Eine Woche, ein Beispiel

8.29. affine paving of quiver flag variety

Here is some personal reflection of the articles:
https://arxiv.org/abs/1804.07736
https://arxiv.org/abs/1909.04907
Now this document is no longer useful. The results have been collected in
arxiv:https://arxiv.org/pdf/2206.00444.pdf

Plan:

① affine paving

	Grassmannian	partial flag variety	strict partial variety
D_4	✓	✓	✓
D_5	✓	✓	✓
D_6	✓	✓	✓
E_6	✓	✓	✓
E_7	✓	✓	✓
E_8	✓	✓	✓

② smooth problem, dimension problem
③ explicit expressions
④ closure & intersection theories, Hasse diagram

The induction process of Grassmannian (affine paving + cellular dec)

M': indecomposable $\qquad\qquad \operatorname{supp}\tilde{\underline{f}} < \operatorname{supp}\underline{f}$

$$\begin{tikzcd} Gr_{\underline{f}}(M) \arrow[r, "\text{weak dec}"] & Gr_{\underline{f}'}(M') \arrow[r, "\text{proj dec}"] \arrow[loop below, "\text{strong dec}"] & Gr_{\tilde{\underline{f}}}(\tilde{M}) \arrow[d, dashed] \\ & & Gr \text{ of smaller quiver} \end{tikzcd}$$

$M' = {}_{234\overset{3}{5}642}$

Remark. 1. This decomposition is not canonical
depend on: order of indecomposable modules; (weak)
choose of projective module. (proj)

2. The amount of calculations grows exponentially.
Doing case-by-case is nearly impossible!
fix a dynkin quiver, you have to choose the directions of arrows,
make the AR-quivers (for all the subquiver) and choose the dim vector.

3. If we can do these three steps for partial flag variety,
then we can get cellular decomposition of part flag var.

4. We know how to compute quotients, but we want to do it easier.

E.g. E_8

$$\begin{array}{ccccccccccccc} & & & & & & & & 8 & & & & \\ & & & & & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 3 & \to & 4 & \to & 5 & \leftarrow & 6 & \leftarrow & 7 \end{array}$$

$$\eta:\ 0 \to 1223\overset{1}{3}21 \longrightarrow 2345\overset{3}{6}42 \longrightarrow 1122\overset{2}{3}21 \longrightarrow 0$$

$$Gr_{\underline{1112\overset{1}{3}21}}(2345\overset{3}{6}42)$$

$$\Big\downarrow \text{strong}$$

$$Gr_{\underline{1112\overset{1}{3}21}}(1223\overset{1}{3}21) \times Gr_{\underline{0000\overset{0}{0}00}}(1122\overset{2}{3}21)$$

$$\amalg$$

$$\vdots$$

$$\amalg\ Gr_{\underline{0000\overset{0}{0}00}}(1223\overset{1}{3}21) \times Gr_{\underline{1112\overset{1}{3}21}}(1122\overset{2}{3}21)$$

fiber $\{*\}$

$\vdots$

$\mathbb{C}^{26}$

$$\begin{aligned} Gr_{\underline{1112\overset{1}{3}21}}(1223\overset{1}{3}21) &\overset{\text{proj}}{=\!=} Gr_{\underline{0001\overset{1}{2}21}}(0112\overset{1}{2}21) \\ &= Gr_{\underline{001\overset{1}{2}21}}(112\overset{1}{2}21) \\ &= Gr_{\underline{01\overset{1}{2}21}}(12\overset{1}{2}21) \\ &= Gr_{\underline{1\overset{1}{2}21}}(1\overset{1}{1}10 \oplus 1\overset{0}{1}11) \\ &= Gr_{\underline{12\overset{1}{2}1}}(01\overset{1}{1}1 \oplus 11\overset{0}{1}1) \\ &\overset{\substack{\text{lazy}\\ \text{proj}}}{=\!=} Gr_{\underline{01\overset{1}{1}1}}(01\overset{1}{1}1 \oplus 00\overset{0}{0}1) \\ &= Gr_{\underline{1\overset{1}{1}1}}(1\overset{1}{1}1 \oplus 0\overset{0}{0}1) \\ &\overset{\text{proj}}{=\!=} Gr_{\underline{0\overset{1}{0}1}}(0\overset{1}{0}1 \oplus 0\overset{0}{0}1) = \mathbb{P}^1 \end{aligned}$$

Another example: E_6

$$\begin{array}{ccccccccc} & & & & 6 & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 3 & \leftarrow & 4 & \leftarrow & 5 \end{array}$$

$$\eta:\ 0 \to 11\overset{1}{2}21 \longrightarrow 11\overset{1}{2}21 \oplus 12\overset{1}{2}11 \longrightarrow 12\overset{1}{2}11 \longrightarrow 0$$

$$Gr_{\underline{11\overset{1}{2}11}}(11\overset{1}{2}21 \oplus 12\overset{1}{2}11) \cong \mathbb{C}^{10} \sqcup \{*\}$$

$$\Big\downarrow \text{weak}$$

$$Gr_{\underline{11\overset{1}{2}11}}(11\overset{1}{2}21) \times Gr_{\underline{00\overset{0}{0}00}}(12\overset{1}{2}11)$$

$$\amalg$$

$$\vdots$$

$$\amalg\ Gr_{\underline{00\overset{0}{0}00}}(11\overset{1}{2}21) \times Gr_{\underline{11\overset{1}{2}11}}(12\overset{1}{2}11)$$

fiber $\{*\}$

$\vdots$

$\mathbb{C}^{10}$

Some unsuccessful tries:

1. For each $N \in Gr_f(M)$, by Krull-Remak-Schmidt Theorem,

$\exists$ indecomposable modules N_i & $t_i \in \mathbb{N}_{>0}$ s.t. $N \cong \bigoplus_i N_i^{\oplus t_i}$

$(\Rightarrow \sum_i t_i \underline{\dim} N_i = f)$

So it's natural to consider

$$Gr_{(N_1^{t_1},\dots,N_r^{t_r})}(M) = \{ N \leq M \mid N \cong \bigoplus_i N_i^{\oplus t_i} \}$$

and we have "explicit expression"

$$\boxed{Gr_{(N_1^{t_1},\dots,N_r^{t_r})}(M) = \left(Hom\left(\bigoplus_i N_i^{\oplus t_i}, M\right) - \{\text{not inj}\}\right) \Big/ Aut\left(\bigoplus_i N_i^{\oplus t_i}\right)}$$

$Hom(\bigoplus_i N_i^{\oplus t_i}, M)$:

$$Hom\left(\bigoplus_i N_i^{\oplus t_i}, M\right) \cong \bigoplus_i Hom(N_i, M)^{\oplus t_i}$$

where the basis of $Hom(N_i, M)$ can be read off from the AR-quiver

(though not easy! any technique for it?)

