Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety : goal

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
https://people.math.harvard.edu/~gross/preprints/sat.pdf

Task. Complete the following tables.

G here is SL_n but not GL_n (to make sure the correctness of K(St))

$K^-(-)$	pt	$\mathcal{B}$ $\quad$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $\quad$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	
Id	$\mathbb{Z}$			$\mathbb{Z}[X^*(T)]/_{I_T}\rtimes\mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)]^W[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)][t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

We use the shorthand.

$K^-(-)$	pt	$\mathcal{B}$ $\quad$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $\quad$ $T^*(\mathcal{B}\times\mathcal{B})$	St
G	$R(T)^W$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$	
Id	$\mathbb{Z}$			$R(T)/_{I_T}\rtimes\mathbb{Z}[W_f]$
$G\times\mathbb{C}^\times$	$R(G)[t^{\pm1}]$			$\mathcal{H}_{ext}$
$B\times\mathbb{C}^\times$	$R(T)[t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$$R(B) = \mathbb{Z}[X^*(T)] = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F))$$

$$R(G) = \mathbb{Z}[X^*(T)]^W \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F))$$

$$R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$$

$$R(G\times\mathbb{C}^\times) = \mathbb{Z}[X^*(T)]^W[t^{\pm1}]$$

$$K^{G\times\mathbb{C}^\times}(St) = \mathcal{H}_{ext} \overset{?}{=} \mathcal{H}(\hat{G}(F), I)$$

Here is an initial example.

$K^-(-)$	pt	$\mathcal{B}$ $\quad$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$ $\quad$ $T^*(\mathcal{B}\times\mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1}, z_2^{\pm1}]/((z_1-z_2)(z_1-z_2^{-1}))$	$\mathbb{Z}[W_{ext}] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]$
B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1}, z]/(z-y)(z-y^{-1})$	$\mathbb{Z}[y^{\pm1}, z_1, z_2]/((z_1-y)(z_1-y^{-1}), (z_2-y)(z_2-y^{-1}))$	
Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1, z_2]/((z_1-1)^2, (z_2-1)^2)$	$R(T)/_{I_T}\rtimes\mathbb{Z}[W_f] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]/(z_w-1)^2$
$SL_2\times\mathbb{C}^\times$	$\mathbb{Z}[x, t^{\pm1}]$			$\mathcal{H}_{ext} = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}, t^{\pm1}]$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[y^{\pm1}, t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

This is our final task. Most of the notations are still not fixed.
G_d here is of type GL.

$K^-(-)$	pt	$\mathcal{F}_{\underline{d}}$ $\widetilde{Rep}_{\underline{d}}(Q)$	$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$\mathcal{Z}_{\underline{d},\underline{d}'}$
G_d	$R(T_d)^{W_d}$	$R(T_d)$	$R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{w'\in W_d} R(T_d)\,[\overline{\Omega^{u,u'}_{w'}}]^{G_d}$	$\bigoplus_{w'\in W_d} R(T_d)\,[Z^{u,u'}_{w'}]^{G_d}$
B_d	$R(T_d)$ $\bigoplus_{w\in W_d} R(G_d)$	$R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{w\in W_d} R(T_d)[\overline{\Omega_w}]^{T_d}$	$R(T_d)\otimes_{R(G_d)}R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{w,w'\in W_d} R(T_d)\,[\overline{\Omega^{u,u'}_{w,w'}}]^{T_d}$	$\bigoplus_{w,w'\in W_d} R(T_d)\,[\overline{\tilde{\Omega}^{u,u'}_{w,w'}}]^{T_d}$
Id	$\mathbb{Z}$	$\bigoplus_{w\in W_d} \mathbb{Z}[\overline{\Omega_w}]$	$\bigoplus_{w,w'\in W_d} \mathbb{Z}\,[\overline{\Omega^{u,u'}_{w,w'}}]$	$\bigoplus_{w,w'\in W_d} \mathbb{Z}\,[\overline{\tilde{\Omega}^{u,u'}_{w,w'}}]$
$G_d\times\mathbb{C}^\times$	$R(G_d)[t^{\pm1}]$	$R(T_d\times\mathbb{C}^\times)$	$\bigoplus_{w'\in W_d} R(T_d\times\mathbb{C}^\times)\,[\overline{\Omega^{u,u'}_{w'}}]^{G_d\times\mathbb{C}^\times}$	$\bigoplus_{w'\in W_d} R(T_d\times\mathbb{C}^\times)\,[Z^{u,u'}_{w'}]^{G_d\times\mathbb{C}^\times}$
$B_d\times\mathbb{C}^\times$	$R(T_d)[t^{\pm1}]$ $\bigoplus_{w\in W_d} R(G_d\times\mathbb{C}^\times)$	$\bigoplus_{w\in W_d} R(T_d\times\mathbb{C}^\times)[\overline{\Omega_w}]^{T_d\times\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(T_d\times\mathbb{C}^\times)\,[\overline{\Omega^{u,u'}_{w,w'}}]^{T_d\times\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(T_d\times\mathbb{C}^\times)\,[\overline{\tilde{\Omega}^{u,u'}_{w,w'}}]^{T_d\times\mathbb{C}^\times}$
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$	$\bigoplus_{w\in W_d} R(\mathbb{C}^\times)[\overline{\Omega_w}]^{\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(\mathbb{C}^\times)[\overline{\Omega^{u,u'}_{w,w'}}]^{\mathbb{C}^\times}$	$\bigoplus_{w,w'\in W_d} R(\mathbb{C}^\times)[\overline{\tilde{\Omega}^{u,u'}_{w,w'}}]^{\mathbb{C}^\times}$

