Modular form

2. computations.

1. Prelude

1.1.2. (a) Show that $\mathrm{Im}(\gamma(\tau)) = \mathrm{Im}(\tau)/|c\tau+d|^2$ for all $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.
(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in \mathrm{SL}_2(\mathbf{Z})$ and $\tau \in \mathcal{H}$.
(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau+d)^2$ for $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}_2(\mathbf{Z})$.

$$\wp(z) = \frac{1}{z^2} + \sum_{z_0 \in \Lambda}{}' \left(\frac{1}{(z-z_0)^2} - \frac{1}{z_0^2} \right)$$

$$= \frac{1}{z^2} + \sum_{z_0 \in \Lambda}{}' \sum_{i=1}^{+\infty} (i+1) \frac{z^i}{z_0^{i+2}}$$

$$= \frac{1}{z^2} + \sum_{i=1}^{+\infty} (i+1) G_{i+2}(\Lambda) z^i$$

$$\Leftarrow \left[\begin{aligned} \frac{1}{(z-z_0)^2} &= \frac{1}{z_0^2}\left(1 + \left(-\frac{z}{z_0}\right)\right)^{-2} \\ &= \frac{1}{z_0^2}\left(1 + \binom{-2}{1}\left(-\frac{z}{z_0}\right) + \binom{-2}{2}\left(-\frac{z}{z_0}\right)^2 + \cdots \right) \\ &= \frac{1}{z_0^2} + \sum_{i=1}^{+\infty} (i+1) \frac{z^i}{z_0^{i+2}} \end{aligned} \right.$$

$$= \frac{1}{z^2} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + O(z^8)$$

$$\Rightarrow \wp'(z) = -2\frac{1}{z^3} + \sum_{i=1}^{+\infty} i(i+1) G_{i+2}(\Lambda) z^{i-1}$$

$$= -2\frac{1}{z^3} + \sum_{i=0}^{+\infty} (i+1)(i+2) G_{i+3}(\Lambda) z^i$$

$$= -2\frac{1}{z^3} + 6G_4 z + 20 G_6 z^3 + 42 G_8 z^5 + O(z^7)$$

$$\Rightarrow (\wp'(z))^2 = 4\wp(z)^3 - 60 G_4 \wp(z) - 140 G_6$$

$$y^2 = 4x^3 - 60G_4 x - 140 G_6 = 4x^3 - g_2 x - g_3$$

Intermediate computation.

$$(\wp'(z))^2 = 4\frac{1}{z^6} - 24 G_4 \frac{1}{z^2} - 80 G_6 + (-168 G_8 + 36 G_4^2) z^2 + O(z^4)$$

$$(\wp(z))^3 = \frac{1}{z^6} + 9 G_4 \frac{1}{z^2} + 15 G_6 + (21 G_8 + 27 G_4^2) z^2 + O(z^4)$$

$$252 G_8 - 108 G_4^2 = 0 \quad \Rightarrow \quad G_8 = \frac{3}{7} G_4^2$$

Rmk. another equation: $y^2 = 4(x-e_1)(x-e_2)(x-e_3)$ $\quad e_1 = \wp(\frac{\omega_1}{2})$ $\quad e_2 = (\frac{\omega_2}{2})$ $\quad e_3 = (\frac{\omega_1+\omega_2}{2})$

$$\Rightarrow \begin{cases} e_1 + e_2 + e_3 = 0 \\ e_1e_2 + e_2e_3 + e_1e_3 = -\frac{1}{4} g_2 = -15 G_4 \\ e_1e_2e_3 = \frac{1}{4} g_3 = 35 G_6 \end{cases}$$

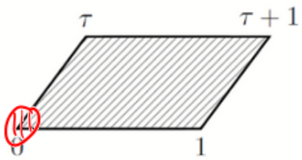
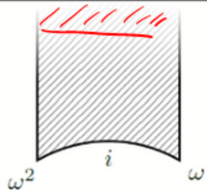
ord	0	$\frac{\omega_1}{2}$	$\frac{\omega_2}{2}$	$\frac{\omega_1+\omega_2}{2}$
y	-3	1	1	1
$x-e_1$	-2	2	0	0
$x-e_2$	-2	0	2	0
$x-e_3$	-2	0	0	2

div(f): encode information of zeros/poles

Ex. $G_4(\rho)=0$ $\quad G_6(i)=0$

$\Rightarrow$ Weierstrass equation of $\mathbb{C}/\mathbb{Z}\oplus\rho\mathbb{Z}$, $\mathbb{C}/\mathbb{Z}\oplus i\mathbb{Z}$

