Eine Woche, ein Beispiel

6.5 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory.
In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categorys here: http://katmat.math.uni-bremen.de/acc/acc.pdf

For Chinese, the theory of category has been summed up in detail in [https://wwli.asia/downloads/books/Al-jabr-1.pdf], Chapter 2-3.

Process

0. Well-know concepts
1. Individual category
   - Complete/Cocomplete/Bicomplete category
   - Cartesian closed category / Closed category
   - Monoidal category = Tensor category
2. Functors between categories
   - Exactness
   - Adjoints
3. Examples of categories
   - Well-known examples
   - $\mathcal{C}at$
   - Hausdorff and compactness
   - Categories in condensed mathematics

Appendix.

Just for the convenience of writing, I would use the following terminologies.

A **class** is a collection of sets.

A **metaclass** is a collection of classes.

A **meta^{2}class** is a collection of metaclasses.

...

For the most time it can be turned to normal sentenses without metaclass.

e.g. Consider a category $\mathcal{C}$.

I want to write it shorter, so I write: Consider $\mathcal{C} \in \mathcal{C}at_{big}$.

But "a proper class can not be written on the left of the symbol '$\in$'", or

"a class is a collection of sets, so no proper class can be contained in a class".

For these silly set-theory problem, I just call $\mathcal{C}at_{big}$ as the metacategory.

For me, it is always quicker to write "consider $M \in Mod(A)$" than "consider an A-module M".

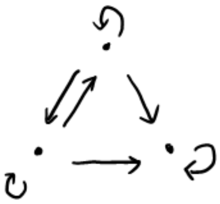
0. well-known concepts

$\mathcal{C}$ is always a category.

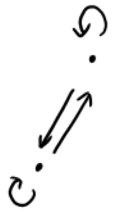
	$Ob(\mathcal{C})$	$Mor(X,Y)$
small	Set	Set
loc. small	—	Set
large	not set or	not set

filtered. directed

cofiltered.

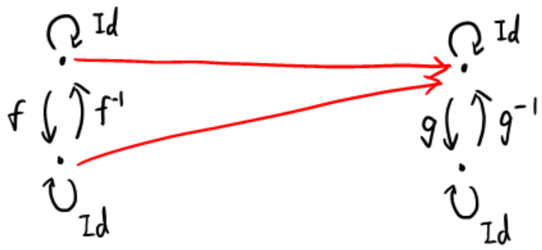


category

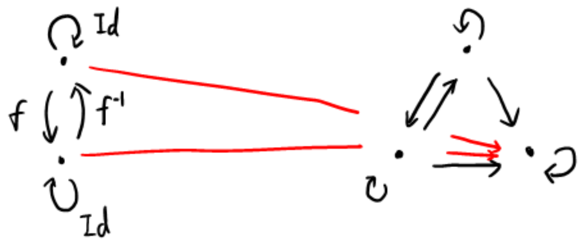


full subcategory +saturated

https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective



equivalence of category



fully faithful

https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/
https://mathoverflow.net/questions/2150/exactness-of-filtered-colimits
https://mathoverflow.net/questions/57099/why-do-filtered-colimits-commute-with-finite-limits

E.g. $\mathrm{Set} \longrightarrow \mathrm{Vect}(k)$ is not an equiv of category.

$$\begin{array}{ccc} S & & k^{\oplus S} \\ \downarrow & \Rightarrow & \downarrow \\ T & & k^{\oplus T} \end{array}$$

E.g. Let L be some alg closure of $\mathbb{Q}$,

$\mathcal{C}_2 := \{\text{alg ext of } \mathbb{Q}\} \leq \mathrm{Field}$

$\mathcal{C}_1 := \{\text{subfield of } L\} \leq \mathrm{Field}$

Then $\mathcal{C}_1 \hookrightarrow \mathcal{C}_2$ is an equiv of category.

1. Individual category

Complete/Cocomplete/Bicomplete category

Def. $\mathcal{C}$ is complete if

$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow C \quad i \longmapsto F_i$,

$\varprojlim_{i\in\Lambda} F_i$ exists (${\varprojlim_{i\in\Lambda}} F_i$ is called the small limit)

$\mathcal{C}$ is cocomplete if

$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow C \quad i \longmapsto F_i$,

$\varinjlim_{i\in\Lambda} F_i$ exists ($\varinjlim_{i\in\Lambda} F_i$ is called the small colimit)

bicomplete = complete + cocomplete

$\mathcal{C}$ is finitely complete if $\forall$ finite limit exists

$\mathcal{C}$ is finitely cocomplete if $\forall$ finite colimit exists.

