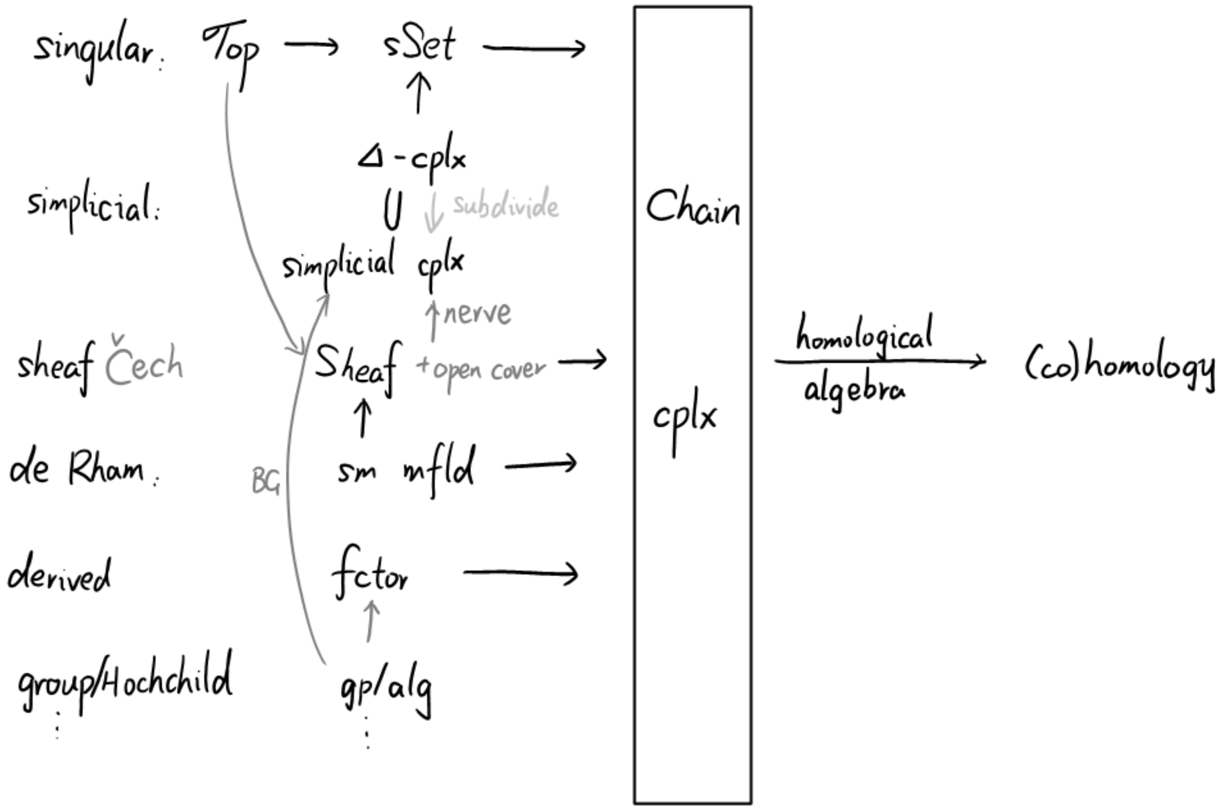
Eine Woche, ein Beispiel
6.25 (co)homology of simplicial set

This document is a continuation of [23.01.09].
https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

Realize Hochschild homology as simplicial homology:
https://arxiv.org/pdf/1802.03076.pdf

1. definition and basic examples

We use α here because
we are considering $X=\Delta^n$ case.
May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha\in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1-d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2-d_0^1+d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{dual} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$Hom_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n}\mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n} Hom_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define
Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset.
If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \mathcal{T}op$ discrete, $X := \mathcal{S}(A) \in sSet$, one can compute

$$\begin{array}{lllllllllll}
C_\bullet(X;G): & 0 & \longleftarrow & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} \cdots \\
C^\bullet(X;G): & 0 & \longrightarrow & \prod_{\alpha\in A} G & \xrightarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} \cdots
\end{array}$$

$$\text{wished}\left\{\begin{array}{lllllllllll}
C^{BM}_\bullet(X;G): & 0 & \longleftarrow & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} \cdots \\
C^\bullet_c(X;G): & 0 & \longrightarrow & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} \cdots
\end{array}\right.$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.
Notice that $\#\Delta'_k = k+2$, so

$$C_\bullet(\Delta';G):\quad 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots$$

basis: $d'_0 \triangleq x_0$; $d'_1 \triangleq x_1$; x_0, x_1, x_2; x_0, x_1, x_2, x_3; x_0, x_1, x_2, x_3, x_4

remember indexes.

