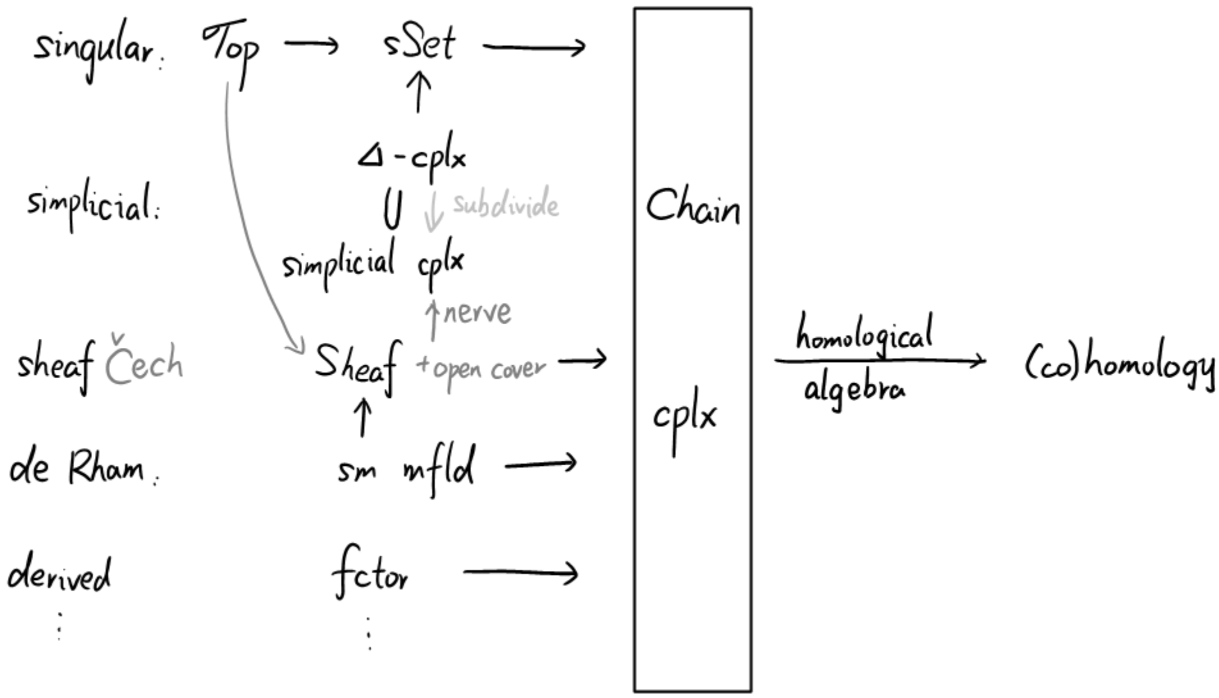
Eine Woche, ein Beispiel
6.25 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

We use α here because we are considering $X=\Delta^n$ case. May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha\in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1-d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2-d_0^1+d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$Hom_{\mathbb{Z}\text{-mod}}\left(\bigoplus_{\alpha\in X_n}\mathbb{Z}, G\right) \cong \prod_{\alpha\in X_n} Hom_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset. If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \mathcal{T}op$ discrete, $X_\cdot = \mathcal{S}(A) \in sSet$, one can compute

$$
\begin{array}{lllllllllll}
C_\cdot(X;G): & 0 & \longleftarrow & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} & \bigoplus_{\alpha\in A} G & \xleftarrow{0} & \bigoplus_{\alpha\in A} G & \xleftarrow{Id} \cdots \\
C^\cdot(X;G): & 0 & \longrightarrow & \prod_{\alpha\in A} G & \xrightarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xrightarrow{Id} \cdots \\
C^{BM}_\cdot(X;G): & 0 & \longleftarrow & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} & \prod_{\alpha\in A} G & \xleftarrow{0} & \prod_{\alpha\in A} G & \xleftarrow{Id} \cdots \\
C_c^\cdot(X;G): & 0 & \longrightarrow & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} & \bigoplus_{\alpha\in A} G & \xrightarrow{0} & \bigoplus_{\alpha\in A} G & \xrightarrow{Id} \cdots
\end{array}
$$

wished $\left\{ C^{BM}_\cdot(X;G),\ C_c^\cdot(X;G) \right.$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.

