Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \quad \mathcal{F}' \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^* \dashv f_*$$
$$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

$$\begin{tikzcd} & \otimes \arrow[dl, dash] \arrow[dr, dash] & \\ f^* \arrow[rr, dash] & & f_! \end{tikzcd}$$

bc: $f^* g_! \cong g'_! f'^*$

$f_* g^! \cong g'^! f'_*$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^! \mathcal{F}) \cong \underline{Hom}(f_! \mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

I: $f^* = f^!$
P: $f_* = f_!$

$p: X \longrightarrow pt$

$H^\bullet(X; \underline{\mathbb{Z}}) := p_* p^* \mathbb{1}$	$H^\bullet(X; \mathcal{F}) := p_* \mathcal{F}$	$= R\Gamma(X; \mathcal{F})$
$H_c^\bullet(X; \underline{\mathbb{Z}}) := p_! p^* \mathbb{1}$	$H_c^\bullet(X; \mathcal{F}) := p_! \mathcal{F}$	
$H_\bullet(X; \underline{\mathbb{Z}}) := p_! p^! \mathbb{1}$	$H_\bullet(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$	$= H_c^\bullet(X; p^! \mathbb{1} \otimes \mathcal{F})$
$H_\bullet^{BM}(X; \underline{\mathbb{Z}}) := p_* p^! \mathbb{1}$	$H_\bullet^{BM}(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$	$= H^\bullet(X; p^! \mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$H_c(X; \mathcal{F}) \otimes H_c(Y; \mathcal{G}) \cong H_c(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

reduced to: $p_{X!} \mathcal{F} \otimes p_{Y!} \mathcal{G} \cong p_!(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G})$

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\quad -^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper $\quad p^! \mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\bullet(X; \underline{\mathbb{Z}})[d] \cong H^\bullet(X; \underline{\mathbb{Z}})^\vee$$

reduced to: $p_* \underline{Hom}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{Hom}(p_! A, B)$