Eine Woche, ein Beispiel

7.4. Polyvector parallelograms

ref: https://pbelmans.ncag.info/blog/2018/11/22/polyvector-parallelogram/#parallelogram

we will make a little varience for the notation, e.g. $g^{i,j} := H^j(X, \Lambda^i T_X)$

Recall: (alg surf)

$\chi(\mathcal{O}_X)\ \chi(\Omega_X)\ \chi(\omega_X)$ $\qquad e := \chi_{top}(X)$

$$\begin{array}{ccccc|c|ccccc}
 & & 1 & & & b_4 & & & h^{2,2} & & \\
 & q & & q & & b_3 & & h^{1,2} & & h^{2,1} & \\
p_g & & h^{1,1} & & p_g & b_2 & h^{0,2} & & h^{1,1} & & h^{2,0} \\
 & q & & q & & b_1 & & h^{0,1} & & h^{1,0} & \\
 & & 1 & & & b_0 & & & h^{0,0} & &
\end{array}$$

p_a

$c_2^2 = e$

$c_1 = K^2 = 12\chi(\mathcal{O}_X) - e$

we also have polyvector parallelograms:

$\chi(\mathcal{O}_X)\ \chi(T_X)\ \chi(\omega_X^\vee)$

$$\begin{array}{ccccc|c|ccccc}
 & & g^{2,2} & & & HH^4 & & & g^{2,2} & & \\
 & g^{1,2} & & g^{2,1} & & HH^3 & & g^{1,2} & & g^{2,1} & \\
p_g & & g^{1,1} & & g^{2,0} & HH^2 & g^{0,2} & & g^{1,1} & & g^{2,0} \\
q & & g^{1,0} & & & HH^1 & & g^{0,1} & & g^{1,0} & \\
 & 1 & & & & HH^0 & & & g^{0,0} & &
\end{array}$$

p_a

e.g. $\dim X = 1$ $\quad$ X cplx alg curve, smooth

$$\begin{array}{ccc} & 0 & \\ 0 & & 3 \\ & 1 & \end{array}\ \mathbb{P}^1
\qquad
\begin{array}{ccc} & 1 & \\ 1 & & 1 \\ & 1 & \end{array}\ \text{ellip curve}
\qquad
\begin{array}{ccc} & 3 & \\ 2 & & 0 \\ & 1 & \end{array}\ \text{genus 2}
\qquad
\begin{array}{ccc} & 6 & \\ 3 & & 0 \\ & 1 & \end{array}\ \text{genus 3}
\qquad
\begin{array}{ccc} & 9 & \\ 4 & & 0 \\ & 1 & \end{array}\ \text{genus 4}
\qquad
\begin{array}{ccc} & 3n-6 & \\ n & & 0 \\ & 1 & \end{array}\ \text{genus } n$$

e.g. $\dim X = 2$ $\quad$ X cplx alg surf, smooth

$$\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 9 \\ & 0 & & 6 & \\ & & 1 & & \end{array} \\ \mathbb{P}^1\times\mathbb{P}^1
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 9 \\ & 0 & & 6 & \\ & & 1 & & \end{array} \\ Bl_1\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 9 \\ & 0 & & 7 & \\ & & 1 & & \end{array} \\ \mathbb{F}_2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 9 \\ & 0 & & 8 & \\ & & 1 & & \end{array} \\ \mathbb{F}_3
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 1 & \\ 0 & & 3 & & 10 \\ & 0 & & 9 & \\ & & 1 & & \end{array} \\ \mathbb{F}_4
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & n-3 & \\ 0 & & n-1 & & n+6 \\ & 0 & & n+5 & \\ & & 1 & & \end{array} \\ \mathbb{F}_n
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 10 \\ & 0 & & 8 & \\ & & 1 & & \end{array} \\ \mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 9 \\ & 0 & & 6 & \\ & & 1 & & \end{array} \\ Bl_1\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 7 \\ & 0 & & 2 & \\ & & 1 & & \end{array} \\ Bl_2\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 6 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \\ Bl_3\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 5 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \\ Bl_4\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 4 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \\ Bl_5\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 6 & & 3 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \\ Bl_6\mathbb{P}^2
\end{array}
\quad
\begin{array}{c}
\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 8 & & 2 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \\ Bl_7\mathbb{P}^2
\end{array}$$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$
K3

$$\begin{array}{ccccc} & & ? & & \\ & ? & & 0 & \\ 0 & & 10 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$
Enriques

