

Eine Woche, ein Beispiel

4.27. homomorphism between Jacobians

[2025.04.20] provides us with many examples and references, and here we do things more theoretically.

Idea:

$$\text{Jac}(C) = \underbrace{H^0(C; \omega_C)^*}_{\text{linear part coherent}} / \underbrace{H_1(C; \mathbb{Z})}_{\text{lattice part constant}}$$

To understand $\text{Jac}(C)$, we need to understand these two parts separately.

For a morphism between two sm proj curves / \mathbb{C} :

$$f: \tilde{C} \longrightarrow C$$

$$\begin{aligned} N_{f,a}: H^0(\tilde{C}; \omega_{\tilde{C}})^* &\longrightarrow H^0(C; \omega_C)^* \\ N_{f,r}: H_1(\tilde{C}; \mathbb{Z}) &\longrightarrow H_1(C; \mathbb{Z}) \\ N_{f,*}: \text{Jac}(\tilde{C}) &\longrightarrow \text{Jac}(C) \end{aligned}$$

$$\begin{aligned} (f^*)_a: H^0(C; \omega_C)^* &\longrightarrow H^0(\tilde{C}; \omega_{\tilde{C}})^* \\ (f^*)_r: H_1(C; \mathbb{Z}) &\longrightarrow H_1(\tilde{C}; \mathbb{Z}) \\ f^*: \text{Jac}(C) &\longrightarrow \text{Jac}(\tilde{C}) \end{aligned}$$

cohom pullback

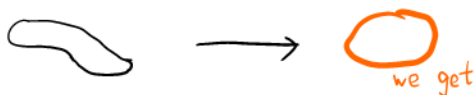
sheaf
origin

$$\begin{aligned} \omega_{\tilde{C}} &\longleftarrow f^* \omega_C \\ f: \pi_{\tilde{C}}^! \mathbb{Z} &\longrightarrow \pi_C^! \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \omega_C &\longleftarrow f_! \omega_{\tilde{C}} \\ \underline{\mathbb{Z}}_C &\longrightarrow f_* \underline{\mathbb{Z}}_{\tilde{C}} \end{aligned}$$

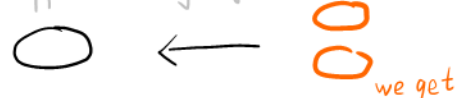
geometric
picture

$$g(f(w))d(f(w)) \longleftarrow g(z)dz$$



$$[q] \longmapsto [f(q)]$$

$$\sum_{f(w)=z} g(w)dz \longleftarrow g(w)dw$$



$$[p] \longmapsto \sum_{f(q)=p} [q]$$

Ex. Show that

$$N_{mf} \circ f^* = [\deg f]: \text{Jac}(C) \longrightarrow \text{Jac}(C)$$

Also,

$$\begin{aligned} N_{mf,a} \circ (f^*)_a &= \deg f \cdot \text{Id}_{H^0(C; \omega_C)^*} \\ N_{mf,r} \circ (f^*)_r &= \deg f \cdot \text{Id}_{H_1(C; \mathbb{Z})} \end{aligned}$$

Hint: use Poincaré duality.

Notations

For an abelian variety A/\mathbb{C} , we want to define

$$t_a, \phi_L, \psi_L, K(L), \Lambda(L), e(L), \mathcal{D}_L, S(Z, W), \delta(Z, W)$$

for $L \in \text{Pic}(A)$, $a \in A$, $Z, W \subset A$ with complementary dim.

$$\begin{array}{ccc} t_a: A & \longrightarrow & A \\ x & \longmapsto & x+a \end{array} \qquad \begin{array}{ccc} \phi_L: A & \longrightarrow & \hat{A} \\ x & \longmapsto & t_x^* L \otimes L^{-1} \end{array}$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & K(L) & \longrightarrow & A & \xrightarrow{\phi_L} & \hat{A} \longrightarrow \text{coker } \phi_L \longrightarrow 0 \\ & & \parallel & & \parallel & & \\ & & \Lambda(L)/\Lambda & & \mathbb{C}^n/\Lambda & & \end{array}$$

