

# Eine Woche, ein Beispiel

## 4.28 naive $\otimes$ -Hom adjunction

- Notation.
- $A$ : associate ring allowed to be non-commutative, contains 1
  - There are two systems to write category of  $A$ -modules:

$$\begin{aligned} \text{Mod}_A &= A\text{-Mod} & \ni {}_A M \\ (\text{Mod}_A)^{\text{op}} &\neq \text{Mod}_{A^{\text{op}}} = \text{Mod}-A = A^{\text{op}}\text{-Mod} & \ni M_A \\ \text{Mod}_{A \otimes B^{\text{op}}} &= A\text{-Mod}-B & \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\text{Mod}_{B \otimes A^{\text{op}}})^{\text{op}} = (B\text{-Mod}-A)^{\text{op}} = (A^{\text{op}}\text{-Mod}-B^{\text{op}})^{\text{op}} \ni {}_B M_A$$

as

$$(\text{Mod}_{A \otimes B^{\text{op}}})^{\overline{\text{op}}} = (A\text{-Mod}-B)^{\overline{\text{op}}}$$

Even though you can identify  $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{\text{op}})$ ,  
 $A^{\text{op}} \notin \text{Ob}(\text{Ring}^{\text{op}})$ ,  $A^{\text{op}}$  is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For  $A$  comm,  $\text{Mod}_A = \text{Mod}_{A^{\text{op}}} \subset \text{Sh}(\text{Spec } A)$ .

In this case, it is desirable to translate algebraic results into geometrical results.  
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that  $A$  is a commutative ring for convenient.

1 definition recall for  $\otimes$  & Hom

2. adjunction

3. comparison between  $\otimes\text{-I Hom}$  &  $f^*\text{-I } f_*$

4. definition recall for  $\otimes$  & Hom

, derived version

5. adjunction

, derived version

6. comparison between  $\otimes\text{-I Hom}$  &  $f^*\text{-I } f_*$

, derived version

1. definition recall for  $\otimes$  & Hom

$$\begin{aligned}\otimes_A : \text{Mod}_{A^{\text{op}}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}} \\ \text{Hom}_A(-, -) : (\text{Mod}_A)^{\text{op}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}}\end{aligned}$$

In general,

$$\begin{aligned}\otimes_B : A\text{-Mod}-B \times B\text{-Mod}-C &\longrightarrow A\text{-Mod}-C \\ \text{Hom}_B(-, -) : (A\text{-Mod}-B)^{\text{op}} \times B\text{-Mod}-C &\longrightarrow A\text{-Mod}-C \\ \text{Hom}_B^A(-, -) : (A\text{-Mod}-B)^{\text{op}} \times B\text{-Mod}-A &\longrightarrow \mathbb{Z}\text{-Mod} \\ \text{Hom}_{B \otimes_{\mathbb{Z}} A^{\text{op}}}(-, -) : (\mathbb{Z}\text{-Mod}-B \otimes_{\mathbb{Z}} A^{\text{op}})^{\text{op}} \times (B \otimes_{\mathbb{Z}} A^{\text{op}}\text{-Mod}-\mathbb{Z})^{\text{op}} &\longrightarrow \mathbb{Z}\text{-Mod}-\mathbb{Z}\end{aligned}$$

$${}_A X_B \cdot {}_B Y_C \cdot {}_C Z_D$$

associativity:  $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity":  $X \otimes_B Y \cong Y \otimes_{B^{\text{op}}} X \quad \text{in } A\text{-Mod}-C = C^{\text{op}}\text{-Mod}-A^{\text{op}}$

"unit":  $A \otimes_A X \cong X \cong X \otimes_B B$

$\text{Hom}_A(A, X) \cong X$

2. adjunction  $\begin{smallmatrix} X_A \\ \downarrow \\ {}_B X_A, {}_C Y_B, {}_C Z_D \end{smallmatrix}$ . we get

$$\text{Hom}_C(Y \otimes_B X, Z) \cong \text{Hom}_B(X, \text{Hom}_C(Y, Z)) \quad \text{in } A\text{-Mod-D.}$$

Reason: both sides equal to the set

$$\{ f: Y \times X \rightarrow Z \mid f(cy, x) = cf(y, bx) \quad \forall b, c \}$$

