Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases

Setting. We work over conn red gp over K. ($G/_K$ conn red)

$\bar{K}$: the seperable closure of K mainly care about $\mathbb{R}$ & p-adic field case.

Γ_K : $= Gal(\bar{K}/K)$ $\sigma \in \Gamma_K$

$$Z^1(W,A) := \left\{ \begin{array}{l} {}^L\varphi : W \longrightarrow A \rtimes W \\ \quad \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{array} \middle| \begin{array}{l} {}^L\varphi : \text{continuous group homo} \\ {}^L\varphi \text{ is a section} \end{array} \right\}$$
$$= \{ \varphi : W \longrightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)^\gamma(\varphi(\gamma')) \}$$

$\varphi \in H^1(W,A)$

$$H^1(W,A) = Z^1(W,A)/A.$$

Def. (Split) G is split if
G has a maximal torus T over K which is split.

Def. (Quasi-split) G is quasi-split if
G has a Borel B over K.

Borel = maximal (Zar-closed) conn sol alg subgp
= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t G/H is projective
= closed subgp containing a Borel.

Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat{Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability
https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf

I make no claim to originality.

1. forms.

Def. $G_1, G_2/K$ are called forms, if

$$\exists\ \alpha : G_{2,\bar{K}} \xrightarrow{\sim} G_{1,\bar{K}} \quad \text{as gps} \qquad \text{not as } \Gamma_K\text{-gps!}$$

α is considered as the information of forms.

Thm.

$$\begin{array}{ccc}
\{K\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_K, \mathrm{Aut}(G_{\bar{K}})) \\
{[G_2,\ \alpha : G_{2,\bar{K}} \longrightarrow G_{\bar{K}}]} & \longmapsto & \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1} \\
G_2 & \longleftarrow\!\shortmid & \varphi
\end{array}$$

$$\begin{tikzcd}
G_{2,\bar{K}} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & G_{\bar{K}} \arrow[d, "\sigma"] \\
G_{2,\bar{K}} \arrow[dr, "\alpha"'] & G_{\bar{K}} \\
& G_{\bar{K}}
\end{tikzcd}$$

$$\Gamma_K \circlearrowright G_{2,\bar{K}} \longrightarrow G_{\bar{K}} \circlearrowleft \Gamma_K,\ \Gamma_K$$

φ_α measures the commutativity between two Γ_K-actions.

$$G_2(K) := \{ g \in G(\bar{K}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_K \}$$

In general, $G_2(R) := \{ g \in G(\bar{K}\otimes_K R) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall\sigma\in\Gamma_K\}$

e.p. $G_2(\bar{K}) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_K} \in \prod_{\sigma\in\Gamma_K} G(\bar{K}) \mid g \in G(\bar{K}) \} \cong G(\bar{K})$

$$G(\bar{K}\otimes_K \bar{K}) \cong G\Big(\bigoplus_{\sigma\in\Gamma_K} \bar{K}\Big) \cong \prod_{\sigma\in\Gamma_K} G(\bar{K})$$

Functorial on K: (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])
Let L/K be finite Galois.

$$\begin{array}{ccccccc}
G_{2,L} & & \{L\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_L, \mathrm{Aut}(G_{\bar{K}})) & & \varphi|_{\Gamma_L} \\
\updownarrow & & \uparrow & & \uparrow & & \updownarrow \\
G_2 & & \{K\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_K, \mathrm{Aut}(G_{\bar{K}})) & & \varphi \\
& & \uparrow & & \uparrow & & \\
& & \{G_2/K : G_{2,L} \cong G_L\} & \longleftrightarrow & H^1(\mathrm{Gal}(L/K), \mathrm{Aut}(G_{\bar{K}})^{\Gamma_L}) & & \\
& & \uparrow & & \uparrow & & \\
& & 1 & & 1 & &
\end{array}$$

$\mathrm{Aut}(G_{\bar{K}})^{\Gamma_L} = \mathrm{Aut}(G_L)$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{G}_m^n) & \subseteq & \mathrm{End}(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow n$

Therefore,

$$H^1(\Gamma_K, \mathrm{Aut}(\mathbb{G}_{m,\bar{K}}^n)) = H^1(\Gamma_K, GL_n(\mathbb{Z}))$$
$$= \mathrm{Hom}_{Grp}(\Gamma_K, GL_n(\mathbb{Z}))/_{GL_n(\mathbb{Z})\text{-conj}}$$
$$\overset{\text{when } K=\mathbb{R}}{=\!=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = Id\}/_{GL_n(\mathbb{Z})\text{-conj}}$$

