

Eine Woche, ein Beispiel

4.28 naive \otimes -Hom adjunction

- Notation.
- A : associate ring allowed to be non-commutative, contains 1
 - There are two systems to write category of A -modules:

$$\begin{aligned} \text{Mod}_A &= A\text{-Mod} & \ni {}_A M \\ (\text{Mod}_A)^{\text{op}} &\neq \text{Mod}_{A^{\text{op}}} = \text{Mod}-A = A^{\text{op}}\text{-Mod} & \ni M_A \\ \text{Mod}_{A \otimes B^{\text{op}}} &= A\text{-Mod}-B & \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\text{Mod}_{B \otimes A^{\text{op}}})^{\text{op}} = (B\text{-Mod}-A)^{\text{op}} = (A^{\text{op}}\text{-Mod}-B^{\text{op}})^{\text{op}} \ni {}_B M_A$$

as

$$(\text{Mod}_{A \otimes B^{\text{op}}})^{\overline{\text{op}}} = (A\text{-Mod}-B)^{\overline{\text{op}}}$$

▽ Even though you can identify $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{\text{op}})$,
 $A^{\text{op}} \notin \text{Ob}(\text{Ring}^{\text{op}})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\text{Mod}_A = \text{Mod}_{A^{\text{op}}} \subset \text{Sh}(\text{Spec } A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for \otimes & Hom

2. adjunction

3. comparison between $\otimes\text{-I Hom}$ & $f^*\text{-I } f_*$

4. definition recall for \otimes & Hom

, derived version

5. adjunction

, derived version

6. comparison between $\otimes\text{-I Hom}$ & $f^*\text{-I } f_*$

, derived version

1. definition recall for \otimes & Hom

$$\begin{aligned}\otimes_A : \text{Mod}_{A^{\text{op}}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}} \\ \text{Hom}_A(-, -) : (\text{Mod}_A)^{\text{op}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}}\end{aligned}$$

In general,

$$\begin{aligned}\otimes_B : A\text{-Mod}-B \times B\text{-Mod}-C &\longrightarrow A\text{-Mod}-C \\ \text{Hom}_B(-, -) : (A\text{-Mod}-B)^{\text{op}} \times B\text{-Mod}-C &\longrightarrow A\text{-Mod}-C \\ \text{Hom}_B^A(-, -) : (A\text{-Mod}-B)^{\text{op}} \times B\text{-Mod}-A &\longrightarrow \mathbb{Z}\text{-Mod} \\ \text{Hom}_{B \otimes_{\mathbb{Z}} A^{\text{op}}}(-, -) : (\mathbb{Z}\text{-Mod}-B \otimes_{\mathbb{Z}} A^{\text{op}})^{\text{op}} \times (B \otimes_{\mathbb{Z}} A^{\text{op}}\text{-Mod}-\mathbb{Z})^{\text{op}} &\longrightarrow \mathbb{Z}\text{-Mod}-\mathbb{Z}\end{aligned}$$

$${}_A X_B \cdot {}_B Y_C \cdot {}_C Z_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{\text{op}}} X \quad \text{in } A\text{-Mod}-C = C^{\text{op}}\text{-Mod}-A^{\text{op}}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\text{Hom}_A(A, X) \cong X$

2. adjunction $\begin{smallmatrix} X_A \\ \downarrow \\ {}_B X_A, {}_C Y_B, {}_C Z_D \end{smallmatrix}$. we get

$$\text{Hom}_C(Y \otimes_B X, Z) \cong \text{Hom}_B(X, \text{Hom}_C(Y, Z)) \quad \text{in } A\text{-Mod-D.}$$

Reason: both sides equal to the set

$$\{ f: Y \times X \rightarrow Z \mid f(cy, x) = cf(y, bx) \quad \forall b, c \}$$

For $A=D=\mathbb{Z}$, fix $Y \in C\text{-Mod-B}$, one gets adjunction factors.

