# Eine Woche, ein Beispiel

## 5.1 Extension of NA local field

F: NA local field

1 List of well-known results
 - in general
 - unramified / totally ramified

2. $\hat{\mathbb{Z}}$ : profinite completion (review)

3. Big picture

4. Henselian ring

5. Cohomological dimension

} not complete, I need time to check the proof

Q. Is there any subfield of $\mathbb{Q}_p$ with finite index?
Can we classify all subfield of $\mathbb{F}_p((t))$ with finite index?

https://math.stackexchange.com/questions/211582/is-there-a-proper-subfield-k-subset-mathbb-r-such-that-mathbb-rk-is-fin

Ref:
Initial motivation comes from
[AY]https://alex-youcis.github.io/localglobalgalois.pdf
which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

[JPS96] Galois cohomology by Jean-Pierre Serre
http://p-adic.com/Local%20Fields.pdf
https://people.clas.ufl.edu/rcrew/files/LCFT.pdf
http://www.mcm.ac.cn/faculty/tianyichao/201409/W020140919372982540194.pdf

## 1. List of well-known results

### In general

F: NA local field  E/F: finite extension

Rmk 1. E is also a NA local field with uniquely extended norm

$$\|x\|: = \|N_{E/F}(x)\|_F^{\frac{1}{n}} \quad \text{resp.} \quad v(x) := \frac{1}{n} v_F(N_{E/F}(x))$$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$ is monogenic, i.e. $\mathcal{O}_E = \mathcal{O}_F[\alpha]$ $\quad \exists \alpha \in \mathcal{O}_E$

Cor. (primitive element thm for NA local field)

$E = F[x]/(g(x))$ $\quad \exists x \in \mathcal{O}_E$, $g(x)$ min poly of $x$.

Rmk: Every separable finite field extension has a primitive element, see wiki:
https://en.wikipedia.org/wiki/Primitive_element_theorem
Separable condition is necessary, see
https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element

Rmk 3. Any finite extension of $\mathbb{Q}_p$ is of form $\mathbb{Q}_p[x]/(g(x))$,
where $g(x) \in \mathbb{Q}[x]$ is an irr poly.
Any finite extension of $\mathbb{F}_q((t))$ is of form $\mathbb{F}_q((t))[x]/(g(x))$
where $g(x) \in \mathbb{F}_q(t)[x]$ is an irr poly.
Both are achieved by Krasner's lemma.

$$v = v_F = \frac{1}{e} v_E \qquad \|\cdot\| = \|\cdot\|_F = \|\cdot\|_E^{\frac{1}{e}} \qquad \mathfrak{p}_F \mathcal{O}_E = \mathfrak{p}_E^e$$

$$\begin{array}{lllll} E & v_E = ev & \|\cdot\|_E = \|\cdot\|^e & \pi_E = \pi_F^{\frac{1}{e}} & v(\pi_E) = \frac{1}{e} \\ \big| \deg n & & & & \\ F & v_F & \|\cdot\|_F & \pi_F & v(\pi_F) = 1 \end{array}$$

## Unramified / totally ramified

Good ref: https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields
It collects the equivalent conditions of unramified/totally ramified field extensions.

| tot ram | wild ram
|         | tame ram
| field ext
| ~~split~~ in local case

When $E/F$ is tot ramified,

$e = n \qquad v(\pi_E) = \frac{1}{n}$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E] \qquad \min(\pi_E) \in \mathcal{O}_F[x]$ is Eisenstein poly.

2. $\hat{\mathbb{Z}}$ = profinite completion of $\mathbb{Z}$ (Recall 2022.2.13 outer auto...)

$$\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l \qquad \hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$\hat{\mathbb{Z}}^\times = \prod_l \mathbb{Z}_l^\times \qquad (\hat{\mathbb{Z}}^\times)^{(p)} := \prod_{l \neq p} \mathbb{Z}_l = (\hat{\mathbb{Z}}^{(p)})^\times$$

Prop. ① $\mathrm{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l = m \\ 0 & l \neq m \end{cases}$ $\quad l, m$ prime.

