Modular form

4 bonus - Lambert series

Ex. $\wp''(z) = 6\wp(z)^2 - 30G_4$ (expansion at 0 or direct calculation)

Ex. $E_6(i) = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{n^5}{e^{2\pi n}-1} = \frac{1}{504}$

$E_4(\rho) = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{n^3}{e^{-2\pi i\rho n}-1} = -\frac{1}{240}$

30° $\quad -2\pi i\rho$

$E_2(i) = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{n}{e^{2\pi n}-1} = \frac{1}{24}\left(1-\frac{3}{\pi}\right)$

In general,

$$E_k(\tau) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)\, q^n = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\frac{n^{k-1}q^n}{1-q^n}$$

Lambert series: $\displaystyle\sum_{n=1}^{\infty}\frac{a(n)\,q^n}{1-q^n} = \sum_{n=1}^{\infty}\Big(\sum_{m|n} a(m)\Big) q^n$ $\qquad a(n)\in\mathbb{C}$

need some convergence conditions

Q: $E_4(i) = ?$

Let $\varpi \in \mathbb{R}_{>0}$ s.t. $\mathbb{C}/\varpi(\mathbb{Z}\oplus i\mathbb{Z}) \xrightarrow{\sim} V(y^2 = 4x^3 - 4x)$, i.e. $4 = 60\,G_4(\varpi(\mathbb{Z}\oplus i\mathbb{Z}))$.

It can be proved [NT II, §9.5, P347, P351] that

$$\varpi \overset{\text{hard}}{=} 2\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(\frac14)^2}{2^{\frac32}\pi^{\frac12}} = \text{semiperimeter of } \infty = 2.62205\ldots$$

$r^2 = \cos(2\theta)$

where $\Gamma(\frac14) = 3.6256099\ldots$

As a comparison,

$$\pi = 2\int_0^1\frac{dx}{\sqrt{1-x^2}} = 3.14159\ldots$$

Ex. show that

$$E_4(i) = 3\frac{\varpi^4}{\pi^4} = \frac{3\Gamma(\frac14)^8}{(2\pi)^6} \Rightarrow \sum_{n=1}^{\infty}\frac{n^3}{e^{2\pi n}-1} = \frac{1}{80}\left(\frac{\varpi}{\pi}\right)^4 - \frac{1}{240}$$

$$G_4(i) = \frac{1}{15}\varpi^4$$

$$G_{4k}(i) = \frac{(2\varpi)^{4k}}{(4k)!}e_k \quad \text{[NT II, § 9.5, P352]}$$

where

k	1	2	3	4	5	⋯
Hurwitz number e_k	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{567}{130}$	$\frac{43659}{170}$	$\frac{392931}{10}$	⋯

Intermediate result: for $m \geq 2$, we have

$$G_{m+4} = \frac{6}{(m+3)(m-2)(m+5)}\sum_{\substack{i+j=m\\ i,j\geq 1}}(i+1)(j+1)\,G_{i+2}\,G_{j+2}$$

$\mathbb{Q}$: How to show that

$$\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n}-1)} = -\frac{\pi}{12} - \frac{1}{2}\log\left(\frac{\varpi}{\sqrt{2}\pi}\right) \ ? \quad [\text{NT II, p301}]$$

Not rigorous: need

$$\text{“}E_0(i)\text{”} = -\frac{6}{\pi}\log\left(\frac{\varpi}{\sqrt{2}\pi}\right) = 1 + \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n(e^{2\pi n}-1)}$$

Thm. [Za p85] (1976, G.V. Chudnovsky)

$\forall\, \tau \in \mathcal{H}$, at least two of three numbers $E_2(\tau), E_4(\tau), E_6(\tau)$ are alg indep.

Cor. $\Gamma(\frac{1}{4})$ is transcendental over $\mathbb{Q}(\pi)$.

Cor. $\forall\, \tau$ CM pt. $E_2(\tau)$ is transcendental over $\mathbb{Q}$,

$$\operatorname{tr\,deg} \mathbb{Q}(E_2(\tau), E_4(\tau), E_6(\tau))/\mathbb{Q} = 2.$$