

Eine Woche, ein Beispiel

## 9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

- vector bundle over manifold
- module over ring

There are hundreds of different versions of it:

### - vector bundle over manifold 几何/几何分析

- diffe mfld • (real) differential v.b. over (real) differential mfld
- Riemann surface • cplx (analytic) line bundle over Riemann surface

### - sheaf over space 代数几何

- scheme theory • locally free sheaf on scheme
- coherent sheaf on scheme

### geo rep theory

- local system over (real/cplx) mfld
- perverse sheaf over Riemann surface (derived)
- simplicial set over category  $\Delta$

### - module over ring 代数

- comm alg • f.g. module over Noetherian commutative ring (with 1)

- rep of grp • group representation over group ( $\rightsquigarrow$  group algebra)

- p-adic rep • smooth representation over unimodular gp ( $\rightsquigarrow$  Hecke algebra  $H(G)$ ) smooth module

- quiver theory • quiver representation over quiver ( $\rightsquigarrow$  path algebra, bound quiver algebra)

- Lie algebra • Lie alg representation over Lie alg ( $\rightsquigarrow$  universal enveloping algebra)

### - Arithmetic Geometry 算数 $\rightsquigarrow$ p进分析

- hermitian line bundle over projective arithmetic variety  $X$

- adelic line bundle over essentially quasi-proj scheme

- over Berkovich analytic space  $X^{\mathrm{an}}$

- over formal scheme  $\mathrm{Spf} A$

- over rigid-analytic space  $K\text{-affinoid space}$

- over adic space  $\mathrm{Spa}(A, A^+)$

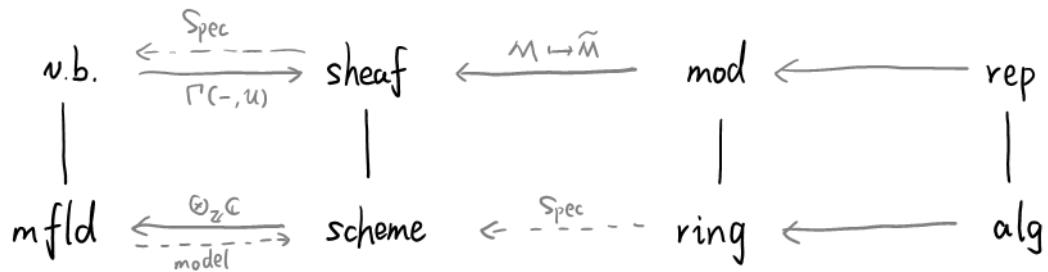
Picture.



① variation (e.g. v.b.  $\rightarrow$  f.b., mfld  $\rightarrow$  CW cplx, sheaf  $\rightarrow$  fct, scheme  $\rightarrow$  stack/adic space,...)

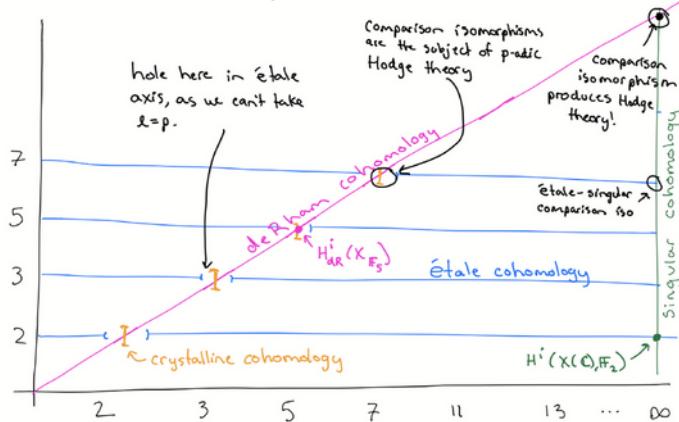
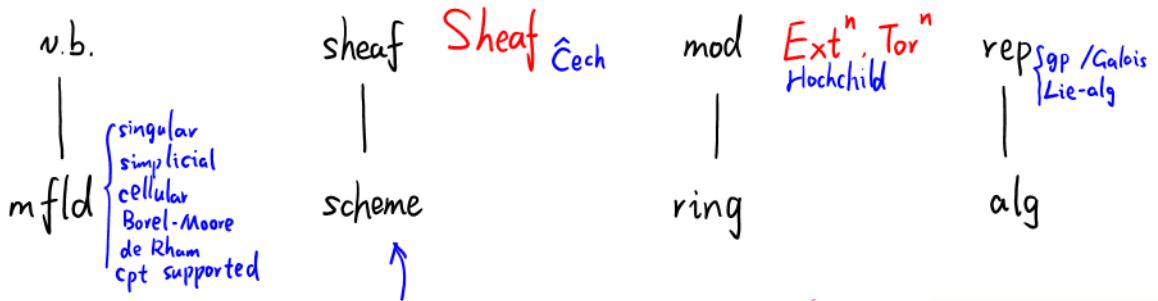
② vertical relation:  $\downarrow$ : v.b as mfld, representable fct, Spec/Proj construction, ...  
 $\uparrow$ : tangent/trivial v.b, structure sheaf, R as R-mod, regular rep, ...

