Eine Woche, ein Beispiel

1.28 conormal bundle

Ref: from [23.11.19]

slogan:

SES	induces	LES,
filtration	induces	spectral sequence.

To expend a little bit,

$$\begin{tikzcd}
 & \text{SES} \arrow[r] & \text{LES} & \\
 & \text{filtration} \arrow[r] & \text{spectral seq} \arrow[dr] & \\
\text{resolution} \arrow[ur, dashed] \arrow[rrr] & & & \begin{array}{l}\text{comparison between} \\ \text{sheaf cohomology} \\ \text{\& other cohomology}\end{array}
\end{tikzcd}$$

Even though "filtration $\Rightarrow$ spectral seq" is the most general statement, people start with "SES $\Rightarrow$ LES" and "acyclic resolution $\Rightarrow$ other coh $\approx$ hyper coh". Let us leave spectral seq in other people's notes.

1. open-closed formalism
2. open cover
3. filtrations from chain complex
4. filtration by $\mathcal{H}^i(\mathcal{F}^\cdot)$
5. filtration by $\mathcal{F}^\cdot$
6. Hodge related filtration

Methods to construct SES: $\begin{cases} \text{check by stalks} \\ \text{filtration by } \mathcal{H}^i(\mathcal{F}) \\ \text{filtration by } \mathcal{F}^i \end{cases}$

method	spectral seq	LES	cohomology/resolution
check by stalks	… for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived fctor	MV	Čech
	coefficient		—
filtration by $\mathcal{H}^i(\mathcal{F})$	Grothendieck		
	Leray-Serre	Gysin	Euler class
			Hodge-Tate
Hodge related; filtration by $\mathcal{F}^i$	Hodge-de Rham		de Rham, Hodge-de Rham
need resolution			Dolbeault $H^p(X,\Omega^q) = H^{p,q}(X)$
to get "another" complex	Frölicher		$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition"
			singular
spectral sequences	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
which	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
I don't know	Green		for Koszul cohomology
	⋮		⋮

For more spectral sequences, see:
https://en.wikipedia.org/wiki/Spectral_sequence
https://github.com/CubicBear/SpectralSequences/blob/main/SpecralSequences.pdf

1. open-closed formalism

‖ related: comparison of $j_!$ & j_*
one-point compactification.

Observe the following pictures:

$$Z \xrightarrow{i} X \xleftarrow{j} \mathcal{U}$$

$$\begin{tikzcd} \mathcal{D}(Z) \arrow[r, "i_* = i_!"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

Black box:

0. We assume some nice conditions.
   e.g. in the category $\mathrm{Haus}^{\text{loc. cpt}}$, and $Z \subset X$ is loc. contractable.
   Under these conditions,
1. $i_* = i_!$, $j^* = j^!$
2. $j_!, i^*, j^*, i_*$ are exact.

Ex. 1. Shows that

$$i^* i_* = i^! i_* = Id_{\mathcal{D}(Z)} \qquad j^* j_! = j^* Rj_* = Id_{\mathcal{D}(\mathcal{U})}$$

$$i^* j_! = 0, \qquad j^* i_* = 0, \qquad i^! Rj_* = 0$$

——: base change
〰〰: check stalkwise.

2. (for category fans)
   $i_*, j_*, j_!$ are fully faithful, and
   $i_*, i^!, j^*, Rj_*$ preserve injectives.

3. One has SES

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0 \qquad (1)$$

Ex for (1).

1. Apply the $R\pi_{X,*}$ to (1), take $\mathcal{F} = \underline{\mathbb{Q}}_X$, what do we get?

In general, what do we get when applying $R\pi_{X,*}$ & $R\pi_{X,!}$?
Discuss 2 spectural cases $\mathcal{F} = \underline{\mathbb{Q}}_X$ $\mathbb{D}_X$ $\mathbb{D}_X := \pi_X^! \underline{\mathbb{Q}}_{\{*\}} = \mathbb{D}_X(\underline{\mathbb{Q}}_X)$

2. Derive from (1) the SES

$$0 \longrightarrow j_!\mathcal{F} \longrightarrow Rj_*\mathcal{F} \longrightarrow i_*i^* Rj_*\mathcal{F} \longrightarrow 0$$

which measures the difference between $j_!\mathcal{F}$ & $j_*\mathcal{F}$.

