Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

Task.

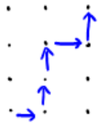
1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. tangent space, Euler class
7. basis of Hecke alg

We may use two examples for the convenience of presentation. Readers can easily distinguish them by the dim vectors.

1. dimension vector

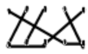
$|d| = 5$

$d = (3,2)$

$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,0 \\ 0,0 \end{pmatrix} =$  $=$  $=$  $\in W_d \backslash \mathbb{W}_d$ or $Min(\mathbb{W}_{|d|}, W_d)$

Young Tableax $\quad$ $r_\varpi = \pi_{\underline{d}}^{-1}(F_\varpi)$

2. Weyl group

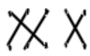
Set	element	special element	others
$\mathbb{W}_{\lvert d \rvert} = S_5$	ϖ	$\varpi_{max} =$ 	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{max} =$ 	$\Pi_d = \{s_1, s_2, s_4\}$
$W_d \backslash \mathbb{W}_{\lvert d \rvert} = S_3 \times S_2 \backslash S_5$	$\varpi, \underline{d}$		$(Comp_d)$
$Min(\mathbb{W}_{\lvert d \rvert}, W_d) = \{$  $, \cdots \}$	u		(Shuffled)

$$\begin{tikzcd} & & & Min(\mathbb{W}_{|d|}, W_d) \arrow[dl] \arrow[d, "\cong"] & \\ 0 \arrow[r] & W_d \arrow[r] & \mathbb{W}_{|d|} \arrow[r] & \mathbb{W}_{|d|} \backslash W_d \arrow[r] & 0 \end{tikzcd}$$

$$\begin{array}{c} u \\ \downarrow \\ \varpi = wu \mapsto \underline{d} \end{array}$$

$\varpi =$ 

$u =$ 

$w =$ 

Another example: $\quad d = (1,2)$ $\quad$ $\begin{array}{c} a \longrightarrow b \\ \langle v_1 \rangle \to \langle v_2, v_3 \rangle \end{array}$

$\varpi = wu$				w	$\underline{d}$, u	order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	B_ϖ	$\varpi B \varpi^{-1}$
Id	Id	$\begin{pmatrix} 1&2&3 \\ 1&2&3 \end{pmatrix}$	$\begin{bmatrix} 1&&\\&1&\\&&1 \end{bmatrix}$		abb	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} *&*&*\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$
t	(23)	$\begin{pmatrix} 1&2&3 \\ 1&3&2 \end{pmatrix}$	$\begin{bmatrix} 1&&\\&&1\\&1& \end{bmatrix}$		abb	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} *&*&*\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$
s	(12)	$\begin{pmatrix} 1&2&3 \\ 2&1&3 \end{pmatrix}$	$\begin{bmatrix} &1&\\1&&\\&&1 \end{bmatrix}$		bab	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} *&&*\\*&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&*\\&*&\\&&* \end{bmatrix}$
ts	(132)	$\begin{pmatrix} 1&2&3 \\ 3&1&2 \end{pmatrix}$	$\begin{bmatrix} &1&\\&&1\\1&& \end{bmatrix}$		bab	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} *&*&\\&*&\\*&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&*&\\&*&\\&&* \end{bmatrix}$
st	(123)	$\begin{pmatrix} 1&2&3 \\ 2&3&1 \end{pmatrix}$	$\begin{bmatrix} &&1\\1&&\\&1& \end{bmatrix}$		bba	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} *&&\\*&*&*\\*&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\*&&* \end{bmatrix}$
sts	(13)	$\begin{pmatrix} 1&2&3 \\ 3&2&1 \end{pmatrix}$	$\begin{bmatrix} &&1\\&1&\\1&& \end{bmatrix}$		bba	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} *&&\\*&*&\\*&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\*&*&\\&&* \end{bmatrix}$

3. alg group & Lie algebra

$\mathbb{G}_{|d|}, \mathbb{B}_{|d|}, \mathbb{T}_{|d|}, \mathbb{N}_{|d|}$ $\mathbb{G}_{|d|} = \mathbb{N}_{|d|}/\mathbb{T}_{|d|}$ $GL_5(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$

