Eine Woche, ein Beispiel

5.15 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory.
In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categorys here: http://katmat.math.uni-bremen.de/acc/acc.pdf

For Chinese, the theory of category has been summed up in detail in [https://wwli.asia/downloads/books/Al-jabr-1.pdf], Chapter 2-3.

Process

0. Well-know concepts
1. Individual category
   - Complete/Cocomplete/Bicomplete category
   - Cartesian closed category / Closed category
   - Monoidal category = Tensor category
2. Functors between categories
   - Exactness
   - Adjoints
3. Examples of categories
   - Well-known examples
   - Cat
   - Hausdorff and compactness
   - Categories in condensed mathematics

Appendix.

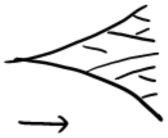
0. Well-know concepts

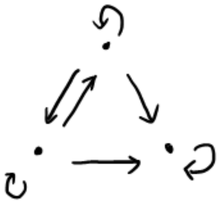
$\mathcal{C}$ is always a category.

	$Ob(\mathcal{C})$		$Mor(X,Y)$
small	Set		Set
loc. small	—		Set
large	not set	or	not set

filtered:

cofiltered:

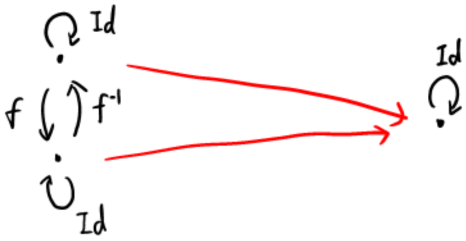




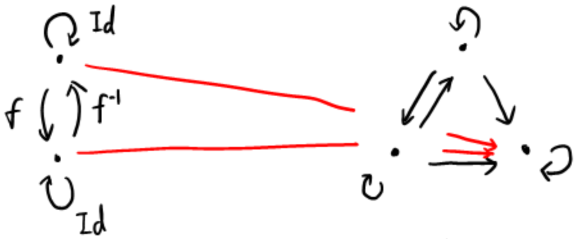
category

full subcategory

https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective



equivalence of category



fully faithful

https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/

1. Individual category

Complete/Cocomplete/Bicomplete category

Def. $\mathcal{C}$ is complete if

$\forall$ small category Λ, $\forall$ fctor $F : \Lambda \longrightarrow C$ $\quad i \longmapsto F_i$,

$\lim\limits_{\overleftarrow{i \in \Lambda}} F_i$ exists $\qquad$ ($\lim\limits_{\overleftarrow{i \in \Lambda}} F_i$ is called the small limit)

$\mathcal{C}$ is cocomplete if

$\forall$ small category Λ, $\forall$ fctor $F : \Lambda \longrightarrow C$ $\quad i \longmapsto F_i$,

$\lim\limits_{\overrightarrow{i \in \Lambda}} F_i$ exists $\qquad$ ($\lim\limits_{\overleftarrow{i \in \Lambda}} F_i$ is called the small colimit)

bicomplete = complete + cocomplete

$\mathcal{C}$ is finitely complete if $\forall$ finite limit exists

$\mathcal{C}$ is finitely cocomplete if $\forall$ finite colimit exists.

Thm. $\mathcal{C}$ is complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & products

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & products

$\mathcal{C}$ is cocomplete $\Leftrightarrow$ $\mathcal{C}$ has coequalizers & coproducts

$\Leftrightarrow$ $\mathcal{C}$ has pushouts & coproducts

$\mathcal{C}$ is finitely complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & finite products

$\Leftrightarrow$ $\mathcal{C}$ has equalizers, binary products & terminal obj

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & terminal obj

For small category $\mathcal{C}$,

$$\begin{tikzcd}
\text{complete} \arrow[rr, Leftrightarrow] \arrow[dr, Rightarrow] & & \text{cocomplete} \arrow[dl, Rightarrow] \\
& \text{thin} \quad (\# \mathrm{Mor}(X,Y) \leq 1) &
\end{tikzcd}$$

The "closedness" of the category is that taking the morphisms between two objects gives another morphism in the same category, rather than the category of sets or some other category.
from: https://math.stackexchange.com/questions/3486846/definition-of-cartesian-closed-category-why-do-we-need-exponential-objects