$K^-(-)$	pt	$\mathcal{F}_d$ $\widetilde{Rep}_d(Q)$	$\mathcal{F}_d \times \mathcal{F}_d$	$\mathcal{Z}_d = \coprod_{\underline{d},\underline{d}'} \mathcal{Z}_{\underline{d},\underline{d}'}$
G_d	$R(T_d)^{W_d}$	$\bigoplus_u R(T_d)$	$\bigoplus_u R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{u,\varpi'} R(T_d)[\overline{O}_{\varpi'}]^{G_d}$	$\bigoplus_{u,\varpi'} R(T_d)\,[Z_{\varpi'}]^{G_d}$
B_d	$R(T_d)$ $\bigoplus_{w\in W_d} R(G_d)$	$\bigoplus_u R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{\varpi\in W_{\lvert d\rvert}} R(T_d)[\overline{O}_\varpi]^{T_d}$	$\bigoplus_u R(T_d)\otimes_{R(G_d)}R(T_d)\otimes_{R(G_d)}R(T_d)$ $\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(T_d)\,[\overline{O_{\varpi,\varpi'}}]^{T_d}$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(T_d)\,[\overline{\tilde{O}_{\varpi,\varpi'}}]^{T_d}$
Id	$\mathbb{Z}$	$\bigoplus_{\varpi\in W_{\lvert d\rvert}} \mathbb{Z}[\overline{O}_\varpi]$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} \mathbb{Z}\,[\overline{O_{\varpi,\varpi'}}]$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} \mathbb{Z}\,[\overline{\tilde{O}_{\varpi,\varpi'}}]$
$G_d\times\mathbb{C}^\times$	$R(G_d)[t^{\pm1}]$	$\bigoplus_u R(T_d\times\mathbb{C}^\times)$	$\bigoplus_{u,\varpi'} R(T_d\times\mathbb{C}^\times)[O_{\varpi'}]^{G_d\times\mathbb{C}^\times}$	$\bigoplus_{u,\varpi'} R(T_d\times\mathbb{C}^\times)\,[Z_{\varpi'}]^{G_d\times\mathbb{C}^\times}$
$B_d\times\mathbb{C}^\times$	$R(T_d)[t^{\pm1}]$ $\bigoplus_{w\in W_d} R(G_d\times\mathbb{C}^\times)$	$\bigoplus_{\varpi\in W_{\lvert d\rvert}} R(T_d\times\mathbb{C}^\times)[\overline{O}_\varpi]^{T_d\times\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(T_d\times\mathbb{C}^\times)\,[\overline{O_{\varpi,\varpi'}}]^{T_d\times\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(T_d\times\mathbb{C}^\times)\,[\overline{\tilde{O}_{\varpi,\varpi'}}]^{T_d\times\mathbb{C}^\times}$
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$	$\bigoplus_{\varpi\in W_{\lvert d\rvert}} R(\mathbb{C}^\times)[\overline{O}_\varpi]^{\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(\mathbb{C}^\times)[\overline{O_{\varpi,\varpi'}}]^{\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi'\in W_{\lvert d\rvert}} R(\mathbb{C}^\times)[\overline{\tilde{O}_{\varpi,\varpi'}}]^{\mathbb{C}^\times}$

