

Eine Woche, ein Beispiel

3.16 Schubert calculus: subvariety with vb

This is a follow up of [2025.02.23].

Goal: relate subvarieties to some vector bundles, so that we can compute their homology class in terms of Chern class (when the dimension is correct).

The Chern class will be dealt with in the next document.

Concretely, we will write subvarieties as:

- the zero set of a section in a v.b.
- the degeneracy loci of a morphism $E \rightarrow F$ among v.b.s
- the preimage of known cycles in Grassmannian
- the subvariety of $\text{Gr}(r, n)$ induced by a very ample bundle (very ample)

1. Known subvarieties and known vector bundles
2. Subvariety as section
3. Subvariety as degeneracy loci
4. Subvariety given by very ample bundle

1. Known subvarieties and known vector bundles

Schubert variety

Recall that the Schubert variety has the expression $w \leftrightarrow (\lambda_1, \dots, \lambda_r)$

$$\begin{aligned}\sum_{\lambda_1, \dots, \lambda_r}(\mathcal{V}) &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+i-\lambda_i} \geq i \quad \forall i\} \\ &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{w_i} \geq i \quad \forall i\} \\ &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{w_i} \leq n - \lambda_i \quad \forall i\}\end{aligned}$$

Especially,

$$\begin{aligned}\sum_{k,s}(\mathcal{V}) &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{n-r+i-k} \leq n - k \quad \forall i \leq s\} \\ &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n - k\} \\ &= \{\Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s\}\end{aligned}$$

For special k, s , one can further simplify the formulas:

	k	1	k	$n-r$
s	$\text{Gr}(r, n)$	$\Lambda + \mathcal{V}_{n-r} \subseteq H \text{ or } \Lambda \cap \mathcal{V}_{n-r} \neq \emptyset$	$\Lambda \cap \mathcal{V}_{n-r+1-k} \neq \emptyset$	$\mathcal{V}_1 \subset \Lambda$
s		$\Lambda + \mathcal{V}_{n-r+s-1} \subseteq H$	$\dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n - k \text{ or } \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s$	$\mathcal{V}_s \subset \Lambda$
r		$\Lambda \subset \mathcal{V}_{n-1}$	$\Lambda \subset \mathcal{V}_{n-k}$	$\{\mathcal{V}_r\}$

Vector bundles on Grassmannian

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q} \rightarrow 0$$

$$0 \rightarrow \mathcal{Q}^\vee \rightarrow \mathcal{O}^{\oplus n} \xrightarrow{\pi_{\mathcal{S}^\vee}} \mathcal{S}^\vee \rightarrow 0$$

\mathcal{S} : Subspace = tautological bundle
 \mathcal{Q} : Quotient = quotient bundle

When $r=1$, $Gr(r, n) = \mathbb{P}^{n-1}$.

$$0 \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{Q} \rightarrow 0$$

$$0 \rightarrow \mathcal{Q}^\vee \rightarrow \mathcal{O}^{\oplus n} \rightarrow \mathcal{O}(1) \rightarrow 0$$

With these basic v.b.s, we can construct more bundles on $Gr(r, n)$.

$$\mathcal{T}_{Gr} = \text{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^\vee \otimes \mathcal{Q} \quad \omega_{Gr}^\vee = \det \mathcal{S}^\vee \otimes \mathcal{Q}$$

$$\Omega_{Gr} = \mathcal{T}_{Gr}^\vee = \text{Hom}(\mathcal{Q}, \mathcal{S}) = \mathcal{Q}^\vee \otimes \mathcal{S} \quad \omega_{Gr} = \det \mathcal{Q}^\vee \otimes \mathcal{S}$$

2. Subvariety as section

Hypersurface and its Fano variety of $(r-1)$ -planes

Let $F \in K[z_1, \dots, z_n]$ be a homo poly of deg d . The hypersurface

$$Y_d := \{F = 0\} \subseteq \mathbb{P}^{n-1}$$

is given as a section of

$$\mathcal{O}(d) = \text{Sym}^d \mathcal{O}(1)$$

In general, the Fano variety of $(r-1)$ -planes ($\cong \mathbb{P}^{r-1}$)

$$F_{r-1}(Y_d) := \{W \in \text{Gr}(r, n) \mid F|_W = 0\} \subseteq \text{Gr}(r, n)$$

is given as a section of $\text{Sym}^d \mathcal{F}^\vee$, through the map

$$\begin{aligned} \text{Sym}^d \pi_{\mathcal{F}^\vee}: \text{Sym}^d (\mathcal{O}^{\oplus n}) &\longrightarrow \text{Sym}^d (\mathcal{F}^\vee) \\ (\text{Sym}^d V^*) \otimes \mathcal{O} & \end{aligned}$$

$$\text{Map of section: } F \otimes 1 \longmapsto s_F = \text{Sym}^d \pi_{\mathcal{F}^\vee}(F \otimes 1)$$

$$\text{Fiberwise, } \text{Sym}^d \pi_{\mathcal{F}^\vee}|_{[W]}: \text{Sym}^d V^* \longrightarrow \text{Sym}^d W^*$$

