# Eine Woche, ein Beispiel

## 11.7. Berkovich space

Ref: Spectral theory and analytic geometry over non-Archimedean fields by Vladimir G. Berkovich(we mainly follow this article)
+courses from Junyi Xie
+An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves

Goal: understand the beautiful picture copied from "An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves"

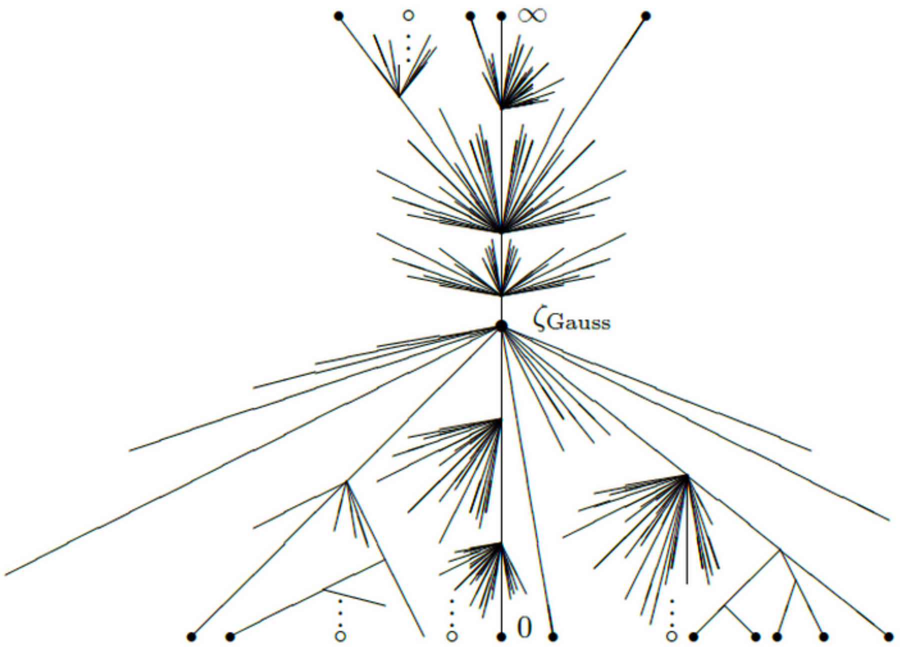



FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

A: comm with 1 (for convenience)

| extra condition | local | | global | closed unit disc | open unit disc |
|---|---|---|---|---|---|
| — | Spec A | affine scheme | scheme | | $\operatorname{Spec} \mathbb{Z}[[T]]$ |
| A: adic ring with f.g. ideal of def | Spf A | affine formal scheme | formal scheme | | $\operatorname{Spf} \mathbb{Z}_p[[T]]$ |
| A: k-affinoid alg, i.e. $A = k\langle T_1, \dots T_n\rangle/I$ | MaxSpec A | k-affinoid space | rigid-analytic space over k | $\operatorname{Max Spec} \mathbb{Q}_p\langle T\rangle$ | $\mathcal{U} = \{ \lvert\cdot\rvert \in \operatorname{MaxSpec} \mathbb{Q}_p\langle T\rangle \mid \lvert T\rvert < 1\}$ |
| $(A, A^+)$: Huber pair | $\operatorname{Spa}(A, A^+)$ | affinoid adic space | adic space | $\operatorname{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ | |
| A: Banach ring | $\mathcal{M}(A)$ | spectrum | Berkovich space | | |
| | | | | | |

Ref of table: Berkeley notes

Rmk. MaxSpec A has only a Grothendieck topology.

k (in k-affinoid space) is a NA field, but can also be generalized to k-Banach alg.

non-archimedean

k: NA field

# 1. Seminorm

**1.1. Def (seminorm of abelian group)** $\|\cdot\|: M \longrightarrow \mathbb{R}_{\geq 0}$ s.t

$\|0\| = 0$ — norm: $\|m\| = 0 \Rightarrow m = 0$

$\|f-g\| \leq \|f\| + \|g\|$ — non-Archimedean: $\|f-g\| \leq \max(\|f\|, \|g\|)$

