

# Eine Woche, ein Beispiel

## 9.4 Hecke algebra for matrix groups

This document is not finished. I need some time to digest and restate them.

I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.

main reference:

[Bump][<http://sporadic.stanford.edu/bump/math263/hecke.pdf>]

[XiongHecke][<https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf>]

[Ginzburg]: Representation Theory and Complex Geometry [<https://link.springer.com/book/10.1007/978-0-8176-4938-8>]

[Willians]: Langlands correspondence and Bezrukavnikov's equivalence

[KalethaTaïbi][<http://www-personal.umich.edu/~kaletha/lncqs.pdf>]

+ The local Langlands conjecture [in sciebo document]

All the references in [https://github.com/ramified/personal\\_handwritten\\_collection/blob/main/modular\\_form/README.md](https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md)

Task. For each double coset decomposition, we want to do.

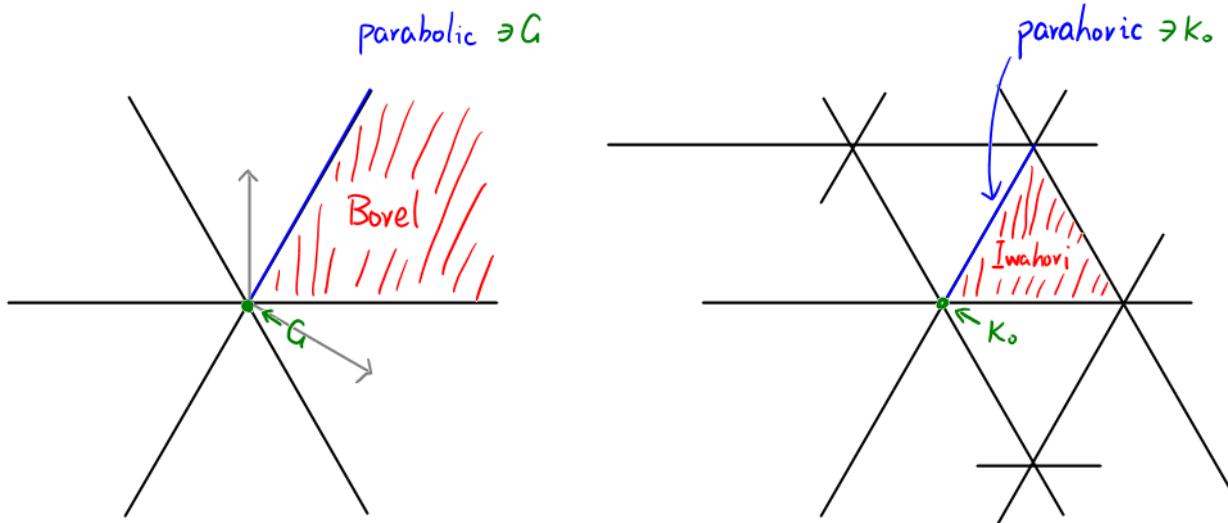
1. decomposition ( $\Gamma \backslash G / \Gamma$  is finite & definition of Hecke alg)
2.  $\mathbb{Z}$ -mod structure, notation
3. alg structure
4. conclusion

<https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of>

	Bruhat	Iwahori affine Bruhat	Cartan
$F$ finite	$G = \bigsqcup_{w \in W} B w B$		Smith normal form
$F$ local	$G = \bigsqcup_{w \in W} B w B$	$G = \bigsqcup_{w \in W_{\text{ext}}} I w I$	$G = \bigsqcup_{\alpha \in T^-} K_\alpha \alpha K_\alpha$
$F$ global	$G = \bigsqcup_{w \in W} B w B$		$GL_+^+(\mathbb{Q}) = \bigsqcup_{\alpha \in T^-} \Gamma \alpha \Gamma$
adèle?			

<https://mathoverflow.net/questions/4547/definitions-of-hecke-alg>

<https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebra>



$$B = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & p^{-1} & p^{-1} \\ p & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & p^{-1} \\ 0 & 0 & p^{-1} \\ p & p & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & p & 0 \end{pmatrix}$$

$$\begin{pmatrix} \pi & -1 \\ p & 1 \end{pmatrix} \in GL_2(\mathbb{Q}) \Rightarrow \begin{pmatrix} \pi & -1 \\ p & 1 \end{pmatrix} \notin I$$

mirabolic: (miracle parabolic)

parahoric (containing an Iwahori subgroup)

<https://mathoverflow.net/questions/24960/why-are-parabolic-subgroups-called-parabolic-subgroups>

For the classical group case, see: <https://math.stackexchange.com/questions/3068424/iwahori-versus-bruhat-decompositions>

▽  $GL_2(\mathbb{Z}_p) \neq \{(a b) \in GL_2(\mathbb{Q}_p) \mid a, b, c, d \in \mathbb{Z}_p\}$ . Instead,

$$GL_2(\mathbb{Z}_p) = \left\{ (a b) \in GL_2(\mathbb{Q}_p) \mid \begin{array}{l} a, b, c, d \in \mathbb{Z}_p \\ \det(a b) \in \mathbb{Z}_p^\times \end{array} \right\}$$

$$\cong \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Similarly,

$$I \cong \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$$

where  $\det \in \mathcal{O}^\times$  is required implicitly.

e.g.  $(\pi) \notin I$ .

Tip: Those matrix decomposition theorems may seem quite frightening in the beginning.  
 In fact, they are just fancy special cases of Gaussian elimination.  
 Remember,

左乘行变换, 右乘列变换  
 multiply a matrix on the left hand side is equiv. to do row operations,  
 multiply a matrix on the right hand side is equiv. to do column operations.

The canonical form usually has entries 0 almost everywhere.  
 To compute the canonical form, we use allowed row/column operations.

E.g.  $G = \bigsqcup_{w \in W} B w B$        $g \sim g' \Leftrightarrow \exists b_1, b_2 \in B \text{ s.t. } g = b_1 g' b_2$   
 e.g.  $q = 7, G = GL_3(\mathbb{F}_7)$

$$\begin{pmatrix} 5 & 1 & 6 \\ 0 & 2 & 4 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} -1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 5 & 1 & 6 \\ 1 & 5 & 3 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 5 & 3 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & -5 & -3 \\ & 1 & 1 \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow{\left( \begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 6 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\left( \begin{smallmatrix} 4 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

E.g.  $G = \bigsqcup_{w \in W_{\text{ext}}} I w I$        $g \sim g' \Leftrightarrow \exists x_1, x_2 \in I \text{ s.t. } g = x_1 g' x_2$

e.g.  $q = 3$        $G = GL_2(\mathbb{Q}_3)$        $I = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \subset GL_2(\mathbb{Z}_3)$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} -3 & 1 \\ & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 1 & \frac{1}{3} \\ -3 & 0 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & \\ & -3 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & \frac{1}{3} \\ -3 & 0 \end{pmatrix} \xrightarrow{\left( \begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 3 & 3^{-1} \end{pmatrix}$$

e.g.  $q = 3$        $G = GL_3(\mathbb{Q}_3)$        $I = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \subset GL_3(\mathbb{Z}_3)$

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & \frac{1}{2} & -\frac{7}{2} \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 2 & \frac{1}{2} & 3 \\ 3 & \frac{1}{2} & 2 \\ 3 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 0 & -6 \\ 0 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

To show the disjointness (i.e. canonical form does not depend on the process), we observe that some properties of  $k \times k$ -minors are preserved under the restricted row/column operations. For  $k=1$  these invariants can be seen easier.

$$\text{E.g. } G = \bigsqcup_{w \in W} BwB$$

$$\xrightarrow{\text{not } 0} \left( \begin{array}{c|c} * & \\ \vdots & \\ * & * \\ \hline 0 & \\ \vdots & \\ 0 & \end{array} \right) \sim \left( \begin{array}{cccccc} 0 & & & * & & & \\ \vdots & & & & & & \\ 1 & 0 & \cdots & \cdots & \cdots & 0 & \\ 0 & & & * & & & \\ \vdots & & & & & & \\ 0 & & & & & & \end{array} \right)$$

$$G = \bigsqcup_{\lambda \in T} K_\lambda K_\lambda$$

$$A = (a_{ij})_{i,j=1}^n \quad e = \underbrace{\min_{i,j} v(a_{ij})}_{\sim} \quad \left( \begin{array}{ccccc} \pi^e & & & 0 & \\ & \ddots & & & \\ 0 & & \ddots & & \end{array} \right)$$

$$G = \bigsqcup_{w \in W_{\text{ext}}} I_w I$$

$$A = (a_{ij})_{i,j=1}^n \quad e = \underbrace{\min_{i,j} v(a_{ij})}_{\sim} \quad i_0 \left( \begin{array}{ccccc} * & & & * & \\ & \vdots & & & \\ 0 & \cdots & \pi^e & \cdots & 0 \\ & & & & \\ * & & & 0 & * \end{array} \right)$$

$$I = \{(i,j) \mid v(a_{ij}) = e\}$$

$(i_0, j_0) \in I$  is in the lower left corner

$$\text{i.e. } \forall (i,j) \in I, \quad \left. \begin{array}{l} i \geq i_0 \\ j \leq j_0 \end{array} \right\} \Rightarrow (i,j) = (i_0, j_0)$$

If you have no clue on the properties of  $k \times k$ -minors for  $k > 1$ , you can see [Bump, Section 9] for the case of  $p$ -adic Cartan decomposition.