We know that

$$\begin{aligned} & F|_W = 0 \\ \Leftrightarrow & \text{Sym}^d \pi_{\mathcal{F}^\vee}|_{[W]}(F) = 0 \\ \Leftrightarrow & s_F = 0, \text{ i.e., } [W] \text{ lies in the zero set of } s_F. \end{aligned}$$

$$\text{E.g. } F_0(Y_d) = Y_d$$

$$F_1(Y_d) \subseteq \text{Gr}(2, n)$$

$$F_m(Y_d) \subseteq \text{Gr}(m+1, 2m+2)$$

or

$$\text{Gr}(m+1, 2m+3)$$

Fano variety of lines

Last } Grassmannian

orthogonal

...

$$\text{Cor. } F_{r-1}(Y_d) \text{ has codimension } \leq \binom{d+r-1}{d} \quad (\text{when non-empty})$$

Tangent line of hypersurfaces [3264, Chap 11]

Let $F = \sum a_I z^I \in K[z_1, \dots, z_n]$ be a homo poly of deg d , which is a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$, and assume that $Y_d = \{F=0\}$ is smooth. We want to describe the locus

$$\begin{aligned}\Gamma' &= \{(p, l) \in \mathbb{P}^{n-1} \times \text{Gr}(2, n) \mid l \text{ is tangent to } Y_d \text{ at } p\} \\ &= \{(p, l) \in \mathbb{P}^{n-1} \times \text{Gr}(2, n) \mid F|_l \text{ has multiplicity } \geq 2 \text{ at } p\}\end{aligned}$$

as a section in the v.b. E over Φ , where

$$\begin{array}{ccc}\Phi &:=& \{(p, l) \in \mathbb{P}^{n-1} \times \text{Gr}(2, n) \mid p \in l\} \\ \beta \curvearrowright && \downarrow \alpha \\ \mathbb{P}^{n-1} & & \text{Gr}(2, n)\end{array}$$

After that, one can describe the locus of tangent lines.

$$\mathcal{L}_*[\Gamma] = \{l \in \text{Gr}(2, n) \mid l \text{ is tangent to } Y_d\}$$

Idea: Fix $p = [1:0:0:\dots:0]$, $l = [*:*:0:\dots:0]$ in Φ . Consider the map

$$\begin{aligned}\varphi_{(p,l)}: H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) &\longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow E|_{(p,l)} = H^0\left(\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{O}_{\mathbb{P}^1}/I_p^2\right) \\ \sum a_I z^I &\longmapsto \sum_k a_k z_0^k z_1^{d-k} \longmapsto \sum_{k \leq 1} a_k z_0^k z_1^{d-k} \\ a_k &= a_{(k, d-k, 0, \dots, 0)}\end{aligned}$$

Then

$$\begin{aligned}\varphi_{(p,l)}(\sum a_I z^I) = 0 &\iff \sum_{k \leq 1} a_k z_0^k z_1^{d-k} = 0 \\ &\iff F|_l \text{ has multiplicity } \geq 2 \text{ at } p.\end{aligned}$$

We want to globalize $\varphi_{(p,l)}$ to maps between v.b.s.

Construction: Define

$$\begin{aligned}\widetilde{\Phi} &:= \Phi \times_{C_{r(2,n)}} \Phi \\ &= \{(p_1, p_2, l) \mid p_1, p_2 \in l\}\end{aligned}$$

$$\begin{array}{ccccc} \Phi & \xrightarrow{\Delta} & \widetilde{\Phi} & \xrightarrow{\pi_1} & \Phi \\ & & \downarrow \pi_2 & & \downarrow \alpha \\ & & \Phi & \xrightarrow{\beta} & \mathbb{P}^{n-1} \\ & & & \downarrow \pi_{\widetilde{\Phi}} & \downarrow \pi_{\mathbb{P}^{n-1}} \\ & & & \Phi & \xrightarrow{\alpha} C_{r(2,n)} \xrightarrow{\pi_{C_r}} \mathcal{F}_*\end{array}$$

