§ 1.1. Structure of finite / local / global field

Road map

	finite field	local field		global field	adéle
		Archi	NA		
base field F $F^\times$ integral ring $\mathcal{O}_F$ units $\mathcal{O}_F^\times$	7 $\mathbb{F}_1$ 1 $\mathbb{F}_p$ For $\mathbb{F}_{1^r}$ $\varepsilon\cdot\mu_r$ μ_{p-1} — — — —	2 $\mathbb{R}$ or $\mathbb{C}$ $\mathbb{R}^+\times\mathbb{Z}/2\mathbb{Z}$ $\mathbb{C}^\times$ — —	3 $\mathbb{Q}_p$ $\mathbb{F}_p((t))$ $\mathbb{Z}_p^\times\times\mathbb{Z}$ $\mathbb{F}_p[[t]]^\times\times\mathbb{Z}$ $\mathbb{Z}_p$ $\mathbb{F}_p[[t]]$ $\mathbb{Z}_p^\times$ $\mathbb{F}_p[[t]]^\times$	4 $\mathbb{Q}$ $\mathbb{F}_p(t)$ $\mathbb{Q}^\times$ $\mathbb{F}_p(t)^\times$ $\mathbb{Z}$ $\mathbb{F}_p[t]$ $\mathbb{Z}/2\mathbb{Z}$ $\mathbb{F}_p^\times$	6 $\mathbb{A}_K$ $\mathbb{I}_K$ K ? $\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$ ari Frob # ext of deg n Spec $\mathcal{O}_F$	$\hat{\mathbb{Z}}$? $\hat{\mathbb{Z}}$? can 1? 1 Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}},1)$ [étale, 2.2.4]	$\mathbb{Z}/2\mathbb{Z}$ Id total order? — 1/0 —	most known choose a lift finite	dream $\xrightarrow{\text{unramified}}$ $\begin{cases}\mathrm{Frob}_{\mathfrak{q}} \\ \mathrm{Frob}_{\mathfrak{p}} \text{ conj class}\end{cases}$ $\xrightarrow{\text{abelian}}$ $\mathrm{Frob}_{\mathfrak{p}}$ $n\neq 1$ inf countable	—
topology topo of $\mathcal{O}_F$ measure	? discrete — ? discrete	Euclidean — Lebesgue	profinite cpt, not discrete $\mu(\mathcal{O}_F)=1$	— — —	restricted K is a lattice in $\mathbb{A}_K$ can be computed

5 Also, discuss

- field extension, norm, trace, ...
- their connection to geometry, ramification theory
- analog with knot theory

Process

1. finite field $\mathbb{F}_q$
2. Archi local field $\mathbb{R}$ or $\mathbb{C}$
3. NA local field
    - Individual structure
    - Field extension
4. global field

    - Dessin d'enfants
5. local and global: connections
    - Basics
    - Traditional point of view
        - Frobenius
        - Application: Quadratic reciprocity
    - Étale point of view
6. local to global: adèle

    - Base field with automorphism
    - Galois extension
7. $\mathbb{F}_1$

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$Gal(\overline{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}}$ as top gps

$Frob_q \longmapsfrom 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $Gal(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $Gal(\mathbb{C}/\mathbb{C}) = Id$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?
  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions.

- Define I_F, P_F
- Construct $I_F / P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $Fr\, \tau\, Fr^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions: (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep} / \mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right)}, \quad \underset{I_{\mathbb{Q}_p}}{\mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}}_p / \widehat{\mathbb{Q}_p^{ur}}\right)}, \quad \underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} / \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)}$$

4. global field

$Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.

$Gal(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D],
we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\}/\text{birational}$$

$Gal(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler. by [GalFun, Thm 3.4.8],

$$Gal(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \widehat{F}(\mathbb{C})$$

↑Free profinite gp on $\mathbb{C}$

Shafarevich's conj: See wiki: Absolute Galois group

$Gal(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite gp

Q: Does $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object/geo realizations?

