

fiber of  $\pi_*, \pi_!, \pi^{-1}, \pi^*$   $\mathcal{G} \quad \mathcal{F}$   
 $\pi: Y \hookrightarrow X$

$$\pi_*: \mathcal{U} \rightarrow X \quad \pi_*: Z \xrightarrow{\text{close}} X$$

$$\pi_* \begin{cases} \mathcal{G}_x & x \in \mathcal{U} \\ 0 & x \notin \mathcal{U} \\ \varinjlim_{x \in V} \mathcal{G}(U \cap V) & x \in \bar{\mathcal{U}} - \mathcal{U} \end{cases} \quad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi_! \begin{cases} \mathcal{G}_x & x \in \mathcal{U} \\ 0 & x \notin \mathcal{U} \end{cases} \quad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi^{-1} \mathcal{F}_y \quad \mathcal{F}_y$$

$$\pi^* \mathcal{F}_y \otimes_{\pi^{-1} \mathcal{O}_{X,Y}} \mathcal{O}_{Y,Y} \quad \mathcal{F}_y \otimes_{\pi^{-1} \mathcal{O}_{X,Y}} \mathcal{O}_{Y,Y}$$

For étale:  $f: X \xrightarrow{\mathcal{G}} Y \xrightarrow{\mathcal{F}}$   $\bar{x} \mapsto \bar{y}$   
 (sheaf)

$$\bar{x} \xrightarrow{u_{\bar{x}}} x \xrightarrow{f} y$$

$$(f^* \mathcal{F})_{\bar{x}} = u_{\bar{x}}^* f^* \mathcal{F}(y) = \mathcal{F}_y$$

If  $f$  q.c.  $(f^* \mathcal{F})_{\bar{x}} \cong \mathcal{F}_y$   
 $(f_* \mathcal{G})_{\bar{y}} \cong \Gamma(\tilde{X}, \tilde{g}^* \mathcal{G})$  where  $\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X \\ \downarrow \wr & & \downarrow f \\ \text{Spec } \mathcal{O}_{Y,\bar{y}}^d & \xrightarrow{g} & Y \end{array}$   
 e.g. for  $f$  finite, we have explicit description of  $f_*$ .

左伴随  $\rightarrow$  右伴随

$$f_! \rightarrow f^* \rightarrow f_* \rightarrow f'$$

自由

忘却

$$-\otimes_A N$$

$$\text{Hom}_A(N, -)$$

$$\Delta$$

$$\varprojlim$$

$$(\ ) \sim$$

$$\varprojlim^*$$

$$f_p$$

$$f^*$$

$$c\text{-Ind}$$

$$\text{Res}$$

$$\text{Res}$$

$$\text{Ind}$$

$$-\otimes_k A^e$$

$$\text{Res}$$

$$\text{sh}(-)$$

$$\text{Res sh} \rightarrow \text{Psh}$$

$$\pi_0$$

$$T \mapsto \coprod_{t \in T} \text{Spec } \mathbb{Z}$$

$$G^{[p]}$$

$$\text{Lie}$$

$$|-|$$

$$S$$

$$\tau_1 = H_0(-)$$

$$N: \text{nerve}$$

$$\beta: \text{Stone-}\check{\text{Cech}}$$

$$U: \text{forget}$$

ad  $\Rightarrow$  右正合

左正合

ad  $\Rightarrow$  与余极限交换  $\varinjlim$

与极限交换  $\varprojlim$

$$\text{coker } f$$

$$\text{ker } f$$

$$\prod A \oplus A$$

$$\prod A \quad A \times_c B$$

$$\text{pushforward}$$

$$\text{pullback}$$

$$\bar{K} = \varinjlim_{L/K} L$$

$$\text{Spec } \bar{K} = \varprojlim_{L/K} \text{Spec } L \quad \text{Gal}(\bar{K}/K) = \varprojlim_{\substack{L \text{ finite} \\ \text{Galois}}} \text{Gal}(L/K)$$

$$\text{Spec } \mathbb{Z}_p = \varinjlim \text{Spec } \mathbb{Z}/p^n \mathbb{Z}$$

completion

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \mathbb{Z}$$

ring point of view

$$\mathcal{F}_p = \varinjlim_v \mathcal{F}(U)$$

stalk

co limit

limit

direct inductive } limit.

inverse limit.

injective

How to memorize:

limit

colimit

$$0 \rightarrow \text{ker} \rightarrow M \rightarrow N \rightarrow \text{coker} \rightarrow 0$$

**Lemma 6.21.3.** Let  $f : X \rightarrow Y$  be a continuous map. There exists a functor  $f_p : PSh(Y) \rightarrow PSh(X)$  which is left adjoint to  $f_*$ . For a presheaf  $\mathcal{G}$  it is determined by the rule

$$f_p \mathcal{G}(U) = \operatorname{colim}_{f(U) \subset V} \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods  $V$  of  $f(U)$  in  $Y$ . The colimits are over directed partially ordered sets. (The restriction mappings of  $f_p \mathcal{G}$  are explained in the proof.)

**Lemma 6.31.4.** Let  $X$  be a topological space. Let  $j : U \rightarrow X$  be the inclusion of an open subset.

- (1) The functor  $j_{p!}$  is a left adjoint to the restriction functor  $j_p$  (see Lemma 6.31.1).
- (2) The functor  $j_!$  is a left adjoint to restriction, in a formula

$$\operatorname{Mor}_{Sh(X)}(j_! \mathcal{F}, \mathcal{G}) = \operatorname{Mor}_{Sh(U)}(\mathcal{F}, j^{-1} \mathcal{G}) = \operatorname{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in  $\mathcal{F}$  and  $\mathcal{G}$ .

- (3) Let  $\mathcal{F}$  be a sheaf of sets on  $U$ . The stalks of the sheaf  $j_! \mathcal{F}$  are described as follows

$$j_! \mathcal{F}_x = \begin{cases} \emptyset & \text{if } x \notin U \\ \mathcal{F}_x & \text{if } x \in U \end{cases}$$

- (4) On the category of presheaves of  $U$  we have  $j_p j_{p!} = \operatorname{id}$ .
- (5) On the category of sheaves of  $U$  we have  $j^{-1} j_! = \operatorname{id}$ .

situation	category $\mathcal{A}$	category $\mathcal{B}$	left adjoint $F: \mathcal{A} \rightarrow \mathcal{B}$	right adjoint $G: \mathcal{B} \rightarrow \mathcal{A}$
A-modules (Ex. 1.5.D)	$Mod_A$	$Mod_A$	$(\cdot) \otimes_A N$	$Hom_A(N, \cdot)$
ring maps $B \rightarrow A$ (Ex. 1.5.E)	$Mod_B$	$Mod_A$	$(\cdot) \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(pre)sheaves on a topological space $X$ (Ex. 2.4.L)	presheaves on $X$	sheaves on $X$	sheafification	forgetful
(semi)groups (§1.5.3)	semigroups	groups	groupification	forgetful
sheaves, $\pi: X \rightarrow Y$ (Ex. 2.7.B)	sheaves on $Y$	sheaves on $X$	$\pi^{-1}$	$\pi_*$
sheaves of abelian groups or $\mathcal{O}$ -modules, open embeddings $\pi: U \hookrightarrow Y$ (Ex. 2.7.G)	sheaves on $U$	sheaves on $Y$	$\pi_!$	$\pi^{-1}$
quasicoherent sheaves, $\pi: X \rightarrow Y$ (Prop. 16.3.6)	$QCoh_Y$	$QCoh_X$	$\pi^*$	$\pi_*$
ring maps $B \rightarrow A$ (Ex. 30.3.A)	$Mod_A$	$Mod_B$	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto$ $Hom_B(A, N)$
quasicoherent sheaves, affine $\pi: X \rightarrow Y$ (Ex. 30.3.B(b))	$QCoh_X$	$QCoh_Y$	$\pi_*$	$\pi_{sh}^!$

Other examples will also come up, such as the adjoint pair  $(\sim, \Gamma_\bullet)$  between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.4).

E.g.  $\text{Spec } \mathbb{C} \longrightarrow \text{Spec } \mathbb{R}$  ring space but not scheme

$$(f^{-1}: \mathcal{R}\text{-mod} \longrightarrow (\text{Spec } \mathbb{C}, \widehat{\mathcal{R}})^{\leftarrow \text{mod}})$$

quasi-coherent:  $f^*: \mathbb{R}\text{-mod} \rightarrow \mathbb{C}\text{-mod} = \bigotimes_{\mathbb{R}} \mathbb{C}$

$$f_* : \mathbb{C}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \quad \text{forget}$$

$$f^* f_*: \mathbb{C}\text{-mod} \rightarrow \mathbb{C}\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C} \leftarrow \mathbb{C}\text{-mod structure}$$

$$f_* f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C}$$

étale:  $f^*: \mathbb{Z}[\mathbb{Z}/\mathbb{Z}] \text{-mod} \rightarrow \mathcal{A}b$  forget

$$f_*: \mathcal{A}b \rightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \text{-mod} = \otimes_{\mathbb{R}} \mathbb{C}$$

$$f^* f_*: \mathcal{A}b \rightarrow \mathcal{A}b \quad - \otimes_{\mathbb{R}} \mathbb{C}$$

$$f_* f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \rightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} = \otimes_{\mathbb{R}} \mathbb{C} \leftarrow \text{Gal}(\mathbb{C}/\mathbb{R})\text{-action}$$

$$\pi: X_{\text{ét}} \longrightarrow X_{\text{zar}}$$

$$\pi^{-1}: Sh(X_{zar}) \longrightarrow Sh(X_{\acute{e}t})$$

$$\pi_*: Sh(X_{\acute{e}t}) \longrightarrow Sh(X_{zar})$$

$$\pi^{-1} \vdash \pi_* \quad \pi_* \pi^{-1} = \text{Id}_{\text{Sh}(X_{\text{zar}})} \Rightarrow \pi^{-1} \text{ fully faithful}$$

