Eine Woche, ein Beispiel

8.21 equivariant cohomology of $\mathbb{P}^1$

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
Other references will be add soon.

1. notations and warnings
2. result
3. computation of completion in practice
4. pt & $\mathbb{P}^1$
5 Euler class

1. notations and warnings

In this document,

$GL_2 = GL_2(\mathbb{C})$ $\quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset GL_2$ $\quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2$ or SL_2

$SL_2 = SL_2(\mathbb{C})$ $\quad \mathbb{C}^\times = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset SL_2$ $\quad \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$

$K_0^G(X) := K_0(Coh^G(X))$

$R(G) := K_0^G(pt) = Rep(G)$

$K_0^G(X)^\wedge_I := \varprojlim_n K_0^G(X)/I^n$

$H_G^*(X;\mathbb{Q}) := H^*(EG \times^G X;\mathbb{Q})$

$S(G) := H_G^*(pt;\mathbb{Q}) = H^*(BG;\mathbb{Q})$

$HP_G^0(X;\mathbb{Q}) := \prod_{n=0}^{+\infty} H_G^n(X;\mathbb{Q}) = H_G^*(X;\mathbb{Q})^\wedge_I$.

To avoid confusion, we don't consider any convolution structure in this document.
we don't consider $G \times \mathbb{C}^\times$-action either
($\mathbb{C}^\times$ is already occupied as a maximal torus of SL_2)

2. result

This time we are not so ambitious. For example, we don't fill in

$$K_0^B(\mathcal{B}\times\mathcal{B}) \cong K_0^G(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) \cong R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$$

just because the result is too long.

We don't want to use these symbols(like x,y,z) in later documents either. If you want to fix a notation, please use the notations in https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.10.23_notation_K%5EG(St).pdf

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Z}[y+y^{-1}]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2]/((z_1-z_2)(z_1-z_2^{-1}))$
	B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1},z]/(z-y)(z-y^{-1})$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$
$G=GL_2$	GL_2	$\mathbb{Z}[y_1+y_2,y_1y_2,\frac{1}{y_1y_2}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},z_1']/((z_1'-z_1)(z_1'-z_2))$
	B	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1}]$	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1},z_1]/((z_1-y_1)(z_1-y_2))$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1',z_2']/((z_1'-1)^2,(z_2'-1)^2)$
$G=SL_n$ or GL_n	G	$R(G)$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w'\in W}R(G)[\overline{\Omega}_{w'}]^G$
	B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w\in W}R(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W}R(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Z}$	$\bigoplus_{w\in W}\mathbb{Z}\cdot[\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W}\mathbb{Z}[\overline{\Omega}_{w,w'}]$

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Q}[b^2]$	$\mathbb{Q}[e]$	$\mathbb{Q}[e_1,e_2]/(e_2^2-e_1^2)$
	B	$\mathbb{Q}[b]$	$\mathbb{Q}[b,e]/(e^2-b^2)$	
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1,e_2]/(e_1^2,e_2^2)$
$G=GL_2$	GL_2	$\mathbb{Q}[b_1+b_2,b_1b_2]$	$\mathbb{Q}[e_1,e_2]$	$\mathbb{Q}[e_1,e_2,e_1']/((e_1'-e_1)(e_1'-e_2))$
	B	$\mathbb{Q}[b_1,b_2]$	$\mathbb{Q}[b_1,b_2,e_1]/((e_1-b_1)(e_1-b_2))$	$e_2'=e_1+e_2-e_1'$
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1',e_2']/(e_1'^2,e_2'^2)$
$G=SL_n$ or GL_n	G	$S(G)$	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w'\in W}S(G)[\overline{\Omega}_{w'}]^G$
	B	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w\in W}S(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W}S(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Q}$	$\bigoplus_{w\in W}\mathbb{Q}[\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W}\mathbb{Q}[\overline{\Omega}_{w,w'}]$

3. computation of completion in practice

Thm (cpl of Noetherian ring by power series)

