

Eine Woche, ein Beispiel

2.23 Schubert calculus: coh of Grassmannian

Ref:

[3264] and [Fulton]

[LW21]: https://www.math.uni-bonn.de/ag/stroppel/Masterarbeit_Wang.pdf

[PV04]: <https://arxiv.org/abs/math/0407170>

[Bartho4]: Barth, Wolf P., Klaus Hulek, Chris A. M. Peters and Antonius Van De Ven. Compact Complex Surfaces.

Igor Pak, Ernesto Vallejo, Combinatorics and geometry of Littlewood-Richardson cones

We will attempt to tackle Schubert calculus in a concise manner. The term "Schubert calculus" is often associated with intersection theory, enumerative geometry, combinatorics, Grassmannians, and more, making it a vast topic. However, I believe its core ideas can be clearly explained in just six hours. I will break the material into several parts:

1. $H^*(Gr(r,n); \mathbb{Z})$ and its combinatorics

2. Relate cycles with v.b.s

cycles in Grassmannian, including:

- cycle class map: $CH^*(Gr(r,n)) \xrightarrow{\sim} H^*(Gr(r,n); \mathbb{Z})$

- incidence variety $\begin{cases} (\text{partial}) \text{ flag variety} \\ \text{Fano variety of planes} \\ \dots \end{cases}$

- a reinterpretation of cycles

3. Chern class calculation

$$X \xrightarrow{f_X} Gr(r, n) \quad \begin{matrix} L \\ / \\ \downarrow \\ S^\vee \end{matrix}$$

Chern class: $c: VB(X) \longrightarrow H^*(X; \mathbb{Z})$

$f_X^*: H^*(Gr(r, \infty); \mathbb{Z}) \longrightarrow H^*(X; \mathbb{Z})$

e.p., $VB(Gr(r, n)) \longrightarrow H^*(Gr(r, n); \mathbb{Z})$

$$\begin{aligned} S^\vee &\longmapsto 1 + \sigma_1 + \sigma_{1^2} + \dots + \sigma_{1^r} \\ T_{Gr} &\longmapsto 1 + n \cdot \sigma_1 + \dots \end{aligned}$$

4. Applications

1. Group structure of $H^*(\mathrm{Gr}(r,n); \mathbb{Z})$
2. Cup product
3. Young diagram formulas
4. Other combinatorial models for LR coefficients

1. Group structure of $H^*(\mathrm{Gr}(r,n); \mathbb{Z})$

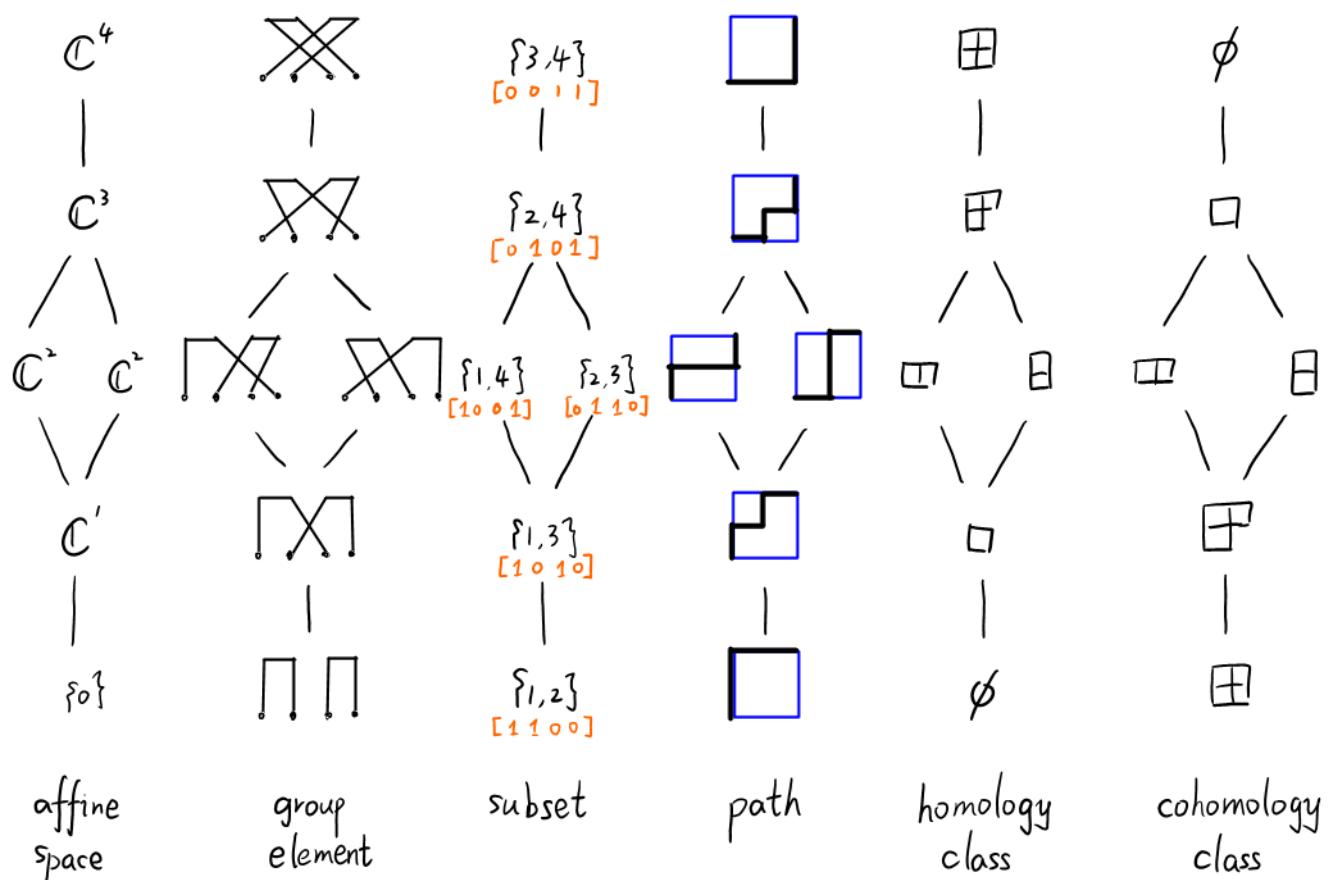
It's well-known that $\mathrm{Gr}(r,n) \cong \mathrm{GL}_n(\mathbb{C})/\mathrm{P}$ has an affine paving w.r.t. $S_n/S_r \times S_{n-r}$:

$$\mathrm{Gr}(r,n) = \bigsqcup_{w \in S_n / S_r \times S_{n-r}} B_w P / P \cong \bigsqcup_{w \in S_n / S_r \times S_{n-r}} \mathbb{C}^{l(w)}$$

$$\# S_n / S_r \times S_{n-r} = \binom{n}{r}$$

We read the diagram from top to bottom, the map from right to left.

E.g. $n=4 \ r=2$



Hint from gp element to homology class.

$$\begin{array}{c} \text{Diagram with red dots labeled } 0 \text{ and } 2 \\ \text{with a crossing} \end{array} \rightsquigarrow (2,0) = \square$$

E.g. $n=5, r=2$

$$\begin{array}{c} \text{Diagram with 5 strands, 2 red dots labeled } 2, 4 \\ \sim \{2,4\} \sim \text{Diagram with blue L-shape} \end{array}$$

$$\begin{array}{c} \text{Diagram with 5 strands, 3 red dots labeled } 3, 5 \\ \sim \{3,5\} \sim \text{Diagram with blue L-shape} \end{array}$$

Ex. compute w_0 -action (left mult) on $S_n/S_r \times S_{n-r}$, where $w_0 = \text{Diagram with 5 strands, 3 red dots labeled } 3, 5$.

2. Cup product

We want to compute intersection number by moving one cycle (so that they intersect transversally)

Lemma 1. $[B^w P/P] = [Bw \circ w P/P]$ in $H^*(G_{r(r,n)}; \mathbb{Z})$.

Proof. $B^w P/P = w_* Bw \circ w P/P \sim Bw_* w P/P$.

Lemma 2.

$$\# (Bw P/P \cap B^- \eta P/P) = \begin{cases} 0 & \eta > w \\ 1 & \eta = w \\ 0 & \eta \neq w \text{ and } l(\eta) = l(w) \\ ? & \text{otherwise} \end{cases}$$

Moreover, when $\eta = w$, $Bw P/P$ and $B^- \eta P/P$ intersect transversally.

Idea: Find a set of representative elements $C_w^+ \cong \mathbb{C}^{l(w)}$ in B , s.t.

$$Bw P/P \xleftarrow{\cong} C_w^+ w P/P \cong C_w^+.$$

Similarly, find a set of representative elements $C_\eta^- \cong \mathbb{C}^{l(w \circ \eta)}$ in B^- , s.t.

$$B^- \eta P/P \xleftarrow{\cong} C_\eta^- \eta P/P \cong C_\eta^-.$$

After that,

$$\begin{aligned} Bw P/P \cap B^- \eta P/P &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_+ w P = c_- \eta P\} \\ &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_-^{-1} c_+ \in \eta P w^{-1}\} \end{aligned}$$

in $C_w^+ \times C_\eta^- \cong \mathbb{C}^{l(w) + l(w \circ \eta)}$ can be written as the zero sets of polynomials (of $\deg \leq 2$)

E.g. $n=5$, $r=2$,

$$\omega = \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} = \begin{pmatrix} & 1 & 1 \\ 1 & & \\ & 1 & \end{pmatrix} = \{35 | 124\} \sim \begin{array}{|c|c|} \hline \text{hom} & \text{cohom} \\ \hline \square & \square \\ \hline \end{array}$$

$$\eta_0 = \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \square \quad \square \quad \square \quad \square \end{array} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ 1 & & \end{pmatrix} = \{13 | 245\} \sim \square \sim \begin{array}{|c|c|} \hline \text{hom} & \text{cohom} \\ \hline \square & \square \\ \hline \end{array}$$

Let $\eta = \eta_0$, we want to describe $B\omega P/P \cap B\eta P/P \subset C_\omega^+ \times C_\eta^-$.
By direct calculation,

$$P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\eta P \omega^{-1} = \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\text{for copy } \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\omega P \omega^{-1} = \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$\eta P \eta^{-1} = \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$C_\omega^+ = \begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \\ 1 & * & 1 \\ 1 & 1 & * \end{pmatrix}$$

$$C_\eta^- = \begin{pmatrix} 1 & & & \\ * & 1 & & \\ * & * & 1 & \\ * & * & * & 1 \\ * & * & * & 1 \end{pmatrix}$$

