

Eine Woche, ein Beispiel

6.18 diagram chasing

Goal: Let's play the game of diagram chasing!

basic: five lemma, snake lemma, SES of complex  $\Rightarrow$  LES of homology

[Vakil] "where there is universal property, there is diagram chasing"

e.p. Chap 1 Category + Adjoints + Spectral sequences

Chap 2 Sheaf on topology space

Please convert everything to Grothendieck topo!

Chap 23 Derived functors

Chap 28 Base change

[J. S. Milne, étale cohomology] Prop II.1.5. one description of the étale sheaf

References for the spectral sequence:

Best naive introduction by Prof. Vakil:

<https://www.3blue1brown.com/content/blog/exact-sequence-picturebook/PuzzlingThroughExactSequences.pdf>

applications of snake lemma and five lemma: [The rising sea2016, 1.7]

applications for algebraic geometry: [The rising sea2016, 23.3]

applications for cohomology group and homotopy group: [GTM82], [Hatcher], [2021.12.12]

ppt with nice pictures: <https://github.com/CubicBear/SpectralSequences/tree/main>

Now: "Fancy objects require a lot of diagram-chasing technique"

- Infinite category
- Stack related
- Condensed objects
- Triangular category, derived category and six-fctors formalism

Extension group

1. Def of  $\text{Ext}_A^n(M, N)$

$$\begin{aligned} E_A(M, N) &= \{0 \rightarrow N \xrightarrow{f} E \xrightarrow{g} M \rightarrow 0\} / \text{equivalence} \\ &= \{\text{proj resolution } P, H^n(\text{Hom}_A(P, N))\} / \text{resolution} \\ &= \{\text{inj resolution } I', H^n(\text{Hom}_A(M, I'))\} / \text{resolution} \\ &= \{\text{derivation}\} / \text{inner derivation} \\ &= \text{Hom}_{D(A\text{-mod})}(M, N[1]) \end{aligned}$$

2. Special module/ring interact with  $\text{Ext}$ ?

$$P \text{ proj} \Leftrightarrow \text{Ext}_A^n(P, -) = 0 \quad \forall n \geq 1 \Leftrightarrow \text{Ext}_A^1(P, -) = 0$$

$$\Leftrightarrow \text{proj dim } P = 0$$

$$I \text{ proj} \Leftrightarrow \text{Ext}_A^n(-, I) = 0 \quad \forall n \geq 1 \Leftrightarrow \text{Ext}_A^1(-, I) = 0$$

$$A \text{ f.d alg} \quad \dim_k \text{Ext}_A^1(S(i), S(j)) = \dim_k \text{Hom}_A(\text{rad}(P(i)), S(j))$$

$$\stackrel{A=kQ/I}{=} |\{a \in Q \mid s(a)=i, t(a)=j\}|$$

Second level of detail.

equivalent of SES  $\begin{array}{ccccccc} & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ & \parallel & \rightarrow & \downarrow & \rightarrow & \parallel & \rightarrow \end{array}$

$\downarrow$   
isomorphic

$\begin{array}{ccccccc} & \rightarrow & \rightarrow & \rightarrow & \rightarrow \\ & \downarrow & \rightarrow & \downarrow & \rightarrow & \downarrow & \rightarrow \end{array}$

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \xrightarrow{P} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \\ & & \parallel & & \parallel & & \downarrow \times 2 \\ 0 & \rightarrow & \mathbb{Z} & \xrightarrow{P} & \mathbb{Z} & \xrightarrow{2 \cdot \pi} & \mathbb{Z}/p\mathbb{Z} \rightarrow 0 \end{array}$$

pushout  
 $\alpha_*$



①



③<sup>\*</sup>

pullback  
 $\beta^*$



+ k-linear  
com  
ass  
0  
 $(\lambda + \mu)a$   
 $(\lambda\mu)a$   
 $1a$   
 $\lambda(a+b)$

$\Rightarrow E_A(M, N)$  ① Def, bifunctor and ③<sup>\*</sup> k-linear space structure ①  $\Rightarrow$  ②  $\Rightarrow$  ③

$$f. \sim g. \Rightarrow H_n(f.) = H_n(g.)$$

$$g.f. \sim \text{Id} \quad f.g. \sim \text{Id} \Rightarrow H_n(C.) = H_n(C')$$

$\Rightarrow \text{Ext}_A^n(M, N)$  ① Def, bifunctor and ③<sup>\*</sup> k-linear space structure ①  $\Rightarrow$  ③  $\Rightarrow$  ②

