# Eine Woche, ein Beispiel

Ref: from [23.11.19]

## 4.28 naive $\otimes$ - Hom adjunction

Notation. - $A$: associate ring   allowed to be non-commutative, contains 1

- There are two systems to write category of $A$-modules.

$$\begin{array}{rcll} \mathrm{Mod}_A & = & A\text{-}\mathrm{Mod} & \ni {}_AM \\ (\mathrm{Mod}_A)^{op} \neq \mathrm{Mod}_{A^{op}} & = & \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} & \ni M_A \\ \mathrm{Mod}_{A\otimes B^{op}} & = & A\text{-}\mathrm{Mod}\text{-}B & \ni {}_AM_B \end{array}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \quad \ni {}_BM_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $Ob(Ring) \cong Ob(Ring^{op})$, $A^{op} \notin Ob(Ring^{op})$, $A^{op}$ is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For $A$ comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset Sh(\mathrm{Spec}\, A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

1. definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$

⋮

6. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-Mod-}B \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-Mod-}B)^{op} \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$

$$\begin{array}{cccc} \mathrm{Hom}_B^A(-,-) : & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}A & \longrightarrow & \mathbb{Z}\text{-Mod} \\ \| & \| \quad\quad\quad \| & & \| \\ \mathrm{Hom}_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) & (\mathbb{Z}\text{-Mod-}B\otimes_{\mathbb{Z}}A^{op})^{op} \times (B\otimes_{\mathbb{Z}}A^{op}\text{-Mod-}\mathbb{Z})^{op} & \longrightarrow & \mathbb{Z}\text{-Mod-}\mathbb{Z} \end{array}$$

$${}_AX_B \ , \ {}_BY_C \ , \ {}_CZ_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{op}} X$ in $A\text{-Mod-}C = C^{op}\text{-Mod-}A^{op}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\mathrm{Hom}_A(A, X) \cong X$

2. adjunction $\quad {}_B X_A,\ {}_C Y_B,\ {}_C Z_D$. we get

$$\mathrm{Hom}_C(Y \otimes_B X,\ Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y,Z)) \qquad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$\{ f: Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c \}$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y,-)"] \end{tikzcd}$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv \mathrm{Hom}$:

$$
\begin{tikzcd}
(A\text{-Mod-}B)^{op} \times (B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "{(\mathrm{Id},\, \mathrm{Hom}_C)}"] \arrow[d, equal] & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}D \arrow[dd, "\mathrm{Hom}_B"] \\
(A\text{-Mod-}B \times B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[d, "{(\otimes_B,\, \mathrm{Id})}"'] & \\
(A\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "\mathrm{Hom}_C"] & A\text{-Mod-}D
\end{tikzcd}
$$

$f^* \dashv f_*$:

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
Y \arrow[r, "f"] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \arrow[r, "{(\mathrm{Id},\, \mathrm{pushforward})}"] \arrow[d, "{(\mathrm{pullback},\, \mathrm{Id})}"'] & Sh(X)^{op} \times Sh(X) \arrow[d, "{\mathrm{Hom}_{Sh(X)}(-,-)}"] \\
Sh(Y)^{op} \times Sh(Y) \arrow[r, "{\mathrm{Hom}_{Sh(Y)}(-,-)}"] & \mathrm{Abel}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathcal{F},\ f,\ \mathcal{G}) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{F},\ f_*\mathcal{G}) \arrow[d, mapsto] \\
(f^*\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] & \mathrm{Hom}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})
\end{tikzcd}
$$

$f_! \dashv f^!$: similar.

