Eine Woche, ein Beispiel
8.28 global field

This note mainly follows [现代数学基础12-数论I：Fermat的梦想和类域论- 日 加藤和也 & 黑川信重-胥鸣伟 & 印林生(译)].
Another reference for complement(and also for non-Chinese reader):
[MIT] https://math.mit.edu/classes/18.785/2015fa/lectures.html

I should have done this in 2021.06.27_adèles_and_idèles. However, I was not familiar with local field at that time.

1. definition
2. adèle ring and idèle group
3. topological properties of $\mathbb{A}_K$ & $\mathbb{I}_K$
4. Tate's thesis

def | fundamental domain | dense
measure | cpt
topo | discrete

1. definition

Def A global field is
- a finite extension of $\mathbb{Q}$ (number field), or
- a finite extension of $\mathbb{F}_p(T)$ (function field)

For an axiomatic definition, see
https://math.stackexchange.com/questions/873666/definition-of-global-field

In this note we denote K for the global field;
when K is used as a cpt open subgp, we denote $\mathbb{E}$ or $\mathbb{F}$ for the global field.

Rmk 1. Ostrowski's thm states that
every non-trivial norm on $\mathbb{Q}$ is equiv to $|\cdot|_p$ or $|\cdot|_\infty$.
In [Thm3, Cor4, https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF(T).pdf],
every non-trivial norm on $\mathbb{F}_p(T)$ equiv to $|\cdot|_\pi$ or $|\cdot|_\infty$ for some monic irr $\pi(T)\in\mathbb{F}_p[T]$

where
$$\left|\tfrac{a}{b}\pi^k\right|_\pi = p^{-\deg\pi\cdot k} \qquad a,b\in\mathbb{F}_p[T],\ \pi\nmid ab \qquad a,b\neq 0$$
$$\left|\tfrac{a}{b}\right|_\infty = p^{\deg a-\deg b} \qquad a,b\in\mathbb{F}_p[T] \qquad a,b\neq 0$$

Ex. Compute $K_v, \mathcal{O}_v$ for $v = 1\cdot|\ |_\infty,\ 1\cdot|\ |_T,\ 1\cdot|\ |_{T-1},\ 1\cdot|\ |_{T^2+1}$ $\qquad K = \mathbb{F}_p(T),\ p=7$

A. $\mathcal{O}_{1\cdot|\ |_\infty} = \mathbb{F}_p[[\frac{1}{T}]]$ $\quad \mathcal{O}_{1\cdot|\ |_T} = \mathbb{F}_p[[T]]$ $\quad \mathcal{O}_{1\cdot|\ |_{T-1}} = \mathbb{F}_p[[T-1]]$

$K_{1\cdot|\ |_\infty} = \mathbb{F}_p((\frac{1}{T}))$ $\quad K_{1\cdot|\ |_T} = \mathbb{F}_p((T))$ $\quad K_{1\cdot|\ |_{T-1}} = \mathbb{F}_p((T-1))$

$\mathcal{O}_K = \mathbb{F}_p[T]$ can not embed in $\mathcal{O}_{1\cdot|\ |_\infty}$, since $\mathbb{F}_p[T] = \mathcal{O}_{\mathbb{P}^1}(\mathbb{A}^1)$.

The prod formula also prohibit $\mathcal{O}_K$ embed to all $\mathcal{O}_v$.

Show that $\mathbb{F}_p((\frac{1}{T}-a)) = \mathbb{F}_p((T-\frac{1}{a}))$ for $a \in \mathbb{F}_p^\times$:

$$\mathbb{F}_p((\tfrac{1}{T}-a)) = \mathbb{F}_p((\tfrac{1-aT}{T})) = \mathbb{F}_p\left(\left(-\tfrac{a}{T}(T-\tfrac{1}{a})\right)\right)$$

$$\cup\!| \qquad\qquad\qquad\qquad \cap\!|$$

$$\mathbb{F}_p\left(\left(-\frac{(T^{-1}-a+a)^{-1}}{a}(\tfrac{1}{T}-a)\right)\right) = \mathbb{F}_p\left(\left(-\tfrac{T}{a}(\tfrac{1}{T}-a)\right)\right) = \mathbb{F}_p((T-\tfrac{1}{a}))$$

$\mathcal{O}_{1\cdot|\ |_{T^2+1}} = \mathbb{F}_p(\alpha)[[T^2+1]]$ $\qquad \alpha^2+1=0$

