Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

Top. space	space	continuous map	Covering of open sets	Sh	cohomology
site = Category + Groth cover	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, $\bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	$U_i \to U_j$ over X	—		
$(Sch/X)_{Zar}$	$Ob(Sch/X)$	$Mor(Sch/X)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		
$(Sch/X)_{ét}$	$Ob(Sch/X)$	$Mor(Sch/X)$	ét + l.f.p		
$(Sch/X)_{sm}$	$Ob(Sch/X)$	$Mor(Sch/X)$	smooth + l.f.p		
$(Sch/X)_{fppf}$	$Ob(Sch/X)$	$Mor(Sch/X)$	f.flat + l.f.p		
$(Sch/X)_{fpqc}$	$Ob(Sch/X)$	$Mor(Sch/X)$	f.flat + f_i^{-1}(q.c) locally q.c		

https://pbelmans.ncag.info/notes/etale-cohomology.pdf
http://homepage.sns.it/vistoli/descent.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

it tells us why we don't have small site for most condition:
https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
Here you can find some informations about comparison between fppf and fpqc topologys:
https://mathoverflow.net/questions/361664/some-basic-questions-on-quotient-of-group-schemes

Thm. ① equiv. of categories

$G_k := Gal(k^{sep}/k)$

$$\mathrm{Sets}((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \mathrm{Disc}\ G_k\text{-Set}$$
$$\mathrm{Ab}((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \mathrm{Disc}\ \mathrm{Mod}_{G_k} \quad (\star)$$

Site $(\mathrm{Spec}\ k)_{ét} \longleftrightarrow$ finite G_k-Set

② $(\star)$ preserve cohomology
$$H^1((\mathrm{Spec}(k))_{ét}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{C})_{ét}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\mathrm{Spec}\ \mathbb{C})$)

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{R})_{ét}$

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{C} \arrow[dr, "\iota^*"'] & \mathrm{Spec}\ \mathbb{C} \arrow[l, "\sigma^*"'] \arrow[d, "\iota^*"] \\ & \mathrm{Spec}\ \mathbb{R} \end{tikzcd} \quad \overset{\mathrm{Spec}}{\leftsquigarrow} \quad \begin{tikzcd} \mathbb{C} \arrow[r, "\sigma"] & \mathbb{C} \\ \mathbb{R} \arrow[u, "\iota"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

σ = conjugation, ι = embedding

$\Big\downarrow \mathcal{F}$

$$\begin{tikzcd} \mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \arrow[r, "\mathcal{F}(\sigma^*)"] & \mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \\ \mathcal{F}(\mathrm{Spec}\ \mathbb{R}) \arrow[u, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"'] & \end{tikzcd} \xrightarrow{\text{Abuse of notation}} \begin{tikzcd} \mathcal{F}(\mathbb{C}) \arrow[r, "\sigma"] & \mathcal{F}(\mathbb{C}) \\ \mathcal{F}(\mathbb{R}) \arrow[u, "\iota"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\quad$ $Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

[$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed:

in this case $0 \to \mathcal{F}(\operatorname{Spec}\mathbb{R}) \to \mathcal{F}(\operatorname{Spec}\mathbb{C}) \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C})$

$$\begin{array}{ccc}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}\left(\operatorname{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}\right) \\
\| & & \| \wr \\
& & \mathcal{F}\left(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\operatorname{Spec}\mathbb{C}\right) \\
& & \| \wr \\
\mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})
\end{array}$$

$\iota_1: x \longmapsto x\otimes 1$ $\quad$ $x\otimes y \longmapsto (xy, x\bar{y})$
$\iota_2: x \longmapsto 1\otimes x$

ι_2: $\operatorname{Spec}\mathbb{C} \xleftarrow{(Id,\sigma)} \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\operatorname{Spec}\mathbb{C}) \xrightarrow{(\mathcal{F}(Id),\mathcal{F}(\sigma))} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})$

Abuse $\rightsquigarrow$ of notation $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$]

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\operatorname{Spec} k, \mathcal{F}) = \mathcal{F}(\operatorname{Spec} k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)}$$

$\mathcal{F}_{k^{sep}} := \varinjlim_{L/k \text{ finite}} \mathcal{F}(\operatorname{Spec} L)$

Ex. descibe the stalk & fiber at $p \in \operatorname{Spec} k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\operatorname{Spec} k, p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts