Eine Woche, ein Beispiel

9.10. ramified covering : alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - Correspondence
   - Valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

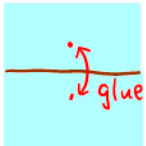
1. alg curve viewed as stack quotient

	Spec $\mathbb{R}$	base change Spec $\mathbb{C}/_{\mathbb{C}}$	Spec $\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	$\emptyset$
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{Id, \tau\}$
$\Gamma_{\mathbb{R}} = Gal(\mathbb{C}/_{\mathbb{R}})$	trivial on pts & fcts	no action	$Id \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. Spec $\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$. As a set,

$$\begin{aligned}
\operatorname{Spec} \mathbb{R}[x] &= \{(x-a) \mid a \in \mathbb{R}\} \cup \{(x^2+bx+c) \mid {b, c \in \mathbb{R} \atop b^2-4c<0}\} \cup \{(0)\} \\
&= \quad \mathbb{R} \quad \cup \quad \mathcal{H} \quad \cup \{(0)\} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})
\end{aligned}$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau \circ x$. Observe that

$$\operatorname{MaxSpec} \mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/_{\Gamma_{\mathbb{R}}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$\operatorname{MaxSpec} \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/_{\Gamma_{\mathbb{F}_p}} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/_{\Gamma_\kappa} \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

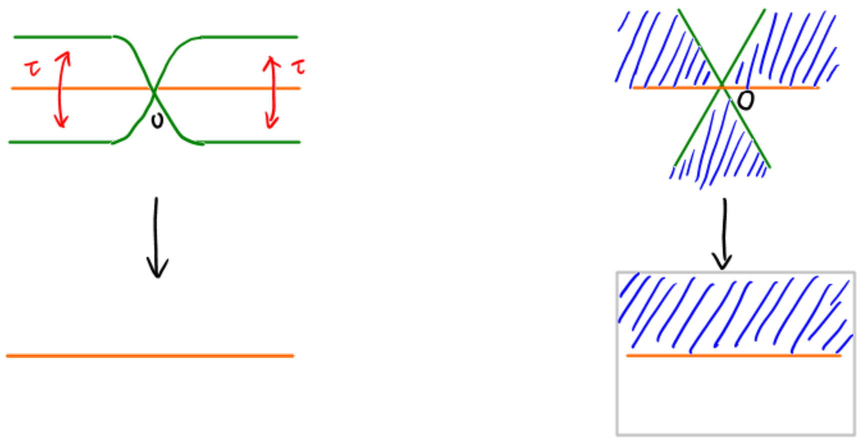
$$\operatorname{Stab}_x(\Gamma_\kappa) = \Gamma_{\kappa'} \quad \Leftrightarrow \quad \text{fiber at } x = \operatorname{Spec} \kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R} \cup \mathcal{H}$	$\mathbb{C}$	$\mathbb{C} \circlearrowleft$ cplx conj
$\mathbb{R}$-pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$-pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \sqcup \mathbb{C}\tau$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

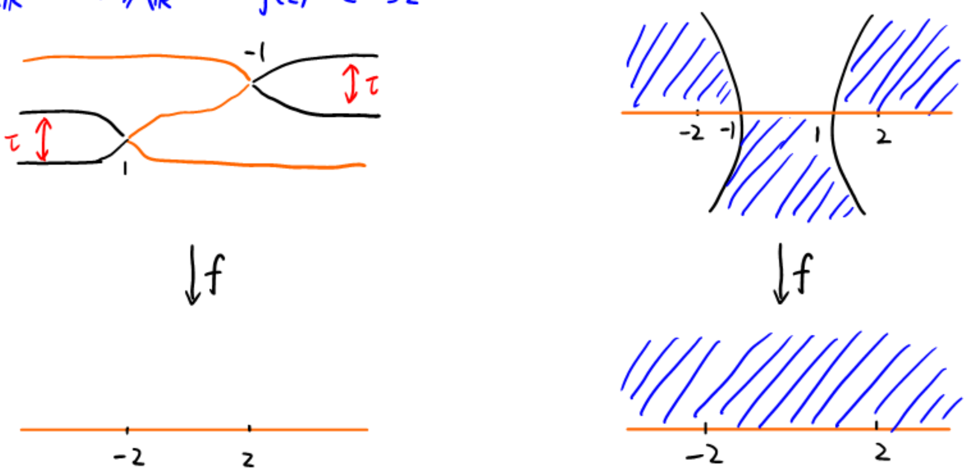
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

