

Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.

[Mum74]: David Mumford. Prym Varieties. I, 1974.

<https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve>

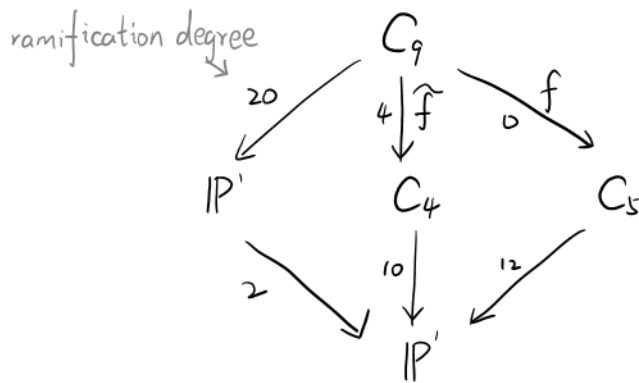
<https://mathoverflow.net/questions/402049/induced-action-on-prym-variety>

Goal: Describe some curves (maybe singular)  $C$  in  $A$ , and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$  has the following covers:  
 $\text{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



where

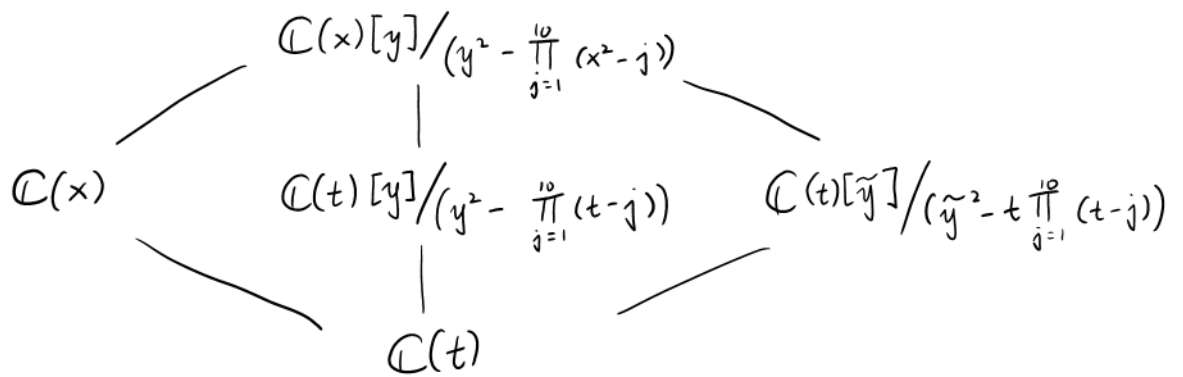
$$C_4 = \{y^2 = \prod_{j=1}^{10} (t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t \prod_{j=1}^{10} (t - j)\}$$

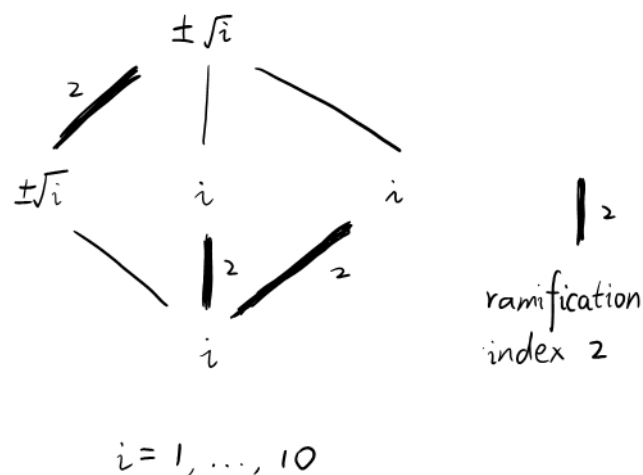
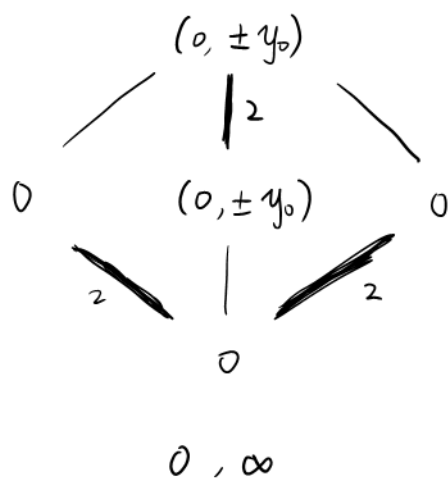
$$t = x^2$$

$$\tilde{y} = xy$$

The crspd field extension:



