Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

Task.

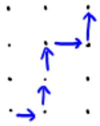
1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. change of basis
7. Convolution product

We may use two examples for the convenience of presentation.
Readers can easily distinguish them by the dim vectors.

1. dimension vector

$|d| = 5$

$d = (3,2)$

$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,0 \\ 0,0 \end{pmatrix} =$  $=$  $=$  $\in W_d \backslash \mathbb{W}_d$ or $Min(\mathbb{W}_{|d|}, W_d)$

Young Tableax $\qquad r_\varpi = \pi_{\underline{d}}^{-1}(F_\varpi)$

2. Weyl group

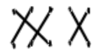
Set	element	special element	others
$\mathbb{W}_{\lvert d\rvert} = S_5$	ϖ, x	$\varpi_{max} =$ 	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{max} =$ 	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_d \backslash \mathbb{W}_{\lvert d\rvert} = S_3 \times S_2 \backslash S_5$	$\varpi, \underline{d}$		$(Comp_d)$
$Min(\mathbb{W}_{\lvert d\rvert}, W_d) = \{$  $, \cdots \}$	u		$(Shuffled)$

$$\begin{tikzcd} & & & Min(\mathbb{W}_{|d|}, W_d) \arrow[dl] \arrow[d, "\cong"] & \\ 0 \arrow[r] & W_d \arrow[r] & \mathbb{W}_{|d|} \arrow[r] & \mathbb{W}_{|d|} \backslash W_d \arrow[r] & 0 \end{tikzcd}$$

$$u \mapsto \underline{d}, \qquad \varpi = wu \longmapsto \underline{d}$$

$\varpi =$ 

$u =$ 

$w =$ 

Another example: $\quad d = (1,2) \qquad \begin{matrix} a \longrightarrow b \\ \langle v_1 \rangle \longrightarrow \langle v_2, v_3 \rangle \end{matrix}$

$\varpi = wu$			w	$\underline{d}$, u	order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	B_ϖ	$\varpi B \varpi^{-1}$
Id	Id	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$		abb	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix}*&*&*\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$
t	(23)	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$		abb	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix}*&*&*\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$
s	(12)	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$		bab	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix}*&&*\\*&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&*\\&*&\\&&*\end{bmatrix}$
ts	(132)	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$		bab	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix}*&*&\\&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&*&\\&*&\\&&*\end{bmatrix}$
st	(123)	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$		bba	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix}*&&\\*&*&*\\*&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&*\\&&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\*&&*\end{bmatrix}$
sts	(13)	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$		bba	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix}*&&\\*&*&\\*&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\&*&\\&*&*\end{bmatrix}$	$\begin{bmatrix}*&&\\*&*&\\&&*\end{bmatrix}$

3. alg group & Lie algebra

$G_{|d|}, \mathbb{B}_{|d|}, \mathbb{T}_{|d|}, \mathbb{N}_{|d|}$ $\quad W_{|d|} = N_{G_{|d|}}(\mathbb{T}_{|d|})/\mathbb{T}_{|d|}$ $\quad GL_5(\mathbb{C})$ $= \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$

G_d, B_d, T_d, N_d $\quad W_d = N_{G_d}(T_d)/T_d$ $\quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right)$

$$\mathbb{B}_\varpi = \varpi \mathbb{B}_{|d|} \varpi^{-1} = \mathrm{Stab}_{G_{|d|}}(F_\varpi)$$

$$B_\varpi \overset{\varpi = wu}{=\!=} w B_d w^{-1} = \mathrm{Stab}_{G_d}(F_\varpi) \qquad N_\varpi = R_u(B_\varpi)$$

