Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

A dictionary for myself:

$\{U_i \to U\}_{i\in\Delta}$ ← may be not jointly surj	sieve
topology	Grothendieck topology
topological space	site
$Sh(X)$	topos
sheaf	sheaf
irr closed set/pts	points

Sieve

Def A fctor $F: \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete fibration if
$\forall c \in \mathcal{C}, b \in \mathcal{B}, g \in Mor(b, F(c))$,
$\exists!\ c' \in \mathcal{C}, h \in Mor(c', c)$ s.t. $F(h) = g$.

$$\begin{tikzcd} c' \arrow[r, "h", "\exists!"'] & c \arrow[d, Rightarrow] \\ b \arrow[r, "g"] & F(c) \end{tikzcd}$$

A fctor $F: \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete opfibration if
$F^{op}: \mathcal{C}^{op} \longrightarrow \mathcal{B}^{op}$ is a discrete fibration.

From[https://arxiv.org/abs/1806.06129]: The left-handed version, now opfibrations, was originally called cofibrations, though this name was rejected to avoid confusing topologists.

Def (sieve in small category)
Let $\mathcal{C}$ be a small category. $S \in Cat/\mathcal{C}$.
S is a sieve in $\mathcal{C}$ if the fctor
$$S \longrightarrow \mathcal{C}$$
is fully faithful and a discrete fibration.
For $c \in \mathcal{C}$, $T \in Cat/(\mathcal{C}/c)$.
T is a sieve on c if the fctor
$$T \longrightarrow \mathcal{C}/c$$
is fully faithful and a discrete fibration.
Viewing T as a fullsubcategory of $\mathcal{C}/c$, this is equivalent to
A sieve on c is a subset $T \subseteq Ob(\mathcal{C}/c)$ s.t.
$(f \circ g: e \to c) \in T$ for any $e, d \in \mathcal{C}, (f: d \to c) \in T, g \in Mor(e, d)$.

Def. Now $\mathcal{C}$ can be any category, $c \in \mathcal{C}$.
A sieve on c is a subclass $T \subseteq Ob(\mathcal{C}/c)$ s.t.
$(f \circ g: e \to c) \in T$ for any $e, d \in \mathcal{C}, (f: d \to c) \in T, g \in Mor(e, d)$.

$$\begin{tikzcd} e \arrow[r, "g"] \arrow[dr, dashed, "f\circ g \in T"'] & d \arrow[d, "f \in T"] \\ & c \end{tikzcd}$$

Let $h_c := Mor_{\mathcal{C}}(-, c): \mathcal{C}^{op} \longrightarrow Set$ be a presheaf on $\mathcal{C}$.
$$c' \longmapsto Mor_{\mathcal{C}}(c', c)$$

Thm. When $\mathcal{C}$ is small, There is a bijection between Sets
$$\{\text{sieves on } c \in \mathcal{C}\} \longleftrightarrow \{\text{subfctors of } h_c\}$$
$$T \longmapsto F_T: \mathcal{C}^{op} \longrightarrow Set$$

$$\begin{tikzcd} d \arrow[d, "a"'] & \{(d \to c) \in T\} \\ d' & \{(d' \to c) \in T\} \arrow[u, "a \circ -"'] \end{tikzcd} \quad a \in Mor_{\mathcal{C}}(d, d') \quad \Rightarrow$$

$$T_F := \coprod_{d \in Ob(\mathcal{C})} F(d) \longleftarrow F \subset h_c$$

Q: How to get a correct statement for this theorem when $\mathcal{C}$ is large?

Grothendieck topology, site and topos

On set theoretic issues: https://stacks.math.columbia.edu/tag/00VI
Ironnically, even though what I can actually understand is the Grothendieck topology over a small category, nearly all the applications I need is the Grothendieck topology over a large category.

Def. A Grothendieck topology $\mathcal{T}$ on a category $\mathcal{C}$ is an assignment

$$\mathcal{T}(-): \mathcal{C} \longrightarrow \mathcal{P}(\{\text{sieves on } c \in \mathcal{C} \text{ for some } c\})$$
$$c \longmapsto \mathcal{T}(c) \subseteq \{\text{sieves on } c\}$$

s.t.

1). (Base change) $\forall g \in Mor_{\mathcal{C}}(d, c)$, $T \in \mathcal{T}(c) \Rightarrow g^* T \in \mathcal{T}(d)$
2) (Local character) Let T be a sieve on $c \in \mathcal{C}$. If $[\exists S \in \mathcal{T}(c) \text{ st } \forall (g: d \to c) \in S,\ g^* T \in \mathcal{T}(d)]$ then $T \in \mathcal{T}(c)$
3) $h_c \in \mathcal{T}(c)$

Def. A site $\mathcal{C} = (\mathcal{C}, \mathcal{T})$ is a category equipped with a Grothendieck topology.
A topos is a category equivalent to $Sh(\mathcal{C})$, where $\mathcal{C}$ is a site.

