

§ 1.1. Structure of finite/local/global field

Road map

	finite field	local field		global field	adéle
		Archi	NA		
base field F	⁷ \mathbb{F}_q <small>For \mathbb{F}_{q^r}</small>	¹ \mathbb{F}_p <small>$\epsilon \cdot \mu_r$</small>	² \mathbb{R} or \mathbb{C} <small>$\mathbb{R}^+ \times \mathbb{Z}/\mathbb{Z}$</small>	³ \mathbb{Q}_p $\mathbb{Z}_p^\times \times \mathbb{Z}$	⁴ \mathbb{Q} \mathbb{Z}/\mathbb{Z}
integral ring \mathcal{O}_F	—	—	—	$\mathbb{F}_p[[t]]$	$\mathbb{F}_p(t)$
units \mathcal{O}_F^\times	—	—	—	\mathbb{Z}_p^\times $\mathbb{Z}_p[[t]]^\times$	\mathbb{Z}/\mathbb{Z} \mathbb{F}_p^\times
$\text{Gal}(F^{\text{sep}}/F)$	$\hat{\mathbb{Z}}?$	$\hat{\mathbb{Z}}$	\mathbb{Z}/\mathbb{Z} total order?	Id	most known choose a lift finite
ari Frob	?	can	—	—	<small>unramified $\xrightarrow{n \neq 1}$ Frob_n Frob_p conj class $\xrightarrow{\text{abelian}}$ Frob_p</small>
# ext of deg n	$1?$	1	1/o	—	inf countable
Spec \mathcal{O}_F	$\text{Spec } \mathbb{F}_q = K(\hat{\mathbb{Z}}, 1)$ [étale, 22.4]	—	—	—	—
topology topo of \mathcal{O}_F	? discrete	Euclidean	—	—	restricted
measure	? discrete	Lebesgue	profinite opt. not discrete $\mu(\mathcal{O}_F) = 1$	—	K is a lattice in A_K can be computed

Also, discuss

- field extension, norm, trace, ...
- their connection to geometry, ramification theory
- analog with knot theory

Process

1. finite field \mathbb{F}_q
2. Archi local field \mathbb{R} or \mathbb{C}
3. NA local field
 - Individual structure
 - Field extension
4. global field
 - Dessin d'enfants
5. local and global: connections
 - Basics
 - Traditional point of view
 - Frobenius
 - Application: Quadratic reciprocity
 - Étale point of view
6. local to global: adèle
 - Base field with automorphism
 - Galois extension
7. \mathbb{F}_1

1. finite field \mathbb{F}_q

Any fin field is of form \mathbb{F}_q , where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

\mathbb{F}_q = the splitting field of $X^q - X$ over \mathbb{F}_p .

$$\begin{aligned} \text{Gal}(\mathbb{F}_q / \mathbb{F}_p) &\cong \widehat{\mathbb{Z}} & \text{as top gps} \\ \text{Frob}_p &\longleftrightarrow 1 \end{aligned}$$

2. Archi local field \mathbb{R} or \mathbb{C}

No difficulty: $\text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $\text{Gal}(\mathbb{C}/\mathbb{C}) = \text{Id}$

\mathbb{C} is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of \mathbb{Q}_p) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, U, U^{(n)}$ in terms of v
- What is the structure of \mathbb{Q}_p^\times ?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?
- Can we classify open subgps of F, F^\times ?
- Give a description of the Haar measure on F and F^\times .

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F .
- Find a wild extension of \mathbb{Q}_p & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of \mathbb{Q}_p " vividly?

We will discuss section 4 in [NAext] together. Some questions.

- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)}$
- Explain why we have $F_r \circ F_r^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions. (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field) ?
- When is the field extension over \mathbb{Q}_p complete?
- Using the result in [NAval], computes the following Galois gps:

$$\text{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep} / \mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right), \quad \text{Gal}\left(\widehat{\mathbb{Q}_p} / \widehat{\mathbb{Q}_p^{ur}}\right), \quad \text{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} / \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)$$

$G_{\mathbb{F}_p((t))}$

$I_{\mathbb{Q}_p}$

$G_{\mathbb{F}_p((t))}$

4. global field

$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.

