Eine Woche, ein Beispiel

1.23 Coxeter group

1. def & realizations
   - def
   - geometrical representation
   - root system
   - polytopes
   - as subgp of S_n
   - as Weyl gp of some Tits system

2. combinatorical results
3. Bruhat order
4. geometrical realization (faithfulness)

Roadmap

gen & relations $\longleftrightarrow$ characteristic properties $\to$ realizations.

wide river

Ref: [Building] Buildings, by Peter Abramenko and Kenneth S. Brown

[Bourbaki] (Elements of Mathematics) Nicolas Bourbaki - Lie Groups and Lie Algebras_ Chapters 4-9-Springer (2002)

[Comb] Combinatorics of Coxeter groups

[Flag] Flag Varieties An Interplay of Geometry, Combinatorics, and Representation Theory

In the first section, we omit technical details, which will be filled in later on. (Mainly: injectivity)

1. def & realizations

def

Def (Coxeter system) (W, S) gp + gen, $m(s,t) \in \mathbb{Z}_{>0} \cup \{+\infty\}$, $m(s,s) = 1$

$$W = \langle s \in S \rangle / \left(s^2 = (st)^{m(s,t)} = 1,\ \forall s,t \in S\right)$$

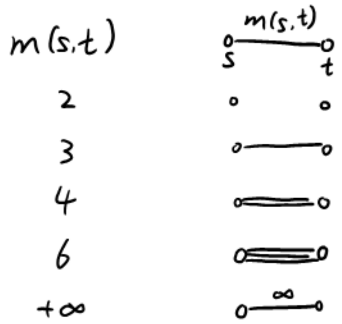
W is a Coxeter gp if $\exists\ S \subseteq W$, (W,S) is a Coxeter system.

E.g.

$$S_n \cong \langle s_i \rangle_i / \left(s_i^2 = (s_i s_j)^2 = (s_i s_{i+1})^3 = 1\right)$$

$|i-j| \geq 2$, and undefined relations (e.g. $(s_{n-1} s_n)^3$) should be removed.

Coxeter graph



Notation

S		simple reflections/transpositions
$l(w) = \min\{r \mid w = s_1 \cdots s_r,\ s_i \in S\}$		length of $w \in W$
$T = \{wsw^{-1} \mid w \in W,\ s \in S\}$		reflections/transpositions

geometrical representation $\quad W \hookrightarrow GL(V_{geo})$

∇ We suppose $|S| < \infty$, which is not necessary (but helpful for concentrating mind)

$$(W, S) \rightsquigarrow (\rho_{geo}, V_{geo}, \langle -,- \rangle) \in \mathrm{Rep}_{\mathbb{R},\mathrm{ortho}}(W)$$

$$V_{geo} := \bigoplus_{s \in S} \mathbb{R}\alpha_s$$

$$\langle -,- \rangle : \quad V_{geo} \otimes V_{geo} \longrightarrow \mathbb{R}$$

$$(\alpha_s, \alpha_t) \longmapsto \begin{cases} -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq +\infty \\ -1 & m(s,t) = +\infty \end{cases}$$

$m(s,t)$	1	2	3	4	5	6	$\cdots$	∞
(α_s, α_t)	1	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{5}+1}{4}$	$-\frac{\sqrt{3}}{2}$	$\cdots$	-1

$$\rho_{geo} : W \longrightarrow GL(V_{geo})$$
$$s \longmapsto r_{\alpha_s}$$

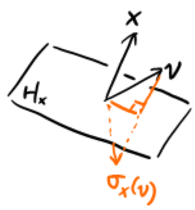
For $x, v \in V_{geo}$, $\langle x, x \rangle = 1$, define

$$r_x(v) = v - 2\langle v, x \rangle x$$

Check: $r_x(x) = -x$

$r_x(v) = v \iff v \in H_x$, where

$H_x = \{ v \in V_{geo} \mid \langle v, x \rangle = 0 \}$



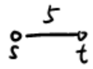
Ex Verify the well-definess.

- $\rho_{geo}(s)$ is linear & orthogonal;
- $\rho_{geo}(\text{relations}) = Id$

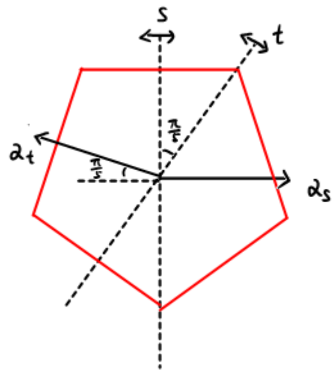
Also, $\langle wv, wv' \rangle = \langle v, v' \rangle$.

