Eine Woche, ein Beispiel

9.10. ramified covering : alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - Correspondence
   - Valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

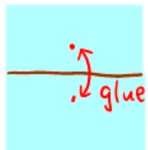
1. alg curve viewed as stack quotient

	$\operatorname{Spec}\mathbb{R}$	base change $\operatorname{Spec}\mathbb{C}/_{\mathbb{C}}$	$\operatorname{Spec}\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	ϕ
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{\mathrm{Id}, \tau\}$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	$\mathrm{Id} \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. $\operatorname{Spec}\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[x]$. As a set,

$$\begin{aligned}\operatorname{Spec}\mathbb{R}[x] &= \{(x-a) \mid a\in\mathbb{R}\} \cup \left\{(x^2+bx+c) \,\middle|\, \begin{smallmatrix} b,c\in\mathbb{R} \\ b^2-4c<0\end{smallmatrix}\right\} \cup \{(0)\} \\ &= \quad \mathbb{R} \quad \cup \quad \mathcal{H} \quad \cup \{(0)\}\end{aligned}$$

$$\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathrm{Mor}_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R}$$

$$\mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) = \mathrm{Mor}_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau\circ x$. Observe that

$$\operatorname{MaxSpec}\mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/\Gamma_{\mathbb{R}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$\operatorname{MaxSpec}\mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/\Gamma_{\mathbb{F}_p} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/\Gamma_\kappa \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

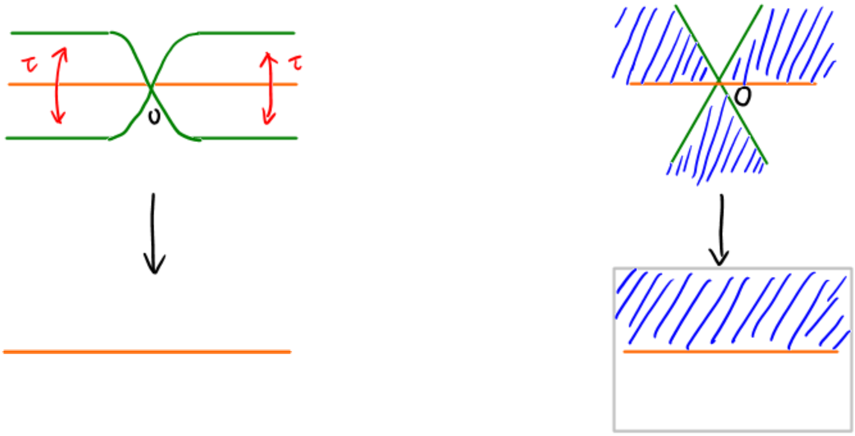
$$\mathrm{Stab}_x(\Gamma_\kappa) = \Gamma_{\kappa'} \quad\Leftrightarrow\quad \text{fiber at } x = \operatorname{Spec}\kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R} \cup \mathcal{H}$	$\mathbb{C}$	$\mathbb{C}$ 2 cplx conj
$\mathbb{R}$-pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$-pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \sqcup \mathbb{C}\tau$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

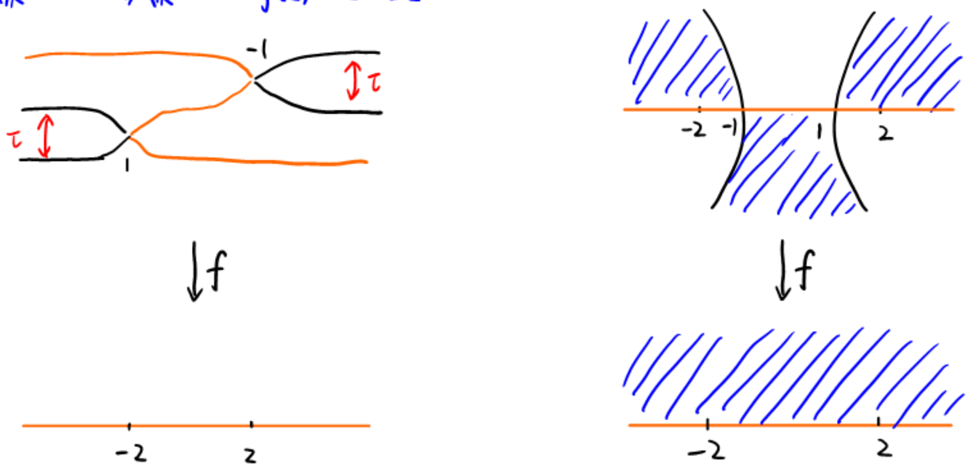
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

