

Local Langlands Correspondence for GL_n

As modifying files in the sciebo folder is prohibited, the corrected version of my portion (with the typo rectified) will be available in the Github directories:

Talk1:

[https://github.com/ramified/personal_handwritten_collection/raw/main/weeklyupdate/2023.04.23_\(non-split\)_reductive_group.pdf](https://github.com/ramified/personal_handwritten_collection/raw/main/weeklyupdate/2023.04.23_(non-split)_reductive_group.pdf)

Talk2(this one):

https://github.com/ramified/personal_handwritten_collection/raw/main/Langlands/GL_case.pdf

F : local field NA local field + \mathbb{R} & \mathbb{C} case

$$\Gamma_F = \text{Gal}(F^{\text{sep}}/F)$$

W_F = Weil group of F

$$\text{NA case: } W_F = \Gamma_F \times_{\mathbb{Z}} \mathbb{Z}$$

$$\mathbb{C} \text{ case: } W_C = \mathbb{C}^\times$$

$$\mathbb{R} \text{ case: } W_{\mathbb{R}} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times \subseteq \mathbb{H}^\times$$

$\text{Rep} = \begin{matrix} \text{sm} \\ \text{irr} \end{matrix} \xrightarrow{\text{rep}}$ For Archimedean case, we only have continuous condition.

$\text{Irr} = \begin{matrix} \text{irr} \\ \text{sm} \end{matrix} \text{ rep}$

$\Pi = \begin{matrix} \text{adm} \\ \text{irr} \end{matrix} \text{ sm rep} \xrightarrow{\text{p-adic}} \text{Irr}_{\mathbb{C}}$

$WD\text{rep} = \text{Weil-Deligne rep}$

1. $GL_n(F)$ for F NA local
2. $GL_n(F)$ for $F = \mathbb{C}$ or \mathbb{R}
3. G nonsplit torus over \mathbb{R}
4. unramified case
5. injectivity

1. $GL_n(F)$ for F NA local

Let us first state the GL_n case for a NA local field F .

Thm (LLC for $GL_n(F)$, Harris-Taylor, Henniart, Scholze)

We have a natural bijection (called the reciprocity map)

$$\text{rec}_{F, GL_n} : \text{Irr}_\mathbb{C}(GL_n(F)) \longleftrightarrow WD_{\substack{n-\dim \\ \text{Frob ss}}}(\mathcal{W}_F)$$

||

$$\left\{ \begin{array}{l} \phi : \mathcal{W}_F \longrightarrow GL_n(\mathbb{C}) \quad \phi(\text{Frob}) \text{ s.s.} \\ + N \in \text{End}(\mathbb{C}^n) \\ + \text{compatibility} \end{array} \right\}$$

$n=1$

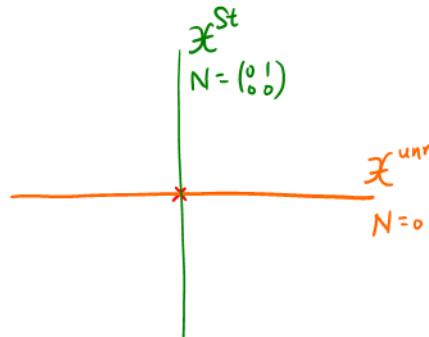
$$\chi : F^\times \longrightarrow \mathbb{C}^\times \iff \chi : \mathcal{W}_F \xrightarrow{\text{ab}} \mathcal{W}_F^\text{ab} \cong F^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$n=2$

1)	$\chi \circ \det$	\iff	$\left(\begin{pmatrix} \chi \cdot 1 & \mathbb{I}_F^{-\frac{1}{2}} \\ \chi \cdot 1 & \mathbb{I}_F^{\frac{1}{2}} \end{pmatrix}, \circ \right)$	*
2)	$n\text{-Ind}_B^{GL_2}(\chi_1, \chi_2)$	\iff	$\left(\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}, \circ \right)$	—
3)	$St \otimes (\chi \circ \det)$	\iff	$\left(\begin{pmatrix} \chi \cdot 1 & \mathbb{I}_F^{-\frac{1}{2}} \\ \chi \cdot 1 & \mathbb{I}_F^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$	
4)	$C\text{-Ind}_{K \times \mathbb{Z}}^{GL_2} \rho$	\iff	don't know how to describe	

Rmk.

action by χ	\iff	$- \otimes (\chi \circ \det)$	\iff	$- \otimes \chi$
unramified	\iff	$V^k \neq \{0\}$	\iff	$\phi _{I_F} = 1_{I_F}, N=0$
	\iff	disc series	\iff	indec rep
	\iff	cuspidal	\iff	irr rep
" n -Ind"	\iff		\oplus	



Let us try to work out $n=1$ case. In that case,

$$\begin{aligned} \text{RHS} &= \left\{ \phi: W_F \rightarrow \mathbb{C}^{\times} \right\} \\ &= \left\{ \phi: W_F^{ab} \rightarrow \mathbb{C}^{\times} \right\} \\ \xrightarrow{\text{Artin}} \left\{ \rho: F^{\times} \rightarrow \mathbb{C}^{\times} \right\} &= \text{LHS} \end{aligned}$$

