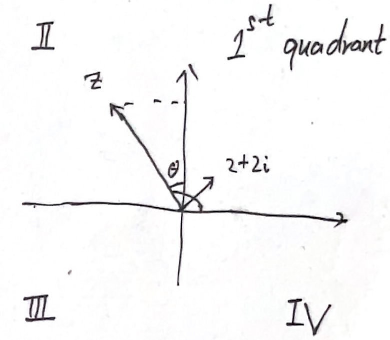
~~Tutorial 4. E~~

Session 4 & Exercise 1, 2.

Answer ALL questions in complete sentences!

Ex 2.1 (a) Compute Re z, Im z for $z = (2+2i)^3$
and write down the polar coordinate of z



A: $z = (2+2i)^3$

$= 8(1+i)^3$

$= 16(1-i)$

Re $z = 16$

Im $z = -16$

$|z| = 16\sqrt{2}$

$\arg z = \theta + \frac{\pi}{2} = \frac{3\pi}{4}$ $\quad \left(\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4}\right)$

In polar coordinate, $z = (16\sqrt{2}, \frac{3\pi}{4})$.

Ex. work out the case for $z = (1-i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$

⚠ For safety, only use $\tan\theta$ for θ acute angle!

[Reason: $a = \tan\theta \not\Leftrightarrow \theta = \arctan a$]

Today: Preliminary

(Set, Pictures and Equivalent descriptions)

Set.

$\{x \in X \mid \text{conditions of } x\}$

This is the most common way to describe a set.

e.g. $\{x \in \mathbb{Z} \mid -2 < x < 3.5\} = \{-1, 0, 1, 2, 3\}$

$A := \{x \in \mathbb{Z}_{>0} \mid -2 < x < 3.5\} = \{ \quad 1, 2, 3\}$

element & subset

$2 \in A$ element $\qquad$ $\{1,2\} \subseteq A$ subset

"Def": We say $X \subseteq Y$, if $[x \in X \Rightarrow x \in Y]$.

Ex. $3 \in A$? $\quad \{3\} \in A$? $\quad \emptyset \in A$?

$3 \subseteq A$? $\quad \{3\} \subseteq A$? $\quad \emptyset \subseteq A$?

~~$\{x \mid x \in A$~~ $[1,3) := \{x \in \mathbb{R} \mid 1 \leq x < 3\}$

$[1, \infty) := \{x \in \mathbb{R} \mid 1 \leq x \quad \}$.

guess: $(-4, 0] := \{ \quad \mid \quad \}$

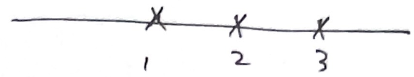
Adv. Ex. Define $A_0 := \emptyset$

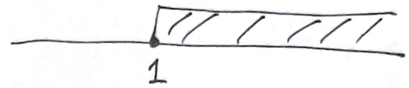
$A_{k+1} := \{x \mid x \subseteq A_k\}$ for $k \in \mathbb{N}$.

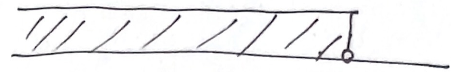
Compute A_1, A_2, A_3.

Draw pictures.

subsets of $\mathbb{R}$.
(number line)

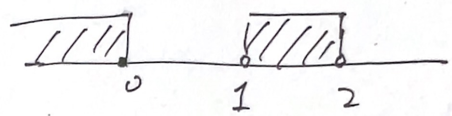
A 

$[1, +\infty)$ 

$(-\infty, 3)$ 

$[1, +\infty) \cap (-\infty, 3)$?

$[1, +\infty) \cup (-\infty, 3)$?

$\{x \in \mathbb{R} \mid \cdots \}$? 

? $[0,4) \cap ([-3,0] \cup (2,\infty))$?

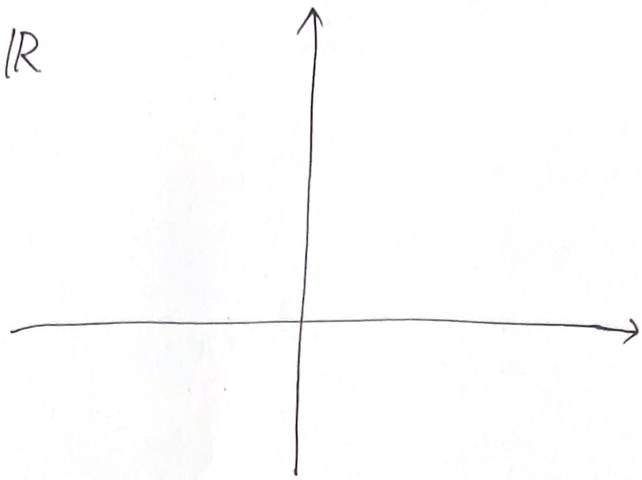
$\cancel{\{x \in \mathbb{R} \mid x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\} = ?}$

$\{x \in \mathbb{R} \mid x = \frac{\pi}{2} + 2k\pi, k \in \mathbb{Z}\}$?

$\cancel{\{x \in \mathbb{N} \mid}$

$\{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{Z}_{>0}\}$?

