

Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

Task.

1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. tangent space, Euler class
7. basis of Hecke alg

We may use two examples for the convenience of presentation.

Readers can easily distinguish them by the dim vectors.

1. dimension vector

$$|\underline{d}| = 5$$

$$\underline{d} = (3, 2)$$

$$\underline{d} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 0 \\ 0, 0 \end{pmatrix} = \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \end{array} = \begin{array}{c} \downarrow \times \times \times \\ \downarrow \times \end{array} \in W_d \backslash W_{\text{id}} \text{ or } \text{Min}(W_{\text{id}}, W_d)$$

$v_{\text{id}} = \pi_{\underline{d}}^{-1}(F_{\text{id}})$

2. Weyl group

Set	element	special element	others
$W_{\text{id}} = S_5$	w, x	$w_{\text{max}} = \begin{array}{ c c c c c } \hline & * & * & * & * \\ \hline * & & & & \\ \hline \end{array}$	$T = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{\text{max}} = \begin{array}{ c c c } \hline * & * & * \\ \hline * & & \\ \hline * & & \\ \hline \end{array}$	$T_d = \{s_1, s_2, s_4\}$
$W_d \backslash W_{\text{id}} = S_3 \times S_2 \backslash S_5$	w, \underline{d}	$\begin{array}{ c c c } \hline * & * & * \\ \hline * & & \\ \hline * & & \\ \hline \end{array}$	(Comp _d)

$$\text{Min}(W_{\text{id}}, W_d) = \left\{ \begin{array}{|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}, \dots \right\} u \quad \begin{array}{|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array} \quad (\text{Shuffled})$$

$$0 \longrightarrow W_d \longrightarrow W_{\text{id}} \longrightarrow W_{\text{id}} \backslash W_d \longrightarrow 0 \quad w = wu \mapsto \underline{d}$$

$\xrightarrow{\text{Min}(W_{\text{id}}, W_d)} \xrightarrow{\cong} u \xrightarrow{\downarrow} \underline{d}$

$w = \begin{array}{|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}$
 $u = \begin{array}{|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}$
 $w = \begin{array}{|c|c|c|} \hline * & * & * \\ \hline * & & \\ \hline * & & \\ \hline \end{array}$

Another example: $\underline{d} = (1, 2)$ $\xrightarrow{a} b$ $\xrightarrow{\langle v_1 \rangle \rightarrow \langle v_2, v_3 \rangle}$

	$w = wu$	w	\underline{d}, u	order of basis	(w)	$l(w)$	B_w	B_{wu}	wB_w^{-1}
Id	$(1^2 3)$	$ \sqcup \sqcup$	$[1, 1]$	$ \sqcup \sqcup$	abb	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$
t	(23)	$ \times \sqcup$	$[1, 1]$	$ \times \sqcup$	abb	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
s	(12)	$X \sqcup$	$[1, 1]$	$ \sqcup \sqcup$	bab	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
ts	(132)	$X \times$	$[1, 1]$	$ \times \sqcup$	bab	2	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
st	(123)	$X \times$	$[1, 1]$	$ \sqcup \sqcup$	bba	2	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
sts	(13)	$* \times$	$[1, 1]$	$ \times \sqcup$	bba	3	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$

3. alg group & Lie algebra

$$G_{\text{Idl}}, B_{\text{Idl}}, T_{\text{Idl}}, N_{\text{Idl}} \quad W_{\text{Idl}} = N_{G_{\text{Idl}}}(\Pi_{\text{Idl}}) / \Pi_{\text{Idl}} \quad GL_5(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$G_d, B_d, T_d, N_d \quad W_d = N_{G_d}(T_d) / T_d \quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$B_\infty = B_{\text{Idl}} \xrightarrow{\omega} B_{\text{Idl}} \omega^{-1} = \text{Stab}_{G_{\text{Idl}}}(F_\infty)$$

$$B_\infty \xrightarrow{\omega = w u} w B_d w^{-1} = \text{Stab}_{G_d}(F_\infty) \quad N_\infty = R_u(B_\infty)$$

