Eine Woche, ein Beispiel

Ref: from [23.11.19]

4.28 naive ⊗ - Hom adjunction

Notation. - A: associate ring allowed to be non-commutative, contains 1
- There are two systems to write category of A-modules.

$$\begin{aligned} & \mathrm{Mod}_A && = A\text{-}\mathrm{Mod} && \ni {}_AM \\ (\mathrm{Mod}_A)^{op} \neq \; & \mathrm{Mod}_{A^{op}} && = \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} && \ni M_A \\ & \mathrm{Mod}_{A\otimes B^{op}} && = A\text{-}\mathrm{Mod}\text{-}B && \ni {}_AM_B \end{aligned}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \ni {}_BM_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $Ob(\mathrm{Ring}) \cong Ob(\mathrm{Ring}^{op})$,
$A^{op} \notin Ob(\mathrm{Ring}^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset Sh(\mathrm{Spec}\,A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for ⊗ & Hom
2. adjunction
3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$
4. definition recall for ⊗ & Hom , derived version
5. adjunction , derived version
6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-Mod-}B \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-Mod-}B)^{op} \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$

$$\begin{array}{cccc} \mathrm{Hom}_B^A(-,-) : & (A\text{-Mod-}B)^{op} \times & B\text{-Mod-}A \longrightarrow & \mathbb{Z}\text{-Mod} \\ \| & \| & \| & \| \\ \mathrm{Hom}_{B\otimes_{\mathbb{Z}} A^{op}}(-,-) & (\mathbb{Z}\text{-Mod-}B\otimes_{\mathbb{Z}} A^{op})^{op} \times & (B\otimes_{\mathbb{Z}} A^{op}\text{-Mod-}\mathbb{Z})^{op} \longrightarrow & \mathbb{Z}\text{-Mod-}\mathbb{Z} \end{array}$$

$${}_A X_B, \quad {}_B Y_C, \quad {}_C Z_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{op}} X$ in $A\text{-Mod-}C = C^{op}\text{-Mod-}A^{op}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\mathrm{Hom}_A(A, X) \cong X$

2. adjunction ${}_B X_A$, ${}_C Y_B$, ${}_C Z_D$. we get

$$\mathrm{Hom}_C(Y \otimes_B X,\ Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y,Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$\{ f: Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c \}$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y, -)"] \end{tikzcd}$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv \mathrm{Hom}$:

$$
\begin{tikzcd}
(A\text{-Mod-}B)^{op} \times (B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "(\mathrm{Id},\, \mathrm{Hom}_C)"] \arrow[d, equal] & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}D \arrow[dd, "\mathrm{Hom}_B"] \\
(A\text{-Mod-}B \times B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[d, "(\otimes_B,\, \mathrm{Id})"'] & \\
(A\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "\mathrm{Hom}_C"] & A\text{-Mod-}D
\end{tikzcd}
$$

$f^* \dashv f_*$:

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$
\begin{tikzcd}
\mathcal{G} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
Y \arrow[r, "f"] & X
\end{tikzcd}
$$

$$
\begin{tikzcd}
Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \arrow[r, "(\mathrm{Id},\, \text{pushforward})"] \arrow[d, "(\text{pullback},\, \mathrm{Id})"'] & Sh(X)^{op} \times Sh(X) \arrow[d, "\mathrm{Hom}_{Sh(X)}(-,-)"] \\
Sh(Y)^{op} \times Sh(Y) \arrow[r, "\mathrm{Hom}_{Sh(Y)}(-,-)"] & \mathrm{Abel}
\end{tikzcd}
$$

$$
\begin{tikzcd}
(\mathcal{F},\ f,\ \mathcal{G}) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{F},\ f_*\mathcal{G}) \arrow[d, mapsto] \\
(f^*\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] & \mathrm{Hom}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})
\end{tikzcd}
$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

Forgetful fctor

Prop. For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, $\exists$ "forgetful fctor"

$$u : S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_RS_S \otimes_S M = \mathrm{Hom}_S({}_SS_R, M)$$

one has adjunction fctors

$$\begin{tikzcd}[column sep=huge] S\text{-Mod} \arrow[r, "\mathrm{Hom}_S({}_SS_R,-)", "u = {}_RS_S \otimes_S -"'] & R\text{-Mod} \arrow[l, bend right=40, "{}_SS_R \otimes_R -"'] \arrow[l, bend left=40, "\mathrm{Hom}_R({}_RS_S,-)"] \end{tikzcd} \qquad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{tikzcd}[column sep=huge] \mathcal{D}(X) \arrow[r, "j^!", "j^*"'] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right=40, "j_!"'] \arrow[l, bend left=40, "Rj_*"] \end{tikzcd}$$

Vague slogan: $u \approx$ "forget the information of Z".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & X.

