

Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>

$$\begin{array}{ccc} G & \xrightarrow{\tau} & \mathcal{F}' \\ \downarrow & \searrow & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

$$\begin{aligned} f^* &\dashv f_* \\ - \otimes \mathcal{F} &\dashv \underline{\mathrm{Hom}}(\mathcal{F}, -) \\ f_! &\dashv f^! \end{aligned}$$

$$\begin{aligned} f^*(- \otimes -) \\ f^*(\mathcal{F} \otimes \mathcal{F}') &\cong f^* \mathcal{F} \otimes f^* \mathcal{F}' \\ f_* \underline{\mathrm{Hom}}(f^* \mathcal{F}, \mathcal{G}) &\cong \underline{\mathrm{Hom}}(\mathcal{F}, f_* \mathcal{G}) \end{aligned}$$

$$\begin{array}{ccc} & \otimes & \\ f^* & \xrightarrow{\quad} & f_! \\ \text{bc: } f^* g_! & \cong & g'_! f'^* \\ f_* g'_! & \cong & g'_! f_* \end{array}$$

proj formula

$$f_!(f^* \mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$$

$$f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^* \mathcal{F}) \cong \underline{\mathrm{Hom}}(f_* \mathcal{G}, \mathcal{F})$$

$$f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') \cong \underline{\mathrm{Hom}}(f^* \mathcal{F}, f^! \mathcal{F}')$$

These extra formulas (compatibilities) come from the upgrade of adjunction formula to internal Hom. To upgrade the adjunction between tensor product and internal Hom, one don't need extra formula, except the association law of tensor product.

$$\begin{aligned} I: f^* &= f^! \\ P: f_* &= f_! \end{aligned}$$

$$p: X \rightarrow pt$$

$$H^i(X; \mathbb{Z}) := p_* p^* \mathbb{1}$$

$$H_c^i(X; \mathbb{Z}) := p_! p^* \mathbb{1}$$

$$H^i(X; \mathbb{Z}) := p_! p^! \mathbb{1}$$

$$H^{BM}_i(X; \mathbb{Z}) := p_* p^! \mathbb{1}$$

$$H^i(X; \mathcal{F}) := p_* \mathcal{F}$$

$$H_c^i(X; \mathcal{F}) := p_! \mathcal{F}$$

$$H^i(X; \mathcal{F}) := p_! (p^! \mathbb{1} \otimes \mathcal{F})$$

$$H^{BM}_i(X; \mathcal{F}) := p_* (p^! \mathbb{1} \otimes \mathcal{F})$$

$$= R\Gamma(X; \mathcal{F})$$

$$= H_c^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

$$= H^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)

$$H_c^i(X; \mathcal{F}) \otimes H_c^j(Y; \mathcal{G}) \cong H_c^{i+j}(X \times Y; \mathcal{F} \otimes \mathcal{G})$$

$$\text{reduced to: } p_{X,*} \mathcal{F} \otimes p_{Y,*} \mathcal{G} \cong p_{!} (p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_! \downarrow & \searrow p & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

App 2. (Poincaré duality)

X : a cpt oriented mfd of dim d , then

$$-^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

proper

$$p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ locally (Verdier duality)}$$

$$p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ globally}$$

$$H^i(X; \mathbb{Z})[d] \cong H^i(X; \mathbb{Z})^\vee$$

$$\text{reduced to: } p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$$

ff. fully faithful
pi. preserve injectives. (if ff)
ie. $\text{inj sheaf} \rightarrow \text{inj sheaf}$

