Tutorial 9 & Ex 8

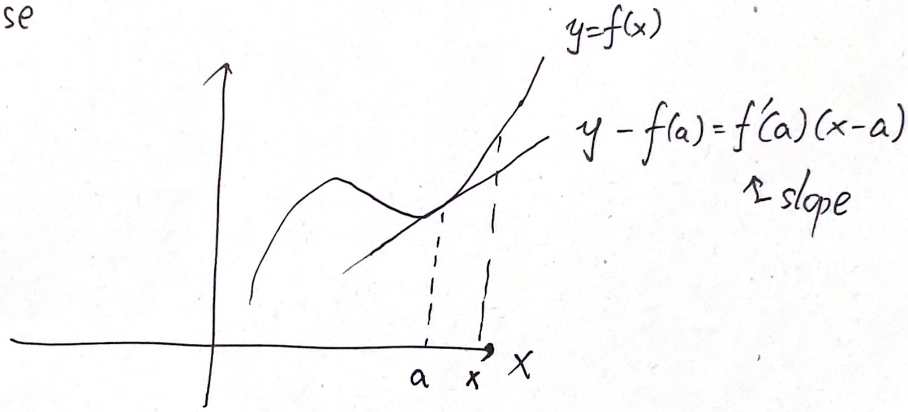
Today we work on differentials.

Questions for Ex 8?

~~1. 1-dim case~~

Slogan: differential $\approx$ approximate fcts by linear fcts.

1. 1-dim case



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{t \to 0} \frac{f(a+t) - f(a)}{t}$$

Ex. For $f(x) = |x|$, compute $f'(x)$

Ex. For $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ compute $f'(x)$.

Ex. For $f: [a,b] \longrightarrow \mathbb{R}$ cont & differentiable in (a,b),

if $f'(x) \equiv 0$, then $f(x) \equiv C$.

Recall: mean value theorem

(Mittelwertsatz der Differentialrechnung)

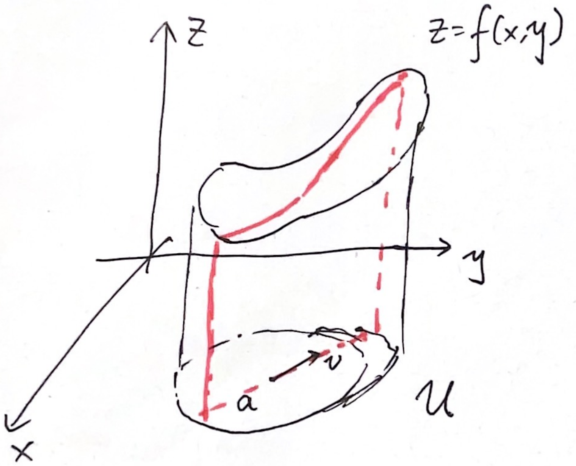
We have two generalizations of 1-dim differential.

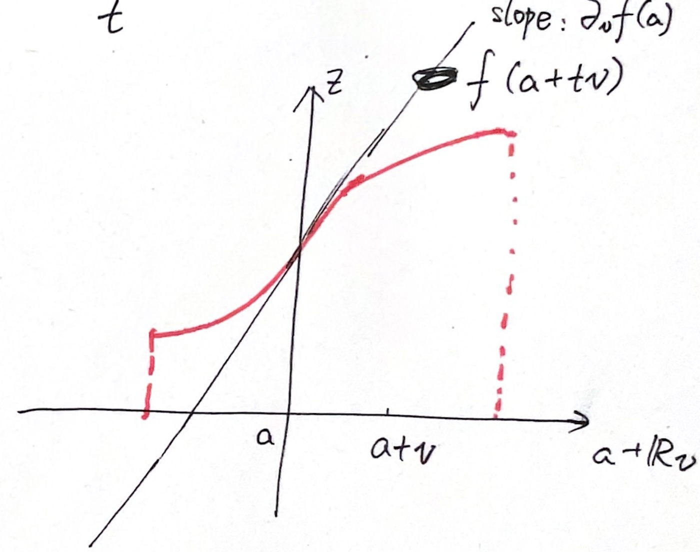
2. Directional derivative (reduced to 1-dim)

Def (Directional derivative, Partial derivative, Jacobi matrix/determinant)

For $\mathcal{U} \subseteq \mathbb{R}^n$ open, $a \in \mathcal{U}$, $v \in \mathbb{R}^n$, $f: \mathcal{U} \longrightarrow \mathbb{R}^m$, ~~define~~ define

$$(\partial_v f)(a) := \lim_{t \to 0} \frac{f(a+tv) - f(a)}{t} \quad \in \mathbb{R}^m \text{ or NaN}$$





E.p. one can define partial derivatives

$$\frac{\partial f}{\partial x_i} := \partial_i f := \partial_{e_i} f : \mathcal{U} \longrightarrow \mathbb{R}^m \qquad e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{the } i\text{-th place}$$

Write $f=(f_1,\dots,f_m)$, denote

$$Df := \left(\frac{\partial f}{\partial x_1},\dots,\frac{\partial f}{\partial x_n}\right) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = \begin{pmatrix} Df_1 \\ \vdots \\ Df_m \end{pmatrix}$$

Df is called the <u>Jacobian matrix</u>.

When $m=n$, $\det Df$ is called the <u>Jacobian determinant</u> of f.

