Coxeter gp $\twoheadleftarrow$ Braid gp $\rightsquigarrow$ Fundamental gp $\rightsquigarrow$ Aut

$\uparrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\uparrow$

Weyl gp $\qquad\qquad\qquad\qquad$ Galois gp

$\uparrow$ core $\qquad\qquad\qquad\qquad$ $\uparrow$

matrix gp / red gp $\qquad\qquad$ $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$

$\uparrow$ $\qquad\qquad\qquad\qquad\qquad\qquad$ $\cup$

"Rep of $G(\mathbb{Q}_p)$" $\longleftrightarrow$ "Rep of $W_{\mathbb{Q}_p}$"

Local Langlands correspondence $\rightsquigarrow$ FLC, GLC ?

Fact: any f.g. gp can be realized as $\pi_1(X)$ for some 4-dim mfld X.

Roadmap

1. Group $\rightsquigarrow$ Representation $\rightsquigarrow$ Geometrical object.

Group	Representation	Geometrical object
structure	character	construction
· set	classification	geo rep
· topology	L-fct	cohomology, intersection, ζ-fct
· root system, Tits system, BT-theory.		

2.

$$\begin{tikzcd} & & \text{Motive} \arrow[d, shift left, "\text{étale cohomology}"] \\ \text{Automorphic world} \arrow[urr, "\text{Shimura variety}"] & & \text{Gal rep} \arrow[ll, dashed, "\text{modularity}"] \arrow[u, dashed, shift left, "\text{FM conj}"] \end{tikzcd}$$

3.

	finite field		local field Archi	local field NA		global field		Adéle
base field	$\mathbb{F}_1$	$\mathbb{F}_p$	$\mathbb{R}$ or $\mathbb{C}$	$\mathbb{Q}_p$	$\mathbb{F}_p((t))$	$\mathbb{Q}$	$\mathbb{F}_p(t)$	$\mathbb{A}_K$
integral ring	—	—	—	$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K

only analog

4. ($G(F)$ - case) $\quad G \quad \mathbb{A}^1 \quad \mathbb{G}_m \quad GL_n \quad$ red gp (G, B, Unipotent, ...)

$\qquad G(F) \quad F \quad F^\times \quad GL_n(F) \quad G(F)$

$\qquad G(\mathbb{A}_K) \quad \mathbb{A}_K \quad I_K \quad GL_n(\mathbb{A}_K) \quad G(\mathbb{A}_K)$

Pseudo-reductive Groups

(both $G(F)$ & Galois)

coefficient ring Λ: $\mathbb{C}$, $\overline{\mathbb{Q}_p}$, $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Z}_p}$, ...

Roughly, need to solve $3 \times 2 \times 8 \times 4 = 192$ cases

\+ much more connections

Usual route: fix 3 & 4, $\boxed{\begin{matrix} 2 \\ G \text{ or } A \end{matrix}} \longrightarrow 1$

Anna's route: fix $4\,(GL_1)$, $\boxed{\begin{matrix} 2 \\ G,\ A,\ M \end{matrix}} \longrightarrow 3 \longrightarrow 1$

Our route: $1 \longrightarrow \boxed{\begin{matrix} 2 \\ G,\ A \end{matrix}} \longrightarrow 3 + 4$

connections in 2 are delayed.

The arrow roughly means:

2 | 1 $\mathbb{1}$

2 | 1 $\mathbb{1}$

Program

§1. No rep

1. Structure of finite/local/global field — Galois gp. Frob, Weil gp
2. Structure of reductive gp (GL_n) — Tits system. BT-theory

§2. 1-dim rep

1. Character of Galois gp — local class field theory
   corresponds to rep of $F^\times$
2. Character of red gp — $\widehat{F^*}$ & $\widehat{F^\times}$, $\widehat{\mathcal{O}_F^*}$ & $\widehat{\mathcal{O}_F^*}$, Hecke character

§3. Rep

1. Galois rep
   - WD-rep
   - $l \neq p$: l-adic monodromy thm
   - $l = p$: Hierarchy of p-adic Galois rep
   - global: Chebotarev density thm
2. Rep theory of red gp
   - fin / NA / $\mathbb{R}$ / $\mathbb{A}_K$
   - preliminary
   - Hecke alg
   - classification (Hierarchy) { principal series, cuspidal, …

§4. Geometrical rep

1. EC — étale cohomology. Fontaine-Mazur conj
2. MF (Moduli space, Shimura variety, Modular curve) — Shimura data, equiv def of MF
3. Flag variety — Rep II

§5. Connections

1. MF $\xrightarrow{\text{ES iso}}$ Gal rep — ES iso, ES relation, Deligne-Serre thm
2. MF $\xleftarrow{\text{modularity}}$ Gal rep — Modularity
3.

$$\begin{tikzcd} S_{K_p} \arrow[r] \arrow[d] & \mathcal{F}l \arrow[d] \\ Igs \arrow[r] & Bun_G \end{tikzcd}$$

"gen fiber of Shimura var" → S_{K_p}

"moduli space of abel var in char p up to quasi iso" → Igs

"moduli of G-bundles on FF curve" → Bun_G

Mingjia's work: HT period map, Torelli theorem

§6. Non-classical Langlands

1. Geometrical Langlands — Fargues-Scholze
2. Categorical geometrical Langlands — Chenji's work

https://mathoverflow.net/questions/56571/a-precise-statement-of-the-categorical-version-of-geometric-langlands-conjecture

Also, in each part:

- Describe L-fction
- Describe connections in section / among sections / with last part.