

Eine Woche, ein Beispiel

1.21. complex multilinear algebra

The title comes from
<http://staff.ustc.edu.cn/~wangzuq/Courses/16F-Manifolds/Notes/Lec16.pdf>

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

M : cplx mfld, $p \in M$

$M_{\mathbb{R}}$: M viewed as smooth mfld, not base change
 better: M_{sm}

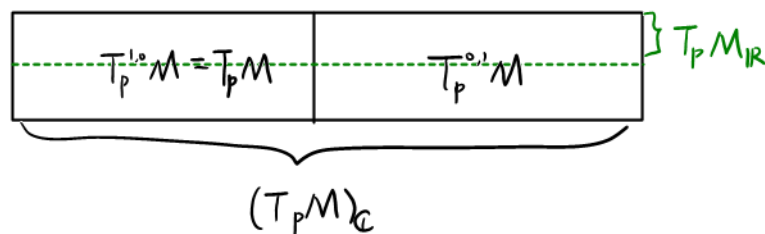
eg. $M = \mathbb{C}$ $p = 0$

Notation	base field	dim	basis	name	[YS20]
$T_p M$	\mathbb{C}	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p M_{\mathbb{R}}$	\mathbb{R}	6	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	real tangent vector	$T_p^{\mathbb{R}} M$
$(T_p M)_{\mathbb{C}} = T_p M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	\mathbb{C}	6	$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ or $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	complexified tangent vector	$T_p^{\mathbb{C}} M$
$T_p^{1,0} M = T_p M$	\mathbb{C}	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p^{0,1} M$	\mathbb{C}	3	$\frac{\partial}{\partial \bar{z}_i}$	anti-holomorphic tangent vector	
$T_p^* M$	\mathbb{C}	3	dz_i	holomorphic 1-form	Ω_p'
$T_p^* M_{\mathbb{R}} \cong \Omega_{\mathbb{R}, p}'$	\mathbb{R}	6	dx_i, dy_i	real 1-form	
$(T_p^* M)_{\mathbb{C}} = T_p^* M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	\mathbb{C}	6	$dz_i, d\bar{z}_i$ or dx_i, dy_i	complexified 1-form	$T_p^{\mathbb{C}} M = A_p'$
$T_p^{1,0,*} M \cong \Omega_p^{1,0} = T_p^{*,0} M$	\mathbb{C}	3	$d\bar{z}_i$	(1,0)-form	$T_p^{1,0,*} M = A_p^{1,0}$
$T_p^{0,1,*} M \cong \Omega_p^{0,1}$	\mathbb{C}	3	dz_i	(0,1)-form	$T_p^{0,1,*} M = A_p^{0,1}$

$\Omega^i, \Omega^{i,j}$ sheaves on M

Rmk. We don't have any natural identification between $T_p M$ & $T_p M_{\mathbb{R}}$.
 Notice that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, $-\frac{1}{2}i$ is not real, so $\frac{\partial}{\partial \bar{z}} \notin T_p M_{\mathbb{R}}$.

although our geometrical intuition of $T_p M$ is often $T_p M_{\mathbb{R}}$,
 $T_p M \cap T_p M_{\mathbb{R}} = \emptyset$ in $(T_p M)_{\mathbb{C}}$.



Reminder: the (induced) almost complex structure is defined as

$$\begin{aligned}
 J: T_p M_{\mathbb{R}} &\longrightarrow T_p M_{\mathbb{R}} \\
 \frac{\partial}{\partial x_i} &\longmapsto \frac{\partial}{\partial y_i} \\
 \frac{\partial}{\partial y_i} &\longmapsto -\frac{\partial}{\partial x_i} \\
 \rightsquigarrow J: T_p M &\longrightarrow T_p M \\
 J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) &= \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\
 J\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i}\right) &= \left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i}\right) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
 \end{aligned}$$

real basis of $(T_p M)_{\mathbb{C}}$:

$$\begin{aligned}
 \mathcal{B}_1 &= \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y} \right\} \\
 \underbrace{\quad}_{\substack{\hookrightarrow T_p M_{\mathbb{R}} \\ \text{Id}}} &\longleftrightarrow \underbrace{\quad}_{\substack{\hookrightarrow i T_p M_{\mathbb{R}} \otimes \mathbb{C} \\ -\text{Id}}}
 \end{aligned}
 \quad
 \begin{cases} \frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \\ \frac{\partial}{\partial y} = \frac{1}{i}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}) \end{cases}
 \quad
 \begin{cases} dx = \frac{1}{2}(dz + d\bar{z}) \\ dy = \frac{1}{2i}(dz - d\bar{z}) \end{cases}$$

$$\begin{aligned}
 \mathcal{B}_2 &= \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, i\frac{\partial}{\partial z}, i\frac{\partial}{\partial \bar{z}} \right\} \\
 \underbrace{\quad}_{\substack{\hookrightarrow T_p M \\ \times i}} &\longleftrightarrow \underbrace{\quad}_{\substack{\hookrightarrow T_p M \otimes \mathbb{C} \\ \times i}}
 \end{aligned}
 \quad
 \begin{cases} \frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \end{cases}
 \quad
 \begin{cases} dz = dx + i dy \\ d\bar{z} = dx - i dy \end{cases}$$

$$\otimes: J, \text{conj}_J$$

$$\otimes: \text{conj}_i$$

$$\otimes: \times i$$

$$\begin{aligned}
J(f\partial_x + g\partial_y) &= f\partial_y - g\partial_x \\
J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_y, -\partial_x, i\partial_y, -i\partial_x) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & 1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
J(f\partial_z + g\partial_{\bar{z}}) &= if\partial_z - ig\partial_{\bar{z}} \\
J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (i\partial_z, -i\partial_{\bar{z}}, -\partial_z, \partial_{\bar{z}}) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_J(f\partial_x + g\partial_y) &= \bar{f}\partial_x - \bar{g}\partial_y \\
\text{conj}_J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, -\partial_y, -i\partial_x, i\partial_y) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_J(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_z + \bar{g}\partial_{\bar{z}} \\
\text{conj}_J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_z, \partial_{\bar{z}}, -i\partial_z, -i\partial_{\bar{z}}) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_i(f\partial_x + g\partial_y) &= \bar{f}\partial_x + \bar{g}\partial_y \\
\text{conj}_i(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, \partial_y, -i\partial_x, -i\partial_y) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_i(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_{\bar{z}} + \bar{g}\partial_z \\
\text{conj}_i(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_{\bar{z}}, \partial_z, -i\partial_{\bar{z}}, -i\partial_z) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}
\end{aligned}$$