$$\begin{array}{ccc} & \xrightarrow{Y \otimes_B -} & \\ B\text{-Mod} & \perp & C\text{-Mod} \\ & \xleftarrow{\text{Hom}_C(Y, -)} & \end{array}$$

slogan: adjunction \approx associativity

$\otimes \dashv \text{Hom}_B$

$$(A\text{-Mod}-B)^{\overline{\text{op}}} \times (B\text{-Mod}-C)^{\overline{\text{op}}} \times C\text{-Mod}-D \xrightarrow{(\text{Id}, \text{Hom}_C)} (A\text{-Mod}-B)^{\overline{\text{op}}} \times B\text{-Mod}-D$$

$$\parallel$$

$$(A\text{-Mod}-B) \times (B\text{-Mod}-C)^{\overline{\text{op}}} \times C\text{-Mod}-D$$

$$(\otimes_B, \text{Id})$$

$$(A\text{-Mod}-C)^{\overline{\text{op}}}$$

$$\times C\text{-Mod}-D \xrightarrow{\text{Hom}_C}$$

$$\downarrow \text{Hom}_B$$

$$A\text{-Mod}-D$$

$f^* \dashv f_*$

$$\text{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

$$Sh(X)^{\text{op}} \times \text{Mor}(Y, X) \times Sh(Y) \xrightarrow{(\text{Id}, \text{pushforward})} Sh(X)^{\text{op}} \times Sh(X)$$

$$(\text{pullback}, \text{Id})$$

$$Sh(Y)^{\text{op}} \times Sh(Y)$$

$$\xrightarrow{\text{Hom}_{Sh(Y)}(-, -)}$$

$$\downarrow \text{Hom}_{Sh(X)}(-, -)$$

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3. comparison between \otimes -Hom & $f^* \dashv f_*$

Forgetful factor

Prop. For ring homo $\begin{array}{c} S \\ \uparrow f \\ R \end{array}$, \exists "forgetful factor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \quad M \mapsto u(M)$$

$$u(M) = {}_R S_S \otimes_S M = \text{Hom}_S(S_R, M)$$

one has adjunction factors

$$\begin{array}{ccccc} & & S_S \otimes_R - & & \\ & \swarrow & \downarrow & \searrow & \\ S\text{-Mod} & \xrightleftharpoons[u=]{{}_R S_S \otimes_S -} & \text{Hom}_S(S_R, -) & \xrightleftharpoons[\downarrow]{} & R\text{-Mod} \\ & \searrow & \downarrow & \swarrow & \\ & & \text{Hom}_R({}_R S_S, -) & & \end{array} \quad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{array}{ccccc} & & j_! & & \\ & \swarrow & \downarrow & \searrow & \\ D(X) & \xrightleftharpoons[j^*]{} & j^! & \xrightleftharpoons[\downarrow]{} & D(U) \\ & \searrow & \downarrow & \swarrow & \\ & & Rj_* & & \end{array}$$

Vague slogan: $u \approx \text{"forget the information of } Z\text{"}$.

In applications, $U \rightarrow X$ is a covering map.
This change the feeling of the size between U & X .

E.g. For finite gps $H \leq G$, one has Res - Ind adjunction:

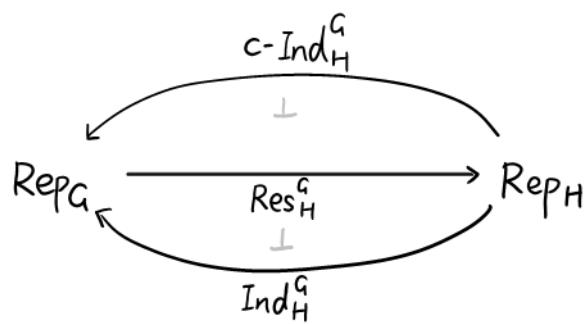
$$\begin{array}{c} \text{Res}_H^G \dashv \text{Ind}_H^G \\ \text{c-Ind}_H^G \dashv \text{Res}_H^G \end{array}$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \leq G \text{ open} \end{cases}$

If one only has $H \leq G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,



Ex. Compare it with the recollement diagram & (3.1).

$$\begin{array}{ccc}
 \mathcal{U} & [*_{\mathcal{H}}] & \\
 \downarrow j & \downarrow & \text{"cover with fiber } G/H \text{"} \\
 X & [*_{\mathcal{G}}] &
 \end{array}$$

translate the following geometrical results into algebraic statements.

