

# Eine Woche, ein Beispiel

## 4.17 preliminary facts of representations of p-adic groups

Main reference: The Local Langlands Conjecture for  $\mathrm{GL}(2)$  by Colin J. Bushnell and Guy Henniart.  
[<https://link.springer.com/book/10.1007/978-3-540-31511-X>]

Process.

1. Basic properties
  - Smoothness
  - Irreducibility and unitary
  - Reduction to smaller cardinal.
2. Examples:  $\mathcal{O}, \mathcal{O}^\times, F, F^\times$
3. Construction of new reps.
  - Special sub & quotient rep
  - Duality
  - Ind and c-Ind
  - Other constructions  $\leftarrow$  Example: mirabolic group  $M$
4. Hecke algebra
5. Intertwining properties  $\leftarrow$  Example:  $\mathrm{GL}_2(\mathbb{Q}_p)$

## 1. Basic properties

### 1.1. Smoothness

$G$ : loc. profinite group

$V$ : cplx v.s.

$$\rho: G \longrightarrow \text{Aut}_{\mathbb{C}}(V) \quad g \mapsto [v \mapsto g.v]$$

Def.  $(\rho, V)$  is smooth if

$$\forall v \in V, \exists K \leq G \text{ cpt open s.t. } k.v = v \quad \forall k \in K$$

$\text{Rep}(G) = \{\text{sm rep of } G\}$  is a full subcategory of  $\{\text{rep of } G\}$ .

Rmk. Any sub/quotient rep of  $(\rho, V) \in \text{Rep}(G)$  is smooth.

$$H \leq G \text{ cpt, } (\rho, V) \in \text{Rep}(G) \Rightarrow (\rho|_H, V) \in \text{Rep}(H)$$

Rmk. For fcts, smoothness has a different meaning.

Recall the definition of  $C^\infty(G)$  &  $C_c^\infty(G)$ :

$$C^\infty(G) := \{f: G \rightarrow \mathbb{C} \mid f \text{ is loc. const}\}$$

$$C_c^\infty(G) := \{f \in C^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$$

## 1.2. Irreducibility and unitary

$$\text{Irr}(G) = \{(\rho, V) \in \text{Rep}(G) \mid \rho \text{ is a irreducible rep}\}$$

$$\widehat{G} = \{(\rho, V) \in \text{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$$

$$\stackrel{[P13]}{=} \{X: G \rightarrow \mathbb{C}^{\times} \mid \ker X \text{ is open}\}$$

$$\stackrel{[C1.6]}{=} \{X: G \rightarrow \mathbb{C}^{\times} \mid X \text{ is continuous}\}$$

Rmk. The notation is slightly different with the original reference.

Rmk.

$$\widehat{G} \subseteq \text{Irr}(G) \subseteq \text{Ind}(G) \subseteq \text{Rep}(G)$$

[P15]

When  $G$  is cpt, or

[P21]  $G/Z(G)$  is cpt with  $G/K$  countable, we get  $\text{Ind}(G) = \text{Irr}(G)$ ;

[P21] when  $G$  is abelian and  $G/K$  is countable,  $\text{Ind}(G) = \widehat{G}$ .

( $\exists K \leq G$  cpt open, countable = at most countable here)

Rmk. A more general result is as follows.

Prop. Let  $(\rho, V) \in \text{Rep}(G)$ ,  $G/K$  countable.  $\exists K \leq G$  cpt open

Suppose  $\rho|_{Z(G)}$  decompose as  $Z(G) \xrightarrow{\chi_w} \mathbb{C}^{\times} \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$ .

Let  $Z(G) \leq K \leq G$   $K \leq G$  open  $K/Z(G)$  is cpt.

Then  $(\rho|_K, V) \in \text{Rep}(K)$  is semisimple.

We will rewrite it as follows.

Prop. Let  $G$  be a loc. cpt gp satisfying the countable Hypothesis.  $Z = Z(G)$ .

For  $(\rho, V) \in \text{Rep}(G) \xrightarrow{\cong} \chi_w$ ,  $K \in \text{Cos}_Z(G)$ ,

$(\rho|_K, V) \in \text{Rep}(K)$  is semisimple.

To prove this we need the following lemma. (when applied, it would be  $K_0 Z(G) \leq K$ )

Lemma. Let  $H \leq G$  open,  $[G:H] < \infty$ ,  $(\rho, V) \in \text{Rep}(G)$ . Then

$\rho$  is  $G$ -semisimple  $\Leftrightarrow \rho|_H$  is  $H$ -semisimple.

Def (multiple of  $\chi$ )  $\rho \xrightarrow{H} \chi$

Let  $H \leq G$ ,  $(\rho, V) \in \text{Rep}(G)$ ,  $\chi \in \widehat{H}$ .

We say  $H$  acts on  $V$  as  $\chi$ .

or  $\rho|_H$  is a multiple of  $\chi$ ,

or (when  $H = Z(G)$ )  $\rho$  admits the central character  $\chi$ .

if  $\rho|_H$  decompose as follows:

$$\rho|_H: H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$$

We may denote  $\chi$  by  $\chi_\rho$  or  $\chi_H$ . When  $H = Z(G)$ ,  $\chi$  is denoted by  $\chi_w$  or  $w\rho$ .

