

Eine Woche, ein Beispiel

9.3. field extension with RS

Goal: construct an equivalence between two categories.

$$\begin{array}{ccc}
 \text{cpt conn} \\
 \downarrow \\
 \text{RS}^{\text{cc}} = \left\{ \begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{l} \text{Obj: } F/C \text{ field ext st.} \\ \text{trdeg}_C F = 1 \\ F/C \text{ f.g. as a field} \\ \text{Mor: morphism as fields/C} \end{array} \right\}^{\text{op}} = \text{field}_{C(t)/C}^{\text{op}} \\
 Y & & M(Y) \\
 \downarrow f & \implies & \uparrow f^* \\
 X & & M(X)
 \end{array}$$

which obeys the following slogan:

(ramified) covering \approx (function) field extension

- Rmk.
1. For requiring F/C f.g. as a field, we avoid examples like $\overline{C(t)}$.
 Do they corresponds to some non-cpt Riemann surface?
 If so, how to enlarge the category RS^{cc} ?
 2. $\text{field}_{C(t)/C}$ means fields over C which are fin ext of $C(t)$ abstractly;
 morphisms don't need to fix $C(t)$.
 Do you have a better name for RS^{cc} and $\text{field}_{C(t)/C}$?

<https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of>
<https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory>

1. field of meromorphic functions
2. Galois covering
3. valuations
4. quadratic extension of $C(x)$: hyperelliptic curve
5. miscellaneous.

1. field of meromorphic functions

Def. For X RS,

$$\begin{aligned} M(X) &= \{\text{meromorphic fcts on } X\} \\ &= \{f: X \rightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{1_\infty\} \\ &\xrightarrow[X \text{ cpt}]{\text{conn}} \{\text{rational fcts on } X\} \end{aligned}$$

Ex. Verify that

$$\begin{aligned} M(\mathbb{C}\mathbb{P}^1) &\cong \mathbb{C}(z) \\ M(\mathbb{C}/\mathbb{Z}[z]) &\cong \text{Frac}(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1))) \end{aligned}$$

Later we will show that, for $X \in RS^{cc}$,

$$\exists \mathbb{C}(x) \hookrightarrow M(X) \quad \text{st.} \quad [M(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f: \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \quad z \mapsto z^3,$$

compute

- 1) $f^*: \mathbb{C}(T) \hookrightarrow \mathbb{C}(S)$ $[\mathbb{C}(S) : \mathbb{C}(T)]$ & a $\mathbb{C}(T)$ -basis
- 2) $\text{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$
- 3) $\mathbb{C}(S)^{\mathbb{Z}/3\mathbb{Z}}$
- 4) $\text{Aut}_f(\mathbb{C}\mathbb{P}^1)$

Ex. For

$$f: \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \quad z \mapsto z + \frac{1}{z},$$

do the same work.

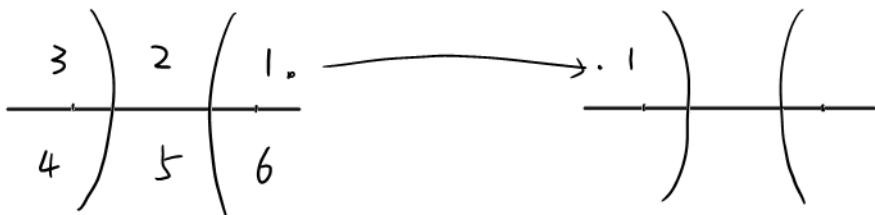
Ex. For

$$f: \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \quad z \mapsto z^3 - 3z,$$

compute the same stuff.

Why isn't $\mathbb{C}(S)/\mathbb{C}(T)$ Galois this time?

Hint.



Prop. For $d \in \mathbb{N}_{>0}$, $f: Y \rightarrow X$ proper holomorphic between conn RSs,
 $[M(Y) : f^*M(X)] = d$.

Cor. For X cpt conn,

$$\exists \mathbb{C}(x) \hookrightarrow M(X) \text{ s.t. } [M(X) : \mathbb{C}(x)] < +\infty$$

In ptc, F/\mathbb{C} f.g as a field. $\text{trdeg}_C F = 1$.

To show the proposition, one need the following black box to find a basis.
Black box (meromorphic fcts separate points)

$X: \text{RS}, \quad x, y \in X, \quad x \neq y, \text{ then}$

$$\exists g \in M(X) \text{ s.t. } g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$

(stronger) $\exists g \in M(X) \text{ s.t. } \text{ord}_x g = -1, \quad g(y) = 0.$

I prefer using Riemann-Roch when X is cpt, and Stein manifold when X is not.

