

# Modular form

## 5. moduli interpretation

- 1 level structure
2. moduli interpretation of  $\Gamma \backslash \mathcal{H}$
3. cplx polarization
4. Siegel moduli space
- 5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S, T \mid S^4 = (ST)^6 = Id \rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}'$	$\{\pm Id\}$	$\langle S, T, (\begin{smallmatrix} 1 & \\ -1 & \end{smallmatrix}) \rangle$
$PSL_2(\mathbb{Z})$	✗	$\mathcal{H}$	$Id$	$\langle S, T \mid S^2 = (ST)^3 = Id \rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}'$	$Id$	$\langle S, T, (\begin{smallmatrix} 1 & \\ -1 & \end{smallmatrix}) \rangle$

can't define  $SL_2/\mathbb{G}_m$

<https://arxiv.org/pdf/1605.07726.pdf>

<https://math.stackexchange.com/questions/1844504/why-is-this-isomorphism-of-pgl2-mathbbz-with-a-coxeter-group-injective>

See [<https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz>] for a higher dimension generalization.

Ex.  $A \leq B \leq C$  gp       $A \triangleleft C \Rightarrow A \triangleleft B$

no other restrictions. i.e. the following cases may happen:

$$\begin{array}{cccccc}
 A \triangleleft B \triangleleft C & A \triangleleft B \leq C & A \triangleleft B \triangleleft C & A \triangleleft B \leq C & A \leq B \triangleleft C & A \leq B \leq C \\
 \vdash \triangleleft \dashv & \vdash \triangleleft \dashv & & & & \\
 \checkmark & \checkmark & C_2 \triangleleft A_4 \triangleleft S_4 & & \checkmark & S_2 \leq S_3 \leq S_4
 \end{array}$$

wiki: congruence subgroup

### 1 level structure

Def (congruence subgp) They're the preimage of some subgp of  $SL_2(\mathbb{Z}/N\mathbb{Z})$ .

$$\begin{array}{ccccc}
 \Gamma(N) & \xrightarrow{\quad} & \{Id\} & & \\
 \cap & & \cap & & \\
 \Gamma_1(N) & \xrightarrow{\quad} & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} & & \\
 \cap & & \cap & & \\
 \Gamma_0(N) & \xrightarrow{\quad} & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} & & \\
 \cap & & \cap & & \\
 \Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{\text{[WWL, Prop 1.4.4]}} & SL_2(\mathbb{Z}/N\mathbb{Z}) & & \\
 \cup & & \cup & & \\
 \Gamma^0(N) & \xrightarrow{\quad} & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} & & \\
 \cup & & \cup & & \\
 \Gamma'(N) & \xrightarrow{\quad} & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} & & 
 \end{array}$$

$\nabla$   $SL_2(\mathbb{Z}/N\mathbb{Z})$  is not  $\mathbb{Z}/N\mathbb{Z}$ -pt of  $SL_2 = \text{Spec } \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}] / (a_{11}a_{22} - a_{12}a_{21} - 1)$ ,  
but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) = SL_2, \mathbb{Z}/N\mathbb{Z}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/N\mathbb{Z} \right\} \text{ s.t. } ad - bc = 1$$

Ex. Verify the following tables (left comes from right)

$\frac{A \triangleleft B}{A}$	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	-	✓	✓	✓
$\Gamma_1(N)$	-	-	✓	✗
$\Gamma_0(N)$	-	-	-	✗
$\Gamma(1)$	-	-	-	-

$\frac{A \triangleleft B}{A}$	$N$	$B$	$G$
$N$	-	✓	✗
$B$	-	-	✗
$G$	-	-	-

Ex. show [WWL, 练习1.4.14]

练习 1.4.14 对所有正整数  $N$ , 证明

$$(\mathrm{SL}(2, \mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p: \text{素数} \\ p|N}} \left(1 - \frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2, \mathbb{Z}) : \Gamma_1(N)) = N^2 \prod_{\substack{p: \text{素数} \\ p|N}} \left(1 - \frac{1}{p^2}\right),$$

$$\begin{aligned} (\mathrm{SL}(2, \mathbb{Z}) : \Gamma_0(N)) &= |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1} \cdot (\mathrm{SL}(2, \mathbb{Z}) : \Gamma_1(N)) \\ &= N \prod_{p|N} \left(1 + \frac{1}{p}\right). \end{aligned}$$

A. Reduced to computation of  $|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})|, |\mathcal{B}(\mathbb{Z}/N\mathbb{Z})|, |\mathcal{N}(\mathbb{Z}/N\mathbb{Z})|$ .

Try  $N=5, 4, 6$  if you don't understand the process.

Notation:  $\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z})_{\text{prim}}^{\oplus 2} / (\mathbb{Z}/N\mathbb{Z})^* \stackrel{[6.3M]}{=} \mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: <https://arxiv.org/pdf/2010.15543v2.pdf>

▽  $\mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}$  is covered by two  $\mathcal{A}_{\mathbb{Z}/N\mathbb{Z}}$ 's [4.5.N],

$\begin{pmatrix} 3 \\ 2 \end{pmatrix} \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathcal{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$ , these do not contradict with each other.

