

Eine Woche, ein Beispiel

12.1 weights of type E

There are already much information in wiki and other references about the exceptional Lie algebra. It is nice, but I always have to check the compatability among different references. In this document, I try to fix a standard coordinate, and state all the combinatorial results without proofs.

We will make a list of the following objects, for E_6 , E_7 and E_8 .

Ref:

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2023. <https://doi.org/10.1017/9781009280020>.

- Weights nearest to the origin
 - some graphs
 - weight lattice
- Simple roots
- Fundamental weights
- Weyl group action

Remark: There is another coordinate system which is written in wiki: del Pezzo surface. We don't use them. There, the different weight spaces are identified, while in our coordinate system, we identify the root lattices.

1. E_6 .

- Weights nearest to the origin

There are two minuscule representations of E_6 . So we just fix one.

affine version

#	typical coordinates	symbol
6	$(1, 0, 0, 0, 0, 0, 1, 0)^T$	ν_i
6	$(1, 0, 0, 0, 0, 0, 0, 1)^T$	$\tilde{\nu}_i$
15	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	ν_{ij}

$$\langle \nu_i, \nu_j \rangle \in \{0, 1, 2\}$$

\uparrow could be $\tilde{\nu}_i$ or ν_{ij} \uparrow edge

weight lattice version

#	typical coordinates	symbol
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, 3, -3)^T$	ν_i
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, -3, 3)^T$	$\tilde{\nu}_i$
15	$\frac{1}{3}(-2, -2, 1, 1, 1, 1, 0, 0)^T$	ν_{ij}

$$\langle \nu_i, \nu_j \rangle \in \{\frac{4}{3}, \frac{1}{3}, -\frac{2}{3}\}$$

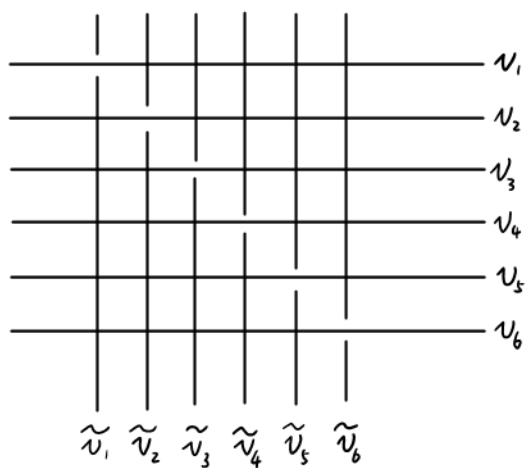
\uparrow could be $\tilde{\nu}_i$ or ν_{ij} \uparrow edge

in $\left\{ \sum_{i=1}^6 z_i = z_7 + z_8 = 0 \right\} \cong \mathbb{R}^6$

The graph constructed is called the Schläfli graph, which has 27 vertices and 216 edges (with HoG Id 1300). This graph is also the configuration graph of 27 lines.

vertices \rightsquigarrow lines
edges \rightsquigarrow intersection points
triangle \rightsquigarrow triangle cut by H \leftarrow only in E_6

Here are some typical subgraphs:

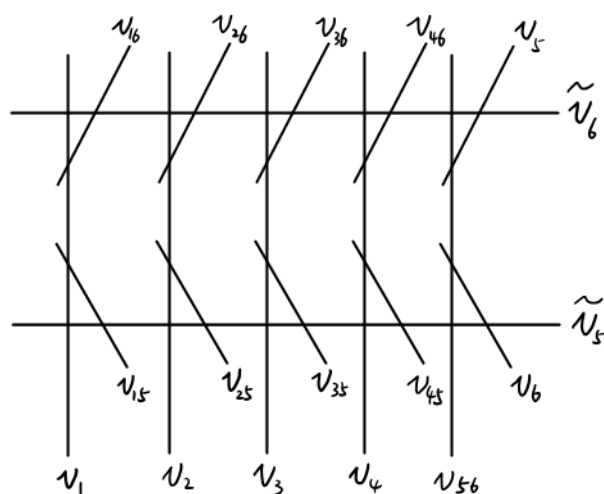


Schläfli double six configuration

$V = 12$ # $E = 30$

HoG Id = 32794

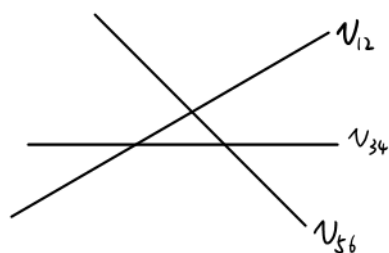
36 many, from [Huy24, Ex 3.6]



(v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6, \dots$)

$V = 17$ # $E = 30 + 20 = 50$

HoG Id = none.

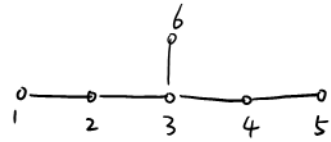


triangle
720 many

Q: For each type of subgraph, how many are they in the Schläfli graph?

I don't know if there are any simple answer for general subgraphs, and I don't know if there are any efficient algorithm for doing this. But this already produces many mysterious combinatorial numbers.

- Simple roots



$$\begin{aligned}
 & \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \} \\
 &= \{ v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5, v_5 - v_6, v_4 - v_{\pm 6} \} \\
 &= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}
 \end{aligned}$$

Ex. Verify that all the 72 roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0, 0)^T$	α_{1-2}
$40 = \binom{6}{3} \cdot 2$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$	$\alpha_{4 \pm 6, 7}$
2	$(0, 0, 0, 0, 0, 0, 1, -1)^T$	α_7

- Fundamental weights

denote by $A = (a_{ij})$ the Cartan matrix, then

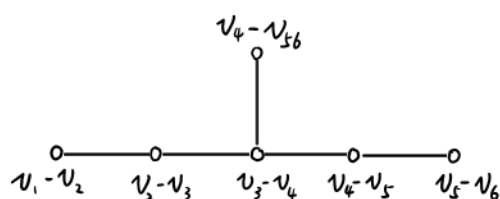
$$\begin{aligned} (\alpha_1, \dots, \alpha_r) &= (\omega_1, \dots, \omega_r) A & \langle \alpha_i, \alpha_j \rangle &= A \\ (\omega_1, \dots, \omega_r) &= (\alpha_1, \dots, \alpha_r) A^{-1} & \langle \omega_i, \omega_j \rangle &= A^{-1} \end{aligned}$$

As a result,

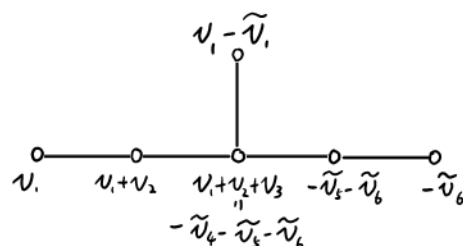
$$\begin{aligned} &\{ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6 \} \\ &= \{ \nu_1, \nu_1 + \nu_2, \nu_1 + \nu_2 + \nu_3, -\tilde{\nu}_5 - \tilde{\nu}_6, -\tilde{\nu}_6, \nu_1 - \tilde{\nu}_1 \} \end{aligned} \quad -\tilde{\nu}_6 \neq \sum_{i=1}^5 \nu_i$$

$$= \left\{ \frac{1}{6} \begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \\ -3 \\ 9 \\ -9 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{6} \begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



α_i



ω_i

- Weyl group action

We know that

$$s_k \alpha_i = \alpha_i - \langle \alpha_k, \alpha_i \rangle \alpha_k \\ = \alpha_i - a_{ki} \alpha_k$$

$$\leadsto s_k(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r) \begin{pmatrix} 1 & & & \\ & \ddots & & \\ -a_{k1} & \dots & 1-a_{kk} & \dots & -a_{kr} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\uparrow (\delta_{ij} - s_{ik} a_{ij})_{ij}$$

In practice, we want to compute s_k -action on coordinates, it's easier to use the formula

$$s_k e_i = e_i - \langle \alpha_k, e_i \rangle \alpha_k$$

E.g. In E_6 -case, when $k=1$, $\alpha = (1, -1, 0, \dots, 0)^T = e_1 - e_2$,

$$s_1 e_1 = e_1 - (e_1 - e_2) = e_2$$

$$s_1 e_2 = e_2 - (-1)(e_1 - e_2) = e_1$$

$$\leadsto s_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & 1 & \ddots & \\ & & & 1 \end{pmatrix}$$

Similarly, $s_k = s_{(k, k+1)}$ for $i = 1, \dots, 5$.

