Eine Woche, ein Beispiel

5.14. modular representation of $\mathbb{Z}/p\mathbb{Z}$

Let $\mathcal{C} = \mathrm{rep}_\Lambda(\mathbb{Z}/p\mathbb{Z}) = \mathrm{mod}(\Lambda[\mathbb{Z}/p\mathbb{Z}])$, where
$\Lambda = \overline{\Lambda}$ is a field with char $\Lambda = p$.

Goal: understand $\mathcal{C}$ in detail.

1. indecomposable representations
2. tensor category structure
3. semisimplification

1. indecomposable representations

We have

$$\Lambda[\mathbb{Z}/p\mathbb{Z}] \cong \Lambda[x]/(x^p-1) \cong \Lambda[x]/(x-1)^p \cong \Lambda[T]/T^p$$

$$\begin{array}{ll}
N(p) & \left(\Lambda^p, \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
\vdots & \\
\uparrow\downarrow & \\
N(2) & \left(\Lambda^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
N(1) & (\Lambda, 0)
\end{array}$$

AR-quiver of $\bullet \circlearrowleft T \,/\, T^p = 0 \;=\; \Lambda[T]/T^p$

https://math.stackexchange.com/questions/368722/what-does-the-group-ring-mathbbzg-of-a-finite-group-know-about-g

2. tensor category structure.

For general ring $A/_{\Lambda}$, there is no tensor structure on $\mathrm{mod}(A)$.
However, for a Hopf algebra $A/_{\Lambda}$, we can construct a natural tensor structure on $\mathrm{mod}(A)$.

Construction.

$$c^{\#}: A \longrightarrow A \otimes_{\Lambda} A \quad \rightsquigarrow \quad \otimes: \mathrm{mod}(A) \times \mathrm{mod}(A) \longrightarrow \mathrm{mod}(A \otimes_{\Lambda} A) \xrightarrow{c^{\#,*}} \mathrm{mod}(A)$$

$$(M, N) \longmapsto M \otimes_{\Lambda} N \longmapsto M \otimes_{\Lambda} N$$

where A acts on $M \otimes_{\Lambda} N$ by

$$A \times M \otimes_{\Lambda} N \longrightarrow M \otimes_{\Lambda} N \qquad a \cdot (m \otimes n) := c^{\#}(a)(m \otimes n) = \sum_i b_i m \otimes c_i n$$

when $c^{\#}(a) = \sum_i b_i \otimes c_i$

$$e^{\#}: A \longrightarrow \Lambda \quad \rightsquigarrow \quad e^{\#,*}: \mathrm{mod}(\Lambda) \longrightarrow \mathrm{mod}(A)$$

$$\Lambda \longmapsto \Lambda$$

$$A \times \Lambda \longrightarrow \Lambda \qquad (a, t) \longmapsto e^{\#}(a) \cdot t$$

$$i^{\#}: A \longrightarrow A^{op} \quad \rightsquigarrow \quad (-)^{\vee}: \mathrm{mod}(A) \xrightarrow{\mathrm{Hom}_{\Lambda}(-, \Lambda)} \mathrm{mod}(A^{op}) \xrightarrow{i^{\#,*}} \mathrm{mod}(A)$$

$$M \longmapsto M^{\vee} \longmapsto M^{\vee}$$

$$A \times M^{\vee} \longrightarrow M^{\vee} \qquad (a, f) \longmapsto f(i^{\#}(a) -)$$

Q: Let A be a Λ-alg.
Given a tensor category structure on $\mathrm{mod}(A)$, can we recover the Hopf algebra on A?
I.e., is the map

$$\left\{ \begin{matrix} \text{Hopf algebra structures} \\ \text{on } A \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{tensor category structures} \\ \text{on } \mathrm{mod}(A) \end{matrix} \right\}$$

inj or surj?

E.g. (tensor category structure of mod $(\Lambda[G])$) $\quad$ G: finite gp

$\mathrm{rep}_\Lambda(G)$ is naturally endowed with $\otimes$-structure:

$$G \circlearrowright M\otimes N \qquad g\cdot(m\otimes n) := gm\otimes gn$$
$$\rightsquigarrow \Lambda[G]\circlearrowright M\otimes N \qquad (\textstyle\sum_i t_i g_i)(m\otimes n) = \sum_i t_i\, g_i(m\otimes n) = \sum_i t_i(g_i m\otimes g_i n) = (\sum_i t_i(g_i\otimes g_i))(m\otimes n)$$

so the Hopf algebra structure on $\Lambda[G]$ should be

$$c^\#: \Lambda[G]\longrightarrow \Lambda[G]\otimes_\Lambda \Lambda[G] \qquad \textstyle\sum_i t_i g_i \longmapsto \sum_i t_i g_i\otimes g_i$$
$$e^\#: \Lambda[G]\longrightarrow \Lambda \qquad \textstyle\sum_i t_i g_i \longmapsto \sum_i t_i$$
$$i^\#: \Lambda[G]\longrightarrow \Lambda[G]^{op} \qquad \textstyle\sum_i t_i g_i \longmapsto \sum_i t_i g_i^{-1}$$

