Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
https://people.math.harvard.edu/~gross/preprints/sat.pdf

Task. Complete the following tables:

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	
Id	$\mathbb{Z}$			$\mathbb{Z}[X^*(T)]/I_T \rtimes \mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)][t^{\pm 1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$			

We use the shorthand.

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$R(T)^W$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$	
Id	$\mathbb{Z}$			$R(T)/I_T \rtimes \mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$R(G)[t^{\pm 1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$R(T)[t^{\pm 1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$			

$$R(B) = \mathbb{Z}[X^*(T)] = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F))$$

$$R(G) = \mathbb{Z}[X^*(T)]^W \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F))$$

$$R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$$

$$R(G\times\mathbb{C}^\times) = \mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$$

$$K^{G\times\mathbb{C}^\times}(St) = \mathcal{H}_{ext} \overset{?}{\neq} \mathcal{H}(\hat{G}(F), I)$$

Here is an initial example.

$K^-(-)$	pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $T^*(\mathcal{B}\times\mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm 1}]$	$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}]/((z_1-z_2)(z_1-z_2^{-1}))$	$\mathbb{Z}[W_{ext}] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm 1}]$
B	$\mathbb{Z}[y^{\pm 1}]$	$\mathbb{Z}[y^{\pm 1}, z]/(z-y)(z-y^{-1})$	$\mathbb{Z}[y^{\pm 1}, z_1, z_2]/((z_1-y)(z_1-y^{-1}), (z_2-y)(z_2-y^{-1}))$	
Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1, z_2]/((z_1-1)^2, (z_2-1)^2)$	$R(T)/I_T \rtimes \mathbb{Z}[W_f] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm 1}]/(z_w-1)^2$
$SL_2\times\mathbb{C}^\times$	$\mathbb{Z}[x, t^{\pm 1}]$			$\mathcal{H}_{ext} = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm 1}, t^{\pm 1}]$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[y^{\pm 1}, t^{\pm 1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$			