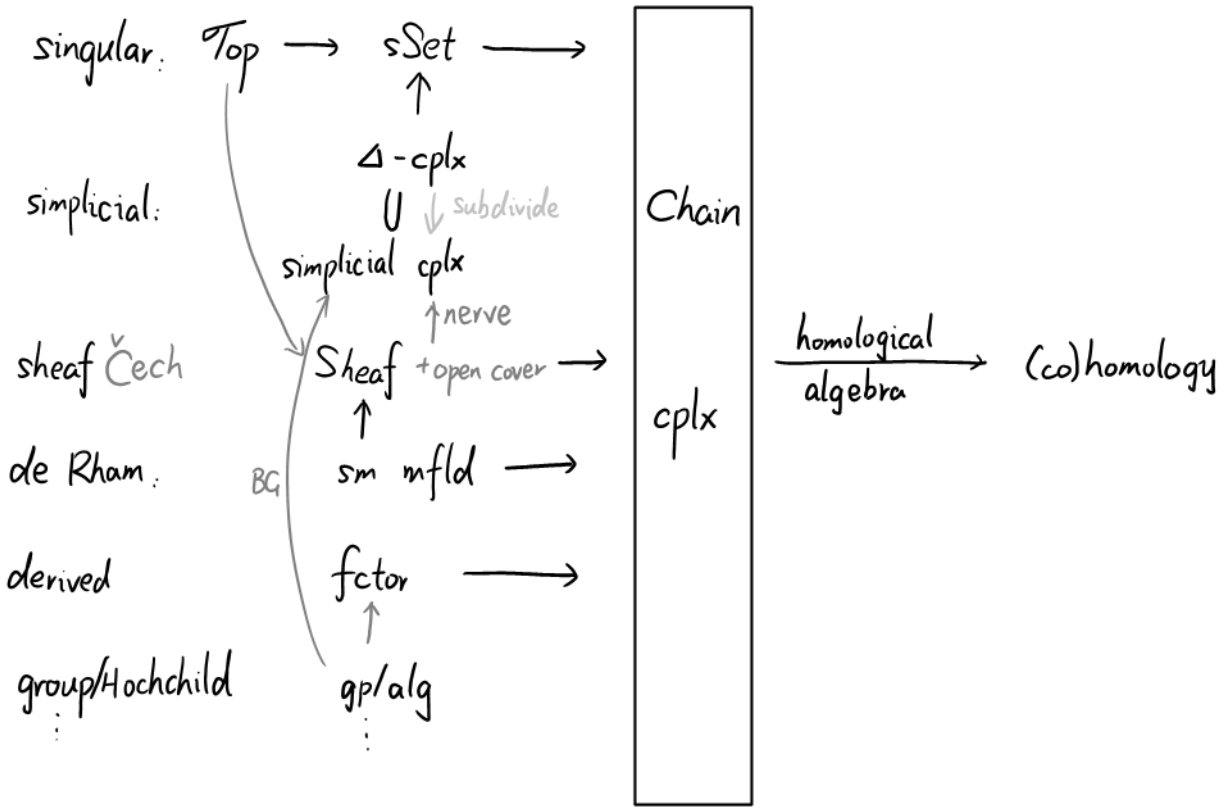
Eine Woche, ein Beispiel

6.25 (co)homology of simplicial set

This document is a continuation of [23.01.09].
https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. relative (co)homology
3. contractable sset has trivial cohomology
4. connection with simplicial complexes
5. more structures
6. connection with sheaf cohomology + derived category

Realize Hochschild homology as simplicial homology:
https://arxiv.org/pdf/1802.03076.pdf

1. definition and basic examples

We use α here because we are considering $X = \Delta^n$ case. May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha \in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha \in X_n} G \qquad 0 \longrightarrow \prod_{\alpha \in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha \in X_1} G \longrightarrow \prod_{\alpha \in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$Hom_{\mathbb{Z}\text{-mod}}\left(\bigoplus_{\alpha \in X_n} \mathbb{Z}, G\right) \cong \prod_{\alpha \in X_n} Hom_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha \in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset. If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \mathcal{T}op$ discrete, $X_\cdot := \mathcal{S}(A) \in sSet$, one can compute

$$
\begin{array}{lccccccccccc}
C_\cdot(X;G): & 0 & \longleftarrow & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} & \cdots \\
C^\cdot(X;G): & 0 & \longrightarrow & \prod_{\alpha\in A} G & \xrightarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} & \cdots \\
\text{wished}\ C^{BM}_\cdot(X;G): & 0 & \longleftarrow & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} & \cdots \\
C^\cdot_c(X;G): & 0 & \longrightarrow & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} & \cdots
\end{array}
$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

