

### § 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections (Characters)
4. L-fct
5. density theorem

Just for convenience, we allow

element  $\in_c$  class    class  $\subset_c$  class     $\{\dots | \dots\}_c$  be a class

We may add  $c$  to emphasize that the family can be a class, instead of set.

1. Galois rep

Setting  $G$ : arbitrary topo gp    e.g.  $G$  any Galois gp

If  $G$  profinite  $\Rightarrow$  open subgps are finite index subgps.

$\Delta$ : top field    e.g.  $\overline{\mathbb{F}_p}, \overline{\mathbb{Q}_p}, \mathbb{C}$ , don't want to mention  $\overline{\mathbb{Z}_p}$  now.

Def (cont Galois rep)  $(\rho, V) \in \text{rep}_{\Delta, \text{cont}}(G)$   
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{cont}$

$\nabla$   $\rho(G)$  can be infinite!    for Gal gp

E.g. When  $\text{char } F \neq l$ , we have  $l$ -adic cyclotomic character

$$\varepsilon_l: \text{Gal}(\overline{F}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \quad \sigma \mapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \quad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute  $\varepsilon_l$  for  $F = \mathbb{F}_p$ .

$$\mathbb{A}: \quad \varepsilon_l: \widehat{\mathbb{Z}} \cong \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \quad 1 \mapsto p$$

$\uparrow$  lift from  $\mathbb{Z} \rightarrow \mathbb{Z}_l^\times$

Ex. Compute  $\varepsilon_l$  for  $F = \mathbb{Q}_p$ .

$$\mathbb{A}: \quad \varepsilon_l: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow \text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p) \longrightarrow \text{Gal}(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p)$$

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\text{IIS}} & \mathbb{Z}_l^\times \\ \text{Frob} \mapsto 1 & \xrightarrow{\quad} & p \end{array}$$

Notice that

$$\begin{aligned} \text{Gal}(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p) &\cong \text{Gal}(\mathbb{F}_p(\zeta_{l^\infty})/\mathbb{F}_p) \cong \varprojlim_k (\mathbb{Z}/l^k \mathbb{Z})^\times \cong \mathbb{Z}_l^\times \\ x \in \widehat{\mathbb{Z}} \text{ fix } \zeta_{l^k}: &\Leftrightarrow \zeta_{l^k}^{p^x} = \zeta_{l^k} \\ &\Leftrightarrow p^x \equiv 1 \pmod{l^k} \end{aligned}$$

Ex. Compute  $\varepsilon_l$  for  $F = \mathbb{Q}_l$ .

Ex: Compute  $\varepsilon_i$  for  $i = \infty$ .

A:  $\varepsilon_i: \text{Gal}(\bar{\mathbb{Q}}_i/\mathbb{Q}_i) \longrightarrow \text{Gal}(\mathbb{Q}_i^{\text{ab}}/\mathbb{Q}_i) \longrightarrow \text{Gal}(\mathbb{Q}(\zeta_i^\infty)/\mathbb{Q}_i)$

$$\widehat{\mathbb{Q}_i^*} \cong \widehat{\mathbb{Z}} \times \mathbb{Z}_i^* \xrightarrow{\pi_{\mathbb{Z}_i^*}} \mathbb{Z}_i^*$$

Rmk. Usually we denote  $Z_l(1)$  as  $Z_l$  with twisted  $G_F$ -action by  $\varepsilon_l$ , i.e.,  
 $(\varepsilon_l, Z_l(1)) \in \text{rep}_{Z_l, \text{cont}}(G_F)$

