Eine Woche, ein Beispiel

9.4 Hecke algebra

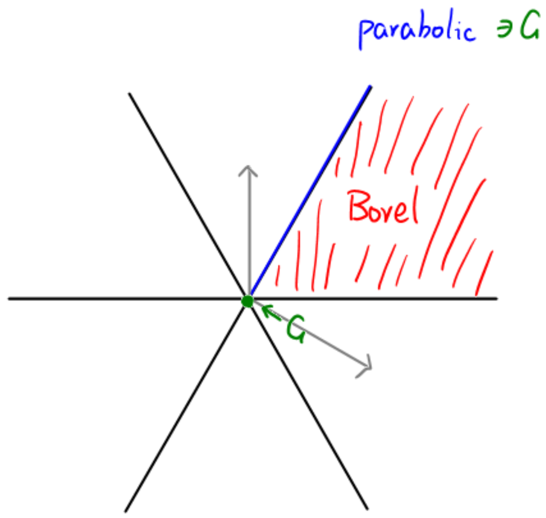
This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]
[XiongHecke][https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf]
All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

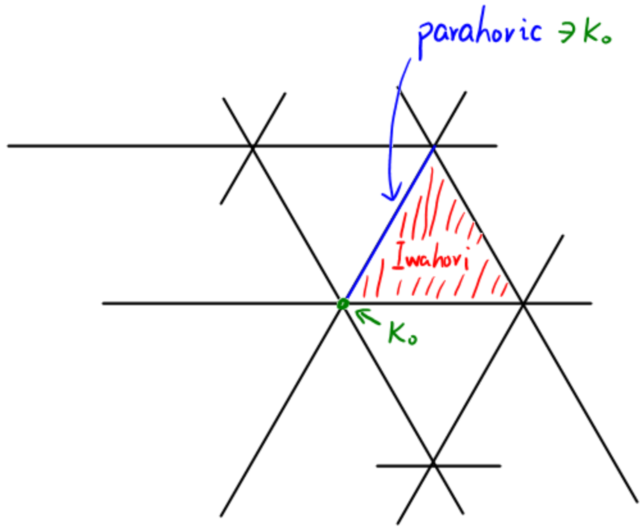
Task. For each double coset decomposition, we want to do:

1. decomposition (& $\Gamma t \Gamma / \Gamma$ is finite & definition of Hecke alg)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsl-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form	Others
F finite	$G = \bigsqcup_{w\in W} BwB$			$\mathcal{H}(S_{m+n}, S_m\times S_n)$ $\mathcal{H}(G\times G, G)$
F local	$G = \bigsqcup_{w\in W} BwB$	$G = \bigsqcup_{w\in W_{ext}} IwI$	$G = \bigsqcup_{\lambda\in T^-} K_0 \lambda K_0$	
F global	$G = \bigsqcup_{w\in W} BwB$		$GL_2^+(\mathbb{Q}) = \bigsqcup_{\lambda\in T^-} \Gamma\lambda\Gamma$	
adèle?				





$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ & * & * \\ * & * & \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$\begin{pmatrix} & 1 \\ p & \end{pmatrix} \in GL_2(\mathcal{O}) \Rightarrow \begin{pmatrix} & 1 \\ p^{1} & \end{pmatrix} \notin I$$

https://mathoverflow.net/questions/4547/definitions-of-hecke-alg
https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebraebras

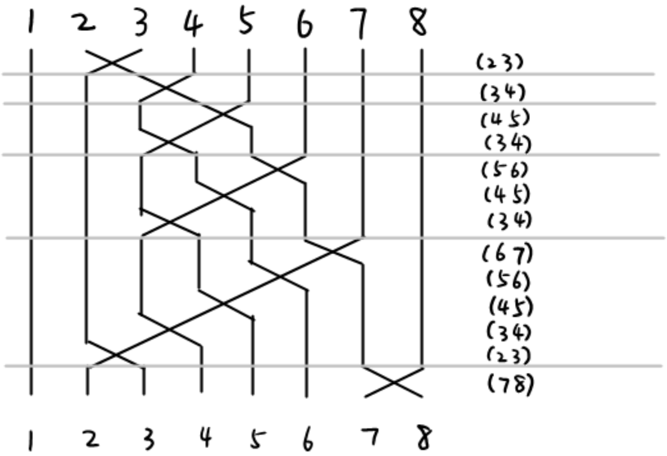
S_n and Tits system

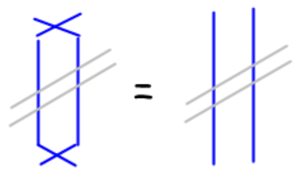
A brief preparation for computations in Bruhat decomposition. $s_i = (i\ i{+}1)$, $1 \le i \le n-1$

E.g. $n=8$, $w_0 = (287)(46) = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 1&8&3&6&5&4&2&7 \end{pmatrix} \in S_8$.

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.





$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0) = 13$ = "inversion number"

$l(s_1 w_0) = 14$ $\quad$ $l(w_0 s_1) = 14$

$l(s_2 w_0) = 12$ $\quad$ $l(w_0 s_2) = 12$

$l(s_3 w_0) = 14$ $\quad$ $l(w_0 s_3) = 14$

$l(s_4 w_0) = 12$ $\quad$ $l(w_0 s_4) = 12$

$l(s_5 w_0) = 12$ $\quad$ $l(w_0 s_5) = 12$

$l(s_6 w_0) = 12$ $\quad$ $l(w_0 s_6) = 14$

$l(s_7 w_0) = 14$ $\quad$ $l(w_0 s_7) = 12$

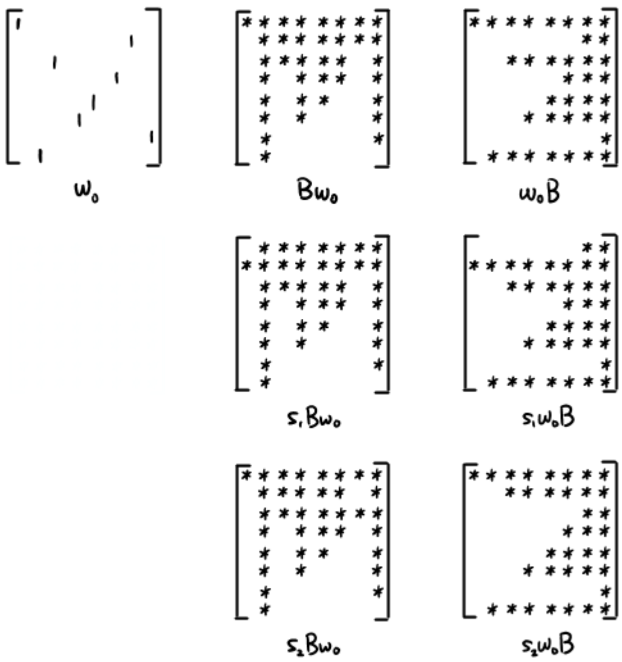
Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

