Un example par jour

4.8 K3 surface: cpt cplx surf s.t $\omega_X \cong \mathcal{O}_X$ $\boxed{H^1(X, \mathcal{O}_X) = 0}$ can be changed by $b_1 = 0$ or $\pi_1(X) = \{*\}$

Today: Fermat quartic surface $X: z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$ $i: X \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$

1. proj smooth alg surf ✓

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to i_* \mathcal{O}_X \to 0$$

Lemma VIII.9 *Let $V \subset \mathbb{P}^n$ be a d-dimensional complete intersection; then $H^i(V, \mathcal{O}_V) = 0$ for $0 < i < d$.* [Beauville]

H^3 $\hookrightarrow \mathbb{C} \to 0$

H^2 $\to 0 \to 0 \to H^2(X, \mathcal{O}_X) \Big\} \cong \mathbb{C}$

H^1 $\to 0 \to 0 \to H^1(X, \mathcal{O}_X) \Big\} \cong 0$

H^0 $0 \to 0 \to \mathbb{C} \to H^0(X, \mathcal{O}_X) \Big\} \cong \mathbb{C}$

$\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4) \xRightarrow{21.5.B \text{ in } [Vakil]} \omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X = \mathcal{O}_X$. $\dfrac{z_0 \, dz_1 \wedge dz_2}{4 z_3^3}$ $z_0 \neq 0$ $z_3 \neq 0$

∴ X is a K3-surface.