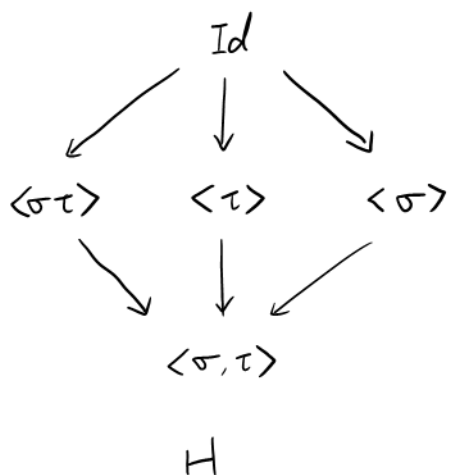
## Ramification information



$$\begin{aligned} \sigma: x &\mapsto -x \\ y &\mapsto -y \\ \text{Prym involution} \end{aligned}$$

$$\begin{aligned} \tau: x &\mapsto -x \\ y &\mapsto y \\ C_4 \text{ involution} \end{aligned}$$

$$\begin{aligned} \sigma\tau: x &\mapsto x \\ y &\mapsto -y \\ \text{hyperelliptic involution} \end{aligned}$$



The geometric signature of  $C_9 \rightarrow \mathbb{P}^1$  is

$$(0; \underbrace{\langle \tau \rangle}_{\text{genus } 0}, \underbrace{\langle \tau \rangle}_{\infty}, \underbrace{\langle \sigma\tau \rangle}_{1}, \dots, \underbrace{\langle \sigma\tau \rangle}_{10})$$

Therefore,

$$\begin{aligned} g_{C_H} &= [G:H](g-1) + 1 + \frac{1}{2} \sum_{j \in I} ([G:H] - |H \setminus G/G_j|) \\ &\stackrel{\text{when } |H|=2}{=} -1 + \frac{1}{2} \sum_{j \in I} (2 - |G/(G_j+H)|) \\ &\stackrel{g=0, G=\langle \sigma, \tau \rangle}{=} \begin{cases} 0, \\ 4, \\ 5, \end{cases} \end{aligned}$$

$H = \langle \sigma\tau \rangle$   
 $H = \langle \tau \rangle$   
 $H = \langle \sigma \rangle$

Local charts for  $f: C_9 \rightarrow C_5$

$$\begin{array}{ccc}
 C_9 & \mathbb{C}[x][y]/(y^2 - \prod_{j=1}^{10} (x^2 - j)) & \longleftrightarrow \mathbb{C}[u][v]/(v^2 - \prod_{j=1}^{10} (1 - ju^2)) \\
 \downarrow f & \uparrow R_{xy} & \uparrow R_{uv} \\
 C_5 & \mathbb{C}[t][\tilde{y}]/(\tilde{y}^2 - t \prod_{j=1}^{10} (t - j)) & \longleftrightarrow \mathbb{C}[s][\tilde{v}]/(\tilde{v}^2 - s \prod_{j=1}^{10} (1 - js)) \\
 & \uparrow R_{t\tilde{y}} & \uparrow R_{s\tilde{v}}
 \end{array}$$

$U_{xy} := \text{Spec } R_{xy}$ : the name of the local chart

$$\begin{array}{ccc}
 \begin{array}{c} x^2 \\ \uparrow \\ t \end{array} & \begin{array}{c} xy \\ \uparrow \\ \tilde{y} \end{array} & \begin{cases} x = \frac{1}{u} \\ y = \frac{v}{u^{10}} \end{cases} \\
 & & \begin{cases} t = \frac{1}{s} \\ \tilde{y} = \frac{\tilde{v}}{s^6} \end{cases}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} u^2 \\ \uparrow \\ s \end{array} & \begin{array}{c} uv \\ \uparrow \\ \tilde{v} \end{array} & \begin{cases} u = \frac{1}{x} \\ v = \frac{y}{x^{10}} \end{cases} \\
 & & \begin{cases} s = \frac{1}{t} \\ \tilde{v} = \frac{\tilde{y}}{t^6} \end{cases}
 \end{array}$$

Q: I saw from [Mum74, p 326] that

$$f^* \mathcal{O}_{C_9} = \mathcal{O}_{C_5} \oplus \eta$$

for some line bundle  $\eta \in \text{Pic}(C_5)$ .

