Eine Woche, ein Beispiel

3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

[svd-notes]: Notes on singular value decomposition for Math 54
https://math.berkeley.edu/~hutching/teach/54-2017/svd-notes.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \circlearrowright X$

discrete $\Rightarrow$ foundamental domain $\qquad \Lambda \circlearrowright \mathbb{C} \qquad SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$

non discrete $\Rightarrow$ stratification by G/G_x $\qquad S^1 \circlearrowright S^2 \qquad \mathbb{C}^\times \circlearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g. $\mathcal{F}lag(V) \cong GL(V)/P$ $\qquad$ e.p. Grassmannian, $\mathbb{P}^n$

$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$

$O(n) := O(n,\mathbb{R})$ $\qquad \rightsquigarrow$ Stiefel mfld [21.11.14]

$SO(n) := SO(n,\mathbb{R})$

$\mathbb{A}^n = \mathbb{A}^n$

$\mathcal{H}^n \cong O^+(1,n)/O(n)$

$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$

$\rightsquigarrow$ Hermitian symmetric space

where $\quad \mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1,\ v_{n+1} > 0\}$

$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad \langle v, w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$

$O(n,1) := \mathrm{Aut}(\mathbb{R}^{n+1}, \langle\ ,\ \rangle) \subseteq GL_{n+1}(\mathbb{R})$

$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

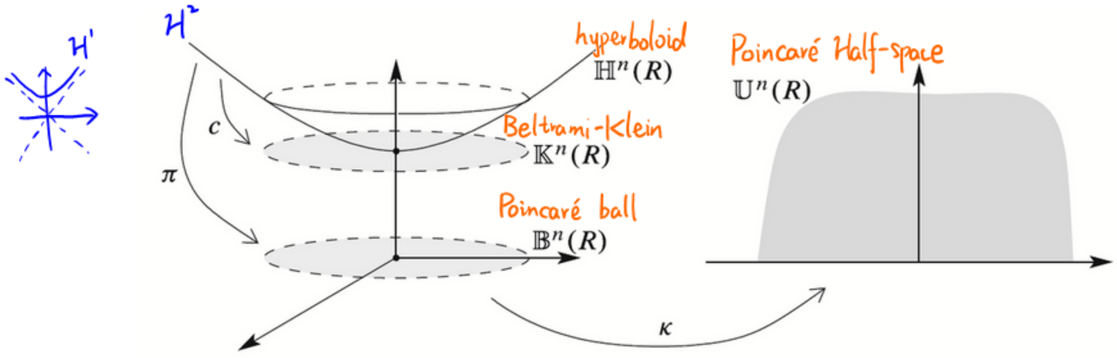


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-01-2

Terminology.

$$\begin{tikzcd}
 & X + G \curvearrowright X \arrow[d] & \\
 & G\text{-space} \arrow[d, "\text{transitive}"] & \\
 & \text{homogeneous space} \arrow[dl, "\substack{\text{Riemannian mfld}\\ G \text{ preserve } g}"'] \arrow[dr, "\substack{\text{Hermitian mfld}\\ G \text{ preserve } g \leftarrow \text{metric}}"] & \\
\text{Riemannian homogeneous space} \arrow[rr, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\substack{\text{geodesic symmetry}\\ \text{globally defined, preserve } g}"'] & & \text{Hermitian homogeneous space} \arrow[d, "\substack{\text{geodesic symmetry}\\ \text{globally defined, preserve } g}"] \\
\text{Symmetric space} \arrow[rr, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\text{transitive on frames}"'] & & \text{Hermitian symmetric space} \\
\text{frame homogeneous space} & &
\end{tikzcd}$$

[JL18, Cor 3.18]: Nice hom space can have compotible Rie mfld structures.

$\mathbb{CP}^n$ $n \geq 1$

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

2. double coset decomposition: schedule

$$G = \bigsqcup_{\alpha\in I} H\alpha K$$

usually, H, K are easier than G.
- comes from (usually) Gauss elimination
- I is the "foundamental domain"
- produces stratifications on G/K and $H\backslash G$ indexed by I.

