Eine Woche, ein Beispiel

3.30 Special loci of $\mathcal{A}_g$

Ref:

I learned this from Podelski Constantin and his articles:

[PCdeg]: The Gauss Map on Theta Divisors with Transversal A_1 Singularities. Journal of Singularities 27 (2024). https://doi.org/10.5427/jsing.2024.27g.
[PCgraph]: The Gauss map for bielliptic Prym varieties. https://doi.org/10.48550/arXiv.2311.13521.

[PP21]: Piotr Pragacz. Prym Varieties and Their Moduli. EMS Press, 2021. https://doi.org/10.4171/114-1/8.

1. a list of special loci
2. invariants

1. a list of special loci

For the moduli space (a stack, but we only care about its coarse moduli space)

$$\mathcal{A}_g = \{(A,\Theta) \text{ ppav} \mid \dim A = g\},$$

we have the following special loci:

$$N_k^{(g)} = \{(A,\Theta) \in \mathcal{A}_g \mid \dim \mathrm{Sing}(\Theta) \geq k\} \qquad \text{Andreotti-Mayer loci}$$

$$G_d^{(g)} = \{(A,\Theta) \in \mathcal{A}_g \mid \deg(\mathbb{P}\Lambda_\Theta \to Gr(g-1,g)) \leq d\} \qquad \text{Gauss loci}$$

$$\mathcal{A}^{\delta}_{t,g-t} = \left\{ (A,\Theta) \in \mathcal{A}_g \;\middle|\; \begin{array}{l} \exists\ A_1, A_2 \text{ abelian variety,} \\ f: A_1 \times A_2 \twoheadrightarrow A \text{ isogeny, s.t.} \\ (f \circ \iota_A)^* \mathcal{O}_A(\Theta) \text{ is of type } \delta \end{array} \right\}$$

$\delta = (a_1, \ldots, a_k)$

$a_i \mid a_{i+1}$

Apart from that, we also have special loci induced from curves.

$$\begin{array}{lll}
\mathcal{H}_g : & \text{moduli space of hyperelliptic curves} & \overset{cl}{\hookrightarrow} \mathcal{A}_g \\
\mathcal{M}_g : & \text{moduli space of curves} & \longrightarrow J_g \subset \mathcal{A}_g \\
\mathcal{R}_{g+1} : & \text{moduli space of Prym pairs } (C, \eta) & \longrightarrow P_g \subset \mathcal{A}_g
\end{array}$$

inj, not closed embeding, not inj for $\overline{\mathcal{M}}_g \longrightarrow J_g$

The following loci in the bielliptic Prym locus may use some notation in the end of the page.

we will omit it in the following def

$$\mathcal{BE}_g = \text{closure of } \{\mathrm{Prym}(\widetilde{C}/C) \mid \mathcal{C} \text{ is bielliptic of genus } g+1\}$$

$$\mathcal{E}_{g,t} = \{\mathrm{Prym}(\widetilde{C}/C) \in \mathcal{BE}_g \mid C' \longrightarrow E \text{ ramified at } 2t \text{ pts}\}$$

$t \leq g-t$ the other one at $(2g-2t)$ pts

$$\mathcal{E}^h_{g,t} = \{\mathrm{Prym}(\widetilde{C}/C) \in \mathcal{E}_{g,t} \mid \mathcal{C}' \text{ hyperelliptic}\}$$

$$\mathcal{S}_\lambda = \left\{ \mathrm{Prym}(\widetilde{C}/C) \in \mathcal{E}_{g,0} \;\middle|\; \begin{array}{l} E \text{ degenerates as } \cup \mathbb{P}^1 \\ \deg(p^{-1}E_i) = \lambda_i \end{array} \right\}$$

$|\lambda| = g$

smooth

$$\mathcal{E}'^k_{g,t} = \{\mathrm{Prym}(\widetilde{C}/C) \in \mathcal{E}'_{g,t} \mid N_{ad}(\widetilde{C}/C) = k\}$$

$$\mathcal{S}^k_\lambda = \{\mathrm{Prym}(\widetilde{C}/C) \in \mathcal{S}_\lambda \mid N_{ad}(\widetilde{C}/C) = k\}$$

$$\begin{tikzcd}
 & \widetilde{C} \arrow[dl, "\text{unramified}"'] \arrow[d] \arrow[dr] & \\
C \arrow[dr, "p"'] & C' \arrow[d] & C'' \arrow[dl] \\
 & E &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & \Xi & \\
W \arrow[r, "\alpha|_W"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \widetilde{\Xi} \arrow[ur, "h", "\substack{\text{gen} \\ 2:1}"'] \arrow[r, "Nm"] \arrow[d, hook] & \{\delta\} \arrow[d, hook] \\
\mathrm{Sym}^t C' \times \mathrm{Sym}^{g-t} C'' \arrow[r, "\alpha"] & \Theta' \times \Theta'' \arrow[r, "Nm"] & \mathrm{Pic}^g(E)
\end{tikzcd}$$

$$\begin{aligned}
N_{ad}(\widetilde{C}/C) &= \#\{\text{additional singularities of } \Theta\} \\
&= \deg\left((h \circ \alpha|_W)_*[W_{sing}]/2\right)
\end{aligned}$$

[PCgraph, p13]

2 invariants

dimension

	2	3	4	5	6	7	8	9	g
$\mathcal{H}_g$	3	5	7	9	11	13	15	17	$2g-1$
$\mathcal{J}_g$	3	6	9	12	15	18	21	24	$3g-3$
$\mathcal{P}_g$	3	6	10	15	18	21	24	27	$3g$
$\mathcal{A}_g$	3	6	10	15	21	28	36	45	$\frac{g(g+1)}{2}$
$\mathcal{E}_{g,t}$ $t=0$				10	12	14	16	18	$2g$
$\mathcal{E}_{g,t}$ $t\geq 1$			8	9	11	13	15	17	$2g-1$

deg Θ at generic pts

	2	3	4	5	6	7	8	9	g
$\mathcal{H}_g$	2	4	8	16	32	64	128	256	2^{g-1}
$\mathcal{J}_g$	2	6	20	70	252	924	3432	12870	$\binom{2g-2}{g-1}$
$\mathcal{P}_g$	2	6	24	120	688	4256	27520	183168	$2^{g-2}\left(C(g-1)+1\right)$
$\mathcal{A}_g$	2	6	24	120	720	5040	40320	362880	$g!$
$\mathcal{E}_{g,t}$ $t=2$			16	60	228	860			
$\mathcal{E}_{g,t}$ $t=0$			20	70	252	924	3432	12870	$\binom{2g-2}{g-1}$
$\mathcal{E}_{g,t}$ $t=1$									

dim SingΘ at generic pts

https://mathoverflow.net/questions/359494/singularities-of-the-theta-divisor-theta

	2	3	4	5	6	7	8	9	g
$\mathcal{H}_g$	sm	0	1	2	3	4	5	6	$g-3$
$\mathcal{J}_g$	sm	sm	0	1	2	3	4	5	$g-4$
$\mathcal{P}_g$	sm	sm	sm	sm					
$\mathcal{A}_g$	sm	sm	sm	sm	sm	sm	sm	sm	sm
possible type D counterexample		0	1	1	2	3	3	4	$\left\lfloor\frac{g-1}{2}\right\rfloor$