

Eine Woche, ein Beispiel

5.28. dual spaces of ∞ -dim v.s.

Ref: <http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-FA/index.html>

$\mathbb{F} = \mathbb{R}$ or \mathbb{C} . What would happen if $\mathbb{F} = \mathbb{C}_p$?

1. def
2. examples

1. def

Def. For any topo v.s. X, Y , define
 $\mathcal{L}(X, Y) := \{L: X \rightarrow Y \mid L \text{ is linear and cont}\}$

The dual space of X is defined as

$$X' := \mathcal{L}(X, \mathbb{F}) = \{L: X \rightarrow \mathbb{F} \mid L \text{ is linear and cont}\}$$

⚠ We follow the notation of analysis in this document.

Other possibilities for the dual space: $X^*, X^\vee, \check{X}, \dots$

Rmk. When X, Y are normed v.s., $\mathcal{L}(X, Y)$ is a normed v.s. with

$$\|L\| = \sup_{\|x\|_X = 1} \|L(x)\|_Y$$

On the other hand, we have the weak $*$ -topology on $\mathcal{L}(X, Y)$:
the weakest topo s.t.

$$\text{ev}_x: \mathcal{L}(X, Y) \longrightarrow Y \quad L \longmapsto L(x)$$

is cont for any $x \in X$.

These two structures are not compatible with each other.

Rmk. By Klein-Milman theorem, we can show that
some Banach spaces are not dual space.

2. initial examples.

For a bounded domain Ω , we have

$$\begin{array}{ccccccc} L^\infty(\Omega) & \subset & \dots & \subset & L^p(\Omega) & \subset & \dots & \subset & L^1(\Omega) \\ & & & & \Downarrow \text{dual} & & & & \\ (L^\infty(\Omega))' & \supset & \dots & \supset & L^q(\Omega) & \supset & \dots & \supset & L^\infty(\Omega) \end{array}$$

For arbitrary domain Ω , we don't have inclusion.

inclusion: cont inj map

<https://math.stackexchange.com/questions/405357/when-exactly-is-the-dual-of-l1-isomorphic-to-l-infty-via-the-natural-map>
<https://math.stackexchange.com/questions/137677/what-is-the-predual-of-l1>

Ex. Show that $(C_0)' = l^1$, $(l^p)' = l^q$, $(l^1)' = l^\infty$ by direct argument.
 Show that $(l^\infty)' \not\cong l^1$.

$$\begin{array}{ccccccc} C_0 & \xhookrightarrow{\text{not dense}} & l^\infty & & l^p & & l^1 \\ & & & \Downarrow \text{dual} & & & \\ l^1 & \longleftarrow & (l^\infty)' & & l^q & & l^\infty \end{array}$$

For $\Omega = \mathbb{R}^n$, we have $\mathcal{S}(\Omega)$ is not defined for $\Omega \stackrel{\text{open}}{\subset} \mathbb{R}^n$, traditionally

$$\begin{array}{ccccc} \mathcal{D}(\Omega) & \subset & \mathcal{S}(\Omega) & \subset & \mathcal{E}(\Omega) \\ & & \Downarrow \text{dual} & & \\ \mathcal{D}'(\Omega) & \supset & \mathcal{S}'(\Omega) & \supset & \mathcal{E}'(\Omega) \end{array}$$

<https://math.stackexchange.com/questions/4730104/is-schwartz-space-canonical-in-any-sense>
 Schwartz Functions on Open Subsets of \mathbb{R}^n : <https://www.math.princeton.edu/events/schwartz-functions-open-subsets-rn-2022-02-28t213000>
 Schwartz functions on real algebraic varieties: <https://arxiv.org/abs/1701.07334>

Rmk. For Hilbert space, $H' \cong H$. e.p. $(H^s(\Omega))' \cong H^s(\Omega)$

For X : cpt Hausdorff space,

$$C(X)' \subset \{\text{signed regular Borel measures}\}$$

⚠ The following illusion is common and confusing:

The dual space of bigger space is bigger/smaller.

