Session 3 and Exercise 1

Any questions for Ex 1?

Suggestion: do the calculation as elegant/simple as possible, if you want to enjoy learning math.

(correctness is enough for the exam)

E.g. Use $z, \bar{z}$ instead of x, y.

Ex 4. (i) $f(z) = |z|^2 = z\bar{z}$

$$\frac{\partial f}{\partial \bar{z}} = z \neq 0 \quad \Rightarrow f \notin \mathcal{O}(\mathbb{C})$$

(ii) $g(z) = \dfrac{x - iy}{x^2+y^2} = \dfrac{\bar{z}}{|z|^2} = \dfrac{\bar{z}}{z\bar{z}} = \dfrac{1}{z}$

$$\frac{\partial g}{\partial \bar{z}} = 0 \quad \Rightarrow g \in \mathcal{O}(\mathbb{C}^\times)$$

(iii) (Use CR-equation in forms of x, y)

Try: $h(z) = \log\sqrt{x^2+y^2} + i\arctan\frac{y}{x}$

$$= \log|z| + i\arg z$$

$$\neq \log(|z|e^{i\arg z})$$

$$= \log z$$

Problem. $\log z$ is a multivalued fct. $(e^{w+2\pi i n} = e^w)$

"Answer": $h(z)$ is a single-valued ~~brach~~ branch of $\log z$, thus it is analytic fct.

domain of h.

Ex 2. Solve $z^n = z_0 = r_0 e^{i\theta_0}$ $\quad n \in \mathbb{Z}_{>0}$

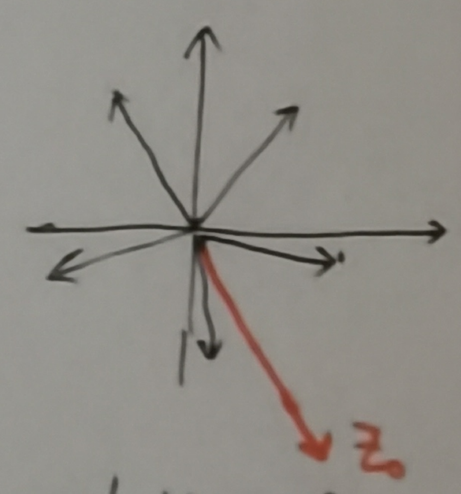
Suppose $z = re^{i\theta}$, then $z^n = r^n e^{in\theta} \Rightarrow \begin{cases} r = \sqrt[n]{r_0} \\ \theta = \dfrac{\theta_0 + 2\pi i k}{n} \end{cases}$ $\quad k \in \mathbb{Z}$

Denote $\zeta_n = e^{\frac{2\pi i}{n}}$, then

$$z = re^{i\theta}$$
$$= r_0^{\frac{1}{n}} e^{\frac{i\theta_0}{n}} \zeta_n^k \qquad k \in \mathbb{Z}$$

Later we denote

$$\sqrt[n]{z_0} := \{ z \in \mathbb{C} \mid z^n = z_0 \}$$
$$= \{ r_0^{\frac{1}{n}} e^{\frac{i\theta_0}{n}} \zeta_n^k \mid k \in \mathbb{Z} \}$$



solution of $z^n = z_0$

(iii) $z^6 + 1 = 0$

$$z^6 = -1 = e^{-\pi i}$$

$$\Rightarrow z = e^{-\frac{i\pi}{6}} \zeta_6^k \qquad k \in \mathbb{Z}$$

(i) (ii). Solve $az^2 + bz + c = 0$ $\quad a, b, c \in \mathbb{C}$, $a \neq 0$

$$a\left(z + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$$

completing the square

$$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

another value is also obtained

choose one randomly

$$z + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

2-value

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

e.g. $z^2 - z + 1 = 0$ $\qquad z^2 - z - 1 = 0$

$$z = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{3}i}{2} \qquad z = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

