Eine Woche, ein Beispiel

12.14 Kuranishi deformation

Ref:

[Car17]: James Carlson, Stefan Müller-Stach, and Chris Peters. Period Mappings and Period Domains. 2nd ed. Cambridge University Press, 2017. https://doi.org/10.1017/9781316995846.

A deformation:

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\{0\} \arrow[r, hook] & S
\end{tikzcd}$$

$T = \operatorname{Spec} A$ is local Artinian

$$\begin{tikzcd}
X_0 \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dd, "f"] \\
& Y \arrow[ur, dashed, "\phi'"] \arrow[dd, "g"'] & \\
\{0\} \arrow[rr] \arrow[dr] & & S \\
& T \arrow[ur, dashed, "\phi"] &
\end{tikzcd}$$

Def. f is complete, if $\forall\ T$, $[g: Y \to T] \in \operatorname{Def}_{X_0}(T)$,
$\exists$ $\phi: T \to S$ s.t (*) commutes.
$\phi': Y \to X$

f is versal, if f is complete, and
when $T = \operatorname{Spec} k[\varepsilon]$, ϕ & ϕ' are unique.

f is universal, if $\forall\ T$, $[g: Y \to T] \in \operatorname{Def}_{X_0}(T)$,
$\exists!$ $\phi: T \to S$ s.t (*) commutes.
$\phi': Y \to X$

universal $\Rightarrow$ versal $\Rightarrow$ complete

Thm (Kuranishi's Thm) [Cor17, Thm 5.2.3]

Let X_0 be a cpt cplx mfld.

Then, $\exists$ cplx space S, s.t.

① $[f: X \longrightarrow S] \in \mathrm{Def}_{X_0}(S)$ versal;

② $\kappa_f : T_0 S \longrightarrow H^1(X_0; T_{X_0})$ is an iso;

③ $\exists$ hol map $\phi : H^1(X_0; T_{X_0}) \dashrightarrow H^2(X_0; T_{X_0})$

with $\phi(0) = 0$, $d\phi_0 = 0$, and

$$S \overset{\text{nbhd of } 0}{\cong} \mathrm{Fiber}(\phi)$$

This $[f: X \longrightarrow S]$ is called the Kuranishi deformation.