What kind of informations do we want to get from lattices? (Goal)

- Basis, rank (Λ), Vol (Λ), sym bilinear forms
- Properties: integral, even, unimodular
- Dual lattice
- Theta fcts $\Rightarrow$ modular form
- $\mathrm{Aut}(\Lambda) := \{ f \in O(n) \mid f \text{ linear}, f(\Lambda) = \Lambda \}$
- $\mathbb{R}^r / \Lambda$: an alg surf?
- as the intersection form of simply-connected mfld of dim 4.
- with Lie alg.

Definitions [edit] [wiki: unimodular lattice]

- A **lattice** is a free abelian group of finite rank with a symmetric bilinear form $(\cdot,\cdot)$.
- The lattice is **integral** if $(\cdot,\cdot)$ takes integer values.
- The **dimension** of a lattice is the same as its rank (as a **Z**-module).
- The **norm** of a lattice element *a* is (a, a).
- A lattice is **positive definite** if the norm of all nonzero elements is positive.
- The **determinant** of a lattice is the determinant of the Gram matrix, a matrix with entries (a_i, a_j), where the elements a_i form a basis for the lattice.
- An integral lattice is **unimodular** if its determinant is 1 or −1.
- A unimodular lattice is **even** or **type II** if all norms are even, otherwise **odd** or **type I**.
- The **minimum** of a positive definite lattice is the lowest nonzero norm.
- Lattices are often embedded in a real vector space with a symmetric bilinear form. The lattice is **positive definite**, **Lorentzian**, and so on if its vector space is.
- The **signature** of a lattice is the signature of the form on the vector space.

Q: Where do we meet the lattices?

A: everywhere which has something to do with f.d linear space. (over $\mathbb{R}$ or $\mathbb{C}$) or f.g. Abelian group

- $H^n(X, \mathbb{Z})/_{\mathrm{Tor}} \subseteq H^n(X, \mathbb{C})$
- in alg num theory: $\mathcal{O}_K \longrightarrow \mathbb{R}^n$ $\qquad$ $\mathcal{O}_K^\times \longrightarrow H \subseteq \mathbb{R}^{r_1+r_2}$
- in Lie algebra: character module $\quad X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^r$
  cocharacter module $\quad X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^r$

Lots of references for this topic:
Sphere Packings, Lattices and Groups
The Sensual (Quadratic) Form
https://www.math.uni-bonn.de/people/gmartin/GitterKugelpackungenSS2022.htmpl

Un example par jour

4.7. lattice in $(\mathbb{R}^n, \langle\ ,\ \rangle)$

Today : E8 lattice

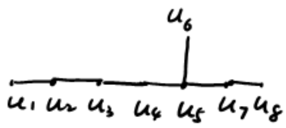
even coordinate system $\Gamma_8 = \{(x_i) \in \mathbb{Z}^8 \cup (\mathbb{Z}+\frac{1}{2})^8 \mid \sum_i x_i \equiv 0 \pmod 2\}$

$\cong$

odd coordinate system $\Gamma_8' = \{(x_i) \in \mathbb{Z}^8 \mid \sum_i x_i \equiv 0 \pmod 2\} \cup \{(x_i) \in \mathbb{Z}^8 \mid \sum_i x_i \equiv 1 \pmod 2\}$

1. $\mathrm{rank}(\Gamma_8) = 8 \qquad \mathrm{Vol}(\Gamma_8) = 1$

$$[u_1, \cdots, u_8] = \begin{bmatrix} 1 & & & & & & & -\frac{1}{2} \\ -1 & 1 & & & & & & -\frac{1}{2} \\ & -1 & 1 & & & & & -\frac{1}{2} \\ & & -1 & 1 & & & & -\frac{1}{2} \\ & & & -1 & 1 & & & -\frac{1}{2} \\ & & & & -1 & 1 & 1 & -\frac{1}{2} \\ & & & & & -1 & 1 & -\frac{1}{2} \\ & & & & & & & -\frac{1}{2} \end{bmatrix} \Rightarrow \langle u_i, u_j \rangle = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & -1 & \\ & & & & -1 & 2 & & \\ & & & & -1 & & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$



E8

$\Rightarrow$ integral, even unimodular lattice.

Prop. If Λ is integral, even unimodular lattice of rank 8, then $\Lambda \cong \Gamma_8$.

We have classification of integral unimodular lattice of low rank.