Eine Woche, ein Beispiel

7.10 Non-Archimedean valued field

See [https://math.stackexchange.com/questions/186326/non-archimedean-fields] for definition and examples.
However, in this document, we only care about field extensions of NA local fields.

In this document, E, F are field extensions over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with extended valuation.
E/F is usually an alg field extension.

Goal.

1. Basic informations
2. Completion
3. Perfection
4. Tilting

$\}$ three operators which do not change the Galois group

1. Basic informations

(F, v): NA valued field

$$\rightsquigarrow \mathcal{O} := \{x \in F \mid v(x) \geq 0\}$$

$$\mathfrak{p} := \{x \in F \mid v(x) > 0\}$$

$$k := \mathcal{O}/\mathfrak{p} \qquad p = \mathrm{char}\, k$$

$$\mathcal{U} := \mathcal{U}^{(0)} := \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\}$$

Prop. (still true)

- $(\mathcal{O}, \mathfrak{p})$ is still a local ring, $\mathcal{O}$ is integral closed.
- F is totally disconnected. $\triangleleft$ 
- Every open ball $B_x(<r)$ is closed and every closed ball $B_x(r)$ is open $\Big\|$ $\{0\}$ is closed but not open in $\mathbb{Q}_p$, $\mathbb{Q}_p - \{0\}$ is open but not closed in $\mathbb{Q}_p$.

⚠ Open ball may be not closed ball! Vice versa. (We never define "ball" alone)

Prop. (New Phenomenon) compared with NA local field

- It's possible that $\mathfrak{p}^2 = \mathfrak{p}$, so the uniformizer π may be not picked.
  Luckily have topological uniformizer $\pi \in \mathfrak{p}$.
  e.g. $K = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$, $\mathcal{O} = \mathbb{Z}_p(p^{\frac{1}{p^\infty}})$, $\pi = p \in \mathfrak{p} = \mathfrak{p}^2$
- k may be not finite
- $\mathcal{O}$ may be not DVR (Noetherian $\Leftrightarrow$ F local field, not dim 1)
  https://math.stackexchange.com/questions/363166/examples-of-non-noetherian-valuation-rings
- $\mathcal{O}$ may be not cpt. $\mathcal{O}^\times$ neither.
- No classification and good enough understanding of the structure (for me)!

2. Completion

Ref: https://math.mit.edu/classes/18.785/2017fa/LectureNotes8.pdf

https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete

A lot of NA valued fields are not complete:

Lemma. E/F an alg extension, F NA local field. Then

$$E \text{ is complete} \iff [E:F] < +\infty$$

Proof: "$\Leftarrow$": $[E:F] < +\infty \Rightarrow E$ NA local field $\Rightarrow E$ is complete

"$\Rightarrow$": If not,

$$E = \bigcup_{\substack{F'/F \text{ finite} \\ F' \subseteq E}} F' \xRightarrow{[E:F] = +\infty \Rightarrow F' \neq F} E \subset E \text{ is of second category}$$

$$E \text{ is complete} \xRightarrow{\text{Baire}} E \subset E \text{ is of first category}$$

$\Rightarrow$ Contradiction!

We usually have 3 ways to complete $\mathcal{O} = \mathcal{O}_F$:

$$\check{\mathcal{O}}_\pi := \varprojlim_n \mathcal{O}/(\pi^n) \qquad \pi\text{-adic completion}$$

$$\check{\mathcal{O}}_\mathfrak{p} := \varprojlim_n \mathcal{O}/(\mathfrak{p}^n) \qquad \mathfrak{p}\text{-adic completion}$$

$$\hat{\mathcal{O}} := \text{completion w.r.t. } \|\cdot\|_F.$$

[Prop 8.11, https://math.mit.edu/classes/18.785/2017fa/LectureNotes8.pdf] tells us, when F is a NA local field, these three completions are equivalent.

Universal property:

Define.

$Ob(\mathrm{Field}_{NAv}) = \{(F, v: F \to \Gamma \cup \{\infty\}) \text{ is a NA valued field}\}/\sim$

$Mor(F, E) = \{f: F \to E \mid f \text{ cont field embedding}\}$

$\mathrm{CplField}_{NAv}$: full subcategory consisting of complete objects.

We get adjoint fctors

$$\begin{tikzcd} \mathrm{CplField}_{NAv} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \mathrm{Field}_{NAv} \arrow[l, bend right, "\text{cpl w.r.t. } \|\cdot\|"'] \end{tikzcd}$$

i.e. $\forall f: F \to E$ cont field embedding, E cpl,
$\exists! \hat{f}: \hat{F} \to E$ s.t. $f = \hat{f} \circ \iota$.

$$\begin{tikzcd} F \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \hat{F} \arrow[d, dashed, "\exists! \hat{f}"] \\ & E \end{tikzcd}$$

Cor. $\hat{\hat{F}} = \hat{F}$.

3. Perfection

Ref: wiki:perfect field
I should also find something with Witt vector in this section.