Eine Woche, ein Beispiel

2.6. six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

A preparation of exams.

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F}\ \mathcal{F}' \\ | & & \backslash\ / \\ Y & \xrightarrow{f} & X \end{array}
\qquad\qquad
\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Goal: $f^* \dashv f_*$

$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$

$f_! \dashv f^!$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

$\otimes$

f^* —— $f_!$

bc: $f^* g_! \cong g'_! f'^{*}$

$f_* g^! \cong g'^! f'_*$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^! \mathcal{F}) \cong \underline{Hom}(f_! \mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

f^*: stalk $\qquad f_* / f_!$: cohomology

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \longrightarrow pt$

$H^{\cdot}(X; \mathbb{Z}) := p_* p^* \mathbb{1}$ $\qquad H^{\cdot}(X; \mathcal{F}) := p_* \mathcal{F}$ $\qquad = R\Gamma(X; \mathcal{F})$

$H^{\cdot}_c(X; \mathbb{Z}) := p_! p^* \mathbb{1}$ $\qquad H^{\cdot}_c(X; \mathcal{F}) := p_! \mathcal{F}$

$H_{\cdot}(X; \mathbb{Z}) := p_! p^! \mathbb{1}$ $\qquad H_{\cdot}(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad = H^{\cdot}_c(X; p^! \mathbb{1} \otimes \mathcal{F})$

$H^{BM}_{\cdot}(X; \mathbb{Z}) := p_* p^! \mathbb{1}$ $\qquad H^{BM}_{\cdot}(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad = H^{\cdot}(X; p^! \mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$H_c(X; \mathcal{F}) \otimes H_c(Y; \mathcal{G}) \cong H_c(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

reduced to: $p_{X,!}\mathcal{F} \otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\qquad -^{\vee} = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^{\cdot}(X; \mathbb{Z})[d] \cong H^{\cdot}(X; \mathbb{Z})^{\vee}$$

reduced to: $p_* \underline{Hom}(A, p^*B \otimes p^! \mathbb{1}) \cong \underline{Hom}(p_! A, B)$

Upgrade: ∞-categories & sym monoidal structure

Idea: $\mathcal{D}_0 : \mathcal{C}^{op} \longrightarrow Cat_\infty$

$$\begin{array}{ccc} X & \longmapsto & D(X) \\ f\downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & D(Y) \end{array}$$

e.g. $X :=$ nice top space, $D(X) :=$ derived category of abelian sheaves over X.

extends to

compatability is encoded!

$$\mathcal{D} : Corr(C, E) \longrightarrow Mon(Cat_\infty)$$

$$[Y \xleftarrow{f} X = X] \longmapsto f^*$$

$$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$$

$$[X \times X \xleftarrow{\Delta} X = X] \longmapsto \otimes$$

Moreover, It factor through

$$Corr(C, E) \longrightarrow LZ_{\mathcal{D}} \longrightarrow Mon(Cat_\infty)$$

Obj: $X \longmapsto X \longmapsto \mathcal{D}(X)$

Mor: $\left[X \xleftarrow{f} Y \xrightarrow{g} Z \right] \longmapsto (f,g)_* \mathbb{1}_Y \in \mathcal{D}(X \times Z)$ kernel $\longmapsto$ FM-transformation

composition = convolution

2-Mor: $\mathcal{E} \to \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$

$$\left[\begin{tikzcd}[column sep=small] & & Z \arrow[dl] \arrow[dr] & & \\ & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \\ & & \ \arrow[d, "F", "\cong"'] & & \\ & & Z \arrow[dll] \arrow[drr] & & \\ X_1 & & & & X_3 \end{tikzcd}\right] \longmapsto \left[\begin{tikzcd} \mathcal{E}_{12} * \mathcal{E}_{23} \arrow[dd, "\cong"'] \\ \\ \mathcal{E}_{13} \end{tikzcd}\right]$$

