The tutorial class for "Math in Phy II"

Hallo everyone.

Before discussing the tasks in the document, let us discuss something unrelated to mathematics.

- Personal information. Course information
  - How to hand in the homework
  - Time schedule (2h-3h, rest: can ask any question, break, finish in advance?)
  - Content
    - e Easy task / Homework? ~~Advanced content~~ topic?
    - videos?
    - g Generalization?
    - a Advanced topics?
    - bird's-eye view?
  - Abbreviations (ask if you don't know!)

~~Ways of course~~

Adv. = advanced.

Eg. = example

Ex. = exercise

w.l.o.g = without loss of generality

i.e. = that means ; equivalently

cplx = complex

mfld = manifold

pt = point

fct = function

~~fctor = functor~~

holo = holomorphic

....

const = constant

Today: Basic knowledge of the cplx plane.

Part I. subsets of $\mathbb{C}$

Part II. fcts on $\mathbb{C}$.

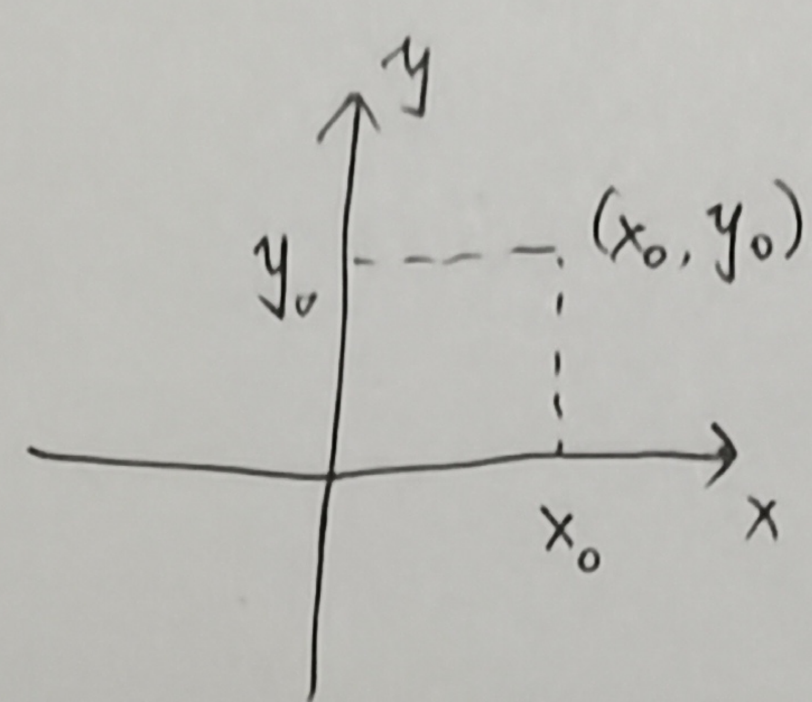
Part III Videos

Part I. subsets of $\mathbb{C}$

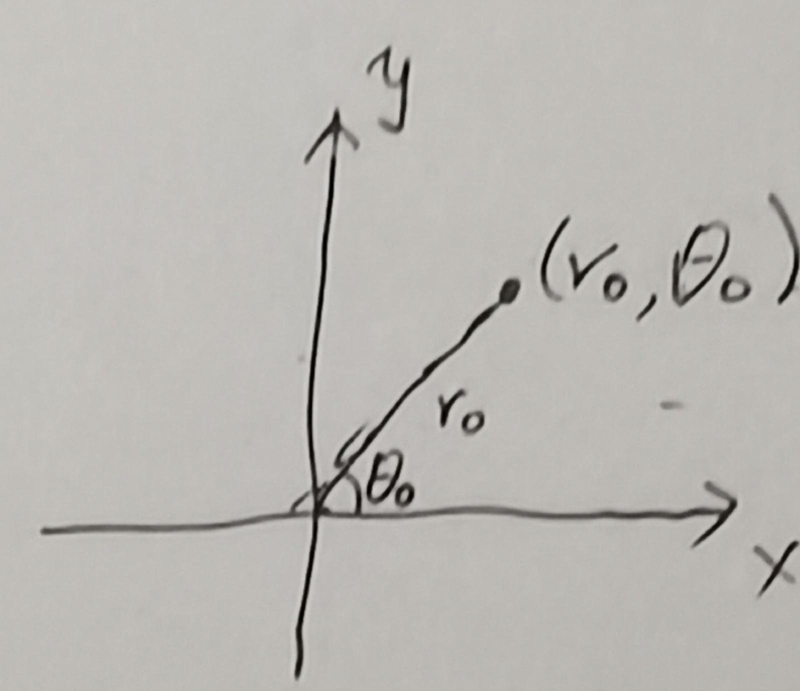
§I.1 ~~Points~~ pts of $\mathbb{C}$

~~Recall th~~

To describe a pt in plane, we have two typical coordinate systems.



