§3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections (Characters)
4. L-fct
5. density theorem

Just for convenience, we allow

element $\in_c$ class class $\subset_c$ class $\{\cdots \mid \cdots\}_c$ be a class

We may add c to emphasize that the family can be a class, instead of set.

1. Galois rep (G ⇝ Γ is better)

<u>Setting</u> G: arbitrary topo gp e.g. G any Galois gp

If G profinite $\Rightarrow$ open subgps are finite index subgps.

Λ: top field e.g. $\overline{\mathbb{F}_p}, \overline{\mathbb{Q}_p}, \mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{cont}}(G)$

$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ cont

⚠ $\rho(G)$ can be infinite! for Gal gp

E.g. When $\mathrm{char}\, F \neq l$, we have l-adic cyclotomic character

$$\varepsilon_l : Gal(F^{sep}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \qquad \sigma \longmapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute ε_l for $F = \mathbb{F}_p$.

A: $$\varepsilon_l : \hat{\mathbb{Z}} \cong Gal(\overline{\mathbb{F}}_p/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \qquad 1 \longmapsto p$$

$\uparrow$ lift from $\mathbb{Z} \to \mathbb{Z}_l^\times$

Ex. Compute ε_l for $F = \mathbb{Q}_p$.

A:
$$\begin{tikzcd}
\varepsilon_l : Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \arrow[r] & Gal(\mathbb{Q}_p^{ur}/\mathbb{Q}_p) \arrow[r, two heads] \arrow[d, phantom, "\cong" rotate=-90] & Gal(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p) \arrow[d, phantom, "\cong" rotate=-90] \\
 & \hat{\mathbb{Z}} \arrow[r] & \mathbb{Z}_l^\times \\
\mathrm{Frob} \arrow[r, mapsto] & 1 \arrow[r, mapsto] & p
\end{tikzcd}$$

Notice that

$$Gal(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p) \cong Gal(\mathbb{F}_p(\zeta_{l^\infty})/\mathbb{F}_p) \cong \varprojlim_k (\mathbb{Z}/l^k\mathbb{Z})^\times \cong \mathbb{Z}_l^\times$$

$x \in \hat{\mathbb{Z}}$ fix ζ_{l^k}: $\Leftrightarrow \zeta_{l^k}^{p^x} = \zeta_{l^k}$

$\Leftrightarrow p^x \equiv 1 \mod l^k$

Ex. Compute ε_l for $F = \mathbb{Q}_l$.

A: $\varepsilon_l : \mathrm{Gal}(\overline{\mathbb{Q}}_l/\mathbb{Q}_l) \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}_l^{ab}/\mathbb{Q}_l) \longrightarrow\!\!\!\!\rightarrow \mathrm{Gal}(\mathbb{Q}_l(\zeta_{l^\infty})/\mathbb{Q}_l)$

$$\begin{array}{ccc} & \mathrm{Gal}(\mathbb{Q}_l^{ab}/\mathbb{Q}_l) & \twoheadrightarrow & \mathrm{Gal}(\mathbb{Q}_l(\zeta_{l^\infty})/\mathbb{Q}_l) \\ & \| \wr & & \| \wr \\ \widehat{\mathbb{Q}_l^\times} \cong & \widehat{\mathbb{Z}} \times \mathbb{Z}_l^\times & \xrightarrow{\pi_{\mathbb{Z}_l^\times}} & \mathbb{Z}_l^\times \end{array}$$

Rmk. Usually we denote $\mathbb{Z}_l(1)$ as $\mathbb{Z}_l$ with twisted Γ_F-action by ε_l, i.e.,

$$(\varepsilon_l, \mathbb{Z}_l(1)) \in \mathrm{rep}_{\mathbb{Z}_l, \mathrm{cont}}(\Gamma_F)$$

We use ε_l to twist reps.

$$V \in \mathrm{Rep}_{\mathbb{Z}_l, \mathrm{cont}}(\Gamma_F) \rightsquigarrow V(j) = V \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(1)^{\otimes j} \in \mathrm{Rep}_{\mathbb{Z}_l, \mathrm{cont}}(\Gamma_F)$$