Then we get

$$\begin{aligned}\gamma: \pi_{\Phi}^* \pi_{\mathbb{P}^{n-1}, *} \mathcal{O}_{\mathbb{P}^{n-1}}(d) &\longrightarrow \pi_{*, *} \pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow \pi_{*, *} (\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \mathcal{O}_{\widetilde{\Phi}/I_{\Delta}^2}) \\ &\quad \parallel \qquad \qquad \qquad \parallel \text{def} \\ \mathcal{O}_{\Phi} \otimes_k H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) &\longrightarrow \mathcal{E}\end{aligned}$$

Reality check:

- ① $(\pi_{\Phi}^* \pi_{\mathbb{P}^{n-1}, *} \mathcal{O}_{\mathbb{P}^{n-1}}(d))|_{(p_0, l_0)} = H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$
- ② $(\pi_{*, *} \pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d))|_{(p_0, l_0)} = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$
- ③ $(\pi_{*, *} (\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \mathcal{O}_{\widetilde{\Phi}/I_{\Delta}^2}))|_{(p_0, l_0)} = E|_{(p_0, l_0)}$

④ Construct the map $\pi_{\Phi}^* \pi_{\mathbb{P}^{n-1}, *} \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow \pi_{*, *} \pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)$

$$\begin{array}{ccccccc} & & \mathbb{P}^{n-1} & & & & \\ & & \searrow & \swarrow & & & \\ & & \mathbb{P}^{n-1} \times \Phi & & \{l_0\} & & \\ & & \searrow & \swarrow & & & \\ \mathcal{O}_{\mathbb{P}^{n-1}}(d) & \parallel & \mathbb{P}^{n-1} & & \{l_0\} & & \\ & & \searrow & \swarrow & & & \\ & & \{l_0\} & & \mathbb{P}^{n-1} & & \\ & & \parallel & & \parallel & & \\ & & \{l_0\} & & \{l_0\} & & \\ & & \parallel & & \parallel & & \\ & & \{l_0\} & & \{l_0\} & & \end{array}$$

①

$$\begin{array}{ccc}
\mathcal{O}_{\mathbb{P}^1}/I_{\Delta}^2 & \xrightarrow{\quad} & \mathcal{O}_{\mathbb{P}^1}(d) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\tilde{\Phi}}/I_{\Delta}^2 & \xrightarrow{\quad} & \{(p, p_0, l_0) \mid p \in l_0\} \cong \mathbb{P}^1 \\
& \searrow & \downarrow \tilde{\Phi} \\
& & \pi_1 \\
\mathcal{O}_{\mathbb{P}^n}(d) & \xrightarrow{\quad} & \{(p_0, l_0)\} \\
\downarrow \beta & \xrightarrow{\quad} & \downarrow \pi_2 \\
\mathbb{P}^{n-1} & \xrightarrow{\quad} & \tilde{\Phi} \\
& \searrow & \downarrow \pi_2 \\
& & \{(p_0, l_0)\}
\end{array}$$

② & ③

$$\begin{array}{ccccc}
\Phi & \xrightarrow{\quad \beta \quad} & & & \\
l_{\Phi} \swarrow & & & & \searrow \tilde{\beta} \\
& \mathbb{P}^{n-1} \times \text{Gr}(2, n) & \xrightarrow{\quad \tilde{\beta} \quad} & \mathbb{P}^{n-1} & \\
\alpha \searrow & \downarrow \tilde{\alpha} & & \downarrow \pi_{\mathbb{P}^{n-1}} & \\
& \text{Gr}(2, n) & \xrightarrow{\quad \pi_{\text{Gr}} \quad} & \{\star\} &
\end{array}$$

$$\pi_{\tilde{\Phi}}^* \pi_{\mathbb{P}^{n-1}, *} = \alpha^* \pi_{\text{Gr}}^* \pi_{\mathbb{P}^{n-1}, *} \longrightarrow \alpha^* \alpha_* \beta^* = \pi_{2,*} \pi_1^* \beta^*$$

where the arrow comes from

$$\pi_{\text{Gr}}^* \pi_{\mathbb{P}^{n-1}, *} = \tilde{\alpha}_* \tilde{\beta}^* \longrightarrow \tilde{\alpha}_* l_{\Phi, *} l_{\Phi}^* \tilde{\beta}^* = \alpha_* \beta^*$$

