

Eine Woche, ein Beispiel

2.4 exercise sheet for six functors

Ref. from [23.11.19]

Review. For X sm mfld, $f: X \rightarrow pt$. compute

$$\begin{array}{ll} Rf_* \mathcal{F} = & f^* \mathcal{Q} = \\ Rf_! \mathcal{F} = & f'_! \mathcal{Q} = \end{array}$$

$f_! \dashv f'$ \approx Poincaré duality
six functor formalism \approx cohomology theory in other categories

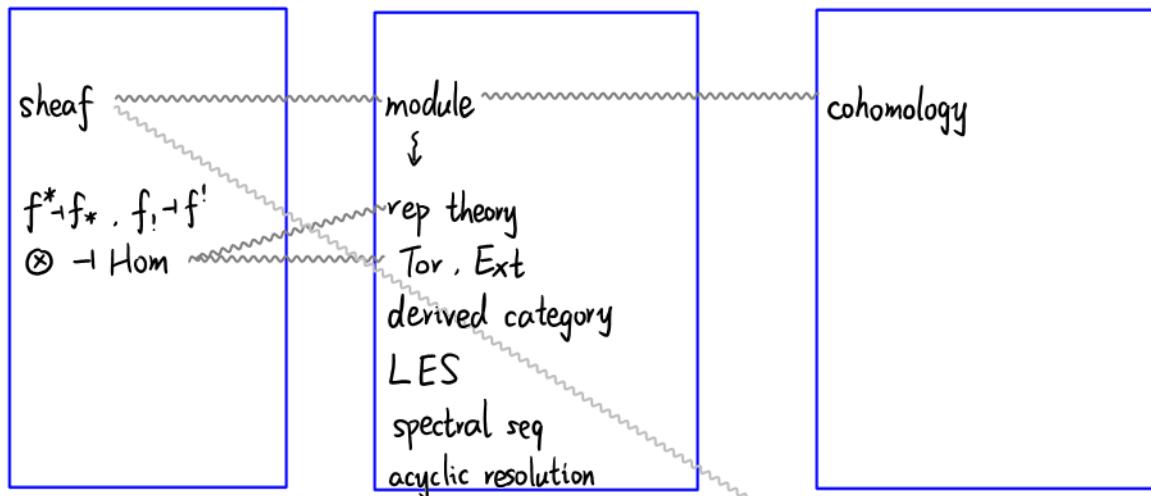
Proper base change formula:

$$f^* Rg_* \cong Rg'_* f'^*$$

$$\begin{array}{ccc} Y & \xrightarrow{f'} & X' \\ g'_! \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

$$g'^* Rf_* \cong Rf'_* g'^! \quad [\text{Hint: Yoneda + adjunction}]$$

$X \rightarrow Y$
between spaces over A
Today: six functor + homological algebra \Rightarrow algebraic topology

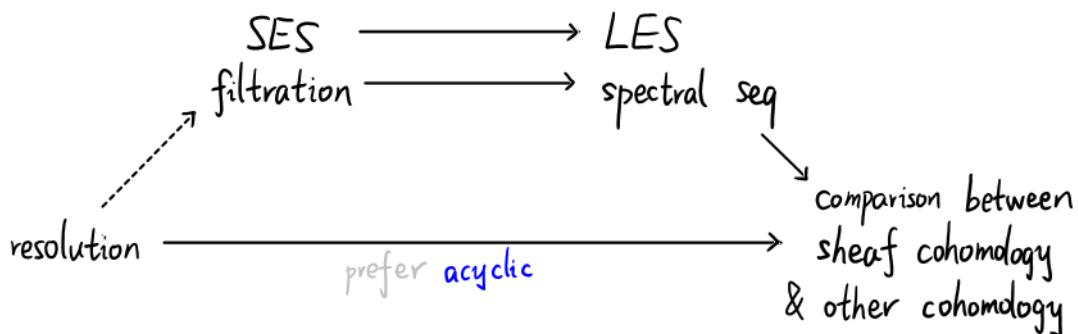


characteristic class
vector bundle
 $\pi_n(X, \cdot)$
spectra
cobordism

slogan:

SES induces LES,
filtration induces spectral sequence.

To expand a little bit,



Even though "filtration \Rightarrow spectral seq" is the most general statement, people start with "SES \Rightarrow LES" and "acyclic resolution \Rightarrow other coh \approx hyper coh".
Let us leave spectral seq in other people's notes.

1. open-closed formalism
2. open cover
3. filtrations from chain complex
4. filtration by $H^i(F)$
5. filtration by F
6. Hodge related filtration

We will assume the Poincaré lemma:

$$H^i(\mathbb{R}^n; \underline{\mathbb{Q}}) \cong \underline{\mathbb{Q}}$$

Since \mathbb{R}^n is contractible, you know how to prove it.

Methods to construct SES: $\left\{ \begin{array}{l} \text{check by stalks} \\ \text{filtration by } H^i(F) \\ \text{filtration by } F^i \end{array} \right.$

method	spectral seq	LES	cohomology/resolution
check by stalks	... for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived functor	MV	Čech open cover
	coefficient		closed cover
filtration by $H^i(F)$	Grothendieck		
	Leray-Serre	Cysin	Euler class Hodge-Tate
			de Rham
filtration by F^i need resolution to get "another" complex	Hodge-de Rham		Hodge-de Rham $H^q(X, \Omega^p) \Rightarrow H^{p+q}(X)$
	Frölicher		Dolbeault $H^q(X, \Omega^p) = H^{p,q}(X)$
			$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition" singular
spectral sequences which I don't know	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
	Green		
	:		
			for Koszul cohomology :

For more spectral sequences, see:

https://en.wikipedia.org/wiki/Spectral_sequence

<https://github.com/CubicBear/SpectralSequences/blob/main/SpectralSequences.pdf>

mixed Hodge theory = using two filtrations of $Rj_* \mathbb{Q}_U$ in the same time
to get two filtrations of $H^i(U; \mathbb{Q})$

1. open-closed formalism

// related: comparison of $j_!$ & j^*
one-point compactification.

