§ 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
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4. L-fct
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1. Galois rep

<u>Setting</u> G: arbitrary topo gp   e.g. G any Galois gp

If G profinite $\Rightarrow$ open subgps are finite index subgps.

Λ: top field   e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

<u>Def</u> (cont Galois rep) $(\rho, V) \in \operatorname{rep}_{\Lambda, \text{cont}}(G)$
$$V \in \operatorname{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{cont}$$

$\bigtriangledown$ $\rho(G)$ can be infinite!   for Gal gp

E.g. When char $F \neq l$, we have l-adic cyclotomic character
$$\varepsilon_l: \operatorname{Gal}(F^{\text{sep}}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \overline{\mathbb{Q}_l^\times} \qquad \sigma \longmapsto \varepsilon_l(\sigma) \quad \text{satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)
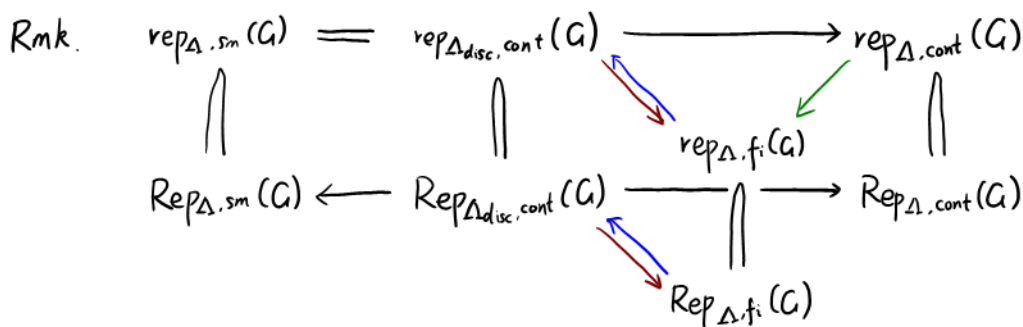
Ex: Compute ε_l for $F = \mathbb{F}_p$.

A: $\varepsilon_l: \hat{\mathbb{Z}} \cong \operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \qquad 1 \longmapsto p$

↑ lift from $\mathbb{Z} \to \mathbb{Z}_l^\times$

Notice the following two definitions don't depend on the topo of Λ.

<u>Def</u> (sm Galois rep) $(\rho, V) \in \operatorname{rep}_{\Lambda, \text{sm}}(G)$
$$V \in \operatorname{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{with open stabilizer.}$$

<u>Def</u> (fin image Galois rep) $(\rho, V) \in \operatorname{rep}_{\Lambda, \text{fi}}(G)$ fi: finite image/ finite index
$$V \in \operatorname{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{with finite image}$$

Rmk.

$\text{rep}_{\Delta,sm}(G) = \text{rep}_{\Delta_{disc},cont}(G) \longrightarrow \text{rep}_{\Delta,cont}(G)$

$\text{rep}_{\Delta,fi}(G)$

$\text{Rep}_{\Delta,sm}(G) \longleftarrow \text{Rep}_{\Delta_{disc},cont}(G) \longrightarrow \text{Rep}_{\Delta,cont}(G)$

$\text{Rep}_{\Delta,fi}(G)$

$\longrightarrow$: if fin index subgps are open

$\longrightarrow$: if G profinite gp   (Only need: open $\Rightarrow$ fin index)

$\longrightarrow$: Artin rep (of profinite gp)

Artin rep. $\Delta = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)

$\exists\, \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U}$ $\Rightarrow$ $H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\|\cdot\| = \|\cdot\|_{max}$, $|\cdot| = |\cdot|_{max}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists m \in \mathbb{N}$, s.t $A^m \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unipotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$.   Take $m \in \mathbb{N}$ s.t $|\lambda^m - 1| > \frac{1}{3}$.

$\frac{1}{3} |v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \text{rep}_{\mathbb{C},cont}(G)$, $\rho(G)$ is finite. G profinite

Proof. Take $\mathcal{U}$ in Lemma 1. then

$\rho^{-1}(\mathcal{U})$ is open $\underset{Lemma 1}{\Longrightarrow}$ $\exists I \leq G_F$ finite index . $\rho(I) \subseteq \mathcal{U}$

$\Longrightarrow \rho(I) = Id$

$\Longrightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more.

