__Last time__ On CondAb, $\otimes$ is "not completed"

$\rightsquigarrow$ new category Solid with

$$\begin{tikzcd} \mathrm{Solid} \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathrm{CondAb} \arrow[l, bend right, "(-)^{\blacksquare}"'] \end{tikzcd}$$

- cpt proj objects $\mathbb{Z}[T]^{\blacksquare}$
- adjunction $\rightsquigarrow$ "completed $\otimes$"

Solid is the ~~full~~ smallest subcategory containing these cpt proj objects.

$$\mathrm{Solid} \overset{\mathrm{def}}{=} \{ A \in \mathrm{CondAb} \mid R\mathrm{Hom}(\mathbb{Z}[S]^{\blacksquare}, A) \simeq R\mathrm{Hom}(\mathbb{Z}[S], A) \}$$

__Today__: prove Thm 5.8.

__Thm__ 5.8. (non-derived part)

1. The full subcategory $\mathrm{Solid} \subset \mathrm{CondAb}$ is a full __abelian__ subcategory (ker, coker, $\oplus$) and is stable under $\varprojlim$, $\varinjlim$ & extensions.
2. $\{ \prod_I \mathbb{Z} \mid I \}$ is a family of cpt proj generators.
3. $\exists$ left adjoint $(-)^{\blacksquare}$ ~~preserving $\varinjlim$, and~~ sending $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^{\blacksquare}$.

Since we're not experienced in the categorical nonsense, let's try to work out some special cases.

Need: $R\mathrm{Hom}(\mathbb{Z}[S]^\square, \mathbb{Z}[A]^\square) = R\mathrm{Hom}(\mathbb{Z}[S], \mathbb{Z}[A]^\square)$

Case 0. $\mathbb{Z}[A]^\square \in \mathrm{Solid}$

Case 1. $\bigoplus_K \mathbb{Z}[A_k]^\square \in \mathrm{Solid}$

Case 2. $C$ is solid, where

$$C: \cdots \to \bigoplus_{K_2} \mathbb{Z}[A_k]^\square \to \cdots \bigoplus_{K_0} \mathbb{Z}[A_k]^\square \to 0$$

Case 3. $\mathrm{Ker}\, f \in \mathrm{Solid}$, where

$$f: \underset{\substack{\| \\ Y_1}}{\bigoplus_I \mathbb{Z}[A_i]^\square} \longrightarrow \underset{\substack{\| \\ Y_2}}{\bigoplus_J \mathbb{Z}[B_j]^\square}$$

Case 4. $\mathrm{coker}\, f \in \mathrm{Solid}$.

~~Case 5. coker $f \in$ Solid, for $f: Y \to$~~

Case 0:

$$
\begin{tikzcd}
\mathrm{RHom}(\mathbb{Z}[S]^{\square}, \mathbb{Z}[A]^{\square}) \arrow[d, equal, "\text{Cor 5.5}"'] & & \mathrm{RHom}(\mathbb{Z}[S], \mathbb{Z}[A]^{\square}) \arrow[d, equal, "\substack{\text{Thm 5.4} \\ \text{RHom comm with prod}}"] \\
\mathrm{RHom}(\prod_I \mathbb{Z}, \mathbb{Z}[A]^{\square}) \arrow[r, phantom, "\sim"] & \mathrm{RHom}(\prod_I \mathbb{R}/\mathbb{Z}[-1], \mathbb{Z}[A]^{\square}) \arrow[r, phantom, "\overset{\text{Thm 4.3}}{\cong}"] & \bigoplus_I \mathbb{Z}[A]^{\square}
\end{tikzcd}
$$

$$\mathrm{RHom}(\mathbb{R}, \mathbb{Z}[A]^{\square}) = \prod_J \mathrm{RHom}(\mathbb{R}, \mathbb{Z}) = 0$$

$$\Rightarrow \mathrm{RHom}(\prod_I \mathbb{R}, \mathbb{Z}[A]^{\square}) = \mathrm{RHom}_{\mathbb{R}}(\prod_I \mathbb{R}, \mathrm{RHom}(\mathbb{R}, \mathbb{Z}[A]^{\square})) = 0$$

