# Modular form
## 2. computations.

# 1. Prelude

**1.1.2.** (a) Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau+d|^2$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.
(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau+d)^2$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.

$$\begin{aligned}
\wp(z) &= \frac{1}{z^2} + \sum_{z_0\in\Lambda}{}' \left(\frac{1}{(z-z_0)^2} - \frac{1}{z_0^2}\right) \\
&= \frac{1}{z^2} + \sum_{z_0\in\Lambda}{}' \sum_{i=1}^{+\infty} (i+1)\frac{z^i}{z_0^{i+2}} \\
&= \frac{1}{z^2} + \sum_{i=1}^{+\infty} (i+1) G_{i+2}(\Lambda) z^i \\
&= \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + O(z^8)
\end{aligned}$$

$$\Leftarrow \left[\begin{aligned}
\frac{1}{(z-z_0)^2} &= \frac{1}{z_0^2}\left(1+\left(-\frac{z}{z_0}\right)\right)^{-2} \\
&= \frac{1}{z_0^2}\left(1+\binom{-2}{1}\left(-\frac{z}{z_0}\right)+\binom{-2}{2}\left(-\frac{z}{z_0}\right)^2+\cdots\right) \\
&= \frac{1}{z_0^2} + \sum_{i=1}^{+\infty}(i+1)\frac{z^i}{z_0^{i+2}}
\end{aligned}\right.$$

$$\begin{aligned}
\Rightarrow \wp'(z) &= -2\frac{1}{z^3} + \sum_{i=1}^{+\infty} i(i+1) G_{i+2}(\Lambda) z^{i-1} \\
&= -2\frac{1}{z^3} + \sum_{i=0}^{+\infty} (i+1)(i+2) G_{i+3}(\Lambda) z^i \\
&= -2\frac{1}{z^3} + 6G_4 z + 20 G_6 z^3 + 42 G_8 z^5 + O(z^7)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow (\wp'(z))^2 &= 4\wp(z)^3 - 60 G_4 \wp(z) - 140 G_6 \\
y^2 &= 4x^3 - 60 G_4 x - 140 G_6 \quad = 4x^3 - g_2 x - g_3
\end{aligned}$$

Intermediate computation.

$$\begin{aligned}
(\wp'(z))^2 &= 4\frac{1}{z^6} - 24 G_4 \frac{1}{z^2} - 80 G_6 + (-168 G_8 + 36 G_4^2) z^2 + O(z^4) \\
(\wp(z))^3 &= \frac{1}{z^6} + 9 G_4 \frac{1}{z^2} + 15 G_6 + (21 G_8 + 27 G_4^2) z^2 + O(z^4)
\end{aligned}$$

$$252 G_8 - 108 G_4^2 = 0 \quad \Rightarrow \quad G_8 = \frac{3}{7} G_4^2$$

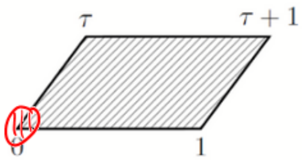
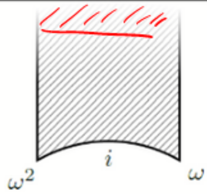
Rmk. another equation: $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ $\quad e_1 = \wp(\frac{\omega_1}{2})$ $\quad e_2 = (\frac{\omega_2}{2})$ $\quad e_3 = (\frac{\omega_1+\omega_2}{2})$


$$\Rightarrow \begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1 e_2 + e_2 e_3 + e_1 e_3 = -\frac{1}{4} g_2 = -15 G_4 \\ e_1 e_2 e_3 = \frac{1}{4} g_3 = 35 G_6 \end{cases}$$

Ex. $G_4(\rho)=0$ $\quad G_6(i)=0$

$\Rightarrow$ Weierstrass equation of $\mathbb{C}/\mathbb{Z}\oplus\rho\mathbb{Z}$, $\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}$

