Eine Woche, ein Beispiel

1.30 Tits system

For many time I want to understand Tits system, I always see the reference to Bourbaki's work. But I believe that the proof can be shown more

Def. (Tits system, BN-pair)
(G, B, N, S) gp + (subgp, subgp) + gen of $W := N/B\cap N$

(TS0) $G = \langle B, N\rangle_{Grp}$
(TS1) $T \lhd N$ where $T := B\cap N$
(TS2) $W = \langle s\in S\rangle_{Grp}$ + s of order 2 $\forall s\in S$ where $W = N/T$
(TS3) $BwB \cdot BsB \subseteq BwsB \cup BwB$
(TS4) $BsB \cdot BsB = B \cup BsB$

Saturated Tits system: e.g. of non-Satuated?
(TS5) $T = \bigcap_{g\in N} gBg^{-1}$ $\subseteq$ is obvious, need $\supseteq$

Notation.
W : Weyl gp of the Tits system
$s\in S$: simple reflections
$l(w) := \min\{r \mid w = s_1\cdots s_r, s_i\in S\}$: length of $w\in W$

For example of Tits system, see wiki or [Prasad, Eg 1.4.3].

Basic results of Tits system

Prop. ① $(BwB)^{-1} = Bw^{-1}B$
② (TS3') $BswB \subseteq BsB\cdot BwB \subseteq BswB \cup BwB$
③ (TS3'') $Bww'B \subseteq BwB\cdot Bw'B \subseteq \bigcup_{l(x)\le l(w)+l(w')} BxB$
④ (TS4') $BwB\cdot BsB\cdot BsB = BwB \cup BwsB$

Proof. ① : by def.
② by (TS3),
$$Bw^{-1}B\cdot BsB \subseteq Bw^{-1}sB \cup Bw^{-1}B$$
Apply $(-)^{-1}$ to both sides.
③ by (TS3).
④ by (TS3) + (TS4),
$$BwB \cup BwsB \subseteq BwB(B\cup BsB) = BwB\cdot BsB\cdot BsB \subseteq BwB \cup BwsB.$$