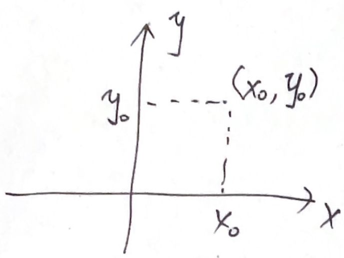
Session 2 & Exercise 0

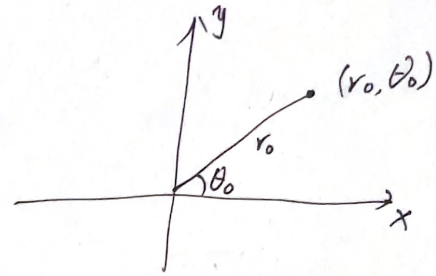
Your handwriting looks nice! Q?

1. pts in $\mathbb{C}$.

To describe a pt in plane, we have two typical coordinate systems.



Cartesian coordinate system



Polar coordinate system $\theta_0 \in (-\pi, \pi]$
shortcomings

Now, in the cplx plane, we can use z_0 to denote a pt (in $\mathbb{C}$)

Define
$$\mathrm{Re}\, z_0 = x_0 \qquad |z_0| = r_0$$
$$\mathrm{Im}\, z_0 = y_0 \qquad \arg z_0 = \theta_0$$

Then we have the relationships

$$\begin{cases} r_0 = \sqrt{x_0^2 + y_0^2} \\ \theta_0 = \arctan \frac{y_0}{x_0} + k\pi \end{cases}$$

k should be determined by case by case!

$$(x_0, y_0) \rightleftarrows (r_0, \theta_0)$$

$$\begin{cases} x_0 = r_0 \cos\theta_0 \\ y_0 = r_0 \sin\theta_0 \end{cases}$$

$$\begin{cases} x_0 = \mathrm{Re}\, z_0 \\ y_0 = \mathrm{Im}\, z_0 \end{cases} \qquad z_0 = x_0 + iy_0 \qquad \begin{cases} r_0 = |z_0| \\ \theta_0 = \arg z_0 \end{cases} \qquad z_0 = r_0 e^{i\theta_0}$$

$$z_0$$

Task 1. Compute $\mathrm{Re}\, z, \mathrm{Im}\, z$, $|z|$, $\arg z$ for

(i) $z = (1-3i)^2$

(ii) $z = \dfrac{5}{3-4i}$

(iii) $z = \left(\dfrac{2-i}{3+2i}\right)^2$

hint: (i) expansion

(ii) $\dfrac{5}{3-4i} = \dfrac{5(3+4i)}{(3-4i)(3+4i)} = \dots$

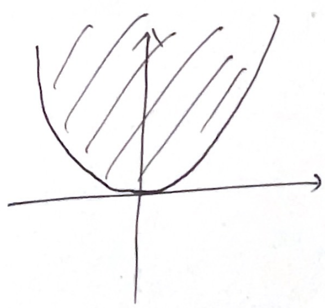
Task 2. draw the set

$\cancel{M_1 = \{z \in \mathbb{C}\}}$

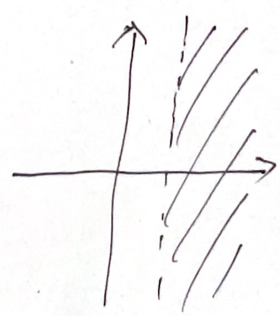
$M_1 = \{(x,y) \in \mathbb{R}^2 \mid y \geq x^2\}$

$M_2 = \{z \in \mathbb{C} \mid \mathrm{Re}\, z > 1\}$

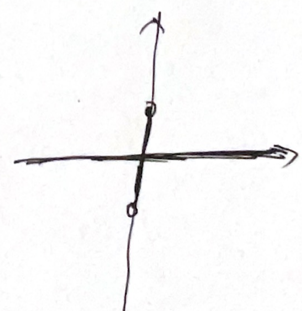
$M_3 = \{z \in \mathbb{C} \mid \arg(1+z^2) = 0\}$