E.g. For finite gps $H \le G$, one has Res - Ind adjunction:

$$\mathrm{Res}^G_H \dashv \mathrm{Ind}^G_H$$
$$c\text{-}\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \le G \text{ open} \end{cases}$

If one only has $H \le G$ closed, then it's possible that $j^! \ne j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $\quad H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{tikzcd}
\mathrm{Rep}_G \arrow[rr, "\mathrm{Res}^G_H"'] & & \mathrm{Rep}_H \arrow[ll, bend right=40, "c\text{-}\mathrm{Ind}^G_H"'] \arrow[ll, bend left=40, "\mathrm{Ind}^G_H"]
\end{tikzcd}$$

$$\perp \qquad \perp$$

Ex. Compare it with the recollement diagram & (3.1).

$$\begin{tikzcd}
\mathcal{U} \arrow[d, "j"] & {[*/H]} \arrow[d, "\text{“cover with fiber } G/H\text{”}"] \\
X & {[*/G]}
\end{tikzcd}$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$

   $c\text{-}\mathrm{Ind}^G_H \longrightarrow \mathrm{Ind}^G_H$ $\qquad\qquad c\text{-}\mathrm{Ind}^G_H = \mathrm{Ind}^G_H$

2. Even though

$$Sh_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \mathrm{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$

   the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\mathrm{Res}^G_{\{*\}} \mathbb{Q} = \mathbb{Q}, \qquad \mathrm{Res}^G_{\{*\}} \mathbb{Q}[G] = \mathbb{Q}[G] \ne \mathbb{Q}$$

⚠ In this example, $j^* R j_* \ne \mathrm{Id}$, $j^! j_! \ne \mathrm{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Compare with i

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{tikzcd} S \\ R \arrow[u, "\tilde{f}"] \end{tikzcd} \qquad M \qquad\qquad \begin{tikzcd} \operatorname{Spec} S \arrow[d, "f"] \\ \operatorname{Spec} R \end{tikzcd} \qquad M$$

$\exists$ "pullback fctor"

$$f^* : R\text{-Mod} \longrightarrow S\text{-Mod} \qquad f^*M = {}_S S_R \otimes_R M$$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{tikzcd} S\text{-Mod} \arrow[rr, "u"] & & R\text{-Mod} \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "\operatorname{Hom}_R({}_R S_S, -)"] \end{tikzcd}$$

compare it with another part of the recollement diagram:

$$\begin{tikzcd} \mathcal{D}(X) \arrow[rr, "i_*"', "i_!"] & & \mathcal{D}(U) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \end{tikzcd}$$

Rmk. u is usually not f. faithful, unless $S = R/I$.

(In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.)

which crspds to closed embedding.

In that case,

$$i^* i_* = \mathrm{Id} : {}_S S_R \otimes_R ({}_R S_S \otimes_S M) \cong M$$

$$i^! i_* = \mathrm{Id} : \operatorname{Hom}_R({}_R S_S, \operatorname{Hom}({}_S S_R, M)) \cong M$$

Slogan: in the comm alg. $\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ is closed embedding.
In general, if
S is an R-idempotent algebra: $S \cong S \otimes_R S$
then $i: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ can be viewed as "closed subset".

E.g. $R_{\mathfrak{p}}$, R/I are idempotent R-algs.
$\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.

Usually R/I is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q: What is the crspd "open subset"?
A: (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. ${}^{L}\otimes \dashv RHom$

	F	RF or LF	R^iF or L^iF	exact fctor
	f^*	f^*	$-$	
	f_*	Rf_*	R^if_*	f_*-acyclic
	$\pi_{X,*}\mathcal{F}$	$\Gamma(X;\mathcal{F})$	$H^i(X;\mathcal{F})$	Γ-acyclic
	$f_!$	$Rf_!$	$R^if_!$	$f_!$-acyclic
	$\pi_{X,!}\mathcal{F}$	$\Gamma_c(X;\mathcal{F})$	$H^i_c(X;\mathcal{F})$	Γ_c-acyclic
	$-$	$f^!$	$\mathcal{H}^i(f^!-)$	
	$-\otimes_R-$	$-{}^{L}\otimes_R-$	$Tor^i_R(-,-)$	flat
	$Hom_R(-,-)$	$RHom_R(-,-)$	$Ext^i_R(-,-)$	injective/projective
e.g. group coh	M_G	$\mathbb{Z}\,{}^{L}\otimes_{\mathbb{Z}[G]} M$	$H_i(G;M)$	
	M^G	$RHom_{\mathbb{Z}[G]}(\mathbb{Z},M)$	$H^i(G;M)$	
e.g. Lie alg coh	$M_{\mathfrak{g}}$	$k\,{}^{L}\otimes_{U\mathfrak{g}} M$	$H_i(\mathfrak{g};M)$	
$\mathfrak{g}/k$: Lie alg	$M^{\mathfrak{g}}$	$RHom_{U\mathfrak{g}}(k,M)$	$H^i(\mathfrak{g};M)$	
e.g. Hochschild coh	$M/[AM]$	$A\,{}^{L}\otimes_{A^e} M$	$HH_i(A,M)$	
	M^A	$RHom_{A^e}(A,M)$	$HH^i(A,M)$	
	$A/[AA]$	$A\,{}^{L}\otimes_{A^e} A$	$HH_i(A)$	
	$Z(A)$	$RHom_{A^e}(A,A)$	$HH^i(A)$	

