

Eine Woche, ein Beispiel

11.6 equivariant K -theory of Steinberg variety: from formula to diagram.

▽ In this document, we always read the diagram from top to bottom.

1. nil Hecke alg

2. $\cdot \rightarrow \cdot$ case

3. $\cdot \curvearrowright \cdot$ case (haven't worked out)

1. nil Hecke alg

Recall that we have an alg homo

$$\mathbb{Z}[e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_d^{\pm 1}] \xrightarrow{\quad} \mathbb{Q}[[\lambda_1, \lambda_2, \dots, \lambda_d]] \supseteq \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

$$e_i \xrightarrow{\quad} e^{\lambda_i}$$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$ for e_i, λ_i , $i \in \{1, \dots, d\}$

Ex 1. define $\partial_i \in \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$ by

$$\partial_i f = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \quad f \in \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

compute $\partial_i \lambda_i$, $\partial_i \lambda_{i+1}$, $\partial_i (\lambda_1^3 \lambda_2 - 3 \lambda_2 \lambda_4 \lambda_5)$.

Ex 2. derive that

$$\partial_i f g = (s_i f) \partial_i g + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} g \quad f \in \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$$

$$f: g \mapsto f \cdot g$$

as operators.

Ex 3. verify that

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ \partial_i \lambda_i = \lambda_{i+1} \partial_i + 1 \end{array} \quad \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ \partial_i \lambda_{i+1} = \lambda_i \partial_i - 1 \end{array}$$

$$\begin{array}{c} \text{Diagram: two crossings sharing a strand} \\ \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \end{array} \quad \begin{array}{c} \text{Diagram: square crossing} \\ \partial_i^2 = 0 \end{array}$$

Ex 1'. define $D_i \in \text{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_i^{\pm}, \dots, e_d^{\pm}])$ by

$$D_i f = \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}}$$

$$= \frac{e_{i+1}f - e_i s_i f}{e_{i+1} - e_i}$$

compute

$$D_i 1 = 1$$

$$D_i e_i = 0$$

$$D_i e_{i+1} = e_i + e_{i+1}$$

$$D_i e_i^{-1} = e_i^{-1} + e_{i+1}^{-1}$$

$$D_i e_{i+1}^{-1} = 0$$

Ex 2'. derive that

$$D_i f g = (s_i f) D_i g + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} g$$

as operators.

Ex 3'. verify that

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ D_i e_i \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ e_{i+1} D_i \end{array} - \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the right one} \\ e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-right strand} \\ D_i e_{i+1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-left strand} \\ e_i D_i \end{array} + \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the right one} \\ e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ D_i e_i^{-1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ e_{i+1}^{-1} D_i \end{array} + \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the left one} \\ e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-right strand} \\ D_i e_{i+1}^{-1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-left strand} \\ e_i^{-1} D_i \end{array} - \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the left one} \\ e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: braid of three strands } D_i D_{i+1} D_i \\ D_i D_{i+1} D_i \end{array} = \begin{array}{c} \text{Diagram: braid of three strands } D_{i+1} D_i D_{i+1} \\ D_{i+1} D_i D_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing twice } D_i^2 \\ D_i^2 \end{array} = \begin{array}{c} \text{Diagram: crossing once } D_i \\ D_i \end{array}$$

Ex 4. Verify that

$$\begin{array}{ccc}
 \text{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_1^{\pm 1}, \dots, e_d^{\pm 1}])_{\text{cpl}} & & \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])_{\text{cpl}} \\
 \cup & & \cup \\
 \langle e_1^{\pm 1}, \dots, e_d^{\pm 1}, D_1, \dots, D_{d-1} \rangle_{\mathbb{Z}\text{-alg, cpl}} & \longrightarrow & \langle \lambda_1, \dots, \lambda_d, \partial_1, \dots, \partial_d \rangle_{\mathbb{Q}\text{-alg, cpl}} \\
 e_i & \xrightarrow{\quad} & e^{\lambda_i} \\
 D_i & \xrightarrow{\quad} & \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} = \frac{\lambda_{i+1} - \lambda_i}{1 - e^{\lambda_{i+1} - \lambda_i}} \partial_i + 1 \\
 \\
 \log e_i & \longleftarrow & \lambda_i \\
 \\
 D_i \frac{1 - \frac{e_i}{e_{i+1}}}{\log \frac{e_i}{e_{i+1}}} = \frac{1 - \frac{e_{i+1}}{e_i}}{\log \frac{e_{i+1}}{e_i}} (D_i - 1) & \longleftarrow & \partial_i
 \end{array}$$

is an alg iso.

Hint.

$$\begin{array}{lcl}
 D_i e_k & = & s_i(e_k) D_i \\
 \Leftrightarrow \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} e^{\lambda_k} & = & s_i(e^{\lambda_k}) \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{1 - e^{\lambda_i - \lambda_{i+1}}} \\
 \Leftrightarrow \partial_i e^{\lambda_k} & = & s_i(e^{\lambda_k}) \partial_i + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{\lambda_i - \lambda_{i+1}}
 \end{array}$$

1. $\bullet \rightarrow \bullet$ case

Recall that we have an alg homo

$$\bigoplus_{u \in M, d \in W_d, W_d} (\mathbb{Z}[e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_{d+1}^{\pm 1}])^u \xrightarrow{\quad} \bigoplus_u (\mathbb{Q}[[\lambda_1, \dots, \lambda_d]])^u \supseteq \bigoplus_u (\mathbb{Q}[\lambda_1, \dots, \lambda_d])^u$$

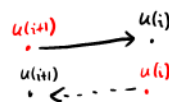
$e_i^u \quad \quad \quad e^{\lambda_i^u}$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$ for e_i^u, λ_i^u , $i \in \{1, \dots, d\}$

$$u s_i = w' u'$$

Ex 1. define $\partial_i^{u,u'} \in \text{End}_{\mathbb{Q}\text{-v.s.}}(\bigoplus_u (\mathbb{Q}[\lambda_1, \dots, \lambda_d])^u)$ by

$$\begin{array}{ll} \textcircled{1} & \partial_i^{u,u} f^u = \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \right)^u & u = u' \\ \textcircled{2} & \partial_i^{u,u'} f^{u'} = (s_i f (\lambda_{i+1} - \lambda_i))^u & u \neq u' \\ \textcircled{3} & \partial_i^{u,u'} f^{u'} = (s_i f)^u & u \neq u' \end{array}$$



For u : ~~λ_i~~ , compute $(\partial_i = \partial_i^{u,u'})$

$$\begin{array}{ccc} \partial_2 1^u & \partial_2 \lambda_2^u & \partial_2 \lambda_3^u \\ \partial_1 1^{u'} & \partial_1 \lambda_1^{u'} & \partial_1 \lambda_2^{u'} \\ \partial_3 1^{u'} & \partial_3 \lambda_3^{u'} & \partial_3 \lambda_4^{u'} \end{array}$$

A: They are

$$\begin{array}{ccc} 0 & 1 & -1 \\ \lambda_2^u - \lambda_1^u & \lambda_2^u (\lambda_2^u - \lambda_1^u) & \lambda_1^u (\lambda_2^u - \lambda_1^u) \\ 1^u & \lambda_4^u & \lambda_3^u \end{array}$$

Ex 2. derive that as operators.

$$\begin{array}{ll} \partial_i^{u,u} f^u = (s_i f)^u \partial_i^{u,u} + \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \right)^u & u = u' \\ \partial_i^{u,u'} f^{u'} = (s_i f)^u \partial_i^{u,u'} & u \neq u' \end{array}$$

Ex 3. verify that

$$\begin{array}{ccc} \textcircled{1} & = & \textcircled{2} \\ \partial_i^{u,u'} \lambda_i^{u'} & = & \lambda_{i+1}^u \partial_i^{u,u'} \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 1: A diamond shape with red lines forming the left and right sides, and black lines forming the top and bottom. The top-left and bottom-right edges are red. Labels 2 and 3 are on the right side.} \\ \partial_i^{u,u'} \partial_i^{u',u} \end{array} & = & \begin{array}{c} \text{Diagram 2: Two vertical lines. The left line has a red dot at the top. The right line has a black dot at the bottom.} \\ \lambda_i^u - \lambda_{i+1}^u \end{array}
 \end{array}
 \qquad
 \begin{array}{ccc}
 \begin{array}{c} \text{Diagram 3: A diamond shape with red lines forming the left and right sides, and black lines forming the top and bottom. The top-right and bottom-left edges are red. Labels 3 and 2 are on the right side.} \\ \partial_i^{u,u'} \partial_i^{u',u} \end{array} & = & \begin{array}{c} \text{Diagram 4: Two vertical lines. The left line has a black dot at the top. The right line has a red dot at the bottom.} \\ \lambda_i^u - \lambda_{i+1}^u \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 5: A hexagon with red lines forming the top, bottom, and right sides. The left side is black. Labels 3, 1, and 3 are on the right side.} \\ \partial_i^{u,u'} \partial_{i+1}^{u',u} \partial_i^{u',u} \end{array} & = & \begin{array}{c} \text{Diagram 6: A hexagon with red lines forming the top, bottom, and left sides. The right side is black. Labels 3, 1, and 2 are on the right side.} \\ \partial_{i+1}^{u,u''} \partial_i^{u',u''} \partial_{i+1}^{u'',u} \end{array} + \begin{array}{c} \text{Diagram 7: Three vertical lines. The middle line is red.} \\ 1^u \end{array}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 8: A hexagon with red lines forming the top, bottom, and left sides. The right side is black. Labels 3, 1, and 2 are on the right side.} \\ \partial_i^{u,u'} \partial_{i+1}^{u',u} \partial_i^{u',u} \end{array} & = & \begin{array}{c} \text{Diagram 9: A hexagon with red lines forming the top, bottom, and right sides. The left side is black. Labels 2, 1, and 3 are on the right side.} \\ \partial_{i+1}^{u,u''} \partial_i^{u',u''} \partial_{i+1}^{u'',u} \end{array} - \begin{array}{c} \text{Diagram 10: Three vertical lines. The middle line is red.} \\ 1^u \end{array}
 \end{array}$$

