

Eine Woche, ein Beispiel

5.11 genus of generalized Fermat curve

- Goal.
1. Find a basis of $H^{p,q}(X)$ by harmonic forms.
 2. Compute the geometric genus of curves

$$C: = \{y^n = x^m - 1\} \subseteq \mathbb{P}^2$$

Rmk: [2024.11.03] try to compute a special case in detail. In this document, more advanced methods are applied, so we don't need to blow up explicitly.
The reference also follows [2024.11.03].

Extra Ref:

Generalised Fermat equation: a survey of solved cases
<https://arxiv.org/abs/2412.11933>

Connection between Fermat curve and hyperelliptic curve:
<https://math.stackexchange.com/questions/3493593/transformation-which-takes-fermat-curve-x^n-y^n-1-to-a-hyperelliptic-curve>

1. Harmonic forms

- Affine plane curve
- Plane curve
- Fermat curve
- Hyperelliptic curve
- generalized Fermat curve
- \mathbb{P}^n
- Hypersurface

2. Riemann - Hurwitz

3. Milnor formula

1. Harmonic forms

Almost all the results in this section come from the answer here:

<https://mathoverflow.net/questions/324812/the-construction-of-a-basis-of-holomorphic-differential-1-forms-for-a-given-plan>

Affine plane curve

Prop. Suppose $C = \{f(x,y) = 0\} \subseteq \mathbb{A}^2$ is a sm curve, then

$$\omega \hat{=} \frac{dx}{f_2(x,y)} = -\frac{dy}{f_1(x,y)}$$

is a global generator of $H^0(C, \Omega')$.

i.e., $\forall \omega' \in H^0(C, \Omega')$, $\omega' = f\omega$ for some $f \in \mathcal{O}_{\text{hol}}(C)$.

Proof. Notice that

$$f_1(x,y)dx + f_2(x,y)dy = 0.$$

When $f_1(x_0, y_0) \neq 0$,

$y: C \rightarrow \mathbb{A}^1$ is a local chart,
 $(x,y) \mapsto y$

$\Rightarrow dy$ is a global generator near (x_0, y_0) .

$\Rightarrow \frac{dy}{f_1(x,y)}$ is a global generator near (x_0, y_0) .

When $f_2(x_0, y_0) \neq 0$,

$x: C \rightarrow \mathbb{A}^1$ is a local chart,
 $(x,y) \mapsto x$

$\Rightarrow dx$ is a global generator near (x_0, y_0) .

$\Rightarrow -\frac{dx}{f_2(x,y)}$ is a global generator near (x_0, y_0) .

Plane curve

Prop. Suppose $C = \{F(x, y, z) = 0\} \subseteq \mathbb{P}^2$ is a sm curve of deg d ,
then $H^0(C, \Omega')$ has a basis

$$\left\{ x^i y^j \frac{dx}{F_2(x, y, 1)} \mid i+j \leq d-3 \right\}$$

Proof Assume $[x:y:1] = [a:b:c]$, i.e., $\begin{cases} x = \frac{a}{c} \\ y = \frac{b}{c} \end{cases}$, then

$$\begin{cases} dx = \frac{cda - adc}{c^2} \\ F_2(x, y, 1) = \frac{1}{c^{d-1}} F_2(a, b, c) \end{cases}$$

Therefore,

$$\begin{aligned} x^i y^j \frac{dx}{F_2(x, y, 1)} &= a^i b^j c^{d-i-j-3} \frac{cda - adc}{F_2(a, b, c)} \\ &= \begin{cases} -b^j c^{d-i-j-3} \frac{dc}{F_2(1, b, c)} & a \equiv 1 \\ -a^i c^{d-i-j-3} w & b \equiv 1 \end{cases} \end{aligned}$$

When $b \equiv 1$, denote

$$w \triangleq \frac{da}{F_3(a, 1, c)} = - \frac{dc}{F_1(a, 1, c)}$$

Since $x F_1(x, y, z) + y F_2(x, y, z) + z F_3(x, y, z) \equiv 0$, we get

$$\begin{aligned} cda - adc &= (c F_3(a, 1, c) + a F_1(a, 1, c)) w \\ &= -F_2(a, 1, c) w \end{aligned}$$

Cor. For the Fermat curve

$$C_d: x^d + y^d = z^d,$$

$$H^0(C, \Omega') = \langle x^i y^j \frac{dx}{dy^{d-1}} \mid i+j \leq d \rangle \cong \mathbb{C}^{\frac{(d-1)(d-2)}{2}}$$