(colorful chalks)

Homogeneous Space, Double Coset Decomposition & Bruhat-Tits Building

<u>speak</u>: preliminary : linear algebra & abstract algebra

<u>ask</u>: How many people know what a homogeneous space is?

<u>answer</u>: Space + action + transitively

$$G \circlearrowright X$$

<u>ask</u>: How to visualize a gp action?

Eg. $S^1 \circlearrowright S^2$ $\qquad$ $\mathbb{C}^\times \circlearrowright \mathbb{CP}^1$ $\quad t.[x:y] = [tx:y]$

<u>ask</u>: How many orbits?

Hint: $\mathbb{CP}^1 \cong S^2$

$$\mathbb{CP}^1 = \mathbb{C} \sqcup \{\infty\} = \mathbb{C}^\times \sqcup \{0\} \sqcup \{\infty\}$$

$$G.x \cong G/\mathrm{Stab}_G(x)$$

<u>conclude</u>: To understand a gp action, basically you need to know

- How many orbits
- The shape of the orbit

repeat: homogeneous space (single orbit)

ask: find a gp action on $\mathbb{CP}^1$ which ...

$\left\{ \begin{array}{l} \text{is transitive} \\ \text{make } \mathbb{CP}^1 \text{ a homo space} \\ \text{has only one orbit} \end{array} \right.$

E.g. $\mathbb{CP}^1 = GL_2(\mathbb{C}) \Big/ \underbrace{\begin{pmatrix} * & * \\ & * \end{pmatrix}}_{=B} = \{ V_1 \subseteq \mathbb{C}^2 \mid \dim V_1 = 1 \}$

~~tell~~

ask: what is the stabilizer?

answer: the upper triangular matrix gp

In rep theory, we call the ... as the Borel subgp.
subgp

denoted by B.

Why care about B: equiv to understand $\mathbb{CP}^1$

$$\mathbb{CP}^2 = GL_3(\mathbb{C}) \Big/ \underbrace{\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}}_{=P} = \{ V_1 \subseteq \mathbb{C}^3 \mid \dim V_1 = 1 \}$$

ask: what's the stabilizer?

answer: ~~the~~ a block upper triangular matrix gp.

a block upper triangular matrix gp is called the Parabolic subgp.

denoted by P.

Similarly,

$$Gr(4,2) = GL_4(\mathbb{C}) \Big/ \left(\begin{array}{c|c} * & * \\ \hline & * \end{array} \right) = \{ V_2 \subseteq \mathbb{C}^4 \mid \dim V_2 = 2 \}$$

speak: these homogeneous spaces are actually moduli spaces, they classify some linear objects.

ask: what does $GL_3(\mathbb{C}) / \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ classify?

$$\underset{\substack{\uparrow \\ \text{flag variety}}}{Flag_3(\mathbb{C})} = GL_3(\mathbb{C}) / \underbrace{\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}}_{=: B} = \underset{\substack{\uparrow \\ \text{(complete) flags}}}{\{0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^3\}}$$

speak: the flag variety generalizes the proj spaces & Grassmannians.

ask: How to understand the structure of $Flag_3(\mathbb{C})$?

introduce the affine paving: written as disjoint union of affine spaces. (usage: compute cohomology, pts counting)

$$\mathbb{C}\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$$
$$\mathbb{C}\mathbb{P}^2 = \mathbb{C}^2 \sqcup \mathbb{C} \sqcup \{0\}$$

ask: Find an affine paving of $Flag_3(\mathbb{C})$?

answer: consider B-orbits on G/B

$$B \circlearrowright G/B \qquad G = GL_3(\mathbb{C})$$

ask How many orbits?

$$\#\{B\text{-orbits}\} = \# B\backslash G/B$$

$$B\backslash G/B = G/\sim \qquad g_1 \sim g_2 \text{ iff } g_1 = b_1 g_2 b_2$$

(double coset: generalization of single coset)

ask How many double cosets?

Thm (Bruhat dec)

$$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} BwB$$

e.g.
$$GL_2(\mathbb{C}) = B\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}B \sqcup B \cdot \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \cdot B$$
$$= B \sqcup B\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}B$$

$w \in S_n$: a permutation matrix, e.g. $\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = (12)(34)$

Cor. $\# \, {}_B\backslash GL_n(\mathbb{C})/_B = n!$

The proof uses Gauss elimination.

