Eine Woche, ein Beispiel

11.6 equivariant K-theory of Steinberg variety: abstract nil Hecke alg

Recall that we have an alg homo

$$\mathbb{Z}[e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_d^{\pm 1}] \longrightarrow \mathbb{Q}[[\lambda_1, \lambda_2, \dots, \lambda_d]] \supseteq \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

$$e_i \longmapsto e^{\lambda_i}$$

for e_i, λ_i, $i \in \{1, \dots, d\}$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$

Ex 1. define $\partial_i \in End_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$ by

$$\partial_i f = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \qquad f \in \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

compute $\partial_i \lambda_i$, $\partial_i \lambda_{i+1}$, $\partial_1(\lambda_1^3 \lambda_2 - 3\lambda_2\lambda_4\lambda_5)$.

Ex 2. derive that

$$\partial_i f g = (s_i f)\partial_i g + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} g$$

$f \in End_{\mathbb{Q}\text{-v.s}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$

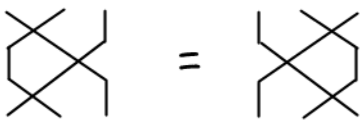
$f: \; g \longmapsto f \cdot g$

as operators.

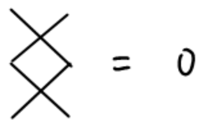
Ex 3. verify that

$$\partial_i \lambda_i = \lambda_{i+1}\partial_i + 1$$

$$\partial_i \lambda_{i+1} = \lambda_i \partial_i - 1$$



$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1}$$



$$\partial_i^2 = 0$$

Ex 1'. define $D_i \in \mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm 1}, \ldots, e_d^{\pm 1}])$ by

$$D_i f = \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}}$$
$$= \frac{e_{i+1} f - e_i s_i f}{e_{i+1} - e_i}$$

compute

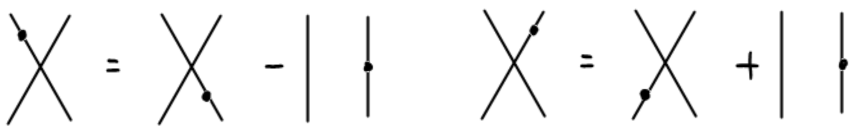
$$D_i 1 = 1$$
$$D_i e_i = 0 \qquad D_i e_i^{-1} = e_i^{-1} + e_{i+1}^{-1}$$
$$D_i e_{i+1} = e_i + e_{i+1} \qquad D_i e_{i+1}^{-1} = 0$$

Ex 2'. derive that

$$D_i f g = (s_i f) D_i g + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} g$$

as operators.

Ex 3'. verify that



$$D_i e_i = e_{i+1} D_i - e_{i+1} \qquad D_i e_{i+1} = e_i D_i + e_{i+1}$$

$$D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \qquad D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1}$$

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \qquad D_i^2 = D_i$$

Ex 4. Verify that

$$
\begin{array}{ccc}
\mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm 1}, \dots, e_d^{\pm 1}])_{cpl} & & \mathrm{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])_{cpl} \\
\cup & & \cup \\
\langle e_1^{\pm 1}, \dots, e_d^{\pm 1}, D_1, \dots, D_{d-1} \rangle_{\mathbb{Z}\text{-alg}, cpl} & \longrightarrow & \langle \lambda_1, \dots, \lambda_d, \partial_1, \dots, \partial_d \rangle_{\mathbb{Q}\text{-alg}, cpl} \\
e_i & \longmapsto & e^{\lambda_i} \\
D_i & \longmapsto & \partial_i \dfrac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} = \dfrac{\lambda_{i+1} - \lambda_i}{1 - e^{\lambda_{i+1} - \lambda_i}} \partial_i + 1 \\
\log e_i & \longleftarrow & \lambda_i \\
D_i \dfrac{1 - \frac{e_i}{e_{i+1}}}{\log \frac{e_i}{e_{i+1}}} = \dfrac{1 - \frac{e_{i+1}}{e_i}}{\log \frac{e_{i+1}}{e_i}} (D_i - 1) & \longleftarrow & \partial_i
\end{array}
$$

is an alg iso.

Hint.

$$
\begin{aligned}
& D_i e_k = s_i(e_k) D_i + \frac{e_k - s_i(e_k)}{1 - \frac{e_i}{e_{i+1}}} \\
\Leftrightarrow\ & \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} e^{\lambda_k} = s_i(e^{\lambda_k}) \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{1 - e^{\lambda_i - \lambda_{i+1}}} \\
\Leftrightarrow\ & \partial_i e^{\lambda_k} = s_i(e^{\lambda_k}) \partial_i + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{\lambda_i - \lambda_{i+1}}
\end{aligned}
$$

This is the recommended sign notation.

Ex 4. Verify that

$$
\begin{array}{ccc}
\mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm 1},\ldots,e_d^{\pm 1}])_{cpl} & & \mathrm{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1,\ldots,\lambda_d])_{cpl} \\
\cup & & \cup \\
\langle e_1^{\pm 1},\ldots,e_d^{\pm 1}, D_1,\ldots,D_{d-1}\rangle_{\mathbb{Z}\text{-alg},cpl} & \longrightarrow & \langle \lambda_1,\ldots,\lambda_d,\partial_1,\ldots,\partial_d\rangle_{\mathbb{Q}\text{-alg},cpl} \\
e_i & \longmapsto & e^{-\lambda_i} \\
D_i & \longmapsto & \partial_i \dfrac{\lambda_i-\lambda_{i+1}}{1-e^{\lambda_{i+1}-\lambda_i}} = \dfrac{\lambda_{i+1}-\lambda_i}{1-e^{\lambda_i-\lambda_{i+1}}}\partial_i + 1 \\
-\log e_i & \longleftarrow & \lambda_i \\
D_i \dfrac{1-\frac{e_{i+1}}{e_i}}{\log \frac{e_i}{e_{i+1}}} = \dfrac{1-\frac{e_i}{e_{i+1}}}{\log\frac{e_{i+1}}{e_i}}(D_i-1) & \longleftarrow & \partial_i
\end{array}
$$

is an alg iso.

Hint.

$$
\begin{aligned}
& D_i e_k = s_i(e_k) D_i + \frac{e_k - s_i(e_k)}{1-\frac{e_{i+1}}{e_i}} \\
\Leftrightarrow\ & \partial_i \frac{\lambda_i-\lambda_{i+1}}{1-e^{\lambda_{i+1}-\lambda_i}} e^{-\lambda_k} = s_i(e^{-\lambda_k})\partial_i \frac{\lambda_i-\lambda_{i+1}}{1-e^{\lambda_{i+1}-\lambda_i}} + \frac{e^{-\lambda_k}-s_i(e^{-\lambda_k})}{1-e^{\lambda_{i+1}-\lambda_i}} \\
\Leftrightarrow\ & \partial_i e^{-\lambda_k} = s_i(e^{-\lambda_k})\partial_i + \frac{e^{-\lambda_k}-s_i(e^{-\lambda_k})}{\lambda_i-\lambda_{i+1}}
\end{aligned}
$$

This is even more confusing.

Need.

$$D_i f = s_i f D_i + \frac{f - s_i f}{1-\frac{e_{i+1}}{e_i}}$$