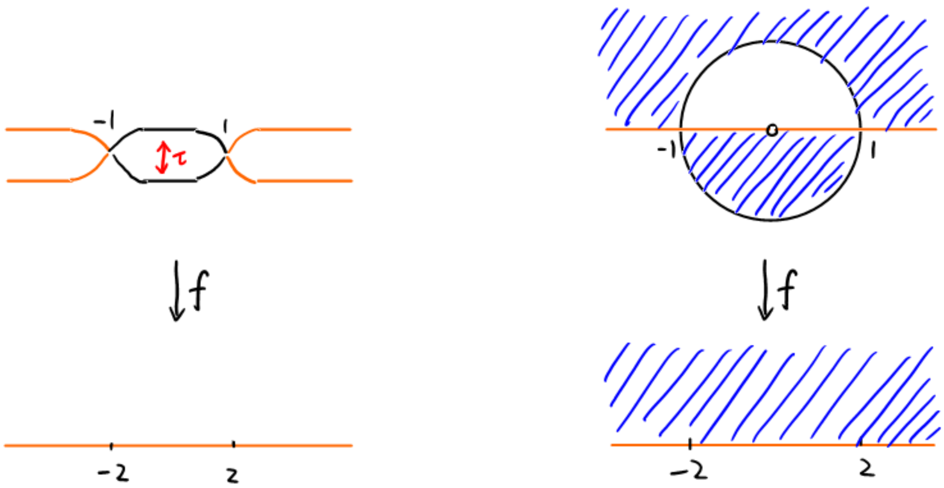
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



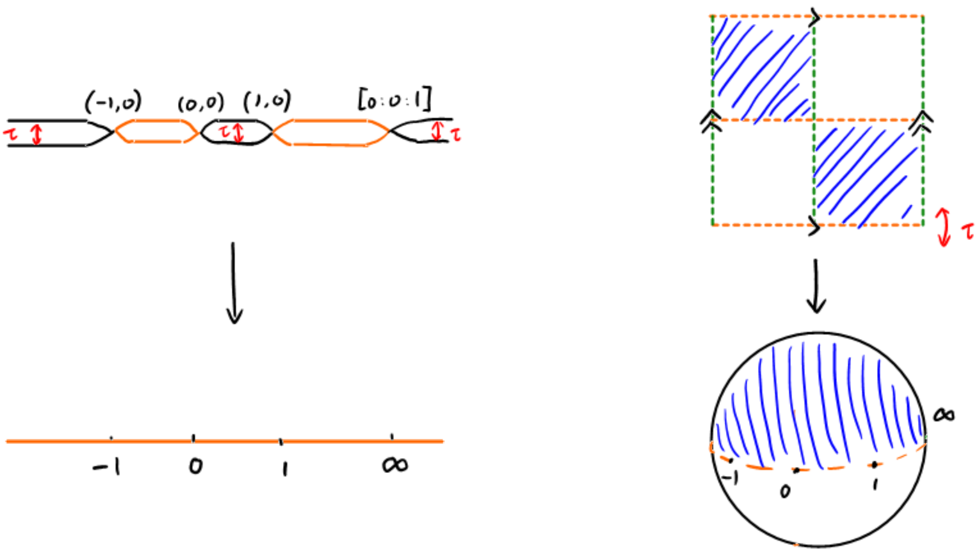
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$



