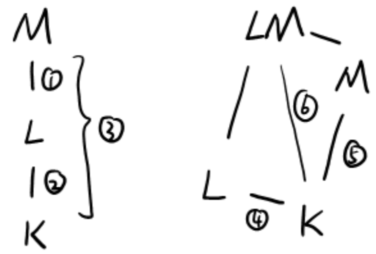


$① \xrightarrow{③\text{ pure ins}} ③$

normal:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
seperable:		① + ② = ③			④ + ⑤ = ⑥
Galois:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
purely inseparable		① + ② = ③			④ + ⑤ = ⑥

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then
F perfect $\Leftrightarrow F^p = F$

$$\begin{tikzcd}
\overline{K} \arrow[d, dash, "\text{closed subgroup}"] \\
L \arrow[d, dash, "\text{quotient group}", "\text{(finite)}"'] \\
K
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{F}_p} \arrow[d, dash, "\mathbb{Z}_l"] \\
\bigcup_{l \nmid a} \mathbb{F}_{p^a} \arrow[d, dash, "\prod_{p \neq l} \mathbb{Z}_p"] \\
\mathbb{F}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{F}_p} \arrow[d, dash, "\prod_{p \neq l} \mathbb{Z}_p"] \\
\bigcup_{n} \mathbb{F}_{p^{l^n}} \arrow[d, dash, "\mathbb{Z}_l"] \\
\mathbb{F}_p
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{\mathbb{F}_p} \arrow[d, dash, "d\hat{\mathbb{Z}}"] \\
\mathbb{F}_q \arrow[d, dash, "\mathbb{Z}/d\mathbb{Z}"] \\
\mathbb{F}_p
\end{tikzcd}$$

$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \quad (q = p^d)$

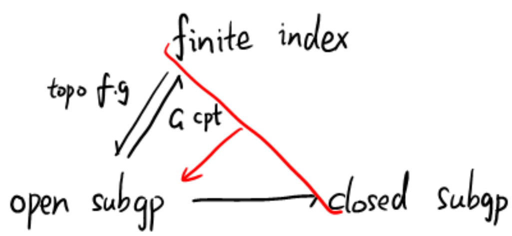
open subgroup $\underset{}{\overset{\{1\} \subseteq \mathbb{Z}_p}{\subseteq}}$ closed subgroup $= \{ Gal(\overline{K}/L) \mid L/k \text{ ext} \} \overset{\mathbb{Z} \subseteq \mathbb{Z}_p}{\subseteq}$ subgroup

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Q: Do we have any finite index gp of $Gal(\overline{k}/k)$ which is not open?

In general,



https://groupprops.subwiki.org/wiki/Closed_subgroup_of_finite_index_implies_open
In a topological group, any closed subgroup of finite index must be an open subgroup.
https://groupprops.subwiki.org/wiki/Open_subgroup_implies_closed
Any open subgroup of a topological group is closed.

https://math.stackexchange.com/questions/4350043/open-subgroup-and-finite-index-subgroup-of-topological-group
https://math.stackexchange.com/questions/1092974/finite-index-subgroup-g-of-mathbbz-p-is-open
https://math.stackexchange.com/questions/3734250/are-normal-subgroups-of-finite-index-in-an-absolute-galois-groups-open

Some wonderful exercises for Galois correspondence:

Let E/F be Galois field ext of deg n, $m \mid n$. prove: $\exists$ subfield ext of deg m.
(Sylow thm & $Z(G) \neq \{1\}$ for a p-gp & classification of f.g. abelian gp)
Cor. For p prime, F field, one can define ${}^p\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^p\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}} \cap \mathbb{R}$)?