Eine Woche, ein Beispiel

9.18 reps of p-adic groups

This is also an unfinished task. I'm afraid that I forget those materials I organized.
main ref: The Local Langlands Conjecture for GL(2)

Now you can see [GL case]: https://github.com/ramified/personal_handwritten_collection/raw/main/Langlands/GL_case.pdf

Process

1. new notations
2. preliminaries
   - group
   - chain order
3. statement of classification (without proof)

realization

4. fin dim

   https://mathoverflow.net/questions/34374/any-finite-dimensional-admissiblesmooth-irreducible-representation-of-gl2-q-p

5. other principal series
   - construction
   - proof
6. Cuspidal reps
7. Applications

1. new notations

I don't want to bother you or make you confused, so I collect my notations here. Often it's not rigorous defined. You can view this section as a dictionary of notations.

from 2022.04.24

F: non-archi local field.

https://math.stackexchange.com/questions/299626/the-center-of-operatornamegln-k

$A = M_{2\times 2}(F)$ $\quad G = GL_2(F)$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $\quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\quad Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = Z(G)$ $\quad S = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$

$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad T^0 = \begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{pmatrix}$ $\quad N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}$ $\quad N_j' = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{pmatrix}$

page	name	symbol	case $e_{\mathfrak{A}} = 1$	case $e_{\mathfrak{A}} = 2$
86	$\mathcal{O}$-lattice chain	$\mathcal{L}$	$\mathfrak{p}^{-1}\oplus\mathfrak{p}^{-1} \subseteq \mathcal{O}\oplus\mathcal{O} \subseteq \mathfrak{p}\oplus\mathfrak{p} \subseteq \cdots$	$\mathfrak{p}^{-1}\oplus\mathcal{O} \subseteq \mathcal{O}\oplus\mathcal{O} \subseteq \mathcal{O}\oplus\mathfrak{p} \subseteq \cdots$
87	$\mathcal{O}$-orders chain order	$\mathfrak{A} = \mathfrak{A}_{\mathcal{L}}$	$\mathfrak{M} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\mathfrak{J} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$
88	prime element	Π	$\begin{pmatrix} \pi & \\ & \pi \end{pmatrix}$	$\begin{pmatrix} & 1 \\ \pi & \end{pmatrix}$
88	Jacobson radical	$\mathrm{Jac}(\mathfrak{A})$	$\mathrm{Jac}(\mathfrak{M}) = \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$	$\mathrm{Jac}(\mathfrak{J}) = \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$
88		$\mathcal{U}_{\mathfrak{A}} = \mathcal{U}_{\mathfrak{A}}^{(0)} = \mathfrak{A}^\times$	$K_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}^\times$	$I_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}^\times = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix}$
88		$\mathcal{U}_{\mathfrak{A}}^{(n)} = 1 + \mathrm{Jac}(\mathfrak{A})^n$ $n \geq 1$	$K_n = 1 + \begin{pmatrix} \mathfrak{p}^n & \mathfrak{p}^n \\ \mathfrak{p}^n & \mathfrak{p}^n \end{pmatrix}$	$I_{2k-1} = 1 + \begin{pmatrix} \mathfrak{p}^k & \mathfrak{p}^{k-1} \\ \mathfrak{p}^k & \mathfrak{p}^k \end{pmatrix}$ $\quad I_{2k} = 1 + \begin{pmatrix} \mathfrak{p}^k & \mathfrak{p}^k \\ \mathfrak{p}^{k+1} & \mathfrak{p}^k \end{pmatrix}$
89		$K_{\mathfrak{A}}$	$K_0 \rtimes \left\langle \begin{pmatrix} \pi & \\ & \pi \end{pmatrix} \right\rangle$	$I_0 \rtimes \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$

Rmk: For the convenience of handwritting and recognization, we use slightly different notations, which are listed as follows:

$\varpi \rightsquigarrow \pi$, $\mathfrak{U} \rightsquigarrow \mathfrak{A}$: I can not write original letters smoothly.

$K \rightsquigarrow K_0$: K is kept to denote arbitrary cpt open subgp.

(Luckily we won't meet any K-theory in this document) For K-theory, there is usually a bracket after it.

$\mathrm{rad}\,\mathfrak{A} \rightsquigarrow \mathrm{Jac}(\mathfrak{A})$: to avoid confusion with other radicals of a ring.

$\mathrm{Jac}(\mathfrak{A})$: Jacobson radical

$\mathfrak{J}$: $= \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$

$U \rightsquigarrow \mathcal{U}$: avoid confusion with union.

$\mathcal{U}_F := \mathcal{U}_F^{(0)} = \mathcal{O}^\times$ $\qquad \mathcal{U}_F^{(n)} := 1 + \mathfrak{p}^n$ $\qquad n \geq 1$

$\mathcal{U}_{\mathfrak{A}} := \mathcal{U}_{\mathfrak{A}}^{(0)} = \mathfrak{A}^\times$ $\qquad \mathcal{U}_{\mathfrak{A}}^{(n)} := 1 + \mathrm{Jac}(\mathfrak{A})^n$ $\qquad n \geq 1$

$\mathcal{U}_{\mathfrak{A}}^{Z} :=$

$\mathcal{U}_{\mathfrak{A}}^{Z,1} :=$

$\mathcal{U}_{\mathfrak{A}}^{Z,2} :=$

Def (level) For $(\rho, V) \in \mathrm{Irr}(G)$,

$$S(\rho) := \left\{ (\mathfrak{A}, n) \;\middle|\; \begin{array}{l} \mathfrak{A} : \text{chain order} \quad n \geq 0 \\ \rho|_{\mathcal{U}_{\mathfrak{A}}^{(n+1)}} \text{ is trivial} \end{array} \right\}$$

(normalized level)

$$l(\rho) := \min \{ n/e_{\mathfrak{A}} \mid (\mathfrak{A}, n) \in S(\rho) \}$$
$$= \min_{(\mathfrak{A}, n)} \{ n/e_{\mathfrak{A}} \mid \rho|_{\mathcal{U}_{\mathfrak{A}}^{(n+1)}} \text{ is trivial} \}$$

$K_0 = GL_2(\mathcal{O}) \supset I_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}^\times = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix}$ — level 0

$I_0 \supset I_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix}$ — 0

$K_1 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & \mathfrak{p} \end{pmatrix} \subset I_1$, $K_1 \supset I_2$ — $\frac{1}{2}$

$I_2 = 1 + \begin{pmatrix} \mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p} \end{pmatrix}$ — $\frac{1}{2}$

$I_2 \supset I_3 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & \mathfrak{p}^2 \end{pmatrix}$ — 1

$K_2 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^2 & \mathfrak{p}^2 \end{pmatrix} \subset I_3$, $K_2 \supset I_4$ — $\frac{3}{2}$

$I_4 = 1 + \begin{pmatrix} \mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^3 & \mathfrak{p}^2 \end{pmatrix}$ — $\frac{3}{2}$, ≥ 2

Cor $l(\pi) = 0 \iff \pi|_{K_1}$ is trivial.

Rmk. $K_n \triangleleft K_0$, $I_n \triangleleft I_0$ are open normal subgps.