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp. so ρ is semisimple. Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same, every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeleton of the Galois gp" in general.

Setting: Λ: NA local field with char $k_\Lambda = l$

Q: What would happen if Λ is only a NA local field?

Finite field

Task. For Λ: NA local field with char $k_\Lambda = l$, understand $rep_{\Lambda, cont}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Lambda)$, TFAE:

⓪. $\hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is a well-defined cont gp homo
$$1 \longmapsto A$$

② $\exists g \in GL_n(\Lambda)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Lambda)$

③ $\det(\lambda I - A) \in \mathcal{O}_\Lambda[\lambda]$, with $\det A \in \mathcal{O}_\Lambda^\times$

A is called bounded in these cases.

Proof
$$① \underset{\text{local}}{\overset{\text{local}}{\rightleftarrows}} ② \underset{\text{local}}{\overset{}{\rightleftarrows}} ③$$

① $\Rightarrow$ ② : $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Lambda)$, which conjugates to $GL_n(\mathcal{O}_\Lambda)$

Another method:

Lemma 1. p, μ: two ways of expressions of gp action

$p: \hat{\mathbb{Z}} \longrightarrow GL_n(\mathbb{Z})$ is cont $\Rightarrow$ $\mu: \hat{\mathbb{Z}} \times \Lambda^n \longrightarrow \Lambda^n$ is cont

$$\left[\begin{array}{l} \Rightarrow \mu: \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{p \times Id_{\Lambda^n}} GL_n(\Lambda) \times \Lambda^n \longrightarrow \Lambda^n \quad \text{is cont.} \\ \Leftarrow ? \quad \text{Is that true?} \end{array} \right]$$

Lemma 2. $\mathcal{L}_1, \mathcal{L}_2$ lattice in $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$ lattice in Λ

$$\left[\begin{array}{l} \mathcal{L}_1 \supseteq (\mathfrak{p}^k)^{\oplus n} \\ \mathcal{L}_2 \supseteq (\mathfrak{p}^k)^{\oplus n} \end{array} \right\} \Rightarrow \# \mathcal{L}_1 + \mathcal{L}_2 / \mathcal{L}_1 < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice} \right]$$

Take $\mathcal{L} = \mathcal{O}_\Lambda^n \subseteq \Lambda^n$, then the stabilizer

$$\begin{aligned} Stab(\mathcal{L}) &= \{g \in \hat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L}\} \\ &= \{g \in \hat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \quad \forall i\} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where

$$\mu_{e_i}: \hat{\mathbb{Z}} \longrightarrow \Lambda^n \qquad g \longmapsto g \cdot e_i \qquad \text{(cont by Lemma 1)}$$

$\underset{\text{Lemma 2}}{\Longrightarrow}$ $\mathcal{L}$ has finite orbit

$\underset{\text{Lemma 2}}{\Longrightarrow}$ $\sum\limits_{\mathcal{L}_i \in \hat{\mathbb{Z}} \cdot \mathcal{L}} \mathcal{L}_i$ is a lattice stabilized by $\mathbb{Z}$.

After conjugation, $\qquad A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \qquad \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

② $\Rightarrow$ ① : w.l.o.g. $A \in GL_n(\mathcal{O}_\Delta)$. Then we get a lift

$$
\begin{array}{ccc}
\hat{\mathbb{Z}} & \xrightarrow{\ \exists! \ \text{cont}\ } & \widehat{GL_n(\mathcal{O}_\Delta)} \cong GL_n(\mathcal{O}_\Delta) \\
\uparrow & & \uparrow \\
\mathbb{Z} & \longrightarrow & GL_n(\mathcal{O}_\Delta)
\end{array}
$$

② $\Rightarrow$ ③ : *Obvious*

③ $\Rightarrow$ ② :

$$\sum\limits_{i \in \mathbb{Z}} A^i \mathcal{L} = \sum\limits_{i=0}^{n-1} A^i \mathcal{L} \quad \text{is a lattice fixed by } A, A^{-1} \ \text{(Lemma 2)}$$

After conjugation, $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \qquad \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

Q. $A, B \in GL_n(\Delta)$ bounded $\Rightarrow$ AB bounded ?