Eine Woche, ein Beispiel

12.1 weights of type E

It feels incomplete to discuss only the type E case without addressing the other classical cases.

Hence, this document serves as a complement to [2024.12.01].

There are some new phenomenons outside type E (which are not essential).

1. The formula becomes

$$2\frac{\langle \varpi_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{ij} \qquad \text{i.e., } \langle \varpi_i \tfrac{2}{\langle \alpha_i, \alpha_i\rangle}, \alpha_j\rangle = \delta_{ij}$$

when $i \neq j$, $\frac{2}{\langle \alpha_i, \alpha_i\rangle}$ won't impact, as $\delta_{ij} = 0$

$$s_k v = v - 2\frac{\langle \alpha_k, v\rangle}{\langle \alpha_k, \alpha_k\rangle}\alpha_k$$

2. $A = (\langle \alpha_i, \alpha_j\rangle)_{i,j}$ is not Cartan matrix. It is $\left(2\frac{\langle \alpha_i, \alpha_j\rangle}{\langle \alpha_i, \alpha_i\rangle}\right)_{i,j}$.

3. The minuscule weight orbit may not generate the whole lattice in type A, B, D

4. The minuscule weight orbit may not be the wts nearest to the origin in type A, B, C, D

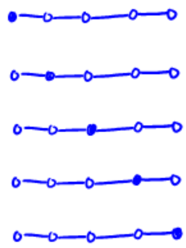
Since the coordinate itself already offers good symmetry(compared with type E case), we will omit many details.

1. A_n E.g. $n=5$

- Weights nearest to the origin

There are n many minuscule representations of A_n:

#	typical coordinates	symbol
$\binom{6}{1}$	$\frac{1}{6}(5,-1,-1,-1,-1,-1)^T$	v_i
$\binom{6}{2}$	$\frac{1}{6}(4,4,-2,-2,-2,-2)^T$	v_{ij}
$\binom{6}{3}$	$\frac{1}{6}(3,3,3,-3,-3,-3)^T$	v_{ijk}
$\binom{6}{4}$	$\frac{1}{6}(2,2,2,2,-4,-4)^T$	v_{ijkl}
$\binom{6}{5}$	$\frac{1}{6}(1,1,1,1,1,-5)^T$	v_{ijklm}



$$|v_i|^2 = \langle v_i, v_i \rangle \in \left\{ \frac{5}{6}, \frac{4}{3}, \frac{3}{2} \right\}$$

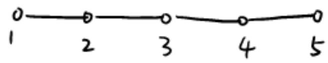
in general, in $\binom{n+1}{k}$, $\langle v_i, v_i \rangle = \frac{k(n+1-k)}{n+1}$ in $\left\{ \sum_{i=1}^{n+1} z_i = 0 \right\} \cong \mathbb{R}^n$

Restrict to the standard rep case,

$$\langle v_i, v_j \rangle \in \left\{ \frac{n}{n+1}, -1 \right\}.$$

The graph has no edges.

- Simple roots



$$
\begin{aligned}
&\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\} \\
&= \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6\} \\
&= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \right\}
\end{aligned}
$$

Ex. Verify that all the $2\binom{n+1}{2} = 30$ roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0)^T$	α_{1-2}

- Fundamental weights

$$
\begin{aligned}
&\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\} \\
&= \{v_1, v_{12}, v_{123}, v_{1234}, v_{12345}\} \\
&= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{6}\begin{pmatrix}4\\4\\-2\\-2\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}3\\3\\3\\-3\\-3\\-3\end{pmatrix}, \frac{1}{6}\begin{pmatrix}2\\2\\2\\2\\-4\\-4\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\} \\
&= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\2\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\1\\1\\1\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\}
\end{aligned}
$$

- Weyl group action

S_{n+1} acts by permutations. Nothing special.

2. D_n E.g. $n=6$ $n>4$ for avoiding special cases.

- Weights nearest to the origin

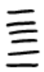
There are 3 minuscule representations of D_n:

	#	typical coordinates		symbol	
vector	$2\cdot 6$	$(1,0,0,0,0,0)^T$		v_i & $-v_i$	
half spin	2^5	$\frac{1}{2}(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1)^T$	odd (neg) sign	$v_{\pm\pm\pm\pm\pm\pm}$	
		$\frac{1}{2}(1,1,1,1,1,-1)^T$		v_-	
	2^5	$\frac{1}{2}(\pm1,\pm1,\pm1,\pm1,\pm1,\pm1)^T$	even (neg) sign	$v_{\pm\pm\pm\pm\pm\pm}$	
		$\frac{1}{2}(1,1,1,1,1,1)^T$		v_+	

in general, $\langle v_i, v_i\rangle \in \{1, \frac{n}{4}\}$ in $\mathbb{R}^n$.

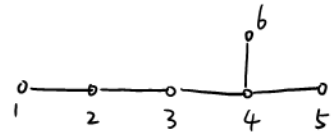
Restrict to the standard rep case,

$$\langle v_i, v_j\rangle \in \{1, 0, -1\}.$$

The graph is .

Here, the weights corresponding to standard reps does not generate all other weights.

– Simple roots



$$
\begin{aligned}
&\{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\ \}\\
&= \{\ v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6,\ v_5 + v_6\ \}\\
&= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix} \right\}
\end{aligned}
$$

Ex. Verify that all the $4\binom{n}{2} = 60$ roots are given by

#	typical coordinates	symbol
60	$(\pm 1, \pm 1, 0, 0, 0, 0)^T$	$\alpha_{\pm i \pm j}$
	$(-1, 1, 0, 0, 0, 0)^T$	α_{-1+2}

– Fundamental weights

$$
\begin{aligned}
&\{\ \varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4,\ \varpi_5,\ \varpi_6\ \}\\
&= \{\ v_1,\ v_1 + v_2,\ \sum_{i=1}^{3} v_i,\ \sum_{i=1}^{4} v_i,\ v_-,\ v_+\ \}\\
&= \left\{ \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\\-1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} \right\}
\end{aligned}
$$

– Weyl group action

$s_k = s_{(k,k+1)}$ for $k = 1, \ldots, n-1$.

$$
s_n = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & & -1 \\ & & & -1 & \end{pmatrix}
$$

$$
W(D_n) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n \subseteq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n
$$

2. D_4

– Weights nearest to the origin

D_4 is more symmetric.

#	typical coordinates	symbol	
$8 = 2 \cdot 4$	$(1, 0, 0, 0)^T$	v_i & $-v_i$	
$8 = 2^3$	$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$ odd sign	$v_{\pm\pm\pm\pm}$	
	$\frac{1}{2}(1, 1, 1, 1)^T$	v_-	
$8 = 2^3$	$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$ even sign	$v_{\pm\pm\pm\pm}$	
	$\frac{1}{2}(1, 1, 1, 1)^T$	v_+	

If not restricted to the standard representation case,

$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1\}.$$