

Eine Woche, ein Beispiel. 5.7. Lie group

Temporary roadmap of Lie Group

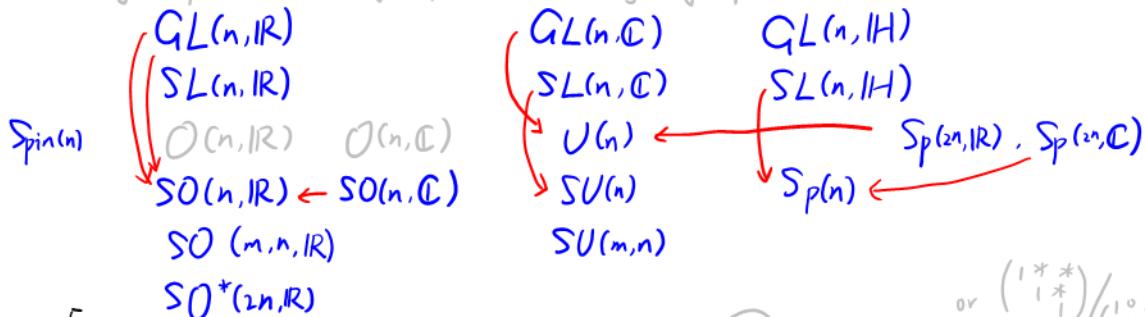
Why is it so difficult to write examples for Lie group?

- Plenty of examples, even restricted to matrix Lie groups:

- List of examples. https://en.wikipedia.org/wiki/Table_of_Lie_groups

[Some important subgroups are not written in the list,
e.g. parabolic subgroups, Heisenberg group, etc.]

$G \rightarrow K$ maximal cpt connected subgroup



[For examples of non matrix Lie group $(\widetilde{SL(2, IR)}, M_p(2n, IR))$, see

<https://math.stackexchange.com/questions/206117/is-there-a-non-matrix-lie-group>

$$\text{or } \begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix} / \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On the contrary, every f.d. Lie alg can be viewed as a matrix Lie algebra by Ado's theorem, see <http://www.math.ubc.ca/~reichst/Ado's-Theorem.pdf>.

- These groups are closely connected to each other.

e.g. the Whitehead tower:

$$\rightarrow \text{Fivebrane}(n) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow \text{O}(n)$$

- Plenty of special phenomena on low-dimensional case:

e.g. $GL(1, IR) \cong IR^*$ is not connected

e.g. $SU(2) \cong S^3$

e.g. $SO(4, IR)$ is not simple

\downarrow
 $SO(3, IR) \cong RP^3$
diff
not as variety

e.g. <https://www.zhihu.com/question/47264301/answer/320472431>

- Fruitful structures on examples

- important subgroups

e.g. maximal torus, unipotent subgroup

Borel subgroup, parabolic subgroup.

discrete subgroup.

radical, unipotent radical, derived group

$$G = [GG]R(G)$$

* centralizer & normalizer of subgroups

This article may give a classification of the cocompact subgroup:
<https://people.uleth.ca/~dave.morris/papers/cocompact.pdf>

One page for defining these groups by pasting from

Anthony W. Knapp - Representation Theory of Semisimple Groups An Overview Based on Examples

Complex groups:

$$\mathrm{GL}(n, \mathbb{C}) = \{\text{nonsingular } n\text{-by-}n \text{ complex matrices}\}$$

$$\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \det g = 1\}$$

$$\mathrm{SO}(n, \mathbb{C}) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid gg^{\mathrm{tr}} = 1\}$$

$$\mathrm{Sp}(n, \mathbb{C}) = \left\{ g \in \mathrm{SL}(2n, \mathbb{C}) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

Compact groups:

~~$$\mathrm{SO}(n, \mathbb{R}) = \mathrm{SO}(n) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g^{\mathrm{tr}} g = 1 \text{ and } g \text{ has real entries}\}$$~~

~~$$\mathrm{U}(n) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{g}^{\mathrm{tr}} g = 1\}$$~~

~~$$\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\}$$~~

~~$$\mathrm{Sp}(n) = \left\{ g \in \mathrm{U}(2n) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$~~

Real noncompact groups: We list G , \mathfrak{g} , K , and \mathfrak{k} for the noncomplex non-compact classical groups.

| G | \mathfrak{g} | K | \mathfrak{k} |
|------------------------------|--------------------------------|--|--|
| $\mathrm{SL}(n, \mathbb{R})$ | $\mathfrak{sl}(n, \mathbb{R})$ | $\mathrm{SO}(n)$ | $\mathfrak{so}(n)$ |
| $\mathrm{SL}(n, \mathbb{H})$ | $\mathfrak{sl}(n, \mathbb{H})$ | $\mathrm{Sp}(n)$ | $\mathfrak{sp}(n)$ |
| $\mathrm{SO}_0(m, n)$ | $\mathfrak{so}(m, n)$ | $\mathrm{SO}(m) \times \mathrm{SO}(n)$ | $\mathfrak{so}(m) \oplus \mathfrak{so}(n)$ |
| $\mathrm{SU}(m, n)$ | $\mathfrak{su}(m, n)$ | $\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ | $\mathfrak{s}(\mathrm{u}(m) \oplus \mathrm{u}(n))$ |
| $\mathrm{Sp}(m, n)$ | $\mathfrak{sp}(m, n)$ | $\mathrm{Sp}(m) \times \mathrm{Sp}(n)$ | $\mathfrak{sp}(m) \oplus \mathfrak{sp}(n)$ |
| $\mathrm{Sp}(n, \mathbb{R})$ | $\mathfrak{sp}(n, \mathbb{R})$ | $\mathrm{U}(n)$ | $\mathfrak{u}(n)$ |
| $\mathrm{SO}^*(2n)$ | $\mathfrak{so}^*(2n)$ | $\mathrm{U}(n)$ | $\mathfrak{u}(n)$. |

$\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{H})$ refer to matrices of determinant one with real and quaternion entries, respectively. $\mathrm{SO}_0(m, n)$, $\mathrm{SU}(m, n)$, and $\mathrm{Sp}(m, n)$ are the linear isometry groups for the Hermitian form

$$|z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2$$

defined over \mathbb{R} , \mathbb{C} , and \mathbb{H} , respectively, with the subscript “0” referring to the identity component. The group $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ for $\mathrm{SU}(m, n)$ is the subgroup of $\mathrm{U}(m) \times \mathrm{U}(n)$ of matrices of determinant one.

