Eine Woche, ein Beispiel
8.28 global field

This note mainly follows [现代数学基础12-数论I：Fermat的梦想和类域论- 日 加藤和也 & 黑川信重-胥鸣伟 & 印林生(译)].
Another reference for complement(and also for non-Chinese reader):
[MIT] https://math.mit.edu/classes/18.785/2015fa/lectures.html

I should have done this in 2021.06.27_adèles_and_idèles. However, I was not familiar with local field at that time.

1. definition
2. adèle ring and idèle group
3. topological properties of $\mathbb{A}_K$ & $\mathbb{I}_K$
4. Tate's thesis

def | fundamental domain | dense
measure | cpt |
topo | discrete |

1. definition

denote by K, hope that won't confuse with cpt open subgroup.

Def A global field is
- a finite extension of $\mathbb{Q}$ (number field), or
- a finite extension of $\mathbb{F}_p(T)$ (function field)

For an axiomatic definition, see
https://math.stackexchange.com/questions/873666/definition-of-global-field

Rmk 1. Ostrowski's thm states that
every non-trivial norm on $\mathbb{Q}$ is equiv to $|\cdot|_p$ or $|\cdot|_\infty$.

In [Thm3, Cor4, https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF(T).pdf],

every non-trivial norm on $\mathbb{F}_p(T)$ equiv to $|\cdot|_\pi$ or $|\cdot|_\infty$ for some monic irr $\pi(T)\in\mathbb{F}_p[T]$

where
$$\left|\tfrac{a}{b}\pi^k\right|_\pi = p^{-\deg\pi\cdot k} \quad a,b\in\mathbb{F}_p[T],\ \pi\nmid ab \quad a,b\neq 0$$
$$\left|\tfrac{a}{b}\right|_\infty = p^{\deg a-\deg b} \quad a,b\in\mathbb{F}_p[T] \quad a,b\neq 0$$

Ex. Compute $K_v, \mathcal{O}_v$ for $v=|\cdot|_\infty, |\cdot|_T, |\cdot|_{T-1}, |\cdot|_{T^2+1}$ $K=\mathbb{F}_p(T),\ p=7$

A. $\mathcal{O}_{|\cdot|_\infty}=\mathbb{F}_p[[\frac{1}{T}]]$ $\mathcal{O}_{|\cdot|_T}=\mathbb{F}_p[[T]]$ $\mathcal{O}_{|\cdot|_{T-1}}=\mathbb{F}_p[[T-1]]$

$K_{|\cdot|_\infty}=\mathbb{F}_p((\frac{1}{T}))$ $K_{|\cdot|_T}=\mathbb{F}_p((T))$ $K_{|\cdot|_{T-1}}=\mathbb{F}_p((T-1))$

$\mathcal{O}_K=\mathbb{F}_p[T]$ can not embed in $\mathcal{O}_{|\cdot|_\infty}$, since $\mathbb{F}_p[T]=\mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1)$.
The prod formula also prohibit $\mathcal{O}_K$ embed to all $\mathcal{O}_v$.

Show that $\mathbb{F}_p((\frac{1}{T}-a))=\mathbb{F}_p((T-\frac{1}{a}))$ for $a\in\mathbb{F}_p^\times$:

$$\mathbb{F}_p((\tfrac{1}{T}-a)) = \mathbb{F}_p((\tfrac{1-aT}{T})) = \mathbb{F}_p\left(\left(-\tfrac{a}{T}(T-\tfrac{1}{a})\right)\right)$$
$$\cup\!\!\mid \qquad\qquad\qquad\qquad \cap\!\!\mid$$
$$\mathbb{F}_p\left(\left(-\frac{(T^{-1}-a+a)^{-1}}{a}(\tfrac{1}{T}-a)\right)\right)=\mathbb{F}_p\left(\left(-\tfrac{T}{a}(\tfrac{1}{T}-a)\right)\right)=\mathbb{F}_p((T-\tfrac{1}{a}))$$

$$\mathcal{O}_{|\cdot|_{T^2+1}} = \mathbb{F}_p(\alpha)[[T^2+1]] \qquad \alpha^2+1=0$$
$$K_{|\cdot|_{T^2+1}} = \mathbb{F}_p(\alpha)((T^2+1))$$

$$\begin{aligned} \mathbb{F}_p[T] &\hookrightarrow \mathbb{F}_p(\alpha)[[T^2+1]] \\ T &\longmapsto \alpha - \tfrac{\alpha}{2}(T^2+1) - \tfrac{\alpha}{8}(T^2+1)^2 - \tfrac{\alpha}{16}(T^2+1)^3 - \tfrac{5\alpha}{128}(T^2+1)^4 - \cdots \\ T^2 &\longmapsto -1 + T^2+1 \end{aligned}$$

