Tutorial class for "Math in Phy II"

Hallo everyone.

Before discussing the tasks and exercises, let us discuss something unrelated to mathematics.

- Personal information
  - email: s6xxzhou@uni-bonn.de
  - website for notes: see eCampus ⇒ Github: ramified
  - whatsapp group?
- Course information
  - How to hand in the homework.
  - Group: 1-3 persons.
  - Time schedule (2h-3h, break, finish in advance?)
  - Welcome: questions, active participation.
  - Content $\begin{cases} \text{Task / Homework} \\ \text{videos?} \\ \text{generalization?} \\ \text{advanced topics?} \\ \text{bird's-eye view?} \end{cases}$ 80% / 20% ? – no devices

From my experience, focusing on Task is a good choice.

- Abbreviations (ask me if you don't know!)

E.g. = example
E.x. = exercise
w.l.o.g = without loss of generality
i.e. = that means; equivalently
TFAE = the following are equivalent
s.t. = such that

pt = point
fct = function
const = constant
v.s. = vector space
iso = isomorphism.

nbhd = neighbourhood
cplx = complex

Today: One (or two) in the following four topics:

- Object: space
- Computation: partial derivative
- Logic: $\varepsilon-\delta$ language
- Application: minimum and maximum.

I could also explain all tasks in today's tutorial, but that would be quite brief and not beneficial for study.

1. Space

1.1. v.s. (vector space / linear space)

Q: What is an $\mathbb{R}$-v.s.?

$(V, +, \cdot)$ s.t. ...

Q: Examples of $\mathbb{R}$-v.s.?

$\mathbb{R}^n$ $\quad$ $\mathbb{C}^n$

$\mathbb{R}^{[a,b]} = \{f: [a,b] \to \mathbb{R}\}$, $L^p([a,b])$, $C([a,b])$, $C^\infty([a,b])$, $\mathbb{R}[x]$

$\mathbb{R}^{\mathbb{N}}$, $l^p(\mathbb{N})$, ...

"sequence of real numbers".

$\dim_{\mathbb{R}}$ is the most important information of v.s. Actually,

$$V \cong W \iff \dim_{\mathbb{R}} V = \dim_{\mathbb{R}} W.$$

Q: What is the dim of those $\mathbb{R}$-v.s.?

1.2. Structures on spaces

inner product	$\rightleftarrows$	norm	$\rightleftarrows$	metric	$\rightleftarrows$	topology
$\langle v, w \rangle$		$\|v\|$		$d(a,b)$		$\mathcal{U}$
	 Parallelogramm		in $\mathbb{R}/\mathbb{C}$ - v.s. + compatability ①		metrizable ②	

angle	length (of a vector)	distance (between two pts)	nbhd (nearness)

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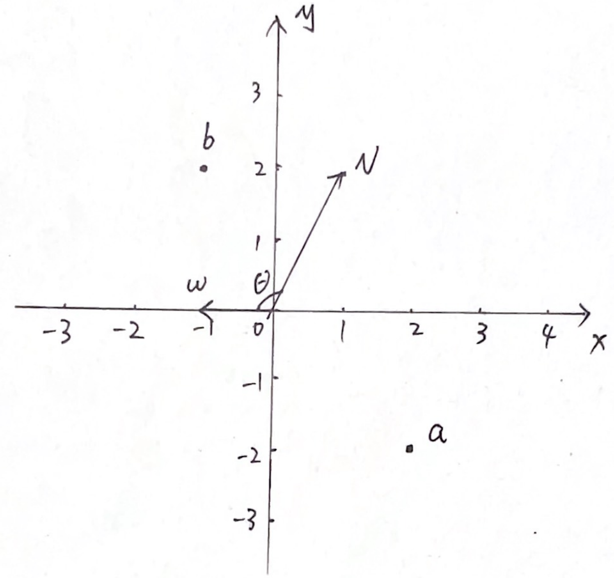
① That means, translation invariant + positively homogeneous

② In $\mathbb{R}/\mathbb{C}$ v.s. case, we need countable local base;

