Eine Woche, ein Beispiel

2.16 lines passing a point

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.
[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

These are perhaps too well-known. But I should record it.

Typical question:

In a hypersurface $X \subset \mathbb{P}^n$,
how many lines $l \cong \mathbb{P}^1$ pass a given point $p \in X$?

Affine version:

In a (conical) hypersurface $X \subset \mathbb{C}^{n+1}$,
how many planes $l \cong \mathbb{C}^2$ contain a given line $p \cong \mathbb{C} \subseteq X$?

1. Method
2. Lines on cubic threefold
3. Lines on quadrics

4. Lines passing through lines

1. Method

Slogan: write down the coordinates explicitly.

w.l.o.g. let $p = [1:0:\cdots:0]$ and $X = \{f=0\}$, where

$$f(z_0, \dots, z_n) = \sum_{i=0}^{d} g_{d-i}(z_1, \dots, z_n)\, z_0^i$$

$g_{d-i}(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$ are homo of degree $d-i$,
and $g_0(z_1, \dots, z_n) = 0$.

Suppose that $l = \langle (1,0,\dots,0), (0, x_1, \dots, x_n) \rangle_{\mathbb{C}\text{-v.s.}}$, then

$$\begin{aligned} & l \subseteq X \\ \Leftrightarrow\ & f(t, x_1, \dots, x_n) \equiv 0 \qquad \forall t \in \mathbb{C} \\ \Leftrightarrow\ & g_i(x_1, \dots, x_n) \equiv 0 \qquad \forall i \in \{1, \dots, d\} \end{aligned}$$

Therefore,

$$\begin{aligned} & \{ l \cong \mathbb{C}^2 \subseteq \mathbb{C}^{n+1} \mid p \in l \subseteq X \} \\ \cong\ & \{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid g_{d-i}(x_1, \dots, x_n) = 0 \quad \forall i \} \\ \cong\ & \{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid \frac{\partial^i f}{\partial z_0^i}(0, x_1, \dots, x_n) = 0 \quad \forall i \} \end{aligned}$$

Here, $\mathbb{CP}^{n-1} = Gr(p^{\perp}, 1)$.

When X is sm at p, $(\nabla f)(p) \neq 0$.
w.l.o.g. let $(\nabla f)(p) = (0, \dots, 1)$, then

$$\begin{cases} T_p X = \{z_n = 0\} \cong \mathbb{C}^n \\ g_1(z_1, \dots, z_n) = z_n \end{cases}$$

In ptc, $p \in l \subseteq X \Rightarrow l \subseteq T_p X$.

2. Lines on cubic threefold

https://math.stackexchange.com/questions/3605767/number-of-lines-passing-a-point-on-a-cubic-threefold

Prop. Generically, there are 6 lines in a cubic threefold passing a given pt.

Proof. w.l.o.g. suppose $p = [1:0:0:0:0]$. $T_pX = \{z_4 = 0\}$, then

$$\{l \mid p \in l \subseteq X\}$$

$$\cong \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{CP}^3 \mid x_4 = g_2(x_1, x_2, x_3, x_4) = g_3(x_1, x_2, x_3, x_4) = 0\}$$
$$\cong \{[x_1 : x_2 : x_3] \in \mathbb{CP}^2 \mid g_2(x_1, x_2, x_3, 0) = g_3(x_1, x_2, x_3, 0) = 0\}$$

has generically 6 pts. (will we get $g_2 | g_3$ all the time for some specific cubic threefold?)