Dessin d'enfants

By [GalFun, Prop 4.7.1 - Rmk 4.7.9], we have an including

$$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow Out(\pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}))$$

induced by $\pi^{ét}_{1,\overline{\mathbb{Q}}} := \pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, $\pi^{ét}_{1,\mathbb{Q}} := \pi_1^{ét}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$

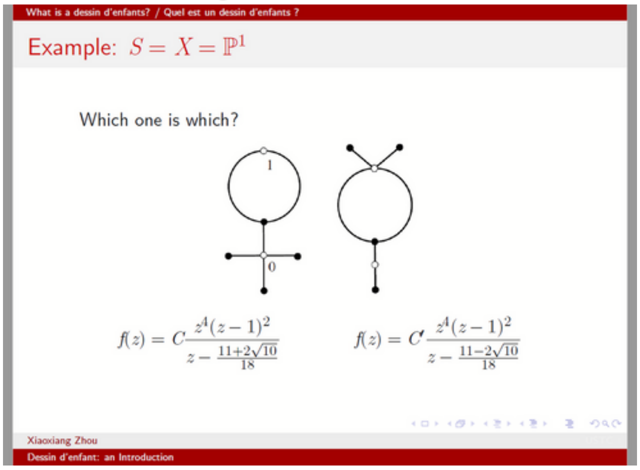
$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[d, equal, "?"'] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] \arrow[d, equal] & \pi^{ét}_{1,\mathbb{Q}} \arrow[r] \arrow[d, "{\substack{\text{conj} \\ g \mapsto g\cdot - \cdot g^{-1}}}"] & Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & Z(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] & Aut(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & Out(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & 1
\end{tikzcd}$$

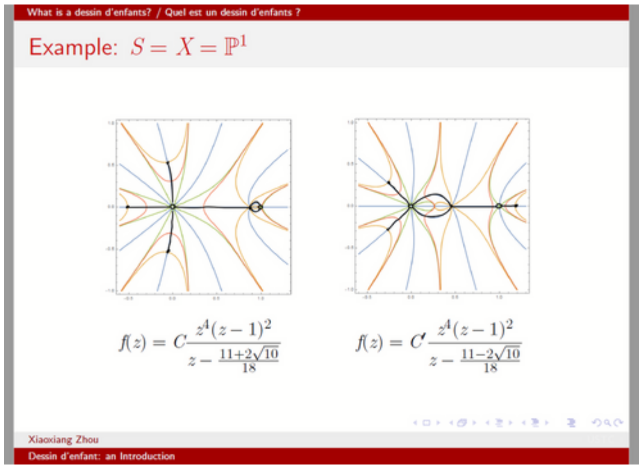
The space $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ is designed for guaranteeing that
$Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Out(\pi^{ét}_{1,\overline{\mathbb{Q}}})$
is inclusion.

Task. Read [Dessin d'enfant] or [Collins],

understand the $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the dessin d'enfants.

- Def of Dessin d'enfact
- Connections with $Out(\pi_{1,\bar{\mathbb{Q}}}^{\acute{e}t})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.





- Can we generalize this to $Gal(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$?

  I don't know how to make a "dessin d'enfant" on alg curves over $\mathbb{F}_p$.

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.

5. local and global connections

Basics

$$\begin{tikzcd} K \arrow[r, hook] & K_v \\ \mathcal{O}_K \arrow[u, phantom, "\cup"] \arrow[r, hook] \arrow[rrd] & \mathcal{O}_v \arrow[u, phantom, "\cup"] \arrow[rd] & \\ & & k_v \end{tikzcd}$$

for any valuation v $\quad (\rightsquigarrow K \hookrightarrow \mathbb{A}_K)$

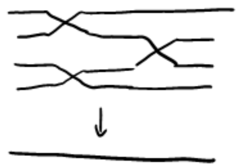
and $\mathcal{O}_v$ is the integral closure of $\mathcal{O}_K$ in K_v (for v fin)

and $\mathcal{O}_K/\mathfrak{p}^r\mathcal{O}_K \cong (\mathcal{O}_K/\mathfrak{p}^r\mathcal{O}_K)_v \cong \mathcal{O}_v/\mathfrak{p}^r\mathcal{O}_v \quad \forall r \in \mathbb{N}_{\geq 0}$

The connections are also compatible with field exts. $\quad$ (L/K Galois)

E.g.