p_i : preserve injectives. (内射)
ie. inj sheaf \leadsto inj sheaf

Just by checking the stalk & taking the dual, one gets

Here, $H_{-1}(S'; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

$$\begin{array}{ccccccc}
 R\Gamma(X, Z; \mathcal{F}) & \longrightarrow & R\Gamma(X; \mathcal{F}) & \longrightarrow & R\Gamma(Z; \mathcal{F}|_Z) & \xrightarrow{+1} & \\
 R\Gamma(X, u; \mathcal{F}) & \longrightarrow & R\Gamma(X; \mathcal{F}) & \longrightarrow & R\Gamma(u; \mathcal{F}|_u) & \xrightarrow{+1} & \\
 R\Gamma_Z^{\text{II}}(X; \mathcal{F}) & & & & & & \\
 \text{When } \mathcal{F} = \underline{\mathbb{Q}}_X, & H^i(X, Z) & \longrightarrow & H^i(X) & \longrightarrow & H^i(Z) & \xrightarrow{+1} \\
 & H^i(X, u) & \longrightarrow & H^i(X) & \longrightarrow & H^i(u) & \xrightarrow{+1} \\
 \text{When } \mathcal{F} = \text{ID}_X, & R\Gamma(X, j_* \underline{\mathbb{Q}}_u) & \longrightarrow & H_i^{\text{BM}}(X) & \longrightarrow & R\Gamma(Z; \text{ID}_X|_Z) & \xrightarrow{+1} \\
 & H_i^{\text{BM}}(Z) & \longrightarrow & H_i^{\text{BM}}(X) & \longrightarrow & H_i^{\text{BM}}(u) & \xrightarrow{+1}
 \end{array}$$
$$\begin{array}{ccccccc} R\Gamma_c^*(U, \mathcal{F}|_U) & \longrightarrow & R\Gamma_c^*(X; \mathcal{F}) & \longrightarrow & R\Gamma_c^*(Z; \mathcal{F}|_Z) & \xrightarrow{+1} & \\ R\Gamma_c^*(Z, i^! \mathcal{F}) & \longrightarrow & R\Gamma_c^*(X; \mathcal{F}) & \longrightarrow & R\Gamma_c^*(X, Rj_*(\mathcal{F}|_U)) & \xrightarrow{+1} & \end{array}$$

When $\mathcal{F} = \underline{\mathbb{Q}}_X$,

$$\begin{array}{ccccccc} H_c^*(U) & \longrightarrow & H_c^*(X) & \longrightarrow & H_c^*(Z) & \xrightarrow{+1} & \\ H_c^*(Z, i^! \underline{\mathbb{Q}}_X) & \longrightarrow & H_c^*(X) & \longrightarrow & H_c^*(X, Rj_* \underline{\mathbb{Q}}_U) & \xrightarrow{+1} & \end{array}$$

When $\mathcal{F} = \text{ID}_X$,

$$\begin{array}{ccccccc} H_*(U) & \longrightarrow & H_*(X) & \longrightarrow & H_*(X, U) & \xrightarrow{+1} & \\ H_*(Z) & \longrightarrow & H_*(X) & \longrightarrow & H_*(X, Z) & \xrightarrow{+1} & \end{array}$$

$$i^! \mathcal{F} \longrightarrow i^* \mathcal{F} \longrightarrow {}^i R j_* j^* \mathcal{F} \xrightarrow{+1}$$

local cohomology compares the difference between stalks and costalks.

Explanation

Exactness & derived

$j_!$ is exact
 i^*, j^* are exact
 i_* is exact

by checking on stalks
in the category $\mathcal{T}op$
when $Z \subset X$ is (strongly) loc. contractable.

j_* is not exact $\rightsquigarrow Rj_*$
 $i^!$ is already derived.

Rmk. strongly loc. contractable: $\forall p \in X, \exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z$ is contractable
loc. contractable: $\forall p \in X, \exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z \subset U_n$ is contractable

E.g. $\{ \text{strongly loc. contractable} \} \subsetneq \{ \text{loc. contractable} \} \not\subset \mathcal{T}op$
CW-cplx, topo mflds
& algebraic varieties (Check?)
Cantor set

See the answer in

<https://math.stackexchange.com/questions/1082601/anr-is-locally-contractible>
for the subtlety of these two definitions.

I don't care. In both cases, the local cohomology vanishes in higher degree, and that's what I want.

For the non-exact functors, there maybe some problems in the composition of derived functors.

<https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof>

<https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de>

E.g. we need to check if $R\pi_{x,*} \circ Rj_* = R\pi_{u,*}$.
Luckily, in the open-closed formalism, we won't meet these problems.

Prop1. Let $e \xrightleftharpoons[G]{F} e'$, assume F is exact. Then

① G preserves injective sheaves;

② $RG(f) \circ RG(g) = RG(f \circ g)$

Proof. ①: by universal property.

②: by adjunction

Prop 2. Let $\mathcal{C} \xrightleftharpoons[G]{F} \mathcal{C}' \xrightleftharpoons[G']{F'} \mathcal{C}''$. Suppose F or F' is exact, then

$$RG \circ RG'(f) = (R(G \circ G'))(f)$$

Proof. By adjunction & Grothendieck-Serre sequence. ($LF' \circ LF = L(F' \circ F)$)
When F' is exact, can use Prop 1 \oplus .

Cor. $R\pi_{x,*} \circ Rj_* = R\pi_{u,*}$.

Rf_* & $Rf_!$ are nice in general.

Reason: $f_!$ sends skyscraper sheaf to skyscraper sheaf, and in general preserve injective sheaves. (need double check)