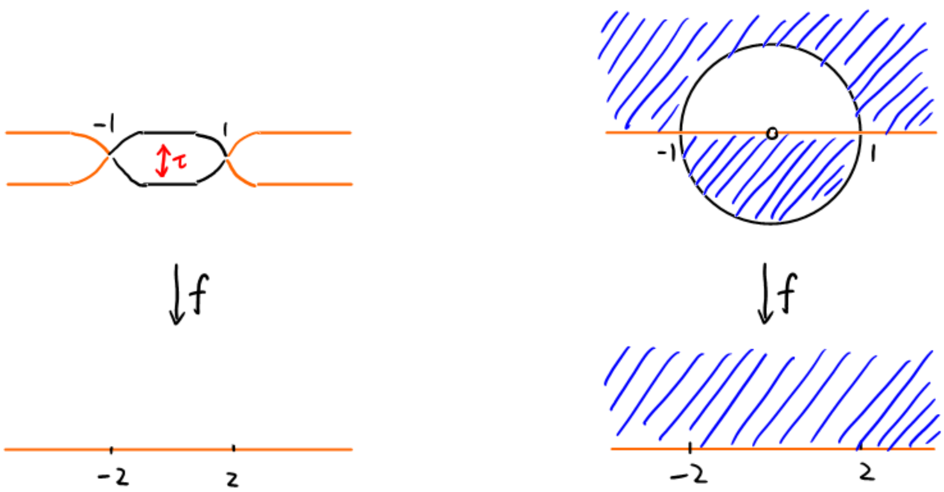
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



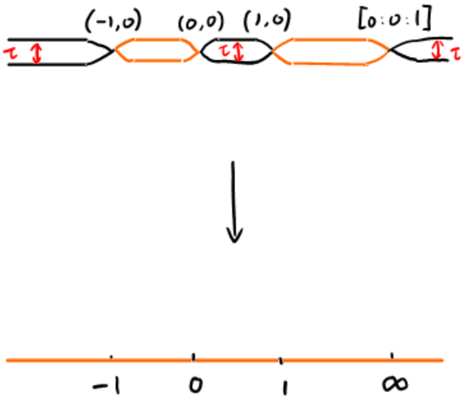
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$

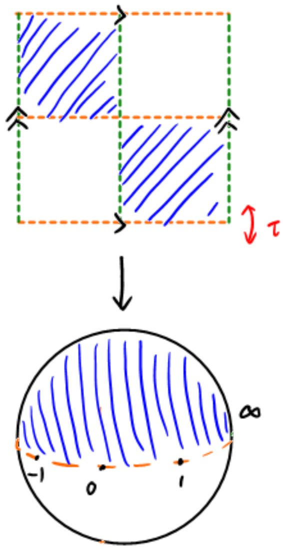


$f: \mathbb{G}_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$



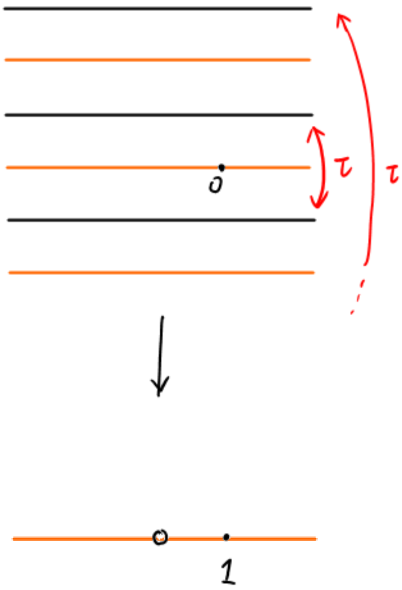
$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ $\quad [x:y:z] \longmapsto [x:z]$ $\quad E_{\mathbb{R}} = \mathrm{Proj}\ \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

