

Eine Woche, ein Beispiel

4.27. homomorphism between Jacobians

[2025.04.20] provides us with many examples and references, and here we do things more theoretically.

Idea:

$$\text{Jac}(C) = H^0(C; \omega_C)^* / H_1(C; \mathbb{Z})$$

linear part
coherent

lattice part
constant

To understand $\text{Jac}(C)$, we need to understand these two parts separately.

For a morphism between two sm proj curves / \mathbb{C} :

$$f: \tilde{C} \longrightarrow C$$

$$\begin{aligned} N_{f,a}: H^0(\tilde{C}; \omega_{\tilde{C}})^* &\longrightarrow H^0(C; \omega_C)^* \\ N_{f,r}: H_1(\tilde{C}; \mathbb{Z}) &\longrightarrow H_1(C; \mathbb{Z}) \\ N_{f,\ell}: \text{Jac}(\tilde{C}) &\longrightarrow \text{Jac}(C) \end{aligned}$$

$$\begin{aligned} (f^*)_a: H^0(C; \omega_C)^* &\longrightarrow H^0(\tilde{C}; \omega_{\tilde{C}})^* \\ (f^*)_r: H_1(C; \mathbb{Z}) &\longrightarrow H_1(\tilde{C}; \mathbb{Z}) \\ f^*: \text{Jac}(C) &\longrightarrow \text{Jac}(\tilde{C}) \end{aligned}$$

cohom pullback

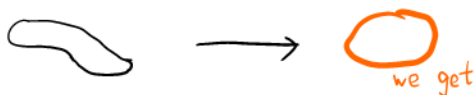
sheaf
origin

$$\begin{aligned} \omega_{\tilde{C}} &\longleftarrow f^* \omega_C \\ f: \pi_{\tilde{C}}^! \mathbb{Z} &\longrightarrow \pi_C^! \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \omega_C &\longleftarrow f_! \omega_{\tilde{C}} \\ \underline{\mathbb{Z}}_C &\longrightarrow f_* \underline{\mathbb{Z}}_{\tilde{C}} \end{aligned}$$

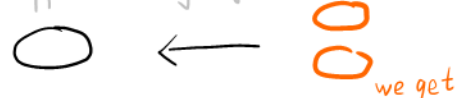
geometric
picture

$$g(f(w))d(f(w)) \longleftarrow g(z)dz$$



$$[q] \longmapsto [f(q)]$$

$$\sum_{f(w)=z} g(w)dz \longleftarrow g(w)dw$$



$$[p] \longmapsto \sum_{f(q)=p} [q]$$

Ex. Show that

$$N_{mf} \circ f^* = [\deg f]: \text{Jac}(C) \longrightarrow \text{Jac}(\tilde{C})$$

Also,

$$\begin{aligned} N_{mf,a} \circ (f^*)_a &= \deg f \cdot \text{Id}_{H^0(C; \omega_C)^*} \\ N_{mf,r} \circ (f^*)_r &= \deg f \cdot \text{Id}_{H_1(C; \mathbb{Z})} \end{aligned}$$

Hint: use Poincaré duality.