M_1



M_2



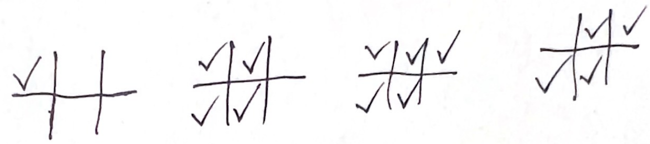
M_3

2. Topology

Task 2 (continue)

Compute M°, M', $\overline{M}$, ∂M

interior, accumulated point, closure, boundary



Def. Let ~~M~~ (X, d) be a metric space, define $(x \in X,\ r \in \mathbb{R}_{>0})$

$$B_r(x) := \{ y \in X \mid d(x,y) < r \}$$

$$B_r(\check{x}) := \{ y \in X \mid d(x,y) < r,\ x \neq y \}$$

For $M \subset X$, we define

$$M^\circ := \{ x \in X \mid \exists r>0,\ B_r(x) \subseteq M \}$$

$$M' := \{ x \in X \mid \forall r>0,\ B_r(\check{x}) \cap M \neq \emptyset \}$$

$$\overline{M} := \{ x \in X \mid \forall r>0,\ B_r(x) \cap M \neq \emptyset \}$$

$$\partial M := \{ x \in X \mid \forall r>0,\ B_r(x) \cap M \neq \emptyset,\ B_r(x) \cap M^c \neq \emptyset \}$$

These definitions look quite difficult to remember, and work with. Let us do another try.

Task 3. Let $V := C([0,1])$

(a) 1) Verify that V is a v.s.

2). Verify that $\|\cdot\|_1$, $\|\cdot\|_\infty$ are two norms on V, where

$$\|f\|_1 = \int_0^1 |f(s)|\, ds$$

$$\|f\|_\infty := \sup_{s\in[0,1]} |f(s)|$$

(b) Show that

1) $\|f\|_1 \leq \|f\|_\infty$

2) If $\{f_n\} \subseteq V$ is a Cauchy seq w.r.t. $\|\cdot\|_\infty$, then

$\{f_n\} \subseteq V$ is a ________________ $\|\cdot\|_1$.

3). The converse is false, i.e.

find a Cauchy seq $\{f_n\} \subseteq V$ w.r.t. $\|\cdot\|_1$

which is not Cauchy w.r.t. $\|\cdot\|_\infty$.

☆ 4). The map

$$\psi : (V, \|\cdot\|_\infty) \longrightarrow (V, \|\cdot\|_1)$$

is continuous.

<u>Hint.</u> The first step is to show, that

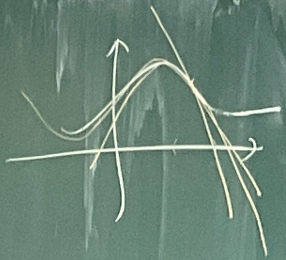
$$\psi^{-1}(B_1(0)) \text{ is open}$$

(c) 1) Show that $(V, \|\cdot\|_1)$ is not complete

2) Show that $(V, \|\cdot\|_\infty)$ is complete.

(r, θ)

$$\begin{aligned} z &= x + iy \\ &= r\cos\theta + ir\sin\theta \\ &= r(\underline{\cos\theta + i\sin\theta}) \\ &= re^{i\theta} \end{aligned}$$



Taylor expansion.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

$$\cos x = 1 - \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\begin{aligned} e^{ix} &= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \cdots \\ &= \cos x + i\sin x \end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

Task 3. Let $V = C([0,1])$.

a) i) Verify that V is a v.s. (vectorspace)

