fiber of $\pi_*, \pi_!, \pi^{-1}, \pi^*$ $\quad \mathcal{G} \quad \mathcal{F}$

$\pi : Y \hookrightarrow X$

$\pi : U \to X$ $\qquad$ $\pi : Z \xrightarrow{\text{close}} X$

$$\pi_* \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin \bar{U} \\ \varprojlim_{x \in V} \mathcal{G}(U \cap V) & x \in \bar{U} - U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi_! \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

π^{-1} $\mathcal{F}_y$ $\qquad$ $\mathcal{F}_y$

π^* $\mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y}$ $\qquad$ $\mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y}$

For étale: $f : X \to Y$ $\quad \bar{x} \mapsto \bar{y}$ $\quad \mathcal{G}$ $\quad \mathcal{F}$

(sheaf)

$\bar{x} \xrightarrow{u_{\bar{x}}} X \xrightarrow{f} Y$

$(f^*\mathcal{F})_{\bar{x}} = u_{\bar{x}}^* f^* \mathcal{F}(Y) = \mathcal{F}_y$

$$(f^*\mathcal{F})_{\bar{x}} \cong \mathcal{F}_y$$

If f q.c. $\quad (f_*\mathcal{G})_{\bar{y}} \cong \Gamma(\widetilde{X}, \tilde{g}^*\mathcal{G})$ where $\begin{array}{ccc} \widetilde{X} & \xrightarrow{\tilde{f}} & X \\ \downarrow & \lrcorner & \downarrow f \\ \operatorname{Spec}\mathcal{O}^{sh}_{Y,\bar{y}} & \xrightarrow{g} & Y \end{array}$

e.g. for f finite, we have explicit description of f_*.

左伴随 ⊣ 右伴随　　$f_! \vdash f^* \vdash f_* \vdash f^!$

自由	忘却
$-\otimes_A N$	$\mathrm{Hom}_A(N,-)$
Δ	$\varprojlim$
$(\ )^{\sim}$	Γ_*
f_p	f^*
Ind	Res
$-\otimes_k A^e$	Res
$sh(-)$	$\mathrm{Res}_{sh\to Psh}$
π_0	$T \mapsto \coprod_{t\in T} \mathrm{Spec}\,\mathbb{Z}$
$G^{[p]}$	Lie
$\lvert - \rvert$	S
$\tau_1 = H_0(-)$	N : nerve

ad ⇒　右正合　　左正合

ad ⇒ 与余极限交换 $\varinjlim$　与极限交换 $\varprojlim$

coker f	ker f
$\coprod A$　$\oplus A$	$\prod A$　$A\times_C B$
pushforward	pullback

$\bar{k} = \varinjlim_{L/k} L$　　$\mathrm{Spec}\,\bar{k} = \varprojlim_{L/k} \mathrm{Spec}\,L$　$Gal(\bar{K}/k) = \varprojlim_{L \text{ finite Galois}} Gal(L/k)$

$\mathrm{Spec}\,\mathbb{Z}_p = \varinjlim_n \mathrm{Spec}\,\mathbb{Z}/p^n\mathbb{Z}$　$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

completion　　ring point of view

$\mathcal{F}_p = \varinjlim_{U} \mathcal{F}(U)$

stalk

co limit

direct
inductive
injective
} limit.

Lemma 6.21.3. *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Lemma 6.31.4. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of presheaves of U we have $j_p j_{p!} = \mathrm{id}$.*

(5) *On the category of sheaves of U we have $j^{-1} j_! = \mathrm{id}$.*

situation	category $\mathscr{A}$	category $\mathscr{B}$	left adjoint $F\colon \mathscr{A} \to \mathscr{B}$	right adjoint $G\colon \mathscr{B} \to \mathscr{A}$
A-modules (Ex. 1.5.D)	Mod_A	Mod_A	$(\cdot) \otimes_A N$	$\mathrm{Hom}_A(N, \cdot)$
ring maps $B \to A$ (Ex. 1.5.E)	Mod_B	Mod_A	$(\cdot) \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(pre)sheaves on a topological space X (Ex. 2.4.L)	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups (§1.5.3)	semigroups	groups	groupification	forgetful
sheaves, $\pi\colon X \to Y$ (Ex. 2.7.B)	sheaves on Y	sheaves on X	π^{-1}	π_*
sheaves of abelian groups or $\mathscr{O}$-modules, open embeddings $\pi\colon U \hookrightarrow Y$ (Ex. 2.7.G)	sheaves on U	sheaves on Y	$\pi_!$	π^{-1}
quasicoherent sheaves, $\pi\colon X \to Y$ (Prop. 16.3.6)	$QCoh_Y$	$QCoh_X$	π^*	π_*
ring maps $B \to A$ (Ex. 30.3.A)	Mod_A	Mod_B	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto \mathrm{Hom}_B(A, N)$
quasicoherent sheaves, affine $\pi\colon X \to Y$ (Ex. 30.3.B(b))	$QCoh_X$	$QCoh_Y$	π_*	$\pi^!_{sh}$

Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.4).

