Preview of Global dimension

Proj. Inj and Global dimension.

$$\begin{aligned}
\text{proj.dim}(M) \leq m &\Leftrightarrow \text{Ext}_A^{p+1}(M,-) = 0 \quad \forall p \geq m \\
&\Leftrightarrow \text{Ext}_A^{m+1}(M,-) = 0 \\
&\Leftrightarrow \text{Ext}_A^{m+1}(M,S) = 0 \quad \forall \text{ simple } S \\
\text{inj.dim}(M) \leq m &\Leftrightarrow \text{Ext}_A^{p+1}(-,M) = 0 \quad \forall p \geq m \\
&\Leftrightarrow \text{Ext}_A^{m+1}(-,M) = 0 \\
&\Leftrightarrow \text{Ext}_A^{m+1}(S,M) = 0 \quad \forall \text{ Simple } S \\
\text{gl.dim}(A) \leq m &\Leftrightarrow \text{Ext}_A^{p+1}(-,-) = 0 \quad \forall p \geq m \\
&\Leftrightarrow \text{Ext}_A^{m+1}(-,-) = 0 \\
&\Leftrightarrow \text{Ext}_A^{m+1}(M,S) = 0 \quad \forall M \text{ cycle, } S \text{ simple} \\
&\overset{A \text{ f.d}}{\Leftrightarrow} \text{Ext}_A^{m+1}(S,S') = 0
\end{aligned}$$

Cor. 1. $\text{gl.dim}(A) = \sup\{\text{proj.dim}(M)\} = \sup\{\text{inj.dim}(M)\}$

2. We totally understand the proj resolution:

$$0 \to U \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

$\text{proj.dim}(M) > n \Leftrightarrow U$ non-proj e.p. $pd(U) = pd(M) - n$

$\text{proj.dim}(M) \leq n \Leftrightarrow U$ proj e.p. $pd(M) = n \Leftrightarrow U \hookrightarrow P$ nonsplit.

Understanding phenomenons in different gl.dim:

gl.dim $A=0$ semisimple, best of all. (In this table, gl.dim $A\neq 0$)

gl.dim A	1	2	3~n	∞
Name	hereditary			
KQ/I: quasi-hereditary label	each	$\exists$ + !	!	each !
Cartan matrix	C_P, C_I, $\langle\ ,\ \rangle_A$ path: $\langle\ ,\ \rangle_Q$			If det $C_P \notin \{1, \cancel{-1}\}$?, then ∞
A: f.g comm k-alg	regular, Kr.dim(R) = gl.dim(R)			not regular
Special ring	path		no loop/cycle quiver	dom.dim(A)=+∞ $\not\Leftrightarrow$? selfinjective ⊃ group alg generalized local ⊃ local
Special properties	ind $\xrightarrow{\neq 0}$ proj $\Rightarrow$ proj $\hookrightarrow$ proj	dom.dim(A) $\leq$ gl.dim(A) = Fin.dim(A) = inj.dim(A)		dom.dim(A) $\overset{\neq\infty \text{ suppose}}{\leq}$ inj.dim(A) fin.dim(A) $\leq$ inj.dim(A) $\overset{?}{=}$ can be done
Conjecture				Finististic, Gorenstein Nakayama

rep.dim(A)

Symmetry: For f.d. ring A,

$$\mathrm{gl.dim}(A) = \mathrm{gl.dim}(A^{op})$$

$$\mathrm{rep.dim}(A) = \mathrm{rep.dim}(A^{op})$$

$$\mathrm{fin.dim}(A) \neq \mathrm{fin.dim}(A^{op})$$

Conj $<\infty \Leftrightarrow <\infty$

$$\mathrm{dom.dim}(A) = \mathrm{dom.dim}(A^{op})$$

$$\mathrm{inj.dim}(A) \overset{?}{=} \mathrm{inj.dim}(A^{op})$$

$$\| \qquad\qquad \|$$

$$\mathrm{proj.dim}(D(A^{op})) \overset{?}{=} \mathrm{proj.dim}(D(A))$$

gl.dim with different I

"$g(Q) < d(Q)$"	
$g(Q) < \infty \Rightarrow d(Q) < \infty$	$g(Q) < \infty$? $d(Q) < \infty$? $d(Q) < \infty \Rightarrow g(Q) < \infty$?
$\exists f: \mathbb{N} \to \mathbb{N}$ s.t $\left.\begin{array}{l} A \cdot \text{f.d} \\ \dim_k(A) \leq d \\ \text{gl.dim}(A) < \infty \end{array}\right\} \Rightarrow \text{gl.dim}(A) \leq f(d)$	If $\text{gl.dim}(KQ/I) < \infty$, then $\text{gl.dim}(KQ/I) < \dim_k(KQ/I)$?
without loop $\Rightarrow \exists I$ s.t $\text{gl.dim}(kQ/I) \leq 2$	

Cartan matrix, Euler form, and Euler form of path alg.

