

Eine Woche, ein Beispiel

1.21. complex multilinear algebra

The title comes from

<http://staff.ustc.edu.cn/~wangzuoq/Courses/16F-Manifolds/Notes/Lec16.pdf>

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

M , cplx mfld, $p \in M$

$M_{\mathbb{R}} : M$ viewed as smooth mfld, not base change
 better: M_{sm}

e.g. $M = \mathbb{C}^3$ $p = 0$

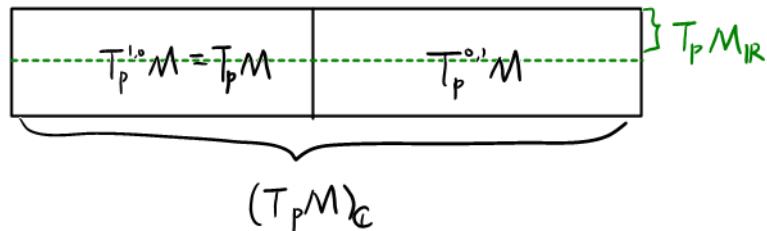
| Notation | base field | dim | basis | name | [YS20] |
|--|--------------|-----|--|---------------------------------|------------------------------|
| $T_p M$ | \mathbb{C} | 3 | $\frac{\partial}{\partial z_i}$ | holomorphic tangent vector | |
| $T_p M_{\mathbb{R}}$ | \mathbb{R} | 6 | $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ | real tangent vector | $T_p^{\mathbb{R}} M$ |
| $(T_p M)_{\mathbb{C}} := T_p M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ | \mathbb{C} | 6 | $\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ or $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$ | complexified tangent vector | $T_p^{\mathbb{C}} M$ |
| $T_p^{1,0} M = T_p M$ | \mathbb{C} | 3 | $\frac{\partial}{\partial z_i}$ | holomorphic tangent vector | |
| $T_p^{0,1} M$ | \mathbb{C} | 3 | $\frac{\partial}{\partial \bar{z}_i}$ | anti-holomorphic tangent vector | |
| $T_p^* M$ | \mathbb{C} | 3 | dz_i | holomorphic 1-form | Ω_p^1 |
| $T_p^* M_{\mathbb{R}} = \Omega_{\mathbb{R}, p}$ | \mathbb{R} | 6 | dx_i, dy_i | real 1-form | |
| $(T_p^* M)_{\mathbb{C}} := T_p^* M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ | \mathbb{C} | 6 | $dz_i, d\bar{z}_i$ or dx_i, dy_i | complexified 1-form | $T_p^{\mathbb{C}} M = A_p^1$ |
| $T_p^{1,0,*} M = \Omega_p^{1,0} = T_p^* M_{\mathbb{C}}$ | \mathbb{C} | 3 | dz_i | $(1,0)$ -form | $T_p^{1,0} M = A_p^{1,0}$ |
| $T_p^{0,1,*} M = \Omega_p^{0,1}$ | \mathbb{C} | 3 | $d\bar{z}_i$ | $(0,1)$ -form | $T_p^{0,1} M = A_p^{0,1}$ |

$\Omega^i, \Omega^{i,j}$ sheaves on M

Rmk. We don't have any natural identification between $T_p M$ & $T_p M_{\mathbb{R}}$.

Notice that $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$, $-\frac{1}{2}i$ is not real. so $\frac{\partial}{\partial \bar{z}} \notin T_p M_{\mathbb{R}}$.

although our geometrical intuition of $T_p M$ is often $T_p M_{\mathbb{R}}$,
 $T_p M \cap T_p M_{\mathbb{R}} = \emptyset$ in $(T_p M)_{\mathbb{C}}$.



Reminder: the (induced) almost complex structure is defined as

$$\begin{aligned}
 J: T_p M_{\mathbb{R}} &\longrightarrow T_p M_{\mathbb{R}} \\
 \frac{\partial}{\partial x_i} &\longmapsto \frac{\partial}{\partial y_i} \\
 \frac{\partial}{\partial y_i} &\longmapsto -\frac{\partial}{\partial x_i} \\
 \rightsquigarrow J: T_p M &\longrightarrow T_p M \\
 J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) &= \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\
 J\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) &= \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) \begin{pmatrix} i & \\ & -i \end{pmatrix}
 \end{aligned}$$

real basis of $(T_p M)_C$:

$$\begin{array}{ccc}
 \mathcal{B}_1 = \left\{ \underbrace{\partial_x, \partial_y}_{G T_p M_{\mathbb{R}}}, \underbrace{i\partial_x, i\partial_y}_{i T_p M_{\mathbb{R}} \mathcal{Q}} \right\} & \xleftrightarrow{\text{Id}} & \left\{ \begin{array}{l} \partial_x = \partial_z + \partial_{\bar{z}} \\ \partial_y = \frac{1}{i}(\partial_z - \partial_{\bar{z}}) \end{array} \right. & dx = \frac{1}{i}(dz + d\bar{z}) \\
 & \xleftrightarrow{-\text{Id}} & \left. \begin{array}{l} \partial_y = \frac{1}{i}(\partial_z - \partial_{\bar{z}}) \\ \partial_x = \partial_z + \partial_{\bar{z}} \end{array} \right. & dy = \frac{1}{2i}(dz - d\bar{z}) \\[10pt]
 \mathcal{B}_2 = \left\{ \underbrace{\partial_z, \partial_{\bar{z}}}_{G T_p M}, \underbrace{i\partial_z, i\partial_{\bar{z}}}_{T^* M \mathcal{Q}} \right\} & \xleftrightarrow{x_i} & \left\{ \begin{array}{l} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{array} \right. & dz = dx + idy \\
 & \xleftrightarrow{x_i} & \left. \begin{array}{l} \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \\ \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \end{array} \right. & d\bar{z} = dx - idy
 \end{array}$$

$\mathcal{Q}: J, \text{conj}_J$

$\mathcal{Q}: \text{conj}_i$

$\mathcal{Q}: x_i$

⚠ The conjugation usually don't preserve cplx subspace.