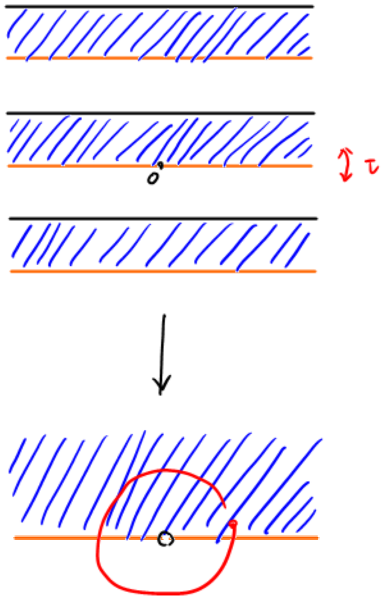




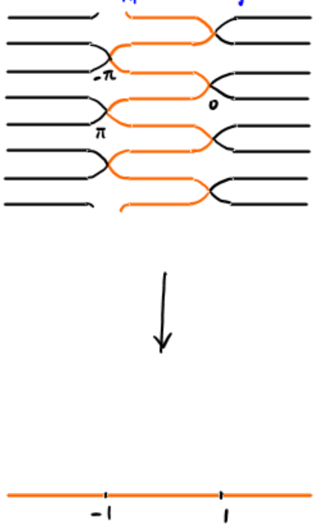
⚠ The following are not alg morphisms!

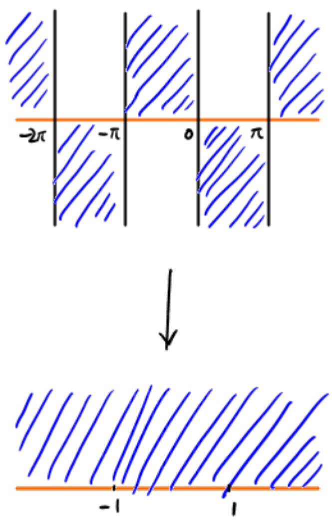
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = e^z$





$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = \cos z$

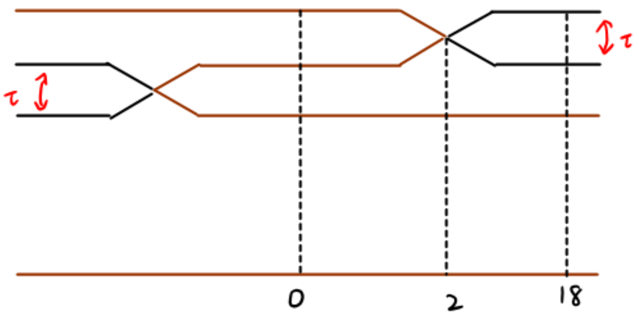




Let's focus on the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = z^3 - 3z$

classical picture



split: $f^{-1}(0) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z-z_0))$

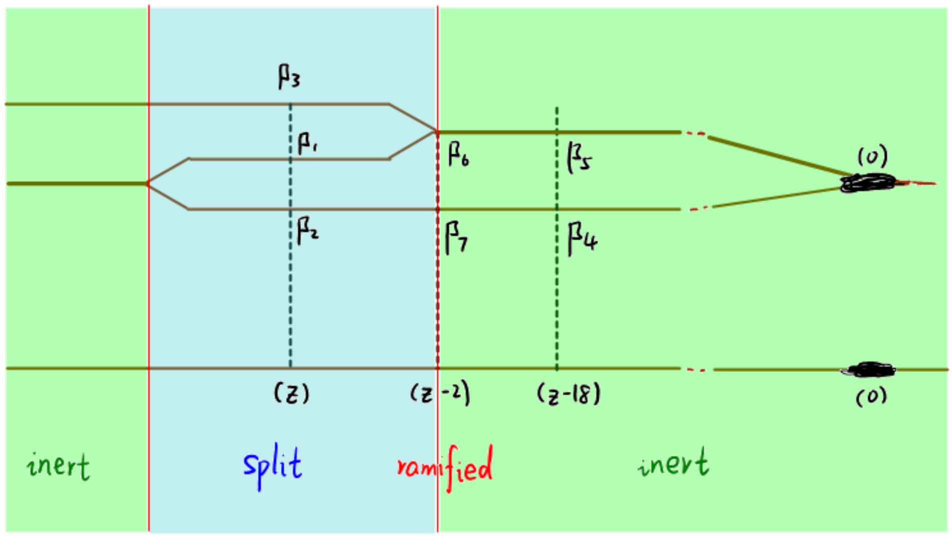
$f^{-1}((z^2+1)) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{R}$

