

# Eine Woche, ein Beispiel

## 10.20 Schur functor: basic formulas

Main reference:

[FH]: William Fulton and Joe Harris. Representation Theory. A First Course.

[Hall]: Brian Hall. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. Graduate Texts in Mathematics. Cham: Springer International Publishing, 2015.

[https://ocw.mit.edu/courses/18-755-lie-groups-and-lie-algebras-ii-spring-2024/resources/mit18\\_755\\_s24\\_lec02\\_pdf/](https://ocw.mit.edu/courses/18-755-lie-groups-and-lie-algebras-ii-spring-2024/resources/mit18_755_s24_lec02_pdf/)

In this document,  $\text{char } \kappa = 0$ ,  $V \in \text{Vect}_\kappa$ .

Schur functor helps us to decompose  $V^{\otimes k}$  by  $S_k$  gp action.  
 $S^\lambda V$  generalize  $\text{Sym}^k V$  &  $\Lambda^k V$ . Moreover,

$$\begin{array}{ccc} \text{Rep}(GL(V)) & = & \text{Rep}(A_{n-1}) \\ \downarrow & & \downarrow \\ S^\lambda V & = & L(\lambda) \end{array} \quad n = \dim V$$

Here,  $\lambda$  has many expressions,  
 e.g. partitions weights

$$\begin{aligned} \lambda &= \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} = (3, 1) & = 2\omega_1 + \omega_2 \\ \lambda &= \dots = (\lambda_1, \lambda_2, \dots) & = \sum m_i \omega_i \end{aligned}$$

$$\{h_{ij}\} = \begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 1 & & \\ \hline \end{array} \quad \text{hook length}$$

1. dimension
2.  $S^\lambda(V \oplus W)$  and ...

# 1. dimension

It can be computed from the Weyl dimension formula:

$$\begin{aligned}
 \dim_{\mathbb{C}} \mathcal{L}(\lambda) &= \frac{\prod_{\alpha \in \Delta^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Delta^+} (\rho, \alpha)} \\
 &\stackrel{\substack{\mathcal{L}(\lambda) \in \text{rep}(A_{n-1}) \\ \lambda = \sum m_i \omega_i}}{=} \frac{(m_1 - 1) \cdots (m_{n-1} + 1) (m_1 + m_2 + 2) (m_2 + m_3 + 2) \cdots (\sum m_i + n - 1)}{\underbrace{1 \cdots 1}_{n-1 \text{ many}} \underbrace{2 \cdots 2}_{n-2 \text{ many}} \cdots n-1} \\
 &= \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j - i} \quad [\text{FH, Thm 6.3 (1)}] \\
 &= \prod_{1 \leq i < j \leq n} \frac{n - i + j}{h_{ij}} \quad [\text{FH, Ex 6.4}]
 \end{aligned}$$

Following [Hall Example 10.23],

$$\Delta^+ = \left\{ \begin{array}{ccccccc} & & & \Sigma \alpha_i & & & \\ & \alpha_1 + \alpha_2 & & \cdots & & \alpha_{n-2} + \alpha_{n-1} & \\ \alpha_1 & \alpha_2 & & \cdots & & & \alpha_{n-1} \end{array} \right\}$$

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} i(n-i) \alpha_i$$

$$= \sum_{i=1}^{n-1} \omega_i$$

$$\chi = \sum_{i=1}^{n-1} \langle \chi, \alpha_i \rangle \omega_i$$

These would be enough to explain the first equality above.