Cartesian coordinate system



Polar coordinate system. $\theta_0 \in (-\pi, \pi]$ shortcomings

Now, in the cplx plane, we can use ~~one sympol symbol z_0~~ z_0 to denote a pt (in $\mathbb{C}$).

Define $\operatorname{Re} z_0 = x_0$ $\quad |z_0| = r_0$

$\operatorname{Im} z_0 = y_0$ $\quad \arg z_0 = \theta_0$

Then we have the relationships

$$\begin{tikzcd}
(x_0, y_0) \arrow[rr, shift left, "\begin{cases} r_0 = \sqrt{x_0^2+y_0^2} \\ \theta_0 = \arctan \frac{y_0}{x_0} \end{cases}"] \arrow[dr, shift left, "z_0 = x_0 + iy_0"] & & (r_0, \theta_0) \arrow[ll, shift left, "\begin{cases} x_0 = r_0\cos\theta_0 \\ y_0 = r_0 \sin\theta_0 \end{cases}"] \arrow[dl, shift left, "z_0 = r_0 e^{i\theta_0}"] \\
& z_0 \arrow[ul, shift left, "\begin{cases} x_0 = \operatorname{Re} z_0 \\ y_0 = \operatorname{Im} z_0 \end{cases}"] \arrow[ur, shift left, "\begin{cases} r_0 = |z_0| \\ \theta_0 = \arg z_0 \end{cases}"] &
\end{tikzcd}$$

To facilitate the computation, we define the cplx conjugation.

Def. ~~(conjugation~~ (cplx conjugation). For $z = x + iy$, define $\bar{z} := x - iy$

Ex. Check that

$$\begin{cases} \operatorname{Re} z = \dfrac{z+\bar z}{2} \\ \operatorname{Im} z = \dfrac{z-\bar z}{2i} \end{cases} \qquad \begin{cases} |z| = \sqrt{z\bar z} \\ \arg z \approx \dfrac{1}{2}\log\dfrac{z}{\bar z} \end{cases}$$

$$z^{-1} = \frac{\bar z}{|z|^2} \qquad\qquad = \arctan \frac{1}{i}\frac{z-\bar z}{z+\bar z}$$

Task 1. Compute $\operatorname{Re} z$, $\operatorname{Im} z$, $|z|$, $\arg z$ for

(i) $z = (1-3i)^2$

(ii) $z = \dfrac{5}{3-4i}$

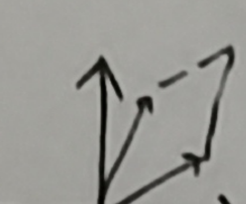
(iii) $z = \left(\dfrac{2-i}{3+2i}\right)^2$

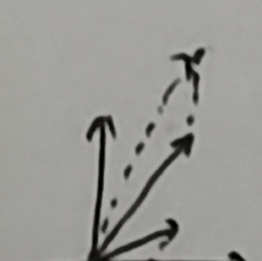
hint. (i) expansion

~~hint.~~ (ii) $\dfrac{5}{3-4i} = \dfrac{5(3+4i)}{(3-4i)(3+4i)} = \cdots$

To verify your calculation, use the formula

$$r_1 e^{i\theta_1} \times r_2 e^{i\theta_2} = r_1 r_2 e^{i\theta_1+\theta_2} \qquad \forall r_i, \theta_i \in \mathbb{R}$$


addition


multiplication.

Adv. ex: define $\mathbb{C} = \mathbb{R}\times\mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}$.

$\cancel{:= \{(x,y) \in \mathbb{R}\times\mathbb{R}\}}$

define $+ : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$

$((x_1,y_1), (x_2,y_2)) \mapsto (x_1+x_2, y_1+y_2)$

$\times : \mathbb{C}\times\mathbb{C} \longrightarrow \mathbb{C}$

$((x_1,y_1), (x_2,y_2)) \mapsto (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$

verifies that $\mathbb{C}$ is a field. Compute $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) := \operatorname{Aut}_{\mathbb{R}\text{-alg}}(\mathbb{C})$.