G_d, B_d, T_d, N_d $G_d = N_d/T_d$ $GL_3(\mathbb{C})\times GL_2(\mathbb{C}) = \begin{pmatrix} * & \\ & * \end{pmatrix}$

$$\mathbb{B}_\varpi = \varpi \mathbb{B}_{|d|}\varpi^{-1} = \mathrm{Stab}_{\mathbb{G}_{|d|}}(F_\varpi)$$

$$B_\varpi \overset{\varpi = wu}{=\!=} w B_d w^{-1} = \mathrm{Stab}_{G_d}(F_\varpi) \qquad N_\varpi = R_u(B_\varpi)$$

For $s\in\Pi$ s.t. $\varpi s \varpi^{-1}\in W_d$ (i.e. $W_d\varpi = W_d\varpi s$), define

$$P_{\varpi,\varpi s} :\overset{\varpi=wu}{=\!=} w(B_d u s u^{-1} B_d \cup B_d)w^{-1} \qquad N_{\varpi,\varpi s} = R_u(B_{\varpi,\varpi s})$$

$$= B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi \qquad = N_\varpi \cap N_{\varpi s}$$

$$M_{\varpi,\varpi s} = N_\varpi / N_{\varpi,\varpi s}$$

Ex. Show that

$$u s_i u^{-1}\in W_d \Rightarrow u s_i u^{-1} = s_{u(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\varpi,\varpi''} := N_\varpi\cap N_{\varpi''}$$

$$M_{\varpi,\varpi''} := N_\varpi/N_{\varpi,\varpi''}$$

Their Lie algebras are collected here.

$\mathbb{g}_{|d|}, \mathbb{b}_{|d|}, \mathbb{t}_{|d|}, \mathbb{n}_{|d|}$

$\mathfrak{g}_d$ $\mathfrak{b}_d$ $\mathfrak{t}_d$ $\mathfrak{n}_d$

$\mathbb{b}_\varpi$ $\mathbb{b}^-_\varpi$ $\mathbb{b}^-_\varpi = \mathbb{b}_{\varpi_{max}\varpi}$

$\mathfrak{b}_\varpi$ $\mathfrak{n}_\varpi$ $\mathfrak{b}^-_\varpi$ $\mathfrak{n}^-_\varpi$ $\mathfrak{b}^-_\varpi = \mathfrak{b}_{w_{max}\varpi}$ ...

$\mathfrak{p}_{\varpi,\varpi s}$ $\mathfrak{n}_{\varpi,\varpi''}$ $\mathfrak{p}^-_{\varpi,\varpi s}$ $\mathfrak{n}^-_{\varpi,\varpi''}$ $\mathfrak{p}^-_{\varpi,\varpi s} = \mathfrak{p}_{w_{max}\varpi,\, w_{max}\varpi s}$ ...

$\mathfrak{m}_{\varpi,\varpi''}$ $\mathfrak{m}^-_{\varpi,\varpi''}$ ...

$$\mathrm{Rep}_d(Q) := \prod_{e\in Q_1}\mathrm{Hom}(V_{s(e)},V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{|d|}$$

in general case, lies in $\mathfrak{g}_{|d|}^{\oplus k}$

$$r_\varpi = \{f\in\mathrm{Rep}_d(Q) \mid f.F_{\varpi,i}\subseteq F_{\varpi,i}\} = \mu_{\underline{d}}\,\pi_{\underline{d}}^{-1}(F_\varpi)$$

$$= \begin{array}{c} \\ v_5 \\ v_4 \end{array}\!\!\begin{array}{c} v_3 \quad v_1 \quad v_2 \\ \begin{pmatrix} * & * & * \\ & & * \end{pmatrix} \end{array} = \begin{array}{c} \\ v_4 \\ v_5 \end{array}\!\!\begin{array}{c} v_1 \quad v_2 \quad v_3 \\ \begin{pmatrix} & * & \\ * & * & * \end{pmatrix} \end{array}$$

$v_{w(i)}$

$$r_{\varpi,\varpi''} = r_\varpi\cap r_{\varpi''}$$

$$\partial_{\varpi,\varpi''} = r_\varpi / r_{\varpi,\varpi''}$$

Later we may twist the group actions.

E.g. $\underline{r}_{\varpi,\varpi'} := r_{\varpi,\varpi\varpi'}$ $r_{\varpi,\varpi''} = \underline{r}_{\varpi,\varpi^{-1}\varpi''}$

4. typical variety

Id corres to

$$\mathcal{F}_{|d|} \cong G_{|d|}/B_{|d|} \qquad F_{Id}$$
$$\mathcal{F}_{\underline{d}} \cong G_d/B_{\underline{d}} \qquad F_u$$
$$\mathcal{F}_{\varpi} \cong G_d/B_{\varpi} \qquad F_{\varpi}$$
$$\mathcal{F}_d := \coprod_{\underline{d}} \mathcal{F}_{\underline{d}} \qquad —$$

$$F_{\varpi} := \varpi(F_{Id}) = F_{\{v_{\varpi(1)}, v_{\varpi(2)}, \dots, v_{\varpi(|d|)}\}}$$
$$= F_{\{v_5, v_3, v_1, v_4, v_2\}}$$

⚠ The action on $\mathcal{F}lag$ is not the same as in http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$$\mathcal{F}_{|d|} \neq \coprod_{d} \mathcal{F}_d$$
$\mathcal{F}_{\varpi} \cong \mathcal{F}_{\underline{d}}$ with different base pt. Base pt makes difference!

$$\mathcal{F}_{|d|} \times \mathcal{F}_{|d|} \qquad F_{Id, Id}$$
$$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \qquad F_{u,u'}$$
$$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'} \qquad F_{\varpi,\varpi'}$$
$$\mathcal{F}_d \times \mathcal{F}_d := \coprod_{\underline{d},\underline{d}'} (\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}) \qquad —$$

$$F_{\varpi,\varpi'} := (F_{\varpi}, F_{\varpi'})$$

$$\begin{tikzcd} \widetilde{Rep}_{\underline{d}}(Q) \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \arrow[d, "\mu_{\underline{d}}"'] \arrow[dr, "\pi_{\underline{d}}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{Rep}_d(Q) \subset Rep_d(Q) \times \mathcal{F}_d \arrow[d, "\mu_d"'] \arrow[dr, "\pi_d"] & \\ Rep_d(Q) & \mathcal{F}_d \end{tikzcd}$$

$\mu_{\underline{d}}^{-1}(M) \cong \mathcal{F}lag_{\underline{d}}(M) \subseteq \mathcal{F}_{\underline{d}}$ is the Springer fiber.

$$\begin{tikzcd} Z_{\underline{d},\underline{d}'} \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \arrow[d, "\mu_{\underline{d},\underline{d}'}"'] \arrow[dr, "\pi_{\underline{d},\underline{d}'}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \end{tikzcd} \qquad \begin{tikzcd} Z_d \subset Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d \arrow[d, "\mu_{d,d}"'] \arrow[dr, "\pi_{d,d}"] & \\ Rep_d(Q) & \mathcal{F}_d \times \mathcal{F}_d \end{tikzcd}$$

$$\widetilde{Rep}_{\underline{d}}(Q) \subseteq Rep_d(Q) \times \mathcal{F}_{\underline{d}} \qquad \widetilde{Rep}_{\varpi}(Q) \cong G_d \times^{B_{\varpi}} r_{\varpi}$$
$$\widetilde{Rep}_d(Q) := \coprod_{\underline{d}} \widetilde{Rep}_{\underline{d}}(Q)$$

$$Z_{\underline{d},\underline{d}'} := \widetilde{Rep}_{\underline{d}}(Q) \times_{Rep_d(Q)} \widetilde{Rep}_{\underline{d}'}(Q) \qquad Z_{\varpi,\varpi'} := Z_{u,u'}$$
$$Z_d = \coprod_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$$
$$= \widetilde{Rep}_d(Q) \times_{Rep_d(Q)} \widetilde{Rep}_d(Q)$$

5. (equivariant) stratifications.

In the following tables, $u w' = \tilde{w}' \tilde{u}$.

$F_\varpi \in \widetilde{Rep}_d(Q)$ means (ρ_0, F_ϖ); $(F_\varpi, F_{\varpi'}) \in \mathcal{Z}_d$ means $(\rho_0, F_\varpi, F_{\varpi'})$.

