Eine Woche, ein Beispiel
10.2 equivariant K-theory of Steinberg variety

Task. Complete the following tables:

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
G	$\mathbb{Z}[x^*(T)]^W$	$\mathbb{Z}[x^*(T)]$	$\mathbb{Z}[x^*(T)]\otimes_{\mathbb{Z}[x^*(T)]^W}\mathbb{Z}[x^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[x^*(T)]$	$\mathbb{Z}[x^*(T)]\otimes_{\mathbb{Z}[x^*(T)]^W}\mathbb{Z}[x^*(T)]$	$\mathbb{Z}[x^*(T)]\otimes_{\mathbb{Z}[x^*(T)]^W}\mathbb{Z}[x^*(T)]\otimes_{\mathbb{Z}[x^*(T)]^W}\mathbb{Z}[x^*(T)]$	
Id	$\mathbb{Z}$			$\mathbb{Z}[x^*(T)]/I_T \rtimes \mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$\mathbb{Z}[x^*(T)]^W[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[x^*(T)][t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

We use the shorthand.

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
G	$R(T)^W$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$	
Id	$\mathbb{Z}$			$R(T)/I_T \rtimes \mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$R(G)[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$R(T)[t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$$R(B) = \mathbb{Z}[x^*(T)] \qquad = \mathcal{H}(\hat{T}(F),\hat{T}(\mathcal{O}_F))$$
$$R(G) = \mathbb{Z}[x^*(T)]^W \qquad \neq \mathcal{H}(\hat{G}(F),\hat{G}(\mathcal{O}_F))$$
$$R(G)[q^{\pm\frac12}] = \mathbb{Z}[x^*(T)]^W[q^{\pm\frac12}] = \mathcal{H}_{sph}[q^{\pm\frac12}]$$
$$R(G\times\mathbb{C}^\times) = \mathbb{Z}[x^*(T)]^W[t^{\pm1}]$$
$$K^{G\times\mathbb{C}^\times}(St) = \mathcal{H}_{ext} \qquad \neq \mathcal{H}(\hat{G}(F), I)$$

Here is an initial example.

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1}]/((z_1-z_2)(z_1-z_2^{-1}))$	$\mathbb{Z}[W_{ext}] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]$
B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1},z]/(z-y)(z-y^{-1})$	$\mathbb{Z}[y^{\pm1},z_1,z_2]/((z_1-y)(z_1-y^{-1}),(z_2-y)(z_2-y^{-1}))$	
Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$	$R(T)/I_T \rtimes \mathbb{Z}[W_f] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]/(z_w-1)^2$
$SL_2\times\mathbb{C}^\times$	$\mathbb{Z}[x,t^{\pm1}]$			$\mathcal{H}_{ext} = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1},t^{\pm1}]$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[y^{\pm1},t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$K^-(-)$	pt	$\mathcal{F}_{\underline{d}}$ $\widetilde{Rep}_{\underline{d}}(\Omega)$	$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$\mathcal{Z}_{\underline{d}} = \coprod_{\underline{d},\underline{d}'} \mathcal{Z}_{\underline{d},\underline{d}'}$
$G_{\underline{d}}$	$R(T_{\underline{d}})^{W_{\underline{d}}}$	$R(T_{\underline{d}})$	$R(T_{\underline{d}}) \otimes_{R(G_{\underline{d}})} R(T_{\underline{d}})$	
$B_{\underline{d}}$	$R(T_{\underline{d}})$	$R(T_{\underline{d}}) \otimes_{R(G_{\underline{d}})} R(T_{\underline{d}})$	$R(T_{\underline{d}}) \otimes_{R(G_{\underline{d}})} R(T_{\underline{d}}) \otimes_{R(G_{\underline{d}})} R(T_{\underline{d}})$	
Id	$\mathbb{Z}$			
$G_{\underline{d}} \times \mathbb{C}^\times$	$R(G_{\underline{d}})[t^{\pm 1}]$			
$B_{\underline{d}} \times \mathbb{C}^\times$	$R(T_{\underline{d}})[t^{\pm 1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$			

Conclusions on the summer vacation.

