Eine Woche, ein Beispiel

2.16 lines passing a point

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.
[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

This document is basically [Huy23, II.2]. Mathematicians have magic playing with this geometrical objects.

Notation in [Huy23]:

affine	proj
V	$\mathbb{P}^{n+1}$
X^{cone}	X
$\mathcal{U}$	P_L
W	L

$X \subseteq \mathbb{P}^4$: cubic threefold

$F(X) \subseteq Gr(5,2)$: moduli space of lines in X

$F_2(X) \subseteq Gr(5,2)$: moduli space of lines of second type in X.

dim & smoothness

$\dim F(X) = 2$:

- incidence variety
- $F(X) = \mathrm{Hilb}_X^{P_{\mathbb{P}^1}(t)} \rightsquigarrow T_L F(X) \cong \mathrm{Hom}(\mathcal{I}_L, i_{L,*}\mathcal{O}_L)$

$$\begin{aligned} T_L F(X) &\cong \mathrm{Hom}(\mathcal{I}_L, i_{L,*}\mathcal{O}_L) \\ &\cong \mathrm{Hom}(i_L^* \mathcal{I}_L, \mathcal{O}_L) \\ &\cong \mathrm{Hom}(\mathcal{I}_L/\mathcal{I}_L^2, \mathcal{O}_L) \\ &\cong \mathrm{Hom}(\mathcal{O}_L, \mathcal{N}_{L/X}) \\ &\cong H^0(L, \mathcal{N}_{L/X}) \end{aligned}$$

https://math.stackexchange.com/questions/239959/conormal-sheaf-morphisms-of-schemes-stacks-project
https://math.stackexchange.com/questions/4899527/why-pullback-of-ideal-sheaf-should-be-the-conormal-sheaf

$\dim F_2(X) = 1$:

- incidence variety to get an upper bound

  use Gauss map to show $\overset{F_2(X)\times X}{\overset{\cup}{\mathbb{I}L_2}} \longrightarrow \overset{X}{\overset{\cup}{q(\mathbb{I}L_2)}}$ is generically finite

- determinance variety to get a lower bound
- work on moduli space of cubic threefolds.

Type of lines

X : sm cubic hypersurface, $L \subseteq X$ a line.
$N_{L/X}$ is a v.b. over $L = \mathbb{P}^1$.
By classification of v.b. over $\mathbb{P}^1$, one can distinguish type of L.

$L = \{z_2 = \cdots = z_n = 0\} = \{[* : * : 0 \cdots 0]\}$

$$\begin{tikzcd}
0 \arrow[r] & N_{L/X} \arrow[r] & N_{L/\mathbb{P}^{n+1}} \arrow[r] \arrow[d, equal] & N_{X/\mathbb{P}^{n+1}}|_L \arrow[r] \arrow[d, equal] & 0 \\
 & & \mathcal{O}_L(1) \otimes_{\mathbb{C}} V/W & \mathcal{O}_L(3) &
\end{tikzcd}$$

$$\rightsquigarrow \quad \begin{tikzcd}
0 \arrow[r] & N_{L/X}(-1) \arrow[r] & \mathcal{O}_L \otimes_{\mathbb{C}} V/W \arrow[r, "\left(\frac{\partial F}{\partial z_2}, \cdots, \frac{\partial F}{\partial z_{n+1}}\right)"] & \mathcal{O}_L(2) \arrow[r] & 0
\end{tikzcd}$$

$$\rightsquigarrow \quad N_{L/X}(-1) = \underbrace{\mathcal{O}_L^{\oplus n-3} \oplus \mathcal{O}_L(-1)^{\oplus 2}}_{\text{first type}} \text{ or } \underbrace{\mathcal{O}_L^{\oplus n-2} \oplus \mathcal{O}_L(-2)}_{\text{second type}} \text{, and}$$

first type $\mathcal{O}_L(-1)^{\oplus 2}$ second type $\mathcal{O}_L \oplus \mathcal{O}_L(-2)$

c.f. [Huy23, Ex. II.2.1, Rmk II.1.16]

first type
second type

$$\begin{tikzcd}
0 \arrow[r] & P_L^{\text{cone}} \arrow[r] \arrow[d] & V \arrow[d] & & \\
 & \substack{n-1 \\ n} & \substack{n+2 \\ n+2} & & \\
0 \arrow[r] & H^0(L, N_{L/X}(-1)) \arrow[r] & V/W \arrow[r, "\left(\frac{\partial F}{\partial z_2}, \cdots \frac{\partial F}{\partial z_{n+1}}\right)"] & S^2(W^*) \arrow[r] & H^1(L, N_{L/X}(-1)) \arrow[r] & 0 \\
 & \substack{n-3 \\ n-2} & \substack{n \\ n} & \substack{3 \\ 3} & \substack{0 \\ 1} &
\end{tikzcd}$$

One can identify the type of L throughout

- $\dim \langle \partial_i F|_L \rangle$ where $\partial_i F|_L \in S^2(W^*)$ Rmk II.2.2
- $P_L^{\text{cone}} = \bigcap_{y \in W} T_y X^{\text{cone}} \simeq \mathbb{C}^{n-1}$ or $\mathbb{C}^n$ Cor II.2.6
- Gauss map γ_X: Ex II.2.10

$$\begin{tikzcd}
\mathbb{P}^{n+1} \arrow[r, "\substack{\text{gen} \\ 2:1}"] & (\mathbb{P}^{n+1})^* \\
X \arrow[u, phantom, "\cup"] \arrow[r, "\substack{\text{gen} \\ 1:1}"] & X^* \arrow[u, phantom, "\cup"] \\
L \arrow[u, phantom, "\cup"] \arrow[r, "\substack{\text{first type: } 1:1 \\ \text{second type: } 2:1}"'] & \gamma_X(L) \arrow[u, phantom, "\cup"]
\end{tikzcd}$$