Eine Woche, ein Beispiel

9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

- vector bundle over manifold
- module over ring

There are hundreds of different versions of it:

- vector bundle over manifold 几何/几何分析
  - diffe mfld · (real) differential v.b. over (real) differential mfld
  - Riemann surface · cplx (analytic) line bundle over Riemann surface
- sheaf over space 代数几何
  - scheme theory · locally free sheaf on scheme
  - · coherent sheaf on scheme
  - geo rep theory · local system over (real/cplx) mfld
  - · perverse sheaf over Riemann surface (derived)
  - · simplicial set over category Δ
- module over ring 代数
  - comm alg · f.g module over Noetherian commutative ring (with 1)
  - rep of grp · group representation over group ($\rightsquigarrow$ group algebra)
  - p-adic rep · smooth representation over unimodular gp ($\rightsquigarrow$ Hecke algebra $\mathcal{H}(G)$) smooth module
  - quiver theory · quiver representation over quiver ($\rightsquigarrow$ path algebra, bound quiver algebra)
  - Lie algebra · Lie alg representation over Lie alg ($\rightsquigarrow$ universal enveloping algebra)
- Arithmetic Geometry 算数→p进分析
  - · hermitian line bundle over projective arithmetic variety $\mathcal{X}$
  - · adelic line bundle over essentially quasi-proj scheme
  - · over Berkovich analytic space X^{an}
  - · over formal scheme $\mathrm{Spf} A$
  - · over rigid-analytic space K-affinoid space
  - · over adic space $\mathrm{Spa}(A, A^+)$

Picture:

$$\begin{tikzcd}
\text{v.b.} \arrow[d, dash] & \text{sheaf} \arrow[d, dash] & \text{mod} \arrow[d, dash] & \text{rep} \arrow[d, dash] \\
\text{mfld} & \text{scheme} & \text{ring} & \text{alg}
\end{tikzcd}$$

① variation (e.g. v.b → f.b, mfld → CW cplx, sheaf → fctor, scheme → stack/adic space, ...)

② vertical relation: ↓: v.b as mfld, representable fct, Spec/Proj construction, ...

↑: tangent/trivial v.b, structure sheaf, R as R-mod, regular rep, ...

③ horizontal relation:

$$\begin{tikzcd}
\text{v.b.} \arrow[d, dash] \arrow[r, shift right, "\Gamma(-,U)"'] & \text{sheaf} \arrow[d, dash] \arrow[l, dashed, shift right, "\mathrm{Spec}"'] & \text{mod} \arrow[d, dash] \arrow[l, "M \mapsto \widetilde{M}"'] & \text{rep} \arrow[d, dash] \arrow[l] \\
\text{mfld} \arrow[r, dashed, shift right, "\text{model}"'] & \text{scheme} \arrow[l, shift right, "\otimes_{\mathbb{Z}} \mathbb{C}"'] & \text{ring} \arrow[l, dashed, "\mathrm{Spec}"'] & \text{alg} \arrow[l]
\end{tikzcd}$$

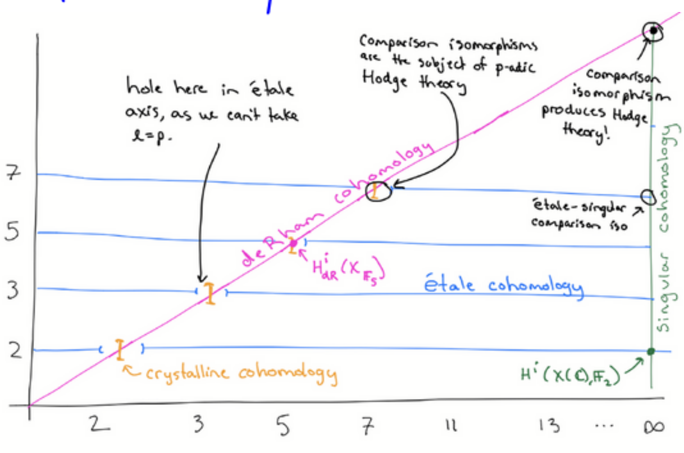
④ homology and cohomology: ⇝ derived category

v.b. — mfld: {singular, simplicial, cellular, Borel-Moore, de Rham, cpt supported}

sheaf — scheme: Sheaf, Čech

mod — ring: Ext^n, Tor^n, Hochchild

rep — alg: {gp / Galois, Lie-alg}



when axes meet: comparison isomorphisms (the "glue" of the "sheaf")

Prof. **Scholze's ICM picture**

https://www.youtube.com/watch?v=5NPFQvdav9o

Remaining (co)homology:

l-adic cohomology
intersection (co)homology
elliptic cohomology
flat cohomology
infinitesimal cohomology

Objects in upper row can be already viewed as element in (co)homology.

eg. v.b. ↔ transition fct ↔ $H^1(X,-)$

One motivation for ∞-category: make a generalizaton from H^1 to H^i

v.b. (differential form, metrics, curvature) — mfld (measure, submfld, homotopy class)

sheaf — scheme (divisor, subvariety, torsor)

mod (Ext, Tor) — ring

rep (L-fct) — alg

The following two pictures comes from here: https://guests.mpim-bonn.mpg.de/gallauer/docs/m6ff.pdf

Coefficients	cohomology groups	
$\mathsf{D}^b_c(X;\mathbb{Q}_\ell)$ *constructible ℓ-adic sheaves*	*ℓ-adic cohomology*	✓
$\mathsf{D}^b_c(X(\mathbb{C});\mathbb{Z})$ *constructible analytic sheaves*	*Betti cohomology*	✓
$\mathsf{D}^b_h(\mathcal{D}_X)$ *holonomic $\mathcal{D}$-modules*	*de Rham cohomology*	✓
$\mathsf{D}^b(\mathrm{Coh}(X))$ *coherent sheaves* ✓	*coherent cohomology*	✓
$\mathsf{D}^b(\mathsf{MHM}(X))$ *mixed Hodge modules*	*absolute Hodge cohomology*	
$\mathsf{DM}(X)$ *Voevodsky motivic sheaves*	*(weight-0) motivic cohomology*	
$\mathsf{SH}(X)$ *stable motivic homotopy sheaves*	*stable motivic (weight-0) cohomotopy groups*	

⑤ Relative point of view (for (co)homology)
Six functors formalism (all are derived)

cohomology	$p_* p^* \mathbb{1}$	$H^\bullet$	
cohomology with compact support	$p_! p^* \mathbb{1}$	$H_c^\bullet$	
homology	$p_! p^! \mathbb{1}$	$H_\bullet$	
Borel-Moore homology	$p_* p^! \mathbb{1}$	$H_\bullet^{BM}$	

$p_* \mathcal{F}$ $\quad H^\cdot(-, \mathcal{F})$

$p_! \mathcal{F}$ $\quad H^\cdot_c(-, \mathcal{F})$

Chern class : from cohomology to cohomology (also for the other Char class)

There are several ways of defining/viewing Chern class.

i) $\mathcal{L} \in \mathrm{Pic}_{\mathbb{C}}(X) \longmapsto c_1(\mathcal{L}) \in H^2(X;\mathbb{Z})$

ii) $H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X;\mathbb{Z})$ by LES

iii) As the coefficient of equation ($CH^*(\mathbb{P}E)$ is a free $CH^*(B)$-module)
Euler class

iv) As the pull back of the universal Chern class in Grassmannian

v) From curvature ; Chern-Weil theory

vi) From Chow group

vii) $\partial\bar{\partial}$, Δ

⑥ moduli problems

$$\begin{tikzcd} \text{v.b.} \arrow[d, dash] & \text{sheaf} \arrow[d, dash] & \text{mod} \arrow[d, dash] & \text{rep} \arrow[d, dash] & & & o \arrow[d, dash] & x\text{-}o \arrow[d, dash] \\ \text{mfld} & \text{scheme} & \text{ring} & \text{alg} & o & x\text{-}o & x & x \end{tikzcd}$$

Three type of geometry:

PDE	elliptic	parabolic	hyperbolic
curvature	+	0	−
genus	0	1	≥ 2
Euler number	−2	0	≥ 2
Kodaira dim	$-\infty$	0	dim X
variety	Fano	Calabi-Yau	general type
filstration	unramified	tame	wild
quiver	Dynkin	affine	strictly wild
condensed	solid	liquid	gaseous

Goal
- structures & invariants
- classifications of
  special v.b, mfld, sub v.b, submfld
- symmetry & quotient
- special functors
- homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n.

(by homotopy group! $\rightsquigarrow$ generalized Picard group?)

Q: Is this group structure natural?

ref: https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres

Thm. $\{\text{rank } m \ \ K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $\qquad K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ rank m / n	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/12\mathbb{Z}$	$(\mathbb{Z}/12\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{R}P^\infty \simeq K(\mathbb{Z}/2\mathbb{Z},1)$

$\pi_{n-1}(GL_m(\mathbb{C}))$ rank m / n	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{C}P^\infty \simeq K(\mathbb{Z},2)$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes$, Λ^k, Sym^k, Res, Ind

For the other spaces:
https://math.stackexchange.com/questions/383838/classifying-vector-bundles
http://www.ms.uky.edu/~guillou/F18/751Notes.pdf
It's still not so explicit.

$\{\text{rank } m \ \ K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_K(m,\infty)]$ $\qquad K = \mathbb{R}, \mathbb{C}$, M: paracompact

Unfinished task: introduce the concept of local systems and compute examples in [https://arxiv.org/pdf/2103.02329.pdf] ,16.3.