For $s \in \Pi$ s.t. $\omega s \omega^{-1} \in W_d$ (i.e. $W_d \omega = W_d \omega s$), define

$$P_{\infty, \omega s} \xrightarrow{\omega = w u} w(B_d u s u^{-1} B_d \cup B_d) w^{-1} \quad N_{\infty, \omega s} = R_u(B_{\infty, \omega s})$$

$$= B_\infty \omega s \omega^{-1} B_\infty \cup B_\infty \quad = N_\infty \cap N_{\infty, \omega s}$$

$$M_{\infty, \omega s} = N_\infty / N_{\infty, \omega s}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{\sigma(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\infty, \omega''} := N_\infty \cap N_\infty$$

$$M_{\infty, \omega''} := N_\infty / N_{\infty, \omega''}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega''}$$

Their Lie algebras are collected here.

$$\mathfrak{g}_{\text{Idl}}, \mathfrak{b}_{\text{Idl}}, \mathfrak{t}_{\text{Idl}}, \mathfrak{n}_{\text{Idl}}$$

$$g_d \quad b_d \quad t_d \quad n_d$$

$$\mathfrak{b}_\infty \quad \bar{\mathfrak{b}}_\infty$$

$$b_\infty \quad n_\infty$$

$$P_{\infty, \omega s} \quad N_{\infty, \omega''}$$

$$m_{\infty, \omega''}$$

$$\bar{\mathfrak{b}}_\infty = \mathfrak{b}_{\omega \max \infty}$$

$$\bar{b}_\infty = b_{w \max \infty}$$

$$\bar{P}_{\infty, \omega s} = P_{w \max \infty, w \max \infty s}$$

$$m_{\infty, \omega''}$$

$$\text{Rep}_d(Q) := \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{\text{Idl}}^{\oplus k}$$

$$V_\infty = \{f \in \text{Rep}_d(Q) \mid f: F_{\infty, i} \subseteq F_{\infty, i}\} = \mu_d \pi_d^{-1}(F_\infty)$$

$$= \begin{smallmatrix} v_3 & v_1 & v_2 \\ v_5 & * & * \\ v_4 & * & * \end{smallmatrix} = \begin{smallmatrix} v_1 & v_2 & v_3 \\ v_4 & * & * \\ v_5 & * & * \end{smallmatrix}$$

$$V_{\omega(i)}$$

$$V_{\infty, \omega''} = V_\infty \cap V_{\infty''}$$

$$J_{\infty, \omega''} = V_\infty / V_{\infty, \omega''}$$

Later we may twist the group actions.

$$\text{E.g. } \underline{r}_{\infty, \omega'} := r_{\infty, \omega \omega'} \quad r_{\infty, \omega''} = \underline{r}_{\infty, \omega^{-1} \omega''}$$

4. typical variety

Id corres to

$$\begin{aligned}
 F_{\text{Id}\text{Id}} &\cong G_{\text{Id}\text{Id}} / B_{\text{Id}\text{Id}} & F_{\text{Id}} \\
 F_{\underline{d}} &\cong G_{\underline{d}} / B_{\underline{d}} & F_u \\
 F_{\infty} &\cong G_{\underline{d}} / B_{\infty} & F_{\omega} \\
 F_{\underline{d}} &= \coprod_{\underline{d}'} F_{\underline{d}'} & - \\
 F_{g\infty} &\cong G_{\underline{d}} / gB_{\infty}g^{-1} & F_{g\infty} \\
 F_{\infty, \cdot} &= \text{Flag}_{\underline{d}}(F_{\text{Id}}) = F_{\{v_{\infty(1)}, v_{\infty(2)}, \dots, v_{\infty(\text{Id})}\}} & \\
 &= F_{\{u_5, u_3, u_1, u_6, u_2\}}
 \end{aligned}$$

✓ The action on Flag is not the same as in

http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_TomaszPrzezdziecki.pdf

$$F_{\text{Id}\text{Id}} \neq \coprod_{\underline{d}} F_{\underline{d}}$$

$F_{\infty} \cong F_{\underline{d}}$ with different base pt. Base pt makes difference!

