# Eine Woche, ein Beispiel

## 7.11 Universal properties

Here we conclude some processes of understanding.

obvious

① Commutative diagram
② Explicitly description
③ Initial object or Final object
④ Adjoint functor or Representable functors

non-trivial

⑤ Geometry
⑥ Functor & Derived functor
⑦ Other properties & Corollary

E.g tensor product    A: comm ring with unit, $M, N \in Mod(A)$

① 
$$\begin{tikzcd} M \times N \arrow[r, "A\text{-bilinear}"] \arrow[dr, "A\text{-bilinear}"'] & M \otimes_A N \in Mod(A) \arrow[d, dashed, "\exists!\ A\text{-mod morphism}"] \\ & T \in Mod(A) \end{tikzcd}$$

we have [illegible] s.t. for any [illegible], ∃! [illegible].

② $$M \otimes_A N = \bigoplus_{(m,n)\in M\times N} A \Big/ \sim$$

e.p when $A=k$ is a field, then $M\otimes_k N$ has a basis.

③ Initial object of
$\{\alpha : M \times N \longrightarrow T \mid T \in Mod(A),\ \alpha: A\text{-bilinear}\}$

④ $Mor_{A\text{-bil}}(M\times N, T) \cong Hom_{A\text{-mod}}(M\otimes_A N, T)$
$\Rightarrow Mor_{A\text{-bil}}(M\times N, -)$ is represented by $M\otimes_A N$.

⑤

$M, N \in Mod(A)$: coherent sheaf of rank $m$ $\otimes$ coherent sheaf of rank $n$ $\overset{\text{is}}{=}$ coherent sheaf of rank $mn$

e.p. $\otimes$ is the multiplication in $Pic(X)$

$N$: A-alg, $M \in Mod(A)$: $-\otimes_A N$ would be pull back of sheaf.

e.p. when $N = A_f$, $\pi^*\mathcal{F} = \mathcal{F}|_{Spec A_f}$

when $N = A/m$, $\pi^*\mathcal{F} = \mathcal{F}_m$

$M, N$: A-alg: Fiber product [Vakil Chap 9]

⑥ $-\otimes-$ is the (A-Mod, A-Mod)-bifunctor
$-\otimes_A N \dashv Hom_A(N, -)$
$\Rightarrow -\otimes_A N$ is right exact
$\Rightarrow -\otimes_A N$ has left derived functor $Tor_i^A(-, N)$

⑦ Flat module, ...

[2023.05.14] $\otimes$-category $\leftsquigarrow$ Hopf algebra $\rightsquigarrow$ group scheme.

e.g. **Kähler differentials**   $k$: comm ring with unit 1   $A$: $k$-alg

① 
$$\begin{tikzcd} A \arrow[r, "k\text{-derivation}"] \arrow[dr, "k\text{-derivation}"'] & \Omega_{A/k} \in Mor(A) \arrow[d, dashed, "\exists!\, A\text{-mod morphism}"] \\ & M \in Mor(A) \end{tikzcd}$$

Hochschild cohomology

where $I := \ker(A \otimes_k A \twoheadrightarrow A)$

② $\Omega_{A/k} = \bigoplus_{a \in A} A\,da / \sim \;\cong I/I^2 \cong I \otimes_{A\otimes A} A$   $\Rightarrow \Omega_{X/k} = \Delta^*(I)$

Proof. If $A = k[x_i]/(f_j)$, then

$A \otimes_k A = k[x_i, y_i] / (f_j(X), f_j(Y))$
$= k[x_i, \Delta_i] / (f_j(X), f_j(X+\Delta))$   $\Delta_i := y_i - x_i$
$\cong A[\Delta_i] / (f_j(X+\Delta))$

$I = \langle \Delta_i \rangle$   $I^2 = \langle \Delta_i \Delta_{i'} \rangle_{i,i'}$

$A \otimes_k A / I^2 \cong A[\Delta_i] / (\sum_i \frac{\partial f_j}{\partial x_i} \Delta_i)_j$

$I/I^2 \cong \langle \Delta_i \rangle / (\sum_i \frac{\partial f_j}{\partial x_i} \Delta_i) \cong \Omega_{A/k}$

③ $A \longrightarrow \Omega_{A/k}$ is the initial object of category $\{ d: A \longrightarrow M \text{ is the derivation} \}$

④ $Der_k(A, M) = Mor_{A\text{-mod}}(\Omega_{A/k}, M)$

$\Rightarrow Der_k(A, -)$ is represented by $\Omega_{A/k}$.  $\Rightarrow$ $Der_k(A,A)$ can be viewed as tangent bundle

⑤ $\Omega_{A/k}$ is a sheaf on $Spec\, A$. It represents the cotangent bundle of $Spec\, A$.

⑥ $A \leftarrow B \leftarrow k$

$\rightsquigarrow i^* \Omega_{B/k} \longrightarrow \Omega_{A/k} \longrightarrow \Omega_{A/B} \longrightarrow 0$   "contravarient functor $\Omega_{-/k}$"

⑦ Let $k$: comm alg with unit   $S, S', R$: $k$-alg
$M$: $S$-mod, $M^2 = 0$ (in $S'$)

<u>Prop 1</u>
$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & S' \arrow[r] & S \arrow[r] & 0 \\ & & & R \arrow[u] \arrow[ul, dashed, "\alpha"] \end{tikzcd}$$

$$\begin{tikzcd} Spec\, S' \arrow[dr, dashed, "\alpha"] \arrow[d, bend right, gray] & \\ Spec\, S \arrow[u, hook] \arrow[r] \arrow[d, gray] & Spec\, R \arrow[dl, gray] \\ Spec\, k & \end{tikzcd}$$

$$\begin{tikzcd} Spec\, S \arrow[r] \arrow[d] & Spec\, R \arrow[d] \\ Spec\, S' \arrow[r] \arrow[ur, dashed, "\alpha"] & Spec\, k \end{tikzcd}$$

then the lift $\alpha$ is $\phi$ or a torsor under $Der_k(R, M) = Mor_{R\text{-mod}}(\Omega_{R/k}, M)$

<u>Prop 2</u> set $S' = S[M]$, $\varphi \in Hom_{k\text{-alg}}(R, S)$, then

$Hom_{S\text{-alg}}(R, S[M]) = Der_k(R, M_\varphi)$

e.p. let $S = R$, then

$Hom_{R\text{-alg}}(R, R[M]) = Der_k(R, M)$

$$\begin{tikzcd} Spec\, R \arrow[r, equal] \arrow[d] & Spec\, R \arrow[d] \\ Spec\, R[M] \arrow[r] \arrow[ur, dashed, "\alpha"] & Spec\, k \end{tikzcd}$$

<u>Cor</u> $A \downarrow k$ formally unramified $\underset{⑦\, Prop\, 1}{\overset{⑦\, Prop\, 2}{\rightleftarrows}}$ $Der_k(A, -) \equiv 0$ $\overset{④}{\Longleftrightarrow}$ $\Omega_{A/k} = 0$ $\underset{③}{\overset{② f: l.f.t}{\rightleftarrows}}$ $\Delta$ is open immersion

$k$ is field

$A \downarrow k$ étale