$f: G_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$

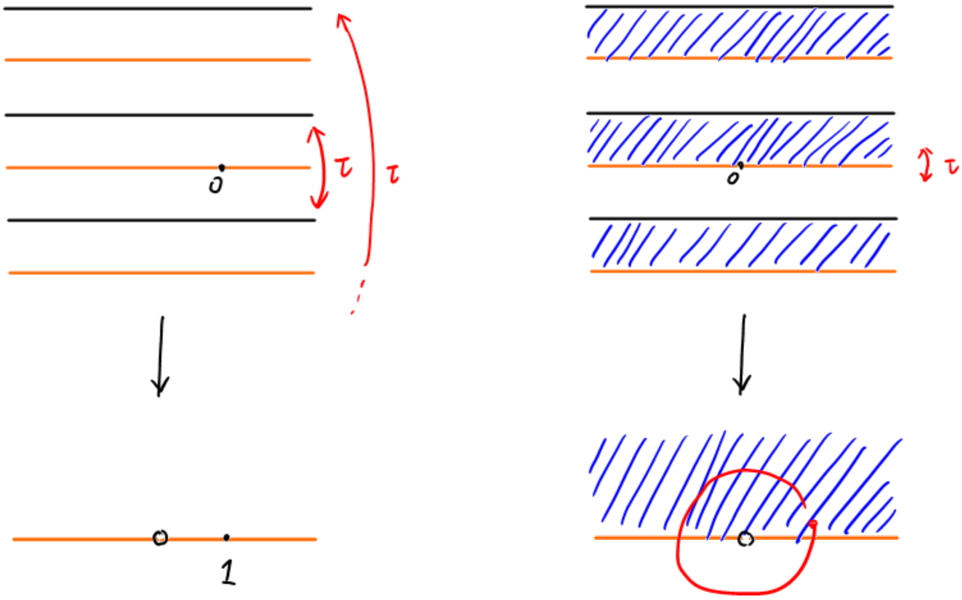


$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}} \qquad [x:y:z] \longmapsto [x:z]$ $\qquad E_{\mathbb{R}} = \mathrm{Proj}\ \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

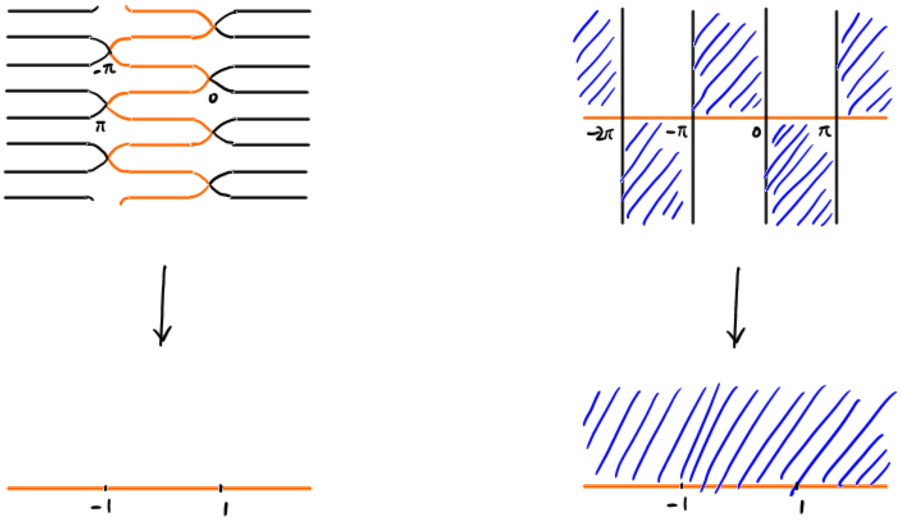


⚠ The following are not alg morphisms!

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = e^z$



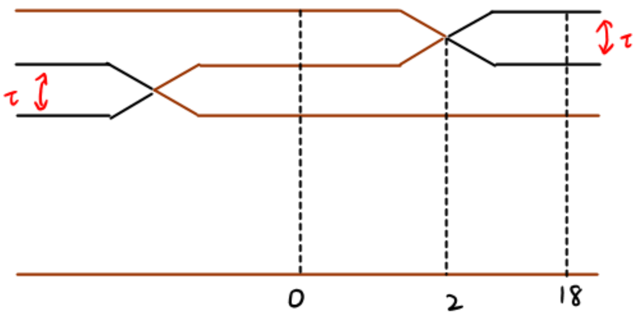
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = \cos z$