In general case, see Encyclopedia of Mathematics: metrizable space

E.g. $\mathbb{R}^2$



$$\langle v, w\rangle = v_1 w_1 + v_2 w_2$$

$$\|v\| = \sqrt{v_1^2 + v_2^2} = \sqrt{\langle v, v\rangle}$$

$$d(a,b) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2} = \|a-b\|$$

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\|\,\|w\|}$$

E.g. $L^2([0,1]) = \left\{ \underset{\text{(measurable)}}{f:[0,1]\longrightarrow \mathbb{R}} \;\middle|\; \int_0^1 f(x)^2 dx < +\infty \right\} / \sim$

$$\langle f, g\rangle_{L^2} := \int_0^1 f(x) g(x) dx$$

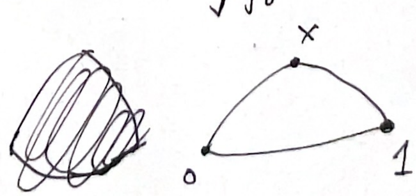
Q: What is:

- the "length" of $x \in L^2([0,1])$?
- the "distance" between x and 1 ?
- the "angle" between x and x^2 ?

Hint: $\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\int_0^1 x \cdot x\, dx} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$

$d(x,1) = \|x-1\| = \sqrt{\int_0^1 (x-1)^2 dx} = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3}$

"Cor"



is an isosceles triangle in $L^2([0,1])$

/aɪ'sasə'liz/

Ex. Find an equilateral triangle in $L^2([0,1])$.

Rmk. In the real inner product space, we have the Cauchy-Schwarz inequality:

$$|\langle v, w\rangle| \leq \|v\| \, \|w\|$$

this guarantees that

$$\cos\theta = \frac{\langle v, w\rangle}{\|v\| \, \|w\|} \in [-1, 1].$$

In $\mathbb{R}^n$, this is equiv to

$$\Big(\sum_i v_i w_i\Big)^2 \leq \Big(\sum_i v_i^2\Big)\Big(\sum_i w_i^2\Big)$$

For a proof, consider $\langle v+tw, v+tw\rangle \geq 0$.

Rmk. In cplx. case, ~~the~~ we still have the Cauchy-Schwarz inequality.

$$|\langle v, w\rangle| \leq \|v\| \, \|w\|$$

In $\mathbb{C}^n$, this is equiv to

$$\Big|\sum_i \bar{v}_i w_i\Big|^2 \leq \Big(\sum_i |v_i|^2\Big)\Big(\sum_i |w_i|^2\Big)$$

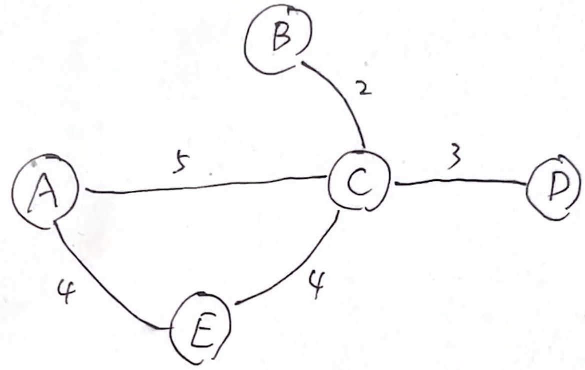
However, the proof becomes more tricky.

For a proof, see wiki: Cauchy-Schwarz inequality.

1.3. Metric

In many cases we are not able to give such nice structures. However, we can still describe the distance.

E.g.



$(X=\{A,B,C,D,E\},\ d)$ e.g. $d(A,D)=8$.

Def. (metric) A metric on a set X is a map

$$d: X\times X \longrightarrow \mathbb{R}$$

s.t.

a) (positive definite) $d(x,y)\geq 0$,
$d(x,y)=0 \iff x=y$

b) (symmetry) $d(x,y)=d(y,x)$

c) (triangle inequality) $d(x,y)+d(y,z)\geq d(x,z)$

Task 1.2. Take $p \in (0, +\infty)$. Determine when μ_p is a metric on $\mathbb{C}^n$, where

$$\mu_p : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{R} \qquad \mu_p(x,y) = \sum_{j=1}^{n} |x_j - y_j|^p$$

Answer: Check the conditions.

e.p. we need

$$\sum_{j=1}^{n} |x_j - y_j|^p + \sum_{j=1}^{n} |y_j - z_j|^p \geq \sum_{j=1}^{n} |x_j - z_j|^p$$

metric ?