3. Shows that

$$H_c^{\cdot}(X) \cong H^{\cdot}(\bar{X}, \{\infty\}; \mathbb{Z})$$

for one pt compactification $\iota: X \hookrightarrow \bar{X}$.

Try to compute $H_c^{\cdot}(\mathbb{R}^n)$ in this way.

It seems that we get only half of the results.

Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathbb{D}_X : D^b(X;\mathbb{Q}) \longrightarrow D^b(X;\mathbb{Q}) \qquad \mathbb{D}_X\mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{D^b(X;\mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\{*\}})$$

We know that

$$\mathbb{D}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\{*\}} \qquad \mathbb{D}_X(\mathcal{F}^\cdot[n]) = (\mathbb{D}_X\mathcal{F}^\cdot)[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \quad \rightsquigarrow \quad \mathbb{D}\mathcal{H}^\cdot \longrightarrow \mathbb{D}\mathcal{G}^\cdot \longrightarrow \mathbb{D}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathbb{D}_X = \mathbb{D}_Y f^* \qquad Rf_* \mathbb{D}_Y = \mathbb{D}_X Rf_!$$

$f: Y \longrightarrow X$

When $\mathcal{F}^\cdot \in D^b(X;\mathbb{Q})$ is constructable, then

$$\mathbb{D}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructable setting,

$$f^* \mathbb{D}_X = \mathbb{D}_Y f^! \qquad Rf_! \mathbb{D}_Y = \mathbb{D}_X Rf_*$$

For exact statements about $\mathbb{D}_X$, see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1} \qquad (2)$$

for $\mathcal{F}^\cdot \in \mathcal{D}^b(X;\mathbb{Q})$ constructable.

Ex for (2). Do the same arguments in "Ex for (1)".

E.g. For a finite graph (as a topo space) X.

$$sk_0 X \xhookrightarrow{i} X \xleftarrow{j} X - sk_0 X \qquad \text{1-cells}$$

$$0 \longrightarrow j_! j^! \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X - sk_0 X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{sk_0 X} \longrightarrow 0$$

Take $R\pi_{X,!}$

$$\begin{tikzcd}
& \overset{=\oplus_e \mathbb{Q}}{H^1_c(X - sk_0 X)} \arrow[r] & H^1_c(X) \arrow[r] & \overset{=0}{H^1_c(sk_0 X)} \arrow[r, "+1"] & {} \\
0 \arrow[r] & \overset{=0}{H^0_c(X - sk_0 X)} \arrow[r] & H^0_c(X) \arrow[r] & H^0_c(sk_0 X) = \oplus_v \mathbb{Q} \arrow[ull, bend right]
\end{tikzcd}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that

$$H^i_c(\mathbb{R}) = \begin{cases} \mathbb{Q} & i = 1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial/cellular cohomology, using the following filtration:

$$\emptyset \subset sk^0 X \subset sk^1 X \subset \cdots \; sk^n X = X$$

Ex. derive the Wang LES for the cpt supp version. over S^1

2. open cover

Ex. For an open cover $X = \mathcal{U}_1 \cap \mathcal{U}_2$, deduce the SES

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus j_! \underline{\mathbb{Q}}_{\mathcal{U}_2} \longleftarrow j_! \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \longleftarrow 0$$
$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_1} \oplus Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_{12}} \xrightarrow{+1} \qquad (3)$$

⚠ We omit the derived symbol and some subscripts in this section. $\mathcal{U}_{12} = \mathcal{U}_1 \cap \mathcal{U}_2$

