Eine Woche, ein Beispiel.

4.23 A naive beginning of affine group scheme

Examples.

① Commutative affine group scheme $\longleftrightarrow$ Commutative, cocommutative Hopf algebra

in general not abelian scheme ← require proper condition

$$
\begin{array}{llllllll}
 & \mathbb{G}_a & k[x] \; (x\mapsto y+z) & \mathbb{G}_m & k[x,x^{-1}] \; (x\mapsto yz) & = D(\mathbb{Z}) & D(G) & G\text{: f.g abel gp} \\
 & \cup\!\!\shortmid & & \cup\!\!\shortmid & & & & \\
(\text{char } k=p) & \alpha_{p^n} & k[x]/x^{p^n} & \mu_n & k[x]/(x^n-1) = k[\mathbb{Z}/n\mathbb{Z}] & = D(\mathbb{Z}/n\mathbb{Z}) & \updownarrow\vee & \text{for finite abel gp} \\
 & \updownarrow\vee & & \updownarrow\vee & & & & \\
 & \alpha_{p^n}^D & & \underline{\mathbb{Z}/n\mathbb{Z}} & k\bar{0}\oplus k\bar{1}\oplus\cdots k\overline{(n-1)} & & \underline{G} & G\text{: finite gp}
\end{array}
$$

$$W_{n,p}^m \subseteq W_{n,p} \qquad \updownarrow\vee \qquad W_{m,p}^n$$

$\alpha_{p^n} = W_{1,p}^n$

$\alpha_{p^n}^D = W_{n,p}^1$

$M_{n,p} = M_{n,p}^D \subseteq W_{n,p}^n$

$$
\begin{array}{ccc}
 & W_{2,p} & \\
/\!\!/ & \cup\!\!\shortmid & \backslash\!\!\backslash \\
\alpha_{p^2} & M_2 & \alpha_{p^2}^D
\end{array}
$$

② (Noncommutative) affine group scheme.

GL_n, SL_n, $\alpha_{p^n}\rtimes\mu_m$, etc...

Goal: As a scheme.

- compute dim (dim $=0 \Rightarrow$ compute $\dim_k R$)
- picture in mind

As an alg:

- compute $R^\times$, $\mathrm{Nil}(R)$, Center, ...
- if $\dim_k R<+\infty$, decide $R^\vee$
- compute

$$
\begin{array}{ccc}
\mathrm{Aut}_{k\text{-Hopf}}(R) & \subseteq & \mathrm{Aut}_{k\text{-alg}}(R) \\
\cap\!\shortmid & & \cap\!\shortmid \\
\mathrm{Hom}_{k\text{-Hopf}}(R) & \subseteq & \mathrm{Hom}_{k\text{-alg}}(R,R)
\end{array}
$$

Relations:

subgroup scheme, quotient scheme, action of group scheme.
classification of finite affine group scheme.

As a reminder for myself

1. Hopf alg structure on $\mathbb{Z}/n\mathbb{Z} = \operatorname{Spec}(k\bar{0} \oplus k\bar{1} \oplus \cdots k\overline{(n-1)})$

ring structure: $(a_0, \dots, a_{n-1})(b_0, \dots, b_{n-1}) \mapsto (a_0 b_0, \dots, a_{n-1} b_{n-1})$

Co-multi $\quad c^\# \quad \bar{l} \longmapsto \sum_{i+j=l} \bar{i} \otimes \bar{j}$

Co-unit $\quad e^\# \quad \bar{l} \longmapsto \begin{cases} 0 & l \neq 0 \\ 1 & l = 0 \end{cases}$

Antipode $\quad i^\# \quad \bar{l} \longmapsto \overline{-l}$

2. $D(G)(S) := \operatorname{Hom}_{\mathbb{Z}}(G, \Gamma^0(S)^\times)$

$\underline{G}(S) := \operatorname{Mor}_{Top}(S, G)$
Zariski, discrete

$W_{2,p}(S) := \Gamma(S) \oplus \Gamma(S)$ [for general, $W_{n,p}(S) = \Gamma(S)^{\oplus n}$ with "similar" ring structure]

$(a_0, a_1) \times (b_0, b_1) = \left(a_0 + b_0,\ a_1 + b_1 + \frac{1}{p}[a_0^p + b_0^p - (a_0 + b_0)^p]\right)$

3. Finite dimension is usually MUCH better then infinite dimension, and most of time $k[x]$ can be constructed as a counterexample.

① The basis of $(k[x])^\vee$ is not $\{(x^i)^*\}_{i \in \mathbb{N}_{\geq 0}}$

$ev_1 \in (k[x])^\vee - \langle (x^i)^* \rangle$ $\qquad ev_1 : f \longmapsto f(1)$

② $(k[x])^{\vee\vee} \supsetneqq k[x]$

$EV_1 \in (\langle (x^i)^* \rangle)^\vee - \langle (x^i)^{**} \rangle$ $\qquad EV_1 : F \longmapsto F(1)$

then extend EV_1 linearly to $(k[x])^{\vee\vee}$

$\Rightarrow EV_1 \in (k[x])^{\vee\vee} - k[x]$

③ $(k[x] \otimes k[y])^\vee \supsetneqq (k[x])^\vee \otimes (k[y])^\vee$

$ev_{1,1} \in (k[x] \otimes k[y])^\vee - (k[x])^\vee \otimes (k[y])^\vee$ $\qquad ev_{1,1} : f \longmapsto f(1,1)$

As a corollary, $k[x]^\vee$ has no natural Hopf alg structure.
induced from the Hopf alg structure of $k[x]$

4. Lemma. A, B Hopf alg $\Rightarrow A \otimes B$ Hopf alg

Cor. $[\mu_n \overset{\vee}{\longleftrightarrow} \mathbb{Z}/n\mathbb{Z}] \Rightarrow [D(G) \overset{\vee}{\longleftrightarrow} \underline{G}]$ $\qquad G$: finite abel group.

Rmk. A, B Hopf alg $\not\Rightarrow A \oplus B$ Hopf alg

[Union of group is no longer group]

General fact: $G^D \cong \underline{\operatorname{Hom}}(G, \mathbb{G}_m)$ $\qquad G$: finite commutative group scheme

5. $\alpha_p \cong \alpha_p^D$ by $\qquad$ Q: compute $\operatorname{Aut}_{k\text{-group Sch}}(\alpha_p)$.

$$k[x]/(x^p) \longrightarrow (k[x]/(x^p))^\vee$$

$$x^m \longmapsto \frac{1}{1 - mx^*} := 1 + mx^* + m^2 (x^2)^* + \cdots$$

Another way: $\quad x^m \longmapsto m!(x^m)^*$

[Verification:

$$\begin{tikzcd}
k[y]/(y^p) \otimes k[z]/(z^p) \arrow[r] \arrow[d] & k[x]/(x^p) \arrow[d] \\
(k[y]/(y^p))^\vee \otimes (k[z]/(z^p))^\vee \arrow[r] & (k[x]/(x^p))^\vee
\end{tikzcd}$$

$(y^i)^* \otimes (z^j)^* \longmapsto \binom{i+j}{i} (x^{i+j})^*$

$$\begin{tikzcd}
y^m \otimes z^n \arrow[r, mapsto] \arrow[d, mapsto] & x^{m+n} \arrow[d, mapsto] \\
\sum_{k=0}^{p-1} m^k (y^k)^* \otimes \sum_{l=0}^{p-1} n^l (z^l)^* \arrow[r, mapsto] \arrow[u, mapsto] \arrow[d, mapsto] & \sum_{k=0}^{p-1} (m+n)^k (x^k)^* \arrow[u, mapsto] \arrow[d, mapsto] \\
m!n!(y^m)^* \otimes (z^n)^* \arrow[r, mapsto] & (m+n)!(x^{m+n})^*
\end{tikzcd}$$
]

Rmk. $\alpha_{p^2} \not\cong \alpha_{p^2}^D$

Proof. char $k = p > 0$ $n, m \in \mathbb{N}^+$,

let $W_{n,p}^m$: $Sch_k \longrightarrow Grp$

$S \longmapsto \{(a_0, a_1, \cdots a_{n-1}) \in W_{n,p}(S) \mid a_0^{p^m} = \cdots = a_{n-1}^{p^m} = 0\} \subseteq W_{n,p}(S)$

then ① $W_{1,p}^m = \alpha_{p^m}$

② $W_{2,p}^1 = \alpha_{p^2}^D$, i.e. $W_{2,p}(S) \overset{\Delta}{=} Hom_{Grp\,Sch/S}(\alpha_{p^2,S}, \mathbb{G}_{m,S})$

[Δ: reduced to $S = Spec\ A$. we have

$W_{2,p}^1(S) \subseteq A \oplus A \longleftrightarrow Hom_{A\text{-}Hopf}(A[t,t^{-1}], A[x]/(x^{p^2})) \subseteq (A[x]/(x^{p^2}))^{\times}$

$(a_1, -(p-1)!\,a_p) \longleftarrow \sum_{i=0}^{p^2-1} a_i x^i$]

α_{p^2} and $W_{2,p}^1$ are different group functors $\Rightarrow \alpha_{p^2} \not\cong \alpha_{p^2}^D$

Rmk. I can still not find a canonical way s.t $\alpha_{p^3}^D = W_{3,p}^1$,
but it's claimed true. In general, it's claimed that $(W_{n,p}^m)^D \cong W_{m,p}^n$.

