Eine Woche, ein Beispiel

8.1 irreducible representation of $GL_2(\mathbb{F}_q)$/$SL_2(\mathbb{F}_q)$, $PGL_2(\mathbb{F}_q)$/$PSL_2(\mathbb{F}_q)$

$q=p^k$, $p\neq 2$
p prime
$k\in\mathbb{N}^+$

Today: $q=3,5,7$

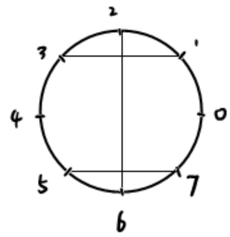
This mainly follows Fulton's book "Representation theory -- a first course".
However, I would recommend "http://math.mit.edu/~etingof/replect.pdf" and "https://zhuanlan.zhihu.com/p/24986434" because of the comfortable notation and less typos.

What can we get from the character table? Answer(in Chinese):https://zhuanlan.zhihu.com/p/27041214

Key ingredient in computing the table:
Ind: We use the Mackey formula many times.
#C_i: We use the conjugation action which is transitive, and then use the orbit-stabilizer theorem to compute it.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \{\pm 1\} \arrow[r] \arrow[d] & SL_2(\mathbb{F}_q) \arrow[r] \arrow[d] & PSL_2(\mathbb{F}_q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{F}_q^\times \arrow[r] \arrow[d, "(-)^2"] & GL_2(\mathbb{F}_q) \arrow[r] \arrow[d, "\det"] & PGL_2(\mathbb{F}_q) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \{a\in\mathbb{F}_q^\times \mid (\frac{a}{q})=1\} \arrow[r, hook] \arrow[d] & \mathbb{F}_q^\times \arrow[r, "(\frac{-}{q})"] \arrow[d] & \{\pm 1\} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

1. $q = 3$



$$\begin{tikzcd} & GL_2(\mathbb{F}_3) & \\ \mathbb{F}_3^\times \arrow[ur] \arrow[r, hook] \arrow[d, phantom, "\cong"] & \mathbb{F}_9^\times \arrow[u] \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\cong"] & \mathbb{F}_3(\sqrt{2})^\times \\ \mathbb{Z}/2\mathbb{Z} \arrow[r, hook] & \mathbb{Z}/8\mathbb{Z} & \end{tikzcd}$$

Id	$\begin{pmatrix}2&2\\1&2\end{pmatrix}$	$\begin{pmatrix}0&2\\1&0\end{pmatrix}$	$\begin{pmatrix}2&1\\2&2\end{pmatrix}$	$\begin{pmatrix}2&\\&2\end{pmatrix}$	$\begin{pmatrix}1&1\\2&1\end{pmatrix}$	$\begin{pmatrix}0&1\\2&0\end{pmatrix}$	$\begin{pmatrix}1&2\\1&1\end{pmatrix}$	$\begin{pmatrix}1&\\&1\end{pmatrix}$
1	$2+\sqrt{2}$	$\sqrt{2}$	$2+2\sqrt{2}$	2	$1+2\sqrt{2}$	$2\sqrt{2}$	$1+\sqrt{2}$	
0	1	2	3	4	5	6	7	8

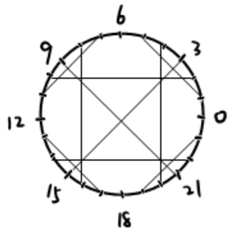
$GL_2(\mathbb{F}_3)/SL_2(\mathbb{F}_3)$ 48/24 $\rho = \zeta_3$

