Eine Woche, ein Beispiel

7.23 trace theorem and Sobolev embedding

This is a continuation of [23.05.28].
In the statement of propositions, all fcts are real-valued fcts.

Prop. For $0 \le k \le n$, $s > \frac{k}{2}$, one can construct cont linear fcts

$$
\begin{array}{ccc}
H^s(\mathbb{R}^n) & \longrightarrow & H^{s-\frac{k}{2}}(\mathbb{R}^{n-k}) \\
\cup & & \cup \\
\mathcal{S}(\mathbb{R}^n) & \longrightarrow & \mathcal{S}(\mathbb{R}^{n-k}) \\
f & \longmapsto & f|_{\{0\}\times\mathbb{R}^{n-k}}
\end{array}
$$

Proof. Denote $V := \mathbb{R}^k$, $W := \mathbb{R}^{n-k}$, then $V \times W = \mathbb{R}^n$, $W \hookrightarrow V \times W$, reduce to show:

$\{0\} \times \mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$

$$\exists\, C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \qquad \|f|_W\|_{H^{s-\frac{k}{2}}} \le \|f\|_{H^s}$$

Step 1. Express $\widehat{f|_W}(\xi_2)$ in terms of $\hat{f}(\xi)$, by using Fourier transform twice.

$$
\left.
\begin{aligned}
f(0, x_2) &= \int_W e^{i\langle x_2, \xi_2\rangle} \widehat{f|_W}(\xi_2)\, d\xi_2 \\
f(0, x_2) &= \int_{V\times W} e^{i\langle x, \xi\rangle} \hat{f}(\xi)\, d\xi \\
&= \int_W e^{i\langle x_2, \xi_2\rangle} \left(\int_V \hat{f}(\xi)\, d\xi_1\right) d\xi_2
\end{aligned}
\right\} \Rightarrow \widehat{f|_W}(\xi_2) = \int_V \hat{f}(\xi)\, d\xi_1
$$

Step 2. Expand.

$$
\begin{aligned}
&\|f|_W\|^2_{H^{s-\frac{k}{2}}} = \|\widehat{f|_W}\|^2_{L^2(W,\, (|\xi_2|^2+1)^{s-\frac{k}{2}} d\xi_2)} \\
&= \int_W \left(\widehat{f|_W}(\xi_2)\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}}\, d\xi_2 \\
&= \int_W \left(\int_V \hat{f}(\xi)\, d\xi\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}}\, d\xi_2
\end{aligned}
$$

$$
\begin{aligned}
&\|f\|^2_{H^s} = \|\hat{f}\|^2_{L^2(V\times W,\, (|\xi|^2+1)^s d\xi)} \\
&= \int_{V\times W} \left(\hat{f}(\xi)\right)^2 (|\xi|^2+1)^s\, d\xi \\
&= \int_W \left(\int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s\, d\xi_1\right) d\xi_2
\end{aligned}
$$

Therefore, the problem reduce to $d\xi_1 \approx d\xi_1$

$$\left(\int_V \hat{f}(\xi)\, d\xi_1\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}} \le C \int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s\, d\xi_1.$$

Step 3. Use Hölder inequality to simplify. Since

$$\left(\int_V \hat{f}(\xi)\, d\xi_1\right)^2 \leqslant \int_V \hat{f}(\xi)^2 (|\xi|^2+1)^s\, d\xi_1 \cdot \int_V (|\xi|^2+1)^{-s}\, d\xi_1 ,$$

the problem reduces to

$$\int_V (|\xi|^2+1)^{-s} d\xi_1 \left(|\xi_2|^2+1\right)^{s-\frac{k}{2}} \le C.$$

Step 4. Compute $\int_V (|\xi|^2+1)^{-s} d\xi_1$ directly.

$$\begin{aligned}
&\int_V (|\xi|^2+1)^{-s} d\xi_1 \\
&= \int_V \frac{1}{(|\xi_1|^2+|\xi_2|^2+1)^s}\, d\xi_1 \\
&\overset{a^2=|\xi_2|^2+1}{\underset{a>0}{=\!=}} \int_V \frac{1}{(|\xi_1|^2+a^2)^s}\, d\xi_1 \\
&= \int_V \frac{1}{(|\frac{\xi_1}{a}|^2+1)^s}\, d\xi_1 \cdot a^{k-2s} \\
&= C a^{k-2s} = C\left(|\xi_2|^2+1\right)^{\frac{k}{2}-s}
\end{aligned}$$

where $C = \int_V (|x|^2+1)^{-s} dx < +\infty$

$$\Rightarrow \quad \int_V (|\xi|^2+1)^{-s} d\xi_1 (|\xi_2|^2+1)^{s-\frac{k}{2}} \le C \qquad \square$$

Rmk. The original C in the proposition can be taken by

$$C = (2\pi)^k \int_{\mathbb{R}} \frac{1}{(|x|^2+1)^s}\, dx .$$