

Eine Woche, ein Beispiel

3.23: Schubert calculus: Chern class over Grassmannian

This is a follow up of [2025.02.23], [2025.03.16].

1. Formulas for tautological bundle
2. Homology class in $Gr(r,n)$

1. Formulas for tautological bundle

Chern class realized as pullback of σ_1 s

Prop. For those v.bs on $Gr(r, n)$, the Chern class is given by

$$\begin{aligned} c(\mathcal{S}) &= 1 - \sigma_1 + \dots + (-1)^r \sigma_{1^r} \\ c(Q) &= 1 + \sigma_1 + \dots + \sigma_k + \dots + \sigma_{n-r} \\ c(\mathcal{S}^\vee) &= 1 + \sigma_1 + \dots + \sigma_{1^r} \\ c(Q^\vee) &= 1 - \sigma_1 + \dots + (-1)^k \sigma_k + \dots + (-1)^{n-k} \sigma_{n-r} \end{aligned}$$

We omit the proof, as there are many equiv definition of Chern class, and I don't know which one to choose.

Cor If $f: X \rightarrow Gr(r, n)$ is induced by $(\mathcal{F}, s_1, \dots, s_n) = (\mathcal{O}_X^{\oplus n} \rightarrow \mathcal{F})$, then

$$\begin{aligned} c_s(\mathcal{F}) &= f^* c_s(\mathcal{S}^\vee) \\ &= f^* \sigma_{1^s} \\ &= f^* \sum_{1^s} (\mathcal{V}^{st}) \\ &= f^* \{ \Delta \subset Gr(r, n) \mid \Delta + \mathcal{V}_{n-r+s-1}^{st} \subseteq H \} \\ &= \{ p \in X \mid (\mathcal{F}|_p)^* + \langle e_1^*, \dots, e_{n-r+s-1}^* \rangle \subseteq \mathcal{K}^{n-1} \} \\ &= \left\{ p \in X \mid \exists (0, \dots, 0, k_{n-r+s}, \dots, k_n) \in \mathcal{K}^n - \{0\}, \text{ s.t. } \right. \\ &\quad \left. k_{n-r+s} s_{n-r+s}(p) + \dots + k_n s_n(p) = 0 \right\} \\ &= \{ p \in X \mid \underbrace{s_{n-r+s}(p), \dots, s_n(p)}_{r-s+1 \text{ many}} \text{ are linear dependent} \} \end{aligned}$$

$$\begin{aligned} (-1)^k s_k(\mathcal{F}) &= (-1)^k \left[\frac{1}{c(\mathcal{F})} \right]_k \\ &= f^* c_k(Q^\vee) \\ &= f^* \sigma_k \\ &= f^* \{ \Delta \subset Gr(r, n) \mid \Delta \cap \mathcal{V}_{n-r+1-k}^{st} \neq \{0\} \} \\ &= \{ p \in X \mid (\mathcal{F}|_p)^* \cap \langle e_1^*, \dots, e_{n-r+1-k}^* \rangle \neq \{0\} \} \\ &= \left\{ p \in X \mid \exists \phi \in (\mathcal{F}|_p)^* - \{0\} \text{ s.t. } \right. \\ &\quad \left. \phi(s_{n-r+2-k}(p)) = \dots = \phi(s_n(p)) = 0 \right\} \\ &= \{ p \in X \mid \underbrace{\langle s_{n-r+2-k}(p), \dots, s_n(p) \rangle}_{r+k-1 \text{ many}} \subseteq \mathcal{F}|_p \text{ is not full} \} \end{aligned}$$

Especially,

$$c_r(\mathcal{F}) = \{p \in X \mid s_n(p) = 0\}$$

\vdots

$$c_i(\mathcal{F}) = \{p \in X \mid \underbrace{s_{n-r+1}(p), \dots, s_n(p)}_{r \text{ many}} \text{ are linear dependent}\}$$

$$= c_i(\Lambda^r \mathcal{F})$$

$$= c_i(\det \mathcal{F})$$

$$- s_i(\mathcal{F}) = c_i(\mathcal{F})$$

\vdots

$$(-1)^{n-r} s_{n-r}(\mathcal{F}) = \{p \in X \mid \langle s_2(p), \dots, s_n(p) \rangle \subset \mathcal{F}|_p \text{ is not full}\}$$

Rmk. $c_s(\mathcal{F}) \neq c_{\text{top}}(\Lambda^{r-s+1} \mathcal{F})$ since

$s_1 \wedge s_2$ (pure wedge) is not a general section in $\Lambda^2 \mathcal{F}$!

Nevertheless, when $s=1$ or r , pure wedge is a general section, so

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F})$$

$$c_r(\mathcal{F}) = c_r(\mathcal{F}).$$

Porteous' formula

Thm [3264, Thm 12.4]

Let X/\mathbb{C} sm $k \in \mathbb{Z}_{\geq 0}$,
 \mathcal{E}, \mathcal{F} : v.b. over X of rank e, f ,
 $\varphi: \mathcal{E} \rightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

$$M_k(\varphi) := \{x \in X \mid \text{rank } \varphi_x \leq k\}$$

remember multiplicity
 $\varphi_x: \mathcal{E}|_x \rightarrow \mathcal{F}|_x$

If $M_k(\varphi) \subset X$ has codim $(e-k)(f-k)$, then

$$[M_k(\varphi)] = \Delta_{f-k}^{e-k} \left[\frac{c(\mathcal{F})}{c(\mathcal{E})} \right] = (-1)^{(e-k)(f-k)} \Delta_{e-k}^{f-k} \left[\frac{c(\mathcal{E})}{c(\mathcal{F})} \right]$$

where

$$\Delta_{f-k}^{e-k}(\gamma) = \begin{vmatrix} \gamma_{f-k} & \cdots & \gamma_{e+f-2k-1} \\ \vdots & \ddots & \vdots \\ \gamma_{f-e+1} & \cdots & \gamma_{f-k} \end{vmatrix}_{(e-k) \times (e-k)}$$

E.g. When $\mathcal{E} = \mathcal{O}_X$,

$$\begin{aligned} [X] &= [M_1(\varphi)] = \Delta_{f-1}^0 [c(\mathcal{F})] = \det 1 = 1 \\ &= \Delta_0^{f-1} \left[\frac{1}{c(\mathcal{F})} \right] = \begin{vmatrix} 1 & & [\frac{1}{c(\mathcal{F})}]_{f-2} \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = 1 \end{aligned}$$

$$\begin{aligned} [V(s)] &= [M_0(\varphi)] = \Delta_f^1 [c(\mathcal{F})] = \det (c_f(\mathcal{F})) = c_f(\mathcal{F}) \\ &= -\Delta_1^f \left[\frac{1}{c(\mathcal{F})} \right] = - \begin{vmatrix} [\frac{1}{c(\mathcal{F})}]_1 & \dots & [\frac{1}{c(\mathcal{F})}]_f \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = c_f(\mathcal{F}) \end{aligned}$$

When $\mathcal{E} = \mathcal{O}_X^{\oplus e}$,

$$\begin{aligned} [X] &= [M_e(\varphi)] = \Delta_{f-e}^0 [c(\mathcal{F})] = 1 \\ [M_{e-1}(\varphi)] &= \Delta_{f-e+1}^1 [c(\mathcal{F})] = c_{f-e+1}(\mathcal{F}) \\ [M_{e-2}(\varphi)] &= \Delta_{f-e+2}^2 [c(\mathcal{F})] = \begin{vmatrix} c_{f-e+2}(\mathcal{F}) & c_{f-e+3}(\mathcal{F}) \\ c_{f-e+1}(\mathcal{F}) & c_{f-e+2}(\mathcal{F}) \end{vmatrix} \\ &\vdots \end{aligned}$$

$$[V(s_1, \dots, s_e)] = [M_0(\varphi)] = \Delta_f^e [c(\mathcal{F})] = \begin{vmatrix} c_f(\mathcal{F}) & \dots & c_{f+e-1}(\mathcal{F}) \\ \vdots & \ddots & \vdots \\ c_{f-e+1}(\mathcal{F}) & \dots & c_f(\mathcal{F}) \end{vmatrix}$$

