Modular form

3. discriminant

① meromorphic MF
③ almost holomorphic MF (殆全纯)
⑤ smooth automorphic form

⑥ Siegel MF
⑦ Hilbert MF

⓪ MF of congruence subgp (同余子群)
④ half-integral weight MF (半整权)

Def. A holo fct $f: \mathcal{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, level $\Gamma := SL_2(\mathbb{Z})$, if.

② quasimodular form

1) $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ $\quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

e.p. $f(\tau+1) = f(\tau)$

2) Write $f(\tau) = \sum_{n \in \mathbb{Z}} a_n (e^{2\pi i \tau})^n$, then $\underbrace{a_n = 0 \text{ for } n<0}$

⑧ p-adic MF

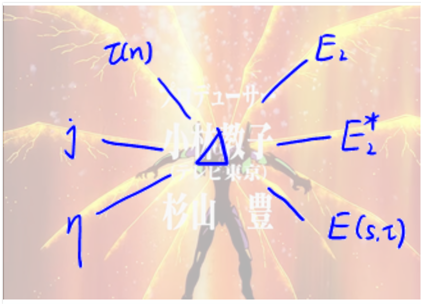
⓪ non-entire MF

① The order I plan to talk about

(For me they become more and more difficult)

Today: Δ, $\tau(n)$, ① j, ② E_2, ③ E_2^*, ④ η

\Delta的性质不可避免地用到了模形式的各类推广，这是今天讨论班的两条主线。



1. Def of Δ
2. relation with EC
3. q-exponential & Ramanujan τ-fct
4. Δ & j-invariant
   - as mero MF
   - as modular invariant
   - q-exponential and moonshine
   - special values and cplx multiplication
5. product formula & E_2
   - automorphy condition for E_2
   - product formula
   - quasimodular form and almost holomorphic MF
6. Δ & Dedekind η-fct
7. Δ & $E(s,\tau)$ ← possiblely to be added here

conclusion

1. Def of Δ

Ex. find $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ s.t. $\Delta \in q + q^2\mathbb{Q}[[q]]$

A. $\Delta(\tau) = \frac{1}{1728}\left(E_4(\tau)^3 - E_6(\tau)^2\right)$

$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + O(q^{11})$$

Cor. $S_*(SL_2(\mathbb{Z})) \cong \Delta\,\mathbb{C}[E_4, E_6] \triangleleft M_*(SL_2(\mathbb{Z}))$

2. relation with EC

Rmk. $\Delta(\tau) = \frac{1}{(2\pi)^{12}}\,\Delta\left(y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \frac{dx}{y}\right)$

which is closely related to the disc of EC.

slogan: any "fct" on moduli space reflects properties of EC.

⚠ The disc depends on "the choice of equations" (in reality, the choice of differential) and can change after the change of variables.

A possible explanation may be found here:
https://math.stackexchange.com/questions/3487698/discriminant-of-elliptic-curve-y2-ax3bx2cxd
(the reason is, \Delta is a function of lattices, and differential gives us a lattice.)

For results in char 2 & 3, see [ECII. Appendix A].

$w = \frac{dx}{2y+a_1x+a_3}$ $\quad y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6$ $\quad \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$

$b_2 = a_1^2+4a_2$

$b_4 = 2a_4+a_1a_3$

$b_6 = a_3^2+4a_6$

$b_8 = a_1^2a_6 + 4a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$

$w = \frac{dx}{y}$ $\quad y^2 = 4x^3+b_2x^2+2b_4x+b_6$ $\quad \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$

$4b_8 = b_2b_6 - b_4^2$

$y^2 = 4x^3 - g_2x - g_3$ $\quad \Delta = g_2^3 - 27g_3^2$ $\quad \Delta_{wiki} = 16\Delta$

$y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ $\quad \Delta = 16\prod_{i<j}(e_i-e_j)^2$

$y^2 = 4x(x-1)(x-\lambda)$ $\quad \Delta = 16\lambda^2(\lambda-1)^2$

$w = a^{\frac{1}{6}}\frac{dx}{y}$ $\quad y^2 = ax^3+bx^2+cx+d$ $\quad \Delta = 16(b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd)$

