Eine Woche, ein Beispiel

4.10. non-Archimedean local field F

wiki: local field
See https://mathoverflow.net/questions/17061/locally-profinite-fields for different definition of local fields. We follow wiki instead.

Classification:

- finite extension of $\mathbb{Q}_p$
- $\mathbb{F}_q((T))$ $\quad (q = p^r)$

Process:

1. Basic structures and results.
2. Topological results.
3. representation of $(F, +)$ and $F^\times$ (next week)

1. Basic structures and results

1.1. None of them is alg closed.

1.2. The natural valuation $v: F \longrightarrow \mathbb{Z}$ is defined. Then

$$\mathcal{O},\ \mathfrak{p},\ k = \mathcal{O}/\mathfrak{p} \qquad p = \mathrm{char}\, k,\quad q = |k| = p^r$$

$$\mathcal{U}^{(0)} = \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\} \qquad \mathcal{U}^{(n)} = 1 + \mathfrak{p}^n \quad n \geq 1$$

are defined, and $\pi \in \mathfrak{p}\mathcal{O}^\times \cong \mathfrak{p} - \mathfrak{p}^2$ is picked.

Moreover, $\mathcal{O}$ is DVR, k is finite,

$$\mathcal{U}^{(0)}/\mathcal{U}^{(1)} \overset{\text{split-iso}}{\cong} k^\times \qquad \mathcal{U}^{(n)}/\mathcal{U}^{(n+1)} \overset{\text{non-canonical}}{\cong} k$$

$$\begin{tikzcd}
 & & & \mu_{q-1} \arrow[dl] \arrow[d, "\cong"] & \\
0 \arrow[r] & \mathcal{U}^{(1)} \arrow[r] & \mathcal{O}^\times \arrow[r] & k^\times \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

$\mu_{q-1} = \{\alpha \in F \mid \alpha^{q-1} = 1\}$

: the Teichmüller lift

$$\Rightarrow \mathcal{O}^\times \cong \mathcal{U}^{(1)} \times \mu_{q-1}.$$

1.3. $$F^\times \cong \langle\pi\rangle \times \mathcal{O}^\times \cong \langle\pi\rangle \times \mu_{q-1} \times \mathcal{U}^{(1)}$$

e.g. when $F = \mathbb{Q}_p$, $\mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p & p \neq 2 \\ \mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2) & p = 2 \end{cases}$

Thm. When $p \geq 3$, $(p\mathbb{Z}_p, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (1 + p\mathbb{Z}_p, \cdot)$ is an iso as topological gps.

2. Topological results.

$\mathcal{O}$ is cpt and profinite group, while F is loc. cpt and loc. profinite group.

Cpt open subgps of $(F,+)$ are $\{\mathfrak{p}^k\}$.

Cpt open subgps of $F^\times$ are not restricted in $\{U^{(k)}\}$,

but $\{U^{(k)}\}$ is a nbhd system of $F^\times$, i.e.,

$\{aU^{(k)}\}_{a\in F^\times}$ is a topological basis of $F^\times$.

$\{\text{open subgps}\} \subseteq \{\text{closed subgps}\}$ for $(F,+)$ and $F^\times$.

Q: Are there any other cpt closed subgp?

A. Yes. e.g. $\{0\} \subseteq (F,+)$ $\{1\} \subseteq F^\times$

Q: Can we classify all cpt closed subgp?

E.g. $\mathbb{Q}_{p^r} := $ the splitting field of $X^q - X$ over $\mathbb{Q}_p$ $q = p^r$

$=$ the unique unramified extension of $\mathbb{Q}_p$ of degree r

$$\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p) \cong \mathbb{Z}/r\mathbb{Z}$$