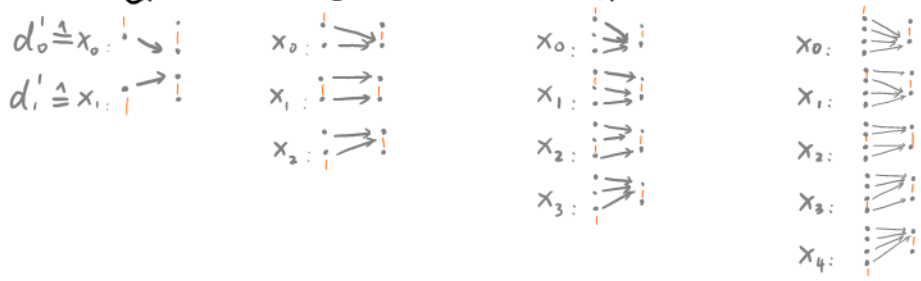
Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.

Notice that $\#\Delta'_k = k+2$, so

$$C_\cdot(\Delta';G): \quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0&1&0 \\ 0&-1&0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1&1&0&0 \\ 0&0&0&0 \\ 0&0&1&1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0&1&0&0&0 \\ 0&-1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&-1&0 \end{pmatrix}} G^{\oplus 5} \cdots$$

basis:

remember indexes:



$$0 = x_0 - x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 - x_1 = x_0 - x_1 \longleftarrow\!\shortmid\; x_1$$
$$0 = x_1 - x_1 \longleftarrow\!\shortmid\; x_2$$

$$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow\!\shortmid\; x_1$$
$$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow\!\shortmid\; x_2$$
$$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow\!\shortmid\; x_3$$
$$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow\!\shortmid\; x_4$$

$$x_0 = x_0 - x_0 + x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 = x_0 - x_1 + x_1 \longleftarrow\!\shortmid\; x_1$$
$$x_2 = x_1 - x_1 + x_2 \longleftarrow\!\shortmid\; x_2$$
$$x_2 = x_2 - x_2 + x_2 \longleftarrow\!\shortmid\; x_3$$

By taking the transpose, one get

$$C^\cdot(\Delta';G): \quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix} 0&0 \\ 1&-1 \\ 0&0 \end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix} 0&0&0&0 \\ 1&-1&0&0 \\ 0&0&0&0 \\ 0&0&1&-1 \\ 0&0&0&0 \end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

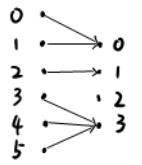
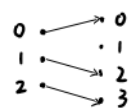
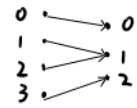
$$H_n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

E.g. 3. When we want to compute $H_n(\Delta^m; G)$ and $H^n(\Delta^m; G)$, we'd better to give elements in $\Delta^m_n \approx \{\text{basis of } C_n(\Delta^m; G)\}$ a better notation, see [23.01.09]
The following table shows some typical element in

$$C_n(\Delta^m; G) = \langle \alpha : [n] \longrightarrow [m] \rangle_{\alpha \in \Delta^m_n}.$$

not confuse with $[n]$

element	picture	list	count	degenerate degree
$\alpha : [5] \to [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	$\Delta^{3,\not{s}3}_5$
$d^3_1 : [2] \to [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		$(0,2,3)$	$[1,0,1,1]$	$\Delta^{3,\theta}_2$
$s^3_1 : [3] \to [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		$(0,1,1,2)$	$[1,2,1]$	$\Delta^{2,\not{s}1}_3$
$\partial\alpha$	——	$(0,0,3,3,3)$ $-(0,0,1,3,3)$	$[2,0,0,3]$ $-[2,1,0,2]$	$\Delta^{3,\not{s}3}_4$ $\Delta^{3,\not{s}2}_4$

e.g. $\partial[2,5,3,4,1,6,0]$

$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$

In this case, $d : C^n(\Delta^m; G) \longrightarrow C^{n+1}(\Delta^m; G)$ is also not hard to describe.

