Eine Woche, ein Beispiel

5.1 Extension of NA local field

F: NA local field

1. List of well-known results
   - in general
   - unramified / totally ramified
2. $\hat{\mathbb{Z}}$ = profinite completion (review)
3. Big picture
4. A detailed discussion concerning proofs.
5. Henselian ring
6. Cohomological dimension.

not complete, I need time to check the proof

7. Bonus: "plane geometry" for $\mathbb{Q}_q$.

Q. Is there any subfield of $\mathbb{Q}_p$ with finite index?
Can we classify all subfield of $\mathbb{F}_p((t))$ with finite index?

https://math.stackexchange.com/questions/211582/is-there-a-proper-subfield-k-subset-mathbb-r-such-that-mathbb-rk-is-fin

Ref:
Initial motivation comes from
[AY]https://alex-youcis.github.io/localglobalgalois.pdf
which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

[JPS96] Galois cohomology by Jean-Pierre Serre
http://p-adic.com/Local%20Fields.pdf
https://people.clas.ufl.edu/rcrew/files/LCFT.pdf
http://www.mcm.ac.cn/faculty/tianyichao/201409/W020140919372982540194.pdf

For existence and uniqueness of extension of valuation, see Theorem 3.2 here:
https://www.dpmms.cam.ac.uk/~ajs1005/ANT/notes_s3-4.pdf

1. List of well-known results

In general

F: NA local field $\quad E/F$: finite extension

Rmk 1. E is also a NA local field with uniquely extended norm

$$\|x\| := \|N_{E/F}(x)\|_F^{\frac{1}{n}} \qquad \text{resp. } v(x) := \tfrac{1}{n} v_F(N_{E/F}(x))$$

E.g.

$\|1-\zeta_n\| = 1$ in $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$ $\quad p\nmid n \qquad v(1-\zeta_n)=0$

$\|1-\zeta_p\| = \frac{1}{\sqrt[p]{p}}$ in $\mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p \qquad v(1-\zeta_p) = \frac{1}{p}$

$\|1-\zeta_5\| = \|(1-\zeta_5)(1-\zeta_5^2)(1-\zeta_5^3)(1-\zeta_5^4)\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \|5\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \frac{1}{\sqrt[5]{5}}$ in $\mathbb{Q}_5(\zeta_5)$

$\|1-\zeta_{p^n}\| = p^{-\frac{1}{\varphi(p^n)}}$ in $\mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p \qquad v(1-\zeta_{p^n}) = \frac{1}{\varphi(p^n)}$

$\Rightarrow 1-\zeta_{p^n}$ is a uniformizer of $\mathbb{Q}_p(\zeta_{p^n})$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$ is monogenic, i.e. $\mathcal{O}_E = \mathcal{O}_F[\alpha]$ $\quad \exists \alpha \in \mathcal{O}_E$

A proof of this may be found here:
https://math.stackexchange.com/questions/3406117/ring-of-integers-of-simple-field-extension-of-local-field-is-monogenic

Cor. (primitive element thm for NA local field)

$E = F[x]/(g(x)) \quad \exists x \in \mathcal{O}_E,\ g(x)$ min poly of x.

Rmk: Every separable finite field extension has a primitive element, see wiki:
https://en.wikipedia.org/wiki/Primitive_element_theorem
Separable condition is necessary, see
https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element

⚠ $\mathcal{O}_E$ may be not a free $\mathcal{O}_F$-module.

See: https://kconrad.math.uconn.edu/blurbs/gradnumthy/notfree.pdf

Rmk 3. Any finite extension of $\mathbb{Q}_p$ is of form $\mathbb{Q}_p[x]/(g(x))$,
where $g(x) \in \mathbb{Q}[x]$ is an irr poly.
Any finite extension of $\mathbb{F}_q((t))$ is of form $\mathbb{F}_q((t))[x]/(g(x))$
where $g(x) \in \mathbb{F}_q(t)[x]$ is an irr poly.
Both are achieved by Krasner's lemma.

From [https://math.mit.edu/classes/18.785/2017fa/LectureNotes11.pdf]:
Remark 11.12. Krasner's lemma is another "Hensel's lemma" in the sense that it characterizes Henselian fields (fraction fields of Henselian rings);

https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete

$$v = v_F = \frac{1}{e} v_E \qquad \|\cdot\| = \|\cdot\|_F = \|\cdot\|_E^{\frac{1}{e}} \qquad \mathfrak{p}_F \mathcal{O}_E = \mathfrak{p}_E^e$$

E	$v_E = ev$	$\|\cdot\|_E = \|\cdot\|^e$	$\pi_E = \pi_F^{\frac{1}{e}}$	$v(\pi_E) = \frac{1}{e}$
$\mid$ deg n				
F	v_F	$\|\cdot\|_F$	π_F	$v(\pi_F) = 1$

Unramified / totally ramified

Good ref: https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields
It collects the equivalent conditions of unramified/totally ramified field extensions.

| tot ram | wild ram
| | tame ram
| field ext
| ~~split~~ in local case

When E/F is tot ramified,

$e = n \qquad v(\pi_E) = \frac{1}{n}$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E] \qquad \min(\pi_E) \in \mathcal{O}_F[x]$ is Eisenstein poly.

Lemma. Let E/F: NA local field, $e = e(E/F)$, $r \in \mathbb{N}_{\geq 0}$. Easy to see

$$\mathfrak{p}_E^{1+r} \cap F = \{x \in F \mid v_E(x) \geq \tfrac{1}{e}(1+r)\}$$
$$\mathfrak{p}_F^{1+[\frac{r}{e}]} = \{x \in F \mid v_F(x) \geq 1 + [\tfrac{r}{e}]\}$$

Then

when E/F is tamely ramified

$$\mathrm{Tr}_{E/F}(\mathfrak{p}_E^{1+r}) \overset{=}{\subset} \mathfrak{p}_E^{1+r} \cap F = \mathfrak{p}_F^{1+[\frac{r}{e}]}$$

Table for $e = 3$: ("proof of lemma")

r	0	1	2	3	4	5	6	7
$\frac{1}{e}(1+r)$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$
$1 + [\frac{r}{e}]$	1	1	1	2	2	2	3	3

E.g. $E/F = \mathbb{Q}_{49}/\mathbb{Q}_7 = \mathbb{Q}_7(\sqrt{3})/\mathbb{Q}_7$ is unramified.

$$\begin{aligned} v(a+b\sqrt{3}) &= \tfrac{1}{2} v(N_{E/F}(a+b\sqrt{3})) \\ &= \tfrac{1}{2} v(a^2-3b^2) \\ &= \tfrac{1}{2}\min(v(a^2), v(b^2)) \\ &= \min(v(a), v(b)) \end{aligned}$$

$a, b \in \mathbb{Q}_7$

$\mathcal{O}_E = \mathbb{Z}_7(\sqrt{3})$ $\quad \mathfrak{p}_E = (7, 7\sqrt{3}) = (7)$ $\quad k_E = \mathbb{Z}_7(\sqrt{3})/(7)$

$\cong \mathbb{Z}_7[\alpha]/(\alpha^2-3, 7) \cong \mathbb{F}_7(\sqrt{3}) \cong \mathbb{F}_{49}$

$\mathfrak{p}_E^{1+r} = (7)^{1+r} = (7^{1+r})$ $\quad Tr_{E/F}(\mathfrak{p}_E^{1+r}) = \mathfrak{p}_F^{1+r} = \mathfrak{p}_E^{1+r} \cap F$ $\quad r \geq 0$

E.g. $E/F = \mathbb{Q}_7(\sqrt{7})/\mathbb{Q}_7$ is tamely ramified.

$$\begin{aligned} v(a+b\sqrt{7}) &= \tfrac{1}{2} v(N_{E/F}(a+b\sqrt{7})) \\ &= \tfrac{1}{2} v(a^2-7b^2) \\ &= \tfrac{1}{2}\min(v(a^2), 1+v(b^2)) \\ &= \min(v(a), \tfrac{1}{2}+v(b)) \end{aligned}$$

$a, b \in \mathbb{Q}_7$

$\mathcal{O}_E = \mathbb{Z}_7(\sqrt{7})$ $\quad \mathfrak{p}_E = (7, \sqrt{7}) = (\sqrt{7})$ $\quad k_E = \mathbb{Z}_7(\sqrt{7})/(\sqrt{7})$

$\cong \mathbb{Z}_7[\alpha]/(\alpha^2-7, \alpha) \cong \mathbb{Z}_7/(7) \cong \mathbb{F}_7$

$$\mathfrak{p}_E^{1+r} = (\sqrt{7})^{1+r} = \begin{cases} (7^{\frac{1+r}{2}}) & r \text{ odd} \\ \sqrt{7}\cdot(7^{\frac{r}{2}}) & r \text{ even} \end{cases}$$

$Tr(7^{\frac{1+r}{2}}) = 2\cdot 7^{\frac{1+r}{2}}$

$Tr(\sqrt{7}\cdot 7^{\frac{r}{2}}) = 0$

$r \geq 0$

So $Tr_{E/F}(\mathfrak{p}_E^{1+r}) = \mathfrak{p}_E^{1+r} \cap F = \mathfrak{p}_F^{1+[\frac{r}{2}]}$.

2. $\hat{\mathbb{Z}}$ = profinite completion of $\mathbb{Z}$ (Recall 2022.2.13 outer auto...)

$$\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l \qquad \hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$\hat{\mathbb{Z}}^\times = \prod_l \mathbb{Z}_l^\times \qquad (\hat{\mathbb{Z}}^\times)^{(p)} := \prod_{l \neq p} \mathbb{Z}_l^\times = (\hat{\mathbb{Z}}^{(p)})^\times$$

Prop. ① $\mathrm{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l = m \\ 0 & l \neq m \end{cases}$ l, m prime.

② $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$

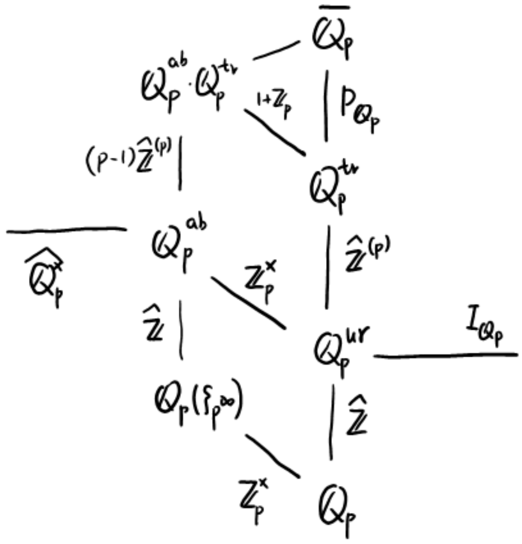
$\mathrm{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^\times$ in the category of profinite gps.

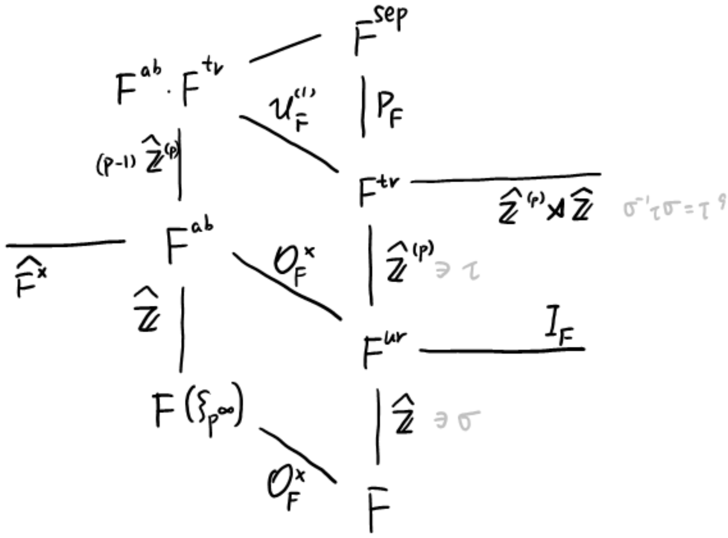
$\mathrm{Aut}(\hat{\mathbb{Z}}^{(p)}) = \hat{\mathbb{Z}}^{\times (p)}$

③ $\mathcal{O}_F, \mathcal{O}_F^\times$ are profinite groups, so $\hat{\mathcal{O}}_F = \mathcal{O}_F \quad \hat{\mathcal{O}}_F^\times = \mathcal{O}_F^\times$.

3. Big picture

Main ref: [AY]https://alex-youcis.github.io/localglobalgalois.pdf





F: NA local field with char $k = p$

unramified $\qquad F^{ur} = \bigcup_{n\geq 1} F(\zeta_{p^n-1}) \underset{\text{Fermat's little thm}}{=\!=} \bigcup_{\substack{n\geq 1\\ p\nmid n}} F(\zeta_n)$

tame ramified $\qquad F^{tr} = F^{ur}\left(\pi_F^{\frac{1}{n}}\Big|_{(n,p)=1}\right)$

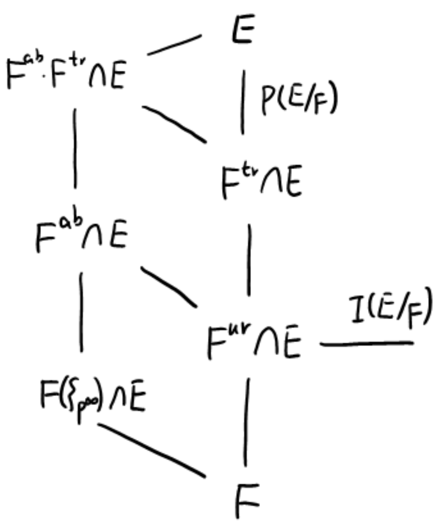
$\qquad = F\left(\pi_F^{\frac{1}{n}}, \zeta_n\Big|_{(n,p)=1}\right)$

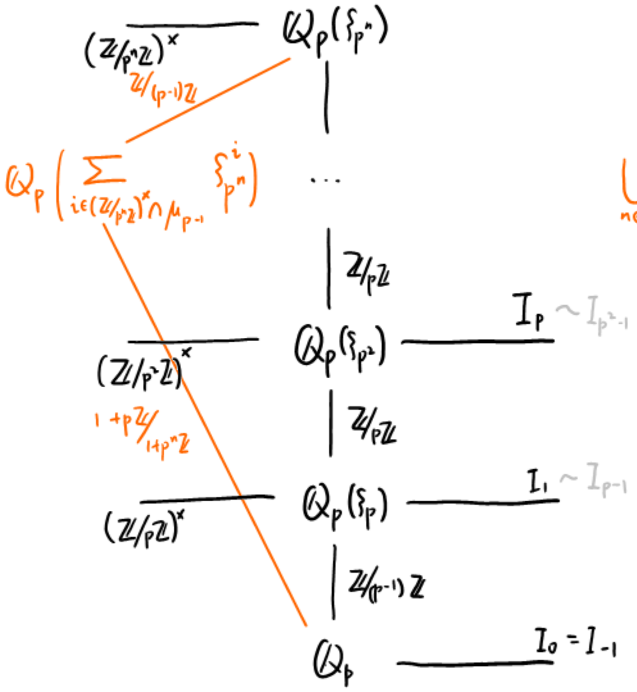
Notice that $\zeta_p \in F^{tr}$!

abelian $\qquad F^{ab} = F(\zeta_\infty) := \bigcup_{n\geq 1} F(\zeta_n)$

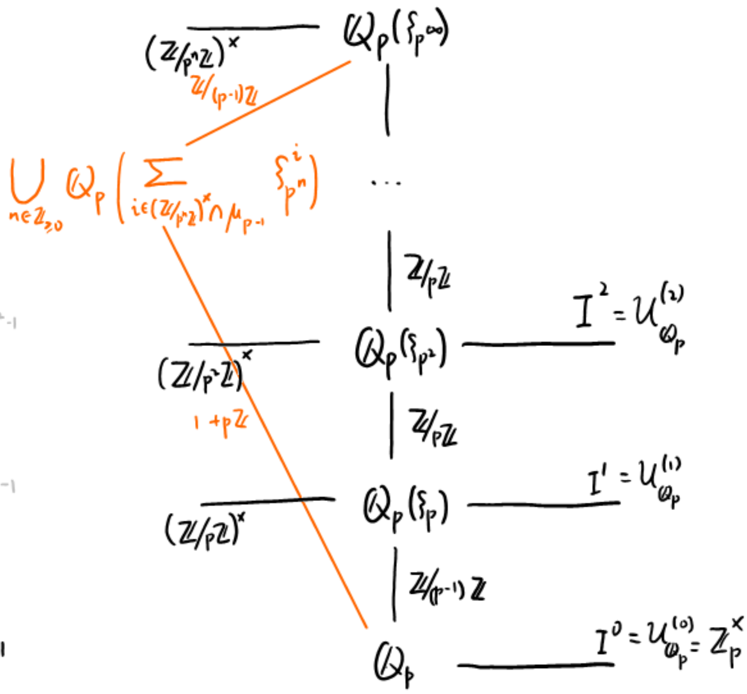
$$F^{ab}\cdot F^{tr} = F\left(\pi_F^{\frac{1}{n}}, \zeta_\infty\Big|_{(n,p)=1}\right)$$

https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions





$$E/F = \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$$



$$E/F = \mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$$

$$\begin{tikzcd} \mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \mathbb{Q}_p(\zeta_p+\zeta_p^{-1}) \arrow[d, dash] \\ \mathbb{Q}_p \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{Q}_p(\zeta_p) \arrow[d, dash] \\ \mathbb{Q}_p(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \mathbb{Q}_p \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{Q}_p(\zeta_p) \arrow[d, dash] \\ \mathbb{Q}_p(\sqrt{-p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \mathbb{Q}_p \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p(\zeta_{p^n}+\zeta_{p^n}^{-1}) \arrow[d, dash] \\ \mathbb{Q}_p^{abwr} := \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\right) \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \\ \mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \mathbb{Q}_p^{abwr} \end{tikzcd}
\qquad
\begin{tikzcd} \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \\ \mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\ \mathbb{Q}_p^{abwr} \end{tikzcd}$$

p odd | $p \equiv 1 \pmod 4$ | $p \equiv 3 \pmod 4$

There are only finite isomorphism classes of degree n extensions of Q_p, see here for a discussion:
https://math.stackexchange.com/questions/1118068/finitely-many-extensions-of-fixed-degree-of-a-local-field

Except for the filstrations as well as cohomology dimensions, the Artin--Schreier theory also gives us a better understanding of the wild inertia group. For example, there are exactly p^2 ramified degree p field extensions of Q_p (for p odd prime). A detailed discussion(and Table 2.1) can be seen here:
https://www.sciencedirect.com/science/article/pii/S0747717105001276?via%3Dihub

For a ref of the Artin--Schreier theory, you can see
https://en.wikipedia.org/wiki/Artin%E2%80%93Schreier_theory
https://math.stackexchange.com/questions/50041/reference-book-for-artin-schreier-theory (gives the proof of x^p-x-a)

Q: How many degree p field extensions of $\mathbb{F}_p((t))$ are there?

Warning:

Even though every degree p field ext of $\mathbb{Q}_p$ can be written of the form

$$\mathbb{Q}_p[x]/(x^p-x-\alpha), \quad \alpha \in \mathbb{Q}_p \quad \alpha \neq \beta^p-\beta \text{ for } \beta \in K$$

it's not feasible to do so when we do for examples (no good parameters)

e.g. It's not easy to find a canonical $\alpha \in \mathbb{Q}_p$ s.t.

$$\mathbb{Q}_p[x]/(x^p-p) \cong \mathbb{Q}_p[y]/(y^p-y-\alpha).$$

4. A detailed discussion concerning proofs.

Let us elaborate our understanding on $\mathrm{Gal}(E/F)$, G_F

(Rather than viewing them as black box)

Setting: E/F be a finite Galois ext of NA local fields.

https://math.stackexchange.com/questions/4385377/ramified-extension-of-local-field-which-is-not-galois

Finite extension

Lemma. $\forall \sigma \in \mathrm{Gal}(E/F)$, $\quad \nu_E(\sigma(-)) = \nu_E(-)$.

e.p. $\sigma(\mathcal{O}_E) = \mathcal{O}_E \quad \sigma(\mathfrak{p}_E) = \mathfrak{p}_E$,

$$\mathrm{Gal}(E/F) \xrightarrow{\sim} \mathrm{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E)$$

[Proof. The valuation of F extends uniquely to E. (use completeness)
In addition, $\nu_E(\sigma(-))$, $\nu_E(-)$ are two valuations.]

Rmk. Using this, one can show that

$$\mathcal{O}_E = \text{integral closure of } \mathcal{O}_F \text{ in } E.$$

Completeness is necessary, see https://math.stackexchange.com/questions/4065594/integer-ring-and-valuation-ring-of-local-fields

Using the lemma, we construct a map

$$\mathrm{Gal}(E/F) \cong \mathrm{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \longrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E) \cong \mathrm{Gal}(\kappa_E/\kappa_F)$$

and extends it to a SES

$$1 \longrightarrow I(E/F) \longrightarrow \mathrm{Gal}(E/F) \longrightarrow \mathrm{Gal}(\kappa_E/\kappa_F) \longrightarrow 1$$

where

$$I(E/F) := \{\sigma \in \mathrm{Gal}(E/F) \mid \sigma(x) \equiv x \mod \mathfrak{p}_E \quad \forall x \in \mathcal{O}_E\}$$

Q. How to show that

$$\mathrm{Gal}(E/F) \longrightarrow \mathrm{Gal}(\kappa_E/\kappa_F)$$

$$\mathrm{Gal}(E/F) \longrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

$$\mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \longrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r})$$

are surjective? (Now: take it as a black box)

In general, define $\quad r \in \mathbb{N}_{\geq 0}$

$$\mathrm{Gal}(E/F) \cong \mathrm{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

we get SES

$$1 \longrightarrow I_r(E/F) \longrightarrow \mathrm{Gal}(E/F) \longrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \longrightarrow 1$$

where

$$I_r(E/F) := \{\sigma \in \mathrm{Gal}(E/F) \mid \sigma(x) \equiv x \mod \mathfrak{p}_E^{r+1} \quad \forall x \in \mathcal{O}_E\}$$

Comparing those constructions, we get a filstration

$$\begin{array}{ccccccccccc}
\cdots \subseteq I_r(E/F) \subseteq \cdots & \subseteq I_2(E/F) & \subseteq & I_1(E/F) & \subseteq & I_0(E/F) & \subseteq & I_{-1}(E/F) \\
& & & \parallel & & \parallel & & \parallel \text{def} \\
& & & P(E/F) & \subseteq & I(E/F) & \subseteq & \mathrm{Gal}(E/F)
\end{array}$$

with $\bigcap_{r\in\mathbb{N}} I_r(E/F) = 1$

and surjections

$$\twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \twoheadrightarrow \cdots \twoheadrightarrow \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{2}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{2}) \twoheadrightarrow \mathrm{Aut}_{\kappa_F\text{-alg}}(\kappa_E) \twoheadrightarrow 1$$

with $\varprojlim_{r\in\mathbb{N}} \mathrm{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \cong \mathrm{Gal}(E/F)$

Prop. We understand filtrations well:

$I_{-1}(E/F)/I_0(E/F) \xrightarrow{\sim} \mathrm{Gal}(\kappa_E/\kappa_F)$ cyclic

$I_0(E/F)/I_1(E/F) \hookrightarrow (\mathcal{O}_E/\mathfrak{p}_E)^\times \cong \kappa_E^\times$ $\quad \sigma \longmapsto \frac{\sigma(\pi)}{\pi}$

i.e. $[\sigma : \pi_E \longmapsto a_1\pi_E + a_2\pi_E^2 + \cdots] \longmapsto a_1$

$I_r(E/F)/I_{r+1}(E/F) \hookrightarrow \mathfrak{p}_E^r/\mathfrak{p}_E^{r+1} \cong \kappa_E$ $\quad \sigma \longmapsto \frac{\sigma(\pi)}{\pi} - 1 \quad r\in\mathbb{N}_{\geq 1}$

i.e. $[\sigma : \pi_E \longmapsto \pi_E + a_r\pi_E^r + \cdots] \longmapsto a_r$ $\quad$ Not cyclic!

Cor. $P(E/F)$ is the Sylow-p-subgp of $I(E/F)$.

Cor. When E/F is tot ramified with deg r, we have an iso

$$I_0(E/F)/I_1(E/F) \cong \mu_r(\kappa_E)$$

In tame case, $\#\mu_r(\kappa_E) = r$.

Absolute Galois gp

⚠ Some naive ideas are surprisingly complicated, so we list possible mistakes here, and don't touch those ideas...

- Don't want to mention $\overline{\mathcal{O}}_E$, since $\overline{E}$ is not a NA local field.
  Also, I don't know if
  the integral closure $\overline{\mathcal{O}}_E = \{x \in \overline{E} \mid v(x) \geq 0\}$
- Can't define $I_{F,r} := \varprojlim_{E/F \text{ fin Gal}} I_r(E/F)$, so we switch to upper indexing.
  In that case,
  $$I_F = I_F^0 \qquad P_F = I_F^{>0} = \bigcup_{v>0} I_F^v$$
  $$\bigcap_{v \in [0,\infty)} I_F^v = 1 \qquad \bigcap_{v'<v} I_F^{v'} = I_F^v$$
  $v, v' \in [0, \infty)$

We define I_F and P_F directly. Denote $\phi_E : G_F \twoheadrightarrow \mathrm{Gal}(E/F)$

$$I_F := \{\sigma \in G_F \mid \phi_E(\sigma) \in I(E/F) \quad \forall\ E/F \text{ fin Gal}\}$$
$$P_F := \{\sigma \in G_F \mid \phi_E(\sigma) \in P(E/F) \quad \forall\ E/F \text{ fin Gal}\}$$

We have SES

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow G_{k_F} \longrightarrow 1, \qquad G_{k_F} \cong \hat{\mathbb{Z}}$$

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(k_E) \longrightarrow 1, \qquad \varprojlim \mu_e(k_E) \cong \hat{\mathbb{Z}}^{(p)}$$

where $\displaystyle\varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(k_E) \cong \varprojlim_{\substack{E = F(\pi_F^{1/m}) \\ (m,p)=1}} \mathbb{Z}/m\mathbb{Z} \cong \hat{\mathbb{Z}}^{(p)}$

Transforming to the field side, we have clear descriptions on F^{ur} & F^{tr}:

$$F^{ur} := F(\zeta_n \mid_{(n,p)=1})$$
$$F^{tr} := F(\pi_F^{\frac{1}{n}}, \zeta_n \mid_{(n,p)=1})$$

Claim: For $F' := F^{ur} \cap E$, $\# E/F' := e$, if $(e,p)=1$, then $E \subseteq F^{tr}$

[Lemma 1.3.2, 1.3.3]: https://kskedlaya.org/cft/sec_localkronweb.html

Try: Only need to show $\pi_E \in F^{tr}$.
Take min poly of π_E over F'

$$f(T) = T^e + a_{e-1}T^{e-1} + \cdots + a_1 T + a_0 \in \mathcal{O}_F[T]$$

$\Rightarrow$ $f(T)$ is an Eisenstein poly

I think that we can use Newton's method to show that, the equation

$$f(T) = 0$$

has a root in F^{tr}.

