Eine Woche, ein Beispiel

1.23 Coxeter group

1. def & realizations
   - def
   - geometrical representation
   - root system
   - polytopes
   - as subgp of S_n
   - as Weyl gp of some Tits system
2. combinatorical results
3. Bruhat order
4. geometrical realization (faithfulness)

Roadmap

gen & relations $\longleftrightarrow$ characteristic properties $\rightarrow$ realizations.

wide river

Ref: [Building] Buildings, by Peter Abramenko and Kenneth S. Brown
[Bourbaki] (Elements of Mathematics) Nicolas Bourbaki - Lie Groups and Lie Algebras_ Chapters 4-9-Springer (2002)
[Comb] Combinatorics of Coxeter groups
[Flag] Flag Varieties An Interplay of Geometry, Combinatorics, and Representation Theory

In the first section, we omit technical details, which will be filled in later on. (Mainly: injectivity)

1. def & realizations

def

Def (Coxeter system) (W,S) gp + gen, $m(s,t) \in \mathbb{Z}_{>0} \cup \{+\infty\}$, $m(s,s)=1$

$$W = \langle s \in S \rangle / (s^2 = (st)^{m(s,t)} = 1,\ \forall s,t \in S)$$

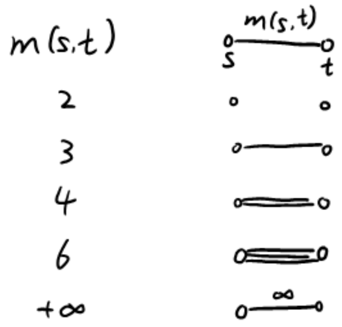
W is a Coxeter gp if $\exists\, S \subseteq W$, (W,S) is a Coxeter system.

E.g.

$$S_n \cong \langle s_i \rangle_i / (s_i^2 = (s_is_j)^2 = (s_i s_{i+1})^3 = 1)$$

$|i-j| \geq 2$, and undefined relations (eg. $(s_{n-1}s_n)^3$) should be removed.

Coxeter graph



Notation		
	S	simple reflections/transpositions
	$l(w) = \min\{r \mid w = s_1 \cdots s_r,\ s_i \in S\}$	length of $w \in W$
	$T = \{wsw^{-1} \mid w \in W,\ s \in S\}$	reflections/transpositions

geometrical representation $W \hookrightarrow GL(V_{geo})$

$\triangledown$ We suppose $|S| < \infty$, which is not necessary (but helpful for concentrating mind)

$$(W, S) \rightsquigarrow (\rho_{geo}, V_{geo}, \langle -,- \rangle) \in \mathrm{Rep}_{\mathbb{R},\mathrm{ortho}}(W)$$

$$V_{geo} := \bigoplus_{s \in S} \mathbb{R}\alpha_s$$

$$\langle -,- \rangle : V_{geo} \otimes V_{geo} \longrightarrow \mathbb{R}$$

$$(\alpha_s, \alpha_t) \longmapsto \begin{cases} -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq +\infty \\ -1 & m(s,t) = +\infty \end{cases}$$

$m(s,t)$	1	2	3	4	5	6	$\cdots$	∞
(α_s, α_t)	1	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{5}+1}{4}$	$-\frac{\sqrt{3}}{2}$	$\cdots$	-1

$$\rho_{geo} : W \longrightarrow GL(V_{geo})$$
$$s \longmapsto r_{\alpha_s}$$

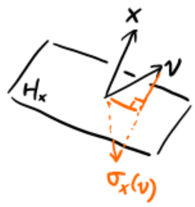
For $x, v \in V_{geo}$, $\langle x, x \rangle = 1$, define

$$r_x(v) = v - 2\langle v, x \rangle x$$

Check: $r_x(x) = -x$

$r_x(v) = v \iff v \in H_x$, where

$H_x = \{ v \in V_{geo} \mid \langle v, x \rangle = 0 \}$



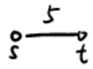
Ex Verify the well-definess.