Notice that $\# \Delta'_k = k+2$, so

$$C_\cdot(\Delta';G): \quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0&1&0 \\ 0&-1&0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1&1&0&0 \\ 0&0&0&0 \\ 0&0&1&1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0&1&0&0&0 \\ 0&-1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&-1&0 \end{pmatrix}} G^{\oplus 5} \cdots$$

basis:

remember indexes:

$d_0' \triangleq x_0$:

$d_1' \triangleq x_1$:

x_0: x_1: x_2:

x_0: x_1: x_2: x_3:

x_0: x_1: x_2: x_3: x_4:

$$0 = x_0 - x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 - x_1 = x_0 - x_1 \longleftarrow\!\shortmid\; x_1$$
$$0 = x_1 - x_1 \longleftarrow\!\shortmid\; x_2$$

$$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow\!\shortmid\; x_1$$
$$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow\!\shortmid\; x_2$$
$$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow\!\shortmid\; x_3$$
$$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow\!\shortmid\; x_4$$

$$x_0 = x_0 - x_0 + x_0 \longleftarrow\!\shortmid\; x_0$$
$$x_0 = x_0 - x_1 + x_1 \longleftarrow\!\shortmid\; x_1$$
$$x_2 = x_1 - x_1 + x_2 \longleftarrow\!\shortmid\; x_2$$
$$x_2 = x_2 - x_2 + x_2 \longleftarrow\!\shortmid\; x_3$$

By taking the transpose, one get

$$C^\cdot(\Delta';G): \quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix} 0&0 \\ 1&-1 \\ 0&0 \end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix} 0&0&0&0 \\ 1&-1&0&0 \\ 0&0&0&0 \\ 0&0&1&-1 \\ 0&0&0&0 \end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$
\begin{tikzcd}
\Delta^1 \arrow[d, "s_0^1"] & C_\cdot(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\,1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1)"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1\,1)"] \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "\mathrm{Id}"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta^0 \arrow[d, "d_0^1"] & C_\cdot(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "\mathrm{Id}"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

s_0^1 d_0^1

s.t. $s^1_{0,*} \circ d^1_{0,*} = \mathrm{Id}_{C_\cdot(\Delta^1;G)}$, $d^1_{0,*} \circ s^1_{0,*} \sim \mathrm{Id}_{C_\cdot(\Delta^1;G)}$.

