# Eine Woche, ein Beispiel
## 6.4. basics of fields

This document is aimed for people who have enough mathematical maturity, but miss the chance and time to study Galois theory. For a (relative) complete study of Galois theory which takes time, please see [GTM167].

1. classical motivation
2. common confusion
3. field extension
4. examples of algebraic closed field

## 1. classical motivation

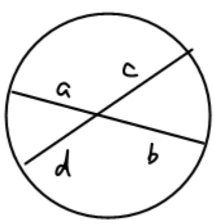
| | ruler-and-compass construction 尺规作图 | | solving higher degree equations 求根公式 | |
|---|---|---|---|---|
| possible | ⬠ 17-gon | $\cos\frac{2\pi}{5}$ $\zeta_5$ <br> $\cos\frac{2\pi}{17}$ $\zeta_{17}$ | $\deg F \le 4$ | $x$: $F(x)=0$ |
| impossible | Squaring the circle <br> Doubling the cube <br> Angle trisection | $\pi$ 化圆为方 <br> $\sqrt[3]{2}$ 倍立方 <br> $x: 4x^3-3x-a=0$ 三等分角 | $\deg F \ge 5$ | $x$: $F(x)=0$ |

Ex. Denote

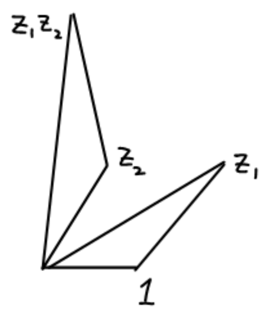
$$F_R := \{z\in\mathbb{C} \mid z \text{ can be drawn by ruler-and-compass, given } 0,1\}$$
$$= \{\text{algebraic constructible complex numbers}\}$$
$$F_{开根} := \{z\in\mathbb{C} \mid z \text{ can be expressed by } +,-,\times,\div, \text{radicals}\}$$

Verify that $F_R$, $F_{开根}$ are fields.

Hint. Verify that $\mathbb{Q}\subseteq F_R$ to get some intuition.




$ab = cd$



Ex. Given $1, a \in \mathbb{R}^+$, try to draw $\sqrt{a}$ by ruler-and-compass.
Argue that why we can draw ⬠ and 17-gons.

Hint. $\cos\frac{2\pi}{5} = \frac{\sqrt{5}}{4} - 1$

$\cos\frac{2\pi}{17} = \frac{1}{16}\left(-1+\sqrt{17}+\sqrt{2(17-\sqrt{17})}+2\sqrt{17+3\sqrt{17}-\sqrt{2(17-\sqrt{17})}-2\sqrt{2(17+\sqrt{17})}}\right)$

Slogan: consider

| | | |
|---|---|---|
| element | $\rightsquigarrow$ | set |
| object | $\rightsquigarrow$ | moduli spaces |
| if $x$ can be realized | $\rightsquigarrow$ | $\{x \mid x \text{ can be realized}\}$ |

2. common confusion

| | Abstract field | Subfield of $\overline{K}$ or $\mathbb{C}$ |
|---|---|---|
| name of category | Field | $\text{Subfield}_{\overline{K}}$ |
| Ob | $\{F : \text{field}\}$ | $\{(F,\iota) \mid \iota : F \hookrightarrow \overline{K}\}$ |
| Mor | $\text{Mor}_{\text{Field}}(F,E) = \{\alpha : F \hookrightarrow E\}$ usually: finitely many elements | $\text{Mor}_{\text{Subfield}}(F,E) = \left\{ \alpha : F \hookrightarrow E \text{ s.t. } \begin{tikzcd} F \arrow[rr, hook, "\alpha"] \arrow[dr] & & E \arrow[dl] \\ & \overline{K} & \end{tikzcd} \right\}$ at most 1 element |
| Examples | $\mathbb{Q}[x]/(x^2+1)$<br>$\mathbb{Q}[x]/(x^3-2)$<br>$\mathbb{Q}(x)$ | $\mathbb{Q}(i)$<br>$\mathbb{Q}(\sqrt[3]{2})$<br>$\mathbb{Q}(\pi)$ |

Common questions: (Which category are we considering for these questions?)
- # {extensions of $K$ of deg 3}
- Automorphism gp of the field.

Abstract fields are not as hard as you may think!

Ex 1). Write down the definition of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$, as well as $\mathbb{Q}(i)$, $\mathbb{Q}(\pi)$

2). Find a $\mathbb{Q}$-basis of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$. Compute the dim.

