Eine Woche, ein Beispiel 4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

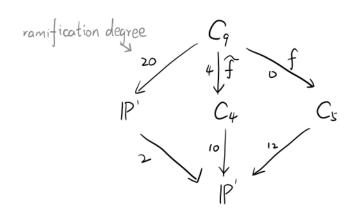
[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.

 $https://math.stackexchange.com/questions/7 \ 10899/prym-variety-associated-to-an-\%c3\%a9 tale-cover-of-degree-2-of-an-hyperelliptic-curve$

https://mathoverflow.net/questions/402049/induced-action-on-prym-variety

Goal: Describe some curves (maybe singular) $\mathrm C$ in A, and describe their degree and the monodromy group.

$$C_q = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$$
 has the following covers:
Aut $(C_q) = \frac{7}{22} \times \frac{7}{22}$



where

$$C_4 = \{ \hat{y}^2 = \prod_{j=1}^{10} (t-j) \}$$

$$C_5 = \{ \hat{y}^2 = t \prod_{j=1}^{10} (t-j) \}$$

The crspd field extension.

$$C(x) \left[\frac{y}{y^{2}} - \frac{1}{1} (x^{2} - \frac{1}{1}) \right]$$

$$= \frac{C(x) \left[\frac{y}{y^{2}} - \frac{1}{1} (x^{2} - \frac{1}{1}) \right]}{C(t) \left[\frac{y}{y^{2}} - \frac{1}{1} (t - \frac{1}{1}) \right]}$$

$$= \frac{C(x) \left[\frac{y}{y^{2}} - \frac{1}{1} (t - \frac{1}{1}) \right]}{C(t)}$$

$$= \frac{C(x) \left[\frac{y}{y^{2}} - \frac{1}{1} (t - \frac{1}{1}) \right]}{C(t) \left[\frac{y}{y^{2}} - \frac{1}{1} (t - \frac{1}{1}) \right]}$$

Global differential forms

Pulling back differential forms give the following maps:

Therefore, $H^{\circ}(C_q; w_{C_q}) \cong \widetilde{f}^* H^{\circ}(C_4; w_{C_4}) \oplus f^* H^{\circ}(C_5; w_{C_5}) \qquad (1)$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which crspds to pulling back of divisors)

However, since $Jac(C) = H^{\circ}(C; \omega_c)^*/_{H,(C; \mathbb{Z})}$, we are working on the dual spaces. The notations are again switched:

$$f^* \longrightarrow N_{mf}$$
 $f^* \longrightarrow f^*$

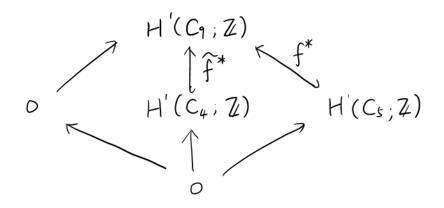
One may get

different meaning compared with (1)!

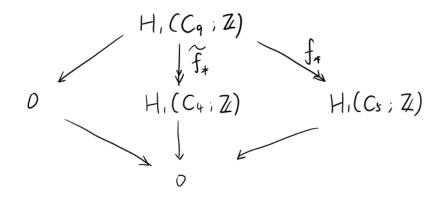
$$H^{\circ}(C_q; \omega_{C_q})^{\dagger} \cong \widetilde{f}^* H^{\circ}(C_4; \omega_{C_4})^{\dagger} \oplus f^* H^{\circ}(C_r; \omega_{C_s})^{\dagger}$$
 (2)

(co) homology class

This page may be easier to understand, and it helps to understand the previous page.



Q: Do we have $H'(C_9; \mathbb{Z}) \cong \widehat{f}^* H'(C_4; \mathbb{Z}) \oplus f^* H'(C_5; \mathbb{Z})$?



Q: Do we have H, (Cq, Z)* = F*H, (C4, Z)* + F* H, (Cs, Z)*?

Curve in Prym variety

$$A = J_{ac}(C_9)/f^*J_{ac}(C_s) \cong Prym(C_9/C_s)$$

Prop O. A is isogenous to Jac(C4);

1. f^* . $Jac(C_s) \longrightarrow Jac(C_q)$ is an isogeny to its image;

π ο AJ_{Cq} is not injective, it factors through C4;
 C4 → A is generically injective;
 C4 → A produces a sm image of A, outside of hon-injective locus.

Idea: observe everything from the tangent space.

Proof. O. Taking the tangent space of (3), one gets

$$0 \longrightarrow H^{\circ}(C_{s}, \omega_{C_{s}})^{*} \xrightarrow{df^{*}} H^{\circ}(C_{q}, \omega_{C_{q}})^{*} \longrightarrow T_{o}A \longrightarrow 0$$

Combined with (2),
$$T_oA \cong H^{\circ}(C_4; \omega_{C_4})^*$$
.

