Eine Woche, ein Beispiel

9.10. ramified covering : alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - field of rational fcts
   - valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

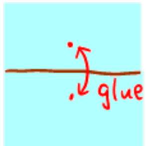
1. alg curve viewed as stack quotient

	base change		
	$\mathrm{Spec}\,\mathbb{R}$	$\mathrm{Spec}\,\mathbb{C}/_{\mathbb{C}}$	$\mathrm{Spec}\,\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	ϕ
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{Id, \tau\}$
$\Gamma_{\mathbb{R}} = Gal(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	$Id \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. $\mathrm{Spec}\,\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \mathrm{Spec}\,\mathbb{R}[x]$. As a set,

$$\begin{aligned}\mathrm{Spec}\,\mathbb{R}[x] &= \{(x-a) \mid a \in \mathbb{R}\} \cup \{(x^2+bx+c) \mid {}^{b,c\in\mathbb{R}}_{b^2-4c<0}\} \cup \{(0)\} \\ &= \quad \mathbb{R} \quad \cup \quad \mathcal{H} \quad \cup \{(0)\} \\ \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) &= \mathrm{Mor}_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R} \\ \mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) &= \mathrm{Mor}_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})\end{aligned}$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau \circ x$. Observe that

$$\mathrm{MaxSpec}\,\mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/\Gamma_{\mathbb{R}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$\mathrm{MaxSpec}\,\mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/\Gamma_{\mathbb{F}_p} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/\Gamma_\kappa \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

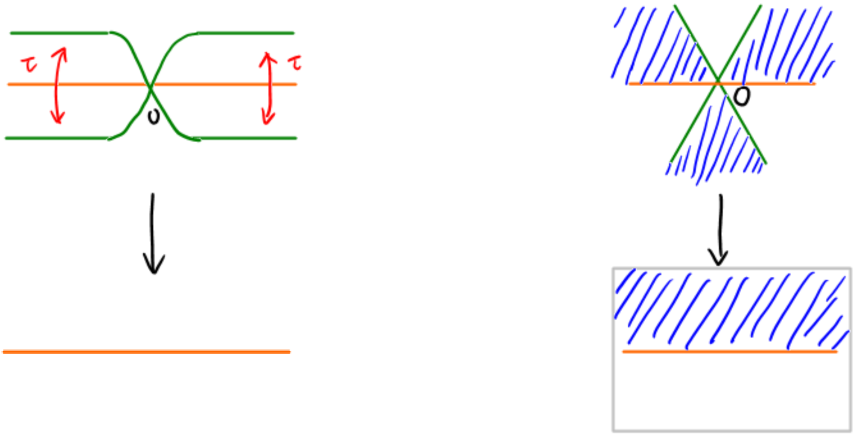
$$\mathrm{Stab}_x(\Gamma_\kappa) = \Gamma_{\kappa'} \quad \Leftrightarrow \quad \text{fiber at } x = \mathrm{Spec}\,\kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R} \cup \mathcal{H}$	$\mathbb{C}$	$\mathbb{C}$ $\circlearrowleft$ cplx conj
$\mathbb{R}$-pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$-pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \sqcup \mathbb{C}\tau$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

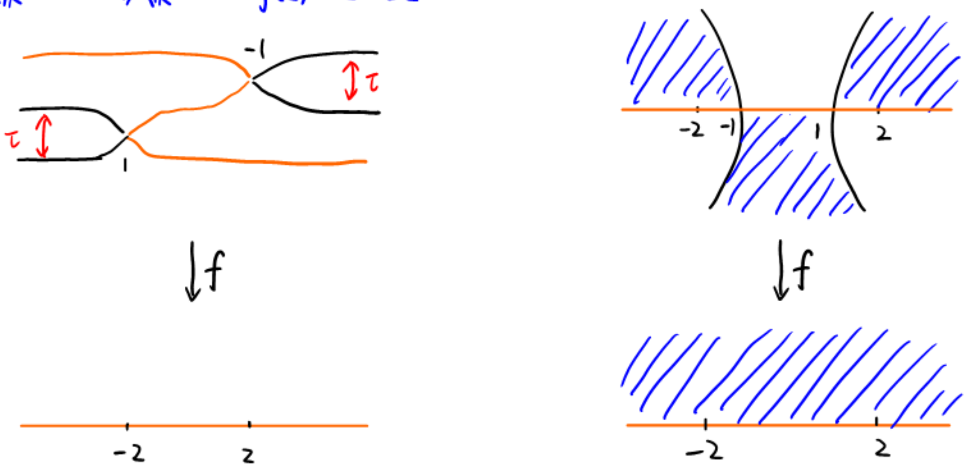
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

