Eine Woche, ein Beispiel

1.14. reminder on Morse theory

1. Calculations of index

E.g.
$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \xrightarrow{\; g = \|\cdot\|^2 \;} \mathbb{R}$$
$$(u_1, u_2) \longmapsto (u_1, u_2, 1)$$
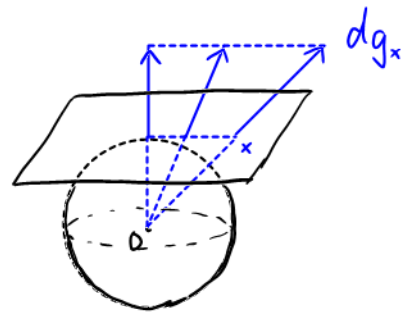$$x = (x_1, x_2, x_3) \longmapsto \langle x, x \rangle = x_1^2 + x_2^2 + x_3^2$$

$$x = (u_1, u_2, 1)$$
$$\frac{\partial x}{\partial u_1} = (1, 0, 0)$$
$$\frac{\partial x}{\partial u_2} = (0, 1, 0)$$
$$\frac{\partial^2 x}{\partial u_i \partial u_j} = (0, 0, 0) \qquad \forall i, j$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$g(x) = \langle x, x \rangle$$
$$dg_x = 2x \, dx = \sum_i 2 x_i \, dx_i$$
$$dg_x(\vec{v}) = 2 \langle x, \vec{v} \rangle \qquad \vec{v} \in T_x \mathbb{R}^3$$

$$f(u) = \langle u, u \rangle + 1$$
$$df_u = 2u \, du = \sum_i 2 u_i \, du_i$$
$$df_u(\vec{v}) = 2 \langle u, \vec{v} \rangle \qquad \vec{v} \in T_u \mathbb{R}^2$$

$$f(u_1, u_2) = u_1^2 + u_2^2 + 1$$
$$\frac{\partial f}{\partial u_1} = 2u_1$$
$$\frac{\partial f}{\partial u_2} = 2u_2$$
$$\left(\frac{\partial^2 f}{\partial u_i \partial u_j} \right)_{i,j} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\frac{\partial f}{\partial u_i} = df_u(\vec{e_i}) = 2 \langle u, \vec{e_i} \rangle = 2 u_i$$
$$\frac{\partial f}{\partial u_i} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u_i} = \sum_j \frac{\partial g}{\partial x_j} \cdot \frac{\partial x_j}{\partial u_i}$$
$$= \sum_j 2 x_j \cdot \delta_{ij}$$
$$= 2 u_i$$

have one critical pt $(0, 0, 1) \in \mathbb{R}^3$, with Morse index 0.

(attach one 0-cell.)

E.g.

$$f: \mathbb{R}^2 \xrightarrow{\;x\;} \mathbb{R}^3 \xrightarrow{\;g=\|\cdot\|^2\;} \mathbb{R}$$

$$(u_1, u_2) \longmapsto (u_1, u_2, u_1^2 + u_2^2 + t)$$

$$x = (x_1, x_2, x_3) \longmapsto \langle x, x \rangle = x_1^2 + x_2^2 + x_3^2$$

$$x = (u_1, u_2, u_1^2 + u_2^2 + t)$$

$$\frac{\partial x}{\partial u_1} = (1, 0, 2u_1)$$

$$\frac{\partial x}{\partial u_2} = (0, 1, 2u_2)$$

$$g_{ij} = \begin{pmatrix} 1+4u_1^2 & 4u_1 u_2 \\ 4u_1 u_2 & 1+4u_2^2 \end{pmatrix}$$

$$\frac{\partial^2 x}{\partial u_i \partial u_j} = \begin{pmatrix} (0,0,2), & (0,0,0) \\ (0,0,0), & (0,0,2) \end{pmatrix}$$

function →
vector → $x \cdot \dfrac{\partial^2 x}{\partial u_i \partial u_j} = \begin{pmatrix} 2x_3 & 0 \\ 0 & 0 \end{pmatrix}$

t = 1 case

$$g(x) = \langle x, x \rangle$$

$$dg_x = 2x\, dx = \sum_i 2x_i\, dx_i$$

$$dg_x(\vec{v}) = 2\langle x, \vec{v} \rangle$$

$$\vec{v} \in T_x \mathbb{R}^3$$

$$f(u) = \langle u, u \rangle + (u_1^2 + u_2^2 + t)^2$$

$$df_u = 2u\, du + 2(u_1^2 + u_2^2 + t)(2u_1 du_1 + 2u_2 du_2)$$

$$= (2u_1^2 + 2u_2^2 + t + 1)(2u\, du)$$

$$df_u(\vec{v}) = (2u_1^2 + 2u_2^2 + t + 1)\, 2\langle u, \vec{v} \rangle$$

$$\vec{v} \in T_u \mathbb{R}^2$$

$$f(u_1, u_2) = u_1^2 + u_2^2 + (u_1^2 + u_2^2 + t)^2$$

$$\frac{\partial f}{\partial u_1} = 2u_1 + 2(u_1^2 + u_2^2 + t)\cdot 2u_1$$

$$\frac{\partial f}{\partial u_2} = 2u_2 + 2(u_1^2 + u_2^2 + t)\cdot 2u_2$$

$= df_u(\vec{e}_1)$
$= df_u(\vec{e}_2)$

$$\left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right)_{i,j} = \begin{pmatrix} 2 + 12u_1^2 + 4(u_2^2 + t) & 8u_1 u_2 \\ 8u_1 u_2 & 2 + 12u_2^2 + 4(u_1^2 + t) \end{pmatrix}$$
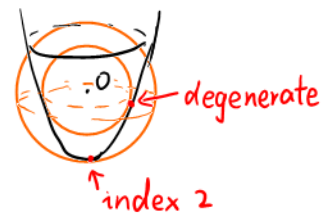
$\underset{u_1 = u_2 = 0}{=\!=\!=} \begin{pmatrix} 2+4t & 0 \\ 0 & 2+4t \end{pmatrix}$

At the critical pt $(0,0,t) \in \mathbb{R}^3$, one has

$$\begin{cases} \text{index } 0 & t > -\frac{1}{2} \\ \text{degenerated critical pt} & t = -\frac{1}{2} \\ \text{index } 2 & t < -\frac{1}{2} \end{cases}$$

degenerate
index 2

$t < -\frac{1}{2}$ case

E.g. Fix $a \in \mathbb{R}^r$.

For

$$f: \mathbb{R}^k \xrightarrow{\ \ x\ \ } \mathbb{R}^r \xrightarrow{\ g = \|\cdot - a\|^2\ } \mathbb{R}$$

$$u \longmapsto x(u)$$

$$x \longmapsto \langle x - a, x - a \rangle$$

one gets

$$\frac{\partial f}{\partial u_i} = \left\langle \frac{\partial}{\partial u_i}(x-a), x-a \right\rangle + \left\langle x-a, \frac{\partial}{\partial u_i}(x-a) \right\rangle$$

$$= 2 \left\langle x-a, \frac{\partial x}{\partial u_i} \right\rangle$$

coordinate calculation:

$$\frac{\partial f}{\partial u_i} = \sum_j \frac{\partial g}{\partial x_j} \frac{\partial x_j}{\partial u_i}$$

$$= \sum_j 2(x_j - a_j) \frac{\partial x_j}{\partial u_i}$$

$$= 2 \left\langle x-a, \frac{\partial x}{\partial u_i} \right\rangle$$

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = 2 \left\langle \frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial u_i} \right\rangle + 2 \left\langle x-a, \frac{\partial x}{\partial u_i \partial u_j} \right\rangle$$

$$= 2 \left(g_{ij} - (a-x) \cdot \vec{l}_{ij} \right)$$

$$\uparrow$$

$$\Phi(u_i, u_j)$$

Q. What happens in the cplx case?