Eine Woche, ein Beispiel

5.12 sheaf version of ⊗ & Hom

1. def of sheaf Hom
2. def of sheaf ⊗

sheaf version of Tensor-Hom adjunction is left in the next document.

Compared with ⊗, Hom is more delicated, and it is harder than you expected.

1. def of sheaf Hom

$$
\begin{array}{ll}
\mathrm{Hom}_A(-,-): & (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod} \qquad A\text{: comm ring} \\
\rightsquigarrow & \\
\mathrm{Hom}_{Sh(X)}(-,-): & Sh(X)^{op} \times Sh(X) \longrightarrow \mathbb{Z}\text{-Mod} \\
\rightsquigarrow & \\
\underline{\mathrm{Hom}}_{Sh(X)}(-,-): & Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X) \\
\rightsquigarrow & \\
R\underline{\mathrm{Hom}}_{\mathcal{D}^+(X)}(-,-): & \mathcal{D}^-(X)^{op} \times \mathcal{D}(X) \longrightarrow \mathcal{D}(X)
\end{array}
$$

non-derived sheaf Hom

Def [Vakil, 2.3.1] For $\mathcal{F}, \mathcal{G} \in Sh(X)$, a morphism of sheave

$$\phi: \mathcal{F} \longrightarrow \mathcal{G}$$

is the data of maps

$$\phi(\mathcal{U}): \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \quad \text{for all } \mathcal{U} \subseteq X \text{ open.}$$

which is compatible with restriction.

We write

$$\phi \in \mathrm{Hom}_{Sh(X)}(\mathcal{F}, \mathcal{G})$$

Similarly, one can define

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

as the set of morphisms in $\mathcal{D}(X)$.

Def [Vakil, 2.3.C] (Sheaf Hom / Internal Hom)

For $\mathcal{F}, \mathcal{G} \in Sh(X)$, one gets a sheaf $\underline{Hom}(\mathcal{F}, \mathcal{G}) \in Sh(X)$ given by

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)(\mathcal{U}) = Hom_{Sh(\mathcal{U})}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}})$$

Cor

$$Hom = \Gamma \circ \underline{Hom} : Sh(X)^{op} \times Sh(X) \xrightarrow{\underline{Hom}} Sh(X) \xrightarrow{\Gamma} Abel$$

⚠ Even though $(\mathcal{F} \otimes \mathcal{G})_p \cong \mathcal{F}_p \otimes \mathcal{G}_p$,

$\underline{Hom}$ does not commute with taking stalks:

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)_p \xrightarrow{\ncong} Hom(\mathcal{F}_p, \mathcal{G}_p)$$

It's neither inj nor surj.

[Left adj comm with limit, $\otimes \dashv \underline{Hom}$].

Ex. Try to compute coefficient $\mathbb{Q}$.

$$\underline{Hom}_{Sh(X)}(\underline{\mathbb{Q}}_X, \mathcal{F}) \cong \mathcal{F}$$
$$\underline{Hom}_{Sh(X)}(j_! \underline{\mathbb{Q}}_{\mathcal{U}}, \mathcal{F}) \cong j_*(\mathcal{F}|_{\mathcal{U}})$$
$$\underline{Hom}_{Sh(\mathbb{C})}(sky_0(\mathbb{Q}), \underline{\mathbb{Q}}_{\mathbb{C}}) \cong 0$$

derived sheaf Hom

Def. For $\mathcal{F}, \mathcal{G} \in Sh(X)$, the derived internal Hom

in general, $\mathcal{F} \in \mathcal{D}(X)^-$, $\mathcal{G} \in \mathcal{D}^+(X)$

$$R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^+(X)$$

is given by

$$\begin{array}{llll} \underline{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \mathcal{I}^\cdot) & \text{when } \mathcal{G} \xrightarrow{\simeq} \mathcal{I}^\cdot & & \text{inj resolution} \\ \underline{Hom}_{\mathcal{C}(X)}(\mathcal{P}^\cdot, \mathcal{G}) & \text{when } \mathcal{F} \xleftarrow{\simeq} \mathcal{P}^\cdot & & \text{proj resolution} \end{array}$$

Here,

$$\underline{Hom} : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$

is extended to the double complex

$\mathcal{C}(X) :=$ complex of sheaves on X, temperate notation

$$\underline{Hom}_{\mathcal{C}(X)} : \mathcal{C}(X)^{op} \times \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$$

Other versions of sheaf Hom

$$\begin{array}{ccc} Hom_A(-,-) & \rightsquigarrow & RHom_A(-,-) \\ \wr\downarrow & & \wr\downarrow \\ Hom_{Sh(X)}(-,-) & \rightsquigarrow & RHom_{\mathcal{D}^+(X)}(-,-) \\ \wr\downarrow & & \wr\downarrow \\ \underline{Hom}_{Sh(X)}(-,-) & \rightsquigarrow & R\underline{Hom}_{\mathcal{D}^+(X)}(-,-) \end{array}$$

$$Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$
$$Hom_{Sh(X)}(\mathcal{F}, \mathcal{G}) = Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}).$$

$$RHom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R\Gamma \circ R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G})$$

