

# Eine Woche, ein Beispiel

## 9.3. field extension with RS

Goal: construct an equivalence between two categories:

$$\begin{array}{ccc}
 \begin{array}{c} \text{cpt conn} \\ \downarrow \\ RS^{cc} = \left\{ \begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array} \right\} \end{array} & \longleftrightarrow & \left\{ \begin{array}{l} \text{Obj: } F/\mathbb{C} \text{ field ext st.} \\ \text{trdeg}_{\mathbb{C}} F = 1 \\ F/\mathbb{C} \text{ f.g. as a field} \\ \text{Mor: morphism as fields}/\mathbb{C} \end{array} \right\}^{\text{op}} = \text{field}_{\mathbb{C}(t)/\mathbb{C}}^{\text{op}} \\
 \begin{array}{c} Y \\ \downarrow f \\ X \end{array} & \implies & \begin{array}{c} \mathcal{M}(Y) \\ \uparrow f^* \\ \mathcal{M}(X) \end{array}
 \end{array}$$

which obeys the following slogan:

(ramified) covering  $\approx$  (function) field extension

- Rmk.
- For requiring  $F/\mathbb{C}$  f.g. as a field, we avoid examples like  $\overline{\mathbb{C}(t)}$ .  
Do they corresponds to some non-cpt Riemann surface?  
If so, how to enlarge the category  $RS^{cc}$ ?
  - $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$  means fields over  $\mathbb{C}$  which are fin ext of  $\mathbb{C}(t)$  abstractly;  
morphisms don't need to fix  $\mathbb{C}(t)$ .  
Do you have a better name for  $RS^{cc}$  and  $\text{field}_{\mathbb{C}(t)/\mathbb{C}}$ ?

<https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of>  
<https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory>

- field of meromorphic functions
- Galois covering
- valuations
- quadratic extension of  $\mathbb{C}(x)$ : hyperelliptic curve
- miscellaneous.

1. field of meromorphic functions

Def. For  $X \in RS$ ,

$$\begin{aligned} \mathcal{M}(X) &:= \{\text{meromorphic fcts on } X\} \\ &= \{f: X \rightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{1_\infty\} \\ &\stackrel{\substack{X \text{ cpt} \\ \text{conn}}}{=} \{\text{rational fcts on } X\} \end{aligned}$$

Ex. Verify that

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$

$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[i]) \cong \text{Frac}(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1)))$$

Later we will show that, for  $X \in RS^{cc}$ ,

$$\exists \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \text{ st. } [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z^3,$$

compute

$$1) f^*: \mathbb{C}(T) \hookrightarrow \mathbb{C}(S) \quad [\mathbb{C}(S) : \mathbb{C}(T)] \text{ \& a } \mathbb{C}(T)\text{-basis}$$

$$2) \text{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$$

$$3) \mathbb{C}(S)^{2/\mathbb{Z}}$$

$$4) \text{Aut}_f(\mathbb{CP}^1)$$

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z + \frac{1}{z},$$

do the same work.

Ex. For

$$f: \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \quad z \mapsto z^3 - 3z,$$

compute the same stuff.

Why isn't  $\mathbb{C}(S)/\mathbb{C}(T)$  Galois this time?

Hint.

$$\begin{array}{c} 3 \quad 2 \\ \hline 4 \quad 5 \end{array} \begin{array}{c} 1 \\ \hline 6 \end{array} \xrightarrow{\quad} \begin{array}{c} 1 \\ \hline \end{array} \begin{array}{c} \end{array}$$

Prop. For  $d \in \mathbb{N}_{>0}$ ,  $f: Y \rightarrow X$  proper holo morphism between conn RSs,  
 $[M(Y): f^*M(X)] = d$ .

Cor. For  $X$  cpt conn,

$$\exists \mathbb{C}(X) \hookrightarrow M(X) \text{ s.t. } [M(X): \mathbb{C}(X)] < +\infty$$

In ptc,  $F/\mathbb{C}$  f.g as a field,  $\text{trdeg}_{\mathbb{C}} F = 1$ .

To show the proposition, one need the following black box to find a basis.  
 Black box (meromorphic fcts separate points)

$X: RS$ ,  $x, y \in X$   $x \neq y$ , then

$$\exists g \in M(X) \text{ s.t. } g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$

(stronger)  $\exists g \in M(X) \text{ s.t. } \text{ord}_x g = -1, \quad g(y) = 0.$

I prefer using Riemann-Roch when  $X$  is cpt, and Stein manifold when  $X$  is not.

Ex. Using the black box, show that,

for  $X: RS$ ,  $\{x_1, \dots, x_n\} \subseteq X$ ,  $\exists g \in M(X)$  s.t.

$$\text{ord}_{x_i} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \dots, n\}$$

$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \quad i, j \in \{2, \dots, n\}$$

Proof of prop

$[M(Y): f^*M(X)] \geq d$ : Fix  $x_0 \in X$  s.t.  $\#f^{-1}(x_0) = d$ . Denote  $f^{-1}(x_0) = \{y_1, \dots, y_d\}$ .

For each  $i$ , let  $g_i \in M(Y)$  be a meromorphic fct s.t.

$$\text{ord}_{x_i} g_i = -1 \quad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$

then  $\{g_1, \dots, g_d\} \subseteq M(Y)$  are  $f^*M(X)$ -linear independent.

Check:  $\text{ord}_{y_i} (\sum f_j g_j) \approx \text{ord}_{y_i} f_i$

$$[M(Y): f^*M(X)] \leq d:$$

$\forall g \in M(Y)$ , need to find  $a_i \in f^*M(X)$  s.t.

$$g^d + a_{d-1} g^{d-1} + \dots + a_0 = 0 \quad \text{in } M(Y)$$

The fcts

$$a_i(z) = (-1)^i \sum_{\{k_1, \dots, k_i\} \subseteq \{1, \dots, d\}} g(z_{k_1}) \dots g(z_{k_d})$$

$$f^{-1}(f(z)) = \{z_1, \dots, z_d\}, \text{ multiplicity is counted}$$

satisfy the conditions.

Use Riemann extension theorem to show  $a_i(z) \in f^*M(X)$ , see [Donaldson, p148].

By primitive element theorem,  $[M(Y): f^*M(X)] \leq d$ .

## 2. Galois covering

Def. Let  $f: Y \rightarrow X$  be a proper hdo map between two conn RSs.  
 $f$  is Galois, if  $M(Y)/f^*M(X)$  is a Galois extension.  
normal normal

Prop.  $f: Y \rightarrow X$  is Galois/normal

- $\Leftrightarrow \deg f = \# \text{Aut}_f(Y)$
- $\Leftrightarrow f^{-1}(x_0)$  is an  $\text{Aut}_f(Y)$ -torsor,  $\forall x_0 \in X - f(\text{Ram}(f))$
- $\Leftrightarrow \text{Aut}_f(Y) \subset f^{-1}(x_0)$  transitively,  $\forall x_0 \in X$
- $\Leftrightarrow Y/\text{Aut}_f(Y) \cong X$ , i.e.  $f$  can be written as  

$$Y \rightarrow Y/G$$

Ex. For  $f: Y \rightarrow X$ , suppose that  
 $[\forall y_1, y_2 \in Y \text{ s.t. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2)$   $\nwarrow$  ramification index  
 Show that  $f$  is Galois by computing  $\# \text{Aut}_f(Y)$ .

[Hint. Use geodesics to divide  $X$  into several smaller triangles.  
 If geodesics are hard, take  $g: X \rightarrow \mathbb{CP}^1$  non-constant,  
 and reduce the problem to  $g \circ f$ .]

This proof is not completely rigorous, and you are encouraged to find a reference to rigorously prove it, or read this discussion on stackexchange:  
<https://math.stackexchange.com/questions/1952655/ramification-index-and-inertia-degree-same-for-all-the-primes-then-is-the-exten>

You may need the following materials for completing the proof, relating with questions about geodesic triangulations.

google: geodesic triangulations

<https://math.stackexchange.com/questions/1661331/proof-of-equivalence-of-conformal-and-complex-structures-on-a-riemann-surface?rq=1>

<https://arxiv.org/pdf/2103.16702.pdf>

(If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.)

copied from p2, in <https://www.mathematik.uni-muenchen.de/~forster/eprints/gaussbonnet.pdf>

<http://czamfirescu.tricube.de/CTZamfirescu-o8.pdf>

[Thm 2] <http://www-fourier.univ-grenoble-alpes.fr/~ycolver/All-Articles/91c.pdf>

<https://mathoverflow.net/questions/138267/what-prevents-a-cover-to-be-galois>

E.g. Consider the covering

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$$

$$z \longmapsto z^3 - 3z$$

This is not a Galois covering. Consider the Galois closure

$$\mathbb{CP}^1$$

$$\downarrow z + \frac{1}{z}$$

$$\mathbb{CP}^1$$

$$\downarrow z^3 - 3z$$

$$\mathbb{CP}^1$$

$$\mathbb{C}(u) = \mathbb{C}(S)[R]/(R^2 + S^2 - 4)$$

$$\uparrow$$

$$\mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3 - 3S - T)$$

$$\uparrow$$

$$\mathbb{C}(T)$$

$$u + \frac{1}{u}$$

$$\uparrow$$

$$S$$

$$S^3 - 3S$$

$$\uparrow$$

$$T$$

Determination of the Galois closure

$$\min(S, \mathbb{C}(T)) = x^3 - 3x - T$$

$$\text{in } \mathbb{C}(T)[x]$$

$$= x^3 - 3x - (S^3 - 3S)$$

$$= (x - S)(x^2 + Sx + S^2 - 3)$$

$$\text{in } \mathbb{C}(S)[x]$$

To decompose the polynomial  $x^2 + Sx + S^2 - 3$ ,  
we have to add root of discriminant:

$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2 - 3)} = \sqrt{3} \sqrt{-S^2 + 4}.$$

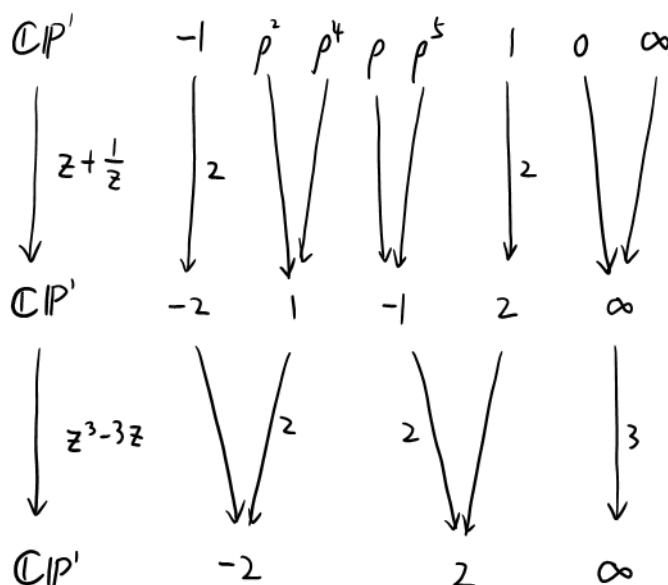
Therefore, the Galois closure of  $\mathbb{C}(S)/\mathbb{C}(T)$  is

$$\mathbb{C}(S)[R]/(R^2 + S^2 - 4) \cong \mathbb{C}\left(\frac{S+iR}{2}\right) \triangleq \mathbb{C}(u)$$

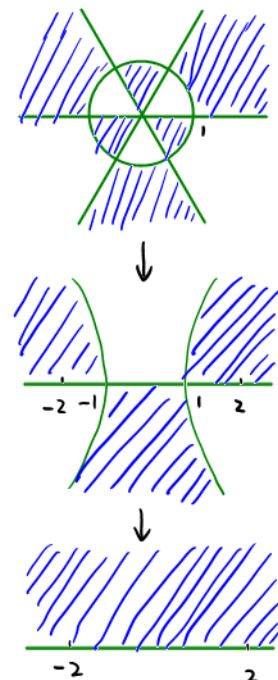
where

$$S = \frac{S+iR}{2} + \frac{S-iR}{2} = u + \frac{1}{u}$$

The picture from the RS side is as follows:



only ramified pts are drawn



affine version

Question:

How to know the genus of the RS corresponding to the Galois closure?

Answer:

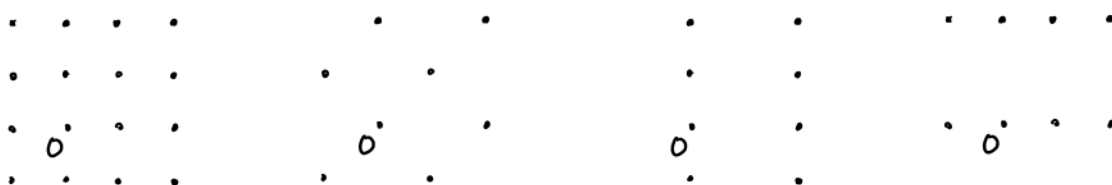
<https://mathoverflow.net/questions/152/how-do-you-see-the-genus-of-a-curve-just-looking-at-its-function-field>

Question:

Do we have any Galois closure whose ramification information is not minimal as we expected?

E.g. 2. For  $E = \mathbb{C}/\Lambda$ , since  $\pi_1(E, o) \cong \mathbb{Z} \oplus \mathbb{Z}$ ,  
 $E$  has three unramified coverings of deg 2.  
 When  $\Lambda = \mathbb{Z}[i]$ , what are the crspd field extensions?