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda, \mathrm{sm}}(\Gamma)$

$V \in \mathrm{vect}_\Lambda$ + $\rho : \Gamma \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda, \mathrm{fi}}(\Gamma)$ fi: finite image / finite index

$V \in \mathrm{vect}_\Lambda$ + $\rho : \Gamma \longrightarrow GL(V)$ with finite image

Rmk.

$$\begin{tikzcd}
\mathrm{rep}_{\Delta,\mathrm{sm}}(\Gamma) \arrow[r, equal] \arrow[dd, phantom, "\cap" rotate=90] & \mathrm{rep}_{\Delta_{\mathrm{disc}},\mathrm{cont}}(\Gamma) \arrow[rr] \arrow[dr, red, shift left] \arrow[dd, phantom, "\cap" rotate=90] & & \mathrm{rep}_{\Delta,\mathrm{cont}}(\Gamma) \arrow[dl, green] \arrow[dd, phantom, "\cap" rotate=90] \\
& & \mathrm{rep}_{\Delta,\mathrm{fi}}(\Gamma) \arrow[ul, blue, shift left] \arrow[dd, phantom, "\cap" rotate=90] & \\
\mathrm{Rep}_{\Delta,\mathrm{sm}}(\Gamma) & \mathrm{Rep}_{\Delta_{\mathrm{disc}},\mathrm{cont}}(\Gamma) \arrow[l] \arrow[rr] \arrow[dr, red, shift left] & & \mathrm{Rep}_{\Delta,\mathrm{cont}}(\Gamma) \\
& & \mathrm{Rep}_{\Delta,\mathrm{fi}}(\Gamma) \arrow[ul, blue, shift left] &
\end{tikzcd}$$

${\color{blue}\to}$: if fin index subgps are open (true when Γ is profinite + topo f.g.)

${\color{red}\to}$: if Γ: profinite gp (Only need: open $\Rightarrow$ fin index)

${\color{green}\to}$: Artin rep (of profinite gp)

https://math.stackexchange.com/questions/1526525/non-open-subgroups-of-finite-index-in-the-idele-class-group-of-a-number-field

Artin rep. $\Delta = (\mathbb{C}, \text{euclidean topo})$ Γ profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\|\cdot\| = \|\cdot\|_{max}$, $|\cdot| = |\cdot|_{max}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists m \in \mathbb{N}$, s.t $A^m \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unipotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$. Take $m \in \mathbb{N}$ s.t $|\lambda^m - 1| > \frac{1}{3}$.

$\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},\mathrm{cont}}(\Gamma)$, $\rho(\Gamma)$ is finite. G profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq \Gamma$ finite index. $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = Id$

$\Rightarrow \rho(\Gamma)$ is finite

Rmk. In general, any real Lie gp admits an open nbhd of 1 containing only {1} as a subgp.

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

   ρ can be viewed as cplx rep of fin gp. so ρ is semisimple.
   Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
   every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
   https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

Setting: Λ: NA local field with char $k_\Lambda = l$

Q: What would happen if Λ is only a NA local field?

Finite field

Goal. For Λ: NA local field with char $k_\Lambda = l$, understand $\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Lambda)$, TFAE:

① $\hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is a well-defined cont gp homo
$1 \longmapsto A$

② $\exists g \in GL_n(\Lambda)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Lambda)$

③ $\det(\lambda I - A) \in \mathcal{O}_\Lambda[\lambda]$, with $\det A \in \mathcal{O}_\Lambda^\times$

A is called bounded in these cases.

Proof

$$① \underset{\text{local}}{\overset{\text{local}}{\rightleftarrows}} ② \underset{\text{local}}{\rightleftarrows} ③$$

① $\Rightarrow$ ②: $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Lambda)$, which conjugates to $GL_n(\mathcal{O}_\Lambda)$

https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl2-mathbb-q-p

Another method:

Lemma 1. ρ, μ: two ways of expressions of gp action

$\rho: \hat{\mathbb{Z}} \to GL_n(\Lambda)$ is cont $\Leftrightarrow$ $\mu: \hat{\mathbb{Z}} \times \Lambda^n \to \Lambda^n$ is cont

$\Rightarrow$: $\mu: \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\rho \times Id_{\Lambda^n}} GL_n(\Lambda) \times \Lambda^n \longrightarrow \Lambda^n$ is cont.