Injectivity: If $f: \bigoplus_i N_i^{\oplus t_i} \longrightarrow M$ is not inj, then $\ker f \neq 0$,

f factors through $\bigoplus_i N_i^{\oplus t_i} / \ker f \longrightarrow M$

$\therefore \{\text{not inj}\} \cong \{ Hom(\bigoplus_i N_i^{\oplus t_i}/T, M) \mid 0 < T \leq \bigoplus_i N_i^{\oplus t_i} \}$

Difficulty: It's doable (but not easy!) to compute the quotient (using SES)
You need to understand all submodules of $\bigoplus_i N_i^{\oplus t_i}$, (by induction...)
and there may be infinite many submodules!

$Aut(\bigoplus_i N_i^{\oplus t_i})$:

Lemma. $Aut(N_i^{\oplus t_i}) \cong GL_{t_i}(\mathbb{C})$

In general, we can imagine $Aut(\bigoplus_i N_i^{\oplus t_i})$ as "quasi upper triangler matrix".

Advantages:

① This decomposition is natural, and doesn't depend on our choices;

② It's easier to get dimensions of $GL_f(M)$ (If $\exists$ inj map) (inj map is open)
e.g. when $g: N \hookrightarrow M$ is a sectional morphism, then $GL_N(M) = \mathbb{CP}^{[N,M]-1}$

Conj: $\dim GL_N(M) = \begin{cases} -\infty & [N,M]=0 \\ [N,M]-1 & \text{otherwise} \end{cases}$ for N, M indecomposable

③ It can be easily generalized to (strict) partial flag variety.

③ It's possible: further decompositions according to the shape of quotients.

Disadvantages:

① It's not affine paving

② Computations are too ugly to write down.

③ Even though it's possible to "fill in the holes", relations among these pieces are still unclear.

2. proj. dec for partial flag variety does not work.

E.g. η: $0 \longrightarrow 11\overset{0}{1}00 \longrightarrow 12\overset{2}{3}21 \longrightarrow 01\overset{1}{1}10 \oplus 00\overset{1}{1}11 \longrightarrow 0$

$$\begin{array}{ccc}
11\overset{0}{1}00 & 12\overset{1}{3}21 & 01\overset{1}{2}21 \\
\cup & \cup & \cup \\
00\overset{0}{0}00 & 01\overset{1}{2}11 & 01\overset{1}{2}11 \\
\cup & \cup & \cup \\
00\overset{0}{0}00 & 01\overset{1}{2}10 & 01\overset{1}{2}10
\end{array}$$

$$\mathrm{Ext}^1(01\overset{1}{1}10 \oplus 00\overset{1}{1}11,\ 11\overset{0}{1}00) = \mathrm{Hom}(11\overset{0}{1}00,\ 11\overset{1}{2}11 \oplus 11\overset{0}{1}10) = \mathbb{C}^2$$

Idea: partial flag variety can be viewed as quiver Grassmannian.

e.g. $\left\{ \begin{array}{c} \mathbb{C}^7 \\ \cup \\ V_3 \\ \cup \\ V_1 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathbb{C}^7 \\ \uparrow \\ \mathbb{C}^7 \\ \text{IU} \\ V_3 \\ \uparrow \\ V_1 \end{array} \begin{array}{l} \\ \\ \\ \text{subrep in } \mathrm{Mod}(kQ_2) \\ Q_2: 1 \to 2 \\ \\ \\ \end{array} \right\}$
$$\begin{tikzcd} V_3 \arrow[r, hook] & \mathbb{C}^7 \\ V_1 \arrow[u] \arrow[r, hook] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{C}^7 \arrow[u, "Id"'] \end{tikzcd}$$

Let $Q: x \longrightarrow y \longleftarrow z \longrightarrow w$

Fix $X \in \mathrm{Mod}(kQ)$

$X: X_x \longrightarrow X_y \longleftarrow X_z \longrightarrow X_w$

$$\left\{ \begin{array}{c} X \\ \text{IU} \\ X_3 \\ \text{IU} \\ X_2 \\ \text{IU} \\ X_1 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \Phi(X): \begin{tikzcd} X_x \arrow[r] & X_y & X_z \arrow[l] \arrow[r] & X_w \\ X_x \arrow[r] \arrow[u] & X_y \arrow[u] & X_z \arrow[l] \arrow[r] \arrow[u] & X_w \arrow[u] \\ X_x \arrow[r] \arrow[u] & X_y \arrow[u] & X_z \arrow[l] \arrow[r] \arrow[u] & X_w \arrow[u] \end{tikzcd} \\ \text{IU} \quad \text{subrep in } \mathrm{Mod}(R) \\ \begin{tikzcd} X_{3x} \arrow[r] & X_{3y} & X_{3z} \arrow[l] \arrow[r] & X_{3w} \\ X_{2x} \arrow[r] \arrow[u] & X_{2y} \arrow[u] & X_{2z} \arrow[l] \arrow[r] \arrow[u] & X_{2w} \arrow[u] \\ X_{1x} \arrow[r] \arrow[u] & X_{1y} \arrow[u] & X_{1z} \arrow[l] \arrow[r] \arrow[u] & X_{1w} \arrow[u] \end{tikzcd} \end{array} \right\}$$

