Eine Woche, ein Beispiel

7.23 trace theorem and Sobolev embedding

This is a continuation of [23.05.28].
In the statement of propositions, all fcts are real-valued fcts.

Prop. For $0 \le k \le n$, $s > \frac{k}{2}$, one can construct cont linear fcts

$$
\begin{array}{ccc}
H^s(\mathbb{R}^n) & \longrightarrow & H^{s-\frac{k}{2}}(\mathbb{R}^{n-k}) \\
\cup & & \cup \\
\mathcal{S}(\mathbb{R}^n) & \longrightarrow & \mathcal{S}(\mathbb{R}^{n-k}) \\
f & \longmapsto & f|_{\{0\}\times\mathbb{R}^{n-k}}
\end{array}
$$

Proof. Denote $V := \mathbb{R}^k$, $W := \mathbb{R}^{n-k}$, then $V \times W = \mathbb{R}^n$, $\underset{\{0\}\times\mathbb{R}^{n-k} \hookrightarrow \mathbb{R}^k\times\mathbb{R}^{n-k}}{W \hookrightarrow V \times W}$, reduce to show:

$$\exists\, C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \qquad \|f|_W\|_{H^{s-\frac{k}{2}}} \le \|f\|_{H^s}$$

Step 1. Express $\widehat{f|_W}(\xi_2)$ in terms of $\hat{f}(\xi)$, by using Fourier transform twice.

$$
\left.
\begin{aligned}
f(0,x_2) &= \int_W e^{i\langle x_2,\xi_2\rangle}\, \widehat{f|_W}(\xi_2)\, d\xi_2 \\
f(0,x_2) &= \int_{V\times W} e^{i\langle x,\xi\rangle}\, \hat{f}(\xi)\, d\xi \\
&= \int_W e^{i\langle x_2,\xi_2\rangle} \left(\int_V \hat{f}(\xi)\, d\xi_1\right) d\xi_2
\end{aligned}
\right\} \Rightarrow \widehat{f|_W}(\xi_2) = \int_V \hat{f}(\xi)\, d\xi_1
$$

Step 2. Expand.

$$
\begin{aligned}
\|f|_W\|^2_{H^{s-\frac{k}{2}}} &= \|\widehat{f|_W}\|^2_{L^2(W,\, (|\xi_2|^2+1)^{s-\frac{k}{2}} d\xi_2)} \\
&= \int_W \left(\widehat{f|_W}(\xi_2)\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}}\, d\xi_2 \\
&= \int_W \left(\int_V \hat{f}(\xi)\, d\xi\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}}\, d\xi_2
\end{aligned}
$$

$$
\begin{aligned}
\|f\|^2_{H^s} &= \|\hat{f}\|^2_{L^2(V\times W,\ (|\xi|^2+1)^s d\xi)} \\
&= \int_{V\times W} \left(\hat{f}(\xi)\right)^2 (|\xi|^2+1)^s\, d\xi \\
&= \int_W \left(\int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s\, d\xi_1\right) d\xi_2
\end{aligned}
$$

Therefore, the problem reduce to $d\xi_1 \approx d\xi_1$

$$\left(\int_V \hat{f}(\xi)\, d\xi_1\right)^2 (|\xi_2|^2+1)^{s-\frac{k}{2}} \le C \int_V (\hat{f}(\xi))^2 (|\xi|^2+1)^s\, d\xi_1.$$

Step 3. Use Hölder inequality to simplify. Since

$$\left(\int_V \hat{f}(\xi)\, d\xi_1\right)^2 \leqslant \int_V \hat{f}(\xi)^2 (|\xi|^2+1)^s\, d\xi_1 \int_V (|\xi|^2+1)^{-s}\, d\xi_1 ,$$

the problem reduces to

$$\int_V (|\xi|^2+1)^{-s} d\xi_1 \left(|\xi_2|^2+1\right)^{s-\frac{k}{2}} \leq C.$$

Step 4. Compute $\int_V (|\xi|^2+1)^{-s} d\xi_1$ directly.

$$\begin{aligned}
&\int_V (|\xi|^2+1)^{-s} d\xi_1 \\
&= \int_V \frac{1}{(|\xi_1|^2+|\xi_2|^2+1)^s}\, d\xi_1 \\
&\overset{a^2=|\xi_2|^2+1}{\underset{a>0}{=\!=}} \int_V \frac{1}{(|\xi_1|^2+a^2)^s}\, d\xi_1 \\
&= \int_V \frac{1}{(|\frac{\xi_1}{a}|^2+1)^s}\, d\xi_1 \cdot a^{k-2s} \\
&= C a^{k-2s} = C\left(|\xi_2|^2+1\right)^{\frac{k}{2}-s}
\end{aligned}$$

where $C = \int_V (|x|^2+1)^{-s} dx < +\infty$

$$\Rightarrow \int_V (|\xi|^2+1)^{-s} d\xi_1 (|\xi_2|^2+1)^{s-\frac{k}{2}} \leq C \qquad \square$$

Rmk. The original C in the proposition can be taken by

$$C = (2\pi)^k \int_{\mathbb{R}} \frac{1}{(|x|^2+1)^s}\, dx .$$

Here, C depends on the definition of the norm $\|\cdot\|_{H^s}$, $\|\cdot\|_{H^{s-\frac{n}{2}}}$. Therefore, usually we don't write down C explicitly.

