Tutorial 7 & Exercise 6

(Blatt 8)

Due to the holiday, there are some problems of indexing numbers.

Today we work on integration & measure, with a special emphasis on convergence.

1. Computation
2. (Absolute) convergence
3. Conditional convergence
4. Lebesque measure

1. Computation

Ex 1. For $\alpha > 0$, compute

$$A(\alpha) = \int_{[0,\alpha]^4} (x+y+z+w)^2 \, dx\,dy\,dz\,dw.$$

Hint. $$\int_{[0,\alpha]^4} x^2 \, dx\,dy\,dz\,dw = \frac{1}{3}\alpha^2 \cdot \alpha \cdot \alpha \cdot \alpha = \frac{1}{3}\alpha^6$$

$$\int_{[0,\alpha]^4} xy \, dx\,dy\,dz\,dw = \frac{1}{2}\alpha^2 \cdot \frac{1}{2}\alpha^2 \cdot \alpha \cdot \alpha = \frac{1}{4}\alpha^6$$

$$A(\alpha) = \int_{[0,\alpha]^4} \sum_{sym} x^2 \, dx\,dy\,dz\,dw + \int_{[0,\alpha]^4} 2\sum_{sym} xy \, dx\,dy\,dz\,dw$$

$$= 4 \cdot \frac{1}{3}\alpha^6 + 2 \cdot \binom{4}{2} \cdot \frac{1}{4}\alpha^6$$

$$= \frac{4}{3}\alpha^6 + 3\alpha^6$$

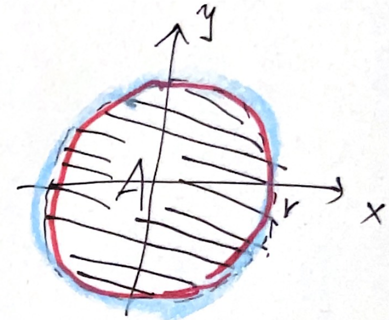
$$= \frac{13}{3}\alpha^6$$

Ex 4. For $r>0$, compute the area of set

$$A=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq r^2\}.$$

(Suppose that you don't know the circular area formula!)

Hint: $\text{area}(A)=\int_A 1\,dx\,dy$



$$=\int_{-r}^{r}\left(\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} 1\,dy\right)dx$$

$$=\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx$$

$$\overset{x=r\sin\theta}{\underset{\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]}{=}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r\cdot r\cos^2\theta\,d\theta$$

$$=2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}\,d\theta$$

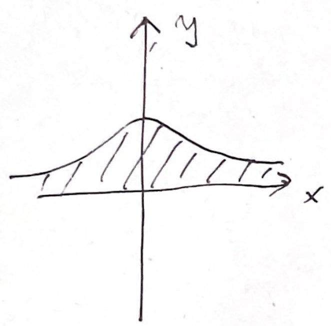
$$=r^2\left(\pi-\frac{1}{2}\sin 2\theta\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right)$$

$$=\pi r^2$$

Task 1. (a) Compute $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2}$

Hint: $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} \xlongequal[\theta\in(-\frac{\pi}{2},\frac{\pi}{2})]{x=\tan\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\frac{1}{\cos^2\theta}\,d\theta}{1+\frac{\sin^2\theta}{\cos^2\theta}} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \pi$

Rigorous: $\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} = \lim_{R\to+\infty} \int_{-R}^{R} \frac{dx}{1+x^2}$



$= \lim_{R\to+\infty} \arctan x \Big|_{-R}^{R}$

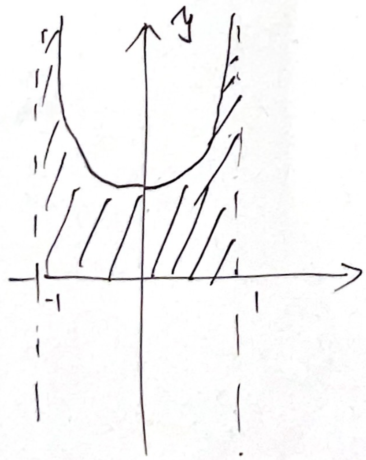
$= 2\lim_{R\to+\infty} \arctan R$

$= \pi$ $\square$

(b) Compute $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$.

Hint. $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} \xlongequal[\theta\in(-\frac{\pi}{2},\frac{\pi}{2})]{x=\sin\theta} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta\,d\theta}{\cos\theta} = \pi.$

Rigorous: $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{R\to 1^-} \int_{-R}^{R} \frac{dx}{\sqrt{1-x^2}}$



$= \lim_{R\to 1^-} \arcsin x \Big|_{-R}^{R}$

$= 2\lim_{R\to+\infty} \arcsin R$

$= \pi$ $\square$

2. (Absolute) convergence.

Lemma: For $\alpha \in \mathbb{R}$,

$$\int_1^{+\infty} x^{-\alpha}\,dx = \begin{cases} +\infty & \alpha \leq 1 \\ \frac{1}{\alpha-1} & \alpha > 1 \end{cases}$$

Def (Convergence of series)

For $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^{\mathbb{N}}$, we say

$\sum_{n=1}^{+\infty} a_n$ converges, if $\lim\limits_{N\to+\infty} \sum_{n=1}^{N} a_n$ $\exists$

$\sum_{n=1}^{+\infty} a_n$ converges absolutely, if $\sum_{n=1}^{+\infty} |a_n|$ $\exists$

$$\text{converge absolutely} \underset{\text{if } a_n \geq 0}{\overset{\longrightarrow}{\longleftarrow}} \text{Converge}$$

Ex. $\sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$ while $\sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$

Method 1: inequality estimation. $\left(\frac{1}{n^2} < \frac{1}{(n-1)n} = \frac{1}{n-1} - \frac{1}{n}\right)$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots \\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{16} + \cdots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots \\
&= +\infty
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{n^2} &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \\
&\leq 1 + \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots \\
&= 2
\end{aligned}$$

Ex 2. Assume we know that

$$\int_{-\infty}^{+\infty} e^{-\frac{1}{2}t^2} dt = \sqrt{2\pi}$$

(1) For $\alpha > 0$, compute

$$A_\alpha := \int_{-\infty}^{\infty} e^{-\alpha t^2} dt$$

(2) Shows that

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-\alpha(x^2+y^2)} dx\, dy = \int_0^{2\pi} \int_0^{+\infty} e^{-\alpha r^2} r\, dr\, d\theta$$

by computing both sides.

A: (1). $A_\alpha = \int_{-\infty}^{+\infty} e^{-\frac{1}{2}(\sqrt{2\alpha}\, t)^2} dt$

$$\overset{s=\sqrt{2\alpha}t}{=\!=} \frac{1}{\sqrt{2\alpha}} \int_{-\infty}^{+\infty} e^{-\frac{1}{2}s^2} ds$$

$$= \frac{1}{\sqrt{2\alpha}} \sqrt{2\pi}$$

$$= \sqrt{\frac{\pi}{\alpha}}$$

(2) $LHS = \int_{-\infty}^{+\infty} e^{-\alpha x^2} dx \cdot \int_{-\infty}^{+\infty} e^{-\alpha y^2} dy$

$$= A_\alpha^2 = \frac{\pi}{\alpha}$$

$RHS = \int_0^{2\pi} d\theta \cdot \int_0^{+\infty} e^{-\alpha r^2} r\, dr$

$$= 2\pi \cdot \frac{1}{2} \int_0^{+\infty} e^{-\alpha r^2} dr^2$$

$$= 2\pi \cdot \frac{1}{2} \cdot \frac{1}{-\alpha} e^{-\alpha r^2} \Big|_0^{+\infty}$$

$$= 2\pi \cdot \frac{1}{2} \cdot \frac{1}{-\alpha} \cdot (-1)$$

$$= \frac{\pi}{\alpha}$$

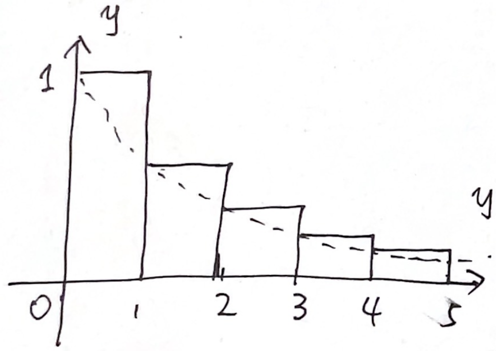
Rigorous. $\displaystyle\sum_{n=2}^{N} \frac{1}{n^2} < \sum_{n=2}^{N}\left(\frac{1}{n-1} - \frac{1}{n}\right)$

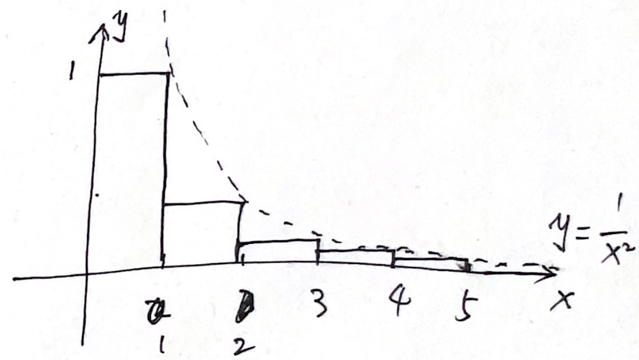
$$= \sum_{n=1}^{N-1} \frac{1}{n} - \sum_{n=2}^{N} \frac{1}{n}$$

$$= 1 - \frac{1}{N}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2} \leqslant 1$$

Method 2. Compare with integration





$$\sum_{n=1}^{\infty} \frac{1}{n} \geqslant \int_0^{+\infty} \frac{1}{x+1}\,dx = +\infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \leqslant 1 + \int_1^{+\infty} \frac{1}{x^2}\,dx = 2 < +\infty$$

More rigorous: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \geqslant \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{1}{x+1}\,dx = \int_0^{+\infty} \frac{1}{x+1}\,dx = +\infty$

Ex. For $\alpha \in \mathbb{R}$, when do we have

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} < +\infty \quad ?$$

(Two methods for $\alpha \leqslant 1$ or $\alpha \geqslant 2$, also)

Task 2.(a). For $\alpha \in \mathbb{R}$, when do we have

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}} < +\infty \quad ?$$

Hint. $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^{\alpha}}\,dx = \begin{cases} +\infty & \alpha \leq 1 \\ \frac{1}{\alpha-1}\ln(\ln 2) & \alpha > 1 \end{cases}$

When $\alpha \leq 1$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}} \geq \int_3^{\infty} \frac{1}{x(\ln x)^{\alpha}}\,dx = +\infty$

When $\alpha > 1$, $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{\alpha}} \leq \int_2^{\infty} \frac{1}{x(\ln x)^{\alpha}}\,dx < +\infty$

3. Conditional convergence

In many cases, $\sum_{n=1}^{\infty} |a_n|$ converges but $\sum_{n=1}^{\infty} a_n$ does not.
We call that $\sum_{n=0}^{\infty} a_n$ <u>converges conditionally</u>.

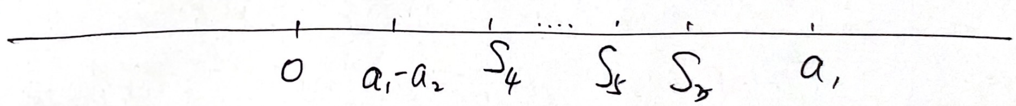
Lemma. $\sum_{n=0}^{\infty} a_n \ \exists \quad \Rightarrow \quad \lim_{n\to\infty} a_n = 0$

e.p. $\lim_{n\to\infty} a_n \neq 0$ or $\nexists \Rightarrow \sum_{n=0}^{\infty} a_n \ \nexists$

Lemma. For $a_n \searrow 0$ (monotonically decreasing with limit 0),

$$\sum_{n=0}^{\infty} (-1)^n a_n \quad \text{converges}$$

Idea.



$\{S_n\}$ converges by the nested intervals theorem.

Task 2. Show that

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^\alpha} \begin{cases} \text{converges absolutely} & \alpha > 1 \\ \text{converges conditionally} & 0 < \alpha \le 1 \\ \text{diverges} & \alpha \le 0 \end{cases}$$

$$\sum_{n=2}^{\infty} (-1)^n \frac{1}{n (\log n)^\alpha} \begin{cases} \text{converges absolutely} & \alpha > 1 \\ \text{converges conditionally} & \alpha \le 1 \end{cases}$$

We also discuss absolute/conditional convergence for <u>improper integrals</u> (wiki it!).

We only discuss one special case.

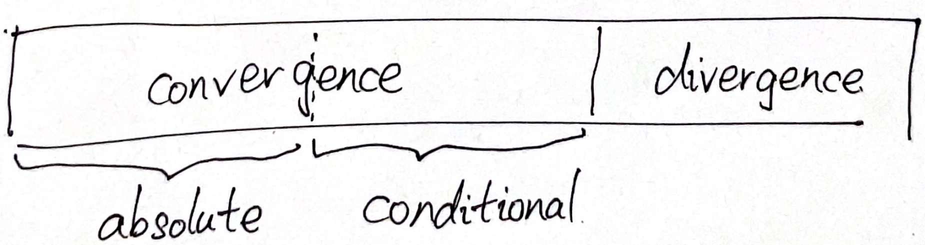
<u>Def.</u> For fct $f: (0,1) \longrightarrow \mathbb{R}$, suppose that

$\forall \delta \in (0,1)$, $f|_{(\delta,1)}$ is Riemann-integrable.

We say

$\int_0^1 f(x)\,dx$ converges, if $\lim_{\delta \to 0^+} \int_\delta^1 f(x)\,dx \ \exists$;

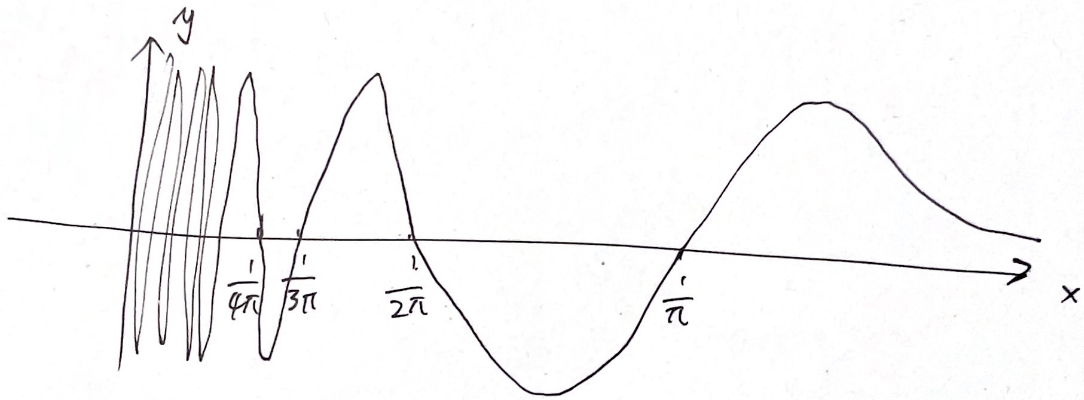
$\int_0^1 f(x)\,dx$ converges absolutely, if $\int_0^1 |f(x)|\,dx$ converges.



