Modular form

5. moduli interpretation

1 level structure
2. moduli interpretation of $\mathcal{H}/\Gamma$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S, T \mid S^4 = (ST)^6 = Id \rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}^-$	$\{\pm Id\}$	$\langle S, T, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S, T \mid S^2 = (ST)^3 = Id \rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}^-$	Id	$\langle S, T, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \rangle$

can't define $SL_2/\mathbb{G}_m$

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A \leq B \leq C$ gp $\quad A \triangleleft C \Rightarrow A \triangleleft B$

no other restrictions. i.e. the following cases may happen:

$A \triangleleft B \triangleleft C$	$A \triangleleft B \leq C$	$A \triangleleft B \triangleleft C$	$A \triangleleft B \leq C$	$A \leq B \triangleleft C$	$A \leq B \leq C$
$\vdash \triangleleft \dashv$	$\vdash \triangleleft \dashv$				
✓	✓	$C_2 \triangleleft A_4 \triangleleft S_4$	✓	✓	$S_2 \leq S_3 \leq S_4$

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$
\begin{array}{ccc}
\Gamma(N) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\Gamma_1(N) & \twoheadrightarrow & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(N) & \twoheadrightarrow & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \cap \\
\Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{[WWL,\ Prop\ 1.4.4]} & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\cup & & \cup \\
\Gamma^0(N) & \twoheadrightarrow & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \\
\cup & & \cup \\
\Gamma^1(N) & \twoheadrightarrow & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}
\end{array}
$$

⚠ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$, but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

Ex. Verify the following tables (left comes from right)

$A \triangleleft B$ (A \ B)	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	–	✓	✓	✓
$\Gamma_1(N)$	–	–	✓	×
$\Gamma_0(N)$	–	–	–	×
$\Gamma(1)$	–	–	–	–

$A \triangleleft B$ (A \ B)	N	B	G
N	–	✓	×
B	–	–	×
G	–	–	–

Ex. show [WWL, 练习1.4.14]

练习 1.4.14 对所有正整数 N, 证明

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma(N)) = N^3 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N^2 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(N)) = \left|(\mathbb{Z}/N\mathbb{Z})^\times\right|^{-1} \cdot (\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N \prod_{p\mid N}\left(1+\frac{1}{p}\right).$$

A. Reduced to computation of $|SL_2(\mathbb{Z}/N\mathbb{Z})|$, $|B(\mathbb{Z}/N\mathbb{Z})|$, $|N(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5,4,6$ if you don't understand the process.

$$\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) := \left((\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\right)_{\text{prim}} / (\mathbb{Z}/N\mathbb{Z})^* \overset{[6.3\,M]}{=\!=\!=} \mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$$

$\because$ $\mathbb{P}^1_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] \in \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathbb{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|SL_2(\mathbb{F}_p)| = p^3 - p$

$|B(\mathbb{F}_p)| = p^2 - p$

$|N(\mathbb{F}_p)| = p$