Eine Woche, ein Beispiel
1.26 Numerical Chern class

Ref:
wiki: Chern class

[BWB21]: Wang, Liao. The Borel–Weil–Bott Theorem in Examples
[GK20]: Frank Gounelas and Alexis Kouvidakis. On Some Invariants of Cubic Fourfolds. European Journal of Mathematics

Nearly all the results are sourced from Wikipedia. I made this document because I tend to mix up the Chern class and the Chern character.
We omit E in notation.
All the results can be check via Macaulay2. You can find my code at
https://github.com/ramified/codelearning/raw/main/Macaulay/numerical_chern_class.txt

$$c(E) = 1 + c_1 + \cdots + c_r \in H^\cdot(X;\mathbb{C})$$
$$= \prod_{i=1}^{r}(1+a_i) \qquad a_i(E) \in H^\cdot(F\ell(E);\mathbb{C})$$

$$c_t(E) = 1 + c_1 t + \cdots + c_r t^r \in H^\cdot(X;\mathbb{C})[t]$$
$$= \prod_{i=1}^{r}(1+a_i t)$$

$$ch(E) = e^{a_1} + \cdots + e^{a_r} \in H^\cdot(X;\mathbb{C})$$
$$= \sum_{k=0}^{+\infty} \frac{1}{k!}(a_1^k + \cdots + a_r^k)$$
$$= \sum_{k=0}^{+\infty} \frac{1}{k!} s_k(c_1, \ldots, c_r)$$
$$= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_2c_1 + 3c_3)$$
$$+ \frac{1}{24}(c_1^4 - 4c_2c_1^2 + 4c_3c_1 + 2c_2^2 - 4c_4) + \cdots$$

$$td(E) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}} \in H^\cdot(X;\mathbb{C})$$
$$= \prod_{i=1}^{r}\left(1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} a_i^{2k}\right)$$
$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2$$
$$+ \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \cdots$$

$$s(E) = \prod_{i=1}^{r} \frac{1}{1+a_i} \in H^\cdot(X;\mathbb{C})$$
$$\hat{=}\ 1 + s_1 + \cdots + s_n$$
$$= 1 - c_1 + (-c_2 + c_1^2) + (-c_3 + 2c_1c_2 - c_1^3)$$
$$+ (-c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + c_1^4) + \cdots$$

$$\sqrt{td(E)} = 1 + \frac{1}{4}c_1 + \frac{1}{24}(c_1^2 + c_2) + \left(\frac{\frac{1}{2}}{2}\right)\left(\frac{1}{2}c_1\right)^2 + \cdots$$
$$= 1 + \frac{1}{4}c_1 + \frac{1}{96}(c_1^2 + 4c_2) + \cdots$$

$$0 \to E \to \widetilde{E} \to E' \to 0$$

$$
\begin{aligned}
c(E \oplus E') &= c(E) \cup c(E') \\
c_t(E \oplus E') &= c_t(E)\, c_t(E') \\
ch(E \oplus E') &= ch(E) + ch(E') \\
td(E \oplus E') &= td(E) \cup td(E') \\
s(E \oplus E') &= s(E) \cup s(E')
\end{aligned}
\qquad\qquad
\begin{aligned}
c(\widetilde{E}) &= c(E) \cup c(E') \\
c_t(\widetilde{E}) &= c_t(E)\, c_t(E') \\
ch(\widetilde{E}) &= ch(E) + ch(E') \\
td(\widetilde{E}) &= td(E) \cup td(E') \\
s(\widetilde{E}) &= s(E) \cup s(E')
\end{aligned}
$$

$$ch(E \otimes E') = ch(E)\, ch(E')$$

Getting $c(E)$ from $ch(E)$:

$$c_k = \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1}\, i!\; ch_i \cdot c_{k-i}$$

$$
\begin{aligned}
c(E) = \; & 1 + ch_1 + \left(\tfrac{1}{2} ch_1^2 - ch_2\right) \\
& + \left(\tfrac{1}{6} ch_1^3 - ch_1 ch_2 + 2ch_3\right) \\
& + \left(\tfrac{1}{24} ch_1^4 - \tfrac{1}{2} ch_1^2 ch_2 + 2ch_1 ch_3 + \tfrac{1}{2} ch_2^2 - 6ch_4\right) + \cdots
\end{aligned}
$$

E.g. $X = \mathbb{P}^1$ $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and
$H \in H^2(\mathbb{P}^1; \mathbb{C})$ as the generator

$$
\begin{aligned}
c(E) &= 1 + aH \\
c_t(E) &= 1 + aHt \\
ch(E) &= 1 + aH \\
td(E) &= 1 + \tfrac{1}{2}aH \\
s(E) &= 1 - aH
\end{aligned}
$$

E.p. $\Omega_{\mathbb{P}^1} = \underset{a=-2}{\omega_{\mathbb{P}^1}} = \mathcal{O}(-2)$ $\quad \underset{a=2}{T_{\mathbb{P}^1}} = \mathcal{O}(2)$

For $E = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, one gets

$$
\begin{aligned}
c(E) &= (1+a_1H) \cup (1+a_2H) &&= 1 + (a_1+a_2)H \\
c_t(E) &= (1+a_1Ht)(1+a_2Ht) &&= 1 + (a_1+a_2)Ht \\
ch(E) &= 1+a_1H + 1+a_2H &&= 2 + (a_1+a_2)H \\
td(E) &= (1+\tfrac{1}{2}a_1H) \cup (1+\tfrac{1}{2}a_2H) &&= 1 + \tfrac{1}{2}(a_1+a_2)H \\
s(E) &= (1-a_1H) \cup (1-a_2H) &&= 1 - (a_1+a_2)H
\end{aligned}
$$

Therefore, these characteristic classes can not distinguish $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

E.g. $X = C$ is of genus g, $E = T_C$, then

$$
\begin{aligned}
c(C) &= 1 + (2-2g)[p] \\
c_t(C) &= 1 + (2-2g)t \\
ch(C) &= 1 + (2-2g)[p] \\
td(C) &= 1 + (1-g)[p] \\
s(C) &= 1 - (2-2g)[p]
\end{aligned}
$$

E.g. $X = \mathbb{P}^2$, $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and
$H \in H^2(\mathbb{P}^2; \mathbb{C})$ as the generator

$$
\begin{aligned}
c(E) &= 1 + aH \\
c_t(E) &= 1 + aHt \\
ch(E) &= 1 + aH + \tfrac{1}{2}a^2H^2 \\
td(E) &= 1 + \tfrac{1}{2}aH + \tfrac{1}{12}a^2H^2 \\
s(E) &= 1 - aH + a^2H^2
\end{aligned}
$$

E.g. $X = \mathbb{P}^n$, $E = T_{\mathbb{P}^n}$, then the Euler sequence

$$
\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} & \longrightarrow & T_{\mathbb{P}^n} & \longrightarrow 0 \\
& \| & & \| & & \| & \\
& \mathrm{Hom}(S, S) & & \mathrm{Hom}(S, \mathcal{O}_{\mathbb{P}^n}) & & \mathrm{Hom}(S, Q) &
\end{array}
$$

tells us that

$$
\begin{aligned}
c(\mathbb{P}^n) &= (1+H)^{n+1} \\
&= 1 + (n+1)H + \binom{n+1}{2}H^2 + \cdots \\
c_t(\mathbb{P}^n) &= 1 + (n+1)Ht + \binom{n+1}{2}H^2t^2 + \cdots \\
ch(\mathbb{P}^n) &= n + (n+1)H + \frac{n+1}{2}H^2 + \frac{n+1}{3!}H^3 + \cdots \\
&= -1 + (n+1)e^H \\
td(\mathbb{P}^n) &= 1 + \tfrac{1}{2}(n+1)H + \tfrac{1}{12}\left((n+1)^2 + \binom{n+1}{2}\right)H^2 + \cdots \\
&= \begin{cases} 1 + \frac{3}{2}H + H^2, & n=2 \\ 1 + 2H + \frac{11}{6}H^2 + H^3, & n=3 \\ 1 + \frac{5}{2}H + \frac{35}{12}H^2 + \frac{25}{12}H^3 + H^4, & n=4 \end{cases} \\
s(\mathbb{P}^n) &= (1+H)^{-n-1} \\
&= 1 - (n+1)H + \binom{n+2}{2}H^2 - \binom{n+3}{3}H^3 + \cdots
\end{aligned}
$$

e.p.

