Modular form
5. moduli interpretation

1 level structure
2. moduli interpretation of $\Gamma\backslash\mathcal{H}$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S,T \mid S^4=(ST)^6=Id\rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	$\{\pm Id\}$	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S,T \mid S^2=(ST)^3=Id\rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	Id	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$

can't define $SL_2/\mathbb{G}_m$

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A\leq B\leq C$ gp $\quad A\triangleleft C\Rightarrow A\triangleleft B$

no other restrictions. i.e. the following cases may happen:

$\underbrace{A\triangleleft B\triangleleft C}_{\triangleleft}$	$\underbrace{A\triangleleft B\leq C}_{\triangleleft}$	$A\triangleleft B\triangleleft C$	$A\triangleleft B\leq C$	$A\leq B\triangleleft C$	$A\leq B\leq C$
✓	✓	$C_2\triangleleft A_4\triangleleft S_4$	✓	✓	$S_2\leq S_3\leq S_4$

wiki: congruence subgroup

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$
\begin{array}{ccc}
\Gamma(N) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\Gamma_1(N) & \twoheadrightarrow & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(N) & \twoheadrightarrow & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \cap \\
\Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{[WWL,\ Prop\ 1.4.4]} \!\!\!\!\rightarrow & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\cup & & \cup \\
\Gamma^0(N) & \twoheadrightarrow & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \\
\cup & & \cup \\
\Gamma^1(N) & \twoheadrightarrow & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}
\end{array}
$$

⚠ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$, but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

⚠ Since $\mathbb{Z}^\times \to (\mathbb{Z}/N\mathbb{Z})^\times$ is not surj in general,
$GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/N\mathbb{Z})$ is not surj.

Ex. Verify the following tables (left comes from right)

$A \triangleleft B$	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	–	✓	✓	✓
$\Gamma_1(N)$	–	–	✓	×
$\Gamma_0(N)$	–	–	–	×
$\Gamma(1)$	–	–	–	–

$A \triangleleft B$	N	B	G
N	–	✓	×
B	–	–	×
G	–	–	–

Ex. show [WWL, 练习1.4.14]

练习 1.4.14 对所有正整数 N, 证明

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma(N)) = N^3 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N^2 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(N)) = |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1}\cdot(\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N\prod_{p\mid N}\left(1+\frac{1}{p}\right).$$

A. Reduced to computation of $|SL_2(\mathbb{Z}/N\mathbb{Z})|$, $|B(\mathbb{Z}/N\mathbb{Z})|$, $|N(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5,4,6$ if you don't understand the process.

Notation: $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z}^{\oplus 2})_{prim}/(\mathbb{Z}/N\mathbb{Z})^* \overset{[6.3\,M]}{=\!=} \mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: https://arxiv.org/pdf/2010.15543v2.pdf

∇ $\mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$\begin{bmatrix}3\\2\end{bmatrix} \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2}\mathbb{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

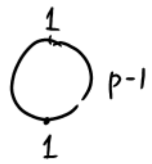
Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different piece of $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|SL_2(\mathbb{F}_p)| = p^3-p$

$|B(\mathbb{F}_p)| = p^2-p$

$|N(\mathbb{F}_p)| = p$

$\#\,\mathbb{F}_p^\times = p-1$



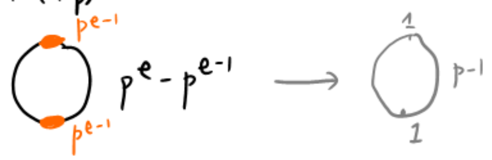
$\mathbb{P}^1(\mathbb{F}_p)$

② $|SL_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e}-p^{3e-2}$

$|B(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e}-p^{2e-1}$

$|N(\mathbb{Z}/p^e\mathbb{Z})| = p^e$

$\#(\mathbb{Z}/p^e\mathbb{Z})^\times = p^e-p^{e-1}$



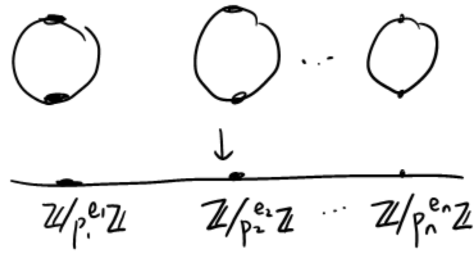
$0 \to p\mathbb{Z}/p^e\mathbb{Z} \to \mathbb{P}^1(\mathbb{Z}/p^e\mathbb{Z}) \longrightarrow \mathbb{P}^1(\mathbb{Z}/p\mathbb{Z}) \to 0$

③ $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p\text{ prime}\\ p\mid N}}\left(1-\frac{1}{p^2}\right)$

$|B(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p\text{ prime}\\ p\mid N}}\left(1-\frac{1}{p}\right)$

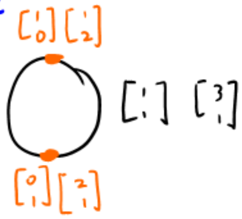
$|N(\mathbb{Z}/N\mathbb{Z})| = N$

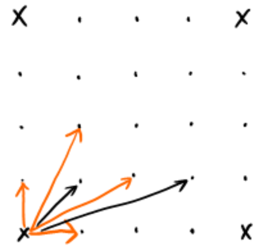
$\#(\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N\prod_{\substack{p\text{ prime}\\ p\mid N}}\left(1-\frac{1}{p}\right)$



$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_{k=1}^{n} \mathbb{P}^1(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$

E.g. $\mathbb{Z}/4\mathbb{Z}$





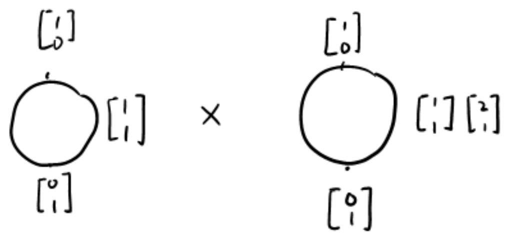
E.g. $\mathbb{Z}/6\mathbb{Z}$

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f\in S_+\\ f \text{ homogeneous}}} \operatorname{Spec}\left(\mathbb{Z}/6\mathbb{Z}[x,y]_f\right)_0$$

e.g. $(x-2,y-3)\lhd \mathbb{Z}/6\mathbb{Z}[x,y]$ is not prime.

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2)\times \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3)$$
$$\cong \mathbb{P}^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2)\times \mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3)$$

Ex. Use [Vakil, 6.3.M] to compute $\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$. Enjoy it!



$\mathbb{P}^1(\mathbb{Z}/6\mathbb{Z})$; $\mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$ \ $\mathbb{P}^1(\mathbb{Z}/3\mathbb{Z})$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix}$	$\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}5\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$

Rmk. The original proof is also good, but less geometrically obvious:

(Now you should understand the geometry in every step)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & SL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[d] & SL_2(\mathbb{F}_p) \arrow[d] & \\
0 \arrow[r] & \underset{p^{4e-4}}{1+pM_2(\mathbb{Z}/p^e\mathbb{Z})} \arrow[r] & GL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[r] \arrow[d] & \overset{(p^2-1)(p^2-p)}{GL_2(\mathbb{F}_p)} \arrow[r] \arrow[d] & 0 \\
 & & \underset{p^e-p^{e-1}}{(\mathbb{Z}/p^e\mathbb{Z})^\times} \arrow[d] & \underset{p-1}{\mathbb{F}_p^\times} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

Finally, use Chinese remainder theorem to get

$$SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^{n} SL_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z}) \qquad \square$$

Ex. do the exactly same thing with SL_2 replaced by GL_2 and PGL_2.

Ex. (hard) explore the Tits building & rep theory of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

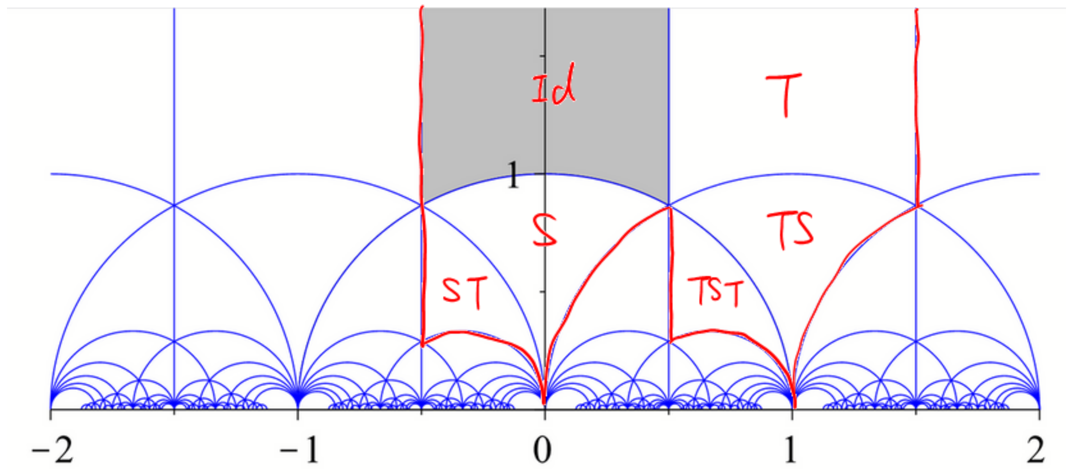
It will be used later on (I believe)

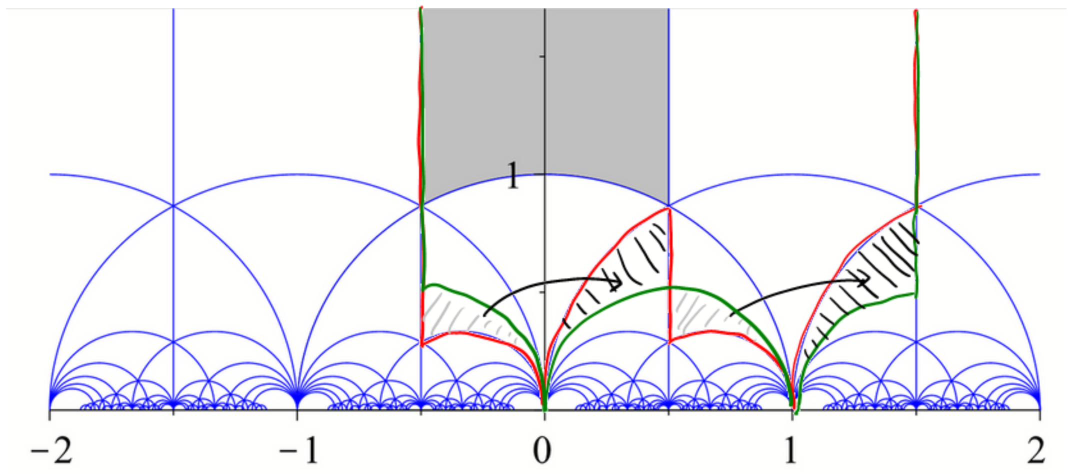
Is the Tits building of $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ functorial?

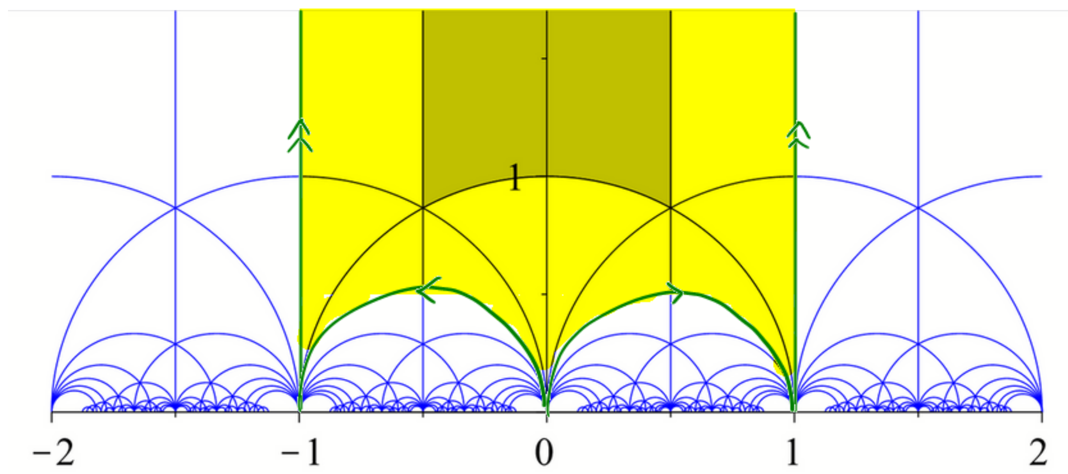
we write left quotient from now on, since it's a left action

Ex. Draw the fundamental domain of $\Gamma(2)\backslash\mathcal{H}$.

Hint. $\Gamma(1)/\Gamma(2) = \{Id, T, S, TS, ST, TST\}$





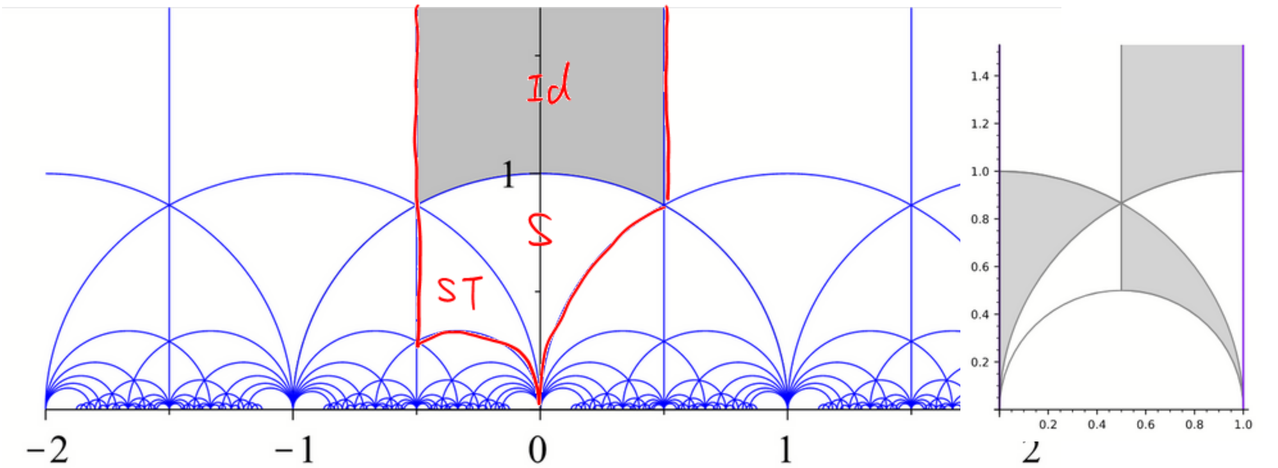


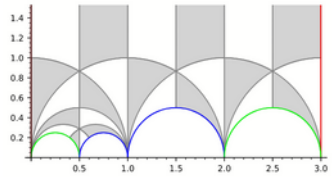
Cor. $\Gamma(2)/\{\pm Id\} = \mathbb{Z} * \mathbb{Z} = \left\langle \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \right\rangle$

Ex. Draw the fundamental domain of $\Gamma_0(2)\backslash\mathcal{H}$. $\Gamma_0(2)=\Gamma_1(2)$

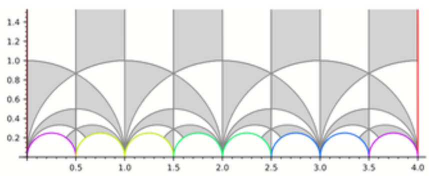
∇ $SL_2(\mathbb{Z}) \curvearrowright \Gamma_0(2)\backslash\mathcal{H}$ is not well-defined. e.g. $S\,i \neq S\,(i+1)$ in $\Gamma_0(2)\backslash\mathcal{H}$.

Hint. $\Gamma_0(2)\backslash\Gamma(1) = \{Id, S, ST\}$

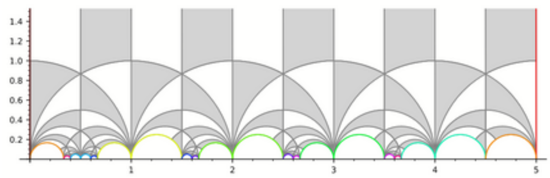




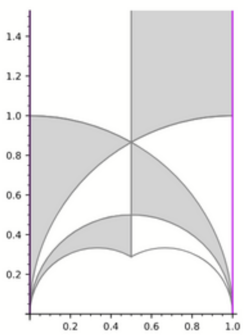
$\Gamma(3)$



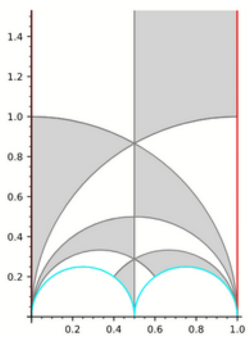
$\Gamma(4)$



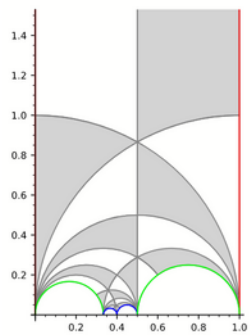
$\Gamma(5)$



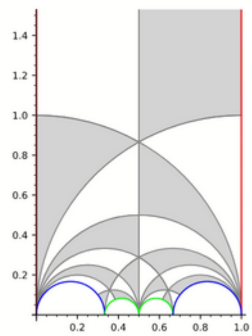
$\Gamma_1(3)=\Gamma_0(3)$



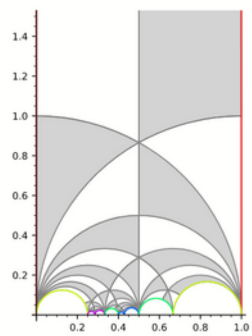
$\pm\Gamma_1(4)=\Gamma_0(4)$



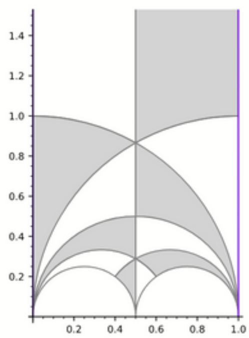
$\Gamma_1(5)$



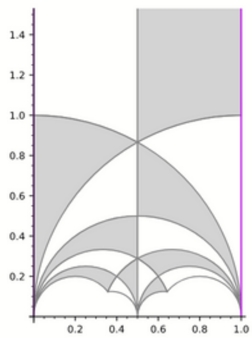
$\pm\Gamma_1(6)=\Gamma_0(6)$



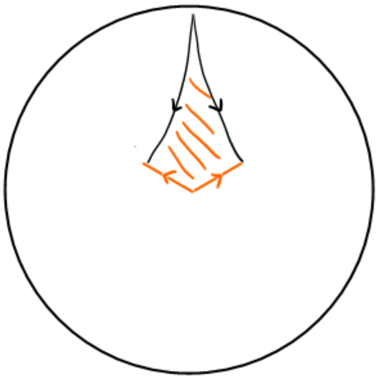
$\Gamma_1(7)$



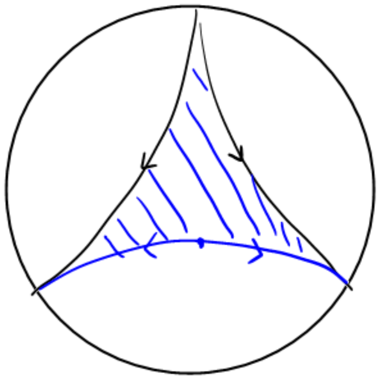
$\Gamma_0(5)$



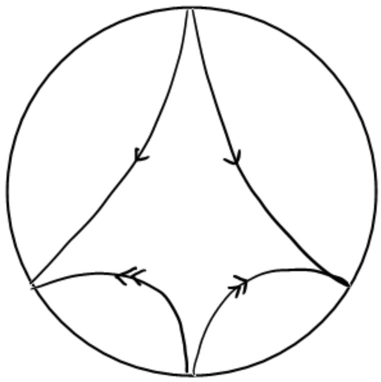
$\Gamma_0(7)$



$\mathcal{H}/SL_2(\mathbb{Z})$



$\mathcal{H}/\Gamma_0(2)$



$\mathcal{H}/\Gamma(2)$

2. moduli interpretation of $\Gamma\backslash\mathcal{H}$

Def. A basis (v_1, v_2) of a lattice $\Lambda \subseteq \mathbb{C}$ is called oriented if $\operatorname{Im}\frac{v_1}{v_2} > 0$.

Def (Weil pairing) [WWL, 注记8.5.9, 定义3.8.9, 练习3.8.10]

For $N \in \mathbb{Z}_{\geq 1}$, $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$, $\operatorname{Im}\frac{u}{v} > 0$, we define the Weil pairing e_N:

$$\begin{array}{ccccc}
 & E[N]\times E[N] & \xrightarrow{e_N} & \mu_N & \\
 & \wr\| & & & \\
a\frac{u}{N}+c\frac{v}{N} & \frac{1}{n}\Lambda/\Lambda \times \frac{1}{n}\Lambda/\Lambda & & & \\
\downarrow & \wr\| & & & \\
 & (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \times (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} & \longrightarrow & \mu_N \cong (\mathbb{Z}/N\mathbb{Z}, +) & \\
\binom{a}{c} & \left(\binom{a}{c}, \binom{b}{d}\right) & \longmapsto & \zeta_N^{\left|\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right|} \longmapsto \left|\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right| &
\end{array}$$

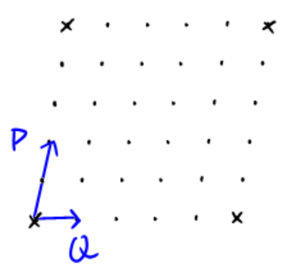
Ex. Let $e_1, e_2 \in E[n]$.

1. e_N is antisymmetric and bilinear.
   $e_N(\gamma(e_1, e_2)) = \zeta_N^{\det\gamma}\, e_N(e_1, e_2) \quad \forall \gamma \in GL_2(\mathbb{Z}/N\mathbb{Z})$
   e.p. e_N only depends on E and N (does not depend on Λ and u, v)
2. $e_N(e_1, e_2) \in \mu_N^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \iff E[N] = \langle e_1, e_2 \rangle_{\mathbb{Z}}$
   $e_N(e_1, e_2) = \zeta_N \longmapsto 1 \iff \exists P, Q \in \frac{1}{N}\Lambda$, $\bar{P} = e_1$, $\bar{Q} = e_2$, (NP, NQ) is an oriented basis of Λ.

Def. (e_1, e_2) is called a pretty oriented basis of $E[N]$. if $e_N(e_1, e_2) = \zeta_N$.

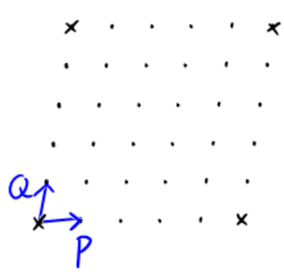
In [KM85], this is called Drinfeld basis.

Ex. $N = 5$



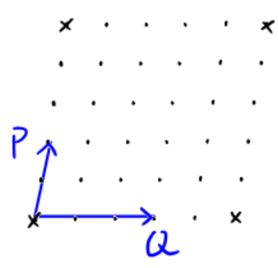
$e_N(\bar{P}, \bar{Q}) = \zeta_5^2$

$(5P, 5Q)$ is not a basis of Λ.



$e_N(\bar{P}, \bar{Q}) = \zeta_5^4 = \zeta_5^{-1}$

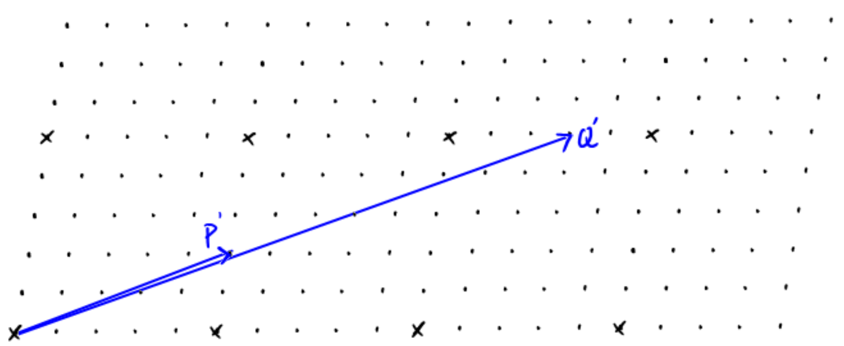
$(5P, 5Q)$ is a basis of Λ, but not an oriented basis.



$e_N(\bar{P}, \bar{Q}) = \zeta_5^6 = \zeta_5$

$(5P, 5Q)$ is not a basis of Λ but $(5P', 5Q')$ is an oriented basis.

$\underset{\substack{\cap \\ SL_2(\mathbb{Z})}}{\begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}} \overset{\text{mod } 5}{\equiv} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$



Recall: For $E = \mathbb{C}/\Lambda$, $E[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$

Main Thm. We have the following moduli interpolations (E: any cplx EC curve)

$$
\begin{tikzcd}
\begin{array}{c}\left\{(E,\alpha) \,\middle|\, \begin{array}{l} \alpha:(\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[N] \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N \end{array}\right\}/\sim \\ \text{E with a pretty oriented basis } (e_1,e_2)\end{array} \arrow[r, "\sim"] \arrow[d, "\beta=\alpha(0,-)"] & \Gamma(N)\backslash\mathcal{H} \arrow[d, "N"] \\
\begin{array}{c}\{(E,\beta) \mid \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]\}/\sim \\ \text{E with an N-torsion pt of E}\end{array} \arrow[r] \arrow[d, "C=\mathrm{Im}\,\beta"] & \Gamma_1(N)\backslash\mathcal{H} \arrow[d, "\begin{cases} \frac{N}{2}\prod_{p\mid N}(1-\frac{1}{p}) & N\neq 2 \\ 1 & N=2 \end{cases}"] \\
\begin{array}{c}\{(E,\mathcal{F}) \mid \mathcal{F}: 0 \subseteq C \subseteq E[N]\}/\sim \\ \text{E with } C\subseteq E[N],\ C\cong \mathbb{Z}/N\mathbb{Z}\end{array} \arrow[r] \arrow[d] & \Gamma_0(N)\backslash\mathcal{H} \arrow[d, "N\prod_{p\mid N}(1+\frac{1}{p})"] \\
\{\text{cplx EC } E\}/\sim \arrow[r] & \Gamma(1)\backslash\mathcal{H}
\end{tikzcd}
$$

Idea. A pretty oriented basis on $E[N]$ gives us a oriented basis on E up to $\Gamma(N)$-action;

coefficient has to be 1, so that (v_1+bv_2, v_2) is a pretty oriented basis.

$$\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}v_1+bv_2\\v_2\end{pmatrix} \quad \Rightarrow \text{ an } n\text{-torsion pt } v_2$$

$$\begin{pmatrix}a&b\\0&d\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}av_1+bv_2\\dv_2\end{pmatrix} \quad \Rightarrow \text{ a flag } 0 \subseteq \underset{\langle v_2\rangle}{\underset{\parallel}{C}} \subseteq \underset{\langle v_1,v_2\rangle}{\underset{\parallel}{E[N]}}$$

Proof. For $\Gamma(N)\backslash\mathcal{H}$,

$$
\begin{aligned}
\text{LHS} &\cong \left\{(\Lambda,\alpha) \,\middle|\, \begin{array}{l}\alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \longrightarrow \frac{1}{N}\Lambda/\Lambda \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda, e_1, e_2) \,\middle|\, \begin{array}{l} e_1, e_2 \in \frac{1}{N}\Lambda/\Lambda \\ e_N(e_1,e_2) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda, z_1, z_2, e_1, e_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_1 = \frac{z_1}{N} \quad e_2 = \frac{z_2}{N} \\ e_N(e_1,e_2) = \zeta_N \end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \left\{(\Lambda, z_1, z_2) \,\middle|\, \begin{array}{l}\Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ (z_1,z_2) \text{ is an oriented basis of } \Lambda\end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \left\{(z_1,z_2) \in (\mathbb{C}-\{0\})^2 \,\middle|\, \mathrm{Im}\,\frac{z_1}{z_2} > 0\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \Gamma(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}
$$

