Eine Woche, ein Beispiel

2.6. six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

A preparation of exams.

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \ \mathcal{F}' \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Goal: $f^* \dashv f_*$

$-\otimes\mathcal{F} \dashv \underline{Hom}(\mathcal{F},-)$

$f_! \dashv f^!$

$$\begin{tikzcd} & \otimes \arrow[dl, dash] \arrow[dr, dash] & \\ f^* \arrow[rr, dash] & & f_! \end{tikzcd}$$

$f^*(-\otimes-)$

$f^*(\mathcal{F}\otimes\mathcal{F}') \cong f^*\mathcal{F}\otimes f^*\mathcal{F}'$

$f_*\underline{Hom}(f^*\mathcal{F},\mathcal{G}) \cong \underline{Hom}(\mathcal{F},f_*\mathcal{G})$

proj formula

$f_!(f^*\mathcal{F}\otimes\mathcal{G}) \cong \mathcal{F}\otimes f_!\mathcal{G}$

$f_*\underline{Hom}(\mathcal{G},f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G},\mathcal{F})$

$f^!\underline{Hom}(\mathcal{F},\mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F},f^!\mathcal{F}')$

bc: $f^*g_! \cong g'_!f'^*$

$f_*g^! \cong g'^! f'_*$

f^*: stalk $\qquad f_*/f_!$: cohomology

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \to pt$

$H^\cdot(X;\mathbb{Z}) := p_*p^*\mathbb{1}$ $\qquad H^\cdot(X;\mathcal{F}) := p_*\mathcal{F}$ $\qquad = R\Gamma(X;\mathcal{F})$

$H^\cdot_c(X;\mathbb{Z}) := p_!p^*\mathbb{1}$ $\qquad H^\cdot_c(X;\mathcal{F}) := p_!\mathcal{F}$

$H_\cdot(X;\mathbb{Z}) := p_!p^!\mathbb{1}$ $\qquad H_\cdot(X;\mathcal{F}) := p_!(p^!\mathbb{1}\otimes\mathcal{F})$ $\qquad = H^\cdot_c(X;p^!\mathbb{1}\otimes\mathcal{F})$

$H^{BM}_\cdot(X;\mathbb{Z}) := p_*p^!\mathbb{1}$ $\qquad H^{BM}_\cdot(X;\mathcal{F}) := p_*(p^!\mathbb{1}\otimes\mathcal{F})$ $\qquad = H^\cdot(X;p^!\mathbb{1}\otimes\mathcal{F})$

App 1. (Künneth formula)

$$H_c(X;\mathcal{F})\otimes H_c(Y;\mathcal{G}) \cong H_c(X\times Y;\mathcal{F}\boxtimes\mathcal{G})$$

reduced to: $p_{X,!}\mathcal{F}\otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F}\otimes p_2^*\mathcal{G})$

$$\begin{tikzcd} X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\qquad -^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-,\mathbb{Z})$

proper

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X;\mathbb{Z})[d] \cong H^\cdot(X;\mathbb{Z})^\vee$$

reduced to: $p_*\underline{Hom}(A, p^*B\otimes p^!\mathbb{1}) \cong \underline{Hom}(p_!A, B)$

Upgrade: ∞-categories & sym monoidal structure

Idea: $\mathcal{D}_0 : \mathcal{C}^{op} \longrightarrow Cat_\infty$

$$\begin{array}{ccc} X & \longmapsto & D(X) \\ f\downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & D(Y) \end{array}$$

extends to

↳ compatability is encoded!

$\mathcal{D} : Corr(C, E) \longrightarrow Mon(Cat_\infty)$

$[Y \xleftarrow{f} X = X] \longmapsto f^*$

$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$

$[X \times X \xleftarrow{\Delta} X = X] \longmapsto \otimes$

Moreover, It factor through

$$Corr(C, E) \longrightarrow LZ_{\mathcal{D}} \longrightarrow Mon(Cat_\infty)$$

Obj: $X \longmapsto X \longmapsto \mathcal{D}(X)$

Mor: $\left[X \xleftarrow{f} Y \xrightarrow{g} Z \right] \longmapsto (f,g)_* \mathbb{1}_Y \in \mathcal{D}(X \times Z)$

kernel $\longmapsto$ FM-transformation

composition = convolution

2-Mor: $\mathcal{E} \to \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$

$$\left[\begin{array}{c} \begin{tikzcd}[column sep=small] & & Z \arrow[dl] \arrow[dr] & & \\ & Y_1 \arrow[dl] \arrow[dr] & & Y_2 \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 \end{tikzcd} \\ \cong \downarrow F \\ \begin{tikzcd}[column sep=small] & Z \arrow[dl] \arrow[dr] & \\ X_1 & & X_3 \end{tikzcd} \end{array} \right] \longmapsto \left[\begin{array}{c} \mathcal{E}_{12} * \mathcal{E}_{23} \\ \cong \Big\downarrow \\ \mathcal{E}_{13} \end{array} \right]$$