Wrong try: $[E : E \cap F^{tr}]$ is p-power, and $[E : E \cap F^{tr}] \mid e$ $\Rightarrow$ $E = E \cap F^{tr}$
$\Rightarrow$ $E \subseteq F^{tr}$

Reason: we don't know yet if F^{tr} is max tame ram ext.

Actually, we can write down the iso explicitly:

$$t_{Fr}: G_F/I_F \longrightarrow \widehat{\mathbb{Z}} \qquad \sigma \longmapsto t_{Fr}(\sigma) \quad \text{satisfying}$$

$\forall$ field ext E/F, $\forall x \in \mathcal{O}_E$, $\qquad \# k_F = q$

$$\sigma(x) \equiv x^{q^{t_{Fr}(\sigma)}} \mod \mathfrak{p}_E$$

Fix compatible generators

$$\zeta = (\zeta_m)_{(m,p)=1} \in \varprojlim_{(m,p)=1} \mu_m(F^{tr}),$$

one can define

$$t_\zeta: I_F/P_F \longrightarrow \widehat{\mathbb{Z}}^{(p)} \qquad \sigma \longmapsto t_\zeta(\sigma) \quad \text{satisfying}$$

$$\frac{\sigma(\pi_F^{\frac{1}{m}})}{\pi_F^{\frac{1}{m}}} = \zeta_m^{t_\zeta(\sigma) \bmod m} \quad \text{in } F^{tr} \qquad \forall (m,p)=1$$

Task. Take $Fr := t_{Fr}^{-1}(1)$ (lift to G_F) $\tau := t_\zeta^{-1}(1)$, $m \in \mathbb{N}_{\geq 1}$.

Show that

$$Fr(\zeta_m) = \zeta_m^q \qquad \tau(\zeta_m) = \zeta_m$$

$$Fr(\pi_F^{\frac{1}{m}}) = \zeta_m^l \, \pi_F^{\frac{1}{m}} \qquad \tau(\pi_F^{\frac{1}{m}}) = \zeta_m \, \pi_F^{\frac{1}{m}} \qquad \exists\, l \in (\mathbb{Z}/m\mathbb{Z})^\times$$

Therefore,

$$Fr\, \tau\, Fr^{-1} = \tau^q \quad \text{in } F(\zeta_m, \pi_F^{\frac{1}{m}})$$

$$\Rightarrow Fr\, \tau\, Fr^{-1} = \tau^q \quad \text{in } F^{tr}.$$

For completeness, one also have the iso

$$t_{ab}: Gal(F^{ab}/F^{ur}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)} \qquad \sigma \longmapsto t_{ab}(\sigma) \quad \text{satisfying}$$

$$\sigma(\zeta_{p^r}) = \zeta_{p^r}^{t_{ab}(\sigma) \bmod p^r} \qquad \forall r \in \mathbb{N}_{\geq 0}$$

5. Henselian ring.

Main ref: https://en.wikipedia.org/wiki/Henselian_ring

R comm with 1 (local in this section)

Def. A local ring (R,m) is Henselian if Hensel's lemma holds, i.e.

$$\begin{array}{ccc} \text{for } P \in R[x] & & \exists f_i \in P[x] \quad ② P = f_1 \cdots f_n \\ \downarrow & & \uparrow ① \\ \bar{P} = g_1 \cdots g_n \in R/m[x] & & g_i \in R/m[x] \end{array}$$

(R,m) is strictly Henselian if additionally $(R/m)^{sep} = R/m$.

E.g. · Fields / Complete Hausdorff local rings are Henselian.

e.p. F, $\mathcal{O}_F$ are Henselian

· R is Henselian $\Leftrightarrow$ $R/Nil(R)$ is Henselian

$\Leftrightarrow$ R/I is Henselian for $\forall I \triangleleft R$

e.p. when $\operatorname{Spec} R = \{*\}$, R is Henselian.

Denote $\quad StrHense \subset Hense \subset locRing \subset CommRing \quad$ full subcategories

zero

$$\begin{tikzcd} StrHense \arrow[r, bend right, "forget"'] & Hense \arrow[l, dashed, bend right, "{[Stack\_OBSL]}"'] \arrow[r, bend right, "forget"'] \arrow[r, phantom, "\perp"] & locRing \arrow[l, bend right, "(-)^h"'] \arrow[ll, bend right=40, "(-)^{sh}"'] \end{tikzcd}$$

Sadly not adjoint?

E.g. $F^h = F \qquad F^{sh} = F^{un}$

Geometrically, Henselian means $\operatorname{Spec} R/m \to \operatorname{Spec} R$ has a section.

6. Cohomological dimension

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

https://mathoverflow.net/questions/349484/what-is-known-about-the-cohomological-dimension-of-algebraic-number-fields

This section is initially devoted to the following result:

Prop. [(7.5.1)] The wild inertia gp P_F is free pro-p-group of countably infinite rank.

See [Galois Theory of p-Extensions, Chap 4] for the definition and construction of free pro-p-groups.

Q: Do we have the adjoint

$$\begin{tikzcd} \text{Pro-p-gp} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \text{Set} \arrow[l, bend right, "(\ )^{\text{free}}"'] \end{tikzcd}$$

?

Now let G: profinite gp
$Mod(G)$: category of discrete G-modules

full subcategory of $Mod(G)$:
- $Mod_t(G)$: torsion
- $Mod_p(G)$: p-torsion
- $Mod_f(G)$: finite

viewed as abelian gp

Lemma For abelian torsion gp X, denote
$$X(p) := \{x \in X \mid x^{p^k} = 1 \quad \exists k \in \mathbb{N}_{>0}\}$$
we have $X = \bigoplus_p X(p)$.

This is trivial when X is finite, but I don't know how to prove this in the general case. It should be not too hard.

Def [(3.3.1)] (cohomological dimension) p prime

$$\begin{aligned}
cd\, G &= \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in Mod_t(G),\ H^i(G,A) \neq 0\} \\
tcd\, G &= \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in Mod(G),\ H^i(G,A) \neq 0\} \\
cd_p\, G &= \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in Mod_t(G),\ H^i(G,A)(p) \neq 0\} \\
tcd_p\, G &= \sup\{i \in \mathbb{N}_{\geq 0} \mid \exists A \in Mod(G),\ H^i(G,A)(p) \neq 0\}
\end{aligned}$$

Prop. (local to global) $cd\, G = \sup_p cd_p\, G$ $\qquad scd\, G = \sup_p scd_p\, G$

Prop. [(3.3.2)] $cd_p G \leq n \iff H^{n+1}(G,A) = 0 \quad \forall$ simple G-mod A with $pA = 0$

e.p. for G: pro-p-gp,
$$cd_p G \leq n \iff H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

E.g. $cd_p \hat{\mathbb{Z}} = 1 \qquad scd_p \hat{\mathbb{Z}} = 2$

Prop. [(3.3.5)] For $H \leq G$ closed,
$$cd_p H \leq cd_p G \qquad scd_p H \leq scd_p G$$
When $p \nmid [G:H]$ or [H open + $cd_p G < +\infty$], the equality holds.
Weaker condition: see [(3.3.5, Serre)]

Cor. G: profinite gp, then
$$cd_p\, G = 0 \iff p \nmid \# G$$

Prop. [(3.5.17)] A pro-p-gp G is free iff $cd\, G \leq 1$.

