# Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases
4. conclusions on various forms

**Setting.** We work over conn red gp over $F$. ($G/F$ conn red)

$\bar{F}$ : the seperable closure of $F$ — mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_F := \mathrm{Gal}(\bar{F}/F)$ — $\sigma \in \Gamma_F$

$$Z^1(W,A) := \left\{ \begin{array}{c} {}^L\varphi : W \longrightarrow A \rtimes W \\ \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{array} \middle| \begin{array}{l} {}^L\varphi : \text{continuous group homo} \\ {}^L\varphi \text{ is a section} \end{array} \right\}$$
$$= \{\varphi : W \longrightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)^{\gamma}(\varphi(\gamma'))\}$$

$\varphi \in H^1(W,A)$

$$H^1(W,A) = Z^1(W,A)/A.$$

**Def. (Split)** $G$ is split if
$G$ has a maximal torus $T$ over $F$ which is split.

**Def. (Quasi-split)** $G$ is quasi-split if
$G$ has a Borel $B$ over $F$.

Borel = maximal (Zar-closed) conn sol alg subgp
= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t $G/H$ is projective
= closed subgp containing a Borel.


Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat{Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability
https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf

https://mathoverflow.net/questions/121959/classification-of-tori-of-gl2-up-to-conjugation


I make no claim to originality.

## 1. forms.

**Def.** $G_1, G_2/F$ are called forms, if
$$\exists\ \alpha : G_{2,\bar F} \xrightarrow{\sim} G_{1,\bar F} \quad \text{as gps} \qquad \text{not as } \Gamma_F\text{-gps!}$$
$\alpha$ is considered as the information of forms.

**Thm.**
$$\{F\text{-forms of } G\}/\sim \ \longleftrightarrow\ H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$$
$$[G_2,\ \alpha : G_{2,\bar F} \to G_{\bar F}] \longmapsto \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1}$$
$$G_2 \longleftarrow\!\!\shortmid\ \varphi$$

$$\begin{array}{ccc} G_{2,\bar F} & \xrightarrow{\alpha} & G_{\bar F} \\ \sigma\downarrow & & \downarrow\sigma \\ G_{2,\bar F} & \xrightarrow{\alpha} & G_{\bar F} \\ & \searrow_{\alpha} & \\ & & G_{\bar F} \end{array}$$

$$\Gamma_F \circlearrowright G_{2,\bar F} \longrightarrow G_{\bar F} \circlearrowleft \Gamma_F,\ \Gamma_F$$

$\varphi_\alpha$ measures the twist between two $\Gamma_F$-actions.

$$G_2(F) := \{ g \in G(\bar F) \mid \underbrace{(\varphi(\sigma)\circ\sigma)}_{\text{induced action}} g = g \quad \forall \sigma \in \Gamma_F \}$$

In general, $G_2(R) := \{ g \in G(\bar F \otimes_F R) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_F \}$

e.p. $G_2(\bar F) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_F} \in \prod_{\sigma\in\Gamma_F} G(\bar F) \mid g \in G(\bar F) \} \cong G(\bar F)$

$$G(\bar F \otimes_F \bar F) \cong G\Big(\bigoplus_{\sigma\in\Gamma_F} \bar F\Big) \cong \prod_{\sigma\in\Gamma_F} G(\bar F)$$

$(G_2, \alpha) \sim (G_2', \alpha')$, if
$\exists\ \beta : G_2 \to G_2'$ as an iso.

$$\begin{array}{ccc} G_{2,\bar F} & \xrightarrow{\alpha} & G_{\bar F} \\ \beta_{\bar F}\downarrow & & \downarrow \alpha'\circ\beta_{\bar F}\circ\alpha^{-1} \\ G'_{2,\bar F} & \xrightarrow{\alpha'} & G_{\bar F} \end{array}$$

Functorial on $F$. (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])
Let $E/F$ be finite Galois.

