Eine Woche, ein Beispiel

3.23: Schubert calculus: Chern class over Grassmannian

This is a follow up of [2025.02.23], [2025.03.16].

1. Formulas for tautological bundle
2. Homology class in $Gr(r,n)$

1. Formulas for tautological bundle

Chern class realized as pullback of σ_{1^s}

Prop. For those v.b.s on $Gr(r,n)$, the Chern class is given by

$$\begin{aligned}
c(S) &= 1 - \sigma_1 + \cdots + (-1)^r \sigma_{1^r} \\
c(Q) &= 1 + \sigma_1 + \cdots + \sigma_k + \cdots + \sigma_{n-r} \\
c(S^\vee) &= 1 + \sigma_1 + \cdots + \sigma_{1^r} \\
c(Q^\vee) &= 1 - \sigma_1 + \cdots + (-1)^k \sigma_k + \cdots + (-1)^{n-k} \sigma_{n-r}
\end{aligned}$$

We omit the proof, as there are many equiv definition of Chern class, and I don't know which one to choose.

Cor If $f: X \longrightarrow Gr(r,n)$ is induced by $(\mathcal{F}, s_1, \ldots, s_n) = (\mathcal{O}_X^{\oplus n} \to \mathcal{F})$, then

$k^n \twoheadrightarrow \mathcal{F}|_p$

$(\mathcal{F}|_p)^* \hookrightarrow k^n$

$$\begin{aligned}
c_s(\mathcal{F}) &= f^* c_s(S^\vee) \\
&= f^* \sigma_{1^s} \\
&= f^* \Sigma_{1^s}(V^{st}) \\
&= f^* \{\Lambda \subset Gr(r,n) \mid \Lambda + V^{st}_{n-r+s-1} \subseteq H\} \\
&= \{p \in X \mid (\mathcal{F}|_p)^* + \langle e_1^*, \ldots, e^*_{n-r+s-1} \rangle \subseteq k^{n-1}\} \\
&= \left\{ p \in X \;\middle|\; \begin{array}{l} \exists\ (0, \ldots, 0, k_{n-r+s}, \ldots, k_n) \in k^n - \{0\}, \text{ s.t.} \\ k_{n-r+s} s_{n-r+s}(p) + \cdots + k_n s_n(p) = 0 \end{array} \right\} \\
&= \{p \in X \mid \underbrace{s_{n-r+s}(p), \ldots, s_n(p)}_{r-s+1 \text{ many}} \text{ are linear dependent}\}
\end{aligned}$$

Especially,

$$\begin{aligned}
c_r(\mathcal{F}) &= \{p \in X \mid s_n(p) = 0\} \\
&\;\;\vdots \\
c_1(\mathcal{F}) &= \{p \in X \mid \underbrace{s_{n-r+1}(p), \ldots, s_n(p)}_{r \text{ many}} \text{ are linear dependent}\} \\
&= c_1(\Lambda^r \mathcal{F}) \\
&= c_1(\det \mathcal{F})
\end{aligned}$$

Rmk. $c_s(\mathcal{F}) \neq c_{top}(\Lambda^{r-s+1}\mathcal{F})$ since
$s_1 \wedge s_2$ (pure wedge) is not a general section in $\Lambda^2 \mathcal{F}$!

Nevertheless, when $s = 1$ or r, pure wedge is a general section, so
$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$ $\quad$ $c_r(\mathcal{F}) = c_r(\mathcal{F})$.

Riemann-Roch

Roughly speaking, Riemann-Roch computes chern class of pushforward.

$$\begin{array}{c} \mathcal{G} \\ | \\ f: Y \longrightarrow X \end{array}$$

$$\begin{aligned}
\text{GRR:}\quad & ch(f_! \mathcal{G})\, td(X) = f_*\left(ch(\mathcal{G})\, td(Y)\right) \\
\text{HRR:}\quad & \chi(Y, \mathcal{G}) = \int_Y ch(\mathcal{G})\, td(Y) \\
& \qquad\quad\; = \left(ch(\mathcal{G})\, td(Y)\right)_{\deg Y}
\end{aligned}$$

RR for surface: $\mathcal{L} = \mathcal{O}(D)$

$$\begin{aligned}
\chi(Y, \mathcal{L}) &= \left[\left(1 + c_1(\mathcal{L}) + \tfrac{1}{2} c_1(\mathcal{L})^2\right)\left(1 + \tfrac{1}{2} c_1(Y) + \tfrac{1}{12}\left(c_1(Y)^2 + c_2(Y)\right)\right)\right]_2 \\
&= \tfrac{1}{2} c_1(\mathcal{L})^2 + \tfrac{1}{2} c_1(\mathcal{L}) c_1(Y) + \tfrac{1}{12}\left(c_1(Y)^2 + c_2(Y)\right) \\
&= \tfrac{1}{2} D(D-K) + \tfrac{1}{12}(K^2 + e)
\end{aligned}$$

$$\Rightarrow \begin{cases} \chi(\mathcal{O}) = \tfrac{1}{12}(K^2 + e) \\ \chi(D) = \chi(\mathcal{O}) + \tfrac{1}{2} D(D-K) \end{cases}$$

RR for curve: $\mathcal{L} = \mathcal{O}(D)$

$$\begin{aligned}
\chi(Y, \mathcal{L}) &= \left[\left(1 + c_1(\mathcal{L})\right)\left(1 + \tfrac{1}{2} c_1(Y)\right)\right]_1 \\
&= c_1(\mathcal{L}) + \tfrac{1}{2} c_1(Y) \\
&= \deg D + 1 - g
\end{aligned}$$

RR for Flag or Grassmannian: Borel-Weil-Bott theorem.

BWB is stronger, because it tells $H^k(Gr(r,n); \mathcal{G})$ for specific k, and it constructs an explicit isomorphism.

[BWB21, Thm 2.4] For a GL_n-regular and dominant (resp. P) weight $\chi \in X^*(T(GL_n))$,

$$H^{l(w)}(Gr(r,n), \mathcal{U}(\chi)) \cong V_{GL_n}(w.\chi) \qquad w.\chi := w(\chi + \rho) - \rho$$

↑ Verma module

[GK20, Sec 3]

$$H^{l(w)}(Gr(r,n), \Sigma_{\chi'} \mathcal{S}^\vee \otimes \Sigma_{\chi''} \mathcal{Q}^\vee) \cong \Sigma_{w.\chi} \mathbb{C}^n$$

Compare HRR with BWB:

$$\begin{aligned}
ch\left(\mathcal{U}(\chi)\ td(Gr(r,n))\right) &= ch\left(\Sigma_{w'} \mathcal{S}^\vee \otimes \Sigma_{w''} \mathcal{Q}^\vee\right)\ td(\mathcal{S}^\vee \otimes \mathcal{Q}) \\
&\overset{?}{=} (-1)^{l(w)} \prod_{1 \le i < j \le n} \frac{(w.\chi)_i - (w.\chi)_j + j - i}{j - i} \\
&= (-1)^{l(w)} \dim V_{GL_n}(w.\chi).
\end{aligned}$$

Porteous' formula

Thm [3264, Thm 12.4]

Let $X/\mathbb{C}$ sm $\quad k \in \mathbb{Z}_{\geq 0}$,

$\mathcal{E}, \mathcal{F}$: v.b. over X of rank e, f,

$\varphi: \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

$$M_k(\varphi) := \{ x \in X \mid \operatorname{rank} \varphi_x \leq k \}$$

remember multiplicity

$\varphi_x : \mathcal{E}|_x \longrightarrow \mathcal{F}|_x$

If $M_k(\varphi) \subset X$ has codim $(e-k)(f-k)$, then

$$[M_k(\varphi)] = \Delta^{e-k}_{f-k}\left[\frac{c(\mathcal{F})}{c(\mathcal{E})}\right] = (-1)^{(e-k)(f-k)} \Delta^{f-k}_{e-k}\left[\frac{c(\mathcal{E})}{c(\mathcal{F})}\right]$$

where

$$\Delta^{e-k}_{f-k}(\gamma) = \begin{vmatrix} \gamma_{f-k} & \cdots & \gamma_{e+f-2k-1} \\ \vdots & \ddots & \vdots \\ \gamma_{f-e+1} & \cdots & \gamma_{f-k} \end{vmatrix}_{(e-k)\times(e-k)}$$

E.g. When $\mathcal{E} = \mathcal{O}_X$,

$$[X] = [M_1(\varphi)] = \Delta^0_{f-1}[c(\mathcal{F})] = \det 1 = 1$$
$$= \Delta^{f-1}_0\left[\frac{1}{c(\mathcal{F})}\right] = \begin{vmatrix} 1 & \cdots & \left[\frac{1}{c(\mathcal{F})}\right]_{f-2} \\ & \ddots & \vdots \\ 0 & & 1 \end{vmatrix} = 1$$

$$[V(s)] = [M_0(\varphi)] = \Delta^1_f[c(\mathcal{F})] = \det(c_f(\mathcal{F})) = c_f(\mathcal{F})$$
$$= -\Delta^f_1\left[\frac{1}{c(\mathcal{F})}\right] = -\begin{vmatrix} \left[\frac{1}{c(\mathcal{F})}\right]_1 & \cdots & \left[\frac{1}{c(\mathcal{F})}\right]_f \\ 1 & \ddots & \vdots \\ 0 & 1 & \left[\frac{1}{c(\mathcal{F})}\right]_1 \end{vmatrix} = c_f(\mathcal{F})$$

When $\mathcal{E} = \mathcal{O}_X^{\oplus e}$,

$$[X] = [M_e(\varphi)] = \Delta^0_{f-e}[c(\mathcal{F})] = 1$$
$$[M_{e-1}(\varphi)] = \Delta^1_{f-e+1}[c(\mathcal{F})] = c_{f-e+1}(\mathcal{F})$$
$$[M_{e-2}(\varphi)] = \Delta^2_{f-e+2}[c(\mathcal{F})] = \begin{vmatrix} c_{f-e+2}(\mathcal{F}) & c_{f-e+3}(\mathcal{F}) \\ c_{f-e+1}(\mathcal{F}) & c_{f-e+2}(\mathcal{F}) \end{vmatrix}$$
$$\vdots$$
$$[V(s_1,\dots,s_e)] = [M_0(\varphi)] = \Delta^e_f[c(\mathcal{F})] = \begin{vmatrix} c_f(\mathcal{F}) & \cdots & c_{f+e-1}(\mathcal{F}) \\ \vdots & \ddots & \vdots \\ c_{f-e+1}(\mathcal{F}) & \cdots & c_f(\mathcal{F}) \end{vmatrix}$$

Furthermore, when $X = Gr(r,n)$, $\mathcal{E} = \mathcal{O}_X^{\oplus e} = \mathcal{O}_X \otimes_k \mathcal{V}_{n-e}^\perp$ and $\mathcal{F} = S^\vee$, we get $f = r$, $c_k(\mathcal{F}) = \sigma_{1^k}$,

$$[M_k(\varphi)] = \Delta^{e-k}_{r-k}[c(\mathcal{F})]$$
$$= \begin{vmatrix} \sigma_{1^{r-k}} & \cdots & \sigma_{1^{e+r-2k-1}} \\ \vdots & \ddots & \vdots \\ \sigma_{1^{r-e+1}} & \cdots & \sigma_{1^{r-k}} \end{vmatrix}_{(e-k)\times(e-k)}$$
$$= \sigma_{(e-k)^{r-k}}$$

In fact, we know that $M_k(\varphi) = \Sigma_{(e-k)^{r-k}}(\mathcal{V})$, since

$$M_k(\varphi) = \{\Lambda \in Gr(r,n) \mid \varphi_\Lambda : \mathcal{V}^\perp \hookrightarrow (\mathbb{C}^n)^* \twoheadrightarrow \Lambda^* \text{ is of rank} \le k\}$$

dual

$$= \{\Lambda \in Gr(r,n) \mid \Lambda \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\mathcal{V} \text{ is of rank} \le k\}$$
$$= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-e} \ge r-k\}$$
$$= \Sigma_{(e-k)^{r-k}}(\mathcal{V})$$

2. Homology class in $Gr(r,n)$

Lines passing planes

E.g. 1. [3264, p131, Question (a)]

For 4 general lines l_1, l_2, l_3, l_4 in $\mathbb{P}^3$, there are 2 lines meet all four.

Reason: In $Gr(2,4)$,

$$
\begin{aligned}
&\# \{ l \in Gr(2,4) \mid l \cap l_i \neq \emptyset, \ \forall i \} \\
&= \deg \sigma_{\square}^{4} \\
&= 2
\end{aligned}
$$

E.g. 2. For 3 general lines l_1, l_2, l_3 in $\mathbb{P}^4$, there is 1 line meet all three.

Reason: In $Gr(2,5)$,

$$
\begin{aligned}
&\# \{ l \in Gr(2,5) \mid l \cap l_i \neq \emptyset, \ \forall i \} \\
&= \deg \sigma_{\square\square}^{3} \\
&= 1
\end{aligned}
$$

One can get further that no line in $\mathbb{P}^5$ passing 3 general lines.

E.g. 3.

For 6 general planes $e_1, \dots, e_6$ in $\mathbb{P}^4$, there are 5 lines passing all these planes.

Reason: In $Gr(2,5)$,

$$
\begin{aligned}
&\# \{ l \in Gr(2,5) \mid l \cap e_i \neq \emptyset, \ \forall i \} \\
&= \deg \sigma_{\square}^{6} \\
&= 5
\end{aligned}
$$

E.g. 4. [3264, p131, Question (a)]

For 4 general $(k-1)$-planes $e_1, e_2, e_3, e_4 \cong \mathbb{P}^{k-1}$ in $\mathbb{P}^{2k-1}$, there are k lines passing all these planes.

Reason: In $Gr(2,2k)$,

$$
\begin{aligned}
&\# \{ l \in Gr(2,2k) \mid l \cap e_i \neq \emptyset, \ \forall i \} \\
&= \deg \sigma_{\underbrace{\square\square\square\square}_{k-1}}^{4} \\
&= k
\end{aligned}
$$