

Eine Woche, ein Beispiel

4.28 naive \otimes -Hom adjunction

Notation: - A : associate ring allowed to be non-commutative, contains 1
 - There are two systems to write category of A -modules:

$$\begin{aligned} \text{Mod}_A &= A\text{-Mod} && \ni {}_A M \\ (\text{Mod}_A)^{\text{op}} &\neq \text{Mod}_{A^{\text{op}}} = \text{Mod}-A = A^{\text{op}}\text{-Mod} && \ni M_A \\ \text{Mod}_{A \otimes B^{\text{op}}} &= A\text{-Mod}-B && \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\text{Mod}_{B \otimes A^{\text{op}}})^{\text{op}} = (B\text{-Mod}-A)^{\text{op}} = (A^{\text{op}}\text{-Mod}-B^{\text{op}})^{\text{op}} \ni {}_B M_A$$

as

$$(\text{Mod}_{A \otimes B^{\text{op}}})^{\text{op}} = (A\text{-Mod}-B)^{\text{op}}$$

⚠ Even though you can identify $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{\text{op}})$,
 $A^{\text{op}} \notin \text{Ob}(\text{Ring}^{\text{op}})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\text{Mod}_A = \text{Mod}_{A^{\text{op}}} \subset \text{Sh}(\text{Spec } A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
 Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1. definition recall for \otimes & Hom
2. adjunction
3. comparison between \otimes -Hom & $f^* \dashv f_*$
4. definition recall for \otimes & Hom , derived version
5. adjunction , derived version
6. comparison between \otimes -Hom & $f^* \dashv f_*$, derived version

1. definition recall for \otimes & Hom

$$\begin{aligned}\otimes_A: \text{Mod}_{A^{\text{op}}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}} \\ \text{Hom}_A(-, -): (\text{Mod}_A)^{\text{op}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}}\end{aligned}$$

In general,

$$\begin{aligned}\otimes_B: A\text{-Mod-}B \times B\text{-Mod-}C &\longrightarrow A\text{-Mod-}C \\ \text{Hom}_B(-, -): (A\text{-Mod-}B)^{\text{op}} \times B\text{-Mod-}C &\longrightarrow A\text{-Mod-}C\end{aligned}$$

$$\begin{aligned}\text{Hom}_B^A(-, -): (A\text{-Mod-}B)^{\text{op}} \times B\text{-Mod-}A &\longrightarrow \mathbb{Z}\text{-Mod} \\ \parallel &\quad \parallel &\quad \parallel &\quad \parallel \\ \text{Hom}_{B \otimes_{\mathbb{Z}} A^{\text{op}}}(-, -): (\mathbb{Z}\text{-Mod-}B \otimes_{\mathbb{Z}} A^{\text{op}})^{\text{op}} \times (B \otimes_{\mathbb{Z}} A^{\text{op}}\text{-Mod-}\mathbb{Z})^{\text{op}} &\longrightarrow \mathbb{Z}\text{-Mod-}\mathbb{Z}\end{aligned}$$

$${}_A X_B, {}_B Y_C, {}_C Z_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{\text{op}}} X$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$$\text{Hom}_A(A, X) \cong X$$

$$\text{in } A\text{-Mod-}C = C^{\text{op}}\text{-Mod-}A^{\text{op}}$$

2. adjunction ${}_B X_A, {}_C Y_B, {}_C Z_D$. we get

$$\text{Hom}_C(Y \otimes_B X, Z) \cong \text{Hom}_B(X, \text{Hom}_C(Y, Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set

$$\{f: Y \times X \longrightarrow Z \mid f(cyb, x) = cf(y, bx) \quad \forall b, c\}$$