Cartesian closed category / Closed category

Def. $\mathcal{C}$ is Cartesian closed if
$\mathcal{C}$ has terminal obj., binary product and exponential, where

a bifctor $F: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$
which is functorial in Y

$$- \times Y \dashv (-)^Y$$

i.e. $\mathrm{Mor}(X\times Y, Z) \cong \mathrm{Mor}(X, Z^Y)$

$\mathcal{C}$ is loc. Cartesian closed if all its slice category is Cartesian closed.

https://ncatlab.org/nlab/show/over+category

Rmk. When $\mathcal{C}$ is loc. Cartesian closed,
$\mathcal{C}$ is Cartesian closed $\Leftrightarrow$ $\mathcal{C}$ has a terminal object.
But $\mathcal{C}$ is Cartesian closed $\not\Rightarrow$ $\mathcal{C}$ is loc. Cartesian closed

For the closed category, we use the definition in https://ncatlab.org/nlab/show/closed+category.

Def A closed category is a category $\mathcal{C}$ together with the following data.

- bifctor $[-,-] : \mathcal{C}^{op}\times C \to \mathcal{C}$ called internal hom-fctor
- $I \in Ob(\mathcal{C})$ called unit object
- $i: Id_{\mathcal{C}} \xrightarrow{\cong} [I,-]$ $\rightsquigarrow$ $i_A: A \xrightarrow{\cong} [I, A]$
- $j_X: I \longrightarrow [X,X]$ extranatural in X
- $L^X_{Y,Z} : [Y,Z] \to [[X,Y],[X,Z]]$ functorial in Y and Z extranatural in X.

- Compatabilities

$$\begin{tikzcd}
I \arrow[r, "j_Y"] \arrow[dr, "j_{[X,Y]}"'] & {[Y,Y]} \arrow[d, "L^X_{Y,Y}"] \\
& {[[X,Y],[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[X,Y]} \arrow[r, "L^X_{XY}"] \arrow[dr, "i_{[X,Y]}"'] & {[[X,X],[X,Y]]} \arrow[d, "{[j_X,1]}"] \\
& {[I,[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[Y,Z]} \arrow[r, "L^I_{YZ}"] \arrow[dr, "{[1,i_Z]}"'] & {[[I\ Y],[I,Z]]} \arrow[d, "{[i_Y,1]}"] \\
& {[Y,[I,Z]]}
\end{tikzcd}$$

$$\begin{tikzcd}
& {[U,V]} \arrow[dl, "L^X_{UV}"'] \arrow[dr, "L^Y_{UV}"] & \\
{[[X,U],[X,V]]} \arrow[d, "L^{[X,Y]}_{[X,U][X,V]}"'] & & {[[Y,U],[Y,V]]} \arrow[d, "{[1,L^X_{YV}]}"] \\
{[[[X,Y],[X,U]],[[X,Y],[X,V]]]} \arrow[rr, "{[L^X_{YU},1]}"] & & {[[Y,U],[[X,Y],[X,V]]]}
\end{tikzcd}$$

$$\begin{aligned} \gamma: \mathrm{Mor}(X,Y) &\longrightarrow \mathrm{Mor}(I,[X,Y]) \\ f &\longmapsto [1,f]\circ j_X \end{aligned}$$ is an iso.

Monoidal category = Tensor category 幺半范畴

Def A monoidal category is a category $\mathcal{C}$ together with the following data.

- bifctor $- \otimes - : \mathcal{C} \times C \longrightarrow \mathcal{C}$
- $I \in Ob(\mathcal{C})$ called unit object
- $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$
- $\lambda_A : I \otimes A \xrightarrow{\cong} A$ lambda: left
- $\rho_A : A \otimes I \xrightarrow{\cong} A$ rho: right

- Compatabilities

$$\begin{tikzcd}
& A\otimes(B\otimes(C\otimes D)) \arrow[dl, "1_A\otimes\alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
A\otimes((B\otimes C)\otimes D) \arrow[d, "\alpha_{A,B\otimes C,D}"'] & \circlearrowright & (A\otimes B)\otimes(C\otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "\alpha_{A,B,C}\otimes 1_A"] & & ((A\otimes B)\otimes C)\otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(I\otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A\otimes\lambda_B"'] & & (A\otimes I)\otimes B \arrow[dl, "\rho_A\otimes 1_B"] \\
& A\otimes B &
\end{tikzcd}$$

For strict monoidal category, we require in addition that $\alpha_{A,B,C}$, λ_A, ρ_A are identities.

