

Eine Woche, ein Beispiel

4.17 preliminary facts of representations of p -adic groups

Main reference: The Local Langlands Conjecture for $GL(2)$ by Colin J. Bushnell Guy Henniart.
[<https://link.springer.com/book/10.1007/3-540-31511-X>]

Process.

1. Basic properties
 - Smoothness
 - Irreducibility and unitary
 - Reduction to smaller cardinal.
2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$
3. Construction of new reps.
 - Special sub & quotient rep
 - Duality
 - Ind and c-Ind
 - Other constructions \leftarrow Example: mirabolic group M
4. Hecke algebra
5. Intertwining properties \leftarrow Example: $GL_2(\mathbb{Q}_p)$

1. Basic properties

1.1. Smoothness

G : loc. profinite group

V : cplx v.s.

$$\rho: G \longrightarrow \text{Aut}_{\mathbb{C}}(V) \quad g \mapsto [v \mapsto g.v]$$

Def. (ρ, V) is smooth if

$$\forall v \in V, \exists K \leq G \text{ cpt open s.t. } k.v \equiv v \quad \forall k \in K$$

$\text{Rep}(G) = \{\text{sm rep of } G\}$ is a full subcategory of $\{\text{rep of } G\}$.

Rmk. Any sub/quotient rep of $(\rho, V) \in \text{Rep}(G)$ is smooth.

$$H \leq G \text{ cpt, } (\rho, V) \in \text{Rep}(G) \Rightarrow (\rho|_H, V) \in \text{Rep}(H)$$

Rmk. For fcts, smoothness has a different meaning.

Recall the definition of $C^\infty(G)$ & $C_c^\infty(G)$:

$$C^\infty(G) := \{f: G \rightarrow \mathbb{C} \mid f \text{ is loc. const}\}$$

$$C_c^\infty(G) := \{f \in C^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$$

1.2. Irreducibility and unitary

$$\text{Irr}(G) = \{(\rho, V) \in \text{Rep}(G) \mid \rho \text{ is a irreducible rep}\}$$

$$\hat{G} = \{(\rho, V) \in \text{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$$

$$\stackrel{[P13]}{=} \{\chi: G \rightarrow \mathbb{C}^\times \mid \ker \chi \text{ is open}\}$$

$$\stackrel{[(1.6)]}{=} \{\chi: G \rightarrow \mathbb{C}^\times \mid \chi \text{ is continuous}\}$$

Rmk. The notation is slightly different with the original reference.

Rmk.

$$\hat{G} \subseteq \text{Irr}(G) \subseteq \text{Rep}(G)$$

When G is cpt, or

$G/Z(G)$ is cpt with G/K countable, we get $\text{Ind}(G) = \text{Irr}(G)$;

when G is abelian and G/K is countable, $\text{Ind}(G) = \hat{G}$.

($\exists K \leq G$ cpt open, countable = at most countable here)

Rmk. A more general result is as follows:

Prop. Let $(\rho, V) \in \text{Rep}(G)$, G/K countable. $\exists K \leq G$ cpt open
 Suppose $\rho|_{Z(G)}$ decompose as $Z(G) \xrightarrow{\chi_w} \mathbb{C}^\times \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$.
 Let $Z(G) \leq H \leq G$ $H \leq G$ open $H/Z(G)$ is cpt.
 Then $(\rho|_H, V) \in \text{Rep}(H)$ is semisimple.

To prove this we need the following lemma. (when applied, it would be $K \cdot Z(G) \leq H$)

Lemma. || Let $H \leq G$ open, $[G:H] < \infty$, $(\rho, V) \in \text{Rep}(G)$. Then
 ρ is G -semisimple $\Leftrightarrow \rho|_H$ is H -semisimple.

Def (Action as character)

Let $H \leq G$, $(\rho, V) \in \text{Rep}(G)$, $\chi \in \hat{H}$.