Now, suppose

$$C_-^{-1} = \begin{pmatrix} 1 & & \\ b_{21} & 1 & \\ b_{41} & b_{43}^{-1} & 1 \\ b_{51} & b_{53} & 1 \end{pmatrix} \quad C_+ = \begin{pmatrix} 1 & a_{13} & a_{15} \\ 1 & a_{23} & a_{25} \\ 1 & a_{45} & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & a_{13} & a_{15} \\ b_{21} & 1 & b_{21}a_{13} + a_{23} & b_{21}a_{15} + a_{25} \\ b_{41} & & 1 & b_{41}a_{13} + b_{43} \\ b_{51} & & b_{51}a_{13} + b_{53} & b_{51}a_{15} + 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in \eta P w^{-1} \Leftrightarrow \begin{cases} b_{21}a_{13} + a_{23} = 0 \\ b_{21}a_{15} + a_{25} = 0 \\ b_{41}a_{13} + b_{43} = 0 \\ b_{41}a_{15} + a_{45} = 0 \\ b_{51}a_{13} + b_{53} = 0 \\ b_{51}a_{15} + 1 = 0 \end{cases}$$

In this case, $BwP/P \cap B^-\eta P/P \cong \mathbb{C}^3 \times \mathbb{C}^\times$.

Now, take $\eta=w$, one suppose that

$$C_-^{-1} = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & b_{43}^{-1} \\ & & 1 \end{pmatrix} \quad C_+ = \begin{pmatrix} 1 & a_{13} & a_{15} \\ 1 & a_{23} & a_{25} \\ 1 & a_{45} & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & a_{13} & a_{15} \\ & 1 & a_{23} \\ & & 1 \\ & & b_{43} \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in w P w^{-1} \Leftrightarrow a_{13} = a_{15} = a_{23} = a_{25} = a_{45} = b_{43} = 0.$$

In this case $BwP/P \cap B^-wP/P = \{*\}$.

Furthermore, one can show the transversality through the tangent argument.

Ex. When $\eta = \omega_0$, verify that

$$B\omega P/P \cap B^{-\omega_0}P/P = \emptyset$$

Generalize this example to prove Lemma 2.

Cor of Lemma 2. When $l(\omega) + l(\omega') = r(n-r)$,

$$\deg ([B\omega P/P] \cup [B\omega' P/P]) = \begin{cases} 1 & \omega = \omega_0 \omega' \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, denote

$$\sigma_\omega := [B\omega P/P] \in H^r(\mathrm{Gr}(r,n); \mathbb{Z})$$

then $\sigma_\omega \sigma_{\omega_0 \omega} = \sigma_{Id}$
 $\sigma_\omega \sigma_\eta = 0 \quad \text{when } l(\omega) + l(\eta) = r(n-r).$

When we view $\omega = \alpha = (\alpha_1, \dots, \alpha_r)$ as the Young diagram in the column class,

$$l(\omega) = r(n-r) - |\alpha|$$

$$\sigma_\omega \stackrel{?}{=} \sigma_\alpha \in H_{l(\omega)}(\mathrm{Gr}(r,n); \mathbb{Z}) \cong H^{|\alpha|}(\mathrm{Gr}(r,n); \mathbb{Z}).$$

For simplicity, we write $\sigma_k = \sigma_{(k,0,\dots,0)}$ and $\sigma_{1^k} = \sigma_{(\underbrace{1,\dots,1}_{k \text{ many}}, 0, \dots, 0)}$.

The moduli interpolation of Schubert variety

To prove the Pieri rule, the method in the proof of Lemma 2 need to be modified. Working with the moduli interpolation of Schubert varieties can help understanding.

E.g. $n=5, r=2$,

$$w = \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \bullet \quad \bullet \quad \bullet \quad \bullet \end{array} = \begin{pmatrix} & 1 & 1 \\ 1 & & \\ & 1 & \end{pmatrix} = \{35|124\} \sim \begin{array}{c} \text{hom} \\ \boxed{} \end{array} \sim \begin{array}{c} \text{cohom} \\ \square \end{array}$$

$$\begin{array}{c} \text{standard} \\ \downarrow \\ \mathcal{V}^{st} : \quad 0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \subset \langle e_1, \dots, e_4 \rangle \subset \langle e_1, \dots, e_5 \rangle \\ \cup \qquad \cup \qquad \cup \qquad \cup \qquad \cup \qquad \cup \\ \langle e_3, e_5 \rangle \cap \mathcal{V}^{st} : \quad 0 = 0 = 0 \subset \langle e_3 \rangle = \langle e_3 \rangle \subset \langle e_3, e_5 \rangle \end{array} \iff wP/P \in G/P \iff w\langle e_1, e_2 \rangle = \langle e_3, e_5 \rangle \in G_{r(2,5)}$$

