Eine Woche, ein Beispiel

5.28. dual spaces of ∞-dim v.s.

Ref:
http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-FA/index.html
https://uni-bonn.sciebo.de/s/2qSqolzpjtJZmEo
https://www.overleaf.com/read/bstfhjyrsjnv

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What would happen if $\mathbb{F} = \mathbb{C}_p$?

1. def
2. examples
3. an overview of spaces in analysis
4. concrete "functions" in $\mathcal{S}'(\mathbb{R}^n)$

4^+. eigenspace in $\mathcal{S}(\mathbb{R}^n)$

1. def

Def. For any topo v.s. X, Y, define
$$\mathcal{L}(X,Y) := \{L : X \to Y \mid L \text{ is linear and cont}\}$$
The dual space of X is defined as
$$X' := \mathcal{L}(X,\mathbb{F}) = \{L : X \to \mathbb{F} \mid L \text{ is linear and cont}\}$$

! We follow the notation of analysis in this document.
Other possibilities for the dual space: $X^*, X^\vee, \check{X}, \dots$

Rmk. When X, Y are normed v.s., $\mathcal{L}(X,Y)$ is a normed v.s. with
$$\|L\| = \sup_{\|x\|_X = 1} \|L(x)\|_Y$$

On the other hand, we have the weak $*$-topology on $\mathcal{L}(X,Y)$: the weakest topo s.t.
$$ev_x : \mathcal{L}(X,Y) \longrightarrow Y \qquad L \longmapsto L(x)$$
is cont for any $x \in X$.

These two structures are not compatible with each other.

Rmk. By Klein-Milman theorem, we can show that some Banach spaces are not dual space.

2. initial examples.

For a bounded domain Ω, we have

$$\begin{array}{ccccccc} L^\infty(\Omega) & \subset \cdots & \subset & L^p(\Omega) & \subset \cdots & \subset & L^1(\Omega) \\ & & & \Downarrow \text{dual} & & & \\ (L^\infty(\Omega))' & \supset \cdots & \supset & L^q(\Omega) & \supset \cdots & \supset & L^\infty(\Omega) \end{array}$$

For arbitrary domain Ω, we don't have inclusion.

inclusion: cont inj map

https://math.stackexchange.com/questions/405357/when-exactly-is-the-dual-of-l1-isomorphic-to-l-infty-via-the-natural-map
https://math.stackexchange.com/questions/137677/what-is-the-predual-of-l1

Ex. Show that $(c_0)' = l^1$, $(l^p)' = l^q$, $(l^1)' = l^\infty$ by direct argument.
Show that $(l^\infty)' \not\cong l^1$.

$$\begin{array}{ccccc} c_0 & \xhookrightarrow{\text{not dense}} & l^\infty & l^p & l^1 \\ & & & \Downarrow \text{dual} & \\ l^1 & \twoheadleftarrow & (l^\infty)' & l^q & l^\infty \end{array}$$

For $\Omega = \mathbb{R}^n$, we have ($\mathcal{S}(\Omega)$ is not defined for $\Omega \overset{\text{open}}{\subset} \mathbb{R}^n$, traditionally)

$$\begin{array}{ccccc} \mathcal{D}(\Omega) & \subset & \mathcal{S}(\Omega) & \subset & \mathcal{E}(\Omega) \\ & & \Downarrow \text{dual} & & \\ \mathcal{D}'(\Omega) & \supset & \mathcal{S}'(\Omega) & \supset & \mathcal{E}'(\Omega) \end{array}$$

https://math.stackexchange.com/questions/4730104/is-schwartz-space-canonical-in-any-sense
Schwartz Functions on Open Subsets of Rn: https://www.math.princeton.edu/events/schwartz-functions-open-subsets-rn-2022-02-28t213000
Schwartz functions on real algebraic varieties: https://arxiv.org/abs/1701.07334

Rmk. For Hilbert space, $H' \cong H$. e.p. $(H^s(\Omega))' \cong H^s(\Omega)$
For X: cpt Hausdorff space,
$C(X)' \subset \{\text{signed regular Borel measures}\}$

⚠ The following illusion is common and confusing:
The dual space of bigger space is bigger/smaller.
Actually, such illusions comes from $f^*: W^* \longrightarrow V^*$ being injective/surjective.
In fin dim case, $\dim V^* = \dim V < \dim W = \dim W^*$;
In dense subspace case, it comes from the uniqueness of cont extension.

