Eine Woche, ein Beispiel

9.3. field extension with RS

Goal: construct an equivalence between two categories:

cpt conn

$$RS^{cc} = \left\{ \begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Obj: } F/\mathbb{C} \text{ field ext s.t.} \\ \quad \operatorname{trdeg}_{\mathbb{C}} F = 1 \\ \quad F/\mathbb{C} \text{ f.g. as a field} \\ \text{Mor: morphism as fields}/\mathbb{C} \end{array} \right\}^{op} = \mathrm{field}^{op}_{\mathbb{C}(t)/\mathbb{C}}$$

$$\begin{array}{ccc} Y & & \mathcal{M}(Y) \\ \downarrow f & \Longrightarrow & \uparrow f^* \\ X & & \mathcal{M}(X) \end{array}$$

which obeys the following slogan:

$$\boxed{\text{(ramified) covering} \approx \text{(function) field extension}}$$

Rmk. 1. For requiring $F/\mathbb{C}$ f.g. as a field, we avoid examples like $\overline{\mathbb{C}(t)}$.

Do they corresponds to some non-cpt Riemann surface?

If so, how to enlarge the category RS^{cc}?

2. $\mathrm{field}_{\mathbb{C}(t)/\mathbb{C}}$ means fields over $\mathbb{C}$ which are fin ext of $\mathbb{C}(t)$ abstractly; morphisms don't need to fix $\mathbb{C}(t)$.

Do you have a better name for RS^{cc} and $\mathrm{field}_{\mathbb{C}(t)/\mathbb{C}}$?

https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of
https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory

1. field of meromorphic functions
2. Galois covering
3. valuations
4. quadratic extension of $\mathbb{C}(x)$: hyperelliptic curve
5. miscellaneous.

1. field of meromorphic functions

Def. For X RS,

$$\mathcal{M}(X) := \{\text{meromorphic fcts on } X\}$$
$$= \{f : X \longrightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{\mathbb{1}_\infty\}$$
$$\overset{X \text{ cpt}}{\underset{\text{conn}}{=}} \{\text{rational fcts on } X\}$$

Ex. Verify that

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$
$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[i]) \cong \operatorname{Frac}\left(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1))\right)$$

Later we will show that, for $X \in RS^{cc}$,

$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3,$$

compute

1) $f^* : \mathbb{C}(T) \hookrightarrow \mathbb{C}(S)$ $\qquad [\mathbb{C}(S) : \mathbb{C}(T)]$ & a $\mathbb{C}(T)$-basis
2) $\operatorname{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$
3) $\mathbb{C}(S)^{\mathbb{Z}/3\mathbb{Z}}$
4) $\operatorname{Aut}_f(\mathbb{CP}^1)$

Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z + \tfrac{1}{z},$$

do the same work.

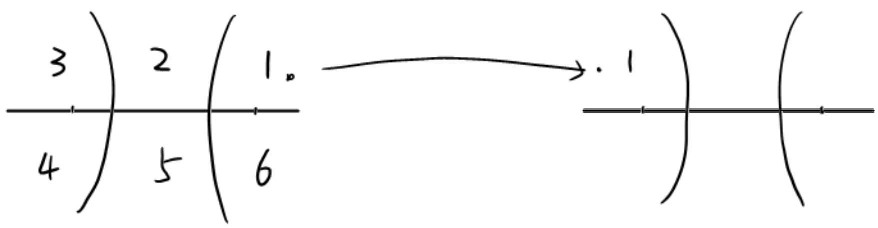
Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z,$$

compute the same stuff.

Why isn't $\mathbb{C}(S)/\mathbb{C}(T)$ Galois this time?

Hint.



Prop. For $d \in \mathbb{N}_{>0}$, $f: Y \to X$ proper holo morphism between conn RSs,

$$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] = d.$$

Cor. For X cpt conn,

$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

In ptc, $F/\mathbb{C}$ f.g as a field, $\mathrm{trdeg}_{\mathbb{C}} F = 1$.

To show the proposition, one need the following black box to find a basis.

