Eine Woche, ein Beispiel

4.17 preliminary facts of representions of p-adic groups

Main reference: The Local Langlands Conjecture for GL(2) by Colin J. BushnellGuy Henniart.
[https://link.springer.com/book/10.1007/3-540-31511-X]

Process.

1. Basic properties
   - Smoothness
   - Irreducibility and unitary
   - Reduction to smaller cardinal.
2. Examples: $\mathcal{O}$, $\mathcal{O}^\times$, F, $F^\times$
3. Construction of new reps.
   - Special sub & quotient rep
   - Duality
   - Ind and c-Ind
   - Other constructions ← Example: mirabolic group M
4. Hecke algebra
5. Intertwining properties. ← Example: $GL_2(\mathbb{Q}_p)$

1. Basic properties

1.1. Smoothness

G: loc. profinite group

V: cplx v.s.

$\rho: G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V) \qquad g \longmapsto [v \longmapsto g.v]$

Def. (ρ, V) is smooth if

$\forall\, v \in V,\ \exists\, K \leqslant G$ cpt open s.t. $k.v = v \quad \forall k \in K$

$\mathrm{Rep}(G) = \{$sm rep of $G\}$ is a full subcategory of $\{$rep of $G\}$.

Rmk. Any sub/quotient rep of $(\rho, V) \in \mathrm{Rep}(G)$ is smooth.

$H \leqslant G$ cpt, $(\rho, V) \in \mathrm{Rep}(G) \Rightarrow (\rho|_H, V) \in \mathrm{Rep}(H)$

Rmk. For fcts, smoothness has a different meaning.

Recall the definition of $C^\infty(G)$ & $C_c^\infty(G)$:

$C^\infty(G) := \{f: G \to \mathbb{C} \mid f$ is loc. const$\}$

$C_c^\infty(G) := \{f \in C^\infty(G) \mid \mathrm{supp}\, f \subset G$ is cpt$\}$

1.2. Irreducibility and unitary

$\mathrm{Irr}(G) = \{(\rho, V) \in \mathrm{Rep}(G) \mid \rho$ is a irreducible rep$\}$

$\widehat{G} = \{(\rho, V) \in \mathrm{Irr}(G) \mid \dim_{\mathbb{C}} V = 1\}$

$\overset{[p13]}{=\!=} \{\chi: G \longrightarrow \mathbb{C}^\times \mid \ker \chi$ is open$\}$

$\overset{[(1.6)]}{=\!=} \{\chi: G \longrightarrow \mathbb{C}^\times \mid \chi$ is continuous$\}$

Rmk. The notation is slightly different with the original reference.

Rmk. $\qquad \widehat{G} \subseteq \mathrm{Irr}(G) \subseteq \mathrm{Rep}(G)$

When [p15] G is cpt, or

[p22] $G/Z(G)$ is cpt with G/K countable, we get $\mathrm{Ind}(G) = \mathrm{Irr}(G)$;

when [p21] G is abelian and G/K is countable, $\mathrm{Ind}(G) = \widehat{G}$.

($\exists\, K \leqslant G$ cpt open, countable = at most countable here)

Rmk. A more general result is as follows.

Prop. Let $(\rho, V) \in \mathrm{Rep}(G)$, G/K countable. $\exists\, K \leqslant G$ cpt open

Suppose $\rho|_{Z(G)}$ decompose as $Z(G) \xrightarrow{\chi_w} \mathbb{C}^\times \xrightarrow{\text{scaler}} \mathrm{Aut}_{\mathbb{C}}(V)$.

Let $Z(G) \leqslant H \leqslant G \qquad H \leqslant G$ open $\qquad H/Z(G)$ is cpt.

Then $(\rho|_H, V) \in \mathrm{Rep}(H)$ is semisimple.

To prove this we need the following lemma. (when applied, it would be $K_0 Z(G) \leqslant H$)

Lemma. Let $H \leqslant G$ open, $[G:H] < +\infty$, $(\rho, V) \in \mathrm{Rep}(G)$. Then

ρ is G-semisimple $\Leftrightarrow$ $\rho|_H$ is H-semisimple.

Def (Action as character)

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $\chi \in \widehat{H}$.

We say H acts on V as χ if $\rho|_H$ decompose as follows:

$$\rho|_H : H \xrightarrow{\chi} \mathbb{C}^\times \xrightarrow{\text{scalar}} \mathrm{Aut}_{\mathbb{C}}(V)$$

We may denote χ by χ_ρ or χ_H. When $H = Z(G)$, χ is denoted by ω_ρ.

Def (Contain irr rep)

Let $H \leq G$, $(\rho, V) \in \mathrm{Rep}(G)$, $(\sigma, W) \in \mathrm{Irr}(H)$.

We say ρ contains σ, or σ occurs in ρ, if

$$\mathrm{Hom}_H(\mathrm{Res}^G_H \rho, \sigma) \neq 0$$

i.e., σ can be realized as a quotient subrep of $\mathrm{Res}^G_H \rho$.

Cor. When H acts on V as χ_ρ, ρ contains χ_ρ.

Def (Unitary operator) V: Hilbert space.

$U \in \mathrm{Aut}_{\mathbb{C}}(V)$ is called an unitary operator if

$$\langle Uv, Uw \rangle = \langle v, w \rangle \qquad \forall v, w \in V$$

Def (Unitary rep) V: Hilbert space.

$(\rho, V) \in \mathrm{Rep}(G)$ is unitary if $\rho(g)$ is an unitary operator $(\forall g \in G)$.

E.g. $\chi \in \widehat{G}$ is unitary if $\mathrm{Im}\,\chi \subseteq S^1$

Rmk. When $G = \bigcup_{\substack{K \leq G \\ \text{cpt open}}} K$, any $\chi \in \widehat{G}$ is unitary.

1.3. Reduction to smaller cardinal

Admissibility

(ρ, V) is admissible if $\dim_{\mathbb{C}} V^K < +\infty$ for $\forall K \leq G$ cpt open.

Countable hypothesis

$\exists / \forall$ $K \leq G$ cpt open, G/K is countable.

Assuming countable hypothesis, we get

$$(\rho, V) \in \mathrm{Irr}(G) \Rightarrow \begin{cases} \dim_{\mathbb{C}} V \text{ is countable} \\ \mathrm{End}_G(V) = \mathbb{C} \\ \rho \text{ acts on } V \text{ as a character } \omega_\rho \end{cases}$$

$$\xrightarrow{G \text{ is abelian}} \dim_{\mathbb{C}} V = 1.$$

2. Examples: $\mathcal{O}, \mathcal{O}^\times, F, F^\times$

Rep of $G = \mathcal{O}, \mathcal{O}^\times, F, F^\times$, where F is a non-archi local field.

In these cases, G is abelian and satisfy the countable hypothesis, so $\mathrm{Ind}(G) = \widehat{G}$, i.e., everything reduced to the classification of characters.

E.g. $G = (\mathcal{O}, +)$

$\forall \chi \in \widehat{\mathcal{O}}$ is trivial on $\mathfrak{p}^k$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \geq 0 \mid \mathfrak{p}^d \subset \ker \chi \}$$

When $\mathrm{level}(\chi) = d$, $\chi : \mathcal{O} \xrightarrow{\pi} \mathcal{O}/\mathfrak{p}^d \longrightarrow \mathbb{C}^\times$ factors through char of $\mathcal{O}/\mathfrak{p}^d$.

E.g. $G = \mathcal{O}^\times$

$\forall \chi \in \widehat{\mathcal{O}^\times}$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi \}$$

When $\mathrm{level}(\chi) = d$, $\chi : \mathcal{O}^\times \xrightarrow{\pi} \mathcal{O}^\times/\mathcal{U}^{(d+1)} \longrightarrow \mathbb{C}^\times$ factors through char of $(\mathcal{O}/\mathfrak{p}^{d+1})^\times$

$$\mathcal{O}^\times/\mathcal{U}^{(d+1)} \cong (\mathcal{O}/\mathfrak{p}^{d+1})^\times$$

Recent advances: Geometrization of continuous characters of Zp [https://msp.org/pjm/2013/261-1/pjm-v261-n1-p05-p.pdf]

E.g. $G = (F, +)$

$\forall \chi \in \widehat{F}$ is trivial on $\mathfrak{p}^k$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \in \mathbb{Z} \mid \mathfrak{p}^d \subset \ker \chi \}$$

Prop (Additive duality)

Fix $\psi \in \widehat{F}$ nontrivial with level d.

We have a gp iso

$$F \longrightarrow \widehat{F} \qquad a \longmapsto \psi(a)\psi(-) \text{ of level } d - v_F(a)$$

(when $a \neq 0$)

Q: Do we have similar result for $\widehat{\mathcal{O}}$?

E.g. $G = F^\times$

$\forall \chi \in \widehat{F^\times}$ is trivial on $\mathcal{U}^{(k)}$. Suppose $\chi \neq 1$.

$$\mathrm{level}(\chi) := \min \{ d \geq 0 \mid \mathcal{U}^{(d+1)} \subset \ker \chi \}$$

Q: Do we have any classification of $F^\times$?

Notation for future: G: loc. profinite gp $\quad Z = Z(G)$

$\mathrm{Cos}(G) := \{$ cpt open subgp K of $G\}$

$\mathrm{Cos}_Z(G) := \{K \leq G \mid K \geq Z,\ K/Z \subset G/Z$ cpt open$\}$

3. Construction of new reps

3.1. Special sub & quotient rep

Def. G: loc. profinite gp $\quad N \triangleleft G$ cpt open

$(\pi, V) \in \mathrm{Rep}(G)$ $\quad \vartheta \in \widehat{N}$, we define sm reps of G:

$$V(N) := \langle v - n.v \rangle_{n \in N,\, v \in V}$$
$$V_N := V/V(N)$$
$$V(\vartheta) := \langle \vartheta(n) v - n.v \rangle_{n \in N,\, v \in V}$$
$$V_\vartheta := V/V(\vartheta)$$

Obviously $V(N) = V(\mathbb{1}_N)$, $V_N = V_{\mathbb{1}_N}$, N acts on $V(\vartheta)$ by ϑ, and

$$0 \longrightarrow V(\vartheta) \longrightarrow V \longrightarrow V_\vartheta \longrightarrow 0 \quad \text{in } \mathrm{Rep}(G)$$

Rmk. (1) Normal subgp gives us plenty of canonical decompositions.

E.g., when $(\rho, V) \in \mathrm{Irr}(G)$, for $\vartheta \in \widehat{N}$, we get

$$\begin{cases} V(\vartheta) = V \\ V_\vartheta = 0 \end{cases} \quad \text{or} \quad \begin{cases} V(\vartheta) = 0 \\ V_\vartheta = V \end{cases}$$

(2) When $(\rho, V) \in \mathrm{Rep}(N)$ is semisimple,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(N) & \longrightarrow & V & \longrightarrow & V_N & \longrightarrow & 0 \\ & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} & & \\ 0 & \longrightarrow & \bigoplus_{\substack{\sigma \in \mathrm{Irr}(N) \\ \sigma \neq \mathbb{1}_N}} V^\sigma & \longrightarrow & \bigoplus_{\sigma \in \mathrm{Irr}(N)} V^\sigma & \longrightarrow & V^{\mathbb{1}_N} & \longrightarrow & 0 \end{array}$$

Assume additionally that

- N is abelian,
- N is the union of an increasing sequence of cpt open subgps.

Then, we have more properties.

Prop. (Integral criterian) [p56 Lemma] $(\rho, V) \in \mathrm{Rep}(N)$, $v \in V$.

$$v \in V(\vartheta) \iff \left[\begin{array}{l} \exists\ N_0 \in \mathrm{Cos}(N) \text{ s.t.} \\ \int_{N_0} \vartheta(n)^{-1} \rho(n) v \, d\mu_N(n) = 0 \end{array} \right]$$

Prop. The fctor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Vect}_{\mathbb{C}} \quad (\pi, V) \longmapsto V_\vartheta$$

is exact. E.g.,

	$V(N)(N) \cong V(N)$	$V(N)_N = 0$
$(\vartheta \neq 1)$	$V(N)(\vartheta) \cong V(N) \cap V(\vartheta)$	$V(N)_\vartheta \cong V_\vartheta$
	$V_N(N) = 0$	$V_{N,N} \cong V_N$
$(\vartheta \neq 1)$	$V_N(\vartheta) \cong V_N$	$V_{N,\vartheta} = 0$

(You can compute $V(\vartheta)$ and V_ϑ also, but I'm lazy.)