Rmk 2. Product formula is still true; that is, for $K = \mathbb{F}_p(T)$

$$|f|_\infty \prod_{\pi \text{ fin}} |f|_\pi = 1 \qquad \forall f \in \mathbb{F}_p(T)^\times$$

Ex. Verify the product formula for other K.

For relationships between local fields and global fields, see: https://alex-youcis.github.io/localglobalgalois.pdf
We only list two results which will be used later :

Let L/K be fin ext of global field. We get two isos as topo ring

$$\begin{array}{ccc} L \otimes_K K_v & \xrightarrow{\cong} & \prod_{i=1}^{g} L_{w_i} \\ \uparrow & & \cup \\ \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_v & \xrightarrow[\text{[MIT, Cor 11.7]}]{\cong} & \prod_{i=1}^{g} \mathcal{O}_{w_i} \end{array}$$

$$\begin{array}{c} w_1 \cdots w_g \\ \backslash\ | \cdots / \\ v \end{array} \qquad \begin{array}{c} L_{w_1} \cdots L_{w_g} \\ \backslash\ | \cdots / \\ K_v \end{array}$$

2. adèle ring and idèle group

Every book begins this topic by restricted product, which is totally correct but a little boring/confusing. Let us derive the restricted product naturally.

global	$\mathbb{A}_K$	$\mathbb{I}_K$	$\mathbb{I}_K^\times$
local	F	$F^\times$	$\mathcal{O}_F^\times$

adèle ring

Def (adèle ring $\mathbb{A}_\mathbb{Q}$) We know that

$$\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1) \subseteq \prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$$

where $\mathbb{Q}$ acts diagonally on $\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$:

$$+ : \mathbb{Q} \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}\right) \longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$$
$$(t, (a_p, a_\infty)) \longmapsto (t + a_p, t + a_\infty)$$

The adèle ring $\mathbb{A}_\mathbb{Q}$ is defined as the orbit of $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$, i.e.

$$\mathbb{A}_\mathbb{Q} := \mathbb{Q} + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)\right)$$
$$= \{(a_v)_v \in \prod_v K_v \mid a_v \in \mathcal{O}_v \text{ for almost all } v\} \triangleq \prod_v{}' K_v$$

We don't define $\mathcal{O}_v$ for $v = |\cdot|_\infty$, but that doesn't matter.

Rmk. You can also replace $[0,1)$ by $\mathbb{R}$ in the definition ($\mathbb{A}_\mathbb{Z} := \prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R}$), then it may happen that

$$t + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R}\right) = t' + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R}\right) \quad \text{for } t \neq t' \in \mathbb{Q}.$$

Rmk. The measure is easy to define while the topo is a bit tricky.

By letting $\mu_p(\mathbb{Z}_p) = 1$, $\mu_\infty([0,1)) = 1$ and give $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$ with the prod measure, the measures on $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}$ are defined.

For the topology on $\mathbb{A}_K$, we take the weakest topo s.t. all the subspaces

$$\prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v \quad = \left(\prod_{\substack{p \in S \\ p \text{ prime}}} \mathbb{Q}_p \times \mathbb{R} \times \prod_{p \notin S} \mathbb{Z}_p\right)$$

(for any S: set of finite places containing all infinite places)

are open, and the subspace topo of $\prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$ coincides with the prod topo.

This topology is a little stronger than the subspace topo of $\mathbb{A}_K \subset \prod_v K_v$, since $\prod_{v \in S} K_v \times \prod_{v \notin S} \mathcal{O}_v$ are not open in this subspace topo.

The same method can be applied to defining the topo of any restricted product.