Eine Woche, ein Beispiel

Ref: from [23.11.19]

4.28 naive $\otimes$ - Hom adjunction

Notation: - A: associate ring allowed to be non-commutative, contains 1

- There are two systems to write category of A-modules.

$$\begin{aligned} & \mathrm{Mod}_A &= A\text{-}\mathrm{Mod} & & \ni {}_A M \\ (\mathrm{Mod}_A)^{op} \neq\ & \mathrm{Mod}_{A^{op}} &= \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} & & \ni M_A \\ & \mathrm{Mod}_{A\otimes B^{op}} &= A\text{-}\mathrm{Mod}\text{-}B & & \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \qquad \ni {}_B M_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $\mathrm{Ob}(\mathrm{Ring}) \cong \mathrm{Ob}(\mathrm{Ring}^{op})$, $A^{op} \notin \mathrm{Ob}(\mathrm{Ring}^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset \mathrm{Sh}(\mathrm{Spec}\, A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$
4. definition recall for $\otimes$ & Hom, derived version
5. adjunction, derived version
6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-}\mathrm{Mod}\text{-}B \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$

$$\begin{array}{cccccc}
\mathrm{Hom}_B^A(-,-) : & (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} & \times & B\text{-}\mathrm{Mod}\text{-}A & \longrightarrow & \mathbb{Z}\text{-}\mathrm{Mod} \\
\parallel & \parallel & & \parallel & & \parallel \\
\mathrm{Hom}_{B\otimes_{\mathbb{Z}} A^{op}}(-,-) & (\mathbb{Z}\text{-}\mathrm{Mod}\text{-}B\otimes_{\mathbb{Z}} A^{op})^{\overline{op}} & \times & (B\otimes_{\mathbb{Z}} A^{op}\text{-}\mathrm{Mod}\text{-}\mathbb{Z})^{\overline{op}} & \longrightarrow & \mathbb{Z}\text{-}\mathrm{Mod}\text{-}\mathbb{Z}
\end{array}$$

$${}_AX_B, \quad {}_BY_C, \quad {}_CZ_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{op}} X$ in $A\text{-}\mathrm{Mod}\text{-}C = C^{op}\text{-}\mathrm{Mod}\text{-}A^{op}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\mathrm{Hom}_A(A, X) \cong X$

2. adjunction $_BX_A$, $_CY_B$, $_CZ_D$. we get

$$\mathrm{Hom}_C(Y \otimes_B X,\ Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y,Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$\{f: Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c\}$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y, -)"] \end{tikzcd}$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv \mathrm{Hom}$:

$$\begin{tikzcd}
(A\text{-Mod-}B)^{op} \times (B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "(\mathrm{Id},\, \mathrm{Hom}_C)"] \arrow[d, equal] & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}D \arrow[dd, "\mathrm{Hom}_B"] \\
(A\text{-Mod-}B \times B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[d, "(\otimes_B,\, \mathrm{Id})"'] & \\
(A\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "\mathrm{Hom}_C"] & A\text{-Mod-}D
\end{tikzcd}$$

$f^* \dashv f_*$:

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \qquad \begin{matrix} \mathcal{G} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \end{matrix}$$

$$\begin{tikzcd}
Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \arrow[r, "(\mathrm{Id},\, \text{pushforward})"] \arrow[d, "(\text{pullback},\, \mathrm{Id})"'] & Sh(X)^{op} \times Sh(X) \arrow[d, "\mathrm{Hom}_{Sh(X)}(-,-)"] \\
Sh(Y)^{op} \times Sh(Y) \arrow[r, "\mathrm{Hom}_{Sh(Y)}(-,-)"] & \mathrm{Abel}
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{F},\ f,\ \mathcal{G}) \arrow[rr, mapsto] \arrow[d, mapsto] & & (\mathcal{F},\ f_*\mathcal{G}) \arrow[d, mapsto] \\
(f^*\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] & \mathrm{Hom}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong & \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})
\end{tikzcd}$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

