

# Eine Woche, ein Beispiel

## 5.1 Extension of NA local field

F: NA local field

1 List of well-known results

- in general

- unramified / totally ramified

2.  $\hat{\mathbb{Z}}$  = profinite completion (review)

3. Big picture

4. A detailed discussion concerning proofs.

5. Henselian ring

} not complete, I need time to check the proof

6. Cohomological dimension

7. Bonus: "plane geometry" for  $\mathbb{Q}_q$ .

Q: Is there any subfield of  $\mathbb{Q}_p$  with finite index?

Can we classify all subfield of  $\mathbb{F}_p((t))$  with finite index?

<https://math.stackexchange.com/questions/211582/is-there-a-proper-subfield-k-subset-mathbb-r-such-that-mathbb-rk-is-fin>

Ref:

Initial motivation comes from

[AY]<https://alex-youcis.github.io/localglobalgalois.pdf>

which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:

[NSW2e]<https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/>

[JPS96] Galois cohomology by Jean-Pierre Serre

<http://p-adic.com/Local%20Fields.pdf>

<https://people.clas.ufl.edu/rcrew/files/LCFT.pdf>

<http://www.mcm.ac.cn/faculty/tianyichao/201409/Wo20140919372982540194.pdf>

For existence and uniqueness of extension of valuation, see Theorem 3.2 here:

[https://www.dpmms.cam.ac.uk/~ajs1005/ANT/notes\\_s3-4.pdf](https://www.dpmms.cam.ac.uk/~ajs1005/ANT/notes_s3-4.pdf)

## 1. List of well-known results

In general

$F$ : NA local field     $E/F$ : finite extension

Rmk 1.  $E$  is also a NA local field with uniquely extended norm

$$\|x\|_v = \|N_{E/F}(x)\|_F^{\frac{1}{n}} \quad \text{resp. } v(x) := \frac{1}{n} v_F(N_{E/F}(x))$$

$$\text{E.g. } \|1 - \zeta_n\| = 1 \text{ in } \mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p \quad p \nmid n \quad v(1 - \zeta_n) = 0$$

$$\|1 - \zeta_p\| = \frac{1}{p} \quad \text{in } \mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p \quad v(1 - \zeta_p) = \frac{1}{p}$$

$$\|1 - \zeta_5\| = \left\| (1 - \zeta_5)(1 - \zeta_5^2)(1 - \zeta_5^3)(1 - \zeta_5^4) \right\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \|1 - \zeta_5\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \frac{1}{\sqrt[5]{5}} \quad \text{in } \mathbb{Q}_5(\zeta_5)$$

$$\|1 - \zeta_{p^n}\| = p^{-\frac{1}{p^{n+1}}} \quad \text{in } \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p \quad v(1 - \zeta_{p^n}) = \frac{1}{p^{n+1}}$$

$\Rightarrow 1 - \zeta_{p^n}$  is a uniformizer of  $\mathbb{Q}_p(\zeta_{p^n})$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$  is monogenic, i.e.  $\mathcal{O}_E = \mathcal{O}_F[\alpha] \quad \exists \alpha \in \mathcal{O}_E$

A proof of this may be found here:

<https://math.stackexchange.com/questions/3406117/ring-of-integers-of-simple-field-extension-of-local-field-is-monogenic>

Cor. (primitive element thm for NA local field)

$$E = F[x]/(g(x)) \quad \exists x \in \mathcal{O}_E, g(x) \text{ min poly of } x.$$

Rmk: Every separable finite field extension has a primitive element, see wiki:

[https://en.wikipedia.org/wiki/Primitive\\_element\\_theorem](https://en.wikipedia.org/wiki/Primitive_element_theorem)

Separable condition is necessary, see

<https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element>

⚠  $\mathcal{O}_E$  may be not a free  $\mathcal{O}_F$ -module.

See: <https://kconrad.math.uconn.edu/blubs/gradnumthy/notfree.pdf>

Rmk 3. Any finite extension of  $\mathbb{Q}_p$  is of form  $\mathbb{Q}_p[x]/(g(x))$ ,

where  $g(x) \in \mathbb{Q}[x]$  is an irr poly.

Any finite extension of  $\mathbb{F}_q(t)$  is of form  $\mathbb{F}_q((t))[x]/(g(x))$

where  $g(x) \in \mathbb{F}_q((t))[x]$  is an irr poly..

Both are achieved by Krasner's lemma.

From [<https://math.mit.edu/classes/18.785/2017fa/LectureNotes11.pdf>]:

Remark 11.12. Krasner's lemma is another "Hensel's lemma" in the sense that it characterizes Henselian fields (fraction fields of Henselian rings);

<https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete>

$$v = v_F = \frac{1}{e} v_E \quad \| \cdot \| = \| \cdot \|_F = \| \cdot \|_E^{\frac{1}{e}} \quad \wp_F \mathcal{O}_E = \wp_E^e$$

$E$	$v_E = ev$	$\  \cdot \ _E = \  \cdot \ ^{e^{-1}}$	$\pi_E = \pi_F^{\frac{1}{e}}$	$v(\pi_E) = \frac{1}{e}$
deg n				
$F$	$v_F$	$\  \cdot \ _F$	$\pi_F$	$v(\pi_F) = 1$

## Unramified / totally ramified

Good ref: [https://en.wikipedia.org/wiki/Finite\\_extensions\\_of\\_local\\_fields](https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields)  
It collects the equivalent conditions of unramified/totally ramified field extensions.

	tot ram	wild ram
		tame ram
	field ext	
		<del>split</del> in local case

When  $E/F$  is tot ramified.

$$e = n \quad v(\pi_E) = \frac{1}{n}$$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E]$      $\min(\pi_E) \in \mathcal{O}_F[x]$  is Eisenstein poly.

Lemma. Let  $E/F$ : NA local field,  $e = e(E/F)$ ,  $r \in \mathbb{N}_{\geq 0}$ . Easy to see

$$\begin{aligned} \wp_E^{1+r} \cap F &= \{x \in F \mid v_E(x) \geq \frac{1}{e}(1+r)\} \\ \wp_F^{1+\lceil \frac{r}{e} \rceil} &= \{x \in F \mid v_F(x) \geq 1 + \lceil \frac{r}{e} \rceil\} \end{aligned}$$

Then

$$Tr_{E/F}(\wp_E^{1+r}) \subseteq \wp_F^{1+\lceil \frac{r}{e} \rceil} \quad \text{when } E/F \text{ is tamely ramified}$$

Table for  $e=3$ : ("proof of lemma")

$r$	0	1	2	3	4	5	6	7
$\frac{1}{e}(1+r)$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$
$1 + \lceil \frac{r}{e} \rceil$	1	1	1	2	2	2	3	3

E.g.  $E/F = \mathbb{Q}_{49}/\mathbb{Q}_7 = \mathbb{Q}_7(\sqrt{3})/\mathbb{Q}_7$  is unramified.

$$\begin{aligned} v(a+b\sqrt{3}) &= \frac{1}{2} v(N_{E/F}(a+b\sqrt{3})) \\ &= \frac{1}{2} v(a^2 - 3b^2) \\ &= \frac{1}{2} \min(v(a^2), v(b^2)) \\ &= \min(v(a), v(b)) \end{aligned} \quad a, b \in \mathbb{Q}_7$$

$$\begin{aligned} \mathcal{O}_E &= \mathbb{Z}_7(\sqrt{3}) & \mathfrak{p}_E &= (7, 7\sqrt{3}) = (7) & k_E &= \mathbb{Z}_7(\sqrt{3})/(7) \\ &&&&&\cong \mathbb{Z}_7[\alpha]/(\alpha^2 - 3, 7) \cong \mathbb{F}_7(\sqrt{3}) \cong \mathbb{F}_{49} \end{aligned}$$

$$\beta_E^{1+r} = (7)^{1+r} = (7^{1+r}) \quad \text{Tr}_{E/F}(\beta_E^{1+r}) = \beta_F^{1+r} = \beta_E^{1+r} \cap F \quad r \geq 0$$

E.g.  $E/F = \mathbb{Q}_7(\sqrt{7})/\mathbb{Q}_7$  is tamely ramified.

$$\begin{aligned} v(a+b\sqrt{7}) &= \frac{1}{2} v(N_{E/F}(a+b\sqrt{7})) \\ &= \frac{1}{2} v(a^2 - 7b^2) \\ &= \frac{1}{2} \min(v(a^2), 1+v(b^2)) \\ &= \min(v(a), \frac{1}{2} + v(b)) \end{aligned} \quad a, b \in \mathbb{Q}_7$$

$$\begin{aligned} \mathcal{O}_E &= \mathbb{Z}_7(\sqrt{7}) & \mathfrak{p}_E &= (7, \sqrt{7}) = (\sqrt{7}) & k_E &= \mathbb{Z}_7(\sqrt{7})/(\sqrt{7}) \\ &&&&&\cong \mathbb{Z}_7[\alpha]/(\alpha^2 - 7, \alpha) \cong \mathbb{Z}_7/(7) \cong \mathbb{F}_7 \end{aligned}$$

$$\beta_E^{1+r} = (\sqrt{7})^{1+r} = \begin{cases} (7^{\frac{1+r}{2}}) & r \text{ odd} \\ \sqrt{7} \cdot (7^{\frac{r}{2}}) & r \text{ even} \end{cases} \quad \text{Tr}(\sqrt{7}^{\frac{1+r}{2}}) = 2 \cdot 7^{\frac{1+r}{2}} \quad r \geq 0$$

$$\text{So } \text{Tr}_{E/F}(\beta_E^{1+r}) = \beta_E^{1+r} \cap F = \beta_F^{1+\lceil \frac{r}{2} \rceil}.$$

2.  $\widehat{\mathbb{Z}} = \text{profinite completion of } \mathbb{Z}$  (Recall 2022.2.13 outer auto...)

$$\widehat{\mathbb{Z}} := \prod_l \mathbb{Z}_l$$

$$\widehat{\mathbb{Z}}^{\times} := \prod_l \mathbb{Z}_l^{\times}$$

$$\widehat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$(\widehat{\mathbb{Z}}^{\times})^{(p)} := \prod_{l \neq p} \mathbb{Z}_l^{\times} = (\widehat{\mathbb{Z}}^{(p)})^{\times}$$

Prop. ①  $\text{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l=m \\ 0 & l \neq m \end{cases} \quad l, m \text{ prime.}$

②  $\text{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^{\times}$

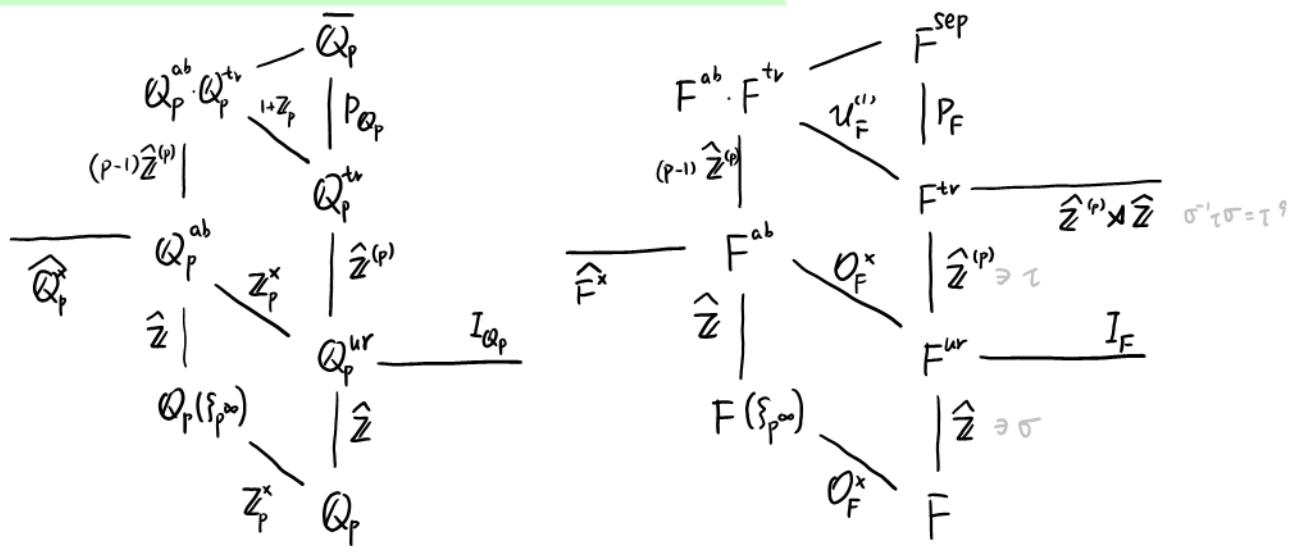
$\text{Aut}(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}^{\times}$  in the category of profinite gps.

$\text{Aut}(\widehat{\mathbb{Z}}^{(p)}) = \widehat{\mathbb{Z}}^{(p)\times}$

③  $\mathcal{O}_F, \mathcal{O}_F^{\times}$  are profinite groups, so  $\widehat{\mathcal{O}}_F = \mathcal{O}_F \quad \widehat{\mathcal{O}}_F^{\times} = \mathcal{O}_F^{\times}$ .