[Γ_K acts on $\mathrm{Aut}(\mathbb{G}_{m,\bar{K}}^n) \subseteq \mathrm{End}(\mathbb{G}_{m,\bar{K}}^n)$ trivially:

see $\bar{K}$-pts, $n=1$:

$$\begin{array}{ccc} \bar{K}^\times & \xrightarrow{\alpha} & \bar{K}^\times \\ \sigma\downarrow & & \downarrow\sigma \\ \bar{K}^\times & \xrightarrow{{}^\sigma\alpha} & \bar{K}^\times \end{array} \qquad \begin{array}{ccc} x & \longmapsto & x^n \\ \downarrow & & \downarrow \\ \sigma(x) & & \sigma(x^n) = \sigma(x)^n \end{array}$$

$\Rightarrow {}^\sigma\alpha = \alpha$]

E.g. $n=1$, $K=\mathbb{R}$

$$H^1(\Gamma_K, \mathrm{Aut}(\mathbb{G}_{m,\bar{K}})) \cong \{1, -1\}$$

$\rightsquigarrow \varphi(\sigma) = (-)^{-1}$

$1 \Downarrow \mathbb{G}_m$, $\quad -1 \Downarrow G = ?$ $SO_{2,\mathbb{R}}$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\} \\ &= \{g \in \mathbb{C}^\times \mid |g| = 1\} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = \mathrm{Spec}\, \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$

Check:
$$\begin{aligned} G(\mathbb{C}) &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1 = 0\} \\ &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1\} \\ &= \left\{(x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \;\middle|\; \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array}\right\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; x,y \in K,\ x^2+y^2 = 1\right\}$$

E.g. $n=2,\ K=\mathbb{R}$

$$H^1(\Gamma_K, \mathrm{Aut}(\mathbb{G}_{m,\bar{K}}^2)) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$\mathbb{G}_m^2 \quad \mathbb{G}_m \times SO_{2,\mathbb{R}} \quad (SO_{2,\mathbb{R}})^2 \quad G = ? \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; (\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$(\varphi(\sigma)\circ\sigma)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \varphi(\sigma) \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar{z}_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\mathbb{G}_m, \quad SO_{2,\mathbb{R}}, \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$1 \quad -1 \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

Fact^{dual}: $\mathrm{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{triv}, \mathbb{Z}_{sign}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

Rmk. Using the same argument, one can show that

$$\{ T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}_{m,\mathbb{F}_{p^2}}^n \} = \text{products of } \mathbb{G}_m, \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}, \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathbb{G}_m$$

The torus G crspd to -1: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$G(\mathbb{F}_p) = \{ g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k \}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid \underset{(a-b\xi)^{-1}}{\varphi(\sigma)(a-b\xi)} = a+b\xi \}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2\varepsilon = 1 \}$$

$$\cong \left\{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p) \right\}$$

3. other cases.

By using the same methods introduced in last section, we can compute the (inner) forms of reductive gps.

	inner forms	outer forms
$\mathbb{G}_m$ $(\mathbb{G}_m)^2$ $(\mathbb{G}_m)^n$	ϕ	SO_2 $SO_2 \times \mathbb{G}_m$, $(SO_2)^2$, $Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ product of lower rank torus
$GL_{2,\mathbb{R}}$ $SL_{2,\mathbb{R}}$ $PGL_{2,\mathbb{R}}$	$\mathbb{H}^\times = GL_1(\mathbb{H}\otimes_{\mathbb{R}} -)$ $\mathbb{H}^{Nm=1} = SU_{2,\mathbb{C}/\mathbb{R}}$?	($U_{2,\mathbb{C}/\mathbb{R},\omega}$ $U(1,1)$) ϕ ϕ
$GL_{n,\mathbb{R}}$ $SL_{n,\mathbb{R}}$ $PGL_{n,\mathbb{R}}$	? $GL^{Nm=1}_{n/2}(\mathbb{H}\otimes_{\mathbb{R}} -)$ when n even ?	? $SU(a, n-a)$ e.p. $SU(2,1)$?
$(SL_2)^2/\mathbb{R}$ $(SL_2)^3/\mathbb{R}$	$SL_2 \times SU_2$, $(SU_2)^2$, ϕ? $(8-1)$ possibilities	$Res_{\mathbb{C}/\mathbb{R}} SL_2$?

← need clarification

?: I have no time to compute / don't know any symbol to represent

Compute $\mathrm{Aut}(G_{\bar{k}})$

Lemma. We understand $\mathrm{Aut}(G_{\bar{k}})$ quite well.

$$G(\bar{k})/Z(G(\bar{k})) = G^{ad}(\bar{k}) \qquad \qquad \mathrm{Aut}(\Psi_0)$$

$$G \qquad 1 \longrightarrow \underset{\mathrm{Inn}(G_{\bar{k}})}{\shortparallel} \longrightarrow \mathrm{Aut}(G_{\bar{k}}) \longrightarrow \underset{\mathrm{Out}(G_{\bar{k}})}{\shortparallel} \longrightarrow 1$$