Why is that true? How to describe  $\eta$ ?

<https://math.stackexchange.com/questions/527194/picard-group-of-a-affine-scheme>  
When  $A$  is an integral domain, the Picard group is known as the class group.

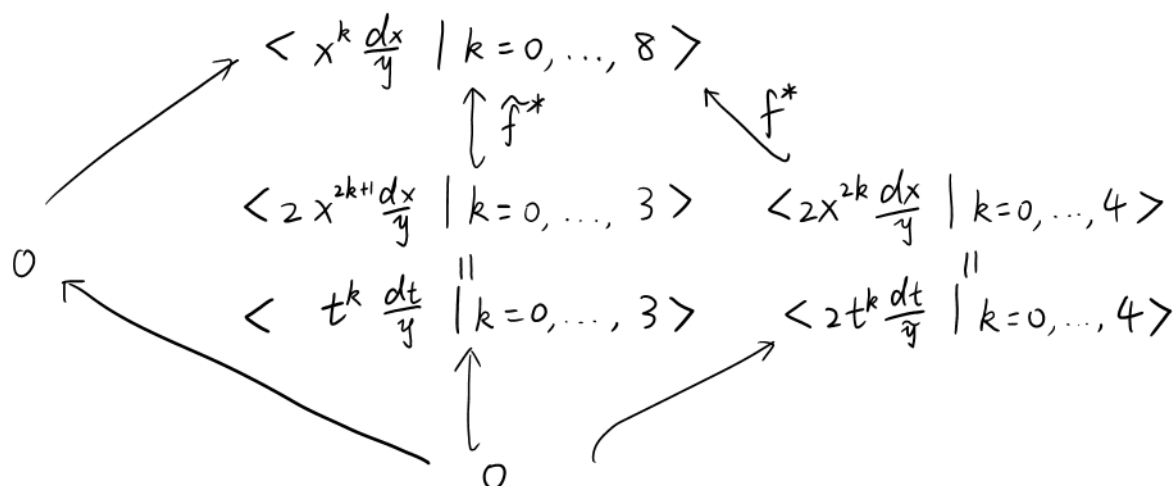
▽  $R_{xy} = \mathbb{C}[x][y]/(y^2 - \prod_{j=1}^{10} (x^2 - j))$  is not a PID, as

$$y^2 = \prod_{j=1}^{10} (x^2 - j)$$

How to compute  $\text{Pic}(\text{Spec } R_{xy})$ ?

## Global differential forms

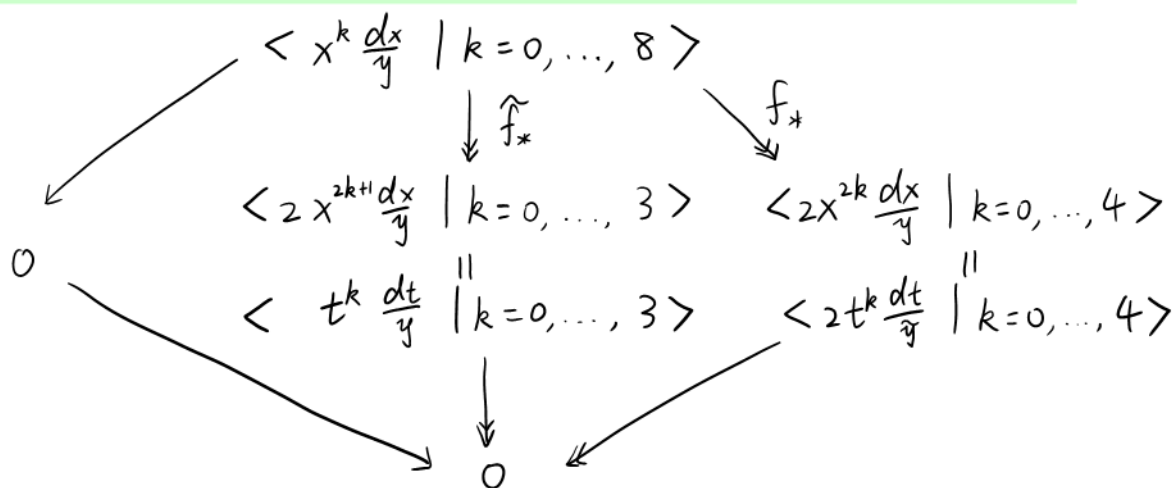
Pulling back differential forms give the following maps:



