Eine Woche, ein Beispiel

7.13. stability manifold of $\mathbb{P}^1$

Ref:

[Okada05]: So Okada, Stability Manifold of P^1

[GKR03]: A. Gorodentscev, S. Kuleshov, A. Rudakov, t-stabilities and t-structures on triangulated categories, https://arxiv.org/abs/math/0312442

[Bri07]: Tom Bridgeland, Stability conditions on triangulated categories, https://arxiv.org/abs/math/0212237

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Huy06]: Huybrechts, D. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr. Oxford: Clarendon Press, 2006

Goal: understand the Bridgeland stability and wall crossing in this toy example.

1. equivalent definitions of stability conditions
2. structure of $\mathrm{Coh}(\mathbb{P}^1)$
3. standard stability
4. exceptional stability.
5. $\mathrm{Stab}(\mathbb{P}^1)$

1. equivalent definitions of stability conditions

Def (locally finite stability condition)

Fix a triangular category $\mathcal{T}$, and denote $K(\mathcal{T})$ as the Grothendieck gp of $\mathcal{T}$.
The set of locally finite stability conditions is defined as

$$\mathrm{Stab}(\mathcal{T}) = \left\{ (Z,\mathcal{P}) \;\middle|\; \begin{array}{ll} Z: K(\mathcal{T}) \longrightarrow \mathbb{C} & \text{(central charge)} \\ \mathcal{P}: \mathbb{R} \longrightarrow \{\text{full additive subcategories of } \mathcal{T}\} & \\ \quad\; \phi \longmapsto \mathcal{P}(\phi) & \text{(slicing)} \\ \text{s.t. } (a)(b)(c)(d) + (e) & \end{array} \right\}$$

(a) (slicing compatible with central charge)
if $E \in \mathcal{P}(\phi)$ then $\dfrac{Z(E)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)
$$\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for } A_j \in \mathcal{P}(\phi_j),\ \phi_1 > \phi_2$$

(d) (HN filtration) HN = Harder-Narashimhan
$\forall E \in \mathcal{T}$, $\exists$ finite seq of real numbers
$$\phi_1 > \phi_2 > \cdots > \phi_n$$
and a filtration
$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\ A_1 \arrow[u, dashed, "+1"] & & & A_n \arrow[u, dashed, "+1"] & \end{tikzcd}$$
s.t. $A_j \in \mathcal{P}(\phi_j)$ $\forall j$.
we define $\phi(E) = \{\phi_1, \ldots, \phi_n\}$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{P}(I)$, $\exists$ a Jordan-Hölder filtration with finite length.
$\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\text{extension-closed}}$

Rmk. For $E \in \mathcal{T}$, $E \neq 0$,
$E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$
$\Leftrightarrow$ the HN filtration of E has length 1
$\overset{\text{def}}{\Longleftrightarrow}$ E is semistable

When E is semistable, define $\phi(E) = \phi$ when $E \in \mathcal{P}(\phi)$.

Lemma 1.1. $\mathcal{P}(\phi)$ is closed under extension.

Proof. Suppose one has one triangle

$$A_1 \longrightarrow E \longrightarrow A_2 \xrightarrow{+1} \tag{1.1}$$

where $A_1, A_2 \in \mathcal{P}(\phi_0)$, we want to show $E \in \mathcal{P}(\phi_0)$.

Suppose $\phi(E) = \{\phi_1, \ldots, \phi_n\}$, $\phi_1 > \cdots > \phi_n$, $n \geq 1$, then

$$\phi_0 > \phi_n \quad \text{or} \quad \phi_1 > \phi_0 \quad \text{or} \quad (\phi_1 = \phi_0,\ n = 1)$$

$$\Downarrow$$

$$E \in \mathcal{P}(\phi_0) \ \checkmark$$

w.l.o.g. assume $\phi_0 > \phi_n$, then $\exists$ triangle

$$B_1 \longrightarrow E \xrightarrow{u} B_2 \xrightarrow{+1}$$

where $u \neq 0$, $B_2 \in \mathcal{P}(\phi_n)$.

Apply $\mathrm{Hom}(-, B_2)$ to (1.1), we get

$$\begin{array}{ccccc}
\mathrm{Hom}(A_1[-1], B_2) & \longleftarrow & \mathrm{Hom}(E[-1], B_2) & \longleftarrow & \mathrm{Hom}(A_2[-1], B_2) \\
\mathrm{Hom}(A_1, B_2) & \longleftarrow & \mathrm{Hom}(E, B_2) & \longleftarrow & \mathrm{Hom}(A_2, B_2) \\
\shortparallel & & \rotatebox{90}{$\in$} & & \shortparallel \\
0 & & u \neq 0 & & 0
\end{array}$$

Contradiction!