Conclusion

复环面 $\mathbb{C}/\Lambda_\tau$	模空间 $\mathcal{H}/SL_2(\mathbb{Z})$
	
$\mathcal{M}(\mathbb{C}/\Lambda_\tau)=\mathbb{C}(\wp,\wp')$	$\mathrm{M}_*(SL_2(\mathbb{Z}))=\mathbb{C}[E_4,E_6]$
椭圆函数	模形式
Weierstrass 函数	Eisenstein 级数

$\mathbb{C}/\Lambda_\tau \xrightarrow{[\wp:\wp':1]} \mathbb{CP}^2 \qquad {}_{SL_2(\mathbb{Z})}\backslash\mathcal{H} \xrightarrow{j=[E_4^3:\Delta]} \mathbb{CP}^1$

Goal: $\mathcal{M}_*(SL_2(\mathbb{Z}))\cong\mathbb{C}[G_4,G_6]$ $\quad$ Any idea? (E_8, zero pt of G_4 or G_6, $\cdots$)

2. q-expansions of G_k (k even)

$q = e^{2\pi i \tau} \Rightarrow dq = 2\pi i q \, d\tau$

$dq^r = r \cdot 2\pi i q^r \, d\tau$

Q: Let $G_k(\tau) = a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$

Compute a_0.

↙ in [Za] $G_k = \frac{1}{2} \cdots$ to delete 2 here

A. $a_0 = \lim_{\operatorname{Im}\tau \to +\infty} G_k(\tau) = \sum_{n \neq 0} \frac{1}{n^k} = 2\zeta(k)$

Idea: Eisenstein fct = "2-dim Riemann zeta fct".

Luckily $\zeta(k)$ ($k>0$, even) are understandable.

Let $B_k, \widetilde{B}_k$ defined by

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} = 1 - \frac{x}{2} + \sum_{k=1}^{\infty} (-1)^{k+1} \widetilde{B}_k \frac{x^{2k}}{(2k)!},$$

then

$$\zeta(2k) = \frac{2^{2k-1}}{(2k)!} \widetilde{B}_k \pi^{2k} = (-1)^{2k+1} \frac{2^{2k-1}}{(2k)!} B_{2k} \pi^{2k} \qquad k \in \mathbb{Z}_{>0}$$

$$\zeta(k) = \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k}\right) \qquad k>0 \text{ even}$$

The following numerical tables are copied from [JP, §4] and wiki:

B_0	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8
1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$
		$\widetilde{B}_1$		$\widetilde{B}_2$		$\widetilde{B}_3$		$\widetilde{B}_4$
		$\frac{1}{6}$		$\frac{1}{30}$		$\frac{1}{42}$		$\frac{1}{30}$

Me	JP	wiki
B_k	b_k	B_k^-
$\widetilde{B}_k$	B_k	—

Examples $\zeta(2) = \frac{\pi^2}{2.3}$, $\zeta(4) = \frac{\pi^4}{2.3^2.5}$, $\zeta(6) = \frac{\pi^6}{3^3.5.7}$,

$\zeta(8) = \frac{\pi^8}{2.3^3.5^2.7}$, $\zeta(10) = \frac{\pi^{10}}{3^5.5.7.11}$, $\zeta(12) = \frac{691\pi^{12}}{3^6.5^3.7^2.11.13}$,

$\zeta(14) = \frac{2\pi^{14}}{3^6.5^2.7.11.13}$.

Thm. Let $k \geq 4$ even. G_k has q-expansion. $\Rightarrow G_k$ is modular form.

Idea: Compute every horizontal line.

Lemma. For $\tau \in \mathbb{C} - \mathbb{Z}$, we have

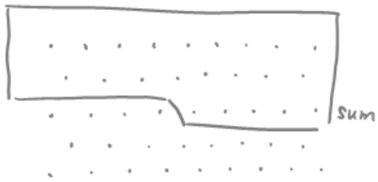
$$\sum_{n \in \mathbb{Z}} \frac{1}{\tau + n} = \frac{\pi}{\tan \pi\tau} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{+\infty} q^r\right)$$

$$\Big\| \text{ seriously}$$

$$\frac{1}{\tau} + \sum_{n=1}^{+\infty} \left(\frac{1}{\tau+n} + \frac{1}{\tau-n}\right)$$