e.g. $d[2,1,0,3] = [3,1,0,3] - [2,1,1,3]$

$d[2,5,3,4,1,6,0]$

$= [3,5,3,4,1,6,0] + [2,5,3,5,1,6,0]$

$- [2,5,3,4,1,7,0] - [2,5,3,4,1,6,1]$

Rmk. The computation of ∂ and d actually comes from the computation of $d_i^{n,*}$ and $d_{i,*}^n$ = the dual of $d_i^{n,*}$ (not $s_i^{n,*}$!).
In general, one can derive the formula of ∂^* & ∂_*.

E.g.

$$\partial^* : C_3(\Delta^m; G) \longrightarrow C_5(\Delta^m; G) \qquad \beta \longmapsto \beta \circ \partial$$

$$[1,2,1] \longmapsto [1,2,1] \circ [2,1,0,3] = [2,1,3]$$

$$d_1^{3,*} : C_3(\Delta^m; G) \longrightarrow C_2(\Delta^m; G) \qquad \beta \longmapsto \beta \circ d_1^3$$

$$[1,2,1] \longmapsto [1,2,1] \circ [1,0,1,1] = [1,1,1]$$

$$s_1^{3,*} : C_2(\Delta^m; G) \longrightarrow C_3(\Delta^m; G) \qquad \beta \longmapsto \beta \circ s_1^3$$

$$[2,1] \longmapsto [2,1] \circ [1,2,1] = [3,1]$$

$$\partial_* : C^5(\Delta^m; G) \longrightarrow C^3(\Delta^m; G)$$

$$[2,1,3] \longmapsto [1,2,1] + [1,1,2]$$

$$[2,2,2] \longmapsto 0$$

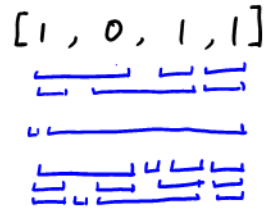
$$[3,3] \longmapsto [3,1] + [2,2]$$

$$d_{1,*}^3 : C^2(\Delta^m; G) \longrightarrow C^3(\Delta^m; G)$$

$$[1,1,1] \longmapsto [2,1,1] + [1,2,1]$$

$$[0,3] \longmapsto [0,4]$$

$$[1,0,1,1] \longmapsto [2,0,1,1] + [1,1,1,1] + [1,0,2,1]$$

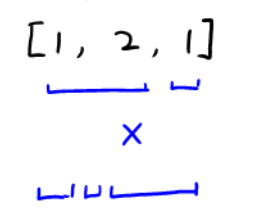


$$s_{1,*}^3 : C^3(\Delta^m; G) \longrightarrow C^2(\Delta^m; G)$$

$$[3,1] \longmapsto [2,1]$$

$$[2,2] \longmapsto 0$$

$$[1,0,3] \longmapsto [1,0,2]$$



2. relative (co)homology

Def For $A \subseteq X$ ssets, denote

$$C_k(X,A;G) := C_k(X;G)/C_k(A;G) = \bigoplus_{\alpha \in X_k - A_k} G$$

⚠ Notice that $X - A$ is not a sset, so we can't write

$$\cancel{C_k(X,A;G) = C_k(X-A;G)}$$

we have $(X/A)_k = X_k/A_k = (X_k - A_k) \sqcup \{*\} \neq X_k - A_k$, so one may write

$$C_k(X,A;G) = \widetilde{C}_k(X/A;G)$$

Ex. Verify that $Y \subset A \subset X$ as ssets

$$C_k(X,\phi;G) = C_k(X;G)$$
$$C_k(X,\{*\};G) = \widetilde{C}_k(X;G)$$
$$C_k(X,X;G) = 0$$
$$0 \longrightarrow C_k(A;G) \longrightarrow C_k(X;G) \longrightarrow C_k(X,A;G) \longrightarrow 0$$
$$0 \longrightarrow C_k(A,Y;G) \longrightarrow C_k(X,Y;G) \longrightarrow C_k(X,A;G) \longrightarrow 0$$

Eg.4. We want to compute $H_n(\Delta^2/\partial\Delta^2;G)$ & $H^n(\Delta^2/\partial\Delta^2;G)$.