Ex. Using the black box, show that,

for $X: \text{RS}, \quad \{x_1, \dots, x_n\} \subseteq X, \quad \exists g \in M(X) \text{ s.t.}$

$$\text{ord}_{x_i} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{1, \dots, n\}$$

$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \quad i, j \in \{1, \dots, n\}$$

Proof of prop

$$[M(Y) : f^*M(X)] \geq d: \text{Fix } x_0 \in X \text{ s.t. } \#f^{-1}(x_0) = d. \text{ Denote } f^{-1}(x_0) = \{y_1, \dots, y_d\}.$$

For each i , let $g_i \in M(Y)$ be a meromorphic fct s.t.

$$\text{ord}_{x_i} g_i = -1 \quad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$

then $\{g_1, \dots, g_d\} \subseteq M(Y)$ are $f^*M(X)$ -linear independent.

Check: $\text{ord}_{y_i} (\sum_j f_i g_j) \approx \text{ord}_{y_i} f_i$

$$[M(Y) : f^*M(X)] \leq d.$$

$\forall g \in M(Y)$, need to find $a_i \in f^*M(X)$ s.t.

$$g^d + a_{d-1}g^{d-1} + \dots + a_0 = 0 \quad \text{in } M(Y)$$

The fcts

$$a_i(z) = (-1)^i \sum_{\{k_1, \dots, k_d\} \subseteq \{1, \dots, d\}} g(z_{k_1}) \dots g(z_{k_d})$$

$f^{-1}(f(z)) = \{z_1, \dots, z_d\}$, multiplicity is counted
satisfy the conditions.

Use Riemann extension theorem to show $a_i(z) \in f^*M(X)$, see [Donaldson, p148].

By primitive element theorem, $[M(Y) : f^*M(X)] \leq d$.

2. Galois covering

Def. Let $f: Y \rightarrow X$ be a proper holo map between two conn RSs.
 f is Galois, if $M(Y)/f^*M(X)$ is a Galois extension.

normal

normal

Prop. $f: Y \rightarrow X$ is Galois/normal

$$\Leftrightarrow \deg f = \# \text{Aut}_f(Y)$$

$$\Leftrightarrow f^{-1}(x_0) \text{ is an } \text{Aut}_f(Y) \text{-torsor}, \quad \forall x_0 \in X - f(\text{Ram}(f))$$

$$\Leftrightarrow \text{Aut}_f(Y) \subset f^{-1}(x_0) \text{ transitively}, \quad \forall x_0 \in X$$

$$\Leftrightarrow Y/\text{Aut}_f(Y) \cong X, \text{ i.e. } f \text{ can be written as}$$

$$Y \xrightarrow{\sim} Y/G$$

Ex. For $f: Y \rightarrow X$, suppose that

$$[\forall y_1, y_2 \in Y \text{ st. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2) \quad \text{ramification index}$$

Show that f is Galois by computing $\# \text{Aut}_f(Y)$.

Hint. Use geodesics to divide X into several smaller triangles.

If geodesics are hard, take $g: X \rightarrow \mathbb{CP}^1$ non-constant, and reduce the problem to $g \circ f$.

This proof is not completely rigorous, and you are encouraged to find a reference to rigorously prove it, or read this discussion on stackexchange:
<https://math.stackexchange.com/questions/1952655/ramification-index-and-inertia-degree-same-for-all-the-primes-then-is-the-exten>

You may need the following materials for completing the proof, relating with questions about geodesic triangulations.

google: geodesic triangulations

<https://math.stackexchange.com/questions/1661331/proof-of-equivalence-of-conformal-and-complex-structures-on-a-riemann-surface?rq=1>
<https://arxiv.org/pdf/1603.16702.pdf>

(If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.)

copied from p2, in <https://www.mathematik.uni-muenchen.de/~forster/eprints/gaussbonnet.pdf>

<http://czamfirescu.tricube.de/CTZamfirescu-o8.pdf>

[Thm 2] <http://www-fourier.univ-grenoble-alpes.fr/~ycolver/All-Articles/91c.pdf>

<https://mathoverflow.net/questions/138267/what-prevents-a-cover-to-be-galois>

E.g. Consider the covering

$$f: \mathbb{C}P^1 \longrightarrow \mathbb{C}P^1 \quad z \mapsto z^3 - 3z$$

This is not a Galois covering. Consider the Galois closure

$$\begin{array}{ccc}
 \mathbb{C}P^1 & \mathbb{C}(U) = \mathbb{C}(S)[R]/(R^2 + S^2 - 4) & U + \frac{1}{U} \\
 \downarrow z + \frac{1}{z} & \uparrow & \uparrow \\
 \mathbb{C}P^1 & \mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3 - 3S - T) & S \quad S^3 - 3S \\
 \downarrow z^3 - 3z & \uparrow & \uparrow \\
 \mathbb{C}P^1 & \mathbb{C}(T) & T
 \end{array}$$