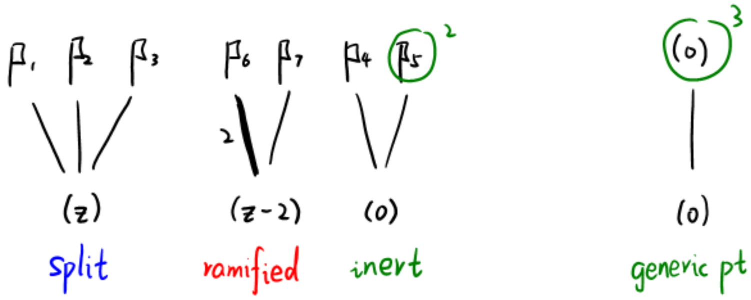
generic point: $f^{-1}((0)) = \operatorname{Spec}\mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$

algebraic picture



$$\begin{array}{ccccc} \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[w] & & w^3-3w \\ f\big\downarrow & & f^*\big\uparrow & & \big\uparrow \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] & & z \end{array}$$



split: $\mathfrak{p} = (z)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w) = (w)(w-\sqrt{3})(w+\sqrt{3}) \triangleq \mathfrak{P}_1 \; \mathfrak{P}_2 \; \mathfrak{P}_3$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$

$\mathfrak{p} = (z^2+1)$, $f^*(\mathfrak{p})\mathbb{R}[w] = ((w^3-3w)^2+1) = (f_1')\,(f_2')\,(f_3') \triangleq \mathfrak{P}_1' \; \mathfrak{P}_2' \; \mathfrak{P}_3'$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$

(partially) inert: $\mathfrak{p} = (z-18)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-18) = (w-3)(w^2+3w+6) \triangleq \mathfrak{P}_4 \; \mathfrak{P}_5$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_4, \mathfrak{P}_5\}$

where $\kappa(\mathfrak{P}_5) = \mathbb{R}[w]/_{(w^2+3w+6)} \cong \mathbb{C}$, $\quad [\kappa(\mathfrak{P}_5):\mathbb{R}] = 2$

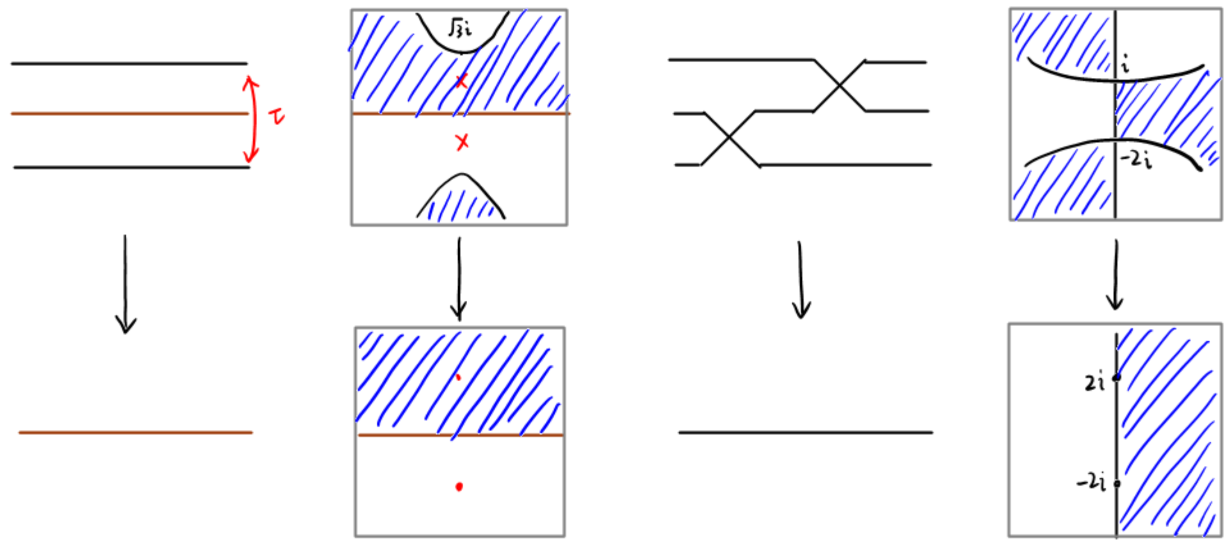
generic point: $\mathfrak{p} = (0)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (0)$ $\qquad f^{-1}(\mathfrak{p}) = \{(0)\}$

where $\kappa((0)) = \mathrm{Frac}\,(\mathbb{R}[w]/_{(0)}) \cong \mathbb{R}(w)$, $\quad [\mathbb{R}(w):\mathbb{R}(z)] = 3$

ramified: $\mathfrak{p} = (z-2)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-2) = (w+1)^2(w-2) \triangleq \mathfrak{P}_6^2 \; \mathfrak{P}_7$ $\qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_4, \mathfrak{P}_5\}$

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = z^3+3z$.