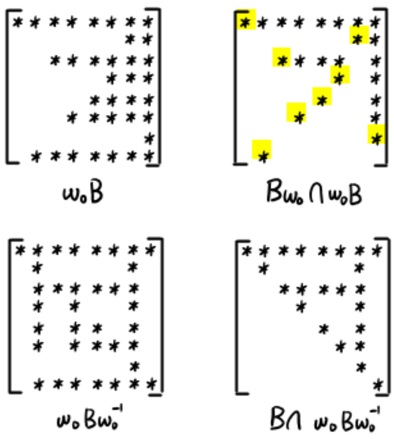
Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

Solution



The following computation will be also computed later on.



finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$ but $BwB \cdot Bw^{-1}B \neq B$ is possible

Ex. Compute $|BwB/B|$ ∇ BwB may not be a group!

Hint: Consider the map

$$\phi : B \twoheadrightarrow BwB/B$$
$$b \longmapsto bwB$$

$$\begin{aligned} \phi(b_1) = \phi(b_2) &\Leftrightarrow b_1 wB = b_2 wB \\ &\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B \\ &\Leftrightarrow b_2^{-1} b_1 \in wBw^{-1} \end{aligned}$$

$$\therefore |BwB/B| = |B|/|wBw^{-1} \cap B| = q^{l(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) := \{ f : G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \ \forall b_1, b_2 \in B, g \in G \}$ where

$$\begin{aligned} (f_1 * f_2)(g) &= \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x) \\ &= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g) \end{aligned}$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G,B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbb{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote $T_w := \mathbb{1}_{BwB}$, $T_i := T_{s_i}$ ($T_{Id} = \mathbb{1}_B$ is the unit of $\mathcal{H}(G,B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G,B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w)\, T_w$$

$$\begin{aligned} (T_u * T_v)(w) &= \frac{1}{|B|} \sum_{yz = w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} \{ (y,z) \in BuB \times BvB \mid yz = w \} \quad \underline{\text{if } w \notin BuB\,BvB} \ 0 \\ &= \frac{1}{|B|} |BuB \cap u Bv^{-1}B| \end{aligned}$$

$$Bs_iB \cdot Bw_oB = \begin{cases} Bs_iw_oB & l(s_iw) = l(w)+1 \\ Bs_iw_oB \cup Bw_oB & l(s_iw) = l(w)-1 \end{cases}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} \mathbb{Z}\cdot T_{s_iw} & l(s_iw) = l(w)+1 \\ \mathbb{Z}\cdot T_{s_iw} + \mathbb{Z}\cdot T_w & l(s_iw) = l(w)-1 \end{cases}$$

Computation of coefficient.

$$|BwB| = |BwB/B| \times |B| = q^{l(w)}\cdot|B|$$

when $l(s_iw) = l(w)+1$,

$$(T_i * T_w)(s_iw) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times BwB \mid yz = s_iw\}$$

$$= \frac{1}{|B|\,|Bs_iwB|}\{(y,z)\in Bs_iB\times BwB \mid yz\in Bs_iwB\}$$

$$= \frac{|Bs_iB|\,|BwB|}{|B|\cdot|Bs_iwB|} = \frac{q^{l(s_i)}\,q^{l(w)}}{q^{l(s_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times Bs_iB \mid yz = Id\}$$

$$= \frac{1}{|B|}\,|Bs_iB| = q$$

$$(T_i * T_i)(s_i) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times Bs_iB \mid yz = s_i\}$$

$$= \frac{1}{|B|\,|Bs_iB|}\{(y,z)\in Bs_iB\times Bs_iB \mid yz\in Bs_iB\}$$

$$= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB\times Bs_iB| - \{(y,z)\in Bs_iB\times Bs_iB \mid yz\in B\}\right)$$

$$= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB|\,|Bs_iB| - |B|\cdot|Bs_iB|\right)$$

$$= q-1$$

when $l(s_iw) = l(w)-1$, we get $l(s_i\cdot s_iw) = l(s_iw)+1$,

$$T_i * T_w = T_i * T_i * T_{s_iw}$$

$$= (qT_{Id} + (q-1)T_i) * T_{s_iw}$$

$$= qT_{s_iw} + (q-1)T_w$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} T_{s_iw} & l(s_iw) = l(w)+1 \\ qT_{s_iw} + (q-1)T_w & l(s_iw) = l(w)-1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$\mathcal{H}(G,B) = \mathbb{Z}\langle T_1,\dots,T_{n-1}\rangle_{alg}$ with relations $\quad (\mathcal{H}(G,B)\cong \mathcal{H}_q(W))$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j|\geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

E.g. For $n=2$, $\mathcal{H}(G,B) \cong \mathbb{Z}[T_1] / (T_1^2-(q-1)T_1-q)$

$$\cong \mathbb{Z}[T_1] / (T_1-q)(T_1+1)$$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1$$

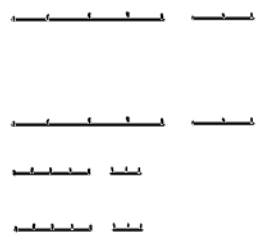
For $n=3$, $\mathcal{H}(G,B) \cong \mathbb{Z}\langle T_1, T_2 \rangle \Big/ \big((T_1-q)(T_1+1), (T_2-q)(T_2+1), T_1T_2T_1 = T_2T_1T_2\big)$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_1T_2 \oplus \mathbb{Z}T_2T_1 \oplus \mathbb{Z}T_1T_2T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_{(123)} \oplus \mathbb{Z}T_{(132)} \oplus \mathbb{Z}T_{(13)}$$

Variation: $\mathcal{H}(S_{m+n}, S_m \times S_n)$ $\qquad m, n \in \mathbb{Z}_{>0}$

E.g. $m+n=8$, $m=5$, $n=3$; $m+n=8$, $m=7$, $n=1$.



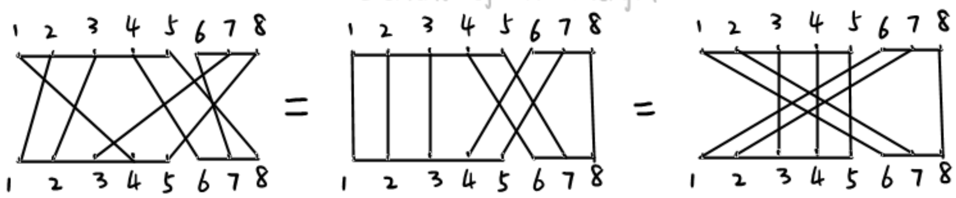
∇ For the convenience of the writing, we denote

$$S_{m,n} := S_m \times S_n$$

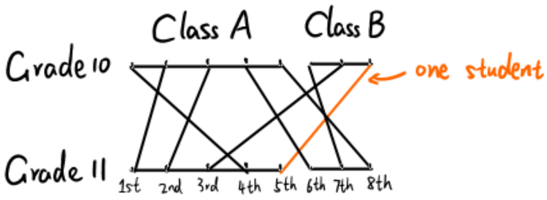
and suppose $m \geq n$.