E.g. When  $\lambda = (3, 1) = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & \\ \hline \end{array} = 2\omega_1 + \omega_2,$

$$\begin{aligned}
 \dim_{\mathbb{C}} \mathcal{L}(\lambda) &= \frac{(2+1)(1+1)(0+1) \cdots (2+1+2)(1+0+2)(0+0+2) \cdots (2+1+0+1)}{1 \cdot 1 \cdot 1 \cdots 2 \cdot 2 \cdot 2 \cdots n-1} \\
 &= \frac{(2+1)(1+1) \cdots (2+1+2)(1+0+2) \cdots (2+1+0+1)}{1 \cdot 1 \cdots 2 \cdot 2 \cdots n-1} \\
 &= \left( \frac{2+1}{\textcircled{1}} \cdot \frac{3+2}{\textcircled{2}} \cdot \frac{3+3}{3} \cdot \cdots \cdot \frac{\textcircled{3+n-1}}{n-1} \right) \cdot \left( \frac{1+1}{\textcircled{1}} \cdot \frac{1+2}{2} \cdot \frac{1+3}{3} \cdots \frac{\textcircled{1+n-2}}{n-2} \right) \\
 &= \frac{(n-1) \cdot n \cdot (n+1) \cdot (n+2)}{1 \cdot 4 \cdot 2 \cdot 1} \\
 &= \frac{\begin{array}{|c|c|c|} \hline n & n+1 & n+2 \\ \hline n-1 & & \\ \hline \end{array}}{\begin{array}{|c|c|c|} \hline 4 & 2 & 1 \\ \hline 1 & & \\ \hline \end{array}}
 \end{aligned}$$

2.  $\mathcal{S}^\lambda(V \oplus W)$  and...

E.g.  $(V \oplus W)^{\otimes 2} = V^{\otimes 2} \oplus (V \otimes W)^{\oplus 2} \oplus W^{\otimes 2}$

$$\begin{aligned} \text{Sym}^2(V \oplus W) &= \text{Sym}^2 V \oplus V \otimes W \oplus \text{Sym}^2 W \\ \Lambda^2(V \oplus W) &= \Lambda^2 V \oplus V \otimes W \oplus \Lambda^2 W \end{aligned}$$



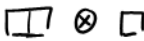




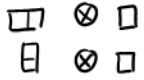
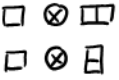






$$(V \oplus W)^{\otimes 3} = V^{\otimes 3} \oplus (V^{\otimes 2} \otimes W)^{\oplus 3} \oplus (V \otimes W^{\otimes 2})^{\oplus 3} \oplus W^{\otimes 3}$$



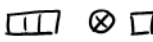



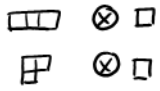
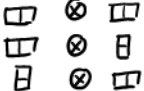


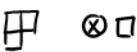




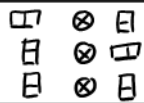




$$\begin{aligned} \text{Sym}^3(V \oplus W) &= \text{Sym}^3 V \oplus \text{Sym}^2 V \otimes W \oplus V \otimes \text{Sym}^2 W \oplus \text{Sym}^3 W \\ \mathcal{S}^\square(V \oplus W) &= \mathcal{S}^\square V \oplus V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2} \oplus \mathcal{S}^\square W \\ \Lambda^3(V \oplus W) &= \Lambda^3 V \oplus \Lambda^2 V \otimes W \oplus V \otimes \Lambda^2 W \oplus \Lambda^3 W \end{aligned}$$

$$\frac{\begin{array}{|c|c|} \hline m+n & m+n+1 \\ \hline m+n-1 & \\ \hline \end{array}}{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array}} = \frac{\begin{array}{|c|c|} \hline m & m+1 \\ \hline m-1 & \\ \hline \end{array}}{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array}} + m^2 n + n m^2 + \frac{\begin{array}{|c|c|} \hline n & n+1 \\ \hline n-1 & \\ \hline \end{array}}{\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & \\ \hline \end{array}}$$

With the help of [FH, Ex 6.11], we can get the following tables:

	$2 + 0$	$1 + 1$	$0 + 2$
$\square \square$	$\square \square$	$\square \otimes \square$	$\square$
$\square$	$\square$	$\square \otimes \square$	$\square$

	$3+0$	$2+1$	$1+2$	$0+3$
				
				
				

	$4+0$	$3+1$	$2+2$	...
				
				
				
				
				

	$5+0$	$4+1$	$3+2$	...

<https://math.stackexchange.com/questions/84103/characters-of-symmetric-and-antisymmetric-powers>