Eine Woche, ein Beispiel

9.10. ramified covering: alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - Correspondence
   - Valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

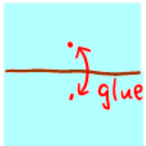
1. alg curve viewed as stack quotient

	Spec $\mathbb{R}$	base change Spec $\mathbb{C}/_{\mathbb{C}}$	Spec $\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	ϕ
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{Id, \tau\}$
$\Gamma_{\mathbb{R}} = Gal(\mathbb{C}/_{\mathbb{R}})$	trivial on pts & fcts	no action	$Id \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. Spec $\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$. As a set,

$$\begin{aligned}
\operatorname{Spec} \mathbb{R}[x] &= \{(x-a) \mid a \in \mathbb{R}\} \cup \{(x^2+bx+c) \mid {\scriptstyle b,c \in \mathbb{R} \atop b^2-4c<0}\} \cup \{(0)\} \\
&= \quad \mathbb{R} \quad \cup \quad \mathcal{H} \quad \cup \{(0)\} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})
\end{aligned}$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau \circ x$. Observe that

$$MaxSpec\, \mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/_{\Gamma_{\mathbb{R}}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$MaxSpec\, \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/_{\Gamma_{\mathbb{F}_p}} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/_{\Gamma_\kappa} \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

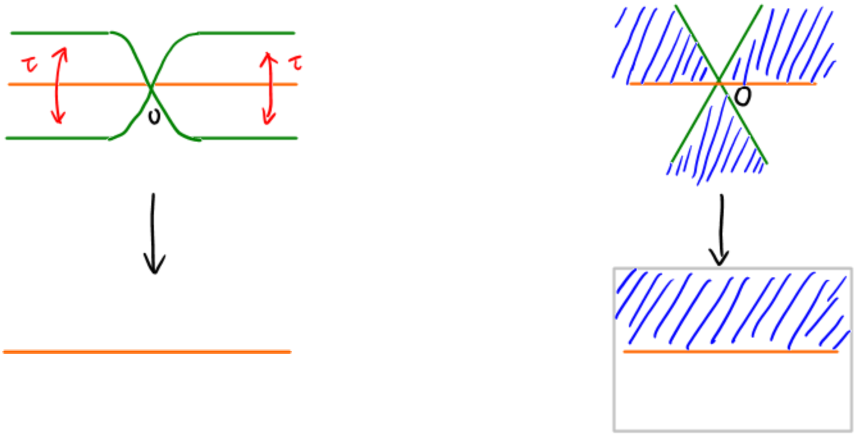
$$Stab_x(\Gamma_\kappa) = \Gamma_{\kappa'} \quad \Leftrightarrow \quad \text{fiber at } x = \operatorname{Spec} \kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R} \cup \mathcal{H}$	$\mathbb{C}$	$\mathbb{C}$ $\circlearrowleft$ cplx conj
$\mathbb{R}$-pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$-pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \sqcup \mathbb{C}\tau$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

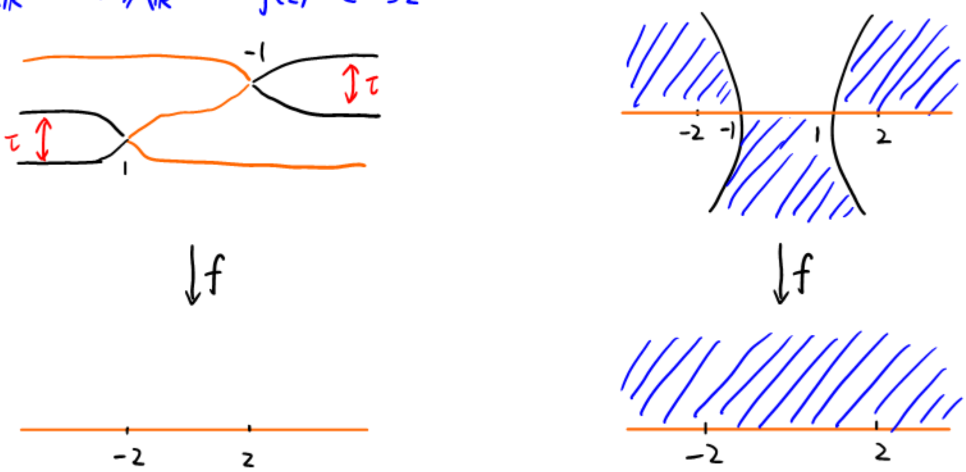
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

