Eine Woche, ein Beispiel

4.13 lattices defining abelian variety

Ref:

[Deb99]: Complex tori and abelian varieties
[Mum74]: Mumford, David, Abelian varieties. Oxford university press Oxford, 1974.
[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.
[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.

This document try to work out [Deb99, p28, Ex (3)]

Claim 1. [Mum74, p35, (1) $\Leftrightarrow$ (4)]

Let $\Lambda \subseteq \mathbb{C}^g$ be a full lattice. Then

$\mathbb{C}^g/\Lambda$ is an abelian variety

$\Leftrightarrow$ $\exists$ an $\mathbb{R}$-bilinear alternating form $\omega: \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{R}$ s.t.

$$\begin{cases} \omega(\Lambda \times \Lambda) \subseteq \mathbb{Z} \\ \omega(x, ix) > 0 \quad \forall x \neq 0 \\ \omega(ix, iy) = \omega(x, y) \end{cases} \qquad (*)$$

i.e. an integral Kähler form

$$\omega(u,v) = \operatorname{Re} h(iu,v) = \operatorname{Im} h(u,v) \qquad h(au,v) = \bar{a}h(u,v)$$
$$= -\operatorname{Re} h(u, iv)$$

From now on. suppose $\Lambda = \langle \lambda_1, \dots, \lambda_{2g} \rangle_{\mathbb{Z}}$, we denote

$$\Pi := (a_{ij})_{i,j=1}^{2g} := (\lambda_1^*, \dots, \lambda_{2g}^*)^T \qquad \Rightarrow \lambda_i^* = \sum a_{ij} e_j^*$$

The matrix Π encodes all information of the lattice (add a basis)

Q: For what kind of conditions of Π, can we find ω satisfying $(*)$?

Let $\omega = \sum_{i<j} c_{ij}\, \lambda_i^* \wedge \lambda_j^*$, then

$$\omega(\Lambda \times \Lambda) \subset \mathbb{Z} \iff c_{ij} \in \mathbb{Z} \qquad \forall i, j.$$

Write $x = \sum x_i e_i$, $y = \sum y_i e_i$, we get

$$\begin{aligned}
\omega(x,y) &= \sum_{k,l} x_k y_l\, \omega(e_k, e_l) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\ \lambda_i^* \wedge \lambda_j^* (e_k, e_l) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\ (a_{ik} a_{jl} - a_{jk} a_{il}) \\
\omega(x, iy) &= \sum_{k,l} x_k y_{l-g} \sum_{i<j} c_{ij}\ (a_{ik} a_{jl} - a_{jk} a_{il}) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\ (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \\
\omega(ix, iy) &= \sum_{k,l} x_{k-g} y_{l-g} \sum_{i<j} c_{ij}\ (a_{ik} a_{jl} - a_{jk} a_{il}) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\ (a_{i(k+g)} a_{j(l+g)} - a_{j(k+g)} a_{i(l+g)})
\end{aligned}$$

Therefore,

$\omega(x, ix) > 0$ is an open condition on Π.

$$\omega(ix, iy) = \omega(x,y) \iff \sum_{i<j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0$$

$$\forall\ k, l \in \{1, \dots, 2g\}$$

Claim 2. $\mathbb{C}^g/\Lambda$ is an abelian variety

$\Lambda = \langle \lambda_1, \dots, \lambda_{2g} \rangle_{\mathbb{Z}}$

$\Pi = (a_{ij})_{i,j=1}^{2g} = (\lambda_1^*, \dots, \lambda_{2g}^*)^T$

$\iff \exists\ c_{ij} \in \mathbb{Z}$ for all $i, j \in \{1, \dots, 2g\}$ s.t.

① $\displaystyle\sum_{i<j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0 \quad \forall\ k, l \in \{1, \dots, 2g\}$

② $\displaystyle\left(\sum_{i<j} c_{ij}\ (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \right)_{k,l=1}^{2g}$ is def positive.

Rmk. The equations in ① are not linear independent.
There are at most $\binom{g}{2} \cdot \frac{4 \cdot 2}{4} = g(g-1)$ equations.

Reason: When $l=k$ or $l=k+g$, we get 0.
e.p. when $g=1$, we get no condition.

When $l \neq k$ & $l \neq k+g$, denote $\{k, l, k+g, l+g\} = \{k_1, k_2, k_3, k_4\}$,

$$a_{ik_1} a_{jk_2} - a_{ik_2} a_{jk_1} - a_{ik_3} a_{jk_4} + a_{ik_4} a_{jk_3}$$

$$\begin{array}{lll} k_1 \leftrightarrow k_2 & k_3 \leftrightarrow k_4 & \text{neg} \\ k_1 \leftrightarrow k_3 & k_2 \leftrightarrow k_4 & \text{neg} \\ k_1 \leftrightarrow k_4 & k_2 \leftrightarrow k_3 & \text{same} \end{array}$$

Cor. Since

$$A_g \subseteq \{\mathbb{C}^n/\Lambda \text{ a.v}\} \cong \{\Pi \in \mathbb{R}^{2g \times 2g} \mid \Pi \text{ satisfies ①②} \underset{\text{for some } (c_{ij})}{}\} / GL_g(\mathbb{C})$$

We know

$$\dim A_g = \tfrac{1}{2}\left((2g)^2 - g(g-1)\right) - g^2 = \tfrac{1}{2} g(g+1).$$

Rmk. [BL04, 4.2]

We are actually reconstructing the Riemann relations.

Let $X = \mathbb{C}^g/\Lambda$, $\quad \Lambda = \langle \lambda_1, \dots, \lambda_{2g} \rangle$, $\quad \lambda_i \in \mathbb{C}^g$

$\Pi = (\lambda_1, \dots, \lambda_{2g})_{g \times 2g} \in M^{g \times 2g}(\mathbb{C})$

For $\omega: \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{R}$ an nondeg alternating form, denote

$$A := (\omega(\lambda_i, \lambda_j))_{2g \times 2g} \in M^{2g \times 2g}(\mathbb{Z}).$$

$\omega(\Lambda \times \Lambda) \subseteq \mathbb{Z}$

Then,

① $\omega(ix, iy) = \omega(x, y) \iff \Pi A^{-1} \Pi^T = 0$

② $\omega(x, ix) > 0 \quad \forall x \neq 0 \iff i \Pi A^{-1} \Pi^H > 0$

$\Pi^H = \overline{\Pi}^T$

With the symplectic basis, $\mathcal{L} = \mathcal{L}(H, \chi)$ is of type D

the Riemann relations become:

① $\Pi_2 D^{-1} \Pi_1^T - \Pi_1 D^{-1} \Pi_2^T = 0$

② $i \Pi_2 D^{-1} \Pi_1^H - i \Pi_1 D^{-1} \Pi_2^H > 0$

For $(\Pi_1, \Pi_2) = (Z, D)$, the Riemann relations become:

① $Z^T = Z$

② $\operatorname{Im} Z > 0$