Eine Woche, ein Beispiel

11.6 equivariant K-theory of Steinberg variety: from formula to diagram.

⚠ In this document, we always read the diagram from top to bottom.

1. nil Hecke alg
2. $\cdot \to \cdot$ case
3. $\cdot \circlearrowleft$ case (haven't worked out

1. nil Hecke alg

Recall that we have an alg homo

$$\mathbb{Z}[e_1^{\pm1}, e_2^{\pm1}, \dots, e_d^{\pm1}] \longrightarrow \mathbb{Q}[[\lambda_1, \lambda_2, \dots, \lambda_d]] \supseteq \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$
$$e_i \longmapsto e^{\lambda_i}$$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$ for e_i, λ_i, $i \in \{1, \dots, d\}$

Ex 1. define $\partial_i \in End_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$ by

$$\partial_i f = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \qquad f \in \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

compute $\partial_i \lambda_i$, $\partial_i \lambda_{i+1}$, $\partial_1(\lambda_1^3\lambda_2 - 3\lambda_2\lambda_4\lambda_5)$.

Ex 2. derive that

$$\partial_i f g = (s_i f)\partial_i g + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} g \qquad \begin{array}{l} f \in End_{\mathbb{Q}\text{-v.s}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d]) \\ f: \ g \longmapsto f \cdot g \end{array}$$

as operators.

Ex 3. verify that

$$\partial_i \lambda_i = \lambda_{i+1}\partial_i + 1 \qquad \partial_i \lambda_{i+1} = \lambda_i \partial_i - 1$$

$$\partial_i \partial_{i+1} \partial_i = \partial_{i+1}\partial_i\partial_{i+1} \qquad \partial_i^2 = 0$$

And the center of $H^*_{G_d}(St; \mathbb{Q})$ is

$$\langle \lambda_1 + \cdots \lambda_n,\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \cdots \lambda_{n-1}\lambda_n,\ \cdots,\ \lambda_1\lambda_2\lambda_3\cdots\lambda_n \rangle_{\mathbb{Q}\text{-alg}}$$
$$= \mathbb{Q}[\lambda_1, \lambda_2, \dots \lambda_n]^{S_n}$$

Ex 1'. define $D_i \in \mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm1}, \dots, e_d^{\pm1}])$ by

$$D_i f = \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}}$$
$$= \frac{e_{i+1} f - e_i s_i f}{e_{i+1} - e_i}$$

compute

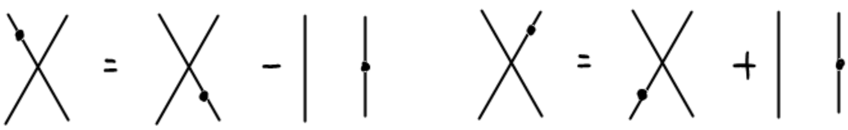
$$D_i 1 = 1$$
$$D_i e_i = 0 \qquad D_i e_i^{-1} = e_i^{-1} + e_{i+1}^{-1}$$
$$D_i e_{i+1} = e_i + e_{i+1} \qquad D_i e_{i+1}^{-1} = 0$$

Ex 2'. derive that

$$D_i f g = (s_i f) D_i g + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} g$$

as operators.

Ex 3'. verify that



$$D_i e_i = e_{i+1} D_i - e_{i+1} \qquad D_i e_{i+1} = e_i D_i + e_{i+1}$$

$$D_i e_i^{-1} = e_{i+1}^{-1} D_i + e_i^{-1} \qquad D_i e_{i+1}^{-1} = e_i^{-1} D_i - e_i^{-1}$$

$$D_i D_{i+1} D_i = D_{i+1} D_i D_{i+1} \qquad D_i^2 = D_i$$

and the center of $K_0^{G_d}(St)$ is
$\mathbb{Z}[e_1^{\pm1}, \dots e_n^{\pm1}]^{S_n}$

