Eine Woche, ein Beispiel

12.12. cohomology group and product structure

Today: Lens space $L(n,q)$
Eilenberg-MacLane space $K(\mathbb{Z},n)$
Grassmannian & Stiefel manifold $V_k(\mathbb{R}^n)$ [Already! in 11.14]
Lie group $SU(n)$, $U(n)$, $Sp(n)$ and $SO(n,\mathbb{R})$

Ref: [GTM, §18 for computation, §14, 15 mainly for theory]
[Jun Hou Fung, the cohomology of Lie groups, url:http://math.uchicago.edu/~may/REU2012/REUPapers/Fung.pdf]
p-local cohomology of K(Z,n): https://www.math.uni-bielefeld.de/documenta/vol-20/09.pdf

The process: 1. find a fiber bundle
2. induce the spectrum sequence
3. compute!
Case 1. can compute $H^i(-,\mathbb{Z})$ directly
$\rightsquigarrow$ know everything

Case 2.
$$\left.\begin{array}{c} H^i(-,\mathbb{Q}) \\ \downarrow \\ H^i(-,\mathbb{F}_p) \end{array}\right\} \Rightarrow H^i(-,\mathbb{Z}) \Rightarrow H_i(-,\mathbb{Z})$$
$\rightsquigarrow$ don't know the prod structure of $H^i(-,\mathbb{Z})$

1. Lens space $L(n,q)$ ($q\in\mathbb{Z}_{>0}$ can be non-prime)

Def $L(n,q) \triangleq S^{2n+1}/(\mathbb{Z}/q\mathbb{Z}\text{-action})$ $L(\infty,q)\triangleq S^\infty/(\mathbb{Z}/q\mathbb{Z}\text{-action})$
e.p. $L(n,2)\cong \mathbb{R}P^{2n+1}$ $L(\infty,q)=K(\mathbb{Z}/q\mathbb{Z},1)$

$$\begin{array}{ccc} \mathbb{Z}/q\mathbb{Z} & \longrightarrow & S^{2n+1} \\ & & \downarrow \\ & & L(n,q) \end{array} \qquad\qquad \begin{array}{ccc} S^1 & \longrightarrow & L(n,q) \\ & & \downarrow \\ & & \mathbb{C}P^n \end{array}$$

$\rightsquigarrow \pi_i(L(n,q))$ $\rightsquigarrow H^\bullet(L(n,q),\mathbb{Z})$

$$H^i(L(n,q),\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \text{ or } 2n+1 \\ \mathbb{Z}/q\mathbb{Z} & i=2,4,\cdots 2n \\ 0 & \text{otherwise.} \end{cases}$$

n \ $H^i(L(n,3),\mathbb{Z})$ \ i	0	1	2	3	4	5	6	7
1	$\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}$	0	0	0	0
2	$\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}$	0	0
3	$\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}$
4	$\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0	$\mathbb{Z}/3\mathbb{Z}$	0

$$H^\cdot(L(n,q),\mathbb{Z}) = \mathbb{Z}[x_2]/(qx_2, x_2^{n+1}) \oplus \mathbb{Z}y$$

$$H^\cdot(L(n,q),\mathbb{F}_p) = \begin{cases} \mathbb{F}_p[y]/(y^2) \cong \mathbb{F}_p \oplus \mathbb{F}_p y & p \nmid q \\ \mathbb{F}_p[x_1]/(x_1^{2n+1}) \oplus \mathbb{F}_p y & p = q \text{ is prime} \end{cases}$$

$$H^\cdot(L(n,q),\mathbb{Q}) = \mathbb{Q}[y]/(y^2) \cong \mathbb{Q} \oplus \mathbb{Q}y$$

2. EM space we know

$$\begin{array}{ccc} K(\mathbb{Z},n-1) & \longrightarrow & PK(\mathbb{Z},n) \\ & & \downarrow \\ & & K(\mathbb{Z},n) \end{array} \qquad\qquad \begin{array}{ccc} \mathbb{C}P^\infty \simeq K(\mathbb{Z},2) & \longrightarrow & PK(\mathbb{Z},3) \\ & & \downarrow \\ & & K(\mathbb{Z},3) \end{array}$$

By the computation in the end, we get.

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	$\mathbb{Z}$	$\mathbb{Z}$													
2	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
3	$\mathbb{Z}$			$\mathbb{Z}$			$\mathbb{F}_2$		$\mathbb{F}_3$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{Z}/10\mathbb{Z}$	$\mathbb{F}_2$	?
4	$\mathbb{Z}$				$\mathbb{Z}$			$\mathbb{F}_2$	$\mathbb{Z}$	$\mathbb{F}_3$		?			
5	$\mathbb{Z}$					$\mathbb{Z}$			$\mathbb{F}_2$		$\mathbb{Z}/6\mathbb{Z}$		?		