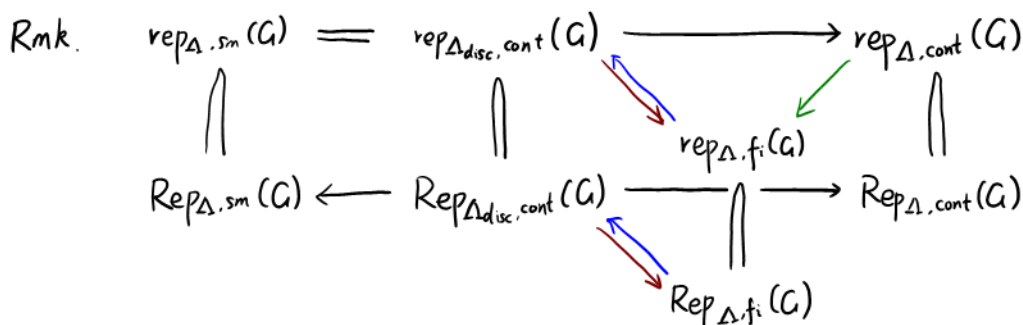
We use  $\varepsilon_i$  to twist reps.

$$V \in \text{Rep}_{\mathbb{Z}_\ell, \text{cont}}(G_F) \rightsquigarrow V(j) = V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes j} \in \text{Rep}_{\mathbb{Z}_\ell, \text{cont}}(G_F)$$

Notice the following two definitions don't depend on the topo of  $\Delta$ .

Def (sm Galois rep)  $(\rho, V) \in \text{rep}_{\Delta, \text{sm}}(G)$   
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{with open stabilizer.}$

Def (fin image Galois rep)  $(\rho, V) \in \text{rep}_{\Delta, \text{fi}}(G)$  with  $\text{fi}$ : finite image / finite index  
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V)$  with finite image



- : if fin index subgps are open (true when  $G$  is profinite + topo f.g.)
- : if  $G$ : profinite gp (Only need: open  $\Rightarrow$  fin index)
- : Artin rep (of profinite gp)

<https://math.stackexchange.com/questions/1526525/non-open-subgroups-of-finite-index-in-the-idele-class-group-of-a-number-field>

Artin rep:  $\Delta = (\mathbb{C}, \text{euclidean topo})$   $G$  profinite

Lemma 1 (No small gp argument)

$\exists U \subset GL_n(\mathbb{C})$  open nbhd of 1 s.t.  
 $\forall H \leq GL_n(\mathbb{C}), H \subseteq U \Rightarrow H = \{\text{Id}\}$ .

Proof. Take  $U = \{A \in GL_n(\mathbb{C}) \mid \|A - \text{Id}\| < \frac{1}{3n}\}$   $\|\cdot\| = \|\cdot\|_{\max}, \|1\| = 1 \cdot \|\max\|$   
 Only need to show,  $\forall A \in GL_n(\mathbb{C}), A \neq \text{Id}, \exists m \in \mathbb{N}$ , s.t.  $A^m \notin U$ .  
 Consider the Jordan form of  $A$ .

Case 1.  $A$  unipotent.

Case 2.  $A$  not unipotent.

$\exists \lambda \neq 1, v \in \mathbb{C}^n \setminus \{0\}$  s.t.  $Av = \lambda v$ . Take  $m \in \mathbb{N}$  s.t.  $|\lambda^m - 1| > \frac{1}{3}$ .

$\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - \text{Id})v\| \leq n \|A^m - \text{Id}\| |v| \Rightarrow \|A^m - \text{Id}\| \geq \frac{1}{3n}$ .

Prop. For  $(\rho, V) \in \text{rep}_{\mathbb{C}, \text{cont}}(G)$ ,  $\rho(G)$  is finite.  $G$  profinite

Proof. Take  $U$  in Lemma 1, then

$\rho^{-1}(U)$  is open  $\Rightarrow \exists I \leq G_F$  finite index,  $\rho(I) \subseteq U$   
 $\xRightarrow{\text{Lemma 1}} \rho(I) = \{\text{Id}\}$   
 $\Rightarrow \rho(G_F)$  is finite

Rmk. In general, any real Lie gp admits an open nbhd of 1 containing only  $\{1\}$  as a subgp.