Rem. The key argument is the Artin map

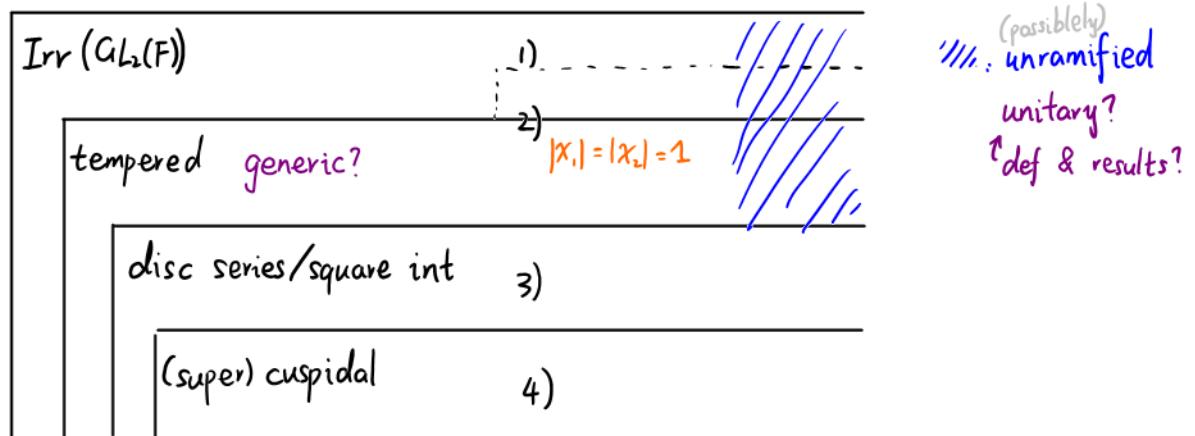
$$W_F^{ab} \cong F^{\times}$$

For $n=2$ case, we still have nice descriptions on both sides. However, it would already take the content of a whole book for us to comprehend the details of this case.

Thm (Langlands classification for $\text{Irr}_{\mathbb{C}}(GL_2(F))$)

$X \mapsto X _{F^{\times}}^{\frac{1}{2}}$	We have a classification of $\text{Irr}_{\mathbb{C}}(GL_2(F))$.	$\chi: K^{\times} \rightarrow \mathbb{C}$
1) 1-dim	$\chi \circ \det$	
2) principal series	$n\text{-Ind}_B^{GL_2}(X_1, X_2)$	$X_1 X_2^{-1} \neq 1, 1^{\pm 1}$
3) a twist of St by χ	$St \otimes (\chi \circ \det)$	
4) supercuspidal rep	$c\text{-Ind}_{KZ}^{GL_2} \rho$	for some $\rho \in \text{Irr}_{\mathbb{C}}(KZ)$

in [Gee]



For $\mathbb{F}_p(t)$ -case, see wiki: Drinfeld module.

$$\text{Rmk. } c\text{-Ind}_U^{G(k)} \rho \in \text{Irr}(G) \rightsquigarrow c\text{-Ind}_U^{G(k)} \rho \text{ sc}$$

ref: Prof. Can's lecture note in Lec 9, p74

Conj. all sc reps arise in this way (true for GL_n)

$$\tilde{\rho} \in \text{Rep}(GL_n(\mathcal{O}_F) \cdot F^\times) \text{ s.t. } \rho \in \text{Cusp}(GL_n(\mathbb{A})) \rightsquigarrow c\text{-Ind}_{GL_n(\mathcal{O}_F) \cdot F^\times}^{GL_n(F)} \rho \in \text{Cusp}(GL_n(F))$$

$\tilde{\rho}|_{1 + \mathfrak{m} M_n(\mathcal{O}_F)} = \text{Id}$

[Thm 1.7] <https://nms.kcl.ac.uk/james.newton/week4.pdf>

<https://mathoverflow.net/questions/101067/are-all-irreducible-supercuspidal-representation-induced-from-compact-mod-center>
<https://math.stackexchange.com/questions/821417/is-supercuspidal-representation-the-same-as-cuspidal-representation>

The Steinberg representation is irreducible:
<https://arxiv.org/pdf/2107.00794.pdf>

2. $GL_n(F)$ for $F = \mathbb{C}$ or \mathbb{R}

For the Archimedean case, we also want to construct such a correspondence. In this case, we have a relatively explicit description on both sides, since the structure of the Weyl gp is easier. Also, we don't need to worry about cuspidal reps here.

For avoiding technical conditions, we only state the LLC for $GL_n(F)$.

$F = \mathbb{R}$ or \mathbb{C} .

Thm (LLC for $GL_n(F)$)

We have a 1-to-1 correspondence

$$\begin{array}{ccc} \pi(GL_n(F))/\sim & \longleftrightarrow & \Phi(GL_{n,F})/\sim \\ \downarrow \cong & & \parallel \text{def} \\ \{\text{Irr adm } (gl_n, K) \text{-modules}\} & & \left\{ \begin{array}{l} \phi: W_F \rightarrow GL_n(\mathbb{C}) \\ \text{semisimple as reps} \end{array} \right\} / GL_n(\mathbb{C})\text{-conj} \end{array}$$

where

$K := O(n)$ or $U(n)$

\sim : up to infinitesimally equivalence
i.e. induce the same (gl_n, K) -modules

For letting $n=1$ case to be true, we have to ask at least
 $W_F^{ab} \cong F^\times$

Also, W_K should be related to Γ_F .

Def (Weil gp for $F = \mathbb{R}, \mathbb{C}$)

$$W_{\mathbb{C}} := \mathbb{C}^\times$$

$$W_{\mathbb{R}} := \mathbb{C}^\times \sqcup_j \mathbb{C}^\times \subset \mathbb{H}^\times$$

$$\text{Ex. } 1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}} \longrightarrow 1$$

$$j^2 = -1 \quad jzj^{-1} = \bar{z} \quad \forall z \in \mathbb{C}^\times$$

$$\Rightarrow \frac{\bar{z}}{z} = jzj^{-1} \in [W_{\mathbb{R}}, W_{\mathbb{R}}]$$

$$\Rightarrow [W_{\mathbb{R}}, W_{\mathbb{R}}] = S'$$

$$\Rightarrow W_{\mathbb{R}}^{ab} \cong (\mathbb{C}^\times \sqcup_j \mathbb{C}^\times)/S' \cong \mathbb{R}_{>0} \sqcup_j \mathbb{R}_{>0} \cong \mathbb{R}^\times$$

By this iso ($W_F^{ab} \cong F^\times$), we have shown the LLC for $n=1$ case abstractly.
To understand more, we must discuss this case in more detail.

$GL_n(F)$	\mathbb{R}	\mathbb{C}
$n=1$	$\mathbb{C} \times \{\pm 1\}$ $i\mathbb{R} \times \{\pm 1\}$	$\mathbb{C} \times \mathbb{Z}$ $i\mathbb{R} \times \mathbb{Z}$
$n=2$	$\mathbb{C} \times N_{>0}$ $i\mathbb{R} \times N_{>0}$...
$n > 2$	\emptyset	\emptyset

... written as direct sum of lower dim reps.
orange: unitary representations. for L-parameters side.