# Constructing new field by adding roots

$$
\begin{array}{r|l}
 & \phantom{1}\,1\,3\,2 \\
102 & 1\,3\,5\,6\,2 \\
 & \underline{1\,0\,2} \\
 & \phantom{1}\,3\,3\,6\,2 \\
 & \phantom{1}\,\underline{3\,0\,6} \\
 & \phantom{1\,3}\,3\,0\,2 \\
 & \phantom{1\,3}\,\underline{2\,0\,2} \\
 & \phantom{1\,3}\,1\,0\,0
\end{array}
\qquad
\begin{array}{r|l}
 & \qquad\qquad\quad x^2+3x+7 \\
x^2-2 & x^4+3x^3+5x^2+6x+2 \\
 & \underline{x^4 \qquad\; -2x^2} \\
 & \qquad 3x^3+7x^2+6x \\
 & \qquad \underline{3x^3 \qquad\;\; -6x} \\
 & \qquad\qquad 7x^2+12x+2 \\
 & \qquad\qquad \underline{7x^2 \qquad\;\; -14} \\
 & \qquad\qquad\qquad 12x+16
\end{array}
$$

$13562 \div 102 = 132 \cdots 100$

$13562 = 102 \times 132 + 100$

$(x^4+3x^3+5x^2+6x+2) \div (x^2-2) = (x^2+3x+7) \cdots (12x+16)$

$x^4+3x^3+5x^2+6x+2 = (x^2-2)(x^2+3x+7)+(12x+16)$

Ex. factorize $x^3+4x^2-7x-10$ in $\mathbb{Q}[x]$ or $\mathbb{F}_3[x]$.

Ex. Let $F = \mathbb{F}_7[x]/(x^3-3)$.

1) Compute $(x^2+1)\cdot(x-1)$, $\frac{1}{x}$,
2) Show that $x^3-3$ is irr in $\mathbb{F}_7[x]$, i.e.

   $x^3-3 = f(x)g(x) \Rightarrow \deg f = 0$ or $\deg g = 0$

   $f, g \in \mathbb{F}_7[x]$
3) Show that $(x^3-3,\ x^2+x+1) = (1)$ in $\mathbb{F}_7[x]$, by Euclidean division.

   In fact, $\mathbb{F}_7[x]$ is ED $\Rightarrow$ PID
4) Compute $(x^2+x+1)^{-1}$ in $F$.
5) Factorize $T^3-3$ in $F[T]$.

Rmk. In fact, $K[T]/(f(T))$ is a field $\Leftrightarrow$ $f(T) \in K[T]$ is irreducible

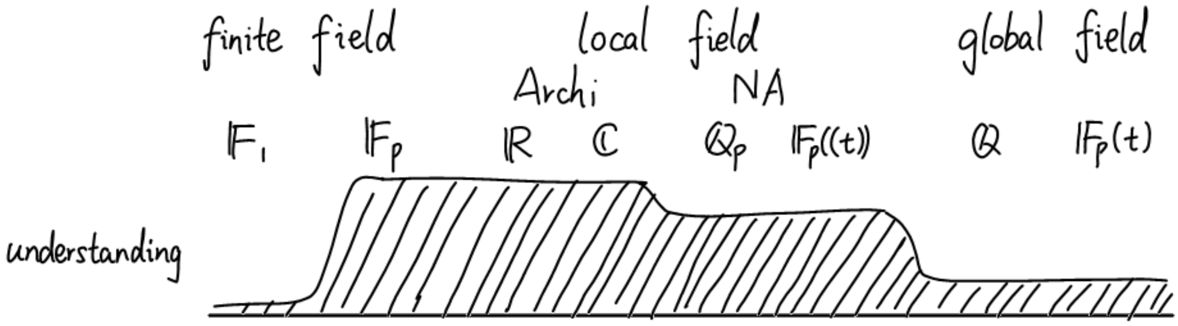
Ex. Let $F = \mathbb{Q}[x]/(x^3-2)$.

1) Compute $\mathrm{Mor}_{\mathrm{Field}}(F, \mathbb{C})$. Are all embeddings real?
2) Discussion: What is the difference between

   $\mathbb{Q}[x]/(x^3-2)$ with $\mathbb{Q}(\sqrt[3]{2})$ ?

## 3. field extension

### Main examples of fields




### Definitions

Def: $E/F$ field extension: $(E, F, \iota: F \hookrightarrow E)$

Def: Base field: $\begin{cases} \mathbb{Q} & \text{char } F = 0 \\ \mathbb{F}_p & \text{char } F = p \end{cases}$

Def. (Algebraic extension)

$E/F$ is alg, if $\forall a \in E$ is alg$/F$, i.e., the following equivalent conditions are true.

1) $\forall a \in E$, $\exists f \in F[x]$, $f \neq 0$, $f(a) = 0$.
2) $\forall a \in E$, $[F(a) : F] < +\infty$.
3) $E = \bigcup_{\substack{F \subset F' \subset E \\ F'/F \text{ finite}}} F'$
4) $\forall a \in E$, $\exists$ f.d. $F$-v.s. $V \subseteq E$ s.t. $aV \subseteq V$.