The group $\mathrm{Sp}(n, \mathbb{R})$ is the subgroup of real matrices in $\mathrm{Sp}(n, \mathbb{C})$ and can be conjugated by a unitary matrix so as to become

$$\{g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} J g = J\};$$

then $\mathrm{SO}^*(2n)$ is the analogous group

$$\left\{ g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}.$$

- decomposition: not easy to prove, but still of great significance.
https://en.wikipedia.org/wiki/Lie_group_decomposition
 - e.g. Jordan - Chevalley decomposition
 - group, mfld, variety, etc. Common questions.

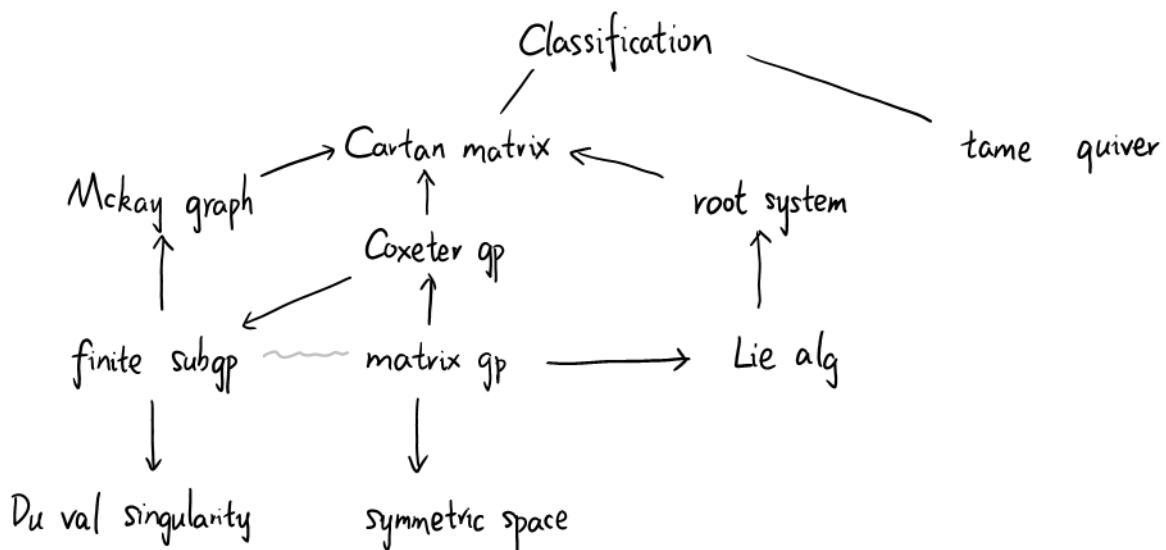
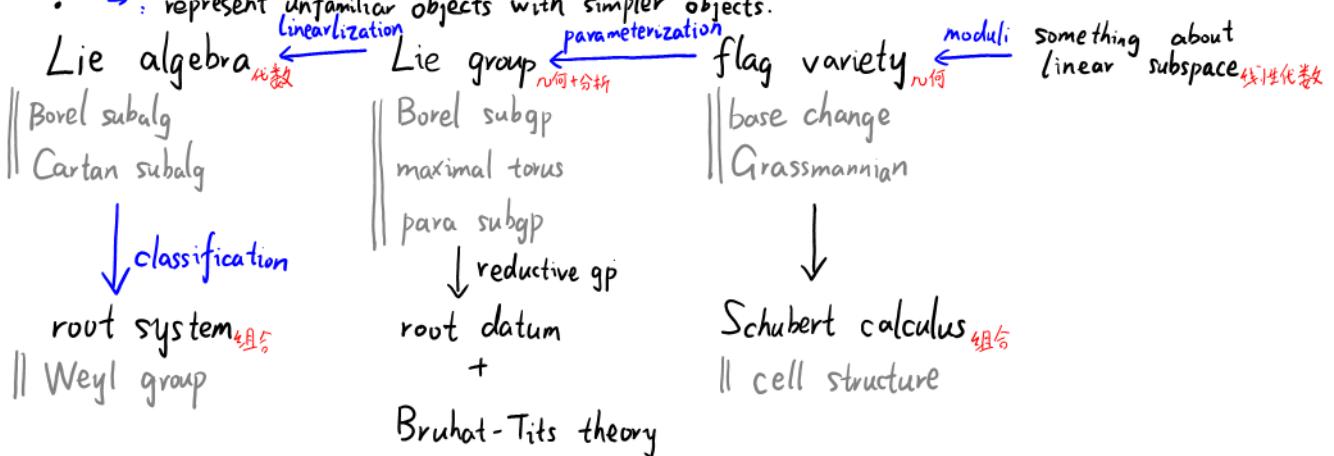
} See a thorough discussion in [23.03.26]

- Hopf algebra structure of homology & cohomology group under mild requirements.

- A large amount of related theories.

- structure theory of Lie group & Lie algebras.

- “ \rightarrow ”: represent unfamiliar objects with simpler objects.