Prop[(7.1.8)] (i) F NA local field with char $k = p$.

$$cd_l(F) = \begin{cases} 2 & \text{if } l \neq \operatorname{char} F, \\ 1 & \text{if } l = \operatorname{char} F. \end{cases}$$

For any E/F field extension s.t. $l^\infty \mid \deg E/F$, $cd_l(E) \leq 1$.

(ii) Fix $n \in \mathbb{N}_{>0}$ s.t. $\operatorname{char} F \nmid n$.

$$H^i(F, \mu_n) = \begin{cases} F^\times/(F^\times)^n & i = 1 \\ \frac{1}{n}\mathbb{Z}/\mathbb{Z} & i = 2 \\ 0 & i \geq 3 \end{cases}$$

[Proof for Prop (7.5.1)

Now $l^\infty \mid \deg F^{tr}/F \overset{(7.1.8)}{\Rightarrow} cd_l(F^{tr}) \leq 1 \quad \forall$ prime l

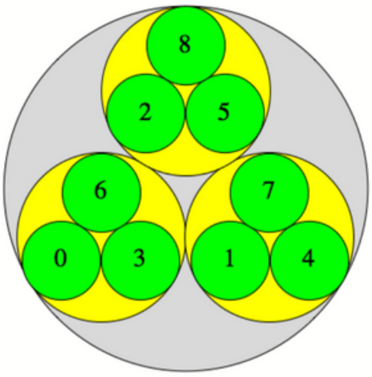
$\Leftrightarrow cd(F^{tr}) \leq 1$

$\Leftrightarrow P_F$ is free pro-p-group. $\square$]

7. Bonus: "plane geometry" for $\mathbb{Q}_9$

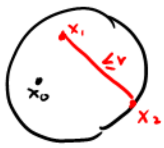
In this section, the picture comes from [https://www.nt.th-koeln.de/fachgebiete/mathe/knospe/p-adic/] by Heiko Knospe.

I want to define:
Compare Q_9 and Q_3(\sqrt(3))
triangle (Actually we just concider 3 points, and they may be "collinear")
disk
sphere
line(in higher dimension, like Q_9 or Q_3(\sqrt(3)))
no angle, no perpendicular, but parallel lines
P^1(Q_3) (should characterize all lines in Q_9 passing through 0, parameterized by a line not passing the origin)
intersections of disks, spheres and lines
sphere packing? Symmetric group of the objects considered? connection with the tree-structures/Bruhat--Tits building?



$\mathbb{Z}_3$

Triangles: $a = a \geq b$



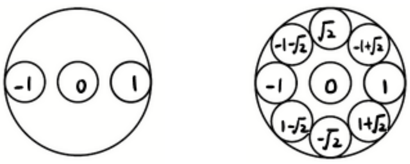
$B_{x_0}(r)$

disks: every pt can be center
(even pts on the edge)

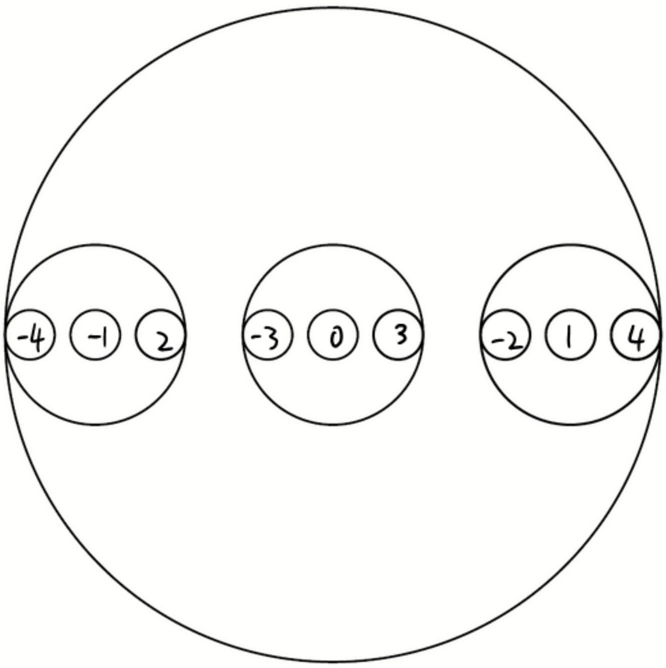
I personally would like to draw it more "compatible with arithmetic":
E.g. $\mathbb{Z}_3$ vs. $\mathbb{Z}_9 = \mathbb{Z}_3(\sqrt{2})$

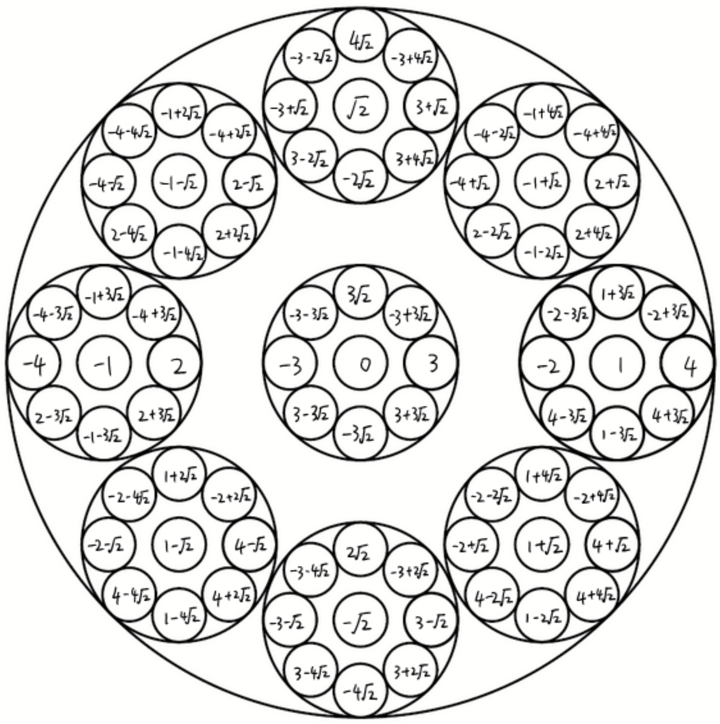
$\left\{ \begin{array}{l} \text{mod } 3 \\ +,\ \text{inverse} \\ \text{multi by } 3 \\ \text{Frobenius} \end{array} \right.$

