

# Eine Woche, ein Beispiel

## 5.4. line bundles on abelian varieties

Ref: follows [2025.04.13].

Most contents in this document can be found in [BL04, Chap 2 and Appendix B].

Goal: For  $A = V/\Lambda$ , identify

$$\begin{array}{ccccc} \text{Pic}(A) & \xlongequal[\text{hidden sheaf argument?}]{\sim} & H^1(\Lambda, H^0(\mathcal{O}_V^*)) & \xlongequal{\text{def}} & \mathcal{P}(\Lambda) \\ \text{algebraic info} & & \text{gp cohom info} & & \text{analytic info} \\ 1 & & a_1: \Lambda \times V \rightarrow \mathbb{C} & & (H, \chi) \\ & & a_1(\lambda, v) = \chi(\lambda) \exp(\pi H(\lambda, v) + \frac{\pi}{2} H(\lambda, \lambda)) & & \end{array}$$

Thm (Appell-Humbert) [BL04, p32]

$$\begin{array}{ccccccc} \{ \chi \} & & \{ (H, \chi) \} & & \{ H \} & \text{where a polarization} \\ \parallel & & \parallel & & \parallel & \swarrow \text{lives} \\ 0 \longrightarrow \text{Hom}(\Lambda, S') \longrightarrow \mathcal{P}(\Lambda) \longrightarrow \text{NS}(A) \longrightarrow 0 \\ \cong \downarrow & & \cong \downarrow & & \parallel & \\ 0 \longrightarrow \text{Pic}^\circ(A) \longrightarrow \text{Pic}(A) \longrightarrow \text{NS}(A) \longrightarrow 0 \\ \uparrow \text{def} & & & & & \\ \hat{A} & & & & & \end{array}$$

where

$$\text{NS}(A) = \left\{ H: V \times V \rightarrow \mathbb{C} \mid \begin{array}{l} H \text{ Hermitian} \\ \text{Im } H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$\mathcal{P}(\Lambda) = \left\{ (H, \chi) \mid \begin{array}{l} H \in \text{NS}(A) \\ \chi: \Lambda \rightarrow S' \text{ semicharacter w.r.t. } H, \text{ i.e.,} \\ \chi(\lambda + \mu) = \chi(\lambda) \chi(\mu) \exp(\pi i \text{Im } H(\lambda, \mu)) \\ \forall \lambda, \mu \in \Lambda \end{array} \right\}$$

## 1. Cohomology of abelian varieties (Betti & Hodge)