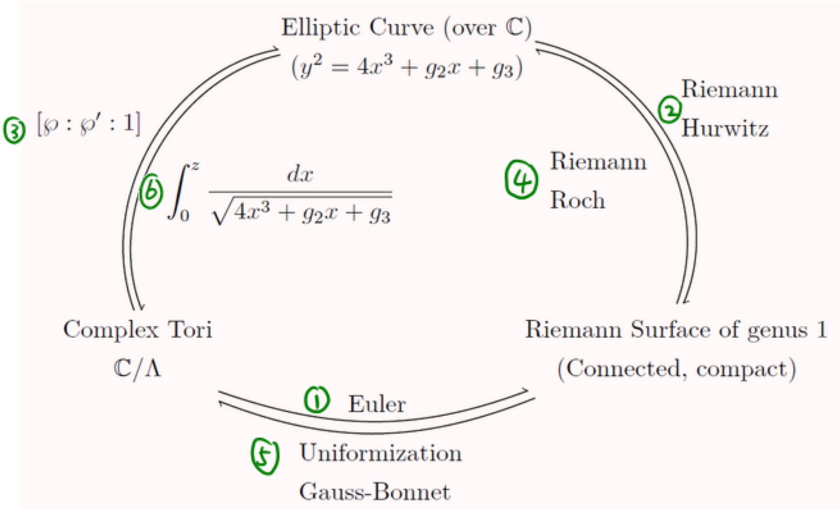
Modular form

1. origin of definition of modular form

1. EC
2. moduli space (from cplx points of view)
3. modular form

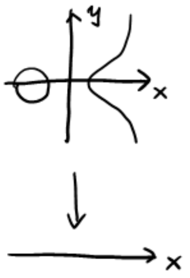
https://www.mathi.uni-heidelberg.de/~otmar/diplom/williams.pdf

1. EC



② RH formula: $f : X \to Y$ $\quad \chi = 2-2g$

$$\chi(X) = \chi(Y)\deg f - \sum_{x \in \mathrm{Ram}(f)} (e(x)-1)$$



$$\begin{tikzcd} E \arrow[d] & {[x:y:z]} \arrow[d] \\ \mathbb{P}^1 & {[x:z]} \end{tikzcd}$$

③④ $\iota : \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$

$$z \longmapsto \begin{cases} [\wp(z) : \wp'(z) : 1] & z \neq 0 \\ [0:1:0] & z = 0 \end{cases}$$

- well-define: $/\Lambda$ + holomorphic
- equation (computation in [WWL, 命题 8.3.2])
- closed embedding [Vakil, 19.2.7, 19.2.10]

⑤ $\tilde{X} \downarrow X$ $\quad \tilde{X}$ RS + $\Gamma \subset \mathrm{Aut}_{RS,\mathrm{tran}}(\tilde{X})$

$\Rightarrow \tilde{X} \cong \mathbb{CP}^1, \mathbb{C}$ or H $\quad \Gamma \subset \mathrm{Aut}_{RS,\mathrm{tran}}(\tilde{X}) \subset \mathrm{Isom}^+(\tilde{X})$

$\Rightarrow$ Riemannian metric with constant curverture on X

$\overset{GB}{\Longrightarrow} \tilde{X} \cong \mathbb{C}$, $X = \mathbb{C}/\Lambda$

⑥ $E \xrightarrow{\cong} H^0(E, \Omega_E)^\vee / \pi_1(E,e)$

$z \longmapsto [\omega \longmapsto \int_{\gamma : 0 \to z} \omega]$

Idea:

$$\begin{tikzcd} \hat{\tilde{E}} \arrow[r, "\sim"] \arrow[d] & H^0(E, \Omega_E)^\vee \arrow[d] \\ E \arrow[r, "\sim"] & H^0(E, \Omega_E)^\vee / \pi_1(E,e) \end{tikzcd}$$

Ex. 1. Discuss 0. Discuss addition structure and their compatabilities.
2. Some computations of $\wp, \wp'$.
3. Describe rational fct field on EC.

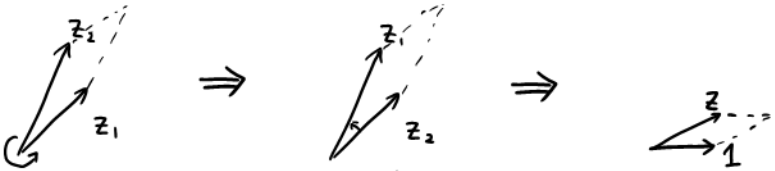
2. moduli space (from cplx points of view)

Origin of $\mathcal{H}/SL_2(\mathbb{Z})$

Lemma. $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda' \Leftrightarrow \Lambda' = z_0 \Lambda$ $\exists\, z_0 \in \mathbb{C}^\times$

Proof. [WWL, 命题 3.8.3, 练习 3.8.4]

Reduced to: Classify lattices (up to cplx scalar)



(up to cplx scalar
up to $SL_2(\mathbb{Z})$-action)

$\gamma\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} := \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

$z \in \mathcal{H}$

(up to $SL_2(\mathbb{Z})$-action)

$\gamma z := ?$

$$\begin{tikzcd}
\left\{(z_1, z_2) \in (\mathbb{C} \setminus \{0\})^2 \,\middle|\, \operatorname{Im} \frac{z_1}{z_2} > 0\right\} \arrow[loop above, "\mathbb{C}^*"] \arrow[loop below, "SL_2(\mathbb{Z})"] \arrow[r] \arrow[dd] & \mathcal{H} \arrow[loop right, "SL_2(\mathbb{Z})"] \arrow[d] \\
 & \mathcal{H}/SL_2(\mathbb{Z}) \arrow[d, equal] \\
\left\{\mathbb{C}/\Lambda \,\middle|\, \Lambda = \mathbb{Z}z_1 \bigoplus \mathbb{Z}z_2\right\} \arrow[loop above, "\mathbb{C}^*"] \arrow[r] & \{\mathbb{C}/\Lambda\}/\sim \arrow[u, bend left, red]
\end{tikzcd}$$

图 2.1 构造模空间/模形式的过程

Description of $\mathcal{H}/SL_2(\mathbb{Z})$

Ex. 1. Special items of $SL_2(\mathbb{Z})$ $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ $S=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$

2. (difficult)  + $SL_2(\mathbb{Z}) = \langle T,S \rangle$ [Za, Prop 1]

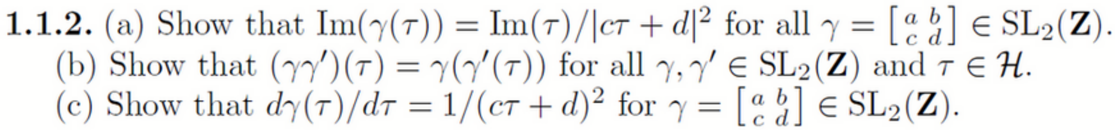
3.  Describe glue, elliptic pts and cusp pt

↑ the corresponding lattices

i: $E_{\mathbb{Z}[i]}: y^2 = x^3+x$ $\phi(x,y)=(-x,iy)$

ρ: $E_{\mathbb{Z}[\rho]}: y^2 = 4x^3-1$ $\phi(x,y)=(\rho x,-y)$

https://math.stackexchange.com/questions/2051526/eisenstein-series-for-hexagonal-lattice?rq=1
http://www.fen.bilkent.edu.tr/~franz/publ/wald.pdf
https://math.stackexchange.com/questions/4043509/how-can-i-calculate-the-eisenstein-series-of-a-complex-lattice
https://mathematica.stackexchange.com/questions/89430/eisenstein-series-in-mathematica



1.1.2. (a) Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau+d|^2$ for all $\gamma = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.
(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma,\gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau+d)^2$ for $\gamma = \left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.

3. Modular form

Def. A holo fct $f: \mathcal{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, lever $\Gamma := SL_2(\mathbb{Z})$ if.

1) $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ $\forall \gamma = \begin{pmatrix}a&b\\c&d\end{pmatrix} \in \Gamma$

e.p. $f(\tau+1) = f(\tau)$

2) Write $f(\tau) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\tau}$, then $a_n = 0$ for $n<0$

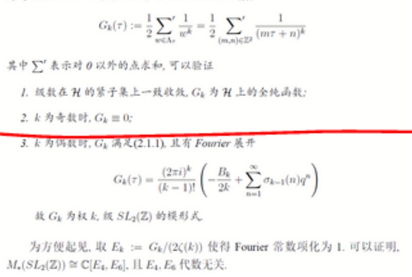
By cplx analysis, this condition is equivalent to

$\exists C>0$ s.t $\{|f(\tau)| \mid \mathrm{Im}\,\tau > C\}$ is bounded.

$M_*(\Gamma) \supseteq S_*(\Gamma)$ ← Cusp form = Spitzenform

Ex. 1. View modular form as fcts on the space of lattices

2. Eisenstein fct.



练习 2.1.1. 对于 $k \in \mathbb{Z}, k > 2$, 定义 *Eisenstein* 函数

$$G_k(\tau) := \frac{1}{2}\sum_{w\in\Lambda_\tau}' \frac{1}{w^k} = \frac{1}{2}\sum_{(m,n)\in\mathbb{Z}^2}' \frac{1}{(m\tau+n)^k}$$

其中 $\sum'$ 表示对 0 以外的点求和, 可以验证

1. 级数在 $\mathcal{H}$ 的紧子集上一致收敛, G_k 为 $\mathcal{H}$ 上的全纯函数;
2. k 为奇数时, $G_k \equiv 0$;
3. k 为偶数时, G_k 满足(2.1.1), 且有 *Fourier* 展开

$$G_k(\tau) = \frac{(2\pi i)^k}{(k-1)!}\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n\right)$$

故 G_k 为权 k, 级 $SL_2(\mathbb{Z})$ 的模形式.

为方便起见, 取 $E_k := G_k/(2\zeta(k))$ 使得 Fourier 常数项化为 1. 可以证明, $M_*(SL_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6]$, 且 E_4, E_6 代数无关.

Next time

3. Δ and j

4. $M_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$