3. an overview of spaces in analysis

$1 < p < +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1$

$\mathbb{1}_{\mathbb{R}}, e^{i\langle a,x\rangle}$

$$
\begin{array}{lccccc}
 & f! & & f!,* & & f* \\
\text{distribution} & \mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
\text{measure} & C'(\mathbb{R}^n) & \subset & (C_0(\mathbb{R}^n)\cap L^p)' & \subset & C_c'(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
 & (L^q_{loc}(\mathbb{R}^n))' & \subseteq & (L^q(\mathbb{R}^n))' & \subseteq & (L^q_c(\mathbb{R}^n))' \\
 & \cup\ \text{bij} & & \cup\ \text{bij} & & \cup\ \text{bij} \\
 & L^p_c(\mathbb{R}^n) & \subseteq & L^p(\mathbb{R}^n) & \subseteq & L^p_{loc}(\mathbb{R}^n) \\
\text{functions} & \cup & & \cup & & \cup \\
 & C_c(\mathbb{R}^n) & \subset & C_0(\mathbb{R}^n)\cap L^p & \subset & C(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
 & \mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n) \\
 & & & \text{for Fourier transform} & & \text{sheaf} \\
 & \text{cpt supported} & & \text{globally bd} & & \text{locally bd}
\end{array}
$$

$\sum_{a\in\mathbb{Z}} \delta_a$

δ_a

$\mathbb{1}_{[0,+\infty)}$

$e^{\varepsilon x}\mathbb{1}_{[0,+\infty)}$

seminorm of the last line:

$\varinjlim_K \mathcal{D}_K(\mathbb{R}^n)$ $\qquad p_{\alpha,\beta}(f) = \|x^\beta \partial^\alpha f\|_\infty$

$p_{\alpha,K}(f) = \sup_{x\in K} \|\partial^\alpha f\|_\infty$ $\qquad p_{\alpha,N}(f) = \|(1+|x|)^N \partial^\alpha f\|_\infty$ $\qquad p_{\alpha,K}(f) = \sup_{x\in K} \|\partial^\alpha f\|_\infty$

measure line: $C'(\mathbb{R}^n)$: fcts of bounded variation

$C_0'(\mathbb{R}^n)$: signed regular Borel measures on $\mathbb{R}^n$.

$(C_c'(\mathbb{R}^n))^+$: Radon measure

Q: is $C_c'(\mathbb{R}^n)$ the signed Radon measure?

https://math.stackexchange.com/questions/4448590/how-to-generalize-riesz-markov-kakutani-representation-theorem-from-c-cx-to
https://math.stackexchange.com/questions/4500358/3-versions-of-riesz-markov-kakutani-theorem

https://math.stackexchange.com/questions/1026961/proof-of-dual-normed-vector-space-is-complete
https://math.stackexchange.com/questions/3587185/completeness-of-the-dual-space-of-a-frechet-space
https://math.stackexchange.com/questions/424876/weak-topology-on-an-infinite-dimensional-normed-vector-space-is-not-metrizable
https://math.stackexchange.com/questions/634818/why-is-the-weak-topology-not-in-general-metrizable

In some theories, measures are defined as bounded distributions. I won't take that point of view.

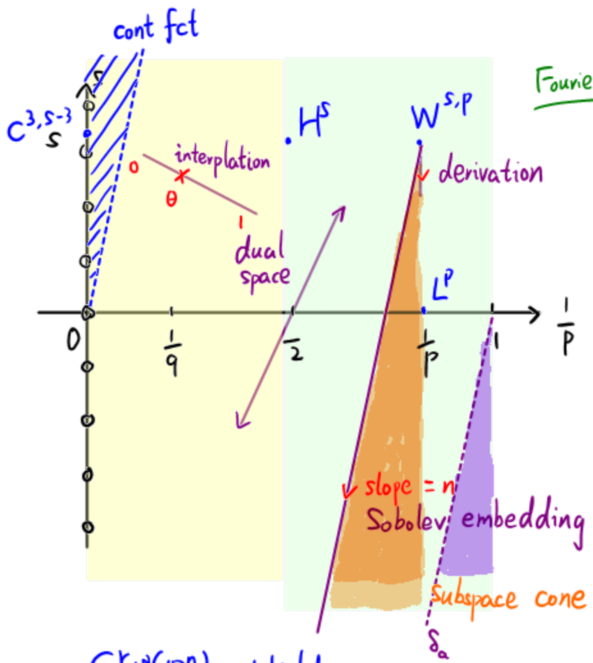
The above diagram has many variations. For example,

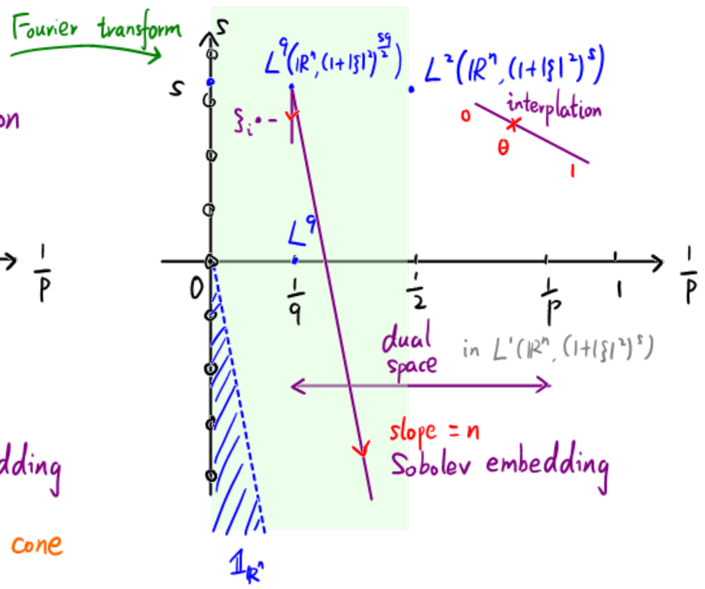
$$\begin{array}{ccccc}
\mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
C'(\mathbb{R}^n) & \subset & C_0'(\mathbb{R}^n) & \subset & C_c'(\mathbb{R}^n) \\
\uparrow & & \uparrow & & \uparrow \\
(L^1_{loc}(\mathbb{R}^n))' & \subset & (L^1(\mathbb{R}^n))' & \subset & (L^1_c(\mathbb{R}^n))' \\
\uparrow \text{bij?} & & \cup \text{ bij} & & \uparrow \text{bij?} \\
L^\infty_c(\mathbb{R}^n) & \subset & L^\infty(\mathbb{R}^n) & \subset & L^\infty_{loc}(\mathbb{R}^n) \\
\uparrow & & \uparrow & & \uparrow \\
C_c(\mathbb{R}^n) & \subset & C_0(\mathbb{R}^n) & \subset & C(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
\mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
\bigcup & & \bigcup & & \bigcup \\
H^{-s}_{loc}(\mathbb{R}^n) & \subset & H^{-s}(\mathbb{R}^n) & \subset & H^{-s}_c(\mathbb{R}^n) \\
\cup \text{ bij?} & & \cup \text{ bij} & & \cup \text{ bij?} \\
H^s_c(\mathbb{R}^n) & \subset & H^s(\mathbb{R}^n) & \subset & H^s_{loc}(\mathbb{R}^n) \\
\bigcup & & \bigcup & & \bigcup \\
\mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
\end{array}$$

https://math.stackexchange.com/questions/221069/why-are-continuous-functions-not-dense-in-l-infty