Thm. $\mathcal{C}$ is complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & products

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & products

$\mathcal{C}$ is cocomplete $\Leftrightarrow$ $\mathcal{C}$ has coequalizers & coproducts

$\Leftrightarrow$ $\mathcal{C}$ has pushouts & coproducts

$\mathcal{C}$ is finitely complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & finite products

$\Leftrightarrow$ $\mathcal{C}$ has equalizers, binary products & terminal obj

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & terminal obj

For small category $\mathcal{C}$,

complete $\Leftrightarrow$ cocomplete

$\Downarrow$ $\quad$ $\Downarrow$

thin ($\# \mathrm{Mor}(X,Y) \leq 1$)

The "closedness" of the category is that taking the morphisms between two objects gives another morphism in the same category, rather than the category of sets or some other category.
from: https://math.stackexchange.com/questions/3486846/definition-of-cartesian-closed-category-why-do-we-need-exponential-objects

Cartesian closed category / Closed category

Def. $\mathcal{C}$ is **Cartesian closed** if
$\mathcal{C}$ has terminal obj, binary product and exponential, where

a bifctor $F: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$
which is functorial in Y

$$- \times Y \dashv (-)^Y$$

i.e. $Mor(X\times Y, Z) \cong Mor(X, Z^Y)$

$\mathcal{C}$ is **loc. Cartesian closed** if all its slice category is Cartesian closed.

https://ncatlab.org/nlab/show/over+category

Rmk. When $\mathcal{C}$ is loc. Cartesian closed,
$\mathcal{C}$ is Cartesian closed $\Leftrightarrow$ $\mathcal{C}$ has a terminal object.
But $\mathcal{C}$ is Cartesian closed $\not\Rightarrow$ $\mathcal{C}$ is loc. Cartesian closed

For the closed category, we use the definition in https://ncatlab.org/nlab/show/closed+category.

Def A **closed category** is a category $\mathcal{C}$ together with the following data.

- bifctor $[-,-] : \mathcal{C}^{op}\times\mathcal{C}\to\mathcal{C}$ — called internal hom-fctor
- $I \in Ob(\mathcal{C})$ — called unit object
- $i: Id_{\mathcal{C}} \xrightarrow{\cong} [I,-]$ $\rightsquigarrow$ $i_A: A \xrightarrow{\cong} [I, A]$
- $j_X: I \longrightarrow [X,X]$ — extranatural in X
- $L^X_{Y,Z}: [Y,Z] \to [[X,Y],[X,Z]]$ — functorial in Y and Z, extranatural in X.
- Compatabilities

$$\begin{tikzcd}
I \arrow[r, "j_Y"] \arrow[dr, "j_{[X,Y]}"'] & {[Y,Y]} \arrow[d, "L^X_{Y,Y}"] \\
& {[[X,Y],[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[X,Y]} \arrow[r, "L^X_{XY}"] \arrow[dr, "i_{[X,Y]}"'] & {[[X,X],[X,Y]]} \arrow[d, "{[j_X,1]}"] \\
& {[I,[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[Y,Z]} \arrow[r, "L^I_{YZ}"] \arrow[dr, "{[1,i_Z]}"'] & {[[I,Y],[I,Z]]} \arrow[d, "{[i_Y,1]}"] \\
& {[Y,[I,Z]]}
\end{tikzcd}$$

$$\begin{tikzcd}
& {[U,V]} \arrow[dl, "L^X_{UV}"'] \arrow[dr, "L^Y_{UV}"] & \\
{[[X,U],[X,V]]} \arrow[d, "L^{[X,Y]}_{[X,U][X,V]}"'] & & {[[Y,U],[Y,V]]} \arrow[d, "{[1,L^X_{YV}]}"] \\
{[[[X,Y],[X,U]],[[X,Y],[X,V]]]} \arrow[rr, "{[L^X_{YU},1]}"] & & {[[Y,U],[[X,Y],[X,V]]]}
\end{tikzcd}$$

$$\begin{aligned}
\gamma: Mor(X,Y) &\longrightarrow Mor(I,[X,Y]) \\
f &\longmapsto [1,f]\circ j_X
\end{aligned}$$
is an iso.