### 3. Big picture

Main ref: [AY] <https://alex-youscis.github.io/localglobalgalois.pdf>



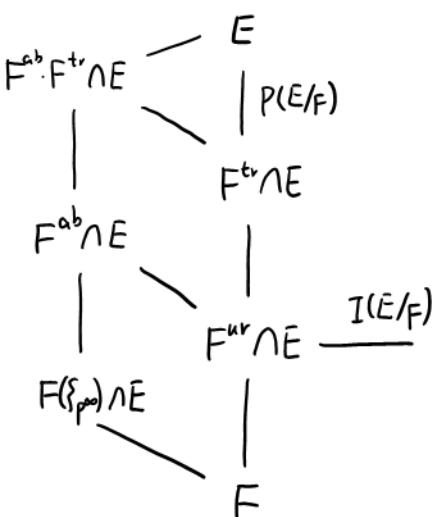
**unramified**  $F^{ur} = \bigcup_{n \geq 1} F(\{s_{p^n}\})$  Fermat's little thm  $\bigcup_{\substack{n \geq 1 \\ p \nmid n}} F(\{s_n\})$  **local field with char k = p**

**tame ramified**  $F^{tr} = F^{ur}(\pi_F^{\frac{1}{n}} |_{(n,p)=1})$   $= F(\pi_F^{\frac{1}{n}}, s_n |_{(n,p)=1})$  Notice that  $s_p \in F^{tr}$ !

**abelian**  $F^{ab} = F(\{s_\infty\}) := \bigcup_{n \geq 1} F(\{s_n\})$

$$F^{ab} \cdot F^{tr} = F(\pi_F^{\frac{1}{n}}, s_\infty |_{(n,p)=1})$$

<https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions>



$$\begin{array}{c}
\mathbb{Q}_p(\{f_p^n\}) \\
| \\
\mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^r\mathbb{Z})^\times \cap \mu_{p-1}} \{f_p^n\}\right) \dots \\
| \quad | \\
\mathbb{Z}/p\mathbb{Z} \quad \mathbb{Q}_p(\{f_{p^2}\}) \xrightarrow{I_p} \mathbb{Z}_{p^{r-1}} \\
| \quad | \\
\mathbb{Z}/p\mathbb{Z} \quad \mathbb{Q}_p(\{f_p\}) \xrightarrow{I_1} \mathbb{Z}_{p^{r-1}} \\
| \quad | \\
\mathbb{Z}/(p^{r-1})\mathbb{Z} \quad \mathbb{Q}_p \xrightarrow{I_0 = I_{-1}} \\
\end{array}$$

$\xrightarrow{\mathbb{Z}/(p^{r-1})\mathbb{Z}}$        $\xrightarrow{1+p\mathbb{Z}/(p^r\mathbb{Z})}$        $\xrightarrow{\mathbb{Z}/(p^{r-1})\mathbb{Z}}$

$$\begin{array}{c}
\mathbb{Q}_p(\{f_p^\infty\}) \\
| \\
\bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^r\mathbb{Z})^\times \cap \mu_{p-1}} \{f_p^n\}\right) \dots \\
| \quad | \\
\mathbb{Z}/p\mathbb{Z} \quad \mathbb{Q}_p(\{f_p\}) \xrightarrow{I^2 = U_{\mathcal{O}_p}^{(1)}} \\
| \quad | \\
\mathbb{Z}/p\mathbb{Z} \quad \mathbb{Q}_p(\{f_{p^2}\}) \xrightarrow{I' = U_{\mathcal{O}_p}^{(1)}} \\
| \quad | \\
\mathbb{Z}/(p^{r-1})\mathbb{Z} \quad \mathbb{Q}_p \xrightarrow{I^0 = U_{\mathcal{O}_p}^{(0)} = \mathbb{Z}_p^\times} \\
\end{array}$$

$\xrightarrow{\mathbb{Z}/(p^{r-1})\mathbb{Z}}$        $\xrightarrow{1+p\mathbb{Z}/(p^r\mathbb{Z})}$        $\xrightarrow{\mathbb{Z}/(p^{r-1})\mathbb{Z}}$

$$E/F = \mathbb{Q}_p(\{f_p^n\})/\mathbb{Q}_p$$

$$E/F = \mathbb{Q}_p(\{f_p^\infty\})/\mathbb{Q}_p$$

$$\begin{array}{ccc}
\mathbb{Q}_p(\{f_p\}) & \mathbb{Q}_p(\{f_p\}) & \mathbb{Q}_p(\{f_p\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & & \\
\mathbb{Q}_p(\{f_p + f_p^{-1}\}) & \mathbb{Q}_p(\{f_p\}) & \mathbb{Q}_p(\{f_p\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Q}_p & \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Q}_p & & \mathbb{Q}_p \\
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Q}_p(\{f_p^\infty\}) & \mathbb{Q}_p(\{f_p^\infty\}) & \mathbb{Q}_p(\{f_p^\infty\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & & \\
\bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p(\{f_p^n + f_{p^n}^{-1}\}) & \mathbb{Q}_p^{abwr}(\{f_p\}) & \mathbb{Q}_p^{abwr}(\{f_p\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & \mathbb{Q}_p^{abwr} & \mathbb{Z}/2\mathbb{Z} \\
\mathbb{Q}_p^{abwr} := \bigcup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^r\mathbb{Z})^\times \cap \mu_{p-1}} \{f_{p^n}\}\right) & & \mathbb{Q}_p^{abwr} \\
\end{array}$$

$$p \text{ odd}$$

$$p \equiv 1 \pmod{4}$$

$$p \equiv 3 \pmod{4}$$

There are only finite isomorphism classes of degree  $n$  extensions of  $\mathbb{Q}_p$ , see here for a discussion:  
<https://math.stackexchange.com/questions/1118068/finitely-many-extensions-of-fixed-degree-of-a-local-field>

Except for the filtrations as well as cohomology dimensions, the Artin-Schreier theory also gives us a better understanding of the wild inertia group. For example, there are exactly  $p^2$  ramified degree  $p$  field extensions of  $\mathbb{Q}_p$  (for  $p$  odd prime). A detailed discussion (and Table 2.1) can be seen here:

<https://www.sciencedirect.com/science/article/pii/S0747717105001276?via%3Dihub>

For a ref of the Artin-Schreier theory, you can see

[https://en.wikipedia.org/wiki/Artin-Schreier\\_theory](https://en.wikipedia.org/wiki/Artin-Schreier_theory)

<https://math.stackexchange.com/questions/50041/reference-book-for-artin-schreier-theory> (gives the proof of  $x^p - x - a$ )

Q: How many degree  $p$  field extensions of  $\mathbb{F}_p((t))$  are there?

Warning:

Even though every degree  $p$  field ext of  $\mathbb{Q}_p$  can be written of the form  
$$\mathbb{Q}_p[x]/(x^p - x - \alpha), \quad \alpha \in \mathbb{Q}_p, \quad \alpha + \beta^p - \beta \text{ for } \beta \in K$$

it's not feasible to do so when we do for examples (no good parameters)

e.g. It's not easy to find a canonical  $\alpha \in \mathbb{Q}_p$  s.t.

$$\mathbb{Q}_p[x]/(x^p - p) \cong \mathbb{Q}_p[y]/(y^p - y - \alpha).$$

#### 4. A detailed discussion concerning proofs.

Let us elaborate our understanding on  $\text{Gal}(E/F)$ ,  $G_F$   
(Rather than viewing them as black box)

Setting:  $E/F$  be a finite Galois ext of NA local fields.

