Eine Woche, ein Beispiel

8.20. diagonalizable group

Ref:

[Vakil]: Vakil, The Rising Sea: Foundations of Algebraic Geometry

[Borel91]: Borel, Linear Algebraic Groups
https://link.springer.com/book/10.1007/978-1-4612-0941-6

[PerrinAG]:
http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/ag-chap3.pdf
http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/ag-chap4.pdf

[Milne]:
[Alggp]: Algebraic Groups, corrected 2022 version.

[Eberhardt23]: lecture notes of "spaces in GRT"
https://jenseberhardt.com/teaching/W2324data/Spaces%20in%20GRT.pdf

In this document, k is a field.

https://mathoverflow.net/questions/12118/what-is-an-algebraic-group-over-a-noncommutative-ring
https://mathoverflow.net/questions/448426/is-diagonalizability-a-local-property

We follow the notation of [Vakil].

gpSch : scheme + gp
AffgpSch : affine + gpSch
gpVar : variety + gp = gpSch_k + reduced + sep + ~~(irr)~~
Alggp : sm + gpVar
AffAlggp : affine + Alggp = linear alg gp
AbVar : (conn) proj + Alggp + geo integral
gpSch_k = gp scheme/k + f.t.

$$\begin{array}{ccc}
\mathrm{gpSch} & \supset & \mathrm{gpVar} \\
\cup & & \cup \\
\mathrm{gpSch}^{sm} & \supset & \mathrm{Alggp} \\
\cup & & \cup \\
\mathrm{AffgpSch}^{sm} & \supset & \mathrm{AffAlggp} \\
 & \mathrm{AntiaffgpSch}^{sm} \supset \mathrm{AbVar} &
\end{array}$$

https://math.stackexchange.com/questions/3237148/how-does-an-affine-algebraic-group-become-a-group-scheme
def of anti-affine group schemes:
https://link.springer.com/chapter/10.1007/978-93-86279-58-3_5
arxiv.org/abs/0710.5211

Chevalley's structure thm. [wiki] For k perfect,
every sm conn alggp is an extension of an abelian variety by sm conn linea alggp.

Three reasons why learning math is hard

1. unusual notation

   E.g. in [Borel 91, P111], A_k is not the base change; instead,

   $$A \cong A_k \otimes_k K$$

   ($K = \bar{k}$ is not specified near the statement neither)

2. same name with different objects.

   E.g. for Variety,

	reduced	irreducible	f.t. + sep
[Vakil], me	✓	× (✓)	✓
wiki	✓	✓	✓
[Borel 91]	✓	×	✓
[PerrinAG]	✓	×	✓
[Milne]	geo reduced	×	✓

   E.g. for algebraic group,

[Vakil], me	gpSch_k	gpSch_k^{sm}	gpVar	Alggp
wiki			Alggp	
[Borel 91]			Alggp	
[PerrinAG]			Alggp	
[Milne]	Alggp	gpVar		

   ◯: where diagonalizable gp is defined (always affine)

   ⚠ In [Milne Alggp], beginning with Chap 9, all gp schemes are affine.

   E.g. for diagonalizable gp.

[Milne], me	diagonalizable gp	multiplicative gp
[Borel 91], wiki	split diagonalizable gp	diagonalizable gp

3. oversimplification.

   E.g. $k = \bar{k}$, or $\mathrm{char}\, k = 0$.

   In [PerrinAG], $k = \bar{k}$;

   in [Eberhard t23], $k = \bar{k}$, $\mathrm{char}\, k = 0$

The results are nicer but can't be referred, and for the most of time these conditions are likely to be missed by readers looking for some results they need.

We mainly follow [Borel91, §8] in the following material,
and [MilneAlggp] for generalization containing nonreduced schemes.

Def. $D \in \mathrm{AffgpSch}_k$ is called diagonalizable, if $X^*(D)^{\Gamma_k}$ generates $k[D]$, where

$$k[D] = \mathcal{O}_D(D)$$
$$X^*(D)^{\Gamma_k} = \mathrm{Mor}_{\mathrm{Alggp}_k}(D, \mathbb{G}_m) = \mathrm{Mor}_{\mathrm{Hopf}_k}(k[t^{\pm1}], k[D]) \subseteq k[D]$$

$D \in \mathrm{AffgpSch}_k$ is called multiplicative (of multiplicative type), if $D_{k^{sep}}$ is diagonalizable.

diagonalizable gp scheme	diagonalizable alg gp
multiplicative gp scheme	multiplicative alg gp

four different objects

Prop. [MilneAlggp, Thm 12.12] [Borel, Prop 8.4]
For $D \in \mathrm{AffgpSch}_k$, TFAE:
1) D is diagonalizable;
2) $D \hookrightarrow \mathbb{G}_{m,k}^{\oplus n}$ for some n;
3) $\mathrm{Ind}_k(D) = \mathrm{Char}_k(D)$ only consider rational reps.