Reason: Spec  $\mathbb{Z}/6\mathbb{Z}$  are two pts. They may lie in different piece of  $\mathcal{A}_{\mathbb{Z}/N\mathbb{Z}}$ .

①  $|\mathrm{SL}_2(\mathbb{F}_p)| = p^3 - p$

$$|\mathcal{B}(\mathbb{F}_p)| = p^2 - p$$

$$|\mathcal{N}(\mathbb{F}_p)| = p$$

$$\# \mathbb{F}_p^\times = p-1$$

②  $|\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e} - p^{3e-2}$

$$|\mathcal{B}(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e} - p^{2e-1}$$

$$|\mathcal{N}(\mathbb{Z}/p^e\mathbb{Z})| = p^e$$

$$\# (\mathbb{Z}/p^e\mathbb{Z})^\times = p^e - p^{e-1}$$

③  $|\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p^2}\right)$

$$|\mathcal{B}(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$$

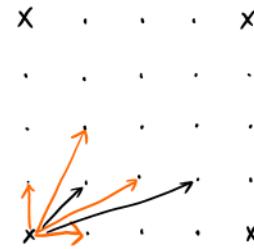
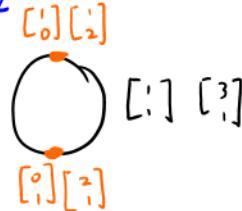
$$|\mathcal{N}(\mathbb{Z}/N\mathbb{Z})| = N$$

$$\# (\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$$

$$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \rightarrow \mathbb{P}'(\mathbb{Z}/p^e\mathbb{Z}) \rightarrow \mathbb{P}'(\mathbb{Z}/p\mathbb{Z})$$

$$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^n \mathbb{P}'(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$$

E.g.  $\mathbb{Z}/4\mathbb{Z}$



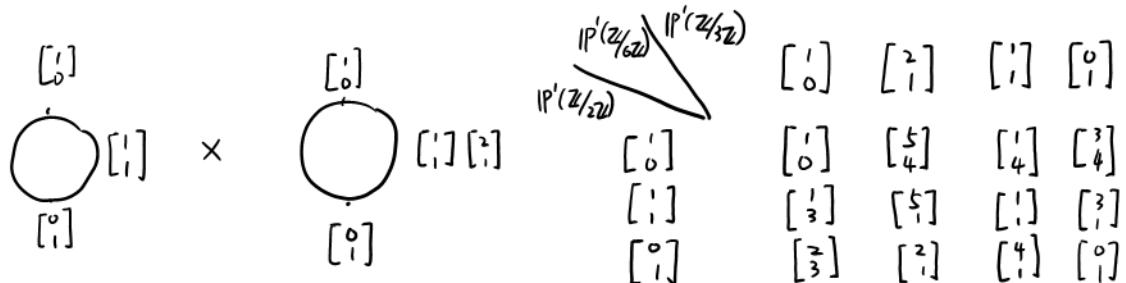
E.g.  $\mathbb{Z}/6\mathbb{Z}$

$$\mathbb{P}_{\mathbb{Z}/6\mathbb{Z}} = \text{Proj } \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f \in S \\ f \text{ homogeneous}}} \text{Spec } (\mathbb{Z}/6\mathbb{Z}[x,y]_f).$$

e.g.  $(x-2, y-3) \triangleleft \mathbb{Z}/6\mathbb{Z}[x,y]$  is not prime.

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) &\cong \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3) \\ &\cong \mathbb{P}_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3) \end{aligned}$$

Ex. Use [Vakil, 6.3.M] to compute  $\mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$ . Enjoy it!



Rmk. The original proof is also good, but less geometrically obvious:

(Now you should understand the geometry in every step)

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ SL_2(\mathbb{Z}/p^e\mathbb{Z}) & & SL_2(\mathbb{F}_p) & & \\ \downarrow & & \downarrow & & \\ 0 \rightarrow 1 + pM_2(\mathbb{Z}/p^e\mathbb{Z}) \xrightarrow{p^{4e-4}} GL_2(\mathbb{Z}/p^e\mathbb{Z}) \rightarrow GL_2(\mathbb{F}_p) \rightarrow 0 & & & & \\ \downarrow & & \downarrow & & \\ (\mathbb{Z}/p^e\mathbb{Z})^\times & & \mathbb{F}_p^\times & & \\ \downarrow p^{e-p^{e-1}} & & \downarrow p^{-1} & & \\ 0 & & 0 & & \end{array}$$

Finally, use Chinese remainder theorem to get

$$SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^r SL_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$$

□

Ex. do the exactly same thing with  $SL_2$  replaced by  $GL_2$  and  $PGL_2$ .

Ex. (hard) explore the Tits building & rep theory of  $SL_2(\mathbb{Z}/N\mathbb{Z})$ .