Def (Contain irr rep)  $\rho \rhd \sigma \quad n = \text{mult}(\rho, \sigma)$

Let  $H \leq G$ ,  $(\rho, V) \in \text{Rep}(G)$ ,  $(\sigma, W) \in \text{Irr}(H)$ .

We say  $\rho$  contains  $\sigma$ , or  $\sigma$  occurs in  $\rho$ , if

$$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \neq 0$$

i.e.,  $\sigma$  can be realized as a quotient subrep of  $\text{Res}_H^G \rho$ .

The multiplicity is defined as

$$\text{mult}(\rho, \sigma) := \dim_{\mathbb{C}} \text{Hom}_H(\text{Res}_H^G \rho, \sigma)$$

Cor. When  $H$  acts on  $V$  as  $\chi_\rho$ ,  $\rho$  contains  $\chi_\rho$ .

Def (Inflation)  $\lambda \xrightarrow{G} \Lambda$

Let  $N \triangleleft G$ . The inflation of  $(\lambda, W) \in \text{Rep}(G/N)$  is defined as

$(\Lambda, W) \in \text{Rep}(G)$ , where

$$\Lambda: G \xrightarrow{\pi} G/N \xrightarrow{\lambda} \text{End}_{\mathbb{C}}(W)$$

Def (Twist)  $\chi_\rho$

Suppose  $G \leq GL_n(F)$ , where  $F$  is a non-archi local field.

For  $\chi \in \widehat{F}^\times$  and  $(\rho, V) \in \text{Rep}(G)$ , we define

$$\chi_\rho := (\chi \circ \det) \otimes_{\mathbb{C}} \rho: G \longrightarrow \text{End}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}} V) = \text{End}_{\mathbb{C}}(V)$$

$$\chi_\rho(g) \cdot v = \chi(\det g) \cdot \rho(g) \cdot v \quad \forall g \in G \quad v \in V$$

as the twist of  $\rho$  by  $\chi$ .

Def (Unitary operator)  $V$ : Hilbert space.

$U \in \text{Aut}_{\mathbb{C}}(V)$  is called an unitary operator if

$$\langle Uv, Uw \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

Def (Unitary rep)  $V$ : Hilbert space.

$(\rho, V) \in \text{Rep}(G)$  is unitary if  $\rho(g)$  is an unitary operator ( $\forall g \in G$ ).

E.p.  $X \in \widehat{G}$  is unitary if  $\text{Im } X \subseteq S'$

Rmk. When  $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$ , any  $X \in \widehat{G}$  is unitary.

### 1.3. Reduction to smaller cardinal

#### Admissibility

Def.  $(\rho, V) \in \text{Rep}(G)$  is admissible if  $\dim_{\mathbb{C}} V^k < +\infty$  for  $\forall K \leq G$  cpt open.

Rmk. Any sub/quotient rep of  $(\rho, V) \in \text{Rep}(G)$  admissible is admissible.

$H \leq G$  cpt.  $(\rho, V) \in \text{Rep}(G)$  admissible

$\Rightarrow (\rho|_H, V) \in \text{Rep}(H)$  is admissible.

#### Countable hypothesis

$\exists / \forall K \leq G$  cpt open,  $G/K$  is countable.

Assuming countable hypothesis, we get

$$(\rho, V) \in \text{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \text{End}_G(V) = \mathbb{C} \\ \rho \text{ acts on } V \text{ as a character } w_p \\ \dim_{\mathbb{C}} V = 1. \end{cases}$$

$\xrightarrow[G \text{ is abelian}]{} \dim_{\mathbb{C}} V = 1.$

2. Examples:  $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

Rep of  $G = \mathcal{O}, \mathcal{O}^\times, F, F^\times$ , where  $F$  is a non-archi local field.

In these cases,  $G$  is abelian and satisfy the countable hypothesis, so  $\text{Ind}(G) = \widehat{G}$ , i.e., everything reduced to the classification of characters.

E.g.  $G = (\mathcal{O}, +)$

$\forall \psi \in \widehat{\mathcal{O}}$  is trivial on  $\mathfrak{p}^k$ . Suppose  $\psi \neq 1$ .

$$\text{level}(\psi) := \min \{d \geq 0 \mid \mathfrak{p}^d \subset \ker \psi\}$$

When  $\text{level}(\psi) = d$ ,  $\psi: \mathcal{O} \xrightarrow{\pi} \mathcal{O}/\mathfrak{p}^d \rightarrow \mathbb{C}^\times$  factors through char of  $\mathcal{O}/\mathfrak{p}^d$ .

E.g.  $G = \mathcal{O}^\times$

$\forall \chi \in \widehat{\mathcal{O}}^\times$  is trivial on  $\mathcal{U}^{(k)}$ . Suppose  $\chi \neq 1$ .

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

When  $\text{level}(\chi) = d$ ,  $\chi: \mathcal{O}^\times \xrightarrow{\pi} \mathcal{O}^\times/\mathcal{U}^{(d+1)} \rightarrow \mathbb{C}^\times$  factors through char of  $(\mathcal{O}/\mathfrak{p}^{d+1})^\times$

$$(\mathcal{O}/\mathfrak{p}^{d+1})^\times$$

Recent advances: Geometrization of continuous characters of  $\mathbb{Z}_p$  [<https://msp.org/pjm/2013/261-1/pjm-v261-n1-p05-p.pdf>]

E.g.  $G = (F, +)$

$\forall \psi \in \widehat{F}$  is trivial on  $\mathfrak{p}^k$ . Suppose  $\psi \neq 1$ .

$$\text{level}(\psi) := \min \{d \in \mathbb{Z} \mid \mathfrak{p}^d \subset \ker \psi\}$$

Prop (Additive duality)

Fix  $\psi \in \widehat{F}$  nontrivial with level  $d$ .

We have a gp iso

$$F \longrightarrow \widehat{F} \quad a \mapsto \psi(a)\psi(-) \text{ of level } d - \nu_F(a) \quad (\text{when } a \neq 0)$$

Q: Do we have similar result for  $\widehat{\mathcal{O}}$ ?

E.g.  $G = F^\times$

$\forall \chi \in \widehat{F}^\times$  is trivial on  $\mathcal{U}^{(k)}$ . Suppose  $\chi \neq 1$ .

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

Q: Do we have any classification of  $F^\times$ ?

A: Yes. Since  $F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times$ , any cont character  $\chi \in \widehat{F}^\times$  is uniquely determined by  $\chi(\pi) \in \mu_\infty$  and  $\chi|_{\mathcal{O}^\times} \in \widehat{\mathcal{O}}^\times$ .

Notation for future:  $G$ : loc. profinite gp  $Z = Z(G)$   
 $\text{Cos}(G) := \{\text{cpt open subgp } K \text{ of } G\}$   
 $\text{Cos}_z(G) := \{K \leq G \mid K \geq Z, K/Z \subset G/Z \text{ cpt open}\}$   
When  $\#Z(G) = +\infty$ ,  $\text{Cos}(G) \cap \text{Cos}_z(G) = \emptyset$ .

### 3. Construction of new reps

#### 3.1. Special sub & quotient rep.

Def.  $G$ : loc. profinite gp  $K \leq G$  cpt open

$(\pi, V) \in \text{Rep}(G)$   $\vartheta \in \widehat{K}$ , we define sm reps of  $K$ :

$$\begin{aligned} V(K) &:= \langle v - k \cdot v \rangle_{k \in K, v \in V} \\ V_K &:= V/V(K) \end{aligned} \quad \left. \begin{array}{l} \text{in } \text{Rep}(B) \text{ if } K \triangleleft B \leq G \\ \text{not confused with } V^N! \end{array} \right.$$

$$V(\vartheta) := \langle \vartheta(k)v - k \cdot v \rangle_{k \in K, v \in V}$$

$$V_\vartheta := V/V(\vartheta)$$

Obviously  $V(K) = V(\mathbb{1}_K)$ ,  $V_K = V_{\mathbb{1}_K}$ ,  $K$  acts on  $V(\vartheta)$  by  $\vartheta$ , and  
 $0 \rightarrow V(\vartheta) \rightarrow V \rightarrow V_\vartheta \rightarrow 0$  in  $\text{Rep}(K)$

Rmk.(1) Given  $(\sigma, V) \in \text{Rep}(K)$ ,  $\vartheta \in \widehat{K}$ , we can view  $V(\vartheta)$  as  $V'(\sigma)$ , where

$$(\sigma', V') := (\vartheta^{-1} \otimes_{\sigma} \sigma, V)$$

(2) Normal subgp gives us plenty of canonical decompositions.

E.g., when  $(\rho, V) \in \text{Irr}(G)$ ,  $K \triangleleft G$ , we get

$$\begin{cases} V(K) = V \\ V_K = 0 \end{cases} \quad \text{or} \quad \begin{cases} V(K) = 0 \\ V_K = V \end{cases}$$

(3) When  $(\rho, V) \in \text{Rep}(K)$  is semisimple,

$$0 \rightarrow V(K) \rightarrow V \rightarrow V_K \rightarrow 0$$

$$0 \rightarrow \bigoplus_{\substack{\sigma \in \text{Irr}(K) \\ \sigma \neq \mathbb{1}_K}}^{11S} V^\sigma \rightarrow \bigoplus_{\sigma \in \text{Irr}(K)}^{11S} V^\sigma \rightarrow V^{\mathbb{1}_K} \rightarrow 0$$

Assume additionally that

- $K$  is abelian,
- $K$  is the union of an increasing sequence of cpt open subgps.

Then, we have more properties.

Prop. (Integral criterian) [p56 Lemma]  $(\sigma, V) \in \text{Rep}(K)$ ,  $v \in V$ .

$$v \in V(\vartheta) \Leftrightarrow \left[ \exists K_0 \in \text{Cos}(K) \text{ s.t } \int_{K_0} \mathcal{J}(k)^{-1} \sigma(k) v \, d\mu_K(k) = 0 \right]$$

Prop. The factor

$$\text{Rep}(G) \longrightarrow \text{Vect}_{\mathbb{C}} \quad (\pi, V) \longmapsto V_{\vartheta}$$

is exact. E.p.,

$$(v \neq 1) \quad \begin{aligned} V(K)(K) &\cong V(K) & V(K)_K &= 0 \\ V(K)(\vartheta) &\cong V(K) \wedge V(\vartheta) & V(K)_{\vartheta} &\cong V_{\vartheta} \end{aligned}$$

$$(v \neq 1) \quad \begin{aligned} V_K(K) &= 0 & V_{K,K} &\cong V_K \\ V_K(\vartheta) &\cong V_K & V_{K,\vartheta} &= 0 \end{aligned}$$

(You can compute  $V(\vartheta)$  and  $V_{\vartheta}$  also, but I'm lazy.)

