Eine Woche, ein Beispiel
12.1 weights of type E

There are already much information in wiki and other references about the exceptional Lie algebra. It is nice, but I always have to check the compatability among different references. In this document, I try to fix a standard coordinate, and state all the combinatorical results without proofs.

We will make a list of the following objects, for E_6, E_7 and E_8.

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2023. https://doi.org/10.1017/9781009280020.

- Weights nearest to the origin
  - some graphs
  - weight lattice

- Simple roots

- Fundamental weights

- Weyl group action

Remark: There is another coordinate system which is written in wiki: del Pezzo surface. We don't use them. There, the different weight spaces are identified, while in our coordinate system, we identify the root lattices.

The order we present:
The order we compute:

②simple roots $\longrightarrow$ ④fundamental wts

③ roots   ① wt orbits
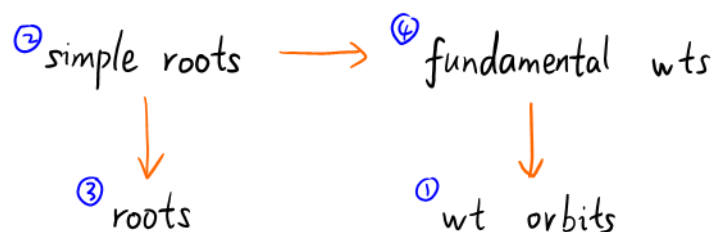
We present in this way, only because we want to express everything in terms of weight orbits.

1. E_6.

– Weights nearest to the origin

There are two minuscule representations of E 6. So we just fix one.

affine version

#	typical coordinates	symbol
6	$(1 , 0 , 0, 0, 0, 0, 1 ,0)^\top$	ν_i
6	$(1 , 0 , 0, 0, 0, 0, 0, 1)^\top$	$\tilde{\nu}_i$
15	$(-\frac{1}{2} , -\frac{1}{2} , \frac{1}{2} , \frac{1}{2} , \frac{1}{2} , \frac{1}{2} ,\frac{1}{2} ,\frac{1}{2})^\top$	ν_{ij}

$$\langle \nu_i , \nu_j \rangle \in \{0, 1, 2\}$$

could be $\tilde{\nu}_i$ or ν_{ij} edge

weight lattice version

#	typical coordinates	symbol
6	$\frac{1}{6}(5 , -1, -1, -1, -1, -1, 3 ,-3)^\top$	ν_i
6	$\frac{1}{6}(5 , -1 , -1, -1, -1, -1, -3, 3)^\top$	$\tilde{\nu}_i$
15	$\frac{1}{3}(-2 , -2 , 1 , 1, 1, 1, 0, 0)^\top$	ν_{ij}

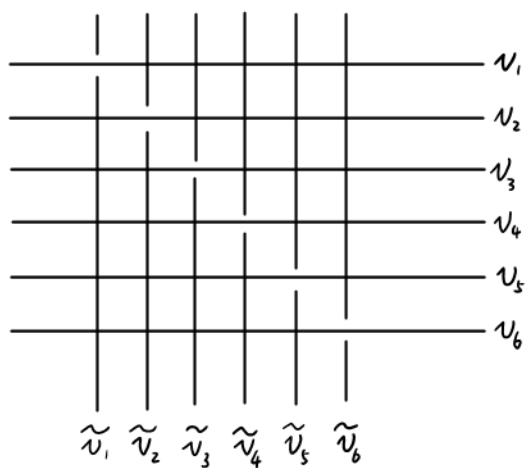$$\langle \nu_i , \nu_j \rangle \in \{\tfrac{4}{3}, \tfrac{1}{3}, -\tfrac{2}{3} \}$$

could be $\tilde{\nu}_i$ or ν_{ij} edge

in $\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \} \cong \mathbb{R}^6$

The graph constructed is called the Schläfli graph, which has 27 vertices and 216 edges (with HoG Id 1300).
This graph is also the configuration graph of 27 lines.

vertices $\longrightarrow$ lines
edges $\rightsquigarrow$ intersection points
triangle $\rightsquigarrow$ triangle cut by H
$\qquad \qquad \qquad \qquad \qquad \nwarrow$ only in E_6

Here are some typical subgraphs:

Schläfli double six configuration
$\# V = 12 \qquad \# E = 30$
HoG Id $= 32794$
36 many, from [Huy24, Ex 3.6]

$v_{16} \quad v_{26} \quad v_{36} \quad v_{46} \quad v_5$

$\tilde{v}_6$

$\tilde{v}_5$

$v_{15} \quad v_{25} \quad v_{35} \quad v_{45} \quad v_6$
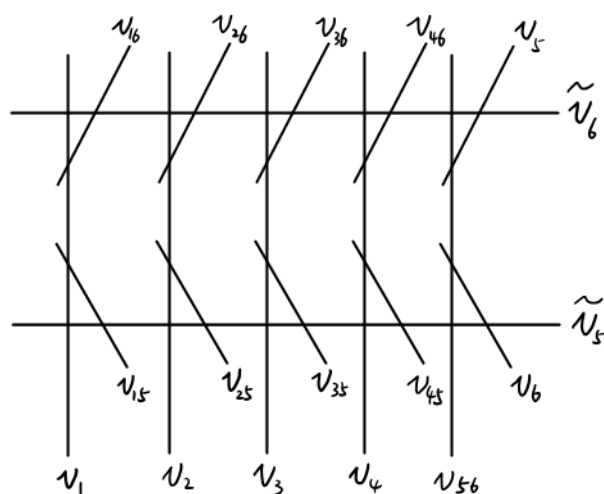
$v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_{56}$
(v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6. \cdots$)
$\# V = 17 \qquad \# E = 30 + 20 = 50$
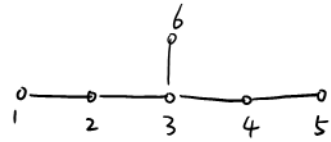HoG Id $=$ none.

v_{12}

v_{34}

v_{56}
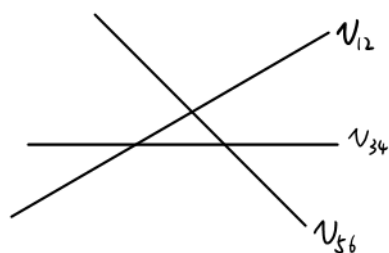
triangle
720 many

Q: For each type of subgraph, how many are they in the Schläfli graph?
I don't know if there are any simple answer for general subgraphs, and I don't know if there are any efficient algorithm
for doing this. But this already produces many mysterious combinatorical numbers.

– Simple roots

$$\{ \alpha_1, \quad \alpha_2, \quad \alpha_3, \quad \alpha_4, \quad \alpha_5, \quad \alpha_6 \}$$
$$= \{ \nu_1-\nu_2, \ \nu_2-\nu_3, \ \nu_3-\nu_4, \ \nu_4-\nu_5, \ \nu_5-\nu_6, \ \nu_4-\nu_{56} \}$$

$$= \left\{
\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} -\tfrac{1}{2} \\ -\tfrac{1}{2} \\ -\tfrac{1}{2} \\ \tfrac{1}{2} \\ \tfrac{1}{2} \\ \tfrac{1}{2} \\ \tfrac{1}{2} \\ -\tfrac{1}{2} \end{pmatrix}
\right\}$$

Ex. Verify that all the 72 roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}
$40 = \binom{6}{3} \cdot 2$	$(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2})^T$	$\alpha_{456,7}$
2	$(0, 0, 0, 0, 0, 0, 1, -1)^T$	α_7

– Fundamental weights
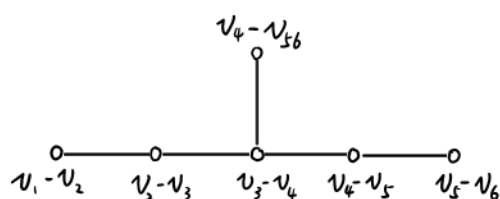
denote by $A = (a_{ij})$ the Cartan matrix, then

$$(\alpha_1, \ldots, \alpha_r) = (\varpi_1, \ldots, \varpi_r) A \qquad \langle \alpha_i, \alpha_j \rangle = A$$
$$(\varpi_1, \ldots, \varpi_r) = (\alpha_1, \ldots, \alpha_r) A^{-1} \qquad \langle \varpi_i, \varpi_j \rangle = A^{-1}$$

As a result,

$$\{ \varpi_1, \quad \varpi_2, \quad \varpi_3, \quad \varpi_4, \quad \varpi_5, \quad \varpi_6 \}$$
$$= \{ v_1, \quad v_1 + v_2, \quad v_1 + v_2 + v_3, \quad -\tilde{v}_5 - \tilde{v}_6, \quad -\tilde{v}_6, \quad v_1 - \tilde{v}_1 \} \qquad -\tilde{v}_6 \neq \sum_{i=1}^{5} v_i$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ -2 \\ 6 \\ -6 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \\ -3 \\ -3 \\ 9 \\ -9 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ -4 \\ 6 \\ -6 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ -6 \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \\ 3 \\ -3 \end{pmatrix}, \ \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

α_i

$$v_4 - v_{56}$$
$$v_1 - v_2 \quad v_2 - v_3 \quad v_3 - v_4 \quad v_4 - v_5 \quad v_5 - v_6$$