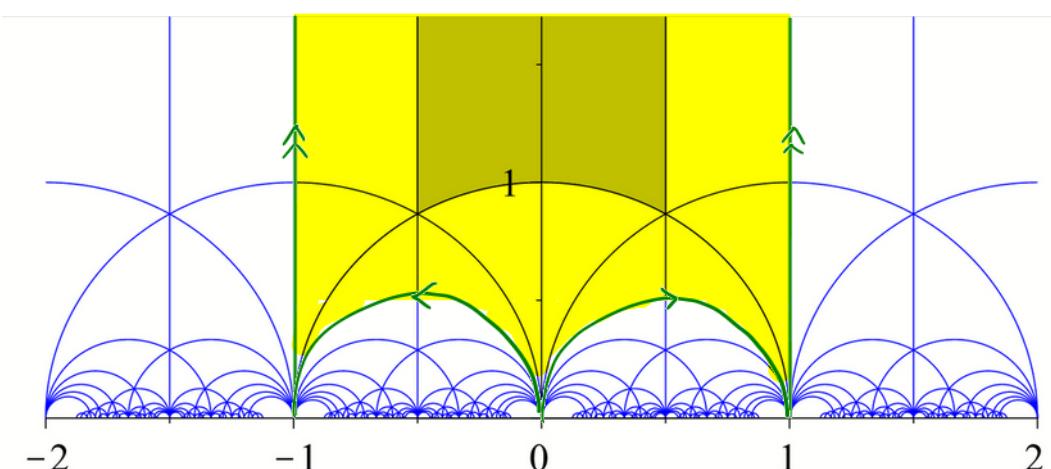
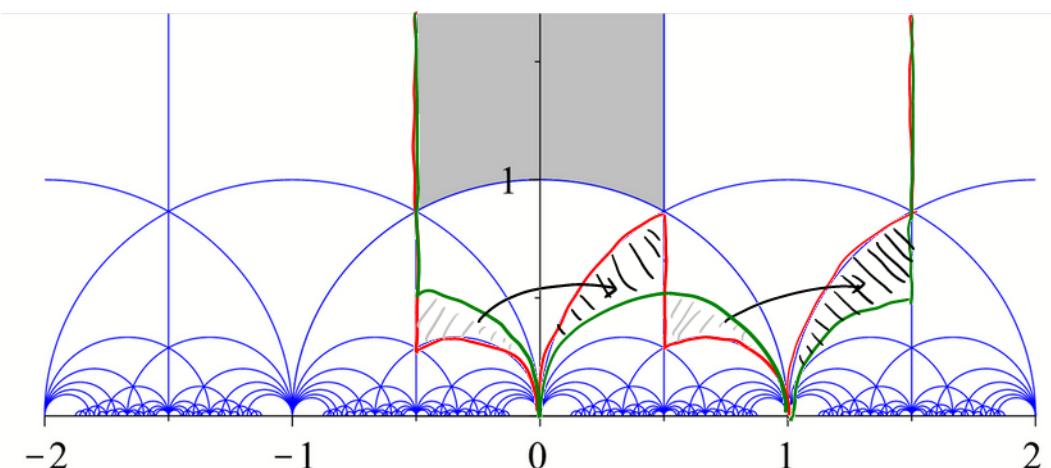
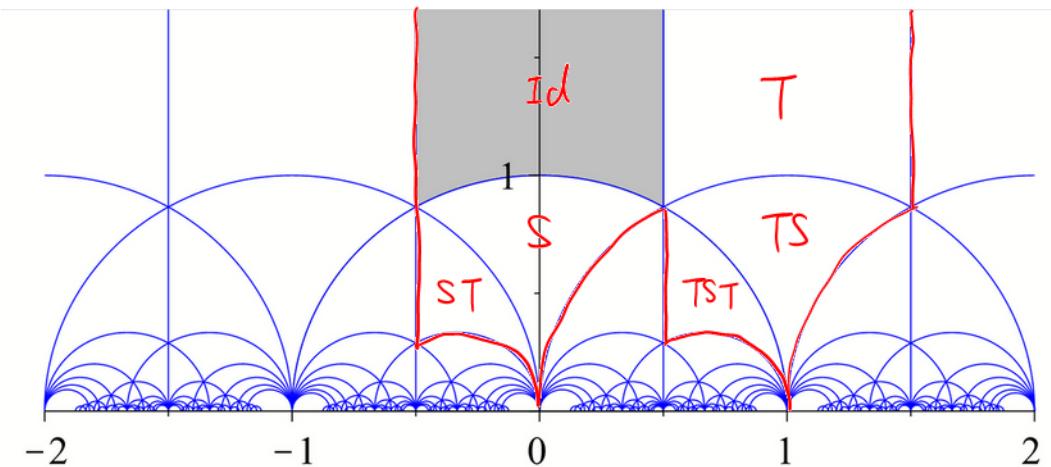
It will be used later on (I believe)

Is the Tits building of  $SL_2(\mathbb{Z}) \rightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$  functorial?