Black box (meromorphic fcts seperate points)

X: RS, $x, y \in X$ $x \neq y$, then

$\exists g \in \mathcal{M}(X)$ s.t. $g(x) \neq g(y)$ $g(x), g(y) \in \mathbb{C}$.

(stronger) $\exists g \in \mathcal{M}(X)$ s.t. $\mathrm{ord}_x g = -1$, $g(y) = 0$.

I prefer using Riemann-Roch when X is cpt, and Stein manifold when X is not.

Ex. Using the black box, show that,

for X: RS, $\{x_1, \cdots, x_n\} \subseteq X$, $\exists g \in \mathcal{M}(X)$ s.t.

$$\mathrm{ord}_{x_1} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \cdots, n\}$$

$$g(x_i) \neq g(x_j) \quad \forall i \neq j,\ i, j \in \{2, \cdots, n\}$$

Proof of prop

$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \geqslant d$: Fix $x_0 \in X$ s.t. $\# f^{-1}(x_0) = d$. Denote $f^{-1}(x_0) = \{y_1, \cdots, y_d\}$.

For each i, let $g_i \in \mathcal{M}(Y)$ be a meromorphic fct s.t.

$$\mathrm{ord}_{x_i} g_i = -1 \qquad g_i(y_j) \in \mathbb{C} \qquad \forall j \neq i,$$

then $\{g_1, \cdots, g_d\} \subseteq \mathcal{M}(Y)$ are $f^*\mathcal{M}(X)$-linear independent.

Check: $\mathrm{ord}_{y_j}(\sum_i f_i g_i) \approx \mathrm{ord}_{y_j} f_j$

$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leqslant d$:

$\forall g \in \mathcal{M}(Y)$, need to find $a_i \in f^*\mathcal{M}(X)$ s.t.

$$g^d + a_{d-1} g^{d-1} + \cdots + a_0 = 0 \qquad \text{in } \mathcal{M}(Y)$$

The fcts

$$a_i(z) = (-1)^i \sum_{\{k_1, \cdots, k_i\} \subseteq \{1, \cdots, d\}} g(z_{k_1}) \cdots g(z_{k_d})$$

$f^{-1}(f(z)) := \{z_1, \cdots, z_d\}$, multiplicity is counted

satisfy the conditions.

Use Riemann extension theorem to show $a_i(z) \in f^*\mathcal{M}(X)$, see [Donaldson, p148].

By primitive element theorem, $[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leq d$.

2. Galois covering

Def. Let $f: Y \to X$ be a proper hdo map between two conn RSs.
f is Galois, if $\mathcal{M}(Y)/f^*\mathcal{M}(X)$ is a Galois extension.
normal normal

Prop. $f: Y \to X$ is Galois/normal
$\Leftrightarrow$ $\deg f = \# \mathrm{Aut}_f(Y)$
$\Leftrightarrow$ $f^{-1}(x_0)$ is an $\mathrm{Aut}_f(Y)$-torsor, $\forall x_0 \in X - f(\mathrm{Ram}(f))$
$\Leftrightarrow$ $\mathrm{Aut}_f(Y) \circlearrowright f^{-1}(x_0)$ transitively, $\forall x_0 \in X$
$\Leftrightarrow$ $Y/\mathrm{Aut}_f(Y) \cong X$, i.e. f can be written as
$$Y \twoheadrightarrow Y/G$$

Ex. For $f: Y \to X$, suppose that
$[\forall y_1, y_2 \in Y \text{ s.t. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2)$ ramification index
Show that f is Galois by computing $\# \mathrm{Aut}_f(Y)$.

[Hint. Use geodesics to divide X into several smaller triangles.
If geodesics are hard, take $g: X \to \mathbb{CP}^1$ non-constant,
and reduce the problem to $g \circ f$.]

This proof is not completely rigorous, and you are encouraged to find a reference to rigorously prove it.