For $s \in \Pi$ s.t. $\varpi s \varpi^{-1} \in W_d$ (i.e. $W_d \varpi = W_d \varpi s$), define

$$P_{\varpi,\varpi s} :\overset{\varpi = wu}{=\!=} w(B_d u s u^{-1} B_d \cup B_d) w^{-1} \qquad N_{\varpi,\varpi s} = R_u(B_{\varpi,\varpi s})$$

$$= B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi \qquad = N_\varpi \cap N_{\varpi s}$$

$$M_{\varpi,\varpi s} = N_\varpi / N_{\varpi,\varpi s}$$

$$= B_\varpi / B_\varpi \cap B_{\varpi s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{u(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\varpi,\varpi''} := N_\varpi \cap N_{\varpi''}$$

$$M_{\varpi,\varpi''} := N_\varpi / N_{\varpi,\varpi''}$$

$$= B_\varpi / B_\varpi \cap B_{\varpi''}$$

Their Lie algebras are collected here.

$\mathfrak{g}_{|d|}, \mathbb{b}_{|d|}, \mathfrak{t}_{|d|}, \mathfrak{n}_{|d|}$

$\mathfrak{g}_d \quad \mathfrak{b}_d \quad \mathfrak{t}_d \quad \mathfrak{n}_d$

$\mathbb{b}_\varpi \qquad\qquad \mathbb{b}^-_\varpi \qquad\qquad \mathbb{b}^-_\varpi = \mathbb{b}_{\varpi_{max}\varpi}$

$\mathfrak{b}_\varpi \quad \mathfrak{n}_\varpi \qquad\qquad \mathfrak{b}^-_\varpi \quad \mathfrak{n}^-_\varpi \qquad\qquad \mathfrak{b}^-_\varpi = \mathfrak{b}_{w_{max}\varpi}$ ...

$\mathfrak{p}_{\varpi,\varpi s} \quad \mathfrak{n}_{\varpi,\varpi''} \qquad\qquad \mathfrak{p}^-_{\varpi,\varpi s} \quad \mathfrak{n}^-_{\varpi,\varpi''} \qquad\qquad \mathfrak{p}^-_{\varpi,\varpi s} = \mathfrak{p}_{w_{max}\varpi,\, w_{max}\varpi s}$ ...

$\mathfrak{m}_{\varpi,\varpi''} \qquad\qquad \mathfrak{m}^-_{\varpi,\varpi''}$ ...

$$\mathrm{Rep}_d(Q) := \prod_{e \in Q_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{|d|}$$

in general case, lies in $\mathfrak{g}_{|d|}^{\oplus k}$

$$\mathfrak{r}_\varpi = \{ f \in \mathrm{Rep}_d(Q) \mid f.F_{\varpi,i} \subseteq F_{\varpi,i} \} = \mu_d \pi_d^{-1}(F_\varpi)$$

$$= \begin{array}{c|ccc} & v_3 & v_1 & v_2 \\ \hline v_5 & * & * & * \\ v_4 & & & * \end{array} = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_4 & & * & \\ v_5 & * & * & * \end{array}$$

$v_{w(i)}$

$$\mathfrak{r}_{\varpi,\varpi''} = \mathfrak{r}_\varpi \cap \mathfrak{r}_{\varpi''}$$

$$\mathfrak{d}_{\varpi,\varpi''} = \mathfrak{r}_\varpi / \mathfrak{r}_{\varpi,\varpi''}$$

Later we may twist the group actions.

E.g. $\underline{\mathfrak{r}}_{\varpi,\varpi'} := \mathfrak{r}_{\varpi,\varpi\varpi'}$ $\quad \mathfrak{r}_{\varpi,\varpi''} = \underline{\mathfrak{r}}_{\varpi,\varpi^{-1}\varpi''}$

4. typical variety

	Id corres to
$\mathcal{F}_{\lvert d\rvert} \cong G_{\lvert d\rvert}/B_{\lvert d\rvert}$	F_{Id}
$\mathcal{F}_{\underline{d}} \cong G_d/B_{\underline{d}}$	F_u
$\mathcal{F}_{\varpi} \cong G_d/B_{\varpi}$	F_{ϖ}
$\mathcal{F}_d := \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$	—
$\mathcal{F}_{g\varpi} \cong G_d/gB_{\varpi}g^{-1}$	$F_{g\varpi}$

$$F_{\varpi} := \varpi(F_{Id}) = F_{\{v_{\varpi(1)}, v_{\varpi(2)}, \cdots, v_{\varpi(\lvert d\rvert)}\}}$$
$$= F_{\{v_5, v_3, v_1, v_4, v_2\}}$$

⚠ The action on $\mathcal{F}lag$ is not the same as in http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$\mathcal{F}_{\lvert d\rvert} \neq \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$

$\mathcal{F}_{\varpi} \cong \mathcal{F}_{\underline{d}}$ with different base pt. Base pt makes difference!

$\mathcal{F}_{\lvert d\rvert} \times \mathcal{F}_{\lvert d\rvert}$	$F_{Id,Id}$
$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_d \times \mathcal{F}_d := \coprod_{\underline{d},\underline{d}'} (\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'})$	—

$$F_{\varpi,\varpi'} := (F_{\varpi}, F_{\varpi'})$$

$$\begin{tikzcd} \widetilde{Rep}_{\underline{d}}(Q) \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \arrow[d, "\mu_{\underline{d}}"'] \arrow[dr, "\pi_{\underline{d}}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{Rep}_d(Q) \subset Rep_d(Q) \times \mathcal{F}_d \arrow[d, "\mu_d"'] \arrow[dr, "\pi_d"] & \\ Rep_d(Q) & \mathcal{F}_d \end{tikzcd}$$

$\mu_{\underline{d}}^{-1}(M) \cong \mathcal{F}lag_{\underline{d}}(M) \subseteq \mathcal{F}_{\underline{d}}$ is the Springer fiber.

$$\begin{tikzcd} Z_{\underline{d},\underline{d}'} \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \arrow[d, "\mu_{\underline{d},\underline{d}'}"'] \arrow[dr, "\pi_{\underline{d},\underline{d}'}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \end{tikzcd} \qquad \begin{tikzcd} Z_d \subset Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d \arrow[d, "\mu_{d,d}"'] \arrow[dr, "\pi_{d,d}"] & \\ Rep_d(Q) & \mathcal{F}_d \times \mathcal{F}_d \end{tikzcd}$$

$$\widetilde{Rep}_{\underline{d}}(Q) \subseteq Rep_d(Q) \times \mathcal{F}_{\underline{d}} \qquad \widetilde{Rep}_{\varpi}(Q) \cong G_d \times^{B_{\varpi}} r_{\varpi}$$
$$\widetilde{Rep}_d(Q) := \bigsqcup_{\underline{d}} \widetilde{Rep}_{\underline{d}}(Q)$$

$$Z_{\underline{d},\underline{d}'} := \widetilde{Rep}_{\underline{d}}(Q) \times_{Rep_d(Q)} \widetilde{Rep}_{\underline{d}'}(Q) \qquad Z_{\varpi,\varpi'} := Z_{u,u'}$$
$$Z_d = \bigsqcup_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$$
$$= \widetilde{Rep}_d(Q) \times_{Rep_d(Q)} \widetilde{Rep}_d(Q)$$

5. (equivariant) stratifications.

In the following tables, $uw' = \tilde{w}'\tilde{u}$.

$F_\varpi \in \widetilde{Rep}_d(Q)$ means (ρ_0, F_ϖ); $(F_\varpi, F_{\varpi'}) \in Z_d$ means $(\rho_0, F_\varpi, F_{\varpi'})$.

⚠ $G\times G$ acts on $\mathcal{F}\times\mathcal{F}$ in a twisted way

e.g. $(g_1,g_2)F_{\varpi,\varpi'} = F_{g_1\varpi,\, g_1\varpi g_2\varpi^{-1}\varpi'}$

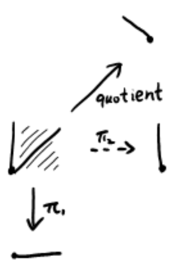
$(g_1,g_2)E_{\varpi,\varpi'} = E_{g_1\varpi,\, g_2\varpi'}$