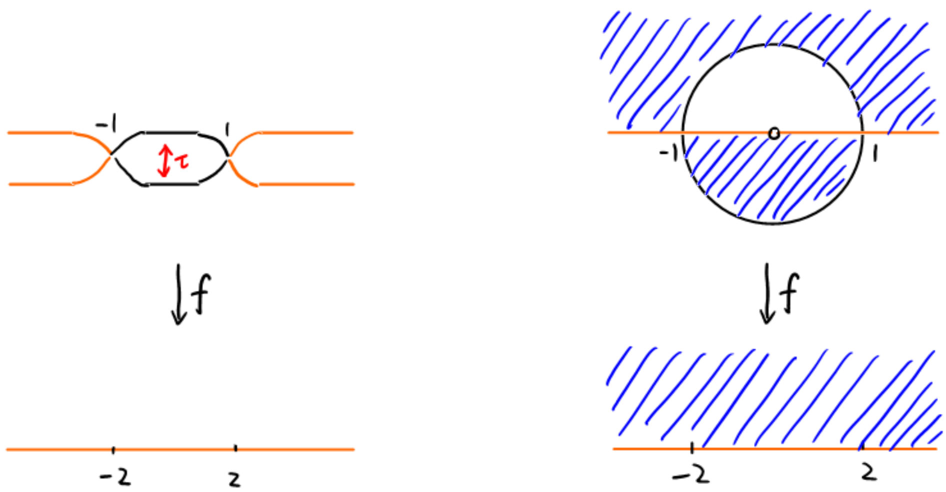
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



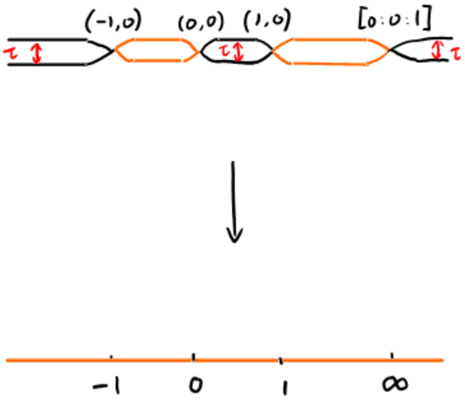
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$

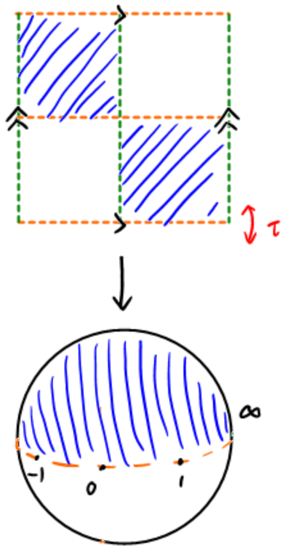


$f: \mathbb{G}_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$



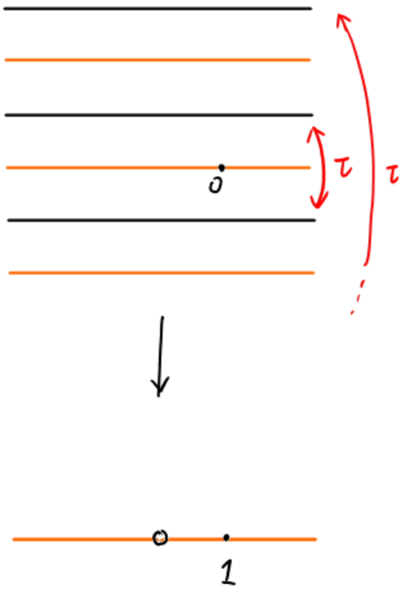
$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ $\quad [x:y:z] \longmapsto [x:z]$ $\quad E_{\mathbb{R}} = \mathrm{Proj}\ \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

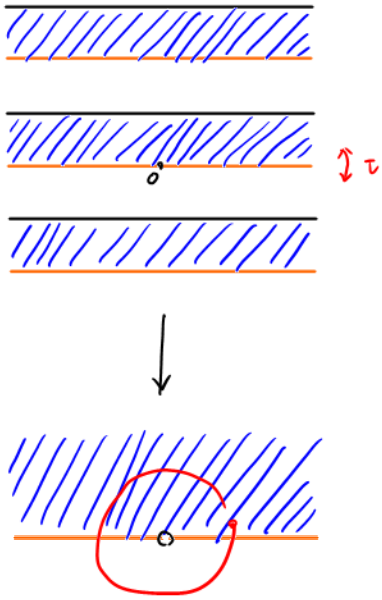




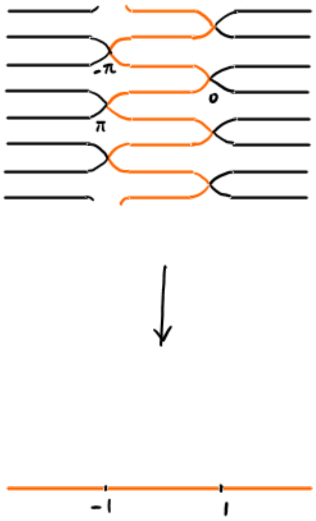
⚠ The following are not alg morphisms!

