

Eine Woche, ein Beispiel

4.10. non-Archimedean local field F

wiki: local field

See <https://mathoverflow.net/questions/17061/locally-profinite-fields> for different definition of local fields. We follow wiki instead.

Classification:

- finite extension of \mathbb{Q}_p
- $\mathbb{F}_q((T))$ ($q = p^r$)

Process:

1. Basic structures and results.
2. Topological results.
3. Haar measure
4. Representation of $(F, +)$ and F^\times (next week)

1. Basic structures and results

1.1. None of them is alg closed.

1.2. The natural valuation $v: F \rightarrow \mathbb{Z} \cup \{\infty\}$ is defined. Then

$\mathcal{O}, \mathfrak{p}, \kappa = \mathcal{O}/\mathfrak{p}$ $p = \text{char } \kappa, q = |\kappa| = p^r$
 $\mathcal{U} = \mathcal{U}^{(0)} = \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\}$ $\mathcal{U}^{(n)} = 1 + \mathfrak{p}^n \quad n \geq 1$
 are defined, and $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$ is picked.

Moreover, \mathcal{O} is DVR, κ is finite,

$$\begin{aligned} \mathcal{U}^{(0)}/\mathcal{U}^{(1)} &\xrightarrow{\text{split iso}} \kappa^\times \\ \mathcal{U}^{(0)}/\mathcal{U}^{(n)} &\xrightarrow{\text{non-split iso}} (\mathcal{O}/\mathfrak{p}^n)^\times \quad n \geq 1 \end{aligned}$$

$$\begin{aligned} \mathcal{U}^{(n)}/\mathcal{U}^{(n+1)} &\xrightarrow{\text{non-canonical}} \kappa \\ \mathcal{U}^{(m+1)}/\mathcal{U}^{(n+1)} &\xrightarrow{\text{non-canonical}} \mathcal{O}/\mathfrak{p}^{m-n} \quad \begin{matrix} n \geq 1 \\ 2n+1 \geq m > n \geq 0 \end{matrix} \end{aligned}$$

$$0 \rightarrow \mathcal{U}^{(n)} \rightarrow \mathcal{O}^\times \xrightarrow{\quad \downarrow \cong \quad} \kappa^\times \rightarrow 0$$

$$\mu_{q-1} = \{a \in F \mid a^{q-1} = 1\}$$

\curvearrowright : the Teichmüller lift

$$\Rightarrow \mathcal{O}^\times \cong \mathcal{U}^{(n)} \times \mu_{q-1}$$

$$1.3. \quad F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times \cong \langle \pi \rangle \times \mu_{q-1} \times \mathcal{U}^{(n)}$$

$$\text{e.g. when } F = \mathbb{Q}_p, \quad \mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p & p \neq 2 \\ \mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2) & p = 2 \end{cases}$$

Thm. When $p \geq 3$, $(p\mathbb{Z}_p, +) \xrightleftharpoons[\log]{\exp} (1+p\mathbb{Z}_p, \cdot)$ is an iso as topological grps.

∇ The topology of F^\times is not the same as the subspace topology $F^\times \subset F$, but is the same as the subspace topology of $F^\times \cong GL_1(F) \cong \{(x, x') \in F^2\} \subset F^2$ (Maybe that is the origin of the topology of the Weil group).

2. Topological results.

$\mathcal{O} = \varprojlim_n \mathcal{O}/\mathfrak{p}^n$ is cpt and profinite group, while F is loc. cpt and loc. profinite group

$\mathcal{O}^\times = \varprojlim_n \mathcal{O}^\times/\mathcal{U}^{(n)}$ is cpt and profinite group, while F^\times is loc. cpt and loc. profinite group

Cpt open subgps of $(F, +)$ are $\{\mathfrak{p}^k\}$.

Cpt open subgps of F^\times are not restricted in $\{\mathcal{U}^{(k)}\}$,

but $\{\mathcal{U}^{(k)}\}$ is a nbhd system of F^\times , i.e.,

$\{a\mathcal{U}^{(k)}\}_{a \in F^\times}$ is a topological basis of F^\times .

$\{\text{open subgps}\} \subseteq \{\text{closed subgps}\}$ for $(F, +)$ and F^\times .

Q: Are there any other cpt closed subgp?

A: Yes. e.g. $\{0\} \subseteq (F, +)$ $\{1\} \subseteq F^\times$

Q: Can we classify all cpt closed subgp?

