Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
https://people.math.harvard.edu/~gross/preprints/sat.pdf

Task. Complete the following tables:

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	
Id	$\mathbb{Z}$			$\mathbb{Z}[X^*(T)]/_{I_T}\rtimes\mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)]^W[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)][t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

We use the shorthand.

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$R(T)^W$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$	
Id	$\mathbb{Z}$			$R(T)/_{I_T}\rtimes\mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$R(G)[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$R(T)[t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$$R(B) = \mathbb{Z}[X^*(T)] = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F))$$

$$R(G) = \mathbb{Z}[X^*(T)]^W \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F))$$

$$R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$$

$$R(G\times\mathbb{C}^\times) = \mathbb{Z}[X^*(T)]^W[t^{\pm1}]$$

$$K^{G\times\mathbb{C}^\times}(St) = \mathcal{H}_{ext} \overset{?}{=} \mathcal{H}(\hat{G}(F), I)$$

Here is an initial example.

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1}, z_2^{\pm1}]/((z_1-z_2)(z_1-z_2^{-1}))$	$\mathbb{Z}[W_{ext}] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]$
B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1}, z]/(z-y)(z-y^{-1})$	$\mathbb{Z}[y^{\pm1}, z_1, z_2]/((z_1-y)(z_1-y^{-1}), (z_2-y)(z_2-y^{-1}))$	
Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1, z_2]/((z_1-1)^2, (z_2-1)^2)$	$R(T)/_{I_T}\rtimes\mathbb{Z}[W_f] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]/(z_w-1)^2$
$SL_2\times\mathbb{C}^\times$	$\mathbb{Z}[x, t^{\pm1}]$			$\mathcal{H}_{ext} = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}, t^{\pm1}]$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[y^{\pm1}, t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

This is our final task. Most of the notations are still not fixed.

$K^-(-)$	pt	$\mathcal{F}_{\underline{d}}$ $\quad \widetilde{Rep}_{\underline{d}}(Q)$	$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$\mathcal{Z}_d = \coprod_{\underline{d},\underline{d}'} \mathcal{Z}_{\underline{d},\underline{d}'}$
G_d	$R(T_d)^{W_d}$	$R(T_d)$ $\bigoplus_{w\in W_d} R(G_d)[\bar{\Omega}_w]^{G_d}$	$R(T_d)\otimes_{R(G_d)} R(T_d)$ $\bigoplus_{w,w'\in W_d} R(G_d)[\bar{\Omega}_{w,w'}]^{G_d}$	
B_d	$R(T_d)$ $\bigoplus_{w\in W_d} R(G_d)$	$R(T_d)\otimes_{R(G_d)} R(T_d)$ $\bigoplus_{w\in W_d} R(T_d)[\bar{\Omega}_w]^{T_d}$	$R(T_d)\otimes_{R(G_d)} R(T_d)\otimes_{R(G_d)} R(T_d)$ $\bigoplus_{w,w'\in W_d} R(T_d)[\bar{\Omega}_{w,w'}]^{T_d}$	
Id	$\mathbb{Z}$	$\bigoplus_{w\in W_d} \mathbb{Z}[\bar{\Omega}_w]$	$\bigoplus_{w,w'\in W_d} \mathbb{Z}[\bar{\Omega}_{w,w'}]$	
$G_d\times\mathbb{C}^\times$	$R(G_d)[t^{\pm1}]$	$\bigoplus_{w\in W_d} R(G_d\times\mathbb{C}^\times)[\bar{\Omega}_w]^{G_d\times\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(G_d\times\mathbb{C}^\times)[\bar{\Omega}_{w,w'}]^{G_d\times\mathbb{C}^\times}$	
$B_d\times\mathbb{C}^\times$	$R(T_d)[t^{\pm1}]$ $\bigoplus_{w\in W_d} R(G_d\times\mathbb{C}^\times)$	$\bigoplus_{w\in W_d} R(T_d\times\mathbb{C}^\times)[\bar{\Omega}_w]^{T_d\times\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(T_d\times\mathbb{C}^\times)[\bar{\Omega}_{w,w'}]^{T_d\times\mathbb{C}^\times}$	
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$	$\bigoplus_{w\in W_d} R(\mathbb{C}^\times)[\bar{\Omega}_w]^{\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(\mathbb{C}^\times)[\bar{\Omega}_{w,w'}]^{\mathbb{C}^\times}$	

Orange: only know the $R(G)$-module structure, and the alg structure is yet not known

light yellow: $R(G_d)$-module + W_d-equiv iso

Conclusions on the summer vacation.

I guess that most part of my tasks coincide with this paper:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf
Sadly, I only found it in the last week of the vacation.

Some possible tasks to work on.