E.g. $\operatorname{Spec}\mathbb{C} \longrightarrow \operatorname{Spec}\mathbb{R}$ ring space but not scheme

$(f^{-1}: \mathbb{R}\text{-mod} \longrightarrow (\operatorname{Spec}\mathbb{C}, \widetilde{\mathbb{R}})\text{-mod})$

quasi-coherent: $f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{C}\text{-mod}$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$

$f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{R}\text{-mod}$ $\quad$ forget

$f^*f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{C}\text{-mod}$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$ ← $\mathbb{C}$-mod structure

$f_*f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{R}\text{-mod}$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$

étale: $f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow Ab$ $\quad$ forget

$f_*: Ab \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod}$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$

$f^*f_*: Ab \longrightarrow Ab$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$

$f_*f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod}$ $\quad -\otimes_{\mathbb{R}}\mathbb{C}$ ← $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action

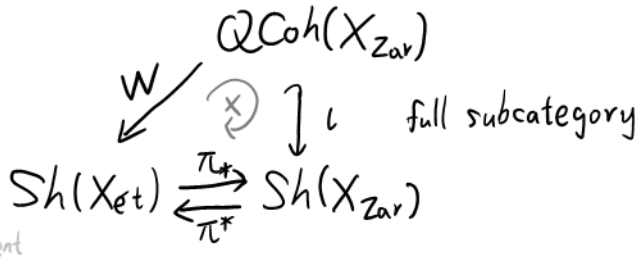
$\pi: X_{ét} \longrightarrow X_{Zar}$

$\pi^{-1}: Sh(X_{Zar}) \longrightarrow Sh(X_{ét})$ $\quad \mathcal{F} \mapsto \left[\pi^{-1}\mathcal{F}: \left(U \xrightarrow{i} X\right) \mapsto \Gamma(U, i^{-1}\mathcal{F})\right]$ Zariski sheafification is not enough...

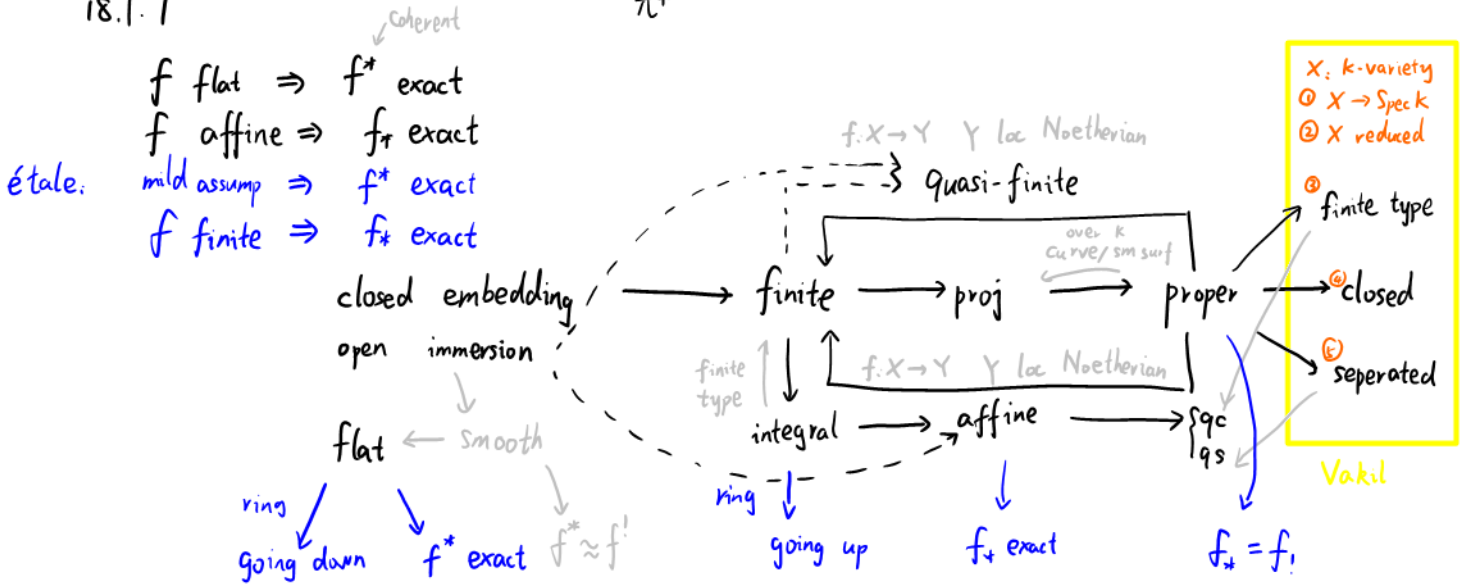
$\pi_*: Sh(X_{ét}) \longrightarrow Sh(X_{Zar})$ $\quad$ forget

