Eine Woche, ein Beispiel

8.17 tropical hypersurface

Ref:

https://arxiv.org/abs/1311.2360

How to draw these tropical curves:
https://mathoverflow.net/questions/328342/how-to-draw-tropical-curves
https://ntiggemann.github.io/coding.html#Plotting_tropical_curves

K: valued field $\quad v: K \longrightarrow \mathbb{R} \cup \{+\infty\}$ $\quad$ most time: $\mathbb{Z} \cup \{+\infty\}$

$X \subseteq \mathbb{A}_K^n$ variety

$$x \in X(K) \quad \Rightarrow \quad -v(x) \in \mathrm{Trop}(X)$$
$$Y \subseteq X \quad \Rightarrow \quad \mathrm{Trop}(Y) \subset \mathrm{Trop}(X)$$

! If we want compatability of $v(x)$ with $\oplus$, then we should define $u \oplus v = \min(u, v)$.
Usually tropical people don't do this, they want

"addition of positive number should go up".

We respect the conventions.

$$v(x+y) \geqslant \min(v(x), v(y))$$
$$v(xy) = v(x) + v(y)$$
$$v(0) = +\infty$$

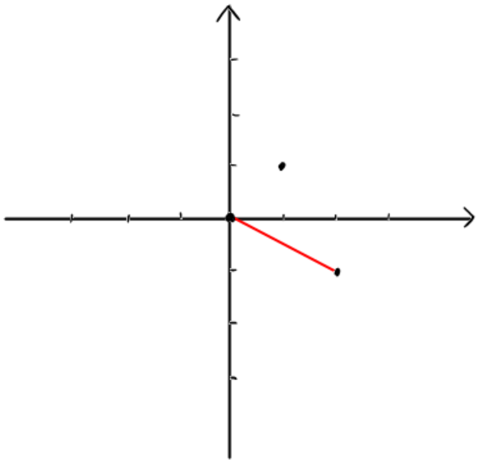
$u \oplus' v = \min(u, v)$
$u \otimes' v = u + v$
$+\infty \quad \mathbb{T} = \mathbb{R} \cup \{+\infty\}$
read from bottom

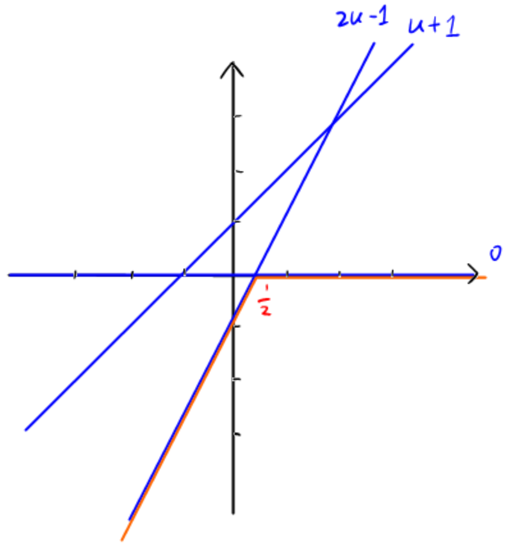
$u \oplus v = \max(u, v)$
$u \otimes' v = u + v$
$-\infty$
read from above

in calculation

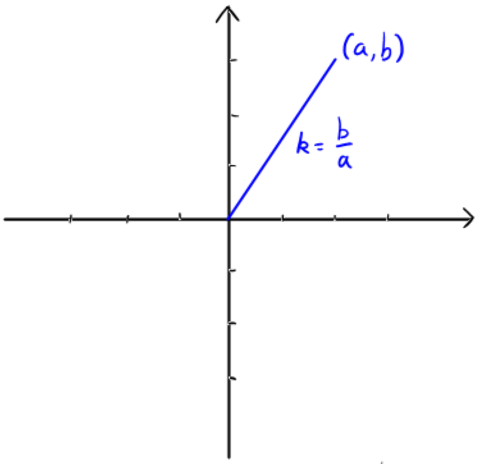
Relation with Newton polygon

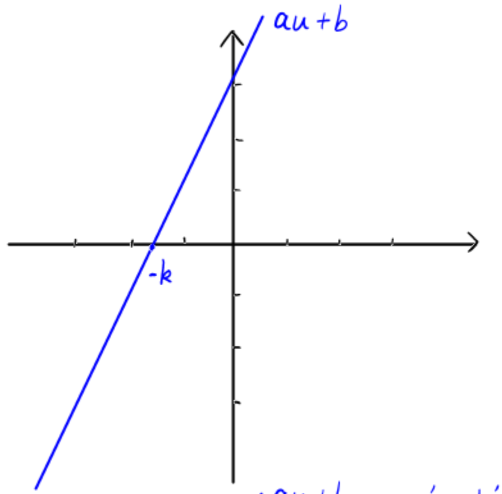
E.g.