Idea: find a canoical form in ${}_B\backslash G/_B$

$\Leftrightarrow$ left / right multiplied by $b \in B$

$\Leftrightarrow$ restricted row / column operators

e.g. $q = 7$ $G = GL_3(\mathbb{F}_7)$

$$\begin{pmatrix} 5 & 1 & 6 \\ 6 & 2 & 4 \\ 0 & 4 & 3 \end{pmatrix} \underset{}{\overset{\begin{pmatrix} 1 & -2 & \\ & -1 & \\ & & 1 \end{pmatrix} \times}{\sim}} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 5 & 3 \\ 0 & 4 & 3 \end{pmatrix} \overset{\times \begin{pmatrix} 1 & -5 & -3 \\ & 1 & \\ & & 1 \end{pmatrix}}{\sim} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

Rmk. Nearly all matrix dec results are special forms of double coset dec.

(and are all proved by "Gauss eliminations")

E.g.

	Bruhat	$GL_n(k) = \bigsqcup_{w\in S_n} BwB$
$\mathbb{R}$	SVD	$GL_n(\mathbb{R}) = \bigsqcup_{a_1\geq\cdots\geq a_n>0} O(n)\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n\end{pmatrix} O(n)$
	QR	$GL_n(\mathbb{R}) = B\cdot O(n)$
$\mathbb{Q}_p$	Cartan	$GL_n(\mathbb{Q}_p) = \bigsqcup_{\substack{a_1\geq\cdots\geq a_n \\ a_i\in\mathbb{Z}}} K\begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n}\end{pmatrix} K$
	Iwahori	$GL_n(\mathbb{Q}_p) = \bigsqcup_{\varpi\in W_{ext}} I\varpi I$
	Iwasawa	$GL_n(\mathbb{Q}_p) = B(\mathbb{Q}_p)\cdot K$

v.b. on $\mathbb{P}^1_\mathbb{C}$ $\quad GL_n(\mathbb{C}[t^{\pm1}]) = \bigsqcup_{\substack{a_1\geq\cdots\geq a_n \\ a_i\in\mathbb{Z}}} GL_n(\mathbb{C}[t^{-1}])\begin{pmatrix} t^{a_1} & & \\ & \ddots & \\ & & t^{a_n}\end{pmatrix} GL_n(\mathbb{C}[t])$

Shimura $\quad GL_2(\mathbb{A}_\mathbb{Q}) = \bigsqcup_{x\in \Gamma_1(N)\backslash\mathcal{H}^\pm} GL_2(\mathbb{Q})\cdot x\cdot\left(\widehat{\Gamma_1(N)}\cdot\mathbb{R}^\times\cdot SO_2\right)$

Here, $K = GL_n(\mathbb{Z}_p)$

$$\begin{array}{ccc} K & \longrightarrow & GL_n(\mathbb{F}_p) \\ \cup & & \cup \\ I & \dashrightarrow & B \end{array}$$

$$W_{ext} = N_G(T)/T(\mathbb{Z}_p) = \begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n}\end{pmatrix} \rtimes S_n$$

Application of double coset dec.

$$G = \bigsqcup_{\alpha \in I} H \alpha K$$

- understand G through H & K
- understand H orbits in G/K, i.e.

$$G/K = \bigsqcup_{\alpha \in I} H\alpha K/K \cong \bigsqcup_{\alpha \in I} H/H_{[\alpha K]} = \bigsqcup_{\alpha \in I} H/H \cap \alpha K \alpha^{-1}$$

- Compute Hecke algebra $\mathcal{H}(H \backslash G / K)$
- compute $N_G(H)$

E.g. $$GL_n(\mathbb{C}) = \bigsqcup_{w \in S_n} BwB$$

$$\Rightarrow GL_3(\mathbb{C})/B = \bigsqcup_{w \in S_3} \underbrace{B/B \cap wBw^{-1}}_{C(w)} = \bigsqcup_{w \in S_3} \mathbb{C}^{l(w)}$$

Here, $l(w)$ = length of w. e.g.  has length 2

$$\begin{tikzcd}
l(w) & & & \\
3 & & (13) \arrow[dl, dash] \arrow[dr, dash] & \\
2 & (123) \arrow[d, dash] \arrow[drr, dash] & & (132) \arrow[d, dash] \arrow[dll, dash] \\
1 & (12) \arrow[dr, dash] & & (23) \arrow[dl, dash] \\
0 & & Id &
\end{tikzcd}$$

$C(w)$: Schubert cell

$\overline{C(w)}$: Schubert variety

Bruhat-Tits Building

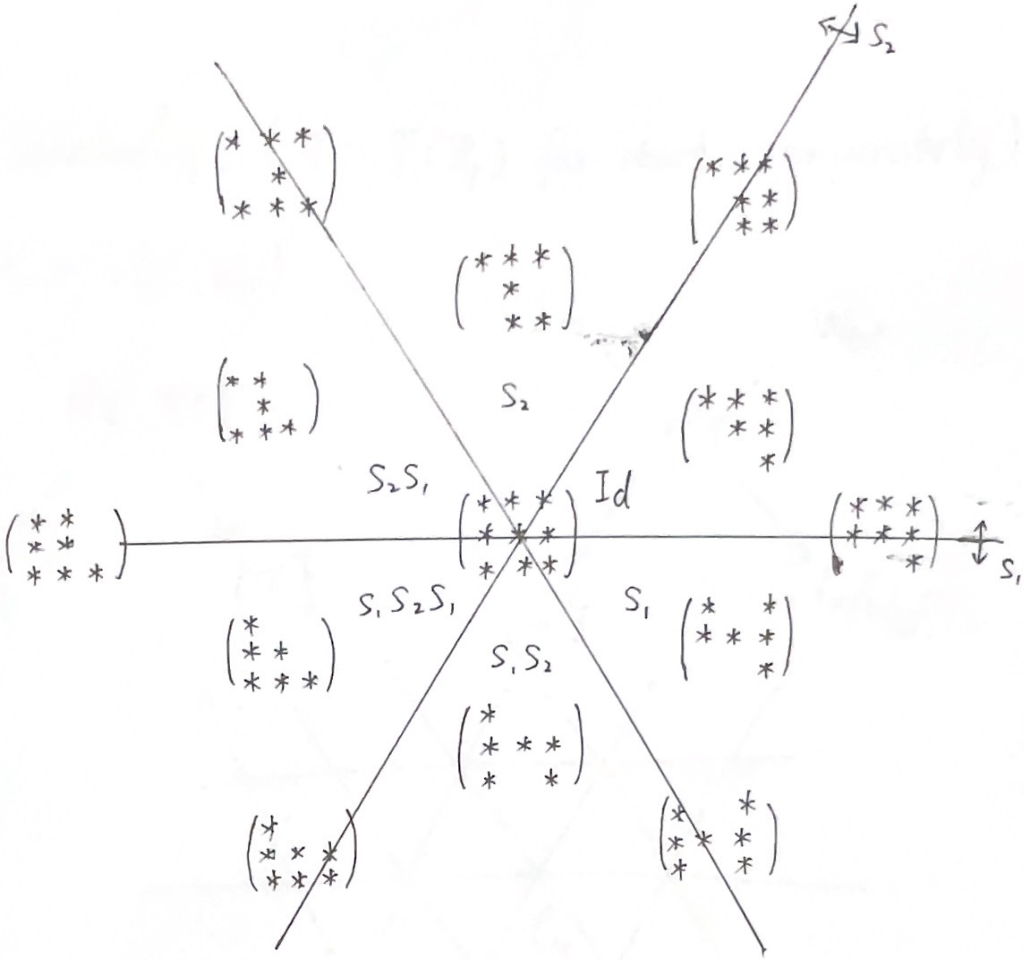
<u>speak</u>: cheat a bit. standard Borel.

$$\{\text{Borel subgp}\} = \{gBg^{-1}\}$$
$$\{\text{Parobolic subgp}\} = \{gPg^{-1}\}$$
$$\{\text{torus}\} = \{gTg^{-1}\} \qquad T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$$

Ex.
$$\{B \subseteq G\} = G/\cancel{B}N_G(B) = G/B$$
$$\{T \subseteq G\} = G/N_G(T) \qquad N_G(T) = \{\text{monoidal matrix}\}$$
$$\{(B,T) \mid B \supset T\} = G/T$$

$$\begin{tikzcd} & \{(B,T)\} \arrow[dl] \arrow[dr] & \\ \{B\} & & \{T\} \end{tikzcd}
\qquad
\begin{tikzcd} B/T \arrow[dr, dash] & & W \arrow[dl, dash] \\ & G/T \arrow[dl] \arrow[dr] & \\ G/B & & G/N_G(T) \end{tikzcd}$$

(B,N)-pair.