$$
\begin{aligned}
c(\mathbb{P}^2) &= 1 + 3H + 3H^2\\
c_t(\mathbb{P}^2) &= 1 + 3Ht + 3H^2t^2\\
ch(\mathbb{P}^2) &= 2 + 3H + \tfrac{3}{2}H^2\\
td(\mathbb{P}^2) &= 1 + \tfrac{3}{2}H + H^2\\
s(\mathbb{P}^2) &= 1 - 3H + 6H^2
\end{aligned}
$$

$$
\begin{aligned}
c(\mathbb{P}^3) &= 1 + 4H + 6H^2 + 4H^3\\
c_t(\mathbb{P}^3) &= 1 + 4Ht + 6H^2t^2 + 4H^3t^3\\
ch(\mathbb{P}^3) &= 3 + 4H + 2H^2 + \tfrac{2}{3}H^3\\
td(\mathbb{P}^3) &= 1 + 2H + \tfrac{11}{6}H^2 + H^3\\
s(\mathbb{P}^3) &= 1 - 4H + 10H^2 - 20H^3
\end{aligned}
$$

https://math.stackexchange.com/questions/998797/the-second-and-third-chern-classes-of-calabi-yau-threefolds

E.g. X = alg K3 surface, $E \in T_X$, then

$$c_1(\Omega_X) = c_1(\omega_X) = c_1(\mathcal{O}_X) = 0 \Rightarrow c_1(T_X) = 0$$

$[p] \in H^4(X;\mathbb{C})$ generator

$$
\begin{aligned}
c(X) &= 1 + 24[p]\\
c_t(X) &= 1 + 24t^2\\
ch(X) &= 2 - 24[p]\\
td(X) &= 1 - 2[p]\\
s(X) &= 1 - 24[p]
\end{aligned}
$$

What's the Schur functor for Chern class? Give me formulas for $c(S^{\lambda}(E))$, for some vector bundle E over X.

Answer from chatgpt:

If E has Chern roots $x_1, \dots, x_r$ and $W(\lambda)$ is the set of weights of the irreducible $GL(r)$–representation of highest weight λ, then

$$c(S^\lambda(E)) = \prod_{w \in W(\lambda)} (1 + a_1x_1 + \cdots + a_rx_r) \qquad (\text{for } w = (a_1,\dots,a_r)),$$

or equivalently,

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1},\dots,e^{x_r}).$$

In particular,

$$c(S^kE) = \prod_{a_1+\cdots+a_r=k} (1 + a_1x_1+\cdots+a_rx_r)$$

and

$$c(\wedge^kE) = \prod_{1 \le i_1 < \cdots < i_k \le r} (1 + x_{i_1}+\cdots+x_{i_k}).$$

Explanation: One very standard answer is to use the splitting principle. Suppose that the vector bundle E has rank r and splits (formally) as a sum of line bundles, so that

$$E = L_1 \oplus L_2 \oplus \cdots \oplus L_r$$

with first Chern classes $x_1, x_2, \dots, x_r$ (so that $c(L_i) = 1 + x_i$). Then any Schur functor S^λ applied to E has, by representation theory, "Chern roots" given by the weights of the irreducible representation of $GL(r)$ with highest weight λ. In other words, if $W(\lambda)$ is the (multi)set of weights – that is, if every weight $w \in W(\lambda)$ is given by an r-tuple $(a_1, \dots, a_r)$, then the corresponding "root" is

$$\langle w, x\rangle = a_1x_1 + \cdots + a_rx_r.$$

Thus the total Chern class of $S^\lambda(E)$ is

$$c(S^\lambda(E)) = \prod_{w \in W(\lambda)} (1 + \langle w, x\rangle).$$

An equivalent formulation is to note that the Chern character of $S^\lambda(E)$ is given by

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1}, \dots, e^{x_r}),$$

where s_λ is the Schur polynomial corresponding to the partition λ. (From ch one may recover the Chern classes via the usual formal relations.)

