Eine Woche, ein Beispiel
8.17 tropical hypersurface

Ref:

https://arxiv.org/abs/1311.2360

How to draw these tropical curves:
https://mathoverflow.net/questions/328342/how-to-draw-tropical-curves
https://ntiggemann.github.io/coding.html#Plotting_tropical_curves

K: valued field $v: K \longrightarrow \mathbb{R} \cup \{+\infty\}$ most time: $\mathbb{Z} \cup \{+\infty\}$

$X \subseteq \mathbb{A}^n_K$ variety

$$x \in X(K) \quad \Rightarrow \quad -v(x) \in \operatorname{Trop}(X)$$
$$Y \subseteq X \quad \Rightarrow \quad \operatorname{Trop}(Y) \subset \operatorname{Trop}(X)$$

⚠ If we want compatability of $v(x)$ with $\oplus$, then we should define $u \oplus v = \min(u, v)$.
Usually tropical people don't do this, they want

"addition of positive number should go up".

We respect the conventions.

$$v(x+y) \geqslant \min(v(x), v(y))$$
$$v(xy) = v(x) + v(y)$$
$$v(0) = +\infty$$

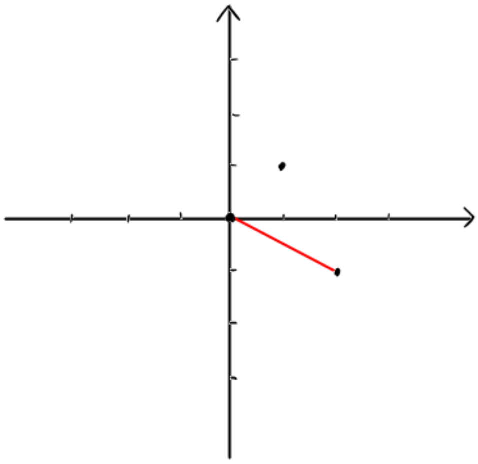
$u \oplus' v = \min(u, v)$
$u \otimes' v = u + v$
$+\infty$ $\mathbb{T} = \mathbb{R} \cup \{+\infty\}$
read from bottom

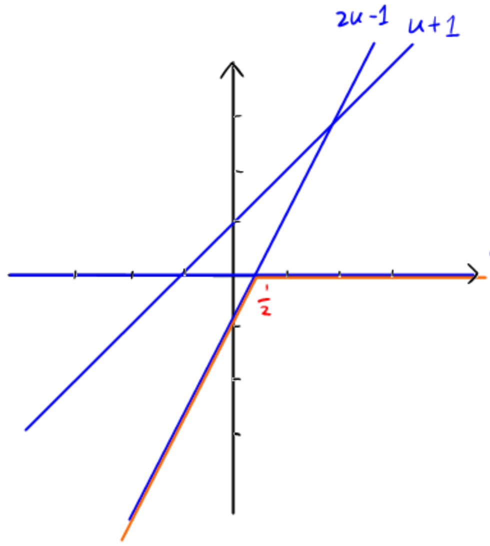
$u \oplus v = \max(u, v)$
$u \otimes' v = u + v$
$-\infty$
read from above

in calculation

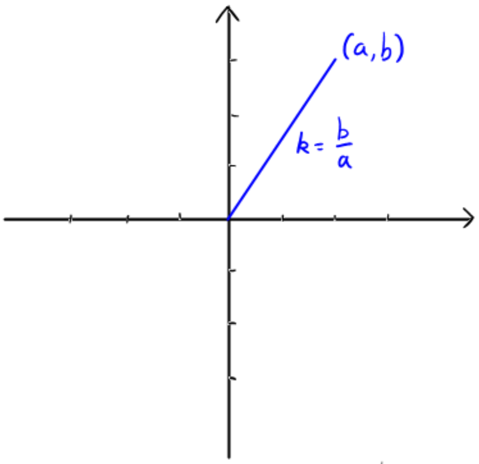
Relation with Newton polygon

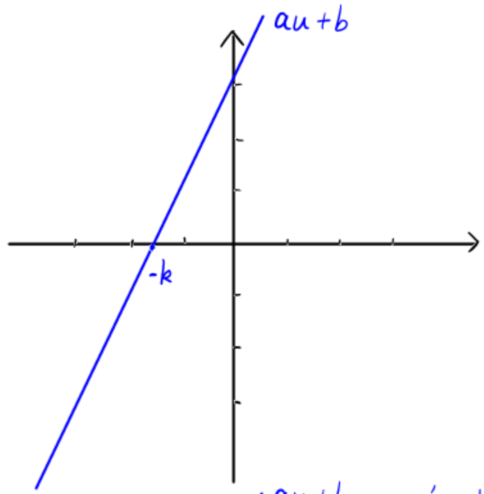
E.g.

