Eine Woche, ein Beispiel

6.12 Condensed

Main Ref: https://people.mpim-bonn.mpg.de/scholze/Course%20Summer%2022.html

That's already so well written. I collect some notations here purely for self-study, and I believe this document is useless for other people.

Other ref:
https://home.mathematik.uni-freiburg.de/arithgeom/abschlussarbeiten/link-bachelor-revised.pdf

Condensed set

Def (pro-étale site $*_{\text{proét}}$)

Category: Prof

Cover: for $S \in \mathrm{Prof}$,

$$\mathrm{Cov}(S) = \left\{ \{S_i \xrightarrow{f_i} S\}_{i\in I} \text{ in Prof} \;\middle|\; \begin{array}{l} I \text{ finite} \\ S = \bigcup_{i\in I} f_i(S_i) \end{array} \right\}$$

Naive def. ⚠ Caveat: Prof is large. Need minor modification.

$$\begin{aligned}
\mathrm{CondSet} &= \mathrm{Sh}(*_{\text{proét}}) \\
&= \left\{ X : \mathrm{Prof}^{op} \longrightarrow \mathrm{Set} \;\middle|\; \begin{array}{l} X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \\ 0 \to X(S) \to X(T) \rightrightarrows X(T\times_S T) \quad T \twoheadrightarrow S \end{array} \right\} \\
&= \{ X : \mathrm{qcProj}^{op} \longrightarrow \mathrm{Set} \mid X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \} \\
\mathrm{CondAb} &= \mathrm{Sh}(*_{\text{proét}}, \mathrm{Ab}) \\
\mathrm{Cond}(\mathcal{C}) &= \mathrm{Sh}(*_{\text{proét}}, \mathcal{C}) \\
\mathrm{Cond}(R) &= \mathrm{Cond}(\mathrm{Mod}(R)) = \mathrm{Sh}(*_{\text{proét}}, \mathrm{Mod}(R)) \qquad R \in \mathrm{Ring}
\end{aligned}$$

when $R \in \mathrm{Cond}(\mathrm{Ring})$, require compatability.

Analytic ring and complete condensed A-module

Def. ⚠ Preliminary

An analytic ring is $\mathcal{A} = (A, \mathcal{M}_A(-), \delta)$, where

- $A \in \mathrm{Cond}(\mathrm{Ring})$
- $\mathcal{M}_A : \mathrm{Prof} \longrightarrow \mathrm{Cond}(A)$ $\qquad S \longmapsto \mathcal{M}_A(S)$
- $\delta_S : S \longrightarrow \mathcal{M}_A(S)$ $\qquad s \longmapsto \delta_s$
- $$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & A \end{tikzcd} \qquad \tilde{f} : \mu \mapsto \int f \mu$$
- $M \in \mathrm{Cond}(A)$ is complete if
$$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & M \end{tikzcd}$$

We require that the full subcategory

$\mathrm{Cond}_{\mathrm{cpl}}(A) := \{\text{complete condensed } A\text{-modules}\} \subseteq \mathrm{Cond}(A)$

should be abelian category.

$0<p\le 1$ afterwards.

Liquid vector spaces. $S\in \mathrm{Prof}$.

$$\mathcal{M}(S) = \{f: C(S;\mathbb{R})\to\mathbb{R} \mid f \text{ cont}\} = \mathbb{R}[S]^{\blacksquare}$$
$$\mathcal{M}(S)_{\ell^p\le c} = \varprojlim_i \mathcal{M}(S_i)_{\ell^p\le c} \subseteq \varprojlim_i \mathbb{R}^{\oplus S_i}$$
$$\mathcal{M}_p(S) = \bigcup_{c>0}\mathcal{M}(S)_{\ell^p\le c}$$
$$\mathcal{M}_{<p}(S) = \bigcup_{0<q<p}\mathcal{M}_q(S)$$

Def. Let $V\in \mathrm{CondAb}$ and $0<p\le 1$. V is *p-liquid* if

$$\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_{<p}(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\
 & V
\end{tikzcd}$$

equiv:
$$\begin{tikzcd}
S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_q(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\
 & V
\end{tikzcd} \qquad \forall q<p$$

equiv:
$$\bigoplus_j \mathcal{M}_{<p}(T_j)\longrightarrow \bigoplus_i \mathcal{M}_{<p}(S_i)\longrightarrow V\longrightarrow 0$$