Determination of the Galois closure

$$\begin{aligned}
 \min(S, \mathbb{C}(T)) &= x^3 - 3x - T && \text{in } \mathbb{C}(T)[x] \\
 &= x^3 - 3x - (S^3 - 3S) \\
 &= (x - S)(x^2 + Sx + S^2 - 3) && \text{in } \mathbb{C}(S)[x]
 \end{aligned}$$

To decompose the polynomial $x^2 + Sx + S^2 - 3$, we have to add root of discriminant.

$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2 - 3)} = \sqrt{3} \sqrt{-S^2 + 4}.$$

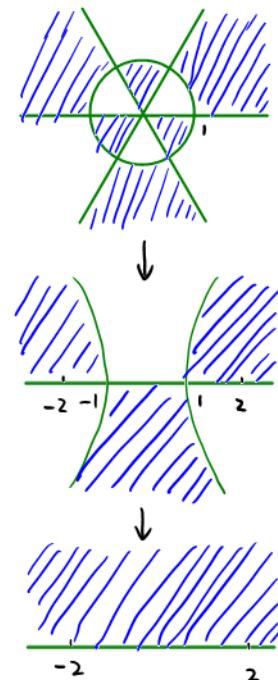
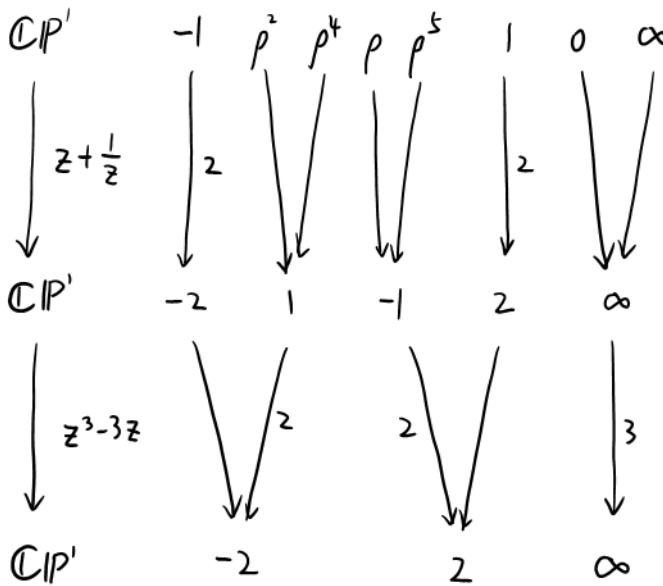
Therefore, the Galois closure of $\mathbb{C}(S)/\mathbb{C}(T)$ is

$$\mathbb{C}(S)[R]/(R^2 + S^2 - 4) \cong \mathbb{C}\left(\frac{S+iR}{2}\right) \stackrel{?}{=} \mathbb{C}(U)$$

where

$$S = \frac{S+iR}{2} + \frac{S-iR}{2} = U + \frac{1}{U}$$

The picture from the RS side is as follows:



only ramified pts are drawn

affine version

Question:

How to know the genus of the RS corresponding to the Galois closure?

Answer:

<https://mathoverflow.net/questions/152/how-do-you-see-the-genus-of-a-curve-just-looking-at-its-function-field>

Question:

Do we have any Galois closure whose ramification information is not minimal as we expected?

E.g. 2. For $E = \mathbb{C}/\Delta$, since $\pi_1(E, \circ) \cong \mathbb{Z} \oplus \mathbb{Z}$,

E has three unramified coverings of deg 2.

When $\Delta = \mathbb{Z}[i]$, what are the crspd field extensions?

There are more deg 2 ramified coverings from the higher genus RS, but we don't discuss them here.

$$\begin{array}{ccccccccc} \cdot & \cdot \\ \cdot & \cdot \\ \vdots & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot \end{array}$$

normalized equation

$$y^2 = x(x+1)(x-1)$$

$$y^2 = x(x+1)(x-1)$$

$$y^2 = 4x^3 - 11x - 7 \quad \Delta = 8$$

$$= (x+1)(4x^2 - 4x - 7)$$

$$= 4(x+1)(x - \frac{1}{2} + \sqrt{2})(x - \frac{1}{2} - \sqrt{2})$$

$$j(2i) = \left(\frac{11}{2}\right)^3 \cdot 1728 = 66^3 \quad g_2(2i) = \frac{11 \Gamma(\frac{1}{4})^8}{2^8 \pi^2} \quad g_3(2i) = \frac{7 \Gamma(\frac{1}{4})^{12}}{2^{12} \pi^3}$$

equation is given by

$$y^2 = 4x^3 - g_2 x - g_3$$

Ex. 1) Show that

$$\text{Aut}_{\text{RS}}(\mathbb{C}/\Delta)[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

no matter \mathbb{C}/Δ has CM or not.