- $\rho_{geo}(s)$ is linear & orthogonal;
- $\rho_{geo}(\text{relations}) = Id$

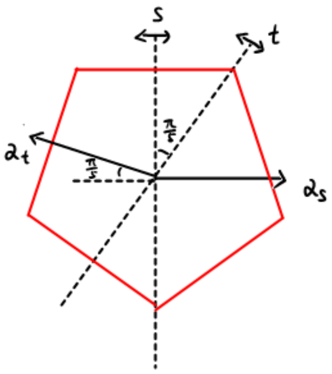
Also, $\langle wv, wv' \rangle = \langle v, v' \rangle$.

Thm. ρ_{geo} is faithful (sketch of proof later on)

E.g. $W = W(I_5)$

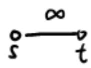


I_5

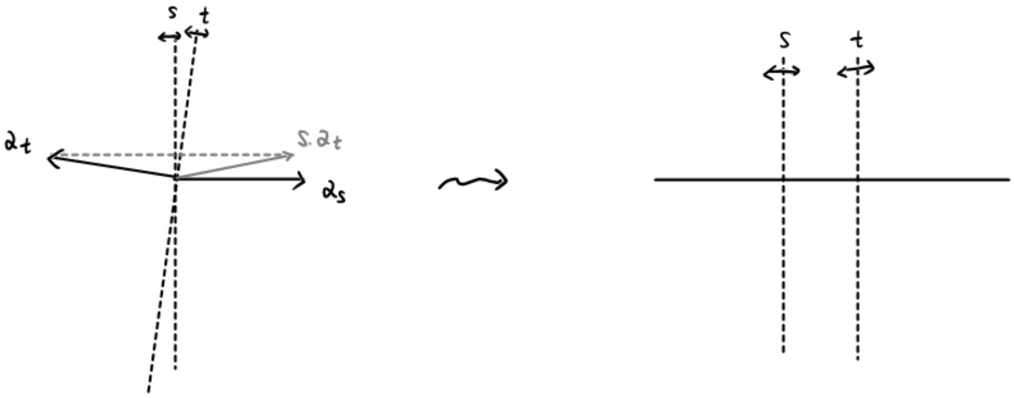


$\rho_{geo}(W) \cong D(5)$ Dihedral gp

$W = W(I_\infty)$



I_∞



$s.\alpha_t = \alpha_t + 2\alpha_s$

$s(\alpha_s, \alpha_t) = (\alpha_s, \alpha_t)\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$

$t(\alpha_s, \alpha_t) = (\alpha_s, \alpha_t)\begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$

$\rho_{geo}(W) \cong \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$

E.g.

H_4

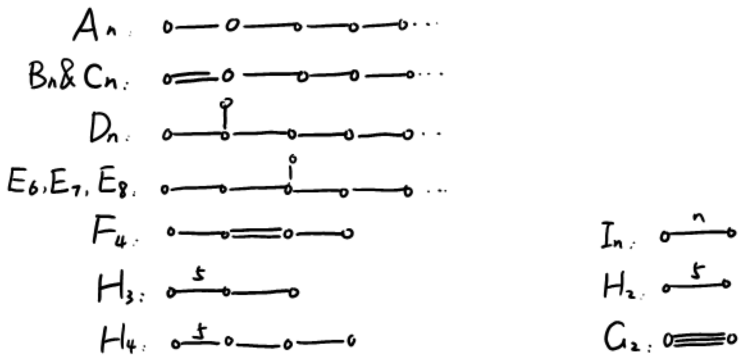
$$\begin{pmatrix} 1 & -\frac{\sqrt{5}+1}{4} & & \\ -\frac{\sqrt{5}+1}{4} & 1 & -\frac{1}{2} & \\ & -\frac{1}{2} & 1 & -\frac{1}{2} \\ & & -\frac{1}{2} & 1 \end{pmatrix}$$ is pos-def

Ex. $\# W < +\infty$

$\overset{?}{\Leftrightarrow}$ The bilinear form $\langle -, - \rangle$ is pos-def

$\Leftrightarrow$ Cartan matrix is pos-def

$\Leftrightarrow$ Coxeter graph is finite disjoint union of following shape.