$$\begin{aligned} \sum_w(\mathcal{V}^{st}) &\stackrel{\Delta}{=} \overline{BwP/P} \\ &= \left\{ \Lambda \in G_{r(2,5)} \mid \begin{array}{l} \dim \Lambda \cap \mathcal{V}_3^{st} \geq 1 \\ \dim \Lambda \cap \mathcal{V}_5^{st} \geq 2 \end{array} \right\} \\ BwP/P &= \left\{ \Lambda \in G_{r(2,5)} \mid \begin{array}{l} \dim \Lambda \cap \mathcal{V}_3^{st} = 1 \\ \dim \Lambda \cap \mathcal{V}_5^{st} = 2 \\ \dim \Lambda \cap \mathcal{V}_2^{st} = 0 \\ \dim \Lambda \cap \mathcal{V}_4^{st} = 1 \end{array} \right\} \end{aligned}$$

Def. For the flag $\mathcal{V} = g\mathcal{V}^{st}$, define

$$\begin{aligned} \sum_w(\mathcal{V}) &= \overline{gBwP/P} \\ &= \left\{ \Lambda \in G_{r(2,5)} \mid \begin{array}{l} \dim \Lambda \cap \mathcal{V}_3 \geq 1 \\ \dim \Lambda \cap \mathcal{V}_5 \geq 2 \end{array} \right\} \end{aligned}$$

General case:

$$\sum_w(\mathcal{V}) = \{ \Lambda \in G_{r(n)} \mid \dim \Lambda \cap \mathcal{V}_{w(i)} \geq i \}$$

Easy to see that $\sum_w(w_0)\mathcal{V}^{st} = \overline{Bw_0wP/P}$.

Lemma 3. Let a, c be Young diagrams which correspond to w, w' st.

$$\begin{cases} |c| = |a| + k \\ a_i \leq c_i \leq a_{i+1} \quad \forall i \end{cases}$$

Then $\sum_a (\mathcal{V}^{\text{st}}) \cap \sum_c (w \circ \mathcal{V}^{\text{st}}) = \underbrace{\mathbb{P} \times \cdots \times \mathbb{P}}_{r \text{ many}}^{w(1)+w(r)-n-1} \times \cdots \times \underbrace{\mathbb{P}}_{w(r)+w'(1)-n-1}$

E.g. $n=5, r=2$, write $\mathcal{V} := \mathcal{V}_{\text{st}}, W = w \circ \mathcal{V}_{\text{st}}$,

$$w = \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array} = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 1 & & 1 & 1 \end{pmatrix} \stackrel{\text{cohom}}{\sim} \{25|134\} \sim \square \leftarrow a = (2,0)$$

$$w' = \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \bullet \bullet \bullet \bullet \bullet \bullet \end{array} = \begin{pmatrix} 1 & & 1 \\ & 1 & \\ 1 & & 1 & 1 \end{pmatrix} \stackrel{\text{cohom}}{\sim} \{24|135\} \sim \boxplus \leftarrow c = (2,1)$$

We want to show $\sum_a (\mathcal{V}) \cap \sum_c (W) \cong \mathbb{P}^0 \times \mathbb{P}'$.

We write

$$A_1 := \mathcal{V}_2 \cap W_4 = \langle v_2 \rangle$$

$$A_2 := \mathcal{V}_5 \cap W_2 = \langle v_4, v_5 \rangle$$

then

$$\begin{aligned} \sum_a (\mathcal{V}) \cap \sum_c (W) &= \text{Gr}(1, A_1) \times \text{Gr}(1, A_2) \\ \Delta &\mapsto (\Delta \cap A_1, \Delta \cap A_2) \\ W_1 \oplus W_2 &\longleftrightarrow (W_1, W_2) \end{aligned}$$

$$\text{Hint: } ①. \Delta \trianglelefteq A_1 + A_2 = A_1 \oplus A_2$$

$$②. \dim \Delta \cap A_i \geq 1$$

$$\begin{aligned} 2 = \dim \Delta &= \dim \Delta \cap (\mathcal{V}_2 + W_4) \\ &= \dim \Delta \cap \mathcal{V}_2 + \dim \Delta \cap W_4 - \dim \Delta \cap A_1, \\ &\geq 1 + 1 - \dim \Delta \cap A_1, \end{aligned}$$

$$③. \dim \Delta \cap A_i = 1, \quad \Delta = \bigoplus \Delta \cap A_i \quad \Delta \subset A$$

$$\begin{aligned} 2 = \dim \Delta &\geq \dim \Delta \cap A \\ &\geq \dim \Delta \cap A_1 + \dim \Delta \cap A_2 \\ &\geq 1 + 1 = 2 \end{aligned}$$

Lemma 4. Let a, c be Young diagrams which crspol to w, w' st.

$$\begin{cases} |c| = |a| + k \\ a_i \leq c_i \leq a_{i+1} \quad \forall i \end{cases}$$

Let $(k, \dots, 0)$ be Young diagram which crspols to w'' .
Let $\mathcal{V}, \mathcal{W}, \mathcal{U}$ be general complete flags in \mathbb{C}^n , then

$$\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) \cap \Sigma_k(\mathcal{U}) = \{\ast\}.$$