$$\begin{array}{ccccccc} G_{2,E} & & \{E\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_E, \mathrm{Aut}(G_{\bar F})) & & \varphi|_{\Gamma_E} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ G_2 & & \{F\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) & & \varphi \\ & & \uparrow & & \uparrow & & \\ & & \{G_2/F : G_{2,E} \cong G_E\} & \longleftrightarrow & H^1(\mathrm{Gal}(E/F), \underbrace{\mathrm{Aut}(G_{\bar F})^{\Gamma_E}}_{=\,\mathrm{Aut}(G_E)}) & & \\ & & \uparrow & & \uparrow & & \\ & & 1 & & 1 & & \end{array}$$

Rmk. We have the classification of connected reductive gps:

$$\begin{array}{lcl} & & \qquad\qquad\qquad\qquad \cong \Psi(G,B,T) \\ \{\text{split red gp}/F\} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) \\ G \text{ split} \quad \{\text{qs red gp}/F\} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) + \Gamma_F\text{-action} \\ & & = (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, Out(G_{\bar F})) \\ \{\text{red gp}/F\} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, Aut(G_{\bar F})) \end{array}$$

To understand the result, the following isos are needed:

$$Aut(G_{\bar F}) \cong Inn(G_{\bar F}) \rtimes Aut(G, B, T, \{u_\alpha\})$$

$$\begin{aligned} Out(G_{\bar F}) &\cong Aut(G, B, T, \{u_\alpha\}) && \text{for embedding} \\ &\cong Aut(\Psi(G, B, T)) && \text{for combinatorics} \end{aligned}$$

Also, by the Hilbert 90, one has

$$H^1(\Gamma_F, Aut(G, B, T, \{u_\alpha\})) \cong H^1(\Gamma_F, Aut(G, B, T))$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} Aut(\mathbb{G}_m^n) & \subseteq & End(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$Hom(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow\!\shortmid n$

Therefore,

$$H^1(\Gamma_F, Aut(\mathbb{G}_{m,\bar{F}}^n)) = H^1(\Gamma_F, GL_n(\mathbb{Z}))$$
$$= Hom_{Grp}(\Gamma_F, GL_n(\mathbb{Z}))/_{GL_n(\mathbb{Z})\text{-conj}}$$
$$\underset{\text{when } F=\mathbb{R}}{=\!=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = Id\}/_{GL_n(\mathbb{Z})\text{-conj}}$$

[ $\Gamma_F$ acts on $Aut(\mathbb{G}_{m,\bar{F}}^n) \subseteq End(\mathbb{G}_{m,\bar{F}}^n)$ trivially:

see $\bar{F}$-pts, $n=1$:

$$\begin{array}{ccc} \bar{F}^\times & \xrightarrow{\alpha} & \bar{F}^\times \\ \sigma\downarrow & & \downarrow\sigma \\ \bar{F}^\times & \xrightarrow{{}^\sigma\alpha} & \bar{F}^\times \end{array} \qquad \begin{array}{ccc} x & \longmapsto & x^n \\ \downarrow & & \downarrow \\ \sigma(x) & & \sigma(x^n) = \sigma(x)^n \end{array}$$

$\Rightarrow {}^\sigma\alpha = \alpha$ ]

E.g. $n=1$, $F=\mathbb{R}$

$$H^1(\Gamma_F, Aut(\mathbb{G}_{m,\bar{F}})) \cong \{1, -1\}$$

$\varphi(\sigma) = (-)^{-1}$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ \mathbb{G}_m & G = ?\ SO_{2,\mathbb{R}} \end{array}$$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\} \\ &= \{g \in \mathbb{C}^\times \mid |g| = 1\} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = Spec\ \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$

Check: 
$$\begin{aligned} G(\mathbb{C}) &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1 = 0\} \\ &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1\} \\ &= \left\{(x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \;\middle|\; \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array}\right\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; x,y \in K,\ x^2+y^2 = 1\right\}$$

E.g. $n=2$, $F=\mathbb{R}$

$$H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}_{m,\bar F}^2)) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\begin{array}{cccc} \Downarrow & \Downarrow & \Downarrow & \Downarrow \\ \mathbb{G}_m^2 & \mathbb{G}_m \times SO_{2,\mathbb{R}} & (SO_{2,\mathbb{R}})^2 & G = ? \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \end{array}$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; (\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$(\varphi(\sigma)\circ\sigma)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \varphi(\sigma) \begin{pmatrix} \bar z_1 \\ \bar z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar z_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\begin{array}{ccc} \mathbb{G}_m, & SO_{2,\mathbb{R}}, & \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \\ \updownarrow & \updownarrow & \updownarrow \\ 1 & -1 & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \updownarrow & \updownarrow & \updownarrow \end{array}$$

Fact$^{dual}$: $\mathrm{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{\mathrm{triv}}, \mathbb{Z}_{\mathrm{sign}}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

Rmk. Using the same argument, one can show that
$\{ T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}_{m,\mathbb{F}_{p^2}}^n \}$ = products of $\mathbb{G}_m$, $\begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}$, $\mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathbb{G}_m$

The torus $G$ crspd to $-1$: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$G(\mathbb{F}_p) = \{ g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k \}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid \varphi(\sigma)(a-b\xi) = a+b\xi \}$$

$$\varphi(\sigma)(a-b\xi) = (a-b\xi)^{-1}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2\varepsilon = 1 \}$$

$$\cong \left\{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p) \right\}$$

3. other cases.

By using the same methods introduced in last section, we can compute the (inner) forms of reductive gps.

| | inner forms | outer forms |
|---|---|---|
| $\mathbb{G}_m$<br>$(\mathbb{G}_m)^2$<br>$(\mathbb{G}_m)^n$ | $\emptyset$ | $SO_2$<br>$SO_2 \times \mathbb{G}_m$, $(SO_2)^2$, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$<br>product of lower rank torus |
| $GL_{2,\mathbb{R}}$<br>$SL_{2,\mathbb{R}}$<br>$PGL_{2,\mathbb{R}}$ | $\mathbb{H}^\times = GL_1(\mathbb{H} \otimes_{\mathbb{R}} -)$<br>$\mathbb{H}^{Nm=1} = SU_{2,\mathbb{C}/\mathbb{R}}$<br>? | $(U_{2,\mathbb{C}/\mathbb{R},\omega} = U(1,1)$ $U(2,0))$<br>$\phi$<br>$\phi$ |
| $GL_{n,\mathbb{R}}$<br>$SL_{n,\mathbb{R}}$<br>$PGL_{n,\mathbb{R}}$ | ?<br>$GL_{n/2}^{Nm=1}(\mathbb{H} \otimes_{\mathbb{R}} -)$ when $n$ even<br>? | $\left(U_{n,\mathbb{C}/\mathbb{R},\omega} = \begin{cases} U(\frac{n}{2},\frac{n}{2}) & n \text{ even} \\ U(\frac{n+1}{2},\frac{n-1}{2}) & n \text{ odd} \end{cases}\quad U(p,q)\right)$ ...<br>$SU(a,n-a)$ e.p. $SU(2,1)$ ← need clarification<br>? |
| $(SL_2)^2/\mathbb{R}$<br>$(SL_2)^3/\mathbb{R}$ | $SL_2 \times SU_2$, $(SU_2)^2$, ... $\phi$?<br>$(8-1)$ possibilities | $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$<br>? |

