Eine Woche, ein Beispiel

2.6. six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

A preparation of exams.

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F}\ \mathcal{F}' \\ Y & \xrightarrow{f} & X \end{array} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Goal:
$$f^* \dashv f_*$$
$$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

$$\begin{tikzcd} & \otimes \arrow[dl, dash] \arrow[dr, dash] & \\ f^* \arrow[rr, dash] & & f_! \end{tikzcd}$$

bc: $f^* g_! \cong g'_! f'^*$

$f_* g^! \cong g'^! f'_*$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

f^*: stalk $\qquad f_* / f_!$: cohomology

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \to pt$

$H^\cdot(X;\mathbb{Z}) := p_* p^* \mathbb{1}$	$H^\cdot(X;\mathcal{F}) := p_* \mathcal{F}$	$= R\Gamma(X;\mathcal{F})$
$H^\cdot_c(X;\mathbb{Z}) := p_! p^* \mathbb{1}$	$H^\cdot_c(X;\mathcal{F}) := p_! \mathcal{F}$	
$H_\cdot(X;\mathbb{Z}) := p_! p^! \mathbb{1}$	$H_\cdot(X;\mathcal{F}) := p_!(p^!\mathbb{1} \otimes \mathcal{F})$	$= H^\cdot_c(X; p^!\mathbb{1} \otimes \mathcal{F})$
$H^{BM}_\cdot(X;\mathbb{Z}) := p_* p^! \mathbb{1}$	$H^{BM}_\cdot(X;\mathcal{F}) := p_*(p^!\mathbb{1} \otimes \mathcal{F})$	$= H^\cdot(X; p^!\mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$H_c(X;\mathcal{F}) \otimes H_c(Y;\mathcal{G}) \cong H_c(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

reduced to: $p_{X,!}\mathcal{F} \otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\qquad -^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X;\mathbb{Z})[d] \cong H^\cdot(X;\mathbb{Z})^\vee$$

reduced to: $p_* \underline{Hom}(A, p^*B \otimes p^!\mathbb{1}) \cong \underline{Hom}(p_!A, B)$

Upgrade: ∞-categories & sym monoidal structure

Idea: $\mathcal{D}_0 : \mathcal{C}^{op} \longrightarrow Cat_\infty$

$$\begin{tikzcd} X \arrow[r, mapsto] \arrow[d, "f"'] \arrow[dr, phantom, "\Rightarrow"] & D(X) \\ Y \arrow[r, mapsto] & D(Y) \arrow[u, "f^*"'] \end{tikzcd}$$

extends to ↳ compatability is encoded!

$\mathcal{D} : Corr(C, E) \longrightarrow Mon(Cat_\infty)$

$[Y \xleftarrow{f} X = X] \longmapsto f^*$

$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$

$[X \times X \xleftarrow{\Delta} X = X] \longmapsto \otimes$

Moreover, It factor through

$$Corr(C,E) \longrightarrow LZ_{\mathcal{D}} \longrightarrow Mon(Cat_\infty)$$

Obj: $\quad X \longmapsto X \longmapsto \mathcal{D}(X)$

Mor: $\quad \left[X \xleftarrow{f} Y \xrightarrow{g} Z \right] \longmapsto (f,g)_* \mathbb{1}_Y \in \mathcal{D}(X \times Z)$

kernel $\longmapsto$ FM-transformation composition = convolution

2-Mor: $\quad \varepsilon \to \varepsilon' \longmapsto \Phi_\varepsilon \longrightarrow \Phi_{\varepsilon'}$

$$\left[\begin{tikzcd}[column sep=small] & & Z \arrow[dl] \arrow[dr] & & \\ & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \\ & & \; \arrow[d, "\cong\ F"] & & \\ & & Z \arrow[dll] \arrow[drr] & & \\ X_1 & & & & X_3 \end{tikzcd}\right] \longmapsto \left[\begin{tikzcd} \varepsilon_{12} * \varepsilon_{23} \arrow[d, "\cong"'] \\ \varepsilon_{13} \end{tikzcd}\right]$$

Goal: framework of ∞-category & $\otimes$ $\quad \rightsquigarrow Corr(C,E)$ & $Corr(C,E)^{\otimes}$

∞- category

$$\Delta \hookrightarrow \mathrm{Fun}(\Delta^{op}, \mathrm{Set}) \stackrel{\Delta}{=} \mathrm{sSet} \supseteq \mathrm{Cat}_\infty \supseteq \mathrm{An}$$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "h"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists K_{n,i}(h)"'] & \end{tikzcd}$$

$\forall\, 0 < i < n$ in Cat_∞

$\forall\, 0 \le i \le n$ in An

Notation

Set	$(0,0)$ - category	set
Cat	$(1,1)$ - category	category
An	$(\infty,0)$ - category	anima / Kan cplx / ∞ - groupoid
Cat_∞	$(\infty,1)$ - category	

Ex. Realize $\mathrm{Corr}(C,E)$ as an ∞ - category.