Conclusion

<table>
<tr><td>复环面 $\mathbb{C}/\Lambda_\tau$</td><td>模空间 $\mathcal{H}/SL_2(\mathbb{Z})$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$\mathcal{M}(\mathbb{C}/\Lambda_\tau)=\mathbb{C}(\wp,\wp')$</td><td>$\mathrm{M}_*(SL_2(\mathbb{Z}))=\mathbb{C}[E_4,E_6]$</td></tr>
<tr><td>椭圆函数</td><td>模形式</td></tr>
<tr><td>Weierstrass 函数</td><td>Eisenstein 级数</td></tr>
</table>

Goal: $M_*(SL_2(\mathbb{Z}))\cong\mathbb{C}[G_4,G_6]$ $\qquad$ Any idea? ($E_8$, zero pt of $G_4$ or $G_6$, $\cdots$)

## 2. q-expansions of $G_k$ (k even)

$q = e^{2\pi i \tau} \Rightarrow dq = 2\pi i q \, d\tau$

$dq^r = r \cdot 2\pi i q^r \, d\tau$

Q: Let $G_k(\tau) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$

Compute $a_0$.

in [Za] $G_k = \frac{1}{2} \cdots$ to delete 2 here

A. $a_0 = \lim_{\operatorname{Im}\tau \to +\infty} G_k(\tau) = \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k)$

Idea. Eisenstein fct = "2-dim Riemann zeta fct".

Luckily $\zeta(k)$ ($k > 0$, even) are understandable.

Let $B_k$, $\widetilde{B}_k$ defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \widetilde{B}_k \frac{x^{2k}}{(2k)!},$$

then

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} \widetilde{B}_k \pi^{2k} = (-1)^{2k+1} \frac{2^{2k-1}}{(2k)!} B_{2k} \pi^{2k} \qquad k \in \mathbb{Z}_{>0}$$

$$\zeta(k) = \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k}\right) \qquad k > 0 \text{ even}$$

The following numerical tables are copied from [JP, §4] and wiki:

| $B_0$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $B_6$ | $B_7$ | $B_8$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $-\frac{1}{2}$ | $\frac{1}{6}$ | $0$ | $-\frac{1}{30}$ | $0$ | $\frac{1}{42}$ | $0$ | $-\frac{1}{30}$ |
| | | $\widetilde{B}_1$ | | $\widetilde{B}_2$ | | $\widetilde{B}_3$ | | $\widetilde{B}_4$ |
| | | $\frac{1}{6}$ | | $\frac{1}{30}$ | | $\frac{1}{42}$ | | $\frac{1}{30}$ |

| Me | JP | wiki |
|---|---|---|
| $B_k$ | $b_k$ | $B_k^-$ |
| $\widetilde{B}_k$ | $B_k$ | — |

*Examples* $\zeta(2) = \dfrac{\pi^2}{2.3}$, $\zeta(4) = \dfrac{\pi^4}{2.3^2.5}$, $\zeta(6) = \dfrac{\pi^6}{3^3.5.7}$,

$\zeta(8) = \dfrac{\pi^8}{2.3^3.5^2.7}$, $\zeta(10) = \dfrac{\pi^{10}}{3^5.5.7.11}$, $\zeta(12) = \dfrac{691\pi^{12}}{3^6.5^3.7^2.11.13}$,

$\zeta(14) = \dfrac{2\pi^{14}}{3^6.5^2.7.11.13}$.

Thm. Let $k \geq 4$ even. $G_k$ has $q$-expansion. $\Rightarrow G_k$ is modular form.

Idea: Compute every horizontal line.

Lemma. For $\tau \in \mathbb{C} - \mathbb{Z}$, we have

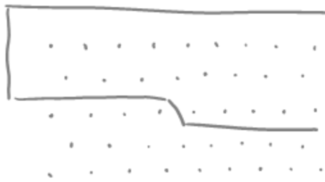
$$\sum_{n\in\mathbb{Z}} \frac{1}{\tau+n} = \frac{\pi}{\tan \pi\tau} = -\pi i \frac{1+q}{1-q} = -2\pi i\left(\frac{1}{2} + \sum_{r=1}^{+\infty} q^r\right)$$

|| seriously

$$\frac{1}{\tau} + \sum_{n=1}^{+\infty}\left(\frac{1}{\tau+n} + \frac{1}{\tau-n}\right)$$

