

Eine Woche, ein Beispiel

6.19. idempotent algebras.

This document want to discuss some basic contents of the course "<https://people.mpim-bonn.mpg.de/scholze/Complex.pdf>", Lecture 5. For me I've never noticed about this special structure before. Hope that you enjoy this small magic. This can be a perfect series of exam questions for the Algebra III in USTC. (better if they have learned computations on tensor products)

Q: Find all (reduced) \mathbb{Z} -algebra A s.t.
 $A \otimes_{\mathbb{Z}} A \cong A$
 as a \mathbb{Z} -alg iso.

A crash recap on [Vakil 9.2] Skip if you know fiber product of schemes!

Ex. $C \in \text{CRing}$, $A, B \in C\text{-Alg} \Rightarrow A \otimes_C B$ is $C\text{-Alg}$, and

$$\begin{array}{ccc} A \otimes_C B & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & C \end{array}$$

is a pushout.

Let $\phi: B \rightarrow A$ be a ring homomorphism. $I \triangleleft B$ S multiplicative set

9.2.A. (Adding an extra variable) $A \otimes_B B[t] \cong A[t]$

9.2.B (Quotient) $A \otimes_B B/I \cong A/\phi(I)$

9.2.F (Localization) $A \otimes_B S^{-1}B \cong [\phi(S)]^{-1}A$

Ex. Compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

Compute fibers of $\text{Spec } \mathbb{Z}[i] \rightarrow \text{Spec } \mathbb{Z}$

$\text{Spec } \mathbb{C}[x] \rightarrow \text{Spec } \mathbb{C}[y]$
 $x^2 \longleftarrow y$

Definition and some cases

Def. Let $R \in \text{Ring}$. $A \in R\text{-Alg}$ is called **idempotent R -algebra** if
 $A \otimes_R A \cong A$ induced by $A \cong R \otimes_R A \rightarrow A \otimes_R A$
as an R -alg iso.

Ex. Verify that $\mathbb{Z}[\frac{1}{b}]$, \mathbb{F}_p , \mathbb{Q} are idempotent \mathbb{Z} -algebras.
Is \mathbb{F}_p^2 idempotent? Is $\mathbb{Z}/p^2\mathbb{Z}$ idempotent? Is \mathbb{Z}_p idempotent?

A new topology on $\text{Spec } A$

Def. (Constructible topology)

$X \in \text{Spec } A$ is called **constructible closed** if
 $\exists f: \text{Spec } B \rightarrow \text{Spec } A \quad \text{Im } f = X$

Ex. Find all constructible closed subset of $\text{Spec } \mathbb{Z}$

Ex. Find all constructible closed subset of $\text{Spec } \mathbb{C}[X]$

Ex. $\{\text{Zariski closed/open subset}\} \subseteq \{\text{constructible closed set}\}$

Fact. [Condensed, Lec 5 Ex 2] Suppose $R \in \mathbf{CRing}$ is Noetherian. Then

Ex. Verify this for $\text{Spec } \mathbb{Z}$.

What constructible closed subset do they correspond?

$\mathbb{C}((X))$ is not $\mathbb{C}[X]$ -idem algs

$$\mathcal{O}(D) = \left\{ \sum_{i=0}^{+\infty} a_i T^i \mid a_i r^i \rightarrow 0 \quad \forall r < 1 \right\} \subseteq \mathbb{C}[[X]]$$

$$\mathcal{O}(\bar{D}) = \bigcap_{r>1} \left\{ \sum_{i=0}^{+\infty} a_i T^i \mid a_i r^i \rightarrow 0 \right\} \subseteq \mathbb{C}[[X]]$$

Lem. A, A' are idem R -algs. Then $\# \text{Mor}_{R\text{-alg}}(A, A') \leq 1$.

[Proof.

$$R \longrightarrow A \xrightarrow{f} A' \cong R \otimes_R A' \longrightarrow A \otimes_R A'$$
$$\begin{array}{ccccccc} -\otimes_R A': & R \otimes_R A' & \xrightarrow{A \otimes_R A'} & A' \otimes_R A' & \cong & R \otimes_R A' \otimes_R A' & \longrightarrow & A \otimes_R A' \otimes_R A' \\ & \parallel_{A'} & & \parallel_{A'} & & & & \parallel_{A \otimes_R A'} \end{array}$$
$$\Rightarrow A \otimes_R A' \cong A' \text{ doesn't depend on } f, \text{ so } f \text{ given by}$$
$$A \longrightarrow A \otimes_R A' \cong A'$$

is unique.]

Cor. $\{\text{idem } R\text{-algs}\}$ is a poset.

Fact. This order is compatible with constructible topology

(Only consider reduced algs. R is Noetherian.)

 $E_x.$

$$\begin{array}{ccc}
 \{\text{reduced idem } R\text{-algs}\} & \xleftrightarrow{1:1} & \{\text{constructible closed subset of } \text{Spec } R\} \\
 A & \longleftrightarrow & Z \\
 A \otimes_R A' & \longleftrightarrow & Z \cap Z' \\
 \ker [A \oplus A' \rightarrow A \otimes_R A'] & \longleftrightarrow & Z \cup Z' \\
 \varinjlim A_i & \longleftrightarrow & \bigcap Z_i \\
 \text{e.g. } Z \subset Z' \text{ iff } A \otimes_R A' \cong A. & &
 \end{array}$$