Eine Woche, ein Beispiel

2.6. six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

A preparation of exams.

$$\mathcal{G} \quad \mathcal{F}\ \mathcal{F}' \qquad Y \xrightarrow{f} X \qquad\qquad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

Goal:
$$f^* \dashv f_*$$
$$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$f^*(-\otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

$\otimes$

f^* —— $f_!$

bc: $f^* g_! \cong g'_! f'^{*}$

$f_* g^! \cong g'^! f'_*$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_!\mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

f^*: stalk $\qquad f_* / f_!$: cohomology

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \longrightarrow pt$

$H^\cdot(X;\mathbb{Z}) := p_* p^* \mathbb{1}$	$H^\cdot(X;\mathcal{F}) := p_*\mathcal{F}$	$= R\Gamma(X;\mathcal{F})$
$H_c^\cdot(X;\mathbb{Z}) := p_! p^* \mathbb{1}$	$H_c^\cdot(X;\mathcal{F}) := p_!\mathcal{F}$	
$H_\cdot(X;\mathbb{Z}) := p_! p^! \mathbb{1}$	$H_\cdot(X;\mathcal{F}) := p_!(p^!\mathbb{1} \otimes \mathcal{F})$	$= H_c^\cdot(X; p^!\mathbb{1} \otimes \mathcal{F})$
$H_\cdot^{BM}(X;\mathbb{Z}) := p_* p^! \mathbb{1}$	$H_\cdot^{BM}(X;\mathcal{F}) := p_*(p^!\mathbb{1} \otimes \mathcal{F})$	$= H^\cdot(X; p^!\mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$H_c(X;\mathcal{F}) \otimes H_c(Y;\mathcal{G}) \cong H_c^\cdot(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ {\scriptstyle p_1}\downarrow & \searrow^{p} & \downarrow{\scriptstyle p_Y} \\ X & \xrightarrow{p_X} & * \end{array}$$

reduced to: $p_{X,!}\mathcal{F} \otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\qquad -^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X;\mathbb{Z})[d] \cong H^\cdot(X;\mathbb{Z})^\vee$$

reduced to: $p_* \underline{Hom}(A, p^*B \otimes p^!\mathbb{1}) \cong \underline{Hom}(p_!A, B)$

Upgrade: ∞-categories & sym monoidal structure

Idea: $\mathcal{D}_0 : \mathcal{C}^{op} \longrightarrow Cat_\infty$

$$\begin{array}{ccc} X & \longmapsto & D(X) \\ f\downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & D(Y) \end{array}$$

e.g. X := nice top space,
$D(X)$:= derived category of abelian sheaves over X.

extends to compatability is encoded!

$\mathcal{D} : Corr(C, E) \longrightarrow Mon(Cat_\infty)$

$[Y \xleftarrow{f} X = X] \longmapsto f^*$

$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$

$[X \times X \xleftarrow{\Delta} X = X] \longmapsto \otimes$

Moreover, It factor through

$$\begin{array}{lccccc} & Corr(C,E) & \longrightarrow & LZ_{\mathcal{D}} & \longrightarrow & Mon(Cat_\infty) \\ \text{Obj:} & X & \longmapsto & X & \longmapsto & \mathcal{D}(X) \end{array}$$

Mor: $\left[\begin{tikzcd}[column sep=small] & Y \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Z \end{tikzcd}\right] \longmapsto$ kernel $\longmapsto$ FM-transformation

$(f,g)_* 1_Y \in \mathcal{D}(X \times Z)$

composition = convolution

2-Mor: $\mathcal{E} \to \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$

$$\left[\begin{tikzcd}[column sep=tiny] & & Z \arrow[dl] \arrow[dr] & & \\ & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \\ & & {} \arrow[d, "F", "\cong"'] & & \\ & & Z \arrow[dll] \arrow[drr] & & \\ X_1 & & & & X_3 \end{tikzcd}\right] \longmapsto \left[\begin{tikzcd} \mathcal{E}_{12} * \mathcal{E}_{23} \arrow[d, "\cong"'] \\ \mathcal{E}_{13} \end{tikzcd}\right]$$