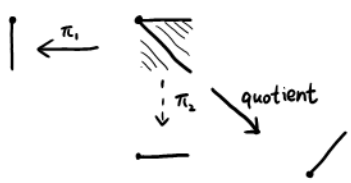
variety / base point (stratification / stabilizer; type)		B-orbit	B×B-orbit	B×G-orbit	G×B-orbit = G×{*}-orbit	Remark
$\mathcal{B}$ F_g	$\mathcal{B}\times\mathcal{B}$ $(F_g, F_{gg'})$	Ω_g $B\cap gBg^{-1}$	$\Omega_{g,g'}$ $(B\cap gBg^{-1})\times(B\cap g'Bg'^{-1})$	$pr_1^{-1}(\Omega_g)$ $(B\cap gBg^{-1})\times g'Bg'^{-1}$	$\Omega_{g'}$ $gBg^{-1}\times(B\cap g'Bg'^{-1})$	 $gBg^{-1}\cap gg'B(gg')^{-1}$
$\mathcal{F}_{\lvert d\rvert}$ F_ϖ	$\mathcal{F}_{\lvert d\rvert}\times\mathcal{F}_{\lvert d\rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_\varpi$	$\mathcal{V}_{\varpi,\varpi'}$ $(\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_\varpi)\times(\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_{\varpi'})$	$pr_1^{-1}(\mathcal{V}_\varpi)$ $(\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_\varpi)\times\mathbb{B}_{\varpi'}$	$\mathcal{V}_{\varpi'}$ $\mathbb{B}_\varpi\times(\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_{\varpi'})$	 $\mathbb{B}_\varpi\cap\mathbb{B}_{\varpi\varpi'}$
$\mathcal{F}_u$ F_{wu}	$\mathcal{F}_u\times\mathcal{F}_{u'}$ $(F_{wu}, F_{ww'u'})$	Ω^u_w $B_d\cap B_w$	$\Omega^{u,u'}_{w,w'}$ $(B_d\cap B_w)\times(B_d\cap B_{w'})$	$pr_{1,u'}^{-1}(\Omega^u_w)$ $(B_d\cap B_w)\times B_{w'}$	$\Omega^{u,u'}_{w'}$ $B_w\times(B_d\cap B_{w'})$	 $B_w\cap B_{ww'}$
$\mathcal{F}_d$ F_ϖ $=$ F_{wu}	$\mathcal{F}_d\times\mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$ $=$ $(F_{wu}, F_{w\tilde{w}'\tilde{u}u'})$	Ω^u_w $B_d\cap B_w$	$\Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$ $(B_d\cap B_w)\times(B_d\cap B_{\tilde{w}'})$	$pr_{1,\tilde{u}u'}^{-1}(\Omega^u_w)$ $(B_d\cap B_w)\times B_{\tilde{w}'}$	$\mathcal{O}^u_{\varpi'} := \Omega^{u,\tilde{u}u'}_{\tilde{w}'}$ $B_w\times(B_d\cap B_{\tilde{w}'})$	compatability $B_w\cap B_{w\tilde{w}'}$
The following may not be single orbit, but derived from the above definition.						
$\mathcal{F}_d$ F_ϖ	$\mathcal{F}_d\times\mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$ $= \Omega^u_w$	$\mathcal{O}_{\varpi,\varpi'}$ $= \Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$	$pr_1^{-1}(\mathcal{O}_\varpi)$ $= \bigsqcup_{u'} pr_{1,u'}^{-1}(\Omega^u_w)$	$\mathcal{O}_{\varpi'}$ $= \bigsqcup_u \mathcal{O}^u_{\varpi'}$	preimage of $\mathcal{F}_d\times\mathcal{F}_d \hookrightarrow \mathcal{F}_{\lvert d\rvert}\times\mathcal{F}_{\lvert d\rvert}$
$\widetilde{Rep}_{\underline{d}}(Q)$ F_{wu}	$Z_{\underline{d},\underline{d}'}$ $(F_{wu}, F_{w'u'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\Omega}^{u,u'}_{w,w'}$	$pr_{1,u'}^{-1}(\widetilde{\Omega}^u_w)$	$\widetilde{\Omega}^{u,u'}_{w'}$	preimage of $Z_{\underline{d},\underline{d}'} \longrightarrow \mathcal{F}_{\underline{d}}\times\mathcal{F}_{\underline{d}'}$
$\widetilde{Rep}_d(Q)$ F_ϖ	Z_d $(F_\varpi, F_{\varpi\varpi'})$				$\widetilde{\mathcal{O}}^u_{\varpi'}$ $= \widetilde{\Omega}^{u,\tilde{u}u'}_{\tilde{w}'}$	preimage of $Z_d \longrightarrow \mathcal{F}_d\times\mathcal{F}_d$
$\widetilde{Rep}_d(Q)$ F_ϖ	Z_d $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$ $= \widetilde{\Omega}^u_w$	$\widetilde{\mathcal{O}}_{\varpi,\varpi'}$ $= \widetilde{\Omega}^{u,\tilde{u}u'}_{w,\tilde{w}'}$	$pr_1^{-1}(\widetilde{\mathcal{O}}_\varpi)$ $= \bigsqcup_{u'} pr_{1,u'}^{-1}(\widetilde{\Omega}^u_w)$	$\widetilde{\mathcal{O}}_{\varpi'}$ $= \bigsqcup_u \widetilde{\mathcal{O}}^u_{\varpi'}$	preimage of $Z_d \longrightarrow \mathcal{F}_d\times\mathcal{F}_d$