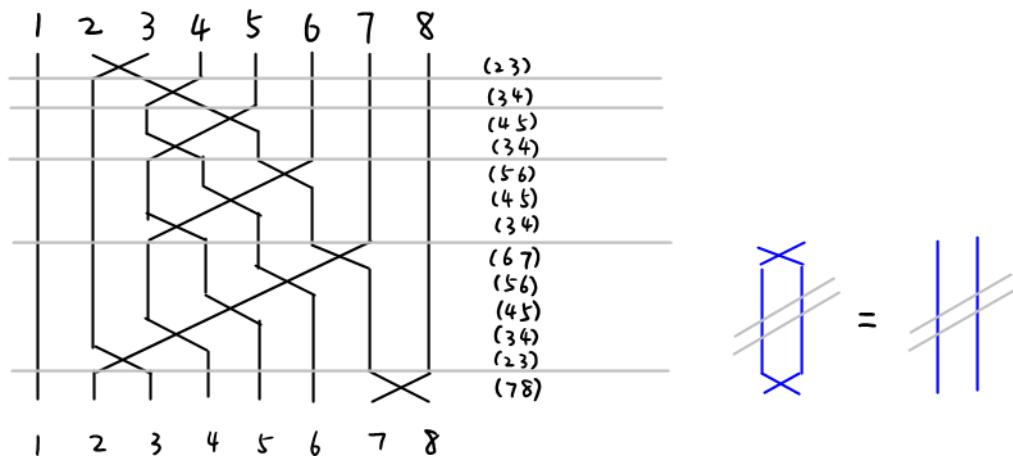
## $S_n$ and Tits system

A brief preparation for computations in Bruhat decomposition.  $s_i = (i \ i+1)$ ,  $1 \leq i \leq n-1$

E.g.  $n=8$ ,  $w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8$ .

Ex. Compute  $l(w_0)$ ,  $l(s_i w_0)$  and  $l(w_0 s_i)$ .

Solution.



$$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$$

$l(w_0) = 13$  = "inversion number"

$$l(s_1 w_0) = 14 \quad l(w_0 s_1) = 14$$

$$l(s_2 w_0) = 12 \quad l(w_0 s_2) = 12$$

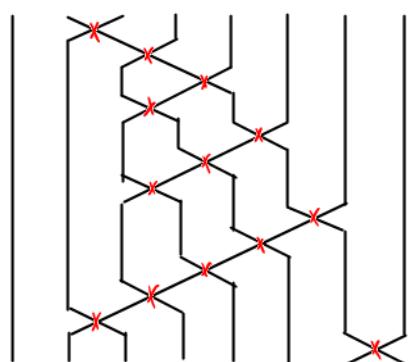
$$l(s_3 w_0) = 14 \quad l(w_0 s_3) = 14$$

$$l(s_4 w_0) = 12 \quad l(w_0 s_4) = 12$$

$$l(s_5 w_0) = 12 \quad l(w_0 s_5) = 12$$

$$l(s_6 w_0) = 12 \quad l(w_0 s_6) = 14$$

$$l(s_7 w_0) = 14 \quad l(w_0 s_7) = 12$$



How to see the length:  
count the intersection number

$$l(w_0) = 13$$

Ex. Let  $G = GL_n(\mathbb{F}_q)$ ,  $B = \begin{pmatrix} * & \cdots & * \\ 0 & \cdots & 0 \end{pmatrix} \leq G$ ,  $T = \begin{pmatrix} * & \cdots & 0 \\ 0 & \cdots & * \end{pmatrix} \leq B$ ,  
 $w_0, s_i \in N(T)$  a lift from  $w_0, s_i \in S_n = N(T)/T$ .  
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0) + 1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0) - 1 \end{cases}$$

Solution

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \quad w_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

The following computation will be also computed later on.

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

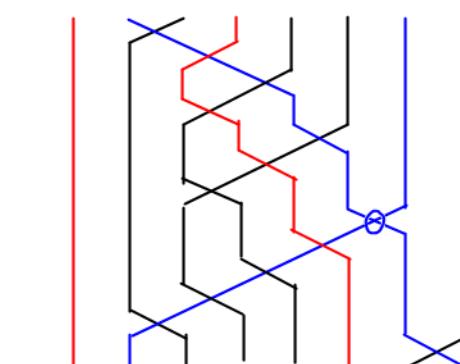
$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0 \cap w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0Bw_0^{-1}$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad B \cap w_0Bw_0^{-1}$$

How to see  $w_0Bw_0^{-1}$ :

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0Bw_0^{-1}$$



finite Bruhat decomposition

Let  $G = GL_n(\mathbb{F}_q)$ ,  $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leq G$ ,  $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \leq B$ ,  
 $w_0, s_i \in N(T)$  a lift from  $w_0, s_i \in S_n = N(T)/T$ .  
(usually take the permutation matrix)

1. decomposition  $G = \bigsqcup_{w \in W} BwB$

Ex.  $(BwB)^{-1} = Bw^{-1}B$  but  $BwB \cdot Bw^{-1}B \neq B$  is possible

Ex. Compute  $|BwB/B|$   $\nabla BwB$  may not be a group!

Hint: Consider the map

$$\phi: B \longrightarrow BwB/B$$

$$b \mapsto b w B$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 w B = b_2 w B$$

$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$

$$\Leftrightarrow b_2^{-1} b_1 \in w B w^{-1}$$

$$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{\ell(w)}$$

We take Haar measure  $\mu$  on  $G$  s.t.  $\mu(B) = 1$ ,  $\mu(pt) = \frac{1}{|B|}$ .

Recall that  $\mathcal{H}(G, B) = \{f: G \rightarrow \mathbb{Z} \mid f(b_1 g b_2) = f(g) \quad \forall b_1, b_2 \in B, g \in G\}$  where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2.  $\mathbb{Z}$ -mod structure, notation

$$\mathcal{H}(G, B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbf{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote  $T_w := \mathbf{1}_{BwB}$ ,  $T_{s_i} := T_{s_i}$  ( $T_{Id} = \mathbf{1}_B$  is the unit of  $\mathcal{H}(G, B)$ )

then  $\{T_w\}_{w \in W}$  is a "basis" of  $\mathcal{H}(G, B)$ .

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) T_w$$

$$\begin{aligned} (T_u * T_v)(w) &= \frac{1}{|B|} \sum_{y, z \in w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} |\{(y, z) \in BwB \times BvB \mid yz = w\}| \quad \text{if } w \notin BwBv^{-1}B \\ &= \frac{1}{|B|} |BwB \cap wBv^{-1}B| \\ &= \frac{1}{|B|} |\{b \in B, wbv^{-1} \in BwB \mid b \in B, wbv^{-1} \in B\}| \end{aligned}$$

$$B_{S_i}B \cdot B_{W_0}B = \begin{cases} B_{S_i}w_0B & l(S_iw) = l(w) + 1 \\ B_{S_i}w_0B \cup B_{W_0}B & l(S_iw) = l(w) - 1 \end{cases}$$

$$\Rightarrow T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{S_i w} & l(S_i w) = l(w) + 1 \\ \mathbb{Z} \cdot T_{S_i w} + \mathbb{Z} \cdot T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Computation of coefficient.

$$|BwB| = |BwB/B| \times |B| = q^{l(w)} \cdot |B|$$

when  $l(S_iw) = l(w) + 1$ ,

$$(T_i * T_w)(S_iw) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz = S_iw \right\}$$

$$= \frac{1}{|B| |B_{S_i}wB|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz \in B_{S_i}wB \right\}$$

$$= \frac{|B_{S_i}B| |BwB|}{|B| \cdot |B_{S_i}wB|} = \frac{q^{l(S_i)} q^{l(w)}}{q^{l(S_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = Id \right\}$$

$$= \frac{1}{|B|} |B_{S_i}B| = q$$

$$(T_i * T_i)(S_i) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = S_i \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B_{S_i}B \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left( |B_{S_i}B \times B_{S_i}B| - \left| \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B \right\} \right| \right)$$

$$= \frac{1}{|B| |B_{S_i}B|} \left( |B_{S_i}B| |B_{S_i}B| - |B| \cdot |B_{S_i}B| \right)$$

$$= q - 1$$

when  $l(S_iw) = l(w) - 1$ , we get  $l(S_i \cdot S_iw) = l(S_iw) + 1$ ,

$$\begin{aligned} T_i * T_w &= T_i * T_i * T_{S_i w} \\ &= (qT_{Id} + (q-1)T_i) * T_{S_i w} \\ &= qT_{S_i w} + (q-1)T_w \end{aligned}$$

$$\Rightarrow T_i * T_w = \begin{cases} T_{S_i w} & l(S_i w) = l(w) + 1 \\ qT_{S_i w} + (q-1)T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$$\mathcal{H}(G, B) = \mathbb{Z}\langle T_1, \dots, T_{n-1} \rangle_{alg} \text{ with relations } (\mathcal{H}(G, B) \subseteq \mathcal{H}_q(W))$$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j| \geq 2$$

Q. How to show that there are no further relations?

A. By comparing the dimensions.

$$\text{E.g. For } n=2, \quad \mathcal{H}(G, B) \cong \mathbb{Z}[T_1] / (T_1^2 - (q-1)T_1 - q)$$

$$\cong \mathbb{Z}[T_1] / (T_1 - q)(T_1 + 1)$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1$$

$$\text{For } n=3, \quad \mathcal{H}(G, B) \cong \mathbb{Z}\langle T_1, T_2 \rangle / ((T_1 - q)(T_1 + 1), (T_2 - q)(T_2 + 1), T_1 T_2 T_1 = T_2 T_1 T_2)$$

$$\stackrel{\mathbb{Z}-\text{mod}}{=} \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_1 T_2 \oplus \mathbb{Z} T_2 T_1 \oplus \mathbb{Z} T_1 T_2 T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_{(12)} \oplus \mathbb{Z} T_{(13)} \oplus \mathbb{Z} T_{(123)}$$

global Cartan decomposition  
1. decomposition

Thm (Elementary divisor thm)  $R$ : PID (In naive proof  $R$  should be ED)

$$M_{2 \times 2}(R) = \coprod_{(b) \subseteq (a)} GL_2(R) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(R)$$

$$\text{Cor } M_{2 \times 2}(\mathbb{Z}) = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 \leq a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det \neq 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det > 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a, b \in \mathbb{Q}_{>0}^\times \\ v_p(a) \leq v_p(b) \quad \forall p}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det > 0}$$

Denote  $\Gamma = SL_2(\mathbb{Z})$ ,

$$\Gamma^- = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \in GL_2^+(\mathbb{Q}) \mid \begin{array}{l} a, b > 0 \\ v_p(a) \leq v_p(b) \quad \forall p \text{ prime} \end{array} \right\} \stackrel{\text{Grp}}{\cong} \mathbb{Q}_{>0}^\times \times (\mathbb{Z}_{>0}, \times)$$

then

$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha \in \Gamma^-} \Gamma \alpha \Gamma$$