? : I have no time to compute / don't know any symbol to represent

: quasi-split gp

## Compute $\mathrm{Aut}(G_{\bar F})$

**Lemma.** We understand $\mathrm{Aut}(G_{\bar F})$ quite well.

| $G$ | $1 \longrightarrow \mathrm{Inn}(G_{\bar F}) = G(\bar F)/Z(G(\bar F)) = G^{ad}(\bar F)$ | $\longrightarrow \mathrm{Aut}(G_{\bar F})$ | $\longrightarrow \mathrm{Out}(G_{\bar F}) = \mathrm{Aut}(\Psi_0)$ | $\longrightarrow 1$ |
| --- | --- | --- | --- | --- |
| $T$ rk $n$ | $1$ | $GL_n(\mathbb{Z})$ | $GL_n(\mathbb{Z})$ | |
| $GL_{2,\mathbb{R}}$ | $PGL_2(\mathbb{C})$ | $PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$ | $\{\pm 1\}$ | |
| $SL_{2,\mathbb{R}}$ | $PGL_2(\mathbb{C})$ | $PGL_2(\mathbb{C})$ | $1$ | |
| $PGL_{2,\mathbb{R}}$ | $PGL_2(\mathbb{C})$ | $PGL_2(\mathbb{C})$ | $1$ | |
| $n \geq 3$ | | | | |
| $GL_{n,\mathbb{R}}$ | $PGL_n(\mathbb{C})$ | $PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$ | $\{\pm 1\}^{\oplus 2}$ | |
| $SL_{n,\mathbb{R}}$ | $PGL_n(\mathbb{C})$ | $PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$ | $\{\pm 1\}$ | |
| $PGL_{n,\mathbb{R}}$ | $PGL_n(\mathbb{C})$ | $PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$ | $\{\pm 1\}$ | |
| $(SL_2)^2/\mathbb{R}$ | $PGL_n(\mathbb{C})^2$ | $PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$ | $\{\pm 1\}$ | |
| $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$ | $PGL_n(\mathbb{C})^2$ | $PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$ | $\{\pm 1\}$ | with different $\Gamma_{\mathbb{R}}$-action |
| $(SL_2)^n/\mathbb{R}$ | $PGL_n(\mathbb{C})^n$ | $PGL_n(\mathbb{C})^n \rtimes S^n$ | $S^n$ | |

Compute $H^1(\Gamma_F, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP] **Theorem 12.7.7** *There is a functorial isomorphism* $\mathrm{H}^1(k, G) \to \pi_1(G)_{\Theta,\mathrm{tor}}$.

[DR09] COROLLARY 2.4.3. *The composition*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

*is a bijection.*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field:

Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
https://arxiv.org/abs/1401.5913

**3.1. Theorem.** *Let $G$, $T_0$, $T$, and $W_0$ be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

*induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.*

global field:

Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
https://arxiv.org/pdf/2303.04120.pdf

E.g. $G = SL_{2,\mathbb{R}}$ , $F = \mathbb{R}$

$$\begin{array}{cccccccccc}
G & & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\
SL_{2,\mathbb{R}} & & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) & & 1 &
\end{array}$$

$$\begin{aligned}
H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(SL_{2,\mathbb{C}})) &= H^1(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C})) \\
&= \{ 1, \quad \omega(\ )\omega^{-1} \}
\end{aligned}
\qquad \omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\begin{array}{cc}
\Downarrow & \Downarrow \\
SL_{2,\mathbb{C}} & G = ?
\end{array}
\qquad \mathbb{H}^{Nm=1}$$

$$\begin{aligned}
G(\mathbb{R}) &= \{ g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}} \} \\
&= \{ g \in SL_2(\mathbb{C}) \mid \omega \bar{g} \omega^{-1} = g \} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \;\middle|\; \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in SL_2(\mathbb{C}) \right\} \\
&= \mathbb{H}^{Nm=1}
\end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$ , $F = \mathbb{R}$

$$\begin{array}{cccccccccc}
G & & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\
GL_{2,\mathbb{R}} & & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) \rtimes \{\pm 1\} & & \{\pm 1\} &
\end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(GL_{2,\mathbb{C}})) = \{ 1, \quad \omega(\ )\omega^{-1}, \quad (\ )^{H,-1}, \quad \omega(\ )^{H,-1}\omega^{-1} \} \qquad \omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$$

$$\begin{array}{cccc}
\Downarrow & \Downarrow & \Downarrow & \Downarrow \\
(GL_{2,\mathbb{C}} & \mathbb{H}^\times) & (\mathcal{U}(2,0) & U_{2,\mathbb{C}/\mathbb{R},\omega})
\end{array}$$

