Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve
https://mathoverflow.net/questions/402049/induced-action-on-prym-variety
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Goal: Describe some curves (maybe singular) C in A, and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{ y^2 = \prod_{j=1}^{10} (x^2 - j) \}$ has the following covers:

genus

$\mathrm{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

ramification degree

$$\begin{tikzcd}
 & C_9 \arrow[dl, "20"'] \arrow[d, "4"', "\tilde{f}"] \arrow[dr, "f", "0"'] & \\
\mathbb{P}^1 \arrow[dr, "2"'] & C_4 \arrow[d, "10"'] & C_5 \arrow[dl, "12"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

where

$$C_4 = \{ y^2 = \prod_{j=1}^{10} (t - j) \}$$

$$C_5 = \{ \tilde{y}^2 = t \prod_{j=1}^{10} (t - j) \}$$

$t = x^2$

$\tilde{y} = xy$

The crspd field extension:

$$\begin{tikzcd}
 & \mathbb{C}(x)[y] / (y^2 - \prod_{j=1}^{10} (x^2 - j)) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{C}(x) \arrow[dr, dash] & \mathbb{C}(t)[y] / (y^2 - \prod_{j=1}^{10} (t - j)) \arrow[d, dash] & \mathbb{C}(t)[\tilde{y}] / (\tilde{y}^2 - t \prod_{j=1}^{10} (t - j)) \arrow[dl, dash] \\
 & \mathbb{C}(t) &
\end{tikzcd}$$

Global differential forms

Pulling back differential forms give the following maps:

$$\begin{tikzcd}
 & \langle x^k \frac{dx}{y} \mid k=0,\dots,8 \rangle & \\
 & \langle 2x^{2k+1}\frac{dx}{y} \mid k=0,\dots,3 \rangle \arrow[u, "\tilde{f}^*"] & \langle 2x^{2k}\frac{dx}{y} \mid k=0,\dots,4 \rangle \arrow[ul, "f^*"'] \\
0 \arrow[uur] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3 \rangle \arrow[u, equal] & \langle 2t^k \frac{dt}{y} \mid k=0,\dots,4 \rangle \arrow[u, equal] \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Therefore,

$$H^0(C_9;\omega_{C_9}) \cong \tilde{f}^* H^0(C_4;\omega_{C_4}) \oplus f^* H^0(C_5;\omega_{C_5}) \tag{1}$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which crspds to pulling back of divisors)

$$\begin{tikzcd}
 & \langle x^k \frac{dx}{y} \mid k=0,\dots,8 \rangle \arrow[ddl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\
 & \langle 2x^{2k+1}\frac{dx}{y} \mid k=0,\dots,3 \rangle & \langle 2x^{2k}\frac{dx}{y} \mid k=0,\dots,4 \rangle \\
0 \arrow[dr] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3 \rangle \arrow[u, equal] \arrow[d] & \langle 2t^k \frac{dt}{y} \mid k=0,\dots,4 \rangle \arrow[u, equal] \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

However, since $\mathrm{Jac}(C) = H^0(C;\omega_C)^* / H_1(C;\mathbb{Z})$, we are working on the dual spaces. The notations are again switched:

$$\begin{aligned} f^* &\rightsquigarrow Nm_f \\ f_* &\rightsquigarrow f^* \end{aligned}$$

One may get

different meaning compared with (1)!

$$H^0(C_9;\omega_{C_9})^* \cong \tilde{f}^* H^0(C_4;\omega_{C_4})^* \oplus f^* H^0(C_5;\omega_{C_5})^* \tag{2}$$

(co)homology class

This page may be easier to understand, and it helps to understand the previous page.

$$\begin{tikzcd} & H^1(C_9;\mathbb{Z}) & \\ 0 \arrow[ur] & H^1(C_4;\mathbb{Z}) \arrow[u, "\tilde{f}^*"] & H^1(C_5;\mathbb{Z}) \arrow[ul, "f^*"'] \\ & 0 \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

Q: Do we have

$$H^1(C_9;\mathbb{Z}) \cong \tilde{f}^* H^1(C_4;\mathbb{Z}) \oplus f^* H^1(C_5;\mathbb{Z}) ?$$

$$\begin{tikzcd} & H_1(C_9;\mathbb{Z}) \arrow[dl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\ 0 \arrow[dr] & H_1(C_4;\mathbb{Z}) \arrow[d] & H_1(C_5;\mathbb{Z}) \arrow[dl] \\ & 0 & \end{tikzcd}$$

Q: Do we have

$$H_1(C_9;\mathbb{Z})^* \cong \tilde{f}^* H_1(C_4;\mathbb{Z})^* \oplus f^* H_1(C_5;\mathbb{Z})^* ?$$

Curve in Prym variety

Define A as the quotient of Jacobians, i.e.,

$$A := Jac(C_9)/_{f^* Jac(C_5)} \cong Prym(C_9/C_5)$$

$$\begin{tikzcd} & & C_9 \arrow[r] \arrow[d, "AJ_{C_9}"] & C_4 \arrow[d] \arrow[r] & Jac(C_4) \arrow[dl, dashed, "\exists ! \text{ isogeny}"] & \\ 0 \arrow[r] & Jac(C_5) \arrow[r, "f^*"] & Jac(C_9) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \quad (3)$$

Prop. 0. A is isogenous to $Jac(C_4)$;
1. $f^*: Jac(C_5) \longrightarrow Jac(C_9)$ is injective;
2. $\pi \circ AJ_{C_9}$ is not injective, it factors through C_4;
3. $C_4 \longrightarrow A$ is generically injective;
4. $C_4 \longrightarrow A$ produces a sm image of A, outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof. 0. Taking the tangent space of (3), one gets

$$0 \longrightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df^*_0} H^0(C_9, \omega_{C_9})^* \longrightarrow T_0 A \longrightarrow 0$$

Combined with (2),

$$T_0 A \cong H^0(C_4; \omega_{C_4})^*.$$

Late we will find a natural isogeny $Jac(C_4) \longrightarrow A$.
What's the degree of this isogeny?

1. Since

$$Nm_f \circ f^* = 2\, Id_{Jac(C_5)} \qquad \& \text{ char } k \neq 2,$$

f^* is injective.

2. For $p_1 = (x_0, y_0)$, $p_2 = (-x_0, y_0)$, we want to show that

$$\int_{\gamma_1 : p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\gamma_2 : p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$\text{LHS} = \int_{\tilde{\gamma}_1 : p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = \text{RHS}.$$

3.

https://mathoverflow.net/questions/68503/has-anyone-studied-the-prym-map-for-double-covers-with-two-ramification-points
https://arxiv.org/abs/1010.4483: It proves that many Prym maps (C->Prym) are generically finite.

Notice : $C_4 \subset \mathrm{Jac}(C_4)$ is only invariant under $p \mapsto -p$,
not invariant under $p \mapsto p + a_0$.

Otherwise, the Gauss map would be cover of deg > 2.
Therefore, after isogeny $C_4 \longrightarrow A$ is still gen inj.
Q: Is this map really inj?

4. $C_4 \hookrightarrow \mathrm{Jac}(C_4)$ is sm, so after isogeny it is still sm outside of non-injective locus.