Eine Woche, ein Beispiel

2.23 Schubert calculus: coh of Grassmannian

Ref:
[3264] and [Fulton]

We will attempt to tackle Schubert calculus in a concise manner. The term "Schubert calculus" is often associated with intersection theory, enumerative geometry, combinatorics, Grassmannians, and more, making it a vast topic. However, I believe its core ideas can be clearly explained in just six hours. I will break the material into several parts:

1. $H^\cdot(Gr(n,r);\mathbb{Z})$ and its combinatorics
2. (inside Grassmannian)
   cycles in Grassmannian, including:
   - cycle class map: $CH^\cdot(Gr(n,r)) \xrightarrow{\sim} H^\cdot(Gr(n,r);\mathbb{Z})$
   - incidence variety $\begin{cases} \text{(partial) flag variety} \\ \text{Fano variety of planes} \\ \cdots \end{cases}$
   - a reintepretation of cycles

$$\begin{array}{ccc} \mathcal{L} & & \mathcal{S}^\vee \\ | & & | \\ X & \xrightarrow{f_{\mathcal{L}}} & Gr(\infty,r) \end{array}$$

3. (outside Grassmannian + v.b.)

Chern class: $\quad c: VB(X) \longrightarrow H^\cdot(X;\mathbb{Z})$

$$f_{\mathcal{L}}^*: H^\cdot(Gr(\infty,r);\mathbb{Z}) \longrightarrow H^\cdot(X;\mathbb{Z})$$

e.g.,
$$\begin{array}{ccc} VB(Gr(n,r)) & \longrightarrow & H^\cdot(Gr(n,r);\mathbb{Z}) \\ \mathcal{S}^* & \longmapsto & 1+\sigma_1+\cdots \\ Q & \longmapsto & 1+\sigma_1+\cdots \\ \mathcal{T}_{Gr} & \longmapsto & 1+n\cdot\sigma_1+\cdots \\ \mathcal{S} & \longmapsto & 1-\sigma_1+\sigma_{1,1}-\sigma_{1,1,1}+\cdots+(-1)^r\sigma_{(1)^r} \end{array}$$

4. Applications

tangent space argument

1. Group structure of $H^{\cdot}(Gr(n,r);\mathbb{Z})$
2. Cup product

1. Group structure of $H^{\cdot}(Gr(n,r);\mathbb{Z})$

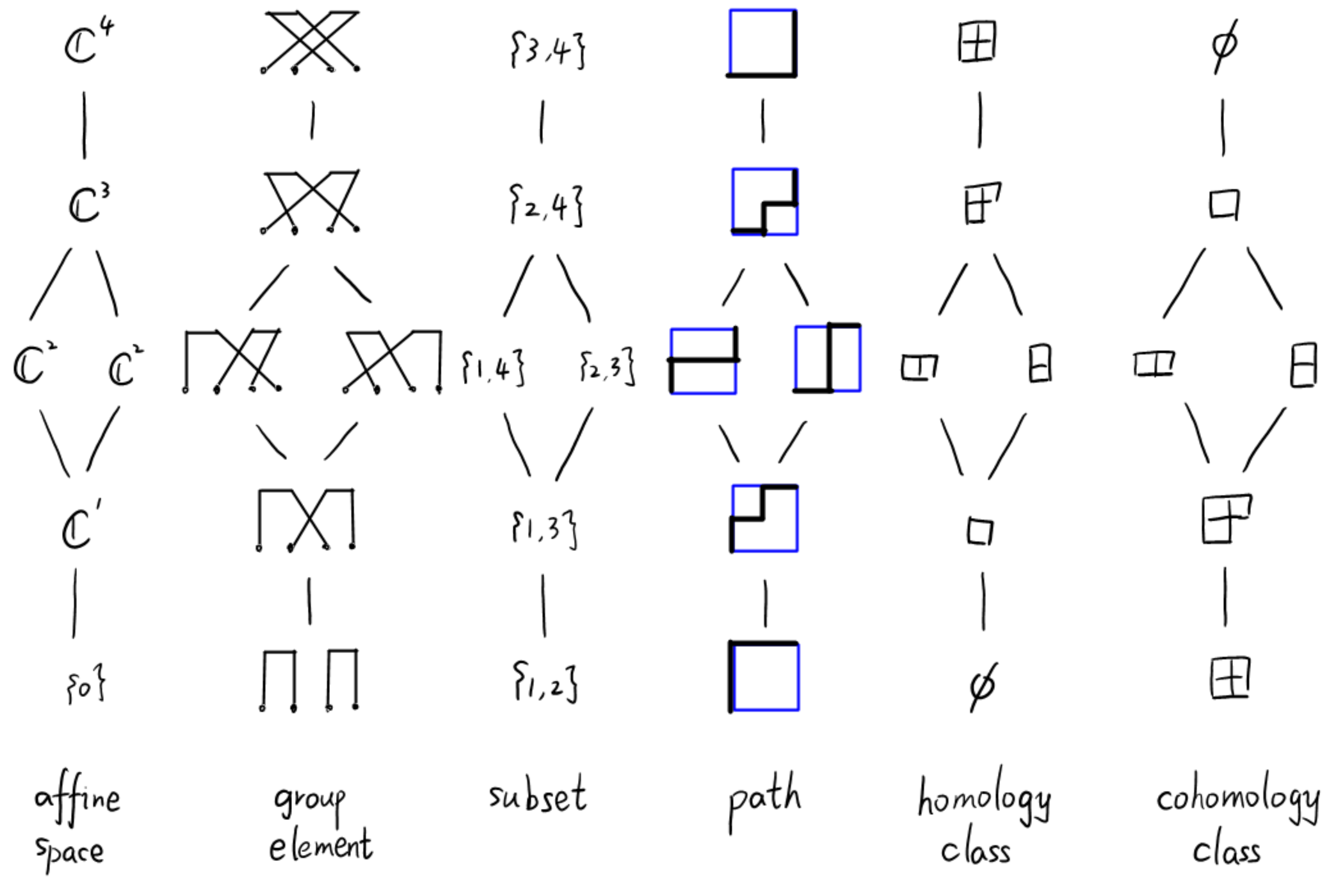
It's well-known that $Gr(n,r) \cong GL_n(\mathbb{C})/P$ has an affine paving w.r.t. $S_n/S_r \times S_{n-r}$:

$$Gr(n,r) = \coprod_{w \in S_n/S_r \times S_{n-r}} BwP/P \cong \coprod_{w \in S_n/S_r\times S_{n-r}} \mathbb{C}^{l(w)}$$

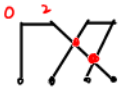
$\# S_n/S_r \times S_{n-r} = \binom{n}{r}$

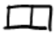
We read the diagram from top to bottom, the map from right to left.

E.g. $n=4$ $r=2$

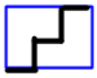


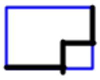
Hint from gp element to homology class:



$\rightsquigarrow$ $(2,0) =$ 

E.g. $n = 5$, $r = 2$

 $\sim \{2,4\} \sim$  $\sim$  $\sim$ 

 $\sim \{3,5\} \sim$  $\sim$  $\sim$ 

Ex. compute w_0-action (left mult) on $S_n / S_r \times S_{n-r}$, where $w_0 =$ .