For  $A=D=\mathbb{Z}$ , fix  $Y \in C\text{-Mod-B}$ , one gets adjunction factors.

$$\begin{array}{ccc} & \xrightarrow{Y \otimes_B -} & \\ B\text{-Mod} & \perp & C\text{-Mod} \\ & \xleftarrow{\text{Hom}_C(Y, -)} & \end{array}$$

slogan: adjunction  $\approx$  associativity

$\otimes \dashv \text{Hom}_B$

$$(A\text{-Mod}-B)^{\overline{\text{op}}} \times (B\text{-Mod}-C)^{\overline{\text{op}}} \times C\text{-Mod}-D \xrightarrow{(\text{Id}, \text{Hom}_C)} (A\text{-Mod}-B)^{\overline{\text{op}}} \times B\text{-Mod}-D$$

$$\parallel$$

$$(A\text{-Mod}-B) \times (B\text{-Mod}-C)^{\overline{\text{op}}} \times C\text{-Mod}-D$$

$$(\otimes_B, \text{Id})$$

$$\text{Hom}_B$$

$$(A\text{-Mod}-C)^{\overline{\text{op}}}$$

$$\times C\text{-Mod}-D \xrightarrow{\text{Hom}_C}$$

$$A\text{-Mod}-D$$

$$\begin{matrix} g & f \\ | & | \\ Y & \xrightarrow{f} X \end{matrix}$$

$f^* \dashv f_*$

$$\text{Hom}(f^*F, G) \cong \text{Hom}(F, f_*G)$$

$$Sh(X)^{\text{op}} \times \text{Mor}(Y, X) \times Sh(Y) \xrightarrow{(\text{Id}, \text{pushforward})} Sh(X)^{\text{op}} \times Sh(X)$$

$$(\text{pullback}, \text{Id})$$

$$\text{Hom}_{Sh(X)}(-, -)$$

$$Sh(Y)^{\text{op}} \times Sh(Y)$$

$$\xrightarrow{\text{Hom}_{Sh(Y)}(-, -)}$$

$$\downarrow$$

$$\downarrow$$

$$\text{Abel}$$

$$(F, f, g) \longmapsto (F, f_*g)$$

$$\downarrow$$

$$(f^*F, g) \longmapsto \text{Hom}_{Sh(Y)}(f^*F, g) \cong \text{Hom}_{Sh(X)}(F, f_*g)$$

$f_! \dashv f^!$ , similar.

3. comparison between  $\otimes$ -Hom &  $f^* \dashv f_*$

Forgetful factor

Prop. For ring homo  $\begin{array}{c} S \\ \uparrow f \\ R \end{array}$ ,  $\exists$  "forgetful factor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \quad M \mapsto u(M)$$

$$u(M) = {}_R S_S \otimes_S M = \text{Hom}_S(S_R, M)$$

one has adjunction factors

$$\begin{array}{ccccc} & & S_S \otimes_R - & & \\ & \swarrow & \downarrow & \searrow & \\ S\text{-Mod} & \xrightleftharpoons[u=]{{}_R S_S \otimes_S -} & \text{Hom}_S(S_R, -) & \xrightleftharpoons[\downarrow]{} & R\text{-Mod} \\ & \searrow & \downarrow & \swarrow & \\ & & \text{Hom}_R({}_R S_S, -) & & \end{array} \quad (3.1)$$

Compare with  $j$

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{array}{ccccc} & & j_! & & \\ & \swarrow & \downarrow & \searrow & \\ D(X) & \xrightleftharpoons[j^*]{} & j^! & \xrightleftharpoons[\downarrow]{} & D(U) \\ & \searrow & \downarrow & \swarrow & \\ & & Rj_* & & \end{array}$$

Vague slogan:  $u \approx \text{"forget the information of } Z\text{"}$ .

In applications,  $U \rightarrow X$  is a covering map.  
This change the feeling of the size between  $U$  &  $X$ .