By taking $\frac{(-1)^{k-1}}{(k-1)!}\frac{d^{k-1}}{d\tau^{k-1}}(-)$, we get Lipschitz's formula: $(k \in \mathbb{Z}_{\geq 2})$

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1} q^r$$



sum

Proof of Thm.

$$\frac{1}{2}G_k(\tau) = \sum_{n=1}^{\infty}\frac{1}{n^k} + \sum_{m=1}^{\infty}\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^k}$$

$$= \zeta(k) + \sum_{m=1}^{\infty}\frac{(-2\pi i)^k}{(k-1)!}\sum_{r=1}^{\infty} r^{k-1} q^{mr}$$

$$= \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

Def

$$E_k(\tau) = \left(-\frac{2k}{B_k}\right)\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

$$\mathcal{G}_k(\tau) = \mathbb{G}_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$

Ex. Compute $\mathbb{G}_k(z)$ and $E_k(z)$. See answer in [Za, P17] [JP, P93].

Rmk. $E_k$ can also be defined as

$$E_k(\tau) := \frac{1}{2}\sum_{\gcd(m,n)=1}\frac{1}{(m\tau+n)^k}$$

3. Degree calculation [Za Prop 2, JP Thm 3]

- def of $\mathrm{ord}_p$, $\mathrm{ord}_\infty$
- statement
- Rmk. modular form can be viewed as a section on the l.b. $\omega^{\otimes \frac{k}{2}}$ above the stack $\mathcal{H}/SL_2(\mathbb{Z})$.

  and this formula computes the degree of some "l.b." "$\omega(\infty)^{\otimes\frac{k}{2}}$" above the compactified space "$(\mathcal{H}/SL_2(\mathbb{Z}))^*$". Realize it?
- Rmk. weight $k$ gives a bound of $\dim M_k(SL_2(\mathbb{Z}))$
- proof by contour integration.

Ex. 0. Compute $\mathrm{ord}_p(E_4)$ and $\mathrm{ord}_p(E_6)$ "again".

1. Bound $M_k(SL_2(\mathbb{Z}))$ when $k$ is small ($\Rightarrow$ [Za, Cor 1])
2. Guess a basis of $M_k(SL_2(\mathbb{Z}))$ and compare the dimension.
3. Show that $E_4$ and $E_6$ are alg indep, thus $M_\cdot(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$

Hint for 3. ① Show $\dim M_{12}(SL_2(\mathbb{Z})) = 2$. || q-expansion, zero or $E_6^2 = \lambda E_4^3 \Rightarrow \frac{E_6}{E_4} \in M_2(SL_2(\mathbb{Z}))$

② Show that if $f_1, f_2 \in M_k(SL_2(\mathbb{Z}))$, $\dim\langle f_1, f_2\rangle_{\mathbb{C}} = 2$, then $f_1$ and $f_2$ are alg indep

If $P(X,Y) = \sum_d P_d(X,Y) \in \mathbb{C}[X,Y]$ st. $P(f_1, f_2) \equiv 0$

$\Rightarrow P_d(f_1, f_2) \equiv 0$

$\Rightarrow f_2^{\,d}\, p_d\left(\frac{f_1}{f_2}\right) \equiv 0$

$\Rightarrow p_d\left(\frac{f_1}{f_2}\right) \equiv 0$

$\Rightarrow \frac{f_1}{f_2} \equiv c$ or $p_d \equiv 0$

③ Show $E_4^3$ and $E_6^2$ are alg indep.

4. Application [Za P18, JS P93]

Ex. From $E_4^2 = E_8$ $E_4 E_6 = E_{10}$ get identities

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{1}{120}\left(\sigma_7(n) - \sigma_3(n)\right)$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m) = \frac{1}{5040}\left(11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n)\right)$$

Next time. begin our generalization of modular form.

$\Delta$, $\tau$, $j$, $\lambda$ $\Gamma(2)$, $E_2$, $E_2'$, $\eta$, ...

$S_k(\Gamma_1)$ ① ③ ④ ⑤ ⑥