

$\zeta_K(s) \rightsquigarrow$ Hecke
 $\zeta(s) \rightsquigarrow$ Dirichlet

A partial conclusion on L-functions

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$$

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \text{ where } S_t = O(t^{1-\delta}) \text{ \& basic convergence}$$

Dirichlet L $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}$
 $\chi: (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow S'$

Hecke L $L(s, \chi) = \sum_a \frac{\chi(a)}{Na^s} = \prod_p \frac{1}{1-\chi(p)Np^{-s}}$

Artin L $L(s, \rho)$

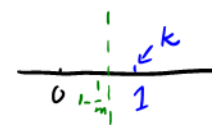
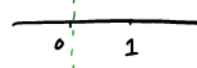
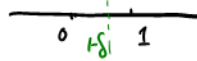
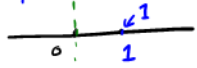
自守 L

$[K:\mathbb{Q}] = m$ $\zeta_K(s) = \sum_{I \in \mathcal{O}_K} N(I)^{-s} = \prod_{p \text{ prime}} \frac{1}{1-N(p)^{-s}}$
 $= \sum_{n=1}^{\infty} \frac{\#\{I \mid N(I)=n\}}{n^s}$

Hasse-Weil $\zeta(X, s) = \exp\left(\sum_{m \geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} q^{-ms}\right)$

X/\mathbb{Z} finite type $\zeta_X(s) = \prod_{x \in |X|} \frac{1}{1-N(x)^{-s}}$

domain (at least)
poles & zeros



equations

Conj
RH

GRH

ERH

$$\zeta_K(s) = \prod_{\chi \in \hat{G}} L(\chi, s) \Rightarrow K = \prod_{\substack{\chi \in \hat{G} \\ \chi \neq \chi_0}} L(\chi, 1)$$

e.g. $K = \mathbb{Q}(\zeta_{12})$ $K \approx 0.3610515$

	0	1	2	3	4	5	6	7	8	9	10	11
$\chi_1 = \text{Id}$	1	1	1	1	1	1	1	1	1	1	1	1
χ_2		1	-1		1	-1		1	-1		1	-1
χ_3		1		-1		1		-1		1		-1
χ_4		1				-1		-1				1

$$K = \left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \dots\right) \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) \left(1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \dots\right)$$

A Dirichlet character χ is called **odd** if $\chi(-1) = -1$ and **even** if $\chi(-1) = 1$. If χ is a Dirichlet character modulo m and $m|m'$, then χ can be lifted to a Dirichlet character modulo m' by pulling back using the projection. A Dirichlet character χ is called **primitive** if it cannot be lifted from Dirichlet character character of smaller modulus. Let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Let χ be a primitive character modulo m . Let

$$\Lambda(s, \chi) = (\pi/m)^{-(s+a)/2} \Gamma\left(\frac{s+a}{2}\right) L(s, \chi).$$

The Dirichlet L -function satisfies the following functional equation:

$$\Lambda(1-s, \bar{\chi}) = \frac{i^a k^{1/2}}{\tau(\chi)} \Lambda(s, \chi),$$

where

$$\tau(\chi) = \sum_{n=1}^m \chi(n) e^{2\pi i n/m}.$$