3. contractable sset has trivial cohomology

Continue of E.g. 2

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$
\begin{tikzcd}
\Delta^1 \arrow[d, "s_0^1"] & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\,1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1)"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d] & \cdots \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "Id"'] & G \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta^0 \arrow[d, "d_0^1"] & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "Id"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] & \cdots \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdot \arrow[d, shift left] \arrow[d, shift right] & \cdot \arrow[d, shift left] \arrow[d, shift right] \\
\cdot \arrow[d] & \cdot \arrow[d] \\
s_0^1 & d_0^1
\end{tikzcd}
$$

s.t. $s_{0,*}^1 \circ d_{0,*}^1 = Id_{C_\cdot(\Delta^1;G)}$, $\quad d_{0,*}^1 \circ s_{0,*}^1 \sim Id_{C_\cdot(\Delta^1;G)}$.

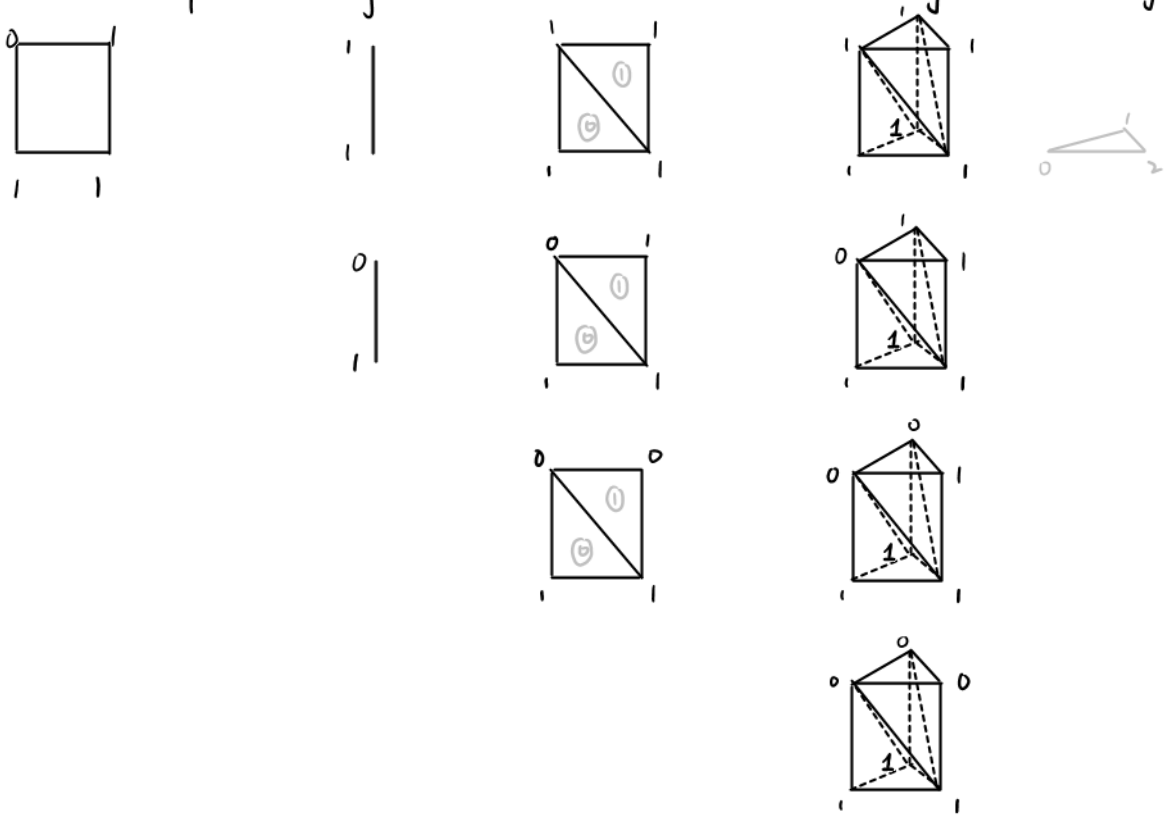
In fact, we have

$$
\begin{tikzcd}
C_\cdot(\Delta^1;G): \arrow[d, Id, "\begin{pmatrix}\cdot\\\cdot\\\cdot\end{pmatrix}_*"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, Id, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dl, dashed, red, "0"'] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, Id, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dl, dashed, red, "\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, Id, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dl, dashed, red, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, Id, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \arrow[dl, dashed, red, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