Λ^n is Haus loc cpt

See [Theorem III.3, III.4]:
https://github.com/lrnmhl/AT1/blob/main/Algebraic_Topology_I___Stefan_Schwede__Bonn__Winter_2021.pdf

$\Leftarrow$: $\rho: \hat{\mathbb{Z}} \to GL_n(\Lambda)$ is cont

$\Leftrightarrow$ $\rho: \hat{\mathbb{Z}} \to M_{n\times n}(\Lambda)$ is cont

$\Leftrightarrow$ $\rho_{ij}: \hat{\mathbb{Z}} \to \Lambda$ is cont $\quad \forall i,j \in \{1,\dots,n\}$

We know that

$\rho_{ij}: \hat{\mathbb{Z}} \xrightarrow{(Id, e_j)} \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\mu} \Lambda^n \xrightarrow{e_i^*} \Lambda$

is cont

linear map between f.d v.s is cont

In this case, e_i^* is projection.

Another $\Leftarrow$: (suggested by Longke Tang)

$$\Leftrightarrow \quad \begin{array}{l} \mu: \hat{\mathbb{Z}} \times \Lambda^n \longrightarrow \Lambda^n \text{ is cont} \\ \hat{\mathbb{Z}} \longrightarrow \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n) \text{ is cont} \\ \hat{\mathbb{Z}} \xrightarrow{\exists!} GL_n(\Lambda) \subset \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n) \end{array}$$

open cpt topo

Only need: $GL_n(\Lambda) \subseteq M_{n\times n}(\Lambda)$, $GL_n(\Lambda) \subset \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n)$ define the same topo on $GL_n(\Lambda)$.

This is hard. Assuming Lemma 1, this can be proved, but then this method can't be a real proof for Lemma 1.

Lemma 2. $\mathcal{L}_1, \mathcal{L}_2$ lattice in $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$ lattice in Λ

$$\left[\left.\begin{array}{l} \mathcal{L}_1 \supseteq (\mathfrak{p}^{k})^{\oplus n} \\ \mathcal{L}_2 \supseteq (\mathfrak{p}^{k'})^{\oplus n} \end{array}\right\} \Rightarrow \# \, \mathcal{L}_1 + \mathcal{L}_2 / \mathcal{L}_1 < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice} \right]$$

Take $\mathcal{L} := \mathcal{O}_\Lambda^n \subseteq \Lambda^n$, then the stabilizer

$$\begin{aligned} \mathrm{Stab}(\mathcal{L}) &= \{ g \in \hat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L} \} \\ &= \{ g \in \hat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \quad \forall i \} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where

$$\mu_{e_i} : \hat{\mathbb{Z}} \longrightarrow \Lambda^n \qquad g \longmapsto g \cdot e_i \qquad \text{(cont by Lemma 1)}$$

$\xRightarrow{\quad}$ $\mathcal{L}$ has finite orbit

$\xRightarrow{\text{Lemma 2}}$ $\sum_{\mathcal{L}_i \in \hat{\mathbb{Z}} \cdot \mathcal{L}} \mathcal{L}_i$ is a lattice stabilized by $\mathbb{Z}$.

After conjugation, $A, A^{-1} \in M^{n\times n}(\mathcal{O}_\Lambda) \Rightarrow A \in GL_n(\mathcal{O}_\Lambda)$

② $\Rightarrow$ ①: w.l.o.g. $A \in GL_n(\mathcal{O}_\Lambda)$. Then we get a lift

$$\begin{tikzcd}
\hat{\mathbb{Z}} \arrow[r, red, "\exists ! \text{ cont}"] & \widehat{GL_n(\mathcal{O}_\Lambda)} \cong GL_n(\mathcal{O}_\Lambda) \\
\mathbb{Z} \arrow[u] \arrow[r] & GL_n(\mathcal{O}_\Lambda) \arrow[u]
\end{tikzcd}$$

② $\Rightarrow$ ③: Obvious

③ $\Rightarrow$ ②:

$$\sum_{i\in\mathbb{Z}} A^i \mathcal{L} = \sum_{i=0}^{n-1} A^i \mathcal{L} \quad \text{is a lattice fixed by } A, A^{-1} \text{ (Lemma 2)}$$

After conjugation, $A, A^{-1} \in M^{n\times n}(\mathcal{O}_\Lambda) \Rightarrow A \in GL_n(\mathcal{O}_\Lambda)$

⚠ $A, B \in GL_n(\Lambda)$ bounded $\not\Rightarrow$ AB bounded

counter eg: (from Longke Tang)

Consider $A = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}$, $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $AB = \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}$.

