Eine Woche, ein Beispiel

3.23: Schubert calculus: Chern class over Grassmannian

This is a follow up of [2025.02.23], [2025.03.16].

1. Formulas for tautological bundle
2. Homology class in $Gr(r,n)$

1. Formulas for tautological bundle

Chern class realized as pullback of σ_{1^s}

Prop. For those v.b.s on $Gr(r,n)$, the Chern class is given by

$$
\begin{aligned}
c(S) &= 1 - \sigma_1 + \cdots + (-1)^r \sigma_{1^r} \\
c(Q) &= 1 + \sigma_1 + \cdots + \sigma_k + \cdots + \sigma_{n-r} \\
c(S^\vee) &= 1 + \sigma_1 + \cdots + \sigma_{1^r} \\
c(Q^\vee) &= 1 - \sigma_1 + \cdots + (-1)^k \sigma_k + \cdots + (-1)^{n-k} \sigma_{n-r}
\end{aligned}
$$

We omit the proof, as there are many equiv definition of Chern class, and I don't know which one to choose.

Cor If $f: X \longrightarrow Gr(r,n)$ is induced by $(\mathcal{F}, s_1, \ldots, s_n) = (\mathcal{O}_X^{\oplus n} \to \mathcal{F})$, then

$k^n \twoheadrightarrow \mathcal{F}|_p$

$(\mathcal{F}|_p)^* \hookrightarrow k^n$

$$
\begin{aligned}
c_s(\mathcal{F}) &= f^* c_s(S^\vee) \\
&= f^* \sigma_{1^s} \\
&= f^* \Sigma_{1^s}(V^{st}) \\
&= f^* \{ \Lambda \subset Gr(r,n) \mid \Lambda + V^{st}_{n-r+s-1} \subseteq H \} \\
&= \{ p \in X \mid (\mathcal{F}|_p)^* + \langle e_1^*, \ldots, e^*_{n-r+s-1} \rangle \subseteq k^{n-1} \} \\
&= \left\{ p \in X \;\middle|\; \begin{array}{l} \exists\ (0, \ldots, 0, k_{n-r+s}, \ldots, k_n) \in k^n - \{0\},\ \text{s.t.} \\ k_{n-r+s} S_{n-r+s}(p) + \cdots + k_n S_n(p) = 0 \end{array} \right\} \\
&= \{ p \in X \mid \underbrace{S_{n-r+s}(p), \ldots, S_n(p)}_{r-s+1 \text{ many}} \text{ are linear dependent} \}
\end{aligned}
$$

Especially,

$$
\begin{aligned}
c_r(\mathcal{F}) &= \{ p \in X \mid s_n(p) = 0 \} \\
&\vdots \\
c_1(\mathcal{F}) &= \{ p \in X \mid \underbrace{s_{n-r+1}(p), \ldots, s_n(p)}_{r \text{ many}} \text{ are linear dependent} \} \\
&= c_1(\Lambda^r \mathcal{F}) \\
&= c_1(\det \mathcal{F})
\end{aligned}
$$

Rmk. $c_s(\mathcal{F}) \neq c_{top}(\Lambda^{r-s+1}\mathcal{F})$ since $s_1 \wedge s_2$ (pure wedge) is not a general section in $\Lambda^2 \mathcal{F}$!

Nevertheless, when $s = 1$ or r, pure wedge is a general section, so
$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ $\quad c_r(\mathcal{F}) = c_r(\mathcal{F})$.

Porteous' formula

Thm [3264, Thm 12.4]

Let $X/\mathbb{C}$ sm $\quad k \in \mathbb{Z}_{\geq 0}$,

$\mathcal{E}, \mathcal{F}$: v.b. over X of rank e, f,

$\varphi: \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

$$M_k(\varphi) := \{ x \in X \mid \operatorname{rank} \varphi_x \leq k \}$$

remember multiplicity

$\varphi_x : \mathcal{E}|_x \longrightarrow \mathcal{F}|_x$

If $M_k(\varphi) \subset X$ has codim $(e-k)(f-k)$, then

$$[M_k(\varphi)] = \Delta^{e-k}_{f-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right] = (-1)^{(e-k)(f-k)} \Delta^{f-k}_{e-k}\left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]$$

where

$$\Delta^{e-k}_{f-k}(\gamma) = \begin{vmatrix} \gamma_{f-k} & \cdots & \gamma_{e+f-2k-1} \\ \vdots & \ddots & \vdots \\ \gamma_{f-e+1} & \cdots & \gamma_{f-k} \end{vmatrix}_{(e-k)\times(e-k)}$$

E.g. When $\mathcal{E}=\mathcal{O}_X$,

$$[X] = [M_1(\varphi)] = \Delta^{0}_{f-1}[c(\mathcal{F})] = \det 1 = 1$$
$$= \Delta^{f-1}_{0}\left[\frac{1}{c(\mathcal{F})}\right] = \begin{vmatrix} 1 & \cdots & \left[\frac{1}{c(\mathcal{F})}\right]_{f-2} \\ & \ddots & \vdots \\ 0 & & 1 \end{vmatrix} = 1$$

$$[V(s)] = [M_0(\varphi)] = \Delta^{1}_{f}[c(\mathcal{F})] = \det(c_f(\mathcal{F})) = c_f(\mathcal{F})$$
$$= -\Delta^{f}_{1}\left[\frac{1}{c(\mathcal{F})}\right] = -\begin{vmatrix} \left[\frac{1}{c(\mathcal{F})}\right]_1 & \cdots & \left[\frac{1}{c(\mathcal{F})}\right]_f \\ 1 & \ddots & \vdots \\ 0 & 1 & \left[\frac{1}{c(\mathcal{F})}\right]_1 \end{vmatrix} = c_f(\mathcal{F})$$

