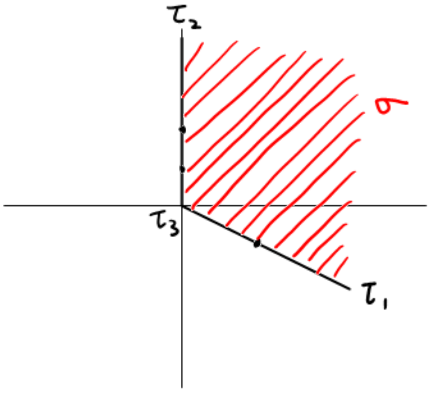
eine Woche, ein Beispiel

4.9. singular surface

Today: $X = \mathrm{Spec}\, \mathbb{C}[a,b,c]/(b^2-ac)$



$U_\sigma = \mathrm{Spec}\,\mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[a,b,c]/(b^2-ac)$

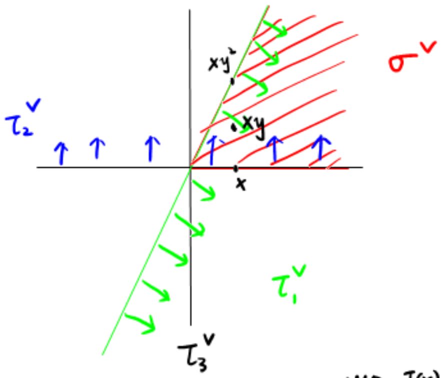
$U_{\tau_1} = \mathrm{Spec}\,\mathbb{C}[x, xy, xy^2, \frac{1}{xy^2}] \cong \mathbb{C}[a,b,c,\frac{1}{c}]/(b^2-ac)$

$U_{\tau_2} = \mathrm{Spec}\,\mathbb{C}[x, xy, xy^2, \frac{1}{x}] \cong \mathbb{C}[a,b,c,\frac{1}{a}]/(b^2-ac)$

$D_1 := V_{\tau_1} = \{xy^2=0, xy=0\} = \{c=b=0\}$

$D_2 = V_{\tau_2} = \{x=0, xy=0\} = \{a=b=0\}$

[$\mathbb{C}[a,b,c]/(b^2-ac)/(c) \cong \mathbb{C}[a,b]/(b^2)$ is not reduced]



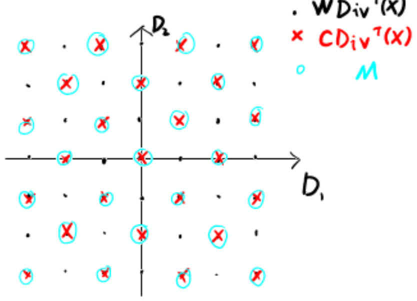
$\mathrm{WDiv}^T(X) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2$

$\mathrm{div}\, a = 2D_2, \quad \mathrm{div}\, b = D_1+D_2, \quad \mathrm{div}\, c = 2D_1$

$\mathrm{CDiv}^T(X) = \{n_1D_1 + n_2D_2 \mid n_1, n_2 \in \mathbb{Z}, n_1+n_2 \in 2\mathbb{Z}\}$

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "{x \mapsto \mathrm{div}\,x,\ y \mapsto \mathrm{div}\,y}"'] \arrow[d, equal] & \mathrm{CDiv}^T(X) \arrow[r] \arrow[d] & \mathrm{Pic}(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathrm{WDiv}^T(X) \arrow[r] & \mathrm{WCl}(X) \arrow[r] & 0
\end{tikzcd}$$

$\rightsquigarrow \mathrm{Pic}(X) \cong 1 \hookrightarrow \mathrm{WCl}(X) \cong \mathbb{Z}/2\mathbb{Z}$

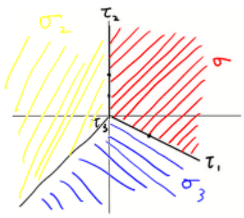


In this situation $\mathrm{Pic}(X) \cong H^2(X,\mathbb{Z})$ since $H^2(X,\mathbb{Z})$ is contractable.

Interesting guess.

- X can not be compactified by adding 1 pt, but can by adding $\mathbb{P}^1_{\mathbb{C}}$ denote it by Y.
- What is $\mathrm{WDiv}^T(Y), \mathrm{CDiv}^T(Y), \mathrm{WCl}(Y), \mathrm{Pic}(Y)$? for $\mathcal{L} \in \mathrm{Pic}(Y)$, how to compute $H^i(Y,\mathcal{L})$?
- Can we develop the intersection theory on Y?

$\varphi: \mathrm{Pic}(Y) \times \mathrm{Pic}(Y) \to \mathbb{Z}$



$Y = \mathrm{Proj}\,\mathbb{C}[a,b,c,T]/(ac-b^2)$

$0 \to \mathbb{Z} \to \mathrm{WDiv}^T(Y) \to \mathrm{WDiv}^T(X) \to 0 \quad \rightsquigarrow \mathrm{WDiv}^T(Y) = \mathbb{Z}^3$

$0 \to \mathbb{Z} \to \mathrm{WCl}(Y) \to \underset{\mathbb{Z}/2\mathbb{Z}}{\underset{\wr|}{\mathrm{WCl}(X)}} \to 0 \quad \rightsquigarrow \mathrm{WCl}(Y) \cong \mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$?

Compute $\mathrm{CDiv}^T(X)$ by the isomorphism:

$$\mathrm{CDiv}^T(X) \cong \ker\Big(\bigoplus_i M/M(\sigma_i) \xRightarrow{\Phi} \bigoplus_{i<j} M/M(\sigma_i \cap \sigma_j)\Big) \qquad M(\sigma_i) := \sigma_i^{\perp} \cap M$$

$$M(\sigma) = \sigma^{\perp} \cap M = 0 \Rightarrow M/M(\sigma) = \mathbb{Z}^2 \qquad M/M(\tau_1) = \mathbb{Z}^2/\mathbb{Z}xy^2$$

$$M(\tau_3) = \tau_3^{\perp} \cap M = M \Rightarrow M/M(\tau_3) = 0 \qquad M/M(\tau_2) = \mathbb{Z}^2/\mathbb{Z}x$$

$$\Phi : \underset{\mathbb{Z}^2}{M/M(\sigma)} \oplus \underset{\mathbb{Z}^2/\mathbb{Z}xy^2}{M/M(\tau_1)} \oplus \underset{\mathbb{Z}/\mathbb{Z}x}{M/M(\tau_2)} \oplus \underset{0}{M/M(\tau_3)} \longrightarrow \underset{\mathbb{Z}^2/\mathbb{Z}xy^2}{M/M(\sigma\cap\tau_1)} \oplus \underset{\mathbb{Z}/\mathbb{Z}x}{M/M(\sigma\cap\tau_2)} \oplus \underset{0}{M/M(\tau_1\cap\tau_2)} \oplus 0 \oplus 0$$

$$\big((x_0, y_0),\ (0, y_1),\ (0, y_2)\big) \longmapsto \big((x_0, y_0 - y_1),\ (x_0, y_0 - y_2)\big)$$

$$\Phi\big((x_0, y_0), (0, y_1), (0, y_2)\big) = 0 \iff y_0 = y_2,\ x_0 = 2(y_0 - y_1)$$

$$\Rightarrow \ker \Phi = \big((2y_0 - 2y_1, y_0), (0, y_1), (0, y_0)\big) \cong \mathbb{Z} \oplus \mathbb{Z}$$

Q: Which one corresponds to $2D_1$, $2D_2$ & $D_1 + D_2$?