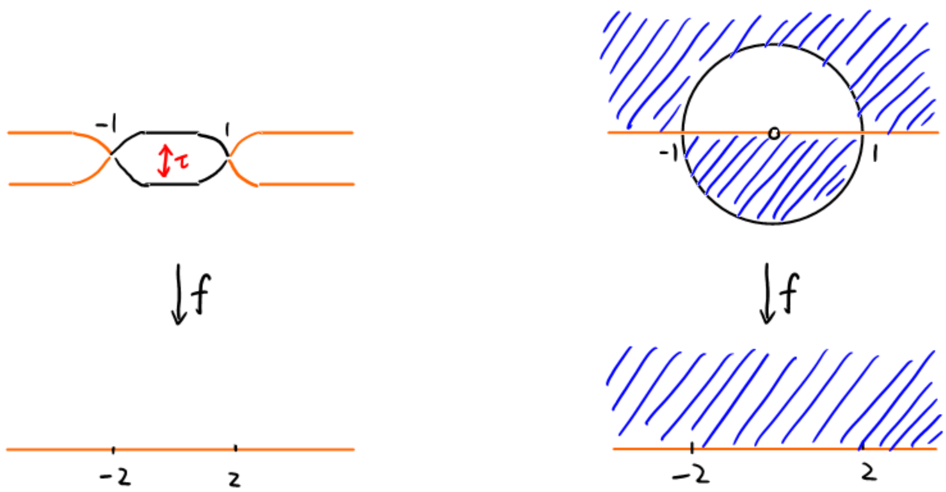
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



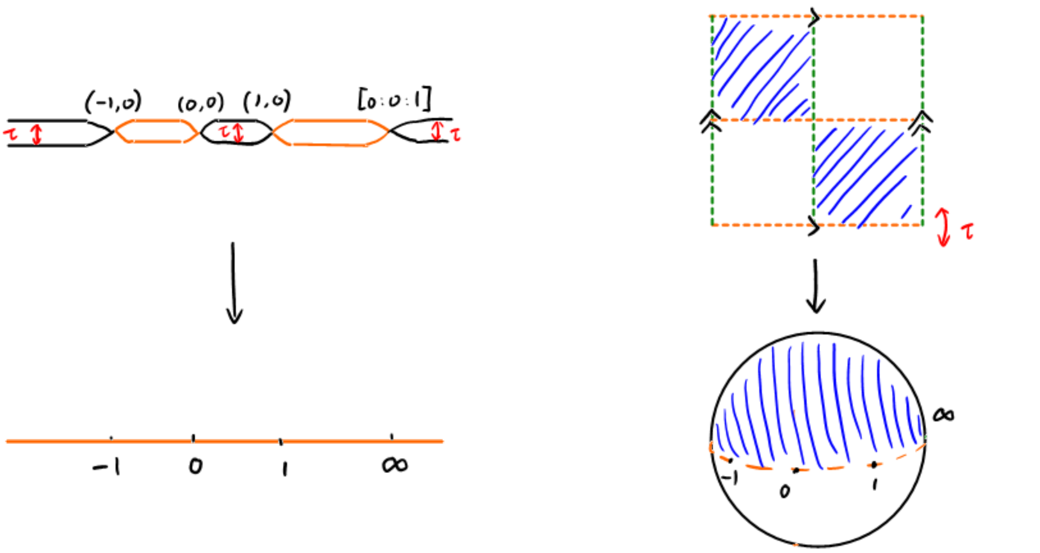
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$



$f: \mathbb{G}_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$

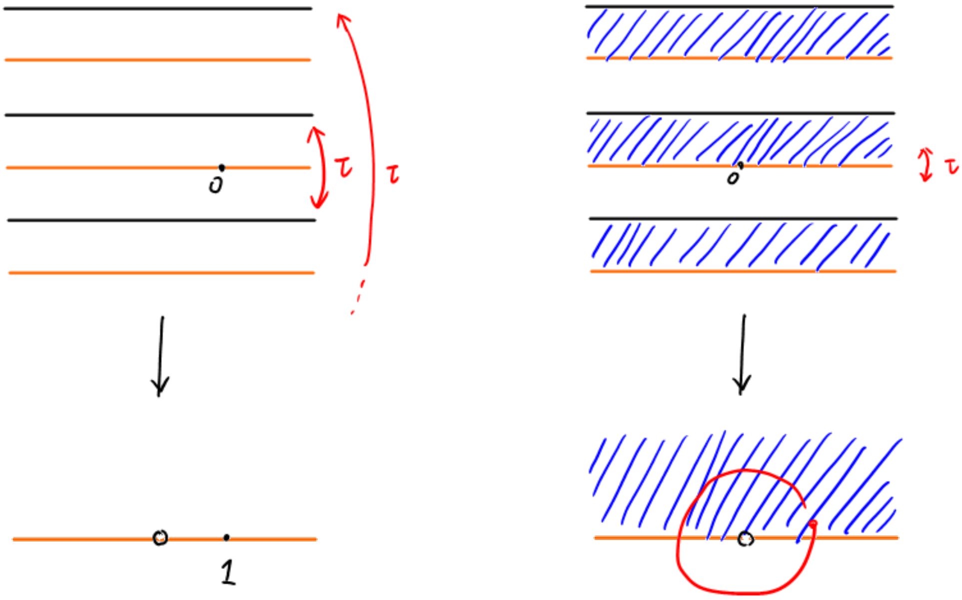


$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ $\quad [x:y:z] \longmapsto [x:z]$ $\quad E_{\mathbb{R}} = \mathrm{Proj}\ \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

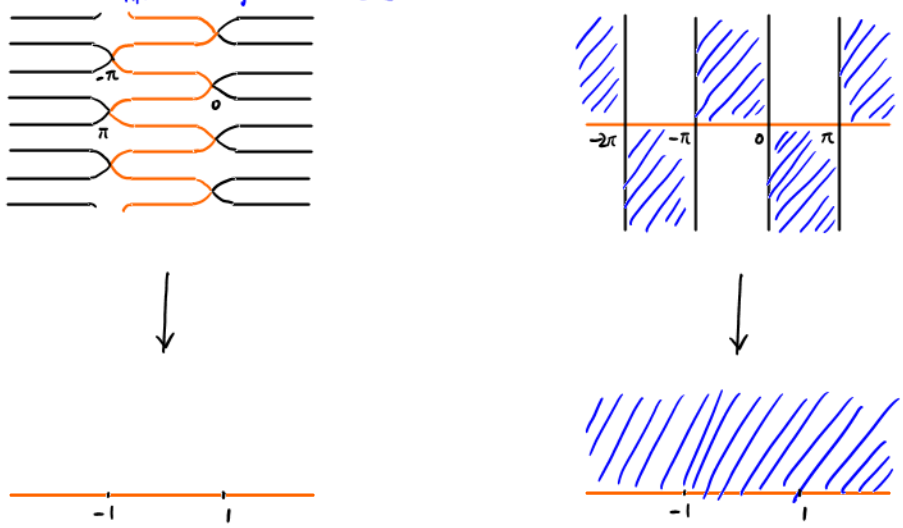


⚠ The following are not alg morphisms!

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = e^z$



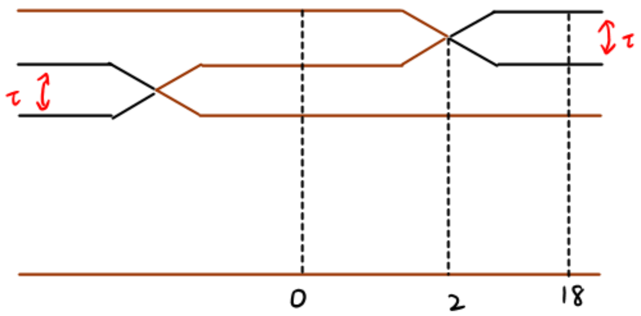
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = \cos z$



Let's focus on the case

$$f : \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = z^3 - 3z$$

classical picture



split: $f^{-1}(0) = \operatorname{Spec} \mathbb{R} \sqcup \operatorname{Spec} \mathbb{R} \sqcup \operatorname{Spec} \mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z - z_0))$