$$\begin{aligned}
 F_{\text{Id}\text{Id}} \times F_{\text{Id}\text{Id}} && F_{\text{Id}, \text{Id}} \\
 F_{\underline{d}} \times F_{\underline{d}'} && F_{u, u'} \\
 F_{\infty} \times F_{\infty'} && F_{\infty, \infty'} \\
 F_{\underline{d}} \times F_{\underline{d}'} &= \coprod_{\underline{d}, \underline{d}'} (F_{\underline{d}} \times F_{\underline{d}'}) & -
 \end{aligned}$$

$$F_{\infty, \infty'} := (F_{\infty}, F_{\infty'})$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$\mu_{\underline{d}}^{-1}(M) \cong \text{Flag}_{\underline{d}}(M) \subseteq F_{\underline{d}}$ is the Springer fiber.

$$\begin{array}{ccc}
 Z_{\underline{d}, \underline{d}'} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}'} \\
 \downarrow M_{\underline{d}, \underline{d}'} & & \downarrow \pi_{\underline{d}, \underline{d}'} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}'}
 \end{array}$$

$$\begin{array}{ccc}
 Z_{\underline{d}} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}, \underline{d}} & & \downarrow \pi_{\underline{d}, \underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}}
 \end{array}$$

$$\begin{array}{c}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \subseteq \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) := \bigsqcup_{\underline{d}} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{array}$$

$$\widetilde{\text{Rep}}_{\infty}(\mathbb{Q}) \cong G_{\underline{d}} \times^{B_{\infty}} r_{\infty}$$

$$\begin{aligned}
 Z_{\underline{d}, \underline{d}'} &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}'}(\mathbb{Q}) \\
 Z_{\underline{d}} &= \bigsqcup_{\underline{d}', \underline{d}''} Z_{\underline{d}, \underline{d}'} \\
 &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{aligned}$$

$$Z_{\infty, \infty'} = Z_{u, u'}$$

5. (equivariant) stratifications.

In the following tables, $uw' = \tilde{w}'\tilde{u}$.

$F_\infty \in \widetilde{\text{Rep}}_d(Q)$ means $(p_0, F_\infty); (F_\infty, F_{\infty'}) \in Z_d$ means $(p_0, F_\infty, F_{\infty'})$.

▽ $G \times G$ acts on $\mathcal{F} \times \mathcal{F}$ in a twisted way

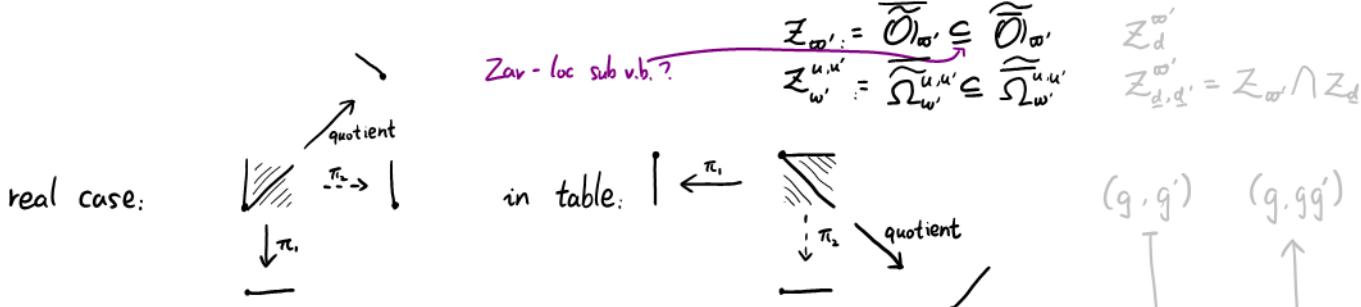
$$\text{e.g. } (g_1, g_2) F_{\infty, \infty'} = F_{g_1 \infty, g_1 \tilde{w} g_2^{-1} \infty'}$$

