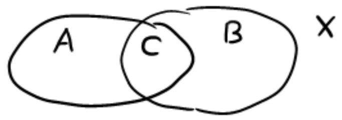
Eine Woche, ein Beispiel

1.9 Excision

Goal: state the homology/cohomology/homotopy excision thm and give applications.
↑ don't want to see the proof, I'm lazy...

The main ref would be Hatcher, and we only consider the singular homology.

Thm (Homology excision thm, [Thm 2.20, Cor 2.24])
Let X space, $A, B \subset X$ subspace, $X = \mathring{A} \cup \mathring{B}$, $C = A \cap B$
or: Let X CW cpk, $A, B \subset X$ subcplx, $X = A \cup B$, $C = A \cap B$



then $\iota_*: H_i(A, C; \mathbb{Z}) \longrightarrow H_i(X, B; \mathbb{Z})$ is iso.

Cor [Prop 2.22] Let (X, A) be an NDR-pair. then
$\pi_*: H_i(X, A; \mathbb{Z}) \longrightarrow \widetilde{H}_i(X/A; \mathbb{Z})$ is iso.

Proof.

$$H_i(X, A; \mathbb{Z}) \overset{LES}{\cong} H_i(X \cup A \times [0,1), A; \mathbb{Z}) \xrightarrow{\cong} H_i(X \cup CA, CA; \mathbb{Z}) \overset{LES}{\cong} \widetilde{H}_i(X \cup CA; \mathbb{Z})$$
$$\sim \widetilde{H}_i(X/A; \mathbb{Z})$$

$X \cup CA \longrightarrow X \cup CA/_{CA} \cong X/A$ is a homotopy equivalence since [Prop 0.17]

- CA is contractable
- (X, A) satisfies HEP $\Rightarrow$ $(X \cup CA, CA)$ satisfies HEP

For the equivalent definitions of NPR-pair, see here:
https://math.stackexchange.com/questions/3547820/neighborhood-deformation-retracts-vs-cofibrations
or Prop A.6 in url:https://www.math.univ-paris13.fr/~schwartz/FIMFA/Ando.pdf

Thm (MV sequence, LES)
Let X space, $\mathcal{U}, V \subseteq X$ open subset, $\mathcal{U} \cup V = X$. Then we get a LES

$$\cdots \to \widetilde{H}_{n+1}(X; \mathbb{Z}) \to \widetilde{H}_n(\mathcal{U} \cap V; \mathbb{Z}) \longrightarrow \widetilde{H}_n(\mathcal{U}; \mathbb{Z}) \oplus \widetilde{H}_n(V; \mathbb{Z}) \longrightarrow \widetilde{H}_n(X; \mathbb{Z}) \to \cdots$$

MV->excision: https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision
excision->MV: https://www.math.ru.nl/~gutierrez/files/homology/Lecture06.pdf
Please be aware of the conditions in the theorems. We have many versions of theorems when we slightly change the conditions, but I don't want to go to the most generality(Actually I don't know the most general condition).

E.g. $H_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \overset{SES}{\cong} H_{n-1}(\partial\Delta^n, \Lambda; \mathbb{Z}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}; \mathbb{Z}) \overset{\text{induction}}{\cong} \mathbb{Z}$.

E.g. The local homology groups $H_n(x) := H_n(X, X \setminus \{x\}; \mathbb{Z})$ can be defined locally, i.e. $H_n(X, X\setminus\{x\}; \mathbb{Z}) \cong H_n(\mathcal{U}, \mathcal{U}\setminus\{x\}; \mathbb{Z})$ for $\{x\} \in \mathcal{U} \overset{\text{open}}{\subseteq} X$.

Given a sm map $f: M \longrightarrow N$ between mflds of dim n, a pt $y \in N$ s.t. $f^{-1}(y) \triangleq \{x_1, \cdots, x_n\}$ is finite, we can define the local degree $\deg_x f \in \mathbb{Z}$ at $x \in f^{-1}(y)$:

$(\mathcal{U} \cap f^{-1}(y) = \{x\})$ $$f_*: \underbrace{H_n(\mathcal{U}, \mathcal{U} - \{x\}; \mathbb{Z})}^{\cong\mathbb{Z}} \longrightarrow \overbrace{H_n(N, N - \{y\}; \mathbb{Z})}^{\cong\mathbb{Z}} \qquad [\mathcal{U}] \mapsto \deg_x f \cdot [N]$$

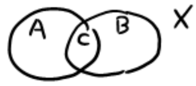
When M, N are cpt, we can also define the global degree $\deg f \in \mathbb{Z}$:

$$f_*: \overbrace{H_n(M; \mathbb{Z})}^{\cong\mathbb{Z}} \longrightarrow \overbrace{H_n(N; \mathbb{Z})}^{\cong\mathbb{Z}} \qquad [M] \mapsto \deg f \cdot [N]$$

we have the equality $\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$.

Thm (Homotopy excision thm [Thm 4.23])

Let X: CW cplx, A, B subcplx, $X = A \cup B$, $C = A \cap B$ nonempty and connected.



If (A, C) is m-connected,

(B, C) is n-connected,

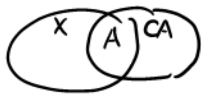
then $\pi_i(A, C, x_0) \longrightarrow \pi_i(X, B, x_0)$ is $\begin{cases} \text{iso} & i < m+n \\ \text{surj} & i = m+n \end{cases}$ $\quad x_0 \in C$

Cor [Prop 4.28] Suppose X, A are CW cplxs, $A \subset X$ is sub cplx.

If (X, A) is r-connected, A is s-connected,

then the map $\pi_{*}: \pi_i(X, A, x_0) \longrightarrow \pi_i(X/A, x_0)$ is $\begin{cases} \text{iso} & i < r+s+1 \\ \text{surj} & i = r+s+1 \end{cases}$

[*Proof.*



$X \cup CA$

$$\pi_i(X, A, x_0) \longrightarrow \pi_i(X \cup CA, CA, x_0) \overset{\text{LES}}{\cong} \pi_i(X \cup CA, x_0) \overset{\text{homotopy}}{\sim} \pi_i(X/A, x_0)$$]

Thm (Freudenthal suspension thm [Cor 4.24])

Let $n \geq 1$, X be an $(n-1)$-connected CW cplx, then the suspension map

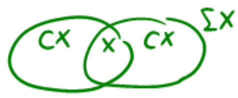
$$\Sigma: \pi_i(X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, x_0)$$
$$[f: S^i \to X] \longmapsto [\Sigma f: S^{i+1} \cong \Sigma S^i \to \Sigma X]$$

is $\begin{cases} \text{iso} & i < 2n-1 \\ \text{surj} & i = 2n-1 \end{cases}$

Rmk. Freudenthal suspension thm $\rightsquigarrow$ concept of the stable homotopy gp.

[*Proof.* $\pi_i(X, x_0) \overset{\text{LES}}{\cong} \pi_{i+1}(CX, X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, CX, x_0) \cong \pi_{i+1}(\Sigma X)$

is $\begin{cases} \text{iso} & i+1 < n+n \\ \text{surj} & i+1 = n+n \end{cases}$



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