E.g. For finite gps  $H \leq G$ , one has Res - Ind adjunction:

$$\begin{array}{c} \text{Res}_H^G \dashv \text{Ind}_H^G \\ \text{c-Ind}_H^G \dashv \text{Res}_H^G \end{array}$$

It can be generalized for  $\begin{cases} G: \text{loc profinite gp}, \\ H \leq G \text{ open} \end{cases}$

If one only has  $H \leq G$  closed, then it's possible that  $j^! \neq j^*$ .

e.g.  $G = GL_2(\mathbb{Q}_p)$   $H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{array}{ccccc} & & c\text{-Ind}_H^G & & \\ & \swarrow & \perp & \searrow & \\ \text{Rep}_G & \xrightarrow{\quad} & \text{Res}_H^G & \xrightarrow{\quad} & \text{Rep}_H \\ & \searrow & \perp & \swarrow & \\ & & \text{Ind}_H^G & & \end{array}$$

Ex. Compare it with the recollement diagram & (3.1).

$$\begin{array}{ccc} U & [\ast/H] & \\ \downarrow j & \downarrow & \text{"cover with fiber } G/H\text{"} \\ X & [\ast/G] & \end{array}$$

translate the following geometrical results into algebraic statements.

1. One has natural factor  $j_! \longrightarrow j_*$ . When  $\# G/H < +\infty$ ,  $j_! = j_*$   
 $c\text{-Ind}_H^G \longrightarrow \text{Ind}_H^G$   $c\text{-Ind}_H^G = \text{Ind}_H^G$

2. Even though

$\text{Sh}_{\mathbb{Q}, v}([\ast/G]) \approx \text{Rep}_G = \mathbb{Q}[G]\text{-Mod}$ ,  
the "structure sheaf" of  $[\ast/G]$  is  $\mathbb{Q}$ , not  $\mathbb{Q}[G]$ .

$$\text{Res}_{\ast/G}^G \mathbb{Q} = \mathbb{Q}, \quad \text{Res}_{\ast/G}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example,  $j^* Rj_* \neq \text{Id}$ ,  $j^! j_! \neq \text{Id}$ .

Until now, we have met three types of six factor formalism: top spaces, A-modules and stacks.

Compare with i

Now, assume  $S, R$  commutative in the scheme setting.

E.g. For ring homo

$$\begin{array}{ccc} S & & \text{Spec } S \\ \uparrow \tilde{f} & & \downarrow f \\ R & \xrightarrow{M} & \text{Spec } R \end{array}$$

$\exists$  "pullback factor"

$$f^*: R\text{-Mod} \longrightarrow S\text{-Mod} \quad f^*M = {}_S S_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{array}{ccccc} & & f^* & & \\ & \swarrow & \downarrow u & \searrow & \\ S\text{-Mod} & \xrightarrow{\quad} & R\text{-Mod} & \xleftarrow{\quad} & \\ & \searrow & \downarrow \perp & \swarrow & \\ & & \text{Hom}_R({}_R S_S, -) & & \end{array}$$

compare it with another part of the recollement diagram:

$$\begin{array}{ccccc} & & i^* & & \\ & \swarrow & \downarrow i_* & \searrow & \\ D(Z) & \xrightarrow{\quad} & D(X) & \xleftarrow{\quad} & \\ & \searrow & \downarrow i_! & \swarrow & \\ & & i^! & & \end{array}$$

Rmk.  $u$  is usually not  $f$ -faithful, unless  $S = R/I$ .

(In fact, only need  $S$  is  $R$ -idempotent, i.e.  $S \cong {}_S S_R S$ .)

which corresponds to closed embedding.

In that case,

$$i^* i_* = \text{Id}: {}_S S_R \otimes_R ({}_{R S} \otimes_S M) \cong M$$

$$i^! i_* = \text{Id}: \text{Hom}_R({}_R S_S, \text{Hom}({}_S S_R, M)) \cong M$$

**Slogan:** in the comm alg,  $\text{Spec } R/I \longrightarrow \text{Spec } R$  is closed embedding.

In general, if

$S$  is an  $R$ -idempotent algebra.  $S \cong S \otimes_R S$

then  $i: \text{Spec } S \longrightarrow \text{Spec } R$  can be viewed as "closed subset".

E.g.  $R_p, R/I$  are idempotent  $R$ -algs.

$\mathbb{Z}[\frac{1}{p}], \mathbb{F}_p, \mathbb{Z}/p\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_{(p)}, \dots$  are idem  $\mathbb{Z}$ -algs.

▽ Usually  $R/I$  is not an derived idem  $R$ -alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves.  $\text{Spec } R/I$  is open instead?

