

Eine Woche, ein Beispiel

9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

- vector bundle over manifold
- module over ring

There are hundreds of different versions of it:

- vector bundle over manifold 几何/几何分析

- diffe mfld • (real) differential v.b. over (real) differential mfld
- Riemann surface • cplx (analytic) line bundle over Riemann surface

- sheaf over space 代数几何

- scheme theory • locally free sheaf on scheme
- coherent sheaf on scheme

- geo rep theory • local system over (real/cplx) mfld

- perverse sheaf over Riemann surface (derived)
- simplicial set over category Δ

- module over ring 代数

- comm alg • f.g. module over Noetherian commutative ring (with 1)

- rep of grp • group representation over group (\rightsquigarrow group algebra)

- p-adic rep • smooth representation over unimodular gp (\rightsquigarrow Hecke algebra $H(G)$) smooth module

- quiver theory • quiver representation over quiver (\rightsquigarrow path algebra, bound quiver algebra)

- Lie algebra • Lie alg representation over Lie alg (\rightsquigarrow universal enveloping algebra)

- Arithmetic Geometry 算数 \rightsquigarrow p进分析

- hermitian line bundle over projective arithmetic variety X

- adelic line bundle over essentially quasi-proj scheme

- over Berkovich analytic space X^{an}

- over formal scheme $\mathrm{Spf} A$

- over rigid-analytic space $K\text{-affinoid space}$

- over adic space $\mathrm{Spa}(A, A^+)$

Picture.

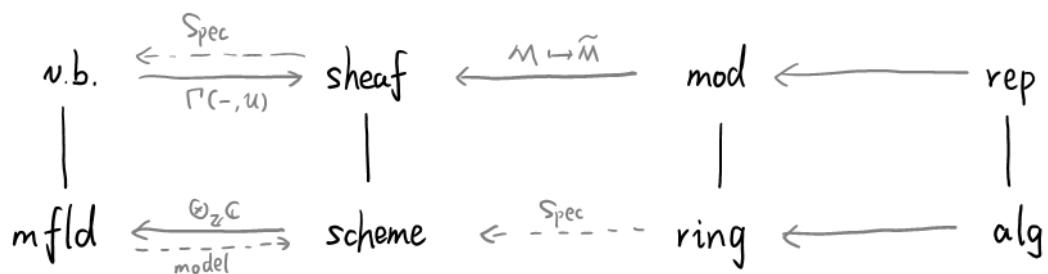


① variation (e.g. v.b. \rightarrow f.b., mfld \rightarrow CW cplx, sheaf \rightarrow fct, scheme \rightarrow stack/adic space,...)

② vertical relation: \downarrow : v.b as mfld, representable fct, Spec/Proj construction, ...

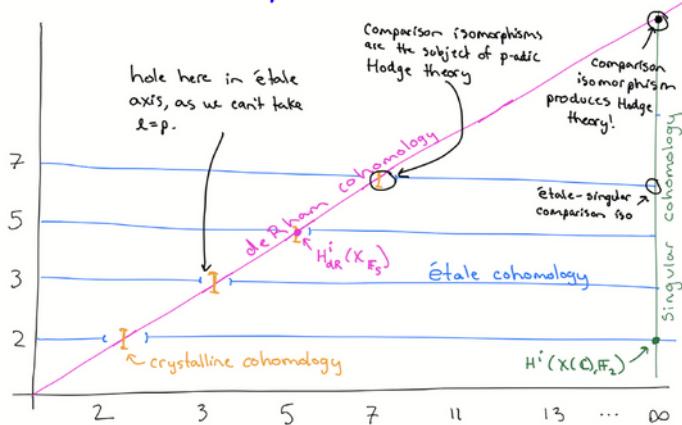
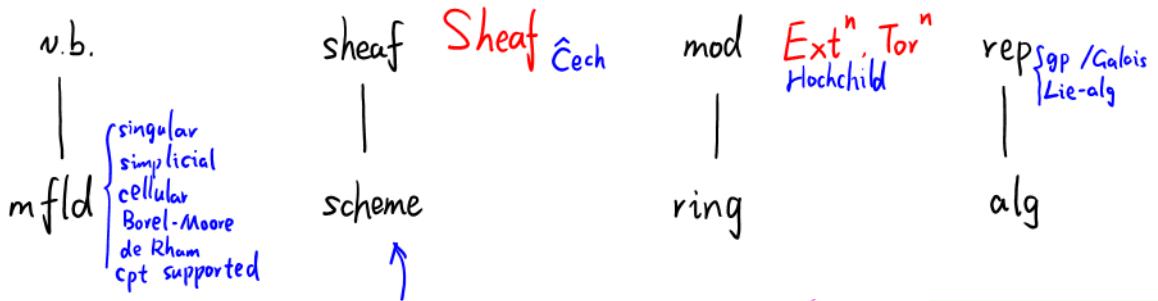
\uparrow : tangent/trivial v.b, structure sheaf, R as R-mod, regular rep, ...