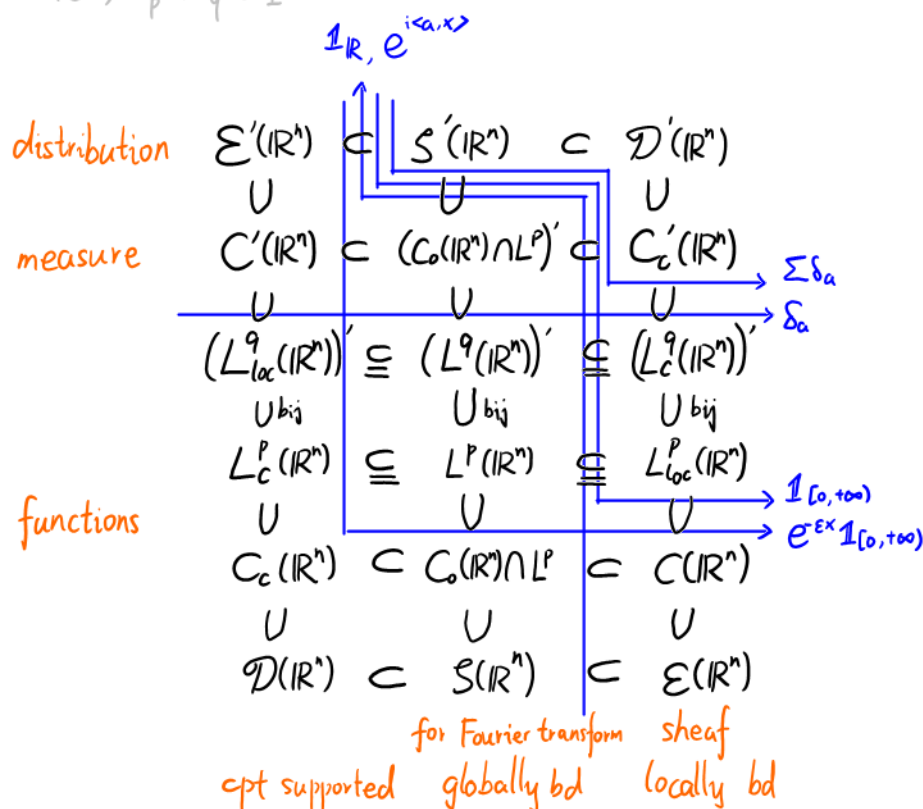
Actually, such illusions comes from $f^*: W^* \rightarrow V^*$ being injective/surjective.

In fin dim case, $\dim V^* = \dim V < \dim W = \dim W^*$;

In dense subspace case, it comes from the uniqueness of cont extension.

3. an overview of spaces in analysis

$$1 < p < +\infty, \frac{1}{p} + \frac{1}{q} = 1$$



seminorm of the last line:

$$p_{\alpha, \beta}(f) = \|x^{\beta} \partial^{\alpha} f\|_{\infty}$$

$$p_{\alpha, N}(f) = \|(1+|x|)^N \partial^{\alpha} f\|_{\infty}$$

$$\lim_k \mathcal{D}_k(\mathbb{R}^n)$$

$$p_{\alpha, k}(f) = \sup_{x \in k} \|\partial^{\alpha} f\|_{\infty}$$

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measure line: $C'(\mathbb{R}^n)$: fcts of bounded variation
 $C_0'(\mathbb{R}^n)$: signed regular Borel measures on \mathbb{R}^n .
 $(C_c'(\mathbb{R}^n))^+$: Radon measure
 Q: is $C_c'(\mathbb{R}^n)$ the signed Radon measure?