E.g. Cartesian monoidal category $\mathcal{C}$. category with finite products

$\otimes = \prod$ $\quad$ I = terminal object

e.g. Set, Cat.

Cocartesian monoidal category $\mathcal{C}$: category with finite coproducts

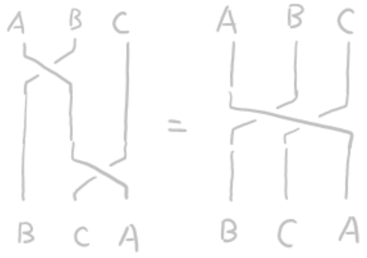
$\otimes = \coprod$ $\quad$ I = initial object

Abelian category is monoidal.

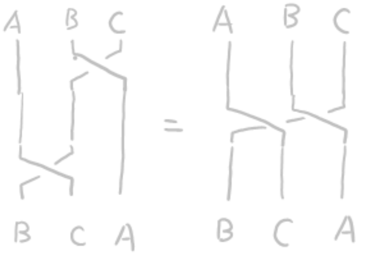
Def (Specializations)

Let $\mathcal{C}$ be a monoidal category.
If in addition we have $\gamma_{A,B}: A\otimes B \longrightarrow B\otimes A$,
then $\mathcal{C}$ is braided monoidal category if

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[dl, "\gamma_{A,B}\otimes 1_C"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"] & & A\otimes(B\otimes C) \arrow[d, "\gamma_{A,B\otimes C}"] \\
B\otimes(A\otimes C) \arrow[dr, "1_B\otimes\gamma_{A,C}"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B\otimes(C\otimes A) &
\end{tikzcd}$$



$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[dl, "1_A\otimes\gamma_{B,C}"'] \arrow[dr, "\alpha^{-1}_{A,B,C}"] & \\
A\otimes(C\otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] & & (A\otimes B)\otimes C \arrow[d, "\gamma_{A\otimes B,C}"] \\
(A\otimes C)\otimes B \arrow[dr, "\gamma_{C,A}\otimes 1_B"'] & & C\otimes(A\otimes B) \arrow[dl, "\alpha^{-1}_{C,A,B}"] \\
 & (C\otimes A)\otimes B &
\end{tikzcd}$$



$$\begin{tikzcd}
I\otimes A \arrow[rr, "\gamma_{I,A}"] \arrow[dr, "\lambda_A"'] & & A\otimes I \arrow[dl, "\rho_A"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[rr, "\gamma_{A,I}"] \arrow[dr, "\rho_A"'] & & I\otimes A \arrow[dl, "\lambda_A"] \\
 & A &
\end{tikzcd}$$

$\mathcal{C}$ is symmetric monoidal category if
$\gamma_{B,A}\circ\gamma_{A,B} = 1_{A\otimes B}$. + $\mathcal{C}$ is braided.

closed monoidal category = closed category + monoidal category
+ compatabilite $-\otimes A \dashv [A,-]$

2. Functors between categories

Exactness

Ref:https://ncatlab.org/nlab/show/exact+functor

Prop/Def For a fctor $F: \mathcal{C} \longrightarrow \mathcal{D}$ between finitely complete categories, TFAE:

- F preserves finite limits,
- F preserves equalizers & finite products,
- F preserves equalizers, binary products & terminal objects,
- F preserves pullbacks & terminal objects,
- $\forall d \in Ob(\mathcal{D})$, the comma category F/d is filtered.

If so, we call F is a left exact fctor.

↳ We require also that $\mathcal{C}$ & $\mathcal{D}$ are finitely complete categories

When $\mathcal{C}, \mathcal{D}$ are abelian categories, this is equivalent to

- F preserves kernels, i.e.
  F sends left exact sequences to left exact sequences.

