Eine Woche, ein Beispiel

8.21 equivariant K-theory of $\mathbb{P}^1$

Let us do a simple case over $\mathbb{P}^1$. It can be generlized "easily" to flag variety, but $\mathbb{P}^1$ is the beginning case of study.

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand

$$\begin{tikzcd}
K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[d] \\
K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2}(pt) \arrow[d] \\
K^{B}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{B}(pt) \arrow[d] \\
K(\mathbb{P}^1) \arrow[r] & K(pt)
\end{tikzcd}$$

where $SL_2 = SL_{2,\mathbb{C}}$, $\quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq SL_{2,\mathbb{C}}$,

$\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} \cong G/B$. $\quad G \underset{\text{mult}}{\overset{\text{left}}{\curvearrowright}} \mathbb{P}^1 \quad \mathbb{C}^\times \overset{\text{trivial}}{\curvearrowright} \mathbb{P}^1$

Notation. For linear alg gp G [Ginz, 5.1],

$K_i^G(X) := K_i(Coh^G(X)) \qquad K^G(X) := K_0^G(X) \qquad K(X) := K^{\{Id\}}(X)$

$R(G) := K^G(pt) = K_0(Coh^G(pt)) = K_0(Rep\, G)$

e.g. $R(\{Id\}) = \mathbb{Z}$, $R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm1}]$, $R(SL_2) \cong \mathbb{Z}[x]$, $R(SL_2\times\mathbb{C}^\times) \cong \mathbb{Z}[x, t^{\pm1}]$

Some furthur discussion of $R(SL_2)$.

$R(SL_2) = \bigoplus_{i\in\mathbb{N}_{\geq 0}} \mathbb{C} x_i$ where x_i represents the $(i+1)$-dim irr rep of SL_2.

As an algebra, $R(SL_2) = \mathbb{C}[x]$ where

$1 = x_0$

$x = x_1$

$x^2 = x_2 + 1$

$x^3 = x_3 + 2x_1$

$x^4 = x_4 + 3x_2 + 2$

$$\begin{array}{ccccccccc}
1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \\
 & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\
 & & 1 & 0 & 3 & 0 & 9 & 0 & 28 \\
 & & & 1 & 0 & 4 & 0 & 14 & 0 \\
 & & & & 1 & 0 & 5 & 0 & 20 \\
 & & & & & 1 & 0 & 6 & 0 \\
 & & & & & & 1 & 0 & 7 \\
 & & & & & & & 1 & 0 \\
 & & & & & & & & 1
\end{array}$$

$x_0 = 1$

$x_1 = x$

$x_2 = x^2 - 1$

$x_3 = x^3 - 2x$

$x_4 = x^4 - 3x^2 + 1$

$$\begin{array}{ccccccccc}
1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\
 & 1 & 0 & -2 & 0 & 3 & 0 & -4 & 0 \\
 & & 1 & 0 & -3 & 0 & 6 & 0 & -10 \\
 & & & 1 & 0 & -4 & 0 & 10 & 0 \\
 & & & & 1 & 0 & -5 & 0 & 15 \\
 & & & & & 1 & 0 & -6 & 0 \\
 & & & & & & 1 & 0 & -7 \\
 & & & & & & & 1 & 0 \\
 & & & & & & & & 1
\end{array}$$

[Ginz, (5.2.4)] $\begin{matrix} G_1 \circlearrowright \\ G_2 \circlearrowright_{\text{trivial}} \end{matrix} X \Rightarrow K^{G_1 \times G_2}(X) \cong K^{G_1}(X) \otimes_{\mathbb{Z}} R(G_2)$

e.g. $K^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1) \cong K^{SL_2}(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(\mathbb{C}^\times) \cong K^{SL_2}(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$

$K^B(\mathbb{P}^1) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(B) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[y^{\pm 1}]$

[Ginz, (5.2.17)]

$$K_i^H(X) \underset{\mathrm{Ind}_H^G}{\overset{\mathrm{Res}_H^G}{\rightleftarrows}} K_i^G(G \times_H X)$$

e.g. $K^{SL_2}(\mathbb{P}^1) \cong K^{SL_2}(SL_2 \times_B pt) \cong K^B(pt) = R(B) = \mathbb{Z}[z^{\pm 1}]$

Q: What is the $R(SL_2)$-module structure on $K^{SL_2}(\mathbb{P}^1)$?