##### Finite extension

Lemma.  $\forall \sigma \in \text{Gal}(E/F), \quad v_E(\sigma(-)) = v_E(-).$

$$\text{e.g. } \sigma(\mathcal{O}_E) = \mathcal{O}_E \quad \sigma(\mathfrak{p}_E) = \mathfrak{p}_E, \\ \text{Gal}(E/F) \xrightarrow{\sim} \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E)$$

[Proof. The valuation of  $F$  extends uniquely to  $E$ . (use completeness)  
In addition,  $v_E(\sigma(-)), v_E(-)$  are two valuations.]

Rmk. Using this, one can show that

$$\mathcal{O}_E = \text{integral closure of } \mathcal{O}_F \text{ in } E.$$

Completeness is necessary, see <https://math.stackexchange.com/questions/4065594/integer-ring-and-valuation-ring-of-local-fields>

Using the lemma, we construct a map

$$\text{Gal}(E/F) \cong \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E) \cong \text{Gal}(k_E/k_F)$$

and extends it to a SES

$$1 \longrightarrow I(E/F) \longrightarrow \text{Gal}(E/F) \longrightarrow \text{Gal}(k_E/k_F) \longrightarrow 1$$

where

$$I(E/F) := \{\sigma \in \text{Gal}(E/F) \mid \sigma(x) \equiv x \pmod{\mathfrak{p}_E} \quad \forall x \in \mathcal{O}_E\}$$

Q. How to show that

$$\begin{aligned} \text{Gal}(E/F) &\longrightarrow \text{Gal}(k_E/k_F) \\ \text{Gal}(E/F) &\longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \\ \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) &\longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^r) \end{aligned}$$

are surjective? (Now: take it as a black box)

In general, define  $r \in \mathbb{N}_{\geq 0}$

$$\text{Gal}(E/F) \cong \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

we get SES

$$1 \longrightarrow I_r(E/F) \longrightarrow \text{Gal}(E/F) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

where

$$I_r(E/F) := \{\sigma \in \text{Gal}(E/F) \mid \sigma(x) \equiv x \pmod{\mathfrak{p}_E^{r+1}} \quad \forall x \in \mathcal{O}_E\}$$

Comparing those constructions, we get a filtration

$$\cdots \subseteq I_r(E/F) \subseteq \cdots \subseteq I_1(E/F) \subseteq I_0(E/F) \subseteq I_{-1}(E/F)$$

||                   ||                   || def

$$P(E/F) \subseteq I(E/F) \subseteq Gal(E/F)$$

with  $\bigcap_{r \in \mathbb{N}} I_r(E/F) = 1$

and surjections

$$\rightarrow Aut_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \rightarrow \cdots \rightarrow Aut_{\mathcal{O}_F/\mathfrak{p}_F^1\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^1) \rightarrow Aut_{k_F\text{-alg}}(k_E) \rightarrow 1$$

with  $\varprojlim_{r \in \mathbb{N}} Aut_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \cong Gal(E/F)$

Prop. We understand filtrations well:

$$\begin{array}{ccc} I_{-1}(E/F) / I_0(E/F) & \xrightarrow{\sim} & Gal(k_E/k_F) \\ I_0(E/F) / I_1(E/F) & \hookrightarrow & (\mathcal{O}_E/\mathfrak{p}_E)^* \cong k_E^* \\ \text{i.e. } [\sigma : \pi_E \mapsto a_1\pi_E + a_2\pi_E^2 + \dots] & \longmapsto & \sigma \mapsto \frac{\sigma(\pi)}{\pi} \\ I_r(E/F) / I_{r+1}(E/F) & \hookrightarrow & (\mathfrak{p}_E^r / \mathfrak{p}_E^{r+1})^* \cong k_E \\ \text{i.e. } [\sigma : \pi_E \mapsto \pi_E + a_r\pi_E^r + \dots] & \longmapsto & \sigma \mapsto \frac{\sigma(\pi)}{\pi} - 1 \quad r \in \mathbb{N}_{\geq 1} \end{array}$$

Cor.  $P(E/F)$  is the Sylow-p-subgp of  $I(E/F)$ .

Cor. When  $E/F$  is tot ramified with deg  $r$ , we have an iso

$$I_0(E/F) / I_1(E/F) \cong \mu_r(k_E)$$

In tame case,  $\#\mu_r(k_E) = r$ .

## Absolute Galois gp

⚠ Some naive ideas are surprisingly complicated, so we list possible mistakes here, and don't touch those ideas...

- Don't want to mention  $\overline{\mathcal{O}_E}$ , since  $\overline{E}$  is not a NA local field.

Also, I don't know if

$$\text{the integral closure } \overline{\mathcal{O}_E} = \{x \in \overline{E} \mid v(x) \geq 0\}$$

- Can't define  $I_{F,r} = \varprojlim_{E/F \text{ fin Gal}} I_r(E/F)$ , so we switch to upper indexing. In that case,

$$I_F = I_F^\circ \quad P_F = I_F^{\geq 0} = \bigcup_{v \geq 0} I_F^v \\ \bigcap_{v \in [0, \infty)} I_F^v = 1 \quad \bigcap_{v' < v} I_F^{v'} = I_F^v \quad v, v' \in [0, \infty)$$

We define  $I_F$  and  $P_F$  directly. Denote  $\phi_E: G_F \rightarrow \text{Gal}(E/F)$

$$I_F := \left\{ \sigma \in G_F \mid \phi_E(\sigma) \in I(E/F) \quad \forall E/F \text{ fin Gal} \right\} \\ P_F := \left\{ \sigma \in G_F \mid \phi_E(\sigma) \in P(E/F) \quad \forall E/F \text{ fin Gal} \right\}$$

We have SES

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \begin{matrix} G_{K_F} \\ \cong \\ \widehat{\mathbb{Z}} \end{matrix} \longrightarrow 1$$

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(K_E) \longrightarrow 1 \quad \cong \widehat{\mathbb{Z}}^{(p)}$$

where  $\varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(K_E) \cong \varprojlim_{\substack{E=F(\pi_F^{\frac{1}{e}}) \\ (m,p)=1}} \mathbb{Z}/m\mathbb{Z} \cong \widehat{\mathbb{Z}}^{(p)}$

Transforming to the field side, we have clear descriptions on  $F^{ur}$  &  $F^{tr}$ .

$$F^{ur} := F(\{\zeta_n \mid_{(n,p)=1}\})$$

$$F^{tr} := F(\pi_F^{\frac{1}{e}}, \{\zeta_n \mid_{(n,p)=1}\})$$

Claim: For  $F' = F^{ur} \cap E$ ,  $\# E/F' = e$ , if  $(e,p)=1$ , then  $E \subseteq F^{tr}$

Try: Only need to show  $\pi_E \in F^{tr}$ .