$\Delta_{wiki} = \frac{1}{16}\Delta$

$w = \frac{dx}{2y}$ $\quad y^2 = x^3+px+q$ $\quad \Delta = 16(-4p^3-27q^2)$

$y^2 = (x-e_1)(x-e_2)(x-e_3)$ $\quad \Delta = 16\prod_{i<j}(e_i-e_j)^2$

$y^2 = x(x-1)(x-\lambda)$ $\quad \Delta = 16\lambda^2(\lambda-1)^2$

$\Delta_{\text{my moduli}} = \Delta_{WWL} = \Delta_{\text{silverman}} = \Delta$

3. q-exponential & Ramanujan τ-fct

$$\Delta(\tau) = \sum_{n=1}^{+\infty} \tau(n) q^n$$

Observation

0) $\tau(1) = 1 \qquad \tau(n) \in \mathbb{Z}$

1) $\tau(nn') = \tau(n)\tau(n')$ for $\gcd(n, n') = 1$

   $\tau(p^{e+1}) = \tau(p)\tau(p^e) - p^{11}\tau(p^{e-1}) \qquad \forall p$ prime, $e \in \mathbb{Z}_{\geq 1}$

   e.g. $\tau(p^2) = \tau(p)^2 - p^{11}$

2) $|\tau(p)| \leq 2p^{\frac{11}{2}} \qquad \forall p$ prime

3) (Lehmer conj) $\forall n \in \mathbb{N}_{\geq 1}$, $\tau(n) \neq 0$

Rmk. 0). the following exercise or $\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ ← will be proved soon

1). by Mordell $\underset{\text{generalize}}{\rightsquigarrow}$ Hecke operators

2). by Deligne, use the Weil conj.

A much weaker version, $|\tau(p)| \leq Cp^6$ can be seen in [Za, Prop 8].

Ex. Show that

$$\tau(n) \in \mathbb{Z}$$

$$\tau(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = m^2, m \text{ odd} \\ 0 \pmod 2 & \text{otherwise} \end{cases}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

Hint. ① Write $A = \sum \sigma_3(n) q^n$, $B = \sum \sigma_5(n) q^n$,

$$\Delta = 5\,\frac{A-B}{2} + B + 100A^2 - 147B^2 + 8000A^3$$

$\Rightarrow$ first two equalities

② Write $G_{12} - \Delta$ as a combination of E_4 & E_6, i.e.

$$G_{12} = \Delta + \frac{691}{156}\left(\frac{E_4^3}{720} + \frac{E_6^2}{1008}\right)$$

The $\tau(n)$ enjoy various congruences modulo 2^{12}, 3^6, 5^3, 7, 23, 691. We quote some special cases (without proof):

(55) $\qquad \tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}$

(56) $\qquad \tau(n) \equiv n\sigma_3(n) \pmod 7$

(57) $\qquad \tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

Ramanujan's congruences [WWL, 例1.4.8]

$\Uparrow$

p-adic modular form

For other examples, and their interpretation in terms of "l-adic representations" see *Sém. Delange-Pisot-Poitou* 1967/68, exposé 14, *Sém. Bourbaki* 1968/69, exposé 355 and Swinnerton-Dyer's lecture at Antwerp (*Lecture Notes*, n° 350, Springer, 1973).

4. Δ & j-invariant

as mero MF

Ex.

$$\left\{\begin{array}{c}\text{entire/non-entire MF}\\ \text{with level } SL_2(\mathbb{Z})\end{array}\right\} \cong \mathbb{C}[E_4, E_6, \Delta^{-1}]$$

$$\left\{\begin{array}{c}\text{meromorphic MF}\\ \text{with level } SL_2(\mathbb{Z})\end{array}\right\} \cong \mathbb{C}(E_4, E_6)$$

$$\left\{\begin{array}{c}\text{meromorphic modular fct}\\ \text{with level } SL_2(\mathbb{Z})\end{array}\right\} \cong \mathbb{C}(j)$$

modular fct
= MF of weight 0

where $$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)^3} = 1728\frac{E_4^3}{E_4^3 - E_6^2}$$

as modular invariant

Ex. $j(\rho)=0 \quad j(i)=1728 \quad j(\infty)=\infty$

$j: (\mathcal{H}/SL_2(\mathbb{Z}))^* \longrightarrow \mathbb{CP}^1$ is an iso of RS.

another road to this result. [WWL, P233]

Cor [WWL, 定理8.2.4] X_1, X_2 : conn cpt RS of genus 1, then

$$X_1 \cong X_2 \Leftrightarrow j(X_1) = j(X_2)$$

Ex. (maybe hard) Let $X_\lambda := V(y^2 = x(x-1)(x-\lambda))$ $\lambda \neq 0,1,\infty$, compute $j(X_\lambda)$.