$$\left\{ \begin{array}{c} X \\ \text{IU} \\ X_3 \\ \text{IU} \\ X_2 \\ \text{IU} \\ X_1 \end{array} \;\middle|\; \forall X_{i+1} \subseteq X_i \right\} \longleftrightarrow \left\{ \begin{array}{c} \Phi(X): \begin{tikzcd} X_x & X_y & X_z & X_w \\ X_x \arrow[u] \arrow[ur] & X_y \arrow[u] & X_z \arrow[u] \arrow[ul] \arrow[ur] & X_w \arrow[u] \\ X_x \arrow[u] \arrow[ur] & X_y \arrow[u] & X_z \arrow[u] \arrow[ul] \arrow[ur] & X_w \arrow[u] \end{tikzcd} \\ \text{IU} \quad \text{subrep in } \mathrm{Mod}(R) \\ \begin{tikzcd} X_{3x} & X_{3y} & X_{3z} & X_{3w} \\ X_{2x} \arrow[u] \arrow[ur] & X_{2y} \arrow[u] & X_{2z} \arrow[u] \arrow[ul] \arrow[ur] & X_{2w} \arrow[u] \\ X_{1x} \arrow[u] \arrow[ur] & X_{1y} \arrow[u] & X_{1z} \arrow[u] \arrow[ul] \arrow[ur] & X_{1w} \arrow[u] \end{tikzcd} \end{array} \right\}$$

$$\boxed{\mathcal{F}l(X) = Gr(\Phi(X))}$$

Notation

Q: Dynkin quiver $\qquad\qquad$ $Q: \ \longrightarrow \ \longleftarrow \ \longrightarrow$

Q_d: new bigger quiver $\qquad\qquad$ Q_3:

$Q_{d,str}$: strict version $\qquad\qquad$ $Q_{3,str}$

We also denote Q_0, Q_1, Q_2 as the vertex, arrow, "virtual arrow" of quiver Q.
I hope this abuse of notation won't bother anybody.

$$R = KQ_d / \text{commutative squares} \quad \text{or} \quad KQ_{d,str} / \text{relations} \qquad d \geq 2$$

$X, Y, S, M, N \in Mod(KQ)$

We always have the SES

$$0 \longrightarrow X \longrightarrow Y \longrightarrow S \longrightarrow 0$$

when X, Y, S emerge in the same time.

$V, U, W, T, T' \in Mod(R)$

We always have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi(X) & \longrightarrow & \Phi(Y) & \longrightarrow & \Phi(S) & \longrightarrow & 0 \\ & & \cong & & & & \cong & & \\ & & V & & & & W & & \end{array}$$

Usually U is the subrepresentation of $\Phi(Y)$ [connected to V & W]

$$0 \longrightarrow V \longrightarrow U \longrightarrow W \longrightarrow 0$$

~~Sometimes U have no special meaning (in Ext-vanishing lemma)~~

$[T, T']^i := \dim_K Ext^i_R(T, T')$ $\quad$ sometimes $[T, T']^i := Ext^i(T, T')$

e.p. $[T, T'] := [T, T']^0 = \dim Hom_R(T, T')$

$$\langle T, T' \rangle_R := \sum_{i=0}^{+\infty} (-1)^i [T, T']^i$$

$$\overset{gl.\dim R \leq 2}{=} [T, T'] - [T, T']^1 + [T, T']^2$$

$$\langle \underline{f}, \underline{g} \rangle_R := \sum_{i \in Q_0} f_i g_i - \sum_{b \in Q_1} f_{s(b)} g_{t(b)} + \sum_{c \in Q_2} f_{s(c)} g_{t(c)} \qquad \underline{f}, \underline{g}: \text{dim vector}$$

We will prove $\langle T, T' \rangle_R = \langle \underline{\dim}\, T, \underline{\dim}\, T' \rangle_R$ in next page.

Ext-vanishing results.

Lemma. (1) $\mathrm{gl.dim}\, R \leq 2$

(2) The functor $\Phi : \mathrm{Mod}(KQ) \longrightarrow \mathrm{Mod}(R)$ is exact

(3) $\mathrm{Hom}_{KQ}(M, N) \cong \mathrm{Hom}_R(\Phi(M), \Phi(N))$

$\mathrm{Ext}^i_{KQ}(M, N) \cong \mathrm{Ext}^i_R(\Phi(M), \Phi(N))$

(4) $\mathrm{proj.dim}\, \Phi(M) \leq 1$, $\mathrm{inj.dim}\, \Phi(N) \leq 1$

(5) $\Phi(X) \supseteq V$, $\Phi(S) \supseteq W$, T in $\mathrm{Mod}(R)$

then $\mathrm{Ext}^2_R(W, T) = 0$ $\quad \mathrm{Ext}^2_R(T, \Phi(X)/V) = 0$

Proof. (1). Just check minimal projective resolution of $S(i)$ in $\mathrm{Mod}(R)$.

Actually, we have a canonical resolution of T in $\mathrm{Mod}(R)$.

[Convince yourself: let $T = S(i)$] [Rep theory class, Thm 7.1]

$$0 \longrightarrow \bigoplus_{\substack{c \in Q_2 \\ = c_1 b_1 \\ = c_2 b_2}} Re_{t(c)} \otimes_k e_{s(c)} T \longrightarrow \bigoplus_{b \in Q_1} Re_{t(b)} \otimes_k e_{s(b)} T \longrightarrow \bigoplus_{i \in Q_0} Re_i \otimes_k e_i T \longrightarrow T \longrightarrow 0$$

$$r \otimes x \longmapsto rc_1 \otimes x + r \otimes b_1 x - rc_1 \otimes x - r \otimes b_2 x$$

$$r \otimes x \longmapsto rx$$

$$r \otimes x \longmapsto rb \otimes x - r \otimes bx$$

$$\begin{tikzcd} \bullet \arrow[r, "c_1"] & \bullet \\ \bullet \arrow[u, "b_1"] \arrow[r, "b_2"'] \arrow[ur, "c"] & \bullet \arrow[u, "c_2"'] \end{tikzcd}$$

by applying $[-, T']$, we get

$$\langle T, T' \rangle_R = \langle \underline{\dim} T, \underline{\dim} T' \rangle_R$$

(2-5). Trivial. leave as an exercise.

Goal: prove that $Gr(\Phi(X))$ has an affine paving.