For $a \in E$, $\mathrm{Min}(a, F) :=$ minimal monic polynomial of $a$ in $F$.

E.g. $\overline{\mathbb{Q}}/\mathbb{Q}$, $\mathbb{Q}(\pi)/\mathbb{Q}$, $\mathbb{C}/\mathbb{Q}$

We mainly consider alg extension, e.p. fin field extension.

Assume: $E/F$ alg

Slogan: Galois $=$ normal $+$ seperable

Def. (Normal extension)

$E/F$ normal, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \subseteq F[x] \subseteq E[x] \text{ splits}}_{a\in E \text{ is normal}}$.

E.g. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ $\quad \mathbb{Q}(\zeta_3)/\mathbb{Q}$ $\quad \mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q}$

Def. (Seperable extension)

$E/F$ sep, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \text{ has no repeated roots in } \overline{F}[x]}_{a\in E \text{ is sep}}$.

E.g. $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$, where $\mathbb{F}_p(T^{\frac{1}{p}}) := \mathbb{F}_p(T)[x]/(x^p - T)$

Rmk. When $\mathrm{char}\, F = 0$ or $\# F < +\infty$, $E/F$ is always seperable.

Def. (Galois extension)

$E/F$ Galois, if $E/F$ is normal and sep. We denote

$$\begin{aligned} \mathrm{Gal}(E/F) &= \mathrm{Aut}_{F\text{-alg}}(E) \\ &= \{\sigma : E \longrightarrow E \mid \sigma|_F = \mathrm{Id}_F\} \end{aligned}$$

Rmk. When $E/F$ finite,

$E/F$ Galois $\Leftrightarrow$ $[E:F] = \# \mathrm{Aut}_{F\text{-alg}}(E)$

E.g. & Exercise: Compute $\# \mathrm{Aut}_{F\text{-alg}}(E)$ for $E/F = \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$.

https://kconrad.math.uconn.edu/blurbs/galoistheory/galoiscorrexamples.pdf

Ex. Read it, and compute

$\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2}, \zeta_3)/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2}, i)/\mathbb{Q})$

$\mathrm{Gal}(F/\mathbb{Q})$ $\quad F$: the splitting field of $x^4 - x^2 - 1$.

I would instead begin with relative easier case:

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\quad \mathrm{Gal}\left(\mathbb{Q}[T]/(T^2-2)^2-2 \,/\mathbb{Q}\right)$

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q})$

After that, do

4.2.3 : $\mathbb{Q}(\sqrt[4]{2}(1+i))/\mathbb{Q}$

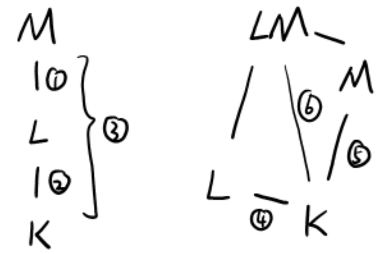
4.1.16: $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$

4.1.4: $\mathbb{Q}(\sqrt{2}, \sqrt{3}, u)/\mathbb{Q}$ $\qquad u^2 = (9 - 5\sqrt{3})(2 - \sqrt{2})$

4.1.7. $F(\alpha)/F$ $\qquad \mathrm{char}\, F = p$, $a \in F$, $x^p - x - a \in F[x]$ irr,

$\alpha^p - \alpha - a = 0$

in [近世代数三百题].




③ pure ins: ① ⟹ ③

normal: ③ ⟹ ① ; ③ ⇏ ② ; ① + ② ⇏ ③ ; ⑥ ⇏ ④ ; ④ + ⑤ ⟹ ⑥

+

seperable: ① + ② = ③ ; ④ + ⑤ = ⑥

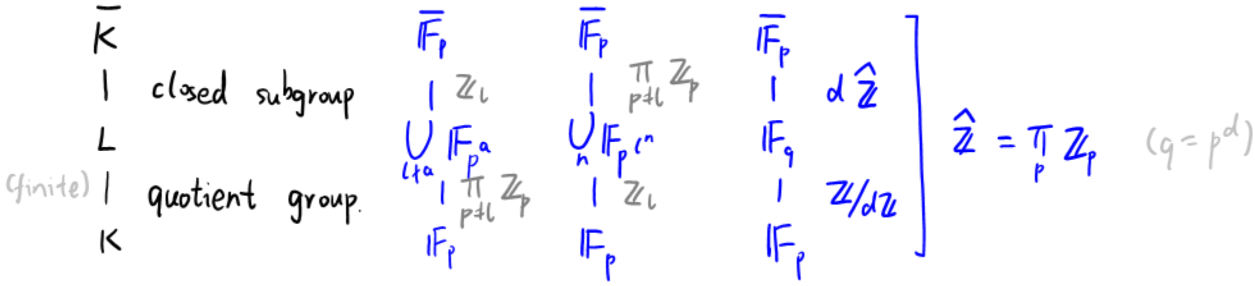
=

Galois: ③ ⟹ ① ; ③ ⇏ ② ; ① + ② ⇏ ③ ; ⑥ ⇏ ④ ; ④ + ⑤ ⟹ ⑥

purely inseparable: ① + ② = ③ ; ④ + ⑤ = ⑥

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then $F$ perfect $\Leftrightarrow F^p = F$