2. $\Big[$ **Prop:** X: K3 surfaces $\Rightarrow$ X is **minimal.**

$P_n = 1$ $n \geq 1$ $\Rightarrow \kappa(X) = 0$

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  2   -20   2        24
          ⋮        ⋮
       1            1
     0   0          0
   1  20  1        22
     0   0          0
       1            1
```

$(b_+, b_-) = (3, 19)$
$c_2 = 24$
$c_1^2 = K^2 = 0$ $\Big]$

Some conclusions on alg topo: ↗ cpt M: closed connected mfld of dim n

	orientable	nonorientable
$H_n(M)$	$\mathbb{Z}$	0
$H^n(M)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H_n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H^n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Moreover, when M is oriented + cpt $\Rightarrow T^p \cong T^{n-p+1}$ $(H^i(X) \cong \mathbb{Z}^{\oplus b_i} \oplus T^i)$

Cor $T_i := H_i(X)$ ↖ def ↳ by universal coefficient thm & Poincaré duality

n	0	1	2	3	4	≥ 5
$H_n(X)$	$\mathbb{Z}$	T	$\mathbb{Z}^{22} \oplus T$	0	$\mathbb{Z}$	0
$H^n(X)$	$\mathbb{Z}$	0	$\mathbb{Z}^{22} \oplus T$	T	$\mathbb{Z}$	0

Claim. $T = 0$, i.e (co)homology has no torsion!

https://math.stackexchange.com/questions/2882059/

$\Big[$ **Proof.** by LES induced by $0 \to \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \to 1$,
only need to prove $\mathrm{Pic}(X)$ is torsion free.
Suppose $D \in \mathrm{Div}(X)$ s.t $nD \equiv 0$ $\Rightarrow \chi(D) = 2$
$\Rightarrow h^0(D) \geq 1$ or $h^0(-D + K_X) = h^2(D) \geq 1$ w.l.o.g suppose $h^0(D) \geq 1$
$\Rightarrow D \sim D'$ where D' is effective
$\xRightarrow[nD \equiv 0]{} D' = 0$ $\Big]$

$$\left[\begin{array}{l}
\text{Another proof. If } H_1(X) \text{ has torsion, then } \exists \, G \lhd H_1(X) \text{ s.t } H_1(X)/_G \cong \mathbb{Z}/_{m\mathbb{Z}}, \\
\text{denote } p: \pi_1(X) \longrightarrow H_1(X), \text{ then } \pi_1(X)/_{p^{-1}(G)} \cong H_1(X)/_G \cong \mathbb{Z}/_{m\mathbb{Z}} \quad m \in \mathbb{N}^+ \\ \hspace{24em} m > 1 \\
\therefore \exists \text{ a nontrivial unramified covering of degree } m \\
\hspace{4em} \Phi: \widehat{X} \longrightarrow X \\
\Rightarrow \begin{cases} K_{\widehat{X}} = 0 \\ \chi_{top}(\widehat{X}) = 24m \end{cases} \Rightarrow \chi(\mathcal{O}_{\widehat{X}}) = \frac{1}{12}(K_{\widehat{X}}^2 + \chi_{top}(\widehat{X})) = 2m > 2 - h^1(\mathcal{O}_{\widehat{X}}) = \chi(\mathcal{O}_{\widehat{X}}) \\
\hspace{30em} \text{Contradiction!}
\end{array}\right]$$

12.2. K3 lattice.

We deduce that a K3 surface has second Betti number $b_2 = 22$. Cup-product equips $H^2(X, \mathbb{Z})$ with the structure of an integral lattice of rank 22. Often this lattice is called **K3 lattice** and denoted by Λ. The following properties of Λ are well-known:

- Λ is unimodular by Poincaré-duality;
- Λ has signature $(3, 19)$ by the topological index theorem;
- Λ is even by Wu's formula since the first Chern class is even. ← one signature:

$C^2 = 2g(C) - 2$ is even

Hence the classification of even unimodular lattices implies that

$$\Lambda \cong U^{\oplus 3} \oplus \not{U}\!\!\not{A} \cdot (-E_8)^{\oplus 2}$$

$\hookleftarrow \widetilde{b}(x,y) := -b(x,y)$ \qquad $U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

Now let us consider the fundamental group of X.

Thm (Lefschetz hyperplane thm) [In wiki there are several proofs].

Let Y: n-dim cplx proj variety

X: hyperplane section of Y

$U: Y - X$ is smooth ← [Maybe we need stronger condition that Y is sm, then the wiki is wrong]

then $H_k(Y, X, \mathbb{Z})$, $H^k(Y, X, \mathbb{Z})$, $\pi_k(Y, X) = 0$ for $k \le n-1$. $k \in \mathbb{N}^+$

Cor.
$$\begin{array}{llll}
& & k < n-1 & k = n \\
H_k(X) \longrightarrow H_k(Y) & \text{is} & \text{iso} & \text{or surj} \\
H^k(Y) \longrightarrow H^k(X) & \text{is} & \text{iso} & \text{or inj} \\
\pi_k(X) \longrightarrow \pi_k(Y) & \text{is} & \text{iso} & \text{or surj}.
\end{array}$$

For X Fermat quartic, $X \hookrightarrow \mathbb{P}_\mathbb{C}^3 \overset{\phi}{\hookrightarrow} \mathbb{P}_\mathbb{C}^{34}$ \quad $Y = \phi(\mathbb{P}_\mathbb{C}^3)$,

$H_1(X) = H_1(Y) = H_1(\mathbb{P}_\mathbb{C}^3) = 0$ \qquad $H^1(X) = 0$ \quad $\pi_1(X) = 0$

$\Rightarrow X$ is simply connected 4-dim mfld.

Rmk. By [GTM82, Thm 17.21], $\pi_2(X) = H_2(X) = \mathbb{Z}^{22}$;

By

$\pi_n(X) = \pi_n(\#^{20} S^2 \times S^3)$ \quad $n \ge 3$, and

n	1	2	3	4	5	6	7
$\pi_n(X)$	0	$\mathbb{Z}^{22}$	$\mathbb{Z}^{252}$	$\mathbb{Z}^{3520} \oplus (\mathbb{Z}/_{2\mathbb{Z}})^{42}$	$\mathbb{Z}^{57960} \oplus$ tor	$\mathbb{Z}^{1020096} \oplus$ tor	$\mathbb{Z}^{22397760} \oplus$ tor

Picard group:

Again, by $\quad 0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$

$$\nearrow H^2(X,\mathbb{Z}) \underset{\cong\mathbb{Z}}{\overset{\cdot 2}{\longrightarrow}} H^2(\mathcal{O}_X)_{\cong\mathbb{C}} \overset{}{\longrightarrow} H^2(\mathcal{O}_X^\times)$$

$$\nearrow H^1(X,\mathbb{Z})_{\cong 0} \longrightarrow H^1(\mathcal{O}_X)_{\cong 0} \longrightarrow H^1(\mathcal{O}_X^\times) = \mathrm{Pic}(X)$$

$$0 \to H^0(X,\mathbb{Z}) \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X^\times)$$
$$\qquad\quad \cong \mathbb{Z} \qquad\qquad \cong \mathbb{C} \qquad\qquad \cong \mathbb{C}^\times$$

$\Rightarrow \mathrm{Pic}(X) = NS(X) \subseteq H^2(X,\mathbb{Z}) = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$

$\Rightarrow \mathrm{Pic}(X) \cong \mathbb{Z}^{\rho(X)}, \ 1 \le \rho(X) \le 20$

<u>Rmk</u>. for X Fermat quartic, $\quad \rho(X) = 20, \quad \mathrm{Pic}(X) \cong (-E_8)^{\oplus 2} \oplus U \oplus (-8)^{\oplus 2} \quad \mathrm{Pic}(X)^\dagger \cong (8)^{\oplus 2}$

[Schütt, Shioda and van Luijk]

$\underline{Q}$: find a metric s.t $\mathrm{Ric}(\rho) \equiv 0$?