Eg. A useful decomposition. $(a, b \in \mathbb{C})$

$$a^n - b^n = \prod_{i=0}^{n-1} (a - \zeta_n^i b)$$

(iii) $z^6 + 1 = 0$ (*) ~~$-1 =$ set b~~

set $z_0 = e^{-\frac{\pi i}{6}}$

$$z^6 + 1 = z^6 - z_0^6$$

$$= (z - z_0)(z - \zeta_6 z_0)(z - \zeta_6^2 z_0)(z - \zeta_6^3 z_0)(z - \zeta_6^4 z_0)(z - \zeta_6^5 z_0)$$

$$(*) \Leftrightarrow \quad (z - z_0)(z - \zeta_6 z_0)(z - \zeta_6^2 z_0)(z - \zeta_6^3 z_0)(z - \zeta_6^4 z)(z - \zeta_6^5 z_0) = 0$$

$$\Leftrightarrow \quad z = \zeta_6^k z_0 \quad \text{for } k \in \{0, \dots 5\}.$$

(iv) $$0 = z^6 + z^3 + 1 = \frac{z^9 - 1}{z^3 - 1}$$

$$= \frac{\prod_{i=0}^{8} (z - \zeta_9^i)}{\prod_{j=0}^{2} (z - \zeta_3^j)}$$

$$= \frac{\prod_{i=0}^{8} (z - \zeta_9^i)}{\prod_{j=0}^{2} (z - \zeta_9^{3j})}$$

$$= (z - \zeta_9)(z - \zeta_9^2)(z - \zeta_9^4)(z - \zeta_9^5)(z - \zeta_9^7)(z - \zeta_9^8)$$

$$\Leftrightarrow \quad z = \zeta_9^k \quad \text{for } k \in \{1, 2, 4, 5, 7, 8\}.$$

Eg. To check a subset (of gp) to be a subgp.

it is enough to check that multiplication & inverse are closed.

Ex 3. (ii) $S^1 \subset \mathbb{C}^\times$ is a subgp

$z_1 \in S^1,\ z_2 \in S^1 \Rightarrow z_1 z_2 \in S^1$	$\lvert z_1\rvert = 1 \quad \lvert z_2\rvert = 1 \Rightarrow \lvert z_1 z_2\rvert = 1$
$1 \in S^1$	$\lvert 1\rvert = 1$
$z \in S^1 \Rightarrow z^{-1} \in S^1$	$\lvert z\rvert = 1 \Rightarrow \lvert z^{-1}\rvert = 1$

(iii) $W_n \subset S^1$ is a subgp

$z_1^n = 1,\ z_2^n = 1 \Rightarrow (z_1 z_2)^n = 1$	
$1^n = 1$	
$z^n = 1 \Rightarrow (z^{-1})^n = 1$	

Rest here

Today: Continue computing the partial derivative

(with emphasis on special differential operators)

Task 1. Fix $n \in \mathbb{Z}$, define

$$I_n : L^2([0,1]) \longrightarrow \mathbb{R} \qquad I_n(u) = \left(\int_0^1 u^2(t)\,dt\right)^{2^n}$$

(i)(ii) For $u \in L^2([0,1])$, $v \in L^2([0,1])$, compute $(D_v I_n)(u)$

position direction $(\nabla I_n)(u)$

(if it exists) (Assume $\|v\| = 1$ to be compatible with the definition in the lecture notes)

Hint. $I_n = I_0^{2^n}$ $\qquad I_0(u) = \langle u, u\rangle$

$$(D_v I_n)(u) = \lim_{s\to 0} \frac{I_0^{2^n}(u+sv) - I_0^{2^n}(u)}{s}$$

$$= \lim_{s\to 0} \frac{(I_0(u+sv))^{2^n} - (I_0(u))^{2^n}}{I_0(u+sv) - I_0(u)} \cdot \frac{I_0(u+sv) - I_0(u)}{s}$$

$$\lim_{s\to 0} \frac{I_0(u+sv)-I_0(u)}{s} = \lim_{s\to 0} \frac{\int_0^1 (u(t)+sv(t))^2\,dt - \int_0^1 (u(t))^2\,dt}{s}$$

$$= \lim_{s\to 0} \frac{s\int_0^1 2u(t)v(t)\,dt + s^2\int_0^1 (v(t))^2\,dt}{s}$$

$$= 2\int_0^1 u(t)v(t)\,dt$$

※ Shorten notations.

$$\lim_{s\to 0} \frac{I_0(u+sv)-I_0(u)}{s} = \lim_{s\to 0} \frac{\langle u+sv, u+sv\rangle - \langle u,u\rangle}{s}$$

$$= \lim_{s\to 0} \frac{s(\langle u,v\rangle + \langle v,u\rangle) + s^2\langle v,v\rangle}{s}$$

$$= 2\langle u,v\rangle$$

$$\lim_{s\to 0} \frac{(I_0(u+sv))^{2^n} - (I_0(u))^{2^n}}{I_0(u+sv)-I_0(u)} \overset{n\geq 0 \text{ or } u\neq 0}{=} \left(z^{2^n}\right)'\Big|_{z=I_0(u)}$$

$$= 2^n z^{2^n-1}\Big|_{z=I_0(u)}$$

$$= 2^n (I_0(u))^{2^n-1}$$

$$= 2^n \langle u,u\rangle^{2^n-1}$$

$$(D_v I_n)(u) = 2^{n+1}\langle u,u\rangle^{2^n-1}\langle u,v\rangle$$

$$= \langle 2^{n+1}\langle u,u\rangle^{2^n-1}u, v\rangle \quad \text{when } n\geq 0 \text{ or } u\neq 0$$

In case of $n<0$, and $u=0$,

$$(D_v I_n)(0) = \lim_{s\to 0} \frac{I_n(sv)-I_n(0)}{s}$$

$$= \lim_{s\to 0} \frac{(s^{2^n})^2 I_n(v)}{s}$$

$$= \begin{cases} \langle v,v\rangle^{2^n} & n=-1 \\ 0 & n<-1,\ v=0 \\ \nexists & n<-1,\ v\neq 0 \end{cases}$$

In conclusion,

$$(D_v I_n)(u) = \begin{cases} \langle 2^{n+1}\langle u,u\rangle^{2^n-1}u, v\rangle & n \geq 0 \text{ or } u \neq 0 \\ \langle v, v\rangle^{2^n} & n=-1, \quad u=0 \\ 0 & n<-1, \quad u=0, v=0 \\ \nexists & n<-1, \quad u=0, v\neq 0 \end{cases}$$

$u=0$

$n=-1$

$$\cancel{\lim_{v\to 0} \frac{|I_n(u+v) - I_n(u) - Av|}{\|v\|} = \lim_{v\to 0} \frac{|\langle u+v, u+v\rangle^{2^n} - \langle u,u\rangle^{2^n} - |I_n(u+v) - I_n(u) - Av|}{|I_0}}$$

When $n \geq 0$ or $u \neq 0$,

$$I_n(u+v) = I_n(u) + (D_v I_n)(u) + o(\|v\|)$$

Set $A \triangleq \langle 2^{n+1}\langle u,u\rangle^{2^n-1}u, -\rangle : L^2([0,1]) \longrightarrow \mathbb{R}$,

$$\lim_{v\to 0} \frac{|I_n(u+v) - I_n(u) - Av|}{\|v\|} = 0.$$

When $n<0$ and $u=0$, can not find $A: L^2([0,1]) \to \mathbb{R}$ s.t.

$$\lim_{v\to 0} \frac{|I_n(u+v) - I_n(u) - Av|}{\|v\|} \quad \text{exists.}$$

In conclusion,

$$(\nabla I_n)(u) = \begin{cases} 2^{n+1}\langle u,u\rangle^{2^n-1}u & n \geq 0 \text{ or } u \neq 0 \\ \nexists & n<0 \text{ and } u=0 \end{cases}$$

(iii)(iv) Determine all u s.t.

- u is a critical pt, i.e. $(\nabla I_n)(u)=0$
- u is a local extremum, i.e. $\exists \mathcal{U}$ open, $u \in \mathcal{U}$ s.t

$$\forall u' \in \mathcal{U}, \quad I_n(u) \leq I_n(v)$$

$$\text{resp. } I_n(u) \geq I_n(v)$$

(iii) $(\nabla I_n)(u)=0 \Leftrightarrow u=0$ and $n \geq 0$

(iv) When ∇I_n is defined $(u \neq 0$ or $n \geq 0)$

u is local extremum $\Rightarrow (\nabla I_n)(u)=0 \Leftrightarrow u=0$ and $n \geq 0$

When $u=0$, $\exists \mathcal{U} := L^2([0,1])$, $\forall u' \in \mathcal{U}$

$$I_n(u') = \left(\int_0^1 (u'(t))^2 dt\right)^{2^n} \geq 0 = I_n(u)$$

$\therefore$ In conclusion,

$$u \text{ is local extremum} \Leftrightarrow u=0$$

Have a rest.