② $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$

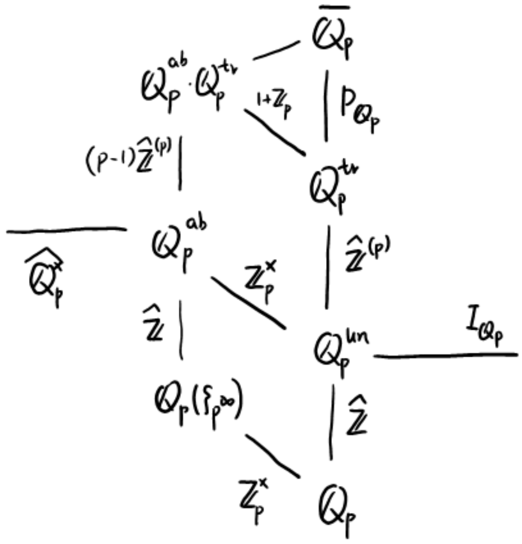
$\mathrm{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^\times$ $\quad$ in the category of profinite gps.

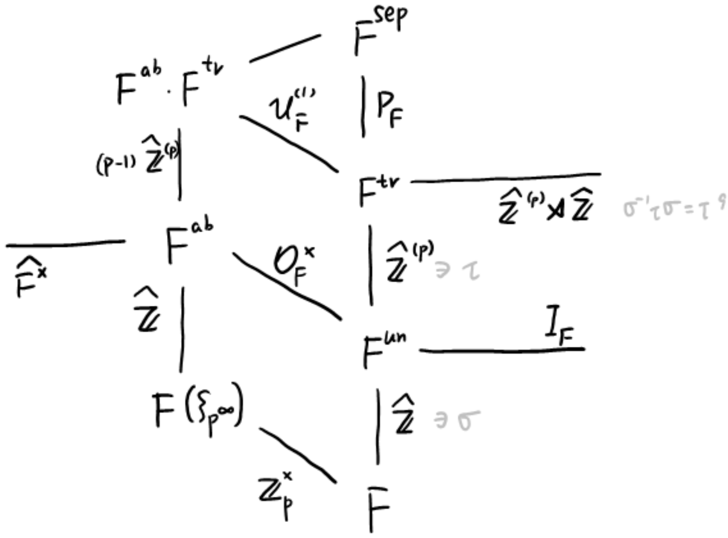
$\mathrm{Aut}(\hat{\mathbb{Z}}^{(p)}) = \hat{\mathbb{Z}}^{\times(p)}$

③ $\mathcal{O}_F, \mathcal{O}_F^\times$ are profinite groups, so $\hat{\mathcal{O}}_F = \mathcal{O}_F \quad \hat{\mathcal{O}}_F^\times = \mathcal{O}_F^\times$.

# 3. Big picture

Main ref: [AY]https://alex-youcis.github.io/localglobalgalois.pdf




F: NA local field with char $k = p$

unramified $\quad F^{un} = \bigcup_{n\geq 1} F(\zeta_{p^n-1}) \underset{\text{Fermat's little thm}}{=} \bigcup_{\substack{n\geq 1\\ p\nmid n}} F(\zeta_n)$

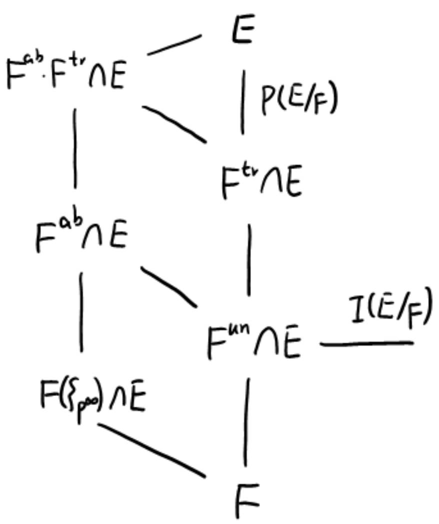
tame ramified $\quad F^{tr} = F^{un}\left(\pi_F^{\frac{1}{n}}\Big|_{(n,p)=1}\right)$

$$= F\left(\pi_F^{\frac{1}{n}}, \zeta_n\Big|_{(n,p)=1}\right)$$

abelian $\quad F^{ab} = F(\zeta_\infty) := \bigcup_{n\geq 1} F(\zeta_n)$

$$F^{ab}\cdot F^{tr} = F\left(\pi_F^{\frac{1}{n}}, \zeta_\infty\Big|_{(n,p)=1}\right)$$

https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions



$$
\begin{tikzcd}[row sep=small]
 & \mathbb{Q}_p(\zeta_{p^n}) \arrow[d, dash] \arrow[dl, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"'] \\
\mathbb{Q}_p\Big(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\Big) & \cdots \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \\
 & \mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \\
 & \mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \\
 & \mathbb{Q}_p
\end{tikzcd}
$$