		scalar		parabolic		hyperbolic	elliptic		
		$\begin{pmatrix}1&\\&1\end{pmatrix}$ 1 $\begin{pmatrix}1&\\&1\end{pmatrix}$ 1	$\begin{pmatrix}2&\\&2\end{pmatrix}$ 1 $\begin{pmatrix}2&\\&2\end{pmatrix}$ 1	$\begin{pmatrix}1&1\\&1\end{pmatrix}$ 8 $\begin{pmatrix}1&1\\&1\end{pmatrix}$ $\begin{pmatrix}1&2\\&1\end{pmatrix}$ 4 4	$\begin{pmatrix}2&1\\&2\end{pmatrix}$ 8 $\begin{pmatrix}2&1\\&2\end{pmatrix}$ $\begin{pmatrix}2&2\\&2\end{pmatrix}$ 4 4	$\begin{pmatrix}1&\\&2\end{pmatrix}$ 12	$\begin{pmatrix}&2\\1&\end{pmatrix}$ 6 $\begin{pmatrix}&2\\1&\end{pmatrix}$ 6	$\begin{pmatrix}1&2\\1&1\end{pmatrix}$ 6	$\begin{pmatrix}2&2\\1&2\end{pmatrix}$ 6
1-dim		1 1 1	1 1 1	1 1 1 1	1 1 1 1	1 -1	1 1 1	1 -1	1 1
Principle series		3 3 3	3 3 3	0 0 0 0	0 0 0 0	1 -1	-1 -1 -1	-1 1	-1 1
	$V_{\lambda,\mathrm{Id}}$	4 2 2	-4 -2 -2	1 $-\rho$ $-\rho^2$ $-\rho^2$ $-\rho$	-1 ρ^2 ρ ρ ρ^2	0	0 0 0	0	0
Complementary series	X_φ X_{φ^5}	2 2 2	-2 -2 -2	-1 -1 -1 -1	1 1 1 1	0 0	0 0 0	$-\sqrt{2}i$ $\sqrt{2}i$	$-\sqrt{2}i$ $\sqrt{2}i$
	X_{φ^2}	2 1 1	2 1 1	-1 ρ ρ^2 ρ^2 ρ	-1 ρ^2 ρ ρ ρ^2	0	2 1 1	0	0

$PGL_2(\mathbb{F}_3)/PSL_2(\mathbb{F}_3) \cong S_4/A_4$ 24/12 $\rho = \zeta_3$

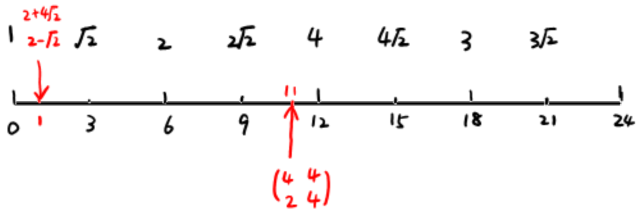
		scalar	parabolic	hyperbolic	elliptic	
		$\begin{pmatrix}1&\\&1\end{pmatrix}$ 1 $\begin{pmatrix}1&\\&1\end{pmatrix}$ 1	$\begin{pmatrix}1&1\\&1\end{pmatrix}$ 8 $\begin{pmatrix}1&1\\&1\end{pmatrix}$ $\begin{pmatrix}1&2\\&1\end{pmatrix}$ 4 4	$\begin{pmatrix}1&\\&2\end{pmatrix}$ 12	$\begin{pmatrix}&2\\1&\end{pmatrix}$ 6 $\begin{pmatrix}&2\\1&\end{pmatrix}$ 6	$\begin{pmatrix}1&2\\1&1\end{pmatrix}$ 6
1-dim		1 1 1	1 1 1 1	1 -1	1 1 1	1 -1
Principle series		3 3 3	0 0 0 0	1 -1	-1 -1 -1	-1 1
Complementary series	X_{φ^2}	2 1 1	-1 ρ ρ^2 ρ^2 ρ	0	2 1 1	0

2. $q=5$



$$
\begin{tikzcd}
 & GL_2(\mathbb{F}_5) & \\
\mathbb{F}_5^\times \arrow[ur] \arrow[r, hook] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{F}_{25}^\times \arrow[u, hook] \arrow[d, phantom, "\cong" rotate=90] & \cong \mathbb{F}_5(\sqrt{2})^\times \\
\mathbb{Z}/5\mathbb{Z} \arrow[r, hook] & \mathbb{Z}/24\mathbb{Z} &
\end{tikzcd}
$$

Id $\begin{pmatrix} & 2 \\ 1 & \end{pmatrix}$ $\begin{pmatrix} 2 & \\ & 2 \end{pmatrix}$ $\begin{pmatrix} & 4 \\ 2 & \end{pmatrix}$ $\begin{pmatrix} 4 & \\ & 4 \end{pmatrix}$ $\begin{pmatrix} & 3 \\ 4 & \end{pmatrix}$ $\begin{pmatrix} 3 & \\ & 3 \end{pmatrix}$ $\begin{pmatrix} & 1 \\ 3 & \end{pmatrix}$



