Let us do a simple case over IP'. It can be generlized "easily" to flag variety. but IP is the beginning case of study.

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand

where
$$SL_{*} = SL_{*}, C$$
, $B = \binom{* *}{*} \subseteq SL_{*}, C$, $P' = P'_{C} \cong G/B$. $G \subseteq P' = C \subseteq P'$ maps are pushback & pullout of $P' \longrightarrow Pt$.

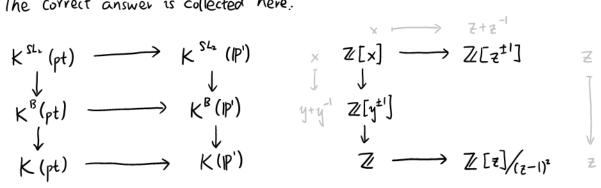
We want to see

- · ring structure, module structure
- · Weyl gp action
- relations

e.g.
$$K^{B}(X) \cong R(B) \otimes_{R(G)} K^{G}(X) \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^{G}(X)$$

 $(K^{B}(X))^{\mathbf{w}} \cong K^{G}(X)$

The correct answer is collected here.



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Notation. For linear alg gp G [Ginz, 5.1], K_i^G(X) = K_i(Coh(X)) K^G(X) = K_o^G(X) K(X) = K^{fids}(X)
                  R(G) := K^{G}(pt) = K_{o}(Coh^{G}(pt)) = K_{o}(Rep G)
            e.g. R(iU_1) = \mathbb{Z}, R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}], R(SL_1) \cong \mathbb{Z}[x], R(SL_2 \times \mathbb{C}^x) \cong \mathbb{Z}[x, t^{\pm 1}]
                                                          = \mathbb{Z}[X^*(T)] = \mathbb{Z}[X^*(T)]^{W_f}
Some further discussion of
                                                R(SL_2)
        R(SL_1) = \bigoplus_{i \in IN_{20}} \mathbb{C} \times_i where \times_i represents the (i+1)-dim irr rep of SL_1.
As an algebra, R(SL_1) = \mathbb{C}[\times] where
                 1 = X6
                 × = x,
                 x'= x2+1
                x^3 = x_3 + 2x,
                x" = X"+3x"+2
                                                                                    10-1010-101
                 x_0 = 1
                 X_1 = X
                                                                                                  1 0-5 0 15
                 x_2 = x^2 - 1
                                                                                                      10-60
                  x_2 = x^3 - 2x
                  x_4 = x^4 - 3x^2 + 1
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V × represents a v.b of vank 2 here!

Q: How to write down all polynomials in R(SLn) which represents an irr rep of SLn?

The important definitions and results in [Ginz, Chap 5] have been collected in this page. Nothing about cohomology theory is short.

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[5.2.5] pullback
[5.2.11] tensor product
[5.2.13] pushforward
[5.216] induction
[5.2.18] reduction
[5.2,20] Convolution
[5.2.26] duality, pairing
[5.3] specialization
[Thm 5.4.17] Thom iso
[Lemma 5.5.1] cellular fibration
[Thm 5.6.1] Kunneth
5.7.1
        Beilinson Resolution
[Thm 5.8.14] Riemann - Roch
[Prop 5 9.3] Devissage
[Rmk 5 118] Lefschetz fixed point formula
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derived pullback gives an alg homo between K-groups, while derived pushforward gives a Z-mod homo between K-groups

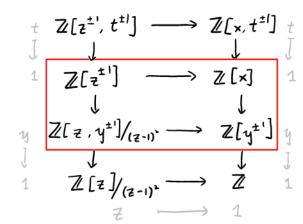
[LCBE, 2.1.1]
$$\textcircled{P}$$
 \textcircled{K} $(P') \cong \mathbb{Z}\mathcal{O}_{P'} \oplus \mathbb{Z}\mathcal{O}_{P'}(1) = \mathbb{Z}[\mathbb{Z}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\frac{1}{2}}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}[\mathbb{Z}^{\frac{1}{2}}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\frac{1}{2}}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{$

