# Eine Woche, ein Beispiel

## 7.13. stability manifold of $\mathbb{P}^1$


Ref:

[Okada05]: So Okada, Stability Manifold of P^1

[GKR03]: A. Gorodentscev, S. Kuleshov, A. Rudakov, t-stabilities and t-structures on triangulated categories, https://arxiv.org/abs/math/0312442

[Bri07]: Tom Bridgeland, Stability conditions on triangulated categories, https://arxiv.org/abs/math/0212237

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Huy06]: Huybrechts, D. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr. Oxford: Clarendon Press, 2006


Goal: understand the Bridgeland stability and wall crossing in this toy example.

1. equivalent definitions of stability conditions
2. structure of $\mathrm{Coh}(\mathbb{P}^1)$
3. standard stability

# 1. equivalent definitions of stability conditions

**Def** (locally finite stability condition)

Fix a triangular category $\mathcal{T}$, and denote $K(\mathcal{T})$ as the Grothendieck gp of $\mathcal{T}$.
The set of locally finite stability conditions is defined as

$$\mathrm{Stab}(\mathcal{T}) = \left\{ (Z, \mathcal{P}) \;\middle|\; \begin{array}{ll} Z : K(\mathcal{T}) \longrightarrow \mathbb{C} & \text{(central charge)} \\ \mathcal{P} : \mathbb{R} \longrightarrow \{\text{full additive subcategories of } \mathcal{T}\} & \\ \quad\;\; \phi \longmapsto \mathcal{P}(\phi) & \text{(slicing)} \\ \text{s.t. } (a)(b)(c)(d) + (e) & \end{array} \right\}$$

(a) (slicing compatible with central charge)
if $E \in \mathcal{P}(\phi)$ then $\dfrac{Z(E)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$ ;

(b) (slicing with shift)
$$\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for } A_j \in \mathcal{P}(\phi_j),\ \phi_1 > \phi_2$$

(d) (HN filtration) HN = Harder-Narashimhan
$\forall E \in \mathcal{T}$, $\exists$ finite seq of real numbers
$$\phi_1 > \phi_2 > \cdots > \phi_n$$
and a filtration
$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[d] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\ & A_1 \arrow[ul, dashed, "+1"] & & A_n \arrow[ul, dashed, "+1"] & \end{tikzcd}$$
s.t. $A_j \in \mathcal{P}(\phi_j)$ $\forall j$.
we define $\phi(E) = \{\phi_1, \ldots, \phi_n\}$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{P}(I)$, $\exists$ a Jordan-Hölder filtration with finite length.
$\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\text{extension-closed}}$

**Rmk.** For $E \in \mathcal{T}$, $E \neq 0$,
$E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$
$\Leftrightarrow$ the HN filtration of $E$ has length 1
$\overset{\text{def}}{\Leftrightarrow}$ $E$ is semistable

When $E$ is semistable, define $\phi(E) = \phi$ when $E \in \mathcal{P}(\phi)$.

Lemma 1.1. $\mathcal{P}(\phi)$ is closed under extension.

Proof. Suppose one has one triangle

$$A_1 \longrightarrow E \longrightarrow A_2 \xrightarrow{+1} \qquad (1.1)$$

where $A_1, A_2 \in \mathcal{P}(\phi_0)$, we want to show $E \in \mathcal{P}(\phi_0)$.

Suppose $\phi(E) = \{\phi_1, \ldots, \phi_n\}$, $\phi_1 > \cdots > \phi_n$, $n \geq 1$, then

$$\phi_0 > \phi_n \quad \text{or} \quad \phi_1 > \phi_0 \quad \text{or} \quad (\underbrace{\phi_1 = \phi_0,\ n = 1}_{\Downarrow})$$

$E \in \mathcal{P}(\phi_0)$ ✓

w.l.o.g. assume $\phi_0 > \phi_n$, then $\exists$ triangle

$$B_1 \longrightarrow E \xrightarrow{u} B_2 \xrightarrow{+1}$$

where $u \neq 0$, $B_2 \in \mathcal{P}(\phi_n)$.

Apply $\mathrm{Hom}(-, B_2)$ to (1.1), we get

$$\cdots \mathrm{Hom}(A_1[-1], B_2) \longleftarrow \mathrm{Hom}(E[-1], B_2) \longleftarrow \mathrm{Hom}(A_2[-1], B_2) \longleftarrow \mathrm{Hom}(A_1, B_2) \longleftarrow \mathrm{Hom}(E, B_2) \longleftarrow \mathrm{Hom}(A_2, B_2) \longleftarrow \cdots$$

$$\mathrm{Hom}(A_1, B_2) = 0, \qquad u \in \mathrm{Hom}(E, B_2),\ u \neq 0, \qquad \mathrm{Hom}(A_2, B_2) = 0$$

Contradiction!