Rmk. [Bri07, Lemma 5.2]
$\mathcal{P}(\phi)$ is an abelian category.

Def (stable sheaf)
Suppose $E \in \mathcal{P}(\phi)$ is semistable.

$$E \text{ is stable} \iff E \in \mathcal{P}(\phi) \text{ is simple}$$

$$\Uparrow$$

$$E \in \mathcal{P}(I) \text{ is simple for some } I \ni \phi$$

The next lemma conclude the behavior of triangles with stability conditions.

Lemma 1.2.

Suppose

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow[+1]{u_3} \qquad (1.2)$$

is a triangle, where $\phi(A_1) = \phi_0$, $\phi(A_2) = \phi_0'$.

(1) If $\phi_0 > \phi_0'$, then

(1.2) is the HN-filtration, so E is not semistable;

(2) If $\phi_0 = \phi_0'$, then $E \in \mathcal{P}(\phi_0)$ by Lemma 1.1;

(3) If $u_3 \neq 0$, then $\widetilde{\phi_0} \leq \phi_0 + 1$.

Rmk.

$$\mathrm{Stab}(\mathcal{T}) \triangleq \left\{ (Z,\phi) \;\middle|\; \begin{array}{l} Z: K(\mathcal{T}) \longrightarrow \mathbb{C} \qquad \text{(central charge)} \\ \phi: \mathcal{T} \longrightarrow \{\text{finite subsets of } \mathbb{R}\} \\ \qquad E \longmapsto \{\phi_1,\dots,\phi_n\} \quad \text{(slicing)} \\ \text{s.t.} \quad (a)(b)(c)(d) + (e) \end{array} \right\}$$

$E \in \mathcal{T}$ is semistable $\overset{\text{def}}{\Longleftrightarrow}$ $\#\phi(E) = 1$

(a) (slicing compatible with central charge)
For E semistable, $\dfrac{Z(E)}{e^{i\pi\phi(E)}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)
$$\phi(E[1]) = \phi(E) + 1$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for} \quad \phi(A_1) > \phi(A_2), \quad A_1, A_2 \text{ semistable}$$

(d) (HN filtration)
$\forall E \in \mathcal{T}$, denote $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 < \cdots < \phi_n$,

$\exists!$ filtration
$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed, "+1"] & & & & A_n \arrow[ul, dashed, "+1"] &
\end{tikzcd}$$
s.t. $\phi(A_i) = \phi_i \quad \forall i$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{T}$ with $\phi(E) \subset I$,
$\exists$ a Jordan-Hölder filtration with finite length.

Prop [Okada 05, Prop 2.3]

$$\mathrm{Stab}(\mathcal{T}) \cong \left\{ (\mathcal{A}, Z) \;\middle|\; \begin{array}{l} \mathcal{A} : \text{heart of } \mathcal{T} \\ Z : K(\mathcal{A}) \longrightarrow \mathbb{C} \\ \quad \text{centered slope-function} \\ \quad \text{with HN property} \end{array} \right\}$$

$$\begin{array}{ccc} (Z, \mathcal{P}) & \longmapsto & (\mathcal{P}((0,1]), Z) \\ (Z, \mathcal{P}) & \longleftarrow\!\!\shortmid & (\mathcal{A}, Z) \end{array}$$

where $\mathcal{P}(\phi) = \{ E \in \mathcal{A} \text{ semistable} \mid \widetilde{\phi}(E) = \phi \} \quad \forall \phi \in (0,1]$

$\widetilde{\phi}(E) = \frac{1}{\pi} \arg Z(E) \in (0,1]$

$E \in \mathcal{A}$ semistable: $\nexists$ dec $0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$ s.t.

$\phi(A_1) > \phi(E) > \phi(A_2)$

2. structure of $\mathrm{Coh}(\mathbb{P}^1)$

Lemma 2.1. On $\mathbb{P}^1$, we have SESs

$$0 \longrightarrow \mathcal{O} \xrightarrow{\times x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(n)^{\oplus n+1} \longrightarrow \mathcal{O}(n+1)^{\oplus n} \longrightarrow 0 \qquad n \geq 0 \qquad (2.1)$$

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{O}(n) \longrightarrow 0 \qquad n \geq 0$$

which induces triangles

$$\mathcal{O}(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}(k)[1] \xrightarrow{+1}$$

$$\mathcal{O}(k+1)^{\oplus k-n}[-1] \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus k-n+1} \xrightarrow{+1} \qquad n \leq k \qquad (2.2)$$

$$\mathcal{O}(k+1)^{\oplus n-k} \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus n-k+1}[1] \xrightarrow{+1} \qquad n \geq k$$

Lemma 2.2. On $\mathbb{P}^1$, we have

$$\mathrm{RHom}(\mathcal{O}, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

$$\mathrm{RHom}(\mathcal{O}, k_p) = \mathbb{C}$$

$$\mathrm{RHom}(k_p, \mathcal{O}) = \mathbb{C}[-1]$$

$$\mathrm{RHom}(k_p, k_q) = \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1], & p = q \\ 0, & p \neq q \end{cases}$$

Sketch of proof

$$\mathrm{RHom}(\mathcal{O}, \mathcal{O}(n)) = H^{\bullet}(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

Then apply $\mathrm{RHom}(\mathcal{O}, -)$, $\mathrm{RHom}(-, \mathcal{O})$, $\mathrm{RHom}(-, k_q)$ to

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow k_p \longrightarrow 0$$

$\square$

The next two lemmas tells us the structure of Coh(P^1).

Lemma 2.3. [GKR03, last line in p16]

$$\forall \mathcal{F} \in Coh(\mathbb{P}^1), \quad \mathcal{F} = \left(\bigoplus_p \Xi_p\right) \oplus \left(\bigoplus_i \mathcal{O}(n_i)\right)$$

finite many

Ξ_p: a torsion sheaf supported at p

Lemma 2.4. [GKR03, Prop 6.3]

$$\forall \mathcal{F}^{\cdot} \in \mathcal{D}^b(Coh(\mathbb{P}^1)), \quad \mathcal{F}^{\cdot} = \bigoplus_i A_i[-i] \quad A_i \in Coh(\mathbb{P}^1)$$

It also works for $\mathcal{D}^b(\mathcal{A})$ where $\operatorname{gldim} \mathcal{A} = 1$.

E.g. 2.5. Since $\mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n)) \cong \mathrm{Ext}^1(k_p,\mathcal{O}) \oplus \mathrm{Ext}^1(k_p,\mathcal{O}(n)) \cong \mathbb{C}^2$, let us describe the extension

$$0 \longrightarrow \mathcal{O}\oplus\mathcal{O}(n) \longrightarrow E \longrightarrow k_p \longrightarrow 0$$

crspd to $(k_1,k_2) \in \mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n))$.

For simplicity, assume that $n>0$ & $k_1,k_2 \neq 0$.

It is defined as pulling back SES:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & k_p \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "\begin{pmatrix} k_1 z & \\ & k_2 z \end{pmatrix}"] & \mathcal{O}(1)\oplus\mathcal{O}(n+1) \arrow[r] & k_p\oplus k_p \arrow[r] & 0
\end{tikzcd} \tag{2.3}$$

Since $\deg E = n+1$, $\mathrm{rank}\, E = 2$, by Lemma 2.4. we get

$$E = \mathcal{O}\oplus\mathcal{O}(n+1), \quad \mathcal{O}(1)\oplus\mathcal{O}(n) \quad \text{or} \quad \mathcal{O}\oplus\mathcal{O}(n)\oplus k_p$$

but which?

We apply $R\mathrm{Hom}(-,\mathcal{O})$ to (2.3):

$$\begin{tikzcd}
0 & \overset{\mathbb{C}^{n-1}}{\overset{\parallel}{\mathrm{Ext}^1(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})}} \arrow[l] & \mathrm{Ext}^1(E,\mathcal{O}) \arrow[l] & \overset{\mathbb{C}}{\overset{\parallel}{\mathrm{Ext}^1(k_p,\mathcal{O})}} \arrow[l] \\
& \underset{\mathbb{C}}{\underset{\parallel}{\mathrm{Hom}(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})}} \arrow[urr, "k_1"] & \mathrm{Hom}(E,\mathcal{O}) \arrow[l] & \underset{0}{\underset{\parallel}{\mathrm{Hom}(k_p,\mathcal{O})}} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$\Rightarrow R\mathrm{Hom}(E,\mathcal{O}) = \mathbb{C}^{n-1}[-1]$

$\Rightarrow E \cong \mathcal{O}(1)\oplus\mathcal{O}(n)$.

Q: How to determine (2.3) completely?