For $A=D=\mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction factors:

$$B\text{-Mod} \begin{array}{c} \xrightarrow{Y \otimes_B -} \\ \perp \\ \xleftarrow{\text{Hom}_C(Y, -)} \end{array} C\text{-Mod}$$

slogan: adjunction \approx associativity

$\otimes \dashv \text{Hom}$:

$$\begin{array}{ccc}
 (A\text{-Mod-}B)^{\text{op}} \times (B\text{-Mod-}C)^{\text{op}} \times C\text{-Mod-}D & \xrightarrow{(\text{Id}, \text{Hom}_C)} & (A\text{-Mod-}B)^{\text{op}} \times B\text{-Mod-}D \\
 \parallel & & \downarrow \text{Hom}_B \\
 (A\text{-Mod-}B \times B\text{-Mod-}C)^{\text{op}} \times C\text{-Mod-}D & & \\
 \downarrow (\otimes_B, \text{Id}) & & \\
 (A\text{-Mod-}C)^{\text{op}} \times C\text{-Mod-}D & \xrightarrow{\text{Hom}_C} & A\text{-Mod-}D
 \end{array}$$

$f^* \dashv f_*$:

$$\text{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$\begin{array}{ccc}
 \mathcal{G} & & \mathcal{F} \\
 \downarrow & & \downarrow \\
 Y & \xrightarrow{f} & X
 \end{array}$$

$$\begin{array}{ccc}
 \text{Sh}(X)^{\text{op}} \times \text{Mor}(Y, X) \times \text{Sh}(Y) & \xrightarrow{(\text{Id}, \text{pushforward})} & \text{Sh}(X)^{\text{op}} \times \text{Sh}(X) \\
 \downarrow (\text{pullback}, \text{Id}) & & \downarrow \text{Hom}_{\text{Sh}(X)}(-, -) \\
 \text{Sh}(Y)^{\text{op}} \times \text{Sh}(Y) & \xrightarrow{\text{Hom}_{\text{Sh}(Y)}(-, -)} & \text{Abel}
 \end{array}$$

$$\begin{array}{ccc}
 (\mathcal{F}, f, \mathcal{G}) & \xrightarrow{\quad} & (\mathcal{F}, f_*\mathcal{G}) \\
 \downarrow & & \downarrow \\
 (f^*\mathcal{F}, \mathcal{G}) & \xrightarrow{\quad} & \text{Hom}_{\text{Sh}(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\text{Sh}(X)}(\mathcal{F}, f_*\mathcal{G})
 \end{array}$$

$f_! \dashv f^!$ similar.

3. comparison between $\otimes \dashv \text{Hom}$ & $f^* \dashv f_*$

Forgetful functor

Prop. For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, \exists "forgetful functor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \quad M \longmapsto u(M)$$

$$u(M) = {}_R S_S \otimes_S M = \text{Hom}_S({}_S S_R, M)$$

one has adjunction functors

$$\begin{array}{ccc} & {}_S S_R \otimes_R - & \\ & \downarrow & \\ S\text{-Mod} & \xrightarrow[\text{red } u = {}_R S_S \otimes_S -]{\text{Hom}_S({}_S S_R, -)} & R\text{-Mod} \\ & \uparrow & \\ & \text{Hom}_R({}_R S_S, -) & \end{array} \quad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{array}{ccc} & j_! & \\ & \downarrow & \\ \mathcal{D}(X) & \xrightarrow[\text{red } j_*]{j^!} & \mathcal{D}(U) \\ & \uparrow & \\ & Rj_* & \end{array}$$

Vague slogan: $u \approx$ "forget the information of Z ".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between \mathcal{U} & X .

E.g. For finite gps $H \leq G$, one has Res-Ind adjunction:

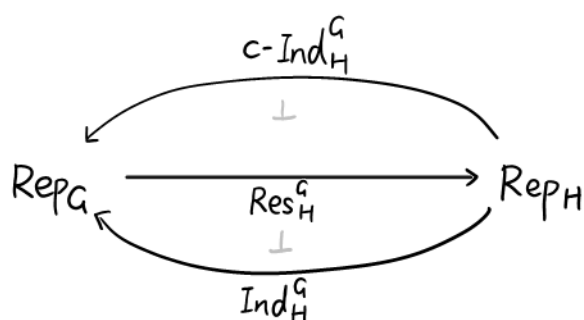
$$\begin{aligned} \text{Res}_H^G &\dashv \text{Ind}_H^G \\ c\text{-Ind}_H^G &\dashv \text{Res}_H^G \end{aligned}$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp,} \\ H \leq G \text{ open} \end{cases}$

If one only has $H \leq G$ closed, then it's possible that $j' \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,



Ex. Compare it with the recollement diagram & (3.1).

$$\begin{array}{ccc} \mathcal{U} & & [* / H] \\ \downarrow j & & \downarrow \text{"cover with fiber } G/H" \\ X & & [* / G] \end{array}$$

translate the following geometrical results into algebraic statements.