Task 3 (a),(b) For $\alpha \in \mathbb{R}$, shows that

$$\int_0^1 \frac{1}{x^\alpha} \operatorname{sign}\left(\sin \frac{1}{x}\right) dx \begin{cases} \text{converges abs} & \alpha < 1 \\ \text{converges con} & \alpha = 1 \\ \text{diverges} & \alpha > 1 \end{cases}$$

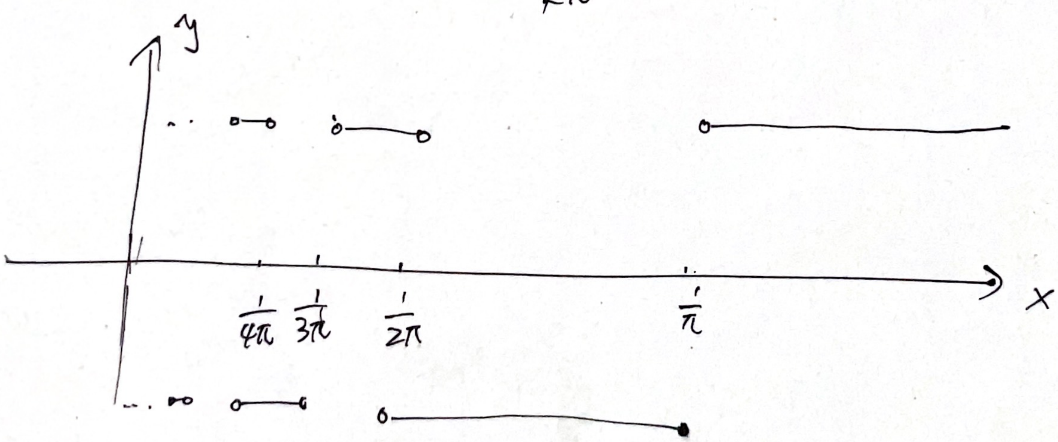
Hint. $\int_0^1 \frac{1}{x^\alpha} dx$ converges $\Leftrightarrow$ $\alpha < 1$



$\sin \frac{1}{x}$ $(x > 0)$

$$\sin \frac{1}{x} = 0 \Leftrightarrow \frac{1}{x} = k\pi, \quad k \in \mathbb{Z} - \{0\}$$

$$\Leftrightarrow x = \frac{1}{k\pi}, \quad k \in \mathbb{Z} - \{0\}$$



$\operatorname{sign}\left(\sin \frac{1}{x}\right)$ $(x > 0)$

$$\int_0^1 \frac{1}{x^\alpha} \operatorname{sign}\left(\sin\frac{1}{x}\right) dx = \int_{\frac{1}{\pi}}^1 \frac{1}{x^\alpha} dx + \underbrace{\sum_{k=1}^{\infty} (-1)^k \int_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}} \frac{1}{x^\alpha} dx}_{A}$$

When $\alpha = 1$,

$$A = \sum_{k=1}^{\infty} (-1)^k \ln x \Big|_{x=\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}$$

$$= \sum_{k=1}^{\infty} (-1)^k \left(\ln\frac{1}{k\pi} - \ln\frac{1}{(k+1)\pi}\right)$$

$$= -\ln\frac{1}{\pi} + \ln\frac{1}{2\pi} + \ln\frac{1}{2\pi} - \ln\frac{1}{3\pi} - \ln\frac{1}{3\pi} + \cdots$$

$$= -\ln\frac{1}{\pi} + \underbrace{2\sum_{k=2}^{\infty} (-1)^k \ln\frac{1}{k\pi}}_{\text{converges.}}$$

When $\alpha > 1$,

$$A = \sum_{k=1}^{\infty} \frac{1}{-\alpha+1} (-1)^k \, x^{-\alpha+1} \Big|_{\frac{1}{(k+1)\pi}}^{\frac{1}{k\pi}}$$

$$= \frac{1}{-\alpha+1} \sum_{k=1}^{\infty} (-1)^k \left((k\pi)^{\alpha-1} - ((k+1)\pi)^{\alpha-1} \right)$$