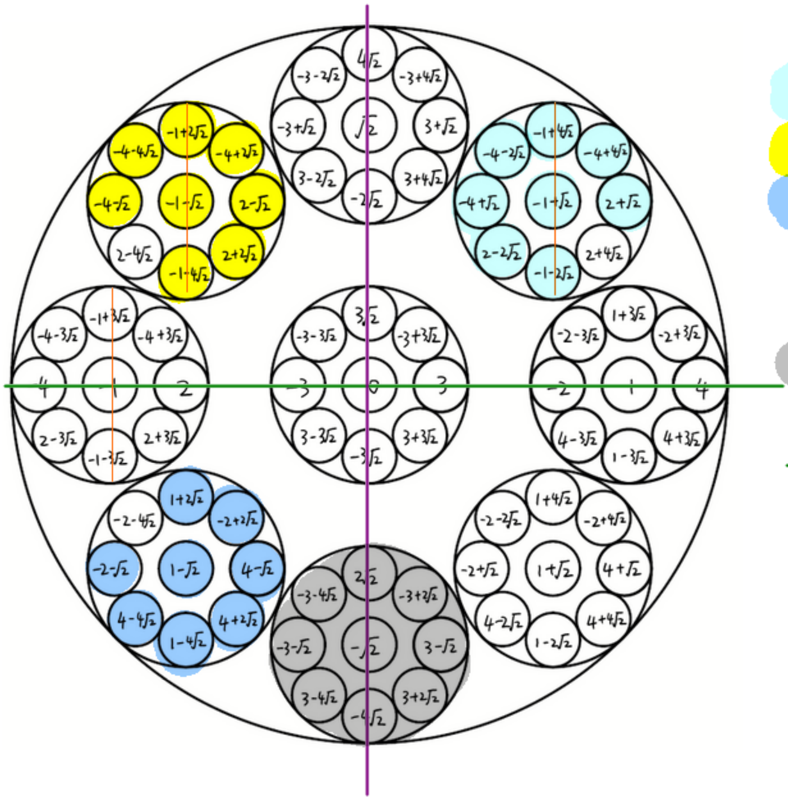
basic buildings:



It's more canonical to use Teichmüller lift rather than 1~p, but I don't do so because of my limited computation ability.







$x_0 = 2+4\sqrt{2}$ $\quad Gal(\mathbb{Q}_9/\mathbb{Q}_3)=\{1,\sigma\}$

$A := \{x\in\mathbb{Q}_9 \mid d(x,x_0)=\frac{1}{3}\}$

σA

$-A$

$x_1 = 2\sqrt{2}$

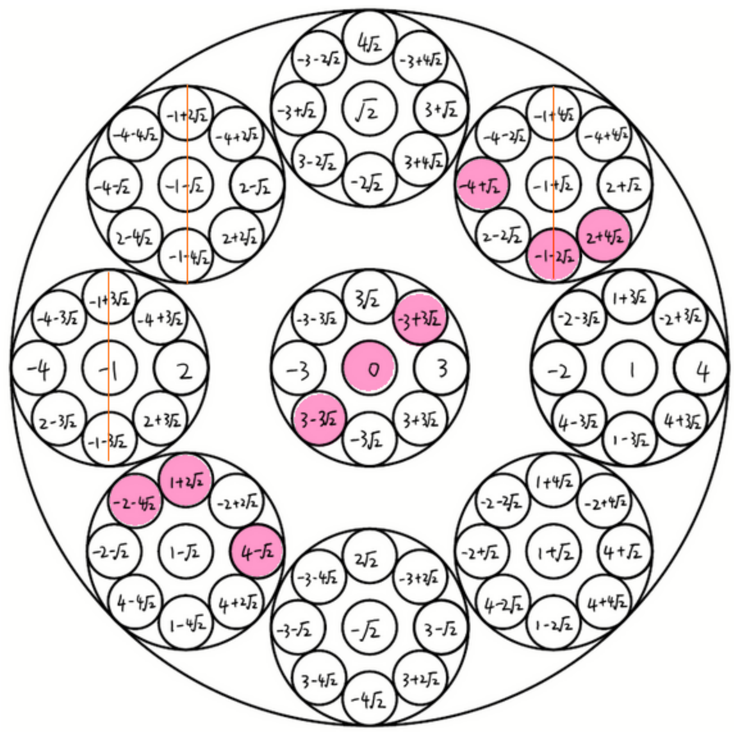
$B := B(x_1,\frac{1}{3})$

$= B(x,\frac{1}{3})$ for $\forall x\in B$

$\mathbb{Q}_3$

$\mathbb{Q}_3$ - v.s. generated by x_1

$\{ax_1 - 1 \mid a\in\mathbb{Q}_3\}$



(smallest) circles containing elements in $\mathbb{Q}_3\cdot x_0$

Observation: for $\forall$ disk $D=\bigsqcup_{i=1}^{9} D_i \subset \mathbb{Q}_9$

if $D\cap \mathbb{Q}_3\cdot x_0\neq\emptyset$, then

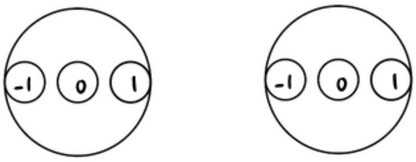
$\#\{i\in\{1,\dots,9\}\mid D_i\cap\mathbb{Q}_3\cdot x_0\neq\emptyset\}=3.$

