Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

$$\begin{array}{c} \mathcal{G} \quad\quad \mathcal{F}\ \mathcal{F}' \\ Y \xrightarrow{f} X \end{array}
\qquad
\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$f^* \dashv f_*$

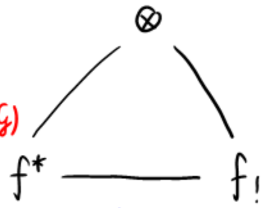
$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$

$f_! \dashv f^!$

$f^*(-\otimes -)$

$f^*(\mathcal{F}\otimes\mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$



bc: $f^* g_! \cong g'_! f'^*$

$f_* g^! \cong g'^! f'_*$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \longrightarrow pt$

$H^\cdot(X;\underline{\mathbb{Z}}) := p_* p^* \mathbb{1}$	$H^\cdot(X;\mathcal{F}) := p_*\mathcal{F}$	$= R\Gamma(X;\mathcal{F})$
$H_c^\cdot(X;\underline{\mathbb{Z}}) := p_! p^* \mathbb{1}$	$H_c^\cdot(X;\mathcal{F}) := p_!\mathcal{F}$	
$H_\cdot(X;\underline{\mathbb{Z}}) := p_! p^! \mathbb{1}$	$H_\cdot(X;\mathcal{F}) := p_!(p^!\mathbb{1}\otimes\mathcal{F})$	$= H_c^\cdot(X; p^!\mathbb{1}\otimes\mathcal{F})$
$H_\cdot^{BM}(X;\underline{\mathbb{Z}}) := p_* p^! \mathbb{1}$	$H_\cdot^{BM}(X;\mathcal{F}) := p_*(p^!\mathbb{1}\otimes\mathcal{F})$	$= H^\cdot(X; p^!\mathbb{1}\otimes\mathcal{F})$

App 1. (Künneth formula)

$$H_c(X;\mathcal{F}) \otimes H_c(Y;\mathcal{G}) \cong H_c^\cdot(X\times Y; \mathcal{F}\boxtimes\mathcal{G})$$

reduced to: $p_{X,!}\mathcal{F} \otimes p_{Y,!}\mathcal{G} \cong p_!(p_1^*\mathcal{F}\otimes p_2^*\mathcal{G})$

$$\begin{tikzcd} X\times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"'] & * \end{tikzcd}$$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then

$-^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^!\mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X;\underline{\mathbb{Z}})[d] \cong H^\cdot(X;\underline{\mathbb{Z}})^\vee$$

reduced to: $p_* \underline{Hom}(A, p^*B \otimes p^!\mathbb{1}) \cong \underline{Hom}(p_! A, B)$

$Z \xrightarrow[\text{close}]{i} X \xleftarrow{j} U \quad \rightsquigarrow$

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(Z) \arrow[r, shift left=3, "\mathrm{pi}\ i_* = i_!\ \mathrm{f.f}"] & \mathcal{D}(X) \arrow[l, shift right=6, "i^*"'] \arrow[l, shift left=6, "\mathrm{pi}\ i^!\ L"] \arrow[r, shift left=0, "\mathrm{pi}\ j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, shift right=6, "j_!\ \mathrm{f.f.}"'] \arrow[l, shift left=6, "\mathrm{pi}\ Rj_*\ L\ \mathrm{f.f}"]
\end{tikzcd}$$

Id Id Id Id

$$\xleftarrow[0]{i^* \quad j_!} \qquad \xrightarrow[0]{i_* \quad j^*} \qquad \xleftarrow[0]{i^! \quad Rj_*}$$

L : left exact (others are exact)

f.f. fully faithful

pi : preserve injectives. (内射)

ie. inj sheaf $\rightsquigarrow$ inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Q} = Or_X[n] \overset{+\text{orientation}}{=\!=\!=} \mathbb{Q}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \xrightarrow{+1}$$

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1}$$

$\triangledown$ https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof
https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de

Therefore, need to check if $R\pi_{X,*} \circ Rj_* = R\pi_{\mathcal{U},*}$ (Why writing derived symbol make less mistakes)

Assume $\triangledown$ makes no problem.

Here, $H_{-1}(S^1; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$R\Gamma(X, Z; \mathcal{F}) \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$

$$\underset{\substack{\| \\ R\Gamma_Z(X;\mathcal{F})}}{R\Gamma(X, \mathcal{U}; \mathcal{F})} \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(\mathcal{U}; \mathcal{F}|_\mathcal{U}) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{Q}_X$,

$$H^\cdot(X, Z) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(Z) \xrightarrow{+1}$$

$$H^\cdot(X, \mathcal{U}) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(\mathcal{U}) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{D}_X$,

$$R\Gamma(X, j_! \mathbb{Q}_\mathcal{U}) \longrightarrow H^{BM}_\cdot(X) \longrightarrow R\Gamma(Z; \mathbb{D}_X|_Z) \xrightarrow{+1}$$

$$H^{BM}_\cdot(Z) \longrightarrow H^{BM}_\cdot(X) \longrightarrow H^{BM}_\cdot(\mathcal{U}) \xrightarrow{+1}$$

Taking $R\pi_{X,!}$

$$R\Gamma_c(\mathcal{U}, \mathcal{F}|_\mathcal{U}) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$

$$R\Gamma_c(Z, i^! \mathcal{F}) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(X, Rj_*(\mathcal{F}|_\mathcal{U})) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{Q}_X$,

$$H^\cdot_c(\mathcal{U}) \longrightarrow H^\cdot_c(X) \longrightarrow H^\cdot_c(Z) \xrightarrow{+1}$$

$$H^\cdot_c(Z, i^! \mathbb{Q}_X) \longrightarrow H^\cdot_c(X) \longrightarrow H^\cdot_c(X, Rj_* \mathbb{Q}_\mathcal{U}) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{D}_X$,

$$H_\cdot(\mathcal{U}) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, \mathcal{U}) \xrightarrow{+1}$$

$$H_\cdot(Z) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, Z) \xrightarrow{+1}$$