*we write left quotient from now on, since it's a left action*

Ex. Draw the fundamental domain of  $\Gamma_{(2)}\backslash \mathcal{H}$ .

Hint.  $\Gamma(1)/\Gamma_{(2)} = \{\text{Id}, T, S, TS, ST, TST\}$

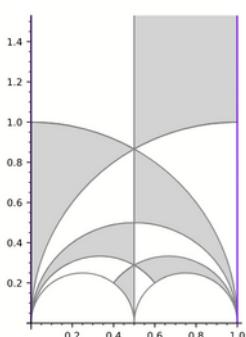
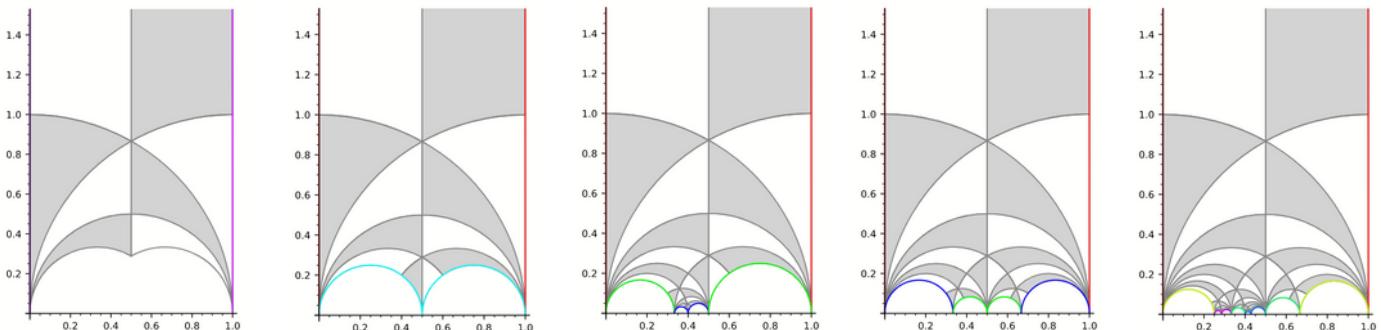
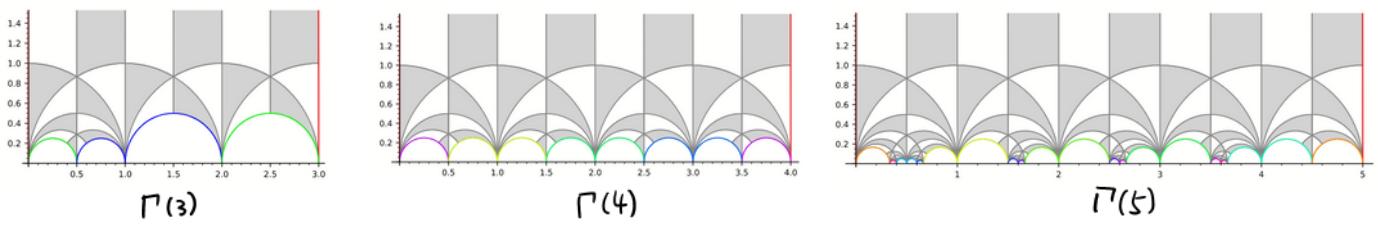
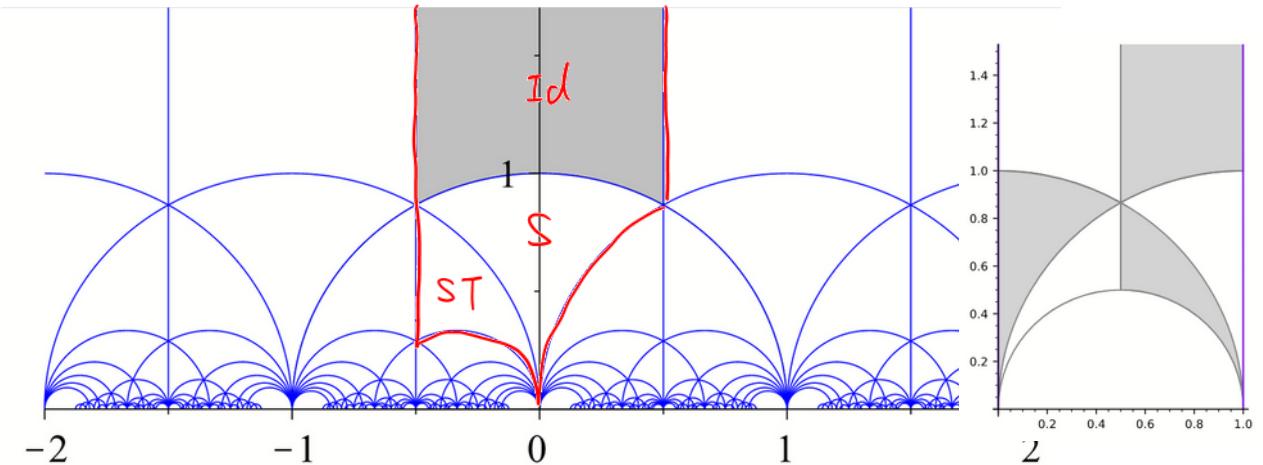


$$\text{Cor. } \Gamma_{(2)} / \{\pm \text{Id}\} = \mathbb{Z} * \mathbb{Z} = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$$

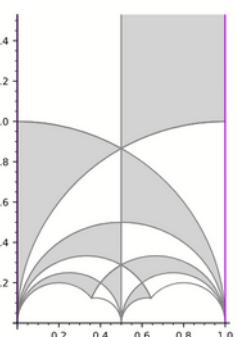
Ex. Draw the fundamental domain of  $\Gamma_0(2)\backslash \mathbb{H}$ .  $\Gamma_0(2) = \Gamma_1(2)$

$\nabla \quad SL_2(\mathbb{Z}) \text{ acts on } \Gamma_0(2)\backslash \mathbb{H} \text{ is not well-defined. e.g. } S_i \neq S_{(i+1)} \text{ in } \Gamma_0(2)\backslash \mathbb{H}$ .

Hint.  $\Gamma_0(2)\backslash \Gamma^{(1)} = \{\text{Id}, S, ST\}$



$\Gamma_0(5)$



$\Gamma_0(7)$

## 2. moduli interpretation of $\mathcal{H}$

Def. A basis  $(v_1, v_2)$  of a lattice  $\Lambda \subseteq \mathbb{C}$  is called **oriented** if  $\text{Im} \frac{v_1}{v_2} > 0$ .

Def (Weil pairing) [WWL, 注记 8.5.9, 定义 3.8.9, 练习 3.8.10]

For  $N \in \mathbb{Z}_{\geq 1}$ ,  $E = \mathbb{C}/\Lambda$ ,  $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$ ,  $\text{Im} \frac{v}{u} > 0$ , we define the Weil pairing  $e_N$ .

$$\begin{array}{ccc}
 E[N] \times E[N] & & \\
 \uparrow \text{is} & & \\
 a \frac{u}{N} + c \frac{v}{N} & \xrightarrow{\text{is}} & \frac{1}{N}\Lambda/\Lambda \times \frac{1}{N}\Lambda/\Lambda \\
 \downarrow & & \\
 \left( \begin{matrix} a \\ c \end{matrix} \right), \left( \begin{matrix} b \\ d \end{matrix} \right) & \xrightarrow{\text{is}} & (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \times (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \\
 & & \xrightarrow{\quad} \mu_N \cong (\mathbb{Z}/N\mathbb{Z}, +) \\
 & & \xrightarrow{\quad} \left\{ \begin{matrix} 1 \\ a \\ c \\ ad \end{matrix} \right\} \xrightarrow{\quad} \left| \begin{matrix} a & b \\ c & d \end{matrix} \right|
 \end{array}$$

Ex. Let  $e_1, e_2 \in E[n]$ .

1.  $e_N$  is antisymmetric and bilinear.

$$e_N(\gamma(e_1, e_2)) = \sum_N^{\det \gamma} e_N(e_1, e_2) \quad \forall \gamma \in GL_2(\mathbb{Z}/N\mathbb{Z})$$

e.p.  $e_N$  only depends on  $E$  and  $N$  (does not depend on  $\Lambda$  and  $u, v$ )

2.  $e_N(e_1, e_2) \in \mu_N^{\times} \cong (\mathbb{Z}/N\mathbb{Z})^{\times} \Leftrightarrow E[N] = \langle e_1, e_2 \rangle_{\mathbb{Z}}$

$$e_N(e_1, e_2) = \sum_N \xrightarrow{\psi} 1 \Leftrightarrow \exists P, Q \in \frac{1}{N}\Lambda, \bar{P} = e_1, \bar{Q} = e_2,$$

$(NP, NQ)$  is an oriented basis of  $\Lambda$ .

Def.  $(e_1, e_2)$  is called a **pretty oriented basis** of  $E[N]$ . if  $e_N(e_1, e_2) = \sum_N$ .

In [KM85], this is called Drinfeld basis.

Ex.  $N=5$

$$\begin{array}{ccc}
 \begin{array}{ccccccccc}
 \times & \cdot & \cdot & \cdot & \cdot & \cdot & \times & & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \end{array} & 
 \begin{array}{ccccccccc}
 \times & \cdot & \cdot & \cdot & \cdot & \cdot & \times & & \\
 \cdot & \\
 \cdot & \\
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 \cdot & \\
 \end{array} & 
 \begin{array}{ccccccccc}
 \times & \cdot & \cdot & \cdot & \cdot & \cdot & \times & & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \cdot & \\
 \end{array} \\
 P \nearrow \quad \swarrow & \quad \alpha \nearrow \quad \swarrow P & \quad P \nearrow \quad \swarrow \alpha
 \end{array}$$

$$e_N(\bar{P}, \bar{Q}) = \sum_5^2$$

$(5P, 5Q)$  is not a basis of  $\Lambda$ .