Therefore,

$$H^0(C_9; \omega_{C_9}) \cong \tilde{f}^* H^0(C_4; \omega_{C_4}) \oplus f^* H^0(C_5; \omega_{C_5}) \quad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which corresponds to pulling back of divisors)



However, since  $\text{Jac}(C) = H^0(C; \omega_C)^* / H_1(C; \mathbb{Z})$ , we are working on the dual spaces. The notations are again switched:

$$\begin{aligned} f^* &\rightsquigarrow N_{mf} \\ f_* &\rightsquigarrow f^* \end{aligned}$$

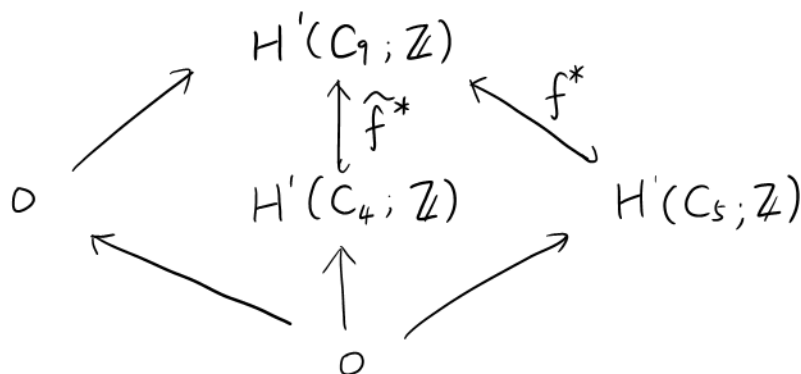
One may get

$$H^0(C_9; \omega_{C_9})^* \cong \tilde{f}^* H^0(C_4; \omega_{C_4})^* \oplus f^* H^0(C_5; \omega_{C_5})^* \quad (2)$$

different meaning compared with (1)!

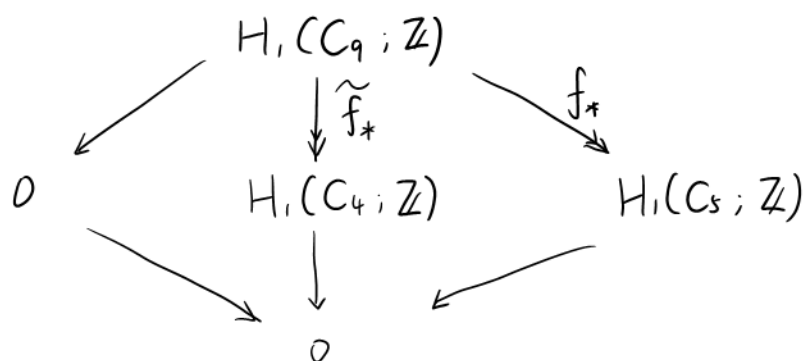
### (co)homology class

This page may be easier to understand, and it helps to understand the previous page.



Q: Do we have

$$H'(C_9; \mathbb{Z}) \cong \tilde{f}^* H'(C_4; \mathbb{Z}) \oplus f^* H'(C_5; \mathbb{Z})?$$



Q: Do we have

$$H_1(C_9; \mathbb{Z})^* \cong \tilde{f}^* H_1(C_4; \mathbb{Z})^* \oplus f^* H_1(C_5; \mathbb{Z})^*?$$

## Curve in Prym variety

Define  $A$  as the quotient of Jacobians, i.e.,

$$A := \text{Jac}(C_9) / f^* \text{Jac}(C_5) \cong \text{Prym}(C_9/C_5)$$

$$\begin{array}{ccccc}
 C_9 & \longrightarrow & C_4 & \xrightarrow{\quad} & \text{Jac}(C_4) \\
 \downarrow A|_{C_9} & & \downarrow & \swarrow \exists! \text{ isogeny} & \\
 \text{Jac}(C_5) & \xrightarrow{f^*} & \text{Jac}(C_9) & \xrightarrow{\pi} & A \longrightarrow 0
 \end{array} \quad (3)$$

- Prop
0.  $A$  is isogenous to  $\text{Jac}(C_4)$ ;
  1.  $f^*: \text{Jac}(C_5) \rightarrow \text{Jac}(C_9)$  is an isogeny to its image;
  2.  $\pi \circ A|_{C_9}$  is not injective, it factors through  $C_4$ ;
  3.  $C_4 \rightarrow A$  is generically injective;
  4.  $C_4 \rightarrow A$  produces a sm image of  $A$ , outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof. 0. Taking the tangent space of (3), one gets

$$0 \rightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df^*} H^0(C_9, \omega_{C_9})^* \rightarrow T_0 A \rightarrow 0$$

Combined with (2),

$$T_0 A \cong H^0(C_4, \omega_{C_4})^*.$$

Late we will find a natural isogeny  $\text{Jac}(C_4) \rightarrow A$ .  
 What's the degree of this isogeny?

1. Since

$$Nm_f \circ f^* = 2 \text{Id}_{\text{Jac}(C_5)}$$

$$\# \text{Ker } f^* < +\infty$$

Is  $f^*$  inj in this case?

2. For  $p_1 = (x_0, y_0)$ ,  $p_2 = (-x_0, y_0)$ , we want to show that

$$\int_{\gamma_1: p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\gamma_2: p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$\text{LHS} = \int_{\gamma_1: p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = \text{RHS}.$$

3.

<https://mathoverflow.net/questions/68503/has-anyone-studied-the-pym-map-for-double-covers-with-two-ramification-points>  
<https://arxiv.org/abs/1010.4483>: It proves that many Prym maps  $(C \rightarrow \text{Prym})$  are generically finite.

Notice:  $C_4 \subset \text{Jac}(C_4)$  is only invariant under  $p \mapsto -p$ ,  
 not invariant under  $p \mapsto p + a_0$ .

Otherwise, the Gauss map would be cover of  $\deg > 2$ .

Therefore, after isogeny  $C_4 \rightarrow A$  is still gen inj.

Q: Is this map really inj?

(4)  $C_4 \hookrightarrow \text{Jac}(C_4)$  is sm, so after isogeny it is still sm  
 outside of non-injective locus.



Rmk. Suppose  $f: \tilde{C} \rightarrow C$  is a **deg 2** (ramified) covering,  
 $\sigma: \tilde{C} \rightarrow \tilde{C}$  the crspd involution,  
 define  $A$  as the quotient

$$\begin{array}{c} \tilde{C} \\ \downarrow AJ_{\tilde{C}} \\ \text{Jac}(C) \xrightarrow{f^*} \text{Jac}(\tilde{C}) \xrightarrow{\pi} A \longrightarrow 0 \end{array}$$

one can identify  $A$  with  $\text{Prym}(\tilde{C}/C) \subset \text{Jac}(\tilde{C})$ , **why?**  
 and the Abel-Prym map is given by

$$\begin{aligned} AP_{\tilde{C}} = \pi \circ AJ_{\tilde{C}}: \quad \tilde{C} &\longrightarrow \text{Jac}(\tilde{C}) \longrightarrow A \\ p &\longmapsto \mathcal{O}_{\tilde{C}}(p - p_0) \longmapsto \mathcal{O}_{\tilde{C}}(p - \sigma(p)) \end{aligned}$$

Therefore, for  $p_1 \neq p_2$ ,

$$\begin{aligned} AP_{\tilde{C}}(p_1) &= AP_{\tilde{C}}(p_2) \\ \Leftrightarrow \mathcal{O}_{\tilde{C}}(p_1 - \sigma(p_1)) &= \mathcal{O}_{\tilde{C}}(p_2 - \sigma(p_2)) \\ \Leftrightarrow \mathcal{O}_{\tilde{C}}(p_1 + \sigma(p_2)) &= \mathcal{O}_{\tilde{C}}(p_2 + \sigma(p_1)) \end{aligned}$$

① When  $p_1, p_2 \in \tilde{C}$  are ramification pts of  $f$ ,  
 i.e.,  $p_1 = \sigma(p_1)$ ,  $p_2 = \sigma(p_2)$

$$AP_{\tilde{C}}(p_1) = AP_{\tilde{C}}(p_2).$$

As a result, when  $f$  is ramified,  $AP_{\tilde{C}}$  is never injective.