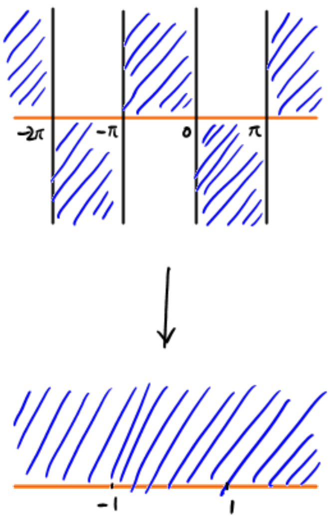
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = e^z$





$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = \cos z$

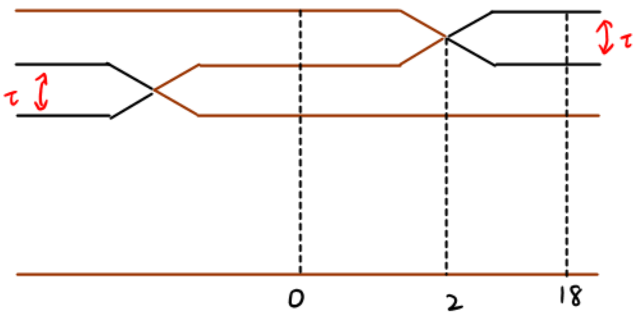




Let's focus on the case

$f : \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = z^3 - 3z$

classical picture



split: $f^{-1}(0) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z-z_0))$

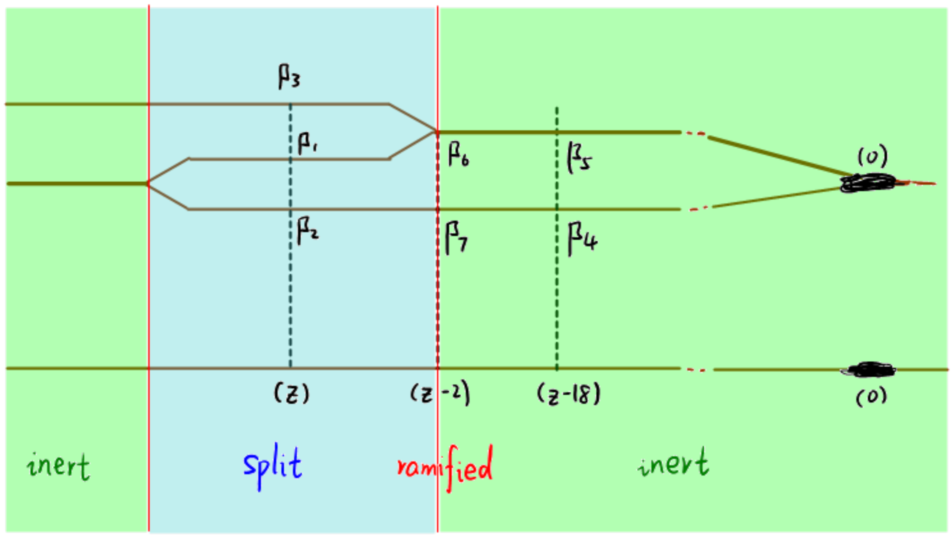
$f^{-1}((z^2+1)) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{R}$

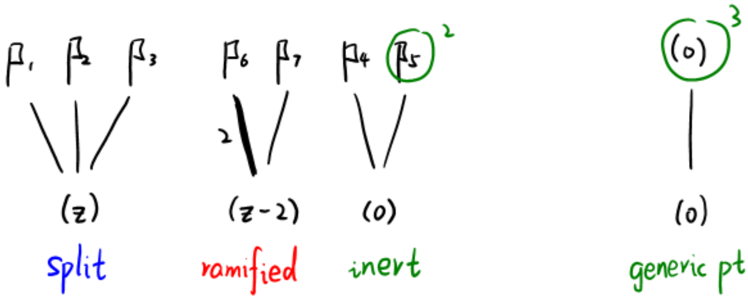
generic point: $f^{-1}((0)) = \operatorname{Spec}\mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$

algebraic picture



$$\begin{array}{ccccc} \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[w] & & w^3-3w \\ f\big\downarrow & & \big\uparrow f^* & & \big\uparrow \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] & & z \end{array}$$



split: $\mathfrak{p} = (z)$, $\quad f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w) = (w)(w-\sqrt{3})(w+\sqrt{3}) \triangleq \mathfrak{P}_1 \quad \mathfrak{P}_2 \quad \mathfrak{P}_3 \qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$

$\mathfrak{p} = (z^2+1)$, $\quad f^*(\mathfrak{p})\mathbb{R}[w] = ((w^3-3w)^2+1) = (f_1')\,(f_2')\,(f_3') \triangleq \mathfrak{P}_1' \quad \mathfrak{P}_2' \quad \mathfrak{P}_3' \qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3\}$

(partially) inert: $\mathfrak{p} = (z-18)$, $\quad f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-18) = (w-3)(w^2+3w+6) \triangleq \mathfrak{P}_4 \quad \mathfrak{P}_5 \qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_4, \mathfrak{P}_5\}$

where $\kappa(\mathfrak{P}_5) = \mathbb{R}[w]/(w^2+3w+6) \cong \mathbb{C}$, $\quad [\kappa(\mathfrak{P}_5):\mathbb{R}] = 2$

