Eine Woche, ein Beispiel

19 Excision

Goal: state the homology/cohomology/homotopy excision thm and give applications.

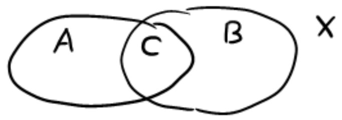
↑ don't want to see the proof, I'm lazy…

The main ref would be Hatcher, and we only consider the singular homology.

Thm (Homology excision thm, [Thm 2.20, Cor 2.24])

Let X space, $A, B \subset X$ subspace, $X = \mathring{A} \cup \mathring{B}$, $C = A \cap B$

or: Let X CW cpx, $A, B \subset X$ subcplx, $X = A \cup B$, $C = A \cap B$



then $\quad \iota_*: H_i(A, C; \mathbb{Z}) \longrightarrow H_i(X, B; \mathbb{Z})$ is iso.

Cor [Prop 2.22] Let (X, A) be an NDR-pair. then

$\pi_*: H_i(X, A; \mathbb{Z}) \longrightarrow \widetilde{H}_i(X/A; \mathbb{Z})$ is iso.

Proof.

$$H_i(X, A; \mathbb{Z}) \overset{LES}{\cong} H_i(X \cup (A \times [0,1)), A; \mathbb{Z}) \xrightarrow{\cong} H_i(X \cup CA, CA; \mathbb{Z}) \overset{LES}{\cong} \widetilde{H}_i(X \cup CA; \mathbb{Z})$$

$$\sim \widetilde{H}_i(X/A; \mathbb{Z})$$

$X \cup CA \longrightarrow (X \cup CA)/CA \cong X/A$ is a homotopy equivalence since [Prop 0.17]

- CA is contractable
- (X, A) satisfies HEP $\Rightarrow$ $(X \cup CA, CA)$ satisfies HEP

For the equivalent definitions of NPR-pair, see here:
https://math.stackexchange.com/questions/3547820/neighborhood-deformation-retracts-vs-cofibrations
or Prop A.6 in url:https://www.math.univ-paris13.fr/~schwartz/FIMFA/Ando.pdf

We also have the third version of excision thm:

Thm. Suppose (X,A), (Y,B) are two relative CW-pairs, $f: X \to Y$ send A to B, and the square

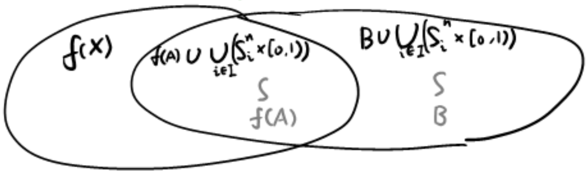
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f|_A"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

is a pushout of spaces, then

$$f_*: H_i(X,A;\mathbb{Z}) \longrightarrow H_i(Y,B;\mathbb{Z}) \text{ is an iso.}$$

Proof. $Y = B \cup_{f,A} X$, i.e. Y is gotten by attaching relative cells of (X,A) to B.

Step 1. If all relative cells of (X,A) are of dim $n+1$, then



$$H_i(Y,B;\mathbb{Z}) \overset{LES}{\cong} H_i\Big(Y,\ B \cup \bigcup_{i\in I}(S_i^n \times [0,1))\Big) \xrightarrow{\cong} H_i\Big(f(X),\ f(A) \cup \bigcup_{i\in I}(S_i^n \times [0,1))\Big) \overset{LES}{\cong} H_i(f(X), f(A);\mathbb{Z})$$

$$H_i(f(X), f(A);\mathbb{Z}) \cong \widetilde{H}_i(f(X)/f(A);\mathbb{Z}) \cong \widetilde{H}_i(X/A;\mathbb{Z}) \cong H_i(X,A;\mathbb{Z})$$

Step 2. Suppose that every square in the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f|_A"'] & C \arrow[r] \arrow[d, "f|_C"] & X \arrow[d, "f"] \\ B \arrow[r] & D \arrow[r] & Y \end{tikzcd}$$

is a pushout of spaces, and

$$(f|_C)_*: H_i(C, A;\mathbb{Z}) \longrightarrow H_i(D,B;\mathbb{Z})$$
$$f_*: H_i(X,C;\mathbb{Z}) \longrightarrow H_i(Y,D;\mathbb{Z})$$

are iso, then by the naturality of LES and 5-lemma,

$$f_*: H_i(X,A;\mathbb{Z}) \longrightarrow H_i(Y,B;\mathbb{Z}) \text{ is iso.}$$

Step 3. Consider the diagram

$$\begin{tikzcd} A = X^{(-1)} \arrow[r] & X^{(0)} \arrow[r] \arrow[d] & \cdots \arrow[r] & X^{(n)} \arrow[r] \arrow[d] & \cdots \\ B = Y^{(-1)} \arrow[r] & Y^{(0)} \arrow[r] & \cdots \arrow[r] & Y^{(n)} \arrow[r] & \cdots \end{tikzcd}$$

and take the colimit.