$p=1$: ✓

$p=2$: $1^2 + 1^2 < 2^2$ ✗

$p>1$: $1^p + 1^p = 2 < 2^p$ ✗

$p=\frac{1}{2}$: $|x_j - z_j|^{\frac{1}{2}} \leq (|x_j - y_j| + |y_j - z_j|)^{\frac{1}{2}}$ ✓

$\leq |x_j - y_j|^{\frac{1}{2}} + |y_j - z_j|^{\frac{1}{2}}$

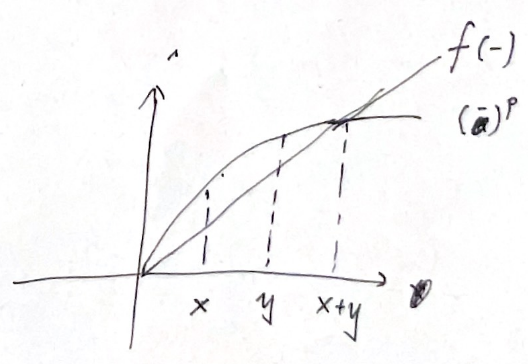
$p<1$: true by a similar argument ✓

Lemma: For $x, y \in \mathbb{R}_{\geq 0}$, $p \leq 1$,

$$(x+y)^p \leq x^p + y^p.$$

Proof. See picture.

$$(x+y)^p = f(x) + f(y)$$
$$\leq x^p + y^p$$



1.4. ~~a~~ Norm

[stackexchange: 465414]

norm $\leftarrow$ norma (Latin): carpenter's square (for measuring a unit)

In some $\mathbb{R}$-v.s. / $\mathbb{C}$-v.s. when the metric is compatible with the v.s. structure, we get the norm.

Def. (norm) A norm on a $\mathbb{R}$-v.s. / $\mathbb{C}$-v.s. is a map

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$

s.t. a) (positive definite) $\|v\| \geq 0$

$\|v\| = 0 \iff v = 0$

b) (positively homogeneous) $\|\alpha v\| = |\alpha| \, \|v\|$ $\alpha \in \mathbb{R}$ or $\mathbb{C}$

c) (triangle inequality) $\|v\| + \|w\| \geq \|v+w\|$

Task 2.1. Verify that $\|\cdot\|_\infty$ is a norm on $\mathbb{R}^n$, where

$$\|\cdot\|_\infty : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \|x\|_\infty := \max_{1\leq j\leq n} |x_j|$$

Similarly, define $(\mathbb{C}^n, \|\cdot\|_\infty)$.

Task 1. Verify that $\|\cdot\|_p$ $(p\geq 1)$ is a norm on $\mathbb{R}^n$ ~~$(p\geq 1)$~~, where

$$\|\cdot\|_p : \mathbb{R}^n \longrightarrow \mathbb{R} \qquad \|x\|_p := \left(|x_1|^p + \cdots + |x_n|^p\right)^{\frac{1}{p}}$$

Similarly, define $(\mathbb{C}^n, \|\cdot\|_p)$

You're allowed to use the Minkowski inequality:

$\forall x_j, y_j \in \mathbb{C}$,

$$\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \geq \left(\sum_{j=1}^n |x_j+y_j|^p\right)^{\frac{1}{p}} \qquad p\geq 1$$

$$\left(\sum_{j=1}^n |x_j|^p\right)^{\frac{1}{p}} + \left(\sum_{j=1}^n |y_j|^p\right)^{\frac{1}{p}} \leq \left(\sum_{j=1}^n |x_j+y_j|^p\right)^{\frac{1}{p}} \qquad p\leq 1$$

