

Eine Woche, ein Beispiel

2.6. six functors

Ref: <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>

A preparation of exams.

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{F}} & \mathcal{F}' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \quad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

Goal:

$$\begin{aligned} f^* &\dashv f_* \\ - \otimes \mathcal{F} &\dashv \underline{\mathrm{Hom}}(\mathcal{F}, -) \\ f_! &\dashv f^! \end{aligned}$$

$$\begin{aligned} f^*(- \otimes -) & \\ f^*(\mathcal{F} \otimes \mathcal{F}') &\cong f^* \mathcal{F} \otimes f^* \mathcal{F}' \\ f_* \underline{\mathrm{Hom}}(f^* \mathcal{F}, \mathcal{G}) &\cong \underline{\mathrm{Hom}}(\mathcal{F}, f_* \mathcal{G}) \end{aligned}$$

$$\begin{array}{ccc} & \otimes & \\ f^* & & f_! \\ \text{bc: } f^* g_! & \cong & g'_! f'^* \\ f_* g'_! & \cong & g'_! f_* \end{array}$$

proj formula

$$\begin{aligned} f_!(f^* \mathcal{F} \otimes \mathcal{G}) &\cong \mathcal{F} \otimes f_! \mathcal{G} \\ f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^* \mathcal{F}) &\cong \underline{\mathrm{Hom}}(f_! \mathcal{G}, \mathcal{F}) \\ f'_! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') &\cong \underline{\mathrm{Hom}}(f^* \mathcal{F}, f'^! \mathcal{F}') \end{aligned}$$

f^* : stalk

$f_* / f_!$: cohomology

$$\begin{aligned} I: f^* &= f^! \\ P: f_* &= f_! \end{aligned}$$

$$p: X \rightarrow \mathrm{pt}$$

$$H^i(X; \mathbb{Z}) := p_* p^* \mathbb{1}$$

$$H_c^i(X; \mathbb{Z}) := p_! p^* \mathbb{1}$$

$$H^i(X; \mathbb{Z}) := p_! p^! \mathbb{1}$$

$$H^{BM}_i(X; \mathbb{Z}) := p_* p^! \mathbb{1}$$

$$H^i(X; \mathcal{F}) := p_* \mathcal{F}$$

$$H_c^i(X; \mathcal{F}) := p_! \mathcal{F}$$

$$H^i(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$$

$$H^{BM}_i(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$$

$$= R\Gamma(X; \mathcal{F})$$

$$= H_c^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

$$= H^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)

$$H_c^i(X; \mathcal{F}) \otimes H_c^j(Y; \mathcal{G}) \cong H_c^{i+j}(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\text{reduced to: } p_{X!} \mathcal{F} \otimes p_{Y!} \mathcal{G} \cong p_!(p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_! \downarrow & \searrow p & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

App 2. (Poincaré duality)

X : a cpt oriented mfd of dim d , then

$$-^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

proper

$$p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ locally (Verdier duality)}$$

$$p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ globally}$$

$$H^i(X; \mathbb{Z})[d] \cong H^i(X; \mathbb{Z})^\vee$$

$$\text{reduced to: } p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$$

Upgrade: ∞ -categories & sym monoidal structure

Idea: $\mathcal{D}_\bullet: \mathcal{C}^{op} \longrightarrow \text{Cat}_\infty$

$$\begin{array}{ccc} X & \longmapsto & \mathcal{D}(X) \\ f \downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & \mathcal{D}(Y) \end{array}$$

extends to \hookrightarrow compatibility is encoded!

$$\mathcal{D}: \text{Corr}(C, E) \longrightarrow \text{Mon}(\text{Cat}_\infty)$$

$$[Y \xleftarrow{f} X = X] \longmapsto f^*$$

$$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$$

$$[X \times X \xleftarrow{\quad} X = X] \longmapsto \otimes$$

Moreover, It factor through

$$\begin{array}{ccccc} \text{Corr}(C, E) & \longrightarrow & \text{LZ}_\mathcal{D} & \longrightarrow & \text{Mon}(\text{Cat}_\infty) \\ \text{Obj: } X & \longmapsto & X & \longmapsto & \mathcal{D}(X) \end{array}$$

$$\text{Mor: } \left[\begin{array}{c} Y \\ X \xleftarrow{f} \quad \searrow g \\ Z \end{array} \right] \longmapsto \text{kernel} \longmapsto \text{FM-transformation} \quad \text{composition} = \text{convolution}$$

$(f, g)_* \mathbb{1}_Y \in \mathcal{D}(X \times Z)$

$$2\text{-Mor: } \mathcal{E} \rightarrow \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$$

$$\left[\begin{array}{c} \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \\ \cong \downarrow F \\ \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \end{array} \right] \mapsto \left[\begin{array}{c} \mathcal{E}_{12} * \mathcal{E}_{23} \\ \downarrow \\ \mathcal{E}_{13} \end{array} \right]$$

Goal: framework of ∞ -category & \otimes

$$\leadsto \text{Corr}(C, E) \text{ \& } \text{Corr}(C, E)^{\otimes}$$

∞ -category

$$\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \text{Set}) \stackrel{\Delta}{=} \text{sSet} \supseteq \text{Cat}_{\infty} \supseteq \text{An}$$

$$\begin{array}{ccc} \Delta^n & \xrightarrow{h} & X \\ \downarrow & \nearrow \exists K_{n,i}(h) & \\ \Delta^n & & \end{array} \quad \begin{array}{ll} \forall 0 \leq i < n & \text{in } \text{Cat}_{\infty} \\ \forall 0 \leq i \leq n & \text{in } \text{An} \end{array}$$

Notation

Set	(0,0) - category	set
Cat	(1,1) - category	category
An	(∞ ,0) - category	anima / Kan cplx / ∞ -groupoid
Cat $_{\infty}$	(∞ ,1) - category	

Ex. Realize $\text{Corr}(C, E)$ as an ∞ -category.

Monoidal structure

In (1,1)-category:

Monoidal structure on \mathcal{C} :

$$\begin{array}{ll} m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} & u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C} \\ (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G} & * \longmapsto 1_{\mathcal{C}} \end{array}$$

Monoidal object in (\mathcal{C}, \otimes) : $X \in \text{Ob}(\mathcal{C})$ with

$$m_X: X \times X \longrightarrow X \quad u_X: 1_{\mathcal{C}} \longrightarrow X$$

In (∞ ,1)-category:

$$(C, \otimes) \stackrel{\text{def}}{\hookrightarrow} \left[\begin{array}{ccc} X: \text{Fin}^{\text{part}} & \longrightarrow & \text{Cat}_{\infty} \\ I & \longmapsto & X(I) \\ \text{comm monoid} & & \end{array} \right] \stackrel{\text{"Straightening"}}{\longleftrightarrow} \left[\begin{array}{ccc} \pi^{\otimes}: Y^{\otimes} & \longrightarrow & \text{Fin}^{\text{part}} \\ \text{coCartesian fibration} & & \\ Y_I^{\otimes} \xrightarrow{\sim} \prod_i Y_i^{\otimes} & & \end{array} \right]$$

\rightsquigarrow See next page for details

where $\text{Ob}(\text{Fin}^{\text{part}}) = \text{Ob}(\text{Fin})$

$$\text{Mor}_{\text{Fin}^{\text{part}}}(I, J) = \{\alpha: I \dashrightarrow J\}$$

$$\text{commutative monoid: } X(I) \xrightarrow{\sim} \prod_i X(i)$$

$$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow (\mathcal{F}, \mathcal{G}) \quad |I|=2$$

coCartesian fibration: see [Def 3.5]