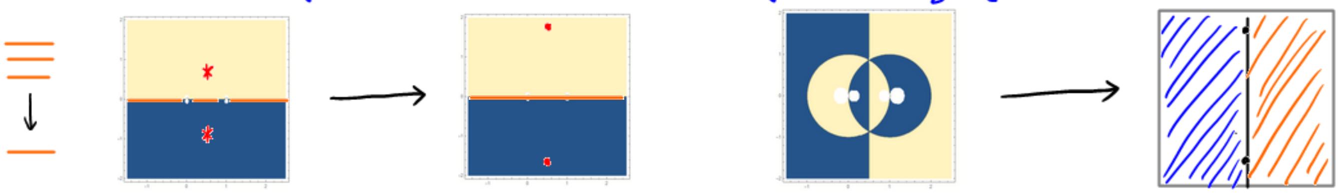


ℝ picture $\qquad\qquad$ iℝ picture

⚠ The ramification pt is outside ℝ. This is not a Galois covering.

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = \dfrac{z^3-3z+1}{z^2-z} - 1.5$



ℝ picture $\qquad\qquad$ iℝ picture

This is a Galois covering, with no inert places (except for the generic pt)

3. Frobenius for alg curve/$\mathbb{R}$

$$\mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \kappa(\mathfrak{q}) = \mathbb{C},\ \kappa(\mathfrak{p}) = \mathbb{R} \\ \{\mathrm{Id}\} & \text{otherwise.} \end{cases}$$

When E/F is Galois, $\mathrm{Spec}\,\mathcal{O}_E/\mathrm{Spec}\,\mathcal{O}_F$ unramified at $\mathfrak{p}$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \mathrm{Frob}_{\mathfrak{q}} & & \longmapsto & & \mathrm{Frob}_{\mathfrak{q}} \end{array}$$

is a subgp of $\mathrm{Gal}(E/F) \cong \mathrm{Aut}(\mathrm{Spv}(E)/\mathrm{Spv}(F))$ Now, just view $\mathrm{Spv}(E) \in \mathrm{AlgCurve}_k$.

Let's try to compute some $\mathrm{Frob}_{\mathfrak{q}}$.

E.g.

$$\begin{array}{cccccccc} \mathbb{A}^1_{\mathbb{R}} & z & \mathbb{R}[\omega] = \mathbb{R}[z^{\frac{1}{2}}] & -1 \quad 1 & i,-i & 0 \\ \downarrow & \downarrow & \uparrow & \setminus\,/ & | & \|\,2 \\ \mathbb{A}^1_{\mathbb{R}} & z^2 & \mathbb{R}[z] & 1 & -1 & 0 \end{array}$$

For $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (\omega - 1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ 1 & & 1 & & \{1, \tau\} \end{array}$$

For $\mathfrak{p} = (z+1)$, $\mathfrak{q} = (\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1, \tau\} & & \{1, \tau\} & & \{1, \tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z+1)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \qquad \omega \longmapsto -\omega$

Not the conjugation, but $\tau(\mathbb{C})|_{i\mathbb{R}}$ coincides with the cplx conj

E.g.

$$\begin{array}{cccccccc} \mathbb{G}_{m,\mathbb{R}} & z & \mathbb{R}[\omega^{\pm 1}] = \mathbb{R}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{\pm 1}\right] & 2 \quad \frac{1}{2} & i,-i & 1 & -1 \\ \downarrow & \downarrow & \uparrow & \setminus\,/ & | & \|\,2 & \|\,2 \\ \mathbb{A}^1_{\mathbb{R}} & z+\frac{1}{z} & \mathbb{R}[z] & \frac{5}{2} & 0 & 2 & -2 \end{array}$$

For $\mathfrak{p} = (z)$, $\mathfrak{q} = (\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1, \tau\} & & \{1, \tau\} & & \{1, \tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \qquad \omega \longmapsto \frac{1}{\omega}$

Not the conjugation, but $\tau(\mathbb{C})|_{S^1}$ coincides with the cplx conj

⚠ $\mathbb{R}(z^{\frac{1}{3}})/\mathbb{R}(z)$ is not Galois at all, so

For $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad z \longmapsto z^3$, $\quad \mathfrak{p} = (z-1)$, $\mathfrak{q} = (w^2+w+1)$,

$$\underbrace{\mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))}_{\{1,\tau\}} \not\twoheadleftarrow \underbrace{\text{“}\mathrm{Gal}(E/F)_{\mathfrak{q}}\text{”}}_{1} \leq \underbrace{\text{“}\mathrm{Gal}(E/F)\text{”}}_{1} \neq \mathbb{Z}/3\mathbb{Z}$$

We will discuss about $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$ in section 4.

