Eine Woche, ein Beispiel

12.7 rationality in algebraic geometry

I hearc these concepts in Jan Lange's talk, so I want to record them.

Ref:

[PP16]: Beauville, Arnaud, Brendan Hassett, Alexander Kuznetsov, and Alessandro Verra. Rationality Problems in Algebraic Geometry. Edited by Rita Pardini and Gian Pietro Pirola. Vol. 2172. Lecture Notes in Mathematics. Springer International Publishing, 2016. https://doi.org/10.1007/978-3-319-46209-7.

[Debo1]: Olivier Debarre. Higher-Dimensional Algebraic Geometry. Universitext, edited by S. Axler, F. W. Gehring, and K. A. Ribet. Springer, 2001. https://doi.org/10.1007/978-1-4757-5406-3.

$$
\begin{tikzcd}
\text{rational} \arrow[r, Rightarrow] & \text{stably rational} \arrow[d, Rightarrow] & & \text{uniruled} + \rho = 1 \arrow[d, Rightarrow, "\text{char } k = 0"] \\
& \text{factor-rational} \arrow[d, Rightarrow] & & \text{Fano} \arrow[d, Rightarrow, "\text{char } k = 0"] \\
& \text{retract rational} \arrow[r, Rightarrow] & \text{unirational} \arrow[r, Rightarrow] & \text{rationally connected} \arrow[d, Rightarrow, "\text{char } k = 0"] \\
& & & \text{uniruled} \arrow[d, Rightarrow] \\
& & & \kappa(X) = -\infty
\end{tikzcd}
$$

This diagram is collected from the following resources:
[PP16, p14, p106]
[Debo1, 5.6]: mainly for Fano => rationally connected
https://mathoverflow.net/questions/66569/uniruled-picard-number-1-fano

In [PP16] everything is over C, [Debo1] is a bit more relaxed. Still, most arrows are true (by checking the definition), so in char p they are still fine.

Variety	Unirational	Rational	Method	Reference
$V_6 \subset \mathbb{P}(1,1,1,2,3)$	?	No	Bir(V)	[Gr]
Quartic double $\mathbb{P}^3$	Yes	No	JV	[V1]
$V_3 \subset \mathbb{P}^4$	"	No	JV	[C-G]
$V_{2,2} \subset \mathbb{P}^5$, $X_5 \subset \mathbb{P}^6$	"	Yes		
Sextic double $\mathbb{P}^3$	?	No	Bir(V)	[I-M]
$V_4 \subset \mathbb{P}^4$	Some	No	Bir(V)	[I-M]
$V_{2,3} \subset \mathbb{P}^5$	Yes	No (generic)	JV, Bir(V)	[B1, P]
$V_{2,2,2} \subset \mathbb{P}^6$	"	No	JV	[B1]
$X_{10} \subset \mathbb{P}^7$	"	No (generic)	JV	[B1]
$X_{12}, X_{16}, X_{18}, X_{22}$	"	Yes		
$X_{14} \subset \mathbb{P}^9$	"	No	JV	[C-G] + [F3][1]

This comes from [PP16, p6].
Now we have better database: https://www.fanography.info/

Here. X: a variety, i.e., an integral scheme of f.t. over k.

Def. [PP16, p4. p13-14]

X is rational	if	$\exists$ birational map	$\mathbb{P}^n \overset{\sim}{\dashrightarrow} X$
X is stably rational	if	$\exists$ birational map	$\mathbb{P}^n \overset{\sim}{\dashrightarrow} X \times \mathbb{P}^k$
X is factor-rational	if	$\exists$ birational map	$\mathbb{P}^n \overset{\sim}{\dashrightarrow} X \times X'$
X is retract rational	if	$\exists$ rational dominant map + a rational section	$\mathbb{P}^n \dashrightarrow X$
X is unirational	if	$\exists$ rational dominant map	$\mathbb{P}^n \dashrightarrow X$

Take a sm proj model V

X is rationally connected if $\forall\, p, q \in X$, $\exists$ a rational curve $C \cong \mathbb{P}^1$ passing p & q

[Deb01, Def 4.1]

X is uniruled if $\exists$ rational dominant map $\mathbb{P}^1 \times X' \dashrightarrow X$