Proof. W.l.o.g. let $\mathcal{V} = V^{st}$, $\mathcal{W} = w_* V^{st}$. [3264, Def 4.4]
We know

$$\begin{aligned} \Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) &= \prod_{i=1}^r \text{Gr}(1, A_i) \\ \Sigma_k(\mathcal{U}) &= \left\{ \Delta \in \text{Gr}(r, n) \mid \dim \Delta \cap U_{n-r+i-k} \geq 1 \right\} \end{aligned}$$

By transversality, $\dim \Delta \cap U_{n-r+i-k} = 1 \Rightarrow \Delta \supset \Delta \cap U_{n-r+i-k}$
Define

$$\psi_i : A \cap U_{n-r+i-k} \subset A \rightarrow A_i$$

Claim: $\Delta \cap A_i = \text{Im } \psi_i$

$$\left[\begin{array}{l} \Delta \cap U_{n-r+i-k} \subset A \cap U_{n-r+i-k} \text{ with equal dim} \\ \Rightarrow \Delta \cap U_{n-r+i-k} = A \cap U_{n-r+i-k} \\ \Rightarrow \text{Im } \psi_i \subseteq \Delta \cap A_i \quad \text{with equal dim} \\ \Rightarrow \text{Im } \psi_i = \Delta \cap A_i \end{array} \right]$$

Therefore, $\Delta = \bigoplus_i \Delta \cap A_i = \bigoplus_i \text{Im } \psi_i$ is uniquely determined. \square

Write Lemma 4 in terms of cohomology class, we get
 Pieri's formula: [3264, Prop 4.9, Thm 4.14]

$$\sigma_a \cdot \sigma_{(k, \dots, 0)} = \sum_{\substack{|c|=|a|+k \\ a_i \leq c_i \leq a_{i-1}}} \sigma_c$$

$$\sigma_a \cdot \sigma_{(\underbrace{1, \dots, 1}_{k\text{-many}}, \dots, 0)} = \sum_{\substack{|c|=|a|+k \\ a_i \leq c_i \leq a_i+1}} \sigma_c$$

E.g. $\sigma_{\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}} \cdot \sigma_{\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}} = \sigma_{\begin{smallmatrix} 3 & 1 \\ 1 & 1 \end{smallmatrix}} + \sigma_{\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix}} + \sigma_{\begin{smallmatrix} 2 & 1 \\ 2 & 1 \end{smallmatrix}} + \sigma_{\begin{smallmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{smallmatrix}}$

$$\sigma_{\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}} \cdot \sigma_{\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}} = \sigma_{\begin{smallmatrix} 3 & 1 \\ 1 & 1 \end{smallmatrix}} + \sigma_{\begin{smallmatrix} 2 & 2 \\ 1 & 1 \end{smallmatrix}}$$

We will play with Young diagrams in the next section.

3. Young diagram formulas

Littlewood - Richardson rule

The Pieri formula can be upgraded to the Littlewood–Richardson rule:

$$\sigma_\lambda \sigma_\mu = \sum N_{\lambda\mu\nu} \sigma_\nu$$

E.g. $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} 1 & 1 \\ 2 \end{smallmatrix} = \begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 1 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 \\ 1 & 1 \\ 1 & 2 \end{smallmatrix}$

$$+ \begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 \\ 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 1 \\ 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 2 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 \\ 1 & 1 \\ 1 & 2 \end{smallmatrix}$$

$$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} 1 & 1 \\ 2 \end{smallmatrix} = \begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 1 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 \\ 1 & 2 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 \\ 1 & 1 \\ 1 & 2 \end{smallmatrix}$$

$$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} 1 & 1 \\ 2 \end{smallmatrix} \cdot \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} = \begin{smallmatrix} 2 & 1 & 1 & 2 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 2 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 2 \end{smallmatrix} + \begin{smallmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 \\ 1 \end{smallmatrix}$$

$$\begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \cdot \begin{smallmatrix} 1 & 1 & 1 \\ 2 \end{smallmatrix} = \begin{smallmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 \end{smallmatrix} \quad \begin{smallmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 2 \end{smallmatrix} \quad \begin{smallmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 \end{smallmatrix} \quad \begin{smallmatrix} 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{smallmatrix}$$

from: https://en.wikipedia.org/wiki/Littlewood%E2%80%93Richardson_rule

The Littlewood–Richardson rule is notorious for the number of errors that appeared prior to its complete, published proof. Several published attempts to prove it are incomplete, and it is particularly difficult to avoid errors when doing hand calculations with it: even the original example in D. E. Littlewood and A. R. Richardson (1934) contains an error.

That's why I don't want to prove it (using only Pieri formula).

Giambelli's formula

This formula expresses σ_λ as polynomials in σ_k .

