Eine Woche, ein Beispiel

4.27. homomorphism between Jacobians

[2025.04.20] provides us with many examples and references, and here we do things more theoretically.

Idea:

$$\mathrm{Jac}(C) = H^0(C;\omega_C)^* / H_1(C;\mathbb{Z})$$

linear part
coherent

lattice part
constant

To understand Jac(C), we need to understand these two parts separately.

For a morphism between two sm proj curves $/\mathbb{C}$:

$$f : \widehat{C} \longrightarrow C$$

$$\begin{aligned} Nm_{f,a} &: H^0(\widehat{C}; \omega_{\widetilde{C}})^* \longrightarrow H^0(C; \omega_C)^* \\ Nm_{f,r} &: H_1(\widehat{C}; \mathbb{Z}) \longrightarrow H_1(C; \mathbb{Z}) \\ Nm_f &: Jac(\widehat{C}) \longrightarrow Jac(C) \end{aligned}$$

$$\begin{aligned} (f^*)_a &: H^0(C; \omega_C)^* \longrightarrow H^0(\widetilde{C}; \omega_{\widetilde{C}})^* \\ (f^*)_r &: H_1(C; \mathbb{Z}) \longrightarrow H_1(\widetilde{C}; \mathbb{Z}) \\ f^* &: Jac(C) \longrightarrow Jac(\widetilde{C}) \end{aligned}$$

cohom pullback

sheaf origin

$$\begin{aligned} \omega_{\widetilde{C}} &\longleftarrow f^*\omega_C \\ f_! \pi^!_{\widetilde{C}} \mathbb{Z} &\longrightarrow \pi^!_C \mathbb{Z} \end{aligned}$$

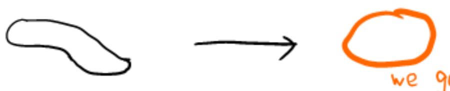
$$\begin{aligned} \omega_C &\longleftarrow f_! \omega_{\widetilde{C}} \quad (= f_* \omega_{\widetilde{C}}) \\ \underline{\mathbb{Z}}_C &\longrightarrow f_* \underline{\mathbb{Z}}_{\widetilde{C}} \end{aligned}$$

$$g(f(\omega))\, d(f(\omega)) \longleftarrow g(z)\, dz$$

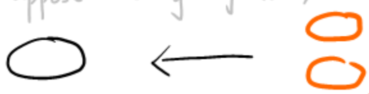
$$\sum_{f(\omega)=z} g(\omega)\, dz \longleftarrow g(\omega)\, d\omega$$

suppose locally $f^*(dz) = d\omega$

geometric picture



we get



we get

$$[q] \longmapsto [f(q)]$$

$$[p] \longmapsto \sum_{f(q)=p} [q]$$

Ex. Show that

$$Nm_f \circ f^* = [\deg f] : Jac(C) \longrightarrow Jac(C)$$

Also,

$$\begin{aligned} Nm_{f,a} \circ (f^*)_a &= \deg f \cdot Id_{H^0(C;\omega_C)^*} \\ Nm_{f,r} \circ (f^*)_r &= \deg f \cdot Id_{H_1(C;\mathbb{Z})} \end{aligned}$$

Hint: use Poincaré duality.

Notations

For an abelian variety $A/\mathbb{C}$, we want to define

$$t_a,\ \phi_{\mathcal{L}},\ \psi_{\mathcal{L}},\ K(\mathcal{L}),\ \Lambda(\mathcal{L}),\ e(\mathcal{L}),\ \mathcal{D}_{\mathcal{L}},\ \varsigma(Z,W),\ \delta(Z,W)$$

for $\mathcal{L}\in \mathrm{Pic}(A)$, $a\in A$, $Z,W\subset A$ with complementary dim.