Rmk. The SES

$$0 \longrightarrow I_{\Delta}/I_{\Delta}^2 \longrightarrow \mathcal{O}_{\tilde{\Phi}}/I_{\Delta}^2 \longrightarrow \mathcal{O}_{\tilde{\Phi}}/I_{\Delta} \longrightarrow 0$$

$$l_{\Delta, *} \mathcal{O}_{\tilde{\Phi}}$$

induces the SES through $\pi_{2,*}(\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes -)$:

$$0 \longrightarrow \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \Omega_{\tilde{\Phi}/\text{Gr}} \longrightarrow \mathcal{E} \longrightarrow \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0$$

Here,

$$\begin{aligned}
& \pi_{2,*}(\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes I_{\Delta}/I_{\Delta}^2) \\
&= \pi_{2,*}(\pi_2^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes I_{\Delta}/I_{\Delta}^2) \\
&= \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \pi_{2,*} I_{\Delta}/I_{\Delta}^2 \\
&= \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \Omega_{\tilde{\Phi}/\text{Gr}}
\end{aligned}$$

Since I_{Δ}/I_{Δ}^2 supported on Δ

Bundles of relative principal parts

The example in the last subsection can be fully generalized, to solve many contact problems.

Setting $\alpha: Y \rightarrow X$ proper sm
 E/Y : v.b.

Define

$$\begin{array}{ccc} \widetilde{Y} & = & Y \times_X Y \\ \downarrow \pi_1 & & \downarrow \alpha \\ Y & \xrightarrow{\quad \alpha \quad} & X \end{array} \qquad \qquad \begin{array}{ccc} F_y & = & \alpha^{-1}(\alpha(y)) \subset Y \times \{y\} \\ \downarrow & & \downarrow \pi_2 \\ F_y & \xrightarrow{\quad l_y \quad} & Y \end{array}$$

and define the bundle of relative m -th order principal parts

$$\begin{aligned} P_{Y/X}^m(E) &= \pi_{2,*} (\pi_1^* E \otimes_{O_{\widetilde{Y}}} O_{\widetilde{Y}} / I_{\Delta}^{m+1}) \\ &= \pi_{2,*} (\pi_2^* E \otimes_{O_{\widetilde{Y}}} O_{\widetilde{Y}} / I_{\Delta}^{m+1}) \\ &= E \otimes_{O_Y} \pi_{2,*} (O_{\widetilde{Y}} / I_{\Delta}^{m+1}) \quad \text{over } Y. \end{aligned}$$

Thm [3264, Thm 11.2]

We have maps among v.b.s

$$\begin{array}{ccc} \pi_Y^* \pi_{Y,*} E & \longrightarrow & \pi_{2,*} \pi_1^* E \longrightarrow \pi_{2,*} (\pi_1^* E \otimes_{O_{\widetilde{Y}}} O_{\widetilde{Y}} / I_{\Delta}^{m+1}) \\ \parallel & & \parallel \text{def} \\ O_Y \otimes_K \Gamma(Y; E) & \longrightarrow & P_{Y/X}^m(E) \end{array}$$

which fiberwise provides linear maps among vector spaces

$$\Gamma(Y; E) \longrightarrow \Gamma(F_y; E|_{F_y}) \longrightarrow P_{Y/X}^m(E)|_y$$

where

$$\begin{aligned}
P_{Y/X}^m(\mathcal{E})|_y &= \mathcal{E}|_y \otimes_K \pi_{F_y,*}(\mathcal{O}_{F_y}/I_y^{m+1}) \\
&= \pi_{F_y,*}(\mathcal{E}|_{F_y} \otimes_{\mathcal{O}_{F_y}} \mathcal{O}_{F_y}/I_y^{m+1}) \\
&= \frac{\Gamma(F_y, \mathcal{L}|_{F_y})}{\Gamma(F_y, \mathcal{L}|_{F_y} \otimes_{\mathcal{O}_{F_y}} I_y^{m+1})} \\
&= \frac{\{\text{germs of sections of } \mathcal{L}|_{F_y} \text{ at } y\}}{\{\text{germs vanishing to order } \geq m+1 \text{ at } y\}}
\end{aligned}$$

Details

$$\pi_Y^* \pi_{Y,*} = \alpha^* \pi_X^* \pi_{X,*} \alpha_* \longrightarrow \alpha^* \alpha_* = \pi_{z,*} \pi_z^*$$

I prefer the derived version. For non-derived version, we need to carefully verify the conditions on the theorem of the cohomology and base change.