Rmk.

Following this slogan, original open/closed subsets are all closed. Also,  $i^{\wedge!}$  is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?

A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. definition recall for  $\otimes$  & Hom

, derived version

$F$	$RF$ or $LF$	$R^iF$ or $L^iF$	exact fator
$f^*$	$f^*$	—	
$f_*$	$Rf_*$	$R^i f_*$	$f_*$ -acyclic
$\pi_{X,*} F$	$\Gamma(X, F)$	$H^i(X, F)$	$\Gamma$ -acyclic
$f_!$	$Rf_!$	$R^i f_!$	$f_!$ -acyclic
$\pi_{X,!} F$	$\Gamma_c(X, F)$	$H_c^i(X, F)$	$\Gamma_c$ -acyclic
—	$f^!$	$H^i(f^! -)$	
$- \otimes_R -$	$- \otimes_R^L -$	$Tor_R^i(-, -)$	flat
$\text{Hom}_R(-, -)$	$R\text{Hom}_R(-, -)$	$\text{Ext}_R^i(-, -)$	injective/projective
$M_G$	$Z \otimes_{Z[G]}^L M$	$H_i(G; M)$	
$M_G^G$	$R\text{Hom}_{Z[G]}(Z, M)$	$H^i(G; M)$	
$M_g$	$x \otimes_{Ug}^L M$	$H_i(g; M)$	
$M_g^g$	$R\text{Hom}_{Ug}(x, M)$	$H^i(g; M)$	
$M/[AM]$	$A \otimes_{Ae}^L M$	$HH_i(A, M)$	
$M^A$	$R\text{Hom}_{Ae}(A, M)$	$HH^i(A, M)$	
$A/[AA]$	$A \otimes_{Ae}^L A$	$HH_i(A)$	
$Z(A)$	$R\text{Hom}_{Ae}(A, A)$	$HH^i(A)$	

e.g. group coh

e.g. Lie alg coh

$g/x$ : Lie alg

e.g. Hochschild coh

For calculations, see:

[23.04.09]: gp coh

[wiki]: Lie algebra coh

[21.05.21]: Hochschild coh

[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.

For  $\text{Hom}(-, A)$ , see <https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m>.

Chenji Fu claimed that  $\text{Hom}(-, A)$  always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.

Related:

<https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod>

<https://mathoverflow.net/questions/29442/what-is-the-opposite-category-of-the-category-of-modules-or-hopf-algebra-repres>

<https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution>

4. definition recall for  $\otimes$  & Hom , derived version

To define  $\otimes^L$  &  $R\text{Hom}$ , one needs to extend factors

$$\begin{array}{ccc} \otimes_A: A\text{-Mod} & \times & A\text{-Mod} \\ \text{Hom}_A(-, -): (A\text{-Mod})^{\text{op}} & \times & A\text{-Mod} \end{array} \longrightarrow A\text{-Mod}$$

to factors on double cplxes.

$\mathcal{C}(A)$ : complex of  $A$ -modules, temperate notation

$$\begin{array}{ccc} \otimes_{\mathcal{C}(A)}: \mathcal{C}(A) & \times & \mathcal{C}(A) \\ \text{Hom}_{\mathcal{C}(A)}(-, -): (\mathcal{C}(A))^{\text{op}} & \times & \mathcal{C}(A) \end{array} \longrightarrow \mathcal{C}(A)$$

But how?

Wishes:

$$\begin{aligned} (M^{\cdot}[i]) \otimes_{\mathcal{C}(A)} (N^{\cdot}[j]) &= (M^{\cdot} \otimes N^{\cdot})[i+j] \\ \text{Hom}_{\mathcal{C}(A)}(M^{\cdot}[-i], N^{\cdot}[j]) &= \text{Hom}(M^{\cdot}, N^{\cdot})[i+j] \end{aligned}$$

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N' & \longrightarrow M^0 \otimes N' & \longrightarrow M^1 \otimes N' & \longrightarrow M^2 \otimes N' & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N^0 & \longrightarrow M^0 \otimes N^0 & \longrightarrow M^1 \otimes N^0 & \longrightarrow M^2 \otimes N^0 & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow M^{-1} \otimes N^{-1} & \longrightarrow M^0 \otimes N^{-1} & \longrightarrow M^1 \otimes N^{-1} & \longrightarrow M^2 \otimes N^{-1} & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
 \end{array}$$

$\text{Tot}(M^\cdot \otimes N^\cdot)$ , the double complex of  $M^\cdot \otimes N^\cdot$ .