Observe the following pictures. (called recollement diagram)

$$\begin{array}{ccccc}
 Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\
 & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\
 D(Z) & \xrightarrow[i_! = i_*]{i^!} & D(X) & \xrightarrow[j^* = j^!]{Rj_*} & D(U) \\
 & \xleftarrow{i^!} & & \xleftarrow{Rj_*} &
 \end{array}$$

Black box:

0. We assume some nice conditions.

e.g. in the category $\text{Haus}^{\text{loc. cpt}}$, and $Z \subset X$ is loc. contractible.

Under these conditions,

1. $i_* = i_!$, $j^* = j^!$ [Achar 21, 1.3.4]

2. $j_!$, i^* , j^* , i_* are exact.

Ex. 1. Shows that [Achar 21, Prop 1.3.9]

$$\underline{i^* i_*} = \underline{i^! i_*} = \text{Id}_{D(Z)} \quad \underline{j^* j_!} = \underline{j^* Rj_*} = \text{Id}_{D(U)}$$

$$\underline{i^* j_!} = 0, \quad \underline{j^* i_*} = 0, \quad \underline{i^! Rj_*} = 0$$

$$i^* Rj_* \cong i^! j_! [1] \quad \text{check this after (1) \& (2)}$$

— base change

~~~~ check stalkwise.

2. (for category fans)

$i_*$ ,  $j_*$ ,  $j_!$  are fully faithful, and

$i_*$ ,  $j^*$ ,  $j^*$  preserve injectives.

$i^!$ ,  $Rj_*$  are not defined for  
the abelian category

<https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints>

3. One has SES

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_! i^* F \longrightarrow 0 \quad (1)$$

You might prefer

$$0 \longrightarrow j_! j^! F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0$$

Well, writing every term by left adjoint factors is better for me.

Ex for (1).

1. Apply the  $R\pi_{X,*}$  to (1), take  $\mathcal{F} = \underline{\mathbb{Q}}_X$ , what do we get?

Define  $R\Gamma(X, \mathbb{Z}; \underline{\mathbb{Q}}) := R\Gamma(X; j_! \underline{\mathbb{Q}})$

In general, what do we get when applying  $R\pi_{X,*}$  &  $R\pi_{X,!}$ ?

Discuss 2 special cases  $\mathcal{F} = \underline{\mathbb{Q}}_X$   $|D_X| = \pi_X^! \underline{\mathbb{Q}}_{\mathbb{R}^{n+1}} = |D_X(\underline{\mathbb{Q}}_X)|$

2. Derive from (1) the triangle

$$j_! \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* i^* Rj_* \mathcal{F} \xrightarrow{+1}$$

which measures the difference between  $j_! \mathcal{F}$  &  $j_* \mathcal{F}$ .

3. Shows that (require:  $\{\infty\} \hookrightarrow \bar{X}$  is loc. contractable)

$$H_c(X) \cong H(\bar{X}, \{\infty\})$$

for one pt compactification  $i: X \hookrightarrow \bar{X}$ .

Try to compute  $H_c(\mathbb{R}^n)$  in this way.

4. [Achar 21, Ex 1.3.7]

Derive from (1) the triangle

$$\xleftarrow{+1} Rj_* \underline{R\text{Hom}}(j^* \mathcal{F}, j^! G) \leftarrow \underline{R\text{Hom}}(\mathcal{F}, G) \leftarrow i^* \underline{R\text{Hom}}(i^* \mathcal{F}, i^! G)$$

which decompose the morphisms by two parts.

It seems that we get only half of the results.

### Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathrm{ID}_X : \mathrm{D}^b(X; \mathbb{Q}) \longrightarrow \mathrm{D}^b(X; \mathbb{Q}) \quad \mathrm{ID}_X \mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{\mathrm{D}^b(X; \mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\mathbb{P}^1})$$

$$R\pi_{X*} \mathrm{ID}_X \mathcal{F}^\cdot = \underline{\mathrm{Hom}}_{\mathrm{D}^b(\mathbb{P}^1; \mathbb{Q})}(\pi_{X*} \mathcal{F}^\cdot, \mathbb{Q})$$

$$= H_c^*(X; \mathcal{F}^\cdot)^*$$

We know that

$$\mathrm{ID}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\mathbb{P}^1}$$

$$\mathrm{ID}_X(\mathcal{F}[n]) = (\mathrm{ID}_X \mathcal{F})[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \rightsquigarrow \mathrm{ID}\mathcal{H}^\cdot \longrightarrow \mathrm{ID}\mathcal{G}^\cdot \longrightarrow \mathrm{ID}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathrm{ID}_X = \mathrm{ID}_Y f^*$$

$$Rf_* \mathrm{ID}_Y = \mathrm{ID}_X Rf_!$$

"left adjoint can be pulled out"

$$f: Y \longrightarrow X$$

When  $\mathcal{F}^\cdot \in \mathrm{D}^b(X; \mathbb{Q})$  is constructible /  $\mathcal{F}$  is  $\pi_X$ -smooth, then

$$\mathrm{ID}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructible setting,

$$f^* \mathrm{ID}_X = \mathrm{ID}_Y f^*$$

$$Rf_* \mathrm{ID}_Y = \mathrm{ID}_X Rf_*$$

For exact statements about  $\mathrm{ID}_X$ , see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^* \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow Rj_* j^* \mathcal{F}^\cdot \xrightarrow{+1} \tag{2}$$