Cor. $\mathrm{rep}_{\Delta,\mathrm{cont}}(\hat{\mathbb{Z}}) \cong \mathrm{rep}^{\mathrm{bdd}}_{\Delta,\mathrm{cont}}(\mathbb{Z})$

↖ full subcategory of $\mathrm{rep}_{\Delta,\mathrm{cont}}(\mathbb{Z})$.

Local field, $p \neq l$

Goal. For Λ : NA local field with $\operatorname{char} k_\Lambda = l$,

F : NA local field with $\operatorname{char} k_F = p \neq l$,

realize cont Galois rep as bounded Weil-Deligne rep. via the following diagrams:

$$\begin{tikzcd}
 & & \mathrm{rep}_{\Lambda,\mathrm{sm}}(W_F) \text{ with } N \arrow[ddl] \\
 & & \cup \\
 & \mathrm{rep}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}_{\Lambda,\mathrm{sm}}(W_F) \\
 & \cup & \cup \\
\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \arrow[r, "\sim"] & \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}^{\mathrm{bdd}}_{\Lambda,\mathrm{sm}}(W_F)
\end{tikzcd}$$

here, "bdd" means $\operatorname{Im}\rho$ are bounded.

Step 1. Realize rep of G_F as rep of W_F.

$$\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \xrightarrow{\sim} \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F)$$

Step 2. Go from cont rep to sm rep.

$$\begin{tikzcd}
 & & \mathrm{rep}_{\Lambda,\mathrm{sm}}(W_F) \arrow[dl, "?"'] \\
 & \mathrm{rep}_{\Lambda,\mathrm{cont}}(W_F) & \\
 & \cup & \\
\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \arrow[r, "\sim"] & \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F) &
\end{tikzcd}$$

$$\Big\Downarrow \ \text{Monodromy}$$

$$\begin{tikzcd}
 & & \mathrm{rep}_{\Lambda,\mathrm{sm}}(W_F) \text{ with } N \arrow[ddl] \\
 & & \cup \\
 & \mathrm{rep}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}_{\Lambda,\mathrm{sm}}(W_F) \\
 & \cup & \\
\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \arrow[r, "\sim"] & \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F) &
\end{tikzcd}$$

Step 3. Boundness is preserved.

$$\begin{tikzcd}
 & & \mathrm{rep}_{\Lambda,\mathrm{sm}}(W_F) \text{ with } N \arrow[ddl] \\
 & & \cup \\
 & \mathrm{rep}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}_{\Lambda,\mathrm{sm}}(W_F) \\
 & \cup & \cup \\
\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \arrow[r, "\sim"] & \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}^{\mathrm{bdd}}_{\Lambda,\mathrm{sm}}(W_F)
\end{tikzcd}$$

In Step 2, $(r, N) \in WDrep_{\Lambda, sm}(W_F)$ should satisfy that

$$r(\sigma) N r(\sigma)^{-1} = (\#\kappa)^{-\nu_F(\sigma)} N \qquad \forall \sigma \in W_F$$

$r: W_F \to GL(V)$
$N \in End(V)$
$\nu_F: W_F \to \mathbb{Z}$

By the monodromy, for $\forall \rho \in rep_{\Lambda, cont}(W_F)$, $\exists N \in End(V)$ s.t. $\exists E/F$ finite,

$$\rho(\sigma) = e^{N t_{\zeta, p}(\sigma)} \qquad \forall \sigma \in I_E.$$

Special cases:

- $\rho(I_F) = Id$ $\rightsquigarrow$ Finite field case (unramified)
- semistable
- 1-dim case
- 2-dim case: Steinberg rep & $N = 0$ case.