Ex. show that

$$\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|$$

and discuss when we get equality.

~~Diff~~

~~Adv. ex.~~

Compatability

	$+$	$\times$	$\lvert z \rvert$	$\arg z$	$\bar{z}$
$+$	field		$\lvert z_1+z_2\rvert \le \lvert z_1\rvert+\lvert z_2\rvert$	—	$\overline{z_1+z_2} = \bar{z}_1+\bar{z}_2$
$\times$			$\lvert z_1 z_2\rvert = \lvert z_1\rvert \lvert z_2\rvert$	$\arg(z_1 z_2) \equiv \arg z_1 + \arg z_2 \pmod{2\pi}$	$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
	field		norm on $\mathbb{C}$	—	$(z \mapsto \bar{z}) \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$

Task 3.

Adv. ex. [Ahlfors, P8-9, P11]

[Ahlfors, ~~like~~ P10]

1). Prove that

$$\sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 \geq \left|\sum_{i=1}^{n} a_i b_i\right|^2 \qquad \text{(Cauchy's inequality)}$$

$a_i, b_i \in \mathbb{C}$

2) [Ahlfors, P9] Prove that $(a, b \in \mathbb{C},\ \bar{a}b \neq 1)$

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1 \iff |a| = 1 \text{ or } |b| = 1$$

An: ~~Hint~~

$$1 = \left|\frac{a-b}{1-\bar{a}b}\right|^2 = \frac{a-b}{1-\bar{a}b} \cdot \frac{\bar{a}-\bar{b}}{1-a\bar{b}} = \frac{|a|^2+|b|^2-\bar{a}b-a\bar{b}}{1+|a|^2|b|^2-\bar{a}b-a\bar{b}}$$

$$\iff |a|^2 + |b|^2 = 1 + |a|^2|b|^2$$

$$\iff (|a|^2-1)(|b|^2-1) = 0$$

$$\iff |a| = 1 \text{ or } |b| = 1$$

Task 3. For a sequence $\{z_n\}_{n\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}$ and $z \in \mathbb{C}$, denote

~~Let~~ $z_n = r_n e^{i\varphi_n}$ $\quad r_n \in [0, +\infty) \quad \varphi_n \in [0, 2\pi)$

$z = r e^{i\varphi}$ $\quad r \in [0, +\infty) \quad \varphi \in [0, 2\pi)$

Shows that

$$\lim_{n\to+\infty} z_n = z \iff \lim_{n\to+\infty} r_n = r \text{ and } \begin{cases} \text{no further condition}, & r = 0 \\ \lim\limits_{n\to+\infty} \varphi_n = \varphi & r \neq 0,\ \varphi \neq 0 \\ \lim\limits_{n\to+\infty} \varphi_n - 2\pi a_n = 0 & r = 0,\ \varphi = 0 \end{cases}$$

for some sequence $a_n \in \{0, 1\}$

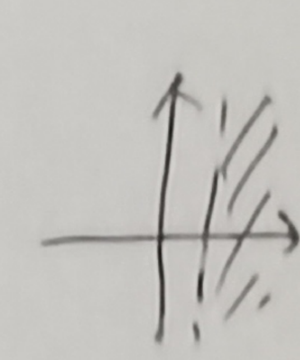
$\Rightarrow$: $|\cdot|$, $\arg : \mathbb{C} - \{z \in \mathbb{C} \mid \operatorname{Im} z = 0,\ \operatorname{Re} z \geq 0\} \to (0, 2\pi)$ are cont.

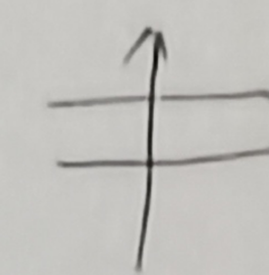
Hint. $\Leftarrow$: $|z_n - z| = |r_n e^{i\varphi_n} - r e^{i\varphi}|$

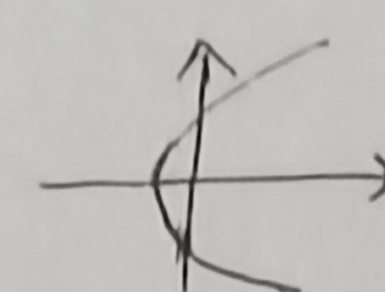
$\leq |r_n e^{i\varphi_n} - r e^{i\varphi_n}| + |r e^{i\varphi_n} - r e^{i\varphi}|$

$= |r_n - r| + r\,|e^{i\varphi_n} - e^{i\varphi}|$

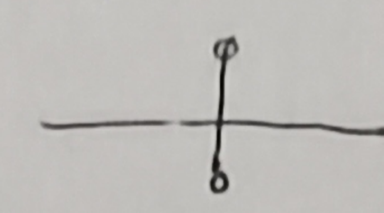
Task 2. Draw the following sets (Fix $c \in \mathbb{R}$)

(i) $\{z \in \mathbb{C} \mid \operatorname{Re} z > c\}$ 

(ii) $\{z \in \mathbb{C} \mid \operatorname{Im} z = c\}$ 

(iii) $\{z \in \mathbb{C} \mid |z| = \operatorname{Re} z + 1\}$ 

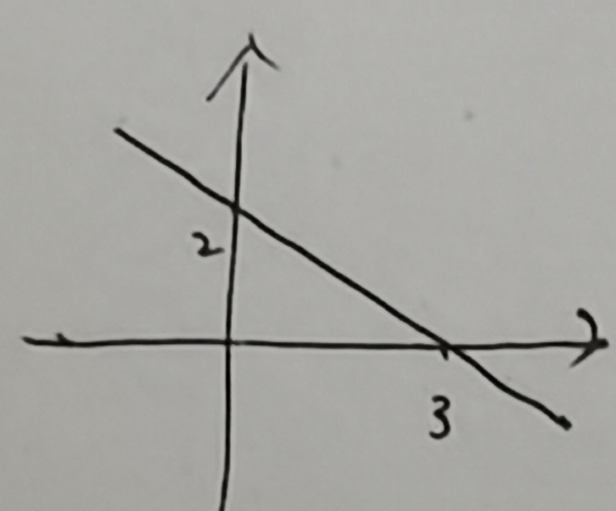
$x^2+y^2 = (x+1)^2$
$y^2 = 2x+1$
$x = \frac{1}{2}y^2 - \frac{1}{2}$

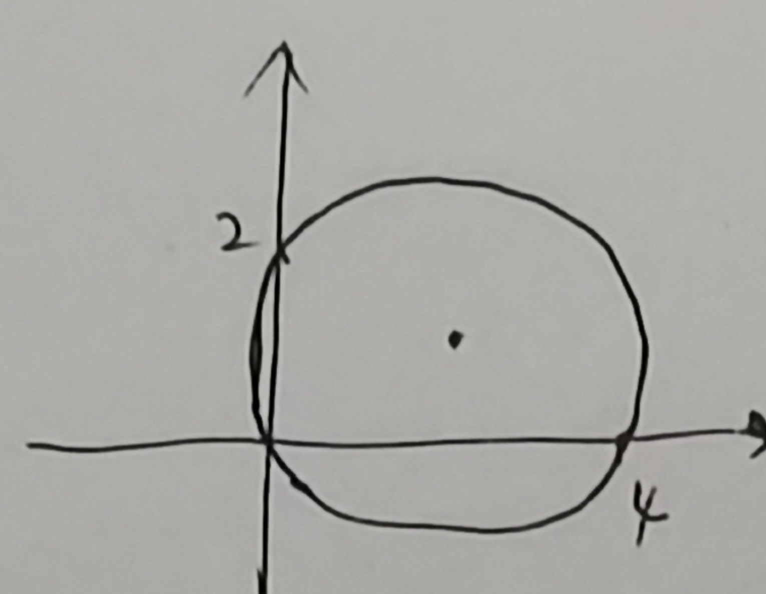
(iv) $\{z \in \mathbb{C} \mid \arg(1+z^2) = 0\}$ 