Relations

$$\begin{tikzcd}
 & \mathrm{Solid} \arrow[d, shift left] & \\
\mathrm{Liq}_p \arrow[r, shift right] & \mathrm{CondAb} \arrow[l, shift right, "(-)^{liq}"'] \arrow[u, shift left, "(\ )^{\blacksquare}"] \arrow[r, shift right] & \mathrm{CondSet} \arrow[l, shift right, "\mathbb{Z}[-]"']
\end{tikzcd}$$

Liq_p	CondAb
· Abelian	· Abelian
· $-\otimes_{\mathbb{R}_{<p}}-$, $-\otimes^{\mathbb{L}}_{\mathbb{R}_{<p}}-$, $\underline{\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$, $\underline{R\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$	· $-\otimes-$, $-\otimes^{\mathbb{L}}-$, $\underline{\mathrm{Hom}}(-,-)$, $\underline{R\mathrm{Hom}}(-,-)$
· flatness	· flatness
· projective objects	· projective objects

Prop 3.4. $M\in \mathrm{CondAb}$ is flat $\iff M(S)$ is torsion free $\forall S\in \mathrm{qcProj}$
$\iff M(S)$ is torsion free $\forall S\in \mathrm{Prof}$

Prop 3.5. $M\in\mathrm{CondAb}$ is cpt proj $\iff M$ is a retract of $\mathbb{Z}[S]$ for some $S\in\mathrm{Extr}$
$\iff M$ is a retract of $\mathbb{Z}[\beta I]$ for some infinite set I with discrete topology
$\Rightarrow M\oplus\mathbb{Z}[\beta I]\simeq \mathbb{Z}[\beta I]$

Recall that for $A,B\in \mathrm{Ob}(\mathcal{C})$, A is a retract of B if $\exists (r,i)$ s.t

$$\begin{tikzcd}
 & A \arrow[dl, "i"'] \arrow[d, "\mathrm{Id}_A"] \\
B \arrow[r, "r"] & A
\end{tikzcd}$$
commutes

Thm 3.14. $\mathcal{M}_{<p}(S) \in \mathrm{Liq}_p$ is flat $\quad \forall S \in \mathrm{Prof}$.

Moreover, $\forall V \in \mathrm{Liq}_p$ qs. we have an iso

$$\mathcal{M}_{<p}(S) \otimes_{\mathbb{R}_{<p}} V \cong \bigcup_{q<p} \bigcup_{\substack{K \subset V \\ q\text{-convex}}} \mathcal{M}_q(S, K)$$

Here, for S finite,

$$\mathcal{M}_q(S, K) = \langle s \otimes k \mid_{s \in S} \rangle_{q\text{-convex hull}} \subseteq \mathbb{R}[S] \otimes_{\mathrm{Cond}(\mathbb{R})} V;$$

for $S = \varprojlim_i S_i$ profinite,

$$\mathcal{M}_q(S, K) = \varprojlim_i \mathcal{M}_q(S_i, K) \qquad \subseteq \mathbb{R}[S] \otimes_{\mathrm{Cond}(\mathbb{R})} V.$$

More examples of p-liquid spaces. $\mathcal{M}_{<p}(X)$, $\ell^{<p}(I)$ and $\mathcal{O}(\mathbb{D})$

Def. ($\mathcal{M}_{<p}(X)$ for $X \in \mathrm{loc.\,Prof}$)

$$\mathcal{M}(X)_{\ell^p \le c} = \{p\text{-measures on } X \text{ with upper bound } C\}$$
$$= \left\{ \mu : \{K \subset X \text{ cpt open}\} \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \mu(K_1 \sqcup K_2) = \mu(K_1) + \mu(K_2) \\ \sum_{i \text{ finite}} |\mu(K_i)|^p \le C \end{array} \right\}$$
$$\subset \prod_{\substack{K \subset X \\ \text{cpt open}}} [-C^{\frac{1}{p}}, C^{\frac{1}{p}}] \quad \text{with product topology}$$

$\Rightarrow \mathcal{M}(X)_{\ell^p \le c} \in \mathrm{CHaus} \le \mathrm{CondSet}$

$$\mathcal{M}_p(X) = \{p\text{-measures on } X\} = \bigcup_{c>0} \mathcal{M}(X)_{\ell^p \le c}$$
$$\mathcal{M}_{<p}(X) = \bigcup_{0<q<p} \mathcal{M}_q(X)$$

Def. Let $I := \mathbb{N}_{\ge 0}$ be a countable set with discrete topo, so $I \in \mathrm{loc.\,Prof}$.

$$\ell(I)_{\ell^p \le c} = \{(x_i)_{i \in I} \in \mathbb{R}^I \mid \textstyle\sum_i |x_i|^p \le C\} \quad \text{with } \ell^p\text{-topology}$$

$\Rightarrow \ell(I)_{\ell^p \le c} \in \mathrm{Top} \longrightarrow \mathrm{CondSet}$

$$\ell^p(I) = \{(x_i)_{i \in I} \in \mathbb{R}^I \mid \textstyle\sum_i |x_i|^p < +\infty\} = \bigcup_{c>0} \ell(I)_{\ell^p \le c}$$
$$\ell^{<p}(I) = \bigcup_{0<q<p} \ell^q(I)$$

We have maps among topological spaces:

$$\begin{array}{ccccccccccc}
\cdots \subset & \mathcal{M}_{\frac{1}{3}}(I) & \subset & \mathcal{M}_1(I) & \subset & \mathcal{M}_2(I) & \subset \cdots \subset & \mathcal{M}_\infty(I) & \subset & \prod_I \mathbb{R} \\
& \cup & & \cup & & \cup & & \cup & & \\
& & & & & & & \ell^\infty(I) & & \\
& & & & & & & \cup & & \\
\cdots \subset & \ell^{\frac{1}{3}}(I) & \subset & \ell^1(I) & \subset & \ell^2(I) & \subset \cdots \subset & c_0(I) & &
\end{array}$$

Def. $\mathcal{O}(\mathbb{D})_{\mathbb{R}} = \left\{ \sum_{i\geq 0} c_i T^i \in \mathbb{R}[[T]] \;\middle|\; c_i r^i \to 0 \quad \forall r<1 \right\}$

$$
\begin{aligned}
&= \left\{ \sum_{i\geq 0} c_i T^i \in \mathbb{R}[[T]] \;\middle|\; c_i r^i \text{ is bounded} \quad \forall r<1 \right\} \\
&\cong \left\{ (c_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; \sum_i |c_i r^i|^p < +\infty \quad \forall r<1 \right\} \\
&= \left\{ (c_i)_{i\in I} \in \mathbb{R}^I \;\middle|\; \sum_i |c_i r_n^i|^p < +\infty \quad \forall n \in \mathbb{N}_{\geq 0} \right\}
\end{aligned}
$$

(Here we choose $r_n = 1-\frac{1}{2n}$ $\quad r_0 = \frac{1}{3}$ $\quad , \mathbb{N} = \mathbb{N}_{\geq 0}$)

$$\cong \varprojlim_n \mathcal{M}_{<p}(\mathbb{N})$$

where

$$\longrightarrow \mathcal{M}_{<p}(\mathbb{N}) \xrightarrow{\lambda^{(2)}} \mathcal{M}_{<p}(\mathbb{N}) \xrightarrow{\lambda^{(1)}} \mathcal{M}_{<p}(\mathbb{N})$$

$$\{b_n^{(1)}\}_{n\in\mathbb{N}} \longmapsto \left\{ b_n^{(1)} \cdot \left(\tfrac{r_0}{r_1}\right)^n \right\}_{n\in\mathbb{N}}$$

$$\{b_n^{(2)}\}_{n\in\mathbb{N}} \longmapsto \left\{ b_n^{(2)} \left(\tfrac{r_1}{r_2}\right)^n \right\}_{n\in\mathbb{N}}$$

[Think in this way:

$$\{c_n r_1^n\}_{n\in\mathbb{N}} \longmapsto \left\{ c_n r_1^n \left(\tfrac{r_0}{r_1}\right)^n \right\}_{n\in\mathbb{N}}$$

$$\{c_n r_2^n\}_{n\in\mathbb{N}} \longmapsto \left\{ c_n r_2^n \left(\tfrac{r_1}{r_2}\right)^n \right\}_{n\in\mathbb{N}}$$
]

We can discuss these examples with $\otimes_{\mathbb{R}_{<p}}$ & flatness.