§ 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections
4. L-fct
5. density theorem

Just for convenience, we allow

element $\in_c$ class   class $\subset_c$ class $\{\cdots | \cdots\}_c$ be a class

We may add c to emphasize that the family can be a class, instead of set.

1. Galois rep
Setting G: arbitrary topo gp   e.g. G any Galois gp
    If G profinite $\Rightarrow$ open subgps are finite index subgps.
    Λ: top field   e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \text{rep}_{\Lambda, \text{cont}}(G)$
    $V \in \text{vect}_\Lambda$ $+$ $\rho: G \longrightarrow GL(V)$ cont

▽ $\rho(G)$ can be infinite!   for Gal gp
    E.g.   When char $F \neq l$, we have l-adic cyclotomic character
    $\varepsilon_l: \text{Gal}(F^{\text{sep}}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \overline{\mathbb{Q}_l^\times}$ $\sigma \longmapsto \varepsilon_l(\sigma)$ satisfying

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

    This is cont by def. (Take usual topo.)
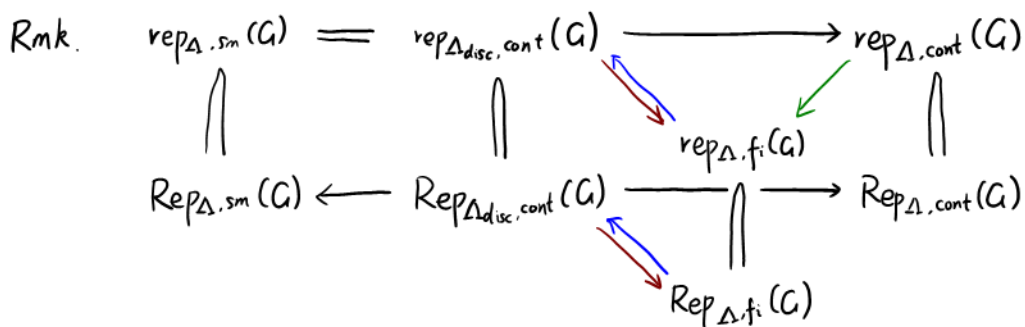   Ex: Compute ε_l for $F = \mathbb{F}_p$.
    A: $\varepsilon_l: \widehat{\mathbb{Z}} \cong \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times$ $1 \longmapsto p$
        $\uparrow$ lift from $\mathbb{Z} \to \mathbb{Z}_l^\times$

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \text{rep}_{\Lambda, \text{sm}}(G)$
    $V \in \text{vect}_\Lambda$ $+$ $\rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \text{rep}_{\Lambda, \text{fi}}(G)$ fi: finite image / finite index
    $V \in \text{vect}_\Lambda$ $+$ $\rho: G \longrightarrow GL(V)$ with finite image

Rmk.

$$\text{rep}_{\Delta,\text{sm}}(G) == \text{rep}_{\Delta_{\text{disc}},\text{cont}}(G) \longrightarrow \text{rep}_{\Delta,\text{cont}}(G)$$

$$\text{rep}_{\Delta,\text{fi}}(G)$$

$$\text{Rep}_{\Delta,\text{sm}}(G) \longleftarrow \text{Rep}_{\Delta_{\text{disc}},\text{cont}}(G) \longrightarrow \text{Rep}_{\Delta,\text{cont}}(G)$$

$$\text{Rep}_{\Delta,\text{fi}}(G)$$

$\longrightarrow$: if fin index subgps are open

$\longrightarrow$: if G profinite gp   (Only need: open $\Rightarrow$ fin index)

$\longrightarrow$: Artin rep (of profinite gp)