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[G_{inz}, (524)] \xrightarrow{G_1G_2} \xrightarrow{G_2G_2} K^{G_1\times G_2}(X) \cong K^{G_1}(X) \otimes_{\mathbb{Z}} R(G_2) \xrightarrow{As \text{ an } R(G_2)-module}
                        e.g. K^{SL_1 \times \mathbb{C}^{\times}}(\mathbb{P}^1) \cong K^{SL_1}(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(\mathbb{C}^{\times}) \cong K^{SL_1}(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]
K^{B}(\mathbb{P}^1) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(B) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[y^{\pm 1}]
[Ginz, (5.2.17)]
                                                                             K_i^H(X) \stackrel{\text{Res}_H^G}{\longleftarrow} K_i^G(G \times_H X)
                 \mathbb{D}_{e,q} K^{sl_1}(\mathbb{P}') \cong K^{sl_1}(SL_2 \times_B pt) \cong K^{g}(pt) = R(B) = \mathbb{Z}[z^{\pm 1}]
 Task, understand Ksl2(IP'): How does equivariant SL2-bundle look like?
  Try: Let \lambda \in X^*(T), C_{\lambda} denotes for B-equivarient 1.b over [pt]. [C_{\lambda}] \in R(B)
                    Step 1. find a local chart of G/B.
                                                G = G \times \text{spt}
                                                                                                                                                                         G. left multiplication
                                                 \begin{bmatrix} x \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ \end{bmatrix}
                                                                                                                                                                                        BGG×Cx:
                                                                                                                                                                                        b(g, x) = (gb^{-1}, bx)
                      Step 2. See the type of lb over G/B. (K^G(P)) \rightarrow K(P))
                                          A' \qquad (x_{i}, s_{i}) \qquad (\begin{bmatrix} x_{i}-1 \\ \vdots & s_{i} \end{bmatrix}, s_{i}) = (\begin{bmatrix} \frac{1}{x_{i}} & 0 \\ \frac{1}{x_{i}} & 1 \end{bmatrix}, \begin{bmatrix} x_{i}-1 \\ 0 & \frac{1}{x_{i}} \end{bmatrix}, s_{i})
A' \qquad (\begin{bmatrix} x_{i}-1 \\ \vdots & s_{i} \end{bmatrix}, s_{i}) \qquad (\begin{bmatrix} \frac{1}{x_{i}} & 0 \\ \vdots & s_{i} \end{bmatrix}, s_{i})
S_{2} = \begin{bmatrix} x_{i} & -1 \\ 0 & \frac{1}{x_{i}} \end{bmatrix}, s_{i} = \lambda(x_{i}) s_{i} \qquad \frac{\lambda = Z_{i} \begin{bmatrix} \frac{1}{x_{i}} & 0 \\ 0 & \frac{1}{x_{i}} \end{bmatrix}}{\lambda + 2} \qquad \chi_{1} s_{i}
1 \times G(s^{1}) \qquad (1 \times (s^{1})^{2}) \qquad \chi_{2} s_{i} \qquad \chi_{3} s_{i}
                                     \mathsf{K}^{\mathsf{G}}(\mathbb{P}') \longrightarrow \mathsf{K}(\mathbb{P}')
                                                       Z → O((-1)
                   Step 3. See the G-action on l.b.

A' \longrightarrow G
(g(x_1), \lambda(g(g,x))s_1)
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In conclusion, we get

$$\begin{array}{cccc} K(\mathbb{b}_{i}) & \longrightarrow & K(\mathbb{b}_{f}) \\ & & & & \downarrow \\ K_{\mathbb{B}}(\mathbb{b}_{i}) & \longrightarrow & K_{\mathbb{B}}(\mathbb{b}_{f}) \\ & & & & \uparrow \\ & & & & \downarrow \\ K_{\mathbb{B}^{r} \times \mathbb{C}_{x}}(\mathbb{b}_{i}) & \longrightarrow & K_{\mathbb{B}^{r} \times \mathbb{C}_{x}}(\mathbb{b}_{f}) \\ & & & & \downarrow \\ & & & & \downarrow \end{array}$$



The difficult part is the middle square. Z[z,y*1]/(z-1) - Z[y*]

Right: by rep theory,

$$Z[x] \longrightarrow Z[y^{\pm 1}] \qquad homo \quad as \quad Z-alg$$

$$x_0 \longmapsto 1$$

$$x_1 \longmapsto y^2 + 1 + y^{-2}$$

$$x_3 \longmapsto y^3 + y + y^{-1} + y^{-3}$$

Up by Borel-Weil-Bott theorem.

$$Z \begin{bmatrix} z^{\pm 1} \end{bmatrix} \xrightarrow{} Z \begin{bmatrix} x \end{bmatrix}$$

$$Z \xrightarrow{} Z \xrightarrow{}$$

Left: by [LW-BWB, Ex 2.6],
$$L_n \supseteq O(-n)$$
, combined with "Up", we get

 $\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2$

e.g. $z^3 \longmapsto -z^3(y+y^{-1})$ (see table below)

 $z \mapsto z^{-1} \quad z^{-1} \quad 1 \quad z \quad z^{2} \quad z^{3} \quad z^{4} \quad \sum_{x=0}^{x} \sum_{x=$

Under these (natural) ring structure,
$$\mathbb{Z}[x,t^{\pm i}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}] \longrightarrow \mathbb{Z}$$
 are homo of rings.

Ex. Generalize to
$$SL_1 \longrightarrow SL_n$$
, $P' \longrightarrow Flag(C')$
 $SL_2 \longrightarrow GL_2$
 $C \longrightarrow FP$ $C^* \longrightarrow FP$
 $Q:$ How to compute $K_i^{SL_1 \times C^*}(P')$ for $i \ge 1$?