Eine Woche, ein Beispiel

1.28 conormal bundle

Ref: from [23.11.19]

slogan:

SES	induces	LES,
filtration	induces	spectral sequence.

To expend a little bit,

$$
\begin{tikzcd}
& \text{SES} \arrow[r] & \text{LES} & \\
& \text{filtration} \arrow[r] & \text{spectral seq} \arrow[dr] & \\
\text{resolution} \arrow[ur, dashed] \arrow[rrr, "\text{prefer acyclic}"'] & & & \text{comparison between sheaf cohomology \& other cohomology}
\end{tikzcd}
$$

Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh". Let us leave spectral seq in other people's notes.

1. open-closed formalism
2. open cover
3. filtrations from chain complex
4. filtration by $\mathcal{H}^i(\mathcal{F}^\cdot)$
5. filtration by $\mathcal{F}^\cdot$
6. Hodge related filtration

Methods to construct SES:
- check by stalks
- filtration by $\mathcal{H}^i(\mathcal{F})$
- filtration by $\mathcal{F}^i$

method	spectral seq	LES	cohomology/resolution
check by stalks	... for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech
	coefficient		—
Hodge related: filtration by $\mathcal{H}^i(\mathcal{F})$	Grothendieck		
	Leray-Serre	Gysin	Euler class
			Hodge-Tate
filtration by $\mathcal{F}^i$ need resolution to get "another" complex	Hodge-de Rham		de Rham, Hodge-de Rham
			Dolbeault $H^p(X,\Omega^q)=H^{p,q}(X)$
	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences which I don't know	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
	Green		for Koszul cohomology
	⋮		⋮

For more spectral sequences, see:
https://en.wikipedia.org/wiki/Spectral_sequence
https://github.com/CubicBear/SpectralSequences/blob/main/SpecralSequences.pdf

1. open-closed formalism

|| related: comparison of $j_!$ & j_*
|| one-point compactification.

Observe the following pictures:

$$
\begin{tikzcd}
Z \arrow[r, "i"] & X & \mathcal{U} \arrow[l, hook', "j"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{D}(Z) \arrow[r, "i_* = i_!"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}
$$

Black box:

0. We assume some nice conditions.
   e.g. in the category $\mathrm{Haus}^{loc.\,cpt}$, and $Z \subset X$ is loc. contractable.
   Under these conditions,
1. $i_* = i_!$, $j^* = j^!$
2. $j_!$, i^*, j^*, i_* are exact.

Ex. 1. Shows that

$$\underset{\sim\sim}{i^* i_*} = \underline{i^! i_*} = Id_{\mathcal{D}(Z)} \qquad \underline{j^* j_!} = \underset{\sim\sim}{j^* Rj_*} = Id_{\mathcal{D}(\mathcal{U})}$$

$$\underset{\sim\sim}{i^* j_!} = 0, \qquad \underset{\sim\sim}{j^* i_*} = 0, \qquad \underline{i^! Rj_*} = 0$$

——— : base change

$\sim\sim\sim$: check stalkwise.

2. (for category fans)
   i_*, j_*, $j_!$ are fully faithful, and
   i_*, $i^!$, j^*, Rj_* preserve injectives.

3. One has SES

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 spectural cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the SES

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow Rj_*\mathcal{F} \longrightarrow i_*i^*Rj_*\mathcal{F} \longrightarrow 0$$

which measures the difference between $j_!\mathcal{F}$ & $j_*\mathcal{F}$.

3. Shows that

$$H_c^{\cdot}(X) \cong H^{\cdot}(\bar{X}, \{\infty\}; \mathbb{Z})$$

for one pt compactification $\iota: X \hookrightarrow \bar{X}$.

Try to compute $H_c^{\cdot}(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X\mathcal{F}^\cdot := \underline{Hom}_{D^b(X;\mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}})$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad \mathbb{D}_X(\mathcal{F}^\cdot[n]) = (\mathbb{D}_X\mathcal{F}^\cdot)[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

$f: Y \to X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then

$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructable setting,

$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad (2)$$

for $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".