As an example, note that when $\lambda = (k)$ the Schur functor is the k_th symmetric power, and its total Chern class may be written as

$$c(S^k(E)) = \prod_{(a_1,\dots,a_r) \in \mathbb{N}^r,\, a_1+\cdots+a_r=k} (1 + a_1x_1 + \cdots + a_rx_r),$$

while when $\lambda = (1^k)$ (that is, a column of k boxes) one has $S^{(1^k)}(E) = \wedge^k E$ and

$$c(\wedge^kE) = \prod_{1 \le i_1 < \cdots < i_k \le r} (1 + x_{i_1} + \cdots + x_{i_k}).$$

Riemann-Roch

Roughly speaking, Riemann-Roch computes chern class of pushforward.

$$\mathcal{G} \,/\, f: Y \longrightarrow X$$

$$\begin{aligned}
\text{GRR:}\quad & ch(f_! \mathcal{G})\, td(X) = f_* \left(ch(\mathcal{G})\, td(Y)\right) \\
\text{HRR:}\quad & \chi(Y, \mathcal{G}) = \int_Y ch(\mathcal{G})\, td(Y) \\
& \qquad\quad = \left(ch(\mathcal{G})\, td(Y)\right)_{\deg Y}
\end{aligned}$$

RR for surface: $\mathcal{L} = \mathcal{O}(D)$

$$\begin{aligned}
\chi(Y, \mathcal{L}) &= \left[\left(1 + c_1(\mathcal{L}) + \tfrac{1}{2} c_1(\mathcal{L})^2\right)\left(1 + \tfrac{1}{2} c_1(Y) + \tfrac{1}{12}\left(c_1(Y)^2 + c_2(Y)\right)\right)\right]_2 \\
&= \tfrac{1}{2} c_1(\mathcal{L})^2 + \tfrac{1}{2} c_1(\mathcal{L})\, c_1(Y) + \tfrac{1}{12}\left(c_1(Y)^2 + c_2(Y)\right) \\
&= \tfrac{1}{2} D(D-K) + \tfrac{1}{12}(K^2 + e)
\end{aligned}$$

$$\Rightarrow \begin{cases} \chi(\mathcal{O}) = \tfrac{1}{12}(K^2 + e) \\ \chi(D) = \chi(\mathcal{O}) + \tfrac{1}{2} D(D-K) \end{cases}$$

E.g. In case $D = -C$,

$$0 \longrightarrow \mathcal{O}(-C) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$\begin{aligned}
\Rightarrow \quad \chi(\mathcal{O}_C) &= \chi(\mathcal{O}) - \chi(D) & &= -\tfrac{1}{2} C(C+K) \\
1 - g(C) &= -\tfrac{1}{2} D(D-K) & & \\
\Rightarrow \quad 2g(C) - 2 &= C(C+K) & &= c_1(\mathcal{L})\left(c_1(\mathcal{L}) + c_1(\omega_X)\right)
\end{aligned}$$

where $\mathcal{L} = \mathcal{O}(C)$

$\mathcal{O}_C \otimes \mathcal{O}_X(C) \not\cong \mathcal{O}_C$, so we only have

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0$$

$$\begin{aligned}
\Rightarrow \quad \chi(\mathcal{O}_C(C)) &= \chi(\mathcal{O}(C)) - \chi(\mathcal{O}) \\
&= \tfrac{1}{2} C(C-K)
\end{aligned}$$

https://math.stackexchange.com/questions/3446039/adjunction-formula-and-riemann-roch-for-surfaces

RR for curve: $\mathcal{L} = \mathcal{O}(D)$

$$\begin{aligned}
\chi(Y, \mathcal{L}) &= \left[\left(1 + c_1(\mathcal{L})\right)\left(1 + \tfrac{1}{2} c_1(Y)\right)\right]_1 \\
&= c_1(\mathcal{L}) + \tfrac{1}{2} c_1(Y) \\
&= \deg D + 1 - g
\end{aligned}$$

RR for Flag or Grassmannian: Borel-Weil-Bott theorem.

