

# Eine Woche, ein Beispiel

## 8.21 equivariant K-theory of $\mathbb{P}^1$



Be careful about the ring structure here!  
Most isomorphisms are not iso as algebras.  
I will revise this document after the vacation.

Let us do a simple case over  $\mathbb{P}^1$ . It can be generalized "easily" to flag variety, but  $\mathbb{P}^1$  is the beginning case of study.

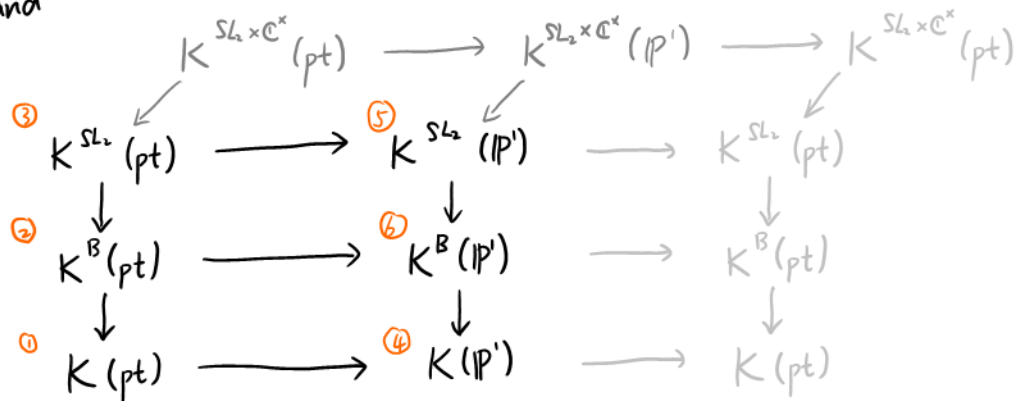
Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand



where  $SL_2 = SL_{2,\mathbb{C}}$ ,  $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq SL_{2,\mathbb{C}}$ ,  
 $\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} \cong G/B$ ,  $G \overset{\text{left}}{\curvearrowright} \mathbb{P}^1$ ,  $\mathbb{C}^\times \overset{\text{trivial}}{\curvearrowright} \mathbb{P}^1$   
 maps are pushback & pullout of  $\mathbb{P}^1 \rightarrow pt$ .

We want to see

- ring structure, module structure
- Weyl gp action
- relations

e.g.  $K^B(X) \cong R(B) \otimes_{R(G)} K^G(X) \cong \mathbb{Z}[W] \otimes_{\mathbb{Z}} K^G(X)$   
 $(K^B(X))^W \cong K^G(X)$

The correct answer is collected here.

$$\begin{array}{ccc}
 K^{SL_2}(pt) & \longrightarrow & K^{SL_2}(P') \\
 \downarrow & & \downarrow \\
 K^B(pt) & \longrightarrow & K^B(P') \\
 \downarrow & & \downarrow \\
 K(pt) & \longrightarrow & K(P')
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} x \\ \downarrow \\ y+y^{-1} \end{array} & \begin{array}{c} \xrightarrow{\quad} \\ \downarrow \\ \downarrow \end{array} & \begin{array}{c} z+z^{-1} \\ \mathbb{Z}[x] \longrightarrow \mathbb{Z}[z^{\pm 1}] \\ \mathbb{Z}[y^{\pm 1}] \\ \mathbb{Z} \longrightarrow \mathbb{Z}[z]/(z-1)^2 \\ \mathbb{Z} \end{array}
 \end{array}$$

Notation. For linear alg qp  $G$  [Ginz. 5.1],

$$K_i^G(X) := K_i(\text{Coh}^G(X)) \quad K^G(X) := K_0^G(X) \quad K(X) := K^{\{\text{Id}\}}(X)$$

$$R(G) := K^G(\text{pt}) = K_0(\text{Coh}^G(\text{pt})) = K_0(\text{Rep } G)$$

①-③ e.g.  $R(\text{Id}) = \mathbb{Z}$ ,  $R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}]$ ,  $R(SL_2) \cong \mathbb{Z}[x]$ ,  $R(SL_2 \times \mathbb{C}^*) \cong \mathbb{Z}[x, t^{\pm 1}]$   
 $= \mathbb{Z}[X^*(T)] \quad = \mathbb{Z}[X^*(T)]^{\text{Wf}}$

Some further discussion of  $R(SL_2)$ .

$$R(SL_2) = \bigoplus_{i \in \mathbb{N}_{\geq 0}} \mathbb{C} x_i \quad \text{where } x_i \text{ represents the } (i+1)\text{-dim irr rep of } SL_2.$$

As an algebra,  $R(SL_2) = \mathbb{C}[x]$  where

$$1 = x_0$$

$$x = x_1$$

$$x^2 = x_2 + 1$$

$$x^3 = x_3 + 2x_1$$

$$x^4 = x_4 + 3x_2 + 2$$

$$x_0 = 1$$

$$x_1 = x$$

$$x_2 = x^2 - 1$$

$$x_3 = x^3 - 2x$$

$$x_4 = x^4 - 3x^2 + 1$$

$$\begin{array}{cccccccc} 1 & 0 & 1 & 0 & 2 & 0 & 5 & 0 & 14 \\ & 1 & 0 & 2 & 0 & 5 & 0 & 14 & 0 \\ & & 1 & 0 & 3 & 0 & 9 & 0 & 28 \\ & & & 1 & 0 & 4 & 0 & 14 & 0 \\ & & & & 1 & 0 & 5 & 0 & 20 \\ & & & & & 1 & 0 & 6 & 0 \\ & & & & & & 1 & 0 & 7 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{array}$$

$$\begin{array}{cccccccc} 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ & 1 & 0 & -2 & 0 & 3 & 0 & -4 & 0 \\ & & 1 & 0 & -3 & 0 & 6 & 0 & -10 \\ & & & 1 & 0 & -4 & 0 & 10 & 0 \\ & & & & 1 & 0 & -5 & 0 & 15 \\ & & & & & 1 & 0 & -6 & 0 \\ & & & & & & 1 & 0 & -7 \\ & & & & & & & 1 & 0 \\ & & & & & & & & 1 \end{array}$$

∇  $x$  represents a v.b of rank 2 here!

Q: How to write down all polynomials in  $R(SL_n)$  which represents an irr rep of  $SL_n$ ?

The important definitions and results in [Ginz, Chap 5] have been collected in this page.  
Nothing about cohomology theory is short.

[5.2.5] pullback

[5.2.11] tensor product

[5.2.13] pushforward

[5.2.16] induction

[5.2.18] reduction

[5.2.20] convolution

[5.2.26] duality, pairing

[5.3] specialization

[Thm 5.4.17] Thom iso

[Lemma 5.5.1] cellular fibration

[Thm 5.6.1] Künneth

[5.7.1] Beilinson Resolution

[Thm 5.8.14] Riemann-Roch

[Prop 5.9.3] Devissage

[Rmk 5.11.8] Lefschetz fixed point formula

$$K^G(G \times_B -) \cong K^B(-)$$

$$K^B(-) \cong K^T(-)$$

derived pullback gives an ab homo between K-groups, while  
derived pushforward gives a  $\mathbb{Z}$ -mod homo between K-groups

$$[LCBE, 2.1.1] \textcircled{4} K(\mathbb{P}^1) \cong \mathbb{Z} \mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{Z} \mathcal{O}_{\mathbb{P}^1}(-1) = \mathbb{Z}[\mathbb{Z}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\pm 1}]/(\mathbb{Z}-1)^2$$

$\nabla$   $\mathbb{Z}$  corresponds to  $\mathcal{O}_{\mathbb{P}^1}(-1)$  here.

Q: How to compute  $K(\mathcal{B})$ ?  $\mathcal{B} = \mathrm{SL}_n/\mathcal{B}$

[Ginz, (5.2.4)]  $\begin{matrix} G \cdot G \\ G_{\text{trivial}} \end{matrix} X \Rightarrow K^{G \times G_1}(X) \cong K^{G_1}(X) \otimes_{\mathbb{Z}} R(G_2)$  As an  $R(G_2)$ -module

e.g.  $K^{SL_2 \times \mathbb{C}^*}(\mathbb{P}^1) \cong K^{SL_2}(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(\mathbb{C}^*) \cong K^{SL_2}(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$   
 $K^B(\mathbb{P}^1) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} R(B) \cong K(\mathbb{P}^1) \otimes_{\mathbb{Z}} \mathbb{Z}[y^{\pm 1}]$

[Ginz, (5.2.17)]

$$K_i^H(X) \xrightleftharpoons[\text{Ind}_H^G]{\text{Res}_H^G} K_i^G(G \times_H X)$$

⊕ e.g.  $K^{SL_2}(\mathbb{P}^1) \cong K^{SL_2}(SL_2 \times_B \text{pt}) \cong K^B(\text{pt}) = R(B) = \mathbb{Z}[z^{\pm 1}]$

Task: understand  $K^{SL_2}(\mathbb{P}^1)$ : How does equivariant  $SL_2$ -bundle look like?