$\Rightarrow E_A(M, N) \rightarrow \text{Ext}_A^1(M, N)$  ① well-defined by resolution & lift & equiv  
② bifunctor  
③ k-linear map

Schanuel's lemma

$$\left. \begin{array}{l} 0 \rightarrow U \rightarrow P \rightarrow M \rightarrow 0 \\ 0 \rightarrow U' \rightarrow P' \rightarrow M \rightarrow 0 \\ P, P' \text{ proj} \end{array} \right\} \Rightarrow U \oplus P' \cong U' \oplus P$$

$0 \rightarrow U \rightarrow X \rightarrow V \rightarrow 0$   $\left\{ \begin{array}{l} \text{non-split} \\ \text{f.d. } A\text{-mod} \end{array} \right. \Rightarrow \dim_k \text{End}_A(X) < \dim_k \text{End}_A(U \oplus V)$   
for f.d.  $A$ -mod  $0 \rightarrow U \rightarrow X \rightarrow V \rightarrow 0$  split  $\Leftrightarrow X \cong U \oplus V$  as  $A$ -module

# Derived category

slogan: complex good, homology bad

Motivated: <https://arxiv.org/pdf/math/0001045.pdf>

Standard reference: S. Gelfand and Yu. Manin, Methods of Homological Algebra, Springer, 1996

we refer this without mention!

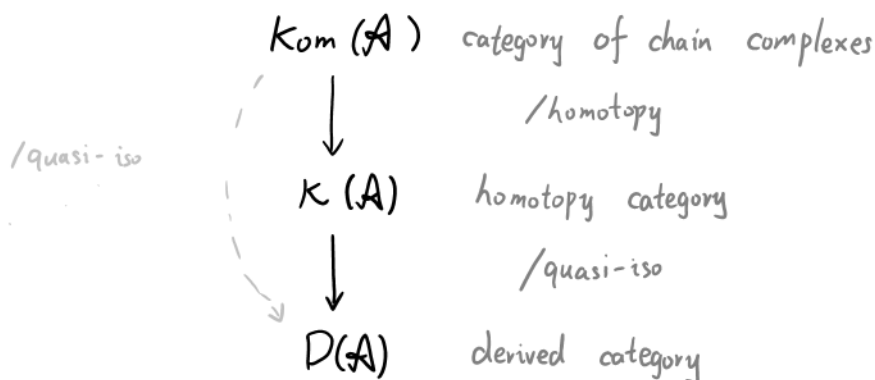
Newly recommended references:

<https://www.math.uni-hamburg.de/home/sosna/triangcat-lect.pdf>

This notes shows all the details of the definition of triangular categories and derived categories, and shows even more informations of derived categories.

<https://guests.mpim-bonn.mpg.de/gallauer/docs/m6ff.pdf>

It introduce the six-functors formalism in detail, even though the exercises are pretty hard.



Remark. 1. For most time we view the category equivalence as "equal".

However, the category defined by universal property is unique under isomorphism.

$$\text{Ob}(\text{Kom}(A)) = \text{Ob}(K(A)) = \text{Ob}(D(A))$$

2. localizing category  $B[S^{-1}]$  does not always have a good description

$$\text{e.g. } D(A) := \text{Kom}(A)[\text{quasi-iso}^{-1}]$$

However, when  $S$  is a localizing class, then we have a good description Lemma III 2.8

$$\text{e.g. } D(A) := K(A)[\text{quasi-iso}^{-1}]$$

Those two definitions define the same category  $D(A)$ .

3.  $D(A)$  is a triangulated category.

To define a distinguished triangle, we denote

$f: K^\bullet \rightarrow L^\bullet$ 	$K^\bullet, L^\bullet$ : complexes	$K^\bullet \xrightarrow{d_K^\bullet} K^\bullet$ $L^\bullet \xrightarrow{d_L^\bullet} L^\bullet$ $d_K^\bullet = d_L^\bullet = d$ to be short
$\text{Cyl}(f) := K^\bullet \oplus K[1] \oplus L^\bullet$ 	$d_{\text{Cyl}(f)} = \begin{bmatrix} d & -1 & \\ & -d & \\ f & & d \end{bmatrix}$	$K^\bullet \oplus K^\bullet \oplus L^\bullet \xrightarrow{\begin{bmatrix} d_K^\bullet & -1 & \\ & -d_K^\bullet & \\ f & & d_L^\bullet \end{bmatrix}} K^\bullet \oplus K^\bullet \oplus L^\bullet$
$C(f) := K[1] \oplus L^\bullet$ 	$d_{C(f)} = \begin{bmatrix} -d & \\ f & d \end{bmatrix}$	$K^\bullet \oplus L^\bullet \xrightarrow{\begin{bmatrix} -d_K^\bullet & \\ f & d_L^\bullet \end{bmatrix}} K^\bullet \oplus L^\bullet$