(3) works for general sheaf

and. induce from (3) the MV sequence:

$$\xleftarrow{+1} H^k_c(X) \longleftarrow H^k_c(\mathcal{U}_1) \oplus H^k_c(\mathcal{U}_2) \longleftarrow H^k_c(\mathcal{U}_1 \cap \mathcal{U}_2)$$
$$H^k(X) \longrightarrow H^k(\mathcal{U}_1) \oplus H^k(\mathcal{U}_2) \longrightarrow H^k(\mathcal{U}_1 \cap \mathcal{U}_2) \xrightarrow{+1}$$

Hint: Apply $R\pi_{X,!}$ & $R\pi_{X,*}$, see [StackProject, 01E9]

Ex. Derived the Wang LES. over S^1

Ex. For an open cover $X = \bigcup_{i \in \Lambda} \mathcal{U}_i$, Λ finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \longleftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{\mathcal{U}_i} \longleftarrow \bigoplus_{i<j} j_! \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longleftarrow \cdots \; j_! \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longleftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i} \longrightarrow \bigoplus_{i<j} Rj_* \underline{\mathbb{Q}}_{\mathcal{U}_i \cap \mathcal{U}_j} \longrightarrow \cdots \; Rj_* \underline{\mathbb{Q}}_{\bigcap_i \mathcal{U}_i} \longrightarrow 0$$

When $\{\mathcal{U}_i\}_{i \in \Lambda}$ is a good cover, $H^\cdot(\mathcal{U}_{i_1 \cdots i_k}) = H^0(\mathcal{U}_{i_1, \cdots i_k})$,

acyclic in AG

one can compute $H^\cdot(X)$ by applying $R\pi_{X,*}$:

$$0 \longrightarrow \bigoplus_i \Gamma(\mathcal{U}_i) \xrightarrow{d^1} \bigoplus_{i<j} \Gamma(\mathcal{U}_i \cap \mathcal{U}_j) \xrightarrow{d^2} \cdots \; \Gamma(\bigcap_i \mathcal{U}_i) \longrightarrow 0$$

$\Downarrow$ Ker/Im

$$H^0(X) \qquad H^1(X) \qquad \cdots \qquad H^{\#\Lambda - 1}(X)$$

Rmk. When X is paracompact & Hausdorff. "limited" Čech = sheaf

e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$\mathcal{F} \longrightarrow \bigoplus_i Rj_* \mathcal{F}|_{\mathcal{U}_i} \qquad \qquad \mathcal{F} \longrightarrow \bigoplus_{x \in X} \mathcal{F}_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

$\# \Lambda = 3$ case:

$$
\begin{tikzcd}[column sep=small]
 & & 0 \arrow[dr] & & 0 & & & 0 \arrow[dr] & & 0 & & \\
 & & & \mathcal{F}_0 \arrow[ur] \arrow[dr] & & & & & \mathcal{F}_2 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[r] & \mathbb{Q}_X \arrow[rr] \arrow[urr] & & \bigoplus_i Rj_*\mathbb{Q}_{\mathcal{U}_i} \arrow[rr, "d^1"] \arrow[dr] & & \bigoplus_i Rj_*\mathbb{Q}_{\mathcal{U}_i\cap\mathcal{U}_j} \arrow[rr, "d^2"] \arrow[urr] & & & \arrow[ll, phantom] Rj_*\mathbb{Q}_{\bigcap_i \mathcal{U}_i} \arrow[rr, "d^3"] & & 0 & \\
 & & & & \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & & & & & \\
 & & & 0 \arrow[ur] & & 0 & & & & & &
\end{tikzcd}
$$

$$\mathcal{F}_2 = Rj_*\mathbb{Q}_{\bigcap_i \mathcal{U}_i} \quad \Rightarrow \quad H^\bullet(\mathcal{F}_2) = \operatorname{Ker} d^3$$

$$
\begin{tikzcd}
 & H^1(\mathcal{F}_1) \arrow[r] & 0 \arrow[r] & H^1(\mathcal{F}_2) \arrow[r, "+1"] & {} \\
0 \arrow[r] & H^0(\mathcal{F}_1) \arrow[r] & \bigoplus_{i<j}\Gamma(\mathcal{U}_i\cap\mathcal{U}_j) \arrow[r, "d^2"] & H^0(\mathcal{F}_2) \arrow[ull, bend right] &
\end{tikzcd}
$$