<u>Rmk</u>. In this course, people use ∇f as the transpose of Df, and call it the gradient.

$$\nabla f := (Df)^T = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = (\nabla f_1 \ \cdots \ \nabla f_m)$$

<u>Task 1</u>. For

$$f: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \qquad f(x) = e_2 + Ax + (x^T B^T x)\, e_1$$

where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\qquad B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$,

compute Df.

A: $$f(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \left((x_1\ x_2\ x_3)\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} x_1 + 2x_2 + 3x_3 \\ 4x_1 + 5x_2 + 6x_3 \end{pmatrix} + \begin{pmatrix} x_2 x_1 + x_3 x_2 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + 2x_2 + 3x_3 + x_2 x_1 + x_3 x_2 \\ 1 + 4x_1 + 5x_2 + 6x_3 \end{pmatrix}$$

$$Df = \begin{pmatrix} 1+x_2 & 2+x_1+x_3 & 3+x_2 \\ 4 & 5 & 6 \end{pmatrix}$$

Task 3. For $\overset{U}{\cancel{\Omega}} \subseteq \mathbb{R}^n$ open and connected, $u \in C^1(\overset{U}{\cancel{\Omega}}; \mathbb{R})$, show that

$$\nabla u(x) = 0 \quad \forall x \in \overset{U}{\cancel{\Omega}} \quad \Rightarrow \quad u \equiv C \text{ in } \cancel{\Omega}\, U$$

Hint: first show it for $\overset{U}{\cancel{\Omega}} = (-1,1)^n$

$$\frac{\partial u}{\partial x_1} \equiv 0 \quad \Rightarrow \quad u(x_1, 0, \dots, 0) = u(0, 0, \dots, 0) \mathrel{\hat{=}} C$$

$$\frac{\partial u}{\partial x_2} \equiv 0 \quad \Rightarrow \quad u(x_1, x_2, \dots, 0) = u(x_1, 0, \dots, 0) = C$$

$$\vdots$$

$$\frac{\partial u}{\partial x_n} \equiv 0 \quad \Rightarrow \quad u(x_1, x_2, \dots, x_n) = u(x_1, x_2, \dots, x_{n-1}, 0) = C$$

Then, show that, for a fixed $x_0 \in \overset{U}{\cancel{\Omega}}$,

$$\{x \in \overset{U}{\cancel{\Omega}} \mid u(x) = u(x_0)\} \text{ is both open and closed in } \overset{U}{\cancel{\Omega}}.$$

$$\Rightarrow \{x \in \overset{U}{\cancel{\Omega}} \mid u(x) = u(x_0)\} = \cancel{\Omega}\, U \qquad \square$$

3. Total derivative (linear fct approximation)

Def For $\mathcal{U} \subseteq \mathbb{R}^n$ open, we say that $f: \mathcal{U} \to \mathbb{R}^m$ is (totally) differentiable at $a \in \mathcal{U}$, if

$\exists A: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear s.t

$$\lim_{x \to a} \frac{\| f(x) - f(a) - A(x-a) \|}{\| x - a \|} = 0$$

cont	$f(x) = f(a) + o(1)$
differentiable	$f(x) = f(a) + A(x-a) + o(\|x-a\|)$

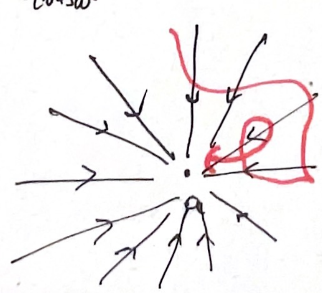
Rmk. If f is differentiable, then $A = (Df)(a)$.

In this case, Df is called the (total) derivative or (total) differential.

Rmk Total derivative is stronger than the directional derivative.

When f is differentiable ~~at a, then~~ $\forall v, w \in \mathbb{R}^n$, $\forall t, s \in \mathbb{R}$

$$D_{tv+sw} f = t D_v f + s D_w f$$

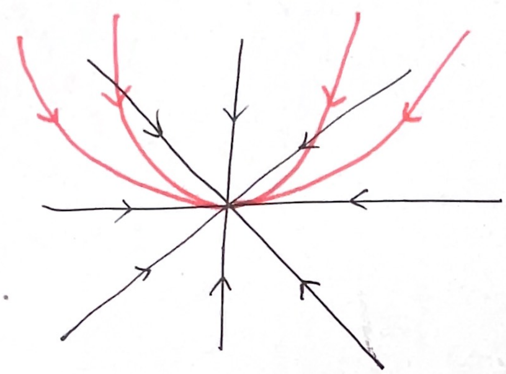


Task 2. Define

$$g(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that $g(x,y)$ is not cont at $(0,0)$, while

$$\lim_{t \to 0} g(tv) = 0 \qquad \forall v = (x_0, y_0) \in \mathbb{R}^2$$



$\rightarrow$: goes to 0

$\rightarrow$: does not go to 0

Similarly, define

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2+y^2} & (x,y) \neq (0,0), \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that $f(x,y)$ is cont.

$$\left[\text{Hint: } \left|\frac{x^2 y}{x^2+y^2}\right| \leq \frac{|x^2+y^2||y|}{|x^2+y^2|} = |y| \text{ or } \left|\frac{x^2 y}{x^2+y^2}\right| \leq \frac{|x| \cdot \frac{1}{2}|x^2+y^2|}{|x^2+y^2|} = \frac{1}{2}|x| \right]$$

Moreover, show that $f(x,y)$ is not differentiable at $(0,0)$, while

$$Df = 0$$

$$D_v f = \cos^2\theta \sin\theta \qquad \forall v = (\cos\theta, \sin\theta) \in \mathbb{R}^2$$