Ex. Verify that  $\Gamma \alpha \Gamma / \Gamma$  is finite, and compute the order.  $\alpha = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \in \Gamma^-$

Hint. See [WWL, 引理 5.1.4].

$$\# \Gamma \alpha \Gamma / \Gamma = \# \Gamma / \Gamma \cap \alpha \Gamma \alpha^{-1} = \# \Gamma / \Gamma_0 \left( \frac{\alpha_1}{\alpha_2} \right) = \# \text{Irr} \left( \frac{\alpha_1}{\alpha_2} \right) = \frac{\alpha_2}{\alpha_1} \prod_{p \mid \frac{\alpha_1}{\alpha_2}} \left( 1 + \frac{1}{p} \right)$$

$$\left[ \alpha \begin{pmatrix} a & \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} a & \frac{a_1}{a_2} b \\ \frac{c}{a_2} c & d \end{pmatrix} \right] \Rightarrow \Gamma \cap \alpha \Gamma \alpha^{-1} = \left( \frac{\mathbb{Z}}{\alpha_1 \mathbb{Z}}, \frac{\mathbb{Z}}{\alpha_2 \mathbb{Z}} \right)_{\det=1} = \Gamma_0 \left( \frac{\alpha_2}{\alpha_1} \right)$$

$$\text{e.g. } \# \Gamma \begin{pmatrix} \alpha_1 & \\ 0 & \alpha_2 \end{pmatrix} \Gamma / \Gamma = 1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma / \Gamma = p+1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix} \Gamma / \Gamma = p^e + p^{e-1}$$

The desired measure can not be realized here, i.e.,

a Haar measure  $\mu$  on  $GL_2^+(\mathbb{Q})$  s.t.  $\mu(\Gamma) = 1$ .

Reason: measure satisfies countable additivity, and  $\Gamma$  is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure  $\rightsquigarrow$  semimeasure)

To e.g.: There is no way to define a Haar measure  $\mu$  on  $\mathbb{Q}$  s.t.  $\mu(\mathbb{Z}) = 1$ .

However, if we only require finite additivity, we can do it.

Def (Semimeasure on  $\mathbb{Q}$ )

For any periodic set  $X \subseteq \mathbb{Q}$  (i.e.,  $\exists m \in \mathbb{Q}_{>0}$  s.t.  $m + X = X$ )  
we set

$$\text{Rmk. 1. } \mu(X) = \frac{1}{m} |X/m\mathbb{Z}| = \frac{1}{m} |X \cap [0, m]|$$

$$\mathbb{Z} \supset \frac{X}{m\mathbb{Z}} \quad |X/m\mathbb{Z}|, |m\mathbb{Z}/m\mathbb{Z}| < +\infty$$

" $m\mathbb{Z}$  are all commensurable gps of  $\mathbb{Z}$ "

2.  $X = \bigsqcup_{\alpha \in \Delta} \alpha + m\mathbb{Z}$  for some  $\Delta \subseteq \mathbb{Q}/m\mathbb{Z}$

" $X$  is a commensurable set of  $\mathbb{Z}$  (when  $\mu(X) < +\infty$ )"

Long A: Def. (Semimeasure on  $GL_2^+(\mathbb{Q})$ )

For any gp  $H \leq GL_2^+(\mathbb{Q})$  which is commensurable with  $\Gamma$

(i.e.,  $\#H/\mathbb{H}\cap\Gamma, \#\Gamma/\mathbb{H}\cap\Gamma$  are finite), set

$$\mu(H) = \frac{|H/\mathbb{H}\cap\Gamma|}{|\Gamma/\mathbb{H}\cap\Gamma|} \stackrel{\text{if } H \leq \Gamma}{=} \frac{1}{|\Gamma/H|} \in \mathbb{Q}_{>0}$$

Similarly we can specify  $\mu$  to any commensurable set  $X \subseteq GL_2^+(\mathbb{Q})$ .

$$\left( \begin{array}{l} \text{i.e., } X = \bigsqcup_{\alpha \in \Delta} \alpha H \text{ for some } H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ X = \bigsqcup_{\alpha \in \Delta'} H' \alpha' \quad \Delta \subseteq GL_2^+(\mathbb{Q})/H, \Delta' \subseteq H' \backslash GL_2^+(\mathbb{Q}) \\ \Delta, \Delta' \text{ finite} \end{array} \right)$$

Rmk: In the most of references the terminology (semi)measure  
is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and  
take the second line as a def of the convolution.

Def. (Hecke alg  $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$ )

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \rightarrow \mathbb{Z} \mid \begin{array}{l} f(\gamma_1 \alpha \gamma_2) = f(\alpha) \quad \forall \gamma_1, \gamma_2 \in \Gamma, \alpha \in GL_2^+(\mathbb{Q}) \\ \#(\text{supp } f)/\Gamma < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} f_1(x) f_2(x^{-1}g) = \sum_{y \in \Gamma \backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1}) f_2(y)$$

2.  $\mathbb{Z}$ -mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha \in \Gamma} \mathbb{Z} \cdot \mathbf{1}_{\Gamma \alpha \Gamma}$$

denote  $T_\alpha := \mathbf{1}_{\Gamma \alpha \Gamma}$

$$\begin{aligned} \lambda \in \mathbb{Q}^\times & \quad R_\lambda := T_{(\lambda)} = \mathbf{1}_{\Gamma(\lambda) \Gamma} = \mathbf{1}_{\lambda \Gamma} \quad (R_1 = \mathbf{1}_\Gamma \text{ is the unit of } \mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)) \\ p \text{ prime, } e \geq 1 & \quad T_{p^e} := T_{(p^e)} = \mathbf{1}_{\Gamma(p^e) \Gamma} \quad T_p := T_{(p)} = \mathbf{1}_{\Gamma(p) \Gamma} \end{aligned}$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in \Gamma} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha \beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x) T_\beta(x^{-1} \gamma) \\ &= \# \left\{ x \in GL_2^+(\mathbb{Q})/\Gamma \mid \begin{array}{l} x \in \Gamma \alpha \Gamma \\ x^{-1} \gamma \in \Gamma \beta \Gamma \end{array} \right\} \\ &= |\Gamma \alpha \Gamma \cap \gamma \Gamma \beta^{-1} \Gamma / \Gamma| \end{aligned}$$

e.p.  $\mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1} g) = f(g \lambda^{-1}) = (f * R_\lambda)(g)$

$$R_\lambda * R_\mu = R_{\lambda \mu}$$

E.g.  $g_{\alpha \beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta|$  where  $|\alpha| := \det \alpha$

The formula above is still not feasible for effective calculation.  
We will derived the easiest way to compute  $g_{\alpha \beta}^\gamma$  in the next page.

$$\text{Suppose } \Gamma_\alpha \Gamma / \Gamma = \{x_1 \Gamma, \dots, x_i \Gamma, \dots\}$$

$$\Gamma_\beta \Gamma / \Gamma = \{y_1 \Gamma, \dots, y_j \Gamma, \dots\}$$

then

$$\begin{aligned} g_{\alpha\beta} &= \sum_{x \in \Gamma_\alpha \cap \Gamma_\beta} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \sum_i T_\beta(x_i^{-1}\gamma) \\ &= \sum_i \mathbf{1}_{x_i^{-1}\gamma \in \Gamma_\beta \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i^{-1}\gamma \in y_j \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{\Gamma_\alpha y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_{y \in \Gamma_\beta \Gamma / \Gamma} \mathbf{1}_{\Gamma_\alpha y \Gamma = \gamma \Gamma} \\ &= \frac{|\Gamma_\alpha \Gamma / \Gamma|}{|\Gamma_\beta \Gamma / \Gamma|} \end{aligned}$$

where

$$\Gamma' := \{\gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_\beta \Gamma\} = \alpha^{-1} \Gamma_\beta \Gamma \beta^{-1} \cap \Gamma$$

$$\Rightarrow \Gamma'_\beta \Gamma / \Gamma = \alpha^{-1} \Gamma_\beta \Gamma \cap \Gamma \beta^{-1} / \Gamma$$

depends on  $\alpha, \beta, \gamma$ .

The rest is a routine work.

$$\text{Ex. } \Gamma('m) \Gamma \cdot \Gamma('n) \Gamma = \Gamma('_{mn}) \Gamma \quad (m, n) = 1$$

$$\Gamma('_{p^e}) \Gamma \cdot \Gamma('_p) \Gamma = \Gamma('_{p^{e+1}}) \Gamma \sqcup \Gamma('_{p^e}) \Gamma \quad p \text{ prime, } e \geq 1$$

[ Hint.  $('m)(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})('n) = (\begin{smallmatrix} a & nb \\ mc & mnd \end{smallmatrix}) \in \Gamma('_{\frac{m}{l}}) \Gamma \quad \text{for } (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z})$  ]

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z} \cdot T_{mn} \\ T_{p^e} * T_p \in \mathbb{Z} \cdot T_{p^{e+1}} + \mathbb{Z} T_{p^{e-1}} R_p \end{cases} \quad (m, n) = 1$$

$$p \text{ prime, } e \geq 1$$

Computation of coefficient:

when  $(m, n) = 1$ ,  $\alpha = (1_m)$ ,  $\beta = (1_n)$ ,  $\gamma = (1_{mn})$ ,

$$\begin{aligned} g_{\alpha\beta}^{\gamma} &= \frac{1}{|\Gamma_{\gamma}\Gamma/\Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \in \Gamma_{\gamma}\Gamma} \\ &= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma_{\beta}\Gamma/\Gamma|}{|\Gamma_{\gamma}\Gamma/\Gamma|} \\ &= 1 \end{aligned}$$

when  $p$  is prime,  $e \geq 1$ ,  $\alpha = (1_{p^e})$ ,  $\beta = (1_p)$ ,  $\gamma_2 = (p_{p^e})$ ,  $\gamma_1 = (1_{p^{e+1}})$ ,

$$\begin{aligned} \Gamma'_2 &\triangleq \left\{ \gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_{\gamma_2} \Gamma \right\} \\ &= \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \gcd(a, pb, p^e c, p^{e+1} d) = p \right\} \\ &= \left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv \left( \begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix} \right) \pmod{p} \right\} \\ &= \Gamma^0(p) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ |\Gamma'_2 \beta \Gamma/\Gamma| &= \left| \Gamma^0(p) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma/\Gamma \right| \\ &= \left| \Gamma^0(p) \right| / \left| \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma^0(p) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \cap \Gamma^0(p) \right| \\ &\stackrel{\text{def}}{=} \left| \Gamma^0(p) \right| / \left| \Gamma^0(p) \right| \\ &= 1 \end{aligned}$$

$$\begin{aligned} \left[ \begin{aligned} \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} &= \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left( \begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left( \begin{smallmatrix} a & b \\ -pc & -pd \end{smallmatrix} \right) \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ &= \left( \begin{smallmatrix} a & b \\ -pc & -pd \end{smallmatrix} \right) \end{aligned} \right] \end{aligned}$$

$$\therefore g_{\alpha\beta}^{\gamma_2} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_2 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_2} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot 1}{p^{e-1} - p^{e-2}}$$

$$g_{\alpha\beta}^{\gamma_1} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_1 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| (|\Gamma_{\beta}\Gamma/\Gamma| - |\Gamma'_1 \beta \Gamma/\Gamma|)}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot (p+1 - 1)}{p^{e+1} - p^e}$$

$$= 1$$

$$4. \text{ Conclusion. } \mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$$

with  $\begin{cases} T_m T_n = T_{mn} \\ T_p^e T_p = T_{p^{e+1}} + p T_{p^{e-1}} R_p \end{cases}$

$(m, n) = 1$   
 $p \text{ prime, } e \geq 1$

By [Hecke, Thm 12],  $\Gamma$  is a Gelfand subgp of  $GL_2^+(\mathbb{Q})$ ,  
thus  $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$  is commutative.  
Gelfand involution:  $\sigma \mapsto \sigma^\top$

Task: generalize it to other congruence subgps.

$p$ -adic Cartan decomposition / not Grothendieck group!  
Set  $G = GL_2(F)$ ,  $K_0 = GL_2(\mathcal{O}_F)$ .

1. decomposition [Bump Prop 35]

$$M_{2 \times 2}(\mathcal{O}_F) = \coprod_{0 \leq e_i \leq e_1 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$M_{2 \times 2}(\mathcal{O}_F)_{\text{det} \neq 0} = \coprod_{0 \leq e_1 \leq e_2 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$G = \coprod_{e_1 \leq e_2} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

Denote  $T^- = \left\{ \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in G \mid e_1 \leq e_2, e_1, e_2 \in \mathbb{Z} \right\} \stackrel{\text{semi gp}}{\cong} \mathbb{Z} \oplus \mathbb{Z}_{\geq 0}$ , then  
 $G = \coprod_{\alpha \in T^-} K_0 \alpha K_0$

Ex. Verify that  $K_0 \alpha K_0 / K_0$  is finite, and compute the order.  $\alpha = \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in T^-$

Hint.

$$\# K_0 \alpha K_0 / K_0 = \# K_0 / K_0 \cap \alpha K_0 \alpha^{-1} = \# K_0 / \Gamma_0(\mathfrak{p}_F^{e_1 - e_2}) = \# \text{IP}'(\mathcal{O}_F / \mathfrak{p}_F^{e_1 - e_2}) = \begin{cases} q^{e_2 - e_1} + q^{e_2 - e_1 - 1} & e_1 < e_2 \\ 1 & e_1 = e_2 \end{cases}$$

$$\left[ \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O}^{e_1 - e_2} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} \Rightarrow K_0 \cap \alpha K_0 \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} = \Gamma_0(\mathfrak{p}_F^{e_2 - e_1}) \right]$$

$$\text{e.g. } \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = 1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q+1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q^e + q^{e-1}$$

Here we use the similar notation in modular form

[[https://github.com/ramified/personal\\_handwritten\\_collection/blob/main/modular\\_form/5.moduli\\_interpretation.pdf](https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/5.moduli_interpretation.pdf)]:

$$\begin{array}{ccc} \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \{ \text{Id} \} \\ \cap & & \cap \\ \text{bal. } \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & N(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & B(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathcal{O}) = K^0 = GL_2(\mathcal{O}_F) & \xrightarrow{\quad} & GL_2(\mathcal{O}_F / \mathfrak{p}_F^e) \end{array}$$

Take the unique Haar measure on  $G$  s.t.  $\mu(K^0) = 1$ , then

$\mu(K^0 \alpha K^0) = \# K^0 \alpha K^0 / K^0$   
 $\mu$  is induced from the measure on coset  $G / K^0$ .

The Hecke algebra has been defined here:

[https://github.com/ramified/personal\\_handwritten\\_collection/blob/main/weeklyupdate/2022.04.17\\_preliminary\\_facts\\_of\\_reps\\_of\\_p-adic\\_groups.pdf](https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.04.17_preliminary_facts_of_reps_of_p-adic_groups.pdf)

We still recall the convolution here:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in G/K_0} f_1(x) f_2(x^{-1}g) = \sum_{y \in K_0 \backslash G} f_1(gy^{-1}) f_2(y)$$

## 2. $\mathbb{Z}$ -mod structure, notation

$$\mathcal{H}(G, K_0) = \bigoplus_{\alpha \in T} \mathbb{Z} \cdot \mathbf{1}_{K_0 \alpha K_0}$$

denote  $T_\alpha := \mathbf{1}_{K_0 \alpha K_0}$

$$\begin{array}{lll} \lambda \in F^\times & R_\lambda := T_{(\lambda)} = \mathbf{1}_{K_0(\lambda) K_0} = \mathbf{1}_{\lambda K_0} & (R_1 = \mathbf{1}_{K_0} \text{ is the unit of } \mathcal{H}(G, K_0)) \\ \epsilon \geq 1 & T_\pi^\epsilon := T_{(\pi^\epsilon)} = \mathbf{1}_{K_0(\pi^\epsilon) K_0} & T_\pi := T_{(\pi)} = \mathbf{1}_{K_0(\pi) K_0} \end{array}$$

## 3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/K_0} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \# \left\{ x \in G/K_0 \mid \begin{array}{l} x \in K_0 \alpha K_0 \\ x^{-1}\gamma \in K_0 \beta K_0 \end{array} \right\} \\ &= |K_0 \alpha K_0 \cap K_0 \beta^{-1} K_0| / |K_0| \end{aligned}$$

$$\text{e.p. } \mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$$

$$R_\lambda * R_\mu = R_{\lambda\mu}$$

$$\text{E.g. } g_{\alpha\beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad \text{where } |\alpha| := \det \alpha$$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^\gamma = \frac{|K_0 \alpha K_0| |K_0 \beta K_0|}{|K_0 \gamma K_0|}$$

where

$$K_0' := \{\gamma' \in K_0 \mid \alpha \gamma' \beta \in K_0 \gamma K_0\} = \alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0$$

$$\Rightarrow |K_0' \beta K_0| / |K_0| = |\alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0| / |K_0|$$

depends on  $\alpha, \beta, \gamma$ .

$$\Rightarrow T_{\pi^\epsilon} T_\pi = T_{\pi^{\epsilon+1}} + q T_{\pi^{\epsilon+1}} R_\pi$$

4. Conclusion (Tamagawa, Satake)

$$\mathcal{H}(G, K^\circ) = \mathbb{Z} [R_\pi^{\pm 1}, T_\pi] \quad \text{with}$$
$$T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q T_{\pi^{e-1}} R_\pi$$

$$\text{e.p. } \mathcal{H}(GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z})) = \bigotimes_{\mathbb{Z}} \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$$

$p$ -adic Iwahori decomposition

We only consider  $GL_n, SL_n, PGL_n$  in this section.  $SL_2$  case is special focused.

$$\text{E.g. } G = SL_2(F) \quad T(F) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad I = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}_{\det=1}$$

$$U(F) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad U(F) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

$$U(O_F) = \begin{pmatrix} 1 & 0_F \\ 0 & 1 \end{pmatrix} \quad U(O_F) = \begin{pmatrix} 1 & 0 \\ 0_F & 1 \end{pmatrix}$$

$$N_G(T(F))/T(F) := W_f$$

$$N_G(T(O_F))/T(O_F) = W_{\text{ext}} \cong \langle s_1, s_0 \rangle_{(s_1^2 = s_0^2 = 1)} \quad s_1 = \begin{pmatrix} 1 & \\ -1 & 1 \end{pmatrix} \quad s_0 = \begin{pmatrix} & \pi \\ -\pi^{-1} & \pi \end{pmatrix}$$

	$\triangle \Phi$ basis roots	root	char	weight	$\triangle \Phi^\vee$ basis coroots	coroot	cochar	coweight	
My notation	$\triangle \Phi$	$Q \subseteq X^* \subseteq P$			$\triangle \Phi^\vee$	$Q^\vee \subseteq X_\tau \subseteq P^\vee$			$W_f \quad W_{\text{aff}} \quad W_{\text{ext}}$
[Ginzburg]	$R$	$Q \subseteq P$			$R^\vee$	$Q^\vee \subseteq P^\vee$			$W \quad W_{\text{aff}}$
[Williams]	$R$	$\subseteq X$			$R^\vee$	$\subseteq X^\vee$			$W_f \quad W \quad W_{\text{ext}}$
[Bump]	$\Phi$	$Q \subseteq P$			$\Phi^\vee$	$Q^\vee \subseteq P^\vee$			$W \quad W_{\text{aff}} \quad \tilde{W}_{\text{aff}}$
									in dual not in dual not in dual
									Iwahori Hecke algebra to $P^\vee$ , not the standard $H(G, I)$
[Kaletha-Taiibi]	$\triangle R$	$\subseteq X$			$R^\vee$	$\subseteq X^\vee$			

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q^\vee & \longrightarrow & W_{\text{aff}} & \xrightarrow{\text{pr}} & W_f \longrightarrow 0 \\ & & \cap & & \Delta & & \\ 0 & \longrightarrow & X_\tau(T) & \longrightarrow & W_{\text{ext}} & \xrightarrow{\text{pr}} & W_f \longrightarrow 0 \end{array}$$