1. One has natural factor $j_! \longrightarrow j^*$. When $\#G/H < +\infty$, $j_! = j^*$
 $c\text{-Ind}_H^G \longrightarrow \text{Ind}_H^G$ $c\text{-Ind}_H^G = \text{Ind}_H^G$

2. Even though

$\text{Sh}_{\text{ét}, S}([* / G]) \approx \text{Rep}_G = \mathbb{Q}[G]\text{-Mod.}$
the "structure sheaf" of $[* / G]$ is \mathbb{Q} , not $\mathbb{Q}[G]$.

$$\text{Res}_{[* / G]}^G \mathbb{Q} = \mathbb{Q}, \quad \text{Res}_{[* / G]}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* R j_* \neq \text{Id}$, $j'_! j_! \neq \text{Id}$.

Until now, we have met three types of six factor formalism: top spaces, A-modules and stacks.

Compare with i

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{array}{ccc} S & & \text{Spec } S \\ \uparrow \tilde{f} & & \downarrow f \\ R & \xrightarrow{M} & \text{Spec } R \end{array}$$

\exists "pullback factor"

$$f^*: R\text{-Mod} \longrightarrow S\text{-Mod} \quad f^*M = {}_S S_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{array}{ccc} & f^* & \\ \swarrow & \perp & \searrow \\ S\text{-Mod} & \xrightarrow{u} & R\text{-Mod} \\ \nwarrow & \perp & \swarrow \\ & \text{Hom}_R({}_R S_S, -) & \end{array}$$

compare it with another part of the recollement diagram:

$$\begin{array}{ccc} & i^* & \\ \swarrow & \perp & \searrow \\ \mathcal{D}(X) & \xrightarrow{i_*} & \mathcal{D}(U) \\ \nwarrow & \perp & \swarrow \\ & i_! & \end{array}$$

Rmk. u is usually not f -faithful, unless $S = R/I$.

(In fact, only need S is R -idempotent, i.e. $S \cong S \otimes_R S$.)

which crspds to closed embedding.

In that case,

$$i^* i_* = \text{Id}: {}_S S_R \otimes_R ({}_R S_S \otimes_S M) \cong M$$

$$i_! i_* = \text{Id}: \text{Hom}_R({}_R S_S, \text{Hom}({}_S S_R, M)) \cong M$$

Slogan: in the comm alg, $\text{Spec } R/I \longrightarrow \text{Spec } R$ is closed embedding.
 In general, if
 S is an R -idempotent algebra: $S \cong S \otimes_R S$
 then $i: \text{Spec } S \longrightarrow \text{Spec } R$ can be viewed as "closed subset".

E.g. $R_{\mathbb{P}}, R/I$ are idempotent R -algs.
 $\mathbb{Z}[\frac{1}{b}], \mathbb{F}_p, \mathbb{Z}/p^2\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \dots$ are idem \mathbb{Z} -algs.

⚠ Usually R/I is not an derived idem R -alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. $\text{Spec } R/I$ is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, $i^!$ is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?

A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. $L\otimes \dashv R\mathrm{Hom}$