$$\begin{aligned}
0 = x_0 - x_0 &\longmapsfrom x_0\\
x_0 - x_1 = x_0 - x_1 &\longmapsfrom x_1\\
0 = x_1 - x_1 &\longmapsfrom x_2
\end{aligned}$$

$$\begin{aligned}
0 = x_0 - x_0 + x_0 - x_0 &\longmapsfrom x_0\\
x_0 - x_1 = x_0 - x_1 + x_1 - x_1 &\longmapsfrom x_1\\
0 = x_1 - x_1 + x_2 - x_2 &\longmapsfrom x_2\\
x_2 - x_3 = x_2 - x_2 + x_2 - x_3 &\longmapsfrom x_3\\
0 = x_3 - x_3 + x_3 - x_3 &\longmapsfrom x_4
\end{aligned}$$

$$\begin{aligned}
x_0 = x_0 - x_0 + x_0 &\longmapsfrom x_0\\
x_0 = x_0 - x_1 + x_1 &\longmapsfrom x_1\\
x_2 = x_1 - x_1 + x_2 &\longmapsfrom x_2\\
x_2 = x_2 - x_2 + x_2 &\longmapsfrom x_3
\end{aligned}$$

By taking the transpose, one get

$$C^\bullet(\Delta';G):\quad 0 \rightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix}0&0\\1&-1\\0&0\end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix}1&0&0\\1&0&0\\0&0&1\\0&0&1\end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix}0&0&0&0\\1&-1&0&0\\0&0&0&0\\0&0&1&-1\\0&0&0&0\end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta';G) = \begin{cases} G & n=0\\ 0 & n>0\end{cases}$$

$$H^n(\Delta';G) = \begin{cases} G & n=0\\ 0 & n>0\end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$
\begin{tikzcd}
\Delta^1 \arrow[d, "s^1"] & C_\cdot(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\,1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1)"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1\,1)"] \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "Id"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta^0 \arrow[d, "d^1_0"] & C_\cdot(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "Id"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

s.t. $s^1_{0,*} \circ d^1_{0,*} = Id_{C_\cdot(\Delta^0;G)}$, $\quad d^1_{0,*} \circ s^1_{0,*} \sim Id_{C_\cdot(\Delta^1;G)}$.

In fact, we have

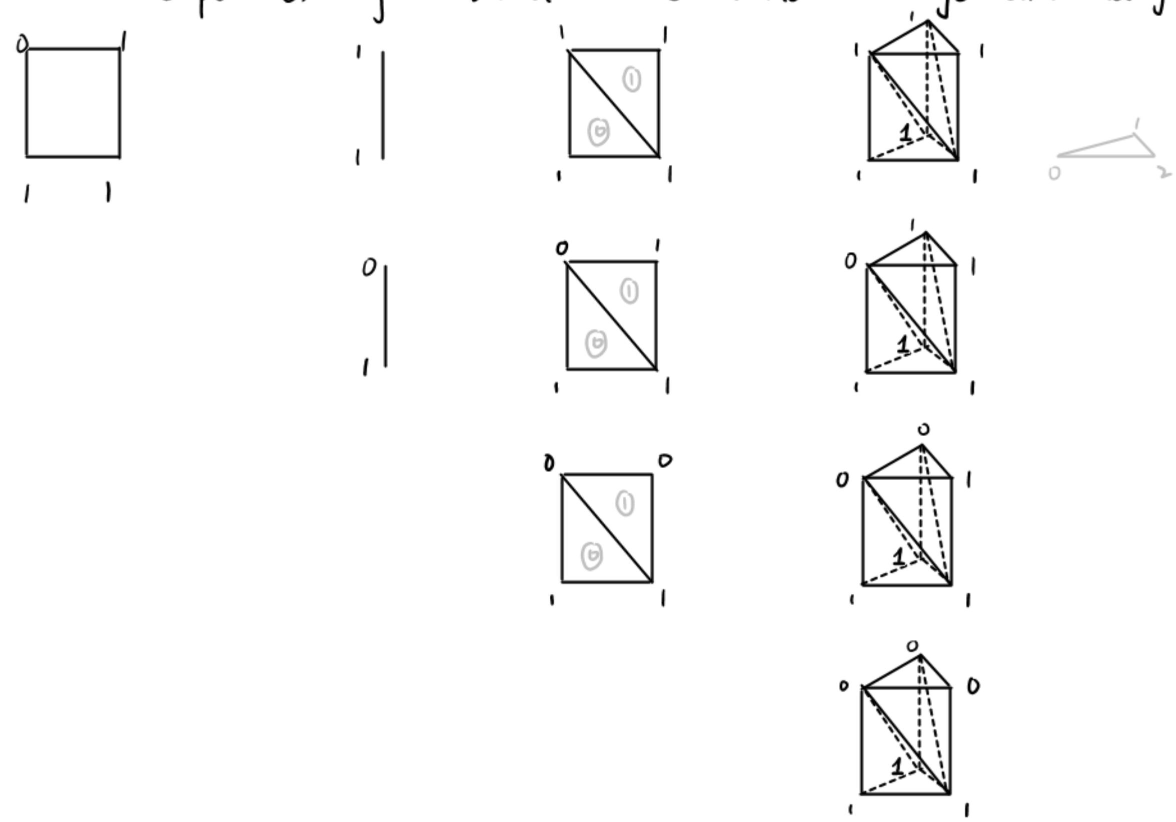
$$
\begin{tikzcd}[column sep=large, row sep=huge]
C_\cdot(\Delta^1;G): \arrow[d, shift right, "Id"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0\\0&0&1\\0&0&-1\\0&0&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

$x_0 \mapsto x_0$
$x_1 \mapsto x_1$

$x_0 \mapsto x_0 - x_0 + x_0 = x_0$
$x_1 \mapsto x_1 - x_1 + x_1 = x_1$
$x_2 \mapsto x_1 - x_2 + x_2 = x_1$
$x_3 \mapsto x_1 - x_2 + x_3$

$x_0 \mapsto x_0 - x_0 = 0$
$x_1 \mapsto x_1 - x_1 = 0$
$x_2 \mapsto x_1 - x_2$

Ex. Observe the picture, try to translate the calculation in geometrical language.

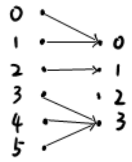
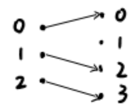
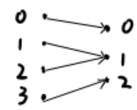


E.g. 3. When we want to compute $H_n(\Delta^m; G)$ and $H^n(\Delta^m; G)$, we'd better to give elements in $\Delta^m_n \approx \{\text{basis of } C_n(\Delta^m; G)\}$ a better notation, see [23.01.09]

The following table shows some typical element in

$$C_n(\Delta^m; G) = \langle a : [n] \longrightarrow [m] \rangle_{a \in \Delta^m_n}.$$

not confuse with [n]

element	picture	list	count	degenerate degree
$a : [5] \longrightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		(0,0,1,3,3,3)	[2,1,0,3]	$\Delta_5^{3,\text{↯}3}$
$d_1^3 : [2] \longrightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		(0,2,3)	[1,0,1,1]	$\Delta_2^{3,\emptyset}$
$s_1^3 : [3] \longrightarrow [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		(0,1,1,2)	[1,2,1]	$\Delta_3^{2,\text{↯}1}$
∂a	—	(0,0,3,3,3) − (0,0,1,3,3)	[2,0,0,3] − [2,1,0,2]	$\Delta_4^{3,\text{↯}3}$ $\Delta_4^{3,\text{↯}2}$

e.g. $\partial[2,5,3,4,1,6,0]$

$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$

In this case, $d : C^n(\Delta^m; G) \longrightarrow C^{n+1}(\Delta^m; G)$ is also not hard to describe.