G	$\mathrm{Inn}(G_{\bar{k}})$	$\mathrm{Aut}(G_{\bar{k}})$	$\mathrm{Out}(G_{\bar{k}})$	
T rk n	1	$GL_n(\mathbb{Z})$	$GL_n(\mathbb{Z})$	
$GL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$SL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$PGL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$n \geq 3$				
$GL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$	$\{\pm 1\}^{\oplus 2}$	
$SL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$PGL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$(SL_2)^2/\mathbb{R}$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	with different Γ_K-action
$(SL_2)^n/\mathbb{R}$	$PGL_n(\mathbb{C})^n$	$PGL_n(\mathbb{C})^n \rtimes S^n$	S^n	

Compute $H^1(\Gamma_k, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP] **Theorem 12.7.7** *There is a functorial isomorphism* $\mathrm{H}^1(k, G) \to \pi_1(G)_{\Theta,\mathrm{tor}}$.

[DR09] COROLLARY 2.4.3. *The composition*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

is a bijection.

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field:

Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
https://arxiv.org/abs/1401.5913

3.1. Theorem. *Let G, T_0, T, and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

global field:

Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
https://arxiv.org/pdf/2303.04120.pdf

$$\{K\text{-forms of } G\} \longleftrightarrow H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$$

Claim:

$$\{K\text{-inner forms of } G\} \longleftrightarrow H^1(\Gamma_K, \mathrm{Inn}(G_{\bar K}))$$

$$\left\{\begin{array}{l} K\text{-forms of } G \\ \text{which are quasi-split} \end{array}\right\} \longleftrightarrow H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}, B, T)) \cong H^1(\Gamma_K, \mathrm{Out}(G_{\bar K}))$$

Q: Do we have

$$\begin{tikzcd}
 & H^1(\Gamma_K, \mathrm{Inn}(G_{\bar K})) \arrow[r, hook] & H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K})) \arrow[r] & H^1(\Gamma_K, \mathrm{Out}(G_{\bar K})) \\
1 \arrow[r] & \underset{\mathrm{Inn}(G_K)}{\overset{\shortparallel}{\mathrm{Inn}(G_{\bar K})^{\Gamma_K}}} \arrow[r] & \underset{\mathrm{Aut}(G_K)}{\overset{\shortparallel}{\mathrm{Aut}(G_{\bar K})^{\Gamma_K}}} \arrow[r] & \underset{\mathrm{Out}(G_K)}{\overset{\shortparallel}{\mathrm{Out}(G_{\bar K})^{\Gamma_K}}} \arrow[ull, bend right]
\end{tikzcd}$$

Give one example s.t. $H^1(\Gamma_K, \mathrm{Inn}(G_{\bar K})) \longrightarrow H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$ is not inj ?

E.g. $G = SL_{2,\mathbb{R}}$, $K = \mathbb{R}$

$$\begin{array}{cccccccccc} G & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar K}) & \longrightarrow & \mathrm{Aut}(G_{\bar K}) & \longrightarrow & \mathrm{Out}(G_{\bar K}) & \longrightarrow & 1 \\ SL_{2,\mathbb{R}} & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) & & 1 & & \end{array}$$

$$\begin{aligned} H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(SL_{2,\mathbb{C}})) &= H^1(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C})) \\ &= \{ 1, \quad \omega(\ )\omega^{-1} \} \end{aligned}$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ SL_{2,\mathbb{C}} & G = {\color{red}?} \end{array} \qquad \mathbb{H}^{N_m = 1}$$

$$\begin{aligned} G(\mathbb{R}) &= \{ g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}} \} \\ &= \{ g \in SL_2(\mathbb{C}) \mid \omega \bar{g} \omega^{-1} = g \} \\ &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \mid \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \} \\ &= \{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in SL_2(\mathbb{C}) \} \\ &= \mathbb{H}^{N_m = 1} \end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$, $K = \mathbb{R}$

$$\begin{array}{cccccccccc} G & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar K}) & \longrightarrow & \mathrm{Aut}(G_{\bar K}) & \longrightarrow & \mathrm{Out}(G_{\bar K}) & \longrightarrow & 1 \\ GL_{2,\mathbb{R}} & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) \rtimes \{\pm 1\} & & \{\pm 1\} & & \end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(GL_{2,\mathbb{C}})) = \{ 1, \quad \omega(\ )\omega^{-1}, \quad (\ )^{H,-1}, \quad \omega(\ )^{H,-1}\omega^{-1} \}$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$$\begin{array}{cccc} \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ {\color{orange}(} GL_{2,\mathbb{C}} & \mathbb{H}^\times {\color{orange})} & {\color{orange}(} \mathcal{U}(1,1) & \mathcal{U}_{2,\mathbb{C}/\mathbb{R},\omega} {\color{orange})} \end{array}$$