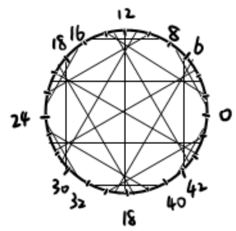
$GL_2(\mathbb{F}_5)$ / $SL_2(\mathbb{F}_5)$ 480/120 $\zeta = \zeta_{24}$

		scalar	parabolic	hyperbolic	elliptic
					3 8 4 11 1
		$\begin{pmatrix}1&\\&1\end{pmatrix}$ …4… $\begin{pmatrix}4&\\&4\end{pmatrix}$	$\begin{pmatrix}1&1\\&1\end{pmatrix}$ …4… $\begin{pmatrix}4&1\\&4\end{pmatrix}$	$\begin{pmatrix}1&\\&2\end{pmatrix}$ …6… $\begin{pmatrix}3&\\&4\end{pmatrix}$	$\begin{pmatrix}&2\\1&\end{pmatrix}$ …10… $\begin{pmatrix}4&4\\2&4\end{pmatrix}$ $\begin{pmatrix}2&3\\4&2\end{pmatrix}$
		1 … 1	24 … 24	30 … 30	20 … 20 20
		$\begin{pmatrix}1&\\&1\end{pmatrix}$ $\begin{pmatrix}4&\\&4\end{pmatrix}$	$\begin{pmatrix}1&1\\&1\end{pmatrix}$ $\begin{pmatrix}1&2\\&1\end{pmatrix}$ $\begin{pmatrix}4&1\\&4\end{pmatrix}$ $\begin{pmatrix}4&2\\&4\end{pmatrix}$	$\begin{pmatrix}2&\\&3\end{pmatrix}$	$\begin{pmatrix}2&4\\2&2\end{pmatrix}$ $\begin{pmatrix}3&4\\2&3\end{pmatrix}$
		1 1	12 12 12 12	30	20 20
1-dim representations		1 1 1 1	1 1 1 1	1 … 1	1 … 1 1
		1 -1 -1 1	1 -1 -1 1	i … i	-i … -i i
		1 1 1 1	1 1 1 1	-1 … -1	-1 … -1 -1
		1 -1 -1 1	1 -1 -1 1	-i … -i	i … i -i
		1 1	1 1 1 1	1	1 … 1
Principle series representations		5 5 5 5	0 0 0 0	1 … 1	-1 … -1 -1
		5 -5 -5 5	0 0 0 0	i … i	i … i -i
		5 5 5 5	0 0 0 0	-1 … -1	1 … 1 1
		5 -5 -5 5	0 0 0 0	-i … -i	-i … -i i
		5 5	0 0 0 0	1	-1 … -1
	$V_{\lambda,Id}$	6 6i -6i 6	1 i -i -1	1+i … -1-i	0 … 0 0
	$V_{\lambda^1,\lambda}$	6 -6i 6i 6	1 -i i -1	-1+i … 1-i	0 … 0 0
	V_{λ^3,λ^2}	6 6i -6i 6	1 i -i -1	-1-i … 1+i	0 … 0 0
	V_{Id,λ^3}	6 -6i 6i 6	1 -i i -1	1-i … -1+i	0 … 0 0
		6 6	1 1 -1 -1	0	0 … 0
	$V_{\lambda^2,Id}$	6 -6 -6 6	1 -1 -1 1	0 … 0	0 … 0 0
	$V_{\lambda^3,\lambda}$	6 6 6 6	1 1 1 1	0 … 0	0 … 0 0
		3 3	$\frac{1+\sqrt5}{2}$ $\frac{1-\sqrt5}{2}$ $\frac{1+\sqrt5}{2}$ $\frac{1-\sqrt5}{2}$	-1	0 0
		3 3	$\frac{1-\sqrt5}{2}$ $\frac{1+\sqrt5}{2}$ $\frac{1-\sqrt5}{2}$ $\frac{1+\sqrt5}{2}$	-1	0 0
Complementary series representations	X_φ	4 4i -4i -4	-1 -i i 1	0 … 0	0 … $-(\zeta^{11}+\zeta^7)$ $-(\zeta+\zeta^5)$
	X_{φ^7}	4 -4i 4i -4	-1 i -i 1	0 … 0	0 … $-(\zeta+\zeta^5)$ $-(\zeta^7+\zeta^{11})$
	$X_{\varphi^{13}}$	4 4i -4i -4	-1 -i i 1	0 … 0	0 … $-(\zeta^{19}+\zeta^{23})$ $-(\zeta^{13}+\zeta^{17})$
	$X_{\varphi^{19}}$	4 -4i 4i -4	-1 i -i 1	0 … 0	0 … $-(\zeta^{13}+\zeta^{17})$ $-(\zeta^{19}+\zeta^{23})$
		4 -4	-1 -1 1 1	0	1 -1
	X_{φ^2}	4 -4 -4 4	1 -1 -1 1	0 … 0	-2i … i -i
	X_{φ^8}	4 4 4 4	1 1 1 1	0 … 0	2 … 1 1
	$X_{\varphi^{14}}$	4 -4 -4 4	1 -1 -1 1	0 … 0	2i … -i i
	X_{φ^4}	4 4 4 4	1 1 1 1	0 … 0	-2 … -1 -1
		4 4	1 1 1 1	0	1 1
	X_{φ^3}	4 -4i -4i 4	-1 i -i 1	0 … 0	0 … 0 0
	X_{φ^9}	4 4i 4i 4	-1 -i i 1	0 … 0	0 … 0 0
		2 -2	$\frac{-1+\sqrt5}{2}$ $\frac{-1-\sqrt5}{2}$ $\frac{1-\sqrt5}{2}$ $\frac{1+\sqrt5}{2}$	0	-1 1
		2 -2	$\frac{-1-\sqrt5}{2}$ $\frac{-1+\sqrt5}{2}$ $\frac{1+\sqrt5}{2}$ $\frac{1-\sqrt5}{2}$	0	-1 1