ϖ_i

$$v_1 - \tilde{v}_1$$
$$v_1 \quad v_1 + v_2 \quad v_1 + v_2 + v_3 \quad -\tilde{v}_5 - \tilde{v}_6 \quad -\tilde{v}_6$$
$$-\tilde{v}_4 - \tilde{v}_5 - \tilde{v}_6$$
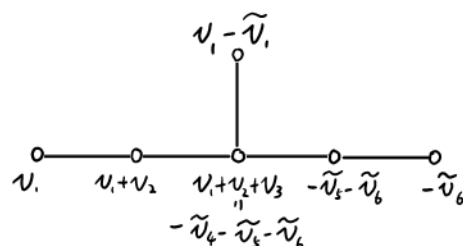
- Weyl group action

We know that

$$s_k \alpha_i = \alpha_i - \langle \alpha_k, \alpha_i \rangle \alpha_k$$
$$= \alpha_i - a_{ki} \alpha_k$$

$$\rightsquigarrow s_k(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r) \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ -a_{k1} & \cdots & 1-a_{kk} & \cdots & -a_{kr} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\left(\delta_{ij} - \delta_{ik} a_{ij} \right)_{i,j}$$

In practice, we want to compute s_k-action on coordinates, it's easier to use the formula

$$s_k e_i = e_i - \langle \alpha_k, e_i \rangle \alpha_k$$

E.g. In E_6-case, when $k=1$, $\alpha = (1, -1, 0, \dots, 0)^T = e_1 - e_2$,

$$s_1 e_1 = e_1 - (e_1 - e_2) = e_2$$
$$s_1 e_2 = e_2 - (-1)(e_1 - e_2) = e_1$$

$$\rightsquigarrow \quad s_1 = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Similarly, $s_k = s_{(k, k+1)}$ for $i = 1, \dots, 5$.

When $k = 6$, $\alpha_k = \frac{1}{2}(-1, -1, -1, 1, 1, 1, 1, -1)^T$,

$$s_6 e_1 = e_1 - (-\frac{1}{2}) \alpha_6 = e_1 + \frac{1}{2} \alpha_6$$
$$= \frac{1}{4}(3, -1, -1, 1, 1, 1, 1, -1)^T$$
$$s_6 e_4 = e_4 - \frac{1}{2} \alpha_6$$
$$= \frac{1}{4}(1, 1, 1, -3, -1, -1, -1, 1)^T$$

$$\rightsquigarrow \quad s_6 = \frac{1}{4} \begin{pmatrix} 3 & -1 & -1 & & 1 & & & -1 \\ -1 & 3 & -1 & & & & & \\ -1 & -1 & 3 & & 3 & -1 & & \\ 1 & & & -1 & 3 & 3 & & 1 \\ -1 & & & 1 & & -1 & 3 & 3 \end{pmatrix}$$
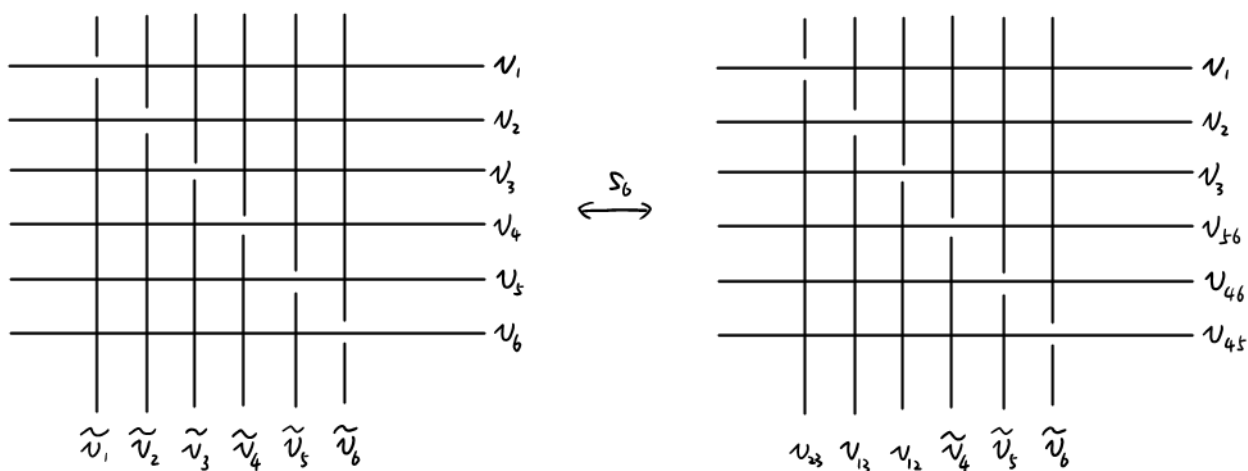
The action of $s_1, \ldots, s_5$ on the Schläfli graph is easy. s_6 is hard.