③ horizontal relation:



For the parallels between curves and quivers, see <https://pbelmans.ncag.info/blog/2022/10/04/new-paper-no-git-quiver-git/>

④ homology and cohomology:  $\rightsquigarrow$  derived category



when axes meet: comparison isomorphisms (the "glue" of the "sheaf")

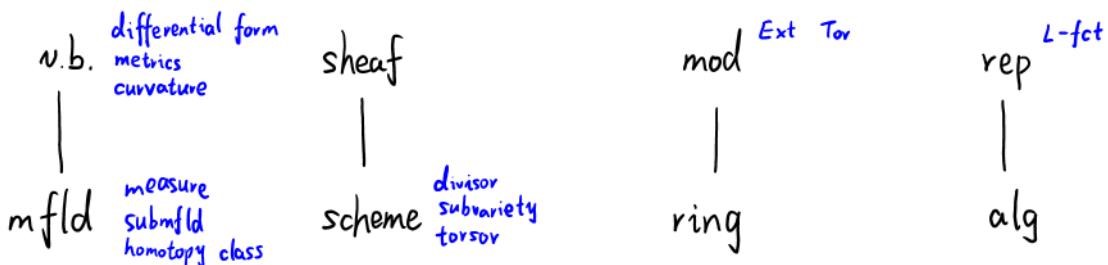
Prof. Scholze's ICM picture

<https://www.youtube.com/watch?v=5NPFQvdavgo>

Objects in upper row can be already viewed as element in (Co)homology.

eg. n.b.  $\leftrightarrow$  transition fit  $\leftrightarrow H^i(X, -)$

One motivation for  $\infty$ -category: make a generalization from  $H^i$  to  $H^i$



The following two pictures comes from here: <https://guests.mpim-bonn.mpg.de/gallauer/docs/meff.pdf>

Coefficients	cohomology groups
$D_c^b(X; \mathbb{Q}_\ell)$ constructible $\ell$ -adic sheaves	$\ell$ -adic cohomology
$D_c^b(X(\mathbb{C}); \mathbb{Z})$ constructible analytic sheaves	Betti cohomology
$D_h^b(\mathcal{D}_X)$ holonomic $\mathcal{D}$ -modules	de Rham cohomology
$D^b(\text{Coh}(X))$ coherent sheaves	coherent cohomology
$D^b(\text{MHM}(X))$ mixed Hodge modules	absolute Hodge cohomology
$DM(X)$ Voevodsky motivic sheaves	(weight-0) motivic cohomology
$SH(X)$ stable motivic homotopy sheaves	stable motivic (weight-0) cohomotopy groups

## ⑤ Relative point of view (for (co)homology) Six functors formalism (all are derived)

cohomology	$p_* p^* \mathbb{1}$	$H^\bullet$	$p_* \mathcal{F} \quad H^\bullet(-, \mathcal{F})$
cohomology with compact support	$p_! p^* \mathbb{1}$	$H_c^\bullet$	$p_! \mathcal{F} \quad H_c^\bullet(-, \mathcal{F})$
homology	$p_! p^! \mathbb{1}$	$H_\bullet$	
Borel-Moore homology	$p_* p^! \mathbb{1}$	$H_\bullet^{BM}$	

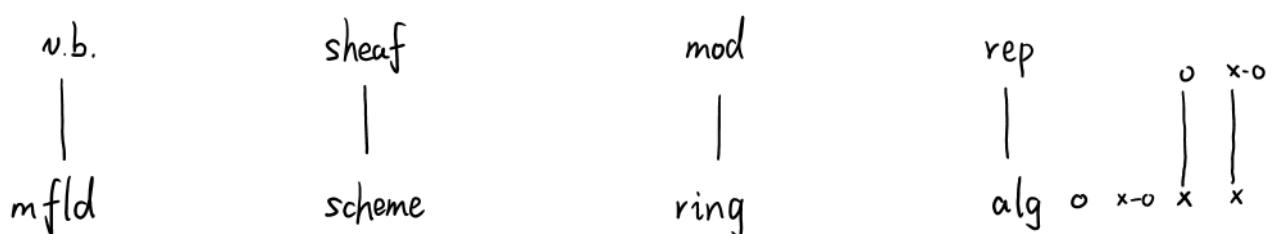
Fourier-Mukai factors

Chern class: from cohomology to cohomology (also for the other Chern class)

There are several ways of defining/viewing Chern class.