$\text{Gal}(\mathbb{F}_p^{\text{sep}}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D], we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\} / \text{birational}$$

$\text{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler: by [GalFun, Thm 3.4.8],

$$\text{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C})$$

↑Free profinite gp on \mathbb{C}

Shafarevich's conj: See wiki: Absolute Galois group

$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{\text{ab}})$ is a free profinite gp

Q: Does $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object/geo realizations?

Dessin d'enfants

By [GalFun, Prop 47.1 - Rmk 47.9], we have an including

$$\begin{array}{ccccccc} \text{induced by } & \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \hookrightarrow & \text{Out}(\pi_1^{\text{ét}}(\mathbb{P}_{\overline{\mathbb{Q}}}^1 - \{0, 1, \infty\})) \\ & \pi_{1, \overline{\mathbb{Q}}}^{\text{ét}} = \pi_1^{\text{ét}}(\mathbb{P}_{\overline{\mathbb{Q}}}^1 - \{0, 1, \infty\}) & , & \pi_{1, \mathbb{Q}}^{\text{ét}} = \pi_1^{\text{ét}}(\mathbb{P}_{\mathbb{Q}}^1 - \{0, 1, \infty\}) & & & \\ & 1 \longrightarrow \pi_{1, \overline{\mathbb{Q}}}^{\text{ét}} & \longrightarrow & \pi_{1, \mathbb{Q}}^{\text{ét}} & \longrightarrow & \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) & \longrightarrow 1 \\ & \downarrow \text{?} & \parallel & & \downarrow \text{conj } g \mapsto g - g^{-1} & & \downarrow \exists! \\ 1 \longrightarrow Z(\pi_{1, \overline{\mathbb{Q}}}^{\text{ét}}) & \longrightarrow & \pi_{1, \overline{\mathbb{Q}}}^{\text{ét}} & \longrightarrow & \text{Aut}(\pi_{1, \overline{\mathbb{Q}}}^{\text{ét}}) & \longrightarrow & \text{Out}(\pi_{1, \overline{\mathbb{Q}}}^{\text{ét}}) \longrightarrow 1 \end{array}$$

The space $\mathbb{P}_{\overline{\mathbb{Q}}}^1 - \{0, 1, \infty\}$ is designed for guaranteeing that
 $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \text{Out}(\pi_{1, \overline{\mathbb{Q}}}^{\text{ét}})$

is inclusion.

Task. Read [Dessin d'enfant] or [Collins],
understand the $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -action on the dessin d'enfants.

- Def of Dessin d'enfant
- Connections with $\text{Out}(\pi_{1,\bar{\mathbb{Q}}}^{\text{\'et}})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.

What is a dessin d'enfants? / Quel est un dessin d'enfants?

Example: $S = X = \mathbb{P}^1$

Which one is which?

$f(z) = C \frac{z^4(z-1)^2}{z - \frac{11+2\sqrt{10}}{18}}$

$f(z) = C' \frac{z^4(z-1)^2}{z - \frac{11-2\sqrt{10}}{18}}$

Xiaoxiang Zhou
Dessin d'enfant: an Introduction

What is a dessin d'enfants? / Quel est un dessin d'enfants?

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- Can we generalize this to $\text{Gal}(\mathbb{F}_p(t)^{\text{sep}}/\mathbb{F}_p(t))$?
I don't know how to make a "dessin d'enfant" on alg curves over \mathbb{F}_p .

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.