for  $\mathcal{F}^\cdot \in \mathrm{D}^b(X; \mathbb{Q})$  constructible.

Rmk. It seems that in [Achar 21, Thm 13.10], (2) is true without assuming  $\mathcal{F}$  is constructible. Do I miss something?

Ex for (2). Do the same arguments in "Ex for (1)", and verifies that  $i^* Rj_* \cong i^! j_![1]$ .

E.g. For a finite graph (as a topo space)  $X$ ,

$$\begin{array}{ccc} \vdots & \text{blue graph} & \vdots \\ \text{sk}_0 X & \xleftarrow{i} & X & \xleftarrow{j} & X - \text{sk}_0 X & \text{1-cells} \end{array}$$

$$0 \longrightarrow j_! j^* \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X - \text{sk}_0 X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{\text{sk}_0 X} \longrightarrow 0$$

Take  $R\pi_{X,!}$ :

$$\underbrace{j_* H_c(X - \text{sk}_0 X) \xrightarrow{\cong \oplus \mathbb{Q}} H_c(X) \longrightarrow H_c(\text{sk}_0 X) \xrightarrow{\cong 0} 0}_{0 \longrightarrow H_c(X - \text{sk}_0 X) \xrightarrow{\cong 0} H_c(X) \longrightarrow H_c(\text{sk}_0 X) \xrightarrow{\cong \oplus \mathbb{Q}}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that

$$H_c^i(\mathbb{R}) = \begin{cases} \mathbb{Q} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial / cellular cohomology, using the following stratification:

$$0 \subset \text{sk}^0 X \subset \text{sk}^1 X \subset \dots \subset \text{sk}^n X = X \quad \downarrow \text{filtration}$$

$$0 \subset i_* \underline{\mathbb{Q}}_{\text{sk}^0 X} \subset i_* \underline{\mathbb{Q}}_{\text{sk}^1 X} \subset \dots \subset \underline{\mathbb{Q}}_{\text{sk}^n X} = \underline{\mathbb{Q}}_X$$

Ex. derive the Wang LES for the cpt supp version over  $S^1$

## 2. open cover

Ex. For an open cover  $X = U_1 \cup U_2$ , deduce the SES

$$0 \leftarrow \underline{\mathbb{Q}}_X \xleftarrow{\quad} j_! \underline{\mathbb{Q}}_{U_1} \oplus j_! \underline{\mathbb{Q}}_{U_2} \xleftarrow{\quad} j_! \underline{\mathbb{Q}}_{U_1 \cap U_2} \leftarrow 0$$

$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_1} \oplus Rj_* \underline{\mathbb{Q}}_{U_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_1 \cap U_2} \xrightarrow{+1} 0 \quad (3)$$

▽ We omit the derived symbol and some subscripts in this section.  $U_{12} = U_1 \cap U_2$

(3) works for general sheaf

and, induce from (3) the MV sequence:

$$\begin{array}{ccccccc} \xleftarrow{+1} & H_c^k(X) & \xleftarrow{\quad} & H_c^k(U_1) \oplus H_c^k(U_2) & \xleftarrow{\quad} & H_c^k(U_1 \cap U_2) & \\ H^k(X) & \longrightarrow & H^k(U_1) \oplus H^k(U_2) & \longrightarrow & H^k(U_1 \cap U_2) & \xrightarrow{+1} & \end{array}$$

Hint: Apply  $R\pi_{X,!}$  &  $R\pi_{X,*}$ , see [StackProject, 01E9]

Ex. Derived the Wang LES over  $S^1$

Ex. For an open cover  $X = \bigcup_{i \in \Lambda} U_i$ ,  $\Lambda$  finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \leftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{U_i} \leftarrow \bigoplus_{i < j} j_! \underline{\mathbb{Q}}_{U_i \cap U_j} \leftarrow \cdots j_! \underline{\mathbb{Q}}_{\cap U_i} \leftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i} \longrightarrow \bigoplus_{i < j} Rj_* \underline{\mathbb{Q}}_{U_i \cap U_j} \longrightarrow \cdots Rj_* \underline{\mathbb{Q}}_{\cap U_i} \longrightarrow 0$$

When  $\{U_i\}_{i \in \Lambda}$  is a good cover,  $H^i(U_{i_1, \dots, i_k}) = H^0(U_{i_1, \dots, i_k})$ ,  
 $\uparrow$  acyclic in AG

one can compute  $H^i(X)$  by applying  $R\pi_{X,*}$ :

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigoplus_i \Gamma(U_i) & \xrightarrow{d^1} & \bigoplus_{i < j} \Gamma(U_i \cap U_j) & \xrightarrow{d^2} & \cdots \Gamma(\bigcap U_i) & \longrightarrow 0 \\ & H^0(X) & & H^1(X) & & \cdots & \\ & & & & \downarrow \text{Ker/Im} & & \\ & & & & & & H^{|\Lambda|-1}(X) \end{array}$$

Rmk. When  $X$  is paracompact & Hausdorff, "limited" Čech = sheaf  
 $\uparrow$  e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$F \longrightarrow \bigoplus_i Rj_* F|_{U_i} \qquad F \longrightarrow \bigoplus_{x \in X} F_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

$\# \Delta = 3$  case:

$$\begin{array}{ccccccc}
 & & \circ & & \circ & & \\
 & & \searrow & & \swarrow & & \\
 & & F_0 & & & & \\