Artin rep: $\Delta = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)
    $\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.
        $\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U}$ $\Rightarrow$ $H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\quad \|\cdot\| = \|\cdot\|_{max}, \ |\cdot| = |\cdot|_{max}$
    Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$. $\exists m \in \mathbb{N}$, s.t $A^m \notin \mathcal{U}$.
    Consider the Jordan form of A.
        Case 1. A unipotent.
        Case 2. A not unipotent.
            $\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$.   Take $m \in \mathbb{N}$ s.t $|\lambda^m - 1| > \frac{1}{3}$.
            $\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \text{rep}_{\mathbb{C},\text{cont}}(G)$, $\rho(G)$ is finite. $\quad G$ profinite
Proof. Take $\mathcal{U}$ in Lemma 1, then
        $\rho^{-1}(\mathcal{U})$ is open $\underset{\text{Lemma1}}{\Longrightarrow}$ $\exists I \leq G_F$ finite index. $\rho(I) \subseteq \mathcal{U}$
                    $\Longrightarrow \rho(I) = Id$
                    $\Longrightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:
    1. ρ is conj to a rep valued in $GL_n(\bar{\mathbb{Q}})$
        [ρ can be viewed as cplx rep of fin gp. so ρ is semisimple.
        Since classifications of irr reps for $\mathbb{C}$ & $\bar{\mathbb{Q}}$ are the same,
        every irr rep is conj to a rep valued in $GL_n(\bar{\mathbb{Q}})$.]
    2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
       https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeleton of the Galois gp" in general.

Setting: Λ: NA local field with char $k_\Lambda = l$

Q: What would happen if Λ is only a NA local field?

Finite field

Goal. For Λ: NA local field with char $k_\Lambda = l$, understand $\text{rep}_{\Lambda,\text{cont}}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Lambda)$, TFAE:

① $\hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is a well-defined cont gp homo
$$1 \longmapsto A$$

② $\exists\, g \in GL_n(\Lambda)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Lambda)$

③ $\det(\lambda I - A) \in \mathcal{O}_\Lambda[\lambda]$, with $\det A \in \mathcal{O}_\Lambda^\times$

A is called bounded in these cases.

Proof
$$① \underset{\text{local}}{\overset{\text{local}}{\rightleftarrows}} ② \underset{\text{local}}{\rightleftarrows} ③$$

① $\Rightarrow$ ② : $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Lambda)$, which conjugates to $GL_n(\mathcal{O}_\Lambda)$

Another method:

Lemma 1. ρ, μ: two ways of expressions of gp action

$\rho : \hat{\mathbb{Z}} \longrightarrow GL_n(\mathbb{Z})$ is cont $\Leftrightarrow$ $\mu : \hat{\mathbb{Z}} \times \Lambda^n \longrightarrow \Lambda^n$ is cont

$\Rightarrow$: $\mu : \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\rho \times Id_{\Lambda^n}} GL_n(\Lambda) \times \Lambda^n \longrightarrow \Lambda^n$ is cont.

Λ^n is Haus loc cpt.

$\Leftarrow$: $\rho : \hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is cont

$\Leftrightarrow$ $\rho : \hat{\mathbb{Z}} \longrightarrow M_{n\times n}(\Lambda)$ is cont

$\Leftrightarrow$ $\rho_{ij} : \hat{\mathbb{Z}} \longrightarrow \Lambda$ is cont $\qquad \forall\, i,j \in \{1,...,n\}$

We know that

$\rho_{ij} : \hat{\mathbb{Z}} \xrightarrow{(Id, e_i)} \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\mu} \Lambda^n \xrightarrow{e_i^*} \Lambda$

is cont

linear map between f.d v.s is cont

In this case, e_i^* is projection.

Another $\Leftarrow$: (suggested by Longke Tang)

$$\Leftrightarrow \quad \mu : \hat{\mathbb{Z}} \times \underset{\overset{\hat{\mathbb{Z}}}{\downarrow}}{\Delta^n} \longrightarrow \Delta^n \quad \text{is cont} \qquad \overset{\text{open cpt topo}}{\swarrow}$$

$$\hat{\mathbb{Z}} \underset{\exists !}{\overset{}{\longrightarrow}} \underset{\cup}{Mor_{Top}}(\Delta^n, \Delta^n) \overset{\nwarrow}{\underset{\text{is cont}}{}}$$

$\longrightarrow GL_n(\Lambda)$

Only need: $GL_n(\Lambda) \subseteq M_{n \times n}(\Lambda)$, $GL_n(\Lambda) \subset Mor_{Top}(\Delta^n, \Delta^n)$
define the same topo on $GL_n(\Lambda)$.

