Eine Woche, ein Beispiel

5.18 theta functions

cohomology of $\mathcal{L} \in \mathrm{Pic}(A)$

Ref: follows [2025.05.04].
Most contents in this document can be found in [BL04, Chap 3].

⚠

[BL04]	My notation
$H(u,v)$	$H(v,u)$
V_1	V_2
V_2	V_1

Rmk: For $H \in NS(A)$ nondegenerate,
when we fix an isotropic dec $V = V_2 \oplus V_1$ s.t. $\Lambda = \Lambda \cap V_2 \oplus \Lambda \cap V_1$, i.e., $H(V_i, V_i) \equiv 0$
we can get a canonical lift

$$\mathcal{L} = \mathcal{L}(H, \chi_0) \in \mathrm{Pic}(A)$$

given by

$$\chi_0(v_1 + v_2) = \exp(\pi i \,\mathrm{Im}\, H(v_1, v_2)).$$

See [BL04, Lemma 3.1.2].

Q: Is that still true when H is not nondegenerate?

Def (characteristic) $c \in V/\Lambda(L) = \mathrm{Im}\, \varphi_A$ is called the char of $\mathcal{L}$, when

$$\chi(v) = \chi_0(v) \exp(2\pi i \,\mathrm{Im}\, H(v, c)) \quad \Leftrightarrow \mathcal{L} \simeq t_c^* \mathcal{L}_0$$

$$= \exp\{2\pi i \,\mathrm{Im}(\tfrac{1}{2} H(v_1, v_2) + H(v, c))\}$$

$$= \exp\{2\pi i \,\mathrm{Im}(\tfrac{1}{2} H(v_1, v_2) + H(v_1, c_2) + H(v_2, c_1)\}$$

$V = V_1 \oplus V_2$

$v = v_1 + v_2$

$c = c_1 + v_2$

We also define B as the $\mathbb{C}$-bilinear extension of $H|_{V_1 \times V_1}$.

Factor of automorphy and theta fcts

Canonical factor of automorphy for $\mathcal{L} = \mathcal{L}(H, \chi)$:

$$a_L : \Lambda \times V \longrightarrow \mathbb{C}^\times$$
$$a_L(\lambda, v) = \chi(u) \exp\left(\pi H(\lambda, v) + \tfrac{\pi}{2} H(\lambda, \lambda)\right)$$

Classical factor of automorphy crspds to other l.b.

$$e_L : \Lambda \times V \longrightarrow \mathbb{C}^\times$$
$$\begin{aligned} e_L(\lambda, v) &= \chi(u) \exp\left(\pi (H-B)(\lambda, v) + \tfrac{\pi}{2} (H-B)(\lambda, v)\right) \\ &= a_L(\lambda, v) \exp\left(-\pi B(\lambda, v) - \tfrac{\pi}{2} B(\lambda, \lambda)\right) \\ &= a_L(\lambda, v) \exp\left(\tfrac{\pi}{2} B(v, v) - \tfrac{\pi}{2} B(\lambda + v, \lambda + v)\right) \end{aligned}$$

Canonical theta fct c: characteristic of $\mathcal{L}$

$$\theta^c(v) = \exp\left(-\pi H(c, v) - \tfrac{\pi}{2} H(c, c) + \tfrac{\pi}{2} B(v + c, v + c)\right)$$
$$\cdot \sum_{\lambda \in \Lambda \cap V_1} \exp\left(\pi (H-B)(\lambda, v + c) - \tfrac{\pi}{2} (H-B)(\lambda, \lambda)\right)$$

$$\theta^c(v + \lambda) = a_L(\lambda, v)\, \theta^c(v)$$

Classical theta fct [BL04, p223] $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^n$

$$\theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(v, Z) = \sum_{l \in \mathbb{Z}^n} \exp\left(\pi i\, (l + \varepsilon_1)^T Z (l + \varepsilon_1) + 2\pi i\, (v + \varepsilon_2)^T (l + \varepsilon_1)\right)$$

$$\theta^{Z\varepsilon_1 + \varepsilon_2}(v) = \exp\left(\tfrac{\pi}{2} B(v, v) - \pi i\, \varepsilon_1^T \varepsilon_2\right) \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(v, Z)$$

$$\begin{aligned} \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(v + \lambda, Z) &= a_L(\lambda, v) \exp\left(-\tfrac{\pi}{2} B(v + \lambda, v + \lambda) + \tfrac{\pi}{2} B(v, v)\right) \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(v, Z) \\ &= e_L(\lambda, v)\, \theta\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix}(v, Z) \end{aligned}$$

Cohomology of $\mathcal{L} \in \mathrm{Pic}(A)$

Suppose $\mathcal{L} = \mathcal{L}(H, \chi)$ is pos def with characteristic c w.r.t. $V \cong V_2 \oplus V_1$.

Def (Shift of theta fct)
For $w \in V$, define

$$\theta_w^c : V \longrightarrow \mathbb{C} \qquad \theta_w^c(v) = a_L(w, v)^{-1} \theta^c(v)$$

One can check that

$$\theta_w^c(v+\lambda) = \exp(2\pi i \,\mathrm{Im} H(\lambda, w))\, a_L(\lambda, v)\, \theta_w^c(v) \qquad \forall \lambda \in \Lambda$$

Prop (basis of $H^0(A; \mathcal{L})$) [BL04, Thm 3.2.7]

$$H^0(A; \mathcal{L}) = \langle \theta_{\bar{w}}^c \mid \bar{w} \in K(L) \cap V_2 \rangle_{\mathbb{C}\text{-v.s.}}$$

As a result,

$$h^0(A; \mathcal{L}) = \mathrm{Pf}(\mathrm{Im} H) = d_1 \cdots d_n.$$

Now suppose $\mathcal{L} = \mathcal{L}(H, \chi)$ is positive semidefinite with type $(d_1, \ldots, d_k, 0, \ldots, 0)$.

