

Eine Woche, ein Beispiel

5.26. 6-fctor formalism toolkit

I. before the toolkit

I.1. $R(F \circ G) = RF \circ RG$?

I.2. construct maps

I.3. roadmap for proofs of topo 6-fctor formalism

II. toolkit

II.1. conditions in topo case

II.2. minimal toolkit

II.3. add-ons (components)

recollement diagram

sm & flat

the hidden fctor $\underline{\text{Hom}}(-, \mathbb{F})$

Künneth fctor \boxtimes

invertible sheaf

descent (inverse problem)

III. after the toolkit (in next document?)

III.1. interpretation for memory

commute

FM transformation

...

III.2. cohomology interpretation

multiplication structure

UCT

cal of coh ring — intersection theory

— Schubert calculus

— Chern class

III.3. Weil conj interpretation

III.4. equiv interpretation

III.5. monodromy, nearby cycles

— topo

— étale

— stack

— in family

I. before the toolkit

In this part, we try to prove formulas in the minimal toolkit (for topological spaces case). To make the toolkit clean and concise, we put most technicality in this part, e.p. the relationship with derived formulas and underived formulas. Readers can skip this part if they don't want to see details.

I.1. $R(F \circ G) = RF \circ RG$? [Achar 21, Table 1.4.1, Def A.6.2]

$$\begin{aligned} f_* &: \text{flasque} \rightsquigarrow \text{flasque} \\ f_! &: \text{c-soft} \rightsquigarrow \text{c-soft} \end{aligned}$$

$$\begin{aligned} R(f \circ g)_* &= Rf_* \circ Rg_* \\ R(f \circ g)_! &= Rf_! \circ Rg_! \end{aligned}$$

$$\otimes : \text{flat} \not\rightsquigarrow \text{flat}$$

$$\begin{aligned} L(\mathcal{F} \otimes \mathcal{G} \otimes -) &\neq (\mathcal{F}^L \otimes -) \circ (\mathcal{G}^L \otimes -) \\ &\parallel \parallel \\ (\mathcal{F} \otimes \mathcal{G})^L \otimes - &\neq (\mathcal{F}^L \otimes \mathcal{G})^L \otimes - \end{aligned}$$

$$\underline{\text{Hom}}(\mathcal{F}, -) : \text{inj} \not\rightsquigarrow \text{inj}$$

$$\begin{aligned} R(\underline{\text{Hom}}(\mathcal{F}, \underline{\text{Hom}}(\mathcal{G}, -))) &\neq R\underline{\text{Hom}}(\mathcal{F}, -) \circ R\underline{\text{Hom}}(\mathcal{G}, -) \\ &\parallel \parallel \\ R\underline{\text{Hom}}(\mathcal{F} \otimes \mathcal{G}, -) &\neq R\underline{\text{Hom}}(\mathcal{F}^L \otimes \mathcal{G}, -) \end{aligned}$$

//A: In needed case $R(F \circ G) = RF \circ RG$, so don't worry about that too much.

I.2. construct maps

These exercises practice your ability to use adjunctions.

In this section, we ignore the derived symbol. I would remind you if you forget.

The reference will be collected in I.3.

Base change (BC): $f^* g_! \cong g'_! f'^*$

① When $g_! = g_*$, $g'_! = g'_*$, construct the map
$$f^* g_! \longrightarrow g'_! f'^*$$

② When $f^* = f^!$, $f'^* = f'^!$, construct the map
$$f^* g_! \longleftarrow g'_! f'^*$$

③ we require that the following diagram commutes:

$$\begin{array}{ccc} f^* g_! & \xrightarrow{\quad \quad \quad} & g'_! f'^* \\ \downarrow & & \downarrow \\ f^* g_! & \longrightarrow & g'_! f'^* \end{array}$$

$f^*(- \otimes -)$: $f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^* \mathcal{F} \otimes f^* \mathcal{F}'$

① construct the map (one side is enough)
$$f^*(\mathcal{F} \otimes \mathcal{F}') \longleftarrow f^* \mathcal{F} \otimes f^* \mathcal{F}'$$

by def of sheaf \otimes .

② show the iso by checking stalks.

③ If you did the underived version of ① & ②, try to work out derived version.

Projection formula: $f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_!\mathcal{G}$

① When $f_! = f_*$, construct the map

$$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \longleftarrow \mathcal{F} \otimes f_!\mathcal{G}$$

② When $f^* = f^!$, construct the map

$$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \longrightarrow \mathcal{F} \otimes f_!\mathcal{G}$$