Rmk. Using the filtration of $\mathcal{O}_{\tilde{Y}}/I_{\Delta}^{m+1}$

$$0 \longleftarrow \mathcal{O}_{\tilde{Y}}/I_{\Delta} \longleftarrow \mathcal{O}_{\tilde{Y}}/I_{\Delta}^2 \longleftarrow \mathcal{O}_{\tilde{Y}}/I_{\Delta}^3 \longleftarrow \dots \longleftarrow \mathcal{O}_{\tilde{Y}}/I_{\Delta}^{m+1}$$

and

$$\begin{aligned}
\pi_{z,*}(I_{\Delta}^m/I_{\Delta}^{m+1}) &= \pi_{z,*}(\text{Sym}^m(I_{\Delta}/I_{\Delta}^2)) \\
&= \text{Sym}^m(\pi_{z,*}(I_{\Delta}/I_{\Delta}^2)) \\
&= \text{Sym}^m \Omega_{Y/X}
\end{aligned}$$

one gets filtration of $P_{Y/X}^m(\mathcal{L})$.

$$0 \longleftarrow \mathcal{L} \longleftarrow P_{Y/X}^0(\mathcal{L}) \longleftarrow P_{Y/X}^1(\mathcal{L}) \longleftarrow \dots \longleftarrow P_{Y/X}^m(\mathcal{L})$$

Therefore,

$$c(P_{Y/X}^m(\mathcal{L})) = \prod_{k=0}^m c(\mathcal{L} \otimes \text{Sym}^k \Omega_{Y/X})$$

E.g. 1 (line with hypersurfaces)

$$\Phi \stackrel{\cong}{\sim} \text{Flag}_{(1,2,n)} \subset \mathbb{P}^{n-1} \times \text{Gr}(2,n)$$

$\beta \searrow \quad \downarrow \alpha$

$$\mathbb{P}^{n-1} \qquad \qquad \qquad \text{Gr}(2,n)$$

Let $Y = \Phi$, $X = \text{Gr}(2,n)$, $L = \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)$, then
the degree d polynomial

$F \in \Gamma(\Phi, \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)) = \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d))$
provides a section of $P_{\Phi/\text{Gr}(2,n)}^m(\beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d))$, and

$$V_{p^m}(F) = \{(p, l) \in \Phi \mid F|_l \text{ has multiplicity } \geq m+1 \text{ at } p\}$$

\nwarrow vanishing set

This recovers the previous subsection.

e.g. (triple points of plane curves)

$$C_d := \{F = 0\} \subset \mathbb{P}^2$$

In E.g. 1. we take $n=3, m=2$, then

$$V_{p^2}(F) = \{(p, l) \in \Phi \mid \text{mult}_p(l, C_d) \geq 3\}$$

$$\beta_*[V_{p^2}(F)] = \{p \in C_d \mid p \text{ is a triple point}\}$$

Rmk: see [3264, 7.5.2, p272] to see the possible real obstruction for calculation.

≥ 3	flex / inflection	triple
≥ 4	hyper flex	quadruple
≥ 5		quintuple
≥ 6		sextuple
higher-order contact points		

E.g. 2 (tangent space of hypersurfaces)

$$\Phi \stackrel{\cong}{\sim} \text{Flag}_{(1, n-1, n)} \subset \mathbb{P}^{n-1} \times \text{Gr}(n-1, n)$$

$\beta \swarrow \quad \downarrow \alpha$

$$\mathbb{P}^{n-1} \qquad \qquad \text{Gr}(n-1, n)$$

Let $Y = \Phi$, $X = \text{Gr}(n-1, n)$, $\mathcal{L} = \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)$, then
the degree d polynomial

$F \in \Gamma(\Phi, \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d)) = \Gamma(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d))$
provides a section of $P_{\Phi/\text{Gr}(n-1, n)}^m(\beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d))$, and $Y_d := \{F=0\}$

$$V_{\rho'}(F) = \{(p, H) \in \Phi \mid T_p Y_d \supset H\}$$

$\swarrow \text{birational equiv} \quad \searrow \text{generic cover}$

$$Y_d \qquad \qquad \text{Gr}(n-1, n)$$

$$V_{\rho''}(F) = \{(p, H) \in \Phi \mid F|_H \text{ has multiplicity } \geq 3 \text{ at } p\}$$

$\swarrow \quad \searrow$

$$Y_d \qquad \qquad \text{Gr}(n-1, n)$$

3. Subvariety as degeneracy loci

Def. (degeneracy loci)

Let X/\mathbb{C} sm $k \in \mathbb{Z}_{\geq 0}$,

\mathcal{E}, \mathcal{F} : v.b. over X of rank e, f ,

$\varphi: \mathcal{E} \rightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

We define the degeneracy loci

$$M_k(\varphi) := \{x \in X \mid \text{rank } \varphi_x \leq k\}$$

remember multiplicity
 $\varphi_x: \mathcal{E}|_x \rightarrow \mathcal{F}|_x$

The expected codimension is $(e-k)(f-k)$.

