

Eine Woche, ein Beispiel

3.13 dual variety

Dual variety is useful in the research of subvarieties of \mathbb{P}^n (and symplectic geometry). We emphasize the embedding here.

Main reference:

<https://arxiv.org/abs/math/0112028v1>

other ref:

Discriminants, Resultants, and Multidimensional Determinants by Israel M. Gelfand, Mikhail M. Kapranov, Andrei V. Zelevinsky.

https://en.wikipedia.org/wiki/Dual_curve

A vivid animation: <https://www.youtube.com/watch?v=HTXpf4jDgYE>

Some pictures: https://www.ima.umn.edu/materials/2006-2007/W9.18-22.06/2203/Piene_190906.pdf

Goal.

1. Definition
2. Basic properties
 - Reflexivity theorem
 - dimension and defect
 - d, g, b, f, δ, k
3. Basic examples

Let $K = \bar{K}$ be a field, V a v.s. of $\dim n+1$.

1. Definition

Def (Dual variety)

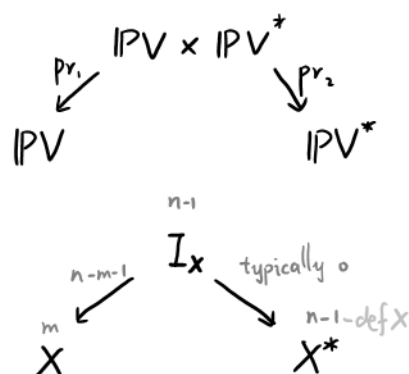
Let $X \subset \mathbb{P}V$: irr proj variety

X_{sm} : smooth locus

$I_X^\circ := \{(z, H) \mid z \in X_{sm}, H \in \mathbb{P}V^*, T_z X \subset H\}$

$I_X := \overline{I_X^\circ}$

Then $X^* := \text{pr}_2(I_X)$ is called the dual variety of X .



$$\mathbb{P}V^* = \mathbb{P}(V^*)$$

$$\dim V = n+1$$

$$\dim X = m$$

$$\text{def } X = \text{codim}_{\mathbb{P}V^*} X^* - 1$$

Relation with symplectic geometry

Def (Lagrangian construction)

Let M be a sm proj irr variety, $Y \subset M$ be any irr subvariety.

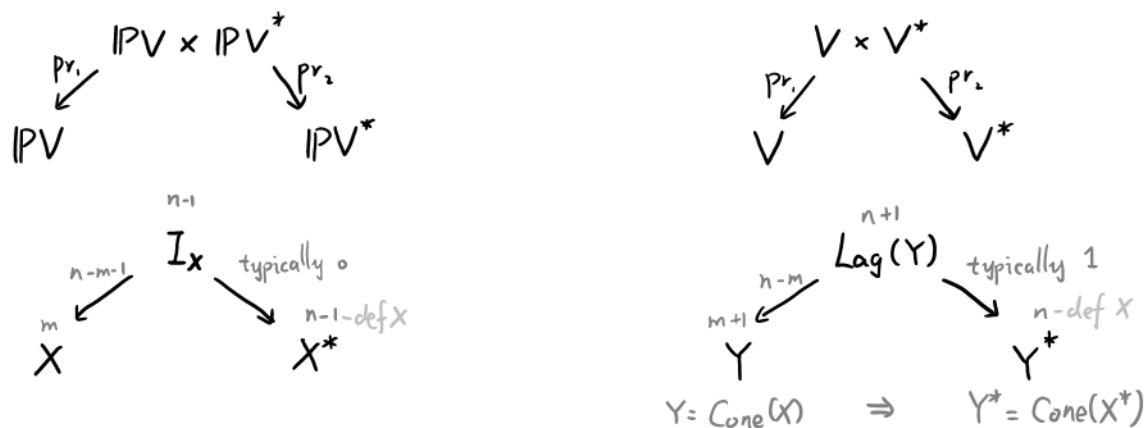
We define

$$\text{Lag}(Y) := \overline{N_{Y, \text{sm}}^* M} \quad (\text{closure in } T^*M)$$

Def. Any set $S \subset T^*M$ is called conical if S is closed under scalar multiplication.

Rmk. [Thm 1.9] $\text{Lag}(Y)$ is a conical Lagrangian subvariety,
and every conical Lagrangian subvariety S is of this form, i.e.
 $S = \text{Lag}(\pi(S))$ $\pi: T^*M \rightarrow M$

Rmk. $\text{Lag}(Y)$ is an analog of I_X , see the following picture:



2. Basic properties

2.1. Thm (Reflexivity thm) $X^{**} = X$

Sketch of proof.

$$\begin{aligned}
 & X \xrightarrow{\cong} X^{**} \\
 \Leftrightarrow & [(z, H) \in I_X^\circ \Leftrightarrow (H, z) \in I_{X^*}^\circ] \\
 \Leftrightarrow & I_X \cong I_{X^*} \quad \text{under the iso } \mathbb{P}V \times \mathbb{P}V^* \xrightarrow{\sim} \mathbb{P}V^* \times \mathbb{P}V^{**} \\
 \Leftrightarrow & \text{Lag}(Y) \cong \text{Lag}(Y^*) \quad \text{where } Y = \text{Cone}(X) \quad Y^* = \text{Cone}(X^*) \\
 & \text{under the iso } T^*V \cong V \times V^* \cong V^* \times V \cong T^*V^*
 \end{aligned}$$

Under this iso, $\text{Lag}(Y)$ is a conical Lagrangian subvariety of T^*V^* , so
 $\text{Lag}(Y) \cong \text{Lag}(\text{pr}_2(\text{Lag}(Y))) \cong \text{Lag}(Y^*)$

2.2. Dimension and defect