$\pi^{-1} \vdash \pi_*$ $\quad \pi_*\pi^{-1} = Id_{Sh(X_{Zar})} \Rightarrow \pi^{-1}$ fully faithful

https://math.stackexchange.com/questions/321633/when-is-a-fully-faithful-functor-an-equivalant-functor

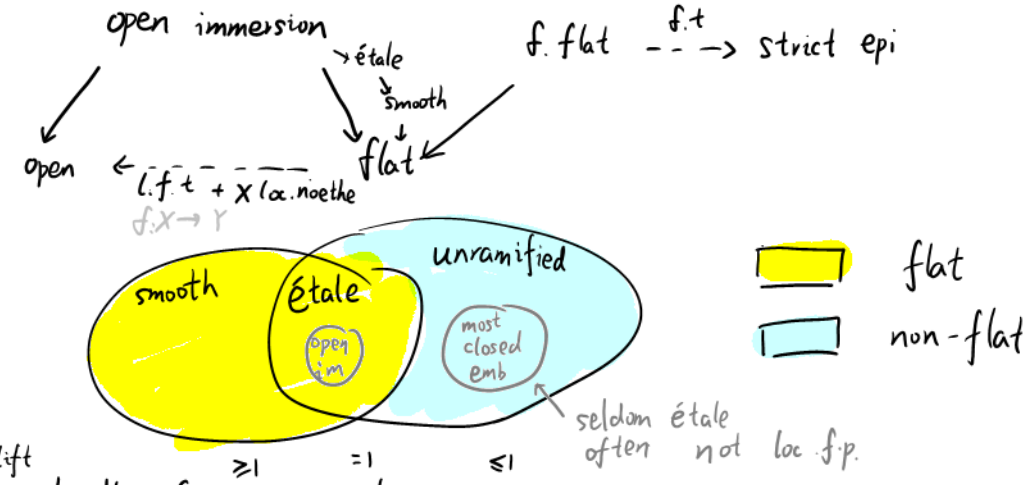


18.1.7



Various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercises 7.3.B(b), 7.3.D, and 10.1.H resp.) are quasiseparated,

https://mathoverflow.net/questions/288466/idea-behind-grothendiecks-proof-that-formally-smooth-implies-flat