- representation theory
 - * group /alg , f.d / inf.d
 - * highest weight theory
 - * Schul - Weyl duality
 - * Borel - Weil - Bott theorem
 - * geometric representation theory
 - * Langlands program
- algebraic topology, Bott periodicity
- algebraic groups.
- ... , see https://en.wikipedia.org/wiki/List_of_Lie_groups_topics

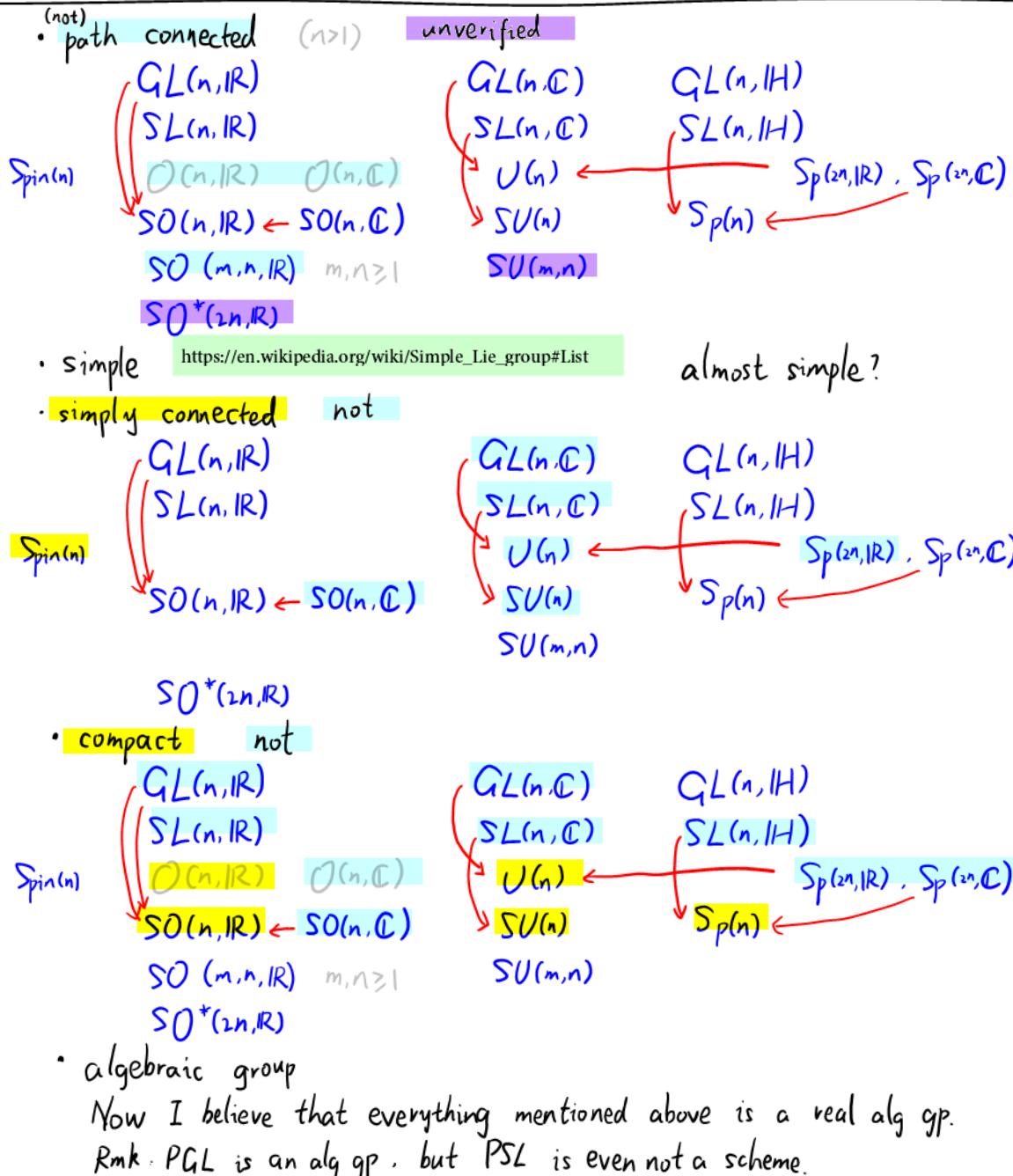
<https://mathoverflow.net/questions/235917/roadmap-to-geometric-representation-theory-leading-to-langlands>

<http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html>

Basic knowledge which has been well-written in [Naive Lie Theory].

- canonical maximal tori
- center

- (1) $Z(\mathrm{SO}(2m)) = \{\pm 1\}$.
- (2) $Z(\mathrm{SO}(2m+1)) = \{1\}$.
- (3) $Z(\mathrm{U}(n)) = \{\omega \mathbf{1} : |\omega| = 1\}$.
- (4) $Z(\mathrm{SU}(n)) = \{\omega \mathbf{1} : \omega^n = 1\}$.
- (5) $Z(\mathrm{Sp}(n)) = \{\pm 1\}$.



In alg top: homotopy groups:

$\text{typo: } \pi_*(U(n)) = \mathbb{Z}$

see

<http://felix.physics.sunysb.edu/~abanov/Teaching/Spring2009/Notes/abanov-cpA1-upload.pdf>

more about $\pi_i(SO(n, \mathbb{R}))$:

<https://ncatlab.org/nlab/show/orthogonal+group#HomotopyGroups>

more about $\pi_i(G)$ (in Chinese):

<https://www.zhihu.com/question/443652195>

more about $\pi_n(G)$:

<https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups>

$\pi_i(SO(m, n))$:

<https://math.stackexchange.com/questions/4126129/homotopy-group-of-op-q>

my try: Cartan decomposition \Rightarrow only consider cpt Lie group.

fibration + low-dimensional understanding, + Bott periodicity

e.g. • $O(1) = S^0$, a two-point discrete space

$U(n) \cong SU(n) \times S^1$

• $SO(1) = \{1\}$

• $SO(2) \cong S^1$

• $SO(3) \cong RP^3$ [3]

• $SO(4)$ is doubly covered by $SU(2) \times SU(2) = S^3 \times S^3$.