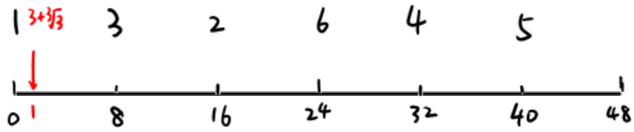
$PGL_2(\mathbb{F}_5)/PSL_2(\mathbb{F}_5) \cong S_5/A_5$ $\quad$ $120/60$ $\quad$ $\xi = \xi_{24}$

		scalar	parabolic	hyperbolic		elliptic		
		$\begin{pmatrix}1&\\&1\end{pmatrix}$ 1 $\begin{pmatrix}1&\\&1\end{pmatrix}$ 1	$\begin{pmatrix}1&1\\&1\end{pmatrix}$ 24 $\begin{pmatrix}1&1\\&1\end{pmatrix}$ $\begin{pmatrix}1&2\\&1\end{pmatrix}$ 12 12	$\begin{pmatrix}1&\\&2\end{pmatrix}$ 30	$\begin{pmatrix}1&\\&4\end{pmatrix}$ 30 $\begin{pmatrix}2&\\&3\end{pmatrix}$ 30	$\begin{pmatrix}&2\\1&\end{pmatrix}$ 20	$\begin{pmatrix}1&2\\1&1\end{pmatrix}$ 20 $\begin{pmatrix}2&4\\2&2\end{pmatrix}$ 20	$\begin{pmatrix}2&2\\1&2\end{pmatrix}$ 20
1-dim		1 1 1	1 1 1 1	1 -1	1 1 1	1 -1	1 1 1	1 -1
Principle series		5 5 5	0 0 0 0	1 -1	1 1 1	-1 1	-1 -1 1	-1 1
	$V_{\chi^3,\lambda}$	6 3 3	1 $\frac{1+\sqrt{5}}{2}$ $\frac{1-\sqrt{5}}{2}$ $\frac{1-\sqrt{5}}{2}$ $\frac{1+\sqrt{5}}{2}$	0	-2 -1 -1	0	0 0 0	0
Complementary series	X_{φ^8} X_{φ^4}	4 4 4	1 1 1 1	0 0	0 0 0	2 -2	1 1 1	1 -1