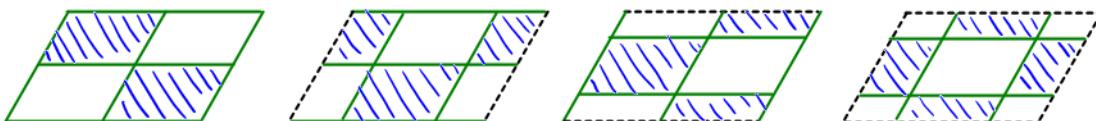
2) We get 7 ramified coverings of deg 2.

$$\pi_\tau: E \longrightarrow E/\langle \tau \rangle \quad \forall \tau \in \text{Aut}_{\text{RS}}(\mathbb{C}/\Delta)[2] - \{\text{Id}\}$$

Which are ramified coverings? Compute the genus & ramification information.

3 unramified , $g(E/\langle \tau \rangle) = 1$

4 ramified at 4 pts , $g(E/\langle \tau \rangle) = 0$



3) Find all index 2 subfields of $M(\mathbb{C}/\Delta)$. hard!

3. valuations.

Q : How to reconstruct the RS of the crspd fct field extension?
i.e., how to give an inverse factor of the fct

$$M(-) : RS^{cc} \longrightarrow \text{field}_{\mathbb{C}(t)/\mathbb{C}}^{\text{op}} \quad x \longmapsto M(x) ?$$

Observation: $\forall X \in RS^{cc}$, $x \in X$, one can define a valuation

$$v_x^X : M(x) \longrightarrow \mathbb{Z} \sqcup \{\infty\} \quad f \longmapsto \deg_x f$$

indicating the order of fcts on x .

If we collect all the valuations on $M(X)$, we may recover the RS X .

Ref. [Perfseminar, L2]

See:

The Zariski-Riemann Space of Valuation Rings by Bruce Olberding
https://link.springer.com/chapter/10.1007/978-3-030-89694-2_21

<https://math.stackexchange.com/questions/188652/finite-extensions-of-rational-functions>
<https://mathoverflow.net/questions/75923/the-space-of-valuations-of-a-function-field>

Def (valuation (Bourbaki) / NA absolute value*) Γ : some tot ordered gp.

For $A \in \text{CRing}$, a valuation of A is a map

$$v : A \longrightarrow \Gamma \sqcup \{\infty\}$$

s.t.

- $v(0) = \infty$

- $v(ab) = v(a) + v(b)$, $v(1) = 0$

- $v(a+b) \geq \min(v(a), v(b))$, with equality if $v(a) \neq v(b)$

↳ which makes v more algebraic rather than analytic
 rigid flexible

If $A \in \text{Field}_{\mathbb{C}}$, we require additionally that $v|_{\mathbb{C}^{\times}} \equiv 0$.

Rmk. The additional assumption on \mathbb{C}^{\times} is natural, as we want

$$v|_{\mathbb{C}^{\times}} : \mathbb{C}^{\times} \longrightarrow \Gamma \sqcup \{\infty\}$$

\uparrow order top

to be a cont gp homo. One gets

$$v(z) = v(|z|) = v(e)^{\ln|z|} \quad \forall z \in \mathbb{C}^{\times}.$$

Moreover, we want v to be an NA absolute value, so $v(z) = 0$.

*

Many people don't use "absolute value" for high rank valuations.

Def (continue) Denote

$$Spv(A) = NAval(A) = \{\text{valuations of } A\}/\sim$$

where

$$v \sim v' \Leftrightarrow \exists v_0: A \rightarrow \Gamma_0 \cup \{\infty\}, \Gamma_0 \rightarrow \Gamma, \Gamma_0 \rightarrow \Gamma' \text{ s.t.}$$

$$\begin{array}{ccc} & v & \Gamma \cup \{\infty\} \\ A & \xrightarrow{v_0} & \Gamma_0 \cup \{\infty\} \\ & v' & \Gamma' \cup \{\infty\} \end{array}$$

commutes

$$\begin{aligned} &\Leftrightarrow \forall x, y \in A, [v(x) \geq v(y) \Leftrightarrow v'(x) \geq v'(y)] \\ &\Leftrightarrow \bar{F}_v = \bar{F}_{v'}, O_v = O_{v'} \end{aligned}$$

My notation
[Perfseminar, L2]

e.g.