We say H acts on V as χ if $\rho|_H$ decompose as follows:

$$\rho|_H: H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$$

We may denote χ by χ_ρ or χ_H . When $H = Z(G)$, χ is denoted by w_ρ .

Def (Contain irr rep)

Let $H \leq G$, $(\rho, V) \in \text{Rep}(G)$, $(\sigma, W) \in \text{Irr}(H)$.

We say ρ contains σ , or σ occurs in ρ , if

$$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \neq 0$$

i.e., σ can be realized as a quotient subrep of $\text{Res}_H^G \rho$.

Cor. When H acts on V as χ_ρ , ρ contains χ_ρ .

Def (Unitary operator) V : Hilbert space.

$U \in \text{Aut}_{\mathbb{C}}(V)$ is called an unitary operator if
 $\langle Uv, Uw \rangle = \langle v, w \rangle \quad \forall v, w \in V$

Def (Unitary rep) V : Hilbert space.

$(\rho, V) \in \text{Rep}(G)$ is unitary if $\rho(g)$ is an unitary operator ($\forall g \in G$).

E.p. $\chi \in \hat{G}$ is unitary if $\text{Im } \chi \subseteq S^1$.

Rmk. When $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$, any $\chi \in \hat{G}$ is unitary.

1.3. Reduction to smaller cardinal

Admissibility

(ρ, V) is admissible if $\dim_{\mathbb{C}} V^k < +\infty$ for $\forall K \leq G$ cpt open.

Countable hypothesis

$\exists / \forall K \leq G$ cpt open, G/K is countable.

Assuming countable hypothesis. we get

$$(\rho, V) \in \text{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \text{End}_G(V) = \mathbb{C} \\ p \text{ acts on } V \text{ as a character } \omega_p \end{cases}$$

$$\xrightarrow{G \text{ is abelian}} \dim_{\mathbb{C}} V = 1.$$

2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

Rep of $G = \mathcal{O}, \mathcal{O}^\times, F, F^\times$, where F is a non-arch local field.

In these cases, G is abelian and satisfy the countable hypothesis, so $\text{Ind}(G) = \hat{G}$, i.e., everything reduced to the classification of characters.

E.g. $G = (\mathcal{O}, +)$

$\forall \chi \in \hat{\mathcal{O}}$ is trivial on \mathfrak{p}^k . Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathfrak{p}^d \subset \ker \chi\}$$

When $\text{level}(\chi) = d$, $\chi: \mathcal{O} \xrightarrow{\pi} \mathcal{O}/\mathfrak{p}^d \rightarrow \mathbb{C}^\times$ factors through char of $\mathcal{O}/\mathfrak{p}^d$.

E.g. $G = \mathcal{O}^\times$

$\forall \chi \in \hat{\mathcal{O}}^\times$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

When $\text{level}(\chi) = d$, $\chi: \mathcal{O}^\times \xrightarrow{\pi} \mathcal{O}^\times/\mathcal{U}^{(d+1)} \rightarrow \mathbb{C}^\times$ factors through char of $(\mathcal{O}/\mathfrak{p}^{d+1})^\times$

Recent advances: Geometrization of continuous characters of \mathbb{Z}_p [<https://msp.org/pjm/2013/261-1/pjm-v261-n1-p05-p.pdf>]

E.g. $G = (F, +)$

$\forall \chi \in \hat{F}$ is trivial on \mathfrak{p}^k . Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \in \mathbb{Z} \mid \mathfrak{p}^d \subset \ker \chi\}$$

Prop (Additive duality)

Fix $\psi \in \hat{F}$ nontrivial with level d .

We have a gp iso

$$F \longrightarrow \hat{F} \quad a \longmapsto \psi(a)\psi(-) \text{ of level } d - v_F(a) \text{ (when } a \neq 0)$$

Q: Do we have similar result for $\hat{\mathcal{O}}$?

E.g. $G = F^\times$

$\forall \chi \in \hat{F}^\times$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

Q: Do we have any classification of F^\times ?

