Eine Woche, ein Beispiel 4.28 naive &- Hom adjunction

Ref: from [23.11.19]

Notation: - A: associate ring allowed to be non-commutative, contains 1 - There are two systems to write category of A-modules.

$$Mod_A = A - Mod$$
 $(Mod_A)^{\circ p} \neq Mod_{A^{\circ p}} = Mod - A = A^{\circ p} - Mod \Rightarrow M_A$
 $Mod_{A \otimes B^{\circ p}} = A - Mod - B \Rightarrow A^{M_B}$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

$$\nabla$$
 Even though you can identify $Ob(Ring) \cong Ob(Ring^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

In this case, it is desirable to translate algebraic results into geometrical results. Q: How to see the geometry of noncommutative rings? It is still vague for me.

- 1 definition recall for ⊗ & Hom
- 2 adjunction
- 3. comparison between ⊗-1 Hom & f*-1 f*

6. comparison between ⊗-1 Hom & f*-1 f*, derived version

1 definition recall for ⊗ & Hom

$$\otimes_A: \operatorname{Mod}_{A^{\circ P}} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$$

 $\operatorname{Hom}_A(-,-): (\operatorname{Mod}_A)^{\circ P} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$

$$\otimes_{B}$$
: $A - Mod - B \times B - Mod - C \longrightarrow A - Mod - C$
 $Hom_{B}(-,-)$: $(A - Mod - B)^{\overline{P}} \times B - Mod - C \longrightarrow A - Mod - C$

$$Hom_{B}^{A}(-,-)$$
: $(A-Mod-B)^{\overline{op}} \times B-Mod-A \longrightarrow \mathbb{Z}-Mod$

$$Hom_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) (\mathbb{Z}-Mod-B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}-Mod-\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}-Mod-\mathbb{Z}$$

$$(X \otimes_{B} Y) \otimes_{C} Z \cong X \otimes_{B} (Y \otimes_{C} Z)$$

$$X \otimes_{B} Y \cong Y \otimes_{B^{op}} X$$

$$A \otimes_{A} X \cong X \cong X \otimes_{B} B$$

$$Hom_{A}(A, X) \cong X$$

in
$$A-Mod-C = C^{op}-Mod-A^{op}$$

2 adjunction BXA, cYB, cZD, we get

 $Homc(Y \otimes_{B} X, Z) \cong Hom_{B}(X, Homc(Y, Z))$ in A-Mod-D.

Reason: both sides equal to the set $f: Y \times X \longrightarrow Z \mid f(cyb,x) = cf(y,bx) \quad \forall b,c$

For A = D = Z, fix $Y \in C$ -Mod-B, one gets adjunction fctors.

slogan: adjunction & associativity

3. comparison between ⊗-1 Hom & f*-1 f*

forgetful fctor

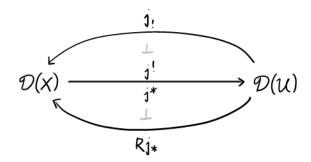
Prop. For ring homo
$$\begin{picture}(1,0) \put(0,0){\line(1,0){150}} \put($$

one has adjunction fctors

djunction fctors
$$S_{R} \otimes_{R} - \frac{\sum_{S_{R} \otimes_{R} - 1}^{S_{R} \otimes_{R}} \otimes_{S_{R} - 1}}{\sum_{S_{R} \otimes_{S} - 1}^{S_{R} \otimes_{S}} \otimes_{S_{R} - 1}} R-Mod \qquad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

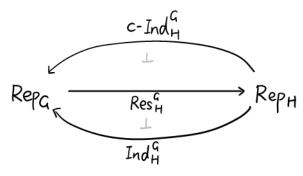


Vague slogan: $u \approx$ "forget the information of Z".

In applications. $U \longrightarrow X$ is a covering map. This change the feeling of the size between U & X.

It can be generalized for
$$G: loc$$
 profinite gp , $H \leq G$ open If one only has $H \leq G$ closed, then it's possible that $j' \neq j^*$. e.g. $G = GL_1(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,



Ex. Compare it with the recollement diagram & (3.1).

$$\mathcal{U}$$
 [*/H]
$$\downarrow j$$
 "cover with fiber G/H"
$$X$$
 [*/G]

translate the following geometrical results into algebraic statements.

1. One has natural fctor
$$j_! \longrightarrow j_*$$
. When $\# G/H < +\infty$, $j_! = j_*$
 $c - Ind_H^G \longrightarrow Ind_H^G$

2. Even though $Sh_{\text{Q-v.s}}([*/G]) \approx \text{Rep}_G = \text{Q[G]-Mod},$ the "structure sheaf" of [*/G] is Q. not Q[G].

$$\operatorname{Res}_{f*}^{G} Q = Q$$
, $\operatorname{Res}_{f*}^{G} Q[G] = Q[G] \neq Q$

 ∇ In this example, $j^*R_{j*} \neq Id$, $j'j! \neq Id$.