For $\Gamma_1(N)\backslash\mathcal{H}$,

$$\begin{aligned}
\text{LHS} &\cong \{(\Lambda,\beta) \mid \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow \tfrac{1}{N}\Lambda/\Lambda\}^{\circlearrowleft \mathbb{C}^\times} \\
&\cong \{(\Lambda, e_2) \mid e_2 \in \tfrac{1}{N}\Lambda/\Lambda,\ \text{order}(e_2)=N\}^{\circlearrowleft \mathbb{C}^\times} \\
&\cong \left\{(\Lambda, z_1, z_2, e_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_2 = \frac{z_2}{N} \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array}\right\}^{\circlearrowleft \mathbb{C}^\times}_{\circlearrowleft \Gamma_1(N)}
\end{aligned}$$

If not, set $(z_1, z_2) \mapsto (-z_1, z_2)$

$$\begin{aligned}
&\cong \left\{(\Lambda, z_1, z_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array}\right\}^{\circlearrowleft \mathbb{C}^\times}_{\circlearrowleft \Gamma_1(N)} \\
&\cong \Gamma_1(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}$$

For $\Gamma_0(N)\backslash\mathcal{H}$,

$$\begin{aligned}
\text{LHS} &\cong \{(\Lambda, C) \mid C \subseteq \tfrac{1}{N}\Lambda/\Lambda,\ C \cong \mathbb{Z}/N\mathbb{Z}\}^{\circlearrowleft \mathbb{C}^\times} \\
&\cong \left\{(\Lambda, z_1, z_2, C) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ C = (\frac{z_2}{N}) \subseteq \frac{1}{N}\Lambda/\Lambda \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array}\right\}^{\circlearrowleft \mathbb{C}^\times}_{\circlearrowleft \Gamma_0(N)} \\
&\cong \left\{(\Lambda, z_1, z_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array}\right\}^{\circlearrowleft \mathbb{C}^\times}_{\circlearrowleft \Gamma_0(N)} \\
&\cong \Gamma_0(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}$$

$\square$

Rmk. If you observe carefully, you will find out that what we prove actually is

$\mathbb{C}^\times = SO_2(\mathbb{R}) \times \mathbb{R}_{>0}$

$$\begin{array}{ccccccc}
\{(E,\alpha)\} & \xrightarrow{\sim} & & & \Gamma(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{(E,\beta)\} & \xrightarrow{\sim} & \Gamma_1^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_1(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_1(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{(E,\mathcal{F})\} & \xrightarrow{\sim} & \Gamma_0^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_0(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_0(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{E\} & \xrightarrow{\sim} & GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & SL_2(\mathbb{Z})\backslash\mathcal{H}
\end{array}$$

where

$$\begin{array}{ccc}
\Gamma_1^{\pm}(N) & \longrightarrow & \begin{pmatrix} 1 & * \\ 0 & \pm 1 \end{pmatrix} \\
\cap & & \cap \\
\Gamma_0^{\pm}(N) & \longrightarrow & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \cap \\
GL_2(\mathbb{Z}) & \twoheadrightarrow & GL_2(\mathbb{Z}/N\mathbb{Z})_{\det=\pm 1} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})
\end{array}$$

This page is by no means complete. Be skeptical about every result here.

Def [Milne. LEC, Def 6.1] Let $Y \to X$ f.flat, G finite, $G \subset \mathrm{Aut}(Y/X)$.
$Y \to X$ is called a Galois covering with gp G if

$$\begin{aligned} G \times Y &\longrightarrow Y \times_X Y \\ (g, y) &\longmapsto (gy, y) \end{aligned}$$

is an iso.

I believe that

without condition that $e_E(\alpha(1,0), \alpha(0,1)) = \zeta_N$

$$\coprod_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathcal{H} \cong \{(E,\alpha)\}/\sim \;\cong \left\{ (E, Y, \alpha) \,\middle|\, \begin{array}{l} E/Y \text{ Galois étale} \\ \alpha: \mathrm{Gal}(E/Y) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{ (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\text{-torsors } E \twoheadrightarrow Y \} / \sim$$

$$\cong \left\{ (E, X, \alpha') \,\middle|\, \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} (\mu_N)^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{ (\mu_N)^{\oplus 2}\text{-torsors } X \twoheadrightarrow E, \; X \text{ connected} \} / \sim$$

$$\Gamma_1(N)\backslash\mathcal{H} \cong \{(E,\beta)\}/\sim \;\cong \left\{ (E, Y, \beta) \,\middle|\, \begin{array}{l} E/Y \text{ Galois étale} \\ \alpha: \mathrm{Gal}(E/Y) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \end{array} \right\} / \sim$$

$$= \{ \mathbb{Z}/N\mathbb{Z}\text{-torsors } E \twoheadrightarrow Y \} / \sim$$

$$\cong \left\{ (E, X, \alpha') \,\middle|\, \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} \mu_N \end{array} \right\} / \sim$$

$$= \{ \mu_N\text{-torsors } X \twoheadrightarrow E, \; X \text{ connected} \} / \sim$$

