Eine Woche, ein Beispiel
3.27 model theory

Model

type --- Structure   Theory ⤹ formula
                                    e.p. sentence
              ⤹
           Language ⤹ symbol

Modern Algebra (H)

Preliminaries

- Logic. set & map, operations on Set. $\left(\begin{array}{l}\cap \cup -^c, \text{ sub/quotient,} \\ \text{two ways of disjoint union} \\ \text{Cartesian product}\end{array}\right)$
  - Russell's paradox
  - Axiomatic set theory (ZFC)
  - type of proof : constructive, algorithm,...

  Ex. graph
  Classify topologies of $\{1,...,n\}$
  First adjunction. $\text{Map}(A \times B, C) \cong \text{Map}(A, \text{Map}(B,C))$

- From $\mathbb{N}$ to $\mathbb{C}$ (the basic of examples, though logically it's not here)
  - Peano axioms. Axiom of induction
  - alg structure, order and topology
  - Completeness axiom.

- Cardinal: the only property of set.
  - naive definition

	alg	total order	topo
$\mathbb{N}$	$(+,\times)$	✓	discrete
$\mathbb{Z}$	$(+,-,\times)$	✓	discrete
$\mathbb{Q}$	$(+,-,\times,\div)$	✓	dense but not complete
$\mathbb{R}$	$(+,-,\times,\div)$	✓	complete
$\mathbb{C}$	$(+,-,\times,\div)$	✗	complete

  - operations on cardinal
  - examples
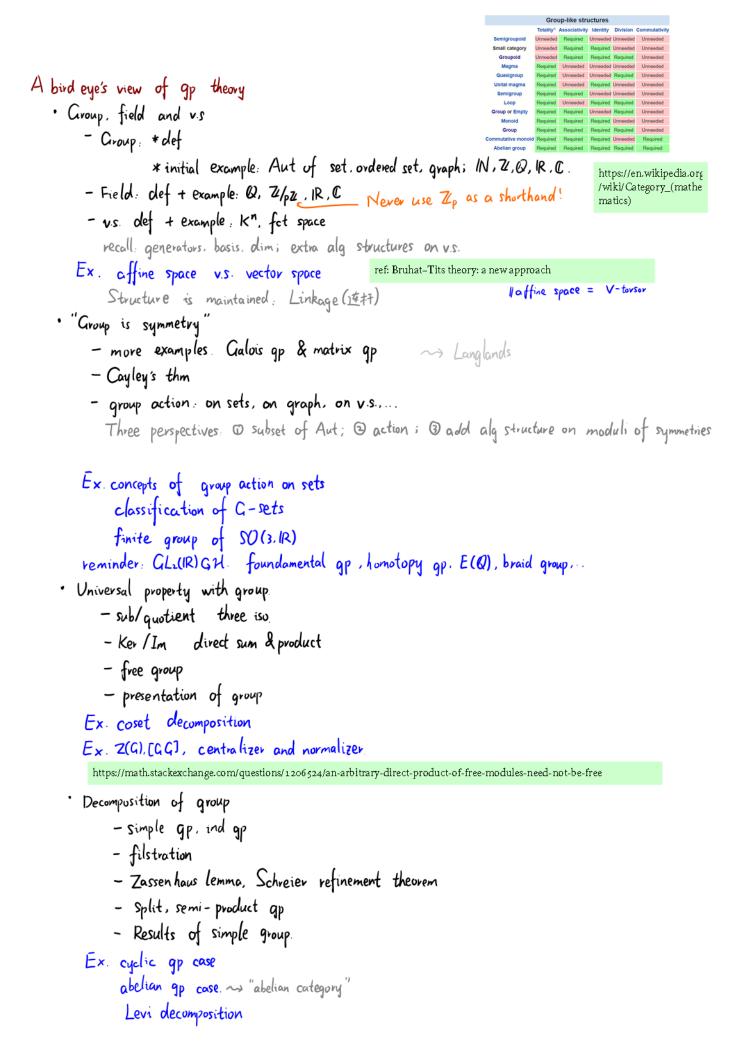 - The continuum hypothesis
  - large cardinal axiom

- Order structure
  - def, operations and properties (partial/total/well order)
  - ordinal. relationship with cardinal.

