Eine Woche, ein Beispiel

7.9. Steenrod algebra

The Steenrod algebra has been studied for decades. As usual, I make no claim to originality.

For calculating the Steenrod algebra in compute, you can get it from this link:
https://math.berkeley.edu/~kruckman/adem/

The original article by José Adem: The Iteration of the Steenrod Squares in Algebraic Topology
https://www.pnas.org/doi/10.1073/pnas.38.8.720

The survey talk(recommend):
http://www.math.uni-bonn.de/people/daniel/2022/algtop/Steenrod_Squares.pdf

A combinatorial method for computing Steenrod squares:
https://www.sciencedirect.com/science/article/pii/S0022404999000067

Chinese collections on Steenrod algebra:
https://www.zhihu.com/question/265308226

Problems in the Steenrod Algebra:
https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=3ba0259a7d1afc849fb1796d5002bc9c7eab1b5a

1. binomial coefficient mod p
2. Adem relations
3. Steenrod algebra

https://en.wikipedia.org/wiki/Adams_operation

1. binomial coefficient mod p

$\binom{m+n}{n}$ mod 2; m \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
9	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
10	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
12	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

period

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

We conclude from table (or Chu-Vandermonde identity) the following practical formula:

Prop. Let $a = \sum_{n\geq 0} a_n 2^n$, $b = \sum_{n \geq 0} b_n 2^n$, $a_n, b_n \in \{0,1\}$. We get

$$\binom{a+b}{a} \equiv 0 \mod 2 \iff \exists\, n \in \mathbb{N}_{\geq 0} \text{ s.t. } a_n = b_n = 1$$

Eg. $a = (11011010100)_2$, $b = (100000110)_2$, then

$$\binom{a+b}{a} \equiv 0 \mod 2 \quad \text{since} \quad \begin{array}{ccccccccccc} 1&1&0&1&1&0&1&0&\boxed{1}&0&0 \\ &&1&0&0&0&0&0&\boxed{1}&1&0 \end{array}$$

Rmk. Similarly, one can show:
for $a = \sum_{n\geq 0} a_n p^n$, $b = \sum_{n \geq 0} b_n p^n$, $a_n, b_n \in \{0, 1, \cdots, p-1\}$,

$$\binom{a+b}{a} \equiv \prod_{n \geq 0} \binom{a_n+b_n}{a_n} \mod p$$

Rmk. It is possible to define $\binom{a+b}{a} \in \mathbb{F}_p$ for $a, b \in \mathbb{Z}[\frac{1}{p}]$.
One may want to:
① Verify if the usual formulas in https://en.wikipedia.org/wiki/Binomial_coefficient work;
② Find a combinatorical explanation of it.

https://en.wikipedia.org/wiki/%CE%9B-ring

$$\lambda^n : \mathbb{Z} \to \mathbb{Z} \qquad x \mapsto \binom{x}{n}$$

is the unique λ-ring on $\mathbb{Z}$.

2. Adem relations

Def (Steenrod squares) see [wiki: Steenrod algebra] for detail.

$$Sq^k : H^*(- ; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{*+k}(- ; \mathbb{Z}/2\mathbb{Z})$$

$$Sq = Sq^0 + Sq^1 + Sq^2 + \cdots \qquad\qquad Sq^0 = Id_{H^*(- ; \mathbb{Z}/2\mathbb{Z})}$$

⚠ $\qquad Sq^3 \neq Sq^1 Sq^1 Sq^1 \qquad Sq \neq Sq^1$

Prop (Adem relations)

For $0 < a < 2b$, we have a formula

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$

$$= \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{(b-a+j-1)+(a-2j)}{a-2j} Sq^{a+b-j} Sq^j$$

Here we list first several terms: $\left(b > \frac{a}{2} \right)$

$$\begin{aligned}
Sq^1 Sq^b &= \binom{b-2+1}{1} Sq^{b+1} \\
Sq^2 Sq^b &= \binom{b-3+2}{2} Sq^{b+2} + \binom{b-2+0}{0} Sq^{b+1} Sq^1 \\
Sq^3 Sq^b &= \binom{b-4+3}{3} Sq^{b+3} + \binom{b-3+1}{1} Sq^{b+2} Sq^1 \\
Sq^4 Sq^b &= \binom{b-5+4}{4} Sq^{b+4} + \binom{b-4+2}{2} Sq^{b+3} Sq^1 + \binom{b-3+0}{0} Sq^{b+2} Sq^2 \\
Sq^5 Sq^b &= \binom{b-6+5}{5} Sq^{b+5} + \binom{b-5+3}{3} Sq^{b+4} Sq^1 + \binom{b-4+1}{1} Sq^{b+3} Sq^2 \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots
\end{aligned}$$

and make a table for later usage:

Sq^aSq^b (Sq^a \ Sq^b)	Sq^0	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5	Sq^6
Sq^0	1	Sq^1	Sq^2	Sq^3	Sq^4	Sq^5	Sq^6
Sq^1	Sq^1	0	Sq^3	0	Sq^5	0	Sq^7
Sq^2	Sq^2	—	Sq^3Sq^1	$Sq^5+Sq^4Sq^1$	$Sq^6+Sq^5Sq^1$	Sq^6Sq^1	Sq^7Sq^1
Sq^3	Sq^3	—	0	Sq^5Sq^1	Sq^7	Sq^7Sq^1	0
Sq^4	Sq^4	—	—	Sq^5Sq^2	$Sq^7Sq^1+Sq^6Sq^2$	$Sq^9+Sq^8Sq^1+Sq^7Sq^2$	$Sq^{10}+Sq^8Sq^2$
Sq^5	Sq^5	—	—	0	Sq^7Sq^2	Sq^9Sq^1	$Sq^{11}+Sq^9Sq^2$
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

possibility of zeros

E.g. When $a=3$,

$$Sq^3Sq^b = \binom{b-4+3}{3} Sq^{b+3} + \binom{b-3+1}{1} Sq^{b+2}Sq^1$$

b	Comparison 11	Comparison 01	Sq^3Sq^b
$2=4+(-2)$	—	—	0
$3=4+(-1)$	—	00	Sq^5Sq^1
$4=4+0$	00	~~01~~	Sq^7
$5=4+1$	~~01~~	10	Sq^7Sq^1
$6=4+2$	~~10~~	~~11~~	0
$7=4+3$	~~11~~	100	Sq^9Sq^1
$8=4+4$	100	~~101~~	Sq^{11}
⋮	⋮	⋮	⋮

periodic

E.g. When $a=4$,

$$Sq^4Sq^b = \binom{b-5+4}{4} Sq^{b+4} + \binom{b-4+2}{2} Sq^{b+3}Sq + \binom{b-3+0}{0} Sq^{b+2}Sq^2$$

b	Comparison			Sq^4Sq^b
	100	010	000	
$3=5+(-2)$	—	—	000	Sq^5Sq^2
$4=5+(-1)$	—	000	001	$Sq^7Sq^1 + Sq^6Sq^2$
$5=5+0$	000	001	010	$Sq^9 + Sq^8Sq^1 + Sq^7Sq^2$
$6=5+1$	001	$\cancel{010}$	011	$Sq^{10} + Sq^8Sq^2$
$7=5+2$	010	$\cancel{011}$	100	$Sq^{11} + Sq^9Sq^2$
$8=5+3$	011	100	101	$Sq^{12} + Sq^{11}Sq^1 + Sq^{10}Sq^2$
$9=5+4$	$\cancel{100}$	101	110	$Sq^{12}Sq^1 + Sq^{11}Sq^2$
$10=5+5$	$\cancel{101}$	$\cancel{110}$	111	$Sq^{12}Sq^2$
$11=5+6$	$\cancel{110}$	$\cancel{111}$	1000	$Sq^{13}Sq^2$
$12=5+7$	$\cancel{111}$	1000	1001	$Sq^{15}Sq^1 + Sq^{14}Sq^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

periodic

E.g. When $a=5$,

$$Sq^5Sq^b = \binom{b-6+5}{5} Sq^{b+5} + \binom{b-5+3}{3} Sq^{b+4}Sq^1 + \binom{b-4+1}{1} Sq^{b+3}Sq^2$$

b	Comparison			Sq^5Sq^b
	101	011	001	
$3=6+(-3)$	—	—	—	0
$4=6+(-2)$	—	—	000	Sq^7Sq^2
$5=6+(-1)$	—	000	$\cancel{001}$	Sq^9Sq^1
$6=6+0$	000	$\cancel{001}$	010	$Sq^{11} + Sq^9Sq^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$11=6+5$	$\cancel{101}$	$\cancel{110}$	$\cancel{111}$	0
$12=6+6$	$\cancel{110}$	$\cancel{111}$	1000	$Sq^{15}Sq^2$
$13=6+7$	$\cancel{111}$	1000	$\cancel{1001}$	$Sq^{17}Sq^1$
$14=6+8$	1000	$\cancel{1001}$	1010	$Sq^{19} + Sq^{17}Sq^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

periodic

a	period begins with	period of $Sq^a Sp^b$
3	2, 6, ⋯	4
4	3, 11, ⋯	8
5	3, 11, ⋯	8
6	4, 12, ⋯	8
7	4, 12, ⋯	8
8	5, 21, ⋯	16
⋮	⋮	⋮

3. Steenrod algebra

Thm. For the Steenrod algebra

$$\mathcal{A} := \left\{ \begin{array}{l} \partial : H^*(- ; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{*+|\partial|}(- ; \mathbb{Z}/2\mathbb{Z}) \\ \text{as a stable cohomology operation} \end{array} \right\},$$

commutes with suspension Σ

we have generators and relations.

$$\mathcal{A} = \mathbb{Z}/2\mathbb{Z}\{Sq^1, Sq^2, Sq^3, \cdots\} / \text{Adem relations}$$

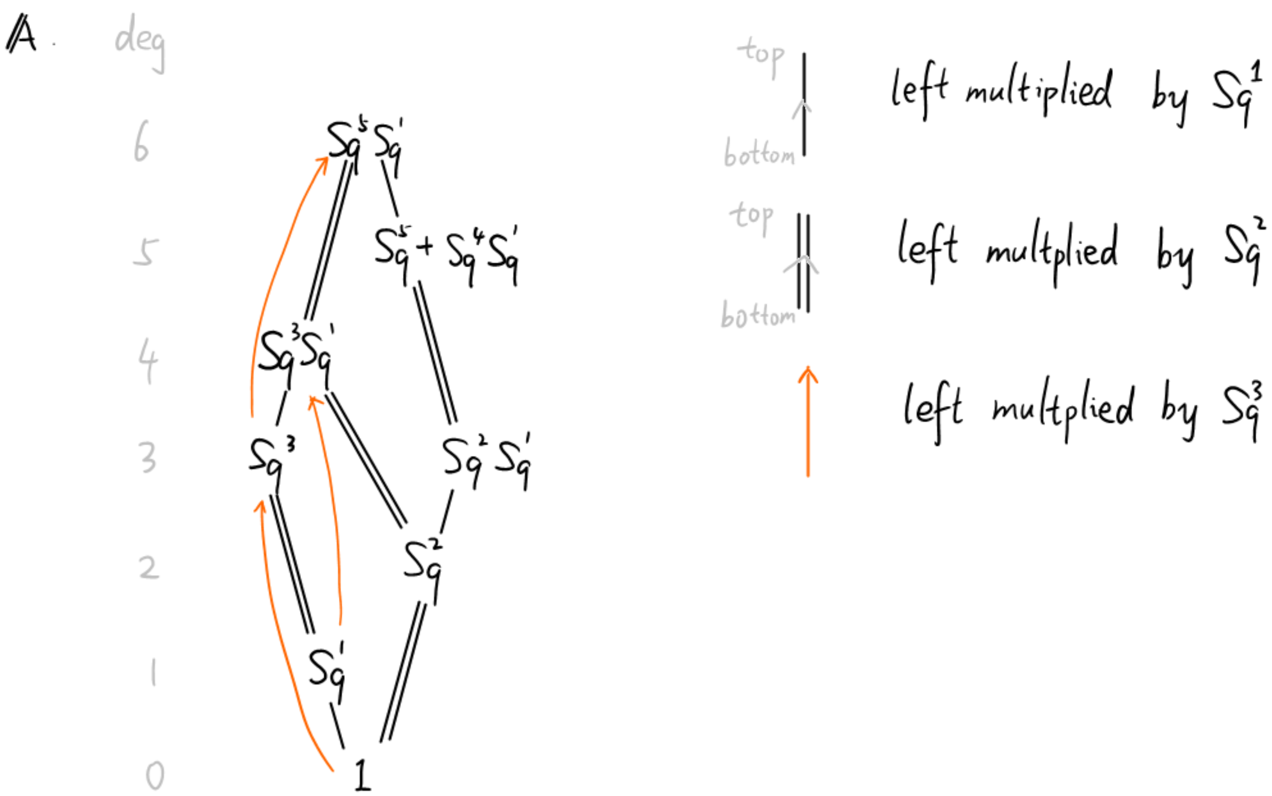
Cor. ① $\mathcal{A} = \langle Sq^1, Sq^2, Sq^3, \cdots \rangle_{\mathbb{Z}/2\mathbb{Z}\text{-alg}}$

② $\mathcal{A}$ has a <u>Serre-Cartan basis</u> (as $\mathbb{Z}/2\mathbb{Z}$-basis)

$$\{ Sq^{i_1} \cdots Sq^{i_n} \mid i_k \geq 2 i_{k+1},\ \forall k \in \{1, \cdots n-1\} \}$$

Ex. Using depth-first search (DFS) or breath-first search (BFS), compute

$$\mathcal{A}(1) := \langle Sq^1, Sq^2 \rangle_{\mathbb{Z}/2\mathbb{Z}\text{-alg}}$$



$$\mathcal{A}(1) \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 8}$$

It reminds me about the Hasse diagram of the Weyl group.

Rmk. In https://math.mit.edu/research/highschool/rsi/documents/2012Shih.pdf, Maurice Shih showed that

$$\mathcal{A}(2) := \langle Sq^1, Sq^2, Sq^4 \rangle_{\mathbb{Z}/2\mathbb{Z}\text{-alg}} \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 64}$$

There is even a schematic for $\mathcal{A}(2)$ in page 17.

E.g. Here we compute $(Sq^4)^k$ ↙compose k times for $k \in \mathbb{N}_{>0}$

For simplicity, denote temporately $[i_1, \dots, i_k] := Sq^{i_1} \cdots Sq^{i_k}$

e.g. $[4,5,7] = Sq^4 Sq^5 Sq^7$

$$
\begin{aligned}
Sq^4 &= [4] \\
(Sq^4)^2 &= [4,4] \\
&= [7,1] + [6,2] \\
(Sq^4)^3 &= [4,4,4] \\
&= [4,7,1] + [4,6,2] \\
&= [9,2,1] + [11,1] + [8,2,2] + [10,2] \\
&= [9,2,1] + [11,1] + [8,3,1] + [10,2] \\
&= [11,1] + [10,2] + [9,2,1] + [8,3,1] \\
(Sq^4)^4 &= [4,4,4,4] \\
&= [4,11,1] + [4,10,2] + [4,9,2,1] + [4,8,3,1] \\
&= [13,2,1] + [12,2,2] + [11,2,2,1] + [12,1,2,1] \\
&\quad + [10,2,3,1] + [11,1,3,1] + [12,3,1] \\
&= [13,2,1] + \cancel{[12,3,1]} + [11,3,1,1] + \cancel{[12,3,1]} \\
&\quad + [10,4,1,1] + [10,5,1] + 0 + [12,3,1] \\
&= [13,2,1] + 0 + 0 + [10,5,1] + [12,3,1] \\
&= [13,2,1] + [12,3,1] + [10,5,1] \\
(Sq^4)^5 &= [4,4,4,4,4] \\
&= [4,13,2,1] + [4,12,3,1] + [4,10,5,1] \\
&= [15,2,2,1] + [16,1,2,1] + [17,2,1] \\
&\quad + [14,2,3,1] + [15,1,3,1] + [12,2,5,1] \\
&= [15,3,1,1] + [16,3,1] + [17,2,1] \\
&\quad + [14,4,1,1] + [14,5,1] + 0 + [12,6,1,1] \\
&= 0 + [16,3,1] + [17,2,1] + 0 + [14,5,1] + 0 \\
&= [17,2,1] + [16,3,1] + [14,5,1] \\
(Sq^4)^6 &= [4,4,4,4,4,4] \\
&= [4,17,2,1] + [4,16,3,1] + [4,14,5,1] \\
&= [19,2,2,1] + [20,1,2,1] \\
&\quad + [18,2,3,1] + [19,1,3,1] + [20,3,1] \\
&\quad + [16,2,5,1] + [18,5,1] \\
&= [19,3,1,1] + \cancel{[20,3,1]} + [18,4,1,1] + \cancel{[18,5,1]} \\
&\quad + 0 + \cancel{[20,3,1]} + [16,6,1,1] + \cancel{[18,5,1]} \\
&= 0 + 0 + 0 \\
&= 0
\end{aligned}
$$

Check this from the calculator: https://math.berkeley.edu/~kruckman/adem/

a	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
order of S_p^a	2	4	3	6	3	4	4	8	3	5	4	7	4	5	5	10	3	7	5	10	4	8
max degree	1	6	6	20	10	18	21	56	18	40	33	72	39	56	60	144	34	108	76	180	63	154

a	23	24	25	26	27	28	29	30														
order of S_p^a	5	8	4	7	5	8	5	6														
max degree	92	168	75	156	108	196	116	150														

Yet not shown in OEIS.