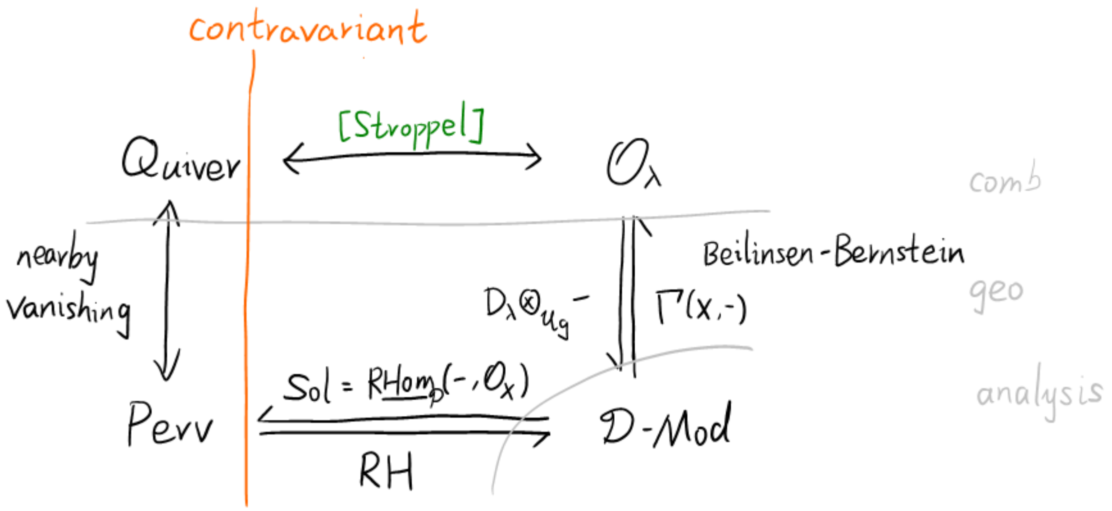
Eine Woche, ein Beispiel

11.10 5 indecomposible representations

This document is the continuation of [2013.11.26]. After the discussion with Renzhi Liang and Aaron, the last piece of the puzzle has been put together.

extra ref:
[Stroppel]: Category O: Quivers and endomorphism rings of projectives
https://www.math.uni-bonn.de/ag/stroppel/Quivers.pdf



$$\begin{tikzcd} \overset{0}{\psi} \arrow[r, bend left, "\mathrm{can}"] & \overset{-2\rho}{\phi} \arrow[l, bend left, "\mathrm{var}"] \end{tikzcd}$$

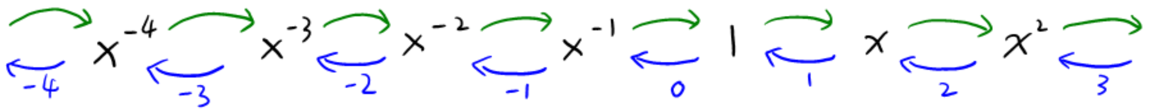
L : irreducible rep
M: Verma module
P: proj rep
I: inj rep

$$\mathrm{var} \circ \mathrm{can} = 0$$

Quiver	$0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{Q}$	$\mathbb{Q} \underset{0}{\overset{0}{\rightleftarrows}} 0$	$\mathbb{Q} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{Q}$	$\mathbb{Q} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{Q}$	$\mathbb{Q} \underset{(1,0)}{\overset{\binom{0}{1}}{\rightleftarrows}} \mathbb{Q}^2$
filtration					
Perv	$i_* \underline{\mathbb{Q}}_{\{\infty\}}$	$\underline{\mathbb{Q}}_{\mathbb{CP}^1}[1]$	$Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	$j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	
alias	IC_0	IC_∞	$I(\psi)$	$P(\psi)$	$P(\phi) = I(\phi)$
$\mathcal{D}$-mod	$A_1/A_1 x$	$A_1/A_1 \partial$	$A_1/A_1 x\partial$	$A_1/A_1 \partial x$	$A_1/A_1 x\partial x$
	$k[\partial]$	$k[x]$	$k[\partial, \partial^{-1}]$	$k[x, x^{-1}]$	
$\mathcal{O}_\lambda$	$L(-2\rho)$ $M(-2\rho)$ $M^*(-2\rho)$	$L(0)$	$M(0)$ $P(0)$	$M(0)^*$ $I(0)$ dual	$P(-2\rho) = I(-2\rho)$

