Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve
https://mathoverflow.net/questions/402049/induced-action-on-prym-variety
.

Goal: Describe some curves (maybe singular) C in A, and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$ has the following covers:

genus

$\mathrm{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

ramification degree

$$\begin{tikzcd}
 & C_9 \arrow[dl, "20"'] \arrow[d, "4"', "\tilde{f}"] \arrow[dr, "0"', "f"] & \\
\mathbb{P}^1 \arrow[dr, "2"'] & C_4 \arrow[d, "10"'] & C_5 \arrow[dl, "12"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

where

$$C_4 = \{y^2 = \prod_{j=1}^{10} (t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t \prod_{j=1}^{10} (t - j)\}$$

$t = x^2$

$\tilde{y} = xy$

The crspd field extension:

$$\begin{tikzcd}[column sep=small]
 & \mathbb{C}(x)[y]/(y^2 - \prod_{j=1}^{10}(x^2 - j)) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{C}(x) \arrow[dr, dash] & \mathbb{C}(t)[y]/(y^2 - \prod_{j=1}^{10}(t - j)) \arrow[d, dash] & \mathbb{C}(t)[\tilde{y}]/(\tilde{y}^2 - t\prod_{j=1}^{10}(t - j)) \arrow[dl, dash] \\
 & \mathbb{C}(t) &
\end{tikzcd}$$

Global differential forms

Pulling back differential forms give the following maps:

$$
\begin{tikzcd}[column sep=small]
& \langle x^k \frac{dx}{y} \mid k=0,\dots,8 \rangle & \\
& \langle 2x^{2k+1} \frac{dx}{y} \mid k=0,\dots,3 \rangle \arrow[u, "\tilde{f}^*"] \arrow[d, equal] & \langle 2x^{2k} \frac{dx}{y} \mid k=0,\dots,4 \rangle \arrow[ul, "f^*"'] \arrow[d, equal] \\
0 \arrow[uur] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3 \rangle & \langle 2t^k \frac{dt}{\tilde{y}} \mid k=0,\dots,4 \rangle \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

Therefore,

$$
H^0(C_9;\omega_{C_9}) \cong \tilde{f}^* H^0(C_4;\omega_{C_4}) \oplus f^* H^0(C_5;\omega_{C_5}) \qquad (1)
$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which crspds to pulling back of divisors)

$$
\begin{tikzcd}[column sep=small]
& \langle x^k \frac{dx}{y} \mid k=0,\dots,8 \rangle \arrow[ddl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\
& \langle 2x^{2k+1} \frac{dx}{y} \mid k=0,\dots,3 \rangle \arrow[d, equal] & \langle 2x^{2k} \frac{dx}{y} \mid k=0,\dots,4 \rangle \arrow[d, equal] \\
0 \arrow[dr] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3 \rangle \arrow[d] & \langle 2t^k \frac{dt}{\tilde{y}} \mid k=0,\dots,4 \rangle \arrow[dl] \\
& 0 &
\end{tikzcd}
$$

However, since $\mathrm{Jac}(C) = H^0(C;\omega_C)^* / H_1(C;\mathbb{Z})$,
we are working on the dual spaces. The notations are again switched:

$$
\begin{aligned}
f^* &\rightsquigarrow Nm_f \\
f_* &\rightsquigarrow f^*
\end{aligned}
$$

One may get

different meaning compared with (1)!

$$
H^0(C_9;\omega_{C_9})^* \cong \tilde{f}^* H^0(C_4;\omega_{C_4})^* \oplus f^* H^0(C_5;\omega_{C_5})^* \qquad (2)
$$

(co)homology class

This page may be easier to understand, and it helps to understand the previous page.

$$\begin{tikzcd}
& H^1(C_9;\mathbb{Z}) & \\
0 \arrow[ur] & H^1(C_4;\mathbb{Z}) \arrow[u, hook, "\tilde{f}^*"'] & H^1(C_5;\mathbb{Z}) \arrow[ul, hook', "f^*"'] \\
& 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Q: Do we have

$$H^1(C_9;\mathbb{Z}) \cong \tilde{f}^* H^1(C_4;\mathbb{Z}) \oplus f^* H^1(C_5;\mathbb{Z}) \ ?$$

$$\begin{tikzcd}
& H_1(C_9;\mathbb{Z}) \arrow[dl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\
0 \arrow[dr] & H_1(C_4;\mathbb{Z}) \arrow[d] & H_1(C_5;\mathbb{Z}) \arrow[dl] \\
& 0 &
\end{tikzcd}$$

Q: Do we have

$$H_1(C_9;\mathbb{Z})^* \cong \tilde{f}^* H_1(C_4;\mathbb{Z})^* \oplus f^* H_1(C_5;\mathbb{Z})^* \ ?$$

Curve in Prym variety

Define A as the quotient of Jacobians, i.e.,

$$A := Jac(C_9)/f^* Jac(C_5) \cong Prym(C_9/C_5)$$

$$\begin{tikzcd} & C_9 \arrow[r] \arrow[d, "AJ_{C_9}"] & C_4 \arrow[d] \arrow[r] & Jac(C_4) \arrow[dl, dashed, "\exists! \text{ isogeny}"] \\ Jac(C_5) \arrow[r, "f^*"] & Jac(C_9) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \quad (3)$$

Prop
0. A is isogenous to $Jac(C_4)$;
1. $f^*: Jac(C_5) \longrightarrow Jac(C_9)$ is generically injective;
2. $\pi \circ AJ_{C_9}$ is not injective, it factors through C_4;
3. $C_4 \longrightarrow A$ is generically injective;
4. $C_4 \longrightarrow A$ produces a sm image of A, outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof. 0. Taking the tangent space of (3), one gets

$$0 \longrightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df^*_0} H^0(C_9, \omega_{C_9})^* \longrightarrow T_0 A \longrightarrow 0$$

Combined with (2),

$$T_0 A \cong H^0(C_4; \omega_{C_4})^*.$$

Late we will find a natural isogeny $Jac(C_4) \longrightarrow A$.
What's the degree of this isogeny?