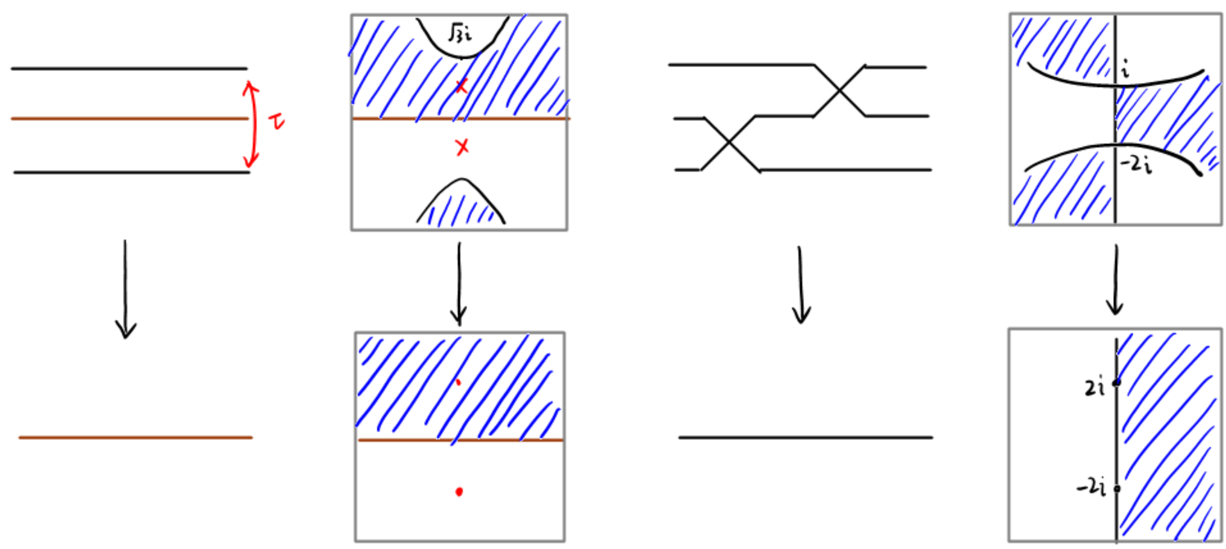
generic point: $\mathfrak{p} = (0)$, $\quad f^*(\mathfrak{p})\mathbb{R}[w] = (0) \qquad f^{-1}(\mathfrak{p}) = \{(0)\}$

where $\kappa((0)) = \mathrm{Frac}(\mathbb{R}[w]/(0)) \cong \mathbb{R}(w)$, $\quad [\mathbb{R}(w):\mathbb{R}(z)] = 3$

ramified: $\mathfrak{p} = (z-2)$, $\quad f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-2) = (w+1)^2(w-2) \triangleq \mathfrak{P}_6^2 \quad \mathfrak{P}_7 \qquad f^{-1}(\mathfrak{p}) = \{\mathfrak{P}_4, \mathfrak{P}_5\}$

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = z^3+3z$.

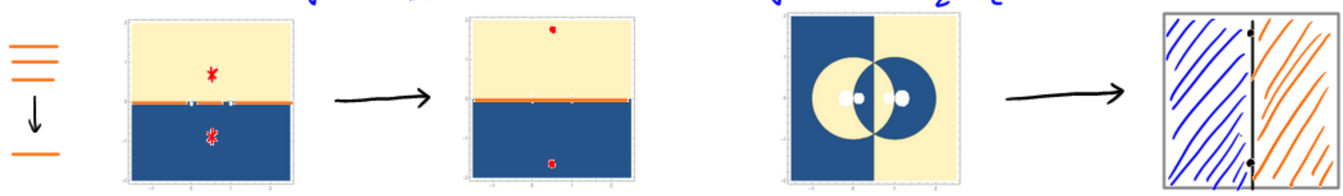


$\mathbb{R}$ picture $\qquad\qquad$ $i\mathbb{R}$ picture

⚠ The ramification pt is outside $\mathbb{R}$. This is not a Galois covering.

Ex. Try to work out the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = \dfrac{z^3-3z+1}{z^2-z} - 1.5$



$\mathbb{R}$ picture $\qquad\qquad$ $i\mathbb{R}$ picture

This is a Galois covering, with no inert places (except for the generic pt)

3. Frobenius for alg curve/$\mathbb{R}$

$$\mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \kappa(\mathfrak{q})=\mathbb{C},\ \kappa(\mathfrak{p})=\mathbb{R} \\ \{\mathrm{Id}\} & \text{otherwise.} \end{cases}$$

When E/F is Galois, $\mathrm{Spec}\,\mathcal{O}_E/\mathrm{Spec}\,\mathcal{O}_F$ unramified at $\mathfrak{p}$,

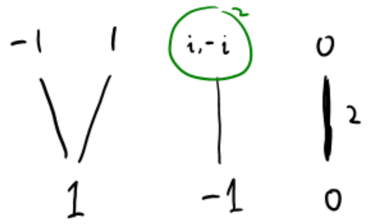
$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \mathrm{Frob}_{\mathfrak{q}} & & \longmapsto & & \mathrm{Frob}_{\mathfrak{q}} \end{array}$$

is a subgp of $\mathrm{Gal}(E/F) \cong \mathrm{Aut}(\mathrm{Spv}(E)/\mathrm{Spv}(F))$ Now, just view $\mathrm{Spv}(E) \in \mathrm{AlgCurve}_k$.