This is hard. Assuming Lemma 1, this can be proved,
but then this method can't be a real proof for Lemma 1.

Lemma 2. $\mathcal{L}_1, \mathcal{L}_2$ lattice in $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$ lattice in Λ

$$\left[\begin{array}{l} \mathcal{L}_1 \supseteq (\mathfrak{p}^{k'})^{\oplus n} \\ \mathcal{L}_2 \supseteq (\mathfrak{p}^{k'})^{\oplus n} \end{array} \right\} \Rightarrow \# {}^{\mathcal{L}_1 + \mathcal{L}_2}/_{\mathcal{L}_1} < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice} \right]$$

Take $\mathcal{L} = \mathcal{O}_\Lambda^n \subseteq \Lambda^n$, then the stabilizer
$$\begin{aligned} Stab(\mathcal{L}) &= \{g \in \hat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L}\} \\ &= \{g \in \hat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \quad \forall i\} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where
$$\mu_{e_i} : \hat{\mathbb{Z}} \longrightarrow \Lambda^n \qquad g \longmapsto g \cdot e_i \qquad \text{(cont by Lemma 1)}$$

$\underset{\text{Lemma 2}}{\Longrightarrow}$ $\mathcal{L}$ has finite orbit

$\Longrightarrow$ $\sum_{\mathcal{L}_i \in \hat{\mathbb{Z}} \cdot \mathcal{L}} \mathcal{L}_i$ is a lattice stabilized by $\mathbb{Z}$.

After conjugation, $\qquad A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \qquad \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

② ⇒ ① : w.l.o.g. $A \in GL_n(\mathcal{O}_\Delta)$. Then we get a lift

$$
\begin{array}{ccc}
\hat{\mathbb{Z}} & \xrightarrow{\exists ! \text{ cont}} & \widehat{GL_n(\mathcal{O}_\Delta)} \cong GL_n(\mathcal{O}_\Delta) \\
\uparrow & & \uparrow \\
\mathbb{Z} & \longrightarrow & GL_n(\mathcal{O}_\Delta)
\end{array}
$$

② ⇒ ③ : *Obvious*

③ ⇒ ② :

$$\sum_{i \in \mathbb{Z}} A^i \mathcal{L} = \sum_{i=0}^{n-1} A^i \mathcal{L} \quad \text{is a lattice fixed by } A, A^{-1} \text{ (Lemma 2)}$$

After conjugation, $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \qquad \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

$\triangledown$ $A, B \in GL_n(\Lambda)$ bounded $\not\Rightarrow$ AB bounded

counter eg: (from Longke Tang)

$$\text{Consider} \quad A = \begin{pmatrix} p & \\ & 1 \end{pmatrix}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}, \quad B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \quad \text{then} \quad AB = \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}.$$

Cor. $\text{rep}_{\Delta, \text{cont}}(\hat{\mathbb{Z}}) \cong \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(\mathbb{Z})$

$\hookleftarrow$ full subcategory of $\text{rep}_{\Delta, \text{cont}}(\mathbb{Z})$.

Local field

Goal. For Λ: NA local field with char $k_\Lambda = l$,

        F: NA local field with char $k_F = p \neq l$,

realize cont Galois rep as bounded Weil-Deligne rep,

via the following diagrams:

$$
\begin{array}{ccc}
 & & \mathrm{rep}_{\Lambda, sm}(W_F) \quad \text{with } N \\
 & & \cup \\
\mathrm{rep}_{\Lambda, cont}(W_F) & \xrightarrow{\ \sim\ } & \mathrm{WDrep}_{\Lambda, sm}(W_F) \\
\cup & & \cup \\
\end{array}
$$

$$
\mathrm{rep}_{\Lambda, cont}(\Gamma_F) \xrightarrow{\ \sim\ } \mathrm{rep}^{bdd}_{\Lambda, cont}(W_F) \xrightarrow{\ \sim\ } \mathrm{WDrep}^{bdd}_{\Lambda, sm}(W_F)
$$

here, "bdd" means $\mathrm{Im}\,\rho$ are bounded.