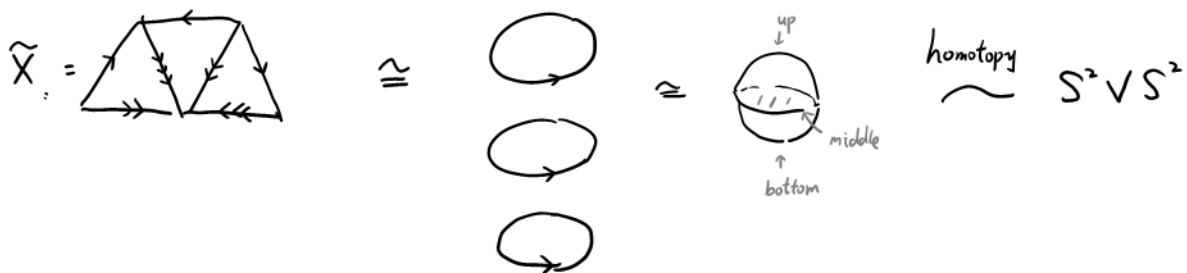


Eine Woche, ein Beispiel
1.16 Whitehead bracket

The story begins with the following naive question: what's the homotopy group of



Answer: the universal covering of X is



so the question reduces to the computation of $\pi_n(S^2 \vee S^2)$.

What is relative easy to do:

$$\pi_n(S^2 \vee S^2, *) \cong \begin{cases} 0 & n=0,1 \\ \mathbb{Z} & n=2 \\ \mathbb{Z}^3 & n=3 \end{cases} \quad \begin{array}{l} \text{by Hurewicz} \\ \text{by [Hatcher Example 4.52]} \end{array}$$

Idea for $\pi_3(S^2 \vee S^2, *) \cong \mathbb{Z}^3$:
By the LES induced by the CW-pair, we get a split SES:
$$0 \rightarrow \pi_{n+1}(S^2 \times S^2, S^2 \vee S^2, *) \rightarrow \pi_n(S^2 \vee S^2, *) \rightarrow \pi_n(S^2 \times S^2, *) \rightarrow 0 \quad \forall n \geq 1$$

By repeatedly applying [Hatcher, Prop 4.28], we get
• $(S^2 \times S^2, S^2 \vee S^2)$ is 3-connected
• $\pi_4(S^2 \times S^2, S^2 \vee S^2, *) \cong \pi_4(S^4, *)$
 $\therefore \pi_3(S^2 \vee S^2, *) \cong \pi_3(S^2, *) \times \pi_3(S^2, *) \times \pi_4(S^4, *) \cong \mathbb{Z}^3$

This problem has been fully solved in some sense, see the first two pages for the description and also the rest for the proof (I'm too lazy to see the proof): <http://nlab-pages.s3.us-east-2.amazonaws.com/nlab/files/Hilton55.pdf>

Finally we get

$$\pi_n(S^2 \vee S^2, *) \cong \pi_n(S^2)^{\oplus 2} \oplus \pi_n(S^3)^{\oplus 1} \oplus \pi_n(S^4)^{\oplus 2} \oplus \pi_n(S^5)^{\oplus 3} \oplus \pi_n(S^6)^{\oplus 6} \oplus \dots$$

Some computations for the future to check: abbreviate $[l_1, [l_1, l_2]]$ as $[1[12]]$

2 weight 1 1 2

1 weight 2 $[12]$

2 weight 3 $[1[12]]$ $[2[12]]$

3 weight 4 $[1[1[12]]]$ $[2[1[12]]]$ $[2[2[12]]]$

6 weight 5 $\begin{cases} 1+4 & [1[1[1[12]]]]_4 & [2[1[1[12]]]]_4 & [2[2[1[12]]]]_4 & [2[2[2[12]]]]_4 \\ 2+3 & [[12][1[12]]]_3 & [[12][2[12]]]_3 \end{cases}$

9 weight 6 $\begin{cases} 1+5 & [1[1[1[1[12]]]]_5 & [2[1[1[1[12]]]]_5 & [2[2[1[1[12]]]]_5 & [2[2[2[1[12]]]]_5 & [2[2[2[2[12]]]]_5 \\ 2+4 & [[12][1[1[12]]]]_4 & [[12][2[1[12]]]]_4 & [[12][2[2[12]]]]_4 \\ 3+3 & [[1[12]][2[12]]]_3 \end{cases}$



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It is more interesting that we have the extra structure for the homotopy group, which gives us a simple way to construct nontrivial element in the higher homotopy groups (maybe very difficult to prove, though): the Whitehead bracket.

See wiki for its definition: https://en.wikipedia.org/wiki/Whitehead_product

results about Whitehead bracket of spheres: <https://mathoverflow.net/questions/315255/whitehead-products-in-homotopy-groups-of-spheres>

Some exercises for myself:

prove that Whitehead bracket is a graded quasi-Lie algebra;

verify if the action of the fundamental group on homotopy groups compatible with the Whitehead bracket;

Q: Suppose $[f] = [g] \in \pi_2(S^2, *)$. Do we have homotopy equivalent between $S^2 \cup_{\varphi_f} D^2$ and $S^2 \cup_{\varphi_g} D^2$?

$$\begin{array}{ccc} \partial D^2 & \hookrightarrow & D^2 \\ f \downarrow & & \downarrow \\ S^2 & \longrightarrow & S^2 \cup_{\varphi_f} D^2 \end{array} \quad \swarrow \text{pushout}$$