E.g. $\mathbb{Q}_{p^r} =$ the splitting field of $X^q - X$ over \mathbb{Q}_p $q = p^r$
 $=$ the unique unramified extension of \mathbb{Q}_p of degree r

$$\text{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \text{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p) \cong \mathbb{Z}/r\mathbb{Z}$$

3. Haar measure

Main reference: The Local Langlands Conjecture for $GL(2)$ by Colin J. Bushnell Guy Henniart.
 [https://link.springer.com/book/10.1007/3-540-31511-X]
 Ref: https://en.wikipedia.org/wiki/Haar_measure

G : loc. profinite gp
 $C_c^\infty(G) := \{f: G \rightarrow \mathbb{C} \mid f \text{ is loc. const}\}$
 $C_c^\infty(G) := \{f \in C_c^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$

Rmk. G has topo basis $\{g_k\}_{k \in \mathbb{N}}$ cpt open.

$\forall f \in C_c^\infty(G), \exists k \in \mathbb{N}$ cpt open, s.t.

$$f = \sum_{g \in G} a_g \mathbb{1}_{g_k} \quad a_g \in \mathbb{C} \quad \#\{g \in G \mid a_g \neq 0\} < +\infty$$

e.g. When $G = (F, +)$, $C_c^\infty(F) = \langle a + \beta^k \rangle_{\substack{a \in F \\ k \in \mathbb{Z}}}$
 when $G = F^\times$, $C_c^\infty(F^\times) = \langle a U^{(k)} \rangle_{\substack{a \in F^\times \\ k \in \mathbb{Z}_{\geq 0}}}$

Def (Left Haar integral & Left Haar measure)

integral: $I: C_c^\infty(G) \rightarrow \mathbb{C}$ s.t.
 • (left invariant) $I(f(g \cdot)) = I(f(\cdot))$
 • (positive) $I(f) \geq 0$

measure: $\mu_G: \mathcal{L}(G) \rightarrow \mathbb{R}$

Lebesgue σ -algebra, see
<https://math.stackexchange.com/question/s/3117419/lebesgue-sigma-algebra>

$\forall f \in C_c^\infty(G) \quad g \in G$

$\forall f \in C_c^\infty(G) \quad f \geq 0$

$S \subset G$ cpt open $\mapsto I(\mathbb{1}_S)$

The domain of I is not extended, so here it is not perfect.

relation/notation: $I(f) = \int_G f(g) d\mu_G(g)$

Rmk. Left Haar measure exists and is unique (up to scalar) on every loc. cpt gp G , see
<https://www.diva-portal.org/smash/get/diva2:1564300/FULLTEXT01.pdf>

Later on, Haar measure = left + right Haar measure.

E.g. Let μ be the Haar measure on F , then

μ^\times is a Haar measure on F^\times , and $(d\mu^\times(x) = \frac{d\mu(x)}{|x|})$
 $\int_{F^\times} f(x) d\mu^\times(x) = \int_F f(x) \frac{d\mu(x)}{|x|} \quad \forall f \in C_c^\infty(F^\times) \subset C_c^\infty(F)$

Let μ be the Haar measure on $A := M_{n \times n}(F)$, then

μ^\times is a Haar measure on $G := GL_n(F)$, and $(d\mu^\times(g) = \frac{d\mu(g)}{|\det g|^n})$

$\int_G f(g) d\mu^\times(g) = \int_A f(g) \frac{d\mu(g)}{|\det g|^n} \quad \forall f \in C_c^\infty(G) \subset C_c^\infty(A)$

Def. Unimodular: left Haar measure = right Haar measure

Rmk. G is cpt $\Rightarrow G$ is unimodular $\Leftrightarrow \delta_G = 1$

G is abelian $\Rightarrow G/Z(G)$ is \Updownarrow unimodular

where $\delta_G : G \rightarrow \mathbb{C}^\times$ is determined by

$$d\mu_G(g^{-1}xg) \stackrel{\text{left inv}}{=} d\mu_G(xg) = \delta_G(g) d\mu_G(x).$$

Actually, $\forall K \leq G$ cpt open, $\delta_G|_K = \mathbb{1}_K$.

e.g. $(F, +), (\mathcal{O}, +), F^\times, \mathcal{O}^\times$ are all unimodular.

e.g. $G = GL_2(\mathbb{Q}_p)$ is unimodular, while

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $M = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ are not unimodular.

It's claimed that every reductive gp over non-arch local field is unimodular, but I don't know the reference.

Any compact, discrete or Abelian locally compact group, as well as any connected reductive or nilpotent Lie group, is unimodular.
from [https://encyclopediaofmath.org/wiki/Unimodular_group]

<https://math.stackexchange.com/questions/323017/are-nilpotent-lie-groups-unimodular>

<https://mathoverflow.net/questions/114704/is-every-subgroup-of-a-connected-unimodular-matrix-lie-group-also-unimodular>

<https://mathoverflow.net/questions/267592/simple-proof-that-a-reductive-group-is-unimodular>