⚠ $G \times G$ acts on $\mathcal{F} \times \mathcal{F}$ in a twisted way

e.g. $(g_1, g_2) F_{\varpi, \varpi'} = F_{g_1 \varpi,\, g_1 \varpi g_2 \varpi^{-1} \varpi'}$

variety / base point \ stratification, stabilizer \ type		B-orbit	B×B-orbit	B×G-orbit	G×B-orbit	Remark
$\mathcal{B}$ F_g	$\mathcal{B} \times \mathcal{B}$ $(F_g, F_{gg'})$	Ω_g $B \cap gBg^{-1}$	$\Omega_{g,g'}$ $(B \cap gBg^{-1}) \times (B \cap g'Bg'^{-1})$	$pr_1^{-1}(\Omega_g)$ $(B \cap gBg^{-1}) \times g'Bg'^{-1}$	$\Omega_{g'}$ $gBg^{-1} \times (B \cap g'Bg'^{-1})$	
$\mathcal{F}_{\lvert d \rvert}$ F_ϖ	$\mathcal{F}_{\lvert d \rvert} \times \mathcal{F}_{\lvert d \rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi$	$\mathcal{V}_{\varpi,\varpi'}$ $(\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi) \times (\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_{\varpi'})$	$pr_1^{-1}(\mathcal{V}_\varpi)$ $(\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi) \times \mathbb{B}_{\varpi'}$	$\mathcal{V}_{\varpi'}$ $\mathbb{B}_\varpi \times (\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_{\varpi'})$	
$\mathcal{F}_u$ F_{wu}	$\mathcal{F}_u \times \mathcal{F}_{u'}$ $(F_{wu}, F_{ww'u'})$	Ω^u_w $B_d \cap B_w$	$\Omega^{u,u'}_{w,w'}$ $(B_d \cap B_w) \times (B_d \cap B_{w'})$	$pr_{1,u'}^{-1}(\Omega^u_w)$ $(B_d \cap B_w) \times B_{w'}$	$\Omega^{u,u'}_{w'}$ $B_w \times (B_d \cap B_{w'})$	
$\mathcal{F}_d$ F_ϖ $\parallel$ F_{wu}	$\mathcal{F}_d \times \mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$ $\parallel$ $(F_{wu}, F_{w\tilde{w}'\tilde{u}u'})$	Ω^u_w $B_d \cap B_w$	$\Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$ $(B_d \cap B_w) \times (B_d \cap B_{\tilde{w}'})$	$pr_{1,\tilde{u}u'}^{-1}(\Omega^u_w)$ $(B_d \cap B_w) \times B_{\tilde{w}'}$	$\mathcal{O}^u_{\varpi'} := \Omega^{u,\tilde{u}u'}_{\tilde{w}'}$ $B_w \times (B_d \cap B_{\tilde{w}'})$	compatability
The following may not be single orbit, but derived from the above definition.						
$\mathcal{F}_d$ F_ϖ	$\mathcal{F}_d \times \mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$ $\parallel$ Ω^u_w	$\mathcal{O}_{\varpi,\varpi'}$ $\parallel$ $\Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$	$pr_1^{-1}(\mathcal{O}_\varpi)$ $\parallel$ $\bigsqcup_{u'} pr_{1,u'}^{-1}(\Omega^u_w)$	$\mathcal{O}_{\varpi'}$ $\parallel$ $\bigsqcup_u \mathcal{O}^u_{\varpi'}$	preimage of $\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\lvert d \rvert} \times \mathcal{F}_{\lvert d \rvert}$
$\widetilde{Rep}_{\underline{d}}(Q)$ F_{wu}	$\mathcal{Z}_{\underline{d},\underline{d}'}$ $(F_{wu}, F_{w'u'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\Omega}^{u,u'}_{w,w'}$	$pr_{1,u'}^{-1}(\widetilde{\Omega}^u_w)$	$\widetilde{\Omega}^{u,u'}_{w'}$	preimage of $\mathcal{Z}_{\underline{d},\underline{d}'} \longrightarrow \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi\varpi'})$				$\widetilde{\mathcal{O}}^u_{\varpi'}$ $\parallel$ $\widetilde{\Omega}^{u,\tilde{u}u'}_{\tilde{w}'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$ $\parallel$ $\widetilde{\Omega}^u_w$	$\widetilde{\mathcal{O}}_{\varpi,\varpi'}$ $\parallel$ $\Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$	$pr_1^{-1}(\widetilde{\mathcal{O}}_\varpi)$ $\parallel$ $\bigsqcup_{u'} pr_{1,u'}^{-1}(\widetilde{\Omega}^u_w)$	$\widetilde{\mathcal{O}}_{\varpi'}$ $\parallel$ $\bigsqcup_u \widetilde{\mathcal{O}}^u_{\varpi'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d \times \mathcal{F}_d$