Some possible tasks to work on.

1. Work out what $K_0^{Id}(\mathcal{B})$ is.

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.
   - Bernstein presentation
   - try to understand the center of $\mathcal{H}(G(F), I)$
   - How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory
   - How can the Hecke algebra be realized as a Hecke algebra?

   ref. [Hecke, Sec 10-17] , [Williamson 11.4 - 12.2]

3. Try to understand what the Hall algebra / Quantum group is.

   ref: [Lec 1-4, Appendix 4, https://arxiv.org/pdf/math/0611617.pdf]

   - understand $\mathcal{H}_{Rep_k^{nil}(Q)}$ where $Q = \cdot \xrightarrow{\cdot} \cdot \circlearrowleft$ [Lec 2-3]
   - understand $\mathcal{H}_{\mathbb{P}^1} \cong \mathcal{U}_v(\widehat{\mathfrak{sl}_2})$ [Lec 4]

$$\cup \qquad\qquad \cup$$
$$\mathcal{H}_{Tor(\mathbb{P}^1)} \cong \bigotimes_{x \in \mathbb{P}^1} \mathcal{H}_{Tor\, x}$$

   - define (Quantum) Kac-Moody / loop algs   [Appendix 4]
   - Why is that
   graded
$$K_0(Rep^{\mathbb{Z}}(R)) = \mathcal{U}_q(n(Q))$$

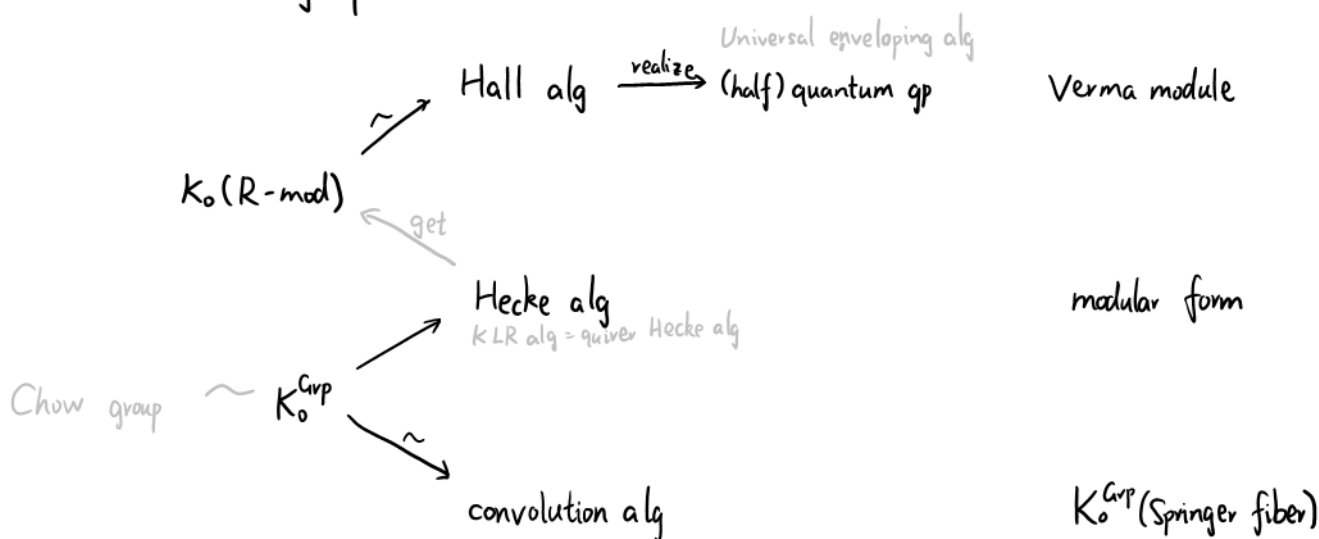   where
$$R = \bigoplus_{\underline{d}} H_\bullet^{G \times \mathbb{C}^\times, BM}(Z_{\underline{d}})$$

   and what is
$$K_0\left(Rep^{\mathbb{Z}}\left(\bigoplus_{\underline{d}} K_0^{G \times \mathbb{C}^\times}(Z_{\underline{d}})\right)\right) ?$$