Thm (MV sequence, LES)

Let X space, $U, V \subseteq X$ open subset, $U \cup V = X$. Then we get a LES

$$\cdots \to \widetilde{H}_{n+1}(X;\mathbb{Z}) \to \widetilde{H}_n(U\cap V;\mathbb{Z}) \longrightarrow \widetilde{H}_n(U;\mathbb{Z}) \oplus \widetilde{H}_n(V;\mathbb{Z}) \longrightarrow \widetilde{H}_n(X;\mathbb{Z}) \to \cdots$$

MV->excision: https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision
excision->MV: https://www.math.ru.nl/~gutierrez/files/homology/Lecture06.pdf
Please be aware of the conditions in the theorems. We have many versions of theorems when we slightly change the conditions, but I don't want to go to the most generality(Actually I don't know the most general condition).

E.g. $H_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \overset{SES}{\cong} H_{n-1}(\partial\Delta^n, \Lambda; \mathbb{Z}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}; \mathbb{Z}) \overset{\text{induction}}{\cong} \mathbb{Z}$.

E.g. The local homology groups $H_n(x) := H_n(X, X \setminus \{x\}; \mathbb{Z})$
can be defined locally, i.e. $H_n(X, X\setminus\{x\}; \mathbb{Z}) \cong H_n(U, U\setminus\{x\}; \mathbb{Z})$ for $\{x\} \in U \overset{\text{open}}{\subseteq} X$.

Given a sm map $f: M \longrightarrow N$ between mflds of dim n,
a pt $y \in N$ s.t. $f^{-1}(y) \triangleq \{x_1, \cdots, x_n\}$ is finite,
we can define the local degree $\deg_x f \in \mathbb{Z}$ at $x \in f^{-1}(y)$:

$(U \cap f^{-1}(y) = \{x\})$
$$f_*: \underbrace{H_n(U, U\setminus\{x\}; \mathbb{Z})}_{\cong\mathbb{Z}} \longrightarrow \underbrace{H_n(N, N\setminus\{y\}; \mathbb{Z})}_{\cong\mathbb{Z}} \qquad [U] \mapsto \deg_x f \cdot [N]$$

When M, N are cpt, we can also define the global degree $\deg f \in \mathbb{Z}$:

$$f_*: \underbrace{H_n(M; \mathbb{Z})}_{\cong\mathbb{Z}} \longrightarrow \underbrace{H_n(N; \mathbb{Z})}_{\cong\mathbb{Z}} \qquad [M] \mapsto \deg f \cdot [N]$$

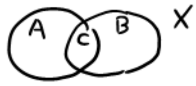
we have the equality $\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$.

Thm (Homotopy excision thm [Thm 4.23])

Let X: CW cplx, A, B subcplx, $X = A \cup B$, $C = A \cap B$ nonempty and connected.

If (A, C) is m-connected,

(B, C) is n-connected,



then $\pi_i(A, C, x_0) \longrightarrow \pi_i(X, B, x_0)$ is $\begin{cases} \text{iso} & i < m+n \\ \text{surj} & i = m+n \end{cases}$ $x_0 \in C$

Cor [Prop 4.28] Suppose X, A are CW cplxs, $A \subset X$ is sub cplx.

If (X, A) is r-connected, A is s-connected,

then the map $\pi_{*}: \pi_i(X, A, x_0) \longrightarrow \pi_i(X/A, x_0)$ is $\begin{cases} \text{iso} & i < r+s+1 \\ \text{surj} & i = r+s+1 \end{cases}$

[Proof.

$$\pi_i(X, A, x_0) \longrightarrow \pi_i(X \cup CA, CA, x_0) \overset{\text{LES}}{\cong} \pi_i(X \cup CA, x_0) \overset{\text{homotopy}}{\sim} \pi_i(X/A, x_0)$$]

Thm (Freudenthal suspension thm [Cor 4.24])

Let $n \geq 1$, X be an $(n-1)$-connected CW cplx, then the suspension map

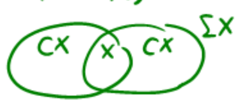
$$\Sigma: \pi_i(X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, x_0)$$
$$[f: S^i \to X] \longmapsto [\Sigma f: S^{i+1} \cong \Sigma S^i \longrightarrow \Sigma X]$$

is $\begin{cases} \text{iso} & i < 2n-1 \\ \text{surj} & i = 2n-1 \end{cases}$

Rmk. Freudenthal suspension thm $\rightsquigarrow$ concept of the stable homotopy gp.

[Proof. $\pi_i(X, x_0) \overset{\text{LES}}{\cong} \pi_{i+1}(CX, X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, CX, x_0) \cong \pi_{i+1}(\Sigma X)$

is $\begin{cases} \text{iso} & i+1 < n+n \\ \text{surj} & i+1 = n+n \end{cases}$



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