# Eine Woche, ein Beispiel

12.17 calculation of NMD

Goal: compute normal Morse data (NMD) $\qquad \{f\geq 0\} \overset{\tilde{i}}{\hookrightarrow} X \overset{\tilde{j}}{\hookleftarrow} \{f<0\}$

$$
\begin{aligned}
\mathrm{NMD}(\mathcal{F}^{\cdot}, S) &= \left(R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathcal{F}^{\cdot}|_{N\cap X})\right)_x \\
&\overset{\substack{S=\{*\} \\ X \text{ is cone} \\ f(x)=0 \\ \text{compatible}}}{=\!=} \left(R\Gamma_{\{f\geq 0\}}(\mathcal{F}^{\cdot})\right)_x \\
&= \iota_x^* \, i^! \mathcal{F}^{\cdot} \\
&= R\Gamma(X, \{f<0\}, \mathcal{F}^{\cdot}) \\
&= \mathrm{Fiber}\left(R\Gamma(X, \mathcal{F}^{\cdot}) \longrightarrow R\Gamma(\{f<0\}, \mathcal{F}^{\cdot})\right) \\
&= \mathrm{Fiber}\left(\mathcal{F}_x \longrightarrow R\Gamma(\iota_x, \mathcal{F}^{\cdot})\right)
\end{aligned}
$$

E.g. $X = \mathbb{CP}^1$ $\quad f : \mathbb{CP}^1 \dashrightarrow \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\quad L_x = \{*\}$ $\quad S = \{\infty\}$

$\infty \longmapsto 0$

| $\mathcal{F}$ | $NMD(\mathcal{F}^\cdot, S)$ | $\mathcal{F}_x$ | $R\Gamma(L_x, \mathcal{F}^\cdot)$ |
|---|---|---|---|
| $i_* \underline{\mathbb{Q}}_{\{\infty\}}$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $0$ |
| $\underline{\mathbb{Q}}_{\mathbb{CP}^1}[1]$ | $0$ | $\mathbb{Q}[1]$ | $\mathbb{Q}[1]$ |
| $Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}[1]$ | $\mathbb{Q}$ | $\mathbb{Q} \oplus \mathbb{Q}[1]$ | $\mathbb{Q}[1]$ |
| $j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1]$ | $\mathbb{Q}$ | $0$ | $\mathbb{Q}[1]$ |
| $P(\phi)$ | $\mathbb{Q}^2$ | $\mathbb{Q}$ | $\mathbb{Q}[1]$ |

E.g. $X = \{z_2^2 = z_1^3\}$ $\quad f : X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\quad L_x = \{a, b\}$ $\quad S = \{0\}$

| $\mathcal{F}$ | $NMD(\mathcal{F}^\cdot, S)$ | $\mathcal{F}_x$ | $R\Gamma(L_x, \mathcal{F}^\cdot)$ |
|---|---|---|---|
| $i_* \underline{\mathbb{Q}}_Z$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $0$ |
| $\underline{\mathbb{Q}}_X[1]$ | $\mathbb{Q}$ | $\mathbb{Q}[1]$ | $\mathbb{Q}^2[1]$ |
| $Rj_* \underline{\mathbb{Q}}_U[1]$ | $\mathbb{Q}^2$ | $\mathbb{Q} \oplus \mathbb{Q}[1]$ | $\mathbb{Q}^2[1]$ |
| $j_! \underline{\mathbb{Q}}_U[1]$ | $\mathbb{Q}^2$ | $0$ | $\mathbb{Q}^2[1]$ |
| $P(\phi)$ | $\mathbb{Q}^3$ | $\mathbb{Q}$ | $\mathbb{Q}^2[1]$ |

E.g. $X = \mathbb{C} \cup_{\{0\}} \mathbb{C} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$

