§2.1. Character of Galois gp

This document is originally prepared for the class field theory, but we don't have time. And also, the ref [**LCFT**] is fantastic(with typos).

Since we discuss §2.1 and §3.1 in the same time, it is preferable to discuss general Galois rep and then restrict to the 1-dim case.

The only speciality of 1-dim case is, the char factor through

$$\mathrm{Gal}(F^{sep}/F) \twoheadrightarrow \mathrm{Gal}(F^{ab}/F) \longrightarrow GL_1(\Lambda),$$

Therefore, the max abel ext F^{ab} plays a role.

fin	✓		
local	local	Kronecker - Weber	$F^{ab} = F(\zeta_\infty)$
global		Kronecker - Weber	$\mathbb{Q}^{ab} = \mathbb{Q}(\zeta_\infty)$

Local Kronecker - Weber

for $\mathbb{Q}_p$: [LCFT, Thm 1.3.4] use Kummer theory

for F: [Allen, Thm 18.3] use Hasse-Arf thm [Allen, Thm 17.16]

Kronecker - Weber

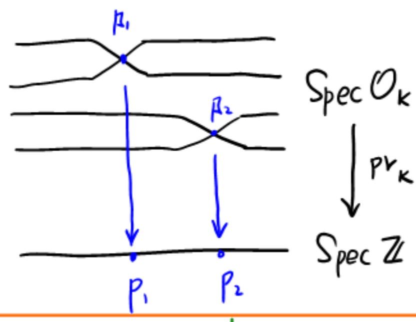
for $\mathbb{Q}$: [LCFT, Thm 1.1.2] use Minkowski's thm

for $\mathbb{Q}(i)$: [Cox x^2+ny^2] use CM Theory

for $\mathbb{F}(t)$: [VS], [Hayes]

https://math.stackexchange.com/questions/2125609/classical-version-and-idelic-version-of-class-field-theory
https://math.stackexchange.com/questions/132006/the-kernel-of-the-reciprocity-map-in-global-class-field-theory

Thm $K/\mathbb{Q}$ fin abelian $\Rightarrow K \subseteq \mathbb{Q}(\zeta_n)$ $\exists n$



Proof.

Step 1. The choice of n:

Denote $\{p_1, \cdots, p_r\}$ as primes over which K ramifies, pick $\mathfrak{p}_i \in pr_K^{-1}(p_i)$.

$\mathrm{Gal}(K_{\mathfrak{p}_i}/\mathbb{Q}_{p_i}) \leq \mathrm{Gal}(K/\mathbb{Q}) \xRightarrow{\text{local KW}} \exists\ n_{p_i} \in \mathbb{N}_{\geq 1}$ s.t. $K_{\mathfrak{p}_i} \subseteq \mathbb{Q}_{p_i}(\zeta_{n_{p_i}})$

Suppose $n_{p_i} = p_i^{e_i} \cdot a_i$, $p_i \nmid a_i$, take $n := \prod_i p_i^{e_i} \in \mathbb{N}_{>0}$.

Step 2 Take $L = K(\zeta_n)$, we will show that $L = \mathbb{Q}(\zeta_n)$. Pick $\mathfrak{q}_i \in pr_{L/K}^{-1}(\mathfrak{p}_i)$.

$$\begin{array}{ccc} L_{\mathfrak{q}} \subseteq \mathbb{Q}_p(\zeta_{n_p}, \zeta_n) & \qquad & L \supseteq \mathbb{Q}(\zeta_n) \\ \Big| \ I_{\mathfrak{q}} & & \Big| \ I := \langle I_{\mathfrak{q}_i} \rangle_i \\ U_{\mathfrak{q}} := \mathbb{Q}_p^{ur} \cap L_{\mathfrak{q}} & & U := L^I \\ \Big| & & \Big| \\ \mathbb{Q}_p & & \mathbb{Q} \end{array}$$

$$|I| \overset{\text{Minko}}{=\!=} [L:\mathbb{Q}] \geq [\mathbb{Q}(\zeta_n):\mathbb{Q}] = \phi(n)$$

$$|I| \leq \prod_i |I_{\mathfrak{q}}| \leq \prod_i \phi(p_i^{e_i}) = \phi(n)$$

$\Rightarrow [L:\mathbb{Q}] = [\mathbb{Q}(\zeta_n):\mathbb{Q}]$, $L = \mathbb{Q}(\zeta_n)$.

Rmk. This argument can not be extended to fct field K, since the residue fields of vals in K may be same (up to iso)

Left: LCFT, Galois cohomology.

Global class field theory

Observe that

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0} \cong \hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \mathrel{\hat{=}} \Gamma_\mathbb{Q}^{ab}$$

In fact, we have Artin reciprocity:

$$\mathrm{Art}: (F^\times \backslash \mathbb{A}_F^\times) / \overline{(F_\infty^\times)^\circ} \cong \Gamma_F^{ab}$$

Ex. What does $-1 \in \hat{\mathbb{Z}}^\times$ corresponds to in $\Gamma_\mathbb{Q}^{ab}$? A: cplx conjugation.

Prop. $(F_\infty^\times)^\circ F^\times$ is closed in $\mathbb{A}_F^\times \iff F = \mathbb{Q}$ or an imaginary quadratic field.

Lemma. For G: top gp, $H \leq G$ open subgp, $A \leq G$.

$$A \subseteq G \text{ closed} \iff A \cap H \subseteq H \text{ closed}$$

Proof.

$$H \leq G \text{ open} \implies H \leq G \text{ closed, so}$$

$$A \subseteq G \text{ closed} \iff A \cap gH \subseteq gH \text{ closed} \quad \forall g \in G$$

$$\iff g^{-1}A \cap H \subseteq H \text{ closed} \quad \forall g \in G$$

$g^{-1}A \cap H$ is a right $A \cap H$-torsor, so $g^{-1}A \cap H \subset H$ is closed. $\square$

Proof of the prop

$$(F_\infty^\times)^\circ F^\times \text{ is closed in } \mathbb{A}_F^\times = F_\infty^\times \cdot \mathbb{A}_{F,fin}^\times$$

$$\iff F^\times \text{ is closed in } \mathbb{A}_{F,fin}^\times = F^\times \cdot \prod_v \mathcal{O}_{F,v}^\times$$

$$\iff \mathcal{O}_F^\times = F^\times \cap \prod_v \mathcal{O}_{F,v}^\times \text{ is closed in } \prod_v \mathcal{O}_{F,v}^\times$$

$$\iff \mathcal{O}_F^\times \text{ is finite}$$

$$\iff F = \mathbb{Q} \text{ or an imaginary quadratic field.}$$

Any closed subgp of profinite gp is profinite, and profinite gp is eithor finite or uncountable, see:
https://math.stackexchange.com/questions/4062798/a-profinite-group-that-is-not-finite-is-not-countable
https://math.stackexchange.com/questions/3165116/direct-proof-that-closed-subgroups-of-profinite-groups-are-profinite

Ex. For $F = \mathbb{Q}(i)$,

$$\mathbb{Q}(i)^\times \backslash \mathbb{A}_{\mathbb{Q}(i)}^\times / \mathbb{C}^\times \cong \prod_{\substack{v \nmid \infty \\ \text{places of } \mathbb{Q}(i)}} \mathcal{O}_v \cong \Gamma_{\mathbb{Q}(i)}^{ab}$$

Q: How to connect this fact with explicit construction of $\mathbb{Q}(i)^{ab}$?

For a statement of the explicit construction, you may read [Cor 9.8] in Moreland's REU paper:
http://math.uchicago.edu/~may/REU2016/REUPapers/Moreland.pdf