Goal: framework of ∞-category & $\otimes$ $\rightsquigarrow Corr(C,E)$ & $Corr(C,E)^{\otimes}$

	f^*	$f_!$	$\otimes$	$\mathcal{F}\otimes -$
Correspondence	$X \xleftarrow{f} Y = Y$	$Y = Y \xrightarrow{f} X$	$X\times X \xleftarrow{\Delta} X = X$	$\mathcal{F}$ on X: $X = X = X$ — Not Correspondence!
FM trans	$Y \xrightarrow{(f,\mathrm{Id})} X\times Y$; $X \leftarrow X\times Y \rightarrow Y$	$Y \xrightarrow{(\mathrm{Id},f)} Y\times X$; $Y \leftarrow Y\times X \rightarrow X$	$X \xrightarrow{\Delta_3} X\times X\times X$; $X\times X \xleftarrow{\pi_{12}} X\times X\times X \xrightarrow{\pi_3} X$	$\mathcal{F}$ on $X \xrightarrow{\Delta} X\times X$, $\Delta_*\mathcal{F}$; $X \leftarrow X\times X \rightarrow X$
X final	1_Y on Y; $X \xleftarrow{f} Y = Y$	1_Y on Y; $Y = Y \xrightarrow{f} X$	$\Delta_{3,*}1_Y$ on $Y\times Y\times Y$; $Y\times Y \xleftarrow{\pi_{12}} Y\times Y\times Y \xrightarrow{\pi_3} Y$	$\Delta_*\mathcal{F}$ on $Y\times Y$; $Y \leftarrow Y\times Y \rightarrow Y$

Ex. Explain base change, projection formula and Poincaré duality.

A:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner"] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^*g_! \cong g'_! f'^{,*} : \mathcal{D}(X') \longrightarrow \mathcal{D}(Y)$$

$$\begin{tikzcd}[column sep=small] & & Y' \arrow[dl, "f'"'] \arrow[dr, "g'"] & & \\ & X' \arrow[dl, equal] \arrow[dr, "g"] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X' & & X & & Y \end{tikzcd} \cong \begin{tikzcd}[column sep=small] & & Y' \arrow[dl, equal] \arrow[dr, equal] & & \\ & Y' \arrow[dl, "f'"'] \arrow[dr, equal] & & Y' \arrow[dl, equal] \arrow[dr, "g'"] & \\ X' & & Y' & & Y \end{tikzcd}$$

$$(*, X') \xrightarrow{g_!} (*, X) \xrightarrow{f^*} (*, Y) \qquad (*, X') \xrightarrow{f'^{,*}} (*, Y') \xrightarrow{g'_!} (*, Y)$$

$$\begin{tikzcd} \mathcal{F} \quad \mathcal{G} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^*(\mathcal{F}\otimes\mathcal{G}) \simeq f^*\mathcal{F}\otimes f^*\mathcal{G} : \mathcal{D}(X)\otimes\mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$$

$$\begin{tikzcd}[column sep=small] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & & \\ & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X\times X & & X & & Y \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & \\ & Y\times Y \arrow[dl, "(f,f)"'] \arrow[dr, equal] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\ X\times X & & Y\times Y & & Y \end{tikzcd}$$

$$(\{1,2\}, (X,X)) \xrightarrow{\Delta^*} (*, X) \xrightarrow{f^*} (*, Y) \qquad (\{1,2\}, (X,X)) \xrightarrow{(f,f)^*} (\{1,2\}, (Y,Y)) \xrightarrow{\Delta^*} (*, Y)$$

$$\begin{tikzcd}
\mathcal{F} \arrow[d, dash] & \mathcal{G} \arrow[d, dash] \\
Y \arrow[r, "f"] & X
\end{tikzcd}
\qquad
f_!(\mathcal{F}\otimes f^*\mathcal{G}) \cong f_!\mathcal{F}\otimes\mathcal{G} : \mathcal{D}(Y)\times\mathcal{D}(X)\longrightarrow\mathcal{D}(X)$$