Notation for future: G : loc. profinite gp $Z = Z(G)$
 $\text{Cos}(G) := \{ \text{cpt open subgp } K \text{ of } G \}$
 $\text{Cos}_Z(G) := \{ K \in \text{Cos}(G) \mid K \geq Z, K/Z \in G/Z \text{ cpt open} \}$

3. Construction of new reps

3.1. Special sub & quotient rep

Def. G : loc. profinite gp $N \triangleleft G$ cpt open
 $(\pi, V) \in \text{Rep}(G)$ $\vartheta \in \hat{N}$, we define sm reps of G :

$$V(N) := \langle v - n.v \rangle_{n \in N, v \in V}$$

$$V_N := V/V(N)$$

$$V(\vartheta) := \langle \vartheta(n)v - n.v \rangle_{n \in N, v \in V}$$

$$V_{\vartheta} := V/V(\vartheta)$$

Obviously $V(N) = V(\mathbb{1}_N)$, $V_N = V_{\mathbb{1}_N}$, N acts on $V(\vartheta)$ by ϑ , and

$$0 \rightarrow V(\vartheta) \rightarrow V \rightarrow V_{\vartheta} \rightarrow 0 \quad \text{in } \text{Rep}(G)$$

Rmk. (1) Normal subgp gives us plenty of canonical decompositions.

E.p., when $(\rho, V) \in \text{Irr}(G)$, for $\vartheta \in \hat{N}$, we get

$$\begin{cases} V(\vartheta) = V \\ V_{\vartheta} = 0 \end{cases} \quad \text{or} \quad \begin{cases} V(\vartheta) = 0 \\ V_{\vartheta} = V \end{cases}$$

(2) When $(\rho, V) \in \text{Rep}(N)$ is semisimple,

$$0 \rightarrow V(N) \rightarrow V \rightarrow V_N \rightarrow 0$$

$$0 \rightarrow \bigoplus_{\substack{\sigma \in \text{Irr}(N) \\ \sigma \neq \mathbb{1}_N}} V^{\sigma} \xrightarrow{\text{IIS}} \bigoplus_{\sigma \in \text{Irr}(N)} V^{\sigma} \xrightarrow{\text{IIS}} V^{\mathbb{1}_N} \rightarrow 0$$

Assume additionally that

- N is abelian,
- N is the union of an increasing sequence of cpt open subgps.

Then, we have more properties.

Prop. (Integral criterion) [p56 Lemma] $(\rho, V) \in \text{Rep}(N)$, $v \in V$,

$$v \in V(\vartheta) \Leftrightarrow \left[\exists N_0 \in \text{Cos}(N) \text{ s.t. } \int_{N_0} \vartheta(n)^{-1} \rho(n) v \, d\mu_N(n) = 0 \right]$$

Prop. The fctor

$$\text{Rep}(G) \longrightarrow \text{Vect}_{\mathbb{C}} \quad (\pi, V) \longmapsto V_{\vartheta}$$

is exact. E.p.,

$$V(N)(N) \cong V(N)$$

$$V(N)_N = 0$$

($\vartheta \neq 1$)

$$V(N)(\vartheta) \cong V(N) \wedge V(\vartheta)$$

$$V(N)_{\vartheta} \cong V_{\vartheta}$$

$$V_N(N) = 0$$

$$V_{N,N} \cong V_N$$

($\vartheta \neq 1$)

$$V_N(\vartheta) \cong V_N$$

$$V_{N,\vartheta} = 0$$

(You can compute $V(\vartheta)$ and V_{ϑ} also, but I'm lazy.)

3.2. Duality

$(\rho, V) \in \text{Rep}(G) \rightsquigarrow (\rho^*, V^*)$ may be not smooth
 $\rightsquigarrow (\check{\rho}, \check{V}) \in \text{Rep}(G)$ is the smooth dual, where
 $\check{V} := \bigcup_{K \in \text{Cos}(G)} (V^*)^K \subset V^*$

$$\text{ev}: \check{V} \times V \rightarrow \mathbb{C} \quad (\check{w}, v) \mapsto \langle \check{w}, v \rangle$$

$$\rightsquigarrow \langle g \cdot \check{w}, v \rangle = \langle \check{w}, g^{-1} \cdot v \rangle$$

Rmk. (1) The contravariant duality fctor

$$\text{Rep}(G) \rightarrow \text{Rep}(G) \quad (\rho, V) \mapsto (\check{\rho}, \check{V})$$

is exact.

exact=>additive: <https://math.stackexchange.com/questions/3039422/in-abelian-categories-is-a-right-left-exact-functor-necessarily-additive>

(2) For $K \in \text{Cos}(G)$, we have iso $\check{V}^K \xrightarrow{\cong} (V^K)^*$ in $\text{Rep}(K)$.

(3) If $(\rho, V) \in \text{Rep}(G)$, $v \in V$, then $\exists \check{w} \in \check{V}$ s.t. $\langle \check{w}, v \rangle \neq 0$.

(4) $\delta: V \rightarrow \check{V}$ is inj, and

\Downarrow δ is iso $\Leftrightarrow \pi$ is admissible

(5) If $(\rho, V) \in \text{Rep}(G)$ is admissible, then

$$(\rho, V) \in \text{Irr}(G) \Leftrightarrow (\check{\rho}, \check{V}) \in \text{Irr}(G)$$

(b) (Bilinear map) Let $(\rho, V), (\sigma, W) \in \text{Rep}(G)$.

$$\mathcal{B}(\rho, \sigma) := \left\{ f: V \times W \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ bilinear} \\ f(gv, gw) = f(v, w) \end{array} \right\}.$$

Then

$$\mathcal{B}(\rho, \sigma) \cong \text{Hom}_G(\rho, \check{\sigma}) \cong \text{Hom}(\sigma, \check{\rho}).$$

3.3. Ind and c-Ind

Definition

Def (Induced representation)

$H \leq G$ **closed**, $(\sigma, W) \in \text{Rep}(H)$, we get

$$\left\{ \begin{array}{l} \text{sm induction} \quad \text{Ind}_H^G \sigma = (\Sigma, X) \in \text{Rep}(G) \\ \text{cpt induction} \quad \text{c-Ind}_H^G \sigma = (\Sigma_c, X_c) \in \text{Rep}(G) \end{array} \right.$$

as follows:

$$\text{Ind}_H^G W = X = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(hg) = \sigma(h)f(g) \\ \exists K \leq G \text{ s.t. } f(gk) = f(g) \end{array} \quad \begin{array}{l} \forall g \in G, h \in H \\ \forall g \in G, k \in K \end{array} \right\}$$

$$\Sigma(g).f = f(-g)$$

$$\text{c-Ind}_H^G W = X_c = \left\{ f \in X \mid \begin{array}{l} \pi(\text{supp } f) \text{ is cpt in } H \backslash G \\ \pi: G \rightarrow H \backslash G \end{array} \right\}$$

$$\Sigma_c(g).f = f(-g)$$

Rmk. (1). (Reality check) When $G=H$,

$$c\text{-Ind}_H^G W = \text{Ind}_H^G W = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(g) = \sigma(g) f(1) \\ \exists k \leq G \text{ s.t. } f(gk) = f(g) \end{array} \right\} \xrightarrow{\cong} W$$

$$f \longmapsto f(1)$$

$$\sigma(-) \cdot w \longleftarrow w$$

(2) Two fcts $\text{Ind}_H^G, c\text{-Ind}_H^G$ are both exact.

(3) Suppose $H \leq G$ open, $[G:H] < +\infty$, $(\sigma, W) \in \text{Irr}(H)$. We get $\text{Ind}_H^G \sigma$ is G -semisimple.