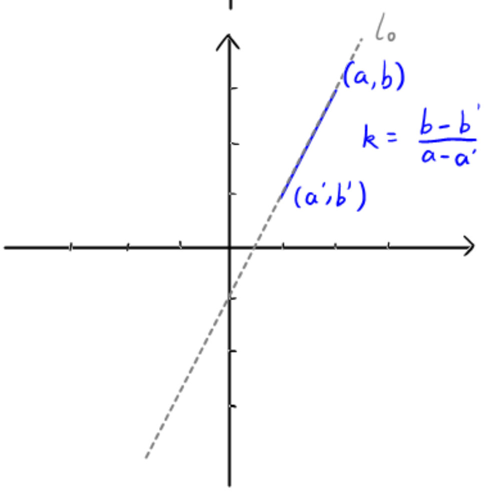


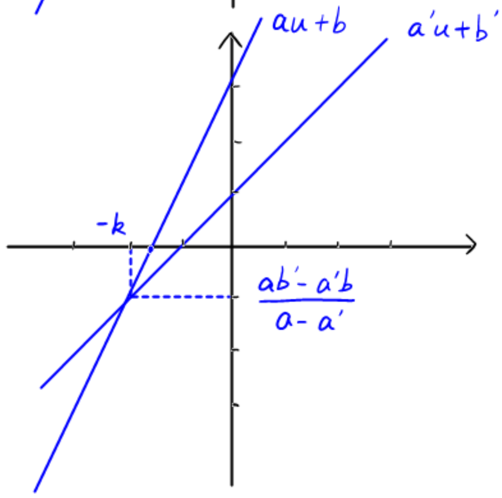


$1 + 5z + \frac{1}{5} z$ $\qquad\qquad$ $0 \oplus' (u+1) \oplus' (2u-1)$



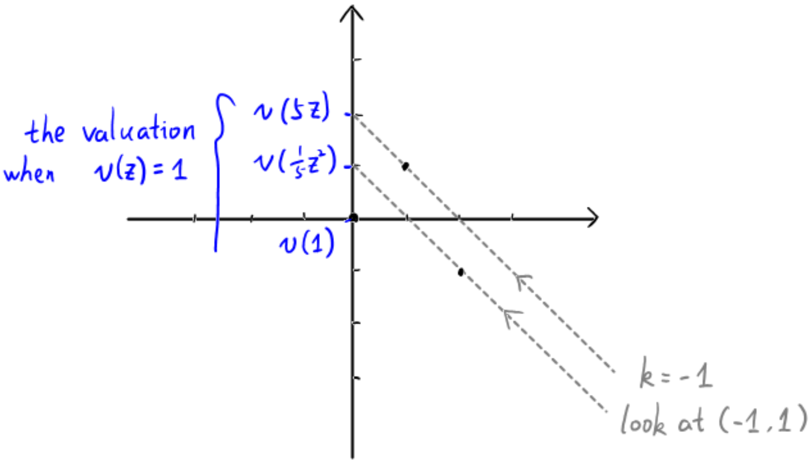






Rmk. $\quad p = (x, y) \in l_0 \iff l_p =$ "$xu+y$" passes through $\left(-k, \frac{ab'-a'b}{a-a'}\right)$

better point of view

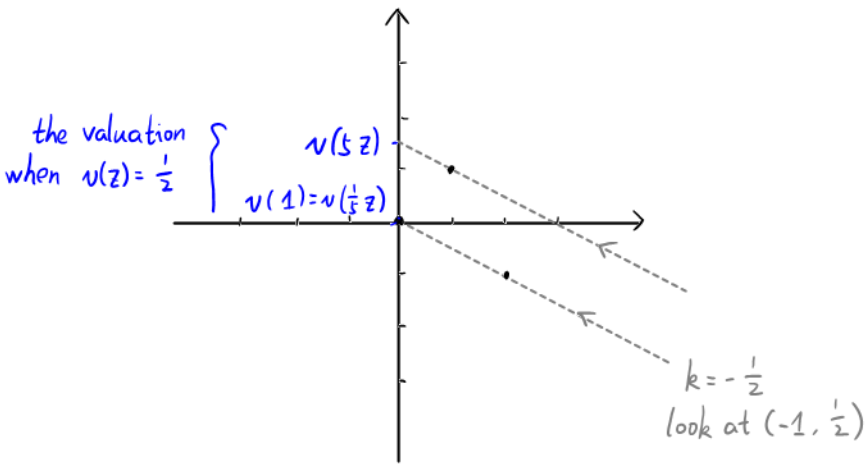


$1 + 5z + \frac{1}{5} z$

A special valuation of z may be seen as a kind of projection.

You can then read the value as though from the markings of a graduated cylinder.

It is curious that mathematicians read numbers from unexpected angles, rather than from the usual horizontal view.



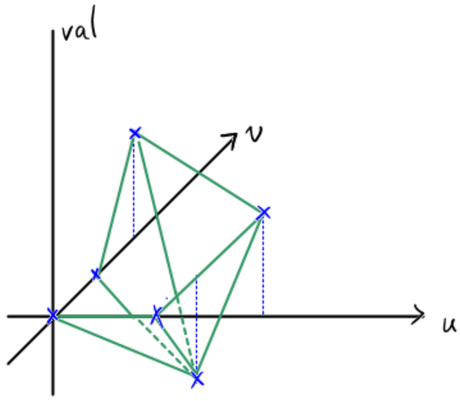
$1 + 5z + \frac{1}{5} z$

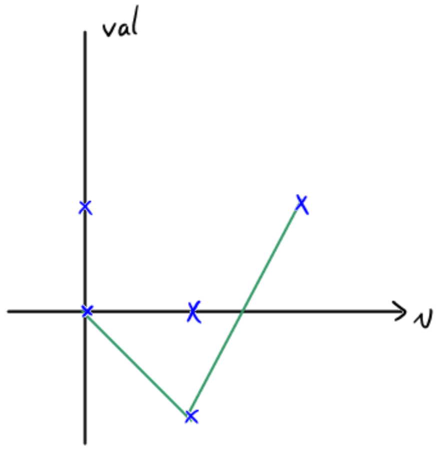
When the two values meet and rest at the very bottom among all values, we have the possibility that v(f)=+\infty.

This happens when the gaze brings the two points into perfect alignment; the negative of the slope of this sightline is v(z).

That line is exactly the lower convex edge of the Newton polygon.

The Newton polygon has a higher version.

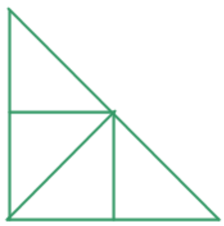


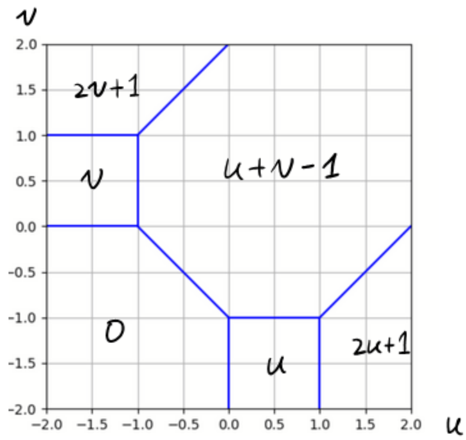


$1+x+y+5x^2+\frac{1}{5}xy+5y^2$

$0 \oplus' u \oplus' v \oplus' (2u+1) \oplus'(u+v-1) \oplus'(2v+1)$

when $u=0$,
we are taking projections.

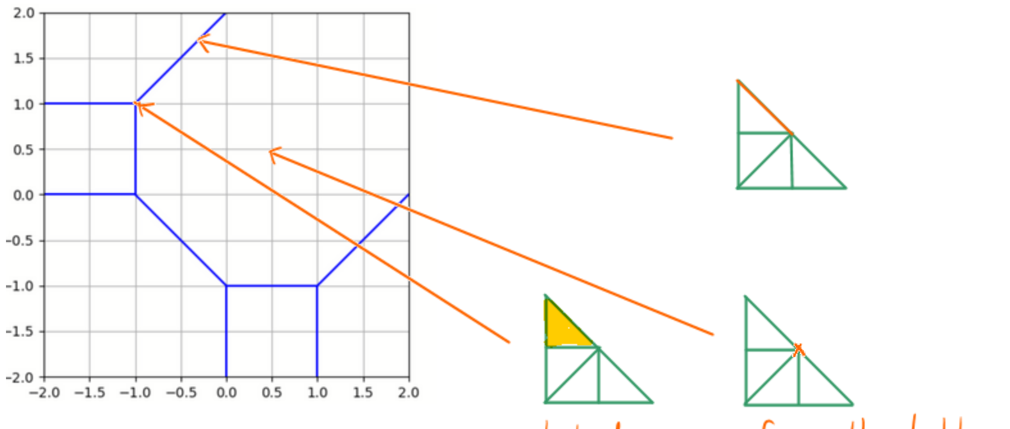




dual subdivisions
i.e., the projection of Newton polygon

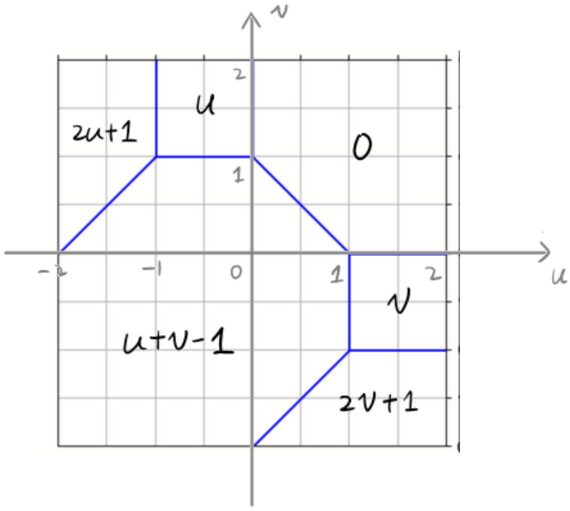
tropical curve (max version)

0+x+y+(-1)*x2+1*xy+(-1)*y2
The software only compute the maximal version.



what do we see from the bottom

⚠ The minimal version:



This is the correct one. but software doesn't produce it automatically.

