

ein Woche, eine Beispiel
 April 16th. examples in algebraic topology

Examples:

Past

closed surface $\dim 2$
 Hopf surface $\dim 4$
 K3 surface

Today

S^n S^∞
 $\mathbb{R}P^n$ $\mathbb{R}P^\infty$
 $\mathbb{C}P^n$ $\mathbb{C}P^\infty$
 ...

Future

Lens space
 Lie group
 Grassmannian mfld, e.g. $G_r(2,4)$
 Moore space
 Eilenberg-MacLane space
 low-dimensional CW-cplx
 ...

Goal.

- compute $H_n(X, \mathbb{Z})$, $H^*(X, \mathbb{Z})$, $\pi_n(X, \mathbb{Z})$ ← Whitehead bracket
- compute characteristic class and applies the results.
- optional question: is X * oriented?

- * a mfld? of $\dim n$
- * a cplx mfld?
- * a Lie group?
- *

proj
 |
 Kähler
 |
 complex

Today: S^n , S^∞ ; $\mathbb{R}P^n$, $\mathbb{R}P^\infty$; $\mathbb{C}P^n$, $\mathbb{C}P^\infty$; ...

$$S^\infty = \bigcup_{n \geq 1} S^n \quad S^n \hookrightarrow S^m \text{ by } (x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0, \dots, 0) \quad \uparrow_m$$

1. relations: fiber bundle

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow S^1 \\ \downarrow \\ \mathbb{R}P^1$$

$$S^1 \longrightarrow S^{2n+1} \\ \downarrow \\ \mathbb{C}P^n \\ [n=1: \text{Hopf fibration}]$$

$$\mathbb{Z}/k\mathbb{Z} \longrightarrow S^{2n+1} \\ \downarrow \\ S^{2n+1}/\mathbb{Z}/k\mathbb{Z} \quad k \in \mathbb{N}^+, k > 1$$

$$\mathbb{Z}/2\mathbb{Z} \longrightarrow S^\infty \\ \downarrow \\ \mathbb{R}P^\infty$$

$$S^1 \longrightarrow S^\infty \\ \downarrow \\ \mathbb{C}P^\infty$$

$$\mathbb{Z}/k\mathbb{Z} \longrightarrow S^\infty \\ \downarrow \\ S^\infty/\mathbb{Z}/k\mathbb{Z}$$

2. (canonical) CW structure.

e.g

# m -cell	0	1	2	3	4	5	$m > 5$
S^5	2	2	2	2	2	2	0
$\mathbb{R}P^5$	1	1	1	1	1	1	0
$\mathbb{C}P^2$	1	0	1	0	1	0	0

$$\Rightarrow \begin{cases} \chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases} \\ \chi(\mathbb{R}P^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \\ \chi(\mathbb{C}P^n) = n+1 \end{cases}$$

3. Homology & Cohomology

homology

$H_i(X, \mathbb{Z})$	0	1	2	3	4	5	$i > 5$
S^5	\mathbb{Z}	0	0	0	0	\mathbb{Z}	0
$\mathbb{R}P^5$	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	\mathbb{Z}	0
$\mathbb{C}P^2$	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	0
$\mathbb{R}P^4$	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0

Cor. $\mathbb{R}P^{2n}$ is nonoriented; $\mathbb{R}P^{2n+1}$, S^n , $\mathbb{C}P^n$ are oriented.

$$S^5: 0 \rightarrow \mathbb{Z}e_1^5 \oplus \mathbb{Z}e_2^5 \rightarrow \mathbb{Z}e_1^4 \oplus \mathbb{Z}e_2^4 \rightarrow \mathbb{Z}e_1^3 \oplus \mathbb{Z}e_2^3 \rightarrow \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2^2 \rightarrow \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \rightarrow \mathbb{Z}e_1^0 \oplus \mathbb{Z}e_2^0 \rightarrow 0$$



$$\begin{array}{lll} e_1^5 \mapsto e_1^4 - e_2^4 & e_1^3 \mapsto e_1^2 - e_2^2 & e_1^1 \mapsto e_1^0 - e_2^0 \\ e_2^5 \mapsto -e_1^4 + e_2^4 & e_2^3 \mapsto -e_1^2 + e_2^2 & e_2^1 \mapsto -e_1^0 + e_2^0 \\ e_1^4 \mapsto e_1^3 + e_2^3 & e_1^2 \mapsto e_1^1 + e_2^1 & \\ e_2^4 \mapsto e_1^3 + e_2^3 & e_2^2 \mapsto e_1^1 + e_2^1 & \end{array}$$