For calculations, see:
[23.04.09]: gp coh
[wiki]: Lie algebra coh
[21.05.21]: Hochschild coh
[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.
For Hom(-,A), see https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m.
Chenji Fu claimed that Hom(-,A) always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.
Related:
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod

https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution

4. definition recall for $\otimes$ & Hom , derived version

To define $^L\otimes$ & RHom, one needs to extend fctors

$$\begin{array}{llll} \otimes_A : & A\text{-Mod} \times A\text{-Mod} & \longrightarrow & A\text{-Mod} \\ \mathrm{Hom}_A(-,-): & (A\text{-Mod})^{op} \times A\text{-Mod} & \longrightarrow & A\text{-Mod} \end{array}$$

to fctors on double cplxes:

$\mathcal{C}(A) :=$ complex of A-modules, temporate notation

$$\begin{array}{llll} \otimes_{\mathcal{C}(A)} : & \mathcal{C}(A) \times \mathcal{C}(A) & \longrightarrow & \mathcal{C}(A) \\ \mathrm{Hom}_{\mathcal{C}(A)}(-,-): & (\mathcal{C}(A))^{op} \times \mathcal{C}(A) & \longrightarrow & \mathcal{C}(A) \end{array}$$

But how?

Wishes:

$$(M^\cdot[i]) \otimes_{\mathcal{C}(A)} (N^\cdot[j]) = (M^\cdot \otimes N^\cdot)[i+j]$$

$$\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot[-i], N^\cdot[j]) = \mathrm{Hom}(M^\cdot, N^\cdot)[i+j]$$

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
{} \arrow[r] & M^{-1}\otimes N^{1} \arrow[r] \arrow[u] & M^{0}\otimes N^{1} \arrow[r] \arrow[u] & M^{1}\otimes N^{1} \arrow[r] \arrow[u] & M^{2}\otimes N^{1} \arrow[r] \arrow[u] & \cdots \\
{} \arrow[r] & M^{-1}\otimes N^{0} \arrow[r] \arrow[u] & M^{0}\otimes N^{0} \arrow[r] \arrow[u] & M^{1}\otimes N^{0} \arrow[r] \arrow[u] & M^{2}\otimes N^{0} \arrow[r] \arrow[u] & \cdots \\
{} \arrow[r] & M^{-1}\otimes N^{-1} \arrow[r] \arrow[u] & M^{0}\otimes N^{-1} \arrow[r] \arrow[u] & M^{1}\otimes N^{-1} \arrow[r] \arrow[u] & M^{2}\otimes N^{-1} \arrow[r] \arrow[u] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

Tot ($M^{\cdot}\otimes N^{\cdot}$), the double complex of $M^{\cdot}\otimes N^{\cdot}$.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
{} \arrow[r] & \mathrm{Hom}(M^{1},N^{1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{0},N^{1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-1},N^{1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-2},N^{1}) \arrow[r] \arrow[u] & \cdots \\
{} \arrow[r] & \mathrm{Hom}(M^{1},N^{0}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{0},N^{0}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-1},N^{0}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-2},N^{0}) \arrow[r] \arrow[u] & \cdots \\
{} \arrow[r] & \mathrm{Hom}(M^{1},N^{-1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{0},N^{-1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-1},N^{-1}) \arrow[r] \arrow[u] & \mathrm{Hom}(M^{-2},N^{-1}) \arrow[r] \arrow[u] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

Tot (Hom($M^{\cdot}, N^{\cdot}$)), the double complex of Hom($M^{\cdot}, N^{\cdot}$).

Def. For $M^\cdot, N^\cdot \in \mathcal{C}(A)$, define

$$M^\cdot \otimes N^\cdot, \quad \mathrm{Hom}_A(M^\cdot, N^\cdot) \in \mathcal{C}(A)$$

by

$$(M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot)^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$\left(\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot)\right)^n = \bigoplus_{i+j=n} \mathrm{Hom}_A(M^{-i}, N^j)$$

and morphisms given by $d + (-1)^j \delta$.