Def. For $(\rho, V) \in rep_{\Lambda, cont}(G_F)$,

semistable: $\rho(I_F) \subseteq \{\text{unipotent matrices}\}$

potentially semistable: $\rho(I_E) \subseteq \{\text{unipotent matrices}\}$ for some E/F fin Galois

$\Leftrightarrow \rho(I) \subseteq \{\text{unipotent matrices}\}$ for some $I \leq I_F$ fin index.

Local field, $p=l$

Goal: make a hierarchy for Galois representations when $p=l$.

Thm (Hodge decomposition)
For $X/\mathbb{Q}$ sm proper variety, $\exists$ iso

reduced

$$\begin{array}{c} H^n_{sing}(X(\mathbb{C});\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \cong \bigoplus_{i+j=n} H^i(X;\Omega^j_{X/\mathbb{C}}) \\ \Big\| \text{(de-Rham comparison)} \\ H^n_{dR}(X/\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C} \end{array}$$

Thm (Hodge-Tate decomposition)
For $F/\mathbb{Q}_p$ NA local field, X/F sm proper variety, $\exists\ \Gamma_F$-equiv iso

$$\underset{\circlearrowleft \Gamma_F}{H^n_{ét}(X_{\overline{F}};\mathbb{Q}_p)\otimes_{\mathbb{Q}_p}\mathbb{C}_p} \cong \bigoplus_{i+j=n} H^i(X,\Omega^j_{X/F})\otimes_F \underset{\circlearrowleft \Gamma_F}{\mathbb{C}_p(-j)}$$

Thm (Tate) Consider the cont coh, then

$$H^i(\Gamma_F,\mathbb{C}_p(j)) = \begin{cases} F, & i=0,1,\quad j=0 \\ 0, & \text{otherwise}. \end{cases}$$

As a Corollary,

$$\mathbb{C}_p^{\Gamma_F} := H^0(\Gamma_F,\mathbb{C}_p) = F,$$

$$\mathrm{Hom}_{\mathrm{Rep}_{\mathbb{C}_p,\mathrm{cont}}(\Gamma_F)}(\mathbb{C}_p(i),\mathbb{C}(j)) \cong H^0(\Gamma_F,\mathbb{C}_p(j-i)) \cong \begin{cases} F, & i=j \\ 0, & i\neq j \end{cases}$$

Def (HT period ring)

$$B_{HT} := \bigoplus_{j\in\mathbb{N}} \mathbb{C}_p(j) = \mathbb{C}_p[t,t^{-1}] \in \mathrm{Rep}_{\mathbb{C}_p,\mathrm{cont}}(\Gamma_F) \quad \text{by}$$

$$\sigma\left(\sum_{i=-\infty}^{+\infty} a_i t^i\right) = \sum_{i=-\infty}^{+\infty} \sigma(a_i)\varepsilon_p^i(\sigma)\, t^i \qquad \rightsquigarrow B_{HT}^{\Gamma_F} = F$$

Cor 1 of Hodge-Tate dec

$$\left(H^n_{ét}(X_{\overline{F}};\mathbb{Q}_p)\otimes_{\mathbb{Q}_p} B_{HT}\right)^{\Gamma_F} \cong \bigoplus_{i+j=n} H^i(X;\Omega^j_{X/F})$$

Def. $V\in \mathrm{rep}_{\mathbb{Q}_p,\mathrm{cont}}(\Gamma_F)$ is called HT (B_{HT}-admissible), if

$$\dim_F (V\otimes_{\mathbb{Q}_p} B_{HT})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$$

By Hodge-Tate dec & Cor 1, $H^n_{ét}(X_{\overline{F}};\mathbb{Q}_p)$ is HT.

Rmk. HT property is stable under subquotients.

Def. For V HT rep, define its HT weight by

$\{\ldots, \underbrace{j, \ldots, j}_{m_j \text{ many}}, \ldots\}$ $\qquad m_j = \dim_F (V \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j))^{\Gamma_F}$

$\qquad$ "$= \dim_F H^{n-j}(X; \Omega^j_{X/F})$"

e.g. $\quad H^i(X; \Omega^j_{X/F}) \cong (H^{i+j}_{\acute{e}t}(X_{\overline{F}}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(j))^{\Gamma_F}$

is HT, with HT weight $\{\underbrace{j, \cdots, j}_{\dim H^i(\ldots) \text{ many}}\}$.