Main tool:
$$\eta:\ \begin{tikzcd} 0 \arrow[r] & \Phi(X) \arrow[r] & \Phi(Y) \arrow[r] & \Phi(S) \arrow[r] & 0 \\ & V \arrow[u, phantom, "\subseteq" rotate=90] & U? & W \arrow[u, phantom, "\subseteq" rotate=90] & \end{tikzcd}$$

$$\boxed{\Psi:\ Gr(\Phi(Y)) \longrightarrow Gr(\Phi(X)) \times Gr(\Phi(S))}$$

(1) decide $Im\ \Psi$

case $\cancel{[S,X]'=0}$ $\eta=0$: surj (by (1a))

case $[S,X]'=1$: (1a) $(V,W)\in Im\Psi \iff Ext^1(\Phi(S),\Phi(X)) \to Ext^1(W,\Phi(X)/V)$, $\eta \mapsto 0$

($\eta\neq 0$ in $Ext^1(S,X)$)

(1b) $\iff [W,\Phi(X)/V]^1=0$

(1c) define $\tilde{X}_S$ & $\tilde{S}^X$

(1d) verify $\tilde{X}_S=\Phi(X_S)$, $\tilde{S}^X=\Phi(S^X)$

(1e) $\iff V\nsubseteq \Phi(X_S)$ or $W\nsupseteq\Phi(S^X)$

(1f) $Im\ \Psi = Gr_f(\Phi(X))\times Gr_g(\Phi(S)) - Gr_f(\Phi(X_S))\times Gr_{g-\dim\Phi(S^X)}(\Phi(S/S^X))$

(2) compute $\Psi^{-1}((V,W))$

(3) verify that Ψ is Zarisky-locally trivial affine bundle.

Proof in details.

(1a) [1, Lemma 21]

$(V,W)\in Im\ \Psi$ iff $Ext^1(\Phi(S),\Phi(X)) \longrightarrow Ext^1(W,\Phi(X)/V)$, $\eta \longmapsto 0$

Proof. by definition,

$$\begin{tikzcd} \eta \in Ext^1(\Phi(S),\Phi(X)) \arrow[d] \\ Ext^1(W,\Phi(X)) \arrow[d] \\ \bar{\eta}\in Ext^1(W,\Phi(X)/V) \end{tikzcd}
\qquad
\begin{tikzcd} 0 \arrow[r] & \Phi(X) \arrow[r] & \Phi(Y) \arrow[r, "\pi"] & \Phi(S) \arrow[r] & 0 \\ 0 \arrow[r] & \Phi(X) \arrow[r] \arrow[u, equal] \arrow[d] & \pi^{-1}(W) \arrow[r] \arrow[u] \arrow[d] & W \arrow[r] \arrow[u] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Phi(X)/V \arrow[r] & \pi^{-1}(W)/V \arrow[r] & W \arrow[r] & 0 \end{tikzcd}$$

$\bar{\eta}=0 \iff \exists\ U \subseteq \pi^{-1}(W)\subseteq \Phi(Y)$ s.t. $\pi(U)=W$, $U\cap\Phi(X)=V$ $\Rightarrow 0\to V\to U\to W\to 0$

$\iff (V,W)\in Im\ \Psi$.

(1b) $$0 \longrightarrow W \longrightarrow \Phi(S) \longrightarrow \Phi(S)/W \longrightarrow 0$$

$[-,\Phi(X)]^i$:

$$[W,\Phi(X)]' \leftarrow [\Phi(S),\Phi(X)]' \qquad [\Phi(S)/W,\Phi(X)]^2 = 0$$

$$0 \longrightarrow V \longrightarrow \Phi(X) \longrightarrow \Phi(X)/V \longrightarrow 0$$

$[W,-]^i$:

$$[W,\Phi(X)]' \longrightarrow [W,\Phi(X)/V]' \qquad [W,V]^2 = 0$$

$$\therefore \quad [\Phi(S),\Phi(X)]' \twoheadrightarrow [W,\Phi(X)]' \twoheadrightarrow [W,\Phi(X)/V]'$$

$$\therefore \quad \bar{\eta} = 0 \iff [W,\Phi(X)/V]' = 0$$

Rmk. $\boxed{[W,\Phi(X)/V]' = 0 \text{ or } 1.}$

(1c) Def. $\widetilde{X_S} := \max \{ V \subseteq \Phi(X) \mid [\Phi(S),\Phi(X)/V]' = 1 \} \subseteq \Phi(X)$

$\widetilde{S^X} := \min \{ W \subseteq \Phi(S) \mid [W,\Phi(X)]' = 1 \} \subseteq \Phi(S)$

$\widehat{X_S}$ & $\widetilde{S^X}$ are well-defined since

Lemma [1, Lemma 27]

(i) Let $V, V' \subset \Phi(X)$ s.t. $[\Phi(S),\Phi(X)/V]' = [\Phi(S),\Phi(X)/V']' = 1$,
then $[\Phi(S),\Phi(X)/V+V']' = 1$.

(ii) Let $W, W' \subset \Phi(S)$ s.t. $[W,\Phi(X)]' = [W',\Phi(X)]' = 1$,
then $[W \cap W',\Phi(X)]' = 1$

proof of lemma. we only prove (i). (ii) is similar.

$$0 \longrightarrow \Phi(X)/V\cap V' \longrightarrow \Phi(X)/V \oplus \Phi(X)/V' \longrightarrow \Phi(X)/V+V' \longrightarrow 0$$

$[\Phi(S),-]^i$:

$$[\Phi(S),\Phi(X)/V\cap V']' \longrightarrow [\Phi(S),\Phi(X)/V]' \oplus [\Phi(S),\Phi(X)/V']' \longrightarrow [\Phi(S),\Phi(X)/V+V']'$$
$$\leq 1 \qquad\qquad\qquad \leq 2 \qquad\qquad\qquad \leq 1$$

$$\Rightarrow [\Phi(S),\Phi(X)/V+V']' = 1$$

(1d) [Lemma 31. (1),(2)]

Let $f: X \to \tau S$ be a non-zero morphism; then $X_S = \ker(f)$

Also, $\Phi(f): \Phi(X) \to \Phi(\tau S)$ is a non-zero morphism; then $\widetilde{X_S} = \ker(\Phi(f))$

$= \Phi(\ker(f)) = \Phi(X_S)$

By similar argument, $\widetilde{S^X} = \Phi(S^X)$.