$=0$

$$
\Rightarrow \quad H^i(\mathcal{F}_1) = \begin{cases} \operatorname{Ker} d^3/\operatorname{Im} d^2, & i=1 \\ \operatorname{Ker} d^2, & i=0 \\ 0, & \text{otherwise} \end{cases}
$$

$$
\begin{tikzcd}
 & H^2(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & H^2(\mathcal{F}_1) \arrow[r, "+1"] & {} \\
 & H^1(\mathcal{F}_0) \arrow[r] & 0 \arrow[r] & H^1(\mathcal{F}_1) \arrow[ull, bend right] & \\
0 \arrow[r] & H^0(\mathcal{F}_0) \arrow[r] & \bigoplus_i \Gamma(\mathcal{U}_i) \arrow[r, "d^1"] & H^0(\mathcal{F}_1) \arrow[ull, bend right] &
\end{tikzcd}
$$

$=0$

$$
\Rightarrow \quad H^i(X) = H^i(\mathcal{F}_0) = \begin{cases} \ker d^3/\operatorname{Im} d^2 & i=2 \\ \ker d^2/\operatorname{Im} d^1 & i=1 \\ \ker d^1 & i=0 \\ 0, & \text{otherwise} \end{cases}
$$

Rmk. When $\{\mathcal{U}_i\}_{i\in\Lambda}$ is not a good cover,
one needs Čech-to-derived functor spectral seq to compute $H^\cdot(X)$.

Rmk. stratification & open cover are two main tools to extract topological information.
They appear with different names in different fields.
Once you realize them, you can apply the six-functor machine to analyze structures.

stratification with extra properties $\begin{cases} \text{CW cplx} \\ \text{triangulization} \\ \text{dessin d'enfant} \\ \text{affine paving} \\ \text{Whitney stratification} \\ \vdots \end{cases}$

Q. How to construct stratifications?
A. For me, there are two efficient methods: $\begin{cases} \text{orbit of gp action} \\ \text{Morse theory} \end{cases}$

That's why some geometrical problems are finally reduced to combinatorical / analytic problems.
Other fields come to geometry by providing stratifications.

In fact, there is only one method:
find a fct $f: X \longrightarrow Y$, and get stratification of X from Y.

E.g.

1. Morse theory $\quad f: X \longrightarrow \mathbb{R}$
2. tessellation $\quad f: \mathcal{H} \longrightarrow \mathcal{H}/\Gamma$
3. semi-continuous fct $\quad f: X \longrightarrow \mathbb{N}_{\geq 0}$ $\quad$ e.g. $f(p) = \dim T_p X$
4. my thesis $\quad f: Gr(X) \longrightarrow Gr(S) \times Gr(X/S)$
5. orbit of gp action $\quad f: X \longrightarrow X/G$

3. filtrations from chain complex [Stack Project, 0118]

Lots of filtrations are obtained just from the naive complex.