[Rmk. The definition of cellular homology uses the (singular) homology of S^1 , so seriously] we can't compute $H_i(S^n, \mathbb{Z})$ by cellular homology.

$$\begin{array}{ccccccccccc} \mathbb{R}P^5: & 0 & \longrightarrow & \mathbb{Z}e^5 & \longrightarrow & \mathbb{Z}e^4 & \longrightarrow & \mathbb{Z}e^3 & \longrightarrow & \mathbb{Z}e^2 & \longrightarrow & \mathbb{Z}e^1 & \longrightarrow & \mathbb{Z}e^0 & \longrightarrow & 0 \\ & & & e^5 \longmapsto 0 & & & & e^3 \longmapsto 0 & & & & e^1 \longmapsto 0 & & & & \\ & & & & & e^4 \longmapsto 2e^3 & & & & e^2 \longmapsto 2e^1 & & & & & & \end{array}$$

$$\begin{array}{ccccccccccc} \mathbb{R}P^4: & 0 & \longrightarrow & \mathbb{Z}e^4 & \longrightarrow & \mathbb{Z}e^3 & \longrightarrow & \mathbb{Z}e^2 & \longrightarrow & \mathbb{Z}e^1 & \longrightarrow & \mathbb{Z}e^0 & \longrightarrow & 0 \\ & & & & & e^3 \longmapsto 0 & & & & e^1 \longmapsto 0 & & & & \\ & & & & & e^4 \longmapsto 2e^3 & & & & e^2 \longmapsto 2e^1 & & & & \end{array}$$

$$\begin{array}{ccccccccccc} \mathbb{C}P^2: & 0 & \longrightarrow & \mathbb{Z}e^4 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}e^2 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}e^0 & \longrightarrow & 0 \\ & & & \mathbb{Z} & & 0 & & \mathbb{Z} & & 0 & & \mathbb{Z} & & \\ & & & & & & & & & & & & & \end{array}$$

Similarly, $H_n(S^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$

$$H_n(\mathbb{R}P^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \quad H_n(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$H_n(\mathbb{C}P^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

cohomology

$H^i(X, \mathbb{Z})$	0	1	2	3	4	5	$i > 5$
S^5	\mathbb{Z}	0	0	0	0	\mathbb{Z}	0
$\mathbb{R}P^5$	\mathbb{Z}	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}	0
$\mathbb{C}P^2$	\mathbb{Z}	0	\mathbb{Z}	0	\mathbb{Z}	0	0
$\mathbb{R}P^4$	\mathbb{Z}	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0

$$\Rightarrow \begin{cases} H^*(\mathbb{R}P^{2n}) = \mathbb{Z}[x]/(2x, x^{n+1}) \\ H^*(\mathbb{R}P^{2n+1}) = \mathbb{Z}[x]/(2x, x^{n+1}) \oplus \mathbb{Z}y \\ H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1}) \end{cases} \quad \begin{array}{l} \deg x = 2 \\ \deg y = 5 \\ \deg t = 1 \end{array} \Rightarrow \begin{cases} H^*(\mathbb{R}P^\infty) = \mathbb{Z}[x]/(2x) \\ H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x] \\ H^*(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[t] \\ H^*(\mathbb{C}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x] \end{cases}$$

prod structure: use Poincaré duality & cellular cohomology, see [May, P153].

$$H^q(\mathbb{C}P^n) \xrightarrow{\sim} H^{q+1}(\mathbb{C}P^{n-1}) \text{ for } q < n$$

<https://math.stackexchange.com/questions/1128712/integral-cohomology-ring-of-real-projective-space>

By spectral sequence: GTM 8.2 Example 14.22, 14.32, Ex 18.4, 18.10

Interlude: LES of homotopy groups

$x_0 \in BCACX$, X top space

Def. relative homotopy group

$$\pi_n(X, A, x_0) = [f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)] \quad n \geq 1 \quad J^{n-1} := \partial I^n - I^{n-1}$$