$$e_N(\bar{P}, \bar{Q}) = \sum_5^4 = \sum_5^{-1}$$

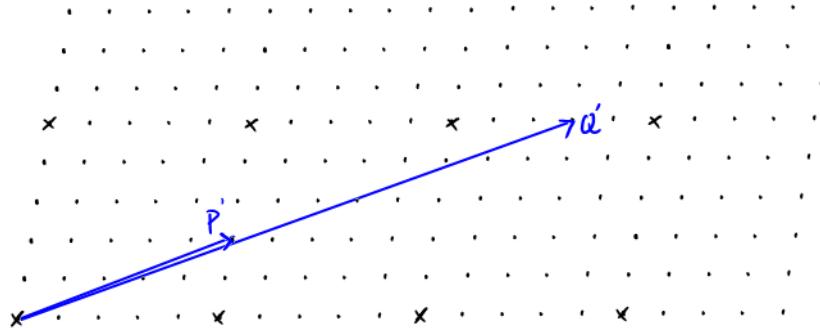
$(5P, 5Q)$  is a basis of  $\Lambda$ , but not an oriented basis.

$$e_N(\bar{P}, \bar{Q}) = \sum_5^6 = \sum_5$$

$(5P, 5Q)$  is not a basis of  $\Lambda$ , but  $(5P', 5Q')$  is an oriented basis.

$$\begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix} \stackrel{\text{mod } 5}{=} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

$\cap$   
 $SL_2(\mathbb{Z})$



Recall: For  $E = C/\Lambda$ ,  $E[N] \cong \frac{1}{N}\Delta/\Lambda \cong \Delta/N\Delta$

Main Thm. We have the following moduli interpolations ( $E$ : any cplx EC curve)

$$\begin{array}{ccc}
 \left\{ (E, \alpha) \mid \begin{array}{l} \alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[N] \\ e_N(\alpha(1,0), \alpha(0,1)) = \delta_N \end{array} \right\} / \sim & \xrightarrow{\sim} & \Gamma(N) \backslash \mathcal{H} \\
 \downarrow \beta = \alpha(0,-) & & \downarrow N \\
 \left\{ (E, \beta) \mid \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N] \right\} / \sim & \longrightarrow & \Gamma(N) \backslash \mathcal{H} \\
 \text{E with a pretty oriented basis } (e_1, e_2) & & \downarrow \begin{cases} \frac{N}{2} \prod_{p|N} (1 - \frac{1}{p}) & N \neq 2 \\ 1 & N=2 \end{cases} \\
 \downarrow & & \downarrow \\
 \left\{ (E, F) \mid F, 0 \subseteq C \subseteq E[N] \right\} / \sim & \longrightarrow & \Gamma_0(N) \backslash \mathcal{H} \\
 \text{E with } C \subseteq E[N], C \cong \mathbb{Z}/N\mathbb{Z} & & \downarrow N \prod_{p|N} (1 + \frac{1}{p}) \\
 \downarrow & & \downarrow \\
 \left\{ \text{cplx EC } E \right\} / \sim & \longrightarrow & \Gamma(1) \backslash \mathcal{H}
 \end{array}$$

Idea. A pretty oriented basis on  $E[N]$  gives us a oriented basis on  $E$  up to  $\Gamma(N)$ -action;  
coefficient has to be 1, so that  $(v_1 + bv_2, v_2)$  is a pretty oriented basis.

$$\begin{aligned}
 \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} v_1 + bv_2 \\ v_2 \end{pmatrix} \Rightarrow \text{an } n\text{-torsion pt } v_2 \\
 \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \begin{pmatrix} av_1 + bv_2 \\ dv_2 \end{pmatrix} \Rightarrow \text{a flag } 0 \subseteq \begin{matrix} C \\ \subset \\ \subset \\ \subset \\ \subset \end{matrix} \subseteq E[N]
 \end{aligned}$$

Proof. For  $\Gamma(N) \backslash \mathcal{H}$ ,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Lambda, \alpha) \mid \begin{array}{l} \alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \longrightarrow \frac{1}{N}\Delta/\Lambda \\ e_N(\alpha(1,0), \alpha(0,1)) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \\
 &\cong \left\{ (\Lambda, e_1, e_2) \mid \begin{array}{l} e_1, e_2 \in \frac{1}{N}\Delta/\Lambda \\ e_N(e_1, e_2) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \\
 &\cong \left\{ (\Lambda, z_1, z_2, e_1, e_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_1 = \frac{z_1}{N}, e_2 = \frac{z_2}{N} \\ e_N(e_1, e_2) = \delta_N \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \mathbb{C}^\times \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \left\{ (\Lambda, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ (z_1, z_2) \text{ is an oriented basis of } \Delta \end{array} \right\} \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \left\{ (z_1, z_2) \in (\mathbb{C} - \{0\})^2 \mid \operatorname{Im} \frac{z_1}{z_2} > 0 \right\} \stackrel{\mathfrak{S}}{\sim} \Gamma(N) \\
 &\cong \Gamma(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

