Un exemple par jour

4.2. the (primary) Hopf surface $X := \mathbb{C}^2 - \{0\} / \mathbb{Z}\gamma$

$$\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^n, \beta z_2) \quad \text{where} \begin{cases} \alpha, \beta \in \mathbb{C}, \ n \in \mathbb{N}^+ \\ 0 < |\alpha| \leq |\beta| < 1 \\ \lambda = 0 \ \text{ or } \ \alpha = \beta^n \end{cases}$$

Today: $\alpha = \beta = \frac{1}{2}$, $\lambda = 0$ applies well for $\alpha = \beta \in \mathbb{R}$, $\lambda = 0$

1 Topology: cpt complex surface

$$\begin{array}{ccccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^2 - \{0\} & & \\ & & \downarrow & & \\ & & \mathbb{CP}^1 & \overset{\mathbb{Z}}{\rightsquigarrow} & \mathbb{T}^2 \longrightarrow X \\ & & \mathbb{R}\wr & & \qquad\quad \downarrow \\ S^1 & \longrightarrow & S^3 & \overset{S^1}{\leftsquigarrow} & \qquad\quad \mathbb{CP}^1 \\ & & \downarrow & & \\ & & S^2 & & \end{array}$$

$$\begin{array}{ccc} \mathbb{C}^* & \longrightarrow & \mathbb{C}^2 - \{0\} \\ \downarrow & & \downarrow \mathbb{Z} \\ \mathbb{T}^2 & \longrightarrow & X \\ \downarrow & & \downarrow S^1 \\ S^1 & \longrightarrow & S^3 \\ & & \downarrow \\ & & S^2 \end{array}$$

2. $X \overset{\text{diffeo}}{\cong} S^3 \times S^1$

$z = [z_1, z_2] \mapsto \left(\left(\frac{z_1}{|z|}, \frac{z_2}{|z|}\right), |z| \text{ in } \mathbb{R}^+/_{2^{\mathbb{Z}}} \right)$

Thm [Kodaira III, Thm 42] X a complex surface.
$X \overset{\text{diffeo}}{\cong} S^3 \times S^1 \iff X$ is a primary Hopf surface.

$b_1 = 1$ $b_2 = 0$
X contains at least one curve $\xRightarrow{[\text{Kodaira II, Thm 34}]}$ X Hopf

$+ \ h^0(\Omega_X) = 0$

[Any $w \in H^0(X, \Omega_X)$ can be written of the form $f_1(z_1,z_2)dz_1 + f_2(z_1,z_2)dz_2$
where $\left.\begin{array}{l} f_1, f_2 \in \mathcal{O}(\mathbb{C}^2 - \{0\}) = \mathcal{O}(\mathbb{C}^2) \\ f_i(2z_1, 2z_2) = \frac{1}{2} f_i(z_1, z_2) \end{array}\right\} \Rightarrow f_1 = f_2 = 0$]

https://math.stackexchange.com/questions/780887/higher-homotopy-group-of-a-product-of-spaces

The Serre duality theorem is also true in complex geometry more generally, for compact complex manifolds that are not necessarily projective complex algebraic varieties.

Cor.:

$$\begin{array}{ccccc|c} 0 & 0 & 0 & & & 0 \\ & & 1 & & & 1 \\ & 0 & & 1 & & 1 \\ 0 & & 0 & & 0 & 0 \\ & 1 & & 0 & & 1 \\ & & 1 & & & 1 \end{array}$$

$c_2 = 0$
$c_1^2 = K^2 = 0$

(co)homology has no torsion

$\pi_n(X) \cong \pi_n(S^1) \oplus \pi_n(S^3)$

π_1	π_2	π_3	π_4	π_5	π_6	π_7	...
$\mathbb{Z}$	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	...

Cor: X is not Kähler.

3. Compute K_X $\psi = \frac{1}{z_1 z_2} dz_1 \wedge dz_2 \in H^0_{\mathcal{M}}(X, \omega_X)$
meromorphic

$C_1 := [z_1 = 0] = \mathbb{C}^*/\mathbb{Z}\gamma_2 \cong \mathbb{C}/\mathbb{Z} \oplus (\frac{1}{2\pi i} \ln 2)\mathbb{Z}$ $(\gamma_2 z_2 = 2 z_2)$

$C := [z_2 = 0] = \mathbb{C}^*/\mathbb{Z}\gamma_1 \cong \mathbb{C}/\mathbb{Z} \oplus (\frac{1}{2\pi i} \ln 2)\mathbb{Z}$ $(\gamma_1 z_1 = 2 z_1)$

$K_X = -C - C_1 \implies P_n = h^0(nK_X) = 0$ for $n \geq 1$ $\Rightarrow \kappa(X) = -\infty$

4. X is an elliptic surface by $\quad \pi: X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, $\quad [z_1, z_2] \longmapsto [z_1 : z_2]$

where C, C_1 are fibers $\Rightarrow C^2 = CC_1 = C_1^2 = 0$

5. $\operatorname{Pic} X = \mathbb{C}/\mathbb{Z} \xrightarrow{z \mapsto e^{2\pi i z}} \mathbb{C}^\times$ by

fiber $\longmapsto$?

$K_X \longmapsto$?