1. One has natural factor $j_! \longrightarrow j^*$. When $\# G/H < +\infty$, $j_! = j^*$
 $c\text{-Ind}_{\mathcal{H}}^G \longrightarrow \text{Ind}_{\mathcal{H}}^G$ $c\text{-Ind}_{\mathcal{H}}^G = \text{Ind}_{\mathcal{H}}^G$

2. Even though

$\text{Sh}_{\emptyset, v}([*_{\mathcal{G}}]) \approx \text{Rep}_{\mathcal{G}} = \mathbb{Q}[G]\text{-Mod}$,
the "structure sheaf" of $[*_{\mathcal{G}}]$ is \mathbb{Q} , not $\mathbb{Q}[G]$.

$$\text{Res}_{\{*\}}^G \mathbb{Q} = \mathbb{Q}, \quad \text{Res}_{\{*\}}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* Rj_* \neq \text{Id}$, $j^! j_! \neq \text{Id}$.

Until now, we have met three types of six functor formalism: top spaces, A-modules and stacks.

<https://mathoverflow.net/questions/258159/yoga-of-six-functors-for-group-representations>

From the answer:

$B(\mathbb{Z}/2\mathbb{Z})$ and $\mathbb{R}\mathbb{P}^\infty$ are homotopy equivalent, but other than that they are completely different objects, and in particular their $D(X)$'s are rather different, and many of the problems mentioned in that answer are just problems with that approach.

Compare with i

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{array}{ccc} S & & \text{Spec } S \\ \uparrow \tilde{f} & & \downarrow f \\ R & \xrightarrow{M} & \text{Spec } R \end{array}$$

\exists "pullback factor"

$$f^*: R\text{-Mod} \longrightarrow S\text{-Mod} \quad f^*M = {}_S S_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{array}{ccccc} & & f^* & & \\ & \swarrow & \downarrow u & \searrow & \\ S\text{-Mod} & \xrightarrow{\quad} & R\text{-Mod} & \xleftarrow{\quad} & \\ & \searrow & \downarrow \perp & \swarrow & \\ & & \text{Hom}_R({}_R S_S, -) & & \end{array}$$

compare it with another part of the recollement diagram:

$$\begin{array}{ccccc} & & i^* & & \\ & \swarrow & \downarrow i_* & \searrow & \\ D(Z) & \xrightarrow{\quad} & D(X) & \xleftarrow{\quad} & \\ & \searrow & \downarrow i_! & \swarrow & \\ & & i^! & & \end{array}$$

Rmk. u is usually not f -faithful, unless $S = R/I$.

(In fact, only need S is R -idempotent, i.e. $S \cong {}_S S_R S$.)

which corresponds to closed embedding.

In that case,

$$i^* i_* = \text{Id}: {}_S S_R \otimes_R ({}_R S_S \otimes_S M) \cong M$$

$$i^! i_* = \text{Id}: \text{Hom}_R({}_R S_S, \text{Hom}({}_S S_R, M)) \cong M$$

Slogan: in the comm alg, $\text{Spec } R/I \longrightarrow \text{Spec } R$ is closed embedding.

In general, if

S is an R -idempotent algebra. $S \cong S \otimes_R S$

then $i: \text{Spec } S \longrightarrow \text{Spec } R$ can be viewed as "closed subset".

E.g. $R_p, R/I$ are idempotent R -algs.

$\mathbb{Z}[\frac{1}{p}], \mathbb{F}_p, \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{(p)}, \dots$ are idem \mathbb{Z} -algs.

▽ Usually R/I is not an derived idem R -alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. $\text{Spec } R/I$ is open instead?

Rmk.

