Eine Woche, ein Beispiel

11.19. Basic sheaf calculation

Goal: Motivate $f_*, f^*, f_!, f^!$, by connecting them with (co)homology theory

After story: $\rightsquigarrow$ calculation of $\mathrm{Perv}_\Delta(\mathbb{CP}^1)$
$\rightsquigarrow$ generalize Morse theory
$\rightsquigarrow$ Characteristic classes/cycles
$\rightsquigarrow$ index theorem

Minor advantages from my talk:

- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    $\rightsquigarrow$ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale) descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \mathrm{Top}$

$\mathrm{Obj}(\mathrm{Sh}(X)) = \{\text{sheaves of abelian gps}\}$

e.g. $\mathrm{Sh}(\{*\}) = \mathrm{Abel}$ $\qquad \mathbb{Q}_{\{*\}} \longleftrightarrow \mathbb{Q}$

0. sheaf
1. f_*, skyscraper sheaf & global sections
2. f^*, constant sheaf & stalks
3. Rf_* & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology

$-\otimes-$ & product structure on cohomology

$\mathrm{Hom}(-,-)$ & Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016
[IHPS] Laurenţiu G. Maxim, Intersection Homology & Perverse Sheaves with Applications to Singularities, 2019

0. Sheaf

Recall the definition of

- presheaf $\mathcal{F}$
- sheaf $\mathcal{F}$
- stalk $\mathcal{F}_x$
- global section $\mathcal{F}(X) = \Gamma(X;\mathcal{F}) = H^0(X;\mathcal{F})$
- cohomology $R^n\Gamma(X;\mathcal{F}) = H^n(X;\mathcal{F})$

https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology
If X is paracompact and Hausdorff, Cech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology
https://math.stackexchange.com/questions/3305512/cech-cohomology-and-the-simplicial-cohomology-of-the-nerve-of-an-open-cover

Recall examples of sheaves:

complicated $\left\{\begin{array}{l} \cdot\ \mathcal{C}_X : \text{sheaf of cont fcts on } X \\ \cdot\ \mathcal{O}_X : \text{structure sheaf on } X \end{array}\right.$ e.g., X: (cplx) mfld, scheme, …

- $\underline{\mathbb{Q}}_X$: constant sheaf on X

- $\mathrm{sky}_p(\mathbb{Q})$: skyscraper sheaf of $p \in X$ on X.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{C}_X)_x \quad \& \quad (\mathrm{sky}_p(\mathbb{Q}))_x .$$

1. f_*, skyscraper sheaf & global sections

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(Y)$, $f : Y \longrightarrow X$ cont

Def. $f_*\mathcal{F} \in Sh(X)$ is given by
$$f_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(f^{-1}(\mathcal{U}))$$
This defines a fctor
$$f_* : Sh(Y) \longrightarrow Sh(X)$$

$$\begin{array}{ccc} \mathcal{F} & & f_*\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_{p,*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$

For $\pi : Y \longrightarrow \{*\}$, $\pi_*\mathcal{F} = \mathcal{F}(Y) = \Gamma(Y;\mathcal{F})$

Ex (hard?) For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, compute $j_*\underline{\mathbb{Q}}_{\mathbb{C}}$.

- ◯ It is a constant sheaf on $\mathbb{CP}^1$.
- ◯ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$.
- ◯ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$.
- ◯ All the above is wrong.
- ◯ I don't know, but I don't want to make a wrong choice.

2. f^*, constant sheaf & stalks

In [Vakil, Chapter 2], it is f^{-1}, the inverse image functor.

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(X)$, $f : Y \longrightarrow X$ cont

Def. $f^*\mathcal{F} \in Sh(Y)$ is given by sheafification of

$$f^{*,\mathrm{pre}}\mathcal{F}(\mathcal{U}) = \varinjlim_{f(\mathcal{U}) \subseteq V} \mathcal{F}(V)$$

$$\begin{tikzcd} f^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ \mathcal{U} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

This defines a fctor

$$f^* : Sh(X) \longrightarrow Sh(Y)$$

Recall:

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (x_p)_p \in \prod_{p \in \mathcal{U}} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall x_0 \in \mathcal{U}, \exists\, \mathcal{U}_{x_0} \subseteq \mathcal{U} \text{ nbhd of } x_0, \\ s \in \mathcal{F}(\mathcal{U}) \text{ s.t.} \\ s_p = x_p \qquad \forall p \in \mathcal{U}_{x_0} \end{array} \right\}$$

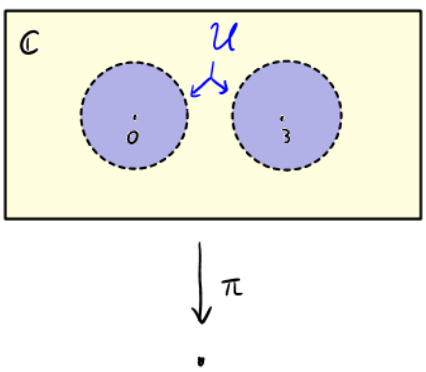
By definition, $(\mathcal{F}^{sh})_p = \mathcal{F}_p$.

Universal property:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f \in Mor_{PSh}"] \arrow[d, "sh"'] & \mathcal{G} \\ \mathcal{F}^{sh} \arrow[ur, dashed, "\exists!\, f^{sh} \in Mor_{Sh}"'] & \end{tikzcd}$$

$\mathcal{G}$: sheaf

Ex. For $\pi : \mathbb{C} \longrightarrow \{*\}, U = B_1(0) \cup B_1(3)$, which one is correct:



- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ All the above is wrong.

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_p^* \mathcal{F} = \mathcal{F}_p$

For $\pi : Y \longrightarrow \{*\}$, $\pi^* \underline{\mathbb{Q}}_{\{*\}} = \underline{\mathbb{Q}}_Y$

For $\mathcal{U} \subset X$ open, $j : \mathcal{U} \longrightarrow X$, $j^* \mathcal{F} = \mathcal{F}|_{\mathcal{U}}$

People generalize the last notation to arbitrary subset:

For $Y \subset X$, $\iota_Y : Y \hookrightarrow X$, $\iota_Y^* \mathcal{F} \triangleq \mathcal{F}|_Y$

Q: For $\mathcal{U} \subset X$ open, how to express $\mathcal{F}(\mathcal{U})$ by fctors?

A:

$$\begin{tikzcd} \mathcal{U} \arrow[rr, hook, "\iota_{\mathcal{U}}"] \arrow[dr, "\pi_{\mathcal{U}}"'] & & X \arrow[dl, "\pi_X"] \\ & \{*\} & \end{tikzcd}$$

$$\mathcal{F}(\mathcal{U}) = \pi_{\mathcal{U},*} \underbrace{\iota_{\mathcal{U}}^* \mathcal{F}}_{\mathcal{F}|_{\mathcal{U}}}$$

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

Prop. One has the adjunction $f^* \dashv f_*$, i.e.,

$$\mathrm{Mor}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G}) \qquad + \text{naturality}$$

[*Hint.* [Vakil, 2.7.B] Show that both side give the same information, i.e.,

$$\phi_{UV} \in \mathrm{Mor}_{Ab}(\mathcal{F}(\mathcal{U}), \mathcal{G}(V)) \qquad \text{for each pair } (V, \mathcal{U}) \text{ s.t. } f(V) \subset \mathcal{U}$$

$+$ compatability]

Cor. f^* is right adjoint, f_* is left adjoint.

Rmk. f^* is an exact functor.

Hint: exactness can be checked on stalks!

After "polished" (because of the structure sheaf), f^* is again only right adjoint.

3. Rf_* & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input $\mathcal{F}$, output a complex (before Ker/Im procedure)
- SES induces LES: for
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
one has
$$\begin{array}{l} H^2(X;\mathcal{F}) \longrightarrow \cdots \\ H^1(X;\mathcal{F}) \longrightarrow H^1(X;\mathcal{G}) \longrightarrow H^1(X;\mathcal{H}) \\ 0 \longrightarrow \underset{\pi_*\mathcal{F}}{\underset{\parallel}{H^0(X;\mathcal{F})}} \longrightarrow \underset{\pi_*\mathcal{G}}{\underset{\parallel}{H^0(X;\mathcal{G})}} \longrightarrow \underset{\pi_*\mathcal{H}}{\underset{\parallel}{H^0(X;\mathcal{H})}} \end{array}$$
$\pi: X \longrightarrow \{*\}$
- can be viewed as right derived fctor of
$$H^0(X,-) = \Gamma(X,-) = \pi_*$$
one gets
$$H^n(X,-) = R^n\Gamma(X,-) = R^n\pi_*$$
We denote the complex (before the Ker/Im procedure) as
$$R\Gamma(X,-) = R\pi_*$$
up to homotopy equiv & quasi-iso, i.e., in the derived category of $\{*\}$.

$$\begin{aligned} \mathcal{D}(X) = \mathcal{D}(Sh(X)) &= \text{“derived category of sheaves over } X\text{”} \\ &= \text{“ complexes of sheaves over } X\text{, up to} \cdots \text{”} \\ &= \{\cdots \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots\} \triangleq \{\mathcal{F}^\bullet\} \end{aligned}$$

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

Def. $Rf_* \mathcal{F}$ = "derived pushforward of $\mathcal{F}$"
$= f_* \mathcal{I}^\bullet$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & Rf_*\mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ & \mathcal{U} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

[Here, $\mathcal{I}^\bullet$ is the injective resolution of $\mathcal{F}$:
$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$
$(\Rightarrow \mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}^\bullet)$]

This defines a fctor
$$Rf_*: \mathcal{D}^b(Y) \longrightarrow \mathcal{D}(X)$$

The derived pushforward is hard to compute.
just like cohomology, and even worse, since we need more information
Luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]
$R^n f_* \mathcal{F}$ is given by the sheafification of
$$(R^n f_*^{pre} \mathcal{F})(\mathcal{U}) = H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$
sometimes omit

e.p. one can compute the stalk
$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in \mathcal{U}} H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] \\ X \end{tikzcd}$$

Cor For $\pi: X \to \{*\}$,
$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For $\pi: \mathbb{CP}^1 \to \{*\}$,
$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in $\mathcal{D}(*)$ are proj, we work over $\mathbb{Q}$]

$$R\pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$
$$= [\underset{}{0} \to \cdots \to \underset{-1}{} \underset{0}{\mathbb{Q}} \to \underset{1}{0} \to \underset{2}{\mathbb{Q}} \to \underset{3}{0} \to \underset{4}{\cdots}]$$

Ex.

For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, what is true about $Rj_*\underline{\mathbb{Q}}_{\mathbb{C}}$?

- ○ $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- ○ $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- ○ $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- ○ $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- ○ What the hell is that?

In fact, $(Rj_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1]$.

$i : \{\infty\} \to \mathbb{CP}^1$ is exact, so $Ri_* = i_*$.

4. $f_!$, extension by zeros & global sections with cpt supp

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & f_!\mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ & \mathcal{U} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

Def. $f_!\mathcal{F} \in Sh(X)$ is given by

$$f_!\mathcal{F}(\mathcal{U}) = \{ s \in \underset{(f_*\mathcal{F})(\mathcal{U})}{\underset{\|}{\mathcal{F}(f^{-1}(\mathcal{U}))}} \mid f|_{supp(s)} : supp(s) \longrightarrow \mathcal{U} \text{ is proper} \}$$

This defines a fctor

$$f_! : Sh(Y) \longrightarrow Sh(X)$$

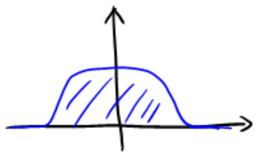
Recall : $supp(s) = \overline{\{ x \in f^{-1}(\mathcal{U}) \mid s_x \neq 0 \}}$

proper : preimage of cpt set is cpt.

Rmk. By def. $(f_!\mathcal{F})(\mathcal{U}) \subseteq (f_*\mathcal{F})(\mathcal{U})$, one has natural transformation $f_! \longrightarrow f_*$.
When f is proper, $f_! = f_*$.

E.g. For $p \in X$. $\iota_p : \{p\} \hookrightarrow X$, $\quad \iota_{p,!}\underline{\mathbb{Q}}_{\{p\}} = \iota_{p,*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$
For $\quad \pi : Y \longrightarrow \{*\}$, $\quad \pi_*\mathcal{F} = \Gamma_c(Y;\mathcal{F}) = H^0_c(Y;\mathcal{F})$

cpt supp fcts on Y



Ex. Do you know what is $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}})$ and $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1})$?

- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}, \quad \Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}.$
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}, \quad \Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0.$
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0, \quad \Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}.$
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0, \quad \Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0.$
- ○ Could you explain the notation again?

E.g. 4.3. For $\mathcal{F} / \mathcal{U} \overset{j}{\hookrightarrow} X$ open. $j_!\mathcal{F}$ is the classical "extension by zero":

$$(j_!\mathcal{F})^{pre}(V) = \begin{cases} \mathcal{F}(\mathcal{U}) & V \subseteq \mathcal{U} \\ 0 & \text{otherwise} \end{cases}$$

e.p.
$$(j_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & p \in \mathcal{U} \\ 0 & p \notin \mathcal{U} \end{cases}$$

In general, [IHPS, p82]

$$(f_!\mathcal{F})_p = \Gamma_c(f^{-1}(p) ; \mathcal{F}|_{f^{-1}(p)})$$

[This comes from the proper base change formula:

$$\iota_p^* f_! \mathcal{F} \cong \pi_! \tilde{\iota}_p^* \mathcal{F}$$

$$\begin{tikzcd} f^{-1}(p) \arrow[r, hook, "\tilde{\iota}_p"] \arrow[d, "\pi"'] & Y \arrow[d, "f"] \\ \{p\} \arrow[r, "\iota_p"] & X \end{tikzcd} \quad Y / \mathcal{F}$$
]

Rmk. In E.g. 4.3, $j_!$ is exact. (Check the stalks!)
In general, $f_!$ is only left adjoint.

e.p. when $f: Y \to X$ is proper, then $f_! = f_*$ is usually not right adjoint.
Notice that $Rf_! \dashv f^!$, and we don't have $f_! \dashv f^!$.

https://math.stackexchange.com/questions/3132036/direct-image-functor-f-left-exact
the same method here argues why f_! is left exact.

Sidemark:
https://math.stackexchange.com/questions/4671873/compare-two-definition-of-rf-derived-pushforward-with-proper-support
it gives another definition of f_! in étale case.

https://en.wikipedia.org/wiki/Borel%E2%80%93Moore_homology
https://mathoverflow.net/questions/249342/two-points-of-view-about-borel-moore-homology