Task 2. For $f \in C^\infty(\mathcal{U})$, define

$$\begin{cases} \dfrac{\partial f}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} - i\dfrac{\partial f}{\partial y}\right) \\ \dfrac{\partial f}{\partial \bar z} = \dfrac{1}{2}\left(\dfrac{\partial f}{\partial x} + i\dfrac{\partial f}{\partial y}\right) \end{cases} \quad (\ast)$$

(comes from the chain's rule)

(1) Verify that

$$\overline{\frac{\partial f}{\partial z}} = \frac{\partial \bar f}{\partial \bar z}$$

$$\frac{\partial(\alpha f + g)}{\partial z} = \alpha\frac{\partial f}{\partial z} + \frac{\partial g}{\partial z} \qquad \frac{\partial(\alpha f + g)}{\partial \bar z} = \alpha\frac{\partial f}{\partial \bar z} + \frac{\partial g}{\partial \bar z}$$

(2). Verify that

$$f \in \mathcal{O}(\mathcal{U}) \Leftrightarrow \frac{\partial f}{\partial \bar z} = 0$$

Hint. substitute the formula $(\ast)$

Recall the CR-equations.

	f	(u, v)
$(z, \bar z)$	$\dfrac{\partial f}{\partial \bar z} = 0$	$\dfrac{\partial u}{\partial \bar z} + i\dfrac{\partial v}{\partial \bar z} = 0$
(x, y)	$\dfrac{\partial f}{\partial x} = \dfrac{1}{i}\dfrac{\partial f}{\partial y}$	$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$, $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$
(r, θ)	$\dfrac{\partial f}{\partial r} = \dfrac{1}{ir}\dfrac{\partial f}{\partial \theta}$	$\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ $\dfrac{\partial v}{\partial r} = -\dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$

$$\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y}$$

$$\overset{\text{CR eq}}{=} \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= 0$$

Ex Verify that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

$$= 2\left(\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial \bar{z}^2}\right)$$

Task 1. $u(x,y) = x^2 + 2axy + by^2$

Determine all $a, b \in \mathbb{R}$, s.t

$\exists f \in \mathcal{O}(\mathbb{C})$, $\operatorname{Re} f = u$?

Hint: If $\operatorname{Re} f = u$, then $\Delta u = 2 + 2b = 0 \Rightarrow b = -1$.

If $b = -1$, we need to determine $f = u + iv$.

for some $c : \mathbb{R} \to \mathbb{R}$ $\cancel{\forall c \in \mathbb{R}}$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow v = 2xy + ay^2 + c(x)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2ax + 2by = -\left(2y + \frac{\partial c}{\partial x}\right)$$

$$\Rightarrow \begin{cases} c(x) = -ax^2 + C' \qquad C' \in \mathbb{R} \\ v = -ax^2 + 2xy + ay^2 + C' \\ f = u + iv \end{cases}$$

$$= x^2 + 2axy - y^2 + i(-ax^2 + 2xy + ay^2) + iC'$$

$$= (x+iy)^2 - ai(x+iy)^2 + iC'$$

$$= (1-ai)z^2 + iC'$$

Check that $f = (1-ai)z^2$ satisfy the requirement.

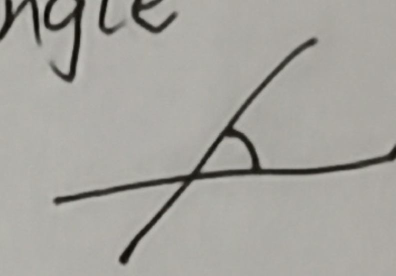
Task 3. Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