In fact, in the middle, we can change various of Sobolev spaces.

https://en.wikipedia.org/wiki/Sobolev_inequality
https://en.wikipedia.org/wiki/Sobolev_space
https://arxiv.org/PS_cache/arxiv/pdf/1104/1104.4345v2.pdf





$C^{r,\alpha}(\mathbb{R}^n)$: Hölder spaces

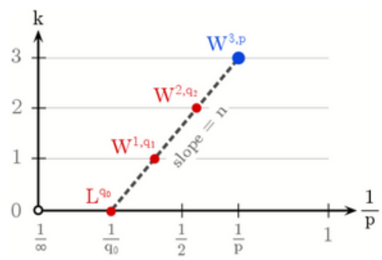
In the right, we have strong Hölder inequality:

$$\|f^{1-\theta} g^{\theta}\|_{E_\theta} \le \|f\|_{E_0}^{1-\theta} \|g\|_{E_1}^{\theta}$$

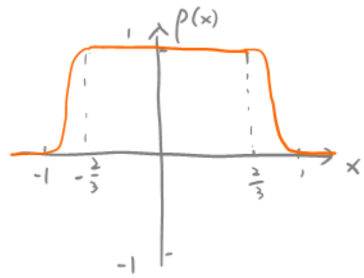
Does this holds for general interpolation spaces?

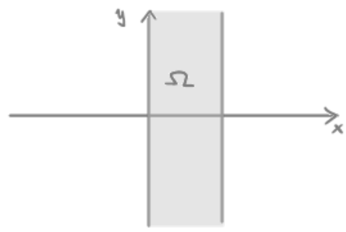
e.p. take $g = 1$, one can get Sobolev embedding

(weak version, $g = \delta_a$ on the left hand side)

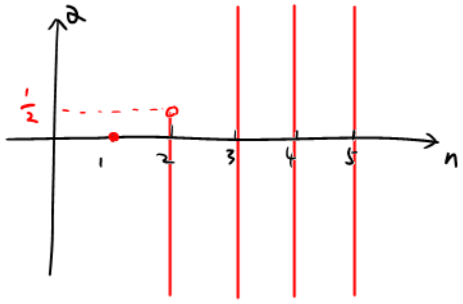


Ex. Draw δ_a, $\rho(|x|)(-\ln|x|)^\alpha$ in the above figure.





known: $\delta_a \in H^{-s}(\mathbb{R}^n)$ for $s > \frac{n}{2}$



when $\rho(|x|)(-\ln|x|)^\alpha \in H^1(\mathbb{R}^n)$

The picture of Sobolev embedding looks similar to Ω in interpolation theorems.

We mainly care about:

1. Element
   - special element e.g. δ
   - (singular) support
2. Set
   - as a set, topo sp
   - best structure? e.g. Fréchet?
   - seq convergence
   - criterion of seminorm / lin fct / map to be cont
3. Map
   - $\to$: cont? inj?
   - $\subset$: dense? (Use regularization/truncate)

     https://math.stackexchange.com/questions/1802755/can-you-recover-a-distribution-from-mollification
   - $\subseteq$: $f: X \hookrightarrow Y$ if $\mathrm{Im} f$ & X have the same topo (topo embedding)
   - cpt operators?
   - Intersection compatible with $\circlearrowleft$? i.e. pullback squares?
4. More structures (add extra dimensions on the diagram)
   - differential
   - $\Omega \subset\subset \Omega'$. sheaf?
   - Fourier transform
   - $\Omega \times \Omega'$ Schwarz kernel

   $\Rightarrow$ integral operators

   $\Rightarrow$ FM transform

Rmk. For $f: X \to Y$ a cont injective map between Fréchet spaces.
f is a topo embedding $\Leftrightarrow$ every cont seminorm on X can be extended to a cont seminorm on Y.

Q. Can we generalize the field from $\mathbb{R}$ or $\mathbb{C}$ to $K = \mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$?

