Be careful about the ring structure here!

Most isomorphisms are not iso as algebras.

I will revise this document after the vacation.

Let us do a simple case over IP'. It can be generlized "easily" to flag variety, but IP' is the beginning case of study.

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

where
$$SL_{z} = SL_{z,C}$$
, $B = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \subseteq SL_{z,C}$, $P' = P'_{c} \cong G/B$. $G G P' = G' G P'$ maps are pushback & pullout of $P' \longrightarrow Pt$.

We want to see

- · ring structure, module structure
- · Weyl gp action
- relations

e.g.
$$K^{B}(X) \cong R(B) \otimes_{R(G)} K^{G}(X) \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^{G}(X)$$

 $(K^{B}(X))^{\mathbf{w}} \cong K^{G}(X)$

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Notation. For linear alg gp G [Ginz, J.1], K_{i}^{G}(X) := K_{i}(Coh(X)) \qquad K^{G}(X) := K_{o}^{G}(X) \qquad K(X) := K^{fid}(X)
R(G) := K^{G}(pt) = K_{o}(Coh^{G}(pt)) = K_{o}(Rep G)
e.g. R(fid) = \mathbb{Z}, R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}], R(SL_{i} \cong \mathbb{Z}[x], R(SL_{i} \times \mathbb{C}^{x}) \cong \mathbb{Z}[x, t^{\pm 1}]
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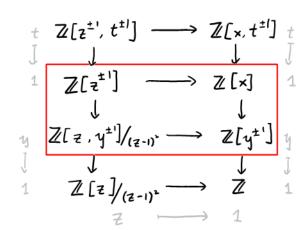
Some further discussion of R(SL2).

 $R(SL_1) = \bigoplus_{i \in \mathbb{N}_{>0}} \mathbb{C} \times_i$ where \times_i represents the (i+1)-dim in rep of SL_2 . As an algebra, $R(SL_2) = \mathbb{C}[\times]$ where

$$X_0 = 1$$
 $X_1 = X$
 $X_2 = X^2 - 1$
 $X_4 = X^4 - 3X^2 + 1$
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[LCBE, 2.1.1]
$$K(P') \cong \mathbb{Z}\mathcal{O}_{P'} \oplus \mathbb{Z}\mathcal{O}_{P'}(1) = \mathbb{Z}[\mathbb{Z}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\pm 1}/(\mathbb{Z}-1)^2]$$
 \mathbb{Z} corresponds to \mathbb{Z} [\mathbb{Z}]/ \mathbb{Z} gives def of pushforward.

In conclusion, we get



The difficult part is the middle square. Z[z,y*1]/(z-1) - Z[y*]

Right: by rep theory, $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}]$ homo as \mathbb{Z} -alg

 $\begin{array}{cccc} x, & \longmapsto & y+y^{-1} \\ x_1 & \longmapsto & y^2+1+y^{-2} \\ x_3 & \longmapsto & y^3+y+y^{-1}+y^{-3} \end{array}$

Up by Borel-Weil-Bott theorem.

$$\mathbb{Z}\left[z^{\pm 1}\right] \longrightarrow \mathbb{Z}[x] \qquad \exists \longmapsto \circ \quad homo \text{ as } \mathbb{Z}[x] \text{-module}$$

$$\downarrow \longmapsto \qquad 1 \qquad \qquad z^2 \longmapsto -1$$

$$\downarrow z^{-1} \qquad \longmapsto \qquad x, \qquad \qquad z^3 \longmapsto -x,$$

$$\downarrow z^{-2} \qquad \longmapsto \qquad x_3 \qquad \qquad z^5 \longmapsto -x_3$$

$$\vdots \qquad \vdots \qquad \vdots$$

Left: by [LW-BWB, Ex 2.6],
$$L_n \cong O(-n)$$
, combined with "Up", we get $\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2$
e.g. $z^3 \longmapsto -z^3(y+y^{-1})$ (see table below)

$$z = z^2 \quad z^{-1} \quad | \quad z \quad z^2 \quad z^3 \quad z^4 \quad | \quad z^{-2} \quad z^{-2} \quad z^{-2} \quad | \quad z^{-2} \quad z^{-2} \quad | \quad z$$

Under these (natural) ring structure,
$$\mathbb{Z}[x,t^{\pm i}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}] \longrightarrow \mathbb{Z}$$
 are homo of rings.

Ex. Generalize to
$$SL_1 \longrightarrow SL_n$$
, $P' \longrightarrow Flag(C')$
 $SL_2 \longrightarrow GL_2$
 $C \longrightarrow FP$ $C^* \longrightarrow FP$
 $Q:$ How to compute $K_i^{SL_1 \times C^*}(P')$ for $i \ge 1$?