R: Noetherian $I := (a_1, \dots, a_n) \trianglelefteq R$, then

$$\begin{aligned} \hat{R_I} &:= \varprojlim_n R/I^n \\ &\cong R[[x_1, \dots, x_n]]/(x_1 - a_1, \dots, x_n - a_n) \\ &\cong R[[a_1, \dots, a_n]] \end{aligned}$$

Ex. $\widehat{\mathbb{Z}[x]_{(x)}} \cong \mathbb{Z}[[x]]$

$$\begin{aligned} \hat{\mathbb{Z}}_{(p)} \cong \mathbb{Z}[[x]]/(x-p) &\xrightarrow{\sim} \mathbb{Z}_p \\ x &\longmapsto p \end{aligned}$$

$\hat{\mathbb{Z}}_{(p^2)} \cong \mathbb{Z}_p$

$\hat{\mathbb{Z}}_{(n)} \cong \prod_{\substack{p \mid n \\ \text{prime}}} \mathbb{Z}_p$

4. pt & $\mathbb{P}^1$

$$\begin{tikzcd}
& ④\, K_0^{GL_2}(pt) \arrow[dl] \arrow[dd] \arrow[rr] & & HP^\circ(GL(2,\infty);\mathbb{Q}) \arrow[dl] \arrow[dd] \\
②\, K_0^{SL_2}(pt) \arrow[rr] \arrow[dd] \arrow[r] & ⑤\, K_0^{SL_2}(\mathbb{P}^1) & HP^\circ(\mathcal{M}^{max}_{\infty\times 2}/SL_2;\mathbb{Q}) \arrow[dd] & \\
& ③\, K_0^{T}(pt) \arrow[dl] \arrow[rr] \arrow[d] & & HP^\circ(\mathbb{CP}^\infty\times\mathbb{CP}^\infty;\mathbb{Q}) \arrow[dl] \\
①\, K_0^{\mathbb{C}^\times}(pt) \arrow[rr] \arrow[r] & ⑥\, K_0^{SL_2}(\mathbb{P}^1) & HP^\circ(\mathbb{CP}^\infty;\mathbb{Q}) &
\end{tikzcd}$$

$$\begin{tikzcd}
& \mathbb{Z}[y_1+y_2, y_1y_2, \frac{1}{y_1y_2}] \arrow[dl] \arrow[dd] \arrow[rr] & & \mathbb{Q}[[b_1+b_2, b_1b_2]] \arrow[dl] \arrow[dd] \\
\mathbb{Z}[y+y^{-1}] \arrow[rr] \arrow[dd] & & \mathbb{Q}[[b^2]] \arrow[dd] & \\
& \mathbb{Z}[y_1^{\pm1}, y_2^{\pm1}] \arrow[dl] \arrow[rr] & & \mathbb{Q}[[b_1, b_2]] \arrow[dl] \\
\mathbb{Z}[y^{\pm1}] \arrow[rr] & & \mathbb{Q}[[b]] &
\end{tikzcd}$$

$$\begin{tikzcd}
& y_1 \arrow[dl, mapsto] \arrow[d, leftrightarrow] & & \\
y \arrow[d, leftrightarrow] & y_2 \arrow[dl, mapsto] & & \\
y^{-1} & y_i \arrow[rr, mapsto] & & e^{b_i} \\
y \arrow[rr, mapsto] & & e^b & b_1 \arrow[dl, mapsto] \arrow[d, leftrightarrow] \\
& & b \arrow[d, leftrightarrow] & b_2 \arrow[dl, mapsto] \\
& & -b &
\end{tikzcd}$$

$\leftrightarrow$: Weyl group action

Later, $c_i := c_i^G$ is a temporate notation,

ch^* is iso after tensored over $\mathbb{Q}$.

$$(ch^*)^{-1}: HP^\circ(BG;\mathbb{Q}) \xrightarrow{\sim} K_0(BG)\otimes_{\mathbb{Z}}\mathbb{Q}$$
$$HP^\circ_G(X;\mathbb{Q}) \xrightarrow{\sim} K_0^G(X)\otimes_{\mathbb{Z}}\mathbb{Q}$$

When I write the inverse map $(ch^*)^{-1}$, remember that the image usually has coefficient in $\mathbb{Q}$.