Take min poly of  $\pi_E$  over  $F'$

$$f(T) = T^e + a_{e-1}T^{e-1} + \dots + a_1T + a_0 \in \mathcal{O}_F[T]$$

$\Rightarrow f(T)$  is an Eisenstein poly

I think that we can use Newton's method to show that, the equation

$$f(T) = 0$$

has a root in  $F^{tr}$ .

Better:  $[E : E \cap F^{tr}]$  is  $p$ -power, and  $[E : E \cap F^{tr}] \mid e \Rightarrow E = E \cap F^{tr}$   
 $\Rightarrow E \subseteq F^{tr}$

Actually, we can write down the iso explicitly:

$$t_{Fr}: G_F/I_F \longrightarrow \widehat{\mathbb{Z}} \quad \sigma \mapsto t_{Fr}(\sigma) \text{ satisfying}$$

$\forall \text{ field ext } E/F, \forall x \in \mathcal{O}_E, \quad \# K_F = q$

$$\sigma(x) \equiv x^{q^{t_{Fr}(\sigma)}} \pmod{p_E}$$

Fix compatible generators

$$\zeta = (\zeta_m)_{(m,p)=1} \in \varprojlim_{(m,p)=1} \mu_m(F^{\text{tr}}),$$

one can define

$$t_\zeta: I_F/p_F \longrightarrow \widehat{\mathbb{Z}}^{(p)} \quad \sigma \mapsto t_\zeta(\sigma) \text{ satisfying}$$

$$\frac{\sigma(\pi_F^{\frac{1}{m}})}{\pi_F^{\frac{1}{m}}} = \zeta_m^{t_\zeta(\sigma) \pmod{m}} \quad \text{in } F^{\text{tr}} \quad \forall (m,p)=1$$

Task. Take  $F_r := t_{Fr}^{-1}(1)$  (lift to  $G_F$ )  $\tau_r := t_\zeta^{-1}(1)$ ,  $m \in \mathbb{N}_{\geq 1}$ .

Show that

$$F_r(\zeta_m) = \zeta_m^q \quad \tau_r(\zeta_m) = \zeta_m$$

$$F_r(\pi_F^{\frac{1}{m}}) = \zeta_m^l \pi_F^{\frac{1}{m}} \quad \tau_r(\pi_F^{\frac{1}{m}}) = \zeta_m \pi_F^{\frac{1}{m}} \quad \exists l \in (\mathbb{Z}/m\mathbb{Z})^\times$$

Therefore,

$$F_r \tau_r F_r^{-1} = \zeta^q \quad \text{in } F(\zeta_m, \pi_F^{\frac{1}{m}})$$

$$\Rightarrow F_r \tau_r F_r^{-1} = \zeta^q \quad \text{in } F^{\text{tr}}.$$

## 5. Henselian ring.

Main ref: [https://en.wikipedia.org/wiki/Henselian\\_ring](https://en.wikipedia.org/wiki/Henselian_ring)

R comm with 1 (local in this section)

Def. A local ring  $(R, \mathfrak{m})$  is Henselian if Hensel's lemma holds, i.e.

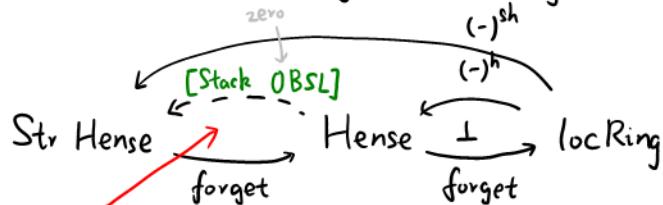
$$\begin{array}{l} \text{for } P \in R[x] \\ \downarrow \\ \bar{P} = g_1, \dots, g_n \in R/m[x] \end{array} \quad \begin{array}{l} \exists f_i \in P[x] \\ \downarrow \Theta \\ g_i \in R/m[x] \end{array}$$

$(R, m)$  is strictly Henselian if additionally  $(R/m)^{\text{sep}} = R/m$ .

- E.g. • Fields/Complete Hausdorff local rings are Henselian.  
 e.p.  $F, \mathcal{O}_F$  are Henselian  
 •  $R$  is Henselian  $\Leftrightarrow R/\text{Nil}(R)$  is Henselian  
 $\Leftrightarrow R/I$  is Henselian for  $V \subset R$   
 e.p. when  $\text{Spec } R = \{\mathfrak{p}\}$ ,  $R$  is Henselian.

Denote  $\text{StrHense} \subset \text{Hense} \subset \text{locRing} \subset \text{CommRing}$

full subcategories



$$\text{E.g. } F^h = F \quad F^{sh} = F^{\text{un}}$$

Sadly not adjoint?

Geometrically, Henselian means  $\text{Spec } R/\mathfrak{m} \rightarrow \text{Spec } R$  has a section.

## 6. Cohomological dimension

main reference for cohomological dimension:  
 [NSW2e] <https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/>

<https://mathoverflow.net/questions/349484/what-is-known-about-the-cohomological-dimension-of-algebraic-number-fields>

This section is initially devoted to the following result.

Prop. [(7.5.1)] The wild inertia gp  $P_F$  is free pro-p-group of countably infinite rank.

See [Galois Theory of p-Extensions, Chap 4] for the definition and construction of free pro-p-groups.

Q: Do we have the adjoint

$$\begin{array}{ccc} \text{Pro-p-gp} & \xrightleftharpoons[\text{forget}]{\perp} & \text{Set} \\ & \xleftarrow{(\ )^{\text{free}}} & \end{array}$$

?

Now let

$$\begin{aligned} G: & \text{ profinite gp} \\ \text{Mod}(G): & \text{ category of discrete } G\text{-modules} \\ \text{full subcategory} \\ \text{of Mod}(G) & \left. \begin{array}{ll} \text{Mod}_t(G): & \text{torsion} \\ \text{Mod}_p(G) & \text{p-torsion} \\ \text{Mod}_f(G) & \text{finite} \end{array} \right\} \text{viewed as abelian gp} \end{aligned}$$

Lemma For abelian torsion gp  $X$ , denote

$$X(p) := \{x \in X \mid x^{p^k} = 1 \quad \exists k \in \mathbb{N}_{>0}\}$$

we have  $X = \bigoplus_p X(p)$ .

This is trivial when  $X$  is finite, but I don't know how to prove this in the general case. It should be not too hard.