Goal: framework of ∞-category & $\otimes$ $\rightsquigarrow$ $Corr(C, E)$ & $Corr(C, E)^{\otimes}$

	f^*	$f_!$	$\otimes$	$\mathcal{F}\otimes -$

Correspondence

$$\begin{tikzcd}[column sep=small] & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X & & Y \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ X\times X & & X \end{tikzcd}\qquad
\boxed{\begin{tikzcd}[column sep=small] & \mathcal{F} \arrow[d, dash] & \\ & X \arrow[dl, equal] \arrow[dr, equal] & \\ X & & X \end{tikzcd}}$$

Not Correspondence!

FM trans

$$\begin{tikzcd}[column sep=small] Y \arrow[dr, "{(f,\mathrm{Id})}"] & & \\ & X\times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] Y \arrow[dr, "{(\mathrm{Id},f)}"] & & \\ & Y\times X \arrow[dl] \arrow[dr] & \\ Y & & X \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] X \arrow[dr, "\Delta_3"] & & \\ & X\times X\times X \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_3"] & \\ X\times X & & X \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] \mathcal{F} \arrow[d, dash] & & \Delta_*\mathcal{F} \arrow[dl, dash] \\ X \arrow[r, "\Delta"] & X\times X \arrow[dl] \arrow[dr] & \\ X & & X \end{tikzcd}$$

X final

$$\begin{tikzcd}[column sep=small] & 1_Y \arrow[d, dash] & \\ & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X & & Y \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] & 1_Y \arrow[d, dash] & \\ & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] & \Delta_{3,*}1_Y \arrow[d, dash] & \\ & Y\times Y\times Y \arrow[dl, "\pi_{12}"'] \arrow[dr, "\pi_3"] & \\ Y\times Y & & Y \end{tikzcd}\qquad
\begin{tikzcd}[column sep=small] & \Delta_*\mathcal{F} \arrow[d, dash] & \\ & Y\times Y \arrow[dl] \arrow[dr] & \\ Y & & Y \end{tikzcd}$$

Ex. Explain base change, projection formula and $f^*(-\otimes -)$.

A:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}\qquad f^*g_! \cong g'_! f'^{,*} : \mathcal{D}(X') \longrightarrow \mathcal{D}(Y)$$

$$\begin{tikzcd}[column sep=small] & & Y' \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\ & X' \arrow[dl, equal] \arrow[dr, "g"] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X' & & X & & Y \end{tikzcd}
\quad\cong\quad
\begin{tikzcd}[column sep=small] & & Y' \arrow[dl, equal] \arrow[dr, equal] & & \\ & Y' \arrow[dl, "f'"'] \arrow[dr, equal] & & Y' \arrow[dl, equal] \arrow[dr, "g'"] & \\ X' & & Y' & & Y \end{tikzcd}$$

$$(*, X') \xrightarrow{g_!} (*, X) \xrightarrow{f^*} (*, Y) \qquad\qquad (*, X') \xrightarrow{f'^{,*}} (*, Y') \xrightarrow{g'_!} (*, Y)$$

$$\begin{tikzcd}[column sep=small] \mathcal{F} \arrow[dr, dash] & & \mathcal{G} \arrow[dl, dash] \\ Y \arrow[r, "f"] & X & \end{tikzcd}\qquad f^*(\mathcal{F}\otimes\mathcal{G}) \simeq f^*\mathcal{F}\otimes f^*\mathcal{G} : \mathcal{D}(X)\otimes\mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$$

$$\begin{tikzcd}[column sep=small] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & & \\ & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X\times X & & X & & Y \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & \\ & Y\times Y \arrow[dl, "{(f,f)}"'] \arrow[dr, equal] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ X\times X & & Y\times Y & & Y \end{tikzcd}$$