2. $q=7$



$$
\begin{tikzcd}
 & GL_2(\mathbb{F}_7) \\
\mathbb{F}_7^\times \arrow[ur] \arrow[r, hook] & \mathbb{F}_{49}^\times \arrow[u, hook] \cong \mathbb{F}_7(\sqrt{2})^\times \\
\mathbb{Z}/6\mathbb{Z} \arrow[u, phantom, "\cong"] \arrow[r, hook] & \mathbb{Z}/48\mathbb{Z} \arrow[u, phantom, "\cong"]
\end{tikzcd}
$$



$GL_2(\mathbb{F}_7)/SL_2(\mathbb{F}_7)$ 2016/336 $\rho=\zeta_6$ $\zeta=\zeta_{48}$

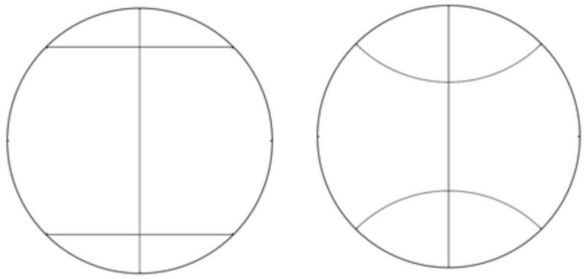
4 18 6 12 26 1

		scalar	parabolic	hyperbolic	elliptic
		$\begin{pmatrix}1&\\&1\end{pmatrix} \overset{6}{\cdots} \begin{pmatrix}6&\\&6\end{pmatrix}$ $1 \quad \cdots \quad 1$ $\begin{pmatrix}1&\\&1\end{pmatrix} \quad \begin{pmatrix}6&\\&6\end{pmatrix}$ $1 \quad 1$	$\begin{pmatrix}1&1\\&1\end{pmatrix} \overset{6}{\cdots} \begin{pmatrix}6&1\\&6\end{pmatrix}$ $48 \quad \cdots \quad 48$ $\begin{pmatrix}1&1\\&1\end{pmatrix}\begin{pmatrix}1&3\\&1\end{pmatrix} \quad \begin{pmatrix}6&1\\&6\end{pmatrix}\begin{pmatrix}6&3\\&6\end{pmatrix}$ $24 \quad 24 \quad 24 \quad 24$	$\begin{pmatrix}1&\\&2\end{pmatrix} \overset{15}{\cdots} \begin{pmatrix}5&\\&6\end{pmatrix}$ $56 \quad \cdots \quad 56$ $\begin{pmatrix}2&\\&4\end{pmatrix}\begin{pmatrix}3&\\&5\end{pmatrix}$ $56 \quad 56$	$\begin{pmatrix}&3\\1&\end{pmatrix} \overset{21}{\cdots} \begin{pmatrix}6&2\\3&6\end{pmatrix}\begin{pmatrix}3&2\\3&3\end{pmatrix}$ $42 \quad \cdots \quad 42 \quad 42$ $\begin{pmatrix}2&3\\1&2\end{pmatrix}\begin{pmatrix}5&3\\1&5\end{pmatrix}\begin{pmatrix}0&2\\3&0\end{pmatrix}$ $42 \quad 42 \quad 42$
1-dim representations		$1 \quad 1 \cdots 1$ $1 \quad \rho^4 \cdots 1$ $1 \quad \rho^2 \cdots 1$ $1 \quad 1 \cdots 1$ $1 \quad \rho^4 \cdots 1$ $1 \quad \rho^2 \cdots 1$ $1 \quad 1$	$1 \quad 1 \cdots 1$ $1 \quad \rho^4 \cdots 1$ $1 \quad \rho^2 \cdots 1$ $1 \quad 1 \cdots 1$ $1 \quad \rho^4 \cdots 1$ $1 \quad \rho^2 \cdots 1$ $1 \quad 1 \quad 1 \quad 1$	$1 \cdots 1$ $\rho^2 \cdots \rho^2$ $\rho^4 \cdots \rho^4$ $1 \cdots 1$ $\rho^2 \cdots \rho^2$ $\rho^4 \cdots \rho^4$ $1 \quad 1$	$1 \cdots 1 \quad 1$ $\rho^4 \cdots \rho^2 \quad \rho$ $\rho^2 \cdots \rho^4 \quad \rho^2$ $1 \cdots 1 \quad -1$ $\rho^4 \cdots \rho^2 \quad \rho^4$ $\rho^2 \cdots \rho^4 \quad \rho^5$ $1 \quad 1 \quad 1$
Principle series representations	$\vdots 6$	$7 \quad 7 \cdots 7$ $\vdots$ $7 \quad 7$	$0 \quad 0 \cdots 0$ $\vdots$ $0 \quad 0 \quad 0 \quad 0$	$1 \cdots 1$ $\vdots$ $1 \quad 1$	$-1 \cdots -1 \quad -1$ $\vdots$ $-1 \quad -1 \quad -1$
