Eine Woche, ein Beispiel

10.5 cohomology of $\mathcal{A}_g$ and $\mathcal{M}_g$

This document aims at a collection of the known results. As usual, I'm not an expert, but often I need to make these documents to clear my brain.

Ref:

[vdG11] Van Der Geer, Gerard. "The Cohomology of the Moduli Space of Abelian Varieties." arXiv:1112.2294. Preprint, arXiv, December 10, 2011. https://doi.org/10.48550/arXiv.1112.2294.

Tautological ring R_i

		0	2	4	6	8	10	12	14	$\cdots$		
	R_1	$\mathbb{Q}$ ϕ	$\mathbb{Q}$ 1									
	R_2	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$ 2	$\mathbb{Q}$ 2+1							
	R_3	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}^2$ 3	$\mathbb{Q}$ 3+1	$\mathbb{Q}$ 3+2	$\mathbb{Q}$ 3+2+1				
	R_4	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}^2$	$\mathbb{Q}^2$ 4	$\mathbb{Q}^2$ 4+1	$\mathbb{Q}^2$ 4+2	$\mathbb{Q}^2$ 4+3 4+2+1	$\mathbb{Q}$ 4+3+1	$\mathbb{Q}$ 4+3+2	$\mathbb{Q}$ 4+3+2+1
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$		
A000009 number of distinct partitions	R_∞	$\mathbb{Q}$ ϕ	$\mathbb{Q}$ 1	$\mathbb{Q}$ 2	$\mathbb{Q}^2$ 3 2+1	$\mathbb{Q}^2$ 4 3+1	$\mathbb{Q}^3$ 5 4+1 3+2	$\mathbb{Q}^4$ 6 5+1 4+2 3+2+1	$\mathbb{Q}^5$ 7 6+1 5+2 4+3 4+2+1	$\cdots$		

Rmk. 1. $\quad R_i \ncong \mathbb{Q}[\lambda_1, \dots, \lambda_i]/(\lambda_1^2, \dots, \lambda_i^2)$

but

$$\mathrm{Gr}\, R_i \cong \mathbb{Q}[\lambda_1, \dots, \lambda_i]/(\lambda_1^2, \dots, \lambda_i^2)$$

In fact,

$$R_i \cong \mathbb{Q}[\lambda_1, \dots, \lambda_i]/\big((1+\lambda_1 + \cdots + \lambda_i)(1 - \lambda_1 + \cdots + (-1)^i) - 1\big)$$

2. In geometry,

$$\lambda_i = c_i(\mathbb{E}) \qquad i = 1, \dots, g$$

is the Chern class of the Hodge bundle $\mathbb{E}$.

$$\begin{array}{c} \mathbb{E} \\ | \\ \mathcal{A}_g \end{array} \qquad \begin{array}{c} T_0^* A \\ \downarrow \\ [A] \end{array}$$

When we view $R_{g-1} \subset CH_{\mathbb{Q}}(\mathcal{A}_g)$, λ_g vanishes;
when we view $R_g \subset CH_{\mathbb{Q}}(\bar{\mathcal{A}}_g^{\,tor})$, λ_g does not vanish.

toroidal compactification in Faltings - Chai.

Chow Rings of $\widetilde{\mathcal{A}}_g$ [vdG11, 7]

$$CH_{\mathbb{Q}}(\widetilde{\mathcal{A}}_1) \cong \mathbb{Q}[\lambda_1]/(\lambda_1^2)$$

$$CH_{\mathbb{Q}}(\widetilde{\mathcal{A}}_2) \cong \mathbb{Q}[\lambda_1, \lambda_2, \sigma_1]/I_2$$

$$I_2 = \left\langle \begin{array}{l} (1+\lambda_1+\lambda_2)(1-\lambda_1+\lambda_2) - 1, \\ \lambda_2\sigma_1, \\ \sigma_1^2 - 22\lambda_1\sigma_1 + 120\lambda_1^2 \end{array} \right\rangle$$

$$CH_{\mathbb{Q}}(\widetilde{\mathcal{A}}_3) \cong \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \sigma_1, \sigma_2]/I_3$$

$$I_3 = \left\langle \begin{array}{l} (1+\lambda_1+\lambda_2+\lambda_3)(1-\lambda_1+\lambda_2-\lambda_3) - 1, \\ \lambda_3\sigma_1, \ \lambda_3\sigma_2, \ \lambda_1^2\sigma_2, \\ \sigma_1^3 - 2016\lambda_3 + 4\lambda_1^2\sigma_1 + 24\lambda_1\sigma_2 - \frac{11}{3}\sigma_2\sigma_1, \\ \sigma_2^2 - 360\lambda_1^3\sigma_1 + 45\lambda_1^2\sigma_1^2 - 15\lambda_1\sigma_2\sigma_1, \\ \sigma_1^2\sigma_2 - 1080\lambda_1^3\sigma_1 - 165\lambda_1^2\sigma_1^2 + 47\lambda_1\sigma_2\sigma_1 \end{array} \right\rangle$$