Rmk. For Artin rep we can speak more:

1.  $\rho$  is conj to a rep valued in  $GL_n(\overline{\mathbb{Q}})$

$\rho$  can be viewed as cplx rep of fin gp, so  $\rho$  is semisimple.  
 Since classifications of irr reps for  $\mathbb{C}$  &  $\overline{\mathbb{Q}}$  are the same,  
 every irr rep is conj to a rep valued in  $GL_n(\overline{\mathbb{Q}})$ .

2.  $\#\{\text{fin subgps in } GL_n(\mathbb{C}) \text{ of "exponent } m"\}$  is bounded, see:  
<https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-n-c>

## 2. Weil-Deligne rep

Now we work over "the skeleton of the Galois gp" in general.

Setting:  $\Delta$ : NA local field with char  $k_\Delta = l$

Q: What would happen if  $\Delta$  is only a NA local field?

## Finite field

Goal: For  $\Delta$ : NA local field with char  $k_\Delta = l$ , understand  $\text{rep}_{\Delta, \text{cont}}(\hat{\mathbb{Z}})$ .

Def/Prop. Let  $A \in GL_n(\Delta)$ , TFAE:

①  $\hat{\mathbb{Z}} \rightarrow GL_n(\Delta)$  is a well-defined cont gp homo  
 $1 \mapsto A$

②  $\exists g \in GL_n(\Delta)$ ,  $gAg^{-1} \in GL_n(\mathcal{O}_\Delta)$

③  $\det(\lambda I - A) \in \mathcal{O}_\Delta[\lambda]$ , with  $\det A \in \mathcal{O}_\Delta^\times$

$A$  is called bounded in these cases.

## Proof

$$\textcircled{1} \xrightleftharpoons[\text{local}]{\text{local}} \textcircled{2} \xrightleftharpoons[\text{local}]{} \textcircled{3}$$

$\textcircled{1} \Rightarrow \textcircled{2}$ :  $\hat{\mathbb{Z}}$  is cpt, so image lies in a max cpt subgp of  $GL_n(\Delta)$ , which conjugates to  $GL_n(\mathcal{O}_\Delta)$

[https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl\\_2-mathbb-q-p](https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl_2-mathbb-q-p)

Another method:

Lemma 1:  $\rho, \mu$ : two ways of expressions of gp action

$\rho: \hat{\mathbb{Z}} \rightarrow GL_n(\Delta)$  is cont  $\Leftrightarrow \mu: \hat{\mathbb{Z}} \times \Delta^n \rightarrow \Delta^n$  is cont

$$\Rightarrow: \mu: \hat{\mathbb{Z}} \times \Delta^n \xrightarrow{\rho \times \text{Id}_{\Delta^n}} GL_n(\Delta) \times \Delta^n \xrightarrow{\quad} \Delta^n \quad \text{is cont.}$$

$\Delta^n \uparrow$  is Haus loc cpt.

See [Theorem III.3, III.4]:

[https://github.com/lrnmhl/AT1/blob/main/Algebraic\\_Topology\\_I\\_Stefan\\_Schwede\\_Bonn\\_Winter\\_2021.pdf](https://github.com/lrnmhl/AT1/blob/main/Algebraic_Topology_I_Stefan_Schwede_Bonn_Winter_2021.pdf)

$\Leftarrow$ :  $\rho: \hat{\mathbb{Z}} \rightarrow GL_n(\Delta)$  is cont

$\Leftrightarrow \rho: \hat{\mathbb{Z}} \rightarrow M_{n \times n}(\Delta)$  is cont

$\Leftrightarrow \rho_{ij}: \hat{\mathbb{Z}} \rightarrow \Delta$  is cont  $\forall i, j \in \{1, \dots, n\}$

We know that

$$\rho_{ij}: \hat{\mathbb{Z}} \xrightarrow{(\text{Id}, e_i)} \hat{\mathbb{Z}} \times \Delta^n \xrightarrow{\mu} \Delta^n \xrightarrow{e_i^*} \Delta$$

is cont

linear map between f.d v.s is cont

In this case,  $e_i^*$  is projection.