E.g. $n=1, F=\mathbb{R}$

$$\left\{ \rho: \mathbb{R}^\times \rightarrow \mathbb{C}^\times \right\} \cong \mathbb{C} \times \{\pm 1\}$$

$\mathbb{R}_{>0}^{\text{ns}} \times \{\pm 1\}$

$$\begin{aligned} x &\mapsto x^t \\ -1 &\mapsto \pm 1 \end{aligned} \quad \rightsquigarrow \begin{cases} \chi_{\text{triv}} \otimes 1 \cdot | \cdot |^t \\ \chi_{\text{sign}} \otimes 1 \cdot | \cdot |^t \end{cases}$$

The characters of W_R are given by

$$W_R \longrightarrow \mathbb{C}^\times$$

\downarrow

$$\mathbb{R}^\times \cong W_R^{\text{ab}}$$

$\exists!$

$$\begin{array}{ccc} z & \xrightarrow{\quad} & |z|^t \\ \downarrow & & \swarrow \\ |z| & & \end{array} \quad \begin{array}{ccc} i & \xrightarrow{\quad} & \pm 1 \\ \downarrow & & \swarrow \\ -1 & & \end{array}$$

e.p. the unitary reps are parameterized by $i\mathbb{R} \times \{\pm 1\}$.

E.g. $n=1, F=\mathbb{C}$

$$\left\{ \rho: \mathbb{C}^\times \rightarrow \mathbb{C}^\times \right\} \cong \mathbb{C} \times \mathbb{Z}$$

$\mathbb{R}_{>0}^{\text{ns}} \times S^1$

$$z = r e^{i\theta} \mapsto r^t e^{i\theta}$$

$$z^{\frac{\mu}{2}} \bar{z}^{\frac{\nu}{2}}$$

↓ reparameterization

$$z \mapsto z^\mu \bar{z}^\nu \quad \{(\mu, \nu) \in \mathbb{C} \times \mathbb{C} \mid \mu - \nu \in \mathbb{Z}\}$$

e.p. the unitary reps are parameterized by $i\mathbb{R} \times \mathbb{Z}$.

E.g. $n=2$, $F=\mathbb{R}$

$$\begin{aligned} \{\phi: W_{\mathbb{R}} &\longrightarrow GL_2(\mathbb{C})\} / \sim \\ z &\longmapsto \begin{pmatrix} z^\mu \bar{z}^\nu & \\ & z^{\mu'} \bar{z}^{\nu'} \end{pmatrix} \end{aligned}$$

①. $\phi = \chi_1 \oplus \chi_2$ $\dim \chi_i = 1$

\rightsquigarrow subquotient of n -Ind $_{\mathbb{R}}^{\mathbb{C}}(\chi_1, \chi_2)$
quotient, when $\operatorname{Re} t_1 \geq \operatorname{Re} t_2$
FD & principal series
finite dim reps.

②. ϕ irreducible.

By linear algebra arguments, i.e. choose a good basis

$$\begin{aligned} \{\phi: W_{\mathbb{R}} &\longrightarrow GL_2(\mathbb{C}) \text{ irr}\} / \sim \cong \mathbb{C} \times \mathbb{N}_{>0} \\ z &\longmapsto \begin{pmatrix} z^\mu \bar{z}^\nu & \\ & z^\nu \bar{z}^\mu \end{pmatrix} \\ j &\longmapsto \begin{pmatrix} & (-1)^{\mu-\nu} \\ 1 & \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \rightsquigarrow DS_1 \otimes |\det(-)|_{\mathbb{R}}^t \\ \downarrow \quad \downarrow \\ \text{discrete series} \quad \chi_{\det} \end{aligned}$$

Rem. In Prof. Caraiani's course, we did the classification of
irr adm $(\mathfrak{gl}_{2,\mathbb{R}}, O(2))$ -modules.
We reproduce it by the LLC!

Details about linear algebras should be put in this page.

Ref here: [Knapp 91, Sec 3]: <https://www.math.stonybrook.edu/~aknapp/pdf-files/motives.pdf>

Step 1. Analyze $\phi|_{\mathbb{C}^\times}$

$$\left. \begin{array}{l} \phi(z) \text{ is diagonalizable} \\ \mathbb{C}^\times \text{ is commutative} \end{array} \right\} \Rightarrow \phi|_{\mathbb{C}^\times} \cong \chi_1 \oplus \chi_2$$

i.e. under some basis $\{u, v\}$,

$$\phi: z \mapsto \begin{pmatrix} z^\mu \bar{z}^\gamma & \\ & z^{\mu'} \bar{z}^{\gamma'} \end{pmatrix} \quad \begin{aligned} \phi(z) \cdot u &= z^\mu \bar{z}^\gamma u \\ \phi(z) \cdot v &= z^{\mu'} \bar{z}^{\gamma'} v \end{aligned}$$

Step 2. Remove decomposable cases:

When $\mu = \mu'$, $\gamma = \gamma'$: (same eigenvalues)

$$\left. \begin{array}{l} \phi(j) \text{ is diagonalizable} \\ \phi(\mathbb{C}^\times) \subset Z(GL_2(\mathbb{C})) \end{array} \right\} \Rightarrow \phi \cong \chi_1 \oplus \chi_2$$

Assume $\mu \neq \mu'$ or $\gamma \neq \gamma'$ now.

$$\begin{aligned} \phi(z) \phi(j) u &= \phi(j) \phi(z) u = z^\gamma \bar{z}^\mu \phi(j) u \\ \Rightarrow \phi(j) u &\text{ is an eigenvector with eigenvalue } z^\gamma \bar{z}^\mu \end{aligned}$$

When $\mu = \gamma$, then

$\mathbb{C}u$ is irr subrep $\Rightarrow \phi \cong \chi_1 \oplus \chi_2$;