$K_{1\cdot|\ |_{T^2+1}} = \mathbb{F}_p(\alpha)((T^2+1))$

$$\begin{aligned} \mathbb{F}_p[T] &\hookrightarrow \mathbb{F}_p(\alpha)[[T^2+1]] \\ T &\longmapsto \alpha - \tfrac{\alpha}{2}(T^2+1) - \tfrac{\alpha}{8}(T^2+1)^2 - \tfrac{\alpha}{16}(T^2+1)^3 - \tfrac{5\alpha}{128}(T^2+1)^4 - \cdots \\ T^2 &\longmapsto -1 + T^2+1 \end{aligned}$$

Rmk 2. Product formula is still true; that is, for $K = \mathbb{F}_p(T)$

$$|f|_\infty \prod_{\pi \text{ fin}} |f|_\pi = 1 \qquad \forall f \in \mathbb{F}_p(T)^\times$$

Ex. Verify the product formula for other K.

For relationships between local fields and global fields, see: https://alex-youcis.github.io/localglobalgalois.pdf
We only list two results which will be used later :

Let L/K be fin ext of global field. We get two isos as topo ring

$$\begin{array}{ccc} L \otimes_K K_v & \xrightarrow{\ \cong\ } & \prod_{i=1}^{g} L_{w_i} \\ \uparrow & & \cup \\ \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_v & \xrightarrow[\text{[MIT, Cor 11.7]}]{\ \cong\ } & \prod_{i=1}^{g} \mathcal{O}_{w_i} \end{array}$$

$w_1 \cdots w_g$ over v $\qquad$ $L_{w_1} \cdots L_{w_g}$ over K_v

2. adèle ring and idèle group

Every book begins this topic by restricted product, which is totally correct but a little boring/confusing. Let us derive the restricted product naturally.

global	$\mathbb{A}_K$	$\mathbb{I}_K$	$\mathbb{I}_K^\times$
local	F	$F^\times$	$\mathcal{O}_F^\times$

adèle ring

Def (adèle ring $\mathbb{A}_\mathbb{Q}$) We know that

$$\left(\prod_{p \text{ prime}} \mathbb{Z}_p\right) \times [0,1) \subseteq \left(\prod_{p \text{ prime}} \mathbb{Q}_p\right) \times \mathbb{R}$$

where $\mathbb{Q}$ acts diagonally on $\prod_{p\text{ prime}} \mathbb{Q}_p \times \mathbb{R}$:

$$+ : \mathbb{Q} \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}\right) \longrightarrow \prod_{p\text{ prime}} \mathbb{Q}_p \times \mathbb{R}$$
$$(t, (a_p, a_\infty)) \longmapsto (t + a_p, t + a_\infty)$$

The adèle ring $\mathbb{A}_\mathbb{Q}$ is defined as the orbit of $\prod_{p\text{ prime}} \mathbb{Z}_p \times [0,1)$, i.e.

$$\mathbb{A}_\mathbb{Q} := \mathbb{Q} + \left(\prod_{p\text{ prime}} \mathbb{Z}_p \times [0,1)\right)$$
$$= \{(a_v)_v \in \prod_v K_v \mid a_v \in \mathcal{O}_v \text{ for almost all } v\} \triangleq \prod_v{}' K_v$$

We don't define $\mathcal{O}_v$ for $v = |\cdot|_\infty$, but that doesn't matter.

Rmk. You can also replace $[0,1)$ by $\mathbb{R}$ in the definition ($\mathbb{A}_\mathbb{Z} := \prod_{p\text{ prime}} \mathbb{Z}_p \times \mathbb{R}$), then it may happen that

$$t + \left(\prod_{p\text{ prime}} \mathbb{Z}_p \times \mathbb{R}\right) = t' + \left(\prod_{p\text{ prime}} \mathbb{Z}_p \times \mathbb{R}\right) \quad \text{for } t \neq t' \in \mathbb{Q}.$$

Rmk. The measure is easy to define while the topo is a bit tricky.

By letting $\mu_p(\mathbb{Z}_p) = 1$, $\mu_\infty([0,1)) = 1$ and give $\prod_{p\text{ prime}} \mathbb{Z}_p \times [0,1)$ with the prod measure, the measures on $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ and $\mathbb{A}_\mathbb{Q}$ are defined.

For the topology on $\mathbb{A}_K$, we take the weakest topo s.t. all the subspaces

$$\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v \qquad = \left(\prod_{\substack{p\in S \\ p \text{ prime}}} \mathbb{Q}_p \times \mathbb{R} \times \prod_{p \notin S} \mathbb{Z}_p\right)$$

(for any S: set of finite places containing all infinite places)

are open, and the subspace topo of $\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v$ coincides with the prod topo.

This topology is a little stronger than the subspace topo of $\mathbb{A}_K \subset \prod_v K_v$, since $\prod_{v\in S} K_v \times \prod_{v\notin S} \mathcal{O}_v$ are not open in this subspace topo.

The same method can be applied to defining the topo of any restricted product.

Ex. Verify that

- $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$ is the fundamental domain of $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$, so

$$\mu\left(\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)\right) = 1 \quad \Rightarrow \mu\left(\mathbb{A}_\mathbb{Q}/\mathbb{Q}\right) = 1$$

Ex. How do they glue with each other?

- $\mathbb{Q} \overset{\Delta}{\hookrightarrow} \mathbb{A}_\mathbb{Q}$ is discrete. (by considering the preimage of $\prod_{p \text{ prime}} \mathbb{Z}_p \times (-\frac{1}{2}, \frac{1}{2})$)
- $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1] \hookrightarrow \mathbb{A}_\mathbb{Q} \twoheadrightarrow \mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is cont

$\Rightarrow$ $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is cpt. $\mathbb{A}_\mathbb{Q}$ is loc. cpt.

- $\mathbb{Q} \overset{\Delta}{\hookrightarrow} \prod'_{p \text{ prime}} \mathbb{Q}_p$, $\quad \mathbb{Q} \overset{\Delta}{\hookrightarrow} \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$ are dense;
- $\mathbb{Z}[\frac{1}{p}] \overset{\Delta}{\hookrightarrow} \mathbb{Q}_p \times \mathbb{R}$, $\quad \{\frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b\} \overset{\Delta}{\hookrightarrow} \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$ are lattices

discrete & quotient is cpt

Ex. define $\mathbb{A}_K$ in general, apply it to $K = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{F}_p(T)$, and compute their measures and fundamental domains.

⚠ From [MIT, #22, p5], $\mu_v(\mathcal{U}) = 2\mu_\mathbb{C}(\mathcal{U})$ for $K_v \cong \mathbb{C}$

Hint. $\mathbb{F}_p[T] \subset \mathbb{F}_p((\frac{1}{T}))$ is a lattice, $\mathbb{F}_p((\frac{1}{T})) = \mathbb{F}_p[T] \oplus \frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]$.

Set $\mu(\mathbb{F}_p[[\frac{1}{T}]]) = 1$, then $\mu(\frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]) = \frac{1}{p}$

$$\Rightarrow \mu\left(\mathbb{A}_{\mathbb{F}_p(T)} / \mathbb{F}_p(T)\right) = \frac{1}{p}.$$

For convenience, we will define

$$\mathbb{A}_{K,fin} = \prod'_{v \text{ fin}} K_v = \hat{K} \qquad \mathbb{A}_{K,inf} = \prod_{v \text{ inf}} K_v \qquad (\mathbb{A}_K = \mathbb{A}_{K,fin} \times \mathbb{A}_{K,inf})$$

in some article, not in our notes

$$\hat{\mathcal{O}}_K = \prod_{v \text{ fin}} \mathcal{O}_v$$

S denotes for any finite set of places containing all infinite places, and
T denotes for any set of places containing all infinite places.
$S, T \neq \emptyset$

idèle group

Def (idèle group $\mathbb{I}_{\mathbb{Q}}$) We know that

$$\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times\right) \times \mathbb{R}_{>0} \subseteq \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times\right) \times \mathbb{R}^\times$$

where $\mathbb{Q}^\times$ acts diagonally on $\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$:

$$\cdot : \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times\right) \longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$$
$$(t \ , \ (a_p \ , \ a_\infty)) \longmapsto (ta_p \ . \ ta_\infty)$$

The idèle group $\mathbb{I}_{\mathbb{Q}}$ is defined as the orbit of $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$, i.e.

$$\begin{aligned}\mathbb{I}_{\mathbb{Q}} :&= \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) \\ &= \{(a_v)_v \in \prod_v K_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \coprod' K_v^\times \\ &= (\textstyle\prod_v' K_v)^\times = \mathbb{A}_{\mathbb{Q}}^\times\end{aligned}$$

In general,

not unique expression

$$\begin{aligned}\mathbb{I}_K &= K^\times \times \left(\prod_{v \text{ fin}} \mathcal{O}_v^\times \times \prod_{v \text{ inf}} K_v^\times\right) \\ &= \{(a_v)_v \in \prod_v K_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \coprod K_v^\times \\ &= (\textstyle\prod_v' K_v)^\times = \mathbb{A}_K^\times\end{aligned}$$

Rmk. The definition of measure and topology are similar.
The topo defined is stronger than the subspace topo $\mathbb{A}_K^\times \subset \mathbb{A}_K$,
since $\prod_{v \in S} K_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$ (for any S) is not open in the subspace topology.

Ex. Verify that

- $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ is the fundamental domain of $\mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times$, so
$$\mu\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) = +\infty \quad \Rightarrow \mu\left(\mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times\right) = +\infty$$
- $\mathbb{Q}^\times \overset{\Delta}{\hookrightarrow} \mathbb{I}_{\mathbb{Q}}$ is discrete. (by considering the preimage of $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$)
- $\mathbb{I}_{\mathbb{Q}}/\mathbb{Q}^\times$ is not cpt, $\mathbb{I}_{\mathbb{Q}}$ is loc. cpt.
- $\mathbb{Q}^\times \overset{\Delta}{\hookrightarrow} \prod'_{p \text{ prime}} \mathbb{Q}_p^\times$ is discrete (by considering the preimage of $\prod_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Z}_p^\times \times (1+7\mathbb{Z}_7)$)
$\mathbb{Q}^\times \overset{\Delta}{\hookrightarrow} \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$ is dense;
- $\mathbb{Z}[\frac{1}{p}] = \pm p^{\mathbb{Z}} \overset{\Delta}{\hookrightarrow} \mathbb{Q}_p^\times \times \mathbb{R}^\times$, $\{\frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b\}^\times = \mathbb{Q}^\times \cap \mathbb{Z}_7^\times \overset{\Delta}{\hookrightarrow} \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$
are discrete.

To remedy the cptness, we introduce the group of 1-idèles.

Def (1-idèles group)

$$\mathbb{I}_{\mathbb{Q}}^{1} := \mathbb{Q}^\times \times \left(\prod_{p\text{ prime}} \mathbb{Z}_p^\times \times \{1\}\right)$$
$$= \{(a_v)_v \in \prod_v{}' K_v^\times \mid \prod_v |a_v|_v = 1\} = \left(\prod_v{}' K_v^\times\right)^1 = \mathbb{A}_{\mathbb{Q}}^{\times,1}$$

In general,

not unique expression

$$\mathbb{I}_K^1 := K^\times \times \left(\prod_{v\text{ fin}}{}' \mathcal{O}_v^\times \times \left(\prod_{v\text{ inf}} K_v^\times\right)^1\right)$$
$$= \{(a_v)_v \in \prod_v{}' K_v^\times \mid \prod_v |a_v|_v = 1\} = \left(\prod_v{}' K_v^\times\right)^1 = \mathbb{A}_K^{\times,1}$$

where

$$\left(\prod_{v\text{ inf}} K_v^\times\right)^1 := \{(a_v)_v \in \prod_{v\text{ inf}} K_v^\times \mid \prod_v |a_v|_v = 1\}$$

We have SESs:

$$0 \longrightarrow \mathbb{I}_K^1 \longrightarrow \mathbb{I}_K \xrightarrow{\|\cdot\|} \mathbb{R}_{>0}^\times \longrightarrow 0 \qquad \text{for } K \text{ number field}$$
$$0 \longrightarrow \mathbb{I}_K^1 \longrightarrow \mathbb{I}_K \xrightarrow{\|\cdot\|} p^{\mathbb{Z}} \longrightarrow 0 \qquad \text{for } K \text{ function field}$$

Rmk [引理6.106][MIT, Lemma 23.8, 23.9]

For measures, I set $\mu(S^1) = 2\pi$, $\mu(\mathbb{Z}_p^\times) = 1$, $\mu(pt) = 1$. I hope they're fine.

The subspace topologies $\mathcal{O}_K^\times \subseteq K^\times$, $\mathcal{O}_K^\times \subseteq K$ coincide. $\mathcal{O}_K^\times \subseteq K$ is closed.

Observation. It's clear if you see

$$\mathbb{I}_K \cong \{(x, x^{-1}) \in \mathbb{A}_K^2\} \subseteq GL_2(\mathbb{A}_K)$$

Ex. Verify that

- $\prod_{p\text{ prime}} \mathbb{Z}_p^\times \times \{1\}$ is the fundamental domain of $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$, so
$$\mu\left(\prod_{p\text{ prime}} \mathbb{Z}_p^\times \times \{1\}\right) = 1 \Rightarrow \mu\left(\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times\right) = 1$$
- $\mathbb{Q}^\times \trianglelefteq \mathbb{I}_{\mathbb{Q}}^1$ is discrete, $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$ is cpt.

Ex. Compute $\mathbb{I}_K$, $\mathbb{I}_K^1$ for $K = \mathbb{Q}(i), \mathbb{Q}(\sqrt{3}), \mathbb{F}_p(t)$.

