Eine Woche, ein Beispiel
3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \circlearrowright X$

discrete $\Rightarrow$ foundamental domain $\quad \Lambda \circlearrowright \mathbb{C} \quad SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$

non discrete $\Rightarrow$ stratification by $G/G_x \quad S^1 \circlearrowright S^2 \quad \mathbb{C}^\times \circlearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g. $\quad \mathrm{Flag}(V) \cong GL(V)/P \qquad$ e.p. Grassmannian, $\mathbb{P}^n$

$$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$$

$O(n) := O(n,\mathbb{R}) \quad \rightsquigarrow$ Stiefel mfld [21.11.14]

$SO(n) := SO(n,\mathbb{R})$

$$\mathbb{A}^n = \mathbb{A}^n$$

$$\mathcal{H}^n \cong O^+(1,n)/O(n)$$

$$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

$\rightsquigarrow$ Hermitian symmetric space

where $\quad \mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1,\ v_{n+1} > 0\}$

$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad \langle v, w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$

$$O(n,1) := \mathrm{Aut}(\mathbb{R}^{n+1}, \langle\ ,\ \rangle) \subseteq GL_{n+1}(\mathbb{R})$$

$$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

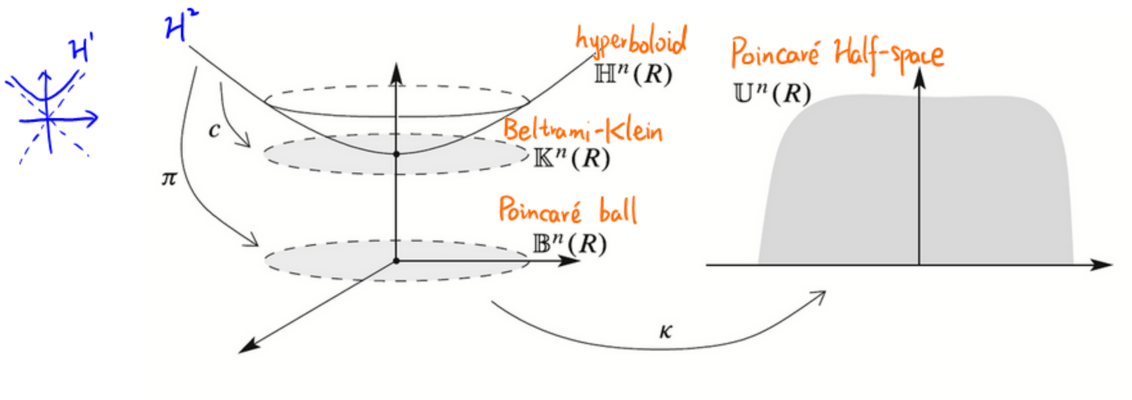

Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-o1-2

Terminology.

$\downarrow$ $X + G \curvearrowright X$

G-space

$\downarrow$ transitive

homogeneous space.

[JL18. Cor 3.18]: Nice hom space can have compatible Rie mfld structures.

Riemannian mfld, G preserve g $\swarrow$ $\qquad$ $\searrow$ Hermitian mfld, G preserve g $\leftarrow$ metric

Riemannian homogeneous space. $\xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}}$ Hermitian homogeneous space.

$\downarrow$ geodesic symmetry globally defined, preserve g $\qquad$ $\downarrow$ geodesic symmetry globally defined, preserve g

Symmetric space $\xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}}$ Hermitian symmetric space

$\downarrow$ transitive on frames $\qquad$ $\mathbb{C}P^n$ $n \geq 1$

frame homogeneous space

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization