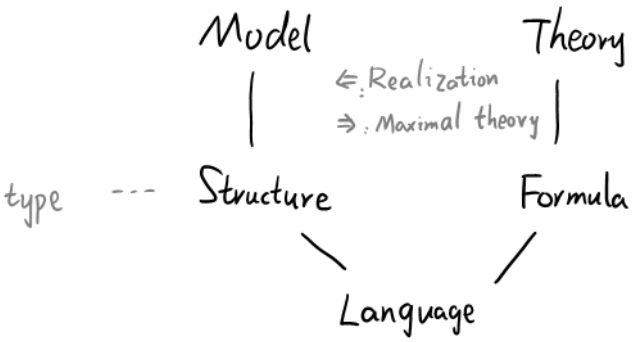
Eine Woche, ein Beispiel

3.27 model theory

Ref: https://philippschlicht.github.io/teaching/files/Lecture.pdf
I heard something from Yilong Zhang, and want to jot down some key points so that I won't be confused next time.



Modern Algebra (H)

Preliminaries

- Logic. set & map, operations on Set. ($\cap \cup -^c$, sub/quotient, two ways of disjoint union Cartesian product)
  - Russell's paradox
  - Axiomatic set theory (ZFC)
  - type of proof : constructive, algorithm, ...

  Ex. graph

  Classify topologies of $\{1, \dots, n\}$

  First adjunction: $\mathrm{Map}(A \times B, C) \cong \mathrm{Map}(A, \mathrm{Map}(B, C))$
- From $\mathbb{N}$ to $\mathbb{C}$ (the basic of examples, though logically it's not here)
  - Peano axioms. Axiom of induction
  - alg structure, order and topology
  - Completeness axiom.
- Cardinal: the only property of set.
  - naive definition

    https://math.stackexchange.com/questions/1712964/attempt-at-proving-the-class-of-all-cardinals-is-a-proper-class

  - operations on cardinal
  - examples
  - The continuum hypothesis
  - large cardinal axiom

	alg	total order	topo
$\mathbb{N}$	$(+, \times)$	✓	discrete
$\mathbb{Z}$	$(+, -, \times)$	✓	discrete
$\mathbb{Q}$	$(+, -, \times, \div)$	✓	dense but not complete
$\mathbb{R}$	$(+, -, \times, \div)$	✓	complete
$\mathbb{C}$	$(+, -, \times, \div)$	✗	complete

- Order structure
  - def, operations and properties (partial/total/well order)
  - ordinal, relationship with cardinal.

  Ex. "well-order" on class of cardinals.

  Classify subpartial ordered set of {subsets of $\{1, \dots, n\}$}, up to iso. $n \le 5$

  the Monoid $\widetilde{R}$, surreals

  poset = partial order set

  coset = 陪集

Group-like structures					
	Totality[1]	Associativity	Identity	Division	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Group or Empty	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Abelian group	Required	Required	Required	Required	Required

A bird eye's view of gp theory

- Group, field and v.s
  - Group: *def
    - * initial example: Aut of set, ordered set, graph; $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
  - Field: def + example: $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{R}, \mathbb{C}$ Never use $\mathbb{Z}_p$ as a shorthand!
  - v.s. def + example: K^n, fct space

    recall: generators, basis, dim; extra alg structures on v.s.

https://en.wikipedia.org/wiki/Category_(mathematics)

Ex. affine space v.s. vector space

ref: Bruhat–Tits theory: a new approach

Structure is maintained: Linkage (连杆)

∥affine space = V-torsor

- "Group is symmetry"
  - more examples. Galois gp & matrix gp
  - Cayley's thm
  - group action: on sets, on graph, on v.s., ...

    Three perspectives: ① subset of Aut; ② action; ③ add alg structure on moduli of symmetries

Ex. concepts of group action on sets

classification of G-sets

finite group of $SO(3,\mathbb{R})$

reminder: $GL_2(\mathbb{R})$ GH. foundamental gp, homotopy gp. $E(\mathbb{Q})$, braid group, ..

- Universal property with group
  - sub/quotient three iso
  - Ker/Im direct sum & product
  - free group
  - presentation of group

Ex. coset decomposition

Ex. Z(G), [G,G], centralizer and normalizer

- Decomposition of group
  - simple gp, ind gp
  - filstration
  - Zassenhaus lemma, Schreier refinement theorem
  - split, semi-product gp
  - Results of simple group.

Ex. cyclic gp case

abelian gp case. ⇝ "abelian category"

Levi decomposition

Down-to-earth analysis.

- f.g. abelian gp + apps.
  Ex. lattice & Crystallographic point gp
  (with/without origin)
  sublattice count
  dual lattice $\Lambda^* := \{ f : \mathbb{R}^n \to \mathbb{R} \text{ linear} \mid \langle f, x \rangle \in \mathbb{Z} \ \forall x \in \Lambda \}$
  lattice in Euclidean space
- Combinatorics related to q-polynomial ← can be quite tricky.
  Ex. finite field with one element
- G act on G
- Sylow thm
- app: classifications of gp of small order. research on specific gp.

Never use the meaning in Model theory!

Ring & module

$R(x)$ $R((x))$ $R\{\{x\}\}$ $R\langle\langle x\rangle\rangle$

- Basic def. e.g. $R[x]$, $R[[x]]$, $R\{x\}$, $R\langle x\rangle$

https://math.stackexchange.com/questions/765787/ring-of-convergent-power-series-in-r-and-c-is-a-local-ring

- Category, k-algs
  Ex. groupoid
- Basic def of modules, abelian category
  E.x. R-algs.
  tensor product, change of basis
  three rep theory
- AG translation, examples.
- Concepts under AG translation
- ED ⇒ PID ⇒ UFD ⇒ domain
  Ex. Elementary divisor thm
  R: PID. $M_{2\times 2}(R) - \{0\} = \coprod_{a|b\in R} GL_2(R)\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(R)$

disjoint for different "$(a)\subset(b)$"

https://en.wikipedia.org/wiki/Smith_normal_form
https://mathoverflow.net/questions/277052/can-one-prove-the-elementary-divisor-theorem-for-pids-by-elementary-matrix-opera
https://math.stackexchange.com/questions/3884394/on-jacobsons-proof-of-the-smith-normal-form-in-a-pid

- classification of f.g. module over PID.

Field and Galois theory (See [GTM167])

can focus more on $\mathbb{F}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, and geometrical point of view.

Many tricky examples to show: https://en.wikipedia.org/wiki/Non-Archimedean_ordered_field

(Reminder) other structures: norm, metric & topo; measure; sheaf.