Thm. ρ_{geo} is faithful (sketch of proof later on)

E.g. $W = W(I_5)$

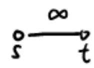


I_5

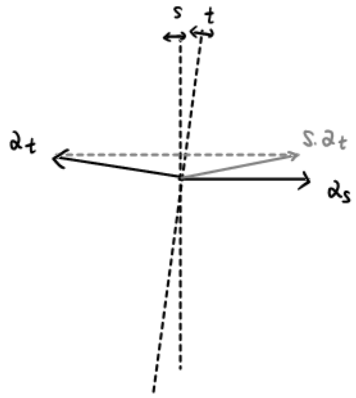


$\rho_{geo}(W) \cong D(5)$ Dihedral gp

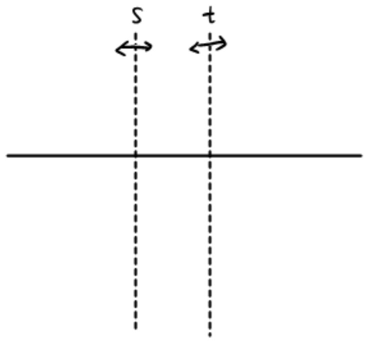
$W = W(I_\infty)$



I_∞



$\rightsquigarrow$



$s.\alpha_t = \alpha_t + 2\alpha_s$

$s(\alpha_s, \alpha_t) = (\alpha_s, \alpha_t)\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

$t(\alpha_s, \alpha_t) = (\alpha_s, \alpha_t)\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

$\rho_{geo}(W) \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

E.g.

H_4

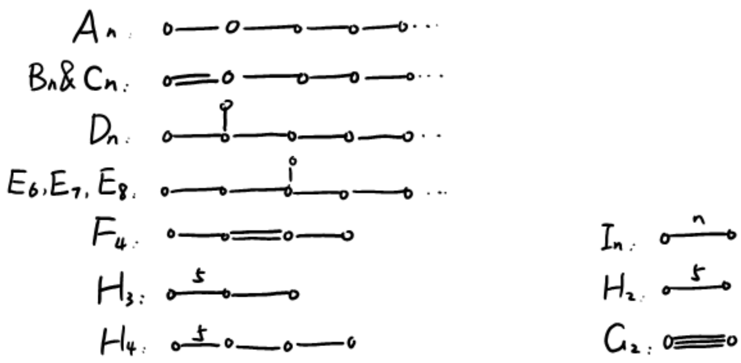
$$\begin{pmatrix} 1 & -\frac{\sqrt{5}+1}{4} & & \\ -\frac{\sqrt{5}+1}{4} & 1 & -\frac{1}{2} & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & -\frac{1}{2} & 1 \end{pmatrix} \text{ is pos-def}$$

Ex. $\#W < +\infty$

$\overset{?}{\Leftrightarrow}$ The bilinear form $\langle -, - \rangle$ is pos-def

$\Leftrightarrow$ Cartan matrix is pos-def

$\Leftrightarrow$ Coxeter graph is finite disjoint union of following shape.


A_n:
B_n & C_n:
D_n:
E_6, E_7, E_8:
F_4:
H_3:
5
H_4:
5
I_n:
n
H_2:
5
G_2:

Root system $\quad W \sim \mathrm{Aut}_R(V_{geo})$

⚠ Not the same as in Lie alg! E.g. here, every root has length 1.
That's why we don't use Φ here.

$$R = \{ v \in V_{geo} \mid v = w.\alpha_s \text{ for some } w \in W, s \in S \}$$

⚠ $\mathcal{T} \xrightarrow{\rho_{geo}} \{ \sigma \in GL(V_{geo}) \mid \sigma = r_x \text{ for some } x \in V_{geo}, \langle x, x \rangle = 1, \sigma(R) = R \}$
can be not surj when the irr root system is not simply laced.
See 1084790 for more details.

⚠ Here, $W \not\cong \mathrm{Aut}(R)$! See example on $W(I_5)$.

Ex. Verify the following properties.

(R1) R spans V_{geo}, does not contain 0

(R2) $-R = R$

(R3) $r_v R = R \quad \forall v \in R$

Define $\quad R^+ := \left(\sum_{s \in S} \mathbb{R}_{\geq 0}\, \alpha_s \right) \cap R \qquad R^- := \left(\sum_{s \in S} \mathbb{R}_{\leq 0}\, \alpha_s \right) \cap R$

one can check $R = R^+ \sqcup R^-$ by hand.

