Eine Woche, ein Beispiel

5.25 cyclic coverings

Main Ref: [Barth04, I.17]

https://archive.ymsc.tsinghua.edu.cn/pacm_download/27/8789-Introduction-cyclic-cover.pdf

[Barth04]: Barth, Wolf P., Klaus Hulek, Chris A. M. Peters and Antonius Van De Ven. Compact Complex Surfaces.

Setting $Y/\mathbb{C}$ conn cplx mfld (or sm integral variety)
$B \in \mathrm{Div}(Y)$ effective
$\mathcal{L} \in \mathrm{Pic}(Y)$ with $\mathcal{L}^{\otimes m} \cong \mathcal{O}_Y(B)$ $\quad m \in \mathbb{N}_{\geq 1}$
$s \leftrightarrow 1$

$$\mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{L}^{-1} \oplus \cdots \oplus \mathcal{L}^{-m+1})$$
$$\|$$

$$\begin{tikzcd}
\rightsquigarrow L \mathrel{\hat{=}} \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{L}^{-1}) \arrow[d, "p"] & \supset & \mathrm{Spec}(\mathrm{Sym}^{\cdot}\mathcal{L}^{-1}/\langle s\rangle) \mathrel{\hat{=}} X \arrow[dll, "f"] \\
Y = \mathrm{Spec}(\mathcal{O}_Y) & &
\end{tikzcd}$$

Define (X, Y, f) as the n-cyclic covering of Y branched along B, determined by $\mathcal{L}$.

Rmk. X has at most singularities over singular pts of B.

Denote $B_1 := f^{-1}(B) \in \mathrm{Div}(X)$, then $\qquad f^*B = mB_1$

$$\begin{array}{lllll|ll}
X & \mathcal{O}_X(B_1) & \mathcal{O}_X(mB_1) & \Omega_X \otimes \pi^*\mathcal{L}^{n-1} & & \mathcal{O}_X & \mathcal{O}_X(B_1) \\
f\downarrow & & & & & & \\
Y & \mathcal{L} & \mathcal{O}_Y(B) & \Omega_Y & & \bigoplus_{j=0}^{n-1} \mathcal{L}^{-j} & \bigoplus_{j=-1}^{n-2} \mathcal{L}^{-j}
\end{array}$$

pullback $\qquad\qquad\qquad$ pushforward

E.g. For a hyperelliptic curve $\mathcal{C}$ with $f: \mathcal{C} \xrightarrow{2:1} \mathbb{P}^1$, $g(C)=g$,

$$B_1 := \{\text{Weierstrass pts of } \mathcal{C}\} = \{p_1, \dots, p_{2g+2}\}$$
$$B := f(B_1)$$

We define

$$\mathcal{L}_0 := f^*\mathcal{O}_{\mathbb{P}^1}(1) \in g^1_2,$$

then

$$\omega_{\mathcal{C}} = f^*\mathcal{O}_{\mathbb{P}^1}(g-1) = \mathcal{L}_0^{g-1},$$

and

$$\begin{array}{cccccccc}
 & & \omega_{\mathcal{C}} & \mathcal{O}_{\mathcal{C}}(B_1) & \mathcal{O}_{\mathcal{C}}(2B_1) & \omega_{\mathcal{C}} \otimes \pi^*\mathcal{L} & & \\
 & & \| & \| & \| & \| & & \\
\mathcal{C} & \mathcal{L}_0 & \mathcal{L}_0^{g-1} & \mathcal{L}_0^{g+1} & \mathcal{L}_0^{2g+2} & \mathcal{L}_0^{-2} & \mathcal{O}_{\mathcal{C}} & \mathcal{O}_{\mathcal{C}}(B_1) \\
{\scriptstyle f}\downarrow & & & & & & & \\
\mathbb{P}^1 & \mathcal{O}(1) & \mathcal{O}(g-1) & \mathcal{O}(g+1) & \mathcal{O}(2g+2) & \mathcal{O}(-2) & \mathcal{O} \oplus \mathcal{O}(-g-1) & \mathcal{O}(g+1) \oplus \mathcal{O} \\
 & & & \| & \| & \| & \| & \| \\
 & & & \mathcal{L} & \mathcal{O}_{\mathbb{P}^1}(B) & \omega_{\mathbb{P}^1} & \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{L}^{-1} & \mathcal{L} \oplus \mathcal{O}_{\mathbb{P}^1}
\end{array}$$

pullback　　　　　　pushforward

$$\operatorname{div} \frac{y}{\prod_{i=g+2}^{2g+2}(x-p_i)} = \sum_{i=1}^{g+1} p_i - \sum_{i=g+2}^{2g+2} p_i$$