

Eine Woche, ein Beispiel

4.24 irreducible representation of the Mirabolic group

Main reference: The Local Langlands Conjecture for $GL(2)$ by Colin J. Bushnell and Guy Henniart.
[https://link.springer.com/book/10.1007/3-540-31511-X]

Process

1. Notations
2. Constructions
3. Classification
4. Applications
 - Computation of $V(N)$, V_N , $V(\psi)$, V_ψ .
 - Dual, Sym^m , \wedge^m , ...
 - Decompose $Res_B^G Rep_B^G Ind_B^G \chi$ (not today, need knowledge of G & B)
 - Trace formula
5. Irr rep of B ?

1. Notations. F : non-arch local field.

<https://math.stackexchange.com/questions/299626/the-center-of-operatornamegl-n-k>

$$A = M_{2 \times 2}(F) \quad G = GL_2(F)$$

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = Z(G) \quad S = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

$$\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad T^0 = \begin{pmatrix} 0^x & 0^x \\ 0 & 0^x \end{pmatrix} \quad N_j = \begin{pmatrix} 1 & p^j \\ 0 & 1 \end{pmatrix} \quad N_j' = \begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix}$$

Temporarily, $P := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(F) \right\} = F \rtimes F^\times = N \rtimes S$ $0 \rightarrow (F, +) \xrightarrow{N} P \xrightarrow{S} F^\times \rightarrow 0$

\uparrow
parabolic subgp

to be short, $Ind = Ind_N^P$, $c\text{-}Ind = c\text{-}Ind_N^P$.

2. Constructions

E.g. 1 (Irr rep from quotient gp)

When $(\rho, \nu) \in \widehat{P}^*$, ρ is the inflation of some $\chi \in \widehat{P/N}^* = \widehat{F^x}^*$, i.e.,

$$\rho: P \rightarrow \mathbb{C}^* \quad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \chi(a)$$

E.g. 2 (Irr rep from subgp)

For every $\nu \in \widehat{N}^* - \{1_N\}$, we claim that $c\text{-Ind } \nu \in \text{Irr}(P)$.

Rmk. For $\nu, \nu' \in \widehat{N}^* - \{1_N\}$, we have an iso $(\exists s \in S, \nu' = \nu(s' - s))$

$$c\text{-Ind } \nu \rightarrow c\text{-Ind } \nu' \quad f \mapsto f(s' - s) \quad \text{in } \text{Rep}(P).$$

So those irr reps in E.g. 2 are iso to each other.

The rest of this section is organized to prove E.g. 2. Step 1, 2 are also used in the next section.

Step 1 If $(\sigma, W) \in \text{Rep}(N)$ is restricted too much, then $W=0$ (in Cor)
 Prop. For $(\sigma, W) \in \text{Rep}(N)$, $\bigcap_{\vartheta \in \hat{N}} W(\vartheta) = \{0\}$.

Cor. (1) For $(\sigma, W) \in \text{Rep}(N)$,

$$W_{\vartheta} = 0 \quad \forall \vartheta \in \hat{N}^* \Rightarrow W = 0$$

(2) When $(\sigma, W) \in \text{Rep}(P)$, since $W_{\vartheta} \cong W_{\vartheta'}$ for $\vartheta, \vartheta' \in \hat{N}^* - \{1_N\}$,

we can further reduce (1) to

$$W_N = 0 \quad W_{\vartheta} = 0 \quad \exists \vartheta \in \hat{N}^* - \{1_N\} \Rightarrow W = 0$$

Proof of Prop. $N=F$ here. Let $w \in W, w \neq 0$, we would find $\vartheta_0 \in \hat{F}^*$ s.t. $w \notin W(\vartheta_0)$.

By the integral criterion,

$$w \notin W(\vartheta) \Leftrightarrow \text{For any } N_0 \in \text{Cos}(F), \int_{N_0} \vartheta(n)^{-1} n \cdot w \, d\mu_N(n) \neq 0$$

$$\Leftrightarrow \text{For any } j \in \mathbb{Z}, \int_{p^j} \vartheta(n)^{-1} n \cdot w \, d\mu_N(n) \neq 0$$

$(\sigma, W) \text{ sm} \Rightarrow \exists j_0 \in \mathbb{Z}$ s.t. $p^{j_0} \cdot w = w$.