(1e) Claim: given $V \subseteq \Phi(X)$ and $W \subseteq \Phi(S)$,

$$[W, \Phi(X)/V]' = 0 \iff V \nsubseteq \Phi(X_S) \text{ or } W \nsupseteq \Phi(S^X)$$

$\Leftarrow$: wlog suppose $V \nsubseteq \Phi(X_S)$, then

$$V \nsubseteq \Phi(X_S) \iff [\Phi(S), \Phi(X)/V]' = 0$$
$$\Rightarrow [W, \Phi(X)/V]' = 0$$

$$\left[\begin{array}{l} 0 \longrightarrow W \longrightarrow \Phi(S) \longrightarrow \Phi(S)/W \longrightarrow 0 \\ [-, \Phi(X)/V]^i: \qquad [\Phi(S)/W, \Phi(X)/V]^2 = 0 \\ \qquad [W, \Phi(X)/V]' \leftarrow [\Phi(S), \Phi(X)/V]' \end{array}\right]$$

$\Rightarrow$: If not, then $V \subseteq \Phi(X_S)$ and $W \supseteq \Phi(S^X)$

then

$$[W, \Phi(X)/V]' \geqslant [\Phi(S^X), \Phi(X)/\Phi(X^S)]'$$
$$= [S^X, X/X^S]' \underset{[1, \text{Lemma } 31,(3)]}{=} 1 \qquad \text{contradiction!}$$

(1f) direct result from (1e).

(2) Recall that $(V, W) \in Im\, \Psi \iff [W, \Phi(X)/V]' = 0$.

In this case, $[W, \Phi(X)/V] = \langle W, \Phi(X)/V \rangle_R = \langle \underline{\dim}\, W, \underline{\dim}\, \Phi(X) - \underline{\dim}\, V \rangle_R$

we want to prove

$$\boxed{\Psi^{-1}((V, W)) \cong Hom_R(W, \Phi(X)/V)} \qquad \text{when } (V, W) \in Im\, \Psi$$

Recall the proof of (1a).

$$\begin{tikzcd} 0 \arrow[r] & \Phi(X) \arrow[r] & \Phi(Y) \arrow[r, "\pi"] & \Phi(S) \arrow[r] & 0 \\ 0 \arrow[r] & \Phi(X) \arrow[r] \arrow[u, equal] \arrow[d] & \pi^{-1}(W) \arrow[r] \arrow[u] \arrow[d] & W \arrow[r] \arrow[u] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Phi(X)/V \arrow[r, hook, "\iota"] & \pi^{-1}(W)/V \arrow[r, "\pi'"] & W \arrow[r] \arrow[l, bend left, "\theta"] & 0 \end{tikzcd}$$

$\eta \in Ext'(\Phi(S), \Phi(X))$

$\downarrow$

$Ext'(W, \Phi(X))$

$\downarrow$

$\bar{\eta} \in Ext'(W, \Phi(X)/V)$

when $(V,W) \in \mathrm{Im}\,\Psi$, $\bar{\eta}$ is split,

- each θ give us an element in $\Psi^{-1}((V,W))$.
- $\forall f \in \mathrm{Hom}_R(W, \Phi(X)/V)$, $\theta + \iota \circ f$ give us another element.
- If we have θ_1, θ_2 are split morphism, then
  $\pi'(\theta_2 - \theta_1) = 0 \quad \Rightarrow \theta_2 - \theta_1 \in \mathrm{Hom}_R(W, \Phi(X)/V)$
- finally, different θ give us different elements in $\Psi^{-1}((V,W))$.

$\therefore \quad \Psi^{-1}((V,W)) = \{\theta : \text{split morphism}\} \circlearrowleft \mathrm{Hom}_R(W, \Phi(X)/V)$

simply transitive.

(3) Just some not so interesting language in AG.

Now we state the central theorems:

$$\Psi : \mathrm{Gr}(\Phi(Y)) \longrightarrow \mathrm{Gr}(\Phi(X)) \times \mathrm{Gr}(\Phi(S))$$
$$\Psi_{\underline{f},\underline{g}} : \Psi^{-1}\left(\mathrm{Gr}_{\underline{f}}(\Phi(X)) \times \mathrm{Gr}_{\underline{g}}(\Phi(S))\right) \longrightarrow \mathrm{Gr}_{\underline{f}}(\Phi(X)) \times \mathrm{Gr}_{\underline{g}}(\Phi(S))$$

Thm. When $[S,X]' = 0$, Ψ is surjective.
Moreover, $\Psi_{\underline{f},\underline{g}}$ is a Zarisky-locally trivial affine bundle of rank $\langle \underline{g}, \underline{\dim}\,\Phi(X) - \underline{f} \rangle_R$.

So we reduce the problem to the case when X is indecomposable.

Thm. When $[S,X]' = 1$,

$$\mathrm{Im}\,\Psi_{\underline{f},\underline{g}} = \left(\mathrm{Gr}_{\underline{f}}(\Phi(X)) \times \mathrm{Gr}_{\underline{g}}(\Phi(S))\right) - \left(\mathrm{Gr}_{\underline{f}}(\Phi(X_S)) \times \mathrm{Gr}_{\underline{g} - \underline{\dim}\,\Phi(S^X)}(\Phi(S/S^X))\right)$$

Moreover, $\Psi_{\underline{f},\underline{g}}$ is a Zarisky-locally trivial affine bundle of rank $\langle \underline{g}, \underline{\dim}\,\Phi(X) - \underline{f} \rangle_R$ over $\mathrm{Im}\,\Psi_{\underline{f},\underline{g}}$.

We will use this theorem for more concrete examples where X_S & S/S^X are easy to describe.