When $\mathcal{E}=\mathcal{O}_X^{\oplus e}$,

$$[X] = [M_e(\varphi)] = \Delta^{0}_{f-e}[c(\mathcal{F})] = 1$$
$$[M_{e-1}(\varphi)] = \Delta^{1}_{f-e+1}[c(\mathcal{F})] = c_{f-e+1}(\mathcal{F})$$
$$[M_{e-2}(\varphi)] = \Delta^{2}_{f-e+2}[c(\mathcal{F})] = \begin{vmatrix} c_{f-e+2}(\mathcal{F}) & c_{f-e+3}(\mathcal{F}) \\ c_{f-e+1}(\mathcal{F}) & c_{f-e+2}(\mathcal{F}) \end{vmatrix}$$
$$\vdots$$
$$[V(s_1,\dots,s_e)] = [M_0(\varphi)] = \Delta^{e}_{f}[c(\mathcal{F})] = \begin{vmatrix} c_f(\mathcal{F}) & \cdots & c_{f+e-1}(\mathcal{F}) \\ \vdots & \ddots & \vdots \\ c_{f-e+1}(\mathcal{F}) & \cdots & c_f(\mathcal{F}) \end{vmatrix}$$

Furthermore, when $X = Gr(r,n)$, $\mathcal{E}=\mathcal{O}_X^{\oplus e} = \mathcal{O}_X \otimes_k \mathcal{V}_{n-e}^{\perp}$ and $\mathcal{F}=S^\vee$, we get $f=r$, $c_k(\mathcal{F}) = \sigma_{1^k}$,

$$[M_k(\varphi)] = \Delta^{e-k}_{r-k}[c(\mathcal{F})]$$
$$= \begin{vmatrix} \sigma_{1^{r-k}} & \cdots & \sigma_{1^{e+r-2k-1}} \\ \vdots & \ddots & \vdots \\ \sigma_{1^{r-e+1}} & \cdots & \sigma_{1^{r-k}} \end{vmatrix}_{(e-k)\times(e-k)}$$
$$= \sigma_{(e-k)^{r-k}}$$

In fact, we know that $M_k(\varphi) = \Sigma_{(e-k)^{r-k}}(\mathcal{V})$, since

$$M_k(\varphi) = \{\Lambda \in Gr(r,n) \mid \varphi_\Lambda : \mathcal{V}^\perp \hookrightarrow (\mathbb{C}^n)^* \overset{\text{dual}}{\twoheadrightarrow} \Lambda^* \text{ is of rank} \le k\}$$
$$= \{\Lambda \in Gr(r,n) \mid \Lambda \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\mathcal{V} \text{ is of rank} \le k\}$$
$$= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-e} \ge r-k\}$$
$$= \Sigma_{(e-k)^{r-k}}(\mathcal{V})$$

2. Homology class in $Gr(r,n)$

Lines passing planes

E.g. 1. [3264, p131, Question (a)]

For 4 general lines l_1, l_2, l_3, l_4 in $\mathbb{P}^3$, there are 2 lines meet all four.

Reason: In $Gr(2,4)$,

$$\begin{aligned} &\# \{ l \in Gr(2,4) \mid l \cap l_i \neq \emptyset, \forall i \} \\ &= \deg \sigma_{\square}^4 \\ &= 2 \end{aligned}$$

E.g. 2. For 3 general lines l_1, l_2, l_3 in $\mathbb{P}^4$, there is 1 line meet all three.

Reason: In $Gr(2,5)$,

$$\begin{aligned} &\# \{ l \in Gr(2,5) \mid l \cap l_i \neq \emptyset, \forall i \} \\ &= \deg \sigma_{\square\square}^3 \\ &= 1 \end{aligned}$$

One can get further that no line in $\mathbb{P}^5$ passing 3 general lines.

E.g. 3.

For 6 general planes $e_1, \cdots, e_6$ in $\mathbb{P}^4$, there are 5 lines passing all these planes.

Reason: In $Gr(2,5)$,

$$\begin{aligned} &\# \{ l \in Gr(2,5) \mid l \cap e_i \neq \emptyset, \forall i \} \\ &= \deg \sigma_{\square}^6 \\ &= 5 \end{aligned}$$

E.g. 4. [3264, p131, Question (a)]

For 4 general $(k-1)$-planes $e_1, e_2, e_3, e_4 \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^{2k-1}$, there are k lines passing all these planes.

Reason: In $Gr(2,2k)$,

$$\begin{aligned} &\# \{ l \in Gr(2,2k) \mid l \cap e_i \neq \emptyset, \forall i \} \\ &= \deg \sigma^4_{\underbrace{\square\square\square\square}_{k-1}} \\ &= k \end{aligned}$$