5. local and global connections

Basics

$$\begin{array}{ccc}
 K \hookrightarrow K_v & \text{for any valuation } v & (\leadsto K \hookrightarrow A_K) \\
 \downarrow & \downarrow & \\
 \mathcal{O}_K \hookrightarrow \mathcal{O}_v & \text{and } \mathcal{O}_v \text{ is the integral closure of } \mathcal{O}_K \text{ in } K_v & (\text{for } v \text{ fin}) \\
 \searrow & & \\
 & k_v & \text{and } \mathcal{O}_K/\mathfrak{p}^r \mathcal{O}_K \cong (\mathcal{O}_K/\mathfrak{p}^r \mathcal{O}_K)_v \cong \mathcal{O}_v/\mathfrak{p}^r \mathcal{O}_v \quad \forall r \in \mathbb{N}_{\geq 0}
 \end{array}$$

The connections are also compatible with field exts. (L/K Galois)
 E.g.

$$\begin{array}{ccccc}
 L & \mathcal{O}_L & q \sim w & \text{---} & \\
 | & \uparrow & | & & \\
 K & \mathcal{O}_K & p \sim v & \text{---} &
 \end{array}$$

$$\mathfrak{p} \mathcal{O}_L = q_1^{e_1} \cdots q_g^{e_g} \quad L \otimes_K K_v \cong \prod_w L_w$$

$$\begin{array}{ccc}
 L \hookrightarrow \prod_w L_w & & L \hookrightarrow \prod_w L_w \\
 N_{LK} \downarrow & \downarrow \prod_w N_{Lw/Kv}^{ew} & \text{Tr}_{LK} \downarrow \sum_w e_w \text{Tr}_{Lw/Kv} \\
 K \hookrightarrow K_v & & K \hookrightarrow K_v
 \end{array}$$

Traditional point of view

$$\begin{array}{ccc}
 K \longrightarrow pt & \cdot & K \\
 \mathcal{O}_v \longrightarrow \text{"loc" curve} & \dashrightarrow & \mathcal{O}_v \\
 \mathcal{O}_K \longrightarrow \text{curve} & \text{---} \bullet & \mathcal{O}_K
 \end{array}$$

Task. Read [Algfungp, 0.2], answer the following questions:

- Understand ramified, inert, split.

- Understand (from geo meaning)

$$\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \bigoplus_{i=1} \mathcal{O}_L/q^{e_i}\mathcal{O}_L$$

- Compute some classical examples

- Compare it with ramified covering in RS.

Frobenius $(L/K \text{ Galois})$

$$1 \rightarrow \text{Gal}(L/K)_w \xrightarrow{\cong} \text{Gal}(L/K)$$

//S

$$\text{Gal}(L_w/K_v)$$

//S

$$1 \rightarrow I(L_w/K_v) \rightarrow \text{Aut}_{O_v\text{-alg}}(O_w) \rightarrow \text{Gal}(K_w/K_v) \rightarrow 1$$

where

$$\text{Gal}(L/K)_w = \{\sigma \in \text{Gal}(L/K) \mid \sigma(q) = q\} \leq \text{Gal}(L/K)$$

By <https://math.stackexchange.com/questions/4131855/frobenius-elements>,

- more conditions \Rightarrow better props of Frob
- ① L/K is unramified at $w \Rightarrow \begin{cases} I(L_w/K_v) = \text{Id} \\ \text{Fr}_w \in \text{Gal}(L/K)_w \text{ well-defined} \end{cases}$
- ② L/K Galois $\Rightarrow \begin{cases} \text{all symbols are meaningful} \\ \forall q, q' \in \text{Spec } O_L, \exists \sigma \in \text{Gal}(L/K), \sigma(q) = q' \\ \text{Fr}_{ob_v} = [\text{Fr}_{ob_w}] \text{ is a conj class in } \text{Gal}(L/K) \end{cases}$
- ③ $L \subseteq K^{ab} \Rightarrow \text{Fr}_{ob_v} \in \text{Gal}(L/K)$

Application: Quadratic reciprocity

Thm. p, l odd primes, $p \neq l$, then

$$\left(\frac{p}{l}\right) \left(\frac{l}{p}\right) = (-1)^{\frac{p-1}{2} \frac{l-1}{2}}$$

$$\left(\frac{-1}{l}\right) = (-1)^{\frac{l-1}{2}} = \begin{cases} 1 & l \equiv 1 \pmod{4} \\ -1 & l \equiv 3 \pmod{4} \end{cases}$$

$$\left(\frac{2}{l}\right) = (-1)^{\frac{l-1}{8}} = \begin{cases} 1 & l \equiv 1, 7 \pmod{8} \\ -1 & l \equiv 3, 5 \pmod{8} \end{cases}$$