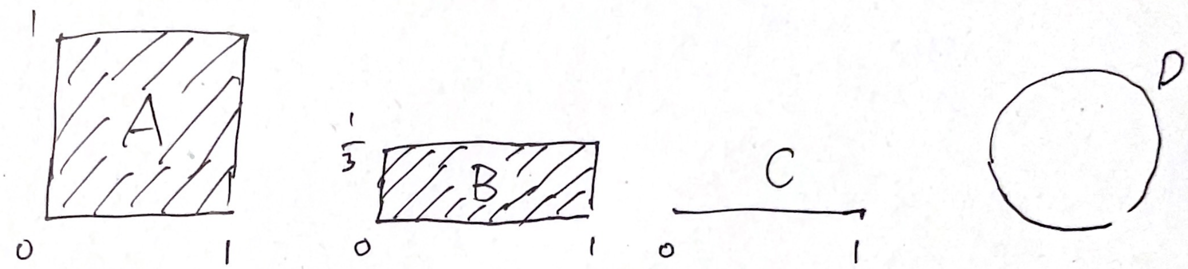
Since $\lim\limits_{k\to+\infty} (k\pi)^{\alpha-1} - ((k+1)\pi)^{\alpha-1} = -\infty$, $A \nexists$, i.e.

$\int_0^1 \frac{1}{x^\alpha} \operatorname{sign}\left(\sin\frac{1}{x}\right) dx$ diverges.

$\square$

4. ~~Various ty~~ Lebesque measure

E.g. in $\mathbb{R}^2$



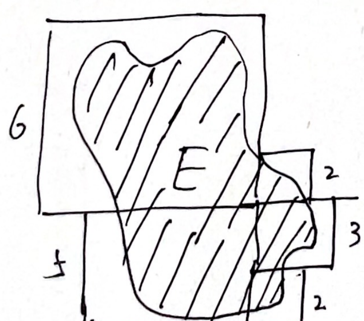
$m(A) = 1$ $\qquad$ $m(B) = \frac{1}{3}$ $\qquad$ $m(C) = 0$ $\qquad$ $m(D) = 0$

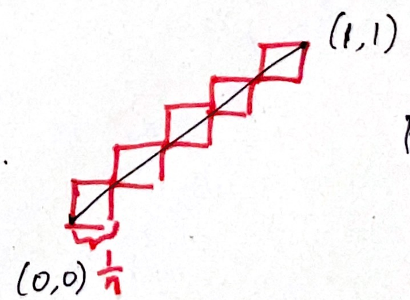
Idea: For lower dimensional space (e.g. curve in $\mathbb{R}^2$), the Lebesque measure is 0.

Def. For $X \subseteq \mathbb{R}^n$ (measurable), define the Lebesgue measure

$$m(X) = \inf \left\{ \sum_{i=1}^{\infty} r_i^n \;\middle|\; \begin{array}{l} X \subseteq \bigcup_{i=1}^{\infty} D_i, \\ D_i : \text{cube with edge length } r_i \end{array} \right\}$$

E.g.





For $E \subset \mathbb{R}^2$,

$$m(E) \leq 6^2 + 5^2 + 2^2 + 3^2 + 2^2 = 78$$

For $F \subset \mathbb{R}^2$,

$$m(F) \leq \underbrace{\left(\tfrac{1}{n}\right)^2 + \left(\tfrac{1}{n}\right)^2 + \cdots + \left(\tfrac{1}{n}\right)^2}_{n \text{ many}} = \frac{1}{n} \quad \forall n \in \mathbb{N}_{>0}$$

$\Rightarrow m(F) = 0.$

Bonus: a list of names of measures.

$$
\begin{array}{ccc}
 & & \text{cplx measure} \\
 & & \cup \\
 & & \text{signed measure} \\
 & & \cup \\
\text{premeasure} & \supset & \text{measure} \\
\boxed{\text{e.g. Jordan measure}} & & \cup \\
 & & \text{Radon measure}
\end{array}
$$

e.g.

Lebesgue measure

on $\mathbb{R}^n$: Hausdorff measure

Dirac measure

on G: Haar measure.