See [https://stacks.math.columbia.edu/tag/010M]. You may get the following results from the argument:

F is a left exact fctor between abelian categories $\Rightarrow$ F preserve binary products

$\Updownarrow$ [oDLP]

F is additive

Def. A contravariant fctor $F: \mathcal{C} \longrightarrow \mathcal{D}$ is called left exact, if

$F: \mathcal{C}^{op} \longrightarrow \mathcal{D}$ is left exact (In ptc, $\mathcal{C}$ is finitely cocomplete, $\mathcal{D}$ is finitely complete)

Similarly, we can define right exactness, and

exact = left exact + right exact

$\overset{\mathcal{C},\mathcal{D}\text{ abelian}}{=\!=}$ sends SES to SES.

Adjoints

$f_! \dashv f^* \dashv f_* \dashv f^!$

	left adjoint ⊣	right adjoint
	free	forget
	$-\otimes_A N$	$\mathrm{Hom}_A(N,-)$
	Δ	$\varprojlim$
	$(\ )^\sim$	Γ_*
	f_p	f^*
	c-Ind	Res
	Res	Ind
	$-\otimes_k A^e$	Res
	sh $(-)$	$\mathrm{Res}_{sh\to Psh}$
	π_0	$T \mapsto \coprod_{t\in T} \mathrm{Spec}\,\mathbb{Z}$
	$G^{[p]}$	Lie
	$\lvert - \rvert$	S
	$\tau_1 = H_0(-)$	N: nerve
	β: Stone-Čech	U: forget
ad ⇒	preserve colimits	preserve limits
⇒	right exacts	left exacts
in (co) complete category	coker f	ker f
	$\coprod A \quad \oplus A$	$\prod A \qquad A\times_C B$
	pushforward	pullback
	coequalizer	equalizer
	$\bar{k} = \varinjlim_{L/K} L$	$\mathrm{Spec}\,\bar{k} = \varprojlim_{L/K} \mathrm{Spec}\,L$

$\mathrm{Gal}(\bar{k}/k) = \varprojlim_{L \text{ finite Galois}} \mathrm{Gal}(L/k)$

$\mathrm{Spec}\,\mathbb{Z}_p = \varinjlim_n \mathrm{Spec}\,\mathbb{Z}/p^n\mathbb{Z}$

completion

$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

ring point of view

$\mathcal{F}_p = \varinjlim_{U} \mathcal{F}(U)$

stalk

co limit	limit
direct inductive injective } limit.	inverse projective } limit.

How to memorize:

$$0 \longrightarrow \overset{\xleftarrow{\text{limit}}}{\ker} \longrightarrow M \longrightarrow N \longrightarrow \overset{\xrightarrow{\text{co limit}}}{\mathrm{coker}} \longrightarrow 0$$

3. Examples of categories

Well-known examples

Set Top Grp Ab Vect(k) Mod(R)

Ring: identity + preserve identity CRing Rng

Field: full subcategory of CRing

0: $Ob(0) = \phi$

1: $Ob(1) = \{*\}$ $Mor(*,*) = \{1_*\}$

$K(2)$: $Ob(K(2)) = \{V, E\}$ $Mor(V,V) = \{1_V\}$ $Mor(E,E) = \{1_E\}$

$Mor(V,E) = \phi$ $Mor(E,V) = \{s, t\}$

$$\begin{tikzcd} E \arrow[loop left, "1_E"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \arrow[loop right, "1_V"] \end{tikzcd}$$

Δ: $Ob(\Delta) = \{[n] := \{0,1,2,\cdots n\} \mid n \geq 0\}$

$Mor([m],[n]) = \{\text{weakly monotone maps}\}$

sSet: $Ob(sSet) = \left\{ X : \Delta^{op} \longrightarrow Set \atop [n] \mapsto X_n \right\}$ $Mor(X,Y) = \left\{ \alpha : \begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"{name=U}] \arrow[r, bend right, "Y"'{name=D}] & Set \arrow[Rightarrow, from=U, to=D, "\alpha"] \end{tikzcd} \right\}$

CHaus: $Ob(CHaus) = \{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} X \}$

https://ncatlab.org/nlab/show/compactum

$Mor(X,Y) = \{ f : X \longrightarrow Y \mid f \text{ cont} \}$

Met: full subcategory of CHaus whose objects are metric spaces.