Def [(3.3.1)] (cohomological dimension)  $p$  prime

$$cd G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A) \neq 0\}$$

$$tcd G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A) \neq 0\}$$

$$cd_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A)(p) \neq 0\}$$

$$tcd_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A)(p) \neq 0\}$$

Prop. (local to global)  $cd G = \sup_p cd_p G$   $scd G = \sup_p scd_p G$

Prop. [(3.3.2)]  $cd_p G \leq n \iff H^{n+1}(G, A) = 0 \quad \forall \text{ simple } G\text{-mod } A \text{ with } pA = 0$

e.g. for  $G$ : pro-p-gp,

$$cd_p G \leq n \iff H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

E.g.  $cd_p \mathbb{Z} = 1$   $scd_p \mathbb{Z} = 2$

Prop. [(3.3.5)] For  $H \leq G$  closed,

$$cd_p H \leq cd_p G \quad scd_p H \leq scd_p G$$

When  $p \nmid [G:H]$  or  $[H \text{ open} + cd_p G < +\infty]$ , the equality holds.

Weaker condition, see [(3.3.5, Serre)]

Cor.  $G$ : profinite gp, then

$$cd_p G = 0 \iff p \nmid \#G$$

Prop. [(3.5.17)] A pro-p-gp  $G$  is free iff  $cd G \leq 1$ .

Prop [7.1.8] (i)  $F$  NA local field with  $\text{char } k = p$ .

$$cd_l(F) = \begin{cases} 2 & \text{if } l \neq \text{char } F, \\ 1 & \text{if } l = \text{char } F. \end{cases}$$

For any  $E/F$  field extension s.t.  $l^{\infty} \mid \deg E/F$ ,  $cd_l(E) \leq 1$ .

(ii) Fix  $n \in \mathbb{N}_{>0}$  s.t.  $\text{char } F \nmid n$ .

$$H^i(F, \mu_n) = \begin{cases} F^\times / (F^\times)^n & i=1 \\ \frac{1}{n} \mathbb{Z}/\mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases}$$

[Proof for Prop 7.5.1]

$$\text{Now } l^{\infty} \mid \deg F^{\text{tr}}/F \stackrel{(7.1.8)}{\Rightarrow} cd_l(F^{\text{tr}}) \leq 1 \quad \forall \text{ prime } l$$

$$\Leftrightarrow cd_l(F) \leq 1$$

$\Leftrightarrow P_F$  is free pro- $p$ -group.

□

## 7. Bonus: "plane geometry" for $\mathbb{Q}_9$

In this section, the picture comes from [<https://www.nt.th-koeln.de/fachgebiete/mathe/knospe/p-adic/>] by Heiko Knospe.

I want to define:

Compare  $\mathbb{Q}_9$  and  $\mathbb{Q}_3(\sqrt{3})$

triangle (Actually we just consider 3 points, and they may be "collinear")

disk

sphere

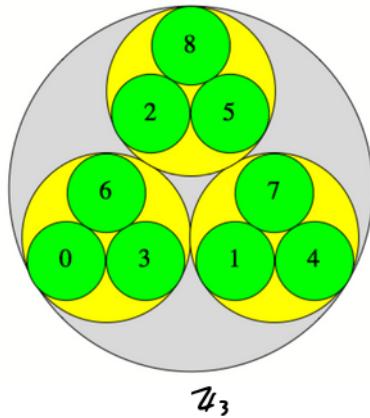
line (in higher dimension, like  $\mathbb{Q}_9$  or  $\mathbb{Q}_3(\sqrt{3})$ )

no angle, no perpendicular, but parallel lines

$P^1(\mathbb{Q}_3)$  (should characterize all lines in  $\mathbb{Q}_9$  passing through 0, parameterized by a line not passing the origin)

intersections of disks, spheres and lines

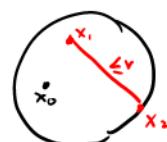
sphere packing? Symmetric group of the objects considered? connection with the tree-structures/Bruhat--Tits building?



$\mathbb{Z}_3$



Triangles:  $a = a \geq b$



disks: every pt can be center  
(even pts on the edge)

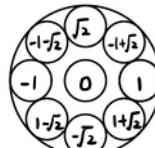
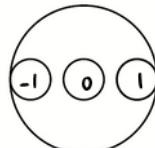
$B_{x_0}(r)$

I personally would like to draw it more "compatible with arithmetic":

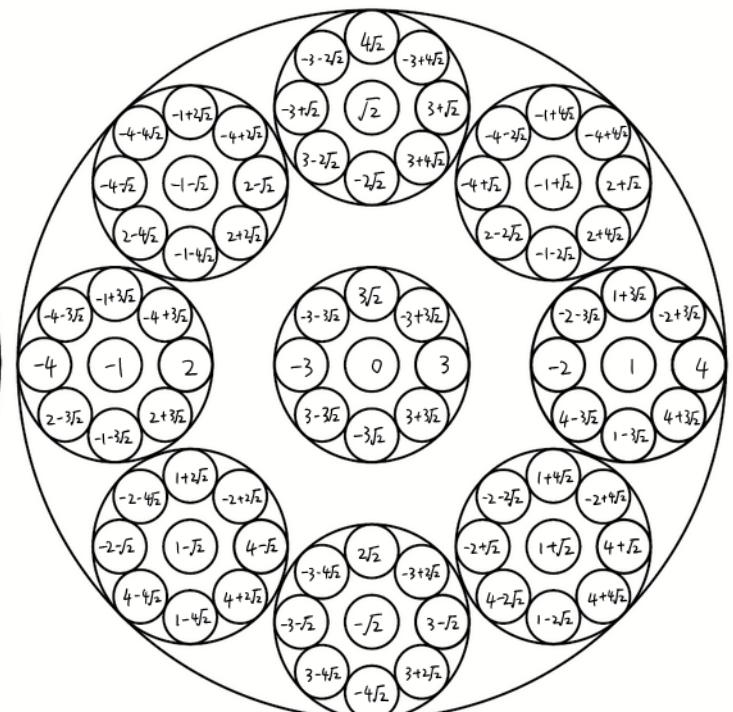
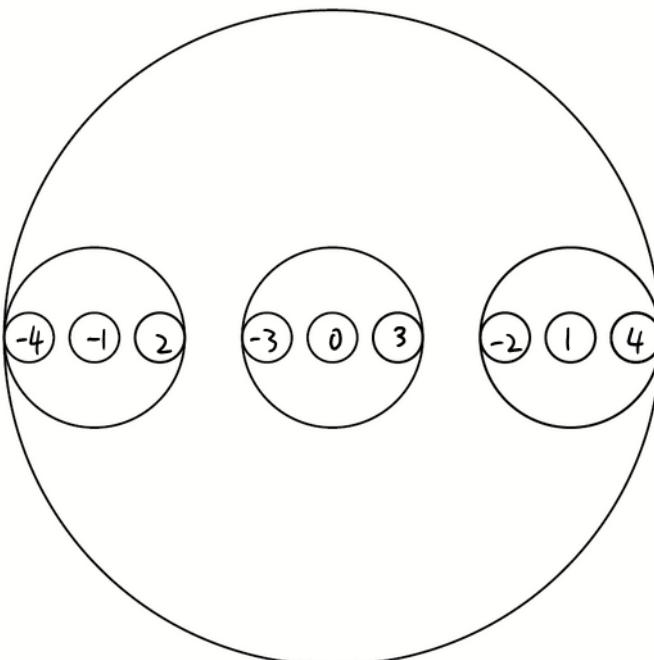
E.g.  $\mathbb{Z}_3$  vs.  $\mathbb{Z}_9 = \mathbb{Z}_3(\sqrt{2})$