E.g. in \mathbb{C}^2 , $\mathbb{C} \cdot (1, i)$ is not preserved by conjugation.

Rmk. $V = \mathbb{C}^n$

$$\begin{array}{ccc} \{\text{conjugations of } V\} & \longleftrightarrow & \{\text{dec } V = W \oplus iW\} \cong GL_n(\mathbb{C})/GL_n(\mathbb{R}) \\ \sigma & \longmapsto & V = V^\sigma \oplus iV^\sigma \end{array}$$

$$(T_p M)_{\mathbb{C}}^{\text{conj.}} = T_p M_{\mathbb{R}}$$

$$(T_p M)_{\mathbb{C}}^{\text{conj.}} = \langle \partial_x, i\partial_y \rangle_{\mathbb{R}} = \langle \partial_z, \partial_{\bar{z}} \rangle_{\mathbb{R}} \quad \text{not stable under } i\text{-action}$$

$$\begin{pmatrix} 1 & \\ & i \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{in } GL_n(\mathbb{C})/GL_n(\mathbb{R})$$

When viewed as cplx v.s.
the natural conjugation on $T_p M$ is induced by conj.

$$\begin{aligned} J(f\partial_x + g\partial_y) &= f\partial_y - g\partial_x \\ J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_y, -\partial_x, i\partial_y, -i\partial_x) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} J(f\partial_z + g\partial_{\bar{z}}) &= if\partial_{\bar{z}} - ig\partial_z \\ J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (i\partial_z, -i\partial_{\bar{z}}, -\partial_z, \partial_{\bar{z}}) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_J(f\partial_x + g\partial_y) &= \bar{f}\partial_x - \bar{g}\partial_y \\ \text{conj}_J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, -\partial_y, -i\partial_x, i\partial_y) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_J(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_z + \bar{g}\partial_{\bar{z}} \\ \text{conj}_J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_z, \partial_{\bar{z}}, -i\partial_z, -i\partial_{\bar{z}}) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_i(f\partial_x + g\partial_y) &= \bar{f}\partial_x + \bar{g}\partial_y \\ \text{conj}_i(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, \partial_y, -i\partial_x, -i\partial_y) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_i(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_{\bar{z}} + \bar{g}\partial_z \\ \text{conj}_i(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_{\bar{z}}, \partial_z, -i\partial_{\bar{z}}, -i\partial_z) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

Hermitian metric

$$\begin{aligned}
 H &= h_{\alpha\beta} dz^\alpha \otimes d\bar{z}^\beta && \text{Hermitian metric } (h_{\alpha\beta}) \in \mathbb{R}^{n \times n} \text{ pos def} \\
 g &= \frac{i}{2} (H + \overline{H}) && \text{Riemannian metric} \\
 \omega &= \frac{i}{2} (H - \overline{H}) && \text{Hermitian form}
 \end{aligned}$$

e.g.

$$\begin{aligned}
 H &= dz \otimes d\bar{z} \\
 &= (dx \otimes dx + dy \otimes dy) - i(dx \otimes dy - dy \otimes dx) = g - i\omega \\
 g &= \frac{i}{2} (dz \otimes d\bar{z} + d\bar{z} \otimes dz) \\
 &= dx \otimes dx + dy \otimes dy \\
 \omega &= \frac{i}{2} (dz \otimes d\bar{z} - d\bar{z} \otimes dz) = i dz \wedge d\bar{z} \\
 &= dx \otimes dy - dy \otimes dx = z dx \wedge dy
 \end{aligned}$$

$$K = -\frac{1}{h} \partial_z \partial_{\bar{z}} \ln h = -\frac{i}{h} \frac{1}{4} \Delta (\ln h) \quad \Delta = \partial_x^2 + \partial_y^2$$

Two methods to show $\mathbb{II}(\partial_z, \partial_{\bar{z}}) = 0$

Method 1.

$$i \mathbb{II}(\partial_z, \partial_{\bar{z}}) = \mathbb{II}(J\partial_z, \partial_{\bar{z}}) = \mathbb{II}(\partial_z, J\partial_{\bar{z}}) = -i \mathbb{II}(\partial_z, \partial_{\bar{z}})$$

Method 2.

$$\begin{aligned}
 \mathbb{II}(\partial_z, \partial_{\bar{z}}) &= \mathbb{II}\left(\frac{1}{2}(J\partial_x - i\partial_y), \frac{1}{2}(J\partial_x + i\partial_y)\right) \\
 &= \mathbb{II}\left(\frac{1}{2}(1-iJ)\partial_x, \frac{1}{2}(1+iJ)\partial_x\right) \\
 &= \frac{1}{2}(1-iJ) \cdot \frac{1}{2}(1+iJ) \mathbb{II}(\partial_x, \partial_x) \\
 &= \frac{1}{4}(1-i^2J^2) \mathbb{II}(\partial_x, \partial_x) \\
 &= 0.
 \end{aligned}$$