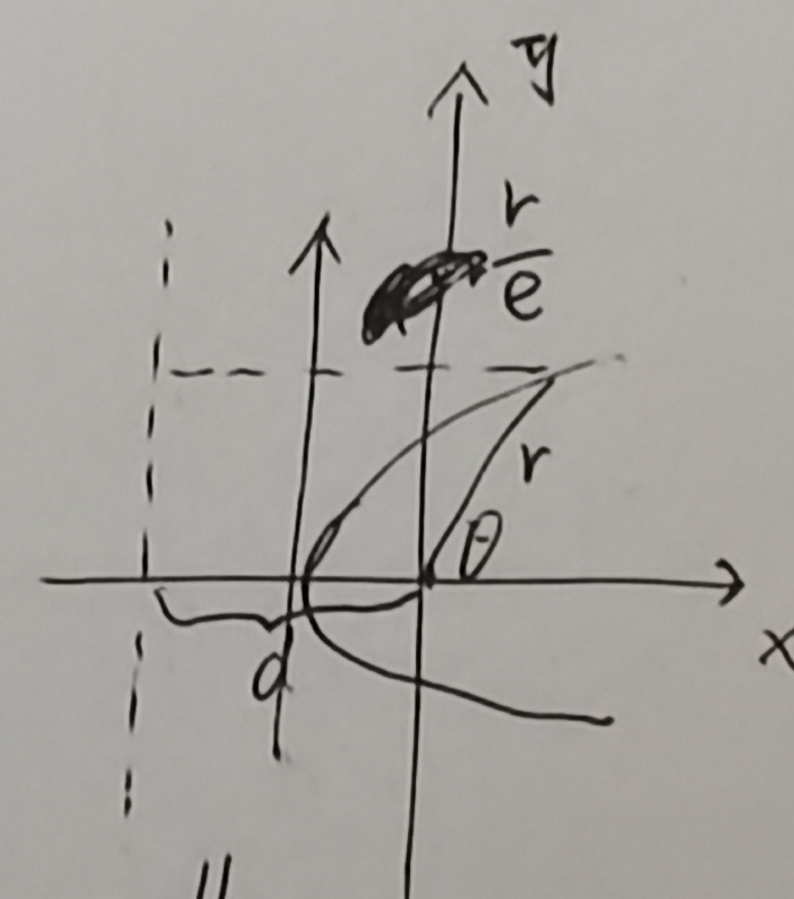
$1+z^2 \in \mathbb{R}_{>0}$
$z^2 \in (-1, +\infty)$

Adv. ex. [Ahlfors, P17]

~~describe~~
Write the equation of







in terms of $z, \bar{z}$

ellipse
parabola
hyperbola

Hint. $r = \frac{de}{1 - e\cos\theta}$

$\Leftrightarrow |z| - e \operatorname{Re} z = de$

$\Leftrightarrow |z| = \frac{e}{2}(2d + z + \bar{z})$

$\Leftrightarrow z\bar{z} = \frac{e^2}{4}(2d + z + \bar{z})^2$

Adv. ex. define a topology on $\mathbb{C}$. (induced by the norm $|\cdot|$)

$\frac{r}{e} - r\cos\theta = d$

Part II fcts on $\mathbb{C}$

§II.1 Overview

holo fct on $\mathbb{C}$

$$\begin{array}{ccccc} \mathbb{C}_z & \subset & \mathbb{C}[z] & \subset & \mathcal{O}(\mathbb{C}) \\ \cap & & \cap & & \cap \\ \mathbb{C}z \oplus \mathbb{C}\bar{z} & \subset & \mathbb{C}[z,\bar{z}] = \mathbb{C}[x,y] & \subset & C^\infty(\mathbb{C}) \subset C^n(\mathbb{C}) \subset \cdots \subset C^1(\mathbb{C}) \subset C(\mathbb{C}) \subset \mathrm{Map}(\mathbb{C},\mathbb{C}) \end{array}$$

Road in this semester

$C^\infty(\mathbb{C})$: sm fct on $\mathbb{C}$; $C(\mathbb{C})$: cont fct

⚠ Please distinguish pts and fcts, even though we may use the same symbol to denote them.

e.g. $z : \mathbb{C} \to \mathbb{C}$ $\quad z_0 \mapsto z_0$ (constant fct)

$\bar{z} : \mathbb{C} \to \mathbb{C}$ $\quad z_0 \mapsto \bar{z}_0$ (conj fct)

are cplx-valued fcts on $\mathbb{C}$, while

$|\cdot| : \mathbb{C} \to \mathbb{R}$ $\quad z_0 \mapsto |z_0|$

$\arg : \mathbb{C} - (-\infty, 0] \to \mathbb{R}$ $\quad z_0 \mapsto \arg(z_0)$

are real-valued fcts on $\mathbb{C}$.

