Eine Woche, ein Beispiel

5.15 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory.
In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

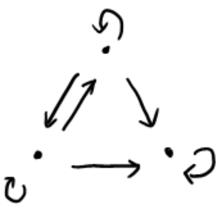
I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categorys here: http://katmat.math.uni-bremen.de/acc/acc.pdf

$\mathcal{C}$ is always a category.

	$Ob(\mathcal{C})$		$Mor(X,Y)$
small	Set		Set
loc. small	—		Set
large	not set	or	not set

filtered:

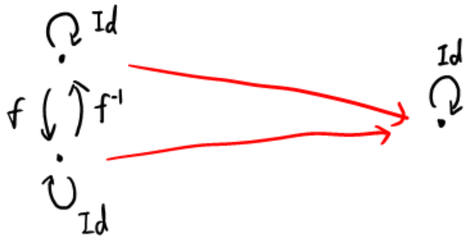
cofiltered:



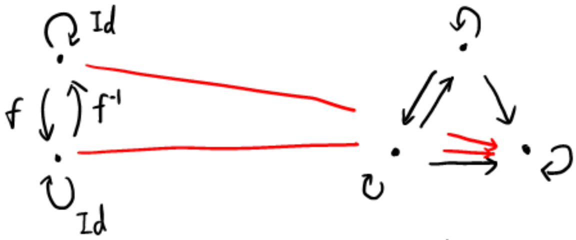
category

full subcategory

https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective



equivalence of category



category

fully faithful

https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/

左伴随 $\dashv$ 右伴随		$f_! \dashv f^* \dashv f_* \dashv f^!$
自由	忘却	
$-\otimes_A N$	$\mathrm{Hom}_A(N,-)$	
Δ	$\varprojlim$	
$(\ )^\sim$	Γ_*	
f_p	f^*	
c-Ind	Res	
Res	Ind	
$-\otimes_k A^e$	Res	
$sh(-)$	$\mathrm{Res}_{sh\to Psh}$	
π_0	$T \mapsto \coprod_{t\in T} \mathrm{Spec}\,\mathbb{Z}$	
$G^{[p]}$	Lie	
$\lvert - \rvert$	S	
$\tau_1 = H_0(-)$	N: nerve	
β: Stone-Čech	U: forget	

ad ⇒ 右正合 | 左正合

ad ⇒ 与余极限交换 $\varinjlim$ | 与极限交换 $\varprojlim$

coker f	ker f	
$\coprod A$ $\quad \oplus A$	$\prod A$ $\quad A\times_C B$	
pushforward	pullback	
coequalizer	equalizer	
$\bar{k} = \varinjlim_{L/K} L$	$\mathrm{Spec}\,\bar{k} = \varprojlim_{L/K} \mathrm{Spec}\,L$	$\mathrm{Gal}(\bar{k}/k) = \varprojlim_{L \text{ finite Galois}} \mathrm{Gal}(L/k)$
$\mathrm{Spec}\,\mathbb{Z}_p = \varinjlim_n \mathrm{Spec}\,\mathbb{Z}/p^n\mathbb{Z}$ completion	$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ ring point of view	

$\mathcal{F}_p = \varinjlim_{U} \mathcal{F}(U)$
stalk

colimit	limit
direct / inductive / injective $\Big\}$ limit.	inverse limit.

How to memorize:

limit ← colimit →

$0 \to \ker \to M \to N \to \mathrm{coker} \to 0$

Complete/Cocomplete/Bicomplete category

Def. $\mathcal{C}$ is **complete** if
$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow C$ $\quad i \longmapsto F_i$,
$\varprojlim_{i\in\Lambda} F_i$ exists $\qquad$ ($\varprojlim_{i\in\Lambda} F_i$ is called the small limit)

$\mathcal{C}$ is **cocomplete** if
$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow C$ $\quad i \longmapsto F_i$,
$\varinjlim_{i\in\Lambda} F_i$ exists $\qquad$ ($\varinjlim_{i\in\Lambda} F_i$ is called the small colimit)

bicomplete = complete + cocomplete
$\mathcal{C}$ is **finitely complete** if $\forall$ finite limit exists
$\mathcal{C}$ is **finitely cocomplete** if $\forall$ finite colimit exists.

Thm. $\mathcal{C}$ is complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & products
$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & products
$\mathcal{C}$ is cocomplete $\Leftrightarrow$ $\mathcal{C}$ has coequalizers & coproducts
$\Leftrightarrow$ $\mathcal{C}$ has pushouts & coproducts
$\mathcal{C}$ is finitely complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & finite products
$\Leftrightarrow$ $\mathcal{C}$ has equalizers, binary products & terminal obj
$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & terminal obj

For small category $\mathcal{C}$,
complete $\Leftrightarrow$ cocomplete
$\Downarrow$ $\qquad$ $\Downarrow$
thin ($\#\mathrm{Mor}(X,Y) \leq 1$)

The "closedness" of the category is that taking the morphisms between two objects gives another morphism in the same category, rather than the category of sets or some other category.
from: https://math.stackexchange.com/questions/3486846/definition-of-cartesian-closed-category-why-do-we-need-exponential-objects

Cartesian closed category / Closed category

Def. $\mathcal{C}$ is Cartesian closed if
$\mathcal{C}$ has terminal obj, binary product and exponential, where

a bifctor $F: \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which is functorial in Y

$$- \times Y \dashv (-)^Y$$

i.e. $\mathrm{Mor}(X\times Y, Z) \cong \mathrm{Mor}(X, Z^Y)$

$\mathcal{C}$ is loc. Cartesian closed if all its slice category is Cartesian closed.

Rmk. When $\mathcal{C}$ is loc. Cartesian closed,
$\mathcal{C}$ is Cartesian closed $\Leftrightarrow$ $\mathcal{C}$ has a terminal object.
But $\mathcal{C}$ is Cartesian closed $\not\Rightarrow$ $\mathcal{C}$ is loc. Cartesian closed

For the closed category, we use the definition in https://ncatlab.org/nlab/show/closed+category.

Def A closed category is a category $\mathcal{C}$ together with the following data.

- bifctor $[-,-]: \mathcal{C}^{op}\times\mathcal{C}\to\mathcal{C}$ called internal hom-fctor
- $I\in Ob(\mathcal{C})$ called unit object
- $i: Id_{\mathcal{C}} \xrightarrow{\cong} [I,-]$ $\rightsquigarrow$ $i_A: A \xrightarrow{\cong} [I,A]$
- $j_X: I \longrightarrow [X,X]$ extranatural in X
- $L^X_{Y,Z}: [Y,Z]\to[[X,Y],[X,Z]]$ functorial in Y and Z, extranatural in X.
- Compatabilities

$$\begin{tikzcd} I \arrow[r, "j_Y"] \arrow[dr, "j_{[X,Y]}"'] & {[Y,Y]} \arrow[d, "L^X_{Y,Y}"] \\ & {[[X,Y],[X,Y]]} \end{tikzcd} \qquad \begin{tikzcd} {[X,Y]} \arrow[r, "L^X_{XY}"] \arrow[dr, "i_{[X,Y]}"'] & {[[X,X],[X,Y]]} \arrow[d, "{[j_X,1]}"] \\ & {[I,[X,Y]]} \end{tikzcd} \qquad \begin{tikzcd} {[Y,Z]} \arrow[r, "L^I_{YZ}"] \arrow[dr, "{[1,i_Z]}"'] & {[[I,Y],[I,Z]]} \arrow[d, "{[i_Y,1]}"] \\ & {[Y,[I,Z]]} \end{tikzcd}$$

$$\begin{tikzcd} & {[U,V]} \arrow[dl, "L^X_{UV}"'] \arrow[dr, "L^Y_{UV}"] & \\ {[[X,U],[X,V]]} \arrow[d, "L^{[X,Y]}_{[X,U][X,V]}"'] & & {[[Y,U],[Y,V]]} \arrow[d, "{[1,L^X_{YV}]}"] \\ {[[[X,Y],[X,U]],[[X,Y],[X,V]]]} \arrow[rr, "{[L^X_{YU},1]}"] & & {[[Y,U],[[X,Y],[X,V]]]} \end{tikzcd}$$

$$\begin{aligned} \gamma: \mathrm{Mor}(X,Y) &\longrightarrow \mathrm{Mor}(I,[X,Y]) \\ f &\longmapsto [1,f]\circ j_X \end{aligned}$$ is an iso.

Monoidal category = Tensor category 幺半范畴

Def A monoidal category is a category $\mathcal{C}$ together with the following data.

- bifctor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- $I \in Ob(\mathcal{C})$ called unit object
- $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$
- $\lambda_A : I \otimes A \xrightarrow{\cong} A$ lambda: left
- $\rho_A : A \otimes I \xrightarrow{\cong} A$ rho: right

- Compatabilities

$$\begin{tikzcd}
 & A \otimes (B \otimes (C \otimes D)) \arrow[dl, "1_A \otimes \alpha_{B,C,D}"'] \arrow[dr, "\alpha_{A,B,C\otimes D}"] & \\
A \otimes ((B \otimes C) \otimes D) \arrow[d, "\alpha_{A,B\otimes C,D}"'] \arrow[rr, phantom, "\circlearrowright"] & & (A \otimes B) \otimes (C \otimes D) \arrow[d, "\alpha_{A\otimes B,C,D}"] \\
(A \otimes (B \otimes C)) \otimes D \arrow[rr, "\alpha_{A,B,C} \otimes 1_A"] & & ((A \otimes B) \otimes C) \otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
A \otimes (I \otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "1_A \otimes \lambda_B"'] & & (A \otimes I) \otimes B \arrow[dl, "\rho_A \otimes 1_B"] \\
 & A \otimes B &
\end{tikzcd}$$

For strict monoidal category, we require in addition that $\alpha_{A,B,C}$, λ_A, ρ_A are identities.

E.g. Cartesian monoidal category $\mathcal{C}$: category with finite products

$\otimes = \prod$ $\qquad$ I = terminal object

e.g. **Set**, **Cat**.

Cocartesian monoidal category $\mathcal{C}$: category with finite coproducts

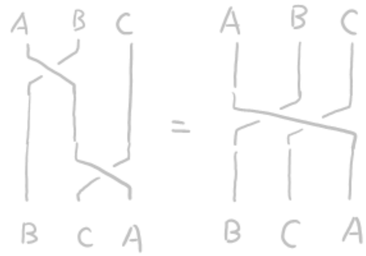
$\otimes = \coprod$ $\qquad$ I = initial object

Abelian category is monoidal.

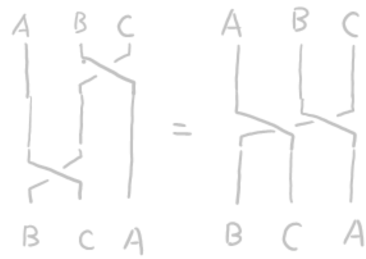
Def (Specializations)

Let $\mathcal{C}$ be a monoidal category.
If in addition we have $\gamma_{A,B}: A\otimes B \longrightarrow B\otimes A$,
then $\mathcal{C}$ is **braided monoidal category** if

$$
\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[dl, "\gamma_{A,B}\otimes 1_C"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"] & & A\otimes(B\otimes C) \arrow[d, "\gamma_{A,B\otimes C}"] \\
B\otimes(A\otimes C) \arrow[dr, "1_B\otimes\gamma_{A,C}"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B\otimes(C\otimes A) &
\end{tikzcd}
$$



$$
\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[dl, "1_A\otimes\gamma_{B,C}"'] \arrow[dr, "\alpha^{-1}_{A,B,C}"] & \\
A\otimes(C\otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] & & (A\otimes B)\otimes C \arrow[d, "\gamma_{A\otimes B,C}"] \\
(A\otimes C)\otimes B \arrow[dr, "\gamma_{C,A}\otimes 1_B"'] & & C\otimes(A\otimes B) \arrow[dl, "\alpha^{-1}_{C,A,B}"] \\
 & (C\otimes A)\otimes B &
\end{tikzcd}
$$



$$
\begin{tikzcd}
I\otimes A \arrow[rr, "\gamma_{I,A}"] \arrow[dr, "\lambda_A"'] & & A\otimes I \arrow[dl, "\rho_A"] \\
 & A &
\end{tikzcd}
\qquad
\begin{tikzcd}
A\otimes I \arrow[rr, "\gamma_{A,I}"] \arrow[dr, "\rho_A"'] & & I\otimes A \arrow[dl, "\lambda_A"] \\
 & A &
\end{tikzcd}
$$

$\mathcal{C}$ is **symmetric monoidal category** if

$$\gamma_{B,A}\circ\gamma_{A,B} = 1_{A\otimes B}.$$

$+$ $\mathcal{C}$ is braided.

closed monoidal category $=$ closed category $+$ monoidal category
$+$ compatabilite $\quad -\otimes A \dashv [A,-]$

A list of categories which I'm interested:

Set **Top** **Grp** **Ab** **Vect(k)** **Mod(R)**

Ring: identity + preserve identity **CRing** **Rng**

Field: full subcategory of CRing

0: $Ob(0) = \emptyset$

1: $Ob(1) = \{*\}$ $Mor(*,*) = \{1_*\}$

K(2): $Ob(K(2)) = \{V, E\}$ $Mor(V,V) = \{1_V\}$ $Mor(E,E) = \{1_E\}$

$Mor(V,E) = \emptyset$ $Mor(E,V) = \{s, t\}$

$$\begin{tikzcd} E \arrow[loop left, "1_E"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \arrow[loop right, "1_V"] \end{tikzcd}$$

Δ: $Ob(\Delta) = \{[n] := \{0,1,2,\dots n\} \mid n \geq 0\}$

$Mor([m],[n]) = \{\text{weakly monotone maps}\}$

sSet: $Ob(sSet) = \left\{ \begin{array}{c} X : \Delta^{op} \longrightarrow Set \\ [n] \longmapsto X_n \end{array} \right\}$ $Mor(X,Y) = \left\{ \alpha : \begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd} \right\}$

CHaus: $Ob(CHaus) = \{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} \; X \}$ https://ncatlab.org/nlab/show/compactum

$Mor(X,Y) = \{ f : X \longrightarrow Y \mid f \text{ cont} \}$

Met: full subcategory of CHaus whose objects are metric spaces.

⚠ For the category of Graph, there've different realizations.

Quiv(C): $Ob(Quiv(\mathcal{C})) = \{ \text{fctor } \Gamma : K(2) \longrightarrow \mathcal{C} \}$

$Mor(\Gamma_1, \Gamma_2) = \left\{ \alpha : \begin{tikzcd} K(2) \arrow[r, bend left, "\Gamma_1"] \arrow[r, bend right, "\Gamma_2"'] & \mathcal{C} \end{tikzcd} \right\}$

Quiv $= Quiv(Set)$

$=$ Category of presheaves on Q^{op}.

$\mathbf{Cat}$ = {the category of small categories} is a 2-category.

$Ob(\mathbf{Cat}) = \{\text{small category } \mathcal{C}\}$

$Mor(\mathcal{C},\mathcal{D})$ is a category by

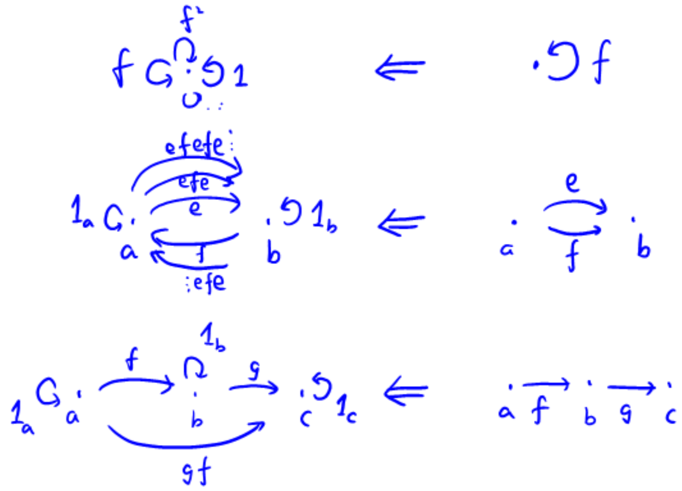
$$Ob(Mor(\mathcal{C},\mathcal{D})) = \{F : \mathcal{C} \to \mathcal{D}\}$$

$$Mor(F,G) = \left\{ \alpha : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \right\}$$

Basic properties of Cat.

1. Initial object 0, Terminal object 1.
2. Cat is loc. small but not small
3. Cat is bicomplete
4. Cat is Cartesian closed but not loc. Cartesian closed
5. Cat is loc. finitely presentable https://ncatlab.org/nlab/show/locally+finitely+presentable+category
6. $\begin{tikzcd} \mathbf{Cat} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\top"] & \mathbf{Quiv} \arrow[l, bend right, "\text{free}"'] \end{tikzcd}$

e.g of "free"



Hausdorff and compactness ← cpt ≈ (quasi)cpt

$$
\begin{array}{cccccc}
 & \mathrm{Haus} & \subset & \mathrm{wHaus} & \subset & \mathcal{T}op \\
 & \cup & & \cup & & \cup \\
 & \mathrm{CGHaus} & \subset & \mathrm{CGwHaus} & \subset & k\mathcal{T}op \\
\text{CW cplx, Met} & \cup & & & & \cup \\
 & \mathrm{loc.CHaus} & & = & & \mathrm{loc.cpt} \\
\text{topo mfld} & \cup & & & & \cup \\
 & \mathrm{CHaus} & & = & & \mathrm{cpt}
\end{array}
$$

Def. $X \in \mathcal{T}op$ is a **weak Hausdorff space** (in wHaus) if
$\forall K \in \mathrm{CHaus}, \forall g: K \to X$ cont, $g(K) \subset X$ is closed.

Def. $X \in \mathcal{T}op$ is **locally compact** (in $\mathrm{loc.cpt}$) if
$\forall p \in X$, $\exists$ cpt nbhd V (i.e. $p \in U \subsetneq V \subseteq X$, $U \subseteq X$ open, V cpt)
$\mathrm{loc.CHaus} = \mathrm{loc.cpt} \cap \mathrm{Haus}$

Def. $X \in \mathcal{T}op$ is a **compactly generated/a k-space** (in $k\mathcal{T}op$) if

- cpt gen in Condensed Math
- Hausdorff-cpt gen/k-space in wiki
- cpt gen/k-space in nlab
- k-space in AT II

$[\mathrm{CGwHaus} = k\mathcal{T}op \cap \mathrm{wHaus}$
$\mathrm{CGHaus} = k\mathcal{T}op \cap \mathrm{Haus}]$

$\forall$ map $f: X \to Y$,
f is cont $\iff$ $K \xrightarrow{g} X \xrightarrow{f} Y$ is cont
$\forall K \in \mathrm{CHaus}$, $g: K \to X$ cont

equivalently,
$\forall A \subseteq X$ subspace,
$A \subseteq X$ is closed $\iff$ $g^{-1}(A) \subseteq K$ is closed
$\forall K \in \mathrm{CHaus}$, $g: K \to X$ cont

When X is Hausdorff, this is equivalent to
$\forall A \subseteq X$ subspace,
$A \subseteq X$ is closed $\iff$ $A \cap K \subseteq K$ is closed
$\forall K \in \mathrm{CHaus}$

Prop. $X \in \mathcal{T}op$, then
X is a k-space $\iff$ $X \cong \coprod_{i \in \Lambda} S_i / \sim$ $\quad S_i \in \mathrm{CHaus}$

Rmk. In the def/prop of $k\mathcal{T}op$, CHaus can be replaced by Prof.

Adjoints

$$
\begin{tikzcd}
\mathrm{Haus} \arrow[rr, hook] & & \mathrm{Top} \arrow[ll, bend right=40, "\beta\text{: Stone-Čech compactification}"', "\perp"] \arrow[d, bend left=50, "\text{Kelley fctor}", "\dashv"'] \\
& \mathrm{CGwHaus} \arrow[r, hook] & \mathrm{kTop} \arrow[u, hook] \arrow[l, bend right=40, "\text{weak Hausdorffication}"', "\perp"]
\end{tikzcd}
$$

Kelley fctor $(-)^{cg} : \mathrm{Top} \longrightarrow \mathrm{kTop}$

$X \longmapsto X^{cg}$ compactly generated

Set: $X^{cg} = X$

Topo: $A \subseteq X^{cg}$ is closed if $g^{-1}(A) \subseteq K$ is closed

$\forall K \in \mathrm{CHaus},\ g: K \to X$ cont

$$
\begin{array}{ccccccccc}
 & & & & & & \mathrm{cpt} & \subset & \mathrm{qc} \\
 & & & & & \cup & & & \cap \\
\mathrm{qcProj} & \subset & \mathrm{Prof} & \subset & \mathrm{qcqs} & \subset & \mathrm{qs} & \subset & \mathrm{CondSet} \\
\wr\Vert & & \wr\Vert & & \wr\Vert & & \wr\Vert & & \bigcup \\
\mathrm{CplBoolAlg}^{op} & & \mathrm{BoolAlg}^{op} & & \mathrm{CHaus} & & \mathrm{Ind(CHaus)} & & \\
 & & & & & & \mathrm{Liq}_p & \subset & \mathrm{CondAb} \\
 & & & & & & \cup & & \cup \\
 & & & & & & \mathrm{Liq}_p(R) & \subset & \mathrm{Cond}(R)
\end{array}
$$

I'm just too lazy to fill in this table. If you know more, tell me and I will fill in, thanks!

Category	cpl	fin cpl	cocpl	fin cocpl	Cartesian closed	closed	monoidal
Set	✓		✓		✓		✓
Top	✓		✓		×		✓
Grp	✓		✓		×		✓
Ab	✓		✓		×		✓
Vect(K)	✓		✓		×		✓
ReCRing Mod(R)	✓		✓		×		✓
Ring	✓		✓				
CRing	✓		✓				
Rng	✓		✓				
Field	×	×	×	×			
0							
1							
K(2)							
Δ	×	×	×	×			
sSet	✓		✓				
CHaus	✓		✓				
Met	×	✓	×	×			
Quiv($\mathcal{C}$)							
Quiv							
Cat	✓		✓		✓		✓
kTop							✓
CGHaus					✓		
CGwHaus	✓		✓		✓		
Prof	✓						