Eine Woche, ein Beispiel

5.11 genus of generalized Fermat curve

Goal. 1. Find a basis of $H^{p,q}(X)$ by harmonic forms.
2. Compute the geometric genus of curves

$$C := \{y^n = x^m - 1\} \subseteq \mathbb{P}^2$$

Rmk: [2024.11.03] try to compute a special case in detail. In this document, more advanced methods are applied, so we don't need to blow up explicitly.
The reference also follows [2024.11.03].

Extra Ref:

Generalised Fermat equation: a survey of solved cases
https://arxiv.org/abs/2412.11933

Connection between Fermat curve and hyperelliptic curve:
https://math.stackexchange.com/questions/3493593/transformation-which-takes-fermat-curve-xnyn-1-to-a-hyperelliptic-curve

1. Harmonic forms

- Affine plane curve
- Plane curve
  Fermat curve
- Hyperelliptic curve
- generalized Fermat curve
- $\mathbb{P}^n$
- Hypersurface

2. Riemann-Hurwitz
3. Milnor formula

1. Harmonic forms

Almost all the results in this section come from the answer here:
https://mathoverflow.net/questions/324812/the-construction-of-a-basis-of-holomorphic-differential-1-forms-for-a-given-plan

Affine plane curve

Prop. Suppose $C = \{f(x,y) = 0\} \subseteq \mathbb{A}^2$ is a sm curve, then

$$\omega \triangleq \frac{dx}{f_2(x,y)} = -\frac{dy}{f_1(x,y)}$$

is a global generator of $H^0(C, \Omega^1)$.

i.e., $\forall\, \omega' \in H^0(C, \Omega^1)$, $\omega' = f\omega$ for some $f \in \mathcal{O}_{hol}(C)$.

Proof. Notice that

$$f_1(x,y)\,dx + f_2(x,y)\,dy = 0.$$

When $f_1(x_0,y_0) \neq 0$,

$$\begin{aligned} y: C &\longrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto y \end{aligned} \quad \text{is a local chart.}$$

$\Rightarrow$ dy is a global generator near (x_0,y_0).

$\Rightarrow$ $\dfrac{dy}{f_1(x,y)}$ is a global generator near (x_0,y_0).

When $f_2(x_0,y_0) \neq 0$,

$$\begin{aligned} x: C &\longrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto x \end{aligned} \quad \text{is a local chart.}$$

$\Rightarrow$ dx is a global generator near (x_0,y_0).

$\Rightarrow$ $-\dfrac{dx}{f_2(x,y)}$ is a global generator near (x_0,y_0).

Plane curve

Prop. Suppose $C = \{F(x,y,z) = 0\} \subseteq \mathbb{P}^2$ is a sm curve of deg d, then $H^0(C, \Omega^1)$ has a basis

$$\left\{ x^i y^j \frac{dx}{F_2(x,y,1)} \;\middle|\; i+j \leqslant d-3 \right\}$$

Proof Assume $[x:y:1] = [a:b:c]$, i.e., $\begin{cases} x = \frac{a}{c} \\ y = \frac{b}{c} \end{cases}$, then

$$\begin{cases} dx = \dfrac{c\,da - a\,dc}{c^2} \\ F_2(x,y,1) = \dfrac{1}{c^{d-1}} F_2(a,b,c) \end{cases}$$

Therefore,

$$\begin{aligned} x^i y^j \frac{dx}{F_2(x,y,1)} &= a^i b^j c^{d-i-j-3} \frac{c\,da - a\,dc}{F_2(a,b,c)} \\ &= \begin{cases} -b^j c^{d-i-j-3} \dfrac{dc}{F_2(1,b,c)} & a \equiv 1 \\ -a^i c^{d-i-j-3}\, \omega & b \equiv 1 \end{cases} \end{aligned}$$

When $b \equiv 1$, denote

$$\omega \triangleq \frac{da}{F_3(a,1,c)} = -\frac{dc}{F_1(a,1,c)}$$

Since $xF_1(x,y,z) + yF_2(x,y,z) + zF_3(x,y,z) = d \cdot F(x,y,z) = 0$,
we get

$$\begin{aligned} c\,da - a\,dc &= (cF_3(a,1,c) + aF_1(a,1,c))\,\omega \\ &= -F_2(a,1,c)\,\omega \end{aligned}$$

Cor. For the Fermat curve

$$C_d:\ x^d + y^d = z^d,$$

$$H^0(C, \Omega^1) = \left\langle x^i y^j \frac{dx}{dy^{d-1}} \;\middle|\; i+j \leqslant d-3 \right\rangle \cong \mathbb{C}^{\frac{(d-1)(d-2)}{2}}$$

[Vakil 21.4.3]

$$\begin{array}{ccccc} H^0(C;\omega_C) & \times & H^1(C;\mathcal{O}_C) & \longrightarrow & H^1(C;\omega_C) \\ \| & & \|\ \text{Čech} & & \|\ \text{Čech} \\ \left\langle \frac{dx}{y}, \dots, x^{g-1}\frac{dx}{y} \right\rangle & \times & \left\langle \frac{y}{x}, \frac{y}{x^2}, \dots, \frac{y}{x^g} \right\rangle & \longrightarrow & \left\langle \frac{dx}{x} \right\rangle \end{array}$$