Let's try to compute some $\mathrm{Frob}_{\mathfrak{q}}$.

E.g.

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{R}} & z & \mathbb{R}[\omega] = \mathbb{R}[z^{\frac{1}{2}}] \\ \downarrow & \downarrow & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z^2 & \mathbb{R}[z] \end{array}$$



For $\mathfrak{p}=(z-1)$, $\mathfrak{q}=(\omega-1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ 1 & & 1 & & \{1,\tau\} \end{array}$$

For $\mathfrak{p}=(z+1)$, $\mathfrak{q}=(\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

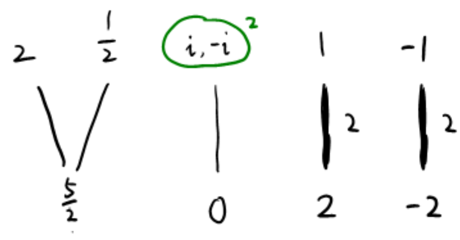
Therefore, $\mathrm{Frob}_{(z+1)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1$ $\quad \omega \longmapsto -\omega$

Not the conjugation but $\tau(\mathbb{C})|_{i\mathbb{R}}$ coincides with the cplx conj

E.g.

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{R}} & z & \mathbb{R}[\omega^{\pm 1}] = \mathbb{R}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{\pm 1}\right] \\ \downarrow & \downarrow & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z+\frac{1}{z} & \mathbb{R}[z] \end{array}$$



For $\mathfrak{p}=(z)$, $\mathfrak{q}=(\omega^2+1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1$ $\quad \omega \longmapsto \frac{1}{\omega}$

Not the conjugation, but $\tau(\mathbb{C})|_{S^1}$ coincides with the cplx conj

⚠ $\mathbb{R}(z^{\frac{1}{3}})/\mathbb{R}(z)$ is not Galois at all, so

For $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad z \longmapsto z^3$, $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (w^2+w+1)$,

$$\underbrace{\mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p}))}_{\{1,\tau\}} \not\simeq \text{“}\underbrace{\mathrm{Gal}(E/F)_{\mathfrak{q}}}_{1}\text{”} \leq \text{“}\underbrace{\mathrm{Gal}(E/F)}_{1}\text{”} \neq \mathbb{Z}/3\mathbb{Z}$$

We will discuss about $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$ in section 4.

Claim. For p odd prime, any deg p extension of $\mathbb{R}(x)$ is not Galois.

This claim is wrong. The field extension

$$\mathbb{R}(x)[T]/(T^3 - xT^2 + (x-3)T + 1) \;/\mathbb{R}(x)$$

is Galois with deg 3. discriminant $\Delta = (x^3-3x+9)^2$ [SerreGT, 1.1]

Wrong proof:

If not, suppose $F/\mathbb{R}(x)$ is a deg p Galois extension, we get the field extension tower in $\overline{\mathbb{R}(x)}$:

$$\begin{tikzcd} & & E \\ F \arrow[urr, dash, "2"] & & \\ & & \mathbb{C}(x) \arrow[uu, dash, "p"'] \\ \mathbb{R}(x) \arrow[uu, dash, "p"] \arrow[urr, dash, "2"'] & & \end{tikzcd}$$

where $\mathrm{Gal}(E/F) \triangleleft \mathrm{Gal}(E/\mathbb{R}(x))$ is a normal subgp of order 2.

By Kummer theory, $E \cong \mathbb{C}(x)[T]/(T^p - f)$ for some $f \in \mathbb{C}(x)$.

~~Since $E/\mathbb{R}(x)$ is Galois, $f \in \mathbb{R}(x)$ (see the example below)~~

When $f \in \mathbb{R}(x)$, one gets

$$\mathrm{Gal}(E/\mathbb{R}(x)) \longhookrightarrow S_p \circlearrowright \{T, \zeta_p T, \dots, \zeta_p^{p-1} T\}$$

Injection: if σ fix T, $\zeta_p T$, then σ fix ζ_p, then $\sigma = Id$.

Since $\# \mathrm{Gal}(E/\mathbb{R}(x)) = 2p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$ or $\mathbb{Z}/2p\mathbb{Z}$.

Since $\mathrm{Gal}(E/\mathbb{R}(x)) \leq S_p$, $\quad \mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$.

However, D_p has no order 2 normal subgp, contradiction!