When $\mu \neq \gamma$, then $\mu' = \gamma$, $\gamma' = \mu$.

under the basis $\{u, \phi(j)u\}$,

$$\begin{aligned} \phi: z &\mapsto \begin{pmatrix} z^\mu \bar{z}^\gamma & \\ & z^\gamma \bar{z}^\mu \end{pmatrix} \\ j &\mapsto \begin{pmatrix} 1 & ? \\ 1 & ? \end{pmatrix} \xrightarrow{j^2 = -1} \begin{pmatrix} 1 & (-1)^{\mu-\gamma} \\ 1 & \end{pmatrix} \end{aligned}$$

Step 3. By the symmetry, we can assume that $\mu - \gamma > 0$.

under the basis $\{\phi(j)u, (-1)^{\mu-\gamma} u\}$,

$$\begin{aligned} \phi: z &\mapsto \begin{pmatrix} z^\gamma \bar{z}^\mu & \\ & z^\mu \bar{z}^\gamma \end{pmatrix} \\ j &\mapsto \begin{pmatrix} & (-1)^{\gamma-\mu} \\ 1 & \end{pmatrix} \end{aligned}$$

$$\left. \begin{aligned} (\det \phi)(z) &= |z|^{\mu+\gamma} \\ (\det \phi)(j) &= (-1)^{\mu-\gamma+1} \\ \phi|_{\mathbb{C}^\times} &\cong \chi_{\mu, \gamma} \oplus \chi_{\gamma, \mu} \end{aligned} \right\} \Rightarrow \begin{aligned} \left(\frac{\mu+\gamma}{2}, |\mu-\gamma| \right) &\in \mathbb{C} \times \mathbb{N}_0 \text{ are determined} \\ &\Rightarrow \text{by the rep } \phi. \end{aligned}$$

□

Rmk. By the similar linear algebra argument, one can show

$$\phi \in \text{Irr}_{\mathbb{C}}(W_{\mathbb{R}}) \rightsquigarrow \dim_{\mathbb{C}} \phi = 1 \text{ or } 2$$

$$\phi \in \text{Irr}_{\mathbb{C}}(W_{\mathbb{C}}) \rightsquigarrow \dim_{\mathbb{C}} \phi = 1$$

By the correspondence, we get classifications of $GL_n(F)$ -reps explicitly.

[Knapp91, p400]: <https://www.math.stonybrook.edu/~aknapp/pdf-files/motives.pdf>

Theorem 1. For $G = GL_n(\mathbb{R})$,

- (a) if the parameters $n_j^{-1}t_j$ of $(\sigma_1, \dots, \sigma_r)$ satisfy

$$n\text{-Ind}_P^{GL_n}(\sigma_1, \dots, \sigma_r) \quad n_1^{-1} \operatorname{Re} t_1 \geq n_2^{-1} \operatorname{Re} t_2 \geq \dots \geq n_r^{-1} \operatorname{Re} t_r, \quad (2.5)$$

then $I(\sigma_1, \dots, \sigma_r)$ has a unique irreducible quotient $J(\sigma_1, \dots, \sigma_r)$,

(b) the representations $J(\sigma_1, \dots, \sigma_r)$ exhaust the irreducible admissible representations of G , up to infinitesimal equivalence,

(c) two such representations $J(\sigma_1, \dots, \sigma_r)$ and $J(\sigma'_1, \dots, \sigma'_r)$ are infinitesimally equivalent if and only if $r' = r$ and there exists a permutation $j(i)$ of $\{1, \dots, r\}$ such that $\sigma'_i = \sigma_{j(i)}$ for $1 \leq i \leq r$.

Q: Find a reference for the statement of $GL_n(\mathbb{C})$.

3. G nonsplit torus over \mathbb{R}

We can also state (and even prove) LLC for nonsplit torus over \mathbb{R} .

I saw the result here [Part V, section 30]: Bill Casselman, Representations of $SL_2(\mathbb{R})$
<https://personal.math.ubc.ca/~cass/research/pdf/Irr.pdf>

For the examples, I try to do computations in a more natural way.

Thm (LLC for G/\mathbb{R} torus)

e.p. $G = \mathbb{G}_{m,\mathbb{R}}, SO_{n,\mathbb{R}}, \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$

We have a 1-to-1 correspondence

$$\pi(G(\mathbb{R})) / \sim \longleftrightarrow \Phi(G_{\mathbb{R}}) / \sim$$

$\parallel \text{def}$

$$\left\{ \begin{array}{l} {}^L\phi: W_{\mathbb{R}} \longrightarrow {}^LG \\ \text{cont gp homo} \end{array} \right. \left. \begin{array}{l} \text{"sec"} \\ \widehat{G}(\mathbb{C}) - \text{conj} \end{array} \right\}$$

where "sec" means, the following diagram commutes:

$$\begin{array}{ccc} W_{\mathbb{R}} & \xrightarrow{{}^L\phi} & {}^LG = \widehat{G}(\mathbb{C}) \times \Gamma_{\mathbb{R}} \\ & \searrow & \swarrow \\ & \Gamma_{\mathbb{R}} & \end{array}$$

Q: How could we state LLC for reductive gp over \mathbb{C} or \mathbb{R} rigorously?