For  $\Gamma_1(N) \backslash \mathcal{H}$ ,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Delta, \beta) \mid \beta : \mathbb{Z}/N\mathbb{Z} \hookrightarrow \frac{1}{N}\Delta/\Delta \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, e_1) \mid e_1 \in \frac{1}{N}\Delta/\Delta, \text{order}(e_1) = N \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, z_1, z_2, e_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_2 = \frac{z_2}{N} \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \xrightarrow{\text{If not set } (z_1, z_2) \mapsto (-z_1, z_2)} \Gamma_1(N) \\
 &\cong \left\{ (\Delta, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_1(N) \\
 &\cong \Gamma_1(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

For  $\Gamma_0(N) \backslash \mathcal{H}$ ,

$$\begin{aligned}
 \text{LHS} &\cong \left\{ (\Delta, C) \mid C \subseteq \frac{1}{N}\Delta/\Delta, C \cong \mathbb{Z}/N\mathbb{Z} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \\
 &\cong \left\{ (\Delta, z_1, z_2, C) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ C = \left(\frac{z_1}{N}\right) \subseteq \frac{1}{N}\Delta/\Delta \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_0(N) \\
 &\cong \left\{ (\Delta, z_1, z_2) \mid \begin{array}{l} \Delta = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \text{Im } \frac{z_1}{z_2} > 0 \end{array} \right\} \stackrel{\mathfrak{S}^{\mathbb{C}^*}}{\sim} \Gamma_0(N) \\
 &\cong \Gamma_0(N) \backslash \mathcal{H} = \text{RHS}
 \end{aligned}$$

□

Rmk. If you observe carefully, you will find out that what we prove actually is

$$\mathbb{C}^* = SO_2(\mathbb{R}) \times \mathbb{R}_{>0}$$

$$\begin{array}{ccc}
 \{ (E, \alpha) \} & \xrightarrow{\sim} & \Gamma(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ (E, \beta) \} & \xrightarrow{\sim} & \Gamma_1^\pm(N) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_1(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_1(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ (E, F) \} & \xrightarrow{\sim} & \Gamma_0^\pm(N) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_0(N) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong \Gamma_0(N) \backslash \mathcal{H} \\
 \downarrow & & \downarrow \\
 \{ E \} & \xrightarrow{\sim} & GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R}) / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong SL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})^+ / SO_2(\mathbb{R}) \cdot \mathbb{R}_{>0} \cong SL_2(\mathbb{Z}) \backslash \mathcal{H}
 \end{array}$$

where

$$\Gamma_1^\pm(N) \longrightarrow \begin{pmatrix} * & * \\ 0 & \pm 1 \end{pmatrix}$$

$$\Gamma_0^\pm(N) \longrightarrow \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$GL_2(\mathbb{Z}) \longrightarrow$$

$$GL_2(\mathbb{Z}/N\mathbb{Z})_{\det=\pm 1} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})$$

This page is by no means complete. Be skeptical about every result here.

**Def [Milne LEC, Def 6.1]** Let  $Y \rightarrow X$  f.flat,  $G$  finite,  $G \subset \text{Aut}(Y/X)$ .

$Y \rightarrow X$  is called a Galois covering with gp  $G$  if

$$G \times Y \longrightarrow Y \times_X Y$$

$$(g, y) \mapsto (gy, y)$$

is an iso.

I believe that

$$\bigsqcup_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)^H \cong \{(E, \alpha)\} / \sim \stackrel{\text{without condition that } E = (\alpha(1,0), \alpha(0,1)) = \mu_N}{=} \left\{ (E, Y, \alpha) \mid \begin{array}{l} E/Y \text{ Galois \'etale} \\ \alpha: \text{Gal}(E/Y) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{(\mathbb{Z}/N\mathbb{Z})^{\oplus 2} - \text{torsors } E \rightarrow Y\} / \sim$$

$$\cong \left\{ (E, X, \alpha') \mid \begin{array}{l} X/E \text{ Galois \'etale} \\ \alpha': \text{Gal}(X/E) \xrightarrow{\sim} (\mu_N)^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{(\mu_N)^{\oplus 2} - \text{torsors } X \rightarrow E, X \text{ connected}\} / \sim$$

$$\Gamma(N)^H \cong \{(E, \beta)\} / \sim \cong \left\{ (E, Y, \beta) \mid \begin{array}{l} E/Y \text{ Galois \'etale} \\ \beta: \text{Gal}(E/Y) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \end{array} \right\} / \sim$$

$$= \{ \mathbb{Z}/N\mathbb{Z} - \text{torsors } E \rightarrow Y\} / \sim$$

$$\cong \left\{ (E, X, \alpha') \mid \begin{array}{l} X/E \text{ Galois \'etale} \\ \alpha': \text{Gal}(X/E) \xrightarrow{\sim} \mu_N \end{array} \right\} / \sim$$