Ex. [3264, Prop 4.16]

Show that (by induction)

$$\sigma_{(\lambda_1, \dots, \lambda_k)} = \begin{vmatrix} \sigma_{\lambda_1} & \cdots & \sigma_{\lambda_1+k-1} \\ & \ddots & \\ & & \sigma_{\lambda_k} \end{vmatrix} \xrightarrow{\text{index } +1}$$

e.p.

$$\sigma_{(1, \dots, 1)} = \begin{vmatrix} \sigma_1 & \cdots & \sigma_k \\ 1 & \ddots & 1 \\ 0 & \ddots & 1 \end{vmatrix} = \left\{ \begin{array}{l} \sigma_1 \\ \sigma_2 - \sigma_1^2 \\ \sigma_3 - \sigma_2 \sigma_1 - (\sigma_2 - \sigma_1^2) \sigma_1 \\ \sigma_4 - \sigma_1 \sigma_3 - (\sigma_2 - \sigma_1^2) \sigma_2 - (\sigma_1, 1, 1) \sigma_1 \\ \vdots \end{array} \right.$$

Relations in $H^*(\mathrm{Gr}(r, n); \mathbb{Z})$

E.g. [3264, Cor 4.10]

$$(1 + \sigma_1 + \dots + \sigma_{n-r}) (1 - \sigma_1 + \dots + (-1)^r \sigma_{1^r}) = 1$$

$$(1 - \sigma_1 + \dots + (-1)^{n-r} \sigma_{n-r}) (1 + \sigma_1 + \dots + \sigma_{1^r}) = 1$$

In $\mathrm{Gr}(5, 2)$, we list the table of products for a hint:

zero Young diagram					
\emptyset	\square	$\begin{smallmatrix} & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ & \\ \square & \end{smallmatrix}$	
\emptyset	\emptyset	\square	$\begin{smallmatrix} & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ & \\ \square & \end{smallmatrix}$
\square	\square	$\square + \square$	$\begin{smallmatrix} & \\ \square & \end{smallmatrix} + \begin{smallmatrix} & \\ & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ & \\ \square & \end{smallmatrix} + \begin{smallmatrix} & \\ & \\ & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ & \\ & \\ \square & \end{smallmatrix}$
\square	\square	\square	$\begin{smallmatrix} & \\ \square & \end{smallmatrix}$	$\begin{smallmatrix} & \\ & \\ \square & \end{smallmatrix}$	-

Thm [3264, Thm 5.26]

$$H^*(\mathrm{Gr}(r, n); \mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_r]/I$$

where

$$c_k = c_k(\mathcal{S}) = (-1)^k \sigma_{1^k}$$

$$I = \left\langle \left(\frac{1}{1+c_1+\dots+c_r} \right)^{\deg=n-r+1}, \dots, \left(\frac{1}{1+c_1+\dots+c_r} \right)^{\deg=n} \right\rangle$$

$$= \langle \sigma_{n-r+1}, \dots, \sigma_n \rangle$$

$$\frac{1}{1+c_1+\dots+c_r} = 1 - (c_1 + \dots + c_r) + (c_1 + \dots + c_r)^2 - \dots$$

$$\stackrel{r \geq 5}{=} 1 - c_1 + (c_1^2 - c_2) + (c_1^3 - 2c_1c_2 + c_3)$$

$$+ (c_1^4 - 3c_1^2c_2 + c_2^2 + 2c_1c_3 - c_4)$$

$$+ (c_1^5 - 4c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 - 2c_2c_3 - 2c_1c_4 + c_5)$$

$$+ \dots$$

$$= 1 + \sigma_1 + (\sigma_1^2 - \sigma_{1^2}) + (-\sigma_1^3 + 2\sigma_1\sigma_{1^2} - \sigma_{1^3}) + \dots$$

Q: How to describe I' in

$$H^*(\mathrm{Gr}(r,n); \mathbb{Z}) \cong \mathbb{Z}[\sigma_1, \dots, \sigma_{n-r}] / I' ?$$