BWB is stronger, because it tells $H^k(Gr(r,n); \mathcal{G})$ for specific k,
and it constructs an explicit isomorphism.

[BWB21, Thm 2.4] For a GL_n-regular and dominant (resp. P) weight $\chi \in X^*(T(GL_n))$,

$$H^{l(w)}(Gr(r,n), \mathcal{U}(\chi)) \cong V_{GL_n}(w.\chi) \qquad w.\chi := w(\chi+\rho)-\rho$$

↑ Verma module

[GK20, Sec 3]

$$H^{l(w)}(Gr(r,n), \Sigma_{\chi'} S^\vee \otimes \Sigma_{\chi''} Q^\vee) \cong \Sigma_{w.\chi} \mathbb{C}^n$$

Compare HRR with BWB:

$$\begin{aligned} ch(\mathcal{U}(\chi)\ td(Gr(r,n))) &= ch(\Sigma_{w'} S^\vee \otimes \Sigma_{w''} Q^\vee)\ td(S^\vee \otimes Q) \\ &\overset{?}{=} (-1)^{l(w)} \prod_{1\le i<j\le n} \frac{(w.\chi)_i - (w.\chi)_j + j - i}{j-i} \\ &= (-1)^{l(w)} \dim V_{GL_n}(w.\chi). \end{aligned}$$

E.g.

$$\chi(\mathbb{P}^1; \mathcal{G}) = ch_1(\mathcal{G}) + H\, ch_0(\mathcal{G})$$
$$\chi(\mathbb{P}^2; \mathcal{G}) = ch_2(\mathcal{G}) + \tfrac{3}{2} H\, ch_1(\mathcal{G}) + H^2 ch_0(\mathcal{G})$$
$$\chi(\mathbb{P}^3; \mathcal{G}) = ch_3(\mathcal{G}) + 2H\, ch_2(\mathcal{G}) + \tfrac{11}{6} H^2 ch_1(\mathcal{G}) + H^3 ch_0(\mathcal{G}).$$

E.g. Let $C \subset \mathbb{P}^2$ be an integral curve of deg d and genus g.

$$\iota: \begin{array}{c} \mathcal{O}_C \\ \rotatebox{90}{$\in$} \\ C \end{array} \longrightarrow \mathbb{P}^2$$

GRR tells us that

$$ch(\iota_*\mathcal{O}_C) \cdot td(\mathbb{P}^2) = \iota_*(\overset{\overset{1}{\shortparallel}}{ch(\mathcal{O}_C)}\, td(C))$$

$$\Rightarrow ch(\iota_*\mathcal{O}_C)(1 + \tfrac{3}{2}H + H^2) = \iota_*(1 + (1-g)[p])$$
$$= [C] + (1-g)[p]$$
$$= dH + (1-g)H^2$$

$$\Rightarrow \quad ch(\iota_*\mathcal{O}_C) = dH + (1-g-\tfrac{3}{2}d)H^2$$
$$= (0, d, 1-g-\tfrac{3}{2}d)$$

Therefore, $c(\iota_*\mathcal{O}_C) = 1 + dH + (\tfrac{1}{2}d^2 + \tfrac{3}{2}d + g - 1)H^2$