Then we have (Lemma III 3.3)

- $\left\{ \begin{array}{l} \textcircled{\circ} \text{ well-defined} \\ \textcircled{1} \text{ SES on row} \\ \textcircled{2} \alpha, \beta: \text{quasi-iso} \end{array} \right.$

$$\begin{array}{ccccccc}
 \emptyset & \longrightarrow & \square & \longrightarrow & \begin{array}{c} \triangle \\ \square \end{array} & \longrightarrow & \begin{array}{c} \diamond \\ \square \end{array} \longrightarrow \emptyset \\
 & & \downarrow \alpha & & \parallel & & \\
 \emptyset & \longrightarrow & \bigcirc & \longrightarrow & \begin{array}{c} \text{cylinder} \\ \square \end{array} & \longrightarrow & \begin{array}{c} \triangle \\ \square \end{array} \longrightarrow \emptyset \\
 & & \parallel & & \downarrow \beta & & \\
 & & \bigcirc & \longrightarrow & \square & & \\
 \\ 
 0 & \longrightarrow & L^\bullet & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & K[1]^\bullet \oplus L & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & K[1]^\bullet \longrightarrow 0 \\
 & & \downarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & & \parallel & & \\
 0 & \longrightarrow & K^\bullet & \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} & K^\bullet \oplus K[1]^\bullet \oplus L^\bullet & \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} & K[1]^\bullet \oplus L \longrightarrow 0 \\
 & & \parallel & & \downarrow [f \circ 1] & & \\
 & & K^\bullet & \xrightarrow{f} & L^\bullet & & 
 \end{array}$$

distinguished triangle:

$$K^\bullet \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} K[1]^\bullet \oplus L \xrightarrow{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} K[1]^\bullet$$

SES: What's your favorite SES?

$$0 \rightarrow I \rightarrow A \rightarrow A/I \rightarrow 0 \quad \text{as } A\text{-mod}$$

$$0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0$$

$$0 \rightarrow I \cap J \rightarrow I \oplus J \rightarrow I + J \rightarrow 0$$

$$0 \rightarrow R/(I \cap J) \rightarrow R/I \oplus R/J \rightarrow R/(I + J) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_X \rightarrow K_X \rightarrow \bigoplus_{x \in X_{\text{closed}}} I_x \rightarrow 0$$

<https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer12.pdf>

$$0 \rightarrow I/I^2 \xrightarrow{\Delta_*^{\parallel} \Omega_X} \mathcal{O}_{X \times X/I^2} \xrightarrow{\Delta_*^{\parallel} \mathcal{O}_X} \mathcal{O}_{X \times X/I} \rightarrow 0$$

<https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer8.pdf>

$$0 \rightarrow I_q \rightarrow D_q \rightarrow \text{Gal}(k_q/k_p) \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_k^* \rightarrow K^* \rightarrow \bigoplus_{\mu \in M_k^*} \mathbb{Z} \rightarrow \text{Cl}(K) \rightarrow 0$$

$$1 \rightarrow Z(G) \rightarrow G \xrightarrow{\text{conj}} \text{Aut}(G) \rightarrow \text{Out}(G) \rightarrow 1$$

exponential  $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_M \rightarrow \mathcal{O}_M^* \rightarrow 1$

generalization: <https://ncatlab.org/nlab/show/exponential+exact+sequence>

Euler  $0 \rightarrow \Omega_{\mathbb{P}_A^n/A} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus (n+1)} \xrightarrow{\text{M. cplx mflld}} \mathcal{O}_{\mathbb{P}_A^n} \rightarrow 0$

$$1 \rightarrow \mathbb{G}_m \xrightarrow{u_{\eta,*}} \mathbb{G}_{m,\eta} \rightarrow \text{Div}(X) \rightarrow 1 \quad u_{\eta}: \eta \rightarrow X_{\text{Spec}(k(X))}$$

$$0 \dashrightarrow f^* \Omega_{X/k} \rightarrow \Omega_{Y/k} \rightarrow \Omega_{Y/X} \rightarrow 0 \quad f: Y \rightarrow X$$

$$0 \dashrightarrow I/I^2 \rightarrow i^* \Omega_{X/k} \rightarrow \Omega_{Z/k} \rightarrow 0$$