For convenience, we define

$$C_K := \mathbb{I}_K/K^\times \qquad C_K^1 := \mathbb{I}_K^1/K^\times$$
$$\mathbb{I}_{K,\text{fin}} := \prod_{v\text{ fin}}{}' K_v^\times \qquad \mathbb{I}_{K,\text{inf}} := \prod_{v\text{ inf}} K_v^\times \qquad (\mathbb{I}_K = \mathbb{I}_{K,\text{fin}} \times \mathbb{I}_{K,\text{inf}})$$

so C_K^1 is cpt, while C_K is loc cpt.

(We've shown this for $K = \mathbb{Q}$.)

⚠ I may use the symbols

$$\mathbb{A}_{K,T} = \prod_{v\in T}{}' K_v \times \prod_{v\notin T} \mathcal{O}_v$$
$$\mathbb{I}_{K,T} = \prod_{v\in T}{}' K_v^\times \times \prod_{v\notin T} \mathcal{O}_v^\times$$
$$\mathbb{I}_{K,T}^1 = \left(\prod_{v\in T}{}' K_v^\times\right)^1 \times \prod_{v\notin T} \mathcal{O}_v^\times$$

to make the result simpler and more symmetrical.

I don't do it now just because I'm lazy.

3. topological properties of $\mathbb{A}_K$ & $\mathbb{I}_K$.

All the properties in this section have been checked for K=Q in the last section(for results concerning S, we checked some examples also). To make everything rigorous and easy to cite (and get some important applications), we make this section.

The roadmap of this section:

$$\begin{array}{ccccc} K \subseteq \mathbb{A}_K \text{ for } K=\mathbb{Q}, \mathbb{F}_p(t) & \Rightarrow & K \subseteq \mathbb{A}_K & \Rightarrow & \mathcal{O}_T \subseteq \prod'_{v \in T} K_v \\ & & \Downarrow & & \\ & & K^\times \subseteq \mathbb{I}_K^1 & \Rightarrow & \mathcal{O}_T^\times \subseteq (\prod'_{v\in T} K_v)^1 \\ & & \Downarrow & & \Downarrow \\ & & \text{class number} & & \text{Dirichlet unit} \end{array}$$

dense comes from the theory of duality, not from lattice.

topo results needed

Def (iso up to cpt gp, Iso_{cpt})

$f: G_1 \longrightarrow G_2 \in Mor(Abel_{Top})$ is called iso up to cpt gp (Iso_{cpt}) if

(1) $G_1/\ker f \cong \operatorname{Im} f$ in $Abel_{Top}$;

(2) $\ker f$, $\operatorname{coker} f$ are cpt.

Def (lattice)

$L \subseteq G$ in $Abel_{Top}$ is called a lattice, if

(1) L is discrete;

(2) G/L is cpt.

⚠ When $G = (\mathbb{R}^n, +)$, this is equiv to a full lattice.

Cor: for $G_1 \xrightarrow{f} G_2 \in Iso_{cpt}$, if G_1 is discrete, then $\operatorname{Im} f$ is a lattice in G_2.

Lemma 1. (1) $G_1 \xrightarrow{f} G_2$, $G_2 \xrightarrow{g} G_3 \in Iso_{cpt}$

$\Rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \in Iso_{cpt}$

(2) $$\begin{array}{c} G_1 \xrightarrow{f} G_2 \in Iso_{cpt} \\ \quad\ \ \text{\rotatebox{90}{$\subseteq$}}/\text{open} \\ \quad\ \ H_2 \end{array} \Rightarrow \begin{array}{ccc} G_1 & \xrightarrow{f} & G_2 \quad \in Iso_{cpt} \\ \text{\rotatebox{90}{$\subseteq$}}/\text{open} & & \text{\rotatebox{90}{$\subseteq$}}/\text{open} \\ f^{-1}(H_2) & \longrightarrow & H_2 \quad \in Iso_{cpt} \end{array}$$

(3) $H \leq G$ in $Abel_{Top}$

$$\begin{array}{ccc} H \text{ is open} & \Leftrightarrow & G/H \text{ is discrete} \\ \Downarrow & & \Downarrow \\ H \text{ is closed} & \Leftrightarrow & G/H \text{ is Hausdorff.} \end{array}$$

lattice

Lemma 2 [6.101] [MIT, Prop 22.10] L/k finite ext of global field. We get an iso of topo rings

$$\begin{aligned} \Phi: L\otimes_k \mathbb{A}_K &\xrightarrow{\cong} \mathbb{A}_L \\ (t, (a_v)_v) &\longmapsto (t a_{w|_k})_w \end{aligned}$$