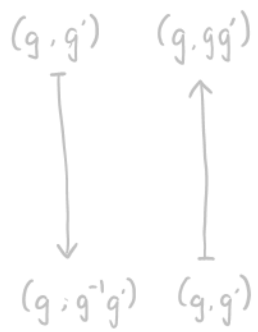
$Z_{\varpi'} := \overline{\widetilde{\mathcal{O}}_{\varpi'}} \subseteq \overline{\widetilde{\mathcal{O}}}_{\varpi'}$ $\qquad Z^{\varpi'}_d$

Zar-loc sub v.b.?

$Z^{u,u'}_{w'} := \overline{\widetilde{\Omega}^{u,u'}_{w'}} \subseteq \overline{\widetilde{\Omega}}{}^{u,u'}_{w'}$ $\qquad Z^{\varpi'}_{\underline{d},\underline{d}'} = Z_{\varpi'} \cap Z_{\underline{d},\underline{d}'}$

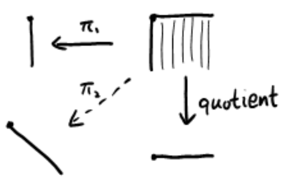
real case: 

in table: 



We want gp action to be compatible with π_1 and the quotient map.

Therefore, we would do a twist.



The following tables may help you to understand the notations.

dim; $\mathbb{B}_{|d|}\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$; $\mathbb{B}_{|d|}\times\mathbb{B}_{|d|}\cdot(F_{\varpi},F_{\varpi\varpi'})$; $\mathbb{B}_{|d|}\cdot F_{\omega}$ ϖ	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

shape; $B_d\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$; $B_d\times B_d\cdot(F_{\varpi},F_{\varpi\varpi'})$; $B_d\cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		O_{Id}	O_{t}	O_{s}	O_{ts}	O_{st}	O_{sts}
$\mathcal{F}_{Id}$	O_{Id}	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	O_{t}	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	O_{s}	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	O_{ts}	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	O_{ts}	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	O_{sts}	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

The following tables may help you to understand the notations.