We want to simplify our notations and use them freely.

- $\partial_{x_i} := \frac{\partial}{\partial x_i}$ $\quad \partial_{x_i}^2 := \frac{\partial^2}{\partial x_i^2}$ $\quad$ (~~sometimes~~ often $\partial_i = \partial_{x_i} = \frac{\partial}{\partial x_i}$)

  $\partial = \frac{\partial}{\partial z}$ $\quad \bar{\partial} = \frac{\partial}{\partial \bar{z}}$, ...

  $u_t := \frac{\partial u}{\partial t}$ $\quad u_{x_i} := \frac{\partial u}{\partial x_i}$, ...

- Matrix $\Leftrightarrow$ sum

  $C = AB \Leftrightarrow c_{ij} = \sum_k a_{ik} b_{kj}$

E.g. $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = \sum_{i=1}^n \partial_i^2$

$$= \left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right)\begin{pmatrix}\frac{\partial}{\partial x_1}\\ \vdots \\ \frac{\partial}{\partial x_n}\end{pmatrix}$$

$$= \text{div} \cdot \nabla$$

Adv. e.x. (sum of powers, Faulhaber polynomials)

$$\sum_{i=1}^n 1 = n$$

$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$(i+1)^3 - i^3 = 3i^2 + 3i + 1$$

$$\sum_{i=1}^n \left((i+1)^3 - i^3\right) = 3\sum_{i=1}^n i^2 + 3\sum_{i=1}^n i + \sum_{i=1}^n 1$$

$$(n+1)^3 - 1 = 3\sum_{i=1}^n i^2 + 3\frac{n(n+1)}{2} + n$$

$$\Rightarrow \sum_{i=1}^n i^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{i=1}^n i^3 = ?$$

$$\sum_{i=1}^n i^k = ?$$

$$\cancel{\sum_{i=1}^n i^k = \sum_{i=0}^{n-1} (i+1)^k = \sum_{i=0}^{n-1} \sum_{j=0}^{k} \binom{k}{j} i^j 1^{k-j}}$$

$A = n^3 + (n-1)^3 + \dots 1$

$$\cancel{= \sum_{j=0}^{k} \binom{k}{j} \sum_{i=0}^{n-1} i^j}$$

$$\cancel{= \sum_{i=0}^{n-1} i^k + \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i=0}^{n-1} i^j}$$

$$\Rightarrow \cancel{n^k = \sum_{j=0}^{k-1} \binom{k}{j} \sum_{i=0}^{n-1} i^k}$$

$$\Rightarrow (n+1)^k =$$

Task 3 Define operators : $C^2(\mathbb{R}\times\mathbb{R}^n) \longrightarrow C^0(\mathbb{R}\times\mathbb{R}^n)$

name	operater	equation
Laplace	$\Delta := \Delta_x = \sum_i \partial_{x_i}^2 = \mathrm{div}\circ\nabla$	$\Delta u = 0$
wave/d'Alembert	$\Box := \partial_t^2 - \Delta_x$	$\frac{\partial^2 u}{\partial t^2} = \Delta u$
heat current	$\partial_t - \Delta := \partial_t - \Delta_x$	$\frac{\partial u}{\partial t} = \Delta u$

