Eine Woche, ein Beispiel

5.14. modular representation of $\mathbb{Z}/p\mathbb{Z}$

Let $\mathcal{C} = \mathrm{rep}_\Lambda(\mathbb{Z}/p\mathbb{Z}) = \mathrm{mod}(\Lambda[\mathbb{Z}/p\mathbb{Z}])$, where
$\Lambda = \overline{\Lambda}$ is a field with char $\Lambda = p$.

Goal: understand $\mathcal{C}$ in detail.

1. indecomposable representations
2. tensor category structure
3.

1. indecomposable representations

We have

$$\Lambda[\mathbb{Z}/p\mathbb{Z}] \cong \Lambda[x]/(x^p-1) \cong \Lambda[x]/(x-1)^p \cong \Lambda[T]/T^p$$

$$\begin{array}{ll}
N(p) & \left(\Lambda^p, \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
\vdots & \\
\uparrow\downarrow & \\
N(2) & \left(\Lambda^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
N(1) & (\Lambda, 0)
\end{array}$$

AR-quiver of $\bullet \circlearrowleft T \,/\, T^p = 0 \;=\; \Lambda[T]/T^p$

https://math.stackexchange.com/questions/368722/what-does-the-group-ring-mathbbzg-of-a-finite-group-know-about-g

2. tensor category structure.

For general ring $A/_{\Lambda}$, there is no tensor structure on mod(A).
However, for a Hopf algebra $A/_{\Lambda}$, we can construct a natural tensor structure on mod(A).

Construction.

$$c^{\#}: A \longrightarrow A \otimes_{\Lambda} A \quad \rightsquigarrow \quad \otimes: \mathrm{mod}(A) \times \mathrm{mod}(A) \longrightarrow \mathrm{mod}(A \otimes_{\Lambda} A) \xrightarrow{c^{\#,*}} \mathrm{mod}(A)$$

$$(M, N) \longmapsto M \otimes_{\Lambda} N \longmapsto M \otimes_{\Lambda} N$$

where A acts on $M \otimes_{\Lambda} N$ by

$$A \times M \otimes_{\Lambda} N \longrightarrow M \otimes_{\Lambda} N \qquad a \cdot (m \otimes n) := c^{\#}(a)(m \otimes n) = \sum_i b_i m \otimes c_i n$$

when $c^{\#}(a) = \sum_i b_i \otimes c_i$

$$e^{\#}: A \longrightarrow \Lambda \quad \rightsquigarrow \quad e^{\#,*}: \mathrm{mod}(\Lambda) \longrightarrow \mathrm{mod}(A)$$

$$\Lambda \longmapsto \Lambda$$

$$A \times \Lambda \longrightarrow \Lambda \qquad (a, t) \longmapsto e^{\#}(a) \cdot t$$

$$i^{\#}: A \longrightarrow A^{op} \quad \rightsquigarrow \quad (-)^{\vee}: \mathrm{mod}(A) \xrightarrow{\mathrm{Hom}_{\Lambda}(-,\Lambda)} \mathrm{mod}(A^{op}) \xrightarrow{i^{\#,*}} \mathrm{mod}(A)$$

$$M \longmapsto M^{\vee} \longmapsto M^{\vee}$$

$$A \times M^{\vee} \longrightarrow M^{\vee} \qquad (a, f) \longmapsto f(i^{\#}(a) -)$$

Q: Let A be a Λ-alg.
Given a tensor category structure on mod(A), can we recover the Hopf algebra on A?
I.e., is the map

$$\left\{ \begin{array}{c} \text{Hopf algebra structures} \\ \text{on } A \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{tensor category structures} \\ \text{on mod}(A) \end{array} \right\}$$

inj or surj?

E.g. (tensor category structure of $\mathrm{mod}(\Lambda[G])$) $\quad$ G: finite gp

$\mathrm{rep}_\Lambda(G)$ is naturally endowed with $\otimes$-structure:

$$G \circlearrowright M\otimes N \qquad g\cdot(m\otimes n) := gm\otimes gn$$

$$\rightsquigarrow \Lambda[G]\circlearrowright M\otimes N \qquad \left(\sum_i t_i g_i\right)(m\otimes n) = \sum_i t_i\, g_i\cdot(m\otimes n) = \sum_i t_i(g_i m\otimes g_i n) = \left(\sum_i t_i(g_i\otimes g_i)\right)(m\otimes n)$$

so the Hopf algebra structure on $\Lambda[G]$ should be

$$c^\#: \Lambda[G]\longrightarrow \Lambda[G]\otimes_\Lambda \Lambda[G] \qquad \sum_i t_i g_i \longmapsto \sum_i t_i g_i\otimes g_i$$

$$e^\#: \Lambda[G]\longrightarrow \Lambda \qquad \sum_i t_i g_i \longmapsto \sum_i t_i$$

$$i^\#: \Lambda[G]\longrightarrow \Lambda[G]^{op} \qquad \sum_i t_i g_i \longmapsto \sum_i t_i g_i^{-1}$$

Verify:

$$G\circlearrowright \Lambda \qquad g.t := t$$

$$\rightsquigarrow \Lambda[G]\circlearrowright\Lambda \qquad \left(\sum_i t_i g_i\right)t = \sum_i t_i(g_i\cdot t) = \sum_i t_i t$$

$$G\circlearrowright M^\vee \qquad g\cdot f := f(g^{-1}\cdot -)$$

$$\rightsquigarrow \Lambda[G]\circlearrowright M^\vee \qquad \left(\sum_i t_i g_i\right)f = \sum_i t_i(g_i\cdot f) = \sum_i t_i f(g_i^{-1}\cdot -) = f\left(\sum_i t_i g_i^{-1} -\right)$$

E.g. (tensor category structure of $\mathrm{mod}(\mathcal{U}(\mathfrak{g}))$) $\quad$ $\mathfrak{g}$: f.d. Lie alg over $\mathbb{C}$

$\mathrm{rep}_{\mathbb{C}}(\mathfrak{g})$ is naturally endowed with $\otimes$-structure:

$$\mathfrak{g}\circlearrowright M\otimes N \qquad X\cdot(m\otimes n) := X.m\otimes n + m\otimes X.n$$

$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M\otimes N \qquad X_1X_2\cdots X_n(m\otimes n) = \sum_{\{1,\dots,k\}=I\sqcup J} (X_I m)\otimes(X_J n) \quad \text{shuffle!}$$

e.p. $[X,Y](m\otimes n) = [X,Y]m\otimes n + m\otimes[X,Y]n$

(For $I=\{i_1,\dots,i_l\}$ fix an order $i_1<i_2<\dots<i_l$, $X_I := X_{i_1}X_{i_2}\cdots X_{i_n}$)

so the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ should be

$$c^\#: \mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})\otimes_{\mathbb{C}}\mathcal{U}(\mathfrak{g}) \qquad X_{\{1,\dots,k\}}\longmapsto \sum_{\{1,\dots,k\}=I\sqcup J} X_I\otimes X_J$$

$$e^\#: \mathcal{U}(\mathfrak{g})\longrightarrow\mathbb{C} \qquad \sum_\alpha t_\alpha X_\alpha \longmapsto t_\emptyset$$

$$i^\#: \mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})^{op} \qquad \sum_\alpha t_\alpha X_\alpha\longmapsto \sum_\alpha(-1)^{|\alpha|}t_\alpha X_\alpha$$

Verify:

$$\mathfrak{g}\circlearrowright\mathbb{C} \qquad X.t := 0$$

$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright\mathbb{C} \qquad \left(\sum_\alpha t_\alpha X_\alpha\right).t = t_\emptyset t$$

$$\mathfrak{g}\circlearrowright M^\vee \qquad X.f := \underset{\text{minus!}}{-}f(X.-)$$

$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M^\vee \qquad \left(\sum_\alpha t_\alpha X_\alpha\right).t = \sum_\alpha t_\alpha(-1)^{|\alpha|}f(X_\alpha.-) = f\left(\sum_\alpha(-1)^{|\alpha|}t_\alpha X_\alpha.-\right)$$

For more examples of Hopf algebras, see wiki: Hopf algebras.