You may need the following materials for completing the proof.

google: geodesic triangulations
https://math.stackexchange.com/questions/1661331/proof-of-equivalence-of-conformal-and-complex-structures-on-a-riemann-surface?rq=1
https://arxiv.org/pdf/2103.16702.pdf
(If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.)
copied from p2, in https://www.mathematik.uni-muenchen.de/~forster/eprints/gaussbonnet.pdf

E.g. Consider the covering

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z$$

This is not a Galois covering. Consider the Galois closure

$$\begin{array}{ccccc}
\mathbb{CP}^1 & & \mathbb{C}(u) = \mathbb{C}(S)[R]/(R^2+S^2-4) & & u + \frac{1}{u} \\
\downarrow z+\frac{1}{z} & & \uparrow & & \uparrow \\
\mathbb{CP}^1 & & \mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3-3S-T) & & S \qquad S^3-3S \\
\downarrow z^3-3z & & \uparrow & & \qquad\quad \uparrow \\
\mathbb{CP}^1 & & \mathbb{C}(T) & & \qquad\quad T
\end{array}$$

Determination of the Galois closure

$$\begin{aligned}
\min(S, \mathbb{C}(T)) &= x^3 - 3x - T && \text{in } \mathbb{C}(T)[x] \\
&= x^3 - 3x - (S^3 - 3S) \\
&= (x - S)(x^2 + Sx + S^2 - 3) && \text{in } \mathbb{C}(S)[x]
\end{aligned}$$

To decompose the polynomial $x^2 + Sx + S^2 - 3$,
we have to add root of discriminant:

$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2-3)} = \sqrt{3}\sqrt{-S^2+4}.$$

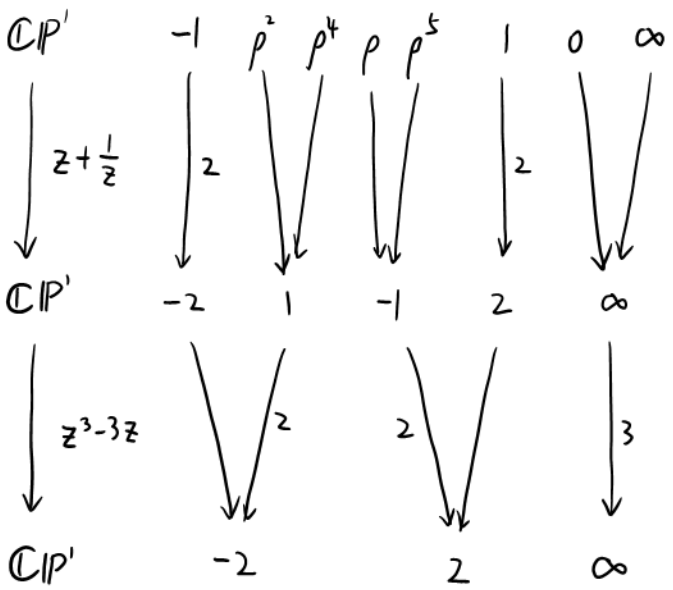
Therefore, the Galois closure of $\mathbb{C}(S)/\mathbb{C}(T)$ is

$$\mathbb{C}(S)[R]/(R^2+S^2-4) \cong \mathbb{C}\left(\frac{S+iR}{2}\right) \triangleq \mathbb{C}(u)$$

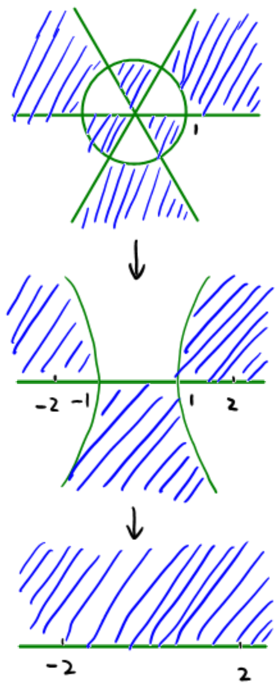
where

$$S = \frac{S+iR}{2} + \frac{S-iR}{2} = u + \frac{1}{u}$$

The picture from the RS side is as follows:



only ramified pts are drawn



affine version