Eine Woche, ein Beispiel

7.9. Steenrod algebra

The Steenrod algebra has been studied for decades. As usual, I make no claim to originality.

For calculating the Steenrod algebra in compute, you can get it from this link:
https://math.berkeley.edu/~kruckman/adem/

The original article by José Adem: The Iteration of the Steenrod Squares in Algebraic Topology
https://www.pnas.org/doi/10.1073/pnas.38.8.720

The survey talk(recommend):
http://www.math.uni-bonn.de/people/daniel/2022/algtop/Steenrod_Squares.pdf

A combinatorial method for computing Steenrod squares:
https://www.sciencedirect.com/science/article/pii/S0022404999000067

Chinese collections on Steenrod algebra:
https://www.zhihu.com/question/265308226

1. binomial coefficient mod p
2. Adem relations

1. binomial coefficient mod p

$\binom{m+n}{n}$ mod 2, $m \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
9	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
10	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
12	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

period

$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

We conclude from table (or Chu-Vandermonde identity) the following practical formula:

Prop. Let $a = \sum_{n \geq 0} a_n 2^n$, $b = \sum_{n \geq 0} b_n 2^n$, $a_n, b_n \in \{0, 1\}$. We get

$$\binom{a+b}{a} \equiv 0 \bmod 2 \iff \exists\, n \in \mathbb{N}_{\geq 0} \text{ s.t. } a_n = b_n = 1$$

Eg. $a = (11011010100)_2$, $b = (100000110)_2$, then

$$\binom{a+b}{a} \equiv 0 \bmod 2 \quad \text{since} \quad \begin{array}{ccccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & \boxed{1} & 0 & 0 \\ & & 1 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 & 0 \end{array}$$

Rmk. Similarly, one can show:
for $a = \sum_{n \geq 0} a_n p^n$, $b = \sum_{n \geq 0} b_n p^n$, $a_n, b_n \in \{0, 1, \dots, p-1\}$,

$$\binom{a+b}{a} \equiv \prod_{n \geq 0} \binom{a_n + b_n}{a_n} \bmod p$$

Rmk. It is possible to define $\binom{a+b}{a} \in \mathbb{F}_p$ for $a, b \in \mathbb{Z}[\frac{1}{p}]$.
One may want to:
① Verify if the usual formulas in https://en.wikipedia.org/wiki/Binomial_coefficient work;
② Find a combinatorical explanation of it.