Eine Woche, ein Beispiel

12.15 Young diagram with vectors

You can check [2024.12.01, 2024.12.08] for the reference, notations(for different weights) and the choice of the coordinates.

Motivation: Write $T \cong \mathbb{G}_m^6$ as the maximal torus of $G(E_6)$,
and $W(E_6) = N(T)/T$ as the Weyl group.
Choose $\beta_1, \dots, \beta_4$ as 4 orthogonal roots in $X_*(T)$.

We constructed a function

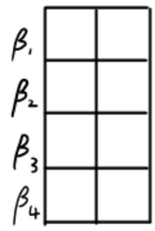
$$f: X^*(T) \longrightarrow \mathbb{R}$$

given by

$$f(\chi) = \sum_{\sigma \in W(E_6)} \langle \sigma(\beta_1), \chi \rangle^2 \langle \sigma(\beta_2), \chi \rangle^2 \langle \sigma(\beta_3), \chi \rangle^2 \langle \sigma(\beta_4), \chi \rangle^2$$

i.e.,

$$f = \sum_{\sigma \in W(E_6)} \sigma\left(\beta_1^2 \beta_2^2 \beta_3^2 \beta_4^2\right)$$



This looks like a "monomial symmetric function of type E_6".

Q: Can we generalize Young diagram to other representations (rather than S_n)?

A: Yes, but we lost some nice properties.

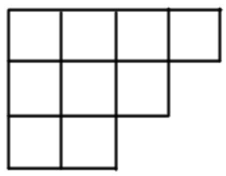
Maybe this generalization is not the "correct" one. I'm glad to hear any new ideas about the question.

1. definition & symmetric function
2. classical results for Weyl group
3. orthogonal roots
4. volume of lattices

1. definition & symmetric function

In this section, let G be a finite group.

Def For $(\rho, V) \in Rep_{\mathbb{C}}(G)$, the Young diagram is some boxes with decoration $\{v_1, v_2, \cdots\} \subseteq V$.



The associated monomial sym fct (on V^*) is given by

$$M_\lambda = \sum_{\sigma \in G} \sigma\left(\prod_i v_i^{k_i}\right) \in \left(Sym^{|\lambda|} V\right)^G$$

E.g. For $G = S_n$, (ρ, V) as the standard rep. and take $v_i = e_i$.
Then, the Young diagram is the usual one,
and the associated monomial sym fct is given by

$$M_\lambda = \sum_{\sigma \in S_n} \sigma\left(m_1^{k_1} \cdots m_t^{k_t}\right) \in \left(Sym^{|\lambda|} V\right)^{S_n}$$

These M_λ's form a basis of $(Sym^{\cdot} V)^{S_n}$,
and the multiplication is given by

https://math.stackexchange.com/questions/395842/decomposition-of-products-of-monomial-symmetric-polynomials-into-sums-of-them

Q: 1. Can we find a basis of $(\mathrm{Sym}^{\cdot} V)^{G}$?

2. Can we define

- H_j: j-th complete sym poly
- M_λ: monomial sym poly
- E_λ: elementary sym poly
- S_λ: Schur poly

and find some algorithm to get coefficients for multiplication?

3. Is this related with the cohomology ring of Grassmannians outside type A?
https://mathoverflow.net/questions/326749/reference-request-grassmannian-and-plucker-coordinates-in-type-b-c-d

4. Can we make these v_i canonical?
One possible way is to require {v_i}_i are orthonormal basis. Do we lost some sym polynomials?
Is it better to choose other bases in A_n and E_6?

2. classical results for Weyl group