-\vec{a}|^2)$$

$$= f(\vec{a}) + (\nabla f)(\vec{a}) \cdot (\vec{x} - \vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T (\mathrm{Hess} f)(\vec{a})(\vec{x}-\vec{a}) + o(|\vec{x}-\vec{a}|^2)$$

dot product

Method 1. $f(x_1, x_2) = e^{-x_1^2 - 2x_2^2}$

$$f_1 = \frac{\partial f}{\partial x_1} = -2x_1 e^{-x_1^2 - 2x_2^2}$$

$$f_2 = \frac{\partial f}{\partial x_2} = -4x_2 e^{-x_1^2 - 2x_2^2}$$

$$f_{11} = \frac{\partial^2 f}{\partial x_1^2} = 4x_1^2 e^{-x_1^2 - 2x_2^2} - 2e^{-x_1^2 - 2x_2^2}$$

$$f_{12} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 4x_1x_2 e^{-x_1^2 - 2x_2^2}$$

$$f_{22} = \frac{\partial^2 f}{\partial x_2^2} = 16x_2^2 e^{-x_1^2 - 2x_2^2} - 4e^{-x_1^2 - 2x_2^2}$$

$\vec{a} = 0$, $f(\vec{x}) = f(0) + (\nabla f)(0) \cdot \vec{x} + \frac{1}{2}\vec{x}^T (\mathrm{Hess} f)(0)\vec{x} + o(|\vec{x}|^2)$

$$= 1 + (0,0)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \frac{1}{2}(x_1, x_2)\begin{pmatrix} -2 & 0 \\ 0 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + o(x_1^2 + x_2^2)$$

$$\cancel{= 1 - x_1^2 - 2y^2 + o(x^2+y^2)}$$

$$= 1 - x_1^2 - 2x_2^2 + o(x_1^2 + x_2^2)$$

Method 2 $\quad e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$

$$e^{-x^2-2y^2} = 1 + (-x^2-2y^2) + o\left((-x^2-2y^2)\right)$$
$$= 1 - x^2 - 2y^2 + o(x^2+y^2)$$

To use this method, you need to be familiar with the following formulas.

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$
$$\ln(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots$$
$$-\ln(1-t) = t + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \cdots$$
$$(1+t)^\alpha = 1 + \alpha t + \binom{\alpha}{2}t^2 + \binom{\alpha}{3}t^3 + \binom{\alpha}{4}t^4 + \cdots$$

$\cdots$

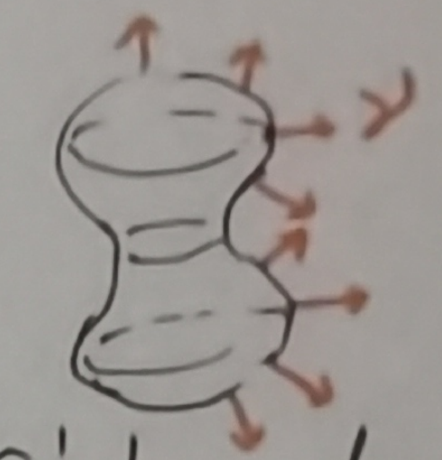
Ex. do it for $\alpha = -1$, $\alpha = \frac{1}{2}$.

E.g. $f(x,y) = \dfrac{1}{x+y+2}$

$$= \frac{1}{2}\,\frac{1}{1+\frac{x+y}{2}}$$
$$= \frac{1}{2}\left(1+\frac{x+y}{2}\right)^{-1}$$
$$= \frac{1}{2}\left[1 - \frac{x+y}{2} + \binom{-1}{2}\left(\frac{x+y}{2}\right)^2 + o\left(\left(\frac{x+y}{2}\right)^2\right)\right]$$
$$= \frac{1}{2}\left[1 - \frac{x+y}{2} + \frac{(-1)(-2)}{2}\,\frac{(x+y)^2}{4}\right] + o(x^2+y^2)$$
$$= \frac{1}{2} - \frac{x+y}{4} + \frac{(x+y)^2}{4} + o(x^2+y^2)$$

Rest.

(This method is usually used when we don't want to compute higher order derivative directly)



Recall the divergence theorem.

Let $\mathcal{U} \subseteq \mathbb{R}^n$ be a bounded open subset with C^1-boundary

For ~~$\vec{F}=(F_1,\dots,F_n)$~~ any vector field $\vec{F}=(F_1,\dots,F_n)$ on $\mathbb{R}^n$,

$$\int_{\mathcal{U}} \nabla\cdot\vec{F}\, dV = \int_{\partial\mathcal{U}} \vec{F}\cdot\vec{\nu}\, dA$$

where $\vec{\nu}=(\nu_1,\dots,\nu_n)^T$ is the outward-pointing unit normal vector at $\partial\mathcal{U}$.

