

Eine Woche, ein Beispiel

7.13. stability manifold of \mathbb{P}^1

Ref:

[Okada05]: So Okada, Stability Manifold of \mathbb{P}^1

[GKR03]: A. Gorodentsev, S. Kuleshov, A. Rudakov, t-stabilities and t-structures on triangulated categories, <https://arxiv.org/abs/math/0312442>

[Bri07]: Tom Bridgeland, Stability conditions on triangulated categories, <https://arxiv.org/abs/math/0212237>

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Huy06]: Huybrechts, D. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr. Oxford: Clarendon Press, 2006

Goal: understand the Bridgeland stability and wall crossing in this toy example.

1. equivalent definitions of stability conditions
2. structure of $Coh(\mathbb{P}^1)$
3. standard stability
4. exceptional stability.
5. $Stab(\mathbb{P}^1)$

1. equivalent definitions of stability conditions

Def (locally finite stability condition)

Fix a triangulated category T , and denote

$K(T)$ as the Grothendieck gp of T .

The set of locally finite stability conditions is defined as

$$Stab(T) = \left\{ (Z, P) \left| \begin{array}{ll} Z: K(T) \rightarrow \mathbb{C} & \text{(central charge)} \\ P: \mathbb{R} \rightarrow \{\text{full additive subcategories of } T\} \\ \phi \mapsto P(\phi) & \text{(slicing)} \\ \text{s.t. (a)(b)(c)(d) + (e)} \end{array} \right. \right\}$$

(a) (slicing compatible with central charge)

if $E \in P(\phi)$ then $\frac{Z(E)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)

$$P(\phi+1) = P(\phi)[1]$$

(c) (inverse order vanishing)

$$\text{Hom}_T(A_1, A_2) = 0 \quad \text{for } A_j \in P(\phi_j), \phi_1 > \phi_2$$

(d) (HN filtration) $HN = \text{Harder-Narasimhan}$

$\forall E \in T, \exists$ finite seq of real numbers
 $\phi_1 > \phi_2 > \dots > \phi_n$

and a filtration

$$0 = E_0 \rightarrow E_1 \rightarrow \dots \rightarrow E_{n-1} \rightarrow E_n = E$$

$\nearrow \downarrow \phi_1$ $\nearrow \downarrow \phi_n$

s.t. $A_j \in P(\phi_j) \quad \forall j$.

we define $\phi(E) = \{\phi_1, \dots, \phi_n\}$.

(e) (loc finite) $\forall t \in \mathbb{R}, \exists I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.

$\forall E \in P(I), \exists$ a Jordan-Hölder filtration with finite length.

$$P(I)_+ = \langle P(\phi) \mid \phi \in I \rangle_{\text{extension-closed}}$$

Rmk. For $E \in T, E \neq 0$,

$E \in P(\phi)$ for some $\phi \in \mathbb{R}$

\Leftrightarrow the HN filtration of E has length 1

$\stackrel{\text{def}}{\Leftrightarrow} E$ is semistable

When E is semistable, define $\phi(E) = \phi$ when $E \in P(\phi)$.

Lemma 1.1. $\mathcal{P}(\phi)$ is closed under extension.

Proof. Suppose one has one triangle

$$A_1 \longrightarrow E \longrightarrow A_2 \quad \xrightarrow{+t} \quad (1.1)$$

where $A_1, A_2 \in P(\phi_0)$, we want to show $E \in P(\phi_0)$.

Suppose $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 > \dots > \phi_n$, $n \geq 1$,
then

$$\phi_0 > \phi_n \quad \text{or} \quad \phi_1 > \phi_0 \quad \text{or} \quad (\phi_1 = \phi_0, n=1) \\ \downarrow \\ E \in \mathcal{P}(\phi_0) \quad \vee$$

w.l.o.g. assume $\phi_0 > \phi_n$, then \exists triangle

$$B_1 \rightarrow E \xrightarrow{u} B_2 \xrightarrow{+1}$$

where $u \neq 0$, $B_2 \in P(\phi_n)$.

Apply $\text{Hom}(-, B_2)$ to (1.1), we get

$$\begin{array}{c} \text{Hom}(A, [-1], B_2) \leftarrow \text{Hom}(E[-1], B_2) \leftarrow \text{Hom}(A_2[-1], B_2), \\ \text{Hom}(A_1, B_2) \leftarrow \text{Hom}(E, B_2) \leftarrow \text{Hom}(A_2, B_2). \end{array}$$

Contradiction!