⚠ For the category of Graph, there're different realizations.

$Quiv(\mathcal{C})$: $Ob(Quiv(\mathcal{C})) = \{ \text{fctor } \Gamma : K(2) \longrightarrow \mathcal{C} \}$

$Mor(\Gamma_1, \Gamma_2) = \left\{ \alpha : \begin{tikzcd} K(2) \arrow[r, bend left, "\Gamma_1"] \arrow[r, bend right, "\Gamma_2"'] & \mathcal{C} \end{tikzcd} \right\}$

$Quiv = Quiv(Set)$

$= \text{Category of presheaves on } Q^{op}.$

Cat

Cat = {the category of small categories} is a 2-category.

$Ob(Cat) = \{\text{small category } \mathcal{C}\}$

$Mor(\mathcal{C},\mathcal{D})$ is a category by

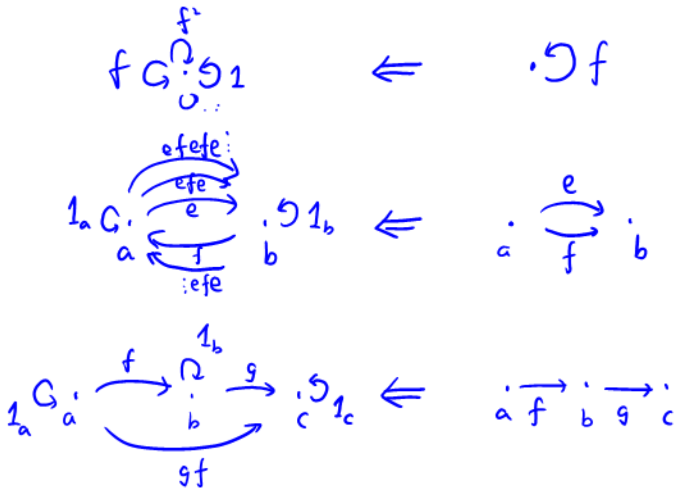
$Ob(Mor(\mathcal{C},\mathcal{D})) = \{F : \mathcal{C} \longrightarrow \mathcal{D}\}$

$$Mor(F,G) = \left\{ \alpha : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \right\}$$

Basic properties of Cat:

1. Initial object 0, Terminal object 1.
2. Cat is loc. small but not small
3. Cat is bicomplete
4. Cat is Cartesian closed but not loc. Cartesian closed
5. Cat is loc. finitely presentable https://ncatlab.org/nlab/show/locally+finitely+presentable+category
6. $$\begin{tikzcd} Cat \arrow[r, bend right, "forget"'] \arrow[r, phantom, "\top"] & Quiv \arrow[l, bend right, "free"'] \end{tikzcd}$$

e.g of "free"



Hausdorff and compactness ← cpt ≈ (quasi)cpt

$$\begin{array}{lccccc}
 & \mathrm{Haus} & \subset & \mathrm{wHaus} & \subset & \mathrm{Top} \\
 & \cup & & \cup & & \cup \\
 & \mathrm{CGHaus} & \subset & \mathrm{CGwHaus} & \subset & \mathrm{kTop} \\
\text{CW cplx, Met} & \cup & & & & \cup \\
 & \text{loc. CHaus} & & = & & \text{loc. cpt} \\
\text{topo mfld} & \cup & & & & \cup \\
 & \mathrm{CHaus} & & = & & \mathrm{cpt}
\end{array}$$

Def. $X \in \mathrm{Top}$ is a weak Hausdorff space (in wHaus) if
$\forall\, K \in \mathrm{CHaus}, \forall\, g : K \to X$ cont, $g(K) \subset X$ is closed.

Def. $X \in \mathrm{Top}$ is locally compact (in loc.cpt) if
$\forall\, p \in X$, $\exists$ cpt nbhd V (i.e. $p \in U \subseteq V \subseteq X$ $U \subseteq X$ open, V cpt)
loc. CHaus = loc. cpt ∩ Haus

see [https://en.wikipedia.org/wiki/Locally_compact_space] for other common definitions which are not equivalent in general.

