ein Woche, eine Beispiel
April 16th.   examples in algebraic topology

Examples:
Past | Today | Future
  closed surface dim 2 | S^n S^∞ | Lens space
  Hopf surface dim 4 | $\mathbb{RP}^n$ $\mathbb{RP}^\infty$ | Lie   group
  K3   surface | $\mathbb{CP}^n$ $\mathbb{CP}^\infty$ | Grassmannian mfld , e.g $Gr(2,4)$
                        | ... |
                        | | Moore space
                        | | Eilenberg - Maclane space
                        | | low-dimensional CW-cplx
                        | | ...

Goal.   · compute $H_n(X,\mathbb{Z})$, $H^*(X,\mathbb{Z})$, $\pi_n(X,\mathbb{Z})$ ← Whitehead bracket
               · compute characteristic class and applies the results.
               · optional question. is   X * oriented?
                                          * a mfld? of dim n
                                          * a cplx mfld? ←── proj
                                          * a Lie group? |
                                          * ......   kähler
                                                                   |
                                                                complex

Today: S^n S^∞; $\mathbb{RP}^n$, $\mathbb{RP}^\infty$; $\mathbb{CP}^n$, $\mathbb{CP}^\infty$; ...

$$S^\infty := \bigcup_{n \geq 1} S^n \qquad S^n \hookrightarrow S^m \quad \text{by} \quad (x_0, \cdots, x_n) \longmapsto (x_0, \cdots, x_n, 0, \cdots, 0)_{\uparrow m}$$

1. relations: fiber bundle

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & S^n \\ & & \downarrow \\ & & \mathbb{RP}^n \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & \mathbb{CP}^n \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/k\mathbb{Z} & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & S^{2n+1}/\mathbb{Z}/k\mathbb{Z} \end{array} \quad k \in \mathbb{N}^+, k > 1$$

$$[n=1: \text{Hopf fibration}]$$

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} & \longrightarrow & S^\infty \\ & & \downarrow \\ & & \mathbb{RP}^\infty \end{array} \qquad \begin{array}{ccc} S^1 & \longrightarrow & S^\infty \\ & & \downarrow \\ & & \mathbb{CP}^\infty \end{array} \qquad \begin{array}{ccc} \mathbb{Z}/k\mathbb{Z} & \longrightarrow & S^\infty \\ & & \downarrow \\ & & S^\infty/\mathbb{Z}/k\mathbb{Z} \end{array}$$

2. (canonical) CW structure.

e.g

# m-cell	0	1	2	3	4	5	$m > 5$
S^5	2	2	2	2	2	2	0
$\mathbb{RP}^5$	1	1	1	1	1	1	0
$\mathbb{CP}^2$	1	0	1	0	1	0	0

$$\Rightarrow \begin{cases} \chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases} \\ \chi(\mathbb{RP}^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \\ \chi(\mathbb{CP}^n) = n+1 \end{cases}$$

3. Homology & Cohomology

homology

$H_i(X, \mathbb{Z})$	0	1	2	3	4	5	$i > 5$
S^5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0
$\mathbb{RP}^5$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0
$\mathbb{CP}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0
$\mathbb{RP}^4$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0

Cor. $\mathbb{RP}^{2n}$ is nonoriented; $\mathbb{RP}^{2n+1}$, S^n, $\mathbb{CP}^n$ are oriented.

S^5: $0 \to \mathbb{Z}e_1^5 \oplus \mathbb{Z}e_2^5 \to \mathbb{Z}e_1^4 \oplus \mathbb{Z}e_2^4 \to \mathbb{Z}e_1^3 \oplus \mathbb{Z}e_2^3 \to \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2^2 \to \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \to \mathbb{Z}e_1^0 \oplus \mathbb{Z}e_2^0 \to 0$