Ex. Let $M^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 3} \underset{0}{\mathbb{Z}}]$, $\quad N^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{0}{\mathbb{Z}}]$

compute $M^\cdot \otimes_{\mathcal{C}(\mathbb{Z})} N^\cdot$ & $\mathrm{Hom}_{\mathcal{C}(\mathbb{Z})}(M^\cdot, N^\cdot)$, and verify that they're complexes.

A:

$$\begin{array}{ccc} 0 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \\ & {\scriptstyle \times 2}\uparrow \qquad \uparrow{\scriptstyle \times 2} \\ -1 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \\ & -1 \qquad\quad 0 \end{array} \rightsquigarrow \begin{array}{c} \left[\mathbb{Z} \xrightarrow{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z} \right] \\ -2 \qquad\quad -1 \qquad\quad 0 \end{array}$$

$\mathrm{Tot}(M^\cdot \otimes N^\cdot)$ $\qquad\qquad$ $M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot$

$$\begin{array}{ccc} 0 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \\ & {\scriptstyle \times 2}\uparrow \qquad \uparrow{\scriptstyle \times 2} \\ -1 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \\ & 0 \qquad\quad 1 \end{array} \rightsquigarrow \begin{array}{c} \left[\mathbb{Z} \xrightarrow{\begin{pmatrix} -3 \\ 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z} \right] \\ -1 \qquad\quad 0 \qquad\quad 1 \end{array}$$

$\mathrm{Tot}(\mathrm{Hom}(M^\cdot, N^\cdot))$ $\qquad\qquad$ $\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot)$

Now, we can define ${}^{L}\otimes$ & RHom:

Def. For $M, N \in A\text{-Mod}$, one can define

$$M \overset{L}{\otimes}_A N := M \otimes_{\mathcal{C}(A)} P^\cdot \quad \text{when } N \xleftarrow{\simeq} P^\cdot \quad \text{flat resolution}$$

in general, $M^\cdot, N^\cdot \in \mathcal{D}^-(A\text{-Mod})$

$$\begin{aligned} \mathrm{RHom}_A(M, N) &:= \mathrm{Hom}_{\mathcal{C}(A)}(M, I^\cdot) \quad \text{when } N \xrightarrow{\simeq} I^\cdot \quad \text{inj resolution} \\ &:= \mathrm{Hom}_{\mathcal{C}(A)}(P^\cdot, N) \quad \text{when } M \xleftarrow{\simeq} P^\cdot \quad \text{proj resolution} \end{aligned}$$

in general, $M^\cdot \in \mathcal{D}^-(A\text{-Mod})$, $N^\cdot \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat

Ex Compute $\mathbb{F}_2 \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{F}_2$ & $\mathrm{RHom}_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $\mathrm{Tor}^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$ & $\mathrm{Ext}^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(A)}(M^\cdot, N^\cdot) &= R^0\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot) \\ \mathrm{Hom}_A(M, N) &= \mathrm{Hom}_{\mathcal{D}(A)}(M^\cdot, N^\cdot) = R^0\mathrm{Hom}_{\mathcal{C}(A)}(M, N). \end{aligned}$$

A wrong proof for "flat $\longrightarrow$ proj"

"Proof": when P is flat,

$$\begin{array}{ccc} P\otimes_A - & \dashv & \mathrm{Hom}_A(P,-) \\ \| & & \\ P\overset{L}{\otimes}_A - & \dashv & R\mathrm{Hom}_A(P,-) \end{array}$$

by the uniqueness of the adjunction, $\mathrm{Hom}_A(P,-) = R\mathrm{Hom}_A(P,-)$, so P is flat.

This is wrong. $\mathbb{Q} \in \mathbb{Z}\text{-Mod}$ is flat but not proj.
In the proof, we only have

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "P\otimes_A -"] \arrow[d, "l_A"'] & A\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_A(P,-)"] \arrow[d, "l_A"] \\ \mathcal{D}(A) \arrow[r, bend left, "P\overset{L}{\otimes}_A -"] & \mathcal{D}(A) \arrow[l, bend left, "R\mathrm{Hom}_A(P,-)"] \end{tikzcd}$$

$$l_A \circ (P\otimes_A -) = (P\overset{L}{\otimes}_A -) \circ l_A.$$

Ex. Compute $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-)$, and shows that

$$l_{\mathbb{Z}} \circ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-) \neq R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-) \circ l_{\mathbb{Z}}.$$

5. adjunction , derived version