

Modular form

5. moduli interpretation

- 1 level structure
2. moduli interpretation of $\Gamma \backslash \mathcal{H}$
3. cplx polarization
4. Siegel moduli space
- 5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	\mathcal{H}	$\{\pm Id\}$	$\langle S, T \mid S^4 = (ST)^6 = Id \rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}'$	$\{\pm Id\}$	$\langle S, T, (\begin{smallmatrix} 1 & \\ -1 & \end{smallmatrix}) \rangle$
$PSL_2(\mathbb{Z})$	✗	\mathcal{H}	Id	$\langle S, T \mid S^2 = (ST)^3 = Id \rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}'$	Id	$\langle S, T, (\begin{smallmatrix} 1 & \\ -1 & \end{smallmatrix}) \rangle$

can't define SL_2/\mathbb{G}_m

<https://arxiv.org/pdf/1605.07726.pdf>

<https://math.stackexchange.com/questions/1844504/why-is-this-isomorphism-of-pgl2-mathbbz-with-a-coxeter-group-injective>

See [<https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz>] for a higher dimension generalization.

Ex. $A \leq B \leq C$ gp $A \triangleleft C \Rightarrow A \triangleleft B$

no other restrictions. i.e. the following cases may happen:

$A \triangleleft B \triangleleft C$	$A \triangleleft B \leq C$	$A \triangleleft B \triangleleft C$	$A \triangleleft B \leq C$	$A \leq B \triangleleft C$	$A \leq B \leq C$
$\vdash \triangleleft \dashv$	$\vdash \triangleleft \dashv$				
✓	✓	$C_2 \triangleleft A_4 \triangleleft S_4$		✓	$S_2 \leq S_3 \leq S_4$

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$\begin{array}{ccccc}
 \Gamma(N) & \xrightarrow{\quad} & \{Id\} & & \\
 \cap & & \cap & & \\
 \Gamma_1(N) & \xrightarrow{\quad} & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} & & \\
 \cap & & \cap & & \\
 \Gamma_0(N) & \xrightarrow{\quad} & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & & \\
 \cap & & \cap & & \\
 \Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{\text{[WWL, Prop 1.4.4]}} & SL_2(\mathbb{Z}/N\mathbb{Z}) & & \\
 \cup & & \cup & & \\
 \Gamma^0(N) & \xrightarrow{\quad} & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} & & \\
 \cup & & \cup & & \\
 \Gamma'(N) & \xrightarrow{\quad} & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} & &
 \end{array}$$

∇ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$ -pt of $SL_2 = \text{Spec } \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}] / (a_{11}a_{22} - a_{12}a_{21} - 1)$,
but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) = SL_2, \mathbb{Z}/N\mathbb{Z}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/N\mathbb{Z} \right\}$$

Ex. Verify the following tables (left comes from right)

$\frac{A \triangleleft B}{A}$	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$	$\frac{A \triangleleft B}{A}$	N	B	G
$\Gamma(N)$	-	✓	✓	✓	N	-	✓	✗
$\Gamma_1(N)$	-	-	✓	✗	B	-	-	✗
$\Gamma_0(N)$	-	-	-	✗	G	-	-	-
$\Gamma(1)$	-	-	-	-				

Ex. show [WWL, 练习1.4.14]

练习 1.4.14 对所有正整数 N , 证明

$$(\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p: \text{素数} \\ p|N}} \left(1 - \frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2, \mathbb{Z}) : \Gamma_1(N)) = N^2 \prod_{\substack{p: \text{素数} \\ p|N}} \left(1 - \frac{1}{p^2}\right),$$

$$\begin{aligned} (\mathrm{SL}(2, \mathbb{Z}) : \Gamma_0(N)) &= |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1} \cdot (\mathrm{SL}(2, \mathbb{Z}) : \Gamma_1(N)) \\ &= N \prod_{p|N} \left(1 + \frac{1}{p}\right). \end{aligned}$$

A. Reduced to computation of $|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|, |\mathcal{B}(\mathbb{Z}/N\mathbb{Z})|, |\mathcal{N}(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5, 4, 6$ if you don't understand the process.

