Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \quad \mathcal{F}' \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd}
\qquad
\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$f^* \dashv f_*$

$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$

$f_! \dashv f^!$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_!\mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

bc: $f^* g_! \cong g'_! f'^{*}$

$f_* g^! \cong g'^! f'_*$

These extra formulas (compatabilities) come from the upgrade of adjunction formula to internal Hom.
To upgrade the adjunction between tensor product and internal Hom, one don't need extra formula, except the association law of tensor product.

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \longrightarrow pt$

$H^{\cdot}(X; \underline{\mathbb{Z}}) := p_* p^* \mathbb{1}$ $\qquad$ $H^{\cdot}(X; \mathcal{F}) := p_* \mathcal{F}$ $\qquad$ $= R\Gamma(X; \mathcal{F})$

$H_c^{\cdot}(X; \underline{\mathbb{Z}}) := p_! p^* \mathbb{1}$ $\qquad$ $H_c^{\cdot}(X; \mathcal{F}) := p_! \mathcal{F}$

$H_{\cdot}(X; \underline{\mathbb{Z}}) := p_! p^! \mathbb{1}$ $\qquad$ $H_{\cdot}(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad$ $= H_c^{\cdot}(X; p^! \mathbb{1} \otimes \mathcal{F})$

$H_{\cdot}^{BM}(X; \underline{\mathbb{Z}}) := p_* p^! \mathbb{1}$ $\qquad$ $H_{\cdot}^{BM}(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad$ $= H^{\cdot}(X; p^! \mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

$$H_c^{\cdot}(X; \mathcal{F}) \otimes H_c^{\cdot}(Y; \mathcal{G}) \cong H_c^{\cdot}(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

reduced to: $p_{X,!}\mathcal{F} \otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$

App 2. (Poincaré duality)

X: a cpt $\underline{\text{oriented}}$ mfld of dim d, then $\qquad$ $-^{\vee} = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^{\cdot}(X; \underline{\mathbb{Z}})[d] \cong H^{\cdot}(X; \underline{\mathbb{Z}})^{\vee}$$

reduced to: $p_* \underline{Hom}(A, p^*B \otimes p^! \mathbb{1}) \cong \underline{Hom}(p_! A, B)$

$$Z \xrightarrow[\text{close}]{i} X \xhookleftarrow{j} U \quad \rightsquigarrow$$

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(Z) \arrow[loop above, "Id"] \arrow[loop below, "Id"] \arrow[r, "\text{pi}\ i_*=i_!\ \text{f.f}" description] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\text{pi}\ i^!"] \arrow[r, "\text{pi}\ j^*=j^!" description] & \mathcal{D}(\mathcal{U}) \arrow[loop above, "Id"] \arrow[loop below, "Id"] \arrow[l, bend right, "j_!\ \text{f.f.}"'] \arrow[l, bend left, "\text{pi}\ Rj_*\ \text{f.f}"]
\end{tikzcd}$$

$$\begin{tikzcd}
{} & {} \arrow[l, "i^* \quad j_!", "0"'] \\
{} \arrow[r, "i_* \quad j^*", "0"'] & {} \\
{} & {} \arrow[l, "i^! \quad Rj_*", "0"']
\end{tikzcd}$$

f.f. fully faithful

pi: preserve injectives. (内射)

ie. inj sheaf ⇝ inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Z} = Or_X[n] \xlongequal{+\text{orientation}} \mathbb{Z}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$\begin{tikzcd}
0 \arrow[r] & j_! j^! \mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & i_* i^* \mathcal{F} \arrow[r] & 0 \\
 & i_! i^! \mathcal{F} \arrow[r] & \mathcal{F} \arrow[r] & Rj_* j^* \mathcal{F} \arrow[r, "+1"] & {}
\end{tikzcd}$$