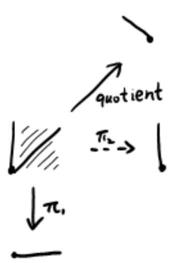
$\mathcal{Z}_{\varpi'} := \overline{\widetilde{\mathcal{O}}_{\varpi'}} \subseteq \overline{\widetilde{\mathcal{O}}}_{\varpi'}$ Zar-loc sub v.b.!

$\mathcal{Z}^{u,u'}_{\varpi'} := \mathcal{Z}_{\varpi'} \cap \mathcal{Z}_{u,u'}$

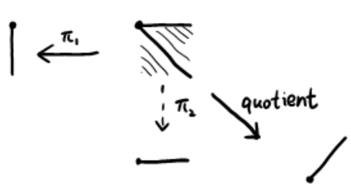
$\mathcal{Z}^{\varpi'}_d$

$\mathcal{Z}^{\varpi'}_{\underline{d},\underline{d}'}$

real case:



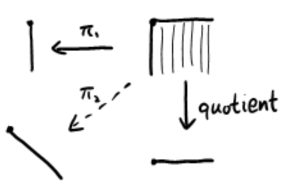
in table:



$$\begin{array}{cc} (g, g') & (g, gg') \\ \downarrow & \uparrow \\ (g, g^{-1}g') & (g, g') \end{array}$$

We want gp action to be compatible with π_1 and the quotient map.

Therefore, we would do a twist.



The following tables may help you to understand the notations.

dim; $\mathbb{B}_{\lvert d \rvert}\cdot F_{\varpi}$ ϖ; $\mathbb{B}_{\lvert d \rvert}\times\mathbb{B}_{\lvert d \rvert}\cdot(F_{\varpi},F_{\varpi\varpi'})$; $\mathbb{B}_{\lvert d \rvert}\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

shape; $B_d\cdot F_{\varpi}$ ϖ; $B_d\times B_d\cdot(F_{\varpi},F_{\varpi\varpi'})$; $B_d\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{Id}$	$\mathcal{O}_{t}$	$\mathcal{O}_{s}$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	$\mathcal{O}_{t}$	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	$\mathcal{O}_{s}$	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	$\mathcal{O}_{ts}$	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	$\mathcal{O}_{sts}$	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

The following tables may help you to understand the notations.

$\varpi = ts$, $\varpi' = s$

dim; $\mathbb{B}_{\lvert d\rvert}\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$; $\mathbb{B}_{\lvert d\rvert}\times\mathbb{B}_{\lvert d\rvert}\cdot(F_{\varpi}, F_{\varpi\varpi'})$; $\mathbb{B}_{\lvert d\rvert}\cdot F_{\omega}$ ϖ	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

$pr_1^{-1}(\vartheta_{ts})$

ϑ_s

shape; $B_d\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$; $B_d\times B_d\cdot(F_{\varpi}, F_{\varpi\varpi'})$; $B_d\cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{Id}$	$\mathcal{O}_{t}$	$\mathcal{O}_{s}$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	$\mathcal{O}_{t}$	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	$\mathcal{O}_{s}$	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	$\mathcal{O}_{ts}$	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	$\mathcal{O}_{sts}$	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

$pr_1^{-1}(\mathcal{O}_{ts})$

$pr_{1,Id}^{-1}(\Omega^{s}_{t})$

$\mathcal{O}_s$

$\Omega^{s,Id}_{Id}$

$\Omega^{s,Id}_{t,Id} = \mathcal{O}_{ts,s}$

6. tangent space, Euler class.