Un example par jour

4.8 K3 surface: cpt cplx surf s.t $\omega_X \cong \mathcal{O}_X$ $H^1(X,\mathcal{O}_X)=0$ ← can be changed by $b_1=0$ or $\pi_1(X)=\{*\}$

Today. Fermat quartic surface $X: z_0^4+z_1^4+z_2^4+z_3^4=0$ $\quad i: X \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$

A nice survey for smooth quartic surfaces: http://www-personal.umich.edu/~jakubw/masterthesis.pdf

1. proj smooth alg surf ✓

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to i_*\mathcal{O}_X \to 0$$

Lemma VIII.9 *Let $V \subset \mathbb{P}^n$ be a d-dimensional complete intersection; then $H^i(V,\mathcal{O}_V)=0$ for $0<i<d$.* [Beauville]

$$\begin{array}{lccccccc}
H^3 & & \to & \mathbb{C} & \to & 0 & & \\
H^2 & & \to & 0 & \to & 0 & \to & H^2(X,\mathcal{O}_X) \cong \mathbb{C} \\
H^1 & & \to & 0 & \to & 0 & \to & H^1(X,\mathcal{O}_X) \cong 0 \\
H^0 & 0 & \to & 0 & \to & \mathbb{C} & \to & H^0(X,\mathcal{O}_X) \cong \mathbb{C}
\end{array}$$

$$\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4) \xRightarrow{21.5.B \text{ in [Vakil]}} \omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X = \mathcal{O}_X. \qquad \frac{z_0\, dz_1 \wedge dz_2}{4 z_3^3} \quad z_0 \neq 0,\ z_3 \neq 0$$

$\therefore X$ is a K3-surface.

2. Prop. X: K3 surfaces $\Rightarrow$ X is minimal.

$P_n = 1 \quad n \geq 1 \Rightarrow \kappa(X)=0$

$$\begin{array}{ccccccc}
2 & -20 & 2 & & & & 24 \\
& & & 1 & & & 1 \\
& & 0 & & 0 & & 0 \\
& 1 & & 20 & & 1 & 22 \\
& & 0 & & 0 & & 0 \\
& & & 1 & & & 1
\end{array}$$

$(b_+,b_-)=(3,19)$

$c_2 = 24$

$c_1^2 = K^2 = 0$

Some conclusions on alg topo: M: closed connected mfld of dim n (↗cpt)

	orientable	nonorientable
$H_n(M)$	$\mathbb{Z}$	0
$H^n(M)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H_n(M,\mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H^n(M,\mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Moreover, when M is oriented + cpt $\Rightarrow$ $T^p \cong T^{n-p+1}$ $\quad (H^i(X) \cong \mathbb{Z}^{\oplus b_i} \oplus T^i)$ ← def

↑ by universal coefficient thm & Poincaré duality

Cor $T_i = H_i(X)$

n	0	1	2	3	4	≥ 5
$H_n(X)$	$\mathbb{Z}$	T	$\mathbb{Z}^{22}\oplus T$	0	$\mathbb{Z}$	0
$H^n(X)$	$\mathbb{Z}$	0	$\mathbb{Z}^{22}\oplus T$	T	$\mathbb{Z}$	0

Claim. $T=0$, i.e. (co)homology has no torsion!

https://math.stackexchange.com/questions/2882059/

Proof. by LES induced by $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 1$,

only need to prove $\mathrm{Pic}(X)$ is torsion free.

Suppose $D \in \mathrm{Div}(X)$ s.t $nD \equiv 0 \Rightarrow \chi(D)=2$

$\Rightarrow h^0(D) \geq 1$ or $h^0(-D+K_X) = h^2(D) \geq 1$ $\quad$ w.l.o.g suppose $h^0(D) \geq 1$

$\Rightarrow D \sim D'$ where D' is effective

$\xRightarrow{nD' \equiv 0} D' = 0$

Another proof. If $H_1(X)$ has torsion, then $\exists\, G \triangleleft H_1(X)$ s.t $H_1(X)/G \cong \mathbb{Z}/m\mathbb{Z}$ $m \in \mathbb{N}^+$, $m > 1$
denote $p: \pi_1(X) \twoheadrightarrow H_1(X)$, then $\pi_1(X)/p^{-1}(G) \cong H_1(X)/G \cong \mathbb{Z}/m\mathbb{Z}$
$\therefore \exists$ a nontrivial unramified covering of degree m
$$\Phi: \widetilde{X} \longrightarrow X$$
$$\Rightarrow \begin{cases} K_{\widetilde{X}} = 0 \\ \chi_{top}(\widetilde{X}) = 24m \end{cases} \Rightarrow \chi(\mathcal{O}_{\widetilde{X}}) = \frac{1}{12}\left(K_{\widetilde{X}}^2 + \chi_{top}(\widetilde{X})\right) = 2m > 2 - h^1(\mathcal{O}_{\widetilde{X}}) = \chi(\mathcal{O}_{\widetilde{X}}),$$
Contradiction!