3.2. Duality

$(\rho, V) \in \mathrm{Rep}(G) \rightsquigarrow (\rho^*, V^*)$ may be not smooth

$\rightsquigarrow (\check{\rho}, \check{V}) \in \mathrm{Rep}(G)$ is the smooth dual, where

$$\check{V} := \bigcup_{K \in \mathrm{Cos}(G)} (V^*)^K \subset V^*$$

$$ev: \check{V} \times V \longrightarrow \mathbb{C} \qquad (\check{w}, v) \longmapsto \langle \check{w}, v \rangle$$

$$\rightsquigarrow \langle g.\check{w}, v \rangle = \langle \check{w}, g^{-1}.v \rangle$$

Rmk. (1) The contravariant duality fctor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(G) \qquad (\rho, V) \longmapsto (\check{\rho}, \check{V})$$

is exact.

exact=>additive: https://math.stackexchange.com/questions/3039422/in-abelian-categories-is-a-right-left-exact-functor-necessarily-additive

(2) For $K \in \mathrm{Cos}(G)$, we have iso $\check{V}^K \xrightarrow{\cong} (V^K)^*$ in $\mathrm{Rep}(K)$.

$\Downarrow$

(3) If $(\rho, V) \in \mathrm{Rep}(G)$, $v \in V$, then $\exists \check{w} \in \check{V}$ s.t. $\langle \check{w}, v \rangle \neq 0$.

$\Downarrow$

(4) $\delta: V \longrightarrow \check{\check{V}}$ is inj, and

$$\delta \text{ is iso} \iff \pi \text{ is admissible}$$

$\Downarrow$

(5) If $(\rho, V) \in \mathrm{Rep}(G)$ is admissible, then

$$(\rho, V) \in \mathrm{Irr}(G) \iff (\check{\rho}, \check{V}) \in \mathrm{Irr}(G)$$

(6) (Bilinear map) Let $(\rho, V), (\sigma, W) \in \mathrm{Rep}(G)$.

$$\mathcal{B}(\rho, \sigma) := \left\{ f: V \times W \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ bilinear} \\ f(gv, gw) = f(v, w) \end{array} \right\}.$$

Then

$$\mathcal{B}(\rho, \sigma) \cong \mathrm{Hom}_G(\rho, \check{\sigma}) \cong \mathrm{Hom}(\sigma, \check{\rho}).$$

3.3. Ind and c-Ind

Definition

Def (Induced representation)

$H \leqslant G$ closed, $(\sigma, W) \in \mathrm{Rep}(H)$, we get

$$\begin{cases} \text{sm induction} & \mathrm{Ind}_H^G \sigma := (\Sigma, X) \in \mathrm{Rep}(G) \\ \text{cpt induction} & \text{c-}\mathrm{Ind}_H^G \sigma := (\Sigma_c, X_c) \in \mathrm{Rep}(G) \end{cases}$$

as follows.

$$\mathrm{Ind}_H^G W := X = \left\{ f: G \longrightarrow W \;\middle|\; \begin{array}{ll} f(hg) = \sigma(h) f(g) & \forall g \in G, h \in H \\ \exists K \leqslant G \text{ s.t. } f(gk) = f(g) & \forall g \in G, k \in K \end{array} \right\}$$

$$\Sigma(g).f = f(-g)$$

$$\text{c-}\mathrm{Ind}_H^G W := X_c = \left\{ f \in X \;\middle|\; \pi(\mathrm{supp}\, f) \text{ is cpt in } H \backslash G \right\} \qquad \pi: G \to H\backslash G$$

$$\Sigma_c(g).f = f(-g)$$

Rmk. (1). (Reality check) When $G = H$,

$$\text{c-Ind}_H^G W = \text{Ind}_H^G W = \left\{ f: G \to W \;\middle|\; \begin{array}{l} f(g) = \sigma(g) f(1) \\ \exists K \leq G \text{ s.t. } f(gk) = f(g) \end{array} \right\} \xrightarrow{\cong} W$$

$$f \longmapsto f(1)$$

$$\sigma(-)\cdot w \longleftarrow w$$

(2) Two fcts Ind_H^G, c-Ind_H^G are both exact.

(3) Suppose $H \leq G$ open. $[G:H] < +\infty$, $(\sigma, W) \in \text{Irr}(H)$. We get $\text{Ind}_H^G \sigma$ is G-semisimple.