Eine Woche, ein Beispiel

3.17 special complex manifolds

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

[Vo02]Voisin, Claire. Hodge Theory and Complex Algebraic Geometry. I. Translated from the French by Leila Schneps. 卷 76. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2002.

Not from "almost mathematics"

$$
\begin{tikzcd}
 & \text{Kähler-Einstein} \arrow[dr, phantom, sloped, "\subset"] & & \\
 & & \text{cplx} \arrow[r, phantom, "\subset"] & \text{almost cplx} \\
 & & \text{Hermitian} \arrow[u, phantom, sloped, "\subset"] \arrow[r, phantom, "\subset"] & \text{almost Hermitian} \arrow[u, phantom, sloped, "\subset"] \\
\text{K3} \arrow[r, phantom, "\subset"] \arrow[drr, bend right, "?"] & \text{Hyperkähler} \arrow[r, phantom, "\subset"] & \text{Kähler} \arrow[u, phantom, sloped, "\subset"] \arrow[r, phantom, "\subset"] & \text{almost Kähler} \arrow[u, phantom, sloped, "\subset"] \\
 & & \text{Kähler-Hodge} \arrow[u, phantom, sloped, "\subset"] & \\
 & & \text{cplx proj variety} \arrow[u, phantom, sloped, "\subset"] &
\end{tikzcd}
$$

When can the Kähler form be understood as special characteristic class? In the following discussion there are some partial answers:
https://mathoverflow.net/questions/197808/line-bundles-over-k%c3%a4hler-hodge-manifolds
Question: are these line bundles defined functorial?

cplx proj variety is Kähler Hodge:
https://mathworld.wolfram.com/KaehlerForm.html
"In the special case of a projective algebraic variety, the Kähler form represents an integral cohomology class."

The following equivalent definitions of Kähler metric come from [Vo02, Theorem 3.13]:

Theorem 3.13 *The following properties are equivalent:*

(i) *The metric h is Kähler.* Hermitian mfld + $d\omega = 0$

(ii) *The complex structure endomorphism I is flat for the Levi-Civita connection. This means that it satisfies*

$$\nabla(I\chi) = I\nabla\chi, \quad \forall\chi \in A^0(T_{X,\mathbb{R}}).$$

(iii) *The Chern connection and the Levi-Civita connection coincide on T_X, identified with $T_{X,\mathbb{R}}$ via the map $\Re$.*

Some information from Prof. Xu:

$$\text{integrable system} \supset \begin{cases} \text{Hitchen integrable system} \\ \text{ALE, ALF, ALG, ALH}, \cdots \\ \text{elliptic K3 surface} \end{cases}$$