In ptc, $\mathbb{A}_K$ is a subring of $\mathbb{A}_L$, $\mathbb{A}_L \cong \mathbb{A}_K^{\oplus [L:K]}$, and we have an iso

$$\begin{tikzcd} L \arrow[r, hook, "\Delta"] & \mathbb{A}_L \\ L\otimes_K K \arrow[u, "\cong"] \arrow[r, hook, "Id_L\otimes_K \Delta"] & L\otimes_K \mathbb{A}_K \arrow[u, "\cong"] \end{tikzcd}$$

Proof. Locally we have

$$L\otimes_K K_v \xrightarrow{\sim} \prod_{i=1}^{g} L_{w_i}$$

[MIT, Cor 11.7]
$$\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_v \xrightarrow{\sim} \prod_{i=1}^{g} \mathcal{O}_{w_i}$$

$$\begin{tikzcd}[column sep=tiny] w_1 \arrow[dr, dash] & \cdots & w_g \arrow[dl, dash] \\ & v & \end{tikzcd} \qquad \begin{tikzcd}[column sep=tiny] L_{w_1} \arrow[dr, dash] & \cdots & L_{w_g} \arrow[dl, dash] \\ & K_v & \end{tikzcd}$$

Since $L\otimes_K -$, $\mathcal{O}_L\otimes_{\mathcal{O}_K} -$ are exact [Stackexchange, 1916457], one get

$$\begin{tikzcd} \mathcal{O}_L \otimes_{\mathcal{O}_K} \prod_{v \text{ fin}} \mathcal{O}_v \arrow[d] \arrow[r, phantom, "\cong"] & \prod_{w \text{ fin}} \mathcal{O}_w \arrow[d, phantom, "\cap"] \\ L\otimes_K \mathbb{A}_K \arrow[r] \arrow[d] & \mathbb{A}_L \arrow[d, phantom, "\cap"] \\ L\otimes_K \prod_v K_v \arrow[r, phantom, "\cong"] & \prod_w K_w \end{tikzcd}$$

which shows the bijection.

(Φ is well-defined, since $a_v \in \mathcal{O}_v \Rightarrow a_v \in \mathcal{O}_w$;
"Φ & Φ^{-1} are cont" should be a routine check (but I don't check it))

Prop 1 [6.78] K is a lattice in $\mathbb{A}_K$.

Proof. We have checked for $K=\mathbb{Q}, \mathbb{F}_p(T)$. The rest comes from Lemma 1.

Prop 2 [6.80(1)] Let T be a set of places of K containing all infinite places, $T\neq\emptyset$.
Let
$$\mathcal{O}_T = \{x\in K \mid x\in\mathcal{O}_v \text{ for } v\notin T\}$$
then $\mathcal{O}_T$ is a lattice in $\prod'_{v\in T} K_v$.

Rmk. For $K/\mathbb{Q}$ of degree n

When $T=\{\text{all places of } K\}$, $\mathcal{O}_T = K$; Application 1.

When $T=\{\text{all inf places of } K\}$, $\mathcal{O}_T = \mathcal{O}_K$ $\Rightarrow$ $\mathcal{O}_K$ is a free $\mathbb{Z}$-module of rank n.

Proof.

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, phantom, "\vee\!/"] & \mathbb{A}_K \arrow[d, phantom, "\vee\!/ \text{ open}"] \\ \mathcal{O}_T \arrow[r] & \prod'_{v\in T} K_v \times \prod_{v\notin T}\mathcal{O}_v \arrow[r, "\pi"] & \prod'_{v\in T} K_v \end{tikzcd}$$

By Lemma 1, $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v\in T} K_v$ is iso up to cpt gp,
and $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v\in T} K_v$ is obviously injective.

Prop 3 [6.82] $K^\times$ is a lattice in $\mathbb{I}_K'$

For a proof, see [Theorem 3.3.6, https://bicmr.pku.edu.cn/~dingyiwen/nt1.pdf]

Prop 4 [6.83] Let T be a set of places containing all infinite places. $T \neq \phi$, let

$$\Big(\prod_{v\in T}{}' \mathbb{R}\Big)^\circ := \Big\{ (c_v)_{v\in T} \in \prod_{v\in T} \mathbb{R} \;\Big|\; \begin{array}{l} c_v = 0 \text{ for almost all } v \\ \sum_{v\in T} c_v = 0 \end{array} \Big\}$$

$$R_T : \mathcal{O}_T^\times \longrightarrow \Big(\prod_{v\in T}{}' \mathbb{R}\Big)^\circ \qquad x \longmapsto (\ln |x|_v)_{v\in T}$$

then R_T makes $R_T(\mathcal{O}_T^\times)$ as a lattice in $(\prod_{v\in T}' \mathbb{R})^\circ$, and $\operatorname{Ker} R_T$ is finite.