$$e_1^5 \longmapsto e_1^4 - e_2^4 \qquad e_1^3 \longmapsto e_1^2 - e_2^2 \qquad e_1^1 \longmapsto e_1^0 - e_2^0$$
$$e_2^5 \longmapsto -e_1^4 + e_2^4 \qquad e_2^3 \longmapsto -e_1^2 + e_2^2 \qquad e_2^1 \longmapsto -e_1^0 + e_2^0$$
$$e_1^4 \longmapsto e_1^3 + e_2^3 \qquad e_1^2 \longmapsto e_1^1 + e_2^1$$
$$e_2^4 \longmapsto e_1^3 + e_2^3 \qquad e_2^2 \longmapsto e_1^1 + e_2^1$$

$$\mathbb{Z} \qquad 0 \qquad 0 \qquad 0 \qquad 0 \qquad \mathbb{Z}$$

[Rmk. The definition of cellular homology uses the (singular) homology of S, so seriously
we can't compute $H_i(S^n, \mathbb{Z})$ by cellular homology.]

$\mathbb{RP}^5$: $0 \longrightarrow \mathbb{Z}e^5 \longrightarrow \mathbb{Z}e^4 \longrightarrow \mathbb{Z}e^3 \longrightarrow \mathbb{Z}e^2 \longrightarrow \mathbb{Z}e^1 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$e^5 \longmapsto 0 \qquad e^3 \longmapsto 0 \qquad e^1 \longmapsto 0$

$e^4 \longmapsto 2e^3 \qquad e^2 \longmapsto 2e^1$

$\mathbb{Z} \qquad 0 \qquad \mathbb{Z}/_2\mathbb{Z} \qquad 0 \qquad \mathbb{Z}/_2\mathbb{Z} \qquad \mathbb{Z}$

$\mathbb{RP}^4$: $0 \longrightarrow \mathbb{Z}e^4 \longrightarrow \mathbb{Z}e^3 \longrightarrow \mathbb{Z}e^2 \longrightarrow \mathbb{Z}e^1 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$e^3 \longmapsto 0 \qquad e^1 \longmapsto 0$

$e^4 \longmapsto 2e^3 \qquad e^2 \longmapsto 2e^1$

$0 \qquad \mathbb{Z}/_2\mathbb{Z} \qquad 0 \qquad \mathbb{Z}/_2\mathbb{Z} \qquad \mathbb{Z}$

$\mathbb{CP}^2$: $0 \longrightarrow \mathbb{Z}e^4 \longrightarrow 0 \longrightarrow \mathbb{Z}e^2 \longrightarrow 0 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$\mathbb{Z} \qquad 0 \qquad \mathbb{Z} \qquad 0 \qquad \mathbb{Z}$

Similarly, $H_n(S^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$

$H_n(\mathbb{RP}^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/_2\mathbb{Z} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \qquad H_n(\mathbb{RP}^\infty, \mathbb{Z}/_2\mathbb{Z}) = \mathbb{Z}/_2\mathbb{Z}$

$H_n(\mathbb{CP}^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$

cohomology

$H^i(X, \mathbb{Z})$	0	1	2	3	4	5	$i > 5$
S^5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0
$\mathbb{RP}^5$	$\mathbb{Z}$	0	$\mathbb{Z}/_2\mathbb{Z}$	0	$\mathbb{Z}/_2\mathbb{Z}$	$\mathbb{Z}$	0
$\mathbb{CP}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0
$\mathbb{RP}^4$	$\mathbb{Z}$	0	$\mathbb{Z}/_2\mathbb{Z}$	0	$\mathbb{Z}/_2\mathbb{Z}$	0	0

$\Rightarrow \begin{cases} H^*(\mathbb{RP}^{2n}) = \mathbb{Z}[x]/_{(2x, x^{n+1})} \\ H^*(\mathbb{RP}^{2n+1}) = \mathbb{Z}[x]/_{(2x, x^{n+1})} \oplus \mathbb{Z}y \\ H^*(\mathbb{CP}^n) = \mathbb{Z}[x]/_{(x^{n+1})} \end{cases}$

