Eine Woche, ein Beispiel

1.26 Numerical Chern class

Ref:
wiki: Chern class

Nearly all the results are sourced from Wikipedia. I made this document because I tend to mix up the Chern class and the Chern character.
We omit E in notation.

$$c(E) = 1 + c_1 + \cdots + c_r \in H^{\cdot}(X;\mathbb{C})$$
$$= \prod_{i=1}^{r} (1+a_i) \qquad a_i(E) \in H^{\cdot}(F\ell(E);\mathbb{C})$$

$$c_t(E) = 1 + c_1 t + \cdots + c_r t^r \in H^{\cdot}(X;\mathbb{C})[t]$$
$$= \prod_{i=1}^{r} (1+a_i t)$$

$$ch(E) = e^{a_1} + \cdots + e^{a_r} \in H^{\cdot}(X;\mathbb{C})$$
$$= \sum_{k=0}^{+\infty} \frac{1}{k!} (a_1^k + \cdots + a_r^k)$$
$$= \sum_{k=0}^{+\infty} \frac{1}{k!} s_k(c_1, \ldots, c_r)$$
$$= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_2c_1 + 3c_3)$$
$$+ \frac{1}{24}(c_1^4 - 4c_2c_1^2 + 4c_3c_1 + 2c_2^2 - 4c_4) + \cdots$$

$$td(E) = \prod_{i=1}^{r} \frac{a_i}{1-e^{-a_i}} \in H^{\cdot}(X;\mathbb{C})$$
$$= \prod_{i=1}^{r} \left(1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} a_i^{2k}\right)$$
$$= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2$$
$$+ \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \cdots$$

$$s(E) = \prod_{i=1}^{r} \frac{1}{1+a_i} \in H^{\cdot}(X;\mathbb{C})$$
$$\hat{=}\ 1 + s_1 + \cdots + s_n$$
$$= 1 - c_1 + (-c_2 + c_1^2) + (-c_3 + 2c_1c_2 - c_1^3)$$
$$+ (-c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + c_1^4) + \cdots$$

$$c(E \oplus E') = c(E) \cup c(E')$$
$$c_t(E \oplus E') = c_t(E)\, c_t(E')$$
$$ch(E \oplus E') = ch(E) + ch(E')$$
$$td(E \oplus E') = td(E) \cup td(E')$$
$$s(E \oplus E') = s(E) \cup s(E')$$

$$ch(E \otimes E') = ch(E)\, ch(E')$$

E.g. $X = \mathbb{P}^1$ $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and

$H \in H^2(\mathbb{P}^1; \mathbb{C})$ as the generator

$$
\begin{aligned}
c(E) &= 1 + aH \\
c_t(E) &= 1 + aHt \\
ch(E) &= 1 + aH \\
td(E) &= 1 + \tfrac{1}{2}aH \\
s(E) &= 1 - aH
\end{aligned}
$$

For $E = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, one gets

$$
\begin{aligned}
c(E) &= (1 + a_1 H) \cup (1 + a_2 H) &&= 1 + (a_1 + a_2)H \\
c_t(E) &= (1 + a_1 Ht)(1 + a_2 Ht) &&= 1 + (a_1 + a_2)Ht \\
ch(E) &= 1 + a_1 H + 1 + a_2 H &&= 2 + (a_1 + a_2)H \\
td(E) &= (1 + \tfrac{1}{2}a_1 H) \cup (1 + \tfrac{1}{2}a_2 H) &&= 1 + \tfrac{1}{2}(a_1 + a_2)H \\
s(E) &= (1 - a_1 H) \cup (1 - a_2 H) &&= 1 - (a_1 + a_2)H
\end{aligned}
$$

Therefore, these characteristic classes can not distinguish $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.