Verify:

$$G\circlearrowright \Lambda \qquad g\cdot t := t$$
$$\rightsquigarrow \Lambda[G]\circlearrowright \Lambda \qquad (\textstyle\sum_i t_i g_i)\, t = \sum_i t_i(g_i\cdot t) = \sum_i t_i t$$

$$G\circlearrowright M^\vee \qquad g\cdot f := f(g^{-1}\cdot -)$$
$$\rightsquigarrow \Lambda[G]\circlearrowright M^\vee \qquad (\textstyle\sum_i t_i g_i) f = \sum_i t_i(g_i\cdot f) = \sum_i t_i f(g_i^{-1}\cdot -) = f(\sum_i t_i g_i^{-1} -)$$

e.p. $\mathrm{Spec}\,\Lambda[\mathbb{Z}/n\mathbb{Z}] \cong \mu_{n,\Lambda}$ as a finite gp scheme.

E.g. (tensor category structure of mod $(\mathcal{U}(\mathfrak{g}))$) $\quad$ $\mathfrak{g}$: f.d. Lie alg over $\mathbb{C}$

$\mathrm{rep}_{\mathbb{C}}(\mathfrak{g})$ is naturally endowed with $\otimes$-structure:

$$\mathfrak{g}\circlearrowright M\otimes N \qquad X\cdot(m\otimes n) := X\cdot m\otimes n + m\otimes X\cdot n$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M\otimes N \qquad X_1X_2\cdots X_n(m\otimes n) = \sum_{\{1,\cdots,k\}=I\sqcup J} (X_I m)\otimes(X_J n) \quad \text{shuffle!}$$

e.p. $[X,Y](m\otimes n) = [X,Y]m\otimes n + m\otimes[X,Y]n$

(For $I=\{i_1,\cdots,i_l\}$ fix an order $i_1<i_2<\cdots<i_l$, $X_I := X_{i_1}X_{i_2}\cdots X_{i_n}$)

so the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ should be

$$c^\#: \mathcal{U}(\mathfrak{g})\longrightarrow \mathcal{U}(\mathfrak{g})\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g}) \qquad X_{\{1,\cdots,k\}}\longmapsto \sum_{\{1,\cdots,k\}=I\sqcup J} X_I\otimes X_J$$
$$e^\#: \mathcal{U}(\mathfrak{g})\longrightarrow \mathbb{C} \qquad \textstyle\sum_\alpha t_\alpha X_\alpha \longmapsto t_\emptyset$$
$$i^\#: \mathcal{U}(\mathfrak{g})\longrightarrow \mathcal{U}(\mathfrak{g})^{op} \qquad \textstyle\sum_\alpha t_\alpha X_\alpha \longmapsto \sum_\alpha (-1)^{|\alpha|} t_\alpha X_\alpha$$

Verify:

$$\mathfrak{g}\circlearrowright \mathbb{C} \qquad X\cdot t := 0$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright \mathbb{C} \qquad (\textstyle\sum_\alpha t_\alpha X_\alpha).t = t_\emptyset t$$

$$\mathfrak{g}\circlearrowright M^\vee \qquad X\cdot f := \overset{\text{minus!}}{-} f(X\cdot -)$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M^\vee \qquad (\textstyle\sum_\alpha t_\alpha X_\alpha).t = \sum_\alpha t_\alpha (-1)^{|\alpha|} f(X_\alpha\cdot -) = f(\sum_\alpha (-1)^{|\alpha|} t_\alpha X_\alpha\cdot -)$$

For more examples of Hopf algebras, see wiki: Hopf algebras.

3. semisimplification.

$\mathrm{Ver}_p := \overline{\mathcal{C}}$ is a fusion category with simple objects.

see wiki, or lecture notes.

$$\overline{N(1)}, \ldots, \overline{N(p-1)}, \text{ denoted as } X_1, \ldots, X_{p-1}.$$

⚠ For $M, N \in \mathrm{Ver}_p$, T acts on $M \otimes N$ by

$$\begin{aligned} T(m \otimes n) &= (x-1)(m \otimes n) \\ &= xm \otimes xn - m \otimes n \\ &= (T+1)m \otimes (T+1)n - m \otimes n \\ &= Tm \otimes Tn + Tm \otimes n + m \otimes Tn \end{aligned}$$

So we don't have $T(m \otimes n) = Tm \otimes Tn$, i.e. T is not a group-like element.

Lemma. In any Ver_p,

$$X_2 \otimes X_i \cong \begin{cases} X_0 \oplus X_2 & i = 1 \\ X_{i-1} \oplus X_{i+1} & 1 < i < p-1 \\ X_{p-2} \oplus X_p & i = p-1 \end{cases}$$

virtual minus sign

If we write $X_2 \otimes X_i = X_{i-1} \oplus X_{i+1}$, we need to assume $X_0 = X_p = 0$, $X_{p+1} = -X_{p-1}, \cdots$

Proof. Let $M = \begin{pmatrix} J_i & J_i \\ & J_i \end{pmatrix}$, $J_i = \mathrm{Id} + N_i$, $N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$,

we need to find the Jordan normal form of M.

$$M - I = \begin{pmatrix} N_i & J_i \\ & N_i \end{pmatrix} = \begin{pmatrix} N_i & \\ & N_i \end{pmatrix} + \begin{pmatrix} & J_i \\ & \end{pmatrix}$$

Since N_i commutes with J_i, $\begin{pmatrix} N_i & \\ & N_i \end{pmatrix}$ commutes with $\begin{pmatrix} & J_i \\ & \end{pmatrix}$,

$$\begin{aligned} (M-I)^l &= \left(\begin{pmatrix} N_i & \\ & N_i \end{pmatrix} + \begin{pmatrix} & J_i \\ & \end{pmatrix} \right)^l \qquad \text{for } l \in \mathbb{N}_{>0} \\ &= \sum_{k=0}^{l} \binom{l}{k} \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^{l-k} \begin{pmatrix} & J_i \\ & \end{pmatrix}^k \\ &= \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^l + l \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^{l-1} \begin{pmatrix} & J_i \\ & \end{pmatrix} \\ &= \begin{pmatrix} N_i^l & l N_i^{l-1} J_i \\ & N_i^l \end{pmatrix} \end{aligned}$$

$$\left.\begin{aligned} \mathrm{rk}(M-I) &\geq 2p-2 \\ (M-I)^i &\neq 0 \\ (M-I)^{i+1} &= 0 \end{aligned}\right\} \Rightarrow M - I \sim \begin{array}{c} \begin{array}{cc} i+1 & i-1 \end{array} \\ \left(\begin{array}{cccc|ccc} 0 & 1 & & & & & \\ & \ddots & \ddots & & & & \\ & & \ddots & 1 & & & \\ & & & 0 & & & \\ \hline & & & & 0 & \ddots & \\ & & & & & \ddots & 1 \\ & & & & & & \end{array}\right) \end{array}$$

$\square$

E.g. $p=2$:

$\otimes$	X_1
X_1	X_1

$p=3$:

$\otimes$	X_1	X_2
X_1	X_1	X_2
X_2	X_2	X_1

$p=5$:

$\otimes$	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_2	$X_1 \oplus X_3$	$X_2 \oplus X_4$	X_3
X_3	X_3	$X_2 \oplus X_4$	$X_1 \oplus X_3$	X_2
X_4	X_4	X_3	X_2	X_1

e.g.
$$\begin{aligned} X_3 \otimes X_4 &= (X_2 \otimes X_2 - X_1) \otimes X_4 \quad \text{virtual minus sign} \\ &= X_2 \otimes (X_2 \otimes X_4) - X_1 \otimes X_4 \\ &= X_2 \otimes X_3 - X_4 \\ &= X_2 \oplus X_4 - X_4 \\ &= X_2 \end{aligned}$$

[To be rigorous, you can compute
$(X_3 \oplus X_1) \otimes X_4 = X_2 \otimes X_2 \otimes X_4 = X_2 \oplus X_4 \Rightarrow X_3 \otimes X_4 = X_2$]

Other cases are similar.

$$\begin{aligned} X_3 \otimes X_3 &= (X_2 \otimes X_2 - X_1) \otimes X_3 \\ &= X_2 \otimes (X_2 \otimes X_3) - X_3 \\ &= X_1 \oplus 2X_3 - X_3 \\ &= X_1 \oplus X_3 \end{aligned}$$

$$\begin{aligned} X_4 \otimes X_4 &= (X_3 \otimes X_2 - X_2) \otimes X_4 \\ &= X_3 \otimes (X_2 \otimes X_4) - X_2 \otimes X_4 \\ &= X_3 \otimes X_3 - X_3 \\ &= X_1 \end{aligned}$$

non-trivial sub $\otimes$-category: $\langle X_1, X_3 \rangle_\oplus$, $\langle X_1, X_4 \rangle_\oplus$.