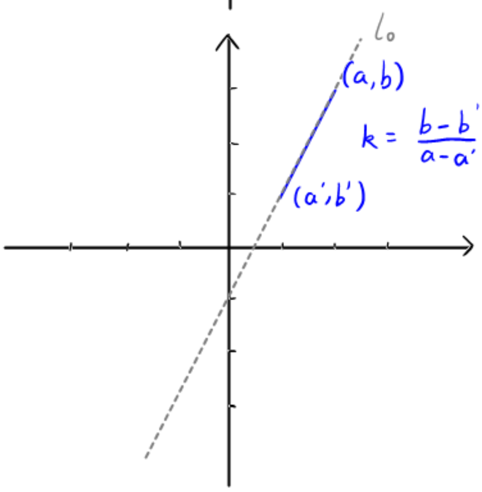


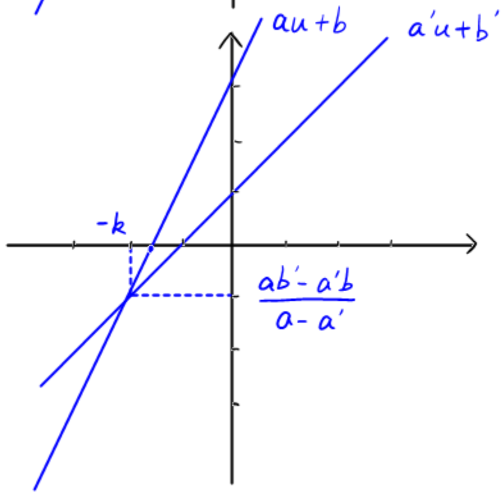


$1+5z+\frac{1}{5}z^2$ $\qquad$ $0 \oplus' (u+1) \oplus' (2u-1)$



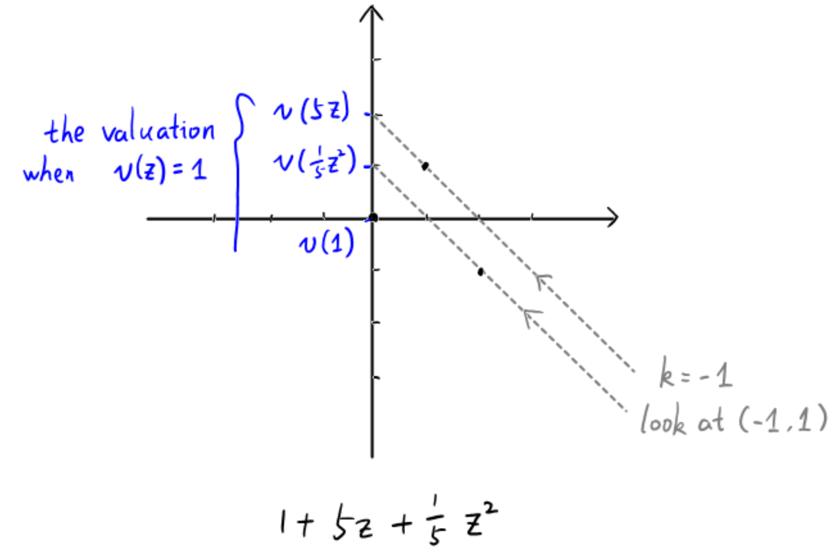






Rmk. $\quad p=(x,y)\in l_0 \iff l_p =$ "$xu+y$" passes through $\left(-k, \frac{ab'-a'b}{a-a'}\right)$

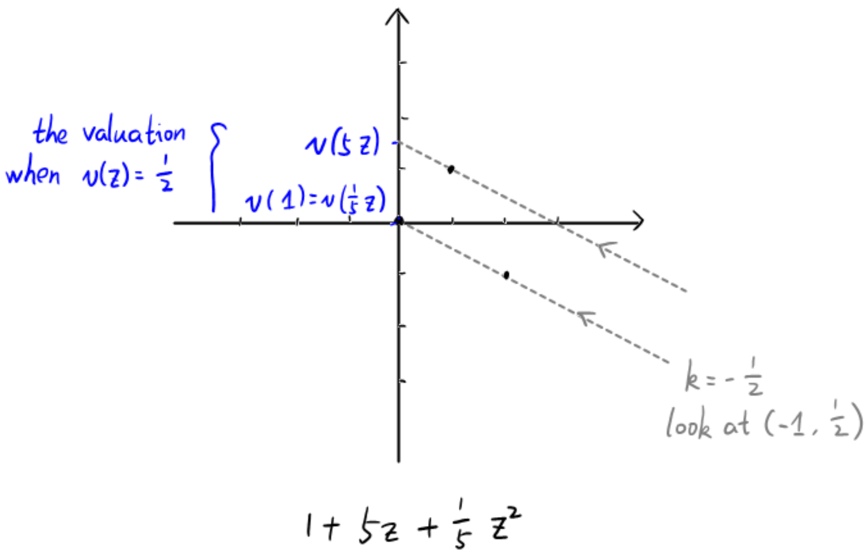
better point of view



A special valuation of z may be seen as a kind of projection.