Following this slogan, original open/closed subsets are all closed. Also, $i^{\wedge!}$ is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?

A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. definition recall for \otimes & Hom

, derived version

F	RF or LF	R^iF or L^iF	exact fator
f^*	f^*	—	
f_*	Rf_*	$R^i f_*$	f_* -acyclic
$\pi_{X,*} F$	$\Gamma(X; F)$	$H^i(X; F)$	Γ -acyclic
$f_!$	$Rf_!$	$R^i f_!$	$f_!$ -acyclic
$\pi_{X,!} F$	$\Gamma_c(X; F)$	$H_c^i(X; F)$	Γ_c -acyclic
—	$f^!$	$H^i(f^! -)$	
$- \otimes_R -$	$- \otimes_R^L -$	$Tor_R^i(-, -)$	flat
$\text{Hom}_R(-, -)$	$R\text{Hom}_R(-, -)$	$\text{Ext}_R^i(-, -)$	injective/projective
M_G	$Z \otimes_{Z[G]}^L M$	$H_i(G; M)$	
M_G^G	$R\text{Hom}_{Z[G]}(Z, M)$	$H^i(G; M)$	
M_g	$x \otimes_{Ug}^L M$	$H_i(g; M)$	
M_g^g	$R\text{Hom}_{Ug}(x, M)$	$H^i(g; M)$	
$M/[AM]$	$A \otimes_{Ae}^L M$	$HH_i(A, M)$	
M^A	$R\text{Hom}_{Ae}(A, M)$	$HH^i(A, M)$	
$A/[AA]$	$A \otimes_{Ae}^L A$	$HH_i(A)$	
$Z(A)$	$R\text{Hom}_{Ae}(A, A)$	$HH^i(A)$	

e.g. group coh

e.g. Lie alg coh

g/x : Lie alg

e.g. Hochschild coh

For calculations, see:

[23.04.09]: gp coh

[wiki]: Lie algebra coh

[21.05.21]: Hochschild coh

[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.

For $\text{Hom}(-, A)$, see <https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m>.

Chenji Fu claimed that $\text{Hom}(-, A)$ always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.

Related:

<https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod>

<https://mathoverflow.net/questions/29442/what-is-the-opposite-category-of-the-category-of-modules-or-hopf-algebra-repres>

<https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution>

4. definition recall for \otimes & Hom , derived version

To define $\overset{\wedge}{\otimes}$ & $R\text{Hom}$, one needs to extend factors

$$\begin{array}{ccc} \otimes_A: A\text{-Mod} & \times & A\text{-Mod} \\ \text{Hom}_A(-, -): (A\text{-Mod})^{\text{op}} & \times & A\text{-Mod} \end{array} \longrightarrow A\text{-Mod}$$

to factors on double cplxes.

$\mathcal{C}(A)$: complex of A -modules, temperate notation

$$\begin{array}{ccc} \otimes_{\mathcal{C}(A)}: \mathcal{C}(A) & \times & \mathcal{C}(A) \\ \text{Hom}_{\mathcal{C}(A)}(-, -): (\mathcal{C}(A))^{\text{op}} & \times & \mathcal{C}(A) \end{array} \longrightarrow \mathcal{C}(A)$$

But how?

Wishes:

$$\begin{aligned} (M^{\cdot}[i]) \otimes_{\mathcal{C}(A)} (N^{\cdot}[j]) &= (M^{\cdot} \otimes N^{\cdot})[i+j] \\ \text{Hom}_{\mathcal{C}(A)}(M^{\cdot}[-i], N^{\cdot}[j]) &= \text{Hom}(M^{\cdot}, N^{\cdot})[i+j] \end{aligned}$$

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N' & \longrightarrow M^0 \otimes N' & \longrightarrow M^1 \otimes N' & \longrightarrow M^2 \otimes N' & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N^0 & \longrightarrow M^0 \otimes N^0 & \longrightarrow M^1 \otimes N^0 & \longrightarrow M^2 \otimes N^0 & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N^{-1} & \longrightarrow M^0 \otimes N^{-1} & \longrightarrow M^1 \otimes N^{-1} & \longrightarrow M^2 \otimes N^{-1} & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
 \end{array}$$

$\text{Tot}(M \otimes N)$, the double complex of $M \otimes N$.