$$(\{1,2\}, (X,X)) \xrightarrow{\Delta^*} (*, X) \xrightarrow{f^*} (*, Y) \qquad (\{1,2\}, (X,X)) \xrightarrow{(f,f)^*} (\{1,2\}, (Y,Y)) \xrightarrow{\Delta^*} (*, Y)$$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & \mathcal{G} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f_!(\mathcal{F}\otimes f^*\mathcal{G}) \cong f_!\mathcal{F}\otimes\mathcal{G} : \mathcal{D}(Y)\times\mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

$$\begin{tikzcd}[column sep=small]
 & & & Y \arrow[dl, equal] \arrow[dr, equal] & & & \\
 & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, equal] \arrow[dr, equal] & & \\
 & Y\times Y \arrow[dl, "(Id,f)"'] \arrow[dr, equal] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, equal] \arrow[dr, "f"] & \\
Y\times X & & Y\times Y & & Y & & X
\end{tikzcd}$$

$$(\{1,2\},(Y,X)) \xrightarrow{(Id,f)^*} (\{1,2\},(Y,Y)) \xrightarrow{\Delta^*} (*,Y) \xrightarrow{f_!} (*,X)$$

$$\begin{tikzcd}[column sep=small]
 & & Y \arrow[dl, "(Id,f)"'] \arrow[dr, "f"] & & \\
 & Y\times X \arrow[dl, equal] \arrow[dr, "(f,Id)"] & & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\
Y\times X & & X\times X & & X
\end{tikzcd}$$

$$(\{1,2\},(Y,X)) \xrightarrow{(f,Id)_!} (\{1,2\},(X,X)) \xrightarrow{\Delta^*} (*,X)$$

∞- category

$$\Delta \hookrightarrow \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \overset{\Delta}{=} \mathrm{sSet} \supseteq \mathrm{Cat}_\infty \supseteq \mathrm{An}$$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "h"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, red, "\exists K_{n,i}(h)"'] & \end{tikzcd}$$

$\forall\, 0 < i < n$ in Cat_∞

$\forall\, 0 \le i \le n$ in An

Notation

Set	$(0,0)$ - category	set
Cat	$(1,1)$ - category	category
An	$(\infty, 0)$ - category	anima / Kan cplx / ∞ - groupoid
Cat_∞	$(\infty, 1)$ - category	

Ex. Realize Corr (C, E) as an ∞ - category.

Monoidal structure

In $(1,1)$- category.

Monoidal structure on $\mathcal{C}$:

$$m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} \qquad u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C}$$

$$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G} \qquad * \longmapsto 1_{\mathcal{C}}$$

Monoidal object in $(\mathcal{C}, \otimes)$. $X \in \mathrm{Ob}(\mathcal{C})$ with

$$m_X: X \times X \longrightarrow X \qquad u_X: 1_{\mathcal{C}} \to X$$

In $(\infty, 1)$ - category:

$$(C, \otimes) \overset{\text{def}}{\rightsquigarrow} \left[\begin{array}{c} \mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto \mathbb{X}(I) \\ \text{comm monoid} \end{array}\right] \overset{\text{"straightening"}}{\leftrightsquigarrow} \left[\begin{array}{c} \pi^{\otimes}: Y^{\otimes} \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ Y^{\otimes}_I \xrightarrow{\sim} \prod_i Y^{\otimes}_i \end{array}\right]$$

$\rightsquigarrow$ See next page for details

where $\mathrm{Ob}(\mathrm{Fin}^{\mathrm{part}}) = \mathrm{Ob}(\mathrm{Fin})$

$\mathrm{Mor}_{\mathrm{Fin}^{\mathrm{part}}}(I, J) = \{\alpha: I \dashrightarrow J\}$

commutative monoid: $\mathbb{X}(I) \xrightarrow{\sim} \prod_i \mathbb{X}(i)$

$$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow (\mathcal{F}, \mathcal{G}) \qquad |I| = 2$$

coCartesian fibration: see [Def 3.5]