$$\begin{tikzcd}[column sep=small]
 & & & Y \arrow[dl, equal] \arrow[dr, equal] & & & \\
 & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, equal] \arrow[dr, equal] & & \\
 & Y\times Y \arrow[dl, "{(Id,f)}"'] \arrow[dr, equal] & & Y \arrow[dl, "\Delta"'] \arrow[dr, equal] & & Y \arrow[dl, equal] \arrow[dr, "f"] & \\
Y\times X & & Y\times Y & & Y & & X
\end{tikzcd}$$

$$(\{1,2\},(Y,X)) \xrightarrow{(Id,f)^*} (\{1,2\},(Y,Y)) \xrightarrow{\Delta^*} (*,Y) \xrightarrow{f_!} (*,X)$$

$$\begin{tikzcd}[column sep=small]
 & & Y \arrow[dl, "{(Id,f)}"'] \arrow[dr, "f"] & & \\
 & Y\times X \arrow[dl, equal] \arrow[dr, "{(f,Id)}"] & & X \arrow[dl, "\Delta"'] \arrow[dr, equal] & \\
Y\times X & & X\times X & & X
\end{tikzcd}$$

$$(\{1,2\},(Y,X)) \xrightarrow{(f,Id)_!} (\{1,2\},(X,X)) \xrightarrow{\Delta^*} (*,X)$$

∞- category

$$\Delta \hookrightarrow \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \overset{\Delta}{=} \mathrm{sSet} \supseteq \mathrm{Cat}_\infty \supseteq \mathrm{An}$$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "h"] \arrow[d] & X \\ \Delta^n \arrow[ur, dashed, "\exists K_{n,i}(h)"'] & \end{tikzcd}$$

$\forall\, 0 < i < n$ in Cat_∞

$\forall\, 0 \le i \le n$ in An

Notation

Set	$(0,0)$ - category	set
Cat	$(1,1)$ - category	category
An	$(\infty, 0)$ - category	anima / Kan cplx / ∞ - groupoid
Cat_∞	$(\infty, 1)$ - category	

Ex. Realize $\mathrm{Corr}(C, E)$ as an ∞ - category.

Monoidal structure

In $(1,1)$- category.

Monoidal structure on $\mathcal{C}$:

$m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ $\qquad u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C}$

$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G}$ $\qquad * \longmapsto 1_{\mathcal{C}}$

Monoidal object in $(\mathcal{C}, \otimes)$. $X \in \mathrm{Ob}(\mathcal{C})$ with

$m_X: X \times X \longrightarrow X$ $\qquad u_X: 1_{\mathcal{C}} \to X$

In $(\infty, 1)$ - category:

$$(C, \otimes) \overset{\text{def}}{\rightsquigarrow} \left[\begin{array}{c} \mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto X(I) \\ \text{comm monoid} \end{array} \right] \overset{\text{“Straightening”}}{\leftrightsquigarrow} \left[\begin{array}{c} \pi^\otimes: Y^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ Y^\otimes_I \xrightarrow{\sim} \prod_i Y^\otimes_i \end{array} \right]$$

$\rightsquigarrow$ See next page for details

where $\mathrm{Ob}(\mathrm{Fin}^{\mathrm{part}}) = \mathrm{Ob}(\mathrm{Fin})$

$\mathrm{Mor}_{\mathrm{Fin}^{\mathrm{part}}}(I, J) = \{ \alpha : I \dashrightarrow J \}$

commutative monoid: $\mathbb{X}(I) \xrightarrow{\sim} \prod_i \mathbb{X}(i)$

$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow\!\shortmid (\mathcal{F}, \mathcal{G})$ $\qquad |I| = 2$

coCartesian fibration: see [Def 3.5]

