Eine Woche, ein Beispiel

Ref: from [23.11.19]

1.28 conormal bundle

slogan: $\boxed{\begin{array}{lll} \text{SES} & \text{induces} & \text{LES,} \\ \text{filtration} & \text{induces} & \text{spectral sequence.} \end{array}}$

To expend a little bit,

$$\begin{tikzcd}
 & \text{SES} \arrow[r] & \text{LES} & \\
 & \text{filtration} \arrow[r] & \text{spectral seq} \arrow[dr] & \\
\text{resolution} \arrow[ur, dashed] \arrow[rrr] & & & \begin{array}{l}\text{comparison between} \\ \text{sheaf cohomology} \\ \text{\& other cohomology}\end{array}
\end{tikzcd}$$

Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh".

Let us leave spectral seq in other people's notes.

Methods to construct SES: $\begin{cases} \text{check by stalks} \\ \text{filtration by } \mathcal{H}^i(\mathcal{F}) \\ \text{filtration by } \mathcal{F}^i \end{cases}$

Hodge related

method	spectral seq	LES	cohomology/resolution
check by stalks	… for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech
filtration by $\mathcal{H}^i(\mathcal{F})$	Grothendieck		
	Leray-Serre	Gysin	Euler class
			Hodge-Tate
filtration by $\mathcal{F}^i$ need resolution to get "another" complex	Hodge-de Rham		de Rham, Hodge-de Rham
			Dolbeault $H^p(X,\Omega^q) = H^{p,q}(X)$
	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences which I don't know	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
	Green		for Koszul cohomology
	⋮		⋮

For more spectral sequences, see:
https://en.wikipedia.org/wiki/Spectral_sequence
https://github.com/CubicBear/SpectralSequences/blob/main/SpecralSequences.pdf

1. open-closed formalism
2. open cover
3. filtration by $\mathcal{H}^i(\mathcal{F})$
4. Hodge related filtration

1. open-closed formalism

related: comparison of $j_!$ & j_*

one-point compactification.

Observe the following pictures:

$$Z \xrightarrow{i} X \xleftarrow{j} \mathcal{U}$$

$$\begin{tikzcd} \mathcal{D}(Z) \arrow[r, "i_* = i_!"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

Black box:

0. We assume some nice conditions.

   e.g. in the category $\mathrm{Haus}^{\mathrm{loc.\,cpt}}$, and $Z \subset X$ is loc. contractable.

   Under these conditions,

1. $i_* = i_!$, $j^* = j^!$

2. $j_!, i^*, j^*, i_*$ are exact.

Ex. 1. Shows that

$$i^* i_* = i^! i_* = Id_{\mathcal{D}(Z)} \qquad j^* j_! = j^* Rj_* = Id_{\mathcal{D}(\mathcal{U})}$$

$$i^* j_! = 0, \qquad j^* i_* = 0, \qquad i^! Rj_* = 0$$

——— : base change

〰〰 : check stalkwise.

2. (for category fans)

   $i_*, j_*, j_!$ are fully faithful, and

   $i_*, i^!, j^*, Rj_*$ preserve injectives.

3. One has SES

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 spectural cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the SES

$$0 \longrightarrow j_! \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* i^* Rj_* \mathcal{F} \longrightarrow 0$$

which measures the difference between $j_! \mathcal{F}$ & $j_* \mathcal{F}$.

3. Shows that

$$H_c^{\cdot}(X) \cong H^{\cdot}(\bar{X}, \{\infty\}; \mathbb{Z})$$

for one pt compactification $\iota : X \hookrightarrow \bar{X}$.

Try to compute $H_c^{\cdot}(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X \mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{D^b(X;\mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}})$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad \mathbb{D}_X(\mathcal{F}^\cdot[n]) = (\mathbb{D}_X \mathcal{F}^\cdot)[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

$f : Y \longrightarrow X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then

$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructable setting,

$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad (2)$$

for $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".