$(\mathbb{Z}/p^n\mathbb{Z})^\times$ — $\mathbb{Q}_p(\zeta_{p^n})$

$(\mathbb{Z}/p^2\mathbb{Z})^\times$, $1+p\mathbb{Z}/1+p^n\mathbb{Z}$ — $\mathbb{Q}_p(\zeta_{p^2})$ — $I_p \sim I_{p^2-1}$

$(\mathbb{Z}/p\mathbb{Z})^\times$ — $\mathbb{Q}_p(\zeta_p)$ — $I_1 \sim I_{p-1}$

$\mathbb{Q}_p$ — $I_0 = I_{-1}$

$$E/F = \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$$

$$
\begin{tikzcd}[row sep=small]
 & \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \arrow[dl, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"'] \\
\bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p\Big(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\Big) & \cdots \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \\
 & \mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \\
 & \mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \\
 & \mathbb{Q}_p
\end{tikzcd}
$$

$(\mathbb{Z}/p^n\mathbb{Z})^\times$ — $\mathbb{Q}_p(\zeta_{p^\infty})$

$(\mathbb{Z}/p^2\mathbb{Z})^\times$, $1+p\mathbb{Z}$ — $\mathbb{Q}_p(\zeta_{p^2})$ — $I^2 = U^{(2)}_{\mathbb{Q}_p}$

$(\mathbb{Z}/p\mathbb{Z})^\times$ — $\mathbb{Q}_p(\zeta_p)$ — $I^1 = U^{(1)}_{\mathbb{Q}_p}$

$\mathbb{Q}_p$ — $I^0 = U^{(0)}_{\mathbb{Q}_p} = \mathbb{Z}_p^\times$

$$E/F = \mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$$

$$
\begin{tikzcd}
\mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p(\zeta_p+\zeta_p^{-1}) \arrow[d, dash] \\
\mathbb{Q}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}_p(\zeta_p) \arrow[d, dash] \\
\mathbb{Q}_p(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}_p(\zeta_p) \arrow[d, dash] \\
\mathbb{Q}_p(\sqrt{-p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p(\zeta_{p^n}+\zeta_{p^n}^{-1}) \arrow[d, dash] \\
\mathbb{Q}_p^{abwr} := \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p\Big(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\Big)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \\
\mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p^{abwr}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \\
\mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p^{abwr}
\end{tikzcd}
$$

$p$ odd $\qquad$ $p\equiv 1 \pmod 4$ $\qquad$ $p \equiv 3 \pmod 4$

4. Henselian ring.

Main ref: https://en.wikipedia.org/wiki/Henselian_ring

$R$ comm with 1 (local in this section)

**Def.** A local ring $(R,m)$ is Henselian if Hensel's lemma holds, i.e.

$$\begin{array}{ccc} \text{for } P \in R[x] & & \exists f_i \in P[x] \quad ② \; P = f_1 \cdots f_n \\ \downarrow & & \uparrow ① \\ \bar{P} = g_1 \cdots g_n \in R/m[x] & & g_i \in R/m[x] \end{array}$$

$(R,m)$ is strictly Henselian if additionally $(R/m)^{sep} = R/m$.

E.g. · Fields / Complete Hausdorff local rings are Henselian.

e.p. $F, \mathcal{O}_F$ are Henselian

· $R$ is Henselian $\Leftrightarrow R/_{Nil(R)}$ is Henselian

$\Leftrightarrow R/_I$ is Henselian for $\forall I \lhd R$

e.p. when $\operatorname{Spec} R = \{*\}$, $R$ is Henselian.

Denote $\quad StrHense \subset Hense \subset locRing \subset CommRing \quad$ full subcategories

zero

$$\begin{tikzcd} Str\,Hense \arrow[r, bend right, "forget"'] & Hense \arrow[l, dashed, bend right, "{[Stack\_OBSL]}"'] \arrow[r, bend right, "forget"'] \arrow[r, phantom, "\perp"] & locRing \arrow[l, bend right, "(-)^h"'] \arrow[ll, bend right=40, "(-)^{sh}"'] \end{tikzcd}$$

Sadly not adjoint ?

E.g. $\quad F^h = F \qquad F^{sh} = F^{un}$

Geometrically, Henselian means $\operatorname{Spec} R/m \to \operatorname{Spec} R$ has a section.

