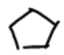
# Eine Woche, ein Beispiel

## 6.4. basics of fields

This document is aimed for people who have enough mathematical maturity, but miss the chance and time to study Galois theory. For a (relative) complete study of Galois theory which takes time, please see [GTM167].

1. classical motivation
2. common confusion
3. field extension
4. examples of algebraic closed field

## 1. classical motivation

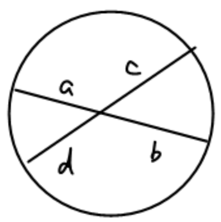
| | ruler-and-compass construction 尺规作图 | | solving higher degree equations 求根公式 | |
|---|---|---|---|---|
| possible |  17-gon | $\cos\frac{2\pi}{5}$ $\zeta_5$ <br> $\cos\frac{2\pi}{17}$ $\zeta_{17}$ | $\deg F \leq 4$ | $x$: $F(x)=0$ |
| impossible | Squaring the circle <br> Doubling the cube <br> Angle trisection | $\pi$ 化圆为方 <br> $\sqrt[3]{2}$ 倍立方 <br> $x: 4x^3-3x-a=0$ 三等分角 | $\deg F \geq 5$ | $x$: $F(x)=0$ |

Ex. Denote

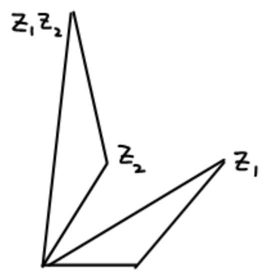
$$F_{\mathcal{R}} := \{z\in\mathbb{C} \mid z \text{ can be drawn by ruler-and-compass, given } 0,1\}$$
$$= \{\text{algebraic constructible complex numbers}\}$$
$$F_{\text{开根}} := \{z\in\mathbb{C} \mid z \text{ can be expressed by } +,-,\times,\div, \text{ radicals}\}$$

Verify that $F_{\mathcal{R}}, F_{\text{开根}}$ are fields.

Hint. Verify that $\mathbb{Q}\subseteq F_{\mathcal{R}}$ to get some intuition.




$ab = cd$



Ex. Given $1, a \in \mathbb{R}^+$, try to draw $\sqrt{a}$ by ruler-and-compass.
Argue that why we can draw ⬠ and 17-gons.

Hint. $\cos\frac{2\pi}{5} = \frac{\sqrt{5}}{4} - 1$

$$\cos\frac{2\pi}{17} = \frac{1}{16}\left(-1+\sqrt{17}+\sqrt{2(17-\sqrt{17})}+2\sqrt{17+3\sqrt{17}-\sqrt{2(17-\sqrt{17})}-2\sqrt{2(17+\sqrt{17})}}\right)$$

Slogan: consider

| | | |
|---|---|---|
| element | $\rightsquigarrow$ | set |
| object | $\rightsquigarrow$ | moduli spaces |
| if $x$ can be realized | $\rightsquigarrow$ | $\{x \mid x \text{ can be realized}\}$ |

2. common confusion

| | Abstract field | Subfield of $\overline{K}$ or $\mathbb{C}$ |
|---|---|---|
| name of category | Field | $\text{Subfield}_{\overline{K}}$ |
| Ob | $\{F : \text{field}\}$ | $\{(F, \iota) \mid \iota : F \hookrightarrow \overline{K}\}$ |
| Mor | $\text{Mor}_{\text{Field}}(F,E) = \{\alpha : F \hookrightarrow E\}$ <br> usually: finitely many elements | $\text{Mor}_{\text{Subfield}}(F,E) = \left\{ \begin{tikzcd} F \arrow[rr, hook, "\alpha"] \arrow[dr, hook] & & E \arrow[dl, hook] \\ & \overline{K} & \end{tikzcd} \text{ s.t. commutes} \right\}$ <br> at most 1 element |
| Examples | $\mathbb{Q}[x]/(x^2+1)$ <br> $\mathbb{Q}[x]/(x^3-2)$ <br> $\mathbb{Q}(x)$ | $\mathbb{Q}(i)$ <br> $\mathbb{Q}(\sqrt[3]{2})$ <br> $\mathbb{Q}(\pi)$ |

Common questions: (Which category are we considering for these questions?)
- $\#\{\text{extensions of } K \text{ of deg } 3\}$
- Automorphism gp of the field.

Abstract fields are not as hard as you may think!

Ex 1). Write down the definition of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$, as well as $\mathbb{Q}(i)$, $\mathbb{Q}(\pi)$
2). Find a $\mathbb{Q}$-basis of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$. Compute the dim.