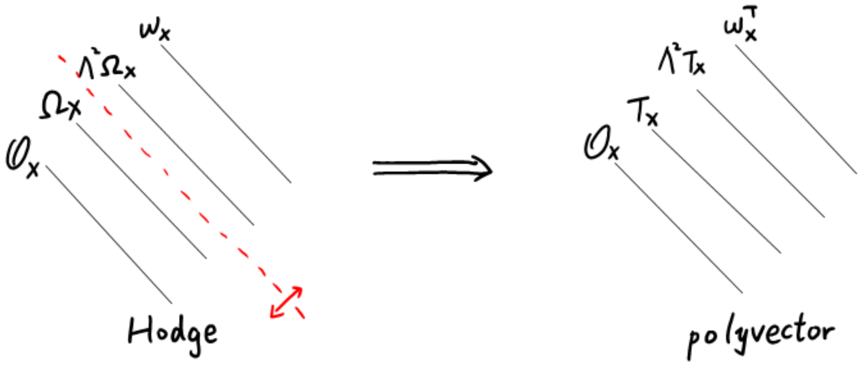
$$\begin{array}{ccccc} & & ? & & \\ & 2 & & 0 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$
abelian

$$\begin{array}{ccccc} & & ? & & \\ & ? & & ? & \\ 0 & & ? & & ? \\ & 1 & & ? & \\ & & 1 & & \end{array}$$
bielliptic

e.g. Calabi-Yao 3-fold

[Use $\Lambda^i\varepsilon \otimes \Lambda^{r-i}\varepsilon \longrightarrow \det\varepsilon \quad \Rightarrow \Lambda^i\varepsilon \cong \Lambda^{r-i}\varepsilon^\vee \otimes \det\varepsilon$

then
$$\mathcal{O}_X = \Lambda^0 T_X \cong \Lambda^3\Omega_X = \omega_X$$
$$T_X = \Lambda^1 T_X \cong \Lambda^2\Omega_X$$
$$\Lambda^2 T_X \cong \Lambda^1\Omega_X = \Omega_X$$
$$\omega_X^\vee = \Lambda^3 T_X \cong \Lambda^0\Omega_X = \mathcal{O}_X$$
]



one case:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \Longrightarrow \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Hodge $\qquad$ polyvector

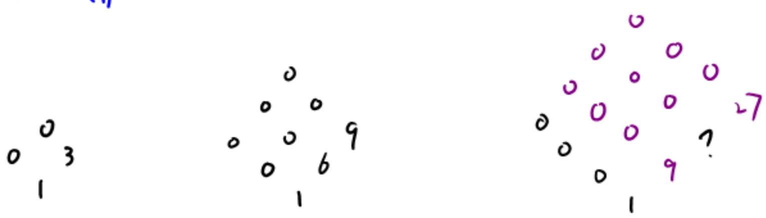
e.g. projective space $\mathbb{P}^n$

$$\begin{array}{ccc} & 0 & \\ 0 & & 3 \\ & 1 & \end{array}$$
$\mathbb{P}^1$

$$\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 0 & & 0 & & 10 \\ & 0 & & 8 & \\ & & 1 & & \end{array}$$
$\mathbb{P}^2$

$\mathbb{P}^0$	1							
$\mathbb{P}^1$	1	3						
$\mathbb{P}^2$	1	8	10					
$\mathbb{P}^3$	1	15	45	35				
$\mathbb{P}^4$	1	24	126	224	126			
$\mathbb{P}^5$	1	35	280	840	1050	462		
$\mathbb{P}^6$	1	48	540	2400	4950	4752	1716	
$\mathbb{P}^7$	1	63	945	5775	17325	27027	21021	6435

Q: Does the combinational identity $\sum_{i=0}^{n+1} (-1)^i \frac{(n+i)!}{i!\,i!\,(n+1-i)!} = 0$ have some easier explanation? (Maybe related to Weyl character formula) e.g. $\mathbb{P}^1 \times \ldots \times \mathbb{P}^1$



$\mathbb{P}^1$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1$ $\qquad$ $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

How to use $\Omega_{A \times B/k} \cong \tilde{\alpha}^* \Omega_{B/k} \oplus \tilde{\beta}^* \Omega_{A/k} \ldots$

Maybe the result can be computed by this package:
https://www2.math.upenn.edu/StringMath2011/notes/Jurke_StringMath2011_talk.pdf