Eine Woche, ein Beispiel

10.27 Schur functor for Hodge modules

Goal: compute $\mathbb{S}^\lambda\left(\begin{smallmatrix} & 1 & \\ g & & g \\ & 1 & \end{smallmatrix}\right)$

$n = 2-2g$

χ(Hodge module)

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \qquad \square \quad n \qquad\qquad\qquad \begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array} \qquad e \quad n$$

$$\begin{array}{ccccc} & & 1 & & \\ & g & & g & \\ \binom{g}{2} & & g^2+1 & & \binom{g}{2} \\ & g & & g & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & g & & g & \\ \binom{g+1}{2} & & g^2+1 & & \binom{g+1}{2} \\ & g & & g & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 1 & & \\ & 2g & & 2g & \\ g^2 & & 2g^2+2 & & g^2 \\ & 2g & & 2g & \\ & & 1 & & \end{array}$$

$$\square\square \quad \binom{n+1}{2} \qquad\qquad \begin{smallmatrix}\square\\\square\end{smallmatrix} \quad \binom{n}{2} \qquad\qquad e^2 \quad n^2$$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & g & & g & & \\ & \binom{g}{2} & & g^2+1 & & \binom{g}{2} & \\ \binom{g}{3} & & g\binom{g}{2}+g & & g\binom{g}{2}+g & & \binom{g}{3} \\ & \binom{g}{2} & & g^2+1 & & \binom{g}{2} & \\ & & g & & g & & \\ & & & 1 & & & \end{array} \Bigg| \begin{array}{ccccccc} & & & 1 & & & \\ & & g & & g & & \\ & g^2 & & 2g^2+1 & & g^2 & \\ \frac{1}{3}(g^3-g) & & g^3+2g & & g^3+2g & & \frac{1}{3}(g^3-g) \\ & g^2 & & 2g^2+1 & & g^2 & \\ & & g & & g & & \\ & & & 1 & & & \end{array}$$

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & \binom{g+1}{2} & & g^2 & & \binom{g+1}{2} & \\ \binom{g+2}{3} & & g\binom{g+1}{2}+g & & g\binom{g+1}{2}+g & & \binom{g+2}{3} \\ & \binom{g+1}{2} & & g^2 & & \binom{g+1}{2} & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \Bigg| \begin{array}{ccccccc} & & & 1 & & & \\ & & 3g & & 3g & & \\ & 3g^2 & & 6g^2+3 & & 3g^2 & \\ g^3 & & 3g^3+6g & & 3g^3+6g & & g^3 \\ & 3g^2 & & 6g^2+3 & & 3g^2 & \\ & & 3g & & 3g & & \\ & & & 1 & & & \end{array}$$

$$\square\square\square \quad \binom{n+2}{3} \qquad \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \quad \frac{(n-1)n(n+1)}{1\cdot 3\cdot 1} \qquad \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} \quad \binom{n}{3} \qquad e^2 \quad n^3$$

Will do it in three steps:

① $\mathbb{S}^\lambda(g, g)$

② $\mathbb{S}^\lambda\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$

③ E.g. $\mathrm{Sym}^n\left(\begin{smallmatrix} & 1 & \\ g & & g \\ & 1 & \end{smallmatrix}\right)$

[MNP13] Murre, Jacob P., Jan Nagel and Chris A. M. Peters. Lectures on the Theory of Pure Motives. Univ. Lect. Ser. Providence, RI: American Mathematical Society (AMS), 2013.

⚠ Because of the super-commutativity rule

$$b \cup a = (-1)^{|a||b|} a \cup b,$$

the dimension may behave quite unexpected.
For example, For M with Hodge numbers $\left(\begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}\right)$,

$$H^{\cdot}(\mathrm{Sym}^2(M)) \cong \Lambda^2 H^{\cdot}(M)$$

has Hodge numbers

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ \binom{g}{2} & & g^2 & & \binom{g}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{array}, \quad \text{not} \quad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ \binom{g+1}{2} & & g^2 & & \binom{g+1}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{array}.$$

In general, when $H^{+}(M) = 0$, then

$$H^{\cdot}(\mathbb{S}^{\lambda} M) \cong \mathbb{S}^{\lambda^T} H^{\cdot}(M)$$

$$\parallel\wedge$$

$$\mathbb{S}^{\lambda} M = \mathbb{S}^{\lambda}(\quad)$$

A special case can be seen in [MNP13, Prop 4.3.3].