Ex 4. Verify that

$$\begin{array}{ccc}
\mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm 1}, \dots, e_d^{\pm 1}])_{cpl} & & \mathrm{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])_{cpl} \\
\cup & & \cup \\
\langle e_1^{\pm 1}, \dots, e_d^{\pm 1}, D_1, \dots, D_{d-1} \rangle_{\mathbb{Z}\text{-alg}, cpl} & \longrightarrow & \langle \lambda_1, \dots, \lambda_d, \partial_1, \dots, \partial_d \rangle_{\mathbb{Q}\text{-alg}, cpl} \\
e_i & \longmapsto & e^{\lambda_i} \\
D_i & \longmapsto & \partial_i \dfrac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} = \dfrac{\lambda_{i+1} - \lambda_i}{1 - e^{\lambda_{i+1} - \lambda_i}} \partial_i + 1 \\
 & & \\
\log e_i & \longleftarrow\!\shortmid & \lambda_i \\
 & & \\
D_i \dfrac{1 - \frac{e_i}{e_{i+1}}}{\log \frac{e_i}{e_{i+1}}} = \dfrac{1 - \frac{e_{i+1}}{e_i}}{\log \frac{e_{i+1}}{e_i}} (D_i - 1) & \longleftarrow\!\shortmid & \partial_i
\end{array}$$

is an alg iso.

Hint.

$$D_i e_k = s_i(e_k) D_i + \frac{e_k - s_i(e_k)}{1 - \frac{e_i}{e_{i+1}}}$$

$$\Leftrightarrow \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} e^{\lambda_k} = s_i(e^{\lambda_k}) \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{1 - e^{\lambda_i - \lambda_{i+1}}}$$

$$\Leftrightarrow \partial_i e^{\lambda_k} = s_i(e^{\lambda_k}) \partial_i + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{\lambda_i - \lambda_{i+1}}$$

1. $\bullet \longrightarrow \bullet$ case

Recall that we have an alg homo

$$\bigoplus_{u\in \mathrm{Mir}(W_{(d)},W_d)} \left(\mathbb{Z}[e_1^{\pm1}, e_2^{\pm1}, \dots, e_{d_1+d_2}^{\pm1}]\right)^u \longrightarrow \bigoplus_u \left(\mathbb{Q}[[\lambda_1,\dots,\lambda_d]]\right)^u \supseteq \bigoplus_u \left(\mathbb{Q}[\lambda_1,\dots,\lambda_d]\right)^u$$

$$e_i^u \longmapsto e^{\lambda_i^u}$$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1,\dots,d-1\}$ for e_i^u, λ_i^u, $i\in\{1,\dots,d\}$

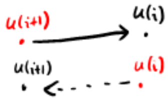
$us_i = w'u'$

Ex 1. define $\partial_i^{u,u'} \in \mathrm{End}_{\mathbb{Q}\text{-v.s.}}\left(\bigoplus_u (\mathbb{Q}[\lambda_1,\dots,\lambda_d])^u\right)$ by

① $\partial_i^{u,u} f^u = \left(\dfrac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right)^u$ $\quad u = u'$

② $\partial_i^{u,u'} f^{u'} = \left(s_i f\,(\lambda_{i+1} - \lambda_i)\right)^u$ $\quad u \neq u'$

③ $\partial_i^{u,u'} f^{u'} = (s_i f)^u$ $\quad u \neq u'$



For u:  , compute $(\partial_i = \partial_i^{u,u'})$

$\partial_2 1^u \quad \partial_2 \lambda_2^u \quad \partial_2 \lambda_3^u$

$\partial_1 1^{u'} \quad \partial_1 \lambda_1^{u'} \quad \partial_1 \lambda_2^{u'}$

$\partial_3 1^{u'} \quad \partial_3 \lambda_3^{u'} \quad \partial_3 \lambda_4^{u'}$

A: They are

$0 \quad 1 \quad -1$

$\lambda_2^u - \lambda_1^u \quad \lambda_2^u(\lambda_2^u - \lambda_1^u) \quad \lambda_1^u(\lambda_2^u - \lambda_1^u)$

