

Eine Woche, ein Beispiel

4.28 naive \otimes -Hom adjunction

Ref: from [23.11.19]

Notation: - A : associate ring allowed to be non-commutative, contains 1
 - There are two systems to write category of A -modules:

$$\begin{array}{llll} \text{Mod}_A & = & A\text{-Mod} & \ni {}_A M \\ (\text{Mod}_A)^{\text{op}} \neq \text{Mod}_{A^{\text{op}}} & = & \text{Mod}-A = A^{\text{op}}\text{-Mod} & \ni M_A \\ \text{Mod}_{A \otimes B^{\text{op}}} & = & A\text{-Mod}-B & \ni {}_A M_B \end{array}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\text{Mod}_{B \otimes A^{\text{op}}})^{\text{op}} = (B\text{-Mod}-A)^{\text{op}} = (A^{\text{op}}\text{-Mod}-B^{\text{op}})^{\text{op}} \ni {}_B M_A$$

as

$$(\text{Mod}_{A \otimes B^{\text{op}}})^{\text{op}} = (A\text{-Mod}-B)^{\text{op}}$$

⚠ Even though you can identify $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{\text{op}})$,
 $A^{\text{op}} \notin \text{Ob}(\text{Ring}^{\text{op}})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\text{Mod}_A = \text{Mod}_{A^{\text{op}}} \subset \text{Sh}(\text{Spec } A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
 Q: How to see the geometry of noncommutative rings? It is still vague for me.

1. definition recall for \otimes & Hom

2. adjunction

3. comparison between \otimes -Hom & $f^* \dashv f_*$

⋮

6. comparison between \otimes -Hom & $f^* \dashv f_*$, derived version

1. definition recall for \otimes & Hom

$$\begin{aligned} \otimes_A: \text{Mod}_{A^{\text{op}}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}} \\ \text{Hom}_A(-, -): (\text{Mod}_A)^{\text{op}} \times \text{Mod}_A &\longrightarrow \text{Mod}_{\mathbb{Z}} \end{aligned}$$

In general,

$$\begin{aligned} \otimes_B: A\text{-Mod-}B \times B\text{-Mod-}C &\longrightarrow A\text{-Mod-}C \\ \text{Hom}_B(-, -): (A\text{-Mod-}B)^{\text{op}} \times B\text{-Mod-}C &\longrightarrow A\text{-Mod-}C \end{aligned}$$

$$\begin{aligned} \text{Hom}_B^A(-, -): (A\text{-Mod-}B)^{\text{op}} \times B\text{-Mod-}A &\longrightarrow \mathbb{Z}\text{-Mod} \\ \parallel &\quad \parallel &\quad \parallel &\quad \parallel \\ \text{Hom}_{B \otimes_{\mathbb{Z}} A^{\text{op}}}(-, -): (\mathbb{Z}\text{-Mod-}B \otimes_{\mathbb{Z}} A^{\text{op}})^{\text{op}} \times (B \otimes_{\mathbb{Z}} A^{\text{op}}\text{-Mod-}\mathbb{Z})^{\text{op}} &\longrightarrow \mathbb{Z}\text{-Mod-}\mathbb{Z} \end{aligned}$$

$${}_A X_B, {}_B Y_C, {}_C Z_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{\text{op}}} X$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$$\text{Hom}_A(A, X) \cong X$$

$$\text{in } A\text{-Mod-}C = C^{\text{op}}\text{-Mod-}A^{\text{op}}$$