$\boxed{\begin{array}{l} \deg x = 2 \\ \deg y = 5 \\ \deg t = 1 \end{array}}$ $\Rightarrow \begin{cases} H^*(\mathbb{RP}^\infty) = \mathbb{Z}[x]/_{(2x)} \\ H^*(\mathbb{CP}^\infty) = \mathbb{Z}[x] \\ H^*(\mathbb{RP}^\infty, \mathbb{Z}/_2\mathbb{Z}) = \mathbb{Z}/_2\mathbb{Z}[t] \\ H^*(\mathbb{CP}^\infty, \mathbb{Z}/_2\mathbb{Z}) = \mathbb{Z}/_2\mathbb{Z}[x] \end{cases}$

prod structure: use Poincaré duality & cellular cohomology. See [May, P153].

$H^{\geq q}(\mathbb{CP}^n) \xrightarrow{\sim} H^{\geq q}(\mathbb{CP}^{n-1})$ for $q < n$

By spectral sequence: GTM 82 Example 14.22, 14.32, Ex 18.4, 18.10

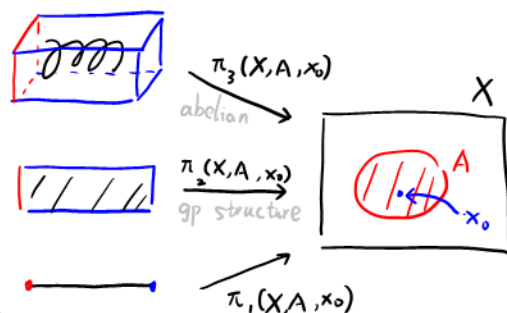**Interlude: LES of homotopy groups** $x_0 \in B \subset A \subset X$, X top space

Def. relative homotopy group
$$\pi_n(X, A, x_0) = \left[f: (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0) \right] \qquad n \geq 1 \quad J^{n-1} := \partial I^n \smallsetminus I^{n-1}$$

Relations: $f \sim g \iff \exists F_t: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ s.t
(denote by $[f] = [g]$) $F_0 = f \qquad F_1 = g$

Lemma: Suppose $f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$,
  $f(I^n) \subseteq A$, then $[f] = [0]$

Thm. we have LES $\leftarrow \ker = \mathrm{Im}$

$\pi_3(X, A, x_0)$ abelian

$\pi_2(X, A, x_0)$ gp structure

$\pi_1(X, A, x_0)$

X, A, x_0

$$\hookrightarrow \pi_2(A, B, x_0) \longrightarrow \pi_2(X, B, x_0) \longrightarrow \pi_2(X, A, x_0) \rightharpoonup$$
$$\hookrightarrow \pi_1(A, B, x_0) \longrightarrow \pi_1(X, B, x_0) \longrightarrow \pi_1(X, A, x_0)$$

Cor. we have LES
$$\pi_2(A, x_0) \longrightarrow \pi_2(X, x_0) \longrightarrow \pi_2(X, A, x_0) \rightharpoonup$$
$$\hookrightarrow \pi_1(A, x_0) \longrightarrow \pi_1(X, x_0) \longrightarrow \pi_1(X, A, x_0) \rightharpoonup$$
$$\hookrightarrow \pi_0(A, x_0) \longrightarrow \pi_0(X, x_0)$$

called Serre fibration

Thm. when $p: E \to B$ has the homotopy lifting property w.r.t. I^k ($\forall k \geq 0$)
  then (denote $b_0 \in B$, $x_0 \in F := p^{-1}(B)$)
$$p_*: \pi_n(E, F, x_0) \longrightarrow \pi_n(B, b_0) \qquad n \geq 1$$
  is an isomorphism.

$$\begin{array}{ccc} I^k \times [0,1] & \longrightarrow & E \ni x_0 \\ \downarrow & {}_{\exists}\nearrow & \downarrow p \\ I^k & \longrightarrow & B \ni b_0 \end{array}$$

