Eine Woche, ein Beispiel

Ref: from [23.11.19]

1.28 conormal bundle

slogan:

SES	induces	LES,
filtration	induces	spectral sequence.

To expend a little bit,

$$\begin{tikzcd}
 & \text{SES} \arrow[rr] & & \text{LES} & \\
 & \text{filtration} \arrow[rr] & & \text{spectral seq} \arrow[dr] & \\
\text{resolution} \arrow[ur, dashed] \arrow[rrrr] & & & & \begin{array}{l}\text{comparison between} \\ \text{sheaf cohomology} \\ \text{\& other cohomology}\end{array}
\end{tikzcd}$$

Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh". Let us leave spectral seq in other people's notes.

Methods to construct SES: $\begin{cases} \text{check by stalks} \\ \text{filtration by } \mathcal{H}^i(\mathcal{F}) \\ \text{filtration by } \mathcal{F}^i \end{cases}$

method	spectral seq	LES	cohomology/resolution
check by stalks	... for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech
filtration by $\mathcal{H}^i(\mathcal{F})$	Grothendieck		
	Leray-Serre	Gysin	~~Euler class~~
			Hodge-Tate
filtration by $\mathcal{F}^i$ need resolution to get "another" complex	Hodge-de Rham		de Rham, Hodge-de Rham
			Dolbeault $H^p(X,\Omega^q)=H^{p,q}(X)$
	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences which I don't know	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
	Green		for Koszul cohomology
	⋮		⋮

Hodge related

For more spectral sequences, see:
https://en.wikipedia.org/wiki/Spectral_sequence
https://github.com/CubicBear/SpectralSequences/blob/main/SpecralSequences.pdf

1. open-closed formalism
2. open cover
3. filtration by $\mathcal{H}^i(\mathcal{F})$
4. Hodge related filtration

1. open-closed formalism

|| related: comparison of $j_!$ & j_*
|| one-point compactification.

Observe the following pictures:

$$Z \xrightarrow{\ i\ } X \xhookleftarrow{\ j\ } \mathcal{U}$$

$$\begin{tikzcd}
\mathcal{D}(Z) \arrow[r, "i_* = i_!" description] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!" description] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}$$

Black box:

0. We assume some nice conditions.
   e.g. in the category $\mathrm{Haus}^{\text{loc. cpt}}$, and $Z \subset X$ is loc. contractable.
   Under these conditions,
1. $i_* = i_!$, $j^* = j^!$
2. $j_!, i^*, j^*, i_*$ are exact.

Ex. 1. Shows that

$$i^* i_* = i^! i_* = Id_{\mathcal{D}(Z)} \qquad j^* j_! = j^* Rj_* = Id_{\mathcal{D}(\mathcal{U})}$$

$$i^* j_! = 0, \qquad j^* i_* = 0, \qquad i^! Rj_* = 0$$

—— : base change

〰〰 : check stalkwise.

2. (for category fans)
   $i_*, j_*, j_!$ are fully faithful, and
   $i_*, i^!, j^*, Rj_*$ preserve injectives.

3. One has SES

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 spectural cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the SES

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow Rj_*\mathcal{F} \longrightarrow i_*i^*Rj_*\mathcal{F} \longrightarrow 0$$

which measures the difference between $j_!\mathcal{F}$ & $j_*\mathcal{F}$.

3. Shows that

$$H_c^\cdot(X) \cong H^\cdot(\bar{X}, \{\infty\}; \mathbb{Z})$$

for one pt compactification $\iota: X \hookrightarrow \bar{X}$.

Try to compute $H_c^\cdot(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X\mathcal{F}^\cdot := \underline{Hom}_{D^b(X;\mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}})$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad \mathbb{D}_X(\mathcal{F}^\cdot[n]) = (\mathbb{D}_X\mathcal{F}^\cdot)[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

$f: Y \to X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then

$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructable setting,

$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad (2)$$

for $\mathcal{F}^\cdot \in \mathcal{D}^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".

E.g. For a finite graph (as a topo space) X.

$$\mathrm{sk}_0 X \xhookrightarrow{\ i\ } X \xleftarrow{\ j\ } X - \mathrm{sk}_0 X \quad \text{1-cells}$$

$$0 \longrightarrow j_! j^! \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X-\mathrm{sk}_0X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{\mathrm{sk}_0X} \longrightarrow 0$$

Take $R\pi_{X,!}$

$$0 \longrightarrow \underbrace{H^0_c(X - \mathrm{sk}_0X)}_{=0} \longrightarrow H^0_c(X) \longrightarrow \underbrace{H^0_c(\mathrm{sk}_0X)}_{=\bigoplus_v \mathbb{Q}} \longrightarrow \underbrace{H^1_c(X - \mathrm{sk}_0X)}_{=\bigoplus_e \mathbb{Q}} \longrightarrow H^1_c(X) \longrightarrow \underbrace{H^1_c(\mathrm{sk}_0X)}_{=0} \xrightarrow{+1}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that
$$H^i_c(\mathbb{R}) = \begin{cases} \mathbb{Q} & i=1 \\ 0 & \text{otherwise} \end{cases}$$
in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial/cellular cohomology, using the following filtration.

$$\emptyset \subset \mathrm{sk}^0 X \subset \mathrm{sk}^1 X \subset \cdots \ \mathrm{sk}^n X = X$$