$$\begin{array}{ccc} L & \mathcal{O}_L & \mathfrak{q} \sim w \\ | & \uparrow & | \\ K & \mathcal{O}_K & \mathfrak{p} \sim v \end{array}$$



$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_g^{e_g} \qquad\qquad L \otimes_K K_v \cong \prod_w L_w$$

$\alpha \in L, \quad \iota_w : L \hookrightarrow L_w$

$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "N_{L/K}"'] & \prod_w L_w \arrow[d, "\prod_w N_{L_w/K_v}^{e_w}"] \\ K \arrow[r, hook] & K_v \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, hook] \arrow[d, "Tr_{L/K}"'] & \prod_w L_w \arrow[d, "\sum_w e_w Tr_{L_w/K_v}"] \\ K \arrow[r, hook] & K_v \end{tikzcd}$$

$$N_{L/K}(\alpha) = \prod_{w|v} N_{L_w/K_v}(\iota_w(\alpha))^{e_w} \qquad Tr_{L/K}(\alpha) = \prod_{w|v} e_w Tr_{L_w/K_v}(\iota_w(\alpha))$$

Traditional point of view

$k \longrightarrow pt$

$\mathcal{O}_v \longrightarrow$ "loc" curve

$\mathcal{O}_K \longrightarrow$ curve

k

$\mathcal{O}_v$

$\mathcal{O}_K$

Task. Read [Algfungp, 0.2], answer the following questions:

- Understand ramified, inert, split.
- Understand (from geo meaning)
$$\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \bigoplus_{i=1}^{g} \mathcal{O}_L/\mathfrak{q}^{e_i}\mathcal{O}_L$$
- Compute some classical examples
- Compare it with ramified covering in RS.

Frobenius (L/K Galois)

$$\begin{tikzcd}
 & 1 \arrow[r] & \mathrm{Gal}(L/K)_w \arrow[r, "\leq"] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Gal}(L/K) & \\
 & & \mathrm{Gal}(L_w/K_v) \arrow[d, phantom, "\cong" rotate=-90] & & \\
1 \arrow[r] & I(L_w/K_v) \arrow[r] & \mathrm{Aut}_{\mathcal{O}_v\text{-alg}}(\mathcal{O}_w) \arrow[r, two heads] & \mathrm{Gal}(k_w/k_v) \arrow[r] & 1
\end{tikzcd}$$

where

$$\mathrm{Gal}(L/K)_w = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma(q) = q\} \leq \mathrm{Gal}(L/K)$$

By https://math.stackexchange.com/questions/4131855/frobenius-elements,

more conditions		better props of Frob
① L/K is unramified at w (True for all except finite w)	$\Rightarrow$	$I(L_w/K_v) = \mathrm{Id}$; $\mathrm{Fr}_w \in \mathrm{Gal}(L/K)_w$ well-defined
② L/K Galois	$\Rightarrow$	all symbols are meaningful; $\forall q, q' \in \mathrm{Spec}\, \mathcal{O}_L$, $\exists \sigma \in \mathrm{Gal}(L/K)$, $\sigma(q) = q'$; $\mathrm{Frob}_v = [\mathrm{Frob}_w]$ is a conj class in $\mathrm{Gal}(L/K)$
③ $L \subseteq K^{ab}$	$\Rightarrow$	$\mathrm{Frob}_v \in \mathrm{Gal}(L/K)$

Application: Quadratic reciprocity

Thm. p, l odd primes, $p \neq l$, then

$$\left(\frac{p}{l}\right)\left(\frac{l}{p}\right) = (-1)^{\frac{p-1}{2}\frac{l-1}{2}}$$

$$\left(\frac{-1}{l}\right) = (-1)^{\frac{l-1}{2}} = \begin{cases} 1 & l \equiv 1 \mod 4 \\ -1 & l \equiv 3 \mod 4 \end{cases}$$

$$\left(\frac{2}{l}\right) = (-1)^{\frac{l^2-1}{8}} = \begin{cases} 1 & l \equiv 1,7 \mod 8 \\ -1 & l \equiv 3,5 \mod 8 \end{cases}$$

Proof Assume $p \equiv 1 \bmod 4$, then $\mathbb{Q}_l(\sqrt{p}) \hookrightarrow \mathbb{Q}_l(\zeta_p)$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_l(\zeta_p)/\mathbb{Q}_l) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_l(\sqrt{p})/\mathbb{Q}_l) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\ \mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{p}{l}\right) \left(\frac{l}{p}\right) \end{tikzcd}$$

$$\begin{aligned} \chi(\mathrm{Frob}_l) = 1 &\iff \mathrm{Frob}_l(\sqrt{p}) = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l} \\ &\iff (\sqrt{p})^l = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l} \\ &\iff \sqrt{p} \in \mathbb{F}_l \\ &\iff \left(\frac{p}{l}\right) = 1 \end{aligned}$$

Assume $p \equiv 3 \bmod 4$, then $\mathbb{Q}_l(\sqrt{-p}) \hookrightarrow \mathbb{Q}_l(\zeta_p)$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_l(\zeta_p)/\mathbb{Q}_l) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_l(\sqrt{-p})/\mathbb{Q}_l) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\ \mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-p}{l}\right) \left(\frac{l}{p}\right) \end{tikzcd}$$

$$\begin{aligned} \chi(\mathrm{Frob}_l) = 1 &\iff \mathrm{Frob}_l(\sqrt{-p}) = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l} \\ &\iff (\sqrt{-p})^l = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l} \\ &\iff \sqrt{-p} \in \mathbb{F}_l \\ &\iff \left(\frac{-p}{l}\right) = 1 \end{aligned}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_l(\zeta_4)/\mathbb{Q}_l) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/4\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_l(\sqrt{-1})/\mathbb{Q}_l) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\ \mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-1}{l}\right) (-1)^{\frac{l-1}{2}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_l(\zeta_8)/\mathbb{Q}_l) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_l(\sqrt{2})/\mathbb{Q}_l) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\ \mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{2}{l}\right) (-1)^{\frac{l^2-1}{8}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_l(\zeta_8)/\mathbb{Q}_l) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_l(\sqrt{-2})/\mathbb{Q}_l) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\ \mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-2}{l}\right) (-1)^{\frac{(l-1)(l+3)}{8}} \end{tikzcd}$$

$\mathbb{Q}$: Quadratic reciprocity for $\mathbb{F}_p(t)$?

Cubic reciprocity ?
[Cox x^2+ny^2, §4.A, 4.B]

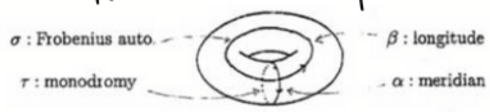
$\left\{\begin{array}{l}\text{primes over } \mathbb{Q}(\zeta_3) \text{ , } \left(\frac{\alpha}{\beta}\right)_3 \quad (\alpha),(\beta) \in \operatorname{Spec}\mathbb{Z}[\zeta_3] \\ \deg 3, \text{ like } \operatorname{Gal}(K/\mathbb{Q}(\sqrt{-23})) \text{ : More difficult}\end{array}\right.$

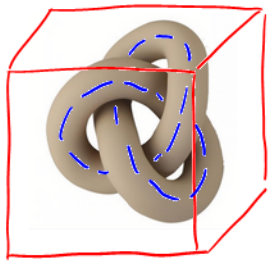
Étale point of view

$k \longrightarrow S^1$

$\mathcal{O}_v \longrightarrow$ tubular nbhd of S^1

$\mathcal{O}_K \longrightarrow$ 3-dim spaces ($\mathbb{R}^3$ when $K = \mathbb{Q}$)





k

$\mathcal{O}_v$

$\mathcal{O}_K$

See [Knotprime Table 1] for more informations.
Also, see this:
http://www.neverendingbooks.org/mazurs-dictionary
算术拓扑的初步理论: https://zhuanlan.zhihu.com/p/563347112

6. local to global: adèle

Recall: Ostrowski's thm & Product formula.

Task. Read [Adèle] and answer the following questions:

- Give a def of $\mathbb{A}_K$ & $\mathbb{I}_K$ (set, topo and measure)
- Verify that
$$K \subseteq \mathbb{A}_K \qquad \mathcal{O}_T \subseteq {\prod_{v\in T}}' K_v$$
$$K^\times \subseteq \mathbb{I}_K^\times \qquad \mathcal{O}_T^\times \subseteq \left({\prod_{v\in T}}' K_v\right)'$$
are lattices. Give fundamental domain in easy cases.
- Deduce the finiteness of class number and Dirichlet unit theorem.