E.g. For a finite graph (as a topo space) X.

$$\mathrm{sk}_0 X \xhookrightarrow{i} X \xleftarrow{j} X - \mathrm{sk}_0 X \quad \text{1-cells}$$

$$0 \longrightarrow j_! j^! \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X - \mathrm{sk}_0 X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{\mathrm{sk}_0 X} \longrightarrow 0$$

Take $R\pi_{X,!}$

$$\begin{tikzcd}
 & \underbrace{H^1_c(X - \mathrm{sk}_0 X)}_{= \bigoplus_e \mathbb{Q}} \arrow[r] & H^1_c(X) \arrow[r] & \underbrace{H^1_c(\mathrm{sk}_0 X)}_{=0} \arrow[r, "+1"] & {} \\
0 \arrow[r] & \underbrace{H^0_c(X - \mathrm{sk}_0 X)}_{=0} \arrow[r] & H^0_c(X) \arrow[r] & \underbrace{H^0_c(\mathrm{sk}_0 X)}_{= \bigoplus_v \mathbb{Q}} \arrow[llu, bend right]
\end{tikzcd}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that

$$H^i_c(\mathbb{R}) = \begin{cases} \mathbb{Q} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial/cellular cohomology, using the following filtration:

$$\emptyset \subset \mathrm{sk}^0 X \subset \mathrm{sk}^1 X \subset \cdots \ \mathrm{sk}^n X = X$$

Ex. derive the Wang LES for the cpt supp version. over S^1

2. open cover

Ex. For an open cover $X = \mathcal{U}_1 \cap \mathcal{U}_2$, deduce the SES

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus j_! \underline{\mathbb{Q}}_{\mathcal{U}_2} \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \longleftarrow 0$$
$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \xrightarrow{+1} \qquad (3)$$

⚠ We omit the derived symbol and some subscripts in this section. $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$

(3) works for general sheaf

and, induce from (3) the MV sequence:

$$\xleftarrow{+1} H^k_c(X) \longleftarrow H^k_c(\mathcal{U}_1) \oplus H^k_c(\mathcal{U}_2) \longleftarrow H^k_c(\mathcal{U}_1 \cap \mathcal{U}_2)$$
$$H^k(X) \longrightarrow H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \longrightarrow H^k(\mathcal{U}_1 \cap \mathcal{U}_2) \xrightarrow{+1}$$

Hint: Apply $R\pi_{X,!}$ & $R\pi_{X,*}$, see [StackProject, 01E9]

Ex. Derived the Wang LES. over S^1

Ex. For an open cover $X = \bigcup_{i \in \Lambda} \mathcal{U}_i$, Λ finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{\mathcal{U}_i} \longleftarrow \bigoplus_{i<j} j_! \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longleftarrow \cdots \; j_! \underline{\mathbb{Q}}_{\cap_i \mathcal{U}_i} \longleftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i} \longrightarrow \bigoplus_{i<j} Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow \cdots \; Rj_* \underline{\mathbb{Q}}_{\cap_i \mathcal{U}_i} \longrightarrow 0$$

When $\{\mathcal{U}_i\}_{i\in\Lambda}$ is a good cover, $H^\cdot(\mathcal{U}_{i_1 \cdots i_k}) = H^0(\mathcal{U}_{i_1, \cdots i_k})$,

acyclic in AG

one can compute $H^\cdot(X)$ by applying $R\pi_{X,*}$:

$$0 \longrightarrow \bigoplus_i \Gamma(\mathcal{U}_i) \xrightarrow{d^1} \bigoplus_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j) \xrightarrow{d^2} \cdots \; \Gamma(\bigcap_i \mathcal{U}_i) \longrightarrow 0$$
$$\Downarrow \mathrm{Ker}/\mathrm{Im}$$
$$H^0(X) \qquad H^1(X) \qquad \cdots \qquad H^{\#\Lambda - 1}(X)$$

Rmk. When X is paracompact & Hausdorff. "limited" Čech = sheaf

e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$\mathcal{F} \longrightarrow \bigoplus_i Rj_* \mathcal{F}|_{\mathcal{U}_i} \qquad \qquad \mathcal{F} \longrightarrow \bigoplus_{x \in X} \mathcal{F}_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