Prop (basis of $H^0(A; \mathcal{L})$) [BL04, Lemma 3.3.2 & Thm 3.3.3]
$K(L)_0$: connected component of $K(L)$

When $\mathcal{L}|_{K(L)_0} \not\cong \mathcal{O}_{K(L)_0}$, $H^0(A; \mathcal{L}) = 0$

When $\mathcal{L}|_{K(L)_0} = \mathcal{O}_{K(L)_0}$, denote $\pi : A \to A/K(L)_0$, then $\exists \bar{\mathcal{L}} \in \mathrm{Pic}(A/K(L)_0)$ pos def with char $\bar{c}$ w.r.t. $V/K(L)_0 \cong \bar{V_2} \oplus \bar{V_1}$
s.t. $\mathcal{L} = \pi^* \bar{\mathcal{L}}$ and

$$H^0(A; \mathcal{L}) \cong H^0(A/K(L)_0; \bar{\mathcal{L}}) \cong \langle \theta_{\bar{w}}^{\bar{c}} \mid \bar{w} \in K(\bar{L}) \cap \bar{V_2} \rangle_{\mathbb{C}\text{-v.s.}}$$

In that case ($\mathcal{L}|_{K(L)_0}$ is trivial),

$$h^0(A; \mathcal{L}) = \mathrm{Pf}_r(\mathrm{Im} H) = d_1 \cdots d_k.$$

Recall that $f^*\mathcal{L}(H,\chi) = \mathcal{L}(f_a^* H, f_r^* \chi)$, so

$$\mathcal{L}|_{K(L)_0} \cong \mathcal{O}_{K(L)_0} \iff \begin{cases} H|_{K(L)_0 \times K(L_0)} \equiv 0 \\ \chi|_{\Lambda(L)_0 \cap \Lambda} \equiv 0 \end{cases}$$

Now suppose that $\mathcal{L} = \mathcal{L}(H,\chi)$ is of type $D = (d_1, \cdots, d_{r+s}, 0, \cdots, 0)$ $d_i > 0$, and the Hermitian form H has r positive and s negative eigenvalues.

Thm [BL04, Thm 3.5.5 & Thm 3.6.1]

$$h^q(A;\mathcal{L}) = \begin{cases} \binom{n-r-s}{q-s} \, \mathrm{Pf}_r(\mathrm{Im}\, H), & \text{if } s \le q \le n-r \ \& \ \mathcal{L}|_{K(L_0)} \cong \mathcal{O}_{K(L_0)} \\ 0, & \text{otherwise} \end{cases}$$

As a result,

$$\chi(A;\mathcal{L}) = (-1)^s \mathrm{Pf}(\mathrm{Im}\, H) = \begin{cases} (-1)^s d_1 \cdots d_n, & H \text{ nondeg} \\ 0, & H \text{ deg} \end{cases}$$

Rmk. 1. When $q \ge s$, we have an iso [BL04, Ex 3.7.(4)]
$$H^q(\mathcal{L}) \cong H^s(\mathcal{L}) \otimes H^{q-s}(\mathcal{O}_{K(L)_0})$$

2. [BL04, Ex 3.7.(7)]
Since $\mathcal{P}_A \in \mathrm{Pic}(A \times \hat{A})$ is nondeg of index n,

$$H^q(A \times \hat{A}; \mathcal{P}_A) = \begin{cases} \mathbb{C}, & \text{if } q = n \\ 0, & \text{otherwise} \end{cases}$$

3. Do you see the shadow of the generic vanishing theorem?
For generic $\mathcal{L} \in \mathrm{Pic}(A)$, $h^q(\mathcal{L}) \equiv 0$.

Thm (Geometric Riemann-Roch) [BL04, Thm 3.6.3 - Lemma 3.6.5]

Suppose that $\mathcal{L} = \mathcal{L}(H, \chi)$ is of type $D = \mathrm{diag}(d_1, \dots, d_n)$. Then

$$c_1(\mathcal{L}) = -\sum_{\nu=1}^{n} d_\nu \, dx_\nu \wedge dy_\nu$$

$$\begin{aligned}
(\mathcal{L}^n) &:= \int_A \Lambda^n c_1(\mathcal{L}) \\
&= (-1)^n \sum_{\sigma \in S_n} \int_A d_{\sigma(1)} \cdots d_{\sigma(n)} \, dx_{\sigma(1)} \wedge dy_{\sigma(1)} \cdots dx_{\sigma(n)} \wedge dy_{\sigma(n)} \\
&= n! \, (-1)^n d_1 \cdots d_n \int_A dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \\
&= \begin{cases} n! \, (-1)^n d_1 \cdots d_n (-1)^{n+s} & \text{when } \mathcal{L} \text{ is nondeg [BL04, Lemma 3.6.4]} \\ 0 & \text{when } \mathcal{L} \text{ is deg since } d_n = 0 \end{cases} \\
&= n! \, (-1)^s d_1 \cdots d_n \\
&= n! \, \chi(A; \mathcal{L})
\end{aligned}$$