Eine Woche, ein Beispiel

9.4 Hecke algebra

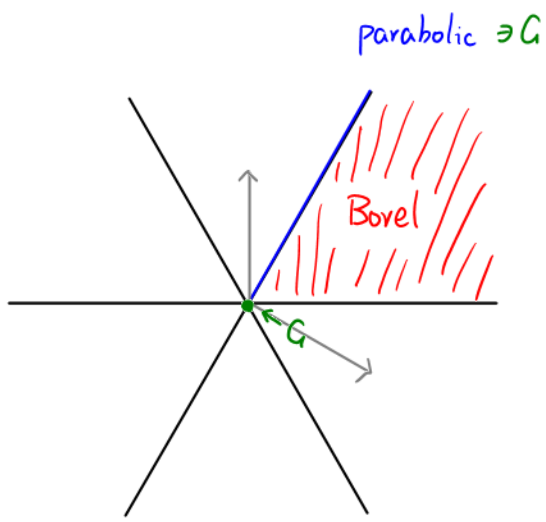
This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]
[XiongHecke][https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf]

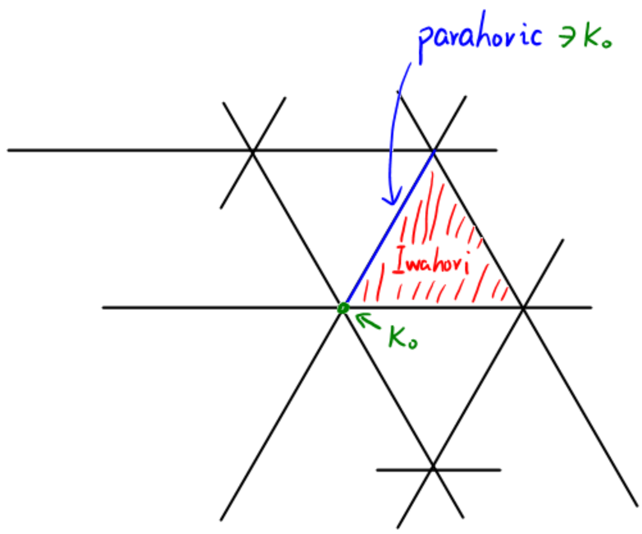
Task. For each double coset decomposition, we want to do:

1. decomposition ($\#\Gamma t\Gamma/\Gamma$ is finite)
2. $\mathbb{Z}$-mod structure , notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form
F finite	$G=\bigsqcup_{w\in W} BwB$		
F local	$G=\bigsqcup_{w\in W} BwB$	$G=\bigsqcup_{w\in W_{ext}} IwI$	$G=\bigsqcup_{t\in T^-} K_0 t K_0$
F global	$G=\bigsqcup_{w\in W} BwB$		$GL_2^+(\mathbb{Q})=\bigsqcup_{t\in T^-}\Gamma t\Gamma$
adèle?			





$$B=\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}=\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\cap\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P=\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}\cap\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix}\cap\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

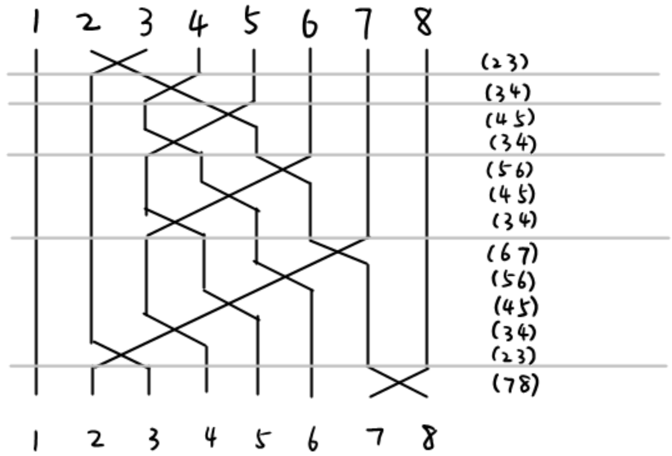
S_n and Tits system

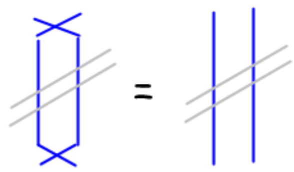
A brief preparation for computations in Bruhat decomposition. $s_i = (i\ i{+}1)$, $1 \le i \le n-1$

E.g. $n=8$, $w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8$.

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.





$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0) = 13$ = "inversion number"

$l(s_1 w_0) = 14 \qquad l(w_0 s_1) = 14$

$l(s_2 w_0) = 12 \qquad l(w_0 s_2) = 12$

$l(s_3 w_0) = 14 \qquad l(w_0 s_3) = 14$

$l(s_4 w_0) = 12 \qquad l(w_0 s_4) = 12$

$l(s_5 w_0) = 12 \qquad l(w_0 s_5) = 12$

$l(s_6 w_0) = 12 \qquad l(w_0 s_6) = 14$

$l(s_7 w_0) = 14 \qquad l(w_0 s_7) = 12$

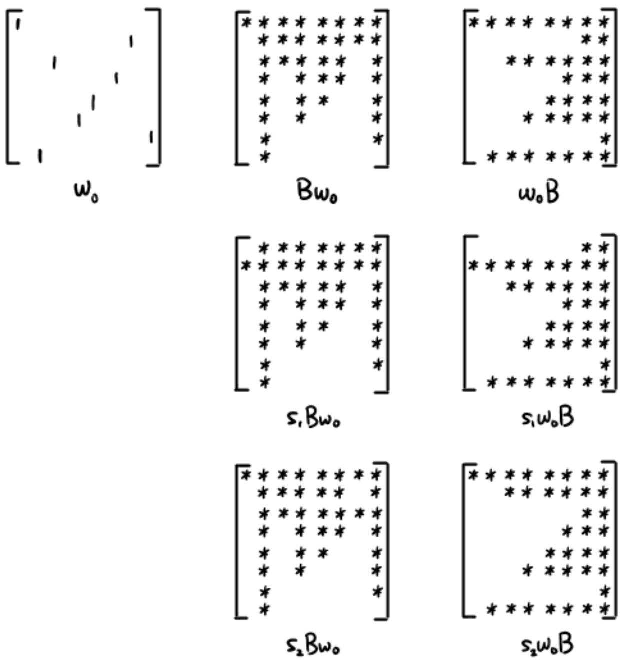
Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \vdots \\ & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

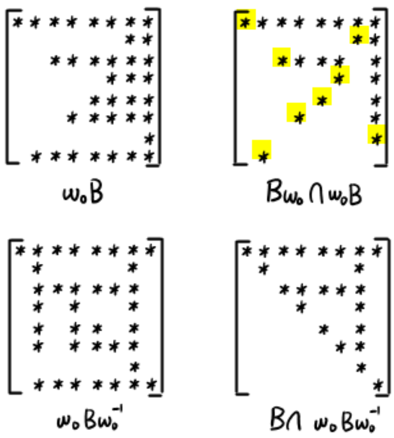
Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

Solution



The following computation will be also computed later on.



finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.

(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$

Ex. Compute $|BwB/B|$

$$\left[\begin{array}{l} \text{Hint: Consider the map} \\ \quad \phi: B \longrightarrow BwB/B \\ \quad\quad b \longmapsto bwB \\ \phi(b_1) = \phi(b_2) \Leftrightarrow b_1 wB = b_2 wB \\ \quad \Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B \\ \quad \Leftrightarrow b_2^{-1} b_1 \in wBw^{-1} \\ \therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{l(w)} \end{array}\right]$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) = \{ f: G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \ \forall b_1, b_2 \in B, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) \, d\mu(x)$$

$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G,B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbb{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote $T_w := \mathbb{1}_{BwB}$, $T_i := T_{s_i}$ ($T_{Id} = \mathbb{1}_B$ is the unit of $\mathcal{H}(G,B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G,B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) \, T_w$$

$$\begin{aligned} (T_u * T_v)(w) &= \frac{1}{|B|} \sum_{yz=w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} \{ (y,z) \in BuB \times BvB \mid yz = w \} \quad \underline{\text{if } w \notin BuBBvB} \ 0 \\ &= \frac{1}{|B|} | BuB \cap uBv^{-1}B | \end{aligned}$$

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw) = l(w)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw) = l(w)-1 \end{cases}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{s_iw} & l(s_iw) = l(w)+1 \\ \mathbb{Z} \cdot T_{s_iw} + \mathbb{Z} \cdot T_w & l(s_iw) = l(w)-1 \end{cases}$$

Computation of coefficient.