Task 1. For $f, g \in C^1(\mathcal{U})$, $i\in\{1,\dots n\}$, shows that

$$\int_{\mathcal{U}} f\,\partial_i g\, dV = \int_{\partial\mathcal{U}} fg\,\nu_i\, dA - \int_{\mathcal{U}} g\,\partial_i f\, dV$$

$$\Leftrightarrow \int_{\mathcal{U}} \partial_i(fg)\, dV = \int_{\partial\mathcal{U}} fg\,\nu_i\, dA$$

$$\Leftrightarrow \int_{\mathcal{U}} \nabla\cdot(fg\,\vec{e}_i)\, dV = \int_{\partial\mathcal{U}} (fg\,\vec{e}_i)\cdot\vec{\nu}\, dA \qquad \square$$

Recall the Stokes' formula.

Let $M \subset \mathbb{R}^3$ be a smooth manifold of dim 2 with C^1-boundary $\gamma := \partial M$

For any vector field $\vec{F} = (F_1, F_2, F_3)$ on $\mathbb{R}^3$,

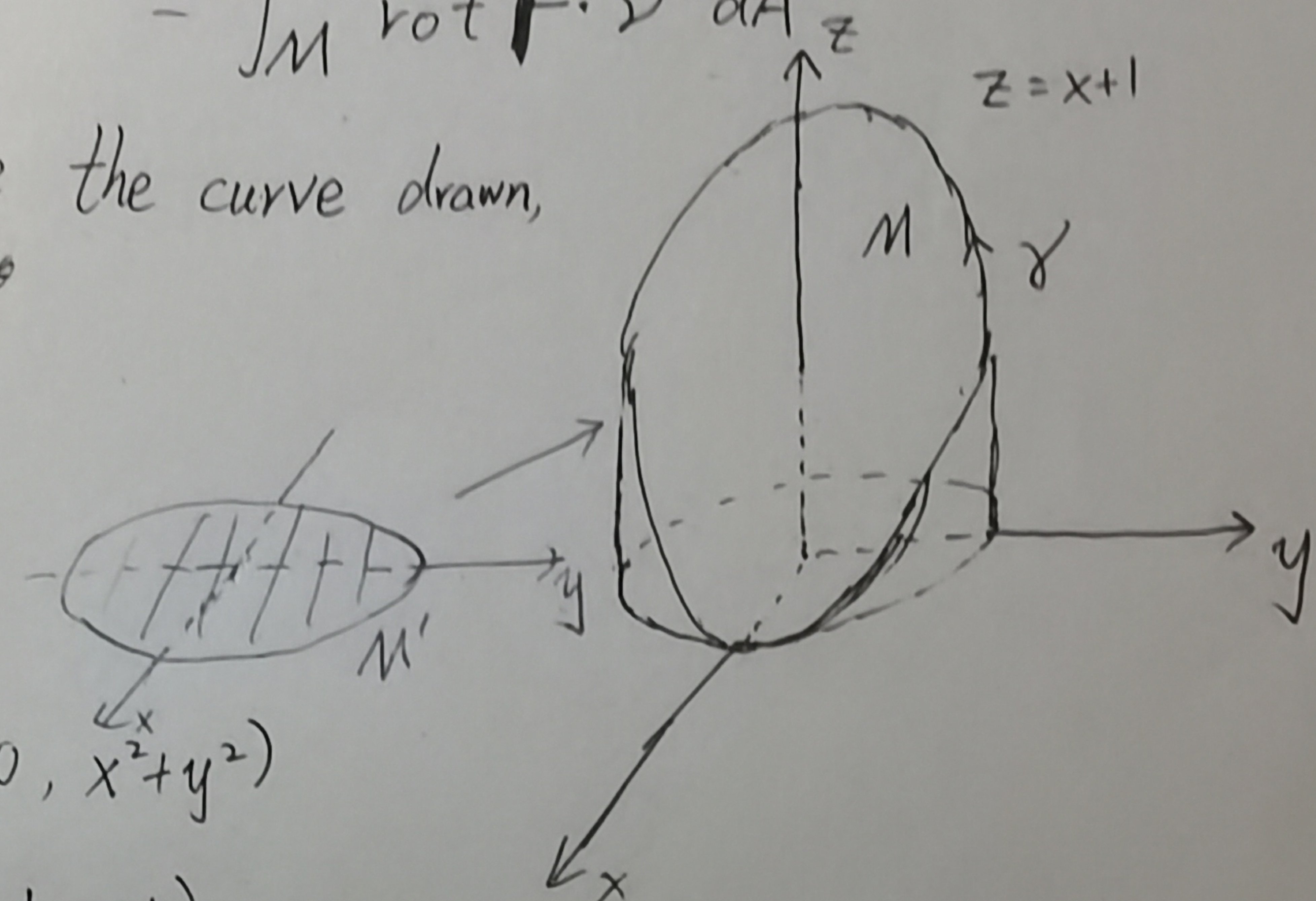
$$\int_\gamma \vec{F} \cdot d\vec{x} = \cancel{\int_M \mathrm{rot}\, \vec{F} \cdot d\vec{A}}$$

