## § 3.1. Galois representation

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### 1. Galois rep

<u>Setting</u> $G$: arbitrary gp  e.g. $G$ any Galois gp

If $G$ profinite $\Rightarrow$ open subgps are finite index subgps.

$\Lambda$: top field  e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{cont}}(G)$

$V \in \mathrm{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V)$ cont

$\triangledown$ $\rho(G)$ can be infinite! for Gal gp

E.g. When $\mathrm{char}\, F \neq p$, we have $p$-adic cyclotomic character

$$\varepsilon_p: \mathrm{Gal}(F^{sep}/F) \longrightarrow \mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p^\times \qquad \sigma \longmapsto \varepsilon_p(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_p(\sigma)} \qquad \forall \zeta \in \mu_{p^\infty}$$

This is cont by def. (Take usual topo.)

Notice the following two definitions don't depend on the topo of $\Lambda$.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$

$V \in \mathrm{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{fi}}(G)$  fi: finite image / finite index

$V \in \mathrm{vect}_\Lambda \quad + \quad \rho: G \longrightarrow GL(V)$ with finite image

Rmk. $\mathrm{rep}_{\Lambda,\mathrm{cont}}(G) \longleftarrow \mathrm{rep}_{\Lambda,\mathrm{fi}}(G) \longleftarrow \mathrm{rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) = \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$

$$\begin{tikzcd}
\mathrm{rep}_{\Lambda,\mathrm{sm}}(G) \arrow[r, equal] \arrow[d, phantom, "\cap"] & \mathrm{rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) \arrow[r, leftrightarrow] \arrow[d, phantom, "\cap"] & \mathrm{rep}_{\Lambda,\mathrm{fi}}(G) \arrow[r, leftrightarrow] \arrow[d, phantom, "\cap"] & \mathrm{rep}_{\Lambda,\mathrm{cont}}(G) \arrow[d, phantom, "\cap"] \\
\mathrm{Rep}_{\Lambda,\mathrm{sm}}(G) \arrow[r] & \mathrm{Rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) \arrow[r, leftrightarrow] & \mathrm{Rep}_{\Lambda,\mathrm{fi}}(G) \arrow[r] & \mathrm{Rep}_{\Lambda,\mathrm{cont}}(G)
\end{tikzcd}$$

$\rightarrow$: if fin index subgps are open

$\rightarrow$: if $G$: profinite gp (Only need: open $\Rightarrow$ fin index)

$\rightarrow$: Artin rep (of profinite gp)

Artin rep. $\Lambda = (\mathbb{C}, \text{euclidean topo})$ $G$ profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{Id\}$.

"Proof" Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\|_{max} < \frac{1}{3}\}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists n \in \mathbb{N}$, s.t $A^n \notin \mathcal{U}$.

Consider the Jordan form of $A$.

Case 1. $A$ unipotent.

Case 2. $A$ not unipotent.

Problem: $\|gAg^{-1}\|_{max} \neq \|A\|_{max}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},\mathrm{cont}}(G)$, $\rho(G)$ is finite. $G$ profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq G_F$ finite index. $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = Id$

$\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. $\rho$ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

$\rho$ can be viewed as cplx rep of fin gp. so $\rho$ is semisimple.
Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

## Finite field

Task. For $\Lambda$: NA local field with char $k_\Lambda = l$, compare

$$\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}}) \longleftrightarrow \mathrm{rep}_{\Lambda,\mathrm{sm}}(\mathbb{Z}) \quad + \text{ extra informations/conditions}$$