Case 1:

$S = \{*\}$ ★

$$
\begin{tikzcd}
\mathrm{RHom}(\mathbb{Z}[S]^{\square}, \bigoplus_K \mathbb{Z}[A_k]^{\square}) \arrow[d, equal, "\text{Cor 5.5}"'] & & \mathrm{RHom}(\mathbb{Z}[S], \bigoplus_K \mathbb{Z}[A_k]^{\square}) \arrow[d, equal, "\text{pscoh}"] \\
\mathrm{RHom}(\prod_I \mathbb{Z}, \bigoplus_K \mathbb{Z}[A_k]^{\square}) \arrow[r, phantom, "\sim"] \arrow[d, dash, "S = \{*\}"] & \mathrm{RHom}(\prod_I \mathbb{R}/\mathbb{Z}[-1], \cancel{\mathbb{Z}[A} \bigoplus_K \mathbb{Z}[A_k]^{\square}) \arrow[d, equal, "\text{pscoh}"] & \bigoplus_K \mathrm{RHom}(\mathbb{Z}[S], \mathbb{Z}[A_k]^{\square}) \arrow[d, equal, "\text{Thm 5.4}"] \\
\bigoplus_K \mathrm{RHom}(\prod_I \mathbb{Z}, \mathbb{Z}[A_k]^{\square}) \arrow[r, phantom, "\sim"] & \bigoplus_K \mathrm{RHom}(\prod_I \mathbb{R}/\mathbb{Z}[1], \mathbb{Z}[A_k]^{\square}) \arrow[r, phantom, "\overset{\text{Thm 4.3}}{\cong}"] & \bigoplus_K \bigoplus_I \mathbb{Z}[A_k]^{\square}
\end{tikzcd}
$$

$$★ \Rightarrow \mathrm{RHom}(\mathbb{R}, \bigoplus_K \mathbb{Z}[A_k]^{\square}) = 0$$

$$\Rightarrow \mathrm{RHom}(\prod_I \mathbb{R}, \bigoplus_K \mathbb{Z}[A_k]^{\square}) = \mathrm{RHom}_{\mathbb{R}}(\prod_I \mathbb{R}, \mathrm{RHom}(\mathbb{R}, \bigoplus_K \mathbb{Z}[A_k]^{\square})) = 0$$

Case 2. bounded: five lemma

unbounded: right adj comm with limit, and

$$C = \varprojlim_{j} \tau^{\geq j} C.$$

Case 3. Reduce to Case 2.

$$\begin{tikzcd}
 & & & 0 \arrow[d] \\
\underbrace{\left( \to \bigoplus_{k_i} \mathbb{Z}[A_k] \to \cdots \bigoplus_{k_0} \mathbb{Z}[A_k] \right)}_{B} \arrow[rrr, "\text{quasi iso}"] \arrow[d] & & & \ker f \arrow[r] \arrow[d] & 0 \\
C \arrow[urrr, dashed, "\exists !"'] & & & Y_1 \arrow[d] \\
 & & & Y_2
\end{tikzcd}$$

Case 2 $\Rightarrow$ $\mathrm{Hom}(\mathbb{Z}[A], Y_i) \cong \mathrm{Hom}(\mathbb{Z}[A]^{\blacksquare}, Y_i)$

$\Rightarrow$ $\mathrm{Hom}(B, Y_i) \cong \mathrm{Hom}(C, Y_i)$

$\Rightarrow$ $\mathrm{Hom}(B, \ker f) \cong \mathrm{Hom}(C, \ker f)$

$\Rightarrow$ $B \simeq \ker f$ is a retract of $C$

$\Rightarrow$ $\ker f \in$ Solid

(The inverse is given by

$$\begin{tikzcd}
\mathrm{RHom}(\mathbb{Z}[S], \ker f) & \\
\mathrm{RHom}(\mathbb{Z}[S], \cancel{\ker f}\, B) \arrow[u, "\cong"] \arrow[r] & \mathrm{RHom}(\mathbb{Z}[S], C) \\
 & \mathrm{RHom}(\mathbb{Z}[S]^{\blacksquare}, C) \arrow[u, "\cong"] \arrow[r] & \mathrm{RHom}(\mathbb{Z}[S]^{\blacksquare}, \ker f)
\end{tikzcd}$$

Case 4. Five lemma.

**Prop.** Let $Q \in \mathrm{CondAb}$. Then

$Q \in \mathrm{Solid} \iff \exists$ SES

$$\bigoplus_I \mathbb{Z}[A_i]^\square \longrightarrow \bigoplus_j \mathbb{Z}[B_j]^\square \longrightarrow Q \longrightarrow 0$$

**Proof.** $\Leftarrow$: Case 4.

$\Rightarrow$:

$$\begin{tikzcd}
\bigoplus_j \mathbb{Z}[B_j] \arrow[r, two heads] \arrow[d] & Q \\
\bigoplus_j \mathbb{Z}[B_j]^\square \arrow[ur, dashed, two heads, "\exists!"']
\end{tikzcd}$$

auto surj.

then repeat.

**Cor** Denote

$$\mathrm{Solid}' = \{A \in \mathrm{CondAb} \mid \mathrm{Hom}(\mathbb{Z}[S]^\square, A) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}[S], A)\}$$

then $\mathrm{Solid} = \mathrm{Solid}'$.

**Proof.**

$$\begin{tikzcd}
Q \in \mathrm{Solid} \arrow[dr, Rightarrow] & & \exists \text{SES} \arrow[ll, Rightarrow, "\text{Case 4}"'] \\
& Q \in \mathrm{Solid}' \arrow[ur, Rightarrow] &
\end{tikzcd}$$

**Proof of Thm 5.8.**

$\mathrm{Hom}(\mathbb{Z}[S], -)$ commute with $\varprojlim$ $\overset{\text{Cor}}{\Longrightarrow}$ ✓ for ker, $\varprojlim$.

Prop $\Rightarrow$ ✓ for ~~coker~~ $\oplus$

$$\begin{tikzcd}
\bigoplus_I \mathbb{Z}[A_i]^\square \arrow[r] \arrow[d] & \bigoplus_J \mathbb{Z}[B_j]^\square \arrow[r] \arrow[d, equal] & \mathrm{coker} f \arrow[r] \arrow[d, equal] & 0 \\
Y_1 \arrow[r] \arrow[d] & \bigoplus_J \mathbb{Z}[B_j]^\square \arrow[r] \arrow[d] & \mathrm{coker} f \arrow[r] \arrow[d, equal] & 0 \\
Y \arrow[r, "f"] & Z \arrow[r] & \mathrm{coker} f \arrow[r] & 0
\end{tikzcd}$$

~~coker~~ five lemma $\Rightarrow$ ✓ for coker, extension

$\oplus$ + coker $\Rightarrow$ ✓ for $\varinjlim$

Pseudocoherence (in Appendix IV)

Def. $X \in \mathcal{A}$ is pseudocoherent if (for filtered colimit)

$$R\mathrm{Hom}(X, \varinjlim_i Y_i) = \varinjlim_i R\mathrm{Hom}(X, Y_i)$$

E.g. $\mathbb{Z}[S]$ is pscoh (cpt proj objects, p12) ← Eds

$\xrightarrow[\text{hypercover}]{\text{simplicial}}$ $\mathbb{Z}[X]$ is pscoh $\forall X$ cpt Haus

$\xrightarrow[\text{Eilenberg-Maclane resolution}]{\text{Thm 4.5}}$ $\underline{X}$ is pscoh $\forall X$ cpt Haus

e.p. $\prod \mathbb{R}/\mathbb{Z}$ is pscoh Even though infinite $\oplus$ is not a filtered colimit,

Rmk. ~~If $X \in \mathcal{A}$ is~~ by def, ~~itself~~

$$\bigoplus_I Y_i = \varinjlim_{\substack{I' \subset I \\ \text{finite}}} \bigoplus_{i \in I'} Y_i,$$ and $\{I' \subset I \mid I' \text{ finite}\}$ is filtered.

"~~SE~~ Compactness"

$S \in \mathrm{CondSet}$ is cpt : $\mathrm{Hom}_{\mathrm{CondSet}}(S, \varinjlim_i Y_i) = \varinjlim_i \mathrm{Hom}_{\mathrm{CondSet}}(S, Y_i)$

$\forall$ fct on $X$ is bounded: $\mathrm{Hom}_{\mathrm{Top}}(X, \mathbb{R}) = \varinjlim_i \mathrm{Hom}_{\mathrm{Top}}(X, [-n, n])$

Prop [SE668905] Let $X$ be a metric space, then

$X$ is cpt $\Leftrightarrow$ $\forall$ fct on $X$ is bounded (real value)

$\Leftrightarrow$ $\mathrm{Hom}_{\mathrm{Top}}(X, \varinjlim_i Y_i) = \varinjlim_i \mathrm{Hom}_{\mathrm{Top}}(X, Y_i)$

commute with colimits.

Thm 5.8 (derived part) $\mathcal{D}(\mathcal{A}_F)$ $\mathcal{D}(\mathcal{A})$

1. The inclusion $\mathcal{D}(\mathrm{Solid}) \subset \mathcal{D}(\mathrm{CondAb})$ is fully faithful.

2. its (essential) images are solid objects of $\mathcal{D}(\mathrm{CondAb})$, and

$C \in \mathcal{D}(\mathrm{CondAb})$ is solid

$$\Updownarrow$$

$H^i(C) \in \mathrm{CondAb}$ is solid

$\mathcal{D}(\mathcal{A}_F)$ piece

$\|$

$\mathcal{D}_F(\mathcal{A})$

$\|$

$\mathcal{D}_F'(\mathcal{A})$ cohom

3. $\exists$ left adjoint [0794]

$(-)^{L\square} = L((-))^{\square}$

$$\begin{tikzcd} \mathcal{D}(\mathrm{Solid}) \arrow[r, bend right, "\subseteq"'] \arrow[r, phantom, "\perp"] & \mathcal{D}(\mathrm{CondAb}) \arrow[l, bend right] \end{tikzcd}$$

For breviation, we write

$\mathcal{A} = \mathrm{CondAb}$, $\mathcal{A}_F = \mathrm{Solid}$,

$\mathcal{A}_0 = \{\mathbb{Z}[S] \mid S \in \mathrm{Prof}\}$

$F : \mathcal{A}_0 \longrightarrow \mathcal{A}_F \subset \mathcal{A}$

$\mathbb{Z}[S] \longmapsto \mathbb{Z}[S]^{\square}$

Proof. For fully faithfulness, need

$$R\mathrm{Hom}_{D(A_F)}(F(X), C) \longrightarrow R\mathrm{Hom}_{D(A)}(F(X), C)$$
$$\| $$
$$R\mathrm{Hom}_{D(A)}(X, C)$$

is an iso.

i.e. $R\mathrm{Hom}(\mathbb{Z}[S]^{\square}, C) \longrightarrow R\mathrm{Hom}(\mathbb{Z}[S], C)$

is iso, so $D(A_F) \subseteq D_F(A)$

$C$ unbounded $\rightsquigarrow$ bounded $\rightsquigarrow$ concentrated in one degree.

upper bound: ✓

lower bound: $\varprojlim_{j} \tau^{\geq j} C$ (Postnikov limit)

$C = Y[0]$, need:

$$R\mathrm{Hom}_{\text{Solid}}(\mathbb{Z}[S]^{\square}, Y) \overset{?}{=} R\mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[S], Y)$$
$$\| \qquad\qquad\qquad\qquad \|$$
$$\mathrm{Hom}_{\text{Solid}}(\mathbb{Z}[S]^{\square}, Y) = \mathrm{Hom}_{\mathrm{CondAb}}(\mathbb{Z}[S], Y)$$

$D_F'(A) \subseteq D_F(A)$: unbounded $\rightsquigarrow$ bounded $\rightsquigarrow$ in one degree.

$D(A_F) \subseteq D_F'(A)$: Solidness is preserved under ker & coker

$D_F(A) \subseteq D(A_F)$: $\{\mathbb{Z}[S]^{\square} \in D_F(A)\}$ are proj generators.