A. $$j(X_\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$$ [WWL, (8.2.6)]

$$j(X_\lambda) = j(X_{\frac{1}{\lambda}}) = j(X_{1-\lambda}) = \cdots$$

Rmk. we will see that λ is also a modular form (of level $\Gamma(2)$)

q-exponential and moonshine [WWL, p65]

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \cdots$$

$$j(\tau) \underset{\text{VOA: vertex operator alg}}{\overset{\infty\text{-dim Lie alg}}{\Longleftrightarrow}} \mathbb{M}$$

zhihu: 313723878

special values and cplx multiplication

Def An EC $E = \mathbb{C}/\Lambda$ admits cplx multiplication (CM) if

$$\exists \lambda \in \mathbb{C} - \mathbb{R}, \qquad \lambda \Lambda \subset \Lambda.$$

$\tau \in \mathcal{H}$ is a CM point if $\mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$ admits CM

E.g. $i, \rho, \sqrt{2} i$ are CM points.

Prop. [Za Prop 22, 23] For any CM pt τ, $j(\tau)$ is an alg number.

E.g. $j(\rho) = 0$ $\qquad j(i) = 1728$ $\qquad j\left(\frac{1+\sqrt{7}i}{2}\right) = -3375$

$j(\sqrt{2} i) = 8000$

$j(2i) = 287496$

$j(3i) = 76771008 + 44330490\sqrt{3}$

We will see from [Za Prop 25] that $j(\tau_0) \in \mathbb{Z}$ for $\tau_0 = \frac{1}{2}(1 + \sqrt{163}\, i)$

$$\Rightarrow q \approx -4 \times 10^{-18}$$

$$\Rightarrow j(\tau_0) = q^{-1} + 744 + O(q) \approx q^{-1} + 744$$

$$\Rightarrow e^{\pi\sqrt{163}} = -q^{-1} \approx 744 - j(\tau_0)$$

$$e^{\pi\sqrt{163}} = 262537412640768743.999\,999\,999\,999\,250\,072\,597\ldots$$

5. product formula & E_2

In this section, we will introduce E_2, and prove the central identity of today's talk.

$$\Delta(\tau) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$$

Def.
$$G_2(\tau) = \sum_{n\neq 0}\frac{1}{n^2} + \sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2}$$
∇ in [Za] $G_2(\tau) = \frac{1}{2}\cdots$

Ex. show that
$$\frac{1}{2}G_2(\tau) = \frac{(2\pi i)^2}{(2-1)!}\left(-\frac{B_2}{2\cdot 2} + \sum_{n=1}^{\infty}\sigma_1(n)q^n\right)$$

Similarly, define
$$E_2(\tau) = \left(-\frac{2\cdot 2}{B_2}\right)\left(-\frac{B_2}{2\cdot 2} + \sum_{n=1}^{\infty}\sigma_1(n)q^n\right)$$
$$\mathcal{G}_2(\tau) = \mathbb{G}_2(\tau) = -\frac{B_2}{2\cdot 2} + \sum_{n=1}^{\infty}\sigma_1(n)q^n$$

for convenience,
$$E_2(\tau) = 1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n = 1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$$

Sadly, $G_2(\tau)$ is no longer a modular form. (quasi MF instead)

automorphy condition for E_2

Prop [Za, Prop 6]
$$G_2(\gamma\tau) = (c\tau+d)^2G_2(\tau) - 2\pi i c(c\tau+d)$$
$$E_2(\gamma\tau) = (c\tau+d)^2E_2(\tau) - \frac{6ic}{\pi}(c\tau+d)$$

Proof. We use deformation. for $\varepsilon>0$, let
$$G_{2,\varepsilon}(\Lambda) = \sum_{z_0\in\Lambda}{}'\frac{1}{z_0^2|z_0|^{2\varepsilon}}$$
then
$$G_{2,\varepsilon}(\lambda\Lambda) = \lambda^{-2}|\lambda|^{-2\varepsilon}G_{2,\varepsilon}(\Lambda)$$
$$\Rightarrow G_{2,\varepsilon}(\gamma\tau) = (c\tau+d)^2|c\tau+d|^{2\varepsilon}G_{2,\varepsilon}(\tau)$$

Claim: $\lim G_{2,\varepsilon}(\tau) = G_2(\tau) - \frac{\pi}{\operatorname{Im}\tau} \overset{\Delta}{=} G_2^*(\tau)$

almost holo MF
$$G_2^*(\tau) = G_2(\tau) - \frac{\pi}{\operatorname{Im}\tau}$$
$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi\operatorname{Im}\tau}$$
$$\mathbb{G}_2^*(\tau) = \mathbb{G}_2(\tau) + \frac{1}{8\pi\operatorname{Im}\tau}$$