The main strategy:

$$\mathbb{S}^{\nu}(V \oplus W) \cong \bigoplus_{\mu, \lambda} N_{\lambda\mu\nu} (\mathbb{S}^{\lambda} V \otimes \mathbb{S}^{\mu} W)$$

Step 1. $\mathbb{S}^\lambda\left(\begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}\right)$

$$\mathbb{S}^\nu\left(\begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}\right) = \bigoplus_{\mu,\lambda} N_{\lambda\mu\nu}\left(\mathbb{S}^\lambda\left(\begin{smallmatrix} & 0 & \\ g & & 0 \\ & 0 & \end{smallmatrix}\right) \otimes \mathbb{S}^\mu\left(\begin{smallmatrix} & 0 & \\ 0 & & g \\ & 0 & \end{smallmatrix}\right)\right)$$

$n = -2g$

χ(Hodge module)

$$\begin{array}{ccc} & 0 & \\ g & & g \\ & 0 & \end{array} \qquad \begin{array}{ccc} & 0 & \\ g & & g \\ & 0 & \end{array}$$

$\yng(1)$ n $\qquad$ e n

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ \binom{g}{2} & & g^2 & & \binom{g}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ \binom{g+1}{2} & & g^2 & & \binom{g+1}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ g^2 & & 2g^2 & & g^2 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$\yng(2)$ $\binom{n+1}{2}$ $\qquad$ $\yng(1,1)$ $\binom{n}{2}$ $\qquad$ e^2 n^2

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ \binom{g}{3} & & g\binom{g}{2} & & g\binom{g}{2} & & \binom{g}{3} \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ \binom{g+1}{3} & & g^3 & & g^3 & & \binom{g+1}{3} \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ \binom{g+2}{3} & & g\binom{g+1}{2} & & g\binom{g+1}{2} & & \binom{g+2}{3} \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array} \quad \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ g^3 & & 3g^3 & & 3g^3 & & g^3 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$\yng(3)$ $\binom{n+2}{3}$ $\qquad$ $\yng(2,1)$ $\dfrac{(n-1)n(n+1)}{1\cdot 3\cdot 1}$ $\qquad$ $\yng(1,1,1)$ $\binom{n}{3}$ $\qquad$ e^2 n^3

Step 1. $\mathbb{S}^\lambda\left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right)$

$$\mathbb{S}^\nu\left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right) = \bigoplus_{\mu,\lambda} N_{\lambda\mu\nu}\left(\mathbb{S}^\lambda\left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 0 & \end{smallmatrix}\right) \otimes \mathbb{S}^\mu\left(\begin{smallmatrix} & 0 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right)\right)$$

$n = 2$

χ(Hodge module)

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

$\square$ $n=2$

$$\begin{array}{ccc} & 1 & \\ 0 & & 0 \\ & 1 & \end{array}$$

e $n=2$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\square\square$ $\binom{n+1}{2}=3$

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$\begin{smallmatrix}\square\\\square\end{smallmatrix}$ $\binom{n}{2}=1$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

e^2 $n^2=4$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

$\square\square\square$ $\binom{n+2}{3}=4$

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$\begin{smallmatrix}\square\square\\\square\end{smallmatrix}$ $\frac{(n-1)n(n+1)}{1\cdot 3\cdot 1}=2$

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}$ $\binom{n}{3}=0$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 3 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 3 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

e^2 $n^3=8$

Step 3. E.g. $\mathrm{Sym}^n \left(\begin{smallmatrix} & 1 & \\ g & & g \\ & 1 & \end{smallmatrix}\right)$ $\quad C^+ \rightsquigarrow \left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right) \quad C^- \rightsquigarrow \left(\begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}\right)$

$$H^\cdot(C^{[k]}) = H^\cdot(\mathrm{Sym}^k C) = H^\bullet\left(\bigoplus_{i+j=k} \mathrm{Sym}^i \left(\begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix}\right) \otimes \mathrm{Sym}^j \left(\begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}\right)\right)$$

$$= \bigoplus_{i+j=k} \left(H^\cdot(\mathrm{Sym}^i(C^+)) \otimes H^\cdot(\mathrm{Sym}^j(C^-))\right)$$