$H^i(K(\mathbb{Z},n),\mathbb{Z})$

3. Lie group.

$$\begin{array}{ccc} SU(n-1) & \longrightarrow & SU(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array} \qquad \begin{array}{ccc} U(n-1) & \longrightarrow & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array} \qquad \begin{array}{ccc} Sp(n-1) & \longrightarrow & Sp(n) \\ & & \downarrow \\ & & S^{4n-1} \end{array}$$

we get **Proposition 1.4.** [JHF]

(1) $H^*(SU(n)) \cong \Lambda[x_3, x_5, \ldots, x_{2n-1}]$.
(2) $H^*(U(n)) \cong \Lambda[x_1, x_3, \ldots, x_{2n-1}]$.
(3) $H^*(Sp(n)) \cong \Lambda[x_3, x_7, \ldots, x_{4n-1}]$.

and $SO(n,\mathbb{R}) \simeq V_{n-1}(\mathbb{R}^n)$ is already computed.

4. Grassmannian

It's showed in [Hatcher, Thm 4D.4] that

$$H^*(Gr_n(\mathbb{R}^\infty);\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[w_1,\cdots,w_n] \qquad \deg w_i = i$$

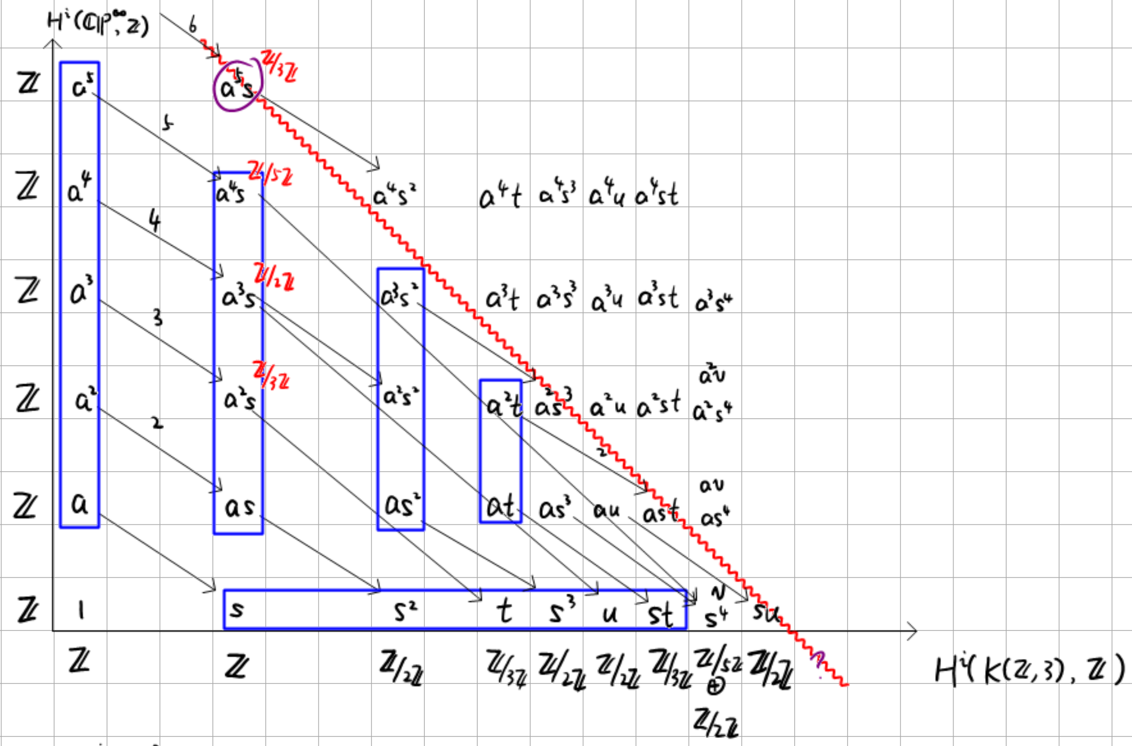
$$H^*(Gr_n(\mathbb{C}^\infty);\mathbb{Z}) \cong \mathbb{Z}[c_1,\cdots,c_n] \qquad \deg c_i = 2i$$

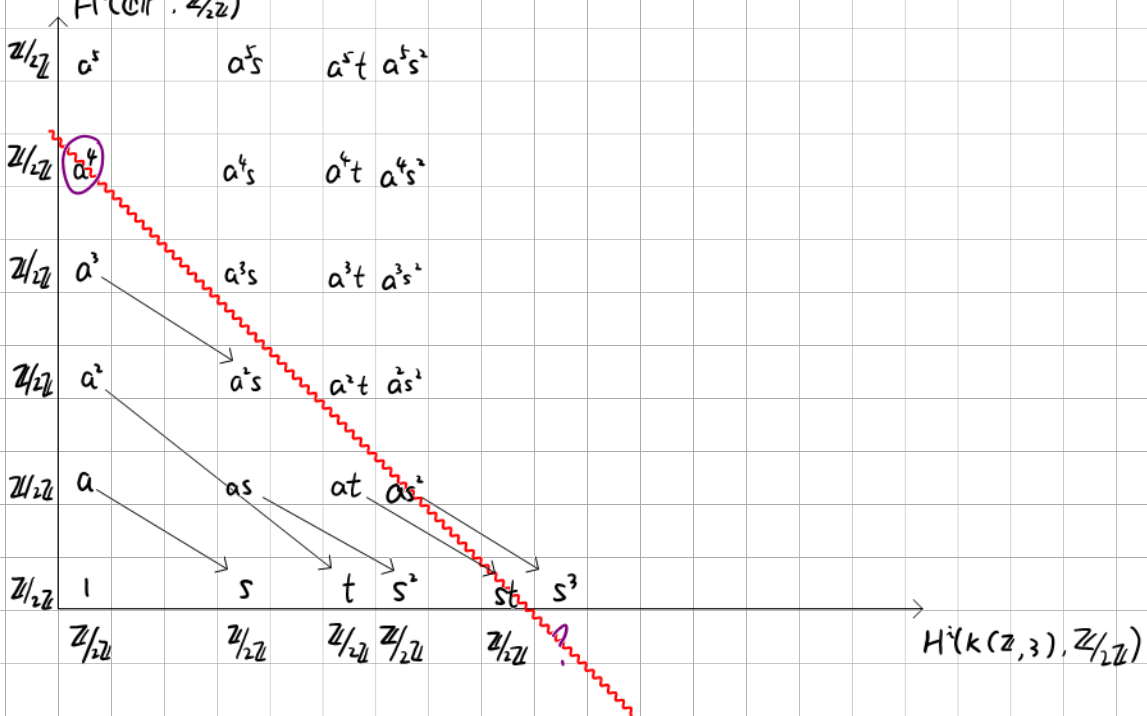
$$H^*(Gr_n(\mathbb{H}^\infty);\mathbb{Z}) \cong \mathbb{Z}[q_1,\cdots,q_n] \qquad \deg q_i = 4i$$