Relations: $f \sim g \Leftrightarrow \exists F_t: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$ s.t.
(denote by $[f] = [g]$) $F_0 = f \quad F_1 = g$

Lemma: Suppose $f: (I^n, \partial I^n, J^{n-1}) \rightarrow (X, A, x_0)$,
 $f(I^n) \subseteq A$, then $[f] = [0]$

Thm. we have $LES \leftarrow \ker = \text{Im}$

$$\hookrightarrow \pi_2(A, B, x_0) \rightarrow \pi_2(X, B, x_0) \rightarrow \pi_2(X, A, x_0)$$

$$\hookrightarrow \pi_1(A, B, x_0) \rightarrow \pi_1(X, B, x_0) \rightarrow \pi_1(X, A, x_0)$$

Cor. we have LES

$$\pi_2(A, x_0) \rightarrow \pi_2(X, x_0) \rightarrow \pi_2(X, A, x_0)$$

$$\hookrightarrow \pi_1(A, x_0) \rightarrow \pi_1(X, x_0) \rightarrow \pi_1(X, A, x_0)$$

$$\hookrightarrow \pi_0(A, x_0) \rightarrow \pi_0(X, x_0)$$

\rightarrow called Serre fibration

Thm. when $p: E \rightarrow B$ has the homotopy lifting property w.r.t. I^k ($\forall k \geq 0$)
then (denote $b_0 \in B, x_0 \in F := p^{-1}(b_0)$)

$$p_*: \pi_n(E, F, x_0) \rightarrow \pi_n(B, b_0) \quad n \geq 1$$

is an isomorphism.

$$\begin{array}{ccc} I^k \times [0, 1] & \xrightarrow{\quad} & E \ni x_0 \\ \downarrow & \searrow \text{red arrow} & \downarrow p \\ I^k & \xrightarrow{\quad} & B \ni b_0 \end{array}$$

Rmk. 1 The proof mainly uses the HLP: homotopy lifting property.

2. Any fiber bundle $p: E \rightarrow B$ is a Serre fibration.

Cor Suppose $p: E \rightarrow B$ is the fiber bundle map, then we have LES
(denote $b_0 \in B, x_0 \in F := p^{-1}(b_0)$)

$$\pi_2(F, x_0) \rightarrow \pi_2(E, x_0) \rightarrow \pi_2(B, b_0)$$

$$\hookrightarrow \pi_1(F, x_0) \rightarrow \pi_1(E, x_0) \rightarrow \pi_1(B, b_0)$$

$$\hookrightarrow \pi_0(F, x_0) \rightarrow \pi_0(E, x_0)$$

4. Homotopy: by LES of fibration, we obtain $n \geq 2$

$$\pi_m(\mathbb{R}P^\infty) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & m=1 \\ \pi_m(S^n) & m>1 \end{cases}$$

$$\pi_m(\mathbb{C}P^\infty) = \begin{cases} 0 & m=1 \\ \mathbb{Z} & m=2 \\ \pi_m(S^{2n+1}) & m>2 \end{cases}$$

$$S^1 \rightarrow S^{2n+1} \rightarrow \mathbb{C}P^n$$

Rmk. S^∞ is contractible by the argument in

<https://mathoverflow.net/questions/198>

Cor. $\mathbb{R}P^\infty$ is of type $K(\mathbb{Z}/2\mathbb{Z}, 1)$

$\mathbb{C}P^\infty$ is of type $K(\mathbb{Z}, 2)$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}	π_{15}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	\mathbb{Z}	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^3	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{12}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_3	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^2	$\mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^2$	\mathbb{Z}_2^2
S^4	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{12}$	\mathbb{Z}_2^2	\mathbb{Z}_2^2	$\mathbb{Z}_{24} \times \mathbb{Z}_3$	\mathbb{Z}_{15}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{120} \times \mathbb{Z}_{12} \times \mathbb{Z}_2$	$\mathbb{Z}_{84} \times \mathbb{Z}_2^5$
S^5	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{14}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{30}	\mathbb{Z}_2	\mathbb{Z}_2^3	$\mathbb{Z}_{72} \times \mathbb{Z}_2$
S^6	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_{60}	$\mathbb{Z}_{24} \times \mathbb{Z}_2$	\mathbb{Z}_2^3
S^7	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	\mathbb{Z}_{120}	\mathbb{Z}_2^3
S^8	0	0	0	0	0	0	0	\mathbb{Z}	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}_{24}	0	0	\mathbb{Z}_2	$\mathbb{Z} \times \mathbb{Z}_{120}$

in GTM 82 (naive)
What I can prove now

$\pi_{15}(S^{15})$

split by the suspension homomorphism

<https://math.stackexchange.com/questions/3969577/how-torsion-arise-in-homotopy-groups-of-spheres>

5. Characteristic class.

We have both tautological vector bundle and tangent bundle for $S^n, \mathbb{R}P^n, \mathbb{C}P^n$.