Rmk. In this case,

$$\begin{aligned} & X_3 \otimes X_3 = X_1 \oplus X_3 \\ \Rightarrow\ & (\mathrm{PFdim}\, X_3)^2 = 1 + \mathrm{PFdim}\, X_3 \\ \Rightarrow\ & \mathrm{PFdim}\, X_3 = \frac{\sqrt{5}+1}{2} \end{aligned}$$

Here, PFdim means the Perron-Frobenius dimension.
Therefore, we find an object with non-integer dimension!

$p=7$:

$\otimes$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	X_4	X_5	X_6
X_2	X_2	$X_1\oplus X_3$	$X_2\oplus X_4$	$X_3\oplus X_5$	$X_4\oplus X_6$	X_5
X_3	X_3	$X_2\oplus X_4$	$X_1\oplus X_3\oplus X_5$	$X_2\oplus X_4\oplus X_6$	$X_3\oplus X_5$	X_4
X_4	X_4	$X_3\oplus X_5$	$X_2\oplus X_4\oplus X_6$	$X_1\oplus X_3\oplus X_5$	$X_2\oplus X_4$	X_3
X_5	X_5	$X_4\oplus X_6$	$X_3\oplus X_5$	$X_2\oplus X_4$	$X_1\oplus X_3$	X_2
X_6	X_6	X_5	X_4	X_3	X_2	X_1

$$
\begin{aligned}
X_3\otimes X_k &= (X_2\otimes X_2 - X_1)\otimes X_k \\
&= X_2\otimes(X_2\otimes X_k) - X_1\otimes X_k \\
&= X_2\otimes(X_{k-1}\oplus X_{k+1}) - X_k \\
&= X_{k-2}\oplus X_k\oplus X_{k+2}
\end{aligned}
$$

$X_7 = X_0 = 0,\ X_8 = -X_6, \ldots\ldots$

$$
\begin{aligned}
X_4\otimes X_k &= (X_3\otimes X_2 - X_2)\otimes X_k \\
&= X_3\otimes(X_2\otimes X_k) - X_2\otimes X_k \\
&= (X_3\otimes X_{k-1})\oplus(X_3\otimes X_{k+1}) - X_{k-1} - X_{k+1} \\
&= X_{k-3}\oplus X_{k-1}\oplus X_{k+1}\oplus X_{k+3}
\end{aligned}
$$

...

non-trivial sub $\otimes$-category : $\langle X_1, X_6\rangle_\oplus$, $\langle X_1, X_3, X_5\rangle_\oplus$

Ex. Check that

$$\mathrm{PFdim}\, X_{p-2} = 2\cos\frac{\pi}{p}$$

PFdim X_k (p \ X_k)	X_1	X_2	X_3	X_4	X_5	X_6	X_7
2	1	—	—	—	—	—	—
3	1	1	—	—	—	—	—
5	1	$2\cos\frac{\pi}{5}$	$2\cos\frac{\pi}{5}$	1	—	—	—
7	1	$2\cos\frac{\pi}{7}$	$1+\frac{1}{2\cos\frac{\pi}{7}-1}$	$1+\frac{1}{2\cos\frac{\pi}{7}-1}$	$2\cos\frac{\pi}{7}$	1	—
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Rmk. (from course) $\quad \mathrm{Ver}_p \cong \overline{\mathrm{Tilt}_\Delta(SL_2)}$, where

semisimplification

$$\mathrm{Tilt}_\Delta(SL_2) = \langle V \rangle_{\oplus,\otimes} = \langle \{V^k\}_{k\geq 0} \rangle_{\otimes}$$

$V := \Delta^2$: standard rep.

In general, for a split conn red gp $G/\mathbb{F}_p$, let $\mathcal{U}$ be the sylow p-subgp of $G(\mathbb{F}_p)$, then

$$\overline{\mathrm{rep}_\Delta(\mathcal{U})} \cong \overline{\mathrm{Tilt}_\Delta(G)}$$

e.g. for $G = GL_n$, $\mathcal{U} = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ is the Heisenberg gp.

Need reference for this remark.