## 4. conclusions on various forms

### $H^1(\Gamma_F, -)$ as parameter space

$$
\begin{tikzcd}[column sep=small]
 & & & 1 \arrow[dr] & & 1 & & \\
1 \arrow[r] & Z(G(\bar F)) \arrow[r] & G(\bar F) \arrow[rr] & & \mathrm{Inn}(G_{\bar F}) \arrow[ur] \arrow[dr] & & & \\
 & & & & & \mathrm{Aut}(G_{\bar F}) \arrow[r] & \mathrm{Out}(G_{\bar F}) \arrow[r] & 1
\end{tikzcd}
$$

$$
\rightsquigarrow H^1(\Gamma_F, Z(G(\bar F))) \to \underbrace{H^1(\Gamma_F, G(\bar F))}_{\text{pure inner twist}} \to \underbrace{H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))}_{\text{inner twist}} \to \underbrace{H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))}_{\text{form}} \to H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))
$$

$$\{F\text{-forms of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$$

$$\{F\text{- inner forms of } G\}/\sim \;\longleftrightarrow\; \mathrm{Im}\left(H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))\right)$$

$$\{F\text{- inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$$

$$\{F\text{-pure inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, G(\bar F))$$

$G$ split:
$$\left\{\begin{matrix}F\text{- forms of } G \\ \text{which are quasi-split}\end{matrix}\right\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) \cong H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))$$
$$H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) = \Gamma_F\text{-actions on } (X^*, \Delta, X_*, \Delta^\vee)$$

**Q:** Do we have

$$
\begin{tikzcd}
 & H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \arrow[r, hook] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[r] & H^1(\Gamma_F, \mathrm{Out}(G_{\bar F})) \\
1 \arrow[r] & \mathrm{Inn}(G_{\bar F})^{\Gamma_F} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Aut}(G_{\bar F})^{\Gamma_F} \arrow[r] \arrow[d, phantom, "\shortparallel"] & \mathrm{Out}(G_{\bar F})^{\Gamma_F} \arrow[d, phantom, "\shortparallel"] \arrow[ull, bend right] \\
 & \mathrm{Inn}(G_F) & \mathrm{Aut}(G_F) & \mathrm{Out}(G_F)
\end{tikzcd}
$$

Give one example s.t. $H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$ is not inj ?

# Categorification of $H^1(\Gamma_F, -)$

These categories are all groupoids.
These $H^1(\Gamma_F, -)$ are all achieved as isomorphism classes.

<table>
<tr><th></th><th>Obj</th><th>Mor $((G_2,\alpha),(G_2',\alpha'))$</th></tr>
<tr>
<td>form<br><br>$H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha"'] \arrow[urr, "\sigma(\alpha)"] & G_{\bar F} & \end{tikzcd}$$

commutes $\forall \sigma \in \Gamma_F$

</td>
<td>

$\beta: G_2 \longrightarrow G_2'$ iso

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes

</td>
</tr>
<tr>
<td>inner form<br><br>$\mathrm{Im}\begin{pmatrix} H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \\ \downarrow \\ H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))\end{pmatrix}$<br><br>full subcategory of "form"</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$

$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.

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<td>

$\beta: G_2 \longrightarrow G_2'$ iso

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes

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<tr>
<td>inner twist<br><br>$H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$<br><br>less isomorphisms compared with inner form</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$

$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.

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$\beta: G_2 \longrightarrow G_2'$ iso

$$\text{s.t } \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

$\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}$ is inner auto.

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<td>pure inner twist<br><br>$H^1(\Gamma_F, G(\bar F))$</td>
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$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F},\ \phi)$
$\phi \in Z^1(\Gamma_F, G(\bar F))$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\phi(\sigma)\text{-conj}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$

commutes

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$(\beta, \delta)$
$\beta: G_2 \longrightarrow G_2'$ iso $\quad \delta \in G(\bar F)$

$$\text{s.t } \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes, and
$\phi_1(\sigma) = \delta^{-1}\phi_2(\sigma)\,\sigma(\delta)$

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