2. def of sheaf $\otimes$

$$\begin{array}{ccccc}
M\otimes - & \dashv & \mathrm{Hom}_A(M,-): & A\text{-Mod} \longrightarrow A\text{-Mod} & A\text{: comm ring} \\
& \rightsquigarrow & & & \\
\mathcal{F}\otimes\pi_X^*(-) & \dashv & \mathrm{Hom}_{Sh(X)}(\mathcal{F},-): & \mathbb{Z}\text{-Mod} \longrightarrow Sh(X) & \\
& \rightsquigarrow & & & \\
\mathcal{F}\otimes - & \dashv & \underline{\mathrm{Hom}}_{Sh(X)}(\mathcal{F},-): & Sh(X) \longrightarrow Sh(X) & \\
& \rightsquigarrow & & & \\
\mathcal{F}^{\cdot}\overset{L}{\otimes} - & \dashv & R\underline{\mathrm{Hom}}_{D^+(X)}(\mathcal{F}^{\cdot},-): & D(X) \longrightarrow D(X) &
\end{array}$$

non-derived sheaf $\otimes$

Def [Vakil 2.6.J] For $\mathcal{F},\mathcal{G}\in Sh(X)$,
$\mathcal{F}\otimes\mathcal{G}\in Sh(X)$ is given by sheafification of

$$\begin{array}{ccc} \mathcal{F} & & \mathcal{G} \\ & \searrow\ \swarrow & \\ & X & \end{array}$$

$$(\mathcal{F}\otimes^{pre}\mathcal{G})(U) = \mathcal{F}(U)\otimes_{\mathbb{Z}}\mathcal{G}(U),$$

this defines a bifctor

$$-\otimes- : Sh(X)\times Sh(X)\longrightarrow Sh(X)$$

Cor. $(\mathcal{F}\otimes\mathcal{G})_p = \varinjlim_{p\in U}\left(\mathcal{F}(U)\otimes_{\mathbb{Z}}\mathcal{G}(U)\right) = \mathcal{F}_p\otimes_{\mathbb{Z}}\mathcal{G}_p$

i.e., $\otimes$ commutes with taking stalks.

Ex. Verify that

$$\begin{aligned}(\mathcal{F}\otimes\mathcal{G})(U) &= (\mathcal{F}|_U\otimes\mathcal{G}|_U)(U) \\ &= \Gamma(\mathcal{F}|_U\otimes\mathcal{G}|_U)\end{aligned}$$

and compare it with the formula

$$\begin{aligned}\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})(U) &= \mathrm{Hom}(\mathcal{F}|_U,\mathcal{G}|_U) \\ &= \Gamma(\underline{\mathrm{Hom}}(\mathcal{F}|_U,\mathcal{G}|_U))\end{aligned}$$

Can't define $\otimes$ in this way though.

In general, one has formula

$$f^*(\mathcal{F}\otimes\mathcal{F}') \cong f^*\mathcal{F}\otimes f^*\mathcal{F}'.$$

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F} \quad \mathcal{F}' \\ | & & \backslash\ / \\ f: Y & \longrightarrow & X \end{array}$$

Combined with formulas

$$f^*\underline{\mathbb{Q}}_X = \underline{\mathbb{Q}}_Y$$
$$\underline{\mathbb{Q}}_X\otimes\mathcal{F} = \mathcal{F},$$

it means that

$$f^*: (Sh(X), \otimes_{Sh(X)}) \longrightarrow (Sh(Y), \otimes_{Sh(Y)})$$

is a (strong) monoidal fctor.

Ex. Try to compute

$$j_!\underline{\mathbb{Q}}_U \otimes_{Sh(X;\mathbb{Q})}\mathcal{F} \cong j_!(\mathcal{F}|_U)$$
$$sky_p(\mathbb{Q}) \otimes_{Sh(\mathbb{C};\mathbb{Q})}\mathcal{F} \cong i_*\mathcal{F}_p$$

In general, one has projection formula

$$f_!(f^*\mathcal{F}\otimes\mathcal{G}) \xleftarrow{\ \cong\ } \mathcal{F}\otimes f_!\mathcal{G}$$ when $\mathcal{F}$ is flat.

So this formula doesn't cover the Ex. (ˇ∩ˇ)
Even so, their proofs are similar: checking stalkwise.

derived sheaf $\otimes$

Def. For $\mathcal{F}, \mathcal{G} \in Sh(X)$, the derived internal Hom

in general, $\mathcal{F}, \mathcal{G} \in \mathcal{D}^-(X)$

$$\mathcal{F} \overset{L}{\otimes}_{\mathcal{D}^-(X)} \mathcal{G} \in \mathcal{D}^-(X)$$

is given by

$$\mathcal{F} \otimes_{\mathcal{C}(X)} \mathcal{P}^{\cdot} \qquad \text{when } \mathcal{G} \xleftarrow{\cong} \mathcal{P}^{\cdot} \quad \text{flat resolution}$$

Here,

$$- \otimes_{Sh(X)} - : Sh(X) \times Sh(X) \longrightarrow Sh(X)$$

is extended to the double complex

$\mathcal{C}(X) :=$ complex of sheaves on X, temperate notation

$$- \otimes_{\mathcal{C}(X)} - : \mathcal{C}(X) \times \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$$

⚠ [KS 90, Def 2.6.2]

To switch from $\mathcal{D}^-(A\text{-Mod})$ to $\mathcal{D}^+(A\text{-Mod})$, we need to require that

$$w.gldim(A) < +\infty. \qquad \text{w.gldim: shortest flat resolution}$$