Rmk. Generically, passing a given pt,
there are 24 lines in a quartic fourfold,
5! lines in a quintic fivefold,
⋮
n! lines in a degree n n-fold.

dim $n-1$ \ dim d	1	2	3	4	5	6	...
0	·	· ·	⋰	⁘	· · · · ·	· · · · · ·	· · ·
1	$\mathbb{P}^1$	twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	$\mathbb{P}^2$	conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	cubic surface	K3 surface			
3	$\mathbb{P}^3$	conical	cubic threefold				
4	$\mathbb{P}^4$	conical					
5	$\mathbb{P}^5$	⋮					

general type ⇑

Fano ⇐ Calabi-Yau

uniruled by $\mathbb{P}^1$ | uniruled by conics

3. Lines on quadrics.

In this case,

$$\{ l \mid p \in l \subseteq X \}$$
$$\cong \{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid x_n = g_2(x_1, \ldots, x_n) = 0 \}$$
$$\cong \{ [x_1 : \cdots : x_{n-1}] \in \mathbb{CP}^{n-2} \mid g_2(x_1, \ldots, 0) = 0 \}$$

is again a quadric of dim $n-3$. (generically) $n \geq 3$

$n = 1, 2$ $\quad \emptyset$ (generically) empty

$F_1(X) = \{ l \subseteq X \text{ lines} \}$

Cor. For $n \geq 3$,

$$\dim F_1(X) = n-3 + n-1 - 1 = 2n-5$$
$$= 2(n-1) - 3$$

dim $n-1$ \ dim d	1	2	3	4	5	6	...
0	·	· ·	· · ·	· · · ·	· · · · ·	· · · · · ·	...
1	0 $\mathbb{P}^1$	$\emptyset$ twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	2 $\mathbb{P}^2$	1 Conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	0 cubic surface				
3	4 $\mathbb{P}^3$	3 Conical	2 cubic threefold				
4	6 $\mathbb{P}^4$	5 conical		3			
5	8 $\mathbb{P}^5$	7 ⋮			4		

general type

Fano | Calabi-Yau

uniruled by $\mathbb{P}^1$ | uniruled by conics

$\dim_{\mathbb{C}} F_1(X)$

In general, one can compute r-planes ($\cong \mathbb{P}^r$) on X passing P:

$$
\begin{aligned}
&\{ e \cong \mathbb{C}^{r+1} \mid p \in e \subseteq X \} \\
\cong\ & \{ e \in Gr(n,r) \mid x_n = g_2(x_1,\dots,x_n) = 0 \quad \forall (x_1,\dots,x_n) \in e \} \\
\cong\ & \{ e \in Gr(n-1,r) \mid g_2(x_1,\dots,x_{n-1},0) = 0 \quad \forall (x_1,\dots,x_{n-1}) \in e \} \\
\cong\ & F_{r-1}(X') \qquad \dim X' = \dim X - 2
\end{aligned}
$$

$\Rightarrow$ when $F_{r-1}(X') \neq \emptyset$ generically,

$$
\begin{aligned}
\dim F_r(X) &= \dim F_{r-1}(X') + \dim^{proj} X - r \\
&= \dim F_{r-1}(X') + (n-1) - r \\
&= \dim F_{r-1}(X') + n - r - 1 \\
&= n-r-1 + ((n-2)-(r-1)-1) + \cdots ((n-2(r-1)) - (r-(r-1)) - 1) \\
&\qquad + \dim F_0(X^{(r)}) \\
&= n-r-1 + (n-r-2) + \cdots + (n-2r) \\
&\qquad + \dim^{proj} X^{(r)} \\
&= \tfrac{1}{2}(2n-3r-1)\, r \quad + n - 2r - 1 \\
&= \tfrac{1}{2}(2n-3r-2)(r+1)
\end{aligned}
$$

dim $n-1$ \ dim d	1	2	3	4	5	6	…
0	·	· ·	· · ·	· · · ·	· · · · ·	· · · · · ·	· · ·
1	$\emptyset$ $\mathbb{P}^1$	$\emptyset$ twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	0 $\mathbb{P}^2$	$\emptyset$ conical $\cong \mathbb{P}^1\times\mathbb{P}^1$	$\emptyset$ cubic surface				
3	3 $\mathbb{P}^3$	$\emptyset$ conical	$\emptyset$ cubic threefold				
4	6 $\mathbb{P}^4$	3 conical		$\emptyset$			
5	9 $\mathbb{P}^5$	6 ⋮			$\emptyset$		

general type $\Uparrow$

Fano $\Leftarrow$ | Calabi-Yau

uniruled by $\mathbb{P}^1$ | uniruled by conics

$3((n-1)-3)$

$\dim_{\mathbb{C}} F_2(X)$

4. Lines passing through lines

Typical question:

In a hypersurface $X \subset \mathbb{P}^n$,
how many lines $l \cong \mathbb{P}^1$ pass a given line $l_0 \subseteq X$?