We have

$$0 \longrightarrow E \longrightarrow \mathcal{O}(1)\oplus\mathcal{O}(n+1)\oplus k_p \longrightarrow k_p \oplus k_p \longrightarrow 0$$

w.l.o.g assume $p=0$, by pulling back to local charts of $\mathbb{P}^1$, we get

$$0 \longrightarrow \underset{\underset{k[z_3]\oplus k[z_4]}{\parallel}}{E|_{\mathbb{A}^1_z}} \longrightarrow k[z_1]\oplus k[z_2]\oplus k[\Delta]/_{\Delta} \longrightarrow k[z_1]/_{(z_1)} \oplus k[z_2]/_{(z_2)} \longrightarrow 0$$

$$0 \longrightarrow \underset{\underset{k[w_1]\oplus k[w_2]}{\parallel}}{E|_{\mathbb{A}^1_w}} \xrightarrow{\cong} k[w_1]\oplus k[w_2] \longrightarrow 0 \longrightarrow 0$$

$$(f(z),\ g(z),\ a) \longmapsto (f(0)+a,\ g(0)+a)$$

$$(f_1(z), f_2(z)) \longmapsto (f_1(z),\ f_1(z)+zf_2(z),\ -f_1(0))$$

$$(f(z),\ g(z),\ a) \overset{T}{\dashrightarrow} (w f(\tfrac{1}{w}),\ w^{n+1} g(\tfrac{1}{w}))$$

Then transition map is given by

$$E|_{\mathbb{A}^1_z} \dashrightarrow E|_{\mathbb{A}^1_w} \qquad (f_1(z), f_2(z)) \dashmapsto (w f_1(\tfrac{1}{w}),\ w^{n+1} f_1(\tfrac{1}{w}) + w^n f_n(\tfrac{1}{w}))$$

i.e.

$$\begin{pmatrix} g_1(w) \\ g_2(w) \end{pmatrix} = \begin{pmatrix} w & \\ w^{n+1} & w^n \end{pmatrix} \begin{pmatrix} f_1(\frac{1}{w}) \\ f_2(\frac{1}{w}) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} h_1(w) \\ h_2(w) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} g_1(w) \\ g_2(w) - w^n g_1(w) \end{pmatrix} = \begin{pmatrix} w & \\ & w^n \end{pmatrix} \begin{pmatrix} f_1(\frac{1}{w}) \\ f_2(\frac{1}{w}) \end{pmatrix}$$

$\Rightarrow E \cong \mathcal{O}(1)\oplus\mathcal{O}(n)$, and (2.3) is

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "{\begin{pmatrix} k_1 z & \\ -k_1 z^n & k_2 \end{pmatrix}}"] \arrow[d, equal] & \mathcal{O}(1)\oplus\mathcal{O}(n) \arrow[r, "{(ev_p,\, 0)}"] \arrow[d, "{\begin{pmatrix} 1 & \\ z^n & z \end{pmatrix}}"] & k_p \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "{\begin{pmatrix} k_1 z & \\ & k_2 z \end{pmatrix}}"] & \mathcal{O}(1)\oplus\mathcal{O}(n+1) \arrow[r] & k_p\oplus k_p \arrow[r] & 0
\end{tikzcd}$$

Shorthand: $\mathrm{Stab}(X) := \mathrm{Stab}(\mathcal{D}^b(\mathrm{Coh}(X)))$ for any variety X.

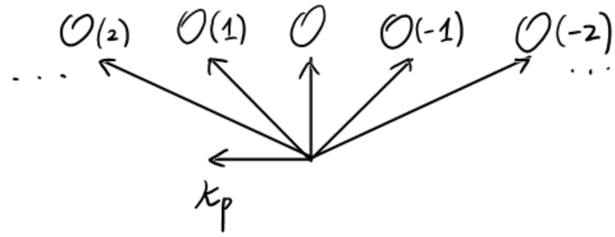
3. standard stability

Goal: classify all stability conditions on $\mathbb{P}^1$
where all the l.b.s and k_p's are semistable
($\Rightarrow$ all torsion sheaves are semistable)

E.g. Consider the slope stability $(Z_0, \mathcal{P}_0)$:

$$Z_0(E) = -\deg E + i \cdot rk(E) \quad \text{e.p.}$$

$$Z_0(\mathcal{O}(n)) = -n+i \qquad \phi(\mathcal{O}(n)) = -\tfrac{1}{\pi}\arg(-n+i)$$
$$Z_0(k_p) = -1 \qquad \phi(k_p) = -1$$



