Eine Woche, ein Beispiel

3.23: Schubert calculus: Chern class over Grassmannian

This is a follow up of [2025.02.23], [2025.03.16].

1. Formulas for tautological bundle
2. Homology class in $Gr(r,n)$

1. Formulas for tautological bundle

Chern class realized as pullback of σ_{1^s}

Prop. For those v.bs on $Gr(r,n)$, the Chern class is given by

$$
\begin{aligned}
c(S) &= 1 - \sigma_1 + \cdots + (-1)^r \sigma_{1^r} \\
c(Q) &= 1 + \sigma_1 + \cdots + \sigma_k + \cdots + \sigma_{n-r} \\
c(S^\vee) &= 1 + \sigma_1 + \cdots + \sigma_{1^r} \\
c(Q^\vee) &= 1 - \sigma_1 + \cdots + (-1)^k \sigma_k + \cdots + (-1)^{n-k} \sigma_{n-r}
\end{aligned}
$$

We omit the proof, as there are many equiv definition of Chern class, and I don't know which one to choose.

Cor If $f: X \longrightarrow Gr(r,n)$ is induced by $(\mathcal{F}, s_1, \dots, s_n) = (\mathcal{O}_X^{\oplus n} \to \mathcal{F})$, then

$k^n \twoheadrightarrow \mathcal{F}|_p$

$(\mathcal{F}|_p)^* \hookrightarrow k^n$

$$
\begin{aligned}
c_s(\mathcal{F}) &= f^* c_s(S^\vee) \\
&= f^* \sigma_{1^s} \\
&= f^* \Sigma_{1^s}(V^{st}) \\
&= f^* \{ \Lambda \subset Gr(r,n) \mid \Lambda + V^{st}_{n-r+s-1} \subseteq H \} \\
&= \{ p \in X \mid (\mathcal{F}|_p)^* + \langle e_1^*, \dots, e_{n-r+s-1}^* \rangle \subseteq k^{n-1} \} \\
&= \left\{ p \in X \;\middle|\; \begin{array}{l} \exists\ (0, \dots, 0, k_{n-r+s}, \dots, k_n) \in k^n - \{0\}, \text{ s.t.} \\ k_{n-r+s} s_{n-r+s}(p) + \cdots + k_n s_n(p) = 0 \end{array} \right\} \\
&= \{ p \in X \mid \underbrace{s_{n-r+s}(p), \dots, s_n(p)}_{r-s+1 \text{ many}} \text{ are linear dependent} \}
\end{aligned}
$$

Especially,

$$
\begin{aligned}
c_r(\mathcal{F}) &= \{ p \in X \mid s_n(p) = 0 \} \\
&\vdots \\
c_1(\mathcal{F}) &= \{ p \in X \mid \underbrace{s_{n-r+1}(p), \dots, s_n(p)}_{r \text{ many}} \text{ are linear dependent} \} \\
&= c_1(\Lambda^r \mathcal{F}) \\
&= c_1(\det \mathcal{F})
\end{aligned}
$$

Rmk. $c_s(\mathcal{F}) \neq c_{top}(\Lambda^{r-s+1} \mathcal{F})$ since

$s_1 \wedge s_2$ (pure wedge) is not a general section in $\Lambda^2 \mathcal{F}$!

Nevertheless, when $s = 1$ or r, pure wedge is a general section, so
$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ $\qquad$ $c_r(\mathcal{F}) = c_r(\mathcal{F})$.

Porteous' formula

Thm [3264, Thm 12.4]

Let $X/\mathbb{C}$ sm $\quad k \in \mathbb{Z}_{\geq 0}$,

$\mathcal{E}, \mathcal{F}$: v.b. over X of rank e, f,

$\varphi : \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

$$M_k(\varphi) := \{ x \in X \mid \operatorname{rank} \varphi_x \leq k \}$$

remember multiplicity

$\varphi_x : \mathcal{E}|_x \longrightarrow \mathcal{F}|_x$

If $M_k(\varphi) \subset X$ has codim $(e-k)(f-k)$, then

$$[M_k(\varphi)] = \Delta^{e-k}_{f-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right] = (-1)^{(e-k)(f-k)} \Delta^{f-k}_{e-k}\left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]$$

where

$$\Delta^{e-k}_{f-k}(\gamma) = \begin{vmatrix} \gamma_{f-k} & \cdots & \gamma_{e+f-2k-1} \\ \vdots & \ddots & \vdots \\ \gamma_{f-e+1} & \cdots & \gamma_{f-k} \end{vmatrix}_{(e-k-1)\times(e-k-1)}$$

2. Homology class in $Gr(r,n)$

Lines passing planes

E.g. 1. [3264, p131, Question (a)]

For 4 general lines l_1, l_2, l_3, l_4 in $\mathbb{P}^3$, there are 2 lines meet all four.

Reason: In $Gr(2,4)$,

$$\begin{aligned} &\# \{ l \in Gr(2,4) \mid l \cap l_i \neq \phi,\ \forall i \} \\ = &\deg \sigma_{\square}^4 \\ = &2 \end{aligned}$$

E.g. 2. For 3 general lines l_1, l_2, l_3 in $\mathbb{P}^4$, there is 1 line meet all three.

Reason: In $Gr(2,5)$,

$$\begin{aligned} &\# \{ l \in Gr(2,5) \mid l \cap l_i \neq \phi,\ \forall i \} \\ = &\deg \sigma_{\square\square}^3 \\ = &1 \end{aligned}$$

One can get further that no line in $\mathbb{P}^5$ passing 3 general lines.

E.g. 3.

For 6 general planes $e_1, \cdots, e_6$ in $\mathbb{P}^4$, there are 5 lines passing all these planes.

Reason: In $Gr(2,5)$,

$$\begin{aligned} &\# \{ l \in Gr(2,5) \mid l \cap e_i \neq \phi,\ \forall i \} \\ = &\deg \sigma_{\square}^6 \\ = &5 \end{aligned}$$

E.g. 4. [3264, p131, Question (a)]

For 4 general $(k-1)$-planes $e_1, e_2, e_3, e_4 \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^{2k-1}$, there are k lines passing all these planes.

Reason: In $Gr(2,2k)$,

$$\begin{aligned} &\# \{ l \in Gr(2,2k) \mid l \cap e_i \neq \phi,\ \forall i \} \\ = &\deg \sigma^4_{\underbrace{\square\square\cdots\square}_{k-1}} \\ = &k \end{aligned}$$