$$
\begin{aligned}
x_0 &\mapsto x_0 \\
x_1 &\mapsto x_1
\end{aligned}
\qquad
\begin{aligned}
x_0 &\mapsto x_0 - x_0 + x_0 = x_0 \\
x_1 &\mapsto x_1 - x_1 + x_1 = x_1 \\
x_2 &\mapsto x_1 - x_2 + x_2 = x_1 \\
x_3 &\mapsto x_1 - x_2 + x_3
\end{aligned}
$$

$$
\begin{aligned}
x_0 &\mapsto x_0 - x_0 = 0 \\
x_1 &\mapsto x_1 - x_1 = 0 \\
x_2 &\mapsto x_1 - x_2
\end{aligned}
$$

Ex. Observe the picture, try to translate the calculation in geometrical language.



4. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\diamond}$ (non-degenerate):

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G)^{\diamond}: & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_\cdot(\Delta';G)^{\diamond}: \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}0\\1\\0\end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

In fact, we have

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)

For $X \in sSet$, define

$\diamond$: non-degenerate

$\lightning$: degenerate

$$X_k^{\diamond} := \{x \in X_k \mid x \text{ non-degernerate}\} = X_k \setminus (sk^{k-1}X)_k$$

$$X_k^{\lightning} := \{x \in X_k \mid x \text{ degenerate}\} = (sk^{k-1}X)_k$$

$$X_k^{\lightning i} := \left\{ x \in X_k \;\middle|\; \begin{array}{l} x = \alpha^*(y) \text{ for some } y \in X_{k-i}^{\diamond} \\ \alpha : [k-i] \to [k] \end{array} \right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$0 = \underbrace{(sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}X)_k}_{X_k^{\lightning}} \underbrace{\overset{X_k^{\lightning 0}}{\subset} (sk^kX)_k}_{X_k^{\diamond}} = X_k$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\diamond} = \bigoplus_{\alpha \in X_n^{\diamond}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\diamond}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\diamond}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\diamond}} G \cdots$$

$$C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots$$

and $H_\cdot(X;G)^{\diamond}$, $H_\cdot(X;G)^{\lightning}$ as crspd homology.

By definition, $$C_\cdot(X;G) \cong C_\cdot(X;G)^{\diamond} \oplus C_\cdot(X;G)^{\lightning}$$

Claim 1. $H_\cdot(X;G)^{\lightning} = 0$, so

$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\diamond}. \qquad (*)$$

Rmk 1. Roughly, $(*)$ says that

singular homology $\approx$ simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expand $C_\cdot(X;G)$ by double complex.

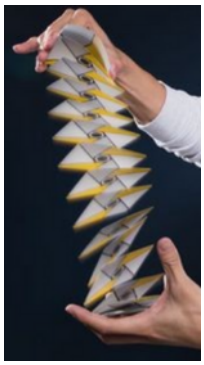
Def (Double complex of $C_\cdot(X;G)$) $\swarrow\!\leftarrow + \leftarrow\!\swarrow = 0$

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0^{\varsigma_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{\varsigma_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{\varsigma_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{\varsigma_3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\varsigma_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{\varsigma_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{\varsigma_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{\varsigma_2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\varsigma_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_1^{\varsigma_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_2^{\varsigma_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_3^{\varsigma_1}} G \arrow[l] \arrow[dl, "0"'] \\
0 & \bigoplus_{\alpha\in X_0^{\phi}} G \arrow[l] & \bigoplus_{\alpha\in X_1^{\phi}} G \arrow[l] & \bigoplus_{\alpha\in X_2^{\phi}} G \arrow[l] & \bigoplus_{\alpha\in X_3^{\phi}} G \arrow[l]
\end{tikzcd}
$$

dim 0 dim 1 dim 2

fold $\Downarrow\Uparrow$ expand

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0} G \arrow[l] & \bigoplus_{\alpha\in X_1} G \arrow[l] & \bigoplus_{\alpha\in X_2} G \arrow[l] & \bigoplus_{\alpha\in X_3} G \arrow[l]
\end{tikzcd}
$$