① $$\begin{array}{ccccccccc}
K_0^{\mathbb{C}^\times}(pt) & \xrightarrow[]{\overset{\text{completion}}{\overset{\shortparallel}{cpl}}} & K_0^{\mathbb{C}^\times}(pt)^\wedge_I & \xrightarrow[]{\overset{\text{Atiyah-Segal}}{\overset{\shortparallel}{\text{AS map}}}} & K_0(B\mathbb{C}^\times) & \xrightarrow{ch^*} & HP^0(B\mathbb{C}^\times;\mathbb{Q}) & \overset{cpl}{\supseteq} & H^*(B\mathbb{C}^\times;\mathbb{Q}) \\
\mathbb{Z}[y^{\pm 1}] & \longrightarrow & \mathbb{Z}[[y-1]] & \longrightarrow & \mathbb{Z}[[c_1^{\mathbb{C}^\times}]] & \longrightarrow & \mathbb{Q}[[b]] & \supset & \mathbb{Q}[b] \\
 & & & & c_1^{\mathbb{C}^\times} & \longmapsto & e^b-1 & & \\
 & & & \mathbb{Q}[[c_1^{\mathbb{C}^\times}]] \ni & \log(1+c_1^{\mathbb{C}^\times}) & \longleftarrow\!\shortmid & b & &
\end{array}$$

② $$\begin{array}{ccccccccc}
K_0^{SL_2}(pt) & \xrightarrow{cpl} & K_0^{SL_2}(pt)^\wedge_I & \xrightarrow{\text{AS map}} & K_0(BSL_2) & \xrightarrow{ch^*} & HP^0(BSL_2;\mathbb{Q}) & \overset{cpl}{\supseteq} & H^*(BSL_2;\mathbb{Q}) \\
\mathbb{Z}[y+y^{-1}] & \longrightarrow & \mathbb{Z}[[y+y^{-1}-2]] & \longrightarrow & \mathbb{Z}[[c_1^{SL_2}]] & \longrightarrow & \mathbb{Q}[[b^2]] & \supset & \mathbb{Q}[b^2] \\
 & & & & c_1^{SL_2} & \longmapsto & e^b+e^{-b}-1 & = & 4\sinh^2\frac{b}{2} \\
 & & & & & & & = & 2\cosh b-2
\end{array}$$

$$\begin{aligned}
4\left(\operatorname{arcsinh}\frac{\sqrt{c_1}}{2}\right)^2 &\longleftarrow\!\shortmid \ b^2 \\
= 4\left(\ln\left(\frac{\sqrt{c_1}}{2}+\sqrt{\frac{c_1}{4}+1}\right)\right)^2 & \\
= \left(\ln\left(1+\frac{c_1}{2}+\sqrt{\frac{c_1^2}{4}+c_1}\right)\right)^2 &
\end{aligned}$$

③ $$\begin{array}{ccccccccc}
K_0^{T}(pt) & \xrightarrow{cpl} & K_0^{T}(pt)^\wedge_I & \xrightarrow{\text{AS map}} & K_0(BT) & \xrightarrow{ch^*} & HP^0(BT;\mathbb{Q}) & \overset{cpl}{\supseteq} & H^*(BT;\mathbb{Q}) \\
\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}] & \longrightarrow & \mathbb{Z}[[y_1-1, y_2-1]] & \longrightarrow & \mathbb{Z}[[c_1^T, c_2^T]] & \longrightarrow & \mathbb{Q}[[b_1,b_2]] & \supset & \mathbb{Q}[b_1,b_2] \\
 & & & & c_i^{\mathbb{C}^\times} & \longmapsto & e^{b_i}-1 & & \\
 & & & & \log(1+c_i^{\mathbb{C}^\times}) & \longleftarrow\!\shortmid & b_i & &
\end{array}$$