$\# \Lambda = 3$ case:

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathcal{F}_0 \arrow[ur] \arrow[dr] & & & & & & \mathcal{F}_2 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & \underline{\mathbb{Q}}_X \arrow[rr] \arrow[urr] & & & \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i} \arrow[rr, "d^1"] \arrow[dr] & & \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i \cap U_j} \arrow[rr, "d^2"] \arrow[urr] & & & & Rj_* \underline{\mathbb{Q}}_{\cap_i U_i} \arrow[r, "d^3"] & 0 & \\
 & & & & & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & & & & & \\
 & & & & 0 \arrow[ur] & & 0 & & & & & &
\end{tikzcd}$$

$$\mathcal{F}_2 = Rj_* \underline{\mathbb{Q}}_{\cap_i U_i} \quad \Rightarrow \quad H^\bullet(\mathcal{F}_2) = \operatorname{Ker} d^3$$

$$\begin{tikzcd}
 & H^1(\mathcal{F}_1) \arrow[r] & 0 \arrow[r] & \overset{=0}{H^1(\mathcal{F}_2)} \arrow[r, "+1"] & {} \\
0 \arrow[r] & H^0(\mathcal{F}_1) \arrow[r] & \bigoplus_{i<j} \Gamma(U_i \cap U_j) \arrow[r, "d^2"] & H^0(\mathcal{F}_2) \arrow[ulll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztotarget.south west) -- (\tikztotarget)}] &
\end{tikzcd}$$

$$\Rightarrow \quad H^i(\mathcal{F}_1) = \begin{cases} \operatorname{Ker} d^3 / \operatorname{Im} d^2, & i = 1 \\ \operatorname{Ker} d^2, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{tikzcd}
 & H^2(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & \overset{=0}{H^2(\mathcal{F}_1)} \arrow[r, "+1"] & {} \\
 & H^1(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & H^1(\mathcal{F}_1) \arrow[ull, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztotarget.south west) -- (\tikztotarget)}] & \\
0 \arrow[r] & H^0(\mathcal{F}_0) \arrow[r] & \bigoplus_i \Gamma(U_i) \arrow[r, "d^1"] & H^0(\mathcal{F}_1) \arrow[ull, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztotarget.south west) -- (\tikztotarget)}] &
\end{tikzcd}$$

$$\Rightarrow \quad H^i(X) = H^i(\mathcal{F}_0) = \begin{cases} \ker d^3 / \operatorname{Im} d^2 & i = 2 \\ \ker d^2 / \operatorname{Im} d^1 & i = 1 \\ \ker d^1 & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When $\{U_i\}_{i \in \Lambda}$ is not a good cover,
one needs Čech-to-derived functor spectral seq to compute $H^\cdot(X)$.

Rmk. stratification & open cover are two main tools to extract topological information.
They appear with different names in different fields.
Once you realize them, you can apply the six-functor machine to analyze structures.

stratification with extra properties
- CW cplx
- triangulization
- dessin d'enfant
- affine paving
- Whitney stratification
- ⋮

Q: How to construct stratifications?
A: For me, there are two efficient methods:
- orbit of gp action
- Morse theory

That's why some geometrical problems are finally reduced to combinatorical/analytic problems.
Other fields come to geometry by providing stratifications.

In fact, there is only one method:
find a fct $f: X \longrightarrow Y$, and get stratification of X from Y.
get better stratification by analyzing f

E.g.
1. Morse theory $\quad f: X \longrightarrow \mathbb{R}$
2. tessellation $\quad f: \mathcal{H} \longrightarrow \mathcal{H}/\Gamma$
3. semi-continuous fct $\quad f: X \longrightarrow \mathbb{N}_{\geq 0}$ $\quad$ e.g. $f(p) = \dim T_pX$
4. my thesis $\quad f: Gr(X) \longrightarrow Gr(S) \times Gr(X/S)$
5. orbit of gp action $\quad f: X \longrightarrow X/G$

3. filtrations from chain complex [Stack Project, 0118]

Lots of filtrations are obtained just from the naive complex.