$1^u \quad \lambda_4^u \quad \lambda_3^u$

Ex 2. derive that as operators.

$$\partial_i^{u,u} f^u = (s_i f)^u \partial_i^{u,u} + \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right)^u \qquad u = u'$$

$$\partial_i^{u,u'} f^{u'} = (s_i f)^u \partial_i^{u,u'} \qquad u \neq u'$$

Ex 3. verify that

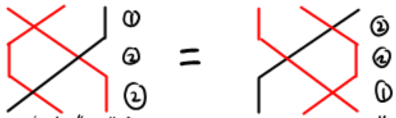
 ③ =  ②

$$\partial_i^{u,u'} \lambda_i^{u'} = \lambda_{i+1}^u \partial_i^{u,u'}$$

$\partial_i^{u,u'}\partial_i^{u',u} = \lambda_i^u - \lambda_{i+1}^u$ $\qquad$ $\partial_i^{u,u'}\partial_i^{u',u} = \lambda_i^u - \lambda_{i+1}^u$

$$\partial_i^{u,u'}\partial_{i+1}^{u',u'}\partial_i^{u',u} = \partial_{i+1}^{u,u''}\partial_i^{u'',u''}\partial_{i+1}^{u'',u} + 1^u$$

$$\partial_i^{u,u'}\partial_{i+1}^{u',u'}\partial_i^{u',u} = \partial_{i+1}^{u,u''}\partial_i^{u'',u''}\partial_{i+1}^{u'',u} - 1^u$$



$$\partial_i^{u,u'}\partial_{i+1}^{u',u''}\partial_i^{u'',u''} = \partial_{i+1}^{u,u}\partial_i^{u,u'}\partial_{i+1}^{u',u''}$$

and the center of $H^*_{G_d}(St;\mathbb{Q})$ is

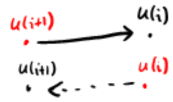
$$\mathbb{Q}[\lambda_1,\dots\lambda_n]^{S_n}$$

Ex 1' define $D_i^{u,u'} \in \mathrm{End}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_u \left(\mathbb{Z}[e_1^{\pm 1},\dots,e_{d_1+d_2}^{\pm 1}]\right)^u\right)$ by

① $D_i^{u,u} f^u = \left(\dfrac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \dfrac{f}{1-\frac{e_i}{e_{i+1}}}\right)^u \qquad u = u'$

② $D_i^{u,u'} f^{u'} = \left(s_i f\left(1-\frac{e_{i+1}}{e_i}\right)\right)^u \qquad u \neq u'$

③ $D_i^{u,u'} f^{u'} = (s_i f)^u \qquad u \neq u'$



For u:  , compute $(D_i = D_i^{u,u'})$

$D_2 1^u \quad D_2 e_2^u \quad D_2 e_3^u \quad D_2 (e_2^u)^{-1} \quad D_2 (e_3^u)^{-1}$

$D_1 1^{u'} \quad D_1 e_1^{u'} \quad D_1 e_2^{u'} \quad D_1 (e_1^{u'})^{-1} \quad D_1 (e_2^{u'})^{-1}$

$D_3 1^{u'} \quad D_3 e_3^{u'} \quad D_3 e_4^{u'} \quad D_3 (e_3^{u'})^{-1} \quad D_3 (e_4^{u'})^{-1}$

A: They are

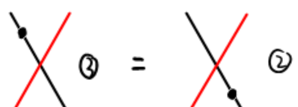
$$\begin{pmatrix} 1 & 0 & e_2+e_3 & e_2^{-1}+e_3^{-1} & 0 \\ 1-\frac{e_2}{e_1} & e_2\left(1-\frac{e_2}{e_1}\right) & e_1\left(1-\frac{e_2}{e_1}\right) & \frac{1}{e_2}\left(1-\frac{e_2}{e_1}\right) & \frac{1}{e_1}\left(1-\frac{e_2}{e_1}\right) \\ 1 & e_4 & e_3 & \frac{1}{e_4} & \frac{1}{e_3} \end{pmatrix}^u$$

Ex 2'. derive that as operators.