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M^1, N') & \longrightarrow \text{Hom}(M^0, N') & \longrightarrow \text{Hom}(M^{-1}, N') & \longrightarrow \text{Hom}(M^{-2}, N') & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M^1, N^0) & \longrightarrow \text{Hom}(M^0, N^0) & \longrightarrow \text{Hom}(M^{-1}, N^0) & \longrightarrow \text{Hom}(M^{-2}, N^0) & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 \longrightarrow \text{Hom}(M^1, N^{-1}) & \longrightarrow \text{Hom}(M^0, N^{-1}) & \longrightarrow \text{Hom}(M^{-1}, N^{-1}) & \longrightarrow \text{Hom}(M^{-2}, N^{-1}) & \longrightarrow \cdots & & \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow
 \end{array}$$

$\text{Tot}(\text{Hom}(M^\cdot, N^\cdot))$ , the double complex of  $\text{Hom}(M^\cdot, N^\cdot)$ .

Def. For  $M^{\cdot}, N^{\cdot} \in \mathcal{C}(A)$ , define

$$M^{\cdot} \otimes N^{\cdot}, \quad \text{Hom}_A(M^{\cdot}, N^{\cdot}) \in \mathcal{C}(A)$$

by

$$(M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot})^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$(\text{Hom}_A(M^{\cdot}, N^{\cdot}))^n = \bigoplus_{i+j=n} \text{Hom}_A(M^{-i}, N^j)$$

and morphisms given by  $d + (-1)^j \delta$ .

⚠  $F^{\cdot} \otimes G^{\cdot} \cong G^{\cdot} \otimes F^{\cdot}$ . However, for the crspcl of elements,  
 $a \otimes b \leftrightarrow b \otimes a$  because of this  $(-1)^j$ .

Ex. Let  $M^{\cdot} = [\begin{matrix} \mathbb{Z} & \xrightarrow{x^3} & \mathbb{Z} \\ -1 & & 0 \end{matrix}]$ ,  $N^{\cdot} = [\begin{matrix} \mathbb{Z} & \xrightarrow{x^2} & \mathbb{Z} \\ -1 & & 0 \end{matrix}]$

compute  $M^{\cdot} \otimes_{\mathcal{C}(\mathbb{Z})} N^{\cdot}$  &  $\text{Hom}_{\mathcal{C}(\mathbb{Z})}(M^{\cdot}, N^{\cdot})$ ,  
and verify that they're complexes.

$$\begin{array}{c} \text{A: } 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad 0 \end{array} \rightsquigarrow \left[ \begin{array}{ccc} \mathbb{Z} & \xrightarrow{(-3)} & \mathbb{Z}^2 & \xrightarrow{(2 \ 3)} & \mathbb{Z} \\ -2 & & -1 & & 0 \end{array} \right]$$

$\text{Tot}(M^{\cdot} \otimes N^{\cdot}) \qquad \qquad M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot}$

$$\begin{array}{c} 0 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ -1 \quad \mathbb{Z} \xrightarrow{x^3} \mathbb{Z} \\ \uparrow x_2 \quad \uparrow x_2 \\ 0 \quad 1 \end{array} \rightsquigarrow \left[ \begin{array}{ccc} \mathbb{Z} & \xrightarrow{(-3)} & \mathbb{Z}^2 & \xrightarrow{(2 \ 3)} & \mathbb{Z} \\ -1 & & 0 & & 1 \end{array} \right]$$

$\text{Tot}(\text{Hom}(M^{\cdot}, N^{\cdot})) \qquad \qquad \text{Hom}_{\mathcal{C}(A)}(M^{\cdot}, N^{\cdot})$

Now, we can define  $\otimes^L$  &  $R\text{Hom}$ :

