Eine Woche, ein Beispiel
4.17 preliminary facts of representions of p-adic groups

Main reference: The Local Langlands Conjecture for GL(2) by Colin J. Bushnell and Guy Henniart.
[https://link.springer.com/book/10.1007/3-540-31511-X]

http://www.math.columbia.edu/~phlee/CourseNotes/p-adicGroups.pdf

Process.

1. Basic properties
   - Smoothness
   - Irreducibility and unitary
   - Reduction to smaller cardinal.
2. Examples: $\mathcal{O}$, $\mathcal{O}^\times$, F, $F^\times$
3. Construction of new reps.
   - Special sub & quotient rep
   - Duality
   - Ind and c-Ind
   - Other constructions ← Example: mirabolic group
4. Hecke algebra
5. Intertwining properties. ← Example: $GL_2(\mathbb{Q}_p)$

1. Basic properties

1.1. Smoothness

G: loc. profinite group

V: cplx v.s.

$\rho: G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V) \qquad g \longmapsto [v \longmapsto g.v]$

Def. (ρ, V) is smooth if

$\forall\, v \in V,\ \exists\, K \leqslant G$ cpt open s.t. $k.v = v \quad \forall k \in K$

$\mathrm{Rep}(G) = \{\text{sm rep of } G\}$ is a full subcategory of $\{\text{rep of } G\}$.

Rmk. Any sub/quotient rep of $(\rho, V) \in \mathrm{Rep}(G)$ is smooth.

$H \leqslant G$ cpt, $(\rho, V) \in \mathrm{Rep}(G) \Rightarrow (\rho|_H, V) \in \mathrm{Rep}(H)$

Rmk. For fcts. smoothness has a different meaning.

Recall the definition of $C^\infty(G)$ & $C_c^\infty(G)$:

$C^\infty(G) := \{f: G \to \mathbb{C} \mid f \text{ is loc. const}\}$

$C_c^\infty(G) := \{f \in C^\infty(G) \mid \operatorname{supp} f \subset G \text{ is cpt}\}$

1.2. Irreducibility and unitary

$$\mathrm{Irr}(G) = \{(\rho, V) \in \mathrm{Rep}(G) \mid \rho \text{ is a irreducible rep}\}$$

$$\widehat{G}^* = \{(\rho, V) \in \mathrm{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$$

$$\overset{[p13]}{=\!=} \{\chi : G \to \mathbb{C}^\times \mid \ker \chi \text{ is open}\}$$

$$\overset{[(1.6)]}{=\!=} \{\chi : G \to \mathbb{C}^\times \mid \chi \text{ is continuous}\}$$

Rmk. The notation is slightly different with the original reference.

Rmk.

$$\widehat{G}^* \subseteq \mathrm{Irr}(G) \subseteq \mathrm{Ind}(G) \subseteq \mathrm{Rep}(G)$$

we add a star to avoid confusing with profinite completions

Indecomposable, not induced or induction

(remind me if I miss it somewhere!)

When [p15] G is cpt, or

[p22] $G/Z(G)$ is cpt with G/K countable, we get $\mathrm{Ind}(G) = \mathrm{Irr}(G)$;

when [p21] G is abelian and G/K is countable, $\mathrm{Ind}(G) = \widehat{G}^*$.

($\exists K \leq G$ cpt open, countable = at most countable here)

Rmk. A more general result is as follows.

Prop. Let $(\rho, V) \in \mathrm{Rep}(G)$, G/K countable. $\exists K \leq G$ cpt open

Suppose $\rho|_{Z(G)}$ decompose as $Z(G) \xrightarrow{\chi_\omega} \mathbb{C}^\times \xrightarrow{\text{scaler}} \mathrm{Aut}_{\mathbb{C}}(V)$.

Let $Z(G) \leq K \leq G$, $K \leq G$ open, $K/Z(G)$ is cpt.

Then $(\rho|_K, V) \in \mathrm{Rep}(K)$ is semisimple.

[We will rewrite it as follows.

Prop. Let G be a loc. cpt gp satisfying the countable Hypothesis. $Z = Z(G)$.

For $(\rho, V) \in \mathrm{Rep}(G) \overset{Z}{\Rightarrow} \chi_\omega$, $K \in \mathrm{Cos}_Z(G)$,

$(\rho|_K, V) \in \mathrm{Rep}(K)$ is semisimple.]

To prove this we need the following lemma. (when applied, it would be $K_0 Z(G) \leq K$)

Lemma. Let $H \leq G$ open, $[G:H] < +\infty$, $(\rho, V) \in \mathrm{Rep}(G)$. Then

ρ is G-semisimple $\Leftrightarrow$ $\rho|_H$ is H-semisimple.

Def (multiple of χ) $\rho \overset{H}{\rightrightarrows} \chi$

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $\chi \in \widehat{H}^*$.

We say H acts on V as χ,

or $\rho|_H$ is a multiple of χ,

or (when $H = Z(G)$) ρ admits the central character χ,

if $\rho|_H$ decompose as follows:

$$\rho|_H : H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \mathrm{Aut}_{\mathbb{C}}(V)$$

We may denote χ by χ_ρ or χ_H. When $H = Z(G)$, χ is denoted by χ_ω or ω_ρ

Def (Contain irr rep) $\rho \overset{n}{\supseteq} \sigma$ $n = \mathrm{mult}(\rho, \sigma)$

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $(\sigma, W) \in \mathrm{Irr}(H)$.

We say ρ contains σ, or σ occurs in ρ, if

$$\mathrm{Hom}_H(\mathrm{Res}_H^G \rho, \sigma) \neq 0$$

i.e., σ can be realized as a quotient subrep of $\mathrm{Res}_H^G \rho$.

The multiplicity is defined as

$$\mathrm{mult}(\rho, \sigma) := \dim_{\mathbb{C}} \mathrm{Hom}_H(\mathrm{Res}_H^G \rho, \sigma)$$

Cor. When H acts on V as χ_ρ, ρ contains χ_ρ.

Def (Inflation) $\lambda \overset{G}{\nearrow} \Lambda$

Let $N \triangleleft G$. The inflation of $(\lambda, W) \in \mathrm{Rep}(G/N)$ is defined as $(\Lambda, W) \in \mathrm{Rep}(G)$, where

$$\Lambda : G \xrightarrow{\pi} G/N \xrightarrow{\lambda} \mathrm{End}_{\mathbb{C}}(W)$$

Def (Twist) $\chi\rho$

Suppose $G \leq GL_n(F)$, where F is a non-archi local field.