A	v	Γ_v	\bar{F}_v	\overline{k}_v	O_v	p_v	k_v
A	v	Γ_v	$p_v = \text{supp}(v) = v^{-1}(\infty)$	$K(p_v)$	R_v	-	-
\mathbb{Q}_p	p-adic	\mathbb{Z}	0	\mathbb{Q}_p	\mathbb{Z}_p	$p\mathbb{Z}_p$	$\mathbb{Z}/p\mathbb{Z}$
\mathbb{Z}	p-adic	\mathbb{Z}	0	\mathbb{Q}	$\mathbb{Z}_{(p)}$	$p\mathbb{Z}_{(p)}$	$\mathbb{Z}/p\mathbb{Z}$
\mathbb{Z}	$1 \cdot _{\mathbb{F}_p}$	0	$p\mathbb{Z}$	\mathbb{F}_p	\mathbb{F}_p	0	\mathbb{F}_p
\mathbb{Z}	v_{triv}	0	0	\mathbb{Q}	\mathbb{Q}	0	\mathbb{Q}

$|\cdot|_\infty \notin Spv(\mathbb{Z})$, since $|\cdot|_\infty$ is Archimedean.

Ex. In this exercise we want to describe $S_{\text{pv}}(\mathbb{C}(z))$.

1). For $v \in S_{\text{pv}}(\mathbb{C}(z))$, suppose $v(z-3) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z+3)}\right)$.

2). For $v \in S_{\text{pv}}(\mathbb{C}(z))$, suppose $v(z-3) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z+3)}\right)$.

3). Define

$$v_{\text{triv}}: \mathbb{C}(z) \longrightarrow 0 \cup \{\infty\} \quad f \neq 0 \mapsto 0$$

Show that $v_{\text{triv}} \in S_{\text{pv}}(\mathbb{C}(z))$.

4) Show that as Sets,

$$S_{\text{pv}}(\mathbb{C}(z)) \cong \{v_{\text{triv}}\} \sqcup \mathbb{C}\mathbb{P}^1$$

$$v_{z_0}^{\mathbb{C}\mathbb{P}^1} \xleftarrow{\quad} z_0$$

Ex. Use the same method to describe $S_{\text{pv}}(\mathbb{Q})$.

Hint. 1. Use the strong triangular inequality, for some $a \neq 0$

$$v(a+a+\dots+a) \geq v(a)$$

$$\parallel$$

$$v(na) = v(n)+v(a)$$

$$\Rightarrow v(n) \geq 0 \quad \forall n \in \mathbb{N}_{\geq 1}.$$

2. For $p \neq p'$ primes, use Bézout's identity to show that

$$[v(p) > 0 \Rightarrow v(p') = 0]$$

3. Conclude that as Sets,

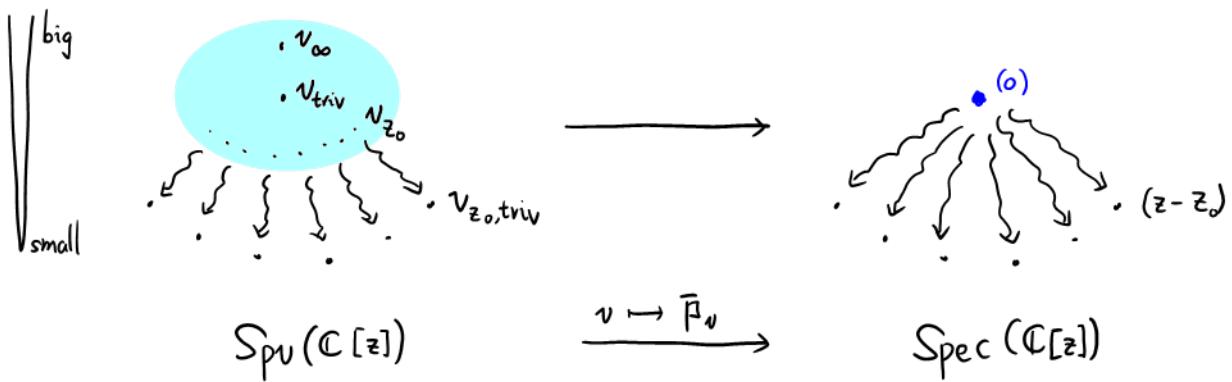
$$S_{\text{pv}}(\mathbb{Q}) \cong \{v_{\text{triv}}\} \sqcup \{\text{primes}\}$$

$$v_p \xleftarrow{\quad} p$$

Ex. In this exercise we want to describe $\text{Spv}(\mathbb{C}[z])$.
 Since $\bar{\mathfrak{P}}_v$ is a prime ideal of $\mathbb{C}[z]$, we get (as Sets)