Ex. i) For $\eta \in \mathrm{Char}_{\mathbb{Z}_p, \mathrm{cont}}(\Gamma_F)$,

η is HT $\Leftrightarrow$ $\exists\, n \in \mathbb{Z}$ s.t. $\varepsilon_p^{-n} \eta$ is potentially unramified

e.p. for $a \in \mathbb{Z}_p$,

$\eta = (\varepsilon_p^{p-1})^a$ is HT $\Leftrightarrow$ $a \in \mathbb{Z}$

ii) For $\eta \in \mathrm{Char}_{\overline{\mathbb{Q}}_p, \mathrm{cont}}(\Gamma_F)$,

η is HT $\Leftrightarrow$ $\exists\, \mathcal{U} \subset F^\times$ open, for each $\tau: F \hookrightarrow \overline{\mathbb{Q}}_p$, $\exists\, n_\tau \in \mathbb{Z}$ s.t. $\forall\, \alpha \in \mathcal{U}$,

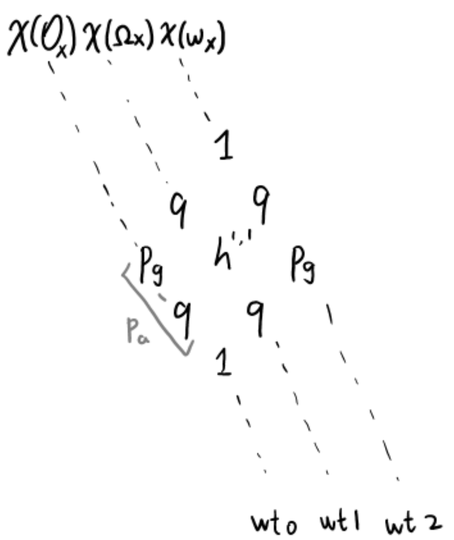
$$(\eta \circ \mathrm{Art}_F)(\alpha) = \prod_{\tau: F \hookrightarrow \overline{\mathbb{Q}}_p} \tau(\alpha)^{-n_\tau}$$

$$F^\times \xrightarrow{\mathrm{Art}_F} W_F^{ab} \longrightarrow \Gamma_F^{ab} \xrightarrow{\eta} \overline{\mathbb{Q}}_p^\times$$

E.g. For $A/\mathbb{Q}$ abelian variety of dim g.

$$H^1(A(\mathbb{C}); \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^0(A, \Omega^1_{A/\mathbb{C}}) \oplus H^1(A, \mathcal{O}_{A/\mathbb{C}})$$
$$H^1_{\acute{e}t}(A_{\overline{\mathbb{Q}}_p}; \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^0(A, \Omega^1_{A/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-1) \oplus H^1(A, \mathcal{O}_{A/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$$

HT wt of $H^1_{\acute{e}t}(A_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$: $\{1, 1, \ldots, 1, 0, 0, \ldots 0\}$



Def/Black box (B_{dR})

B_{dR}/F is a filtered ring s.t.

$$\mathrm{gr}(B_{dR}) = B_{HT} \qquad B_{dR}^{\Gamma_F} = F$$

Thm (de Rham comparison)

$$H^n_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR} \cong H^n_{dR}(X/F) \otimes_F B_{dR}$$

$$\rightsquigarrow \quad (H^n_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} \cong H^n_{dR}(X/F)$$

$$\dim_F (H^n_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} = \dim_F H^n_{dR}(X/F) = \dim_{\mathbb{Q}_p} H^n_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p).$$