④ $$\begin{array}{ccccccccc}
K_0^{GL_2}(pt) & \xrightarrow{cpl} & K_0^{GL_2}(pt)^\wedge_I & \xrightarrow{\text{AS map}} & K_0(BGL_2) & \xrightarrow{ch^*} & HP^0(BGL_2;\mathbb{Q}) & \overset{cpl}{\supseteq} & H^*(BGL_2;\mathbb{Q}) \\
\mathbb{Z}[y_1+y_2, y_1y_2, \frac{1}{y_1y_2}] & \longrightarrow & \mathbb{Z}[[y_1+y_2-2, y_1y_2-1]] & \longrightarrow & \mathbb{Z}[[c_1^{GL_2}, c_2^{GL_2}]] & \longrightarrow & \mathbb{Q}[[b_1+b_2, b_1b_2]] & \supset & \mathbb{Q}[b_1+b_2, b_1b_2] \\
 & & & & c_1^{GL_2} & \longmapsto & e^{b_1}+e^{b_2}-1 & & \\
 & & & & c_2^{GL_2} & \longmapsto & e^{b_1+b_2}-1 & & \\
 & & & & \log(1+c_2^{GL_2}) & \longleftarrow\!\shortmid & b_1+b_2 & & \\
 & & & & \log(1+y_1-1)\log(1+y_2-1) & \longleftarrow\!\shortmid & b_1b_2 & &
\end{array}$$

$$= \sum_{k=2}^{+\infty} \sum_{\substack{n+m=k \\ n,m\geq 1}} \frac{(-1)^k}{nm}(y_1-1)^n(y_2-1)^m$$

$$= \cdots$$

To facilitate the computation, use the notation

$$\begin{aligned}
c_3^{GL_2} &= (y_1-1)(y_2-1) \\
&= (y_1y_2-1)-(y_1+y_2-2) \\
&= c_2^{GL_2} - c_1^{GL_2}
\end{aligned}$$

$$
\begin{array}{ccccccccc}
5 & K_0^{SL_2}(\mathbb{P}^1) & \xrightarrow{cpl} & K_0^{SL_2}(\mathbb{P}^1)^\wedge_I & \xrightarrow{AS} & K_0(ESL_2 \times^{SL_2} \mathbb{P}^1) & \xrightarrow{ch^*} & HP^\circ_{SL_2}(\mathbb{P}^1;\mathbb{Q}) & \overset{cpl}{\supset} & H^*_{SL_2}(\mathbb{P}^1;\mathbb{Q}) \\
& \mathbb{Z}[z^{\pm 1}] & \longrightarrow & \mathbb{Z}[[z-1]] & \longrightarrow & \mathbb{Z}[[c_1]] & \longrightarrow & \mathbb{Q}[[\ell]] & \supset & \mathbb{Q}[\ell] \\
& & & & & c_1 & \longmapsto & e^{\ell}-1 & & \\
& & & & & \log(1+c_1) & \longleftarrow\!\shortmid & \ell & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
6 & K_0^{\mathbb{C}^\times}(\mathbb{P}^1) & \xrightarrow{cpl} & K_0^{\mathbb{C}^\times}(\mathbb{P}^1)^\wedge_I & \xrightarrow{AS} & K_0(E\mathbb{C}^\times \times^{\mathbb{C}^\times} \mathbb{P}^1) & \xrightarrow{ch^*} & HP^\circ_{\mathbb{C}^\times}(\mathbb{P}^1;\mathbb{Q}) & \overset{cpl}{\supset} & H^*_{\mathbb{C}^\times}(\mathbb{P}^1;\mathbb{Q}) \\
& \mathbb{Z}[y^{\pm 1}, z]/((z-y)(z-y^{-1})) & \longrightarrow & \mathbb{Z}[[y-1, z-1]]/\ldots & \longrightarrow & \mathbb{Z}[[c_1, c_2]]/((c_1-c_2)(c_1c_2+c_1+c_2)) & \longrightarrow & \mathbb{Q}[[b,\ell]]/(\ell^2-b^2) & \supset & \mathbb{Q}[b,\ell]/(\ell^2-b^2) \\
& & & & & c_1 & \longmapsto & e^{b}-1 & & \\
& & & & & c_2 & \longmapsto & e^{\ell}-1 & & \\
& & & & & \log(1+c_1) & \longleftarrow\!\shortmid & b & & \\
& & & & & \log(1+c_2) & \longleftarrow\!\shortmid & \ell & &
\end{array}
$$

5. Euler class

At first glance, chern class seems to be an exponential map.