$$\begin{aligned}\text{Spv}(\mathbb{C}[z]) &\cong \bigsqcup_{\mathfrak{p} \in \text{Spec } \mathbb{C}[z]} \text{Spv}(\text{Frac}(\mathbb{C}[z]/\mathfrak{p})) \\ &\cong \text{Spv}(\mathbb{C}(z)) \sqcup \bigsqcup_{z_0 \in \mathbb{C}} \text{Spv}(\mathbb{C}[z]_{(z-z_0)}) \\ &\cong \{v_{\text{triv}}\} \sqcup \mathbb{C}P^1 \sqcup \mathbb{C}\end{aligned}$$

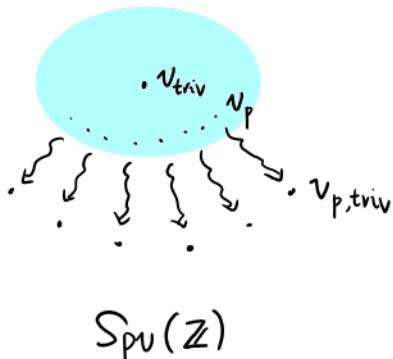
Here, $v \leq v'$ iff $\exists \Gamma_v \xrightarrow{\Gamma_v'} \Gamma_{v'}$, $v(a) \geq v'(a) \quad \forall a \in A$
 e.g. $v_\infty \geq v_0$ in $\text{Spv}(\mathbb{C}[z])$, while they are incomparable in $\text{Spv}(\mathbb{C}(z))$.



$z_0 \neq 0$			
0	1	\bar{z}	$z - \bar{z}_0$
v_∞	∞	0	-1
v_{triv}	∞	0	0
v_{z_0}	∞	0	1
$v_{z_0, \text{triv}}$	∞	0	∞

The generic point contains information about the curves.
 This philosophy becomes clearer when working with Spv . We observe that the (preimage of the) generic point inherits all the closed points, even those outside the local affine chart.

Ex. Use the same method to compute $\text{Spv}(\mathbb{Z})$.



Q: How to understand $\text{Spv}(F)$, for $F = \mathbb{C}(x)[y]/(y^2 - x(x+1)(x-1))$?

Idea: Use the restriction map

$$\begin{array}{c} \text{Spv}(F) \\ \downarrow \pi \\ \text{Spv}(\mathbb{C}(x)) \end{array}$$

we only need to understand the fiber at each pt.

1) When $\pi(v) = v_{\text{triv}}$,

$$\begin{aligned} v(y^2) &= v(x(x+1)(x-1)) \\ &= v_{\text{triv}}(x(x+1)(x-1)) = 0 \end{aligned} \Rightarrow v(y) = 0$$

for $f \in \mathbb{C}(x) - \{0\}$,

$$\begin{aligned} v(y+f) + v(y-f) &= v(x(x+1)(x-1) - f^2) \\ &= v_{\text{triv}}(x(x+1)(x-1) - f^2) = 0 \end{aligned} \quad \left. \right\} \Rightarrow v(y+f) = v(y-f) = 0$$

$$v(2f) = v_{\text{triv}}(f) = 0$$

for $f_1 \in \mathbb{C}(x) - \{0\}$, $f_2 \in \mathbb{C}(x)$,

$$v(f_1 y + f_2) = v(f_1) + v(y + \frac{f_2}{f_1}) = 0.$$

Therefore, $\pi^{-1}(v_{\text{triv}}) = v_{\text{triv}}$.

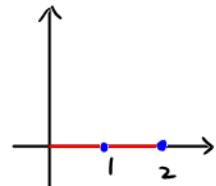
Shorter: for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation

$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

where

$$v(-2f) = v(f^2 - x(x+1)(x-1)) = 0.$$

Therefore, $v(z_0) = 0$.



Newton polygon

2). When $\pi(v) = v_1$,

$$v(y^2) = v_1(x(x+1)(x-1)) = 1 \Rightarrow v(y) = \frac{1}{2}$$

for $f_1, f_2 \in \mathbb{C}(x)$,

$$v(f_1 y + f_2) = \min(v_1(f_1) + \frac{1}{2}, v_1(f_2)) \text{ is uniquely determined.}$$