See [BVA 92, Theorem 1.18]

E.g. For $G = \mathrm{G}_{m,\mathrm{IR}}$, \rtimes becomes \times , and

$$\begin{aligned}\text{RHS} &= \{\phi: W_{\mathrm{IR}} \longrightarrow \mathbb{C}^{\times}\} \\ &= \{\rho: \mathrm{IR}^{\times} \longrightarrow \mathbb{C}^{\times}\} = \text{LHS}.\end{aligned}$$

□

E.g. For $G = \mathrm{SO}_{2,\mathrm{IR}}$, we get

$$\widehat{G}(\mathbb{C}) = \mathbb{C}^{\times} \quad {}^L G = \mathbb{C}^{\times} \rtimes \Gamma_{\mathrm{IR}}$$

where Γ_{IR} acts on \mathbb{C}^{\times} by

$$\Gamma_{\mathrm{IR}} \times \mathbb{C}^{\times} \longrightarrow \mathbb{C}^{\times} \quad (\sigma, z) \mapsto \sigma z = z^{-1}$$

$$\begin{aligned}\text{RHS} &= \{{}^L \phi: W_{\mathrm{IR}} \longrightarrow \mathbb{C}^{\times} \rtimes \Gamma_{\mathrm{IR}} \text{ cont "sec"}\} / \mathbb{C}^{\times}\text{-conj} \\ &= \{{}^L \phi: W_{\mathrm{IR}} \longrightarrow \mathbb{C}^{\times} \text{ cocycle}\} / \text{twisted } \mathbb{C}^{\times}\text{-conj}\end{aligned}$$

By mimicking the proof in split torus case, one computes

$$\phi(\bar{z}) = \phi(jzj^{-1}) = \phi(j)\phi(z)\phi(j)^{-1} = \phi(z) = \phi(z)^{-1}$$

$$\Rightarrow \phi(|z|) = \phi(jzj^{-1}z) = 1$$

$$\Rightarrow {}^L \phi: W_{\mathrm{IR}} \longrightarrow \mathbb{C}^{\times} \rtimes \Gamma_{\mathrm{IR}}$$

$$\downarrow \quad \quad \quad \exists!$$

$$S' \sqcup_j S' \cong W_{\mathrm{IR}} / \langle jzj^{-1}z \rangle_{z \in \mathbb{C}^{\times}}$$

Therefore,

$$\begin{aligned}\text{RHS} &= \{{}^L \phi: W_{\mathrm{IR}} \longrightarrow \mathbb{C}^{\times} \rtimes \Gamma_{\mathrm{IR}} \text{ cont "sec"}\} / \mathbb{C}^{\times}\text{-conj} \\ &= \{{}^L \phi: S' \sqcup_j S' \longrightarrow \mathbb{C}^{\times} \rtimes \Gamma_{\mathrm{IR}} \text{ cont "sec"}\} / \mathbb{C}^{\times}\text{-conj} \\ &\stackrel{\text{green arrow}}{=} \{{}^L \rho: S' \longrightarrow \mathbb{C}^{\times} \text{ cont}\} = \text{LHS}\end{aligned}$$

For the effect of $\mathbb{C}^{\times}\text{-conj}$,

$$\begin{aligned}(z, \text{Id}) (\phi(j), \sigma) (z^{-1}, \text{Id}) &= (z \phi(j) \sigma z^{-1}, \sigma) \\ &= (z \phi(j) z, \sigma) \\ &= (z^2 \phi(j), \sigma)\end{aligned}$$

So we can assume $\phi(j) = 1$, and remove $\mathbb{C}^{\times}\text{-conj}$ action.

□

E.g. For $G = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$, we get

$$\widehat{G}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \quad {}^L G = (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_{\mathbb{R}}$$

where $\Gamma_{\mathbb{R}}$ acts on $\mathbb{C}^\times \times \mathbb{C}^\times$ by

$$\Gamma_{\mathbb{R}} \times (\mathbb{C}^\times \times \mathbb{C}^\times) \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \quad (\sigma, (z_1, z_2)) \mapsto \sigma(z_1, z_2) = (z_2, z_1)$$

$$\begin{aligned} \text{RHS} &= \left\{ {}^L \phi : W_{\mathbb{R}} \longrightarrow (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_{\mathbb{R}} \text{ cont "sec"} \right\} / (\mathbb{C}^\times \times \mathbb{C}^\times)\text{-conj} \\ &= \left\{ \phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \text{ cocycle} \right\} / \text{twisted } (\mathbb{C}^\times \times \mathbb{C}^\times)\text{-conj} \end{aligned}$$

By the formula

$$\phi(\bar{z}) = \phi(j z j^{-1}) = \overline{\phi(z)}$$

we get

$$\phi(z) = (z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu).$$

By $(\mathbb{C}^\times \times \mathbb{C}^\times)$ -conj, we can assume

$$\phi(j) = (1, 1)$$

Therefore,

$$\begin{aligned} \text{RHS} &= \left\{ \rho : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \right. \\ &\quad \left. z \mapsto (z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu) \right\} \\ &= \left\{ \rho : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \text{ cont} \right\} \\ &= \left\{ \rho : (\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})(\mathbb{R}) \longrightarrow \mathbb{C}^\times \text{ cont} \right\} = \text{LHS}. \quad \square \end{aligned}$$

One may compute

$$\begin{aligned} &((z_1, z_2), \text{Id}) ((a, b), \sigma) ((z_1^{-1}, z_2^{-1}), \text{Id}) \\ &= ((z_1 a \overline{(z_2)}^\nu, z_2 b \overline{(z_1)}^\mu), \sigma) \\ &= ((z_1^\mu a, z_2^\nu b), \sigma) \end{aligned}$$

Fun game: you have already some examples of LLC. (GL_n + torus) Try to make some comparisons and find some functoriality results!

e.g. $\mathbb{G}_{m,\mathbb{R}} \hookrightarrow GL_2, \mathbb{R} \Rightarrow \pi_1(\mathbb{R}^\times) \leftarrow \pi_1(GL_2(\mathbb{R}))$

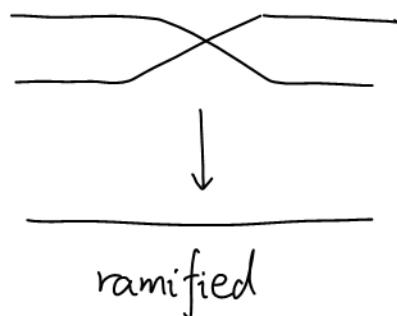
$SO_2, \mathbb{R} \hookrightarrow \pi_1(\mathbb{S}^1) \leftarrow$

e.g. $\pi_1(\mathbb{G}_m(\mathbb{C})) = \pi_1((\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}})(\mathbb{R}))$

Task left: work out some examples of torus over F , where F : local NA field

4. unramified case.