If so, $G_2(\gamma\tau) - \frac{\pi}{2\operatorname{Im}\gamma\tau} = (c\tau+d)^2\left(G_2(\tau) - \frac{\pi}{\operatorname{Im}\tau}\right)$
$$\Rightarrow G_2(\gamma\tau) = (c\tau+d)^2G_2(\tau) - \pi\left(\frac{(c\tau+d)^2}{\operatorname{Im}\tau} - \frac{1}{\operatorname{Im}\gamma\tau}\right)$$
$$= (c\tau+d)^2G_2(\tau) - 2\pi i c(c\tau+d)$$

Proof of claim Idea: 配凑法

$$\frac{1}{2}G_{2,\varepsilon}(\tau)=\sum_{n=1}^{\infty}\frac{1}{n^{2+2\varepsilon}}+\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}}\frac{1}{(m\tau+n)^2|m\tau+n|^{2\varepsilon}}$$

$\lim_{\varepsilon\to 0}$ comes in!

$$\frac{1}{2}G_{2,\varepsilon}-\sum_{m=1}^{+\infty}I_\varepsilon(m\tau)=\sum_{n=1}^{+\infty}\frac{1}{n^{2+2\varepsilon}}+\sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\left(\frac{1}{(m\tau+n)^2|m\tau+n|^{2\varepsilon}}-\int_n^{n+1}\frac{dt}{(m\tau+t)^2|m\tau+t|^{2\varepsilon}}\right)\quad(*)$$

where

$$I_\varepsilon(\tau)=\int_{-\infty}^{\infty}\frac{dt}{(\tau+t)^2|\tau+t|^{2\varepsilon}}\quad\left(\text{defined for all }\varepsilon>-\tfrac{1}{2}\right)\qquad I(\varepsilon)=\int_{-\infty}^{\infty}\frac{ds}{(s+i)^2|s^2+1|^{2\varepsilon}}$$

$$I_\varepsilon(x+iy)=\int_{-\infty}^{\infty}\frac{dt}{(x+t+iy)^2((x+t)^2+y^2)^{\varepsilon}}=\frac{1}{y^{1+2\varepsilon}}I(\varepsilon)$$

$$\Rightarrow\sum_{m=1}^{+\infty}I_\varepsilon(m\tau)=\frac{1}{y^{1+2\varepsilon}}\zeta(1+2\varepsilon)I(\varepsilon)$$
$$=\frac{1}{y^{1+2\varepsilon}}\left(\frac{1}{2\varepsilon}+O(1)\right)\left(-\pi\varepsilon+O(\varepsilon^2)\right)\xrightarrow{\varepsilon\to 0}-\frac{\pi}{2y}$$

Take $\lim_{\varepsilon\to 0}$ in $(*)$, the claim is proved. □

product formula

Now set $\quad\underline{\Delta}(\tau)=q\prod_{n=1}^{\infty}(1-q^n)^{24}$

We want $\underline{\Delta}=\Delta$. Reduce to $\underline{\Delta}(\gamma\tau)=(c\tau+d)^{12}\underline{\Delta}(\tau)$.

Lemma. $\frac{1}{2\pi i}\frac{d}{d\tau}\log\underline{\Delta}(\tau)=E_2(\tau)\qquad E_2(\gamma\tau)=(c\tau+d)^2E_2(\tau)-\frac{6ic}{\pi}(c\tau+d)$

$\frac{1}{2\pi i}\frac{d}{ds}\Big|_{s=\tau}\log\underline{\Delta}(s)$

Proof

$$\frac{1}{2\pi i}\frac{d}{d\tau}\log\left(\frac{\underline{\Delta}(\gamma\tau)}{(c\tau+d)^{12}\underline{\Delta}(\gamma)}\right)=\frac{1}{(c\tau+d)^2}E_2(\gamma\tau)-E_2(\tau)-\frac{12}{2\pi i}\frac{c}{c\tau+d}=0$$

$$\Rightarrow\underline{\Delta}(\gamma\tau)=C(\gamma)(c\tau+d)^{12}\underline{\Delta}(\tau)\qquad\forall\gamma\in SL_2(\mathbb{Z})$$

Reduced to $C(T)=C(S)=1$. □

Rmk. [NT II §9.2, p312] lists 5 methods to prove $\underline{\Delta}\in S_{12}(SL_2(\mathbb{Z}))$.
We adopt Method 5, which is also used in [Serre, Za, WWL, 228].

