

Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{F}} & \mathcal{F}' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

$$\begin{aligned} f^* &\dashv f_* \\ - \otimes \mathcal{F} &\dashv \underline{\mathrm{Hom}}(\mathcal{F}, -) \\ f_! &\dashv f^! \end{aligned}$$

$$\begin{aligned} f^*(- \otimes -) \\ f^*(\mathcal{F} \otimes \mathcal{F}') &\cong f^* \mathcal{F} \otimes f^* \mathcal{F}' \\ f_* \underline{\mathrm{Hom}}(f^* \mathcal{F}, \mathcal{G}) &\cong \underline{\mathrm{Hom}}(\mathcal{F}, f_* \mathcal{G}) \end{aligned}$$

$$\begin{array}{ccc} & \otimes & \\ f^* & & f_! \\ \text{bc: } f^* g_! & \cong & g'_! f'^* \\ f_* g'_! & \cong & g'_! f_* \end{array}$$

$$\begin{aligned} \text{proj formula} \\ f_!(f^* \mathcal{F} \otimes \mathcal{G}) &\cong \mathcal{F} \otimes f_! \mathcal{G} \\ f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^* \mathcal{F}) &\cong \underline{\mathrm{Hom}}(f_! \mathcal{G}, \mathcal{F}) \\ f'_! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') &\cong \underline{\mathrm{Hom}}(f^* \mathcal{F}, f'^* \mathcal{F}') \end{aligned}$$

$$\begin{aligned} I: f^* &= f^! \\ P: f_* &= f_! \end{aligned}$$

$$p: X \rightarrow pt$$

$$H^i(X; \mathbb{Z}) := p_* p^* \mathbb{1}$$

$$H_c^i(X; \mathbb{Z}) := p_! p^* \mathbb{1}$$

$$H^i(X; \mathbb{Z}) := p_! p^! \mathbb{1}$$

$$H^{BM}_i(X; \mathbb{Z}) := p_* p^! \mathbb{1}$$

$$H^i(X; \mathcal{F}) := p_* \mathcal{F}$$

$$H_c^i(X; \mathcal{F}) := p_! \mathcal{F}$$

$$H^i(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$$

$$H^{BM}_i(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$$

$$= R\Gamma(X; \mathcal{F})$$

$$= H_c^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

$$= H^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)

$$H_c^i(X; \mathcal{F}) \otimes H_c^j(Y; \mathcal{G}) \cong H_c^{i+j}(X \times Y; \mathcal{F} \otimes \mathcal{G})$$

$$\text{reduced to: } p_{X!} \mathcal{F} \otimes p_{Y!} \mathcal{G} \cong p_!(p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_1 \downarrow & \searrow p & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

App 2. (Poincaré duality)

X : a cpt oriented mfd of dim d , then

$$\begin{aligned} \text{proper} \quad p^! \mathbb{Z} &\cong \mathbb{Z}[d] \text{ locally (Verdier duality)} \\ p^! \mathbb{Z} &\cong \mathbb{Z}[d] \text{ globally} \end{aligned}$$

$$-^{\vee} = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

$$H^i(X; \mathbb{Z})[d] \cong H^i(X; \mathbb{Z})^{\vee}$$

$$\text{reduced to: } p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$$

$$Z \xrightarrow[i_{\text{close}}]{i} X \xleftarrow{j} U \rightsquigarrow \begin{array}{c} \text{Id} \swarrow \xleftarrow{i^*} \\ \mathcal{D}(Z) \xrightarrow[p_i]{i_* = i^!} \mathcal{D}(X) \xrightarrow[p_i]{j_*} \mathcal{D}(U) \\ \text{Id} \swarrow \xleftarrow{i^!} \end{array} \quad \begin{array}{c} \text{Id} \swarrow \xleftarrow{j^*} \\ \mathcal{D}(X) \xrightarrow[p_i]{j_*} \mathcal{D}(U) \xrightarrow[p_i]{j_*} \mathcal{D}(U) \\ \text{Id} \swarrow \xleftarrow{j^!} \end{array} \quad \begin{array}{c} \text{Id} \swarrow \xleftarrow{i^*} \\ \mathcal{D}(U) \xrightarrow[p_i]{j_*} \mathcal{D}(U) \xrightarrow[p_i]{j_*} \mathcal{D}(U) \\ \text{Id} \swarrow \xleftarrow{j^!} \end{array}$$