CIP: by https://en.wikipedia.org/wiki/Chern_class

$$c(\mathbb{CP}^n) \stackrel{\text{def}}{=} c(T\mathbb{CP}^n) = c(\mathcal{O}_{\mathbb{CP}^n}(1))^{n+1} = (1+a)^{n+1},$$

where a is the canonical generator of the cohomology group $H^2(\mathbb{CP}^n, \mathbb{Z})$;

tautological bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$: $c(\mathcal{O}_{\mathbb{CP}^n}(-1)) = 1 - a$

Cor. $T\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(-1)$ are not spin; \mathbb{CP}^n is not a boundary.

IRIPⁿ: similarly, $w(\gamma_n') = 1+t$ $w(\text{IRIP}^n) = w(\gamma_n')^{n+1} = (1+t)^{n+1}$

Cor. γ_n' is not orientable;

$TIRIP^n$ is orientable only when $n \equiv 1 \pmod{2}$;

TIRIP^n is spin only when $n \equiv 3 \pmod{4}$ or $n = 1$.

S^n : Lemma. $\pi^*: H^n(\mathbb{R}P^n, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{j} H^n(S^n, \mathbb{Z}/2\mathbb{Z})$ is zero.

Proof by computation.

$C(\mathbb{R}P^5, \mathbb{Z}/2\mathbb{Z})$ $0 \longrightarrow e^5 \longrightarrow e^4 \longrightarrow e^3 \longrightarrow e^2 \longrightarrow e^1 \longrightarrow e^0 \longrightarrow 0$
 $\quad \quad \quad \uparrow \begin{smallmatrix} e^5 \mapsto e^5 \\ e_2^5 \mapsto e^5 \end{smallmatrix} \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow \quad \quad \quad \uparrow$
 $C(S^5, \mathbb{Z}/2\mathbb{Z})$ $0 \longrightarrow e^5, e_2^5 \longrightarrow e_1^4, e_2^4 \longrightarrow e_1^3, e_2^3 \longrightarrow e_1^2, e_2^2 \longrightarrow e_1^1, e_2^1 \longrightarrow e_1^0, e_2^0 \longrightarrow 0$
 $\quad \quad \quad \begin{smallmatrix} e_1^5 \mapsto e_1^4 - e_2^4 \\ e_2^5 \mapsto -e_1^4 + e_2^4 \end{smallmatrix}$

$C^*(\mathbb{R}P^5, \mathbb{Z}/2\mathbb{Z})$ $0 \longleftarrow e^{5*} \longleftarrow e^{4*} \longleftarrow e^{3*} \longleftarrow e^{2*} \longleftarrow e^{1*} \longleftarrow e^{0*} \longleftarrow 0$
 $\quad \quad \quad \downarrow \begin{smallmatrix} e^{5*} \mapsto e^{5*} - e_2^{5*} \\ e_1^{5*} \mapsto e^{5*} \end{smallmatrix} \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$
 $C^*(S^5, \mathbb{Z}/2\mathbb{Z})$ $0 \longleftarrow e^{5*}, e_2^{5*} \longleftarrow e_1^{4*}, e_2^{4*} \longleftarrow e_1^{3*}, e_2^{3*} \longleftarrow e_1^{2*}, e_2^{2*} \longleftarrow e_1^{1*}, e_2^{1*} \longleftarrow e_1^{0*}, e_2^{0*} \longleftarrow 0$
 $\quad \quad \quad \begin{smallmatrix} e_1^{5*} - e_2^{5*} \longleftarrow e_1^{4*} \\ -e_1^{5*} + e_2^{5*} \longleftarrow e_2^{4*} \end{smallmatrix}$

btw. when n is odd, $H^n(\mathbb{R}P^n, \mathbb{Z}) \longrightarrow H^n(S^n, \mathbb{Z})$
 $\quad \quad \quad \parallel \quad \quad \quad \parallel$
 $\quad \quad \quad \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}$

Cor. $\omega(\gamma'_{n, S^n}) = \pi^* \omega(\gamma'_n, \mathbb{R}P^n) = 1$

$$\omega(TS^n) = \pi^* \omega(T\mathbb{R}P^n) = 1$$

γ_n, S^n, TS^n are spin, $S^n = \partial D^n$.