When $k=6$, $\alpha_k = \frac{1}{2}(-1, -1, -1, 1, 1, 1, 1, -1)^T$,

$$s_6 e_1 = e_1 - (-\frac{1}{2}) \alpha_6 = e_1 + \frac{1}{2} \alpha_6 \\ = \frac{1}{4}(3, -1, -1, 1, 1, 1, 1, -1)^T$$

$$s_6 e_4 = e_4 - \frac{1}{2} \alpha_6 \\ = \frac{1}{4}(1, 1, 1, -3, -1, -1, -1, 1)^T$$

$$\leadsto s_6 = \frac{1}{4} \begin{pmatrix} 3 & -1 & & & & & \\ -1 & 3 & & & & & \\ & & 3 & & & & \\ & & & 3 & & & \\ & & & & 3 & & \\ & & & & & 3 & \\ & & & & & & 3 \\ -1 & -1 & -1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

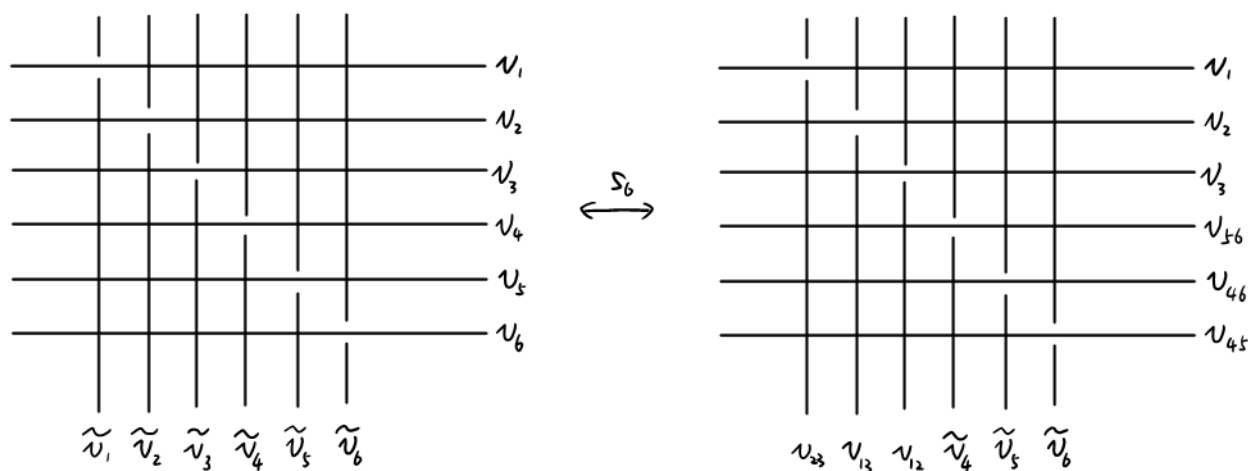
The action of s_1, \dots, s_5 on the Schläfli graph is easy. s_6 is hard.

E.g.

$$\begin{aligned}
 s_6 v_1 &= s_6 (e_1 + e_7) = e_1 + e_7 = v_1 & s_6 v_2 &= v_2 & s_6 v_3 &= v_3 \\
 s_6 v_4 &= s_6 (e_4 + e_7) = \frac{1}{2}(1, 1, 1, 1, -1, -1, 1, 1)^T = v_{56} & s_6 v_5 &= v_{46} & s_6 v_6 &= v_{45} \\
 s_6 \tilde{v}_1 &= s_6 (e_1 + e_8) = \frac{1}{2}(1, -1, -1, 1, 1, 1, 1, 1)^T = v_{23} & s_6 \tilde{v}_2 &= v_{13} & s_6 \tilde{v}_3 &= v_{12} \\
 s_6 \tilde{v}_4 &= s_6 (e_4 + e_8) = e_4 + e_8 = \tilde{v}_4 & s_6 \tilde{v}_5 &= \tilde{v}_5 & s_6 \tilde{v}_6 &= \tilde{v}_6
 \end{aligned}$$

