§4.2. Modular form

https://github.com/ramified/personal_handwritten_collection/tree/main/modular_form
https://github.com/ramified/personal_tex_collection/blob/main/KleinAG_2023Sep_Talk2/KleinAG_talk2_LC_XiaoxiangZhou.pdf

I only add some left materials here. If needed, the content here will move to other documents.

Shimura

$$Sh_G(\mathcal{U}_{fin})(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q},fin})/\mathcal{U}_{fin}\mathcal{U}_{\infty}$$

$\mathcal{A}_{cusp}(GL_2, \omega)$ = space of cusp auto forms on $GL_2(\mathbb{A}_{\mathbb{Q}})$ with central char ω.

Rmk. When ω is unitary, i.e., $\omega : \mathbb{Q}^\times\backslash\mathbb{A}_{\mathbb{Q}}^\times \longrightarrow S^1$,

$$\mathcal{A}_{cusp}(GL_2, \omega) \subseteq L^2_{cusp}(GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}}); \omega)$$

has dense degree, where

$$\langle \phi, \phi' \rangle_{L^2} = \int_{GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}})} \phi(g)\, \phi'(g)\, dg$$

Hierarchy:

Siegel $\Rightarrow$ PEL $\Rightarrow$ Hodge $\Rightarrow$ abelian

Rmk.

$$GL_2(\mathbb{R}) \xrightarrow{\cong} \{\text{lattice in } \mathbb{C}\} = \left\{(z_1, z_2) \in \mathbb{C}^2 \,\middle|\, \operatorname{Im}\frac{z_1}{z_2} \neq 0\right\} \xrightarrow{\cong} \mathcal{H}^{\pm} \times \mathbb{C}^{\times}$$

$$\mathrm{Id} \longleftrightarrow \begin{pmatrix} i \\ 1 \end{pmatrix} \qquad (z_1, z_2) \longmapsto \left(\frac{z_1}{z_2}, z_2\right)$$

$$\begin{pmatrix} y_1 & x_1 \\ y_2 & x_2 \end{pmatrix} \longleftrightarrow \begin{pmatrix} x_1 + y_1 i \\ x_2 + y_2 i \end{pmatrix} \qquad (\tau z_2, z_2) \longleftarrow (\tau, z_2)$$

Why confusion?

Reason: 1. We want to normalize z_2 to 1, focusing on the first variable.

2. When we write a basis, for the most time we writing 1 in the beginning.

$$T_{Id} GL_2(\mathbb{R}) = \langle \partial_{x_\tau}, \partial_{y_\tau}, \partial_{x_z}, \partial_{y_z} \rangle$$

$$\partial_{x_\tau} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \partial_{y_\tau} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \partial_{x_z} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial_{y_z} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$= \langle \partial_\tau, \bar{\partial}_\tau, \partial_z, \bar{\partial}_z \rangle$$

$$\partial_\tau = \frac{1}{2}\begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\partial}_\tau = \frac{1}{2}\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}, \quad \partial_z = \frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad \bar{\partial}_z = \frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$= \langle \partial_\tau, \partial_z \rangle \oplus \langle \bar{\partial}_\tau, \bar{\partial}_z \rangle$$

$$= \langle E \quad Z-H \rangle \oplus \langle F, Z+H \rangle$$

where

$$E = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$F = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$E = \gamma e \gamma^{-1} \quad \text{where} \quad \gamma = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Why choosing γ: $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = \gamma \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} \gamma^{-1}$

Rmk. There are three different definitions for fcts over lattices, which confused me several times.

$$
\begin{aligned}
F: \quad GL_2(\mathbb{R}) \cong \{\text{lattices in } \mathbb{C}\} &\longrightarrow \mathbb{C} \\
(z_1, z_2)^T &\longmapsto F(z_1, z_2) \\
g &\longmapsto F(g)
\end{aligned}
$$

$F_1(z_1, z_2) = F_1(g) := f(g(i))$	$= f\left(\frac{z_1}{z_2}\right)$	Naive def	$\mathcal{H} = SL_2/SO_2$
$F_2(z_1, z_2) = z_2^{-k} F_2\left(\frac{z_1}{z_2}, 1\right)$	$= z_2^{-k} f\left(\frac{z_1}{z_2}\right)$	My def	hol fcts on lattice
$F_3(z_1, z_2) = F_3(g)$	$= \det(g)^{k-1} z_2^{-k} f\left(\frac{z_1}{z_2}\right)$	Hecke alg def	for Shimura data

Their behavior is concluded in the following table:

	$F(\gamma(z_1, z_2))$	$F\left(g\begin{pmatrix} c & -s \\ s & c \end{pmatrix}\right)$	$F(tz_1, tz_2)$
F_1	$j(\gamma, g(i))^k F(z_1, z_2)$	invariant	invariant
F_2	invariant	$e^{-ik\theta} F(g)$ $(c+si)^{-k} F(g)$	$t^{-k} F(z_1, z_2)$
F_3	invariant ($\det\gamma = 1$)	$e^{-ik\theta} F(g)$ $(c+si)^{-k} F(g)$	$t^{k-2} F(z_1, z_2)$
	$\gamma \in \Gamma(N)$	$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \in SO_2$	$t \in \mathbb{R}_{>0}$