### 3.2. Duality

$(\rho, V) \in \text{Rep}(G)$   $\rightsquigarrow (\rho^*, V^*)$  may be not smooth

$\rightsquigarrow (\check{\rho}, \check{V}) \in \text{Rep}(G)$  is the smooth dual, where  
 $\check{V} := \bigcup_{K \in \text{Cos}(G)} (V^*)^K \subset V^*$

$$\text{ev}: \check{V} \times V \rightarrow \mathbb{C} \quad (\check{w}, v) \mapsto \langle \check{w}, v \rangle$$

$$\rightsquigarrow \langle g \cdot \check{w}, v \rangle = \langle \check{w}, g^{-1} \cdot v \rangle$$

Rmk. (1) The contravariant duality functor

$$\text{Rep}(G) \rightarrow \text{Rep}(G) \quad (\rho, V) \mapsto (\check{\rho}, \check{V})$$

is exact.

exact=>additive: <https://math.stackexchange.com/questions/3039422/in-abelian-categories-is-a-right-left-exact-functor-necessarily-additive>

(2) For  $K \in \text{Cos}(G)$ , we have iso  $\check{V}^K \xrightarrow{\cong} (V^K)^*$  in  $\text{Rep}(K)$ .

(3) If  $(\rho, V) \in \text{Rep}(G)$ ,  $v \in V$ , then  $\exists \check{w} \in \check{V}$  s.t.  $\langle \check{w}, v \rangle \neq 0$ .

(4)  $\delta: V \rightarrow \check{V}$  is inj. and

$\downarrow$   
 $\delta$  is iso  $\Leftrightarrow \pi$  is admissible

(5) If  $(\rho, V) \in \text{Rep}(G)$  is admissible, then

$$(\rho, V) \in \text{Irr}(G) \Leftrightarrow (\check{\rho}, \check{V}) \in \text{Irr}(G)$$

(6) (Bilinear map) Let  $(\rho, V), (\sigma, W) \in \text{Rep}(G)$

$$\mathcal{S}(\rho, \sigma) := \left\{ f: V \times W \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ bilinear} \\ f(gv, gw) = f(v, w) \end{array} \right\}.$$

Then

$$\mathcal{S}(\rho, \sigma) \cong \text{Hom}_G(\rho, \check{\sigma}) \cong \text{Hom}(\sigma, \check{\rho}).$$

### 3.3. Ind and c-Ind

#### Definition

Def (Induced representation)

$H \leq G$  closed,  $(\sigma, W) \in \text{Rep}(H)$ , we get

{ sm induction  $\text{Ind}_H^G \sigma = (\Sigma, X) \in \text{Rep}(G)$

{ cpt induction  $c\text{-Ind}_H^G \sigma = (\Sigma_c, X_c) \in \text{Rep}(G)$

as follows:

$$\text{Ind}_H^G W = X = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(hg) = \sigma(h)f(g) \\ \exists K \leq G \text{ s.t. } f(gk) = f(g) \end{array} \quad \begin{array}{l} \forall g \in G, h \in H \\ \forall g \in G, k \in K \end{array} \right\}$$

$$\Sigma(g). f = f(-g)$$

$$c\text{-Ind}_H^G W = X_c = \left\{ f \in X \mid \begin{array}{l} \pi(\text{supp } f) \text{ is cpt in } H \backslash G \\ \pi: G \rightarrow H \backslash G \end{array} \right\}$$

$$\Sigma_c(g). f = f(-g)$$

Rmk. (1). (Reality check) When  $G = H$ ,

$$c\text{-} \text{Ind}_H^G W = \text{Ind}_H^G W = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(g) = \sigma(g) f(1) \\ \exists k \in G \text{ s.t. } f(gk) = f(g) \end{array} \right\} \xrightarrow{\cong} W$$

$$\begin{aligned} f &\longmapsto f(1) \\ \sigma(-) \cdot w &\longleftarrow w \end{aligned}$$

(2) Two facts  $\text{Ind}_H^G$ ,  $c\text{-}\text{Ind}_H^G$  are both exact.

(3) Suppose  $H \leq G$  open,  $[G:H] < +\infty$ ,  $(\sigma, W) \in \text{Irr}(H)$ . We get  $\text{Ind}_H^G \sigma$  is  $G$ -semisimple.