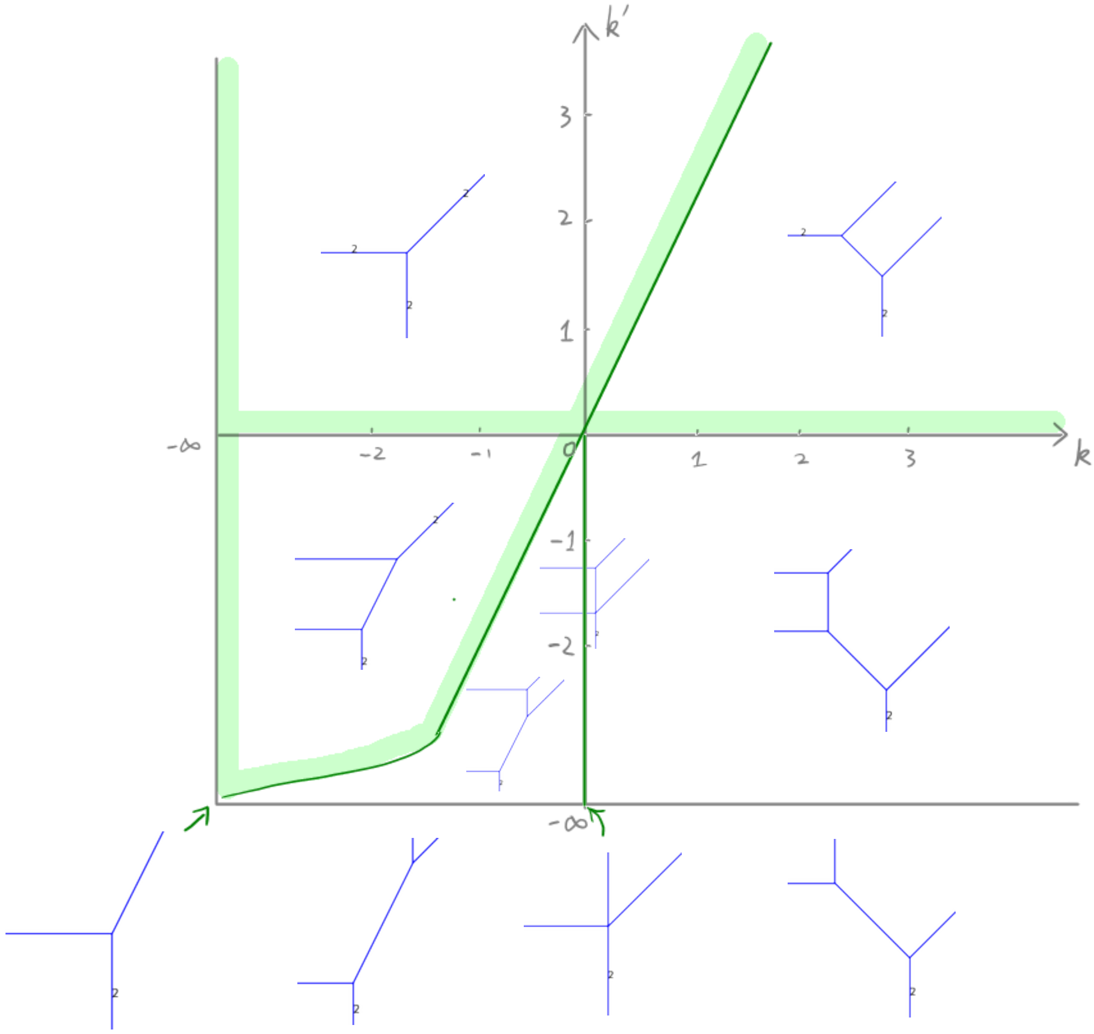
E.g. We want to draw the tropical curve crspding to

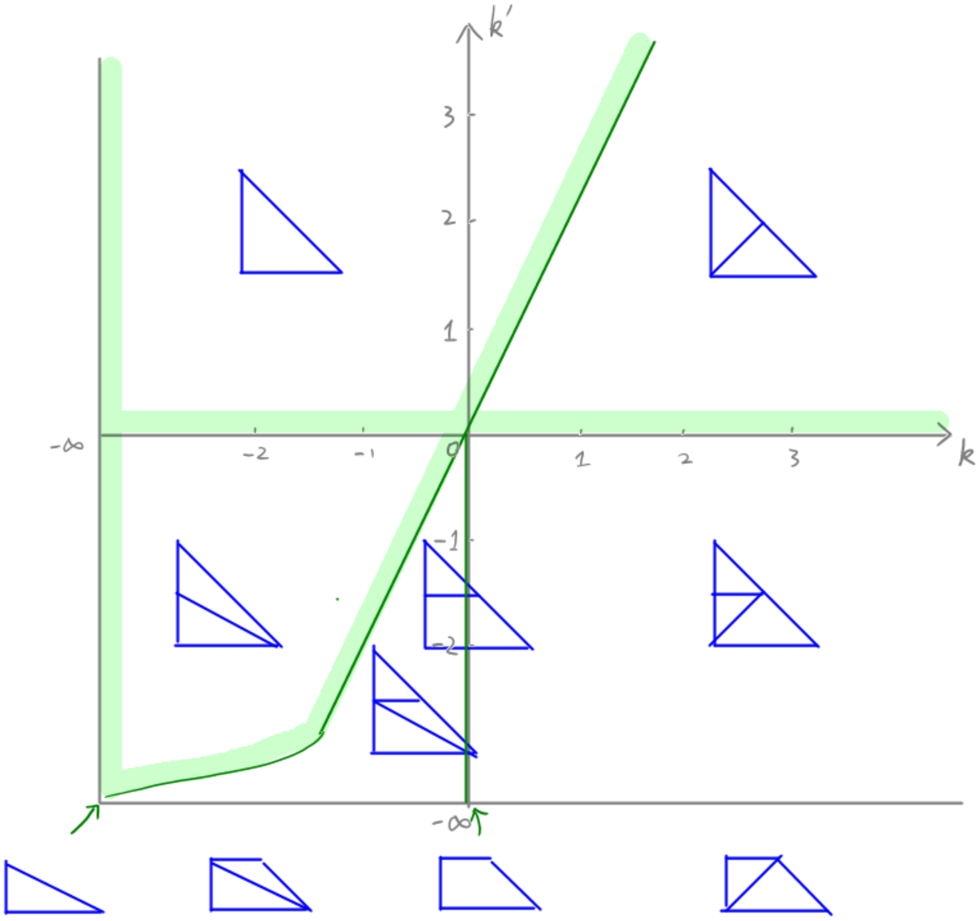
$$1 + x^{-1} + y^{-1} + x^{-2} + \zeta^{-k}x^{-1}y^{-1} + \zeta^{-k'}y^{-2}$$

$$0 \oplus' (-u) \oplus' (-v) \oplus' (-2u) \oplus' (-u-v-k) \oplus' (-2v-k')$$

$$0 \oplus u \oplus v \oplus 2u \oplus u+v+k \oplus 2v+k'$$

0+x+y+x2+(k)xy+(k')y2





k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞

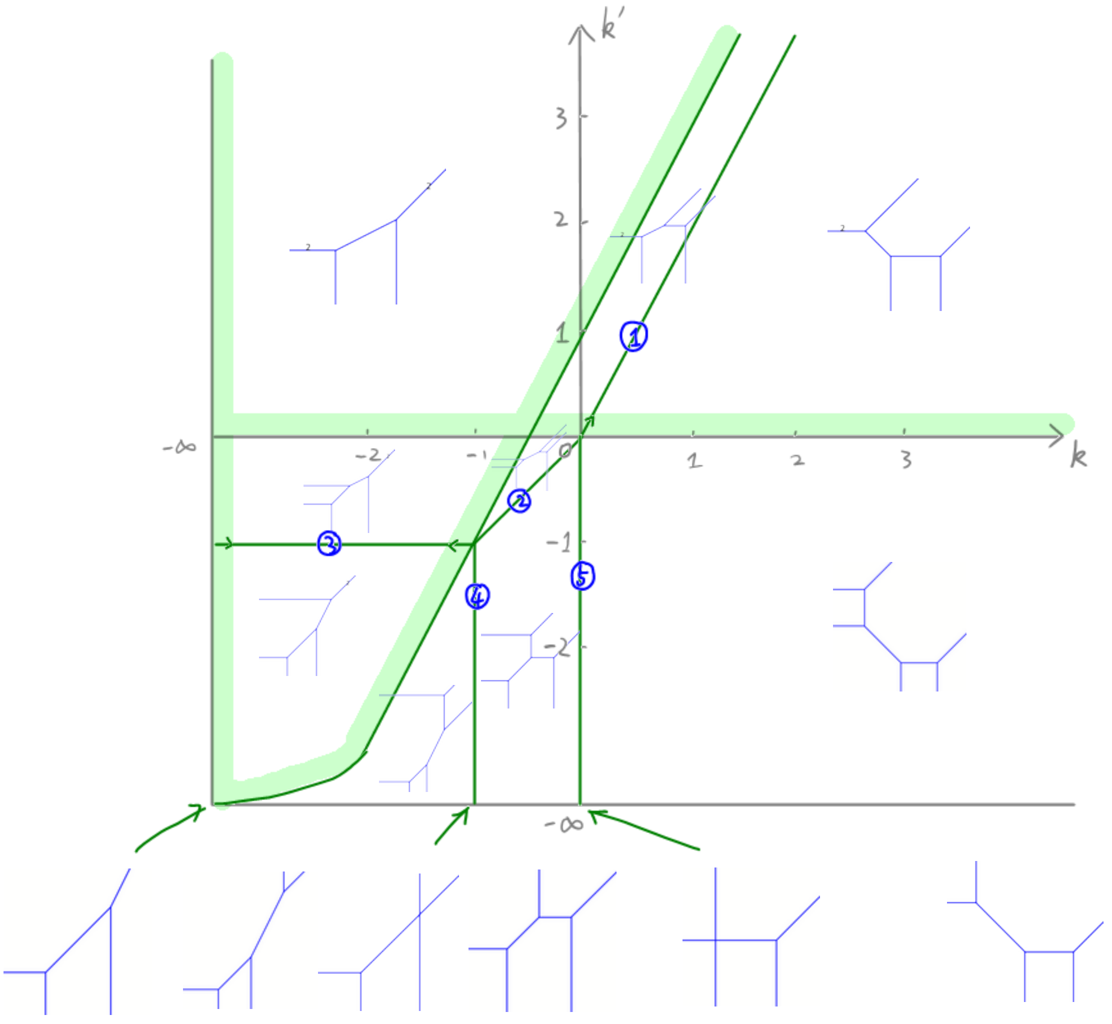
E.g. We want to draw the tropical curve crspding to

