Eine Woche, ein Beispiel

11.19. Basic sheaf calculation

Goal: Motivate $f_*$, $f^*$, $f_!$, $f^!$, by connecting them with (co)homology theory

After story: $\rightsquigarrow$ calculation of $\mathrm{Perv}_\Delta(\mathbb{CP}^1)$
$\rightsquigarrow$ generalize Morse theory
$\rightsquigarrow$ Characteristic classes / cycles
$\rightsquigarrow$ index theorem

Minor advantages from my talk:
- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    $\rightsquigarrow$ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale) descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \mathcal{T}op$
$\mathrm{Obj}(Sh(X)) = \{\text{sheaves of abelian gps}\}$
e.g. $Sh(\{*\}) = \mathcal{A}bel$ $\qquad \underline{\mathbb{Q}}_{\{*\}} \longleftrightarrow \underline{\mathbb{Q}}$

0. sheaf
1. $f_*$, skyscraper sheaf & global sections
2. $f^*$, constant sheaf & stalks
3. $Rf_*$ & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology
Poincaré duality.

# 0. Sheaf

Recall the definition of

- presheaf $\mathcal{F}$
- sheaf $\mathcal{F}$
- stalk $\mathcal{F}_x$
- global section $\mathcal{F}(X) = \Gamma(X;\mathcal{F}) = H^0(X;\mathcal{F})$
- cohomology $R^n\Gamma(X;\mathcal{F}) = H^n(X;\mathcal{F})$

https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology
If X is paracompact and Hausdorff, Cech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

Recall examples of sheaves:

complicated $\left\{ \begin{array}{l} \cdot\ \mathcal{C}_X: \text{ sheaf of cont fcts on } X \\ \cdot\ \mathcal{O}_X: \text{ structure sheaf on } X \end{array} \right.$ e.g., X: (cplx) mfld, scheme, ⋯

- $\underline{\mathbb{Q}}_X$: constant sheaf on $X$
- $sky_p(\mathbb{Q})$: skyscraper sheaf of $p \in X$ on $X$.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{C}_X)_x \quad \& \quad (sky_p(\mathbb{Q}))_x .$$