

# Eine Woche, ein Beispiel

## 9.3. field extension with RS

Goal: construct an equivalence between two categories:

$$\begin{array}{ccc}
 \begin{array}{c} \text{cpt conn} \\ \downarrow \\ RS^{cc} = \left\{ \begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array} \right\} \end{array} & \longleftrightarrow & \left\{ \begin{array}{l} \text{Obj: } F/\mathbb{C} \text{ field ext st.} \\ \text{trdeg}_{\mathbb{C}} F = 1 \\ F/\mathbb{C} \text{ f.g. as a field} \\ \text{Mor: morphism as fields}/\mathbb{C} \end{array} \right\}^{\text{op}} = \text{field}_{\mathbb{C}(t)/\mathbb{C}}^{\text{op}} \\
 \begin{array}{c} Y \\ \downarrow f \\ X \end{array} & \implies & \begin{array}{c} \mathcal{M}(Y) \\ \uparrow f^* \\ \mathcal{M}(X) \end{array}
 \end{array}$$

which obeys the following slogan:

(ramified) covering  $\approx$  (function) field extension

- Rmk.
- For requiring  $F/\mathbb{C}$  f.g. as a field, we avoid examples like  $\overline{\mathbb{C}(t)}$ .  
Do they corresponds to some non-cpt Riemann surface?  
If so, how to enlarge the category  $RS^{cc}$ ?
  - $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$  means fields over  $\mathbb{C}$  which are fin ext of  $\mathbb{C}(t)$  abstractly;  
morphisms don't need to fix  $\mathbb{C}(t)$ .  
Do you have a better name for  $RS^{cc}$  and  $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$ ?

<https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of>  
<https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory>

- field of meromorphic functions
- Galois covering
- valuations
- quadratic extension of  $\mathbb{C}(x)$ : hyperelliptic curve
- miscellaneous.

# 1. field of meromorphic functions

Def. For  $X \in RS$ ,

$$\begin{aligned} \mathcal{M}(X) &:= \{\text{meromorphic fcts on } X\} \\ &= \{f: X \rightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{1_\infty\} \\ &\stackrel{\substack{X \text{ cpt} \\ \text{conn}}}{=} \{\text{rational fcts on } X\} \end{aligned}$$

Ex. Verify that

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$

$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[1]) \cong \text{Frac}(\mathbb{C}[x, y]/(y^2 - x(x+1)(x-1)))$$

Later we will show that, for  $X \in RS^{cc}$ ,

$$\exists \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \text{ st. } [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z^3,$$

compute

$$1) f^*: \mathbb{C}(T) \hookrightarrow \mathbb{C}(S) \quad [\mathbb{C}(S) : \mathbb{C}(T)] \text{ \& a } \mathbb{C}(T)\text{-basis}$$

$$2) \text{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$$

$$3) \mathbb{C}(S)^{2/\mathbb{Z}}$$

$$4) \text{Aut}_f(\mathbb{CP}^1)$$

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z + \frac{1}{z},$$

do the same work.

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z^3 - 3z,$$

compute the same stuff.

Why isn't  $\mathbb{C}(S)/\mathbb{C}(T)$  Galois this time?

Hint.

$$\begin{array}{c} 3 \\ \hline 4 \end{array} \begin{array}{c} 2 \\ \hline 5 \end{array} \begin{array}{c} 1 \\ \hline 6 \end{array} \longrightarrow \begin{array}{c} 1 \\ \hline 1 \end{array} \begin{array}{c} \phantom{1} \\ \hline \phantom{1} \end{array} \begin{array}{c} \phantom{1} \\ \hline \phantom{1} \end{array}$$

Prop. For  $d \in \mathbb{N}_{>0}$ ,  $f: Y \rightarrow X$  proper holo morphism between conn RSs,  
 $[M(Y): f^*M(X)] = d$ .

Cor. For  $X$  cpt conn,

$$\exists \mathbb{C}(X) \hookrightarrow M(X) \text{ s.t. } [M(X): \mathbb{C}(X)] < +\infty$$

In ptc,  $F/\mathbb{C}$  f.g as a field,  $\text{trdeg}_{\mathbb{C}} F = 1$ .

