Eine Woche, ein Beispiel

9.3. field extension with RS

Goal: construct an equivalence between two categories:

$$RS^{cc} = \left\{\begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{Obj: } F/\mathbb{C} \text{ field ext s.t.} \\ \quad \operatorname{trdeg}_{\mathbb{C}} F = 1 \\ \quad F/\mathbb{C} \text{ f.g as a field} \\ \text{Mor: morphism as fields}/\mathbb{C} \end{array}\right\}^{op} = \text{field}^{op}_{\mathbb{C}(t)/\mathbb{C}}$$

cpt conn

$$\begin{array}{ccc} Y & & \mathcal{M}(Y) \\ \big\downarrow f & \Longrightarrow & \big\uparrow f^* \\ X & & \mathcal{M}(X) \end{array}$$

which obeys the following slogan:

$$\boxed{\text{(ramified) covering} \quad \approx \quad \text{(function) field extension}}$$

Rmk. 1. For requiring $F/\mathbb{C}$ f.g. as a field, we avoid examples like $\overline{\mathbb{C}(t)}$.

Do they corresponds to some non-cpt Riemann surface?

If so, how to enlarge the category RS^{cc}?

2. $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$ means fields over $\mathbb{C}$ which are fin ext of $\mathbb{C}(t)$ abstractly; morphisms don't need to fix $\mathbb{C}(t)$.

Do you have a better name for RS^{cc} and $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$?

https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of
https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory

1. field of meromorphic functions
2. Galois covering
3. valuations
4. quadratic extension of $\mathbb{C}(x)$: hyperelliptic curve
5. miscellaneous.

1. field of meromorphic functions

Def. For X RS,

$$\begin{aligned}\mathcal{M}(X) &:= \{\text{meromorphic fcts on } X\} \\ &= \{f : X \longrightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{\mathbb{1}_\infty\} \\ &\overset{X \text{ cpt}}{\underset{\text{conn}}{=}} \{\text{rational fcts on } X\}\end{aligned}$$

Ex. Verify that

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$

$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[i]) \cong \mathrm{Frac}\left(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1))\right)$$

Later we will show that, for $X \in RS^{cc}$,

$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3,$$

compute

1) $f^* : \mathbb{C}(T) \hookrightarrow \mathbb{C}(S)$ $\quad [\mathbb{C}(S) : \mathbb{C}(T)]$ & a $\mathbb{C}(T)$-basis
2) $\mathrm{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$
3) $\mathbb{C}(S)^{\mathbb{Z}/3\mathbb{Z}}$
4) $\mathrm{Aut}_f(\mathbb{CP}^1)$

Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z + \frac{1}{z},$$

do the same work.

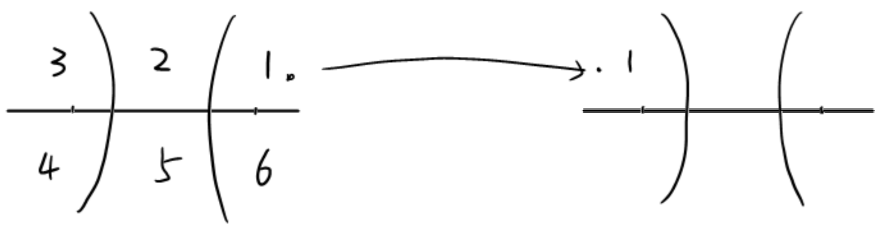
Ex. For

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z,$$

compute the same stuff.

Why isn't $\mathbb{C}(S)/\mathbb{C}(T)$ Galois this time?

Hint.



Prop. For $d \in \mathbb{N}_{>0}$, $f: Y \to X$ proper holo morphism between conn RSs,
$$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] = d.$$

Cor. For X cpt conn,
$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$
In ptc, $F/\mathbb{C}$ f.g as a field, $\operatorname{trdeg}_{\mathbb{C}} F = 1$.

To show the proposition, one need the following black box to find a basis.
Black box (meromorphic fcts seperate points)
$$X: RS, \quad x, y \in X \quad x \neq y, \text{ then}$$
$$\exists g \in \mathcal{M}(X) \quad \text{s.t.} \quad g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$
(stronger) $\exists\ g \in \mathcal{M}(X)$ s.t. $\operatorname{ord}_x g = -1$, $g(y) = 0$.

I prefer using Riemann-Roch when X is cpt, and Stein manifold when X is not.

Ex. Using the black box, show that,
for X: RS, $\{x_1, \cdots, x_n\} \subseteq X$, $\exists g \in \mathcal{M}(X)$ s.t.
$$\operatorname{ord}_{x_1} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \cdots, n\}$$
$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \quad i, j \in \{2, \cdots, n\}$$

Proof of prop
$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \geq d$: Fix $x_0 \in X$ s.t. $\# f^{-1}(x_0) = d$. Denote $f^{-1}(x_0) = \{y_1, \cdots, y_d\}$.
For each i, let $g_i \in \mathcal{M}(Y)$ be a meromorphic fct s.t.
$$\operatorname{ord}_{x_i} g_i = -1 \quad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$
then $\{g_1, \cdots, g_d\} \subseteq \mathcal{M}(Y)$ are $f^*\mathcal{M}(X)$-linear independent.
Check: $\operatorname{ord}_{y_j}(\sum_i f_i g_i) \approx \operatorname{ord}_{y_j} f_j$

$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leq d$:
$\forall g \in \mathcal{M}(Y)$, need to find $a_i \in f^*\mathcal{M}(X)$ s.t.
$$g^d + a_{d-1} g^{d-1} + \cdots + a_0 = 0 \quad \text{in } \mathcal{M}(Y)$$
The fcts
$$a_i(z) = (-1)^i \sum_{\{k_1, \cdots, k_i\} \subseteq \{1, \cdots, d\}} g(z_{k_1}) \cdots g(z_{k_d})$$
$$f^{-1}(f(z)) := \{z_1, \cdots, z_d\}, \text{ multiplicity is counted}$$
satisfy the conditions.
Use Riemann extension theorem to show $a_i(z) \in f^*\mathcal{M}(X)$, see [Donaldson, p148].
By primitive element theorem, $[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leq d$.