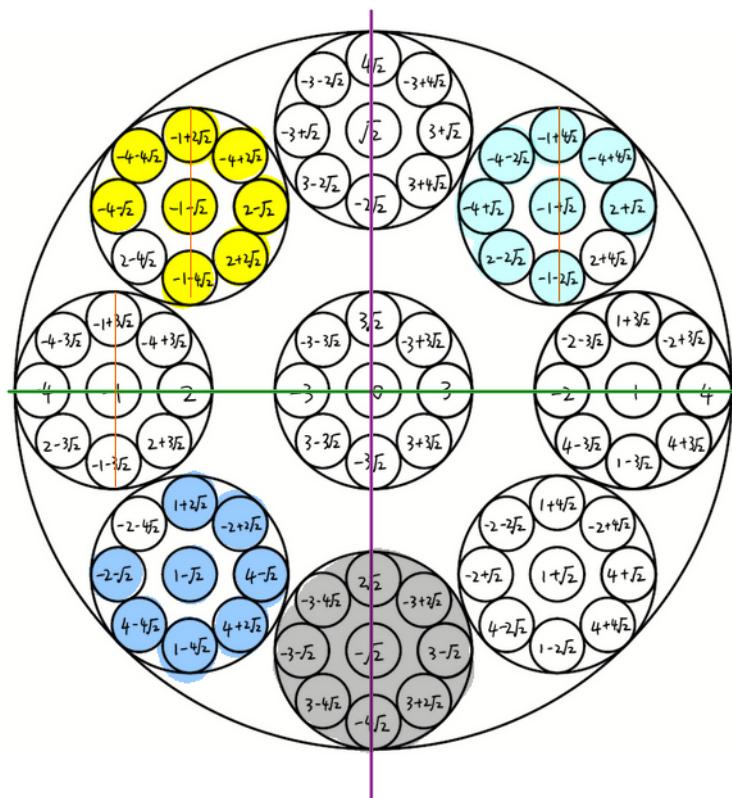
$\left. \begin{array}{l} \text{mod } 3 \\ +, \text{ inverse} \\ \text{multi by } 3 \\ \text{Frobenius} \end{array} \right\}$

basic buildings:



It's more canonical to use Teichmüller lift rather than 1~p, but I don't do so because of my limited computation ability.





$$x_0 = 2 + 4\sqrt{2} \quad \text{Gal}(\mathbb{Q}_9/\mathbb{Q}_3) = \{1, \sigma\}$$

$$A := \{x \in \mathbb{Q}_9 \mid d(x, x_0) = \frac{1}{3}\}$$

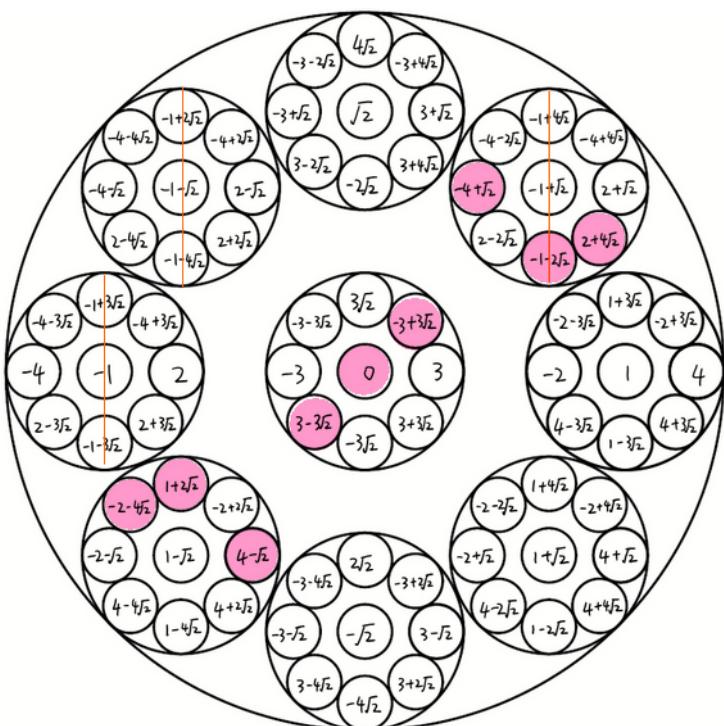
●  $\sigma A$   
●  $-A$

$$x_1 = 2\sqrt{2}$$

$$B := B(x_1, \frac{1}{3})$$

$$= B(x, \frac{1}{3}) \text{ for } \forall x \in B$$

—  $\mathbb{Q}_3$     —  $\mathbb{Q}_3$ -v.s. generated by  $x_1$   
—  $\{ax_1 - 1 \mid a \in \mathbb{Q}_3\}$



(smallest)  
circles containing elements in  $\mathbb{Q}_3 \cdot x_0$

Observation: for  $\forall$  disk  $D = \bigcup_{i=1}^9 D_i \subset \mathbb{Q}_9$   
if  $D \cap (\mathbb{Q}_3 \cdot x_0) \neq \emptyset$ , then  
 $\#\{i \in \{1, \dots, 9\} \mid D_i \cap (\mathbb{Q}_3 \cdot x_0) \neq \emptyset\} = 3$ .