### Frobenius Reciprocity

Thm.

condition		$(\rho, V) \in \text{Rep}(G)$ , $(\sigma, W) \in \text{Rep}(H)$
$H \leq G$ closed	$\text{Res}_H^G \vdash \text{Ind}_H^G$	$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \cong \text{Hom}_G(\rho, \text{Ind}_H^G \sigma)$
$H \leq G$ open	$c\text{-}\text{Ind}_H^G \vdash \text{Res}_H^G$	$\text{Hom}_G(c\text{-}\text{Ind}_H^G \sigma, \rho) \cong \text{Hom}_H(\sigma, \text{Res}_H^G \rho)$

E.p. we have two canonical map in  $\text{Rep}(H)$ :

$$\begin{array}{lll} H \leq G \text{ closed} & \varepsilon_W: \text{Ind}_H^G W \rightarrow W & f \mapsto f(1) \\ H \leq G \text{ open} & \eta_W: W \rightarrow c\text{-}\text{Ind}_H^G W & w \mapsto \begin{cases} f_w: h \in H \mapsto h \cdot w \\ x \notin H \mapsto 0 \end{cases} \end{array}$$

### Duality Theorem [p32]

Let  $H \leq G$  closed, define

$$\delta_{H \setminus G} = \delta_H^{-1} \delta_{G|H}: H \rightarrow \mathbb{R}_{>0}^*$$

then  $\delta_{H \setminus G} \in \widehat{H}$ , and  $V(\sigma, W) \in \text{Rep}(H)$ , we have

$$(c\text{-}\text{Ind}_H^G \sigma)^* \cong \text{Ind}_H^G (\delta_{H \setminus G} \otimes \check{\sigma}).$$

### 3.4. Other constructions. (Concerned closely with Hecke algebra)

-  $\otimes$ , Sym, and  $\Lambda$ .

It's difficult for me to work out.

-  $C_c^\infty(G)$ ,  $C_c^\infty(G) \in \text{Rep}(G \times G)$

$C_c^\infty(G)$  will play a central role in Hecke alg. We focus on  $C_c^\infty(G)$  here.

Def. (matrix coefficient set  $C(\rho)$ )

Let  $(\rho, V) \in \text{Rep}(G)$ ,  $v \in V$ ,  $w \in \check{V}$ .

$$\mathbb{I}: \check{V} \otimes V \longrightarrow C^\infty(G)$$

$$w \otimes v \longmapsto \begin{bmatrix} \gamma_{w \otimes v}: G \longrightarrow \mathbb{C} \\ \gamma_{w \otimes v}(g) = w(g \cdot v) = \langle w, \rho(g)v \rangle \end{bmatrix}$$

is a morphism in  $\text{Rep}(G \times G)$ .

$$C(\rho) = \text{Im } \mathbb{I} \subseteq C^\infty(G)$$

Rmk.  $\forall \gamma \in C(\rho)$ ,  $\text{Supp } \gamma$  is invariant under translation by  $\mathbb{Z}$ .

Def. ( $\gamma$ -cuspidal rep)

Let  $G$  be a **unimodular** loc. profinite gp. We define

$$\text{Cusp}_c(G) = \left\{ (\rho, V) \in \text{Irr}(G) \mid \begin{array}{l} \text{supp}(\gamma)/\mathbb{Z} \subseteq G/\mathbb{Z} \text{ is cpt} \\ \text{for every } \gamma \in C(\rho) \end{array} \right\}$$

Prop. [p70] (1) Every  $(\rho, V) \in \text{Cusp}_c(G)$  is admissible

(2) Suppose

- $(\rho, V) \in \text{Irr}(G)$  is admissible
- $\exists \gamma \in C(\rho)$ ,  $\text{supp}(\gamma)/\mathbb{Z}$  is cpt.

Then,

•  $\mathbb{I}: V^* \otimes V \longrightarrow C(\pi)$  is an iso

•  $(\pi, V) \in \text{Cusp}_c(G)$

[Idea of proof: Use dual and cardinal argument.]

[Hecke algebra is needed which will be introduced later]

Rmk.  $\gamma$ -Cuspidal has the property of projectivity and Schur's orthogonality relation, see [p74].

#### 4. Hecke algebra

Assume  $G$  is unimodular, and fix an Haar measure  $\mu$ .

**Basic, alg - mod - rep**

Def. (Hecke algebra  $H(G)$ )

$$H(G) := (C_c^\infty(G), *)$$

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

Rmk In general,  $H(G)$  has no unit element.

However,  $H(G)$  has idempotent elements

$$e_K := \frac{1}{\mu(K)} \mathbb{1}_K \quad \text{for } K \in \text{Cos}(G)$$

$$e_\sigma \in H(G) \quad \text{for } (\sigma, W) \in \text{Irr}(K), K \in \text{Cos}(G)$$

$$e_\sigma(g) = \begin{cases} \frac{\dim_{\mathbb{C}} W}{\mu(K)} \text{tr}(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

well-defined  
since  $\dim_{\mathbb{C}} W < +\infty$

$$\text{e.p. } \mathbb{1}_K \in H(G), e_{\mathbb{1}_K} = e_K.$$

Rmk. When  $G$  is finite,  $H(G) \cong \mathbb{C}[G]$  is the path algebra.

Def. (Smooth  $H(G)$ -module)

Be careful: the scalar multiplication is also denoted by  $*$ .

An  $H(G)$ -module  $M$  is smooth if

$$H(G) * M = M$$

$$\Leftrightarrow \forall m \in M, \exists K \in \text{Cos}(G) \text{ s.t. } e_K * m = m$$

$\text{Mod}(H(G)) = \{\text{sm } H(G)\text{-modules}\}$  is a full subcategory of  $\{\text{H}(G)\text{-modules}\}$ .