The above diagram has many variations. For example,

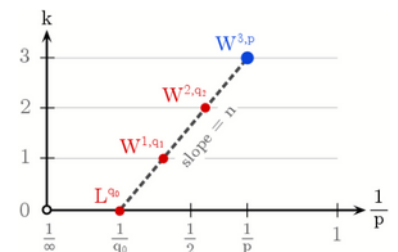
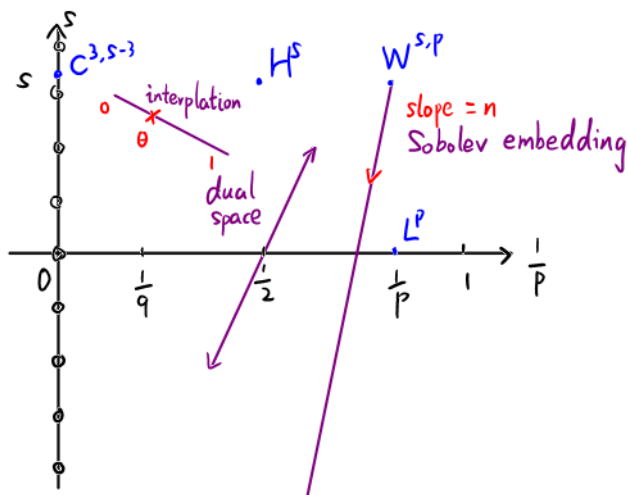
$$\begin{array}{ccccc}
 \mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
 \cup & & \cup & & \cup \\
 \mathcal{C}'(\mathbb{R}^n) & \subset & \mathcal{C}_0'(\mathbb{R}^n) & \subset & \mathcal{C}_c'(\mathbb{R}^n) \\
 \uparrow & & \uparrow & & \uparrow \\
 (L_{loc}^1(\mathbb{R}^n))' & = & (L^1(\mathbb{R}^n))' & = & (L_c^1(\mathbb{R}^n))' \\
 \uparrow \text{bij?} & & \cup \text{bij} & & \uparrow \text{bij?} \\
 L_c^\infty(\mathbb{R}^n) & \subset & L^\infty(\mathbb{R}^n) & \subset & L_{loc}^\infty(\mathbb{R}^n) \\
 \uparrow & & \uparrow & & \uparrow \\
 \mathcal{C}_c(\mathbb{R}^n) & \subset & \mathcal{C}_0(\mathbb{R}^n) & \subset & \mathcal{C}(\mathbb{R}^n) \\
 \cup & & \cup & & \cup \\
 \mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
 \end{array}$$

$$\begin{array}{ccccc}
 \mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
 \cup & & \cup & & \cup \\
 H_{loc}^{-s}(\mathbb{R}^n) & \subset & H^{-s}(\mathbb{R}^n) & \subset & H_c^{-s}(\mathbb{R}^n) \\
 \cup \text{bij?} & & \cup \text{bij} & & \cup \text{bij?} \\
 H_c^s(\mathbb{R}^n) & \subset & H^s(\mathbb{R}^n) & \subset & H_{loc}^s(\mathbb{R}^n) \\
 \cup & & \cup & & \cup \\
 \mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
 \end{array}$$

<https://math.stackexchange.com/questions/221069/why-are-continuous-functions-not-dense-in-l-infty>

In fact, in the middle, we can change various of Sobolev spaces.

https://en.wikipedia.org/wiki/Sobolev_inequality
https://en.wikipedia.org/wiki/Sobolev_space
https://arxiv.org/PS_cache/arxiv/pdf/1104/1104.4345v2.pdf



$C^{r,\alpha}(\mathbb{R}^n)$: Hölder spaces

We mainly care about:

1. Element - special element e.g. δ
 - (singular) support

2. Set - as a set, topo sp
 - best structure? e.g. Fréchet?
 - seq convergence

3. Map - criterion of seminorm / lin fct / map to be cont

- \rightarrow : cont? inj?
 - \subset : dense? (Use regularization/truncate)

<https://math.stackexchange.com/questions/1802755/can-you-recover-a-distribution-from-mollification>

- \subseteq : $f: X \hookrightarrow Y$ if $\text{Im } f$ & X have the same topo (topo embedding)

- cpt operators?

- Intersection compatible with $\uparrow \downarrow$? i.e. pullback squares?

4. More structures (add extra dimensions on the diagram)

- differential

- $\Omega \subset \subset \Omega'$ sheaf?

- Fourier transform

- $\Omega \times \Omega'$ Schwarz kernel

} \Rightarrow integral operators
 \Rightarrow FM transform

Rmk. For $f: X \rightarrow Y$ a cont injective map between Fréchet spaces.

f is a topo embedding \Leftrightarrow every cont seminorm on X can be extended to a cont seminorm on Y .

Q: Can we generalize the field from \mathbb{R} or \mathbb{C} to $K = \mathbb{C}_p = \widehat{\mathbb{Q}_p}$?