Session 7 & Ex 5.

Today we focus on the convergence of power series.

The following three properties are quite useful.

$\{a_i\}$ sequence of cplx number

Prop. $\sum_{i=0}^{\infty} a_i \; \exists \Rightarrow \lim_{i\to\infty} a_i = 0$

$\exists : \in (-\infty, +\infty)$

$\sum_{i=0}^{\infty} a_i = +\infty \Rightarrow \sum_{i=0}^{\infty} a_i \nexists$

Prop. (Cauchy's convergence test)

$$\sum_{i=0}^{\infty} a_i \; \exists \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}_{>0} \text{ s.t } \forall m > n > N,$$

$$\left| \sum_{i=n}^{m} a_i \right| < \varepsilon$$

Prop (sum of geometric sequence)

$$1 + r + r^2 + \cdots + r^k = \frac{1 - r^{k+1}}{1 - r}$$

Task 2. (ii) Show that, for $x_0 \in \mathbb{C}$ s.t $|x_0| > 1$,

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i} x_0^i \nexists$$

Proof. Suppose $\sum_{i=1}^{\infty} \frac{(-1)^i}{i} x_0^i \; \exists \Rightarrow \lim_{i\to+\infty} \frac{(-1)^i}{i} x_0^i = 0$

$$\Rightarrow \lim_{i\to+\infty} \frac{1}{i} |x_0|^i = 0$$

but $\lim_{i\to+\infty} \frac{1}{i} |x_0|^i = +\infty$ contradiction!

(i) Let $r<1$ Show that. for $x_0 \in B_0(r) \subseteq \mathbb{C}$.

$$\sum_{i=1}^{\infty} \frac{(-1)^i}{i} x_0^i \ \exists.$$

Proof.
$$\left|\sum_{i=n}^{m} \frac{(-1)^i}{i} x_0^i\right| \leq \sum_{i=n}^{m} \frac{1}{i}|x_0|^i$$
$$\leq \sum_{i=n}^{m} r^i$$
$$= r^n \sum_{i=0}^{m-n} r^i$$
$$= r^n \frac{1-r^{m-n+1}}{1-r}$$
$$\leq r^n \frac{1}{1-r} \xrightarrow{n \to +\infty} 0.$$

Actually we can generalize this process a lot.

$\overline{\lim_{i}}$: upper limit, superior limit

Thm (Cauchy - Hadamard theorem)

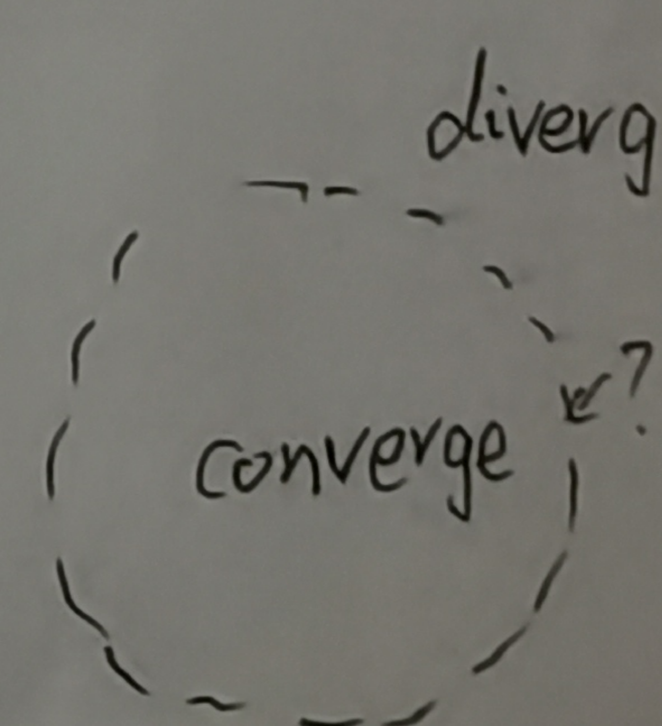
For $a_i \in \mathbb{C}$, let $R = \dfrac{1}{\overline{\lim\limits_{i}} |a_i|^{\frac{1}{i}}} \in [0,+\infty]$

1). for $z \in \mathbb{C}$, $|z|>R$, the series $\sum\limits_{i=0}^{+\infty} a_i z^i \ \nexists$

2). ~~Insid~~ The series $\sum\limits_{i=0}^{+\infty} a_i z^i$ converges absolutely inside $B_0(R)$,

and converges uniformly on every cpt subset of $B_0(R)$.

R is called the radius of convergence.