$$= \{ \mu_N - \text{torsors } X \rightarrow E, X \text{ connected}\} / \sim$$

$$\cong \{(\Lambda, \Lambda') \mid \Lambda \subset \mathbb{Z}/N\mathbb{Z} \} / \sim$$

$$\cong \left\{ (E, Y) \mid \begin{array}{l} E/Y \text{ Galois \'etale} \\ \text{Gal}(E/Y) \cong \mathbb{Z}/N\mathbb{Z} \end{array} \right\} / \sim$$

$$= \{ \mathbb{Z}/N\mathbb{Z} - \text{isogeny } E \rightarrow Y\}$$

$$\cong \left\{ (E, X) \mid \begin{array}{l} X/E \text{ Galois \'etale} \\ \text{Gal}(X/E) \cong \mu_N \end{array} \right\} / \sim$$

$$= \{ \mu_N - \text{isogeny } X \rightarrow E, X \text{ connected}\} / \sim$$

My confusion: In [<https://arxiv.org/pdf/1510.05687.pdf>] (and its historical version), it's claimed that

$$\bigsqcup_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)^H \cong \{(E, \alpha)\} / \sim \stackrel{\text{without condition that } E = (\alpha(1,0), \alpha(0,1)) = \mu_N}{=} \left\{ (E, X, \alpha') \mid \begin{array}{l} X/E \text{ Galois \'etale} \\ \alpha': \text{Gal}(X/E) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} / \sim$$

$$= \{(\mathbb{Z}/N\mathbb{Z})^{\oplus 2} - \text{torsors } X \rightarrow E\} / \sim$$

Q: • For  $K = \mathbb{C}$ , the  $\mathbb{Z}/N\mathbb{Z}$ -torsor is exactly the  $\mu_N$ -torsor, but which one is correct when it concerns with the arithmetical moduli?

- Is  $\mathbb{Z}/N\mathbb{Z}$ -torsor  $X \rightarrow E$  representable as a scheme over  $\text{Spec } \mathbb{Z}[\frac{1}{6N}]$ ?
- How to translate the condition "connected" in the arithmetical moduli?

Ex. For an elliptic curve  $E/\mathbb{C}$ , compute the number of connected  $\mathbb{Z}/N\mathbb{Z}$ -torsors over  $E$ .  
(For the torsors, the  $\mathbb{Z}/N\mathbb{Z}$ -action is also considered as information)

Explain the ramification of map  $\Gamma_1(N)\backslash \mathcal{H} \rightarrow \Gamma_1(1)\backslash \mathcal{H}$ .

For moduli interpretations of noncongruence groups, see [<https://arxiv.org/pdf/1510.05687.pdf>] (I'm skeptical about this article, the results are vaguely true, while some assumptions are not explicitly written)

Rmk. In this remark we discuss about modular curve in AG. The main ref is [KM85] (if not mentioned), and lecture note [<https://www.math.uni-bonn.de/people/mihatsch/21u22%20WS/moduli/>] is also referred. This page will be moved to [[https://github.com/ramified/moduli\\_in\\_algebraic\\_geometry](https://github.com/ramified/moduli_in_algebraic_geometry)] when I fully understand it.

Even though the font is less preferable, many beautiful figures and tables are included in [KM85].

¶ In [KM85, p146],  $\Gamma(N) \subseteq \text{bal.}\Gamma_1(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \Gamma(1) = GL_2(\mathbb{Z})$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad \begin{pmatrix} * & * \\ * & * \end{pmatrix} = GL_2(\mathbb{Z}/N\mathbb{Z})$$

[4.7.0] If  $\mathcal{P}$  is relatively rep., affine over  $(E/\mathbb{Q})$ , and rigid

$\Rightarrow \mathcal{P}$  is rep.

$\Rightarrow$  [Cor(4.7.1)] Suppose  $\mathcal{P}$ : relative rep., affine and étale over  $(E/\mathbb{Q})$ , and rigid

$\Rightarrow \mathcal{P}$  is rep by a sm affine curve over  $\mathbb{Z}$ .

rigidity: [4.7.2] at least for  $[\Gamma(N)]$

rel. rep: [3.6.0] (relative representability thm) (relative: fix  $E/S$ )

étale: [3.6.1] when  $S \in \text{Sch}_{\mathbb{Z}[\frac{1}{N}]}$ .  $X \rightarrow E$  is étale

[5.1.1].  $[\Gamma(N)]$ ,  $[\Gamma_1(N)]$  &  $[\Gamma_0(N)]$  are

- relatively rep over  $(E/\mathbb{Q})$ ,
- finite & flat/ $(E/\mathbb{Q})$  of constant rank  $\geq 1$
- regular
- $- \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{N}]$  is finite étale over  $(E/\mathbb{Z}[\frac{1}{N}])$

E.g. [5.1.11] rep of  $\Gamma(3)$  in  $\text{Sch}_{\mathbb{Z}[\frac{1}{3}]}$ .