Rmk. 1 The proof mainly uses the HLP: homotopy lifting property.
  2. Any fiber bundle $p: E \to B$ is a Serre fibration.
Cor Suppose $p: E \to B$ is the fiber bundle map. then we have LES
  (denote $b_0 \in B$, $x_0 \in F := p^{-1}(B)$)
$$\pi_2(F, x_0) \longrightarrow \pi_2(E, x_0) \longrightarrow \pi_2(B, b_0) \rightharpoonup$$
$$\hookrightarrow \pi_1(F, x_0) \longrightarrow \pi_1(E, x_0) \longrightarrow \pi_1(B, b_0) \rightharpoonup$$
$$\hookrightarrow \pi_0(F, x_0) \longrightarrow \pi_0(E, x_0)$$

4. Homotopy: by LES of fibration, we obtain $n \geq 2$

$$\pi_m(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/_2\mathbb{Z} & m = 1 \\ \pi_m(S^n) & m > 1 \end{cases} \qquad \pi_m(\mathbb{CP}^n) = \begin{cases} 0 & m = 1 \\ \mathbb{Z} & m = 2 \\ \pi_m(S^{2n+1}) & m > 2 \end{cases}$$

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n$$

Rmk. S^∞ is contractable by the argument in   https://mathoverflow.net/questions/198
Cor. $\mathbb{RP}^\infty$ is of type $K(\mathbb{Z}/_2\mathbb{Z}, 1)$
  $\mathbb{CP}^\infty$ is of type $K(\mathbb{Z}, 2)$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
S^0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

in GTM 82
What I can prove now
by Hopf fibration

$\pi_7(S^7)$

5. Characteristic class

We have both tautological vector bundle and tangent bundle for S^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$.

$\mathbb{CP}^n$: by https://en.wikipedia.org/wiki/Chern_class ,

$$c(\mathbb{CP}^n) \overset{\text{def}}{=} c(T\mathbb{CP}^n) = c(\mathcal{O}_{\mathbb{CP}^n}(1))^{n+1} = (1+a)^{n+1},$$

where a is the canonical generator of the cohomology group $H^2(\mathbb{CP}^n, \mathbb{Z})$;

tautological bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$: $c(\mathcal{O}_{\mathbb{CP}^n}(-1)) = 1-a$

Cor. $T\mathbb{CP}^n$, $\mathcal{O}_{\mathbb{CP}^n}(-1)$ are not spin; $\mathbb{CP}^n$ is not a boundary.

$\mathbb{RP}^n$: similarly, $w(\gamma_n') = 1+t$ $\quad$ $w(\mathbb{RP}^n) = w(\gamma_n')^{n+1} = (1+t)^{n+1}$

Cor. γ_n' is not orientable ;
$\quad$ $T\mathbb{RP}^n$ is orientable only when $n \equiv 1 \bmod 2$;
$\quad$ $T\mathbb{RP}^n$ is spin only when $n \equiv 3 \bmod 4$ or $n = 1$.

S^n: Lemma. $\pi^* : H^n(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^n(S^n, \mathbb{Z}/2\mathbb{Z})$ is zero.

Proof. by computation.

$C_\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \qquad 0 \longrightarrow e^5 \longrightarrow e^4 \longrightarrow e^3 \longrightarrow e^2 \longrightarrow e^1 \longrightarrow e^0 \longrightarrow 0$

$e_1^5 \mapsto e^5$
$e_2^5 \mapsto e^5$

$C_\cdot(S^5, \mathbb{Z}/2\mathbb{Z}) \qquad 0 \longrightarrow e_1^5, e_2^5 \longrightarrow e_1^4, e_2^4 \longrightarrow e_1^3, e_2^3 \longrightarrow e_1^2, e_2^2 \longrightarrow e_1^1, e_2^1 \longrightarrow e_1^0, e_2^0 \longrightarrow 0$

$e_1^5 \longmapsto e_1^4 - e_2^4$
$e_2^5 \longmapsto -e_1^4 + e_2^4$

$C^\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \qquad 0 \longleftarrow e^{5*} \longleftarrow e^{4*} \longleftarrow e^{3*} \longleftarrow e^{2*} \longleftarrow e^{1*} \longleftarrow e^{0*} \longleftarrow 0$

