Eine Woche, ein Beispiel
6.19. idempotent algebras.

This document want to discuss some basic contents of the course "https://people.mpim-bonn.mpg.de/scholze/Complex.pdf", Lecture 5.
For me I've never noticed about this special structure before. Hope that you enjoy this small magic.
This can be a perfect series of exam questions for the Algebra III in USTC. (better if they have learned computations on tensor products)

Q: Find all (reduced) $\mathbb{Z}$-algebra A s.t.

$$A \otimes_{\mathbb{Z}} A \cong A$$

as a $\mathbb{Z}$-alg iso.

A crash recap on [Vakil 9.2] Skip if you know fiber product of schemes!

Ex. $C \in CRing$, $A, B \in C\text{-}Alg$ $\Rightarrow$ $A \otimes_C B$ is C-Alg. and

$$\begin{array}{ccc} A\otimes_C B & \longleftarrow & A \\ \uparrow & & \uparrow \\ B & \longleftarrow & C \end{array}$$

is a pushout.

Let $\phi: B \longrightarrow A$ be a ring homomorphism. $I \lhd B$ S multiplicative set

9.2.A. (Adding an extra variable) $\quad A \otimes_B B[t] \cong A[t]$

9.2.B (Quotient) $\quad A \otimes_B B/I \cong A/\phi(I)$

9.2.F (Localization) $\quad A \otimes_B S^{-1}B \cong [\phi(S)]^{-1}A$

Ex. Compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

Compute fibers of $\operatorname{Spec} \mathbb{Z}[i] \longrightarrow \operatorname{Spec} \mathbb{Z}$ $\qquad \operatorname{Spec} \mathbb{C}[x] \longrightarrow \operatorname{Spec} \mathbb{C}[y]$

$x^2 \longleftarrow y$

Definition and some cases

Def. Let $R \in \mathrm{Ring}$. $A \in R\text{-}\mathrm{Alg}$ is called **idempotent R-algebra** if

$$A \otimes_R A \cong A \qquad \text{induced by } A \cong R \otimes_R A \longrightarrow A \otimes_R A$$

as an R-alg iso.

Ex. Verify that $\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Q}$ are idempotent $\mathbb{Z}$-algebras.
Is $\mathbb{F}_{p^2}$ idempotent? Is $\mathbb{Z}/p^2\mathbb{Z}$ idempotent? Is $\mathbb{Z}_p$ idempotent?

A new topology on Spec A

Def. (Constructible topology)

$X \subseteq \operatorname{Spec} A$ is called constructible closed if

$$\exists\, f : \operatorname{Spec} B \longrightarrow \operatorname{Spec} A \qquad \operatorname{Im} f = X$$

Ex. Find all constructible closed subset of $\operatorname{Spec} \mathbb{Z}$

Ex. Find all constructible closed subset of $\operatorname{Spec} \mathbb{C}[X]$

Ex. $\{$Zariski closed / open subset$\} \subseteq \{$constructible closed set$\}$

Central result I want to prove

Conj. [Condensed, Lec 5 Ex 2] Suppose $R \in \mathrm{CRing}$ is Noetherian. Then

$$\{(\text{reduced}) \text{ idem } R\text{-algs}\} \xleftrightarrow{1:1} \{\text{constructible closed subset of } \operatorname{Spec} R\}$$
$$A \longmapsto \operatorname{Im}(\operatorname{Spec} A \longrightarrow \operatorname{Spec} R)$$

may be false

Ex. Verify this for $\operatorname{Spec} \mathbb{Z}$.

They may be not idem $\mathbb{C}[x]$-alg?

Ex. Verify that $\mathbb{C}[x]/(x-a)$, $\mathbb{C}(x)$, $\mathbb{C}[x, \frac{1}{x}]$, $\mathbb{C}[[x]]$, $\mathcal{O}(D)$, $\mathcal{O}(\bar{D})$ are idem $\mathbb{C}[x]$-algs.
What constructible closed subset do they correspond?

$\mathbb{C}((x))$ is not $\mathbb{C}[x]$-idem algs

$\mathcal{O}(D) = \{\sum_{i=0}^{+\infty} a_i T^i \mid a_i r^i \to 0 \ \forall r < 1\} \subseteq \mathbb{C}[[x]]$

$\mathcal{O}(\bar{D}) = \bigcap_{r>1} \{\sum_{i=0}^{+\infty} a_i T^i \mid a_i r^i \to 0\} \subseteq \mathbb{C}[[x]]$

Lem. A, A' are idem R-algs. Then $\# \operatorname{Mor}_{R\text{-alg}}(A, A') \leq 1$.

Proof.
$$R \longrightarrow A \xrightarrow{f} A' \cong R \otimes_R A' \longrightarrow A \otimes_R A'$$
$$-\otimes_R A': \quad \underset{A'}{\overset{\parallel}{R \otimes_R A'}} \longrightarrow A \otimes_R A' \xrightarrow{f \otimes_R A'} \underset{A'}{\overset{\parallel}{A' \otimes_R A'}} \cong R \otimes_R A' \otimes_R A' \longrightarrow \underset{A \otimes_R A'}{\overset{\parallel}{A \otimes_R A' \otimes_R A'}}$$
$\Rightarrow A \otimes_R A' \cong A'$ doesn't depend on f, so f given by
$$A \longrightarrow A \otimes_R A' \cong A'$$
is unique.

Cor. $\{\text{idem } R\text{-algs}\}$ is a poset.

Fact. This order is compatible with constructible topology

(Only consider reduced algs. R is Noetherian.)

may be false → Ex.

$$\{(\text{reduced}) \text{ idem } R\text{-algs}\} \xleftrightarrow{1:1} \{\text{constructible closed subset of } \operatorname{Spec} R\}$$
$$A \longleftrightarrow Z$$
$$A \otimes_R A' \longleftrightarrow Z \cap Z'$$
$$\operatorname{Ker}[A \oplus A' \longrightarrow A \otimes_R A'] \longleftrightarrow Z \cup Z'$$
$$\varinjlim_i A_i \longleftrightarrow \bigcap_i Z_i$$

e.p. $Z \subset Z'$ iff $A \otimes_R A' \cong A$.