 & \nearrow & & \searrow & & & \\
 0 \rightarrow & Q_X \rightarrow & \bigoplus_i Rj_* \mathbb{Q}_{U_i} & \xrightarrow{d'} & \bigoplus_i Rj_* \mathbb{Q}_{U_i \cap U_j} & \xrightarrow{d''} & Rj_* \mathbb{Q}_{\cap U_i} \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & F_1 & & F_2 & & \\
 & & \nearrow & & \searrow & & \\
 & & \circ & & \circ & & \\
 \end{array}$$

$$F_2 = Rj_* \mathbb{Q}_{\cap U_i} \Rightarrow H^*(F_2) = \ker d^3$$

$$\begin{array}{ccccc}
 \underbrace{H^*(F_1)}_{0 \rightarrow H^0(F_1) \rightarrow \bigoplus_{i<j} \Gamma(U_i \cap U_j) \xrightarrow{d''}} & \longrightarrow & 0 & \longrightarrow & H^*(F_2) \xrightarrow{+1} \\
 & & & & \curvearrowright H^0(F_2) \\
 & & & & 
 \end{array}$$

$$\Rightarrow H^i(F_1) = \begin{cases} \ker d^3 / \text{Im } d^2, & i=1 \\ \ker d^2, & i=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{ccccc}
 \underbrace{H^*(F_0)}_{0 \rightarrow H^0(F_0) \rightarrow \bigoplus_i \Gamma(U_i) \xrightarrow{d'}} & \longrightarrow & 0 & \longrightarrow & H^*(F_1) \xrightarrow{+1} \\
 & & & & \curvearrowright H^1(F_0) \\
 & & & & 
 \end{array}$$

$$\Rightarrow H^i(X) = H^i(F_0) = \begin{cases} \ker d^3 / \text{Im } d^2 & i=2 \\ \ker d^2 / \text{Im } d^1 & i=1 \\ \ker d^1 & i=0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When  $\{U_i\}_{i \in \Lambda}$  is not a good cover,  
one needs Čech-to-derived functor spectral seq to compute  $H^i(X)$ .

Rmk. Closed cover can be computed in a similar way.

Ex. For an closed cover  $X = Z_1 \cup Z_2$ , deduce the SES

$$0 \rightarrow \underline{\mathbb{Q}}_X \xrightarrow{i_1^*} i_1^*\underline{\mathbb{Q}}_{Z_1} \oplus i_2^*\underline{\mathbb{Q}}_{Z_2} \xrightarrow{i_2^*} i_2^*\underline{\mathbb{Q}}_{Z_2} \rightarrow 0$$

$$\xleftarrow{+!} \underline{\mathbb{Q}}_X \xleftarrow{i_* i^!} i_* i^! \underline{\mathbb{Q}}_X \oplus i_* i^! \underline{\mathbb{Q}}_X \xleftarrow{i_* i^!} i_* i^! \underline{\mathbb{Q}}_X \quad (3')$$

In general, one would get

$$0 \rightarrow F \xrightarrow{i_1^*} i_1^* F \oplus i_2^* F \xrightarrow{i_2^*} i_2^* F \rightarrow 0$$

$$\xleftarrow{+!} DF \xleftarrow{i_* i^!} i_* i^! DF \oplus i_* i^! DF \xleftarrow{i_* i^!} i_* i^! DF$$

$\downarrow$  different  $i$  here...

Ex. Use the closed cover, try to compute

$$H_{\text{BM}}^*(Z; \mathbb{Q}) \quad Z := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 z_3 = 0\} = \begin{array}{c} \text{a cube} \\ \text{with a hole} \end{array}$$

or  $Z = \{z_1, z_2, z_3 \mid (z_1 + z_2 + z_3) = 0\}$

In the first case,  $\mathbb{C}^3 - Z \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  You can use this fact to double check results.

closed cover of  $Z \approx$  "excessive cover" of  $\mathbb{C}^3 - Z$   
 $\neq$  open cover of  $\mathbb{C}^3 - Z$   
no relation

Rmk. stratification & open cover are two main tools to extract topological information.  
 They appear with different names in different fields.  
 Once you realize them, you can apply the six-functor machine to analyze structures.

stratification with extra properties

|   |                        |
|---|------------------------|
| { | CW cplx                |
|   | triangulization        |
|   | dessin d'enfant        |
|   | affine paving          |
|   | Whitney stratification |
|   | ⋮                      |

Q. How to construct stratifications?

A. For me, there are two efficient methods:

|   |                    |
|---|--------------------|
| { | orbit of gp action |
|   | Morse theory       |

That's why some geometrical problems are finally reduced to combinatorical / analytic problems.  
 Other fields come to geometry by providing stratifications.

In fact, there is only one method:

find a fct  $f: X \rightarrow Y$ , and get stratification of  $X$  from  $Y$ .  
 get better stratification by analyzing  $f$

|      |                        |                                                                 |
|------|------------------------|-----------------------------------------------------------------|
| E.g. | 1. Morse theory        | $f: X \rightarrow \mathbb{R}$                                   |
|      | 2. tessellation        | $f: H \rightarrow H/\Gamma$                                     |
