

Eine Woche, ein Beispiel

8.6. Kottwitz set

This document is a continuation of [23.08.06].
Reorganized from Luozi Shi (and his partners)'s talk.

Recall that $\widehat{\mathbb{Q}_p^{ur}} = \text{Frac}(\widehat{\mathbb{Z}_p^{ur}})$, $\widehat{\mathbb{Z}_p^{ur}} = W(\mathbb{F}_p)$. Here " $\widehat{}$ " is completion w.r.t. valuation.

Setting. In this document, F is a NA local field,

$$L := \widehat{F^{ur}} \stackrel{\text{if } F: p\text{-adic}}{=} \text{Frac}(W(\bar{\kappa}_F) \otimes_{W(\kappa_F)} \mathcal{O}_F).$$

Def For G/F conn reductive, the Kottwitz set $B(G)$ is defined as

$$\begin{aligned} B(G) &:= H'(W_F, G_{\bar{F}}) \\ &\cong H'(\langle \sigma \rangle, G_L) \quad \text{by Inf-Res seq \& } H'(I_F, G_{\bar{F}}) = 0 \\ &\cong G(L)/\sigma\text{-twisted } G(L)\text{-conj} \\ &\stackrel{\text{when } G = GL_n}{\cong} \text{Isoc}/\sim \\ &\quad \quad \quad \uparrow \text{Isocrystals.} \end{aligned}$$

Rmk. By Hilbert 90, $H'(\Gamma_F, GL_n, \bar{F}) = \{1\}$. In most cases,

$$H'(\Gamma_F, G_{\bar{F}}) \not\cong H'(W_F, G_{\bar{F}})$$

[even though $H'(\Gamma_F, G_{\bar{F}}) \cong G(F) \cong H^0(W_F, G_{\bar{F}})$, we take different resolutions.]

E.g. $B(G_m) \cong \mathbb{Z}$

Proof. The map

$$\beta: \mathcal{O}_L^\times \longrightarrow \mathcal{O}_L^\times \quad x \longmapsto x \cdot \sigma(x)^{-1}$$

is surjective, since

$$\beta_1: \kappa_L^\times \longrightarrow \kappa_L^\times \quad x \longmapsto x^{1-q}$$

$$\beta_2: \kappa_L \longrightarrow \kappa_L \quad x \longmapsto x - x^q$$

are surjective. ($\kappa_L \cong \bar{\mathbb{F}_p}$ is alg closed)

Then the well-defined morphism

$$v: L^\times / \text{Im } \beta \longrightarrow \mathbb{Z} \quad x \longmapsto v(x)$$

is injective, thus an iso.

Ex. Check that the SES

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow G_m \xrightarrow{(-)^*} G_m \rightarrow 1$$

induce LES in gp cohomology:

$$0 \rightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\quad} F^{\times} \xrightarrow{(-)^2} F^{\times}$$

where

$$\begin{aligned} B(\underline{\mathbb{Z}/2\mathbb{Z}}) & \stackrel{\text{def}}{=} H^*(W_F, \mathbb{Z}/2\mathbb{Z}) & W_F \hookrightarrow \mathbb{Z}/2\mathbb{Z} \text{ trivially} \\ & = \text{Hom}(W_F, \mathbb{Z}/2\mathbb{Z}) \\ & \cong \{ H \triangleleft W_F \text{ closed with index } 2 \} \cup \{0\} \\ & \cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 2 \end{cases} \end{aligned}$$

This LES gives us to compute number of finite extensions. *eg with prime degree.*
One gets

$$F^x \xrightarrow{(-)^n} F^x \hookrightarrow B(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{0} \dots$$

E.g. $B(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{Z} \times \mathbb{Z} = X_*(\mathbb{G}_m \times \mathbb{G}_m)$

Proof. $B(G_m \times G_m) \cong G_m(L) \times G_m(L) / \sigma\text{-twisted } G_m(L) \times G_m(L)\text{-conj}$

$$\cong G_m(L)/\sim \times G_m(L)/\sim$$

$$\cong \mathbb{Z} \times \mathbb{Z}$$

Rmk. In general, for a torus T/\mathbb{F} ,
 $B(T) \cong X_*(T)_{\Gamma_F} \cong X^*(\widehat{T}^{\Gamma_F})$

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E.g. For E/F any Galois extension of deg 2, $\text{Gal}(E/F) = \{1, \sigma\}$.

When $T = \text{Res}_{E/F} G_{m,E}$, $B(T) \cong \mathbb{Z}$.

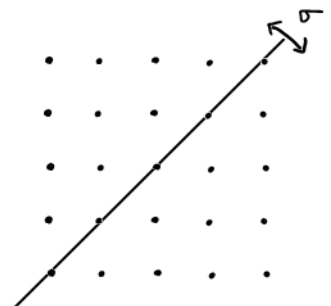
$$X_*(\Gamma) \cong \mathbb{Z} \times \mathbb{Z} \quad \sigma: \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} b \\ a \end{pmatrix}$$

$$X_*(T)_{\Gamma_F} \cong \mathbb{Z} \times \mathbb{Z} / \langle \sigma(v) - v \rangle \xrightarrow[\sim]{+} \mathbb{Z}$$

$$\widehat{T} \cong \mathbb{C}^x \times \mathbb{C}^x \quad \sigma: \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{pmatrix} b \\ a \end{pmatrix}$$

$$\begin{array}{ccc} U & & \uparrow \Delta \\ \widehat{T}^{\mathbb{F}} & \cong & \mathbb{C}^{\times} \end{array}$$

$$X^*(\varprojlim \Gamma_F) \cong \mathbb{Z}$$



When $T = \text{Res}'_{E/F} G_{m,E}$, $B(T) \cong \mathbb{Z}/2\mathbb{Z}$
 $X_*(T) \cong \mathbb{Z}$ $\sigma: a \mapsto -a$
 $X_*(T)_{\Gamma_F} \cong \mathbb{Z}/\langle \sigma(a) - a \rangle \cong \mathbb{Z}/2\mathbb{Z}$

$\hat{T} \cong \mathbb{G}^x$ $\sigma: a \mapsto a^{-1}$
 $\downarrow \quad \downarrow$
 $\hat{T}_{\Gamma_F} \cong \mu_2(\mathbb{G}^x) \cong \mathbb{Z}/2\mathbb{Z}$
 $X^*(\hat{T}_{\Gamma_F}) \cong \mathbb{Z}/2\mathbb{Z}$

Ex. Check that the SES

$$1 \longrightarrow \text{Res}'_{E/F} G_{m,E} \longrightarrow \text{Res}_{E/F} G_{m,E} \longrightarrow G_{m,F} \longrightarrow 1$$

induce LES in gp cohomology:

$$\begin{array}{ccccc} & \xrightarrow{\quad \text{"}\mathbb{Z}/2\mathbb{Z}\text{"} \quad} & \xrightarrow{\quad \text{"}\mathbb{Z}\text{"} \quad} & \xrightarrow{\quad \text{"}\mathbb{Z}\text{"} \quad} & \\ & B(\text{Res}'_{E/F} G_{m,E}) & \xrightarrow{0} & B(\text{Res}_{E/F} G_{m,E}) & \xrightarrow{x^2} B(G_{m,F}) \\ & \searrow & & & \nearrow \\ 0 & \longrightarrow & H^0(W_F, (\text{Res}'_{E/F} G_{m,E})_{\bar{F}}) & \longrightarrow & H^0(W_F, (\text{Res}_{E/F} G_{m,E})_{\bar{F}}) \xrightarrow{\text{Norm}} H^0(W_F, G_{m,\bar{F}}) \\ & & \text{"}\text{Res}'_{E/F} G_{m,E}\text{"}(F) & & \text{"}E^x\text{"} \quad \text{"}F^x\text{"} \end{array}$$

Aside: $(\text{Res}'_{E/F} G_{m,E})(F) = \ker [(\text{Res}_{E/F} G_{m,E})(F) \xrightarrow{\text{Norm}} G_m(F)]$
 $= \ker [E^x \xrightarrow{\text{Norm}} F^x]$
 $= \{x \in E^x \mid x \sigma(x) = 1\}$