Try: Let  $\lambda \in X^*(T)$ ,  $\mathbb{C}_\lambda$  denotes for  $B$ -equivariant l.b over  $\{\text{pt}\}$ .  $[\mathbb{C}_\lambda] \in R(B)$

Step 1. find a local chart of  $G/B$ .

$$\begin{array}{ccc} & \xrightarrow{G \times \mathbb{C}_\lambda} & G \\ & \downarrow & \downarrow \\ \mathbb{A}^1 & \longrightarrow & G/B = G \times_B \{\text{pt}\} \\ \begin{bmatrix} 1 \\ x_2 \end{bmatrix} & \longmapsto & \begin{bmatrix} 1 & 0 \\ x_2 & 1 \end{bmatrix} \\ \begin{bmatrix} x_1 \\ 1 \end{bmatrix} & \longmapsto & \begin{bmatrix} x_1 & -1 \\ 1 & 0 \end{bmatrix} \end{array}$$

$$\begin{array}{ccc} & \xrightarrow{B \times \mathbb{C}_\lambda} & G \\ & \downarrow & \downarrow \\ G/B & \longrightarrow & G \times_B \{\text{pt}\} \\ & \downarrow & \downarrow \\ G/B & \longrightarrow & G \times_B \{\text{pt}\} \end{array}$$

$G$ : left multiplication  
 $B \subset G \times \mathbb{C}_\lambda$   
 $b(g, x) = (gb^{-1}, bx)$

Step 2. See the type of l.b over  $G/B$ . ( $K^G(\mathbb{P}^1) \rightarrow K(\mathbb{P}^1)$ )

$$\begin{array}{ccc} \mathbb{A}^1 & \searrow & (x_1, s_1) \\ & G/B & \searrow \\ \mathbb{A}^1 & \nearrow & (x_2, s_2) \end{array}$$

$$(x_1, s_1) \mapsto ([\begin{smallmatrix} x_1 & -1 \\ 1 & 0 \end{smallmatrix}], s_1) = ([\begin{smallmatrix} 1 & 0 \\ x_1 & 1 \end{smallmatrix}], [\begin{smallmatrix} x_1 & -1 \\ 0 & x_1 \end{smallmatrix}], s_1)$$

$$(x_2, s_2) \mapsto ([\begin{smallmatrix} 1 & 0 \\ x_2 & 1 \end{smallmatrix}], s_2)$$

$$\therefore s_2 = [\begin{smallmatrix} x_1 & -1 \\ 0 & x_1 \end{smallmatrix}] \cdot s_1 = \lambda(x_1) s_1 \quad \underline{\lambda = z_1 [\begin{smallmatrix} 1 & 0 \\ 0 & z_1^{-1} \end{smallmatrix}] \mapsto t} \quad x_1, s_1$$

$$\therefore K^G(\mathbb{P}^1) \rightarrow K(\mathbb{P}^1)$$

Step 3. See the  $G$ -action on l.b.

$$\begin{array}{ccc} \mathbb{A}^1 & \longrightarrow & G \\ \downarrow g & & \downarrow g \\ \mathbb{A}^1 & \longrightarrow & G \end{array} \quad \begin{array}{ccc} (x_1, s_1) & \mapsto & ([\begin{smallmatrix} x_1 & -1 \\ 1 & 0 \end{smallmatrix}], s_1) \\ \downarrow & & \downarrow \\ (g(x_1), \lambda(cx+d)s_1) & & ([\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}] [\begin{smallmatrix} x_1 & -1 \\ 1 & 0 \end{smallmatrix}], s_1) \\ \parallel & & \parallel \\ (g(x_1), \lambda(g(x_1)s_1)) & & ([\begin{smallmatrix} g(x_1) & -1 \\ 1 & 0 \end{smallmatrix}], [\begin{smallmatrix} cx+d & -d \\ 0 & cx+d \end{smallmatrix}]^{-1} \cdot s_1) \end{array}$$

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$

Q: What is the  $R(SL_2)$ -module structure on  $K^{SL_2}(P')$ ?

$$x \cdot - : \mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z^{\pm 1}]$$

My answer: the action is induced by  $R(SL_2) \rightarrow R(B) \hookrightarrow K^{SL_2}(P')$

$$\begin{array}{ccccc} \times & R(SL_2) & \times & K^{SL_2}(P') & \longrightarrow & K^{SL_2}(P') \\ \downarrow & \downarrow & & \parallel_S & & \parallel_S \\ z+z^{-1} & R(B) & \times & R(B) & \xrightarrow{\text{multi}} & R(B) \end{array}$$

$$\begin{array}{ccc} \text{so} & \times \cdot - : \mathbb{Z}[z^{\pm 1}] & \longrightarrow \mathbb{Z}[z^{\pm 1}] \\ & f & \longmapsto (z+z^{-1}) \cdot f \end{array}$$