Affine version:

In a (conical) hypersurface $X \subset \mathbb{C}^{n+1}$,
how many planes $l \cong \mathbb{C}^2$ intersect a given plane $l_0 \cong \mathbb{C} \subseteq X$ non-trivially?

w.l.o.g. let $l_0 = [* : * : 0 : \cdots : 0]$ and $X = \{f = 0\}$, where

$$f(z_0, \ldots, z_n) = \sum_{\substack{i,j \\ i+j \le d}} a_{ij}(z_2, \ldots, z_n)\, z_0^i z_1^j$$

$$a_{ij}(z_1, \ldots, z_n) \in \mathbb{C}[z_1, \ldots, z_n] \quad \text{are homo of degree } d-i-j$$

and

$$l_0 \subseteq X \iff a_{ij}(z_1, \ldots, z_n) = 0 \quad \text{when } d = i+j$$

Therefore,

$$f(z_0, \ldots, z_n) = \sum_{\substack{i,j \\ i+j < d}} a_{ij}(z_2, \ldots, z_n)\, z_0^i z_1^j .$$

We want to restrict f to a plane $e = \mathbb{P}^2$ containing l.
Suppose $e = \{z_i = k_i w \mid i = 2, \ldots, n\}$ for some $k_i \in \mathbb{C}$, then

$$f|_e = \sum_{\substack{i,j \\ i+j<d}} a_{ij}(k_2 w, \ldots, k_n w)\, z_0^i z_1^j$$

$$= \sum_{\substack{i,j \\ i+j<d}} a_{ij}(k_2, \ldots, k_n)\, z_0^i z_1^j\, w^{d-i-j}$$

$$= \omega\left(\sum_{\substack{i,j,k \\ i+j+k=d-1}} a_{ij}(k_2, \dots, k_n)\, z_0^i z_1^j \omega^k\right)$$

That means, $X \cap e = l \cup \mathcal{C}$
for some curve $\mathcal{C}$ of degree $d-1$.

When $d = 3$, $\mathcal{C}$ is a conic, and

$$\sum_{\substack{i,j,k \\ i+j+k=d-1}} a_{ij}(k_2, \dots, k_n)\, z_0^i z_1^j \omega^k = (z_0, z_1, \omega) \underbrace{\begin{pmatrix} a_{20} & \frac{a_{11}}{2} & \frac{a_{10}}{2} \\ \frac{a_{11}}{2} & a_{02} & \frac{a_{01}}{2} \\ \frac{a_{10}}{2} & \frac{a_{01}}{2} & a_{00} \end{pmatrix}}_{= A} \begin{pmatrix} z_0 \\ z_1 \\ \omega \end{pmatrix}$$

Moreover,

$\mathcal{C}$ is singular	$\Leftrightarrow$	$\det A = 0$
$\mathcal{C}$ splits as two distinct lines	$\Leftrightarrow$	$\operatorname{rk} A = 2$
$\mathcal{C}$ splits as two identity lines	$\Leftrightarrow$	$\operatorname{rk} A = 1$.

Notice that $\det A \in \mathbb{C}[k_2, \dots, k_n]$ is a homo poly of degree 5,
so gives a hyperplane in $\mathbb{CP}^{n-2}$ with degree 5.

Rmk.

For a smooth cubic threefold, would we find a plane, such that the intersection is a union of line and two identified line?
No. Since the cubic threefold is smooth, all its plane sections must be reduced curves.