Late we will find a natural isogeny $Jac(C_4) \longrightarrow A$. What's the degree of this isogeny?

1. Since

$$Ker f^* < +\infty$$
 Is f^* inj in this case?

2. For
$$p_1 = (x_0, y_0)$$
, $p_2 = (-x_0, y_0)$, we want to show that
$$\int_{\mathcal{X}_1: p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\mathcal{X}_2: p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$LHS = \int_{\mathcal{X}_1: p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = RHS.$$

3.

https://mathoverflow.net/questions/68503/has-anyone-studied-the-prym-map-for-double-covers-with-two-ramification-points https://arxiv.org/abs/1010.4483: It proves that many Prym maps (C->Prym) are generically finite.

Notice: $C_4 \subset Jac(C_4)$ is only invariant under $p \mapsto -p$, not invariant under $p \mapsto p + a_0$

Otherwise, the Gauss map would be cover of deg >2. Therefore, after isogeny $C_4 \longrightarrow A$ is still gen inj. Q: Is this map really inj?

(4) C4 → Jac(C4) is sm, so after isogeny it is still sm outside of non-injective locus.

Rmk. Suppose $f: \widehat{C} \longrightarrow C$ is a deg 2 (ramified) covering, $\sigma: \widehat{C} \longrightarrow \widehat{C}$ the crspd involution, define A as the quotient

$$\begin{array}{c}
\widetilde{C} \\
\downarrow AJ_{\widetilde{c}} \\
Jac(C) \xrightarrow{f^*} Jac(\widetilde{C}) \xrightarrow{\pi} A \longrightarrow 0
\end{array}$$

one can identify A with $Prym(\widetilde{C}/C) \subset Jac(\widetilde{C})$, why? and the Abel-Prym map is given by

$$\begin{array}{cccc} \mathsf{AP\tilde{c}} = & \pi \circ \mathsf{AJ\tilde{c}} \colon \stackrel{\sim}{\mathsf{C}} & \longrightarrow & \mathsf{Jac}(\stackrel{\sim}{\mathsf{C}}) & \longrightarrow & \mathsf{A} \\ & \mathsf{p} & \longmapsto & \mathcal{O}_{\stackrel{\sim}{\mathsf{C}}}(\mathsf{p}-\mathsf{p_0}) & \longmapsto & \mathcal{O}_{\stackrel{\sim}{\mathsf{C}}}(\mathsf{p}-\sigma(\mathsf{p})) \end{array}$$

Therefore, for p, # P2,

$$AP_{\widetilde{c}}(p_{1}) = AP_{\widetilde{c}}(p_{2})$$

$$\Leftrightarrow O_{\widetilde{c}}(p_{1} - \sigma(p_{1})) = O_{\widetilde{c}}(p_{2} - \sigma(p_{2}))$$

$$\Leftrightarrow O_{\widetilde{c}}(p_{1} + \sigma(p_{2})) = O_{\widetilde{c}}(p_{2} + \sigma(p_{1}))$$

O When $p_1, p_2 \in \widehat{C}$ are ramification pts of f, i.e., $p_1 = \sigma(p_1)$, $p_2 = \sigma(p_2)$ $A P_{\widehat{C}}(p_1) = A P_{\widehat{C}}(p_2)$. As a result, when f is ramified, $AP_{\widehat{C}}$ is never injective.

② Now assume APE is not inj. $AP_{c}(p_{1}) = AP_{c}(p_{2})$. When $p_{1} \neq \sigma(p_{1})$ or $p_{2} \neq \sigma(p_{2})$,

$$\widetilde{C} \longrightarrow |P'|$$
.

3 In the example,

$$\widetilde{C} = C_q, \quad C = C_s,$$

$$\sigma \colon \times \longmapsto -x$$

$$y \mapsto y$$
 C_4 involution

$$\sigma\tau\colon \times \longmapsto \times$$

we can show directly that $AP_{C_9}(\tau(p)) = AP_{C_9}(p)$. This gives a second proof for Prop (2).

$$AP_{C_9}^{\vee}(\tau(p)) = AP_{C_9}(p)$$

Reason.