|      | 3. semi-continuous fct | $f: X \rightarrow \mathbb{N}_{\geq 0}$ e.g. $f(p) = \dim T_p X$ |
|      | 4. my master thesis    | $f: Gr(X) \rightarrow Gr(S) \times Gr(X/S)$                     |
|      | 5. orbit of gp action  | $f: X \rightarrow [X/G] \rightarrow X/G$                        |

### 3. filtrations from chain complex [Stack Project, 0118]

Lots of filtrations are obtained just from the naive complex.

"chopping pork ribs"

Consider a chain complex  $C$ :

$$\dots \xrightarrow{d^{-2}} C^{-2} \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^0} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots$$

One can make a "stupid" truncation

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \end{array}$$

which is denoted by

$$0 \longrightarrow \sigma_{\geq 0} C \longrightarrow C \longrightarrow \sigma_{\leq -1} C \longrightarrow 0$$

One can also make a canonical truncation

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & \ker d' & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \text{coker } d^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \end{array}$$

which is denoted by

$$0 \longrightarrow \tau_{\leq 0} C \longrightarrow C \longrightarrow \tau_{\geq 1} C \longrightarrow 0$$

Using these truncations, one can easily construct filtrations:

$$0 \subset \cdots \subset \tau_{\geq 1} C \overset{C'[-1]}{\subset} \tau_{\geq 0} C \overset{C'}{\subset} \tau_{\geq -1} C \subset \cdots \subset C$$

$$0 \subset \cdots \subset \tau_{\leq -1} C \overset{H^0(C)}{\subset} \tau_{\leq 0} C \overset{H^0(C)[1]}{\subset} \tau_{\leq 1} C \subset \cdots \subset C$$

Rmk. 1. These two filtrations have opposite directions!

(a striking feature for me)

2. The "stupid" truncation extracts pieces of the chain cplx, while the canonical truncation extracts cohomology. ( $\text{Ker}/\text{Im}$ )  
Therefore, only the canonical truncation can be defined on the derived category.

This information is culmulated in the standard/natural t-structure  $(D_{\leq 0}, D_{\geq 0})$ .

One has adjoint factors:

$$\begin{array}{ccccc} & l_{\leq 0} & & T_{\geq 1} & \\ D_{\leq 0} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & D & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & D_{\geq 1} \\ \end{array}$$

where  $l_{\leq 0}, l_{\geq 1}$  are natural inclusions.

The following notations are from: <https://ncatlab.org/nlab/show/t-structure>

$D_{\leq 0}$ : t-co-connective objects

$D_{\geq 0}$ : t-connective objects

$\tau_{\geq 0}$ : connective cover

Let's apply these filtrations!

#### 4. filtration by $H^i(F)$

Assume that  $B$  is loc. contractable.  $\Leftarrow$  Can we remove this condition?

(+  $B$  simply connected to simplify the argument)

Ex. Suppose that  $\pi: E \rightarrow B$  is an oriented  $S^k$ -bundle.

Analyze  $R\pi_* \underline{\mathbb{Q}}_E$ , and apply  $R\pi_{B,*}$  to get the Gysin sequence.

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[+1]{ev_\pi \wedge}$$

Hint. consider the easy case ( $B = \{*\}$ ,  $E \cong S^k \times B$ ) to see the shape of  $R\pi_* \underline{\mathbb{Q}}_E$ .

Q. Why does  $\pi_*: D^b(E) \rightarrow D^b(B)$  takes injective objects to  $\pi_{B,*}$ -acyclic objects?

A. [Vo02, p195] By definition,  $\pi_*$  takes flasque sheaves to flasque sheaves.

Similarly,  $\pi_!$  takes c-soft sheaves to c-soft sheaves.

Rmk. 1. Here we can't use the "stupid" truncation,

because  $R\pi_* \underline{\mathbb{Q}}_E$  lies in the derived category.

2. You can generalize it to fiber bundle. then

you will get the Leray-Serre spectral sequence.

Think how the following conditions simplify the final results.

①  $\pi$  is oriented  $S^k$ -bundle

②  $B$  is simply-connected

③ (Leray-Hirsch)  $H^*(E; \mathbb{Q}) \rightarrow H^*(F; \mathbb{Q})$  is surjection

④  $\pi_!(B)$  acts on  $H^*(F)$  trivially.

3. This is also a special case of Grothendieck-Serre spectral sequence.

▽  $R^i \pi_* \underline{\mathbb{Q}}_E = H^i(R\pi_* \underline{\mathbb{Q}}_E)$

$\neq$  the  $i$ -th term in the "chain cplx"  $R\pi_* \underline{\mathbb{Q}}_E$  (undefined)

## 5. filtration by $\mathcal{F}$

For a single sheaf  $\mathcal{F}$ , we have no way to produce filtrations.