$f: X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1+z_2} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\qquad L_x = \{a, b\}$ $\qquad S = \{0\}$

| $\mathcal{F}$ | $NMD(\mathcal{F}^\bullet, S)$ | $\mathcal{F}_x$ | $R\Gamma(L_x, \mathcal{F}^\bullet)$ |
|---|---|---|---|
| $i_* \mathbb{Q}_Z$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $0$ |
| $\mathbb{Q}_X[1]$ | $\mathbb{Q}$ | $\mathbb{Q}[1]$ | $\mathbb{Q}^2[1]$ |
| $Rj_* \mathbb{Q}_U[1]$ | $\mathbb{Q}^2$ | $\mathbb{Q}^2 \oplus \mathbb{Q}^2[1]$ | $\mathbb{Q}^2[1]$ |
| $j_! \mathbb{Q}_U[1]$ | $\mathbb{Q}^2$ | $0$ | $\mathbb{Q}^2[1]$ |
| $\pi^! \mathbb{Q}[-1]$ | $\mathbb{Q}$ | $\mathbb{Q} \oplus \mathbb{Q}^2[1]$ | $\mathbb{Q}^2[1]$ |
| $IC(\mathbb{Q}_U[1])$ | $0$ | $\mathbb{Q}^2[1]$ | $\mathbb{Q}^2[1]$ |

E.g. $X = X_3$ $\qquad f: X \hookrightarrow \mathbb{C}^3 \xrightarrow{z_3} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\qquad L_x = \mathbb{C}^\times$ $\qquad S = \{0\}$

| $\mathcal{F}$ | $NMD(\mathcal{F}^\bullet, S)$ | $\mathcal{F}_x$ | $R\Gamma(L_x, \mathcal{F}^\bullet)$ |
|---|---|---|---|
| $i_* \mathbb{Q}_Z$ | $\mathbb{Q}$ | $\mathbb{Q}$ | $0$ |
| $\mathbb{Q}_X[2] = \pi^! \mathbb{Q}[-2]$ | $\mathbb{Q}$ | $\mathbb{Q}[2]$ | $\mathbb{Q}[1] \oplus \mathbb{Q}[2]$ |
| $Rj_* \mathbb{Q}_U[2]$ | $\mathbb{Q} \oplus \mathbb{Q}[-1]$ | $\mathbb{Q}[2] \oplus \mathbb{Q}[-1]$ | $\mathbb{Q}[1] \oplus \mathbb{Q}[2]$ |
| $j_! \mathbb{Q}_U[2]$ | $\mathbb{Q} \oplus \mathbb{Q}[1]$ | $0$ | $\mathbb{Q}[1] \oplus \mathbb{Q}[2]$ |
| $IC(\mathbb{Q}_U[2])$ | $\mathbb{Q}$ | $\mathbb{Q}[2]$ | $\mathbb{Q}[1] \oplus \mathbb{Q}[2]$ |

**Setting** $M$ : analytic mfld e.g. $M = \mathbb{C}^n$ or $\mathbb{CP}^n$

$X \subset M$ analytic variety of $\dim_{\mathbb{C}} X = m$

$\mathcal{S}: \emptyset \subset \{o\} \subset X$ where $o$ is the only singularity

$x_0 \in X - \{o\}$

$\mathcal{F} \in Perv_{\mathcal{S}}(X)$

$\mathcal{L} := \mathcal{F}|_{X-\{o\}}[-m]$ local system on $X - \{o\}$ with rank $r$

Special case: $\mathcal{F} = IC(\mathcal{L}[m])$

**Task**: Compute the following quantities.

$$
\begin{tikzcd}
{\text{(1)}\,NMD(\mathcal{F}, \{o\})} \arrow[d] & & {\text{(3)}\,NMD(\mathcal{F}, X-\{o\})} \arrow[d] & & \\
{\text{(2)}\,m(\{o\})} \arrow[dr] & & {\text{(4)}\,m(X-\{o\})} \arrow[dl] & \text{(6)}\,Eu_X \arrow[d] & Eu_{\{o\}} \arrow[d] \\
& \text{(5)}\,CC(\mathcal{F}^{\cdot}) \arrow[dr] & & \text{(7)}\,\deg(\Lambda_X) \arrow[dl] & \deg(\Lambda_{\{o\}}) \arrow[dll] \\
& & \text{(8)}\,\chi_{IC}(X) & &
\end{tikzcd}
$$

Here we use notations in https://arxiv.org/abs/2105.13069v2. 6-8 comes from my supervisor's notation, if needed I should find some references for the definition.

① See the examples before

② $m(\{0\}) = \chi(NMD(\mathcal{F}, \{0\}))$

③ $NMD(\mathcal{F}, X-\{0\}) \cong \mathcal{F}_{x_0} \cong \mathbb{Q}^r[m]$

④ $$\begin{aligned} m(X-\{0\}) &= (-1)^{\dim_{\mathbb{C}}(X-\{0\})} \chi(NMD(\mathcal{F}, X-\{0\})) \\ &= (-1)^m \cdot (-1)^m \cdot r \\ &= r \end{aligned}$$

⑤ $$\begin{aligned} CC(\mathcal{F}) &= m(X-\{0\}) \, [\overline{T^*_{X-\{0\}}M}] + m(\{0\}) \, [\overline{T^*_{\{0\}}M}] \\ &= r \, [T^*_X M] + m(\{0\}) \, [T^*_{\{0\}}M] \\ &= r \, \Lambda_X + m(\{0\}) \, \Lambda_{\{0\}} \end{aligned}$$

recall: $[T^*_X M] = [\overline{T^*_{X-\{0\}}M}]$ $\Lambda_{\overline{S}} := [\overline{T^*_S M}]$

⑥ Need to check the definition.

For $X \subset \mathbb{C}^2$ cuspidal cubic, $\mathrm{Sing}(X) = \{p_0\}$,

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X - \{p_0\} \\ 2 & p = p_0 \end{cases}$$

In general. from my memory it looks like:

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X_{sm} \\ \geq 1 & p \in X - X_{sm} \end{cases}$$

⑦ $\deg(\Lambda_X) := \#(\Lambda_X \cdot \Lambda_M)$ in $T^*M$

$= (-1)^m \chi(X, Eu_X)$

$= (-1)^m \left(\chi(X-\{0\}) \cdot Eu_X(x_0) + \chi(\{0\}) \cdot Eu_X(0)\right)$

$= (-1)^m \left(\chi(X-\{0\}) + Eu_X(0)\right)$ $= -2$ for $X = M = \mathbb{CP}^1$

$\deg(\Lambda_{\{0\}}) := \#(\Lambda_{\{0\}} \cdot \Lambda_M)$

$= \chi(\{0\}, Eu_{\{0\}})$

$= \chi(\{0\}) \cdot Eu_{\{0\}}(0)$

$= 1$

⑧ $(-1)^m \chi_{IC}(X) = \deg(CC(\mathcal{F}))$ Here, $\mathcal{F} := IC(\underline{\mathbb{Q}}_{X-\{0\}}[m])$, $r = 1$

$= \deg(r\Lambda_X + m(\{0\})\Lambda_{\{0\}})$

$= r \cdot \deg \Lambda_X + m(\{0\}) \deg \Lambda_0$

$= \deg \Lambda_X + m(\{0\})$

$\Rightarrow \quad \chi_{IC}(X) = \chi(X-\{0\}) + Eu_X(0) + (-1)^m m(\{0\})$

| $X$ | $\chi(X-\{0\})$ | $Eu_X(0)$ | $m(\{0\})$ | $\chi(X)$ |
|---|---|---|---|---|
| $\mathbb{C}$ | 0 | 1 | 0 | 1 |
| $\{y^2 = x^3\}$ | 0 | 2 | 1 | 1 |