Thm We have the equiv of categories.

$$\text{Rep}(G) \longrightarrow \text{Mod}(H(G))$$

$$(\rho, V) \longmapsto V$$

$$f * v = \int_G f(g) (\rho(g).v) d\mu(g)$$

for  $f \in H(G), v \in V$

$$(\rho, M) \longleftarrow M$$

$$\rho(g).m = \frac{1}{\mu(K)} (\mathbb{1}_{gK} * m)$$

for  $g \in G, m \in M,$

$K \in \text{Cos}(G)$  s.t.  $e_K * m = m$

[ Idea:  $G \dashrightarrow H(G)$   
 $g \dashrightarrow \frac{1}{\mu(K)} \mathbb{1}_{gK} \quad K \ll G$  ]

$$\text{E.p. } e_K * v \in V^K \text{ and } \rho(1).m = m$$

Prop [4.3, P37] Let  $(\rho, V) \in \text{Rep}(G), K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K)$ , then

$$e_\sigma * - : V \longrightarrow V^\sigma \quad \text{in } \text{Rep}(K)$$

is a projection.

Task. Assume  $G$  is unimodular. Redo everything in the language of  $\text{Mod}(H(G))$ .

Interesting topic:

The Hecke algebra of a reductive p-adic group: a view from noncommutative geometry [https://www.imj-prg.fr/preprints/389.pdf]

## $H(G, K)$ : an analog of $\mathbb{C}[K \backslash G / K]$

Def. Let  $K \in \text{Cos}(G)$ ,  $(\sigma, W) \in \text{Irr}(K)$ , we define

$$H_1(G, \sigma) := e_\sigma * H(G) * e_\sigma$$

when  $\sigma \in K$   $\{ f \in C_c^\infty(G) \mid f(k_1 g k_2) = \sigma(k_1, k_2)^{-1} f(g) \quad \forall g \in G, k_1, k_2 \in K \}$

$$\begin{aligned} H(G, K) &:= H_1(G, \mathbf{1}_K) \\ &= e_K * H(G) * e_K \\ &= \{ f \in C_c^\infty(G) \mid f(k_1 g k_2) = f(g) \quad \forall g \in G, k_1, k_2 \in K \} \\ &= C_c^\infty(K \backslash G / K) \quad \text{though undefined} \end{aligned}$$

Rmk.  $H_1(G, \sigma)$  has unit  $e_\sigma$ , while  $H(G, K)$  has unit  $e_K$ .

When  $\# K \backslash G / K < +\infty$ ,  $H(G, K) = \mathbb{C}[K \backslash G / K]$ .

Def (Smooth  $H_1(G, \sigma)$ -module =  $H_1(G, \sigma)$ -module)

An  $H_1(G, \sigma)$ -module  $M$  is smooth if

$$\begin{aligned} H(G, \sigma) * M &= M \\ \Leftrightarrow \forall m \in M, e_\sigma * m &= m \\ \Leftrightarrow \checkmark \end{aligned}$$

Q: How can we view  $H_1(G, \sigma)$ -module as some special reps?

Prop. Let  $(\rho, V) \in \text{Rep}(G)$ ,  $K \in \text{Cos}(G)$ . We get SES

$$\begin{array}{ccccccc} 0 & \longrightarrow & V(K) & \longrightarrow & V & \longrightarrow & V_K \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel s \\ 0 & \longrightarrow & V(K) & \longrightarrow & V & \xrightarrow{e_K * -} & V^K \longrightarrow 0 \end{array} \quad \begin{array}{l} \text{in } \text{Rep}(K) \\ (\text{in } \text{Rep}(B)) \\ \text{if } K \triangleleft B \leq G \end{array}$$

Moreover,  $V^K \in \text{Mod}(H(G, K))$ .

Prop. Fix  $K \in \text{Cos}(G)$ . Then

$$\begin{aligned} \{(\rho, V) \in \text{Irr}(G) \mid V^K \neq 0\} &\xrightleftharpoons{\cong} \text{Irr}(H(G, K)) \\ (\rho, V) &\longmapsto V^K = e_K * V \\ r_{(G)} \otimes_{H(G, K)} M &\in \text{Mod}(H(G)) \longleftarrow M \end{aligned}$$

## $\sigma$ -Spherical Hecke algebra

Def. Suppose  $K \in \text{Cos}(G)$ ,  $(\sigma, W) \in \text{Irr}(K)$ .

The  $\sigma$ -spherical Hecke alg/intertwining alg  $\mathcal{H}(G, \sigma)$  is defined as

$$\mathcal{H}(G, \sigma) := \left\{ f: G \rightarrow \text{End}_{\mathbb{C}}(W) \mid \begin{array}{l} \text{supp } f \text{ is cpt} \\ f(k_1 g k_2) = \sigma(k_1^{-1}) f(g) \sigma(k_2) \quad \forall g \in G, k_1, k_2 \in K \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

Rmk.  $\mathcal{H}(G, \sigma)$  is ass with unit

$$E_{\sigma}: G \rightarrow \text{End}_{\mathbb{C}}(W) \quad E_{\sigma}(g) = \begin{cases} \frac{1}{\mu(K)} \sigma(g)^{-1} & g \in K \\ 0 & g \notin K \end{cases}$$

and we have

$$\mathcal{H}(G, \sigma) \cong \mathcal{H}_1(G, \sigma) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(W) \quad \text{as } \mathbb{C}\text{-alg.}$$