Forgetful fctor

Prop. For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \mathrm{Hom}_S({}_S S_R, M)$$

one has adjunction fctors

$$\begin{tikzcd}
S\text{-Mod} \arrow[rr, "\mathrm{Hom}_S({}_S S_R, -)", "u = {}_R S_S \otimes_S -"'] & & R\text{-Mod} \arrow[ll, bend right, "{}_S S_R \otimes_R -"'] \arrow[ll, bend left, "\mathrm{Hom}_R({}_R S_S, -)"]
\end{tikzcd} \tag{3.1}$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{tikzcd}
\mathcal{D}(X) \arrow[rr, "j^!", "j^*"'] & & \mathcal{D}(\mathcal{U}) \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "Rj_*"]
\end{tikzcd}$$

Vague slogan: $u \approx$ "forget the information of Z".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & X.

E.g. For finite gps $H \le G$, one has Res - Ind adjunction:

$$\mathrm{Res}^G_H \dashv \mathrm{Ind}^G_H$$
$$\text{c-}\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \le G \text{ open} \end{cases}$

If one only has $H \le G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{tikzcd} \mathrm{Rep}_G \arrow[rr, "\mathrm{Res}^G_H"] & & \mathrm{Rep}_H \arrow[ll, bend right, "\text{c-}\mathrm{Ind}^G_H"'] \arrow[ll, bend left, "\mathrm{Ind}^G_H"] \end{tikzcd}$$

Ex. Compare it with the recollement diagram & (3.1).

$$\begin{tikzcd} \mathcal{U} \arrow[d, "j"] & [*/H] \arrow[d, "\text{“cover with fiber } G/H\text{”}"] \\ X & [*/G] \end{tikzcd}$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$

$\text{c-}\mathrm{Ind}^G_H \longrightarrow \mathrm{Ind}^G_H$ $\qquad$ $\text{c-}\mathrm{Ind}^G_H = \mathrm{Ind}^G_H$

2. Even though

$$\mathrm{Sh}_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \mathrm{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$

the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\mathrm{Res}^G_{\{*\}} \mathbb{Q} = \mathbb{Q}, \qquad \mathrm{Res}^G_{\{*\}} \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* R j_* \neq \mathrm{Id}$, $j^! j_! \neq \mathrm{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Compare with i

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{tikzcd} S & & \operatorname{Spec} S \arrow[d, "f"] \\ R \arrow[u, "\tilde{f}"] & M & \operatorname{Spec} R & M \end{tikzcd}$$

$\exists$ "pullback fctor"

$$f^* : R\text{-Mod} \longrightarrow S\text{-Mod} \qquad f^*M = {}_SS_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{tikzcd} S\text{-Mod} \arrow[rr, "u"] & & R\text{-Mod} \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "\operatorname{Hom}_R({}_RS_S, -)"] \end{tikzcd}$$

compare it with another part of the recollement diagram:

$$\begin{tikzcd} \mathcal{D}(X) \arrow[rr, "i_*"', "i_!"] & & \mathcal{D}(U) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \end{tikzcd}$$

Rmk. u is usually not f.faithful, unless $S = R/I$.

(In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.)

which crspds to closed embedding.

In that case,

$$i^* i_* = \mathrm{Id} : {}_SS_R \otimes_R ({}_RS_S \otimes_S M) \cong M$$

$$i^! i_* = \mathrm{Id} : \operatorname{Hom}_R({}_RS_S, \operatorname{Hom}({}_SS_R, M)) \cong M$$

Slogan: in the comm alg. Spec $R/I \longrightarrow$ Spec R is closed embedding.
In general, if
S is an R-idempotent algebra: $S \cong S \otimes_R S$
then i: Spec $S \longrightarrow$ Spec R can be viewed as "closed subset".

E.g. $R_{\mathfrak{p}}$, R/I are idempotent R-algs.
$\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.