e.g. $d[2,1,0,3] = [3,1,0,3] - [2,1,1,3]$

$d[2,5,3,4,1,6,0]$

$= [3,5,3,4,1,6,0] + [2,5,3,5,1,6,0]$

$- [2,5,3,4,1,7,0] - [2,5,3,4,1,6,1]$

Rmk. The computation of ∂ and d actually comes from the computation of $d_i^{n,*}$ and $d^n_{i,*}$ = the dual of $d_i^{n,*}$ (not $s_i^{n,*}$!). In general, one can derive the formula of ∂^* & ∂_*.

E.g.

$\partial^*: C_3(\Delta^m; G) \longrightarrow C_5(\Delta^m; G) \qquad \beta \longmapsto \beta \circ \partial$

$[1,2,1] \longmapsto [1,2,1] \circ [2,1,0,3] = [2,1,3]$

$d_1^{3,*}: C_3(\Delta^m; G) \longrightarrow C_2(\Delta^m; G) \qquad \beta \longmapsto \beta \circ d_1^3$

$[1,2,1] \longmapsto [1,2,1] \circ [1,0,1,1] = [1,1,1]$

$s_1^{3,*}: C_2(\Delta^m; G) \longrightarrow C_3(\Delta^m; G) \qquad \beta \longmapsto \beta \circ s_1^3$

$[2,1] \longmapsto [2,1] \circ [1,2,1] = [3,1]$

$\partial_*: C^5(\Delta^m; G) \longrightarrow C^3(\Delta^m; G) \qquad [2,1,0,3]$

$[2,1,3] \longmapsto [1,2,1] + [1,1,2]$

$[2,2,2] \longmapsto 0 \qquad \times$

$[3,3] \longmapsto [3,1] + [2,2]$

$d^3_{1,*}: C^2(\Delta^m; G) \longrightarrow C^3(\Delta^m; G) \qquad [1,0,1,1]$

$[1,1,1] \longmapsto [2,1,1] + [1,2,1]$

$[0,3] \longmapsto [0,4]$

$[1,0,1,1] \longmapsto [2,0,1,1] + [1,1,1,1] + [1,0,2,1]$

$s^3_{1,*}: C^3(\Delta^m; G) \longrightarrow C^2(\Delta^m; G) \qquad [1,2,1]$

$[3,1] \longmapsto [2,1]$

$[2,2] \longmapsto 0 \qquad \times$

$[1,0,3] \longmapsto [1,0,2]$

2. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\diamond}$:

non-degenerate

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G)^{\diamond}: & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_\cdot(\Delta';G)^{\diamond}: \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

In fact, we have

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red, "\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)
For $X \in sSet$, define

$\Diamond$: non-degenerate
$\lightning$: degenerate

$$
\begin{aligned}
X_k^{\Diamond} &:= \{ x \in X_k \mid x \text{ non-degenerate} \} &&= X_k \setminus (sk^{k-1}X)_k \\
X_k^{\lightning} &:= \{ x \in X_k \mid x \text{ degenerate} \} &&= (sk^{k-1}X)_k \\
X_k^{\lightning i} &:= \left\{ x \in X_k \;\middle|\; x = \alpha^*(y) \text{ for some } \begin{array}{l} y \in X_{k-i}^{\Diamond} \\ \alpha : [k-i] \to [k] \end{array} \right\} &&= (sk^{k-i}X)_k - (sk^{k-i-1}X)_k
\end{aligned}
$$

$$
0 = \underbrace{(sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}X)_k}_{X_k^{\lightning}} \overset{X_k^{\lightning 0}}{\subset} \underbrace{(sk^kX)_k}_{X_k^{\Diamond}} = X_k
$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$
C_n(X;G)^{\Diamond} = \bigoplus_{\alpha \in X_n^{\Diamond}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\Diamond}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\Diamond}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\Diamond}} G \cdots
$$

$$
C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots
$$

and $H_\cdot(X;G)^{\Diamond}$, $H_\cdot(X;G)^{\lightning}$ as crspd homology.