Prop.  $c\text{-Ind}_K^G \sigma$  is an  $\mathcal{H}(G, \sigma)$ -module defined by

$$\mathcal{H}(G, \sigma) \times c\text{-Ind}_K^G \sigma \rightarrow c\text{-Ind}_K^G \sigma \quad (\phi, f) \mapsto \phi * f$$

where

$$\phi * f(g) := \int_G \phi(x) \cdot f(x^{-1}g) d\mu(x)$$

We have an iso

$$\mathcal{H}(G, \sigma) \longrightarrow \text{End}_G(c\text{-Ind}_K^G W) \quad \phi \longrightarrow \phi + -$$

$$\mathcal{H}(G, \sigma) \longleftarrow \text{Hom}_K(W, c\text{-Ind}_K^G W) \quad \left[ \begin{array}{l} \phi: G \rightarrow \text{End}_{\mathbb{C}} W \\ g \mapsto [w \mapsto \psi_w(g)] \end{array} \right] \longleftarrow \psi$$

Generalization.  $K \in \text{Cos}_Z(G)$  Fix  $\chi \in \widehat{Z}$ .

Rmk.  $G$  is unimodular  $\Rightarrow G/Z$  is unimodular. Denote  $\mu_{G/Z}$  as Haar measure of  $G/Z$ .

Def. (Hecke algebra)  $H_\chi(G)$

$$C_{c,\chi}^\infty(G) = \left\{ f \in C^\infty(G) \mid \begin{array}{l} f(zg) = \chi(z)^{-1} f(g) \\ \text{supp } f/Z \text{ is cpt} \end{array} \forall z \in Z, g \in G \right\}$$

$$H_\chi(G) = (C_{c,\chi}^\infty(G), *)$$

$$(f_1 * f_2)(g) := \int_{G/Z} f_1(x) f_2(x^{-1}g) d\mu_{G/Z}(x)$$

Rmk  $H_{1_Z}(G) = H(G/Z)$ .

$H_\chi(G)$  has idempotent elements

$$e_\sigma \in H_\chi(G) \quad \text{for } (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi, K \in \text{Cos}(G)$$

$$e_\sigma(g) = \begin{cases} \frac{\dim_{\mathbb{C}} W}{\mu_{G/Z}(K_Z)} \text{tr}(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

Def. (Smooth  $H_\chi(G)$ -module)

An  $H_\chi(G)$ -module  $M$  is smooth if

$$H_\chi(G) * M = M$$

$$\Leftrightarrow \forall m \in M, \exists K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi \text{ s.t. } e_\sigma * m = m.$$

Thm We have the equiv of categories.

$$\{(p, V) \in \text{Rep}(G) \mid p \xrightarrow{\cong} \chi\} \longrightarrow \text{Mod}(H_\chi(G))$$

$$(p, V) \longmapsto V$$

$$f * v = \int_{G/Z} f(g) (p(g).v) d\mu_{G/Z}(g)$$

$$\text{for } f \in H_\chi(G), v \in V$$

$$(p, M) \longleftrightarrow M$$

$$p(g).m = e_\chi(g^{-1}) * m$$

for  $g \in G, m \in M$ ,

$K \in \text{Cos}_Z(G)$   $\chi \in \widehat{K} \xrightarrow{\cong} \chi$  s.t.  $e_\chi * m = m$

E.p.  $e_\sigma * v \in V^\sigma$  and  $p(1).m = m$

Prop Let  $(p, V) \in \text{Rep}(G), K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi$ , then

$$e_\sigma * - : V \longrightarrow V^\sigma \quad \text{in } \text{Rep}(K)$$

is a projection.



## 5. Intertwining properties.

In this subsection,  $G$  is an unimodular loc. profinite gp satisfying the countable hypothesis, and  $Z = Z(G)$ .

### Definition

Def. Let  $K_1, K_2 \in \text{Cos}(G)/\text{Cos}_Z(G)$ ,  $\sigma_1 \in \text{Irr}(K_1)$ ,  $\sigma_2 \in \text{Irr}(K_2)$ ,  $g \in G$ .

$$K_1^g := g^{-1}K_1g \quad \rho_1^g = \rho_1(g - g^{-1}) \in \text{Irr}(K_1^g)$$

We call  $g$  intertwines  $\sigma_1$  with  $\sigma_2$  if

$$\text{Hom}_{K_1^g \cap K_2}(\sigma_1^g, \sigma_2) \neq 0$$

usually missed

$$\text{Notation. } \text{itw}_G(\sigma_1, \sigma_2) := \{g \in G \mid g \text{ intertwines } \sigma_1 \text{ with } \sigma_2\}$$

$$\text{itw}_G(\sigma) := \text{itw}_G(\sigma, \sigma)$$

$\sigma_1$  intertwines with  $\sigma_2$  if  $\text{itw}_G(\sigma_1, \sigma_2) \neq \emptyset$

