Eine Woche, ein Beispiel

4.10. non-Archimedean local field F

wiki: local field
See https://mathoverflow.net/questions/17061/locally-profinite-fields for different definition of local fields. We follow wiki instead.

Classification:

- finite extension of $\mathbb{Q}_p$
- $\mathbb{F}_q((T))$ $\quad (q=p^r)$

Process:

1. Basic structures and results.
2. Topological results.
3. Haar measure
4. Representation of $(F,+)$ and $F^\times$ (next week)

1. Basic structures and results

1.1. None of them is alg closed.

1.2. The natural valuation $v: F \longrightarrow \mathbb{Z}$ is defined. Then

$\mathcal{O},\ \mathfrak{p},\ k = \mathcal{O}/\mathfrak{p}$ $\qquad p = \operatorname{char} k,\quad q = |k| = p^r$

$\mathcal{U} = \mathcal{U}^{(0)} = \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\}$ $\qquad \mathcal{U}^{(n)} = 1 + \mathfrak{p}^n \quad n \geq 1$

are defined, and $\pi \in \mathfrak{p}\mathcal{O}^\times \cong \mathfrak{p} - \mathfrak{p}^2$ is picked.

Moreover, $\mathcal{O}$ is DVR, k is finite,

$$\mathcal{U}^{(0)}/\mathcal{U}^{(1)} \overset{\text{split iso}}{\cong} k^\times \qquad \mathcal{U}^{(n)}/\mathcal{U}^{(n+1)} \overset{\text{non-canonical}}{\cong} k \quad n \geq 1$$

$$\mathcal{U}^{(0)}/\mathcal{U}^{(n)} \overset{\text{non split iso}}{\cong} (\mathcal{O}/\mathfrak{p}^n)^\times \quad n \geq 1 \qquad \mathcal{U}^{(m)}/\mathcal{U}^{(n+1)} \overset{\text{non-canonical}}{\cong} \mathcal{O}/\mathfrak{p}^{m-n} \quad 2n+1 \geq m > n \geq 0$$

$$\begin{tikzcd}
 & & & \mu_{q-1} \arrow[dl] \arrow[d, "\cong"] & \\
0 \arrow[r] & \mathcal{U}^{(1)} \arrow[r] & \mathcal{O}^\times \arrow[r] & k^\times \arrow[r] \arrow[l, bend right] & 0
\end{tikzcd}$$

$\mu_{q-1} = \{a \in F \mid a^{q-1} = 1\}$

$\curvearrowleft$: the Teichmüller lift

$$\Rightarrow \mathcal{O}^\times \cong \mathcal{U}^{(1)} \times \mu_{q-1}.$$

1.3. $F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times \cong \langle \pi \rangle \times \mu_{q-1} \times \mathcal{U}^{(1)}$

e.g. when $F = \mathbb{Q}_p$, $\mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p & p \neq 2 \\ \mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2) & p = 2 \end{cases}$

Thm. When $p \geq 3$, $(p\mathbb{Z}_p, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (1+p\mathbb{Z}_p, \cdot)$ is an iso as topological gps.

2. Topological results.

$\mathcal{O} = \varprojlim_n \mathcal{O}/\mathfrak{p}^n$ is cpt and profinite group, while F is loc. cpt and loc. profinite group.
$\mathcal{O}^\times = \varprojlim_n \mathcal{O}^\times/\mathcal{U}^{(n)}$ is cpt and profinite group, while $F^\times$ is loc. cpt and loc. profinite group.

Cpt open subgps of $(F,+)$ are $\{\mathfrak{p}^k\}$.
Cpt open subgps of $F^\times$ are not restricted in $\{\mathcal{U}^{(k)}\}$,
but $\{\mathcal{U}^{(k)}\}$ is a nbhd system of $F^\times$, i.e.,
$\{a\,\mathcal{U}^{(k)}\}_{a\in F^\times}$ is a topological basis of $F^\times$.

$\{\text{open subgps}\} \subseteq \{\text{closed subgps}\}$ for $(F,+)$ and $F^\times$.
Q: Are there any other cpt closed subgp?
A. Yes. e.g. $\{0\} \subseteq (F,+)$ $\{1\} \subseteq F^\times$
Q: Can we classify all cpt closed subgp?