Rmk. [Bri07, Lemma 5.2]

$\mathcal{P}(\phi)$ is an abelian category.

Def (stable sheaf)

Suppose $E \in P(\phi)$ is semistable.

E is stable $\Leftrightarrow E \in P(\phi)$ is simple

$E \in \mathcal{P}(I)$ is simple for some $I \ni \phi$

The next lemma conclude the behavior of triangles with stability conditions.

Lemma 1.2.

Suppose

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow{+1} \quad (1.2)$$

is a triangle, where $\phi(A_1) = \phi_0$, $\phi(A_2) = \phi'_0$.

(1) If $\phi_0 > \phi'_0$, then

(1.2) is the HN-filtration, so E is not semistable;

(2) If $\phi_0 = \phi'_0$, then $E \in \mathcal{P}(\phi_0)$ by Lemma 1.1;

(3) If $u_3 \neq 0$, then $\tilde{\phi}_0 \leq \phi_0 + 1$.

Rmk.

$$\text{Stab}(\mathcal{T}) \cong \left\{ (Z, \phi) \mid \begin{array}{l} Z: k(\mathcal{T}) \rightarrow \mathbb{C} \\ \phi: \mathcal{T} \rightarrow \{\text{finite subsets of } \mathbb{R}\} \\ E \mapsto \{\phi_1, \dots, \phi_n\} \end{array} \right. \begin{array}{l} (\text{central charge}) \\ \text{s.t. (a)(b)(c)(d)} + (\text{e}) \\ (\text{slicing}) \end{array} \left. \right\}$$

$E \in \mathcal{T}$ is semistable $\stackrel{\text{def}}{\iff} \# \phi(E) = 1$

(a) (slicing compatible with central charge)

For E semistable, $\frac{Z(E)}{e^{i\pi\phi(E)}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)

$$\phi(E[1]) = \phi(E) + 1$$

(c) (inverse order vanishing)

$$\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for} \quad \phi(A_1) > \phi(A_2), \quad A_1, A_2 \text{ semistable}$$

(d) (HN filtration)

$\forall E \in \mathcal{T}$, denote $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 < \dots < \phi_n$,

$\exists!$ filtration

$$0 = E_0 \longrightarrow E_1 \longrightarrow \dots \longrightarrow E_{n-1} \longrightarrow E_n = E$$

$\nearrow \wedge_i \quad \searrow \wedge_i$

s.t. $\phi(A_i) = \phi_i \quad \forall i$.

(e) (loc finite) $\forall t \in \mathbb{R}, \exists I = (t - \varepsilon, t + \varepsilon) \subseteq \mathbb{R}$ s.t.

$\forall E \in \mathcal{T}$ with $\phi(E) \subset I$,

\exists a Jordan-Hölder filtration with finite length.

Prop [Okada 05, Prop 2.3]

$$\text{Stab}(\mathcal{T}) \cong \left\{ (\mathbb{A}, Z) \mid \begin{array}{l} \mathbb{A} : \text{heart of } \mathcal{T} \\ Z : K(\mathbb{A}) \xrightarrow{\sim} \mathbb{C} \\ \text{centered slope-function} \\ \text{with HN property} \end{array} \right\}$$

$$(Z, P) \longleftrightarrow (P((0, 1]), Z)$$

$$(Z, P) \longleftrightarrow (\mathbb{A}, Z)$$

where $P(\phi) = \{E \in \mathbb{A} \text{ semistable} \mid \tilde{\phi}(E) = \phi\}$ $\forall \phi \in (0, 1]$

$$\tilde{\phi}(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1]$$

$E \in \mathbb{A}$ semistable: $\# \text{dec } o \rightarrow A_1 \rightarrow E \rightarrow A_2 \rightarrow o$ s.t.
 $\phi(A_1) > \phi(E) > \phi(A_2)$

2. structure of $Coh(\mathbb{P}^1)$

Lemma 2.1. On \mathbb{P}^1 , we have SESs

$$\begin{aligned} 0 \longrightarrow \mathcal{O} &\xrightarrow{\times x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x \longrightarrow 0 \\ 0 \longrightarrow \mathcal{O} &\longrightarrow \mathcal{O}(n)^{\oplus n+1} \xrightarrow{\quad} \mathcal{O}(n+1)^{\oplus n} \longrightarrow 0 \quad n \geq 0 \quad (2.1) \\ 0 \longrightarrow \mathcal{O}(-1)^{\oplus n} &\longrightarrow \mathcal{O}^{\oplus n+1} \xrightarrow{\quad} \mathcal{O}(n) \longrightarrow 0 \quad n \geq 0 \end{aligned}$$

which induces triangles

$$\begin{aligned} \mathcal{O}(k+1) &\longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}(k)[1] \xrightarrow{+1} \\ \mathcal{O}(k+1)^{\oplus k-n}[-1] &\longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus k-n+1}[1] \xrightarrow{+1} \quad n \leq k \quad (2.2) \\ \mathcal{O}(k+1)^{\oplus n-k} &\longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus n-k+1}[1] \xrightarrow{+1} \quad n \geq k \end{aligned}$$

Lemma 2.2. On \mathbb{P}^1 , we have

$$\begin{aligned} R\text{Hom}(\mathcal{O}, \mathcal{O}(n)) &= \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases} \\ R\text{Hom}(\mathcal{O}, \kappa_p) &= \mathbb{C} \\ R\text{Hom}(\kappa_p, \mathcal{O}) &= \mathbb{C}[-1] \\ R\text{Hom}(\kappa_p, \kappa_q) &= \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1], & p = q \\ 0, & p \neq q \end{cases} \end{aligned}$$

Sketch of proof

$$R\text{Hom}(\mathcal{O}, \mathcal{O}(n)) = H^*(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

Then apply $R\text{Hom}(\mathcal{O}, -)$, $R\text{Hom}(-, \mathcal{O})$, $R\text{Hom}(-, \kappa_q)$ to

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow \kappa_p \longrightarrow 0$$

□

The next two lemmas tells us the structure of $\text{Coh}(\mathbb{P}^1)$.

Lemma 2.3. [GKR03, last line in p16]

$$\forall \mathcal{F} \in \text{Coh}(\mathbb{P}^1), \quad \mathcal{F} = (\bigoplus_p \mathbb{E}_p) \oplus (\bigoplus_i \mathcal{O}(n_i))$$

finite many

\mathbb{E}_p : a torsion sheaf supported at p

Lemma 2.4. [GKR03, Prop 6.3] [Huy06, Cor 3.15]

$$\forall \mathcal{F} \in \mathcal{D}^b(\text{Coh}(\mathbb{P}^1)), \quad \mathcal{F} = \bigoplus_i A_i[-i] \quad A_i \in \text{Coh}(\mathbb{P}^1)$$

It also works for $\mathcal{D}^b(\mathbb{A})$ where $\text{gldim } \mathbb{A} = 1$.

E.g. 25. Since $\text{Ext}^1(k_p, \mathcal{O} \oplus \mathcal{O}(n)) \cong \text{Ext}^1(k_p, \mathcal{O}) \oplus \text{Ext}^1(k_p, \mathcal{O}(n)) \cong \mathbb{C}^2$, let us describe the extension

$$0 \longrightarrow \mathcal{O} \oplus \mathcal{O}(n) \longrightarrow E \longrightarrow k_p \longrightarrow 0$$

correspond to $(k_1, k_2) \in \text{Ext}^1(k_p, \mathcal{O} \oplus \mathcal{O}(n))$.

For simplicity, assume that $n > 0$ & $k_1, k_2 \neq 0$.

It is defined as pulling back SES.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(n) & \longrightarrow & E & \longrightarrow & k_p \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \Delta \\ 0 & \longrightarrow & \mathcal{O} \oplus \mathcal{O}(n) & \xrightarrow{(k_1, \mathbb{C} \atop k_2, \mathbb{C})} & \mathcal{O}(1) \oplus \mathcal{O}(n+1) & \longrightarrow & k_p \oplus k_p \longrightarrow 0 \end{array} \quad (2.3)$$

Since $\deg E = n+1$, rank $E = 2$, by Lemma 2.4. we get

$$E = \mathcal{O} \oplus \mathcal{O}(n+1), \quad \mathcal{O}(1) \oplus \mathcal{O}(n) \quad \text{or} \quad \mathcal{O} \oplus \mathcal{O}(n) \oplus k_p$$

but which?

We apply $R\text{Hom}(-, \mathcal{O})$ to (2.3).

$$\begin{array}{ccccccccc} 0 & \leftarrow & \text{Ext}^1(\mathcal{O} \oplus \mathcal{O}(n), \mathcal{O}) & \leftarrow & \text{Ext}^1(E, \mathcal{O}) & \leftarrow & \text{Ext}^1(k_p, \mathcal{O}) & \hookrightarrow & k_p \\ & & \overbrace{\mathcal{C}^{n-1}}^{\parallel} & & \downarrow & & \text{Ext}^1(k_p, \mathcal{O}) & \overset{\mathbb{C}}{\parallel} & \\ & & \text{Hom}(\mathcal{O} \oplus \mathcal{O}(n), \mathcal{O}) & \leftarrow & \text{Hom}(E, \mathcal{O}) & \leftarrow & \text{Hom}(k_p, \mathcal{O}) & \leftarrow & 0 \\ & & \mathcal{C} & & & & \mathbb{C} & & \parallel \end{array}$$

$$\Rightarrow R\text{Hom}(E, \mathcal{O}) = \mathbb{C}^{n-1}[-1]$$

$$\Rightarrow E \cong \mathcal{O}(1) \oplus \mathcal{O}(n).$$

Q: How to determine (2.3) completely?

we have

$$0 \rightarrow E \rightarrow \mathcal{O}(1) \oplus \mathcal{O}(n+1) \oplus k_p \rightarrow k_p \oplus k_p \rightarrow 0$$

w.l.o.g assume $p=0$, by pulling back to local charts of \mathbb{P}^1 , we get

$$0 \rightarrow E|_{A'_z} \longrightarrow \kappa[z] \oplus \kappa[z_2] \oplus \kappa[\Delta] \xrightarrow{\quad} \kappa[z]/_{(z_1)} \oplus \kappa[z_2]/_{(z_2)} \longrightarrow 0$$

$\kappa[z_3] \oplus \kappa[z_4]$

$$0 \rightarrow E|_{A'_w} \xrightarrow{\cong} \kappa[w] \oplus \kappa[w_2] \longrightarrow 0 \longrightarrow 0$$

$\kappa[w_1] \oplus \kappa[w_3]$

$$\begin{aligned} (f_1(z), f_2(z)) &\longmapsto (f_1(0) + a, g_1(0) + a) \\ (f_1(z), f_2(z)) &\longmapsto (f_1(z), f_1(z) + zf_2(z), -f_1(0)) \\ (f(z), g(z), a) &\\ &\downarrow^T \\ (\omega f(\bar{\omega}), \omega^{n+1}g(\bar{\omega})) & \end{aligned}$$

Then transition map is given by

$$E|_{A'_z} \dashrightarrow E|_{A'_w} \quad (f_1(z), f_2(z)) \dashrightarrow (\omega f_1(\bar{\omega}), \omega^{n+1}f_1(\bar{\omega}) + \omega^n f_2(\bar{\omega}))$$

i.e.

$$\begin{aligned} \begin{pmatrix} g_1(\omega) \\ g_2(\omega) \end{pmatrix} &= \begin{pmatrix} \omega & \\ \omega^{n+1} & \omega^n \end{pmatrix} \begin{pmatrix} f_1(\bar{\omega}) \\ f_2(\bar{\omega}) \end{pmatrix} \\ \Rightarrow \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \end{pmatrix} &\stackrel{\text{def}}{=} \begin{pmatrix} g_1(\omega) \\ g_2(\omega) - \omega^n g_1(\omega) \end{pmatrix} = \begin{pmatrix} \omega & \\ & \omega^n \end{pmatrix} \begin{pmatrix} f_1(\bar{\omega}) \\ f_2(\bar{\omega}) \end{pmatrix} \end{aligned}$$

$\Rightarrow E \cong \mathcal{O}(1) \oplus \mathcal{O}(n)$, and (2.3) is

$$\begin{aligned} 0 \rightarrow \mathcal{O} \oplus \mathcal{O}(n) &\xrightarrow{\begin{pmatrix} k_1 z & \\ -k_1 z^n & k_2 \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(n) \xrightarrow{(\text{ev}_p, 0)} k_p \longrightarrow 0 \\ 0 \rightarrow \mathcal{O} \oplus \mathcal{O}(n) &\xrightarrow{\begin{pmatrix} k_1 z & \\ k_2 z & \end{pmatrix}} \mathcal{O}(1) \oplus \mathcal{O}(n+1) \longrightarrow k_p \oplus k_p \longrightarrow 0 \end{aligned}$$

$\parallel \quad \downarrow \left(\frac{1}{z^n} z\right) \quad \downarrow \Delta$

Shorthand: $\text{Stab}(X) := \text{Stab}(\mathcal{D}^b(\text{Coh}(X)))$ for any variety X .

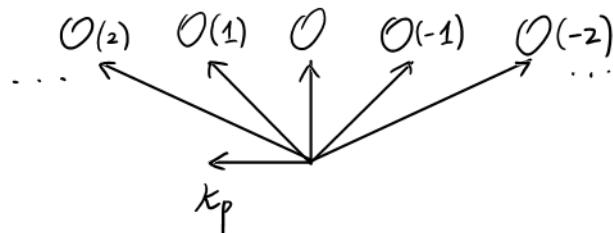
3. standard stability

Goal: classify all stability conditions on \mathbb{P}'
 where all the l.b.s and x_p 's are semistable
 $(\Rightarrow$ all torsion sheaves are semistable)

E.g. Consider the slope stability (Z_0, P_0) :

$$Z_0(E) = -\deg E + i \cdot \text{rk}(E) \quad \text{e.p.}$$

$$\begin{aligned} Z_0(\mathcal{O}(n)) &= -n + i & \phi(\mathcal{O}(n)) &= -\frac{i}{\pi} \arg(-n+i) \\ Z_0(x_p) &= -1 & \phi(x_p) &= -1 \end{aligned}$$



Def. \mathbb{C} acts on $\text{Stab}(\mathbb{P}')$ via rotating the Z -plane:

$$\begin{aligned} \mathbb{C} \times \text{Stab}(\mathbb{P}') &\longrightarrow \text{Stab}(\mathbb{P}') & z = x + iy \\ z \cdot (Z, P) &= (e^z Z, P(\cdot - \frac{y}{\pi})) \\ z \cdot (Z, \phi) &= (e^z Z, \phi(\cdot) - \frac{y}{\pi}) \end{aligned}$$

Rmk 3.3. This action changes the heart but
 preserve the (semi)stability of sheaves, i.e.,

$$E \text{ is (semi)stable in } (Z, P) \Leftrightarrow E \text{ is (semi)stable in } z \cdot (Z, P).$$

Now denote

$$\text{Stab}_{\text{st}}(\mathbb{P}') \stackrel{\text{def}}{=} \left\{ (Z, P) \in \text{Stab}(\mathbb{P}') \mid \text{all } O(n) \text{ & } x_p \text{'s semistable} \right\}$$

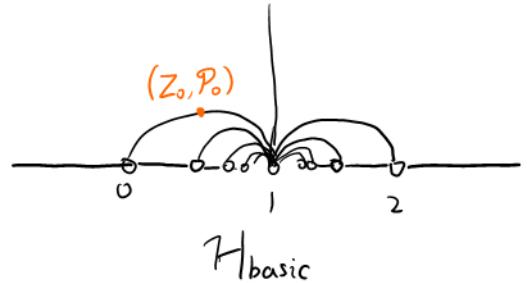
standard

by Rmk 3.3, it is a \mathbb{C} -fiber bundle over

$$\text{Stab}_{\text{st}'}(\mathbb{P}') \stackrel{\text{def}}{=} \left\{ (Z, P) \in \text{Stab}_{\text{st}}(\mathbb{P}') \mid Z(O(-1)) = 1, \phi(O(-1)) = 0 \right\}$$

Prop. $\text{Stab}_{\text{st}'}(\mathbb{P}') \cong \mathcal{H} \cup \mathbb{R} - \left\{ \{1 \pm \frac{1}{n} \mid n \in \mathbb{N}_> \} \cup \{1\} \right\} \cong \mathcal{H}_{\text{basic}}$

$$(Z, P) \mapsto Z(O)$$



Proof Step 1. $Z(O) \in \mathcal{H}_{\text{basic}}$.

$$\text{Hom}(O(-1), O) \cong \mathbb{C}^* \neq 0 \Rightarrow \phi(O) \geq 0$$

$$O \rightarrow x_p \rightarrow O(-1)[1] \xrightarrow{+1}$$

is not an HN-filtration

$$\Rightarrow \phi(O) \leq 1$$

$$0 \neq Z(O(n)) = (n+1)Z(O) - nZ(O(-1)) \Rightarrow Z(O) \notin \left\{ 1 \pm \frac{1}{n} \mid n \in \mathbb{N}_> \right\} \cup \{1\}$$

Step 2. For each $z \in \mathcal{H}_{\text{basic}}$, construct $(Z, P) \in \text{Stab}_{\text{st}'}(\mathbb{P}')$ s.t. $Z(O) = z$.

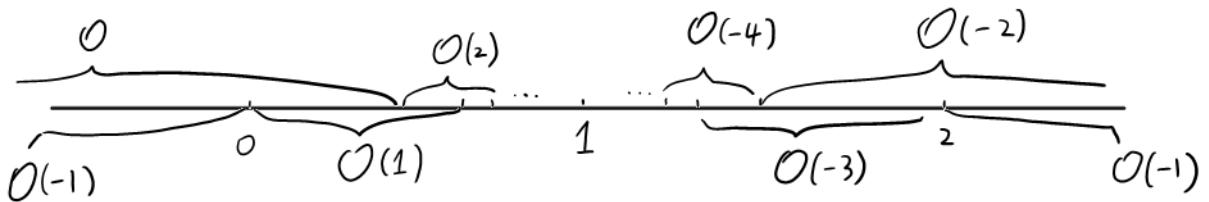
$$\text{Take } Z(O(n)) = (n+1)z - n$$

$$Z(x_p) = z - 1,$$

that (Z, P) determines $(Z, P) \in \text{Stab}_{\text{st}'}(\mathbb{P}')$

Rmk. Assume $(Z, P) \in \text{Stab}_{\text{st}}(\mathbb{P}')$.

When $Z(O) \in \mathcal{H}$, all $O(n)$ & x_p are stable;
 when $Z(O) \in (-\infty, 0)$, only O & $O(-1)$ are stable.
 In general,



Def. \mathbb{Z} acts on $\text{Stab}(\mathbb{P}')$ by tensoring with $O(-n)$.

$$\mathbb{Z} \times \text{Stab}(\mathbb{P}') \longrightarrow \text{Stab}(\mathbb{P}')$$

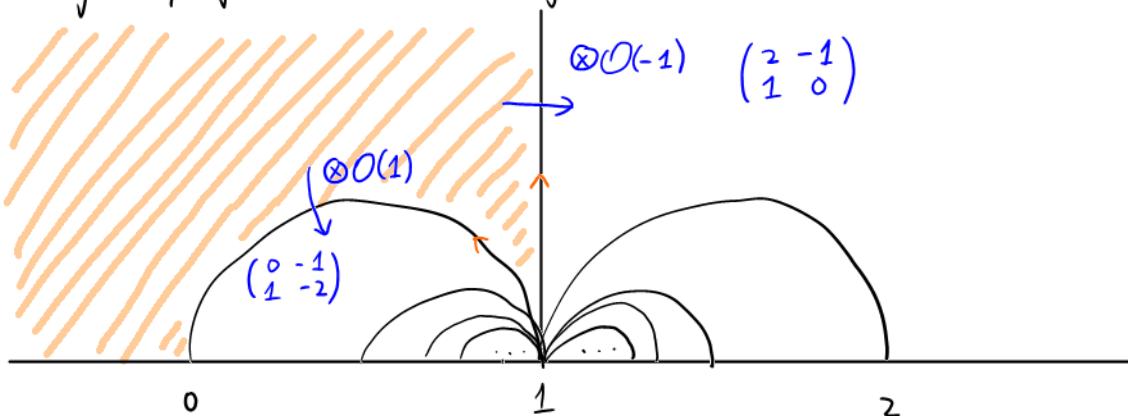
$$n \cdot (Z, P) = (Z(- \otimes O(-n)), n * P)$$

$$\text{Ob}((n * P)(\phi)) = \{E \in \mathcal{T} \mid E \otimes O(-n) \in P(\phi)\}$$

$$n \cdot (Z, \phi) = (Z(- \otimes O(-n)), \phi(- \otimes O(-n)))$$

Rmk. $\text{Stab}_{\text{st}}(\mathbb{P}')$ has a fundamental domain w.r.t. \mathbb{Z} -action.

After projection, it is as follows:



4. exceptional stability.

Fix a stability condition $(\mathcal{Z}, \mathcal{P})$ s.t.
some l.b. or x_p is not semistable.
Denote it as E , then we get

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow{+1} \quad (4.1)$$

with A_2 semistable, $\text{Hom}_{\mathcal{T}}(A_1[n], A_2) = 0 \quad \forall n \geq 0$.

