

# Eine Woche, ein Beispiel

## 11.19. Basic sheaf calculation

Goal: Motivate  $f_*$ ,  $f^*$ ,  $f_!$ ,  $f'$ , by connecting them with (co)homology theory

- After story:
- ~> calculation of  $\text{Perv}_{\Delta}(\text{CIP}')$
  - ~> generalize Morse theory
  - ~> Characteristic classes / cycles
  - ~> index theorem

Minor advantages from my talk:

- offers examples for derived category.  
(more geometrical compared with examples about quiver reps)
- the first step toward 6-factor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)  
~> understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...  
covering: (étale) descent, ramification, ...

Three types: closed immersion, submersion, covering.

Usual setting:  $X \in \text{Top}$

$\text{Ob}(\text{Sh}(X)) = \{\text{sheaves of abelian qps}\}$

e.g.  $\text{Sh}(f_*) = \text{Abel}$

$\mathbb{Q}_{\text{perf}} \longleftrightarrow \mathbb{Q}$

In this document, we assume  $H^k(X; \mathbb{Q}) = H_{\text{sing}}^k(X; \mathbb{Q})$  in the exercises,  
and assume readers are familiar with calculations of  $H_{\text{sing}}^k(X; \mathbb{Q})$ .

0. sheaf

- 1.  $f_*$ , skyscraper sheaf & global sections
- 2.  $f^*$ , constant sheaf & stalks
- 3.  $Rf_*$  & cohomology
- 4.  $f_!$  & global sections with cpt supp
- 5.  $Rf_!$  & cohomology with cpt supp
- 6.  $f'$  & homology
  - $\otimes$  - & product structure on cohomology
  - $\text{Hom}(-, -)$  & Poincaré duality.

Ref: (also suited for [24.02.04])

- [Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016
- [IHPS] Laurențiu G. Maxim, Intersection Homology & Perverse Sheaves with Applications to Singularities, 2019
- [BI86] Birger Iversen, Cohomology of Sheaves, 1986  
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<https://people.math.wisc.edu/~lmaxim/vanishing.pdf>
- [MS21]:Laurențiu G. Maxim, Jörg Schürmann, Constructible sheaf complexes in complex geometry and Applications  
<https://arxiv.org/abs/2105.13069>
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- [Cl17]Dustin Clausen. Algebraic de Rham Cohomology. <https://sites.google.com/view/algebraicderham/home>.
- [KS90] Masaki Kashiwara, Pierre Schapira. Sheaves on Manifolds. With a Short History “Les Débuts de La Théorie Des Faisceaux” by Christian Houzel. Grundlehren Math. Wiss. Berlin etc.: Springer-Verlag, 1990.  
  
<https://mathoverflow.net/questions/32689/how-should-a-homotopy-theorist-think-about-sheaf-cohomology>

## 0. Sheaf

Recall the definition of

- |                  |   |
|------------------|---|
| • presheaf       | $\mathcal{F}$                                     |
| • sheaf          | $\mathcal{F}$                                     |
| • stalk          | $\mathcal{F}_x$                                   |
| • global section | $\Gamma(X; \mathcal{F}) = H^0(X; \mathcal{F})$    |
| • cohomology     | $R^n\Gamma(X; \mathcal{F}) = H^n(X; \mathcal{F})$ |

<https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology>  
If  $X$  is paracompact and Hausdorff, Čech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

<https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology>

<https://math.stackexchange.com/questions/3305512/cech-cohomology-and-the-simplicial-cohomology-of-the-nerve-of-an-open-cover>

Recall examples of sheaves:

- complicated
- |   |
|---|
| • $\mathcal{E}_X$ : sheaf of cont fcts on $X$                                   |
| • $\mathcal{O}_X$ : structure sheaf on $X$ e.g., $X$ : (cplx) mfld, scheme, ... |
| • $\underline{\mathbb{Q}}_X$ : constant sheaf on $X$                            |
| • $\text{sky}_p(\mathbb{Q})$ : skyscraper sheaf of $p \in X$ on $X$ .           |

Ex. For  $X = \mathbb{C}$  as cplx mfld,  $x=0$ , compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{E}_X)_x \quad \& \quad (\text{sky}_p(\mathbb{Q}))_x.$$

1.  $f_*$ , skyscraper sheaf & global sections

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

Def.  $f_* \mathcal{F} \in \text{Sh}(X)$  is given by

$$f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

This defines a factor

$$f_*: \text{Sh}(Y) \longrightarrow \text{Sh}(X)$$

$$\begin{array}{ccc} \mathcal{F} & & f_* \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ | & & | \\ U & & U \end{array}$$

E.g. For  $p \in X$ ,  $i_p: \{p\} \hookrightarrow X$ ,  $i_{p*} \underline{\mathbb{Q}}_{\{p\}} = \text{sky}_p(\mathbb{Q})$   
 For  $\pi: Y \rightarrow \{*\}$ ,  $\pi_* \mathcal{F} = \mathcal{F}(Y) = \Gamma(Y; \mathcal{F})$

Ex (hard?)

For  $j: \mathbb{C} \rightarrow \mathbb{CP}^1$ , compute  $j_* \underline{\mathbb{Q}}_{\mathbb{C}}$ .

- It is a constant sheaf on  $\mathbb{CP}^1$ .
- It is not a constant sheaf on  $\mathbb{CP}^1$ , and  $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$ .
- It is not a constant sheaf on  $\mathbb{CP}^1$ , and  $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$ .
- All the above is wrong.
- I don't know, but I don't want to make a wrong choice.

2.  $f^*$ , constant sheaf & stalks

In [Vakil, Chapter 2], it is  $f^{-1}$ , the inverse image functor.

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(X)$ ,  $f: Y \rightarrow X$  cont

Def.  $f^*\mathcal{F} \in \text{Sh}(Y)$  is given by sheafification of

$$f^{*,\text{pre}}\mathcal{F}(U) = \varinjlim_{f(u) \in V} \mathcal{F}(V)$$

This defines a factor

$$f^*: \text{Sh}(X) \longrightarrow \text{Sh}(Y)$$

$$\begin{array}{ccc} f^*\mathcal{F} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ \cup & & \cup \\ U & & \mathcal{U} \end{array}$$

Recall:

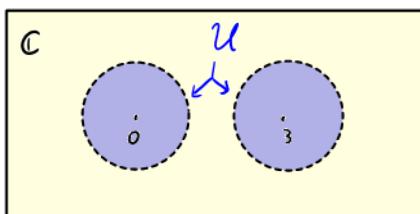
$$\mathcal{F}^{\text{sh}}(U) = \left\{ (x_p)_p \in \prod_{p \in U} \mathcal{F}_p \mid \begin{array}{l} \forall x_0 \in U, \exists U_{x_0} \subseteq U \text{ nbhd of } x_0, \\ s \in \mathcal{F}(U) \text{ st. } \\ s_p = x_p \quad \forall p \in U_{x_0} \end{array} \right\}$$

By definition,  $(\mathcal{F}^{\text{sh}})_p = \mathcal{F}_p$ .

Universal property:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{fcMor}_{\text{PSH}}} & \mathcal{G} \\ \text{sh} \downarrow & \text{G} \nearrow & \mathcal{G} \text{ : sheaf} \\ \mathcal{F}^{\text{sh}} & \dashv \exists! f^{\text{sh}} \in \text{Mor}_{\text{Sh}} & \end{array}$$

Ex. For  $\pi: \mathbb{C} \rightarrow \{\ast\}$ ,  $U = B_1(0) \cup B_1(3)$ , which one is correct:



$$\downarrow \pi$$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2.$

All the above is wrong.

E.g. For  $p \in X$ ,  $\iota_p : \{p\} \hookrightarrow X$ ,  $\iota_p^* \mathcal{F} = \mathcal{F}_p$   
 For  $\pi : Y \rightarrow \{\ast\}$ ,  $\pi^* \underline{\mathcal{Q}}_{\{\ast\}} = \underline{\mathcal{Q}}_Y$   
 For  $U \subset X$  open,  $j : U \hookrightarrow X$ ,  $j^* \mathcal{F} = \mathcal{F}|_U$

People generalize the last notation to arbitrary subset:  
 For  $Y \subset X$ ,  $\iota_Y : Y \hookrightarrow X$ ,  $\iota_Y^* \mathcal{F} = \mathcal{F}|_Y$

Q: For  $U \subset X$  open, how to express  $\mathcal{F}(U)$  by factors?

A:

$$\mathcal{U} \xrightarrow{\text{inj}} X$$

$$\pi_{\mathcal{U}} \downarrow \quad \downarrow \pi_X$$

$\{*\}$

$$\mathcal{F}(\mathcal{U}) = \pi_{\mathcal{U}, *} \underbrace{\iota^* \mathcal{F}}_{\mathcal{F}_{\mathcal{U}}}$$

Prop. One has the adjunction  $f^* \dashv f_*$ , i.e.,

$$\begin{array}{ccc} G & & F \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

$$\text{Mor}_{\text{Sh}(Y)}(f^*F, G) \cong \text{Mor}_{\text{Sh}(X)}(F, f_*G) \quad + \text{naturality}$$

[Hint. [Vakil, 2.7.B] Show that both side give the same information, i.e.,

$$\phi_{uv} \in \text{Mor}_{\text{Ab}}(F(U), G(V)) \quad \text{for each pair } (V, U) \\ \text{s.t. } f(V) \subset U \\ + \text{compatibility}$$

Cor.  $f^*$  is right exact,  $f_*$  is left exact.

Rmk.  $f^*$  is an exact functor.

Hint: exactness can be checked on stalks!

⚠ After "polished" (because of the structure sheaf),  $f^*$  is again only right adjoint.

Application.

$$\text{Hom}_{\text{Sh}(X; \mathbb{Q})}(\underline{\mathbb{Q}}_X, F) \cong \Gamma(X; F)$$

$$\text{Hom}_{\text{Sh}(X; \mathbb{Q})}(F, \iota_{p,*}\mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}\text{-v.s.}}(F_p, \mathbb{Q}) \cong F_p^*$$

$$\text{Hom}_{\text{Sh}(U; \mathbb{Q})}(F|_U, G) \cong \text{Hom}_{\text{Sh}(X; \mathbb{Q})}(F, \iota_{U,*}G)$$

where  $\text{Sh}(X; \mathbb{Q}) \subset \text{Sh}(X)$ : sheaves of  $\mathbb{Q}$ -v.s.

$\text{Sh}(X; \mathbb{F}_4) \subset \text{Sh}(X; \mathbb{F}_2)$  is not  $f$ .faithful

$$\begin{array}{ccc} G & & F \\ | & & | \\ U & \xrightarrow{\iota_U} & X \end{array}$$

### 3. $Rf_*$ & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input  $\mathcal{F}$ , output a complex (before Ker/Im procedure)
- SES induces LES: for

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

one has

$$\rightarrow H^2(X; \mathcal{F}) \longrightarrow \dots$$

$$\rightarrow H^1(X; \mathcal{F}) \longrightarrow H^1(X; \mathcal{G}) \longrightarrow H^1(X; \mathcal{H}) \rightarrow$$

$$0 \rightarrow H^0(X; \mathcal{F}) \longrightarrow H^0(X; \mathcal{G}) \longrightarrow H^0(X; \mathcal{H}) \rightarrow$$

$$\pi''_{*\mathcal{F}}$$

$$\pi''_{*\mathcal{G}}$$

$$\pi''_{*\mathcal{H}}$$

$$\pi: X \rightarrow \{*\}$$

- can be viewed as right derived factor of

$$H^0(X, -) = \Gamma(X, -) = \pi_*$$

one gets

$$H^n(X, -) = R^n \Gamma(X, -) = R^n \pi_*$$

We denote the complex (before the Ker/Im procedure) as

$$R\Gamma(X, -) = R\pi_*$$

up to homotopy equiv & quasi-iso, i.e., in the derived category of  $\{*\}$ .

$$\begin{aligned} \mathcal{D}(X) = \mathcal{D}(\text{Sh}(X)) &= \text{"derived category of sheaves over } X \text{"} \\ &= \text{"complexes of sheaves over } X, \text{ up to ...}" \\ &= \left\{ \dots \rightarrow \mathcal{F}^{-2} \rightarrow \mathcal{F}^{-1} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots \right\}_{\sim} \hat{=} \{\mathcal{F}^n\} \end{aligned}$$

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

Def.  $Rf_* \mathcal{F}$  = "derived pushforward of  $\mathcal{F}$ "  
 $= f_* \mathcal{I}'$

[Here,  $\mathcal{I}'$  is the injective resolution of  $\mathcal{F}$ :  
 $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$   
 $(\Rightarrow \mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}')$ ]

$$\begin{array}{ccc} \mathcal{F} & Rf_* \mathcal{F} \\ | & | \\ Y & \xrightarrow{f} & X \\ \cup & & \cup \\ U & & U \end{array}$$

This defines a functor

$$Rf_*: \mathcal{D}^+(Y) \longrightarrow \mathcal{D}^+(X)$$

in fact, can be  $\mathcal{D}(Y) \rightarrow \mathcal{D}(X)$   
(nontrivial)

The derived pushforward is hard to compute.

just like cohomology, and even worse, since we need more information  
luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]

$$R^n f_* \mathcal{F} \text{ is given by the sheafification of } (R^n f_* \mathcal{F})^{\text{pre}}(U) = H^n(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

$\hookleftarrow$  sometimes omit

e.p. one can compute the stalk

$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in U} H^n(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

$\mathcal{F}$

|

Cor For  $\pi: X \rightarrow \mathbb{P}_*$ ,

$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For  $\pi: \mathbb{CP}^1 \rightarrow \mathbb{P}_*$ ,

$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in  $\mathcal{D}(*)$  "split", we work over  $\mathbb{Q}$ ]

$$R \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= \left[ \underset{-1}{0} \rightarrow \dots \rightarrow \underset{0}{\mathbb{Q}} \rightarrow \underset{1}{0} \rightarrow \underset{2}{\mathbb{Q}} \rightarrow \underset{3}{0} \rightarrow \underset{4}{\dots} \right]$$

*Ex.* For  $j : \mathbb{C} \rightarrow \mathbb{CP}^1$ , what is true about  $Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}$ ?

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$

What the hell is that?

In fact,  $(Rj_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1]$ .

$i : \mathbb{P}^1 \rightarrow \mathbb{CP}^1$  is exact, so  $Ri_* = i_*$ .

Upgrade formulas to derived version

$$\begin{aligned} \text{Hom}(f^* \mathcal{F}, G) &\cong \text{Hom}(\mathcal{F}, f_* G) \\ \rightsquigarrow \text{Hom}(f^* \mathcal{F}, G) &\cong \text{Hom}(f^* \mathcal{F}, I) \\ &\cong \text{Hom}(\mathcal{F}, f_* I) \\ &\cong \text{Hom}(\mathcal{F}, Rf_* G) \end{aligned}$$

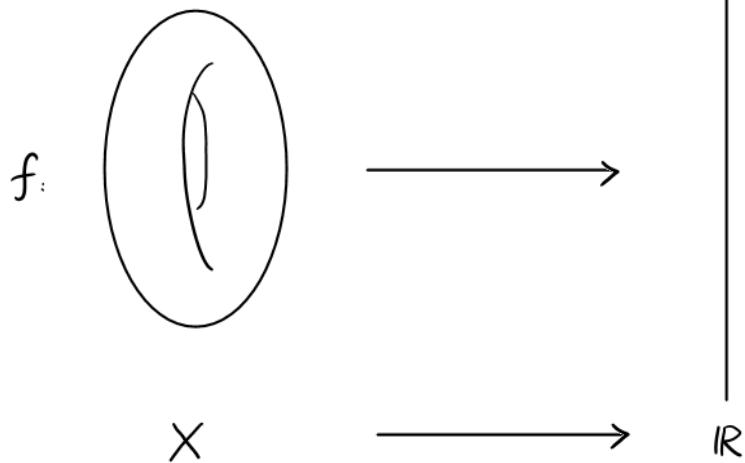
Is this argument correct?

try:  $\text{Hom}_{\mathcal{D}^+(X)}(f^* \mathcal{F}, I) = \text{Hom}_{K^+(X)}(f^* \mathcal{F}, I)$   
 $\cong \text{Hom}_{K^+(X)}(\mathcal{F}, f_* I)$

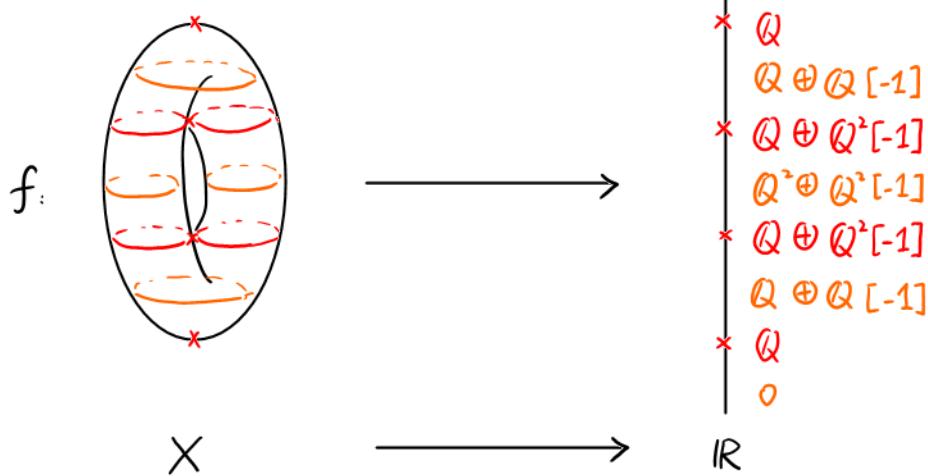
$f^* g_! \cong g'_! f'^*$        $\xrightarrow{f^*, f'^* \text{ exact}} f^* Rg_! \cong Rg'_! f'^*$

Can we upgrade the projection formula  $\& f^*(- \otimes -)$  to derived version?

Ex. Try to compute  $Rf_* \underline{\mathbb{Q}}_X$ , where



A:



Hint: You may need to use the flat base change:

$$R\pi_* \widehat{f}^* \cong f^* R\pi_{F,*}$$

$$\Rightarrow \pi_* \widehat{f}^* \cong f^* \pi_{F,*}$$

$$R^i \pi_{F,*} (\underline{\mathbb{Q}} \boxtimes F) \cong \underline{\mathbb{Q}}_B \otimes_{\underline{\mathbb{Q}}} H^i(F, F)$$

$$\begin{array}{ccc} B \times F & \xrightarrow{\widehat{f}} & F \\ \pi \downarrow & & \downarrow \pi_F \\ B & \xrightarrow{f \text{ flat}} & pt \end{array}$$

4.  $f_!$ , extension by zeros & global sections with cpt supp

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

$X, Y \in \text{Haus}^{\text{loc cpt}}$  of fin coh dim [IHPS, P81]  $\rightsquigarrow f_! \mathcal{F}$  is a sheaf  
+ base change

$$\begin{array}{ccc} \mathcal{F} & & f_! \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \downarrow \\ & & U \\ & & \downarrow \\ & & U \end{array}$$

Def.  $f_! \mathcal{F} \in \text{Sh}(X)$  is given by

$$f_! \mathcal{F}(U) = \left\{ s \in \mathcal{F}(f^{-1}(U)) \mid \begin{array}{l} f|_{\text{supp}(s)}: \text{supp}(s) \rightarrow U \text{ is proper} \\ (f_* \mathcal{F})(U) \end{array} \right\}$$

This defines a functor

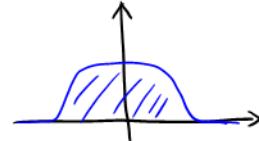
$$f_!: \text{Sh}(Y) \rightarrow \text{Sh}(X)$$

$$\text{Recall: } \text{supp}(s) = \overline{\{x \in f^{-1}(U) \mid s_x \neq 0\}} = \{x \in f^{-1}(U) \mid s_x \neq 0\}$$

proper: preimage of cpt set is cpt. (in nice setting)

Rmk. By def.  $(f_! \mathcal{F})(U) \subseteq (f_* \mathcal{F})(U)$ , one has natural transformation  $f_! \rightarrow f_*$ .  
When  $f$  is proper,  $f_! = f_*$ .

E.g. For  $p \in X$ ,  $\iota_p: \{p\} \hookrightarrow X$ ,  $\iota_{p,!} \underline{\mathbb{Q}}_{\{p\}} = \iota_{p,*} \underline{\mathbb{Q}}_{\{p\}} = \text{sky}_p(\mathbb{Q})$   
For  $\pi: Y \rightarrow \{*\}$ ,  $\pi_! \mathcal{F} = \Gamma_c(Y, \mathcal{F}) = H^0_c(Y, \mathcal{F})$   
 $\uparrow$  cpt supp facts on  $Y$



Ex.

Do you know what is  $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}})$  and  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1})$ ?

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$ .

Could you explain the notation again?

E.g. 4.3. For  $U \xrightarrow{f} X$  open,  $j_! F$  is the classical "extension by zero":

$$(j_! F)^{\text{pre}}(V) = \begin{cases} F(V) & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

e.p.  $(j_! F)_p = \begin{cases} F_p & p \in U \\ 0 & p \notin U \end{cases}$

In general, [IHPS, p82] [KS90, Prop 2.5.2]

$$(f_! F)_p = \Gamma_c(f^{-1}(p); F|_{f^{-1}(p)})$$

This comes from the proper base change formula:

$$l_p^* f_! F \cong \pi_* \tilde{l}_p^* F$$

For the proof, see [KS90, Prop 2.5.11].

$$\begin{array}{ccc} f^{-1}(p) & \xrightarrow{\tilde{l}_p} & Y^F \\ \pi \downarrow & \perp & \downarrow f \\ \{p\} & \xrightarrow{l_p} & X \end{array}$$

Rmk. In E.g. 4.3,  $j_!$  is exact. (Check the stalks!)

In general,  $f_!$  is only left exact.

e.p. when  $f: Y \rightarrow X$  is proper, then  $f_! = f_*$  is usually not right exact.  
Notice that  $Rf_! \dashv f^!$ , and we don't have  $f_! \dashv f^!$ .

<https://math.stackexchange.com/questions/3132036/direct-image-functor-f-left-exact>  
the same method here argues why  $f_!$  is left exact.

Sidemark:

<https://math.stackexchange.com/questions/4671873/compare-two-definition-of-rf-derived-pushforward-with-proper-support>  
it gives another definition of  $f_!$  in étale case.

<https://math.stackexchange.com/questions/842473/soft-sheaves-adapted-to-f>

## 5. $Rf_!$ & cohomology with cpt supp

Just like  $Rf_*$ , we derive the factor

$$H_c^0(X, -) = \Gamma_c(X, -) = \pi_!$$

to get

$$H_c^n(X, -) = R^n \Gamma_c(X, -) = R^n \pi_!$$

$X$   
 $\downarrow \pi$   
 $\{*\}$

Def.  $Rf_! \mathcal{F} =$  "derived proper pushforward of  $\mathcal{F}$ "

$$= f_! \mathcal{I}^*$$

Here,  $\mathcal{I}^*$  is the injective resolution of  $\mathcal{F}$ .

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$$

(⇒  $\mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}^*$ )

$$\begin{array}{ccc} \mathcal{F} & Rf_! \mathcal{F} \\ | & | \\ Y & \xrightarrow{f} & X \\ \cup & & \cup \\ U & & U \end{array}$$

This defines a factor

$$Rf_! : \mathcal{D}^+(Y) \longrightarrow \mathcal{D}^+(X)$$

$\mathcal{F}$

|

Cor For  $\pi: X \rightarrow \{*\}$ ,

$$R^n \pi_! \mathcal{F} = H_c^n(X; \mathcal{F})$$

E.g. For  $\pi: \mathbb{C}\mathbb{P}^1 \rightarrow \{*\}$ ,

$$R^n \pi_! \underline{\mathbb{Q}}_{\mathbb{C}\mathbb{P}^1} = H_c^n(\mathbb{C}\mathbb{P}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in  $\mathcal{D}(*)$  are proj, we work over  $\mathbb{Q}$ ]

$$R \pi_! \underline{\mathbb{Q}}_{\mathbb{C}\mathbb{P}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= \left[ \underset{-1}{0} \rightarrow \dots \rightarrow \underset{0}{\mathbb{Q}} \rightarrow \underset{1}{0} \rightarrow \underset{2}{\mathbb{Q}} \rightarrow \underset{3}{0} \rightarrow \underset{4}{\mathbb{Q}} \rightarrow \dots \right]$$

$\mathbb{C}\mathbb{P}^1 \rightsquigarrow \mathbb{C}$ , what would happen?

For  $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$ , what is true about  $Rj_! \underline{\mathbb{Q}}_{\mathbb{C}}$ ?

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}.$

This question is too easy for me. Ask more difficult questions next time!

In fact,  $j_!$  is exact, so  $(Rj_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = (j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0$ .

## 6. $f^!$ & homology

The correct title should actually be:  
 $f^!$ , orientation sheaf & costalks  
 homology is just some special (sheaf) cohomology.

Now we come to the hard part, since  $f^!$  can only be defined over the derived category. That's why there's no derived symbol for  $f^!$ .

"Def"  $f^!$ , if exists, should be given by  
 the right adjoint of  $Rf_!$ :

$$Rf_! \dashv f^!$$

Remember:

$$\begin{aligned} f^* &\dashv Rf_* \\ Rf_! &\dashv Rf^! \\ M \otimes - &\dashv R\text{Hom}(M, -) \end{aligned}$$

$Rf^!$ : in some cases

Hard exercise: for  $\pi: \mathbb{R}^n \rightarrow \{\ast\}$ , shows that  
 $\pi^! \underline{\mathbb{Q}} \cong \underline{\mathbb{Q}}_{\mathbb{R}^n}[n]$

by using the def of  $\pi^!$ .

Black box: (reduced from the Hard exercise)

For  $M$  n-dim mfld (without boundary),  $\pi: M \rightarrow \{\ast\}$ ,

$$\pi^! \underline{\mathbb{Q}} \cong \mathcal{O}_M[n] \quad \mathcal{O}_M: \underline{\mathbb{Q}}\text{-orientation sheaf of } M$$

e.g. when  $M$  is oriented, fixing an orientation  $\mathcal{O}_M \cong \underline{\mathbb{Q}}_M$ , one gets

$$\pi^! \underline{\mathbb{Q}} \cong \underline{\mathbb{Q}}_M[n].$$

What does the adjunction tell us?

<https://mathoverflow.net/questions/404706/how-duality-follows-from-a-six-functor-formalism>

Poincaré duality, version 1

$$\underline{\text{Hom}}_{\mathcal{D}^+(\mathbb{Y}, \mathbb{Q})}(Rf_! \mathcal{F}, G) \cong \underline{Rf_*} \underline{\text{Hom}}_{\mathcal{D}^+(X, \mathbb{Q})}(\mathcal{F}, f^! G)$$

$$\begin{array}{ccc} \mathcal{F} & & G \\ | & & | \\ X & \xrightarrow{f} & Y \end{array}$$

Now, take  $\mathbb{Y} = \{\ast\}$ ,  $G = \mathbb{Q}$ ,  $X$ : n-dim mfld, one gets

$$\begin{aligned} \underline{\text{Hom}}_{\mathcal{D}^+(\ast, \mathbb{Q})}(R\Gamma_c(X; \mathcal{F}), \mathbb{Q}) &\cong \underline{Rf_*} \underline{\text{Hom}}_{\mathcal{D}^+(X, \mathbb{Q})}(\mathcal{F}, \mathcal{O}_{X,n}) \\ R\Gamma_c(X; \mathcal{F})^* &\cong \underline{Rf_*} \underline{\text{Hom}}_{\mathcal{D}^+(X, \mathbb{Q})}(\mathcal{F}, \mathcal{O}_{X,n})[n] \end{aligned}$$

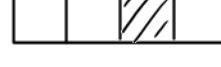
Take  $\mathcal{F} = \mathcal{O}_X$ , one gets

$$R\Gamma_c(X; \mathbb{Q})^* \cong R\Gamma(X; \mathcal{O}_X)[n]$$

Take  $(-i)$ 's cohomology, one gets

$$H_c^i(X; \mathbb{Q})^* \cong H^{n-i}(X; \mathcal{O}_X)$$

E.g.  $n=3$   $i=1$   $X=S^2 \times \mathbb{R}$

	$R\Gamma_c(X; \mathcal{O}_X)^*$	$H^{-i}(R\Gamma_c(X; \mathcal{O}_X)^*)$
	$R\Gamma_c(X; \mathcal{O}_X)$	$H_c^i(X; \mathcal{O}_X)^*$ IS
	$R\Gamma(X; \mathbb{Q})$	$H^{n-i}(X; \mathbb{Q})$ IS PD
	$R\Gamma(X; \mathbb{Q})[n]$	$H^{-i}(R\Gamma(X; \mathbb{Q})[n])$
-3 -2 -1 0 1 2 3		

An illusion for index shift

Take  $\mathcal{F} = \mathcal{O}_X$ , one gets

$$\begin{aligned} R\Gamma_c(X; \mathcal{O}_X)^* &\cong \underline{\text{Hom}}_{\mathcal{D}(X, \mathbb{Q})}(\mathcal{O}_X, \mathcal{O}_X)[n] \\ &\cong \underline{\text{Hom}}_{\mathcal{D}(X, \mathbb{Q})}(\mathbb{Q}, \mathcal{O}_X^\vee \otimes \mathcal{O}_X)[n] \\ &\cong R\Gamma(X; \mathbb{Q})[n] \end{aligned}$$

Take  $(-i)$ 's cohomology, one gets

$$(H_c^i(X; \mathcal{O}_X))^* \cong H^{n-i}(X; \mathbb{Q})$$

These are the Poincaré duality.

Rmk. 1. When  $X$  is cpt, one gets  $R\pi_! = R\pi_*$ ,  $H^i = H_c^i$ ,  
 $(H^i(X; \mathbb{Q}))^{**} \cong (H^{n-i}(X; \mathcal{O}_{\nu_X}))^* \cong H^i(X; \mathbb{Q})$   
So  $\dim_{\mathbb{Q}} H^i(X; \mathbb{Q}) < +\infty$ .

<https://math.stackexchange.com/questions/35779/what-can-be-said-about-the-dual-space-of-an-infinite-dimensional-real-vector-spa>

$H^i(X; \mathbb{Q})$  has an upper bound given by the triangulation of  $X$ .

2. In fact, these isos come from the non-degenerate  
bilinear pairing maps (why?)

$$\begin{aligned} U: H_c^i(X; \mathbb{Q}) \times H^{n-i}(X; \mathcal{O}_{\nu_X}) &\longrightarrow H_c^n(X; \mathcal{O}_{\nu_X}) \cong \bigoplus_{\text{comp}} \mathbb{Q} \xrightarrow{\text{sum}} \mathbb{Q} \\ U: H^i(X; \mathbb{Q}) \times H_c^{n-i}(X; \mathcal{O}_{\nu_X}) &\longrightarrow H_c^n(X; \mathcal{O}_{\nu_X}) \cong \bigoplus_{\text{comp}} \mathbb{Q} \xrightarrow{\text{sum}} \mathbb{Q} \end{aligned}$$

## Poincaré duality, version 2 & 3

Def. Let  $X$  be a mfld of dim  $n$ , the homology of  $X$  are defined as

$$H_{-i}^{BM}(X; \mathbb{Q}) = (R\pi_* \pi^! \mathbb{Q})_i = H^i(X; \mathcal{O}_X[n]) = H^{n+i}(X; \mathcal{O}_X)$$

$$H_{-i}(X; \mathbb{Q}) = (R\pi_* \pi^! \mathbb{Q})_i = H_c^i(X; \mathcal{O}_X[n]) = H_c^{n+i}(X; \mathcal{O}_X)$$

This is Poincaré duality, version 2.

[https://en.wikipedia.org/wiki/Borel-Moore\\_homology](https://en.wikipedia.org/wiki/Borel-Moore_homology)

Definitions are not equivalent when  $X = \mathbb{Z}$ , but fine for good spaces

<https://mathoverflow.net/questions/277069/what-is-homology-anyway>

<https://mathoverflow.net/questions/249342/two-points-of-view-about-borel-moore-homology>

Ex. Do you know what is  $H_*(C; \mathbb{Q})$  &  $H_*^{BM}(C; \mathbb{Q})$ ?

	-3	-2	-1	0	1	2	3	
A	0	$\mathbb{Q}$	0	0	0	0	0	$R\pi_* \pi^! \mathbb{Q}$
B	0	0	0	$\mathbb{Q}$	0	0	0	$H_*(C; \mathbb{Q})$
C	0	0	0	0	0	$\mathbb{Q}$	0	$H_*^{BM}(C; \mathbb{Q})$

Combining two versions, one gets Poincaré duality, version 3: (UCT)

$$\begin{aligned} (H_{n-i}(X; \mathbb{Q}))^* &\cong H^{n-i}(X; \mathcal{O}_{\mathcal{R}X}) = H_i^{BM}(X; \mathbb{Q}) \\ (H_{n-i}(X; \mathbb{Q}))^* &= (H_c^i(X; \mathcal{O}_{\mathcal{R}X}))^* \cong H^{n-i}(X; \mathbb{Q}) \end{aligned}$$

$$\begin{array}{ccccc} & H^i(X; \mathcal{O}_{\mathcal{R}X}) & & H_c^i(X; \mathcal{O}_{\mathcal{R}X}) & \\ & \parallel & \nearrow n-i,* & \parallel & \\ H^i(X; \mathbb{Q}) & \parallel & \nearrow n-i,* & \parallel & H_c^i(X; \mathbb{Q}) \\ & \parallel & * & \parallel & \\ & H_i^{BM}(X; \mathbb{Q}) & \nearrow n-i & \parallel & H_i(X; \mathbb{Q}) \\ & \parallel & * & \parallel & \\ H_i^{BM}(X; \mathbb{Q}) & \longrightarrow & H_i(X; \mathcal{O}_{\mathcal{R}X}) & & \end{array}$$



- $\leftrightarrow$  : compact
- $\swarrow$  : oriented
- $\Downarrow$  : def ( $n-i$ -shift)

- $\square$  : cohomological guy  
(six functor formalism)
- $\square \rightarrowtail$  : topology guy  
(no sheaf cohomology)

e.g. in oriented mfld case, it becomes

$$\begin{array}{ccc} H^i(X; \mathbb{Q}) & \xrightarrow{n-i,*} & H_c^i(X; \mathbb{Q}) \\ n-i \parallel & & n-i \parallel \\ H_i^{BM}(X; \mathbb{Q}) & & H_i(X; \mathbb{Q}) \end{array}$$

in cpt oriented mfld case, it becomes

$$\begin{array}{c} H^i(X; \mathbb{Q}) \xrightarrow{n-i,*} \\ \parallel n-i \\ H_i(X; \mathbb{Q}) \end{array}$$

## Poincaré duality for non-mflds

Def. For  $X \in \text{Top}$ ,  $\pi: X \rightarrow \{\ast\}$ , if  $\pi^! Q$  is well-defined, then the homology of  $X$  are defined as

$$H_{-}^{BM}(X; Q) = R\pi_* \pi^! Q = H^i(X; \pi^! Q)$$

$$H_{-}(X; Q) = R\pi_* \pi^! Q = H_c^i(X; \pi^! Q)$$

We will compute  $f^{\wedge !}$  for many cases next time.

Remember, when we compute new  $f^{\wedge !}$ 's, we discover new versions of Poincaré duality.

E.g. In [BI86, V.2.8, Example V.2.9., VI 3.2], it claims that

For  $(X, \partial X)$  a mfld with boundary,  $\partial X \xrightarrow{i_{\partial X}} X \xleftarrow{i_u} U$

$U := X - \partial X$ , one has

$$\pi_X^! Q = i_{U,!} \pi_U^! Q$$

With this in hand, one derives the Poincaré-Lefschetz duality:

$$H_c^i(X; Q)^* \cong \underline{H^{n-i}(X, \partial X; i_{U,*} \Omega_U)}$$

*when X is oriented*

$$\underline{H^{n-i}(X, \partial X; Q)}$$

<https://math.stackexchange.com/questions/2212857/poincare-lefschetz-duality-from-poincare-lefschetz-alexander-duality>  
I include this link just for remembering the name of these dualities.