  Ex. "well·order" on class of cardinals.
  Classify subpartial ordered set of $\{\text{subsets of } \{1,..,n\}\}$, up to iso. $n \leq 5$
  the Monoid $\widetilde{R}$, surreals

poset = partial order set
coset = 陪集

A bird eye's view of gp theory

- Group, field and v.s
  - Group: * def
      * initial example: Aut of set, ordered set, graph; $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
  - Field: def + example: $\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}, \mathbb{R}, \mathbb{C}$ Never use $\mathbb{Z}_p$ as a shorthand!
  - v.s. def + example: K^n, fct space
    recall: generators, basis, dim; extra alg structures on v.s.

  Ex. affine space v.s. vector space
    Structure is maintained: Linkage (连杆)

      $\|$ affine space = V-torsor

- "Group is symmetry"
  - more examples: Galois gp & matrix gp $\leadsto$ Langlands
  - Cayley's thm
  - group action: on sets, on graph, on v.s.,...
    Three perspectives: ① subset of Aut; ② action; ③ add alg structure on moduli of symmetries

  Ex. concepts of group action on sets
      classification of G-sets
      finite group of $SO(3, \mathbb{R})$
  reminder: $GL_2(\mathbb{R}) \circlearrowright \mathcal{H}$. foundamental gp, homotopy gp. $E(\mathbb{Q})$, braid group,...

- Universal property with group.
  - sub/quotient   three iso.
  - Ker/Im   direct sum & product
  - free group
  - presentation of group
  Ex. coset decomposition
  Ex. $Z(G), [G,G]$, centralizer and normalizer

- Decomposition of group
  - simple gp, ind gp
  - filstration
  - Zassenhaus lemma, Schreier refinement theorem
  - Split, semi-product gp
  - Results of simple group.
  Ex. cyclic gp case
      abelian gp case. $\leadsto$ "abelian category"
          Levi decomposition

Group-like structures					
	Totality	Associativity	Identity	Division	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Group or Empty	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Abelian group	Required	Required	Required	Required	Required

Down-to-earth analysis.

- f.g. abelian gp + apps.

  Ex. lattice & Crystallographic point gp
  ← with/without origin

    sublattice count

    dual lattice $\Lambda^* = \{ f : \mathbb{R}^n \to \mathbb{R} \text{ linear} \mid <f, x> \in \mathbb{Z} \quad \forall x \in \Lambda \}$

    lattice in Euclidean space

  ex. Show that for f.g. abelian gp G, $f \in \text{End}_{Abel}(G)$,

         f surj $\Rightarrow$ f isomorphism.

  Find an example when G is not f.g. $(S^1, \mathbb{Q}/\mathbb{Z}, \bigoplus_{\mathbb{N}} \mathbb{Z})$

- S_n

  begin with ghost leg (鬼脚图)

- Combinatorics related to q-polynomial ← can be quite tricky.

  Ex. finite field with one element

- G act on G

- Sylow thm

- app: classifications of gp of small order. research on specific gp.

- Coxeter group

Ring & module

$R(x)$ $R((x))$ $R\{\{x\}\}$ $R\langle\langle x\rangle\rangle$

- Basic def. e.g. $R[x]$, $R[[x]]$, $R\{x\}$, $R\langle x\rangle$

- Category, k-algs
    Ex. groupoid
- Basic def of modules, abelian category
    Ex. R-algs.
        tensor product, change of basis
        three rep theory
- AG translation, examples.
- Concepts under AG translation
- ED $\Rightarrow$ PID $\Rightarrow$ UFD $\Rightarrow$ domain

    Ex. Elementary divisor thm
        R: PID. $M_{2\times 2}(R) - \{0\} = \coprod_{a|b \in R} GL_2(R) \begin{pmatrix} a \\ & b \end{pmatrix} GL_2(R)$

- classification of f.g. module over PID.

Field and Galois theory (See [GTM 167])

    can focus more on $\mathbb{F}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, and geometrical point of view.

(Reminder) other structures: norm, metric & topo; measure; sheaf.