$$\begin{bmatrix}(\mathcal{C},\otimes)\\ \text{monoidal}\\ \infty\text{-cat}\end{bmatrix} \overset{\text{def}}{\leftrightsquigarrow} \begin{bmatrix} \mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto \mathbb{X}(I) \\ \text{comm monoid}\end{bmatrix} \overset{\text{"straightening"}}{\leftrightsquigarrow} \begin{bmatrix}\pi^\otimes: \mathbb{Y}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ \mathbb{Y}^\otimes_I \xrightarrow{\sim} \prod_i \mathbb{Y}^\otimes_i\end{bmatrix}$$

$$(\mathcal{C},\otimes) \longmapsto \mathcal{C}^{(-)}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty$$

$$\begin{array}{ccccc} I & \longmapsto & \mathcal{C}^I := \prod_{i\in I}\mathcal{C} & \quad & (X_i)_{i\in I} \\ \downarrow{\scriptstyle \alpha} & \Rightarrow & \downarrow & & \downarrow \\ J & & \mathcal{C}^J & & \left(\bigotimes_{i\in\alpha^{-1}(j)} X_i\right)_{j\in J} \end{array}$$

$$\longmapsto \quad \pi^\otimes: \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}}$$

$$\mathcal{C}^\otimes = \{(I,(X_i)_i) \mid I\in\mathrm{Fin}^{\mathrm{part}}, X_i\in\mathcal{C}\} \qquad (I,(X_i)_i) \longmapsto I$$

$$\mathrm{Mor}_{\mathcal{C}^\otimes}((I,X_i),(J,Y_j)) = \left\{\alpha: I \dashrightarrow J,\ \left\{\bigotimes_{i\in\alpha^{-1}(j)} X_i \to Y_j\right\}_j\right\}$$

$$(\mathbb{X}(*), \mathbb{X}(\{1,2\}\to\{*\})) \longleftarrow\!\shortmid\ \mathbb{X} \longmapsto \mathbb{X}^\otimes = \{(I,A) \mid I\in\mathrm{Fin}^{\mathrm{part}}, A\in\mathbb{X}(I)\}$$

$$\mathrm{Mor}((I,A),(J,B)) = \left\{\begin{array}{l}\alpha: I \dashrightarrow J\\ f: (\mathbb{X}(\alpha))A \longrightarrow B \quad \text{in } \mathbb{X}(J)\end{array}\right\}$$

$$\mathbb{Y}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \longleftarrow\!\shortmid\ \mathbb{Y}^\otimes$$

$$\begin{array}{ccc} I & \longmapsto & \mathbb{Y}^I := \mathbb{Y}^\otimes_I \\ \downarrow{\scriptstyle \alpha} & \Rightarrow & \downarrow \text{ loc. coCartesian lift} \\ J & & \mathbb{Y}^J \end{array}$$

$$(\pi^{\otimes^{-1}}(*), \text{loc. coCartesian lift of } \{1,2\}\to\{*\}) \longleftarrow\!\shortmid\ \mathbb{Y}^\otimes$$

$\mathbb{X}$ encodes the monoidal structure

$$\begin{tikzcd}[column sep=small] & \{12,3\} \arrow[dr] & \\ \{1,2,3\} \arrow[ur] \arrow[dr] & & * \\ & \{1,23\} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \mathcal{C}\times\mathcal{C} \arrow[dr] & \\ \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[ur] \arrow[dr] & & \mathcal{C} \\ & \mathcal{C}\times\mathcal{C} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & (X\otimes Y, Z) \arrow[r, mapsto] & (X\otimes Y)\otimes Z \arrow[d, phantom, "\wr"] \\ (X,Y,Z) \arrow[ur, mapsto] \arrow[dr, mapsto] & & X\otimes(Y\otimes Z) \\ & (X, Y\otimes Z) \arrow[ur, mapsto] & \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \{1,2\} \arrow[d] \arrow[dr] & \\ \{1,2\} \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathcal{C}\times\mathcal{C} \arrow[d] \arrow[dr] & \\ \mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] (X,Y) \arrow[d] \arrow[r, mapsto] & X\otimes Y \arrow[d, phantom, "\wr"] \\ (Y,X) \arrow[r, mapsto] & Y\otimes X \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] \{1\} \arrow[d] \arrow[dr] & \\ \{1,2\} \arrow[r] & * \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] \mathcal{C} \arrow[d] \arrow[dr] & \\ \mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] X \arrow[d] \arrow[r, mapsto] & X \arrow[d, phantom, "\wr"] \\ (X, 1_{\mathcal{C}}) \arrow[r, mapsto] & X\otimes 1_{\mathcal{C}} \end{tikzcd}$$