Let's focus on the case

$f : \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = z^3 - 3z$

classical picture



split: $f^{-1}(0) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z-z_0))$

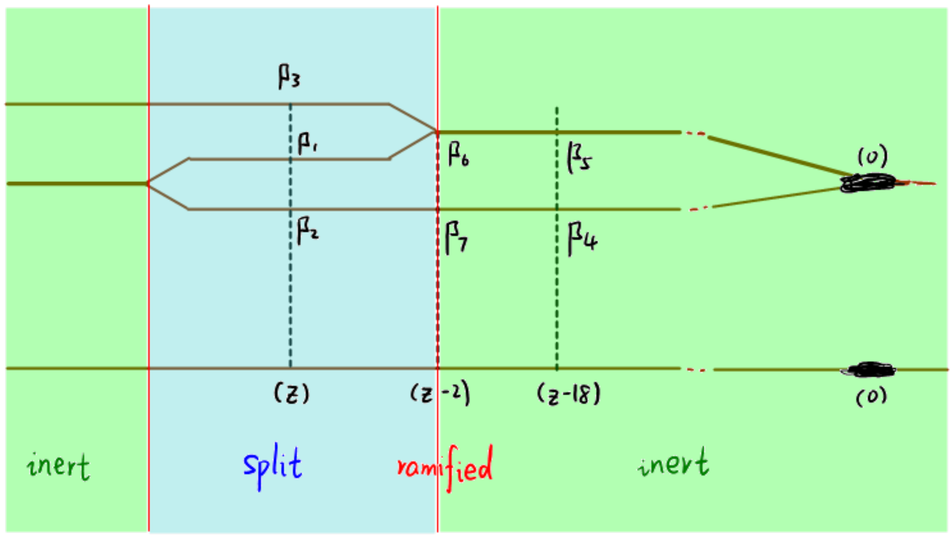
$f^{-1}((z^2+1)) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{R}$

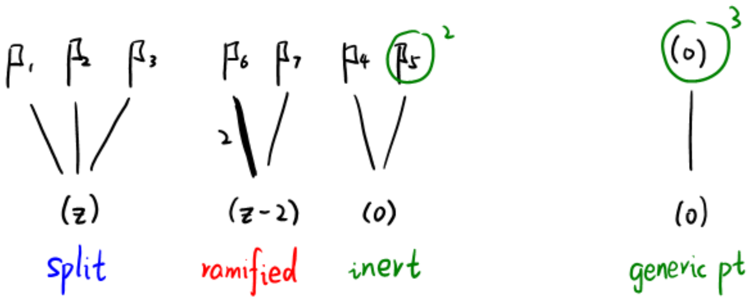
generic point: $f^{-1}((0)) = \operatorname{Spec}\mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$

algebraic picture



$$\begin{array}{ccccc} \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[w] & & w^3-3w \\ f\big\downarrow & & \big\uparrow f^* & & \big\uparrow \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] & & z \end{array}$$



split: $\mathfrak{p} = (z)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w) = (w)(w-\sqrt{3})(w+\sqrt{3})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangleq\ \mathfrak{p}_1\quad \mathfrak{p}_2\quad \mathfrak{p}_3$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$

$\mathfrak{p} = (z^2+1)$, $f^*(\mathfrak{p})\mathbb{R}[w] = ((w^3-3w)^2+1) = (f_1')\ (f_2')\ (f_3')$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangleq\ \mathfrak{p}_1'\quad \mathfrak{p}_2'\quad \mathfrak{p}_3'$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$

(partially) inert: $\mathfrak{p} = (z-18)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-18) = (w-3)(w^2+3w+6)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangleq\ \mathfrak{p}_4\quad \mathfrak{p}_5$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{p}_4, \mathfrak{p}_5\}$

where $\kappa(\mathfrak{p}_5) = \mathbb{R}[w]/_{(w^2+3w+6)} \cong \mathbb{C}$, $[\kappa(\mathfrak{p}_5):\mathbb{R}] = 2$

generic point: $\mathfrak{p} = (0)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (0)$ $\qquad f^{-1}(\mathfrak{p}) = \{(0)\}$

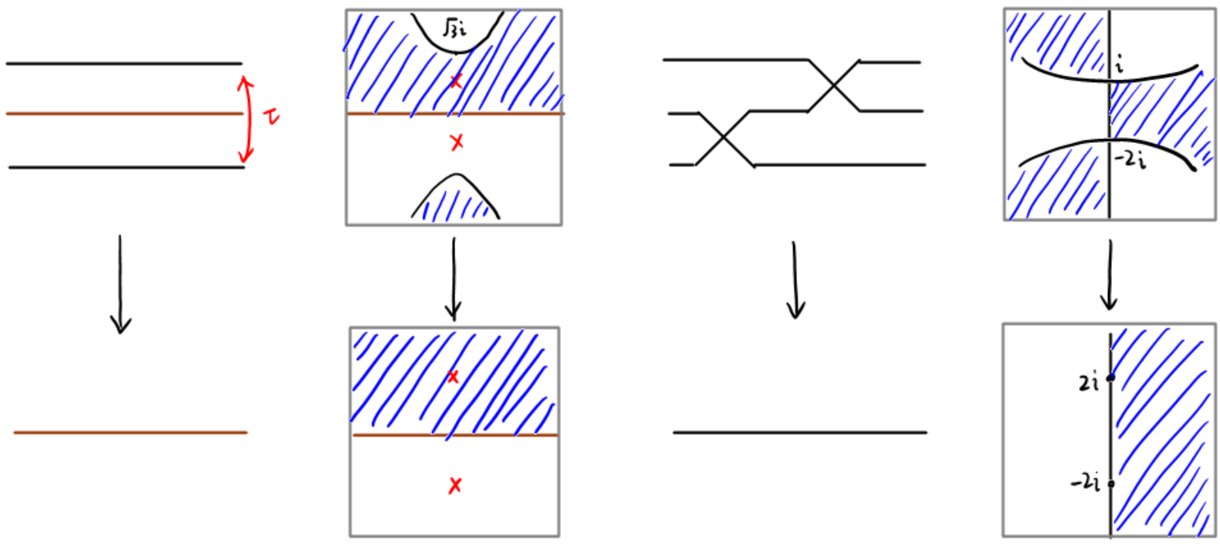
where $\kappa((0)) = \mathrm{Frac}(\mathbb{R}[w]/_{(0)}) \cong \mathbb{R}(w)$, $[\mathbb{R}(w):\mathbb{R}(z)] = 3$

ramified: $\mathfrak{p} = (z-2)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-2) = (w+1)^2(w-2)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\triangleq\ \mathfrak{p}_6^2\quad \mathfrak{p}_7$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{p}_4, \mathfrak{p}_5\}$

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = z^3+3z$.



$\mathbb{R}$ picture $\qquad\qquad\qquad\qquad i\mathbb{R}$ picture

⚠ The ramification pt is outside $\mathbb{R}$.

This is not a Galois covering.

3. Frobenius for alg curve/$\mathbb{R}$

$$Gal(\kappa(q)/\kappa(p)) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \kappa(q)=\mathbb{C},\ \kappa(p)=\mathbb{R} \\ \{Id\} & \text{otherwise.} \end{cases}$$

When E/F is Galois, $\mathrm{Spec}\,\mathcal{O}_E/\mathrm{Spec}\,\mathcal{O}_F$ unramified at p,

$$\begin{array}{ccc} Gal(\kappa(q)/\kappa(p)) \cong Gal(E/F)_q & \le & Gal(E/F) \\ \mathrm{Frob}_q & \longmapsto & \mathrm{Frob}_q \end{array}$$

is a subgp of $Gal(E/F) \cong Aut(Spv(E)/Spv(F))$ Now, just view $Spv(E) \in AlgCurve_k$.

