

Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.

[Mum74]: David Mumford. Prym Varieties. I, 1974.

<https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve>

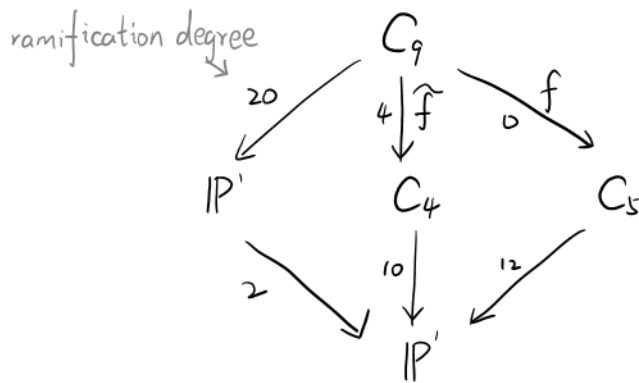
<https://mathoverflow.net/questions/402049/induced-action-on-prym-variety>

Goal: Describe some curves (maybe singular) C in A , and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$ has the following covers:
 $\text{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



where

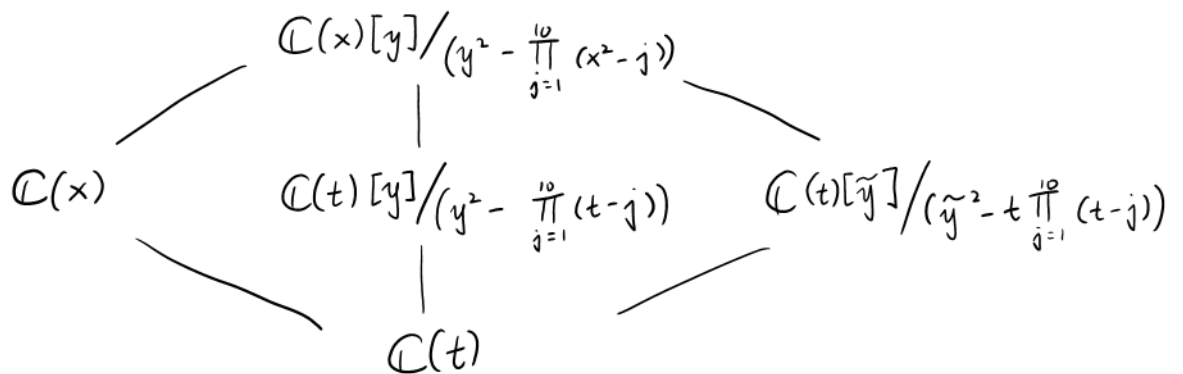
$$C_4 = \{y^2 = \prod_{j=1}^{10} (t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t \prod_{j=1}^{10} (t - j)\}$$

$$t = x^2$$

$$\tilde{y} = xy$$

The crspd field extension:



Local charts for $f: C_9 \rightarrow C_5$

$$\begin{array}{ccc}
 C_9 & \mathbb{C}[x][y]/(y^2 - \prod_{j=1}^{10} (x^2 - j)) & \longleftrightarrow \mathbb{C}[u][v]/(v^2 - \prod_{j=1}^{10} (1 - ju^2)) \\
 \downarrow f & \uparrow R_{xy} & \uparrow R_{uv} \\
 C_5 & \mathbb{C}[t][\tilde{y}]/(\tilde{y}^2 - t \prod_{j=1}^{10} (t - j)) & \longleftrightarrow \mathbb{C}[s][\tilde{v}]/(\tilde{v}^2 - s \prod_{j=1}^{10} (1 - js)) \\
 & \uparrow R_{t\tilde{y}} & \uparrow R_{s\tilde{v}}
 \end{array}$$