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 1$$

$$0 \longrightarrow \underset{=\mathbb{Z}}{H^0(X,\mathbb{Z})} \longrightarrow \underset{=\mathbb{C}}{H^0(X,\mathcal{O}_X)} \longrightarrow H^0(X,\mathcal{O}_X^\times) = \mathbb{C}/\mathbb{Z}$$

$$\longrightarrow \underset{=\mathbb{Z}}{H^1(X,\mathbb{Z})} \xrightarrow{\iota} \underset{=\mathbb{C}}{H^1(X,\mathcal{O}_X)} \longrightarrow H^1(X,\mathcal{O}_X^\times) = \operatorname{Pic} X = \mathbb{C}/\mathbb{Z}$$

$$\longrightarrow \underset{=0}{H^2(X,\mathbb{Z})} \longrightarrow \underset{=0}{H^2(X,\mathcal{O}_X)} \longrightarrow \cdots$$

explain

for cpt complex surface,

$$\begin{array}{ccccccccccc} 0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^\times & \to & 1 & & \\ & & \downarrow & & \| & & \downarrow & & & & \\ 0 & \to & \mathbb{C} & \to & \mathcal{O}_X & \xrightarrow{d} & \Omega_X & \xrightarrow{d} & \omega_X & \to & 1 \end{array}$$

$\Downarrow$

[$\iota: H^1(X,\mathbb{Z}) \hookrightarrow H^1(X,\mathbb{C}) \xrightarrow{\simeq} H^1(X,\mathcal{O}_X)$, $\quad \mathbb{Z} \hookrightarrow \mathbb{C} \xrightarrow{\simeq} \mathbb{C}$ is canonical embedding]

Rmk. ① $H^i_{mfld}(X,\mathbb{Z}) \cong H^i(X,\mathbb{Z})$ but $H^i_{Sch}(X,\mathbb{Z}) \not\cong H^i(X,\mathbb{Z})$

② exp is only defined over cplx mfld, but not on the general scheme.

③ the GAGA only applies to coherent sheaf, but $\mathbb{Z}$ is not coherent.

$\mathbb{C}^g/\mathbb{Z}^{2g-1}$ or $\prod_{i=1}^{g} \mathbb{C}/\Lambda_i$?

f.g. since [for cpt mfld X, $H^i(X,\mathbb{Z})$ or $H_i(X,\mathbb{Z})$ f.g]

$$\Rightarrow 1 \longrightarrow \operatorname{Pic}^0 X \longrightarrow \operatorname{Pic} X \longrightarrow NS(X) \longrightarrow 1$$

$\operatorname{Pic}^0 X = H^1(X,\mathcal{O}_X)/\operatorname{Im}[H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X)]$

$NS(X) = \ker[H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)]$

grey remarks are specially for a cpt complex surface.

Remark 3.3.3 Often, the image of the map $c_1 : \operatorname{Pic}(X) \to H^2(X,\mathbb{R}) \subset H^2(X,\mathbb{C})$ is called the *Néron—Severi group* $\mathrm{NS}(X)$ of the manifold X. It spans a finite dimensional real vector space $\mathrm{NS}(X)_\mathbb{R} = \mathrm{NS}(X) \otimes \mathbb{R} \subset H^2(X,\mathbb{R}) \cap H^{1,1}(X)$, where the inclusion is strict in general. The Lefschetz theorem above thus says that the natural inclusion $\mathrm{NS}(X) \subset H^{1,1}(X,\mathbb{Z})$ is an equality.

If X is projective, yet another description of the Néron–Severi group can be given. Then, $\mathrm{NS}(X)$ is the quotient of $\operatorname{Pic}(X)$ by the subgroup of numerically trivial line bundles. A line bundle L is called numerically trivial if L is of degree zero on any curve $C \subset X$.

[Huybrechts]

https://mathoverflow.net/questions/349661

Q: $\rho(X) = h^{1,1}$?

In general (may not be over $\mathbb{C}$), we only have

$$NS(X) \hookrightarrow H^2(X,\mathbb{Z}) \Rightarrow \rho(X) \le b_2$$

but over $\mathbb{C}$, we have

$$NS(X)/_{Tor} \cong \operatorname{Im}(H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C})) \cap H^{1,1}(X) = H^{1,1}(X,\mathbb{Z}) \Rightarrow \rho(X) \le h^{1,1}$$

and we may have $\rho(X) < h^{1,1}$, e.g K3 surfaces.

Cor: Except the fibers of π, there are no other curves, no sections, no (-1)-curve $\Rightarrow$ X minimal surface

Intersection matrix is ϕ.

nef cone: ϕ [for any non-proj complex surface]

effective cone: $\mathbb{C}^\times = \mathrm{Pic}(X)$

Q: $\mathrm{Aut}_1 X := \{f: X \to X \text{ holomorphic}\}$? [It's better to consider first $\mathrm{Aut}_1(\mathbb{C}^2 - 0) = \mathrm{Aut}_{1, \text{fix } 0}(\mathbb{C}^2)$]

$$\mathrm{Aut}_2 X := \left\{ f: \begin{tikzcd} X \arrow[rr] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\ & \mathbb{P}^1 & \end{tikzcd} \text{ holo} \right\} ?$$

$\mathcal{M}(X) := \{\text{meromorphic functions on } X\}$? $\mathrm{tr\,deg}_{\mathbb{C}} \mathcal{M}(X) = ?$

Ref:
[Kodaira *]: On the Structure of Compact Complex Analytic Surfaces, *;
[Huybrechts]: Complex Geometry