$$(g_1, g_2) E_{\infty, \infty'} = E_{g_1 \infty, g_2 \tilde{w} \infty'}$$

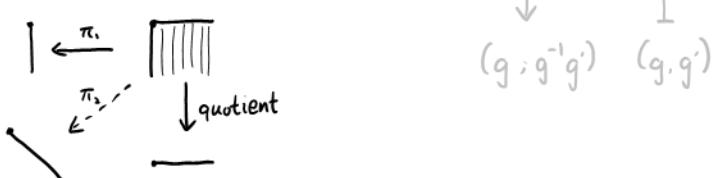
variety base point	stratification stabilizer	type	B-orbit	$B \times B$ -orbit	$B \times G$ -orbit	$G \times B$ -orbit	Remark $G \times \{*\}$ -orbit
\mathcal{B}	$\mathcal{B} \times \mathcal{B}$		Ω_g	$\Omega_{g, g'}$	$\text{pr}_1^{-1}(\Omega_g)$	$\Omega_{g'}$	
F_g	(F_g, F_{gg})		$B \cap gBg^{-1}$	$(B \cap gBg^{-1}) \times (B \cap gBg^{-1})$	$(B \cap gBg^{-1}) \times g'Bg'^{-1}$	$gBg^{-1} \times (B \cap g'Bg'^{-1})$	$gBg^{-1} \cap gg'B(gg')^{-1}$
F_{ldl}	$F_{\text{ldl}} \times F_{\text{ldl}}$		\mathcal{V}_∞	$\mathcal{V}_{\infty, \infty'}$	$\text{pr}_1^{-1}(\mathcal{V}_\infty)$	$\mathcal{V}_{\infty'}$	
F_∞	$(F_\infty, F_{\infty\infty})$		$B_{\text{ldl}} \cap B_\infty$	$(B_{\text{ldl}} \cap B_\infty) \times (B_{\text{ldl}} \cap B_\infty)$	$(B_{\text{ldl}} \cap B_\infty) \times B_{\infty'}$	$B_\infty \times (B_{\text{ldl}} \cap B_\infty)$	$B_\infty \cap B_{\infty\infty'}$
F_u	$F_u \times F_{u'}$		Ω_w^u	$\Omega_{w, w'}^{u, u'}$	$\text{pr}_{1,u}^{-1}(\Omega_w^u)$	$\Omega_{w'}^{u, u'}$	
F_{wu}	(F_{wu}, F_{wwu})		$B_d \cap B_w$	$(B_d \cap B_w) \times (B_d \cap B_w)$	$(B_d \cap B_w) \times B_{w'}$	$B_w \times (B_d \cap B_w)$	$B_w \cap B_{wwu}$
F_d	$F_d \times F_d$		Ω_w^u	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\text{pr}_{1,u}^{-1}(\Omega_w^u)$	$\Omega_{\infty'}^{u, u'} = \Omega_{\tilde{w}}^{u, \tilde{u} u'}$	compatibility
F_∞	$(F_\infty, F_{\infty\infty})$		$B_d \cap B_w$	$(B_d \cap B_w) \times (B_d \cap B_{\tilde{w}})$	$(B_d \cap B_w) \times B_{\tilde{w}'}$	$B_w \times (B_d \cap B_{\tilde{w}})$	$B_w \cap B_{\infty\infty'}$
F_{wu}	$(F_{wu}, F_{wwu\tilde{w}'})$						

The following may not be single orbit, but derived from the above definition.

\mathcal{F}_d	$\mathcal{F}_d \times \mathcal{F}_d$	\mathcal{O}_∞	$\mathcal{O}_{\infty, \infty'}$	$\text{pr}_1^{-1}(\mathcal{O}_\infty)$	$\mathcal{O}_{\infty'}$	preimage of
F_∞	$(F_\infty, F_{\infty\infty})$	Ω_w^u	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\sqcup \text{pr}_{1,u}^{-1}(\Omega_w^u)$	$\sqcup \Omega_{\tilde{w}}^{u, \tilde{u} u'}$	$\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\text{ldl}} \times \mathcal{F}_{\text{ldl}}$
$\widetilde{\text{Rep}}_d(Q)$	$Z_{d, d'}$	$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, w'}^{u, u'}$	$\text{pr}_{1,u}^{-1}(\widetilde{\Omega}_w^u)$	$\widetilde{\Omega}_{w'}^{u, u'}$	preimage of
F_{wu}	(F_{wu}, F_{wwu})					$Z_{d, d'} \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{\text{Rep}}_d(Q)$	Z_d				$\widetilde{\Omega}_\infty^u$	preimage of
F_∞	$(F_\infty, F_{\infty\infty})$				$\widetilde{\Omega}_{\infty'}^u$	$Z_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{\text{Rep}}_d(Q)$	Z_d	$\widetilde{\mathcal{O}}_\infty$	$\widetilde{\mathcal{O}}_{\infty, \infty'}$	$\text{pr}_1^{-1}(\widetilde{\mathcal{O}}_\infty)$	$\widetilde{\mathcal{O}}_{\infty'}$	preimage of
F_∞	$(F_\infty, F_{\infty\infty})$	$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\sqcup \text{pr}_{1,u}^{-1}(\widetilde{\Omega}_w^u)$	$\sqcup \widetilde{\Omega}_{\tilde{w}}^{u, \tilde{u} u'}$	$Z_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$