There are more deg 2 ramified coverings from the higher genus RS, but we don't discuss them here.



normalized  
equation

$$y^2 = x(x+1)(x-1)$$

$$y^2 = x(x+1)(x-1)$$

$$\begin{aligned} y^2 &= 4x^3 - 11x - 7 \quad \Delta = 8 \\ &= (x+1)(4x^2 - 4x - 7) \\ &= 4(x+1)\left(x - \frac{1}{2} + \sqrt{2}\right)\left(x - \frac{1}{2} - \sqrt{2}\right) \end{aligned}$$

$$j(2i) = \left(\frac{11}{2}\right)^3 \cdot 1728 = 66^3$$

$$g_2(2i) = \frac{11 \Gamma(\frac{1}{4})^8}{2^8 \pi^2}$$

$$g_3(2i) = \frac{7 \Gamma(\frac{1}{4})^{12}}{2^{12} \pi^3}$$

equation is given by

$$y^2 = 4x^3 - g_2x - g_3$$

Ex. 1) Show that

$$\text{Aut}_{\text{RS}}(\mathbb{C}/\Lambda)[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

no matter  $\mathbb{C}/\Lambda$  has CM or not.

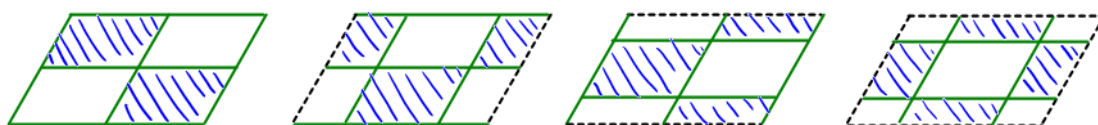
2) We get 7 ramified coverings of deg 2.

$$\pi_\tau: E \longrightarrow E/\langle \tau \rangle \quad \forall \tau \in \text{Aut}_{\text{RS}}(\mathbb{C}/\Lambda)[2] - \{\text{Id}\}$$

Which are ramified coverings? Compute the genus & ramification information.

$$3 \text{ unramified}, \quad g(E/\langle \tau \rangle) = 1$$

$$4 \text{ ramified at 4 pts}, \quad g(E/\langle \tau \rangle) = 0$$



3) Find all index 2 subfields of  $M(\mathbb{C}/\Lambda)$ . *hard!*

### 3. valuations.

Q. How to reconstruct the RS of the crspd fct field extension?  
i.e., how to give an inverse fctor of the fctor

$$M(-): RS^c \longrightarrow \text{field}_{\mathbb{C}(t)/\mathbb{C}}^{\text{op}} \quad X \longmapsto M(X) \quad ?$$

Observation:  $\forall X \in RS^c$ ,  $x \in X$ , one can define a valuation

$$v_x^X: M(X) \longrightarrow \mathbb{Z} \sqcup \{\infty\} \quad f \longmapsto \deg_x f$$

indicating the order of fcts on  $x$ .

If we collect all the valuations on  $M(X)$ , we may recover the RS  $X$ .

Ref. [Perfseminar, L2]

Def (valuation (Bourbaki) / NA absolute value\*)  $\Gamma$ : some tot ordered gp.

For  $A \in \mathbb{C}\text{Ring}$ , a valuation of  $A$  is a map

$$v: A \longrightarrow \Gamma \cup \{\infty\}$$

s.t.

- $v(0) = \infty$
- $v(ab) = v(a) + v(b)$ ,  $v(1) = 0$
- $v(a+b) \geq \min(v(a), v(b))$ , with equality if  $v(a) \neq v(b)$

↳ which makes  $v$  more algebraic rather than analytic  
rigid flexible

If  $A \in \text{Field}_{\mathbb{C}}$ , we require additionally that  $v|_{\mathbb{C}^*} \equiv 0$ .

Rmk. The additional assumption on  $\mathbb{C}^*$  is natural, as we want

$$v|_{\mathbb{C}^*}: \mathbb{C}^* \longrightarrow \Gamma \cup \{\infty\}$$

↑ order top

to be a cont gp homo. One gets

$$v(z) = v(|z|) = v(e)^{|\ln|z||} \quad \forall z \in \mathbb{C}^*$$

Moreover, we want  $v$  to be an NA absolute value, so  $v(z) = 0$ .

\* Many people don't use "absolute value" for high rank valuations.

See:

The Zariski-Riemann Space of Valuation Rings by Bruce Olberding

[https://link.springer.com/chapter/10.1007/978-3-030-89694-2\\_21](https://link.springer.com/chapter/10.1007/978-3-030-89694-2_21)

Lec 2: Spaces of valuations

<http://math.stanford.edu/~conrad/Perfseminar/>

The other part also worth reading, if you have problem reading the berkeley notes.

<https://math.stackexchange.com/questions/188652/finite-extensions-of-rational-functions>



## 5. miscellaneous.

- genus of a fct field  $F$ ?
- non-cpt RS, infinite covering
- Spv for higher dimensional varieties, high rank pts
- RS/scheme structures reconstruction
- gp structures on valuations of  $\mathbb{C}[x,y,z]/(y^2z - x(x-z)(x+z))$
- cyclic extension
- maximal abelian extension/unramified extension/unramified outside some places

<https://math.stackexchange.com/questions/2836916/what-are-the-abelian-extensions-of-bbb-cx>