You can then read the value as though from the markings of a graduated cylinder.

It is curious that mathematicians read numbers from unexpected angles, rather than from the usual horizontal view.

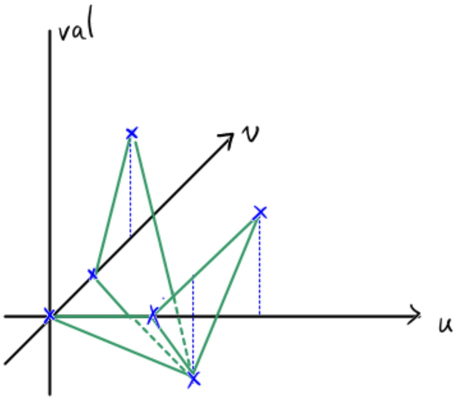


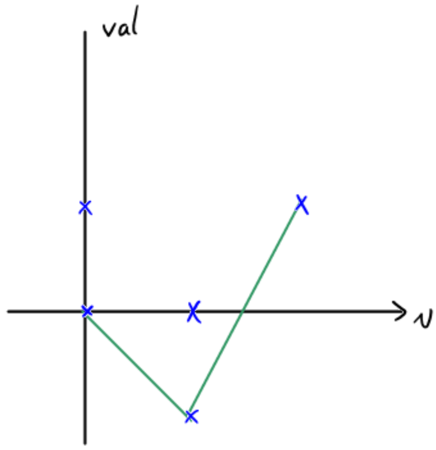
When the two values meet and rest at the very bottom among all values, we have the possibility that v(f)=+\infty.

This happens when the gaze brings the two points into perfect alignment; the negative of the slope of this sightline is v(z).

That line is exactly the lower convex edge of the Newton polygon.

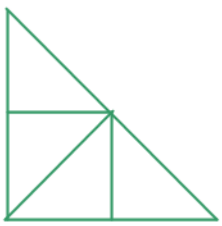
The Newton polygon has a higher version.

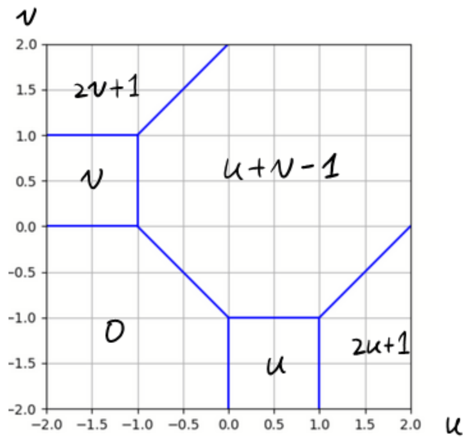




$$1 + x + y + 5x^2 + \tfrac{1}{5}xy + 5y^2$$
$$0 \oplus' u \oplus' v \oplus' (2u+1) \oplus' (u+v-1) \oplus' (2v+1)$$

when $u = 0$,
we are taking projections.

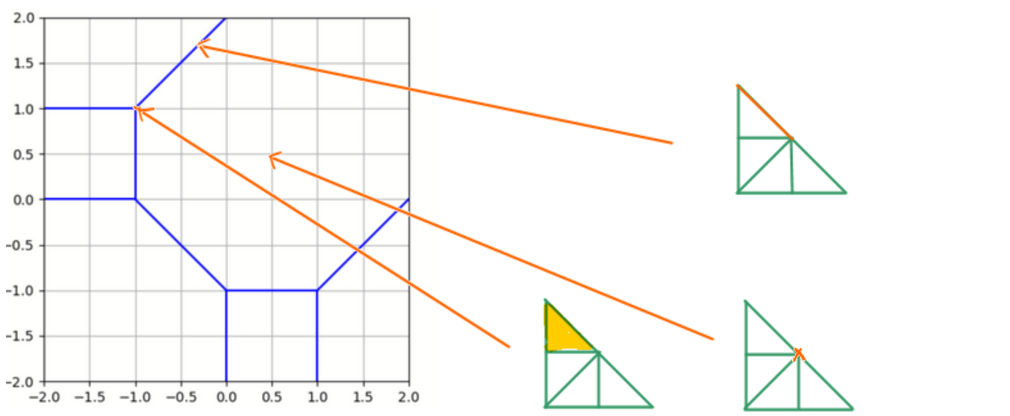




dual subdivisions
i.e., the projection of Newton polygon

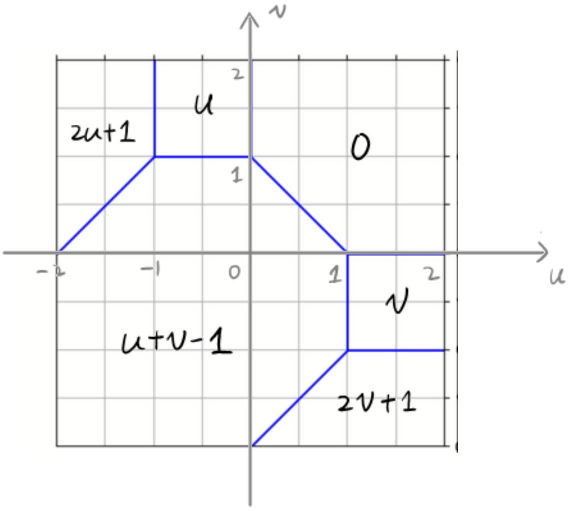
tropical curve (max version)

0+x+y+(-1)*x2+1*xy+(-1)*y2
The software only compute the maximal version.



what do we see from the bottom

⚠ The minimal version:



This is the correct one. but software doesn't produce it automatically.