We want gp action to be compatible with π_i and the quotient map. Therefore, we would do a twist.



The following tables may help you to understand the notations.

\dim	$B_{\text{Id}} \cdot F_{\text{var}}$	0	1	1	2	2	3
	$B_{\text{Id}} \times B_{\text{Id}} \cdot (F_{\text{var}}, F_{\text{var}})$	\mathcal{V}_{Id}	\mathcal{V}_t	\mathcal{V}_s	\mathcal{V}_{ts}	\mathcal{V}_{st}	\mathcal{V}_{sts}
	$B_{\text{Id}} \cdot F_w$	\mathcal{V}_{Id}	$\mathcal{V}_{\text{Id},\text{Id}}$	$\mathcal{V}_{\text{Id},t}$	$\mathcal{V}_{\text{Id},s}$	$\mathcal{V}_{\text{Id},ts}$	$\mathcal{V}_{\text{Id},st}$
0							
1							
2							
3							

\dim	$B_d \cdot F_{\text{var}}$	0	1	2	3	4	5
	$B_d \times B_d \cdot (F_{\text{var}}, F_{\text{var}})$	\mathcal{O}_{Id}	\mathcal{O}_t	\mathcal{O}_s	\mathcal{O}_{ts}	\mathcal{O}_{st}	\mathcal{O}_{sts}
	$B_d \cdot F_w$	\mathcal{O}_{Id}	$\mathcal{O}_{\text{Id},\text{Id}}$	$\mathcal{O}_{\text{Id},t}$	$\mathcal{O}_{\text{Id},s}$	$\mathcal{O}_{\text{Id},ts}$	$\mathcal{O}_{\text{Id},st}$
0							
1							
2							
3							
4							
5							

The following tables may help you to understand the notations.

$$\omega = ts, \omega' = s$$

\dim	$B_{Id} \cdot F_{ts}$	0	1	1	2	2	3	$\text{pr}_i^{-1}(\mathcal{V}_{ts})$
	$B_{Id} \times B_{Id} \cdot (F_{ts}, F_{ts})$	\mathcal{V}_{Id}	\mathcal{V}_t	\mathcal{V}_s	\mathcal{V}_{ts}	\mathcal{V}_{st}	\mathcal{V}_{sts}	
	$B_{Id} \cdot F_{ts}$	\mathcal{V}_{Id}	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
0		\mathcal{V}_{Id}	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
1		\mathcal{V}_t	$\mathcal{V}_{t,t}$	$\mathcal{V}_{t,Id}$	$\mathcal{V}_{t,ts}$	$\mathcal{V}_{t,s}$	$\mathcal{V}_{t,sts}$	$\mathcal{V}_{t,st}$
1		\mathcal{V}_s	$\mathcal{V}_{s,s}$	$\mathcal{V}_{s,st}$	$\mathcal{V}_{s,Id}$	$\mathcal{V}_{s,sts}$	$\mathcal{V}_{s,t}$	$\mathcal{V}_{s,ts}$
2		\mathcal{V}_{ts}	$\mathcal{V}_{ts,st}$	$\mathcal{V}_{ts,s}$	$\mathcal{V}_{ts,sts}$	$\mathcal{V}_{ts,Id}$	$\mathcal{V}_{ts,ts}$	$\mathcal{V}_{ts,t}$
2		\mathcal{V}_{st}	$\mathcal{V}_{st,ts}$	$\mathcal{V}_{st,sts}$	$\mathcal{V}_{st,t}$	$\mathcal{V}_{st,st}$	$\mathcal{V}_{st,Id}$	$\mathcal{V}_{st,s}$
3		\mathcal{V}_{sts}	$\mathcal{V}_{sts,sts}$	$\mathcal{V}_{sts,ts}$	$\mathcal{V}_{sts,st}$	$\mathcal{V}_{sts,t}$	$\mathcal{V}_{sts,s}$	$\mathcal{V}_{sts,Id}$