Another  $\Leftarrow$  : (suggested by Longke Tang)

$$\Leftrightarrow \begin{array}{ccc} \mu: \widehat{\mathbb{Z}} \times \widehat{\mathbb{Z}} \times \Lambda^n & \longrightarrow & \Lambda^n \text{ is cont} \\ & \searrow & \text{open cpt topo} \\ & \widehat{\mathbb{Z}} & \xrightarrow{\exists!} \text{Mor}_{\text{Top}}(\Lambda^n, \Lambda^n) \text{ is cont} \\ & & \text{GL}_n(\Lambda) \end{array}$$

Only need:  $\text{GL}_n(\Lambda) \subseteq \text{M}_{n \times n}(\Lambda)$ ,  $\text{GL}_n(\Lambda) \subset \text{Mor}_{\text{Top}}(\Lambda^n, \Lambda^n)$   
define the same topo on  $\text{GL}_n(\Lambda)$ .

This is hard. Assuming Lemma 1, this can be proved,  
but then this method can't be a real proof for Lemma 1.

Lemma 2.  $\mathcal{L}_1, \mathcal{L}_2$  lattice in  $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$  lattice in  $\Lambda$

$$\left[ \begin{array}{l} \mathcal{L}_1 \supseteq (\mathfrak{p}^{k_1})^{\oplus n} \\ \mathcal{L}_2 \supseteq (\mathfrak{p}^{k_2})^{\oplus n} \end{array} \right] \Rightarrow \# \mathcal{L}_1 + \mathcal{L}_2 / \mathcal{L}_1 < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice}$$

Take  $\mathcal{L}_1 = \mathcal{O}_{\Lambda}^n \subseteq \Lambda^n$ , then the stabilizer

$$\begin{aligned} \text{Stab}(\mathcal{L}) &= \{g \in \widehat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L}\} \\ &= \{g \in \widehat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \ \forall i\} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where

$$\mu_{e_i}: \widehat{\mathbb{Z}} \longrightarrow \Lambda^n \quad g \mapsto g \cdot e_i \quad (\text{cont by Lemma 1})$$

$\Rightarrow \mathcal{L}$  has finite orbit  
 $\xRightarrow{\text{Lemma 2}} \sum_{i \in \mathbb{Z}} \mathcal{L}_i$  is a lattice stabilized by  $\mathbb{Z}$ .

After conjugation,  $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

②  $\Rightarrow$  ①: w.l.o.g.  $A \in GL_n(\mathcal{O}_\Delta)$ . Then we get a lift

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\exists! \text{ cont}} & \widehat{GL_n(\mathcal{O}_\Delta)} \cong GL_n(\mathcal{O}_\Delta) \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & GL_n(\mathcal{O}_\Delta) \end{array}$$

②  $\Rightarrow$  ③: Obvious

③  $\Rightarrow$  ②:  $\sum_{i \in \mathbb{Z}} A^i \mathcal{L} = \sum_{i=0}^{n-1} A^i \mathcal{L}$  is a lattice fixed by  $A, A^{-1}$  (Lemma 2)

After conjugation,  $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

$\nabla A, B \in GL_n(\Delta)$  bounded  $\not\Rightarrow AB$  bounded  
 counter eg: (from Longke Tang)

Consider  $A = \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}^{-1}$ ,  $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ , then  $AB = \begin{pmatrix} p & p^{-1} \\ 1 & 1 \end{pmatrix}$ .

Cor.  $\text{rep}_{\Delta, \text{cont}}(\widehat{\mathbb{Z}}) \cong \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(\mathbb{Z})$   
 $\hookrightarrow$  full subcategory of  $\text{rep}_{\Delta, \text{cont}}(\mathbb{Z})$ .