$$H_*^{G_d}(\mathcal{Z}_{\underline{d},\underline{d}}) \cong \bigotimes_{d_i} NH_{d_i}$$

$$K^{G_d}(\mathcal{Z}_{\underline{d},\underline{d}}) \cong R(T_d)\otimes_{R(G_d)}R(T_d) \cong \bigotimes_{d_i} R(T_{d_i})\otimes_{R(G_{d_i})}R(T_{d_i})$$

Black: know the alg structure under tensor prod
Grey: know the alg structure under tensor prod, which is not prefered
red: know the alg structure under convolution prod
Orange: only know the $R(Grp)$-module structure, and the alg structure is yet not known
light yellow: $R(G_d)$-module + W_d-equiv iso

Conclusions on the summer vacation.

I guess that most part of my tasks coincide with this paper:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf
Sadly, I only found it in the last week of the vacation.

Some possible tasks to work on.

1. Work out what $K_0^{Id}(\mathcal{B})$ is.

ref:
In [3264], the author computes the Chow group of G(2,4).
https://pbelmans.ncag.info/blog/2018/08/22/rank-flag-varieties/
http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/BGG-SchubCells-Usp.pdf

The module structure is easy, see
[https://math.stackexchange.com/questions/1012699/when-does-a-smooth-projective-variety-x-have-a-free-grothendieck-group]

For the cohomology of flag variety, see [GTM86, Prop 21.17].

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.
   - Bernstein presentation
   - try to understand the center of $\mathcal{H}(G(F), I)$
   - How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory
   - How can the Hecke algebra be realized as a Hecke algebra?

ref. [Hecke, Sec 10-17], [Williamson 11.4-12.2]

3. Try to understand what the Hall algebra / Quantum group is.

ref: [Lec 1-4, Appendix 4, https://arxiv.org/pdf/math/0611617.pdf]

- understand $\mathcal{H}_{Rep_k^{nil}(Q)}$ where $Q = \cdot \quad \cdot \to \cdot \quad \cdot \circlearrowleft$ [Lec 2-3]
- understand [Lec 4]

$$\begin{array}{ccc} \mathcal{H}_{\mathbb{P}^1} & \cong & \mathcal{U}_v(\widehat{sl_2}) \\ \cup & & \cup \\ \mathcal{H}_{Tor(\mathbb{P}^1)} & \cong & \bigotimes_{x \in \mathbb{P}^1} \mathcal{H}_{Tor\, x} \end{array}$$

- define (Quantum) Kac-Moody / loop algs [Appendix 4]
- Why is that

graded

$$K_0(Rep^{\mathbb{Z}}(R)) = \mathcal{U}_q(n(Q))$$

where

$$R = \bigoplus_{\underline{d}} H_\cdot^{G \times \mathbb{C}^\times, BM}(Z_{\underline{d}})$$

and what is

$$K_0\left(Rep^{\mathbb{Z}}\left(\bigoplus_{\underline{d}} K_0^{G \times \mathbb{C}^\times}(Z_{\underline{d}})\right)\right) ?$$

4. Work out the big picture

$$
\begin{tikzcd}
 & & \text{Hall alg} \arrow[r, "\text{realize}"] & \text{(half) quantum gp} & \text{Verma module} \\
 & K_0(R\text{-mod}) \arrow[ur, "\sim"] & & & \\
 & & \text{Hecke alg} \arrow[ul, "\text{get}"'] & & \text{modular form} \\
\text{Chow group} \arrow[r, phantom, "\sim"] & K_0^{Grp} \arrow[ur] \arrow[dr, "\sim"] & & & \\
 & & \text{convolution alg} & & K_0^{Grp}(\text{Springer fiber})
\end{tikzcd}
$$

Universal enveloping alg

KLR alg = quiver Hecke alg

Nil Hecke ring

5. A closer check of Satake iso

K_0	combinations	Hecke alg
$R(B) =$	$\mathbb{Z}[X^*(T)]$	$= \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F))$
$R(G) =$	$\mathbb{Z}[X^*(T)]^W$	$\neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F))$
$R(G)[q^{\pm\frac{1}{2}}] =$	$\mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}]$	$= \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$
$R(G \times \mathbb{C}^*) =$	$\mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$	
$R(T) \otimes_{R(G)} R(T) =$	NH_n $c\,End_{\mathbb{Z}}(\mathbb{Z}[X^*(T)])$	
$K^{G\times\mathbb{C}^*}(St) =$	$\mathcal{H}_{ext}$	$\overset{?}{=} \mathcal{H}(\hat{G}(F), I)$

$G = GL_n$

It's claimed by my schoolmate that

$$K_0(Perv_B(G/B)) \cong \mathcal{H}(G, B)$$

↑ sym monoidal structure induced from the convolution

then, what is

$$K_0^B(\mathcal{B}) \cong ?$$

$$K_0^{Id}(\mathcal{B}) \cong ?$$

$$? \cong \mathcal{H}(S_{m+n}, S_m \times S_n)$$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces.
(e.g. in the case of $\cdot$, $\cdot\circlearrowleft$, $\cdot \to \cdot$)

G, B, T

$\mathcal{B}$, $T^*\mathcal{B}$, St

$\mathfrak{g}$, $\tilde{\mathfrak{g}}$, $\mathfrak{g}^{sr}$, $\tilde{\mathfrak{g}}^{sr}$, $\mathcal{N}$, $\tilde{\mathcal{N}}$, $\mathfrak{h}$, $\mathfrak{n}$, $\mathfrak{g}_h$

$\tilde{\mathfrak{g}}^h$, $\mathcal{O}_h$, Λ_w^h

7. Try to understand what Kazhdan-Lusztig polynomials are [KL2], and
- Compute the transformation matrix between
$\{[T_w^*], w \in W_f\}$ and $\{[\Lambda_w^h], w \in W_f\}$? [KaSai]?
- understand what standard/crystal basis is
- understand the relationship between KL poly and crystal basis
- see if it is related to two basis in Rep(G) (irr reps & multiplicative basis)

Answer from my schoolmate:

$$\begin{array}{ccccc} \text{standard basis} & \xleftrightarrow{\text{KL-poly}} & \text{canonical basis} & \overset{\text{fix } q}{\rightsquigarrow} & \text{crystal basis} \\ \{[T_w^*], w \in W_f\} & & \{[\Lambda_w^h], w \in W_f\} & & \\ \text{irr reps} & & \text{multiplicative basis} & & \end{array}$$

8. Try to understand the module part, i.e.,
- numbers of components of the Springer fiber
- how does $K_0^{Grp}(St)$ act on K_0^{Grp}(Springer fiber) also act on $K_0^{Grp}(Rep_d(Q))$
- does that occupy "all rep" of $K_0^{Grp}(St)$

9. Ways of finding multiplication structure

1 By direct computation (with techniques)

Hecke algebra — double coset calculus

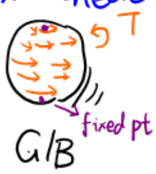
2. By formulas as alg-isos

$K_0^G(\mathcal{B})$ — induction formula

3. By geometrical computation

cohomology — cup product ? de Rham calculus

index theorem



Chow group — intersection theory

4. By deformation (indirect)

$H^{BM}_{top}(St)$

$K_0^{G\times\mathbb{C}^*}(St)$

How to get a clear intersection formula of Grothendieck? It's claimed in the introduction of [https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf], but I can not find any other references about this excess intersection formula.
How is it related to the Riemann-Roch theorem?

10. Different views on the double coset

$$B\backslash G/B = (*/B)\times_{*/G}(*/B)$$

- as a set
- as flag variety quotient B-action
- as a stack
- groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $\mathcal{H}(G,B)$, $\mathcal{H}_{sph}$, ($\mathcal{H}_{aff}$)	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG, ...	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:

- 3 Seminars (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
  - L-parameters
  - moduli in AG
  - some following developments of the modular form (different type of gps, Hecke operators, ...)
  - reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.