## 3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

### Forgetful fctor

**Prop.** For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \mathrm{Hom}_S({}_S S_R, M)$$

one has adjunction fctors

$$\begin{tikzcd}
S\text{-Mod} \arrow[rr, "{\mathrm{Hom}_S({}_S S_R,-)}", "{u = {}_R S_S \otimes_S -}"'] & & R\text{-Mod} \arrow[ll, bend right, "{}_S S_R \otimes_R -"'] \arrow[ll, bend left, "{\mathrm{Hom}_R({}_R S_S,-)}"]
\end{tikzcd} \qquad (3.1)$$

### Compare with $j$

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[rr, "j^!", "j^*"'] & & \mathcal{D}(\mathcal{U}) \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "Rj_*"]
\end{tikzcd}$$

Vague slogan: $u \approx$ "forget the information of $Z$".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & $X$.

**E.g.** For finite gps $H \le G$, one has Res - Ind adjunction:

$$\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$$
$$c\text{-}\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \le G \text{ open} \end{cases}$

If one only has $H \le G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{tikzcd} \mathrm{Rep}_G \arrow[rr, "\mathrm{Res}_H^G" description] & & \mathrm{Rep}_H \arrow[ll, bend right=50, "c\text{-}\mathrm{Ind}_H^G"'] \arrow[ll, bend left=50, "\mathrm{Ind}_H^G"] \end{tikzcd}$$

**Ex.** Compare it with the recollement diagram & (3.1).

$$\begin{tikzcd} \mathcal{U} \arrow[d, "j"] & {[*/H]} \arrow[d, "\text{“cover with fiber } G/H\text{”}"] \\ X & {[*/G]} \end{tikzcd}$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$

   $c\text{-}\mathrm{Ind}_H^G \longrightarrow \mathrm{Ind}_H^G$ $\qquad$ $c\text{-}\mathrm{Ind}_H^G = \mathrm{Ind}_H^G$

2. Even though

$$Sh_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \mathrm{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$

the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\mathrm{Res}_{\{*\}}^G \mathbb{Q} = \mathbb{Q}, \qquad \mathrm{Res}_{\{*\}}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* R j_* \neq \mathrm{Id}$, $j^! j_! \neq \mathrm{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

## Compare with $i$

Now, assume $S, R$ commutative in the scheme setting.

E.g. For ring homo

$$\begin{tikzcd} S \\ R \arrow[u, "\tilde{f}"] \arrow[r, dash, "M"] & {} \end{tikzcd} \qquad \begin{tikzcd} \operatorname{Spec} S \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, dash, "M"] & {} \end{tikzcd}$$

$\exists$ "pullback fctor"

$$f^* : R\text{-Mod} \longrightarrow S\text{-Mod} \qquad f^*M = {}_S S_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{tikzcd} S\text{-Mod} \arrow[rr, "u"] & & R\text{-Mod} \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "\operatorname{Hom}_R({}_R S_S, -)"] \end{tikzcd}$$

compare it with another part of the recollement diagram:

$$\begin{tikzcd} \mathcal{D}(X) \arrow[rr, "i_*"', "i_!"] & & \mathcal{D}(U) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \end{tikzcd}$$

Rmk. $u$ is usually not f.faithful, unless $S = R/I$.

(In fact, only need $S$ is $R$-idempotent, i.e. $S \cong S \otimes_R S$.)

which crspds to closed embedding.

In that case,

$$i^* i_* = \mathrm{Id} : {}_S S_R \otimes_R ({}_R S_S \otimes_S M) \cong M$$

$$i^! i_* = \mathrm{Id} : \operatorname{Hom}_R({}_R S_S, \operatorname{Hom}({}_S S_R, M)) \cong M$$

**Slogan:** in the comm alg. $\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ is closed embedding.
In general, if
$S$ is an $R$-idempotent algebra: $S \cong S \otimes_R S$
then $i: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ can be viewed as "closed subset".

**E.g.** $R_{\mathfrak{p}}$, $R/I$ are idempotent $R$-algs.
$\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.

⚠ Usually $R/I$ is not an derived idem $R$-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

**Rmk.** Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q: What is the crspd "open subset"?
A: (possibly) the Verdier quotient.
We will come back to this after we derive everything.