[BL04, Prop 2.4.8]

$$\begin{array}{ccc} L \text{ is nondegenerate} & \iff & \# K(L) < +\infty \\ H = \langle -, - \rangle \text{ nondeg} & \iff & \text{coker } \phi_L = 0 \end{array}$$

[BL04, 4.1, 5.1]

When L is pos def, $c_1(L) \in H^2(A; \mathbb{Z})$ is called a polarization
In this case, $\# K(L) < +\infty$ & $\text{coker } \phi_L = 0$, let

$$\begin{array}{l} K(L) = \mathbb{Z}/d_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/d_n\mathbb{Z} \quad d_1 | \dots | d_n, \\ \text{denote } e(L) = d_n, \text{ then we can define} \end{array}$$

$$\begin{array}{ccc} \psi_L: \hat{A} & \longrightarrow & A \\ \phi_L(x) & \longmapsto & e(L) \cdot x \end{array}$$

Check: ① ψ_L is well-defined,

$$\text{② } \psi_L \circ \phi_L = [e(L)]_A, \quad \phi_L \circ \psi_L = [e(L)]_{\hat{A}}.$$

For $f: A \longrightarrow A$, $\mathcal{L} \in \text{Pic}(A)$,

$$D_{\mathcal{L}}: \text{End}(A) \longrightarrow \text{Pic}(A)$$

$$f \longmapsto (f + \text{Id}_A)^* \mathcal{L} \otimes f^* \mathcal{L}^{-1} \otimes \mathcal{L}^{-1}$$

[BL04, 5.4] For $Z, W \in A$ with $Z \cdot W = \sum_i [x_i] \in \text{CH}_0(A)$,
we define

$$S(Z, W) := \sum_i x_i \in A$$

$$\begin{array}{ll} S(Z, W): A \longrightarrow A & x \longmapsto S(Z \cdot (t_x^* W - W)) \\ \text{e.p. } \delta(Z, c_1(\mathcal{L})): A \longrightarrow A & x \longmapsto S(Z \cdot c_1(\phi_{\mathcal{L}}(x))) \end{array}$$

Fact. $\delta(Z, W) \in \text{End}(A)$.

Given A/\mathbb{C} : abelian variety
 $\mathcal{L} \in \text{Pic}(A)$ a polarization
 $f \in \text{End}_{\mathbb{Q}}(A)$ $\iota_{A'}: A' \hookrightarrow A$

we want to define the Rosati involution $f' \in \text{End}(A)$
 and the norm-endomorphism $N_{A'}: A \rightarrow A' \subset A$

$$f': A \xrightarrow{\phi_{\mathcal{L}}} \hat{A} \xrightarrow{\hat{f}} \hat{A} \xrightarrow{\phi_{\mathcal{L}}^{-1}} A$$

$$e(A') = e(\mathcal{L}|_{A'}) \quad \psi_{\mathcal{L}|_{A'}} := e(A') \phi_{\mathcal{L}|_{A'}}^{-1}$$

$$N_{A'}: A \xrightarrow{\phi_{\mathcal{L}}} \hat{A} \xrightarrow{\hat{\iota}_{A'}} \hat{A}' \xrightarrow{\psi_{\mathcal{L}|_{A'}}} A'$$

Nondegeneracy

Z integral

[Ar85, p6]

$Z \subset \mathbb{P}^r$ is non-degenerate, if
 $Z \not\subset H$ for any hyperplane $H \subseteq \mathbb{P}^r$.

[BL04, 11.8, p341]

$Z \subset A$, Z generates A , if
 $Z \not\subset A'$ for any subabelian variety $A' \subsetneq A$.

[JKLM23, p10]

$Z \subset A$ is non-degenerate, if
 $\forall \pi: A \rightarrow A'$, $\pi|_Z$ is surj or gen fin map.

For Z divisor,

$Z \subset A$ nondeg $\Leftrightarrow \mathcal{O}(Z)$ is ample

For Z curve,

$Z \subset A$ nondeg $\Leftrightarrow Z$ generates A .

[JKLM23, p42]

$\xLeftrightarrow{Z \text{ smooth}}$

$N_{Z/A}$ ample