# Eine Woche, ein Beispiel

## 11.19. Basic sheaf calculation

Goal: Motivate $f_*, f^*, f_!, f^!$, by connecting them with (co)homology theory

After story: $\rightsquigarrow$ calculation of $\mathrm{Perv}_\Delta(\mathbb{C}\mathbb{P}^1)$
$\rightsquigarrow$ generalize Morse theory
$\rightsquigarrow$ Characteristic classes / cycles
$\rightsquigarrow$ index theorem

Minor advantages from my talk:

- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    $\rightsquigarrow$ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale) descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \mathrm{Top}$
$\mathrm{Obj}(Sh(X)) = \{\text{sheaves of abelian gps}\}$
e.p. $Sh(\{*\}) = \mathrm{Abel}$ $\quad \underline{\mathbb{Q}}_{\{*\}} \longleftrightarrow \underline{\mathbb{Q}}$

0. sheaf
1. $f_*$, skyscraper sheaf & global sections
2. $f^*$, constant sheaf & stalks
3. $Rf_*$ & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology
   Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016

# 0. Sheaf

Recall the definition of

- presheaf $\mathcal{F}$
- sheaf $\mathcal{F}$
- stalk $\mathcal{F}_x$
- global section $\mathcal{F}(X) = \Gamma(X;\mathcal{F}) = H^0(X;\mathcal{F})$
- cohomology $R^n\Gamma(X;\mathcal{F}) = H^n(X;\mathcal{F})$

https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology
If X is paracompact and Hausdorff, Cech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

Recall examples of sheaves:

complicated
- $\mathcal{C}_X$ : sheaf of cont fcts on X
- $\mathcal{O}_X$ : structure sheaf on X  e.g., X: (cplx) mfld, scheme, ...
- $\underline{\mathbb{Q}}_X$ : constant sheaf on X
- $\mathrm{sky}_p(\mathbb{Q})$ : skyscraper sheaf of $p \in X$ on X.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{C}_X)_x \qquad \& \quad (\mathrm{sky}_p(\mathbb{Q}))_x .$$

## 1. $f_*$, skyscraper sheaf & global sections

**Setting** $X, Y \in Top$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

**Def.** $f_*\mathcal{F} \in Sh(X)$ is given by

$$f_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(f^{-1}(\mathcal{U}))$$

This defines a fctor

$$f_*: Sh(Y) \longrightarrow Sh(X)$$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & f_*\mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ & \mathcal{U} \arrow[u, phantom, "\cup" rotate=90] \end{tikzcd}$$

E.g. For $p \in X$, $\iota_p: \{p\} \hookrightarrow X$, $\quad \iota_{p,*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$

For $\quad \pi: Y \longrightarrow \{*\}$, $\quad \pi_*\mathcal{F} = \mathcal{F}(Y) = \Gamma(Y; \mathcal{F})$

## 2. $f^*$, constant sheaf & stalks

In [Vakil, Chapter 2], it is $f^{-1}$, the inverse image functor.

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(X)$, $f: Y \longrightarrow X$ cont

Def. $f^*\mathcal{F} \in Sh(Y)$ is given by sheafification of

$$\begin{array}{ccc} f^*\mathcal{F} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ \cup & & \\ \mathcal{U} & & \end{array}$$

$$f^{*,pre}\mathcal{F}(\mathcal{U}) = \varinjlim_{f(\mathcal{U})\subseteq V} \mathcal{F}(V)$$

This defines a fctor

$$f^*: Sh(X) \longrightarrow Sh(Y)$$

Recall:

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (x_p)_p \in \prod_{p\in\mathcal{U}} \mathcal{F}_p \;\middle|\; \right.$$