Method	name	make use of	fct equation of
1	η-fct	pentagonal number thm	Poisson sum of ψ_x
2	Kronecker limit formula	$\frac{\partial E}{\partial s}(0,\tau)=\frac{1}{6}\log((\operatorname{Im}\tau)^6\|\Delta(\tau)\|)$	$E(s,\tau)=\frac{1}{2}\sum_{\gcd(c,d)=1}\frac{(\operatorname{Im}\tau)^s}{\|c\tau+d\|^{2s}}$ convergent for $\operatorname{Re}s>1$
3	Siegel	η-fct	contour integral of $f_\nu(z)=\frac{1}{z}\cot(\nu z)\cot\left(\frac{\nu z}{i\tau}\right)$
4	Weil	$\sum_{n=1}^{\infty}\log(1-q^n)=\sum_{n=1}^{\infty}c_nq^n$, $\sum_{n=1}^{\infty}c_nn^{-s}=-\zeta(s)\zeta(s+1)$	$\zeta(s)$
5	Hurwitz method	$E_2=\frac{1}{2\pi i}\frac{d}{dz}\log\Delta(z)$	E_2

quasimodular form and almost holomorphic MF

We will return to this topic of quasi/almost holo MF in [Za, Sec 5, e.p. 5.3].
We only list results here:

quasi MF $\qquad \widetilde{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_2, E_4, E_6]$

almost holo MF $\qquad \widehat{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_2^*, E_4, E_6]$

Rmk. $\widetilde{M}_*(SL_2(\mathbb{Z}))$ is closed under differential.

e.g. $\quad E_2' = \frac{1}{12}(E_2^2 - E_4)$

$$\Rightarrow \sum_{m=1}^{n-1} \sigma_1(m)\sigma_1(n-m) = \frac{1}{12}\left(5\sigma_3(n) - (6n-1)\sigma_1(n)\right)$$

6. Δ & Dedekind η-fct

Def.
$$\eta(\tau) := \Delta(\tau)^{\frac{1}{24}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n) = e^{\frac{2\pi i\tau}{24}} \prod_{n=1}^{\infty}\left(1-e^{2\pi i n\tau}\right)$$

Ex. show that
$$\eta(\tau+1) = e^{\frac{2\pi i}{24}}\eta(\tau)$$
$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)$$

Ex. compute q-expansions of $\eta(\tau)$.

see wiki: Pentagonal number theorem if you don't work out the proof.
Another proof uses \theta-function.(i.e., equivalent with Jacobi triple product)

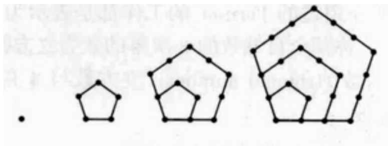


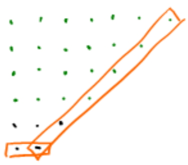
图 0.2 n 角数

A. $$\eta(\tau) = \sum_{n=1}^{+\infty} \chi_{12}(n)\, q^{\frac{n^2}{24}} = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \cdots$$
$$= q^{\frac{1}{24}} \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{2}} = q^{\frac{1}{24}}\left(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \cdots\right)$$

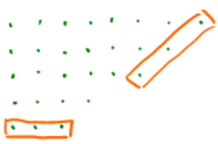
n	1	5	7	11
$\chi_{12}(n)$	1	-1	-1	1

"Proof"

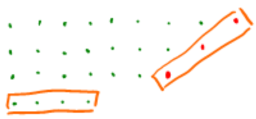
$\Rightarrow$



$27 = 8+7+5+4+3 \qquad 27 = 7+6+5+4+3+2$



$\Rightarrow$



$28 = 8+7+6+4+3 \qquad 28 = 9+8+7+4$

Conclusion

type	symbol	EC	relation with Δ	q-expansion	furthur study in
$-$	Δ, τ	Δ: disc	$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n$	$= q - 24q^2 + 252q^3 - \cdots$	Sec 4
①	j	modular inv	$j = \frac{E_4^3}{\Delta}$	$j(\tau) = q^{-1} + 744 + 196884q + \cdots$	Sec 6
②,③	E_2, E_2^*	$-$	$E_2 = \frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta(\tau)$	$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - \cdots$	Sec 5
④	η	$-$	$\eta = \Delta^{\frac{1}{24}}$	$\eta(\tau) = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \cdots$	Sec 3

	∞	i	ρ
Δ	0	$\frac{1}{2^6} \frac{\overline{\omega}^{12}}{\pi^{12}}$	?
j	∞	1728	0
E_2	1	$\frac{3}{\pi}$	$\frac{2\sqrt{3}}{\pi}$
η	0	$\frac{1}{\sqrt{2}} \frac{\overline{\omega}}{\pi}$	?