$\varpi = ts$, $\varpi' = s$

dim / $\mathbb{B}_{\lvert d\rvert}\cdot F_{\varpi'}$ / $\mathbb{B}_{\lvert d\rvert}\times\mathbb{B}_{\lvert d\rvert}\cdot(F_{\varpi}, F_{\varpi\varpi'})$ $\varpi\varpi'$ / $\mathbb{B}_{\lvert d\rvert}\cdot F_{\varpi}$ ϖ	0 ϑ_{Id}	1 ϑ_t	1 ϑ_s	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}	$pr_1^{-1}(\vartheta_{ts})$
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$	ϑ_s
1 ϑ_t	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$	
1 ϑ_s	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$	
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$	
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$	
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$	

shape / $B_d\cdot F_{\varpi'}$ / $B_d\times B_d\cdot(F_{\varpi}, F_{\varpi\varpi'})$ $\varpi\varpi'$ / $B_d\cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_s$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{Id}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	$\mathcal{O}_t$	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_s$	$\mathcal{O}_s$	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	$\mathcal{O}_{ts}$	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	$\mathcal{O}_{sts}$	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

$pr_1^{-1}(\mathcal{O}_{ts})$ $pr_{1,Id}^{-1}(\Omega^s_t)$ $\Omega^{s,Id}_{t,Id} = \mathcal{O}_{ts,s}$

$\mathcal{O}_s$ $\Omega^{s,Id}_{Id}$

6. Change of basis

§6.1 Two basis

Def Let $Y \subset X$ be G-equiv closed subvariety, X proj.

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_Y"] \arrow[d, "\pi_Y"] & X \\ pt \end{tikzcd}$$

with same notation,

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{S(G)} \in H_G^*(X;\mathbb{Q})$$

By cellular fibration lemma,

$$\begin{tikzcd} K_0^{T_d}(\mathcal{F}_d) \arrow[r, phantom, "\cong"] \arrow[d, equal] & K_0^{G_d}(\mathcal{F}_d \times \mathcal{F}_d) \arrow[r, phantom, "\cong"] \arrow[d, equal] & K_0^{G_d}(Z_d) \arrow[d, equal] \arrow[dd] \\ \bigoplus_{\varpi' \in W_{|d|}} R(T_d)[\overline{Q}_{\varpi'}]^{T_d} \arrow[r, phantom, "\cong"] & \bigoplus_{\varpi' \in W_{|d|}} R(T_d)[\overline{O}_{\varpi'}]^{G_d} \arrow[r, phantom, "\cong"] & \bigoplus_{\varpi' \in W_{|d|}} R(T_d)[Z_{\varpi'}]^{G_d} \\ & \cong & K_0^{T_d}(Z_d) \arrow[d, equal] \\ & & \bigoplus_{\varpi,\varpi' \in W_{|d|}} R(T_d)[\overline{\widetilde{O}}_{\varpi,\varpi'}]^{T_d} \end{tikzcd}$$

as $R(T_d)$-modules.

⚠ There is no evidence if $[Z_{\varpi'}]^{G_d}$ will be maped to $\bigoplus_{\varpi \in W_{|d|}} [\overline{\widetilde{O}}_{\varpi,\varpi'}]^{T_d}$.
Luckily, the horizontal line sends generators to generators.

[Hint: Consider the following commutative diagram:

$$\begin{tikzcd} & \mathcal{F}_d \arrow[r, "(F_e, Id)"] & \mathcal{F}_d \times \mathcal{F}_d \arrow[r, "(p_0, Id)"] & Z_d \\ \overline{O}_{\varpi'} \arrow[ur] \arrow[r, hook] \arrow[d] & \overline{O}_{\varpi'} \arrow[ur] \arrow[r, hook] \arrow[d] & \overline{Z}_{\varpi'} \arrow[ur] \arrow[d] \\ pt \arrow[r, equal] & pt \arrow[r, equal] & pt \end{tikzcd}$$

]

To do linear alg, we take

$$\mathcal{R}(G) := \mathrm{Frac}(R(G)) \qquad \mathcal{K}_0^G(X) := K_0^G(X) \otimes_{R(T_d)} \mathcal{R}(T_d)$$
$$\mathcal{S}(G) := \mathrm{Frac}(S(G)) \qquad \mathcal{H}_G^*(X;\mathbb{Q}) := H_G^*(X;\mathbb{Q}) \otimes_{S(T_d)} \mathcal{S}(T_d)$$