Def: $\mathbb{C}$ acts on $\mathrm{Stab}(\mathbb{P}^1)$ via rotating the Z-plane:

$$\mathbb{C} \times \mathrm{Stab}(\mathbb{P}^1) \longrightarrow \mathrm{Stab}(\mathbb{P}^1) \qquad z = x+iy$$
$$z.(Z, \mathcal{P}) = (e^z Z, \mathcal{P}(\cdot - \tfrac{y}{\pi}))$$
$$z.(Z, \phi) = (e^z Z, \phi(\cdot) - \tfrac{y}{\pi})$$

Rmk 3.3. This action changes the heart but preserve the (semi)stability of sheaves, i.e.,

$$E \text{ is (semi)stable in } (Z, \mathcal{P}) \iff E \text{ is (semi)stable in } z.(Z, \mathcal{P}).$$

Now denote

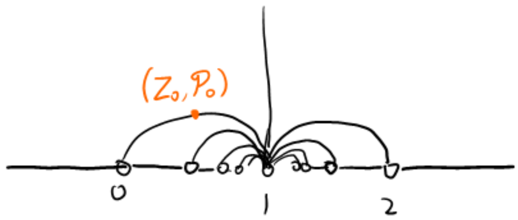
$$\mathrm{Stab}_{st}(\mathbb{P}^1) \overset{\text{def}}{=} \{(Z,\mathcal{P}) \in \mathrm{Stab}(\mathbb{P}^1) \mid \text{all } \mathcal{O}(n) \ \& \ k_p\text{'s semistable}\}$$

↖standard

by Rmk 3.3, it is a $\mathbb{C}$-fiber bundle over

$$\mathrm{Stab}_{st'}(\mathbb{P}^1) \overset{\text{def}}{=} \{(Z,\mathcal{P}) \in \mathrm{Stab}_{st}(\mathbb{P}^1) \mid Z(\mathcal{O}(-1)) = 1, \phi(\mathcal{O}(-1)) = 0\}$$

Prop.
$$\begin{aligned} \mathrm{Stab}_{st'}(\mathbb{P}^1) &\cong \mathcal{H} \sqcup \mathbb{R} - \left\{ \{1 \pm \tfrac{1}{n} \mid n \in \mathbb{N}_{>0}\} \sqcup \{1\} \right\} \triangleq \mathcal{H}_{basic} \\ (Z,\mathcal{P}) &\longmapsto Z(\mathcal{O}) \end{aligned}$$



$\mathcal{H}_{basic}$

Proof Step 1. $Z(\mathcal{O}) \in \mathcal{H}_{basic}$.

$\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong \mathbb{C}^2 \neq 0 \quad \Rightarrow \quad \phi(\mathcal{O}) \geq 0$

$\mathcal{O} \longrightarrow k_p \longrightarrow \mathcal{O}(-1)[1] \xrightarrow{+1}$ is not an HN-filtration $\quad \Rightarrow \quad \phi(\mathcal{O}) \leq 1$

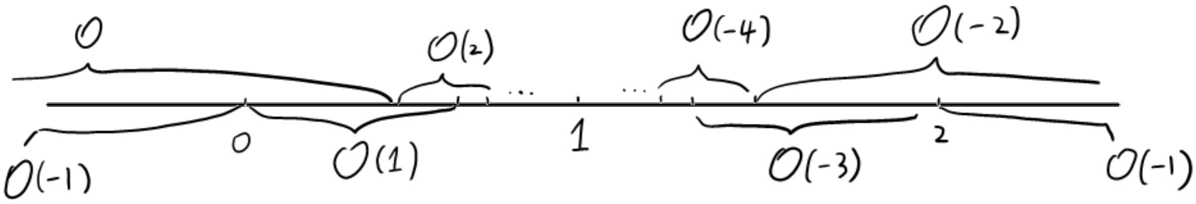
$0 \neq Z(\mathcal{O}(n)) = (n+1) Z(\mathcal{O}) - n Z(\mathcal{O}(-1)) \quad \Rightarrow Z(\mathcal{O}) \notin \{1 \pm \frac{1}{n} \mid n \in \mathbb{N}_{>0}\} \sqcup \{1\}$

Step 2. For each $z \in \mathcal{H}_{basic}$, construct ! $(Z,\mathcal{P}) \in \mathrm{Stab}_{st'}(\mathbb{P}^1)$ s.t. $Z(\mathcal{O}) = z$.

Take
$$\begin{aligned} Z(\mathcal{O}(n)) &= (n+1)z - n \\ Z(k_p) &= z - 1, \end{aligned}$$
that ! determines $(Z,\mathcal{P}) \in \mathrm{Stab}_{st'}(\mathbb{P}^1)$

Rmk. Assume $(Z,\mathcal{P}) \in \mathrm{Stab}_{st}(\mathbb{P}^1)$.

When $Z(\mathcal{O}) \in \mathcal{H}$, all $\mathcal{O}(n)$ & k_p are stable;
when $Z(\mathcal{O}) \in (-\infty, 0)$, only $\mathcal{O}$ & $\mathcal{O}(-1)$ are stable.
In general,



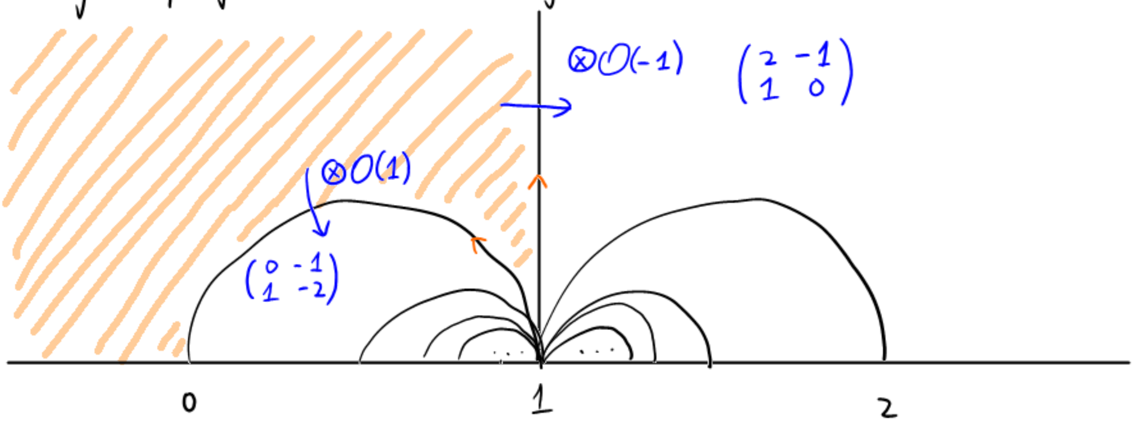
Def. $\mathbb{Z}$ acts on $\mathrm{Stab}(\mathbb{P}^1)$ by tensoring with $\mathcal{O}(-n)$.

$$\begin{aligned} \mathbb{Z} \times \mathrm{Stab}(\mathbb{P}^1) &\longrightarrow \mathrm{Stab}(\mathbb{P}^1) \\ n \cdot (Z, \mathcal{P}) &= \left(Z(- \otimes \mathcal{O}(-n)),\ n * \mathcal{P}\right) \end{aligned}$$

$$\mathrm{Ob}((n*\mathcal{P})(\phi)) = \{E \in \mathcal{T} \mid E \otimes \mathcal{O}(-n) \in \mathcal{P}(\phi)\}$$

$$n \cdot (Z, \phi) = \left(Z(- \otimes \mathcal{O}(-n)),\ \phi(- \otimes \mathcal{O}(-n))\right)$$

Rmk. $\mathrm{Stab}_{st}(\mathbb{P}^1)$ has a fundamental domain w.r.t. $\mathbb{Z}$-action. After projection, it is as follows:



4. exceptional stability.

Fix a stability condition $(Z, \mathcal{P})$ s.t.
some l.b. or k_p is not semistable.
Denote it as E, then we get

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow{+1} \qquad (4.1)$$

with A_2 semistable, $\mathrm{Hom}_{\mathcal{T}}(A_1[n], A_2) = 0 \quad \forall n \geq 0$.

Lemma 4.1. [GKR03, Rmk 6.8]
Assume that we have a triangle

$$A_1 \longrightarrow E \xrightarrow{(f,0)} B_1 \oplus B_2 \xrightarrow{+1} \qquad B_2 \neq 0$$

then $A_1 \cong C \oplus B_2[-1]$, $\mathrm{Hom}_{\mathcal{T}}(A_1[1], B_1 \oplus B_2) \neq 0$.

