Eine Woche, ein Beispiel

3.16 Schubert calculus: subvariety with vb

This is a follow up of [2025.02.23].

Goal: relate subvarieties to some vector bundles, so that we can compute their homology class in terms of Chern class (when the dimension is correct).

The Chern class will be dealt with in the next document.

Concretely, we will write subvarieties as:

- the zero set of a section in a v.b.
- the degeneracy loci of a morphism $\mathcal{E} \longrightarrow \mathcal{F}$ among v.bs
- the preimage of known cycles in Grassmannian
- ...

1. Known subvarieties and known vector bundles
2. Subvariety as section
3. Subvariety as degeneracy loci

1. Known subvarieties and known vector bundles

Schubert variety

Recall that the Schubert variety has the expression $\omega \overset{\text{cohom}}{\longleftrightarrow} (\lambda_1,\dots,\lambda_r)$

$$
\begin{aligned}
\Sigma_{\lambda_1,\dots,\lambda_r}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+i-\lambda_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{\omega_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{\omega_i} \leq n-\lambda_i \quad \forall i\}
\end{aligned}
$$

Especially,

$$
\begin{aligned}
\Sigma_{k^s}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+i-k} \leq n-k \quad \forall i \leq s\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s\}
\end{aligned}
$$

For special k, s, one can further simplify the formulas:

s \ k ($Gr(r,n)$)	1	k	$n-r$
1	$\Lambda + \mathcal{V}_{n-r} \subseteq H$ or $\Lambda \cap \mathcal{V}_{n-r} \neq \{0\}$	$\Lambda \cap \mathcal{V}_{n-r+1-k} \neq \{0\}$	$\mathcal{V}_1 \subset \Lambda$
s	$\Lambda + \mathcal{V}_{n-r+s-1} \subseteq H$	$\dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k$ or $\dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s$	$\mathcal{V}_s \subset \Lambda$
r	$\Lambda \subset \mathcal{V}_{n-1}$	$\Lambda \subset \mathcal{V}_{n-k}$	$\{\mathcal{V}_r\}$