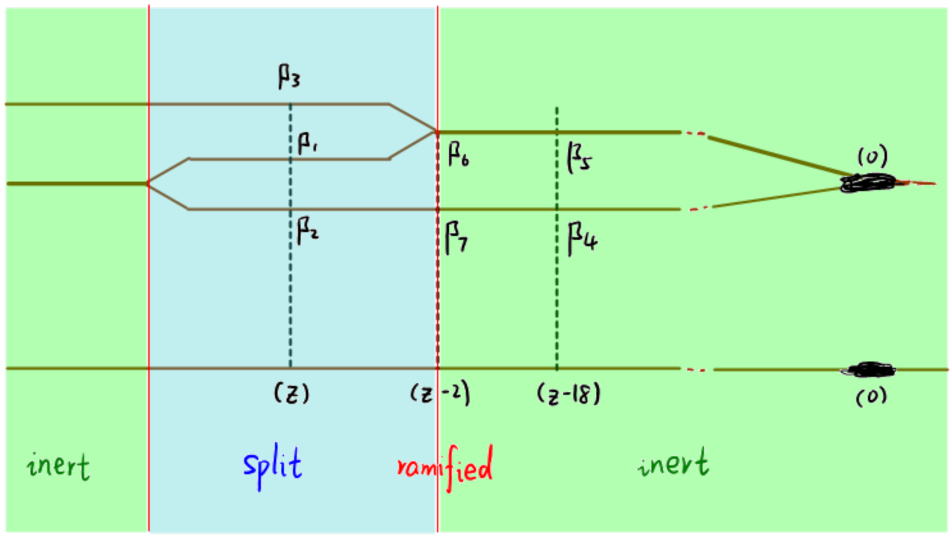
$f^{-1}((z^2+1)) = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec} \mathbb{C} \sqcup \operatorname{Spec} \mathbb{R}$

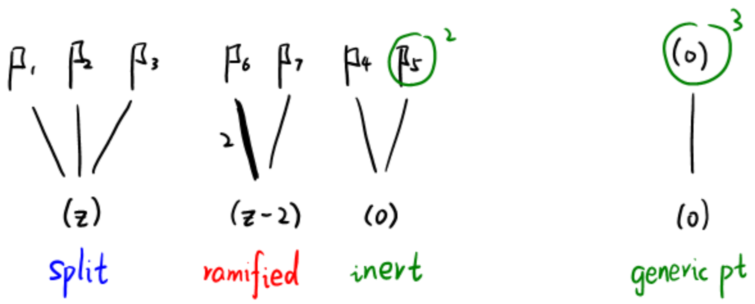
generic point: $f^{-1}((0)) = \operatorname{Spec} \mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec} \mathbb{R} \sqcup \operatorname{Spec} \mathbb{R}$

algebraic picture



$$\begin{array}{ccccc} \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[w] & & w^3 - 3w \\ f \downarrow & & \uparrow f^* & & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] & & z \end{array}$$



split: $\mathfrak{p}=(z)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w) = (w)(w-\sqrt{3})(w+\sqrt{3}) \triangleq \mathfrak{p}_1\ \mathfrak{p}_2\ \mathfrak{p}_3$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$

$\mathfrak{p}=(z^2+1)$, $f^*(\mathfrak{p})\mathbb{R}[w] = ((w^3-3w)^2+1) = (f_1')(f_2')(f_3') \triangleq \mathfrak{p}_1'\ \mathfrak{p}_2'\ \mathfrak{p}_3'$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$

(partially) inert: $\mathfrak{p}=(z-18)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-18) = (w-3)(w^2+3w+6) \triangleq \mathfrak{p}_4\ \mathfrak{p}_5$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_4,\mathfrak{p}_5\}$

where $\kappa(\mathfrak{p}_5)=\mathbb{R}[w]/(w^2+3w+6)\cong\mathbb{C}$, $[\kappa(\mathfrak{p}_5):\mathbb{R}]=2$