shape	$B_d \cdot F_{ts}$	\mathcal{F}_{Id}	\mathcal{F}_s	\mathcal{F}_{st}	$\text{pr}_i^{-1}(\mathcal{O}_{ts})$	$\text{pr}_{i,Id}^{-1}(\Omega_t^s)$	$\Omega_{t,Id}^{s,Id} = \mathcal{O}_{ts,s}$
	$B_d \times B_d \cdot (F_{ts}, F_{ts})$	\mathcal{O}_{Id}	\mathcal{O}_t	\mathcal{O}_s	\mathcal{O}_{ts}	\mathcal{O}_{st}	\mathcal{O}_{sts}
\mathcal{F}_{Id}	\mathcal{O}_{Id}	$\Omega_{Id,Id}^{Id,Id}$	$\Omega_{Id,t}^{Id,Id}$	$\Omega_{Id,s}^{Id,s}$	$\Omega_{Id,t}^{Id,s}$	$\Omega_{Id,ts}^{Id,Id}$	$\Omega_{Id,st}^{Id,Id}$
	\mathcal{O}_t	$\Omega_{t,t}^{Id,Id}$	$\Omega_{t,Id}^{Id,Id}$	$\Omega_{t,s}^{Id,s}$	$\Omega_{t,Id}^{Id,s}$	$\Omega_{t,ts}^{Id,Id}$	$\Omega_{t,st}^{Id,Id}$
\mathcal{F}_s	\mathcal{O}_s	$\Omega_{Id,Id}^{s,Id}$	$\Omega_{Id,t}^{s,Id}$	$\Omega_{Id,Id}^{s,s}$	$\Omega_{Id,t}^{s,s}$	$\Omega_{Id,ts}^{s,Id}$	$\Omega_{Id,st}^{s,Id}$
	\mathcal{O}_{ts}	$\Omega_{t,t}^{s,Id}$	$\Omega_{t,Id}^{s,Id}$	$\Omega_{t,t}^{s,s}$	$\Omega_{t,Id}^{s,s}$	$\Omega_{t,ts}^{s,Id}$	$\Omega_{t,st}^{s,Id}$
\mathcal{F}_{st}	\mathcal{O}_{ts}	$\Omega_{Id,Id}^{st,Id}$	$\Omega_{Id,t}^{st,Id}$	$\Omega_{Id,Id}^{st,s}$	$\Omega_{Id,t}^{st,s}$	$\Omega_{Id,ts}^{st,Id}$	$\Omega_{Id,st}^{st,Id}$
	\mathcal{O}_{sts}	$\Omega_{t,t}^{st,Id}$	$\Omega_{t,Id}^{st,Id}$	$\Omega_{t,t}^{st,s}$	$\Omega_{t,Id}^{st,s}$	$\Omega_{t,ts}^{st,Id}$	$\Omega_{t,st}^{st,Id}$

6. Change of basis

§6.1. Two basis

Def Let $Y \subset X$ be G -equiv closed subvariety, X proj.

