

# Eine Woche, ein Beispiel

## 9.18 reps of $p$ -adic groups

This is also an unfinished task. I'm afraid that I forget those materials I organized.  
main ref: The Local Langlands Conjecture for  $GL(2)$

### Process

1. new notations
2. preliminaries
  - group
  - chain order
3. statement of classification (without proof)
4. fin dim
  - <https://mathoverflow.net/questions/34374/any-finite-dimensional-admissible-smooth-irreducible-representation-of-gl2-q-p>
5. other principal series
  - construction
  - proof
6. Cuspidal reps
7. Applications

realization

# 1. new notations

I don't want to bother you or make you confused, so I collect my notations here. Often it's not rigorous defined. You can view this section as a dictionary of notations.

from 2022.04.24

$F$ : non-arch local field.

$A = M_{2 \times 2}(F)$   $G = GL_2(F)$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$   $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$   $N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$   $Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = Z(G)$   $S = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$

$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$   $T^0 = \begin{pmatrix} 0^x & 0^x \\ 0 & 0^x \end{pmatrix}$   $N_j = \begin{pmatrix} 1 & p^j \\ 0 & 1 \end{pmatrix}$   $N'_j = \begin{pmatrix} 1 & 0 \\ p^j & 1 \end{pmatrix}$

<https://math.stackexchange.com/questions/299626/the-center-of-operatornamegl-n-k>

page	name	symbol	case $e_A = 1$	case $e_A = 2$
86	$\mathcal{O}$ -lattice chain	$\mathcal{L}$	$p^{-1}\mathcal{O} p^{-1} \subseteq \mathcal{O} \subseteq p\mathcal{O} p \subseteq \dots$	$p^{-1}\mathcal{O} \subseteq \mathcal{O} \subseteq \mathcal{O} \subseteq p\mathcal{O} p \subseteq \dots$
87	$\mathcal{O}$ -orders chain order	$\mathcal{A} = \mathcal{A}_{\mathcal{L}}$	$m = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\mathcal{T} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} \end{pmatrix}$
88	prime element	$\pi$	$(\pi \ \pi)$	$(\pi \ 1)$
88	Jacobson radical	$\text{Jac}(\mathcal{A})$	$\text{Jac}(m) = \begin{pmatrix} p & p \\ p & p \end{pmatrix}$	$\text{Jac}(\mathcal{T}) = \begin{pmatrix} p & \mathcal{O} \\ p & p \end{pmatrix}$
88		$\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}^{(0)} = \mathcal{A}^*$	$K_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}^x$	$I_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} \end{pmatrix}^x = \begin{pmatrix} \mathcal{O}^x & \mathcal{O} \\ p & \mathcal{O} \end{pmatrix}$
88		$\mathcal{U}_{\mathcal{A}}^{(n)} = 1 + \text{Jac}(\mathcal{A})^n$ $n \geq 1$	$K_n = 1 + \begin{pmatrix} p^n & p^n \\ p^n & p^n \end{pmatrix}$	$I_{2k-1} = 1 + \begin{pmatrix} p^k & p^{k-1} \\ p^k & p^k \end{pmatrix}$ $I_{2k} = 1 + \begin{pmatrix} p^k & p^k \\ p^{k+1} & p^k \end{pmatrix}$
89		$K_{\mathcal{A}}$	$K_0 \rtimes \langle (\pi \ \pi) \rangle$	$I_0 \rtimes \begin{pmatrix} p & \mathcal{O} \\ p & p \end{pmatrix}$

Rmk: For the convenience of handwriting and recognition, we use slightly different notations, which are listed as follows:

$\bar{w} \rightsquigarrow \pi$ ,  $\mathcal{U} \rightsquigarrow \mathcal{A}$ : I can not write original letters smoothly.

$K \rightsquigarrow K_0$ :  $K$  is kept to denote arbitrary cpt open subgp.

(Luckily we won't meet any  $K$ -theory in this document) For  $K$ -theory, there is usually a bracket after it.

$\text{rad} \mathcal{A} \rightsquigarrow \text{Jac}(\mathcal{A})$ : to avoid confusion with other radicals of a ring.

$\text{Jac}(\mathcal{A})$ : Jacobson radical

$$\mathcal{T} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p & \mathcal{O} \end{pmatrix}$$

$\mathcal{U} \rightsquigarrow \mathcal{U}$ : avoid confusion with union.

$$\mathcal{U}_F := \mathcal{U}_F^{(0)} = \mathcal{O}^*$$

$$\mathcal{U}_F^{(n)} = 1 + p^n$$

$n \geq 1$

$$\mathcal{U}_{\mathcal{A}} := \mathcal{U}_{\mathcal{A}}^{(0)} = \mathcal{A}^*$$

$$\mathcal{U}_{\mathcal{A}}^{(n)} = 1 + \text{Jac}(\mathcal{A})^n$$

$n \geq 1$

$$\mathcal{U}_2^Z :=$$

$$\mathcal{U}_2^{Z,1} :=$$

$$\mathcal{U}_2^{Z,2} :=$$

Def (level) For  $(\rho, V) \in \text{Irr}(G)$ ,

$$S(\rho) := \left\{ (A, n) \mid \begin{array}{l} A: \text{chain order } n \geq 0 \\ \rho|_{U_A^{(n+1)}} \text{ is trivial} \end{array} \right\}$$

(normalized level)

$$\begin{aligned} l(\rho) &:= \min \{ n/e_A \mid (A, n) \in S(\rho) \} \\ &= \min_{(A, n)} \{ n/e_A \mid \rho|_{U_A^{(n+1)}} \text{ is trivial} \} \end{aligned}$$

$$K_0 = GL_2(\mathcal{O}) \supset$$

$$I_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}^* = \begin{pmatrix} \mathcal{O}^* & \mathcal{O}^* \\ \mathfrak{p} & \mathcal{O}^* \end{pmatrix}$$

level

1

0

↓

$$K_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} \subset I_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$$

0

1/2

$$I_2 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{pmatrix} \supset$$

↑

1

$$K_2 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^2 & \mathfrak{p}^2 \end{pmatrix} \subset I_3 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^2 & \mathfrak{p}^2 \end{pmatrix}$$

↓

3/2

$$I_4 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^3 & \mathfrak{p}^2 \end{pmatrix} \supset$$

↑

> 2

$$\text{Cor } l(\pi) = 0 \Leftrightarrow \pi|_{K_1} \text{ is trivial.}$$