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M', N') & \longrightarrow \text{Hom}(M^0, N') & \longrightarrow \text{Hom}(M^{-1}, N') & \longrightarrow \text{Hom}(M^{-2}, N') & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M', N^0) & \longrightarrow \text{Hom}(M^0, N^0) & \longrightarrow \text{Hom}(M^{-1}, N^0) & \longrightarrow \text{Hom}(M^{-2}, N^0) & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M', N^{-1}) & \longrightarrow \text{Hom}(M^0, N^{-1}) & \longrightarrow \text{Hom}(M^{-1}, N^{-1}) & \longrightarrow \text{Hom}(M^{-2}, N^{-1}) & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
 \end{array}$$

$\text{Tot}(\text{Hom}(M, N))$, the double complex of $\text{Hom}(M, N)$.

Def. For $M^{\cdot}, N^{\cdot} \in \mathcal{C}(A)$, define

$$M^{\cdot} \otimes N^{\cdot}, \quad \text{Hom}_A(M^{\cdot}, N^{\cdot}) \in \mathcal{C}(A)$$

by

$$(M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot})^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$(\text{Hom}_A(M^{\cdot}, N^{\cdot}))^n = \bigoplus_{i+j=n} \text{Hom}_A(M^{-i}, N^j)$$

and morphisms given by $d + (-1)^j \delta$.

⚠ $F^{\cdot} \otimes G^{\cdot} \cong G^{\cdot} \otimes F^{\cdot}$. However, for the crspcl of elements,
 $a \otimes b \leftrightarrow b \otimes a$ because of this $(-1)^j$.

Ex. Let $M^{\cdot} = [\begin{matrix} \mathbb{Z} & \xrightarrow{x^3} & \mathbb{Z} \\ -1 & & 0 \end{matrix}]$, $N^{\cdot} = [\begin{matrix} \mathbb{Z} & \xrightarrow{x^2} & \mathbb{Z} \\ -1 & & 0 \end{matrix}]$

compute $M^{\cdot} \otimes_{\mathcal{C}(\mathbb{Z})} N^{\cdot}$ & $\text{Hom}_{\mathcal{C}(\mathbb{Z})}(M^{\cdot}, N^{\cdot})$,
and verify that they're complexes.

$$\begin{array}{c} \text{A: } 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad 0 \end{array} \rightsquigarrow \left[\begin{array}{ccc} \mathbb{Z} & \xrightarrow{(-3)} & \mathbb{Z}^2 & \xrightarrow{(2 \ 3)} & \mathbb{Z} \\ -2 & & -1 & & 0 \end{array} \right]$$

$\text{Tot}(M^{\cdot} \otimes N^{\cdot}) \qquad \qquad M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot}$

$$\begin{array}{c} 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ 0 \quad 1 \end{array} \rightsquigarrow \left[\begin{array}{ccc} \mathbb{Z} & \xrightarrow{(-3)} & \mathbb{Z}^2 & \xrightarrow{(2 \ 3)} & \mathbb{Z} \\ -1 & & 0 & & 1 \end{array} \right]$$

$\text{Tot}(\text{Hom}(M^{\cdot}, N^{\cdot})) \qquad \qquad \text{Hom}_{\mathcal{C}(A)}(M^{\cdot}, N^{\cdot})$

Now, we can define \otimes^L & $R\text{Hom}$:

Def. For $M, N \in A\text{-Mod}$, one can define

$$M \otimes_A^L N = M \otimes_{e(A)} P \quad \text{when } N \xrightarrow{\cong} P \quad \text{flat resolution}$$

in general, $M, N \in \mathcal{D}^-(A\text{-Mod})$

$$\begin{aligned} R\text{Hom}_A(M, N) &:= \text{Hom}_{e(A)}(M, I) && \text{when } N \xrightarrow{\cong} I && \text{inj resolution} \\ &:= \text{Hom}_{e(A)}(P, N) && \text{when } M \xleftarrow{\cong} P && \text{proj resolution} \end{aligned}$$

in general, $M \in \mathcal{D}^-(A\text{-Mod}), N \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