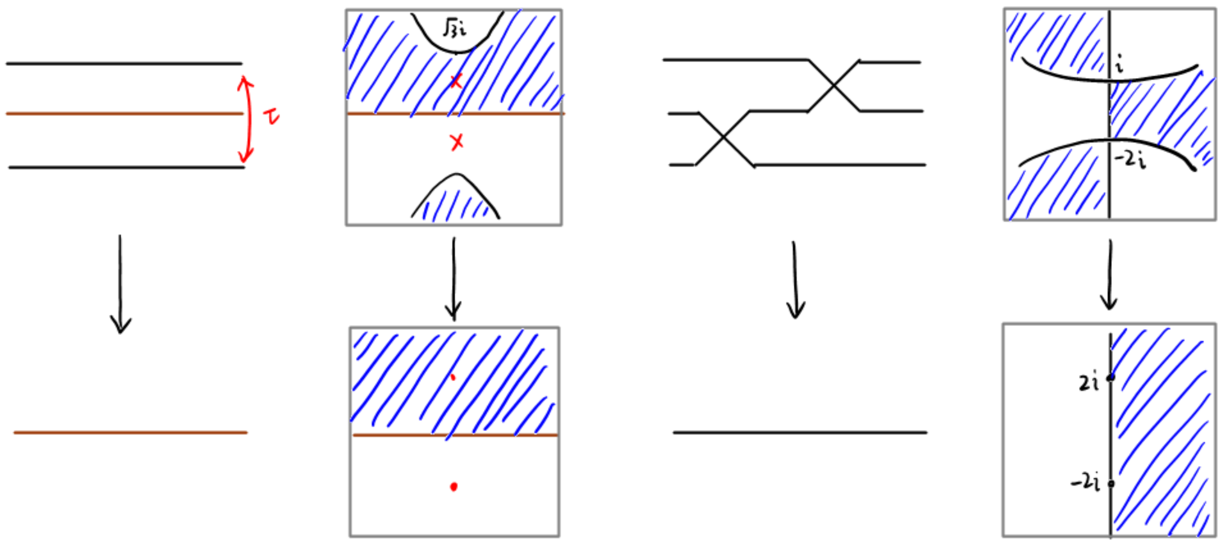
generic point: $\mathfrak{p}=(0)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (0)$ $\qquad f^{-1}(\mathfrak{p})=\{(0)\}$

where $\kappa((0))=\mathrm{Frac}(\mathbb{R}[w]/(0))\cong\mathbb{R}(w)$, $[\mathbb{R}(w):\mathbb{R}(z)]=3$

ramified: $\mathfrak{p}=(z-2)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-2) = (w+1)^2(w-2) \triangleq \mathfrak{p}_6^2\ \mathfrak{p}_7$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_4,\mathfrak{p}_5\}$

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z)=z^3+3z$.

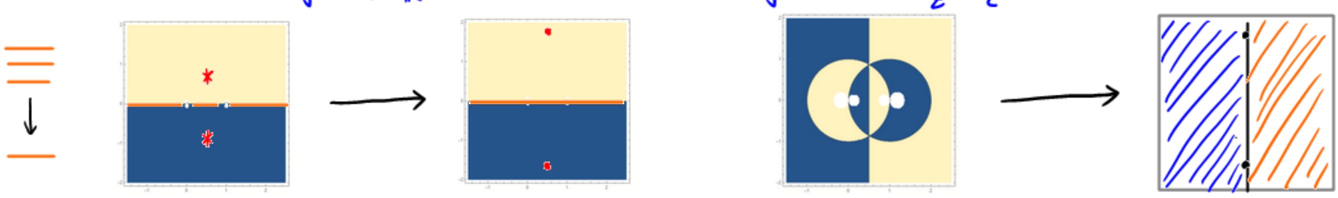


ℝ picture $\qquad$ iℝ picture

⚠ The ramification pt is outside ℝ. This is not a Galois covering.

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z)=\dfrac{z^3-3z+1}{z^2-z}-1.5$



ℝ picture $\qquad$ iℝ picture

This is a Galois covering, with no inert places (except for the generic pt)

3. Frobenius for alg curve/$\mathbb{R}$

$$\mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \kappa(\mathfrak{q}) = \mathbb{C},\ \kappa(\mathfrak{p}) = \mathbb{R} \\ \{\mathrm{Id}\} & \text{otherwise.} \end{cases}$$

When E/F is Galois, $\mathrm{Spec}\, \mathcal{O}_E / \mathrm{Spec}\, \mathcal{O}_F$ unramified at $\mathfrak{p}$,

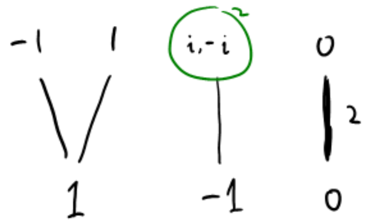
$$\begin{array}{ccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) \cong \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \mathrm{Frob}_{\mathfrak{q}} & \longmapsto & \mathrm{Frob}_{\mathfrak{q}} \end{array}$$

is a subgp of $\mathrm{Gal}(E/F) \cong \mathrm{Aut}(\mathrm{Spv}(E)/\mathrm{Spv}(F))$ Now, just view $\mathrm{Spv}(E) \in \mathrm{AlgCurve}_{k}$.

