

Eine Woche, ein Beispiel

4.17 preliminary facts of representations of p-adic groups

Main reference: The Local Langlands Conjecture for $\mathrm{GL}(2)$ by Colin J. Bushnell and Guy Henniart.
[<https://link.springer.com/book/10.1007/978-3-540-31511-X>]

Process.

1. Basic properties

- Smoothness
- Irreducibility and unitary
- Reduction to smaller cardinal.

2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

3. Construction of new reps.

- Special sub & quotient rep
- Duality
- Ind and c-Ind
- Other constructions ← Example: mirabolic group

4. Hecke algebra

5. Intertwining properties ← Example: $\mathrm{GL}_2(\mathbb{Q}_p)$

1. Basic properties

1.1. Smoothness

G : loc. profinite group

V : cplx v.s.

$$\rho: G \longrightarrow \text{Aut}_{\mathbb{C}}(V) \quad g \mapsto [v \mapsto g.v]$$

Def. (ρ, V) is smooth if

$$\forall v \in V, \exists K \leq G \text{ cpt open s.t. } k.v = v \quad \forall k \in K$$

$\text{Rep}(G) = \{\text{sm rep of } G\}$ is a full subcategory of $\{\text{rep of } G\}$.

Rmk. Any sub/quotient rep of $(\rho, V) \in \text{Rep}(G)$ is smooth.

$$H \leq G \text{ cpt, } (\rho, V) \in \text{Rep}(G) \Rightarrow (\rho|_H, V) \in \text{Rep}(H)$$

Rmk. For fcts, smoothness has a different meaning.

Recall the definition of $C^\infty(G)$ & $C_c^\infty(G)$.

$$C^\infty(G) := \{f: G \rightarrow \mathbb{C} \mid f \text{ is loc. const}\}$$

$$C_c^\infty(G) := \{f \in C^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$$

1.2. Irreducibility and unitary

$$\text{Irr}(G) = \{(\rho, V) \in \text{Rep}(G) \mid \rho \text{ is a irreducible rep}\}$$

$$\widehat{G} = \{(\rho, V) \in \text{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$$

$$\stackrel{[P13]}{=} \{X: G \rightarrow \mathbb{C}^{\times} \mid \ker X \text{ is open}\}$$

$$\stackrel{[C1.6]}{=} \{X: G \rightarrow \mathbb{C}^{\times} \mid X \text{ is continuous}\}$$

Rmk. The notation is slightly different with the original reference.

Rmk.

$$\widehat{G} \subseteq \text{Irr}(G) \subseteq \text{Ind}(G) \subseteq \text{Rep}(G)$$

[P15]

When G is cpt, or

[P21] $G/Z(G)$ is cpt with G/K countable, we get $\text{Ind}(G) = \text{Irr}(G)$;

[P21] when G is abelian and G/K is countable, $\text{Ind}(G) = \widehat{G}$.

($\exists K \leq G$ cpt open, countable = at most countable here)

Rmk. A more general result is as follows.

Prop. Let $(\rho, V) \in \text{Rep}(G)$, G/K countable. $\exists K \leq G$ cpt open

Suppose $\rho|_{Z(G)}$ decompose as $Z(G) \xrightarrow{\chi_w} \mathbb{C}^{\times} \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$.

Let $Z(G) \leq K \leq G$ $K \leq G$ open $K/Z(G)$ is cpt.

Then $(\rho|_K, V) \in \text{Rep}(K)$ is semisimple.

We will rewrite it as follows.

Prop. Let G be a loc. cpt gp satisfying the countable Hypothesis. $Z = Z(G)$.

For $(\rho, V) \in \text{Rep}(G) \xrightarrow{\cong} \chi_w$, $K \in \text{Cos}_Z(G)$,

$(\rho|_K, V) \in \text{Rep}(K)$ is semisimple.

To prove this we need the following lemma. (when applied, it would be $K_0 Z(G) \leq K$)

Lemma. Let $H \leq G$ open, $[G:H] < \infty$, $(\rho, V) \in \text{Rep}(G)$. Then

ρ is G -semisimple $\Leftrightarrow \rho|_H$ is H -semisimple.

Def (multiple of χ) $\rho \xrightarrow{H} \chi$

Let $H \leq G$, $(\rho, V) \in \text{Rep}(G)$, $\chi \in \widehat{H}$.

We say H acts on V as χ .

or $\rho|_H$ is a multiple of χ ,

or (when $H = Z(G)$) ρ admits the central character χ .

if $\rho|_H$ decompose as follows:

$$\rho|_H: H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \text{Aut}_{\mathbb{C}}(V)$$

We may denote χ by χ_ρ or χ_H . When $H = Z(G)$, χ is denoted by χ_w or $w\rho$.

Def (Contain irr rep) $\rho \rhd \sigma \quad n = \text{mult}(\rho, \sigma)$

Let $H \leq G$, $(\rho, V) \in \text{Rep}(G)$, $(\sigma, W) \in \text{Irr}(H)$.

We say ρ contains σ , or σ occurs in ρ , if

$$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \neq 0$$

i.e., σ can be realized as a quotient subrep of $\text{Res}_H^G \rho$.

The multiplicity is defined as

$$\text{mult}(\rho, \sigma) := \dim_{\mathbb{C}} \text{Hom}_H(\text{Res}_H^G \rho, \sigma)$$

Cor. When H acts on V as χ_ρ , ρ contains χ_ρ .

Def (Inflation) $\lambda \xrightarrow{G} \Lambda$

Let $N \triangleleft G$. The inflation of $(\lambda, W) \in \text{Rep}(G/N)$ is defined as $(\lambda, W) \in \text{Rep}(G)$, where

$$\Lambda: G \xrightarrow{\pi} G/N \xrightarrow{\lambda} \text{End}_{\mathbb{C}}(W)$$

Def (Twist) χ_ρ

Suppose $G \leq GL_n(F)$, where F is a non-archi local field.