Since we only care about  $H^i(\mathcal{F})$ , we may replace  $\mathcal{F}$  by another complex  $C^\bullet$ , which is usually achieved by resolution:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots \quad \text{exact}$$

i.e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \cdots \\ & & \downarrow \varepsilon & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 \xrightarrow{d^2} \cdots \end{array} \quad \text{iso in } \mathcal{D}(X)$$

Then, one can use "stupid" truncation to get filtrations, and finally spectral sequences.  
(The canonical truncation won't give you anything new)

E.p. When  $C^\bullet$  is  $\Gamma$ -acyclic, the spectral seq deg. and  
 $H^i(X; \mathcal{F}) \cong H^i(\Gamma(C^\bullet))$

E.g. The spectral seqs in Section 1,2 are special cases.

## Coefficient spectral seq

E.g. For  $r \in \mathbb{Z}_{>0}$ , consider the SES

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{x^r} \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow 0$$

which induces the iso

$$\underline{\mathbb{Z}/r\mathbb{Z}} \cong [\rightarrow \underline{\mathbb{Z}} \xrightarrow{x^r} \underline{\mathbb{Z}} \longrightarrow]$$

the triangle

$$\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}[1] \xrightarrow{+1}$$

and the LES

$$H^k(X; \mathbb{Z}) \longrightarrow H^k(X, \underline{\mathbb{Z}/r\mathbb{Z}}) \longrightarrow H^{k+1}(X, \underline{\mathbb{Z}}) \xrightarrow{+1}$$

Ex. Compute  $H^k(\mathbb{R}\mathbb{P}^2; \underline{\mathbb{Z}/8\mathbb{Z}})$  in this way.

$$A: H^k(\mathbb{R}\mathbb{P}^2; \underline{\mathbb{Z}/8\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}/8\mathbb{Z}}, & k=0 \\ \underline{\mathbb{Z}/2\mathbb{Z}}, & k=1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Rmk. In general, one can get coefficient spectral seq [FF16, 20.5.B, p318] through the coefficient seq.  $G$  abelian

$$\dots \xrightarrow{\partial_2} \underline{G}_1 \xrightarrow{\partial_1} \underline{G}_0 \xrightarrow{\epsilon} \underline{G} \longrightarrow 0$$

Q. Can we recover the inflation - restriction seq in this way?

$$0 \longrightarrow H^*(G/H, M^H) \xrightarrow{\text{Inf}} H^*(G, M) \xrightarrow{\text{Res}} H^*(H, M)$$

A: I think no. We haven't introduce  ${}^L\otimes$  &  $R\text{Hom}$ .

## Singular cohomology

We present a defective proof in [Voo2, Theorem 4.47].

The bug has been discussed in the stack exchange:

<https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology>

|                 |                     |                             |                                      |
|-----------------|---------------------|-----------------------------|--------------------------------------|
| [Stackexchange] | $\mathcal{S}^k$     | $\widetilde{\mathcal{S}}^k$ | $\mathcal{S}^k(X)_0$                 |
| [Voo2]          | $C_{\text{sing}}^k$ | $C_{\text{sing}}^k$         | $C_{\text{sing}}^k(U, \mathbb{Z})_0$ |

$X \in \mathbf{Top}$ .

Def The presheaf  $\mathcal{S}^k \in \text{Psh}(X)$  is defined as

$$\begin{aligned}\mathcal{S}^k(U) &= \text{singular } k\text{-cochains on } U \\ &= \text{Hom}_{\text{Abel}}(\text{Sing}_k(U), \mathbb{Z})\end{aligned}$$

where

$$\begin{aligned}\text{Sing}_k(U) &= \text{singular } k\text{-chains on } U \\ &= \langle \sigma : \Delta^k \rightarrow U \rangle_{\text{free, Abel}}\end{aligned}$$

and  $\widetilde{\mathcal{S}}^k := (\mathcal{S}^k)^{\text{sh}}$  is called the sheaf of singular cochains.

Def

$$H_{\text{sing}}^k(X, \mathbb{Z}) := H^k(\mathcal{S}(X))$$

Thm For  $X$  loc contractible,

$$H^k(X, \mathbb{Z}) \cong H_{\text{sing}}^k(X, \mathbb{Z})$$

Sketch of the wrong proof: Still hard. I don't check details.

1. Construct a complex of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathcal{S}}^0 \longrightarrow \widetilde{\mathcal{S}}^1 \longrightarrow \widetilde{\mathcal{S}}^2 \longrightarrow \dots \quad (\star)$$

2. Show that  $(\star)$  is exact when  $X$  is loc contractible.  $\Rightarrow \mathbb{Z} \cong \widetilde{\mathcal{S}}^0$

3. Show that  $\widetilde{\mathcal{S}}^k$  is flabby, then

$$H^k(X, \mathbb{Z}) \cong H^k(\widetilde{\mathcal{S}}^k(X))$$

Reduced to show:  $H^k(\widetilde{\mathcal{S}}^k(X)) \cong H^k(\mathcal{S}^k(X))$