Monoidal structure

In $(1,1)$- category.

Monoidal structure on $\mathcal{C}$:

$m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ $\qquad u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C}$

$(\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G}$ $\qquad * \longmapsto 1_{\mathcal{C}}$

Monoidal object in $(\mathcal{C}, \otimes)$. $X \in \mathrm{Ob}(\mathcal{C})$ with

$m_X: X \times X \longrightarrow X$ $\qquad u_X: 1_{\mathcal{C}} \to X$

In $(\infty,1)$ - category:

$$(C, \otimes) \overset{\text{def}}{\longleftrightarrow} \left[\begin{array}{c} \mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto X(I) \\ \text{comm monoid} \end{array}\right] \overset{\text{“Straightening”}}{\longleftrightarrow} \left[\begin{array}{c} \pi^\otimes: \Upsilon^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ \Upsilon^\otimes_I \xrightarrow{\sim} \prod_i \Upsilon^\otimes_i \end{array}\right]$$

⇝ See next page for details

where $\mathrm{Ob}(\mathrm{Fin}^{\mathrm{part}}) = \mathrm{Ob}(\mathrm{Fin})$

$\mathrm{Mor}_{\mathrm{Fin}^{\mathrm{part}}}(I, J) = \{\alpha: I \dashrightarrow J\}$

commutative monoid: $\mathbb{X}(I) \xrightarrow{\sim} \prod_i \mathbb{X}(i)$

$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow (\mathcal{F}, \mathcal{G}) \qquad |I| = 2$

coCartesian fibration: see [Def 3.5]

$$\left[\begin{array}{c}(\mathcal{C},\otimes)\\ \text{monoidal}\\ \infty\text{-cat}\end{array}\right] \overset{\text{def}}{\leftrightsquigarrow} \left[\begin{array}{c}\mathbb{X}: \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty\\ I \longmapsto \mathbb{X}(I)\\ \text{comm monoid}\end{array}\right] \overset{\text{“straightening”}}{\leftrightsquigarrow} \left[\begin{array}{c}\pi^\otimes: \mathbb{Y}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}}\\ \text{coCartesian fibration}\\ \mathbb{Y}^\otimes_I \xrightarrow{\sim} \prod_i \mathbb{Y}^\otimes_i\end{array}\right]$$

$$\begin{tikzcd}
(\mathcal{C},\otimes) \arrow[r, mapsto] & \mathcal{C}^{(-)}: \mathrm{Fin}^{\mathrm{part}} \arrow[r] & \mathrm{Cat}_\infty \\
& I \arrow[r, mapsto] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{C}^I := \prod_{i\in I}\mathcal{C} \arrow[d] \\
& J & \mathcal{C}^J
\end{tikzcd}
\qquad
\begin{array}{c}(X_i)_{i\in I}\\ \downarrow\\ \left(\bigotimes_{i\in\alpha^{-1}(j)} X_i\right)_{j\in J}\end{array}$$

$$\longmapsto \quad \pi^\otimes: \mathcal{C}^\otimes \longrightarrow \mathrm{Fin}^{\mathrm{part}}$$

$$\mathcal{C}^\otimes = \{(I,(X_i)_i) \mid I \in \mathrm{Fin}^{\mathrm{part}}, X_i \in \mathcal{C}\} \qquad (I,(X_i)_i) \longmapsto I$$

$$\mathrm{Mor}_{\mathcal{C}^\otimes}((I,X_i),(J,Y_j)) = \left\{\alpha: I \dashrightarrow J,\ \left\{\bigotimes_{i\in\alpha^{-1}(j)} X_i \longrightarrow Y_j\right\}_j\right\}$$

$$(\mathbb{X}(*), \mathbb{X}(\{1,2\}\to\{*\})) \longleftarrow\!\mapsfrom\ \mathbb{X} \longmapsto \mathbb{X}^\otimes = \{(I,A) \mid I \in \mathrm{Fin}^{\mathrm{part}}, A \in \mathbb{X}(I)\}$$

$$\mathrm{Mor}((I,A),(J,B)) = \left\{\begin{array}{l}\alpha: I \dashrightarrow J\\ f: (\mathbb{X}(\alpha))A \longrightarrow B \quad \text{in } \mathbb{X}(J)\end{array}\right\}$$