- seminorm $\Rightarrow$ topology

  Prop. $(M, \|\cdot\|)$ is Hausdorff $\Leftrightarrow$ $\|\cdot\|$ is norm

  Def (equivalence of norm)

- sub, quotient, homomorphism

  Def (restricted seminorm)

  Def. (residue seminorm) $\pi: (M, \|\cdot\|_M) \longrightarrow M/N$ induce the seminorm on $M/N$:

  $$\|\bar{m}\|_{M/N} := \inf_{\pi(m') = \bar{m}} \|m'\|_M$$

  Def (bounded / admissible) $\varphi: (M, \|\cdot\|_M) \longrightarrow (N, \|\cdot\|_N)$

  - bounded: $\exists C > 0$, $\|\varphi(m)\|_N \leq C\|m\|_M$
  - admissible: $\bar{\varphi}: (M/\ker\varphi, \|\cdot\|_{quo}) \longrightarrow (\mathrm{Im}\,\varphi, \|\cdot\|_{res})$ induces equivalence of norm.

**1.2. Def (seminorm of ring** non-comm, with 1**)**: seminorm group +

$\|1\| = 1$ — norm — $\xrightarrow{+\text{completed}}$ Banach ring

$\|fg\| \leq \|f\|\,\|g\|$

power-multi: $\|f^n\| = \|f\|^n$

multiplicative: $\|fg\| = \|f\|\,\|g\|$ $\Rightarrow$ valuation

- quotient, $\prod_{\text{infinite}}$, $\mathcal{A}\langle r^{-1}T\rangle$, ...
- comparison among norms: bounded.
- Def related to valuation field.

**1.3. Def (seminorm of $\mathcal{A}$-module, where $\mathcal{A}$: normed ring)**

seminorm group + $\exists C > 0$, $\|fm\| \leq C\|f\|\,\|m\|$

- $\hat{\otimes}_{\mathcal{A}}$

seminormed ring

$\uparrow$ — $(\mathbb{Z}, |\ |_p)$, $(\mathbb{Q}, |\ |_\infty)$, $(\mathbb{Q}, |\ |_p)$

Banach ring

$\uparrow$ — $(R, \text{triv})$ R: not field, $(\mathbb{Z}, |\ |_\infty)$, $(\mathbb{C}\langle r^{-1}T\rangle, \|\ \|_{ind})$

valuation field

| Arch field | non-Arch field |
|---|---|
| $(\mathbb{R}, |\ |_\infty)$ $(\mathbb{R}/\mathbb{C}, |\ |_\infty^\varepsilon)$ $0 < \varepsilon \leq 1$ | $(\mathbb{F}_q, \text{triv})$ $(K, \text{triv})$ |
| $(\mathbb{C}, |\ |_\infty)$ | $(\mathbb{Q}_p, |\cdot|_p)$ |

## 2. Affine case

suppose $\mathcal{A}$: Banach ring  comm + 1

$\mathcal{M}(\mathcal{A}) := \{\text{bounded mult seminorms on } \mathcal{A}\}$

with top basis generated by $U_{m,(a,b)} := \{\|\cdot\| \in \mathcal{M}(\mathcal{A}) \mid \|m\| \in (a,b)\}$

$m \in \mathcal{A},\ (a,b) \subseteq \mathbb{R}$

$\mathcal{M}(\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)) := \{\text{mult seminorms on } \mathcal{A}\}$

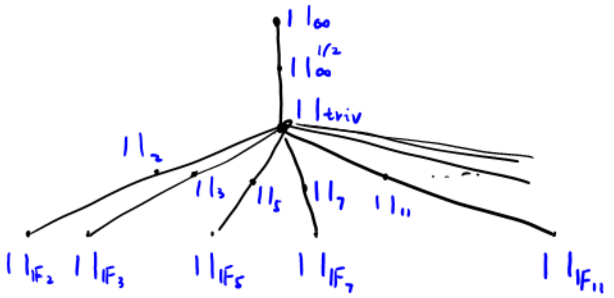
E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