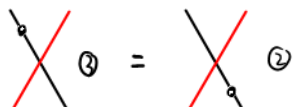
$$D_i^{u,u} f^u = (s_i f)^u D_i^{u,u} + \left(\frac{f - s_i f}{1-\frac{e_i}{e_{i+1}}}\right)^u \qquad u = u'$$

$$D_i^{u,u'} f^{u'} = (s_i f)^u D_i^{u,u'} \qquad u \neq u'$$

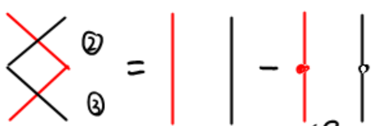
Ex 3'. verify that



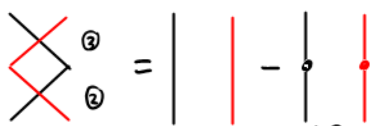
$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$



$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$



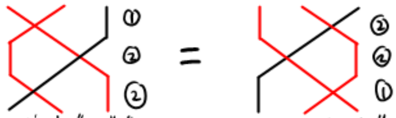
$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}}\right)^u$



$D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i}\right)^u$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u',u''} D_{i+1}^{u'',u} + \left(\frac{e_{i+2}}{e_{i+1}}\right)^u$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \left(\frac{e_{i+1}}{e_i}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

and the center of $K_0^{G_d}(St)$ is

$$\mathbb{Z}[e_1^{\pm 1}, \dots e_n^{\pm 1}]^{S_n}$$

Ex 4. Verify that

$$\begin{array}{ccc}
\mathrm{End}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_u \left(\mathbb{Z}[e_1^{\pm 1}, \dots, e_{|d|}^{\pm 1}]\right)^u\right)_{cpl} & & \mathrm{End}_{\mathbb{Q}\text{-v.s.}}\left(\bigoplus_u \left(\mathbb{Q}[\lambda_1, \dots, \lambda_{d_1+d_2}]\right)^u\right)_{cpl} \\
\cup & & \cup \\
\langle (e_i^u)^{\pm 1}, D_i^{u,u'} \rangle_{\mathbb{Z}\text{-alg}, cpl} & \longrightarrow & \langle \lambda_i^u, \partial_i^{u',u} \rangle_{\mathbb{Q}\text{-alg}, cpl} \\
e_i^u & \longmapsto & e^{\lambda_i^u} \\
① \quad D_i^{u,u} & \longmapsto & \partial_i^{u,u}\left(\dfrac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}}\right)^u = \left(\dfrac{\lambda_{i+1} - \lambda_i}{1 - e^{\lambda_{i+1} - \lambda_i}}\right)^u \partial_i^{u,u} + 1^u \\
② \quad D_i^{u,u'} & \longmapsto & \partial_i^{u,u'}\left(\dfrac{1 - e^{\lambda_i - \lambda_{i+1}}}{\lambda_i - \lambda_{i+1}}\right)^{u'} = \left(\dfrac{1 - e^{\lambda_{i+1} - \lambda_i}}{\lambda_{i+1} - \lambda_i}\right)^u \partial_i^{u,u'} \\
③ \quad D_i^{u,u'} & \longmapsto & \partial_i^{u,u'}
\end{array}$$

is an alg iso.

1. $\cdot \longrightarrow \cdot$ case

Recall that we have an alg homo

$$\bigoplus_{u \in M_{in}(W_{|d|}, W_d)} \left(\mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_{d_1+d_2}^{\pm 1}]\right)^u \longrightarrow \bigoplus_u \left(\mathbb{Q}[[t]][[\lambda_1, \dots, \lambda_d]]\right)^u \supseteq \bigoplus_u \left(\mathbb{Q}[t][\lambda_1, \dots, \lambda_d]\right)^u$$

$$e_i^u \longmapsto e^{\lambda_i^u}$$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$ for e_i^u, λ_i^u, $i \in \{1, \dots, d\}$