Frobenius Reciprocity

Thm.

condition		$(\rho, V) \in \text{Rep}(G), (\sigma, W) \in \text{Rep}(H)$
$H \leq G$ closed	$\text{Res}_H^G \vdash \text{Ind}_H^G$	$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \cong \text{Hom}_G(\rho, \text{Ind}_H^G \sigma)$
$H \leq G$ open	$c\text{-Ind}_H^G \vdash \text{Res}_H^G$	$\text{Hom}_G(c\text{-Ind}_H^G \sigma, \rho) \cong \text{Hom}_H(\sigma, \text{Res}_H^G \rho)$

E.p. we have two canonical map in $\text{Rep}(H)$:

$$H \leq G \text{ closed} \quad \varepsilon_W: \text{Ind}_H^G W \rightarrow W$$

$$H \leq G \text{ open} \quad \eta_W: W \rightarrow c\text{-Ind}_H^G W$$

$$f \mapsto f(1)$$

$$w \mapsto \begin{bmatrix} f_w: h \in H \mapsto h.w \\ x \notin H \mapsto 0 \end{bmatrix}$$

Duality Theorem [p32]

Let $H \leq G$ closed, define

$$\delta_{H \setminus G} = \delta_H^{-1} \delta_G|_H: H \rightarrow \mathbb{R}_{>0}^\times$$

then $\delta_{H \setminus G} \in \hat{H}$, and $\forall (\sigma, W) \in \text{Rep}(H)$, we have

$$(c\text{-Ind}_H^G \sigma)^\vee \cong \text{Ind}_H^G (\delta_{H \setminus G} \otimes \check{\sigma}).$$

3.4. Other constructions. (Concerned closely with Hecke algebra)

- \otimes , Sym, and \wedge .

It's difficult for me to work out.

- $C^\infty(G)$, $C_c^\infty(G) \in \text{Rep}(G \times G)$

$C_c^\infty(G)$ will play a central role in Hecke alg. We focus on $C^\infty(G)$ here.

Def. (matrix coefficient set $C(\rho)$)

Let $(\rho, V) \in \text{Rep}(G)$, $v \in V$, $\check{w} \in \check{V}$.

$$\begin{aligned} \mathbb{I}: \check{V} \otimes V &\longrightarrow C^\infty(G) \\ \check{w} \otimes v &\longmapsto \left[\begin{array}{l} \chi_{\check{w} \otimes v}: G \longrightarrow \mathbb{C} \\ \chi_{\check{w} \otimes v}(g) = w(g.v) = \langle w, \rho(g)v \rangle \end{array} \right] \end{aligned}$$

is a morphism in $\text{Rep}(G \times G)$.

$$C(\rho) = \text{Im } \mathbb{I} \subseteq C^\infty(G)$$

Rmk. $\forall \chi \in C(\rho)$, $\text{Supp } \chi$ is invariant under translation by Z .

Def. (χ -cuspidal rep)

Let G be a **unimodular** loc. profinite gp. We define

$$\text{Cusp}_c(G) = \left\{ (\rho, V) \in \text{Irr}(G) \mid \begin{array}{l} \text{supp}(\chi)/Z \subseteq G/Z \text{ is cpt} \\ \text{for every } \chi \in C(\rho) \end{array} \right\}$$

Prop. [p 70] (1) Every $(\rho, V) \in \text{Cusp}_c(G)$ is admissible

(2) Suppose

- $(\rho, V) \in \text{Irr}(G)$ is admissible
- $\exists \chi \in C(\rho)$, $\text{supp}(\chi)/Z$ is cpt.

Then,

- $\mathbb{I}: V^* \otimes V \longrightarrow C(\pi)$ is an iso
- $(\pi, V) \in \text{Cusp}_c(G)$

[Idea of proof: Use dual and cardinal argument.
Hecke algebra is needed which will be introduced later]

Rmk. χ -Cuspidal has the property of projectivity and Schur's orthogonality relation, see [p 74].

4. Hecke algebra