Here, $H_{-1}(S^1;\mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$\begin{tikzcd}
R\Gamma(X,Z;\mathcal{F}) \arrow[r] & R\Gamma(X;\mathcal{F}) \arrow[r] & R\Gamma(Z;\mathcal{F}|_Z) \arrow[r, "+1"] & {} \\
R\Gamma(X,\mathcal{U};\mathcal{F}) \arrow[r] \arrow[d, equal] & R\Gamma(X;\mathcal{F}) \arrow[r] & R\Gamma(\mathcal{U};\mathcal{F}|_{\mathcal{U}}) \arrow[r, "+1"] & {} \\
R\Gamma_Z(X;\mathcal{F}) & & &
\end{tikzcd}$$

When $\mathcal{F} = \underline{\mathbb{Q}}_X$,

$$\begin{tikzcd}
H^{\cdot}(X,Z) \arrow[r] & H^{\cdot}(X) \arrow[r] & H^{\cdot}(Z) \arrow[r, "+1"] & {} \\
H^{\cdot}(X,\mathcal{U}) \arrow[r] & H^{\cdot}(X) \arrow[r] & H^{\cdot}(\mathcal{U}) \arrow[r, "+1"] & {}
\end{tikzcd}$$

When $\mathcal{F} = \mathbb{D}_X$,

$$\begin{tikzcd}
R\Gamma(X, j_!\underline{\mathbb{Q}}_{\mathcal{U}}) \arrow[r] & H_{\cdot}^{BM}(X) \arrow[r] & R\Gamma(Z;\mathbb{D}_X|_Z) \arrow[r, "+1"] & {} \\
H_{\cdot}^{BM}(Z) \arrow[r] & H_{\cdot}^{BM}(X) \arrow[r] & H_{\cdot}^{BM}(\mathcal{U}) \arrow[r, "+1"] & {}
\end{tikzcd}$$

Taking $R\pi_{X,!}$

$$\begin{tikzcd}
R\Gamma_c(\mathcal{U},\mathcal{F}|_{\mathcal{U}}) \arrow[r] & R\Gamma_c(X;\mathcal{F}) \arrow[r] & R\Gamma_c(Z;\mathcal{F}|_Z) \arrow[r, "+1"] & {} \\
R\Gamma_c(Z, i^!\mathcal{F}) \arrow[r] & R\Gamma_c(X;\mathcal{F}) \arrow[r] & R\Gamma_c(X, Rj_*(\mathcal{F}|_{\mathcal{U}})) \arrow[r, "+1"] & {}
\end{tikzcd}$$

When $\mathcal{F} = \underline{\mathbb{Q}}_X$,

$$\begin{tikzcd}
H_c^{\cdot}(\mathcal{U}) \arrow[r] & H_c^{\cdot}(X) \arrow[r] & H_c^{\cdot}(Z) \arrow[r, "+1"] & {} \\
H_c^{\cdot}(Z, i^!\underline{\mathbb{Q}}_X) \arrow[r] & H_c^{\cdot}(X) \arrow[r] & H_c^{\cdot}(X, Rj_*\underline{\mathbb{Q}}_{\mathcal{U}}) \arrow[r, "+1"] & {}
\end{tikzcd}$$

When $\mathcal{F} = \mathbb{D}_X$,

$$\begin{tikzcd}
H_{\cdot}(\mathcal{U}) \arrow[r] & H_{\cdot}(X) \arrow[r] & H_{\cdot}(X,\mathcal{U}) \arrow[r, "+1"] & {} \\
H_{\cdot}(Z) \arrow[r] & H_{\cdot}(X) \arrow[r] & H_{\cdot}(X,Z) \arrow[r, "+1"] & {}
\end{tikzcd}$$

$$\begin{tikzcd} i^!\mathcal{F} \arrow[r] & i^*\mathcal{F} \arrow[r] & i^*Rj_*j^*\mathcal{F} \arrow[r, "+1"] & {} \end{tikzcd}$$

local cohomology compares the difference between stalks and costalks.