<https://mathoverflow.net/questions/270762/the-mittag-leffler-condition-as-necessary-and-sufficient?rq=1>  
<https://math.stackexchange.com/questions/4260668/mittag-leffler-condition-and-exact-sequence-of-inverse-systems>

$$0 \rightarrow \varprojlim_n A_n \rightarrow \prod_{n=1}^{\infty} A_n \xrightarrow{\text{Id-shift}} \prod_{n=1}^{\infty} A_n \rightarrow \varprojlim_n^1 A_n \rightarrow 0$$

$$Z \xrightarrow[\text{close}]{i} X \xleftarrow{j} U \rightsquigarrow \begin{array}{ccc} \text{Id} \swarrow & i^* & \nwarrow \\ D(\text{Sh}(Z_{\text{ét}})) & \xrightarrow[\text{Id}]{\substack{\pi_i \circ i^* \text{ f.f.} \\ \pi_i \circ i^! \text{ L}}} & D(\text{Sh}(X_{\text{ét}})) \\ \text{Id} \swarrow & i^* & \nwarrow \end{array} \quad \begin{array}{ccc} j_! \text{ f.f.} & & \\ \pi_i \circ j^* & \xrightarrow[\text{Id}]{\text{f.f.}} & D(\text{Sh}(U_{\text{ét}})) \\ \pi_i \circ j_* & \xrightarrow[\text{Id}]{\text{f.f.}} & \end{array} \quad \begin{array}{ccc} i^* & j_! & \\ 0 & & \\ i_* & j^* & \\ 0 & & \\ i^! & Rj_* & \\ 0 & & \end{array}$$

$L$ : left exact (others are exact)

f.f. fully faithful

$\pi_i$ : preserve injectives. (inj)

ie. inj sheaf  $\leadsto$  inj sheaf

$$\mathcal{F}_1 \xleftarrow{i^*} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(\mathcal{F}_1, \mathcal{F}_2, \alpha) \xrightarrow{j^*} \mathcal{F}_2$$

$$\ker \alpha \xleftarrow{i^!} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(0, \mathcal{F}_2, 0) \xleftarrow{j_!} \mathcal{F}_2$$

$$\mathcal{F}_1 \xrightarrow{i^*} (\mathcal{F}_1, 0, 0)$$

$$(i^* j_* \mathcal{F}_2, \mathcal{F}_2, \text{Id}) \xleftarrow{Rj_*} \mathcal{F}_2$$

$$0 \rightarrow I \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Z \rightarrow 1$$

$$\leadsto 0 \rightarrow \tilde{I} \rightarrow \mathcal{O}_{X \times X} \rightarrow \Delta_* \mathcal{O}_X \rightarrow 1$$

$$\leadsto 0 \rightarrow \Omega_{X/k} \rightarrow \Delta^* \mathcal{O}_{X \times X} \rightarrow \Delta^* \Delta_* \mathcal{O}_X \rightarrow 1$$

$$i_! i^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1}$$

~~$\mathcal{F}$  is supported on  $Z \Leftrightarrow \mathcal{H}_Z^0(\mathcal{F}) = \mathcal{F} \Leftrightarrow j_* j^* \mathcal{F} = 0$~~  ?  $i^!$  is only defined on derived category?

$$0 \rightarrow j_! j^! \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* i^* \mathcal{F} \rightarrow 0$$

$j_! :$

<https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer3.pdf>

Check on stalks.

Kummer sequence  $1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 1$   
 $0 \longleftarrow k[x]/(x^n-1) \longleftarrow k[x, x^{-1}] \longleftarrow k[x, x^{-1}]$

$k/\mathbb{F}_p$   $1 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F:=(-)^p} \mathbb{G}_a \longrightarrow 1$   
 $0 \longleftarrow k[x]/(x^p) \longleftarrow k[x] \longleftarrow k[x]$

Artin-Schreier sequence  $1 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{F-\text{Id}} \mathbb{G}_a \longrightarrow 1$   
 $0 \longleftarrow k[x]/(x^p-x) \longleftarrow k[x] \longleftarrow k[x]$

	Zariski	étale	fppf
$\mu_n$	x	✓ when $n \in \mathbb{P}(x, \mathcal{O}_x)^\times$ x in general	✓
$\alpha_p$	x	x in general	✓
$\mathbb{Z}/p\mathbb{Z}$	x	✓	✓