4. Work out the big picture

Hall alg $\xrightarrow{\text{realize}}$ (half) quantum gp   Verma module

$K_0(R\text{-mod})$

$\sim$ _get_

Hecke alg   modular form

KLR alg = quiver Hecke alg

Chow group $\sim$ K_0^{Grp}

$\sim$

convolution alg $K_0^{Grp}(\text{Springer fiber})$

5. A closer check of Satake iso

K_0 combinations   Hecke alg

$R(B) = \mathbb{Z}[X^*(T)]$ $= \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F))$

$R(G) = \mathbb{Z}[X^*(T)]^W$ $\neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F))$

$R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$

$R(G \times \mathbb{C}^*) = \mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$

$K^{G\times\mathbb{C}^*}(St) = \mathcal{H}_{ext}$ $\neq \mathcal{H}(\hat{G}(F), I)$

It's claimed by my schoolmate that

$K_0(\text{Perv}_B(G/B)) \cong \mathcal{H}(G, B)$

↑
sym monoidal structure
induced from the convolution

then, what is

$K_0^B(\mathcal{B}) \cong$?

$K_0^{Id}(\mathcal{B}) \cong$?

? $\cong \mathcal{H}(S_{m+n}, S_m \times S_n)$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces.
   (e.p. in the case of $\cdot$, $\cdot\circlearrowleft$, $\cdot\rightarrow\cdot$)

   G, B, T

   $\mathcal{B}$, $T^*\mathcal{B}$, St

   $\mathfrak{g}$, $\widetilde{\mathfrak{g}}$, $\mathfrak{g}^{sv}$, $\widetilde{\mathfrak{g}}^{sv}$, $\mathcal{N}$, $\widetilde{\mathcal{N}}$, $\mathfrak{h}$, n

   $\widetilde{\mathfrak{g}}^h$, $\mathcal{O}_h$, Λ_w^h

7. Try to understand what Kazhdan-Lusztig polynomials are [KL2], and
   - Compute the transformation matrix between
     $\{[T_w^*], w \in W_f\}$ and $\{[\Lambda_w^h], w \in W_f\}$?   [KaSai]?
   - understand what standard / crystal basis is
   - understand the relationship between KL poly and crystal basis
   - see if it is related to two basis in $\text{Rep}(G)$ (irr reps & multiplicative basis)

8. Try to understand the module part, i.e.,
   - numbers of components of the Springer fiber
   - how does $K_0^{Grp}(St)$ act on $K_0^{Grp}(\text{Springer fiber})$ also act on $K_0^{Grp}(\text{Rep}_{pol}(Q))$
   - does that occupy "all rep" of $K_0^{Grp}(St)$

9. Ways of finding multiplication structure
   1 By direct computation (with techniques)
        Hecke algebra   double coset calculus
   2. By formulas as alg-isos
        $K_0^G(\mathcal{B})$ induction formula
   3. By geometrical computation
        cohomology   cup product ? de Rham calculus
        Chow group   intersection theory
   4. By deformation (indirect)
        $H_{top}^{BM}(St)$
        $K_0^{G\times\mathbb{C}^*}(St)$

10. Different views on the double coset
     $$B\backslash G/B = (*/B) \times_{*/G} (*/B)$$

     - as a set
     - as flag variety quotient B-action
     - as a stack
     - groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $H(G, B)$, H_{sph}, (H_{aff})	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG,....	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:
- 3 Seminars   (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
  - · L-parameters
  - · moduli in AG
  - · some following developments of the modular form (different type of gps, Hecke operators...)
  - · reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.