$$= \int_M \mathrm{rot}\, \vec{F} \cdot \vec{\nu}\, dA$$

Task 2. Let $\gamma \subset \mathbb{R}^3$ be the curve drawn,

$\vec{F} = (-x^2 y, xy^2, z^3)$

Compute $\int_\gamma \vec{F} \cdot d\vec{x}$



Hint. $\mathrm{rot}\, \vec{F} = (0, 0, x^2+y^2)$

$\vec{\nu} = \frac{1}{\sqrt{2}}(1, 0, 1)$

$\mathrm{rot}\, \vec{F} \cdot \vec{\nu} = \frac{1}{\sqrt{2}}(x^2+y^2)$

$$\int_\gamma \vec{F} \cdot d\vec{x} = \int_M \frac{1}{\sqrt{2}}(x^2+y^2)\, dA$$

$$= \int_{M'} \frac{1}{\sqrt{2}}(x^2+y^2)\sqrt{2}\, dx\, dy$$

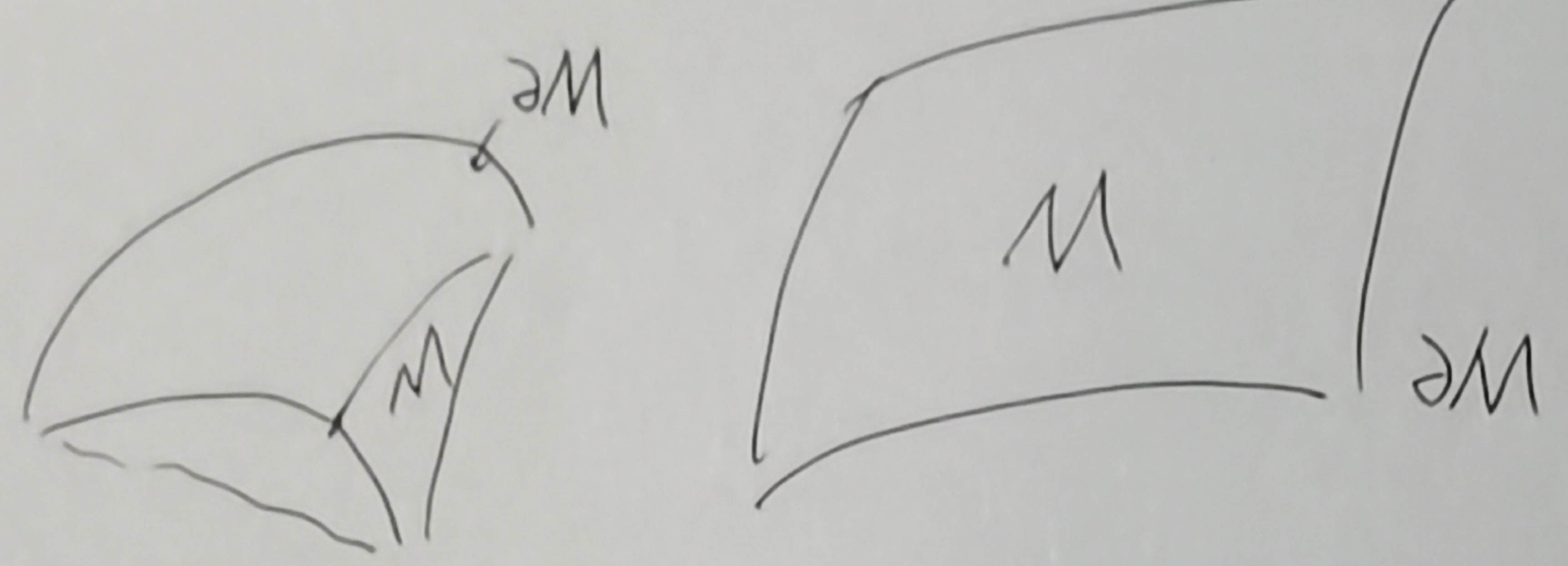
$$= \int_0^1 \int_0^{2\pi} \frac{1}{\sqrt{2}} r^2 \cdot \sqrt{2} \cdot r\, dr\, d\theta$$