6. cplx mfld

$\mathbb{C}P^n$ is undoubtedly proj cplx mfld.

$\mathbb{R}P^{2n-1}, S^{2n-1}$ are not cplx mflds since they're of odd dim.

$\mathbb{R}P^{2n}$ is not cplx mfld since it's not orientable.

$S^n (n > 6), S^4$ are not cplx mflds, see <https://mathoverflow.net/questions/11664/complex-structure-on-s-n>

whether S^6 is a cplx mfld is still an open problem, see

<https://mathoverflow.net/questions/1973/is-there-a-complex-structure-on-the-6-sphere>

related problems: is the cplx structure of $\mathbb{C}P^n$ unique? Still open, see

<https://mathoverflow.net/questions/382442/is-the-complex-structure-of-mathbb-cp-n-unique>

7. Lie group: $S^1, S^3, \mathbb{R}P^1, \mathbb{R}P^3$, we have $\mathbb{R}P^1 \cong S^1$ and

$$S^3 \cong SU_2 \cong \{g \in \mathbb{H} \mid |g| = 1\}$$

$$\begin{array}{ccc} \downarrow \pi & \searrow \pi' & \\ \mathbb{R}P^3 \cong SO_3 & & \end{array}$$

<https://math.stackexchange.com/questions/4065801/collecting-proofs-so3-cong-bbb-r-p3>
But a better way to see it is here: https://www.youtube.com/watch?v=ACZC_XEyg9U

for S^n : <https://math.stackexchange.com/questions/12453/is-there-an-easy-way-to-show-which-spheres-can-be-lie-groups>

for $\mathbb{R}P^n$: lemma: a Lie/topological group structure lifts to a covering space

proof: see <https://math.stackexchange.com/questions/5391/covering-of-a-topological-group-is-a-topological-group>

Cor: $\mathbb{R}P^n (n > 3)$ is not a Lie group

for $\mathbb{C}P^n$: lemma: for the connected Lie group G , $\pi_2(G) = 0$ $\pi_3(G)$ has no torsion!

proof: see <https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-group>

Cor: $\mathbb{C}P^n$ is not a Lie group.

different proof of this cor: <https://math.stackexchange.com/questions/3043483/lie-group-structure-on-the-complex-projective-space>

Interesting results during the ways of searching

Lemma: a cpt Lie group is either abelian \Rightarrow torus
or nonabelian & have nonzero H^3 .

See <https://math.stackexchange.com/questions/3421788/topological-lie-group-structure-on-projective-spaces>

Lemma: every compact Lie group has zero Euler characteristic since it is parallelizable

See <https://math.stackexchange.com/questions/829928/can-s2-be-turned-into-a-topological-group/>

8. suspensions \sim : homotopy equivalence

$$S^n \sim \Sigma S^{n-1}$$

$$\mathbb{R}P^1 \cong S^1 \sim \Sigma S^0$$

$$\mathbb{C}P^1 \cong S^2 \sim \Sigma S^1$$

For $n \geq 2$, any $X \in \text{Top}$,

$$\mathbb{R}P^n, \mathbb{C}P^n \not\sim \Sigma X$$

<https://math.stackexchange.com/questions/4445986/proof-that-mathbb{R}P^2-is-not-the-suspension-of-a-space-x>

Q: How much do we know about the suspension of $\mathbb{R}P^n, \mathbb{C}P^n$?

A:

<https://math.stackexchange.com/questions/2583182/topology-of-suspension-of-real-projective-space>

9. boundary

Pontryagin & Thom proved that a closed smooth manifold is null-cobordant if and only if all of its Stiefel-Whitney numbers are zero.
<https://math.stackexchange.com/questions/1385708/examples-of-manifolds-that-are-not-boundaries>