Eine Woche, ein Beispiel
4.28   naive   ⊗ - Hom adjunction

right:

Notation: - A: associate ring   *allowed to be non-commutative, contains 1*
   - There are two systems to write category of A-modules.

$$\text{Mod}_A \quad = \quad A - \text{Mod} \qquad\qquad \ni \, {}_A M$$
$$(\text{Mod}_A)^{op} \neq \text{Mod}_{A^{op}} \quad = \qquad \text{Mod} - A \; = A^{op} - \text{Mod} \quad \ni \, M_A$$
$$\text{Mod}_{A \otimes B^{op}} = \quad A - \text{Mod} - B \qquad\qquad \ni \, {}_A M_B$$

highlight: In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For   convenience, we write
$$\left(\text{Mod}_{B \otimes A^{op}}\right)^{op} = (B - \text{Mod} - A)^{op} = \left(A^{op} - \text{Mod} - B^{op}\right)^{op} \qquad\qquad \ni \, {}_B M_A$$

as
$$\left(\text{Mod}_{A \otimes B^{op}}\right)^{\overline{op}} = (A - \text{Mod} - B)^{\overline{op}}$$

▽ Even though you can identify $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{op})$,
   $A^{op} \notin \text{Ob}(\text{Ring}^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

   - For A comm, $\text{Mod}_A = \text{Mod}_{A^{op}} \subset \text{Sh}(\text{Spec } A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

1 definition recall for ⊗ & Hom
2. adjunction
3. comparison between ⊗⊣Hom & $f^* \dashv f_*$
   ⋮
6. comparison between ⊗⊣Hom & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : \quad A\text{-}\mathrm{Mod}\text{-}B \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$
$$\mathrm{Hom}_B(-,-) : \quad (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$

$$\mathrm{Hom}_B^A(-,-) : \quad (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}A \longrightarrow \mathbb{Z}\text{-}\mathrm{Mod}$$
$$\| \qquad\qquad \| \qquad\qquad\qquad \| \qquad\qquad\qquad\qquad \|$$
$$\mathrm{Hom}_{B\otimes_{\mathbb{Z}} A^{op}}(-,-) \quad (\mathbb{Z}\text{-}\mathrm{Mod}\text{-}B\otimes_{\mathbb{Z}} A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}} A^{op}\text{-}\mathrm{Mod}\text{-}\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}\text{-}\mathrm{Mod}\text{-}\mathbb{Z}$$

$${}_A X_B \;,\; {}_B Y_C \;,\; {}_C Z_D$$

associativity: $\qquad (X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $\qquad X \otimes_B Y \cong Y \otimes_{B^{op}} X \qquad\qquad$ in $A\text{-}\mathrm{Mod}\text{-}C = C^{op}\text{-}\mathrm{Mod}\text{-}A^{op}$

"unit": $\qquad\qquad A \otimes_A X \cong X \cong X \otimes_B B$

$\qquad\qquad\qquad \mathrm{Hom}_A(A, X) \cong X$

2. adjunction $_B X_A$, $_C Y_B$, $_C Z_D$. we get

$$\operatorname{Hom}_C (Y \otimes_B X, Z) \cong \operatorname{Hom}_B (X, \operatorname{Hom}_C (Y, Z)) \qquad \text{in } A\text{-}Mod\text{-}D.$$

Reason: both sides equal to the set
$$\{ f : Y \times X \longrightarrow Z \mid f(cyb, x) = c\, f(y, bx) \quad \forall b, c \}$$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-}Mod\text{-}B$, one gets adjunction fctors:

$$B\text{-}Mod \underset{\operatorname{Hom}_C (Y, -)}{\overset{Y \otimes_B -}{\underset{\perp}{\rightleftarrows}}} C\text{-}Mod$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv$ Hom:

$$(A\text{-}Mod\text{-}B)^{\overline{op}} \times (B\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D \xrightarrow{(Id, Hom_C)} (A\text{-}Mod\text{-}B)^{\overline{op}} \times B\text{-}Mod\text{-}D$$

$$\parallel$$

$$(A\text{-}Mod\text{-}B \times B\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D$$