② Now assume  $AP_{\tilde{C}}$  is not inj,  $AP_{\tilde{C}}(p_1) = AP_{\tilde{C}}(p_2)$ .  
 When  $p_1 \neq \sigma(p_1)$  or  $p_2 \neq \sigma(p_2)$ ,

$\mathcal{L} := \mathcal{O}_{\tilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\tilde{C}}(p_2 + \sigma(p_1))$   
 is a l.b. with  $\deg 2$  &  $\text{rk} \geq 1$ , so  $\tilde{C}$  must be hyperelliptic,  
 and  $\mathcal{L}$  induces a degree 2 map

$$\tilde{C} \longrightarrow \mathbb{P}^1.$$

③ In the example,

$$\tilde{C} = C_9, \quad C = C_5,$$

$$\begin{aligned} \sigma: \quad x &\mapsto -x \\ y &\mapsto -y \\ \text{Prym involution} \end{aligned}$$

$$\begin{aligned} \tau: \quad x &\mapsto -x \\ y &\mapsto y \\ C_4 \text{ involution} \end{aligned}$$

$$\begin{aligned} \sigma\tau: \quad x &\mapsto x \\ y &\mapsto -y \\ \text{hyperelliptic involution} \end{aligned}$$

we can show directly that

$$AP_{C_9}(\tau(p)) = AP_{C_9}(p).$$

This gives a second proof for Prop (2).

Reason:

$$\begin{aligned} \mathcal{O}_{C_9}(\tau(p) + \sigma(p)) &= \mathcal{O}_{C_9}(\tau(p + \sigma\tau(p))) \\ &= \mathcal{O}_{C_9}(p + \sigma\tau(p)) \end{aligned}$$

$\sigma\tau = \tau\sigma$   
 $\sigma\tau$  is an  
hyperelliptic involution

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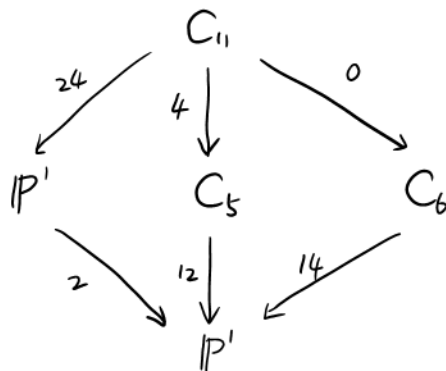
## Gauss map

Taking the Gauss map of (3), one gets

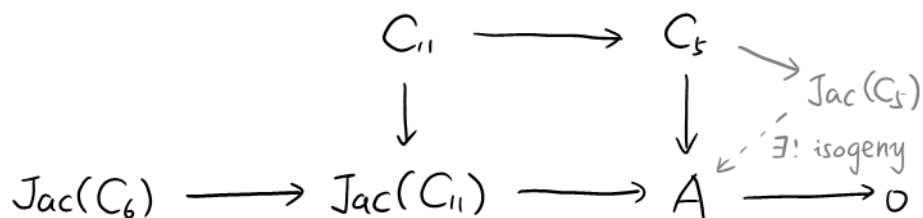
$$\begin{array}{ccc}
 C_9 & \xrightarrow[\quad 4]{\quad 2:1 \quad} & C_4 \\
 \downarrow \substack{2:1 \\ 20} & & \downarrow \substack{2:1 \\ 10} \\
 R_1 & \xrightarrow[\substack{\text{ram } a^t \\ 0, \infty}]{\quad 2:1 \quad} & R_2 \\
 \downarrow & & \downarrow \\
 \mathbb{P}^8 & \dashrightarrow & \mathbb{P}^3 \\
 [\alpha_0, \dots, \alpha_8] & \longmapsto & [a_1 : a_3 : a_5 : a_7]
 \end{array}$$