⚠ I didn't find any ref for ramified L -packets for real red gps,
so use at your own risk!



$$I_F \subseteq \Gamma_F \text{ Inertia subgp}$$



$$f(x) = \begin{cases} n|x| & n=2 \\ |x|^{2-n} & n>2 \end{cases}$$

spherical

$$S' \subseteq \mathbb{C}^\times \text{ max cpt subgp}$$

Compare: for F' local field,

$$\begin{array}{ccccccc} 1 & \rightarrow & I_{F'} & \rightarrow & W_{F'} & \longrightarrow & \mathbb{Z} \rightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \rightarrow & \mathcal{O}_{F'}^\times & \rightarrow & F'^\times & \longrightarrow & \mathbb{Z} \rightarrow 1 \end{array}$$

$$\begin{array}{ccccccc} 1 & \rightarrow & S' \sqcup_j S' & \rightarrow & W_{\mathbb{R}} & \longrightarrow & \mathbb{R}_{>0} \rightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \rightarrow & \{\pm 1\} & \rightarrow & \mathbb{R}^\times & \longrightarrow & \mathbb{R}_{>0} \rightarrow 1 \end{array}$$

$$\begin{array}{ccccccc} 1 & \rightarrow & S' & \rightarrow & W_{\mathbb{C}} & \longrightarrow & \mathbb{R}_{>0} \rightarrow 1 \\ & & \parallel & & \parallel & & \parallel \\ 1 & \rightarrow & S' & \rightarrow & \mathbb{C}^\times & \longrightarrow & \mathbb{R}_{>0} \rightarrow 1 \end{array}$$

where $\mathcal{O}_{F'}^\times \subseteq F'^\times$, $\{\pm 1\} \subseteq \mathbb{R}^\times$, $S' \subseteq \mathbb{C}^\times$ are maximal compact subgps.

Slogan. Inertia gp can be viewed as the preimage of maximal compact subgp.

Def. Let $F = \mathbb{R}$ or \mathbb{C} . We say that

$\phi \in \Phi(GL_n, F)$ is unramified, if $\begin{cases} \phi(S' \sqcup jS') = 1_{n \times n} \\ \phi(S') = 1_{n \times n} \end{cases}$ $F = \mathbb{R}$
 $F = \mathbb{C}$

$\rho \in \Pi(GL_n(F))$ is unramified, if the cusp d $\phi \in \Phi(GL_n, F)$ is unramified.
One get

$$\begin{array}{ccc} \Pi(GL_n(F))/\sim & \longleftrightarrow & \Phi(GL_n, F)/\sim \\ \cup & & \cup \\ \Pi_{ur}(GL_n(F))/\sim & \longleftrightarrow & \Phi_{ur}(GL_n, F)/\sim \end{array}$$

Later, we will see that $K = O(n)$ or $U(n)$

$$\Pi_{ur}(GL_n(F)) = \{(p, V) \in \Pi(GL_n(F)) \mid V^k \neq 0\}$$

In general, G : conn red gp

for G/\mathbb{R} , $\phi \in \Phi(G)$ is unramified $\Leftrightarrow \phi'(S' \sqcup jS') = 1_G^\wedge$ for some $\phi' \sim \phi$

for G/\mathbb{C} , $\phi \in \Phi(G)$ is unramified $\Leftrightarrow \phi(S') = 1_G^\wedge$

Q: Do we have

$$\Pi_{ur}(G(F)) = \{(p, V) \in \Pi(G(F)) \mid V^k \neq 0 \text{ for some } K \text{ max cpt subgp}\}?$$

$GL_n(F)$	\mathbb{R}	\mathbb{C}
$n=1$	$\mathbb{C} \times \{\pm 1\}$ $i\mathbb{R} \times \{\pm 1\}$ $\mathbb{C} \times \{1\}$	$\mathbb{C} \times \mathbb{Z}$ $i\mathbb{R} \times \mathbb{Z}$ $\mathbb{C} \times \{1\}$
$n=2$	$\mathbb{C} \times N_{>0}$ $i\mathbb{R} \times N_{>0}$	$\mathbb{C}^{\oplus n}/S^n\text{-action}$
$n > 2$	ϕ	ϕ

orange: unitary representations. for L-parameters side.

blue: unramified L-parameters

E.g. $n=1, F=\mathbb{R}$

All the unramified L-parameters are given by

$$\begin{array}{ccc} W_{\mathbb{R}} & \longrightarrow & \mathbb{C}^{\times} \\ \downarrow & & \\ \mathbb{R}_{>0} & & \end{array} \quad \begin{array}{ccc} z & \downarrow & |z|^t \\ & \nearrow & \\ |z| & & \end{array} \quad \begin{array}{ccc} j & \downarrow & 1 \\ & \nearrow & \\ 1 & & \end{array}$$

i.e. $\Phi_{ur}(\mathbb{G}_m, \mathbb{R})/\sim \cong \mathbb{C} \times \{1\}$
 $\rightsquigarrow \Pi_{ur}(\mathbb{G}_m(\mathbb{R}))/\sim \cong \{1 \cdot 1^t \mid t \in \mathbb{C}\}$

E.g. $n=1, F=\mathbb{C}$

All the unramified L-parameters are given by

$$\begin{array}{ccc} W_{\mathbb{C}} & \longrightarrow & \mathbb{C}^{\times} \\ \downarrow & & \\ \mathbb{R}_{>0} & & \end{array} \quad \begin{array}{ccc} z & \downarrow & |z|^t \\ & \nearrow & \\ |z| & & \end{array}$$

i.e. $\Phi_{ur}(\mathbb{G}_m, \mathbb{C})/\sim \cong \mathbb{C} \times \{1\}$
 $\rightsquigarrow \Pi_{ur}(\mathbb{G}_m(\mathbb{C}))/\sim \cong \{1 \cdot 1^t \mid t \in \mathbb{C}\}$

E.g. $n=2, F=\mathbb{R}$

All the unramified L-parameters are given by

$$\begin{array}{ccc} W_{\mathbb{R}} & \longrightarrow & GL_2(\mathbb{C}) \\ \downarrow & & \\ \mathbb{R}_{>0} & & \end{array} \quad \begin{array}{ccc} z & \downarrow & (|z|^{t_1}, |z|^{t_2}) \\ & \nearrow & \\ |z| & & \end{array} \quad \begin{array}{ccc} j & \downarrow & (1, 1) \\ & \nearrow & \\ 1 & & \end{array}$$

i.e. $\Phi_{ur}(GL_2, \mathbb{R})/\sim \cong \mathbb{C} \times \mathbb{C}/S^1\text{-action}$
 $\rightsquigarrow \Pi_{ur}(\mathbb{G}_m(\mathbb{R}))$ are FD or principal series.
 In ptc, they're all the subquotients of $n\text{-Ind}_{\mathcal{B}}^G(1 \cdot 1^{t_1}, 1 \cdot 1^{t_2})$.