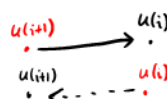
$$\begin{array}{ccc}
 \begin{array}{c} \text{Diagram 11: A hexagon with red lines forming the top, bottom, and right sides. The left side is black. Labels 1, 2, and 2 are on the right side.} \\ \partial_i^{u,u'} \partial_{i+1}^{u',u} \partial_i^{u',u} \end{array} & = & \begin{array}{c} \text{Diagram 12: A hexagon with red lines forming the top, bottom, and left sides. The right side is black. Labels 2, 2, and 1 are on the right side.} \\ \partial_{i+1}^{u,u''} \partial_i^{u',u''} \partial_{i+1}^{u'',u} \end{array}
 \end{array}$$

Ex 1' define $D_i^{u,u'} \in \text{End}_{\mathbb{Z}\text{-mod}}(\oplus_u (\mathbb{Z}[e_i^{\pm 1}, \dots, e_{d_1+d_2}^{\pm 1}])^u)$ by

$$\textcircled{1} D_i^{u,u} f^u = \left(\frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \right)^u \quad u = u'$$

$$\textcircled{2} D_i^{u,u'} f^{u'} = \left(s_i f \left(1 - \frac{e_{i+1}}{e_i} \right) \right)^u \quad u \neq u'$$

$$\textcircled{3} D_i^{u,u'} f^{u'} = (s_i f)^u \quad u \neq u'$$



For u : ~~XXXX~~, compute $(D_i = D_i^{u,u'})$

$$\begin{array}{ccccc} D_2 1^u & D_2 e_2^u & D_2 e_3^u & D_2 (e_2^u)^{-1} & D_2 (e_3^u)^{-1} \\ D_1 1^{u'} & D_1 e_1^{u'} & D_1 e_2^{u'} & D_1 (e_1^{u'})^{-1} & D_1 (e_2^{u'})^{-1} \\ D_3 1^{u'} & D_3 e_3^{u'} & D_3 e_4^{u'} & D_3 (e_3^{u'})^{-1} & D_3 (e_4^{u'})^{-1} \end{array}$$

A: They are

$$\begin{pmatrix} 1 & 0 & e_2 + e_3 & e_2^{-1} + e_3^{-1} & 0 \\ 1 - \frac{e_2}{e_1} & e_2(1 - \frac{e_2}{e_1}) & e_1(1 - \frac{e_2}{e_1}) & \frac{1}{e_2}(1 - \frac{e_2}{e_1}) & \frac{1}{e_1}(1 - \frac{e_2}{e_1}) \\ 1 & e_4 & e_3 & \frac{1}{e_4} & \frac{1}{e_3} \end{pmatrix}^u$$

Ex 2' derive that as operators.

$$D_i^{u,u} f^u = (s_i f) D_i^{u,u} + \left(\frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} \right)^u \quad u = u'$$

$$D_i^{u,u'} f^{u'} = (s_i f) D_i^{u,u'} \quad u \neq u'$$

Ex 3' verify that

$$\textcircled{1} \text{ (diagram with red X) } = \textcircled{2} \text{ (diagram with red X)} \\ D_i^{u,u'} e_i^{u'} = e_{i+1}^{u'} D_i^{u,u'}$$

$$\textcircled{1} \text{ (diagram with red X) } = \textcircled{2} \text{ (diagram with red X)} \\ D_i^{u,u'} (e_i^{u'})^{-1} = (e_{i+1}^{u'})^{-1} D_i^{u,u'}$$

$$\textcircled{1} \text{ (diagram with red X) } = \textcircled{2} \text{ (diagram with red X)} \\ D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_i}{e_{i+1}} \right)^u$$

$$\textcircled{1} \text{ (diagram with red X) } = \textcircled{2} \text{ (diagram with red X)} \\ D_i^{u,u'} D_i^{u',u} = 1^u - \left(\frac{e_{i+1}}{e_i} \right)^u$$

$$\begin{array}{c} \text{Diagram 1} \\ D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u} \end{array} = \begin{array}{c} \text{Diagram 2} \\ D_{i+1}^{u,u'} D_i^{u',u''} D_{i+1}^{u'',u} \end{array} + \begin{array}{c} \text{Diagram 3} \\ \left(\frac{e_{i+1}}{e_i} \right)^u \end{array}$$

$$\begin{array}{c} \text{Diagram 4} \\ D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u} \end{array} = \begin{array}{c} \text{Diagram 5} \\ D_{i+1}^{u,u'} D_i^{u',u''} D_{i+1}^{u'',u} \end{array} - \begin{array}{c} \text{Diagram 6} \\ \left(\frac{e_{i+1}}{e_i} \right)^u \end{array}$$

$$\begin{array}{c} \text{Diagram 7} \\ D_i^{u,u'} D_{i+1}^{u',u''} D_i^{u'',u} \end{array} = \begin{array}{c} \text{Diagram 8} \\ D_{i+1}^{u,u'} D_i^{u',u''} D_{i+1}^{u'',u} \end{array}$$