Proof.

$$\begin{array}{ccccccc} K^\times & \overset{\Delta}{\hookrightarrow} & \mathbb{I}_K' & & & & \\ \text{∨/ open} & & \text{∨/ open} & & & & \\ \mathcal{O}_T^\times & \overset{\Delta}{\hookrightarrow} & \big(\prod_{v\in T}' K_v^\times\big) \times \prod_{v\notin T} \mathcal{O}_v^\times & \xrightarrow{\pi} & \big(\prod_{v\in T}' K_v^\times\big)' & \longrightarrow & \big(\prod_{v\in T}' \mathbb{R}\big)^\circ \\ & & & & (a_v)_{v\in T} & \longmapsto & (\ln |a_v|_v)_{v\in T} \end{array}$$

Rmk. R_T is in general not injective.

Application 2.

$$\operatorname{Ker} R_T = \{\text{root of unity in } K\} \leftarrow \text{the unity root gp of } K$$
$$= \{x \in K \mid x^n = 1,\ \exists n \in \mathbb{N}_{\geq 1}\}$$

where "$\subseteq$" comes from the finiteness of $\operatorname{Ker} R_T$.

As a corollary, the unity root gp of K is finite.

Application 3. Suppose $K/\mathbb{Q}$ is a number field, $\# T < +\infty$. We get SES

$$1 \longrightarrow \operatorname{Ker} R_T \longrightarrow \mathcal{O}_T^\times \longrightarrow \mathbb{Z}^{\oplus(\# T - 1)} \longrightarrow 1$$

When $T = \{\text{all inf places}\}$, we get Dirichlet unit theorem.

Application 4. For $K/\mathbb{Q}$, define

$$\begin{aligned} \mathbb{I}_K &\longrightarrow I(K) \\ (a_v)_v &\longmapsto \prod_{\substack{v \text{ fin} \\ v = v_\mathfrak{p}}} \mathfrak{p}^{v(a_v)} \end{aligned}$$

we get SES's:

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d] & \overset{\supseteq \mathcal{U}}{\prod_{v \text{ fin}} \mathcal{O}_v^\times \times \prod_{v \text{ inf}} K_v^\times} \arrow[r] \arrow[d] & \bar{\mathcal{U}} \arrow[d] \arrow[dddll, bend left] & \\
1 \arrow[r] & K^\times \arrow[r] \arrow[d] & \mathbb{I}_K \arrow[r] \arrow[d] & C_K \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P(K) \arrow[r] \arrow[d] & I(K) \arrow[r] \arrow[d] & Cl(K) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

$$\Rightarrow \quad 1 \longrightarrow \mathcal{O}_K^\times \longrightarrow K^\times \longrightarrow I(K) \longrightarrow Cl(K) \longrightarrow 1$$

By replacing $\mathbb{I}_K$ by $\mathbb{I}_K'$, we get

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{O}_K^\times \arrow[r] \arrow[d] & \overset{\supseteq \mathcal{U}'}{\prod_{v \text{ fin}} \mathcal{O}_v^\times \times \left(\prod_{v \text{ inf}} K_v^\times\right)'} \arrow[r] \arrow[d] & \bar{\mathcal{U}}' \arrow[d] \arrow[dddll, bend left] & \\
1 \arrow[r] & K^\times \arrow[r] \arrow[d] & \mathbb{I}_K' \arrow[r] \arrow[d] & C_K' \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & P(K) \arrow[r] \arrow[d] & I(K) \arrow[r] \arrow[d] & Cl(K) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

$$\left.\begin{array}{l} C_K' \text{ cpt} \Rightarrow Cl(K) \text{ is cpt} \\ \prod_{v \text{ fin}} \mathcal{O}_v^\times \times \left(\prod_{v \text{ inf}} K_v^\times\right)' \subseteq \mathbb{I}_K' \text{ is open} \Rightarrow I(K) \text{ is discrete} \Rightarrow Cl(K) \text{ is discrete} \end{array}\right\}$$

$\Rightarrow Cl(K)$ is finite. $\square$

dense.

Work over the category of loc. cpt Abelian gp Abel_{lc}.