E.g.

$s_6 \nu_1 = s_6(e_1 + e_7) = e_1 + e_7 = \nu_1.$

$s_6 \nu_4 = s_6(e_4 + e_7) = \frac{1}{2}(1,1,1,1,-1,-1,1,1)^T = \nu_{56}$

$s_6 \widetilde{\nu_1} = s_6(e_1 + e_8) = \frac{1}{2}(1,-1,-1,1,1,1,1,1)^T = \nu_{23}$

$s_6 \widetilde{\nu_4} = s_6(e_4 + e_8) = e_4 + e_8 = \widetilde{\nu_4}$

$s_6 \nu_2 = \nu_2 \qquad s_6 \nu_3 = \nu_3$

$s_6 \nu_5 = \nu_{46} \qquad s_6 \nu_6 = \nu_{45}$

$s_6 \widetilde{\nu_2} = \nu_{13} \qquad s_6 \widetilde{\nu_3} = \nu_{12}$

$s_6 \widetilde{\nu_5} = \widetilde{\nu_5} \qquad s_6 \widetilde{\nu_6} = \widetilde{\nu_6}$

The rest are easy to determine through the Schläfli double six configuration.

e.g. $s_6 \nu_{14} = \nu_{14}$

2. E_7.

— Weights nearest to the origin

There is just one minuscule representations of E_7.

<u>integer version</u>

#	typical coordinates	symbol
28	$(3 , 3 , -1,-1,-1,-1,-1,-1)^\top$	ν_{ij}
28	$(-3, -3 , 1, 1, 1, 1, 1, 1)^\top$	$\widetilde{\nu}_{ij} = -\nu_{ij}$

$$\langle \nu_i, \nu_j \rangle \in \{24, 8, -8, -24\}$$
$\underset{edge}{}$

<u>weight lattice version</u>

#	typical coordinates	symbol
28	$\frac{1}{4}(3 , 3 , -1,-1,-1,-1,-1,-1)^\top$	ν_{ij}
28	$\frac{1}{4}(-3, -3 , 1, 1, 1, 1, 1, 1)^\top$	$\widetilde{\nu}_{ij} = -\nu_{ij}$

$$\langle \nu_i, \nu_j \rangle \in \{\tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}\}$$
$\underset{edge}{}$

in $\quad \left\{ \sum_{i=1}^{8} z_i = 0 \right\} \cong \mathbb{R}^7$

$$56 = 7 + \binom{7}{2} + \binom{7}{5} + 7$$

vertices $\longrightarrow$ lines

edges $\rightsquigarrow$ intersection points
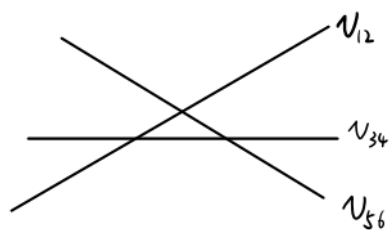
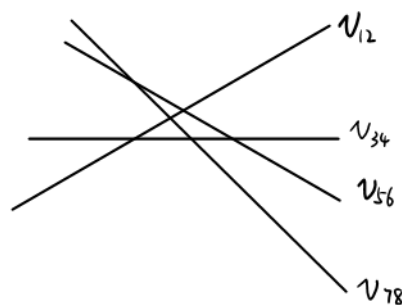triangle $\rightsquigarrow$ triangle

Here are some typical subgraphs:

N_{18}
N_{28}
N_{38}
N_{48}
V_{58}
V_{68}
V_{78}

$\tilde{N}_{18} \ \tilde{V}_{28} \ \tilde{V}_{38} \ \tilde{V}_{48} \ \tilde{V}_{58} \ \tilde{V}_{68} \ \tilde{V}_{78}$

"double seven configuration"
#V = 14   #E = 42
HoG Id = 50584

$\{ V_{ij} \}_{i,j}$

V_{16} intersect with $V_{25}, V_{35}, V_{45}, V_6$. $\cdots$
#V = 28   #E = 210
HoG Id = 50698.

N_{12}
N_{34}
N_{56}

triangle
4032 many
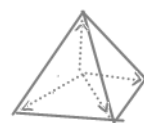
N_{12}
N_{34}
V_{56}
V_{78}
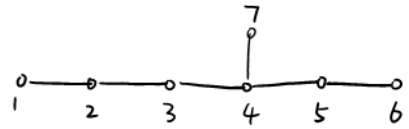
1008 many

in (-1)-curves setting,

$$\langle V_i, V_j \rangle \in \left\{ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\}$$

intersection number: -1   0   1   2

– Simple roots

$$\{\,\alpha_1\,,\quad \alpha_2\,,\quad \alpha_3\,,\quad \alpha_4\,,\quad \alpha_5\,,\quad \alpha_6\,,\quad \alpha_7\,\}$$

$$=\{\,\nu_{18}-\nu_{28},\ \nu_{28}-\nu_{38},\ \nu_{38}-\nu_{48},\ \nu_{48}-\nu_{58},\ \nu_{58}-\nu_{68},\ \nu_{68}-\nu_{78},\ -\nu_{12}-\nu_{34}\,\}$$

$$=\left\{
\begin{pmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\1\\-1\\0\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\1\\-1\\0\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{pmatrix},
\begin{pmatrix}0\\0\\0\\0\\0\\1\\-1\\0\end{pmatrix},
\begin{pmatrix}-\tfrac12\\-\tfrac12\\-\tfrac12\\-\tfrac12\\\tfrac12\\\tfrac12\\\tfrac12\\\tfrac12\end{pmatrix}
\right\}$$

Ex. Verify that all the 126 roots are given by

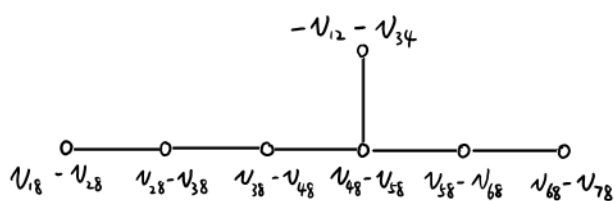| #	typical coordinates	symbol
$56 = 8\cdot 7$	$(\,1\,,\,-1\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\,)^T$	α_{1-2}
$70 = \binom{8}{4}$	$(-\tfrac12\,,-\tfrac12\,,-\tfrac12\,,-\tfrac12\,,\,\tfrac12\,,\,\tfrac12\,,\,\tfrac12\,,\,\tfrac12\,)^T$	$\alpha_{5,6,7,8}$

Fundamental weights

For convenient, denote

$$v_{j-k} = v_{ij} - v_{ik} = e_j - e_k \quad \text{for some } i = j, k.$$

$$\{\, w_1, \quad w_2, \quad w_3, \quad w_4, \quad w_5, \quad w_6, \quad w_7 \,\}$$
$$= \left\{\, v_{18}, \quad v_{18}+v_{28}, \quad v_{18}+v_{28}+v_{38}, \quad \sum_{i=0}^{4} v_{i8}, \quad v_{6-8}+v_{7-8}, \quad v_{7-8}, \quad v_{78}+v_{7-8} \,\right\}$$

$$= \left\{\; \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}2\\2\\-2\\-2\\-2\\-2\\-2\\6\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\-4\\-4\\-4\\12\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\-4\\-4\\8\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\0\\-4\\4\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$

$$= \left\{\; \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix},\;
\frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix},\;
\begin{pmatrix}0\\0\\0\\0\\-1\\-1\\-1\\3\end{pmatrix},\;
\begin{pmatrix}0\\0\\0\\0\\0\\-1\\-1\\2\end{pmatrix},\;
\begin{pmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{pmatrix},\;
\frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$

Dynkin diagram (left):

$-v_{12} - v_{34}$ (top node)

$v_{18}-v_{28} \;-\; v_{28}-v_{38} \;-\; v_{38}-v_{48} \;-\; v_{48}-v_{58} \;-\; v_{58}-v_{68} \;-\; v_{68}-v_{78}$

α_i

Dynkin diagram (right):

$v_{78}+v_{7-8}$ (top node)

$v_{18} \;-\; v_{18}+v_{28} \;-\; \sum_{i=1}^{3} v_{i8} \;-\; \sum_{i=1}^{4} v_{i8} \;-\; v_{6-8}-v_{7-8} \;-\; v_{7-8}$
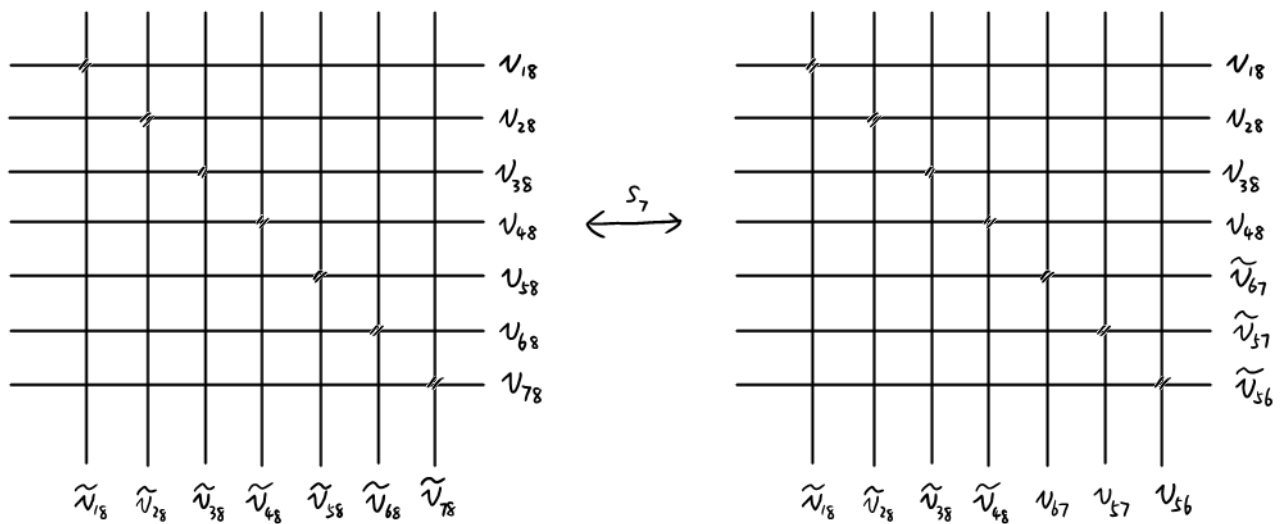
w_i
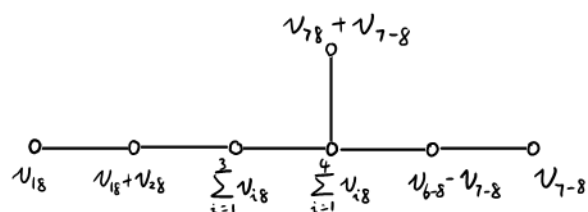
– Weyl group action

Using the similar methods like E_6, we get

$$S_k = S_{(k, k+1)} \qquad \text{for} \quad i = 1, \cdots, 6$$

$$S_7 = \frac{1}{4} \left(\begin{array}{cc|cc} \begin{matrix} 3 & & -1 \\ & 3 & \\ -1 & & 3 \end{matrix} & \begin{matrix} 1 \end{matrix} \\ \hline \begin{matrix} & 1 & \end{matrix} & \begin{matrix} 3 & & -1 \\ & 3 & \\ -1 & & 3 \end{matrix} \end{array} \right)$$

$$S_7 \, v_{ij} = \begin{cases} v_{ij} & \text{if} \quad i \in \{1, 2, 3, 4\}, \quad j = \{5, 6, 7, 8\} \\ \widetilde{v}_{kl} & \text{if} \quad \{i, j, k, l\} = \{1, 2, 3, 4\} \text{ or } \{5, 6, 7, 8\} \end{cases}$$

v_{18}, v_{28}, v_{38}, v_{48}, v_{58}, v_{68}, v_{78}

$\widetilde{v}_{18}$ $\widetilde{v}_{28}$ $\widetilde{v}_{38}$ $\widetilde{v}_{48}$ $\widetilde{v}_{58}$ $\widetilde{v}_{68}$ $\widetilde{v}_{78}$

$\xleftrightarrow{S_7}$

v_{18}, v_{28}, v_{38}, v_{48}, $\widetilde{v}_{67}$, $\widetilde{v}_{57}$, $\widetilde{v}_{56}$

$\widetilde{v}_{18}$ $\widetilde{v}_{28}$ $\widetilde{v}_{38}$ $\widetilde{v}_{48}$ v_{67} v_{57} v_{56}

2. E_8.

— Weights nearest to the origin

There is no minuscule representations of E_8, the 240 weights are roots.

weight lattice version   (allow negative roots)

#	typical coordinates	symbol	inverse
$56 = 2 \cdot \binom{8}{2}$	$(1 , 1 , 0 , 0 , 0 , 0 , 0 , 0)^\top$	α_{12}	$\tilde{\alpha}_{12}$
$56 = 8 \cdot 7$	$(1 , -1 , 0 , 0 , 0 , 0 , 0 , 0)^\top$	α_{1-2}	$\tilde{\alpha}_{2-1}$
2	$(-\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2})$	ν_ϕ	$\tilde{\nu}_\phi$
$56 = 2 \cdot \binom{8}{2}$	$(\frac{1}{2} , \frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2})^\top$	ν_{12}	$\tilde{\nu}_{12}$
$70 = \binom{8}{4}$	$(\frac{1}{2} , \frac{1}{2} , \frac{1}{2} , \frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2} , -\frac{1}{2})^\top$	ν_{1234}	ν_{5678}

$$\langle \nu_i , \nu_j \rangle \in \{ 2 , 1 , 0 , -1 , 2 \} \qquad \text{in } \mathbb{R}^8$$

$\uparrow$ edge

shorter:

#	typical coordinates	symbol	
$112 = 4 \cdot 28$	$(\pm 1 , \pm 1 , 0 , 0 , 0 , 0 , 0 , 0)^\top$	$\alpha_{\pm i \pm j}$	
$128 = 2^7$	$(\pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2} , \pm\frac{1}{2})^\top$	even sign	ν_I

Q: $\text{Aut}(\Gamma) = W(E_8)$?

in (-1)-curves setting,

$$\langle v_i, v_j \rangle \in \{2, 1, 0, -1, -2\}$$

intersection number: $\qquad -1 \quad 0 \quad 1 \quad 2 \quad 3$

If we allow multiple edges, then I believe $\text{Aut}(\Gamma_{mult}) = W(E_8)$.

α_{12}
α_{1-2}
α_{34}
α_{3-4}
α_{56}
α_{5-6}
α_{78}
α_{7-8}

$\tilde{\alpha}_{12} \quad \alpha_{2-1} \quad \tilde{\alpha}_{34} \quad \alpha_{4-3} \quad \tilde{\alpha}_{56} \quad \alpha_{6-5} \quad \tilde{\alpha}_{78} \quad \alpha_{8-7}$
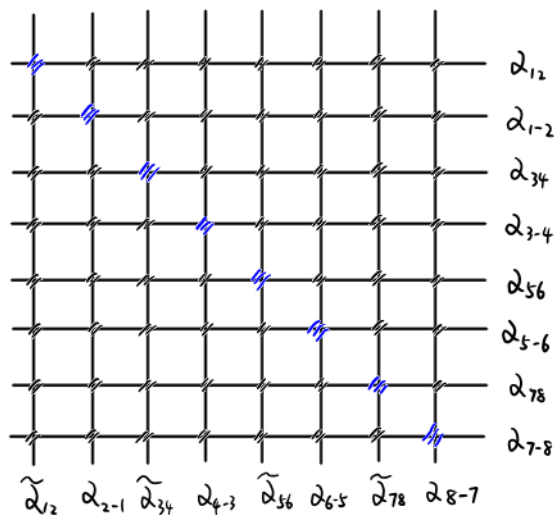
"double eight configuration"
$\#V = 16 \qquad \#E = 0$

– Simple roots

Dynkin diagram (top right): nodes labeled 1—2—3—4—5—6—7 in a line, with node 8 branching up from node 5.
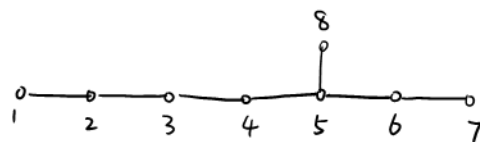
$$\{\; \alpha_1 \;,\; \alpha_2 \;,\; \alpha_3 \;,\; \alpha_4 \;,\; \alpha_5 \;,\; \alpha_6 \;,\; \alpha_7 \;,\; \alpha_8 \;\}$$

$$= \{\; \alpha_{1-2} \;,\; \alpha_{2-3} \;,\; \alpha_{3-4} \;,\; \alpha_{4-5} \;,\; \alpha_{5-6} \;,\; \alpha_{67} \;,\; \nu_\phi \;,\; \alpha_{6-7} \;\}$$

$$= \left\{
\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix},
\begin{pmatrix} -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \\ -\tfrac12 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}
\right\}$$

– Fundamental weights

$$\{\; \varpi_1 \;,\; \varpi_2 \;,\; \varpi_3 \;,\; \varpi_4 \;,\; \varpi_5 \;,\; \varpi_6 \;,\; \varpi_7 \;,\; \varpi_8 \;\}$$

$$= \left\{ \alpha_{1-8}\,,\; \alpha_{1-8}-\alpha_{2-8}\,,\; \sum_{i=1}^{3}\alpha_{i-8}\,,\; \sum_{i=1}^{4}\alpha_{i-8}\,,\; \sum_{i=1}^{5}\alpha_{i-8}\,,\; \begin{array}{c}\alpha_{1-8}+\alpha_{2-8}\\ +\,\alpha_{3-8}+\nu_{4567}\end{array}\,,\; \alpha_{1-8}-\alpha_{18}\,,\; \alpha_{1-8}-\alpha_{18}-\nu_{78} \right\}$$

$$= \left\{
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},
\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix},
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix},
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -4 \end{pmatrix},
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ -5 \end{pmatrix},
\begin{pmatrix} \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ -\tfrac72 \end{pmatrix},
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix},
\begin{pmatrix} \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ \tfrac12 \\ -\tfrac12 \\ -\tfrac52 \end{pmatrix}
\right\}$$

Bottom left diagram: linear chain with nodes labeled α_{1-2}, α_{2-3}, α_{3-4}, α_{4-5}, α_{5-6}, α_{67}, ν_ϕ; with branch node α_{6-7} above the α_{5-6} node.

