Eine Woche, ein Beispiel
3.27 model theory

Model   Theory

| ⇐ : Realization

⇒ : Maximal theory |

type   - - -   Structure   Formula

Language

Modern Algebra (H)

Preliminaries

- Logic. set & map, operations on Set. ($\cap \cup -^c$, sub/quotient, two ways of disjoint union, Cartesian product)
  - Russell's paradox
  - Axiomatic set theory (ZFC)
  - type of proof : constructive, algorithm,...

  Ex. graph

  Classify topologies of $\{1, \cdots, n\}$

  First adjunction. $\text{Map}(A \times B, C) \cong \text{Map}(A, \text{Map}(B, C))$

- From $\mathbb{N}$ to $\mathbb{C}$ (the basic of examples, though logically it's not here)
  - Peano axioms. Axiom of induction
  - alg structure, order and topology
  - Completeness axiom.

- Cardinal : the only property of set.
  - naive definition

	alg	total order	topo
$\mathbb{N}$	$(+, \times)$	✓	discrete
$\mathbb{Z}$	$(+, -, \times)$	✓	discrete
$\mathbb{Q}$	$(+, -, \times, \div)$	✓	dense but not complete
$\mathbb{R}$	$(+, -, \times, \div)$	✓	complete
$\mathbb{C}$	$(+, -, \times, \div)$	✗	complete

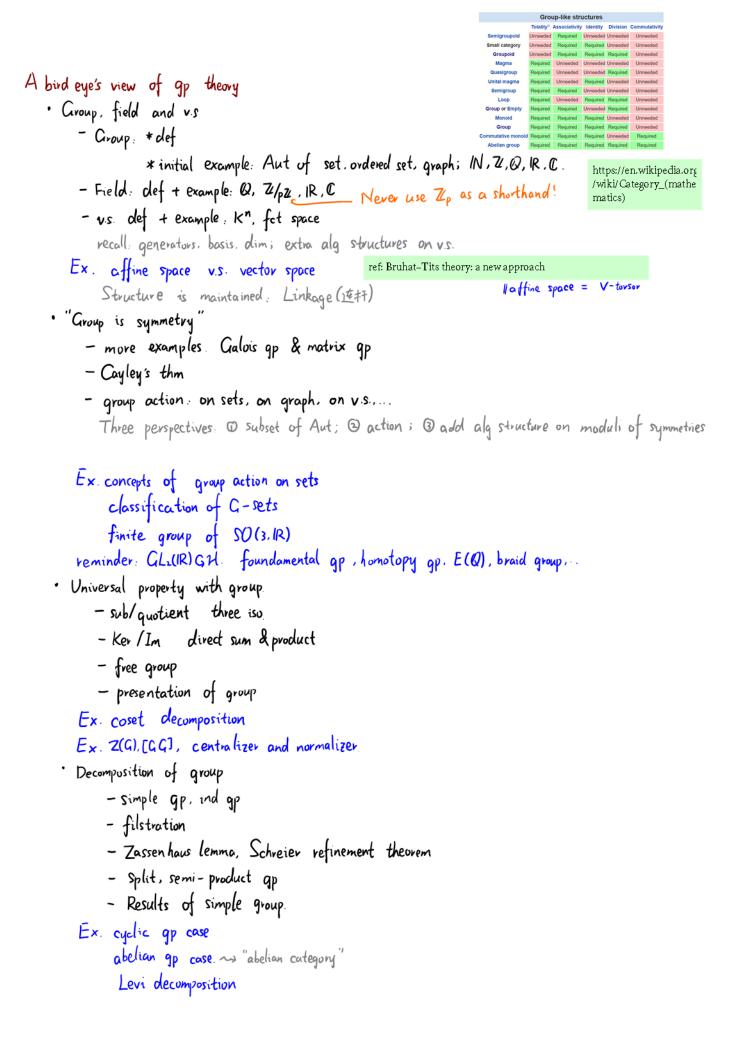 - operations on cardinal
  - examples
  - The continuum hypothesis
  - large cardinal axiom

- Order structure
  - def, operations and properties (partial/total/well order)
  - ordinal, relationship with cardinal.

  Ex. "well-order" on class of cardinals.

  Classify subpartial ordered set of $\{$subsets of $\{1,..,n\}\}$, up to iso. $n \leq 5$

  the Monoid $\widetilde{R}$, surreals

poset = partial order set
coset = 陪集

A bird eye's view of gp theory

- Group, field and v.s
    - Group: * def
        * initial example: Aut of set, ordered set, graph; $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
    - Field: def + example: $\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, \mathbb{R}, \mathbb{C}$ Never use $\mathbb{Z}_p$ as a shorthand!
    - v.s def + example: K^n, fct space
        recall: generators, basis, dim; extra alg structures on v.s.

  Ex. affine space v.s. vector space
        Structure is maintained: Linkage (连接)

https://en.wikipedia.org/wiki/Category_(mathematics)

ref: Bruhat–Tits theory: a new approach

‖ affine space = V-torsor

- "Group is symmetry"
    - more examples: Galois gp & matrix gp
    - Cayley's thm
    - group action: on sets, on graph, on v.s.,...
        Three perspectives: ① subset of Aut; ② action; ③ add alg structure on moduli of symmetries

  Ex. concepts of group action on sets
        classification of G-sets
        finite group of $SO(3, \mathbb{R})$
  reminder: $GL_2(\mathbb{R})$, $G \backslash H$. foundamental gp, homotopy gp. $E(\mathbb{Q})$, braid group,...

- Universal property with group.
    - sub/quotient   three iso.
    - Ker/Im   direct sum & product
    - free group
    - presentation of group

  Ex. coset decomposition
  Ex. $Z(G), [G,G]$, centralizer and normalizer.

- Decomposition of group
    - simple gp, ind gp
    - filstration
    - Zassenhaus lemma, Schreier refinement theorem
    - Split, semi-product gp
    - Results of simple group.

  Ex. cyclic gp case
        abelian gp case. ⤳ "abelian category"
        Levi decomposition

Down-to-earth analysis.

- f.g. abelian gp + apps.
  Ex. lattice & Crystallographic point gp
    ↖ with/without origin
      sublattice count
      dual lattice $\quad \Lambda^* = \{ f : \mathbb{R}^n \to \mathbb{R} \text{ linear} \mid \langle f, x \rangle \in \mathbb{Z} \quad \forall x \in \Lambda \}$
      lattice in Euclidean space
- Combinatorics related to q-polynomial ← can be quite tricky.
  Ex. finite field with one element
- G act on G
- Sylow thm
- app: classifications of gp of small order. research on specific gp.

Ring & module

$R(x)$ $R((x))$ $R\{\{x\}\}$ $R\langle\langle x\rangle\rangle$

- Basic def. e.g. $R[x]$, $R[[x]]$, $R\{x\}$, $R\langle x\rangle$

- Category, k-algs
  - Ex. groupoid
- Basic def of modules, abelian category
  - E.x. R-algs.
    - tensor product, change of basis
    - three rep theory
- AG translation, examples.
- Concepts under AG translation
- ED ⇒ PID ⇒ UFD ⇒ domain

  disjoint for different "$(a) \subset (b)$"

  - Ex. Elementary divisor thm

    R: PID. $M_{2\times2}(R) - \{0\} = \coprod\limits_{a\mid b \in R} GL_2(R) \begin{pmatrix} a \\ & b \end{pmatrix} GL_2(R)$

- classification of f.g. module over PID.

Field and Galois theory (See [GTM 167])

can focus more on $\mathbb{F}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, and geometrical point of view.

(Reminder) other structures: norm, metric & topo; measure; sheaf.