Therefore, $\pi^{-1}(v_1) = \{v_{(1,0)}\}$, where $f_1, f_2 \in \mathbb{C}(x)$

$$v_{(1,0)}(f_1 y + f_2) = \min(v_1(f_1) + \frac{1}{2}, v_1(f_2))$$

Idea: $y = \sqrt{x-1} (\text{---})^{\frac{1}{2}} + \frac{3\sqrt{x-1}}{4} (\text{---})^{\frac{3}{2}} - \frac{5\sqrt{x-1}}{32} (\text{---})^{\frac{5}{2}} + \frac{35\sqrt{x-1}}{128} (\text{---})^{\frac{7}{2}} + \dots \in \mathbb{C}((\sqrt{x-1}))$

<https://calculator-online.net/power-series-calculator/>

In a similar way, one may show that

$$\#\pi^{-1}(v_0), \#\pi^{-1}(v_{-1}) = 1.$$

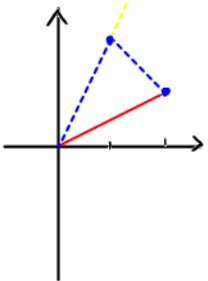
E.p. $v(y+f) = \min(\frac{1}{2}, v_1(f))$.

Argument by Newton polygon: denote $a = v_1(f)$ temporarily
for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation
$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

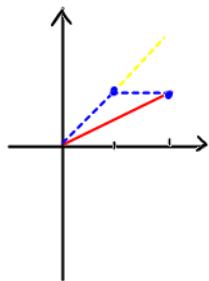
where

$$v(-2f) = a$$

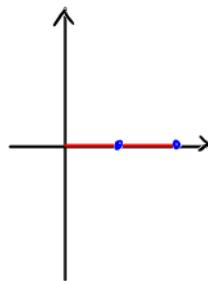
$$v(f^2 - x(x+1)(x-1)) = \min(1, 2a)$$



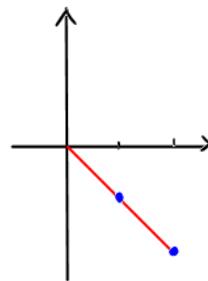
$a=2$ case



$a=1$ case



$a=0$ case



$a=-1$ case

Therefore, $v(z_0) = \min(\frac{1}{2}, v_1(f))$.

3). When $\pi(v) = v_\infty$,

$$v(y^2) = v_\infty(x(x+1)(x-1)) = -3 \Rightarrow v(y) = -\frac{3}{2}$$

for $f_1, f_2 \in \mathbb{C}(x)$,

$$v(f_1y + f_2) = \min(v_\infty(f_1) - \frac{3}{2}, v_\infty(f_2)) \text{ is uniquely determined.}$$

Therefore, $\#\pi^{-1}(v_\infty) = 1$

[Idea: $y = x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{8}x^{-\frac{5}{2}} - \frac{1}{16}x^{\frac{9}{2}} - \frac{5}{128}x^{\frac{13}{2}} - \dots \in \mathbb{C}((x^{-\frac{1}{2}}))$]

E.p. $v(y+f) = \min(-\frac{3}{2}, v_\infty(f))$.

Argument by Newton polygon: denote $a = v_\infty(f)$ temporarily

for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation
$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

where

$$v(-2f) = a$$

$$v(f^2 - x(x+1)(x-1)) = \min(-3, 2a)$$

Therefore, $v(z_0) = \min(-\frac{3}{2}, v_\infty(f))$.

4). When $\pi(v) = v_3$,

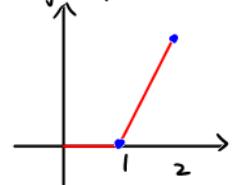
$$v(y^2) = v_3(x(x+1)(x-1)) = 0 \Rightarrow v(y) = 0$$

for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation $a = v_3(f)$

$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

where

$$\begin{aligned} v(-2f) &= a \\ v(f^2 - x(x+1)(x-1)) &\geq \min(0, 2a) \quad \text{with equality if } a \neq 0 \\ v(z_0) \text{ is only not determined when} \\ \begin{cases} a = 0 \\ v(f^2 - x(x+1)(x-1)) > 0 \end{cases} \\ \Leftrightarrow f(3)^2 &= 24 \\ \Leftrightarrow f(3) &= \pm\sqrt{24} \\ \Leftrightarrow v_3(f - \sqrt{24}) &> 0 \quad \text{or} \quad v_3(f + \sqrt{24}) > 0 \\ \Leftrightarrow f(x)^2 - 24 &= (x-3)^k g(x) \quad \text{for some } k > 0, g(x) \in \mathbb{C}(x), x-3 \nmid g \end{aligned}$$



bad case

Take $f_0(x) = x - 3 + \sqrt{24}$. Suppose that $v(y + f_0) = 1$, $v(y - f_0) = 1$, we need to determine $v(y + f)$.