The rest are easy to determine through the Schläfli double six configuration.

e.g. $s_6 v_{14} = v_{14}$



2. E_7 .

- Weights nearest to the origin

There is just one minuscule representations of E_7 .

integer version

#	typical coordinates	symbol
28	$(3, 3, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$(-3, -3, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij} = -v_{ij}$

$$\langle v_i, v_j \rangle \in \{24, 8, -8, -24\}$$

↑
edge

weight lattice version

#	typical coordinates	symbol
28	$\frac{1}{4}(3, 3, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$\frac{1}{4}(-3, -3, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij} = -v_{ij}$

$$\langle v_i, v_j \rangle \in \left\{ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\}$$

↑
edge

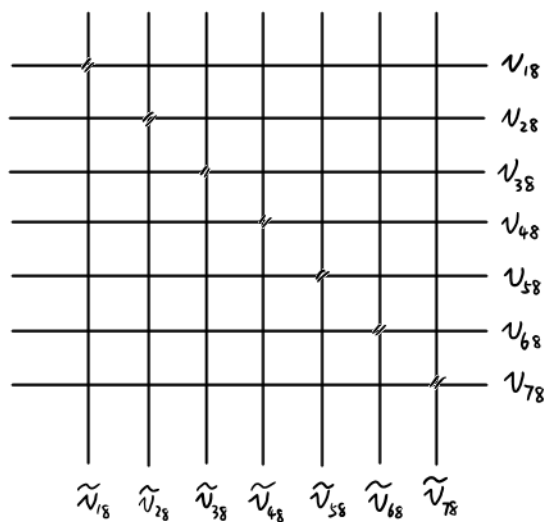
in $\left\{ \sum_{i=1}^8 z_i = 0 \right\} \cong \mathbb{R}^7$

The graph constructed is called the Gosset graph, which has 56 vertices and 756 edges (with HoG Id 1114).
This graph is also the configuration graph of 56 (-1) -curves on P^2 blowing up 7 points.

$$56 = 7 + \binom{7}{2} + \binom{7}{5} + 7$$

vertices \rightsquigarrow lines
edges \rightsquigarrow intersection points
triangle \rightsquigarrow triangle

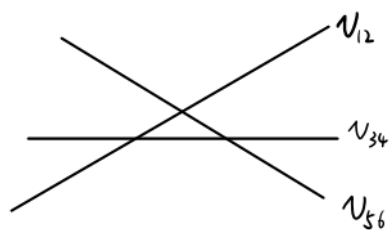
Here are some typical subgraphs:



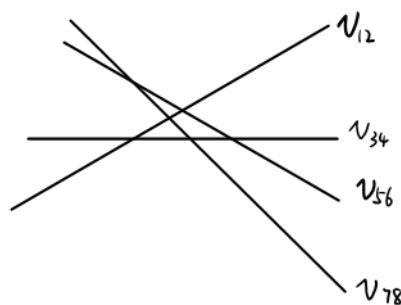
$$\{v_{ij}\}_{i,j}$$

"double seven configuration"
V = 14 # E = 42
HoG Id = 50584

v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6, \dots$
V = 28 # E = 210
HoG Id = 50698.



triangle
4032 many



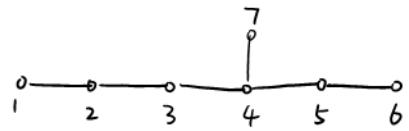
1008 many



in (-1) -curves setting,

$\langle v_i, v_j \rangle \in \left\{ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\}$
intersection number: -1 0 1 2

- Simple roots



$$\begin{aligned}
 & \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \} \\
 &= \{ \nu_{18} - \nu_{28}, \nu_{28} - \nu_{38}, \nu_{38} - \nu_{48}, \nu_{48} - \nu_{58}, \nu_{58} - \nu_{68}, \nu_{68} - \nu_{78}, -\nu_{12} - \nu_{34} \} \\
 &= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}
 \end{aligned}$$