F	RF or LF	R^iF or L^iF	exact fctor
f^* f_* $\pi_{X,*}\mathcal{F}$ $f_!$ $\pi_{X,!}\mathcal{F}$ $-$	f^* Rf_* $\Gamma(X;\mathcal{F})$ $Rf_!$ $\Gamma_c(X;\mathcal{F})$ $f_!$	$-$ R^if_* $H^i(X;\mathcal{F})$ $R^if_!$ $H_c^i(X;\mathcal{F})$ $H^i(f_! -)$	f^* -acyclic Γ -acyclic $f_!$ -acyclic Γ_c -acyclic
$-\otimes_R-$ $\mathrm{Hom}_R(-,-)$ M_G M^G $M_{\mathfrak{g}}$ $M^{\mathfrak{g}}$ $M/[AM]$ M^A $A/[AA]$ $Z(A)$	$-\overset{L}{\otimes}_R-$ $R\mathrm{Hom}_R(-,-)$ $Z^L\otimes_{Z[G]}M$ $R\mathrm{Hom}_{Z[G]}(Z,M)$ $x^L\otimes_{U_{\mathfrak{g}}}M$ $R\mathrm{Hom}_{U_{\mathfrak{g}}}(x,M)$ $A^L\otimes_{A^e}M$ $R\mathrm{Hom}_{A^e}(A,M)$ $A^L\otimes_{A^e}A$ $R\mathrm{Hom}_{A^e}(A,A)$	$\mathrm{Tor}_R^i(-,-)$ $\mathrm{Ext}_R^i(-,-)$ $H_i(G;M)$ $H^i(G;M)$ $H_i(\mathfrak{g};M)$ $H^i(\mathfrak{g};M)$ $HH_i(A,M)$ $HH^i(A,M)$ $HH_i(A)$ $HH^i(A)$	flat injective/projective

e.g. group coh

e.g. Lie alg coh

\mathfrak{g}/x : Lie alg

e.g. Hochschild coh

For calculations, see:

[23.04.09]: gp coh

[wiki]: Lie algebra coh

[21.05.21]: Hochschild coh

[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.

For $\mathrm{Hom}(-,A)$, see <https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m>.

Chenji Fu claimed that $\mathrm{Hom}(-,A)$ always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.

Related:

<https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod>

<https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod>

<https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution>

4. definition recall for \otimes & Hom , derived version

To define ${}^L\otimes$ & $R\text{Hom}$, one needs to extend functors

$$\begin{array}{lcl} \otimes_A: & A\text{-Mod} & \times A\text{-Mod} \longrightarrow A\text{-Mod} \\ \text{Hom}_A(-, -): & (A\text{-Mod})^{\text{op}} & \times A\text{-Mod} \longrightarrow A\text{-Mod} \end{array}$$

to functors on double complexes.

$\mathcal{C}(A)$: = complex of A -modules, temperate notation

$$\begin{array}{lcl} \otimes_{\mathcal{C}(A)}: & \mathcal{C}(A) & \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A) \\ \text{Hom}_{\mathcal{C}(A)}(-, -): & (\mathcal{C}(A))^{\text{op}} & \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A) \end{array}$$

But how?

Wishes:

$$\begin{aligned} (M[i]) \otimes_{\mathcal{C}(A)} (N[j]) &= (M \otimes N)[i+j] \\ \text{Hom}_{\mathcal{C}(A)}(M[-i], N[j]) &= \text{Hom}(M, N)[i+j] \end{aligned}$$

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & & \vdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & M^{-1} \otimes N' & \longrightarrow & M^0 \otimes N' & \longrightarrow & M^1 \otimes N' & \longrightarrow & M^2 \otimes N' & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & M^{-1} \otimes N^0 & \longrightarrow & M^0 \otimes N^0 & \longrightarrow & M^1 \otimes N^0 & \longrightarrow & M^2 \otimes N^0 & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & M^{-1} \otimes N^{-1} & \longrightarrow & M^0 \otimes N^{-1} & \longrightarrow & M^1 \otimes N^{-1} & \longrightarrow & M^2 \otimes N^{-1} & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow
\end{array}$$

$\text{Tot}(M' \otimes N')$, the double complex of $M' \otimes N'$.

$$\begin{array}{ccccccc}
& \vdots & & \vdots & & \vdots & & \vdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \text{Hom}(M', N') & \longrightarrow & \text{Hom}(M^0, N') & \longrightarrow & \text{Hom}(M^{-1}, N') & \longrightarrow & \text{Hom}(M^{-2}, N') & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \text{Hom}(M', N^0) & \longrightarrow & \text{Hom}(M^0, N^0) & \longrightarrow & \text{Hom}(M^{-1}, N^0) & \longrightarrow & \text{Hom}(M^{-2}, N^0) & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\longrightarrow & \text{Hom}(M', N^{-1}) & \longrightarrow & \text{Hom}(M^0, N^{-1}) & \longrightarrow & \text{Hom}(M^{-1}, N^{-1}) & \longrightarrow & \text{Hom}(M^{-2}, N^{-1}) & \longrightarrow \dots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow
\end{array}$$

$\text{Tot}(\text{Hom}(M', N'))$, the double complex of $\text{Hom}(M', N')$.