$$\begin{bmatrix} (\mathcal{C},\otimes) \\ \text{monoidal} \\ \infty\text{-cat} \end{bmatrix} \overset{\text{def}}{\longleftrightarrow} \begin{bmatrix} \mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto \mathbb{X}(I) \\ \text{comm monoid} \end{bmatrix} \overset{\text{"straightening"}}{\longleftrightarrow} \begin{bmatrix} \pi^\otimes: \mathbb{Y}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ \mathbb{Y}^\otimes_I \xrightarrow{\sim} \prod_i \mathbb{Y}^\otimes_i \end{bmatrix}$$

$$(\mathcal{C},\otimes) \longmapsto \mathcal{C}^{(-)}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty$$

$$\begin{tikzcd} I \arrow[r, mapsto] \arrow[d, dashed, "\alpha"] & \mathcal{C}^I := \prod_{i\in I} \mathcal{C} \arrow[d] \\ J \arrow[r, phantom, "\Rightarrow"] & \mathcal{C}^J \end{tikzcd} \qquad \begin{tikzcd} (X_i)_{i\in I} \arrow[d, mapsto] \\ \left(\bigotimes_{i\in\alpha^{-1}(j)} X_i\right)_{j\in J} \end{tikzcd}$$

$$\longmapsto \quad \pi^\otimes: \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}}$$

$$\mathcal{C}^\otimes = \{(I,(X_i)_i) \mid I\in\mathrm{Fin}^{\mathrm{part}}, X_i\in\mathcal{C}\} \qquad (I,(X_i)_i) \longmapsto I$$

$$\mathrm{Mor}_{\mathcal{C}^\otimes}((I,X_i),(J,Y_j)) = \left\{\alpha: I \dashrightarrow J,\ \left\{\bigotimes_{i\in\alpha^{-1}(j)} X_i \longrightarrow Y_j\right\}_j\right\}$$

$$(\mathbb{X}(*), \mathbb{X}(\{1,2\}\to\{*\})) \longleftarrow\!\shortmid\ \mathbb{X} \longmapsto \mathbb{X}^\otimes = \{(I,A) \mid I\in\mathrm{Fin}^{\mathrm{part}}, A\in\mathbb{X}(I)\}$$

$$\mathrm{Mor}((I,A),(J,B)) = \left\{\begin{array}{l} \alpha: I \dashrightarrow J \\ f: (\mathbb{X}(\alpha))A \longrightarrow B \quad \text{in } \mathbb{X}(J) \end{array}\right\}$$

$$\mathbb{Y}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \longleftarrow\!\shortmid\ \mathbb{Y}^\otimes$$

$$\begin{tikzcd} I \arrow[r, mapsto] \arrow[d, dashed, "\alpha"] & \mathbb{Y}^I := \mathbb{Y}^\otimes_I \arrow[d, "\text{loc. coCartesian lift}"] \\ J \arrow[r, phantom, "\Rightarrow"] & \mathbb{Y}^J \end{tikzcd}$$

$$(\pi^{\otimes^{-1}}(*), \text{loc. coCartesian lift of } \{1,2\}\to\{*\}) \longleftarrow\!\shortmid\ \mathbb{Y}^\otimes$$

$\mathbb{X}$ encodes the monoidal structure

$$\begin{tikzcd}[column sep=small] & \{12,3\} \arrow[dr] & \\ \{1,2,3\} \arrow[ur] \arrow[dr] & & * \\ & \{1,23\} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathcal{C}\times\mathcal{C} \arrow[dr] & \\ \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[ur] \arrow[dr] & & \mathcal{C} \\ & \mathcal{C}\times\mathcal{C} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (X\otimes Y, Z) \arrow[r, mapsto] & (X\otimes Y)\otimes Z \arrow[d, phantom, "\wr\wr"] \\ (X,Y,Z) \arrow[ur, mapsto] \arrow[dr, mapsto] & & X\otimes(Y\otimes Z) \\ & (X, Y\otimes Z) \arrow[ur, mapsto] & \end{tikzcd}$$