A_n:
B_n & C_n:
D_n:
E_6, E_7, E_8:
F_4:
H_3:
5
H_4:
5
I_n:
n
H_2:
5
G_2:

Root system $\quad W \sim \mathrm{Aut}_R(V_{geo})$

⚠ Not the same as in Lie alg! E.g. here, every root has length 1.
That's why we don't use Φ here.

$$R = \{ v \in V_{geo} \mid v = w \cdot \alpha_s \text{ for some } w \in W, s \in S \}$$

⚠ $\mathcal{T} \xrightarrow{\rho_{geo}} \{ \sigma \in GL(V_{geo}) \mid \sigma = r_x \text{ for some } x \in V_{geo}, \langle x, x \rangle = 1, \sigma(R) = R \}$
can be not surj when the irr root system is not simply laced.
See 1084790 for more details.

⚠ Here, $W \ncong \mathrm{Aut}(R)$! See example on $W(I_5)$.

Ex. Verify the following properties.

(R1) R spans V_{geo}, does not contain 0

(R2) $-R = R$

(R3) $r_v R = R \quad \forall v \in R$

Define $\quad R^+ := \left(\sum_{s \in S} \mathbb{R}_{\geq 0} \alpha_s \right) \cap R \qquad R^- := \left(\sum_{s \in S} \mathbb{R}_{\leq 0} \alpha_s \right) \cap R$

one can check $R = R^+ \sqcup R^-$ by hand.

Lemma. $\quad r_{w \cdot \alpha_s} = \rho_{geo}(wsw^{-1}) \qquad w \in W, s \in S$

Proof.
$$\begin{aligned} r_{w \cdot \alpha_s}(x) &= x - 2\langle w \cdot \alpha_s, x \rangle w \cdot \alpha_s \\ &= w \cdot (w^{-1}x - 2\langle \alpha_s, w^{-1}x \rangle \alpha_s) \\ &= w \cdot \sigma_{\alpha_s}(w^{-1}x) \\ &= \rho_{geo}(wsw^{-1})\, x. \end{aligned}$$

Prop. We have bijection
$$\begin{aligned} R &\longrightarrow \mathcal{T} \times \{\pm 1\} \\ w \cdot \alpha_s &\longmapsto (wsw^{-1}, \eta(w;s)) \end{aligned}$$

$R^+ \longleftrightarrow \mathcal{T} \times \{+1\}$

$R^- \longleftrightarrow \mathcal{T} \times \{-1\}$

where $\quad \eta(s;t) = \begin{cases} -1 & s = t \\ 1 & s \neq t \end{cases} \qquad \eta(w'w;t) = \eta(w'; wtw^{-1})\, \eta(w;t)$

For the well-definess of η, we postpone to next section.

See https://math.stackexchange.com/questions/1084790/weyl-groups-correspondence-of-reflections-and-roots and [Building, Prop 1.113].

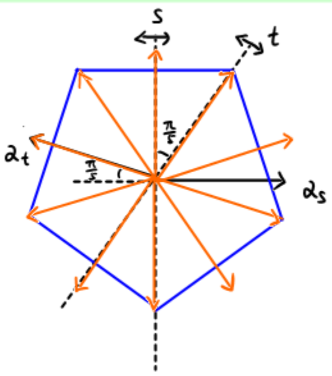
Polytopes $\quad W \cong \mathrm{Aut}(\text{Polytopes})$ $\quad$ (fundamental domain, chambers)

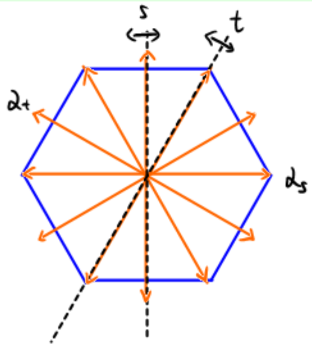
⚠ For Dynkin-Coxeter graph.

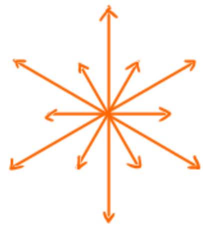
Others can be viewed as mosaic in spaces with constant curv ≤ 0.

It comes from the geo rep.