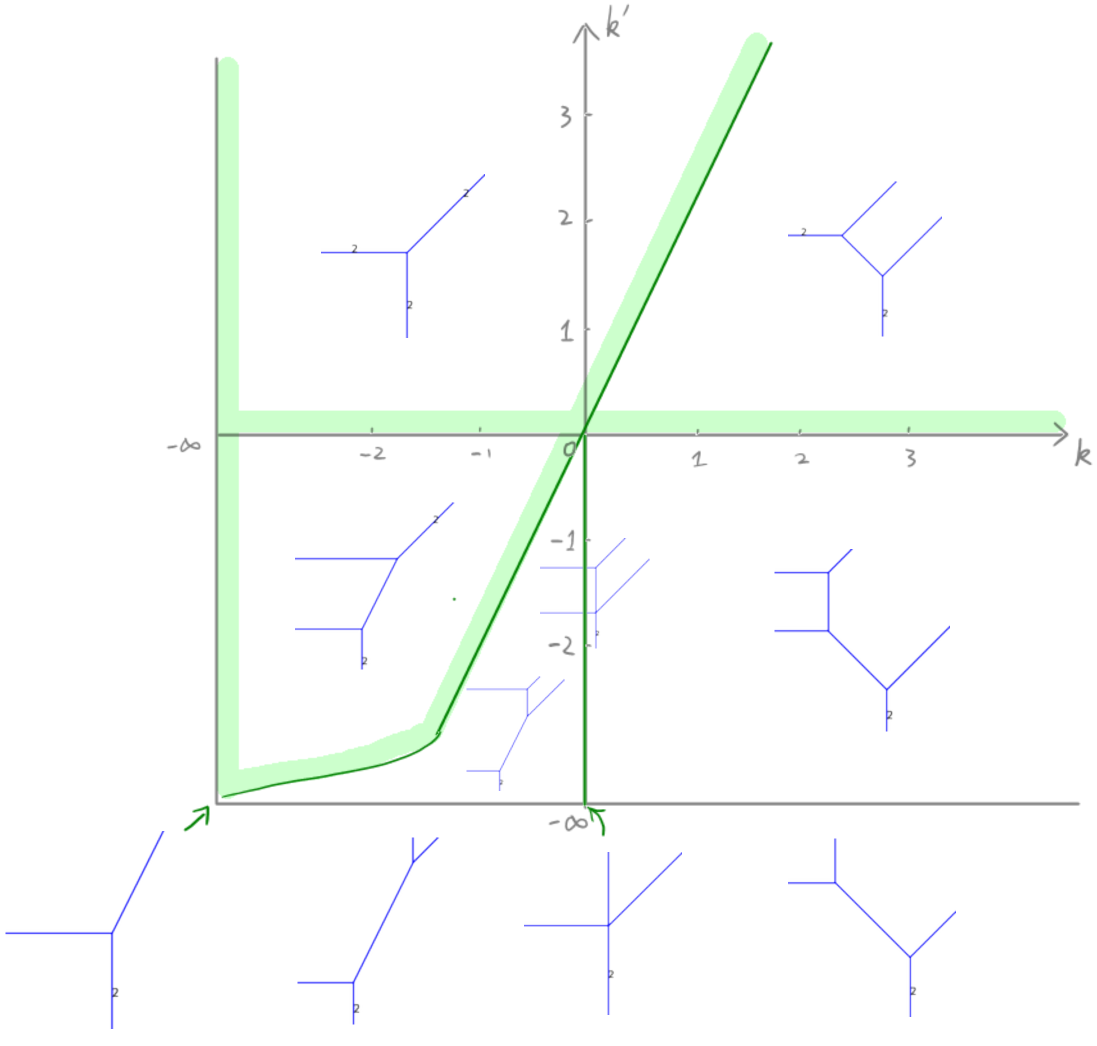
E.g. We want to draw the tropical curve crspding to