define M_n: $S \mapsto \{(a_0, a_1, \cdots a_{n-1}) \in W_{n,p}^n(S) \mid a_0^p = 0, a_i^p = a_{i-1}\ i \in \{1, \dots, n-1\}\} \subseteq W_{n,p}^n(S)$,

then $M_2 = Spec\ k[x_0, x_1]/(x_0^p, x_1^p - x_0)$

$= Spec\ k[x_1]/(x_1^{p^2})$ $\quad z_1 \longmapsto x_1 + y_1 + \frac{1}{p}\{x_1^{p^2} + y_1^{p^2} - (x_1^p + y_1^p)^p\}$

Don't forget the Hopf alg structure! So it's not α_{p^2}.

It's claimed that $M_n^D \cong M_n$, even though I can't figure it out.

Fix a field K. If we understand $\mathrm{Gal}(K^{sep}/K)$ well, then

- we understand finite étale group schemes $/K$ well.
- ep. when K is of char 0, then we finish our classification of finite group schemes.

[Martin] **Corollary 3.12.** *Let K be a field with separable closure K^{sep} and absolute Galois group $G := \mathrm{Gal}(K^{sep}/K)$. There is an equivalence of categories*

$$\begin{array}{rcl} (\textit{Finite étale group schemes over } K) & \leftrightarrow & (\textit{Finite groups with continuous } G\textit{-action}) \\ H & \mapsto & H(K^{sep}) \\ \mathrm{Spec}\,(\mathrm{Hom}_{(G-Set)}(H, K^{sep})) & \leftarrow\!\shortmid & H. \end{array}$$

For affine group scheme $G = \operatorname{Spec} R$, $\Omega_{R/k} \cong m/m^2 \otimes_k R$:

	G	R	m	$\Omega_{R/k}$		dim	$\operatorname{Lie}(G)$	$[p]$	$\operatorname{Der}_k(R,R)$
Lie bracket trivial; étale	$\underline{G}$	$k\oplus k\oplus\cdots k$	0	0		0	0	0	0
	$\mathbb{G}_a$	$k[x]$	(x)	$k[x]dx$		1	$k\frac{\partial}{\partial x}$	0	$k[x]\frac{\partial}{\partial x}$
	$\mathbb{G}_m$	$k[x,x^{-1}]$	$(x-1)$	$k[x,x^{-1}]d(x-1)$		1	$kx\frac{\partial}{\partial x}$	$\left(\lambda x\frac{\partial}{\partial x}\right)^{[p]} = \lambda^p x\frac{\partial}{\partial x}$	$k[x,x^{-1}]\frac{\partial}{\partial x}$
	α_{p^n}	$k[x]/x^{p^n}$	(x)	$k[x]/x^{p^n}dx$		1	$k\frac{\partial}{\partial x}$	0	$k[x]/x^{p^n}\frac{\partial}{\partial x}$
	μ_n	$k[x]/(x^n-1)$	$(x-1)$	$k[x]/(x^n-1)\,d(x-1)$	$p\mid n$	1	$kx\frac{\partial}{\partial x}$	$\left(\lambda x\frac{\partial}{\partial x}\right)^{[p]} = \lambda^p x\frac{\partial}{\partial x}$	$k[x]/(x^n-1)\frac{\partial}{\partial x}$
étale				0	$p\nmid n$	0	0	0	0

G	ht	G^0	G^{red} (k is perfect)
$\underline{G}$	∞	1	$\underline{G}$
$\mathbb{G}_a$	∞	$\mathbb{G}_a$	$\mathbb{G}_a$
$\mathbb{G}_m$	∞	$\mathbb{G}_m$	$\mathbb{G}_m$
α_{p^n}	n	α_{p^n}	$\underline{Id}$
μ_n	$\begin{cases} N & \text{if } n = p^N \\ \infty & \text{otherwise} \end{cases}$	$\begin{cases} \mu_{p^{\nu_0}} & \text{if } n = p^{\nu_0}q_1^{\nu_1}\cdots q_s^{\nu_s} \\ 1 & \text{if char } k = 0 \end{cases}$	$\begin{cases} \mu_{n/p^{\nu_0}} & \text{if } n = p^{\nu_0}q_1^{\nu_1}\cdots q_s^{\nu_s} \\ \mu_n & \text{if char } k = 0 \end{cases}$

G	$\# G$	$G[F]$
$\underline{G}$	$\# G$	G
$\mathbb{G}_a$	—	α_p
$\mathbb{G}_m$	—	μ_p
α_{p^n}	p^n	α_p
μ_n	n	$\begin{cases} \mu_m & p\mid m \\ \operatorname{Spec} k & p\nmid m \end{cases}$