

Un exemple par jour

4.6. nonorientable closed surfaces without boundary

$$\tilde{\Sigma}_l := \underbrace{\mathbb{R}P^2 \# \dots \# \mathbb{R}P^2}_l$$

Today: Klein bottle  $K = \mathbb{R}P^2 \# \mathbb{R}P^2$   $\mathbb{T}^2 \# \mathbb{R}P^2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$   $\tilde{\Sigma}_l$  embedded in  $\mathbb{R}^4$ .

nonorientable  $\Rightarrow$   $\left\{ \begin{array}{l} \text{cannot be embedded in } \mathbb{R}^3 \\ \text{can't be realized as a Lie group.} \end{array} \right.$

orientation double cover of degree 2  $\pi: \Sigma_{l-1} \rightarrow \tilde{\Sigma}_l$  [compute  $\chi$ ]

$$\Rightarrow \pi_n(X) \cong 0 \text{ when } n \geq 2, l \geq 2$$

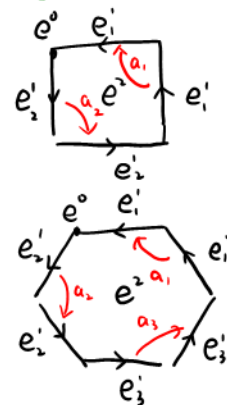
$$\text{btw } \pi_1(X) \cong \langle e_1, \dots, e_l \mid (e_1)^2 \dots (e_l)^2 \rangle \quad \pi_n \checkmark$$

cellular homology  $0 \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow 0$

$$\begin{array}{ccccc} \mathbb{Z}e^2 & \xrightarrow{\oplus_{i=1}^l \mathbb{Z}e_i} & \mathbb{Z}e^1 & \xrightarrow{\mathbb{Z}e^0} & \mathbb{Z}e^0 \\ e^2 & \xrightarrow{\quad \quad \quad} & 2\sum e_i & \xrightarrow{\quad \quad \quad} & 0 \end{array}$$

$$\Rightarrow$$

$n$	0	1	2	$n \geq 2$
$H_n(X)$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus l-1} \oplus \mathbb{Z}/2\mathbb{Z}$	0	0
$H^n(X)$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus l-1}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H_n(X, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus l}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H^n(X, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus l}$	$\mathbb{Z}/2\mathbb{Z}$	0



$$\Rightarrow H^*(X) = ?$$

$$\Rightarrow H^*(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{\oplus l} \oplus \mathbb{Z}/2\mathbb{Z}$$

$$\begin{array}{c|cc} x & a_1 & \dots & a_l \\ \hline a_1 & 1 & & 0 \\ \vdots & & \ddots & \\ a_l & 0 & & 1 \end{array} \leftarrow a_1^2$$

SW-class

$$w(T\tilde{\Sigma}_l) = 1 + w_1(T\tilde{\Sigma}_l) + w_2(T\tilde{\Sigma}_l)^2 \quad w(T\tilde{\Sigma}_l) = a_1 + \dots + a_l \in H^1(\tilde{\Sigma}_l, \mathbb{Z}/2\mathbb{Z})$$

$$w_2 = w_1^2 = \begin{cases} 1 & l = 2n-1 \\ 0 & l = 2n \end{cases} \quad n \in \mathbb{N}^+$$

Cor.  $T\tilde{\Sigma}_l$  non orientable

Cor.  $\tilde{\Sigma}_l$  is a boundary  $\Leftrightarrow l = 2n$ .

Q: find  $M$  st.  $\partial M = \tilde{\Sigma}_l$ ?