Lemma 4.1. [GKR03, Rmk 6.8]

Assume that we have a triangle

$$A_1 \longrightarrow E \xrightarrow{(f, o)} B_1 \oplus B_2 \xrightarrow{+1} \quad B_2 \neq 0$$

then $A_1 \cong C \oplus B_2[-1]$, $\text{Hom}_{\mathcal{T}}(A_1[1], B_1 \oplus B_2) \neq 0$.

Proof. try

$$\begin{array}{ccccccc} E & \xrightarrow{f} & B_1 & \dashrightarrow & C & \xrightarrow{+1} & \\ 0 & \xrightarrow{(f, o)} & B_2 & \longrightarrow & B_2 & \xrightarrow{+1} & \\ \rightsquigarrow & E & \xrightarrow{(f, o)} & B_1 \oplus B_2 & \longrightarrow & C \oplus B_2 & \xrightarrow{+1} \end{array}$$

Similarly, if it is
 $B_3 \oplus B_4 \xrightarrow{(f, o)} E \longrightarrow A_2 \xrightarrow{+1} \quad B_4 \neq 0$
then $\text{Hom}_{\mathcal{T}}((B_3 \oplus B_4)[1], A_2) \neq 0$

Therefore, we can assume (4.1) are of form

$$\begin{aligned} A_1 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{n_i \geq 0} \mathcal{O}(n_i) \oplus \bigoplus_{m_i \leq -2} \mathcal{O}(m_i)[1] \oplus \bigoplus_P \Xi_P & \xrightarrow{+1} \\ A_1 \longrightarrow x_p \longrightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \oplus \Xi'_P \oplus \Xi''_P[1] & \xrightarrow{+1} \end{aligned}$$

Lemma 4.2. [Okada 06, Lemma 3.1(c)]

If $\text{Hom}_T(A_1[n], A_2) = 0 \quad \forall n \geq 0$ & A_2 semistable,
 $\forall \phi_i \in \phi(A_1), \phi_i < \phi(A_2)$ (i.e. HN filtration condition)
then (4.1) are of form (2.2).

Check it! E.g. 2.5 is one example to determine A_{-1} . It seems easy but turns out to be extremely hard. There are too many cases to discuss.

Here is a try:

$$x_p \longrightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \oplus \Xi'_p \oplus \Xi''_p[1] \longrightarrow A_1[1] \xrightarrow{+1}$$

Apply $\text{Hom}(-, \mathcal{O}(k))$:

$$\text{Hom}(x_p[-3], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_2[-3], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_1[-2], \mathcal{O}(k)) \curvearrowright$$

$$\text{Hom}(x_p[-2], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_2[-2], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_1[-1], \mathcal{O}(k)) \curvearrowright$$

$$\text{Hom}(x_p[-1], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_2[-1], \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_1, \mathcal{O}(k)) \curvearrowright$$

$$\text{Hom}(x_p, \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_2, \mathcal{O}(k)) \xleftarrow{\approx 0} \text{Hom}(A_1[1], \mathcal{O}(k)) \curvearrowright$$

$$\Rightarrow \text{Hom}(A_1[n], \mathcal{O}(k)) = 0 \quad \forall n \neq 0, -1$$

$$\Rightarrow A_1[1] \text{ is of form } \bigoplus_{\tilde{n}_i} \mathcal{O}(\tilde{n}_i)[1] \oplus \tilde{\Xi}'_p \oplus \tilde{\Xi}''_p[1]$$

Notice that $\phi(\Xi_p') = \phi(\Xi_p'') = \dots \phi(x_p)$.

If $\Xi_p' \neq 0$, then Ξ_p' is semistable $\Rightarrow x_p$ is semistable,
contradiction!

So $\Xi_p' = 0$.

Take $k \ll 0$, $\Rightarrow k < n_i \& k < \tilde{n}_i \quad \forall n_i, \tilde{n}_i$
one gets

$$\mathbb{C} \leftarrow \text{Ext}'(\Xi_p', \mathcal{O}(k))_{\leq 0} \leftarrow \text{Ext}'(\Xi_p', \mathcal{O}(k)) \leftarrow 0$$

$$\Rightarrow \Xi_p' = 0.$$

Similarly, $\Xi_p'' = \Xi_p''' = 0$, (4.1) is reduced to

$$\bigoplus_{n_i} \mathcal{O}(\tilde{n}_i) \rightarrow x_p \rightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \xrightarrow{+1}$$

the rest is easy.

Idea: ① determine the shape of A,

② verify that x_p is not semistable.