12.2. **K3 lattice.** We deduce that a K3 surface has second Betti number $b_2 = 22$. Cup-product equips $H^2(X, \mathbb{Z})$ with the structure of an integral lattice of rank 22. Often this lattice is called **K3 lattice** and denoted by Λ. The following properties of Λ are well-known:

- Λ is unimodular by Poincaré-duality;
- Λ has signature $(3, 19)$ by the topological index theorem;
- Λ is even by Wu's formula since the first Chern class is even. ← one Signature: $C^2 = 2g(C) - 2$ is even

Hence the classification of even unimodular lattices implies that
$$\Lambda \cong U^{\oplus 3} \oplus \cancel{E_8}(-E_8)^{\oplus 2} \qquad U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$
$\leftarrow \widetilde{b}(x,y) := -b(x,y)$

Now let us consider the fundamental group of X.

Thm (Lefschetz hyperplane thm) [In wiki there are several proofs].
Let Y: n-dim cplx proj variety
X: hyperplane section of Y
U: $Y - X$ is smooth
then $H_k(Y, X, \mathbb{Z})$, $H^k(Y, X, \mathbb{Z})$, $\pi_k(Y, X) = 0$ for $k \le n-1$. $k \in \mathbb{N}^+$

Cor.

		$k<n-1$		$k=n-1$
$H_k(X) \longrightarrow H_k(Y)$	is	iso	or	surj
$H^k(Y) \longrightarrow H^k(X)$	is	iso	or	inj
$\pi_k(X) \longrightarrow \pi_k(Y)$	is	iso	or	surj.

For X Fermat quartic, $X \xhookrightarrow{i} \mathbb{P}^3_{\mathbb{C}} \xhookrightarrow{\phi} \mathbb{P}^{34}_{\mathbb{C}}$ $Y = \phi(\mathbb{P}^3_{\mathbb{C}})$.
$H_1(X) = H_1(Y) = H_1(\mathbb{P}^3_{\mathbb{C}}) = 0$ $H^1(X) = 0$ $\pi_1(X) = 0$
$\Rightarrow X$ is simply connected 4-dim mfld.

Rmk. By [GTM82, Thm 17.21], $\pi_2(X) = H_2(X) = \mathbb{Z}^{22}$;
By https://mathoverflow.net/questions/186119/what-are-the-higher-homotopy-groups-of-a-k3-suface
$\pi_n(X) = \pi_n(\#^{20} S^2 \times S^3)$ $n \geq 3$, and

	1	2	3	4	5	6	7
$\pi_n(X)$	0	$\mathbb{Z}^{22}$	$\mathbb{Z}^{252}$	$\mathbb{Z}^{3520} \oplus (\mathbb{Z}/2\mathbb{Z})^{42}$	$\mathbb{Z}^{57960} \oplus$ tor	$\mathbb{Z}^{1020096} \oplus$ tor	$\mathbb{Z}^{22397760} \oplus$ tor

It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry.
-Solomon Lefschetz

Picard group:

Again, by $0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$

$$0 \to \underbrace{H^0(X,\mathbb{Z})}_{\cong \mathbb{Z}} \to \underbrace{H^0(\mathcal{O}_X)}_{\cong \mathbb{C}} \to \underbrace{H^0(\mathcal{O}_X^\times)}_{\cong \mathbb{C}^\times} \to \underbrace{H^1(X,\mathbb{Z})}_{\cong 0} \longrightarrow \underbrace{H^1(\mathcal{O}_X)}_{\cong 0} \to \underbrace{H^1(\mathcal{O}_X^\times)}_{= \mathrm{Pic}(X)} \to \underbrace{H^2(X,\mathbb{Z})}_{\cong \mathbb{Z}^{22}} \longrightarrow \underbrace{H^2(\mathcal{O}_X)}_{\cong \mathbb{C}} \to H^2(\mathcal{O}_X^\times)$$

$\Rightarrow \mathrm{Pic}(X) = NS(X) \subseteq H^2(X,\mathbb{Z}) = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$

$\Rightarrow \mathrm{Pic}(X) \cong \mathbb{Z}^{\rho(X)}, 1 \leq \rho(X) \leq 20$

Rmk. for X Fermat quartic, $\rho(X) = 20$, $\mathrm{Pic}(X) \cong (-E_8)^{\oplus 2} \oplus U \oplus (-8)^{\oplus 2}$ $\mathrm{Pic}(X)^\perp \cong (8)^{\oplus 2}$

[Schütt, Shioda and van Luijk]

Q. find a metric s.t $\mathrm{Ric}(\rho) \equiv 0$?