Ref:
https://syntopia.github.io/Polytopia/polytopes.html
https://www3.mpifr-bonn.mpg.de/staff/pfreire/polyhedra/index.html
https://www.mdpi.com/2073-8994/11/3/391/pdf?version=1552904082 (Some vague pictures of 5D polytopes)

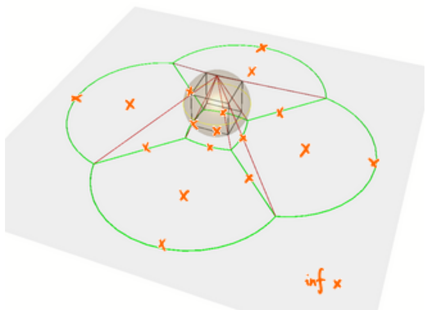






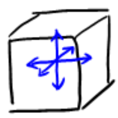
G_2

Ex. draw roots in , . (Bad picture for  !)



as subgp of S_n $\quad$ strand description

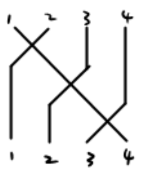
⚠ For type A~D, since they have "nice" shapes of polytopes.

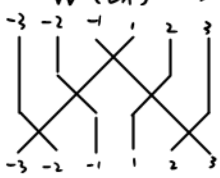
$W(A_3) \xrightarrow{\cong} S_4$ $\qquad W(B_3) \hookrightarrow S_6$ $\qquad W(D_3) \hookrightarrow S_6$

$W(A_n) \cong S_{n+1}$ $\qquad W(B_n) \hookrightarrow S_{2n}$ $\qquad W(D_n) \hookrightarrow S_{2n}$

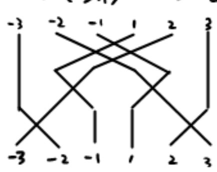
Standard



σ_1
σ_2
σ_3

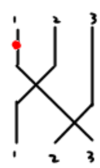


σ_0
σ_1
σ_2

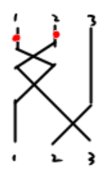


σ_0'
σ_1
σ_2

with dot



σ_0
σ_1
σ_2



σ_0'
σ_1
σ_2

as Weyl gp of some Tits system (later)

Ex. for the section

1. Verify the gen & rel in each case.
2. Describe element, reflection, simple reflection, length, roots, ... in each realization.

   e.g. how to see $|T| = l(w_0)$?

3. (Finite) group study:
   - $\#G$
   - conj class
   - $Z(G)$, $[G,G]$
   - char table (Rep theory)
   - simple?
   - subgp, quotient, central series, ...

4. Generalize everything to affine diagram.

   e.g. find a strand description of $\widehat{A}_n$.

2. combinatorical results

Lemma. For $(W,S)\in\mathrm{Cosgp}$, $\exists!$ gp homo
$$\mathrm{sgn}: W \longrightarrow \{\pm1\}$$
$$s \longmapsto -1$$
s.t. $\mathrm{sgn}(w) = (-1)^{l(w)}$ $\forall w\in W$

Cor. $\forall w\in W,\ s\in S,\quad l(ws)\equiv l(sw)\equiv l(w)+1 \mod 2$

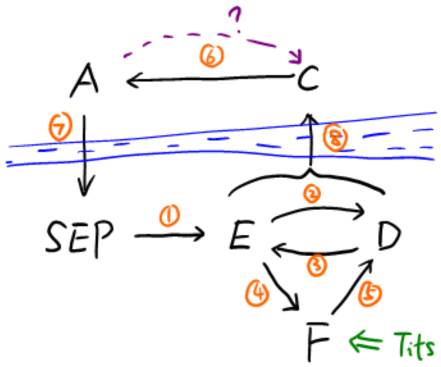
In ptc. $\quad l(ws)\neq l(w)$

Setting In this section, W is a gp, S is a set of gen of order 2.

Still,
$$l(w) := \min\{r \mid w = s_1\cdots s_r,\ s_i\in S\} \qquad \text{length of } w\in W$$
$$\mathcal{T} := \{wsw^{-1} \mid w\in W,\ s\in S\} \qquad \text{reflections / transpositions}$$

We have $l(w^{-1}) = l(w)$, but it is possible that $l(ws)=l(w)$ now.

