Eine Woche, ein Beispiel

3.9. semi-orthogonal dec outside Kuznetsov component

Ref:

[MS19]: Emanuele Macrì, Benjamin Schmidt, Lectures on Bridgeland Stability, https://arxiv.org/abs/1607.01262
[GR87]: A. L. Gorodentsev, A. N. Rudakov, Exceptional vector bundles on projective spaces

Well, the main part of this document aims to solve some results in [MS19].

Recall that [MS19]

$$\begin{aligned} ch^B(E) &= ch(E)\cdot e^{-B} \qquad [p14] \\ &= ch_0(E) + (ch_1(E) - B\cdot ch_0(E)) \\ &\quad + \left(ch_2(E) - B\cdot ch_1(E) + \frac{B^2}{2} ch_0(E)\right) + \cdots \end{aligned}$$

$$\begin{aligned} Z_{\omega,B}(E) &= -\int_X e^{-B+i\omega} ch(E) \qquad [p31] \\ &= \left(-ch_2^B(E) + \frac{\omega^2}{2} ch_0^B(E)\right) + i\omega\cdot ch_1^B(E) \end{aligned}$$

$$\begin{aligned} \nu_{\omega,B}(E) &= -\frac{\operatorname{Re} Z_{\omega,B}(E)}{\operatorname{Im} Z_{\omega,B}(E)} \\ &= \frac{ch_2^B(E) - \frac{\omega^2}{2}\cdot ch_0^B(E)}{\omega\cdot ch_1^B(E)} \end{aligned}$$

$$\begin{aligned} Z_{\alpha,\beta}(E) &= Z_{\alpha H, B_0+\beta H}(E) \qquad [p37] \\ \nu_{\alpha,\beta}(E) &= \nu_{\alpha H, B_0+\beta H}(E) \end{aligned}$$

Therefore,

$$\begin{aligned} Z_{\alpha,\beta}(E) &= Z_{\alpha H, B_0+\beta H}(E) \\ &= -\int_X e^{-B_0-\beta H + i\alpha H} ch(E) \\ &= -\int_X e^{-B_0 + i(\alpha+i\beta)H} ch(E) \\ &= \left(-ch_2^{B_0}(E) + \frac{(\alpha+i\beta)^2H^2}{2} ch_0^{B_0}(E)\right) + i(\alpha+i\beta)H ch_1^{B_0}(E) \\ &\overset{\substack{B=0 \\ z=\alpha+i\beta}}{\underset{\text{in } \mathbb{P}^2}{=\!=}} \left(-ch_2(E) + \frac{z^2}{2} ch_0(E)\right) + iz\, ch_1(E) \end{aligned}$$

[MS19, Ex 7.3]

$$\begin{aligned} ch(I_Z) &= 1-4H^2 &&= (1,\ 0,\ -4) \\ ch(\mathcal{O}(-2)) &= 1-2H+2H^2 &&= (1,\ -2,\ 2) \end{aligned}$$

Therefore,

$$\begin{aligned} Z_{\alpha,\beta}(I_Z) &= 4+\frac{z^2}{2} &&= \frac{1}{2}(z^2+8) \\ Z_{\alpha,\beta}(\mathcal{O}(-2)) &= -2+\frac{z^2}{2}-2iz &&= \frac{1}{2}(z-2i)^2 \end{aligned}$$

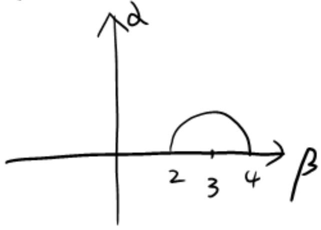
$$0 = \operatorname{Im} \frac{8+z^2}{(z-2i)^2}$$

$$\Leftrightarrow \quad (8+z^2)(\bar{z}^2+4i\bar{z}-4) - (8+\bar{z}^2)(z^2-4iz-4) = 0$$

$$\Leftrightarrow \quad (\bar{z}+z)\left(12(\bar{z}-z)+32i+4iz\bar{z}\right) = 0$$

$$\Leftarrow \quad |z|^2 - 6\operatorname{Im} z + 8 = 0$$

$$\Leftrightarrow \quad \alpha^2+(\beta-3)^2 = 1$$



Or: $8+(\alpha+i\beta)^2 \sim (\alpha+i\beta-2i)^2$

$$\begin{aligned} 8+(\alpha+i\beta)^2 &= 8+\alpha^2-\beta^2+2\alpha\beta\cdot i \\ (\alpha+i\beta-2i)^2 &= \alpha^2-(\beta-2)^2+2\alpha(\beta-2)i \end{aligned}$$

$$\Leftrightarrow \quad (8-\alpha^2-\beta^2)(\cancel{2\alpha}(\beta-2)) = (\alpha^2-(\beta-2)^2)\cancel{2\alpha}\beta$$

$$\Leftarrow \quad -2(8-\alpha^2-\beta^2) + 4\beta(3-\beta) = 0$$

$$\Leftrightarrow \quad \alpha^2+(\beta-3)^2 = 1$$

Lemma. [GR87, 4.2]

Let $\mathcal{A}$ be an abelian category, and $F, E, G \in \mathcal{A}$. Assume that we have a SES

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

with $[F,G]^0 = [G,F]^2 = 0$. Then

$$[E,E]' \geqslant [F,F]' + [G,G]'.$$

Proof

$$\begin{tikzcd}
 & 0 \arrow[d] & & & & & \\
0 & {[F,F]'} \arrow[l] \arrow[d] \arrow[dr, "\exists", "\wr"'] & & {[E,F]'} \arrow[ll] \arrow[d] & & {[G,F]'} \arrow[ll] \arrow[d] & \\
 & & \mathrm{Im}\,g \arrow[dl, "\wr"] & & & & \\
 & {[F,E]'} \arrow[d] & & {[E,E]'} \arrow[ll, "g"'] \arrow[d] \arrow[ul] & & {[G,E]'} \arrow[ll, "g"'] \arrow[d] \arrow[dl] & \\
 & & & & \mathrm{Im}\,f \arrow[ul] \arrow[dr, "\exists", "\wr"'] & & \\
 & {[F,G]'} & & {[E,G]'} \arrow[ll] & & {[G,G]'} \arrow[ll] \arrow[d] & 0 \arrow[l] \\
 & & & & & 0 &
\end{tikzcd}$$

$$\begin{aligned}
[E,E]' &= \dim \mathrm{Im}\,g + \dim \mathrm{Im}\,f \\
&\geqslant [F,F]' + [G,G]'
\end{aligned}$$