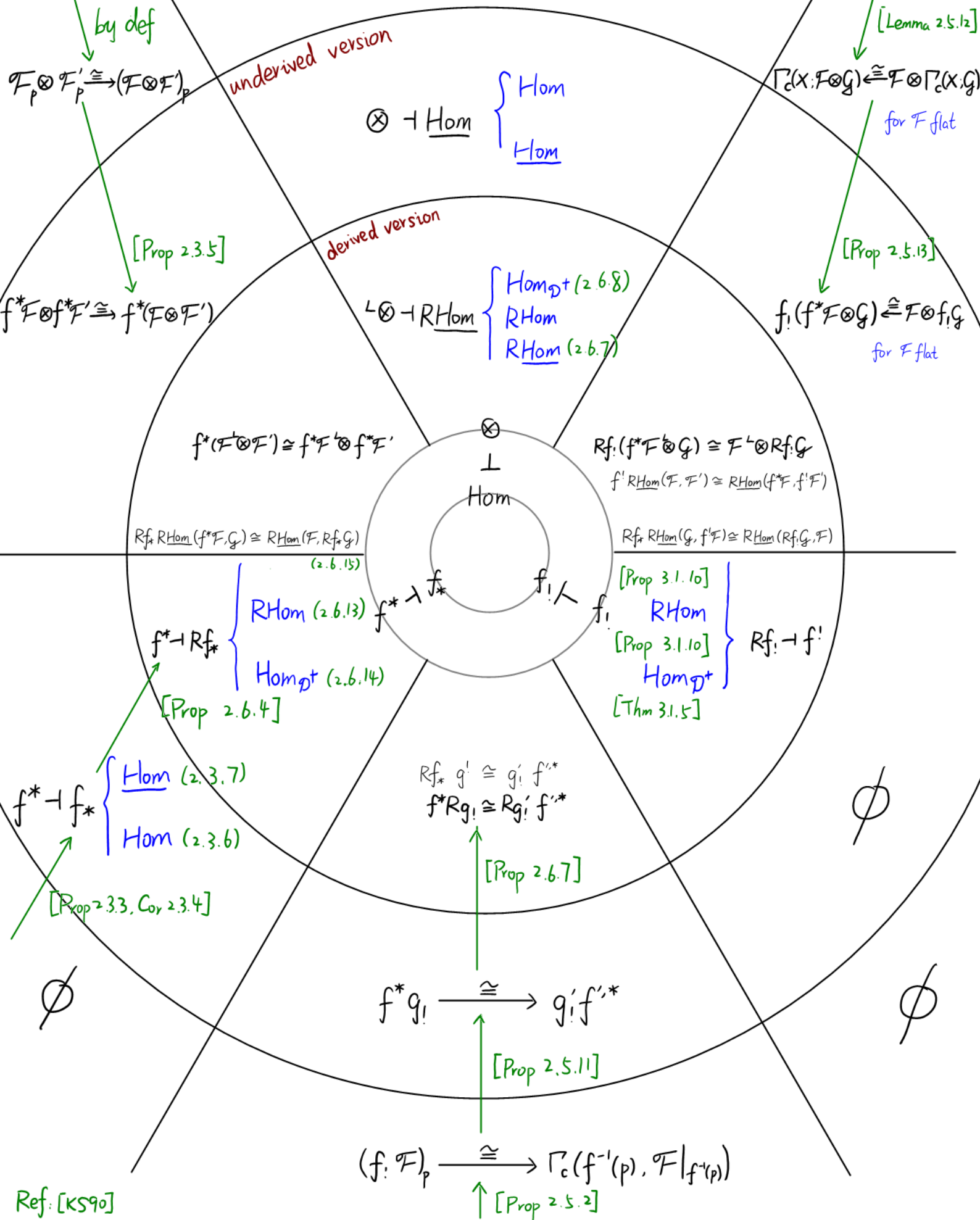
③ we require that the following diagram commutes:

$$\begin{array}{ccc} f_!(f^*\mathcal{F} \otimes \mathcal{G}) & \xleftarrow{\quad \quad \quad} & \mathcal{F} \otimes f_!\mathcal{G} \\ \downarrow & & \downarrow \\ f_!(f^*\mathcal{F} \otimes \mathcal{G}) & \longleftarrow & \mathcal{F} \otimes f_!\mathcal{G} \end{array}$$

I. 3. roadmap for proofs of topo 6-factor formalism

stalkwise true

$\otimes \dashv \text{Hom}$ for abelian gp



stalkwise true

$\otimes \dashv \text{Hom}$ for abelian gp

underived version

$\otimes \dashv \underline{\text{Hom}}$

$$\Gamma_c(X; \mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes \Gamma_c(X; \mathcal{G})$$

for \mathcal{F} flat

derived version

${}^L\otimes \dashv \underline{\text{RHom}}$

$$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_!\mathcal{G}$$

for \mathcal{F} flat

$$f^*\mathcal{F} \otimes f^*\mathcal{F}' \cong f^*(\mathcal{F} \otimes \mathcal{F}')$$

$$f^*(\mathcal{F}' \otimes \mathcal{F}) \cong f^*\mathcal{F}' \otimes f^*\mathcal{F}$$

\otimes

\perp

Hom

$$Rf_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes Rf_!\mathcal{G}$$

$$f'^*\text{RHom}(\mathcal{F}, \mathcal{F}') \cong \text{RHom}(f^*\mathcal{F}, f'^*\mathcal{F}')$$

$$Rf_*\text{RHom}(f^*\mathcal{F}, \mathcal{G}) \cong \text{RHom}(\mathcal{F}, Rf_*\mathcal{G})$$

$$Rf_*\text{RHom}(\mathcal{G}, f'^*\mathcal{F}) \cong \text{RHom}(Rf_*\mathcal{G}, \mathcal{F})$$

$$f^* \dashv f_*$$

$$f_! \dashv f^!$$

$$f^* \dashv Rf_*$$

$$Rf_! \dashv f^!$$

$$f^* \dashv f_*$$

$$Rf_*g' \cong g'_!f'^*$$

$$f^*Rg'_! \cong Rg'_!f'^*$$

$$f^*g'_! \xrightarrow{\cong} g'_!f'^*$$

$$(f; \mathcal{F})_p \xrightarrow{\cong} \Gamma_c(f^{-1}(p), \mathcal{F}|_{f^{-1}(p)})$$

The figure shows the routes for upgrading isomorphisms, while you can also downgrade isomorphisms.

To move from derived to underived categories, take the 0 -th cohomology. To go from underived categories to basic formulas, take the stalks.