③ horizontal relation:



For the parallels between curves and quivers, see <https://pbelmans.ncag.info/blog/2022/10/04/new-paper-no-git-quiver-git/>

④ homology and cohomology: \rightsquigarrow derived category



when axes meet: comparison isomorphisms (the "glue" of the "sheaf")

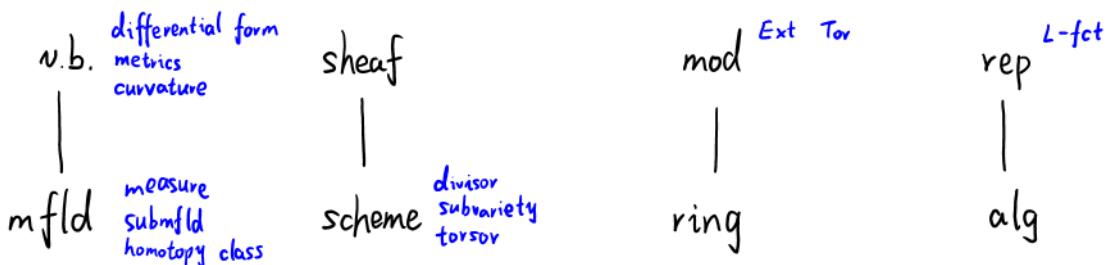
Prof. Scholze's ICM picture

<https://www.youtube.com/watch?v=5NPFQvdavgo>

Objects in upper row can be already viewed as element in (Co)homology.

eg. v.b. \leftrightarrow transition fit $\leftrightarrow H^i(X, -)$

One motivation for ∞ -category: make a generalization from H^i to H^i



The following two pictures comes from here: <https://guests.mpim-bonn.mpg.de/gallauer/docs/meff.pdf>

Coefficients	cohomology groups
$D_c^b(X; \mathbb{Q}_\ell)$ constructible ℓ -adic sheaves	ℓ -adic cohomology
$D_c^b(X(\mathbb{C}); \mathbb{Z})$ constructible analytic sheaves	Betti cohomology
$D_h^b(\mathcal{D}_X)$ holonomic \mathcal{D} -modules	de Rham cohomology
$D^b(\text{Coh}(X))$ coherent sheaves	coherent cohomology
$D^b(\text{MHM}(X))$ mixed Hodge modules	absolute Hodge cohomology
$DM(X)$ Voevodsky motivic sheaves	(weight-0) motivic cohomology
$SH(X)$ stable motivic homotopy sheaves	stable motivic (weight-0) cohomotopy groups

⑤ Relative point of view (for (co)homology) Six functors formalism (all are derived)

cohomology	$p_* p^* \mathbb{1}$	H^\bullet	$p_* \mathcal{F} \quad H^\bullet(-, \mathcal{F})$
cohomology with compact support	$p_! p^* \mathbb{1}$	H_c^\bullet	$p_! \mathcal{F} \quad H_c^\bullet(-, \mathcal{F})$
homology	$p_! p^! \mathbb{1}$	H_\bullet	
Borel-Moore homology	$p_* p^! \mathbb{1}$	H_\bullet^{BM}	

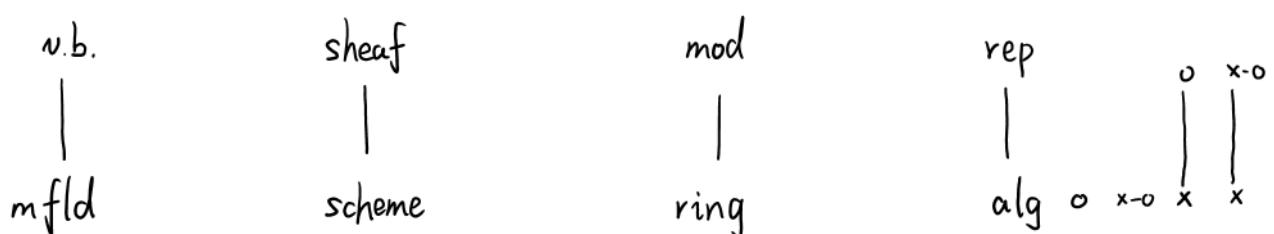
Fourier-Mukai factors

Chern class: from cohomology to cohomology (also for the other Chern class)

There are several ways of defining/viewing Chern class.

