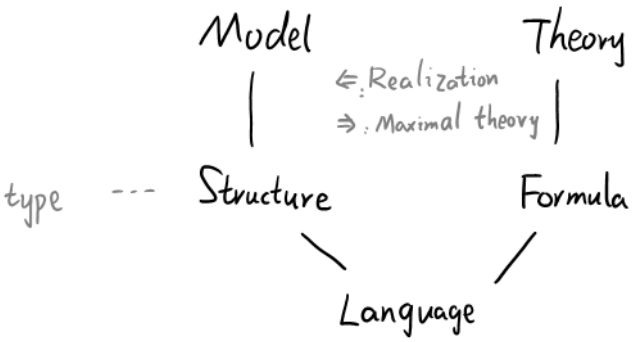
Eine Woche, ein Beispiel

3.27 model theory

Ref: https://philippschlicht.github.io/teaching/files/Lecture.pdf
I heard something from Yilong Zhang, and want to jot down some key points so that I won't be confused next time.



Modern Algebra (H)

Preliminaries

- Logic. set & map, operations on Set. ($\cap \cup -^c$, sub/quotient, two ways of disjoint union, Cartesian product)
  - Axiomatic set theory (ZFC)
  - Russell's paradox
  - type of proof : constructive, algorithm, ...

  Ex. graph

  Classify topologies of $\{1, \dots, n\}$

  First adjunction: $\mathrm{Map}(A \times B, C) \cong \mathrm{Map}(A, \mathrm{Map}(B, C))$
- From $\mathbb{N}$ to $\mathbb{C}$ (the basic of examples, though logically it's not here)
  - Peano axioms. Axiom of induction
  - alg structure, order and topology
  - Completeness axiom.
- Cardinal: the only property of set.
  - naive definition

    https://math.stackexchange.com/questions/1712964/attempt-at-proving-the-class-of-all-cardinals-is-a-proper-class

	alg	total order	topo
$\mathbb{N}$	$(+, \times)$	✓	discrete
$\mathbb{Z}$	$(+, -, \times)$	✓	discrete
$\mathbb{Q}$	$(+, -, \times, \div)$	✓	dense but not complete
$\mathbb{R}$	$(+, -, \times, \div)$	✓	complete
$\mathbb{C}$	$(+, -, \times, \div)$	✗	complete

  - operations on cardinal
  - examples
  - The continuum hypothesis
  - large cardinal axiom
- Order structure
  - def, operations and properties (partial/total/well order)
  - ordinal, relationship with cardinal.

  Ex. "well-order" on class of cardinals.

  Classify subpartial ordered set of {subsets of $\{1, \dots, n\}$}, up to iso. $n \le 5$

  the Monoid $\widetilde{R}$, surreals

  poset = partial order set

  coset = 陪集

Group-like structures					
	Totality[1]	Associativity	Identity	Division	Commutativity
Semigroupoid	Unneeded	Required	Unneeded	Unneeded	Unneeded
Small category	Unneeded	Required	Required	Unneeded	Unneeded
Groupoid	Unneeded	Required	Required	Required	Unneeded
Magma	Required	Unneeded	Unneeded	Unneeded	Unneeded
Quasigroup	Required	Unneeded	Unneeded	Required	Unneeded
Unital magma	Required	Unneeded	Required	Unneeded	Unneeded
Semigroup	Required	Required	Unneeded	Unneeded	Unneeded
Loop	Required	Unneeded	Required	Required	Unneeded
Group or Empty	Required	Required	Unneeded	Required	Unneeded
Monoid	Required	Required	Required	Unneeded	Unneeded
Group	Required	Required	Required	Required	Unneeded
Commutative monoid	Required	Required	Required	Unneeded	Required
Abelian group	Required	Required	Required	Required	Required

https://en.wikipedia.org/wiki/Category_(mathematics)

A bird eye's view of gp theory

- Group, field and v.s
  - Group: *def
    - * initial example: Aut of set, ordered set, graph; $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
  - Field: def + example: $\mathbb{Q}$, $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{R}, \mathbb{C}$
  - v.s. def + example: K^n, fct space

    recall: generators, basis, dim; extra alg structures on v.s.

  Ex. affine space v.s. vector space

  Structure is maintained: Linkage (连杆)

  ref: Bruhat–Tits theory: a new approach

  // affine space = V-torsor

- "Group is symmetry"
  - more examples. Galois gp & matrix gp
  - Cayley's thm
  - group action: on sets, on graph, on v.s., ...

    Three perspectives: ① subset of Aut; ② action; ③ add alg structure on moduli of symmetries

  Ex. concepts of group action on sets

  classification of G-sets

  finite group of $SO(3,\mathbb{R})$

  reminder: $GL_2(\mathbb{R})$ GH. foundamental gp, homotopy gp. $E(\mathbb{Q})$, braid group, ..
- Universal property with group
  - sub/quotient three iso
  - Ker/Im direct sum & product
  - free group
  - presentation of group

  Ex. coset decomposition

  Ex. Z(G), [G,G], centralizer and normalizer
- Decomposition of group
  - simple gp, ind gp
  - filstration
  - Zassenhaus lemma, Schreier refinement theorem
  - Split, semi-product gp
  - Results of simple group.

  Ex. cyclic gp case

  abelian gp case. ⇝ "abelian category"

  Levi decomposition

Down-to-earth analysis.

- f.g. abelian gp + apps.
  Ex. lattice & Crystallographic point gp
  (with/without origin)
  sublattice count
  dual lattice $\Lambda^* := \{ f : \mathbb{R}^n \to \mathbb{R} \text{ linear} \mid \langle f, x \rangle \in \mathbb{Z} \ \forall x \in \Lambda \}$
  lattice in Euclidean space
- Combinatorics related to q-polynomial ← can be quite tricky.
  Ex. finite field with one element
- G act on G
- Sylow thm
- app: classifications of gp of small order. research on specific gp.

Ring & module

$R(x)$ $R((x))$ $R\{\{x\}\}$ $R\langle\langle x\rangle\rangle$

- Basic def. e.g. $R[x]$, $R[[x]]$, $R\{x\}$, $R\langle x\rangle$
- Category, k-algs

  Ex. groupoid
- Basic def of modules, abelian category

  E.x. R-algs.

  tensor product, change of basis

  three rep theory
- AG translation, examples.
- Concepts under AG translation
- ED $\Rightarrow$ PID $\Rightarrow$ UFD $\Rightarrow$ domain

  Ex. Elementary divisor thm

  R: PID. $\quad M_2(R) - \{0\} = \coprod_{a|b\in R} GL_2(R)\begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(R)$
- classification of f.g. module over PID.

https://math.stackexchange.com/questions/765787/ring-of-convergent-power-series-in-r-and-c-is-a-local-ring

Field and Galois theory (See [GTM167])

can focus more on $\mathbb{F}_p$, $\mathbb{Q}_p$, $\mathbb{F}_p((t))$, and geometrical point of view.

(Reminder) other structures: norm, metric & topo; measure; sheaf.