E.g. $\mathbb{C}(z)[T]/(T^3-(z-i))$ over $\mathbb{R}(z)$ is not Galois, since

$$\begin{tikzcd} \mathbb{C}(T) \arrow[d, dash] \\ \mathbb{C}(z) \arrow[d, dash] \\ \mathbb{R}(z) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] 0 \arrow[d, dash, "3"'] & & z_0 \arrow[d, dash] & \rho z_0 \arrow[dl, dash] & \rho^2 z_0 \arrow[dll, dash] \\ i \arrow[dr, dash] & & -i \arrow[dl, dash] & & \\ & (z^2+1) & & & \end{tikzcd}$$

This example is not general enough. For example,

$\mathbb{C}(z)[T]/(T^3 - \frac{z-i}{z+i})$ over $\mathbb{R}(z)$ can be Galois

Q. For $F/\mathbb{R}(x)$ Galois extension, is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by its order 2 elements?

I call it as the "weaked version of Chebotarev's density theorem for $\mathbb{P}^1_{\mathbb{R}}$".

A. No.

We could not expect the density theorem to be true in the real case,

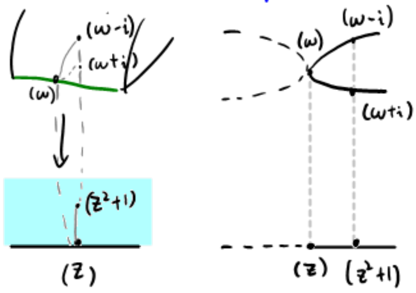
since in S_3 case the order 3 conj class can never be reached by a single Frob.

For a possible direct and brutal method to this question, use the result in this link:
math.stackexchange.com/questions/318690/absolute-galois-group-of-mathbbrt
How is Z/3Z realized as the quotient group of this group? (better: compatible with the field extension mentioned above)

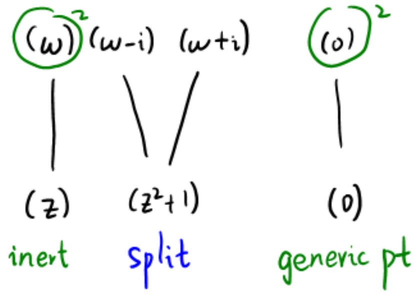
4. complexify is a ramified covering by non geometrical connected spaces

E.x. $f: \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ is an unramified covering of alg curves$/\mathbb{R}$.

$$\begin{array}{ccc} \mathbb{A}^1_{\mathbb{C}} & & \mathbb{C}[w] \\ \downarrow f & & \uparrow f^* \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] \end{array}$$



along $i\mathbb{R}$



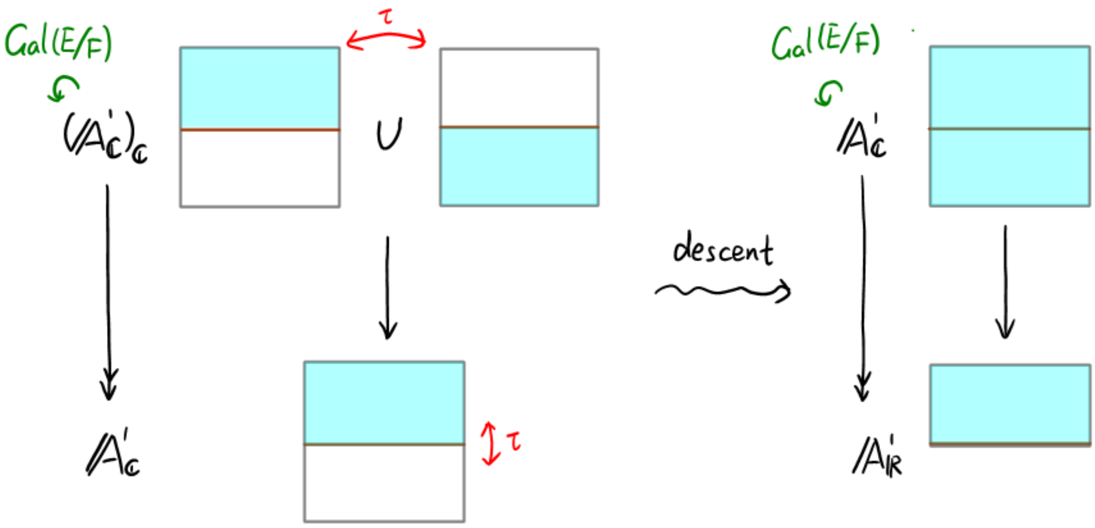
This is an unramified covering.

As an $\mathbb{R}$-scheme, $\mathbb{A}^1_{\mathbb{C}}$ is not geo connected.

$$\begin{array}{ccc} \overset{\mathrm{Gal}(E/F)\ \circlearrowleft\ \ \circlearrowleft\Gamma_{\mathbb{R}}}{\mathbb{C}[w]\otimes_{\mathbb{R}}\mathbb{C}} & \cong & \overset{\mathrm{Gal}(E/F)\ \Gamma_{\mathbb{R}}}{\mathbb{C}[w]\oplus\mathbb{C}[w]} \\ \uparrow & & \uparrow (\mathrm{Id},\sigma) \\ \mathbb{R}[z]\otimes_{\mathbb{R}}\mathbb{C}\ \circlearrowleft\Gamma_{\mathbb{R}} & \cong & \Gamma_{\mathbb{R}}\circlearrowleft\ \mathbb{C}[z] \end{array}$$

$f(w)\otimes_{\mathbb{R}} a \longmapsto (a f(w), \bar{a} f(w))$

$f(z)\otimes_{\mathbb{R}} a \longmapsto a f(z)$



For $\mathfrak{p}=(z)$, $\mathfrak{q}=(w)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z)} = \tau: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}): \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1$$
$$w_1 \longmapsto \overline{w_1}$$
$$w_2 \longmapsto \overline{w_2}$$

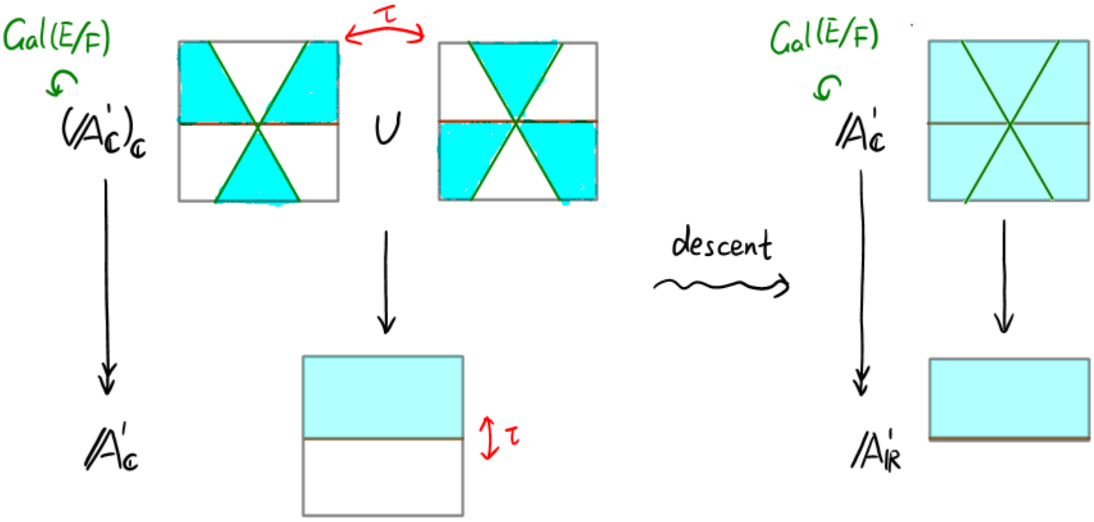
Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R}\cup\mathbb{R}}$ coincides with the cplx conj (switch)

Ex. Try to work out $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$, and compute Frobenius elements.

Recall: $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) \cong S_3 \circlearrowright \{\sqrt[3]{2}, \rho\sqrt[3]{2}, \rho^2\sqrt[3]{2}\}$.

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)) \cong S_3 \circlearrowright \{\underset{1}{z^{\frac{1}{3}}}, \underset{2}{\rho z^{\frac{1}{3}}}, \underset{3}{\rho^2 z^{\frac{1}{3}}}\}$

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z))$	Id	(23)	(12)	(13)	(123)	(132)
$z^{\frac{1}{3}} \longmapsto$	$z^{\frac{1}{3}}$	$z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$
$\rho \longmapsto$	ρ	ρ^2	ρ^2	ρ^2	ρ	ρ
$\mathbb{CP}^1 \ni a \longmapsto$	a	$\bar{a}$	$\rho^2\bar{a}$	$\rho\bar{a}$	$\rho^2 a$	ρa
geometry	Id					



For $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (w-1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1, \tau\} & & \{1, (23)\} & & S_3 \end{array}$$

Therefore, $\mathrm{Frob}_{(w-1)} = \tau_{(23)} : \mathbb{P}'_{\mathbb{C}} \longrightarrow \mathbb{P}'_{\mathbb{C}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}' \sqcup \mathbb{CP}' \longrightarrow \mathbb{CP}' \sqcup \mathbb{CP}'$

$w_1 \longmapsto \bar{w}_1$

$w_2 \longmapsto \bar{w}_2$

Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R}\cup\mathbb{R}}$ coincides with the cplx conj (switch)

Similarly, $\mathrm{Frob}_{(w-\rho)} = \tau_{(13)} : \mathbb{P}'_{\mathbb{C}} \longrightarrow \mathbb{P}'_{\mathbb{C}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}' \sqcup \mathbb{CP}' \longrightarrow \mathbb{CP}' \sqcup \mathbb{CP}'$

$w_1 \longmapsto \rho\bar{w}_1$

$w_2 \longmapsto \rho\bar{w}_2$

Not the conjugation, but $\tau(\mathbb{C})|_{\rho^2\mathbb{R}\cup\rho^2\mathbb{R}}$ coincides with the cplx conj (switch)

In this case, $\mathrm{Gal}(E/F)$ is generated by all $\mathrm{Frob}_{(z-z_0)}$.