Claim. For p odd prime, any deg p extension of $\mathbb{R}(x)$ is not Galois.

This claim is wrong. The field extension

$$\mathbb{R}(x)[T]/(T^3 - xT^2 + (x-3)T + 1) \;/\mathbb{R}(x)$$

is Galois with deg 3. discriminant $\Delta = (x^3-3x+9)^2$ [SerreGT, 1.1]

Wrong proof:

If not, suppose $F/\mathbb{R}(x)$ is a deg p Galois extension, we get the field extension tower in $\overline{\mathbb{R}(x)}$:

$$\begin{tikzcd}
 & & E \arrow[dd, dash, "p"] \\
F \arrow[urr, dash, "2"] \arrow[dd, dash, "p"'] & & \\
 & & \mathbb{C}(x) \\
\mathbb{R}(x) \arrow[urr, dash, "2"'] & &
\end{tikzcd}$$

where $\mathrm{Gal}(E/F) \lhd \mathrm{Gal}(E/\mathbb{R}(x))$ is a normal subgp of order 2.

By Kummer theory, $E \cong \mathbb{C}(x)[T]/(T^p - f)$ for some $f \in \mathbb{C}(x)$.

~~Since $E/\mathbb{R}(x)$ is Galois, $f \in \mathbb{R}(x)$ (see the example below)~~

When $f \in \mathbb{R}(x)$, one gets

$$\mathrm{Gal}(E/\mathbb{R}(x)) \longhookrightarrow S_p \circlearrowright \{T, \zeta_p T, \ldots, \zeta_p^{p-1} T\}$$

Injection: if σ fix T, $\zeta_p T$, then σ fix ζ_p, then $\sigma = \mathrm{Id}$.

Since $\#\mathrm{Gal}(E/\mathbb{R}(x)) = 2p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$ or $\mathbb{Z}/2p\mathbb{Z}$.

Since $\mathrm{Gal}(E/\mathbb{R}(x)) \leq S_p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$.

However, D_p has no order 2 normal subgp, contradiction!

E.g. $\mathbb{C}(z)[T]/(T^3 - (z-i))$ over $\mathbb{R}(z)$ is not Galois, since

$$\begin{tikzcd}
\mathbb{C}(T) \arrow[d, dash] \\
\mathbb{C}(z) \arrow[d, dash] \\
\mathbb{R}(z)
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
0 \arrow[d, dash, "3"'] & & z_0 \arrow[d, dash] & \rho z_0 \arrow[dl, dash] & \rho^2 z_0 \arrow[dll, dash] \\
i \arrow[dr, dash] & & -i \arrow[dl, dash] & & \\
 & (z^2+1) & & &
\end{tikzcd}$$

This example is not general enough. For example,

$$\mathbb{C}(z)[T]/\left(T^3 - \frac{z-i}{z+i}\right) \text{ over } \mathbb{R}(z) \text{ can be Galois}$$

Q. For $F/\mathbb{R}(x)$ Galois extension, is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by its order 2 elements?

I call it as the "weaked version of Chebotarev's density theorem for $\mathbb{P}^1_{\mathbb{R}}$".

A: No.

We could not expect the density theorem to be true in the real case,

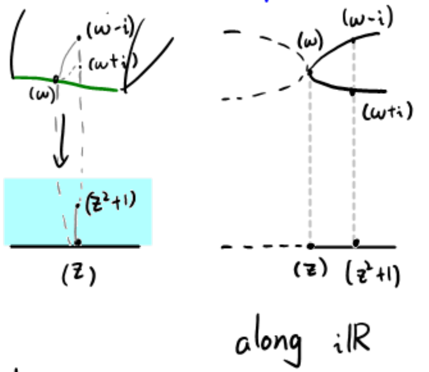
since in S_3 case the order 3 conj class can never be reached by a single Frob.