Base field with automorphism

We know that

	finite field	local field: Archi	local field: NA	global field	Adèle
base field F	$\mathbb{F}_p$	$\mathbb{R}$	$\mathbb{Q}_p$ $\quad \mathbb{F}_p((t))$	$\mathbb{Q}$ $\quad \mathbb{F}_p(t)$	$\mathbb{A}_K$
	$\mathrm{Aut}_{ring}(\mathbb{F}_p)=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{R})=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{Q}_p)=1$ $\mathrm{Aut}_{top\ ring}(\mathbb{F}_p((t)))\neq 1$	$\mathrm{Aut}_{ring}(\mathbb{Q})=1$ $\mathrm{Aut}_{ring}(\mathbb{F}_p(t))\neq 1$	 $\mathrm{Aut}_{ring}(\mathbb{A}_{\mathbb{F}_p(t)})\neq 1$

Q: Do we have $\mathrm{Aut}_{ring}(\mathbb{A}_{\mathbb{Q}})=1$?

A: Yes. See [LCFT, Ex 6.3.6]. I don't understand this proof.

Galois extension

Setting: L/K fin ext of global field

Recall that we have an iso
$$L\otimes_K \mathbb{A}_K \xrightarrow{\cong} \mathbb{A}_L \text{ of topo rings with compatible embedding of } L$$
$\rightsquigarrow$ $\mathbb{A}_K \subseteq \mathbb{A}_L$ subring, $\mathbb{A}_L \cong \mathbb{A}_K^{\oplus[L:K]}$ as $\mathbb{A}_K$-module.

Lemma [LCFT, Ex 6.3.2]
$$\text{integral closure of } K \text{ in } \mathbb{A}_L = L$$

Proof Reduce to
$$\text{integral closure of } L \text{ in } \mathbb{A}_L \subseteq L$$
If $\exists\, x\in \mathbb{A}_L - L$ which is integral over L, then $L(x)/L$ is a fin field ext in $\mathbb{A}_L$, and
$$\#\{\mathfrak{q}\in \operatorname{Spec}\mathcal{O}_L \mid \mathfrak{q} \text{ do not split completely}\}$$
$$\overset{?}{\leq} \#\{\mathfrak{q}\in \operatorname{Spec}\mathcal{O}_L \mid x_{\mathfrak{q}} \notin \mathcal{O}_{\mathfrak{q}}\} < +\infty.$$
But fin nontrivial field ext have inf many non split primes. ↯

7. $\mathbb{F}_1$

This is better explained in §1.2. Anyhow, it is still a "field".

⚠ It is always better to think $\# \mathbb{F}_1 = 1 + \varepsilon$, where $\varepsilon \ll 1$.

In that way you can "see object at different level", like

$\# \mathbb{F}_1^\times = \varepsilon$ $\qquad$ $\mathbb{F}_1^\times$ is not empty!

Slogan: "Infinitesimal is only visible when constant level is zero"

This phenomenon already happens when we learn integrals:

$$\int x^n = \begin{cases} \frac{1}{n+1} x^{n+1} + C & n \neq -1 \\ \log x + C & n = -1 \end{cases}$$

Here, $\log x$ is that "infinitesimal".

See
wiki: https://en.wikipedia.org/wiki/Field_with_one_element
nlab: https://ncatlab.org/nlab/show/field+with+one+element

It is desirable to define

- The "field extension" $\mathbb{F}_{1^n}/\mathbb{F}_1$ of deg n $\quad$ + $\mathbb{F}_{1^n}^\times \simeq \varepsilon \mu_n$
- The "Galois group" $\mathrm{Gal}(\overline{\mathbb{F}}_1/\mathbb{F}_1) \simeq \widehat{\mathbb{Z}}$
- The "v.s. over $\mathbb{F}_1$"

Q: Do we have $\mathbb{Q}_1$?

A: See https://mathoverflow.net/questions/309664/what-is-mathbbq-1-the-field-of-1-adic-numbers