$$\begin{tikzcd} \{1,2\} \arrow[d] \arrow[dr] & \\ \{1,2\} \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}\times\mathcal{C} \arrow[d] \arrow[dr] & \\ \mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} (X,Y) \arrow[d, mapsto] \arrow[r, mapsto] & X\otimes Y \arrow[d, phantom, "\wr\wr"] \\ (Y,X) \arrow[r, mapsto] & Y\otimes X \end{tikzcd}$$

$$\begin{tikzcd} \{1\} \arrow[d] \arrow[dr] & \\ \{1,2\} \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C} \arrow[d] \arrow[dr] & \\ \mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, mapsto] \arrow[r, mapsto] & X \arrow[d, phantom, "\wr\wr"] \\ (X, 1_{\mathcal{C}}) \arrow[r, mapsto] & X\otimes 1_{\mathcal{C}} \end{tikzcd}$$

$I \xrightarrow{\alpha} J$

$$\begin{tikzcd} 1 \arrow[r] & 1 \\ 2 \arrow[ur] & \end{tikzcd}$$

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

2 $u_{\mathcal{C}} : 1 \to \mathcal{C}$.

3. $\mathcal{C} \to 1$ no information

Fctor: (lax) sym monoidal fctors

Special case: $[F : (\mathcal{C}, \otimes) \longrightarrow (\mathcal{D}, \times)] \longleftrightarrow \left[\begin{array}{l} F : \mathcal{C}^{\otimes} \longrightarrow \mathcal{D} \\ \quad \text{with conditions} \end{array}\right]$

Ex. Realize $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, and show $f^*(- \otimes -)$, bc & proj formula.

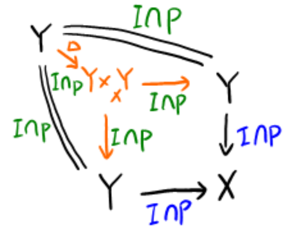
Why is $f_! : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ $\mathcal{D}(Y)$-linear?

Category	Object	Morphism
$\mathcal{C}$	$X \quad Y$	$X \longrightarrow Y$
$\mathcal{C}^{op}$	$X \quad Y$	$X \leftarrow Y$ in $\mathcal{C}$
$(\mathcal{C}^{op})^{\sqcup}$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha : I \dashrightarrow J$, $\{\bigotimes^{op}_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j\}_j$ in $\mathcal{C}^{op}$ $=$ $\{\prod_{i \in \alpha^{-1}(j)} X_i \longleftarrow Y_j\}_j$ in $\mathcal{C}$
$((\mathcal{C}^{op})^{\sqcup})^{op}$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha : I \leftarrow J$, $\{\prod_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j\}_j$ in $\mathcal{C}$

Construction "Uniqueness of $f_!$"

Const 1. $f: Y \longrightarrow X$ $\quad f \in I \cap P \quad \Rightarrow f_! \cong f_*$

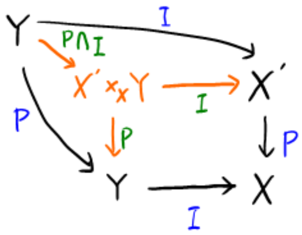
By induction,



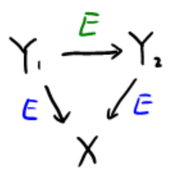
= Initial case

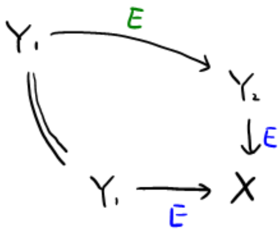
= Deduced case

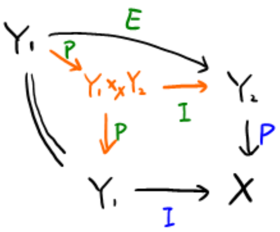
Const 2



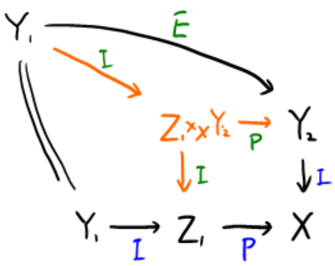
Const 3. E satisfies 2-out-of-3, i.e.