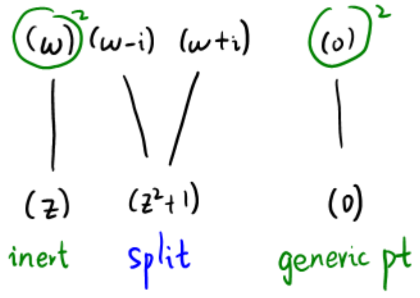
For a possible direct and brutal method to this question, use the result in this link:
math.stackexchange.com/questions/318690/absolute-galois-group-of-mathbbrt
How is Z/3Z realized as the quotient group of this group? (better: compatible with the field extension mentioned above)

4. complexify is a ramified covering by non geometrical connected spaces

E.x. $f: \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ is an unramified covering of alg curves/$\mathbb{R}$.

$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} \arrow[d, "f"] & \mathbb{C}[\omega] \\ \mathbb{A}^1_{\mathbb{R}} & \mathbb{R}[z] \arrow[u, "f^*"'] \end{tikzcd}$$





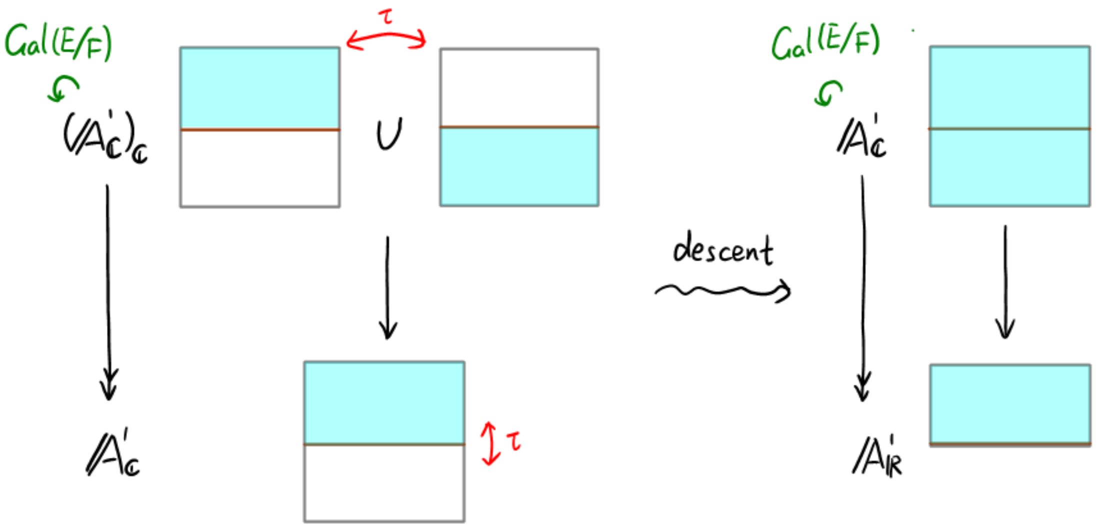
This is an unramified covering.

As an $\mathbb{R}$-scheme, $\mathbb{A}^1_{\mathbb{C}}$ is not geo connected.

$$\begin{tikzcd} \overset{\circlearrowleft\, Gal(E/F)}{\mathbb{C}[\omega]\otimes_{\mathbb{R}}\mathbb{C}} \circlearrowleft \Gamma_{\mathbb{R}} & \cong & \overset{Gal(E/F)\,\Gamma_{\mathbb{R}}}{\mathbb{C}[\omega]\oplus\mathbb{C}[\omega]} \\ \mathbb{R}[z]\otimes_{\mathbb{R}}\mathbb{C} \circlearrowleft \Gamma_{\mathbb{R}} \arrow[u] & \cong & \Gamma_{\mathbb{R}} \circlearrowleft\, \mathbb{C}[z] \arrow[u, "(Id,\sigma)"'] \end{tikzcd}$$

$f(\omega)\otimes_{\mathbb{R}} a \longmapsto (a f(\omega), \bar{a} f(\omega))$

$f(z)\otimes_{\mathbb{R}} a \longmapsto a f(z)$



For $p=(z)$, $q=(\omega)$,

$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \leq & Gal(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $Frob_{(z)} = \tau : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$\tau(\mathbb{C}): \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1$

$\omega_1 \longmapsto \overline{\omega_1}$

$\omega_2 \longmapsto \overline{\omega_2}$

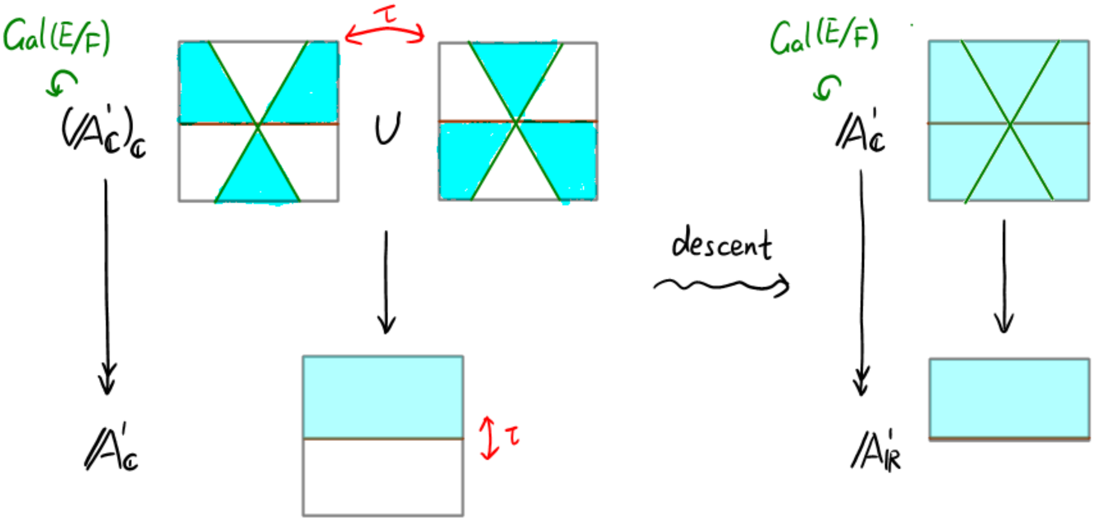
Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R}\cup\mathbb{R}}$ coincides with the cplx conj (switch)

Ex. Try to work out $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$, and compute Frobenius elements.

Recall: $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) \cong S_3 \circlearrowright \{\sqrt[3]{2}, \rho\sqrt[3]{2}, \rho^2\sqrt[3]{2}\}$.

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)) \cong S_3 \circlearrowright \{\underset{1}{z^{\frac{1}{3}}}, \underset{2}{\rho z^{\frac{1}{3}}}, \underset{3}{\rho^2 z^{\frac{1}{3}}}\}$

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z))$	Id	(23)	(12)	(13)	(123)	(132)
$z^{\frac{1}{3}} \longmapsto$	$z^{\frac{1}{3}}$	$z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$
$\rho \longmapsto$	ρ	ρ^2	ρ^2	ρ^2	ρ	ρ
$\mathbb{CP}^1 \ni a \longmapsto$	a	$\bar{a}$	$\rho^2\bar{a}$	$\rho\bar{a}$	$\rho^2 a$	ρa
geometry	Id					



For $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (w-1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1, \tau\} & & \{1, (23)\} & & S_3 \end{array}$$

Therefore, $\mathrm{Frob}_{(w-1)} = \tau_{(23)}: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}): \mathbb{CP}^1 \sqcup \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \sqcup \mathbb{CP}^1$$
$$w_1 \longmapsto \bar{w}_1$$
$$w_2 \longmapsto \bar{w}_2$$

Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R} \cup \mathbb{R}}$ coincides with the cplx conj (switch)

Similarly, $\mathrm{Frob}_{(w-\rho)} = \tau_{(13)}: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}): \mathbb{CP}^1 \sqcup \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \sqcup \mathbb{CP}^1$$
$$w_1 \longmapsto \rho\bar{w}_1$$
$$w_2 \longmapsto \rho\bar{w}_2$$

Not the conjugation, but $\tau(\mathbb{C})|_{\rho^2\mathbb{R} \cup \rho^2\mathbb{R}}$ coincides with the cplx conj (switch)

In this case, $\mathrm{Gal}(E/F)$ is generated by all $\mathrm{Frob}_{(z-z_0)}$.

6. alg curve over $\mathbb{F}_p$: miscellaneous.

- $\# X(\mathbb{F}_p)$, $\# X(\mathbb{F}_{p^2})$, $\cdots$ $\rightsquigarrow$ L-fcts, heights, ...
- Computation of Frob.
- Chebotarev density theorem: give a proof.
- hyperelliptic curve over $\mathbb{F}_2$: unexpected ones
- $p = 1$: What would happen then?
- Shtukas (in Langlands, though).