1. Since

$$Nm_f \circ f^* = 2\, Id_{Jac(C_5)}$$

f^* is gen inj. Is f^* inj in this case?

2. For $p_1 = (x_0, y_0)$, $p_2 = (-x_0, y_0)$, we want to show that

$$\int_{\gamma_1 : p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\gamma_2 : p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$LHS = \int_{\tilde{\gamma}_1 : p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = RHS.$$

3.

https://mathoverflow.net/questions/68503/has-anyone-studied-the-prym-map-for-double-covers-with-two-ramification-points
https://arxiv.org/abs/1010.4483: It proves that many Prym maps (C->Prym) are generically finite.

Notice: $C_4 \subset Jac(C_4)$ is only invariant under $p \mapsto -p$,
not invariant under $p \mapsto p + a_0$.

Otherwise, the Gauss map would be cover of deg > 2.
Therefore, after isogeny $C_4 \longrightarrow A$ is still gen inj.
Q: Is this map really inj?

(4) $C_4 \hookrightarrow Jac(C_4)$ is sm, so after isogeny it is still sm outside of non-injective locus.

Rmk. Suppose $f: \widetilde{C} \longrightarrow C$ is a deg 2 (ramified) covering,
$\sigma: \widetilde{C} \longrightarrow \widetilde{C}$ the crspd involution,
define A as the quotient

$$\begin{tikzcd} & \widetilde{C} \arrow[d, "AJ_{\widetilde{C}}"] & & \\ Jac(C) \arrow[r, "f^*"] & Jac(\widetilde{C}) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

one can identify A with $Prym(\widetilde{C}/C) \subset Jac(\widetilde{C})$, why?
and the Abel-Prym map is given by

$$AP_{\widetilde{C}} = \pi \circ AJ_{\widetilde{C}}: \widetilde{C} \longrightarrow Jac(\widetilde{C}) \longrightarrow A$$
$$p \longmapsto \mathcal{O}_{\widetilde{C}}(p - p_0) \longmapsto \mathcal{O}_{\widetilde{C}}(p - \sigma(p))$$

Therefore, for $p_1 \neq p_2$,

$$\begin{aligned} & AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2) \\ \Leftrightarrow\ & \mathcal{O}_{\widetilde{C}}(p_1 - \sigma(p_1)) = \mathcal{O}_{\widetilde{C}}(p_2 - \sigma(p_2)) \\ \Leftrightarrow\ & \mathcal{O}_{\widetilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\widetilde{C}}(p_2 + \sigma(p_1)) \end{aligned}$$

① When $p_1, p_2 \in \widetilde{C}$ are ramification pts of f,
i.e., $p_1 = \sigma(p_1)$, $p_2 = \sigma(p_2)$

$$AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2).$$

As a result, when f is ramified, $AP_{\widetilde{C}}$ is never injective.

② Now assume $AP_{\widetilde{C}}$ is not inj, $AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2)$.
When $p_1 \neq \sigma(p_1)$ or $p_2 \neq \sigma(p_2)$,

$$\mathcal{L} := \mathcal{O}_{\widetilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\widetilde{C}}(p_2 + \sigma(p_1))$$

is a l.b. with deg 2 & $rk \geq 1$, so $\widetilde{C}$ must be hyperelliptic,
and $\mathcal{L}$ induces a degree 2 map

$$\widetilde{C} \longrightarrow \mathbb{P}^1.$$

③ In the example,

$$\widetilde{C} = C_9, \quad C = C_5,$$

$\sigma: \begin{array}{l} x \longmapsto -x \\ y \longmapsto -y \end{array}$ Prym involution

$\tau: \begin{array}{l} x \longmapsto -x \\ y \longmapsto y \end{array}$ C_4 involution

$\sigma\tau: \begin{array}{l} x \longmapsto x \\ y \longmapsto -y \end{array}$ hyperelliptic involution

we can show directly that

$$AP_{C_9}(\tau(p)) = AP_{C_9}(p).$$

This gives a second proof for Prop (2).

Reason:

$$\mathcal{O}_{C_9}(\tau(p) + \sigma(p)) = \mathcal{O}_{C_9}(\tau(p + \sigma\tau(p))) = \mathcal{O}_{C_9}(p + \sigma\tau(p))$$

$\sigma\tau = \tau\sigma$

$\sigma\tau$ is an hyperelliptic involution

Gauss map

Taking the Gauss map of (3), one gets

$$\begin{tikzcd}
C_9 \arrow[r, "2:1"', "4"] \arrow[d, "2:1"', "20"] & C_4 \arrow[d, "2:1", "10"'] \\
R_1 \arrow[r, "2:1", "\text{ram at } 0,\infty"'] \arrow[d] & R_2 \arrow[d] \\
\mathbb{P}^8 \arrow[r, dashed] & \mathbb{P}^3
\end{tikzcd}$$

$$[a_0, \cdots, a_8] \longmapsto [a_1 : a_3 : a_5 : a_7]$$

$\Rightarrow \deg_A C_4 = 6, \quad Gal(\gamma) = S_6 = W(C_3).$