Let's try to compute some $\mathrm{Frob}_{\mathfrak{q}}$

E.g.

$$\begin{array}{cccc} \mathbb{A}^1_{\mathbb{R}} & z & & \mathbb{R}[\omega] = \mathbb{R}[z^{\frac{1}{2}}] \\ \downarrow & \downarrow & & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z^2 & & \mathbb{R}[z] \end{array}$$



For $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (\omega - 1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ 1 & & 1 & & \{1, \tau\} \end{array}$$

For $\mathfrak{p} = (z+1)$, $\mathfrak{q} = (\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ \{1, \tau\} & & \{1, \tau\} & & \{1, \tau\} \end{array}$$

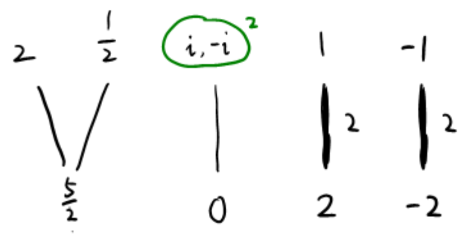
Therefore, $\mathrm{Frob}_{(z+1)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad \omega \longmapsto -\omega$

Not the conjugation but $\tau(\mathbb{C})|_{i\mathbb{R}}$ coincides with the cplx conj

E.g.

$$\begin{array}{cccc} \mathbb{G}_{m,\mathbb{R}} & z & & \mathbb{R}[\omega^{\pm 1}] = \mathbb{R}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{\pm 1}\right] \\ \downarrow & \downarrow & & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z + \frac{1}{z} & & \mathbb{R}[z] \end{array}$$



For $\mathfrak{p} = (z)$, $\mathfrak{q} = (\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ \{1, \tau\} & & \{1, \tau\} & & \{1, \tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z+1)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad \omega \longmapsto \frac{1}{\omega}$

Not the conjugation but $\tau(\mathbb{C})|_{S^1}$ coincides with the cplx conj

⚠ $\mathbb{R}(z^{\frac{1}{3}})/\mathbb{R}(z)$ is not Galois at all, so

For $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $z \longmapsto z^3$, $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (w^2+w+1)$,

$$\underset{\{1,\tau\}}{\mathrm{Gal}(k(\mathfrak{q})/k(\mathfrak{p}))} \not\simeq \underset{1}{\text{“}\mathrm{Gal}(E/F)_{\mathfrak{q}}\text{”}} \leq \underset{1}{\text{“}\mathrm{Gal}(E/F)\text{”}} \neq \mathbb{Z}/3\mathbb{Z}$$

We will discuss about $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$ in section 4.

Claim. For p odd prime, any deg p extension of $\mathbb{R}(x)$ is not Galois.

This claim is wrong. The field extension

$$\mathbb{R}(x)[T]/(T^3 - xT^2 + (x-3)T + 1) \;/\mathbb{R}(x)$$

is Galois with deg 3. discriminant $\Delta = (x^3-3x+9)^2$ [SerreGT, 1.1]

Wrong proof:

If not, suppose $F/\mathbb{R}(x)$ is a deg p Galois extension, we get the field extension tower in $\overline{\mathbb{R}(x)}$:

$$\begin{tikzcd} & E \arrow[dl, dash, "2"'] \arrow[d, dash, "p"] \\ F \arrow[d, dash, "p"'] & \mathbb{C}(x) \arrow[dl, dash, "2"] \\ \mathbb{R}(x) & \end{tikzcd}$$

where $\mathrm{Gal}(E/F) \triangleleft \mathrm{Gal}(E/\mathbb{R}(x))$ is a normal subgp of order 2.

By Kummer theory, $E \cong \mathbb{C}(x)[T]/(T^p - f)$ for some $f \in \mathbb{C}(x)$.