$$t_a : A \longrightarrow A \qquad\qquad \phi_{\mathcal{L}} : A \longrightarrow \hat{A}$$
$$x \longmapsto x+a \qquad\qquad x \longmapsto t_x^*\mathcal{L}\otimes\mathcal{L}^{-1}$$

$$0 \longrightarrow K(\mathcal{L}) \longrightarrow A \xrightarrow{\phi_{\mathcal{L}}} \hat{A} \longrightarrow \mathrm{coker}\,\phi_{\mathcal{L}} \longrightarrow 0$$

$$K(\mathcal{L}) = \Lambda(\mathcal{L})/\Lambda, \qquad A = \mathbb{C}^n/\Lambda$$

[BL04, Prop 2.4.8]

$\mathcal{L}$ is nondegenerate $\iff$ $\# K(\mathcal{L}) < +\infty$

$H=\langle -,-\rangle$ nondeg $\iff$ $\mathrm{coker}\,\phi_{\mathcal{L}} = 0$

[BL04, 4.1, 5.1]

When $\mathcal{L}$ is pos def, $c_1(\mathcal{L})\in H^2(A;\mathbb{Z})$ is called a polarization
In this case, $\# K(\mathcal{L}) < +\infty$ & $\mathrm{coker}\,\phi_{\mathcal{L}} = 0$, let

$$K(\mathcal{L}) = \mathbb{Z}/d_1\mathbb{Z}\oplus\cdots\oplus\mathbb{Z}/d_n\mathbb{Z} \qquad d_1|\cdots|d_n,$$

denote $e(\mathcal{L}) = d_n$, then we can define

$$\psi_{\mathcal{L}} : \hat{A} \longrightarrow A$$
$$\phi_{\mathcal{L}}(x) \longmapsto e(\mathcal{L})\cdot x$$

Check: ① $\psi_{\mathcal{L}}$ is well-defined,
② $\psi_{\mathcal{L}}\circ\phi_{\mathcal{L}} = [e(\mathcal{L})]_A$, $\quad \phi_{\mathcal{L}}\circ\psi_{\mathcal{L}} = [e(\mathcal{L})]_{\hat{A}}$.

For $f: A \longrightarrow A$, $\mathcal{L} \in \mathrm{Pic}(A)$,

$$D_{\mathcal{L}} : \mathrm{End}(A) \longrightarrow \mathrm{Pic}(A)$$
$$f \longmapsto (f+\mathrm{Id}_A)^*\mathcal{L} \otimes f^*\mathcal{L}^{-1} \otimes \mathcal{L}^{-1}$$

[BL04, 5.4] For $Z, W \subset A$ with $Z \cdot W = \sum_i [x_i] \in CH_0(A)$, we define

$$S(Z,W) := \sum_i x_i \in A$$

$$\delta(Z, W) : A \longrightarrow A \qquad x \longmapsto S(Z \cdot (t_x^* W - W))$$

e.p. $\delta(Z, c_1(\mathcal{L})) : A \longrightarrow A \qquad x \longmapsto S(Z \cdot c_1(\phi_{\mathcal{L}}(x)))$

Fact. $\delta(Z, W) \in \mathrm{End}(A)$.

Nondegeneracy

Z integral

[Ar85, p6] $Z \subset \mathbb{P}^r$ is non-degenerate, if $Z \nsubseteq H$ for any hyperplane $H \subseteq \mathbb{P}^r$.

[BL04, 11.8, p341] $Z \subset A$, Z generates A, if $Z \nsubseteq A'$ for any subabelian variety $A' \subseteq A$.

[JKLM23, p10] $Z \subset A$ is non-degenerate, if $\forall \pi : A \to A'$, $\pi|_Z$ is surj or gen fin map.

For Z divisor,

$$Z \subset A \text{ nondeg} \iff \mathcal{O}(Z) \text{ is ample}$$

For Z curve,

$$Z \subset A \text{ nondeg} \iff Z \text{ generates } A.$$

[JKLM23, p42] $\overset{Z \text{ smooth}}{\iff}$ $N_{Z/A}$ ample