Adv. Ex. Compute M°, ∂M, M', $\overline{M}$, $\{$isolated pts of $M\}$ for M (in the last three examples) fcts on $\mathbb{R}$



$f(x) = x$	$f(x) = \lvert x \rvert$	$f(x) = e^x$
$f(x) = 3x+1$	$f(x) = [x]$	$f(x) = \ln x \quad (x>0)$
$f(x) = \frac{1}{x}$		$f(x) = \sin x$
$f(x) = x^2$		$f(x) = \cos x$
$f(x) = x^3$		$f(x) = \tan x$
$\vdots$		$\vdots$
Junior high school		Senior high school.

$$\left.\begin{array}{l} f(x)=\sin x \\ f(x)=\cos x \\ f(x)=\tan x \end{array}\right\} \text{arc}\ldots ?$$

Ex. Read from the picture

$\{x \in \mathbb{R} \mid e^x = 1\}$

$\{x \in \mathbb{R} \mid e^x > 1\}$

$\{x \in \mathbb{R} \mid \sin x = 1\}$

$\{x \in \mathbb{R} \mid x^2 - 1 \geq 0\}$

$\{x \in \mathbb{R} \mid \sin x \geq \cos x\}$

Rmk. What we draw is actually the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = f(x)\}$$

Q: How to draw

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > f(x)\}$$

e.g. $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > \sin x\}$?

Ex. $\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > x+1,\ y \leq 3x+1\}$

$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > e^x,\ y \leq 3\}$

$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y > e^x \text{ or } y \leq 3\}$

$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y^2 > x\}$

Equivalent descriptions.

$f, g \in \mathbb{R}, \quad a \in \mathbb{R}$

$$f = g \iff a+f = a+g \iff f-g=0$$

$$f = g \not\Leftrightarrow af = ag$$

When $a \neq 0$,

$$f = g \iff af = ag$$

Ex. solve equations $ax+b=0 \quad (a, b \in \mathbb{R})$

i.e. describe the set $\{x \in \mathbb{R} \mid ax+b=0\}$

A: $\quad ax+b = 0$

$\iff \quad ax = -b \quad ☆$

When $a \neq 0$, $\quad ☆ \iff x = -\frac{b}{a}$ $\quad$ solution space $= \{-\frac{b}{a}\}$

When $a = 0$, $\quad ☆ \iff 0 = -b$

When $b \neq 0$, $\quad$ ———— $= \emptyset$

When $b = 0$ $\quad$ ———— $= \mathbb{R}$

$$f^2 = g \not\Leftrightarrow f = \sqrt{g}$$

When $g>0$, $f^2 = g \Leftrightarrow f = \sqrt{g}$ or $f = -\sqrt{g}$

When $g=0$, $f^2 = g \Leftrightarrow f = 0$

When $g<0$, $f^2 = g \Leftrightarrow$ impossible.

Ex. solve equations $ax^2+bx+c=0$, $a,b,c \in \mathbb{R}$.

$= \cdot \Rightarrow \geq$: What ~~w~~ would happen?

Useful:

$$f > g \Leftrightarrow f-g > 0$$

$$f = g \Leftrightarrow f-g = 0$$

$$f < g \Leftrightarrow f-g < 0$$

Resolve:

$$\{(x,y) \in \mathbb{R} \times \mathbb{R} \mid x^2+y^2>1\}$$

Ex. 2.2, 2.4.

Describe M°, ∂M, $\overline{M}$, M', {isolated pts of M}.

for

a) $M_1 = \{ z \in \mathbb{C} \mid \sin(\operatorname{Re} z) = 1 \}$

b) $M_2 = \{ z \in \mathbb{C} \mid \operatorname{Im} z < 2 \}$

c) $M_3 = \{ z \in \mathbb{C} \mid |z| = \operatorname{Re}(z) + 1 \}$

$M_A = \{0\} \cup [1,2) \subseteq \mathbb{R}$

$M_B = \{0\} \cup [1,2) \subseteq \mathbb{C}$

$M_C = (\{0\} \cup [1,2)) \cap \mathbb{Q} \subseteq \mathbb{R}$

$M_D = \{ p/q \mid p,q \in \mathbb{N}_{>0},\ |p-q| = 1 \} \subseteq \mathbb{R}$

Task 1. For $\alpha \in \mathbb{R}$, we define

$$f_\alpha : \mathbb{R} - \{0\} \longrightarrow \mathbb{R} \qquad f_\alpha(x) = |x|^\alpha \arctan\left(\frac{1}{x}\right)$$

a) Compute $\underline{\lim}_{x \to 0} f_\alpha(x)$ and $\overline{\lim}_{x \to 0} f_\alpha(x)$.

b). Find all $\alpha \in \mathbb{R}$ s.t. $\lim_{x \to 0^+} f_\alpha(x)$ exists.

(i.e. write down $\{\alpha \in \mathbb{R} \mid \lim_{x \to 0^+} f_\alpha(x) \; \exists\}$)

Similarly, find $\{\alpha \in \mathbb{R} \mid \lim_{x \to 0^-} f_\alpha(x) \; \exists\}$

c). Find ~~all~~ all $\alpha \in \mathbb{R}$ s.t.

f_α can be extended to a $g \in C(\mathbb{R})$.

Determine the g in this case.

Task 2. Define two fcts on $\mathbb{R}$.

$$\mathrm{sign}(x) := \begin{cases} \frac{x}{|x|} & x \in \mathbb{R} \setminus \{0\} \\ 0 & x = 0 \end{cases}$$

$$g(x) := \begin{cases} \cos\left(\frac{1}{x}\right) & x \in \mathbb{R} - \{0\} \\ 0 & x = 0 \end{cases}$$

Find out the uncontinuous pts of sign & g, i.e.

$$\{ x_0 \in \mathbb{R} \mid \mathrm{sign}(x) \text{ is not cont at } x_0 \}.$$