For $\chi \in \widehat{F^\times}^*$ and $(\rho, V) \in \mathrm{Rep}(G)$, we define

$$\chi\rho := (\chi \circ \det) \otimes_{\mathbb{C}} \rho : G \longrightarrow \mathrm{End}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}} V) = \mathrm{End}_{\mathbb{C}}(V)$$

$$\chi\rho(g).v = \chi(\det g) \cdot \rho(g).v \qquad \forall g \in G \quad v \in V$$

as the twist of ρ by χ.

Def (Unitary operator) V: Hilbert space.

$U \in \mathrm{Aut}_{\mathbb{C}}(V)$ is called an unitary operator if

$$\langle Uv, Uw \rangle = \langle v, w \rangle \qquad \forall v, w \in V$$

Def (Unitary rep) V: Hilbert space.

$(\rho, V) \in \mathrm{Rep}(G)$ is unitary if $\rho(g)$ is an unitary operator $(\forall g \in G)$.

E.g. $\chi \in \widehat{G}^*$ is unitary if $\mathrm{Im}\,\chi \subseteq S^1$

Rmk. When $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$, any $\chi \in \widehat{G}^*$ is unitary.

1.3. Reduction to smaller cardinal

Admissibility

Def. $(\rho, V) \in \mathrm{Rep}(G)$ is admissible if $\dim_{\mathbb{C}} V^K < +\infty$ for $\forall K \leq G$ cpt open.

Rmk. Any sub/quotient rep of $(\rho, V) \in \mathrm{Rep}(G)$ admissible is admissible.

$H \leq G$ cpt, $(\rho, V) \in \mathrm{Rep}(G)$ admissible

$$\Rightarrow (\rho|_H, V) \in \mathrm{Rep}(H) \text{ is admissible.}$$

Countable hypothesis

$\exists/\forall$ $K \leq G$ cpt open, G/K is countable.

Assuming countable hypothesis, we get

$$(\rho, V) \in \mathrm{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \mathrm{End}_G(V) = \mathbb{C} \\ \rho \text{ acts on } V \text{ as a character } \omega_\rho \end{cases}$$

$$\xRightarrow{G \text{ is abelian}} \dim_{\mathbb{C}} V = 1.$$

Finite dimension

K: cpt (profinite) gp, $(\sigma, W) \in \mathrm{Irr}(K) \rightsquigarrow \dim_{\mathbb{C}} W < +\infty$

Assuming countable hypothesis of $Z(G)$. If $K \leq G$ open,

$K \geq Z(G)$, $K/Z(G)$ is cpt, $(\sigma, W) \in \mathrm{Irr}(K) \rightsquigarrow \dim_{\mathbb{C}} W < +\infty$.

2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

Rep of $G = \mathcal{O}, \mathcal{O}^\times, F, F^\times$, where F is a non-archi local field.

In these cases, G is abelian and satisfy the countable hypothesis, so $\mathrm{Ind}(G) = \widehat{G}^*$, i.e., everything reduced to the classification of characters.

E.g. $G = (\mathcal{O}, +)$

$\forall \psi \in \widehat{\mathcal{O}}^*$ is trivial on $\mathfrak{p}^k$. Suppose $\psi \neq 1$.

$$\mathrm{level}(\psi) := \min \{ d \geq 0 \mid \mathfrak{p}^d \subset \ker \psi \}$$

When $\mathrm{level}(\psi) = d$, $\psi : \mathcal{O} \xrightarrow{\pi} \mathcal{O}/\mathfrak{p}^d \longrightarrow \mathbb{C}^\times$ factors through char of $\mathcal{O}/\mathfrak{p}^d$.

E.g. $G = \mathcal{O}^\times$

$\forall \chi \in \widehat{\mathcal{O}^\times}^*$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi \}$$

When $\mathrm{level}(\chi) = d$, $\chi : \mathcal{O}^\times \xrightarrow{\pi} \mathcal{O}^\times/\mathcal{U}^{(d+1)} \longrightarrow \mathbb{C}^\times$ factors through char of $(\mathcal{O}/\mathfrak{p}^{d+1})^\times$

$\mathcal{O}^\times/\mathcal{U}^{(d+1)} \cong (\mathcal{O}/\mathfrak{p}^{d+1})^\times$

Recent advances: Geometrization of continuous characters of Zp [https://msp.org/pjm/2013/261-1/pjm-v261-n1-p05-p.pdf]

E.g. $G = (F, +)$

$\forall \psi \in \widehat{F}^*$ is trivial on $\mathfrak{p}^k$. Suppose $\psi \neq 1$.

$$\mathrm{level}(\psi) := \min \{ d \in \mathbb{Z} \mid \mathfrak{p}^d \subset \ker \psi \}$$

Prop (Additive duality)

Fix $\psi \in \widehat{F}^*$ nontrivial with level d.

We have a gp iso

$F \longrightarrow \widehat{F}^*$ $\quad a \longmapsto \psi(a)\psi(-)$ of level $d - v_F(a)$ (when $a \neq 0$)

Q: Do we have similar result for $\widehat{\mathcal{O}}$?

E.g. $G = F^\times$

$\forall \chi \in \widehat{F^\times}^*$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi \}$$

Q: Do we have any classification of $F^\times$?

A: Yes. Since $F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times$, any cont character $\chi \in \widehat{F^\times}^*$ is uniquely determined by $\chi(\pi) \in \mu_\infty$ and $\chi|_{\mathcal{O}^\times} \in \widehat{\mathcal{O}^\times}^*$.

Notation for future: G: loc. profinite gp $\quad Z = Z(G)$

$\mathrm{Cos}(G) := \{$ cpt open subgp K of $G\}$

$\mathrm{Cos}_Z(G) := \{K \leq G \mid K \geq Z,\ K/Z \subset G/Z$ cpt open$\}$

When $\# Z(G) = +\infty$, $\mathrm{Cos}(G) \cap \mathrm{Cos}_Z(G) = \phi$.