Actually, chern class induces ring isomorphism $(+ \to +,\ \times \to \times)$

At first glance, Euler class seems to be a termwise -log map. $(\times \to +)$

Actually,

in one monomial $\quad 1+eu(\mathcal{L}_1\otimes\mathcal{L}_2) = (1+eu\,\mathcal{L}_1)(1+eu\,\mathcal{L}_2) \quad \times \to (1+\cdot)^{\times}$

for sum among monomials. $\quad eu(E_1\oplus E_2) = eu(E_1)\,eu(E_2) \quad + \to \times$

Let us see some examples of Euler class.

E.g.

$$K_0^{GL_n}(\mathcal{B}) \longrightarrow HP^0_{GL_n}(\mathcal{B};\mathbb{Q}) \supset H^*_{GL_n}(\mathcal{B};\mathbb{Q})$$

$$\mathbb{Z}[y_1^{\pm1},\dots,y_n^{\pm1}] \longrightarrow \mathbb{Q}[[b_1,\dots,b_n]] \supset \mathbb{Q}[b_1,\dots,b_n]$$

$$\begin{tikzcd} y_i \arrow[r, mapsto] \arrow[d, mapsto] \arrow[drr, mapsto] & e^{b_i} \arrow[dr, dashed] & \\ y_i - 1 & & b_i \end{tikzcd}$$

$\log y_i = \log(1+(y_i-1)) \approx y_i - 1$

$$\begin{tikzcd} \prod_i y_i^{k_i} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[drr, mapsto] & e^{\sum_i k_i b_i} \arrow[dr, dashed] & \\ (\prod_i y_i^{k_i}) - 1 & & \sum k_i b_i \end{tikzcd}$$

$$\begin{tikzcd} \frac{y_2}{y_1}+\frac{y_3}{y_1}+\frac{y_3}{y_2} \arrow[r, mapsto] \arrow[d, mapsto] \arrow[drr, mapsto] & e^{b_2-b_1}+e^{b_3-b_1}+e^{b_3-b_2} \arrow[dr, dashed] & \\ (\frac{y_2}{y_1}-1)(\frac{y_3}{y_1}-1)(\frac{y_3}{y_2}-1) & & (b_2-b_1)(b_3-b_1)(b_3-b_2) \end{tikzcd}$$

Q: What is right definition of $eu(\mathcal{T})$?

$eu(\mathcal{T}) = \sum_{i=0}^{+\infty}(-1)^i[\Lambda^i\mathcal{T}]$

$eu(\frac{y_2}{y_1}) = 1 - \frac{y_2}{y_1}$

compatible with Euler characteristic.

$e(X) = \sum_{i=0}^{+\infty}(-1)^i H^i(X;\mathbb{Q})$

will induce

$$D_i f = sf D_i + \frac{f-sf}{1-\frac{e_{i+1}}{e_i}}$$

p3: https://arxiv.org/pdf/math/0405333.pdf

or

$eu(\mathcal{T}) = \sum_{i=0}^{+\infty}(-1)^{i+1}[\Lambda^i\mathcal{T}]$?

$eu(\frac{y_2}{y_1}) = \frac{y_2}{y_1} - 1$

compatible with log map:

$\log(y_i) = (y_i-1) - \frac{(y_i-1)^2}{2} + \frac{(y_i-1)^3}{3} - \dots$

will induce

$$D_i f = sf D_i - \frac{f-sf}{1-\frac{e_{i+1}}{e_i}}$$

1.15: https://arxiv.org/pdf/math/0309168.pdf
p50: https://link.springer.com/content/pdf/10.1007/b10326.pdf
p93: http://sporadic.stanford.edu/bump/math263/hecke.pdf

reasons for each possibility

In 22.13, Another definition is mensioned: https://pages.uoregon.edu/ddugger/kgeom.pdf
p75: https://www.sciencedirect.com/science/article/pii/0022404994900884
It's also the definition in [Ginzburg, Cor 5.11.3]

$eu(\frac{y_2}{y_1}) = 1 - \frac{y_1}{y_2}$?