<https://math.stackexchange.com/questions/4322028/three-ways-to-prove-that-projective-modules-are-flat>

Ex Compute $\mathbb{F}_2 \otimes_{\mathbb{Z}}^L \mathbb{F}_2$ & $R\text{Hom}_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $\text{Tor}_{\mathbb{Z}}^1(\mathbb{F}_2, \mathbb{F}_2)$ & $\text{Ext}_{\mathbb{Z}}^1(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$\text{Hom}_{\mathcal{D}^+(A)}(M, N) = R^0 \text{Hom}_{e(A)}(M, N)$$

$$\text{Hom}_A(M, N) = \text{Hom}_{\mathcal{D}^+(A)}(M, N) = R^0 \text{Hom}_{e(A)}(M, N).$$

⚠ [KS 90, Def 2.6.2]

To switch from $\mathcal{D}^-(X)$ to $\mathcal{D}^+(X)$, we need to require that

$$\text{w.gldim}(A) < +\infty. \quad \text{w.gldim: shortest flat resolution}$$

In our case, $\text{w.gldim}(Q) = 0 < +\infty$. So don't worry.

A wrong proof for "flat \rightarrow proj"

"Proof": when P is flat,

$$\begin{aligned} P \otimes_A - &\dashv \text{Hom}_A(P, -) \\ P \otimes_A^L - &\dashv R\text{Hom}_A(P, -) \end{aligned}$$

by the uniqueness of the adjunction, $\text{Hom}_A(P, -) = R\text{Hom}_A(P, -)$,
so P is flat.

This is wrong. $Q \in \mathbb{Z}\text{-Mod}$ is flat but not proj.
In the proof, we only have

$$\begin{array}{ccc} A\text{-Mod} & \begin{array}{c} \xrightarrow{P \otimes_A -} \\ \xleftarrow{\text{Hom}_A(P, -)} \end{array} & A\text{-Mod} \\ \downarrow l_A & & \downarrow l_A \\ \mathcal{D}(A) & \begin{array}{c} \xrightarrow{P \otimes_A^L -} \\ \xleftarrow{R\text{Hom}_A(P, -)} \end{array} & \mathcal{D}(A) \end{array}$$

$$l_A \circ (P \otimes_A -) = (P \otimes_A^L -) \circ l_A.$$

Ex. Compute $R\text{Hom}_{\mathbb{Z}}(Q, -)$, and shows that

$$l_{\mathbb{Z}} \circ \text{Hom}_{\mathbb{Z}}(Q, -) \neq R\text{Hom}_{\mathbb{Z}}(Q, -) \circ l_{\mathbb{Z}}.$$

5. adjunction

, derived version

Prop. For A comm ring, $L, M, N \in A\text{-Mod}$, we get
 $L, M \in \mathcal{D}^-(A)$, $N \in \mathcal{D}^+(A)$ in general

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

Proof.

$$\text{Hom}_A(M \otimes_A N, L) \cong \text{Hom}_A(N, \text{Hom}_A(M, L))$$

\Downarrow take $(-)^n$

$$\text{Hom}_{\mathcal{E}(A)}(M \otimes_{\mathcal{E}(A)} N, L) \cong \text{Hom}_{\mathcal{E}(A)}(N, \text{Hom}_{\mathcal{E}(A)}(M, L))$$