$$\begin{aligned} v(y + f) &= v((y + f_0) - (f_0 - \sqrt{24}) + (f - \sqrt{24})) \\ &= \begin{cases} v(f^2 - x(x+1)(x-1)), & v_3(f - \sqrt{24}) > 0 \Leftrightarrow f(3) = \sqrt{24} \\ 0, & v_3(f - \sqrt{24}) = 0 \Leftrightarrow f(3) \neq \sqrt{24} \end{cases} \end{aligned}$$

Ex. Verify that this is indeed a valuation.

$$[\text{Idea: } y = \pm \left(2\sqrt{6} + \frac{13\sqrt{6}}{12}(x-3) + \frac{47\sqrt{6}}{576}(x-3)^2 - \frac{35\sqrt{6}}{13824}(x-3)^3 + \dots \right) \in \mathbb{C}((x-3))]$$

In conclusion, $\pi^{-1}(v_3) = \{v_{(3, \sqrt{24})}, v_{(3, -\sqrt{24})}\}$, which are determined by $v(y + f_0)$, and

$$v_{(3, \sqrt{24})}(y + f) = \begin{cases} \min(0, v(f)), & f(3)^2 \neq 24 \\ v(f^2 - x(x+1)(x-1)), & f(3) = \sqrt{24} \\ 0, & f(3) = -\sqrt{24} \end{cases}$$

$$v_{(3, -\sqrt{24})}(y + f) = \begin{cases} \min(0, v(f)), & f(3)^2 \neq 24 \\ v(f^2 - x(x+1)(x-1)), & f(3) = -\sqrt{24} \\ 0, & f(3) = \sqrt{24} \end{cases}$$

Rmk The Newton polygon is a powerful tool for understanding ramified coverings from a function field perspective.
The crucial point is that, the valuation of the solution is determined by the valuation of the coefficient.

Ex. Try to understand $\mathrm{Spv}(F)$, for $F = \mathbb{C}(x)[y]/(y^2 - x(x+1)(x-1)(x-2))$.

What is the genus of the crspd RS?

Notice that $\#\pi^{-1}(v_\infty) = 1$ this time.

Ex. Do the same stuff for $F = \mathbb{C}(x)[y]/(x - (y^3 - 3y))$.

Q: How to see the genus of the crspd RS directly through the field, instead through the ramified covering?

Q. (RS/scheme structures reconstruction)

How to recover the (Zariski) topology of $\text{Spv}(F)$?

- Compute $\text{div } f$
- Find local charts
- Understand the rational subset

$$U\left(\frac{T}{s}\right) = \{v \in \text{Spv}(F) \mid |t(v)| \leq |s(v)| \neq 0, \text{ for all } t \in T\}$$

4. quadratic extension of $\mathbb{C}(x)$: hyperelliptic curve

It's not always the case that the RS side is easier than the field extension side. In this section, we'll explore a toy example. This example has a straightforward and direct description on the field extension side, but reveals a rich geometrical structure on the RS side.

Prop. All the quadratic equations of $\mathbb{C}(x)$ are iso to

$$\mathbb{C}(x)[T]/(T^2 - f(x))$$

for some $f(x) \in \mathbb{C}[x] - \{0\}$ monic square-free.

Hint. Completing the square. Notice that $\text{char } \mathbb{C} = 0$.

Def. The crspd RS is called the hyperelliptic curve.

Rmk. The crspd RS of $T^n - f(x)$ is called the superelliptic curve, see wiki.

Roughly,

$$\begin{aligned} \text{hyperelliptic curve} &\Leftrightarrow \deg 2 \text{ extension of } \mathbb{C}(x) \\ \text{superelliptic curve} &\Leftrightarrow \text{cyclic extension of } \mathbb{C}(x) \\ &\quad \text{Kummer since } \zeta_n \in \mathbb{C}^\times \end{aligned}$$

Using the method in the last section, one can compute the genus

$$g = \left\lfloor \frac{\deg f - 1}{2} \right\rfloor$$

for hyperelliptic curve

For the genus of the Superelliptic curve, see wiki: https://en.wikipedia.org/wiki/Superelliptic_curve

Q: How could we identify superelliptic curve just from the information of ramification?

e.g. for a Galois covering $f: X \rightarrow \mathbb{P}^1$ of deg m ,

$\text{ex } |m$ for all $x \in X$,

can we conclude that X is superelliptic?

5. miscellaneous.

- non-cpt RS, infinite covering
- Spv for higher dimensional varieties, high rank pts
- gp structures on valuations of $\mathbb{C}[x,y,z]/(y^2z - x(x-z)(x+z))$
- maximal abelian extension/unramified extension/unramified outside some places

<https://math.stackexchange.com/questions/2836916/what-are-the-abelian-extensions-of-bbb-cx>