For $\chi \in \widehat{F}^\times$ and $(\rho, V) \in \text{Rep}(G)$, we define

$$\chi_\rho := (\chi \circ \det) \otimes_{\mathbb{C}} \rho: G \longrightarrow \text{End}_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}} V) = \text{End}_{\mathbb{C}}(V)$$

$$\chi_\rho(g) \cdot v = \chi(\det g) \cdot \rho(g) \cdot v \quad \forall g \in G \quad v \in V$$

as the twist of ρ by χ .

Def (Unitary operator) V : Hilbert space.

$U \in \text{Aut}_{\mathbb{C}}(V)$ is called an unitary operator if

$$\langle Uv, Uw \rangle = \langle v, w \rangle \quad \forall v, w \in V$$

Def (Unitary rep) V : Hilbert space.

$(\rho, V) \in \text{Rep}(G)$ is unitary if $\rho(g)$ is an unitary operator ($\forall g \in G$).

E.p. $X \in \widehat{G}$ is unitary if $\text{Im } X \subseteq S'$

Rmk. When $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$, any $X \in \widehat{G}$ is unitary.

1.3. Reduction to smaller cardinal

Admissibility

Def. $(\rho, V) \in \text{Rep}(G)$ is admissible if $\dim_{\mathbb{C}} V^k < +\infty$ for $\forall K \leq G$ cpt open.

Rmk. Any sub/quotient rep of $(\rho, V) \in \text{Rep}(G)$ admissible is admissible.

$H \leq G$ cpt, $(\rho, V) \in \text{Rep}(G)$ admissible

$\Rightarrow (\rho|_H, V) \in \text{Rep}(H)$ is admissible.

Countable hypothesis

$\exists / \forall K \leq G$ cpt open, G/K is countable.

Assuming countable hypothesis, we get

$$(\rho, V) \in \text{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \text{End}_G(V) = \mathbb{C} \\ \rho \text{ acts on } V \text{ as a character } w_p \\ \dim_{\mathbb{C}} V = 1. \end{cases}$$

$\xrightarrow[G \text{ is abelian}]{} \dim_{\mathbb{C}} V = 1.$

Finite dimension

K : cpt (profinite) gp, $(\sigma, W) \in \text{Irr}(K) \rightsquigarrow \dim_{\mathbb{C}} W < +\infty$

Assuming countable hypothesis of $Z(G)$. If $K \leq G$ open,

$K \geq Z(G)$, $K/Z(G)$ is cpt, $(\sigma, W) \in \text{Irr}(K) \rightsquigarrow \dim_{\mathbb{C}} W < +\infty$.

2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

Rep of $G = \mathcal{O}, \mathcal{O}^\times, F, F^\times$, where F is a non-archi local field.

In these cases, G is abelian and satisfy the countable hypothesis, so $\text{Ind}(G) = \widehat{G}$, i.e., everything reduced to the classification of characters.

E.g. $G = (\mathcal{O}, +)$

$\forall \psi \in \widehat{\mathcal{O}}$ is trivial on \mathfrak{p}^k . Suppose $\psi \neq 1$.

$$\text{level}(\psi) := \min \{d \geq 0 \mid \mathfrak{p}^d \subset \ker \psi\}$$

When $\text{level}(\psi) = d$, $\psi: \mathcal{O} \xrightarrow{\pi} \mathcal{O}/\mathfrak{p}^d \rightarrow \mathbb{C}^\times$ factors through char of $\mathcal{O}/\mathfrak{p}^d$.

E.g. $G = \mathcal{O}^\times$

$\forall \chi \in \widehat{\mathcal{O}}^\times$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

When $\text{level}(\chi) = d$, $\chi: \mathcal{O}^\times \xrightarrow{\pi} \mathcal{O}^\times/\mathcal{U}^{(d+1)} \rightarrow \mathbb{C}^\times$ factors through char of $(\mathcal{O}/\mathfrak{p}^{d+1})^\times$

$$(\mathcal{O}/\mathfrak{p}^{d+1})^\times$$

Recent advances: Geometrization of continuous characters of \mathbb{Z}_p [<https://msp.org/pjm/2013/261-1/pjm-v261-n1-p05-p.pdf>]

E.g. $G = (F, +)$

$\forall \psi \in \widehat{F}$ is trivial on \mathfrak{p}^k . Suppose $\psi \neq 1$.

$$\text{level}(\psi) := \min \{d \in \mathbb{Z} \mid \mathfrak{p}^d \subset \ker \psi\}$$

Prop (Additive duality)

Fix $\psi \in \widehat{F}$ nontrivial with level d .

We have a gp iso

$$F \longrightarrow \widehat{F} \quad a \mapsto \psi(a)\psi(-) \text{ of level } d - \nu_F(a) \quad (\text{when } a \neq 0)$$

Q: Do we have similar result for $\widehat{\mathcal{O}}$?

E.g. $G = F^\times$

$\forall \chi \in \widehat{F}^\times$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\text{level}(\chi) := \min \{d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi\}$$

Q: Do we have any classification of F^\times ?

A: Yes. Since $F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times$, any cont character $\chi \in \widehat{F}^\times$ is uniquely determined by $\chi(\pi) \in \mu_\infty$ and $\chi|_{\mathcal{O}^\times} \in \widehat{\mathcal{O}}^\times$.

Notation for future: G : loc. profinite gp $Z = Z(G)$
 $\text{Cos}(G) := \{\text{cpt open subgp } K \text{ of } G\}$
 $\text{Cos}_z(G) := \{K \leq G \mid K \geq Z, K/Z \subset G/Z \text{ cpt open}\}$
When $\#Z(G) = +\infty$, $\text{Cos}(G) \cap \text{Cos}_z(G) = \emptyset$.