Prop. [MilneAlggp, Thm 12.18, Cor 12.21] [Borel, Prop 8.4'] [PerrinAG, Prop 3.3.2, Thm 4.1.8]
For $D \in \mathrm{AffgpSch}_k$, TFAE:
1) D is multiplicative;
2) D is comm & $\mathrm{Hom}(D, \mathbb{G}_a) = 0$;
3) D is comm & $k[D]$ is coétale;

Moreover, if D is sm, 1) – 3) are equiv to 4) & 4'):
4) D is comm & all $g \in D(k^{sep})$ are semisimple; i.e., $g_u = 1$
4') $\exists D' \overset{\text{dense}}{\subseteq} D(k^{sep})$, D' is comm & all $g \in D'(k^{sep})$ are semisimple.

Fact. [MilneAlggp, Thm 12.9 (a), Thm 12.23, Thm 11.45] [Borel, p119]
For κ field, we have the equiv of categories:

$$\left\{\begin{array}{l}\text{Obj: multiplicative gp } D/\kappa \\ \text{Mor: } \kappa\text{-morphisms}\end{array}\right\} \longrightarrow \left\{\begin{array}{l}\text{Obj: f.g. } \mathbb{Z}\text{-mod} \\ \qquad \text{with cont } \Gamma_\kappa\text{-action} \\ \text{Mor: } \Gamma_\kappa\text{-equiv } \mathbb{Z}\text{-mod homo}\end{array}\right\}^{op}$$

$$\cup\!\!\mid \qquad\qquad\qquad\qquad \cup\!\!\mid$$

$$\left\{\begin{array}{l}\text{Obj: multiplicative gp } D/\kappa \\ \text{Mor: } \kappa\text{-morphisms}\end{array}\right\} \longrightarrow \{\text{f.g. } \mathbb{Z}\text{-mod}\}^{op}$$

$$G \longmapsto X^*(G) := \mathrm{Hom}_{\mathrm{gpSch}_{\kappa^{sep}}}(G_{\kappa^{sep}}, \mathbb{G}_{m,\kappa^{sep}})$$
$$D(M) := \mathrm{Spec}(\kappa[M]) \longleftarrow M$$

e.g.
$$\mu_{n,\kappa} \longleftrightarrow \mathbb{Z}/n\mathbb{Z}$$
$$\mathbb{G}_{m,\kappa} \longleftrightarrow \mathbb{Z}$$

Q: How far can this equiv of category can be generated when we replace κ by a general ring R?

E.x. Using the equiv of category, verify that

$$\mathrm{Hom}(\mu_{n,\kappa}, \mu_{m,\kappa}) = \mathbb{Z}/\mathrm{lcm}(n,m)\mathbb{Z} \qquad \mathrm{Aut}(\mu_{n,\kappa}) = (\mathbb{Z}/n\mathbb{Z})^\times$$
$$X^*(\mu_{n,\kappa}) = \mathrm{Hom}(\mu_{n,\kappa}, \mathbb{G}_m) = \mathbb{Z}/n\mathbb{Z}$$
$$X_*(\mu_{n,\kappa}) = \mathrm{Hom}(\mathbb{G}_m, \mu_{n,\kappa}) = 0$$
$$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z} \qquad \mathrm{Aut}(\mathbb{G}_m) = \{\pm 1\}$$

Rmk. In [Borel, p119], the author assumes that diagonalizable gps are reduced, so μ_p ($p = \mathrm{char}\ \kappa$) is removed, and one only gets f.g. $\mathbb{Z}$-mod without p-torsion on the right hand side.
In fact, one has [MilneAlggp, Rmk 12.5]

$$D(M) \text{ is reduced} \Leftrightarrow M \text{ has no } p\text{-torsion}$$
$$\text{smooth} \Leftrightarrow$$
$$D(M) \text{ is connected} \Leftrightarrow M \text{ has no } l\text{-torsion}, \ \forall\, l \neq p$$
$$D(M) \text{ is sm \& conn} \Leftrightarrow M \text{ is free}$$
$$\Leftrightarrow D(M) \text{ is a torus}$$

Ex. Construct the following (non-split) SES of gp schemes.

Kummer sequence :

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 1$$

$$1 \longrightarrow \mathrm{Res}'_{E/F}\mathbb{G}_m \longrightarrow \mathrm{Res}_{E/F}\mathbb{G}_m \xrightarrow{\text{Norm}} \mathbb{G}_m \longrightarrow 1$$

$$1 \longrightarrow \mathbb{G}_m \longrightarrow \mathrm{Res}_{E/F}\mathbb{G}_m \longrightarrow \mathrm{Res}'_{E/F}\mathbb{G}_m \longrightarrow 1$$

Also, verify that

$$\mathrm{End}(SO_{2,\mathbb{R}}) = \mathbb{Z} \qquad \mathrm{Aut}(SO_{2,\mathbb{R}}) = \mathbb{Z}/2\mathbb{Z}$$

$$\mathrm{End}(\mathrm{Res}_{E/F}\mathbb{G}_m) = \mathbb{Z}^{\oplus 2} \qquad \mathrm{Aut}(\mathrm{Res}_{E/F}\mathbb{G}_m) = \mathbb{Z}/2\mathbb{Z}^{\oplus 2}$$