§ 1.1. Structure of finite / local / global field

Road map

	finite field	local field		global field	Adèle
		Archi	NA		
base field F	6 $\mathbb{F}_1$ (For $\mathbb{F}_{1^r}$) 1 $\mathbb{F}_p$	2 $\mathbb{R}$ or $\mathbb{C}$	3 $\mathbb{Q}_p$ $\mathbb{F}_p((t))$	4 $\mathbb{Q}$ $\mathbb{F}_p(t)$	5 $\mathbb{A}_K$
$F^\times$	$\varepsilon \cdot \mu_r$ μ_{p-1}	$\mathbb{R}^+ \times \mathbb{Z}/2\mathbb{Z}$ $\mathbb{C}^\times$	$\mathbb{Z}_p^\times \times \mathbb{Z}$ $\mathbb{F}_p[[t]]^\times \times \mathbb{Z}$	$\mathbb{Q}^\times$ $\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	— —	—	$\mathbb{Z}_p$ $\mathbb{F}_p[[t]]$	$\mathbb{Z}$ $\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	— —	—	$\mathbb{Z}_p^\times$ $\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$ $\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$	$\hat{\mathbb{Z}}$? $\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$ Id	most known	dream	
ari Frob	? can	total order? —	choose a lift	? ?	
# ext of deg n	1? 1	1/0	finite	$n \neq 1$ inf countable	
Spec $\mathcal{O}_F$	Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}}, 1)$ [étale, 2.2.4]	—	· · · · ·	—	
topology	? discrete	Euclidean	profinite	—	restricted
topo of $\mathcal{O}_F$	—	—	cpt, not discrete	—	K is a lattice in $\mathbb{A}_K$
measure	? discrete	Lebesgue	$\mu(\mathcal{O}_F) = 1$	—	can be computed

Also, discuss

- field extension, norm, trace, ...
- their connection to geometry, ramification theory
- analog with knot theory

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.
$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.
$Gal(\bar{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}}$ as top gps
$Frob_p \longleftrightarrow 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $Gal(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $Gal(\mathbb{C}/\mathbb{C}) = Id$
$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field
Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure
Task. Read [NAlocal], answer the following questions:
- Describe $\mathcal{O}, \pi, \kappa, U, U^{(n)}$ in terms of v
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?
  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension
Task. Read [NAext], answer the following questions:
- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions.
- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $Fr \, \tau \, Fr^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions: (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep}/\mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right), \quad \mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}_p}}/\widehat{\mathbb{Q}_p^{ur}}\right), \quad \mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}/\mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)$$

$G_{\mathbb{F}_p((t))}$ $I_{\mathbb{Q}_p}$ $G_{\mathbb{F}_p((t))}$