2). $\|\cdot\|_1, \|\cdot\|_\infty$ are norms

where $\|f\|_1 = \int_0^1 |f(s)|\,ds$

$\|f\|_\infty = \sup_{s\in[0,1]} |f(s)|$

c) 1) Show that $(V, \|\cdot\|_1)$ is not cpl

☆ 2) ______ $(V, \|\cdot\|_\infty)$ is cpl (complete)

b). Show that

1) $\|f\|_1 \le \|f\|_\infty$

2) If $\{f_n\} \subseteq V$ is Cauchy w.r.t $\|\cdot\|_\infty$ (with respect to)

$\Rightarrow$ ______________ $\|\cdot\|_1$

3) $\Leftarrow$ is false

☆ 4) The map

$\psi: (V, \|\cdot\|_\infty) \longrightarrow (V, \|\cdot\|_1)$

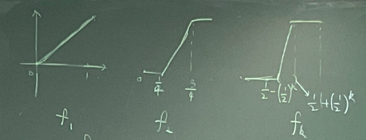
is cont

$(\mathbb{Q}, d)$ / $(\mathbb{R}, d)$

$\mathbb{Q}$: 1, $1+\frac{1}{2}$, $1+\frac{1}{2}+\frac{1}{3!}$

$e = 2.718\ldots$

2, 2.7, 2.71, 2.718



$$f_k(x) = \begin{cases} 0 & 0 \le x < \frac{1}{2} - (\frac{1}{2})^k \\ \text{linear} & \frac{1}{2}-(\frac{1}{2})^k \le x \le \\ 1 & 1 \ge x > \frac{1}{2} + (\frac{1}{2})^k \end{cases}$$

$\|f_m - f_n\|_1 \quad m<n$

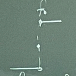
$= \int_0^1 |f_m(x) - f_n(x)|\,dx$

$= \int_{\frac{1}{2}-(\frac{1}{2})^m}^{\frac{1}{2}+(\frac{1}{2})^m} |f_m(x)-f_n(x)|\,dx$

$\le 2\cdot(\frac{1}{2})^m$

$\times$ If $f_n \to g_0 \in C([0,1]) \Leftarrow$

$f_0 = g_0$ $\times$



$f_n \longrightarrow f_0$

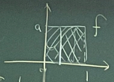
$$\|f_m - f_n\|_\infty = \max_{x\in[0,1]} |f_m(x) - f_n(x)|$$
$$\ge \left| f_m\left(\tfrac{1}{2} - (\tfrac{1}{2})^n\right) - f_n\left(\tfrac{1}{2}-(\tfrac{1}{2})^n\right)\right|$$
$$= \left| f_m\left(\tfrac{1}{2} - (\tfrac{1}{2})^n\right)\right| > \frac{1}{4}$$

c) 1) $\forall \varepsilon > 0\ \exists N > 0$ s.t $\forall m,n > N$,

$\|f_m - f_n\| < \varepsilon$

$\forall \varepsilon > 0$, $\exists N > 0$ s.t $\sup_{m,n>N} \|f_m - f_n\| < \varepsilon$

$\lim_{N\to\infty} \sup_{m,n>N} \|f_m - f_n\| = 0$



$\forall \varepsilon > 0\ \exists N =$ ___ s.t $\forall m,n > N$, $\|f_m - f_n\|_1 \le 2\cdot(\frac{1}{2})^N$

2) $\{f_n\}$ is Cauchy w.r.t. $\|\cdot\|_\infty$

$\Longleftrightarrow \lim_{k\to+\infty} \sup_{n,m>k} \|f_n - f_m\|_\infty = 0$

$\|f_n - f_m\|_1 \le \|f_n - f_m\|_\infty$

$\Rightarrow 0 \le \lim_{k\to+\infty} \sup_{n,m>k} \|f_n - f_m\|_1 \le \lim\sup \|f_n - f_m\|_\infty = 0$

$\Longleftrightarrow \{f_n\}$ is Cauchy w.r.t. $\|\cdot\|_1$