1. decomposition

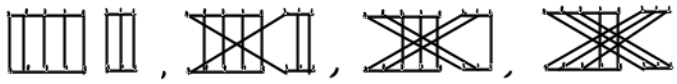
E.g. random element $\qquad$ canonical form for draw by hand element of minimal length $\qquad$ canonical form for computation



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 6 & 8 & 7 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 4 & 5 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 3 & 4 & 5 & 1 & 2 & 8 \end{pmatrix} \text{ in } S_{5,3}\backslash S_8 / S_{5,3}$$



A vivid explanation: students are distributed into different classes on the basis of their order.
The teacher only care about the stability of the class, i.e. how many people move from Class A to Class B as time went by.

$\therefore$ $S_{5,3}\backslash S_8 / S_{5,3} = \{$  $\}$

$[\varpi_0] = [Id]$, $[\varpi_1]$, $[\varpi_2]$, $[\varpi_3]$

e.g. $\varpi_1 = (16) \in S_8$
$[\varpi_1] \in S_{5,3}\backslash S_8 / S_{5,3}$

In general, $\quad S_{m+n} = \coprod_{i=0}^{n} S_{m,n}\, \varpi_i\, S_{m,n}$

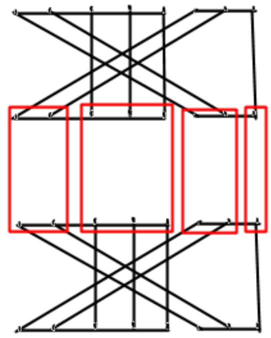
Ex. Compute $\quad |S_{m,n}\, \varpi_i\, S_{m,n} / S_{m,n}|$

A. $|S_{m,n}\, \varpi_i\, S_{m,n} / S_{m,n}| = |S_{m,n} / \varpi_i S_{m,n} \varpi_i^{-1} \cap S_{m,n}|$

$$= |S_{m,n} / S_{i,m-i} \times S_{i,n-i}|$$

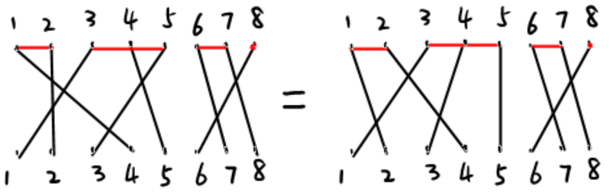
$$= \frac{m!\, n!}{i!\,(m-i)!\; i!\,(n-i)!}$$

$$= \binom{m}{i}\binom{n}{i}$$

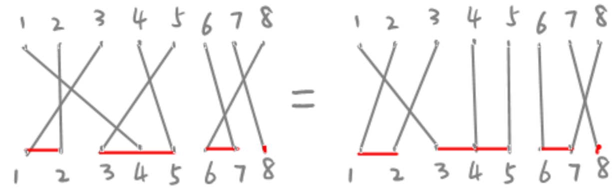


$\varpi_i S_{m,n} \varpi_i^{-1} \cap S_{m,n}$

E.g. (canonical form)



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 5 & 7 & 8 & 6 \end{pmatrix} \quad \text{in } S_{5,3} / S_{2,3} \times S_{2,1}$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 4 & 5 & 8 \end{pmatrix} \quad \text{in } S_{2,3} \times S_{2,1} \backslash S_{5,3}$$

Recall that $\mathcal{H}(G,H) := \{ f : G \to \mathbb{Z} \mid f(h_1 g h_2) = f(g) \ \forall h_1, h_2 \in H, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1} g) \, d\mu(x)$$

$$= \frac{1}{|H|} \sum_{x \in G} f_1(x) f_2(x^{-1} g)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(S_{m+n}, S_{m,n}) = \bigoplus_{i=0}^{n} \mathbb{Z} \cdot \mathbb{1}_{S_{m,n} \varpi_i S_{m,n}} = \mathbb{Z}^{\oplus(n+1)}$$

denote $T_i := \mathbb{1}_{S_{m,n} \varpi_i S_{m,n}}$ $\qquad$ ($T_0 = \mathbb{1}_{S_{m,n}}$ is the unit of $\mathcal{H}(S_{m+n}, S_{m,n})$)

3. alg structure

E.g. $\mathcal{H}(S_8, S_7) = \mathbb{Z} \oplus \mathbb{Z} T = \mathbb{Z}[T] / (T-7)(T+1)$

$$g^{Id}_{\varpi_1, \varpi_1} = \frac{1}{|S_7|} \{ (y, z) \in S_7 \varpi_1 S_7 \times S_7 \varpi_1 S_7 \mid yz = 1 \}$$

$$= \frac{|S_7 \varpi_1 S_7|}{|S_7|}$$

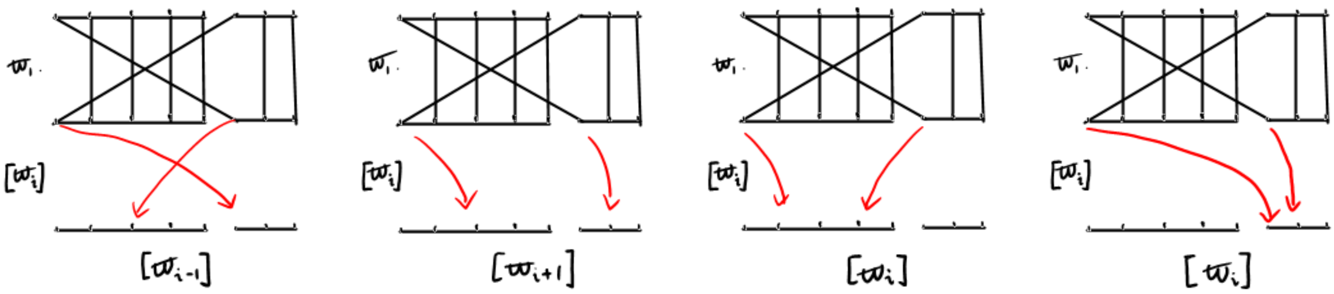
$$= \frac{8! - 7!}{7!} = 7$$

$$g^{\varpi_1}_{\varpi_1,\varpi_1} = \frac{1}{|S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz = \varpi_1\}$$
$$= \frac{1}{|S_7||S_7\varpi_1 S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz \in S_7\varpi_1 S_7\}$$
$$= \frac{1}{|S_7||S_7\varpi_1 S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz \notin S_7\}$$
$$= \frac{\cancel{|S_7\varpi_1 S_7|}\,|S_7\varpi_1 S_7| - \cancel{|S_7\varpi_1 S_7|}\,|S_7|}{|S_7|\,\cancel{|S_7\varpi_1 S_7|}}$$
$$= \binom{7}{1}\binom{1}{1} - 1$$
$$= 6$$

In general, $\mathcal{H}(S_{m+1}, S_m) = \mathbb{Z}[T]/(T-m)(T+1)$.