α_i

Bottom right diagram: linear chain with nodes labeled α_{1-8}, $\alpha_{1-8}+\alpha_{2-8}$, $\sum_{i=1}^{3}\alpha_{i-8}$, $\sum_{i=1}^{4}\alpha_{i-8}$, $\sum_{i=1}^{5}\alpha_{i-8}$, $\sum_{i=1}^{3}\alpha_{i-8}+\nu_{4567}$, $\alpha_{1-8}-\alpha_{18}$; with branch node $\alpha_{1-8}-\alpha_{18}-\nu_{78}$ above.

ϖ_i
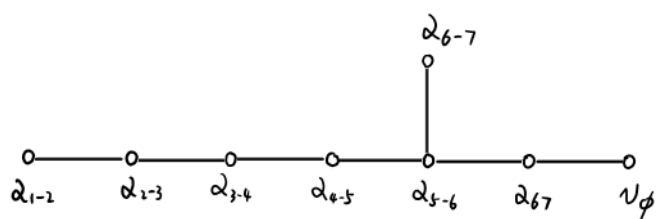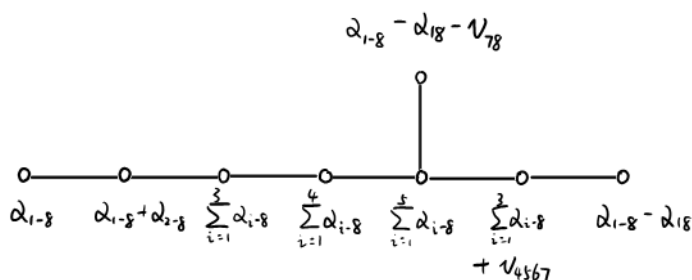
– Weyl group action

$$S_k = S_{(k, k+1)} \qquad \text{for} \quad i = 1, \cdots, 5$$

$$S_6 = \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & \ddots & \\ & & -1 & -1 \\ & & & 1 \end{pmatrix} \qquad S_7 = \frac{1}{4}\begin{pmatrix} 3 & & -1 \\ & \ddots & \\ -1 & & 3 \end{pmatrix} \qquad S_8 = S_{(6,7)}$$

Ex. Check the S_7-action on roots are given by

$$S_7(\alpha_{ij}) = \nu_{ij} \qquad\qquad S_7(\nu_\phi) = -\nu_\phi$$
$$S_7(\alpha_{i-j}) = \alpha_{i-j} \qquad\qquad S_7(\nu_{ij}) = \alpha_{ij}$$
$$\qquad\qquad\qquad\qquad S_7(\nu_{ijkl}) = \nu_{ijkl}$$

4. Comparison among different root systems.

Rmk. For the root lattice,

$$E_8 = \left\{ z_i \in \mathbb{Z}^8 \cup (\mathbb{Z}+\tfrac{1}{2})^8 \;\middle|\; \sum_{i=1}^{8} z_i \equiv 0 \mod 2 \right\}$$

$$E_7 = E_8 \cap \left\{ \sum_{i=1}^{8} z_i = 0 \right\}$$

$$E_6 = E_8 \cap \left\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \right\}$$

E_8 S_1 S_2 S_3 S_4 S_5 S_6 S_7 S_8

$$S_6 = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & & -1 \\ & & -1 & \end{pmatrix} \qquad S_7 = \frac{1}{4}\begin{pmatrix} 3 & & -1 \\ & \ddots & \\ -1 & & 3 \end{pmatrix} \qquad S_8 = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$$

E_7 S_1 S_2 S_3 S_4 S_5 S_6 S_7

$$S_6 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \qquad S_7 = \frac{1}{4}\left(\begin{array}{ccc|c} 3 & 3 & -1 & \\ -1 & 3 & 3 & 1 \\ \hline & 1 & & \begin{smallmatrix} 3 & 3 & -1 \\ -1 & 3 & 3 \end{smallmatrix} \end{array}\right)$$

E_6 S_1 S_2 S_3 S_4 S_5 S_6

$$\frac{1}{4}\left(\begin{array}{c|c|c} \begin{smallmatrix}3 & & \\ & 3 & 3 \\ -1 & & 3\end{smallmatrix} & 1 & -1 \\ \hline & \begin{smallmatrix}3 & 3 & \\ & 3 & -1 \\ -1 & 3\end{smallmatrix} & 1 \\ \hline 1 & & \begin{smallmatrix}3 & \\ 3 & 3 \\ -1 & 3\end{smallmatrix} \\ -1 & 1 & 3 \end{array}\right)$$