

Modular form

2. computations.

1. Prelude

- 1.1.2. (a) Show that $\text{Im}(\gamma(\tau)) = \text{Im}(\tau)/|c\tau + d|^2$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbf{Z})$.
 (b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \text{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
 (c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbf{Z})$.

$$\begin{aligned} \wp(z) &= \frac{1}{z^2} + \sum'_{z_0 \in \Lambda} \left(\frac{1}{(z-z_0)^2} - \frac{1}{z_0^2} \right) \\ &= \frac{1}{z^2} + \sum'_{z_0 \in \Lambda} \sum_{i=1}^{+\infty} (i+1) \frac{z^i}{z_0^{i+2}} \\ &= \frac{1}{z^2} + \sum_{i=1}^{+\infty} (i+1) G_{i+2}(\Lambda) z^i \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \frac{1}{(z-z_0)^2} &= \frac{1}{z_0^2} \left(1 + \left(-\frac{z}{z_0}\right) \right)^{-2} \\ &= \frac{1}{z_0^2} \left(1 + \binom{-1}{1} \left(-\frac{z}{z_0}\right) + \binom{-2}{2} \left(-\frac{z}{z_0}\right)^2 + \dots \right) \\ &= \frac{1}{z_0^2} + \sum_{i=1}^{+\infty} (i+1) \frac{z^i}{z_0^{i+2}} \end{aligned}$$

$$= \frac{1}{z^2} + 3G_4 z^1 + 5G_6 z^2 + 7G_8 z^3 + \mathcal{O}(z^4)$$

$$\Rightarrow \wp'(z) = -2 \frac{1}{z^3} + \sum_{i=1}^{+\infty} i(i+1) G_{i+2}(\Lambda) z^{i-1}$$

$$= -2 \frac{1}{z^3} + \sum_{i=0}^{+\infty} (i+1)(i+2) G_{i+3}(\Lambda) z^i$$

$$= -2 \frac{1}{z^3} + 6G_4 z + 20G_6 z^2 + 42G_8 z^3 + \mathcal{O}(z^4)$$

$$\Rightarrow (\wp'(z))^2 = 4 \wp(z)^3 - 60 G_4 \wp(z) - 140 G_6$$

$$y^2 = 4x^3 - 60G_4 x - 140G_6 = 4x^3 - g_2 x - g_3$$

Intermediate computation.

$$(\wp'(z))^2 = 4 \frac{1}{z^6} - 24G_4 \frac{1}{z^4} - 80G_6 + (-168G_8 + 36G_4^2) z^2 + \mathcal{O}(z^4)$$

$$(\wp(z))^3 = \frac{1}{z^6} + 9G_4 \frac{1}{z^4} + 15G_6 + (21G_8 + 27G_4^2) z^2 + \mathcal{O}(z^4)$$

$$252G_8 - 108G_4^2 = 0 \Rightarrow G_8 = \frac{3}{7} G_4^2$$

Rmk. another equation: $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ $e_1 = \wp(\frac{\omega_1}{2})$ $e_2 = \wp(\frac{\omega_2}{2})$ $e_3 = \wp(\frac{\omega_1+\omega_2}{2})$

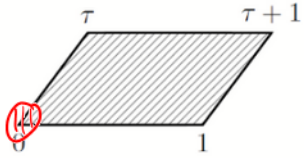
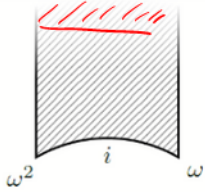
$$\Rightarrow \begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1 e_2 + e_1 e_3 + e_2 e_3 = -\frac{1}{4} g_2 = -15G_4 \\ e_1 e_2 e_3 = \frac{1}{4} g_3 = 35G_6 \end{cases}$$

$$\begin{array}{ccc} & \nearrow \omega_2 & \\ & \omega_1 & \\ & \searrow & \end{array}$$

Ex. $G_4(\rho) = 0$ $G_6(i) = 0$

\Rightarrow Weierstrass equation of $\mathbb{C}/\mathbb{Z} \oplus \rho\mathbb{Z}$, $\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z}$

Conclusion

复环面 \mathbb{C}/Λ_τ	模空间 $\mathcal{H}/SL_2(\mathbb{Z})$
	
$\mathcal{M}(\mathbb{C}/\Lambda_\tau) = \mathbb{C}(\wp, \wp')$	$\mathcal{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$
椭圆函数	模形式
Weierstrass 函数	Eisenstein 级数

Goal: $\mathcal{M}_*(SL_2(\mathbb{Z})) \cong \mathbb{C}[G_4, G_6]$

Any idea? (E_8 , zero pt of G_4 or $G_6 \dots$)

2. q -expansions of G_k (k even) $q = e^{2\pi i \tau} \Rightarrow dq = 2\pi i q d\tau$
 $dq^r = r \cdot 2\pi i q^r d\tau$

Q: Let $G_k(\tau) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \dots$

Compute a_0 .

↓ in $[Za]$ $G_k = \frac{1}{2} \dots$ to delete 2 here

A. $a_0 = \lim_{Im \tau \rightarrow +\infty} G_k(\tau) = \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k)$

Idea: Eisenstein fct = "2-dim Riemann zeta fct".

Luckily $\zeta(k)$ ($k > 0$, even) are understandable.

Let B_k, \tilde{B}_k defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \tilde{B}_k \frac{x^{2k}}{(2k)!},$$

then

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} \tilde{B}_k \pi^{2k} = (-1)^{2k+1} \frac{2^{2k-1}}{(2k)!} B_{2k} \pi^{2k} \quad k \in \mathbb{Z}_{>0}$$

$$\zeta(k) = \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k}\right)$$

$k > 0$ even

The following numerical tables are copied from [JP, §4] and wiki:

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	Me	JP	wiki
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	B_k	b_k	B_k^-
		\tilde{B}_1		\tilde{B}_2		\tilde{B}_3		\tilde{B}_4	\tilde{B}_k	B_k	—
		$\frac{1}{6}$		$\frac{1}{30}$		$\frac{1}{42}$		$\frac{1}{30}$			

Examples $\zeta(2) = \frac{\pi^2}{2 \cdot 3}, \zeta(4) = \frac{\pi^4}{2 \cdot 3^2 \cdot 5}, \zeta(6) = \frac{\pi^6}{3^3 \cdot 5 \cdot 7},$

$$\zeta(8) = \frac{\pi^8}{2 \cdot 3^3 \cdot 5^2 \cdot 7}, \zeta(10) = \frac{\pi^{10}}{3^5 \cdot 5 \cdot 7 \cdot 11}, \zeta(12) = \frac{691 \pi^{12}}{3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13},$$

$$\zeta(14) = \frac{2\pi^{14}}{3^6 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13}.$$

Thm. Let $k \geq 4$ even. G_k has q -expansion. $\Rightarrow G_k$ is modular form.

Idea: Compute every horizontal line.

Lemma. For $\tau \in \mathbb{C} - \mathbb{Z}$, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{\tau + n} = \frac{\pi}{\tan \pi \tau} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{+\infty} q^r \right)$$

// seriously

$$\frac{1}{\tau} + \sum_{n=1}^{+\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n} \right)$$

By taking $\frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\tau^{k-1}} (-)$, we get Lipschitz's formula: ($k \in \mathbb{Z}_{\geq 2}$)

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r$$

$$\begin{aligned} \text{Proof of Thm. } \frac{1}{2} G_k(\tau) &= \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau+n)^k} \\ &= f(k) + \sum_{m=1}^{\infty} \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \\ &= \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right) \end{aligned}$$



Def

$$E_k(\tau) = \left(-\frac{2k}{B_k} \right) \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right)$$

$$g_k(\tau) = G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Ex. Compute $G_k(z)$ and $E_k(z)$. See answer in [Za, P17] [JP, P93].

Rmk. E_k can also be defined as

$$E_k(\tau) := \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{1}{(m\tau+n)^k}$$

$$\begin{aligned} E_4(z) &= 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \\ E_6(z) &= 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \\ E_8(z) &= 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n \\ E_{10}(z) &= 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n \\ E_{12}(z) &= 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n \\ E_{14}(z) &= 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n. \end{aligned}$$

$$\begin{aligned} E_4(\tau) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n} \\ E_6(\tau) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n} \\ E_8(\tau) &= 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n} \end{aligned}$$

3. Degree calculation [Za Prop 2, JP Thm 3]

- def of ord_p , ord_∞
- statement
- Rmk. modular form can be viewed as a section on the l.b. $\omega^{\otimes \frac{k}{2}}$ above the stack $\mathcal{H}/\text{SL}_2(\mathbb{Z})$.
and this formula computes the degree of some "l.b." $\omega(\infty)^{\otimes \frac{k}{2}}$ above the compactified space $(\mathcal{H}/\text{SL}_2(\mathbb{Z}))^*$. Realize it?
- Rmk. weight k gives a bound of $\dim \mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$
- proof by contour integration.

Ex. 0. Compute $\text{ord}_p(E_4)$ and $\text{ord}_p(E_6)$ "again".

1. Bound $\mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$ when k is small (\Rightarrow [Za, Cor 1])
2. Guess a basis of $\mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$ and compare the dimension.
3. Show that E_4 and E_6 are alg indep, thus $\mathcal{M}_*(\text{SL}_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$
Hint for 3. ① Show $\dim \mathcal{M}_{12}(\text{SL}_2(\mathbb{Z})) = 2$. \parallel q -expansion, zero or $E_6^2 = \lambda E_4^3 \Rightarrow \frac{E_6}{E_4} \in \mathcal{M}_2(\text{SL}_2(\mathbb{Z}))$
- ② Show that if $f_1, f_2 \in \mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$, $\dim \langle f_1, f_2 \rangle_{\mathbb{C}} = 2$, then f_1 and f_2 are alg indep
If $P(X, Y) = \sum p_d(X, Y) \in \mathbb{C}[X, Y]$ st. $P(f_1, f_2) \equiv 0$
 $\Rightarrow p_d(f_1, f_2) \equiv 0$
 $\Rightarrow f_2^d p_d(\frac{f_1}{f_2}) \equiv 0$
 $\Rightarrow p_d(\frac{f_1}{f_2}) \equiv 0$
 $\Rightarrow \frac{f_1}{f_2} \equiv c$ or $p_d \equiv 0$
- ③ Show E_4^3 and E_6^2 are alg indep.

4. Application [Za P18, JS P93]

Ex. From $E_4^2 = E_8$ $E_4 E_6 = E_{10}$ get identities

$$\sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n-m) = \frac{1}{120} (\sigma_7(n) - \sigma_3(n))$$

$$\sum_{m=1}^{n-1} \sigma_3(m) \sigma_5(n-m) = \frac{1}{5040} (11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n))$$

Next time. begin our generalization of modular form.

$$\Delta, \tau, j, E_2, E_2', \eta, \dots \quad \lambda \Gamma(2),$$

$\Sigma_n(\Gamma)$ ① ② ③ ④ ⑤