- i)  $L \in \text{Pic}(X) \rightarrow c(L) \in H^2(X; \mathbb{Z})$
- ii)  $H^*(X, \mathcal{O}_X^\times) \rightarrow H^*(X; \mathbb{Z})$  by LES
- iii) As the coefficient of equation ( $\text{CH}^*(\text{PE})$  is a free  $\text{CH}^*(B)$ -module)  
Euler class
- iv) As the pull back of the universal Chern class in Grassmannian
- v) From curvature; Chern-Weil theory
- vi) From Chow group
- vii)  $\partial\bar{\partial}, \Delta$

## ⑥ moduli problems



As a comparision, see the picture made by other people:  
A possibly nice introduction of the first column is here:  
[https://irma-web1.math.unistra.fr/~loday/PAPERS/A-O-C\(Lille2012\).pdf](https://irma-web1.math.unistra.fr/~loday/PAPERS/A-O-C(Lille2012).pdf)

SPIN	ALGEBRA	GEOMETRY	LINEAR PHYSICS	NON-LINEAR PHYSICS	COMPLEX-SYMPLECTIC DUALITY
1	modules	vector bundles	Maxwell equ.	Yang-Mills	Donaldson-Uhlenbeck-Yau
2	algebras	manifolds	Linear gravity equ.	Einstein gravity	Calabi-Yau theorem
3	operads	? (moduli spaces)	Rarita-Schwinger equ.	? (CFT)	? (Mirror symmetry)
	combinad				

Three type of geometry:

PDE	elliptic	parabolic	hyperbolic
curvature	+	0	-
genus	0	1	$\geq 2$
Euler number	-2	0	$\geq 2$
Kodaira dim	$-\infty$	0	$\dim X$
variety	Fano	Calabi-Yau	general type
filtration	unramified	tame	wild
quiver	Dynkin	affine	strictly wild
condensed	solid	liquid	gaseous

- Goal
- structures & invariants
  - classifications of
    - special v.b., mfld, subv.b., submfld
  - symmetry & quotient
  - special functors
  - homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on  $S^n$

(by homotopy group!  $\rightarrow$  generalized Picard group?)

Q: Is this group structure natural?

ref: <https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres>  
<https://www.math.ru.nl/~gutierrez/files/k-theory/Lecture06.pdf>

Thm.  $\{ \text{rank } m \text{ K-v.b. over } S^n \} \longleftrightarrow \pi_{n-1}(GL_m(\mathbb{K}))$   $\mathbb{K} = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ rank $n$	1	2	3	4	5	6	$>6$
$S^1$ 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$S^2$ 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$S^3$ 3	0	0	0	0	0	0	0
$S^4$ 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{(2)}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$S^5$ 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{(2)}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
$S^6$ 6	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{(2)}$	0	0	0

$$\mathbb{R}\mathbb{P}^\infty \cong \mathbb{K}(\mathbb{Z}/2\mathbb{Z}, 1)$$

$\pi_{n-1}(GL_m(\mathbb{C}))$ rank $n$	1	2	3	4	5	6	$>6$
$S^1$ 1	0	0	0	0	0	0	0
$S^2$ 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$S^3$ 3	0	0	0	0	0	0	0
$S^4$ 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$S^5$ 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
$S^6$ 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$$\mathbb{C}\mathbb{P}^\infty \cong \mathbb{K}(\mathbb{Z}, 2)$$

Problems. Describe the special bundles, e.g.  $TS^n$

Describe the operations, e.g. dual,  $\oplus, \otimes, \wedge^k, \text{Sym}^k, \text{Res}, \text{Ind}$

For the other spaces:

<https://math.stackexchange.com/questions/383838/classifying-vector-bundles>

<http://www.ms.uky.edu/~guillou/F18/751Notes.pdf>

It's still not so explicit.

<https://mathoverflow.net/questions/92337/total-spaces-of-ts2-and-s2-times-r2-not-homeomorphic>

$\{ \text{rank } m \text{ K-v.b. over } M \} \longleftrightarrow [M, \text{Gr}_k(m, \infty)]$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$

$M$ : paracompact

Unfinished task: introduce the concept of local systems and compute examples in [<https://arxiv.org/pdf/2103.02329.pdf>] , 16.3.

Here I collect some stuffs which may be useful for my PhD study.

Motivic Homotopy: <https://www.zhihu.com/question/555338503>

Nearby cycles and Bloch-Kato conjecture: <https://zhuanlan.zhihu.com/p/461885915>