Direct argument shows that

$$T_i * T_1 \in \begin{cases} \mathbb{Z}\cdot T_{i-1} + \mathbb{Z}\cdot T_i + \mathbb{Z}\cdot T_{i+1} & 0<i<n \\ \cancel{\mathbb{Z}\cdot T_0} + \mathbb{Z}\cdot T_1 & i=0 \\ \mathbb{Z}\cdot T_{n-1} + \mathbb{Z}\cdot T_n & i=n \end{cases}$$



Computation of the coefficient.

$$g^{\varpi_{i-1}}_{\varpi_i,\varpi_1} = \frac{1}{|S_{m,n}|}\#\{(y,z)\in S_{m,n}\varpi_i S_{m,n} \times S_{m,n}\varpi_1 S_{m,n} \mid yz = \varpi_{i-1}\} \qquad (0<i\leq n)$$
$$= \frac{1}{|S_{m,n}||S_{m,n}\varpi_{i-1}S_{m,n}|}\#\{(y,z)\in S_{m,n}\varpi_i S_{m,n} \times S_{m,n}\varpi_1 S_{m,n} \mid yz \in S_{m,n}\varpi_{i-1}S_{m,n}\}$$
$$= \frac{|S_{m,n}\varpi_i S_{m,n}|}{|S_{m,n}||S_{m,n}\varpi_{i-1}S_{m,n}|}\#\{z\in S_{m,n}\varpi_1 S_{m,n} \mid \varpi_i z \in S_{m,n}\varpi_{i-1}S_{m,n}\}$$
$$= \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,-)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}||S_{m,n}\varpi_{i-1}S_{m,n}|}$$

$$g^{\varpi_{i+1}}_{\varpi_i,\varpi_1} = \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,+)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}||S_{m,n}\varpi_{i+1}S_{m,n}|} \qquad (0\leq i<n)$$

$$g^{\varpi_i}_{\varpi_i,\varpi_1} = \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,0)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}||S_{m,n}\varpi_i S_{m,n}|}$$

where

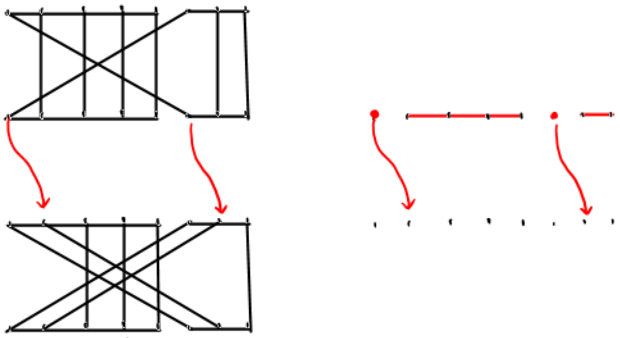
$$S_{m,n}^{(i,-)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_{i-1} S_{m,n} \}$$
$$S_{m,n}^{(i,+)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_{i+1} S_{m,n} \}$$
$$S_{m,n}^{(i,0)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_i S_{m,n} \}$$

Recall that

$$S_{m,n}^{(i,-)} \varpi_1 S_{m,n} / S_{m,n} \subset S_{m,n} \varpi_1 S_{m,n} / S_{m,n} \quad \text{as left coset}$$

By the following picture, for $g \in S_{m,n}$,

$$g \in S_{m,n}^{-} \iff \begin{cases} g(1) \in \{1, 2, \cdots, i\} \\ g(m+1) \in \{m+1, m+2, \cdots, m+i\} \end{cases}$$



$$\therefore \quad \left| S_{m,n}^{(i,-)} \varpi_1 S_{m,n} / S_{m,n} \right| = i^2$$

Similarly,

$$\left| S_{m,n}^{(i,+)} \varpi_1 S_{m,n} / S_{m,n} \right| = (m-i)(n-i)$$
$$\left| S_{m,n}^{(i,0)} \varpi_1 S_{m,n} / S_{m,n} \right| = i(n-i) + (m-i)i$$
$$= i(m+n-2i)$$

$$\therefore \quad g_{\varpi_i, \varpi_1}^{\varpi_{i-1}} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,-)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_{i-1} S_{m,n}|} \qquad (0 < i \leq n)$$

$$= \frac{\binom{m}{i} \binom{n}{i} i^2}{\binom{m}{i-1} \binom{n}{i-1}} \qquad \binom{m}{n} = \frac{m-i+1}{i} \binom{m}{i-1}$$

$$= (m-i+1)(n-i+1)$$

$$g_{\varpi_i, \varpi_1}^{\varpi_{i+1}} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,+)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_{i+1} S_{m,n}|} \qquad (0 \leq i < n)$$

$$= \frac{\binom{m}{i} \binom{n}{i} (m-i)(n-i)}{\binom{m}{i+1} \binom{n}{i+1}}$$

$$= (i+1)^2$$

$$g_{\varpi_i, \varpi_1}^{\varpi_i} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,0)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_i S_{m,n}|}$$

$$= \left| S_{m,n}^{(i,0)} \varpi_1 S_{m,n} / S_{m,n} \right|$$

$$= i(m+n-2i)$$

Therefore,

$$T_i * T_1 = \begin{cases} (m-i+1)(n-i+1)\,T_{i-1} + i(m+n-2i)\,T_i + (i+1)^2\,T_{i+1}, & 0<i<n \\ T_1, & i=0 \\ (m-n+1)\,T_{n-1} + n(m-n)\,T_n, & i=n \end{cases}$$

4. Conclusion

By [Hecke, Prop 6], $S_{m,n}$ is a Gelfand subgp of S_{m+n}.
thus $\mathcal{H}(S_{m+n}, S_{m,n})$ is commutative.
Gelfand involution: $\sigma \longmapsto \sigma^T$ $\quad (S_{m+n} \hookrightarrow GL_{m+n}(K))$

Possible extension: compute $\mathcal{H}(S_{m+n+l}, S_m \times S_n \times S_l)$.