$x \cdot - : \mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z^{\pm 1}]$

My answer: the action is induced by $R(SL_2) \longrightarrow R(B) \circlearrowright K^{SL_2}(\mathbb{P}^1)$

$$\begin{array}{cccccc} x & R(SL_2) & \times & K^{SL_2}(\mathbb{P}^1) & \longrightarrow & K^{SL_2}(\mathbb{P}^1) \\ \downarrow & \downarrow & & \wr\wr & & \wr\wr \\ z+z^{-1} & R(B) & \times & R(B) & \xrightarrow{\text{multi}} & R(B) \end{array}$$

so $x \cdot - \quad \mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z^{\pm 1}]$

$f \longmapsto (z + z^{-1}) \cdot f$

[LCBE, 2.1.1] $K(\mathbb{P}^1) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}^1}(1) = \mathbb{Z}[z]/(z-1)^2 = \mathbb{Z}[z^{\pm 1}]/(z-1)^2$

⚠ z corresponds to $\mathcal{O}_{\mathbb{P}^1}(-1)$ here.

[Ginz, 5.2.13] gives def of pushforward.

In conclusion, we get

$$
\begin{tikzcd}
K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[d] \\
K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2}(pt) \arrow[d] \\
K^{B}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{B}(pt) \arrow[d] \\
K(\mathbb{P}^1) \arrow[r] & K(pt)
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[d, mapsto] & \mathbb{Z}[z^{\pm1}, t^{\pm1}] \arrow[r] \arrow[d] & \mathbb{Z}[x, t^{\pm1}] \arrow[d] & t \arrow[d, mapsto] \\
1 & \mathbb{Z}[z^{\pm1}] \arrow[r] \arrow[d] & \mathbb{Z}[x] \arrow[d] & 1 \\
y \arrow[d, mapsto] & \mathbb{Z}[z, y^{\pm1}]/(z-1)^2 \arrow[r] \arrow[d] & \mathbb{Z}[y^{\pm1}] \arrow[d] & y \arrow[d, mapsto] \\
1 & \mathbb{Z}[z]/(z-1)^2 \arrow[r] & \mathbb{Z} & 1 \\
& w \arrow[r, mapsto] & 1 &
\end{tikzcd}
$$

The difficult part is the middle square.

Down: $\mathbb{Z}[z, y^{\pm1}]/(w-1)^2 \longrightarrow \mathbb{Z}[y^{\pm1}]$

$$
\begin{aligned}
z &\longmapsto 1 \\
y &\longmapsto y \\
y^{-1} &\longmapsto y^{-1}
\end{aligned}
$$

Right: by rep theory,

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm1}]$ homo as $\mathbb{Z}$-alg

$$
\begin{aligned}
x_0 &\longmapsto 1 \\
x_1 &\longmapsto y+y^{-1} \\
x_2 &\longmapsto y^2+1+y^{-2} \\
x_3 &\longmapsto y^3+y+y^{-1}+y^{-3} \\
&\ \vdots \qquad\qquad \vdots
\end{aligned}
$$

Up: by Borel-Weil-Bott theorem,

$\mathbb{Z}[z^{\pm1}] \longrightarrow \mathbb{Z}[x]$ homo as $\mathbb{Z}[x]$-module

$$
\begin{aligned}
1 &\longmapsto 1 & \qquad z &\longmapsto 0 \\
z^{-1} &\longmapsto x_1 & z^2 &\longmapsto -1 \\
z^{-2} &\longmapsto x_2 & z^3 &\longmapsto -x_1 \\
z^{-3} &\longmapsto x_3 & z^4 &\longmapsto -x_2 \\
& & z^5 &\longmapsto -x_3 \\
\vdots & \qquad \vdots & \vdots & \qquad \vdots
\end{aligned}
$$

Left: by [LW-BWB, Ex 2.6], $\mathcal{L}_n \cong \mathcal{O}(-n)$, combined with "Up", we get

$\mathbb{Z}[z^{\pm1}] \longrightarrow \mathbb{Z}[z, y^{\pm1}]/(z-1)^2$

e.g. $z^3 \longmapsto -z^3(y+y^{-1})$ (see table below)

z	z^{-2}	z^{-1}	1	z	z^2	z^3	z^4	z^m
x	x_2	x_1	1	0	-1	$-x_1$	$-x_2$	$\begin{cases} -x_{m-2} & m\geq 2 \\ 0 & m=1 \\ x_{-m} & m\leq 0 \end{cases}$
y	y^2+1+y^{-2}	$y+y^{-1}$	1	0	-1	$-y-y^{-1}$	$-y^2-1-y^{-2}$	$-\dfrac{y^m-y^{-m+2}}{y^2-1}$

Under these (natural) ring structure,

$$\mathbb{Z}[x, t^{\pm 1}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm 1}] \longrightarrow \mathbb{Z}$$

are homo of rings.

Ex. Generalize to

- $SL_2 \rightsquigarrow SL_n$, $\mathbb{P}^1 \rightsquigarrow \mathcal{F}lag(\mathbb{C}^n)$
- $SL_2 \rightsquigarrow GL_2$
- $\mathbb{C} \rightsquigarrow \mathbb{F}_p$ $\quad \mathbb{C}^\times \rightsquigarrow \mathbb{F}_p^\times$

Q: How to compute $K_i^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1)$ for $i \geq 1$?