Rmk. We know that, for $x_1, x_2 \in \widehat{\mathbb{R}^{\times}}$,
 $n\text{-Ind}_{\mathcal{B}}^G(x_1, x_2)$ has an $\mathcal{O}_2(\mathbb{R})$ -fixed vector

$$\Leftrightarrow f: GL_2(\mathbb{R}) \longrightarrow \mathbb{C}^{\times}$$

$$g = bk \longmapsto x_1(b_1) x_2(b_2) \left| \frac{b_1}{b_2} \right|^{\frac{1}{2}}$$

$k \in \mathcal{O}_2(\mathbb{R})$
 $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in \mathcal{B}(\mathbb{R})$

is well-defined

$$\Leftrightarrow x_1(-1) = x_2(-1) = 1$$

Ex. Check that in NA local field case, $x_1, x_2 \in \widehat{F^{\times}}$

$$\Leftrightarrow n\text{-Ind}_{\mathcal{B}}^G(x_1, x_2)$$
 has an $GL_2(\mathcal{O}_F)$ -fixed vector

$$\Leftrightarrow f: GL_2(F) \longrightarrow \mathbb{C}^{\times}$$

$$g = bk \longmapsto x_1(b_1) x_2(b_2) \left| \frac{b_1}{b_2} \right|^{\frac{1}{2}}$$

$k \in GL_2(\mathcal{O}_F)$
 $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in \mathcal{B}(F)$

is well-defined

$$\Leftrightarrow x_1(b_1) x_2(b_2) = 1$$

$$\Leftrightarrow x_1, x_2 \text{ are unramified.}$$

$\forall b_1, b_2 \in \mathcal{O}_F^{\times}$

E.g. For $G = SO_{2, \mathbb{R}}$, all the unramified L-parameters are given by

$$\begin{array}{ccc} W_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{C}^{\times} \\ \downarrow & \nearrow & \\ \mathbb{R}_{>0} & & \end{array} \quad \begin{array}{ccc} z & \xrightarrow{|z|} & |z|^t \\ \downarrow & \searrow & \\ |z| & & \end{array} \quad \begin{array}{ccc} j & \xrightarrow{1} & 1 \\ \downarrow & \nearrow & \\ 1 & & \end{array}$$

with $\phi(|z|^2) = \phi(jzj^{-1}z) = 1 \Rightarrow t=0$
 i.e. $\Phi_{ur}(SO_{2, \mathbb{R}})/_{\sim} \cong \{1_{W_{\mathbb{R}}}\}$
 $\rightsquigarrow \Pi_{ur}(SO_{2, \mathbb{R}})/_{\sim} \cong \{1_{SO_{2, \mathbb{R}}}\}$

□

E.g. For $G = Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{R}}$, all the unramified L-parameters are given by

$$\begin{array}{ccc} W_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{C}^{\times} \times \mathbb{C}^{\times} \\ \downarrow & \nearrow & \\ \mathbb{R}_{>0} & & \end{array} \quad \begin{array}{ccc} z & \xrightarrow{|z|} & (|z|^{t_1}, |z|^{t_2}) \\ \downarrow & \searrow & \\ |z| & & \end{array} \quad \begin{array}{ccc} j & \xrightarrow{(1,1)} & (1,1) \\ \downarrow & \nearrow & \\ 1 & & \end{array}$$

with $\phi(\bar{z}) = \phi(jzj^{-1}) = \sigma(\phi(z)) \Rightarrow t_1 = t_2$
 i.e. $\Phi_{ur}(Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{R}})/_{\sim} \cong \mathbb{C}$
 $\rightsquigarrow \Pi_{ur}(Res_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m, \mathbb{R}})/_{\sim} \cong \{1 \cdot |t| \mid t \in \mathbb{C}\}$

□

5. injectivity $\widehat{G} = \widehat{G}(\mathbb{C})$ $\widehat{T} = \widehat{T}(\mathbb{C})$

In this section, we show that

$$\pi_0(\overline{S_\phi}) = \{\ast\} \quad \text{for } \phi \in \Phi(G), \quad G = GL_{n,\mathbb{R}} \text{ or torus}/\mathbb{R}$$

where

$$S_\phi := C_G({}^L\phi(W_{\mathbb{R}})) \cap \widehat{G}$$

$$Z(\widehat{G})^\Gamma := C_G({}^L\phi) \cap \widehat{G} \quad \text{if } G \text{ split} \quad Z(\widehat{G})$$

$$\overline{S_\phi} = S_\phi / Z(\widehat{G})^\Gamma$$

$GL_{n,\mathbb{C}}$ or torus/ \mathbb{C} are easier.

