

# Eine Woche, ein Beispiel

## 1.21. complex multilinear algebra

The title comes from

<http://staff.ustc.edu.cn/~wangzuoq/Courses/16F-Manifolds/Notes/Lec16.pdf>

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:

[http://maths.nju.edu.cn/~yshi/BICMR\\_ComplexGeometry.pdf](http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf)

$M$ , cplx mfld,  $p \in M$

$M_{\mathbb{R}} : M$  viewed as smooth mfld, not base change  
better:  $M_{sm}$

e.g.  $M = \mathbb{C}^3$   $p = 0$

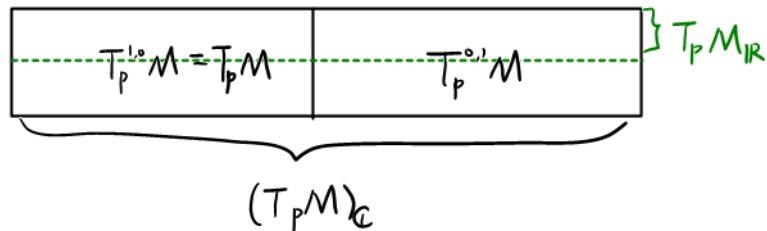
Notation	base field	dim	basis	name	[YS20]
$T_p M$	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p M_{\mathbb{R}}$	$\mathbb{R}$	6	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	real tangent vector	$T_p^{\mathbb{R}} M$
$(T_p M)_{\mathbb{C}} := T_p M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}$ or $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	complexified tangent vector	$T_p^{\mathbb{C}} M$
$T_p^{1,0} M = T_p M$	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p^{0,1} M$	$\mathbb{C}$	3	$\frac{\partial}{\partial \bar{z}_i}$	anti-holomorphic tangent vector	
$T_p^* M$	$\mathbb{C}$	3	$dz_i$	holomorphic 1-form	$\Omega_p^1$
$T_p^* M_{\mathbb{R}} = \Omega_{\mathbb{R}, p}$	$\mathbb{R}$	6	$dx_i, dy_i$	real 1-form	
$(T_p^* M)_{\mathbb{C}} := T_p^* M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$dz_i, d\bar{z}_i$ or $dx_i, dy_i$	complexified 1-form	$T_p^{\mathbb{C}} M = A_p^1$
$T_p^{1,0,*} M = \Omega_p^{1,0} = T_p^* M_{\mathbb{C}}$	$\mathbb{C}$	3	$dz_i$	$(1,0)$ -form	$T_p^{1,0,*} M = A_p^{1,0}$
$T_p^{0,1,*} M = \Omega_p^{0,1}$	$\mathbb{C}$	3	$d\bar{z}_i$	$(0,1)$ -form	$T_p^{0,1,*} M = A_p^{0,1}$

$\Omega^i, \Omega^{i,j}$  sheaves on  $M$

Rmk. We don't have any natural identification between  $T_p M$  &  $T_p M_{\mathbb{R}}$ .

Notice that  $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} (\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})$ ,  $-\frac{1}{2}i$  is not real. so  $\frac{\partial}{\partial \bar{z}} \notin T_p M_{\mathbb{R}}$ .

although our geometrical intuition of  $T_p M$  is often  $T_p M_{\mathbb{R}}$ ,  
 $T_p M \cap T_p M_{\mathbb{R}} = \emptyset$  in  $(T_p M)_{\mathbb{C}}$ .



Reminder: the (induced) almost complex structure is defined as

$$\begin{aligned}
 J: T_p M_{\mathbb{R}} &\longrightarrow T_p M_{\mathbb{R}} \\
 \frac{\partial}{\partial x_i} &\longmapsto \frac{\partial}{\partial y_i} \\
 \frac{\partial}{\partial y_i} &\longmapsto -\frac{\partial}{\partial x_i} \\
 \rightsquigarrow J: T_p M &\longrightarrow T_p M \\
 J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) &= \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \\
 J\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) &= \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) \begin{pmatrix} i & \\ & -i \end{pmatrix}
 \end{aligned}$$

real basis of  $(T_p M)_C$ :

$$\begin{aligned}
 \mathcal{B}_1 = \left\{ \underbrace{\partial_x, \partial_y}_{G T_p M_{\mathbb{R}}}, \underbrace{i\partial_x, i\partial_y}_{i T_p M_{\mathbb{R}} \mathcal{D}} \right\} \\
 \text{with } \xrightarrow{\text{Id}} \quad \xleftarrow{\text{-Id}}
 \end{aligned}$$

$$\begin{cases} \partial_x = \partial_z + \partial_{\bar{z}} \\ \partial_y = \frac{1}{2i}(\partial_z - \partial_{\bar{z}}) \end{cases} \quad \begin{aligned} dx &= \frac{1}{2}(dz + d\bar{z}) \\ dy &= \frac{1}{2i}(dz - d\bar{z}) \end{aligned}$$

$$\begin{aligned}
 \mathcal{B}_2 = \left\{ \underbrace{\partial_z, \partial_{\bar{z}}}_{G T_p M}, \underbrace{i\partial_z, i\partial_{\bar{z}}}_{T^* M \mathcal{D}} \right\} \\
 \text{with } \xleftarrow{x_i} \quad \xrightarrow{x_i}
 \end{aligned}$$

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \quad \begin{aligned} dz &= dx + idy \\ d\bar{z} &= dx - idy \end{aligned}$$

$\mathcal{D}$ :  $J$ ,  $\text{conj}_J$

$\mathcal{D}$ :  $\text{conj}_i$

$\mathcal{D}$ :  $x_i$

$$\begin{aligned} J(f\partial_x + g\partial_y) &= f\partial_y - g\partial_x \\ J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_y, -\partial_x, i\partial_y, -i\partial_x) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} J(f\partial_z + g\partial_{\bar{z}}) &= if\partial_{\bar{z}} - ig\partial_z \\ J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (i\partial_z, -i\partial_{\bar{z}}, -\partial_z, \partial_{\bar{z}}) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_J(f\partial_x + g\partial_y) &= \bar{f}\partial_x - \bar{g}\partial_y \\ \text{conj}_J(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, -\partial_y, -i\partial_x, i\partial_y) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_J(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_z + \bar{g}\partial_{\bar{z}} \\ \text{conj}_J(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_z, \partial_{\bar{z}}, -i\partial_z, -i\partial_{\bar{z}}) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_i(f\partial_x + g\partial_y) &= \bar{f}\partial_x + \bar{g}\partial_y \\ \text{conj}_i(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, \partial_y, -i\partial_x, -i\partial_y) \\ &= (\partial_x, \partial_y, i\partial_x, i\partial_y) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \text{conj}_i(f\partial_z + g\partial_{\bar{z}}) &= \bar{f}\partial_{\bar{z}} + \bar{g}\partial_z \\ \text{conj}_i(\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) &= (\partial_{\bar{z}}, \partial_z, -i\partial_{\bar{z}}, -i\partial_z) \\ &= (\partial_z, \partial_{\bar{z}}, i\partial_z, i\partial_{\bar{z}}) \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 \end{pmatrix} \end{aligned}$$