Fix $g \in C^2(\mathbb{R})$, $v \in \mathbb{R}^n \setminus \{0\}$, $c := |v| > 0$, define

$$w(t,x) = g(v\cdot x - ct)$$
$$k(t,x) = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

verify that

$$\Box w = 0 \qquad (\partial_t - \Delta) k = 0$$

Proof $\partial_t w = -c g'(v\cdot x - ct)$

$$\partial_t^2 w = c^2 g''(v\cdot x - ct)$$
$$\partial_{x_i} w = v_i g'(v\cdot x - ct)$$
$$\partial_{x_i}^2 w = v_i^2 g''(v\cdot x - ct)$$
$$\Box w = \partial_t^2 w - \sum_i \partial_{x_i}^2 w = 0$$
$$\partial_t k = \left(-\frac{n}{2}\right) t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \cdot \left(\frac{|x|^2}{4t^2}\right)$$
$$= -\frac{n}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{|x|^2}{4} t^{-\frac{n}{2}-2} e^{-\frac{|x|^2}{4t}}$$
$$\partial_{x_i} k = t^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}} \cdot \left(-\frac{2x_i}{4t}\right)$$
$$= -\frac{1}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} \cdot x$$
$$\partial_{x_i}^2 k = -\frac{1}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + -\frac{1}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} \cdot \left(-\frac{2x_i}{4t}\right) x_i$$
$$= -\frac{1}{2} t^{-\frac{n}{2}-1} e^{-\frac{|x|^2}{4t}} + \frac{x_i^2}{4} t^{-\frac{n}{2}-2} e^{-\frac{|x|^2}{4t}}$$

$(\partial_t - \Delta) k = 0$

Task 2. Fix $n \in \mathbb{Z}_{\geq 2}$, let $\mathcal{U} := \mathbb{R}^n - \{0\} \subset \mathbb{R}^n$. shows that

$$\Phi_n : \mathcal{U} \longrightarrow \mathbb{R} \qquad \Phi_n(x) = \begin{cases} \log|x| & n=2 \\ |x|^{2-n} & n>2 \end{cases}$$

satisfies

$$\Delta \Phi_n = 0$$

Method 1. (Student's try, for $n=2$ or $n>2$)

$n=2$: $\quad \Phi_2(x) = \log|x| = \frac{1}{2}\log(x_1^2+x_2^2)$

$$\partial_i \Phi_2 = \frac{x_i}{x_1^2+x_2^2}$$

$$\partial_i^2 \Phi_2 = \frac{1}{x_1^2+x_2^2} - \frac{x_i \cdot 2x_i}{(x_1^2+x_2^2)^2}$$

$$\Delta \Phi_2 = \sum_{i=1}^{2} \partial_i^2 \Phi_2 = 0$$

$n>2$: $\quad \Phi_n(x) = |x|^{2-n} = (x_1^2+\cdots+x_n^2)^{1-\frac{n}{2}}$

$$\partial_i \Phi_n = (1-\tfrac{n}{2})(x_1^2+\cdots+x_n^2)^{-\frac{n}{2}} \cdot 2x_i$$

$$= (2-n)\, x_i (x_1^2+\cdots+x_n^2)^{-\frac{n}{2}}$$

$$\partial_i^2 \Phi_n = (2-n)\left[(x_1^2+\cdots+x_n^2)^{-\frac{n}{2}} - n x_i^2 (x_1^2+\cdots+x_n^2)^{-\frac{n}{2}-1}\right]$$

$$\Delta \Phi_n = \sum_i \partial_i^2 \Phi_n = 0$$

~~Method 2.~~

Method 2.

$$\partial_i |x| = \partial_i (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$
$$= \frac{1}{2}(x_1^2 + \cdots + x_n^2)^{-\frac{1}{2}} \cdot 2x_i$$
$$= \frac{x_i}{|x|}$$

$n=2$: $\partial_i \Phi_2 = \partial_i(\log|x|) = \frac{1}{|x|}\partial_i|x| = \frac{x_i}{|x|^2} = x_i|x|^{-2}$

$$\partial_i^2 \Phi_2 = |x|^{-2} + x_i(-2|x|^{-3})\partial_i|x|$$
$$= |x|^{-2} - 2x_i^2|x|^{-4}$$

$$\Delta\Phi_2 = \partial_1^2\Phi_2 + \partial_2^2\Phi_2 = 2|x|^{-2} - 2x_1^2|x|^{-4} - 2x_2^2|x|^{-4} = 0$$

$n>2$: $\partial_i \Phi_n = (2-n)|x|^{1-n}\partial_i|x|$
$$= (2-n)x_i|x|^{-n}$$
$$\partial_i^2\Phi_n = (2-n)\left[|x|^{-n} + x_i(-n|x|^{-n-1})\partial_i|x|\right]$$
$$= (2-n)\left[|x|^{-n} - nx_i^2|x|^{-n-2}\right]$$

$$\Delta\Phi_n = \sum_i \partial_i^2\Phi_n = (2-n)\left[n|x|^{-n} - n\left(\sum_i x_i^2\right)|x|^{-n-2}\right] = 0$$

Method 3.