Application

One point compactification

$$\begin{tikzcd} X \arrow[rr, hook, "\iota"] \arrow[dr, "\pi"'] & & \overline{X} \arrow[dl, "\overline{\pi}"] \\ & \{*\} & \end{tikzcd}$$

$$\begin{aligned} H_c^{\cdot}(X) &= R\pi_! \pi^* \mathbb{Z} \\ &= R\overline{\pi}_! \iota_! \iota^* \pi^* \mathbb{Z} \\ &= R\overline{\pi}_* (\iota_! \iota^! \underline{\mathbb{Z}}_{\overline{X}}) \\ &= H^{\cdot}(\overline{X}, \{\infty\}; \mathbb{Z}) \end{aligned}$$

$$\begin{aligned} H_{\cdot}^{BM}(X) &= R\pi_* \pi^! \mathbb{Z} \\ &= R\overline{\pi}_* \iota_* \iota^! \overline{\pi}^! \mathbb{Z} \\ &= R\overline{\pi}_* (\iota_* \iota^* \overline{\pi}^! \mathbb{Z}) \\ &= \operatorname{cone} (R\overline{\pi}_* i_! i^! \overline{\pi}^! \mathbb{Z} \longrightarrow R\overline{\pi}_* \overline{\pi}^! \mathbb{Z}) \\ &= H_{\cdot}(\overline{X}, \{\infty\}; \mathbb{Z}) \end{aligned}$$

Originally, this is another def of cpt supp coh & BM homology.

Vector bundle with 6-functors.

Goal: Define Thom class & Euler class as in [GTM82, §6]

[GTM82]: Raoul Bott , Loring W. Tu, Differential Forms in Algebraic Topology, 1982
https://link.springer.com/book/10.1007/978-1-4757-3951-0

$$\begin{tikzcd}
H_c^{\cdot}(F) & H_{cv}^{\cdot}(E) \arrow[l] \arrow[d, "\cong"] \arrow[dr, bend left, "s^*"] & \\
& H^{\cdot-r}(B) & H^{\cdot}(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
\text{generator} \in H_c^r(F) & \Phi \in H_{cv}^r(E) \arrow[l, mapsto] \arrow[d, mapsto] \arrow[dr, bend left, mapsto] & \\
& 1 \in H^0(B) & eu_B \in H^r(B)
\end{tikzcd}$$

Setting

$\pi : E \longrightarrow B$ oriented v.b. with fiber $F \cong \mathbb{R}^r$

β_F : one point compactification $(\mathbb{R}^n \subset S^n)$

β_E : fiberwise compactification

$\pi_X : X \longrightarrow \{*\}$

$$\begin{tikzcd}
& \bar{F} \arrow[rr, "\bar{\iota}_F"] & & \bar{E} \arrow[dl, "\bar{\pi}"] \\
F \arrow[ur, hook, "\beta_F"] \arrow[rr, hook, "\iota_F"] & & E \arrow[ur, hook, "\beta_E"'] \arrow[d, "\pi"'] & \\
& & B &
\end{tikzcd}$$

Def

$$\begin{aligned}
H_{cv}^{\cdot}(E) :\hat{=}\ & H^{\cdot}(\bar{E}, \bar{E} - E) \\
=\ & R\pi_{\bar{E},*}\, \beta_{E,!}\, \beta_E^{!}\, \underline{\mathbb{Z}}_{\bar{E}} \\
=\ & R\pi_{B,*}\, (R\bar{\pi}_!\, \beta_{E,!})(\beta_E^*\, \underline{\mathbb{Z}}_{\bar{E}}) \\
=\ & R\pi_{B,*}\, R\pi_!\, \underline{\mathbb{Z}}_E
\end{aligned}$$

compact vertical

Ex. Construct the following canonical maps by six functors.