Task 2.2, 2.3. For $(V_i, \|\cdot\|_{V_i})$, $i\in\{1,\dots,n\}$, $p\geq 1$, construct different norms on $V = V_1\times\cdots\times V_n$:

$$\|u\|_{V,\infty} := \max_{1\leq j\leq n} \|u_j\|_{V_j}$$

$$\|u\|_{V,p} := \left(\|u_1\|_{V_1}^p + \cdots + \|u_n\|_{V_n}^p\right)^{\frac{1}{p}}$$

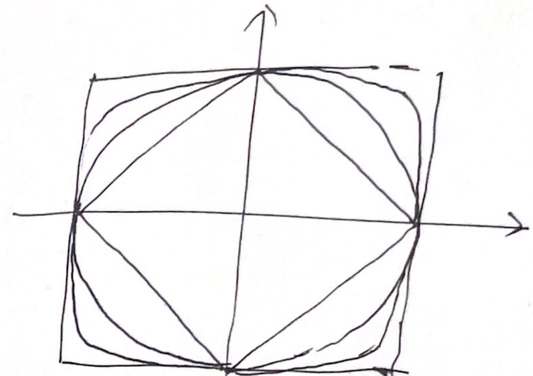
Verify that $\|\cdot\|_{V,\infty}$, $\|\cdot\|_{V,p}$ are norms.

For $V_i = \mathbb{R}$, It reduces to the last task.

Q. How to "see" those norms?

A. By drawing the unit sphere $\{x \in V \mid \|x\| = 1\}$

Ex. Draw $\{x \in \mathbb{R}^2 \mid \|x\|_p = 1\}$ for $p = 1, 2, 4, \infty$.



We skip the topology part for time reason.

(X, d) metric space

open ball

$B_r(x) := \{y \in X \mid d(x,y) < r\}$

closed ball

$B_r[x] := \{y \in X \mid d(x,y) \leq r\}$

↑ temporary symbol, from wiki.

2. Cauchy series. (completeness)

~~People want to have natural description~~ interet

~~Def. Let (X,d) be a metric space.~~

~~We know~~ that

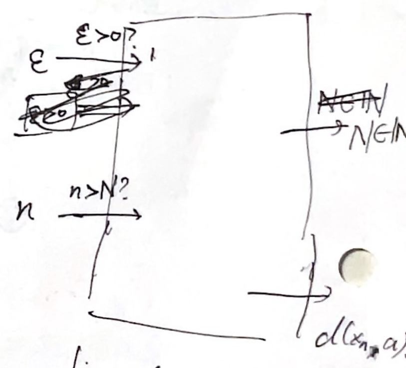
People want to do analysis on spaces. For that reason, we ~~want~~ need to define convergence ; and require that ~~series~~ sequence which looks converge should really converge.

Let (X,d) be a metric space.

Def. Let $a \in X$, $\{x_n\}_{n\in\mathbb{N}}$ a ~~series~~ sequence of pts in X.

We say $\{x_n\}_{n\in\mathbb{N}}$ converges to a, if

$$\left[\ \forall \varepsilon>0,\ \exists N\in\mathbb{N}\ \text{s.t}\ \forall n\geq N,\quad d(x_n,a)<\varepsilon\ \right]$$



~~In this~~

We also say the limit of $\{x_n\}_n$ is a, i.e. $\lim_{n\to\infty} x_n = a$.

~~If~~ ~~Def.~~

Ex. We say $\{x_n\}_{n\in\mathbb{N}}$ does not converge to a, if

$$\left[\ \exists\varepsilon>0,\ \forall N\in\mathbb{N}\ \exists n > N,\ \text{s.t}\quad d(x_n,a)\geq\varepsilon\ \right]$$

$X=\mathbb{R}$ ~~$(X,d)=(\mathbb{R},d)$~~

Ex. $\lim_{n\to\infty}\frac{1}{n}=0$ $\quad \lim_{n\to\infty}(-1)^n\ \nexists$, ~~$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.~~

~~$(X,d)=(\mathbb{Q},d)$~~ $X=\mathbb{Q}$ ~~$\lim_{n\to\infty}(1+\frac{1}{n})^n\ \nexists$.~~

~~(X,d)~~ $X=\mathbb{R}-\{0\}$ $\lim_{n\to\infty}\frac{1}{n}\ \nexists$

Now we define the Cauchy sequence, i.e. the seq which looks converge.