$$= \int_0^1 r^3\, dr \cdot \int_0^{2\pi} d\theta$$

$$= \frac{\pi}{2}$$

□

Now let us talk about the Stokes' formula



$\dim M = n$

$\omega \in \Omega^{n-1}(M)$

$$\int_M d\omega = \int_{\partial M} \omega|_{\partial M}$$

with compatible orientation of M & ∂M.

$$\iota : \partial M \hookrightarrow M$$

$$\rightsquigarrow \iota^* : \Omega^{n-1}(M) \longrightarrow \Omega^{n-1}(\partial M)$$

$$\omega|_{\partial M} := \iota^* \omega \in \Omega^{n-1}(\partial M)$$

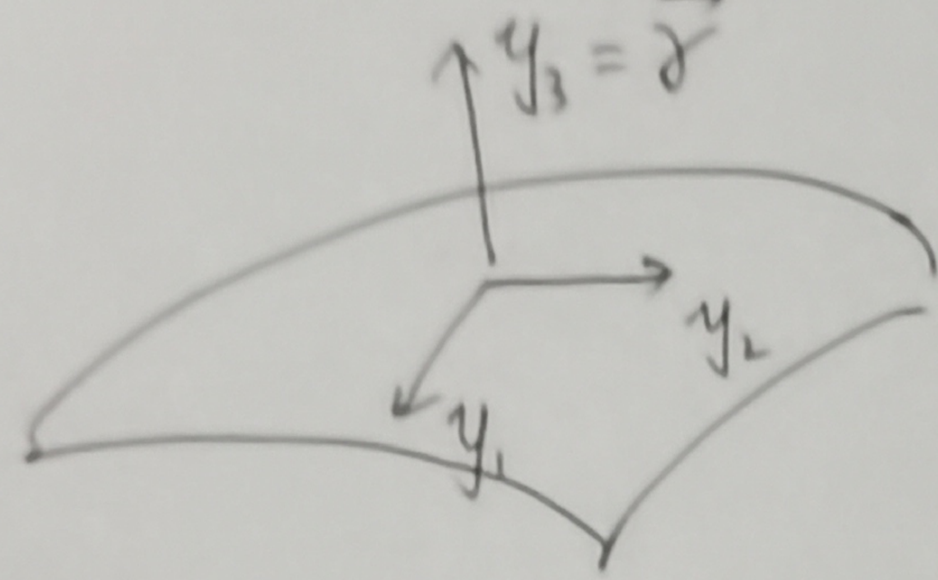
Stokes $\Rightarrow$ divergence in dim 3.

Take $M = \mathcal{U}$,

$$\omega = F_1 \, dx_2 \wedge dx_3 + F_2 \, dx_3 \wedge dx_1 + F_3 \, dx_1 \wedge dx_2$$

$$\Rightarrow d\omega = \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3} \right) dx_1 \wedge dx_2 \wedge dx_3 = \nabla \cdot \vec{F} \, dV$$

$$\begin{aligned}
\omega|_{\partial \mathcal{U}} &= F_1 \, \iota^*(dx_2 \wedge dx_3) + F_2 \, \iota^*(dx_3 \wedge dx_1) + F_3 \, \iota^*(dx_1 \wedge dx_2) \\
&= (F_1 \nu_1 + F_2 \nu_2 + F_3 \nu_3) \, dA \\
&= \vec{F} \cdot \vec{\nu} \, dA.
\end{aligned}$$



$$\iota^*: \Omega^2(M) \longrightarrow \Omega^2(\partial M)$$

$$dy_1 \wedge dy_2 \longmapsto dy_1 \wedge dy_2 = dA$$ ← not confuse with matrix!

