Eine Woche, ein Beispiel

7.18 irreducible representation of semisimple Lie alg

today: $sl_2(\mathbb{C})$ & $sl_3(\mathbb{C})$

Goal.

1. Get some informations of irr rep
   - dim
   - weight space + dim
   - realization (eg. $\mathrm{Sym}^n V$, $\Lambda^n V, \dots$)

2. Understand why "each irr rep corresponds to each highest weight vector".

1. $sl_2(\mathbb{C})$

Notations:

the split basis $\quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad v_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad v_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

the compact basis of $sl_2(\mathbb{C})$ $\quad K = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad X_+ = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \quad X_- = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

the Casimir element $\quad \Omega = -\frac{K^2}{4} + \frac{X_+X_-}{2} + \frac{X_-X_+}{2}$

$$h.v_+ = 2v_+ \qquad h.v_- = -2v_- \qquad [v_+v_-] = h$$

$$\mathrm{ad}\, h = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \mathrm{ad}\, v_+ = \begin{bmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathrm{ad}\, v_- = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Killing form: $\kappa(x,y) = tr(\mathrm{ad}\,x \cdot \mathrm{ad}\,y)$

$$\begin{array}{c|c} & h \quad v_+ \quad v_- \\ \hline \begin{matrix} h \\ v_+ \\ v_- \end{matrix} & \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix} \end{array}$$

$\kappa(-,-)$ for $sl_2(\mathbb{C})$

Lie bracket structure on $\Lambda^{\bullet} sl_2(\mathbb{C}) = \mathbb{C} \oplus sl_2(\mathbb{C}) \oplus \Lambda^2 sl_2(\mathbb{C}) \oplus \Lambda^3 sl_2(\mathbb{C})$:
of degree -1

$[\downarrow, \rightarrow]$	1	h	e	f	$e\wedge f$	$h\wedge f$	$h\wedge e$	$h\wedge e\wedge f$
1	0	0	0	0	0	0	0	0
h	0	0	$2e$	$2f$	0	$-2h\wedge f$	$2h\wedge e$	0
e	0	$-2e$	0	h	$-h\wedge e$	$-2e\wedge f$	0	0
f	0	$-2f$	h	0	$-h\wedge f$	0	$-2e\wedge f$	0
$e\wedge f$	0	0	$h\wedge e$	$h\wedge f$	0	0	0	0
$h\wedge f$	0	$2h\wedge f$	$2e\wedge f$	0	0	0	$-4h\wedge e\wedge f$	0
$h\wedge e$	0	$-2h\wedge e$	0	$2e\wedge f$	0	$-4h\wedge e\wedge f$	0	0
$h\wedge e\wedge f$	0	0	0	0	0	0	0	0

Representations: $\mathrm{Sym}^n V \qquad V \cong \mathbb{C}^2$: standard representation

e.g. $n = 3$

$$h \mapsto \begin{bmatrix} 3 & & & \\ & 1 & & \\ & & -1 & \\ & & & -3 \end{bmatrix} \quad v_+ \mapsto \begin{bmatrix} 0 & 1 & & \\ & 0 & 2 & \\ & & 0 & 3 \\ & & & 0 \end{bmatrix} \quad v_- \mapsto \begin{bmatrix} 0 & & & \\ 3 & 0 & & \\ & 2 & 0 & \\ & & 1 & 0 \end{bmatrix}$$

e.g. the adjoint representation is $\mathrm{Sym}^2 V$

$$\begin{tikzcd} 2v_- \arrow[r, bend left, "/2"] & h \arrow[l, bend left] \arrow[r, bend left] & -2v_+ \arrow[l, bend left, "/2"] \end{tikzcd} \qquad \begin{tikzcd} y^2 \arrow[r, bend left, "/2"] & xy \arrow[l, bend left] \arrow[r, bend left] & x^2 \arrow[l, bend left, "/2"] \end{tikzcd}$$

2. $\mathfrak{sl}_3(\mathbb{C})$

Ref: I would recommend the book "Representation Theory -- a First Course" by Fulton.[Lecture 11-13].

Actually, if you just want to find the answer, then the website "https://www.jgibson.id.au/lievis/" can satisfy most of your requirement. And also if you want to draw the rank 2 root diagrams, then "https://ctan.org/pkg/rank-2-roots" may be a good choice.