https://en.wikipedia.org/wiki/Orthogonal_group

| Homotopy groups of orthogonal groups | | | | | | | | |
|--------------------------------------|---------|---------|------------------|--------------------|--------------------|-----------------------|--------------------|--------------------|
| π_1 | π_2 | π_3 | π_4 | π_5 | π_6 | π_7 | π_8 | π_9 |
| $SO(2)$ | Z | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $SO(3)$ | $[Z_2]$ | 0 | Z | Z_2 | Z_2 | Z_{12} | Z_2 | Z_2 |
| $SO(4)$ | Z_2 | $[0]$ | $(Z)^{\times 2}$ | $(Z_2)^{\times 2}$ | $(Z_2)^{\times 2}$ | $(Z_{12})^{\times 2}$ | $(Z_2)^{\times 2}$ | $(Z_2)^{\times 2}$ |
| $SO(5)$ | Z_2 | 0 | $[Z]$ | Z_2 | Z_2 | 0 | Z | 0 |
| $SO(6)$ | Z_2 | 0 | Z | $[0]$ | Z | 0 | Z | Z_{24} |
| $SO(n), n > 6$ | Z_2 | 0 | Z | 0 | 0 | 0 | 0 | 0 |

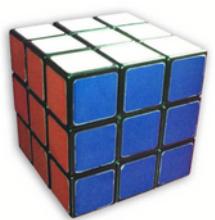
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my understanding

fibration: [Hatcher Example 4.54, 4.55]

space of n -frames in \mathbb{R}^k

$$\begin{array}{ccccccc}
 V_{n-m}(\mathbb{R}^{k-m}) & \rightarrow & V_n(\mathbb{R}^k) & O(n, \mathbb{R}) & \rightarrow & O(n+1, \mathbb{R}) & U(n) \rightarrow U(n+1) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 V_m(\mathbb{R}^k) & & & S^n & & & S^{2n+1} \\
 & & & & \downarrow & & \\
 & & & & S^n & & \\
 SO(n, \mathbb{R}) & \rightarrow & SO(n+1, \mathbb{R}) & & & & \\
 & & & & & & \\
 SO(n, \mathbb{R}) & \rightarrow & O(n, \mathbb{R}) & & \mathbb{Z}/2\mathbb{Z} & \rightarrow & Spin(n) \\
 & & & \downarrow & & & \downarrow \\
 & & & \mathbb{Z}/2\mathbb{Z} & & & SO(n)
 \end{array}$$



reductive group
 G



Borel subgroup
 B



maximal torus
 T

Structure theory of Lie group

Ref:

<http://www.math.columbia.edu/~maki/sumi/cld/reductivegroups.pdf>

← notations, results, examples, pictures.

Today:

$$G = GL(2, \mathbb{C})$$

center

$$Z(G) = \mathbb{C}^{\times} \quad \pi_1(G) = \mathbb{Z}$$

radical

$$\text{rad}(G) := R(G) = \mathbb{C}^{\times}$$

reductive

$$\text{Unipotent rad} \quad \text{rad}_u(G) := R_u(G) = \{I\}$$

$$\text{rank } G = 2$$

canonical maximal torus $T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$C_G(T) = T$$

$$N_G(T) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cup \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$W(G) = S_2 \quad \text{Weyl group} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$X^* = X^*(T) = \langle \varepsilon_i : (t_1 t_2) \mapsto t_i \rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \alpha \mapsto \alpha(g^{-1}g)$$

$$X_{\neq} = X_{\neq}(T) = \left\langle \begin{array}{l} \varepsilon_{1,*} : t \mapsto (t_1) \\ \varepsilon_{2,*} : t \mapsto (1, t) \end{array} \right\rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \lambda \mapsto g \lambda(-)g^{-1}$$

$$\Phi = \{\varepsilon_1 - \varepsilon_2, -(\varepsilon_1 + \varepsilon_2)\} := \{\alpha_{12}, -\alpha_{12}\}$$

$$\Phi^V = \{\alpha_{12}^V, -\alpha_{12}^V\} = \{\varepsilon_{1,*} - \varepsilon_{2,*}, -(\varepsilon_{1,*} + \varepsilon_{2,*})\}$$

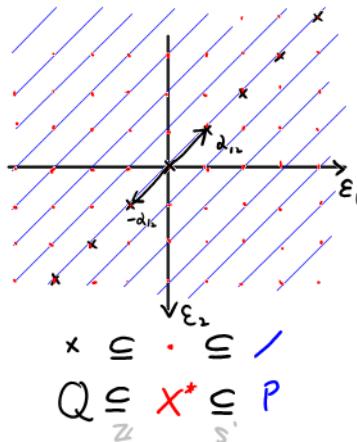
$r_?$ = action of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $X^* \xrightarrow{\quad} X^*$

| | |
|-------------------------------------|-----------------|
| $\varepsilon_1 \xrightarrow{\quad}$ | ε_1 |
| $\varepsilon_2 \xrightarrow{\quad}$ | ε_2 |

$r_?$ is reflection and $r_? = r_{\alpha_{12}}$

$x - r_?(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_1 \varepsilon_2 - x_2 \varepsilon_1 = (x_1 - x_2) \alpha_{12} \quad x := x_1 \varepsilon_1 + x_2 \varepsilon_2$

$\langle -, \alpha_{12}^V \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^V = \varepsilon_{1,*} - \varepsilon_{2,*}$



$$G = SL(2, \mathbb{C})$$

$$Z(G) = \{\pm 1\} \quad \pi_1(G) = \{*\}$$

$$\text{rad}(G) = \{I\} \Rightarrow \text{semisimple}$$

$$\text{rad}_u(G) = \{I\}$$

$$\text{rank } G = 1$$

canonical maximal torus $T = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$

$$C_G(T) = T$$

$$N_G(T) = \{t' \mid t' \sim t\}$$

$$W(G) = S_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

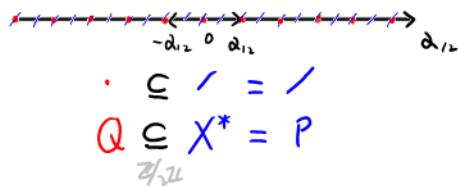
$$X^* = \langle \varepsilon_i : (t_{-t}) \mapsto t \rangle_{\mathbb{Z}} \quad \text{with } \alpha \mapsto \alpha(g^{-1} - g) \quad \alpha_1 = 2\varepsilon_1!$$