③ use $\text{Hom}(-, \mathcal{O}(k))$ to control the rest terms.

Lemma 4.3. $\exists k$ s.t. $\mathcal{O}(k)$ & $\mathcal{O}(k+1)$ are semistable,
and other l.b. or torsion sheaves are not semistable.

Proof Take

$$E_0 \in \left\{ E \mid \begin{array}{l} E = \mathcal{O}(n) \text{ or } \mathbb{K}_p \\ E \text{ not semistable} \end{array} \right\}$$

with minimal HN-filtration length, then

$$A_1 \longrightarrow E_0 \longrightarrow A_2 \xrightarrow{+1}$$

is of form (2.2) by Lemma 4.2, so $\mathcal{O}(k)$ is semistable.

Since $\mathcal{O}(k+1)$ has smaller HN-filtration length than E_0 ,
 $\mathcal{O}(k+1)$ is semistable.

$$\Rightarrow \phi(\mathcal{O}(k+1)) > \phi(\mathcal{O}(k)) + 1$$

\Rightarrow all triangles in (2.2) are HN-filtrations

\Rightarrow all other l.b. or torsion sheaves are not semistable.

5. $\text{Stab}(\mathbb{P}')$

Now we can describe $\text{Stab}(\mathbb{P}')$ in a relatively satisfied way.
Denote

$$\text{Stab}_{\text{ex},k}(\mathbb{P}') = \left\{ (Z, \mathcal{P}) \in \text{Stab}(\mathbb{P}') \mid \begin{array}{l} \mathcal{O}(k), \mathcal{O}(k+1) \text{ are semistable} \\ x_p \text{ is not} \end{array} \right\}$$

by Rmk 3.3, $\text{Stab}_{\text{ex},-1}(\mathbb{P}')$ is a \mathbb{C} -fiber bundle over

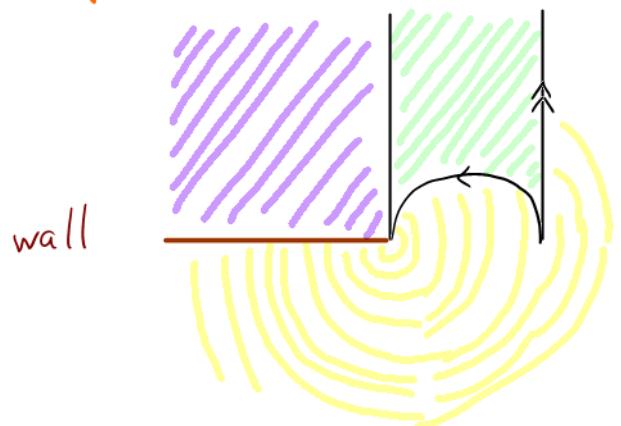
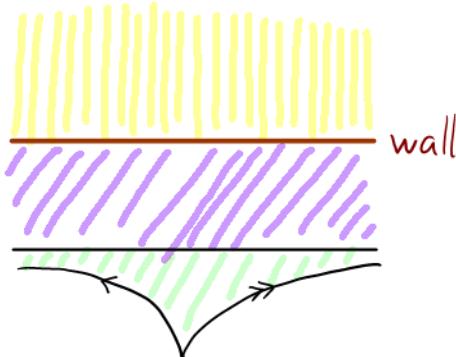
$$\text{Stab}_{\text{ex},-1}(\mathbb{P}') \stackrel{\text{def}}{=} \left\{ (Z, \mathcal{P}) \in \text{Stab}_{\text{ex},-1}(\mathbb{P}') \mid Z(\mathcal{O}(-1)) = 1, \phi(\mathcal{O}(-1)) = 0 \right\}$$

$$\text{Prop. } \text{Stab}_{\text{ex},-1}(\mathbb{P}') \cong \left\{ x + iy \in \mathbb{H} \mid y > \pi \right\} \xrightarrow{\text{exp}} \mathbb{C}^\times$$

$$(Z, \phi) \mapsto \ln |Z(O)| + i\pi \phi(O) \mapsto Z(O)$$

Proof. Reduce to construct (Z, ϕ) where
 $\phi(O) > 1$ and $\frac{Z(O)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$.

In conclusion: (A fundamental domain of $\text{Stab}(\mathbb{P}')/\mathbb{Z}(\mathbb{C})$)



$$\Rightarrow \text{Stab}(\mathbb{P}') \cong \mathbb{C}^2.$$