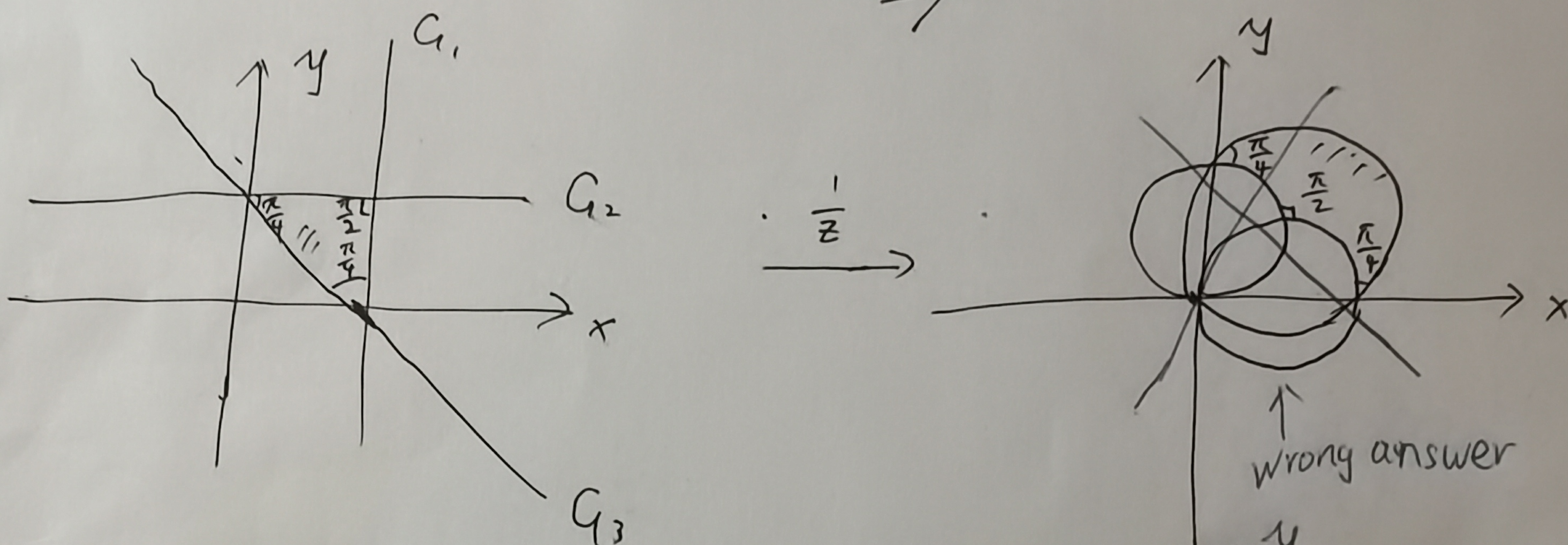
(1) Show that $f: \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is biholomorphic.
$z \longmapsto \frac{1}{z} = w$

Hint: By definition.

(2) die Gerade = straight line

der Schnittwinkel = cutting angle





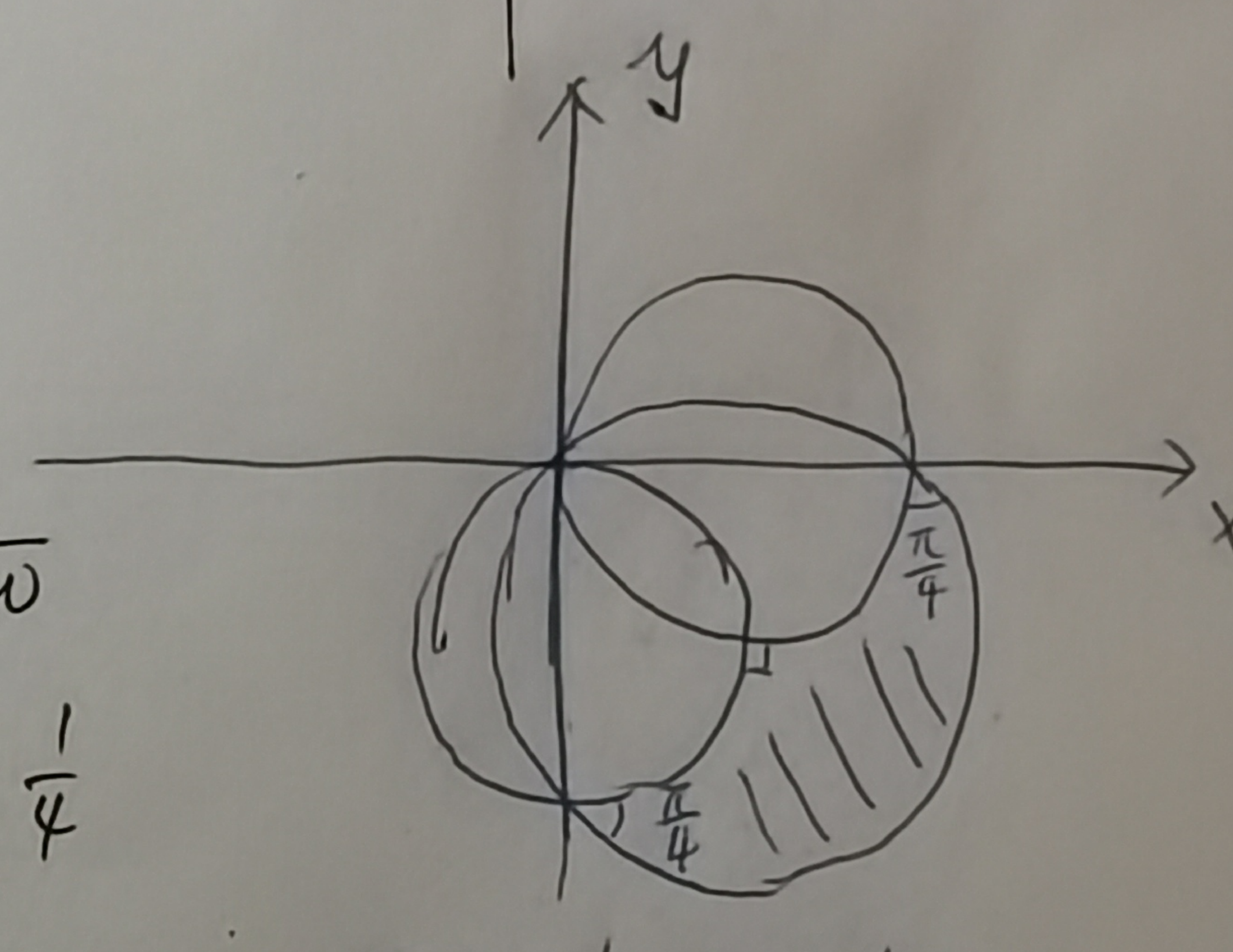
(3) $\operatorname{Re} z = 1 \Rightarrow \operatorname{Re} \frac{1}{w} = 1$