Proof. try

$$\begin{array}{ccccccc} E & \xrightarrow{f} & B_1 & \dashrightarrow & C & \xrightarrow{+1} \\ 0 & \longrightarrow & B_2 & \longrightarrow & B_2 & \xrightarrow{+1} \\ \rightsquigarrow \quad E & \xrightarrow{(f,0)} & B_1 \oplus B_2 & \longrightarrow & C \oplus B_2 & \xrightarrow{+1} \end{array}$$

Similarly, if it is $B_3 \oplus B_4 \xrightarrow{\binom{f}{0}} E \longrightarrow A_2 \xrightarrow{+1} \qquad B_4 \neq 0$

then $\mathrm{Hom}_{\mathcal{T}}((B_3 \oplus B_4)[1], A_2) \neq 0$

Therefore, we can assume (4.1) are of form

$$A_1 \longrightarrow \mathcal{O} \longrightarrow \bigoplus_{n_i \geq 0} \mathcal{O}(n_i) \oplus \bigoplus_{m_i \leq -2} \mathcal{O}(m_i)[1] \oplus \bigoplus_p \Xi_p \xrightarrow{+1}$$

$$A_1 \longrightarrow k_p \longrightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \oplus \Xi_p' \oplus \Xi_p''[1] \xrightarrow{+1}$$

Lemma 4.2. [Okada 06, Lemma 3.1 (c)]

If $\mathrm{Hom}_{\mathcal{T}}(A_1[n], A_2) = 0 \quad \forall n \geq 0$ & A_2 semistable,

$\forall \phi_i \in \phi(A_1),\ \phi_i < \phi(A_2)$ (i.e. HN filtration condition)

then (4.1) are of form (2.2).

Check it! E.g. 2.5 is one example to determine A_1. It seems easy but turns out to be extremely hard. There are too many cases to discuss.

Here is a try:

$$k_p \longrightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \oplus \Xi_p' \oplus \Xi_p''[1] \longrightarrow A_1[1] \xrightarrow{+1}$$

Apply $\mathrm{Hom}(-, \mathcal{O}(k))$:

$$\begin{array}{l}
\underbrace{\mathrm{Hom}(k_p[-3], \mathcal{O}(k))}_{=0} \longleftarrow \underbrace{\mathrm{Hom}(A_2[-3], \mathcal{O}(k))}_{=0} \longleftarrow \mathrm{Hom}(A_1[-2], \mathcal{O}(k)) \\
\underbrace{\mathrm{Hom}(k_p[-2], \mathcal{O}(k))}_{=0} \longleftarrow \mathrm{Hom}(A_2[-2], \mathcal{O}(k)) \longleftarrow \mathrm{Hom}(A_1[-1], \mathcal{O}(k)) \\
\underbrace{\mathrm{Hom}(k_p[-1], \mathcal{O}(k))}_{=\mathbb{C}} \longleftarrow \mathrm{Hom}(A_2[-1], \mathcal{O}(k)) \longleftarrow \mathrm{Hom}(A_1, \mathcal{O}(k)) \\
\underbrace{\mathrm{Hom}(k_p, \mathcal{O}(k))}_{=0} \longleftarrow \underbrace{\mathrm{Hom}(A_2, \mathcal{O}(k))}_{=0} \longleftarrow \mathrm{Hom}(A_1[1], \mathcal{O}(k))
\end{array}$$

$\Rightarrow \mathrm{Hom}(A_1[n], \mathcal{O}(k)) = 0 \quad \forall n \neq 0, -1$

$\Rightarrow A_1[1]$ is of form $\bigoplus_{\tilde{n}_i} \mathcal{O}(\tilde{n}_i)[1] \oplus \tilde{\Xi}_p' \oplus \tilde{\Xi}_p''[1]$

Notice that $\phi(\Xi_p') = \phi(\Xi_p'') = \cdots \phi(\kappa_p)$.

If $\Xi_p' \neq 0$, then Ξ_p' is semistable $\Rightarrow \kappa_p$ is semistable, contradiction!

So $\Xi_p' = 0$.

Take $k << 0$, $\Rightarrow$ $k < n_i$ & $k < \tilde{n}_i$ $\forall\, n_i, \tilde{n}_i$

one gets

$$\mathbb{C} \longleftarrow \underbrace{\mathrm{Ext}^1(\Xi_p', \mathcal{O}(k))}_{=0} \longleftarrow \mathrm{Ext}^1(\tilde{\Xi}_p', \mathcal{O}(k)) \longleftarrow 0$$

$\Rightarrow \tilde{\Xi}_p' = 0$.

Similarly, $\Xi_p'' = \tilde{\Xi}_p'' = 0$, (4.1) is reduced to

$$\bigoplus_{\tilde{n}_i} \mathcal{O}(\tilde{n}_i) \longrightarrow \kappa_p \longrightarrow \bigoplus_{n_i} \mathcal{O}(n_i)[1] \xrightarrow{+1}$$

the rest is easy.