Def. $V \in \mathrm{rep}_{\mathbb{Q}_p,\mathrm{cont}}(\Gamma_F)$ is called de Rham (B_{dR} - admissible), if

$$\dim_F (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$$

Rmk. For $V \in \mathrm{rep}_{\mathbb{Q}_p,\mathrm{cont}}(\Gamma_F)$,

$V = H^n_{\acute{e}t}(X_{\overline{F}}, \mathbb{Q}_p)$ for some proper sm variety X/F

$$\Downarrow$$

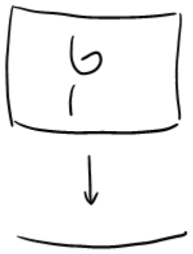
$$\dim_F (V \otimes_{\mathbb{Q}_p} B_{dR})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$$

$$\Downarrow$$

$$\dim_F (V \otimes_{\mathbb{Q}_p} B_{HT})^{\Gamma_F} = \dim_{\mathbb{Q}_p} V$$

Left: (local) Fontaine Mazur, see:
mathoverflow.net/questions/340152/failure-of-local-fontaine-mazur

Geometry.



$$\begin{array}{ccc} \mathfrak{X} & X & \overline{X} \\ | & | & | \\ \mathcal{O}_F & F & \kappa_F \end{array}$$

When char $F \neq p$,

$\mathfrak{X}/\mathcal{O}_F$ proper sm

$\Rightarrow H^i_{\text{ét}}(X_{\overline{F}};\mathbb{Q}_p) \cong H^i_{\text{ét}}(\overline{X}_{\overline{\kappa}_F};\mathbb{Q}_p) \in \mathrm{rep}_{\mathbb{Q}_p,\text{cont}}(G_F) \cong \mathrm{WDrep}^{\text{bdd}}_{\mathbb{Q}_p,\text{sm}}(W_F)$

$\mathfrak{X}/\mathcal{O}_F$ proper + semi-stable reduction

$\Rightarrow H^i_{\text{ét}}(X_{\overline{F}};\mathbb{Q}_p) \in \mathrm{WDrep}^{\text{bdd}}_{\mathbb{Q}_p,\text{sm}}(W_F)$ is semistable (i.e. r is unramified)

When char $F = p$, by [Gee, Thm 2.23],

X/F proper sm + good/semistable reduction

$\Rightarrow H^i_{\text{ét}}(X_{\overline{F}};\overline{\mathbb{Q}}_p)$ is crystalline/semistable.

Hierarchy

pot = potential

	{crystalline} $\subsetneq$	{semistable} $\subsetneq$	{de-Rham} $\subsetneq$	{HT}
	$\cap$	$\not\cap$	$\Vert$	$\Vert$
	{pot crystalline} $\subsetneq$	{pot semistable} =	{pot de-Rham} =	{pot HT}
coming from	good red	semistable red	dR comparison	HT dec
compare with $l \neq p$	unramified reps	$\rho(I_F)$ unipotent	all reps	—
WD rep	r unramified	r unramified	defined	—
$WD(\rho) = (r, N)$	$N = 0$		HT weights	—
1-dim case				
$F = \mathbb{Q}_p$	$\rho\vert_{I_F} = \varepsilon_p^n$		$\rho\vert_{I_F} = \psi\varepsilon_p^n$ $n \in \mathbb{Z}$, ψ finite order	
F: general			$\varepsilon_p \rightsquigarrow$ Lubin-Tate characters	
$\Lambda = \overline{\mathbb{Q}}_p$	$(\chi \circ \mathrm{Art}_F)(\alpha) = \prod_\tau \tau(\alpha)^{-n_\tau}$ $\forall \alpha \in \mathcal{O}_F^\times$		$(\chi \circ \mathrm{Art}_F)(\alpha) = \prod_\tau \tau(\alpha)^{-n_\tau}$ $\exists\, \mathcal{U} \overset{\text{open}}{<} F^\times$ $\forall \alpha \in \mathcal{U}$	

https://mathoverflow.net/questions/111760/a-natural-way-of-thinking-of-the-definition-of-an-artin-l-function

4.

References:
https://en.wikipedia.org/wiki/Dirichlet_character

在算术几何中格罗藤迪克的l-进上同调(l-adic cohomology)可以看作对于函数域(function field)上的L-函数(L-function)的一种范畴化：
a) 函数方程(functional equation)对应庞伽莱对偶(Poincare duality)
b) 欧拉分解(Euler factorisation)对应迹公式(trace formula)
c) 解析延拓(analytic continuation)对应有限性(finitude)

from https://www.zhihu.com/question/31823394/answer/54820421