\Downarrow $\text{Hom}_A(M, -)$ preserves inj modules
for M flat

$$R\text{Hom}_A(M \otimes_A^L N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

$$\begin{aligned} R\text{Hom}_A(M \otimes_A^L N, L) &= R\text{Hom}_A(P \otimes_{\mathcal{E}(A)} N, L) & M &\xleftarrow{\cong} P \text{ flat} \\ &= \text{Hom}_{\mathcal{E}(A)}(P \otimes_{\mathcal{E}(A)} N, I) & L &\xrightarrow{\cong} I \text{ inj} \\ &= \text{Hom}_{\mathcal{E}(A)}(N, \text{Hom}_{\mathcal{E}(A)}(P, I)) & \text{adj in } \mathcal{E}(A) \\ &= R\text{Hom}_A(N, \text{Hom}_{\mathcal{E}(A)}(P, I)) & \text{Hom}_{\mathcal{E}(A)}(P, I) \text{ is inj} \\ &= R\text{Hom}_A(N, R\text{Hom}_A(P, I)) & I \text{ is inj} \\ &= R\text{Hom}_A(N, R\text{Hom}_A(M, L)) \end{aligned}$$

□

⚠ We don't have

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, \text{Hom}_A(M, L))$$

Find a counterexample?

Take $N = A$, reduce to:

$$R\text{Hom}_A(M, L) \cong \text{Hom}_A(M, L)$$

then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between $\otimes^L \text{Hom}$ & $f^* - f_*$, derived version

Now it's time to extend Section 3 to derived version!
 Notice that most factors in (3.1) are already exact.

$$\begin{array}{ccc}
 & S\mathcal{S}_R \otimes_R^L - & \\
 & \perp & \\
 \mathcal{D}(S\text{-Mod}) & \xleftarrow{\quad u = R\mathcal{S}_S \otimes_S - \quad} & \mathcal{D}(R\text{-Mod}) \\
 & \perp & \\
 & R\text{Hom}_R(R\mathcal{S}_S, -) &
 \end{array} \quad (6.1)$$

Compare with j

In the extreme cases where S is a free R -Mod,
 e.g. $S = k[G]$, $R = k[H]$

all the factors in (3.1) are exact.

Compare with i

(6.1) can be rewritten as

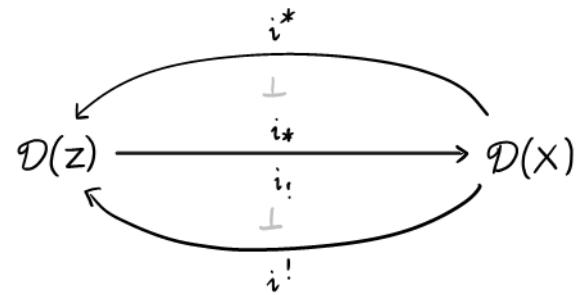
$$\begin{array}{ccc}
 & Lf^* & \\
 & \perp & \\
 \mathcal{D}(S\text{-Mod}) & \xrightarrow{\quad u \quad} & \mathcal{D}(R\text{-Mod}) \\
 & \perp & \\
 & R\text{Hom}_R(R\mathcal{S}_S, -) &
 \end{array} \quad (6.2)$$

where the "pullback factor"

$$Lf^*: \mathcal{D}(R\text{-Mod}) \longrightarrow \mathcal{D}(S\text{-Mod}) \quad Lf^*M = S\mathcal{S}_R \otimes_R^L M$$

is now derived.

compare it with i -part of the recollement diagram:



To guarantee that

$$\begin{cases} u \text{ is faithful} \\ i^* i_* = i_! i^* = \text{Id} \end{cases}$$

one has to require that $S \cong S \otimes_R^L S$.

e.g. $S = R_p$ or $S^{-1}R$

Slogan: Zariski open subsets are closed!

To extend (6.1) to the whole recollement diagram, one has to define the Verdier quotient.

Verdier quotient

Def. For $\mathcal{C} \subset \mathcal{D}$ triangulated categories,

the Verdier quotient \mathcal{D}/\mathcal{C} is the triangulated category s.t.

$$\begin{array}{ccccc} & & \xrightarrow{o} & & \\ \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}/\mathcal{C} \\ & & \searrow o & \downarrow \exists! & \\ & & & & \mathcal{C} \end{array}$$

full subcategory:

<https://math.stackexchange.com/questions/1285761/why-is-it-an-equivalent-definition-of-a-triangulated-full-subcategory>

Rmk: Verdier quotient always exist.