3. Construction of new reps

3.1. Special sub & quotient rep.

Def. G : loc. profinite gp $K \leq G$

$(\pi, V) \in \text{Rep}(G)$ $\forall \vartheta \in \widehat{K}$, we define sm reps of K :

$$\begin{aligned} V(K) &:= \langle v - k \cdot v \rangle_{k \in K, v \in V} \\ V_K &:= V/V(K) \end{aligned} \quad \left. \begin{array}{l} \text{in } \text{Rep}(B) \text{ if } K \triangleleft B \leq G \\ \text{not confused with } V^N! \end{array} \right.$$

$$V(\vartheta) := \langle \vartheta(k)v - k \cdot v \rangle_{k \in K, v \in V}$$

$$V_{\vartheta} := V/V(\vartheta)$$

Obviously $V(K) = V(\mathbb{1}_K)$, $V_K = V_{\mathbb{1}_K}$, K acts on $V(\vartheta)$ by ϑ , and
 $0 \rightarrow V(\vartheta) \rightarrow V \rightarrow V_{\vartheta} \rightarrow 0$ in $\text{Rep}(K)$

Rmk.(1) Given $(\sigma, V) \in \text{Rep}(K)$, $\vartheta \in \widehat{K}$, we can view $V(\vartheta)$ as $V'(\sigma)$, where

$$(\sigma', V') := (\vartheta^{-1} \otimes_{\sigma} \sigma, V)$$

(2) Normal subgp gives us plenty of canonical decompositions.

E.g., when $(\rho, V) \in \text{Irr}(G)$, $K \triangleleft G$, we get

$$\begin{cases} V(K) = V \\ V_K = 0 \end{cases} \quad \text{or} \quad \begin{cases} V(K) = 0 \\ V_K = V \end{cases}$$

(3) When $(\rho, V) \in \text{Rep}(K)$ is semisimple,

$$0 \rightarrow V(K) \rightarrow V \rightarrow V_K \rightarrow 0$$

$$0 \rightarrow \bigoplus_{\substack{\sigma \in \text{Irr}(K) \\ \sigma \neq \mathbb{1}_K}}^{11S} V^{\sigma} \rightarrow \bigoplus_{\sigma \in \text{Irr}(K)}^{11S} V^{\sigma} \rightarrow V^{\mathbb{1}_K} \rightarrow 0$$

Assume additionally that

- K is abelian,
- K is the union of an increasing sequence of cpt open subgps.

Then, we have more properties.

Prop. (Integral criterian) [p56 Lemma] $(\sigma, V) \in \text{Rep}(K)$, $v \in V$.

$$v \in V(\vartheta) \Leftrightarrow \left[\exists K_0 \in \text{Cos}(K) \text{ s.t } \int_{K_0} \mathcal{J}(k)^{-1} \sigma(k) v \, d\mu_K(k) = 0 \right]$$

Prop. The factor

$$\text{Rep}(G) \longrightarrow \text{Vect}_{\mathbb{C}} \quad (\pi, V) \longmapsto V_{\vartheta}$$

is exact. E.p.,

$$(v \neq 1) \quad \begin{aligned} V(K)(K) &\cong V(K) & V(K)_K &= 0 \\ V(K)(\vartheta) &\cong V(K) \wedge V(\vartheta) & V(K)_{\vartheta} &\cong V_{\vartheta} \end{aligned}$$

$$\begin{aligned} V_K(K) &= 0 & V_{K,K} &\cong V_K \\ (v \neq 1) \quad V_K(\vartheta) &\cong V_K & V_{K,\vartheta} &= 0 \end{aligned}$$

(You can compute $V(\vartheta)$ and V_{ϑ} also, but I'm lazy.)

3.2. Duality

$(\rho, V) \in \text{Rep}(G)$ $\rightsquigarrow (\rho^*, V^*)$ may be not smooth
 $\rightsquigarrow (\check{\rho}, \check{V}) \in \text{Rep}(G)$ is the smooth dual, where
 $\check{V} := \bigcup_{K \in \text{Cos}(G)} (V^*)^K \subset V^*$

$$\text{ev}: \check{V} \times V \rightarrow \mathbb{C} \quad (\check{w}, v) \mapsto \langle \check{w}, v \rangle$$

$$\rightsquigarrow \langle g \cdot \check{w}, v \rangle = \langle \check{w}, g^{-1} \cdot v \rangle$$

Rmk. (1) The contravariant duality functor

$$\text{Rep}(G) \rightarrow \text{Rep}(G) \quad (\rho, V) \mapsto (\check{\rho}, \check{V})$$

is exact.

exact=>additive: <https://math.stackexchange.com/questions/3039422/in-abelian-categories-is-a-right-left-exact-functor-necessarily-additive>

(2) For $K \in \text{Cos}(G)$, we have iso $\check{V}^K \xrightarrow{\cong} (V^K)^*$ in $\text{Rep}(K)$.

(3) If $(\rho, V) \in \text{Rep}(G)$, $v \in V$, then $\exists \check{w} \in \check{V}$ s.t. $\langle \check{w}, v \rangle \neq 0$.

