§ 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
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4. L-fct
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1. Galois rep

<u>Setting</u> G: arbitrary topo gp e.g. G any Galois gp
If G profinite $\Rightarrow$ open subgps are finite index subgps.
Λ: top field e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda, \mathrm{cont}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ cont

$\triangledown$ $\rho(G)$ can be infinite! for Gal gp
E.g. When char $F \neq l$, we have l-adic cyclotomic character

$$\varepsilon_l : \mathrm{Gal}(F^{sep}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \qquad \sigma \longmapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute ε_l for $F = \mathbb{F}_p$.
A: $\varepsilon_l : \hat{\mathbb{Z}} \cong \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \qquad 1 \longmapsto p$
lift from $\mathbb{Z} \longrightarrow \mathbb{Z}_l^\times$

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda, \mathrm{sm}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda, \mathrm{fi}}(G)$ fi: finite image / finite index
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with finite image

Rmk.

$$
\begin{tikzcd}
\mathrm{rep}_{\Lambda,\mathrm{sm}}(G) \arrow[r, equal] & \mathrm{rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) \arrow[rr] \arrow[dr, red, shift left] & & \mathrm{rep}_{\Lambda,\mathrm{cont}}(G) \arrow[dl, green] \\
& & \mathrm{rep}_{\Lambda,\mathrm{fi}}(G) \arrow[ul, blue, shift left] & \\
\mathrm{Rep}_{\Lambda,\mathrm{sm}}(G) \arrow[uu, phantom, sloped, "\subset"] & \mathrm{Rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) \arrow[l] \arrow[rr] \arrow[dr, red, shift left] \arrow[uu, phantom, sloped, "\subset"] & & \mathrm{Rep}_{\Lambda,\mathrm{cont}}(G) \arrow[uu, phantom, sloped, "\subset"] \\
& & \mathrm{Rep}_{\Lambda,\mathrm{fi}}(G) \arrow[ul, blue, shift left] \arrow[uu, phantom, sloped, "\subset"] &
\end{tikzcd}
$$

$\to$: if fin index subgps are open

$\to$: if G: profinite gp (Only need: open $\Rightarrow$ fin index)

$\to$: Artin rep (of profinite gp)

Artin rep. $\Lambda = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\quad \|\cdot\| = \|\cdot\|_{max}$, $|\cdot| = |\cdot|_{max}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists m \in \mathbb{N}$, s.t $A^m \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unipotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$. Take $m \in \mathbb{N}$ s.t $|\lambda^m - 1| > \frac{1}{3}$.

$\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},\mathrm{cont}}(G)$, $\rho(G)$ is finite. $\quad G$ profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq G_F$ finite index, $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = Id$

$\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp, so ρ is semisimple.
Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

Setting: Λ : NA local field with char $k_\Lambda = l$

Q: What would happen if Λ is only a NA local field?

Finite field

Task. For Λ: NA local field with char $k_\Lambda = l$, understand $\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Lambda)$, TFAE:

① $\hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is a well-defined cont gp homo
$\quad 1 \longmapsto A$

② $\exists\, g \in GL_n(\Lambda)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Lambda)$

③ $\lambda I - A \in \mathcal{O}_\Lambda[\lambda]$, with $\det A \in \mathcal{O}_\Lambda^\times$

A is called bounded in these cases.

Proof $\quad ① \underset{\text{local}}{\overset{\text{local}}{\rightleftarrows}} ② \underset{\text{local}}{\rightleftarrows} ③$

① ⇒ ②: $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Lambda)$, which conjugates to $GL_n(\mathcal{O}_\Lambda)$

https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl2-mathbb-q-p

Another method:

Lemma 1. ρ, μ: two ways of expressions of gp action

$\rho: \hat{\mathbb{Z}} \to GL_n(\mathbb{Z})$ is cont $\Rightarrow$ $\mu: \hat{\mathbb{Z}} \times \Lambda^n \to \Lambda^n$ is cont

[⇒: $\mu: \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\rho \times Id_{\Lambda^n}} GL_n(\Lambda) \times \Lambda^n \longrightarrow \Lambda^n$ is cont.
⇐ ? Is that true?]

Lemma 2. L_1, L_2 lattice in Λ^n ⇒ $L_1 + L_2$ lattice in Λ

$$\left[\begin{array}{l} L_1 \supseteq (p^{k})^{\oplus n} \\ L_2 \supseteq (p^{k})^{\oplus n} \end{array}\right\} \Rightarrow \#\, {}^{L_1+L_2}/_{L_1} < +\infty \Rightarrow L_1 + L_2 \text{ is a lattice} \Big]$$

Take $L := \mathcal{O}_\Lambda^n \subseteq \Lambda^n$, then the stabilizer

$$\begin{aligned} \mathrm{Stab}(L) &= \{g \in \hat{\mathbb{Z}} \mid g.L = L\} \\ &= \{g \in \hat{\mathbb{Z}} \mid g.e_i \in L \quad \forall i\} \\ &= \bigcap_i \mu_{e_i}^{-1}(L) \end{aligned}$$

is open, where

$\mu_{e_i}: \hat{\mathbb{Z}} \longrightarrow \Lambda^n \qquad g \longmapsto g.e_i \qquad$ (cont by Lemma 1)

$\Longrightarrow \mathcal{L}$ has finite orbit

$\overset{\text{Lemma 2}}{\Longrightarrow} \sum_{\mathcal{L}_i \in \hat{\mathbb{Z}} \cdot \mathcal{L}} \mathcal{L}_i$ is a lattice stabilized by $\mathbb{Z}$.

② ⇒ ①: w.l.o.g. $A \in GL_n(\mathcal{O}_\Lambda)$. Then we get a lift

$$\begin{tikzcd} \hat{\mathbb{Z}} \arrow[r, "\exists ! \text{ cont}"] & \widehat{GL_n(\mathcal{O}_\Lambda)} \cong GL_n(\mathcal{O}_\Lambda) \\ \mathbb{Z} \arrow[u] \arrow[r] & GL_n(\mathcal{O}_\Lambda) \arrow[u] \end{tikzcd}$$

② ⇒ ③: Obvious

③ ⇒ ②: $V = \Lambda^n$ can be viewed as a $\Lambda[T]$-module, and we have classifications of $\Lambda[T]$-module. The problem reduces to

$$\left[\begin{array}{l} \text{For } \Lambda[T]\text{-module } M := \bigoplus_i \Lambda[T]/(f_i(T)) \quad \text{with } \prod_i f_i(T) \in \mathcal{O}_\Lambda[T], \ \prod_i f_i(0) \in \mathcal{O}_\Lambda^\times, \\ \text{find a } \mathcal{O}_\Lambda[T]\text{-module } \underset{\operatorname{rank}_{\mathcal{O}_\Lambda} \mathcal{L} = n}{\mathcal{L} \subseteq M} \quad \text{s.t. } \mathcal{L} \otimes_{\mathcal{O}_\Lambda} \Lambda \cong M. \end{array}\right]$$