Def. For $M', N' \in \mathcal{C}(A)$, define

$$M' \otimes N', \quad \text{Hom}_A(M', N') \in \mathcal{C}(A)$$

by

$$(M' \otimes_{\mathcal{C}(A)} N')^n = \bigoplus_{i+j=n} M'^i \otimes_A N'^j$$

$$(\text{Hom}_{\mathcal{C}(A)}(M', N'))^n = \bigoplus_{i+j=n} \text{Hom}_A(M'^{-i}, N'^j)$$

and morphisms given by $d + (-1)^j \delta$.

Ex. Let $M' = \begin{bmatrix} \mathbb{Z} & \xrightarrow{x^3} & \mathbb{Z} \\ -1 & & 0 \end{bmatrix}$, $N' = \begin{bmatrix} \mathbb{Z} & \xrightarrow{x^2} & \mathbb{Z} \\ -1 & & 0 \end{bmatrix}$

compute $M' \otimes_{\mathcal{C}(\mathbb{Z})} N'$ & $\text{Hom}_{\mathcal{C}(\mathbb{Z})}(M', N')$,
and verify that they're complexes.

$$\begin{array}{c} \mathbb{A}: \quad 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \quad \quad \uparrow x^2 \quad \quad \uparrow x^2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \quad -1 \quad \quad 0 \\ \text{Tot}(M' \otimes N') \end{array} \rightsquigarrow \begin{array}{c} \left[\mathbb{Z} \xrightarrow{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 3 \end{pmatrix}} \mathbb{Z} \right] \\ -2 \quad \quad -1 \quad \quad 0 \\ M' \otimes_{\mathcal{C}(A)} N' \end{array}$$

$$\begin{array}{c} 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \quad \quad \uparrow x^2 \quad \quad \uparrow x^2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \quad 0 \quad \quad 1 \\ \text{Tot}(\text{Hom}(M', N')) \end{array} \rightsquigarrow \begin{array}{c} \left[\mathbb{Z} \xrightarrow{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 3 \end{pmatrix}} \mathbb{Z} \right] \\ -1 \quad \quad 0 \quad \quad 1 \\ \text{Hom}_{\mathcal{C}(A)}(M', N') \end{array}$$

Now, we can define ${}^L\otimes$ & $R\text{Hom}$.

Def. For $M, N \in A\text{-Mod}$, one can define

$$M {}^L\otimes_A N := M \otimes_{e(A)} P' \quad \text{when } N \xleftarrow{\cong} P' \text{ flat resolution}$$

in general, $M', N' \in \mathcal{D}^-(A\text{-Mod})$

$$R\text{Hom}_A(M, N) := \text{Hom}_{e(A)}(M, I') \quad \text{when } N \xrightarrow{\cong} I' \text{ inj resolution}$$

$$:= \text{Hom}_{e(A)}(P', N) \quad \text{when } M \xleftarrow{\cong} P' \text{ proj resolution}$$

in general, $M' \in \mathcal{D}^-(A\text{-Mod}), N' \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

<https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat>

Ex Compute $\mathbb{F}_2 {}^L\otimes_{\mathbb{Z}} \mathbb{F}_2$ & $R\text{Hom}_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $\text{Tor}_{\mathbb{Z}}^i(\mathbb{F}_2, \mathbb{F}_2)$ & $\text{Ext}_{\mathbb{Z}}^i(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$\text{Hom}_{\mathcal{D}^+(A)}(M', N') = R^0 \text{Hom}_{e(A)}(M', N')$$

$$\text{Hom}_A(M, N) = \text{Hom}_{\mathcal{D}^+(A)}(M', N') = R^0 \text{Hom}_{e(A)}(M, N).$$