E.g. $G = GL_{n,\mathbb{R}}$. We get

$$\widehat{G} = \mathbb{C}^\times \quad Z(\widehat{G})^\Gamma = Z(\widehat{G}) = \mathbb{C}^\times.$$

For $\phi \in \Phi(GL_{n,\mathbb{R}})$,

$$S_\phi = \mathbb{C}^\times \quad \overline{S_\phi} = \{\ast\} \quad \pi_0(\overline{S_\phi}) = \{\ast\}.$$

Lemma. In $GL_2(\mathbb{C})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff b=c=0 \text{ or } x=y$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y & x \\ z & w \end{pmatrix} = \begin{pmatrix} y & x \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff a=d, by=cx$$

i.e. $C_{GL_2(\mathbb{C})} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{cases} T(\mathbb{C}) & x \neq y \\ GL_2(\mathbb{C}) & x=y \end{cases}$

$$C_{GL_2(\mathbb{C})} \begin{pmatrix} y & x \\ z & w \end{pmatrix} = \left\{ \begin{pmatrix} a & bx \\ ty & a \end{pmatrix} \mid a, t \in \mathbb{C}, a^2 \neq txy \right\}$$

E.g. $G = GL_{2,\mathbb{R}}$. We get

$$\widehat{G} = GL_2(\mathbb{C}) \quad Z(\widehat{G})^\Gamma = Z(\widehat{G}) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \cong \mathbb{C}^\times$$

For $\phi = \chi_i \oplus \chi_i$ corresponding to $(t_i, \varepsilon_i), (t_2, \varepsilon_2)$,

we write

$${}^L\phi: W_{\mathbb{R}} \longrightarrow GL_2(\mathbb{C}) \times \Gamma_{\mathbb{R}}$$

$$z \longmapsto \left(\begin{pmatrix} |z|^{t_i} & 0 \\ 0 & |z|^{t_2} \end{pmatrix}, 1 \right)$$

$$j \longmapsto \left(\begin{pmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_2 \end{pmatrix}, \sigma \right)$$

$$\Rightarrow S_\phi = \begin{cases} \widehat{G} & t_1 = t_2, \varepsilon_1 = \varepsilon_2, \\ \widehat{T} & \text{otherwise,} \end{cases}$$

$$\Rightarrow S_\phi = \begin{cases} \widehat{G}/\mathbb{C}^\times & t_1 = t_2, \varepsilon_1 = \varepsilon_2, \\ \widehat{T}/\mathbb{C}^\times & \text{otherwise,} \end{cases}$$

$$\Rightarrow \pi_0(\overline{S_\phi}) = \{\ast\}$$

For ϕ simple, we write $\mu - \nu > 0$

$$\begin{aligned}\langle \phi : W_{IR} &\longrightarrow GL_2(\mathbb{C}) \times \Gamma_R \\ z &\longmapsto \left(\begin{pmatrix} z^\mu \bar{z}^\nu & \\ & \bar{z}^\nu \bar{z}^\mu \end{pmatrix}, 1 \right) \\ j &\longmapsto \left(\begin{pmatrix} & (-1)^{\mu-\nu} \\ 1 & \end{pmatrix}, \sigma \right)\end{aligned}$$

$$\Rightarrow S_\phi \cong \mathbb{C}^\times, \overline{S}_\phi \cong \mathbb{C}^\times / \mathbb{C}^\times = \{*\}, \pi_0(\overline{S}_\phi) = \{*\}$$

To compute the $GL_{n,IR}$ case, one may want to compute

$$C_{GL_n(\mathbb{C})} \left(\text{diag}((1, 1), \dots, (1, 1), (1, -1), \dots, (1, -1), 1, \dots, 1) \right)$$

Luckily, by observation one can conclude that $\pi_0(S_\phi) = \{*\}, \pi_0(\overline{S}_\phi) = \{*\}$ without really computing the centralizer.

If you really struggle to compute it, use

Centralizer of $\begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix} = \begin{pmatrix} * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \\ * & * & * & & * & * & * & * \end{pmatrix}$

and

$$(1, 1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad (1, -1) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

E.g. $G = GL_{n,IR}$. We get
 $\widehat{G} = GL_n(\mathbb{C})$ $Z(\widehat{G})^R = Z(\widehat{G}) = \begin{pmatrix} t & & & \\ & \ddots & & \\ & & t & \\ & & & t \end{pmatrix} \cong \mathbb{C}^\times$
 $\Rightarrow \pi_0(S_\phi) = \{*\}$ $\pi_0(\overline{S}_\phi) = \{*\}$.

Eg. $G = SO_{2, \mathbb{R}}$. We get

$$\widehat{G} = \mathbb{C}^\times \quad Z(\widehat{G}) = \mathbb{C}^\times \quad Z(\widehat{G})^\Gamma \cong \{z \in \mathbb{C}^\times \mid z^{-1} = z\} \cong \{\pm 1\}$$

For $\psi: W_R \longrightarrow \mathbb{C}^\times \rtimes \Gamma_R$

$$\begin{aligned} z &\mapsto (|z|^t, 1) \\ j &\mapsto (1, \sigma) \end{aligned}$$

we can compute

$$\begin{aligned} S_\psi &= \left\{ \omega \in \mathbb{C}^\times \mid (|z|^t, 1)(\omega, 1) = (\omega, 1)(|z|^t, 1) \right. \\ &\quad \left. (1, \sigma)(\omega, 1) = (\omega, 1)(1, \sigma) \right\} \\ &= \left\{ \omega \in \mathbb{C}^\times \mid \omega^{-1} = \omega \right\} \\ &= \{\pm 1\} \\ \Rightarrow \bar{S}_\psi &= \{*\}, \quad \pi_0(\bar{S}_\psi) = \{*\}. \end{aligned}$$

!! Here $\pi_0(S_\psi) \neq \{*\}$.

Eg. $G = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}_{m, \mathbb{C}}$. We get

$$\begin{aligned} \widehat{G} &= \mathbb{C}^\times \times \mathbb{C}^\times \quad Z(\widehat{G}) = \widehat{G} \quad Z(\widehat{G})^\Gamma \cong \{(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times \mid z_1 = z_2\} \\ &\cong \{(z, z) \in \mathbb{C}^\times \times \mathbb{C}^\times\} \\ &\cong \mathbb{C}^\times \text{ diagonal.} \end{aligned}$$

For $\psi: W_R \longrightarrow (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_R$

$$\begin{aligned} z &\mapsto ((z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu), 1) \\ j &\mapsto ((1, 1), \sigma) \end{aligned}$$

we get finally

$$S_\psi = Z(\widehat{G})^\Gamma \cong \mathbb{C}^\times \quad \bar{S}_\psi = \{*\} \quad \pi_0(\bar{S}_\psi) = \{*\}.$$

□