(4) $\delta: V \rightarrow \check{V}$ is inj. and

\downarrow
 δ is iso $\Leftrightarrow \pi$ is admissible

(5) If $(\rho, V) \in \text{Rep}(G)$ is admissible, then

$$(\rho, V) \in \text{Irr}(G) \Leftrightarrow (\check{\rho}, \check{V}) \in \text{Irr}(G)$$

(6) (Bilinear map) Let $(\rho, V), (\sigma, W) \in \text{Rep}(G)$

$$\mathcal{G}(\rho, \sigma) := \left\{ f: V \times W \rightarrow \mathbb{C} \mid \begin{array}{l} f \text{ bilinear} \\ f(gv, gw) = f(v, w) \end{array} \right\}.$$

Then

$$\mathcal{G}(\rho, \sigma) \cong \text{Hom}_G(\rho, \check{\sigma}) \cong \text{Hom}(\sigma, \check{\rho}).$$

3.3. Ind and c-Ind

Definition

Def (Induced representation)

$H \leq G$ closed, $(\sigma, W) \in \text{Rep}(H)$, we get

$\left\{ \begin{array}{ll} \text{sm induction} & \text{Ind}_H^G \sigma = (\Sigma, X) \in \text{Rep}(G) \\ \text{cpt induction} & \text{c-Ind}_H^G \sigma = (\Sigma_c, X_c) \in \text{Rep}(G) \end{array} \right.$

as follows:

$$\text{Ind}_H^G W = X = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(hg) = \sigma(h)f(g) \\ \exists K \leq G \text{ s.t. } f(gk) = f(g) \end{array} \quad \begin{array}{l} \forall g \in G, h \in H \\ \forall g \in G, k \in K \end{array} \right\}$$

$$\Sigma(g). f = f(-g)$$

$$\text{c-Ind}_H^G W = X_c = \left\{ f \in X \mid \begin{array}{l} \pi(\text{supp } f) \text{ is cpt in } H \backslash G \\ \pi: G \rightarrow H \backslash G \end{array} \right\}$$

$$\Sigma_c(g). f = f(-g)$$

Rmk. (1). (Reality check) When $G = H$,

$$c\text{-} \text{Ind}_H^G W = \text{Ind}_H^G W = \left\{ f: G \rightarrow W \mid \begin{array}{l} f(g) = \sigma(g) f(1) \\ \exists k \in G \text{ s.t. } f(gk) = f(g) \end{array} \right\} \xrightarrow{\cong} W$$

$$\begin{aligned} f &\longmapsto f(1) \\ \sigma(-) \cdot w &\longleftarrow w \end{aligned}$$

(2) Two facts Ind_H^G , $c\text{-}\text{Ind}_H^G$ are both exact.

(3) Suppose $H \leq G$ open. $[G:H] < +\infty$, $(\sigma, W) \in \text{Irr}(H)$. We get
 $\text{Ind}_H^G \sigma$ is G -semisimple.

Frobenius Reciprocity

Thm.

condition		$(\rho, V) \in \text{Rep}(G)$, $(\sigma, W) \in \text{Rep}(H)$
$H \leq G$ closed	$\text{Res}_H^G \vdash \text{Ind}_H^G$	$\text{Hom}_H(\text{Res}_H^G \rho, \sigma) \cong \text{Hom}_G(\rho, \text{Ind}_H^G \sigma)$
$H \leq G$ open	$c\text{-}\text{Ind}_H^G \vdash \text{Res}_H^G$	$\text{Hom}_G(c\text{-}\text{Ind}_H^G \sigma, \rho) \cong \text{Hom}_H(\sigma, \text{Res}_H^G \rho)$

E.p. we have two canonical map in $\text{Rep}(H)$:

$$\begin{array}{lll} H \leq G \text{ closed} & \varepsilon_W: \text{Ind}_H^G W \rightarrow W & f \mapsto f(1) \\ H \leq G \text{ open} & \eta_W: W \rightarrow c\text{-}\text{Ind}_H^G W & w \mapsto \begin{cases} f_w: h \in H \mapsto h.w \\ x \notin H \mapsto 0 \end{cases} \end{array}$$

Duality Theorem [p32]

Let $H \leq G$ closed, define

$$\delta_{H \setminus G} = \delta_H^{-1} \delta_{G|H}: H \rightarrow \mathbb{R}_{>0}^*$$

then $\delta_{H \setminus G} \in \widehat{H}$, and $V(\sigma, W) \in \text{Rep}(H)$, we have

$$(c\text{-}\text{Ind}_H^G \sigma)^* \cong \text{Ind}_H^G (\delta_{H \setminus G} \otimes \check{\sigma}).$$

3.4. Other constructions. (Concerned closely with Hecke algebra)

- \otimes , Sym, and Λ .

It's difficult for me to work out.

- $C_c^\infty(G)$, $C_c^\infty(G) \in \text{Rep}(G \times G)$

$C_c^\infty(G)$ will play a central role in Hecke alg. We focus on $C_c^\infty(G)$ here.

Def. (matrix coefficient set $C(\rho)$)

Let $(\rho, V) \in \text{Rep}(G)$, $v \in V$, $w \in \check{V}$.

$$\mathbb{I}: \check{V} \otimes V \longrightarrow C_c^\infty(G)$$

$$w \otimes v \longmapsto \begin{bmatrix} \gamma_{w \otimes v}: G \longrightarrow \mathbb{C} \\ \gamma_{w \otimes v}(g) = w(g \cdot v) = \langle w, \rho(g)v \rangle \end{bmatrix}$$

is a morphism in $\text{Rep}(G \times G)$.

$$C(\rho) := \text{Im } \mathbb{I} \subseteq C_c^\infty(G)$$

Rmk. $\forall \gamma \in C(\rho)$, $\text{Supp } \gamma$ is invariant under translation by \mathbb{Z} .

Def. (γ -cuspidal rep)

Let G be a unimodular loc. profinite gp. We define

$$\text{Cusp}_c(G) = \left\{ (\rho, V) \in \text{Irr}(G) \mid \begin{array}{l} \text{supp}(\gamma)/\mathbb{Z} \subseteq G/\mathbb{Z} \text{ is cpt} \\ \text{for every } \gamma \in C(\rho) \end{array} \right\}$$

Prop. [p70] (1) Every $(\rho, V) \in \text{Cusp}_c(G)$ is admissible

(2) Suppose

- $(\rho, V) \in \text{Irr}(G)$ is admissible
- $\exists \gamma \in C(\rho)$, $\text{supp}(\gamma)/\mathbb{Z}$ is cpt.

Then,

• $\mathbb{I}: V^* \otimes V \longrightarrow C(\pi)$ is an iso

• $(\pi, V) \in \text{Cusp}_c(G)$

[Idea of proof: Use dual and cardinal argument.]