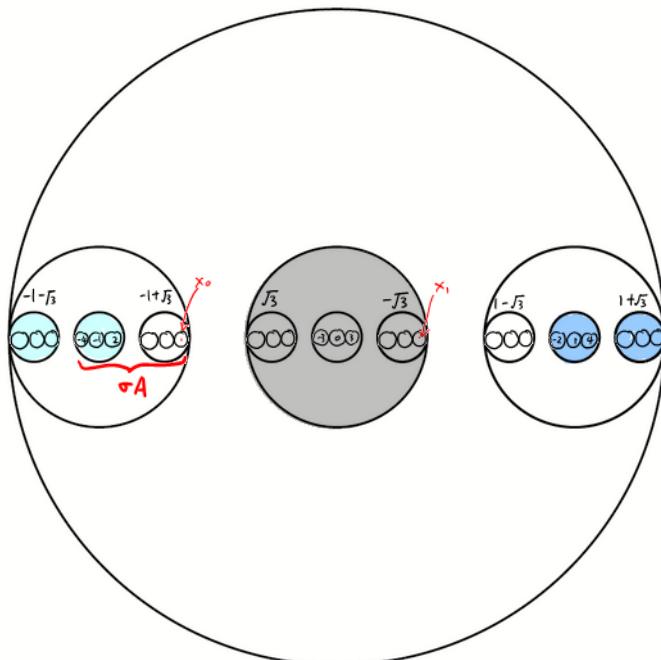
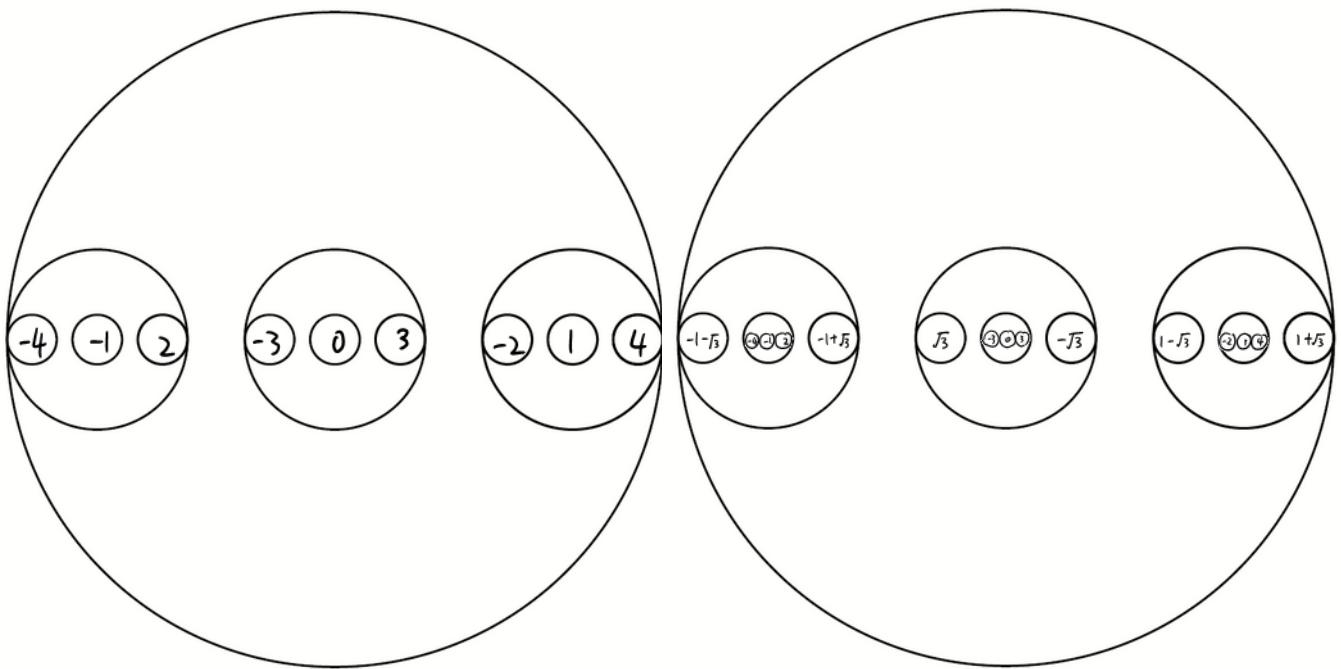
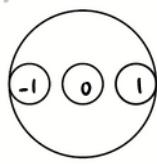
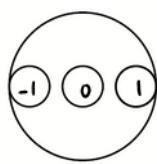
Q: Can we recover  $\mathbb{Q}_3$ -v.s V from the set

$$\left\{ D \subset \mathbb{Q}_9 \mid \begin{array}{l} D = B_x(r) \text{ for some } x \in \mathbb{Q}_9, r \geq 1 \\ D \cap V \neq \emptyset \end{array} \right\}?$$

E.g.  $\mathbb{Z}_3$  vs.  $\mathbb{Z}_3(\sqrt{3})$

$$\|\cdot\| = \|\cdot\|_{\mathbb{Z}_3} = \|\cdot\|_{\mathbb{Z}_3(\sqrt{3})}^{\frac{1}{2}}$$

basic buildings:



$$x_0 = 2 + \sqrt{3} \quad \text{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q}_3) = \{1, \sigma\}$$

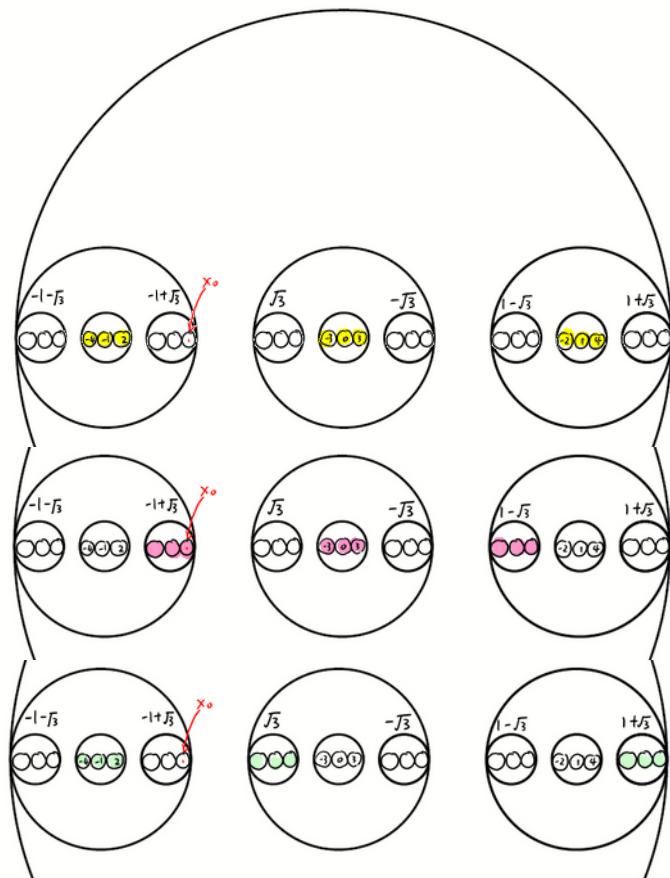
$A := \{x \in \mathbb{Q}_9 \mid d(x, x_0) = \frac{1}{3}\}$

$\sigma A$

$-A$

$$x_1 = 2\sqrt{3}$$

$B := B(x_1, \frac{1}{\sqrt{3}})$   
 $= B(x, \frac{1}{\sqrt{3}}) \text{ for } \forall x \in B$



smallest

- circles containing elements in  $Q_3 \cdot 1$
- circles containing elements in  $Q_3 \cdot x_0$
- circles containing elements in  $\{2x_0 - 1\}$

Observation: for  $\forall$  disk  $D = \bigcup_{i=1}^3 D_i \subset Q_3(\sqrt{3})$

if  $D \cap Q_3 \cdot x_0 \neq \emptyset$ , then  
 $\#\{i \in \{1, \dots, 3\} \mid D_i \cap Q_3 \cdot x_0 \neq \emptyset\} = 3$ .

Tasks for interesting readers: figure out all the cases of quadratic extension of  $Q_2$ .

