Eine Woche, ein Beispiel

2.13. outer automorphism

We do something very elementary but tricky, and will later find out its connection to the advanced topic, like Teichmüller space.

1. outer automorphism group Out(G) / automorphism group Aut(G)

Ref:
https://en.wikipedia.org/wiki/Outer_automorphism_group
https://en.wikipedia.org/wiki/Automorphisms_of_the_symmetric_and_alternating_groups

Def. Let G be a group. We have a LES

$$1 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\text{conj}} \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1$$

where $Z(G)$ is the center of G
$\mathrm{Aut}(G)$ is the automorphism of G
$\mathrm{Inn}(G) := \mathrm{Im}(\text{conj})$ is the inner automorphism of G
$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the outer automorphism of G.

E.g. $G = \mathbb{Z}$, $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$, $\mathrm{Out}(\mathbb{Z}) = \{\pm 1\}$
$G = \mathbb{Z}/m\mathbb{Z}$, see https://zhuanlan.zhihu.com/p/97195375 ← typo: $@ \Rightarrow 2$ $(m \geq 2)$
an easy result is that $\#\mathrm{Out}(\mathbb{Z}/m\mathbb{Z}) = \varphi(m)$.

E.g. $G = S_n$, $(n \in \mathbb{N}_{>0})$

$$\mathrm{Aut}(S_n) = \begin{cases} S_n & n \neq 2, 6 \\ \{*\} & n = 2 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\mathrm{Out}(S_n) = \begin{cases} \{*\} & n \neq 6 \\ \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$G = A_n$

$$\mathrm{Aut}(A_n) = \begin{cases} S_n & n \neq 2, 3, 6 \\ \{*\} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 3 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\mathrm{Out}(A_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \neq 2, 3, 6 \\ \{*\} & n = 2 \text{ or } 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

For a reference of the proof and constructions of the exotic outer automorphism of S_6, see wiki and here:
https://wordpress.nmsu.edu/pamorand/files/2018/10/AutGroups.pdf
For Chinese you can also see here: https://zhuanlan.zhihu.com/p/24764617
They are elementary and everybody who have learned something about Sylow's theorem should be able to follow the proofs.

{elements in conj class $\boxed{}$ = (123)}

exotic automorphism