Thm Let $(\mathcal{C}, \otimes)$, $(\mathcal{D}, \otimes)$ be two Cartesian closed category,

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "f"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "g"] \end{tikzcd} \qquad \mathcal{F} \in \mathcal{C}, \quad \mathcal{G} \in \mathcal{D}$$

$$f(\mathcal{F} \otimes \mathcal{F}') \cong f\mathcal{F} \otimes f\mathcal{F}' \qquad \forall \mathcal{F}, \mathcal{F}' \in Ob(\mathcal{C})$$

From the adjuction, we know that

$$Hom_{\mathcal{D}}(f\mathcal{F}, \mathcal{G}) \cong Hom_{\mathcal{C}}(\mathcal{F}, g\mathcal{G}) \quad \text{as a set}$$

We can upgrade it as iso in internal Hom.

$$g\,\underline{Hom}_{\mathcal{D}}(f\mathcal{F}, \mathcal{G}) \cong \underline{Hom}_{\mathcal{C}}(\mathcal{F}, g\mathcal{G})$$

Proof. $\forall \mathcal{H} \in Ob(\mathcal{C})$,

$$\begin{aligned} & Hom_{\mathcal{C}}(\mathcal{H}, g\,\underline{Hom}_{\mathcal{D}}(f\mathcal{F}, \mathcal{G})) \\ \cong\ & Hom_{\mathcal{D}}(f\mathcal{H}, \underline{Hom}_{\mathcal{D}}(f\mathcal{F}, \mathcal{G})) \\ \cong\ & Hom_{\mathcal{D}}(f\mathcal{H} \otimes f\mathcal{F}, \mathcal{G}) \\ \cong\ & Hom_{\mathcal{D}}(f(\mathcal{H} \otimes \mathcal{F}), \mathcal{G}) \\ \cong\ & Hom_{\mathcal{C}}(\mathcal{H} \otimes \mathcal{F}, g\mathcal{G}) \\ \cong\ & Hom_{\mathcal{C}}(\mathcal{H}, \underline{Hom}_{\mathcal{C}}(\mathcal{F}, g\mathcal{G})) \end{aligned}$$

By Yoneda lemma, we get iso. $\square$

Monoidal category = Tensor category　　幺半范畴

Def A monoidal category is a category $\mathcal{C}$ together with the following data.

- bifctor $-\otimes- : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- $I \in Ob(\mathcal{C})$　　called unit object
- $\alpha_{A,B,C} : A\otimes(B\otimes C) \xrightarrow{\cong} (A\otimes B)\otimes C$
- $\lambda_A : I\otimes A \xrightarrow{\cong} A$　　lambda: left
- $\rho_A : A\otimes I \xrightarrow{\cong} A$　　rho: right
- Compatabilities

$$\begin{tikzcd}[column sep=small]
 & A\otimes(B\otimes(C\otimes D)) \arrow[dl, "1_A\otimes\alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
A\otimes((B\otimes C)\otimes D) \arrow[d, "\alpha_{A,B\otimes C,D}"'] & \circlearrowright & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha_{A,B,C}\otimes 1_A"] & & ((A\otimes B)\otimes C)\otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(I\otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A\otimes\lambda_B"'] & & (A\otimes I)\otimes B \arrow[dl, "\rho_A\otimes 1_B"] \\
 & A\otimes B &
\end{tikzcd}$$

For strict monoidal category, we require in addition that $\alpha_{A,B,C}$, λ_A, ρ_A are identities.

E.g. Cartesian monoidal category $\mathcal{C}$: category with finite products

$\otimes = \prod$　　　I = terminal object

e.g. **Set**, **Cat**.

Cocartesian monoidal category $\mathcal{C}$: category with finite coproducts

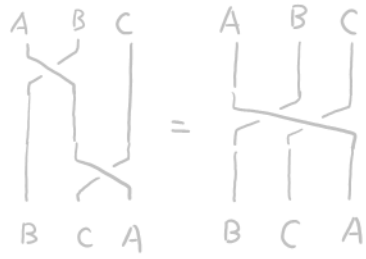
$\otimes = \coprod$　　　I = initial object

Abelian category is monoidal.