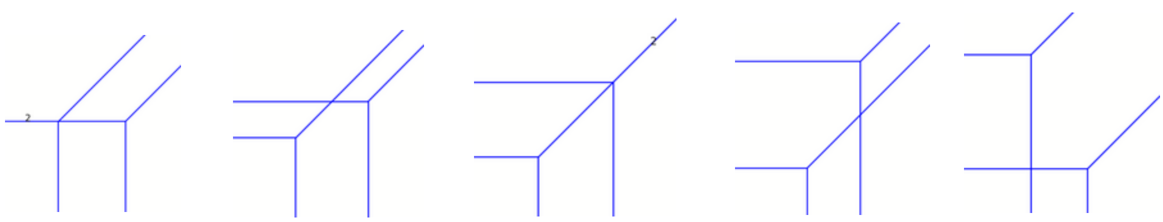
$$1 + x^{-1} + y^{-1} + \tfrac{1}{5}x^{-2} + 5^{-k}x^{-1}y^{-1} + 5^{-k'}y^{-2}$$

$$0 \oplus' (-u) \oplus' (-v) \oplus' (-2u+1) \oplus' (-u-v-k) \oplus' (-2v-k')$$

$$0 \oplus u \oplus v \oplus (2u-1) \oplus u+v+k \oplus 2v+k'$$

0+x+y+(-1)x2+(k)xy+(k')y2





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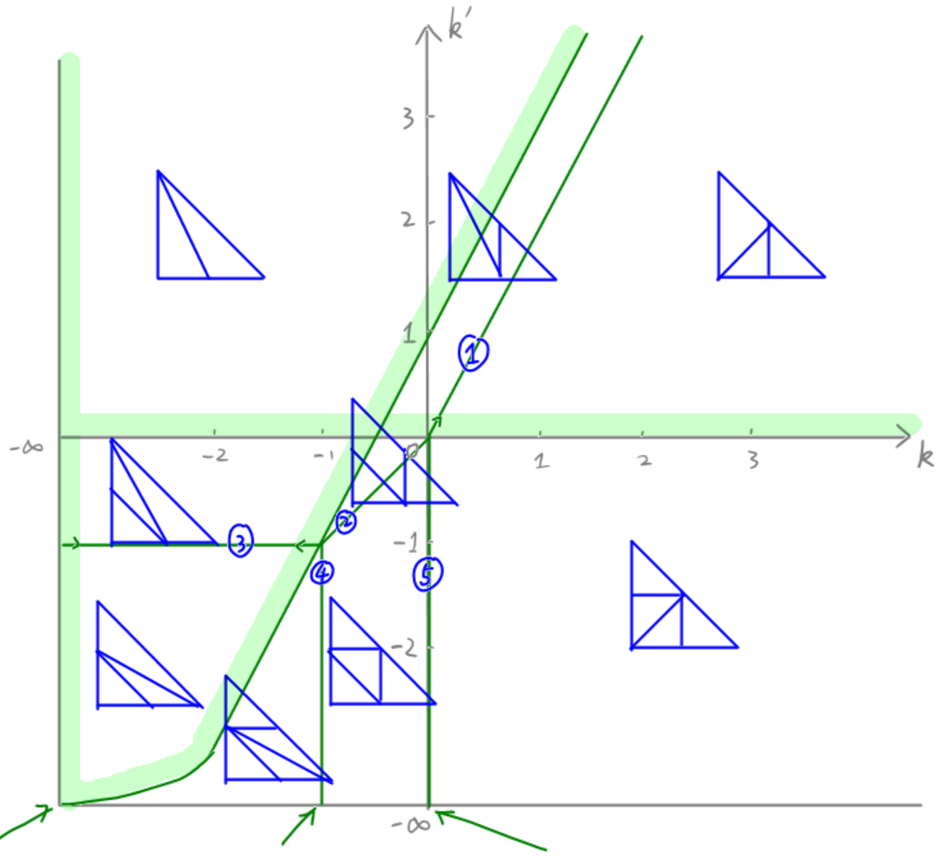