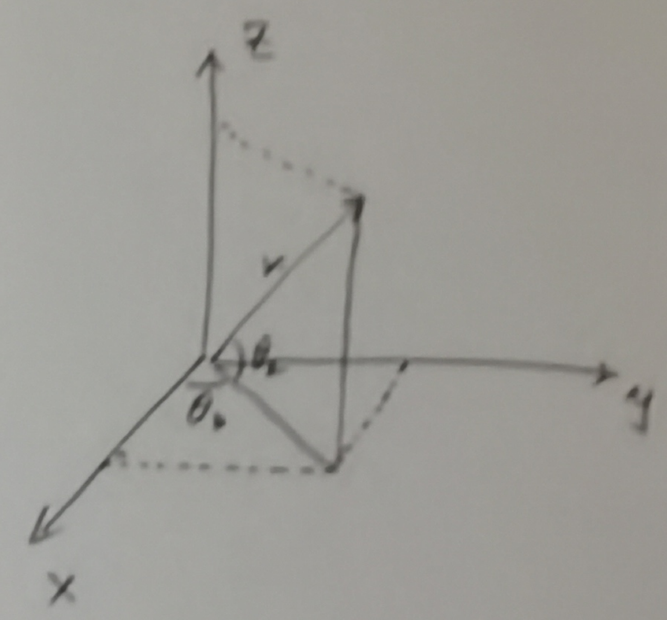
$$\Delta = \begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} & n=2 \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} & n=3 \\ \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\Delta_{S^{n-1}} & n\geq 2 \end{cases}$$

where $\Delta_{S^{n-1}} = \sum_{i,j=2}^{n} a_{ij}\,\partial_{\theta_i}\partial_{\theta_j} + \sum_{i=2}^{n} b_i\,\partial_{\theta_i}$ for some $a_{ij}, b_i \in C^\infty(U)$

Now $\Phi_n(x) = \begin{cases} \log r & n=2 \\ r^{2-n} & n>2 \end{cases}$

$$\Delta\Phi_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\log r\right) = 0$$
$$\Delta\Phi_n = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}r^{2-n}\right) = 0$$

$$\begin{cases} x = r\cos\theta_2\cos\theta_3 \\ y = r\cos\theta_2\sin\theta_3 \\ z = r\sin\theta_2 \end{cases}$$



In general,

$$\begin{cases} x_1 = r\cos\theta_2\cos\theta_3 \cdots \cos\theta_n \\ x_2 = r\cos\theta_2\cos\theta_3 \cdots \sin\theta_n \\ x_3 = r\cos\theta_2 \cdots \sin\theta_{n-1} \\ \vdots \\ x_n = r\sin\theta_2 \end{cases}$$

$$\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta_2} \\ \frac{\partial}{\partial \theta_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} & \frac{\partial z}{\partial r} \\ \frac{\partial x}{\partial \theta_2} & \frac{\partial y}{\partial \theta_2} & \frac{\partial z}{\partial \theta_2} \\ \frac{\partial x}{\partial \theta_3} & \frac{\partial y}{\partial \theta_3} & \frac{\partial z}{\partial \theta_3} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_2\cos\theta_3 & \cos\theta_2\sin\theta_3 & \sin\theta_2 \\ -r\sin\theta_2\cos\theta_3 & -r\sin\theta_2\sin\theta_3 & r\cos\theta_2 \\ -r\cos\theta_2\sin\theta_3 & r\cos\theta_2\cos\theta_3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & r & \\ & & r\cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_2\cos\theta_3 & \cos\theta_2\sin\theta_3 & \sin\theta_2 \\ -\sin\theta_2\cos\theta_3 & -\sin\theta_2\sin\theta_3 & \cos\theta_2 \\ -\sin\theta_3 & \cos\theta_3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

