1.28 conormal bundle

Slogan:

SES   induces   LES,
filtration induces   spectral sequence.

To expend a little bit,

$$SES \longrightarrow LES$$
$$filtration \longrightarrow spectral\ seq$$

resolution $\longrightarrow$ Comparison between sheaf cohomology & other cohomology
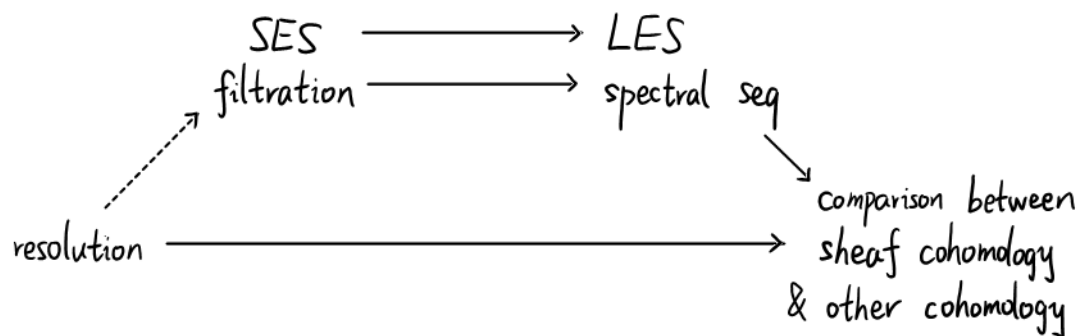
Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh". Let us leave spectral seq in other people's notes.

Methods to construct SES: $\begin{cases} \text{check by stalks} \\ \text{filtration by } \mathcal{H}^i(F) \\ \text{filtration by } F^i \end{cases}$

method	spectral seq	LES	cohomology/resolution
check by stalks	... for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech
filtration by $\mathcal{H}^i(F)$	Grothendieck		
	Leray-Serre	Gysin	Euler class
			Hodge-Tate
filtration by F^i	Hodge-de Rham		de Rham, Hodge-de Rham
need resolution to get "another" complex			Dolbeault $\quad H^p(X, \Omega^q) = H^{p,q}(X)$
	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences

which

I don't know | Adams
Atiyah-Hirzebruch
Bar
Bockstein
Cartan-Leray
Eilenberg-Moore
Green
$\vdots$ | | for stable homotopy gp
for top K-theory
for group
for group homology

for Koszul cohomology
$\vdots$ |

Hodge related (bracket spanning "filtration by $\mathcal{H}^i(F)$" and "filtration by F^i")

1. open-closed formalism
2. open cover
3. filtration by $\mathcal{H}^i(F)$
4. Hodge related filtration

1. open-closed formalism

Observe the following pictures:

$$Z \xrightarrow{\;\;i\;\;} X \xleftarrow{\;\;j\;\;} U$$

$$\mathcal{D}(Z) \;\;\begin{array}{c}\xleftarrow{\;\;i^*\;\;} \\ \xrightarrow{\;i_* = i_!\;} \\ \xleftarrow{\;\;i^!\;\;}\end{array}\; \mathcal{D}(X) \;\;\begin{array}{c}\xleftarrow{\;\;j_!\;\;} \\ \xrightarrow{\;j^* = j^!\;} \\ \xleftarrow{\;\;Rj_*\;\;}\end{array}\; \mathcal{D}(U)$$

Black box:

0. We assume some nice conditions.
   e.g. in the category $\mathrm{Haus}^{\mathrm{loc.\,cpt}}$, and $Z \subset X$ is loc. contractable.
   Under these conditions,

1. $i_* = i_!$, $j^* = j^!$
2. $j_!, i^*, j^*, i_*$ are exact.

Ex. 1. Shows that
$$\underset{\sim}{i^* i_*} = \underline{i^! i_*} = Id_{\mathcal{D}(Z)} \qquad \underline{j^* j_!} = \underset{\sim}{j^* R j_*} = Id_{\mathcal{D}(U)}$$

$$\underset{\sim}{\underline{i^* j_!}} = 0, \qquad \underset{\sim}{j^* i_*} = 0, \qquad \underline{i^! R j_*} = 0$$

—— : base change

〰〰 : check stalkwise.

2. (for category fans)
   $i_*, j_*, j_!$ are fully faithful, and
   $i_*, i^!, j^*, Rj_*$ preserve injectives.