$(\otimes_B, Id) \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow Hom_B$

$$(A\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D \xrightarrow{Hom_C} A\text{-}Mod\text{-}D$$

$$\begin{array}{ccc} & \mathcal{G} & \mathcal{F} \\ & | & | \\ & Y & X \end{array}$$

$$Y \xrightarrow{f} X$$

$f^* \dashv f_*$:

$$Hom(f^*\mathcal{F}, \mathcal{G}) \cong Hom(\mathcal{F}, f_*\mathcal{G})$$

$$Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \xrightarrow{(Id, \text{pushforward})} Sh(X)^{op} \times Sh(X)$$

$(\text{pullback}, Id) \downarrow \qquad\qquad\qquad\qquad\qquad \downarrow Hom_{Sh(X)}(-,-)$

$$Sh(Y)^{op} \times Sh(Y) \xrightarrow{Hom_{Sh(Y)}(-,-)} Abel$$

$$(\mathcal{F}, \quad f, \quad \mathcal{G}) \longmapsto (\mathcal{F}, f_*\mathcal{G})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$(f^*\mathcal{F}, \mathcal{G}) \longmapsto Hom_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \text{Hom}$ & $f^* \dashv f_*$

Prop. For ring homo $\underset{R}{\overset{S}{\uparrow}} f$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \text{Hom}_S({}_S S_R, M)$$

one has adjunction fctors

$$\text{S-Mod} \underset{\substack{ \\ \text{Hom}_R({}_R S_S, -)}}{\overset{\substack{{}_S S_R \otimes_R - \\ \perp \\ \text{Hom}_S({}_S S_R, -) \\ u = {}_R S_S \otimes_S - \\ \perp}}{\rightleftarrows}} \text{R-Mod} \qquad (3.1)$$

Now, we compare (3.1) with part of the recollement diagram:

$$\mathcal{D}(X) \underset{\substack{ \\ Rj_*}}{\overset{\substack{j_! \\ \perp \\ j^! \\ j^* \\ \perp}}{\rightleftarrows}} \mathcal{D}(U)$$

Vague slogan: $u \approx$ "forget the information of Z".

In applications. $U \longrightarrow X$ is a covering map. This change the feeling of the size between U & X.

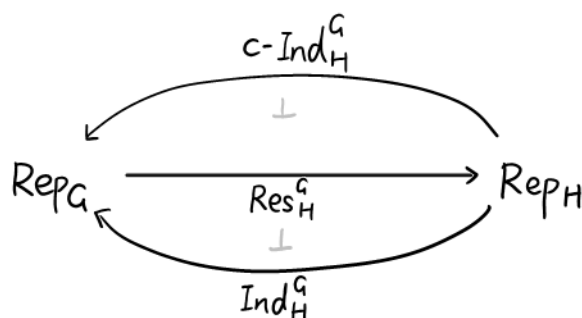**E.g.** For finite gps $H \leq G$, one has Res-Ind adjunction:
$$\text{Res}_H^G \dashv \text{Ind}_H^G$$
$$c\text{-}\text{Ind}_H^G \dashv \text{Res}_H^G$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \leq G \text{ open} \end{cases}$

If one only has $H \leq G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\text{c-Ind}_H^G$$
$$\perp$$
$$\text{Rep}_G \xleftarrow{\quad} \text{Res}_H^G \xrightarrow{\quad} \text{Rep}_H$$
$$\perp$$
$$\text{Ind}_H^G$$

Ex. Compare it with the recollement diagram & (3.1).

$$\mathcal{U} \qquad\qquad [*/H]$$
$$\downarrow j \qquad\qquad \downarrow \text{"cover with fiber } G/H\text{"}$$
$$X \qquad\qquad [*/G]$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\#G/H < +\infty$, $j_! = j_*$.
   $$c\text{-}\text{Ind}_H^G \longrightarrow \text{Ind}_H^G \qquad\qquad c\text{-}\text{Ind}_H^G = \text{Ind}_H^G$$

2. Even though
   $$\text{Sh}_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \text{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$
   the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

   $$\text{Res}_{\{*\}}^G \mathbb{Q} = \mathbb{Q}, \qquad \text{Res}_{\{*\}}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

🔻 In this example, $j^* R j_* \neq \text{Id}$, $j^! j_! \neq \text{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{array}{ccc} S & & \mathrm{Spec}\, S \\ \uparrow \tilde{f} & M & \downarrow f \qquad M \\ R & & \mathrm{Spec}\, R \end{array}$$

$\exists$ "pullback fctor"
$$f^* : R\text{-Mod} \longrightarrow S\text{-Mod} \qquad f^*M = {}_sS_R \otimes_R M$$
This is also called the base change.