Ex. Verify that all the 126 roots are given by

#	typical coordinates	symbol
$56 = 8 \cdot 7$	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}
$70 = \binom{8}{4}$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	$\alpha_{5,6,7,8}$

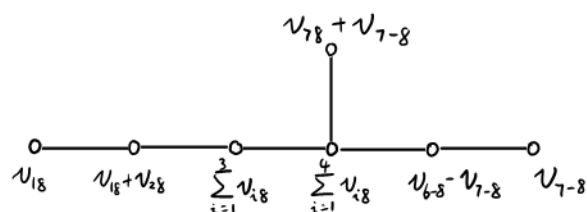
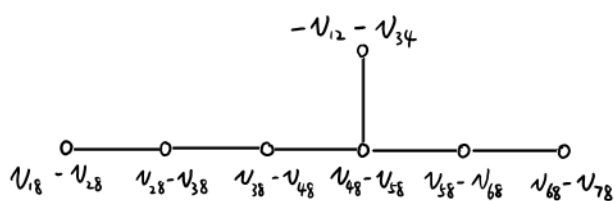
- Fundamental weights

For convenient, denote

$$v_{j-k} = v_{ij} - v_{ik} = e_j - e_k \quad \text{for some } i=j, k.$$

$$\begin{aligned} & \{ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7 \} \\ &= \left\{ v_{18}, v_{18} + v_{28}, v_{18} + v_{28} + v_{38}, \sum_{i=0}^4 v_{i8}, v_{6-8} + v_{7-8}, v_{7-8}, v_{78} + v_{7-8} \right\} \end{aligned}$$

$$\begin{aligned} &= \left\{ \frac{1}{4} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 2 \\ 2 \\ -2 \\ -2 \\ -2 \\ -2 \\ -2 \\ 6 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \\ -3 \\ -3 \\ -3 \\ 9 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -4 \\ -4 \\ -4 \\ 12 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4 \\ -4 \\ 8 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -4 \\ 4 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 7 \end{pmatrix} \right\} \\ &= \left\{ \frac{1}{4} \begin{pmatrix} 3 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \\ -3 \\ -3 \\ -3 \\ 9 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{4} \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 7 \end{pmatrix} \right\} \end{aligned}$$



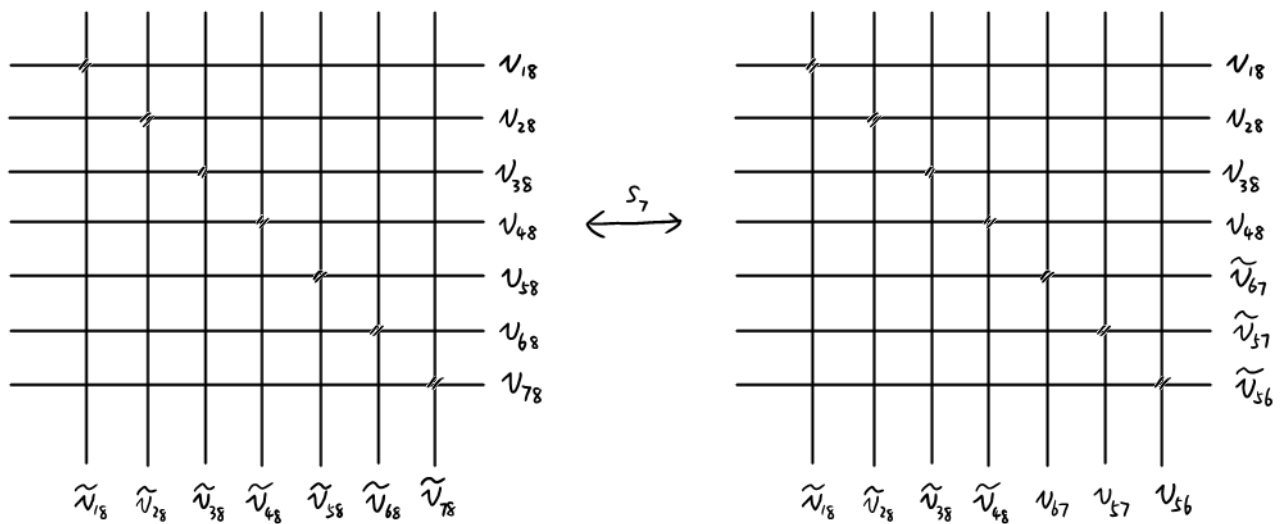
- Weyl group action

Using the similar methods like E_6, we get

$$S_k = S_{(k, k+1)} \quad \text{for } i = 1, \dots, 6$$

$$S_7 = \frac{1}{4} \left(\begin{array}{ccc|ccc} 3 & 3 & -1 & & & \\ & 3 & 3 & & & \\ -1 & & 3 & & & \\ \hline & & & 3 & 3 & -1 \\ 1 & & & -1 & 3 & 3 \end{array} \right)$$

$$S_7 \nu_{ij} = \begin{cases} \nu_{ij} & \text{if } i \in \{1, 2, 3, 4\}, j = \{5, 6, 7, 8\} \\ \tilde{\nu}_{kl} & \text{if } \{i, j, k, l\} = \{1, 2, 3, 4\} \text{ or } \{5, 6, 7, 8\} \end{cases}$$



2. E_8 .

- Weights nearest to the origin

There is no minuscule representations of E_8 , the 240 weights are roots.

weight lattice version (allow negative roots)

#	typical coordinates	symbol	inverse
$56 = 2 \cdot \binom{8}{2}$	$(1, 1, 0, 0, 0, 0, 0, 0)^T$	α_{12}	$\tilde{\alpha}_{12}$
$56 = 8 \cdot 7$	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}	α_{2-1}
2	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	ν_ϕ	$\tilde{\nu}_\phi$
$56 = 2 \cdot \binom{8}{2}$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	ν_{12}	$\tilde{\nu}_{12}$
$70 = \binom{8}{4}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	ν_{1234}	ν_{5678}

$$\langle \nu_i, \nu_j \rangle \in \{2, 1, 0, -1, 2\} \quad \text{in } \mathbb{R}^8$$

↑
edge

shorter:

#	typical coordinates	symbol
$112 = 4 \cdot 28$	$(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)^T$	$\alpha_{\pm i \pm j}$
$128 = 2^7$	$(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})^T$ even sign	ν_I

We call the constructed graph as the E_8 -Gosset graph. It has 240 vertices and $126 \cdot 240 / 2 = 15120$ edges, with no HoG Id.

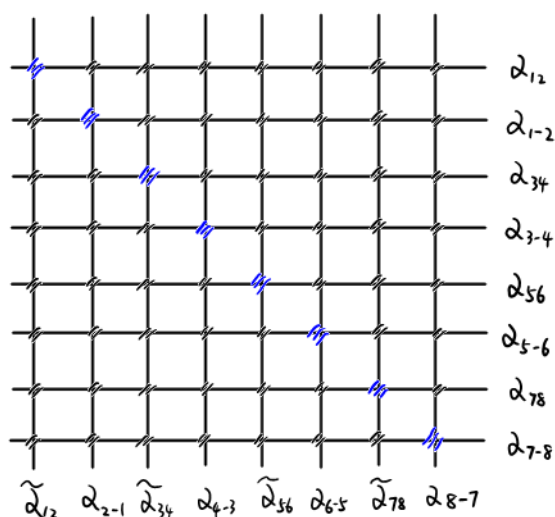
Q: $\text{Aut}(\Gamma) = W(E_8)$?

in (-1) -curves setting,

intersection number: $\langle v_i, v_j \rangle \in \{2, 1, 0, -1, -2\}$
 $\begin{matrix} & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{matrix}$

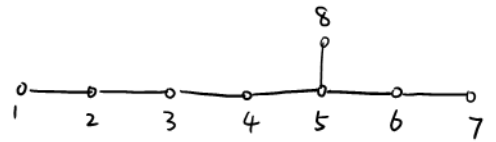
If we allow multiple edges, then I believe $\text{Aut}(\Gamma_{\text{mult}}) = W(E_8)$.