$\mathcal{C}_B$ chambre

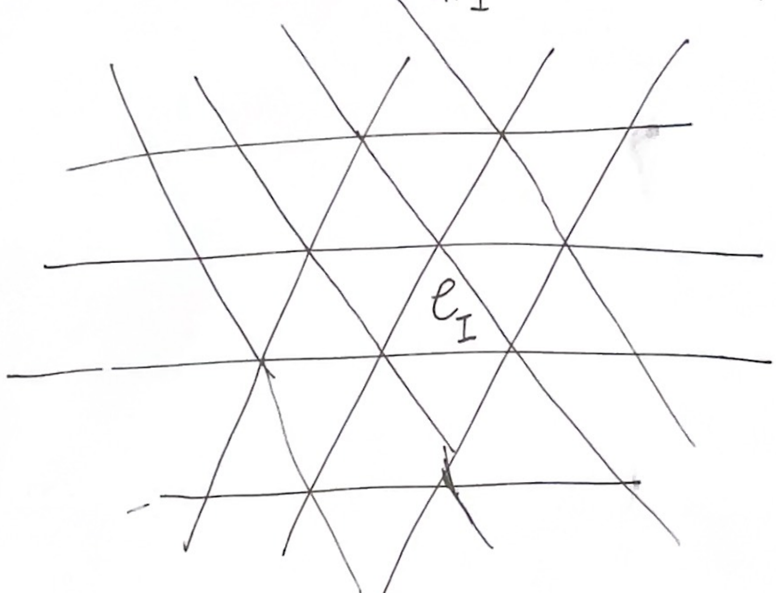
$\mathcal{A}_T = \bigcup_{B \supset T} \mathcal{C}_B$ apartment

$\mathcal{B} = \bigcup_{B} \mathcal{C}_B$ building

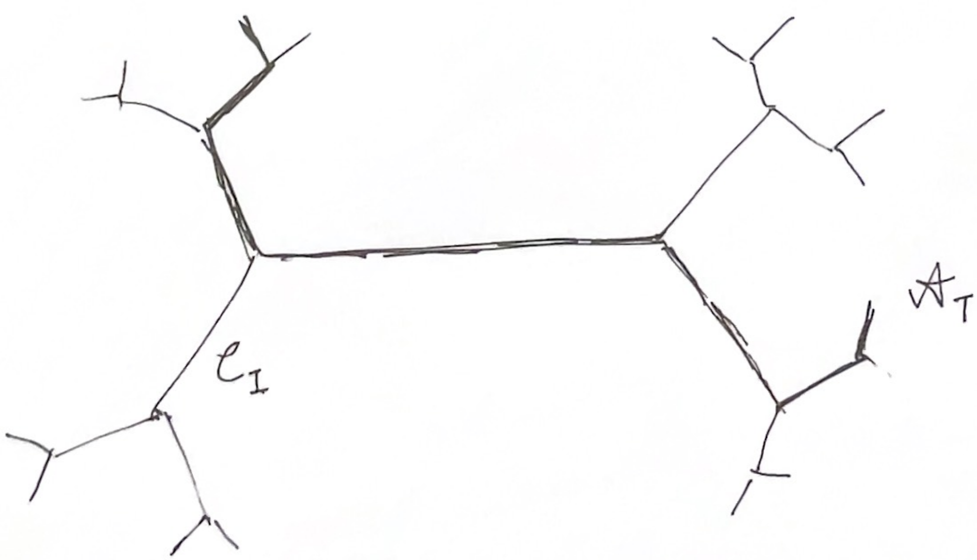
Similarly, ($T = T(\mathbb{Z}_p)$ for short, temporately)

For $G = SL_n(\mathbb{Q}_p)$,

$$
\begin{tikzcd}
 & \{(I,T)\} \arrow[dl] \arrow[dr] & \\
\{I\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
I/T \arrow[dr, dash] & & W_{ext} \arrow[dl, dash] \\
 & G/T \arrow[dl] \arrow[dr] & \\
G/I & & G/N_G(T)
\end{tikzcd}
$$



$SL_3(\mathbb{Q}_p)$-case, $\mathcal{A}_T$



$SL_2(\mathbb{Q}_2)$-case, $\mathcal{B}$