Furthermore, when $X = \text{Gr}(r, n)$, $\mathcal{E} = \mathcal{O}_X^{\oplus e} = \mathcal{O}_X \otimes_k \mathcal{V}_{n-e}^\perp$ and $\mathcal{F} = \mathcal{S}^\vee$, we get $f=r$, $c_k(\mathcal{F}) = \sigma_1^k$,

$$\begin{aligned} [M_k(\varphi)] &= \Delta_{r-k}^{e-k} [c(\mathcal{F})] \\ &= \begin{vmatrix} \sigma_1^{r-k} & \dots & \sigma_1^{e+r-2k-1} \\ \vdots & \ddots & \vdots \\ \sigma_1^{r-e+1} & \dots & \sigma_1^{r-k} \end{vmatrix}_{(e-k) \times (e-k)} \\ &= \sigma_{(e-k)^{r-k}} \end{aligned}$$

In fact, we know that $M_k(\varphi) = \sum_{(e-k)^{r-k}}(\mathcal{V})$, since

$$\begin{aligned} M_k(\varphi) &= \{ \Lambda \in \text{Gr}(r, n) \mid \varphi_\Lambda: \mathcal{V}^\perp \hookrightarrow (\mathbb{C}^n)^* \xrightarrow{\text{dual}} \Lambda^* \text{ is of rank } \leq k \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \Lambda \hookrightarrow \mathbb{C}^n \rightarrow \mathbb{C}^n / \mathcal{V} \text{ is of rank } \leq k \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{n-e}^\perp \geq r-k \} \\ &= \sum_{(e-k)^{r-k}}(\mathcal{V}) \end{aligned}$$

Harris - Tu formula

Ref: J. Harris., and L. Tu. Chern Numbers of Kernel and Cokernel Bundles. Inventiones Mathematicae 75

Still, one defines

X/\mathbb{C} sm $k \in \mathbb{Z}_{\geq 0}$,

\mathcal{E}, \mathcal{F} : v.b. over X of rank e, f ,

$\varphi: \mathcal{E} \rightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

$$\mathcal{M}_k \hat{=} \mathcal{M}_k(\varphi) := \{x \in X \mid \text{rank } \varphi_x \leq k\} \quad \text{remember multiplicity}$$

$\varphi_x: \mathcal{E}|_x \rightarrow \mathcal{F}|_x$

Restrict φ to \mathcal{M}_k , one gets LES of coh sheaves:

$$0 \rightarrow \mathcal{K}_k \rightarrow \mathcal{E}|_{\mathcal{M}_k} \rightarrow \mathcal{F}|_{\mathcal{M}_k} \rightarrow \mathcal{J}_k \rightarrow 0$$

\uparrow kernel \uparrow cokernel, but \mathcal{J}_k looks like curve

⚠ Since we won't use stalk in this document, we abbreviate

$$\mathcal{K}_x := \mathcal{K}_k|_x, \quad \mathcal{E}_x := \mathcal{E}|_x, \dots$$

to save time and energy.

Prop. For $x \in \mathcal{M}_k - \mathcal{M}_{k-1}$,

$$T_x \mathcal{M}_k = \{\psi \in \text{Hom}(\mathcal{E}_x, \mathcal{F}_x) \mid \psi(\mathcal{K}_x) \subset \text{Im } \varphi_x\}$$

$$N_{\mathcal{M}_k/\mathcal{M}, x} = N_x \mathcal{M}_k = \text{Hom}(\mathcal{K}_x, \mathcal{J}_x)$$

$$0 \rightarrow \mathcal{K}_x \xrightarrow{e-k} \mathcal{E}_x \xrightarrow{e} \mathcal{F}_x \xrightarrow{f} \mathcal{J}_x \xrightarrow{f-k} 0$$

$\searrow \quad \nearrow$
 $\mathcal{K}_x \xrightarrow{k} \text{Im } \varphi_x \rightarrow 0$

$$\begin{pmatrix} \mathcal{E}_x & \mathcal{K}_x \\ \mathcal{F}_x & \mathcal{J}_x \end{pmatrix} \begin{pmatrix} \text{---} \\ \text{---} \end{pmatrix}$$

Thm. When M is cpt and $M_{k-1} = \emptyset$,

(1) \mathcal{K}_k & \mathcal{J}_k are v.b.s

(2) $N_{M_k/M} = \mathcal{K}_k^\vee \otimes \mathcal{J}_k$

(3) We know $c(\mathcal{K}_p)$: define

$$c_l := c_l(\mathcal{F}|_{M_k} - \mathcal{G}|_{M_k})$$

$$:= \sum_{i+j=l} c_i(\mathcal{F}|_{M_k}) c_j(-\mathcal{G}|_{M_k})$$

$$= \sum_{i+j=l} c_i(\mathcal{F}|_{M_k}) \left[\frac{1}{c(\mathcal{G}|_{M_k})} \right]_j$$

$-\varepsilon \neq \varepsilon^\vee$

$$x_1^{i_1} \cdots x_{e-k}^{i_{e-k}} = \left| \begin{array}{cccc} c_{e-k+i_1} & \cdots & \cdots & \cdots \\ & \ddots & & \\ & & c_{e-k+i_{e-k}} & \\ & \cdots & \cdots & c_{e-k+i_{e-k}} \end{array} \right|_{(e-k) \times (e-k)}$$

index +1
→

then

$$c_t(\mathcal{K}) = \prod_{i=1}^{e-k} (1 + x_i t)$$

(4) We can compute $c(M_k)$:

$$\left. \begin{array}{c} c(\mathcal{K}_k) \\ \vdots \\ c(\mathcal{J}_k) \end{array} \right\} \rightsquigarrow \left. \begin{array}{c} c(TM) \\ c(N_{M_k/M}) \end{array} \right\} \rightsquigarrow c(TM_k)$$

2. Homology class in $Gr(r,n)$

Lines passing planes

E.g. 1. [3264, p131, Question(a)]

For 4 general lines l_1, l_2, l_3, l_4 in \mathbb{P}^3 , there are 2 lines meet all four.

Reason: In $Gr(2,4)$,

$$\begin{aligned} & \# \{l \in Gr(2,4) \mid l \cap l_i \neq \emptyset, \forall i\} \\ &= \deg \sigma_{\square}^4 \\ &= 2 \end{aligned}$$

E.g. 2. For 3 general lines l_1, l_2, l_3 in \mathbb{P}^4 , there is 1 line meet all three.

Reason: In $Gr(2,5)$,

$$\begin{aligned} & \# \{l \in Gr(2,5) \mid l \cap l_i \neq \emptyset, \forall i\} \\ &= \deg \sigma_{\square}^3 \\ &= 1 \end{aligned}$$

One can get further that no line in \mathbb{P}^5 passing 3 general lines.

E.g. 3.

For 6 general planes e_1, \dots, e_6 in \mathbb{P}^5 , there are 5 lines passing all these planes.

Reason: In $Gr(2,5)$,

$$\begin{aligned} & \# \{l \in Gr(2,5) \mid l \cap e_i \neq \emptyset, \forall i\} \\ &= \deg \sigma_{\square}^6 \\ &= 5 \end{aligned}$$

E.g. 4. [3264, p131, Question(a)]

For 4 general $(k-1)$ -planes $e_1, e_2, e_3, e_4 \cong \mathbb{P}^{k-1}$ in \mathbb{P}^{2k-1} , there are k lines passing all these planes.

Reason: In $Gr(2,2k)$,

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