Let's try to compute some Frob_q

E.g.
$$\begin{array}{cccccccc} \mathbb{A}^1_{\mathbb{R}} & z & \mathbb{R}[\omega]=\mathbb{R}[z^{\frac{1}{2}}] & -1 \quad 1 & (i,-i)^2 & 0 \\ \downarrow & \downarrow & \uparrow & \diagdown \ \diagup & | & \|^{2} \\ \mathbb{A}^1_{\mathbb{R}} & z^2 & \mathbb{R}[z] & 1 & -1 & 0 \end{array}$$

For $p=(z-1)$, $q=(\omega-1)$,
$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ 1 & & 1 & & \{1,\tau\} \end{array}$$

For $p=(z+1)$, $q=(\omega^2+1)$,
$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z+1)} = \tau: \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where
$\tau(\mathbb{C}): \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1$ $\quad \omega \longmapsto -\omega$

Not the conjugation but $\tau(\mathbb{C})|_{i\mathbb{R}}$ coincides with the cplx conj

E.g.
$$\begin{array}{cccccccc} \mathbb{G}_{m,\mathbb{R}} & z & \mathbb{R}[\omega^{\pm1}]=\mathbb{R}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{\pm1}\right] & 2 \quad \frac{1}{2} & (i,-i)^2 & 1 & -1 \\ \downarrow & \downarrow & \uparrow & \diagdown \ \diagup & | & \|^{2} & \|^{2} \\ \mathbb{A}^1_{\mathbb{R}} & z+\frac{1}{z} & \mathbb{R}[z] & \frac{5}{2} & 0 & 2 & -2 \end{array}$$

For $p=(z)$, $q=(\omega^2+1)$,
$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z+1)} = \tau: \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where
$\tau(\mathbb{C}): \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1$ $\quad \omega \longmapsto \frac{1}{\omega}$

Not the conjugation, but $\tau(\mathbb{C})|_{S^1}$ coincides with the cplx conj

⚠ $\mathbb{R}(z^{\frac{1}{3}})/\mathbb{R}(z)$ is not Galois at all, so

For $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \quad z \longmapsto z^3$, $\beta = (z-1)$, $q = (\omega^2+\omega+1)$,

$$\underset{\{1,\tau\}}{\mathrm{Gal}(k(q)/k(\beta))} \not\cong \underset{1}{\text{“}\mathrm{Gal}(E/F)_q\text{”}} \leq \underset{1}{\text{“}\mathrm{Gal}(E/F)\text{”}} \neq \mathbb{Z}/3\mathbb{Z}$$

We will discuss about $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$ in section 4.

Claim. For p odd prime, any deg p extension of $\mathbb{R}(x)$ is not Galois.

Proof given by Zhuoni Chi:

If not, suppose $F/\mathbb{R}(x)$ is a deg p Galois extension, we get the field extension tower in $\overline{\mathbb{R}(x)}$:

$$\begin{tikzcd} & & E \\ F \arrow[urr, dash, "2"] & & \\ & & \mathbb{C}(x) \arrow[uu, dash, "p"'] \\ \mathbb{R}(x) \arrow[uu, dash, "p"] \arrow[urr, dash, "2"'] & & \end{tikzcd}$$

where $\mathrm{Gal}(E/F) \lhd \mathrm{Gal}(E/\mathbb{R}(x))$ is a normal subgp of order 2.

One gets

$$\mathrm{Gal}(E/\mathbb{R}(x)) \hookrightarrow S_p \circlearrowright \{T, \xi_p T, \ldots, \xi_p^{p-1} T\}$$

Injection: if σ fix T, $\xi_p T$, then σ fix ξ_p, then $\sigma = Id$.

Since $\#\mathrm{Gal}(E/\mathbb{R}(x)) = 2p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$ or $\mathbb{Z}/2p\mathbb{Z}$.

Since $\mathrm{Gal}(E/\mathbb{R}(x)) \leq S_p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$.

However, D_p has no order 2 normal subgp, contradiction! $\square$

Q. For $F/\mathbb{R}(x)$ Galois extension, is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by its order 2 elements?

Is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by all Frobenius elements?

I call it as the “weaked version of Chebotarev's density theorem for $\mathbb{P}^1_{\mathbb{R}}$”.

We could not expect the density theorem to be true in the real case, since in S_3 case the order 3 conj class can never be reached by a single Frob.

Possible direct and brutal method to this question: use the result in this link:
math.stackexchange.com/questions/318690/absolute-galois-group-of-mathbbrt