Consider a chain complex $C^\cdot$:

$$\cdots \xrightarrow{d^{-2}} C^{-2} \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^{0}} C^{0} \xrightarrow{d^{1}} C^{1} \xrightarrow{d^{2}} C^{2} \xrightarrow{d^{3}} \cdots$$

One can make a "stupid" truncation

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] & C^{-1} \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \sigma_{\geq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \sigma_{\leq -1} C^\cdot \longrightarrow 0$$

One can also make a canonical truncation

$$\begin{tikzcd}
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & \operatorname{Ker} d^1 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, "d^{-2}"] & C^{-2} \arrow[r, "d^{-1}"] \arrow[d] & C^{-1} \arrow[r, "d^0"] \arrow[d] & C^0 \arrow[r, "d^1"] \arrow[d] & C^1 \arrow[r, "d^2"] \arrow[d] & C^2 \arrow[r, "d^3"] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \operatorname{coker} d^0 \arrow[r, "d^1"] & C^1 \arrow[r, "d^2"] & C^2 \arrow[r, "d^3"] & \cdots
\end{tikzcd}$$

which is denoted by

$$0 \longrightarrow \tau_{\leq 0} C^\cdot \longrightarrow C^\cdot \longrightarrow \tau_{\geq 1} C^\cdot \longrightarrow 0$$

Using these truncations, one can easily construct filtrations:

$$0 \subset \cdots \subset \sigma_{\geq 1} C^{\cdot} \overset{C^{\cdot}[-1]}{\subset} \sigma_{\geq 0} C^{\cdot} \overset{C^{0}}{\subset} \sigma_{\geq -1} C^{\cdot} \subset \cdots \subset C^{\cdot}$$

$$0 \subset \cdots \subset \tau_{\leq -1} C^{\cdot} \overset{\mathcal{H}^0(C^{\cdot})}{\subset} \tau_{\leq 0} C^{\cdot} \overset{\mathcal{H}^1(C^{\cdot})[1]}{\subset} \tau_{\leq 1} C^{\cdot} \subset \cdots \subset C^{\cdot}$$

Rmk. 1. These two filtrations have opposite directions!
(a striking feature for me)
2. The "stupid" truncation extracts pieces of the chain cplx, while the canonical truncation extracts cohomology. (Ker/Im)
Therefore, only the canonical truncation can be defined on the derived category.
This information is culmulated in the standard/natural t-structure $(\mathcal{D}_{\leq 0}, \mathcal{D}_{\geq 0})$.
One has adjoint fctors:

$$\begin{tikzcd} \mathcal{D}_{\leq 0} \arrow[r, bend left, "\iota_{\leq 0}"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "\tau_{\leq 0}"] \arrow[r, bend left, "\tau_{\geq 1}"] \arrow[r, phantom, "\perp"] & \mathcal{D}_{\geq 1} \arrow[l, bend left, "\iota_{\geq 0}"] \end{tikzcd}$$

The following notations are from: https://ncatlab.org/nlab/show/t-structure

$\mathcal{D}_{\leq 0}$: t-co-connective objects
$\mathcal{D}_{\geq 0}$: t - connective objects
$\tau_{\geq 0}$: connective cover

Let's apply these filtrations!

4. filtration by $\mathcal{H}^i(\mathcal{F}^\cdot)$

Ex. Suppose that $\pi : E \longrightarrow B$ is an oriented S^k-bundle.
Analyze $R\pi_* \mathbb{Q}_E$, and apply $R\pi_{B,*}$ to get the Gysin sequence:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[+1]{eu_\pi \wedge}$$

Q: Why does $\pi_* : \mathcal{D}^b(E) \longrightarrow \mathcal{D}^b(B)$ takes injective objects to $\pi_{B,*}$-acyclic objects?

Rmk. 1. Here we can't use the "stupid" truncation, because $R\pi_* \mathbb{Q}_E$ lies in the derived category.

2. You can generalize it to fiber bundle. then you will get the Leray-Serre spectral sequence.
Think how the following conditions simplify the final results.
① π is oriented S^k-bundle
② B is simply-connected
③ (Leray-Hirsch) $H^*(E;\mathbb{Q}) \longrightarrow H^*(F;\mathbb{Q})$ is surjection
④ $\pi_1(B)$ acts on $H^*(F)$ trivially.

3. This is also a special case of Grothendieck-Serre spectral sequence.