To be exact,

$$G/K = \bigsqcup_{\alpha\in I} H\alpha K/K \cong \bigsqcup_{\alpha\in I} H/H_{[\alpha K]} = \bigsqcup_{\alpha\in I} H/(H\cap\alpha K\alpha^{-1})$$

$$H\backslash G = \bigsqcup_{\alpha\in I} H\backslash H\alpha K \cong \bigsqcup_{\alpha\in I} K_{[H\alpha]}\backslash K = \bigsqcup_{\alpha\in I} (K\cap\alpha^{-1}H\alpha)\backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$
$K_{[H\alpha]}$: stabilizer of K on $[H\alpha] \in H\backslash G$

$$\# \; H/H\cap\alpha K\alpha^{-1} = \#\left\{\begin{matrix}\text{single cosets } [gK] \\ \text{in one double coset } H\alpha K\end{matrix}\right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad H\backslash G \qquad \text{individually}$$

- can be viewed as stack quotient.
$[*/G]$: groupoid

$$H\backslash G/K \overset{\text{def}}{=} [*/H] \times_{[*/G]} [*/K] \quad \text{with groupoid structure}$$

$$H_H^*(G/K) \cong H^*(H\backslash G/K) \cong H_K^*(H\backslash G)$$

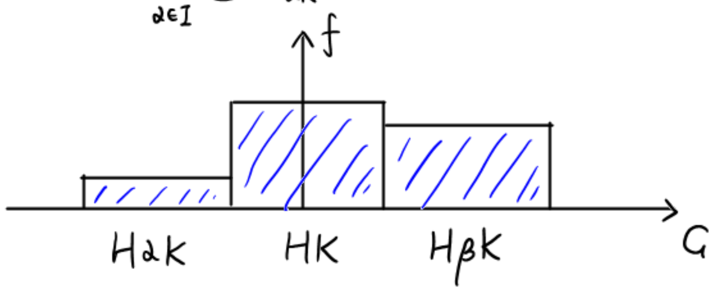
slogan: the (equiv) cohomology of G/K and $H\backslash G$ are connected.

- Hecke algebra $\mathcal{H}(H\backslash G/K)$

for $H=K$. You can also do $\mathcal{H}(H_1\backslash G/H_2) \hookrightarrow \bigoplus_{i,j=1}^{2} \mathcal{H}(H_i\backslash G/H_j)$

$\mathcal{H}(H\backslash G/K)$: reasonable subspaces of

$$\mathbb{C}[H\backslash G/K] \underset{\text{"o-dim"}}{=} \{ f: G \to \mathbb{C} \mid f(hgk) = f(g) \quad \forall h\in H, g\in G, k\in K \}$$
$$\supseteq \bigoplus_{\alpha\in I} \mathbb{C}\, \mathbb{1}_{H\alpha K}$$



with reasonable convolution structure

$$*: \mathcal{H}(H_1\backslash G/H_2) \times \mathcal{H}(H_2\backslash G/H_3) \longrightarrow \mathcal{H}(H_1\backslash G/H_3)$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

Vague: $\mathcal{H}(H\backslash G/K) \sim H^*(H\backslash G/K)$ should be a type of cohomology

$\mathcal{H}(G) \overset{G \text{ fin}}{=\!=} \mathbb{C}[G]$

$\mathcal{H}(K\backslash G/K) \cong (\mathrm{End}(c\text{-}\mathrm{Ind}_K^G \mathbb{1}_K))^{op}$ should be a type of base ring

Generalize: $\mathrm{Ind}_H^G \chi \approx \mathcal{H}_\chi(H\backslash G/K) \subseteq \{ f: G\to\mathbb{C} \mid f(hgk) = \chi(h) f(g) \}$

depth of χ

3. examples (after [22.09.04])

Works over:

- list of possibilities
- moduli intepretation
- typical examples

finite field, $GL_n(\mathbb{F}_q)$ (Applies to any field κ, actually)

