

Un exemple par jour

4.8 K3 surface: cpt cplx surf s.t $\omega_X \cong \mathcal{O}_X$

can be changed by $b_i=0$ or $\pi_1(X)=\{*\}$

$$H^1(X, \mathcal{O}_X) = 0$$

Today: Fermat quartic surface $X: z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$

$$i: X \hookrightarrow \mathbb{P}^3$$

A nice survey for smooth quartic surfaces: <http://www-personal.umich.edu/~jakubw/masterthesis.pdf>

1. proj smooth alg surf ✓

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-4) \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow i_* \mathcal{O}_X \rightarrow 0$$

Lemma VIII.9 Let $V \subset \mathbb{P}^n$ be a d -dimensional complete intersection; then $H^i(V, \mathcal{O}_V) = 0$ for $0 < i < d$. [Beauville]

$$\begin{array}{l} H^3 \\ H^2 \\ H^1 \\ H^0 \end{array} \quad \begin{array}{c} \xrightarrow{\quad} \mathbb{C} \rightarrow 0 \\ \xrightarrow{\quad} 0 \rightarrow 0 \rightarrow H^2(X, \mathcal{O}_X) \cong \mathbb{C} \\ \xrightarrow{\quad} 0 \rightarrow 0 \rightarrow H^1(X, \mathcal{O}_X) \cong 0 \\ 0 \rightarrow 0 \rightarrow \mathbb{C} \rightarrow H^0(X, \mathcal{O}_X) \cong \mathbb{C} \end{array}$$

$$\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4) \xrightarrow{\text{2.5.B in [Vakil]}} \omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_X = \mathcal{O}_X$$

$$\frac{z_0 dz_1 \wedge dz_2}{4z_3^3} \quad \begin{array}{l} z_0 \neq 0 \\ z_3 \neq 0 \end{array}$$

$\therefore X$ is a K3-surface.

2. Prop. X : K3 surfaces $\Rightarrow X$ is minimal.

$$P_n = 1 \quad n \geq 1 \Rightarrow K(X) = 0$$

$$\begin{array}{cccc} 2 & -20 & 2 & 24 \\ & & 1 & 1 \\ & & 0 & 0 \\ (b_+, b_-) = (3, 19) & 1 & 20 & 1 \\ & 0 & 0 & 0 \\ c_2 = 24 & & 1 & 1 \\ c_1^2 = K^2 = 0 & & & \end{array}$$

Some conclusions on alg topo:

M : closed connected mfd of dim n

	orientable	nonorientable
$H_n(M)$	\mathbb{Z}	0
$H^n(M)$	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$
$H_n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H^n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Moreover, when M is oriented +cpt

$$\Rightarrow T^p \cong T^{n-p+1} \quad (H^i(X) \cong \mathbb{Z}^{\oplus b_i} \oplus T^i)$$

by universal coefficient thm & Poincaré duality

Cor. $T := H_1(X)$

n	0	1	2	3	4	≥ 5
$H_n(X)$	\mathbb{Z}	T	$\mathbb{Z}^{22} \oplus T$	0	\mathbb{Z}	0
$H^n(X)$	\mathbb{Z}	0	$\mathbb{Z}^{22} \oplus T$	T	\mathbb{Z}	0

Claim. $T = 0$, i.e. $H_1(X)$ has no torsion!

<https://math.stackexchange.com/questions/2882059/>

Proof. by LES induced by $0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \rightarrow 1$,
only need to prove $\text{Pic}(X)$ is torsion free.

Suppose $D \in \text{Div}(X)$ s.t. $nD \equiv 0 \Rightarrow \chi(D) = 2$

$$\Rightarrow h^0(D) \geq 1 \text{ or } h^0(-D + K_X) = h^2(D) \geq 1 \quad \text{w.l.o.g. suppose } h^0(D) \geq 1$$

$$\Rightarrow D \sim D' \text{ where } D' \text{ is effective}$$

$$\Rightarrow D' = 0$$

Another proof. If $H_1(X)$ has torsion, then $\exists G \triangleleft H_1(X)$ s.t. $H_1(X)/G \cong \mathbb{Z}/m\mathbb{Z}$
 denote $p: \pi_1(X) \rightarrow H_1(X)$, then $\pi_1(X)/p^{-1}(G) \cong H_1(X)/G \cong \mathbb{Z}/m\mathbb{Z}$ $m \in \mathbb{N}^+$
 $m > 1$
 $\therefore \exists$ a nontrivial unramified covering of degree m
 $\Phi: \tilde{X} \rightarrow X$
 $\Rightarrow \begin{cases} K_{\tilde{X}} = 0 \\ \chi_{top}(\tilde{X}) = 24m \end{cases} \Rightarrow \chi(\mathcal{O}_{\tilde{X}}) = \frac{1}{12} (K_{\tilde{X}}^2 + \chi_{top}(\tilde{X})) = 2m > 2 - h^1(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_{\tilde{X}})$
 Contradiction!