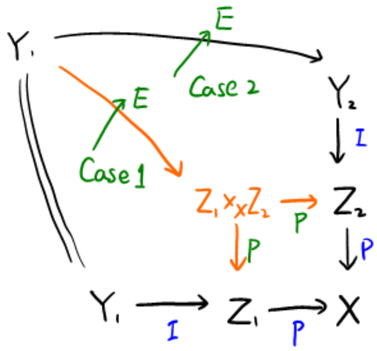




Case 1

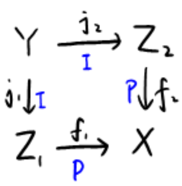


Case 2

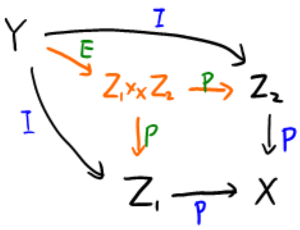


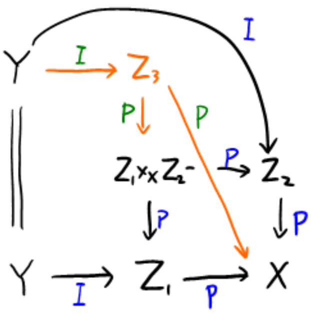
Case 3

Const 4.



want: $f_{1,*} j_{1,!} \cong f_{2,*} j_{2,!}$





Construction

$$
\begin{aligned}
\mathrm{Corr}(C,E) &= \delta_+^* \mathrm{Corr}(C,E)_{ct,co} \\
&\overset{\mathrm{Fun}(-,E)}{\simeq} \delta^* \mathrm{Corr}(C,E)_{ct,co} \\
&\simeq \delta^* \mathrm{Corr}(C,I,P)_{ct,co,co} \\
&\longleftarrow \delta^* \mathrm{Corr}(C,I,P)_{ct,ct,ct}
\end{aligned}
$$

$$
\begin{tikzcd}
\Delta^n \times \Delta^n \times \Delta^n \arrow[r] & \mathrm{Cat}_\infty \\
\Delta^n \times \Delta^n \arrow[r] \arrow[dr] & \mathrm{Fun}(\Delta^n, \mathrm{Cat}_\infty) \\
& \mathrm{Fun}^L(\Delta^n, \mathrm{Cat}_\infty) \arrow[u, phantom, "\cup" description] \arrow[d] \\
& \mathrm{Fun}((\Delta^n)^{op}, \mathrm{Cat}_\infty)
\end{tikzcd}
$$

f-smooth (= f-admissible) $\qquad\qquad\qquad f : Y \longrightarrow X$

$$\begin{tikzcd}[column sep=small] & A \arrow[d, dash] & \\ & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & B \arrow[d, dash] & \\ & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X & & Y \end{tikzcd}$$

$$F : f_!(A \otimes -) \quad \dashv \quad G : B \otimes f^* -$$
$$\dashv \quad Hom(A, f^! -)$$

① $\quad B \otimes f^* - \cong Hom(A, f^! -)$

App 1. $\Delta_! 1_Y$ cpt $\Rightarrow$ A cpt

[Proof. $Hom(\Delta_! 1_Y, B \otimes f^* -) \cong Hom(A, -)$ preserves filtered colimit.]