Definition 7.33. Let NH_m denote the NilHecke ring, i.e., the unital ring of endomorphisms of $k[y(1), ..., y(m)]$ generated by multiplication with $y(1), ..., y(m)$ and Demazure operators

$f \cdot g + f \cdot \partial_l g$

$$\partial_l(f) = \frac{f - s_l f}{y(l) - y(l+1)},$$

$$\partial_l(fg) = \frac{fg - s_l(fg)}{y_l - y_{l+1}} = \partial_l f \cdot g + f \cdot \partial_l g$$

for $1 \leq l \leq m-1$, where s_l is the transposition switching $y(l)$ and $y(l+1)$. The endomorphisms which act by multiplication with $y(1), ..., y(m)$ generate a subring which is canonically isomorphic to $k[y(1), ..., y(m)]$. Moreover, it is well-known that the ring of endomorphisms which act by mutiplication by a symmetric polynomial equals the centre of NH_m.

Lemma 11.14. *Let $\partial_{\overline{y},l}$ denote the Demazure operator*

$$\partial_{\overline{y},l} : f \mapsto \frac{f - s_l(f)}{x_{\overline{y}}(l+1) - x_{\overline{y}}(l)},$$

Not compatible!

Example 11.28 (NilHecke ring). Set $\mathbf{I} = \{i\}$, $\mathbf{H} = \varnothing$ and $\underline{\mathbf{d}} = ni$. Then $\mathbb{W}_{\mathbf{d}} = W_{\underline{\mathbf{d}}} \cong \mathfrak{S}_n$, $|Y_{\underline{\mathbf{d}}}| = 1$, $Y_{\underline{\mathbf{d}}} = \{\overline{y}\}$, where $\overline{y} = (i, i, ..., i)$, $G_{\underline{\mathbf{d}}} = \mathbb{G}_{\mathbf{d}} \cong \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{Rep}_{\underline{\mathbf{d}}} = \{0\}$. Moreover, $\widetilde{\mathcal{F}}_{\underline{\mathbf{d}}} = \mathcal{F}_{\underline{\mathbf{d}}} = \mathcal{F}_{\overline{y}}$, $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{F}_{\overline{y}}) = k[x_{\overline{y}}(1), ..., x_{\overline{y}}(n)]$ and $\mathcal{Z}_{\underline{\mathbf{d}}} = \mathcal{F}_{\overline{y}} \times \mathcal{F}_{\overline{y}}$. Since for each $s_l \in \Pi$, we have $s_l(\overline{y}) = \overline{y}$, the elements $\sigma_{\overline{y}}(l)$ always act as Demazure operators. Hence $H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}})$ is the ring of endomorphisms of $k[x_{\overline{y}}(1), ..., x_{\overline{y}}(n)]$ generated by endomorphisms $\varkappa_{\overline{y}}(l)$ which act by multiplication with $x_{\overline{y}}(l)$ and Demazure operators $\sigma_{\overline{y}}(l)$. Therefore

$$H_*^{G_{\underline{\mathbf{d}}}}(\mathcal{Z}_{\underline{\mathbf{d}}}) \cong NH_n,$$

$$\mathrm{Ind}_T^{P_s}(e^\lambda) = \frac{e^\lambda - e^{s \cdot \lambda}}{1 - e^{-\alpha_s}} = \frac{e^{\lambda + \alpha_s/2} - e^{s(\lambda) - \alpha_s/2}}{e^{\alpha_s/2} - e^{-\alpha_s/2}}$$

This is quite confusing.

$$\frac{e^\lambda - e^{s(\lambda)}}{1 - e^{-\alpha_s}} = \frac{e^{\alpha_s/2}}{e^{\alpha_s/2}} \frac{e^\lambda - e^{s(\lambda)}}{1 - e^{-\alpha_s}} = \frac{e^{\lambda + \alpha_s/2} - e^{s(\lambda) + \alpha_s/2}}{e^{\alpha_s/2} - e^{-\alpha_s/2}}$$