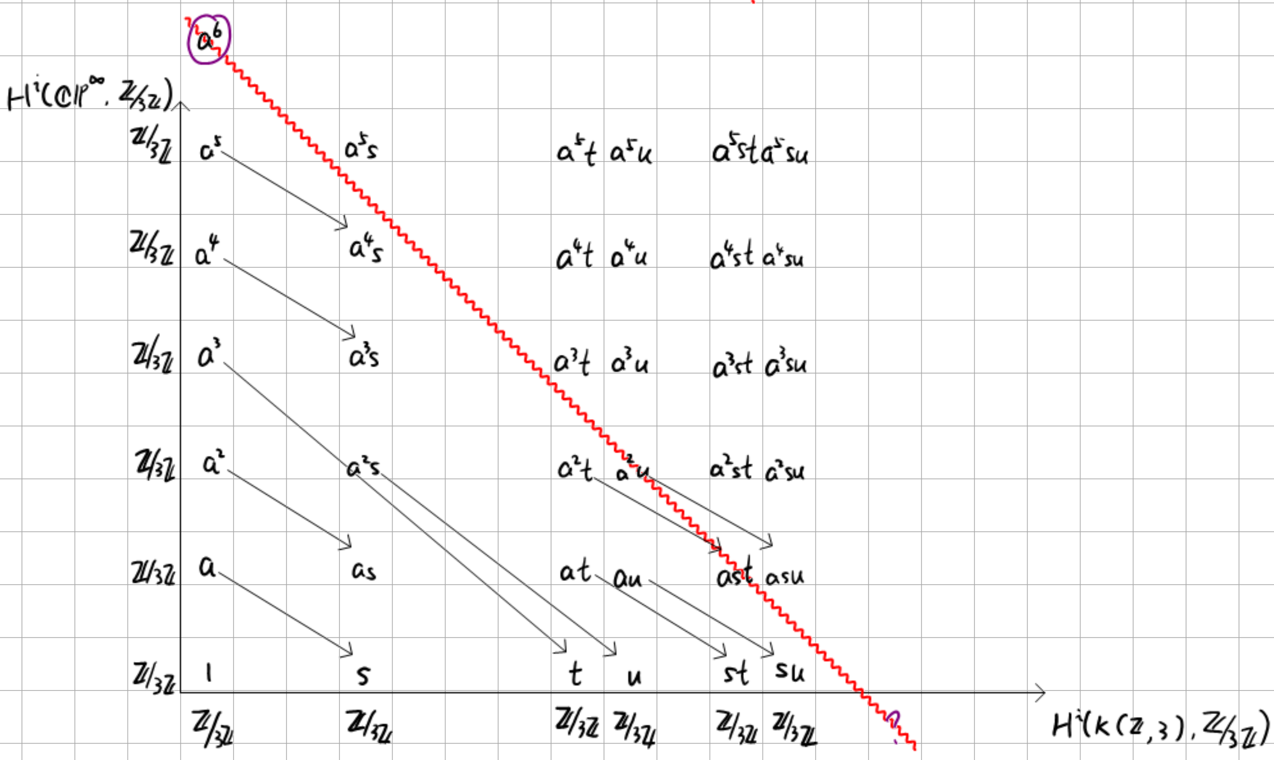
$$H^*(Gr_n(\mathbb{R}^\infty);\mathbb{Z}[\tfrac{1}{2}]) \cong \mathbb{Z}[\tfrac{1}{2}][p_1,\cdots,p_{[\frac{n}{2}]}] \qquad \deg p_i = 4i$$

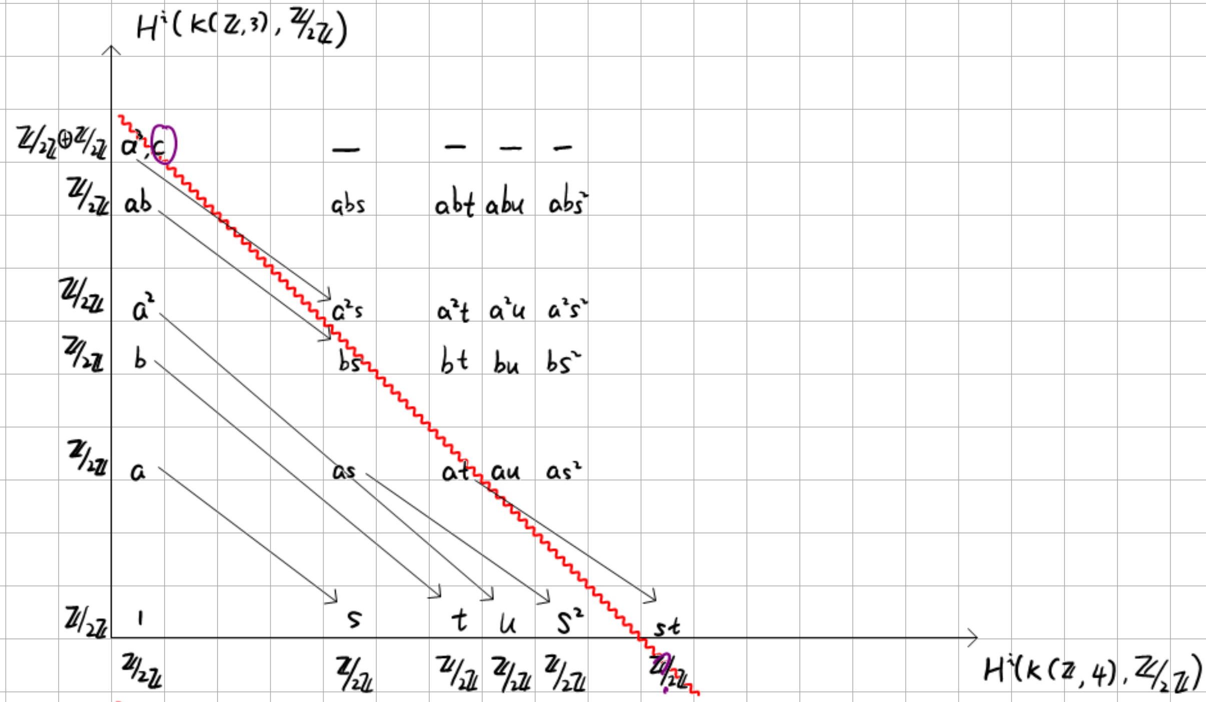
Rmk. This also gives us a way to define Chern class, SW class & Pontrjagin class.

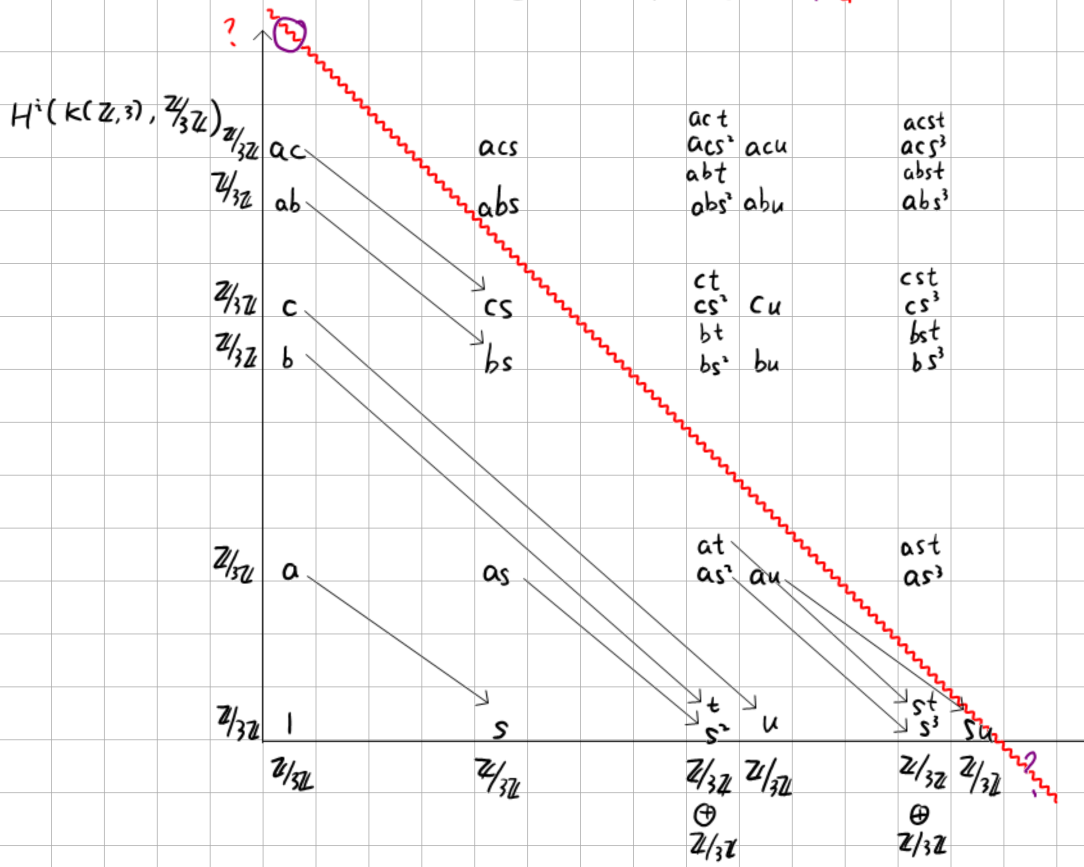
https://mathoverflow.net/questions/195398/integral-or-rational-cohomology-of-real-grassmannians
https://www.math.stonybrook.edu/~markmclean/MAT566/lecture7.pdf