For $j \in \mathbb{Z}$, let $W_j := \langle w \rangle_{\text{Rep}(p^j)}$. We will define $\vartheta_0 \in \hat{F}^*$ inductively, i.e.

$$\vartheta_0|_{p^{j_0}} \rightsquigarrow \vartheta_0|_{p^j} \rightsquigarrow \vartheta_0$$

(1) $\vartheta_0|_{p^{j_0}} = 1_{p^{j_0}}$, then

$$\int_{p^{j_0}} \vartheta_0(n)^{-1} n \cdot w \, d\mu_N(n) = \mu_N(p^{j_0}) \cdot w \neq 0$$

for $j \geq j_0$

(2) Suppose $\vartheta_0|_{p^{j+1}} \triangleq \eta_{j+1}$ is defined s.t. $W_{j+1}^{\eta_{j+1}} = 0$.

We define $\vartheta_0|_{p^j} \triangleq \eta_j$ s.t.

$$\textcircled{1} \eta_j|_{p^{j+1}} = \eta_{j+1}$$

$$\textcircled{2} W_j^{\eta_j} \neq 0$$

$$\textcircled{3} e_{\eta_j} * w = \int_{p^j} \eta_j(n)^{-1} n \cdot w \, d\mu_N(n) \neq 0.$$

$$0 \neq \langle W_{j+1}^{\eta_{j+1}} \rangle_{\text{Rep}(p^j)} \subset W_j$$

$$\Rightarrow \exists \eta_j \in \hat{p}^{j*} \text{ contained in } \langle W_{j+1}^{\eta_{j+1}} \rangle_{\text{Rep}(p^j)}$$

$$\Rightarrow \textcircled{1}, \textcircled{2}$$

$$\Rightarrow e_{\eta_j} * - : W_j \longrightarrow W_j^{\eta_j}$$

is not 0

$$x \longmapsto \int_{N_j} \eta_j(n)^{-1} n \cdot x \, d\mu_N(n)$$

$$\xrightarrow{W_j = \langle w \rangle_{\text{Rep}(p^j)}} e_{\eta_j} * w \neq 0$$

(3) Let $\vartheta_0(n) = \eta_j(n)$ ($n \in p^j, j \in \mathbb{Z}$). Then ϑ_0 is well-defined (by $\textcircled{1}$), and satisfy

$$\int_{p^j} \vartheta_0(n)^{-1} n \cdot w \, d\mu_N(n) \neq 0 \quad \forall j \in \mathbb{Z}$$

□

[$N \triangleleft P$ closed but not open.
 ↳ so we only have $\varepsilon_v: \text{Ind } \mathcal{V} \rightarrow \mathbb{C}$]

Step 2 We show that $c\text{-Ind } \mathcal{V}$ is heavily restricted.

Prop. \label{prop:jacqofind} Let $\mathcal{V} \in \widehat{N}^* - \{1_N\}$.

$$(1) \quad (c\text{-Ind } \mathcal{V})(N) = (\text{Ind } \mathcal{V})(N) = c\text{-Ind } \mathcal{V}$$

$$(c\text{-Ind } \mathcal{V})_N = 0$$

$$(\text{Ind } \mathcal{V})_N \cong \text{Ind } \mathcal{V} / c\text{-Ind } \mathcal{V}$$

$$(2) \quad (c\text{-Ind } \mathcal{V})(\mathcal{V}) \cong \ker \varepsilon_{\mathcal{V}} \cap c\text{-Ind } \mathcal{V}$$

$$(\text{Ind } \mathcal{V})(\mathcal{V}) \cong \ker \varepsilon_{\mathcal{V}}$$

$$(c\text{-Ind } \mathcal{V})_{\mathcal{V}} \cong \mathbb{C}$$

$$(\text{Ind } \mathcal{V})_{\mathcal{V}} \cong \mathbb{C}$$

Proof. (1). $(c\text{-Ind } \mathcal{V})(N) \subset (\text{Ind } \mathcal{V})(N) \subset c\text{-Ind } \mathcal{V}$: by direct computation.

$c\text{-Ind } \mathcal{V} \subset (c\text{-Ind } \mathcal{V})(N)$: find generators of $c\text{-Ind } \mathcal{V}$, and verify it.