Even though we can realize real-valued fcts as cplx-valued fcts, it is not meaningful for us, because it is not holomorphic fct (unless it is a const)

~~Prop.~~

Thm (Little Picard theorem)

For $f \in \mathcal{O}(\mathbb{C})$ non-constant,

$$\mathbb{C} - \mathrm{Im} f = \phi \text{ or } \{pt\}.$$

On the contrary, the cplx-valued fcts can be written as two real-valued fcts and this is useful. $(f(z) = u(z) + iv(z))$

E.g. $z = x + iy$ $\quad e^z = \cancel{\cos z + i \sin z}$

$\bar{z} = x - iy$ $\quad e^x \cos y + i e^x \sin y$

Adv. ex. write down a basis of

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C},\mathbb{C}) = \mathbb{R}z \oplus \mathbb{R}iz \oplus \mathbb{R}\bar{z} \oplus \mathbb{R}i\bar{z}$$

$$\cup \qquad\qquad \cup$$

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C},\mathbb{C}) = \mathbb{C}z$$

§ II 2.2. Def of $\mathcal{O}(\mathcal{U})$

Now recall the def of holo fct.

Def. Suppose $\mathcal{U} \overset{\text{open}}{\subseteq} \mathbb{C}$. A fct $f: \mathcal{U} \to \mathbb{C}$ is ~~holo~~ cplx differentiable at $z_0 \in \mathcal{U}$, if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \;\;=:\; \overset{\Delta}{=} f'(z_0)$$

exists. ~~Denote~~

~~$f: \mathcal{U} \to \mathbb{C}$ is holo~~

~~$f'(z) = \lim_{z\to z_0}$~~

$f: \mathcal{U} \to \mathbb{C}$ is called a holo fct if f is ~~holo~~ cplx differentiable at $\forall z_0 \in \mathcal{U}$,

and denote

$$\mathcal{O}(\mathcal{U}) := \{\text{holo fct } f: \mathcal{U} \to \mathbb{C}\}$$

Adv. ex.

Verify that $\mathcal{O}(\mathcal{U})$ is a comm ring with unit.

Find out all maximal ideals of $\mathcal{O}(\mathcal{U})$.

(Ring $\approx$ abstraction of fct space)

Ex. Shows that $\bar{z} \notin \mathcal{O}(\mathbb{C})$ by definition. (Pic in wiki)

Hint: let $f(z) = \bar{z}$,

$$\lim_{z\to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{\bar{z}}{z} = \lim_{z\to 0} e^{2i\arg z}$$ does not exist.

Ex. Is the fct $f(z) = |z|^2$ cplx differentiable at $z_0 \in \mathbb{C}$? $f \in \mathcal{O}(\mathbb{C})$?

"Equivalent definition" of holo fct

Thm [Ahlfors, p26]. ~~Let $f(z) = u(x,y) +$~~ ~~$u(z)+iv(z)$~~ U

~~$f : U \to \mathbb{C}$~~ ~~$f(z) = u(z)+iv(z)$~~

Suppose $u(x,y),\ v(x,y) \in C^1(U)$ (i.e. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ $\exists$ & cont),

~~then~~

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{CCR-equation)}$$

then $f(x,y) := u(x,y) + iv(x,y)$ is a holo fct on U.

Check.

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

if f' $\exists$, $f' = \frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y} \Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

[Ahlfors, p27]

Adv. ex. Show that $f \in \mathcal{O}(U)$ ($f = u+iv$) $\Rightarrow$ $\Delta u = \Delta v = 0$, ($\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$)

Recall three type of coordinates. (x,y), (r,θ), $(z,\bar{z})$

Task 4

~~Claim~~. ~~The~~ CR-equation $\iff$ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \varphi}$ $\qquad$ $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \varphi}$

$$\iff \frac{\partial f}{\partial \bar{z}} \equiv 0$$

Computation. $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \Rightarrow \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \qquad \frac{\partial v}{\partial r} = ?$$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

$$\frac{\partial v}{\partial \varphi} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \varphi} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \varphi} \qquad \frac{\partial u}{\partial \varphi} = ?$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta$$

$\begin{cases} z = x+iy \\ \bar{z} = x-iy \end{cases} \Rightarrow \begin{cases} x = \frac{1}{2}(z+\bar{z}) \\ y = \frac{1}{2i}(z-\bar{z}) \end{cases} \Rightarrow \begin{pmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial \bar{z}} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2i} & -\frac{1}{2i} \end{pmatrix}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right)$$

~~Since yo~~

Ex. Shows that $f(z) = |z|^2$ is not holomorphic.

Since you're not familiar with the chain's rule, let's stop here.

Part II Videos.

Q. How can we visualize the ~~l~~ cplx fcts?