12.2. **K3 lattice.** We deduce that a K3 surface has second Betti number $b_2 = 22$.
 Cup-product equips $H^2(X, \mathbb{Z})$ with the structure of an integral lattice of rank 22.
 Often this lattice is called **K3 lattice** and denoted by Λ . The following properties
 of Λ are well-known:

- Λ is unimodular by Poincaré-duality;
- Λ has signature $(3, 19)$ by the topological index theorem;
- Λ is even by Wu's formula since the first Chern class is even.

← one signature:

$$C^2 = 2g(C) - 2 \text{ is even}$$

Hence the classification of even unimodular lattices implies that

$$\Lambda \cong U^3 \oplus \mathbb{H}(-E_8)^{\oplus 2}$$

$$U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$$

$$\tilde{b}(x, y) = -b(x, y)$$

Now let us consider the fundamental group of X .

Thm (Lefschetz hyperplane thm) [In wiki there are several proofs].

Let Y : n -dim cplx proj variety
 X : hyperplane section of Y
 U : $Y - X$ is smooth

then $H_k(Y, X, \mathbb{Z}), H^k(Y, X, \mathbb{Z}), \pi_k(Y, X) = 0$ for $k \leq n-1$. $k \in \mathbb{N}^+$

Cor.

$$\begin{array}{llll} H_k(X) \rightarrow H_k(Y) & \text{is} & \begin{array}{cc} k < n-1 & k = n \\ \text{iso} & \text{or} & \text{surj} \end{array} \\ H^k(Y) \rightarrow H^k(X) & \text{is} & \begin{array}{cc} \text{iso} & \text{or} & \text{inj} \end{array} \\ \pi_k(X) \rightarrow \pi_k(Y) & \text{is} & \begin{array}{cc} \text{iso} & \text{or} & \text{surj} \end{array} \end{array}$$

For X Fermat quartic, $X \hookrightarrow \mathbb{P}_{\mathbb{C}}^3 \xrightarrow{\phi} \mathbb{P}_{\mathbb{C}}^{3*}$ $Y = \phi(\mathbb{P}_{\mathbb{C}}^3)$

$$H_1(X) = H_1(Y) = H_1(\mathbb{P}_{\mathbb{C}}^3) = 0 \quad H^1(X) = 0 \quad \pi_1(X) = 0$$

$\Rightarrow X$ is simply connected 4-dim mfd.

Rmk. By [GTM82, Thm 17.21], $\pi_2(X) = H_2(X) = \mathbb{Z}^{22}$:

By <https://mathoverflow.net/questions/186119/what-are-the-higher-homotopy-groups-of-a-k3-surface>

$$\pi_n(X) = \pi_n(\#^{20} S^2 \times S^3) \quad n \geq 3, \text{ and}$$

	1	2	3	4	5	6	7
$\pi_n(X)$	0	\mathbb{Z}^{22}	\mathbb{Z}^{251}	$\mathbb{Z}^{3520} \oplus (\mathbb{Z}/2\mathbb{Z})^{42}$	$\mathbb{Z}^{57960} \oplus \text{tor}$	$\mathbb{Z}^{1020096} \oplus \text{tor}$	$\mathbb{Z}^{22397760} \oplus \text{tor}$

It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry.

-Solomon Lefschetz

Picard group.

Again, by

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$$

$$\begin{array}{c} \hookrightarrow H^2(X, \mathbb{Z}) \xrightarrow{\cong} H^2(\mathcal{O}_X) \xrightarrow{\cong} H^2(\mathcal{O}_X^\times) \\ \hookrightarrow H^1(X, \mathbb{Z}) \xrightarrow{\cong} H^1(\mathcal{O}_X) \xrightarrow{\cong} H^1(\mathcal{O}_X^\times) = \text{Pic}(X) \\ 0 \rightarrow H^0(X, \mathbb{Z}) \xrightarrow{\cong} H^0(\mathcal{O}_X) \xrightarrow{\cong} H^0(\mathcal{O}_X^\times) = \mathbb{C}^\times \end{array}$$

$$\Rightarrow \text{Pic}(X) = \text{NS}(X) \subseteq H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$$

$$\Rightarrow \text{Pic}(X) \cong \mathbb{Z}^{\rho(X)}, 1 \leq \rho(X) \leq 20$$

Rmk. for X Fermat quartic, $\rho(X)=20$, $\text{Pic}(X) \cong (-E_8)^{\oplus 2} \oplus U \oplus (-8)^{\oplus 2}$ $\text{Pic}(X)^\dagger \cong (8)^{\oplus 2}$

[Schütt, Shioda and van Luijk]

Q: find a metric s.t $\text{Ric}(\rho) \equiv 0$?