- subgps can be

Borel	max split torus	unipotent
B	T	N
parabolic	Levi	unipotent
P	L	M
	nonsplit torus	
	T'	

$$\square \backslash GL_n(\kappa) / \square$$

- moduli intepretation $V = \kappa^{\oplus n}$

$$G/B = \{ \text{cpl flags in } V \}$$

$$G/T = \{ (V_i)_{i=1}^n \mid V = \bigoplus_{i=1}^n V_i ,\ \dim V_i = 1 \}$$

$$G/N = \left\{ (\mathcal{F}, m_i) \;\middle|\; \begin{array}{l} \mathcal{F}: 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = V \quad \text{cpl} \\ 0 \neq m_i \in M_i/M_{i-1} \end{array} \right\}$$

$$G/P = \{ \text{flags in } V \}$$

$$G/L = \{ (V_i)_{i \in I} \mid V = \bigoplus_{i \in I} V_i \}$$

$$G/M = \left\{ (\mathcal{F}, \mathcal{B}_i) \;\middle|\; \begin{array}{l} \mathcal{F}: 0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_d = V \\ \mathcal{B}_i: \text{ a basis of } M_i/M_{i-1} \end{array} \right\}$$

Rmk. We have a fiber bundle

$$\begin{array}{ccc} \mathbb{A}^{\oplus \binom{n}{2}} \cong B/N & \longrightarrow & G/N \\ & & \downarrow \\ & & G/B \end{array}$$

which makes G/N a $\mathbb{A}^{\oplus \binom{n}{2}}$-torsor over G/B

⚠ G/N is not a $\underbrace{\text{rk } \binom{n}{2} \text{ v.b}}_{\text{i.e. } GL_{\binom{n}{2}}(\kappa)\text{-torsor}}$ over G/B, so G/N can be affine space.

- E.g. Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

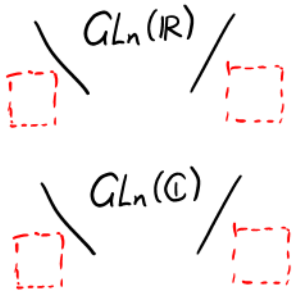
  - Gauss elimination gives "$\subseteq$",
    while the observation of process gives "$\sqcup$" (Something is invariant)
  - the "fundamental domain" W has a gp structure, and crsp to B-orbits of G/B.
    gp structure comes from Tits system
  - produces an affine paving of G/B, and the Zariski topo gives Bruhat order
    works also for Euclidean topo, $\kappa = \mathbb{R}$ or $\mathbb{C}$.
  - ${}_{B}\backslash^{G}/_{B} = [{}^{*}/_{B}] \times_{[{}^{*}/_{G}]} [{}^{*}/_{B}]$, with
    $$H_B^*(G/B) \cong H_T^*(G/B) \cong \bigoplus_{w} H_T^*(pt)$$ [my master thesis]
  - $\mathcal{H}(G, B)$: see [22.09.04]

- possible exercise:
  - Work out
    $${}_{T}\backslash^{G}/_{B}$$
    $${}_{P_1}\backslash^{G}/_{P_2} \quad {}_{GL_m \times GL_n}\backslash^{GL_{m+n}}/_{GL_m \times GL_n} \quad {}_{S_m \times S_n}\backslash^{S_{m+n}}/_{S_m \times S_n}$$ [22.11.13]
    $${}_{\mathbb{F}_{q^n}^{\times}}\backslash^{GL_n(\mathbb{F}_q)}/_{B}, \quad \ldots$$
    $\kappa = \mathbb{F}_1$; $GL_n \rightsquigarrow$ other gps
  - Computation of cardinals.

Archi field, $\mathbb{R}$ or $\mathbb{C}$

- subgps can be

	nearly affine		cpt
Borel	max split torus	unipotent	
B	T	N	$O(n)$
parabolic	Levi	unipotent	or $SO(n)$
P	L	M	
	nonsplit torus		$\mathcal{U}(n) := \mathcal{U}_{\mathbb{C}/\mathbb{R}}(n)$
	T'		or $SU(n)$



+ real & cplx

https://mathoverflow.net/questions/249313/real-orbits-on-flag-varieties

⚠ $M^{sym}_{n\times n}(\mathbb{R})$, $M^{sym,>0}_{n\times n}(\mathbb{R})$ are not gps!