$$= \bigoplus_{i+j=k} \left(\mathrm{Sym}^i H^\cdot(C^+) \otimes \Lambda^j H^\cdot(C^-)\right)$$

$$= \bigoplus_{i+j=k} \left(\mathrm{Sym}^i H^+(C) \otimes \Lambda^j H^-(C)\right)$$

$$\mathrm{Sym}^0(C^+) = 1 \qquad \mathrm{Sym}^0(C^-) = 1$$

$$\mathrm{Sym}^1(C^+) = \begin{smallmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{smallmatrix} \qquad \mathrm{Sym}^1(C^-) = \begin{smallmatrix} & 0 & \\ g & & g \\ & 0 & \end{smallmatrix}$$

$$\mathrm{Sym}^2(C^+) = \begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \mathrm{Sym}^2(C^-) = \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ \binom{g}{2} & & g^2 & & \binom{g}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{array}$$

$$\mathrm{Sym}^3(C^+) = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \mathrm{Sym}^3(C^-) = \begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ \binom{g}{3} & & \binom{g}{1}\binom{g}{2} & & \binom{g}{1}\binom{g}{2} & & \binom{g}{3} \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

Cor. When $p+q \leq n$,

$$h^{p,q}(C^{[n]}) = \sum_{0 \leq k \leq \min(p,q)} \binom{g}{p-k}\binom{g}{q-k}$$

$$= \binom{g}{p}\binom{g}{q} + h^{p-1,q-1}(C^{[n]})$$

Similarly, for $\Lambda^n \begin{pmatrix} & 1 & \\ g & & g \\ & 1 & \end{pmatrix}$, $\quad C^+ \rightsquigarrow \begin{pmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{pmatrix} \quad C^- \rightsquigarrow \begin{pmatrix} & 0 & \\ g & & g \\ & 0 & \end{pmatrix}$

$$
\begin{aligned}
H^\cdot(\Lambda^k C) &= H^\bullet\left(\bigoplus_{i+j=k} \Lambda^i \begin{pmatrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{pmatrix} \otimes \Lambda^j \begin{pmatrix} & 0 & \\ g & & g \\ & 0 & \end{pmatrix} \right) \\
&= \bigoplus_{i+j=k} \left(H^\cdot(\Lambda^i(C^+)) \otimes H^\cdot(\Lambda^j(C^-)) \right) \\
&= \bigoplus_{i+j=k} \left(\Lambda^i H^\cdot(C^+) \otimes \mathrm{Sym}^j H^\cdot(C^-) \right) \\
&= \bigoplus_{i+j=k} \left(\Lambda^i H^+(C) \otimes \mathrm{Sym}^j H^-(C) \right)
\end{aligned}
$$

$$\Lambda^0(C^+) = 1 \qquad\qquad \Lambda^0(C^-) = 1$$

$$\Lambda^1(C^+) = \begin{matrix} & 1 & \\ 0 & & 0 \\ & 1 & \end{matrix} \qquad\qquad \Lambda^1(C^-) = \begin{matrix} & 0 & \\ g & & g \\ & 0 & \end{matrix}$$

$$\Lambda^2(C^+) = \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{matrix} \qquad\qquad \Lambda^2(C^-) = \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ \binom{g+1}{2} & & g^2 & & \binom{g+1}{2} \\ & 0 & & 0 & \\ & & 0 & & \end{matrix}$$

$$\Lambda^3(C^+) = \begin{matrix} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{matrix}$$

$$\Lambda^3(C^-) = \begin{matrix} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ \binom{g+2}{3} & & g\binom{g+1}{2} & & g\binom{g+1}{2} & & \binom{g+2}{3} \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{matrix}$$

Geometric information

$$
\begin{tikzcd}
X^n \arrow[ddr, bend right, "n!:1"'] & & \\
& X^{[n]} \arrow[d, "\text{blow up}"] & \text{Hilbert scheme} \\
& X^{(n)} \arrow[d] & \text{symmetric product} \\
& \mathrm{Alb}(X) &
\end{tikzcd}
$$

$$
\begin{aligned}
\mathfrak{h}(X^{[2]}) &\cong S^2\mathfrak{h}(X) \oplus \bigoplus_{i=1}^{n-1} \mathfrak{h}(X)(-i) \\
&\overset{n=1}{=\!=} S^2\mathfrak{h}(X)
\end{aligned}
$$