$$\begin{tikzcd}
\mathbb{Y}: \mathrm{Fin}^{\mathrm{part}} \arrow[r] & \mathrm{Cat}_\infty & \mathbb{Y}^\otimes \arrow[l, mapsto] \\
I \arrow[r, mapsto] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "\Rightarrow"] & \mathbb{Y}^I := \mathbb{Y}^\otimes_I \arrow[d, "\text{loc. coCartesian lift}"] & \\
J & \mathbb{Y}^J &
\end{tikzcd}$$

$$(\pi^{\otimes^{-1}}(*), \text{loc. coCartesian lift of } \{1,2\}\to\{*\}) \longleftarrow\!\mapsfrom\ \mathbb{Y}^\otimes$$

$\mathbb{X}$ encodes the monoidal structure

$$\begin{tikzcd}[column sep=small]
& \{12,3\} \arrow[dr] & \\
\{1,2,3\} \arrow[ur] \arrow[dr] & & * \\
& \{1,23\} \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& \mathcal{C}\times\mathcal{C} \arrow[dr] & \\
\mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[ur] \arrow[dr] & & \mathcal{C} \\
& \mathcal{C}\times\mathcal{C} \arrow[ur] &
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
& (X\otimes Y,Z) \arrow[r, mapsto] & (X\otimes Y)\otimes Z \arrow[d, phantom, "\wr\wr"] \\
(X,Y,Z) \arrow[ur, mapsto] \arrow[dr, mapsto] & & X\otimes(Y\otimes Z) \\
& (X, Y\otimes Z) \arrow[ur, mapsto] &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\{1,2\} \arrow[d] \arrow[dr] & \\
\{1,2\} \arrow[r] & *
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\mathcal{C}\times\mathcal{C} \arrow[d] \arrow[dr] & \\
\mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C}
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
(X,Y) \arrow[d] \arrow[r, mapsto] & X\otimes Y \arrow[d, phantom, "\wr\wr"] \\
(Y,X) \arrow[r, mapsto] & Y\otimes X
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\{1\} \arrow[d] \arrow[dr] & \\
\{1,2\} \arrow[r] & *
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
\mathcal{C} \arrow[d] \arrow[dr] & \\
\mathcal{C}\times\mathcal{C} \arrow[r] & \mathcal{C}
\end{tikzcd}
\quad
\begin{tikzcd}[column sep=small]
X \arrow[d, mapsto] \arrow[r, mapsto] & X \arrow[d, phantom, "\wr\wr"] \\
(X, 1_{\mathcal{C}}) \arrow[r, mapsto] & X\otimes 1_{\mathcal{C}}
\end{tikzcd}$$

$I \xrightarrow{\alpha} J$

$$\begin{tikzcd} 1 \arrow[r] & 1 \\ 2 \arrow[ur] & \end{tikzcd}$$

$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

2 $u_{\mathcal{C}} : 1 \to \mathcal{C}.$

3. $\mathcal{C} \to 1$ no information

Fctor: (lax) sym monoidal fctors

Special case: $[F : (C, \otimes) \longrightarrow (D, \times)] \longleftrightarrow \left[F : C^{\otimes} \longrightarrow D \text{ with conditions} \right]$

Ex. Realize $\mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, and show $f^*(- \otimes -)$, bc & proj formula.

Why is $f_! : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ $\mathcal{D}(Y)$-linear?

Category	Object	Morphism
$\mathcal{C}$	$X \quad Y$	$X \longrightarrow Y$
$\mathcal{C}^{op}$	$X \quad Y$	$X \leftarrow Y$ in $\mathcal{C}$
$(\mathcal{C}^{op})^{\sqcup}$	$(I, (X_i)_i),\ (J, (Y_j)_j)$	$\alpha : I \dashrightarrow J,\ \{ \bigotimes^{op}_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j \}_j$ in $\mathcal{C}^{op}$ $=$ $\{ \prod_{i \in \alpha^{-1}(j)} X_i \longleftarrow Y_j \}_j$ in $\mathcal{C}$
$((\mathcal{C}^{op})^{\sqcup})^{op}$	$(I, (X_i)_i),\ (J, (Y_j)_j)$	$\alpha : I \leftarrow\!\!- J,\ \{ \prod_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j \}_j$ in $\mathcal{C}$

Construction "Uniqueness of $f_!$"

Const 1. $f: Y \longrightarrow X$ $\quad f \in I\cap P \quad \Rightarrow f_! \cong f_*$

By induction,

$$\begin{tikzcd}
Y \arrow[dr, "\Delta"] \arrow[drr, bend left, equal, "I\cap P"] \arrow[ddr, bend right, equal, "I\cap P"'] & & \\
& Y\times_X Y \arrow[r, "I\cap P"] \arrow[d, "I\cap P"] & Y \arrow[d, "I\cap P"] \\
& Y \arrow[r, "I\cap P"] & X
\end{tikzcd}$$

= Initial case

= Deduced case

Const 2

$$\begin{tikzcd}
Y \arrow[dr, "P\cap I"] \arrow[drr, bend left, "I"] \arrow[ddr, bend right, "P"'] & & \\
& X'\times_X Y \arrow[r, "I"] \arrow[d, "P"] & X' \arrow[d, "P"] \\
& Y \arrow[r, "I"] & X
\end{tikzcd}$$

Const 3. E satisfies 2-out-of-3, i.e.

$$\begin{tikzcd}
Y_1 \arrow[rr, "E"] \arrow[dr, "E"'] & & Y_2 \arrow[dl, "E"] \\
& X &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_1 \arrow[drr, bend left, "E"] \arrow[dd, equal] & & \\
& & Y_2 \arrow[d, "E"] \\
Y_1 \arrow[rr, "E"] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
Y_1 \arrow[dr, "P"] \arrow[drr, bend left, "E"] \arrow[ddr, bend right, equal] & & \\
& Y_1\times_X Y_2 \arrow[r, "I"] \arrow[d, "P"] & Y_2 \arrow[d, "P"] \\
& Y_1 \arrow[r, "I"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_1 \arrow[dr, "I"] \arrow[drr, bend left, "E"] \arrow[dd, equal] & & \\
& Z_1\times_X Y_2 \arrow[r, "P"] \arrow[d, "I"] & Y_2 \arrow[d, "I"] \\
Y_1 \arrow[r, "I"] & Z_1 \arrow[r, "P"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y_1 \arrow[rr, bend left, "E"] \arrow[dr, "E"] \arrow[dd, equal] & & Y_2 \arrow[d, "I"] \\
& Z_1\times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\
Y_1 \arrow[r, "I"] & Z_1 \arrow[r, "P"] & X
\end{tikzcd}$$

Case 1 $\qquad$ Case 2 $\qquad$ Case 3

Case 1 $\quad$ Case 2

Const 4.

$$\begin{tikzcd}
Y \arrow[r, "j_2"', "I"] \arrow[d, "j_1"', "I"] & Z_2 \arrow[d, "f_2", "P"'] \\
Z_1 \arrow[r, "f_1", "P"'] & X
\end{tikzcd}$$

want: $f_{1,*}\, j_{1,!} \cong f_{2,*}\, j_{2,!}$

$$\begin{tikzcd}
Y \arrow[dr, "E"] \arrow[drr, bend left, "I"] \arrow[ddr, bend right, "I"'] & & \\
& Z_1\times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\
& Z_1 \arrow[r, "P"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[r, "I"] \arrow[rrd, bend left, "I"] \arrow[dd, equal] & Z_3 \arrow[d, "P"] \arrow[rdd, "P"] & \\
& Z_1\times_X Z_2 \arrow[r, "P"] \arrow[d, "P"] & Z_2 \arrow[d, "P"] \\
Y \arrow[r, "I"] & Z_1 \arrow[r, "P"] & X
\end{tikzcd}$$

Construction

$$
\begin{aligned}
\mathrm{Corr}(C,E) &= \delta_+^* \mathrm{Corr}(C,E)_{ct,co} \\
&\overset{\mathrm{Fun}(-,\mathcal{E})}{\sim} \delta^* \mathrm{Corr}(C,E)_{ct,co} \\
&\sim \delta^* \mathrm{Corr}(C,I,P)_{ct,co,co} \\
&\longleftarrow \delta^* \mathrm{Corr}(C,I,P)_{ct,ct,ct}
\end{aligned}
$$

$$
\begin{tikzcd}
\Delta^n \times \Delta^n \times \Delta^n \arrow[r] & \mathrm{Cat}_\infty \\
\Delta^n \times \Delta^n \arrow[r] \arrow[dr] & \mathrm{Fun}(\Delta^n, \mathrm{Cat}_\infty) \\
 & \mathrm{Fun}^L(\Delta^n, \mathrm{Cat}_\infty) \arrow[u, phantom, "\cup"] \arrow[d] \\
 & \mathrm{Fun}((\Delta^n)^{op}, \mathrm{Cat}_\infty)
\end{tikzcd}
$$