Task 13

Measurable fct & (square) integrable fct.

~~sa~~ stackexchange

↓

(SE 905415)

Def. $f: \mathcal{U} \to \mathbb{R}$ is measurable, if

$\forall (a,b) \subseteq \mathbb{R}$, $f^{-1}((a,b))$ is measurable

~~sa~~ equiv, $\forall$ $I \subseteq \mathbb{R}$ ~~sa~~ open, $f^{-1}(\overset{I}{\cancel{(a,b)}})$ ——

—— measurable, ——

Task 2. Find $f: [0,1] \to \mathbb{R}$ which is not measurable.

Task 1. (~~i,ii~~ 3 (iii).

For $f: \mathcal{U} \to \mathbb{R}$ continuous, show that f is measurable.

e.p. for

$f_\alpha: B_1(\vec{0}) \to \mathbb{R}$

$x \mapsto |x|^{-\alpha}$

$g: [0,2\pi) \to \mathbb{R}$

$x \mapsto \sinh(x) = \frac{e^x - e^{-x}}{2}$

show f_α, g are measurable.

Task 1. (ii.) For what α, $f_\alpha \in L^2(B_1(0))$?

$d=3$

$$\int_{B_1(0)} |f_\alpha|^2 d\mu = \int_0^1 4\pi r^2 \cdot r^{-2\alpha} dr = \begin{cases} \text{finite} & \alpha < \frac{3}{2} \\ \infty & \alpha \geq \frac{3}{2} \end{cases}$$

$$\int_{B_1(0)} |f_\alpha|^2 d\mu = \int_0^1 C r^{d-1} r^{-2\alpha} dr = \begin{cases} \text{finite} & \alpha < \frac{d}{2} \\ \infty & \alpha \geq \frac{3}{2} \end{cases}$$

Task 3. (ii) Show that $g \in L^2((0,2\pi))$.

$$\int_0^{2\pi} |g(t)|^2 dt \leq \int_0^{2\pi} |\sinh(2\pi)|^2 dt = 2\pi \cdot |\sinh(2\pi)|^2$$

Task 3 (iii) Write $g(x) \sim \sum_{m\in\mathbb{Z}} a_m e^{imx}$, determine a_m.

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} g(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{(1-in)x} - e^{(-1-in)x}}{2}\, dx$$

$$= \frac{1}{2\pi}\frac{1}{2}\left(\frac{e^{(1-in)x}}{1-in} - \frac{e^{(-1-in)x}}{-1-in} \right)\Bigg|_0^{2\pi}$$

$$= \frac{1}{2\pi}\frac{1}{2}\left(\frac{e^{(1-in)2\pi}}{1-in} + \frac{e^{(-1-in)2\pi}}{1+in} - \left(\frac{1}{1+in} + \frac{1}{1-in} \right)\right)$$

$$= \frac{1}{2\pi}\frac{1}{2}\left(\frac{e^{2\pi}(1+in) + e^{-2\pi}(1-in)}{1+n^2} - \frac{2}{1+n^2} \right)$$

$$= \frac{1}{2\pi}\frac{1}{1+n^2}\left[-1 + \cosh(2\pi) + in\sinh(2\pi)\right]$$

Task 3 (iv). $\sum_{m\in\mathbb{Z}} a_m e^{imx} \xrightarrow{\text{uniformly}} g(x)$?

No. $[f_i \text{ cont}, f_i \rightrightarrows f] \Rightarrow f \text{ cont}$

But g is not cont at 0.