# Eine Woche, ein Beispiel

## 1.28 conormal bundle

1. conceptions describing the singularity
2. smooth mfld case

## 1. conceptions describing the singularity

tangent cone $\subseteq$ $\bigcup$ limit of tangent spaces $\subseteq$ tangent space

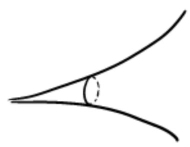
in Grassmannian, for affine variety

e.g.

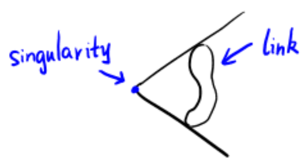
$\{w^2 = z^3\}$ $\quad$ $\{w^2 = 0\}$ $= \{ v \in \mathbb{C}^2 \mid \langle v, dw \rangle = 0 \}$ $\subseteq \mathbb{C}^2$

$\{z_1^2 + z_2^3 + z_3^4 = 0\}$ $\quad$ $\{z_1^2 = 0\}$ $= \left\{ v \in \mathbb{C}^3 \;\middle|\; \begin{array}{l} \langle v, a\,dz_1 + b\,dz_2 \rangle = 0 \\ \text{for some } (a,b) \in \mathbb{C}^2 - 0 \end{array} \right\}$ $\subseteq \mathbb{C}^3$

Apart from these conceptions, we have topological cone:



$\{w^2 = z^3\}$
with link $S^1 \subseteq S^3$




$\{z_1^2 + z_2^3 + z_3^4 = 0\}$
with link $S^3/\mathbb{Z}/3\mathbb{Z} \subseteq S^5$

2. smooth mfld case

<u>Def</u> For $X \subseteq V$ an immersion of smooth mflds, $p \in X$, the normal space $N_pX$ at $p$ is defined as

$$N_pX := \frac{T_pV}{T_pX}$$

$$\overset{\text{Riemannian metric}}{=\!=\!=\!=} (T_pX)^{\perp}$$

(also denoted as $(T_{V/X})_p, (T_XV)_p, \ldots$)

and the conormal space $(T_X^*V)_p$ at $p$ is defined as

$$\begin{aligned}(T_X^*V)_p &:= \ker [T_p^*V \longrightarrow T_p^*X] \\ &= \{\alpha \in T_p^*V \mid \alpha(\vec{v}) = 0 \quad \forall \vec{v} \in T_pX\} \\ &= (T_pX)^{\perp}\end{aligned}$$

(also denoted as $(T^*_{V/X})_p, N_p^*X, \ldots$)

One has SESs

$$0 \longrightarrow T_pX \longrightarrow T_pV \longrightarrow N_pX \longrightarrow 0$$

$$0 \longrightarrow (T_X^*V)_p \longrightarrow T_p^*V \longrightarrow T_p^*X \longrightarrow 0$$

As v.b.,

$$0 \longrightarrow TX \longrightarrow TV \longrightarrow NX \longrightarrow 0$$

$$0 \longrightarrow T_X^*V \longrightarrow T^*V \longrightarrow T^*X \longrightarrow 0$$

Rmk. When $X$ is defined by equations, then
$(T_X^* V)_p$ is easier to compute than other spaces
e.g. tangent space $T_p X$

E.g. remove $0 \in V$ to avoid singularity
For $V = \mathbb{C}^3$,
$X = x^2 + y^3 + z^5 = 0$, at $p = (x_0, y_0, z_0) \in X$,

$$
\begin{aligned}
(T_X^* V)_p &= \langle 2x_0 dx + 3y_0^2 dy + 5z_0^4 dz \rangle_{\mathbb{C}} \\
&= \langle (df)_p \rangle_{\mathbb{C}}
\end{aligned}
$$

[ where $f : V \longrightarrow \mathbb{C}$
$(x, y, z) \longmapsto x^2 + y^3 + z^5$

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V \arrow[d, "f"] \\
\{0\} \arrow[r, hook] & \mathbb{C}
\end{tikzcd}
$$
]

$$T_p^* X \cong \mathbb{C}^3 / \langle (df)_p \rangle_{\mathbb{C}}$$

$$
\begin{aligned}
T_p X &= \{ \vec{v} \in T_p V \mid \alpha(\vec{v}) = 0 \quad \forall \alpha \in (T_X^* V)_p \} \\
&= \{ v_x \partial_x + v_y \partial_y + v_z \partial_z \mid 2x_0 v_x + 3y_0^2 v_y + 5z_0^4 v_z = 0 \} \\
&\overset{p \neq 0}{\cong} \mathbb{C}^2
\end{aligned}
$$

$$N_p X \cong \mathbb{C}^3 / T_p X$$

In conclusion, conormal space is more natural when spaces are defined by equations.