4. concrete "functions" in $\mathcal{S}'(\mathbb{R}^n)$

math.stackexchange.com/questions/4263808/expression-for-dirac-delta-deltaxy

① (singular) support

Check for δ_0, δ_Δ, $\mathbb{1}_{[0,+\infty)}$, $\ln x \cdot \mathbb{1}_{(0,1)}$, $\cdots$

② limit/density

$$\lim_{n\to+\infty} \cos(nx) = 0 \qquad \text{View } \cos(nx) \in \mathcal{S}'(\mathbb{R})$$

$$\lim_{n\to+\infty} \mathbb{1}_{[-n,n]} = \mathbb{1}_{\mathbb{R}}$$

$$\lim_{\varepsilon\to 0^+} e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)} = \mathbb{1}_{[0,+\infty)} \qquad \lim_{\varepsilon\to+\infty} e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)} = 0$$

For $\rho \in L^1(\mathbb{R})$, $\int_{\mathbb{R}} \rho(x)\,dx = 1$,

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\rho\left(\frac{x}{\lambda}\right) = \delta_0 \qquad \lim_{\lambda\to+\infty} \frac{1}{\lambda}\rho\left(\frac{x}{\lambda}\right) = 0 \qquad \text{in } \mathcal{S}'(\mathbb{R})$$

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda}\rho\left(\frac{y-x}{\lambda}\right) = \frac{1}{\sqrt{2}}\delta_\Delta \qquad \text{in } \mathcal{S}'(\mathbb{R}^2)$$

In general, for $\rho \in L^1(\mathbb{R}^n)$, $\int_{\mathbb{R}^n} \rho(x)\,dx = 1$,

$$\lim_{\lambda\to 0^+} \frac{1}{\lambda^n}\rho\left(\frac{x-x_0}{\lambda}\right) = \delta_{x_0} \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

③ derivative

$$\mathbb{1}_{[0,+\infty)}'(x) = \delta_0(x) = \text{“}\int_{\mathbb{R}} e^{-ix\xi}\,d\xi\text{”}$$

$$\delta_0'(x) = \text{“}\int_{\mathbb{R}} -i\xi\, e^{-ix\xi}\,d\xi\text{”}$$

let $f(x) = \ln x \cdot \mathbb{1}_{(0,1)}(x)$, then

$$\langle f', \varphi\rangle = \int_0^1 \frac{\varphi(x)-\varphi(0)}{x}\,dx$$

$$\langle f'', \varphi\rangle = \varphi'(0) + \varphi(0) + \int_0^1 \frac{\varphi(x)-\varphi(0)-x\varphi'(0)}{x^2}\,dx$$

④ Fourier transform

I. "$\widehat{\delta_a}(\xi)$" $= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}\delta_a(x)\,dx = e^{-i\langle a,\xi\rangle} \in \mathcal{S}'(\mathbb{R}^n) - \mathcal{S}(\mathbb{R}^n)$

rigorously:

$$\begin{aligned}\langle \mathcal{F}\delta_a, \psi\rangle &= \langle \delta_a, \mathcal{F}\psi\rangle \\ &= (\mathcal{F}\psi)(a) \\ &= \int_{\mathbb{R}^n} e^{-i\langle x,a\rangle}\psi(x)\,dx\end{aligned}$$

II. "$\widehat{e^{i\langle a,x\rangle}}(\xi)$" $= \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} e^{i\langle a,x\rangle}dx$

$$\begin{aligned}&= \int_{\mathbb{R}^n} e^{i\langle x,\xi-a\rangle}dx \\ &= (2\pi)^n \int_{\mathbb{R}^n} e^{i\langle x,\xi-a\rangle}\,dx \quad \text{"="}\ \delta_a(\xi)\end{aligned}$$

rigorously:

$$\begin{aligned}\langle \mathcal{F}e^{i\langle a,-\rangle}, \psi\rangle &= \langle e^{i\langle a,-\rangle}, \mathcal{F}\psi\rangle \\ &= \int_{\mathbb{R}^n} e^{i\langle a,\xi\rangle}\hat{\psi}(\xi)\,d\xi \\ &= (2\pi)^n \int_{\mathbb{R}^n} e^{i\langle a,\xi\rangle}\hat{\psi}(\xi)\,d\xi \\ &= (2\pi)^n\psi(a)\end{aligned}$$