② $\qquad B \cong Hom(A, f^! 1_X)$

$p_2^* B \otimes p_1^* - \cong Hom(p_2^* A, p_1^! -)$ $\qquad$ [Verdier's diagonal trick]

Prop. A is f-smooth $\Leftrightarrow$ $p_2^* B \otimes p_1^* A \cong Hom(p_2^* A, p_1^! A)$ $\quad$ (2b)

where $B \cong Hom(A, f^! 1_X)$ $\quad$ (2a)

$\Rightarrow$: ✓

$\Leftarrow$: Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = Id$,

A is f-smooth $\Leftrightarrow$ $Hom(A, 1_X) \otimes A \cong Hom(A, A)$

$\Leftrightarrow$ A is dualizable

App 3. When $A = 1_Y$, $B = f^! 1_X$,

1_Y is f-smooth $\Leftrightarrow$ $p_2^* f^! 1_X \cong p_1^! 1_Y$

$\xLeftrightarrow{+ f^! 1_X \text{ inv}}$ f is coh smooth

Using this, one can prove results on coh étale.

Write $B = \mathbb{D}_f(A)$. we get $\mathbb{D}_f(\mathbb{D}_f(A)) \cong A$. $\quad$ (adjunction is symmetric in A & B).

$B = SD_f(A)$ $\qquad w_f := SD_f(1_Y)$

smooth dual

f-proper (f-coadmissible) $f: Y \longrightarrow X$

$$\begin{tikzcd}[column sep=small] & A \arrow[d, dash] & & & & B \arrow[d, dash] & \\ & Y \arrow[dl, "f"'] \arrow[dr, equal] & & & & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y & & Y & & X \end{tikzcd}$$

$$F: A \otimes f^* - \quad \dashv \quad G: f_!(B \otimes -)$$
$$\dashv \quad f_* Hom(A, -)$$

① $f_!(B \otimes -) \cong f_* Hom(A, -)$

App 1. 1_X cpt $\Rightarrow$ A cpt

[Proof. $Hom(1_X, f_!(B \otimes -)) \cong Hom(A, -)$ preserves filtered colimit.]

② $\quad p_{1,!}(p_2^* B \otimes -) \cong p_{1,*} Hom(p_2^* A, -)$ [Verdier's diagonal trick]

$\quad B \cong p_{1,*} Hom(p_2^* A, \Delta_! 1_Y)$

Prop. A is f-proper $\iff$ $f_!(B \otimes A) \cong f_* Hom(A, A)$ ②b

where $B \cong p_{1,*} Hom(p_2^* A, \Delta_! 1_Y)$ ②a

$\Rightarrow$: ✓

$\Leftarrow$: Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = Id$,

A is f-proper $\iff$ $Hom(A, 1_X) \otimes A \cong Hom(A, A)$

$\iff$ A is dualizable

App 3. When $A = 1_Y$, $B = p_{1,*} \Delta_! 1_Y$

1_Y is f-proper $\iff$ $f_! p_{1,*} \Delta_! 1_Y \cong f_* 1_Y$

Using this, one can prove results on coh proper.

Write $B = \mathbb{D}_f^{pro}(A)$. we get $\mathbb{D}_f^{pro}(\mathbb{D}_f^{pro}(A)) \cong A$. (adjunction is symmetric in A & B).

$B = P\mathbb{D}_f(A) \qquad \omega_f := P\mathbb{D}_f(1_Y)$

↑ proper dual

When $\Delta_! = \Delta_*$, $\mathbb{D}_f^{pro} = Hom(-, 1_Y)$ is the naive dual.

Relations

open immersion $\longrightarrow$ coh smooth $\underset{\text{if } f^! 1_X \text{ inv}}{\rightleftarrows}$ 1_Y is f-sm $\overset{\text{if } \Delta \text{ coh étale, } f \text{ is n-truncated}}{\rightleftarrows}$ coh étale

proper $\xrightarrow{\text{if } \Delta_! = \Delta_*}$ 1_Y is f-proper $\overset{\text{if } \Delta \text{ coh proper, } f \text{ is n-truncated}}{\rightleftarrows}$ coh proper