Lemma. $\quad r_{w.\alpha_s} = \rho_{geo}(w s w^{-1}) \qquad w \in W, s \in S$

Proof.
$$\begin{aligned} r_{w.\alpha_s}(x) &= x - 2\langle w.\alpha_s, x \rangle\, w.\alpha_s \\ &= w \cdot (w^{-1} x - 2 \langle \alpha_s, w^{-1} x \rangle \alpha_s) \\ &= w \cdot \sigma_{\alpha_s}(w^{-1} x) \\ &= \rho_{geo}(w s w^{-1})\, x. \end{aligned}$$

Prop. We have bijection

$$\begin{aligned} R &\longrightarrow \mathcal{T} \times \{\pm 1\} \\ w.\alpha_s &\longmapsto (w s w^{-1}, \eta(w; s)) \end{aligned}$$

$R^+ \longleftrightarrow \mathcal{T} \times \{+1\}$

$R^- \longleftrightarrow \mathcal{T} \times \{-1\}$

where $\quad \eta(s; t) = \begin{cases} -1 & s = t \\ 1 & s \neq t \end{cases} \qquad \eta(w'w; t) = \eta(w'; w t w^{-1})\, \eta(w; t)$

For the well-defines of η, we postpone to next section.

See https://math.stackexchange.com/questions/1084790/weyl-groups-correspondence-of-reflections-and-roots and [Building, Prop 1.113].

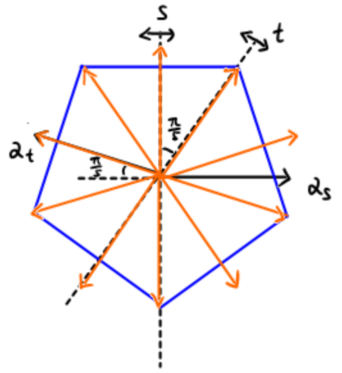
Polytopes $\quad W \cong \mathrm{Aut}(\text{Polytopes})$ $\quad$ (fundamental domain, chambers)

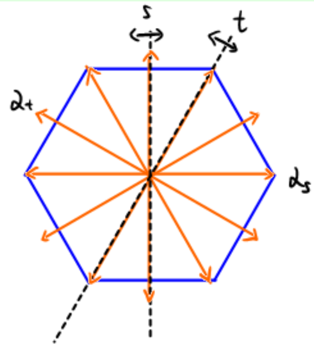
$\triangledown$ For Dynkin - Coxeter graph.

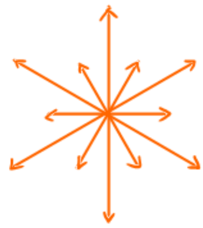
Others can be viewed as mosaic in spaces with constant curv ≤ 0.

It comes from the geo rep.

Ref:
https://syntopia.github.io/Polytopia/polytopes.html
https://www3.mpifr-bonn.mpg.de/staff/pfreire/polyhedra/index.html
https://www.mdpi.com/2073-8994/11/3/391/pdf?version=1552904082 (Some vague pictures of 5D polytopes)

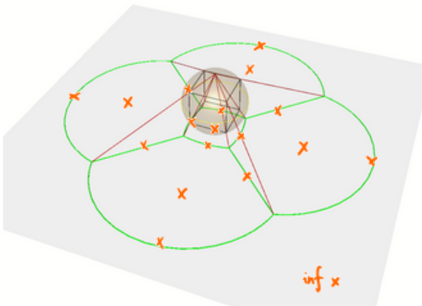






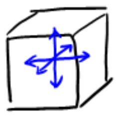
G_2

Ex. draw roots in ⟁ , ▢ . $\quad$ (Bad picture for ⟁ !)



as subgp of S_n $\quad$ strand description

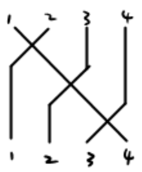
$\triangledown$ For type $A \sim D$, since they have "nice" shapes of polytopes.