The rest of the section is devoted to compute $F_{m,n} \in \mathbb{Z}[T]$ s.t

$$\mathcal{H}(S_{m+n}, S_{m,n}) \cong \mathbb{Z}[T]/(F_{m,n}) \qquad T = T_1$$

Appendix: "Linear algebra"

Set $v_i = T_i$, $w_i = T^i$

First cases:

$w_0 = 1 = T_0 = v_0$

$w_1 = T = T_1 = v_1$

$w_2 = T^2 = Tv_1 = mnv_0 + (m+n-2)v_1 + 4v_2$

$w_3 = T^3 = T(mnv_0 + (m+n-2)v_1 + 4v_2) = \dots$

$$\text{Define } \mathcal{A}: \begin{array}{ccc} \mathcal{H}(S_{m+n}, S_{m,n}) & \longrightarrow & \mathcal{H}(S_{m+n}, S_{m,n}) \\ f & \longmapsto & f * T \end{array}$$

Then $\mathcal{A}(w_i) = w_{i+1}$

$$\mathcal{A}(v_0, \dots, v_n) = (v_0, \dots, v_n) \begin{bmatrix} 0 & mn & & & & & \\ 1 & (m+n-2) & (m-1)(n-1) & & & & \\ & 4 & 2(m+n-4) & (m-2)(n-2) & & & \\ & & 9 & 3(m+n-6) & (m-3)(n-3) & & \\ & & & 16 & \ddots & \ddots & \\ & & & & \ddots & \ddots & m-n+1 \\ & & & & & n^2 & n(m-n) \end{bmatrix}_{(n+1)\times(n+1)}$$

Therefore, if $w_i = \sum_j a_{ij} v_j$, then

$$w_{i+1} = \mathcal{A}(w_i) = \sum_j a_{ij} \mathcal{A}(v_j)$$

$$(w_0,\cdots,w_n)=(v_0,\cdots,v_n)\begin{bmatrix} 1 & 0 & mn & * & * & \cdots \\ & 1 & m+n-2 & * & * & \cdots \\ & & 4 & * & * & \cdots \\ & & & 4\cdot 9 & * & \cdots \\ & & & & 4\cdot 9\cdot 16 & \cdots \\ & & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}$$

after tensored over $\mathbb{Q}$, $(w_0,\cdots,w_n)$ become a basis, and

$$\mathcal{A}(w_0,\cdots,w_n)=(w_0,\cdots,w_n)\begin{bmatrix} & & & -c_0 \\ 1 & & & \vdots \\ & 1 & & \vdots \\ & & \ddots & -c_{n-1} \\ & & 1 & -c_n \end{bmatrix}_{(n+1)\times(n+1)}$$

where $F_{m,n}(T)=b_{n+1}(T^{n+1}+c_nT^n+c_{n-1}T^{n-1}+\cdots c_0)\in\mathbb{Z}[T]$ $\quad c_i\in\mathbb{Q}$

$b_{n+1}:=\operatorname{lcm}(\text{denominators of } c_i)$

least common multiple

Therefore, the problem reduces to the computation of

$$\begin{aligned}&T^{n+1}+c_nT^n+c_{n-1}T^{n-1}+\cdots+c_0\\ &=\text{char poly of } \mathcal{A}\\ &=\text{char poly of }\begin{bmatrix} 0 & mn & & \\ 1 & m+n-2 & (m-1)(n-1) & \\ & 4 & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}_{(n+1)\times(n+1)}\end{aligned}$$

Since $c_i\in\mathbb{Z}$, we get $b_{n+1}=1$, i.e.

$$F_{m,n}(T)=\text{char poly of }\mathcal{A}.$$

Fix $m\geq n$, denote $n\geq k\geq 0$,

$$\beta_k^T=[0,\cdots,0,1]\in\mathbb{Z}^k,$$

$$A_k:=A_{m,n,k}:=\begin{bmatrix} 0 & mn & & & \\ 1 & m+n-2 & (m-1)(n-1) & & \\ & 4 & \ddots & \ddots & \\ & & \ddots & \ddots & (m-k+1)(n-k+1) \\ & & & k^2 & k(m+n-2k) \end{bmatrix}_{(k+1)\cdot(k+1)}$$

$$\overset{\text{if } k\geq 1}{=\!=}\begin{bmatrix} A_{m,n,k-1} & (m-k+1)(n-k+1)\beta_k \\ k^2\beta_k^T & k(m+n-2k)\end{bmatrix}$$

e.p. $\mathcal{A}=A_{m,n,n}$

$$\lambda I - A_k = \begin{bmatrix} \lambda I - A_{k-1} & -(m-k+1)(n-k+1)\beta_k \\ -k^2\beta_k^T & \lambda - k(m+n-2k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -k^2\beta_k^T(\lambda I - A_{k-1})^{-1} & 1 \end{bmatrix} \begin{bmatrix} \lambda I - A_{k-1} & -(m-k+1)(n-k+1)\beta_k \\ 0 & -k^2(m-k+1)(n-k+1)\beta_k^T(\lambda I - A)^{-1}\beta_k \\ & +\lambda - k(m+n-2k) \end{bmatrix}$$

$$\beta_k^T(\lambda I - A_{k-1})^{-1}\beta_k = \left((\lambda I - A_{k-1})^{-1}\right)_{k,k} = \begin{cases} \dfrac{\det(\lambda I - A_{k-2})}{\det(\lambda I - A_{k-1})} & k \geq 1 \\ \lambda^{-1} & k = 1 \end{cases}$$

Hint:

$$B^{-1} = \frac{1}{\det B}\begin{bmatrix} B_{11} & -B_{12} & B_{13} & \cdots \\ -B_{21} & B_{22} & & \\ B_{31} & & & -B_{(n-1)n} \\ \vdots & & -B_{n(n-1)} & B_{n,n} \end{bmatrix}$$

Denote $\mathrm{Det}_k := \mathrm{Det}(\lambda I - A_k)$, then

$$\mathrm{Det}_k = \mathrm{Det}_{k-1}\left(-k^2(m-k+1)(n-k+1)\frac{\mathrm{Det}_{k-2}}{\mathrm{Det}_{k-1}} + \lambda - k(m+n-2k)\right)$$

$$= (\lambda - k(m+n-2k))\,\mathrm{Det}_{k-1} - k^2(m-k+1)(n-k+1)\,\mathrm{Det}_{k-2} \quad (\text{for } k \geq 2)$$

$$\mathrm{Det}_0 = \lambda I - A_0 = \lambda$$

$$\mathrm{Det}_1 = \lambda I - A_1 = \lambda^2 - (m+n-2)\lambda - mn$$

$$F_{m,n}(\lambda) = \mathrm{Det}_n$$

$\square$

global Cartan decomposition

1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2\times 2}(R) = \coprod_{(b)\subseteq(a)} GL_2(R)\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(R)$$

