§3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections
4. L-fct
5. density theorem

Just for convenience, we allow

element $\in_c$ class   class $\subset_c$ class $\{\cdots|\cdots\}_c$ be a class

We may add c to emphasize that the family can be a class, instead of set.

1. Galois rep

<u>Setting</u> G: arbitrary topo gp e.g. G any Galois gp

If G profinite $\Rightarrow$ open subgps are finite index subgps.

Λ: top field e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{cont}}(G)$

$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ cont

⚠ $\rho(G)$ can be infinite! for Gal gp

E.g. When char $F \neq l$, we have l-adic cyclotomic character

$$\varepsilon_l: Gal(F^{sep}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \qquad \sigma \longmapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute ε_l for $F = \mathbb{F}_p$.

A: $\varepsilon_l: \hat{\mathbb{Z}} \cong Gal(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \qquad 1 \longmapsto p$

↑ lift from $\mathbb{Z} \to \mathbb{Z}_l^\times$

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$

$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{fi}}(G)$ fi: finite image / finite index

$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with finite image

Rmk.

$$\begin{tikzcd}
\mathrm{rep}_{\Lambda,sm}(G) \arrow[r, phantom, "="] & \mathrm{rep}_{\Lambda_{disc},cont}(G) \arrow[rr] \arrow[dr, shift left, red] & & \mathrm{rep}_{\Lambda,cont}(G) \arrow[dl, green] \\
& & \mathrm{rep}_{\Lambda,fi}(G) \arrow[ul, shift left, blue] & \\
\mathrm{Rep}_{\Lambda,sm}(G) \arrow[uu, phantom, "\bigcap" description] & \mathrm{Rep}_{\Lambda_{disc},cont}(G) \arrow[l] \arrow[rr] \arrow[uu, phantom, "\bigcap" description] \arrow[dr, shift left, red] & & \mathrm{Rep}_{\Lambda,cont}(G) \arrow[uu, phantom, "\bigcap" description] \\
& & \mathrm{Rep}_{\Lambda,fi}(G) \arrow[ul, shift left, blue] \arrow[uu, phantom, "\bigcap" description] &
\end{tikzcd}$$

$\rightarrow$: if fin index subgps are open

$\rightarrow$: if G: profinite gp (Only need: open $\Rightarrow$ fin index)

$\rightarrow$: Artin rep (of profinite gp)

Artin rep. $\Lambda = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\quad \|\cdot\| = \|\cdot\|_{max}$, $|\cdot| = |\cdot|_{max}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists m \in \mathbb{N}$, s.t. $A^m \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unpotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$. Take $m \in \mathbb{N}$ s.t. $|\lambda^m - 1| > \frac{1}{3}$.

$\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},cont}(G)$, $\rho(G)$ is finite. $\quad G$ profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq G_F$ finite index, $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = Id$

$\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp, so ρ is semisimple.
Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

Setting: Λ: NA local field with char $k_\Lambda = l$

Q: What would happen if Λ is only a NA local field?

Finite field

Goal. For Λ: NA local field with char $k_\Lambda = l$, understand $\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Lambda)$, TFAE:

① $\hat{\mathbb{Z}} \longrightarrow GL_n(\Lambda)$ is a well-defined cont gp homo
$1 \longmapsto A$

② $\exists\, g \in GL_n(\Lambda)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Lambda)$

③ $\det(\lambda I - A) \in \mathcal{O}_\Lambda[\lambda]$, with $\det A \in \mathcal{O}_\Lambda^\times$

A is called bounded in these cases.

Proof $\quad$ ① $\underset{\text{local}}{\overset{\text{local}}{\rightleftarrows}}$ ② $\underset{\text{local}}{\rightleftarrows}$ ③

① $\Rightarrow$ ②: $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Lambda)$, which conjugates to $GL_n(\mathcal{O}_\Lambda)$

https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl2-mathbb-q-p

Another method:

Lemma 1. ρ, μ: two ways of expressions of gp action

$\rho: \hat{\mathbb{Z}} \to GL_n(\mathbb{Z})$ is cont $\Leftrightarrow$ $\mu: \hat{\mathbb{Z}} \times \Lambda^n \to \Lambda^n$ is cont

$\Rightarrow$: $\mu: \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\rho \times Id_{\Lambda^n}} GL_n(\Lambda) \times \Lambda^n \longrightarrow \Lambda^n$ is cont.

Λ^n is Haus loc cpt

See [Theorem III.3, III.4]:
https://github.com/lrnmhl/AT1/blob/main/Algebraic_Topology_I___Stefan_Schwede__Bonn__Winter_2021.pdf

$\Leftarrow$: $\quad \rho: \hat{\mathbb{Z}} \to GL_n(\Lambda)$ is cont

$\Leftrightarrow \rho: \hat{\mathbb{Z}} \to M_{n\times n}(\Lambda)$ is cont

$\Leftrightarrow \rho_{ij}: \hat{\mathbb{Z}} \to \Lambda$ is cont $\quad \forall\, i,j \in \{1,\dots,n\}$

We know that

$\rho_{ij}: \hat{\mathbb{Z}} \xrightarrow{(Id, e_j)} \hat{\mathbb{Z}} \times \Lambda^n \xrightarrow{\mu} \Lambda^n \xrightarrow{e_i^*} \Lambda$

is cont

linear map between f.d v.s is cont

In this case, e_i^* is projection.

Another $\Leftarrow$: (suggested by Longke Tang)

$$\mu : \hat{\mathbb{Z}} \times \Lambda^n \longrightarrow \Lambda^n \text{ is cont}$$

$$\Leftrightarrow \quad \hat{\mathbb{Z}} \longrightarrow \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n) \text{ is cont}$$

open cpt topo

$$\hat{\mathbb{Z}} \xrightarrow{\exists !} GL_n(\Lambda) \subseteq \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n)$$

Only need: $GL_n(\Lambda) \subseteq M_{n\times n}(\Lambda)$, $GL_n(\Lambda) \subset \mathrm{Mor}_{\mathrm{Top}}(\Lambda^n, \Lambda^n)$ define the same topo on $GL_n(\Lambda)$.

This is hard. Assuming Lemma 1, this can be proved, but then this method can't be a real proof for Lemma 1.

Lemma 2. $\mathcal{L}_1, \mathcal{L}_2$ lattice in $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$ lattice in Λ

$$\left[\left.\begin{array}{l} \mathcal{L}_1 \supseteq (p^{k'})^{\oplus n} \\ \mathcal{L}_2 \supseteq (p^{k'})^{\oplus n} \end{array}\right\} \Rightarrow \# \mathcal{L}_1 + \mathcal{L}_2 / \mathcal{L}_1 < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice} \right]$$

Take $\mathcal{L} := \mathcal{O}_\Lambda^n \subseteq \Lambda^n$, then the stabilizer

$$\begin{aligned} \mathrm{Stab}(\mathcal{L}) &= \{ g \in \hat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L} \} \\ &= \{ g \in \hat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \quad \forall i \} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where

$$\mu_{e_i} : \hat{\mathbb{Z}} \longrightarrow \Lambda^n \qquad g \longmapsto g \cdot e_i \qquad \text{(cont by Lemma 1)}$$

$\Longrightarrow \mathcal{L}$ has finite orbit

$\overset{\text{Lemma 2}}{\Longrightarrow} \sum_{\mathcal{L}_i \in \hat{\mathbb{Z}} \cdot \mathcal{L}} \mathcal{L}_i$ is a lattice stabilized by $\mathbb{Z}$.

After conjugation, $A, A^{-1} \in M^{n\times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

② $\Rightarrow$ ①: w.l.o.g. $A \in GL_n(\mathcal{O}_\Delta)$. Then we get a lift

$$\begin{tikzcd} \hat{\mathbb{Z}} \arrow[r, "\exists ! \text{ cont}"] & \widehat{GL_n(\mathcal{O}_\Delta)} \cong GL_n(\mathcal{O}_\Delta) \\ \mathbb{Z} \arrow[u] \arrow[r] & GL_n(\mathcal{O}_\Delta) \arrow[u] \end{tikzcd}$$

② $\Rightarrow$ ③: Obvious

③ $\Rightarrow$ ②:

$$\sum_{i\in\mathbb{Z}} A^i \mathcal{L} = \sum_{i=0}^{n-1} A^i \mathcal{L} \text{ is a lattice fixed by } A, A^{-1} \text{ (Lemma 2)}$$

After conjugation, $A, A^{-1} \in M^{n\times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

⚠ $A, B \in GL_n(\Lambda)$ bounded $\not\Rightarrow$ AB bounded

counter eg: (from Longke Tang)

Consider $A = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \begin{pmatrix} p & \\ & 1 \end{pmatrix}^{-1}$, $B = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then $AB = \begin{pmatrix} p & \\ & p^{-1} \end{pmatrix}$.

Cor. $\mathrm{rep}_{\Delta,\mathrm{cont}}(\hat{\mathbb{Z}}) \cong \mathrm{rep}^{\mathrm{bdd}}_{\Delta,\mathrm{cont}}(\mathbb{Z})$

↖ full subcategory of $\mathrm{rep}_{\Delta,\mathrm{cont}}(\mathbb{Z})$.

Local field

Goal. For Λ: NA local field with char $k_\Lambda = l$,

F: NA local field with char $k_F = p \neq l$,

realize cont Galois rep as bounded Weil-Deligne rep.

via the following diagrams:

$$\begin{tikzcd}
 & & \mathrm{rep}_{\Lambda,\mathrm{sm}}(W_F) \text{ with } N \arrow[dl] \arrow[d, phantom, "\cup"] \\
 & \mathrm{rep}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] \arrow[d, phantom, "\cup"] & \mathrm{WDrep}_{\Lambda,\mathrm{sm}}(W_F) \arrow[d, phantom, "\cup"] \\
\mathrm{rep}_{\Lambda,\mathrm{cont}}(\Gamma_F) \arrow[r, "\sim"] & \mathrm{rep}^{\mathrm{bdd}}_{\Lambda,\mathrm{cont}}(W_F) \arrow[r, "\sim"] & \mathrm{WDrep}^{\mathrm{bdd}}_{\Lambda,\mathrm{sm}}(W_F)
\end{tikzcd}$$

here, "bdd" means $\mathrm{Im}\,\rho$ are bounded.