Eine Woche, ein Beispiel

5.18 theta functions

cohomology of $\mathcal{L} \in \mathrm{Pic}(A)$

Ref: follows [2025.05.04].
Most contents in this document can be found in [BL04, Chap 3].

[BL04]	My notation
$H(u,v)$	$H(v,u)$
V_1	V_2
V_2	V_1

Rmk: For $H \in NS(A)$ nondegenerate,
when we fix an isotropic dec $V = V_2 \oplus V_1$, i.e., $H(V_i, V_i) \equiv 0$
we can get a canonical lift
$$\mathcal{L} = \mathcal{L}(H, \chi_0) \in \mathrm{Pic}(A)$$
given by
$$\chi_0(v_1 + v_2) = \exp(\pi i \,\mathrm{Im}\, H(v_1, v_2)).$$
See [BL04, Lemma 3.1.2].

Q: Is that still true when H is not nondegenerate?

Def (characteristic) $c \in V/\Lambda(L) = \mathrm{Im}\, \varphi_A$ is called the char of $\mathcal{L}$, when

$$\chi(v) = \chi_0(v) \exp(2\pi i \,\mathrm{Im}\, H(v, c)) \quad \Leftrightarrow \mathcal{L} \cong t_c^* \mathcal{L}_0$$
$$= \exp\{2\pi i \,\mathrm{Im}(\tfrac{1}{2} H(v_1, v_2) + H(v, c))\}$$
$$= \exp\{2\pi i \,\mathrm{Im}(\tfrac{1}{2} H(v_1, v_2) + H(v_1, c_2) + H(v_2, c_1)\}$$

$V = V_1 \oplus V_2$

$v = v_1 + v_2$

$c = c_1 + v_2$

We also define B as the $\mathbb{C}$-bilinear extension of $H|_{V_1 \times V_1}$.

Factor of automorphy and theta fcts

Canonical factor of automorphy for $\mathcal{L} = \mathcal{L}(H, \chi)$:

$$a_L : \Lambda \times V \longrightarrow \mathbb{C}^\times$$
$$a_L(\lambda, v) = \chi(u) \exp\left(\pi H(\lambda, v) + \tfrac{\pi}{2} H(\lambda, \lambda)\right)$$

Classical factor of automorphy crspds to other l.b.

$$e_L : \Lambda \times V \longrightarrow \mathbb{C}^\times$$
$$\begin{aligned} e_L(\lambda, v) &= \chi(u) \exp\left(\pi (H-B)(\lambda, v) + \tfrac{\pi}{2}(H-B)(\lambda, v)\right) \\ &= a_L(\lambda, v) \exp\left(-\pi B(\lambda, v) - \tfrac{\pi}{2} B(\lambda, \lambda)\right) \\ &= a_L(\lambda, v) \exp\left(\tfrac{\pi}{2} B(v, v) - \tfrac{\pi}{2} B(\lambda + v, \lambda + v)\right) \end{aligned}$$

Canonical theta fct c: characteristic of $\mathcal{L}$

$$\theta^c(v) = \exp\left(-\pi H(c, v) - \tfrac{\pi}{2} H(c, c) + \tfrac{\pi}{2} B(v+c, v+c)\right) \cdot \sum_{\lambda \in \Lambda \cap V_1} \exp\left(\pi (H-B)(\lambda, v+c) - \tfrac{\pi}{2}(H-B)(\lambda, \lambda)\right)$$

$$\theta^c(v + \lambda) = a_L(\lambda, v)\, \theta^c(v)$$

Classical theta fct [BL04, p223] $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^n$

$$\theta\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}(v, Z) = \sum_{l \in \mathbb{Z}^n} \exp\left(\pi i (l + \varepsilon_1)^T Z (l + \varepsilon_1) + 2\pi i (v + \varepsilon_2)^T (l + \varepsilon_1)\right)$$

$$\theta^{Z\varepsilon_1 + \varepsilon_2}(v) = \exp\left(\tfrac{\pi}{2} B(v, v) - \pi i\, \varepsilon_1^T \varepsilon_2\right) \theta\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}(v, Z)$$

$$\begin{aligned} \theta\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}(v + \lambda, Z) &= a_L(\lambda, v) \exp\left(-\tfrac{\pi}{2} B(v+\lambda, v+\lambda) + \tfrac{\pi}{2} B(v, v)\right) \theta\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}(v, Z) \\ &= e_L(\lambda, v)\, \theta\begin{bmatrix}\varepsilon_1 \\ \varepsilon_2\end{bmatrix}(v, Z) \end{aligned}$$