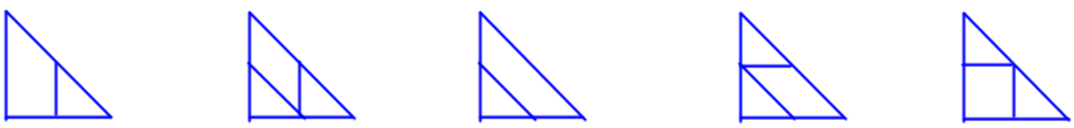

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The above examples tell us the structure of the tropical hypersurface

$$0 \oplus u \oplus v \oplus 2u+k'' \oplus u+v+k \oplus 2v+k' \quad \text{in } \mathbb{R}^5.$$

The stratification has some wall-crossing behavior: the tropical curve change the topology when it cross over the wall.

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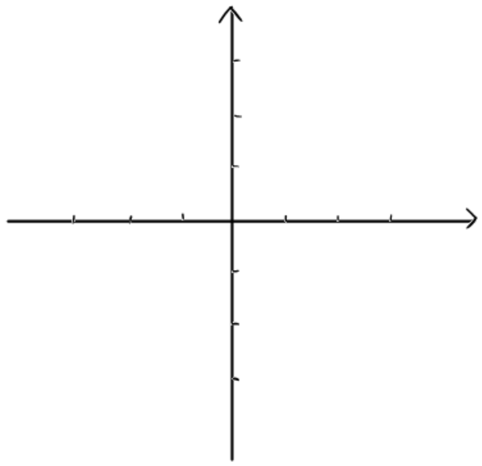
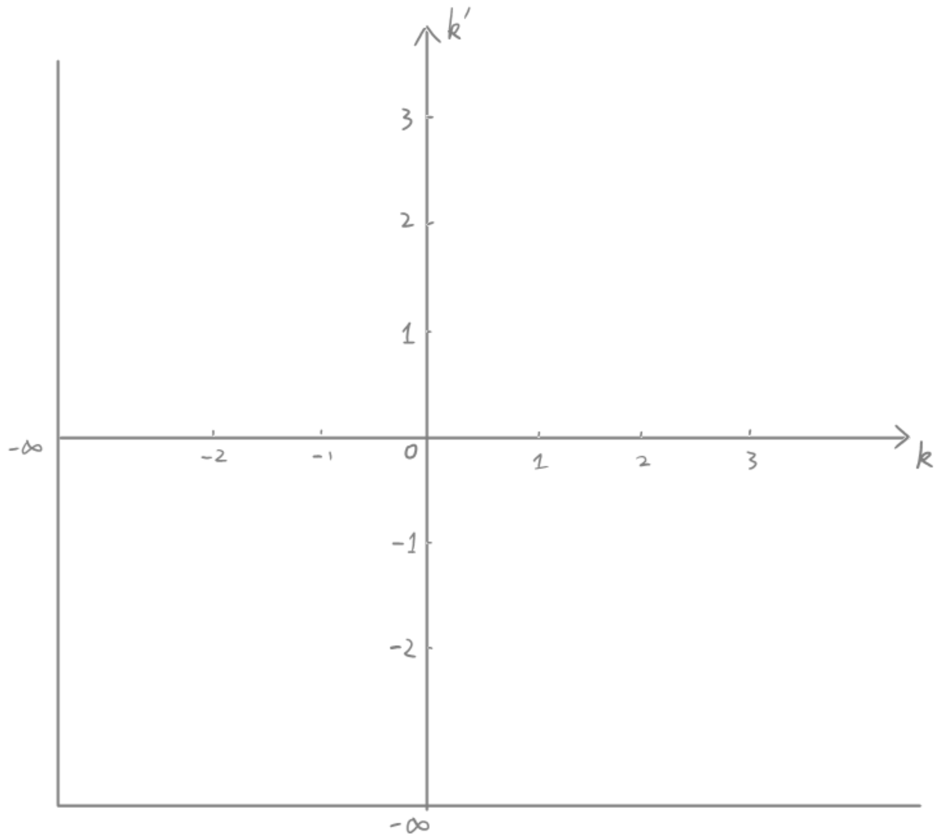
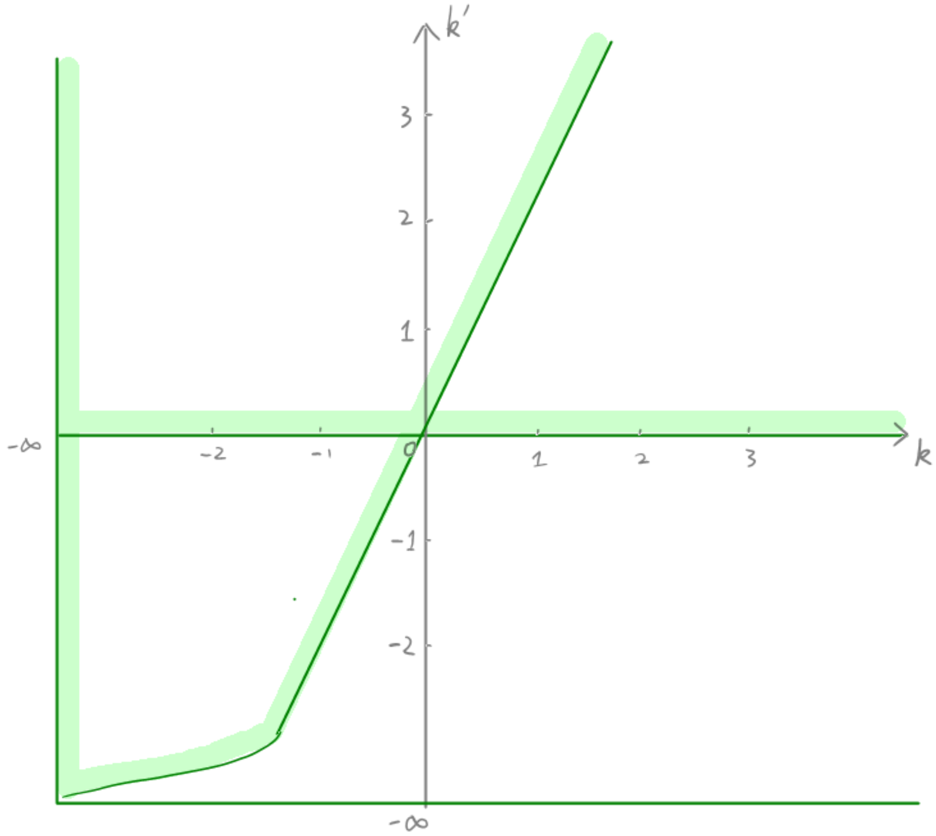
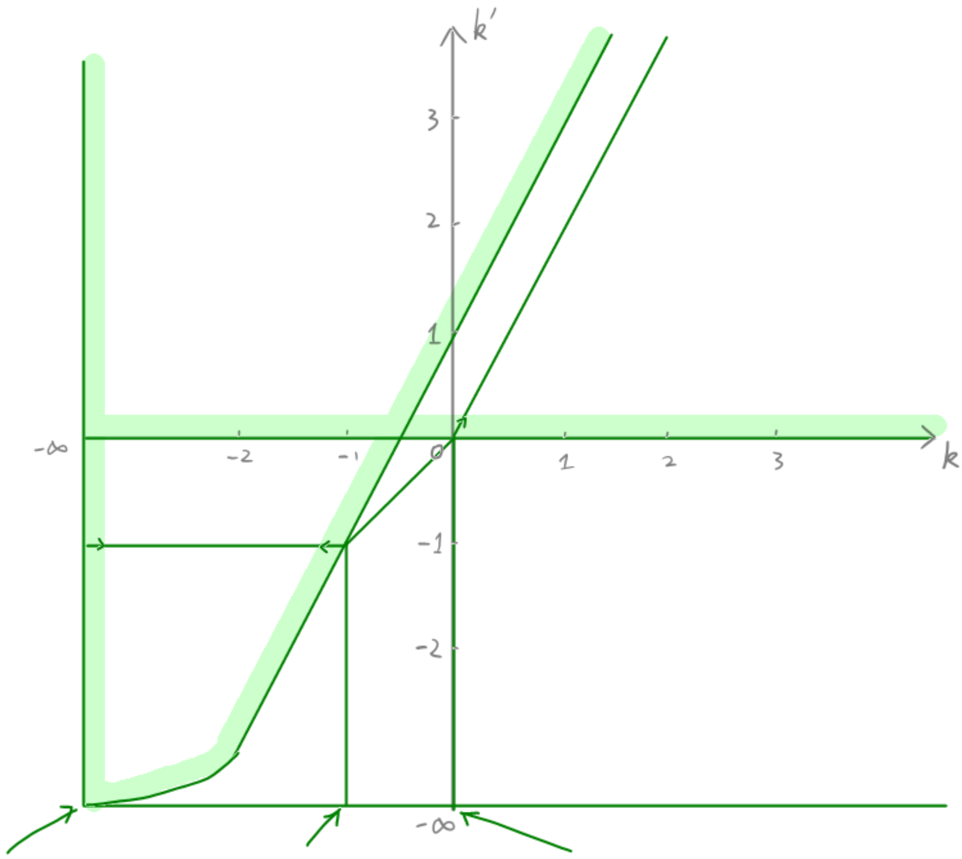






k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞



k'
3
2
1
-∞
-2
-1
0
1
2
3
k
-1
-2
-∞