Local field,  $p \neq l$

Goal. For  $\Delta$ : NA local field with  $\text{char } K_\Delta = l$ ,

$F$ : NA local field with  $\text{char } K_F = p \neq l$ ,

realize cont Galois rep as bounded Weil-Deligne rep.  
via the following diagrams:

$$\begin{array}{ccccc} & & \text{rep}_{\Delta, \text{sm}}(W_F) \text{ with } N & & \\ & & \cup & & \\ & \swarrow & & \searrow & \\ & \text{rep}_{\Delta, \text{cont}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}(W_F) & \\ & \cup & & \cup & \\ \text{rep}_{\Delta, \text{cont}}(\Gamma_F) & \xrightarrow{\sim} & \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}^{\text{bdd}}(W_F) \end{array}$$

here, "bdd" means  $\text{Im } \rho$  are bounded.

Step 1. Realize rep of  $G_F$  as rep of  $W_F$ .

$$\text{rep}_{\Delta, \text{cont}}(\Gamma_F) \xrightarrow{\sim} \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F)$$

Step 2. Go from cont rep to sm rep.

$$\begin{array}{ccccc} & & \text{rep}_{\Delta, \text{cont}}(W_F) & \xleftarrow{?} & \text{rep}_{\Delta, \text{sm}}(W_F) \\ & & \cup & & \\ \text{rep}_{\Delta, \text{cont}}(\Gamma_F) & \xrightarrow{\sim} & \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F) & & \\ & & \Downarrow \text{Monodromy} & & \\ & & \text{rep}_{\Delta, \text{cont}}(W_F) & \xleftarrow{?} & \text{rep}_{\Delta, \text{sm}}(W_F) \text{ with } N \\ & & \cup & & \\ \text{rep}_{\Delta, \text{cont}}(\Gamma_F) & \xrightarrow{\sim} & \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}(W_F) \end{array}$$

Step 3. Boundness is preserved.

$$\begin{array}{ccccc} & & \text{rep}_{\Delta, \text{sm}}(W_F) \text{ with } N & & \\ & & \cup & & \\ & \swarrow & & \searrow & \\ & \text{rep}_{\Delta, \text{cont}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}(W_F) & \\ & \cup & & \cup & \\ \text{rep}_{\Delta, \text{cont}}(\Gamma_F) & \xrightarrow{\sim} & \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}^{\text{bdd}}(W_F) \end{array}$$

In Step 2,  $(r, N) \in \text{WDrep}_{\Delta, \text{sm}}(W_F)$  should satisfy that

$$r(\sigma) N r(\sigma)^{-1} = (\#x)^{-v_F(\sigma)} N \quad \forall \sigma \in W_F$$

$$r: W_F \rightarrow \text{GL}(V)$$

$$N \in \text{End}(V)$$

$$v_F: W_F \rightarrow \mathbb{Z}$$

By the monodromy, for  $\forall \rho \in \text{rep}_{\Delta, \text{cont}}(W_F), \exists N \in \text{End}(V)$  s.t.  $\exists E/F$  finite,  $\forall \sigma \in I_E$ .

$$\rho(\sigma) = e^{N \cdot t_{\sigma, \rho}(\sigma)}$$

Special cases:

- $\rho(I_F) = \text{Id} \rightsquigarrow$  Finite field case (unramified)
- semistable
- 1-dim case
- 2-dim case: Steinberg rep &  $N=0$  case.

Def. For  $(\rho, V) \in \text{rep}_{\Delta, \text{cont}}(G_F)$ ,

$$\begin{aligned} \text{semistable: } \rho(I_F) &\subseteq \{\text{unipotent matrices}\} \\ \text{potentially semistable: } \rho(I_E) &\subseteq \{\text{unipotent matrices}\} \text{ for some } E/F \text{ fin Galois} \\ &\Leftrightarrow \rho(I) \subseteq \{\text{unipotent matrices}\} \text{ for some } I \leq I_F \text{ fin index.} \end{aligned}$$



Local field,  $p=l$

Goal: make a hierarchy for Galois representations when  $p=l$ .

Thm (Hodge decomposition)

For  $X/\mathbb{Q}$  sm proper variety,  $\exists$  iso

$$H_{\text{sing}}^n(X(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong \bigoplus_{i+j=n} H^i(X; \Omega_{X/\mathbb{C}}^j)$$

$\uparrow$  reduced  
 $\parallel$  (de-Rham comparison)

$$H_{\text{dR}}^n(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

Thm (Hodge-Tate decomposition)

For  $F/\mathbb{Q}_p$  NA local field,  $X_F$  sm proper variety,  $\exists \Gamma_F$ -equiv iso

$$H_{\text{ét}}^n(X_F; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega_{X/F}^j) \otimes_F \mathbb{C}_p(-j)$$

$\uparrow \Gamma_F$                        $\downarrow \Gamma_F$

Thm (Tate) Consider the cont coh, then

$$H^i(\Gamma_F, \mathbb{C}_p(j)) = \begin{cases} F, & i=0, 1, \quad j=0 \\ 0, & \text{otherwise.} \end{cases}$$

As a Corollary,

$$\mathbb{C}_p^{\Gamma_F} = H^0(\Gamma_F, \mathbb{C}_p) = F,$$

$$\text{Hom}_{\text{Rep}_{\mathbb{C}_p, \text{cont}}(\Gamma_F)}(\mathbb{C}_p(i), \mathbb{C}_p(j)) \cong H^0(\Gamma_F, \mathbb{C}_p(j-i)) \cong \begin{cases} F, & i=j \\ 0, & i \neq j \end{cases}$$

Def (HT period ring)

$$B_{\text{HT}} := \bigoplus_{j \in \mathbb{N}} \mathbb{C}_p(j) = \mathbb{C}_p[t, t^{-1}] \in \text{Rep}_{\mathbb{C}_p, \text{cont}}(\Gamma_F) \quad \text{by}$$

$$\sigma\left(\sum_{i=-\infty}^{+\infty} a_i t^i\right) = \sum_{i=-\infty}^{+\infty} \sigma(a_i) \varepsilon_p^i(\sigma) t^i \quad \rightsquigarrow B_{\text{HT}}^{\Gamma_F} = F$$

Cor 1 of Hodge-Tate dec

$$(H_{\text{ét}}^n(X_F; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}})^{\Gamma_F} \cong \bigoplus_{i+j=n} H^i(X; \Omega_{X/F}^j)$$

Def.  $V \in \text{rep}_{\mathbb{Q}_p, \text{cont}}(\Gamma_F)$  is called HT ( $B_{\text{HT}}$ -admissible), if

$$\dim_F (V \otimes_{\mathbb{Q}_p} B_{\text{HT}})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$$

By Hodge-Tate dec & Cor 1,  $H_{\text{ét}}^n(X_F; \mathbb{Q}_p)$  is HT.

Rmk. HT property is stable under subquotients.

Def. For  $V$  HT rep, define its HT weight by  
 $\{ \dots, \underbrace{j, \dots, j}_{m_j \text{ many}}, \dots \}$   $m_j = \dim_F (V^{\otimes_{\mathbb{Q}_p} \mathbb{C}_p(j)})^{\Gamma_F}$   
 $" = \dim_F H^{n-1}(X; \Omega_{X/F}^j)"$

e.g.  $H^i(X; \Omega_{X/F}^j) \cong (H_{\text{ét}}^{i+j}(X_F; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j))^{\Gamma_F}$   
 is HT, with HT weight  $\{ \underbrace{j, \dots, j}_{\dim H^i(\dots) \text{ many}} \}$ .

Ex. i) For  $\eta \in \text{Char}_{\mathbb{Z}_p, \text{cont}}(\Gamma_F)$ ,

$\eta$  is HT  $\Leftrightarrow \exists n \in \mathbb{Z}$  s.t.  $\varepsilon_p^{-n} \eta$  is potentially unramified

e.p. for  $a \in \mathbb{Z}_p$ ,

$\eta = (\varepsilon_p^{-1})^a$  is HT  $\Leftrightarrow a \in \mathbb{Z}$

ii) For  $\eta \in \text{Char}_{\overline{\mathbb{Q}_p}, \text{cont}}(\Gamma_F)$ ,