Consider a chain complex $C^\cdot$:

$$\cdots \xrightarrow{d^{-2}} C^{-2} \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^{0}} C^{0} \xrightarrow{d^{1}} C^{1} \xrightarrow{d^{2}} C^{2} \xrightarrow{d^{3}} \cdots$$

One can make a "stupid" truncation

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] & C^{-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \sigma_{\geq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \sigma_{\leq -1} C^\cdot \longrightarrow 0$$

One can also make a canonical truncation

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & \operatorname{Ker} d^1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^0 \arrow[r, "d^1"] & C^1 \arrow[r, "d^2"] & C^2 \arrow[r, "d^3"] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \tau_{\leq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \tau_{\geq 1} C^\cdot \longrightarrow 0$$

Using these truncations, one can easily construct filtrations:

$$0 \subset \cdots \subset \sigma_{\geq 1} C^{\cdot} \overset{C^{\cdot}[-1]}{\subset} \sigma_{\geq 0} C^{\cdot} \overset{C^{0}}{\subset} \sigma_{\geq -1} C^{\cdot} \subset \cdots \subset C^{\cdot}$$

$$0 \subset \cdots \subset \tau_{\leq -1} C^{\cdot} \overset{\mathcal{H}^{0}(C^{\cdot})}{\subset} \tau_{\leq 0} C^{\cdot} \overset{\mathcal{H}^{1}(C^{\cdot})[1]}{\subset} \tau_{\leq 1} C^{\cdot} \subset \cdots \subset C^{\cdot}$$

Rmk. 1. These two filtrations have opposite directions!
(a striking feature for me)

2. The "stupid" truncation extracts pieces of the chain cplx, while the canonical truncation extracts cohomology. (Ker/Im)

Therefore, only the canonical truncation can be defined on the derived category.

This information is culmulated in the standard/natural t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$.

One has adjoint fctors:

$$\begin{tikzcd} \mathcal{D}_{\leq 0} \arrow[r, bend left, "\iota_{\leq 0}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\tau_{\leq 0}"] \arrow[r, bend left, "\tau_{\geq 1}"] \arrow[r, phantom, "\perp"] & \mathcal{D}_{\geq 1} \arrow[l, bend left, "\iota_{\geq 0}"] \end{tikzcd}$$

The following notations are from: https://ncatlab.org/nlab/show/t-structure

$\mathcal{D}_{\leq 0}$: t-co-connective objects
$\mathcal{D}_{\geq 0}$: t - connective objects
$\tau_{\geq 0}$: connective cover

Let's apply these filtrations!

4. filtration by $\mathcal{H}^i(\mathcal{F}^\cdot)$

Ex. Suppose that $\pi : E \longrightarrow B$ is an oriented S^k-bundle.
Analyze $R\pi_* \mathbb{Q}_E$, and apply $R\pi_{B,*}$ to get the Gysin sequence:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[+1]{eu_\pi \wedge}$$

Q. Why does $\pi_* : \mathcal{D}^b(E) \to \mathcal{D}^b(B)$ takes injective objects to π_{B*}-acyclic objects?
A. [Vo02, p195] By definition, π_* takes flasque sheaves to flasque sheaves.

Rmk. 1. Here we can't use the "stupid" truncation,
because $R\pi_* \mathbb{Q}_E$ lies in the derived category.
2. You can generalize it to fiber bundle. then
you will get the Leray-Serre spectral sequence.
Think how the following conditions simplify the final results.
① π is oriented S^k-bundle
② B is simply-connected
③ (Leray-Hirsch) $H^*(E;\mathbb{Q}) \longrightarrow H^*(F;\mathbb{Q})$ is surjection
④ $\pi_1(B)$ acts on $H^*(F)$ trivially.