$e^{5*} \mapsto e_1^{5*} + e_2^{5*}$

$C^\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \qquad 0 \longleftarrow e_1^{5*}, e_2^{5*} \longleftarrow e_1^{4*}, e_2^{4*} \longleftarrow e_1^{3*}, e_2^{3*} \longleftarrow e_1^{2*}, e_2^{2*} \longleftarrow e_1^{1*}, e_2^{1*} \longleftarrow e_1^{0*}, e_2^{0*} \longleftarrow 0$

$e_1^{5*} - e_2^{5*} \longleftarrow e_1^{4*}$
$-e_1^{5*} + e_2^{5*} \longleftarrow e_2^{4*}$

btw. when n is odd, $H^n(\mathbb{RP}^n, \mathbb{Z}) \longrightarrow H^n(S^n, \mathbb{Z})$
$\qquad\qquad \| \qquad\qquad\qquad \|$
$\qquad\qquad \mathbb{Z} \overset{\times 2}{\longrightarrow} \mathbb{Z}$

Cor. $w(\gamma_{n,S^n}') = \pi^* w(\gamma_n', \mathbb{RP}^n) = 1$
$\qquad w(TS^n) = \pi^* w(T\mathbb{RP}^n) = 1$
$\qquad \gamma_{n,S^n}', TS^n$ are spin, $S^n = \partial D^n$.

6. Cplx mfld

$\mathbb{C}P^n$ is undoubtedly proj cplx mfld.

$\mathbb{R}P^{2n-1}$, S^{2n-1} are not cplx mflds since they're of odd dim.

$\mathbb{R}P^{2n}$ is not cplx mfld since it's not orientable.

S^n $(n>6)$, S^4 are not cplx mflds. see https://mathoverflow.net/questions/11664/complex-structure-on-sn

whether S^6 is a cplx mfld is still an open problem, see
https://mathoverflow.net/questions/1973/is-there-a-complex-structure-on-the-6-sphere

related problems: is the cplx structure of $\mathbb{C}P^n$ unique? Still open, see
https://mathoverflow.net/questions/382442/is-the-complex-structure-of-mathbb-cpn-unique

7. Lie group:

S^1, S^3, $\mathbb{R}P^1$, $\mathbb{R}P^3$, we have $\mathbb{R}P^1 \cong S^1$ and
$$S^3 \cong SU_2 \cong \{q \in \mathbb{H} \mid |q| = 1\}$$
$\downarrow \pi \quad \downarrow \pi'$

$\mathbb{R}P^3 \cong SO_3$

https://math.stackexchange.com/questions/4065801/collecting-proofs-so3-cong-bbb-r-p3
But a better way to see it is here: https://www.youtube.com/watch?v=ACZC_XEyg9U

for S^n: https://math.stackexchange.com/questions/12453/is-there-an-easy-way-to-show-which-spheres-can-be-lie-groups

for $\mathbb{R}P^n$: lemma: a Lie/topological group structure lifts to a covering space
   proof: see https://math.stackexchange.com/questions/5391/covering-of-a-topological-group-is-a-topological-group
   Cor: $\mathbb{R}P^n$ $(n>3)$ is not a Lie group

for $\mathbb{C}P^n$: lemma: for the connected Lie group G, $\pi_2(G) = 0$ $\pi_3(G)$ has no torsion!
   proof: see https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups
   Cor. $\mathbb{C}P^n$ is not a Lie group.
   different proof of this Cor: https://math.stackexchange.com/questions/3043483/lie-group-structure-on-the-complex-projective-space

Interesting results during the ways of searching
   Lemma: a cpt Lie group is either abelian $\Rightarrow$ torus
                                     or nonabelian & have nonzero H^3.

   See   https://math.stackexchange.com/questions/3421788/topological-lie-group-structure-on-projective-spaces
   Lemma   every compact Lie group has zero Euler characteristic since it is parallelizable
   See   https://math.stackexchange.com/questions/829928/can-s2-be-turned-into-a-topological-group/