Category + Groth cover	space open sets	continuous map	Covering of open sets	Sh	cohomology
site	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I}, \bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	full sub of Sch/X	—		H
Sch_{Zar}	$Ob(Sch)$	$Mor(Sch)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		$H_{ét}$
$Sch_{ét}$	$Ob(Sch)$	$Mor(Sch)$	ét + l.f.p		
Sch_{sm}	$Ob(Sch)$	$Mor(Sch)$	smooth + l.f.p		
Sch_{fppf}	$Ob(Sch)$	$Mor(Sch)$	f flat + l.f.p		
Sch_{fpqc}	$Ob(Sch)$	$Mor(Sch)$	f flat + f_i^{-1}(q.c) locally q.c		
X/k, $W_n := W_n(k)$ $Cris(X/W_n)$	$\left\{(U,V,i,\delta) \,\middle\vert\, \begin{array}{l} U \subseteq X \text{ open} \\ \cdots \\ \delta\text{: PD-thickening of } U \end{array}\right\}$	$\left\{(\iota,f) \,\middle\vert\, \begin{array}{l} \iota: U \xrightarrow{\text{open}} U' \\ f: V \to V' \\ \text{compatible with PD} \end{array}\right\}$	$\left\{\begin{array}{c}(U_i,V_i,i_i,\delta_i) \\ \downarrow \\ (U,V,i,\delta)\end{array} \,\middle\vert\, \{U_i\} \text{ cover of } U\right\}$		$H^i_{cris}(X/W_n, -)$

(recommended)https://sites.math.washington.edu/~jarod/moduli.pdf
https://pbelmans.ncag.info/notes/etale-cohomology.pdf
http://homepage.sns.it/vistoli/descent.pdf
(crystalline site)http://page.mi.fu-berlin.de/castillejo/docs/crystalline_cohomology.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

it tells us why we don't have small site for most condition:
https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
Here you can find some informations about comparison between fppf and fpqc topologys:
https://mathoverflow.net/questions/361664/some-basic-questions-on-quotient-of-group-schemes

Thm. ① equiv. of categories

$$\mathrm{Sets}((\operatorname{Spec} k)_{\text{ét}}) \longleftrightarrow \mathrm{Disc}\ G_k\text{-Set}$$
$$\mathrm{Ab}((\operatorname{Spec} k)_{\text{ét}}) \longleftrightarrow \mathrm{Disc}\ \mathrm{Mod}_{G_k} \qquad (\star)$$

$G_k := \mathrm{Gal}(k^{sep}/k)$

$(\operatorname{Spec} k)_{\text{ét}} \overset{\text{Site}}{\longleftrightarrow} \overset{\text{finite}}{G_k\text{-Set}}$

② $(\star)$ preserve cohomology
$$H^1((\operatorname{Spec}(k))_{\text{ét}}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\operatorname{Spec} \mathbb{C})_{\text{ét}}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\operatorname{Spec}\mathbb{C})$)

Ex. describe sheaf on $(\operatorname{Spec} \mathbb{R})_{\text{ét}}$

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \arrow[dr, "\iota^*"'] & \operatorname{Spec}\mathbb{C} \arrow[l, "\sigma^*"'] \arrow[d, "\iota^*"] & \arrow[r, leftsquigarrow, "\operatorname{Spec}"] & {} & \mathbb{C} \arrow[r, "\sigma"] & \mathbb{C} \\
& \operatorname{Spec}\mathbb{R} & & & \mathbb{R} \arrow[u, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

σ = conjugation, ι = embedding

$\wr\ \mathcal{F}$

$$\begin{tikzcd}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[r, "\mathcal{F}(\sigma^*)"] & \mathcal{F}(\operatorname{Spec}\mathbb{C}) & \arrow[r, Rightarrow, "\text{Abuse of notation}"] & {} & \mathcal{F}(\mathbb{C}) \arrow[r, "\sigma"] & \mathcal{F}(\mathbb{C}) \\
\mathcal{F}(\operatorname{Spec}\mathbb{R}) \arrow[u, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"'] & & & & \mathcal{F}(\mathbb{R}) \arrow[u, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\qquad Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

[$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed.

in this case $0 \to \mathcal{F}(\operatorname{Spec}\mathbb{R}) \to \mathcal{F}(\operatorname{Spec}\mathbb{C}) \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{F}(\operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C})$

$$\begin{tikzcd}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[r] \arrow[dd, equal] & \mathcal{F}(\operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}\left(\operatorname{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}\right) \arrow[d, phantom, "\cong"] \\
 & \mathcal{F}\left(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\operatorname{Spec}\mathbb{C}\right) \arrow[d, phantom, "\cong"] \\
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[r] & \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})
\end{tikzcd}$$

$\iota_1: x \longmapsto x\otimes 1$
$\iota_2: x \longmapsto 1\otimes x$
$x\otimes y \longmapsto (xy, x\bar{y})$

ι_2: $\operatorname{Spec}\mathbb{C} \xleftarrow{(Id,\sigma)} \operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\operatorname{Spec}\mathbb{C}) \xrightarrow{(\mathcal{F}(Id),\mathcal{F}(\sigma))} \mathcal{F}(\operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})$

Abuse $\rightsquigarrow$ of notation $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$]

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\operatorname{Spec} k, \mathcal{F}) = \mathcal{F}(\operatorname{Spec} k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)}$$

$\mathcal{F}_{k^{sep}} := \varinjlim_{L/k \text{ finite}} \mathcal{F}(\operatorname{Spec} L)$

Ex. descibe the stalk & fiber at $p \in \operatorname{Spec} k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\operatorname{Spec} k, p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts