Eine Woche, ein Beispiel

9.10. ramified covering: alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - field of rational fcts
   - valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

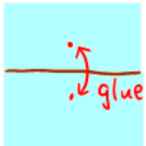
1. alg curve viewed as stack quotient

	Spec $\mathbb{R}$	base change Spec $\mathbb{C}/_{\mathbb{C}}$	Spec $\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	$\emptyset$
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{Id, \tau\}$
$\Gamma_{\mathbb{R}} = Gal(\mathbb{C}/_{\mathbb{R}})$	trivial on pts & fcts	no action	$Id \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. Spec $\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec} \mathbb{R}[x]$. As a set,

$$\begin{aligned} \operatorname{Spec} \mathbb{R}[x] &= \{(x-a) \mid a \in \mathbb{R}\} \cup \{(x^2+bx+c) \mid \substack{b,c \in \mathbb{R} \\ b^2-4c<0}\} \cup \{(0)\} \\ &= \quad \mathbb{R} \quad \cup \quad \mathcal{H} \quad \cup \{(0)\} \\ \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R} \\ \mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C}) \end{aligned}$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau \circ x$. Observe that

$$\operatorname{MaxSpec} \mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/_{\Gamma_{\mathbb{R}}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$\operatorname{MaxSpec} \mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/_{\Gamma_{\mathbb{F}_p}} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/_{\Gamma_\kappa} \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

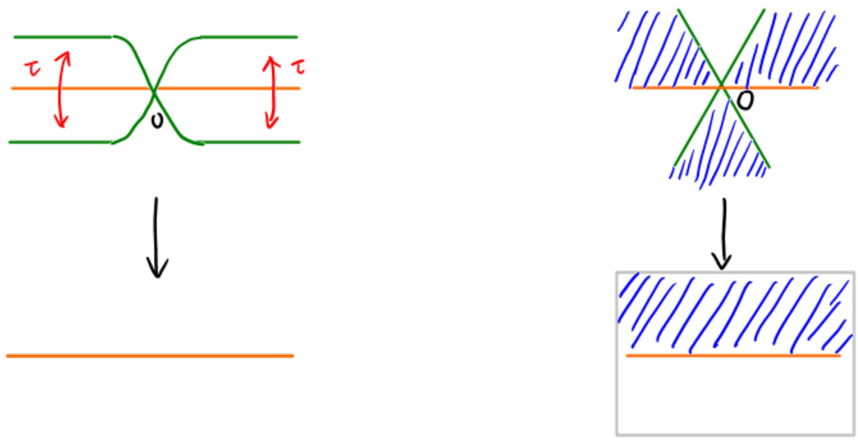
$$\operatorname{Stab}_x(\Gamma_\kappa) = \Gamma_{\kappa'} \iff \text{fiber at } x = \operatorname{Spec} \kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R} \cup \mathcal{H}$	$\mathbb{C}$	$\mathbb{C}$ 2 cplx conj
$\mathbb{R}$ - pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$ - pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C} \sqcup \mathbb{C}_\tau$
$\Gamma_{\mathbb{R}} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

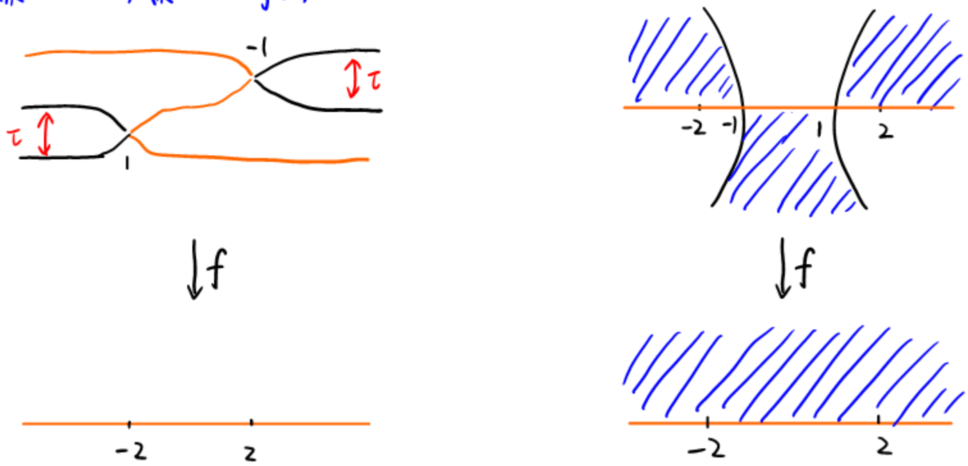
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

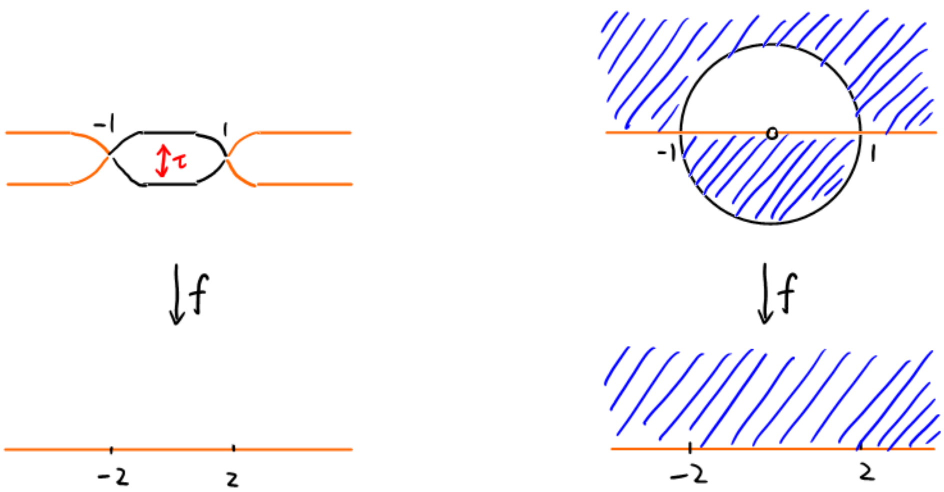
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



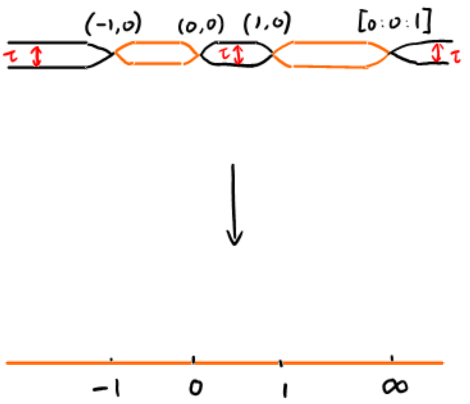
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$

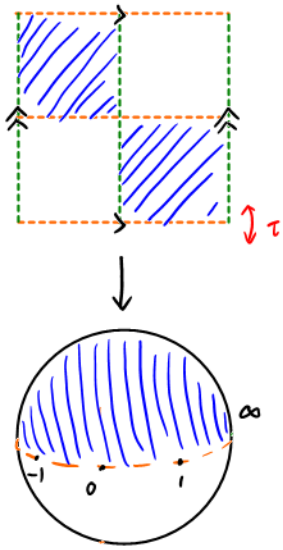


$f: \mathbb{G}_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$



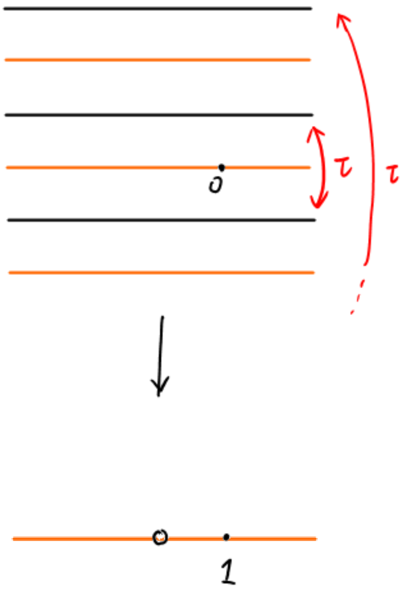
$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ $\quad [x:y:z] \longmapsto [x:z]$ $\quad E_{\mathbb{R}} = \operatorname{Proj} \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

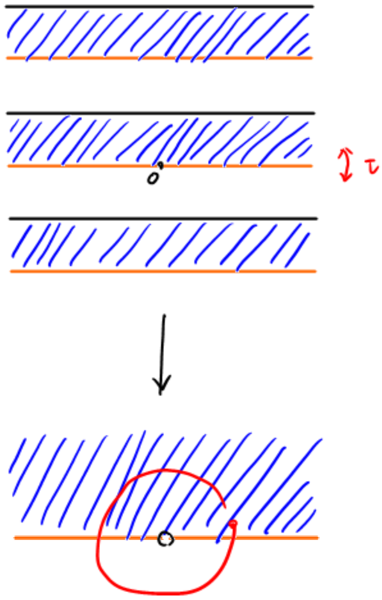




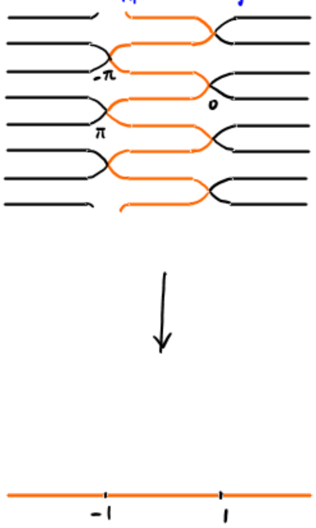
⚠ The following are not alg morphisms!

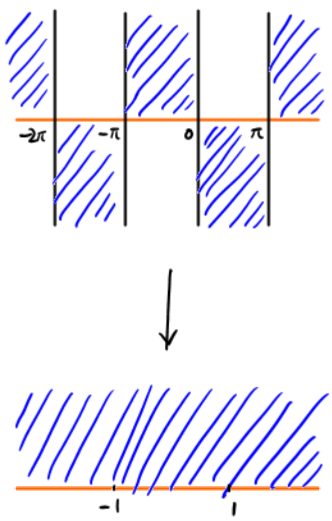
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = e^z$





$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = \cos z$

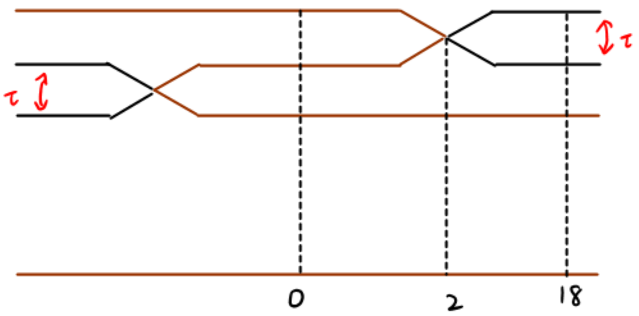




Let's focus on the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = z^3 - 3z$

classical picture



split: $f^{-1}(0) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z-z_0))$

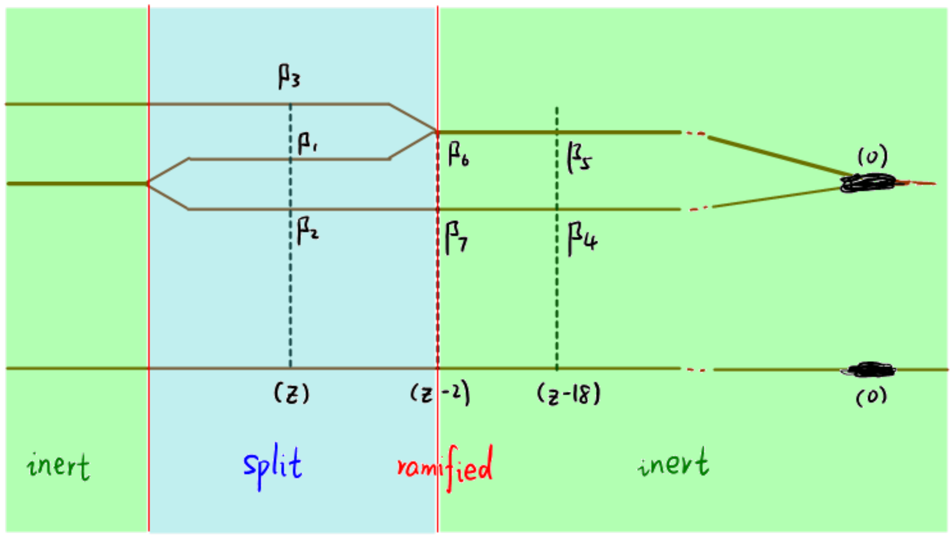
$f^{-1}((z^2+1)) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{R}$

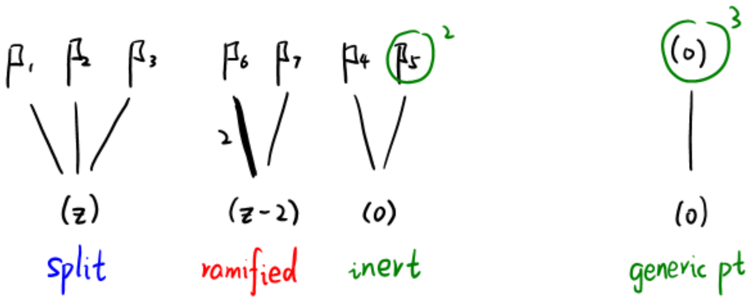
generic point: $f^{-1}((0)) = \operatorname{Spec}\mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$

algebraic picture



$$\begin{array}{ccccc} \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[w] & & w^3-3w \\ \downarrow f & & \uparrow f^* & & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & & \mathbb{R}[z] & & z \end{array}$$



split: $\mathfrak{p}=(z)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w) = (w)(w-\sqrt{3})(w+\sqrt{3})$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \triangleq \ \mathfrak{p}_1 \quad \mathfrak{p}_2 \quad \mathfrak{p}_3$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$

$\mathfrak{p}=(z^2+1)$, $f^*(\mathfrak{p})\mathbb{R}[w] = ((w^3-3w)^2+1) = (f_1')(f_2')(f_3')$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \triangleq \ \mathfrak{p}_1' \quad \mathfrak{p}_2' \quad \mathfrak{p}_3'$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_1,\mathfrak{p}_2,\mathfrak{p}_3\}$

(partially) inert: $\mathfrak{p}=(z-18)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-18) = (w-3)(w^2+3w+6)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \triangleq \ \mathfrak{p}_4 \quad \mathfrak{p}_5$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_4,\mathfrak{p}_5\}$

where $\kappa(\mathfrak{p}_5) = \mathbb{R}[w]/_{(w^2+3w+6)} \cong \mathbb{C}$, $\quad [\kappa(\mathfrak{p}_5):\mathbb{R}] = 2$

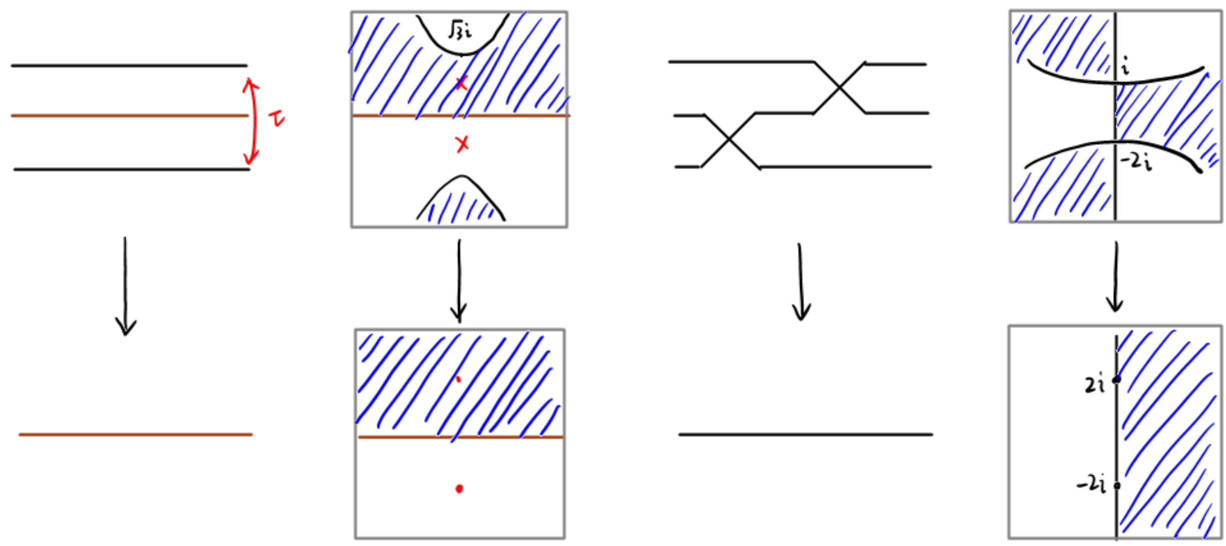
generic point: $\mathfrak{p}=(0)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (0)$

$f^{-1}(\mathfrak{p})=\{(0)\}$

where $\kappa((0)) = \mathrm{Frac}\left(\mathbb{R}[w]/_{(0)}\right) \cong \mathbb{R}(w)$, $\quad [\mathbb{R}(w):\mathbb{R}(z)]=3$

ramified: $\mathfrak{p}=(z-2)$, $f^*(\mathfrak{p})\mathbb{R}[w] = (w^3-3w-2) = (w+1)^2(w-2)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \triangleq \ \mathfrak{p}_6^2 \quad \mathfrak{p}_7$ $\qquad f^{-1}(\mathfrak{p})=\{\mathfrak{p}_4,\mathfrak{p}_5\}$

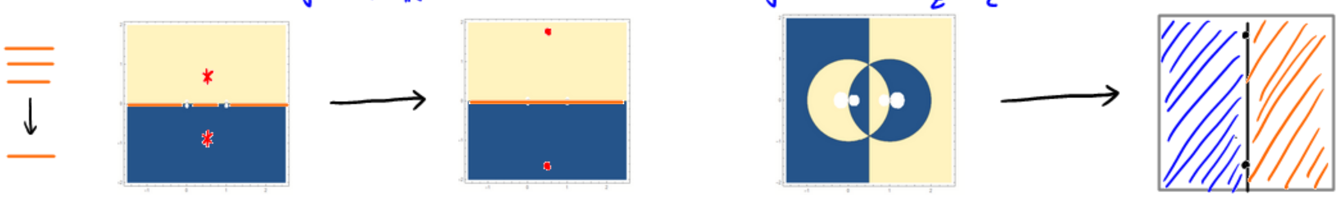
Ex. Try to work out the case
$$f:\ \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = z^3+3z.$$



$\mathbb{R}$ picture $\qquad\qquad\qquad\qquad$ $i\mathbb{R}$ picture

⚠ The ramification pt is outside $\mathbb{R}$. This is not a Galois covering.

Ex. Try to work out the case
$$f:\ \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}} \qquad f(z) = \frac{z^3-3z+1}{z^2-z} - 1.5$$



$\mathbb{R}$ picture $\qquad\qquad\qquad\qquad$ $i\mathbb{R}$ picture

This is a Galois covering, with no inert places (except for the generic pt)

3. Frobenius for alg curve/$\mathbb{R}$

$$Gal(\kappa(q)/\kappa(p)) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & \text{if } \kappa(q)=\mathbb{C},\ \kappa(p)=\mathbb{R} \\ \{Id\} & \text{otherwise.}\end{cases}$$

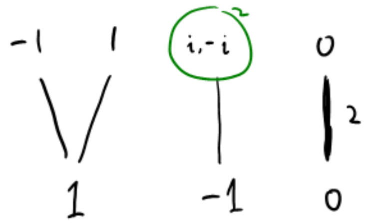
When E/F is Galois, $\operatorname{Spec} O_E/\operatorname{Spec} O_F$ unramified at p,

$$\begin{array}{ccc} Gal(\kappa(q)/\kappa(p)) \cong Gal(E/F)_q & \le & Gal(E/F) \\ Frob_q & \longmapsto & Frob_q \end{array}$$

is a subgp of $Gal(E/F) \cong Aut(Spv(E)/Spv(F))$ Now, just view $Spv(E) \in AlgCurve_k$.

Let's try to compute some $Frob_q$

E.g. $\begin{array}{ccc} \mathbb{A}^1_{\mathbb{R}} & z & \mathbb{R}[\omega] = \mathbb{R}[z^{\frac{1}{2}}] \\ \downarrow & \downarrow & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z^2 & \mathbb{R}[z] \end{array}$



For $p=(z-1)$, $q=(\omega-1)$,

$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ 1 & & 1 & & \{1,\tau\} \end{array}$$

For $p=(z+1)$, $q=(\omega^2+1)$,

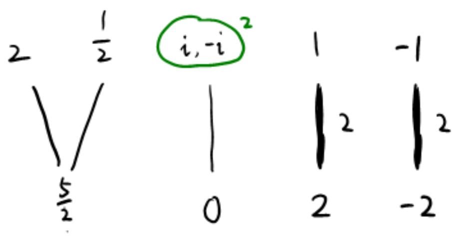
$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $Frob_{(z+1)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad \omega \longmapsto -\omega$

Not the conjugation but $\tau(\mathbb{C})|_{i\mathbb{R}}$ coincides with the cplx conj

E.g. $\begin{array}{ccc} \mathbb{G}_{m,\mathbb{R}} & z & \mathbb{R}[\omega^{\pm 1}] = \mathbb{R}\left[\left(\frac{z+\sqrt{z^2-4}}{2}\right)^{\pm 1}\right] \\ \downarrow & \downarrow & \uparrow \\ \mathbb{A}^1_{\mathbb{R}} & z+\frac{1}{z} & \mathbb{R}[z] \end{array}$



For $p=(z)$, $q=(\omega^2+1)$,

$$\begin{array}{ccccc} Gal(\kappa(q)/\kappa(p)) & \cong & Gal(E/F)_q & \le & Gal(E/F) \\ \| & & \| & & \| \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $Frob_{(z)} = \tau : \mathbb{P}^1_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$, where

$\tau(\mathbb{C}) : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad \omega \longmapsto \frac{1}{\omega}$

Not the conjugation but $\tau(\mathbb{C})|_{S^1}$ coincides with the cplx conj

⚠ $\mathbb{R}(z^{\frac{1}{3}})/\mathbb{R}(z)$ is not Galois at all, so

For $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$, $z \longmapsto z^3$, $\mathfrak{p} = (z-1)$, $\mathfrak{q} = (w^2+w+1)$,

$$\underbrace{\mathrm{Gal}(k(\mathfrak{q})/k(\mathfrak{p}))}_{\{1,\tau\}} \not\cong \underbrace{``\mathrm{Gal}(E/F)_{\mathfrak{q}}"}_{1} \leq \underbrace{``\mathrm{Gal}(E/F)"}_{1} \neq \mathbb{Z}/3\mathbb{Z}$$

We will discuss about $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$ in section 4.

~~Claim. For p odd prime, any deg p extension of $\mathbb{R}(x)$ is not Galois.~~

This claim is wrong. The field extension

$$\mathbb{R}(x)[T]/(T^3 - xT^2 + (x-3)T + 1) \;/\; \mathbb{R}(x)$$

is Galois with deg 3. discriminant $\Delta = (x^3-3x+9)^2$ [SerreGT, 1.1]

Wrong proof:

If not, suppose $F/\mathbb{R}(x)$ is a deg p Galois extension, we get the field extension tower in $\overline{\mathbb{R}(x)}$:

$$\begin{tikzcd}
 & E \arrow[dl, dash, "2"'] \arrow[dd, dash, "p"] \\
F \arrow[dd, dash, "p"'] & \\
 & \mathbb{C}(x) \arrow[dl, dash, "2"] \\
\mathbb{R}(x) &
\end{tikzcd}$$

where $\mathrm{Gal}(E/F) \triangleleft \mathrm{Gal}(E/\mathbb{R}(x))$ is a normal subgp of order 2.

By Kummer theory, $E \cong \mathbb{C}(x)[T]/(T^p - f)$ for some $f \in \mathbb{C}(x)$.

~~Since $E/\mathbb{R}(x)$ is Galois, $f \in \mathbb{R}(x)$ (see the example below)~~

When $f \in \mathbb{R}(x)$, one gets

$$\mathrm{Gal}(E/\mathbb{R}(x)) \longhookrightarrow S_p \circlearrowright \{T, \zeta_p T, \ldots, \zeta_p^{p-1} T\}$$

Injection: if σ fix T, $\zeta_p T$, then σ fix ζ_p, then $\sigma = \mathrm{Id}$.

Since $\# \mathrm{Gal}(E/\mathbb{R}(x)) = 2p$, $\mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$ or $\mathbb{Z}/2p\mathbb{Z}$.

Since $\mathrm{Gal}(E/\mathbb{R}(x)) \leq S_p$, $\mathrm{Gal}(E/\mathbb{R}(x)) \cong D_p$.

However, D_p has no order 2 normal subgp, contradiction!

E.g. $\mathbb{C}(z)[T]/(T^3 - (z-i))$ over $\mathbb{R}(z)$ is not Galois, since

$$\begin{tikzcd}[row sep=small]
\mathbb{C}(T) \arrow[d, dash] & & 0 \arrow[d, dash, "3"'] & z_0 \arrow[dr, dash] & \rho z_0 \arrow[d, dash] & \rho^2 z_0 \arrow[dl, dash] \\
\mathbb{C}(z) \arrow[d, dash] & & i \arrow[dr, dash] & & -i \arrow[dl, dash] & \\
\mathbb{R}(z) & & & (z^2+1) & &
\end{tikzcd}$$

This example is not general enough. For example,

$\mathbb{C}(z)[T]/(T^3 - \frac{z-i}{z+i})$ over $\mathbb{R}(z)$ can be Galois

Q. For $F/\mathbb{R}(x)$ Galois extension, is $\mathrm{Gal}(F/\mathbb{R}(x))$ generated by its order 2 elements?

I call it as the "weaked version of Chebotarev's density theorem for $\mathbb{P}^1_{\mathbb{R}}$".

A. No.

We could not expect the density theorem to be true in the real case,

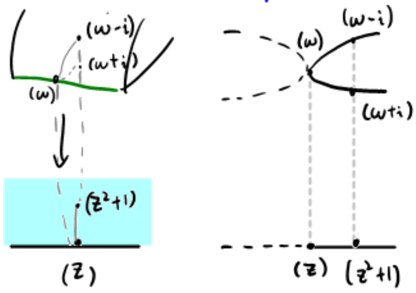
since in S_3 case the order 3 conj class can never be reached by a single Frob.

For a possible direct and brutal method to this question, use the result in this link:
math.stackexchange.com/questions/318690/absolute-galois-group-of-mathbbrt
How is Z/3Z realized as the quotient group of this group? (better: compatible with the field extension mentioned above)

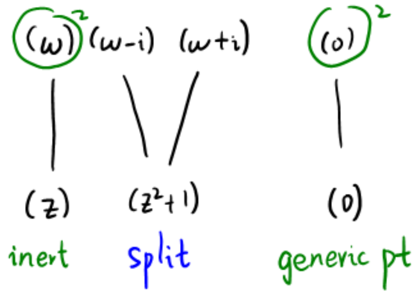
4. complexify is a ramified covering by non geometrical connected spaces

E.x. $f: \mathbb{A}^1_{\mathbb{C}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ is an unramified covering of alg curves/$\mathbb{R}$.

$$\begin{tikzcd} \mathbb{A}^1_{\mathbb{C}} \arrow[d, "f"] & \mathbb{C}[w] \\ \mathbb{A}^1_{\mathbb{R}} & \mathbb{R}[z] \arrow[u, "f^*"'] \end{tikzcd}$$



along $i\mathbb{R}$



This is an unramified covering.

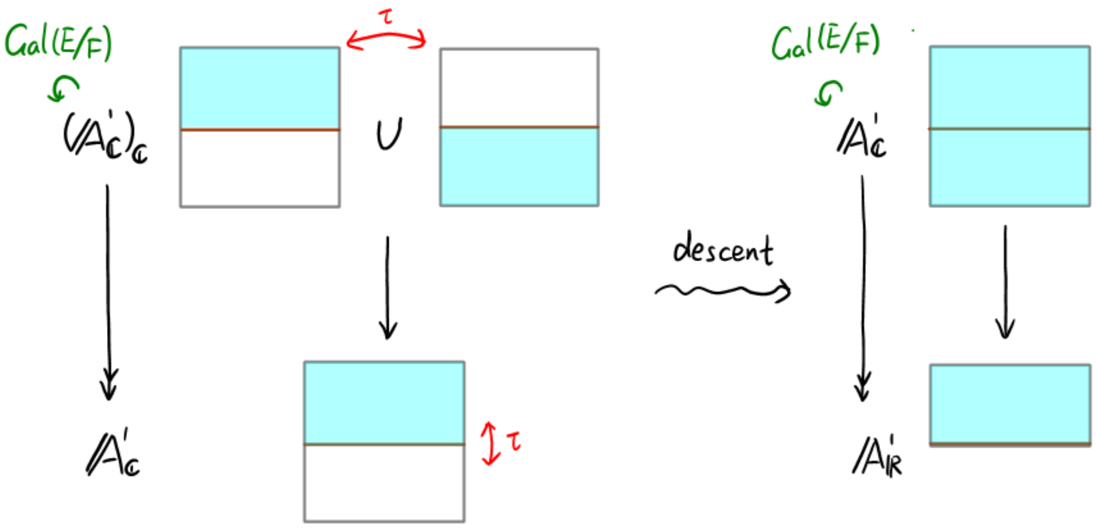
As an $\mathbb{R}$-scheme, $\mathbb{A}^1_{\mathbb{C}}$ is not geo connected.

$$\begin{tikzcd} \mathbb{C}[\omega]\otimes_{\mathbb{R}}\mathbb{C} \arrow[r, phantom, "\cong"] & \mathbb{C}[\omega]\oplus\mathbb{C}[\omega] \\ \mathbb{R}[z]\otimes_{\mathbb{R}}\mathbb{C} \arrow[u] \arrow[r, phantom, "\cong"] & \mathbb{C}[z] \arrow[u, "(Id,\sigma)"'] \end{tikzcd}$$

$\mathrm{Gal}(E/F)$ $\circlearrowleft$ $\mathbb{C}[\omega]\otimes_{\mathbb{R}}\mathbb{C}$ $\circlearrowleft \Gamma_{\mathbb{R}}$; $\mathrm{Gal}(E/F)\,\Gamma_{\mathbb{R}}$ on $\mathbb{C}[\omega]\oplus\mathbb{C}[\omega]$; $\circlearrowleft \Gamma_{\mathbb{R}}$ on $\mathbb{R}[z]\otimes_{\mathbb{R}}\mathbb{C}$; $\Gamma_{\mathbb{R}} \circlearrowleft$ on $\mathbb{C}[z]$

$$f(\omega)\otimes_{\mathbb{R}} a \longmapsto (a f(\omega), \bar{a} f(\omega))$$

$$f(z)\otimes_{\mathbb{R}} a \longmapsto a f(z)$$



For $p=(z)$, $q=(\omega)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(q)/\kappa(p)) & \cong & \mathrm{Gal}(E/F)_q & \leq & \mathrm{Gal}(E/F) \\ \parallel & & \parallel & & \parallel \\ \{1,\tau\} & & \{1,\tau\} & & \{1,\tau\} \end{array}$$

Therefore, $\mathrm{Frob}_{(z)} = \tau : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1$$

$$\omega_1 \longmapsto \overline{\omega_1}$$

$$\omega_2 \longmapsto \overline{\omega_2}$$

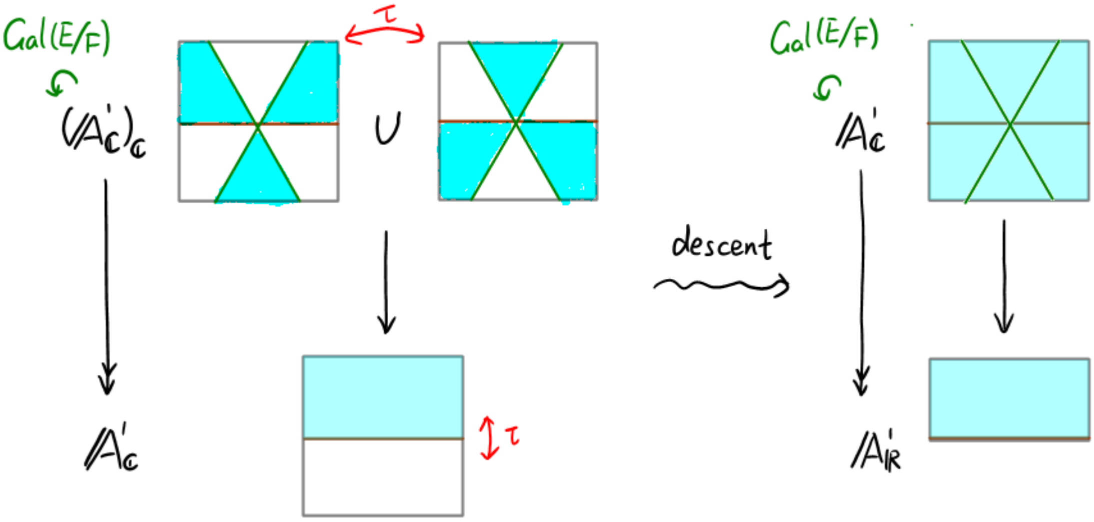
Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R}\cup\mathbb{R}}$ coincides with the cplx conj (switch)

Ex. Try to work out $\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)$, and compute Frobenius elements.

Recall : $\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}) \cong S_3 \circlearrowright \{\sqrt[3]{2}, \rho\sqrt[3]{2}, \rho^2\sqrt[3]{2}\}$.

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z)) \cong S_3 \circlearrowright \{\underset{1}{z^{\frac{1}{3}}}, \underset{2}{\rho z^{\frac{1}{3}}}, \underset{3}{\rho^2 z^{\frac{1}{3}}}\}$

$\mathrm{Gal}(\mathbb{C}(z^{\frac{1}{3}})/\mathbb{R}(z))$	Id	(23)	(12)	(13)	(123)	(132)
$z^{\frac{1}{3}} \longmapsto$	$z^{\frac{1}{3}}$	$z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$	$\rho^2 z^{\frac{1}{3}}$
$\rho \longmapsto$	ρ	ρ^2	ρ^2	ρ^2	ρ	ρ
$\mathbb{C}\mathbb{P}^1 \ni a \longmapsto$	a	$\bar{a}$	$\rho^2\bar{a}$	$\rho\bar{a}$	$\rho^2 a$	ρa
geometry	Id					



For $\mathfrak{p}=(z-1)$, $\mathfrak{q}=(w-1)$,

$$\begin{array}{ccccc} \mathrm{Gal}(\kappa(\mathfrak{q})/\kappa(\mathfrak{p})) & \cong & \mathrm{Gal}(E/F)_{\mathfrak{q}} & \leq & \mathrm{Gal}(E/F) \\ \| & & \| & & \| \\ \{1, \tau\} & & \{1, (23)\} & & S_3 \end{array}$$

Therefore, $\mathrm{Frob}_{(w-1)} = \tau_{(23)} : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1$$
$$w_1 \longmapsto \bar{w}_1$$
$$w_2 \longmapsto \bar{w}_2$$

Not the conjugation, but $\tau(\mathbb{C})|_{\mathbb{R}\cup\mathbb{R}}$ coincides with the cplx conj (switch)

Similarly, $\mathrm{Frob}_{(w-\rho)} = \tau_{(13)} : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$, where

$$\tau(\mathbb{C}) : \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1 \longrightarrow \mathbb{C}\mathbb{P}^1 \sqcup \mathbb{C}\mathbb{P}^1$$
$$w_1 \longmapsto \rho\bar{w}_1$$
$$w_2 \longmapsto \rho\bar{w}_2$$

Not the conjugation, but $\tau(\mathbb{C})|_{\rho^2\mathbb{R}\cup\rho^2\mathbb{R}}$ coincides with the cplx conj (switch)

In this case, $\mathrm{Gal}(E/F)$ is generated by all $\mathrm{Frob}_{(z-z_0)}$.

5. alg curves and function fields

In this section, we follow the same route as in [2023.09.03].
The following theorem generalize the goal in [2023.09.03].

zero, to avoid confusion with O.

Thm 1 [Stack Project, 0BXX, Thm 53.2.6]

For k field, we have an equiv of categories

$$\mathrm{AlgCurve}_k = \left\{ \begin{array}{l} \text{Obj: sm proj curves}/k \\ \text{Mor: non-const alg morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Obj: } F/k \text{ field ext s.t.} \\ \quad \mathrm{trdeg}_k F = 1 \\ \quad F/k \text{ f.g as a field} \\ \text{Mor: morphism as fields}/k \end{array} \right\}^{op} = \mathrm{field}_{k(t)/k}^{op}$$

f.g. field ext F/k of transcendence deg 1.

> term in Stack Project
> curve/k = 1-dim variety/k
> variety/k = f.t./k + integral + seperated
> require irreducible
> don't require closed

field of rational functions

Def. For $X \in \mathrm{AlgCurve}_k$,

$$\mathcal{M}(X) := \{\text{rational fcts on } X\}$$

Ex. Verify that $\mathcal{M}(\mathbb{P}^1_k) \cong k(z)$

Rmk By the proof of Thm 1 (or [Vakil, Thm 11.2.1]) & Noether Normalization [Vakil, 11.2.4],

1) $\mathcal{M}(X) \in \mathrm{field}_{k(t)/k}$

2) $\exists\ k(x) \hookrightarrow \mathcal{M}(X)$ s.t. $[\mathcal{M}(X) : k(x)] < +\infty$

Def. [Vakil, Def 11.2.2]

For $f: Y \dashrightarrow X$ a dominant rational map of irr varieties/k with $\dim Y = \dim X$,

$$\deg f := [\mathcal{M}(Y) : f^*\mathcal{M}(X)].$$

Similarly, people can see if a ramified covering is Galois.

Write it down rigorously, using the language of geometry.

valuations

Let's try to compute $\mathrm{Spv}(A)$ for some $A \in k\text{-Alg}$. (require $v|_{k^\times} \equiv 0$)

Ex. In this exercise we want to describe $\mathrm{Spv}(\mathbb{R}(z))$.

1). For $v \in \mathrm{Spv}(\mathbb{R}(z))$, suppose $v(z-3) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$z^2+1 = (z-3)(z+3)+10$

$\Rightarrow v(z^2+1) = \min\big(v(z-3)+v(z+3),\ v(10)\big) = 0$

For $v \in \mathrm{Spv}(\mathbb{R}(z))$, suppose $v(z^2+1) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$v(z^2+1) = 1 \quad \Rightarrow \quad v(z^2)=0 \quad \Rightarrow \quad v(z)=0$

$\Rightarrow v(z^2+1-(\pi^2+1)) = 0 \Rightarrow \left.\begin{array}{l} v(z+\pi)+v(z-\pi)=0 \\ v(2\pi)=0 \end{array}\right\} \Rightarrow v(z-\pi)=0$

$\Rightarrow \quad v(z^2-9)=0 \quad \Rightarrow \left.\begin{array}{l} v(z+3)+v(z-3)=0 \\ v(6)=0 \end{array}\right\} \Rightarrow v(z-3)=0$

Similarly, other irr polynomials have valuation 0.

2). For $v \in \mathrm{Spv}(\mathbb{C}(z))$, suppose $v(z-3) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$z^2+1 = (z-3)(z+3)+10$

$\Rightarrow v(z^2+1) = \min\big(v(z-3)+v(z+3),\ v(10)\big) = -2$

For $v \in \mathrm{Spv}(\mathbb{C}(z))$, suppose $v(z^2+1) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z^2+1)}\right)$.

$v(z^2+1) = -1 \quad \Rightarrow \quad v(z^2-9) = -1 \quad \Rightarrow \left.\begin{array}{l} v(z+3)+v(z-3)=-1 \\ v(6)=0 \end{array}\right\} \Rightarrow v(z-3) = -\frac{1}{2}$

3). Define

$$v_{triv}: \mathbb{R}(z) \longrightarrow 0 \cup \{\infty\} \qquad f \neq 0 \longmapsto 0$$

Show that $v_{triv} \in \mathrm{Spv}(\mathbb{R}(z))$.

4) Show that as Sets,

$$\mathrm{Spv}(\mathbb{R}(z)) \cong \{v_{triv}\} \sqcup \{\text{closed pts of } \mathbb{P}^1_{\mathbb{R}}\}$$

$$v^{\mathbb{P}^1_{\mathbb{R}}}_{z_0} \longleftarrow z_0$$

and we have comm diagrams.

$$\begin{array}{ccc} \mathrm{Spv}(\mathbb{C}(z)) & \cong & \{v_{triv}\} \sqcup \mathbb{C}\mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathrm{Spv}(\mathbb{R}(z)) & \cong & \{v_{triv}\} \sqcup \{\text{closed pts of } \mathbb{P}^1_{\mathbb{R}}\} \end{array}$$

Ex. Using the same method, try to show that $\quad v|_{k^\times} \equiv 0$

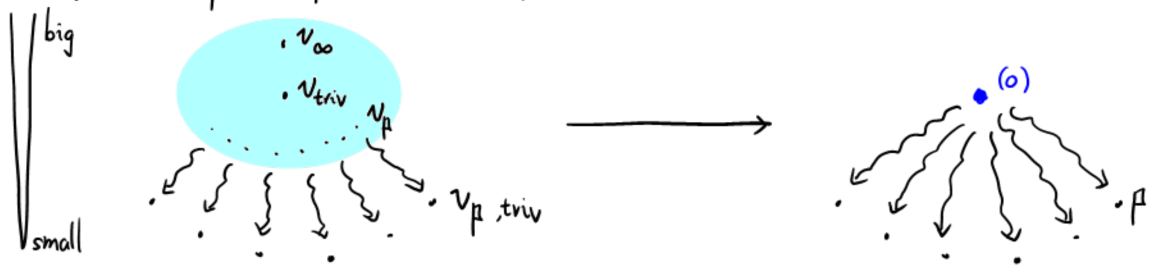
$$\mathrm{Spv}_k(k(z)) \cong \{v_{triv}\} \sqcup \{\text{closed pts of } \mathbb{P}^1_k\}$$

Hint: pick the lowest degree polynomial f s.t. $v(f) \neq 0$.

Q: How to show that, for $X \in \mathrm{AlgCurve}_k$,

$$\mathrm{Spv}_k(M(X)) \cong \{v_{triv}\} \sqcup \{\text{closed pts of } X\} ?$$

Ex. Try to compute $\mathrm{Spv}(\mathbb{R}[z])$ & $\mathrm{Spv}_k(k[z])$.



$$\mathrm{Spv}(k[z]) \xrightarrow{\ v \,\mapsto\, \mathfrak p_v\ } \mathrm{Spec}(k[z])$$

Q: How to understand $\mathrm{Spv}(F)$, for $F = \mathbb{R}(x)[y]/(y^2 - x(x+1)(x-1))$?
As usual, this reduces to understand the fiber of the map

$$\begin{tikzcd} & \mathrm{Spv}(F\otimes_{\mathbb{R}}\mathbb{C}) \arrow[dl] \arrow[d] \\ \mathrm{Spv}(F) \arrow[d] & \mathrm{Spv}(\mathbb{C}(x)) \arrow[dl] \\ \mathrm{Spv}(\mathbb{R}(x)) & \end{tikzcd}$$

We only mention the case 4), as other cases have no essential difference.

4). When $\pi(v) = v_{-3}$,

$$v(y^2) = v_3(x(x+1)(x-1)) = 0 \quad \Rightarrow \quad v(y) = 0$$

for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation $\quad a = v_3(f)$

$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

where

$$v(-2f) = a$$

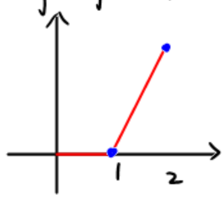
$$v(f^2 - x(x+1)(x-1)) \geq \min(0, 2a) \quad \text{with equality if } 0 \neq 2a$$

$v(z_0)$ is only not determined when

$$\begin{cases} a = 0 \\ v(f^2 - x(x+1)(x-1)) > 0 \end{cases}$$

$$\Leftrightarrow \quad f(-3)^2 = -24$$

$$\Leftrightarrow \quad \emptyset$$



bad case

Therefore, $\#\pi^{-1}(v_3) = 1$, where

$$v(y+f) = \min(0, v(f)).$$

Notice that $\mathcal{O}_{v_{-3}}/\mathfrak p_{v_{-3}} \cong \mathbb{R}$ while $\mathcal{O}_v/\mathfrak p_v \cong \mathbb{C}$

Hint: $\mathcal{O}_v = (\mathbb{R}[x,y]/(y^2 - x(x+1)(x-1)))_v$,

$\mathfrak p_v = (x+3, y^2+24)$.

When $\pi(v) = v_{(x^2+1)}$,

$$v(y^2) = v_{(x^2+1)}(x(x+1)(x-1)) = 0 \quad \Rightarrow \quad v(y) = 0$$

for $f \in \mathbb{C}(x) - \{0\}$, $z_0 := y + f$ satisfies the equation $\quad a = v_3(f)$

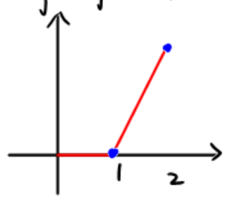
$$z^2 - 2fz + f^2 - x(x+1)(x-1) = 0$$

where

$$v(-2f) = a$$

$$v(f^2 - x(x+1)(x-1)) \geq \min(0, 2a) \quad \text{with equality if } 0 \neq 2a$$

$v(z_0)$ is only not determined when



bad case

$$\begin{cases} a = 0 \\ v(f^2 - x(x+1)(x-1)) > 0 \end{cases}$$

$$\Leftrightarrow \quad f(i)^2 = -2i \qquad f \in \mathbb{R}(x) \subseteq \mathbb{C}(x)$$

$$\Leftrightarrow \quad f(i) = \pm\sqrt{-2i}$$

$$\Leftrightarrow \quad v_i(f - \sqrt{-2i}) > 0 \quad \text{or} \quad v_i(f + \sqrt{-2i}) > 0 \qquad v_i \in \mathrm{Spv}(\mathbb{C}(x))$$

Take $f_0(x) = x - 1$. Suppose that $v(y + f_0) = 1$, $v(y - f_0) = 0$, we need to determine $v(y+f)$.

$$v(y+f) = v(\underbrace{(y+f_0)}_{1} + (f - f_0))$$

$$= \begin{cases} v(f^2 - x(x+1)(x-1)), & f(i) = f_0(i) = i-1 \\ 0, & f(i) \neq f_0(i) = i-1 \end{cases}$$

Ex. Verify that this is indeed a valuation.

In $\mathbb{R}[x,y]$,

$$(x^2+1, y^2 - x(x+1)(x-1)) = (x^2+1, y^2+2x) = (x^2+1, (y+x-1)(y-x+1))$$

$$= (x^2+1, y+x-1)(x^2+1, y-x+1)$$

In conclusion, $\# \pi^{-1}(v_{(x^2+1)}) = 2$, where

$$\begin{tikzcd} & \mathrm{Spv}(F \otimes_{\mathbb{R}} \mathbb{C}) \arrow[dl] \arrow[d] \\ \mathrm{Spv}(F) \arrow[d] & \mathrm{Spv}(\mathbb{C}(x)) \arrow[dl] \\ \mathrm{Spv}(\mathbb{R}(x)) & \end{tikzcd} \qquad \begin{tikzcd} & v_{(i,1-i)}\ v_{(-i,1+i)} \atop v_{(i,-1+i)}\ v_{(-i,-1-i)} \arrow[dl] \arrow[d] \\ v_{(x^2+1, y+x-1)} \atop v_{(x^2+1, y-x+1)} \arrow[d] & v_i,\ v_{-i} \arrow[dl] \\ v_{(x^2+1)} & \end{tikzcd}$$

Q: How to show that, for E/F fin field extension in $\mathrm{field}_{\kappa(z)/\kappa}$, the map

$$\mathrm{Spv}(E) \longrightarrow \mathrm{Spv}(F)$$

is surj?

Maybe comes from the fact that $\quad Y, X \in \mathrm{AlgCurve}_{\kappa}$

$$f: Y \longrightarrow X \text{ nonconstant} \quad \Rightarrow \quad f \text{ is surj.}$$

Ex. Try to understand the hyperelliptic curve over $\mathbb{R}$ or $\mathbb{F}_p$. $(p > 2)$.
What is the dim of the moduli space?

6. alg curve over $\mathbb{F}_p$: miscellaneous.

- $\# X(\mathbb{F}_p)$, $\# X(\mathbb{F}_{p^2})$, $\cdots$ $\rightsquigarrow$ L-fcts, heights, ...
- Computation of Frob.
- Chebotarev density theorem: give a proof.
- hyperelliptic curve over $\mathbb{F}_2$: unexpected ones
- how is ramified covering compatible with adèle?
- $p = 1$: What would happen then?
- Shtukas, (in Langlands, though).