	$V_{\lambda,Id}$ $\vdots 6$	$8 \quad 8\rho^2 \cdots -8$ $\vdots$ $8 \quad -8$	$1 \quad \rho^2 \cdots -1$ $\vdots$ $1 \quad 1 \quad -1 \quad -1$	$\rho \cdots -\rho$ $\vdots$ $-1 \quad 1$	$0 \cdots 0 \quad 0$ $\vdots$ $0 \quad 0 \quad 0$
	$V_{\lambda^2,Id}$ $\vdots 6$	$8 \quad 8\rho^4 \cdots 8$ $\vdots$ $8 \quad 8$	$1 \quad \rho^4 \cdots 1$ $\vdots$ $1 \quad 1 \quad 1 \quad 1$	$\rho^5 \cdots \rho^5$ $\vdots$ $-1 \quad -1$	$0 \cdots 0 \quad 0$ $\vdots$ $0 \quad 0 \quad 0$
	$V_{\lambda^3,Id}$ $\vdots 3$	$8 \quad 8 \cdots -8$ $\vdots$ $4 \quad -4$ $4 \quad -4$	$1 \quad 1 \cdots -1$ $\vdots$ $\frac{1+\sqrt{7}i}{2} \quad \frac{1-\sqrt{7}i}{2} \quad \frac{-1+\sqrt{7}i}{2} \quad \frac{-1-\sqrt{7}i}{2}$ $\frac{1-\sqrt{7}i}{2} \quad \frac{1+\sqrt{7}i}{2} \quad \frac{-1-\sqrt{7}i}{2} \quad \frac{-1+\sqrt{7}i}{2}$	$2 \cdots -2$ $\vdots$ $1 \quad -1$ $1 \quad -1$	$0 \cdots 0 \quad 0$ $\vdots$ $0 \quad 0 \quad 0$ $0 \quad 0 \quad 0$
Complementary series representations	X_{φ} $\vdots 6$	$6 \quad 6\rho^2 \cdots -6$ $\vdots$ $6 \quad -6$	$-1 \quad -\rho^2 \cdots 1$ $\vdots$ $-1 \quad -1 \quad 1 \quad 1$	$0 \cdots 0$ $\vdots$ $0 \quad 0$	$0 \cdots -(\zeta^{26}+\zeta^{38}) \quad -(\zeta+\zeta^{7})$ $\vdots$ $\sqrt{2} \quad -\sqrt{2} \quad 0$
	X_{φ^2} $\vdots 6$	$6 \quad 6\rho^4 \cdots 6$ $\vdots$ $6 \quad 6$	$-1 \quad -\rho^4 \cdots -1$ $\vdots$ $-1 \quad -1 \quad 1 \quad 1$	$0 \cdots 0$ $\vdots$ $0 \quad 0$	$-2\rho \cdots 0 \quad -(\zeta^2+\zeta^{14})$ $\vdots$ $0 \quad 0 \quad 2$
	X_{φ^3} $\vdots 6$	$6 \quad 6 \cdots -6$ $\vdots$ $6 \quad -6$	$-1 \quad -1 \cdots 1$ $\vdots$ $-1 \quad -1 \quad 1 \quad 1$	$0 \cdots 0$ $\vdots$ $0 \quad 0$	$0 \cdots \sqrt{2} \quad -(\zeta^3+\zeta^{16})$ $\vdots$ $-\sqrt{2} \quad \sqrt{2} \quad 0$
	X_{φ^4} $\vdots 3$	$6 \quad 6\rho^2 \cdots 6$ $\vdots$ $3 \quad 3$ $3 \quad 3$	$-1 \quad -\rho^2 \cdots -1$ $\vdots$ $\frac{-1+\sqrt{7}i}{2} \quad \frac{-1-\sqrt{7}i}{2} \quad \frac{-1-\sqrt{7}i}{2} \quad \frac{-1+\sqrt{7}i}{2}$ $\frac{-1-\sqrt{7}i}{2} \quad \frac{-1+\sqrt{7}i}{2} \quad \frac{-1+\sqrt{7}i}{2} \quad \frac{-1-\sqrt{7}i}{2}$	$0 \cdots 0$ $\vdots$ $0 \quad 0$ $0 \quad 0$	$-2\rho^2 \cdots -2\rho \quad 0$ $\vdots$ $1 \quad 1 \quad -1$ $1 \quad 1 \quad -1$