For $R(T_d)$-mod $K_0^G(X)$, $S(T_d)$-mod $H_G^*(X;\mathbb{Q})$

Define
$$\psi_\varpi := [\{F_\varpi\}]^{T_d} = (i_\varpi)_* 1_{R(T_d)} \in K_0^{T_d}(\mathcal{F}_d)$$
$$\psi_{\varpi,\varpi'} := [\{(\rho_0, F_\varpi, F_{\varpi'})\}]^{T_d} = (i_{\varpi,\varpi'})_* 1_{R(T_d)} \in K_0^{T_d}(\mathcal{Z}_d)$$
$$\psi_{\varpi,\varpi'} := [\{(\rho_0, F_\varpi, F_{\varpi\varpi'})\}]^{T_d}$$

We get two $\mathcal{R}(T_d)$-basis: (ψ_ϖ is $\mathcal{R}(T_d)$-basis, by Localization theorem.)

$$\begin{array}{ccc} K_0^{T_d}(\mathcal{F}_d) & \longrightarrow & K_0^{T_d}(\mathcal{Z}_d) \\ [\overline{\mathcal{O}_{\varpi'}}]^{T_d} & & [\overline{\mathcal{O}_{\varpi,\varpi'}}]^{T_d} \\ \psi_{\varpi'} & & \psi_{\varpi,\varpi'} \end{array}$$

standard basis: for stratification
canonical basis: for convolution

Localization thm [Thm 10.1]

Let $i: X^{T_d} \hookrightarrow X$, X is smooth.

$$\mathcal{K}_0^{T_d}(X^{T_d}) \xrightarrow{i_*} \mathcal{K}_0^{T_d}(X) \xrightarrow{i^*} \mathcal{K}_0^{T_d}(X^{T_d})$$
$$\mathcal{H}_{T_d}^*(X^{T_d};\mathbb{Q}) \xrightarrow{i_*} \mathcal{H}_{T_d}^*(X;\mathbb{Q}) \xrightarrow{i^*} \mathcal{H}_{T_d}^*(X^{T_d};\mathbb{Q})$$

are isos as $\mathcal{R}(T_d)$ or $\mathcal{S}(T_d)$-module.

Q: The Steinberg variety $\mathcal{Z}_d$ is usually not smooth.
How to show that $\{\psi_{\varpi,\varpi'}\}$ forms a basis?

Guess: apply localization thm to $\mathcal{F}_d \times \mathcal{F}_d$ first.

§ 6.2 tangent space, Euler class.

Def (tangent space of fixed pts. in $R(T_d)$)

$$\Lambda_{\varpi} := T_{F_\varpi} \mathcal{F}_d \cong T_{Id}(G_d/B_\varpi) \cong \mathfrak{g}_d/\mathfrak{b}_\varpi = \mathfrak{n}_\varpi^-$$

$$\widetilde{\Lambda}_{\varpi} := T_{(\rho_0,F_\varpi)} \widetilde{Rep}_d(Q) \cong T_0 \mathfrak{r}_\varpi \oplus T_{F_\varpi}\mathcal{F}_d = \mathfrak{r}_\varpi \oplus \mathfrak{n}_\varpi^-$$

$$\Lambda^x_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})} \overline{\mathcal{O}}_x$$

$$\widetilde{\Lambda}^x_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})} Z_x \cong T_0 \mathfrak{r}_{\varpi,\varpi'} \oplus T_{(F_\varpi,F_{\varpi'})} \overline{\mathcal{O}}_x = \mathfrak{r}_{\varpi,\varpi'} \oplus \Lambda^x_{\varpi,\varpi'}$$

Hint:
$$\begin{tikzcd} \overline{\mathcal{O}}_x \arrow[r, "{(\rho_0,Id)}"] \arrow[dr, equal] & Z_x \arrow[d] \\ & \overline{\mathcal{O}}_x \end{tikzcd}$$

$$T_{x_0}(\times) = T_{x_0}(/) \oplus T_{x_0}(\backslash)$$

$$\underline{\Lambda}^x_{\varpi,\varpi'} := T_{(F_\varpi,F_{\varpi\varpi'})} \overline{\mathcal{O}}_x$$

Rmk. It is still not easy to express $\widetilde{\Lambda}^x_{\varpi,\varpi'}$ as Lie alg.