$\Rightarrow$  Standard isomorphism

$$W_{\text{aff}} \cong Q^\vee \rtimes W_f$$

$$W_{\text{ext}} \cong X_\tau(T) \rtimes W_f$$

Q: How does  $W_{\text{ext}}$  act on

$$\pi_*(T) = X_*(T) \cong T(F)/T(O_F) \cong \mathbb{Z}^{\text{rank } G} \subseteq \mathbb{Z}^n . ?$$

$$\left[ \lambda : t \mapsto \begin{pmatrix} t^{e_1} \\ \vdots \\ t^{e_n} \end{pmatrix} \right] \leftrightarrow \begin{pmatrix} \pi^{e_1} & & \\ & \ddots & \\ & & \pi^{e_n} \end{pmatrix} \leftrightarrow \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

A: Denote

$$\text{pr}: W_{\text{ext}} \longrightarrow W_f \quad \begin{pmatrix} * & * & * \\ * & * & * \\ -1 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$s: W_f \longleftrightarrow W_{\text{ext}} \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} * & * & * \\ * & * & * \\ -1 & 1 & 1 \end{pmatrix}$$

*This is canonical embedding! We will omit s to simplify the formula.*

Then,  $W_{\text{ext}}$  acts on  $X_*(T)$  by

$$\begin{aligned} W_{\text{ext}} \times X_*(T) &\longrightarrow X_*(T) \\ N_G(T(F)) / T(O_F) \times T(F) / T(O_F) &\longrightarrow T(F) / T(O_F) \\ (w, \lambda) &\mapsto w \cdot \lambda := \underbrace{w \lambda s(\text{pr}(w))^{-1}}_{\substack{w = \mu x u \\ \text{in } \mathbb{Z}^n \\ x \rightarrow +}} \underbrace{\mu u \lambda u^{-1}}_{\mu + u \lambda u^{-1}} \end{aligned} \quad \mu \in X_*(T), u \in W_f$$

So roughly speaking,  $W_{\text{ext}}$  acts on  $X_*(T)$  by a rotation/flip followed with a translation.

We will denote  $w = t_\mu u$  when we view  $w \in W_{\text{ext}}$  as operators on  $X_*(T)$ .

◻  $w \neq r_{\alpha, k}$  in many cases.

Recall that for  $\alpha \in X^*(T)$ ,  $k \in \mathbb{Z}$ , we can define

$$\begin{aligned} \psi &= \alpha + k : X_*(T) \rightarrow \mathbb{R} & \lambda &\mapsto \langle \lambda, \alpha + k \rangle = \langle \lambda, \alpha \rangle + k \\ r_\psi &= r_{\alpha, k} : X_*(T) \rightarrow X_*(T) & \lambda &\mapsto \lambda - \langle \lambda, \alpha + k \rangle \alpha^\vee \\ &&&= \lambda - \langle \lambda, \alpha \rangle \alpha^\vee - k \alpha^\vee \end{aligned}$$

$$\begin{aligned} H_\psi &= H_{\alpha, k} = \{ \lambda \in X_*(T) \mid r_{\alpha, k}(\lambda) = \lambda \} \\ &= \{ \lambda \in X_*(T) \mid \langle \lambda, \psi \rangle = 0 \} \subseteq X_*(T) \end{aligned}$$

The set

$$\Psi := \{ \psi : X_*(T) \rightarrow \mathbb{R} \mid \psi = \alpha + k \text{ for some } \alpha \in \Phi, k \in \mathbb{Z} \}$$

is an affine root system, and  $W_{\text{ext}}$  acts on  $\Psi$  by

$$W_{\text{ext}} \times \Psi \rightarrow \Psi \quad (w, \psi) \mapsto \psi \circ w^{-1}$$

Sign is useless

E.g. For  $G = SL_2$ ,

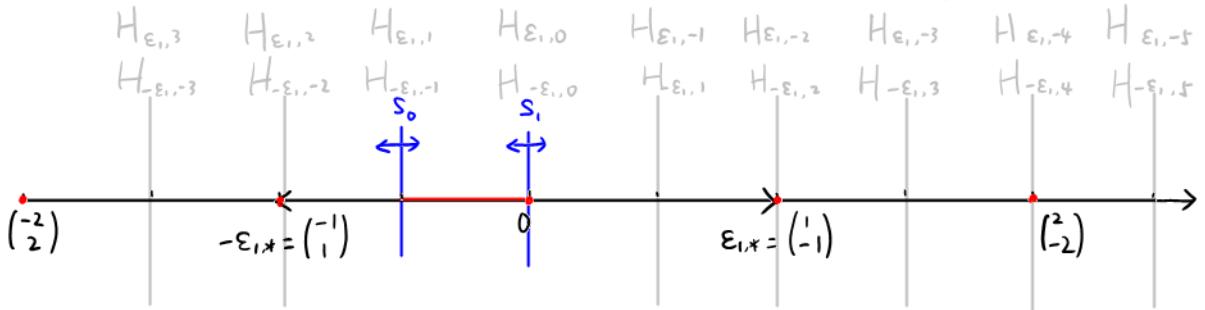
$$\begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ -\lambda_1 - e \end{pmatrix}$$

e.p.  $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \\ \lambda_1 \end{pmatrix}$

$$\begin{pmatrix} -\pi^{-e} & \pi^e \\ & \pi^{-e} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 + e \\ \lambda_1 - e \end{pmatrix}$$

$$\begin{pmatrix} -\pi & \pi^e \\ & \pi^{-e} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 - 1 \\ \lambda_1 + 1 \end{pmatrix}$$

Let  $s_1 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $s_0 = \begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix}$ , then  $s_1 s_0 = \begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix} = \varepsilon_{1,*} \cdot \text{Id}$   
 Never mind symbol!  $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in T(\mathcal{O}_F)$



shortest matrix	$(-\pi^2, \pi^{-2})$	$(\pi^{-1}, \pi)$	$(-\pi, \pi^{-1})$	$(1, 1)$	$(-1, 1)$	$(\pi, \pi^{-1})$	$(-\pi^{-1}, \pi)$	$(\pi^2, \pi^{-2})$	$(-\pi^{-2}, \pi^2)$
operator	$t_{-2\varepsilon_{1,*}} s_1$	$t_{-\varepsilon_{1,*}}$	$t_{-\varepsilon_{1,*}} s_1$	$\text{Id}$	$s_1$	$s_1 s_0$	$(s_1 s_0) s_1$	$(s_1 s_0)^2$	$(s_1 s_0)^2 s_1$
reflections	$r_{\varepsilon_{1,2}}$		$r_{\varepsilon_{1,1}}$		$r_{\varepsilon_{1,0}}$		$r_{\varepsilon_{1,-1}}$		$r_{\varepsilon_{1,-2}}$
length	3	2	1	0	1	2	3	4	5

Inahori  $\begin{pmatrix} 0 & \beta^{-3} \\ \beta^4 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^{-2} \\ \beta^3 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^{-1} \\ \beta^2 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^3 \\ \beta & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^3 \\ \beta^2 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^4 \\ \beta^3 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^4 \\ \beta^4 & 0 \end{pmatrix}$

parahoric  $\begin{pmatrix} 0 & \beta^{-3} \\ \beta^3 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^{-2} \\ \beta^2 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^{-1} \\ \beta & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^3 \\ \beta^2 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^3 \\ \beta^3 & 0 \end{pmatrix}$   $\begin{pmatrix} 0 & \beta^4 \\ \beta^4 & 0 \end{pmatrix}$

$$\bullet = \bullet \subseteq \overset{\circ}{P} \\ Q^\vee = X_\ast \subseteq P^\vee$$

E.g. For  $G = \mathrm{PGL}_2$ ,  $\begin{pmatrix} \pi^e & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ \lambda_2 \end{pmatrix}$

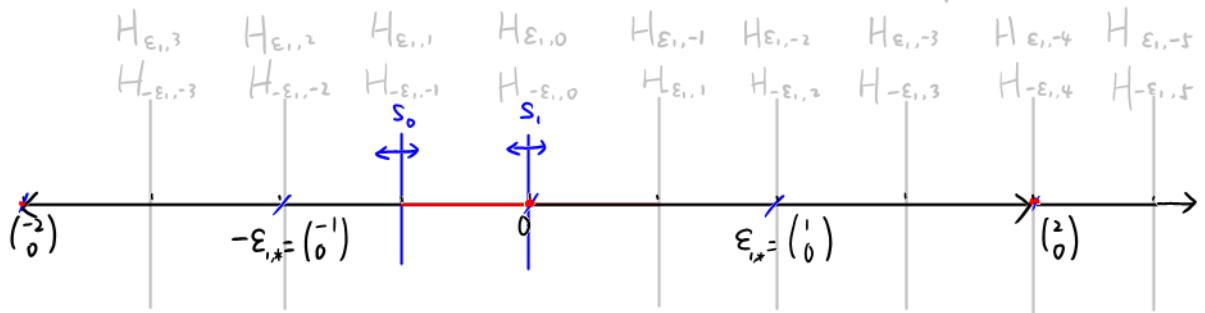
$$\text{e.g. } \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$$

$$\begin{pmatrix} -\pi^e & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + e \end{pmatrix}$$

$$\begin{pmatrix} -\pi & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + 1 \end{pmatrix}$$

Let  $s_i = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ ,  $s_o = \begin{pmatrix} -\pi & \\ & 1 \end{pmatrix}$ , then  $s_i s_o = \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} = \varepsilon_i^* \cdot \mathrm{Id}$

Never mind symbol!  $(^{-1} -1) \in T(\mathcal{O}_F)$



shortest matrix	$s_o s_i s_o$	$s_o s_i$	$s_o$	$\mathrm{Id}$	$s_i$	$s_i s_o$	$s_i s_o s_i$	$s_i s_o s_i s_o$	$s_i s_o s_i s_o s_i$
	$(-\pi^e)^2$	$(s_o s_i)^{-1}$	$(s_o s_i)^2$	$\mathrm{Id}$	$s_i$	$s_i s_o$	$(s_o s_i) s_i$	$(s_o s_i)^2$	$(s_o s_i)^3 s_i$
operator	$t_{-2\varepsilon_1, *} s_i$	$t_{-\varepsilon_1, *} s_i$	$t_{-\varepsilon_1, *} s_i$	$\mathrm{Id}$	$s_i$	$t_{\varepsilon_1, *}$	$t_{\varepsilon_1, *} s_i$	$t_{2\varepsilon_1, *} s_i$	$t_{2\varepsilon_1, *} s_i$
reflections	$r_{\varepsilon_1, 2}$		$r_{\varepsilon_1, 1}$		$r_{\varepsilon_1, 0}$		$r_{\varepsilon_1, -1}$		$r_{\varepsilon_1, -2}$
	$r_{\varepsilon_1, -2}$		$r_{-\varepsilon_1, -1}$		$r_{-\varepsilon_1, 0}$		$r_{-\varepsilon_1, 1}$		$r_{-\varepsilon_1, 2}$
length	3	2	1	0	1	2	3	4	5
Inahori	$\left( \begin{smallmatrix} 0 & p^{-1} \\ p^2 & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & p^2 \\ p^{-1} & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & p^3 \\ p^{-2} & 0 \end{smallmatrix} \right)$
parahoric		$\left( \begin{smallmatrix} 0 & p^{-1} \\ p & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)$			$\left( \begin{smallmatrix} 0 & p \\ p & 0 \end{smallmatrix} \right)$		$\left( \begin{smallmatrix} 0 & p^2 \\ p^{-2} & 0 \end{smallmatrix} \right)$

$$\bullet \subseteq \wedge = \wedge$$

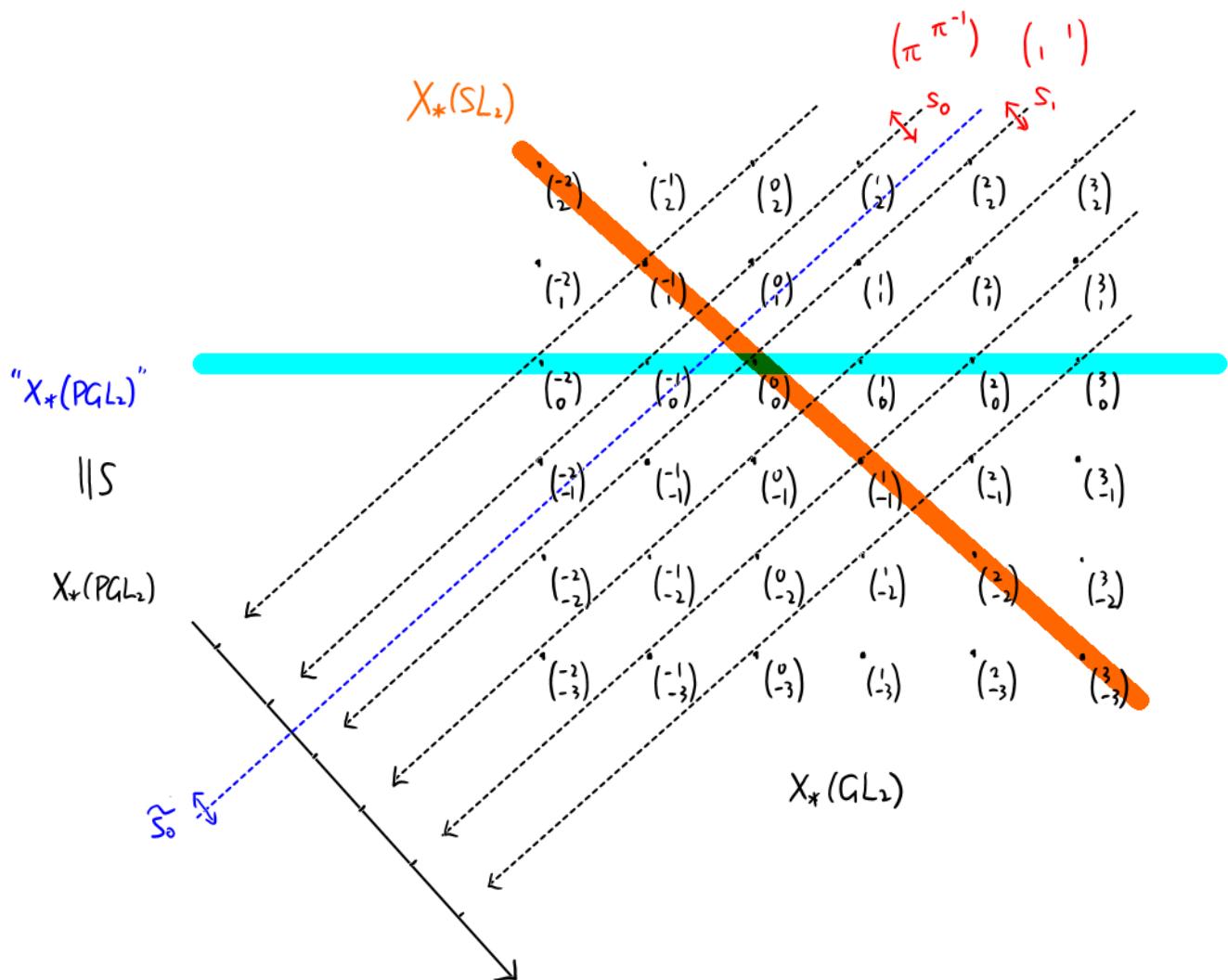
$$Q \subseteq X_* = P^*$$

Here, we still require  $x \in \left( \begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix} \right)$  satisfies  $\det x \in \mathcal{O}^\times$ , so

$$\left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \notin \left( \begin{smallmatrix} 0 & 0 \\ p & 0 \end{smallmatrix} \right)$$

You see also that  $\left( \begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix} \right) \in N_G(T(\mathcal{O}_F)) - T(\mathcal{O}_F)$ .

A vivid picture describing relationships among  $X_*(GL_2)$ ,  $X_*(SL_2)$ ,  $X_*(PSL_2)$



$$X_*(SL_2) \hookrightarrow X_*(GL_2)$$

$$\downarrow$$
  

$$X_*(PGL_2)$$

$$W_{\text{ext}}(SL_2) \hookrightarrow W_{\text{ext}}(GL_2)$$

$$\downarrow$$
  

$$W_{\text{ext}}(PGL_2)$$

$$(\pi') = S_1 \circ (('_0) + \dots)$$

$$= \left( \begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix} \right) + \tilde{S}_0 \circ \dots$$

In \$GL\_n\$ case,

\$W\_{\text{ext}}\$ is not the usual affine Weyl gp.

In both cases \$(SL\_2 \& PGL\_2)\$, \$W\_{\text{ext}} = X\_\* \sqcup X\_\*\$'s, as set.

We will focus on the case \$G = SL\_2\$.

1. decomposition.

$$\begin{aligned} G &= \bigsqcup_{w \in W_{\text{ext}}} IwI \\ &= \bigsqcup_{e \in \mathbb{Z}} I(\pi^e \pi^{-e})I \sqcup \bigsqcup_{e \in \mathbb{Z}} I(-\pi^{-e} \pi^e)I \\ &= \bigsqcup_{e \in \mathbb{Z}} I(s, s_0)^e I \sqcup \bigsqcup_{e \in \mathbb{Z}} I(s, s_0)^e s, I \end{aligned}$$

Ex. Verify that \$IwI/I\$ is finite, and compute the order.

Hint. \$\# IwI/I = \# I/I \cap wIw^{-1}\$

\$\# I/I \cap wIw^{-1}\$

$$\text{When } w = (\pi^e \pi^{-e}), \quad wIw^{-1} = \begin{pmatrix} \pi^e & \pi^{-e} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \begin{pmatrix} \pi^{-e} & \pi^e \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & p^{-e} \\ p^{1-e} & 0 \end{pmatrix}$$

$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} & e \geq 0 \\ \begin{pmatrix} 0 & 0 \\ p^{1-e} & 0 \end{pmatrix} & e < 0 \end{cases}$$

$$\begin{matrix} q^{2e} \\ q^{-2e} \end{matrix}$$

$$\text{When } w = (-\pi^{-e} \pi^e), \quad wIw^{-1} = \begin{pmatrix} -\pi^{-e} \pi^e & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \begin{pmatrix} -\pi^{-e} \pi^e & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & p^{2e+1} \\ p^{-e} & 0 \end{pmatrix}$$

$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} 0 & p^{2e+1} \\ p^{-e} & 0 \end{pmatrix} & e \geq 0 \\ \begin{pmatrix} 0 & 0 \\ p^{-e} & 0 \end{pmatrix} & e < 0 \end{cases}$$

$$\begin{matrix} q^{2e+1} \\ q^{-2e-1} \\ q \end{matrix}$$

In conclusion,

$$\# IwI/I = \# I/I \cap wIw^{-1} = q^{l(w)}$$

□

Take the unique Haar measure on \$G\$ s.t. \$\mu(I) = 1\$, then

$$\mu(IwI) = \# IwI/I = q^{l(w)}$$

\$\mu\$ is induced from the measure on coset \$G/I\$.

2. \$\mathbb{Z}\$-mod structure, notation.

$$\mathcal{H}(G, I) = \bigoplus_{w \in W_{\text{ext}}} \mathbb{Z} \cdot \mathbf{1}_{IwI}$$

denote \$T\_w = \mathbf{1}\_{IwI}

\$T\_{Id} = \mathbf{1}\_I\$ is the unit of \$\mathcal{H}(G, I)

$$T_{s_0} := T_s, \quad T_{s_0^{-1}} := T_{s_0}$$

$$e \in \mathbb{Z} \quad S_{(s_0)^e} := \overline{T_{(\pi^e \pi^{-e})}} = T_{(s, s_0)^e} = T_{(s, s_0)^e}$$

The general notation is \$S\_\lambda\$, for \$\lambda \in X\_\*(T)

### 3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in W_{\text{ext}}} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/I} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \# \left\{ x \in G/I \mid \begin{array}{l} x \in I\alpha I \\ x^{-1}\gamma \in I\beta I \end{array} \right\} \\ &= |I\alpha I \cap \gamma I\beta^{-1} I|_I \end{aligned}$$

e.p.  $\mathbf{1}_I * f = f * \mathbf{1}_I = f$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^\gamma = \frac{|I\alpha I|_I |I\alpha\beta^{-1} I|_I}{|I\gamma I|_I}$$

where

$$\begin{aligned} I_{\alpha\beta}^\gamma &:= \{w' \in I \mid \alpha w' \beta \in I\gamma I\} = \alpha^{-1} I \gamma I \beta^{-1} \cap I \\ &\Rightarrow I_{\alpha\beta}^\gamma I/I = \alpha^{-1} I \gamma I \cap I \beta^{-1} I/I \end{aligned}$$

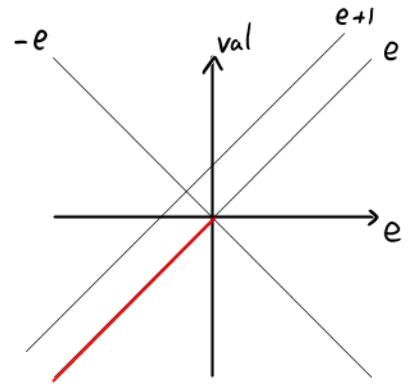
depends on  $\alpha, \beta, \gamma$ .

In the following computation, the minus sign is not important, so we ignore it.

E.g. When we write  $(+, +)$ , we actually mean  $(-, +)$ .

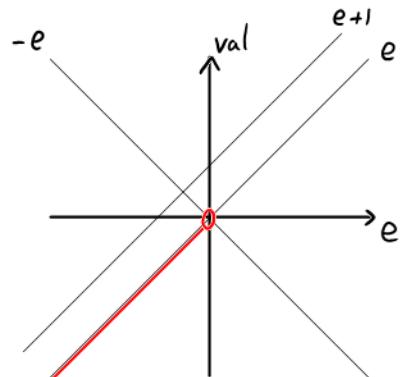
$$(1') \left( \begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} \pi^e & \\ & \pi^{-e} \end{smallmatrix} \right) = \left( \begin{smallmatrix} \pi^{e+1} c & \pi^e d \\ \pi^e a & \pi^{-e} b \end{smallmatrix} \right)$$

$\left( \begin{smallmatrix} \pi^e & \pi^{-e} \\ \pi^e & \end{smallmatrix} \right)$	$e \leq 0$
$\left( \begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^e \end{smallmatrix} \right)$	$e > 0, \text{val}(b) = 0$
$\left( \begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^e \end{smallmatrix} \right)$	$e > 0, \text{val}(b) > 0$



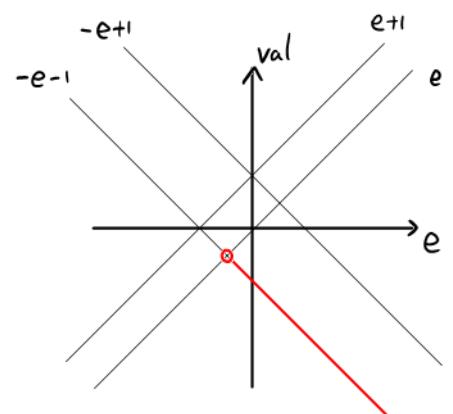
$$(1') \left( \begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} & \pi^e \\ \pi^{-e} & \end{smallmatrix} \right) = \left( \begin{smallmatrix} \pi^e d & \pi^{e+1} c \\ \pi^{-e} b & \pi^e a \end{smallmatrix} \right)$$

$\left( \begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^e \end{smallmatrix} \right)$	$e < 0$
$\left( \begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^{-e} \end{smallmatrix} \right)$	$e \geq 0, \text{val}(b) = 0$
$\left( \begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^e \end{smallmatrix} \right)$	$e \geq 0, \text{val}(b) > 0$



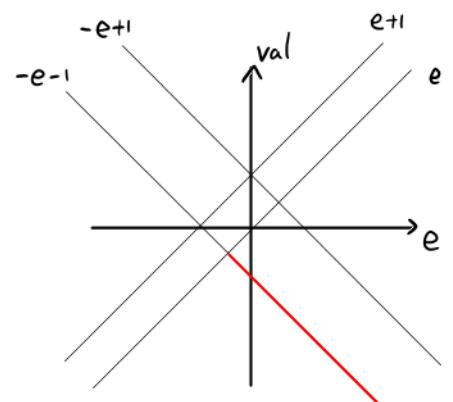
$$(\pi^{-1}) \left( \begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} \pi^e & \\ & \pi^{-e} \end{smallmatrix} \right) = \left( \begin{smallmatrix} \pi^e c & \pi^{e-1} d \\ \pi^{-e+1} a & \pi^{-e+1} b \end{smallmatrix} \right)$$

$\left( \begin{smallmatrix} \pi^{e+1} & \pi^{-e-1} \\ \pi^e & \end{smallmatrix} \right)$	$e \geq 0$
$\left( \begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^{-e-1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) = 0$
$\left( \begin{smallmatrix} \pi^e & \pi^{-e-1} \\ & \pi^{e+1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) > 0$



$$(\pi^{-1}) \left( \begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left( \begin{smallmatrix} & \pi^e \\ \pi^{-e} & \end{smallmatrix} \right) = \left( \begin{smallmatrix} \pi^{e-1} d & \pi^e c \\ \pi^{-e+1} b & \pi^{e+1} a \end{smallmatrix} \right)$$

$\left( \begin{smallmatrix} \pi^{-e-1} & \pi^{e+1} \\ & \pi^e \end{smallmatrix} \right)$	$e \geq 0$
$\left( \begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^{-e-1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) = 0$
$\left( \begin{smallmatrix} \pi^{-e-1} & \pi^{e+1} \\ & \pi^{e+1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) > 0$



In conclusion,

$$T_i * T_\omega \in \left\{ \begin{array}{l} \mathbb{Z} T_{s_i \omega} \\ \mathbb{Z} T_{s_i \omega} + \mathbb{Z} T_\omega \end{array} \right.$$

$$l(s_i \omega) = l(\omega) + 1$$

$$l(s_i \omega) = l(\omega) - 1$$

In the following computation,  $\alpha = s_i$ ,  $\beta = w$ ,  $\gamma = s_i w$  or  $\gamma = w$ .

When  $l(s_i w) = l(w) + 1$ ,

$$g_{s_i, w}^{s_i w} = \frac{|I s_i I/I| |I w I/I|}{|I s_i w I/I|}$$

$$= \frac{q^{l(s_i)} q^{l(w)}}{q^{l(s_i w)}}$$

$$= 1$$

When  $l(s_i w) = l(w) - 1$  and  $i = 1$ ,  $w = (\pi^e \pi^{-e})$ ,  $e > 0$  or  $w = (\pi^{-e} \pi^e)$ ,  $e \geq 0$

$$I_{s_i, w}^{s_i w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_{s_i, w}^w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} \# I_{s_i, w}^{s_i w} w I/I &= \# I_{\alpha \beta}^\gamma / (I_{\alpha \beta}^\gamma \cap w I w^{-1}) & \gamma = s_i w \\ &= \# I_{\alpha \beta}^\gamma / (I \cap w I w^{-1}) \\ &= \frac{\# I/I \cap w I w^{-1}}{\# I/I_{\alpha \beta}^\gamma} \\ &= q^{l(w)-1} \\ g_{s_i, w}^{s_i w} &= \frac{|I s_i I/I| |I_{s_i, w}^{s_i w} w I/I|}{|I s_i w I/I|} \\ &= \frac{q^{l(s_i)} q^{l(w)-1}}{q^{l(s_i w)}} \\ &= q \\ g_{s_i, w}^w &= \frac{|I s_i I/I| |I_{s_i, w}^w w I/I|}{|I w I/I|} \\ &= \frac{|I s_i I/I| (|I w I/I| - |I_{s_i, w}^{s_i w} w I/I|)}{|I w I/I|} \\ &= \frac{q^{l(s_i)} (q^{l(w)} - q^{l(w)-1})}{q^{l(w)}} \\ &= q - 1 \end{aligned}$$

When  $l(s_i w) = l(w) - 1$  and  $i=0$ ,  $w = (\pi^e \pi^{-e})$ ,  $e < 0$  or  $w = (\pi^{-e} \pi^e)$ ,  $e < 0$

$$I_{s_i w}^{s_i w} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p^2 & \mathcal{O} \end{pmatrix}, \quad I_{s_i, w}^w = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p-p^2 & \mathcal{O} \end{pmatrix}.$$

$$\begin{aligned} \# I_{s_i w}^{s_i w} wI/I &= \# I_{\alpha\beta}^\gamma / (I_{\alpha\beta}^\gamma \cap wIw^{-1}) & \gamma = s_i w \\ &= \# I_{\alpha\beta}^\gamma / (I \cap wIw^{-1}) \\ &= \frac{\# I/I \cap wIw^{-1}}{\# I/I_{\alpha\beta}^\gamma} \\ &= q^{l(w)-1} \\ g_{s_i w}^{s_i w} &= \frac{|I_{s_i} I/I| |I_{s_i w}^{s_i w} wI/I|}{|I_{s_i w} I/I|} \\ &= \frac{q^{l(s_i)} q^{l(w)-1}}{q^{l(s_i w)}} \\ &= q \\ g_{s_i w}^w &= \frac{|I_{s_i} I/I| |I_{s_i, w}^w wI/I|}{|I_w I/I|} \\ &= \frac{|I_{s_i} I/I| (|I_w I/I| - |I_{s_i, w}^{s_i w} wI/I|)}{|I_w I/I|} \\ &= \frac{q^{l(s_i)} (q^{l(w)} - q^{l(w)-1})}{q^{l(w)}} \\ &= q-1 \end{aligned}$$

In conclusion,

$$T_i * T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w) + 1 \\ q T_{s_i w} + (q-1) T_w & l(s_i w) = l(w) - 1 \end{cases}$$

#### 4. Conclusion.

$$\mathcal{H}(G, I) = \mathbb{Z}\{T_0, T_i\}$$

with

$$T_i * T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w) + 1 \\ q T_{s_i w} + (q-1) T_w & l(s_i w) = l(w) - 1 \end{cases}$$

Bernstein presentation of  $\mathcal{H}_q(G, I)$

To continue, we now work on  $\mathbb{Z}[q^{\pm 1}]$ -coefficient Iwahori Hecke algebra

$$\begin{aligned} \mathcal{H}_q(G, I) &= \mathcal{H}(G, I) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}] \\ &= \mathcal{H}(G, I) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(qx-1) \end{aligned}$$

▽ The map

$$X_*(T) \longrightarrow \mathcal{H}_q(G, I)^{\times} \quad \lambda \mapsto T_{\lambda}$$

is not a gp homomorphism, i.e.

$$\mathbb{Z}[q^{\pm 1}][X_*(T)] \hookrightarrow \mathcal{H}_q(G, I) \quad \lambda \mapsto T_{\lambda}$$

is not a  $\mathbb{Z}[q^{\pm 1}]$ -alg homomorphism.