~~Since $E/\mathbb{R}(x)$ is Galois, $f \in \mathbb{R}(x)$ (see the example below)~~

When $f \in \mathbb{R}(x)$, one gets

$$\mathrm{Gal}(E/\mathbb{R}(x)) \hookrightarrow S_p \circlearrowright \{T, \zeta_p T, \ldots, \zeta_p^{p-1} T\}$$

Injection: if σ fix T, $\zeta_p T$, then σ fix ζ_p, then $\sigma = \mathrm{Id}$.

Since $\#\mathrm{Gal}(E/\mathbb{R}(x)) = 2p$, $\mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$ or $\mathbb{Z}/2p\mathbb{Z}$.

Since $\mathrm{Gal}(E/\mathbb{R}(x)) \leq S_p$, $\mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$.

However, D_p has no order 2 normal subgp, contradiction!

E.g. $\mathbb{C}(z)[T]/(T^3 - (z-i))$ over $\mathbb{R}(z)$ is not Galois, since

$$\begin{tikzcd} \mathbb{C}(T) \arrow[d, dash] \\ \mathbb{C}(z) \arrow[d, dash] \\ \mathbb{R}(z) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] 0 \arrow[d, dash, "3"'] & & z_0 \arrow[d, dash] & \rho z_0 \arrow[dl, dash] & \rho^2 z_0 \arrow[dll, dash] \\ i \arrow[dr, dash] & & -i \arrow[dl, dash] & & \\ & (z^2+1) & & & \end{tikzcd}$$

This example is not general enough. For example,

$\mathbb{C}(z)[T]/(T^3 - \frac{z-i}{z+i})$ over $\mathbb{R}(z)$ can be Galois

Q. For $F/\mathbb{R}(x)$ Galois extension, is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by its order 2 elements?

I call it as the "weaked version of Chebotarev's density theorem for $\mathbb{P}^1_{\mathbb{R}}$".

A: No.

We could not expect the density theorem to be true in the real case,

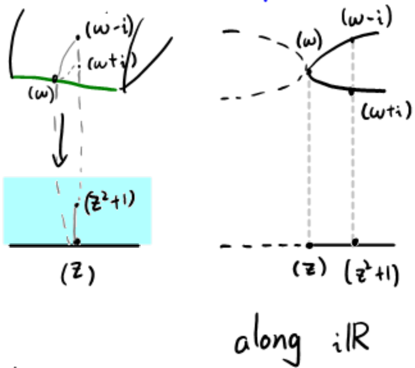
since in S_3 case the order 3 conj class can never be reached by a single Frob.

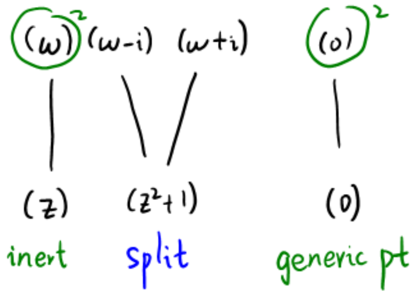
For a possible direct and brutal method to this question, use the result in this link:
math.stackexchange.com/questions/318690/absolute-galois-group-of-mathbbrt
How is Z/3Z realized as the quotient group of this group? (better: compatible with the field extension mentioned above)

4. complexify is a ramified covering by non geometrical connected spaces

Ex. $f: \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ is an unramified covering of alg curves$/\mathbb{R}$.

$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} \arrow[d, "f"] & \mathbb{C}[w] \\ \mathbb{A}^1_{\mathbb{R}} & \mathbb{R}[z] \arrow[u, "f^*"'] \end{tikzcd}$$





This is an unramified covering.

As an $\mathbb{R}$-scheme, $\mathbb{A}^1_{\mathbb{C}}$ is not geo connected.

$$\begin{tikzcd} \mathbb{C}[w]\otimes_{\mathbb{R}}\mathbb{C} \arrow[r, phantom, "\cong"] & \mathbb{C}[w]\oplus\mathbb{C}[w] & f(w)\otimes_{\mathbb{R}} a \arrow[r, mapsto] & (a f(w), \bar{a} f(w)) \\ \mathbb{R}[z]\otimes_{\mathbb{R}}\mathbb{C} \arrow[u] \arrow[r, phantom, "\cong"] & \mathbb{C}[z] \arrow[u, "(Id,\sigma)"'] & f(z)\otimes_{\mathbb{R}} a \arrow[r, mapsto] & a f(z) \end{tikzcd}$$