By definition, $\quad C_\cdot(X;G) \cong C_\cdot(X;G)^{\Diamond} \oplus C_\cdot(X;G)^{\lightning}$

Claim 1. $H_\cdot(X;G)^{\lightning} = 0$, so

$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\Diamond}. \qquad (*)$$

Rmk 1. Roughly, $(*)$ says that

singular homology $\approx$ simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expand $C_\cdot(X;G)$ by double complex.

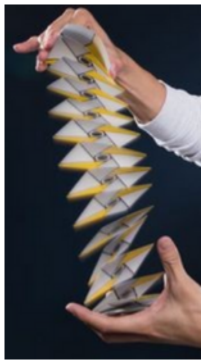
Def (Double complex of $C_\cdot(X;G)$) $\swarrow\!\leftarrow + \leftarrow\!\swarrow = 0$

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0^{S_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{S_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{S_3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{S_3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{S_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{S_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{S_2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{S_2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{S_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_1^{S_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_2^{S_1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_3^{S_1}} G \arrow[l] \arrow[dl, "0"'] \\
0 & \bigoplus_{\alpha\in X_0^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_1^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_2^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_3^{\emptyset}} G \arrow[l]
\end{tikzcd}
$$

$0 \Leftarrow$ dim 0   dim 1   dim 2

fold $\Downarrow \Uparrow$ expand

$$
0 \longleftarrow \bigoplus_{\alpha\in X_0} G \longleftarrow \bigoplus_{\alpha\in X_1} G \longleftarrow \bigoplus_{\alpha\in X_2} G \longleftarrow \bigoplus_{\alpha\in X_3} G
$$



fold / expend

Eg. For $X=\Delta^1$, we have double complex

$$
\begin{tikzcd}
 & & & & & \cdots \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0,x_4\rangle \arrow[l] \arrow[dl, "0"'] & \cdots \arrow[l, "\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0,x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] & \langle x_1,x_2,x_3\rangle \arrow[l, "\begin{pmatrix}1&0&0\\0&0&-1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}"'] & \\
0 & 0 \arrow[l] \arrow[dl] & \langle x_0,x_2\rangle \arrow[l] \arrow[dl, "0"'] & \langle x_1,x_2\rangle \arrow[l, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \arrow[dl, "0"'] & 0 \arrow[l] \arrow[dl] & \\
0 & \langle x_0,x_1\rangle \arrow[l] & \langle x_1\rangle \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l] & 0 \arrow[l] &
\end{tikzcd}
$$

$0 \Leftarrow$

fold $\Downarrow \Uparrow$ expand

$$
0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots
$$

Claim 2. We have chain homotopy equivalence between the following two cplx:

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_n^{\spadesuit}} G \arrow[l] \arrow[d, equal] & \bigoplus_{\alpha\in X_{n+1}^{\spadesuit 1}} G \arrow[l, "0"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+2}^{\spadesuit 2}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+3}^{\spadesuit 3}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] \\
0 & \bigoplus_{\alpha\in X_n^{\spadesuit}} G \arrow[l] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"']
\end{tikzcd}
\quad (**)
$$

i.e. (**) is exact on all terms except $\bigoplus_{\alpha\in X_n^{\spadesuit}} G$.

Proof idea of Claim 2 for $X=\Delta^m$.

(can be generalized to arbitrary subset of Δ^m, e.g. for any triangulation)

Q: How far can we weaken our conditions on X?