$U_{xy} := \text{Spec } R_{xy}$: the name of the local chart

$$\begin{array}{ccc}
 \begin{array}{c} x^2 \\ \uparrow \\ t \end{array} & \begin{array}{c} xy \\ \uparrow \\ \tilde{y} \end{array} & \begin{cases} x = \frac{1}{u} \\ y = \frac{v}{u^{10}} \end{cases} \\
 \begin{array}{c} \tilde{y}^2 \\ \uparrow \\ \tilde{y} \end{array} & \begin{array}{c} \tilde{y}^2 - t \prod_{j=1}^{10} (t - j) \\ \uparrow \\ \tilde{y}^2 - t \prod_{j=1}^{10} (t - j) \end{array} & \begin{cases} t = \frac{1}{s} \\ \tilde{y} = \frac{\tilde{v}}{s^6} \end{cases}
 \end{array}$$

Q: I saw from [Mum74, p 326] that

$$f^* \mathcal{O}_{C_9} = \mathcal{O}_{C_5} \oplus \eta$$

for some line bundle $\eta \in \text{Pic}(C_5)$.

Why is that true? How to describe η ?

<https://math.stackexchange.com/questions/527194/picard-group-of-a-affine-scheme>
When A is an integral domain, the Picard group is known as the class group.

▽ $R_{xy} = \mathbb{C}[x][y]/(y^2 - \prod_{j=1}^{10} (x^2 - j))$ is not a PID, as

$$y^2 = \prod_{j=1}^{10} (x^2 - j)$$

How to compute $\text{Pic}(\text{Spec } R_{xy})$?

Global differential forms

Pulling back differential forms give the following maps:

$$\begin{array}{ccccc}
 & & \langle x^k \frac{dx}{y} \mid k=0, \dots, 8 \rangle & & \\
 & \nearrow & \uparrow \tilde{f}^* & \nwarrow f^* & \\
 & & \langle 2x^{2k+1} \frac{dx}{y} \mid k=0, \dots, 3 \rangle & & \langle 2x^{2k} \frac{dx}{y} \mid k=0, \dots, 4 \rangle \\
 0 & \nwarrow & \parallel & \nearrow & \\
 & & \langle t^k \frac{dt}{y} \mid k=0, \dots, 3 \rangle & & \langle 2t^k \frac{dt}{y} \mid k=0, \dots, 4 \rangle \\
 & \nearrow & \downarrow & \nwarrow & \\
 & & 0 & &
 \end{array}$$

Therefore,

$$H^0(C_9; \omega_{C_9}) \cong \tilde{f}^* H^0(C_4; \omega_{C_4}) \oplus f^* H^0(C_5; \omega_{C_5}) \quad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which corresponds to pulling back of divisors)

$$\begin{array}{ccccc}
 & & \langle x^k \frac{dx}{y} \mid k=0, \dots, 8 \rangle & & \\
 & \nwarrow & \downarrow \tilde{f}_* & \searrow f_* & \\
 & & \langle 2x^{2k+1} \frac{dx}{y} \mid k=0, \dots, 3 \rangle & & \langle 2x^{2k} \frac{dx}{y} \mid k=0, \dots, 4 \rangle \\
 0 & \nwarrow & \parallel & \nearrow & \\
 & & \langle t^k \frac{dt}{y} \mid k=0, \dots, 3 \rangle & & \langle 2t^k \frac{dt}{y} \mid k=0, \dots, 4 \rangle \\
 & \nearrow & \downarrow & \nwarrow & \\
 & & 0 & &
 \end{array}$$

However, since $\text{Jac}(C) = H^0(C; \omega_C)^* / H_1(C; \mathbb{Z})$, we are working on the dual spaces. The notations are again switched:

$$\begin{array}{ccc}
 f^* & \rightsquigarrow & N_{mf} \\
 f_* & \rightsquigarrow & f^*
 \end{array}$$