fold / expend

Eg. For $X = \Delta^1$, we have double complex

$$
\begin{tikzcd}
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_4\rangle \arrow[l] \arrow[dl, "0"'] & \cdots \arrow[l, "\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}1&0&0\\0&0&-1\end{pmatrix}"'] & \langle x_1, x_2, x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}"'] & \\
0 & 0 \arrow[l] \arrow[dl] & \langle x_0, x_2\rangle \arrow[l] \arrow[dl, "0"'] & \langle x_1, x_2\rangle \arrow[l] \arrow[dl, "0"'] & 0 \arrow[l] \arrow[dl] & \\
0 & \langle x_0, x_1\rangle \arrow[l] & \langle x_1\rangle \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l] & 0 \arrow[l] &
\end{tikzcd}
$$

fold $\Downarrow\Uparrow$ expand

$$
\begin{tikzcd}
0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Claim 2. We have chain homotopy equivalence between the following two cplx.

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_n^{\Delta}} G \arrow[l] \arrow[d, equal] & \bigoplus_{\alpha\in X_{n+1}^{S_1}} G \arrow[l, "0"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+2}^{S_2}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+3}^{S_3}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] \\
0 & \bigoplus_{\alpha\in X_n^{\Delta}} G \arrow[l] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"']
\end{tikzcd} \qquad (**)$$

i.e. $(**)$ is exact on all terms except $\bigoplus_{\alpha\in X_n^{\Delta}} G$.

Proof idea of Claim 2 for $X=\Delta^m$.

(can be generalized to arbitrary subsset of Δ^m, e.p. for any triangulation)

Q: How far can we weaken our conditions on X?

Does this proof work for $\partial\Delta^m$? (No, I think)

$$\begin{tikzcd}
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{S_{k-1}}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k}^{S_k}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k+1}^{S_{k+1}}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \cdots \arrow[l] \\
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{S_{k-1}}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k}^{S_k}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k+1}^{S_{k+1}}} G \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Define

$$s[\underbrace{a_1,\cdots a_l}_{\{0,1\}}, \underbrace{a_{l+1}}_{\{0,1\}},\cdots,a_m] = \begin{cases} (-1)^{i}[a_1,\cdots,a_l,a_{l+1}+1,\cdots,a_m], & a_{k+1} \text{ even} \\ 0, & a_{k+1} \text{ odd} \end{cases}$$

$i=\sum_{j=1}^{l} a_j$

Ex. Check that s is a homotopy.

e.g. $X=\Delta^3$, $n=2$, $k=3$ $\Rightarrow m=3$, $n+k=5$

$$\begin{tikzcd}
-[2,1,0,2] & [2,1,0,3] \arrow[l, mapsto] \arrow[dr] & \\
 & -[3,1,0,2] \arrow[ur, mapsto] & \\
 & [2,1,0,3] + [3,1,0,2] & [3,1,0,3] \arrow[l, mapsto]
\end{tikzcd}$$

$X=\Delta^6$, $n=5$, $k=15$ $\Rightarrow m=6$, $n+k=20$

$$\begin{tikzcd}
[2,4,3,4,1,6,0] & [2,5,3,4,1,6,0] \arrow[l, mapsto] \arrow[dr, mapsto] & \\
-[2,5,2,4,1,6,0] \arrow[dr, mapsto] & & \\
 & [3,4,3,4,1,6,0] - [3,5,2,4,1,6,0] & \\
 & [2,5,3,4,1,6,0] - [3,4,3,4,1,6,0] + [3,5,2,4,1,6,0] & [3,5,3,4,1,6,0] \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
{[4,4]} & {[4,5]} \arrow[l, mapsto] \arrow[dr, mapsto] & \\
& {[5,4]} \arrow[ur, mapsto] & \\
& {\begin{matrix}[4,5]\\-[5,4]\end{matrix}} \arrow[r, mapsto] & {[5,5]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{\begin{matrix}[2,5]\\-[3,4]\end{matrix}} & {[3,5]} \arrow[l, mapsto] \arrow[dr, mapsto] & \\
& {[3,5]} \arrow[ur, mapsto] & \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$