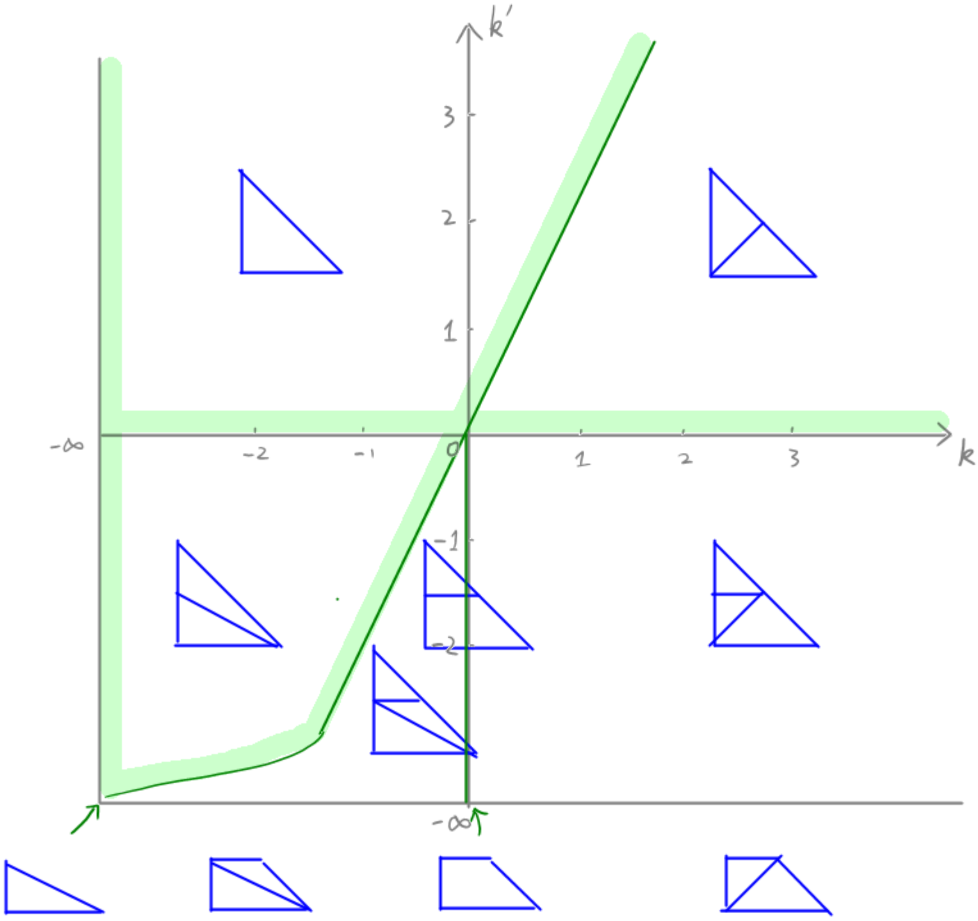
$$1 + x^{-1} + y^{-1} + x^{-2} + \zeta^{-k}x^{-1}y^{-1} + \zeta^{-k'}y^{-2}$$

$$0 \oplus' (-u) \oplus' (-v) \oplus' (-2u) \oplus' (-u-v-k) \oplus' (-2v-k')$$

$$0 \oplus u \oplus v \oplus 2u \oplus u+v+k \oplus 2v+k'$$

0+x+y+x2+(k)xy+(k')y2





k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞

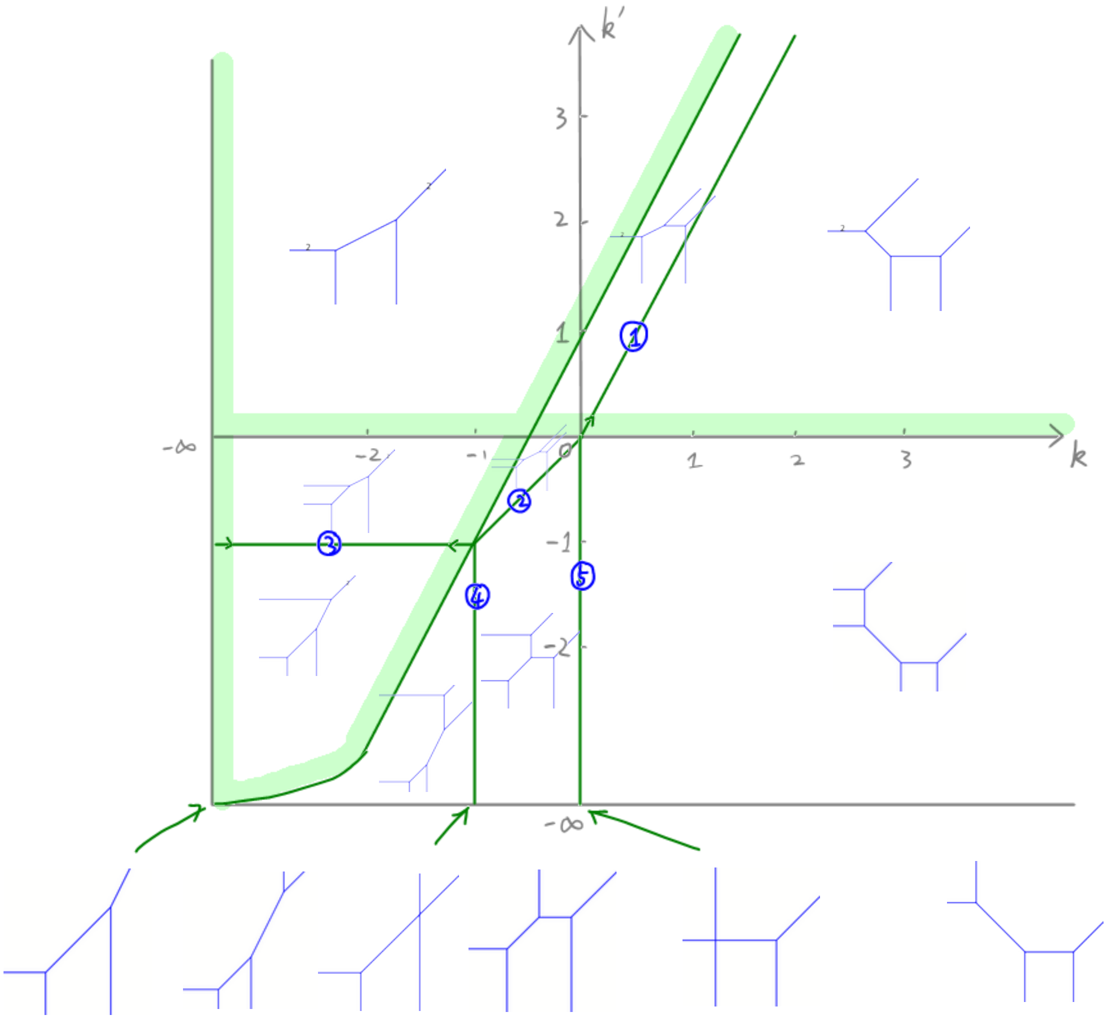
E.g. We want to draw the tropical curve crsp'ding to

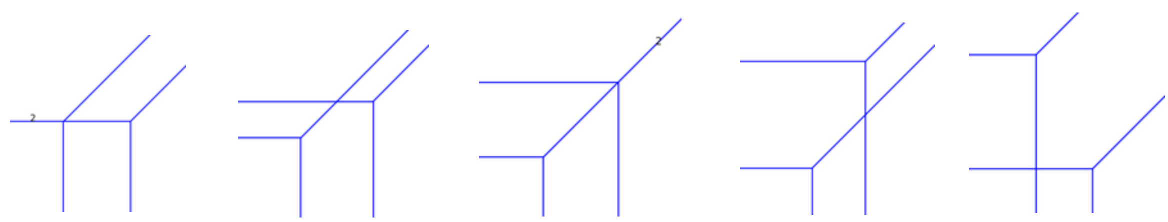
$$1 + x^{-1} + y^{-1} + \frac{1}{5}x^{-2} + 5^{-k}x^{-1}y^{-1} + 5^{-k'}y^{-2}$$

$$0 \oplus' (-u) \oplus' (-v) \oplus' (-2u+1) \oplus' (-u-v-k) \oplus' (-2v-k')$$

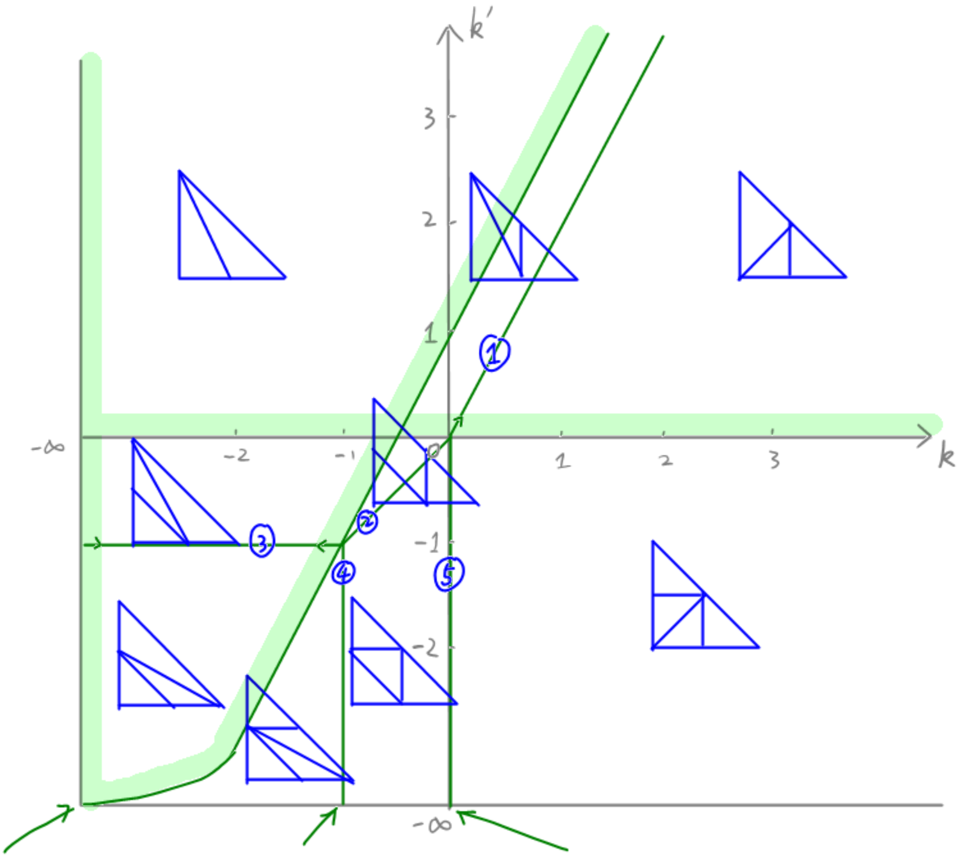
$$0 \oplus u \oplus v \oplus (2u-1) \oplus u+v+k \oplus 2v+k'$$