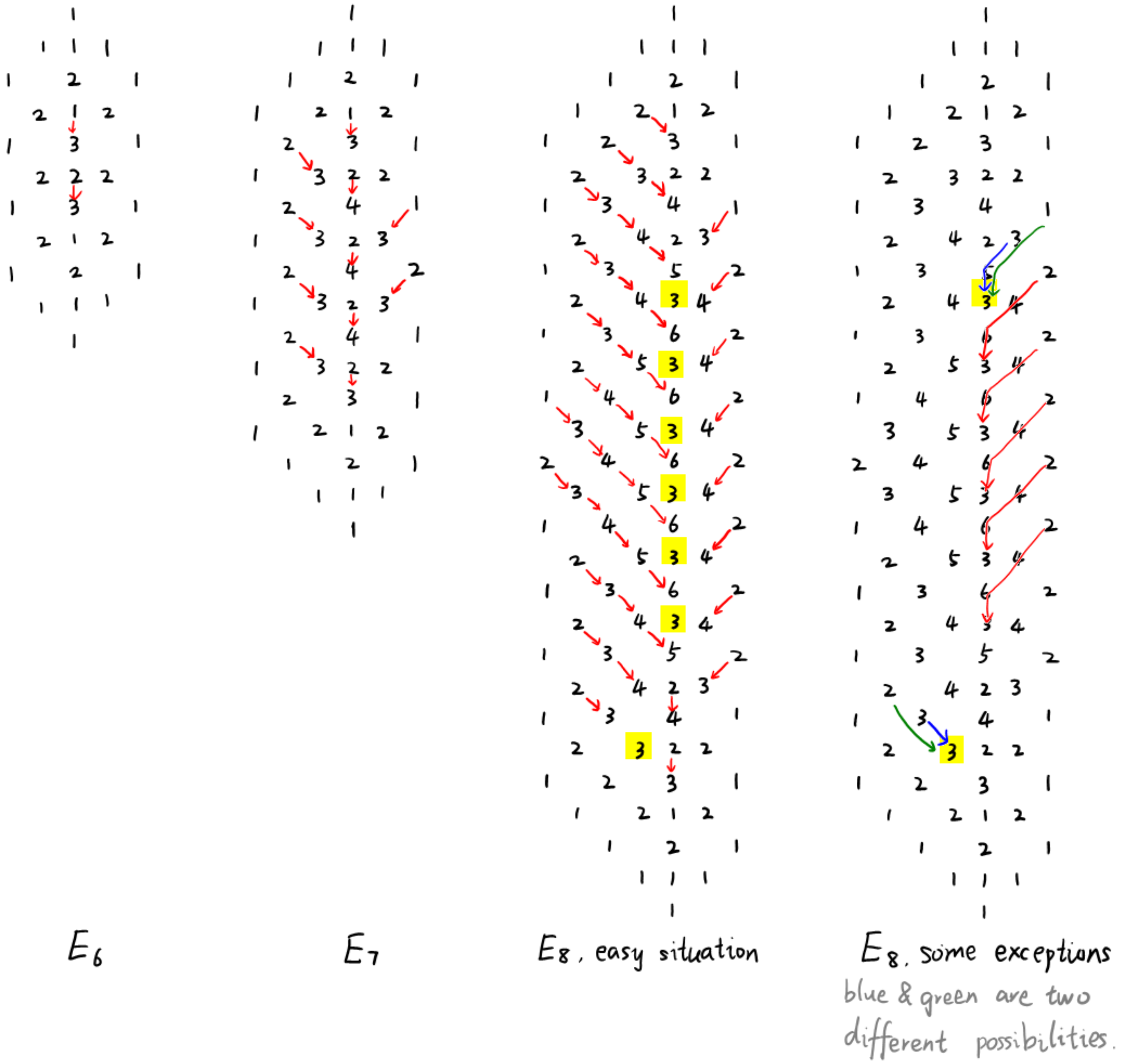
$Q \in \{E_6, E_7, E_8\}$

Let $M \in \mathrm{Mod}(kQ)$. denote $\mathrm{ord}(M) := \max_{i \in Q_0} \dim M_i$.

e.g. when $\underline{\dim} M = {}_{1\,2\,3\,4\,3\,1}^{\;\;\;\;\;\;2}$, then $\mathrm{ord}(M) = 4$

Fix Y to be indecomposable. When $\mathrm{ord}(Y) \le 2$, then $Gr(Y)$ has an affine paving. When $\mathrm{ord}(Y) > 2$, we use the following figure to give X:

$X \longrightarrow Y$



All the morphisms $X \longrightarrow Y$ we chose are examples of minimal sectional monos.

Def. given two indecomposable Q–representations X and Y, a sectional morphism $f : X \to Y$ is the composition

$$f : X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{t-1} \xrightarrow{f_t} X_t = Y \tag{12}$$

of irreducible morphisms f_i's such that $X_{i-2} \ncong \tau X_i$;

We notice that f is either mono or epi

We say that f is a *minimal sectional mono* if it is mono and each morphism $X_i \to Y$ is epi for $i = 1, 2, \cdots, t$.

Let $\quad 0 \to X \to Y \to S := Y/X \to 0 \quad$ where $X \hookrightarrow Y$ is min sec mono.

Lemma. We have the following values:

$[M,N]$ / $[M,N]'$ (row M, column N)	X	Y	S
X	1 / 0	1 / 0	0 / 0
Y	0 / 0	1 / 0	1 / 0
S	0 / 1	0 / 0	1 / 0

Proof. we know $\quad [X,X] = [Y,Y] = 1 \qquad [X,X]' = [Y,Y]' = 0$

by def of min sec mono, we get

$$[X,Y] = 1 \quad [Y,X] = 0 \qquad [X,Y]' = [Y,X]' = 0$$

then apply $[Y,-]$, $[-,S]$, $[X,-]$, $[-,X]$, $[-,Y]$ to $\eta : 0 \to X \to Y \to S \to 0$.

Cor. S is indecomposable & rigid;

$[S,X]' = 1$, so X_S & S^X are well-defined.

Claim: $S^X = S$

$$\begin{tikzcd}
\iota^*\eta & 0 \arrow[r] & X \arrow[r, hook] \arrow[d, equal] & Y' \arrow[r] \arrow[d, "\eta"] & S \arrow[r] \arrow[d, "\iota"] & 0 \\
\eta & 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

If $\quad Y' = Y_1' \oplus Y_2' \oplus \cdots Y_n'$, then $\quad X \to Y_i' \hookrightarrow Y \quad \Rightarrow Y_i' = X$ or $X \overset{\circ}{\to} Y_i'$ or $Y_i' = Y$

$\exists i \quad Y_i' = X$: $\iota^*\eta$ split

$\exists i \quad Y_i' = Y$: η iso $\Rightarrow$ ι iso

$\forall i \quad X \overset{\circ}{\to} Y_i'$: $X \overset{\circ}{\to} Y'$

$\}$ Contradiction!

Description of X_S

Let $E \longrightarrow X$ be the minimal right almost split morphism ending in X.
then $E = E' \oplus \tau X_1$

[1, Lemma 36]

Claim: when Y is not proj, $X_S \cong \ker(E \xrightarrow{\text{non-zero}} \tau Y) \cong E' \oplus \ker(\tau X_1 \longrightarrow \tau Y)$;
when Y is proj, $X_S \cong E$.