In conclusion, we get

$$\begin{array}{ccc}
 K^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1) & \longrightarrow & K^{SL_2 \times \mathbb{C}^\times}(pt) \\
 \downarrow & & \downarrow \\
 K^{SL_2}(\mathbb{P}^1) & \longrightarrow & K^{SL_2}(pt) \\
 \downarrow & & \downarrow \\
 K^B(\mathbb{P}^1) & \longrightarrow & K^B(pt) \\
 \downarrow & & \downarrow \\
 K(\mathbb{P}^1) & \longrightarrow & K(pt)
 \end{array}$$

$$\begin{array}{ccc}
 \mathbb{Z}[z^{\pm 1}, t^{\pm 1}] & \longrightarrow & \mathbb{Z}[x, t^{\pm 1}] \\
 \downarrow & & \downarrow \\
 \mathbb{Z}[z^{\pm 1}] & \longrightarrow & \mathbb{Z}[x] \\
 \downarrow & & \downarrow \\
 \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2 & \longrightarrow & \mathbb{Z}[y^{\pm 1}] \\
 \downarrow & & \downarrow \\
 \mathbb{Z}[z]/(z-1)^2 & \longrightarrow & \mathbb{Z} \\
 \downarrow & & \downarrow \\
 \mathbb{Z} & \longrightarrow & 1
 \end{array}$$

The difficult part is the middle square.

Down:

$$\begin{array}{ccc}
 \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2 & \longrightarrow & \mathbb{Z}[y^{\pm 1}] \\
 z & \longmapsto & 1 \\
 y & \longmapsto & y \\
 y^{-1} & \longmapsto & y^{-1}
 \end{array}$$

Right: by rep theory,

$$\begin{array}{ccc}
 \mathbb{Z}[x] & \longrightarrow & \mathbb{Z}[y^{\pm 1}] \\
 x_0 & \longmapsto & 1 \\
 x_1 & \longmapsto & y + y^{-1} \\
 x_2 & \longmapsto & y^2 + 1 + y^{-2} \\
 x_3 & \longmapsto & y^3 + y + y^{-1} + y^{-3} \\
 \vdots & & \vdots
 \end{array}$$

homo as  $\mathbb{Z}$ -alg

Up: by Borel-Weil-Bott theorem,

$$\begin{array}{ccc}
 \mathbb{Z}[z^{\pm 1}] & \longrightarrow & \mathbb{Z}[x] \\
 1 & \longmapsto & 1 \\
 z^{-1} & \longmapsto & x_1 \\
 z^{-2} & \longmapsto & x_2 \\
 z^{-3} & \longmapsto & x_3 \\
 \vdots & & \vdots
 \end{array}$$

$$\begin{array}{ccc}
 z & \longmapsto & 0 \\
 z^2 & \longmapsto & -1 \\
 z^3 & \longmapsto & -x_1 \\
 z^4 & \longmapsto & -x_2 \\
 z^5 & \longmapsto & -x_3 \\
 \vdots & & \vdots
 \end{array}$$

homo as  $\mathbb{Z}[x]$ -module.

Left: by [LW-BWB, Ex 2.6],  $L_n \cong \mathcal{O}(-n)$ , combined with "Up", we get

$$\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2$$

e.g.  $z^3 \longmapsto -z^3(y+y^{-1})$  (see table below)

$z$	$z^{-2}$	$z^{-1}$	$1$	$z$	$z^2$	$z^3$	$z^4$
$x$	$x_2$	$x_1$	$1$	$0$	$-1$	$-x_1$	$-x_2$
$y$	$y^2+1+y^{-2}$	$y+y^{-1}$	$1$	$0$	$-1$	$-y-y^{-1}$	$-y^2-1-y^{-2}$

$$\begin{cases} -x_{m-2} \\ 0 \\ x_{-m} \end{cases} z^m \begin{matrix} m \geq 2 \\ m=1 \\ m \leq 0 \end{matrix} = \frac{y^m - y^{-m+2}}{y^2 - 1}$$

Under these (natural) ring structure,

$$\mathbb{Z}[x, t^{\pm 1}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm 1}] \longrightarrow \mathbb{Z}$$

are homo of rings.

Ex. Generalize to

$$\bullet SL_2 \rightsquigarrow SL_n, \quad \mathbb{P}^1 \rightsquigarrow \text{Flag}(\mathbb{C}^n)$$

$$\bullet SL_2 \rightsquigarrow GL_2$$

$$\bullet \mathbb{C} \rightsquigarrow \mathbb{F}_p \quad \mathbb{C}^\times \rightsquigarrow \mathbb{F}_p^\times$$

Q: How to compute  $K_i^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1)$  for  $i \geq 1$ ?