5. alg curves and function fields

In this section, we follow the same route as in [2023.09.03].
The following theorem generalize the goal in [2023.09.03].

zero, to avoid confusion with O.

Thm 1. [Stack Project, 0BXX, Thm 53.2.6]
For k field, we have an equiv of categories

$$\mathrm{AlgCurve}_k = \left\{\begin{array}{l} \text{Obj: sm proj curves}/k \\ \text{Mor: non-const alg morphisms} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{Obj: } F/k \text{ field ext s.t.} \\ \qquad \mathrm{trdeg}_k F = 1 \\ \qquad F/k \text{ f.g as a field} \\ \text{Mor: morphism as fields}/k \end{array}\right\}^{op} = \mathrm{field}^{op}_{k(t)/k}$$

f.g. field ext F/k of transcendence deg 1.

> term in Stack Project
> curve$/k$ = 1-dim variety$/k$
> variety$/k$ = f.t.$/k$ + integral + seperated
> require irreducible
> don't require closed

field of rational functions

Def. For $X \in \mathrm{AlgCurve}_k$,

$$\mathcal{M}(X) := \{\text{rational fcts on } X\}$$

Ex. Verify that $\mathcal{M}(\mathbb{P}^1_k) \cong k(z)$

Rmk By the proof of Thm 1 (or [Vakil, Thm 11.2.1]) & Noether Normalization [Vakil, 11.2.4],

1) $\mathcal{M}(X) \in \mathrm{field}_{k(t)/k}$
2) $\exists\ k(x) \hookrightarrow \mathcal{M}(X)$ s.t. $[\mathcal{M}(X) : k(x)] < +\infty$

Def. [Vakil, Def 11.2.2]
For $f: Y \dashrightarrow X$ a dominant rational map of irr varieties$/k$ with $\dim Y = \dim X$,

$$\deg f := [\mathcal{M}(Y) : f^*\mathcal{M}(X)].$$

Similarly, people can see if a ramified covering is Galois.
Write it down rigorously, using the language of geometry.

valuations

Let's try to compute $Spv(A)$ for some $A \in k\text{-}Alg$. (require $v|_{k^\times} \equiv 0$)

Ex. In this exercise we want to describe $Spv(\mathbb{R}(z))$.

1). For $v \in Spv(\mathbb{R}(z))$, suppose $v(z-3) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$z^2+1 = (z-3)(z+3)+10$

$\Rightarrow v(z^2+1) = \min\big(v(z-3)+v(z+3),\ v(10)\big) = 0$

For $v \in Spv(\mathbb{R}(z))$, suppose $v(z^2+1) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$v(z^2+1) = 1 \Rightarrow v(z^2)=0 \Rightarrow v(z)=0$

$\Rightarrow v(z^2+1-(\pi^2+1)) = 0 \Rightarrow \left.\begin{array}{l} v(z+\pi)+v(z-\pi)=0 \\ v(2\pi)=0 \end{array}\right\} \Rightarrow v(z-\pi)=0$

$\Rightarrow v(z^2-9)=0 \Rightarrow \left.\begin{array}{l} v(z+3)+v(z-3)=0 \\ v(6)=0 \end{array}\right\} \Rightarrow v(z-3)=0$

Similarly, other irr polynomials have valuation 0.

2). For $v \in Spv(\mathbb{C}(z))$, suppose $v(z-3) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$z^2+1 = (z-3)(z+3)+10$

$\Rightarrow v(z^2+1) = \min\big(v(z-3)+v(z+3),\ v(10)\big) = -2$

For $v \in Spv(\mathbb{C}(z))$, suppose $v(z^2+1) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$v(z^2+1) = -1 \Rightarrow v(z^2-9) = -1 \Rightarrow \left.\begin{array}{l} v(z+3)+v(z-3)=-1 \\ v(6)=0 \end{array}\right\} \Rightarrow v(z-3) = -\frac{1}{2}$

3). Define

$$v_{triv}: \mathbb{R}(z) \longrightarrow 0 \cup \{\infty\} \qquad f \neq 0 \longmapsto 0$$

Show that $v_{triv} \in Spv(\mathbb{R}(z))$.

4) Show that as Sets,

$$Spv(\mathbb{P}(z)) \cong \{v_{triv}\} \sqcup \{\text{closed pts of } \mathbb{P}^1_{\mathbb{R}}\}$$

$$v^{\mathbb{P}^1_{\mathbb{R}}}_{z_0} \longleftarrow z_0$$

6. alg curve over $\mathbb{F}_p$: miscellaneous.

- $\# X(\mathbb{F}_p)$, $\# X(\mathbb{F}_{p^2})$, … $\rightsquigarrow$ L-fcts, heights, …
- Computation of Frob.
- Chebotarev density theorem: give a proof.
- hyperelliptic curve over $\mathbb{F}_2$: unexpected ones
- how is ramified covering compatible with adèle?
- $p = 1$: What would happen then?
- Shtukas, (in Langlands, though).