$us_i = w'u'$

Ex 1. define $\partial_i^{u,u'} \in End_{\mathbb{Q}\text{-v.s.}}\left(\bigoplus_u \left(\mathbb{Q}[t][\lambda_1, \dots, \lambda_d]\right)^u\right)$ by

① $\partial_i^{u,u} f^u = \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right)^u$ $u = u'$

② $\partial_i^{u,u'} f^{u'} = \left(s_i f \, (\lambda_{i+1} - \lambda_i - t)\right)^u$ $u \neq u'$ $\quad u(i+1) \longrightarrow u(i)$

③ $\partial_i^{u,u'} f^{u'} = (s_i f)^u$ $u \neq u'$ $\quad u(i+1) \dashleftarrow u(i)$

For u:  , compute $(\partial_i = \partial_i^{u,u'})$

$\partial_2 1^u$	$\partial_2 \lambda_2^u$	$\partial_2 \lambda_3^u$
$\partial_1 1^{u'}$	$\partial_1 \lambda_1^{u'}$	$\partial_1 \lambda_2^{u'}$
$\partial_3 1^{u'}$	$\partial_3 \lambda_3^{u'}$	$\partial_3 \lambda_4^{u'}$

A: They are

0	1	-1
$\lambda_2^u - \lambda_1^u - t$	$\lambda_2^u(\lambda_2^u - \lambda_1^u - t)$	$\lambda_1^u(\lambda_2^u - \lambda_1^u - t)$
1^u	λ_4^u	λ_3^u

Ex 2. derive that as operators.

$$\partial_i^{u,u} f^u = (s_i f)^u \partial_i^{u,u} + \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}}\right)^u \qquad u = u'$$

$$\partial_i^{u,u'} f^{u'} = (s_i f)^u \partial_i^{u,u'} \qquad u \neq u'$$

Ex 3. verify that

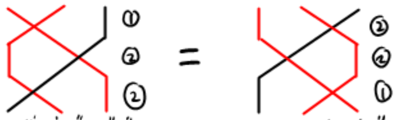
 ③ =  ②

$$\partial_i^{u,u'} \lambda_i^{u'} = \lambda_{i+1}^u \partial_i^{u,u'}$$

$$\partial_i^{u,u'}\partial_i^{u',u} = \lambda_i^u - \lambda_{i+1}^u - t\,1^u \qquad \partial_i^{u,u'}\partial_i^{u',u} = \lambda_i^u - \lambda_{i+1}^u - t\,1^u$$

$$\partial_i^{u,u'}\partial_{i+1}^{u',u'}\partial_i^{u',u} = \partial_{i+1}^{u,u''}\partial_i^{u'',u''}\partial_{i+1}^{u'',u} + 1^u$$

$$\partial_i^{u,u'}\partial_{i+1}^{u',u'}\partial_i^{u',u} = \partial_{i+1}^{u,u''}\partial_i^{u'',u''}\partial_{i+1}^{u'',u} - 1^u$$



$$\partial_i^{u,u'}\partial_{i+1}^{u',u''}\partial_i^{u'',u''} = \partial_{i+1}^{u,u}\partial_i^{u,u'}\partial_{i+1}^{u',u''}$$

and the center of $H^*_{G_a\times\mathbb{C}^*}(St;\mathbb{Q})$ is

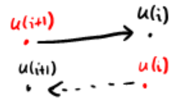
$$\mathbb{Q}[t][\lambda_1,\dots\lambda_n]^{S_n}$$

Ex 1' define $D_i^{u,u'} \in \mathrm{End}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_u \left(\mathbb{Z}[q^{\pm1}][e_1^{\pm1}, \dots, e_{d_1+d_2}^{\pm1}]\right)^u \right)$ by

① $D_i^{u,u} f^u = \left(\dfrac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \dfrac{f}{1 - \frac{e_i}{e_{i+1}}} \right)^u \qquad u = u'$

② $D_i^{u,u'} f^{u'} = \left(s_i f \left(1 - \dfrac{e_{i+1}}{e_i q}\right)\right)^u \qquad u \neq u'$