E.g. $\mathbb{Q}_{p^r} :=$ the splitting field of $X^q - X$ over $\mathbb{Q}_p$ $q = p^r$
$=$ the unique unramified extension of $\mathbb{Q}_p$ of degree r

$\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p) \cong \mathbb{Z}/r\mathbb{Z}$

3. Haar measure

Main reference: The Local Langlands Conjecture for GL(2) by Colin J. BushnellGuy Henniart.
[https://link.springer.com/book/10.1007/3-540-31511-X]
Ref: https://en.wikipedia.org/wiki/Haar_measure

G: loc. profinite gp

$C^\infty(G) := \{ f: G \to \mathbb{C} \mid f \text{ is loc. const} \}$

$C_c^\infty(G) := \{ f \in C^\infty(G) \mid \operatorname{supp} f \subset G \text{ is cpt} \}$

Rmk. G has topo basis $\{gK\}_{\substack{g\in G \\ K \leq G \text{ cpt open}}}$.

$\forall f \in C_c^\infty(G)$, $\exists K \leq G$ cpt open, s.t.

$$f = \sum_{g\in G} a_g \mathbb{1}_{KgK} \qquad a_g \in \mathbb{C} \qquad \#\{g\in G \mid a_g \neq 0\} < +\infty$$

e.g. When $G = (F,+)$, $C_c^\infty(F) = \langle a + \beta^k \rangle_{\substack{a\in F \\ k\in\mathbb{Z}}}$

when $G = F^\times$, $C_c^\infty(F^\times) = \langle a U^{(k)} \rangle_{\substack{a\in F^\times \\ k\in\mathbb{Z}_{\geq 0}}}$

Def (Left Haar integral & Left Haar measure)

integral: $I: C_c^\infty(G) \longrightarrow \mathbb{C}$ s.t

- (left invarient) $I(f(g\cdot)) = I(f(\cdot))$ $\qquad \forall f \in C_c^\infty(G) \quad g \in G$
- (positive) $I(f) \geq 0$ $\qquad \forall f \in C_c^\infty(G) \quad f \geq 0$

measure: $\mu_G : \mathcal{L}(G) \longrightarrow \mathbb{R}$ $\qquad S \subset G$ cpt open $\mapsto I(\mathbb{1}_S)$

Lebesque σ-algebra, see https://math.stackexchange.com/questions/3117419/lebesgue-sigma-algebra

The domain of I is not extended, so here it is not perfect.

. relation/notation: $I(f) = \int_G f(g)\, d\mu_G(g)$

Rmk.

Left Haar measure exists and is unique(up to scalar) on every loc. cpt gp G, see https://www.diva-portal.org/smash/get/diva2:1564300/FULLTEXT01.pdf

Later on, Haar measure = left + right Haar measure.

E.g. Let μ be the Haar measure on F, then $\mu^\times$ is a Haar mesure on $F^\times$, and $\left(d\mu^\times(x) = \frac{d\mu(x)}{\|x\|}\right)$

$$\int_{F^\times} f(x)\, d\mu^\times(x) = \int_F f(x) \frac{d\mu(x)}{\|x\|} \qquad \forall f \in C_c^\infty(F^\times) \subset C_c^\infty(F)$$

Let μ be the Haar measure on $A := M_{n\times n}(F)$, then $\mu^\times$ is a Haar measure on $G := GL_n(F)$, and $\left(d\mu^\times(g) = \frac{d\mu(g)}{\|\det g\|^n}\right)$

$$\int_G f(g)\, d\mu^\times(g) = \int_A f(g) \frac{d\mu(g)}{\|\det g\|^n} \qquad \forall f \in C_c^\infty(G) \subset C_c^\infty(A)$$

Def Unimodular: left Haar measure = right Haar measure

Rmk. G is cpt $\Rightarrow$ G is unimodular $\Leftrightarrow \delta_G = 1$

G is abelian $\Rightarrow$ $G/Z(G)$ is $\Updownarrow$ unimodular

where $\delta_G : G \longrightarrow \mathbb{C}^\times$ is determined by

$d\mu_G(g^{-1}xg) \overset{\text{left inv}}{=\!=} d\mu_G(xg) = \delta_G(g)\, d\mu_G(x)$.

Actually, $\forall K \leq G$ cpt open, $\delta_G|_K = \mathbb{1}_K$.

e.g. $(F, +), (\mathcal{O}, +), F^\times, \mathcal{O}^\times$ are all unimodular.

e.g. $G = GL_2(\mathbb{Q}_p)$ is unimodular, while

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $M = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ are not unimodular.

https://math.stackexchange.com/questions/323017/are-nilpotent-lie-groups-unimodular
https://mathoverflow.net/questions/114704/is-every-subgroup-of-a-connected-unimodular-matrix-lie-group-also-unimodular