$$I \xrightarrow{\alpha} J$$

$$\begin{matrix} 1 & \longrightarrow & 1 \\ 2 & \nearrow & \end{matrix}$$

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

(2) $u_{\mathcal{C}} : 1 \to \mathcal{C}$.

(3). $\mathcal{C} \to 1$ no information

Fctor: (lax) sym monoidal fctors

Special case: $[F : (\mathcal{C}, \otimes) \longrightarrow (D, \times)] \longleftrightarrow \left[\begin{matrix} F : \mathcal{C}^{\otimes} \longrightarrow D \\ \text{with conditions} \end{matrix}\right]$

Ex. Realize $\mathrm{Corr}(\mathbb{C}, E)^{\otimes}$, and show $f^*(- \otimes -)$, bc & proj formula.

Why is $f_! : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ $\mathcal{D}(Y)$-linear?

Category	Object	Morphism
$\mathcal{C}$	$X \quad Y$	$X \longrightarrow Y$
$\mathcal{C}^{op}$	$X \quad Y$	$X \leftarrow Y$ in $\mathcal{C}$
$(\mathcal{C}^{op})^{\sqcup}$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha : I \to J$, $\{\bigotimes^{op}_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j\}_j$ in $\mathcal{C}^{op}$ $\parallel$ $\{\prod_{i \in \alpha^{-1}(j)} X_i \longleftarrow Y_j\}_j$ in $\mathcal{C}$
$((\mathcal{C}^{op})^{\sqcup})^{op}$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha : I \leftarrow J$, $\{\prod_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j\}_j$ in $\mathcal{C}$

Construction "Uniqueness of $f_!$"

Const 1. $f: Y \longrightarrow X$ $\quad f \in I \cap P \quad \Rightarrow f_! \cong f_*$

By induction,

$$\begin{tikzcd} Y \arrow[dr, "\Delta"] \arrow[drr, bend left, equal, "I\cap P"] \arrow[ddr, bend right, equal, "I\cap P"'] & & \\ & Y \times_X Y \arrow[r, "I\cap P"] \arrow[d, "I\cap P"] & Y \arrow[d, "I\cap P"] \\ & Y \arrow[r, "I\cap P"'] & X \end{tikzcd}$$

= Initial case

= Deduced case

Const 2

$$\begin{tikzcd} Y \arrow[dr, "P\cap I"] \arrow[drr, bend left, "I"] \arrow[ddr, bend right, "P"'] & & \\ & X' \times_X Y \arrow[r, "I"] \arrow[d, "P"] & X' \arrow[d, "P"] \\ & Y \arrow[r, "I"'] & X \end{tikzcd}$$

Const 3. E satisfies 2-out-of-3, i.e.

$$\begin{tikzcd} Y_1 \arrow[r, "E"] \arrow[dr, "E"'] & Y_2 \arrow[d, "E"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[rr, bend left, "E"] \arrow[dr, equal] & & Y_2 \arrow[d, "E"] \\ & Y_1 \arrow[r, "E"'] & X \end{tikzcd}$$

$$\begin{tikzcd} Y_1 \arrow[dr, "P"] \arrow[drr, bend left, "E"] \arrow[ddr, bend right, equal] & & \\ & Y_1 \times_X Y_2 \arrow[r, "I"] \arrow[d, "P"] & Y_2 \arrow[d, "P"] \\ & Y_1 \arrow[r, "I"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[dr, "I"] \arrow[drr, bend left, "E"] \arrow[ddr, bend right, equal] & & & \\ & Z_1 \times_X Y_2 \arrow[r, "P"] \arrow[d, "I"] & Y_2 \arrow[d, "I"] & \\ Y_1 \arrow[r, "I"'] & Z_1 \arrow[r, "P"'] & X & \end{tikzcd} \qquad \begin{tikzcd} Y_1 \arrow[rr, bend left, "E" , "\text{Case 2}"'] \arrow[dr, "E", "\text{Case 1}"'] \arrow[ddr, bend right, equal] & & Y_2 \arrow[d, "I"] \\ & Z_1 \times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\ Y_1 \arrow[r, "I"'] & Z_1 \arrow[r, "P"'] & X \end{tikzcd}$$