Cor
$$M_{2\times 2}(\mathbb{Z}) = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0\leq a\leq b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det\neq 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\leq b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det> 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\leq b}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a,b\in\mathbb{Q}^\times_{>0}\\ v_p(a)\leq v_p(b)\ \forall p}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}SL_2(\mathbb{Z}) \qquad GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det>0}$$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$T^- = \left\{ \begin{pmatrix} a & \\ & b\end{pmatrix}\in GL_2^+(\mathbb{Q}) \;\middle|\; \begin{array}{l} a,b>0 \\ v_p(a)\leq v_p(b) \quad \forall p \text{ prime}\end{array}\right\} \overset{\mathrm{Grp}}{\cong} \mathbb{Q}^\times_{>0}\times(\mathbb{Z}_{>0},\times)$$

then
$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha\in T^-}\Gamma\alpha\Gamma$$

Ex. Verify that $\Gamma\alpha\Gamma/\Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix}\alpha_1 & \\ & \alpha_2\end{pmatrix}\in T^-$

Hint. See [WWL, 引理5.1.4].

$$\#\Gamma\alpha\Gamma/\Gamma = \#\Gamma/\Gamma\cap\alpha\Gamma\alpha^{-1} = \#\Gamma/\Gamma_0\left(\tfrac{\alpha_2}{\alpha_1}\right) = \#\mathbb{P}^1\left(\tfrac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_2}{\alpha_1}\prod_{p\mid\frac{\alpha_2}{\alpha_1}}\left(1+\frac{1}{p}\right)$$

$$\left[\alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}\alpha^{-1} = \begin{pmatrix} a & \frac{\alpha_1}{\alpha_2}b\\ \frac{\alpha_2}{\alpha_1}c & d\end{pmatrix} \quad\Rightarrow \Gamma\cap\alpha\Gamma\alpha^{-1} = \begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ \frac{\alpha_2}{\alpha_1}\mathbb{Z} & \mathbb{Z}\end{pmatrix}_{\det=1} = \Gamma_0\left(\tfrac{\alpha_2}{\alpha_1}\right)\right]$$

e.g. $\#\Gamma\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix}1 & 0\\ 0 & p\end{pmatrix}\Gamma/\Gamma = p+1$, $\#\Gamma\begin{pmatrix}1 & 0\\ 0 & p^e\end{pmatrix}\Gamma/\Gamma = p^e+p^{e-1}$

The desired measure can not be realized here, i.e.,

a Haar measure μ on $GL_2^+(\mathbb{Q})$ s.t. $\mu(\Gamma)=1$.

Reason: measure satisfies countable additivity, and Γ is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure $\rightsquigarrow$ semimeasure)

Toy e.g.: There is no way to define a Haar measure μ on $\mathbb{Q}$ s.t. $\mu(\mathbb{Z})=1$.

However, if we only require finite additivity, we can do it:

Def (Semimeasure on $\mathbb{Q}$)

For any periodic set $X \subseteq \mathbb{Q}$ (i.e., $\exists m \in \mathbb{Q}_{>0}$ s.t. $m+X=X$)

we set

$$\mu(X) := \frac{1}{m}\left|X/m\mathbb{Z}\right| = \frac{1}{m}\left|X\cap[0,m)\right|$$

Rmk. 1. $\mathbb{Z} \supset \mathbb{Z}\cap m\mathbb{Z} \subset m\mathbb{Z}$ $\quad \left|\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}\right|,\ \left|m\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}\right| < +\infty$

"$m\mathbb{Z}$ are all commensurable gps of $\mathbb{Z}$"

2. $X = \bigsqcup_{\alpha\in\Lambda} \alpha + m\mathbb{Z}$ for some $\Lambda \subseteq \mathbb{Q}/m\mathbb{Z}$

"X is a commensurable set of $\mathbb{Z}$ (when $\mu(X)<+\infty$)"

Long A: Def. (Semimeasure on $GL_2^+(\mathbb{Q})$)

For any gp $H \leq GL_2^+(\mathbb{Q})$ which is commensurable with Γ (i.e., $\#H/H\cap\Gamma$, $\#\Gamma/H\cap\Gamma$ are finite), set

$$\mu(H) = \frac{|H/H\cap\Gamma|}{|\Gamma/H\cap\Gamma|} \overset{\text{if } H\leq\Gamma}{=\!=} \frac{1}{|\Gamma/H|} \in \mathbb{Q}_{>0}$$

Similarly we can specify μ to any commensurable set $X \subseteq GL_2^+(\mathbb{Q})$.

$$\left(\begin{array}{lll} \text{i.e.,} \quad X = \bigsqcup_{\alpha\in\Lambda} \alpha H & \text{for some} & H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ \quad\quad\ \ X = \bigsqcup_{\alpha\in\Lambda'} H'\alpha & & \Lambda \subseteq GL_2^+(\mathbb{Q})/H,\ \Lambda' \subseteq H'\backslash GL_2^+(\mathbb{Q}) \\ & & \Lambda, \Lambda' \text{ finite} \end{array}\right)$$

Rmk: In the most of references the terminology (semi)measure is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and take the second line as a def of the convolution.

Def. (Hecke alg $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma)$)

$$\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \to \mathbb{Z} \;\middle|\; \begin{array}{l} f(\gamma_1\alpha\gamma_2) = f(\alpha) \quad \forall \gamma_1,\gamma_2\in\Gamma,\ \alpha\in GL_2^+(\mathbb{Q}) \\ \#\left(\operatorname{supp} f/\Gamma\right) < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \sum_{x\in GL_2^+(\mathbb{Q})/\Gamma} f_1(x) f_2(x^{-1}g) = \sum_{y\in\Gamma\backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1}) f_2(y)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha \in T^-} \mathbb{Z} \cdot \mathbb{1}_{\Gamma\alpha\Gamma}$$

denote $T_\alpha := \mathbb{1}_{\Gamma\alpha\Gamma}$

$\lambda \in \mathbb{Q}^\times$ $\quad R_\lambda := T_{\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)\Gamma} = \mathbb{1}_{\lambda\Gamma}$ $\quad$ ($R_1 = \mathbb{1}_\Gamma$ is the unit of $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$)

p prime, $e \geq 1$ $\quad T_{p^e} := T_{\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)\Gamma}$ $\quad T_p := T_{\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)\Gamma}$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T^-} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) &= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma) \\ &= \#\left\{ x \in GL_2^+(\mathbb{Q})/\Gamma \;\middle|\; \begin{array}{l} x \in \Gamma\alpha\Gamma \\ x^{-1}\gamma \in \Gamma\beta\Gamma \end{array} \right\} \\ &= \left| \Gamma\alpha\Gamma \cap \gamma\Gamma\beta^{-1}\Gamma/\Gamma \right| \end{aligned}$$

e.p. $\quad \mathbb{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$

$R_\lambda * R_\mu = R_{\lambda\mu}$

E.g. $\quad g_{\alpha\beta}^{\gamma} \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad$ where $|\alpha| := \det \alpha$

The formula above is still not feasible for effective calculation. We will derived the easiest way to comute $g_{\alpha\beta}^{\gamma}$ in the next page.

