Eine Woche, ein Beispiel

2.16 lines passing a point

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.
[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

These are perhaps too well-known. But I should record it.

Typical question:

In a hypersurface $X \subset \mathbb{P}^n$,
how many lines $l \cong \mathbb{P}^1$ pass a given point $p \in X$?

Affine version:

In a (conical) hypersurface $X \subset \mathbb{C}^{n+1}$,
how many planes $l \cong \mathbb{C}^2$ contain a given line $p \cong \mathbb{C} \subseteq X$?

1. Method
2. Lines on cubic threefold
3. Lines on quadrics

1. Method

Slogan: write down the coordinates explicitly.

w.l.o.g. let $p = [1:0:\cdots:0]$ and $X = \{f=0\}$, where

$$f(z_0, \dots, z_n) = \sum_{i=0}^{d} g_{d-i}(z_1, \dots, z_n)\, z_0^i$$

$g_{d-i}(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$ are homo of degree $d-i$,
and $g_0(z_1, \dots, z_n) = 0$.

Suppose that $l = \langle (1,0,\dots,0), (0, x_1, \dots, x_n) \rangle_{\mathbb{C}\text{-v.s.}}$, then

$$\begin{aligned} & l \subseteq X \\ \Leftrightarrow\ & f(t, x_1, \dots, x_n) \equiv 0 \qquad \forall t \in \mathbb{C} \\ \Leftrightarrow\ & g_i(x_1, \dots, x_n) \equiv 0 \qquad \forall i \in \{1, \dots, d\} \end{aligned}$$

Therefore,

$$\begin{aligned} & \{ l \cong \mathbb{C}^2 \subseteq \mathbb{C}^{n+1} \mid p \in l \subseteq X \} \\ \cong\ & \{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid g_{d-i}(x_1, \dots, x_n) = 0 \quad \forall i \} \\ \cong\ & \{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid \frac{\partial^i f}{\partial z_0^i}(0, x_1, \dots, x_n) = 0 \quad \forall i \} \end{aligned}$$

Here, $\mathbb{CP}^{n-1} = Gr(p^\perp, 1)$.

When X is sm at p, $(\nabla f)(p) \neq 0$.
w.l.o.g. let $(\nabla f)(p) = (0, \dots, 1)$, then

$$\begin{cases} T_p X = \{z_n = 0\} \cong \mathbb{C}^n \\ g_1(z_1, \dots, z_n) = z_n \end{cases}$$

In ptc, $p \in l \subseteq X \Rightarrow l \subseteq T_p X$.

2. Lines on cubic threefold

https://math.stackexchange.com/questions/3605767/number-of-lines-passing-a-point-on-a-cubic-threefold

Prop. Generically, there are 6 lines in a cubic threefold passing a given pt.

Proof. w.l.o.g. suppose $p=[1:0:0:0:0]$, $T_pX=\{z_4=0\}$, then

$$\{l \mid p\in l\subseteq X\}$$

$$\cong \{[x_1:x_2:x_3:x_4]\in\mathbb{CP}^3 \mid x_4 = g_2(x_1,x_2,x_3,x_4)=g_3(x_1,x_2,x_3,x_4)=0\}$$
$$\cong \{[x_1:x_2:x_3]\in\mathbb{CP}^2 \mid g_2(x_1,x_2,x_3,0)=g_3(x_1,x_2,x_3,0)=0\}$$

has generically 6 pts.

Rmk. Generically, passing a given pt,
there are 24 lines in a quartic fourfold,
5! lines in a quintic fivefold,
⋮
$n!$ lines in a degree n n-fold.

dim $n-1$ \ dim d	1	2	3	4	5	6	...
0	·	· ·	∴	∷	· · · · ·	· · · · · ·	...
1	$\mathbb{P}^1$	twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	$\mathbb{P}^2$	conical $\cong\mathbb{P}^1\times\mathbb{P}^1$	cubic surface				
3	$\mathbb{P}^3$	conical	cubic threefold				
4	$\mathbb{P}^4$	conical					
5	$\mathbb{P}^5$	⋮					

general type ⇑

Fano ⇐ | Calabi-Yau

uniruled by $\mathbb{P}^1$ | uniruled by conics

3. Lines on quadrics.

In this case,

$$
\begin{aligned}
&\{ l \mid p \in l \subseteq X \} \\
\cong\ &\{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} \mid x_n = g_2(x_1, \ldots, x_n) = 0 \} \\
\cong\ &\{ [x_1 : \cdots : x_{n-1}] \in \mathbb{CP}^{n-2} \mid g_2(x_1, \ldots, 0) = 0 \}
\end{aligned}
$$

is again a quadric of dim $n-3$ (generically) $n \geq 3$

$n = 1, 2$ $\quad \emptyset \quad$ (generically) empty

$F_1(X) = \{ l \subseteq X \text{ lines} \}$

Cor. For $n \geq 3$,

$$
\begin{aligned}
\dim F_1(X) &= n-3 + n-1 - 1 = 2n-5 \\
&= 2(n-1) - 3
\end{aligned}
$$

dim $n-1$ \ dim d	1	2	3	4	5	6	...
0	·	· ·	∴	∷	· · · · ·	· · · · · ·	...
1	0 $\mathbb{P}^1$	$\emptyset$ twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	2 $\mathbb{P}^2$	1 conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	0 cubic surface				
3	4 $\mathbb{P}^3$	3 conical	2 cubic threefold				
4	6 $\mathbb{P}^4$	5 conical		3			
5	8 $\mathbb{P}^5$	7 ⋮			4		

general type $\Uparrow$

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$\dim_{\mathbb{C}} F_1(X)$

In general, one can compute r-planes ($\cong \mathbb{P}^r$) on X passing P:

$$
\begin{aligned}
& \{ e \cong \mathbb{C}^{r+1} \mid p \in e \subseteq X \} \\
\cong\ & \{ e \in Gr(n,r) \mid x_n = g_2(x_1, \dots, x_n) = 0 \quad \forall (x_1, \dots, x_n) \in e \} \\
\cong\ & \{ e \in Gr(n-1,r) \mid g_2(x_1, \dots, x_{n-1}, 0) = 0 \quad \forall (x_1, \dots, x_{n-1}) \in e \} \\
\cong\ & F_{r-1}(X') \qquad \dim X' = \dim X - 2
\end{aligned}
$$

$\Rightarrow$ when $F_{r-1}(X') \neq \emptyset$ generically,

$$
\begin{aligned}
\dim F_r(X) &= \dim F_{r-1}(X') + \dim^{proj} X - r \\
&= \dim F_{r-1}(X') + (n-1) - r \\
&= \dim F_{r-1}(X') + n - r - 1 \\
&= n-r-1 + ((n-2)-(r-1)-1) + \cdots ((n-2(r-1)) - (r-(r-1)) - 1) \\
&\qquad + \dim F_0(X^{(r)}) \\
&= n-r-1 + (n-r-2) + \cdots + (n-2r) \\
&\qquad + \dim^{proj} X^{(r)} \\
&= \tfrac{1}{2}(2n-3r-1)\, r \quad + n - 2r - 1 \\
&= \tfrac{1}{2}(2n-3r-2)(r+1)
\end{aligned}
$$

dim $n-1$ \ dim d	1	2	3	4	5	6	…
0	·	· ·	∴	∷	· · · · ·	· · · · · ·	…
1	$\emptyset$ $\mathbb{P}^1$	$\emptyset$ twistor $\mathbb{P}^1$	$g=1$	$g=6$	$g=10$	$g=15$	$g=\frac{d(d-1)}{2}$
2	0 $\mathbb{P}^2$	$\emptyset$ conical $\cong \mathbb{P}^1 \times \mathbb{P}^1$	$\emptyset$ cubic surface				
3	3 $\mathbb{P}^3$	$\emptyset$ conical	$\emptyset$ cubic threefold				
4	6 $\mathbb{P}^4$	3 conical		$\emptyset$			
5	9 $\mathbb{P}^5$	6 ⋮			$\emptyset$		

$3((n-1)-3)$

$\dim_{\mathbb{C}} F_2(X)$

general type

Fano

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uniruled by $\mathbb{P}^1$

uniruled by conics