Def. For  $M, N \in A\text{-Mod}$ , one can define

$$M \otimes_A^L N = M \otimes_{e(A)} P \quad \text{when } N \xrightarrow{\cong} P \quad \text{flat resolution}$$

in general,  $M, N \in \mathcal{D}^-(A\text{-Mod})$

$$\begin{aligned} R\text{Hom}_A(M, N) &:= \text{Hom}_{e(A)}(M, I) && \text{when } N \xrightarrow{\cong} I && \text{inj resolution} \\ &:= \text{Hom}_{e(A)}(P, N) && \text{when } M \xleftarrow{\cong} P && \text{proj resolution} \end{aligned}$$

in general,  $M \in \mathcal{D}^-(A\text{-Mod}), N \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

<https://math.stackexchange.com/questions/4322028/three-ways-to-prove-that-projective-modules-are-flat>

Ex Compute  $\mathbb{F}_2 \otimes_{\mathbb{Z}}^L \mathbb{F}_2$  &  $R\text{Hom}_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$ ,  
and get  $\text{Tor}_{\mathbb{Z}}^1(\mathbb{F}_2, \mathbb{F}_2)$  &  $\text{Ext}_{\mathbb{Z}}^1(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$\text{Hom}_{\mathcal{D}^+(A)}(M, N) = R^0 \text{Hom}_{e(A)}(M, N)$$

$$\text{Hom}_A(M, N) = \text{Hom}_{\mathcal{D}^+(A)}(M, N) = R^0 \text{Hom}_{e(A)}(M, N).$$

⚠ [KS 90, Def 2.6.2]

To switch from  $\mathcal{D}^-(X)$  to  $\mathcal{D}^+(X)$ , we need to require that

$$\text{w.gldim}(A) < +\infty. \quad \text{w.gldim: shortest flat resolution}$$

In our case,  $\text{w.gldim}(Q) = 0 < +\infty$ . So don't worry.

A wrong proof for "flat  $\rightarrow$  proj"

"Proof": when  $P$  is flat,

$$\begin{aligned} P \otimes_A - &\dashv \text{Hom}_A(P, -) \\ P \otimes_A^L - &\dashv R\text{Hom}_A(P, -) \end{aligned}$$

by the uniqueness of the adjunction,  $\text{Hom}_A(P, -) = R\text{Hom}_A(P, -)$ ,  
so  $P$  is flat.

This is wrong.  $Q \in \mathbb{Z}\text{-Mod}$  is flat but not proj.  
In the proof, we only have

$$\begin{array}{ccc} A\text{-Mod} & \begin{array}{c} \xrightarrow{P \otimes_A -} \\ \xleftarrow{\text{Hom}_A(P, -)} \end{array} & A\text{-Mod} \\ \downarrow l_A & & \downarrow l_A \\ \mathcal{D}(A) & \begin{array}{c} \xrightarrow{P \otimes_A^L -} \\ \xleftarrow{R\text{Hom}_A(P, -)} \end{array} & \mathcal{D}(A) \end{array}$$

$$l_A \circ (P \otimes_A -) = (P \otimes_A^L -) \circ l_A.$$

Ex. Compute  $R\text{Hom}_{\mathbb{Z}}(Q, -)$ , and shows that

$$l_{\mathbb{Z}} \circ \text{Hom}_{\mathbb{Z}}(Q, -) \neq R\text{Hom}_{\mathbb{Z}}(Q, -) \circ l_{\mathbb{Z}}.$$

## 5. adjunction

, derived version

Prop. For  $A$  comm ring,  $L, M, N \in A\text{-Mod}$ , we get  
 $L, M \in \mathcal{D}^-(A)$ ,  $N \in \mathcal{D}^+(A)$  in general

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

Proof.

$$\text{Hom}_A(M \otimes_A N, L) \cong \text{Hom}_A(N, \text{Hom}_A(M, L))$$

$\Downarrow$  take  $(-)^n$

$$\text{Hom}_{\mathcal{E}(A)}(M \otimes_{\mathcal{E}(A)} N, L) \cong \text{Hom}_{\mathcal{E}(A)}(N, \text{Hom}_{\mathcal{E}(A)}(M, L))$$

