Eine Woche, ein Beispiel
9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

> — vector bundle over manifold
> — module over ring

There are hundreds of different versions of it.

— vector bundle over manifold 几何/几何分析
diffe mfld · (real) differential v.b. over (real) differential mfld
Riemann surface · cplx (analytic) line bundle over Riemann surface
— sheaf over space 代数几何
scheme theory · locally free sheaf on scheme
· coherent sheaf on scheme
geo rep theory · local system over (real/cplx) mfld
· perverse sheaf over Riemann surface (derived)
— module over ring 代数
comm alg · f.g module over Noetherian commutative ring (with 1)
rep of grp · group representation over group (⤳ group algebra)
quiver theory · quiver representation over quiver (⤳ path algebra, bound quiver algebra)
Lie algebra · Lie alg representation over Lie alg (⤳ universal enveloping algebra)

Goal — structures & invariants
— classifications of
special v.b, mfld, subv.b. submfld
— symmetry & quotient
— special functors
— homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n.
(by homotopy group! $\leadsto$ generalized Picard group?)
Q: Is this group structure natural?

Thm. $\{\text{rank } m \ K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ \ rank m / n	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/12\mathbb{Z}$	$(\mathbb{Z}/12\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{R}P^\infty \cong K(\mathbb{Z}/2\mathbb{Z}, 1)$

$\pi_{n-1}(GL_m(\mathbb{C}))$ \ rank m / n	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$

Problems. Describe the special bundles, e.g. TS^n
Describe the operations, e.g. dual, $\oplus, \otimes, \wedge^k, \text{Sym}^k, \text{Res}, \text{Ind}$

$\{\text{rank } m \ K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_K(m, \infty)]$ $K = \mathbb{R}, \mathbb{C}$
M: paracompact