In general, $(\partial_{\theta_i})_{i=0}^{n-1} = R\,Q\,(\partial_{x_k})_{k=1}^{n}$ where $R = \begin{pmatrix} 1 & & & \\ & r & & \\ & & r\cos\theta_2 & \\ & & & \ddots \\ & & & & r\cos\theta_2 \cdots \cos\theta_{n-1} \end{pmatrix}$

$Q^TQ = QQ^T = Id$

$$(\partial_{x_k}) = Q^{-1}R^{-1}(\partial_{\theta_i})$$
$$= Q^TR^{-1}(\partial_{\theta_i})$$

In terms of summation,

$$\partial_{\theta_i} = \sum_k r_{ii} q_{ik} \partial_{x_k} = r_{ii} \sum_k q_{ik} \partial_{x_k}$$

$$\partial_{x_k} = \sum_i q_{ik} r_{ii}^{-1} \partial_{\theta_i}$$

$$\partial_{x_k}^2 = \sum_i q_{ik} r_{ii}^{-1} \partial_{\theta_i} \left(\sum_j q_{jk} r_{jj}^{-1} \partial_{\theta_j} \right)$$

$$= \sum_{i,j} q_{ik} r_{ii}^{-1} (q_{jk} r_{jj}^{-1}) \partial_{\theta_i} \partial_{\theta_j} + \sum_{i,j} q_{ik} r_{ii}^{-1} \partial_{\theta_i} (q_{jk} r_{jj}^{-1}) \cdot \partial_{\theta_j}$$

$$\sum_k \partial_{x_k}^2 = \sum_{i,j} \underbrace{\left(\sum_k q_{ik} q_{jk} \right)}_{= (QQ^T)_{ij} = \delta_{ij}} r_{ii}^{-1} r_{jj}^{-1} \partial_{\theta_i} \partial_{\theta_j} + \sum_j \left(\sum_{i,k} q_{ik} r_{ii}^{-1} \partial_{\theta_i} (q_{jk} r_{jj}^{-1}) \right) \partial_{\theta_j}$$

$$= \sum_i r_{ii}^{-2} \partial_{\theta_i}^2 + \sum_j \left(\sum_{i,k} q_{ik} r_{ii}^{-1} \partial_{\theta_i} (q_{jk} r_{jj}^{-1}) \right) \partial_{\theta_j}$$

For $n > 3$, only need to show that

$$\sum_{i,k} q_{ik} r_{ii}^{-1} \partial_{\theta_i} (q_{ik} r_{ii}^{-1}) = \frac{n-1}{\theta_1} \qquad \theta_1 = r$$

$$r_{11} = 1, \quad q_{1k} = \frac{\partial x_k}{\partial \theta_1} = \frac{x_k}{\theta_1}$$

$$\partial_{\theta_i} q_{ik} = \partial_{\theta_i} \frac{x_k}{\theta_1} = (\partial_{\theta_i} x_k) \theta_1^{-1} + x_k (-1) \theta_1^{-2} \delta_{1i}$$

$$= r_{ii} q_{ik} \theta_1^{-1} - x_k \theta_1^{-2} \delta_{1i}$$

$$LHS = \sum_{i,k} q_{ik} r_{ii}^{-1} (r_{ii} q_{ik} \theta_1^{-1} - x_k \theta_1^{-2} \delta_{1i})$$

$$= \theta_1^{-1} \sum_i \underbrace{\left(\sum_k q_{ik} q_{ik} \right)}_{(QQ^T)_{ii} = 1} - \theta_1^{-2} \sum_k x_k \cdot q_{1k} r_{11}^{-1}$$

$$= \theta_1^{-1} \sum_i 1 - \theta_1^{-2} \sum_k q_{1k} q_{1k} \theta_1$$

$$= \theta_1^{-1} \sum_i 1 - \theta_1^{-1} = \frac{n-1}{\theta_1}$$

Ex. Compute $n=3$ case, ~~and write d~~

Ex. ~~Con~~ Given a general expression of $\Delta_{S^{n-1}}$ in terms of g_{ik}, $r_{i,j}$, d_{ij}.