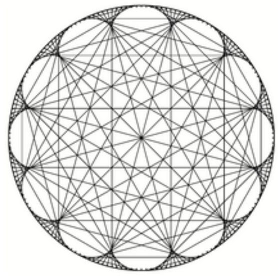
$PGL_2(\mathbb{F}_7)/PSL_2(\mathbb{F}_7)$ $336/168$ $\rho=\zeta_6$ $\zeta=\zeta_{48}$

		scalar	parabolic		hyperbolic			elliptic			
									41	18	45
		$\begin{pmatrix}1&\\&1\end{pmatrix}$	$\begin{pmatrix}1&1\\&1\end{pmatrix}$		$\begin{pmatrix}1&\\&2\end{pmatrix}$	$\begin{pmatrix}1&\\&3\end{pmatrix}$	$\begin{pmatrix}1&\\&6\end{pmatrix}$	$\begin{pmatrix}&3\\1&\end{pmatrix}$	$\begin{pmatrix}1&3\\1&1\end{pmatrix}$	$\begin{pmatrix}2&3\\1&2\end{pmatrix}$	$\begin{pmatrix}3&3\\1&3\end{pmatrix}$
		1	48		56	56	28	21	42	42	42
	(red)	$\begin{pmatrix}1&\\&1\end{pmatrix}$	$\begin{pmatrix}1&1\\&1\end{pmatrix}$	$\begin{pmatrix}1&3\\&1\end{pmatrix}$	$\begin{pmatrix}2&\\&4\end{pmatrix}$			$\begin{pmatrix}&2\\3&\end{pmatrix}$		$\begin{pmatrix}2&3\\1&2\end{pmatrix}$	
	(red)	1	24	24	56			21		42	
1-dim		1	1		1	1	1	1	1	1	1
		1	1		1	-1	-1	1	-1	1	-1
	(red)	1	1	1	1			1		1	
Principle series		7	0		1	1	1	-1	-1	-1	-1
		7	0		1	-1	-1	-1	1	-1	1
	(red)	7	0	0	1			-1		-1	
	V_{λ^2,λ^4}	8	1		-1	-1	2	0	0	0	0
	V_{λ^1,λ^5}	8	1		-1	1	-2	0	0	0	0
	(red)	8	1	1	-1			0		0	
Complementary series	$X_{\varphi^{18}}$	6	-1		0	0	0	2	$\sqrt{2}$	0	$-\sqrt{2}$
	$X_{\varphi^{6}}$	6	-1		0	0	0	2	$-\sqrt{2}$	0	$\sqrt{2}$
	(red)	6	-1	-1	0			2		0	
	$X_{\varphi^{12}}$	6	-1		0	0	0	-2	0	2	0
	(red)	3	$\frac{-1+\sqrt{7}i}{2}$	$\frac{-1-\sqrt{7}i}{2}$	0			1		1	
	(red)	3	$\frac{-1-\sqrt{7}i}{2}$	$\frac{-1+\sqrt{7}i}{2}$	0			1		1	

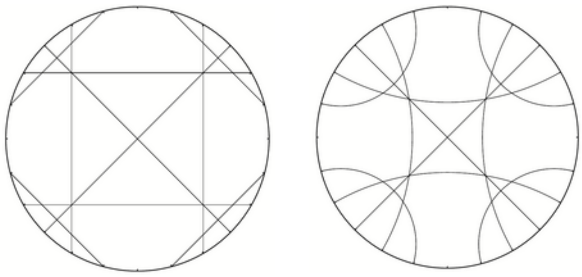
Bonus: Frobenius map in $\mathbb{F}_{q^2}/\mathbb{F}_q$