3. One has SES
$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 spectural cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the SES
$$ 0 \longrightarrow j_! \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* i^* Rj_* \mathcal{F} \longrightarrow 0 $$
which measures the difference between $j_! \mathcal{F}$ & $j_* \mathcal{F}$.

3. Shows that
$$ H_c^{\cdot}(X) \cong H^{\cdot}(\overline{X}, \{\infty\} ; \mathbb{Z}) $$
for one pt compactification $\iota : X \hookrightarrow \overline{X}$.

Try to compute $H_c^{\cdot}(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X \mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{D^b(X;\mathbb{Q})}\left(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}}\right)$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad\qquad \mathbb{D}_X(\mathcal{F}[n]) = (\mathbb{D}_X \mathcal{F})[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad\qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

$f: Y \longrightarrow X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then
$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$
Therefore, in the constructable setting,
$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad\qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

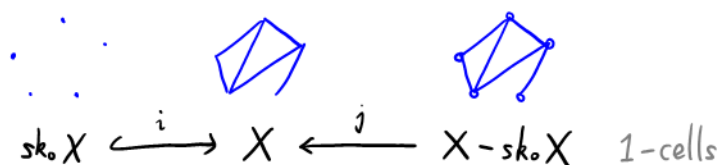For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad\qquad (2)$$

for $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".

E.g. For a finite graph (as a topo space) X.

$$sk_0X \overset{i}{\hookrightarrow} X \overset{j}{\longleftarrow} X - sk_0X \quad \text{1-cells}$$

$$0 \longrightarrow j_! j^! \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X-sk_0X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{sk_0X} \longrightarrow 0$$

Take $R\pi_{X,!}$

$$\overset{= \underset{e}{\oplus}\mathbb{Q}}{H^1_c(X - sk_0X)} \longrightarrow H^1_c(X) \longrightarrow \overset{=0}{H^1_c(sk_0X)} \overset{+1}{\longrightarrow}$$

$$0 \longrightarrow \overset{=0}{H^0_c(X - sk_0X)} \longrightarrow H^0_c(X) \longrightarrow \overset{= \underset{v}{\oplus}\mathbb{Q}}{H^0_c(sk_0X)}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that

$$H^i_c(\mathbb{R}) = \begin{cases} \mathbb{Q} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial/cellular cohomology, using the following filtration.

$$0 \subset sk^0X \subset sk^1X \subset \cdots \ sk^nX = X$$

Ex. derive the Wang LES for the cpt supp version. over S^1

2. open cover

Ex. For an open cover $X = U_1 \cap U_2$, deduce the SES

$$0 \longleftarrow \underline{\mathbb{Q}}_X \longleftarrow j_! \underline{\mathbb{Q}}_{U_1} \oplus j_! \underline{\mathbb{Q}}_{U_2} \longleftarrow j_! \underline{\mathbb{Q}}_{U_{12}} \longleftarrow 0$$

$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_1} \oplus Rj_* \underline{\mathbb{Q}}_{U_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_{12}} \xrightarrow{+1} \qquad (3)$$

▽ We omit the derived symbol and some subscripts in this section. $\quad U_{12} = U_1 \cap U_2$

(3) works for general sheaf

and. induce from (3) the MV sequence:

$$\xleftarrow{+1} H_c^k(X) \longleftarrow H_c^k(U_1) \oplus H_c^k(U_2) \longleftarrow H_c^k(U_1 \cap U_2)$$

$$H^k(X) \longrightarrow H^k(U_1) \oplus H^k(U_2) \longrightarrow H^k(U_1 \cap U_2) \xrightarrow{+1}$$

Hint: Apply $R\pi_{X,!}$ & $R\pi_{X,*}$, see [StackProject, 01E9]

Ex. Derived the Wang LES. over S^1

Ex. For an open cover $X = \bigcup_{i \in \Lambda} U_i$, Λ finite, deduce the exact seq

$$0 \longleftarrow \underline{\mathbb{Q}}_X \longleftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{U_i} \longleftarrow \bigoplus_{i<j} j_! \underline{\mathbb{Q}}_{U_i \cap U_j} \longleftarrow \cdots j_! \underline{\mathbb{Q}}_{\cap U_i} \longleftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i} \longrightarrow \bigoplus_{i<j} Rj_* \underline{\mathbb{Q}}_{U_i \cap U_j} \longrightarrow \cdots Rj_* \underline{\mathbb{Q}}_{\cap U_i} \longrightarrow 0$$

When $\{U_i\}_{i \in \Lambda}$ is a good cover, $H^{\cdot}(U_{i_1 \cdots i_k}) = H^0(U_{i_1 \cdots i_k})$,

