Eine Woche, ein Beispiel

3.13 dual variety

Dual variety is useful is the research of subvarieties of P^n (and symplectic geometry). We emphasize the embedding here.

Main reference:
https://arxiv.org/abs/math/0112028v1
other ref:
Discriminants, Resultants, and Multidimensional Determinants by Israel M. GelfandMikhail M. KapranovAndrei V. Zelevinsky.
https://en.wikipedia.org/wiki/Dual_curve

A vivid animation: https://www.youtube.com/watch?v=HTXpf4jDgYE
Some pictures: https://www.ima.umn.edu/materials/2006-2007/W9.18-22.06/2203/Piene_190906.pdf

Goal.

1. Definition
2. Basic properties
   - Reflexivity theorem
   - dimension and defect
   - d, g, b, f, δ, k
3. Basic examples

Let $K = \bar{K}$ be a field, V a v.s. of dim $n+1$.

1. Definition

Def (Dual variety)

Let $X \subset \mathbb{P}V$: irr proj variety

X_{sm} : smooth locus

$I_X^\circ := \{(z, H) \mid z \in X_{sm},\ H \in \mathbb{P}V^*,\ T_z X \subset H\}$

$I_X := \overline{I_X^\circ}$.

Then $X^* := pr_2(I_X)$ is called the dual variety of X.

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd}$$

$\mathbb{P}V^* = \mathbb{P}(V^*)$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\operatorname{def} X}{X^*} \end{tikzcd}$$

$\dim V = n+1$

$\dim X = m$

$\operatorname{def} X = \operatorname{codim}_{\mathbb{P}V^*} X^* - 1$

Relation with symplectic geometry

Def (Lagrangian construction)

Let M be a sm proj irr variety, $Y \subset M$ be any irr subvariety.
We define

$$Lag(Y) := \overline{N^*_{Y_{sm}} M} \quad \text{(closure in } T^*M)$$

Def. Any set $S \subset T^*M$ is called conical if S is closed under scalar multiplication.

Rmk. [Thm 1.9] $Lag(Y)$ is a conical Lagrangian subvariety, and every conical Lagrangian subvariety S is of this form, i.e.

$$S = Lag(\pi(S)) \qquad \pi : T^*M \longrightarrow M$$

Rmk. $Lag(Y)$ is an analog of I_X. see the following picture:

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd} \qquad \begin{tikzcd} & V \times V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ V & & V^* \end{tikzcd}$$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\operatorname{def} X}{X^*} \end{tikzcd} \qquad \begin{tikzcd} & \overset{n+1}{Lag(Y)} \arrow[dl, "n-m"'] \arrow[dr, "\text{typically } 1"] & \\ \overset{m+1}{Y} & & \overset{n-\operatorname{def} X}{Y^*} \end{tikzcd}$$

$$Y = Cone(X) \quad \Rightarrow \quad Y^* = Cone(X^*)$$

2. Basic properties

2.1. Thm (Reflexivity thm) $X^{**} = X$

Sketch of proof.

$$X \xrightarrow{\cong} X^{**}$$

$$\Leftrightarrow \quad [(z,H) \in I_X^o \Leftrightarrow (H,z) \in I_{X^*}^o]$$

$$\Leftrightarrow \quad I_X \cong I_{X^*} \quad \text{under the iso } \mathbb{P}V \times \mathbb{P}V^* \xrightarrow{\sim} \mathbb{P}V^* \times \mathbb{P}V^{**}$$

$$\Leftrightarrow \quad Lag(Y) \cong Lag(Y^*) \quad \text{where } Y := Cone(X) \quad Y^* := Cone(X^*)$$

under the iso $\quad T^*V \cong V \times V^* \cong V^* \times V \cong T^*V^*$

Under this iso, $Lag(Y)$ is a conical Lagrangian subvariety of T^*V^*, so

$$Lag(Y) \cong Lag(pr_2(Lag(Y))) \cong Lag(Y^*)$$

2.2. Dimension and defect