

Eine Woche, ein Beispiel

1.21. complex multilinear algebra

The title comes from
<http://staff.ustc.edu.cn/~wangzuq/Courses/16F-Manifolds/Notes/Lec16.pdf>

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

M : cplx mfld, $p \in M$

$M_{\mathbb{R}}$: M viewed as smooth mfld, not base change
 better: M_{sm}

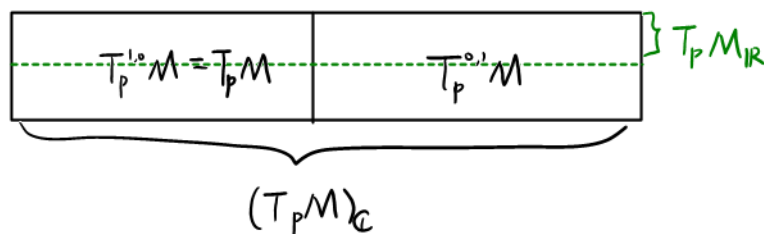
eg. $M = \mathbb{C}$ $p = 0$

Notation	base field	dim	basis	name	[YS20]
$T_p M$	\mathbb{C}	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p M_{\mathbb{R}}$	\mathbb{R}	6	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	real tangent vector	$T_p^{\mathbb{R}} M$
$(T_p M)_{\mathbb{C}} = T_p M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	\mathbb{C}	6	$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ or $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	complexified tangent vector	$T_p^{\mathbb{C}} M$
$T_p^{1,0} M = T_p M$	\mathbb{C}	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p^{0,1} M$	\mathbb{C}	3	$\frac{\partial}{\partial \bar{z}_i}$	anti-holomorphic tangent vector	
$T_p^* M$	\mathbb{C}	3	dz_i	holomorphic 1-form	Ω_p'
$T_p^* M_{\mathbb{R}} \cong \Omega_{\mathbb{R}, p}'$	\mathbb{R}	6	dx_i, dy_i	real 1-form	
$(T_p^* M)_{\mathbb{C}} = T_p^* M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	\mathbb{C}	6	$dz_i, d\bar{z}_i$ or dx_i, dy_i	complexified 1-form	$T_p^{\mathbb{C}} M = A_p'$
$T_p^{1,0,*} M \cong \Omega_p^{1,0} = T_p^{*,0} M$	\mathbb{C}	3	dz_i	(1,0)-form	$T_p^{1,0,*} M = A_p^{1,0}$
$T_p^{0,1,*} M \cong \Omega_p^{0,1}$	\mathbb{C}	3	$d\bar{z}_i$	(0,1)-form	$T_p^{0,1,*} M = A_p^{0,1}$

$\Omega^i, \Omega^{i,j}$ sheaves on M

Rmk. We don't have any natural identification between $T_p M$ & $T_p M_{\mathbb{R}}$.
 Notice that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$, $-\frac{1}{2}i$ is not real, so $\frac{\partial}{\partial \bar{z}} \notin T_p M_{\mathbb{R}}$.

although our geometrical intuition of $T_p M$ is often $T_p M_{\mathbb{R}}$,
 $T_p M \cap T_p M_{\mathbb{R}} = \emptyset$ in $(T_p M)_{\mathbb{C}}$.



Reminder: the (induced) almost complex structure is defined as

$$\begin{aligned}
 J: T_p M_{\mathbb{R}} &\longrightarrow T_p M_{\mathbb{R}} \\
 \frac{\partial}{\partial x_i} &\longmapsto \frac{\partial}{\partial y_i} \\
 \frac{\partial}{\partial y_i} &\longmapsto -\frac{\partial}{\partial x_i} \\
 \leadsto J: T_p M &\longrightarrow T_p M \\
 J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) &= \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
 J\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i}\right) &= \left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_i}\right) \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}
 \end{aligned}$$

real basis of $(T_p M)_{\mathbb{C}}$:

$$\begin{aligned}
 \mathcal{B}_1 &= \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, i\frac{\partial}{\partial x}, i\frac{\partial}{\partial y} \right\} \\
 &\xrightarrow[\text{Id}]{\text{Id}} i T_p M_{\mathbb{R}} \cong T_p M_{\mathbb{R}} \\
 \mathcal{B}_2 &= \left\{ \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, i\frac{\partial}{\partial z}, i\frac{\partial}{\partial \bar{z}} \right\} \\
 &\xrightarrow[\text{xi}]{\text{xi}} T_p M \xrightarrow[\text{xi}]{\text{xi}} T_p M \\
 \begin{aligned}
 \left\{ \begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{2}(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}) \\ \frac{\partial}{\partial y} &= \frac{1}{2i}(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}) \end{aligned} \right. & \quad \begin{aligned} dx &= \frac{1}{2}(dz + d\bar{z}) \\ dy &= \frac{1}{2i}(dz - d\bar{z}) \end{aligned} \\
 \left\{ \begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}) \end{aligned} \right. & \quad \begin{aligned} dz &= dx + i dy \\ d\bar{z} &= dx - i dy \end{aligned}
 \end{aligned}
 \end{aligned}$$

$\mathcal{Q}: J, \text{conj}_J$
 $\mathcal{Q}: \text{conj}_i$
 $\mathcal{Q}: \text{xi}$

⚠ The conjugation usually don't preserve cplx subspace.

