Preview: module of finite length. (f.l)

Generalization:
- f.d k-linear space → f.l module
- commutative ring → noncommutative

Structure of f.l. modules.

- Jordan – Hölder Thm: filtration $\overset{\text{reduced to}}{\rightsquigarrow}$ simple modules

Theorem 9.3 (Jordan-Hölder). *Assume that a module V has a composition series of length s. Then the following hold:*

(i) *Any filtration of V has length at most s and can be refined to a composition series;*

(ii) *All composition series of V have length s.*

(iii) uniqueness of simple modules (fil is not unique, 单模也不一定随意调位)

- Krull-Remak-Schmidt Thm: direct sum $\rightsquigarrow$ indecomposable modules

Corollary 11.5 (Krull-Remak-Schmidt Theorem). *Let $V_1, \dots, V_n$ be modules with local endomorphism rings, and let $W_1, \dots, W_m$ be indecomposable modules. If*

$$\bigoplus_{i=1}^{n} V_i \cong \bigoplus_{j=1}^{m} W_j$$

then $n = m$ and there exists a permutation π such that $V_i \cong W_{\pi(i)}$ for all $1 \le i \le n$.

So people need to do the research about
- simple modules — easy since $S \to S'$ is 0 or iso
- indec modules $\overset{\text{f.l}}{\rightleftharpoons}$ End (M) is local, but Mor (M, M')?
- filtration — homology alg, Ext, etc...

Often some Thm can be generalized to non f.l. modules, but they're usually techniqal. But it's still useful to have examples in different conditions.

mod over alg $\left\{ \begin{array}{l} \text{• Quiver} \\ \text{• } \mathbb{C}[G] \text{, where } G \text{ is a finite group} \\ \text{• matrix algebra, e.g } A = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ & \mathbb{Q} \end{pmatrix} \end{array} \right.$

$\rightsquigarrow$ Temperley-Lieb algebras

- $A = K[[X,Y]]/(XY)$, mod (A): Gelfand-Ponomarev modules

$\cup$

- string modules — indecomposable, $\mathrm{Mor}(M(v), M(w))$ computable in some sense.

Need to see from eg: structure of given module (e.g. $_AA$)

Details for each section.

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E.x. 计算合成列，见9.3.3

Cor. Invariant of M: $l(M)$, $[M:S]$

They're addictive for SES, thus.

① $l(V) = \sum_{i=1}^{t} l(U_i/U_{i-1})$. $\quad l(U_1 \oplus U_2) = l(U_1) + l(U_2)$

② $l(U_1) + l(U_2) = l(U_1 + U_2) + l(U_1 \cap U_2)$.

③ $f: V \to W$ $\begin{cases} l(V) = l(\mathrm{Ker}(f)) + l(\mathrm{Im}(f)) \\ l(W) = l(\mathrm{Im}(f)) + l(\mathrm{Cok}(f)) \end{cases}$

10. local, idem, invertible, nil

Def (local ring) TFAE. $(1 \neq 0)$

- If $r \in R$, then r or $1 - r$ is invertible. → sometime the key for theorems. e.g. when $\forall r$ is invert or nilpotent, then R is a local ring
- *R contains a unique maximal left ideal;*

Rem. local ring $\Rightarrow \begin{cases} \mathrm{idem}(R) = \{0,1\} \\ \text{极大左理想} = J(R),\ R/J(R) \text{ 为可除环} \end{cases}$

End (M) local $\underset{☆ f.l.}{\rightleftarrows}$ M indecomposable

$\Updownarrow$ Prop 2.5

☆: fitting lemma $\quad$ idem $(\mathrm{End}(M)) = \{0,1\}$

E.g.1 | 10.5.3. $A: k[T]$ $M = N(\infty)$
$\mathrm{End}(N(\infty)) = k[[T]]$ is local

E.g.2 | $A: k[T]$ $M = {}_A A$ is indecom
but $l({}_A A) = +\infty$
$\mathrm{End}({}_A A) \cong k[T]$ is not local

11. **Corollary 11.4** (Cancellation Theorem). *Let V, X_1, X_2 be modules with*

$$V \oplus X_1 \cong V \oplus X_2.$$

If $\mathrm{End}(V)$ *is a local ring, then*

$$X_1 \cong X_2.$$

Corollary 11.5 (Krull-Remak-Schmidt Theorem). *Let $V_1, \dots, V_n$ be modules with local endomorphism rings, and let $W_1, \dots, W_m$ be indecomposable modules. If*

$$\bigoplus_{i=1}^{n} V_i \cong \bigoplus_{j=1}^{m} W_j$$

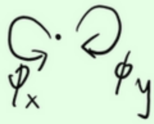
then $n = m$ and there exists a permutation π such that $V_i \cong W_{\pi(i)}$ for all $1 \le i \le n$.

For non f.l module, Cor 11.4 may fail, see 11.3.3

12. GP-module: $= \text{mod}(K[[X,Y]]/(XY))$

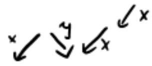
A 2-module (V, ϕ_x, ϕ_y) is called a **Gelfand-Ponomarev module** if the following hold:

- V is finite-dimensional;
- ϕ_x and ϕ_y are nilpotent;
- $\phi_x\phi_y = 0 = \phi_y\phi_x$.



$\phi_x \in End(V)$ is not iso

Ex. string module. Are just difficult linear algebra.



- indecomposable
- $Mor(M(v), M(w))$ have a canonical basis

13. Consider a chain of maps between indecom modules M_i, we get Harada-Sai lemma.
    - connect $Mor(M_i, M_{i+1})$ with $l(M_i)$
    - A "generalization" of fitting lemma
    - can be achieved by string module.