By taking $\frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{d\tau^{k-1}}(-)$, we get Lipschitz's formula. $(k \in \mathbb{Z}_{\geq 2})$

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r$$



Proof of Thm.
$$\frac{1}{2} G_k(\tau) = \sum_{n=1}^{\infty} \frac{1}{n^k} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m\tau+n)^k}$$

$$= \zeta(k) + \sum_{m=1}^{\infty} \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^{mr}$$

$$= \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right)$$

Def
$$E_k(\tau) = \left(-\frac{2k}{B_k}\right)\left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n\right)$$

$$\mathcal{G}_k(\tau) = \mathbb{G}_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

Ex. Compute $\mathbb{G}_k(\tau)$ and $E_k(\tau)$. See answer in [Za, P17] [JP, P93].

Rmk. E_k can also be defined as

$$E_k(\tau) := \frac{1}{2} \sum_{\gcd(m,n)=1} \frac{1}{(m\tau+n)^k}$$

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$E_8(z) = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

$$E_{10}(z) = 1 - 264 \sum_{n=1}^{\infty} \sigma_9(n) q^n$$

$$E_{12}(z) = 1 + \frac{65520}{691} \sum_{n=1}^{\infty} \sigma_{11}(n) q^n$$

$$E_{14}(z) = 1 - 24 \sum_{n=1}^{\infty} \sigma_{13}(n) q^n.$$

$$E_4(\tau) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}$$

$$E_6(\tau) = 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1-q^n}$$

$$E_8(\tau) = 1 + 480 \sum_{n=1}^{\infty} \frac{n^7 q^n}{1-q^n}$$

3. Degree calculation [Za Prop 2, JP Thm 3]

- def of ord_p, ord_∞
- statement
- Rmk. modular form can be viewed as a section on the l.b. $\omega^{\otimes \frac{k}{2}}$ above the stack $\mathcal{H}/SL_2(\mathbb{Z})$.

  and this formula computes the degree of some "l.b." "$\omega(\infty)^{\otimes\frac{k}{2}}$" above the compactified space "$(\mathcal{H}/SL_2(\mathbb{Z}))^*$". Realize it?
- Rmk. weight k gives a bound of $\dim M_k(SL_2(\mathbb{Z}))$
- proof by contour integration.

Ex. 0. Compute $\mathrm{ord}_p(E_4)$ and $\mathrm{ord}_p(E_6)$ "again".

1. Bound $M_k(SL_2(\mathbb{Z}))$ when k is small ($\Rightarrow$ [Za, Cor 1])
2. Guess a basis of $M_k(SL_2(\mathbb{Z}))$ and compare the dimension.
3. Show that E_4 and E_6 are alg indep, thus $M_\cdot(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$

   Hint for 3. ① Show $\dim M_{12}(SL_2(\mathbb{Z})) = 2$. ‖ q-expansion, zero or $E_6^2 = \lambda E_4^3 \Rightarrow \frac{E_6}{E_4} \in M_2(SL_2(\mathbb{Z}))$

   ② Show that if $f_1, f_2 \in M_k(SL_2(\mathbb{Z}))$, $\dim \langle f_1, f_2\rangle_{\mathbb{C}} = 2$, then f_1 and f_2 are alg indep

   If $P(X,Y) = \sum_d P_d(X,Y) \in \mathbb{C}[X,Y]$ s.t. $P(f_1, f_2) \equiv 0$

   $\Rightarrow P_d(f_1, f_2) \equiv 0$

   $\Rightarrow f_2^{\,d}\, P_d(\frac{f_1}{f_2}) \equiv 0$

   $\Rightarrow p_d(\frac{f_1}{f_2}) \equiv 0$

   $\Rightarrow \frac{f_1}{f_2} \equiv c$ or $p_d \equiv 0$

   ③ Show E_4^3 and E_6^2 are alg indep.

4. Application [Za P18, JS P93]

Ex. From $E_4^2 = E_8$ $E_4 E_6 = E_{10}$ get identities

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{1}{120}\left(\sigma_7(n) - \sigma_3(n)\right)$$

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_5(n-m) = \frac{1}{5040}\left(11\sigma_9(n) - 21\sigma_5(n) + 10\sigma_3(n)\right)$$

Next time. begin our generalization of modular form.

Δ, τ, j, E_2, E_2', η, ... λ $\Gamma(2)$,

$S_n(\Gamma_1)$ ① ② ③ ④ ⓪