Instead, if we twist it (e.g. for non-dominant part), then the map

$$\theta: \mathbb{Z}[q^{\pm 1}][X_*(T)] \hookrightarrow \mathcal{H}_q(G, I) \quad \begin{aligned} \lambda \text{ dominant} &\mapsto q^{-\frac{l(\lambda)}{2}} T_{\lambda} \\ \lambda = \lambda_1 \lambda_2^{-1} &\mapsto q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1} \end{aligned}$$

in usual reference,  $\lambda = \lambda_1 - \lambda_2$  to indicate  $X_*(T)$  is commutative.

is a well-defined  $\mathbb{Z}[q^{\pm 1}]$ -alg homomorphism.

Prop  $\theta$  is a well-defined  $\mathbb{Z}[q^{\pm 1}]$ -alg homomorphism.

"Proof".

Well-defined. If  $\lambda = \lambda_1 \lambda_2^{-1} = \lambda'_1 \lambda'^{-1}_2$ , then

$\lambda_1, \lambda_2, \lambda'_1, \lambda'_2$  dominant

$$\begin{aligned} \cdot \lambda_1 \lambda_2^{-1} = \lambda'_1 \lambda_2 &\Rightarrow l(\lambda_1) + l(\lambda_2) = l(\lambda'_1) + l(\lambda_2) \\ &\Rightarrow q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} = q^{-\frac{l(\lambda'_1) - l(\lambda_2)}{2}} \end{aligned}$$

$$\cdot T_{\lambda_1} T_{\lambda_2}^{-1} = T_{\lambda_1} T_{\lambda_2} T_{\lambda_2}^{-1} T_{\lambda_1}^{-1} = T_{\lambda_1 \lambda_2} T_{\lambda_2}^{-1} \quad !!$$

$$\begin{aligned} T_{\lambda'_1} T_{\lambda'_2}^{-1} &= T_{\lambda'_1} T_{\lambda_2} T_{\lambda_2}^{-1} T_{\lambda'_2}^{-1} = T_{\lambda'_1 \lambda_2} T_{\lambda_2}^{-1} \\ \cdot q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1} &= q^{-\frac{l(\lambda'_1) - l(\lambda_2)}{2}} T_{\lambda'_1} T_{\lambda_2}^{-1} \end{aligned}$$

multiplication. If  $\lambda = \lambda_1 \lambda_2^{-1}$ ,  $\mu = \mu_1 \mu_2^{-1}$ , then

$\lambda_1, \lambda_2, \mu_1, \mu_2$  dominant

- $\lambda\mu = (\lambda_1\mu_1)(\lambda_2\mu_2)^{-1}$
- $T_{\lambda_1} T_{\mu_1} = T_{\mu_1} T_{\lambda_2} \Rightarrow T_{\mu_1} T_{\lambda_2}^{-1} = T_{\lambda_2}^{-1} T_{\mu_1}$
- $T_{\lambda_1\mu_1} T_{\lambda_2\mu_2}^{-1} = T_{\lambda_1} T_{\mu_1} T_{\lambda_2}^{-1} T_{\mu_2}^{-1}$   
 $= T_{\lambda_1} T_{\mu_1} T_{\lambda_2}^{-1} T_{\mu_2}^{-1}$   
 $= T_{\lambda_1} T_{\lambda_2}^{-1} T_{\mu_1} T_{\mu_2}^{-1}$
- $q^{-\frac{l(\lambda_1\mu_1) - l(\lambda_2\mu_2)}{2}} T_{\lambda_1\mu_1} T_{\lambda_2\mu_2}^{-1} = q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1} q^{-\frac{l(\mu_1) - l(\mu_2)}{2}} T_{\mu_1} T_{\mu_2}^{-1}$  □

▽ People prefer to write multiplication in  $X_*(T)$  as addition.

e.g.  $\lambda = \lambda_1 - \lambda_2$ ,  $\lambda = 3\varepsilon_{1,*}$

Be careful that these additions are not additions in  $\mathbb{Z}[q^{\pm 1}][X_*(T)]$ .

to avoid the clash of terminology, use multiplication symbol;

to make it easier to digest, use addition symbol.

Prop. (Bernstein presentation) For  $\lambda \in X_*(T)$ , we have

$$\theta(\lambda) * T_i - T_i * \theta(-\lambda) = (q-1) \frac{\theta(\lambda) - \theta(-\lambda)}{1 - \theta(-\varepsilon_{i,*})} \quad (\star)$$

Proof 1. If  $\lambda, \lambda'$  satisfy  $(\star)$ , then  $\lambda + \lambda'$  satisfies  $(\star)$ .

$$\begin{aligned} & \theta(\lambda + \lambda') * T_i - T_i * \theta(-\lambda - \lambda') \\ &= \theta(\lambda)(\theta(\lambda') T_i - T_i \theta(-\lambda')) + (\theta(\lambda) T_i - T_i \theta(-\lambda)) \theta(-\lambda') \\ &= (q-1) \frac{\theta(\lambda)(\theta(\lambda') - \theta(-\lambda'))}{1 - \theta(-\varepsilon_{i,*})} + (q-1) \frac{(\theta(\lambda) - \theta(-\lambda)) \theta(-\lambda')}{1 - \theta(-\varepsilon_{i,*})} \\ &= (q-1) \frac{\theta(\lambda + \lambda') - \theta(-(\lambda + \lambda'))}{1 - \theta(-\varepsilon_{i,*})} \end{aligned}$$

2.  $\lambda = 0$  satisfies  $(\star)$ . Obviously.

3. If  $\lambda$  satisfies  $(\star)$ , then  $-\lambda$  satisfies  $(\star)$

$$\begin{aligned} & \theta(-\lambda) * T_i - T_i * \theta(\lambda) \\ &= \theta(-\lambda)(T_i \theta(-\lambda) - \theta(\lambda) T_i) \theta(\lambda) \\ &= \theta(-\lambda) \left( (q-1) \frac{-\theta(\lambda) + \theta(-\lambda)}{1 - \theta(-\varepsilon_{i,*})} \right) \theta(\lambda) \\ &= (q-1) \frac{\theta(-\lambda) - \theta(\lambda)}{1 - \theta(-\varepsilon_{i,*})} \end{aligned}$$

4.  $\lambda = \varepsilon_{i,*}$  satisfies  $(\star)$ . Therefore,  $(\star)$  is true for any  $\lambda \in X_*(T)$ .

$$RHS = (q-1) \frac{\theta(\lambda) - \theta(-\lambda)}{1 - \theta(-\lambda)} = (q-1)(1 + \theta(\lambda))$$

$$T_i^2 = (q-1) T_i + q \Rightarrow q T_i^{-1} = T_i - (q-1)$$

$$\begin{aligned} LHS &= q^{-1} T_\lambda T_i - q T_i T_\lambda^{-1} \\ &= q^{-1} [T_i T_0 T_i - T_i (q T_0^{-1})(q T_i^{-1})] \\ &= q^{-1} [T_i T_0 T_i - T_i (T_0 - (q-1))(T_i - (q-1))] \\ &= q^{-1} [(q-1) T_i (T_0 + T_i) - (q-1)^2 T_i] \\ &= \frac{q-1}{q} [T_i T_0 + T_i^2 - (q-1) T_i] \\ &= \frac{q-1}{q} (T_i T_0 + q) \\ &= (q-1)(1 + \theta(\lambda)) \end{aligned}$$

In conclusion, denote

$$\begin{aligned} \mathbb{H} &= \langle \theta(\lambda) |_{\lambda \in X_*(T)} \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} = \theta(\mathbb{Z}[q^{\pm 1}][X_*(T)]) \subseteq \mathcal{H}_q(G, I) \\ &= X_*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}] \\ &= \langle \theta(\varepsilon_{i,*}), \theta(-\varepsilon_{i,*}) \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} \\ &= \mathbb{Z}[q^{\pm 1}][\lambda_0^{\pm 1}] \end{aligned}$$

$$\lambda_0 = \varepsilon_{i,*}$$

$$\mathcal{H}_q(W_f) = \mathcal{H}(W_f) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}]$$

Then

$$\begin{aligned} \mathcal{H}_q(G, I) &\cong \mathbb{H} \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{H}_q(W_f) \quad \text{as left } \mathbb{H}\text{-module} \\ &\cong \bigoplus_{\omega \in W_f} \mathbb{H} \cdot T_\omega \end{aligned}$$

Guess.  $\mathcal{H}_q(G, I) \cong \langle \theta(\varepsilon_{i,*})^{\pm 1}, T_i \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} \subseteq \text{End}_{\mathbb{Z}[q^{\pm 1}]\text{-mod}}(\mathbb{H})$

where

$$T_i * \theta(\lambda) = (q-1) \left( \frac{\theta(\lambda)}{1-\theta(-\varepsilon_{i,*})} + \frac{\theta(-\lambda)}{1-\theta(\varepsilon_{i,*})} \right) \quad \theta(\lambda) \in \mathbb{H}$$

Center of  $\mathcal{H}_q(G, I)$

Q. What is the center of  $\mathcal{H}_q(G, I)$ ?

A. It is

$$\begin{aligned} \mathbb{H}^{w_f} &= \langle \theta(\varepsilon_{i,*}) + \theta(-\varepsilon_{i,*}) \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} \\ &= \mathbb{Z}[q^{\pm 1}][\lambda_0 + \lambda_0^{-1}] \end{aligned}$$

$$\lambda_0 = \varepsilon_{i,*}$$

I believe that people can get this result by direct computation.

See [Bump, Theorem 23] for a proof.