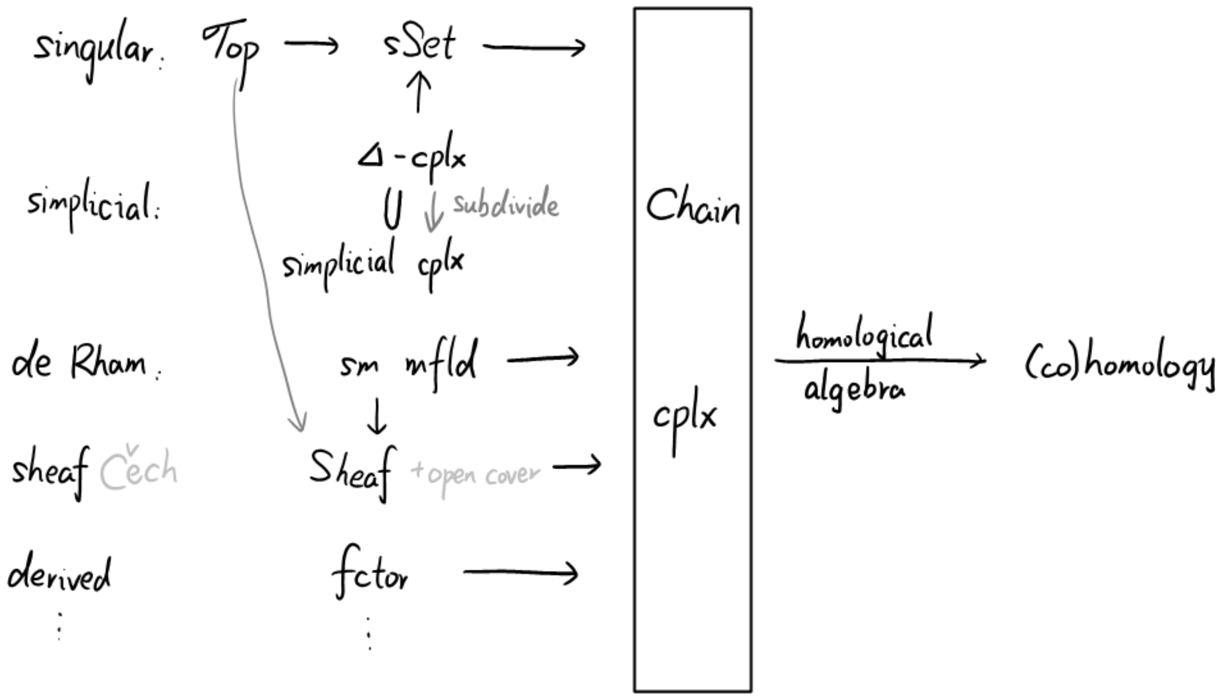
Eine Woche, ein Beispiel
6.24 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet $\longrightarrow$ chain cplx - - → (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha\in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1-d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2-d_0^1+d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n}\mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n}\mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

E.g. 1 For $A \in Top$, $X := \mathcal{S}(A) \in sSet$, one can compute

$$C_\bullet(X;G):\quad 0 \longleftarrow \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C^\bullet(X;G):\quad 0 \longrightarrow \prod_{\alpha\in A} G \xrightarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \cdots$$

$$C_\bullet^{BM}(X;G):\quad 0 \longleftarrow \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C_c^\bullet(X;G):\quad 0 \longrightarrow \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \cdots$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H_n^{BM}(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H_c^n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta^1;G)$ & $H^n(\Delta^1;G)$.
Notice that $\#\Delta^1_k = k+2$, so

$$C_\bullet(\Delta^1;G):\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots$$

basis: $d'_0 \triangleq x_0$, $d'_1 \triangleq x_1$

remember indexes.

