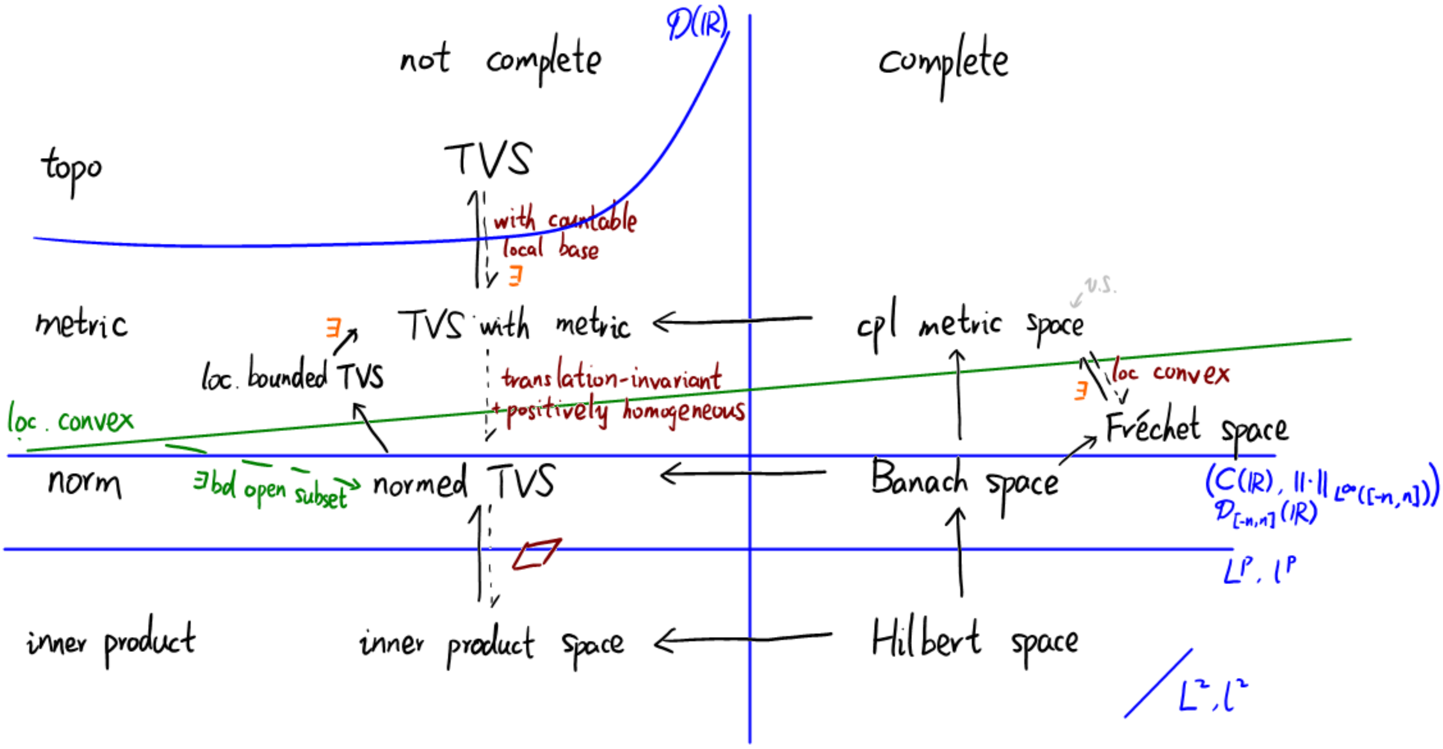
Eine Woche, ein Beispiel

4.30 TVS = topological vector space

Ref:
Lec 1-7: http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-FA/index.html

In this document, we don't worry about extra structure here, and we assume Hausdorff.



$C[0,1] \subset L^2[0,1]$

∃: exists a metric

metrizable TVS = TVS with countable local base

seminormable TVS = TVS with convex local base = LCTVS (locally convex)

normable TVS = TVS with a convex bounded nbhd

F-space = TVS + ∃ cpl tran-inv metric

https://math.stackexchange.com/questions/3356421/example-of-a-locally-convex-topological-vector-space-which-is-not-metrizable
https://math.stackexchange.com/questions/854355/a-locally-convex-space-is-metrizable-if-and-only-if-it-is-first-countable
https://math.stackexchange.com/questions/1070034/the-weak-star-topology-is-completely-hausdorff-in-particular-hausdorff
https://math.stackexchange.com/questions/1809719/weak-and-weak-star-topologies-are-locally-convex

Fréchet space = countable sep seminorms $\{p_n\}$ + ∃ cpl tran-inv metric
or: the induced metric is cpl.

= loc convex + ∃ cpl tran-inv metric

Rmk. There are two definitions of boundedness in metrizable TVS X, and they coincide if the metric is translation-invariant.

Def. (boundness for TVS)

$E \subset X$ is bounded if $\forall\, 0 \in \mathcal{U}$ open, $\exists\, s > 0$ s.t. $\forall\, t > s$, $E \subset t\,\mathcal{U}$

In this case,

$$E \text{ is bounded} \iff \left[\begin{array}{c} \forall \{x_n\} \subset E,\quad \{\alpha_n\} \subseteq \mathbb{R} \text{ or } \mathbb{C}, \\ \alpha_n \to 0 \Rightarrow \alpha_n x_n \to 0 \end{array}\right]$$

Def. (boundness for metric space)

Fix $x_0 \in X$. boundness does not depend on x_0.

$E \subset X$ is bounded if $\exists\, r > 0$, $E \subset B_{x_0}(r)$.

Rmk. 1. For loc convex TVS with metric, all open balls are convex.

2. cpl metric space $\Rightarrow$ 2^{nd} category.

1^{st} category set

https://math.stackexchange.com/questions/1237159/understanding-the-definition-of-nowhere-dense-sets-in-abbotts-understanding-ana

Def. A closed subset $A \subset X$ is nowhere dense if
A^c is dense in X, i.e., $A^\circ = \phi$

A^c dense $\qquad$ A nowhere dense $\qquad$ A closed

Def. $A \subset X$ is of 1^{st} category, if

$$A \subset \bigcup_{i \in \mathbb{Z}_{>0}} A_i \qquad \text{for some } A_i = \overline{A_i} \text{ nowhere dense.}$$