E.g. When $\mathcal{E} = \mathcal{O}_X$, we know $e=1$,

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F}) \quad \varphi \leftrightarrow s$$

$$M_1(\varphi) = X, \quad M_0(\varphi) = V(s)$$

\uparrow vanishing set in X

Therefore, the degeneracy loci generalizes the section of v.b..

E.g. When $\mathcal{E} = \mathcal{O}_X^{\oplus e}$,

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F})^{\oplus e} \quad \varphi \leftrightarrow (s_1, \dots, s_e)$$

$$M_e(\varphi) = X$$

$$M_{e-1}(\varphi) = \{x \in X \mid s_1(x), \dots, s_e(x) \text{ are linear dependent}\}$$

$$M_k(\varphi) = \{x \in X \mid \dim \langle s_i(x) \rangle_i \leq k\}$$

$$M_0(\varphi) = V(s_1, \dots, s_e)$$

Flag variety

E.g.

$$\begin{aligned}\sum_{k'}^{\text{intersect}} &:= \left\{ (V, V') \in \text{Gr}(r, n) \times \text{Gr}(r', n) \mid \dim V \cap V' \geq k' \right\} \\ &= \left\{ (V, V') \in \text{Gr}(r, n) \times \text{Gr}(r', n) \mid \dim V + V' \leq r + r' - k' \right\} \\ &= \left\{ (V, V') \mid V \oplus V' \rightarrow \mathbb{C}^n \text{ is of rank } \leq r + r' - k' \right\} \\ &= M_{r+r'-k'} (\varphi : \pi_1^{-1} \mathcal{S} \oplus \pi_2^{-1} \mathcal{S}' \rightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

The expected codimension is

$$(r + r' - (r + r' - k'))(n - (r + r' - k')) = k'(n + k' - r - r')$$

When $\begin{cases} k' \leq \min(r, r') \\ n + k' - r - r' \geq 0 \end{cases}$, $\sum_{k'}^{\text{intersect}}$ has the expected codimension.

In general, one can define

$$\begin{aligned}\sum_k^{\text{sum}} &:= \left\{ (V_i)_i \in \prod_i \text{Gr}(r_i, n) \mid \dim \sum_i V_i \leq k \right\} \\ &= M_k (\varphi : \bigoplus_i \pi_i^{-1} \mathcal{S}_i \rightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

with the expected codimension $(\sum_i r_i - k)(n - k)$.

When $\begin{cases} k \geq \max\{r_i\}_i \\ k \leq n \end{cases}$, \sum_k^{sum} has expected codimension.

A more general case (also generalize [3264, Ex 12.11, Ex 12.9]).

Let $X : \text{sm proj}$, $\mathcal{F}_i \subset \mathcal{E}$ are v.b.s
 $\text{rank } \mathcal{F}_i = r_i \quad n$

$$\begin{aligned}\sum_k^{\text{sum}} &:= \left\{ p \in X \mid \dim \sum_i \mathcal{F}_i|_p \leq k \right\} \\ &= \left\{ p \in X \mid \bigoplus_i \mathcal{F}_i|_p \rightarrow \mathcal{E}|_p \text{ is of rank } \leq k \right\} \\ &= M_k (\varphi : \bigoplus_i \mathcal{F}_i \rightarrow \mathcal{E})\end{aligned}$$

The general partial flag variety can be express as the degeneracy loci.

$$\begin{aligned}
 \text{E.g. } \text{Flag}_{r_1, r_2, r_3}(\mathbb{C}^n) &= \{ o \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^n \mid \dim V_i = r_i \} \\
 &= \{ (V_1, V_2, V_3) \in \prod_i \text{Gr}(r_i, n) \mid \dim V_i + V_{i+1} \leq r_{i+1} \} \\
 &= M_{r_2+r_3} \left(\begin{array}{c|c} \pi_1^{-1} S_1 \oplus \pi_2^{-1} S_2 & O^{\oplus n} \\ \hline \pi_2^{-1} S_2 \oplus \pi_3^{-1} S_3 & O^{\oplus n} \end{array} \right) \\
 &= \pi_{12}^{-1} \sum_{r_2}^{\text{sum}} \cap \pi_{23}^{-1} \sum_{r_3}^{\text{sum}}
 \end{aligned}$$

Ramification locus [Barth 04 I.16]