$$\Leftrightarrow \frac{1}{w} + \frac{1}{\bar{w}} = 2$$

$$\Leftrightarrow w + \bar{w} = 2w\bar{w}$$

$$\Leftrightarrow (w - \tfrac{1}{2})(\bar{w} - \tfrac{1}{2}) = \tfrac{1}{4}$$

$$\Leftrightarrow |w - \tfrac{1}{2}|^2 = \tfrac{1}{4}$$



correct answer
(check. $\frac{1}{i} = -i$)

Rmk. • linear fractional transformation

$$f(z) := \frac{az+b}{cz+d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C})$$

maps circle to circle.

More properties can be found in [Ahlfors, Chap 3.3]

• The holo fct is also called the conformal map, see [Ahlfors, Chap 3.2]

Task 4 schief = skew

gradient operator ∇

curl operator curl (rot)

divergence operator div

($U \subseteq \mathbb{R}^n$ open)

$$0 \to \Omega^0(U) \xrightarrow{\nabla} \Omega^1(U) \xrightarrow{\text{rot}} \Omega^2(U) \to \cdots \to \Omega^{n-1}(U) \xrightarrow{\text{div}} \Omega^n(U) \to 0$$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

$$d(f dx_I) = df \wedge dx_I = \sum \frac{\partial f}{\partial x_i} dx_i \wedge dx_I$$

For $f \in \underset{(C^\infty(U))^n}{\Omega^1(U)}$, define

$$\text{curl} f = \begin{pmatrix} (\text{curl} f)_{1,1} & \cdots & (\text{curl} f)_{1,n} \\ \vdots & & \vdots \\ (\text{curl} f)_{n,1} & \cdots & (\text{curl} f)_{n,n} \end{pmatrix} \in M_{n \times n}(C^\infty(U))$$

where $(\text{curl} f)_{i,j} = \partial_i f_j - \partial_j f_i = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$

$n=2$ $\quad \text{rot} f = \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \quad \in C^\infty(U)$

$n=3$ $\quad \text{rot} f = \nabla \times f$

$$= \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ f_1 & f_2 & f_3 \\ e_1 & e_2 & e_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) \in C^\infty(U)^{\oplus}$$

$V^{\otimes 2} = Sym^{2}(V) \oplus \Lambda^2 V$

Shows that

(1) $(curl\, f)^T = -curl\, f$

(2) $(d=2)$ $\quad rot\, f = \frac{1}{2}\left((curl\, f)_{12} - (curl\, f)_{21}\right)$

(3) $\exists\, w: \mathbb{R}^{3\times 3} \longrightarrow \Lambda^2 \mathbb{R}^3$ s.t

$$\begin{cases} rot\, f = w(curl(f)) \\ A v = w(A) \times v \end{cases} \qquad \forall A \in M_{3\times 3}(\mathbb{R}), v \in \mathbb{R}^3$$

(4) $\dim \Lambda^2 \mathbb{R}^d = \frac{d(d-1)}{2}.$