$\Downarrow$   $\text{Hom}_A(M, -)$  preserves inj modules  
for  $M$  flat

$$R\text{Hom}_A(M \otimes_A^L N, L) \cong R\text{Hom}_A(N, R\text{Hom}_A(M, L))$$

$$\begin{aligned} R\text{Hom}_A(M \otimes_A^L N, L) &= R\text{Hom}_A(P \otimes_{\mathcal{E}(A)} N, L) & M &\xleftarrow{\cong} P \text{ flat} \\ &= \text{Hom}_{\mathcal{E}(A)}(P \otimes_{\mathcal{E}(A)} N, I) & L &\xrightarrow{\cong} I \text{ inj} \\ &= \text{Hom}_{\mathcal{E}(A)}(N, \text{Hom}_{\mathcal{E}(A)}(P, I)) & \text{adj in } \mathcal{E}(A) \\ &= R\text{Hom}_A(N, \text{Hom}_{\mathcal{E}(A)}(P, I)) & \text{Hom}_{\mathcal{E}(A)}(P, I) \text{ is inj} \\ &= R\text{Hom}_A(N, R\text{Hom}_A(P, I)) & I \text{ is inj} \\ &= R\text{Hom}_A(N, R\text{Hom}_A(M, L)) \end{aligned}$$

□

⚠ We don't have

$$R\text{Hom}_A(M \otimes_A N, L) \cong R\text{Hom}_A(N, \text{Hom}_A(M, L))$$

Find a counterexample?

Take  $N = A$ , reduce to:

$$R\text{Hom}_A(M, L) \cong \text{Hom}_A(M, L)$$

then take  $A = \mathbb{Z}$ ,  $M = L = \mathbb{Z}/2\mathbb{Z}$ .

6. comparison between  $\otimes^L \text{Hom}$  &  $f^* - f_*$ , derived version

Now it's time to extend Section 3 to derived version!  
 Notice that most factors in (3.1) are already exact.

$$\begin{array}{ccc}
 & S\mathcal{S}_R \otimes_R^L - & \\
 & \perp & \\
 \mathcal{D}(S\text{-Mod}) & \xleftarrow{\quad u = R\mathcal{S}_S \otimes_S - \quad} & \mathcal{D}(R\text{-Mod}) \\
 & \perp & \\
 & R\text{Hom}_R(R\mathcal{S}_S, -) &
 \end{array} \quad (6.1)$$

Compare with j

In the extreme cases where  $S$  is a free  $R$ -Mod,  
 e.g.  $S = k[G]$ ,  $R = k[H]$

all the factors in (3.1) are exact.

Compare with i

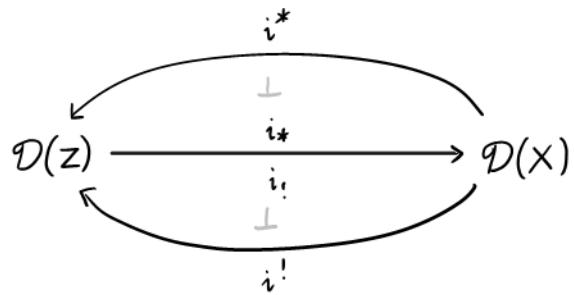
(6.1) can be rewritten as

$$\begin{array}{ccc}
 & Lf^* & \\
 & \perp & \\
 \mathcal{D}(S\text{-Mod}) & \xrightarrow{\quad u \quad} & \mathcal{D}(R\text{-Mod}) \\
 & \perp & \\
 & R\text{Hom}_R(R\mathcal{S}_S, -) &
 \end{array} \quad (6.2)$$

where the "pullback factor"

$$Lf^*: \mathcal{D}(R\text{-Mod}) \longrightarrow \mathcal{D}(S\text{-Mod}) \quad Lf^*M = S\mathcal{S}_R \otimes_R^L M$$

compare it with  $i$ -part of the recollement diagram:



To guarantee that

$$\begin{cases} u \text{ is faithful} \\ i^* i_* = i_! i^* = \text{Id} \end{cases}$$

one has to require that  $S \cong S \otimes_R^L S$ .

e.g.  $S = R_p$  or  $S^{-1}R$

Slogan: Zariski open subsets are closed!

To extend (6.1) to the whole recollement diagram, one has to define the Verdier quotient.

## Verdier quotient

Def. For  $\mathcal{C} \subset \mathcal{D}$  triangulated categories,

the Verdier quotient  $\mathcal{D}/\mathcal{C}$  is the triangulated category s.t.

$$\begin{array}{ccccc} & & \xrightarrow{o} & & \\ \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}/\mathcal{C} \\ & & \searrow o & \downarrow \exists! & \\ & & & & \mathcal{C} \end{array}$$

full subcategory:

<https://math.stackexchange.com/questions/1285761/why-is-it-an-equivalent-definition-of-a-triangulated-full-subcategory>

Rmk: Verdier quotient always exist.