III. "$\widehat{\mathbb{1}_{[0,+\infty)}}(\xi)$" $= \int_{\mathbb{R}} e^{-ix\xi}\,\mathbb{1}_{[0,+\infty)}\,dx$

$$\begin{aligned}&= \int_0^{+\infty} e^{-ix\xi}dx \\ &= \frac{e^{-ix\xi}\big|_0^{+\infty}}{-i\xi} = \frac{1}{i\xi} \quad \text{"}\in\text{"}\ \mathcal{S}'(\mathbb{R}) - \mathcal{S}(\mathbb{R})\end{aligned}$$

This is wrong. Rigorously:

$$\begin{aligned}\langle \mathcal{F}\,\mathbb{1}_{[0,+\infty)}, \psi\rangle &= \langle \mathbb{1}_{[0,+\infty)}, \mathcal{F}\psi\rangle \\ &= \int_0^{+\infty}\hat{\psi}(\xi)\,d\xi = ?\end{aligned}$$

<u>Idea</u>: $e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)} \xrightarrow{\mathcal{S}'(\mathbb{R})} \mathbb{1}_{[0,+\infty)} \Rightarrow \widehat{e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)}} \xrightarrow{\mathcal{S}'(\mathbb{R})} \widehat{\mathbb{1}_{[0,+\infty)}}$

Since $e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)} \in L^1(\mathbb{R})$, one can directly compute

$$\begin{aligned}\widehat{e^{-\varepsilon x}\mathbb{1}_{[0,+\infty)}}(\xi) &= \int_{\mathbb{R}} e^{-ix\xi}e^{-\varepsilon x}\,\mathbb{1}_{[0,+\infty)}\,dx \\ &= \int_0^{+\infty} e^{-(\varepsilon+i\xi)x}\,dx \\ &= \frac{1}{-(\varepsilon+i\xi)}\, e^{-(\varepsilon+i\xi)x}\Big|_0^{+\infty} \\ &= \frac{1}{\varepsilon+i\xi}\end{aligned}$$

Take $\varepsilon \to 0^+$,

$$\widehat{\mathbb{1}_{[0,+\infty)}} = \partial \log|\cdot| - i\pi\cdot\delta_0$$

IV. (The Dirac comb)

$$\widehat{\sum_{a\in\mathbb{Z}}\delta_a} = C\sum_{b\in 2\pi\mathbb{Z}}\delta_b$$

in general,

I am a bit lazy to rewrite these exercises. I will do that when I really need these examples.

Exercise 4.4 (The Dirac comb). Let

$$\Gamma := \mathbb{Z}\omega_1 + \ldots + \mathbb{Z}\omega_n \subset \mathbb{R}^n$$

be a lattice, where $\{\omega_j\}_{j=1,\ldots,n}$ is a basis of $\mathbb{R}^n$. We consider the fundamental domain

$$F_\Gamma := \left\{ \sum_{j=1}^{n} \lambda_j \omega_j \mid 0 \le \lambda_j \le 1 \right\} \subset \mathbb{R}^n,$$

and we note that $\mathrm{vol}(\mathbb{R}^n/\Gamma) = \mathrm{vol}(F_\Gamma)$. Denote by

$$\Gamma^\perp := \left\{ a \in \mathbb{R}^n \mid e^{i\langle a,\gamma\rangle} = 1 \text{ for all } \gamma \in \Gamma \right\}$$

the dual lattice. (For example, if $\Gamma = \mathbb{Z}\cdot\lambda \subset \mathbb{R}$ for some $\lambda \neq 0$, then $\Gamma^\perp = \mathbb{Z}\cdot\frac{2\pi}{\lambda}$.) The *Dirac comb* for the lattice Γ is defined as

$$T := \sum_{\gamma\in\Gamma} \delta_\gamma \in \mathcal{S}'(\mathbb{R}^n),$$

where δ_γ is the Dirac distribution centered at the point $\gamma \in \mathbb{R}^n$.