ff. fully faithful
 p_i : preserve injectives. (内射)
 ie. inj sheaf \leadsto inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Q} = \mathcal{O}_{X[n]} \xrightarrow{+\text{orientation}} \underline{\mathbb{Q}}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$\begin{array}{ccccccc} 0 \longrightarrow & j_! j^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_* i^* \mathcal{F} & \xrightarrow{+1} \\ & i_! i^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & Rj_* j^* \mathcal{F} & \xrightarrow{+1} \end{array}$$



<https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof>
<https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de>

Therefore, need to check if $R\pi_{X,*} Rj_* = R\pi_{U,*}$ (Why writing derived symbol make less mistakes)

Assume ∇ makes no problem.

Here, $H_{-1}(S'; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$\begin{array}{ccccccc} R\Gamma(X, Z; \mathcal{F}) & \longrightarrow & R\Gamma(X; \mathcal{F}) & \longrightarrow & R\Gamma(Z; \mathcal{F}|_Z) & \xrightarrow{+1} \\ R\Gamma(X, U; \mathcal{F}) & \longrightarrow & R\Gamma(X; \mathcal{F}) & \longrightarrow & R\Gamma(U; \mathcal{F}|_U) & \xrightarrow{+1} \\ R\Gamma_Z(X; \mathcal{F}) & & & & & \\ \text{When } \mathcal{F} = \underline{\mathbb{Q}}_X, & H^i(X, Z) & \longrightarrow & H^i(X) & \longrightarrow & H^i(Z) & \xrightarrow{+1} \\ & H^i(X, U) & \longrightarrow & H^i(X) & \longrightarrow & H^i(U) & \xrightarrow{+1} \\ \text{When } \mathcal{F} = \text{ID}_X, & R\Gamma(X, j_* \underline{\mathbb{Q}}_U) & \longrightarrow & H_i^{BM}(X) & \longrightarrow & R\Gamma(Z; \text{ID}_X|_Z) & \xrightarrow{+1} \\ & H_i^{BM}(Z) & \longrightarrow & H_i^{BM}(X) & \longrightarrow & H_i^{BM}(U) & \xrightarrow{+1} \end{array}$$

Taking $R\pi_{X,!}$

$$\begin{array}{ccccccc} R\Gamma_c(U, \mathcal{F}|_U) & \longrightarrow & R\Gamma_c(X; \mathcal{F}) & \longrightarrow & R\Gamma_c(Z; \mathcal{F}|_Z) & \xrightarrow{+1} \\ R\Gamma_c(Z, i^! \mathcal{F}) & \longrightarrow & R\Gamma_c(X; \mathcal{F}) & \longrightarrow & R\Gamma_c(X, Rj_* (\mathcal{F}|_U)) & \xrightarrow{+1} \\ & & & & & \\ \text{When } \mathcal{F} = \underline{\mathbb{Q}}_X, & H_c^i(U) & \longrightarrow & H_c^i(X) & \longrightarrow & H_c^i(Z) & \xrightarrow{+1} \\ & H_c^i(Z, i^! \underline{\mathbb{Q}}_X) & \longrightarrow & H_c^i(X) & \longrightarrow & H_c^i(X, Rj_* \underline{\mathbb{Q}}_U) & \xrightarrow{+1} \\ \text{When } \mathcal{F} = \text{ID}_X, & H_i(U) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, U) & \xrightarrow{+1} \\ & H_i(Z) & \longrightarrow & H_i(X) & \longrightarrow & H_i(X, Z) & \xrightarrow{+1} \end{array}$$