Eine Woche, ein Beispiel
9.4 Hecke algebra

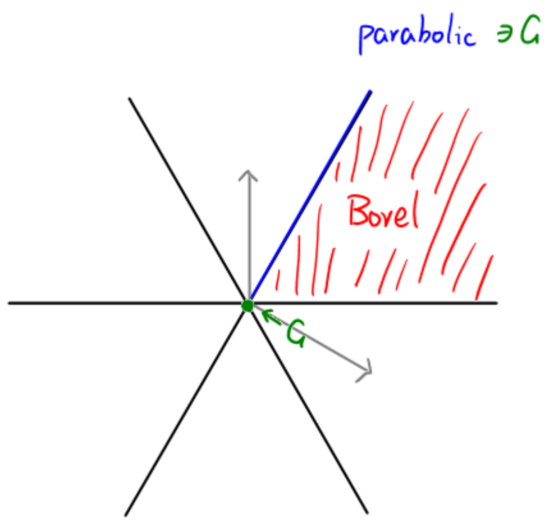
This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]

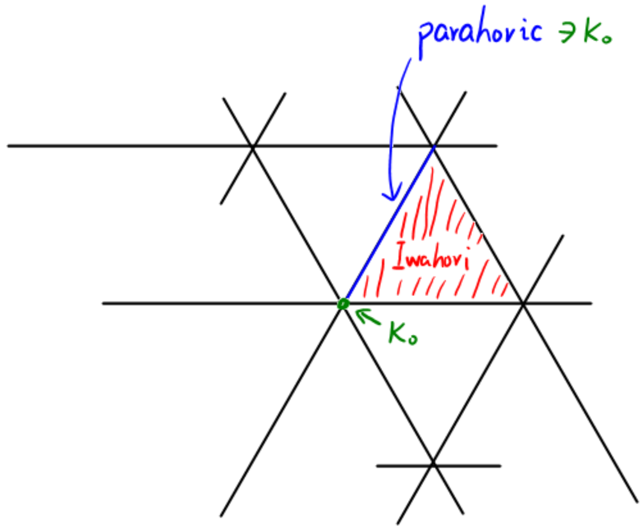
Task. For each double coset decomposition, we want to do:

1. decomposition (& $\Gamma t \Gamma / \Gamma$ is finite)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form
F finite	$G = \bigsqcup_{w \in W} BwB$		
F local	$G = \bigsqcup_{w \in W} BwB$	$G = \bigsqcup_{w \in W_{ext}} IwI$	$G = \bigsqcup_{t \in T^-} K_0 t K_0$
F global	$G = \bigsqcup_{w \in W} BwB$		$GL_2^+(\mathbb{Q}) = \bigsqcup_{t \in T^-} \Gamma t \Gamma$
adèle?			





$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

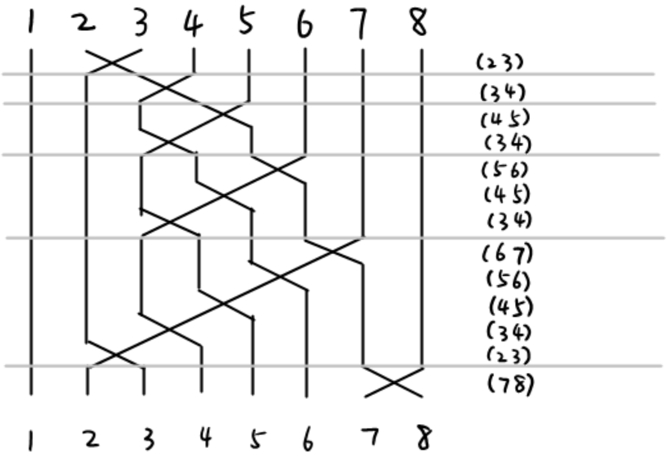
S_n and Tits system

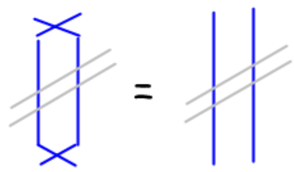
A brief preparation for computations in Bruhat decomposition. $s_i=(i\ i{+}1)$, $1\le i\le n-1$

E.g. $n=8$, $w_0=(287)(46)=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&8&3&6&5&4&2&7\end{pmatrix}\in S_8$.

Ex. Compute $l(w_0)$, $l(s_iw_0)$ and $l(w_0s_i)$.

Solution.





$w_0=(78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0)=13$ = "inversion number"

$l(s_1w_0)=14$ $\quad$ $l(w_0s_1)=14$

$l(s_2w_0)=12$ $\quad$ $l(w_0s_2)=12$

$l(s_3w_0)=14$ $\quad$ $l(w_0s_3)=14$

$l(s_4w_0)=12$ $\quad$ $l(w_0s_4)=12$

$l(s_5w_0)=12$ $\quad$ $l(w_0s_5)=12$

$l(s_6w_0)=12$ $\quad$ $l(w_0s_6)=14$

$l(s_7w_0)=14$ $\quad$ $l(w_0s_7)=12$

Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

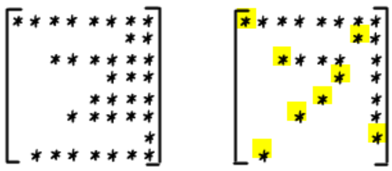
Solution

w_0 $\quad$ Bw_0 $\quad$ w_0B

s_1Bw_0 $\quad$ s_1w_0B

s_2Bw_0 $\quad$ s_2w_0B

The following computation will be also computed later on.



w_0B $\quad$ $Bw_0 \cap w_0B$

$w_0Bw_0^{-1}$ $\quad$ $B \cap w_0Bw_0^{-1}$

finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \coprod_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$

Ex. Compute $|BwB/B|$

Hint: Consider the map

$$\phi : B \longrightarrow BwB/B$$
$$b \longmapsto bwB$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 wB = b_2 wB$$
$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$
$$\Leftrightarrow b_2^{-1} b_1 \in wBw^{-1}$$

$$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{l(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) := \{ f : G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \quad \forall b_1, b_2 \in B, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$