<https://math.stackexchange.com/questions/321633/when-is-a-fully-faithful-functor-a-n-equivalent-functor>

$$Q\text{Coh}(X_{\text{zar}})$$

$\swarrow$   $\otimes$   $\downarrow$  full subcategory

$$Sh(X_{\text{ét}}) \xrightleftharpoons[\pi^*]{\pi_*} Sh(X_{\text{zar}})$$

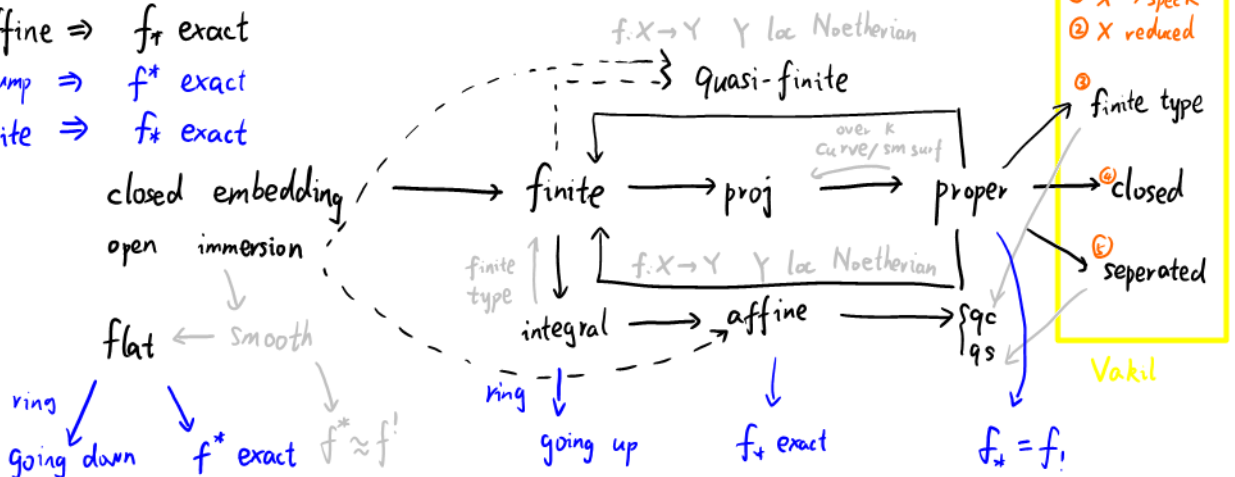
18.1.7

$$f \text{ flat} \Rightarrow f^* \text{ exact}$$
$$f \text{ affine} \Rightarrow f_{\tau} \text{ exact}$$

mild assumpt  $\Rightarrow f^*$  exact

$$f \text{ finite} \Rightarrow f_* \text{ exact}$$

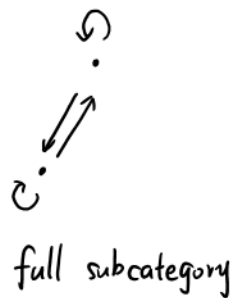
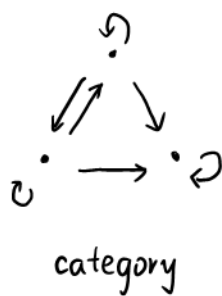
étale:



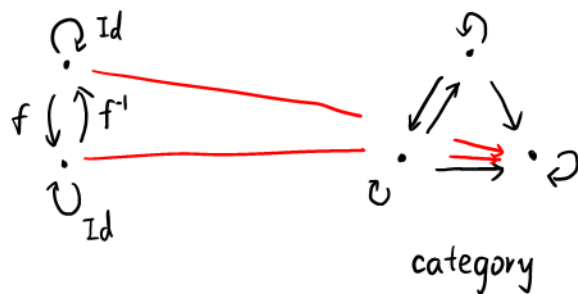
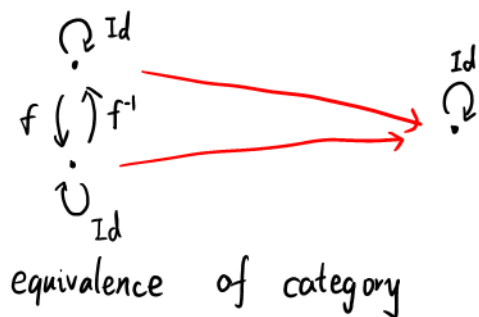
Various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercises 7.3.B(b), 7.3.D, and 10.1.H resp.) are quasiseparated,







<https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective>



fully faithful

<https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/>