

## Eine Woche, ein Beispiel

### 2.16 lines passing a point

Ref:

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

These are perhaps too well-known. But I should record it.

Typical question:

In a hypersurface  $X \subset \mathbb{P}^n$ ,  
how many lines  $l \cong \mathbb{P}^1$  pass a given point  $p \in X$ ?

Affine version:

In a (conical) hypersurface  $X \subset \mathbb{C}^{n+1}$ ,  
how many planes  $l \cong \mathbb{C}^2$  contain a given line  $p \cong \mathbb{C} \subseteq X$ ?

1. Method
2. Lines on cubic threefold
3. Lines on quadrics
4. Lines passing through lines

# 1. Method

Slogan: write down the coordinates explicitly.

w.l.o.g. let  $p = [1:0:\dots:0]$  and  $X = \{f=0\}$ , where

$$f(z_0, \dots, z_n) = \sum_{i=0}^d g_{d-i}(z_1, \dots, z_n) z_0^i$$

$g_{d-i}(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$  are homo of degree  $d-i$ ,  
and  $g_0(z_1, \dots, z_n) = 0$ .

Suppose that  $l = \langle (1, 0, \dots, 0), (0, x_1, \dots, x_n) \rangle_{\mathbb{C}\text{-v.s.}}$ , then

$$l \subseteq X$$

$$\Leftrightarrow f(t, x_1, \dots, x_n) \equiv 0 \quad \forall t \in \mathbb{C}$$

$$\Leftrightarrow g_i(x_1, \dots, x_n) \equiv 0 \quad \forall i \in \{1, \dots, d\}$$

Therefore,

$$\begin{aligned} & \{ l \cong \mathbb{C}^2 \subseteq \mathbb{C}^{n+1} \mid p \in l \subseteq X \} \\ & \cong \{ [x_1: \dots: x_n] \in \mathbb{C}P^{n-1} \mid g_{d-i}(x_1, \dots, x_n) = 0 \quad \forall i \} \\ & \cong \{ [x_1: \dots: x_n] \in \mathbb{C}P^{n-1} \mid \frac{\partial f}{\partial z_0^i}(0, x_1, \dots, x_n) = 0 \quad \forall i \} \end{aligned}$$

Here,  $\mathbb{C}P^{n-1} = Gr(p^\perp, 1)$ .

When  $X$  is sm at  $p$ ,  $(\nabla f)(p) \neq 0$ .

w.l.o.g. let  $(\nabla f)(p) = (0, \dots, 1)$ , then

$$\begin{cases} T_p X = \{z_n = 0\} \cong \mathbb{C}^n \\ g_i(z_1, \dots, z_n) = z_n \end{cases}$$

In ptc,  $p \in l \subseteq X \Rightarrow l \subseteq T_p X$ .

## 2. Lines on cubic threefold

<https://math.stackexchange.com/questions/3605767/number-of-lines-passing-a-point-on-a-cubic-threefold>

Prop. Generically, there are 6 lines in a cubic threefold passing a given pt.

Proof. w.l.o.g. suppose  $p = [1:0:0:0:0]$ ,  $T_p X = \{z_4 = 0\}$ , then

$$\{l \mid p \in l \subseteq X\}$$

$$\cong \{[x_1 : x_2 : x_3 : x_4] \in \mathbb{CP}^3 \mid x_4 = g_2(x_1, x_2, x_3, x_4) = g_3(x_1, x_2, x_3, x_4) = 0\}$$

$$\cong \{[x_1 : x_2 : x_3] \in \mathbb{CP}^2 \mid g_2(x_1, x_2, x_3, 0) = g_3(x_1, x_2, x_3, 0) = 0\}$$

has generically 6 pts. (will we get  $g_2, g_3$  all the time for some specific cubic threefold?)

Rmk. Generically, passing a given pt,  
there are 24 lines in a quartic fourfold,  
5! lines in a quintic fivefold,

⋮

$n!$  lines in a degree  $n$   $n$ -fold.

$\begin{matrix} \text{dim} & \text{dim } d \\ n-1 & \end{matrix}$	1	2	3	4	5	6	...
0	.	..	...	...	...	...	...
1	$\mathbb{P}^1$	twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g = \frac{d(d-1)}{2}$
2	$\mathbb{P}^2$	conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	cubic surface	K3 surface			
3	$\mathbb{P}^3$	conical	cubic threefold				
4	$\mathbb{P}^4$	conical	cubic fourfold				
5	$\mathbb{P}^5$	⋮					

general type  
↑

Fano ← Calabi-Yau

uniruled by  $\mathbb{P}^1$  | uniruled by conics

### 3. Lines on quadrics.

In this case,

$$\begin{aligned} & \{l \mid p \in l \subseteq X\} \\ & \cong \{[x_1 : \dots : x_n] \in \mathbb{CP}^{n-1} \mid x_n = g_2(x_1, \dots, x_n) = 0\} \\ & \cong \{[x_1 : \dots : x_{n-1}] \in \mathbb{CP}^{n-2} \mid g_2(x_1, \dots, 0) = 0\} \end{aligned}$$

is again a quadric of dim  $n-3$ . (generically)  $n \geq 3$

$n=1,2$   $\emptyset$  (generically) empty

$$F_1(X) = \{l \subseteq X \text{ lines}\}$$

Cor. For  $n \geq 3$ ,

$$\begin{aligned} \dim F_1(X) &= n-3 + n-1 - 1 = 2n-5 \\ &= 2(n-1) - 3 \end{aligned}$$

$\begin{smallmatrix} \text{dim } d \\ \text{dim } n-1 \end{smallmatrix}$	1	2	3	4	5	6	...
0	.	..	...	...	...	...	...
1	<sup>0</sup> $\mathbb{P}^1$	<sup>1</sup> $\text{twistor } \mathbb{P}^1$	<sup>0</sup> $g=1$	$g=6$	$g=10$	$g=15$	$g = \frac{d(d-1)}{2}$
2	<sup>2</sup> $\mathbb{P}^2$	<sup>1</sup> $\text{conical } \cong \mathbb{P}^1 \times \mathbb{P}^1$	<sup>0</sup> $\text{cubic surface}$				
3	<sup>4</sup> $\mathbb{P}^3$	<sup>3</sup> $\text{conical}$	<sup>2</sup> $\text{cubic threefold}$				
4	<sup>6</sup> $\mathbb{P}^4$	<sup>5</sup> $\text{conical}$	<sup>4</sup> $\text{cubic fourfold}$	<sup>3</sup>			
5	<sup>8</sup> $\mathbb{P}^5$	<sup>7</sup> $\vdots$	<sup>6</sup>	<sup>5</sup>	<sup>4</sup>		

general type  $\uparrow$

Fano  $\leftarrow$  Calabi-Yau

uniruled by  $\mathbb{P}^1$  | uniruled by conics

$\dim_{\mathbb{C}} F_1(X)$

In general, one can compute  $r$ -planes ( $\cong \mathbb{P}^r$ ) on  $X$  passing  $P$ .

$$\begin{aligned}
 & \{e \cong \mathbb{C}^{r+1} \mid p \in e \subseteq X\} \\
 & \cong \{e \in \text{Gr}(n, r) \mid x_n = g_2(x_1, \dots, x_n) = 0 \quad \forall (x_1, \dots, x_n) \in e\} \\
 & \cong \{e \in \text{Gr}(n-1, r) \mid g_2(x_1, \dots, x_{n-1}, 0) = 0 \quad \forall (x_1, \dots, x_{n-1}) \in e\} \\
 & \cong F_{r-1}(X') \quad \dim X' = \dim X - 2
 \end{aligned}$$

$\Rightarrow$  when  $F_{r-1}(X') \neq \emptyset$  generically,

$$\begin{aligned}
 \dim F_r(X) &= \dim F_{r-1}(X') + \dim^{\text{proj}} X - r \\
 &= \dim F_{r-1}(X') + (n-1) - r \\
 &= \dim F_{r-1}(X') + n - r - 1 \\
 &= n - r - 1 + ((n-2) - (r-1) - 1) + \dots + ((n-2(r-1)) - (r-(r-1)) - 1) \\
 &\quad + \dim F_0(X^{(r)}) \\
 &= n - r - 1 + (n - r - 2) + \dots + (n - 2r) \\
 &\quad + \dim^{\text{proj}} X^{(r)} \\
 &= \frac{1}{2} (2n - 3r - 1) r + n - 2r - 1 \\
 &= \frac{1}{2} (2n - 3r - 2)(r + 1)
 \end{aligned}$$

$\begin{smallmatrix} \text{dim } d \\ \text{dim } n-1 \end{smallmatrix}$	1	2	3	4	5	6	...
0	.	..	...	...	...	...	...
1	$\emptyset$ $\mathbb{P}^1$	$\emptyset$ twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g = \frac{d(d-1)}{2}$
2	$^0$ $\mathbb{P}^2$	$\emptyset$ conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	$\emptyset$ cubic surface				
3	$^3$ $\mathbb{P}^3$	$\emptyset$ conical	$\emptyset$ cubic threefold				
4	$^6$ $\mathbb{P}^4$	$^3$ conical	cubic fourfold	$\emptyset$			
5	$^9$ $\mathbb{P}^5$	$^6$ :			$\emptyset$		

general type  $\uparrow$

$3(n-1) - 3$       uniruled by  $\mathbb{P}^1$       uniruled by conics      Fano  $\Leftarrow$  Calabi-Yau

$\dim_{\mathbb{C}} F_2(X)$

#### 4. Lines passing through lines

Typical question:

In a hypersurface  $X \subset \mathbb{P}^n$ ,  
how many lines  $l \cong \mathbb{P}^1$  pass a given line  $l_0 \in X$ ?

Affine version:

In a (conical) hypersurface  $X \subset \mathbb{C}^{n+1}$ ,  
how many planes  $l \cong \mathbb{C}^2$  intersect a given plane  $l_0 \cong \mathbb{C} \subseteq X$  non-trivially?

w.l.o.g. let  $l_0 = [* : * : 0 : \dots : 0]$  and  $X = \{f=0\}$ , where

$$f(z_0, \dots, z_n) = \sum_{\substack{i,j \\ i+j \leq d}} a_{ij}(z_2, \dots, z_n) z_0^i z_1^j$$

and  $a_{ij}(z_2, \dots, z_n) \in \mathbb{C}[z_2, \dots, z_n]$  are homo of degree  $d-i-j$

$$l_0 \subseteq X \iff a_{ij}(z_2, \dots, z_n) = 0 \text{ when } d = i+j$$

Therefore,

$$f(z_0, \dots, z_n) = \sum_{\substack{i,j \\ i+j < d}} a_{ij}(z_2, \dots, z_n) z_0^i z_1^j.$$

We want to restrict  $f$  to a plane  $e = \mathbb{P}^2$  containing  $l$ .  
Suppose  $e = \{z_i = k_i w \mid i=2, \dots, n\}$  for some  $k_i \in \mathbb{C}$ , then

$$\begin{aligned} f|_e &= \sum_{\substack{i,j \\ i+j < d}} a_{ij}(k_2 w, \dots, k_n w) z_0^i z_1^j \\ &= \sum_{\substack{i,j \\ i+j < d}} a_{ij}(k_2, \dots, k_n) z_0^i z_1^j w^{d-i-j} \end{aligned}$$

$$= \omega \left( \sum_{\substack{i,j,k \\ i+j+k=d-1}} a_{ij}(k_2, \dots, k_n) z_0^i z_1^j \omega^k \right)$$

That means,  $X \cap e = l \cup C$   
for some curve  $C$  of degree  $d-1$ .

When  $d=3$ ,  $C$  is a conic, and

$$\sum_{\substack{i,j,k \\ i+j+k=d-1}} a_{ij}(k_2, \dots, k_n) z_0^i z_1^j \omega^k = (z_0, z_1, \omega) \begin{pmatrix} a_{20} & \frac{a_{11}}{2} & \frac{a_{10}}{2} \\ \frac{a_{11}}{2} & a_{02} & \frac{a_{01}}{2} \\ \frac{a_{10}}{2} & \frac{a_{01}}{2} & a_{00} \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \\ \omega \end{pmatrix}$$

$\overset{A}{\parallel}$

Moreover,

$$\begin{array}{ll} C \text{ is singular} & \Leftrightarrow \det A = 0 \\ C \text{ splits as two distinct lines} & \Leftrightarrow \text{rk } A = 2 \\ C \text{ splits as two identity lines} & \Leftrightarrow \text{rk } A = 1. \end{array}$$

Notice that  $\det A \in \mathbb{C}[k_2, \dots, k_n]$  is a homo poly of degree 5, so gives a hyperplane in  $\mathbb{CP}^{n-2}$  with degree 5.

Rmk.

For a smooth cubic threefold, would we find a plane, such that the intersection is a union of line and two identified line? No. Since the cubic threefold is smooth, all its plane sections must be reduced curves.

Therefore,

$$\{l \subset X \mid l \cap l_0 = p \text{ for some } p\} \subseteq F \subseteq \text{Gr}(n+1, 2) = C_{l_0}$$

$\swarrow$   $\begin{smallmatrix} 2:1 \\ \text{unramified?} \end{smallmatrix}$

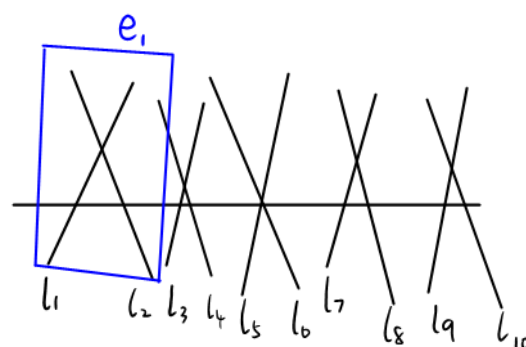
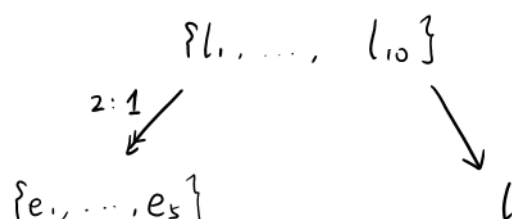
$\{\det A = 0\} \subseteq \mathbb{CP}^{n-2} \subseteq \text{Gr}(n+1, 3)$   
containing  $l$

$\searrow$

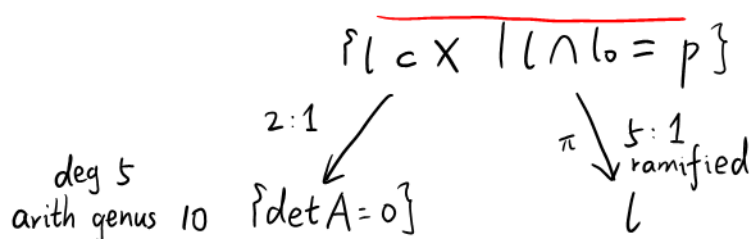
$l \subseteq \text{Gr}(n+1, 1)$

$\begin{array}{c} l \\ \swarrow \quad \searrow \\ l+l_0 \quad l \cap l_0 \end{array}$

E.g.  $n=3$  cubic surface case



E.g.  $n=4$  cubic threefold case



arith genus 19

Q: Let  $f: X \rightarrow Y$  be a map of algebraic curves (maybe singular), where  $f$  is an unramified covering of degree 2. The arithmetic genus of  $Y$  is 10, so what is the range of the arithmetic genus of  $X$ ?

A: (guess) still 19. A geometrical intuition may confirm the RH formula still applies.

A by chatgpt: The arithmetic genus of  $X$  can be any integer between 9 and 19 (in particular, if  $Y$  is smooth then  $p_a(X) = 19$ , while for a maximally nodal  $Y$  one may have  $p_a(X) = 9$ ).

However, when  $Y$  is allowed to be singular the situation is a bit subtler. (Here we assume  $Y$  is an irreducible projective curve with only nodal singularities, so that  $p_a(Y) = g(\tilde{N}) + \delta$  where  $\delta$  is the number of nodes and  $\tilde{N}$  is the normalization.) An "admissible" unramified double cover  $f: X \rightarrow Y$  will then lift to an étale double cover of the normalizations (so that the geometric genus of  $X$  is  $2g(\tilde{N}) - 1$ ). Moreover, over a node of  $Y$  there are two possibilities. In local terms one may have either "non-splitting" – so that a node of  $Y$  gives rise to a single node in  $X$  – or "splitting" – so that the pre-image consists of two nodes. Writing  $r$  for the number of nodes of  $Y$  that split (with  $0 \leq r \leq \delta$ ) one can show that

$$p_a(X) = 2p_a(Y) - 1 - \delta + r.$$

Since for an (irreducible) nodal curve one always has  $\delta \leq 10$  and  $r$  can vary between 0 and  $\delta$ , the highest value is obtained when  $\delta = 0$  (or when every node splits, so  $r = \delta$ ) giving  $p_a(X) = 19$ , while the lowest occurs when  $Y$  is as singular as possible (e.g.  $\delta = 10$ ) with no splitting ( $r = 0$ ), so that  $p_a(X) = 19 - 10 = 9$ .

When  $\{\det A = 0\}$  is sm,  $\pi$  is ramified at 46 pts (counted with multiplicity)

Q: Is  $C_{l_0}$  connected?

∇ In [Huy23, Ex 1.16], it is claimed that  $g(C_{l_0}) = 11$ . Therefore, we must have computed something wrong.