# 5. Cohomological dimension

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

https://mathoverflow.net/questions/349484/what-is-known-about-the-cohomological-dimension-of-algebraic-number-fields

This section is initially devoted to the following result:

Prop. [(7.5.1)] The wild inertia gp $P_F$ is free pro-$p$-group of countably infinite rank.

See [Galois Theory of p-Extensions, Chap 4] for the definition and construction of free pro-p-groups.

Q: Do we have the adjoint

$$\begin{tikzcd} \text{Pro-}p\text{-gp} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \text{Set} \arrow[l, bend right, "(\ )^{\text{free}}"'] \end{tikzcd}$$

?

Now let $G$: profinite gp
$\mathrm{Mod}(G)$: category of discrete $G$-modules

full subcategory of $\mathrm{Mod}(G)$:
- $\mathrm{Mod}_t(G)$: torsion
- $\mathrm{Mod}_p(G)$: $p$-torsion
- $\mathrm{Mod}_f(G)$: finite

} viewed as abelian gp

Lemma For abelian torsion gp $X$, denote

$$X(p) := \{x \in X \mid x^{p^k} = 1 \quad \exists k \in \mathbb{N}_{>0}\}$$

we have $X = \bigoplus_p X(p)$.

This is trivial when X is finite, but I don't know how to prove this in the general case. It should be not too hard.

Def [(3.3.1)] (cohomological dimension) $p$ prime

$$\mathrm{cd}\, G = \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \mathrm{Mod}_t(G),\ H^i(G,A) \neq 0\}$$
$$\mathrm{scd}\, G = \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \mathrm{Mod}(G),\ H^i(G,A) \neq 0\}$$
$$\mathrm{cd}_p\, G = \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \mathrm{Mod}_t(G),\ H^i(G,A)(p) \neq 0\}$$
$$\mathrm{scd}_p\, G = \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \mathrm{Mod}(G),\ H^i(G,A)(p) \neq 0\}$$

Prop. (local to global) $\mathrm{cd}\, G = \sup_p \mathrm{cd}_p G$ $\qquad$ $\mathrm{scd}\, G = \sup_p \mathrm{scd}_p G$

Prop. [(3.3.2)] $\mathrm{cd}_p G \leq n \iff H^{n+1}(G,A) = 0 \quad \forall$ simple $G$-mod $A$ with $pA = 0$

e.p. for $G$: pro-$p$-gp,

$$\mathrm{cd}_p G \leq n \iff H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

E.g. $\mathrm{cd}_p \hat{\mathbb{Z}} = 1$ $\qquad$ $\mathrm{scd}_p \hat{\mathbb{Z}} = 2$

Prop. [(3.3.5)] For $H \leq G$ closed,

$$\mathrm{cd}_p H \leq \mathrm{cd}_p G \qquad \mathrm{scd}_p H \leq \mathrm{scd}_p G$$

When $p \nmid [G:H]$ or $[H$ open $+$ $\mathrm{cd}_p G < +\infty]$, the equality holds.

Weaker condition: see [(3.3.5, Serre)]

Cor. $G$: profinite gp, then

$$\mathrm{cd}_p G = 0 \iff p \nmid \# G$$

Prop. [(3.5.17)] A pro-$p$-gp $G$ is free iff $\mathrm{cd}\, G \leq 1$.

Prop[(7.1.8)] (i) $F$ NA local field with char $k = p$.

$$cd_l(F) = \begin{cases} 2 & \text{if } l \neq \operatorname{char} F, \\ 1 & \text{if } l = \operatorname{char} F. \end{cases}$$

For any $E/F$ field extension s.t. $l^\infty | \deg E/F$, $cd_l(E) \leq 1$.

(ii) Fix $n \in \mathbb{N}_{>0}$ s.t. $\operatorname{char} F \nmid n$.

$$H^i(F, \mu_n) = \begin{cases} F^\times/(F^\times)^n & i = 1 \\ \frac{1}{n}\mathbb{Z}/\mathbb{Z} & i = 2 \\ 0 & i \geq 3 \end{cases}$$

[Proof for Prop (7.5.1)

Now $l^\infty | \deg F^{tr}/F$ $\overset{(7.1.8)}{\Rightarrow}$ $cd_l(F^{tr}) \leq 1$ $\forall$ prime $l$

$\Leftrightarrow$ $cd(F^{tr}) \leq 1$

$\Leftrightarrow$ $P_F$ is free pro-p-group. □]