Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \quad \mathcal{F}' \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$f^* \dashv f_*$$
$$- \otimes \mathcal{F} \dashv \underline{\mathrm{Hom}}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{\mathrm{Hom}}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{\mathrm{Hom}}(\mathcal{F}, f_*\mathcal{G})$

$$\begin{tikzcd} & \otimes \arrow[dl, dash] \arrow[dr, dash] & \\ f^* \arrow[rr, dash] & & f_! \end{tikzcd}$$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^! \mathcal{F}) \cong \underline{\mathrm{Hom}}(f_! \mathcal{G}, \mathcal{F})$

$f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') \cong \underline{\mathrm{Hom}}(f^*\mathcal{F}, f^!\mathcal{F}')$

bc: $f^* g_! \cong g'_! f'^*$

$f_* g^! \cong g'^! f'_*$

These extra formulas (compatabilities) come from the upgrade of adjunction formula to internal Hom.
To upgrade the adjunction between tensor product and internal Hom, one don't need extra formula, except the association law of tensor product.

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \to pt$

$H^\cdot(X; \underline{\mathbb{Z}}) := p_* p^* \mathbb{1}$	$H^\cdot(X; \mathcal{F}) := p_* \mathcal{F}$	$= R\Gamma(X; \mathcal{F})$
$H^\cdot_c(X; \underline{\mathbb{Z}}) := p_! p^* \mathbb{1}$	$H^\cdot_c(X; \mathcal{F}) := p_! \mathcal{F}$	
$H_\cdot(X; \underline{\mathbb{Z}}) := p_! p^! \mathbb{1}$	$H_\cdot(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$	$= H^\cdot_c(X; p^! \mathbb{1} \otimes \mathcal{F})$
$H^{BM}_\cdot(X; \underline{\mathbb{Z}}) := p_* p^! \mathbb{1}$	$H^{BM}_\cdot(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$	$= H^\cdot(X; p^! \mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

$$H^\cdot_c(X; \mathcal{F}) \otimes H^\cdot_c(Y; \mathcal{G}) \cong H^\cdot_c(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

reduced to: $p_{X,!} \mathcal{F} \otimes p_{Y,!} \mathcal{G} \cong p_!(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G})$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\quad -^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X; \underline{\mathbb{Z}})[d] \cong H_\cdot(X; \underline{\mathbb{Z}})^\vee$$

reduced to: $p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$

$$Z \xrightarrow[\text{close}]{i} X \xhookleftarrow{j} U \quad \rightsquigarrow$$

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(Z) \arrow[r, "\text{pi}\ i_* = i_!\ \text{f.f}"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\text{pi}\ i^!"] \arrow[r, "\text{pi}\ j^* = j^!"] & \mathcal{D}(U) \arrow[l, bend right, "j_!\ \text{f.f.}"'] \arrow[l, bend left, "\text{pi}\ Rj_*\ \text{f.f}"]
\end{tikzcd}$$

Id, Id, Id, Id

$i^* j_! = 0$, $j^* i_* = 0$, $i^! Rj_* = 0$

f.f. fully faithful

pi: preserve injectives. (内射)

i.e. inj sheaf $\rightsquigarrow$ inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Q} = Or_X[n] \overset{+\text{orientation}}{=\!=\!=} \mathbb{Q}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \xrightarrow{+1}$$

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1}$$

Here, $H_{-1}(S^1; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$R\Gamma(X, Z; \mathcal{F}) \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$

$$R\Gamma(X, U; \mathcal{F}) \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(U; \mathcal{F}|_U) \xrightarrow{+1}$$

$$\overset{\shortparallel}{R\Gamma_Z(X; \mathcal{F})}$$

When $\mathcal{F} = \mathbb{Q}_X$, $H^\cdot(X, Z) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(Z) \xrightarrow{+1}$

$H^\cdot(X, U) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(U) \xrightarrow{+1}$

When $\mathcal{F} = \mathbb{D}_X$, $R\Gamma(X, j_! \mathbb{Q}_U) \longrightarrow H_\cdot^{BM}(X) \longrightarrow R\Gamma(Z; \mathbb{D}_X|_Z) \xrightarrow{+1}$

$H_\cdot^{BM}(Z) \longrightarrow H_\cdot^{BM}(X) \longrightarrow H_\cdot^{BM}(U) \xrightarrow{+1}$

Taking $R\pi_{X,!}$

$$R\Gamma_c(U, \mathcal{F}|_U) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$

$$R\Gamma_c(Z, i^! \mathcal{F}) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(X, Rj_*(\mathcal{F}|_U)) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{Q}_X$, $H_c^\cdot(U) \longrightarrow H_c^\cdot(X) \longrightarrow H_c^\cdot(Z) \xrightarrow{+1}$

$H_c^\cdot(Z, i^! \mathbb{Q}_X) \longrightarrow H_c^\cdot(X) \longrightarrow H_c^\cdot(X, Rj_* \mathbb{Q}_U) \xrightarrow{+1}$

When $\mathcal{F} = \mathbb{D}_X$, $H_\cdot(U) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, U) \xrightarrow{+1}$

$H_\cdot(Z) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, Z) \xrightarrow{+1}$

Explanation

Exactness & derived

$j_!$ is exact — by checking on stalks

i^*, j^* are exact — in the category Top

i_* is exact — when $Z \subset X$ is (strongly) loc. contractable.

j_* is not exact $\rightsquigarrow Rj_*$

i^* is already derived.

Rmk: strongly loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p s.t. $U_n \cap Z$ is contractable

loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p s.t. $U_n \cap Z \subset U_n$ is contractable

E.g. $\{\text{strongly loc. contractable}\} \subseteq \{\text{loc. contractable}\} \not\subseteq \text{Top}$

CW-cplx, topo mflds & algebraic varieties (Check?) — Cantor set

See the answer in
https://math.stackexchange.com/questions/1082601/anr-is-locally-contractible
for the subtlety of these two definitions.

I don't care. In both cases, the local cohomology vanishes in higher degree, and that's what I want.

For the non-exact functors, there maybe some problems in the composition of derived functors.

https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof
https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de

E.g. we need to check if $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.

Luckily, in the open-closed formalism, we won't meet these problems.

Prop1. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ ($\perp$), assume F is exact. Then

① G preserves injective sheaves;

② $RG(f) \circ RG(g) = RG(f \circ g)$

Proof. ①: by universal property.

②. by adjunction

Prop 2. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C}''$. Suppose F or F' is exact, then

$$RG \circ RG'(f) = (R(G \circ G'))(f)$$

Proof. By adjunction & Grothendieck-Serre sequence. ($LF' \circ LF = L(F' \circ F)$)
When F' is exact, can use Prop 1 ①.

Cor. $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.
Rf_* are nice in general, while $f^!$ should be a bit careful
(but $j^!$, $i^!$ are nice, when Z is loc. contractable)