1. Work out what $K_0^{Id}(\mathcal{B})$ is.

ref:
In [3264], the author computes the Chow group of G(2,4).
https://pbelmans.ncag.info/blog/2018/08/22/rank-flag-varieties/

The module structure is easy, see
[https://math.stackexchange.com/questions/1012699/when-does-a-smooth-projective-variety-x-have-a-free-grothendieck-group]

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.
   - Bernstein presentation
   - try to understand the center of $\mathcal{H}(G(F), I)$
   - How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory
   - How can the Hecke algebra be realized as a Hecke algebra?

ref. [Hecke, Sec 10-17], [Williamson 11.4-12.2]

3. Try to understand what the Hall algebra / Quantum group is.

ref: [Lec 1-4, Appendix 4, https://arxiv.org/pdf/math/0611617.pdf]

- understand $\mathcal{H}_{Rep_k^{nil}(Q)}$ where $Q = \cdot \quad \cdot \to \cdot \quad \cdot \circlearrowleft$ [Lec 2-3]
- understand [Lec 4]

$$\begin{array}{ccc} \mathcal{H}_{\mathbb{P}^1} & \cong & \mathcal{U}_v(\widehat{sl_2}) \\ \cup & & \cup \\ \mathcal{H}_{Tor(\mathbb{P}^1)} & \cong & \bigotimes_{x \in \mathbb{P}^1} \mathcal{H}_{Tor\, x} \end{array}$$

- define (Quantum) Kac-Moody / loop algs [Appendix 4]
- Why is that

graded

$$K_0(Rep^{\mathbb{Z}}(R)) = \mathcal{U}_q(n(Q))$$

where

$$R = \bigoplus_{\underline{d}} H_\cdot^{G \times \mathbb{C}^\times, BM}(Z_{\underline{d}})$$

and what is

$$K_0\left(Rep^{\mathbb{Z}}\left(\bigoplus_{\underline{d}} K_0^{G \times \mathbb{C}^\times}(Z_{\underline{d}})\right)\right) ?$$

4. Work out the big picture

$$
\begin{tikzcd}
 & & \text{Hall alg} \arrow[r, "\text{realize}"] & \overset{\text{Universal enveloping alg}}{\text{(half) quantum gp}} & \text{Verma module} \\
 & K_0(R\text{-mod}) \arrow[ur, "\sim"] & & & \\
 & & \underset{\text{KLR alg = quiver Hecke alg}}{\text{Hecke alg}} \arrow[ul, "\text{get}"'] & & \text{modular form} \\
\text{Chow group} \arrow[r, phantom, "\sim"] & K_0^{Grp} \arrow[ur] \arrow[dr, "\sim"] & & & \\
 & & \text{convolution alg} & & K_0^{Grp}(\text{Springer fiber})
\end{tikzcd}
$$

5. A closer check of Satake iso

$$
\begin{array}{rcl}
K_0 & \text{combinations} & \text{Hecke alg} \\
R(B) = & \mathbb{Z}[X^*(T)] & = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F)) \\
R(G) = & \mathbb{Z}[X^*(T)]^W & \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
R(G)[q^{\pm\frac{1}{2}}] = & \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] & = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}] \\
R(G \times \mathbb{C}^\times) = & \mathbb{Z}[X^*(T)]^W[t^{\pm 1}] & \\
K^{G\times\mathbb{C}^\times}(St) = & \mathcal{H}_{ext} & \overset{?}{=} \mathcal{H}(\hat{G}(F), I)
\end{array}
$$

It's claimed by my schoolmate that

$$
\underset{\substack{\uparrow \\ \text{sym monoidal structure} \\ \text{induced from the convolution}}}{K_0(Perv_B(G/B))} \cong \mathcal{H}(G, B)
$$

then, what is

$$
\begin{array}{ccc}
K_0^B(\mathcal{B}) & \cong & ? \\
K_0^{Id}(\mathcal{B}) & \cong & ? \\
? & \cong & \mathcal{H}(S_{m+n}, S_m \times S_n)
\end{array}
$$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces:
(e.p. in the case of $\cdot$, $\cdot\circlearrowleft$, $\cdot\to\cdot$)

G, B, T

$\mathcal{B}, T^*\mathcal{B}, St$

$\mathfrak{g}, \tilde{\mathfrak{g}}, \mathfrak{g}^{sr}, \tilde{\mathfrak{g}}^{sr}, \mathcal{N}, \tilde{\mathcal{N}}, \mathfrak{h}, \mathfrak{n}$

$\tilde{\mathfrak{g}}^h_h, \mathcal{O}_h, \Lambda^h_w$

7. Try to understand what Kazhdan-Lusztig polynomials are [KL2], and
   - Compute the transformation matrix between
     $\{[T^*_w], w \in W_f\}$ and $\{[\Lambda^h_w], w \in W_f\}$? [KaSai]?
   - understand what standard/crystal basis is
   - understand the relationship between KL poly and crystal basis
   - see if it is related to two basis in Rep(G) (irr reps & multiplicative basis)

8. Try to understand the module part, i.e.,
   - numbers of components of the Springer fiber
   - how does $K_0^{Grp}(St)$ act on K_0^{Grp}(Springer fiber) also act on $K_0^{Grp}(Rep_{pd}(Q))$
   - does that occupy "all rep" of $K_0^{Grp}(St)$

9. Ways of finding multiplication structure
   1. By direct computation (with techniques)
      Hecke algebra double coset calculus
   2. By formulas as alg-isos
      $K_0^G(\mathcal{B})$ induction formula
   3. By geometrical computation
      cohomology cup product ? de Rham calculus
      Chow group intersection theory
   4. By deformation (indirect)
      $H^{BM}_{top}(St)$
      $K_0^{G\times\mathbb{C}^*}(St)$

10. Different views on the double coset

$$B\backslash G/B = (*/B)\times_{*/G}(*/B)$$

   - as a set
   - as flag variety quotient B-action
   - as a stack
   - groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $\mathcal{H}(G,B)$, $\mathcal{H}_{sph}$, ($\mathcal{H}_{aff}$)	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG, ...	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:

- 3 Seminars (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
  - L-parameters
  - moduli in AG
  - some following developments of the modular form (different type of gps, Hecke operators, ...)
  - reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.