Eine Woche, ein Beispiel

6.2. Roof structure for moduli of pairs

Setting $\mathcal{C}$: category e.g. Top

Def A roof in $\mathcal{C}$ is a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$X, Y, Z \in \mathcal{C}$
$f \in Mor_{\mathcal{C}}(Z, X)$, $g \in Mor_{\mathcal{C}}(Z, Y)$

"Equivalently", this can be written as

when $X \times Y$ $\exists$

$$\begin{tikzcd} Z \arrow[r] \arrow[dr, bend right, "f"'] \arrow[drr, bend right, "g"] & X \times Y \arrow[d] \arrow[dr] & \\ & X & Y \end{tikzcd}$$

Z is called as "incidence space" in some references, and a roof is also called as a incidence structure or a correspondence.

Roofs are used in many different areas.

E.g. - construct derived category by "quotienting out quasi-isos"

- define $\mathrm{Corr}(C, E)$ in abstract 6-fctor formalism
- define Fourier-Mukai transformation

$$\Phi_{\mathcal{F}} = g_! \circ (\mathcal{F} \otimes -) \circ f^*$$

$$\begin{tikzcd} \mathcal{F} \arrow[dr, dash] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

- ???

In most cases, roofs are used in understanding the moduli of pairs:

E.g.
$$X = \{x\text{'s}\}$$
$$Y = \{y\text{'s}\}$$
$$Z = \{(x,y) \in X \times Y \mid \phi(x,y) = \mathrm{True}\}$$

$\phi : X \times Y \longrightarrow \{0, \underset{\text{True}}{1}\}$

Then the figure

$$\begin{tikzcd} \{y \mid \phi(x_0, y)\} \arrow[dr, dash] & & \{x \mid \phi(x, y_0)\} \arrow[dl, dash] \\ & \{(x,y) \mid \phi(x,y)\} \arrow[dl] \arrow[dr] & \\ \{x\} & & \{y\} \end{tikzcd}$$

$$\begin{tikzcd} f^{-1}(x_0) \arrow[dr, dash] & & g^{-1}(y_0) \arrow[dl, dash] \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

presents many moduli spaces in a clear way.

E.p., one can describe Z by stratifications through f and g.

1. (B,N)-pair & buildings.

$$\begin{tikzcd} & (\text{building}, G) \arrow[dl, "\phi"'] \arrow[d, "\text{forget}"] \arrow[dl, phantom, shift left=4ex, "G\circlearrowright"] \\ \{(B,N)\text{-pair}\} \arrow[r, dashed, "\Phi"'] & \text{building} \end{tikzcd}$$

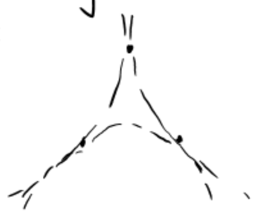
$G \circlearrowright \mathcal{B}$ simplicially

ϕ: Fix $C \in \mathcal{C}$, $A \in \mathcal{A}$, then
$B := \mathrm{Stab}_C(G)$, $N := \mathrm{Stab}_A(G)$

ϕ is usually not inj/surj. see wiki: Building (math)

⚠ There are some problems in wiki:

- How to define the parabolic subgp of $GL_2(\mathbb{Q}_p)$?
  $I \subsetneqq \{A \in GL_2(\mathbb{Q}_p) \mid v(\det A) = 0\} \subsetneqq GL_2(\mathbb{Q}_p)$

- In the (B,N)-pair case (of $SL_2(\mathbb{F}_2)$),
  is $SL_2(\mathbb{F}_2) \subset SL_2(\mathbb{F}_2)$ a maximal parabolic subgp?
  - If true, then the building X has only 1 vertice;
  - If false, then $\mathcal{A}_{T_0} = S^0$, $\mathcal{B} =$ 

E.g. For a (B, N)-pair,

$$
\begin{tikzcd}
 & \{(B,T) \mid B \supset T\} \arrow[dl] \arrow[dr] & \\
\{B\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B/T \arrow[dr, dash] & & W \arrow[dl, dash] \\
 & G/T \arrow[dl] \arrow[dr] & \\
G/B & & G/N
\end{tikzcd}
$$

$\dashrightarrow$ building $\mathcal{B} = \bigcup_T \mathcal{A}_T = \bigcup_B \mathcal{C}_B$

E.g. For $G = SL_n(F)$, $T = T(\mathcal{O}_F)$,

$$
\begin{tikzcd}
 & \{(I,T) \mid I \subset T\} \arrow[dl] \arrow[dr] & \\
\{I\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
I/T \arrow[dr, dash] & & W_{ext} \arrow[dl, dash] \\
 & G/T \arrow[dl] \arrow[dr] & \\
G/I & & G/N_G(T)
\end{tikzcd}
$$

$\rightsquigarrow$ Bruhat-Tits building $\mathcal{B} = \bigcup_T \mathcal{A}_T = \bigcup_I \mathcal{C}_I$

In many cases, the (B, N)-pair can't give us a building.

Roadmap:

$$\{(I, T(\mathcal{O}_F)) \text{ in } G(\mathcal{O}_F)\}$$
$$(I, N_{G(\mathcal{O}_F)}(T(\mathcal{O}_F))) \text{ - pair}$$

$$\{(B(x_F), T(x_F)) \text{ in } G(x_F)\}$$
$$(B(x_F), N_{G(x_F)}(T(x_F))) \text{ - pair}$$

$$\{(I, T(\mathcal{O}_F)) \text{ in } G(F)\}$$
$$\shortparallel$$
$$\text{“}\{(I, T(F)) \text{ in } G(F)\}\text{”} \quad I \not\supseteq T(F)$$
$$(I, N_{G(F)}(T(F))) \text{ - pair}$$