- moduli intepretation

$V := \mathbb{R}^{\oplus n}$ In $\mathbb{C}^{\oplus n}$ case, replace inner product by Hermittian prod.

$$G/O(n) \;\cong\; \{\text{inner products on } V\} \;\cong\; M^{sym,>0}_{n\times n}(\mathbb{R})$$

$$g = (v_1, \cdots, v_n) \longmapsto \langle\, ,\, \rangle \text{ s.t. } \{v_1,\cdots,v_n\} \text{ is an ortho basis} \longmapsto (\langle e_i, e_j\rangle)^n_{i,j=1}$$

$v_i = g e_i$ i.e. $\langle x,y\rangle := x^T (g^{-1})^T g^{-1} y$

$$g \longmapsto (g^{-1})^T g^{-1}$$

as G-sets, where

$g\cdot x := gx$ $\quad g\cdot\langle\, ,\, \rangle := \langle g^{-1}-, g^{-1}-\rangle$ $\quad g\cdot A := (g^{-1})^T A g^{-1}$

i.e. $\langle gx, gy\rangle_g = \langle x,y\rangle$

action on inner product

Rmk. We actually get the polar decomposition here. not hard, but not obvious

$$GL_n(\mathbb{R}) = M^{sym,>0}_{n\times n}(\mathbb{R})\, O(n) \qquad GL_n(\mathbb{C}) = M^{herm,>0}_{n\times n}(\mathbb{C})\, \mathcal{U}(n)$$

Eg. $\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO(2)$

$\cong \{\text{inner products on } V\}/\text{scalars}$

$\cong M^{sym,>0}_{n\times n}(\mathbb{R})/\text{scalars}$

$\cong \{\text{max cpt subgps of } GL_2(\mathbb{R})\}$

Lemma 1,2

Lemma 1. cpt subgps are conj to a subgp of $O(2)$.

[Idea of proof. $K \hookrightarrow GL_2(\mathbb{R}) \circlearrowright \mathcal{H}$ maps bounded set to bounded set
$\Rightarrow K$ preserves one pt in $\mathcal{H}$]

Lemma 2. $g O(2) g^{-1} = O(2) \iff g \in O(2)\cdot\mathbb{R}_{>0}$

[Idea of proof. use $G \circlearrowright \mathcal{H}$ or SVD ← shown later]

– E.g. singular value decomposition (SVD) [svd-notes]

$$GL_n(\mathbb{R}) = \bigsqcup_{a_1 \geq a_2 \cdots a_n > 0} O(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} O(n)$$

$$GL_n(\mathbb{C}) = \bigsqcup_{a_1 \geq a_2 \cdots a_n > 0} U(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} U(n)$$

- "$\subseteq$", lazy proof.
  When $A \in GL_n(\mathbb{R})$ is symmetric, $A \underset{\sim}{\overset{O(n)\text{-conj}}{}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\quad \lambda_i \in \mathbb{R}^\times$.
  When $A \in GL_n(\mathbb{C})$ is normal matrix, $A \underset{\sim}{\overset{U(n)\text{-conj}}{}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\quad \lambda_i \in \mathbb{C}^\times$
  One can then use polar dec to show SVD.
- "$\sqcup$", algorithm.
  Suppose $A = U \Sigma V^T \in O(n) \Sigma O(n)$ $\quad \Sigma = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. $\quad a_i \in \mathbb{R}_{>0}$
  Observe that

$$A^T A = V \Sigma^T \Sigma V^T = V \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} V^{-1}$$

  $\Rightarrow$ eigenvalues of $A^T A$ tell us Σ.
- "$\subseteq$", algorithm. [svd-notes, Thm 3.2]

$$A^T A = V \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} V^{-1} \qquad a_i \in \mathbb{R}_{>0} \qquad A^T A (v_1, \cdots, v_n) = (v_1, \cdots, v_n) \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix}$$

  Take $\Sigma := \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, $\quad U = AV\Sigma^{-1}$, then $U \in O(n)$, $\quad A = U \Sigma V^T$.
- "$\sqcup$", geometry.

$$a_1 = \max_{v \neq 0} \frac{\|Av\|}{\|v\|} \qquad \|\cdot\|: \text{2-norm}$$

$$a_k = \min_{\substack{V \subseteq \mathbb{C}^n \\ \dim k-1}} \max_{\substack{v \perp V \\ v \neq 0}} \frac{\|Av\|}{\|v\|}$$

