Eine Woche, ein Beispiel

11.7. Berkovich space

Ref: Spectral theory and analytic geometry over non-Archimedean fields by Vladimir G. Berkovich(we mainly follow this article)
+courses from Junyi Xie
+An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves

Goal: understand the beautiful picture copied from "An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves"

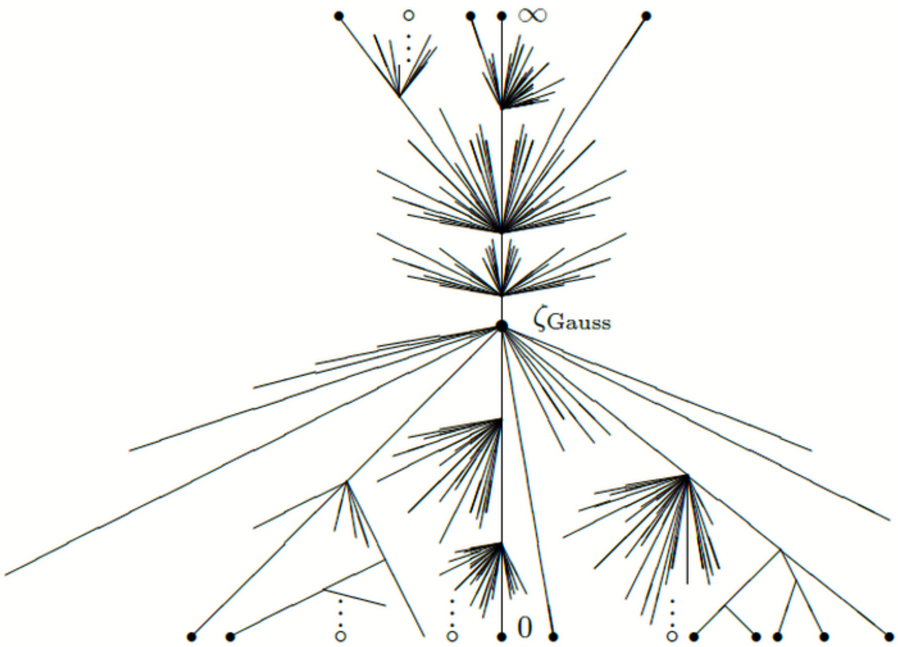


FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

1. Seminorm

1.1. Def (seminorm of abelian group) $\|\cdot\|: M \longrightarrow \mathbb{R}_{\geq 0}$ s.t

$\|0\| = 0$ — norm: $\|m\| = 0 \Rightarrow m = 0$

$\|f-g\| \leq \|f\| + \|g\|$ — non-Archimedean: $\|f-g\| \leq \max(\|f\|, \|g\|)$

- seminorm $\Rightarrow$ topology

  Prop. $(M, \|\cdot\|)$ is Hausdorff $\Leftrightarrow$ $\|\cdot\|$ is norm

  Def (equivalence of norm)

- sub, quotient, homomorphism

  Def (restricted seminorm)

  Def. (residue seminorm) $\pi: (M, \|\cdot\|_M) \longrightarrow M/N$ induce the seminorm on M/N:

  $$\|\bar{m}\|_{M/N} := \inf_{\pi(m')=\bar{m}} \|m'\|_M$$

  Def (bounded / admissible) $\varphi: (M, \|\cdot\|_M) \longrightarrow (N, \|\cdot\|_N)$

  - bounded: $\exists C > 0$, $\|\varphi(m)\|_N \leq C\|m\|_M$
  - admissible: $\bar{\varphi}: (M/\ker\varphi, \|\cdot\|_{quo}) \longrightarrow (\mathrm{Im}\varphi, \|\cdot\|_{res})$ induces equivalence of norm.

1.2. Def (seminorm of ring non-comm, with 1): seminorm group +

$\|1\| = 1$ — norm

$\|fg\| \leq \|f\| \|g\|$

power-multi: $\|f^n\| = \|f\|^n$

multiplicative: $\|fg\| = \|f\|\|g\|$

norm $\xrightarrow{+\text{completed}}$ Banach ring

power-multi, multiplicative $\Rightarrow$ valuation

- quotient, $\prod_{\text{infinite}}$, $\mathcal{A}\langle r^{-1}T\rangle$, ...
- comparison among norms: bounded.
- Def related to valuation field.

1.3. Def (seminorm of $\mathcal{A}$-module, where $\mathcal{A}$: normed ring)

seminorm group + $\exists C > 0$, $\|fm\| \leq C\|f\| \|m\|$

- $\hat{\otimes}_{\mathcal{A}}$

seminormed ring

↑ — $(\mathbb{Z}, |\ |_p)$, $(\mathbb{Q}, |\ |_\infty)$, $(\mathbb{Q}, |\ |_p)$

Banach ring

↑ — (R, triv) R: not field, $(\mathbb{Z}, |\ |_\infty)$, $(\mathbb{C}\langle r^{-1}T\rangle, \|\ \|_{ind})$

valuation field

Arch field	non-Arch field
$(\mathbb{R}, |\ |_\infty)$ $(\mathbb{R}/\mathbb{C}, |\ |_\infty^{\varepsilon})$ $0<\varepsilon\leq 1$	$(\mathbb{F}_q, \text{triv})$ (K, triv)
$(\mathbb{C}, |\ |_\infty)$	$(\mathbb{Q}_p, |\cdot|_p)$

2. Affine case

suppose $\mathcal{A}$: Banach ring comm + 1

$\mathcal{M}(\mathcal{A}) := \{\text{bounded mult seminorms on } \mathcal{A}\}$

with top basis generated by $U_{m,(a,b)} := \{\|\cdot\| \in \mathcal{M}(\mathcal{A}) \mid \|m\| \in (a,b)\}$

$m \in \mathcal{A}$, $(a,b) \subseteq \mathbb{R}$

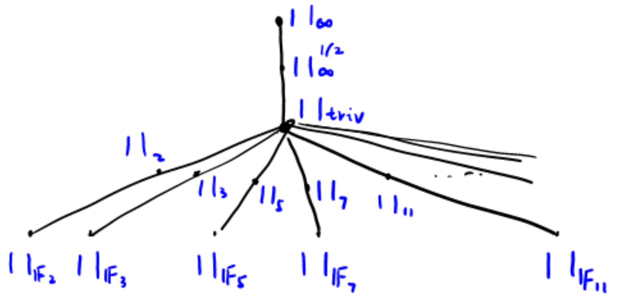
E.g. $\mathcal{A} = (\mathbb{Z}, |\,|_\infty)$

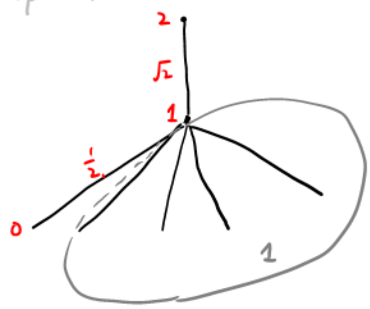
We have

$$\mathcal{M}(\mathbb{Z}, |\,|_\infty) = \left\{ \begin{array}{ll} |\,|_{triv} := \text{trivial norm} & \\ |\,|_p^t : t \in (0, +\infty] & |m|_{\mathbb{F}_p} := |m|_p^\infty = \begin{cases} 0 & p \mid m \\ 1 & p \nmid m \end{cases} \\ |\,|_\infty^\varepsilon : \varepsilon \in (0,1] & \end{array} \right\}$$

$|\,|_{triv} = |\,|_p^0 = |\,|_\infty^0$

Picture:



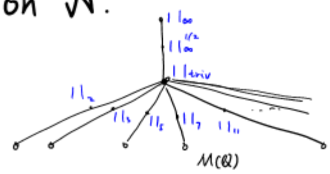


value of 2.

From this picture, we want to get:
Bound relations among seminorms
Topology properties: Hausdorff? compact?
Residue field, injection and contraction
... See next page

Rmk. It's better when we consider all the mult seminorms on $\mathcal{A}$.

I don't want to choose a norm on $\mathcal{A}$ delibaratelly.



E.g. $\mathcal{A} = \mathbb{Q}$.

E.g. $\mathcal{A} = \mathbb{F}_q$ $\quad \mathcal{M}(\mathbb{F}_q) = \{triv\}$

E.g. $\mathcal{A} = \mathbb{R}/\mathbb{C}$ $\quad$ reasonable seminorms are $\|\cdot\|_\infty^\varepsilon$, $\varepsilon \in [0,1]$.

Do we have any other seminorms?

E.g. $\mathcal{A} = \mathbb{Q}_p$ $\quad$ reasonable seminorms are $\|\cdot\|_p^t$, $\varepsilon \in [0, +\infty]$.

Do we have any other seminorms?

E.g. $\mathcal{A} = \mathbb{C}_p$

E.g. $\mathcal{A} = \mathbb{C}[x]$ $\quad$ If we only consider the norm which restricted to $\mathbb{C}$ is $|\,|_\infty$, we would get $\mathbb{C}$.

What would happen in the other cases?

I'm very happy to do the homework one years ago.

E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

Try to answer the following questions:

- Set
  - $\mathcal{M}(\mathbb{Z}) =$ ✓
  - partial order ⇝ bound order
  - Picture ✓
  - maximal / minimal norm — max: $\|\cdot\|_{\mathbb{F}_p}$ min $\|\cdot\|_\infty$
  - Berkovich structure of $\|\cdot\| \in \mathcal{M}(\mathbb{Z})$?
- Topo
  - Close set — not contain $|\ |_{triv}$: normal way + contain only finite $\|\cdot\|_p^t$; contain $|\ |_{triv}$: normal way
  - Open set — not contain $|\ |_{triv}$: normal way; contain $|\ |_{triv}$: normal way + contain all $\|\cdot\|_p^t$ except finite p
  - Topo properties: connected? ✓ ($X = Y \sqcup Z$) Hausdorff? ✓ (quasi)compact? ✓
  - irreducible? ✗ ($X = Y \cup Z$)

$(\mathcal{M}(\mathbb{Z}), \text{graph})$

$\downarrow$

$(\mathcal{M}(\mathbb{Z}), \text{weak})$ is continuous

weak top is a little weaker then graph top

Def. $p \in X$ is a closed pt iff $\{p\}$ is closed

Then every pt is closed pt

The definitions of Residue field, injection and contraction follows from [3.1.1, https://arxiv.org/abs/2105.13587v3]