x_0, x_1, x_2 ; x_0, x_1, x_2, x_3 ; x_0, x_1, x_2, x_3, x_4

$$0 = x_0 - x_0 \longleftarrow x_0$$
$$x_0 - x_1 = x_0 - x_1 \longleftarrow x_1$$
$$0 = x_1 - x_1 \longleftarrow x_2$$

$$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow x_0$$
$$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow x_1$$
$$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow x_2$$
$$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow x_3$$
$$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow x_4$$

$$x_0 = x_0 - x_0 + x_0 \longleftarrow x_0$$
$$x_0 = x_0 - x_1 + x_1 \longleftarrow x_1$$
$$x_2 = x_1 - x_1 + x_2 \longleftarrow x_2$$
$$x_2 = x_2 - x_2 + x_2 \longleftarrow x_3$$

By taking the transpose, one get

$$C^\bullet(\Delta^1;G):\quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix}0&0\\1&-1\\0&0\end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix}1&0&0\\1&0&0\\0&0&1\\0&0&1\end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix}0&0&0&0\\1&-1&0&0\\0&0&0&0\\0&0&1&-1\\0&0&0&0\end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta^1;G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta^1;G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$\begin{tikzcd}
\Delta^1 \arrow[d, "s_0^1"] & C_\cdot(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\ 1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0&1&0\\ 0&-1&0 \end{pmatrix}"'] \arrow[d, "(1\ 1\ 1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1&1&0&0\\ 0&0&0&0\\ 0&0&1&1 \end{pmatrix}"'] \arrow[d, "(1\ 1\ 1\ 1)"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix} 0&1&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&0 \end{pmatrix}"'] \arrow[d, "(1\ 1\ 1\ 1\ 1)"] \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "\mathrm{Id}"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^0 \arrow[d, "d_0^1"] & C_\cdot(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix} 1\\0 \end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix} 1\\0\\0 \end{pmatrix}"] & G \arrow[l, "\mathrm{Id}"'] \arrow[d, "\begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix} 1\\0\\0\\0\\0 \end{pmatrix}"] \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0&1&0\\ 0&-1&0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1&1&0&0\\ 0&0&0&0\\ 0&0&1&1 \end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix} 0&1&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&0 \end{pmatrix}"']
\end{tikzcd}$$

$s_0^1 \qquad d_0^1$

s.t. $\quad s^1_{0,*} \circ d^1_{0,*} = \mathrm{Id}_{C_\cdot(\Delta^1;G)}, \qquad d^1_{0,*} \circ s^1_{0,*} \sim \mathrm{Id}_{C_\cdot(\Delta^1;G)}$.

In fact, we have

$$\begin{tikzcd}
C_\cdot(\Delta^1;G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "(\cdot)_*"] & 0 \arrow[dr, dashed, "0"'] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 1&0\\ 0&0 \end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix} 1&0\\ 0&1\\ 0&0 \end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0&1&0\\ 0&-1&0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 1&1&1\\ 0&0&0\\ 0&0&0 \end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&0&-1\\ 0&0&0 \end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1&1&0&0\\ 0&0&0&0\\ 0&0&1&1 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 1&1&1&1\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix} 1&0&0&0\\ 0&1&1&1\\ 0&0&0&-1\\ 0&0&0&1\\ 0&0&0&0 \end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix} 0&1&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 1&1&1&1&1\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{pmatrix}"] \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0&1&0\\ 0&-1&0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1&1&0&0\\ 0&0&0&0\\ 0&0&1&1 \end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix} 0&1&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&0 \end{pmatrix}"']
\end{tikzcd}$$

$x_0 \mapsto x_0$

$x_1 \mapsto x_1$

$x_0 \mapsto x_0 - x_0 + x_0 = x_0$

$x_1 \mapsto x_1 - x_1 + x_1 = x_1$

$x_2 \mapsto x_1 - x_2 + x_2 = x_1$

$x_3 \mapsto x_1 - x_2 + x_3$

$x_0 \mapsto x_0 - x_0 = 0$

$x_1 \mapsto x_1 - x_1 = 0$

$x_2 \mapsto x_1 - x_2$

Ex. Observe the picture, try to translate the calculation in geometrical language.