0+x+y+(-1)x2+(k)xy+(k')y2





1 2 3 4 5



k'
k
3
2
1
0
-1
-2
-∞
-2
-1
1
2
3
-∞
1
2
3
4
5

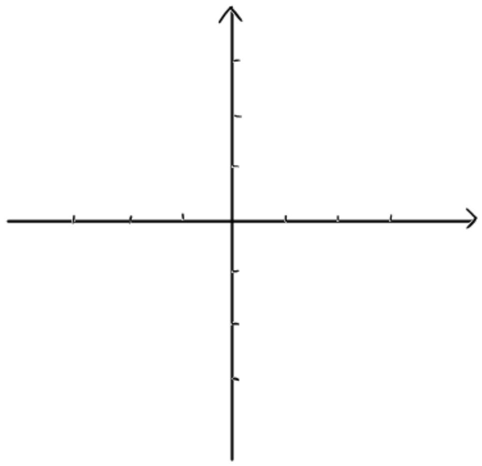
1
2
3
4
5

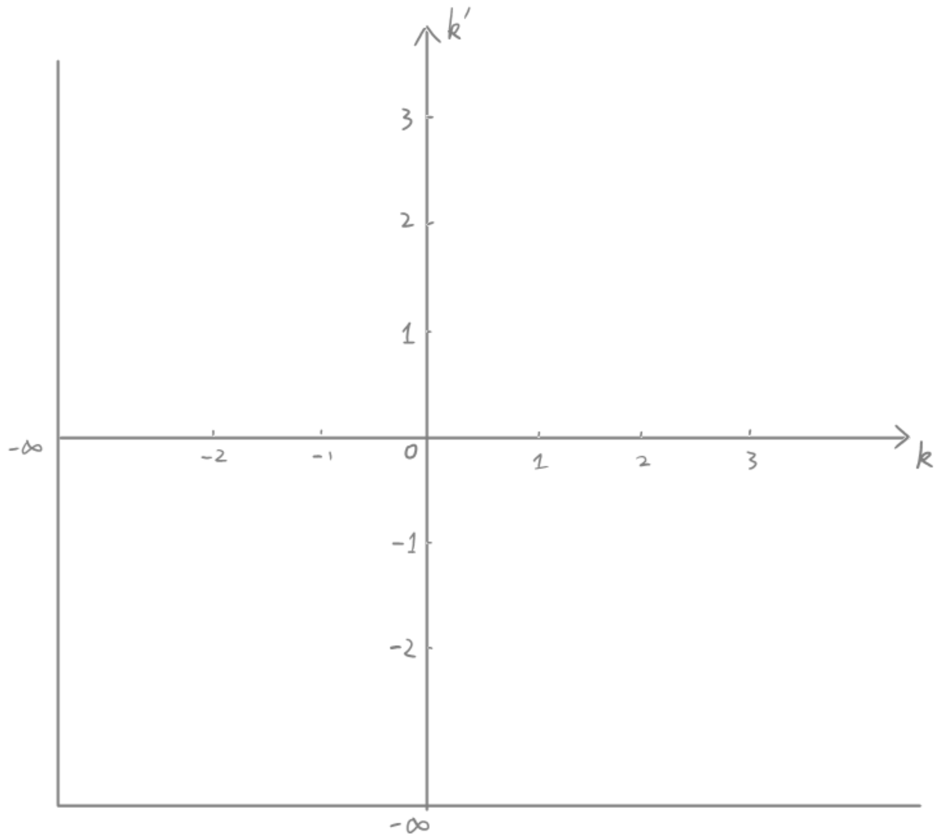
The above examples tell us the structure of the tropical hypersurface