$$X_t = \langle \varepsilon_i^t : t \mapsto (t_{-t}) \rangle_{\mathbb{Z}} \quad \text{with } \lambda \mapsto g \cdot \lambda(-) \cdot g^{-1}$$

$$\Phi = \{\alpha_{12}, -\alpha_{12}\} = \{2\varepsilon_1, -2\varepsilon_1\}$$

$$\vec{\Phi}^V = \{d_{1+}^V, -d_{12}^V\} = \{\varepsilon_{1,+} - \varepsilon_{1,+}\}$$

$$\boxed{\begin{aligned} r_1 &= \text{action of } (\downarrow^1): X^* \longrightarrow X^* \quad \varepsilon_i \mapsto -\varepsilon_i \\ x - r_1(x) &= x_0 \varepsilon_i - x_0(-\varepsilon_i) = x_0 \alpha_{12} \xrightarrow{r_1 \text{ ref}} + r_2 = \nu_{12} \quad x_i = x_0 \varepsilon_i \\ \langle -, \alpha_{12}^V \rangle: x_0 \varepsilon_i &\mapsto x_0 \quad \Rightarrow \alpha_{12}^V = \varepsilon_{1,2} \end{aligned}}$$



$G = GL(3, \mathbb{C})$

$$\begin{aligned} Z(G) &= \mathbb{C}^\times & \pi_1(G) &= \mathbb{Z} \\ \text{rad}(G) &= \mathbb{C}^\times & \left. \begin{aligned} \text{rad}(G) &= \{I\} \end{aligned} \right\} &\Rightarrow \text{reductive} \\ \text{rank } G &= 3 & &\text{but not semi} \end{aligned}$$

canonical maximal torus $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \text{monomial matrixs}$

$W(G) = S_3$

$$X^* = \langle \varepsilon_i, \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}}$$

$$X_* = \langle \varepsilon_i^*, t \mapsto (t, t, t) \rangle_{\mathbb{Z}}$$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$$\left[\begin{array}{l} r_{\alpha_{12}} = \text{action of } \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, X^* \rightarrow X^* \\ \begin{array}{l} \varepsilon_1 \mapsto \varepsilon_2 \\ \varepsilon_2 \mapsto \varepsilon_1 \\ \varepsilon_3 \mapsto \varepsilon_3 \end{array} \end{array} \right]$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 \\ = (x_1 - x_2) \alpha_{12}$$

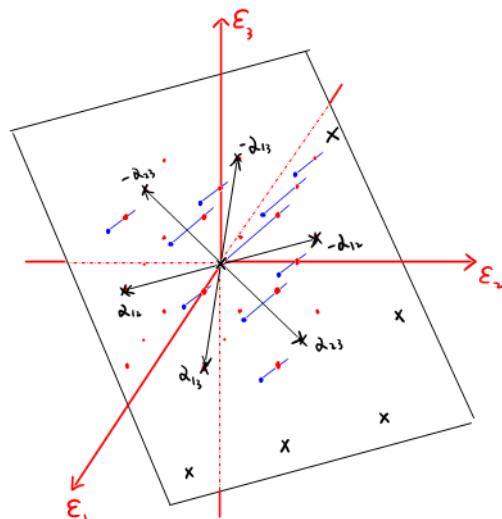
$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \Rightarrow \alpha_{12}^\vee = \varepsilon_{1,*} - \varepsilon_{2,*}$$

Similarly, $\alpha_{23}^\vee = \varepsilon_{2,*} - \varepsilon_{3,*}$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$\text{where } \alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$$

$$\begin{aligned} \varepsilon &\mapsto \varepsilon, \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \\ &= \varepsilon, \begin{pmatrix} t_2 & t_1 & t_3 \end{pmatrix} = \varepsilon_2 \end{aligned}$$



$$x \subseteq \cdot \subseteq /$$

$$Q \subseteq X^* \subseteq P$$

$G = SL(3, \mathbb{C})$

$$Z(G) = \mathbb{Z}/3\mathbb{Z} \quad \pi_1(G) = \langle \gamma \rangle$$

$\text{rad}(G) = \langle I \rangle \Rightarrow \text{semisimple}$

$\text{rad}_u(G) = \langle I \rangle$

rank $G = 2$

canonical maximal torus $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \text{monomial matrixs}$

$W(G) = S_3$

$$X^* = \langle \varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}} / \mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$X_* = \langle \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^* : \varepsilon_1^* - \varepsilon_2^* - \varepsilon_3^* \rangle_{\mathbb{Z}}$$

$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$: dual basis of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$r_{\alpha_{12}}$ = action of $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$, $X^* \rightarrow X^*$

$$\begin{aligned} \varepsilon_1 &\mapsto \varepsilon_2 \\ \varepsilon_2 &\mapsto \varepsilon_1 \\ \varepsilon_3 &\mapsto \varepsilon_3 \end{aligned}$$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

where $\alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$

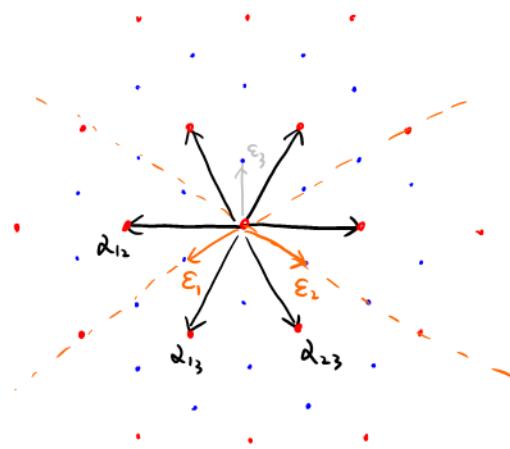
$$\begin{aligned} \varepsilon_1 &\mapsto \varepsilon_1 \left(\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \\ &= \varepsilon_1 \begin{pmatrix} t_2 & t_1 & t_3 \end{pmatrix} = \varepsilon_2 \end{aligned}$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 \quad x = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3$$

$$= (x_1 - x_2) \alpha_{12}$$

$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_{1,*} - \varepsilon_{2,*}$$