Goal: framework of ∞-category & $\otimes$ $\rightsquigarrow Corr(C, E)$ & $Corr(C, E)^{\otimes}$

∞- category

$$\Delta \hookrightarrow \mathrm{Fun}(\Delta^{op}, Set) \triangleq sSet \supseteq Cat_\infty \supseteq An$$

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "h"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists K_{n,i}(h)"'] & \end{tikzcd}$$

$\forall\, 0<i<n$ in Cat_∞

$\forall\, 0\leq i\leq n$ in An

Notation

Set	$(0,0)$ - category	set
Cat	$(1,1)$ - category	category
An	$(\infty, 0)$ - category	anima / Kan cplx / ∞ - groupoid
Cat_∞	$(\infty,1)$ - category	

Ex. Realize $Corr(C,E)$ as an ∞ - category.

Monoidal structure

In $(1,1)$- category:

Monoidal structure on $\mathcal{C}$:

$$m_{\mathcal{C}}: \mathcal{C}\times\mathcal{C} \to \mathcal{C} \qquad u_{\mathcal{C}}: 1 \to \mathcal{C}$$

$$(\mathcal{F},\mathcal{G}) \mapsto \mathcal{F}\otimes\mathcal{G} \qquad * \mapsto 1_{\mathcal{C}}$$

Monoidal object in $(\mathcal{C},\otimes)$: $X \in Ob(\mathcal{C})$ with

$$m_X: X\times X \to X \qquad u_X: 1_{\mathcal{C}} \to X$$

In $(\infty,1)$ - category:

$$(C,\otimes) \overset{\text{def}}{\leftrightsquigarrow} \left[\begin{array}{c} X: Fin^{part} \longrightarrow Cat_\infty \\ \quad I \longmapsto X(I) \\ \text{comm monoid} \end{array}\right] \overset{\text{“Straightening”}}{\leftrightsquigarrow} \left[\begin{array}{c} \pi^\otimes: Y^\otimes \longrightarrow Fin^{part} \\ \text{coCartesian fibration} \\ Y^\otimes_I \xrightarrow{\sim} \prod_i Y^\otimes_i \end{array}\right]$$

⇝ See next page for details

where $Ob(Fin^{part}) = Ob(Fin)$

$Mor_{Fin^{part}}(I,J) = \{\alpha: I \dashrightarrow J\}$

commutative monoid: $X(I) \xrightarrow{\sim} \prod_i X(i)$

$\mathcal{F}\boxtimes\mathcal{G} \leftarrow\!\shortmid (\mathcal{F},\mathcal{G}) \qquad |I|=2$

coCartesian fibration: see [Def 3.5]

$$\begin{bmatrix} (\mathcal{C}, \otimes) \\ \text{monoidal } \infty\text{-cat} \end{bmatrix} \overset{\text{def}}{\longleftrightarrow} \begin{bmatrix} X : \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \\ I \longmapsto X(I) \\ \text{comm monoid} \end{bmatrix} \overset{\text{"straightening"}}{\longleftrightarrow} \begin{bmatrix} \pi^{\otimes} : Y^{\otimes} \longrightarrow \mathrm{Fin}^{\mathrm{part}} \\ \text{coCartesian fibration} \\ Y_I^{\otimes} \xrightarrow{\sim} \prod_i Y_i^{\otimes} \end{bmatrix}$$

$$\begin{tikzcd} (\mathcal{C}, \otimes) \arrow[r, mapsto] & \mathcal{C}^{(-)} : \mathrm{Fin}^{\mathrm{part}} \longrightarrow \mathrm{Cat}_\infty \end{tikzcd}$$

$$\begin{tikzcd} I \arrow[r, mapsto] \arrow[d, "\alpha"] & \mathcal{C}^I := \mathcal{C}^{\otimes I} = \bigotimes_{i \in I} \mathcal{C} \arrow[d] \\ J \arrow[r, phantom, "\Rightarrow"] & \mathcal{C}^J \end{tikzcd}$$

$$\begin{tikzcd} (X_i)_{i \in I} \arrow[d, mapsto] & = \bigotimes_i X_i \arrow[d, mapsto] \\ \left(\bigotimes_{i \in \alpha^{-1}(j)} X_i\right)_{j \in J} & = \bigotimes_j \left(\bigotimes_{i \in \alpha^{-1}(j)} X_i\right) \end{tikzcd}$$

$$\longmapsto \pi^{\otimes} :$$