However, we still know some special cases:

$$\begin{aligned}
\underline{\Lambda}^x_{\varpi,x} &:= T_{(F_\varpi,F_{\varpi x})} \overline{\mathcal{O}}_x \\
&= T_{(F_\varpi,F_{\varpi x})} \mathcal{O}_x \\
&= T_{(F_\varpi,F_{\varpi x})} \mathcal{O}^u_x \\
&= T_{Id}\, G_d/B_\varpi \cap B_{\varpi x} \\
&= \mathfrak{g}_d - \mathfrak{b}_\varpi \cap \mathfrak{b}_{\varpi x} \\
&= \mathfrak{g}_d - \mathfrak{b}_\varpi + \mathfrak{b}_\varpi/(\mathfrak{b}_\varpi \cap \mathfrak{b}_{\varpi x}) \\
&= \mathfrak{n}_\varpi^- + \underline{\mathfrak{m}}_{\varpi,x}
\end{aligned}$$

($\underline{\mathfrak{m}}_{\varpi,Id} = 0$. For $s \in \Pi$, $\varpi s \varpi^{-1} \notin W_d$, we have $\underline{\mathfrak{m}}_{\varpi,s} = 0$)

Now suppose $\varpi s \varpi^{-1} \in W_d$.

$$\Lambda^s_{\varpi,\varpi s} = \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi,\varpi s}$$

$$\begin{aligned}
\Lambda^s_{\varpi,\varpi} &:= T_{(F_\varpi,F_\varpi)} \overline{\mathcal{O}}_s \\
&= T_{(Id,Id)}\, G_d/B_\varpi \times P_{\varpi,\varpi s}/B_\varpi \\
&= \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi s,\varpi}
\end{aligned}$$

$$\widetilde{\Lambda}^s_{\varpi,\varpi s} = \mathfrak{r}_{\varpi,\varpi s} \oplus \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi,\varpi s}$$

$$\widetilde{\Lambda}^s_{\varpi,\varpi} = \mathfrak{r}_{\varpi,\varpi} \oplus \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi s,\varpi}$$

6.3. transition matrix, localization formula

Thm. (Localization formula) [Thm 10.2]

Suppose $Y \subset X$ is T-equivariant, $\alpha \in \mathcal{K}_0^T(X)$, X smooth.

$X^T := \{x_1, \cdots, x_m\}$, $i_k : \{x_k\} \hookrightarrow X$, then

$$\alpha = \sum_{k=1}^{m} \varepsilon_k (i_k)_* (i_k)^*(\alpha) \qquad \varepsilon_k = (T_{x_k} X)^{-1} \in R(T)$$

e.p.
$$[Y]^T = \sum_{k=1}^{m} \varepsilon_k (i_k)_*\left((i_k)^* [Y]^T \cdot 1_{R(T)}\right)$$
$$= \sum_{k=1}^{m} \varepsilon_k \left((i_k)^* [Y]^T\right) (i_k)_* 1_{R(T)}$$
$$= \sum_{k=1}^{m} \varepsilon_k \left((i_k)^* [Y]^T\right) [x_k]^T$$
$$[X]^T = \sum_{k=1}^{m} \varepsilon_k [x_k]^T$$

Suppose $Y^T = \{x_1, \cdots, x_n\}$, $\iota_k : \{x_k\} \hookrightarrow Y$, then

$$[Y]^T = \sum_{k=1}^{n} \beta_k [x_k]^T \qquad \beta_k = \varepsilon_k \cdot (i_k)^*[Y]^T$$

When Y is sm at x_k,

$$\begin{cases} \beta_k = (T_{x_k} Y)^{-1} \\ (i_k)^*[Y]^T = T_{x_k} X \cdot (T_{x_k} Y)^{-1} \end{cases}$$