$G \in \mathrm{Abel}_{lc}$

$\widehat{G}^{*,u} := \{\chi : G \longrightarrow S^1 \text{ cont}\}$
↑ unitary

$\widehat{G}^{*} := \{\chi : G \longrightarrow \mathbb{C}^\times \text{ cont}\}$

$\widehat{K} :=$ "profinite completion of K"

$V^* :=$ "dual space of V"

Thm (Pontrjagin dual) For $G \in \mathrm{Abel}_{lc}$,

$$\begin{aligned} \phi : G &\xrightarrow{\sim} \widehat{\widehat{G}^{*,u}}^{*,u} \\ g &\longmapsto \left[\begin{aligned} \phi_g : \widehat{G}^{*,u} &\longrightarrow S^1 \\ \chi &\longmapsto \chi(g) \end{aligned} \right] \end{aligned}$$

is an iso (as topo abelian gp)

$\chi^k \neq Id \quad \forall k \in \mathbb{N}_{\geq 1}$

Thm. 1. For K local field, take $\chi \in \widehat{K}^{*,u}$ not root of unit,

$$\begin{aligned} K &\xrightarrow{\sim} \widehat{K}^{*,u} \\ x &\longmapsto \chi(x-) \end{aligned}$$

is an iso

$\Rightarrow \chi \in \widehat{\mathbb{A}_K}^{*,u}$

2. For K global field, take $\chi \in \widehat{\mathbb{A}_K/K}^{*,u}$ not root of unit.

$$\begin{aligned} K &\xrightarrow{\sim} \widehat{\mathbb{A}_K/K}^{*,u} \\ x &\longmapsto \chi(x-) \\ \mathbb{A}_K &\xrightarrow{\sim} \widehat{\mathbb{A}_K}^{*,u} \\ x &\longmapsto \chi(x-) \end{aligned}$$

are isos.

3. For K global field, T, take $\chi \in \widehat{\left(\prod'_{v \in T} K_v\right)/\mathcal{O}_T}^{*,u}$ not root of unit.

$$\mathcal{O}_T \xrightarrow{\sim} \widehat{\left(\prod'_{v \in T} K_v\right)/\mathcal{O}_T}^{*,u}$$

$$\prod'_{v \in T} K_v \xrightarrow{\sim} \widehat{\prod'_{v \in T} K_v}^{*,u}$$

are isos.

Lemma. For $f \in \mathrm{Mor}_{\mathrm{Abel}_{lc}}(G_1, G_2)$,

$$\mathrm{Im} f \subseteq G_2 \text{ is dense} \iff \widehat{f}^{*,u} : \begin{array}{ccc} \widehat{G_2}^{*,u} & \longrightarrow & \widehat{G_1}^{*,u} \\ \chi & \longmapsto & \chi(f(-)) \end{array} \text{ is inj.}$$

Proof. Let $H = \overline{\mathrm{Im} f}$.

$$\begin{aligned} & \mathrm{Im} f \subseteq G_2 \text{ is dense} \\ \iff & H = G_2 \\ \iff & G_2/H = \{\mathrm{Id}\} \\ \iff & \widehat{G_2/H}^{*,u} = \{\mathrm{Id}\} \\ \iff & \widehat{f}^{*,u} \text{ is inj.} \end{aligned}$$

Prop 5 [6.79] Let $T \subsetneq \{\text{places of } K\}$, then the image of

$$K \longrightarrow \prod_{v \in T}{}' K_v$$

is dense.

Proof. Reduce to show, for any w,

$$\begin{aligned} & \text{the image of } K \longrightarrow \prod_{v \neq w}{}' K_v \text{ is dense} \\ \iff & \widehat{\prod_{v \neq w}{}' K_v}^{*,u} \longrightarrow \widehat{K}^{*,u} \text{ is inj} \\ & \qquad \text{\rotatebox{90}{$\cong$}} \qquad\qquad\qquad \text{\rotatebox{90}{$\cong$}} \\ \iff & \prod_{v \neq w}{}' K_v \longrightarrow \mathbb{A}_K/K \text{ is inj} \end{aligned}$$

□

Prop 6 [6.80(2)] Suppose $T' \subsetneq T$, then the image of

$$\mathcal{O}_T \longrightarrow \prod_{v \in T'}{}' K_v$$

is dense.

Proof. Reduce to show, for any $w \in T$,

$$\begin{aligned} & \text{the image of } \mathcal{O}_T \longrightarrow \prod_{v \in T - \{w\}}{}' K_v \text{ is dense.} \\ \iff & \widehat{\prod_{v \in T-\{w\}}{}' K_v}^{*,u} \longrightarrow \widehat{\mathcal{O}_T}^{*,u} \text{ is inj} \\ & \qquad \text{\rotatebox{90}{$\cong$}} \qquad\qquad\qquad \text{\rotatebox{90}{$\cong$}} \\ \iff & \prod_{v \in T-\{w\}}{}' K_v \longrightarrow \prod_{v \in T}{}' K_v / \mathcal{O}_T \text{ is inj} \end{aligned}$$

□