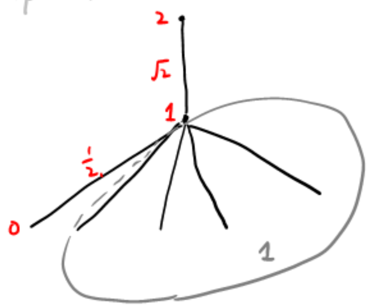
We have

$$\mathcal{M}(\mathbb{Z}, |\ |_\infty) = \left\{ \begin{array}{ll} |\ |_{triv} := \text{trivial norm} & \\ |\ |_p^t \ : \ t \in (0, +\infty] & |m|_{\mathbb{F}_p} := |m|_p^\infty = \begin{cases} 0 & p \mid m \\ 1 & p \nmid m \end{cases} \\ |\ |_\infty^\varepsilon \ : \ \varepsilon \in (0,1] & \end{array} \right\}$$

$|\ |_{triv} = |\ |_p^0 = |\ |_\infty^0$

Picture:

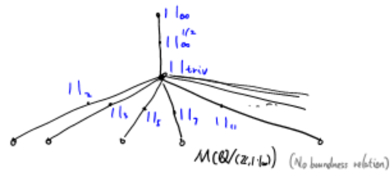



value of 2.

From this picture, we want to get:
Bound relations among seminorms
Topology properties: Hausdorff? compact?
Residue field, injection and contraction
... See next page

Rmk. When we do not identify the norm, we mean $\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)$.




E.g. $\mathcal{A} = (\mathbb{Q}, \|\cdot\|_{any})$, $\mathcal{M}(\mathcal{A}) = \{*\}$

E.g. $\mathcal{A} = (\mathbb{F}_q, \|\cdot\|_{triv})$  $\mathcal{M}(\mathbb{F}_q) = \{*\}$

E.g. $\mathcal{A} = \mathbb{R}/\mathbb{C}$  reasonable seminorms are $\|\cdot\|_\infty^\varepsilon$, $\varepsilon \in [0,1]$.

Do we have any other seminorms?

E.g. $\mathcal{A} = \mathbb{Z}_p$  reasonable seminorms are $\|\cdot\|_p^t$, $\varepsilon \in [0, +\infty]$. ($\mathcal{A} = \mathbb{Q}_p$ is also interesting)

Do we have any other seminorms?

E.g. $\mathcal{A} = \mathbb{C}_p$

E.g. $\mathcal{A} = \mathbb{C}[x]$  If we only consider the norm which restricted to $\mathbb{C}$ is $|\ |_\infty$, we would get $\mathbb{C}$.

What would happen in the other cases?

E.g. $\mathcal{A} = \mathbb{C}_p\langle r^{-1}T\rangle$ or $\mathbb{P}^1_{\mathbb{C}_p}$

E.g. $\mathcal{A} = (\mathbb{Z}[i], \|\ \|_\infty)$

I'm very happy to do the homework one years ago.

E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

Try to answer the following questions:

- Set
  - $\mathcal{M}(\mathbb{Z})$ = ✓
  - partial order ⇝ bound order
  - Picture ✓
  - maximal / minimal seminorm — max: $\|\cdot\|_{\mathbb{F}_p}$, min $\|\cdot\|_\infty$
  - Berkovich structure of $\|\cdot\| \in \mathcal{M}(\mathbb{Z})$ ?
  - Archi or non Archi?

