Eine Woche, ein Beispiel

2.13. outer automorphism

We do something very elementary but tricky, and will later find out its connection to the advanced topic, like Teichmüller space.

1. outer automorphism group $Out(G)$ / automorphism group $Aut(G)$

Ref:
https://en.wikipedia.org/wiki/Outer_automorphism_group
https://en.wikipedia.org/wiki/Automorphisms_of_the_symmetric_and_alternating_groups

Def. Let G be a group. We have a LES

$$1 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\text{conj}} Aut(G) \longrightarrow Out(G) \longrightarrow 1$$

where $Z(G)$ is the center of G
$Aut(G)$ is the automorphism of G
$Inn(G) := Im(\text{conj})$ is the inner automorphism of G
$Out(G) := Aut(G)/Inn(G)$ is the outer automorphism of G.

E.g. $G = \mathbb{Z}$, $Aut(\mathbb{Z}) = \{\pm 1\}$, $Out(\mathbb{Z}) = \{\pm 1\}$

$G = \mathbb{Z}/m\mathbb{Z}$, see https://zhuanlan.zhihu.com/p/97195375 ← typo. ② ⇒ 2 $(m \geq 2)$

an easy result is that $\# Out(\mathbb{Z}/m\mathbb{Z}) = \varphi(m)$.

E.g. $G = S_n$, $(n \in \mathbb{N}_{>0})$

$$Aut(S_n) = \begin{cases} S_n & n \neq 2, 6 \\ \{*\} & n = 2 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$Out(S_n) = \begin{cases} \{*\} & n \neq 6 \\ \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$G = A_n$

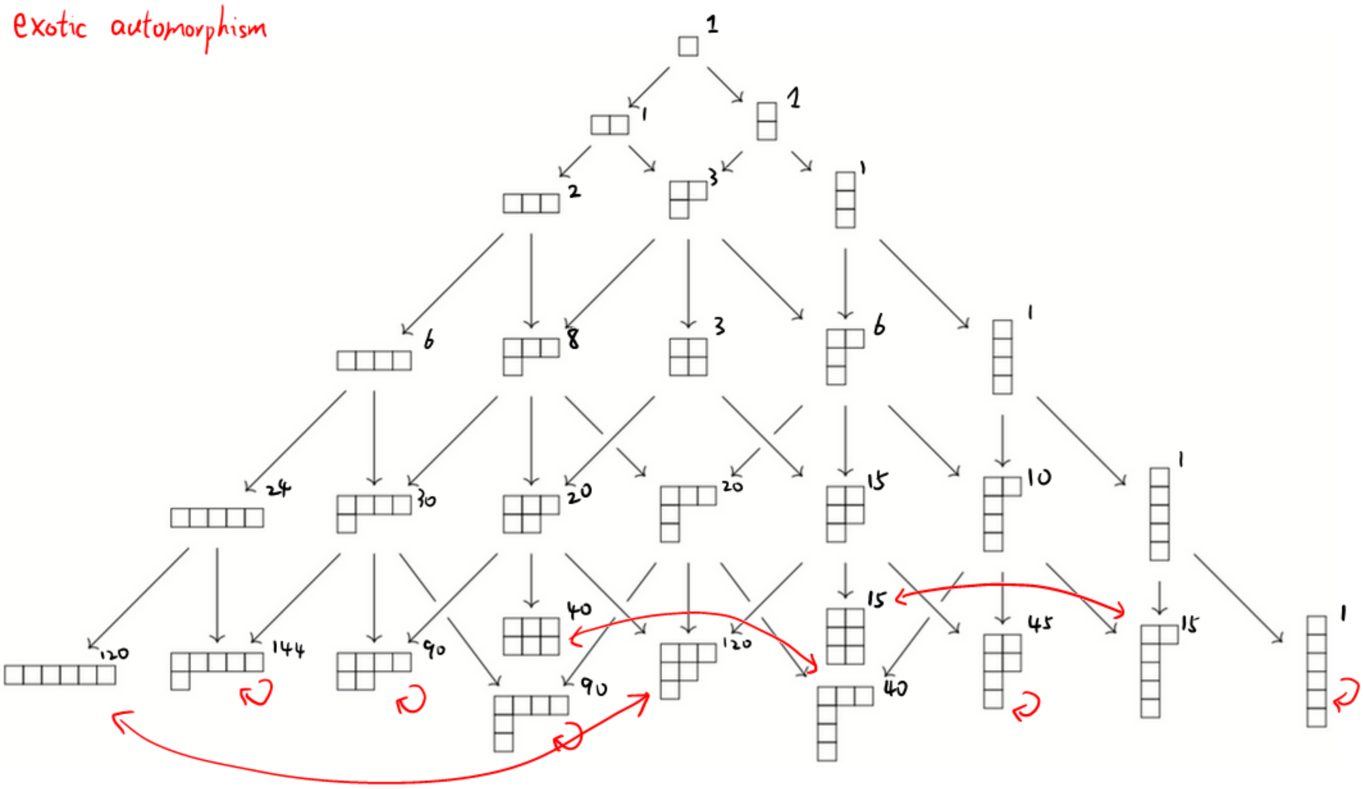
$$Aut(A_n) = \begin{cases} S_n & n \neq 2, 3, 6 \\ \{*\} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 3 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$Out(A_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \neq 2, 3, 6 \\ \{*\} & n = 2 \text{ or } 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

For a reference of the proof and constructions of the exotic outer automorphism of S_6, see wiki and here:
https://wordpress.nmsu.edu/pamorand/files/2018/10/AutGroups.pdf
For Chinese you can also see here: https://zhuanlan.zhihu.com/p/24764617
They are elementary and everybody who have learned something about Sylow's theorem should be able to follow the proofs.

$\{$elements in conj class [diagram] $= (123)\}$

exotic automorphism



E.g. $G = PSL(2, \mathbb{F}_7) \cong GL(3, \mathbb{F}_2)$

$Aut(PSL(2, \mathbb{F}_7)) \cong PGL(2, \mathbb{F}_7)$ $\qquad$ $Out(PSL(2, \mathbb{F}_7)) \cong \{\pm 1\}$

Statement:
https://mathoverflow.net/questions/348440/what-is-the-outer-automorphism-group-of-operatornamesl2-mathbbf-q
For the other lie group, e.g. group in wiki: https://en.wikipedia.org/wiki/Projective_linear_group,
there is a general theory for its outer automorphism group, please see this book: (Even though I'm not so interested now)
https://www.cambridge.org/core/journals/canadian-journal-of-mathematics/article/automorphisms-of-finite-linear-groups/16c
23F257E0F21D57873B1450E9F15E4

E.g. $F_n :=$ free group generated by $a_1, \dots, a_n$

$F_n \longrightarrow F_n/[F_n, F_n] \cong \mathbb{Z}_n \quad \rightsquigarrow \quad Out(F_n) \longrightarrow GL(n, \mathbb{Z})$

It's claimed that $Out(F_2) \cong GL(2, \mathbb{Z})$.

Left: f.g. abelian group, like $\mathbb{Z}^n$. $(Aut(\mathbb{Z}^n) \cong Out(\mathbb{Z}^n) \cong GL(n, \mathbb{Z}))$

$(\mathbb{Z}/8\mathbb{Z})^{\oplus 3}$