Case 1 $\qquad$ Case 2 $\qquad$ Case 3

Const 4.

$$\begin{tikzcd} Y \arrow[r, "j_2", "I"'] \arrow[d, "j_1"', "I"] & Z_2 \arrow[d, "P"', "f_2"] \\ Z_1 \arrow[r, "f_1", "P"'] & X \end{tikzcd}$$

want: $f_{1,*}\, j_{1,!} \cong f_{2,*}\, j_{2,!}$

$$\begin{tikzcd} Y \arrow[dr, "E"] \arrow[drr, bend left, "I"] \arrow[ddr, bend right, "I"'] & & \\ & Z_1 \times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\ & Z_1 \arrow[r, "P"'] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "I"] \arrow[dd, equal] \arrow[drr, bend left, "I"] & Z_3 \arrow[d, "P"] \arrow[ddr, "P"] & \\ & Z_1 \times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\ Y \arrow[r, "I"'] & Z_1 \arrow[r, "P"'] & X \end{tikzcd}$$

Construction

$$
\begin{aligned}
\mathrm{Corr}(C,E) &= \delta_+^* \mathrm{Corr}(C,E)_{ct,co} \\
&\overset{\mathrm{Fun}(-,\mathcal{E})}{\sim} \delta^* \mathrm{Corr}(C,E)_{ct,co} \\
&\sim \delta^* \mathrm{Corr}(C,I,P)_{ct,co,co} \\
&\longleftarrow \delta^* \mathrm{Corr}(C,I,P)_{ct,ct,ct}
\end{aligned}
$$

$$
\begin{tikzcd}
\Delta^n \times \Delta^n \times \Delta^n \arrow[r] & \mathrm{Cat}_\infty \\
\Delta^n \times \Delta^n \arrow[r] \arrow[dr] & \mathrm{Fun}(\Delta^n, \mathrm{Cat}_\infty) \\
 & \mathrm{Fun}^L(\Delta^n, \mathrm{Cat}_\infty) \arrow[u, phantom, "\cup"] \arrow[d] \\
 & \mathrm{Fun}((\Delta^n)^{op}, \mathrm{Cat}_\infty)
\end{tikzcd}
$$

f-smooth ($= f$-admissible) $\qquad\qquad f: Y \longrightarrow X$

$$
\begin{tikzcd}
 & A \arrow[d, dash] & \\
 & Y \arrow[dl, equal] \arrow[dr, "f"] & \\
Y & & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & B \arrow[d, dash] & \\
 & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\
X & & Y
\end{tikzcd}
$$

$$F: f_!(A \otimes -) \dashv G: B \otimes f^* -$$
$$\dashv \mathrm{Hom}(A, f^! -)$$

① $\quad B \otimes f^* - \cong \mathrm{Hom}(A, f^! -)$

App 1. $\Delta_! 1_Y$ cpt $\Rightarrow$ A cpt

[Proof. $\mathrm{Hom}(\Delta_! 1_Y, B \otimes f^* -) \cong \mathrm{Hom}(A, -)$ preserves filtered colimit.]