$\eta$  is HT  $\Leftrightarrow \exists U \subset F^\times$  open, for each  $\tau: F \hookrightarrow \overline{\mathbb{Q}_p}$ ,  $\exists n_\tau \in \mathbb{Z}$  s.t.  $\forall \alpha \in U$ ,  
 $(\eta \circ \text{Art}_F)(\alpha) = \prod_{\tau: F \hookrightarrow \overline{\mathbb{Q}_p}} \tau(\alpha)^{-n_\tau}$

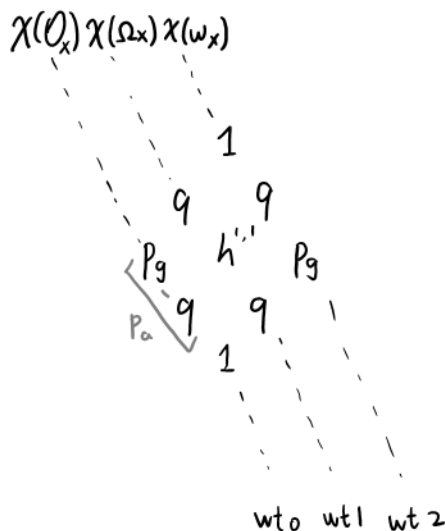
$$F^\times \xrightarrow{\text{Art}_F} W_F^{\text{ab}} \longrightarrow \Gamma_F^{\text{ab}} \xrightarrow{\eta} \overline{\mathbb{Q}_p}^\times$$

E.g. For  $A/\mathbb{Q}$  abelian variety of dim  $g$ ,

$$H^i(A(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i(A, \Omega_{A/\mathbb{C}}) \oplus H^i(A, \mathcal{O}_{A/\mathbb{C}})$$

$$H_{\text{ét}}^i(A_{\overline{\mathbb{Q}_p}}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^i(A, \Omega_{A/\mathbb{Q}_p}^j) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-i) \oplus H^i(A, \mathcal{O}_{A/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

HT wt of  $H_{\text{ét}}^i(A_{\overline{\mathbb{Q}_p}}; \mathbb{Q}_p)$ :  $\{ 1, 1, \dots, 1, 0, 0, \dots, 0 \}$



Def/Black box (B<sub>dR</sub>)

B<sub>dR</sub>/F is a filtered ring s.t.  
 $\text{gr}(B_{dR}) = B_{HT}$        $B_{dR}^{\Gamma_F} = F$

Thm (de Rham comparison)

$$\begin{aligned} H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} &\cong H_{dR}^n(X/F) \otimes_F B_{dR} \\ \leadsto (H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} &\cong H_{dR}^n(X/F) \\ \dim_F (H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} &= \dim_F H_{dR}^n(X/F) = \dim_{\mathbb{Q}_p} H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_p). \end{aligned}$$

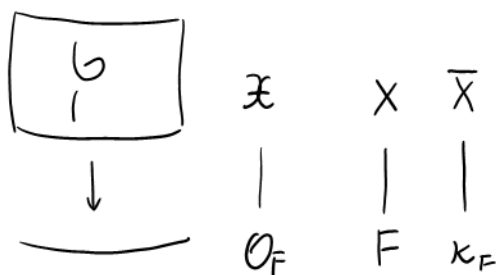
Def.  $V \in \text{rep}_{\mathbb{Q}_p, \text{cont}}(\Gamma_F)$  is called de Rham (B<sub>dR</sub>-admissible), if  
 $\dim_F (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$

Rmk. For  $V \in \text{rep}_{\mathbb{Q}_p, \text{cont}}(\Gamma_F)$ ,  
 $V = H_{\text{ét}}^n(X_{\overline{F}}, \mathbb{Q}_p)$  for some proper sm variety  $X/F$

$$\begin{aligned} &\Downarrow \\ \dim_F (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} &= \dim_{\mathbb{Q}_p} V \\ &\Downarrow \\ \dim_F (V \otimes_{\mathbb{Q}_p} B_{HT})^{\Gamma_F} &= \dim_{\mathbb{Q}_p} V \end{aligned}$$