Ex. the A_1-module structure of $k[x,x^{-1}]$

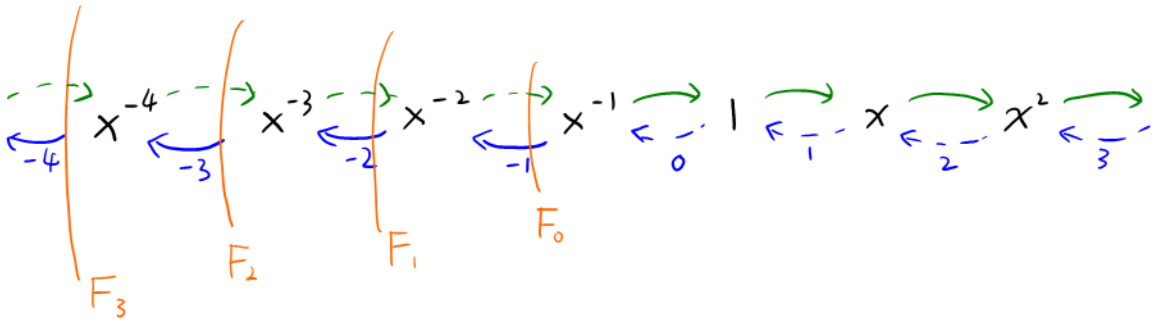
Basis with action:



$\rightarrow$: x-action

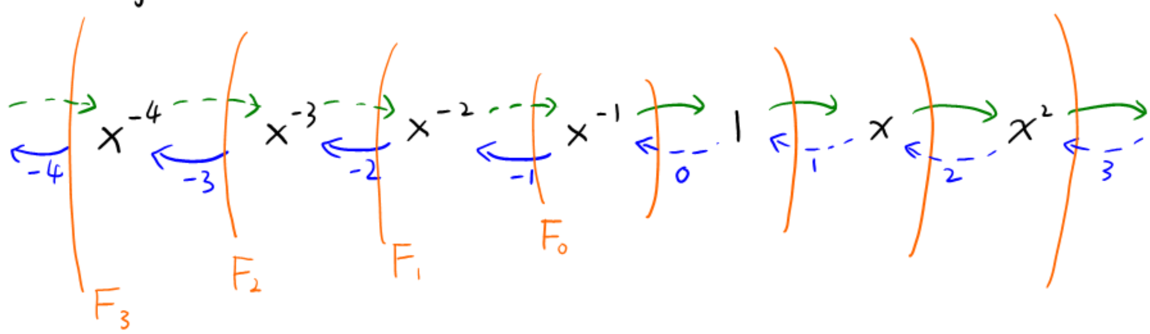
$\rightarrow$: ∂-action

Order filtration:



$$\Rightarrow \operatorname{gr}_{F_\cdot^{ord}} k[x,x^{-1}] = k[x,\partial]/x\partial$$

Bernstein filtration:



$$\Rightarrow \operatorname{gr}_{F_\cdot^{B}} k[x,x^{-1}] = k[x,\partial]/x\partial$$