# Constructing new field by adding roots

$$\begin{array}{r} 132 \\ 102\overline{)\,13562} \\ \underline{102\phantom{00}} \\ 3362 \\ \underline{306\phantom{0}} \\ 302 \\ \underline{202} \\ 100 \end{array}
\qquad\qquad
\begin{array}{r} x^2+3x+7 \\ x^2-2\overline{)\,x^4+3x^3+5x^2+6x+2} \\ \underline{x^4\phantom{+3x^3}-2x^2\phantom{+6x+2}} \\ 3x^3+7x^2+6x\phantom{+2} \\ \underline{3x^3\phantom{+7x^2}-6x\phantom{+2}} \\ 7x^2+12x+2 \\ \underline{7x^2\phantom{+12x}-14} \\ 12x+16 \end{array}$$

$$13562 \div 102 = 132 \cdots 100 \qquad (x^4+3x^3+5x^2+6x+2)\div(x^2-2) = (x^2+3x+7)\cdots(12x+16)$$

$$13562 = 102 \times 132 + 100 \qquad x^4+3x^3+5x^2+6x+2 = (x^2-2)(x^2+3x+7)+(12x+16)$$

Ex. factorize $x^3+4x^2-7x-10$ in $\mathbb{Q}[x]$ or $\mathbb{F}_3[x]$.

Ex. Let $F = \mathbb{F}_7[x]/(x^3-3)$.

1) Compute $(x^2+1)\cdot(x-1)$, $\frac{1}{x}$,

2) Show that $x^3-3$ is irr in $\mathbb{F}_7[x]$, i.e.

$$x^3-3 = f(x)g(x) \Rightarrow \deg f = 0 \text{ or } \deg g = 0$$

$f, g \in \mathbb{F}_7[x]$

3) Show that $(x^3-3,\ x^2+x+1) = (1)$ in $\mathbb{F}_7[x]$, by Euclidean division.

In fact, $\mathbb{F}_7[x]$ is ED $\Rightarrow$ PID

4) Compute $(x^2+x+1)^{-1}$ in $F$.

5) Factorize $T^3-3$ in $F[T]$.

Rmk. In fact, $K[T]/(f(T))$ is a field $\Leftrightarrow$ $f(T) \in K[T]$ is irreducible

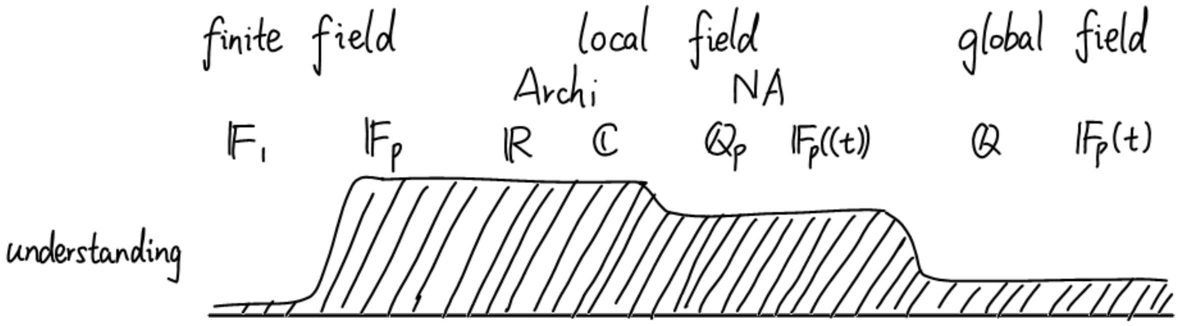
Ex. Let $F = \mathbb{Q}[x]/(x^3-2)$.

1) Compute $\mathrm{Mor}_{\text{Field}}(F, \mathbb{C})$. Are all embeddings real?

2) Discussion: What is the difference between $\mathbb{Q}[x]/(x^3-2)$ with $\mathbb{Q}(\sqrt[3]{2})$ ?

## 3. field extension

### Main examples of fields




### Definitions

Def: $E/F$ field extension: $(E, F, \iota: F \hookrightarrow E)$

Def: Base field: $\begin{cases} \mathbb{Q} & \text{char } F = 0 \\ \mathbb{F}_p & \text{char } F = p \end{cases}$

Def. (Algebraic extension)

$E/F$ is alg, if $\forall a \in E$ is alg$/F$, i.e., the following equivalent conditions are true.

1) $\forall a \in E$, $\exists f \in F[x]$, $f \neq 0$, $f(a) = 0$.
2) $\forall a \in E$, $[F(\alpha) : F] < +\infty$.
3) $E = \bigcup_{\substack{F \subset F' \subset E \\ F'/F \text{ finite}}} F'$
4) $\forall a \in E$, $\exists$ f.d. $F$-v.s. $V \subseteq E$ s.t $aV \subseteq V$.