Does this proof work for $\partial\Delta^m$? (No, I think)

$$
\begin{tikzcd}
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{\spadesuit k-1}} G \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \bigoplus_{\alpha\in X_{n+k}^{\spadesuit k}} G \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dl, "s"] & \bigoplus_{\alpha\in X_{n+k+1}^{\spadesuit k+1}} G \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dl, "s"] & \cdots \arrow[l] \arrow[dl, "s"] \\
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{\spadesuit k-1}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k}^{\spadesuit k}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k+1}^{\spadesuit k+1}} G \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

Define

$$
s[\underbrace{a_1,\cdots a_l}_{\in\{0,1\}}, \underbrace{a_{l+1}}_{\notin\{0,1\}},\cdots,a_m] = \begin{cases} (-1)^{i}[a_1,\cdots,a_l, a_{l+1}+1,\cdots,a_m], & a_{l+1} \text{ even} \\ 0, & a_{l+1} \text{ odd} \end{cases}
$$

$i=\sum_{j=1}^{l} a_j$

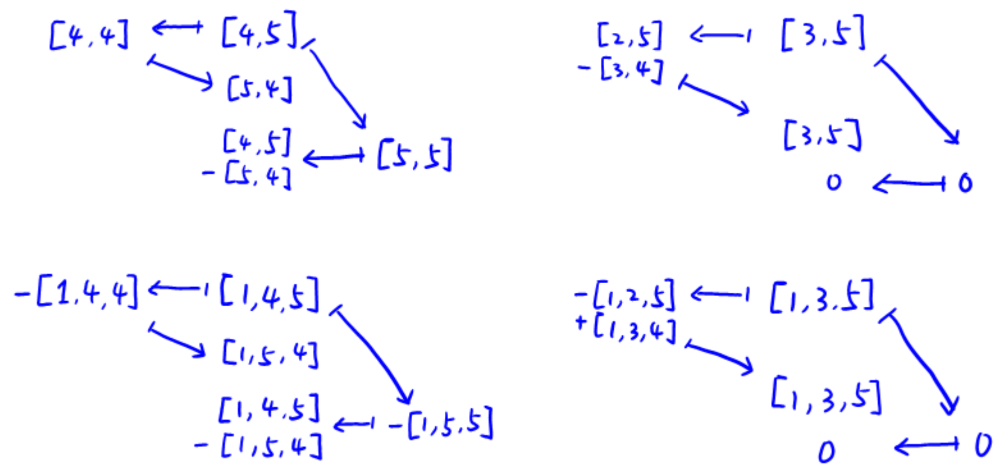
Ex. Check that s is a homotopy.

e.g. $X=\Delta^3$, $n=2$, $k=3$ $\Rightarrow m=3$, $n+k=5$

$$
\begin{tikzcd}
-[2,1,0,2] \arrow[dr, mapsto] & {[2,1,0,3]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
 & -[3,1,0,2] & \\
 & {[2,1,0,3]+[3,1,0,2]} & {[3,1,0,3]} \arrow[l, mapsto]
\end{tikzcd}
$$

$X=\Delta^6$, $n=5$, $k=15$ $\Rightarrow m=6$, $n+k=20$

$$
\begin{tikzcd}
{[2,4,3,4,1,6,0]} & {[2,5,3,4,1,6,0]} \arrow[l, mapsto] \arrow[dddr, mapsto] & \\
-[2,5,2,4,1,6,0] \arrow[dr, mapsto] & & \\
 & {[3,4,3,4,1,6,0]-[3,5,2,4,1,6,0]} & \\
 & {[2,5,3,4,1,6,0]-[3,4,3,4,1,6,0]+[3,5,2,4,1,6,0]} & {[3,5,3,4,1,6,0]} \arrow[l, mapsto]
\end{tikzcd}
$$

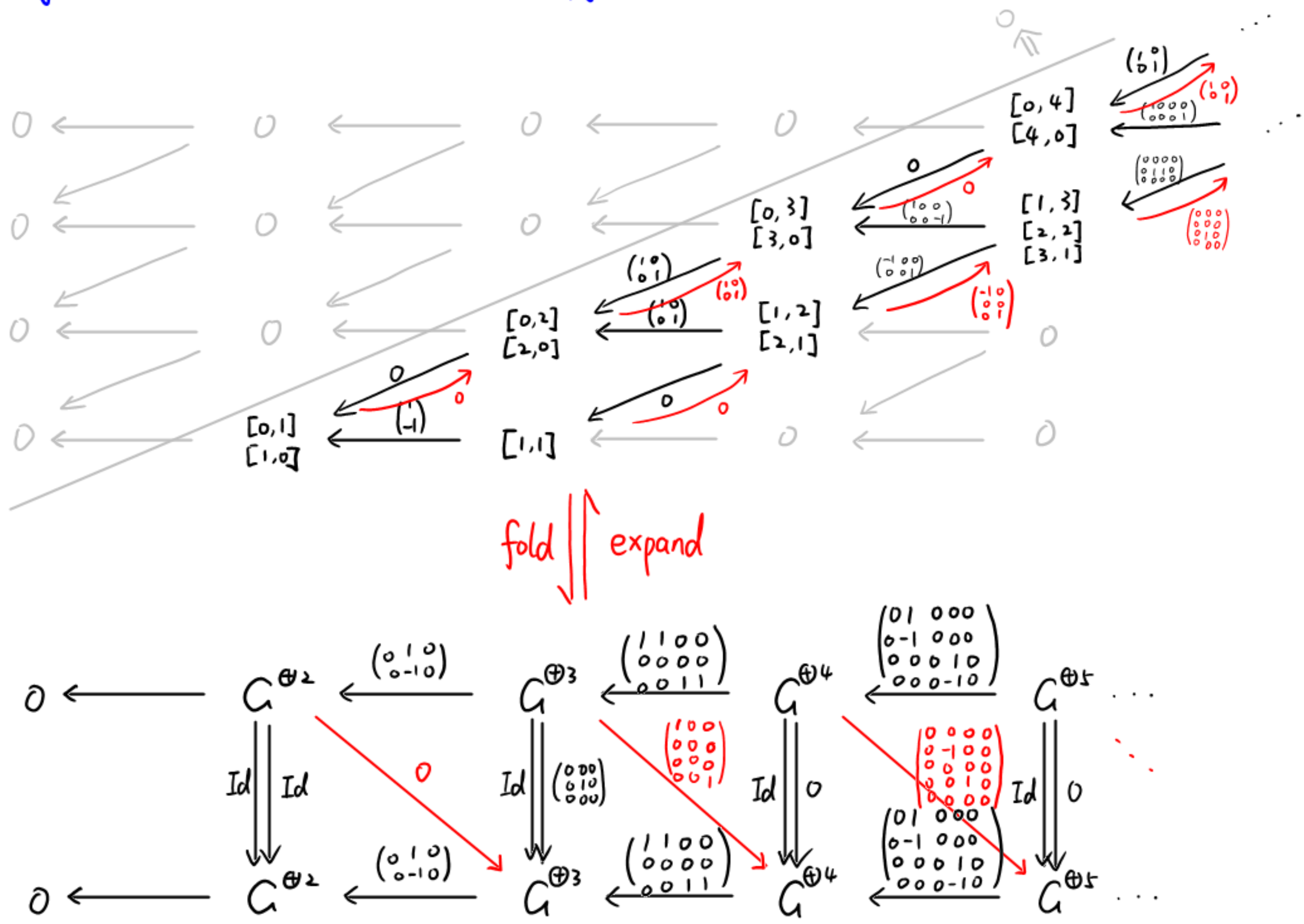


In conclusion,

$$\text{Claim 2} \Rightarrow \text{Claim 1} \Rightarrow \text{Rmk 1}$$

Coming back to Eg.2, one can now find a homotopy without guess.

Eg. For $X = \Delta^1$, we have homotopy



Ex. Check that (I believe that this argument also works for general sset X)

① $\underset{\xleftarrow{\partial^{\leftarrow}}}{\overset{s}{\nearrow}} + \overset{\xleftarrow{\partial^{\leftarrow}}}{\nearrow_s} = 0$

② the collected s is a homotopy.

3. more structures

math.stackexchange.com/questions/2559705/cup-product-why-do-we-need-to-consider-cohomology-with-coefficients-in-a-ring
https://arxiv.org/pdf/2105.08025.pdf
https://users.math.msu.edu/users/ruiterj2/math/Documents/Spring%202019/Galois%20Cohomology%20Seminar%20Week%203.pdf

When $G=R$ is a k-alg, the product structure on $C^\cdot(X;R)$ is defined by (cup product)

wedge in de Rham

$\cup \approx \wedge$

bad symbol compatability...

$$\cup: \quad C^i(X;R)\otimes C^j(X;R) \xrightarrow{d_{i,j,*}\otimes d'_{i,j,*}} C^{i+j}(X;R)\otimes C^{i+j}(X;R) \to C^{i+j}(X;R\otimes R) \xrightarrow{\text{multiply}} C^{i+j}(X;R)$$

$$f_1 \otimes f_2 \longmapsto (f_1\circ d_{i,j,*})\otimes(f_2\circ d'_{i,j,*})$$

$$f\otimes g \longmapsto fg$$

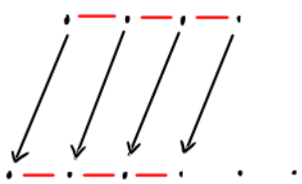
the $C^\cdot(X;R)$-module structure on $C_\cdot(X;G)$ is defined by (cap product)

$$\cap: \quad C^i(X;R)\otimes C_{i+j}(X;R) \xrightarrow{Id\otimes d^*_{i,j}(-)\otimes d'^*_{i,j}(-)} C^i(X;R)\otimes C_i(X;R)\otimes C_j(X;R) \to R\otimes C_j(X;R\otimes R) \xrightarrow{\text{multiply}} C_j(X;R)$$

$$f\otimes \alpha \longmapsto f(d^*_{i,j}(\alpha))\otimes d'^*_{i,j}(\alpha)$$

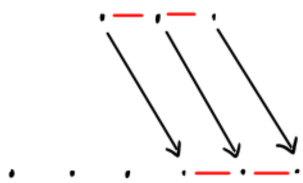
$$r\otimes\beta \longmapsto r\beta$$

where $(i=3, j=2)$



$d_{i,j}: [i] \longrightarrow [i+j]$

$d_{i,j} = [\underbrace{1,\dots,1}_{i+1 \text{ many}},\underbrace{0,\dots,0}_{j \text{ many}}]$



$d'_{i,j}: [j] \longrightarrow [i+j]$

$d'_{i,j} = [\underbrace{0,\dots,0}_{i \text{ many}},\underbrace{1,\dots,1}_{j+1 \text{ many}}]$

$\otimes$ are over k. Notice that

$$\begin{cases} C_i(X;R) = \bigoplus_{\alpha\in X_i} R \\ C^i(X;R) = \prod_{\alpha\in X_i} R \end{cases}$$

are bi R-modules.

https://math.stackexchange.com/questions/4439483/applications-of-the-cup-product-before-descending-to-cohomology

Ex. Show that:

- $C^{\cdot}(X;R)$ is a dga,
- $H^{\cdot}(X;R)$ is a graded R-alg. (graded commutative)
- $H_{\cdot}(X;R)$ is a graded $H^{\cdot}(X;R)$-module.

Hint. Once the following key formulas are verified, we are done.

For $a \in C^p(X;R)$, $b \in C^q(X;R)$,

$$d^{p+q}(a \cup b) = d^p(a) \cup b + (-1)^p a \cup d^p(b)$$

$$a \cup b - (-1)^{pq} b \cup a = (-1)^{p+q-1}\left(d^{p+q-1}(a \cup_1 b) - d^p(a) \cup_1 b - (-1)^p a \cup_1 d^q(b)\right)$$

where

$a \cup_1 b \in C^{p+q-1}(X;R)$:

$$a \cup_1 b\,(x) = \sum_{i=0}^{p-1} (-1)^{(p-i)(q+1)}\, a\left(\delta^{out,*}_{i,(p,q)}(x)\right) \cdot b\left(\delta^{in,*}_{i,(p,q)}(x)\right)$$

For the definition of $\delta^{out}_{i,(p,q)}$, $\delta^{in}_{i,(p,q)}$, see [23.01.09. Table 1].