$$\text{Similarly, } \alpha_{23}^\vee = \varepsilon_{2,*} - \varepsilon_{3,*}$$



$$\begin{aligned} Q &\subseteq X^* = P \\ &\subseteq \mathbb{Z}/3\mathbb{Z} \end{aligned}$$

$$G = \mathrm{PGL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C}) / \mu_3$$

$$Z(G) = \{\mathbf{I}\} \quad \pi_1(G) = \mathbb{Z}/3\mathbb{Z}$$

$\mathrm{rad}(G) = \{\mathbf{I}\} \Rightarrow$ semisimple

$$\mathrm{rad}_u(G) = \{\mathbf{I}\}$$

rank $G = 2$

$$\mathbb{C}^* \cong \mathbb{C}_{\mu_3}^* \times \mathbb{C}^*$$

$$t^3 \leftrightarrow \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \times \begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}$$

canonical maximal torus $T = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} / \mu_3$

$$C_G(T) = T$$

$N_G(T)$ = monomial matrixs / μ_3

$$W(G) = S_3$$

$$X^* = \{n_1 \varepsilon_1 + n_2 \varepsilon_2 + n_3 \varepsilon_3 \mid n_i \in \mathbb{Z}, 3 \mid \sum n_i\} /_{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \mathbb{Z}}$$

$$= \langle \alpha_{12}, \alpha_{23} \rangle_{\mathbb{Z}}$$

$\varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \rightarrow t_i$ is no longer well-defined

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$X_* = \{n_1 \varepsilon_{1,*} + n_2 \varepsilon_{2,*} + n_3 \varepsilon_{3,*} \mid n_i \in \frac{1}{3}\mathbb{Z}, \sum n_i = 0\}$$

$$= \left\langle \frac{1}{3} \varepsilon_{1,*} + \frac{1}{3} \varepsilon_{2,*} - \frac{2}{3} \varepsilon_{3,*}, \varepsilon_{3,*} - \varepsilon_{2,*} \right\rangle_{\mathbb{Z}}$$

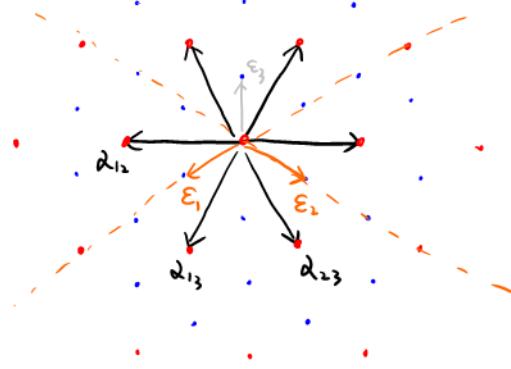
$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$: dual basis of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

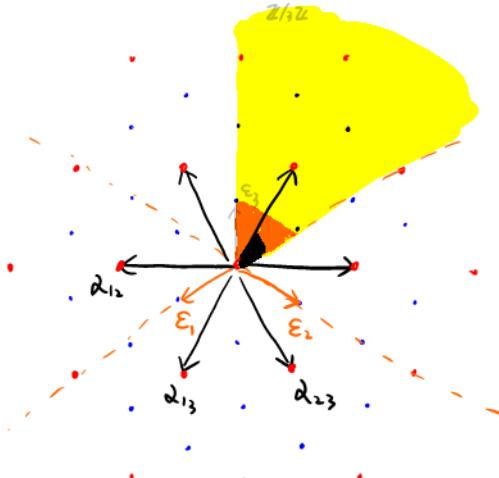
$$\alpha_{ij,*} = \varepsilon_i - \varepsilon_j$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

where $\alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$



$$Q = X^* \subseteq P$$



fundamental domain of

Weyl group []
affine Weyl group []
extended affine Weyl group []

$$G = \mathrm{Sp}(4, \mathbb{C}) = \{ A \in GL(4, \mathbb{C}) \mid A^T M A = M \}$$

$M = \begin{pmatrix} & & -1 & 1 \\ & -1 & & \\ -1 & & & \\ & & 1 & -1 \end{pmatrix}$

$Z(G) = \{\pm I\}$ $\pi_1(G) = \mathbb{Z}_2$ $\text{rad}(G) = \{I\} \Rightarrow \text{semisimple}$ $\text{rad}_u(G) = \{I\}$ $\text{rank } G = 2$

canonical maximal torus $T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3^{-1} & \\ & & & t_4^{-1} \end{pmatrix}$

$C_G(T) = T$ $N_G(T) = \text{monomial matrixs}$ $W(G) = D_4 \subseteq S_4$

$$\mathrm{sp}(4, \mathbb{C}) = \{ A \in M^{4 \times 4}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

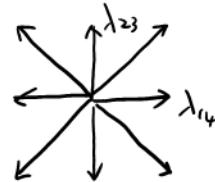
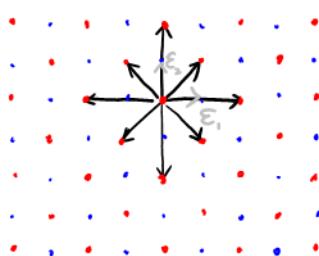
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & -a_{13} & & \\ & & a_{22} & a_{12} \\ & & -a_{21} & -a_{11} \end{pmatrix} \in M^{4 \times 4}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{14} \oplus \mathbb{Z}\lambda_{23}$$

$$\Phi = \{\pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23}\} = \{\pm (\varepsilon_1 - \varepsilon_2), \pm (\varepsilon_1 + \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$$

$$\Phi^\vee = \{\pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee\} = \{\pm (\lambda_{14} - \lambda_{23}), \pm (\lambda_{14} + \lambda_{23}), \pm \lambda_{14}, \pm \lambda_{23}\}$$



$$\Phi \subseteq Q \subseteq X^* = P$$

$\mathbb{Z}/2\mathbb{Z}$

$$\Phi^\vee$$

$$G = SO(5, \mathbb{C}) \cong \{ A \in SL(5, \mathbb{C}) \mid A^T M A = M \}$$

$$Z(G) = \{ Id \} \quad \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$$

$\text{rad}(G) = \{ I \} \Rightarrow \text{semisimple}$

$$\text{rad}_u(G) = \{ I \}$$

$$\text{rank } G = 2$$

$$M = \begin{pmatrix} & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}$$

canonical maximal torus

$$C_G(T) = T$$

$N_G(T)$ = monomial matrixs

$$W(G) = D_4$$

$$T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3 & \\ & & & t_4^{-1} \end{pmatrix}$$

$$SO(5, \mathbb{C}) = \{ A \in M^{5 \times 5}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