4. Consider the exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^k(X)_0 \longrightarrow \mathcal{S}^k(X) \longrightarrow \widetilde{\mathcal{S}}^k(X) \longrightarrow \mathcal{S}^k(X)^0 \longrightarrow 0$$

shows that:

$$\textcircled{1} \quad \mathcal{S}^k(X)^0 = 0 \quad \text{This is wrong. We ignore this bug...}$$

$$\textcircled{2} \quad \mathcal{S}^k(X)_0 = \left\{ \alpha \in \mathcal{S}^k(U) \mid \begin{array}{l} \exists \text{ open covering } \{U_i\}_i \text{ of } U \text{ s.t.} \\ \alpha|_{U_i} = 0 \quad \forall i \end{array} \right\}$$

$$\textcircled{3} \quad H^k(\mathcal{S}^k(X)_0) = 0$$

Finally,  $H^k(X, \mathbb{Z}) \cong H^k(\widetilde{\mathcal{S}}^k(X)) \cong H^k(\mathcal{S}^k(X)) \stackrel{\text{def}}{=} H_{\text{sing}}^k(X, \mathbb{Z})$ .

Q. Can one shows  $H^k(\mathcal{S}^k(X)^0) = 0$  to replace the bug in 4.①?

## 6. Hodge related filtration

### Smooth de Rham resolution

The classical story is the real de Rham cohomology.

Thm For  $X$  mfld, one has iso

$$H^k(X; \mathbb{R}) \cong H_{dR}^k(X; \mathbb{R}) := H^k(\Omega_X(X))$$

Here,

$\Omega_X^k$  = sheaf of smooth k-forms with coefficient  $\mathbb{R}$

[https://en.wikipedia.org/wiki/De\\_Rham\\_cohomology#Proof](https://en.wikipedia.org/wiki/De_Rham_cohomology#Proof)

### Sketch of proof

1. Construct a cplx of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \Omega_X^2 \xrightarrow{d^2} \dots (*)$$

2. Show that  $(*)$  is exact by Poincaré lemma. [Cl17, Lec1, Lemma 5]

3. Show that  $\Omega_X^i$  is  $\Gamma$ -acyclic [Cl17, Lec1, Lemma 4], then

$$H^k(X; \mathbb{R}) \cong H^k(\Omega_X(X)) \stackrel{\text{def}}{=} H_{dR}^k(X; \mathbb{R})$$

POU  $\Rightarrow$  fine  $\xrightarrow{\text{paracpt}}$  acyclic

Rmk. You can change coefficient from  $\mathbb{R}$  to  $\mathbb{C}$ , where

$$\begin{aligned} \Omega_{X,C}^k &= \text{sheaf of smooth k-forms with coefficient } \mathbb{C} \\ &= \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} = \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} \\ &\quad \uparrow \text{it can be viewed as } \mathbb{R}\text{-module} \end{aligned}$$

then

$$H^k(X; \mathbb{C}) \cong H_{dR}^k(X; \mathbb{C}) := H^k(\Omega_{X,C}(X)).$$

Now we step into the world of cplx mfld.

▽ Hereafter,  $X$ : cplx mfld,

$$\Omega_X^k = \text{sheaf of holomorphic } k\text{-forms} = \Omega_X^{k,0}$$

$$= \{ f dz_1 \wedge \dots \wedge dz_k \mid f: X \rightarrow \mathbb{C} \text{ holomorphic} \}$$

$\uparrow_{\text{no } d\bar{z}}$

while

$$A_{sm}^k = \Omega_{X_{sm}}^k = \text{sheaf of smooth } k\text{-forms}$$

$$= \{ f dx_1 \wedge \dots \wedge dx_k \mid f: X \rightarrow \mathbb{R} \text{ smooth} \}$$

$\uparrow d\bar{z} \text{ is allowed}$

$$A^k := \Omega_{X_{sm}, \mathbb{C}}^k = \{ f dx_1 \wedge \dots \wedge dx_k \mid f: X \rightarrow \mathbb{C} \text{ smooth} \} = \bigoplus_{p+q=k} A^{p,q}$$

$$\begin{array}{ll} \Omega: \text{holomorphic} & \Omega^k = \Omega^{k,0} \\ A: C^\infty & A^k = \bigoplus_{p+q=k} A^{p,q} \end{array}$$

This abuse of notation confuses me a lot.

For example, we still have the resolution (called the holomorphic de Rham resolution)

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \Omega_X^2 \xrightarrow{d^2} \dots \quad (**)$$

but  $\Omega_X^i$  is no longer  $\Gamma$ -acyclic, and

$$\dim_C \Gamma(X; \Omega_X^k) < +\infty \quad \text{for } X \text{ proj.}$$

$$H^i(X; \Omega_X^p) \longrightarrow H^i(X; \Omega_X^{p+1}) \text{ is 0-map} \quad \text{for } X \text{ proj.}$$

Find reference for the 0-map. ( $E_1$ -degeneration)

[Car17, Prop 6.3.3] shows the 0-map using the dim argument + Hodge decomposition.

Do we have any elegant proof to replace it?

[Vo02, p214]

1. Let  $S$  be a compact complex surface.

- (a) Show that the Frölicher spectral sequence of  $S$  degenerates at  $E_3$  for degree reasons. What are the possibly non-zero differentials  $d_2$ ?

▽

The differential map

$$d: \Omega_X^k \longrightarrow \Omega_X^{k+1}$$

is a map between coherent sheaves, but not a morphism in  $Coh(X)$ .  
i.e.,

$$d \notin \text{Hom}_{\mathcal{O}_X}(\Omega_X^k, \Omega_X^{k+1}) \cong \text{Hom}_{\mathcal{O}_X^{\text{an}}}(\Omega_X^{k, \text{an}}, \Omega_X^{k+1, \text{an}})$$

## Holomorphic de Rham resolution & Hodge-Tate

For simplicity, let's consider a proj curve  $X/\mathbb{C}$ .

