Eine Woche, ein Beispiel
1.9. simplicial set

Ref:
[sSet]http://www.math.uni-bonn.de/~schwede/sset_vs_spaces.pdf
[6-Fctor]https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

Some visual pictures can be seen here:
https://arxiv.org/pdf/0809.4221.pdf

Today: The category sSet and $\partial\Delta^n$, Λ^n_i, $sk^m X$, $\Delta^n/\partial\Delta^n$, $Hom(X,Y) \in Ob(sSet)$

Def $[n] = \{0, 1, \dots, n\}$ $\quad n \geq 0$

The simplex category Δ is defined by

$$Ob(\Delta) = \{[n] \mid n \geq 0\}$$

$$Mor_\Delta([m],[n]) = \{f : [m] \longrightarrow [n] \text{ weakly increasing}\}$$

The category of simplicial sets sSet is defined by

$$sSet = Fun(\Delta^{op}, Set)$$

Notation in $Mor(\Delta)$: $d_i^n : [n-1] \longrightarrow [n]$ miss i $\quad 0 \leq i \leq n$

$s_i^n : [n] \longrightarrow [n-1]$ contracts i

$\Delta \hookrightarrow sSet \quad [n] \longmapsto \Delta^n := Mor_\Delta(-, [n])$

e.p. $\Delta^n_k = Mor_\Delta([k],[n])$

read from down to top

Rmk. In Δ we don't have finite colimit, while in $sSet = Fun(\Delta^{op}, Set)$ we have finite colimit because Set is (complete +) cocomplete.

For a construction, see
https://math.stackexchange.com/questions/3837844/limits-and-colimits-are-computed-pointwise-in-functor-categories

Notice that

$$\partial\Delta^n, \Lambda^n_i, sk^m\Delta^n, \Delta^n/\partial\Delta^n \in sSet - \Delta$$

Conclusion: sSet is a Grothendieck topos.

It is Cartesian closed, complete and cocomplete.

Rmk. ([sSet]) If you have strong enough geometrical background, you will find out the adjoint pair

$$\begin{tikzcd} sSet \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & Top \arrow[l, bend left, "S"] \end{tikzcd}$$

quite useful, where

$$|X| := \left(\coprod_{n\geq 0} X_n \times \nabla^n\right)/\sim$$

$$S(A)_n := Mor_{Top}(\nabla^n, A)$$

$$\alpha^* : S(A)_n \longrightarrow S(A)_m \qquad x \longmapsto x \circ S(\alpha)$$

$\alpha : [m] \to [n]$

$S(\alpha) : \nabla^m \longrightarrow \nabla^n$

Moreover, we have eqvlc of categories

$$\begin{tikzcd} & Ho(Top_{cw}) \arrow[d] \\ sSet[weq^{-1}] \arrow[ur, "|-|"] & Top[weq^{-1}] \arrow[l, "S"'] \end{tikzcd}$$

where

Top_{cw} is the full subcategory of Top with objects the top spaces admitting a CW-cplx structure, and

$Ho(Top_{cw})$ is the homotopy category of CW cplxes.

Q: Do we have the following comm diag: (as eqvlc of categories)

$$\begin{tikzcd} sSet[weq^{-1}] \arrow[r, "|-|"] & Ho(Top_{cw}) \arrow[d] \\ An \arrow[u, hook] & Top[weq^{-1}] \arrow[l, "S"'] \end{tikzcd}$$

Q: For $\mathcal{C} \in Cat_\infty \subseteq sSet$, how to view $\mathcal{C}$ as a topo space?

e.p. compute $\pi_n(\mathcal{C})$?

Roughly, we have three ways to define/determine a set:

1. By writing down their def directly; — brutal force
2. By universal property (pullback, pushforward, ...) — abstract construction
3. By its geometrical realization — name

Let us see how they're compatible with each other.