Suppose $\Gamma\alpha\Gamma/\Gamma = \{x_1\Gamma, \cdots, x_i\Gamma, \cdots\}$
$\Gamma\beta\Gamma/\Gamma = \{y_1\Gamma, \cdots, y_j\Gamma, \cdots\}$

then

$$
\begin{aligned}
g_{\alpha\beta}^{\gamma} &= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma) \\
&= \sum_i T_\beta(x_i^{-1}\gamma) \\
&= \sum_i \mathbb{1}_{x_i^{-1}\gamma \in \Gamma\beta\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i^{-1}\gamma \in y_j\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i y_j \Gamma = \gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{x_i y_j \in \Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma x_i y_j \Gamma = \Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma\alpha y_j \Gamma = \Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|} \sum_{y \in \Gamma\beta\Gamma/\Gamma} \mathbb{1}_{\Gamma\alpha y\Gamma = \Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma'\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}
\end{aligned}
$$

where

$$\Gamma' := \{\gamma' \in \Gamma \mid \alpha\gamma'\beta \in \Gamma\gamma\Gamma\} = \alpha^{-1}\Gamma\gamma\Gamma\beta^{-1} \cap \Gamma$$
$$\Rightarrow \Gamma'\beta\Gamma/\Gamma = \alpha^{-1}\Gamma\gamma\Gamma \cap \Gamma\beta\Gamma/\Gamma$$

depends on α, β, γ.

The rest is a routine work.

Ex. $\Gamma\begin{pmatrix}1 & \\ & m\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1 & \\ & n\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1 & \\ & mn\end{pmatrix}\Gamma$ $\quad (m,n)=1$

$\Gamma\begin{pmatrix}1 & \\ & p^e\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1 & \\ & p^{e+1}\end{pmatrix}\Gamma \sqcup \Gamma\begin{pmatrix}p & \\ & p^e\end{pmatrix}\Gamma$ $\quad p$ prime, $e \geq 1$

[Hint. $\begin{pmatrix}1 & \\ & m\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & \\ & n\end{pmatrix} = \begin{pmatrix}a & nb\\ mc & mnd\end{pmatrix} \in \Gamma\begin{pmatrix}l & \\ & \frac{mn}{l}\end{pmatrix}\Gamma$ for $\begin{pmatrix}a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$
$l = \gcd(a, nb, mc, mnd)$]

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z}\cdot T_{mn} & (m,n)=1 \\ T_{p^e} * T_p \in \mathbb{Z}\cdot T_{p^{e+1}} + \mathbb{Z}\, T_{p^{e-1}} R_p & p \text{ prime}, e \geq 1 \end{cases}$$

Computation of coefficient:

when $(m,n)=1$, $\alpha = \begin{pmatrix} 1 & \\ & m \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & \\ & n \end{pmatrix}$, $\gamma = \begin{pmatrix} 1 & \\ & mn \end{pmatrix}$.

$$g_{\alpha\beta}^{\gamma} = \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{x_i y_j \in \Gamma\gamma\Gamma}$$

$$= \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}$$

$$= 1$$

when p is prime, $e \geq 1$, $\alpha = \begin{pmatrix} 1 & \\ & p^e \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & \\ & p \end{pmatrix}$, $\gamma_2 = \begin{pmatrix} p & \\ & p^e \end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 1 & \\ & p^{e+1} \end{pmatrix}$,

$$\Gamma_2' \triangleq \{\gamma' \in \Gamma \mid \alpha\gamma'\beta \in \Gamma\gamma_2\Gamma\}$$

$$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \gcd(a, pb, p^e c, p^{e+1} d) = p\right\}$$

$$= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 0 & * \\ * & * \end{pmatrix} \bmod p\right\}$$

$$= \Gamma^0(p) \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$|\Gamma_2'\beta\Gamma/\Gamma| = \left|\Gamma^0(p) \begin{pmatrix} & p \\ -1 & \end{pmatrix} \Gamma/\Gamma\right|$$

$$= |\Gamma^0(p)| \Big/ \left|\begin{pmatrix} & p \\ -1 & \end{pmatrix} \Gamma^0(p) \begin{pmatrix} & p \\ -1 & \end{pmatrix}^{-1} \cap \Gamma^0(p)\right|$$

$$= |\Gamma^0(p)| / |\Gamma^0(p)|$$

$$= 1$$

$$\left[\begin{aligned} \begin{pmatrix} & p \\ -1 & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & p \\ -1 & \end{pmatrix}^{-1} &= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} 1 & \\ & p \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & \\ & p \end{pmatrix}^{-1}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}^{-1} \\ &= \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} a & \frac{1}{p}b \\ pc & d \end{pmatrix}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}^{-1} \\ &= \begin{pmatrix} pc & d \\ -a & -\frac{1}{p}b \end{pmatrix}\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\ &= \begin{pmatrix} d & -pc \\ -\frac{1}{p}b & a \end{pmatrix} \end{aligned}\right]$$

$\therefore$

$$g_{\alpha\beta}^{\gamma_2} = \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_2'\beta\Gamma/\Gamma|}{|\Gamma\gamma_2\Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot 1}{p^{e-1} - p^{e-2}}$$

$$= p$$

$$g_{\alpha\beta}^{\gamma_1} = \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_1'\beta\Gamma/\Gamma|}{|\Gamma\gamma_1\Gamma/\Gamma|}$$

$$= \frac{|\Gamma\alpha\Gamma/\Gamma|\,\left(|\Gamma\beta\Gamma/\Gamma| - |\Gamma_2'\beta\Gamma/\Gamma|\right)}{|\Gamma\gamma_1\Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot (p+1-1)}{p^{e+1} - p^e}$$

$$= 1$$

4. Conclusion. $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$

with $\begin{cases} T_m T_n = T_{mn} & (m,n)=1 \\ T_{p^e} T_p = T_{p^{e+1}} + p T_{p^{e-1}} R_p & p \text{ prime},\ e \geq 1 \end{cases}$

Task: generalize it to other congruence subgps.

p-adic Cartan decomposition ↙ not Grothendieck group!

Set $G = GL_2(F)$, $K_0 = GL_2(\mathcal{O}_F)$.