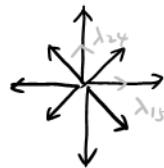
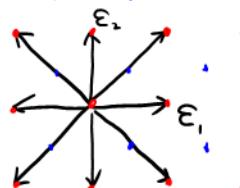
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ & 0 & a_{14} & & \\ & & 0 & a_{23} & -a_{13} \\ & & & a_{32} - a_{23} & a_{12} \\ 0 & & & & a_{41} - a_{31} \\ & & & & a_{21} - a_{11} \end{pmatrix} \in M^{5 \times 5}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_5, \varepsilon_2 + \varepsilon_4, \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{15} \oplus \mathbb{Z}\lambda_{24}$$

$$\Phi = \{ \pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23} \} = \{ \pm (\varepsilon_1 - \varepsilon_2), \pm \varepsilon_1, \pm (\varepsilon_1 + \varepsilon_2), \pm \varepsilon_2 \}$$

$$\Phi^\vee = \{ \pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee \} = \{ \pm (\lambda_{15} - \lambda_{24}), \pm 2\lambda_{15}, \pm (\lambda_{15} + \lambda_{24}), \pm 2\lambda_{24} \}$$



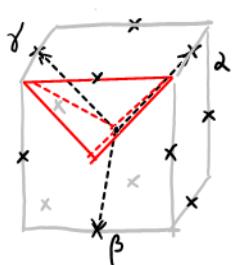
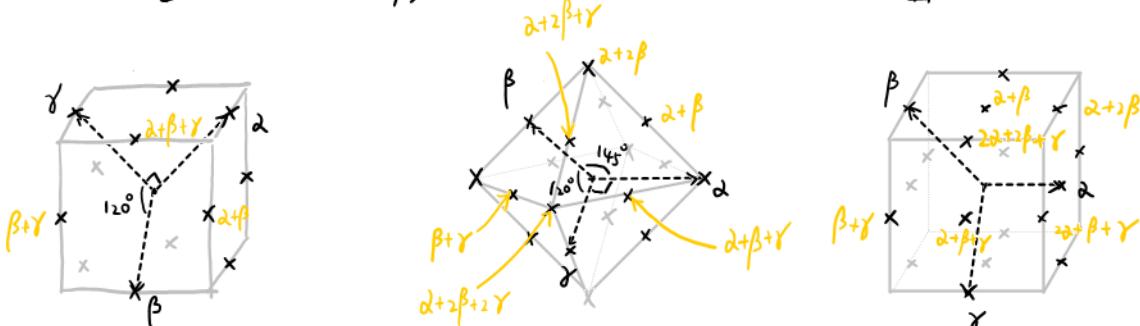
$$\Phi \subseteq Q = X^* \subseteq P$$

$\mathbb{Z}/2\mathbb{Z}$

$$\Phi^\vee$$

Now let's enjoy the beauty of low-dim root system!

| | α | β | $\alpha + \beta$ | $\alpha + 2\beta$ | $2\alpha + \beta$ | $2\alpha + 3\beta$ |
|----------------------|---|---|--|--|--------------------------|--------------------|
| Weyl group dimension | $SL(2, \mathbb{C}), A_1$ $S_2, 2$ 3 | $SL(3, \mathbb{C}), A_2$ $S_3, 6$ 8 | $Sp(4, \mathbb{C}), C_2$ $D_4, 8$ 10 | $SO(5, \mathbb{C}), B_2$ $D_4, 8$ 10 | G_2 $D_6, 12$ 14 | |
| $\pi_1(\mathbb{P})$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}$ | |



| | |
|-------------------------------|--------------------------|
| $\text{Weyl group dimension}$ | $SL(4, \mathbb{C}), A_3$ |
| $\pi_1(\Phi)$ | $S_4, 24$ |
| | 15 |
| | $\mathbb{Z}/4\mathbb{Z}$ |

| Φ | $ \Phi $ | $ \Phi^{\leq} $ | I | D | $ W $ |
|----------------------|-----------|-----------------|-----------|-------|--------------|
| A_n ($n \geq 1$) | $n(n+1)$ | | | $n+1$ | $(n+1)!$ |
| B_n ($n \geq 2$) | $2n^2$ | $2n$ | 2 | 2 | $2^n n!$ |
| C_n ($n \geq 3$) | $2n^2$ | $2n(n-1)$ | 2^{n-1} | 2 | $2^n n!$ |
| D_n ($n \geq 4$) | $2n(n-1)$ | | | 4 | $2^{n-1} n!$ |
| E_6 | 72 | | | 3 | 51840 |
| E_7 | 126 | | | 2 | 2903040 |
| E_8 | 240 | | | 1 | 696729600 |
| F_4 | 48 | 24 | 4 | 1 | 1152 |
| G_2 | 12 | 6 | 3 | 1 | 12 |

| | |
|----------------|---|
| Coxeter number | 1 |
| ↓ | 1 |
| h | 1 |
| n+1 | 2 |
| 2n | 3 |
| 2n | 3 |
| 2(n-1) | 3 |
| 12 | 3 |
| 18 | 3 |
| 30 | 3 |
| 12 | 3 |
| 6 | 3 |

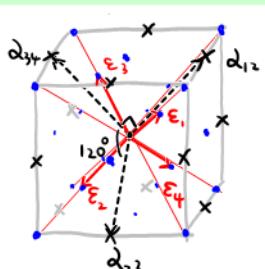
The Weyl groups for classical groups:

- The Weyl group of $SU(n)$ is still $S(n)$.
 - The Weyl group of $SO(2l+1)$ is $G(l)$, the group of permutations φ of the set $\{-l, \dots, -1, 1, \dots, l\}$ with $\varphi(-k) = -\varphi(k)$ for all $1 \leq k \leq l$.
 - The Weyl group of $SO(2l)$ is the subgroup $SG(l)$ of $G(l)$ that consists of even permutations.
 - The Weyl group of $Sp(n)$ is still $G(n)$.