(0) Cartan matrix conj.

$$\mathrm{gl.dim}(A) < \infty \quad \Rightarrow \quad \det C_P \in \{1, \overset{?}{\cancel{-1}}\}$$

(1) For A: $\mathrm{gl.dim}(A) < \infty$

$$\langle X, Y\rangle_A := \sum_{k=0}^{d} (-1)^k \dim_k \mathrm{Ext}_A^k (X, Y)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} [X : S(i)][Y : S(j)] \langle S(i), S(j)\rangle_A$$

corresponding matrix $(C_p)^{T^{-1}} S_A = S_A C_I^{-1}$

(2) For $A = KQ$ $\quad Q_0 = \{1, \cdots, n\}$

$$\langle -, -\rangle_Q : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z} \qquad (x, y) \mapsto \sum_{i=1}^{n} x_i y_i - \sum_{b \in Q_1} x_{s(b)} y_{t(b)}$$

$$\langle X, Y\rangle_Q = \langle \underline{\dim}_Q (X), \underline{\dim}_Q (Y)\rangle_Q = \langle X, Y\rangle_A$$

$$q_Q (X) := q_Q(\underline{\dim}_Q (X)) = \langle X, X\rangle_A$$

(3) Relations: $\langle X, Y\rangle_A = \langle X, Y\rangle_Q$ but $\langle x, y\rangle_A \neq \langle x, y\rangle_Q$

$\Rightarrow$ different matrix

also, $\underline{\dim}\, X \neq \underline{\dim}_Q (X)$

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
\underline{\dim}\, X \arrow[d, "{\langle\ ,\ \rangle_A}"'] & & \underline{\dim}_Q (X) \arrow[d, "{\langle\ ,\ \rangle_Q}"] \\
\langle \underline{\dim} (X), \underline{\dim} (Y)\rangle_A & = & \langle \underline{\dim}_Q (X), \underline{\dim}_Q (Y)\rangle_Q
\end{tikzcd}$$

After that, we works on $gl.dim(A)=\infty$, A f.d.

No loop $\quad \exists S,\ Ext^1_A(S,S)\neq 0 \Rightarrow gl.dim(S)=\infty$

$\Rightarrow proj.dim(S)=\infty$

$\overset{?}{\not\Rightarrow} Ext^m_A(S,S)\neq 0$ infinite m

Fin.dim for monomial $+$ $gl.dim<+\infty$ (Quite a lot!)

$Fin.dim(A)\neq fin.dim(A)\neq fin.dim(A^{op})$

$=0?$ $\quad fin.dim(A)=0 \iff Hom_A(D(A_A),S)\neq 0 \quad \forall S$

same Loewy length $\Rightarrow$ $Fin.dim(A)=0$

generalized local

local

If right, then a lot of other conjs are right.

$$
\begin{tikzcd}
\text{Finitistic Dimension Conj.} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Gorenstein Symmetry Conj.} & \\
\text{Strong Nakayama Conj.} \arrow[d, Rightarrow] & & \text{almost complete tilting modules have only finitely many complements} \arrow[d, Rightarrow] \\
\text{Generalized Nakayama Conj.} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{all selforthogonal generators are projective} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \text{projective almost complete tilting modules have only finitely many complements} \\
\text{Nakayama Conj.} \arrow[r, Leftrightarrow] & \text{all selforthogonal generator-cogenerators are projective} &
\end{tikzcd}
$$

$\forall M,\ \exists i$ $Ext^i(M_A,A)\neq 0$

$\forall S,\ \exists i,$ $Ext^i(S,{}_AA)\neq 0$

$D(A)\in add(\oplus I^i)$

selfinj $=$ domdim ∞

$Ext^i(M,M)=0 \quad \forall i\geq 1$

$$
\begin{tikzcd}
\{(A,M) \mid M \text{ gen-cogen}\}/\sim \arrow[r, leftrightarrow] & \{B \mid \mathrm{dom}(B) \geq 2\} \\
\{A \mid A \text{ rep-fin}\}/\sim \arrow[r, leftrightarrow] \arrow[u, hook] & \{B \mid \mathrm{dom}(B) \geq 2 \geq \mathrm{gl.dim}(B)\}/\sim \arrow[u, hook]
\end{tikzcd}
$$

$$(A, M) \longmapsto \Gamma_M := \mathrm{End}_A(M)^{op}$$

$$(\mathrm{End}_B(Q)^{op}, \mathrm{Hom}_B(Q, D(B_B))) \longmapsfrom B$$

$\Downarrow$ generalization

$$\{(A,M) \mid M \text{ } n\text{-cluster-tilting}\}/\sim \;\longleftrightarrow\; \{B \mid \mathrm{dom}(B) \geq n+1 \geq \mathrm{gl.dim}(B)\}/\sim$$

Let $M \in \mathrm{mod}(A)$ $\quad B = \Gamma_M$

$$\mathrm{Hom}_A(M, -) : \mathrm{mod}(A) \longrightarrow \mathrm{mod}(B)$$

$$
\rightsquigarrow \text{equivalence} \left[
\begin{array}{lll}
 & & \mathrm{add}(M) \to \mathrm{proj}(B) \\
\text{If} & M \text{ proj} & C_P \to \mathrm{mod}(B) \\
 & M \text{ proj gen} & \mathrm{mod}(A) \to \mathrm{mod}(B) \\
 & M \text{ gen-cogen} & \mathrm{inj}(A) \to \mathrm{proj}\text{-}\mathrm{inj}(B)
\end{array}
\right.
$$

$\mathrm{rep.dim}(A) := \min\{\mathrm{gl.dim}(\Gamma_M) \mid M \text{ gen-cogen}\}$

- $0 \Leftrightarrow$ semisimple
- $1 \Leftrightarrow \nexists$
- $\leq 2 \Leftrightarrow$ rep finite
- $\leq 3 \Rightarrow \mathrm{findim}(A) < \infty$
- $n \quad \exists$
- $\infty \Leftarrow \nexists$ Moreover, we can always find an upper bound for specific A.