- i) $L \in \text{Pic}(X) \rightarrow c(L) \in H^2(X; \mathbb{Z})$
- ii) $H^*(X, \mathcal{O}_X^\times) \rightarrow H^2(X; \mathbb{Z})$ by LES
- iii) As the coefficient of equation ($\text{CH}^*(\text{PE})$ is a free $\text{CH}^*(B)$ -module)
Euler class
- iv) As the pull back of the universal Chern class in Grassmannian
- v) From curvature; Chern-Weil theory
- vi) From Chow group
- vii) $\partial\bar{\partial}, \Delta$

⑥ moduli problems



As a comparision, see the picture made by other people:
A possibly nice introduction of the first column is here:
[https://irma-web1.math.unistra.fr/~loday/PAPERS/A-O-C\(Lille2012\).pdf](https://irma-web1.math.unistra.fr/~loday/PAPERS/A-O-C(Lille2012).pdf)

SPIN	ALGEBRA	GEOMETRY	LINEAR PHYSICS	NON-LINEAR PHYSICS	COMPLEX-SYMPLECTIC DUALITY
1	modules	vector bundles	Maxwell equ.	Yang-Mills	Donaldson-Uhlenbeck-Yau
2	algebras	manifolds	Linear gravity equ.	Einstein gravity	Calabi-Yau theorem
3	operads	? (moduli spaces)	Rarita-Schwinger equ.	? (CFT)	? (Mirror symmetry)

combinad

Three type of geometry:

PDE	elliptic	parabolic	hyperbolic
curvature	+	0	-
genus	0	1	≥ 2
Euler number	-2	0	≥ 2
Kodaira dim	$-\infty$	0	$\dim X$
variety	Fano	Calabi-Yau	general type
filtration	unramified	tame	wild
quiver	Dynkin	affine	strictly wild
condensed	solid	liquid	gaseous

- Goal
- structures & invariants
 - classifications of
 - special v.b., mfld, subv.b., submfld
 - symmetry & quotient
 - special functors
 - homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n

(by homotopy group! \rightarrow generalized Picard group?)

Q: Is this group structure natural?

ref: <https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres>
<https://www.math.ru.nl/~gutierrez/files/k-theory/Lecture06.pdf>

Thm. $\{ \text{rank } m \text{ K-v.b. over } S^n \} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ rank m	1	2	3	4	5	6	>6
n	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2	2	0	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3	3	0	0	0	0	0	0
S^4	4	0	0	\mathbb{Z}	$\mathbb{Z}^{(2)}$	\mathbb{Z}	\mathbb{Z}
S^5	5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{(2)}$	$\mathbb{Z}/2\mathbb{Z}$	0
S^6	6	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{(2)}$	0	0

$$\mathbb{R}\mathbb{P}^\infty \cong K(\mathbb{Z}/2\mathbb{Z}, 1)$$

$\pi_{n-1}(GL_m(\mathbb{C}))$ rank m	1	2	3	4	5	6	>6
n	1	0	0	0	0	0	0
S^1	1	0	0	0	0	0	0
S^2	2	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
S^3	3	0	0	0	0	0	0
S^4	4	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
S^5	5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0
S^6	6	0	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}

$$\mathbb{C}\mathbb{P}^\infty \cong K(\mathbb{Z}, 2)$$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes, \wedge^k, \text{Sym}^k, \text{Res}, \text{Ind}$

For the other spaces:

<https://math.stackexchange.com/questions/383838/classifying-vector-bundles>

<http://www.ms.uky.edu/~guillou/F18/751Notes.pdf>

It's still not so explicit.

<https://mathoverflow.net/questions/92337/total-spaces-of-ts2-and-s2-times-r2-not-homeomorphic>

$\{ \text{rank } m \text{ K-v.b. over } M \} \longleftrightarrow [M, \text{Gr}_k(m, \infty)]$ $K = \mathbb{R}, \mathbb{C}$

M : paracompact

Unfinished task: introduce the concept of local systems and compute examples in [<https://arxiv.org/pdf/2103.02329.pdf>], 16.3.

Corollary 3.3.2 *We have*

$$\text{Vect}_r(S_n) \simeq \pi_{n-1}(\text{GL}(r, \mathbb{C})).$$

Corollary 3.3.3 *Let X be a compact topological space and SX the suspension of X . Then*

$$\text{Vect}_r(SX) \simeq [X, \text{GL}(r, \mathbb{C})].$$

In Theorem 3.4.1, X is a compact Riemann surface.

Theorem 3.4.1 (a) *The group $\text{Pic}(X)$ is isomorphic to \mathbb{Z} .*

(b) *The map*

$$L \mapsto L \oplus \mathbb{C}_X^{r-1} : \text{Pic}(X) \longrightarrow \text{Vect}_r(X)$$

is bijective.

Here I collect some stuffs which may be useful for my PhD study.

Motivic Homotopy: <https://www.zhihu.com/question/555338503>

Nearby cycles and Bloch-Kato conjecture: <https://zhuanlan.zhihu.com/p/461885915>