Generators: $\{f_{a,j} \in C^\infty(P) \mid a \in F^\times, j \geq 1\}$, where

$$f_{a,j}(g) = \begin{cases} \mathcal{V} \begin{pmatrix} 1 & x \\ 0 & i \end{pmatrix} & g = \begin{pmatrix} 1 & x \\ 0 & i \end{pmatrix} \begin{pmatrix} a u & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & g \notin N \cdot \begin{pmatrix} a U_F^{(i)} & 0 \\ 0 & 1 \end{pmatrix} \end{cases} \quad \exists x \in F, u \in U_F^{(i)}$$

↑ Informal: think it as $F \rtimes a U_F^{(i)}$

Verify $f_{a,j} \in (c\text{-Ind } \mathcal{V})(N)$. Let $d = \text{level}(\mathcal{V})$.

$$\Rightarrow \mathcal{V}|_{P^d} = 1_{P^d}, \mathcal{V}|_{P^{d-1}} \neq 1_{P^{d-1}}$$

Let $y_0 \in P^{d-1}$ s.t. $\mathcal{V} \begin{pmatrix} 1 & y_0 \\ 0 & i \end{pmatrix} \triangleq c \neq 1$, and $x_0 = \frac{y_0}{i}$. We get

$$\mathcal{V} \begin{pmatrix} 1 & a U_F^{(i)} x_0 \\ 0 & i \end{pmatrix} = \mathcal{V} \begin{pmatrix} 1 & y_0 U_F^{(i)} \\ 0 & i \end{pmatrix} \equiv c \neq 1$$

$$\Rightarrow f_{a,j} = \frac{1}{i-c} (f_{a,j} - \begin{pmatrix} 1 & x_0 \\ 0 & i \end{pmatrix} \cdot f_{a,j}) \in (c\text{-Ind } \mathcal{V})(N).$$

Other results are then obvious.

(2)

$$\begin{array}{ccccccc}
 0 & \longrightarrow & (c\text{-Ind } \mathcal{V})(\mathcal{V}) & \longrightarrow & c\text{-Ind } \mathcal{V} & \longrightarrow & (c\text{-Ind } \mathcal{V})_{\mathcal{V}} \longrightarrow 0 \\
 & & \textcircled{1} \swarrow & & \parallel & & \textcircled{2} \swarrow \downarrow \textcircled{0} \\
 0 & \longrightarrow & \text{Ker } \varepsilon_{\mathcal{V}} \cap c\text{-Ind } \mathcal{V} & \xrightarrow{\quad} & c\text{-Ind } \mathcal{V} & \longrightarrow & \mathbb{C} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & (\text{Ind } \mathcal{V})(\mathcal{V}) & \longrightarrow & \text{Ind } \mathcal{V} & \longrightarrow & (\text{Ind } \mathcal{V})_{\mathcal{V}} \longrightarrow 0 \\
 & & \downarrow \textcircled{4} & & \parallel & & \parallel \textcircled{3} \\
 0 & \longrightarrow & \text{Ker } \varepsilon_{\mathcal{V}} & \longrightarrow & \text{Ind } \mathcal{V} & \longrightarrow & \mathbb{C} \longrightarrow 0
 \end{array}$$

$$\begin{aligned} \textcircled{\textcircled{c}}: & \quad 0 \rightarrow c\text{-Ind}\mathcal{V} \rightarrow \text{Ind}\mathcal{V} \rightarrow \text{Ind}\mathcal{V}/_c\text{-Ind}\mathcal{V} \rightarrow 0 \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \cong (\text{Ind}\mathcal{V})_N \\ \rightsquigarrow & \quad 0 \rightarrow (c\text{-Ind}\mathcal{V})_{\mathcal{H}} \rightarrow (\text{Ind}\mathcal{V})_{\mathcal{H}} \rightarrow (\text{Ind}\mathcal{V})_{N,\mathcal{H}} \rightarrow 0 \\ \rightsquigarrow & \quad \textcircled{\textcircled{c}} \text{ is isu} \end{aligned}$$

①: To verify that $\text{Ker } \varepsilon_{\mathcal{V}} \cap c\text{-Ind } \mathcal{V} \subset (c\text{-Ind } \mathcal{V})(\mathcal{W})$,
we only need to show the generators of $\text{Ker } \varepsilon_{\mathcal{V}} \cap c\text{-Ind } \mathcal{V}$ belong to $(c\text{-Ind } \mathcal{V})(\mathcal{W})$.

Generators: $\{f_{a,j} \in C^\infty(P) \mid a \in F^\times - U_F^{(j)}, j \geq 1\}$

Verify $f_{\alpha, j} \in (c\text{-Ind } \mathcal{V})(\mathcal{V})$: Let $d = \text{level}(\mathcal{V})$, $j_0 := v_F(\alpha-1) < j$.

Let $y_0 \in \mathbb{P}^{d-1}$ s.t. $\mathcal{V}(\begin{smallmatrix} 1 & y_0 \\ 0 & 1 \end{smallmatrix}) = c \neq 1$, and $x_0 = \frac{y_0}{a-1}$. We get

$$v_F(\alpha x_0 \beta^j) \geq v_F(a) + d - 1 - j_0 + j \geq \begin{cases} v_F(a) + d \geq d, & v_F(a) \geq 0 \\ v_F(a) + d - j_0 = d, & v_F(a) < 0 \end{cases}$$

$$\begin{aligned}
 & \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) - \vartheta \left(\begin{smallmatrix} 1 & a U_F^{(i)} x_0 \\ 0 & 1 \end{smallmatrix} \right) \\
 &= \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) - \vartheta \left(\begin{smallmatrix} 1 & a x_0 \\ 0 & 1 \end{smallmatrix} \right) \vartheta \left(\begin{smallmatrix} 1 & a x_0 p^j \\ 0 & 1 \end{smallmatrix} \right) \\
 &\stackrel{U_F(a x_0 p^j) \geq 0}{=} \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) - \vartheta \left(\begin{smallmatrix} 1 & a x_0 \\ 0 & 1 \end{smallmatrix} \right) \\
 &= \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) - \vartheta \left(\begin{smallmatrix} 1 & (a^{-1}) x_0 \\ 0 & 1 \end{smallmatrix} \right) \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) \\
 &= (1-c) \vartheta \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix} \right) \neq 0
 \end{aligned}$$

$$\Rightarrow f_{a,j} = \frac{1}{(1-c)\psi\left(\begin{smallmatrix} x_0 \\ 0 & 1 \end{smallmatrix}\right)} \left(\mathcal{V}\left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix}\right) f_{a,j} - \left(\begin{smallmatrix} 1 & x_0 \\ 0 & 1 \end{smallmatrix}\right) \cdot f_{a,j} \right) \in (c\text{-Ind } \mathcal{V})(\mathcal{V}).$$

Finally, $(1) \text{ iso} \Rightarrow (2) \text{ iso} \Rightarrow (3) \text{ iso} \Rightarrow (4) \text{ iso}$.

□

Step 3 Finally we can prove that $c\text{-Ind } \vartheta \in \text{Irr}(P)$. $\forall \vartheta \in \widehat{N}^* - \{1_N\}$.

Proof. Let $V \leq c\text{-Ind } \vartheta$ in $\text{Rep}(P)$, we show that $V=0$ or $c\text{-Ind } \vartheta/V=0$.

$$0 \longrightarrow V \longrightarrow c\text{-Ind } \vartheta \longrightarrow c\text{-Ind } \vartheta/V \longrightarrow 0$$

$$\Rightarrow \begin{cases} 0 \longrightarrow V_N \xrightarrow{\cong} (c\text{-Ind } \vartheta)_N \xrightarrow{\cong} (c\text{-Ind } \vartheta/V)_N \longrightarrow 0 \\ 0 \longrightarrow V_{\vartheta} \xrightarrow{\cong} (c\text{-Ind } \vartheta)_{\vartheta} \xrightarrow{\cong} (c\text{-Ind } \vartheta/V)_{\vartheta} \longrightarrow 0 \end{cases}$$

$$\begin{array}{l} \rightarrow V_{\vartheta} = 0 \xRightarrow{V_N=0} V = 0 \\ \text{or} \\ \rightarrow (c\text{-Ind } \vartheta/V)_{\vartheta} = 0 \xRightarrow{(c\text{-Ind } \vartheta/V)_N=0} c\text{-Ind } \vartheta/V = 0 \end{array}$$

□

3. Classification.

We will prove that the two examples in the last section are all irr reps of P .

Lemma. Let $(\rho, V) \in \text{Rep}(P)$, we get

$$\begin{array}{ccc} V & \xrightarrow{\pi_*} & \text{Ind}_N^P V_{\vartheta} \\ \cup & & \cup \\ V(N) & \xrightarrow[\cong]{\pi_*|_{V(N)}} & (\text{Ind}_N^P V_{\vartheta})(N) \cong c\text{-Ind}_N^P V_{\vartheta}. \end{array} \quad \text{induced by} \quad \text{Res}_N^P V \xrightarrow{\pi} V_{\vartheta}$$

Proof. Denote $W = \ker \pi_*|_{V(N)}$, $W' = \text{Coker } \pi_*|_{V(N)}$, we get LES

$$\begin{aligned} 0 &\rightarrow W \rightarrow V(N) \rightarrow (\text{Ind}_N^P V_{\vartheta})(N) \rightarrow W' \rightarrow 0 \\ \Rightarrow \begin{cases} 0 \rightarrow W_N \rightarrow V(N)_N \xrightarrow{\cong} (\text{Ind}_N^P V_{\vartheta})(N)_N \rightarrow W'_N \rightarrow 0 \\ 0 \rightarrow W_{V_{\vartheta}} \rightarrow V(N)_{V_{\vartheta}} \xrightarrow[\cong]{V_{\vartheta}} (\text{Ind}_N^P V_{\vartheta})(N)_{V_{\vartheta}} \xrightarrow[\cong]{(\text{Ind}_N^P V_{\vartheta})_{V_{\vartheta}} \cong V_{\vartheta}} W'_{V_{\vartheta}} \rightarrow 0 \end{cases} \\ \Rightarrow \begin{cases} W_N = 0 & W'_N = 0 \\ W_{V_{\vartheta}} = 0 & W'_{V_{\vartheta}} = 0 \end{cases} \Rightarrow W = 0, W' = 0 \Rightarrow \pi_*|_{V(N)} \text{ is iso.} \quad \square \end{aligned}$$

Thm. Let $(\rho, V) \in \text{Irr}(P)$. Fix $\vartheta \in \widehat{N}^* \setminus \{1_N\}$

(1) When $V(N) = 0$, $\rho \in \widehat{P}^*$ is the inflation of some $\chi \in \widehat{P/N}^* = \widehat{F}^*$;

(2) When $V(N) = V$, $V \cong c\text{-Ind}_N^P \vartheta$.

Proof. When $V(N) = 0$, $\rho|_N = \text{Id}_V \Rightarrow \exists \chi \in \text{Irr}(P/N) = \widehat{P/N}^*$, $\chi \xrightarrow{P} \rho$.

When $V(N) = V$, $V = V(N) \xrightarrow{\text{Lemma}} c\text{-Ind}_N^P V_{\vartheta} \in \text{Irr}(P)$

$\Rightarrow \dim_{\mathbb{C}} V_{\vartheta} = 1$, i.e., $V_{\vartheta} \cong \vartheta$ in $\text{Rep}(N)$

$\Rightarrow V \cong c\text{-Ind}_N^P \vartheta$. □

4. Applications.

4.1. Computation of $V(N), V_N, V(\vartheta), V_\vartheta$. $(p, V) \in \text{Irr}(P)$

For $p = c\text{-Ind } \vartheta$ $\vartheta \in \widehat{N}^* - \{1_N\}$, we have computed in [prop. jacq of ind].

For $p_X: P \longrightarrow \mathbb{C}^\times$ $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mapsto \chi(a)$, we know that

$$\begin{array}{ll} V(N) = 0 & V_N \cong \mathbb{C} \\ V(\vartheta) \cong \mathbb{C} & V_\vartheta = 0 \end{array} \quad \forall \vartheta \in \widehat{N}^* - \{1_N\}.$$

4.2. Dual, $\text{Sym}^m, \wedge^m, \dots$

$N \leq P$ closed, N is unimodular, while P is not. $\delta_P|_N = 1_N$, so

$$(c\text{-Ind } \vartheta)^\vee \cong \text{Ind}(\delta_P|_N \otimes \check{\vartheta}) \cong \text{Ind } \check{\vartheta} \cong \text{Ind } \vartheta$$

Q: $\delta_P = ?$ (lazy to compute it.)

Future use for rep of GL₂:

<https://hal.archives-ouvertes.fr/hal-00576943v2/document>

<https://math.berkeley.edu/~nadler/274spring2017.html>

<https://zhuanlan.zhihu.com/p/24986434>

<https://zhuanlan.zhihu.com/p/27041214>