- Topo
  - Close set: not contain $|\ |_{triv}$: normal way + contain only finite $\|\cdot\|_p^t$; contain $|\ |_{triv}$: normal way
  - Open set: not contain $|\ |_{triv}$: normal way; contain $|\ |_{triv}$: normal way + contain all $\|\cdot\|_p^t$ except finite p


$(\mathcal{M}(\mathbb{Z}), \text{graph})$ → $(\mathcal{M}(\mathbb{Z}), \text{weak})$ is continuous

weak top is a little weaker then graph top

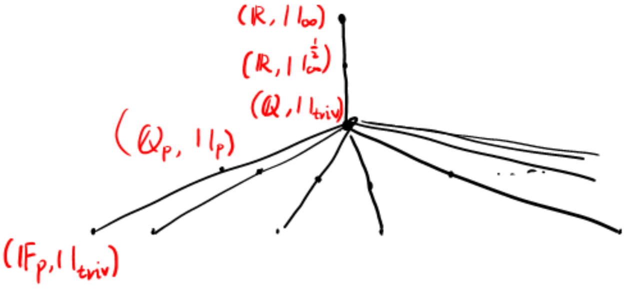
- Topo properties: connected? ✓ ($X = Y \sqcup Z$) Hausdorff? ✓ (quasi)compact? ✓

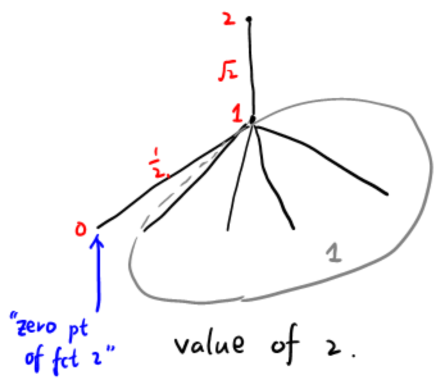
  irreducible? ✗ ($X = Y \cup Z$)

Def. $p \in X$ is a closed pt iff $\{p\}$ is closed

Then every pt is closed pt

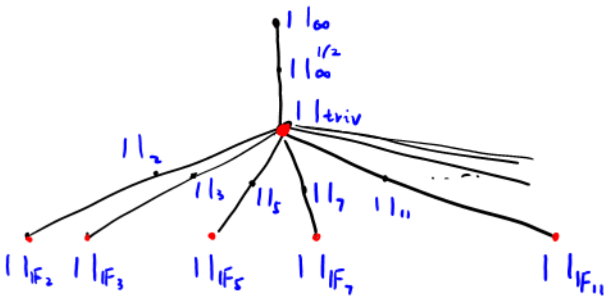
The definitions of Residue field, injection and contraction follows from [3.1.1, https://arxiv.org/abs/2105.13587v3]

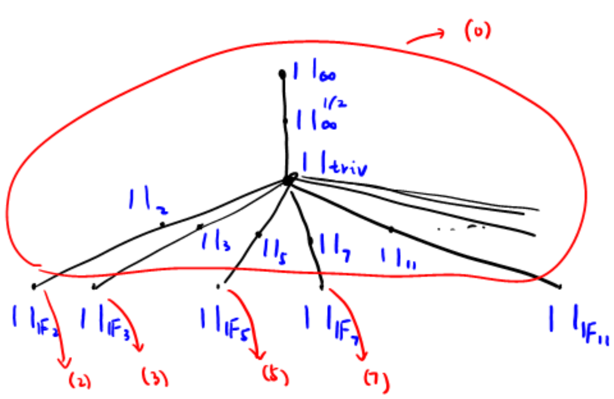


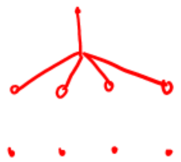

value of 2.

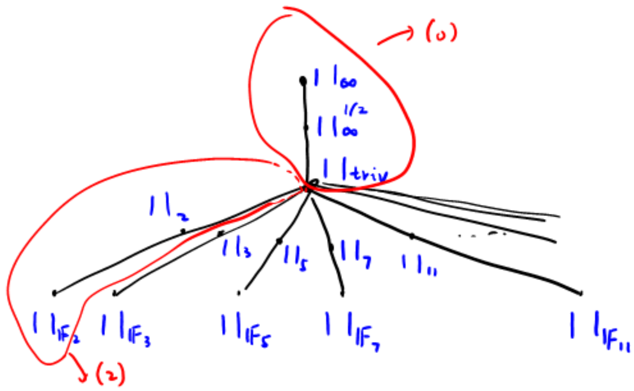
Residue field






k: kernel map (contraction)






r: reduction map (specialization)