Road map



A. Action [Building, p65]
C Coxeter
D DP = Deletion property
E EP = Exchange property
F Folding condition [Building, p79]

(Coxeter) (W,S) is a Coxeter system

(SEP) $w = s_1\ldots s_r,\quad s_i\in S,\quad t\in\mathcal{T},\quad l(tw)<l(w)$

$\Rightarrow \quad tw = s_1\cdots\hat{s}_i\cdots s_r \qquad \exists i$

(EP) $w = s_1\ldots s_r,\quad s_i\in S,\quad t\in S,\quad l(tw)<l(w)$

$\Rightarrow \quad tw = s_1\cdots\hat{s}_i\cdots s_r \qquad \exists i$

(DP) $w = s_1\cdots s_r \quad s_i\in S \qquad l(w)<r$

$\Rightarrow \quad w = s_1\cdots\hat{s}_i\cdots\hat{s}_j\cdots s_r \qquad \exists i,j$

(Folding) For $w\in W$ $s,t\in S$ s.t. $l(tw)=l(w)+1$, $l(ws)=l(w)+1$,

$\Rightarrow \quad l(tws)=l(w)+2$ or $tws=w$

(Action) $\exists \rho: W \circlearrowright T\times\{\pm1\}$ s.t. $\forall s\in S$,
$$\rho_s(t,\varepsilon) = \begin{cases} (s,-\varepsilon) & s=t \\ (sts,\ \varepsilon) & s\neq t \end{cases}$$

In ptc, $\rho_w(t,\varepsilon) = (wtw^{-1},\eta(w;t)\varepsilon)$ where
$$\eta(s;t) = \begin{cases} -1 & s=t \\ 1 & s\neq t \end{cases}$$
$$\eta(w'w;t) = \eta(w';wtw^{-1})\,\eta(w;t)$$

Def. (Reduced expression)

$\omega = s_1 \cdots s_r$ is reduced, if $l(\omega) = r$.

① Obvious

② Choose i maximal s.t. $s_i \cdots s_r$ is not reduced.

$$\Longrightarrow l(s_i s_{i+1} \cdots s_r) < l(s_{i+1} \cdots s_r)$$

$$\overset{(EP)}{\Longrightarrow} \quad s_i \cdots s_r = s_{i+1} \cdots \hat{s}_j \cdots s_r$$

$$\Longrightarrow \quad s_1 \cdots s_r = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$$

③ $l(tw) < l(w) \leq r$

$$\overset{(DP)}{\Longrightarrow} \quad tw = t s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r \quad \text{or} \quad s_1 \cdots \hat{s}_i \cdots s_r$$

$\Downarrow$ $l(w) \leq r-2$

Then use induction on r.

④ Take $w = s_1 \cdots s_r$. If $l(tws) \neq l(w) + 2$, then $l(tws) < l(ws)$

$$\overset{(EP)}{\Longrightarrow} \quad tws = s_1 \cdots \hat{s}_j \cdots s_r s \qquad \text{or} \qquad s_1 \cdots s_r$$

$\Downarrow$ $tw = s_1 \cdots \hat{s}_j \cdots s_r$ ↯ $\qquad$ $\Downarrow$ $tws = w$

⑤ By using induction on r, we can assume

$l(s_1 \cdots s_{r-1}) = l(s_2 \cdots s_r) = r-1$. Obviously $l(s_2 \cdots s_{r-1}) = r-2$.

Since $l(s_1 s_2 \cdots s_{r-1} \cdot s_r) \neq l(s_2 \cdots s_r) + 2$, $\quad s_1 \cdots s_r = s_2 \cdots s_{r-1}$

⑥ By direct calculation.