Now, (3.1) can be rewritten as

$$f^*$$
$$\perp$$
$$u$$
$$S\text{-Mod} \longrightarrow R\text{-Mod}$$
$$\perp$$
$$\mathrm{Hom}_R\,({}_RS_S, -)$$

Compare it with another part of the recollement diagram:

$$i^*$$
$$\perp$$
$$i_*$$
$$\mathcal{D}(X) \xrightarrow{\quad i_! \quad} \mathcal{D}(U)$$
$$\perp$$
$$i^!$$

Rmk. u is usually not f. faithful, unless $S = R/I$.
   (In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.)
   which crspds to closed embedding.
   In that case,
$$i^* i_* = \mathrm{Id}: \quad {}_sS_R \otimes_R ({}_RS_S \otimes_S M) \cong M$$
$$i^! i_* = \mathrm{Id}: \quad \mathrm{Hom}_R\,({}_RS_S, \mathrm{Hom}({}_sS_R, M)) \cong M$$

Slogan: in the comm alg, $\text{Spec } R/I \longrightarrow \text{Spec } R$ is closed embedding.
In general, if
S is an R-idempotent algebra: $S \cong S \otimes_R S$
then $i: \text{Spec } S \longrightarrow \text{Spec } R$ can be viewed as "closed subset".

E.g. $R_{\mathfrak{p}}$, R/I are idempotent R-algs.
$\mathbb{Z}[\frac{1}{b}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.
$\triangledown$ Usually R/I is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?
A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. $^L\otimes \dashv RHom$

F	RF or LF	R^iF or L_iF	exact fctor
f^*	f^*	—	f_*-acyclic
f_*	Rf_*	$R^i f_*$	Γ-acyclic
$\pi_{x,*}\mathcal{F}$	$\Gamma(X;\mathcal{F})$	$H^i(X;\mathcal{F})$	$f_!$-acyclic
$f_!$	$Rf_!$	$R^i f_!$	Γ_c-acyclic
$\pi_{x,!}\mathcal{F}$	$\Gamma_c(X;\mathcal{F})$	$H^i_c(X;\mathcal{F})$	
—	$f^!$	$H^i(f^!-)$	
$-\otimes_R-$	$-\,{}^L\!\otimes_R-$	$Tor^R_i(-,-)$	flat
$Hom_R(-,-)$	$RHom_R(-,-)$	$Ext^i_R(-,-)$	injective/projective
M_G	$\mathbb{Z}\,{}^L\!\otimes_{\mathbb{Z}[G]}M$	$H_i(G;M)$	
M^G	$RHom_{\mathbb{Z}[G]}(\mathbb{Z},M)$	$H^i(G;M)$	
$M_{\mathfrak{g}}$	$x\,{}^L\!\otimes_{u\mathfrak{g}}M$	$H_i(\mathfrak{g};M)$	
$M^{\mathfrak{g}}$	$RHom_{u\mathfrak{g}}(x,M)$	$H^i(\mathfrak{g};M)$	
$M/[AM]$	$A\,{}^L\!\otimes_{A^e}M$	$HH_i(A,M)$	
M^A	$RHom_{A^e}(A,M)$	$HH^i(A,M)$	
$A/[AA]$	$A\,{}^L\!\otimes_{A^e}A$	$HH_i(A)$	
$Z(A)$	$RHom_{A^e}(A,A)$	$HH^i(A)$	

e.g. group coh

e.g. Lie alg coh

$\mathfrak{g}/\kappa$: Lie alg

e.g. Hochschild coh