E.g. (continue) Using the SES

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_{\mathbb{P}^2} \longrightarrow \iota_*\mathcal{O}_C \longrightarrow 0$$

we get

$$c(\mathcal{I}_C) = 1 - dH + (\tfrac{1}{2}d^2 + 1 - g - \tfrac{3}{2}d)H^2$$
$$ch(\mathcal{I}_C) = 1 - dH - (1-g-\tfrac{3}{2}d)H^2$$
$$= (1, -d, g + \tfrac{3}{2}d - 1)$$

E.g. [MS19, Ex 9.8]
Let $C \subset \mathbb{P}^3$ be an integral curve of deg d and genus g.

$$\begin{array}{c} \mathcal{O}_C \\ | \\ \iota : C \longrightarrow \mathbb{P}^3 \end{array}$$

GRR tells us that

$$ch(\iota_* \mathcal{O}_C) \cdot td(\mathbb{P}^3) = \iota_* (\overset{1}{\overset{\shortparallel}{ch(\mathcal{O}_C)}} \, td(C))$$

$$\begin{aligned} \Rightarrow \; ch(\iota_* \mathcal{O}_C)(1 + 2H + \tfrac{11}{6}H^2 + H^3) &= \iota_* (1 + (1-g)[p]) \\ &= [C] + (1-g)[p] \\ &= dH^2 + (1-g)H^3 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad ch(\iota_* \mathcal{O}_C) &= dH^2 + (1-g-2d)H^3 \\ &= (0, 0, d, 1-g-2d) \end{aligned}$$

Therefore, $\quad c(\iota_* \mathcal{O}_C) = 1 - dH^2 + 2(1-g-2d)H^3.$

E.g. (continue) Using the SES

$$0 \longrightarrow I_C \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \iota_* \mathcal{O}_C \longrightarrow 0$$

we get

$$\begin{aligned} c(I_C) &= 1 + dH^2 - 2(1-g-2d)H^3 \\ ch(I_C) &= 1 - dH^2 - (1-g-2d)H^3 \\ &= (1, 0, -d, 2d+g-1). \end{aligned}$$

Mukai vector & Mukai pairing

Def Let $X/\mathbb{C}$ be a sm proj variety, and $E \in D^b(X)$.
Define

$$
\begin{aligned}
v(E) &:= ch(E)\sqrt{td(X)} \qquad \in H^{\cdot}(X;\mathbb{C}) \\
&= \Big(ch_0(E),\ ch_1(E) + \tfrac{1}{4}c_1(X)ch_0(E), \\
&\qquad ch_2(E) + \tfrac{1}{4}c_1(X)ch_1(E) + \tfrac{1}{96}\left(c_1^2(X) + 4c_2(X)\right)ch_0(E),\ \dots \Big)
\end{aligned}
$$

as the Mukai vector, and the Mukai pairing
also called Euler pairing

$$
\begin{aligned}
\langle -, - \rangle &: H^*(X;\mathbb{C}) \times H^*(X;\mathbb{C}) \longrightarrow \mathbb{C} \\
\langle v, v' \rangle &= \int_X e^{\frac{c_1(X)}{2}} \cdot (v^\vee \cdot v')
\end{aligned}
$$

where

$$
v^\vee = \sum_j (\sqrt{-1})^{2j} v_j \qquad v_j \in H^{2j}(X;\mathbb{C})
$$

Check [Huy06, p133] for this definition.
https://math.stackexchange.com/questions/4346782/the-mukai-pairing

So

$$
\begin{aligned}
\langle v(E), v(F) \rangle &= \int_X ch(E^\vee)ch(F)td(X) \\
&= \chi(X, E^\vee \otimes F) \qquad = \sum_j (-1)^j \dim Hom(E, F[j]) \\
&= \chi(E, F)
\end{aligned}
$$