Proof Assume $p \equiv 1 \pmod{4}$, then $\mathbb{Q}(\sqrt{p}) \hookrightarrow \mathbb{Q}(\zeta_p)$

$$\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$$

$$\downarrow$$

$$\text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \xrightarrow{\chi} \{-1\}$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto l$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto \left(\frac{l}{p}\right)$$

$$\chi(\text{Frob}_l) = 1 \Leftrightarrow \text{Frob}_l(\sqrt{p}) = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l}$$

$$\Leftrightarrow (\sqrt{p})^l = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l}$$

$$\Leftrightarrow \sqrt{p} \in \mathbb{F}_l$$

$$\Leftrightarrow \left(\frac{p}{l}\right) = 1$$

Assume $p \equiv 3 \pmod{4}$, then $\mathbb{Q}(\sqrt{p}) \hookrightarrow \mathbb{Q}(\zeta_p)$

$$\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong (\mathbb{Z}/p\mathbb{Z})^\times$$

$$\downarrow$$

$$\text{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \xrightarrow{\chi} \{-1\}$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto l$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto \left(\frac{-p}{l}\right)$$

$$\chi(\text{Frob}_l) = 1 \Leftrightarrow \text{Frob}_l(\sqrt{-p}) = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l}$$

$$\Leftrightarrow (\sqrt{-p})^l = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l}$$

$$\Leftrightarrow \sqrt{-p} \in \mathbb{F}_l$$

$$\Leftrightarrow \left(\frac{-p}{l}\right) = 1$$

$$\text{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q}) \cong (\mathbb{Z}/4\mathbb{Z})^\times$$

$$\downarrow$$

$$\text{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \xrightarrow{\chi} \{-1\}$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto l$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto \left(\frac{-1}{l}\right)$$

$$\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$$

$$\downarrow$$

$$\text{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \xrightarrow{\chi} \{-1\}$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto l$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto \left(\frac{2}{l}\right)$$

$$\text{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$$

$$\downarrow$$

$$\text{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \xrightarrow{\chi} \{-1\}$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto l$$

$$\downarrow$$

$$\text{Frob}_l \longmapsto \left(\frac{(-1)(l+3)}{8}\right)$$

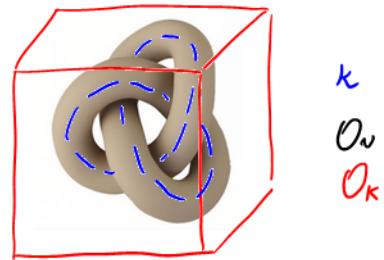
Q: Quadratic reciprocity for $\mathbb{F}_p(t)$?

Cubic reciprocity?
[Cox x^3+ny^3 , § 4.A, 4.B]

{ primes over $\mathbb{Q}(\zeta_3)$
 $\deg 3$, like $\text{Gal}(\mathbb{K}/\mathbb{Q}(\sqrt{-3}))$. More difficult

Étale point of view

$K \rightarrow S'$
 $\mathcal{O}_v \rightarrow$ tubular nbhd of S'
 $\mathcal{O}_K \rightarrow$ 3-dim spaces (\mathbb{R}^3 when $K = \mathbb{Q}$)



See [Knotprime Table 1] for more informations.

6. local to global: adèle

Recall: Ostrowski's thm & Product formula.

Task. Read [Adèle] and answer the following questions:

- Give a def of \mathbb{A}_K & \mathbb{I}_K (set, topo and measure)
- Verify that

$$\begin{aligned} K \subseteq \mathbb{A}_K &\quad \mathcal{O}_T \subseteq \prod'_{v \in T} K_v \\ K^\times \subseteq \mathbb{I}_K^\times &\quad \mathcal{O}_T^\times \subseteq (\prod'_{v \in T} K_v)^\times \end{aligned}$$

are lattices. Give fundamental domain in easy cases.