In fact, we have

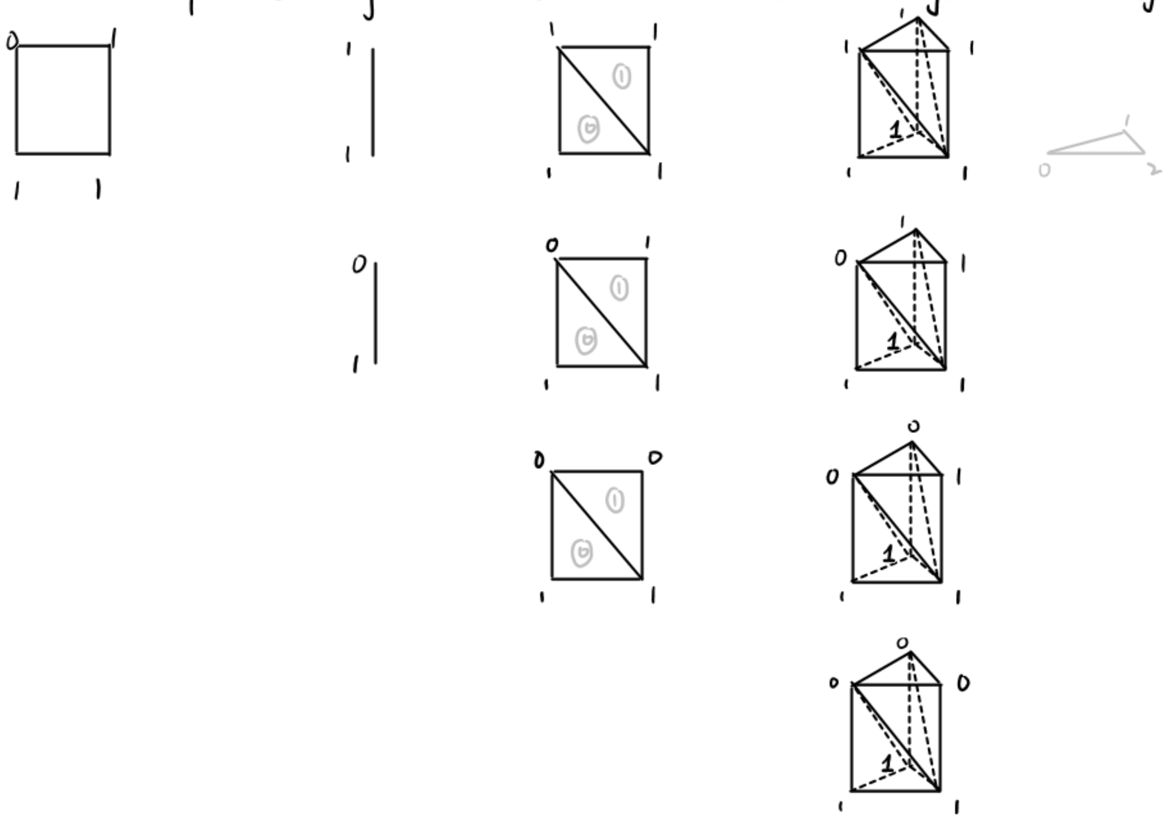
$$
\begin{tikzcd}
C_\cdot(\Delta^1;G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0\\0&0&1\\0&0&-1\\0&0&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

$$x_0 \mapsto x_0$$
$$x_1 \mapsto x_1$$

$$x_0 \mapsto x_0 - x_0 + x_0 = x_0$$
$$x_1 \mapsto x_1 - x_1 + x_1 = x_1$$
$$x_2 \mapsto x_1 - x_2 + x_2 = x_1$$
$$x_3 \mapsto x_1 - x_2 + x_3$$

$$x_0 \mapsto x_0 - x_0 = 0$$
$$x_1 \mapsto x_1 - x_1 = 0$$
$$x_2 \mapsto x_1 - x_2$$

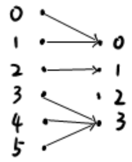
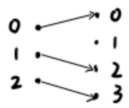
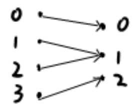
Ex. Observe the picture, try to translate the calculation in geometrical language.



E.g. 3. When we want to compute $H_n(\Delta^m; G)$ and $H^n(\Delta^m; G)$, we'd better to give elements in $\Delta^m_n \approx \{\text{basis of } C_n(\Delta^m; G)\}$ a better notation. The following table shows some typical element in

$$C_n(\Delta^m; G) = \langle \alpha : [n] \longrightarrow [m] \rangle_{\alpha \in \Delta^m_n}.$$

not confuse with $[n]$

element	picture	list	count	degenerate degree
$\alpha : [5] \longrightarrow [3]$ $0 \longmapsto 0$ $1 \longmapsto 0$ $2 \longmapsto 1$ $3 \longmapsto 3$ $4 \longmapsto 3$ $5 \longmapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	$\Delta_5^{3,\{3}$
$d_1^3 : [2] \longrightarrow [3]$ $0 \longmapsto 0$ $1 \longmapsto 2$ $2 \longmapsto 3$		$(0,2,3)$	$[1,0,1,1]$	$\Delta_2^{3,\emptyset}$
$s_1^3 : [3] \longrightarrow [2]$ $0 \longmapsto 0$ $1 \longmapsto 1$ $2 \longmapsto 1$ $3 \longmapsto 2$		$(0,1,1,2)$	$[1,2,1]$	$\Delta_3^{2,\{1}$
$\partial\alpha$	——	$(0,0,3,3,3)$ $-(0,0,1,3,3)$	$[2,0,0,3]$ $-[2,1,0,2]$	$\Delta_4^{3,\{3}$ $\Delta_4^{3,\{2}$

e.g. $\partial[2,5,3,4,1,6,0]$

$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$

In this case, $d : C^n(\Delta^m; G) \longrightarrow C^{n+1}(\Delta^m; G)$ is also not hard to describe.

e.g. $d[2,1,0,3] = [3,1,0,3] - [2,1,1,3]$

$d[2,5,3,4,1,6,0]$

$= [3,5,3,4,1,6,0] + [2,5,3,5,1,6,0]$

$- [2,5,3,4,1,7,0] - [2,5,3,4,1,6,1]$

2. connection with simplicial complexes.

non-degenerate

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\diamond}$:

$$
\begin{tikzcd}[column sep=large]
C_\cdot(\Delta';G): \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G)^{\diamond}: & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
C_\cdot(\Delta';G)^{\diamond}: \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}0\\1\\0\end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

In fact, we have

$$
\begin{tikzcd}[column sep=large, row sep=huge]
C_\cdot(\Delta';G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)

For $X \in sSet$, define

$\varnothing$: non-degenerate

$\lightning$: degenerate

$$X_k^{\varnothing} := \{x \in X_k \mid x \text{ non-degerate}\} = X_k \smallsetminus (sk^{k-1}X)_k$$
$$X_k^{\lightning} := \{x \in X_k \mid x \text{ degenerate}\} = (sk^{k-1}X)_k$$
$$X_k^{\lightning i} := \left\{x \in X_k \;\middle|\; x = \alpha^*(y) \text{ for some } \underset{\alpha:[k-i]\to[k]}{y \in X_{k-i}^{\varnothing}}\right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$\underbrace{0 = (sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}X)_k}_{X_k^{\lightning}} \underbrace{\overset{X_k^{\lightning 0}}{\subset} (sk^kX)}_{X_k^{\varnothing}} = X_k$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\varnothing} = \bigoplus_{\alpha \in X_n^{\varnothing}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\varnothing}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\varnothing}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\varnothing}} G \cdots$$

$$C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots$$

and $H_\cdot(X;G)^{\varnothing}$, $H_\cdot(X;G)^{\lightning}$ as crspd homology.

By definition, $\quad C_\cdot(X;G) \cong C_\cdot(X;G)^{\varnothing} \oplus C_\cdot(X;G)^{\lightning}$

Claim 1. $H_\cdot(X;G)^{\lightning} = 0$, so

$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\varnothing}. \qquad (*)$$

Rmk 1. Roughly, $(*)$ says that

singular homology $\approx$ simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expend $C_\cdot(X;G)$ by double complex.

Def (Double complex of $C.(X;G)$) $\swarrow\leftharpoonup + \leftharpoonup\swarrow = 0$

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0^{s3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{s3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{s3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{s3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{s2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{s2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{s2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{s2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{s1}} G \arrow[l] \arrow[dl, "0"] & \bigoplus_{\alpha\in X_1^{s1}} G \arrow[l] \arrow[dl, "0"] & \bigoplus_{\alpha\in X_2^{s1}} G \arrow[l] \arrow[dl, "0"] & \bigoplus_{\alpha\in X_3^{s1}} G \arrow[l] \arrow[dl, "0"] \\
0 & \bigoplus_{\alpha\in X_0^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_1^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_2^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_3^{\emptyset}} G \arrow[l]
\end{tikzcd}
$$

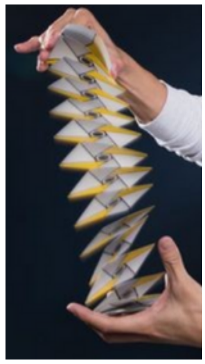
dim 0

dim 1

dim 2

fold $\Downarrow\Uparrow$ expand

$$
0 \leftarrow \bigoplus_{\alpha\in X_0} G \leftarrow \bigoplus_{\alpha\in X_1} G \leftarrow \bigoplus_{\alpha\in X_2} G \leftarrow \bigoplus_{\alpha\in X_3} G
$$



fold / expend

Eg. For $X = \Delta'$, we have double complex

$$
\begin{tikzcd}
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_4\rangle \arrow[l] \arrow[dl, "0"'] & \cdots \arrow[l, "\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}"] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] & \langle x_1, x_2, x_3\rangle \arrow[l, "\begin{pmatrix}1&0&0\\0&0&-1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}"'] & \cdots \arrow[ul, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & \langle x_0, x_2\rangle \arrow[l] \arrow[dl, "0"'] & \langle x_1, x_2\rangle \arrow[l, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \arrow[dl, "0"] & 0 \arrow[l] \arrow[dl] & \\
0 & \langle x_0, x_1\rangle \arrow[l] & \langle x_1\rangle \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l] & 0 \arrow[l] &
\end{tikzcd}
$$

fold $\Downarrow\Uparrow$ expand

$$
0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots
$$

Claim 2. We have chain homotopy equivalence between the following two cplx:

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_n^{\diamond}} G \arrow[l] \arrow[d, equal] & \bigoplus_{\alpha\in X_{n+1}^{\S 1}} G \arrow[l, "0"'] \arrow[d, bend right, "0"'] & \bigoplus_{\alpha\in X_{n+2}^{\S 2}} G \arrow[l, "\partial^{\swarrow}"'] \arrow[d, bend right, "0"'] & \bigoplus_{\alpha\in X_{n+3}^{\S 3}} G \arrow[l, "\partial^{\swarrow}"'] \arrow[d, bend right, "0"'] & (**) \\
0 & \bigoplus_{\alpha\in X_n^{\diamond}} G \arrow[l] & 0 \arrow[l] \arrow[u, bend right, "0"'] & 0 \arrow[l] \arrow[u, bend right, "0"'] & 0 \arrow[l] \arrow[u, bend right, "0"']
\end{tikzcd}$$

i.e. (**) is exact on all terms except $\bigoplus_{\alpha\in X_n^{\diamond}} G$.

Proof idea of Claim 2 for $X=\Delta^m$. (can be generalized to arbitrary X)

$$\begin{tikzcd}
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{\S k-1}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] & \bigoplus_{\alpha\in X_{n+k}^{\S k}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dl, "s"] & \bigoplus_{\alpha\in X_{n+k+1}^{\S k+1}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dl, "s"] & \cdots \arrow[l] \arrow[dl, "s"] \\
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{\S k-1}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k}^{\S k}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k+1}^{\S k+1}} G \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Define

$$s[\underbrace{a_1,\cdots a_l}_{\not\ni\{0,1\}},\ \underset{\in\{0,1\}}{a_{l+1}},\cdots,a_m] = \begin{cases} (-1)^{i}[a_1,\cdots,a_l,a_{l+1}+1,\cdots,a_m], & a_{k+1}\text{ even} \\ 0, & a_{k+1}\text{ odd}\end{cases}$$

$i=\sum_{j=1}^{l} a_j$

Ex. Check that s is a homotopy.

e.g. $X=\Delta^3$, $n=2$, $k=3$ $\Rightarrow m=3$, $n+k=5$

$-[2,1,0,2] \longleftarrow\!\shortmid [2,1,0,3]$

$-[2,1,0,2] \longmapsto -[3,1,0,2]$

$[2,1,0,3] \longmapsto [3,1,0,3]$

$[2,1,0,3] + [3,1,0,2] \longleftarrow\!\shortmid [3,1,0,3]$

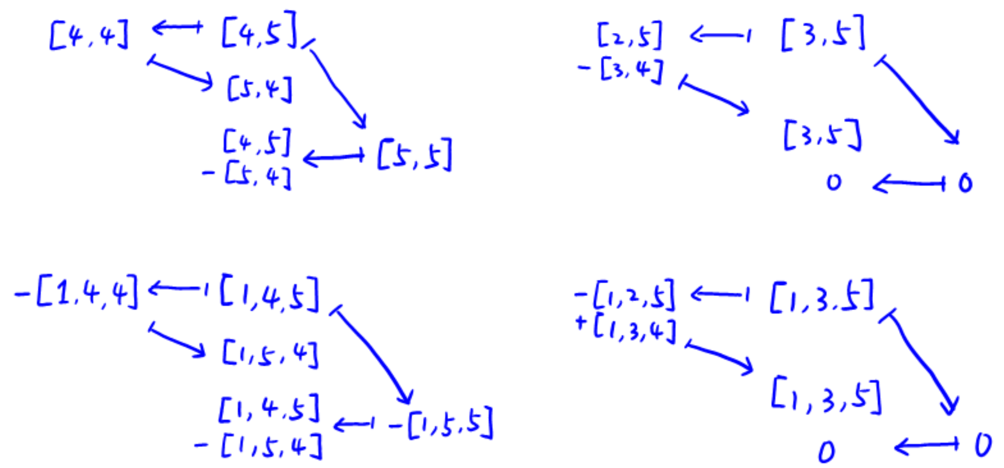
$X=\Delta^6$, $n=5$, $k=15$ $\Rightarrow m=6$, $n+k=20$

$[2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] \longleftarrow\!\shortmid [2,5,3,4,1,6,0]$

$[2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] \longmapsto [3,4,3,4,1,6,0] - [3,5,2,4,1,6,0]$

$[2,5,3,4,1,6,0] \longmapsto [3,5,3,4,1,6,0]$

$[2,5,3,4,1,6,0] - [3,4,3,4,1,6,0] + [3,5,2,4,1,6,0] \longleftarrow\!\shortmid [3,5,3,4,1,6,0]$

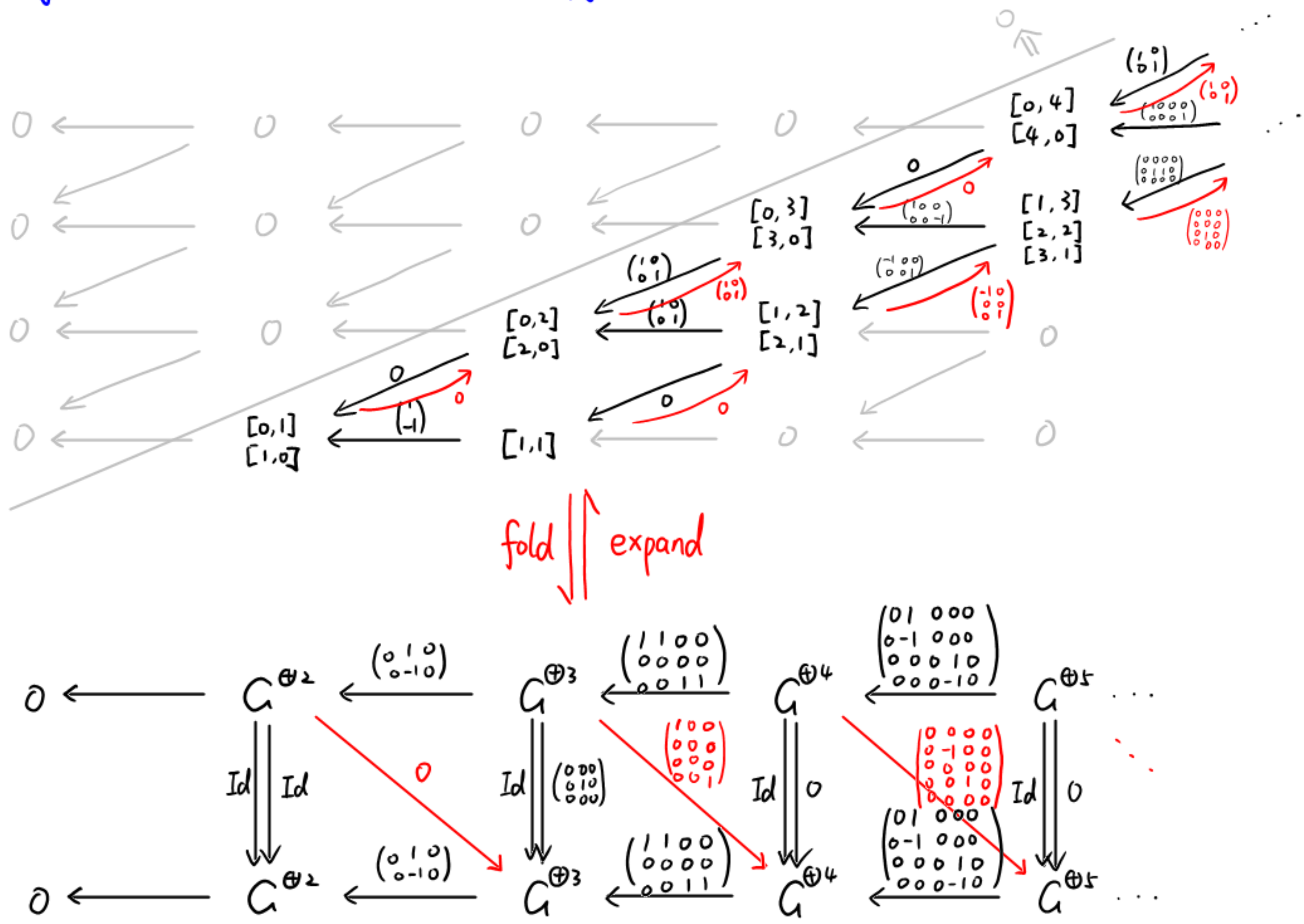


In conclusion,

$$\text{Claim 2} \Rightarrow \text{Claim 1} \Rightarrow \text{Rmk 1}$$

Coming back to Eg.2, one can now find a homotopy without guess.

Eg. For $X=\Delta^1$, we have homotopy



Ex. Check that (I believe that this argument also works for general sset X)

① $\overset{s}{\nearrow}\,\underset{\partial^{\leftarrow}}{\longleftarrow} + \overset{\partial^{\leftarrow}}{\longleftarrow}\,\underset{s}{\nearrow} = 0$

② the collected s is a homotopy.

3. more structures

math.stackexchange.com/questions/2559705/cup-product-why-do-we-need-to-consider-cohomology-with-coefficients-in-a-ring

When $G = R$ is a k-alg, the product structure on $C^{\cdot}(X;R)$ is defined by

$$C^i(X;R)\otimes C^j(X;R) \xrightarrow{d_{i,j,*}\otimes d'_{i,j,*}} C^{i+j}(X;R)\otimes C^{i+j}(X;R) \longrightarrow C^{i+j}(X;R\otimes R) \xrightarrow{\text{multiply}} C^{i+j}(X;R)$$

$$f_1 \otimes f_2 \longmapsto (f_1\circ d_{i,j,*})\otimes(f_2\circ d'_{i,j,*})$$

$$f \otimes g \longmapsto fg$$

the $C^{\cdot}(X;R)$-module structure on $C_{\cdot}(X;G)$ is defined by

$$C^i(X;R)\otimes C_{i+j}(X;R) \xrightarrow{Id\otimes d^*_{i,j}(-)\otimes d'^*_{i,j}(-)} C^i(X;R)\otimes C_i(X;R)\otimes C_j(X;R) \longrightarrow R\otimes C_j(X;R\otimes R) \xrightarrow{\text{multiply}} C_j(X;R)$$

$$f \otimes \alpha \longmapsto f(d^*_{i,j}(\alpha))\otimes d'^*_{i,j}(\alpha)$$

$$r \otimes \beta \longmapsto r\beta$$

where $(i=3, j=2)$

$d_{i,j} : [i] \longrightarrow [i+j]$ $\qquad$ $d'_{i,j} : [j] \longrightarrow [i+j]$

$d_{i,j} = [\underbrace{1,\dots,1}_{i+1 \text{ many}},\underbrace{0,\dots,0}_{j \text{ many}}]$ $\qquad$ $d'_{i,j} = [\underbrace{0,\dots,0}_{i \text{ many}},\underbrace{1,\dots,1}_{j+1 \text{ many}}]$

$\otimes$ are over k. Notice that

$$\begin{cases} C_i(X;R) = \bigoplus_{\alpha\in X_i} R \\ C^i(X;R) = \prod_{\alpha\in X_i} R \end{cases}$$

are bi R-modules.

Ex. Show that

- $C^{\cdot}(X;R)$ is a dga,
- $H^{\cdot}(X;R)$ is a graded R-alg,
- $H_{\cdot}(X;R)$ is a graded $H^{\cdot}(X;R)$-module.