For $a \in E$, $\mathrm{Min}(a, F) :=$ minimal monic polynomial of $a$ in $F$.

E.g. $\overline{\mathbb{Q}}/\mathbb{Q}$, $\mathbb{Q}(\pi)/\mathbb{Q}$, $\mathbb{C}/\mathbb{Q}$

We mainly consider alg extension, e.p. fin field extension.

Assume: $E/F$ alg

Slogan: Galois = normal + seperable

Def. (Normal extension)

$E/F$ normal, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \subseteq F[x] \subseteq E[x] \text{ splits}}_{a\in E \text{ is normal}}$.

E.g. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ $\quad \mathbb{Q}(\zeta_3)/\mathbb{Q}$ $\quad \mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$

Def. (Seperable extension)

$E/F$ sep, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \text{ has no repeated roots in } \bar{F}[x]}_{a\in E \text{ is sep}}$.

E.g. $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$, where $\mathbb{F}_p(T^{\frac{1}{p}}) := \mathbb{F}_p(T)[x]/(x^p - T)$

Rmk. When $\operatorname{char} F = 0$ or $\# F < +\infty$, $E/F$ is always seperable.

Def. (Galois extension)

$E/F$ Galois, if $E/F$ is normal and sep. We denote

$$\begin{aligned} \mathrm{Gal}(E/F) &= \mathrm{Aut}_{F\text{-alg}}(E) \\ &= \{\sigma : E \longrightarrow E \mid \sigma|_F = \mathrm{Id}_F\} \end{aligned}$$

Rmk. When $E/F$ finite,

$E/F$ Galois $\Leftrightarrow$ $[E:F] = \# \mathrm{Aut}_{F\text{-alg}}(E)$

E.g. & Exercise. Compute $\#\mathrm{Aut}_{F\text{-alg}}(E)$ for $E/F = \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$.

https://kconrad.math.uconn.edu/blurbs/galoistheory/galoiscorrexamples.pdf

Ex. Read it, and compute

$\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q})$

$\mathrm{Gal}(F/\mathbb{Q})$ $\quad$ $F$: the splitting field of $x^4 - x^2 - 1$.

I would instead begin with relative easier case:

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\quad \mathrm{Gal}\left(\mathbb{Q}[T]/(T^2-2)^2-2 \,/\mathbb{Q}\right)$

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q})$

After that, do

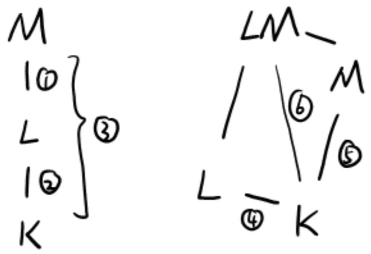
4.2.3 : $\mathbb{Q}(\sqrt[4]{2}(1+i))/\mathbb{Q}$

4.1.16: $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$

4.1.4: $\mathbb{Q}(\sqrt{2}, \sqrt{3}, u)/\mathbb{Q}$ $\qquad u^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$

4.1.7. $F(\alpha)/F$ $\qquad \operatorname{char} F = p$, $a \in F$, $x^p - x - a \in F[x]$ irr, $\alpha^p - \alpha - a = 0$

in [近世代数三百题].




③ pure ins: ① ⟹ ③

normal: ③ ⇒ ① ③ ⇏ ② ①+② ⇏ ③ ⑥ ⇏ ④ ④+⑤ ⇒ ⑥

+

seperable: ① + ② = ③ ④ + ⑤ = ⑥

‖

Galois: ③ ⇒ ① ③ ⇏ ② ①+② ⇏ ③ ⑥ ⇏ ④ ④+⑤ ⇒ ⑥

purely inseparable ① + ② = ③ ④ + ⑤ = ⑥

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then
$F$ perfect $\Leftrightarrow F^p = F$

$\bar{K}$ | closed subgroup, $L$ | quotient group, $K$ (finite) $q$

$\overline{\mathbb{F}_p}$ | $\mathbb{Z}_l$ $\bigcup_{l \nmid a} \mathbb{F}_{p^a}$ | $\prod_{p \nmid l} \mathbb{Z}_p$ $\mathbb{F}_p$

$\overline{\mathbb{F}_p}$ | $\prod_{p\nmid l} \mathbb{Z}_p$ $\bigcup_n \mathbb{F}_{p^{l^n}}$ | $\mathbb{Z}_l$ $\mathbb{F}_p$