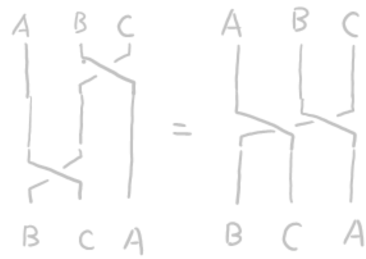
Def (Specializations)

Let $\mathcal{C}$ be a monoidal category.
If in addition we have $\gamma_{A,B}: A\otimes B \longrightarrow B\otimes A$,
then $\mathcal{C}$ is braided monoidal category if

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[dl, "\gamma_{A,B}\otimes 1_C"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"] & & A\otimes(B\otimes C) \arrow[d, "\gamma_{A,B\otimes C}"] \\
B\otimes(A\otimes C) \arrow[dr, "1_B\otimes\gamma_{A,C}"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B\otimes(C\otimes A) &
\end{tikzcd}$$



$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[dl, "1_A\otimes\gamma_{B,C}"'] \arrow[dr, "\alpha^{-1}_{A,B,C}"] & \\
A\otimes(C\otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] & & (A\otimes B)\otimes C \arrow[d, "\gamma_{A\otimes B,C}"] \\
(A\otimes C)\otimes B \arrow[dr, "\gamma_{C,A}\otimes 1_B"'] & & C\otimes(A\otimes B) \arrow[dl, "\alpha^{-1}_{C,A,B}"] \\
 & (C\otimes A)\otimes B &
\end{tikzcd}$$



$$\begin{tikzcd}
I\otimes A \arrow[rr, "\gamma_{I,A}"] \arrow[dr, "\lambda_A"'] & & A\otimes I \arrow[dl, "\rho_A"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[rr, "\gamma_{A,I}"] \arrow[dr, "\rho_A"'] & & I\otimes A \arrow[dl, "\lambda_A"] \\
 & A &
\end{tikzcd}$$

$\mathcal{C}$ is symmetric monoidal category if
$\gamma_{B,A}\circ\gamma_{A,B} = 1_{A\otimes B}$. + $\mathcal{C}$ is braided.

closed monoidal category = closed category + monoidal category
+ compatabilite $-\otimes A \dashv [A,-]$

2. Functors between categories

Exactness

Ref:https://ncatlab.org/nlab/show/exact+functor

Prop/Def For a fctor $F: \mathcal{C} \longrightarrow \mathcal{D}$ between finitely complete categories. TFAE:

- F preserves finite limits,
- F preserves equalizers & finite products,
- F preserves equalizers, binary products & terminal objects,
- F preserves pullbacks & terminal objects,
- $\forall d \in Ob(\mathcal{D})$, the comma category F/d is filtered.

If so, we call F is a left exact fctor.

We require also that $\mathcal{C}$ & $\mathcal{D}$ are finitely complete categories

When $\mathcal{C}, \mathcal{D}$ are abelian categories, this is equivalent to

- F preserves kernels, i.e.

  F sends left exact sequences to left exact sequences.

See [https://stacks.math.columbia.edu/tag/010M]. You may get the following results from the argument:

F is a left exact fctor between abelian categories $\Rightarrow$ F preserve binary products

$\Updownarrow$ [oDLP]

F is additive

Def. A contravariant fctor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is called left exact, if

$F: \mathcal{C}^{op} \longrightarrow \mathcal{D}$ is left exact (In ptc, $\mathcal{C}$ is finitely cocomplete, $\mathcal{D}$ is finitely complete)

Similarly, we can define right exactness, and

exact = left exact + right exact

$\overset{\mathcal{C},\mathcal{D}\text{ abelian}}{=\!=}$ sends SES to SES.

Adjoints

For more adjoints, see §3 on examples.

left adjoint $\dashv$	right adjoint
free	forget
$-\otimes_A N$	$\mathrm{Hom}_A(N,-)$
Δ	$\varprojlim$
$(\ )^\sim$	Γ_*
f_p	f^*
c-Ind	Res
Res	Ind
$-\otimes_k A^e$	Res
sh $(-)$	$\mathrm{Res}_{sh\to Psh}$
π_0	$T\mapsto \coprod_{t\in T}\mathrm{Spec}\,\mathbb{Z}$
$G^{[p]}$	Lie
$\lvert - \rvert$	S
$\tau_1 = H_0(-)$	N: nerve
$W_{\mathbb{Z}_p}$	$(-)^b$
$k^{oa}\text{-mod} \rightleftarrows k^{o}\text{-mod}$ (with $\perp$) $\quad (-)^a$	$(-)_*$
inclusion $\mathcal{D}^{\le n}\hookrightarrow \mathcal{D}$	$\tau^{\le n}$
$\tau^{\ge n}$	$\mathcal{D}^{\ge n}\hookrightarrow \mathcal{D}$

$\Phi_{P_L} \dashv \Phi_P \dashv \Phi_{P_R}$

$P_L := P^\vee \otimes p^*\omega_Y[\dim(Y)]$

$P_R := P^\vee \otimes q^*\omega_X[\dim(X)]$

f^*	f_*
$f_!$	$f^!$

$$\begin{tikzcd} Z \arrow[r, "i_*"] & X \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{U} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

$i_* = i_!$ $\quad j^* = j^!$

ad $\Rightarrow$ preserve colimits | preserve limits

$\Rightarrow$ right exacts | left exacts

in (co) complete category

coker f | ker f

$\coprod A$ $\quad \oplus A$ | $\prod A$ $\quad A \times_C B$

pushforward | pullback

coequalizer | equalizer

$\bar{k} = \varinjlim_{L/k} L$ | $\operatorname{Spec} \bar{k} = \varprojlim_{L/k} \operatorname{Spec} L$ $\quad Gal(\bar{k}/k) = \varprojlim_{L \text{ finite Galois}} Gal(L/k)$

$\operatorname{Spec} \mathbb{Z}_p = \varinjlim_n \operatorname{Spec} \mathbb{Z}/p^n\mathbb{Z}$ $\quad \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

completion $\quad$ ring point of view

$\mathcal{F}_p = \varinjlim_{U} \mathcal{F}(U)$

stalk

co limit | limit

direct, inductive, injective $\}$ limit. | inverse, projective $\}$ limit.

How to memorize:

$$0 \longrightarrow \underset{\xleftarrow{\text{limit}}}{\ker} \longrightarrow M \longrightarrow N \longrightarrow \underset{\xrightarrow{\text{colimit}}}{\operatorname{coker}} \longrightarrow 0$$

notations	$\bigoplus_I R$	$\prod_I R$
initial example	$\bigoplus_{\mathbb{N}_{\geq 0}} \mathbb{R} = \mathbb{R}[\mathbb{N}_{\geq 0}] \sim \mathbb{R}[x]$	$\prod_{\mathbb{N}_{\geq 0}} \mathbb{R} = Map(\mathbb{N}_{\geq 0}, \mathbb{R}) \sim \{\text{number seq}\}$
generalization topology as R-module	group algebra algebra homology projective	functional space analysis cohomology not injective
GAGA finite cpt (proj)	$\bigoplus_I R$ $\quad\cong$ proj cplx varieties $\quad\cong$	$\prod_I R$ proj cplx analytic spaces
bilinear pairing cap product	$\prod_I R \times \bigoplus_I R \longrightarrow R$ $(\{r_i\}, \{s_i\}) \longmapsto \sum_i r_i s_i$	

https://math.stackexchange.com/questions/320444/why-isnt-an-infinite-direct-product-of-copies-of-bbb-z-a-free-module
https://math.stackexchange.com/questions/1965000/direct-product-of-injective-modules-is-injective-stucked-in-proof

3. Examples of categories

Well-known examples

$\mathcal{S}et$ $\quad \mathcal{T}op$ $\quad \mathcal{G}rp$ $\quad \mathcal{A}b$ $\quad \mathrm{Vect}(k)$ $\quad \mathrm{Mod}(R)$

Ring: identity + preserve identity + $0 \in \mathrm{Ring}$ $\qquad$ CRing $\quad$ Rng

Field: full subcategory of CRing

$\mathbf{0}$: $\quad Ob(0) = \emptyset$

$\mathbf{1}$: $\quad Ob(1) = \{*\} \quad Mor(*,*) = \{1_*\}$

https://unapologetic.wordpress.com/2007/05/24/cardinals-and-ordinals-as-categories/

In general, for a set X, one can construct a category $\mathcal{C}_X$ as follows:

$$Ob(\mathcal{C}_X) = X \qquad Mor(x,y) = \begin{cases} \emptyset, & x \neq y \\ \{1_x\}, & x = y \end{cases}$$

$K(2)$: $\quad Ob(K(2)) = \{V, E\} \qquad Mor(V,V) = \{1_V\} \quad Mor(E,E) = \{1_E\}$