Here are some typical subgraphs:



"double eight configuration"
 $\# V = 16$ $\# E = 0$

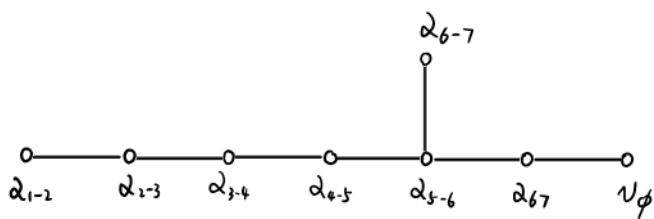
- Simple roots



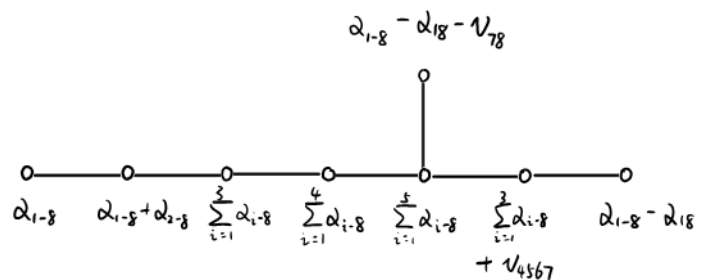
$$\begin{aligned}
 & \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8 \} \\
 &= \{ \alpha_{1-2}, \alpha_{2-3}, \alpha_{3-4}, \alpha_{4-5}, \alpha_{5-6}, \alpha_{6-7}, \nu_\phi, \alpha_{6-7} \} \\
 &= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}
 \end{aligned}$$

- Fundamental weights

$$\begin{aligned}
 & \{ \omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8 \} \\
 &= \{ \alpha_{1-8}, \alpha_{1-8}-\alpha_{2-8}, \sum_{i=1}^2 \alpha_{i-8}, \sum_{i=1}^4 \alpha_{i-8}, \sum_{i=1}^5 \alpha_{i-8}, \alpha_{1-8}+\alpha_{2-8}, \alpha_{1-8}-\alpha_{2-8}, \alpha_{1-8}-\alpha_{2-8}-\nu_{78} \} \\
 &= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ -5 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{7}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{5}{2} \end{pmatrix} \right\}
 \end{aligned}$$



α_i



ω_i

- Weyl group action

Using the similar methods like E_6, we get

$$s_k = s_{(k, k+1)} \quad \text{for } i = 1, \dots, 5$$

$$s_6 = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix} \quad s_7 = \frac{1}{4} \begin{pmatrix} 3 & & & & -1 \\ & 3 & & & \\ & & 3 & & \\ & & & 3 & \\ -1 & & & & 3 \end{pmatrix} \quad s_8 = s_{(6,7)}$$

Ex. Check the s_7 -action on roots are given by

$$\begin{aligned} s_7(\alpha_{ij}) &= \nu_{ij} \\ s_7(\alpha_{i-j}) &= \alpha_{i-j} \end{aligned}$$

$$\begin{aligned} s_7(\nu_\phi) &= -\nu_\phi \\ s_7(\nu_{ij}) &= \alpha_{ij} \\ s_7(\nu_{ijkl}) &= \nu_{ijkl} \end{aligned}$$

4. Comparison among different root systems.

Rmk. For the root lattice,

$$E_8 = \{ z_i \in \mathbb{Z}_8^8 \cup (\mathbb{Z} + \frac{1}{2})^8 \mid \sum_{i=1}^8 z_i \equiv 0 \pmod{2} \}$$

$$E_7 = E_8 \cap \left\{ \sum_{i=1}^8 z_i = 0 \right\}$$

$$E_6 = E_8 \cap \left\{ \sum_{i=1}^8 z_i = z_7 + z_8 = 0 \right\}$$

$$E_8 \quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad s_6 \quad s_7 \quad s_8$$

$$E_7 \quad S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6 \quad S_7$$

$$E_6 \quad S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad S_6$$

$$\frac{1}{4} \left(\begin{array}{c|c|c} 3 & 3 & 1 \\ -1 & 3 & -1 \\ \hline 1 & 3 & -1 \\ -1 & 1 & 3 \end{array} \right)$$