3. This is also a special case of Grothendieck-Serre spectral sequence.

5. filtration by $\mathcal{F}^\bullet$

For a single sheaf F, we have no way to produce filtrations.
Since we only care about H^i(F), we may replace F by another complex C^*, which is usually achieved by resolution:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \cdots \quad \text{exact}$$

i.e.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, "\varepsilon"] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ 0 \arrow[r] & C^0 \arrow[r, "d^1"] & C^1 \arrow[r, "d^2"] & C^2 \arrow[r, "d^3"] & \cdots \end{tikzcd} \quad \text{iso in } \mathcal{D}(X)$$

Then, one can use "stupid" truncation to get filtrations, and finally spectral sequences.
(The canonical truncation won't give you anything new)

E.p. When $C^\bullet$ is π_*-acyclic, the spectral seq deg, and
$H^i(X;\mathcal{F}) \cong \mathcal{H}^i(C^\bullet)$.

E.g. The spectral seqs in Section 1,2 are special cases.

Coefficient spectral seq

E.g. For $r \in \mathbb{Z}_{>0}$, consider the SES

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{\times r} \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow 0$$

which induces the iso

$$\underline{\mathbb{Z}/r\mathbb{Z}} \cong [\longrightarrow \underset{-1}{\underline{\mathbb{Z}}} \xrightarrow{\times r} \underset{0}{\underline{\mathbb{Z}}} \longrightarrow]$$

the triangle

$$\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}[1] \xrightarrow{+1}$$

and the LES

$$H^k(X;\mathbb{Z}) \longrightarrow H^k(X;\mathbb{Z}/r\mathbb{Z}) \longrightarrow H^{k+1}(X;\mathbb{Z}) \xrightarrow[+1]{\times r}$$

Ex. Compute $H^k(\mathbb{RP}^2;\mathbb{Z}/8\mathbb{Z})$ in this way.

A: $H^k(\mathbb{RP}^2;\mathbb{Z}/8\mathbb{Z}) = \begin{cases} \mathbb{Z}/8\mathbb{Z}, & k=0 \\ \mathbb{Z}/2\mathbb{Z}, & k=1,2 \\ 0, & \text{otherwise.} \end{cases}$

Rmk. In general, one can get coefficient spectral seq [FF16, 20.5.B, p318] through the coefficient seq: G abelian

$$\cdots \xrightarrow{\partial_2} \underline{G_1} \xrightarrow{\partial_1} \underline{G_0} \xrightarrow{\varepsilon} \underline{G} \longrightarrow 0$$

Q: Can we recover the inflation-restriction seq in this way?

$$0 \longrightarrow H^1(G/H, M^H) \xrightarrow{\mathrm{Inf}} H^1(G,M) \xrightarrow{\mathrm{Res}} H^1(H,M)$$

A: I think no. We haven't introduce $\overset{L}{\otimes}$ & $R\underline{\mathrm{Hom}}$.

Singular cohomology

We present a defective proof in [Vo02, Theorem 4.47].
The bug has been discussed in the stack exchange:
https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology

[Stackexchange]	$\mathcal{S}^k$	$\tilde{\mathcal{S}}^k$	$\mathcal{S}^k(X)_0$
[Vo02]	C^q_{sing}	$\mathcal{C}^q_{sing}$	$C^q_{sing}(\mathcal{U}, \mathbb{Z})_0$

$X \in \mathcal{T}op$.

<u>Def</u> The presheaf $\mathcal{S}^k \in Psh(X)$ is defined as

$$\mathcal{S}^k(\mathcal{U}) = \text{singular } k\text{-cochains on } \mathcal{U} = Hom_{Abel}(Sing_k(\mathcal{U}); \mathbb{Z})$$

where

$$Sing_k(\mathcal{U}) = \text{singular } k\text{-chains on } \mathcal{U} = \langle \sigma : \Delta^k \longrightarrow \mathcal{U} \rangle_{free.Abel}$$

and $\tilde{\mathcal{S}}^k := (\mathcal{S}^k)^{sh}$ is called the sheaf of singular cochains.

<u>Def</u>

$$H^k_{sing}(X;\mathbb{Z}) := \mathcal{H}^k(\mathcal{S}^\cdot(X))$$

<u>Thm</u> For X loc contractible,

$$H^k(X;\mathbb{Z}) \cong H^k_{sing}(X;\mathbb{Z})$$

Sketch of the wrong proof: Still hard. I don't check details.

1. Construct a complex of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\mathcal{S}}^0 \longrightarrow \tilde{\mathcal{S}}^1 \longrightarrow \tilde{\mathcal{S}}^2 \longrightarrow \cdots \qquad (*)$$

2. Show that $(*)$ is exact when X is loc contractible. $\Rightarrow \mathbb{Z} \simeq \tilde{\mathcal{S}}^\cdot$
3. Show that $\tilde{\mathcal{S}}^k$ is flabby, then

$$H^k(X;\mathbb{Z}) \cong \mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X))$$

Reduced to show: $\mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X)) \cong \mathcal{H}^k(\mathcal{S}^\cdot(X))$

4. Conside the exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^\cdot(X)_0 \longrightarrow \mathcal{S}^\cdot(X) \longrightarrow \tilde{\mathcal{S}}^\cdot(X) \longrightarrow \mathcal{S}^\cdot(X)^0 \longrightarrow 0$$

shows that:

① $\mathcal{S}^\cdot(X)^0 = 0$ ⚠ This is wrong. We ignore this bug...

② $\mathcal{S}^\cdot(X)_0 = \left\{ \alpha \in \mathcal{S}^\cdot(\mathcal{U}) \;\middle|\; \begin{array}{l} \exists \text{ open covering } \{\mathcal{U}_i\}_i \text{ of } \mathcal{U} \text{ st.} \\ \alpha|_{\mathcal{U}_i} = 0 \quad \forall i \end{array} \right\}$

③ $\mathcal{H}^k(\mathcal{S}^\cdot(X)_0) = 0$

Finally,

$$H^k(X;\mathbb{Z}) \cong \mathcal{H}^k(\tilde{\mathcal{S}}^\cdot(X)) \cong \mathcal{H}^k(\mathcal{S}^\cdot(X)) \overset{def}{=} H^k_{sing}(X;\mathbb{Z}).$$

Q. Can one shows $\mathcal{H}^k(\mathcal{S}^\cdot(X)^0) = 0$ to replace the bug in 4.①?

6. Hodge related filtration

Smooth de Rham resolution

The classical story is the real de Rham cohomology.

Thm For X mfld, one has iso

$$H^k(X;\underline{\mathbb{R}}) \cong H^k_{dR}(X;\mathbb{R}) := \mathcal{H}^k(\Omega_X^\bullet)$$

Here,

$$\Omega_X^k = \text{sheaf of smooth } k\text{-forms with coefficient } \mathbb{R}$$

https://en.wikipedia.org/wiki/De_Rham_cohomology#Proof

Sketch of proof

1. Construct a cplx of sheaves

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega_X^0 \xrightarrow{d^1} \Omega_X^1 \xrightarrow{d^2} \Omega_X^2 \xrightarrow{d^3} \cdots \qquad (\star)$$

2. Show that $(\star)$ is exact by Poincaré lemma. [Cl17, Lec 1, Lemma 5]

3. Show that Ω_X^i is π_*-acyclic [Cl17, Lec 1, Lemma 4], then

$$H^k(X;\underline{\mathbb{R}}) \cong \mathcal{H}^k(\Omega_X^\bullet) \overset{\text{def}}{=\!=} H^k_{dR}(X;\mathbb{R})$$

POU $\Rightarrow$ fine $\xRightarrow{\text{paracpt}}$ acyclic

Rmk. You can change coefficient from $\mathbb{R}$ to $\mathbb{C}$, where

$$\begin{aligned} \Omega_{X,\mathbb{C}}^k &= \text{sheaf of smooth } k\text{-forms with coefficient } \mathbb{R} \\ &= \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} \overset{?}{=} \Omega_X^k \otimes_{\underline{\mathbb{R}}} \underline{\mathbb{C}} \end{aligned}$$

it can be viewed as $\underline{\mathbb{R}}$-module?

then

$$H^k(X;\underline{\mathbb{C}}) \cong H^k_{dR}(X;\mathbb{C}) := \mathcal{H}^k(\Omega_{X,\mathbb{C}}^\bullet).$$