

Eine Woche, ein Beispiel

5.19. Weierstrass point

references:

https://en.wikipedia.org/wiki/Weierstrass_point

https://en.wikipedia.org/wiki/Inflection_point

Klein quartic has 24 inflection points:

https://www.uio.no/studier/emner/matnat/math/MAT2000/v24/projects/2023_the_klein_quartic_and_its_n_weierstrass_points.pdf

curve of genus >0 don't have single simple pole:

<https://math.stackexchange.com/questions/2841459/finding-a-meromorphic-function-on-a-compact-riemann-surface-with-prescribed-zero>

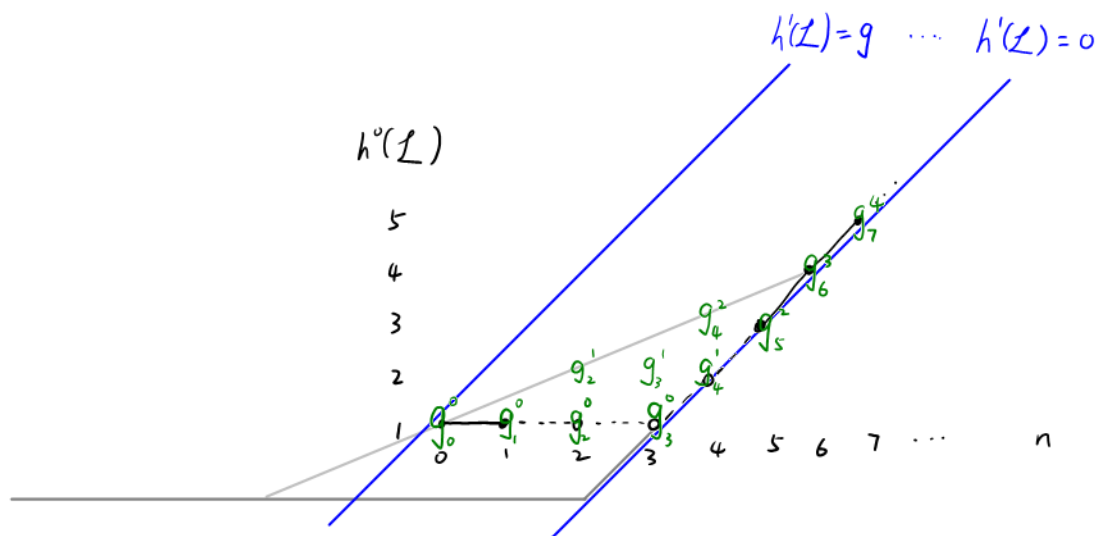
Setting: C : proj sm curve / \mathbb{C} $\bar{\mathbb{C}} = \mathbb{C}$, $\text{char } \mathbb{C} = 0$

$h^0(\mathcal{O}(nP))$ \ n		0	1	2	3	4	5	6	7	8	$g(g^2-1)$
$g(C)$											
0		1	2	3	4	5	6	7	8	9	0
1		1	1	2	3	4	5	6	7	8	0
2		1	1	?	2	3	4	5	6	7	6
3		1	1	?	?	?	3	4	5	6	24
4		1	1	?	?	?	?	?	4	5	60
\vdots		\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
$g=3$:	non-Weierstrass	1	1	1	1	2	3	4	5	6	\emptyset
	general quartic	1	1	1	2	2	3	4	5	6	1×24
	W: e.g. Klein quartic	1	1	1	2	3	3	4	5	6	2×12
	W: Fermat quartic	1	1	2	2	3	3	4	5	6	3×8
	W: hyperelliptic case										

by Clifford's thm, $h^0(\mathcal{O}(nP)) \leq \frac{n}{2} + 1$,
so the hyperelliptic case reaches the limit.

Finiteness of Weierstrass point:

<https://math.stackexchange.com/questions/4719889/is-this-proof-that-the-number-of-weierstrass-points-on-a-compact-riemann-surface>

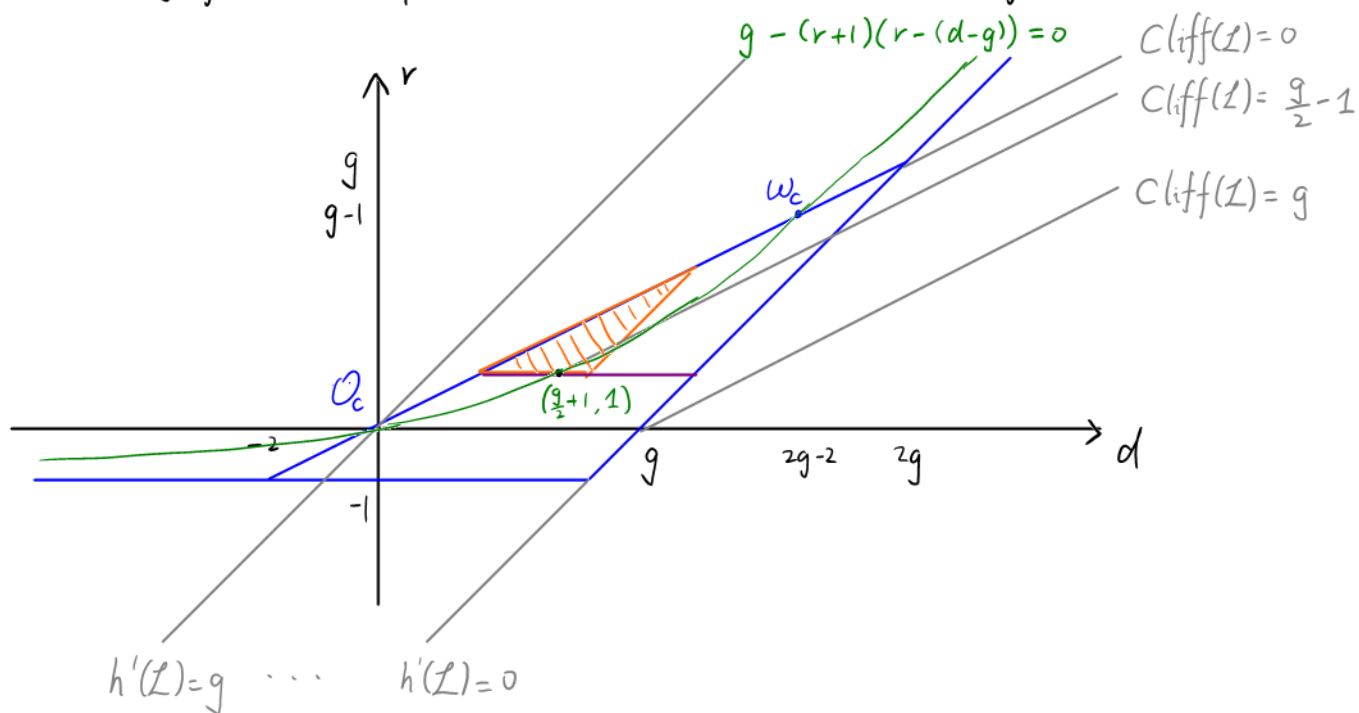


$$(V, \mathcal{L}) \in g_{\deg \mathcal{L}}^{\dim V}, \text{ where } \mathcal{O}_X \otimes_k V \subset \mathcal{O}_X \otimes_k H^0(X, \mathcal{L}) \rightarrow \mathcal{L}$$

$$g_{\deg \mathcal{L}}^{\dim V} \cong \mathbb{P}^{\dim V - 1} \hookrightarrow C^{\deg \mathcal{L}}$$

First $g_{\deg \mathcal{L}}^{\dim V}$: moduli of linear systems

Second $g_{\deg \mathcal{L}}^{\dim V}$: one special linear system = moduli of divisors

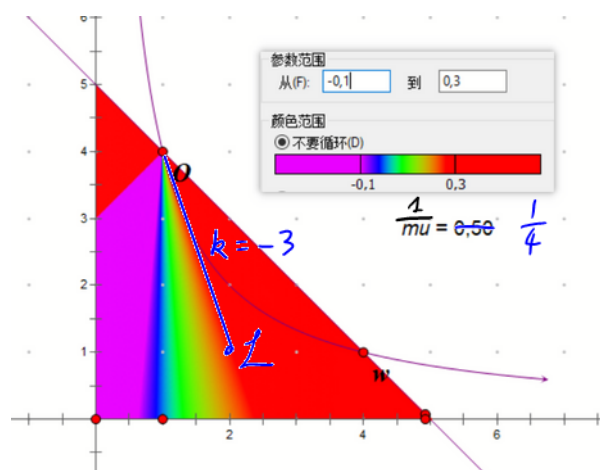
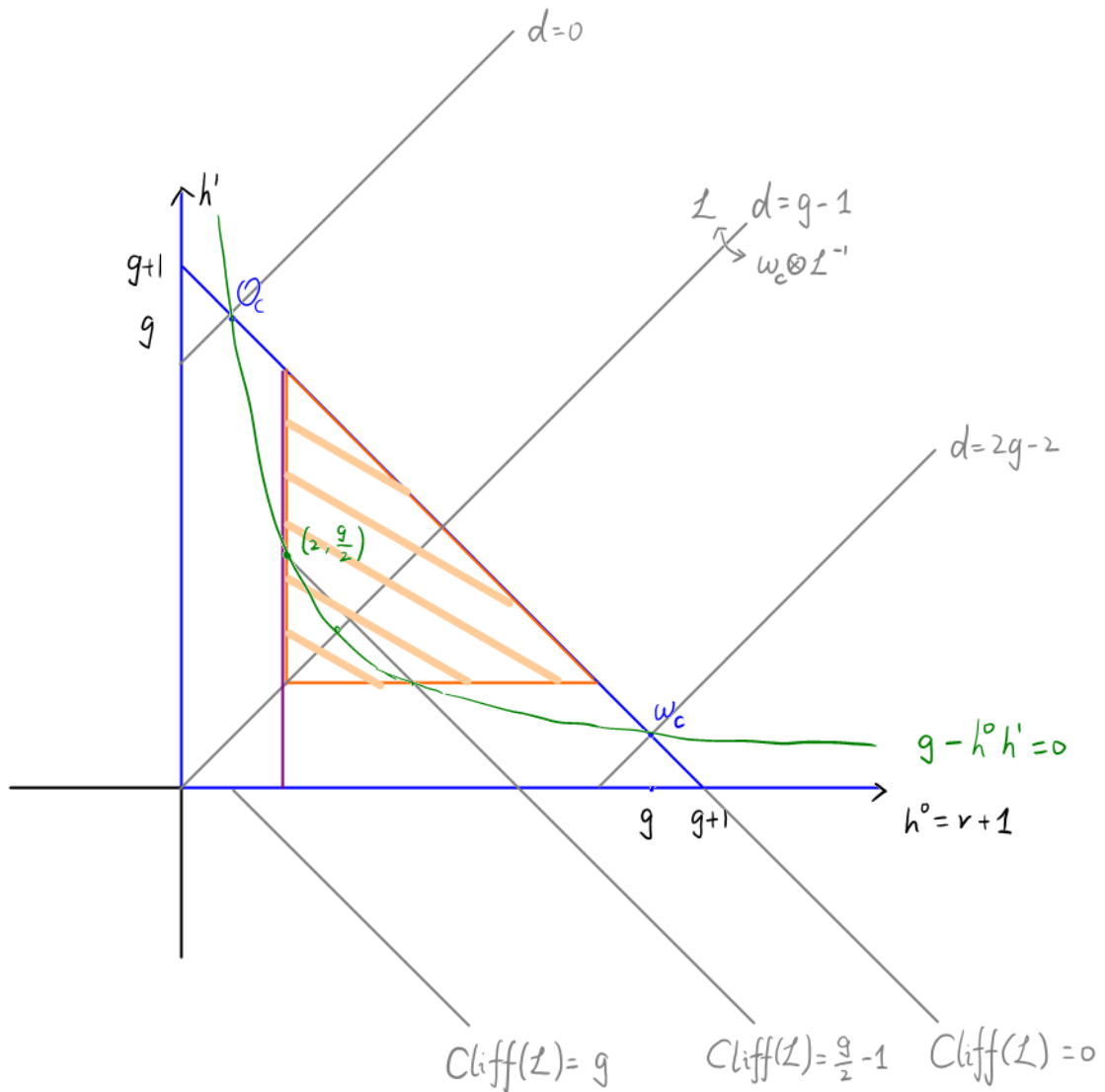


$$\text{Cliff}(C) := \min \{ \text{Cliff}(\mathcal{L}) \mid h^0(\mathcal{L}), h^1(\mathcal{L}) \geq 2 \} \quad \text{when } g \geq 4$$

$$\text{gon}(C) := \min \{ \deg(\mathcal{L}) \mid r(\mathcal{L}) = 1 \}$$

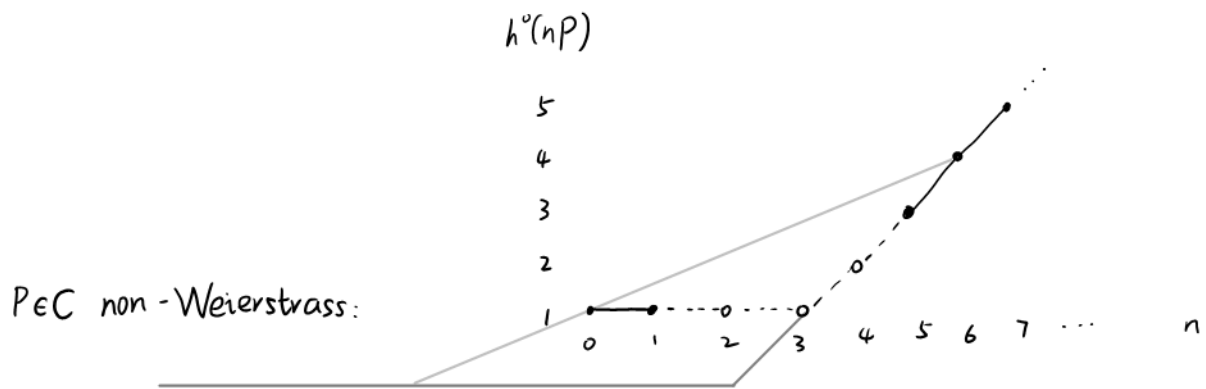
$$\underline{\text{Cor}} \quad \text{gon}(C) \geq \text{Cliff}(C) + 2$$

$$" = " \Leftrightarrow \text{Clifford dim of } C \text{ is } 1.$$



$$\mu := \frac{\deg(L)}{r(L)} = \frac{h^0 - h^1 + g - 1}{h^0 - 1} = 1 - k$$

$k = \text{slope of line passing } Q_c \& L$



$g=3$ case

$$\Leftrightarrow h^0(\mathcal{O}(gP)) = 1$$

$P \in C$ Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(gP)) \geq 2$$

$\Leftrightarrow \exists f \in K(C)$, f has a single pole at P ,
with $\text{ord}_P(f) \geq -g$

$$\Leftrightarrow h^0(K - gP) \geq 1$$

e.p.

$g=2$: $P \in C$ Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(2P)) = 2$$

$\Leftrightarrow \exists f \in K(C)$, f has a single double pole at P

$g=3$: $P \in C$ Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(3P)) \geq 2$$

$\Leftrightarrow \exists f \in K(C)$, f has a single triple pole at P

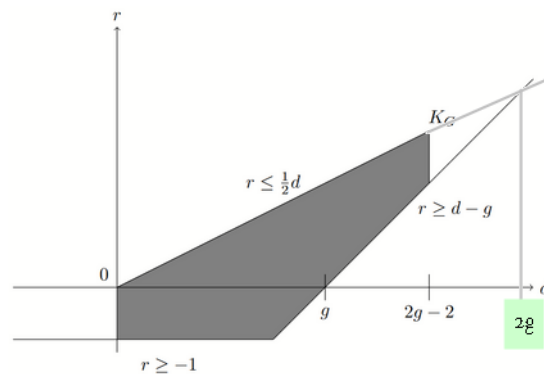


FIGURE 1. Possibilities for the degree and rank of a divisor.

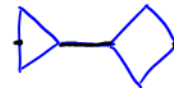
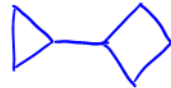
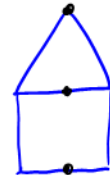
A tropical version for analog. I could draw it also for the classical AG cases, but I'm lazy.
Photo comes from [MA 764: Chip Firing, Lec 9]:
<https://www.ms.uky.edu/~dhje223/MA%20764%20Spring%202019.html>

A case in tropical algebraic curve where the "Weierstrass points are not finite":

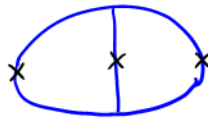
$$g = 2$$



"Weierstrass pts":
 $\text{rk}(z_p) = 1$



comparison between tropical & classical.



See this article for more examples of Weierstrass points on tropical alg curves:

<https://arxiv.org/pdf/2303.07729>

See [Theorem 1.7] which computes the total weight of the Weierstrass locus: $d - r + rg$.

When $D=K$, $d=2g-2$, $r=g-1$, the total weight is g^2-1 .

Notice that the definition of weight is slightly changed.

The Dhar's burning algorithm is mainly used for eliminating negative divisors.

Step1: blow (burn negative divisors)

Step2: suck (attract positive divisors)

This process looks like the process when I suck the river snail, therefore, I call it as "嗦田螺算法". It's an effective algorithm in determining if a divisor is effective.

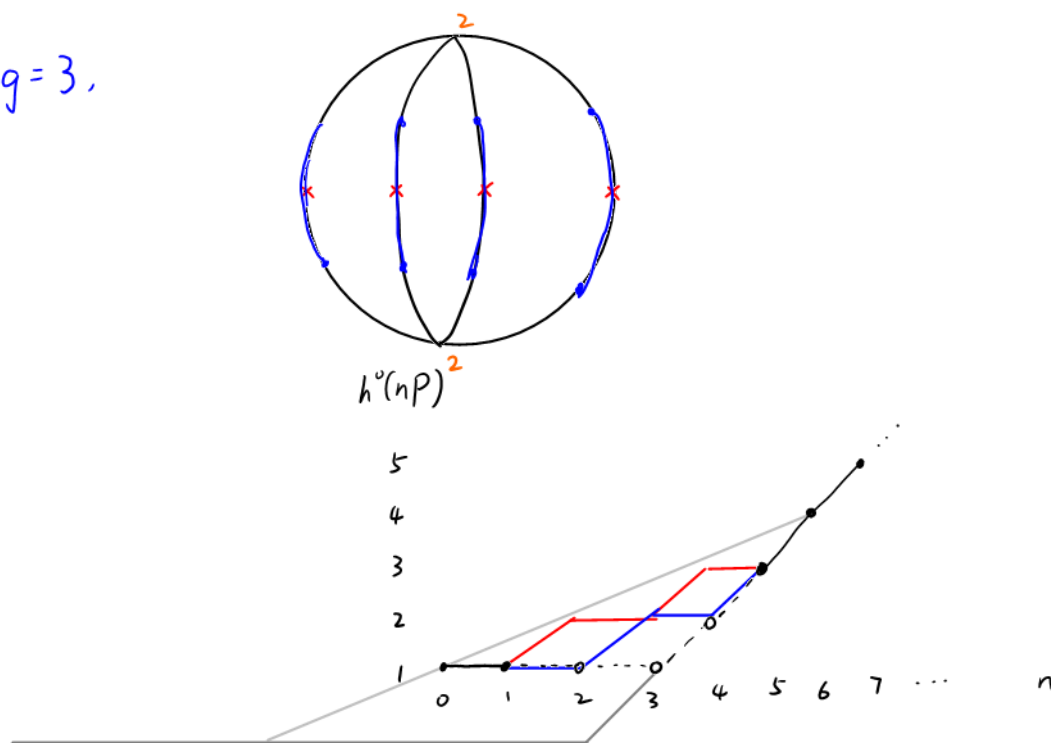
Some differences between classical algebraic curves and tropical algebraic curves:

We have Dhar's burning algorithm for tropical algebraic curves, which is not so explicit in classical case. (Maybe I'm wrong; the hyperelliptic curves can be seen in [Theorem 4.1.6]: <https://algant.eu/documents/theses/dipiazza.pdf>)

We can also divide K into two canonical parts.

In classical algebraic curves, the Weierstrass point is finite, which is not true in tropical algebraic curves.

E.g. $g=3$,



$g=3$ case

By Riemann-Roch,

$$r(kP) = r(K - kP) + k - g + 1$$

$$\underline{\underline{g=3}} \quad \begin{cases} r(K - 2P) & k=2 \\ r(K - 3P) + 1 & k=3 \\ r(K - 4P) + 2 & k=4 \end{cases}$$