Eine Woche, ein Beispiel

11.14. Stiefel manifold

Goal: • understand some homotopy gp of $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$
• metric on Stiefel mfld ⇝ geodesic, volumn, ...
• cellular structure
• dim, some fiber bundle, with Lie gp,
• measure

Ref: https://en.wikipedia.org/wiki/Stiefel_manifold

For the description of metric, see https://math.stackexchange.com/questions/1371410/geodesic-of-stiefel-manifold
For the cellular structure, see https://math.stackexchange.com/questions/58041/cell-structure-on-stiefel-manifolds

Orthogonal basis = 标正基.

1. fiber bundle

$$O(k,\mathbb{R}) \to V_k(\mathbb{R}^n) \to Gr_k(\mathbb{R}^n) \qquad V_{k-l}(\mathbb{R}^{n-l}) \to V_k(\mathbb{R}^n) \to V_l(\mathbb{R}^n) \qquad S^{n-k} \to V_k(\mathbb{R}^n) \to V_{k-1}(\mathbb{R}^n)$$

$$U(k) \to V_k(\mathbb{C}^n) \to Gr_k(\mathbb{C}^n) \qquad V_{k-l}(\mathbb{C}^{n-l}) \to V_k(\mathbb{C}^n) \to V_l(\mathbb{C}^n) \qquad S^{2n-2k+1} \to V_k(\mathbb{C}^n) \to V_{k-1}(\mathbb{C}^n)$$

2. homotopy gp

Ref: https://people.math.ethz.ch/~jagnaw/Seminar_Notes/Obstruction_theory_Stiefel_Whitney_classes.pdf

Lemma 5 The homotopy groups of the Stiefel manifold $V_k(\mathbb{R}^n)$ for $l \leq n-k$ are

$$\pi_l(V_k(\mathbb{R}^n)) = \begin{cases} 0 & \text{if } l < n-k \\ \mathbb{Z} & \text{if } l = n-k \text{ and } k = 1 \\ \mathbb{Z} & \text{if } l = n-k \text{ is even} \\ \mathbb{Z}_2 & \text{if } l = n-k \text{ is odd and } k \neq 1. \end{cases}$$

ref:http://math.uchicago.edu/~may/REU2012/REUPapers/Fung.pdf
in[Lemma 1.10], the author proved this result by the elementary argument. Really nice!

For the references on
https://projecteuclid.org/journals/journal-of-the-institute-of-polytechnics-osaka-city-university-series-a-mathematics/volume-6/issue-1/On-the-homotopy-groups-of-Stiefel-manifolds/ojm/1353054734.pdf
https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-71/issue-4/Some-homotopy-groups-of-Stiefel-manifolds/bams/1183527242.full
https://www.maths.ed.ac.uk/~v1ranick/papers/paechter5.pdf
they all concern only with the stable homotopy group. So in general it's quite difficult to compute the other homotopy groups.

E.g. $n=5$ $\pi_i(V_k(\mathbb{R}^5))$

i \ k	0	$\{*\}$	1	S^1	2	3	4	$SO(5)$	5	$O(5)$
1	0		0		0	0	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	
2	0		0		0	$\mathbb{Z}$	0		0	
3	0		0		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$	
4	0		$\mathbb{Z}$			$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	
5	0		$\mathbb{Z}/2\mathbb{Z}$			$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	
6	0		$\mathbb{Z}/2\mathbb{Z}$			0	0		0	
7	0		$\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$			$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$	
8	0		$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$			0	0		0	

E.g. $n=6$ $\pi_i(V_k(\mathbb{R}^6))$

i \ k	0	$\{*\}$	1	S^1	2	3	4	5	$SO(6)$	6	$O(6)$
1	0		0		0	0	0	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	
2	0		0		0	0	$\mathbb{Z}$	0		0	
3	0		0		0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$	
4	0		0		$\mathbb{Z}$		0	0		0	
5	0		$\mathbb{Z}$				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$	
6	0		$\mathbb{Z}/2\mathbb{Z}$				0	0		0	
7	0		$\mathbb{Z}/2\mathbb{Z}$				$\mathbb{Z}$	$\mathbb{Z}$		$\mathbb{Z}$	
8	0		$\mathbb{Z}/24\mathbb{Z}$				$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$		$\mathbb{Z}/24\mathbb{Z}$	

E.g. $n=5$ $H^i(V_k(\mathbb{R}^5),\mathbb{Z})$

i \ k	0	1	2	3	4	5
	$\{*\}$	S^4			$SO(5)$	$O(5)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$
1	0	0	0	0	0	0
2	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
3	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}$
4	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
5	0	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
6	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
7	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z})$
8	0	0	0	0	0	0
9	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
10	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$

E.g. $n=6$ $H^i(V_k(\mathbb{R}^6),\mathbb{Z})$

i \ k	0	1	2	3	4	5	6
	$\{*\}$	S^5				$SO(6)$	$O(6)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	
1							
2					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
3						$\mathbb{Z}$	
4			$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$
5		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	EVERY
6					$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
7				$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$	
8						$\mathbb{Z}$	
9			$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$	THING
10						$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	
11					$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
12				$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	
13							
14					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
15						$\mathbb{Z}$	