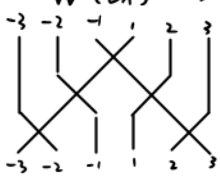
$W(A_3) \xhookrightarrow{\cong} S_4$ $\quad$ $W(B_3) \hookrightarrow S_6$ $\quad$ $W(D_3) \hookrightarrow S_6$

$W(A_n) \cong S_{n+1}$ $\quad$ $W(B_n) \hookrightarrow S_{2n}$ $\quad$ $W(D_n) \hookrightarrow S_{2n}$

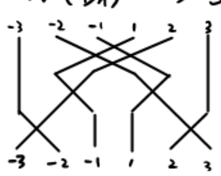
Standard



σ_1
σ_2
σ_3

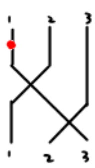


σ_0
σ_1
σ_2

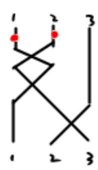


σ_0'
σ_1
σ_2

with dot



σ_0
σ_1
σ_2



σ_0'
σ_1
σ_2

as Weyl gp of some Tits system (later)

Ex. for the section

1. Verify the gen & rel in each case.
2. Describe element, reflection, simple reflection, length, roots, ... in each realization.

   e.g. how to see $|T| = \ell(w_0)$?

3. (Finite) group study:
   - $\# G$
   - conj class
   - $Z(G)$, $[G,G]$
   - char table (Rep theory)
   - simple?
   - subgp, quotient, central series, ...

4. Generalize everything to affine diagram.

   e.g. find a strand description of $\widehat{A}_n$.

2. combinatorical results

Lemma. For $(W,S) \in \mathrm{Cosgp}$, $\exists !$ gp homo

$$\begin{aligned} \mathrm{sgn}: W &\longrightarrow \{\pm 1\} \\ s &\longmapsto -1 \end{aligned}$$

s.t. $\mathrm{sgn}(w) = (-1)^{l(w)}$ $\quad \forall w \in W$

Cor. $\forall w \in W,\ s \in S,\quad l(ws) \equiv l(sw) \equiv l(w)+1 \mod 2$

In ptc, $\quad l(ws) \neq l(w)$

Setting In this section, W is a gp, S is a set of gen of order 2.

Still,

$$l(w) := \min\{r \mid w = s_1 \cdots s_r,\ s_i \in S\} \qquad \text{length of } w \in W$$

$$\mathcal{T} := \{wsw^{-1} \mid w \in W,\ s \in S\} \qquad \text{reflections / transpositions}$$

We have $l(w^{-1}) = l(w)$, but it is possible that $l(ws) = l(w)$ now.

Road map

$$\begin{tikzcd}
A \arrow[d, "7"'] & & C \arrow[ll, "6"'] & \\
SEP \arrow[r, "1"] & E \arrow[rr, bend left, "2"] \arrow[dr, "4"'] & {} \arrow[u, "8"'] & D \arrow[ll, bend left, "3"] \\
& & F \arrow[ur, "5"'] &
\end{tikzcd}$$

$F \Leftarrow$ Tits

A	Action	[Building, p65]
C	Coxeter	
D	DP = Deletion property	
E	EP = Exchange property	
F	Folding condition	[Building, p79]

(Coxeter) $\quad (W,S)$ is a Coxeter system

(SEP) $\quad w = s_1 \ldots s_r,\ s_i \in S,\ t \in \mathcal{T},\quad l(tw) < l(w)$

$$\Rightarrow \quad tw = s_1 \cdots \hat{s_i} \cdots s_r \qquad \exists i$$

(EP) $\quad w = s_1 \ldots s_r,\ s_i \in S,\ t \in S,\quad l(tw) < l(w)$

$$\Rightarrow \quad tw = s_1 \cdots \hat{s_i} \cdots s_r \qquad \exists i$$

(DP) $\quad w = s_1 \cdots s_r \quad s_i \in S \qquad l(w) < r$

$$\Rightarrow \quad w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r \qquad \exists i,j$$

(Folding) For $w \in W$ $s,t \in S$ s.t. $l(tw) = l(w)+1$, $l(ws) = l(w)+1$,

$$\Rightarrow \quad l(tws) = l(w)+2 \quad \text{or} \quad tws = w$$

(Action) $\exists \rho: W \circlearrowright T \times \{\pm 1\}$ s.t. $\forall s \in S$,

$$\rho_s(t, \varepsilon) = \begin{cases} (s, -\varepsilon) & s = t \\ (sts, \varepsilon) & s \neq t \end{cases}$$

In ptc, $\quad \rho_w(t,\varepsilon) = (wtw^{-1}, \eta(w;t)\varepsilon)$ where

$$\eta(s;t) = \begin{cases} -1 & s = t \\ 1 & s \neq t \end{cases}$$

$$\eta(w'w;t) = \eta(w';wtw^{-1})\,\eta(w;t)$$