Notation: $\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z})_{\text{prim}}^{\oplus 2} / (\mathbb{Z}/N\mathbb{Z})^* \stackrel{[6.3M]}{=} \mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: <https://arxiv.org/pdf/2010.15543v2.pdf>

▽ $\mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathcal{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathcal{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different piece of $\mathcal{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|\mathrm{SL}_2(\mathbb{F}_p)| = p^3 - p$

$$|\mathcal{B}(\mathbb{F}_p)| = p^2 - p$$

$$|\mathcal{N}(\mathbb{F}_p)| = p$$

$$\# \mathbb{F}_p^\times = p-1$$

② $|\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e} - p^{3e-2}$

$$|\mathcal{B}(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e} - p^{2e-1}$$

$$|\mathcal{N}(\mathbb{Z}/p^e\mathbb{Z})| = p^e$$

$$\# (\mathbb{Z}/p^e\mathbb{Z})^\times = p^e - p^{e-1}$$

③ $|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p^2}\right)$

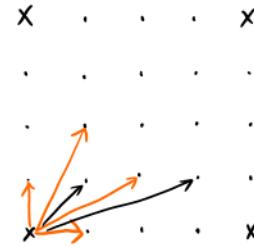
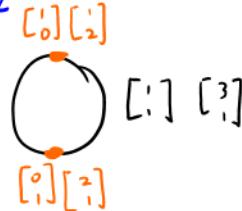
$$|\mathcal{B}(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$$

$$|\mathcal{N}(\mathbb{Z}/N\mathbb{Z})| = N$$

$$\# (\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$$

$$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^n \mathbb{P}'(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$$

E.g. $\mathbb{Z}/4\mathbb{Z}$



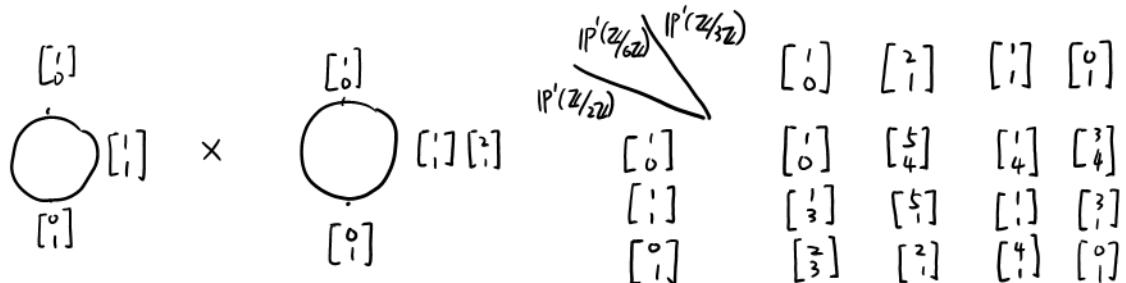
E.g. $\mathbb{Z}/6\mathbb{Z}$

$$\mathbb{P}_{\mathbb{Z}/6\mathbb{Z}} = \text{Proj } \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f \in S \\ f \text{ homogeneous}}} \text{Spec } (\mathbb{Z}/6\mathbb{Z}[x,y]_f).$$

e.g. $(x-2, y-3) \triangleleft \mathbb{Z}/6\mathbb{Z}[x,y]$ is not prime.

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) &\cong \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3) \\ &\cong \mathbb{P}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3) \end{aligned}$$

Ex. Use [Vakil, 6.3.M] to compute $\mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$. Enjoy it!