Ex. derive the Wang LES for the cpt supp version. over S^1

2. open cover

Ex. For an open cover $X = \mathcal{U}_1 \cap \mathcal{U}_2$, deduce the SES

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus j_! \underline{\mathbb{Q}}_{\mathcal{U}_2} \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \longleftarrow 0$$
$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \xrightarrow{+1} \qquad (3)$$

⚠ We omit the derived symbol and some subscripts in this section. $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$

(3) works for general sheaf

and. induce from (3) the MV sequence:

$$\xleftarrow{+1} H_c^k(X) \longleftarrow H_c^k(\mathcal{U}_1) \oplus H_c^k(\mathcal{U}_2) \longleftarrow H_c^k(\mathcal{U}_1 \cap \mathcal{U}_2)$$
$$H^k(X) \longrightarrow H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \longrightarrow H^k(\mathcal{U}_1 \cap \mathcal{U}_2) \xrightarrow{+1}$$

Hint: Apply $R\pi_{X,!}$ & $R\pi_{X,*}$, see [StackProject, 01E9]

Ex. Derived the Wang LES. over S^1

Ex. For an open cover $X = \bigcup_{i \in \Lambda} \mathcal{U}_i$, Λ finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{\mathcal{U}_i} \longleftarrow \bigoplus_{i<j} j_! \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longleftarrow \cdots \quad j_! \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longleftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i} \longrightarrow \bigoplus_{i<j} Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow \cdots \quad Rj_* \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longrightarrow 0$$

When $\{\mathcal{U}_i\}_{i \in \Lambda}$ is a good cover, $H^\cdot(\mathcal{U}_{i_1 \cdots i_k}) = H^0(\mathcal{U}_{i_1, \cdots i_k})$.
acyclic in AG

one can compute $H^\cdot(X)$ by applying $R\pi_{X,*}$:

$$0 \longrightarrow \bigoplus_i \Gamma(\mathcal{U}_i) \xrightarrow{d^1} \bigoplus_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j) \xrightarrow{d^2} \cdots \quad \Gamma(\bigcap_i \mathcal{U}_i) \longrightarrow 0$$
$$\Downarrow \mathrm{Ker}/\mathrm{Im}$$
$$H^0(X) \qquad H^1(X) \qquad \cdots \qquad H^{\#\Lambda - 1}(X)$$

Rmk. When X is paracompact & Hausdorff. "limited" Čech = sheaf
e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$\mathcal{F} \longrightarrow \bigoplus_i Rj_* \mathcal{F}|_{\mathcal{U}_i} \qquad \mathcal{F} \longrightarrow \bigoplus_{x \in X} \mathcal{F}_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

$\# \Lambda = 3$ case:

$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathcal{F}_0 \arrow[ur] \arrow[dr] & & & & & \mathcal{F}_2 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & \mathbb{Q}_X \arrow[rr] \arrow[urr] & & & \bigoplus_i Rj_*\mathbb{Q}_{U_i} \arrow[rr, "d^1"] \arrow[dr] & & \bigoplus_i Rj_*\mathbb{Q}_{U_i \cap U_j} \arrow[rrr, "d^2"] \arrow[urr] & & & Rj_*\mathbb{Q}_{\bigcap_i U_i} \arrow[rr, "d^3"] & & 0 \\
 & & & & & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & & & & \\
 & & & & 0 \arrow[ur] & & 0 & & & & &
\end{tikzcd}$$

$$\mathcal{F}_2 = Rj_*\mathbb{Q}_{\bigcap_i U_i} \quad \Rightarrow \quad H^{\cdot}(\mathcal{F}_2) = \operatorname{Ker} d^3$$

$$\begin{array}{ccccccc}
 & & H^1(\mathcal{F}_1) & \longrightarrow & 0 & \longrightarrow & \overset{=0}{H^1(\mathcal{F}_2)} \xrightarrow{+1} \\
0 & \longrightarrow & H^0(\mathcal{F}_1) & \longrightarrow & \bigoplus_{i<j} \Gamma(U_i \cap U_j) & \xrightarrow{d^2} & H^0(\mathcal{F}_2)
\end{array}$$

$$\Rightarrow \quad H^i(\mathcal{F}_1) = \begin{cases} \operatorname{Ker} d^3 / \operatorname{Im} d^2, & i = 1 \\ \operatorname{Ker} d^2, & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{ccccccc}
 & & H^2(\mathcal{F}_0) & \longrightarrow & 0 & \longrightarrow & \overset{=0}{H^2(\mathcal{F}_1)} \xrightarrow{+1} \\
 & & H^1(\mathcal{F}_0) & \longrightarrow & 0 & \longrightarrow & H^1(\mathcal{F}_1) \\
0 & \longrightarrow & H^0(\mathcal{F}_0) & \longrightarrow & \bigoplus_i \Gamma(U_i) & \xrightarrow{d^1} & H^0(\mathcal{F}_1)
\end{array}$$

$$\Rightarrow \quad H^i(X) = H^i(\mathcal{F}_0) = \begin{cases} \operatorname{Ker} d^3 / \operatorname{Im} d^2 & i = 2 \\ \operatorname{Ker} d^2 / \operatorname{Im} d^1 & i = 1 \\ \operatorname{Ker} d^1 & i = 0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When $\{U_i\}_{i \in \Lambda}$ is not a good cover,
one needs Čech-to-derived functor spectral seq to compute $H^{\cdot}(X)$.