Ex compute R for $\frac{1}{1-z} = \sum_{i=0}^{+\infty} z^i$

Suppose $a_i \neq 0 \ \forall i$

Lemma 1. If $\underset{i}{\underrightarrow{\lim}} \left|\frac{a_i}{a_{i+1}}\right| = \rho \in [0,+\infty]$, then

$$\frac{1}{\underset{i}{\underrightarrow{\lim}} |a_i|^{\frac{1}{i}}} = \rho \in [0,+\infty]$$

Hint. $\underset{i}{\underrightarrow{\lim}} |a_i|^{-\frac{1}{i}} = e^{-\underset{i}{\underrightarrow{\lim}} \frac{\ln|a_i|}{i}} = e^{-\underset{i}{\underrightarrow{\lim}} (\ln|a_{i+1}| - \ln|a_i|)}$

$$= e^{\underset{i}{\underrightarrow{\lim}} \ln\left(\left|\frac{a_i}{a_{i+1}}\right|\right)} = \underset{i}{\underrightarrow{\lim}} \left|\frac{a_i}{a_{i+1}}\right| = \rho$$

Lemma 2. (Stirling's formula)

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right)$$

$$\Rightarrow \lim_{n\to+\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1$$

Task 1. Determine the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^3}{(3n)!} z^n \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)!} z^{2n-1}$$

Hint. $$\lim_{n\to} \left(\frac{\frac{(n!)^3}{(3n)!}}{\frac{((n+1)!)^3}{(3(n+1))!}} \right) = \lim_{n\to} \frac{(3n+3)(3n+2)(3n+1)}{(n+1)^3} = 27$$

$\Rightarrow R = 27$

$$\lim_{n\to} \left(\frac{\frac{1}{(2n-1)!}}{\frac{1}{(2n+1)!}} \right) = \lim_{n\to} (2n+1)(2n) = +\infty$$

$$\lim_{n\to} \left(\frac{1}{(2n-1)!} \right)^{-\frac{1}{2n-1}} = \left(\lim_{n\to} \left(\frac{\frac{1}{(2n-1)!}}{\frac{1}{(2n+1)!}} \right) \right)^{\frac{1}{2}} = +\infty$$

$\Rightarrow R = +\infty$

Or: Stirling's formula.

For $|z| = R$, we have no general statement on the convergence of the series.

However, we can still compute some cases.

Task 3. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$

(1) $\forall z_0 \in S^1$, $\sum_{k=0}^{\infty} k z_0^k \nexists$

$$\lim_{k \to +\infty} k z_0^k \nexists \Rightarrow \sum_{k=0}^{\infty} k z_0^k \nexists$$

(2) $\forall z_0 \in S^1$, $\sum_{k=1}^{\infty} \frac{1}{k^2} z_0^k \exists$

$$\begin{aligned}
\left| \sum_{k=n}^{m} \frac{1}{k^2} z_0^k \right| &\leq \sum_{k=n}^{m} \frac{1}{k^2} \\
&\leq \sum_{k=n}^{m} \frac{1}{k(k-1)} \\
&= \sum_{k=n}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) \\
&= \frac{1}{n-1} - \frac{1}{m} \\
&= \frac{1}{n-1} \xrightarrow{n \to +\infty} 0
\end{aligned}$$

(3) $\forall z_0 \in S^1 - \{1\}$, $\sum_{k=1}^{\infty} \frac{1}{k} z_0^k$ $\exists$

$$\left| \sum_{k=n}^{m} \frac{1}{k} z_0^k \right| = \left| \sum_{k=n}^{m} \frac{1}{k} \frac{z_0^{k+1} - z_0^k}{z_0 - 1} \right|$$

$$= \frac{1}{|z_0 - 1|} \left| \sum_{k=n+1}^{m+1} \frac{1}{k-1} z_0^k - \sum_{k=n}^{m} \frac{1}{k} z_0^k \right|$$

$$= \frac{1}{|z_0 - 1|} \left| \left(\sum_{k=n}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) z_0^k \right) + \frac{1}{m} z_0^{m+1} - \frac{1}{n} z_0^{n+1} \right|$$

$$\leq \frac{1}{|z_0 - 1|} \left[\sum_{k=n}^{m} \left(\frac{1}{k-1} - \frac{1}{k} \right) + \frac{1}{m} + \frac{1}{n} \right]$$

$$= \frac{1}{|z_0 - 1|} \left[\frac{1}{n-1} - \frac{1}{m} + \frac{1}{m} + \frac{1}{n} \right]$$

$$= \frac{1}{|z_0 - 1|} \left[\frac{1}{n-1} + \frac{1}{n} \right] \xrightarrow{n \to +\infty} 0.$$