Another name: Arr

$$Mor(V,E) = \emptyset \qquad Mor(E,V) = \{s, t\}$$

$$\begin{tikzcd} E \arrow[loop left, "1_E"] \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \arrow[loop right, "1_V"] \end{tikzcd}$$

Δ: $\quad Ob(\Delta) = \{[n] := \{0,1,2,\cdots n\} \mid n \geq 0\}$

$$Mor([m],[n]) = \{\text{weakly monotone maps}\}$$

sSet: $\quad Ob(sSet) = \left\{ X : \Delta^{op} \longrightarrow Set \atop [n] \longmapsto X_n \right\} \qquad Mor(X,Y) = \left\{ \alpha : \begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd} \right\}$

CHaus: $\quad Ob(CHaus) = \{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} \; X \}$

https://ncatlab.org/nlab/show/compactum

$$Mor(X,Y) = \{ f : X \longrightarrow Y \mid f \text{ cont} \}$$

Met: full subcategory of CHaus whose objects are metric spaces.

⚠ For the category of Graph, there're different realizations.

$\mathrm{Quiv}(\mathcal{C})$: $\quad Ob(\mathrm{Quiv}(\mathcal{C})) = \{ \text{fctor } \Gamma : K(2) \longrightarrow \mathcal{C} \}$

$$Mor(\Gamma_1, \Gamma_2) = \left\{ \alpha : \begin{tikzcd} K(2) \arrow[r, bend left, "\Gamma_1"] \arrow[r, bend right, "\Gamma_2"'] & \mathcal{C} \end{tikzcd} \right\}$$

$\mathcal{Q}uiv \quad = \mathrm{Quiv}(\mathcal{S}et)$

$\qquad\quad$ = Category of presheaves on $\mathcal{Q}^{op}$.

Cat

$\mathbf{Cat}$ = {the category of small categories} is a 2-category.

$Ob(\mathbf{Cat})$ = {small category $\mathcal{C}$}

$Mor(\mathcal{C},\mathcal{D})$ is a category by

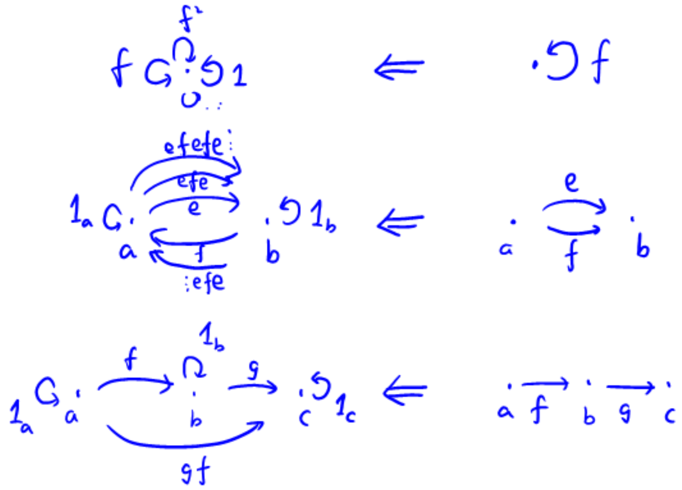
$$Ob(Mor(\mathcal{C},\mathcal{D})) = \{ F: \mathcal{C} \longrightarrow \mathcal{D} \}$$

$$Mor(F,G) = \left\{ \alpha : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \right\}$$

Basic properties of Cat:

1. Initial object 0, Terminal object 1.
2. Cat is loc. small but not small
3. Cat is bicomplete
4. Cat is Cartesian closed but not loc. Cartesian closed
5. Cat is loc. finitely presentable https://ncatlab.org/nlab/show/locally+finitely+presentable+category
6. $\begin{tikzcd} \mathbf{Cat} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\top"] & \mathbf{Quiv} \arrow[l, bend right, "\text{free}"'] \end{tikzcd}$

e.g of "free"



https://math.stackexchange.com/questions/750731/is-there-a-category-of-categories

Inspired by this question and answer, I will use metacategory to indicate $\mathbf{Cat}_{big}$, where

$Ob(\mathbf{Cat}_{big})$ = {all categories} as a metaclass

$Mor(\mathcal{C},\mathcal{D})$ is a metacategory by

$$Ob(Mor(\mathcal{C},\mathcal{D})) = \{ F: \mathcal{C} \longrightarrow \mathcal{D} \}$$ as a metaclass

$$Mor(F,G) = \left\{ \alpha : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \right\}$$ as a metaclass

Hausdorff and compactness ← cpt ≈ (quasi)cpt

$$
\begin{array}{ccccc}
\mathrm{Haus} & \subset & \mathrm{wHaus} & \subset & \mathrm{Top} \\
\cup & & \cup & & \cup \\
\mathrm{CGHaus} & \subset & \mathrm{CGwHaus} & \subset & \mathrm{kTop} \\
\cup & & & & \cup \\
\mathrm{loc.CHaus} & = & = & = & \mathrm{loc.cpt} \\
\cup & & & & \cup \\
\mathrm{CHaus} & = & = & = & \mathrm{cpt}
\end{array}
$$

CW cplx, Met

topo mfld

Def. $X \in \mathrm{Top}$ is a **weak Hausdorff space** (in wHaus) if $\forall\, K \in \mathrm{CHaus}, \forall\, g: K \to X$ cont, $g(K) \subset X$ is closed.

Def. $X \in \mathrm{Top}$ is **locally compact** (in $\mathrm{loc.cpt}$) if $\forall\, p \in X$, $\exists$ cpt nbhd V (i.e. $p \in U \subseteq V \subseteq X$, $U \subseteq X$ open, V cpt)

$\mathrm{loc.CHaus} = \mathrm{loc.cpt} \cap \mathrm{Haus}$

see [https://en.wikipedia.org/wiki/Locally_compact_space] for other common definitions which are not equivalent in general.

Def. $X \in \mathrm{Top}$ is a **compactly generated / a k-space** (in kTop) if

- cpt gen in Condensed Math
- Hausdorff-cpt gen / k-space in wiki
- cpt gen / k-space in nlab
- k-space in AT II

$\left[\begin{array}{l} \mathrm{CGwHaus} = \mathrm{kTop} \cap \mathrm{wHaus} \\ \mathrm{CGHaus} = \mathrm{kTop} \cap \mathrm{Haus} \end{array}\right]$

https://mathoverflow.net/questions/47702/why-the-w-in-cgwh-compactly-generated-weakly-hausdorff-spaces

$\forall$ map $f: X \to Y$,

f is cont $\Leftrightarrow$ $K \xrightarrow{g} X \xrightarrow{f} Y$ is cont $\forall\, K \in \mathrm{CHaus}$, $g: K \to X$ cont

equivalently,

$\forall\, A \subseteq X$ subspace,

$A \subseteq X$ is closed $\Leftrightarrow$ $g^{-1}(A) \subseteq K$ is closed $\forall\, K \in \mathrm{CHaus}$, $g: K \to X$ cont

When X is Hausdorff, this is equivalent to

$\forall\, A \subseteq X$ subspace,

$A \subseteq X$ is closed $\Leftrightarrow$ $A \cap K \subseteq K$ is closed $\forall\, K \in \mathrm{CHaus}$

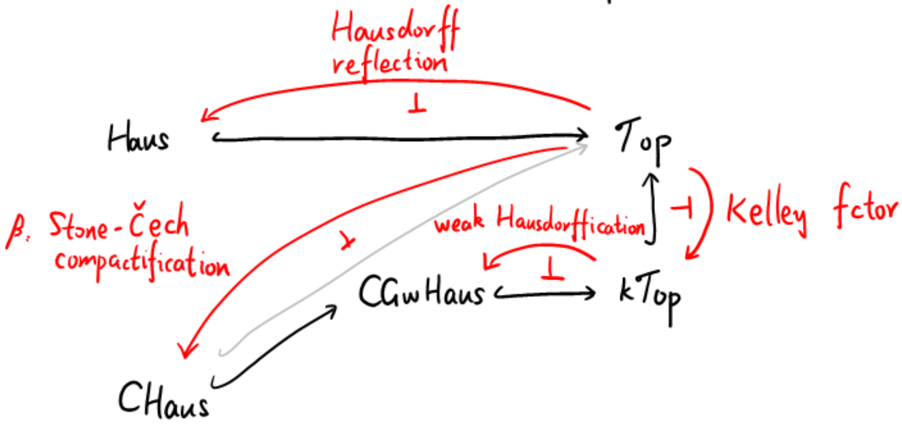
Prop. $X \in \mathrm{Top}$, then

X is a k-space $\Leftrightarrow$ $X \cong \coprod_{i \in \Lambda} S_i / \sim$ $\quad S_i \in \mathrm{CHaus}$

Rmk. In the def/prop of kTop, CHaus can be replaced by Prof.

Adjoints

⚠ We assume the Zorn's lemma, so that $\beta\mathbb{N}$ exists.



Kelley fctor $(-)^{cg}: \mathcal{T}op \longrightarrow k\mathcal{T}op$

$X \longmapsto X^{cg}$ compactly generated

Set: $X^{cg} = X$

Topo: $A \subseteq X^{cg}$ is closed if $g^{-1}(A) \subseteq K$ is closed $\forall K \in \mathrm{CHaus},\ g: K \to X$ cont

Definition and properties of Hausdorff reflection:
https://ncatlab.org/nlab/show/Hausdorff+space#HausdorffReflections

Detaied verification of the Hausdorff reflection:
https://math.stackexchange.com/questions/471238/construction-of-a-hausdorff-space-from-a-topological-space

More beautiful pictures of adjoint functors: https://arxiv.org/abs/2112.13654

Categories in condensed mathematics

$$
\begin{array}{ccccccccc}
 & & & & \mathrm{cpt} & \subset & \mathrm{qc} & & \\
 & & & \cup & & & & \cap & \\
\mathrm{qcProj} & \subset & \mathrm{Prof} & \subset & \mathrm{qcqs} & \subset & \mathrm{qs} & \subset & \mathrm{CondSet} \\
\cong & & \cong & & \cong & & \cong & & \bigcup \\
\mathrm{CplBoolAlg}^{op} & & \mathrm{BoolAlg}^{op} & & \mathrm{CHaus} & & \mathrm{Ind(CHaus)} & & \\
 & & & & & & & & \\
 & & & & \mathrm{Liq}_p & \subset & \mathrm{CondAb} & \supset & \mathrm{Solid} \\
 & & & & \cup & & \cup & & \\
 & & & & \mathrm{Liq}_p(\mathbb{R}) & \subset & \mathrm{Cond}(\mathbb{R}) & &
\end{array}
$$

https://ncatlab.org/nlab/show/Stone+duality
https://math.unice.fr/tacl/assets/2019/contributed/s3/1/6-nurakunov-stronkowski.pdf

Adjoints

$$
\begin{tikzcd}
\mathrm{Solid} \arrow[drr] & & \\
\mathrm{Liq}_p \arrow[rr, shift right] & & \mathrm{CondAb} \arrow[ll, bend right, "(-)^{liq}"'] \arrow[ll, phantom, "\perp"] \arrow[ull, bend right, "(\ )^{\blacksquare}"'] \arrow[ull, phantom, "\perp"] \arrow[rr, shift right] & & \mathrm{CondSet} \arrow[ll, bend right, "\mathbb{Z}[-]"'] \arrow[ll, phantom, "\perp"]
\end{tikzcd}
$$

$$\mathbb{Z}[X]^{liq} \longleftarrow\!\shortmid \mathbb{Z}[X] \longleftarrow\!\shortmid X$$

$\overset{X \text{ loc. prof}}{=\!=}$ $\bigcup_{\substack{S \subset X \\ S \text{ prof}}} \mathcal{M}_{<p}(S)$

$= $ sheafification of $[S \mapsto \mathbb{Z}[X(S)]]$

$\overset{X \text{ prof}}{=\!=}$ $\mathcal{M}_{<p}(X)$

$\overset{X = \{*\}}{=\!=}$ $\mathbb{R}$

Appendix

I'm just too lazy to fill in this table. If you know more, tell me and I will fill in, thanks!

ReCRing

Category	cpl	fin cpl	cocpl	fin cocpl	Cartesian closed	closed	monoidal
Set	✓		✓		✓		✓
Top	✓		✓		×		✓
Grp	✓		✓		×		✓
Ab	✓		✓		×		✓
Vect(K)	✓		✓		×		✓
Mod(R)	✓		✓		×		✓
Ring	✓		✓				
CRing	✓		✓				
Rng	✓		✓				
Field	×	×	×	×			
0							
1	✓		✓		✓	✓	✓
K(2)							
Δ	×	×	×	×			
sSet	✓		✓				
CHaus	✓		✓				
Met	×	✓	×	×			
Quiv($\mathcal{C}$)							
Quiv							
Cat	✓		✓		✓		✓
kTop							✓
CGHaus					✓		
CGwHaus	✓		✓		✓		
Prof	✓						