Def. We say $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, if

$$\left[\begin{array}{l} \forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \forall n,m > N, \\ \qquad d(x_n, x_m) < \varepsilon \end{array}\right]$$

Ex. We say $\{x_n\}_{n\in\mathbb{N}}$ is not a Cauchy sequence, if

$$\left[\begin{array}{l} \exists \varepsilon > 0,\ \forall N \in \mathbb{N} \text{ s.t. } \exists n,m > N, \\ \qquad d(x_n, x_m) \geq \varepsilon \end{array}\right]$$

Ex. Is $\{\frac{1}{n}\}_{n\in\mathbb{N}}$ a Cauchy sequence in $\mathbb{R}-\{0\}$?

Is $\{(1+\frac{1}{n})^n\}_{n\in\mathbb{N}}$ a Cauchy seq in $\mathbb{Q}$?

Is $\{(-1)^n\}_{n\in\mathbb{N}}$ a Cauchy seq in $\mathbb{R}$?

Ex. ~~Cauchy sequen~~

$\lim_{n\to\infty} x_n \ \exists \quad \Rightarrow \quad \{x_n\}$ is a Cauchy sequence

$\not\Leftarrow$

completeness

Task 4. Take $\{x_n\}$ a sequence, $\{x_{s_n}\}$ a subsequence of $\{x_n\}$ in (X, d).

$(1 \leq s_1 < s_2 < \cdots < s_n < \cdots)$

a) If $\{x_n\}$ is Cauchy $\Rightarrow$ $\{x_{s_n}\}$ is Cauchy

b) $\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} x_{s_n} = a$

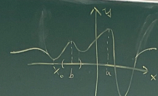
Def. (X, d), $f: X \to \mathbb{R}$ $\quad x_0 \in X$

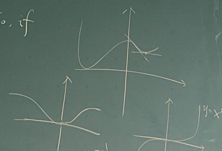
We say f takes a (strict) global max at x_0, if

$\forall x \in X \quad f(x_0) \geq f(x)$

f ~~(strict)~~ at x_0, if

$\exists r > 0, \forall x \in B_r(x_0) \quad f(x_0) \geq f(x)$



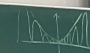


$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

Task 3 For $\alpha \in \mathbb{R}$

$f_\alpha: \mathbb{R}^2 \to \mathbb{R} \quad f_\alpha(x,y) := \sin(x^2 + \alpha y^2)$

$\sin t = t - \frac{t^3}{3!} + o(t^4)$



$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

a) Taylor expansion of f_α

$f_\alpha(x,y) = x^2 + \alpha y^2 - \frac{(x^2+\alpha y^2)^3}{3!} + o((x^2+\alpha y^2)^4)$

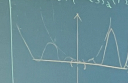
$= x^2 + \alpha y^2 + o(x^2 + y^2)$

b) All $\alpha \in \mathbb{R}$ s.t.

f_α is a $\begin{cases} \text{critical pt} \\ \text{local min} \\ \text{local max} \end{cases}$ at $(0,0)$.

c) $\begin{cases} \text{local min} \\ \text{global min} \end{cases}$ at $\left(\sqrt{\frac{3\pi}{2}}, 0\right)$

d) $\det\left(\mathrm{Hess}_\alpha\left(\sqrt{\frac{3\pi}{2}}, 0\right)\right)$



$a \in X$ $\quad [\ \forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n > N$,

$d(x_n, a) < \varepsilon\ ]$

$\lim_{n \to \infty} x_n = a$

Ex. $\{x_n\}_{n \in \mathbb{N}}$ does not converge to a, if

[ ]

$d(x_n, x_m) < \varepsilon$]

Ex. $\lim_{n \to \infty} x_n \ \exists \implies \{x_n\}$ is Cauchy

$\not\Longleftarrow$ complete $\quad (0,1)$