$\overline{\mathbb{F}_p}$ | $d\hat{\mathbb{Z}}$ $\mathbb{F}_q$ | $\mathbb{Z}/d\mathbb{Z}$ $\mathbb{F}_p$

$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ $(q = p^d)$

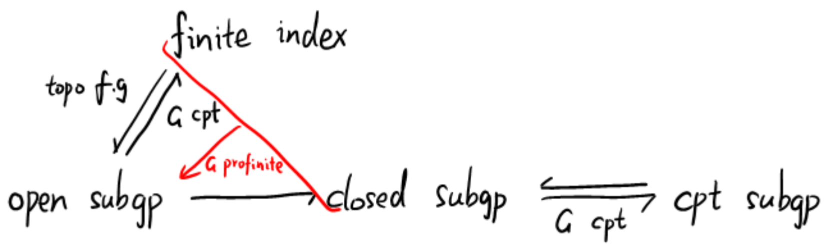
open subgroup $\overset{\{1\}\subseteq \mathbb{Z}_p}{\subsetneq}$ closed subgroup $= \{Gal(\bar{K}/L) \mid L/k \text{ ext}\} \overset{\mathbb{Z}\subseteq\mathbb{Z}_p}{\subsetneq}$ subgroup

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Q: Do we have any finite index gp of $Gal(\bar{k}/k)$ which is not open?

In general,




https://groupprops.subwiki.org/wiki/Closed_subgroup_of_finite_index_implies_open
In a topological group, any closed subgroup of finite index must be an open subgroup.
https://groupprops.subwiki.org/wiki/Open_subgroup_implies_closed
Any open subgroup of a topological group is closed.
https://math.stackexchange.com/questions/1526525/non-open-subgroups-of-finite-index-in-the-idele-class-group-of-a-number-field

https://math.stackexchange.com/questions/4350043/open-subgroup-and-finite-index-subgroup-of-topological-group
https://math.stackexchange.com/questions/1092974/finite-index-subgroup-g-of-mathbbz-p-is-open
https://math.stackexchange.com/questions/3734250/are-normal-subgroups-of-finite-index-in-an-absolute-galois-groups-open
https://math.stackexchange.com/questions/83355/how-to-prove-that-a-compact-set-in-a-hausdorff-topological-space-is-closed

https://math.stackexchange.com/questions/4062798/a-profinite-group-that-is-not-finite-is-not-countable
https://math.stackexchange.com/questions/3165116/direct-proof-that-closed-subgroups-of-profinite-groups-are-profinite

Some wonderful exercises for Galois correspondence:

Let $E/F$ be Galois field ext of deg $n$, $m|n$. prove: $\exists$ subfield ext of deg $m$.
( Sylow thm & $Z(G) \neq \{1\}$ for a p-gp & classification of f.g. abelian gp)
Cor. For p prime, F field, one can define ${}^{p}\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^{p}\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}} \cap \mathbb{R}$)?

4. examples of algebraic closed field

① $\overline{\mathbb{Q}} \overset{\pi}{\underset{\bar{\mathbb{Z}}}{\subsetneq}} \mathbb{C} \overset{t}{\subsetneq} \bigcup_n \mathbb{C}((t^{\frac{1}{n}})) = \overline{\mathbb{C}((t))}$ $\mathbb{C}[[t]]$

Puiseux series

$\sum_{n=0}^{+\infty} p^n$ $\quad \cap$

$\overline{\mathbb{Q}_p} \underset{\bar{\mathbb{Z}}_p}{\subsetneq} \mathbb{C}_p$

$\overset{\shortparallel}{\mathbb{Q}_p \cdot \overline{\mathbb{Q}}}$

② char $K = p$: $\quad \overline{\mathbb{F}_q} = \overline{\mathbb{F}_p}$ $\quad (\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \hat{\mathbb{Z}})$

Task. ① Prove they are alg closed. ($\mathbb{C}$, $\bigcup_n \mathbb{C}((t^{\frac{1}{n}}))$, $\mathbb{C}_p$)

② Find an element in each "⊂".

$Q_p \cap \bar{Q}$:$https://math.stackexchange.com/questions/1280053/explicit-description-of-bbb-q-p-cap-bar-bbb-q

$C_p \setminus \bar{Q_p}$:$https://math.stackexchange.com/questions/123925/is-the-algebraic-closure-of-a-p-adic-field-complete

https://math.stackexchange.com/questions/2430665/algebraic-closure-of-q-p-is-composite-of-bar-mathbbq-and-mathbbq-p

https://math.stackexchange.com/questions/2153580/transcendental-numbers-in-mathbbq-p