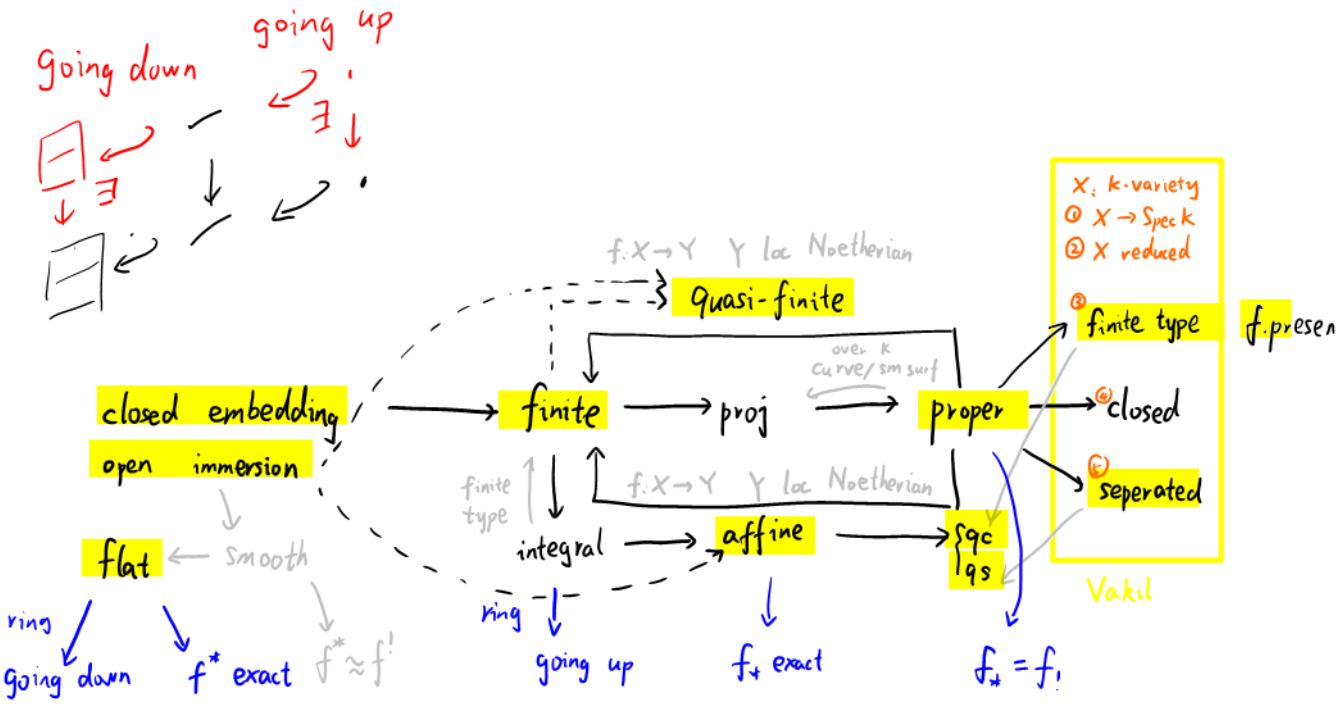


Rmk locally f.p is crucial.

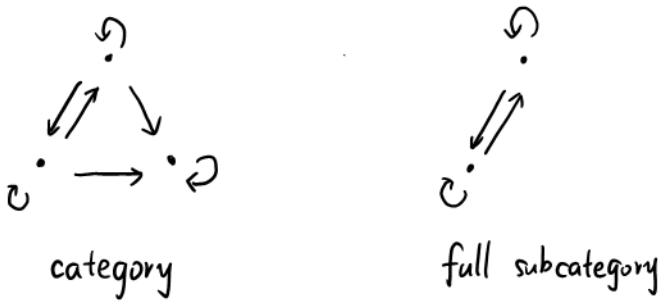
e.g. $\mathbb{C}[t^\lambda : \lambda \in \mathbb{Q}_{>0}] \longrightarrow \mathbb{C}$ is formally étale but not flat
$t^\lambda \longmapsto 0$

https://rankeya.people.uic.edu/formallyunramifiedetale.pdf

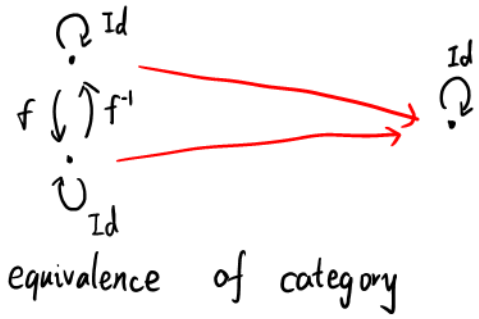
$\mathbb{F}_p[T] \longrightarrow (\mathbb{F}_p[T])^{sep}$ is formally unramified, not formally étale but flat.

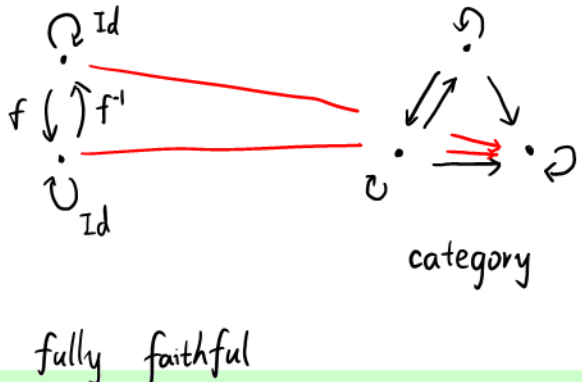


: properties which satisfy the f.flat descent.



https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective





https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/