⑦ Lemma. $\eta(\omega; t) = -1 \iff l(\omega t) < l(\omega)$

$\eta(\omega; t) = 1 \iff l(\omega t) > l(\omega)$

[Proof. Suppose $l(\omega) = r$, $\omega = s_1 \cdots s_r$.

$$\eta(\omega; t) = -1 \overset{\text{def}}{\Longrightarrow} t = s_r \cdots s_j \cdots s_r \quad \exists j$$

$$\Longrightarrow \omega t = s_1 \cdots \hat{s}_j \cdots s_r$$

$$\Longrightarrow l(\omega t) < l(\omega)$$

$$\eta(\omega; t) = 1 \Longrightarrow \eta(\omega t; t) = \eta(\omega; t)\,\eta(t; t) = -1$$

$$\Longrightarrow l(\omega t \cdot t) < l(\omega t) \quad]$$

So $l(\omega t) < l(\omega) \Rightarrow \eta(\omega; t) = -1 \overset{\text{def}}{\Longrightarrow} \omega t = s_1 \cdots \hat{s}_j \cdots s_r \quad \exists j$.

⑧ For (W,S), define Coxeter gp (W_{cox}, S) s.t.

$$m(s,t) = \min\left(\left\{ r\in\mathbb{Z}_{\geq 0} \,\middle|\, (st)^r = Id \text{ in } W\right\} \cup \{+\infty\}\right)$$

then

$$W_{cox} = \langle s\in S\rangle \Big/ \left((st)^{m(s,t)} \quad s,t\in S\right)$$

and we have surj map

$$\Phi : W_{cox} \twoheadrightarrow W$$

Need: injectivity.

⚠ Later, all elements s_i are in W_{cox},

$$l(s_1\cdots s_r) = l(\Phi(s_1\cdots s_r)) \neq l_{cox}(s_1\cdots s_r)$$

Don't know yet

and also

$$\Phi((st)^r) = Id \Leftrightarrow r(s,t) \mid r \Leftrightarrow (st)^r = Id \qquad (\star)$$

Claim 1. $\forall r\in\mathbb{Z}_{\geq 0}$, if $\Phi(s_1\cdots s_{2r}) = Id$, then $s_1\cdots s_{2r} = Id$

i.e. if $\Phi(s_1\cdots s_r) = \Phi(s_1'\cdots s_r')$, then $s_1\cdots s_r = s_1'\cdots s_r'$

Proof of Claim 1.

Use induction on r. $r=0$ ✓

By induction, assume $s_1 \neq s_1'$

By (DP) & induction, assume $s_1\cdots s_r$, $s_1'\cdots s_r'$ are reduced

$$l(s_1's_1\cdots s_r) = l(s_2'\cdots s_r') = r-1 < r = l(s_1\cdots s_r)$$

$$\overset{(EP)}{\Longrightarrow} \Phi(s_1's_1\cdots s_r) = \Phi(s_1\cdots\hat{s}_j\cdots s_r) \qquad \exists j$$

$$\Longrightarrow \Phi(s_1's_1\cdots s_{j-1}) = \Phi(s_1\cdots s_j)$$

If we show $s_1's_1\cdots s_{j-1} = s_1\cdots s_j$, (For $j<r$, get by induction)

$$\Phi(s_1's_1\cdots\hat{s}_j\cdots s_r) = \Phi(s_1\cdots s_r) = \Phi(s_1'\cdots s_r')$$

$$\Longrightarrow \Phi(s_1\cdots\hat{s}_j\cdots s_r) = \Phi(s_2'\cdots s_r')$$

$$\overset{\text{induction}}{\Longrightarrow} s_1\cdots\hat{s}_j\cdots s_r = s_2'\cdots s_r'$$