Rmk. [Bri07, Lemma 5.2]
$\mathcal{P}(\phi)$ is an abelian category.

Def (stable sheaf)
Suppose $E \in \mathcal{P}(\phi)$ is semistable.

$$E \text{ is stable} \iff E \in \mathcal{P}(\phi) \text{ is simple}$$

$$\Uparrow$$

$$E \in \mathcal{P}(I) \text{ is simple for some } I \ni \phi$$

The next lemma conclude the behavior of triangles with stability conditions.

Lemma 1.2.

Suppose

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow[+1]{u_3} \qquad (1.2)$$

is a triangle, where $\phi(A_1) = \phi_0$, $\phi(A_2) = \phi_0'$.

(1) If $\phi_0 > \phi_0'$, then

(1.2) is the HN-filtration, so $E$ is not semistable;

(2) If $\phi_0 = \phi_0'$, then $E \in \mathcal{P}(\phi_0)$ by Lemma 1.1;

(3) If $u_3 \neq 0$, then $\widetilde{\phi_0} \leq \phi_0 + 1$.

Rmk.

$$\mathrm{Stab}(\mathcal{T}) \triangleq \left\{ (Z,\phi) \;\middle|\; \begin{array}{l} Z: K(\mathcal{T}) \longrightarrow \mathbb{C} \quad \text{(central charge)} \\ \phi: \mathcal{T} \longrightarrow \{\text{finite subsets of } \mathbb{R}\} \\ \quad\;\; E \longmapsto \{\phi_1, \dots, \phi_n\} \quad \text{(slicing)} \\ \text{s.t. } \text{(a)(b)(c)(d)} + \text{(e)} \end{array} \right\}$$

$E \in \mathcal{T}$ is semistable $\overset{\text{def}}{\Longleftrightarrow}$ $\# \phi(E) = 1$

(a) (slicing compatible with central charge)
For $E$ semistable, $\dfrac{Z(E)}{e^{i\pi\phi(E)}} \in \mathbb{R}_{>0}$ ;

(b) (slicing with shift)

$$\phi(E[1]) = \phi(E) + 1$$

(c) (inverse order vanishing)

$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for} \quad \phi(A_1) > \phi(A_2), \; A_1, A_2 \text{ semistable}$$

(d) (HN filtration)
$\forall E \in \mathcal{T}$, denote $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 < \dots < \phi_n$,

$\exists !$ filtration

$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\ & A_1 \arrow[ul, dashed, "+1"] & & & A_n \arrow[ul, dashed, "+1"] \end{tikzcd}$$

s.t. $\phi(A_i) = \phi_i \quad \forall i$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{T}$ with $\phi(E) \subset I$,
$\exists$ a Jordan-Hölder filtration with finite length.

Prop [Okada 05, Prop 2.3]

$$\mathrm{Stab}(\mathcal{T}) \cong \left\{ (\mathcal{A}, Z) \;\middle|\; \begin{array}{l} \mathcal{A} : \text{heart of } \mathcal{T} \\ Z : K(\mathcal{A}) \longrightarrow \mathbb{C} \\ \quad \text{centered slope-function} \\ \quad \text{with HN property} \end{array} \right\}$$

$$\begin{array}{ccc} (Z, \mathcal{P}) & \longmapsto & (\mathcal{P}((0,1]), Z) \\ (Z, \mathcal{P}) & \longleftarrow\!\!| & (\mathcal{A}, Z) \end{array}$$

where $\mathcal{P}(\phi) = \{ E \in \mathcal{A} \text{ semistable} \mid \tilde{\phi}(E) = \phi \} \quad \forall \phi \in (0, 1]$

$\tilde{\phi}(E) = \frac{1}{\pi} \arg Z(E) \in (0,1]$

$E \in \mathcal{A}$ semistable: $\nexists$ dec $0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$ s.t.