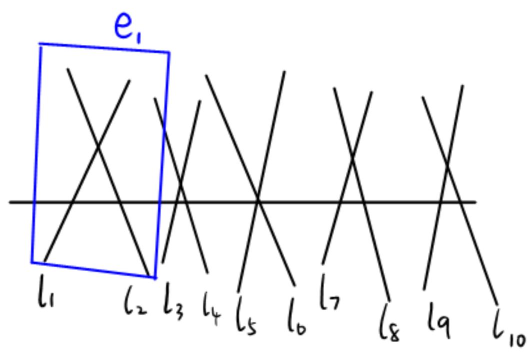
Therefore,

$$\begin{tikzcd}[column sep=small] & \{l \subset X \mid l \cap l_0 = \overset{\text{for some } p}{p}\} \subseteq F \subseteq Gr(n+1,2) \arrow[dl, "2:1 \text{ unramified}"'] \arrow[dr] & \\ \{\det A = 0\} \subseteq \underset{\text{containing } l}{\mathbb{CP}^{n-2}} \subseteq Gr(n+1,3) & & l \subseteq Gr(n+1,1) \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & l \arrow[dl] \arrow[dr] & \\ l + l_0 & & l \cap l_0 \end{tikzcd}$$

E.g. $n=3$ cubic surface case

$$\begin{tikzcd}[column sep=small] & \{l_1, \dots, l_{10}\} \arrow[dl, "2:1"'] \arrow[dr] & \\ \{e_1, \dots, e_5\} & & l \end{tikzcd}$$



E.g. $n=4$ cubic threefold case

arith genus 19

$$\begin{tikzcd}[column sep=small] & \{l \subset X \mid l \cap l_0 = p\} \arrow[dl, "2:1"'] \arrow[dr, "\pi"', "\substack{5:1 \\ \text{ramified}}"] & \\ \{\det A = 0\} & & l \end{tikzcd}$$

deg 5
arith genus 10

Q: Let \$f:X->Y\$ be a map of algbraic curves(maybe singular), where \$f\$ is a unramified covering of degree 2. The arithmetic genus of \$Y\$ is 10, so what is the range of the arithmetic genus of \$X\$?

A: (guess) still 19. A geometrical intuition may confirm the RH formula still applies.
A by chatgpt: The arithmetic genus of X can be any integer between 9 and 19 (in particular, if Y is smooth then $p_a(X) = 19$, while for a maximally nodal Y one may have $p_a(X) = 9$).

However, when Y is allowed to be singular the situation is a bit subtler. (Here we assume Y is an irreducible projective curve with only nodal singularities, so that $p_a(Y)=g(\tilde{N}) + \bar{\delta}$ where $\bar{\delta}$ is the number of nodes and $\tilde{N}$ is the normalization.) An "admissible" unramified double cover $f : X \longrightarrow Y$ will then lift to an étale double cover of the normalizations (so that the geometric genus of X is $2g(\tilde{N}) - 1$). Moreover, over a node of Y there are two possibilities. In local terms one may have either "non-splitting" – so that a node of Y gives rise to a single node in X – or "splitting" – so that the pre-image consists of two nodes. Writing r for the number of nodes of Y that split (with $0 \le r \le \bar{\delta}$) one can show that

$$p_a(X) = 2p_a(Y) - 1 - \bar{\delta} + r.$$

Since for an (irreducible) nodal curve one always has $\bar{\delta} \le 10$ and r can vary between 0 and $\bar{\delta}$, the highest value is obtained when $\bar{\delta} = 0$ (or when every node splits, so $r = \bar{\delta}$) giving $p_a(X) = 19$, while the lowest occurs when Y is as singular as possible (e.g. $\bar{\delta} = 10$) with no splitting ($r = 0$), so that $p_a(X) = 19 - 10 = 9$.

When $\{\det A = 0\}$ is sm, π is ramified at 46 pts (counted with multiplicity)

Q: Is $\{l \subset X \mid l \cap l_0 = p \text{ for some } p\}$ connected ?