$$\Longrightarrow s_1\cdots s_j\cdots s_r = s_1's_2'\cdots s_r'$$

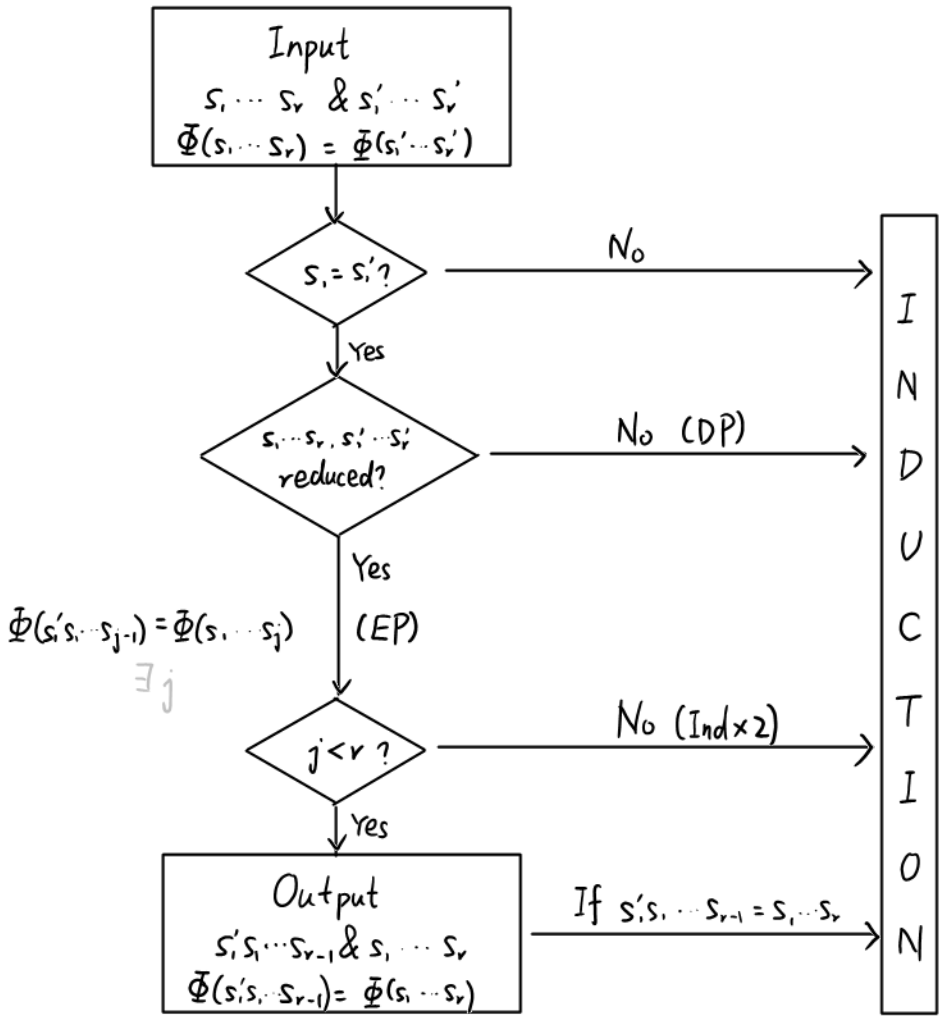
If $j=r$, we reduce the Claim 1 to Claim 2.

Claim 2. If $\Phi(s_1's_1\cdots s_{r-1}) = \Phi(s_1\cdots s_r)$, then $s_1's_1\cdots s_{r-1} = s_1\cdots s_r$.

Take this reduction repeatedly, we get reduced to

Claim 2. If $\Phi(s_1's_1s_1'\cdots) = \Phi(s_1s_1's_1\cdots)$, then $s_1's_1s_1'\cdots = s_1s_1's_1\cdots$

This is $(\star)$.



Rmk. The claim is stronger than we stated here.

By using the same method, we can show that

Prop. (Matsumoto) $\forall\, r \in \mathbb{Z}_{\geq 0}$, if

$\Phi(s_1 \cdots s_r) = \Phi(s_1' \cdots s_r')$, $\quad l(s_1 \cdots s_r) = r$. $\quad$ i.e. $\Phi(s_1) \cdots \Phi(s_r)$ is reduced

then $s_1 \cdots s_r \sim s_1' \cdots s_r'$ by braid relations.

Q. Try to show @ directly.

Possible idea: mimic the proof of faithfulness of geo rep.

Rmk. By (DP),

$w = s_1 \cdots s_r$ is reduced $\Leftrightarrow$ $w = s_1 \cdots s_r$ can't be shorter

i.e. $w = s_1 \cdots \hat{s}_{i_1} \cdots \hat{s}_{i_{r'}} \cdots s_r \Rightarrow r' = 0$