cpt kähler is enough

Where do we use kähler condition?

### Hodge de-Rham

$$\underline{\mathcal{C}} \cong [\Omega^0 \xrightarrow{0} \Omega^1]_1$$

"stupid" filtration degenerates to the triangle

$$\begin{array}{ccccccc} \Omega^1[-1] & \longrightarrow & \underline{\mathcal{C}} & \longrightarrow & \Omega^0 & \xrightarrow{+1} & \\ & & \curvearrowright H^1(X; \Omega^1) & \longrightarrow & H^2(X; \mathbb{C}) & \longrightarrow & H^2(X; \Omega^0) \\ & & \curvearrowright H^0(X; \Omega^1) & \longrightarrow & H^1(X; \mathbb{C}) & \longrightarrow & H^1(X; \Omega^0) \\ & & \curvearrowright H^1(X; \Omega^1) & \xrightarrow{=0} & H^0(X; \mathbb{C}) & \longrightarrow & H^0(X; \Omega^0) \end{array}$$

$\rightsquigarrow$  Hodge filtration:

$$0 \longrightarrow H^0(X; \Omega^1) \longrightarrow H^1(X; \mathbb{C}) \longrightarrow H^1(X; \Omega^0) \longrightarrow 0$$

Hodge Tate similar to Euler class (I'm foolish in this field)

$$\nu: X_{\text{proét}} \xrightarrow{\widehat{\mathcal{O}_X}} X_{\text{ét}}$$

$$\begin{array}{ccccccc} R^0 \nu_* \widehat{\mathcal{O}_X} & \longrightarrow & R \nu_* \widehat{\mathcal{O}_X} & \longrightarrow & R^1 \nu_* \widehat{\mathcal{O}_X}[-1] & \xrightarrow{+1} & \\ \parallel & & \parallel & & \parallel & & \\ \Omega_X^0 & & & & \Omega_X^1[-1] & & \\ & & & & & & \\ & & \curvearrowright H^1(X; \Omega^0) & \xrightarrow{=0} & H^2(X; \mathbb{C}) & \longrightarrow & H^2(X; \Omega^1) \\ & & \curvearrowright H^1(X; \Omega^0) & \longrightarrow & H^1(X; \mathbb{C}) & \longrightarrow & H^0(X; \Omega^1) \\ & & & & & & \xrightarrow{0} \\ & & & & H^0(X; \Omega^0) & \longrightarrow & H^1(X; \Omega^1) \end{array}$$

$$\rightsquigarrow 0 \longrightarrow H^0(X; \Omega^0) \longrightarrow H^1(X; \mathbb{C}) \longrightarrow H^0(X; \Omega^1) \longrightarrow 0$$

"Euler class of  $\nu$ " is trivial?

Q. Combining these two filtrations, does it give Hodge decomposition? in the p-adic sense

Hodge decomposition in the algebraic surface case.

$$\begin{array}{ccc}
 \chi(\mathcal{O}_X) \chi(\Omega_X) \chi(\omega_X) & & e := \chi_{\text{top}}(X) \\
 \downarrow & & \downarrow \\
 \begin{matrix} 1 \\ q \\ Pg \\ Pg \end{matrix} & \begin{matrix} 1 \\ q \\ h' \\ 1 \end{matrix} & \begin{matrix} b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{matrix} \\
 \begin{matrix} c_2 = e \\ c_1 = K^2 = 12\chi(\mathcal{O}_X) - e \end{matrix} & \begin{matrix} q \\ 1 \end{matrix} & \begin{matrix} h^{2,2} \\ h^{1,2} \quad h^{2,1} \\ h^{0,2} \quad h^{1,1} \quad h^{2,0} \\ h^{0,1} \quad h^{1,0} \\ h^{0,0} \end{matrix}
 \end{array}$$

## Dolbeault resolution [Vo02, p94]

For  $\pi_E: E \rightarrow X$  holo v.b., let

$\mathcal{A}_E^{p,q} = \text{sheaf of smooth } (p,q)\text{-forms with coefficient as sections in } E$

$\mathcal{E} = \text{sheaf of holo sections of } E \triangleq \Omega_E^{0,0}$

One can construct a cplx of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{A}_E^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}_E^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}_E^{0,2} \xrightarrow{\bar{\partial}} \dots \quad (***)$$

Show that  $(***)$  is exact, and  $\mathcal{A}_E^{p,q}$  is  $\pi_*$ -acyclic.

$$\Rightarrow H^k(X; \mathcal{E}) \cong H^k(\mathcal{A}_E^{0,\cdot}(X))$$

E.g. When  $\mathcal{E} = \Omega_X^p$ ,  $\mathcal{A}_E^{p,q} = \mathcal{A}^{p,q}$ , one gets

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots \quad (***)'$$

$$\Rightarrow H^q(X; \Omega_X^p) \cong H^q(\mathcal{A}_E^{0,\cdot}(X)) \stackrel{\text{def}}{=} H_{\bar{\partial}}^{p,q}(X)$$

Rmk.  $(*)$ ,  $(**)$  &  $(***)'$  are related by the following double complex:

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow \bar{\partial} \\
 0 & \longrightarrow & \mathcal{A}^{0,1} & \xrightarrow{\partial} & \mathcal{A}^{1,1} & \xrightarrow{\partial} & \mathcal{A}^{2,1} \xrightarrow{\partial} \mathcal{A}^{3,1} \xrightarrow{\partial} \dots \\
 & \uparrow \bar{\partial} \\
 0 & \longrightarrow & \mathcal{A}^{0,0} & \xrightarrow{\partial} & \mathcal{A}^{1,0} & \xrightarrow{\partial} & \mathcal{A}^{2,0} \xrightarrow{\partial} \mathcal{A}^{3,0} \xrightarrow{\partial} \dots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$