Def. $X \in \mathrm{Top}$ is a compactly generated/a k-space (in kTop) if

cpt gen in Condensed Math
Hausdorff-cpt gen/k-space in wiki
cpt gen/k-space in nlab
k-space in AT II

[CGwHaus = kTop ∩ wHaus
CGHaus = kTop ∩ Haus]

$\forall$ map $f : X \to Y$,
f is cont $\Leftrightarrow$ $K \xrightarrow{g} X \xrightarrow{f} Y$ is cont
$\forall\, K \in \mathrm{CHaus}$, $g : K \to X$ cont

equivalently,
$\forall\, A \subseteq X$ subspace,
$A \subseteq X$ is closed $\Leftrightarrow$ $g^{-1}(A) \subseteq K$ is closed
$\forall\, K \in \mathrm{CHaus}$, $g : K \to X$ cont

When X is Hausdorff, this is equivalent to
$\forall\, A \subseteq X$ subspace,
$A \subseteq X$ is closed $\Leftrightarrow$ $A \cap K \subseteq K$ is closed
$\forall\, K \in \mathrm{CHaus}$

Prop. $X \in \mathrm{Top}$, then
X is a k-space $\Leftrightarrow$ $X \cong \coprod_{i \in \Lambda} S_i / \sim$ $S_i \in \mathrm{CHaus}$

Rmk. In the def/prop of kTop, CHaus can be replaced by Prof.

Adjoints

$$\begin{tikzcd}
\mathrm{Haus} \arrow[rr, hook] \arrow[rr, phantom, "\perp", shift left=3] & & \mathcal{T}op \arrow[ll, bend right, "\beta\text{: Stone-Čech compactification}"'] \arrow[d, bend left, "\text{Kelley fctor}"] \arrow[d, phantom, "\dashv"] \\
& \mathrm{CGwHaus} \arrow[r, hook] \arrow[r, phantom, "\perp", shift left=3] & k\mathcal{T}op \arrow[u, hook] \arrow[l, bend right, "\text{weak Hausdorffication}"']
\end{tikzcd}$$

Kelley fctor $(-)^{cg}: \mathcal{T}op \longrightarrow k\mathcal{T}op$

$X \longmapsto X^{cg}$ compactly generated

Set: $X^{cg} = X$

Topo: $A \subseteq X^{cg}$ is closed if $g^{-1}(A) \subseteq K$ is closed

$\forall K \in \mathrm{CHaus}$, $g: K \longrightarrow X$ cont

Categories in condensed mathematics

$$\begin{tikzcd}
 & & & \mathrm{cpt} \arrow[r, phantom, "\subset"] \arrow[dl, phantom, "\cup"] & \mathrm{qc} \arrow[d, phantom, "\cap"] \\
\mathrm{qcProj} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & \mathrm{Prof} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & \mathrm{qcqs} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & \mathrm{qs} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cong"] & \mathrm{CondSet} \arrow[dd, phantom, "\bigcup"] \\
\mathrm{CplBoolAlg}^{op} & \mathrm{BoolAlg}^{op} & \mathrm{CHaus} & \mathrm{Ind(CHaus)} & \\
 & & & \mathrm{Liq}_p \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cup"] & \mathrm{CondAb} \arrow[d, phantom, "\cup"] \\
 & & & \mathrm{Liq}_p(R) \arrow[r, phantom, "\subset"] & \mathrm{Cond}(R)
\end{tikzcd}$$

Appendix

I'm just too lazy to fill in this table. If you know more, tell me and I will fill in, thanks!

Category	cpl	fin cpl	cocpl	fin cocpl	Cartesian closed	closed	monoidal
Set	✓		✓		✓		✓
Top	✓		✓		×		✓
Grp	✓		✓		×		✓
Ab	✓		✓		×		✓
Vect(K)	✓		✓		×		✓
Mod(R)	✓		✓		×		✓
Ring	✓		✓				
CRing	✓		✓				
Rng	✓		✓				
Field	×	×	×	×			
0							
1							
$K(2)$							
Δ	×	×	×	×			
sSet	✓		✓				
CHaus	✓		✓				
Met	×	✓	×	×			
Quiv($\mathcal{C}$)							
Quiv							
Cat	✓		✓		✓		✓
kTop							✓
CGHaus					✓		
CGwHaus	✓		✓		✓		
Prof	✓						

ReCRing