Eine Woche, ein Beispiel

9.4 Hecke algebra

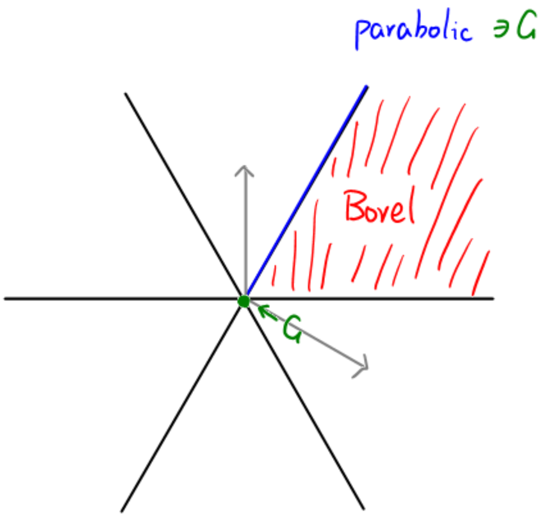
This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]

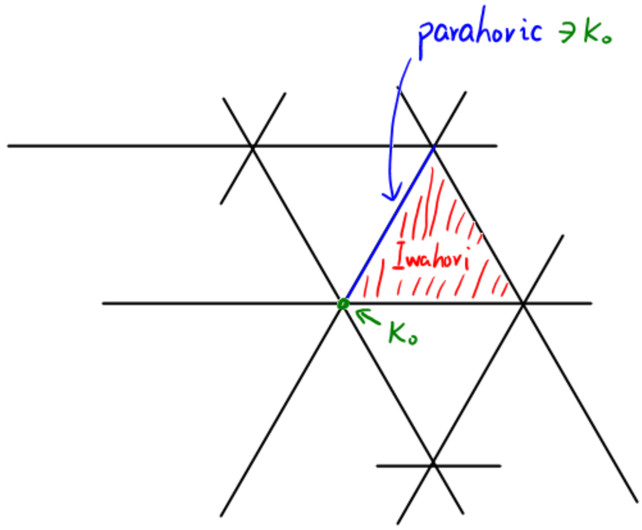
Task. For each double coset decomposition, we want to do:

1. decomposition (& $\Gamma t \Gamma/\Gamma$ is finite)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form
F finite	$G = \bigsqcup_{w\in W} BwB$		
F local	$G = \bigsqcup_{w\in W} BwB$	$G = \bigsqcup_{w\in W_{ext}} IwI$	$G = \bigsqcup_{t\in T^-} K_0 t K_0$
F global	$G = \bigsqcup_{w\in W} BwB$		$GL_2^+(\mathbb{Q}) = \bigsqcup_{t\in T^-} \Gamma t \Gamma$
adèle?			





$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

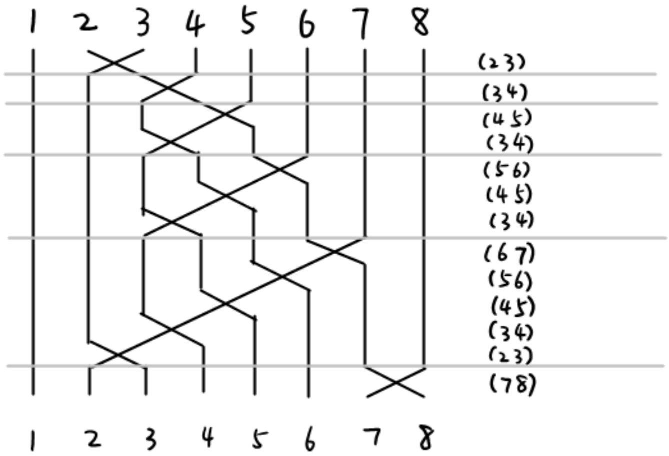
S_n and Tits system

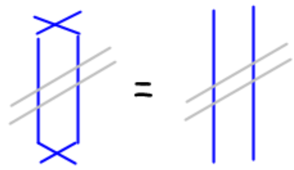
A brief preparation for computations in Bruhat decomposition. $s_i = (i\ i{+}1)$, $1 \le i \le n-1$

E.g. $n = 8$, $w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8$.

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.





$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0) = 13$ = "inversion number"

$l(s_1 w_0) = 14$	$l(w_0 s_1) = 14$
$l(s_2 w_0) = 12$	$l(w_0 s_2) = 12$
$l(s_3 w_0) = 14$	$l(w_0 s_3) = 14$
$l(s_4 w_0) = 12$	$l(w_0 s_4) = 12$
$l(s_5 w_0) = 12$	$l(w_0 s_5) = 12$
$l(s_6 w_0) = 12$	$l(w_0 s_6) = 14$
$l(s_7 w_0) = 14$	$l(w_0 s_7) = 12$

Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

Solution

w_0

Bw_0

w_0B

s_1Bw_0

s_1w_0B

s_2Bw_0

s_2w_0B

The following computation will be also computed later on.

w_0B

$Bw_0 \cap w_0B$

$w_0Bw_0^{-1}$

$B \cap w_0Bw_0^{-1}$

finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,
$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \coprod_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$

Ex. Compute $|BwB/B|$

Hint: Consider the map

$$\phi : B \twoheadrightarrow BwB/B$$
$$b \longmapsto bwB$$

$$\begin{aligned}\phi(b_1) = \phi(b_2) &\Leftrightarrow b_1 wB = b_2 wB \\ &\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B \\ &\Leftrightarrow b_2^{-1} b_1 \in wBw^{-1}\end{aligned}$$

$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{l(w)}$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) = \{ f : G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \ \forall b_1, b_2 \in B, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) \, d\mu(x)$$
$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G,B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbb{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote $T_w := \mathbb{1}_{BwB}$, $T_i := T_{s_i}$ ($T_{Id} = \mathbb{1}_B$ is the unit of $\mathcal{H}(G,B)$)
then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G,B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) \, T_w$$

$$\begin{aligned}(T_u * T_v)(w) &= \frac{1}{|B|} \sum_{yz = w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} \{ (y,z) \in BuB \times BvB \mid yz = w \} \quad \underline{\text{if } w \notin BuBBvB} \ 0 \\ &= \frac{1}{|B|} |BuB \cap uBv^{-1}B|\end{aligned}$$

$$Bs_iB \cdot Bw_oB = \begin{cases} Bs_iw_oB & l(s_iw) = l(w)+1 \\ Bs_iw_oB \cup Bw_oB & l(s_iw) = l(w)-1 \end{cases}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{s_iw} & l(s_iw) = l(w)+1 \\ \mathbb{Z} \cdot T_{s_iw} + \mathbb{Z} \cdot T_w & l(s_iw) = l(w)-1 \end{cases}$$

Computation of coefficient.

$$|BwB| = |BwB/B| \times |B| = q^{l(w)} \cdot |B|$$

when $l(s_iw) = l(w)+1$,

$$\begin{aligned}(T_i * T_w)(s_iw) &= \frac{1}{|B|}\{(y,z) \in Bs_iB \times BwB \mid yz = s_iw\} \\ &= \frac{1}{|B|\,|Bs_iwB|}\{(y,z) \in Bs_iB \times BwB \mid yz \in Bs_iwB\} \\ &= \frac{|Bs_iB|\,|BwB|}{|B| \cdot |Bs_iwB|} = \frac{q^{l(s_i)}\,q^{l(w)}}{q^{l(s_iw)}} = 1\end{aligned}$$

$$\begin{aligned}(T_i * T_i)(Id) &= \frac{1}{|B|}\{(y,z) \in Bs_iB \times Bs_iB \mid yz = Id\} \\ &= \frac{1}{|B|}\,|Bs_iB| = q\end{aligned}$$

$$\begin{aligned}(T_i * T_i)(s_i) &= \frac{1}{|B|}\{(y,z) \in Bs_iB \times Bs_iB \mid yz = s_i\} \\ &= \frac{1}{|B|\,|Bs_iB|}\{(y,z) \in Bs_iB \times Bs_iB \mid yz \in Bs_iB\} \\ &= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB \times Bs_iB| - \{(y,z) \in Bs_iB \times Bs_iB \mid yz \in B\}\right) \\ &= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB|\,|Bs_iB| - |B| \cdot |Bs_iB|\right) \\ &= q-1\end{aligned}$$

when $l(s_iw) = l(w)-1$, we get $l(s_i \cdot s_iw) = l(s_iw)+1$,

$$\begin{aligned}T_i * T_w &= T_i * T_i * T_{s_iw} \\ &= (qT_{Id} + (q-1)T_i) * T_{s_iw} \\ &= qT_{s_iw} + (q-1)T_w\end{aligned}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} T_{s_iw} & l(s_iw) = l(w)+1 \\ qT_{s_iw} + (q-1)T_w & l(s_iw) = l(w)-1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$\mathcal{H}(G,B) = \mathbb{Z}\langle T_1, \dots, T_{n-1}\rangle_{alg}$ with relations $\quad (\mathcal{H}(G,B) \cong \mathcal{H}_q(W))$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j| \geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

E.g. For $n=2$, $\mathcal{H}(G,B) \cong \mathbb{Z}[T_1] / (T_1^2-(q-1)T_1-q)$

$$\cong \mathbb{Z}[T_1] / (T_1-q)(T_1+1)$$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1$$

For $n=3$, $\mathcal{H}(G,B) \cong \mathbb{Z}\langle T_1, T_2 \rangle \big/ \big((T_1-q)(T_1+1), (T_2-q)(T_2+1), T_1T_2T_1 = T_2T_1T_2\big)$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_1T_2 \oplus \mathbb{Z}T_2T_1 \oplus \mathbb{Z}T_1T_2T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_{(123)} \oplus \mathbb{Z}T_{(132)} \oplus \mathbb{Z}T_{(13)}$$