Eg. 1. $\Delta^n_k = Mor_\Delta([k],[n]) = \{ x : [k] \longrightarrow [n] \text{ weakly increasing} \}$

$$
\begin{aligned}
|\Delta^n| &= \left(\coprod_k \Delta^n_k \times \nabla^k \right) / \sim \\
&\sim (\Delta^n_n \times \nabla^n)/\sim \\
&\sim \nabla^n
\end{aligned}
$$

Eg. 2. $\Delta^{n-1}_{(i)} := Im\,(d^n_i : \Delta^{n-1} \longrightarrow \Delta^n)$ in sSet

$\Rightarrow (\Delta^{n-1}_{(i)})_k = \{ x \in \Delta^n_k \mid \exists\, y \in \Delta^{n-1}_k \text{ s.t. } x = d^n_i \circ y \}$

$$
\begin{aligned}
|\Delta^{n-1}_{(i)}| &= \left(\coprod_k (\Delta^{n-1}_{(i)})_k \times \nabla^k \right)/\sim \\
&\sim \left((\Delta^{n-1}_{(i)})^{nondeg}_{n-1} \times \nabla^{n-1} \right)/\sim \\
&\sim (Sd^n_i)(\nabla^{n-1})
\end{aligned}
$$

Denote $|\Delta^{n-1}_{(i)}|$ by $\nabla^{n-1}_{(i)}$, i.e. $\nabla^{n-1}_{(i)} := Im\,(Sd^n_i : \nabla^{n-1} \longrightarrow \nabla^n)$

Eg. 3. $(\partial\Delta^n)_k = \{ x \in \Delta^n_k \mid x \text{ is not surj} \}$

$\partial\Delta^n = \bigcup_{i=0}^n \Delta^{n-1}_{(i)} = \text{colimit of } \cdots$

e.g. $\partial\Delta^2 = \left[\text{colimit of } \begin{tikzcd} \Delta^0 \arrow[d] \arrow[dr] & \Delta^0 \arrow[dl] \arrow[dr] & \Delta^0 \arrow[dl] \arrow[d] \\ \Delta^1 & \Delta^1 & \Delta^1 \end{tikzcd} \right]$

$$
\begin{aligned}
|\partial\Delta^n| &= \left(\coprod_k (\partial\Delta^n)_k \times \nabla^k \right)/\sim \\
&\sim \left(Mor^{nondeg}_\Delta([n-1],[n]) \times \nabla^{n-1} \right)/\sim \\
&\sim \left(\coprod_{i=0}^n Sd^n_i(\nabla^{n-1}) \right)/\sim \\
&= \bigcup_{i=0}^n \nabla^{n-1}_{(i)} \\
&= \partial\nabla^n
\end{aligned}
$$

Eg. 3. $(\Lambda^n_i)_k = \left\{ x \in \Delta^n_k \;\middle|\; x = \alpha^*(y) \text{ for some } y \in \Delta^n_{n-1} \text{ and } \alpha : [k] \to [n-1] \right\}$

$y \neq d^n_i$

$\Lambda^n_i = \bigcup_{j \neq i} \Delta^{n-1}_{(j)}$ = colimit of $\cdots$

e.g. $\Lambda^2_1 = \left[\text{colimit of} \quad \begin{tikzcd} & & \Delta^0 \arrow[d, dashed, "d^1_0"] \\ & \Delta^0 \arrow[r, "d^1_1"] \arrow[d, "d^1_0"'] & \Delta^1 \\ \Delta^0 \arrow[r, dashed, "d^1_1"] & \Delta^1 & \end{tikzcd} \right]$

$= \Delta^1 \sqcup_{\Delta^0} \Delta^1$

ex. write down $(X \sqcup_Y Z)_k$ for $X, Y, Z \in sSet$

$$
\begin{aligned}
|\Lambda^n_i| &= \Big(\coprod_k (\Lambda^n_i)_k \times \nabla^k \Big) / \sim \\
&\sim \big((\Lambda^n_i)^{nondeg}_{n-1} \times \nabla^{n-1} \big) / \sim \\
&\sim \Big(\coprod_{j \neq i} (Sd^n_j)(\nabla^{n-1}) \Big) / \sim \\
&\sim \bigcup_{j \neq i} \nabla^{n-1}_{(j)}
\end{aligned}
$$

Eg. 5. $(sk^m \Delta^n)_k = \left\{ x \in \Delta^n_k \;\middle|\; x = \alpha^*(y) \text{ for some } y \in \Delta^n_m \text{ and } \alpha : [k] \to [m] \right\}$

$sk^m \Delta^n = \bigcup_{\beta : [m] \to [n]} \beta(\Delta^m)$ = colimit of $\cdots$

$$
\begin{aligned}
|sk^m \Delta^n| &= \Big(\coprod_k (sk^m \Delta^n)_k \times \nabla^k \Big) / \sim \\
&\sim \big((sk^m \Delta^n)^{nondeg}_m \times \nabla^m \big) / \sim \\
&\sim \big(Mor^{nondeg}([m],[n]) \times \nabla^m \big) / \sim \\
&\sim \bigcup_{\beta : [m] \to [n]} (S\beta)(\nabla^m)
\end{aligned}
$$

Eg. 6. $(\Delta^n / \partial\Delta^n)_k = \Delta^n_k / (\partial\Delta^n)_k = \Delta^n_k / \sim$

$x \sim y \iff x, y \in (\partial\Delta^n)_k$ or $x = y$

Universal property:

$$
\begin{tikzcd}
\partial\Delta^n \arrow[r] \arrow[drr, dashed, "\text{contract to one pt}"'] & \Delta^n \arrow[r] \arrow[dr] & \Delta^n / \partial\Delta^n \arrow[r] \arrow[d, dashed, "\exists !"] & 0 \\
& & X &
\end{tikzcd}
$$

$$
\begin{aligned}
|\Delta^n/\partial\Delta^n| &= \Big(\coprod_k (\Delta^n/\partial\Delta^n)_k \times \nabla^k\Big)/\sim \\
&\sim \Big(((\Delta^n/\partial\Delta^n)_n^{\text{nondeg}} \times \nabla^n\Big)/\sim \\
&\sim \nabla^n/\sim \\
&\sim \nabla^n/\partial\nabla^n
\end{aligned}
$$

Eg. 7 $\quad (\mathrm{Hom}(X,Y))_n = \mathrm{Hom}_{sSet}(\Delta^n \times X, Y)$

$$\alpha^*: \mathrm{Hom}_{sSet}(\Delta^n \times X, Y) \longrightarrow \mathrm{Hom}_{sSet}(\Delta^m \times X, Y)$$

for $\alpha: [m] \to [n]$
$\alpha: \Delta^m \to \Delta^n$

$$
\begin{tikzcd}
sSet \arrow[r, bend left, "- \times X"] \arrow[r, phantom, "\perp"] & sSet \arrow[l, bend left, "\mathrm{Hom}(X,-)"]
\end{tikzcd}
$$

"Proof"

$$
\begin{aligned}
\mathrm{Hom}_{sSet}(Z, \mathrm{Hom}(X,Y)) &\cong \{ g_m : Z_m \longrightarrow \mathrm{Hom}_{sSet}(\Delta^m \times X, Y) + \cdots \} \\
&\cong \{ h_{m,k} : Z_m \times \Delta^m_k \times X_k \longrightarrow Y_k + \cdots \} \\
&\cong \{ h_k : Z_k \times X_k \longrightarrow Y_k + \cdots \} \\
&\cong \mathrm{Hom}_{sSet}(Z \times X, Y)
\end{aligned}
$$

Cor. $\quad \mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \mathrm{Hom}(X,Y))$

e.g. $\quad \mathrm{Hom}(\Delta^0, Y) \cong Y \qquad (\mathrm{Hom}(\Delta^n, Y))_m \cong (\mathrm{Hom}(\Delta^m, Y))_n$

$\mathrm{Hom}(X, \Delta^0) \cong \Delta^0 \qquad$ e.p. $(\mathrm{Hom}(\Delta^n, Y))_0 \cong Y_n$

$$
\begin{aligned}
|\mathrm{Hom}(X,Y)| &= \Big(\coprod_k \mathrm{Hom}_{sSet}(\Delta^k \times X, Y) \times \nabla^k\Big)/\sim \\
&= ?
\end{aligned}
$$

Remaining: Compute $\# (\mathrm{Hom}(\Delta^n, \Delta^m))_k$
Compute $(\mathrm{Hom}(\Delta^n/\partial\Delta^n, Y))_k$. How is it related to Y_{k+n} or $\pi_n(|Y|)$?
How to see the geometrical realization of $\mathrm{Hom}(X,Y)$,
e.p. in these examples?