Eine Woche, ein Beispiel

10.13 grading rules

In this document, I want to remember the rules of grading.
For example, for the graded Jacobi identity,

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]]$$

How to recover the coefficient?

Step 1. write the ungraded version:

$$[a,[b,c]] = [[a,b],c] + \qquad [b,[a,c]]$$

Step 2 write the grading elements in order:

$$a\ b\ c \qquad a\ b\ c \qquad\qquad b\ a\ c$$

Step 3. Switch them to the same order, and get coefficient:

$$a\ b\ c \qquad a\ b\ c \qquad (-1)^{|a||b|}\, b\ a\ c$$

Ex. try the graded Jacobi identity:

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0,$$

graded version of categories

	ungraded		graded		dg = differential graded
(vague)	$\mathcal{C}$	$\rightsquigarrow$	$gr(\mathcal{C})$	$\rightsquigarrow$	$dg(\mathcal{C})$
	$Vect_k$	$\rightsquigarrow$	graded v.s.	$\rightsquigarrow$	$C^\cdot(Vect_k)$ = dg v.s. $d^2=0$
	k-Alg commutivity	$\rightsquigarrow$	graded alg graded commutivity	$\rightsquigarrow$	DGA_k graded Leibniz rule $d(ab) = da\cdot b + (-1)^{\lvert a\rvert} a\, db$
	Lie_k anti commutivity Jacobi	$\rightsquigarrow$	graded Lie alg graded anti commutivity graded Jacobi	$\rightsquigarrow$	$DGLA_k$ $d[a,b] = [da,b] + (-1)^{\lvert a\rvert}[a,db]$
	Mod_A	$\rightsquigarrow$	graded A-module A can be graded alg	$\rightsquigarrow$	dg A-module A can be DGA

E.g. Any $(A^\cdot, d) \in DGA$ can be naturally viewed as a DGLA, where

$$[-,-]: A^\cdot \times A^\cdot \longrightarrow A^\cdot$$
$$(a\ ,\ b) \longmapsto [a,b] := ab - (-1)^{\lvert b\rvert\lvert a\rvert} ba$$

E.g. For any $(V^\cdot, \partial) \in dg(Vect_k)$, $(End(V^\cdot), d) \in DGA$, where

$$df = [\partial, f] = \partial f - (-1)^{\lvert f\rvert} f \partial$$