## Eine Woche, ein Beispiel 8.21 equivariant K-theory of P'

Let us do a simple case over IP'. It can be generlized "easily" to flag variety, but IP' is the beginning case of study.

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

where 
$$SL_{*} = SL_{*}, C$$
,  $B = (*, *) \subseteq SL_{*}, C$ ,  $B = (*, *) \subseteq S$ 

We want to see

- · ring structure, module structure
- · Weyl gp action
- relations

e.g. 
$$K^{B}(X) \cong R(B) \otimes_{R(G)} K^{G}(X) \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^{G}(X)$$
  
 $(K^{B}(X))^{\mathbf{w}} \cong K^{G}(X)$ 

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Notation. For linear alg gp G [Ginz, J.1], K_{i}^{G}(X) := K_{i}(Coh(X)) \qquad K^{G}(X) := K_{o}^{G}(X) \qquad K(X) := K^{fid}(X)
R(G) := K^{G}(pt) = K_{o}(Coh^{G}(pt)) = K_{o}(Rep G)
e.g. R(fid) = \mathbb{Z}, R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}], R(SL_{i} \cong \mathbb{Z}[x], R(SL_{i} \times \mathbb{C}^{x}) \cong \mathbb{Z}[x, t^{\pm 1}]
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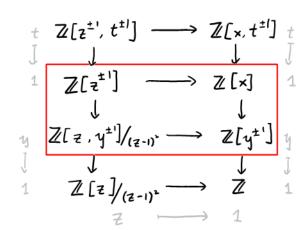
Some further discussion of R(SL2).

 $R(SL_1) = \bigoplus_{i \in \mathbb{N}_{>0}} \mathbb{C} \times_i$  where  $\times_i$  represents the (i+1)-dim irr rep of  $SL_2$ . As an algebra,  $R(SL_2) = \mathbb{C}[\times]$  where

$$X_0 = 1$$
 $X_1 = X$ 
 $X_2 = X^2 - 1$ 
 $X_4 = X^4 - 3X^2 + 1$ 
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[LCBE, 2.1.1] 
$$K(P') \cong \mathbb{Z}\mathcal{O}_{P'} \oplus \mathbb{Z}\mathcal{O}_{P'}(1) = \mathbb{Z}[\mathbb{Z}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\pm 1}/(\mathbb{Z}-1)^2]$$
 $\mathbb{Z}$  corresponds to  $\mathbb{Z}$ [ $\mathbb{Z}$ ]/ $\mathbb{Z}$  gives def of pushforward.

In conclusion, we get



The difficult part is the middle square. Z[z,y\*1]/(z-1) - Z[y\*]

Right: by rep theory,  $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}]$  homo as  $\mathbb{Z}$ -alg

 $\begin{array}{cccc} x, & \longmapsto & y+y^{-1} \\ x_1 & \longmapsto & y^2+1+y^{-2} \\ x_3 & \longmapsto & y^3+y+y^{-1}+y^{-3} \end{array}$ 

Up by Borel-Weil-Bott theorem.

$$\mathbb{Z}\left[z^{\pm 1}\right] \longrightarrow \mathbb{Z}[x] \qquad \exists \longmapsto \circ \quad homo \text{ as } \mathbb{Z}[x] \text{-module}$$

$$\downarrow \longmapsto \qquad 1 \qquad \qquad z^2 \longmapsto -1$$

$$\downarrow z^{-1} \qquad \longmapsto \qquad x, \qquad \qquad z^3 \longmapsto -x,$$

$$\downarrow z^{-2} \qquad \longmapsto \qquad x_3 \qquad \qquad z^5 \longmapsto -x_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

Left: by [LW-BWB, Ex 2.6], 
$$L_n \cong O(-n)$$
, combined with "Up", we get  $\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2$   
e.g.  $z^3 \longmapsto -z^3(y+y^{-1})$  (see table below)

$$z = z^2 \quad z^{-1} \quad | \quad z \quad z^2 \quad z^3 \quad z^4 \quad | \quad z^{-2} \quad z^{-2} \quad z^{-2} \quad | \quad z^{-2} \quad z^{-2} \quad | \quad z$$

Under these (natural) ring structure, 
$$\mathbb{Z}[x,t^{\pm i}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}] \longrightarrow \mathbb{Z}$$
 are homo of rings.

Ex. Generalize to 
$$SL_1 \longrightarrow SL_n$$
,  $P' \longrightarrow Flag(C')$   
 $SL_2 \longrightarrow GL_2$   
 $C \longrightarrow FP$   $C^* \longrightarrow FP$   
 $Q:$  How to compute  $K_i^{SL_1 \times C^*}(P')$  for  $i \ge 1$ ?