③ $D_i^{u,u'} f^{u'} = (s_i f)^u \qquad u \neq u'$



For u:  , compute $(D_i = D_i^{u,u'})$

$D_2 1^u \quad D_2 e_2^u \quad D_2 e_3^u \quad D_2 (e_2^u)^{-1} \quad D_2 (e_3^u)^{-1}$

$D_1 1^{u'} \quad D_1 e_1^{u'} \quad D_1 e_2^{u'} \quad D_1 (e_1^{u'})^{-1} \quad D_1 (e_2^{u'})^{-1}$

$D_3 1^{u'} \quad D_3 e_3^{u'} \quad D_3 e_4^{u'} \quad D_3 (e_3^{u'})^{-1} \quad D_3 (e_4^{u'})^{-1}$

A: They are

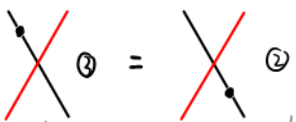
$$\begin{pmatrix} 1 & 0 & e_2 + e_3 & e_2^{-1} + e_3^{-1} & 0 \\ 1 - \frac{e_2}{e_1 q} & e_2\left(1 - \frac{e_2}{e_1 q}\right) & e_1\left(1 - \frac{e_2}{e_1 q}\right) & \frac{1}{e_2}\left(1 - \frac{e_2}{e_1 q}\right) & \frac{1}{e_1}\left(1 - \frac{e_2}{e_1 q}\right) \\ 1 & e_4 & e_3 & \frac{1}{e_4} & \frac{1}{e_3} \end{pmatrix}^u$$

Ex 2'. derive that as operators.

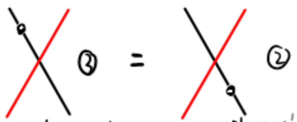
$$D_i^{u,u} f^u = (s_i f)^u D_i^{u,u} + \left(\frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} \right)^u \qquad u = u'$$

$$D_i^{u,u'} f^{u'} = (s_i f)^u D_i^{u,u'} \qquad u \neq u'$$

Ex 3'. verify that

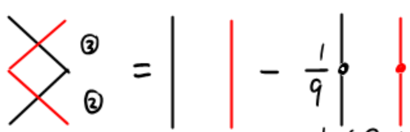


$D_i^{u,u'} e_i^{u'} = e_{i+1}^u D_i^{u,u'}$

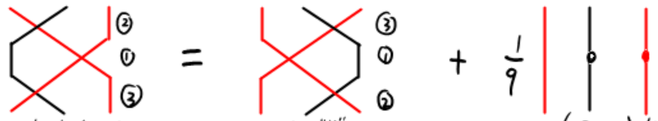


$D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^u)^{-1} D_i^{u,u'}$

$D_i^{u,u'} D_i^{u',u} = 1^u - \frac{1}{q}\left(\frac{e_i}{e_{i+1}}\right)^u$

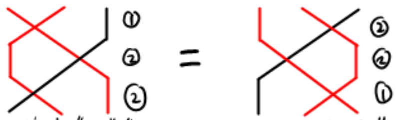


$D_i^{u,u'} D_i^{u',u} = 1^u - \frac{1}{q}\left(\frac{e_{i+1}}{e_i}\right)^u$



$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u''u''} D_{i+1}^{u'',u} + \frac{1}{q}\left(\frac{e_{i+2}}{e_{i+1}}\right)^u$$

$$D_i^{u,u'} D_{i+1}^{u',u'} D_i^{u',u} = D_{i+1}^{u,u''} D_i^{u'',u''} D_{i+1}^{u'',u} - \frac{1}{q}\left(\frac{e_{i+1}}{e_i}\right)^u$$



$$D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u''} = D_{i+1}^{u,u} D_i^{u,u'} D_{i+1}^{u',u''}$$

and the center of $K_0^{G_d \times \mathbb{C}^\times}(St)$ is

$$\mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \ldots e_n^{\pm 1}]^{S_n}$$