3. Construction of new reps

3.1. Special sub & quotient rep.

Def. G: loc. profinite gp $\quad K \leq G$

$(\rho, V) \in \mathrm{Rep}(G)$ $\quad \vartheta \in \widehat{K}^*$, we define sm reps of K:

$$V(K) := \langle v - k.v \rangle_{k \in K,\, v \in V}$$

$$V_K := V/V(K)$$

} in $\mathrm{Rep}(B)$ if $K \triangleleft B \leq G$

not confused with V^N!

$$V(\vartheta) := \langle \vartheta(k) v - k.v \rangle_{k \in K,\, v \in V}$$

$$V_\vartheta := V/V(\vartheta)$$

Obviously $V(K) = V(\mathbb{1}_K)$, $V_K = V_{\mathbb{1}_K}$, V_ϑ is a multiple of ϑ, and

$$0 \to V(\vartheta) \to V \to V_\vartheta \to 0 \quad \text{in } \mathrm{Rep}(K)$$

Rmk (1) For $K \triangleleft B \leq G$, $g \in B$, $\vartheta \in \widehat{K}^*$, $\vartheta' = \vartheta(g^{-1} \cdot g) \in \widehat{K}^*$, we get isos

$$V(\vartheta) \longrightarrow V(\vartheta') \qquad V_\vartheta \longrightarrow V_{\vartheta'} \qquad \text{in } \mathrm{Rep}(K)$$

$$v \longmapsto g.v \qquad\qquad v \longmapsto g.v$$

(2) Given $(\sigma, V) \in \mathrm{Rep}(K)$, $\vartheta \in \widehat{K}^*$, we can view $V(\vartheta)$ as $V'(K)$, where

$$(\sigma', V') := (\vartheta^{-1} \otimes_{\mathbb{C}} \sigma, V)$$

(3) Normal subgp gives us plenty of canonical decompositions.

E.g., when $(\rho, V) \in \mathrm{Irr}(G)$, $K \triangleleft G$, we get

$$\begin{cases} V(K) = V \\ V_K = 0 \end{cases} \quad \text{or} \quad \begin{cases} V(K) = 0 \\ V_K = V \end{cases}$$

(4) When $(\rho, V) \in \mathrm{Rep}(K)$ is semisimple,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(K) & \longrightarrow & V & \longrightarrow & V_K & \longrightarrow & 0 \\ & & \wr\wr & & \wr\wr & & \wr\wr & & \\ 0 & \to & \bigoplus_{\substack{\sigma \in \mathrm{Irr}(K) \\ \sigma \neq \mathbb{1}_K}} V^\sigma & \longrightarrow & \bigoplus_{\sigma \in \mathrm{Irr}(K)} V^\sigma & \longrightarrow & V^{\mathbb{1}_K} & \longrightarrow & 0 \end{array}$$

Assume additionally that

- K is abelian,
- K is the union of an increasing sequence of cpt open subgps.

Then, we have more properties.

Prop. (Integral criterian) [p56 Lemma] $(\sigma, V) \in \mathrm{Rep}(K)$, $v \in V$.

$$v \in V(\vartheta) \Leftrightarrow \left[\begin{array}{l} \exists \ K_0 \in \mathrm{Cos}(K) \ \text{ s.t } \\ \int_{K_0} \vartheta(k)^{-1} \sigma(k) v \ d\mu_K(k) = 0 \end{array} \right]$$

Prop. The fctor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \quad (\pi, V) \longmapsto V_\vartheta$$

is exact. E.p.,

	$V(K)(K) \cong V(K)$	$V(K)_K = 0$
$(\vartheta \neq 1)$	$V(K)(\vartheta) \cong V(K) \cap V(\vartheta)$	$V(K)_\vartheta \cong V_\vartheta$
	$V_K(K) = 0$	$V_{K,K} \cong V_K$
$(\vartheta \neq 1)$	$V_K(\vartheta) \cong V_K$	$V_{K,\vartheta} = 0$

(You can compute $V(\vartheta)$ and V_ϑ also, but I'm lazy.)

3.2. Duality

$(\rho, V) \in \mathrm{Rep}(G) \rightsquigarrow (\rho^*, V^*)$ may be not smooth $\quad (g^{-1}-)$

$\rightsquigarrow (\check{\rho}, \check{V}) \in \mathrm{Rep}(G)$ is the smooth dual, where

$$\check{V} := \bigcup_{K \in \mathrm{Cos}(G)} (V^*)^K \subset V^*$$

$$ev : \check{V} \times V \longrightarrow \mathbb{C} \qquad (\check{\omega}, v) \longmapsto \langle \check{\omega}, v \rangle$$

$\rightsquigarrow \langle g.\check{\omega}, v \rangle = \langle \check{\omega}, g^{-1}.v \rangle$

Rmk. (0) When $\rho \in \widehat{G}$, $\rho^* \cong \check{\rho} \cong \rho(-^{-1})$ in $\widehat{G}^*$.

(1) The contravariant duality fctor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(G) \qquad (\rho, V) \longmapsto (\check{\rho}, \check{V})$$

is exact.

exact=>additive: https://math.stackexchange.com/questions/3039422/in-abelian-categories-is-a-right-left-exact-functor-necessarily-additive

(2) For $K \in \mathrm{Cos}(G)$, we have iso $\check{V}^K \xrightarrow{\cong} (V^K)^*$ in $\mathrm{Rep}(K)$.

$\Downarrow$

(3) If $(\rho, V) \in \mathrm{Rep}(G)$, $v \in V$, then $\exists \check{\omega} \in \check{V}$ s.t. $\langle \check{\omega}, v \rangle \neq 0$.

$\Downarrow$

(4) $\delta : V \longrightarrow \check{\check{V}}$ is inj, and

$$\delta \text{ is iso} \iff \pi \text{ is admissible}$$

$\Downarrow$

(5) If $(\rho, V) \in \mathrm{Rep}(G)$ is admissible, then

$$(\rho, V) \in \mathrm{Irr}(G) \iff (\check{\rho}, \check{V}) \in \mathrm{Irr}(G)$$

(6) (Bilinear map) Let $(\rho, V), (\sigma, W) \in \mathrm{Rep}(G)$.

$$\mathcal{B}(\rho, \sigma) := \left\{ f : V \times W \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ bilinear} \\ f(gv, gw) = f(v, w) \end{array} \right\}.$$

Then

$$\mathcal{B}(\rho, \sigma) \cong \mathrm{Hom}_G(\rho, \check{\sigma}) \cong \mathrm{Hom}(\sigma, \check{\rho}).$$

3.3. Ind and c-Ind

Definition

Def (Induced representation) $(-g)$

$H \leq G$ closed, $(\sigma, W) \in \mathrm{Rep}(H)$, we get

$$\begin{cases} \text{sm induction} & \mathrm{Ind}_H^G \sigma := (\Sigma, X) \in \mathrm{Rep}(G) \\ \text{cpt induction} & \text{c-}\mathrm{Ind}_H^G \sigma := (\Sigma_c, X_c) \in \mathrm{Rep}(G) \end{cases}$$

as follows.

$$\mathrm{Ind}_H^G W := X = \left\{ f : G \longrightarrow W \;\middle|\; \begin{array}{ll} f(hg) = \sigma(h) f(g) & \forall g \in G, h \in H \\ \exists K \in \mathrm{Cos}(G) \text{ s.t. } f(gk) = f(g) & \forall g \in G, k \in K \end{array} \right\}$$

$$\Sigma(g).f := f(-g)$$

$$\text{c-}\mathrm{Ind}_H^G W := X_c = \left\{ f \in X \;\middle|\; \pi(\mathrm{supp}\, f) \text{ is cpt in } H \backslash G \right\} \qquad \pi : G \to H\backslash G$$

$$\Sigma_c(g).f := f(-g)$$

Rmk. (0) This action uses $(-g)$ rather than $(g^{-1}-)$ which looks ugly. Can we fix it?

(1). (Reality check) When $G = H$,

$$\text{c-}\mathrm{Ind}_H^G W = \mathrm{Ind}_H^G W = \left\{ f : G \to W \;\middle|\; \begin{array}{l} f(g) = \sigma(g) f(1) \\ \exists K \leq G \text{ s.t. } f(gk) = f(g) \end{array} \right\} \xrightarrow{\cong} W$$

$$f \longmapsto f(1)$$
$$\sigma(-)\cdot w \longleftarrow w$$

(2) Two fcts Ind_H^G, $\text{c-}\mathrm{Ind}_H^G$ are both exact.

(3) Suppose $H \leq G$ open, $[G : H] < +\infty$, $(\sigma, W) \in \mathrm{Irr}(H)$. We get $\mathrm{Ind}_H^G \sigma$ is G-semisimple.

Frobenius Reciprocity

Thm.

condition		$(\rho, V) \in \mathrm{Rep}(G)$, $(\sigma, W) \in \mathrm{Rep}(H)$
$H \leq G$ closed	$\mathrm{Res}_H^G \vdash \mathrm{Ind}_H^G$	$\mathrm{Hom}_H(\mathrm{Res}_H^G \rho, \sigma) \cong \mathrm{Hom}_G(\rho, \mathrm{Ind}_H^G \sigma)$
$H \leq G$ open	$\text{c-}\mathrm{Ind}_H^G \vdash \mathrm{Res}_H^G$	$\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_H^G \sigma, \rho) \cong \mathrm{Hom}_H(\sigma, \mathrm{Res}_H^G \rho)$

E.p. we have two canonical map in $\mathrm{Rep}(H)$:

$H \leq G$ closed $\quad \varepsilon_W : \mathrm{Ind}_H^G W \longrightarrow W \qquad f \longmapsto f(1)$

$H \leq G$ open $\quad \eta_W : W \longrightarrow \text{c-}\mathrm{Ind}_H^G W \qquad w \longmapsto \left[f_w : \begin{array}{l} h \in H \mapsto h.w \\ x \notin H \mapsto 0 \end{array} \right]$

Duality Theorem [p32]

Let $H \leq G$ closed, define

$$\delta_{H\backslash G} := \delta_H^{-1} \delta_G|_H : H \longrightarrow \mathbb{R}_{>0}^\times$$

then $\delta_{H\backslash G} \in \widehat{H}^*$, and $\forall (\sigma, W) \in \mathrm{Rep}(H)$, we have

$$(\text{c-}\mathrm{Ind}_H^G \sigma)^\vee \cong \mathrm{Ind}_H^G (\delta_{H\backslash G} \otimes \check{\sigma}).$$

↳ Does it come from the non-compactibility of Ind & dual? Will it be resolved when $G \curvearrowright V^*$ by $(-g)$?

3.4. Other constructions. (Concerned closely with Hecke algebra)

- $\otimes$, Sym, and $\wedge^{\cdot}$.

  It's difficult for me to work out.

- $C^\infty(G)$, $C_c^\infty(G) \in \mathrm{Rep}(G\times G)$

  $C_c^\infty(G)$ will play a central role in Hecke alg. We focus on $C^\infty(G)$ here.

  Def. (matrix coefficient set $C(\rho)$)

  Let $(\rho, V) \in \mathrm{Rep}(G)$. $v \in V$, $\check{w} \in \check{V}$.

$$\Psi : \check{V} \otimes V \longrightarrow C^\infty(G)$$
$$\check{w} \otimes v \longmapsto \left[\begin{array}{l} \gamma_{\check{w}\otimes v} : G \longrightarrow \mathbb{C} \\ \gamma_{\check{w}\otimes v}(g) = w(g.v) = \langle w, \rho(g)v \rangle \end{array}\right]$$

  is a morphism in $\mathrm{Rep}(G\times G)$.

$$C(\rho) = \mathrm{Im}\,\Psi \subseteq C^\infty(G)$$

  Rmk. $\forall \gamma \in C(\rho)$, Supp γ is invariant under translation by Z.

  Def. (γ-cuspidal rep)

  Let G be a unimodular loc. profinite gp. We define

$$\mathrm{Cusp}_c(G) = \left\{ (\rho, V) \in \mathrm{Irr}(G) \;\middle|\; \begin{array}{l} \mathrm{supp}(\gamma)/Z \subseteq G/Z \text{ is cpt} \\ \text{for every } \gamma \in C(\rho) \end{array} \right\}$$

  Prop. [p70] (1) Every $(\rho, V) \in \mathrm{Cusp}_c(G)$ is admissible

  (2) Suppose

  - $(\rho, V) \in \mathrm{Irr}(G)$ is admissible
  - $\exists\ \gamma \in C(\rho)$, $\mathrm{supp}(\gamma)/Z$ is cpt.

  Then,

  - $\Psi : V^* \otimes V \longrightarrow C(\pi)$ is an iso
  - $(\pi, V) \in \mathrm{Cusp}_c(G)$

  [Idea of proof: Use dual and cardinal argument.
  Hecke algebra is needed which will be introduced later]

  Rmk. γ-Cuspidal has the property of projectivity and Schur's orthogonality relation, see [p74].

4. Hecke algebra

This should be a better ref than the Bushnell's book: http://virtualmath1.stanford.edu/~conrad/JLseminar/Notes/L2.pdf

Assume G is unimodular, and fix an Haar measure μ.

Basic: alg - mod - rep

Def. (Hecke algebra $\mathcal{H}(G)$)

$$\mathcal{H}(G) := (C_c^\infty(G), *)$$

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

Rmk In general, $\mathcal{H}(G)$ has no unit element.

However, $\mathcal{H}(G)$ has idempotent elements

$e_K := \frac{1}{\mu(K)} \mathbb{1}_K$ for $K \in Cos(G)$

$e_\sigma \in \mathcal{H}(G)$ for $(\sigma, W) \in Irr(K)$, $K \in Cos(G)$

Since $\mathbb{1}_{gK} = \mathbb{1}_K(g^{-1} -)$

$$e_\sigma(g) = \begin{cases} \frac{\dim_{\mathbb{C}} W}{\mu(K)} \operatorname{tr}(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

well-defined since $\dim_{\mathbb{C}} W < +\infty$

e.g. $\mathbb{1}_K \in \widehat{K}^*$, $e_{\mathbb{1}_K} = e_K$.

Rmk. When G is finite, $\mathcal{H}(G) \cong \mathbb{C}[G]$ is the path algebra.

Def. (Smooth $\mathcal{H}(G)$-module)

Be careful: the scalar multiplication is also denoted by $*$.

An $\mathcal{H}(G)$-module M is smooth if

$$\mathcal{H}(G) * M = M$$

$$\Leftrightarrow \forall m \in M, \quad \exists K \in \mathrm{Cos}(G) \quad \text{s.t.} \quad e_K * m = m$$

$\mathrm{Mod}(\mathcal{H}(G)) = \{\text{sm } \mathcal{H}(G)\text{-modules}\}$ is a full subcategory of $\{\mathcal{H}(G)\text{-modules}\}$.

Rmk. The $\mathbb{C}$-v.s. structure is inherited from the smooth structure:

$$\mathbb{C} \times (\mathcal{H}(G) * M) \longrightarrow \mathcal{H}(G) * M \qquad k.(f*m) = (kf)*m$$

i.e., any sm $\mathcal{H}(G)$-module is a $\mathbb{C}$-v.s.

Thm We have the equiv of categories.

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Mod}(\mathcal{H}(G))$$

$$(\rho, V) \longmapsto V$$

$$f * v = \int_G f(g)\, (\rho(g).v)\, d\mu(g)$$

for $f \in \mathcal{H}(G)$, $v \in V$

$$(\rho, M) \longleftarrow M$$

$$\rho(g).m = \frac{1}{\mu(K)} (\mathbb{1}_{gK} * m)$$

for $g \in G$, $m \in M$,
$K \in \mathrm{Cos}(G)$ s.t. $e_K * m = m$

[Idea: $G \leftarrow - - \rightarrow \mathcal{H}(G)$
$g \mapsto - - \rightarrow \frac{1}{\mu(K)} \mathbb{1}_{gK}$ $K << G$]

E.p. $e_K * v \in V^K$ and $\rho(1).m = m$

Prop [4.3, P37] Let $(\rho, V) \in \mathrm{Rep}(G)$, $K \in \mathrm{Cos}(G)$, $(\sigma, W) \in \mathrm{Irr}(K)$, then

$$e_\sigma * - : V \longrightarrow V^\sigma \qquad \text{in } \mathrm{Rep}(K)$$

is a projection.

Task. Assume G is unimodular. Redo everything in the language of $\mathrm{Mod}(\mathcal{H}(G))$.

Interesting topic:
The Hecke algebra of a reductive p-adic group: a view from noncommutative geometry [https://www.imj-prg.fr/preprints/389.pdf]

$\mathcal{H}(G,K)$: an analog of $\mathbb{C}[K\backslash G/K]$

Def. Let $K \in \mathrm{Cos}(G)$, $(\sigma, W) \in \mathrm{Irr}(K)$, we define

$$\mathcal{H}_1(G,\sigma) := e_\sigma * \mathcal{H}(G) * e_\sigma \underset{\text{when } \sigma\in\hat{K}}{=\!=} \{f \in C_c^\infty(G) \mid f(k_1 g k_2) = \sigma(k_1 k_2)^{-1} f(g) \quad \forall g\in G,\ k_1,k_2\in K\}$$

$$\begin{aligned}\mathcal{H}(G,K) :&= \mathcal{H}_1(G, \mathbb{1}_K)\\ &= e_K * \mathcal{H}(G) * e_K\\ &= \{f\in C_c^\infty(G) \mid f(k_1 g k_2) = f(g) \quad \forall g\in G,\ k_1,k_2\in K\}\\ &= C_c^\infty(K\backslash G/K) \quad \text{though undefined}\end{aligned}$$

Rmk. $\mathcal{H}_1(G,\sigma)$ has unit e_σ, while $\mathcal{H}(G,K)$ has unit e_K.

When $\# K\backslash G/K < +\infty$, $\mathcal{H}(G,K) = \mathbb{C}[K\backslash G/K]$.

Def (Smooth $\mathcal{H}_1(G,\sigma)$-module = $\mathcal{H}_1(G,\sigma)$-module)

An $\mathcal{H}_1(G,\sigma)$-module M is smooth if

$$\mathcal{H}(G,\sigma) * M = M$$
$$\Leftrightarrow \forall m\in M,\ e_\sigma * m = m$$
$$\Leftrightarrow \checkmark$$

Q: How can we view $\mathcal{H}_1(G,\sigma)$-module as some special reps?

Prop. Let $(\rho, V) \in \mathrm{Rep}(G)$, $K\in \mathrm{Cos}(G)$. We get SES

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(K) & \longrightarrow & V & \longrightarrow & V_K & \longrightarrow & 0 \\ & & \| & & \| & & \wr\| & & \\ 0 & \longrightarrow & V(K) & \longrightarrow & V & \xrightarrow{e_K * -} & V^K & \longrightarrow & 0 \end{array}$$

in $\mathrm{Rep}(K)$

(in $\mathrm{Rep}(B)$ if $K \triangleleft B \leq G$)

Moreover, $V^K \in \mathrm{Mod}(\mathcal{H}(G,K))$.

Prop. Fix $K\in\mathrm{Cos}(G)$. Then

$$\begin{array}{ccc} \{(\rho,V)\in\mathrm{Irr}(G) \mid V^K\neq 0\} & \overset{\hat{=}}{\longleftrightarrow} & \mathrm{Irr}(\mathcal{H}(G,K)) \\ (\rho, V) & \longmapsto & V^K = e_K * V \\ \left(\mathcal{H}(G)\otimes_{\mathcal{H}(G,K)} M\right)/X \in \mathrm{Mod}(\mathcal{H}(G)) & \longleftarrow & M \end{array}$$

Where X is the maximal subrep of $\mathcal{H}(G)\otimes_{\mathcal{H}(G,K)} M$ st. $\underset{X^K}{\underbrace{X\cap(e_K\otimes M)}} = 0$

In ptc, $\{(\rho,V)\in\mathrm{Rep}(G) \text{ gen. by } V^K\} \overset{\hat{=}}{\longleftrightarrow} \mathrm{Rep}(\mathcal{H}(G,K))$

σ- spherical Hecke algebra

Def. Suppose $K \in Cos(G)$, $(\sigma, W) \in Irr(K)$.

The σ-spherical Hecke alg / intertwining alg $\mathcal{H}(G,\sigma)$ is defined as

$$\mathcal{H}(G,\sigma) := \left\{ f: G \longrightarrow End_{\mathbb{C}}(W) \;\middle|\; \begin{array}{l} \text{supp } f \text{ is cpt} \\ f(k_1 g k_2) = \sigma(k_2)^{-1} f(g) \sigma(k_1)^{-1} \quad \forall g \in G, k_1, k_2 \in K \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_G f_2(x^{-1}g)\, f_1(x)\, d\mu(x)$$

Rmk. $\mathcal{H}(G,\sigma)$ is ass with unit

$$E_\sigma: G \longrightarrow End_{\mathbb{C}}(W) \qquad E_\sigma(g) = \begin{cases} \frac{1}{\mu(K)} \sigma(g)^{-1} & g \in K \\ 0 & g \notin K \end{cases}$$

and we have

$$\mathcal{H}(G,\sigma) \cong \mathcal{H}_1(G,\sigma) \otimes_{\mathbb{C}} End_{\mathbb{C}}(W) \quad \text{as } \mathbb{C}\text{-alg.}$$

Prop. $c\text{-}Ind_K^G \sigma$ is an $\mathcal{H}(G,\check{\sigma})$-module defined by

$$\mathcal{H}(G,\check{\sigma}) \times c\text{-}Ind_K^G \sigma \longrightarrow c\text{-}Ind_K^G \sigma \qquad (\phi, f) \longmapsto \phi * f$$

where

$$\phi * f(g) := \int_G \phi(x) . f(x^{-1}g)\, d\mu(x)$$

We have an iso

$$\begin{array}{ccc} \mathcal{H}(G,\check{\sigma}) \longrightarrow End_G(c\text{-}Ind_K^G W) & \phi & \longmapsto \phi * - \\ \quad\quad \cong\big\downarrow -\circ\eta_W & & \\ \mathcal{H}(G,\check{\sigma}) \longleftarrow Hom_K(W, c\text{-}Ind_K^G W) & \left[\begin{array}{l} \phi: G \to End_{\mathbb{C}} W \\ g \mapsto [f \mapsto f(\psi_f(g))] \end{array} \right] & \longleftarrow \psi \end{array}$$

Moreover, When $\sigma \in \widehat{K}^*$, $W = \mathbb{C}$, the two actions are compatative. $(\rho, V) \in Rep(G)$

$$\begin{array}{ccc} \mathcal{H}(G,\check{\sigma}) & \cong & End_G(c\text{-}Ind_K^G \mathbb{C}) \\ \circlearrowleft & & \circlearrowleft \\ V^{\check{\sigma}} & \cong & Hom_G(c\text{-}Ind_K^G \mathbb{C}, V) \end{array}$$

$$f * v = \int_G f(g)\, g.v\, d\mu(g)$$

In the lecture it is claimed that $\left(End_G(c\text{-}Ind_K^G \mathbb{1}_K)\right)^{op} \cong \mathcal{H}(G,K)$.
So it's very possible that it is wrong here.

Do we have the iso $\left(End_G(W)\right)^{op} \cong End_G(\check{W})$?
idea: when W is admissible,

$$End_G(W) \cong W \otimes \check{W} \cong \check{W} \otimes \check{\check{W}} \cong End_G(\check{W}) \quad \text{as } G\text{-rep.}$$

Interesting topic:
The spherical Hecke algebra for affine Kac-Moody groups
[https://annals.math.princeton.edu/wp-content/uploads/annals-v174-n3-p05-p.pdf]

Generalization. $K \in \mathrm{Cos}_Z(G)$ Fix $\chi \in \widehat{Z}^*$.

Rmk. G is unimodular $\Rightarrow$ G/Z is unimodular. Denote $\mu_{G/Z}$ as Haar measure of G/Z.

Def. (Hecke algebra $\mathcal{H}_\chi(G)$)

$$C^\infty_{c,\chi}(G) = \left\{ f \in C^\infty(G) \,\middle|\, \begin{array}{l} f(zg) = \chi(z)^{-1} f(g) \quad \forall z \in Z,\ g \in G \\ \mathrm{supp} f/Z \text{ is cpt} \end{array} \right\}$$

$$\mathcal{H}_\chi(G) := (C^\infty_{c,\chi}(G), *)$$

$$(f_1 * f_2)(g) := \int_{G/Z} f_1(x) f_2(x^{-1}g)\, d\mu_{G/Z}(x)$$

Rmk $\mathcal{H}_{1_Z}(G) = \mathcal{H}(G/Z)$.

$\mathcal{H}_\chi(G)$ has idempotent elements

$$e_\sigma \in \mathcal{H}_\chi(G) \qquad \text{for } (\sigma, W) \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi,\ K \in \mathrm{Cos}(G)$$

$$e_\sigma(g) = \begin{cases} \dfrac{\dim_{\mathbb{C}} W}{\mu_{G/Z}(K/Z)} \mathrm{tr}\,(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

Def. (Smooth $\mathcal{H}_\chi(G)$-module)

An $\mathcal{H}_\chi(G)$-module M is smooth if

$$\mathcal{H}_\chi(G) * M = M$$

$$\Leftrightarrow \forall m \in M,\ \exists K \in \mathrm{Cos}_Z(G),\ (\sigma, W) \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi \text{ s.t. } e_\sigma * m = m.$$

<u>Thm</u> We have the equiv of categories.

$$\{(\rho, V) \in \mathrm{Rep}(G) \mid \rho \overset{Z}{\rightrightarrows} \chi\} \longrightarrow \mathrm{Mod}(\mathcal{H}_\chi(G))$$

$$(\rho, V) \longmapsto V$$

$$f * v = \int_{G/Z} f(g)\, (\rho(g) \cdot v)\, d\mu_{G/Z}(g)$$

for $f \in \mathcal{H}_\chi(G)$, $v \in V$

$$(\rho, M) \longleftarrow M$$

$$\rho(g) \cdot m = e_\psi(g^{-1} -) * m$$

for $g \in G$, $m \in M$,

$K \in \mathrm{Cos}_Z(G)$ $\psi \in \widehat{K} \overset{Z}{\rightrightarrows} \chi$ s.t. $e_\psi * m = m$

E.p. $e_\sigma * v \in V^\sigma$ and $\rho(1).m = m$

Prop Let $(\rho, V) \in \mathrm{Rep}(G) \overset{Z}{\rightrightarrows} \chi$, $K \in \mathrm{Cos}_Z(G)$, $(\sigma, W) \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi$, then

$$e_\sigma * - : V \longrightarrow V^\sigma \qquad \text{in } \mathrm{Rep}(K)$$

is a projection.

Def. Let $K \in \mathrm{Cos}_Z(G)$, $(\sigma, W) \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi$, we define

$$\mathcal{H}_{1,\chi}(G,\sigma) := e_\sigma * \mathcal{H}_\chi(G) * e_\sigma$$
$$\underset{\text{when } \sigma \in \widehat{K}^*}{=} \{ f \in C^\infty_{c,\chi}(G) \mid f(k_1 g k_2) = \sigma(k_1 k_2)^{-1} f(g) \quad \forall g \in G,\ k_1, k_2 \in K \}$$

Rmk. $\mathcal{H}_{1,\chi}(G,\sigma)$ has unit e_σ.

Smooth $\mathcal{H}_{1,\chi}(G,\sigma)$-module $=$ $\mathcal{H}_{1,\chi}(G,\sigma)$-module

Prop. Fix $K \in \mathrm{Cos}_Z(G)$, $\sigma \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi$. Then

$$\{(\rho, V) \in \mathrm{Irr}(G) \overset{Z}{\rightrightarrows} \chi \mid V^\sigma \neq 0\} \overset{\cong}{\longleftrightarrow} \mathrm{Irr}(\mathcal{H}_{1,\chi}(G,\sigma))$$
$$(\rho, V) \longmapsto V^\sigma = e_\sigma * V$$
$$\left(\mathcal{H}_\chi(G) \otimes_{\mathcal{H}_{1,\chi}(G,\sigma)} M\right)/X \in \mathrm{Irr}(\mathcal{H}_\chi(G)) \longleftarrow M$$

Where X is the maximal subrep of $\mathcal{H}_\chi(G) \otimes_{\mathcal{H}_{1,\chi}(G,\sigma)} M$ s.t. $X \cap \underbrace{(e_\sigma \otimes M)}_{= X^\sigma} = 0$

In ptc, $\{(\rho, V) \in \mathrm{Rep}(G) \overset{Z}{\rightrightarrows} \chi \text{ gen by } V^\sigma\} \longleftrightarrow \mathrm{Rep}(\mathcal{H}_{1,\chi}(G,\sigma))$

Def. Suppose $K \in \mathrm{Cos}_Z(G)$, $(\sigma, W) \in \mathrm{Irr}(K) \overset{Z}{\rightrightarrows} \chi$

The σ-spherical Hecke alg / intertwining alg $\mathcal{H}_\chi(G,\sigma)$ is defined as

$$\mathcal{H}_\chi(G,\sigma) := \left\{ f : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W) \;\middle|\; \begin{array}{l} f(k_1 g k_2) = \sigma(k_2)^{-1} f(g) \sigma(k_1)^{-1} \quad \forall g \in G,\ k_1, k_2 \in K \\ \mathrm{supp}\, f/Z \text{ is cpt} \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{G/Z} f_2(x^{-1} g) f_1(x)\, d\mu_{G/Z}(x)$$

Rmk. $\mathcal{H}(G,\sigma)$ is ass with unit

$$E_\sigma : G \longrightarrow \mathrm{End}_{\mathbb{C}}(W) \qquad E_\sigma(g) = \begin{cases} \frac{1}{\mu_{G/Z}(K/Z)} \sigma(g)^{-1} & g \in K \\ 0 & g \notin K \end{cases}$$

and we have

$$\mathcal{H}_\chi(G,\sigma) \cong \mathcal{H}_{1,\chi}(G,\sigma) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(W) \quad \text{as } \mathbb{C}\text{-alg}.$$

Prop. $c\text{-}\mathrm{Ind}_K^G \sigma$ is an $\mathcal{H}_\chi(G,\check{\sigma})$-module defined by

$$\mathcal{H}_\chi(G,\check{\sigma}) \times c\text{-}\mathrm{Ind}_K^G \sigma \longrightarrow c\text{-}\mathrm{Ind}_K^G \sigma \qquad (\phi, f) \longmapsto \phi * f$$

where

$$\phi * f(g) := \int_{G/Z} \phi(x)[f(x^{-1} -)]\, d\mu_{G/Z}$$

We have an iso

$$\mathcal{H}_\chi(G,\check{\sigma}) \longrightarrow \mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G W) \qquad \phi \longmapsto \phi * -$$
$$\mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G W) \underset{\cong}{\xrightarrow{- \circ \eta_W}} \mathrm{Hom}_K(W, c\text{-}\mathrm{Ind}_K^G W)$$
$$\mathcal{H}_\chi(G,\check{\sigma}) \longleftarrow \mathrm{Hom}_K(W, c\text{-}\mathrm{Ind}_K^G W) \qquad \left[\begin{array}{l} \phi : G \to \mathrm{End}_{\mathbb{C}} \check{W} \\ \quad g \mapsto [f \mapsto f(\psi(g))] \end{array}\right] \longleftarrow \psi$$

Moreover, when $\sigma \in \widehat{K}^{*} \overset{Z}{\rightrightarrows} \chi$, $W = \mathbb{C}$, the two actions are compatative. $(\rho, V) \in \mathrm{Rep}(G)$

$$\begin{array}{ccc} \mathcal{H}_\chi(G,\check{\sigma}) & \cong & \mathrm{End}_G(c\text{-}\mathrm{Ind}_K^G \mathbb{C}) \\ \circlearrowright & & \circlearrowright \\ V^{\check{\sigma}} & \cong & \mathrm{Hom}_G(c\text{-}\mathrm{Ind}_K^G \mathbb{C}, V) \end{array}$$

$$f * v = \int_G f(g)\, g.v\, d\mu(g)$$

Maybe wrong.

5. Intertwining properties

In this subsection, G is an unimodular loc. profinite gp satisfying the countable hypothesis, and $Z = Z(G)$.

Definition

Def. Let $K_1, K_2 \in Cos(G)/Cos_Z(G)$, $\sigma_1 \in Irr(K_1)$, $\sigma_2 \in Irr(K_2)$, $g \in G$.

$K_1^g := g^{-1} K_1 g \qquad \rho_1^g = \rho_1(g - g^{-1}) \in Irr(K_1^g)$

We call g intertwines σ_1 with σ_2 if

$$Hom_{K_1^g \cap K_2}(\sigma_1^g, \sigma_2) \neq 0$$

usually missed

Notation. $itw_G(\sigma_1, \sigma_2) := \{ g \in G \mid g \text{ intertwines } \sigma_1 \text{ with } \sigma_2 \}$

$itw_G(\sigma) := itw_G(\sigma_1, \sigma_2)$

σ_1 intertwines with σ_2 if $itw_G(\sigma_1, \sigma_2) \neq \phi$

E.g. When $K_1 = K_2 = K$, $g \in N_K(G)$, $\sigma_1, \sigma_2 \in Irr(K)$, then

$$g \in itw(\sigma_1, \sigma_2) \Leftrightarrow Hom_K(\sigma_1^g, \sigma_2) \neq 0$$
$$\Leftrightarrow \sigma_1^g \cong \sigma_2$$

Rmk. $g \in itw(\sigma_1, \sigma_2) \Rightarrow k_1 g k_2 \in itw(\sigma_1, \sigma_2)$ for $\forall\, k_1 \in K_1,\ k_2 \in K_2$.

Basic results

Prop 1. Suppose $(\rho, V) \in Irr(G)$, $(\sigma_1, W_1) \in Irr(K_1)$, $(\sigma_2, W_2) \in Irr(K_2)$,

$$\begin{array}{ccc} & (\rho, V) & \\ {}^{\text{in } Rep(K_1)}\cup & & \cup^{\text{in } Rep(K_2)} \\ (\sigma_1, W_1) & & (\sigma_2, W_2) \end{array}$$

Then $itw(\sigma_1, \sigma_2) \neq \phi$.

Proof. We know $V^{\sigma_1} \neq 0$ in $Rep(K_1)$, $V^{\sigma_2} \neq 0$ in $Rep(K_2)$.

$$\left.\begin{array}{l} V^{\sigma_1} \neq 0 \overset{(\rho,V)\in Irr(G)}{\Longrightarrow} \langle V^{\sigma_1^g} \rangle_{g\in G} \text{ spans } V \\ \pi_{\sigma_2}: V \longrightarrow V^{\rho_2} \text{ is nonzero in } Rep(K_2) \end{array}\right\} \Rightarrow \begin{array}{l} \exists g \in G,\ \pi_{\sigma_2}|_{V^{\sigma_1^g}}: V^{\sigma_1^g} \longrightarrow V^{\sigma_2} \\ \text{is nonzero in } Rep(K_1^g \cap K_2). \end{array}$$

Prop 2. Let $K \in Cos(G)$, $\sigma \in Irr(K)$, $g \in G$, then

$$g \in itw(\sigma) \Leftrightarrow \exists f \in \mathcal{H}_1(G, \sigma) \quad s.t. \quad f|_{KgK} \neq 0$$

Prop 3. Let $K \in Cos_Z(G)$, $(\sigma, W) \in Irr(K)$, $g \in G$, then

$$g \in itw(\sigma) \Leftrightarrow \exists f \in \mathcal{H}(G, \sigma) \quad s.t. \quad supp\, f = KgK$$

Moreover, we have

$$\begin{array}{ccc} \{ f \in \mathcal{H}(G,\sigma) \mid supp\, f \subseteq KgK \} & \xrightarrow{\cong} & Hom_{K^g \cap K}(\sigma, \sigma^g) \subseteq End_{\mathbb{C}}(W) \\ f & \longmapsto & f(g) \\ f: f(x) = \begin{cases} \sigma(k_1) h \sigma(k_2)^{-1} & x = k_1 g k_2 \\ 0 & x \notin KgK \end{cases} & \longleftarrow & h \end{array}$$

Q: Any generalization of Prop 2 & Prop 3? $Cos(G) \to Cos_Z(G)$ or $Cos_Z(G) \to Cos(G)$?

c-Ind with cuspidal rep

Thm. [11.4, P80] (Construction method for cuspidal reps)

Suppose $\forall (\rho, V) \in \mathrm{Irr}(G)$ is admissible,
$K \in \mathrm{Cos}_Z(G)$, $(\sigma, W) \in \mathrm{Irr}(K)$ and $\mathrm{itw}(\sigma) = K$.
Then $c\text{-}\mathrm{Ind}_K^G \sigma \in \mathrm{Cusp}_c(G)$.

Rmk. (itw is key ingredient)
Let $K \in \mathrm{Cos}_Z(G)$, $(\sigma, W) \in \mathrm{Irr}(K)$ and $\mathrm{itw}(\sigma) \supsetneq K$.
Then $c\text{-}\mathrm{Ind}_K^G \sigma \notin \mathrm{Irr}(G)$

(Since $\mathrm{End}_G(c\text{-}\mathrm{Ind}\,\sigma) \cong \mathcal{H}(G, \sigma)$ has $\dim > 1$)

Rmk. In the theorem, if we assume further that $\delta_{K \backslash G} = \mathbb{1}_K$, then

$$
\begin{aligned}
& c\text{-}\mathrm{Ind}_K^G \sigma \in \mathrm{Irr}(G) \\
\xRightarrow{\text{adm}}\ & (c\text{-}\mathrm{Ind}_K^G \sigma)^\vee \in \mathrm{Irr}(G) \\
\Rightarrow\ & \mathrm{Ind}_K^G(\check{\sigma}) \overset{\delta_{K\backslash G} = \mathbb{1}_K}{\cong} (c\text{-}\mathrm{Ind}_K^G \sigma)^\vee \in \mathrm{Irr}(G) \\
\Rightarrow\ & c\text{-}\mathrm{Ind}_K^G(\check{\sigma}) = \mathrm{Ind}_K^G(\check{\sigma}) \\
\Rightarrow\ & \mathrm{Ind}_K^G(\sigma) \cong (c\text{-}\mathrm{Ind}_K^G(\check{\sigma}))^\vee \cong (\mathrm{Ind}_K^G(\check{\sigma}))^\vee \cong c\text{-}\mathrm{Ind}_K^G(\sigma)
\end{aligned}
$$