Compare with: https://en.wikipedia.org/wiki/Min-max_theorem
(Courant–Fischer–Weyl min-max principle)

- the "fundamental domain"

$$I = \{(a_1, \cdots, a_n) \in \mathbb{R}_{>0}^{\oplus n} \mid a_1 \geq a_2 \geq \cdots \geq a_n\} = \bigsqcup_{\substack{(k, (n_1, \dots, n_k)) \\ \sum n_i = n}} I_{n_1, \cdots, n_k}$$

$$I_{n_1, \cdots, n_k} = \{(\underbrace{a_1, \cdots, a_1}_{n_1}, \cdots, \underbrace{a_k, \cdots a_k}_{n_k}) \in \mathbb{R}_{>0}^{\oplus n} \mid a_1 > a_2 \cdots > a_k\}$$

is an n-dim real mfld, with boundary $I - I_{1, \cdots, 1}$.

- produces a foliation of $GL_n(\mathbb{R})/O(n)$ or $GL_n(\mathbb{C})/U(n)$ indexed by I, with each piece iso to

$$O(n)/\Sigma O(n)\Sigma^{-1}\cap O(n) \cong O(n)/O(n_1)\times\cdots O(n_k) \cong GL_n(\mathbb{R})/L$$
$$U(n)/\Sigma U(n)\Sigma^{-1}\cap U(n) \cong U(n)/U(n_1)\times\cdots U(n_k) \underset{\uparrow}{\cong} GL_n(\mathbb{C})/L$$

QR dec

Space	$\dim_{\mathbb{R}}$	Space	$\dim_{\mathbb{R}}$
$GL_n(\mathbb{R})$	n^2	$GL_n(\mathbb{C})$	$2n^2$
$O(n)$	$\frac{n(n-1)}{2}$	$U(n)$	n^2
$GL_n(\mathbb{R})/O(n)$	$\frac{n(n+1)}{2}$	$GL_n(\mathbb{C})/U(n)$	n^2
$GL_n(\mathbb{R})/L$	$\begin{smallmatrix} * & \\ * & * \end{smallmatrix}$	$GL_n(\mathbb{C})/L$	$\begin{smallmatrix} * & \\ * & * \end{smallmatrix}$ $\times 2$
$I_{n_1,\dots,n_k}$	k	$I_{n_1,\dots,n_k}$	k

E.g. The $SO(2)$-orbit on $\mathcal{H} := SL_2(\mathbb{R})/SO(2)$ is as follows:

$\mathcal{H}$

- stack quotient: not discussed yet
- [Getz, 3.3] https://mathoverflow.net/questions/301410/what-is-the-archimedean-hecke-algebra

$$\mathcal{H}(GL_n(\mathbb{R}), O(n)) := \left\{ f: GL_n(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ distributions} \\ \operatorname{supp} f \subseteq O(n) \\ f: \text{bi } O(n)\text{-finite} \end{array} \right\}$$
$$\overset{!}{\neq} \left\{ f: GL_n(\mathbb{R}) \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ sm, supp } f \text{ cpt,} \\ f(k_1 g k_2) = f(g) \;\; \forall k_1, k_2 \in O(n) \end{array} \right\}$$

bi $O(n)$-finite: $\langle f \rangle_{(O(n), O(n))\text{-module}} \subseteq \{\text{Distributions on } GL_n(\mathbb{R})\}$ is of fin dim.

– E.g. QR decomposition

ortho upper

We write "RQ dec" instead.

$$GL_n(\mathbb{R}) = B\cdot O(n) = \coprod_{t_i\in\{\pm 1\}} N \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} O(n)$$

$$GL_n(\mathbb{C}) = B\cdot \mathcal{U}(n) = \coprod_{\substack{t_i\in\mathbb{C} \\ |t_i|=1}} N \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mathcal{U}(n)$$

