Eine Woche, ein Beispiel

1.12 monodromy of the Gauss map

Ref:
[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

This is written for presentations. I may forget some important hints, so I collect the process here.

$A \cong \mathbb{C}^g/\Lambda$

$Z \subset A$ sm of dim r e.g. $C \subset \mathrm{Jac}(C)$

$\rightsquigarrow$ Gauss map

tautological bd

$$\begin{array}{ccccc} \phi: & Z \ni TZ & \longrightarrow & Gr(g,r) \ni S & \qquad z \longmapsto [T_z Z] \\ & \cup & & \cup & \\ & \{p_1,\dots,p_d\} & \longrightarrow & Gr(g-1,r) & \end{array}$$

$$d = \deg \Lambda_Z = \phi^*\sigma_{(1)^r} = (-1)^r \phi^* c_r(S) = (-1)^r c_r(TZ)$$

$$\begin{array}{ccc} I_{mon} \subset Z \times Gr(g,g-1) & \subset A \times Gr(g,g-1) \;\; (*) & \\ & \longrightarrow & Gr(g,r) \times Gr(g,g-1) \\ \downarrow & & \\ Gr(g,g-1) & & \end{array}$$

$\rightsquigarrow$ monodromy gp

$$Gal(\gamma) := \left(\text{mon gp of } I_{mon} \longrightarrow Gr(g,g-1)\right)$$

Q: How can we compute $Gal(\gamma)$?

E.g. 1. $Z = \mathcal{C}$ non-hyperelliptic, $Gr(g,1) = \mathbb{P}^{g-1}$,

$$\begin{tikzcd} \phi = |\omega_C| : \mathcal{C} \arrow[r] & \mathbb{P}^{g-1} \\ \{p_1, \dots, p_{2g-2}\} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & H \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

$$Gal(\gamma) = S_{2g-2} = W(A_{2g-3})$$

E.g. 2. $Z = \mathcal{C}$ hyperelliptic, $Gr(g,1) = \mathbb{P}^{g-1}$.

$$\begin{tikzcd} \phi = |\omega_C| : \mathcal{C} \arrow[r] & \mathbb{P}^1 \arrow[r] & \mathbb{P}^{g-1} \\ \{p_1, \dots, p_{2g-2}\} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & \{z_1, \dots, z_{g-1}\} \arrow[u, phantom, "\cup" rotate=90] \arrow[r] & H \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

$$Gal(\gamma) = S_{g-1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{g-1} = W(C_{g-1})$$

E.g. 3. X : cubic threefold $\subseteq \mathbb{P}^4$

$F = F(X)$: Fano surface $\subseteq Gr(5,2)$

$F \subset Alb(F)$ ← dim = 5

$$\begin{tikzcd} \rightsquigarrow \phi : F \arrow[r] & Gr(5,2) \\ F(X \cap H) \arrow[u, phantom, "\cup" rotate=90] \arrow[r] \arrow[d, phantom, "=" rotate=90] & Gr(4,2) \arrow[u, phantom, "\cup" rotate=90] \\ \{p_1, \dots, p_{27}\} & \end{tikzcd}$$

$$Gal(\gamma) = Aut(\text{Schläfli graph}) = W(E_6)$$

Q: Can we find $Z \subset A$ sm s.t. $Gal(\gamma) = W(E_7)$?