Rmk. The factor $e^{\frac{c_1(X)}{2}}$ comes from [Huy 06, p133]:

$$
v(E^\vee) = v(E)^\vee \cdot e^{\frac{c_1(X)}{2}}
$$

E.g. When $X=C$ is a curve of genus g, $c(C)=1+(2-2g)[p]$, so

$$\begin{aligned} v(E) &= \left(ch_0(E),\ ch_1(E)+\tfrac{1}{2}(1-g)\,ch_0(E)\right) \\ &= \left(rk,\ deg+\tfrac{1}{2}(1-g)\,rk\right) \end{aligned}$$

$$\begin{aligned} \langle v(E), v(F)\rangle &= \Big(\big(1+(1-g)[p]\big)\big(v_0(E)-v_1(E)\big)\big(v_0(F)+v_1(F)\big)\Big)_{deg=1} \\ &= v_0(E)v_1(F)-v_1(E)v_0(F)+(1-g)\,v_0(E)v_0(F) \\ &= r(E)\,d(F)-d(E)\,r(F)+(1-g)\,r(E)\,r(F) \end{aligned}$$

E.g. When $X=\mathbb{P}^2$, $c(\mathbb{P}^2)=1+3H+3H^2$, so

$$\begin{aligned} v(E) &= \left(ch_0(E),\ ch_1(E)+\tfrac{3}{4}H\,ch_0(E),\ ch_2(E)+\tfrac{3}{4}H\,ch_1(E)+\tfrac{7}{32}H^2ch_0(E)\right) \\ &= ch(E)\cdot\begin{pmatrix} 1 & \frac{3}{4}H & \frac{7}{32}H^2 \\ & 1 & \frac{3}{4}H \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \langle v(E), v(F)\rangle &= \Big(\big(1+\tfrac{3}{2}H+\tfrac{9}{8}H^2\big)\cdot\big(v_0(E)-v_1(E)+v_2(E)\big)\big(v_0(F)-v_1(F)+v_2(F)\big)\Big)_{deg=2} \\ &= v_0(E)v_2(F)-v_1(E)v_1(F)+v_2(E)v_0(F) \\ &\quad+\tfrac{3}{2}H\big(v_0(E)v_1(F)-v_1(E)v_0(F)\big) \\ &\quad+\tfrac{9}{8}H^2v_0(E)v_0(F) \\ &= H^2ch_0(E)ch_0(F)-\tfrac{3}{2}H\,ch_1(E)ch_0(F)+\tfrac{3}{2}H\,ch_0(E)ch_1(F) \\ &\quad+ch_2(E)ch_0(F)-ch_1(E)ch_1(F)+ch_0(E)ch_2(F) \\ &= \big(H\,ch_0(E),\ ch_1(E),\ H^{-1}ch_2(E)\big)\begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ -\frac{3}{4} & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} H\,ch_0(F) \\ ch_1(F) \\ H^{-1}ch_2(F) \end{pmatrix} \\ &= \big(ch_0(E),\ ch_1(E),\ ch_2(E)\big)\begin{pmatrix} H^2 & \frac{3}{4}H & \frac{1}{2} \\ -\frac{3}{4}H & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}\begin{pmatrix} ch_0(F) \\ ch_1(F) \\ ch_2(F) \end{pmatrix} \end{aligned}$$

E.g. When X is a K3 surface, $c(X)=1+24[p]$, so

$$\begin{aligned} v(E) &= \big(ch_0(E),\ ch_1(E),\ ch_2(E)+ch_0(E)\big) \\ \langle v(E), v(F)\rangle &= \Big(\big(v_0(E)-v_1(E)+v_2(E)\big)\big(v_0(F)+v_1(F)+v_2(F)\big)\Big)_{deg=2} \\ &= v_0(E)v_2(F)-v_1(E)v_1(F)+v_2(E)v_0(F) \\ &= ch_0(E)ch_2(F)-ch_1(E)ch_1(F)+ch_2(E)ch_0(F)+2ch_0(E)ch_0(F) \\ &= \big(ch_0(E),\ ch_1(E),\ ch_2(E)\big)\begin{pmatrix} 2[p] & & \frac{1}{2} \\ & -1 & \\ \frac{1}{2} & & \end{pmatrix}\begin{pmatrix} ch_0(F) \\ ch_1(F) \\ ch_2(F) \end{pmatrix} \end{aligned}$$