Proof. ① when X is not proj, $\mathrm{Ext}^1(X, \tau X)$ is generated by SES

$$\xi:\ 0 \longrightarrow \tau X \longrightarrow E \longrightarrow X \longrightarrow 0$$

Consider the pullback of $f: X \longrightarrow \tau S$.

$$\begin{tikzcd}
{?} \in \mathrm{Ext}^1(X,\tau X) \arrow[r, phantom, "\overset{\text{dual}}{\cong}"] & \mathrm{Hom}(X,X) \arrow[d, "f_* \in \mathrm{Hom}(X,\tau S)"] \\
0 \neq \tau\eta \in \mathrm{Ext}^1(\tau S,\tau X) \arrow[u, "f^*"] \arrow[r, phantom, "\overset{\text{dual}}{\cong}"] & \mathrm{Hom}(X,\tau S)
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & \tau X \arrow[r] \arrow[d, equal] & {??} \arrow[r] \arrow[d, "\iota"] & X \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \tau X \arrow[r] & \tau Y \arrow[r] & \tau S \arrow[r] & 0
\end{tikzcd}$$

$f_*: \mathrm{Hom}(X,X) \longrightarrow \mathrm{Hom}(X,\tau S)$ is isomorphism

$\Rightarrow f^*: \mathrm{Ext}^1(\tau S,\tau X) \longrightarrow \mathrm{Ext}^1(X,\tau X)$ is isomorphism

$\Rightarrow ? = t\xi \quad (t \in k^\times)$, $?? = E$

$\Rightarrow X_S = \ker f = \ker \iota = \ker(E \xrightarrow{\text{non-zero}} \tau Y) = E' \oplus \ker(\tau X_1 \to \tau Y)$

② when $X = P(k)$ is proj, then $E = \mathrm{rad}(X) = \bigoplus_{k \to j} P(j)$
& SES $0 \longrightarrow E \longrightarrow X \longrightarrow S(k) \longrightarrow 0$

②a when Y is not proj, we have comm diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, dashed, "\exists\iota"] & X \arrow[r] \arrow[d, "f"] & S(k) \arrow[r] \arrow[d, dashed, "\exists\ \text{inj}"] & 0 \\
0 \arrow[r] & \tau Y \arrow[r] & \tau S \arrow[r] & I(k) \arrow[r] & 0
\end{tikzcd}$$

$\nu^{-1}(X)$

$\Rightarrow X_S = \ker f = \ker \iota$

②b when $Y = P(j)$ is proj, we have comm diagram

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, dashed, "\exists\iota"] & X \arrow[r] \arrow[d, "f"] & S(k) \arrow[r] \arrow[d, dashed, "\exists\ \text{inj}"] & 0 \\
0 \arrow[r] & 0 \arrow[r] & \tau S \arrow[r, "\sim"] & I(k) \arrow[r] & 0
\end{tikzcd}$$

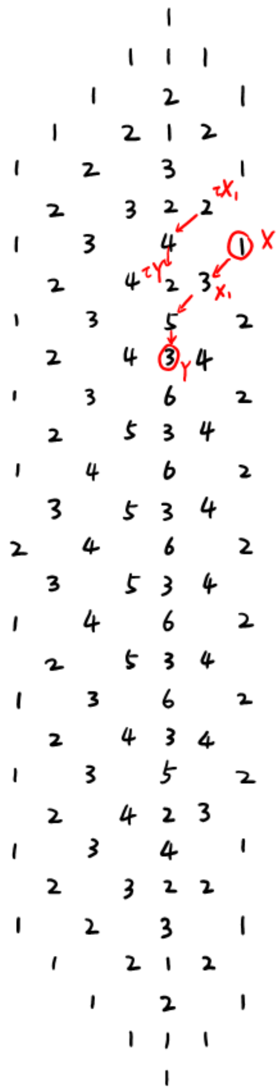
$\nu^{-1}(X)$

$\Rightarrow X_S = \ker f = \ker \iota$

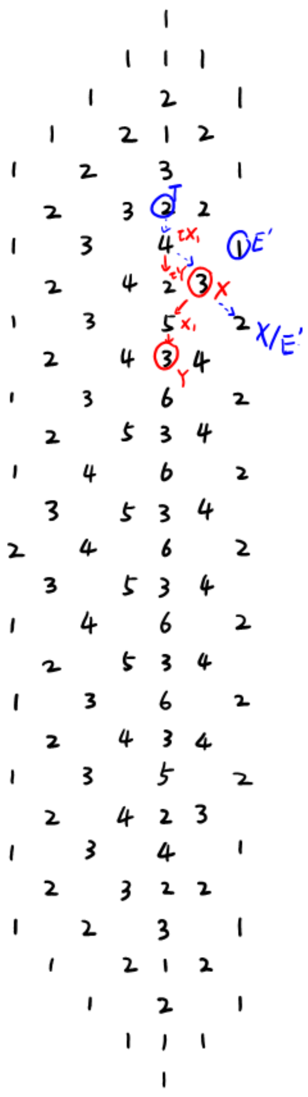
Now we can prove that $Gr(M)$ has an affine paving.

E.g.

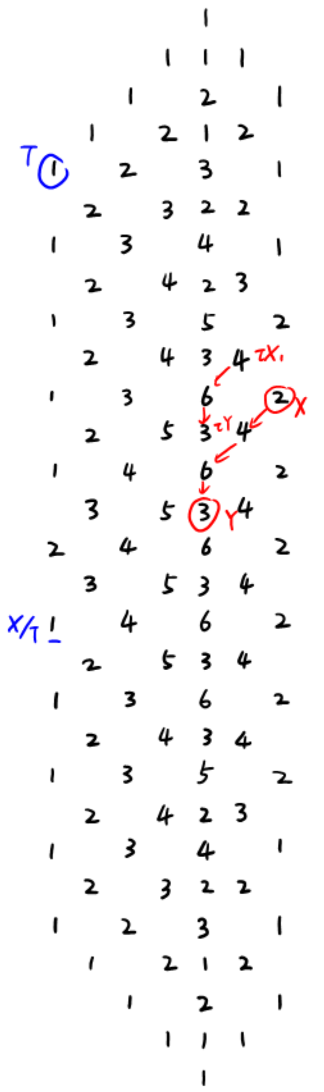
$$\begin{smallmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{smallmatrix}$$



$$\begin{smallmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{smallmatrix}$$



$$\begin{smallmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{smallmatrix}$$