E.g. in \mathbb{C}^2 , $\mathbb{C} \cdot (1, i)$ is not preserved by conjugation.

Rmk. $V = \mathbb{C}^n$

$$\begin{array}{ccc} \{\text{conjugations of } V\} & \longleftrightarrow & \{\text{dec } V = W \oplus iW\} \cong GL_n(\mathbb{C})/GL_n(\mathbb{R}) \\ \sigma & \longmapsto & V = V^\sigma \oplus iV^\sigma \end{array}$$

$$(T_p M)_{\mathbb{C}}^{\text{conj}_i} = T_p M_{\mathbb{R}}$$

$$(T_p M)_{\mathbb{C}}^{\text{conj}_J} = \langle \partial_x, i\partial_y \rangle_{\mathbb{R}} = \langle \partial_z, \partial_{\bar{z}} \rangle_{\mathbb{R}} \quad \text{not stable under } i\text{-action}$$

$$\begin{pmatrix} 1 & \\ & i \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & \frac{1}{2}i \end{pmatrix} \quad \text{in } GL_n(\mathbb{C})/GL_n(\mathbb{R})$$

When viewed as cplx v.s.
the natural conjugation on $T_p^{p,q} M$ is induced by conj_i .

$$\begin{aligned}
J(f\partial_x + g\partial_y) &= f\partial_y - g\partial_x \\
J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_y, -\partial_x, i\partial_y, -i\partial_x) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & 1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
J(f\partial_z + g\partial_{\bar{z}}) &= if\partial_z - ig\partial_{\bar{z}} \\
J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (i\partial_z, -i\partial_{\bar{z}}, -\partial_z, \partial_{\bar{z}}) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_J(f\partial_x + g\partial_y) &= \bar{f}\partial_x - \bar{g}\partial_y \\
\text{conj}_J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, -\partial_y, -i\partial_x, i\partial_y) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_J(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_z + \bar{g}\partial_{\bar{z}} \\
\text{conj}_J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_z, \partial_{\bar{z}}, -i\partial_z, -i\partial_{\bar{z}}) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_i(f\partial_x + g\partial_y) &= \bar{f}\partial_x + \bar{g}\partial_y \\
\text{conj}_i(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, \partial_y, -i\partial_x, -i\partial_y) \\
&= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\text{conj}_i(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_{\bar{z}} + \bar{g}\partial_z \\
\text{conj}_i(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_{\bar{z}}, \partial_z, -i\partial_{\bar{z}}, -i\partial_z) \\
&= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}$$

Hermitian metric

$$\begin{aligned}
 H &= h_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta && \text{Hermitian metric } (h_{\alpha\beta}) \in \mathbb{R}^{n \times n} \text{ pos def} \\
 g &= \frac{1}{2} (H + \bar{H}) && \text{Riemannian metric} \\
 \omega &= \frac{i}{2} (H - \bar{H}) && \text{Hermitian form}
 \end{aligned}$$

e.g.

$$\begin{aligned}
 H &= dz \otimes d\bar{z} \\
 &= (dx \otimes dx + dy \otimes dy) - i(dx \otimes dy - dy \otimes dx) = g - i\omega \\
 g &= \frac{1}{2} (dz \otimes d\bar{z} + d\bar{z} \otimes dz) \\
 &= dx \otimes dx + dy \otimes dy \\
 \omega &= \frac{i}{2} (dz \otimes d\bar{z} - d\bar{z} \otimes dz) = i dz \wedge d\bar{z} \\
 &= dx \otimes dy - dy \otimes dx = 2 dx \wedge dy
 \end{aligned}$$

$$\kappa = -\frac{1}{h} \partial_z \partial_{\bar{z}} \ln h = -\frac{1}{h} \frac{1}{4} \Delta (\ln h) \quad \Delta = \partial_x^2 + \partial_y^2$$

Two methods to show $\mathbb{I}(\partial_z, \partial_{\bar{z}}) = 0$

Method 1.

$$i \mathbb{I}(\partial_z, \partial_{\bar{z}}) = \mathbb{I}(J\partial_z, \partial_{\bar{z}}) = \mathbb{I}(\partial_z, J\partial_{\bar{z}}) = -i \mathbb{I}(\partial_z, \partial_{\bar{z}})$$

Method 2.

$$\begin{aligned}
 \mathbb{I}(\partial_z, \partial_{\bar{z}}) &= \mathbb{I}\left(\frac{1}{2}(\partial_x - i\partial_y), \frac{1}{2}(\partial_x + i\partial_y)\right) \\
 &= \mathbb{I}\left(\frac{1}{2}(1 - iJ)\partial_x, \frac{1}{2}(1 + iJ)\partial_x\right) \\
 &= \frac{1}{2}(1 - iJ) \cdot \frac{1}{2}(1 + iJ) \mathbb{I}(\partial_x, \partial_x) \\
 &= \frac{1}{4}(1 - i^2 J^2) \mathbb{I}(\partial_x, \partial_x) \\
 &= 0.
 \end{aligned}$$