$\phi(A_1) > \phi(E) > \phi(A_2)$

## 2. structure of $\mathrm{Coh}(\mathbb{P}^1)$

**Lemma 2.1.** On $\mathbb{P}^1$, we have SESs

$$0 \longrightarrow \mathcal{O} \xrightarrow{\times x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(n)^{\oplus n+1} \longrightarrow \mathcal{O}(n+1)^{\oplus n} \longrightarrow 0 \qquad n \geq 0 \qquad (2.1)$$

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{O}(n) \longrightarrow 0 \qquad n \geq 0$$

which induces triangles

$$\mathcal{O}(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}(k)[1] \xrightarrow{+1}$$

$$\mathcal{O}(k+1)^{\oplus k-n}[-1] \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus k-n+1} \xrightarrow{+1} \qquad n \leq k \qquad (2.2)$$

$$\mathcal{O}(k+1)^{\oplus n-k} \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus n-k+1}[1] \xrightarrow{+1} \qquad n \geq k$$

**Lemma 2.2.** On $\mathbb{P}^1$, we have

$$R\mathrm{Hom}(\mathcal{O}, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

$$R\mathrm{Hom}(\mathcal{O}, k_p) = \mathbb{C}$$

$$R\mathrm{Hom}(k_p, \mathcal{O}) = \mathbb{C}[-1]$$

$$R\mathrm{Hom}(k_p, k_q) = \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1], & p = q \\ 0, & p \neq q \end{cases}$$

*Sketch of proof*

$$R\mathrm{Hom}(\mathcal{O}, \mathcal{O}(n)) = H^\bullet(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

Then apply $R\mathrm{Hom}(\mathcal{O}, -)$, $R\mathrm{Hom}(-, \mathcal{O})$, $R\mathrm{Hom}(-, k_q)$ to

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow k_p \longrightarrow 0$$

$\square$

The next two lemmas tells us the structure of Coh(P^1).

**Lemma 2.3.** [GKR03, last line in p16]

$$\forall \mathcal{F} \in \mathrm{Coh}(\mathbb{P}^1), \quad \mathcal{F} = \left(\bigoplus_p \Xi_p\right) \oplus \left(\bigoplus_i \mathcal{O}(n_i)\right)$$

finite many

$\Xi_p$: a torsion sheaf supported at $p$

**Lemma 2.4.** [GKR03, Prop 6.3]

$$\forall \mathcal{F}^\cdot \in \mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1)), \quad \mathcal{F}^\cdot = \bigoplus_i A_i[-i] \quad A_i \in \mathrm{Coh}(\mathbb{P}^1)$$

It also works for $\mathcal{D}^b(\mathcal{A})$ where $\mathrm{gldim}\,\mathcal{A} = 1$.

E.g 2.5. Since $\mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n)) \cong \mathrm{Ext}^1(k_p,\mathcal{O}) \oplus \mathrm{Ext}^1(k_p,\mathcal{O}(n)) \cong \mathbb{C}^2$, let us describe the extension

$$0 \longrightarrow \mathcal{O}\oplus\mathcal{O}(n) \longrightarrow E \longrightarrow k_p \longrightarrow 0$$

crspd to $(k_1,k_2) \in \mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n))$.

For simplicity, assume that $n>0$ & $k_1, k_2 \neq 0$.

It is defined as pulling back SES:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & k_p \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "\begin{pmatrix} k_1 z & \\ & k_2 z\end{pmatrix}"] & \mathcal{O}(1)\oplus\mathcal{O}(n+1) \arrow[r] & k_p\oplus k_p \arrow[r] & 0
\end{tikzcd} \tag{2.3}$$

Since $\deg E = n+1$, $\operatorname{rank} E = 2$, by Lemma 2.4. we get

$$E = \mathcal{O}\oplus\mathcal{O}(n+1), \quad \mathcal{O}(1)\oplus\mathcal{O}(n) \quad \text{or} \quad \mathcal{O}\oplus\mathcal{O}(n)\oplus k_p$$

but which?

We apply $R\mathrm{Hom}(-,\mathcal{O})$ to (2.3):

$$\begin{tikzcd}
0 & \underset{\mathbb{C}^{n-1}}{\mathrm{Ext}^1(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})} \arrow[l] & \mathrm{Ext}^1(E,\mathcal{O}) \arrow[l] & \underset{\mathbb{C}}{\mathrm{Ext}^1(k_p,\mathcal{O})} \arrow[l] \\
& \underset{\mathbb{C}}{\mathrm{Hom}(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})} \arrow[urr, "k_1"] & \mathrm{Hom}(E,\mathcal{O}) \arrow[l] & \underset{0}{\mathrm{Hom}(k_p,\mathcal{O})} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$\Rightarrow R\mathrm{Hom}(E,\mathcal{O}) = \mathbb{C}^{n-1}[-1]$

$\Rightarrow E \cong \mathcal{O}(1)\oplus\mathcal{O}(n)$.

Q: How to determine (2.3) completely?

We have

$$0 \longrightarrow E \longrightarrow \mathcal{O}(1)\oplus\mathcal{O}(n+1)\oplus k_p \longrightarrow k_p\oplus k_p \longrightarrow 0$$

w.l.o.g assume $p=0$, by pulling back to local charts of $\mathbb{P}^1$, we get

$$\begin{tikzcd}
0 \arrow[r] & E|_{\mathbb{A}^1_z} \arrow[r] \arrow[d, phantom, "\parallel"] & k[z_1]\oplus k[z_2]\oplus k[\Delta]/\Delta \arrow[r] & k[z_1]/(z_1)\oplus k[z_2]/(z_2) \arrow[r] & 0 \\
& k[z_3]\oplus k[z_4] & & & \\
0 \arrow[r] & E|_{\mathbb{A}^1_w} \arrow[r, "\cong"] \arrow[d, phantom, "\parallel"] & k[w_1]\oplus k[w_2] \arrow[r] & 0 \arrow[r] & 0 \\
& k[w_1]\oplus k[w_2] & & &
\end{tikzcd}$$

$$(f_1(z), f_2(z)) \longmapsto (f_1(z), f_1(z)+zf_2(z), -f_1(0))$$

$$(f(z), g(z), a) \longmapsto (f(0)+a, g(0)+a)$$

$$(f(z), g(z), a) \overset{T}{\dashrightarrow} \left(wf(\tfrac{1}{w}), w^{n+1}g(\tfrac{1}{w})\right)$$

Then transition map is given by

$$E|_{\mathbb{A}^1_z} \dashrightarrow E|_{\mathbb{A}^1_w} \qquad (f_1(z), f_2(z)) \mapsto \left(wf_1(\tfrac{1}{w}), w^{n+1}f_1(\tfrac{1}{w})+w^n f_n(\tfrac{1}{w})\right)$$

i.e.
$$\begin{pmatrix} g_1(w) \\ g_2(w)\end{pmatrix} = \begin{pmatrix} w & \\ w^{n+1} & w^n\end{pmatrix}\begin{pmatrix} f_1(\frac{1}{w}) \\ f_2(\frac{1}{w})\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} h_1(w) \\ h_2(w)\end{pmatrix} \overset{def}{=} \begin{pmatrix} g_1(w) \\ g_2(w)-w^n g_1(w)\end{pmatrix} = \begin{pmatrix} w & \\ & w^n\end{pmatrix}\begin{pmatrix} f_1(\frac{1}{w}) \\ f_2(\frac{1}{w})\end{pmatrix}$$

$\Rightarrow E \cong \mathcal{O}(1)\oplus\mathcal{O}(n)$, and (2.3) is

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "{\begin{pmatrix} k_1 z & \\ -k_1 z^n & k_2\end{pmatrix}}"] \arrow[d, equal] & \mathcal{O}(1)\oplus\mathcal{O}(n) \arrow[r, "{(ev_p, 0)}"] \arrow[d, "{\begin{pmatrix} 1 & \\ z^n & z\end{pmatrix}}"] & k_p \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "{\begin{pmatrix} k_1 z & \\ & k_2 z\end{pmatrix}}"] & \mathcal{O}(1)\oplus\mathcal{O}(n+1) \arrow[r] & k_p\oplus k_p \arrow[r] & 0
\end{tikzcd}$$

Shorthand: $\mathrm{Stab}(X) := \mathrm{Stab}(\mathcal{D}^b(\mathrm{Coh}(X)))$ for any variety $X$.

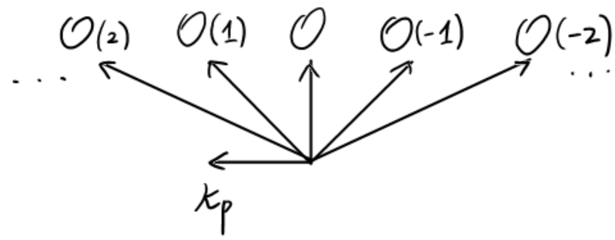
## 3. standard stability

Goal: classify all stability conditions on $\mathbb{P}^1$
where all the l.b.s and $k_p$'s are semistable
($\Rightarrow$ all torsion sheaves are semistable)

E.g. Consider the slope stability $(Z_0, \mathcal{P}_0)$:

$$Z_0(E) = -\deg E + i \cdot \mathrm{rk}(E) \quad \text{e.p.}$$

$$Z_0(\mathcal{O}(n)) = -n + i \qquad \phi(\mathcal{O}(n)) = -\tfrac{1}{\pi} \arg(-n+i)$$
$$Z_0(k_p) = -1 \qquad \phi(k_p) = -1$$




Def: $\mathbb{C}$ acts on $\mathrm{Stab}(\mathbb{P}^1)$ via rotating the $Z$-plane:

$$\mathbb{C} \times \mathrm{Stab}(\mathbb{P}^1) \longrightarrow \mathrm{Stab}(\mathbb{P}^1) \qquad z = x + iy$$
$$z . (Z, \mathcal{P}) = (e^z Z, \mathcal{P}(\cdot - \tfrac{y}{\pi}))$$
$$z . (Z, \phi) = (e^z Z, \phi(\cdot) - \tfrac{y}{\pi})$$

Rmk 3.3. This action changes the heart but preserve the (semi)stability of sheaves, i.e.,

$$E \text{ is (semi)stable in } (Z, \mathcal{P}) \Leftrightarrow E \text{ is (semi)stable in } z.(Z, \mathcal{P}).$$

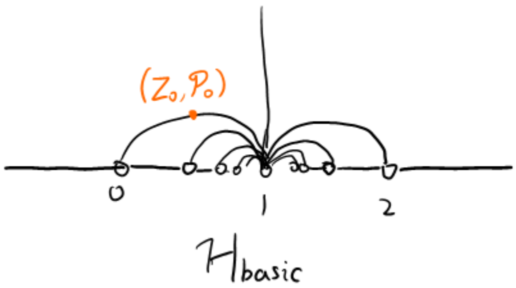
Now denote

$$\mathrm{Stab}_{st}(\mathbb{P}^1) \overset{\mathrm{def}}{=} \{(Z,\mathcal{P}) \in \mathrm{Stab}(\mathbb{P}^1) \mid \text{all } \mathcal{O}(n) \ \& \ k_p\text{'s semistable}\}$$

by Rmk 3.3, it is a $\mathbb{C}$-fiber bundle over

$$\mathrm{Stab}_{st'}(\mathbb{P}^1) \overset{\mathrm{def}}{=} \{(Z,\mathcal{P}) \in \mathrm{Stab}_{st}(\mathbb{P}^1) \mid Z(\mathcal{O}(-1)) = 1, \phi(\mathcal{O}(-1)) = 0\}$$

Prop.
$$\begin{array}{ccc} \mathrm{Stab}_{st'}(\mathbb{P}^1) & \cong & \mathcal{H} \sqcup \mathbb{R} - \left\{ \{1 \pm \frac{1}{n} \mid n \in \mathbb{N}_{>0}\} \sqcup \{1\} \right\} \triangleq \mathcal{H}_{basic} \\ (Z,\mathcal{P}) & \longmapsto & Z(\mathcal{O}) \end{array}$$




Proof Step 1. $Z(\mathcal{O}) \in \mathcal{H}_{basic}$.

$$\mathrm{Hom}(\mathcal{O}(-1), \mathcal{O}) \cong \mathbb{C}^2 \neq 0 \qquad \Rightarrow \phi(\mathcal{O}) \geq 0$$

$$\mathcal{O} \longrightarrow k_p \longrightarrow \mathcal{O}(-1)[1] \xrightarrow{+1} \qquad \Rightarrow \phi(\mathcal{O}) \leq 1$$
is not an HN-filtration

$$0 \neq Z(\mathcal{O}(n)) = (n+1)Z(\mathcal{O}) - nZ(\mathcal{O}(-1)) \qquad \Rightarrow Z(\mathcal{O}) \notin \{1 \pm \tfrac{1}{n} \mid n \in \mathbb{N}_{>0}\} \sqcup \{1\}$$

Step 2. For each $z \in \mathcal{H}_{basic}$, construct ! $(Z,\mathcal{P}) \in \mathrm{Stab}_{st'}(\mathbb{P}^1)$ s.t. $Z(\mathcal{O}) = z$.

Take
$$Z(\mathcal{O}(n)) = (n+1)z - n$$
$$Z(k_p) = z - 1,$$
that ! determines $(Z,\mathcal{P}) \in \mathrm{Stab}_{st'}(\mathbb{P}^1)$