1. decomposition [Bump Prop 3.5]

$$M_{2\times 2}(\mathcal{O}_F) = \coprod_{0\le e_1\le e_2\le +\infty} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

$$M_{2\times 2}(\mathcal{O}_F)_{\det\neq 0} = \coprod_{0\le e_1\le e_2<+\infty} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

$$G = \coprod_{e_1\le e_2} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

Denote $T^- = \left\{\begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} \in G \,\middle|\, e_1\le e_2,\ e_1,e_2\in\mathbb{Z}\right\} \overset{\text{semi gp}}{\cong} \mathbb{Z}\oplus\mathbb{Z}_{\ge 0}$, then

$$G = \coprod_{\alpha\in T^-} K_0\alpha K_0$$

Ex. Verify that $K_0\alpha K_0/K_0$ is finite, and compute the order. $\alpha = \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix}\in T^-$

Hint.

$$\# K_0\alpha K_0/K_0 = \# K_0/K_0\cap\alpha K_0\alpha^{-1} = \# K_0/\Gamma_0(\mathfrak{p}^{e_2-e_1}) = \# \mathbb{P}^1(\mathcal{O}_F/\mathfrak{p}_F^{e_2-e_1}) = \begin{cases} q^{e_2-e_1}+q^{e_2-e_1-1} & e_1<e_2 \\ 1 & e_1=e_2\end{cases}$$

$$\left[\alpha\begin{pmatrix}a&b\\c&d\end{pmatrix}\alpha^{-1} = \begin{pmatrix}\mathcal{O} & \mathfrak{p}^{e_1-e_2}\\ \mathfrak{p}^{e_2-e_1} & \mathcal{O}\end{pmatrix} \Rightarrow K_0\cap\alpha K_0\alpha^{-1} = \begin{pmatrix}\mathcal{O}&\mathcal{O}\\ \mathfrak{p}^{e_2-e_1} & \mathcal{O}\end{pmatrix} = \Gamma_0(\mathfrak{p}^{e_2-e_1})\right]$$

e.g. $\#\Gamma\begin{pmatrix}\pi&0\\0&\pi\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix}1&0\\0&\pi\end{pmatrix}\Gamma/\Gamma = q+1$, $\#\Gamma\begin{pmatrix}1&0\\0&\pi^e\end{pmatrix}\Gamma/\Gamma = q^e+q^{e-1}$

Here we use the similar notation in modular form [https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/5.moduli_interpretation.pdf]:

$$\begin{array}{ccc}
\Gamma(\mathfrak{p}^e) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\mathrm{bal}.\Gamma_1(\mathfrak{p}^e) & \twoheadrightarrow & N(\mathcal{O}_F/\mathfrak{p}_F^e) = \begin{pmatrix}1&*\\0&1\end{pmatrix} \\
\cap & & \cap \\
\Gamma_1(\mathfrak{p}^e) & \twoheadrightarrow & \begin{pmatrix}1&*\\&*\end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(\mathfrak{p}^e) & \twoheadrightarrow & B(\mathcal{O}_F/\mathfrak{p}_F^e) = \begin{pmatrix}*&*\\0&*\end{pmatrix} \\
\cap & & \cap \\
\Gamma(\mathcal{O}) = K^0 = GL_2(\mathcal{O}_F) & \twoheadrightarrow & GL_2(\mathcal{O}_F/\mathfrak{p}_F^e)
\end{array}$$

Take the unique Haar measure on G s.t. $\mu(K^0)=1$, then

$$\mu(K^0tK^0) = \# K^0tK^0/K^0$$

μ is induced from the measure on coset G/K^0.

The Hecke algebra has been defined here:
https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.04.17_preliminary_facts_of_reps_of_p-adic_groups.pdf
We still recall the convolution here:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \sum_{x \in G/K_0} f_1(x) f_2(x^{-1}g) = \sum_{y \in K_0 \backslash G} f_1(gy^{-1}) f_2(y)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G, K_0) = \bigoplus_{\alpha \in T^-} \mathbb{Z} \cdot \mathbb{1}_{K_0 \alpha K_0}$$

denote $T_\alpha := \mathbb{1}_{K_0 \alpha K_0}$

$\lambda \in F^\times$ $\quad R_\lambda := T_{\begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}} = \mathbb{1}_{K_0 \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix} K_0} = \mathbb{1}_{\lambda K_0}$ $\quad$ ($R_1 = \mathbb{1}_{K_0}$ is the unit of $\mathcal{H}(G, K_0)$)

$e \geq 1$ $\quad T_{\pi^e} := T_{\begin{pmatrix} 1 & \\ & \pi^e \end{pmatrix}} = \mathbb{1}_{K_0 \begin{pmatrix} 1 & \\ & \pi^e \end{pmatrix} K_0}$ $\quad T_\pi := T_{\begin{pmatrix} 1 & \\ & \pi \end{pmatrix}} = \mathbb{1}_{K_0 \begin{pmatrix} 1 & \\ & \pi \end{pmatrix} K_0}$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T^-} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/K_0} T_\alpha(x) T_\beta(x^{-1}\gamma)$$

$$= \# \left\{ x \in G/K_0 \;\middle|\; \begin{matrix} x \in K_0 \alpha K_0 \\ x^{-1}\gamma \in K_0 \beta K_0 \end{matrix} \right\}$$

$$= \left| K_0 \alpha K_0 \cap \gamma K_0 \beta^{-1} K_0 / K_0 \right|$$

e.p. $\quad \mathbb{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1} g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$

$$R_\lambda * R_\mu = R_{\lambda\mu}$$

E.g. $\quad g_{\alpha\beta}^{\gamma} \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad$ where $|\alpha| := \det \alpha$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^{\gamma} = \frac{|K_0 \alpha K_0 / K_0|\, |K_0' \beta K_0 / K_0|}{|K_0 \gamma K_0 / K_0|}$$

where

$$K_0' := \{\gamma' \in K_0 \mid \alpha \gamma' \beta \in K_0 \gamma K_0\} = \alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0$$

$$\Rightarrow K_0' \beta K_0 / \Gamma = \alpha^{-1} K_0 \gamma K_0 \cap K_0 \beta K_0 / K_0$$

depends on α, β, γ.

$$\Rightarrow T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q\, T_{\pi^{e-1}} R_\pi$$

4. Conclusion (Tamagawa, Satake)

$$\mathcal{H}(G, K^\circ) = \mathbb{Z}[R_\pi^{\pm 1}, T_\pi] \quad \text{with}$$

$$T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q\, T_{\pi^{e-1}} \cdot R_\pi$$

e.p. $\mathcal{H}(GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z})) = \bigotimes_{\mathbb{Z}} \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$