open subgroup $\subsetneq$ closed subgroup $= \{Gal(\bar{K}/L) \mid L/k \text{ ext}\} \subsetneq$ subgroup

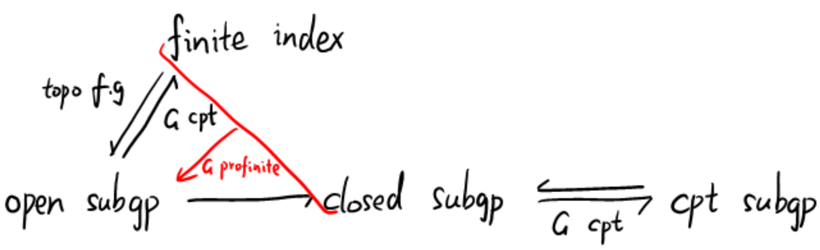
($\{1\} \subseteq \mathbb{Z}_p$ ; $\mathbb{Z} \subseteq \mathbb{Z}_p$)

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Q: Do we have any finite index gp of $Gal(\bar{k}/k)$ which is not open?

In general,




https://groupprops.subwiki.org/wiki/Closed_subgroup_of_finite_index_implies_open
In a topological group, any closed subgroup of finite index must be an open subgroup.
https://groupprops.subwiki.org/wiki/Open_subgroup_implies_closed
Any open subgroup of a topological group is closed.

https://math.stackexchange.com/questions/4350043/open-subgroup-and-finite-index-subgroup-of-topological-group
https://math.stackexchange.com/questions/1092974/finite-index-subgroup-g-of-mathbbz-p-is-open
https://math.stackexchange.com/questions/3734250/are-normal-subgroups-of-finite-index-in-an-absolute-galois-groups-open
https://math.stackexchange.com/questions/83355/how-to-prove-that-a-compact-set-in-a-hausdorff-topological-space-is-closed

https://math.stackexchange.com/questions/4062798/a-profinite-group-that-is-not-finite-is-not-countable
https://math.stackexchange.com/questions/3165116/direct-proof-that-closed-subgroups-of-profinite-groups-are-profinite

Some wonderful exercises for Galois correspondence:

Let $E/F$ be Galois field ext of deg $n$, $m \mid n$. prove: $\exists$ subfield ext of deg $m$.
(Sylow thm & $Z(G) \neq \{1\}$ for a $p$-gp & classification of f.g. abelian gp)
Cor. For $p$ prime, $F$ field, one can define ${}^p\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^p\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}} \cap \mathbb{R}$)?

## 4. examples of algebraic closed field

① $\overline{\mathbb{Q}} \overset{\pi}{\underset{\bar{\mathbb{Z}}}{\subsetneq}} \mathbb{C} \overset{t}{\subsetneq} \bigcup_n \mathbb{C}((t^{\frac{1}{n}})) = \overline{\mathbb{C}((t))}$ $\mathbb{C}[[t]]$

$\sum_{n=0}^{+\infty} p^n$ $\quad \cap$ Puiseux series

$\overline{\mathbb{Q}_p} \underset{\bar{\mathbb{Z}}_p}{\subsetneq} \mathbb{C}_p$

$\mathbb{Q}_p \cdot \overline{\mathbb{Q}}$

② $\operatorname{char} K = p$ : $\overline{\mathbb{F}_q} = \overline{\mathbb{F}_p}$ $\left( \operatorname{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \hat{\mathbb{Z}} \right)$

Task. ① Prove they are alg closed. $(\mathbb{C}, \bigcup_n \mathbb{C}((t^{\frac{1}{n}})), \mathbb{C}_p)$

② Find an element in each "⊊".

$Q_p \cap \bar{Q}$:$https://math.stackexchange.com/questions/1280053/explicit-description-of-bbb-q-p-cap-bar-bbb-q

$C_p \setminus \bar{Q_p}$:$https://math.stackexchange.com/questions/123925/is-the-algebraic-closure-of-a-p-adic-field-complete

https://math.stackexchange.com/questions/2430665/algebraic-closure-of-q-p-is-composite-of-bar-mathbbq-and-mathbbq-p

https://math.stackexchange.com/questions/2153580/transcendental-numbers-in-mathbbq-p