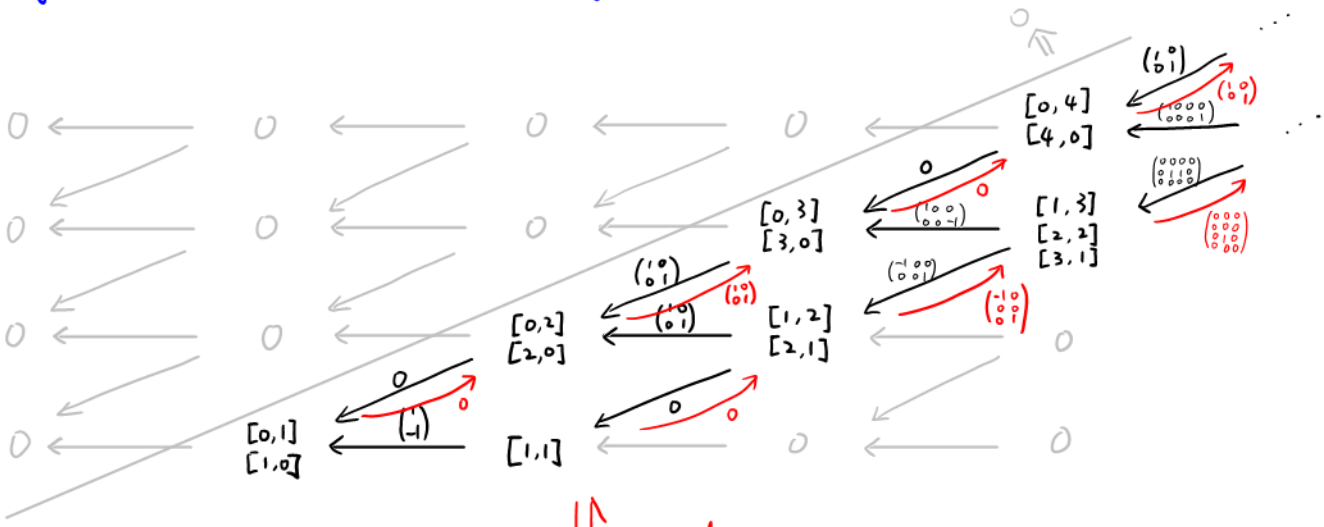
$$\begin{tikzcd}
{-[1,4,4]} & {[1,4,5]} \arrow[l, mapsto] \arrow[dr, mapsto] & \\
& {[1,5,4]} \arrow[ur, mapsto] & \\
& {\begin{matrix}[1,4,5]\\-[1,5,4]\end{matrix}} \arrow[r, mapsto] & {-[1,5,5]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{\begin{matrix}-[1,2,5]\\+[1,3,4]\end{matrix}} & {[1,3,5]} \arrow[l, mapsto] \arrow[dr, mapsto] & \\
& {[1,3,5]} \arrow[ur, mapsto] & \\
0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$

In conclusion,

$$\text{Claim 2} \Rightarrow \text{Claim 1} \Rightarrow \text{Rmk 1}$$

Coming back to E.g.2, one can now find a homotopy without guess.

Eg. For $X = \Delta^1$, we have homotopy



fold ⇓ ⇑ expand

$$\begin{tikzcd}
0 & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}"] \arrow[dr, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, "\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}$$

Ex. Check that (I believe that this argument also works for general sset X)

① $\begin{tikzcd} \cdot \arrow[r, shift left, "s"] & \cdot \arrow[l, shift left, "\partial^{\leftarrow}"] \end{tikzcd} + \begin{tikzcd} \cdot & \cdot \arrow[l, shift right, "\partial^{\leftarrow}"'] \arrow[r, shift left, "s"] & \cdot \end{tikzcd} = 0$

② the collected s is a homotopy.

5. more structures

math.stackexchange.com/questions/2559705/cup-product-why-do-we-need-to-consider-cohomology-with-coefficients-in-a-ring
https://arxiv.org/pdf/2105.08025.pdf
https://users.math.msu.edu/users/ruiterj2/math/Documents/Spring%202019/Galois%20Cohomology%20Seminar%20Week%203.pdf

When $G = R$ is a k-alg, the product structure on $C^{\cdot}(X;R)$ is defined by
(cup product)

wedge in de Rham

$\cup \approx \wedge$

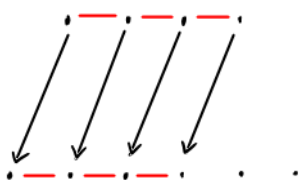
bad symbol compatability...