$$|BwB| = |BwB/B| \times |B| = q^{l(w)} \cdot |B|$$

when $l(s_iw) = l(w)+1$,

$$(T_i * T_w)(s_iw) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times BwB \mid yz = s_iw\}$$

$$= \frac{1}{|B|\,|Bs_iwB|} \{(y,z) \in Bs_iB \times BwB \mid yz \in Bs_iwB\}$$

$$= \frac{|Bs_iB|\,|BwB|}{|B| \cdot |Bs_iwB|} = \frac{q^{l(s_i)}\, q^{l(w)}}{q^{l(s_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz = Id\}$$

$$= \frac{1}{|B|} |Bs_iB| = q$$

$$(T_i * T_i)(s_i) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz = s_i\}$$

$$= \frac{1}{|B|\,|Bs_iB|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz \in Bs_iB\}$$

$$= \frac{1}{|B|\,|Bs_iB|} \left(|Bs_iB \times Bs_iB| - \{(y,z) \in Bs_iB \times Bs_iB \mid yz \in B\}\right)$$

$$= \frac{1}{|B|\,|Bs_iB|} \left(|Bs_iB|\,|Bs_iB| - |B| \cdot |Bs_iB|\right)$$

$$= q-1$$

when $l(s_iw) = l(w)-1$, we get $l(s_i \cdot s_iw) = l(s_iw)+1$,

$$T_i * T_w = T_i * T_i * T_{s_iw}$$

$$= (qT_{Id} + (q-1)T_i) * T_{s_iw}$$

$$= qT_{s_iw} + (q-1)T_w$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} T_{s_iw} & l(s_iw) = l(w)+1 \\ qT_{s_iw} + (q-1)T_w & l(s_iw) = l(w)-1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$\mathcal{H}(G,B) = \mathbb{Z}\langle T_1, \dots, T_{n-1}\rangle_{alg}$ with relations $\quad (\mathcal{H}(G,B) \cong \mathcal{H}_q(W))$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j| \geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

E.g. For $n=2$, $\mathcal{H}(G,B) \cong \mathbb{Z}[T_1] / (T_1^2-(q-1)T_1-q)$

$$\cong \mathbb{Z}[T_1] / (T_1-q)(T_1+1)$$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1$$

For $n=3$, $\mathcal{H}(G,B) \cong \mathbb{Z}\langle T_1, T_2 \rangle \Big/ \big((T_1-q)(T_1+1), (T_2-q)(T_2+1), T_1T_2T_1 = T_2T_1T_2\big)$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_1T_2 \oplus \mathbb{Z}T_2T_1 \oplus \mathbb{Z}T_1T_2T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_{(123)} \oplus \mathbb{Z}T_{(132)} \oplus \mathbb{Z}T_{(13)}$$

global Cartan decomposition

1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2\times 2}(R) = \coprod_{(b)\subseteq(a)} GL_2(R)\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(R)$$

Cor
$$M_{2\times 2}(\mathbb{Z}) = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0\leq a\leq b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det\neq 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\leq b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det> 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\leq b}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a,b\in\mathbb{Q}^\times_{>0}\\ v_p(a)\leq v_p(b)\ \forall p}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix}SL_2(\mathbb{Z}) \qquad GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det>0}$$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$T^- = \left\{ \begin{pmatrix} a & \\ & b\end{pmatrix} \in GL_2^+(\mathbb{Q}) \;\middle|\; \begin{array}{l} a,b>0 \\ v_p(a)\leq v_p(b) \quad \forall p \text{ prime}\end{array}\right\} \overset{\text{Grp}}{\cong} \mathbb{Q}^\times_{>0}\times(\mathbb{Z}_{>0},\times)$$

then
$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha\in T^-} \Gamma\alpha\Gamma$$

Ex. Verify that $\Gamma\alpha\Gamma/\Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix} \alpha_1 & \\ & \alpha_2\end{pmatrix}\in T^-$

Hint. See [LWW, 引理5.1.4].

$$\#\Gamma\alpha\Gamma/\Gamma = \#\Gamma/\Gamma\cap\alpha\Gamma\alpha^{-1} = \#\Gamma/\Gamma_0\left(\tfrac{\alpha_2}{\alpha_1}\right) = \#\mathbb{P}^1\left(\tfrac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_2}{\alpha_1}\prod_{p\mid\frac{\alpha_2}{\alpha_1}}\left(1+\frac{1}{p}\right)$$

$$\left[\alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}\alpha^{-1} = \begin{pmatrix} a & \frac{\alpha_1}{\alpha_2}b\\ \frac{\alpha_2}{\alpha_1}c & d\end{pmatrix} \quad\Rightarrow\quad \Gamma\cap\alpha\Gamma\alpha^{-1} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\\ \frac{\alpha_2}{\alpha_1}\mathbb{Z} & \mathbb{Z}\end{pmatrix}_{\det=1} = \Gamma_0\left(\tfrac{\alpha_2}{\alpha_1}\right)\right]$$

e.g. $\#\Gamma\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}\Gamma/\Gamma = p+1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p^e\end{pmatrix}\Gamma/\Gamma = p^e+p^{e-1}$