$$\begin{tikzcd}
H^{\cdot}(\bar{F}, \bar{F} - F) \arrow[d, phantom, "\cong"] & H^{\cdot}(\bar{E}, \bar{E} - E) \arrow[l] \arrow[d, phantom, "\cong"] \\
H_c^{\cdot}(F) & H_{cv}^{\cdot}(E) \arrow[l] \arrow[d, "\sim"] \\
& H^{\cdot-r}(B)
\end{tikzcd}
\qquad
\begin{tikzcd}
R\pi_{\bar{F},*}\, \beta_{F,!}\, \underline{\mathbb{Z}}_F \arrow[d, phantom, "\cong"] & R\pi_{\bar{E},*}\, \beta_{E,!}\, \underline{\mathbb{Z}}_E \arrow[l] \arrow[d, phantom, "\cong"] \\
R\pi_{F,!}\, \underline{\mathbb{Z}}_F & R\pi_{B,*}\, R\pi_!\, \underline{\mathbb{Z}}_E \arrow[l] \arrow[d, "\sim"] \\
& R\pi_{B,*}\, \underline{\mathbb{Z}}_B[-r]
\end{tikzcd}$$

Lemma. $R\pi_! \underline{\mathbb{Z}}_E \cong \underline{\mathbb{Z}}_B[-r]$. As a result, $H^{\cdot}_{cv}(E) \cong H^{\cdot-r}(B)$.

Proof.

$$\begin{aligned} R\pi_! \underline{\mathbb{Z}}_E &= R\pi_! \pi^* \underline{\mathbb{Z}}_B && \text{expand} \\ &\cong R\pi_! \pi^! \underline{\mathbb{Z}}_B[-r] && \text{Verdier duality, } \pi \text{ is a v.b.} \\ &\xrightarrow{\sim} \underline{\mathbb{Z}}_B[-r] && \text{adjunction, iso comes from} \\ & && \quad H_{\cdot}(\mathbb{R}^r;\mathbb{Z}) \cong \mathbb{Z} \text{ (check locally)} \end{aligned}$$

Explanation

Exactness & derived

$j_!$ is exact　　by checking on stalks
i^*, j^* are exact　　in the category Top
i_* is exact　　when $Z \subset X$ is (strongly) loc. contractable.

j_* is not exact $\rightsquigarrow$ Rj_*
$i^!$ is already derived.

Rmk. strongly loc. contractable: $\forall\, p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z$ is contractable
loc. contractable: $\forall\, p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z \subset U_n$ is contractable

E.g. $|$ {strongly loc.contractable} $\subseteq$ {loc. contractable} $\subsetneq$ Top
CW-cplx, topo mflds　　Cantor set
& algebraic varieties (Check?)

See the answer in
https://math.stackexchange.com/questions/1082601/anr-is-locally-contractible
for the subtlety of these two definitions.

I don't care. In both cases, the local cohomology vanishes in higher degree, and that's what I want.

For the non-exact functors, there maybe some problems in the composition of derived functors.

https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof
https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de

E.g. we need to check if $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.
Luckily, in the open-closed formalism, we won't meet these problems.

Prop 1. Let $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, bend left, "G"] \end{tikzcd}$, assume F is exact. Then

① G preserves injective sheaves;
② $RG(f) \circ RG(g) = RG(f \circ g)$

Proof. ①: by universal property.
②. by adjunction

Prop 2. Let $$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, bend left, "G"] \arrow[r, bend left, "F'"] \arrow[r, phantom, "\perp"] & \mathcal{C}'' \arrow[l, bend left, "G'"] \end{tikzcd}$$. Suppose F or F' is exact, then

$$RG \circ RG'(f) = (R(G \circ G'))(f)$$

Proof. By adjunction & Grothendieck-Serre sequence. $(LF' \circ LF = L(F' \circ F))$
When F' is exact, can use Prop 1 ①.

Cor. $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.
Rf_* & $Rf_!$ are nice in general.

Reason: f_! sends skyscraper sheaf to skyscraper sheaf, and in general preserve injective sheaves. (need double check)