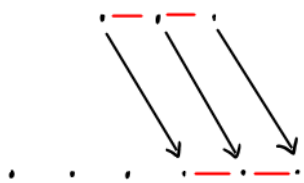
$$\begin{tikzcd}
 & C^{i+j}(X;R) \otimes C^{i+j}(X;R) \arrow[dr] & & \\
\cup: \; C^i(X;R) \otimes C^j(X;R) \arrow[ur, "d_{i,j,*} \otimes d'_{i,j,*}"] & & C^{i+j}(X;R \otimes R) \arrow[r, "\text{multiply}"] & C^{i+j}(X;R) \\
f_1 \otimes f_2 \arrow[rr, mapsto] & & (f_1 \circ d_{i,j,*}) \otimes (f_2 \circ d'_{i,j,*}) & \\
 & & f \otimes g \arrow[r, mapsto] & fg
\end{tikzcd}$$

the $C^{\cdot}(X;R)$-module structure on $C_{\cdot}(X;G)$ is defined by
(cap product)

$$\begin{tikzcd}
 & C^i(X;R) \otimes C_i(X;R) \otimes C_j(X;R) \arrow[dr] & & \\
\cap: \; C^i(X;R) \otimes C_{i+j}(X;R) \arrow[ur, "Id \otimes d^*_{i,j}(-) \otimes d'^*_{i,j}(-)"] & & R \otimes C_j(X;R \otimes R) \arrow[r, "\text{multiply}"] & C_j(X;R) \\
f \otimes \alpha \arrow[rr, mapsto] & & f(d^*_{i,j}(\alpha)) \otimes d'^*_{i,j}(\alpha) & \\
 & & r \otimes \beta \arrow[r, mapsto] & r\beta
\end{tikzcd}$$

where $(i=3, j=2)$





$d_{i,j} : [i] \longrightarrow [i+j]$

$d_{i,j} = [\underbrace{1, \dots, 1}_{i+1 \text{ many}}, \underbrace{0, \dots, 0}_{j \text{ many}}]$

$d'_{i,j} : [j] \longrightarrow [i+j]$

$d'_{i,j} = [\underbrace{0, \dots, 0}_{i \text{ many}}, \underbrace{1, \dots, 1}_{j+1 \text{ many}}]$

$\otimes$ are over k. Notice that

$$\begin{cases} C_i(X;R) = \bigoplus_{\alpha \in X_i} R \\ C^i(X;R) = \prod_{\alpha \in X_i} R \end{cases}$$

are bi R-modules.

https://math.stackexchange.com/questions/4439483/applications-of-the-cup-product-before-descending-to-cohomology

Ex: Show that:

- $C^{\cdot}(X;R)$ is a dga,
- $H^{\cdot}(X;R)$ is a graded R-alg. (graded commutative)
- $H_{\cdot}(X;R)$ is a graded $H^{\cdot}(X;R)$-module.

Hint. Once the following key formulas are verified, we are done.

For $a \in C^p(X;R)$, $b \in C^q(X;R)$,

$$d^{p+q}(a \cup b) = d^p(a) \cup b + (-1)^p a \cup d^p(b)$$

$$a \cup b - (-1)^{pq} b \cup a = (-1)^{p+q-1}\left(d^{p+q-1}(a \cup_1 b) - d^p(a) \cup_1 b - (-1)^p a \cup_1 d^q(b)\right)$$

where

$$a \cup_1 b \in C^{p+q-1}(X;R):$$

$$a \cup_1 b\,(x) = \sum_{i=0}^{p-1} (-1)^{(p-i)(q+1)}\, a\left(\delta^{out,*}_{i,(p,q)}(x)\right) \cdot b\left(\delta^{in,*}_{i,(p,q)}(x)\right)$$

For the definition of $\delta^{out}_{i,(p,q)}$, $\delta^{in}_{i,(p,q)}$, see [23.01.09, Table 1].