H^i(CP^∞, Z)
H^j(K(Z,3), Z)
Z/3Z
Z/5Z
Z/2Z
Z/3Z
1
a
a²
a³
a⁴
a⁵
s
s²
t
s³
u
st
s⁴
su
Z
Z
Z/2Z
Z/3Z
Z/2Z
Z/2Z
Z/3Z
Z/5Z ⊕ Z/2Z
Z/2Z
6
5
4
3
2
?


H^i(CP^∞, Z/2Z)
H^j(K(Z,3), Z/2Z)
1
a
a²
a³
a⁴
a⁵
s
t
s²
st
s³
Z/2Z
?


H^i(CP^∞, Z/3Z)
H^j(K(Z,3), Z/3Z)
a⁶
1
a
a²
a³
a⁴
a⁵
s
t
u
st
su
Z/3Z
?

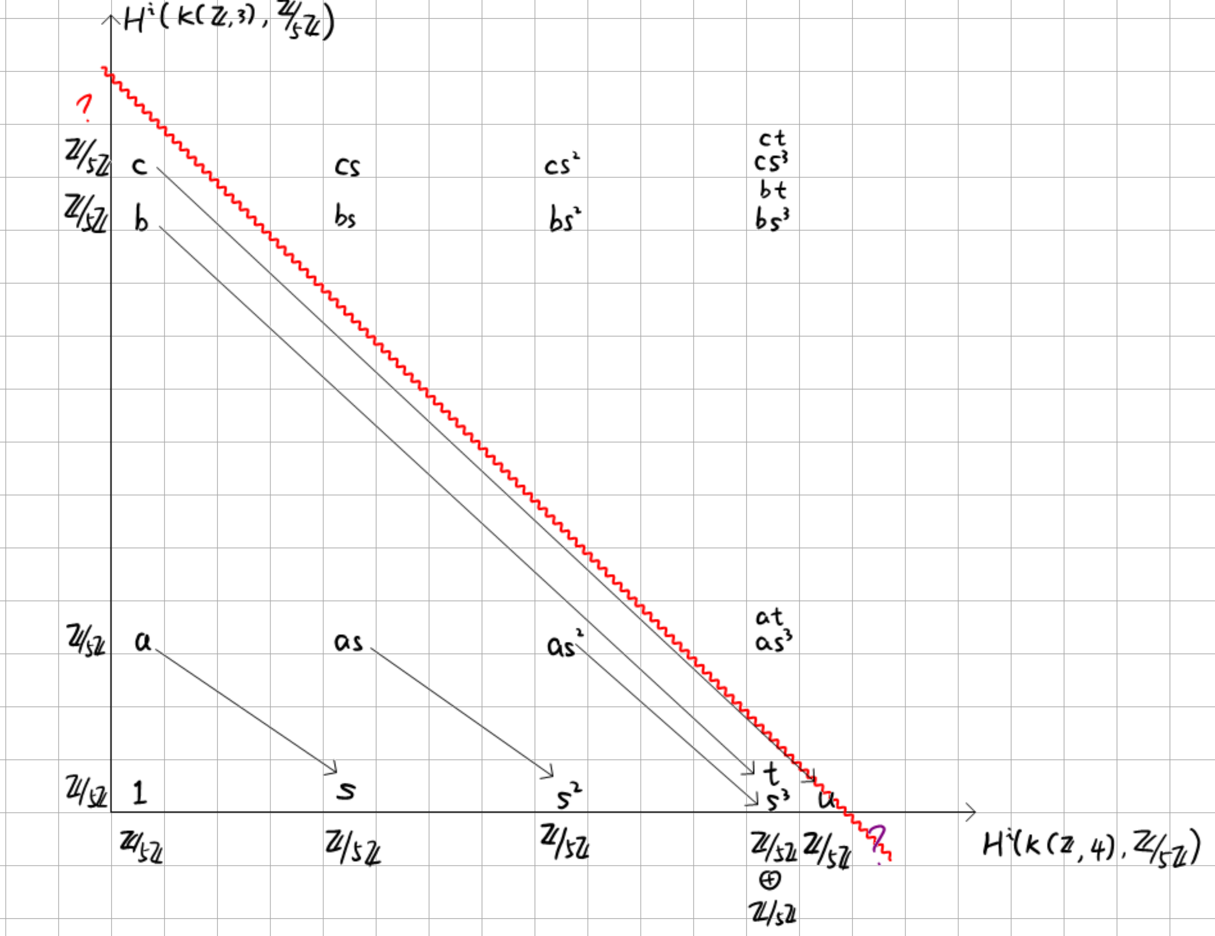




$H^i(K(\mathbb{Z},4), \mathbb{Z}/3\mathbb{Z})$

$H^8(K(\mathbb{Z},4), \mathbb{Z}/2\mathbb{Z}) \neq \mathbb{Z}/9\mathbb{Z}$

by considering $H^*(K(\mathbb{Z},4), \mathbb{Q})$



$\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}/2\mathbb{Z}$ $\mathbb{Z}$ $\mathbb{Z}/3\mathbb{Z}$. ? $H^i(K(\mathbb{Z},4), \mathbb{Z})$

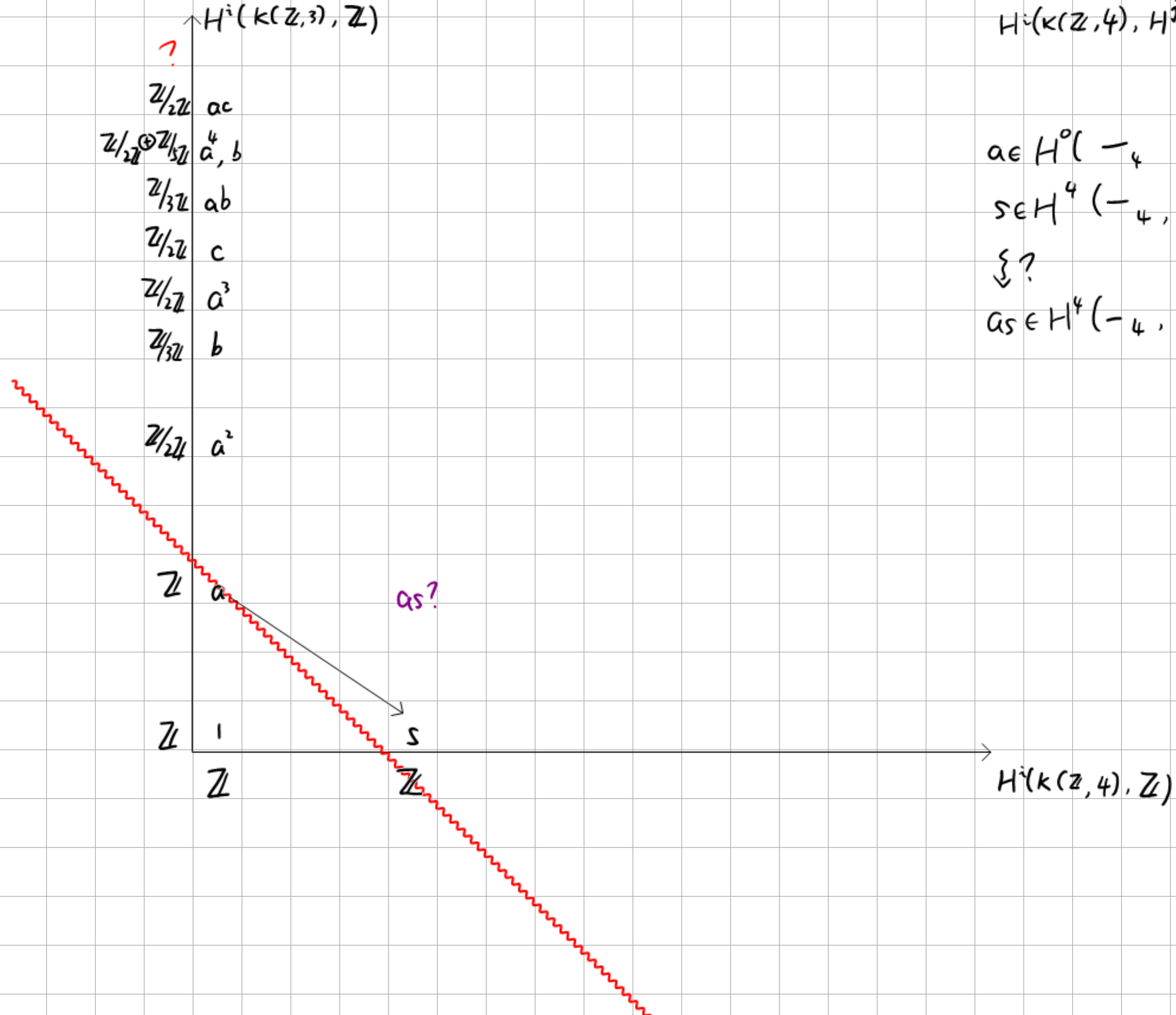
$H^i(K(\mathbb{Z},4), H^j(K(\mathbb{Z},3), \mathbb{Z}))$

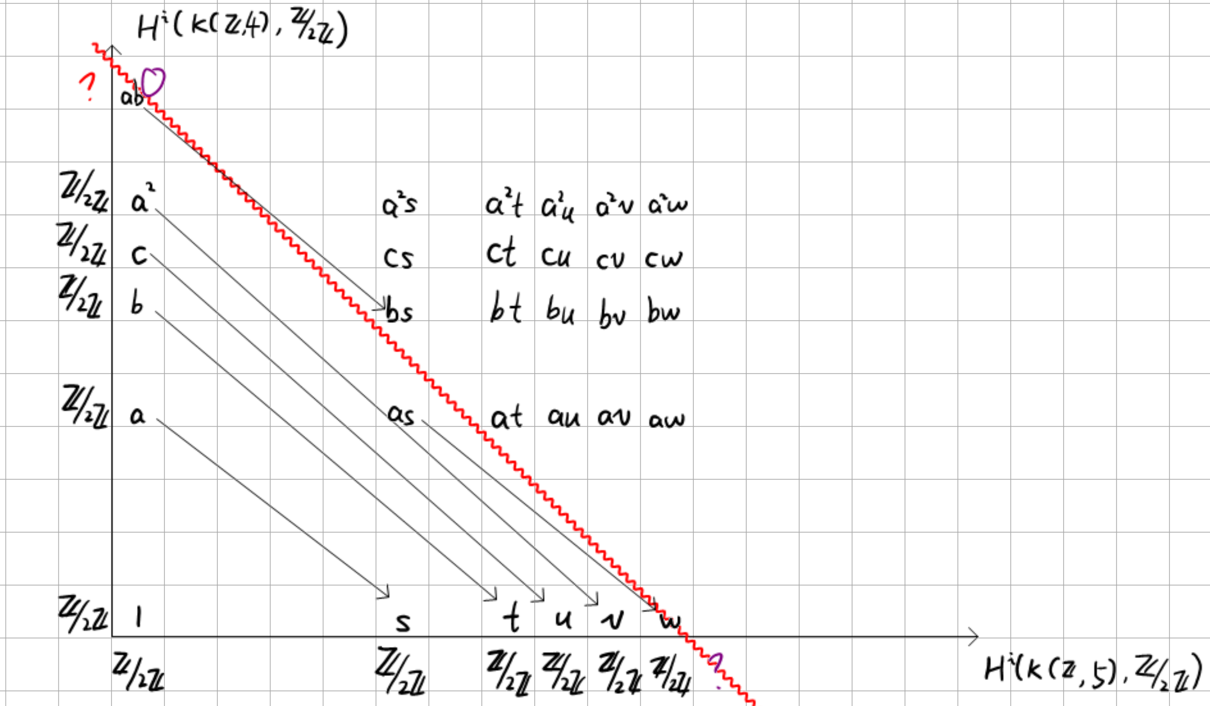
$a \in H^0(-_4, H^3(-_3, \mathbb{Z}))$

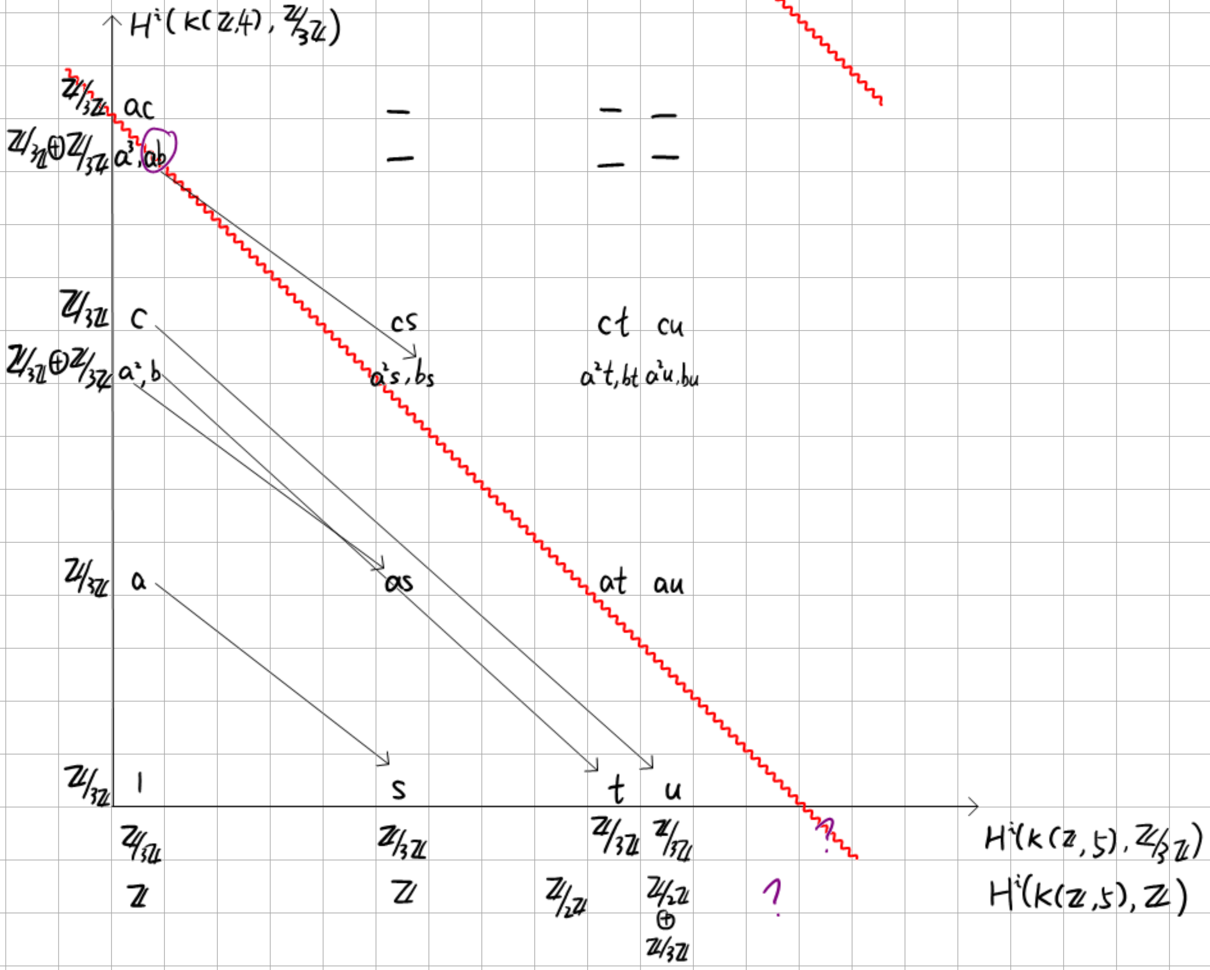
$s \in H^4(-_4, H^0(-_3, \mathbb{Z}))$

$\rightsquigarrow$?

$as \in H^4(-_4, H^3(-_3, \mathbb{Z}))$



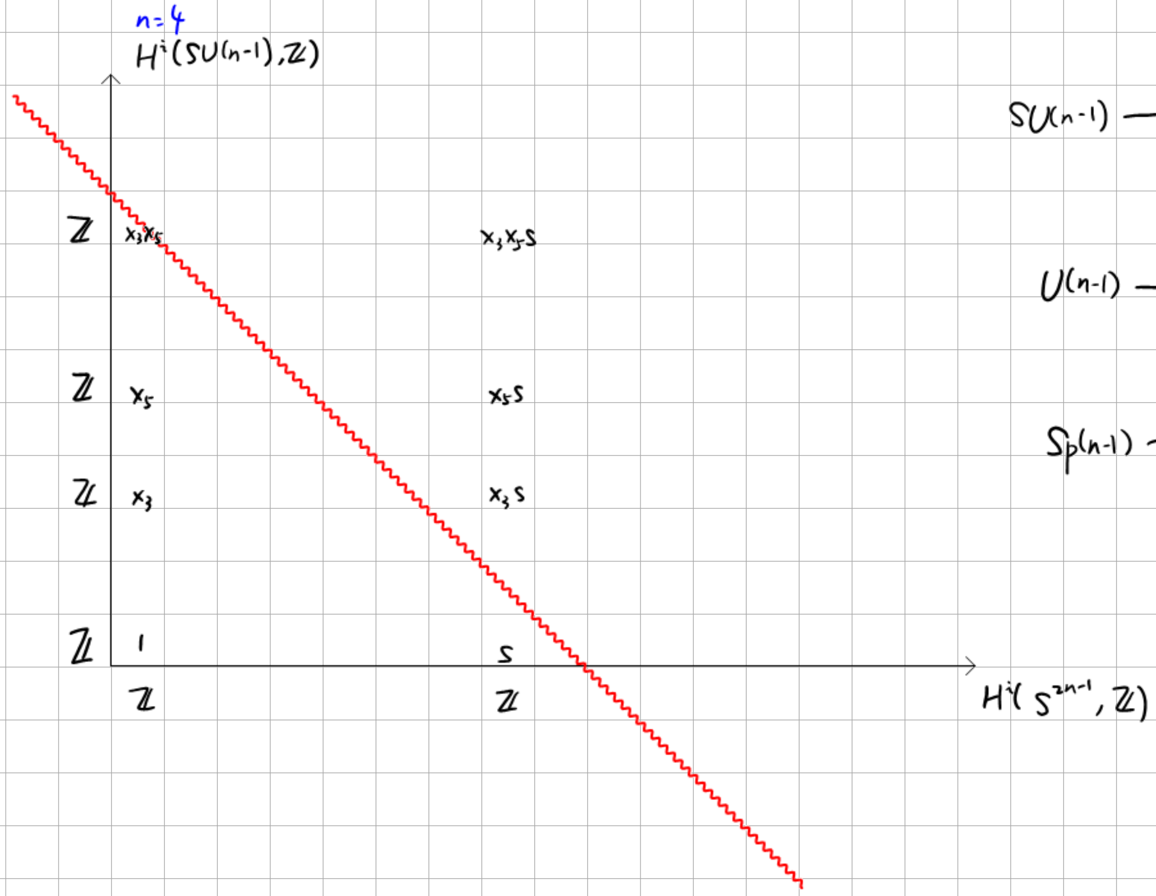




Conclusion:

$n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	$\mathbb{Z}$	$\mathbb{Z}$													
2	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
3	$\mathbb{Z}$			$\mathbb{Z}$			$\mathbb{F}_2$		$\mathbb{F}_3$	$\mathbb{F}_2$	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{Z}/10\mathbb{Z}$	$\mathbb{F}_2$	?
4	$\mathbb{Z}$				$\mathbb{Z}$			$\mathbb{F}_2$	$\mathbb{Z}$	$\mathbb{F}_3$		?			
5	$\mathbb{Z}$					$\mathbb{Z}$			$\mathbb{F}_2$		$\mathbb{Z}/6\mathbb{Z}$		?		

$H^i(K(\mathbb{Z},n),\mathbb{Z})$



$$\begin{tikzcd} SU(n-1) \arrow[r] & SU(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

$$\begin{tikzcd} U(n-1) \arrow[r] & U(n) \arrow[d] \\ & S^{2n-1} \end{tikzcd}$$

$$\begin{tikzcd} Sp(n-1) \arrow[r] & Sp(n) \arrow[d] \\ & S^{4n-1} \end{tikzcd}$$