$$\mathcal{O}_{C_q}(\tau(p) + \sigma(p)) = \mathcal{O}_{C_q}(\tau(p + \sigma\tau(p))) \qquad \sigma\tau = \mathcal{O}_{C_q}(p + \sigma\tau(p)) \qquad \sigma\tau$$

hyperelliptic involution_

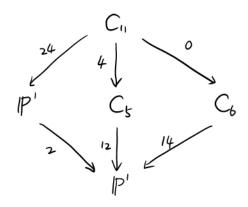
Gauss map

Taking the Gauss map of (3), one gets

$$\begin{array}{ccc}
C_{q} & \xrightarrow{2:1} & C_{4} \\
& & \downarrow^{2:1} & \downarrow^{2:1} \\
R_{1} & \xrightarrow{\text{ram at}} & R_{2} \\
& \downarrow^{0,\infty} & \downarrow \\
& \downarrow^{8} & \xrightarrow{---} & \downarrow^{3} \\
[\alpha_{0}, \dots, \alpha_{8}] & & \downarrow^{-} & [\alpha_{1}: \alpha_{3}: \alpha_{5}: \alpha_{7}]
\end{array}$$

 \Rightarrow deg_A $C_4 = 6$, $Gal(x) = S_6 = W(C_3)$.

E.g. 2. $C_n = \int_{j=1}^{n} (x^2 - j)^2$ has the following covering.



Curves in Jacobians: (A = Prym (C1/C6)?)

$$C_{II} \longrightarrow C_{f}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad$$

Gauss map:

$$C_{11} \xrightarrow{2:1} C_{6}$$

$$2:1 \downarrow \qquad \qquad \downarrow 2:1 \downarrow \qquad \downarrow 14$$

$$R_{1} \xrightarrow{z:1} R_{2}$$

$$Q_{1} \xrightarrow{z:1} R_{2}$$

$$Q_{2} \xrightarrow{1} R_{3}$$

$$R_{2} \xrightarrow{0, \infty} Q_{3} \xrightarrow{0} Q_{4}$$

$$Q_{1} \xrightarrow{0} Q_{2} \xrightarrow{0} Q_{3}$$

$$Q_{2} \xrightarrow{0} Q_{3} \xrightarrow{0} Q_{4}$$

$$Q_{3} \xrightarrow{0} Q_{4} \xrightarrow{0} Q_{5} = Q_{4} \xrightarrow{0} Q_{5}$$

$$Q_{4} \xrightarrow{0} Q_{5} = Q_{4} \xrightarrow{0} Q_{5} = Q_{4} \xrightarrow{0} Q_{5}$$

$$Q_{5} \xrightarrow{0} Q_{5} = Q_{5} \xrightarrow{0} Q_{5} = Q$$

E.g. 3.
$$C_{5} = \int y^{2} = (x^{3} + x + 2)^{4} + 1$$

$$\int \int_{8}^{3:1} 8$$

$$E = \int y^{2} = t^{4} + 1$$

Let us write down the change of variables for f:

$$C_{5} \qquad C(x)[y]/(y^{2}-((x^{3}+x+2)^{4}+1)) \cong C(u)[v']/(v'^{2}-((u+u^{2}+2u^{3})^{4}+u'^{2}))$$

$$E \qquad C(t)[y]/(y^{2}-(t^{4}+1)) \cong C(s)[v]/(v^{2}-(u+s^{4}))$$

$$X^{3}+X+2 \qquad y \qquad \begin{cases} x=\frac{1}{u} \\ y=\frac{v'}{u^{6}} \end{cases} \qquad \begin{cases} u=\frac{1}{x} & u^{3} & v' \\ v'=\frac{y}{x^{6}} & 1+u^{2}+2u^{3} \end{cases} \qquad \begin{cases} v' \\ (1+u^{2}+2u^{3})^{2} \end{cases}$$

$$t \qquad y \qquad \begin{cases} t=\frac{1}{s} \\ y=\frac{v}{s^{2}} \end{cases} \qquad \begin{cases} s=\frac{1}{t} \\ v=\frac{y}{t^{2}} \end{cases}$$

Also, we write down the local coordinate charts of f:

$$C[x][y]/(y^{2}-((x^{3}+x+2)^{4}+1)) \leftarrow \rightarrow C[u][v']/(v'^{2}-((1+u^{2}+2u^{3})^{4}+u'^{2}))$$

$$C[x][y]/(y^{2}-((x^{3}+x+2)^{4}+1)) \leftarrow \rightarrow C[x][v]/(v'^{2}-((1+u^{2}+2u^{3})^{4}+u'^{2}))$$

$$C[x][y]/(y^{2}-((x^{3}+x+2)^{4}+1)) \leftarrow \rightarrow C[x][v]/(v'^{2}-((1+x^{4})))$$

Curves in Jacobians: $A = P_{rym}(C_s/E)$? $C_s \downarrow AJc_s \qquad \downarrow AJc_s$

Prop. 1. $\# \ker f^* < +\infty$; 2. $\pi \circ AJc_s : C_s \longrightarrow A$ is gen injective; 3. The image of $C_s \longrightarrow A$ is smooth outside the non-injective locus.