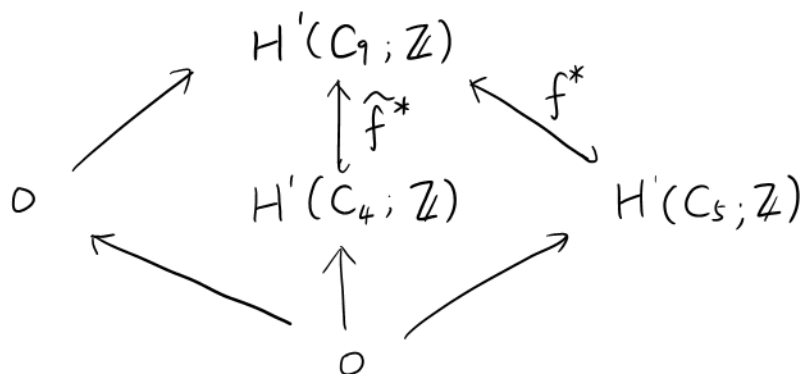
One may get

$$H^0(C_9; \omega_{C_9})^* \cong \tilde{f}^* H^0(C_4; \omega_{C_4})^* \oplus f^* H^0(C_5; \omega_{C_5})^* \quad (2)$$

different meaning compared with (1)!

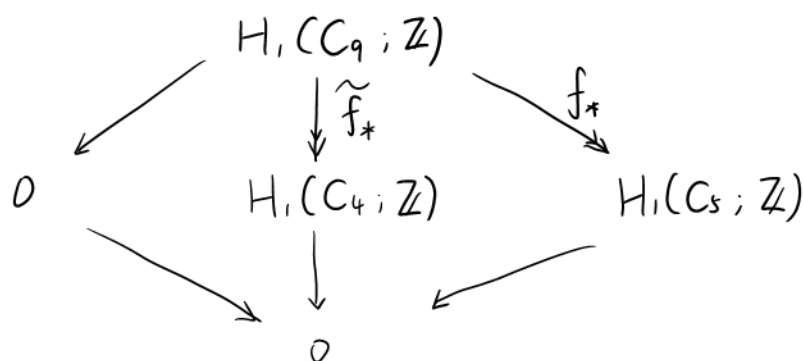
(co)homology class

This page may be easier to understand, and it helps to understand the previous page.



Q: Do we have

$$H'(C_9; \mathbb{Z}) \cong \tilde{f}^* H'(C_4; \mathbb{Z}) \oplus f^* H'(C_5; \mathbb{Z})?$$



Q: Do we have

$$H_1(C_9; \mathbb{Z})^* \cong \tilde{f}^* H_1(C_4; \mathbb{Z})^* \oplus f^* H_1(C_5; \mathbb{Z})^*?$$

Curve in Prym variety

Define A as the quotient of Jacobians. i.e.,

$$A := \text{Jac}(C_9) / f^* \text{Jac}(C_5) \cong \text{Prym}(C_9/C_5)$$

$$\begin{array}{ccccc}
 C_9 & \longrightarrow & C_4 & \xrightarrow{\quad} & \text{Jac}(C_4) \\
 \downarrow A|_{C_9} & & \downarrow & \nearrow \exists! \text{ isogeny} & \\
 \text{Jac}(C_5) & \xrightarrow{f^*} & \text{Jac}(C_9) & \xrightarrow{\pi} & A \longrightarrow 0
 \end{array} \quad (3)$$

- Prop
0. A is isogenous to $\text{Jac}(C_4)$;
 1. $f^*: \text{Jac}(C_5) \rightarrow \text{Jac}(C_9)$ is an isogeny to its image;
 2. $\pi \circ A|_{C_9}$ is not injective, it factors through C_4 ;
 3. $C_4 \rightarrow A$ is generically injective;
 4. $C_4 \rightarrow A$ produces a sm image of A , outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof. 0. Taking the tangent space of (3), one gets

$$0 \rightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df^*} H^0(C_9, \omega_{C_9})^* \rightarrow T_0 A \rightarrow 0$$

Combined with (2),

$$T_0 A \cong H^0(C_4, \omega_{C_4})^*.$$

Late we will find a natural isogeny $\text{Jac}(C_4) \rightarrow A$.
 What's the degree of this isogeny?

1. Since

$$N_{m_f} \circ f^* = 2 \text{Id}_{\text{Jac}(C_5)}$$

$$\# \text{Ker } f^* < +\infty$$

Is f^* inj in this case?

2. For $p_1 = (x_0, y_0)$, $p_2 = (-x_0, y_0)$, we want to show that

$$\int_{\gamma_1: p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\gamma_2: p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$\text{LHS} = \int_{\gamma_1: p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = \text{RHS}.$$

3.

<https://mathoverflow.net/questions/68503/has-anyone-studied-the-ptym-map-for-double-covers-with-two-ramification-points>
<https://arxiv.org/abs/1010.4483>: It proves that many Prym maps $(C \rightarrow \text{Prym})$ are generically finite.

Notice: $C_4 \subset \text{Jac}(C_4)$ is only invariant under $p \mapsto -p$,
 not invariant under $p \mapsto p + a_0$.

Otherwise, the Gauss map would be cover of $\deg > 2$.

Therefore, after isogeny $C_4 \rightarrow A$ is still gen inj.

Q: Is this map really inj?

(4) $C_4 \hookrightarrow \text{Jac}(C_4)$ is sm, so after isogeny it is still sm
 outside of non-injective locus.

Rmk. Suppose $f: \tilde{C} \rightarrow C$ is a **deg 2** (ramified) covering,
 $\sigma: \tilde{C} \rightarrow \tilde{C}$ the crspd involution,
 define A as the quotient

$$\begin{array}{c} \tilde{C} \\ \downarrow AJ_{\tilde{C}} \\ \text{Jac}(C) \xrightarrow{f^*} \text{Jac}(\tilde{C}) \xrightarrow{\pi} A \longrightarrow 0 \end{array}$$

one can identify A with $\text{Prym}(\tilde{C}/C) \subset \text{Jac}(\tilde{C})$, **why?**
 and the Abel-Prym map is given by

$$\begin{aligned} AP_{\tilde{C}} = \pi \circ AJ_{\tilde{C}}: \quad \tilde{C} &\longrightarrow \text{Jac}(\tilde{C}) \longrightarrow A \\ p &\longmapsto \mathcal{O}_{\tilde{C}}(p - p_0) \longmapsto \mathcal{O}_{\tilde{C}}(p - \sigma(p)) \end{aligned}$$

Therefore, for $p_1 \neq p_2$,

$$\begin{aligned} AP_{\tilde{C}}(p_1) &= AP_{\tilde{C}}(p_2) \\ \Leftrightarrow \mathcal{O}_{\tilde{C}}(p_1 - \sigma(p_1)) &= \mathcal{O}_{\tilde{C}}(p_2 - \sigma(p_2)) \\ \Leftrightarrow \mathcal{O}_{\tilde{C}}(p_1 + \sigma(p_2)) &= \mathcal{O}_{\tilde{C}}(p_2 + \sigma(p_1)) \end{aligned}$$

① When $p_1, p_2 \in \tilde{C}$ are ramification pts of f ,
 i.e., $p_1 = \sigma(p_1)$, $p_2 = \sigma(p_2)$

$$AP_{\tilde{C}}(p_1) = AP_{\tilde{C}}(p_2).$$

As a result, when f is ramified, $AP_{\tilde{C}}$ is never injective.

② Now assume $AP_{\tilde{C}}$ is not inj., $AP_{\tilde{C}}(p_1) = AP_{\tilde{C}}(p_2)$.
 When $p_1 \neq \sigma(p_1)$ or $p_2 \neq \sigma(p_2)$,

$\mathcal{L} := \mathcal{O}_{\tilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\tilde{C}}(p_2 + \sigma(p_1))$
 is a l.b. with $\deg 2$ & $\text{rk} \geq 1$, so \tilde{C} must be hyperelliptic,
 and \mathcal{L} induces a degree 2 map

$$\tilde{C} \longrightarrow \mathbb{P}^1.$$

③ In the example,

$$\tilde{C} = C_9, \quad C = C_5,$$

$$\begin{aligned} \sigma: \quad x &\mapsto -x \\ y &\mapsto -y \\ \text{Prym involution} \end{aligned}$$

$$\begin{aligned} \tau: \quad x &\mapsto -x \\ y &\mapsto y \\ C_4 \text{ involution} \end{aligned}$$

$$\begin{aligned} \sigma\tau: \quad x &\mapsto x \\ y &\mapsto -y \\ \text{hyperelliptic involution} \end{aligned}$$

we can show directly that

$$AP_{C_9}(\tau(p)) = AP_{C_9}(p).$$

This gives a second proof for Prop (2).

Reason:

$$\begin{aligned} \mathcal{O}_{C_9}(\tau(p) + \sigma(p)) &= \mathcal{O}_{C_9}(\tau(p + \sigma\tau(p))) \\ &= \mathcal{O}_{C_9}(p + \sigma\tau(p)) \end{aligned}$$

$\sigma\tau = \tau\sigma$
 $\sigma\tau$ is an
hyperelliptic involution

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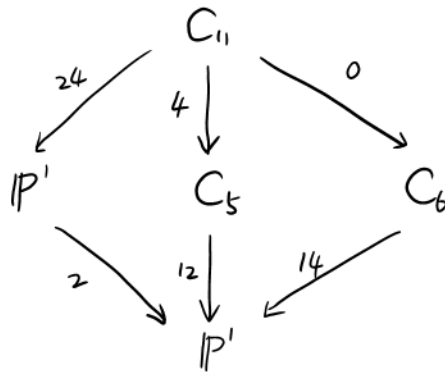
Gauss map

Taking the Gauss map of (3), one gets

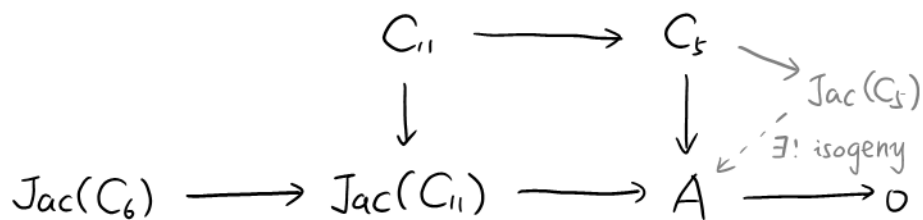
$$\begin{array}{ccc}
 C_9 & \xrightarrow[4]{2:1} & C_4 \\
 \downarrow \substack{2:1 \\ 20} & & \downarrow \substack{2:1 \\ 10} \\
 R_1 & \xrightarrow[0, \infty]{\substack{2:1 \\ ram \ a^t}} & R_2 \\
 \downarrow & & \downarrow \\
 \mathbb{P}^8 & \dashrightarrow & \mathbb{P}^3 \\
 [\alpha_0, \dots, \alpha_8] & \longmapsto & [a_1 : a_3 : a_5 : a_7]
 \end{array}$$

$$\Rightarrow \deg_A C_4 = 6, \quad \text{Gal}(\gamma) = S_6 = W(C_3).$$

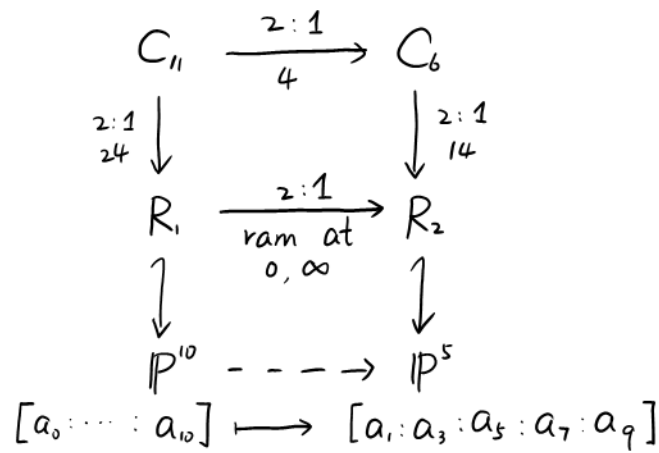
E.g. 2. $C_{11} = \{y^2 = \prod_{j=1}^{12} (x^2 - j)\}$ has the following covering:



Curves in Jacobians: ($A = \text{Prym}(C_{11}/C_6)$?)



Gauss map:



$$\Rightarrow \deg_A C_6 = 14, \quad \text{Gal}(\gamma) = S_{14} = W(C_7).$$

E.g. 3.

$$C_5 = \{y^2 = (x^3 + x + 2)^4 + 1\}$$

$$f \downarrow \begin{matrix} 3:1 \\ 8 \end{matrix}$$

$$E = \{y^2 = t^4 + 1\}$$

Let us write down the change of variables for f :

$$\begin{array}{ccc}
 C_5 & \mathbb{C}(x)[y]/(y^2 - (x^3 + x + 2)^4 + 1) & \cong \mathbb{C}(u)[v']/(v'^2 - ((1 + u^2 + 2u^3)^4 + u'^2)) \\
 \downarrow & \uparrow & \uparrow \\
 E & \mathbb{C}(t)[y]/(y^2 - (t^4 + 1)) & \cong \mathbb{C}(s)[v]/(v^2 - (1 + s^4))
 \end{array}$$

$$\begin{array}{ccccc}
 \begin{array}{c} x^3 + x + 2 \\ \updownarrow \\ t \end{array} & \begin{array}{c} y \\ \updownarrow \\ y \end{array} & \begin{cases} x = \frac{1}{u} \\ y = \frac{v'}{u^6} \end{cases} & \begin{cases} u = \frac{1}{x} \\ v' = \frac{y}{x^6} \end{cases} & \begin{array}{c} \frac{u^3}{1 + u^2 + 2u^3} \\ \updownarrow \\ s \end{array} & \begin{array}{c} \frac{v'}{(1 + u^2 + 2u^3)^2} \\ \updownarrow \\ v \end{array} \\
 & & \begin{cases} t = \frac{1}{s} \\ y = \frac{v}{s^2} \end{cases} & \begin{cases} s = \frac{1}{t} \\ v = \frac{y}{t^2} \end{cases} & &
 \end{array}$$

Also, we write down the local coordinate charts of f :

$$\begin{array}{ccc}
 C_5 & \mathbb{C}[x][y]/(y^2 - (x^3 + x + 2)^4 + 1) & \longleftrightarrow \mathbb{C}[u][v']/(v'^2 - ((1 + u^2 + 2u^3)^4 + u'^2)) \\
 \downarrow & \uparrow & \uparrow \\
 E & \mathbb{C}[t][y]/(y^2 - (t^4 + 1)) & \longleftrightarrow \mathbb{C}[s][v]/(v^2 - (1 + s^4))
 \end{array}$$

Curves in Jacobians:

$$A = \text{Prym}(C_s/E) \text{ ?}$$

$$\begin{array}{ccccc} & C_s & & & \\ & \downarrow A|_{C_s} & \searrow & & \\ E & \xrightarrow{f^*} & \text{Jac}(C_s) & \xrightarrow{\pi} & A \longrightarrow 0 \end{array}$$

- Prop.
1. $\# \ker f^* < +\infty$;
 2. $\pi \circ A|_{C_s} : C_s \longrightarrow A$ is gen injective ;
 3. The image of $C_s \longrightarrow A$ is smooth outside the non-injective locus.