⚠ [KS 90, Def 2.6.2]

To switch from $\mathcal{D}^-(X)$ to $\mathcal{D}^+(X)$, we need to require that

w.gldim $(A) < +\infty$. w.gldim: shortest flat resolution

A wrong proof for "flat \rightarrow proj"

"Proof": when P is flat,

$$\begin{array}{ccc} P \otimes_A - & \dashv & \text{Hom}_A(P, -) \\ \parallel & & \\ P^L \otimes_A - & \dashv & \text{RHom}_A(P, -) \end{array}$$

by the uniqueness of the adjunction, $\text{Hom}_A(P, -) = \text{RHom}_A(P, -)$,
so P is flat.

This is wrong. $\mathbb{Q} \in \mathbb{Z}\text{-Mod}$ is flat but not proj.
In the proof, we only have

$$\begin{array}{ccc} A\text{-Mod} & \begin{array}{c} \xrightarrow{P \otimes_A -} \\ \xleftarrow{\text{Hom}_A(P, -)} \end{array} & A\text{-Mod} \\ \downarrow l_A & & \downarrow l_A \\ \mathcal{D}(A) & \begin{array}{c} \xrightarrow{P^L \otimes_A -} \\ \xleftarrow{\text{RHom}_A(P, -)} \end{array} & \mathcal{D}(A) \end{array}$$

$$l_A \circ (P \otimes_A -) = (P^L \otimes_A -) \circ l_A.$$

Ex. Compute $\text{RHom}_{\mathbb{Z}}(\mathbb{Q}, -)$, and shows that

$$l_{\mathbb{Z}} \circ \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -) \neq \text{RHom}_{\mathbb{Z}}(\mathbb{Q}, -) \circ l_{\mathbb{Z}}.$$

5. adjunction

, derived version

Prop. For A comm ring, $L, M, N \in A\text{-Mod}$, we get
 $L', M' \in \mathcal{D}^-(A)$, $N' \in \mathcal{D}^+(A)$ in general

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

Proof.

$$\text{Hom}_A(M \otimes_A N, L) \cong \text{Hom}_A(N, \text{Hom}_A(M, L))$$

\Downarrow take $(-)^*$

$$\text{Hom}_{\mathcal{C}(A)}(M \otimes_{\mathcal{C}(A)} N, L) \cong \text{Hom}_{\mathcal{C}(A)}(N, \text{Hom}_{\mathcal{C}(A)}(M, L))$$

\Downarrow $\text{Hom}_A(M, -)$ preserves inj modules
 for M flat wiki: injective module

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

$$\begin{aligned} R\text{Hom}_A(M \otimes_A N, L) &= R\text{Hom}_A(P' \otimes_{\mathcal{C}(A)} N, L) \\ &= \text{Hom}_{\mathcal{C}(A)}(P' \otimes_{\mathcal{C}(A)} N, I') \\ &= \text{Hom}_{\mathcal{C}(A)}(N, \text{Hom}_{\mathcal{C}(A)}(P', I')) \\ &= R\text{Hom}_A(N, \text{Hom}_{\mathcal{C}(A)}(P', I')) \\ &= R\text{Hom}_A(N, R\text{Hom}_A(P', I')) \\ &= R\text{Hom}_A(N, R\text{Hom}_A(M, L)) \end{aligned}$$

$M' \xleftarrow{\cong} P'$ flat
 $L' \xrightarrow{\cong} I'$ inj
 adj in $\mathcal{C}(A)$
 $\text{Hom}_{\mathcal{C}(A)}(P', I')$ is inj
 I' is inj

□

⚠ We don't have

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, \text{Hom}_A(M, L))$$

Find a counterexample?

Take $N = A$, reduce to:

$$R\text{Hom}_A(M, L) \cong \text{Hom}_A(M, L)$$

then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between $\otimes\text{-Hom}$ & $f^* \dashv f_*$, derived version

Will write: verdier category

interpretation of \mathfrak{z} cohomology (gp + Lie alg + Hochschild)