- Deduce the finiteness of class number and Dirichlet unit theorem.

Base field with automorphism

We know that

	finite field	local field	global field	Adèle
base field F	\mathbb{F}_p	\mathbb{R}	\mathbb{Q}_p , $\mathbb{F}_p((t))$	\mathbb{Q} , $\mathbb{F}_p(t)$, \mathbb{A}_K
$\text{Aut}_{\text{ring}}(\mathbb{F}_p) = 1$	$\text{Aut}_{\text{top ring}}(\mathbb{R}) = 1$	$\text{Aut}_{\text{top ring}}(\mathbb{Q}_p) = 1$	$\text{Aut}_{\text{ring}}(\mathbb{Q}) = 1$	
			$\text{Aut}_{\text{top ring}}(\mathbb{F}_p((t))) \neq 1$	$\text{Aut}_{\text{ring}}(\mathbb{F}_p(t)) \neq 1$, $\text{Aut}_{\text{ring}}(\mathbb{A}_{\mathbb{F}_p(t)}) \neq 1$

Q: Do we have $\text{Aut}_{\text{ring}}(\mathbb{A}_\mathbb{Q}) = \{\text{Id}\}$?

A: Yes. See [LCFT, Ex 6.3.6]. I don't understand this proof.

Galois extension

Setting: L/K fin ext of global field

Recall that we have an iso

$$\begin{aligned} L \otimes_K \mathbb{A}_K &\xrightarrow{\cong} \mathbb{A}_L \text{ of topo rings with compatible embedding of } L \\ \hookrightarrow \mathbb{A}_K &\subseteq \mathbb{A}_L \text{ subring, } \mathbb{A}_L \cong \mathbb{A}_K^{\oplus [L:K]} \text{ as } \mathbb{A}_K\text{-module.} \end{aligned}$$

Lemma [LCFT, Ex 6.3.2]

Proof Reduce to integral closure of K in $\mathbb{A}_L = L$

If $\exists x \in \mathbb{A}_L - L$ which is integral over L , then

$L(x)/L$ is a fin field ext in \mathbb{A}_L , and

$\#\{q \in \text{Spec } \mathcal{O}_L \mid q \text{ do not split completely}\}$

$\leq \#\{q \in \text{Spec } \mathcal{O}_L \mid x_q \notin \mathcal{O}_q\} < \infty$.

But fin nontrivial field ext have inf many non split primes. \diamond

7. \mathbb{F}_1

This is better explained in §1.2. Anyhow, it is still a "field".

⚠ It is always better to think $\#\mathbb{F}_1 = 1 + \varepsilon$, where $\varepsilon \ll 1$.

In that way you can "see object at different level", like

$$\#\mathbb{F}_1^\times = \varepsilon \quad \mathbb{F}_1^\times \text{ is not empty!}$$

Slogan: "Infinitesimal is only visible when constant level is zero"

This phenomenon already happens when we learn integrals.

$$\int x^n = \begin{cases} \frac{1}{n+1} x^{n+1} + C & n \neq -1 \\ \log x + C & n = -1 \end{cases}$$

Here, $\log x$ is that "infinitesimal".

See

wiki: https://en.wikipedia.org/wiki/Field_with_one_element
nlab: <https://ncatlab.org/nlab/show/field+with+one+element>

It is desirable to define

- The "field extension" $\mathbb{F}_1^n/\mathbb{F}_1$ of deg n + $\mathbb{F}_1^n \cong \varepsilon \mu_n$
- The "Galois group" $\text{Gal}(\bar{\mathbb{F}}_1/\mathbb{F}_1) \cong \hat{\mathbb{Z}}$
- The "v.s. over \mathbb{F}_1 "

Q: Do we have \mathbb{Q}_1 ?

A: See <https://mathoverflow.net/questions/309664/what-is-mathbbq-1-the-field-of-1-adic-numbers>