⚠ Usually R/I is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q: What is the crspd "open subset"?
A: (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. ${}^{L}\otimes \dashv RHom$

	F	RF or LF	R^iF or L^iF	exact fctor
	f^*	f^*	—	
	f_*	Rf_*	R^if_*	f_*-acyclic
	$\pi_{X,*}\mathcal{F}$	$\Gamma(X;\mathcal{F})$	$H^i(X;\mathcal{F})$	Γ-acyclic
	$f_!$	$Rf_!$	$R^if_!$	$f_!$-acyclic
	$\pi_{X,!}\mathcal{F}$	$\Gamma_c(X;\mathcal{F})$	$H^i_c(X;\mathcal{F})$	Γ_c-acyclic
	—	$f^!$	$\mathcal{H}^i(f^!-)$	
	$-\otimes_R-$	$-{}^{L}\otimes_R-$	$\mathrm{Tor}^i_R(-,-)$	flat
	$\mathrm{Hom}_R(-,-)$	$R\mathrm{Hom}_R(-,-)$	$\mathrm{Ext}^i_R(-,-)$	injective/projective
e.g. group coh	M_G	$\mathbb{Z}\,{}^{L}\otimes_{\mathbb{Z}[G]}M$	$H_i(G;M)$	
	M^G	$R\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M)$	$H^i(G;M)$	
e.g. Lie alg coh	$M_{\mathfrak{g}}$	$k\,{}^{L}\otimes_{U\mathfrak{g}}M$	$H_i(\mathfrak{g};M)$	
$\mathfrak{g}/k$: Lie alg	$M^{\mathfrak{g}}$	$R\mathrm{Hom}_{U\mathfrak{g}}(k,M)$	$H^i(\mathfrak{g};M)$	
e.g. Hochschild coh	$M/[AM]$	$A\,{}^{L}\otimes_{A^e}M$	$HH_i(A,M)$	
	M^A	$R\mathrm{Hom}_{A^e}(A,M)$	$HH^i(A,M)$	
	$A/[AA]$	$A\,{}^{L}\otimes_{A^e}A$	$HH_i(A)$	
	$Z(A)$	$R\mathrm{Hom}_{A^e}(A,A)$	$HH^i(A)$	

For calculations, see:
[23.04.09]: gp coh
[wiki]: Lie algebra coh
[21.05.21]: Hochschild coh
[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.
For Hom(-,A), see https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m.
Chenji Fu claimed that Hom(-,A) always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.
Related:
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod

https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution

4. definition recall for $\otimes$ & Hom , derived version

To define $^L\otimes$ & RHom, one needs to extend fctors

$$\begin{aligned}\otimes_A &: A\text{-Mod} \times A\text{-Mod} \longrightarrow A\text{-Mod}\\ \mathrm{Hom}_A(-,-) &: (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod}\end{aligned}$$

to fctors on double cplxes:

$\mathcal{C}(A) :=$ complex of A-modules, temporate notation

$$\begin{aligned}\otimes_{\mathcal{C}(A)} &: \mathcal{C}(A) \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A)\\ \mathrm{Hom}_{\mathcal{C}(A)}(-,-) &: (\mathcal{C}(A))^{op} \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A)\end{aligned}$$

But how?

Wishes:

$$\begin{aligned}(M^\cdot[i]) \otimes_{\mathcal{C}(A)} (N^\cdot[j]) &= (M^\cdot \otimes N^\cdot)[i+j]\\ \mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot[-i], N^\cdot[j]) &= \mathrm{Hom}(M^\cdot, N^\cdot)[i+j]\end{aligned}$$

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
 \arrow[r] & M^{-1}\otimes N^{1} \arrow[u] \arrow[r] & M^{0}\otimes N^{1} \arrow[u] \arrow[r] & M^{1}\otimes N^{1} \arrow[u] \arrow[r] & M^{2}\otimes N^{1} \arrow[u] \arrow[r] & \cdots \\
 \arrow[r] & M^{-1}\otimes N^{0} \arrow[u] \arrow[r] & M^{0}\otimes N^{0} \arrow[u] \arrow[r] & M^{1}\otimes N^{0} \arrow[u] \arrow[r] & M^{2}\otimes N^{0} \arrow[u] \arrow[r] & \cdots \\
 \arrow[r] & M^{-1}\otimes N^{-1} \arrow[u] \arrow[r] & M^{0}\otimes N^{-1} \arrow[u] \arrow[r] & M^{1}\otimes N^{-1} \arrow[u] \arrow[r] & M^{2}\otimes N^{-1} \arrow[u] \arrow[r] & \cdots \\
 & \arrow[u] & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}
$$

Tot ($M^{\cdot}\otimes N^{\cdot}$), the double complex of $M^{\cdot}\otimes N^{\cdot}$.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
 \arrow[r] & Hom(M^{1}, N^{1}) \arrow[u] \arrow[r] & Hom(M^{0}, N^{1}) \arrow[u] \arrow[r] & Hom(M^{-1}, N^{1}) \arrow[u] \arrow[r] & Hom(M^{-2}, N^{1}) \arrow[u] \arrow[r] & \cdots \\
 \arrow[r] & Hom(M^{1}, N^{0}) \arrow[u] \arrow[r] & Hom(M^{0}, N^{0}) \arrow[u] \arrow[r] & Hom(M^{-1}, N^{0}) \arrow[u] \arrow[r] & Hom(M^{-2}, N^{0}) \arrow[u] \arrow[r] & \cdots \\
 \arrow[r] & Hom(M^{1}, N^{-1}) \arrow[u] \arrow[r] & Hom(M^{0}, N^{-1}) \arrow[u] \arrow[r] & Hom(M^{-1}, N^{-1}) \arrow[u] \arrow[r] & Hom(M^{-2}, N^{-1}) \arrow[u] \arrow[r] & \cdots \\
 & \arrow[u] & \arrow[u] & \arrow[u] & \arrow[u] &
\end{tikzcd}
$$

Tot ($Hom(M^{\cdot}, N^{\cdot})$), the double complex of $Hom(M^{\cdot}, N^{\cdot})$.

Def. For $M^\cdot, N^\cdot \in \mathcal{C}(A)$, define

$$M^\cdot \otimes N^\cdot, \quad \mathrm{Hom}_A(M^\cdot, N^\cdot) \in \mathcal{C}(A)$$

by

$$(M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot)^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$(\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot))^n = \bigoplus_{i+j=n} \mathrm{Hom}_A(M^{-i}, N^j)$$

and morphisms given by $d + (-1)^j \delta$.

Ex. Let $M^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 3} \underset{0}{\mathbb{Z}}]$, $\quad N^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{0}{\mathbb{Z}}]$

compute $M^\cdot \otimes_{\mathcal{C}(\mathbb{Z})} N^\cdot$ & $\mathrm{Hom}_{\mathcal{C}(\mathbb{Z})}(M^\cdot, N^\cdot)$, and verify that they're complexes.

A:

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[r, "\times 3"] & \mathbb{Z} \\ -1 & \mathbb{Z} \arrow[u, "\times 2"] \arrow[r, "\times 3"] & \mathbb{Z} \arrow[u, "\times 2"'] \\ & -1 & 0 \end{tikzcd} \rightsquigarrow \left[\underset{-2}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{-1}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{0}{\mathbb{Z}} \right]$$

$\mathrm{Tot}(M^\cdot \otimes N^\cdot)$ $\qquad\qquad$ $M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot$

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[r, "\times 3"] & \mathbb{Z} \\ -1 & \mathbb{Z} \arrow[u, "\times 2"] \arrow[r, "\times 3"] & \mathbb{Z} \arrow[u, "\times 2"'] \\ & 0 & 1 \end{tikzcd} \rightsquigarrow \left[\underset{-1}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{0}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{1}{\mathbb{Z}} \right]$$

$\mathrm{Tot}(\mathrm{Hom}(M^\cdot, N^\cdot))$ $\qquad\qquad$ $\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot)$

Now, we can define $\overset{L}{\otimes}$ & RHom:

Def. For $M, N \in A\text{-Mod}$, one can define

$$M \overset{L}{\otimes}_A N := M \otimes_{e(A)} P^\cdot \quad \text{when } N \xleftarrow{\simeq} P^\cdot \quad \text{flat resolution}$$

in general, $M^\cdot, N^\cdot \in \mathcal{D}^-(A\text{-Mod})$

$$RHom_A(M, N) := Hom_{e(A)}(M, I^\cdot) \quad \text{when } N \xrightarrow{\simeq} I^\cdot \quad \text{inj resolution}$$
$$:= Hom_{e(A)}(P^\cdot, N) \quad \text{when } M \xleftarrow{\simeq} P^\cdot \quad \text{proj resolution}$$

in general, $M^\cdot \in \mathcal{D}^-(A\text{-Mod})$, $N^\cdot \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat

Ex Compute $\mathbb{F}_2 \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{F}_2$ & $RHom_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $Tor^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$ & $Ext^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$Hom_{\mathcal{D}(A)}(M^\cdot, N^\cdot) = R^0 Hom_{e(A)}(M^\cdot, N^\cdot)$$
$$Hom_A(M, N) = Hom_{\mathcal{D}(A)}(M^\cdot, N^\cdot) = R^0 Hom_{e(A)}(M, N).$$

⚠ [KS 90, Def 2.6.2]
To switch from $\mathcal{D}^-(X)$ to $\mathcal{D}^+(X)$, we need to require that

$$w.gldim(A) < +\infty. \qquad \text{w.gldim: shortest flat resolution}$$

A wrong proof for "flat $\longrightarrow$ proj"

"Proof": when P is flat,

$$\begin{array}{ccc} P\otimes_A - & \dashv & \mathrm{Hom}_A(P,-) \\ \parallel & & \\ P\overset{L}{\otimes}_A - & \dashv & R\mathrm{Hom}_A(P,-) \end{array}$$

by the uniqueness of the adjunction, $\mathrm{Hom}_A(P,-) = R\mathrm{Hom}_A(P,-)$, so P is flat.

This is wrong. $\mathbb{Q}\in\mathbb{Z}$-Mod is flat but not proj.
In the proof, we only have

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "P\otimes_A -"] \arrow[d, "\iota_A"'] & A\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_A(P,-)"] \arrow[d, "\iota_A"] \\ \mathcal{D}(A) \arrow[r, bend left, "P\overset{L}{\otimes}_A -"] & \mathcal{D}(A) \arrow[l, bend left, "R\mathrm{Hom}_A(P,-)"] \end{tikzcd}$$

$$\iota_A\circ(P\otimes_A -) = (P\overset{L}{\otimes}_A -)\circ\iota_A.$$

Ex. Compute $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-)$, and shows that

$$\iota_{\mathbb{Z}}\circ\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-) \neq R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-)\circ\iota_{\mathbb{Z}}.$$

5. adjunction , derived version

Prop. For A comm ring, $L, M, N \in A\text{-Mod}$, we get
$L^\cdot, M^\cdot \in \mathcal{D}^-(A)$, $N^\cdot \in \mathcal{D}^+(A)$ in general

$$R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

Proof.

$$\mathrm{Hom}_A(M \otimes_A N, L) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

$\Downarrow$ take $(-)^n$

$$\mathrm{Hom}_{\mathcal{C}(A)}(M \otimes_{\mathcal{C}(A)} N, L) \cong \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(M, L))$$

$\Downarrow$ $\mathrm{Hom}_A(M, -)$ preserves inj modules for M flat

wiki: injective module

$$R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

$$\begin{aligned}
R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) &= R\mathrm{Hom}_A(P^\cdot \otimes_{\mathcal{C}(A)} N, L) && M^\cdot \xleftarrow{\cong} P^\cdot \text{ flat} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(P^\cdot \otimes_{\mathcal{C}(A)} N, I^\cdot) && L^\cdot \xrightarrow{\cong} I^\cdot \text{ inj} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^\cdot, I^\cdot)) && \text{adj in } \mathcal{C}(A) \\
&= R\mathrm{Hom}_A(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^\cdot, I^\cdot)) && \mathrm{Hom}_{\mathcal{C}(A)}(P^\cdot, I^\cdot) \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(P^\cdot, I^\cdot)) && I^\cdot \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))
\end{aligned}$$

□

⚠ We don't have

$$R\mathrm{Hom}_A(M \otimes_A N, L) \cong R\mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

Find a counterexample?

Take $N = A$, reduce to:

$$R\mathrm{Hom}_A(M, L) \cong \mathrm{Hom}_A(M, L)$$

then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

Will write: verdier category
interpretation of 3 cohomology (gp + Lie alg + Hochschild)