$$\Rightarrow \deg_A C_4 = 6, \quad \text{Gal}(\gamma) = S_6 = W(C_3).$$

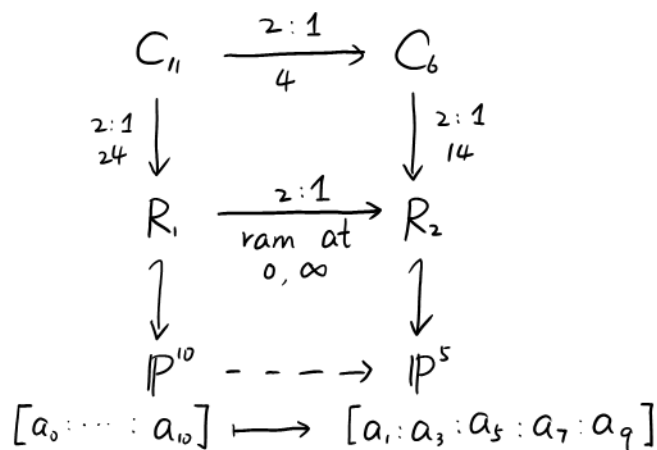
E.g. 2.  $C_{11} = \{y^2 = \prod_{j=1}^{12} (x^2 - j)\}$  has the following covering:



Curves in Jacobians: ( $A = \text{Prym}(C_{11}/C_6)$ ?)



Gauss map:



$$\Rightarrow \deg_A C_6 = 14, \quad \text{Gal}(\gamma) = S_{14} = W(C_7).$$

E.g. 3.

$$C_5 = \{y^2 = (x^3 + x + 2)^4 + 1\}$$

$$f \downarrow \begin{smallmatrix} 3:1 \\ 8 \end{smallmatrix}$$

$$E = \{y^2 = t^4 + 1\}$$

Let us write down the change of variables for  $f$ :

$$\begin{array}{ccc}
 C_5 & \mathbb{C}(x)[y]/(y^2 - ((x^3 + x + 2)^4 + 1)) & \cong \mathbb{C}(u)[v']/(v'^2 - ((1 + u^2 + 2u^3)^4 + u'^2)) \\
 \downarrow & \uparrow & \uparrow \\
 E & \mathbb{C}(t)[y]/(y^2 - (t^4 + 1)) & \cong \mathbb{C}(s)[v]/(v^2 - (1 + s^4))
 \end{array}$$
  

$$\begin{array}{ccc}
 \begin{array}{c} x^3 + x + 2 \\ \updownarrow \\ t \end{array} & \begin{array}{c} y \\ \updownarrow \\ y \end{array} & \begin{array}{c} \begin{cases} x = \frac{1}{u} \\ y = \frac{v'}{u^6} \end{cases} \\ \begin{cases} t = \frac{1}{s} \\ y = \frac{v}{s^2} \end{cases} \end{array}
 \end{array}$$
  

$$\begin{array}{ccc}
 \begin{array}{c} u^3 \\ \updownarrow \\ s \end{array} & \begin{array}{c} \frac{v'}{(1 + u^2 + 2u^3)^2} \\ \updownarrow \\ v \end{array} & \begin{array}{c} \begin{cases} u = \frac{1}{x} \\ v' = \frac{y}{x^6} \end{cases} \\ \begin{cases} s = \frac{1}{t} \\ v = \frac{y}{t^2} \end{cases} \end{array}
 \end{array}$$

Also, we write down the local coordinate charts of  $f$ :

$$\begin{array}{ccc}
 C_5 & \mathbb{C}[x][y]/(y^2 - ((x^3 + x + 2)^4 + 1)) & \longleftrightarrow \mathbb{C}[u][v']/(v'^2 - ((1 + u^2 + 2u^3)^4 + u'^2)) \\
 \downarrow & \uparrow & \uparrow \\
 E & \mathbb{C}[t][y]/(y^2 - (t^4 + 1)) & \longleftrightarrow \mathbb{C}[s][v]/(v^2 - (1 + s^4))
 \end{array}$$

Curves in Jacobians:

$$A = \text{Prym}(C_s/E) \text{ ?}$$

$$\begin{array}{ccccc} & C_s & & & \\ & \downarrow AJ_{C_s} & \searrow & & \\ E & \xrightarrow{f^*} & \text{Jac}(C_s) & \xrightarrow{\pi} & A \longrightarrow 0 \end{array}$$

- Prop.
1.  $\# \ker f^* < +\infty$  ;
  2.  $\pi \circ AJ_{C_s} : C_s \longrightarrow A$  is gen injective ;
  3. The image of  $C_s \longrightarrow A$  is smooth outside the non-injective locus.