$3 \to 3$ surj:

$$Gr(Y) \longrightarrow Gr(X) - Gr(X_S)$$
or
$$Gr(X) \times Gr(S)$$

$Gr(X) = 0$ or $\{*\}$. □

$3 \to 3$ inj:

$$Gr(Y) \to Gr(X) - Gr(T \oplus E')$$
or
$$Gr(X) \times Gr(S)$$

$$Gr(X) - Gr(T\oplus E') \to Gr(E') - Gr(\overset{0}{\overset{\shortparallel}{E'_{X/E'}}})$$
or
$$Gr(E') \times (Gr(X/E') - Gr(T))$$

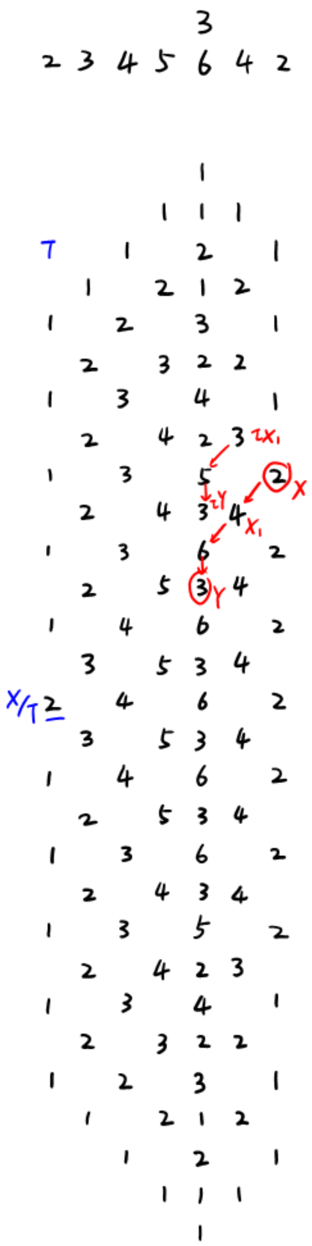
$$Gr(X/E') - Gr(T) \to Gr(T) - Gr(\overset{0}{\overset{\shortparallel}{T_{(X/E)/T}}})$$
or
$$Gr(T) \times \underbrace{Gr((X/E')/T)}_{= 0 \text{ or } \{*\}}$$

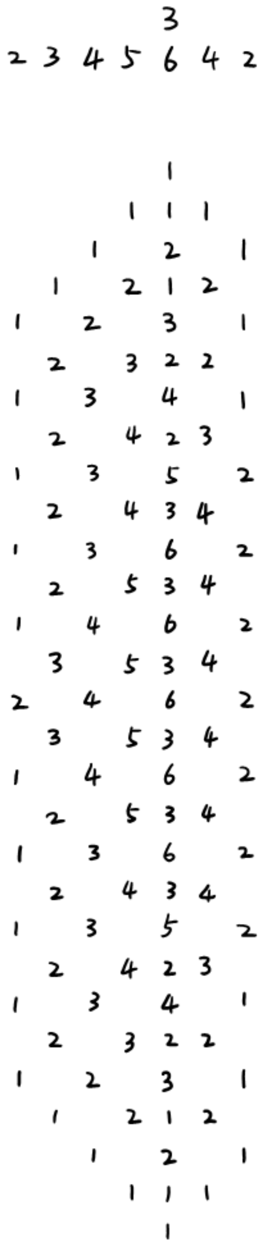
$$Gr(Y) \to Gr(X) - Gr(T)$$
or
$$Gr(X) \times Gr(S)$$

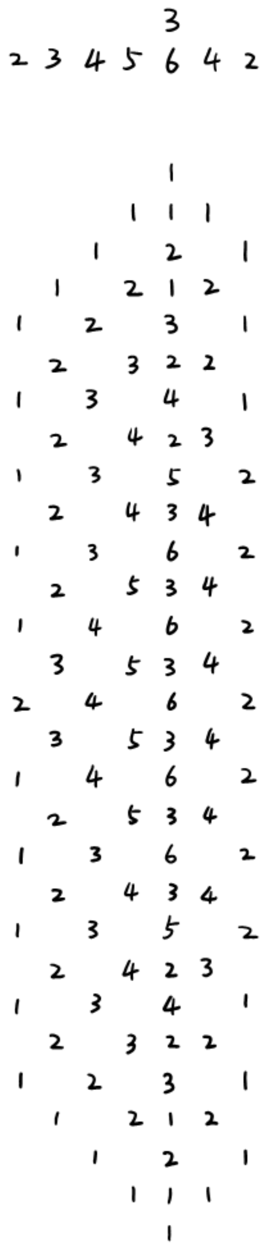
$$Gr(X) - Gr(T) \to Gr(T) - Gr(\cdots)$$
or
$$Gr(T) \times Gr(X/T)$$

$Gr(T) = 0$ or $\{*\}$. □

Try yourself!







$3 \to 3$ inj:

$$\mathrm{Gr}(Y) \to \mathrm{Gr}(X) \times \mathrm{Gr}(S)$$

$3 \to 3$ suj:

$$\mathrm{Gr}(Y) \to \mathrm{Gr}(X) - \mathrm{Gr}(T)$$

or

$$\mathrm{Gr}(X) \times \mathrm{Gr}(S)$$

$$\mathrm{Gr}(X) - \mathrm{Gr}(T) \to \mathrm{Gr}(T) - \mathrm{Gr}(\ldots)$$

or

$$\mathrm{Gr}(T) \times \mathrm{Gr}(X/T)$$

$\mathrm{Gr}(T)$: 0 or $\{*\}$ $\square$

Looking forward: by using similar argument, we may generalize to affine quiver if

- We understand quasi-finite rep better.

  e.g. Can we find indec modules X, S s.t

  $$0 \to X \to \text{quasi} \to S \to 0 \quad \text{exact?}$$

- We carefully examine [1, Section 6].

It's very possible that $\widetilde{A}$ & $\widetilde{D}$ type is true, since $\mathrm{ord}(S) \leq 2$.