E.g. When  $K_1 = K_2 = K$ ,  $g \in N_K(G)$ ,  $\sigma_1, \sigma_2 \in \text{Irr}(K)$ , then

$$g \in \text{itw}(\sigma_1, \sigma_2) \Leftrightarrow \text{Hom}_K(\sigma_1^g, \sigma_2) \neq 0$$

$$\Leftrightarrow \sigma_1^g \cong \sigma_2$$

$$\text{Rmk. } g \in \text{itw}(\sigma_1, \sigma_2) \Rightarrow k_1 g k_2 \in \text{itw}(\sigma_1, \sigma_2) \quad \text{for } \forall k_1 \in K_1, k_2 \in K_2.$$

### Basic results

Prop 1. Suppose  $(\rho, V) \in \text{Irr}(G)$ ,  $(\sigma_1, W_1) \in \text{Irr}(K_1)$ ,  $(\sigma_2, W_2) \in \text{Irr}(K_2)$ ,

$$\bigcup_{\text{in Rep}(K_1)} (\rho, V) \bigcup_{\text{in Rep}(K_2)} (\sigma_2, W_2)$$

$$(\sigma_1, W_1) \quad (\sigma_2, W_2)$$

Then  $\text{itw}(\sigma_1, \sigma_2) \neq \emptyset$ .

[Proof. We know  $V^{\sigma_1} \neq 0$  in  $\text{Rep}(K_1)$ ,  $V^{\sigma_2} \neq 0$  in  $\text{Rep}(K_2)$ .  
 $V^{\sigma_1} \neq 0 \xrightarrow{(\rho, V) \in \text{Irr}(G)} \langle V^{\sigma_1} \rangle_{g \in G} \text{ spans } V \quad \left. \begin{array}{l} \exists g \in G, \pi_{\sigma_2}|_{V^{\sigma_1}}, V^{\sigma_1} \longrightarrow V^{\sigma_2} \\ \pi_{\sigma_2}: V \longrightarrow V^{\sigma_2} \text{ is nonzero in } \text{Rep}(K_2) \end{array} \right\} \text{ is nonzero in } \text{Rep}(K_1^g \cap K_2).$

Prop 2. Let  $K \in \text{Cos}(G)$ ,  $\sigma \in \text{Irr}(K)$ ,  $g \in G$ , then

$$g \in \text{itw}(\sigma) \Leftrightarrow \exists f \in \mathcal{H}(G, \sigma) \text{ s.t. } f|_{Kgk} \neq 0$$

Prop 3 Let  $K \in \text{Cos}_Z(G)$ ,  $(\sigma, W) \in \text{Irr}(K)$ ,  $g \in G$ , then

$$g \in \text{itw}(\sigma) \Leftrightarrow \exists f \in \mathcal{H}(G, \sigma) \text{ s.t. } \text{supp } f = KgK$$

Moreover, we have

$$\{f \in \mathcal{H}(G, \sigma) \mid \text{supp } f \subseteq KgK\} \xrightarrow{\cong} \text{Hom}_{K^g \cap K}(\sigma, \sigma^g) \subseteq \text{End}_{\mathbb{C}}(W)$$

$$f \mapsto f(g)$$

$$f: f(x) = \begin{cases} \sigma(k)^{-1} h \sigma(k) & x = k_1 g k_2 \\ 0 & x \notin KgK \end{cases}$$

Q: Any generalization of Prop 2 & Prop 3?  $\text{Cos}(G) \rightarrow \text{Cos}_Z(G)$  or  $\text{Cos}_Z(G) \rightarrow \text{Cos}(G)$ ?

## $c\text{-Ind}$ with cuspidal rep

Thm. [11.4, P80] (Construction method for cuspidal reps)

Suppose  $\forall (\rho, V) \in \text{Irr}(G)$  is admissible,

$K \in \text{Cos}_Z(G)$ ,  $(\sigma, W) \in \text{Irr}(K)$  and  $\text{itw}(\sigma) = K$ .

Then  $c\text{-Ind}_K^G \sigma \in \text{Cusp}_c(G)$ .

Rmk. ( $\text{itw}$  is key ingredient)

Let  $K \in \text{Cos}_Z(G)$ ,  $(\sigma, W) \in \text{Irr}(K)$  and  $\text{itw}(\sigma) \neq K$ .

Then  $c\text{-Ind}_K^G \sigma \notin \text{Irr}(G)$

(Since  $\text{End}_G(c\text{-Ind}_K \sigma) \cong \mathbb{H}(G, \sigma)$  has  $\dim > 1$ )

Rmk. In the theorem, if we assume further that  $\delta_{K \backslash G} = \mathbb{1}_K$ , then

$$c\text{-Ind}_K^G \sigma \in \text{Irr}(G)$$

$$\xrightarrow{\text{adm}} (c\text{-Ind}_K^G \sigma)^\vee \in \text{Irr}(G)$$

$$\Rightarrow \text{Ind}_K^G(\sigma) \stackrel{\delta_{K \backslash G} = \mathbb{1}_K}{\cong} (c\text{-Ind}_K^G \sigma)^\vee \in \text{Irr}(G)$$

$$\Rightarrow c\text{-Ind}_K^G(\sigma) = \text{Ind}_K^G(\sigma)$$

$$\Rightarrow \text{Ind}_K^G(\sigma) \cong (c\text{-Ind}_K^G(\sigma))^\vee \cong (\text{Ind}_K^G(\sigma))^\vee \cong c\text{-Ind}_K^G(\sigma)$$