For an explicit construction, see

<https://stacks.math.columbia.edu/tag/05RA>

Ex. Show the following isomorphism:

$$\begin{array}{ccccc} Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\ & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\ \mathcal{D}(Z) & \xrightarrow[i_!=i_*]{j^*} & \mathcal{D}(X) & \xrightleftharpoons[j^*=j_!]{Rj_*} & \mathcal{D}(U) \\ & \xleftarrow{i^!} & & \xleftarrow{} & \end{array}$$

$$\mathcal{D}(X)/_{j_!}\mathcal{D}(U) \cong \mathcal{D}(Z)$$

$$\mathcal{D}(X)/_{i^*}\mathcal{D}(Z) \cong \mathcal{D}(U)$$

$$\mathcal{D}(X)/_{Rj_*}\mathcal{D}(U) \cong \mathcal{D}(Z)$$

restricted to  $\pi_X$ -sm sheaves.

Hint: take the middle one as an example:

$$\begin{array}{ccccc} \mathcal{D}(Z) & \xrightarrow{i_!} & \mathcal{D}(X) & \xrightarrow{j^*} & \mathcal{D}(U) \\ & \searrow o & \downarrow F & \downarrow \exists! G & \\ & & & & \mathcal{C} \end{array}$$

Uniqueness:  $G \circ j^* = F \Rightarrow G = F \circ j_! = F \circ Rj_*$

Existence: Let  $G = F \circ j_!$ . Apply  $F$  to

$$j_! j^* F \xrightarrow{F} i^* F \xrightarrow{+1} i_! i^* F \xrightarrow{+1} F$$

get  $F \circ j_! \circ j^* = F$ .

Now, we can extend (6.2) to a full recollement:  
 (use stable  $\infty$ -category to guarantee that cocone is a factor)

$$\begin{array}{ccccc}
 Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\
 \downarrow L i^* & & \downarrow j_! & & \\
 \mathcal{D}(S\text{-Mod}) & \xrightarrow{\begin{matrix} i_! = i_* \\ R i^! \end{matrix}} & \mathcal{D}(R\text{-Mod}) & \xrightarrow{\begin{matrix} j^* = j^! \\ R j_* \end{matrix}} & \mathcal{D}(R\text{-Mod}) / \mathcal{D}(S\text{-Mod}) \\
 \downarrow & & \downarrow & & \\
 & & & & 
 \end{array}$$

$L i^* = S \otimes_R^L -$        $j_! = \text{cocone } [- \longrightarrow S \otimes_R^L -]$   
 $i_! = u(-)$        $j^* = \text{projection}$   
 $R i^! = R\text{Hom}_R(S, -)$        $R j_* = R\text{Hom}_R(\text{cocone } [R \xrightarrow{\sim} S], -)$

Rmk: For any half recollement diagram (satisfying nec conditions), we can extend it to a full recollement diagram through the Verdier quotient. We will prove that rigorously in the 6-fator formalism toolkit.

## geometrical realizations of representations

	representations realized as sheaves on some space	stack	formulas
gp	equiv sheaves		$\text{Rep}(G) \rightarrow \text{Sh}([*/G]; \underline{\mathbb{Q}})$
Lie alg	D-modules	flag variety	$\text{Mod}(g, X_\lambda) \cong \text{Mod}_{\text{qc}}^e(D_\lambda)$
Hochschild	Differential, like	Hodge-Tate	$A\text{-Mod}-A \rightarrow ?$
some quiver	perverse sheaves	stratified space	$\text{Rep}(Q) \rightarrow \text{Perv}_s(X)$

Remark for quiver, there's no "structure sheaf/rep" of a given quiver. Therefore, we don't discuss about quiver cohomology.

Realize homogeneous vector bundle on Hermitian symmetric variety as quiver representations:  
<https://arxiv.org/abs/math/0403307>

The reference for Lie algebra cohomology is [HT07, Corollary 11.2.6(p274), p298 and Chapter 11]. Aaron told me about this story, and wrote this equivalence in detail in his master thesis.