To show the proposition, one need the following black box to find a basis.  
 Black box (meromorphic fcts separate points)

$X: RS$ ,  $x, y \in X$   $x \neq y$ , then

$$\exists g \in M(X) \text{ s.t. } g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$

(stronger)  $\exists g \in M(X) \text{ s.t. } \text{ord}_x g = -1, \quad g(y) = 0.$

I prefer using Riemann-Roch when  $X$  is cpt, and Stein manifold when  $X$  is not.

Ex. Using the black box, show that,

for  $X: RS$ ,  $\{x_1, \dots, x_n\} \subseteq X$ ,  $\exists g \in M(X)$  s.t.

$$\text{ord}_{x_i} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \dots, n\}$$

$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \quad i, j \in \{2, \dots, n\}$$

Proof of prop

$[M(Y): f^*M(X)] \geq d$ : Fix  $x_0 \in X$  s.t.  $\#f^{-1}(x_0) = d$ . Denote  $f^{-1}(x_0) = \{y_1, \dots, y_d\}$ .

For each  $i$ , let  $g_i \in M(Y)$  be a meromorphic fct s.t.

$$\text{ord}_{x_i} g_i = -1 \quad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$

then  $\{g_1, \dots, g_d\} \subseteq M(Y)$  are  $f^*M(X)$ -linear independent.

Check:  $\text{ord}_{y_i} (\sum f_j g_j) \approx \text{ord}_{y_i} f_i$

$$[M(Y): f^*M(X)] \leq d:$$

$\forall g \in M(Y)$ , need to find  $a_i \in f^*M(X)$  s.t.

$$g^d + a_{d-1} g^{d-1} + \dots + a_0 = 0 \quad \text{in } M(Y)$$

The fcts

$$a_i(z) = (-1)^i \sum_{\{k_1, \dots, k_i\} \subseteq \{1, \dots, d\}} g(z_{k_1}) \dots g(z_{k_d})$$

$$f^{-1}(f(z)) = \{z_1, \dots, z_d\}, \text{ multiplicity is counted}$$

satisfy the conditions.

Use Riemann extension theorem to show  $a_i(z) \in f^*M(X)$ , see [Donaldson, p148].

By primitive element theorem,  $[M(Y): f^*M(X)] \leq d$ .

## 2. Galois covering

Def. Let  $f: Y \rightarrow X$  be a proper hdo map between two conn RSs.  
 $f$  is  $\underset{\text{normal}}{\text{Galois}}$ , if  $M(Y)/f^*M(X)$  is a  $\underset{\text{normal}}{\text{Galois extension}}$ .

Prop.  $f: Y \rightarrow X$  is Galois/normal

$$\Leftrightarrow \deg f = \# \text{Aut}_f(Y)$$

$$\Leftrightarrow f^{-1}(x_0) \text{ is an } \text{Aut}_f(Y)\text{-torsor,} \quad \forall x_0 \in X - f(\text{Ram}(f))$$

$$\Leftrightarrow \text{Aut}_f(Y) \subset f^{-1}(x_0) \text{ transitively,} \quad \forall x_0 \in X$$

$$\Leftrightarrow Y/\text{Aut}_f(Y) \cong X, \text{ i.e. } f \text{ can be written as}$$

$$Y \rightarrow Y/G$$

Ex. For  $f: Y \rightarrow X$ , suppose that  
 $[\forall y_1, y_2 \in Y \text{ s.t. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2)$   $\nwarrow$  ramification index  
 Show that  $f$  is Galois by computing  $\# \text{Aut}_f(Y)$ .

[Hint. Use geodesics to divide  $X$  into several smaller triangles.  
 If geodesics are hard, take  $g: X \rightarrow \mathbb{CP}^1$  non-constant,  
 and reduce the problem to  $g \circ f$ .]

This proof is not completely rigorous, and you are encouraged to find a reference to rigorously prove it.

You may need the following materials for completing the proof.

google: geodesic triangulations

<https://math.stackexchange.com/questions/1661331/proof-of-equivalence-of-conformal-and-complex-structures-on-a-riemann-surface?rq=1>

<https://arxiv.org/pdf/2103.16702.pdf>

(If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.)

copied from p2, in <https://www.mathematik.uni-muenchen.de/~forster/eprints/gaussbonnet.pdf>

E.g. Consider the covering

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$

$$z \longmapsto z^3 - 3z$$

This is not a Galois covering. Consider the Galois closure

$$\begin{array}{ccccc}
 \mathbb{CP}^1 & & \mathbb{C}(u) = \mathbb{C}(S)[R]/(R^2 + S^2 - 4) & & u + \frac{1}{u} \\
 \downarrow z + \frac{1}{z} & & \uparrow & & \uparrow \\
 \mathbb{CP}^1 & & \mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3 - 3S - T) & & S \quad S^3 - 3S \\
 \downarrow z^3 - 3z & & \uparrow & & \uparrow \\
 \mathbb{CP}^1 & & \mathbb{C}(T) & & T
 \end{array}$$

Determination of the Galois closure

$$\min(S, \mathbb{C}(T)) = x^3 - 3x - T \quad \text{in } \mathbb{C}(T)[x]$$

$$= x^3 - 3x - (S^3 - 3S)$$

$$= (x - S)(x^2 + Sx + S^2 - 3) \quad \text{in } \mathbb{C}(S)[x]$$

To decompose the polynomial  $x^2 + Sx + S^2 - 3$ , we have to add root of discriminant:

$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2 - 3)} = \sqrt{3} \sqrt{-S^2 + 4}.$$

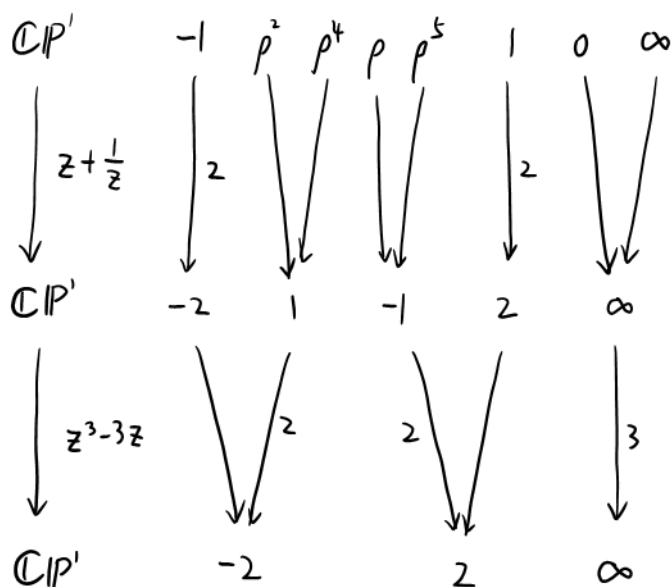
Therefore, the Galois closure of  $\mathbb{C}(S)/\mathbb{C}(T)$  is

$$\mathbb{C}(S)[R]/(R^2 + S^2 - 4) \cong \mathbb{C}\left(\frac{S+iR}{2}\right) \triangleq \mathbb{C}(u)$$

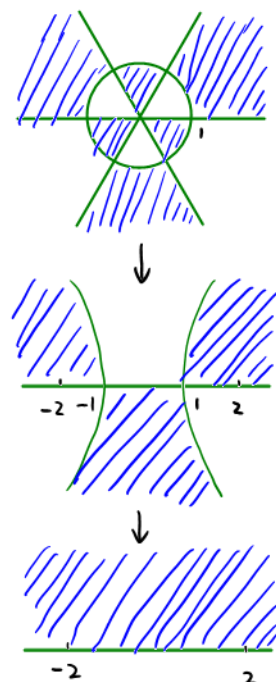
where

$$S = \frac{S+iR}{2} + \frac{S-iR}{2} = u + \frac{1}{u}$$

The picture from the RS side is as follows:



only ramified pts are drawn



affine version

Questions:

How to know the genus of the RS corresponding to the Galois closure?

Do we have any Galois closure whose ramification information is not minimal as we expected?

E.g. 2. For  $E = \mathbb{C}/\Lambda$ , since  $\pi_1(E, o) \cong \mathbb{Z} \oplus \mathbb{Z}$ ,  
 $E$  has three unramified coverings of deg 2.  
When  $\Lambda = \mathbb{Z}[i]$ , what are the crspd field extensions?

There are more deg 2 ramified coverings from the higher genus RS, but we don't discuss them here.