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

with same notation,

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{S(G)} \in H_q^*(X; \mathbb{Q})$$

$$\begin{array}{ccc} Y & \xhookrightarrow{\iota_Y} & X \\ & \downarrow \pi_Y & \\ & pt & \end{array}$$

By cellular fibration lemma,

$$\begin{array}{ccccccc} K_0^{Td}(F_d) & \cong & K_0^{Cd}(F_d \times F_d) & \cong & K_0^{Cd}(Z_d) \\ \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Td} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Cd} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[Z_\omega]^{Cd} \\ & & & & & & \downarrow \\ & & & & & \parallel S & K_0^{Td}(Z_d) \\ & & & & & & \\ & & & & & & \oplus_{\omega, \omega' \in W_{td}} R(T_d)[\bar{\mathcal{O}}_{\omega, \omega'}]^{Td} \end{array}$$

as $R(T_d)$ -modules.

⚠ There is no evidence if $[Z_\omega]^{Cd}$ will be mapped to $\oplus_{\omega \in W_{td}} [\bar{\mathcal{O}}_{\omega, \omega'}]^{Td}$. Luckily, the horizontal line sends generators to generators.

Hint: Consider the following commutative diagram:

$$\begin{array}{ccccc} F_d & \xrightarrow{(F_d, Id)} & F_d \times F_d & \xrightarrow{(p_0, Id)} & Z_d \\ \bar{\mathcal{O}}_\omega & \nearrow & \nearrow & \nearrow & \\ pt & = & pt & = & pt \end{array}$$

To do linear alg., we take

$$\begin{aligned} \mathcal{R}(G) &:= \text{Frac } (R(G)) \\ \mathcal{S}(G) &:= \text{Frac } (S(G)) \end{aligned}$$

$$\begin{aligned} \mathcal{K}_o^G(X) &= K_o^G(X) \otimes_{R(T_d)} \mathcal{R}(T_d) \\ \mathcal{H}_G^*(X; \mathbb{Q}) &= H_G^*(X; \mathbb{Q}) \otimes_{S(T_d)} S(T_d) \end{aligned}$$

For $R(T_d)$ -mod $K_o^G(X)$, $S(T_d)$ -mod $H_G^*(X; \mathbb{Q})$

$$\begin{aligned} \text{Define } \psi_\infty &:= [\{F_\infty\}]^{T_d} = (i_\infty)_* 1_{R(T_d)} \in K_o^{T_d}(\mathcal{F}_d) \\ \psi_{\infty, \infty'} &:= [\{f(p_0, F_\infty, F_{\infty'})\}]^{T_d} = (i_{\infty, \infty'})_* 1_{R(T_d)} \in K_o^{T_d}(\mathbb{Z}_d) \\ \psi_{\infty, \infty'} &:= [\{f(p_0, F_\infty, F_{\infty'})\}]^{T_d} \end{aligned}$$

We get two $\mathcal{R}(T_d)$ -basis. (ψ_∞ is $\mathcal{R}(T_d)$ -basis, by Localization theorem.)

$$\begin{array}{ccc} K_o^{T_d}(\mathcal{F}_d) & \longrightarrow & K_o^{T_d}(\mathbb{Z}_d) \\ [\overline{O_\infty}]^{T_d} & & [\overline{O_{\infty, \infty'}}]^{T_d} \\ \psi_\infty & & \psi_{\infty, \infty'} \end{array} \quad \begin{array}{l} \text{standard basis: for stratification} \\ \text{canonical basis: for convolution} \end{array}$$

Localization thm [Thm 10.1]

Let $i: X^{T_d} \hookrightarrow X$, X is smooth.

$$\begin{array}{ccccc} \mathcal{K}_o^{T_d}(X^{T_d}) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X^{T_d}) \\ \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) \end{array}$$

are isos as $\mathcal{R}(T_d)$ or $\mathcal{S}(T_d)$ -module.

Q: The Steinberg variety Z_d is usually not smooth.

How to show that $\{\psi_{\infty, \infty'}\}$ forms a basis?

Guess: apply localization thm to $T_d \times T_d$ first.

§ 6.2. tangent space, Euler class.

Def (tangent space of fixed pts. in $R(T_d)$)

$$\begin{aligned} \Lambda_\infty &:= T_{F_\infty} \mathcal{F}_d \\ \widetilde{\Lambda}_\infty &:= T_{(p_0, F_\infty)} \widetilde{\text{Rep}}_d(\mathbb{Q}) \\ \Lambda_{\infty, \infty'} &:= \end{aligned}$$