[Hecke algebra is needed which will be introduced later]

Rmk. γ -Cuspidal has the property of

projectivity and Schur's orthogonality relation, see [p74].

4. Hecke algebra

Assume G is unimodular, and fix an Haar measure μ .

Basic, alg - mod - rep

Def. (Hecke algebra $H(G)$)

$$H(G) := (C_c^\infty(G), *)$$

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

Rmk In general, $H(G)$ has no unit element.

However, $H(G)$ has idempotent elements

$$e_K := \frac{1}{\mu(K)} \mathbf{1}_K \quad \text{for } K \in \text{Cos}(G)$$

$$e_\sigma \in H(G) \quad \text{for } (\sigma, W) \in \text{Irr}(K), K \in \text{Cos}(G)$$

$$e_\sigma(g) = \begin{cases} \frac{\dim_{\mathbb{C}} W}{\mu(K)} \text{tr}(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

well-defined
since $\dim_{\mathbb{C}} W < +\infty$

$$\text{e.p. } \mathbf{1}_K \in H(G), e_{\mathbf{1}_K} = e_K.$$

Rmk. When G is finite, $H(G) \cong \mathbb{C}[G]$ is the path algebra.

Def. (Smooth $H(G)$ -module)

Be careful: the scalar multiplication is also denoted by $*$.

An $H(G)$ -module M is smooth if

$$H(G) * M = M$$

$$\Leftrightarrow \forall m \in M, \exists K \in \text{Cos}(G) \text{ s.t. } e_K * m = m$$

$\text{Mod}(H(G)) = \{\text{sm } H(G)\text{-modules}\}$ is a full subcategory of $\{\text{H}(G)\text{-modules}\}$.

Thm We have the equiv of categories.

$$\text{Rep}(G) \longrightarrow \text{Mod}(H(G))$$

$$(\rho, V) \longmapsto V$$

$$f * v = \int_G f(g) (\rho(g).v) d\mu(g)$$

for $f \in H(G), v \in V$

$$(\rho, M) \longleftarrow M$$

$$\rho(g).m = \frac{1}{\mu(K)} (\mathbf{1}_{gK} * m)$$

for $g \in G, m \in M,$

$K \in \text{Cos}(G)$ s.t. $e_K * m = m$

[Idea: $G \dashrightarrow H(G)$
 $g \dashrightarrow \frac{1}{\mu(K)} \mathbf{1}_{gK} \quad K \ll G$]

$$\text{E.p. } e_K * v \in V^K \text{ and } \rho(1).m = m$$

Prop [4.3, P37] Let $(\rho, V) \in \text{Rep}(G), K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K)$, then

$$e_\sigma * - : V \longrightarrow V^\sigma \quad \text{in } \text{Rep}(K)$$

is a projection.

Task. Assume G is unimodular. Redo everything in the language of $\text{Mod}(H(G))$.

Interesting topic:

The Hecke algebra of a reductive p-adic group: a view from noncommutative geometry [https://www.imj-prg.fr/preprints/389.pdf]

$\mathcal{H}(G, K)$: an analog of $\mathbb{C}[K \backslash G / K]$

Def. Let $K \in \text{Cos}(G)$, $(\sigma, W) \in \text{Irr}(K)$, we define

$$\mathcal{H}_\sigma(G, \sigma) := e_\sigma * \mathcal{H}(G) * e_\sigma$$

when $\sigma \in K$ $\{ f \in C_c^\infty(G) \mid f(k_1 g k_2) = \sigma(k_1, k_2)^{-1} f(g) \quad \forall g \in G, k_1, k_2 \in K \}$

$$\begin{aligned} \mathcal{H}(G, K) &:= \mathcal{H}_\sigma(G, \mathbf{1}_K) \\ &= e_K * \mathcal{H}(G) * e_K \\ &= \{ f \in C_c^\infty(G) \mid f(k_1 g k_2) = f(g) \quad \forall g \in G, k_1, k_2 \in K \} \\ &= C_c^\infty(K \backslash G / K) \quad \text{though undefined} \end{aligned}$$

Rmk. $\mathcal{H}_\sigma(G, \sigma)$ has unit e_σ , while $\mathcal{H}(G, K)$ has unit e_K .

When $\# K \backslash G / K < +\infty$, $\mathcal{H}(G, K) = \mathbb{C}[K \backslash G / K]$.

Def (Smooth $\mathcal{H}_\sigma(G, \sigma)$ -module = $\mathcal{H}_\sigma(G, \sigma)$ -module)

An $\mathcal{H}_\sigma(G, \sigma)$ -module M is smooth if

$$\begin{aligned} \mathcal{H}(G, \sigma) * M &= M \\ \Leftrightarrow \forall m \in M, e_\sigma * m &= m \\ \Leftrightarrow &\checkmark \end{aligned}$$

Q: How can we view $\mathcal{H}_\sigma(G, \sigma)$ -module as some special reps?

Prop. Let $(\rho, V) \in \text{Rep}(G)$, $K \in \text{Cos}(G)$. We get SES

$$\begin{array}{ccccccc} 0 & \longrightarrow & V(K) & \longrightarrow & V & \longrightarrow & V_K \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel s \\ 0 & \longrightarrow & V(K) & \longrightarrow & V & \xrightarrow{e_K * -} & V^K \longrightarrow 0 \end{array} \quad \begin{array}{l} \text{in } \text{Rep}(K) \\ (\text{in } \text{Rep}(B)) \\ \text{if } K \triangleleft B \leq G \end{array}$$

Moreover, $V^K \in \text{Mod}(\mathcal{H}(G, K))$.