Rmk. The original proof is also good, but less geometrically obvious:

(Now you should understand the geometry in every step)

$$\begin{array}{ccccc} & 0 & & 0 & \\ & \downarrow & & \downarrow & \\ & \mathbb{S}\mathbb{L}_2(\mathbb{Z}/p^e\mathbb{Z}) & & \mathbb{S}\mathbb{L}_2(\mathbb{F}_p) & \\ & \downarrow & & \downarrow & \\ 0 \rightarrow 1 + pM_2(\mathbb{Z}/p^e\mathbb{Z}) & \xrightarrow{p^{4e-4}} & \mathbb{G}\mathbb{L}_2(\mathbb{Z}/p^e\mathbb{Z}) & \rightarrow & \mathbb{G}\mathbb{L}_2(\mathbb{F}_p) \rightarrow 0 \\ & & \downarrow & & \downarrow \\ & (\mathbb{Z}/p^e\mathbb{Z})^\times & & \mathbb{F}_p^\times & \\ & \downarrow p^{e-p^{e-1}} & & \downarrow p^{-1} & \\ & 0 & & 0 & \end{array}$$

Finally, use Chinese remainder theorem to get

$$\mathbb{S}\mathbb{L}_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^r \mathbb{S}\mathbb{L}_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$$

□

Ex. do the exactly same thing with $\mathbb{S}\mathbb{L}_2$ replaced by $\mathbb{G}\mathbb{L}_2$ and $\mathbb{P}\mathbb{G}\mathbb{L}_2$.

Ex. (hard) explore the Tits building & rep theory of $\mathbb{S}\mathbb{L}_2(\mathbb{Z}/N\mathbb{Z})$.

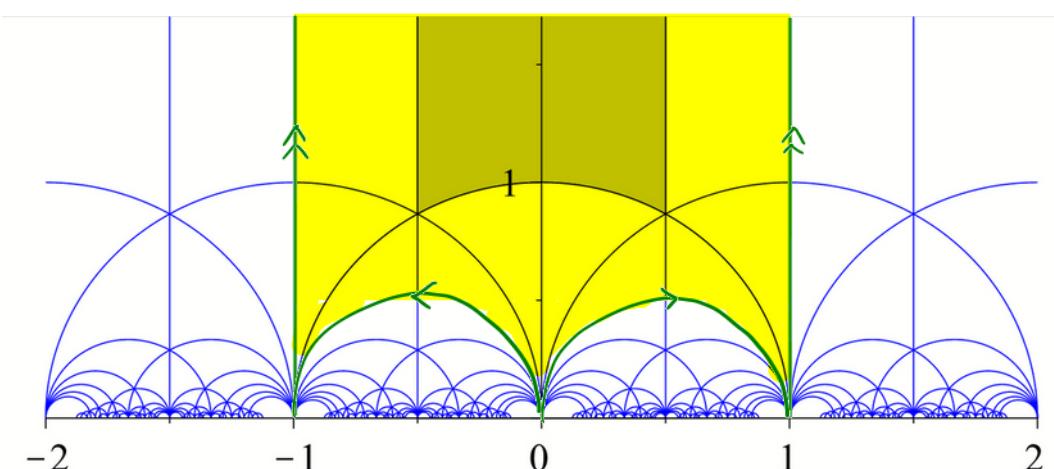
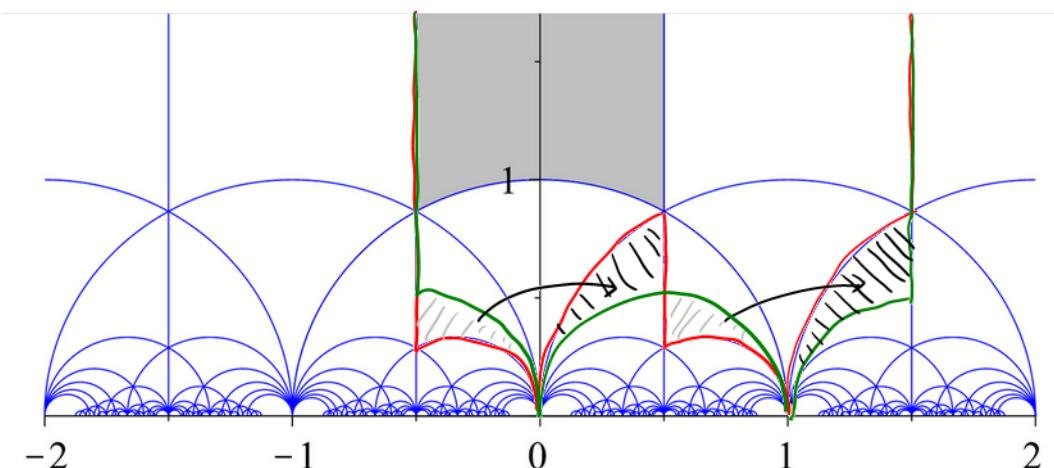
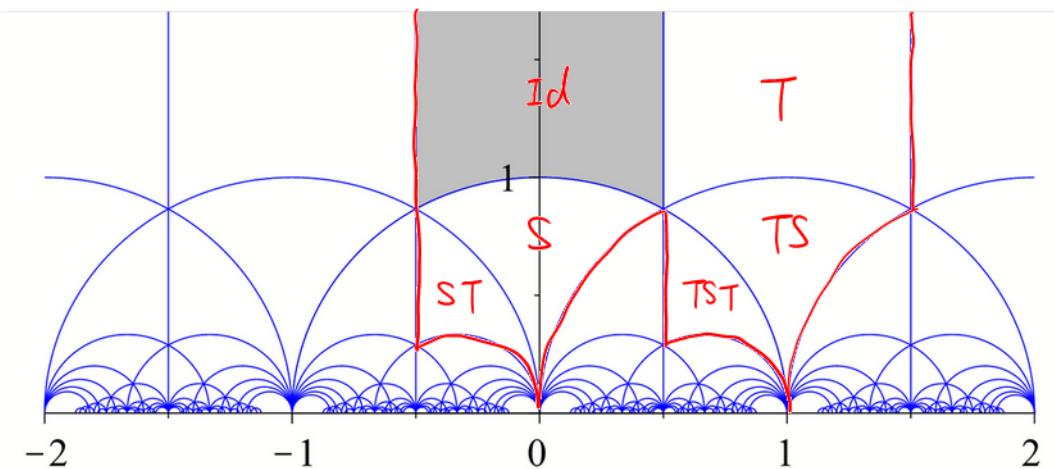
It will be used later on (I believe)

Is the Tits building of $\mathbb{S}\mathbb{L}_2(\mathbb{Z}) \rightarrow \mathbb{S}\mathbb{L}_2(\mathbb{Z}/N\mathbb{Z})$ functorial?

we write left quotient from now on, since it's a left action

Ex. Draw the fundamental domain of $\Gamma_{(2)} \backslash \mathcal{H}$.

Hint. $\Gamma(1)/\Gamma_{(2)} = \{\text{Id}, T, S, TS, ST, TST\}$

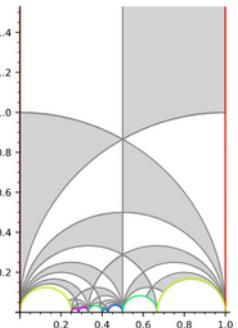
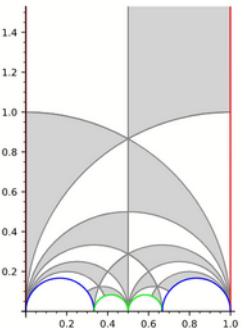
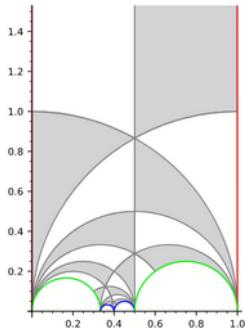
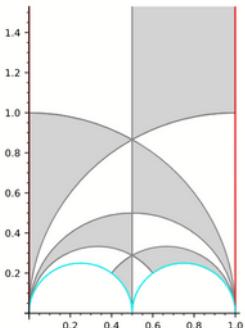
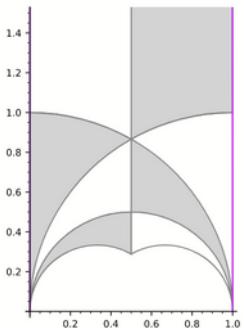
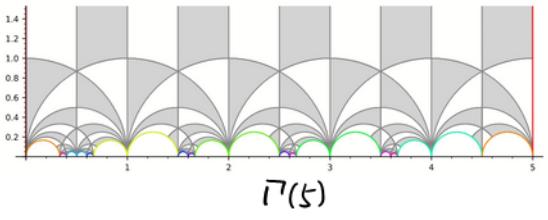
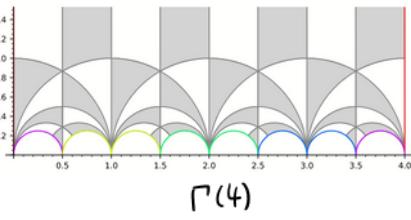
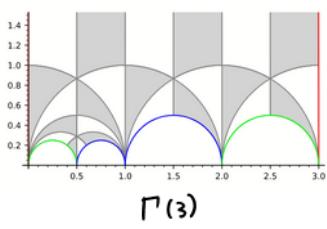
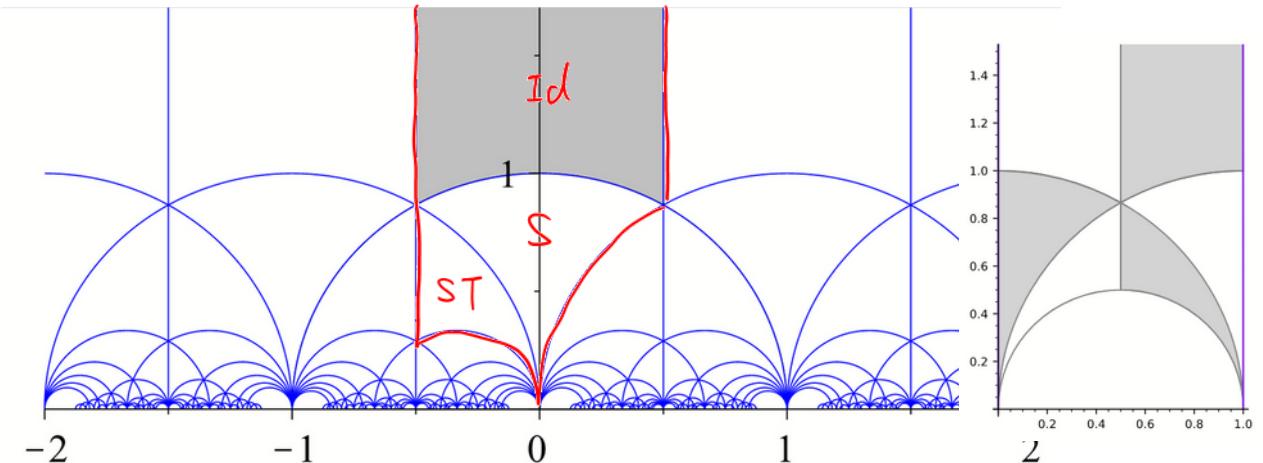


$$\text{Cor. } \Gamma_{(2)} / \{\pm \text{Id}\} = \mathbb{Z} * \mathbb{Z} = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$$

Ex. Draw the fundamental domain of $\Gamma_0(2)\backslash \mathbb{H}$. $\Gamma_0(2) = \Gamma_1(2)$

$\nabla \quad SL_2(\mathbb{Z}) \cap \Gamma_0(2)\backslash \mathbb{H}$ is not well-defined. e.g. $S_i \neq S_{(i+1)}$ in $\Gamma_0(2)\backslash \mathbb{H}$.

Hint. $\Gamma_0(2)\backslash \Gamma^{(1)} = \{\text{Id}, S, ST\}$



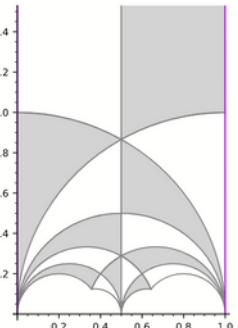
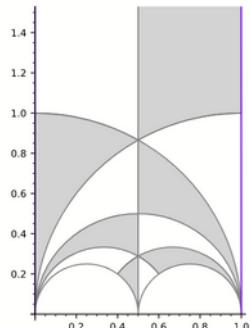
$$\Gamma_1(3) = \Gamma_0(3)$$

$$\Gamma_1(4) = \Gamma_0(4)$$

$$\Gamma_1(5)$$

$$\Gamma_1(6) = \Gamma_0(6)$$

$$\Gamma_1(7)$$



$$\Gamma_0(5)$$

$$\Gamma_0(7)$$

2. moduli interpretation of \mathbb{P}^1/\mathbb{H}

Def. A basis (v_1, v_2) of a lattice $\Lambda \subseteq \mathbb{C}$ is called **oriented** if $\text{Im } \frac{v_1}{v_2} > 0$.

Def (Weil pairing) [WWL, 注记 8.5.9, 定义 3.8.9, 练习 3.8.10]

For $N \in \mathbb{Z}_{\geq 1}$, $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$, $\text{Im } \frac{v}{u} > 0$, we define the Weil pairing e_N .

$$\begin{array}{ccc}
 E[N] \times E[N] & & \\
 \uparrow \text{is} & & \\
 a \frac{u}{N} + c \frac{v}{N} & \xrightarrow{\text{is}} & \frac{1}{N}\Lambda/\Lambda \times \frac{1}{N}\Lambda/\Lambda \\
 \downarrow & & \\
 \left(\begin{matrix} a \\ c \end{matrix} \right), \left(\begin{matrix} b \\ d \end{matrix} \right) & \xrightarrow{\text{is}} & (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \times (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \\
 & & \xrightarrow{\quad} \mu_N \cong (\mathbb{Z}/N\mathbb{Z}, +) \\
 & & \xrightarrow{\quad} \begin{matrix} \{1^a 2^b\} \\ N \end{matrix} \xrightarrow{\quad} \begin{matrix} |ab| \\ |cd| \end{matrix}
 \end{array}$$

Ex. Let $e_1, e_2 \in E[n]$.

1. e_N is antisymmetric and bilinear.

$$e_N(\gamma(e_1, e_2)) = \sum_N^{\det \gamma} e_N(e_1, e_2) \quad \forall \gamma \in GL_2(\mathbb{Z}/N\mathbb{Z})$$

e.p. e_N only depends on E and N (does not depend on Λ and u, v)

2. $e_N(e_1, e_2) \in \mu_N^{\times} \cong (\mathbb{Z}/N\mathbb{Z})^{\times} \Leftrightarrow E[N] = \langle e_1, e_2 \rangle_{\mathbb{Z}}$

$$e_N(e_1, e_2) = \sum_N \xrightarrow{\psi} 1 \Leftrightarrow \exists P, Q \in \frac{1}{N}\Lambda, \bar{P} = e_1, \bar{Q} = e_2,$$

(NP, NQ) is an oriented basis of Λ .

Def. (e_1, e_2) is called a **pretty oriented basis** of $E[N]$. if $e_N(e_1, e_2) = \sum_N$.

In [KM85], this is called Drinfeld basis.

Ex. $N=5$

$$\begin{array}{ccc}
 \begin{matrix} \times & \cdot & \cdot & \cdot & \cdot & \times \end{matrix} &
 \begin{matrix} \times & \cdot & \cdot & \cdot & \cdot & \times \end{matrix} &
 \begin{matrix} \times & \cdot & \cdot & \cdot & \cdot & \times \end{matrix} \\
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} &
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} &
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \\
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} &
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} &
 \begin{matrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{matrix} \\
 \begin{matrix} P \\ \uparrow \\ Q \end{matrix} &
 \begin{matrix} Q \\ \uparrow \\ P \end{matrix} &
 \begin{matrix} P \\ \uparrow \\ Q \end{matrix}
 \end{array}$$

$$e_N(\bar{P}, \bar{Q}) = \sum_5^2$$

$(5P, 5Q)$ is not a basis of Λ .

$$e_N(\bar{P}, \bar{Q}) = \sum_5^4 = \sum_5^{-1}$$

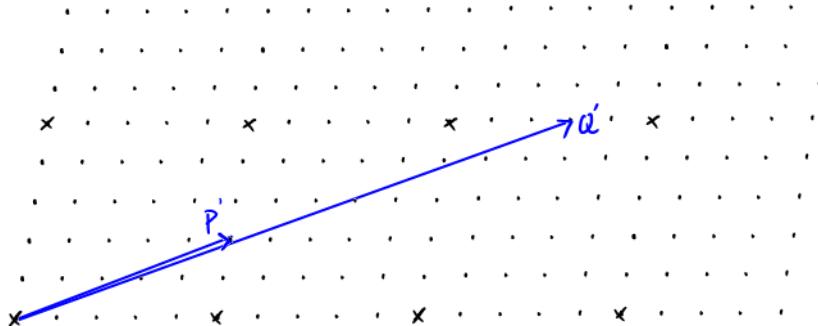
$(5P, 5Q)$ is a basis of Λ , but not an oriented basis.

$$e_N(\bar{P}, \bar{Q}) = \sum_5^6 = \sum_5$$

$(5P, 5Q)$ is not a basis of Λ but $(5P', 5Q')$ is an oriented basis.

$$\begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix} \stackrel{\text{mod } 5}{=} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

\cap
 $SL_2(\mathbb{Z})$



Recall: For $E = C/\Lambda$, $E[N] \cong \frac{1}{N}\Delta/\Lambda \cong \Delta/N\Delta$

Main Thm. We have the following moduli interpolations (E : any cplx EC curve)

$$\begin{array}{ccc}
 \left\{ (E, \alpha) \mid \begin{array}{l} \alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[N] \\ e_N(\alpha(1,0), \alpha(0,1)) = \delta_N \end{array} \right\} / \sim & \xrightarrow{\sim} & \Gamma(N) \backslash \mathcal{H} \\
 \downarrow \text{E with a pretty oriented basis } (e_1, e_2) & & \downarrow N \\
 \left\{ (E, \beta) \mid \begin{array}{l} \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N] \\ e_N(\beta(1,0), \beta(0,1)) = \delta_N \end{array} \right\} / \sim & \xrightarrow{\sim} & \Gamma(N) \backslash \mathcal{H} \\
 \downarrow \beta = \alpha(0, -) & & \downarrow \begin{cases} \frac{N}{2} \prod_{p|N} (1 - \frac{1}{p}) & N \neq 2 \\ 1 & N=2 \end{cases} \\
 \left\{ (E, F) \mid \begin{array}{l} F, 0 \subseteq C \subseteq E[N] \\ e_N(F, 0) = \delta_N \end{array} \right\} / \sim & \xrightarrow{\sim} & \Gamma_0(N) \backslash \mathcal{H} \\
 \downarrow \text{E with } C \subseteq E[N], C \cong \mathbb{Z}/N\mathbb{Z} & & \downarrow N \prod_{p|N} (1 + \frac{1}{p}) \\
 \left\{ \text{cplx EC } E \right\} / \sim & \xrightarrow{\sim} & \Gamma(1) \backslash \mathcal{H}
 \end{array}$$

Idea. A pretty oriented basis on $E[N]$ gives us a oriented basis on E up to $\Gamma(N)$ -action;
coefficient has to be 1, so that $(v_1 + bv_2, v_2)$ is a pretty oriented basis.

$$\begin{aligned}
 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} v_1 + bv_2 \\ v_2 \end{pmatrix} \Rightarrow \text{an } n\text{-torsion pt } v_2 \\
 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} av_1 + bv_2 \\ dv_2 \end{pmatrix} \Rightarrow \text{a flag } 0 \subseteq \begin{matrix} C \\ \subset \\ \subset \\ \subset \end{matrix} \subseteq E[N]
 \end{aligned}$$

Proof. For $\Gamma(N) \backslash \mathcal{H}$,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Lambda, \alpha) \mid \begin{array}{l} \alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \longrightarrow \frac{1}{N}\Delta/\Lambda \\ e_N(\alpha(1,0), \alpha(0,1)) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \\
 &\cong \left\{ (\Lambda, e_1, e_2) \mid \begin{array}{l} e_1, e_2 \in \frac{1}{N}\Delta/\Lambda \\ e_N(e_1, e_2) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \\
 &\cong \left\{ (\Lambda, z_1, z_2, e_1, e_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_1 = \frac{z_1}{N}, e_2 = \frac{z_2}{N} \\ e_N(e_1, e_2) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \left\{ (\Lambda, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ (z_1, z_2) \text{ is an oriented basis of } \Delta \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \left\{ (z_1, z_2) \in (\mathbb{C} - \{0\})^2 \mid \operatorname{Im} \frac{z_1}{z_2} > 0 \right\} \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \Gamma(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

For $\Gamma_1(N) \backslash \mathcal{H}$,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Delta, \beta) \mid \beta : \mathbb{Z}/N\mathbb{Z} \hookrightarrow \frac{1}{N}\Delta/\Delta \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, e_1) \mid e_1 \in \frac{1}{N}\Delta/\Delta, \text{order}(e_1) = N \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, z_1, z_2, e_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_2 = \frac{z_2}{N} \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \xrightarrow{\text{If not set } (z_1, z_2) \mapsto (-z_1, z_2)} \Gamma_1(N) \\
 &\cong \left\{ (\Delta, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_1(N) \\
 &\cong \Gamma_1(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

For $\Gamma_0(N) \backslash \mathcal{H}$,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Delta, C) \mid C \subseteq \frac{1}{N}\Delta/\Delta, C \cong \mathbb{Z}/N\mathbb{Z} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, z_1, z_2, C) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ C = \left(\frac{z_1}{N}\right) \subseteq \frac{1}{N}\Delta/\Delta \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_0(N) \\
 &\cong \left\{ (\Delta, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_0(N) \\
 &\cong \Gamma_0(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

□

Rmk. If you observe carefully, you will find out that what we prove actually is

$$\mathbb{C}^\times = SO_2(\mathbb{R}) \times \mathbb{R}_{>0}$$

$$\begin{array}{ccc}
 \{ (E, \alpha) \} & \xrightarrow{\sim} & \Gamma(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ (E, \beta) \} & \xrightarrow{\sim} & \Gamma_1^\pm(N) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_1(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_1(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ (E, F) \} & \xrightarrow{\sim} & \Gamma_0^\pm(N) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_0(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_0(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ E \} & \xrightarrow{\sim} & GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong SL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong SL_2(\mathbb{Z}) \backslash \mathcal{H}
 \end{array}$$

where

$$\Gamma_1^\pm(N) \longrightarrow \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}$$

$$\Gamma_0^\pm(N) \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$GL_2(\mathbb{Z}) \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$GL_2(\mathbb{Z}/N\mathbb{Z})_{\det=\pm 1} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$$