Idea: ① determine the shape of A,

② verify that κ_p is not semistable.

③ use $\mathrm{Hom}(-, \mathcal{O}(k))$ to control the rest terms.

Lemma 4.3. $\exists\ k$ s.t. $\mathcal{O}(k)$ & $\mathcal{O}(k+1)$ are semistable,
and other l.b. or torsion sheaves are not semistable.

Proof Take

$$E_0 \in \left\{ E \;\middle|\; \begin{array}{l} E = \mathcal{O}(n) \text{ or } k_p \\ E \text{ not semistable} \end{array} \right\}$$

with minimal HN-filtration length, then

$$A_1 \longrightarrow E_0 \longrightarrow A_2 \xrightarrow{+1}$$

is of form (2.2) by Lemma 4.2, so $\mathcal{O}(k)$ is semistable.

Since $\mathcal{O}(k+1)$ has smaller HN-filtration length then E_0, $\mathcal{O}(k+1)$ is semistable.

$\Rightarrow \phi(\mathcal{O}(k+1)) > \phi(\mathcal{O}(k)) + 1$

$\Rightarrow$ all triangles in (2.2) are HN-filtrations

$\Rightarrow$ all other l.b. or torsion sheaves are not semistable.

5. Stab($\mathbb{P}^1$)

Now we can describe Stab($\mathbb{P}^1$) in a relatively satisfied way.
Denote

$$\mathrm{Stab}_{ex,k}(\mathbb{P}^1) = \left\{ (Z,\mathcal{P}) \in \mathrm{Stab}(\mathbb{P}^1) \;\middle|\; \begin{array}{l} \mathcal{O}(k),\mathcal{O}(k+1) \text{ are semistable} \\ x_p \text{ is not} \end{array} \right\}$$

by Rmk 3.3, $\mathrm{Stab}_{ex,-1}(\mathbb{P}^1)$ is a $\mathbb{C}$-fiber bundle over

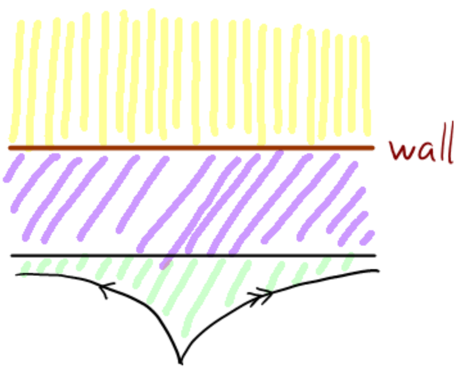
$$\mathrm{Stab}_{ex',-1}(\mathbb{P}^1) \overset{def}{=} \{ (Z,\mathcal{P}) \in \mathrm{Stab}_{ex,-1}(\mathbb{P}^1) \mid Z(\mathcal{O}(-1)) = 1, \phi(\mathcal{O}(-1)) = 0 \}$$

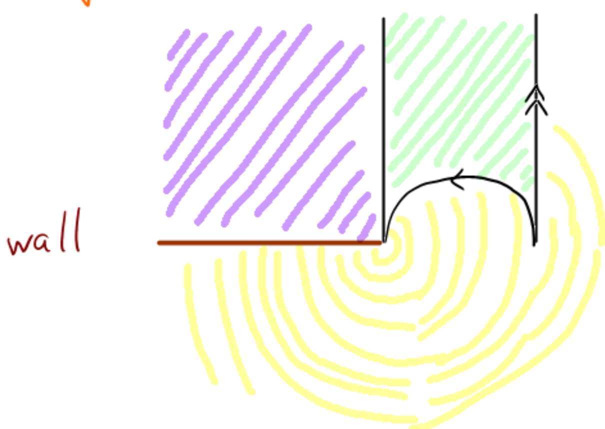
Prop. $\mathrm{Stab}_{ex',-1}(\mathbb{P}^1) \cong \{ x+iy \in \mathcal{H} \mid y > \pi \} \xrightarrow{\exp} \mathbb{C}^\times$

$$(Z,\phi) \longmapsto \ln|Z(\mathcal{O})| + i\pi\phi(\mathcal{O}) \longmapsto Z(\mathcal{O})$$

Proof. Reduce to construct (Z,ϕ) where
$\phi(\mathcal{O}) > 1$ and $\frac{Z(\mathcal{O})}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$.

In conclusion: (A foundamental domain of $\mathrm{Stab}(\mathbb{P}^1)/\mathbb{Z}(\mathbb{C})$)





$\Rightarrow$ $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$.