② $\qquad B \cong \mathrm{Hom}(A, f^! 1_X)$

$p_2^* B \otimes p_1^* - \cong \mathrm{Hom}(p_2^* A, p_1^! -)$ $\qquad$ [Verdier's diagonal trick]

Prop. A is f-smooth $\Longleftrightarrow$ $p_2^* B \otimes p_1^* A \cong \mathrm{Hom}(p_2^* A, p_1^! A)$ $\quad$ (2b)

where $\quad B \cong \mathrm{Hom}(A, f^! 1_X)$ $\quad$ (2a)

$\Rightarrow$: ✓

$\Leftarrow$: Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = \mathrm{Id}$,

A is f-smooth $\Longleftrightarrow$ $\mathrm{Hom}(A, 1_X) \otimes A \cong \mathrm{Hom}(A, A)$

$\Longleftrightarrow$ A is dualizable

App 3. When $A = 1_Y$, $B = f^! 1_X$,

1_Y is f-smooth $\Longleftrightarrow$ $p_1^* f^! 1_X \cong p_1^! 1_Y$

$\xLeftrightarrow{+ f^! 1_X \text{ inv}}$ f is coh smooth

Using this, one can prove results on coh étale.

Write $B = \mathbb{D}_f(A)$. we get $\mathbb{D}_f(\mathbb{D}_f(A)) \cong A$. (adjunction is symmetric in A & B)

$B = SD_f(A) \qquad w_f := SD_f(1_Y)$

smooth dual

f-proper (f-coadmissible) $f: Y \to X$

$$\begin{tikzcd} & A \arrow[d, dash] & \\ & Y \arrow[dl, "f"'] \arrow[dr, equal] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & B \arrow[d, dash] & \\ & Y \arrow[dl, equal] \arrow[dr, "f"] & \\ Y & & X \end{tikzcd}$$

$F: A \otimes f^* - \quad \dashv \quad G: f_!(B \otimes -)$
$\qquad\qquad\qquad\quad \dashv \quad f_* Hom(A, -)$

① $f_!(B \otimes -) \simeq f_* Hom(A, -)$

App 1. 1_X cpt $\Rightarrow$ A cpt

[Proof. $Hom(1_X, f_!(B \otimes -)) \cong Hom(A, -)$ preserves filtered colimit.]

② $\quad p_{1,!}(p_2^* B \otimes -) \cong p_{1,*} Hom(p_2^* A, -)$ [Verdier's diagonal trick]

$\qquad B \cong p_{1,*} Hom(p_2^* A, \Delta_! 1_Y)$

Prop. A is f-proper $\Leftrightarrow$ $f_!(B \otimes A) \simeq f_* Hom(A, A)$ ②b

where $B \simeq p_{1,*} Hom(p_2^* A, \Delta_! 1_Y)$ ②a

$\Rightarrow$: ✓

$\Leftarrow$: Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = Id$,

A is f-proper $\Leftrightarrow$ $Hom(A, 1_X) \otimes A \simeq Hom(A, A)$

$\Leftrightarrow$ A is dualizable

App 3. When $A = 1_Y$, $B = p_{1,*} \Delta_! 1_Y$

1_Y is f-proper $\Leftrightarrow$ $f_! p_{1,*} \Delta_! 1_Y \simeq f_* 1_Y$

Using this, one can prove results on coh proper.

Write $B = \mathbb{D}_f^{pro}(A)$. we get $\mathbb{D}_f^{pro}(\mathbb{D}_f^{pro}(A)) \simeq A$. (adjunction is symmetric in A & B)

$B = P\mathbb{D}_f(A) \qquad \omega_f := P\mathbb{D}_f(1_Y)$

proper dual

When $\Delta_! = \Delta_*$, $\mathbb{D}_f^{pro} = Hom(-, 1_Y)$ is the naive dual.

Relations

open immersion $\longrightarrow$ coh smooth $\underset{\text{if } f^! 1_X \text{ inv}}{\rightleftarrows}$ 1_Y is f-sm $\underset{}{\overset{\text{if } \Delta \text{ coh étale, } f \text{ is n-truncated}}{\rightleftarrows}}$ coh étale

proper $\xrightarrow{\text{if } \Delta_! = \Delta_*}$ 1_Y is f-proper $\overset{\text{if } \Delta \text{ coh proper, } f \text{ is n-truncated}}{\rightleftarrows}$ coh proper