Eine Woche, ein Beispiel

Ref: from [23.11.19]

4.28 naive $\otimes$ - Hom adjunction

Notation. - A: associate ring allowed to be non-commutative, contains 1
- There are two systems to write category of A-modules.

$$\begin{aligned} & \mathrm{Mod}_A = A\text{-}\mathrm{Mod} & \ni {}_A M \\ (\mathrm{Mod}_A)^{op} \neq \ & \mathrm{Mod}_{A^{op}} = \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} & \ni M_A \\ & \mathrm{Mod}_{A\otimes B^{op}} = A\text{-}\mathrm{Mod}\text{-}B & \ni {}_A M_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \ni {}_B M_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $Ob(\mathrm{Ring}) \cong Ob(\mathrm{Ring}^{op})$,
$A^{op} \notin Ob(\mathrm{Ring}^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset Sh(\mathrm{Spec}\, A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

1 definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$
⋮
6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

1.definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-}\mathrm{Mod}\text{-}B \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}C \longrightarrow A\text{-}\mathrm{Mod}\text{-}C$$

$$\mathrm{Hom}_B^A(-,-) : (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}} \times B\text{-}\mathrm{Mod}\text{-}A \longrightarrow \mathbb{Z}\text{-}\mathrm{Mod}$$
$$\parallel \qquad \parallel \qquad \parallel \qquad \parallel$$
$$\mathrm{Hom}_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) \; (\mathbb{Z}\text{-}\mathrm{Mod}\text{-}B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}\text{-}\mathrm{Mod}\text{-}\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}\text{-}\mathrm{Mod}\text{-}\mathbb{Z}$$

$${}_AX_B, \quad {}_BY_C, \quad {}_CZ_D$$

associativity: $(X\otimes_B Y)\otimes_C Z \cong X\otimes_B(Y\otimes_C Z)$

"commutativity": $X\otimes_B Y \cong Y\otimes_{B^{op}} X$ in $A\text{-}\mathrm{Mod}\text{-}C = C^{op}\text{-}\mathrm{Mod}\text{-}A^{op}$

"unit": $A\otimes_A X \cong X \cong X\otimes_B B$

$\mathrm{Hom}_A(A,X) \cong X$

2. adjunction ${}_B X_A$, ${}_C Y_B$, ${}_C Z_D$. we get

$$\mathrm{Hom}_C(Y \otimes_B X,\ Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y, Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$\{ f : Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c \}$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y, -)"] \end{tikzcd}$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv \mathrm{Hom}$:

$$\begin{tikzcd}
(A\text{-Mod-}B)^{op} \times (B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "(\mathrm{Id},\, \mathrm{Hom}_C)"] \arrow[d, equal] & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}D \arrow[dd, "\mathrm{Hom}_B"] \\
(A\text{-Mod-}B \times B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[d, "(\otimes_B,\, \mathrm{Id})"'] & \\
(A\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "\mathrm{Hom}_C"] & A\text{-Mod-}D
\end{tikzcd}$$

$f^* \dashv f_*$:

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G}) \qquad \begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$$\begin{tikzcd}
Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \arrow[r, "(\mathrm{Id},\, \text{pushforward})"] \arrow[d, "(\text{pullback},\, \mathrm{Id})"'] & Sh(X)^{op} \times Sh(X) \arrow[d, "\mathrm{Hom}_{Sh(X)}(-,-)"] \\
Sh(Y)^{op} \times Sh(Y) \arrow[r, "\mathrm{Hom}_{Sh(Y)}(-,-)"] & \mathrm{Abel}
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{F},\ f,\ \mathcal{G}) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{F},\ f_*\mathcal{G}) \arrow[d, mapsto] \\
(f^*\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] & \mathrm{Hom}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})
\end{tikzcd}$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

forgetful fctor

Prop. For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \mathrm{Hom}_S({}_S S_R, M)$$

one has adjunction fctors

$$\begin{tikzcd} S\text{-Mod} \arrow[rr, "\mathrm{Hom}_S({}_S S_R, -)", "u = {}_R S_S \otimes_S -"'] & & R\text{-Mod} \arrow[ll, bend right, "{}_S S_R \otimes_R -"'] \arrow[ll, bend left, "\mathrm{Hom}_R({}_R S_S, -)"] \end{tikzcd} \qquad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{tikzcd} \mathcal{D}(X) \arrow[rr, "j^!", "j^*"'] & & \mathcal{D}(\mathcal{U}) \arrow[ll, bend right, "j_!"'] \arrow[ll, bend left, "Rj_*"] \end{tikzcd}$$

Vague slogan: $u \approx$ "forget the information of Z".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & X.

E.g. For finite gps $H \leq G$, one has Res - Ind adjunction:

$$\mathrm{Res}_H^G \dashv \mathrm{Ind}_H^G$$
$$\text{c-}\mathrm{Ind}_H^G \dashv \mathrm{Res}_H^G$$

It can be generalized for $\begin{cases} G: \text{ loc profinite gp}, \\ H \leq G \text{ open} \end{cases}$

If one only has $H \leq G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{tikzcd} \mathrm{Rep}_G \arrow[rr, "\mathrm{Res}_H^G" description] & & \mathrm{Rep}_H \arrow[ll, bend right, "\text{c-}\mathrm{Ind}_H^G"'] \arrow[ll, bend left, "\mathrm{Ind}_H^G"] \end{tikzcd}$$

Ex. Compare it with the recollement diagram & (3.1).

$$\begin{tikzcd} \mathcal{U} \arrow[d, "j"] & [*/H] \arrow[d, "\text{``cover with fiber } G/H\text{''}"] \\ X & [*/G] \end{tikzcd}$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$

$\text{c-}\mathrm{Ind}_H^G \longrightarrow \mathrm{Ind}_H^G$ $\text{c-}\mathrm{Ind}_H^G = \mathrm{Ind}_H^G$

2. Even though

$$Sh_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \mathrm{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$

the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\mathrm{Res}_{\{*\}}^G \mathbb{Q} = \mathbb{Q}, \qquad \mathrm{Res}_{\{*\}}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* R j_* \neq \mathrm{Id}$, $j^! j_! \neq \mathrm{Id}$.