$$\Gamma_0(N)\backslash\mathcal{H} \cong \{(E, \mathcal{F})\}/\sim \;\cong \{ (\Lambda, \Lambda') \mid \Lambda \overset{\mathbb{Z}/N\mathbb{Z}}{\subset} \Lambda' \}^{\circlearrowleft \mathbb{C}^\times} / \sim$$

$$\cong \left\{ (E, Y) \,\middle|\, \begin{array}{l} E/Y \text{ Galois étale} \\ \mathrm{Gal}(E/Y) \cong \mathbb{Z}/N\mathbb{Z} \end{array} \right\} / \sim$$

$$= \{ \mathbb{Z}/N\mathbb{Z}\text{-isogeny } E \twoheadrightarrow Y \}$$

$$\cong \left\{ (E, X) \,\middle|\, \begin{array}{l} X/E \text{ Galois étale} \\ \mathrm{Gal}(X/E) \cong \mu_N \end{array} \right\} / \sim$$

$$= \{ \mu_N\text{-isogeny } X \twoheadrightarrow E, \; X \text{ connected} \} / \sim$$

My confusion: In [https://arxiv.org/pdf/1510.05687.pdf](and its historical version), it's claimed that

without condition that $e = (\alpha(1,0), \alpha(0,1)) = \zeta_N$

$$\coprod_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathcal{H} \cong \{(E,\alpha)\}/\sim \;\cong \left\{ (E, X, \alpha') \,\middle|\, \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{ (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\text{-torsors } X \twoheadrightarrow E \} / \sim$$

Q:
- For $K = \mathbb{C}$, the $\mathbb{Z}/N\mathbb{Z}$-torsor is exactly the μ_N-torsor, but which one is correct when it concerns with the arithmetical moduli?
- Is $\mathbb{Z}/N\mathbb{Z}$-torsor $X \twoheadrightarrow E$ representable as a scheme over $\mathrm{Spec}\, \mathbb{Z}[\frac{1}{6N}]$?
- How to translate the condition "connected" in the arithmetical moduli?

Ex. For an elliptic curve $E/\mathbb{C}$, compute the number of connected $\mathbb{Z}/N\mathbb{Z}$-torsors over E.

(For the torsors, the $\mathbb{Z}/N\mathbb{Z}$-action is also considered as information)

Explain the ramification of map $_{\Gamma_1(N)}\backslash\mathcal{H} \longrightarrow {}_{\Gamma(1)}\backslash\mathcal{H}$.

For moduli interpretations of noncongruence groups, see [https://arxiv.org/pdf/1510.05687.pdf] (I'm skeptical about this article, the results are vaguely true, while some assumptions are not explicitly written)

Rmk. In this remark we discuss about modular curve in AG. The main ref is [KM85] (if not mentioned), and lecture note [https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/] is also refered. This page will be moved to [https://github.com/ramified/moduli_in_algebraic_geometry] when I fully understand it.

Even though the font is less preferable, many beautiful figures and tables are included in [KM85].

∇ In [KM85, p146], $\Gamma(N) \subseteq \text{bal.}\Gamma_1(N) \overset{2}{\subseteq} \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \Gamma(1) = GL_2(\mathbb{Z})$

$\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $\quad \begin{pmatrix}1&*\\0&1\end{pmatrix}$ $\varphi(N)$ $\begin{pmatrix}1&*\\0&*\end{pmatrix}$ $\quad \begin{pmatrix}*&*\\0&*\end{pmatrix}$ $\quad \begin{pmatrix}*&*\\ *&*\end{pmatrix} = GL_2(\mathbb{Z}/N\mathbb{Z})$

$N \neq 2$

$\downarrow$ not surj when $N \neq 2,3,4,6$

[(4.7.0)] If $\mathcal{P}$ is relatively rep, affine over (Ell), and rigid
$\Rightarrow \mathcal{P}$ is rep.

$\Rightarrow$[Cor(4.7.1)] Suppose $\mathcal{P}$: relative rep, affine and étale over (Ell), and rigid
$\Rightarrow \mathcal{P}$ is rep by a sm affine curve over $\mathbb{Z}$.

rigidity: [(2.7.2)] at least for $[\Gamma(N)]$
rel. rep: [(3.6.0)] (relative representability thm) (relative: fix E/S)
étale: [(3.6.1)] when $S \in Sch_{\mathbb{Z}[\frac{1}{N}]}$. $X \longrightarrow E$ is étale

[5.1.1]. $[\Gamma(N)]$, $[\Gamma_1(N)]$ & $[\Gamma_0(N)]$ are
- relatively rep over (Ell),
- finite & flat$/(Ell)$ of constant rank ≥ 1
- regular
- $-\otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{N}]$ is finite étale over $(Ell/\mathbb{Z}[\frac{1}{N}])$

E.g. [(2.1.11)] rep of $\Gamma(3)$ in $Sch_{\mathbb{Z}[\frac{1}{3}]}$.