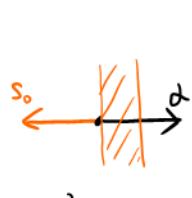
<http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2027.pdf>

https://en.wikipedia.org/wiki/Boot_system

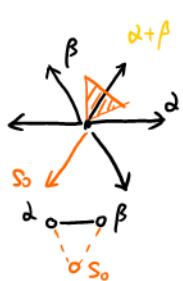
A better table can be found here: https://www.jgibson.id.au/jievis/tables_fin_aff/



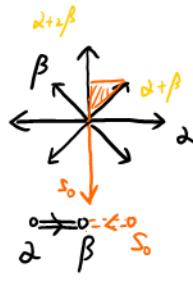
Now let's construct some associated extended Dynkin diagrams!



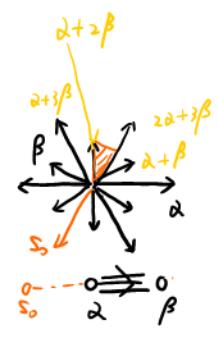
\tilde{A}_1



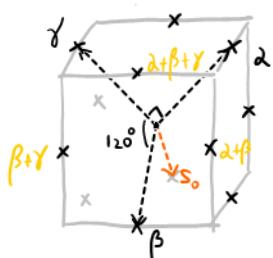
\tilde{A}_2



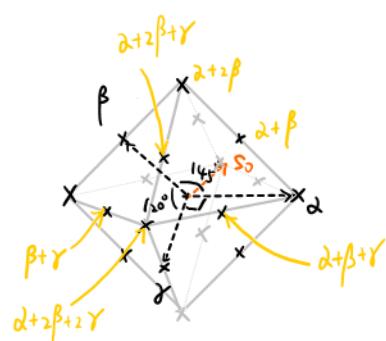
\tilde{C}_2



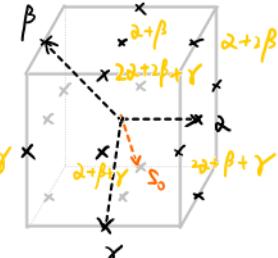
\tilde{G}_2



\tilde{A}_3

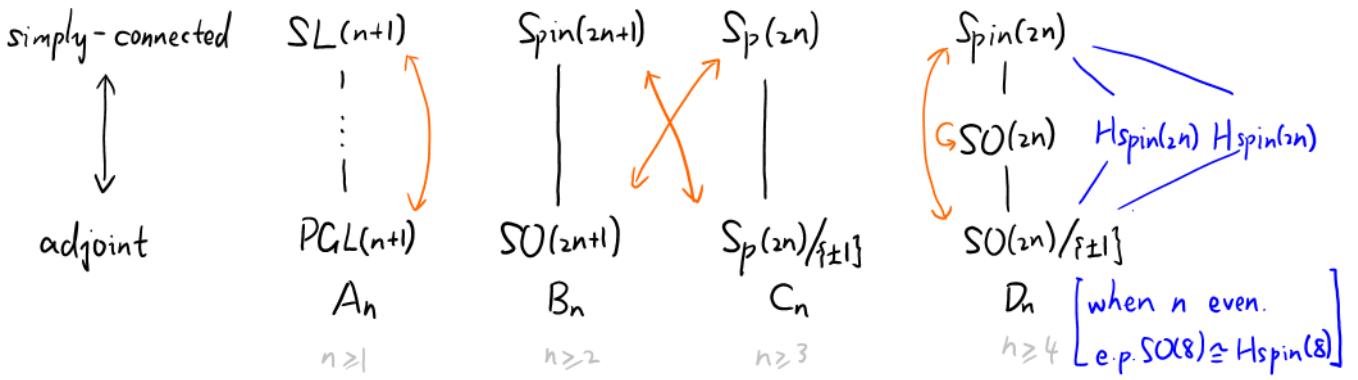


\tilde{C}_3



\tilde{B}_3

$k = \overline{k}$, $\text{char } k = 0$.



The following table comes from [linear algebraic groups and finite groups of Lie type, p72].
Errata for this book: https://www.mathematik.uni-kl.de/~malle/download/MT_errata.pdf

Table 9.2 *Isogeny types of simple algebraic groups*

| Φ | $\Lambda(\Phi)$ | G_{sc} | G_{ad} | in between |
|----------------------|------------------|----------------------|-------------------------|-------------------------------------|
| $A_{n-1}, n \geq 2$ | Z_n | SL_n | PGL_n | $\text{SL}_n/Z_d \ (d n)$ |
| $B_n, n \geq 2$ | Z_2 | Spin_{2n+1} | SO_{2n+1} | — |
| $C_n, n \geq 2$ | Z_2 | Sp_{2n} | PCSp_{2n} | — |
| $D_n, n \geq 3$ odd | Z_4 | Spin_{2n} | PCO_{2n}° | SO_{2n} |
| $D_n, n \geq 4$ even | $Z_2 \times Z_2$ | Spin_{2n} | PCO_{2n}° | $\text{SO}_{2n}, \text{HSpin}_{2n}$ |
| G_2 | 1 | G_2 | | — |
| F_4 | 1 | F_4 | | — |
| E_6 | Z_3 | $(E_6)_{\text{sc}}$ | $(E_6)_{\text{ad}}$ | — |
| E_7 | Z_2 | $(E_7)_{\text{sc}}$ | $(E_7)_{\text{ad}}$ | — |
| E_8 | 1 | E_8 | | — |