Un exemple par jour

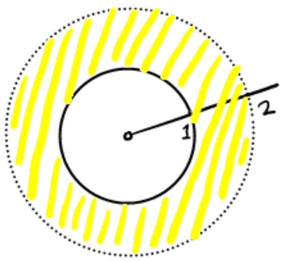
4.1. the complex torus of form $\mathbb{C}^\times/\mathbb{Z}\gamma$

$$\mathcal{C} := \mathbb{C}^\times/\mathbb{Z}\gamma \overset{\text{topo}}{=\!=} \mathbb{T}^2 \text{ is a cpt Riemannian surface of genus } 1.$$

$\gamma \in \mathrm{Aut}(\mathbb{C}^\times)$ $\gamma(z) = az$ $a \in \mathbb{C}^\times$ $|a| > 1$

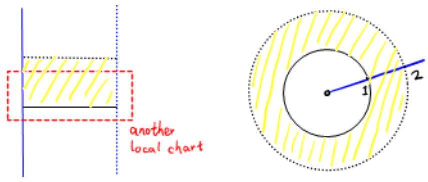
Today: $a = 2$

1. fundamental set:



$\Rightarrow$ only need 2 local chart

2.
$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z} & \hookrightarrow & \mathbb{C} & \xrightarrow{f:\, z \mapsto e^{2\pi i z}} & \mathbb{C}^\times & \to 1 \\ & \downarrow + \frac{1}{2\pi i}\ln 2 & & \downarrow + \frac{1}{2\pi i}\ln 2 & & \downarrow \times 2 & \\ 0 \to & \mathbb{Z} + \frac{1}{2\pi i}\ln 2 & \to & \mathbb{C} & \longrightarrow & \mathbb{C}^\times & \to 1 \end{array}$$



$\mathbb{C} \xrightarrow{f} \mathbb{C}^\times$

$$\mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \Rightarrow \mathbb{C}^\times/\mathbb{Z}\gamma = \mathbb{C}/(\mathbb{Z} \oplus \tfrac{1}{2\pi i}\ln 2\, \mathbb{Z})$$

better: $a = e^{2\pi} \approx 535.49$

$a = e^{-2\pi i w} \approx -230.765$

3. line bundle on $\mathcal{C}$

$b \in \mathbb{C}^\times$ $\quad \mathcal{L}_b := \mathbb{C}^\times \times \mathbb{C} \,/\, (z,\xi) \sim (2z, b\xi)$

$$\begin{array}{c} \mathcal{L}_b \\ \downarrow \\ \mathcal{C} = \mathbb{C}^\times/z \sim 2z \end{array}$$

$\Rightarrow$ ① $\mathcal{L}_b \in \mathrm{Pic}_0(C)$; $\quad$ (cont. deformation $\mathcal{L}_b \sim \mathcal{L}_1 \cong \mathcal{O}_C$)

(naive, base pt $1 \in \mathbb{C}^\times/\mathbb{Z}\gamma$)

② $\mathrm{Pic}_0(C) \cong C = \mathbb{C}^\times/\mathbb{Z}\gamma$

$\mathcal{L}_b \longmapsto b$

Reduced to: find a section s on $\mathcal{L}_b$ st $\operatorname{div} s = [b] - [1]$

Reduced to: find a meromorphic functions g on $\mathbb{C}^\times$ s.t

① $g(2z) = b\,g(z)$ $\qquad b \in \mathbb{C}^\times$, $b \neq 2^k$; e.g. $b = 3$

② g has simple poles on 2^n, and simple zeros on $2^n b$ $\quad n \in \mathbb{Z}$

$b = e^{2\pi i c}$, $c \in \mathbb{C}$

$\tau := \frac{1}{2\pi i}\ln 2$

$w(z) = \frac{1}{2\pi i}\ln z$

$$g(z) := \frac{\theta\left[\begin{smallmatrix}1\\1-2c\end{smallmatrix}\right](w(z),\tau)}{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](w(z),\tau)} \quad \text{is the required one.}$$

Blue — example
Orange — more than this example
Red — important results
Purple — I don't know the answer/proof
Green — sketsch of proof : in a minimal way
Grey — some supplementary explanation. Unimportant assumptions.
Hell grey — explanation on well-known notations.
Brown — small title in subsections.

My symbol collection set

		Mathbb	Mathrsf/Mathcal		
A	a	$\mathbb{A}$ adèles	$\mathcal{A}$		α
B	b	$\mathbb{B}$	$\mathcal{B}$ building		β
C	c	$\mathbb{C}$ cplx number	$\mathcal{C}$ category	Γ gp graph	γ
D	d constant	$\mathbb{D}$	$\mathcal{D}$ Poincare disk	Δ diag embedding	δ
E	e	$\mathbb{E}$	$\mathcal{E}$		ε
F	f	$\mathbb{F}$ finite field	$\mathcal{F}$ sheaf		ζ
G group	g	$\mathbb{G}$ gp scheme	$\mathcal{G}$ $\mathfrak{g}$: Lie alg, upper half plane		η
H	h constant	$\mathbb{H}$	$\mathcal{H}$ Hecke alg	Θ	θ
I ideal	i	$\mathbb{I}$	$\mathcal{I}$ ideal of sheaf		ι injection
J	j	$\mathbb{J}$	$\mathcal{J}$		κ
K cos/base field	$k \leftarrow k$	$\mathbb{K}$	$\mathcal{K}$	Λ lattice	λ
L	l	$\mathbb{L}$	$\mathcal{L}$		μ
M module	m	$\mathbb{M}$	$\mathcal{M}$		ν root of unity (ζ/ω)
N	n	$\mathbb{N}$ natural number	$\mathcal{N}$		ξ constant
O	o	$\mathbb{O}$	$\mathcal{O}$ structure sheaf	Π multi	π uniformizer projection
P	p	$\mathbb{P}$ proj space	$\mathcal{P}$ Weierstrass $\wp$: ell fct		$\rho \leftarrow \varrho$
Q	q	$\mathbb{Q}$ rational number	$\mathcal{Q}$	Σ sum	σ
R ring	r	$\mathbb{R}$ real number	$\mathcal{R}$		τ
S	s	$\mathbb{S}$	-	Φ	φ
T	t	$\mathbb{T}$	$\mathcal{T}$		χ character
$U \leftarrow U$	u	$\mathbb{U}$	-	Ψ	ψ
V v.s.	v	$\mathbb{V}$	-	Ω	ω
W witt vector	w	$\mathbb{W}$	-		
X	x	$\mathbb{X}$	$\mathcal{X}$		
$Y = Y$	y	$\mathbb{Y}$	-		
Z	z	$\mathbb{Z}$ integer	$\mathcal{Z}$		

Green: number / basic stuffs in senior high school

Orange: scheme - related

Darkyellow: advanced algebra

Don't use them simultaneously!

Don't mix: w/ω, ζ/ξ, $k/\kappa/\mathcal{K}$, $1/l/\iota$, $\times/x/\chi/\mathcal{X}$, φ/ψ