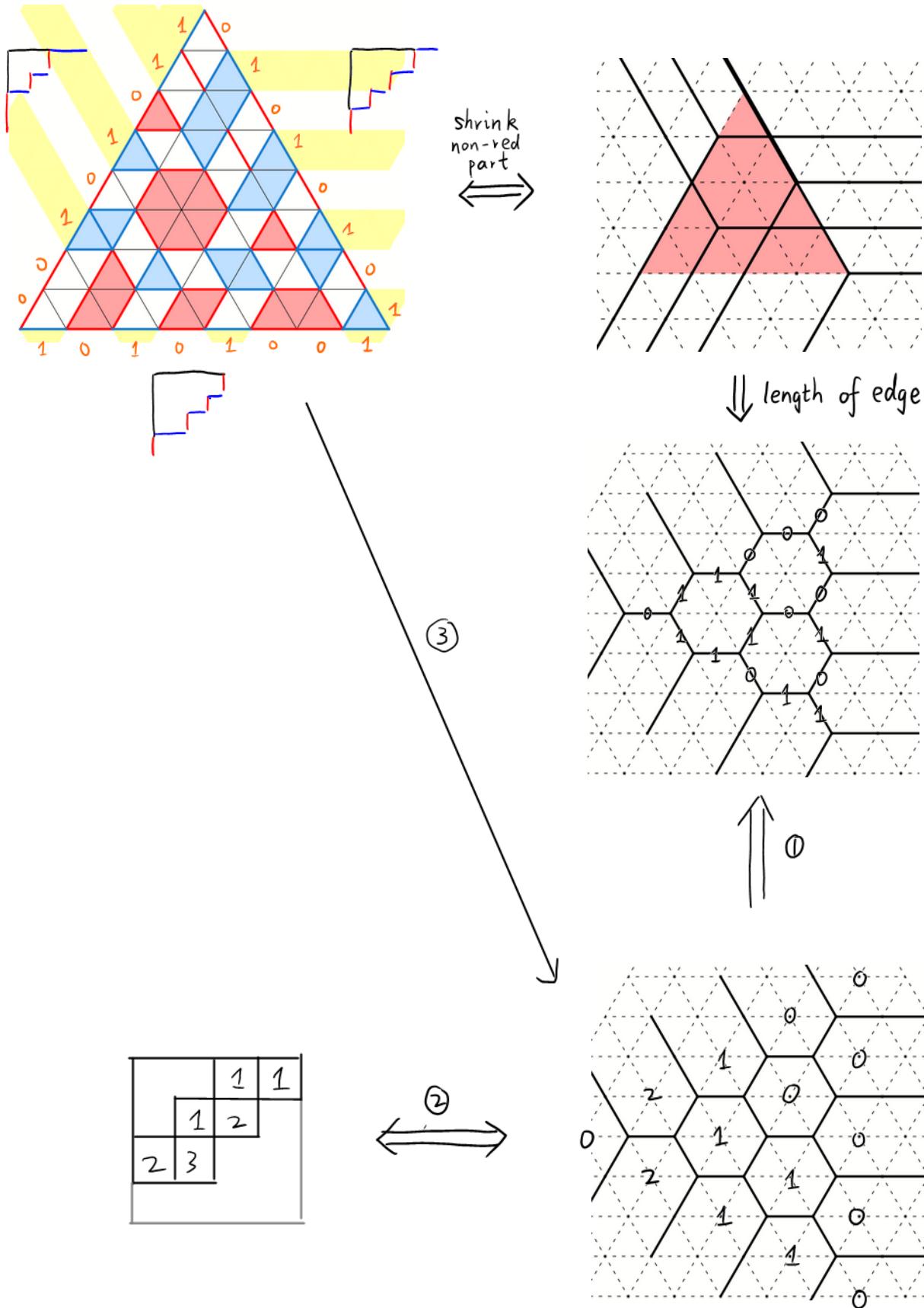
Guess: by duality [3264, Ex 4.31],

$$\begin{aligned} I' &= \left\langle \left(\frac{1}{1 - \sigma_1 + \dots + (-1)^{r-r} \sigma_{n-r}} \right)^{\deg = r+1}, \dots, \left(\frac{1}{1 - \sigma_1 + \dots + (-1)^{n-r} \sigma_{n-r}} \right)^{\deg = n} \right\rangle \\ &= \langle \sigma_1^{r+1}, \dots, \sigma_1^n \rangle \end{aligned}$$

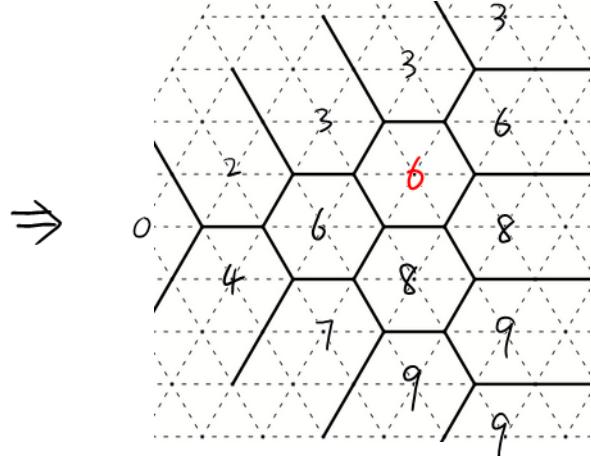
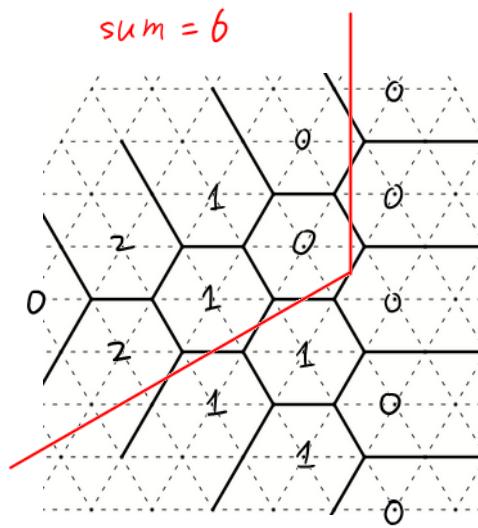
4. Other combinatorial models for LR coefficients

I would highly recommend to read [PV04] before this section. It clarifies the correspondence clearly.