Show that its Fourier transform equals

$$\widehat{T} = C_\Gamma \cdot \sum_{a\in\Gamma^\perp} \delta_a,$$

for some constant $C_\Gamma > 0$, and determine this constant explicitly.

Hint: Prove for $\varphi \in \mathcal{D}(\mathbb{R}^n)$ the Poisson summation formula:

$$\sum_{\gamma\in\Gamma} \varphi(\gamma) = C_\Gamma \sum_{a\in\Gamma^\perp} \widehat{\varphi}(a),$$

by considering the function $\Phi(x) := \sum_{\gamma\in\Gamma} \varphi(x+\gamma)$. You may use (without proof) that $\{e_a(x)\}_{a\in\Gamma^\perp}$ is an orthonormal basis of $L^2(\mathbb{R}^n/\Gamma)$, where

$$e_a(x) := \frac{1}{\sqrt{\mathrm{vol}\, F_\Gamma}} e^{i\langle a,x\rangle}.$$

You may also use that the Fourier series $\sum_{a\in\Gamma^\perp}\langle e_a, f\rangle e_a$ of a smooth Γ-periodic function f converges uniformly.

4^+. eigenspace in $\mathcal{S}(\mathbb{R}^n)$

Exercise 5.1. The purpose of this exercise is to construct the eigenspaces of the Fourier transform. This is closely related to the quantum mechanical harmonic oscillator as well as to the theory of the Hermite polynomials. Recall that for $\varphi_0(x) := e^{-x^2/2}$ the Fourier transform is given by $\widehat{\varphi_0} = 2\pi\varphi_0$. In other words, φ_0 is an eigenvector of the Fourier transform with the eigenvalue 2π. Let $D := \frac{d}{dx} + x$, $D^* := -\frac{d}{dx} + x$. Put

$$\varphi_n(x) := (D^*)^n \varphi_0(x) =: H_n(x)\cdot e^{-x^2/2}.$$

Verify the following facts:

(a) $H_n(x)$ is a polynomial of degree n with leading term $2^n x^n$. Furthermore, $H_n(-x) = (-1)^n H_n(x)$.

(b) $\widehat{\varphi_n} = 2\pi(-i)^n\varphi_n$.

(c) $(-\partial_x^2 + x^2)\varphi_n = (2n+1)\varphi_n$.
Hint: $-\partial_x^2 + x^2 = DD^* - \mathrm{Id} = D^*D + \mathrm{Id}$.

(d) $H_n(x) = (-1)^n e^{x^2}\partial_x^n e^{-x^2}$.

(e) $\langle\varphi_n, \varphi_m\rangle_{L^2(\mathbb{R})} = \sqrt{\pi}\cdot 2^n \cdot n! \cdot \delta_{nm}$.

(f) $\varphi_n, n = 0,1,2,\ldots,$ is an orthogonal basis of the Hilbert space $L^2(\mathbb{R})$. This basis consists of *joint* eigenvectors for the Fourier transform and the harmonic oscillator $-\partial_x^2 + x^2$!
Hint: For $f \in L^2(\mathbb{R})$ expand the Fourier transform of $f\cdot e^{-x^2/2}$ into a power series.

(a) (Generating Function)

$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n, \quad t,x \in \mathbb{C}.$$

(b) (Binomial identities) The following identities which are reminiscent of the Binomial Theorem hold[1]:

$$H_n(ax+by) = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \cdot H_k(x)\cdot H_{n-k}(y), \quad \text{for } a^2+b^2 = 1, \tag{1}$$

$$H_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} \cdot H_k(x)\cdot y^{n-k}. \tag{2}$$

(c) (Explicit Formula)

$$H_n(x) = \sum_{0\le k\le n/2} \frac{(-1)^k n! 2^{n-2k}}{k!(n-2k)!} x^{n-2k}.$$