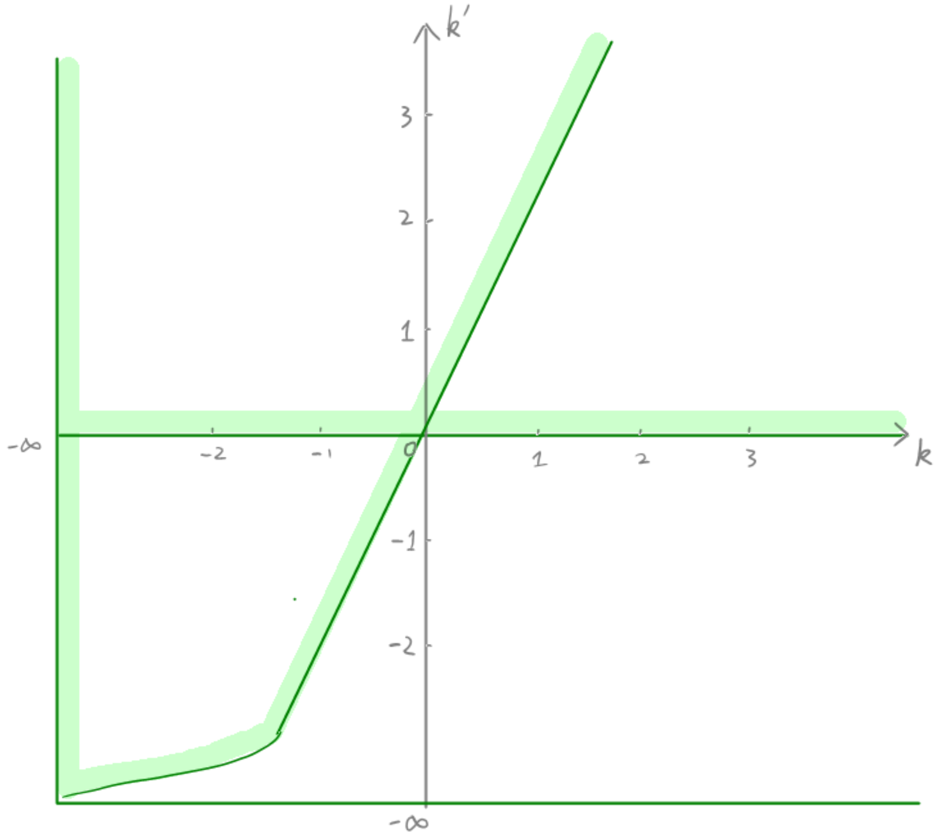
$$0 \oplus u \oplus v \oplus 2u+k'' \oplus u+v+k \oplus 2v+k' \quad \text{in } \mathbb{R}^5.$$

The stratification has some wall-crossing behavior: the tropical curve change the topology when it cross over the wall.

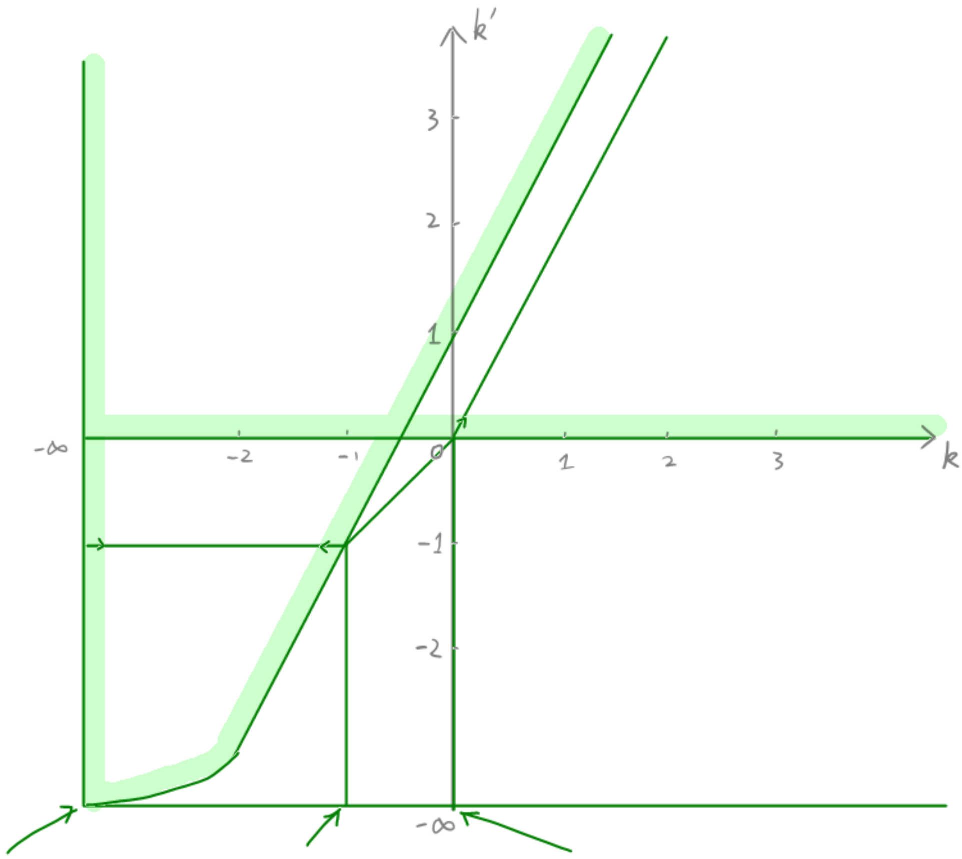
Left for future copy







k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞



k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