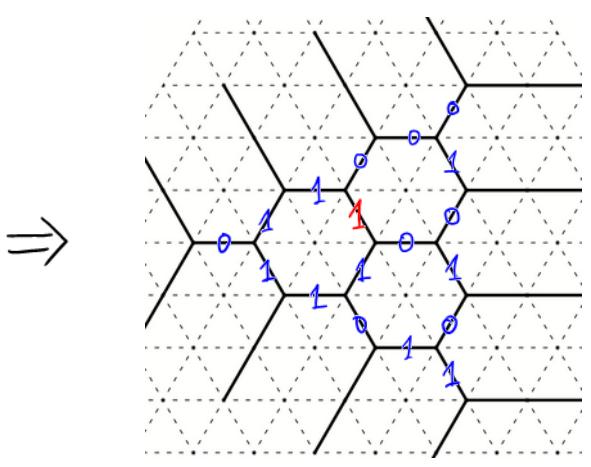
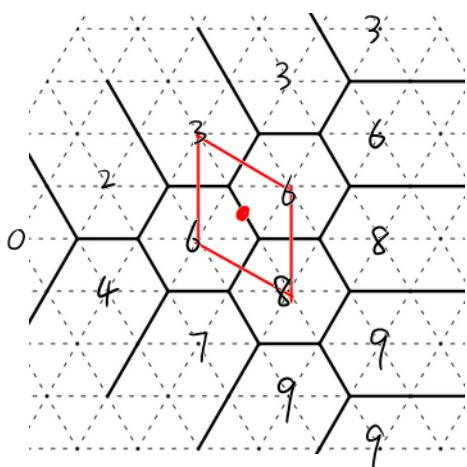
The main goal is the correspondence among these combinatorial models, especially the Knutson--Tao puzzle and the Young diagram.



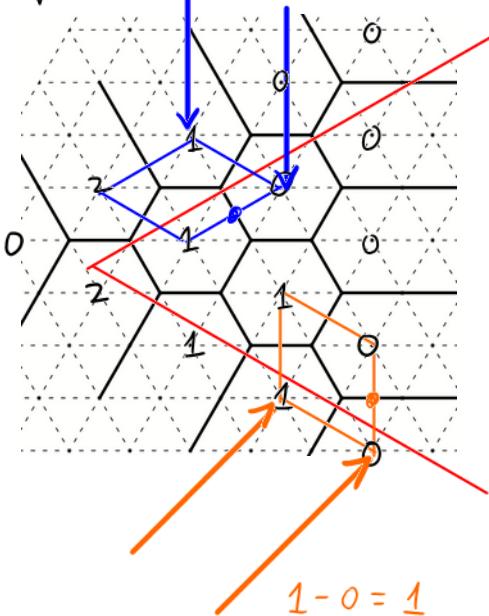
①:



$$(6+6) - (3+8) = 1$$



Direct argument: $1 - 0 = 1$



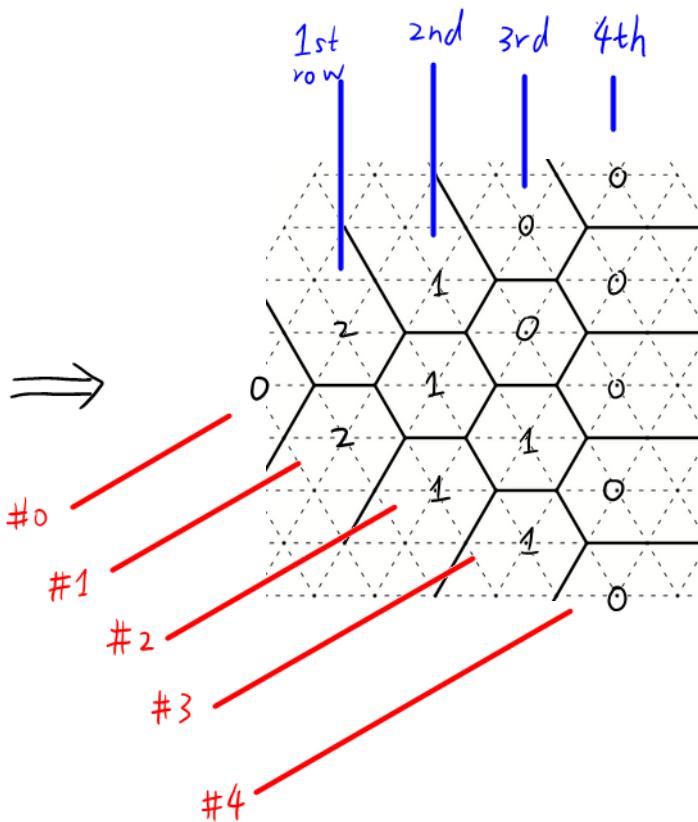
0	0	0
0	1	1
1	1	0
1	0	1
1	0	1

↑: Read the black one gives

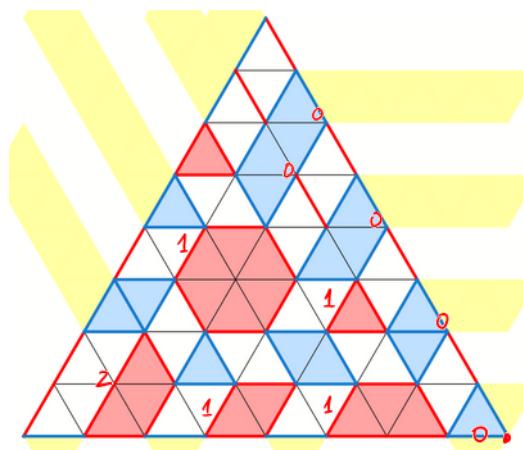
$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}$$
, and the rest is easy.

② : \Rightarrow

0	0	1	1
0	1	2	
2	3		



③ \Downarrow



reading the length of the NW side gives the numbers.