Prop. For $n, k \geq 0$, $s > k + \frac{n}{2}$, one can construct cont linear fcts.

$$\begin{array}{ccc} H^s(\mathbb{R}^n) & \longrightarrow & C_0^k(\mathbb{R}^n) \\ \cup & & \cup \\ \mathcal{S}(\mathbb{R}^n) & = & \mathcal{S}(\mathbb{R}^n) \end{array}$$

Proof. Only show the $k=0$ case, then $s > \frac{n}{2}$.
Once we show:

$$\exists C > 0 \text{ s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \quad \|f\|_{L^\infty} \leq C \|f\|_{H^s},$$

by the completeness of $C_0^0(\mathbb{R}^n)$, the embedding is then well-defined.

$$\begin{aligned} \|f\|_{L^\infty}^2 &\leq \|\hat{f}\|_{L^1}^2 \\ &= \left(\int_{\mathbb{R}^n} \hat{f}(\xi)\, d\xi\right)^2 \\ &\leq \int_{\mathbb{R}^n} (|\xi|^2+1)^{-s} d\xi \int_{\mathbb{R}^n} (\hat{f}(\xi))^2 (|\xi|^2+1)^s d\xi \\ &= C^2 \|f\|_{H^s}^2 \end{aligned}$$

Here, $C = \sqrt{\int_{\mathbb{R}} \frac{1}{(|\xi|^2+1)^s} d\xi} < +\infty$ $\square$

Prop (in final exam). For $n \geq 0$, $s > \frac{n}{2}$, one can construct cont linear fcts.

$$\begin{array}{ccc} L^1(\mathbb{R}^n) & \longrightarrow & H^{-s}(\mathbb{R}^n) \\ \cup & & \cup \\ \mathcal{S}(\mathbb{R}^n) & = & \mathcal{S}(\mathbb{R}^n) \end{array}$$

Proof. Reduced to show:

$$\exists C > 0 \quad \text{s.t. } \forall f \in \mathcal{S}(\mathbb{R}^n), \quad \|f\|_{H^{-s}} \leq \|f\|_{L^1}.$$

$$\begin{aligned} \|f\|_{H^{-s}}^2 &= \|\hat{f}\|_{L^2(\mathbb{R}^n, (|\xi|^2+1)^{-s} d\xi)} \\ &= \int_{\mathbb{R}^n} (\hat{f}(\xi))^2 (|\xi|^2+1)^{-s} d\xi \\ &\leq \int_{\mathbb{R}^n} (|\xi|^2+1)^{-s} d\xi \ \|\hat{f}\|_{L^\infty}^2 \\ &\leq \int_{\mathbb{R}^n} (|\xi|^2+1)^{-s} d\xi \ \|\hat{\hat{f}}\|_{L^1}^2 \\ &= \int_{\mathbb{R}^n} (|\xi|^2+1)^{-s} d\xi \cdot (2\pi)^n \|f\|_{L^1}^2 \\ &= C^2 \|f\|_{L^1}^2 \end{aligned}$$

Here,

$$C = \sqrt{\int_{\mathbb{R}^n} \frac{1}{(|\xi|^2+1)^s} d\xi} \cdot (2\pi)^{\frac{n}{2}} < +\infty$$

The general case is as follows.

Exercise 6.3. Let $n = p + q$, $1 \leq p \leq n$, and write $\mathbb{R}_x^n := \mathbb{R}_{x'}^p \oplus \mathbb{R}_{x''}^q$ (i.e., for $x \in \mathbb{R}^n$ let $x' := (x_1, \ldots, x_p)$ and $x'' := (x_{p+1}, \ldots, x_n)$). Furthermore, let $s > k + p/2$ for some $k \in \mathbb{Z}_+$.

(a) Show that the assignment

$$H^s(\mathbb{R}^n) \ni f \mapsto \left(x' \mapsto f(x', \cdot) \in H^{s-p/2}(\mathbb{R}^q)\right)$$

is a well-defined continuous linear map $H^s(\mathbb{R}^n) \to C_0^k(\mathbb{R}^p, H^{s-p/2}(\mathbb{R}^q))$.

(b) Identify the Trace Theorem and the Sobolev Embedding Theorem as special cases.