Left (local) Fontaine Mazur, see:  
[mathoverflow.net/questions/340152/failure-of-local-fontaine-mazur](https://mathoverflow.net/questions/340152/failure-of-local-fontaine-mazur)

# Geometry.



When  $\text{char } F \neq p$ ,

$$\mathcal{X}/\mathcal{O}_F \text{ proper sm} \\ \Rightarrow H_{\text{ét}}^i(X_{\bar{F}}; \mathbb{Q}_p) \cong H_{\text{ét}}^i(\bar{X}_{\bar{K}_F}; \mathbb{Q}_p) \in \text{rep}_{\mathbb{Q}_p, \text{cont}}(G_F) \cong \text{WDrep}_{\mathbb{Q}_p, \text{sm}}^{\text{bdd}}(W_F)$$

$$\mathcal{X}/\mathcal{O}_F \text{ proper + semi-stable reduction} \\ \Rightarrow H_{\text{ét}}^i(X_{\bar{F}}; \mathbb{Q}_p) \in \text{WDrep}_{\mathbb{Q}_p, \text{sm}}^{\text{bdd}}(W_F) \text{ is semistable (i.e. } r \text{ is unramified)}$$

When  $\text{char } F = p$ , by [Gee, Thm 2.23],

$$X/F \text{ proper sm + good/semistable reduction} \\ \Rightarrow H_{\text{ét}}^i(X_{\bar{F}}; \bar{\mathbb{Q}}_p) \text{ is crystalline/semistable.}$$

Hierarchy  $\text{pot} = \text{potential}$

	$\{\text{crystalline}\} \subsetneq \{\text{semistable}\} \subsetneq \{\text{de-Rham}\} \subsetneq \{\text{HT}\}$ $\cap \quad \cap \quad \parallel \quad \parallel$ $\{\text{pot crystalline}\} \subsetneq \{\text{pot semistable}\} = \{\text{pot de-Rham}\} = \{\text{pot HT}\}$			
coming from compare with $\ell \neq p$ WD rep $\text{WD}(\rho) = (r, N)$	good red unramified reps $r$ unramified $N = 0$	semistable red $\rho _{I_F}$ unipotent $r$ unramified	dR comparison all reps defined HT weights	HT dec — — —
1-dim case $F = \mathbb{Q}_p$ $F$ : general $\Delta = \bar{\mathbb{Q}}_p$	$\rho _{I_F} = \varepsilon_p^n$ $(\chi \circ \text{Art}_F)(\alpha) = \prod_{\tau} \tau(\alpha)^{-n_{\tau}} \quad \forall \alpha \in \mathcal{O}_F^{\times}$		$\rho _{I_F} = \psi \varepsilon_p^n \quad n \in \mathbb{Z}, \psi \text{ finite order}$ $\varepsilon_p \rightsquigarrow \text{Lubin-Tate characters}$ $(\chi \circ \text{Art}_F)(\alpha) = \prod_{\tau} \tau(\alpha)^{-n_{\tau}} \quad \exists U \stackrel{\text{open}}{\subset} F^{\times} \quad \forall \alpha \in U$	

<https://mathoverflow.net/questions/111760/a-natural-way-of-thinking-of-the-definition-of-an-artin-l-function>

4.

References:

[https://en.wikipedia.org/wiki/Dirichlet\\_character](https://en.wikipedia.org/wiki/Dirichlet_character)

在算术几何中格罗藤迪克的 $l$ -进上同调( $l$ -adic cohomology)可以看作对于函数域(function field)上的 $L$ -函数( $L$ -function)的一种范畴化:

- a) 函数方程(functional equation)对应庞加莱对偶(Poincare duality)
- b) 欧拉分解(Euler factorisation)对应迹公式(trace formula)
- c) 解析延拓(analytic continuation)对应有限性(finitude)

from <https://www.zhihu.com/question/31823394/answer/54820421>