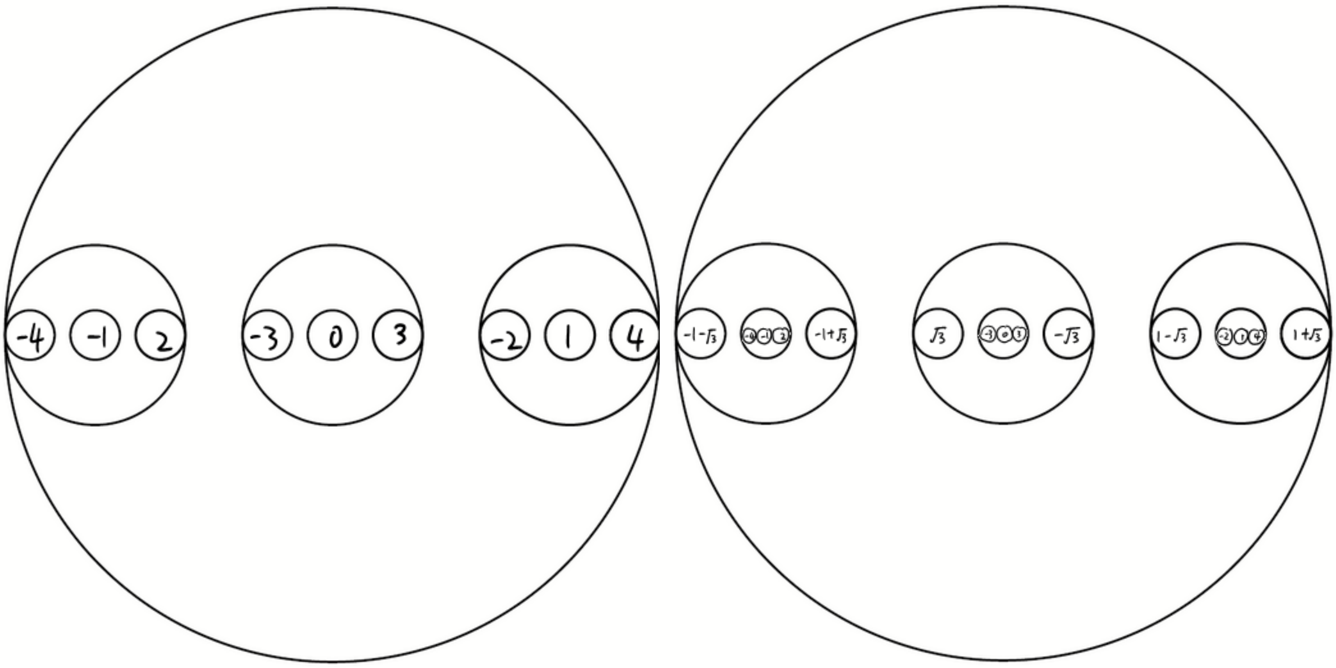
Q: Can we recover $\mathbb{Q}_3$-v.s V from the set

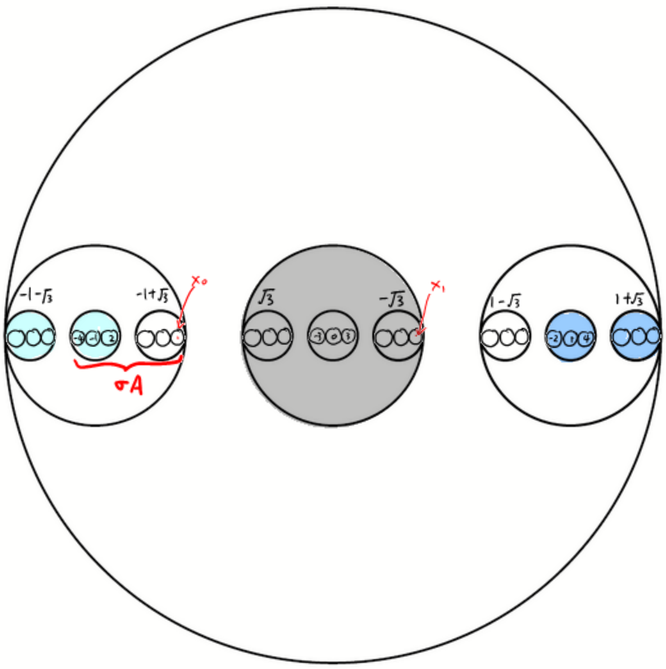
$\left\{D\subset\mathbb{Q}_9 \;\middle|\; \begin{array}{l} D=B_x(r) \text{ for some } x\in\mathbb{Q}_9, r\geq 1 \\ D\cap V\neq\emptyset \end{array}\right\}$?

E.g. $\mathbb{Z}_3$ vs. $\mathbb{Z}_3(\sqrt{3})$ $\qquad \|\cdot\| = \|\cdot\|_{\mathbb{Z}_3} = \|\cdot\|^{\frac{1}{2}}_{\mathbb{Z}_3(\sqrt{3})}$

basic buildings:







$x_0 = 2+\sqrt{3}$ $\qquad Gal(\mathbb{Q}_3(\sqrt{3})/\mathbb{Q}_3) = \{1, \sigma\}$

$A := \{ x \in \mathbb{Q}_9 \mid d(x, x_0) = \frac{1}{3} \}$

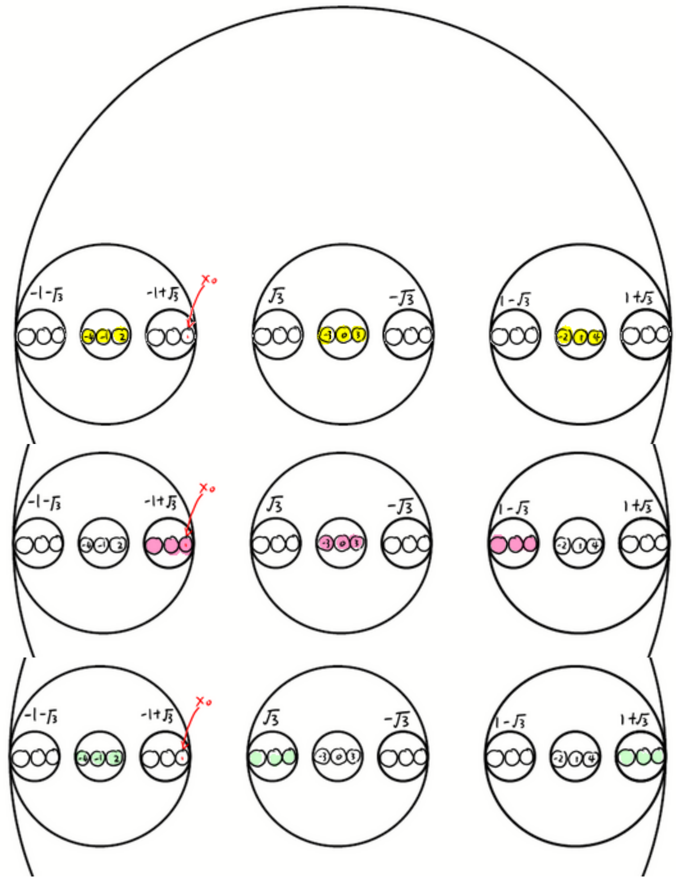
σA

$-A$

$x_1 = 2\sqrt{3}$

$B := B(x_1, \frac{1}{\sqrt{3}})$

$= B(x, \frac{1}{\sqrt{3}})$ for $\forall x \in B$



smallest

circles containing elements in $\mathbb{Q}_3 \cdot 1$

circles containing elements in $\mathbb{Q}_3 \cdot x_0$

circles containing elements in $\{\alpha x_0 - 1\}$

Observation: for $\forall$ disk $D = \bigsqcup_{i=1}^{3} D_i \subset \mathbb{Q}_3(\sqrt{3})$

if $D \cap \mathbb{Q}_3 \cdot x_0 \neq \emptyset$, then

$\# \{ i \in \{1, \cdots 3\} \mid D_i \cap \mathbb{Q}_3 \cdot x_0 \neq \emptyset \} = 3.$

Tasks for interesting readers: figure out all the cases of quadratic extension of $\mathbb{Q}_2$.