Eine Woche, ein Beispiel

8.21 equivariant K-theory of $\mathbb{P}^1$

Be careful about the ring structure here!
Most isomorphisms are not iso as algebras.
I will revise this document after the vacation.

Let us do a simple case over $\mathbb{P}^1$. It can be generlized "easily" to flag variety, but $\mathbb{P}^1$ is the beginning case of study.

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand

$$
\begin{tikzcd}
 & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[rr] \arrow[dl] & & K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[rr] \arrow[dl] & & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[dl] \\
③\ K^{SL_2}(pt) \arrow[rr] \arrow[d] & & ⑤\ K^{SL_2}(\mathbb{P}^1) \arrow[rr] \arrow[d] & & K^{SL_2}(pt) \arrow[d] & \\
②\ K^{B}(pt) \arrow[rr] \arrow[d] & & ⑥\ K^{B}(\mathbb{P}^1) \arrow[rr] \arrow[d] & & K^{B}(pt) \arrow[d] & \\
①\ K(pt) \arrow[rr] & & ④\ K(\mathbb{P}^1) \arrow[rr] & & K(pt) &
\end{tikzcd}
$$

where $SL_2 = SL_{2,\mathbb{C}}$, $\quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq SL_{2,\mathbb{C}}$,

$\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} \cong G/B$. $\quad G \underset{\text{mult}}{\overset{\text{left}}{\curvearrowright}} \mathbb{P}^1 \quad \mathbb{C}^\times \overset{\text{trivial}}{\curvearrowright} \mathbb{P}^1$

maps are pushback & pullout of $\mathbb{P}^1 \longrightarrow pt$.

We want to see

- ring structure, module structure
- Weyl gp action
- relations

e.g. $\quad K^B(X) \cong R(B) \otimes_{R(G)} K^G(X) \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^G(X)$

$(K^B(X))^W \cong K^G(X)$

The correct answer is collected here.

$$
\begin{tikzcd}
K^{SL_2}(pt) \arrow[r] \arrow[d] & K^{SL_2}(\mathbb{P}^1) \arrow[d] \\
K^{B}(pt) \arrow[r] \arrow[d] & K^{B}(\mathbb{P}^1) \arrow[d] \\
K(pt) \arrow[r] & K(\mathbb{P}^1)
\end{tikzcd}
\qquad
\begin{tikzcd}
 & x \arrow[r, mapsto] & z+z^{-1} & \\
x \arrow[d, mapsto] & \mathbb{Z}[x] \arrow[r] \arrow[d] & \mathbb{Z}[z^{\pm 1}] & z \arrow[dd, mapsto] \\
y+y^{-1} & \mathbb{Z}[y^{\pm 1}] \arrow[d] & & \\
 & \mathbb{Z} \arrow[r] & \mathbb{Z}[z]/(z-1)^2 & z
\end{tikzcd}
$$

Notation. For linear alg$_{\mathbb{C}}$ gp G [Ginz, 5.1],

$$K_i^G(X) := K_i(\mathrm{Coh}^G(X)) \qquad K^G(X) := K_0^G(X) \qquad K(X) := K^{\{Id\}}(X)$$

$$R(G) := K^G(pt) = K_0(\mathrm{Coh}^G(pt)) = K_0(\mathrm{Rep}\, G)$$

①-③ e.g. $R(\{Id\}) = \mathbb{Z}$, $R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}]$, $R(SL_2) \cong \mathbb{Z}[x]$, $R(SL_2 \times \mathbb{C}^\times) \cong \mathbb{Z}[x, t^{\pm 1}]$

$\qquad\qquad = \mathbb{Z}[X^*(T)] \qquad\qquad = \mathbb{Z}[X^*(T)]^{W_f}$

Some furthur discussion of $R(SL_2)$.

$R(SL_2) = \bigoplus_{i \in \mathbb{N}_{\geq 0}} \mathbb{C} x_i$ where x_i represents the $(i+1)$-dim irr rep of SL_2.

As an algebra, $R(SL_2) = \mathbb{C}[x]$ where

$1 = x_0$

$x = x_1$

$x^2 = x_2 + 1$

$x^3 = x_3 + 2x_1$

$x^4 = x_4 + 3x_2 + 2$

1	0	1	0	2	0	5	0	14
	1	0	2	0	5	0	14	0
		1	0	3	0	9	0	28
			1	0	4	0	14	0
				1	0	5	0	20
					1	0	6	0
						1	0	7
							1	0
								1

$x_0 = 1$

$x_1 = x$

$x_2 = x^2 - 1$

$x_3 = x^3 - 2x$

$x_4 = x^4 - 3x^2 + 1$

1	0	-1	0	1	0	-1	0	1
	1	0	-2	0	3	0	-4	0
		1	0	-3	0	6	0	-10
			1	0	-4	0	10	0
				1	0	-5	0	15
					1	0	-6	0
						1	0	-7
							1	0
								1

⚠ x represents a v.b of rank 2 here!

Q: How to write down all polynomials in $R(SL_n)$ which represents an irr rep of SL_n?

The important definitions and results in [Ginz, Chap 5] have been collected in this page.
Nothing about cohomology theory is short.

[5.2.5] pullback
[5.2.11] tensor product
[5.2.13] pushforward
[5.2.16] induction
[5.2.18] reduction
[5.2.20] convolution
[5.2.26] duality, pairing
[5.3] specialization
[Thm 5.4.17] Thom iso
[Lemma 5.5.1] cellular fibration
[Thm 5.6.1] Künneth
[5.7.1] Beilinson Resolution
[Thm 5.8.14] Riemann-Roch
[Prop 5.9.3] Devissage
[Rmk 5.11.8] Lefschetz fixed point formula

derived pullback gives an alg homo between K-groups, while
derived pushforward gives a $\mathbb{Z}$-mod homo between K-groups

[LCBE, 2.1.1] ④ $K(\mathbb{P}^1) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}^1}(1) = \mathbb{Z}[z]/(z-1)^2 = \mathbb{Z}[z^{\pm 1}]/(z-1)^2$

∇ z corresponds to $\mathcal{O}_{\mathbb{P}^1}(-1)$ here.

Q: How to compute $K(\mathcal{B})$? $\mathcal{B} = SL_n/B$

[Ginz, (5.2.4)] $\begin{matrix} G_1 \circlearrowright \\ G_2 \circlearrowright_{\text{trivial}} \end{matrix} X \Rightarrow K^{G_1\times G_2}(X) \cong K^{G_1}(X)\otimes_{\mathbb{Z}} R(G_2)$ As an $R(G_2)$-module

e.g. $K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \cong K^{SL_2}(\mathbb{P}^1)\otimes_{\mathbb{Z}} R(\mathbb{C}^\times) \cong K^{SL_2}(\mathbb{P}^1)\otimes_{\mathbb{Z}}\mathbb{Z}[t^{\pm1}]$

$K^{B}(\mathbb{P}^1) \cong K(\mathbb{P}^1)\otimes_{\mathbb{Z}} R(B) \cong K(\mathbb{P}^1)\otimes_{\mathbb{Z}}\mathbb{Z}[y^{\pm1}]$

[Ginz, (5.2.17)]

$$K_i^H(X) \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} K_i^G(G\times_H X)$$

(5) e.g. $K^{SL_2}(\mathbb{P}^1) \cong K^{SL_2}(SL_2\times_B pt) \cong K^B(pt) = R(B) = \mathbb{Z}[z^{\pm1}]$

Task: understand $K^{SL_2}(\mathbb{P}^1)$: How does equivariant SL_2-bundle look like?

Try: Let $\lambda\in X^*(T)$, $\mathbb{C}_\lambda$ denotes for B-equivariant l.b over $\{pt\}$. $[\mathbb{C}_\lambda]\in R(B)$

Step 1. find a local chart of G/B.

$$\begin{tikzcd} & G \arrow[d] & = & G\times\{pt\} \arrow[d] \\ \mathbb{A}^1 \arrow[ur, dashed] \arrow[r] & G/B & = & G\times_B\{pt\} \end{tikzcd}$$

$G\times\mathbb{C}_\lambda$

$\begin{bmatrix}1\\x_2\end{bmatrix} \longmapsto \begin{bmatrix}1&0\\x_2&1\end{bmatrix}$

$\begin{bmatrix}x_1\\1\end{bmatrix} \longmapsto \begin{bmatrix}x_1&-1\\1&0\end{bmatrix}$

$G\overset{B}{\times}\mathbb{C}_\lambda$

$$\begin{tikzcd} G\circlearrowright G \circlearrowleft B \arrow[d] & = & G\circlearrowright G \overset{B}{\times}\{pt\} \arrow[d] \\ G\circlearrowright G/B & = & G\circlearrowright G\times_B\{pt\} \end{tikzcd}$$

G: left multiplication

$B \circlearrowright G\times\mathbb{C}_\lambda$:

$b(g,x) = (gb^{-1}, bx)$

Step 2. See the type of l.b over G/B. $(K^G(\mathbb{P}^1)\to K(\mathbb{P}^1))$

$$\begin{tikzcd} \mathbb{A}^1 \arrow[dr] & \\ & G/B \\ \mathbb{A}^1 \arrow[ur] & \end{tikzcd}$$

$(x_1,s_1) \searrow \left(\begin{bmatrix}x_1&-1\\1&0\end{bmatrix}, s_1\right) = \left(\begin{bmatrix}1&0\\\frac{1}{x_1}&1\end{bmatrix}, \begin{bmatrix}x_1&-1\\0&\frac{1}{x_1}\end{bmatrix}.s_1\right)$

$(x_2,s_2) \nearrow \left(\begin{bmatrix}1&0\\x_2&1\end{bmatrix}, s_2\right)$

$\therefore\ s_2 = \begin{bmatrix}x_1&-1\\0&\frac{1}{x_1}\end{bmatrix}.s_1 = \lambda(x_1)s_1$ $\quad\underline{\underline{\lambda = z: \begin{bmatrix}t&0\\0&t^{-1}\end{bmatrix}\mapsto t}}\quad x_1 s_1$

$\therefore\ K^G(\mathbb{P}^1) \longrightarrow K(\mathbb{P}^1)$

$z \longmapsto \mathcal{O}_{\mathbb{P}^1}(-1)$

Step 3. See the G-action on l.b. $g = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in G$

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r] \arrow[d, dashed, "g\cdot"] & G \arrow[d, "g\cdot"] \\ \mathbb{A}^1 \arrow[r] & G \end{tikzcd}$$

$(x_1,s_1) \longmapsto \left(\begin{bmatrix}x_1&-1\\1&0\end{bmatrix}, s_1\right)$

$\downarrow$

$\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}x_1&-1\\1&0\end{bmatrix}, s_1\right)$

$\shortparallel$

$\left(\begin{bmatrix}g(x)&-1\\1&0\end{bmatrix}, \begin{bmatrix}cx+d&-d\\0&(cx+d)^{-1}\end{bmatrix}.s_1\right)$

$(g(x_1), \lambda(cx+d)s_1)$

$\shortparallel$

$(g(x_1), \lambda(j(g,x))s_1)$

Q: What is the $R(SL_2)$-module structure on $K^{SL_2}(\mathbb{P}^1)$?

$$\times \cdot - : \mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z^{\pm 1}]$$

My answer: the action is induced by $R(SL_2) \longrightarrow R(B) \circlearrowright K^{SL_2}(\mathbb{P}^1)$

$$\begin{tikzcd}
\times \arrow[d, mapsto] & R(SL_2) \arrow[d] & \times & K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d, phantom, "\cong"] & K^{SL_2}(\mathbb{P}^1) \arrow[d, phantom, "\cong"] \\
z+z^{-1} & R(B) & \times & R(B) \arrow[r, "\text{multi}"] & R(B)
\end{tikzcd}$$

so

$$\begin{aligned} \times \cdot - \quad \mathbb{Z}[z^{\pm 1}] &\longrightarrow \mathbb{Z}[z^{\pm 1}] \\ f &\longmapsto (z+z^{-1}) \cdot f \end{aligned}$$

In conclusion, we get

$$
\begin{tikzcd}
K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[d] \\
K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2}(pt) \arrow[d] \\
K^{B}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{B}(pt) \arrow[d] \\
K(\mathbb{P}^1) \arrow[r] & K(pt)
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[d, mapsto] & \mathbb{Z}[z^{\pm1}, t^{\pm1}] \arrow[r] \arrow[d] & \mathbb{Z}[x, t^{\pm1}] \arrow[d] & t \arrow[d, mapsto] \\
1 & \mathbb{Z}[z^{\pm1}] \arrow[r] \arrow[d] & \mathbb{Z}[x] \arrow[d] & 1 \\
y \arrow[d, mapsto] & \mathbb{Z}[z, y^{\pm1}]/(z-1)^2 \arrow[r] \arrow[d] & \mathbb{Z}[y^{\pm1}] \arrow[d] & y \arrow[d, mapsto] \\
1 & \mathbb{Z}[z]/(z-1)^2 \arrow[r] & \mathbb{Z} & 1 \\
 & z \arrow[r, mapsto] & 1 &
\end{tikzcd}
$$

The difficult part is the middle square.

Down: $\mathbb{Z}[z, y^{\pm1}]/(z-1)^2 \longrightarrow \mathbb{Z}[y^{\pm1}]$

$$
\begin{aligned}
z &\longmapsto 1 \\
y &\longmapsto y \\
y^{-1} &\longmapsto y^{-1}
\end{aligned}
$$

Right: by rep theory,

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm1}]$ homo as $\mathbb{Z}$-alg

$$
\begin{aligned}
x_0 &\longmapsto 1 \\
x_1 &\longmapsto y+y^{-1} \\
x_2 &\longmapsto y^2+1+y^{-2} \\
x_3 &\longmapsto y^3+y+y^{-1}+y^{-3} \\
&\ \vdots \qquad\qquad \vdots
\end{aligned}
$$

Up: by Borel-Weil-Bott theorem,

$\mathbb{Z}[z^{\pm1}] \longrightarrow \mathbb{Z}[x]$ homo as $\mathbb{Z}[x]$-module

$$
\begin{aligned}
1 &\longmapsto 1 & \qquad z &\longmapsto 0 \\
z^{-1} &\longmapsto x_1 & z^2 &\longmapsto -1 \\
z^{-2} &\longmapsto x_2 & z^3 &\longmapsto -x_1 \\
z^{-3} &\longmapsto x_3 & z^4 &\longmapsto -x_2 \\
\vdots &\qquad \vdots & z^5 &\longmapsto -x_3 \\
& & \vdots &\qquad \vdots
\end{aligned}
$$

Left: by [LW-BWB, Ex 2.6], $\mathcal{L}_n \cong \mathcal{O}(-n)$, combined with "Up", we get

$\mathbb{Z}[z^{\pm1}] \longrightarrow \mathbb{Z}[z, y^{\pm1}]/(z-1)^2$

e.g. $z^3 \longmapsto -z^3(y+y^{-1})$ (see table below)

z	z^{-2}	z^{-1}	1	z	z^2	z^3	z^4	z^m
x	x_2	x_1	1	0	-1	$-x_1$	$-x_2$	$\begin{cases} -x_{m-2} & m\geq 2 \\ 0 & m=1 \\ x_{-m} & m\leq 0 \end{cases}$
y	y^2+1+y^{-2}	$y+y^{-1}$	1	0	-1	$-y-y^{-1}$	$-y^2-1-y^{-2}$	$-\dfrac{y^m-y^{-m+2}}{y^2-1}$

Under these (natural) ring structure,

$$\mathbb{Z}[x, t^{\pm 1}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm 1}] \longrightarrow \mathbb{Z}$$

are homo of rings.

Ex. Generalize to

- $SL_2 \rightsquigarrow SL_n$, $\mathbb{P}^1 \rightsquigarrow \mathcal{F}lag(\mathbb{C}^n)$
- $SL_2 \rightsquigarrow GL_2$
- $\mathbb{C} \rightsquigarrow \mathbb{F}_p$ $\mathbb{C}^\times \rightsquigarrow \mathbb{F}_p^\times$

Q: How to compute $K_i^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1)$ for $i \geq 1$?