$$dy_2 \wedge dy_3 \longmapsto 0$$

$$dy_3 \wedge dy_1 \longmapsto 0$$

$$(y_1, y_2, y_3) = (x_1, x_2, x_3) A \qquad A = \begin{pmatrix} * & * & \gamma_1 \\ * & * & \gamma_2 \\ * & * & \gamma_3 \end{pmatrix} \qquad AA^T = A^TA = Id$$

$$(x_1, x_2, x_3) = (y_1, y_2, y_3) B \qquad B = A^{-1} = A^T$$

$$\cancel{dx_1 = b_{11} d}$$

$$\Rightarrow (dx_1, dx_2, dx_3) = (dy_1, dy_2, dy_3)\, B$$

$$dx_1 \wedge dx_2 = (b_{11} b_{22} - b_{12} b_{21})\, dy_1 \wedge dy_2 + \cdots$$

$$= a_{33}\, dy_1 \wedge dy_2 + \cdots$$

$$= \gamma_3\, dy_1 \wedge dy_2 + \cdots$$

$$\Rightarrow \iota^*(dx_1 \wedge dx_2) = \gamma_3\, dy_1 \wedge dy_2$$

$$\cancel{= \gamma_3\, dA}$$

$$= \nu_3\, dA$$

Stokes $\Rightarrow$ Stokes in dim 2

Take $M = M$, $\phi : M \hookrightarrow \mathbb{R}^3$

$$\omega' = F_1 dx_1 + F_2 dx_2 + F_3 dx_3 \in \Omega^1(\mathbb{R}^3)$$

$$\omega = \phi^* \omega' \in \Omega^1(M)$$

$$\Rightarrow d\omega = d\phi^* \omega' = \phi^*(d\omega')$$

$$= \phi^*(\mathrm{rot}\,\vec{F})$$

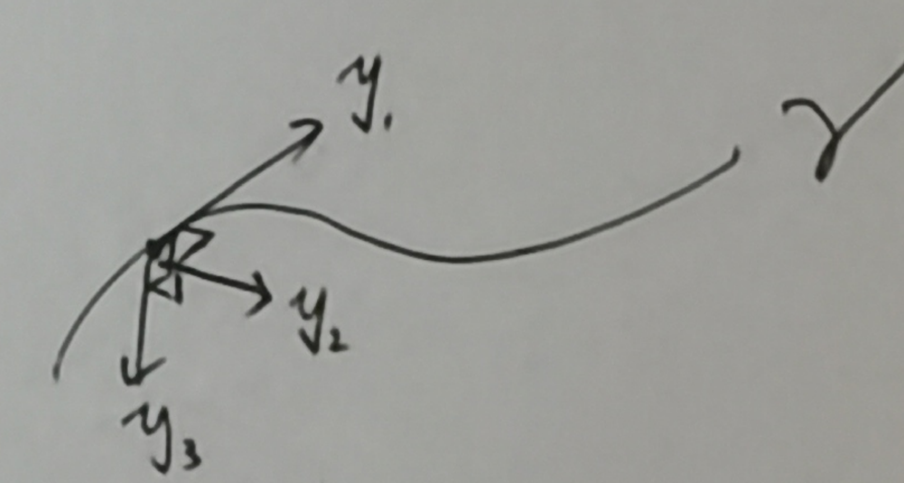
$$= \mathrm{rot}\,\vec{F} \cdot \vec{\gamma}\, dA$$

$$\omega|_\gamma = \iota^* \omega$$

$$= \iota^* \phi^* \omega'$$

$$= (\phi \circ \iota)^* \omega'$$

$$= \vec{F} \cdot d\vec{x}$$



$$\gamma \xhookrightarrow{\iota} M \xhookrightarrow{\phi} \mathbb{R}^3.$$

$$\rightsquigarrow (\iota \circ \phi)^* : \Omega^1(\mathbb{R}^3) \longrightarrow \Omega^1(\gamma)$$

$$dy_1 \longmapsto dy_1$$

$$dy_2 \longmapsto 0$$

$$dy_3 \longmapsto 0$$

Recall that

$$(dx_1, dx_2, dx_3) = (dy_1, dy_2, dy_3)\, B$$

$$\sum_i F_i\, dx_i = (F_1 b_{11} + F_2 b_{12} + F_3 b_{13})\, dy_1 + \cdots$$

$$= (F_1 a_{11} + F_2 a_{21} + F_3 a_{31})\, dy_1 + \cdots$$

$$= \vec{F} \cdot \vec{y}_1\, dy_1 + \cdots$$

$$\Rightarrow (\phi \circ \iota)^* \omega' = \vec{F} \cdot \vec{y}_1\, dy_1$$

$$= \vec{F} \cdot d\vec{x}$$

$\square$