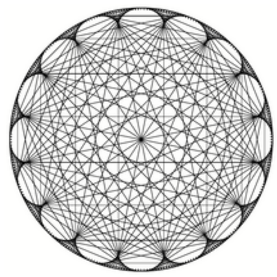
$\mathbb{F}_9/\mathbb{F}_3$



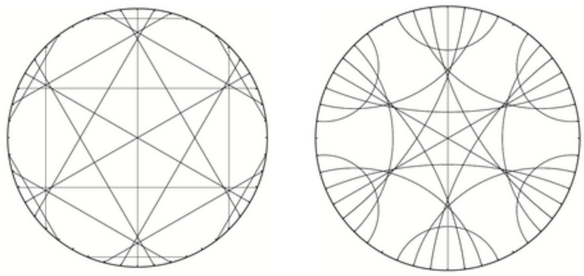
$\mathbb{F}_{169}/\mathbb{F}_{13}$



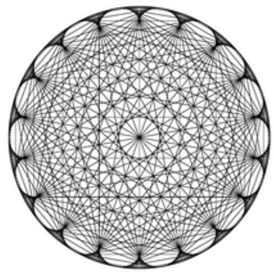
$\mathbb{F}_{25}/\mathbb{F}_5$



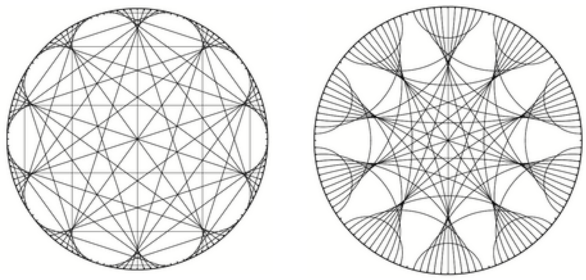
$\mathbb{F}_{289}/\mathbb{F}_{17}$



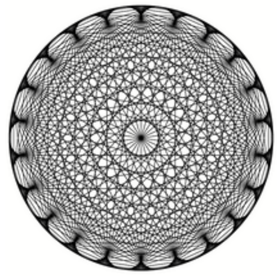
$\mathbb{F}_{49}/\mathbb{F}_7$



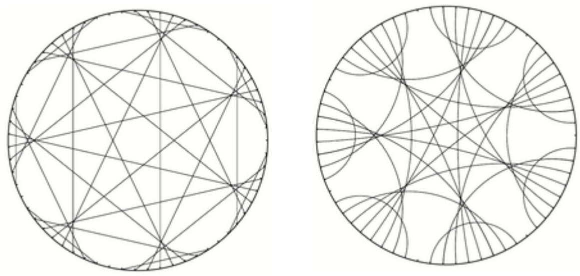
$\mathbb{F}_{361}/\mathbb{F}_{19}$



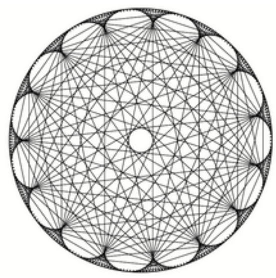
$\mathbb{F}_{121}/\mathbb{F}_{11}$



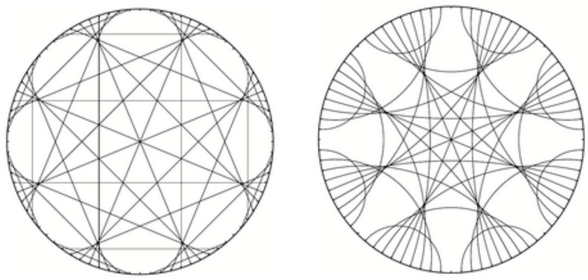
$\mathbb{F}_{529}/\mathbb{F}_{23}$



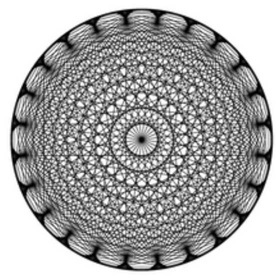
$\mathbb{F}_{64}/\mathbb{F}_8$



$\mathbb{F}_{256}/\mathbb{F}_{16}$



$\mathbb{F}_{81}/\mathbb{F}_9$



$\mathbb{F}_{625}/\mathbb{F}_{25}$