Prop. Fix $K \in \text{Cos}(G)$. Then

$$\begin{aligned} \{(\rho, V) \in \text{Irr}(G) \mid V^K \neq 0\} &\xrightleftharpoons[\sim]{} \text{Irr}(\mathcal{H}(G, K)) \\ (\rho, V) &\longmapsto V^K = e_K * V \\ (\mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} M) /_{\tilde{X}} &\in \text{Mod}(\mathcal{H}(G)) \longleftarrow M \end{aligned}$$

Where X is the maximal subrep

of $\mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} M$ st. $X \cap (e_K \otimes M) = 0$

In ptc, $\{(\rho, V) \in \text{Rep}(G) \text{ gen. by } V^K\} \xrightleftharpoons[\sim]{} \text{Rep}(\mathcal{H}(G, K))$

σ -Spherical Hecke algebra

Def. Suppose $K \in \text{Cos}(G)$, $(\sigma, W) \in \text{Irr}(K)$.

The σ -spherical Hecke alg/intertwining alg $\mathcal{H}(G, \sigma)$ is defined as

$$\mathcal{H}(G, \sigma) := \left\{ f: G \rightarrow \text{End}_{\mathbb{C}}(W) \mid \begin{array}{l} \text{supp } f \text{ is cpt} \\ f(k_1 g k_2) = \sigma(k_1^{-1}) f(g) \sigma(k_2) \quad \forall g \in G, k_1, k_2 \in K \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_G f_1(x^{-1}g) f_2(x) d\mu(x)$$

Rmk. $\mathcal{H}(G, \sigma)$ is ass with unit

$$E_{\sigma}: G \rightarrow \text{End}_{\mathbb{C}}(W) \quad E_{\sigma}(g) = \begin{cases} \frac{1}{\mu(K)} \sigma(g)^{-1} & g \in K \\ 0 & g \notin K \end{cases}$$

and we have

$$\mathcal{H}(G, \sigma) \cong \mathcal{H}_1(G, \sigma) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(W) \quad \text{as } \mathbb{C}\text{-alg.}$$

Prop. $c\text{-Ind}_K^G \sigma$ is an $\mathcal{H}(G, \sigma)$ -module defined by

$$\mathcal{H}(G, \sigma) \times c\text{-Ind}_K^G \sigma \rightarrow c\text{-Ind}_K^G \sigma \quad (\phi, f) \mapsto \phi * f$$

where

$$\phi * f(g) := \int_G \phi(x) \cdot f(x^{-1}g) d\mu(x)$$

We have an iso

$$\mathcal{H}(G, \sigma) \xrightarrow{\cong} \text{End}_G(c\text{-Ind}_K^G W) \quad \phi \mapsto \phi + -$$

$$\mathcal{H}(G, \sigma) \xleftarrow{\cong} \text{Hom}_K(W, c\text{-Ind}_K^G W) \quad \left[\begin{array}{l} \phi: G \rightarrow \text{End}_{\mathbb{C}} W \\ g \mapsto [f \mapsto f(g)] \end{array} \right] \xleftarrow{\cong} \psi$$

Moreover, When $\sigma \in \widehat{K}$, $\sigma = \check{\sigma}$, $W = \mathbb{C}$, the two actions are compatible. $(p, V) \in \text{Rep}(G)$

$$\mathcal{H}(G, \sigma) \cong \text{End}_G(c\text{-Ind}_K^G \mathbb{C})$$

$$\Downarrow \qquad \Downarrow$$

$$V^{\sigma} \cong \text{Hom}_G(c\text{-Ind}_K^G \mathbb{C}, V)$$

$$f * v = \int_G f(g) g \cdot v d\mu(g)$$

Generalization. $K \in \text{Cos}_Z(G)$ Fix $\chi \in \widehat{Z}$.

Rmk. G is unimodular $\Rightarrow G/Z$ is unimodular. Denote $\mu_{G/Z}$ as Haar measure of G/Z .

Def. (Hecke algebra) $H_\chi(G)$

$$C_{c,\chi}^\infty(G) = \left\{ f \in C^\infty(G) \mid \begin{array}{l} f(zg) = \chi(z)^{-1} f(g) \\ \text{supp } f/Z \text{ is cpt} \end{array} \forall z \in Z, g \in G \right\}$$

$$H_\chi(G) := (C_{c,\chi}^\infty(G), *)$$

$$(f_1 * f_2)(g) := \int_{G/Z} f_1(x) f_2(x^{-1}g) d\mu_{G/Z}(x)$$

Rmk $H_{1_Z}(G) = H(G/Z)$.

$H_\chi(G)$ has idempotent elements

$$e_\sigma \in H_\chi(G) \quad \text{for } (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi, K \in \text{Cos}(G)$$

$$e_\sigma(g) = \begin{cases} \frac{\dim_{\mathbb{C}} W}{\mu_{G/Z}(K_Z)} \text{tr}(\sigma(g)^{-1}) & g \in K \\ 0 & g \notin K \end{cases}$$

Def. (Smooth $H_\chi(G)$ -module)

An $H_\chi(G)$ -module M is smooth if

$$H_\chi(G) * M = M$$

$$\Leftrightarrow \forall m \in M, \exists K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi \text{ s.t. } e_\sigma * m = m.$$

Thm We have the equiv of categories.

$$\{(p, V) \in \text{Rep}(G) \mid p \xrightarrow{\cong} \chi\} \longrightarrow \text{Mod}(H_\chi(G))$$

$$(p, V) \longmapsto V$$

$$f * v = \int_{G/Z} f(g) (p(g).v) d\mu_{G/Z}(g)$$

$$\text{for } f \in H_\chi(G), v \in V$$

$$(p, M) \longleftrightarrow M$$

$$p(g).m = e_\chi(g^{-1}) * m$$

for $g \in G, m \in M$,

$K \in \text{Cos}_Z(G)$ $\chi \in \widehat{K} \xrightarrow{\cong} \chi$ s.t. $e_\chi * m = m$

E.p. $e_\sigma * v \in V^\sigma$ and $p(1).m = m$

Prop Let $(p, V) \in \text{Rep}(G) \xrightarrow{\cong} \chi, K \in \text{Cos}(G), (\sigma, W) \in \text{Irr}(K) \xrightarrow{\cong} \chi$, then

$$e_\sigma * - : V \longrightarrow V^\sigma \quad \text{in } \text{Rep}(K)$$

is a projection.

Def. Let $K \in \text{Cos}_z(G)$, $(\sigma, W) \in \text{Irr}(K) \xrightarrow{z} X$, we define

$$\begin{aligned} H_{i,x}(G, \sigma) &= e_\sigma * H_x(G) * e_\sigma \\ &\stackrel{\text{when } \sigma \in K}{=} \{ f \in C_c^\infty(G) \mid f(k_1 g k_2) = \sigma(k_1) f(g) \sigma(k_2) \quad \forall g \in G, k_1, k_2 \in K \} \end{aligned}$$

Rmk. $H_{i,x}(G, \sigma)$ has unit e_σ .

Smooth $H_{i,x}(G, \sigma)$ -module = $H_{i,x}(G, \sigma)$ -module

Prop. Fix $K \in \text{Cos}_z(G)$, $\sigma \in \text{Irr}(K) \xrightarrow{z} X$. Then

$$\begin{aligned} \{ (\rho, V) \in \text{Irr}(G) \xrightarrow{z} X \mid V^\sigma \neq 0 \} &\xleftrightarrow{\cong} \text{Irr}(H_{i,x}(G, \sigma)) \\ (\rho, V) &\mapsto V^\sigma = e_\sigma * V \\ (H_x(G) \otimes_{H_{i,x}(G, \sigma)} M) /_{X^\sigma} &\xleftarrow{\cong} M \end{aligned}$$

Where X is the maximal subrep

of $H_x(G) \otimes_{H_{i,x}(G, \sigma)} M$ s.t. $X \cap (e_\sigma \otimes M) = 0$

$$\text{In ptc, } \{ (\rho, V) \in \text{Rep}(G) \xrightarrow{z} X \text{ gen by } V^\sigma \} \longleftrightarrow \text{Rep}(H_{i,x}(G, \sigma))$$

Def. Suppose $K \in \text{Cos}_z(G)$, $(\sigma, W) \in \text{Irr}(K) \xrightarrow{z} X$

The σ -spherical Hecke alg/intertwining alg $H_x(G, \sigma)$ is defined as

$$\begin{aligned} H_x(G, \sigma) &= \left\{ f: G \longrightarrow \text{End}_{\mathbb{C}}(W) \mid \begin{array}{l} f(k_1 g k_2) = \sigma(k_1) f(g) \sigma(k_2) \quad \forall g \in G, k_1, k_2 \in K \\ \text{supp } f \text{ is cpt} \end{array} \right\} \\ (f_1 * f_2)(g) &= \int_{G/K} f_1(x^{-1}g) f_2(x) d\mu_{G/K}(x) \end{aligned}$$

Rmk. $H_x(G, \sigma)$ is ass with unit

$$E_\sigma: G \longrightarrow \text{End}_{\mathbb{C}}(W) \quad E_\sigma(g) = \begin{cases} \frac{1}{\mu_{G/K}(K)} \sigma(g)^{-1} & g \in K \\ 0 & g \notin K \end{cases}$$

and we have

$$H_x(G, \sigma) \cong H_{i,x}(G, \sigma) \otimes_{\mathbb{C}} \text{End}_{\mathbb{C}}(W) \quad \text{as } \mathbb{C}\text{-alg.}$$

Prop. $c\text{-Ind}_K^G \sigma$ is an $H_x(G, \sigma)$ -module defined by

$$H_x(G, \sigma) \times c\text{-Ind}_K^G \sigma \rightarrow c\text{-Ind}_K^G \sigma \quad (\phi, f) \mapsto \phi * f$$

where

$$\phi * f(g) = \int_{G/K} \phi(x) [f(x^{-1} \cdot)] d\mu_{G/K}$$

We have an iso

$$H_x(G, \sigma) \longrightarrow \text{End}_G(c\text{-Ind}_K^G W) \quad \phi \mapsto \phi + -$$

$$H_x(G, \sigma) \longleftarrow \text{Hom}_K(W, c\text{-Ind}_K^G W) \quad \begin{bmatrix} \phi: G \rightarrow \text{End}_{\mathbb{C}} W \\ g \mapsto [f \mapsto f(\phi(g))] \end{bmatrix} \longleftarrow \psi$$

Moreover, When $\sigma \in \widehat{K} \xrightarrow{z} X$, $\sigma = \check{\sigma}$, $W = \mathbb{C}$, the two actions are compatible: $(\rho, V) \in \text{Rep}(G)$

$$H_x(G, \sigma) \cong \text{End}_G(c\text{-Ind}_K^G \mathbb{C})$$

$$\square \quad \square$$

$$V^\sigma \cong \text{Hom}_G(c\text{-Ind}_K^G \mathbb{C}, V)$$

$$f * v = \int_G f(g) g \cdot v d\mu(g)$$

5. Intertwining properties.

In this subsection, G is an unimodular loc. profinite gp satisfying the countable hypothesis, and $Z = Z(G)$.

Definition

Def. Let $K_1, K_2 \in \text{Cos}(G)/\text{Cos}_Z(G)$, $\sigma_1 \in \text{Irr}(K_1)$, $\sigma_2 \in \text{Irr}(K_2)$, $g \in G$.

$$K_1^g := g^{-1}K_1g \quad \rho_1^g = \rho_1(g - g^{-1}) \in \text{Irr}(K_1^g)$$

We call g intertwines σ_1 with σ_2 if

$$\text{Hom}_{K_1^g \cap K_2}(\sigma_1^g, \sigma_2) \neq 0$$

usually missed

$$\text{Notation. } \text{itw}_G(\sigma_1, \sigma_2) := \{g \in G \mid g \text{ intertwines } \sigma_1 \text{ with } \sigma_2\}$$

$$\text{itw}_G(\sigma) := \text{itw}_G(\sigma, \sigma)$$

σ_1 intertwines with σ_2 if $\text{itw}_G(\sigma_1, \sigma_2) \neq \emptyset$

E.g. When $K_1 = K_2 = K$, $g \in N_K(G)$, $\sigma_1, \sigma_2 \in \text{Irr}(K)$, then

$$g \in \text{itw}(\sigma_1, \sigma_2) \Leftrightarrow \text{Hom}_K(\sigma_1^g, \sigma_2) \neq 0$$

$$\Leftrightarrow \sigma_1^g \cong \sigma_2$$

$$\text{Rmk. } g \in \text{itw}(\sigma_1, \sigma_2) \Rightarrow k_1 g k_2 \in \text{itw}(\sigma_1, \sigma_2) \quad \text{for } \forall k_1 \in K_1, k_2 \in K_2.$$

Basic results

Prop 1. Suppose $(\rho, V) \in \text{Irr}(G)$, $(\sigma_1, W_1) \in \text{Irr}(K_1)$, $(\sigma_2, W_2) \in \text{Irr}(K_2)$,

$$\left(\begin{matrix} (\rho, V) \\ \cup \\ \text{in } \text{Rep}(K_1) \end{matrix} \right) \cup \left(\begin{matrix} (\rho, V) \\ \cup \\ \text{in } \text{Rep}(K_2) \end{matrix} \right)$$

$$(\sigma_1, W_1) \quad (\sigma_2, W_2)$$

Then $\text{itw}(\sigma_1, \sigma_2) \neq \emptyset$.

[Proof. We know $V^{\sigma_1} \neq 0$ in $\text{Rep}(K_1)$, $V^{\sigma_2} \neq 0$ in $\text{Rep}(K_2)$.
 $V^{\sigma_1} \neq 0 \xrightarrow{(\rho, V) \in \text{Irr}(G)} \langle V^{\sigma_1} \rangle_{g \in G} \text{ spans } V \quad \left\{ \begin{array}{l} \exists g \in G, \pi_{\sigma_2}|_{V^{\sigma_1}}, V^{\sigma_1} \xrightarrow{\pi_{\sigma_2}} V^{\sigma_2} \\ \pi_{\sigma_2}: V \rightarrow V^{\sigma_2} \text{ is nonzero in } \text{Rep}(K_2) \end{array} \right\} \Rightarrow \text{itw}(\sigma_1, \sigma_2) \neq \emptyset$

Prop 2. Let $K \in \text{Cos}(G)$, $\sigma \in \text{Irr}(K)$, $g \in G$, then

$$g \in \text{itw}(\sigma) \Leftrightarrow \exists f \in \mathcal{H}(G, \sigma) \text{ s.t. } f|_{Kgk} \neq 0$$

Prop 3 Let $K \in \text{Cos}_Z(G)$, $(\sigma, W) \in \text{Irr}(K)$, $g \in G$, then

$$g \in \text{itw}(\sigma) \Leftrightarrow \exists f \in \mathcal{H}(G, \sigma) \text{ s.t. } \text{supp } f = KgK$$

Moreover, we have

$$\{f \in \mathcal{H}(G, \sigma) \mid \text{supp } f \subseteq KgK\} \xrightarrow{\cong} \text{Hom}_{KgK}(\sigma, \sigma^g) \subseteq \text{End}_C(W)$$

$$f \mapsto f(g)$$

$$f: f(x) = \begin{cases} \sigma(k)^{-1} h \sigma(k) & x = k_1 g k_2 \\ 0 & x \notin KgK \end{cases}$$

Q: Any generalization of Prop 2 & Prop 3? $\text{Cos}(G) \rightarrow \text{Cos}_Z(G)$ or $\text{Cos}_Z(G) \rightarrow \text{Cos}(G)$?

$c\text{-Ind}$ with cuspidal rep

Thm. [11.4, P80] (Construction method for cuspidal reps)

Suppose $\forall (\rho, V) \in \text{Irr}(G)$ is admissible,

$K \in \text{Cos}_Z(G)$, $(\sigma, W) \in \text{Irr}(K)$ and $\text{itw}(\sigma) = K$.

Then $c\text{-Ind}_K^G \sigma \in \text{Cusp}_c(G)$.

Rmk. (itw is key ingredient)

Let $K \in \text{Cos}_Z(G)$, $(\sigma, W) \in \text{Irr}(K)$ and $\text{itw}(\sigma) \neq K$.

Then $c\text{-Ind}_K^G \sigma \notin \text{Irr}(G)$

(Since $\text{End}_G(c\text{-Ind}_K \sigma) \cong \mathbb{H}(G, \sigma)$ has $\dim > 1$)

Rmk. In the theorem, if we assume further that $\delta_{K \backslash G} = \mathbb{1}_K$, then

$$c\text{-Ind}_K^G \sigma \in \text{Irr}(G)$$

$$\xrightarrow{\text{adm}} (c\text{-Ind}_K^G \sigma)^\vee \in \text{Irr}(G)$$

$$\Rightarrow \text{Ind}_K^G(\sigma) \stackrel{\delta_{K \backslash G} = \mathbb{1}_K}{\cong} (c\text{-Ind}_K^G \sigma)^\vee \in \text{Irr}(G)$$

$$\Rightarrow c\text{-Ind}_K^G(\sigma) = \text{Ind}_K^G(\sigma)$$

$$\Rightarrow \text{Ind}_K^G(\sigma) \cong (c\text{-Ind}_K^G(\sigma))^\vee \cong (\text{Ind}_K^G(\sigma))^\vee \cong c\text{-Ind}_K^G(\sigma)$$