Since $\partial x \cdot \frac{1}{x} = (x\partial + 1) \cdot \frac{1}{x} = 0$, we get

$$k[x, x^{-1}] \cong A_1 / A_1 \partial x \cong A_1 / A_1 (x\partial + 1)$$

We have a SES:

$$\begin{tikzcd}
0 \arrow[r] & k[x] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & k[x, x^{-1}] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & k[x, x^{-1}] / k[x] \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & 0 \\
0 \arrow[r] & A_1 / A_1 \partial \arrow[r, "\cdot x"] & A_1 / A_1 \partial x \arrow[r] & A_1 / A_1 x \arrow[r] & 0
\end{tikzcd}$$

$$0 \longleftarrow \underline{\mathbb{Q}}_{\mathbb{C}\mathbb{P}^1}[1] \longleftarrow j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1] \longleftarrow i_* \underline{\mathbb{Q}} \longleftarrow 0$$

It does not split.

Restrict to $D_{\mathbb{C}^\times} = k[x, x^{-1}]\langle\partial\rangle$:

$$D_{\mathbb{C}^\times} / D_{\mathbb{C}^\times} \partial x = D_{\mathbb{C}^\times} / D_{\mathbb{C}^\times} \partial = k[x, x^{-1}]$$

$M^*(0)$ | does not extend to $\mathbb{P}^1$

Lie algebra action:

$$\begin{tikzcd}
{} \arrow[r, bend left, "-5"] & x^{-4} \arrow[l, bend left, "4"] \arrow[loop above, "-8"] \arrow[r, bend left, "-4"] & x^{-3} \arrow[l, bend left, "3"] \arrow[loop above, "-6"] \arrow[r, bend left, "-3"] & x^{-2} \arrow[l, bend left, "2"] \arrow[loop above, "-4"] \arrow[r, bend left, "-2"] & x^{-1} \arrow[l, bend left, "1"] \arrow[loop above, "-2"] \arrow[r, bend left, "-1"] & 1 \arrow[l, bend left, dashed, "0"] \arrow[loop above, "0"] \arrow[r, bend left, dashed, "0"] & x \arrow[l, bend left, "-1"] \arrow[loop above, "2"] \arrow[r, bend left, "1"] & x^2 \arrow[l, bend left, "-2"] \arrow[loop above, "4"] \arrow[r, bend left, "2"] & {} \arrow[l, bend left, "-3"]
\end{tikzcd}$$

$\longrightarrow$: ∂_e-action
$\longrightarrow$: ∂_f-action
$\longrightarrow$: ∂_h-action

$$\begin{cases} \partial_e = x^2 \partial \\ \partial_f = -\partial \\ \partial_h = 2x\partial \end{cases}$$

Perverse sheaf interpolation:

$$R\Gamma(\mathbb{A}^1; \mathcal{P}) = R\mathrm{Hom}_{A_1}(k[x, x^{-1}], \mathcal{O}_{\mathbb{A}^1})$$
$$= R\mathrm{Hom}_{A_1}(A_1/A_1\partial x, A_1/A_1\partial)$$
$$= R\mathrm{Hom}_{A_1}([\underset{-1}{A_1} \xrightarrow{\partial x} \underset{0}{A_1}], A_1/A_1\partial)$$
$$= [\underset{1}{A_1/A_1\partial} \xleftarrow{\partial x \cdot} \underset{0}{A_1/A_1\partial}]$$
$$= [\underset{1}{k[x]} \xleftarrow{\partial x \cdot} \underset{0}{k[x]}]$$
$$= 0$$

$$R\Gamma(\mathbb{G}_m, \mathcal{P}) = [\underset{1}{k[x, x^{-1}]} \xleftarrow{\partial x \cdot} \underset{0}{k[x, x^{-1}]}]$$
$$= k \oplus k[-1]$$

For $\mathfrak{p} \in \mathrm{MaxSpec}\, k[x, x^{-1}]$,

$$\mathcal{P}_{\mathfrak{p}} \neq [k[x, x^{-1}]_{\mathfrak{p}} \xleftarrow{\partial x \cdot} k[x, x^{-1}]_{\mathfrak{p}}]$$

$\mathcal{P}_{\mathfrak{p}} = k$?