Let $Y, X/\mathbb{C}$: sm of dim n , $f: Y \rightarrow X$ finite.
The ramification divisor of f is defined as

$$\begin{aligned} R &= \left\{ y \in Y \mid T_y f: T_y Y \rightarrow T_{f(y)} X \text{ is not surj} \right\} \\ &= \left\{ y \in Y \mid f^*: T_{f(y)}^* X \rightarrow T_y^* Y \text{ is not surj} \right\} \\ &= \left\{ y \in Y \mid \text{rank } \varphi_y \leq n-1, \text{ where } \varphi_y: f^* \Omega_X \xrightarrow{\varphi} \Omega_Y \right\} \\ &= M_{n-1}(f^* \Omega_X \rightarrow \Omega_Y) \end{aligned}$$

with the expected codim $(n-(n-1))(n-(n-1)) = 1$

Rmks. 1. R may have multiplicity, which is also counted in the degeneracy loci.
Recall that, for the zero set of section, we also count the multiplicity

2. Since

$\mathbb{C}^n \rightarrow \mathbb{C}^n$ is of $\text{rk} \leq n-1 \Leftrightarrow \det \mathbb{C}^n \rightarrow \det \mathbb{C}^n$ is zero,
we get

$$\begin{aligned} R &= M_0(f^* \omega_X \rightarrow \omega_Y) \\ \omega_Y &= f^* \omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R) \quad \rightsquigarrow \text{Hurwitz formula} \end{aligned}$$

$$0 \rightarrow f^* \omega_X \rightarrow \omega_Y \rightarrow L_{R,*} \mathcal{O}_R \rightarrow 0$$

3. I guess that we can generalize to f generic finite,
then we can get ramification locus + special fiber part.

How to distinguish these two locus?

Guess: for those special fibers, the pushforward will give us
zero cycle. Can we use that?

4. For Y, X sm variety of $\dim Y, \dim X$,
when $f: Y \rightarrow X$ is a closed embedding, we get:

$$0 \rightarrow N_{Y/X}^\vee \rightarrow f^* \Omega_X \rightarrow \Omega_Y \rightarrow 0$$

In this case, $\varphi: f^* \Omega_X \rightarrow \Omega_Y$ is always surj,
so the degeneracy loci is meaningless.

4. Subvariety given by very ample bundle

For X/\mathbb{C} sm proj, \mathcal{F}/X v.b. of rank r , assume that $(\mathcal{F}, s_1, \dots, s_n)$ provides an embedding $\phi_{\mathcal{F}}: X \rightarrow \text{Gr}(r, n)$, we want to compute $[X] \in H_{2\dim_{\mathbb{C}} X} \text{Gr}(r, n)$.

Rmk 1.

This is different from the previous construction. Here, the cycle we want to compute is the pushforward, not the pullback. By Poincaré duality in $\text{Gr}(r, n)$, the pushforward can be computed using the pullback of special cycles in $\text{Gr}(r, n)$.

Rmk 2. We know

$$\begin{aligned} \mathcal{L} \text{ is very ample} &\Leftrightarrow (\mathcal{L}, \Gamma(X; \mathcal{L})) \text{ induces embedding} \\ \mathcal{F} \text{ is very ample} &\Rightarrow (\mathcal{F}, \Gamma(X; \mathcal{F})) \text{ induces embedding} \end{aligned}$$

See p73 in

R Hartshorne, Ample vector bundles

https://www.numdam.org/item/PMIHES_1966__29__63_0.pdf

E.g. When \mathcal{L}/X induces an embedding $\phi_{\mathcal{L}}: X \rightarrow \mathbb{P}^{n-1}$,

$$\begin{aligned} [X] &= \phi_{\mathcal{L}}^* ([H]^{\dim X}) [\mathbb{P}^{\dim X}] \\ &= \deg \mathcal{L} \cdot [\mathbb{P}^{\dim X}]. \end{aligned}$$

Therefore,

$$\text{compute } X \Leftrightarrow \text{compute } \deg \mathcal{L}.$$

Special embeddings

Let us include several special embeddings here.

E.g. For $\iota_X: X \hookrightarrow \mathrm{Gr}(r, n)$, ι_X is induced by $\iota_X^* \mathcal{S}^\vee$.

E.g. (Veronese embedding)

For $d \geq 1$, $\mathcal{O}_{\mathbb{P}^n}(d)$ induces the Veronese embedding $\mathrm{Sym}^d \mathcal{O}^{\oplus n} \rightarrow \mathrm{Sym}^d \mathcal{O}(1)$

$$\begin{aligned} \nu_d: \mathbb{P}^n &\longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \\ [z_0: \dots: z_n] &\longmapsto [z_0^{i_0} \dots z_n^{i_n}]_{\sum i_k = d} \end{aligned}$$

of degree d^n .

It describes when the deg d hypersurfaces in \mathbb{P}^n degenerates as a (non-reduced) hyperplane.

E.g. (Segre embedding)

$\mathcal{L} := \mathcal{O}_{\mathbb{P}^m}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(1)$ induces the Segre embedding $\mathcal{O}^{\oplus(m+1)} \boxtimes \mathcal{O}^{\oplus(n+1)} \rightarrow \mathcal{L}$

$$\begin{aligned} \phi_1: \mathbb{P}^m \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{(m+1)(n+1)-1} \\ ([x_0: \dots: x_m], [y_0: \dots: y_n]) &\longmapsto [x_i y_j]_{i,j} \end{aligned}$$

of degree $\binom{m+n}{m}$.

E.g. (Plücker embedding)

In $\mathrm{Gr}(r, n)$, $\det \mathcal{S}^\vee$ induces the Plücker embedding

$$\begin{aligned} \iota_{\mathrm{Gr}}: \mathrm{Gr}(r, n) &\longrightarrow \mathbb{P}(\Lambda^r \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{r}-1} \\ W = \langle w_1, \dots, w_r \rangle &\longmapsto w_1 \wedge \dots \wedge w_r \end{aligned}$$

of degree $(r(n-r))! \prod_{i=0}^{r-1} \frac{i!}{(n-r+i)!}$ [3264, Prop 4.12]

Not very ample case

There are cases where L is not very ample.

E.g. $\text{Sym}^2 \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$ induces the map

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 6} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 9} \\ \parallel & & \parallel \\ \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array}$$

$$\phi: \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8 \dashrightarrow \mathbb{P}^5$$

$$([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) \mapsto \begin{bmatrix} x_0 y_0 & \dots & x_0 y_2 \\ \vdots & \ddots & \vdots \\ x_2 y_0 & \dots & x_2 y_2 \end{bmatrix} \mapsto \begin{bmatrix} x_0 y_0 : x_0 y_1 + x_1 y_0 : x_0 y_2 + x_2 y_0 \\ \vdots : \vdots : \vdots \\ x_1 y_1 : x_1 y_2 + x_2 y_1 \\ \vdots : \vdots \\ x_2 y_2 \end{bmatrix}$$

$\text{Im } \phi$ describes when the quadric curve degenerates as union of two lines.
Since $\phi: \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \text{Im } \phi$ is a 2:1 ramified cover,

$$\phi^*([H]^4) = 6 \Rightarrow \text{Im } \phi \subset \mathbb{P}^5 \text{ is of degree 3.}$$

E.g. [3264, 2.2.1]

$$\begin{array}{ccc} \text{Sym}^3 \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \text{Sym}^2 \mathcal{O}_{\mathbb{P}^5}^{\oplus 3} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \\ \parallel & & \parallel \\ \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 10} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 18} \longrightarrow \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \end{array}$$

induces the map

$$\phi: \mathbb{P}^2 \times \mathbb{P}^5 \longrightarrow \mathbb{P}^9 \dashrightarrow \mathbb{P}^9$$

$$(F_1, F_2) \longmapsto F_1 F_2$$

$$\sum x_i z_i \quad \sum_{i,j} y_{ij} z_i z_j \quad \sum_{i,j,k} x_i y_{ijk} z_i z_j z_k$$

$\text{Im } \phi$ describes when the cubic curve contains a line.
Since $\phi: \mathbb{P}^2 \times \mathbb{P}^5 \longrightarrow \text{Im } \phi$ is generically injective,

$$\phi^*([H]^7) = \binom{7}{2} = 21 \Rightarrow \text{Im } \phi \subset \mathbb{P}^9 \text{ is of degree 21.}$$

E.g. [3264, 2, 2, 2]

$$\begin{array}{ccccc} \text{Sym}^3 \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \parallel \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 10} & \longrightarrow & \parallel \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 27} & \longrightarrow & \parallel \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array}$$

induces the map

$$\phi: \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^6 \dashrightarrow \mathbb{P}^9 \quad (F_1, F_2, F_3) \mapsto F_1 F_2 F_3$$

$\text{Im } \phi$ describes when the cubic curve degenerates as union of three lines.
Since $\phi: \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \text{Im } \phi$ is a 6:1 ramified cover,

$$\phi^*([H]^6) = \binom{6}{4} \binom{4}{2} = \frac{6!}{2!2!2!} = 90 \Rightarrow \text{Im } \phi \subset \mathbb{P}^9 \text{ is of degree 15.}$$