- Gauss elemination by B: Gram-Schmidt process
  Gauss elemination by $O(n)$: rotation s.t. $Av_i \in \langle e_1,\cdots,e_i\rangle$
- the "fundamental domain" is a single pt
- $GL_n(\mathbb{R})/O(n) \cong B/B\cap O(n) \overset{\text{as gp}}{\cong} \mathbb{R}_{>0}^{\oplus n} \oplus \mathbb{R}^{\oplus\binom{n}{2}}$
  $GL_n(\mathbb{C})/\mathcal{U}(n) \cong B/B\cap \mathcal{U}(n) \overset{\text{as gp}}{\cong} \mathbb{R}_{>0}^{\oplus n} \oplus \mathbb{C}^{\oplus\binom{n}{2}}$
  $B\backslash GL_n(\mathbb{R}) \cong B\cap O(n)\backslash O(n) \cong \{\pm 1\}^{\oplus n}\backslash O(n)$ is cpt
  $B\backslash GL_n(\mathbb{C}) \cong B\cap \mathcal{U}(n)\backslash \mathcal{U}(n) \cong (S^1)^{\oplus n}\backslash \mathcal{U}(n)$ is cpt

Rmk. As a Corollary, we know the (higher) homotopy gp of $B\backslash GL_n(\mathbb{R})$.
It's foundamental gp is still hard to construct.

e.g.

$$\pi_1\left(B\backslash GL_n(\mathbb{R})\right) \cong \begin{cases} \{Id\} & n=1 \\ \mathbb{Z} & n=2 \\ 1 \to \mathbb{Z}/2\mathbb{Z} \to ? \to (\mathbb{Z}/2\mathbb{Z})^{\oplus n} \to 1 & n>2 \end{cases}$$

The fundamental group of a real flag manifold
https://www.researchgate.net/publication/222792895_The_fundamental_group_of_a_real_flag_manifold

From this ref [Thm 1.1 + §5.2], we see

$$\pi_1(B\backslash GL_n(\mathbb{R})) \cong \langle t_{\alpha_1},\dots, t_{\alpha_{n-1}}\rangle \Big/ \begin{pmatrix} t_{\alpha_i}t_{\alpha_{i+1}} = t_{\alpha_{i+1}}t_{\alpha_i}^{-1},\ t_{\alpha_{i+1}}t_{\alpha_i} = t_{\alpha_i}t_{\alpha_{i+1}}^{-1}, \\ t_{\alpha_i}t_{\alpha_j} = t_{\alpha_j}t_{\alpha_i} \quad |i-j|\geq 2 \end{pmatrix}$$

e.p. $\pi_1(B\backslash GL_2(\mathbb{R})) \cong \langle t \rangle$

$\pi_1(B\backslash GL_3(\mathbb{R})) \cong \langle t,s\rangle / (tsts^{-1}, stst^{-1})$

$\cong \langle t,s\rangle / (t^4=1, s^2=t^2, sts^{-1}=t^{-1}) \cong Q_8$

cohomology rings of real flag manifolds are also well understood:

On the cohomology rings of real flag manifolds: Schubert cycles:
https://link.springer.com/article/10.1007/s00208-021-02237-z

- $H^*_{O(n)}(B\backslash GL_n(\mathbb{R})) \cong H^*_{O(n)}(B\cap O(n)\backslash O(n)) \cong H^*_{B\cap O(n)}(pt)$
  $H^*_{\mathcal{U}(n)}(B\backslash GL_n(\mathbb{C})) \cong H^*_{\mathcal{U}(n)}(B\cap \mathcal{U}(n)\backslash \mathcal{U}(n)) \cong H^*_{B\cap \mathcal{U}(n)}(pt)$

- Possible ex: work out

$SO(n) \backslash SL_n(\mathbb{R}) / SO(n)$

$O(n) \backslash GL_n(\mathbb{R}) / N$, $O(n) \backslash GL_n(\mathbb{R}) / T$, $O(n) \backslash GL_n(\mathbb{R}) / P$,

$GL_n(\mathbb{R}) \backslash GL_n(\mathbb{C}) / B$, ...

$B \backslash SO(n+1) / SO(n)$ $\rightsquigarrow$ Q: Can we find a good stratification of S^n in this way?

Borel of $SO(n)$

https://math.stackexchange.com/questions/466998/what-are-the-borels-parabolics-of-the-orthogonal-or-symplectic-groups

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization