Eine Woche, ein Beispiel

12.17 calculation of NMD

Goal: compute normal Morse data (NMD) $\{f\geq 0\} \overset{\tilde{i}}{\hookrightarrow} X \overset{\tilde{j}}{\hookleftarrow} \{f<0\}$

$$
\begin{aligned}
NMD(\mathcal{F}^{\cdot}, S) &= (R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathcal{F}^{\cdot}|_{N\cap X}))_x \\
&\overset{\substack{S=\{*\} \\ X \text{ is cone} \\ f(x)=0}}{\underset{\text{compatible}}{=\!=}} (R\Gamma_{\{f\geq 0\}}(\mathcal{F}^{\cdot}))_x \\
&= \iota_x^* i^! \mathcal{F}^{\cdot} \\
&= R\Gamma(X, \{f<0\}, \mathcal{F}^{\cdot}) \\
&= \mathrm{Fiber}\left(R\Gamma(X, \mathcal{F}^{\cdot}) \longrightarrow R\Gamma(\{f<0\}, \mathcal{F}^{\cdot})\right) \\
&= \mathrm{Fiber}\left(\mathcal{F}_x \longrightarrow R\Gamma(\iota_x, \mathcal{F}^{\cdot})\right)
\end{aligned}
$$

1. low dimensional case
2. quadratic hypersurface
3. du val singularity
4. other quantities

Ref:

https://bastian.rieck.me/blog/posts/2019/morse_theory/

https://oldbookstonew.blogspot.com/
Contains the following books:

[MilnorMT]: Morse Theory by Milnor
[MilnorCC]: Characteristic Classes by Stasheff and Milnor

[MilnorSing]: singular points of complex hypersurfaces by Milnor

[Maxim20]:notes on vanishing cycles and applications
https://people.math.wisc.edu/~lmaxim/vanishing.pdf

1. low dimensional case

E.g. $X = \mathbb{CP}^1$ $\quad f: \mathbb{CP}^1 \dashrightarrow \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\quad L_x = \{*\}$ $\quad S = \{\infty\}$

$\infty \longmapsto 0$

$\mathcal{F}$	$NMD(\mathcal{F}^{\cdot}, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^{\cdot})$
$i_* \underline{\mathbb{Q}}_{\{\infty\}}$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_{\mathbb{CP}^1}[1]$	0	$\mathbb{Q}[1]$	$\mathbb{Q}[1]$
$Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}[1]$
$j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	$\mathbb{Q}$	0	$\mathbb{Q}[1]$
$P(\phi)$	$\mathbb{Q}^2$	$\mathbb{Q}$	$\mathbb{Q}[1]$

E.g. $X = \{z_2^2 = z_1^3\}$ $\quad f: X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\quad L_x = \{a, b\}$ $\quad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^{\cdot}, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^{\cdot})$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[1]$	$\mathbb{Q}$	$\mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$Rj_* \underline{\mathbb{Q}}_{\mathcal{U}}[1]$	$\mathbb{Q}^2$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$j_! \underline{\mathbb{Q}}_{\mathcal{U}}[1]$	$\mathbb{Q}^2$	0	$\mathbb{Q}^2[1]$
$P(\phi)$	$\mathbb{Q}^3$	$\mathbb{Q}$	$\mathbb{Q}^2[1]$

E.g. $X = \mathbb{C} \cup_{\{0\}} \mathbb{C} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$

$f : X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1 + z_2} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\qquad L_x = \{a, b\}$ $\qquad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^\cdot, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^\cdot)$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[1]$	$\mathbb{Q}$	$\mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$Rj_* \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	$\mathbb{Q}^2 \oplus \mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$
$j_! \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	0	$\mathbb{Q}^2[1]$
$\pi^! \mathbb{Q}[-1]$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$
$IC(\underline{\mathbb{Q}}_U[1])$	0	$\mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$

E.g. $X = X_3$ $\qquad f : X \hookrightarrow \mathbb{C}^3 \xrightarrow{z_3} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\qquad L_x = \mathbb{C}^\times$ $\qquad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^\cdot, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^\cdot)$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[2] = \pi^! \mathbb{Q}[-2]$	$\mathbb{Q}$	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$Rj_* \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[2] \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$j_! \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	0	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$IC(\underline{\mathbb{Q}}_U[2])$	$\mathbb{Q}$	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$

2. quadratic hypersurface

This table is computed by Lefschetz hyperplane theorem and Chern class.

This table is computed by open-closed formalism. (Q-coefficient)
Using the Morse theory, one can show that (A variant of [Maxim20, Example 2.18]

This table is computed by spectral sequence and Euler class.
Using the Morse theory, one can show that

To compute the stalk of IC sheaf, one truncates in the middle.
Z-coefficient cohomology need more work on Euler class.

After truncation, only the red one remains.
After truncation, nothing remains.

3. du val singularity

https://math.stackexchange.com/questions/40351/what-are-the-finite-subgroups-of-su-2c

Name	$R(x,y,z)$	gp G	$\#G$	G/G'	det(Cartan)	
A_n	$x^2+y^2+z^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$	
D_n	$x^2+y^2z+z^{n-1}$	$\mathbb{BD}_{2(n-2)} = \mathrm{Dic}_{n-2}$	$4(n-2)$	$\begin{cases} \mathbb{Z}/4\mathbb{Z}, & n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & n \text{ even} \end{cases}$	4	dicyclic
E_6	$x^2+y^3+z^4$	$\mathbb{BT} \cong SL_2(\mathbb{F}_3)$	24	$\mathbb{Z}/3\mathbb{Z}$	3	
E_7	$x^2+y^3+yz^3$	$\mathbb{BO} \cong 2.S_4^-$	48	$\mathbb{Z}/2\mathbb{Z}$	2	(48,28)
E_8	$x^2+y^3+z^5$	$\mathbb{BD} \cong SL_2(\mathbb{F}_5)$	120	1	1	

$\mathcal{U}$ = link	0	1	2	3
$H^\cdot(\mathcal{U};\mathbb{Z})$	$\mathbb{Z}$	0	G/G'	$\mathbb{Z}$
$H_\cdot(\mathcal{U};\mathbb{Z})$	$\mathbb{Z}$	G/G'	0	$\mathbb{Z}$

L_x homotopic equiv to $\begin{cases} S^1 & A_n \,\&\, D_n \\ S^1 \vee S^1 & E_6, E_7, E_8 \end{cases}$

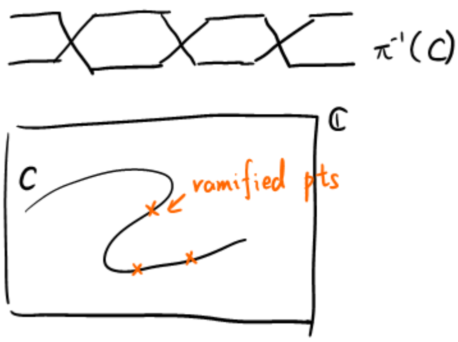
$\Rightarrow H^\cdot(L_x;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & A_n \,\&\, D_n \\ \mathbb{Z} \oplus \mathbb{Z}^2[-1] & E_6, E_7, E_8 \end{cases}$

$\Rightarrow H^\cdot(\mathcal{U}, L_x;\mathbb{Q}) = \begin{cases} \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] & A_n \,\&\, D_n \\ \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] & E_6, E_7, E_8 \end{cases}$

$\Rightarrow NMD(X; IC(\mathbb{Q}_{\mathcal{U}}[2])) = \begin{cases} \mathbb{Q} & A_n \,\&\, D_n \\ \mathbb{Q}^2 & E_6, E_7, E_8 \end{cases}$

Three different arguments for $L_x \sim S^1$ or $S^1 \vee S^1$:

1. Morse index [MilnorSing, Theorem 6.5 & 5.11]
2. Riemann surface, contract to $\pi^{-1}(C)$
3. Join construction, see [Maxim 20, Example 2.18]



These singularities can be used to understand 2-dimensional weighted projective spaces.
For weighted projective spaces, the local charts are of form C^n quotient cyclic group.

The topology of cone is still easy to compute by spectral sequence. For the result, see:
http://www.map.mpim-bonn.mpg.de/Fake_lens_spaces
However, the equations become harder to get, and we don't know the topology of the link.
I just believe that there should be a answer for all these singularities.

4. other quantities

Setting $\mathcal{M}$: analytic mfld e.g. $\mathcal{M} = \mathbb{C}^n$ or $\mathbb{CP}^n$

$X \subset \mathcal{M}$ analytic variety of $\dim_{\mathbb{C}} X = m$

$\mathcal{S}$: $\phi \subset \{0\} \subset X$ where 0 is the only singularity

$x_0 \in X - \{0\}$

$\mathcal{F} \in Perv_{\mathcal{S}}(X)$

$\mathcal{L} := \mathcal{F}|_{X-\{0\}}[-m]$ local system on $X - \{0\}$ with rank r

Special case: $\mathcal{F} = IC(\mathcal{L}[m])$

Task: Compute the following quantities.

$$\begin{tikzcd}
(1)\ NMD(\mathcal{F}, \{0\}) \arrow[d] & & (3)\ NMD(\mathcal{F}, X-\{0\}) \arrow[d] & & \\
(2)\ m(\{0\}) \arrow[dr] & & (4)\ m(X-\{0\}) \arrow[dl] & (6)\ Eu_X \arrow[d] & Eu_{\{0\}} \arrow[d] \\
& (5)\ CC(\mathcal{F}) \arrow[dr] & & (7)\ deg(\Lambda_X) \arrow[dl] & deg(\Lambda_{\{0\}}) \arrow[dll] \\
& & (8)\ \chi_{IC}(X) & &
\end{tikzcd}$$

Here we use notations in https://arxiv.org/abs/2105.13069v2. 6-8 comes from my supervisor's notation, if needed I should find some references for the definition.

① See the examples before

② $m(\{0\}) = \chi(NMD(\mathcal{F}, \{0\}))$

③ $NMD(\mathcal{F}, X-\{0\}) \cong \mathcal{F}_{x_0} \cong \mathbb{Q}^r[m]$

④ $$m(X-\{0\}) = (-1)^{\dim_{\mathbb{C}}(X-\{0\})} \chi(NMD(\mathcal{F}, X-\{0\}))$$
$$= (-1)^m \cdot (-1)^m \cdot r$$
$$= r$$

⑤ $$CC(\mathcal{F}) = m(X-\{0\}) \, [\overline{T^*_{X-\{0\}}M}] + m(\{0\}) \, [\overline{T^*_{\{0\}}M}]$$
$$= r \, [T^*_X M] + m(\{0\}) \, [T^*_{\{0\}}M]$$
$$= r \, \Lambda_X + m(\{0\}) \, \Lambda_{\{0\}}$$

recall: $[T^*_X M] = [\overline{T^*_{X-\{0\}}M}]$ $\Lambda_{\overline{S}} := [\overline{T^*_S M}]$

⑥ Need to check the definition.

For $X \subset \mathbb{C}^2$ cuspidal cubic, $\mathrm{Sing}(X) = \{p_0\}$,

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X - \{p_0\} \\ 2 & p = p_0 \end{cases}$$

In general, from my memory it looks like:

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X_{sm} \\ \geq 1 & p \in X - X_{sm} \end{cases}$$

⑦ $\deg(\Lambda_X) := \#(\Lambda_X \cdot \Lambda_M)$ in T^*M

$$= (-1)^m \chi(X, Eu_X)$$

$$= (-1)^m \left(\chi(X - \{0\}) \cdot Eu_X(x_0) + \chi(\{0\}) \cdot Eu_X(0)\right)$$

$$= (-1)^m \left(\chi(X-\{0\}) + Eu_X(0)\right)$$ $= -2$ for $X = M = \mathbb{C}\mathbb{P}^1$

$\deg(\Lambda_{\{0\}}) := \#(\Lambda_{\{0\}} \cdot \Lambda_M)$

$$= \chi(\{0\}, Eu_{\{0\}})$$

$$= \chi(\{0\}) \cdot Eu_{\{0\}}(0)$$

$$= 1$$

⑧ $(-1)^m \chi_{IC}(X) = \deg(CC(\mathcal{F}))$ Here, $\mathcal{F} := IC(\underline{\mathbb{Q}}_{X-\{0\}}[m])$, $r = 1$

$$= \deg(r\Lambda_X + m(\{0\})\Lambda_{\{0\}})$$

$$= r \cdot \deg \Lambda_X + m(\{0\}) \deg \Lambda_0$$

$$= \deg \Lambda_X + m(\{0\})$$

$$\Rightarrow \quad \chi_{IC}(X) = \chi(X - \{0\}) + Eu_X(0) + (-1)^m m(\{0\})$$

X	$\chi(X-\{0\})$	$Eu_X(0)$	$m(\{0\})$	$\chi(X)$
$\mathbb{C}$	0	1	0	1
$\{y^2 = x^3\}$	0	2	1	1