↑ acyclic in AG

one can compute $H^{\cdot}(X)$ by applying $R\pi_{X,*}$:

$$0 \longrightarrow \bigoplus_i \Gamma(U_i) \xrightarrow{d^1} \bigoplus_{i<j} \Gamma(U_i \cap U_j) \xrightarrow{d^2} \cdots \Gamma(\cap U_i) \longrightarrow 0$$

$$\Downarrow \text{Ker}/\text{Im}$$

$$H^0(X) \qquad\qquad H^1(X) \qquad \cdots \qquad H^{\#\Lambda-1}(X)$$

Rmk. When X is paracompact & Hausdorff, "limited" Čech = sheaf

e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$\mathcal{F} \longrightarrow \bigoplus_i Rj_* \mathcal{F}|_{U_i} \qquad\qquad \mathcal{F} \longrightarrow \bigoplus_{x \in X} \mathcal{F}_x$$

$\#\Delta = 3$ case:

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i} \xrightarrow{d^1} \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i \cap U_j} \xrightarrow{d^2} Rj_* \underline{\mathbb{Q}}_{\cap_i U_i} \xrightarrow{d^3} 0$$

with $\mathcal{F}_0$, $\mathcal{F}_1$, $\mathcal{F}_2$ the stages of the resolution.

$$\mathcal{F}_2 = Rj_* \underline{\mathbb{Q}}_{\cap_i U_i} \quad \Rightarrow \quad H^\bullet(\mathcal{F}_2) = \ker d^3$$

$$\curvearrowright H^1(\mathcal{F}_1) \longrightarrow 0 \longrightarrow H^1(\mathcal{F}_2) \xrightarrow[\ =0\ ]{+1}$$

$$0 \longrightarrow H^0(\mathcal{F}_1) \longrightarrow \bigoplus_{i<j} \Gamma(U_i \cap U_j) \xrightarrow{d^2} H^0(\mathcal{F}_2)$$

$$\Rightarrow \quad H^i(\mathcal{F}_1) = \begin{cases} \ker d^3 / \operatorname{Im} d^2, & i = 1 \\ \ker d^2, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\curvearrowright H^2(\mathcal{F}_0) \longrightarrow 0 \longrightarrow H^2(\mathcal{F}_1) \xrightarrow[\ =0\ ]{+1}$$

$$\curvearrowright H^1(\mathcal{F}_0) \longrightarrow 0 \longrightarrow H^1(\mathcal{F}_1)$$

$$0 \longrightarrow H^0(\mathcal{F}_0) \longrightarrow \bigoplus_i \Gamma(U_i) \xrightarrow{d^1} H^0(\mathcal{F}_1)$$

$$\Rightarrow \quad H^i(X) = H^i(\mathcal{F}_0) = \begin{cases} \ker d^3 / \operatorname{Im} d^2 & i = 2 \\ \ker d^2 / \operatorname{Im} d^1 & i = 1 \\ \ker d^1 & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When $\{U_i\}_{i \in \Lambda}$ is not a good cover, one needs Čech-to-derived functor spectral seq to compute $H^\bullet(X)$.

Rmk. stratification & open cover are two main tools to extract topological information.
They appear with different names in different fields.
Once you realize them, you can apply the six-functor machine to analyze structures.

$$\text{stratification with extra properties} \begin{cases} \text{CW cplx} \\ \text{triangulization} \\ \text{dessin d'enfant} \\ \text{affine paving} \\ \text{Whitney stratification} \\ \vdots \end{cases}$$

Q. How to construct stratifications?
A: For me, there are two efficient methods. $\begin{cases} \text{orbit of gp action} \\ \text{Morse theory} \end{cases}$

That's why some geometrical problems are finally reduced to combinatorical / analytic problems.
Other fields come to geometry by providing stratifications.

In fact, there is only one method:
find a fct $f: X \longrightarrow Y$, and get stratification of X from Y.

E.g. 1. Morse theory $f: X \longrightarrow \mathbb{R}$
      2. tessellation $f: \mathcal{H} \longrightarrow \mathcal{H}/\Gamma$
      3. semi-continuous fct $f: X \longrightarrow \mathbb{N}_{\geq 0}$ e.g. $f(p) = \dim T_p X$
      4. my thesis $f: Gr(X) \longrightarrow Gr(S) \times Gr(X/S)$
      5. orbit of gp action $f: X \longrightarrow X/G$