For an explicit construction, see

<https://stacks.math.columbia.edu/tag/o5RA>

Ex. Show the following isomorphism:

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\ & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}(Z) & \xrightarrow[i_!=i_*]{j^*} & \mathcal{D}(X) & \xrightleftharpoons[j^*=j_!]{Rj_*} & \mathcal{D}(U) \\ & \xleftarrow{i^!} & & \xleftarrow{} & \end{array}$$

$$\mathcal{D}(X)/_{j_!}\mathcal{D}(U) \cong \mathcal{D}(Z)$$

$$\mathcal{D}(X)/_{i^*}\mathcal{D}(Z) \cong \mathcal{D}(U)$$

$$\mathcal{D}(X)/_{Rj_*}\mathcal{D}(U) \cong \mathcal{D}(Z)$$

restricted to π_X -sm sheaves.

Hint: take the middle one as an example:

$$\begin{array}{ccccc} \mathcal{D}(Z) & \xrightarrow{i_!} & \mathcal{D}(X) & \xrightarrow{j^*} & \mathcal{D}(U) \\ & \searrow o & \downarrow F & \downarrow \exists! G & \\ & & & & \mathcal{C} \end{array}$$

Uniqueness: $G \circ j^* = F \Rightarrow G = F \circ j_! = F \circ Rj_*$

Existence: Let $G = F \circ j_!$. Apply F to

$$j_! j^* F \rightarrow F \rightarrow i_! i^* F \xrightarrow{+1} \\ \text{get } F \circ j_! \circ j^* = F.$$

Now, we can extend (6.2) to a full recollement:
 (use stable ∞ -category to guarantee that cocone is a factor)

$$\begin{array}{ccccc}
 Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\
 & \xleftarrow{L_i^*} & & \xleftarrow{j_!} & \\
 \mathcal{D}(S\text{-Mod}) & \xrightarrow{\begin{matrix} i_! = i_* \\ R_i^! \end{matrix}} & \mathcal{D}(R\text{-Mod}) & \xrightarrow{\begin{matrix} j^* = j^! \\ Rj_* \end{matrix}} & \mathcal{D}(R\text{-Mod}) / \mathcal{D}(S\text{-Mod}) \\
 & \xleftarrow{} & & \xleftarrow{} &
 \end{array}$$

$L_i^* = S \otimes_R^L -$ $j_! = \text{cocone } [- \longrightarrow S \otimes_R^L -]$
 $i_! = u(-)$ $j^* = \text{projection}$
 $R_i^! = R\text{Hom}_R(S, -)$ $Rj_* = R\text{Hom}_R(\text{cocone } [R \xrightarrow{\sim} S], -)$

Rmk: For any half recollement diagram (satisfying nec conditions), we can extend it to a full recollement diagram through the Verdier quotient. We will prove that rigorously in the 6-fator formalism toolkit.

geometrical realizations of representations

	representations realized as sheaves on some space	stack	formulas
gp	equiv sheaves		$\text{Rep}(G) \rightarrow \text{Sh}([*/G]; \underline{\mathbb{Q}})$
Lie alg	D-modules	flag variety	$\text{Mod}(g, X_\lambda) \cong \text{Mod}_{\text{qc}}^e(D_\lambda)$
Hochschild	Differential, like	Hodge-Tate	$A\text{-Mod}-A \rightarrow ?$
some quiver	perverse sheaves	stratified space	$\text{Rep}(Q) \rightarrow \text{Perv}_s(X)$

Remark for quiver, there's no "structure sheaf/rep" of a given quiver. Therefore, we don't discuss about quiver cohomology.

Realize homogeneous vector bundle on Hermitian symmetric variety as quiver representations:
<https://arxiv.org/abs/math/0403307>

The reference for Lie algebra cohomology is [HT07, Corollary 11.2.6(p274), p298 and Chapter 11]. Aaron told me about this story, and wrote this equivalence in detail in his master thesis.