

Eine Woche, ein Beispiel

3.26. double coset decomposition

Double coset decompositions are quite impressive!

This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:

Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gerodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
<https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf>

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
<https://www-users.cse.umn.edu/~kwlanc/articles/intro-sh-ex.pdf>

[svd-notes]: Notes on singular value decomposition for Math 54
<https://math.berkeley.edu/~hutching/teach/54-2017/svd-notes.pdf>

<https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf>
<https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html>

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on \mathbb{P}^1 .

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \curvearrowright X$

discrete	\Rightarrow fundamental domain	$\Delta \subset \mathbb{C}$	$SL_2(\mathbb{Z}) \subset H$
non discrete	\Rightarrow stratification by G/G_x	$S' \subset S^2$	$C^\times \subset \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

$$\text{E.g. } \text{Flag}(V) \cong GL(V)/P \quad \text{e.g. Grassmannian, } \mathbb{P}^n$$

$$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$$

$$\begin{aligned} O(n) &:= O(n, \mathbb{R}) \\ SO(n) &:= SO(n, \mathbb{R}) \end{aligned} \quad \rightsquigarrow \text{Stiefel mfld} \quad [21.11.14]$$

$$A^n = A^n$$

$$H^n \cong O(1, n)/O(n)$$

$$H^n \cong GL_2(\mathbb{R})/O_{2, -1} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

\rightsquigarrow Hermitian symmetric space

$$\begin{aligned} \text{where } H^n &:= \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v \rangle = -1, v_{n+1} > 0\} \\ \langle , \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} &\longrightarrow \mathbb{R} \quad \langle v, w \rangle = v^T \begin{pmatrix} 1 & & \\ & \ddots & \\ & & -1 \end{pmatrix} w \end{aligned}$$

$$O(n, 1) := \text{Aut}(\mathbb{R}^{n+1}, \langle , \rangle) \subseteq GL_{n+1}(\mathbb{R})$$

$$O^+(n, 1) := \{g \in O(n, 1) \mid gH^n \subset H^n\}$$

For more informations about H^n , see [JL18, P62-67].

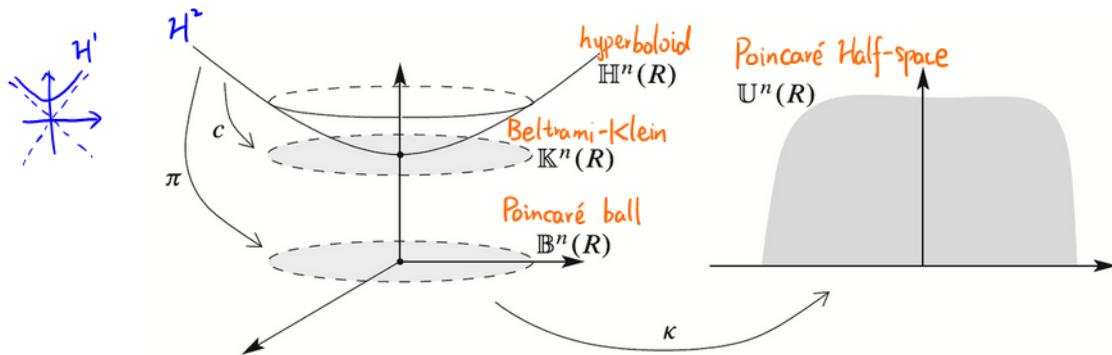
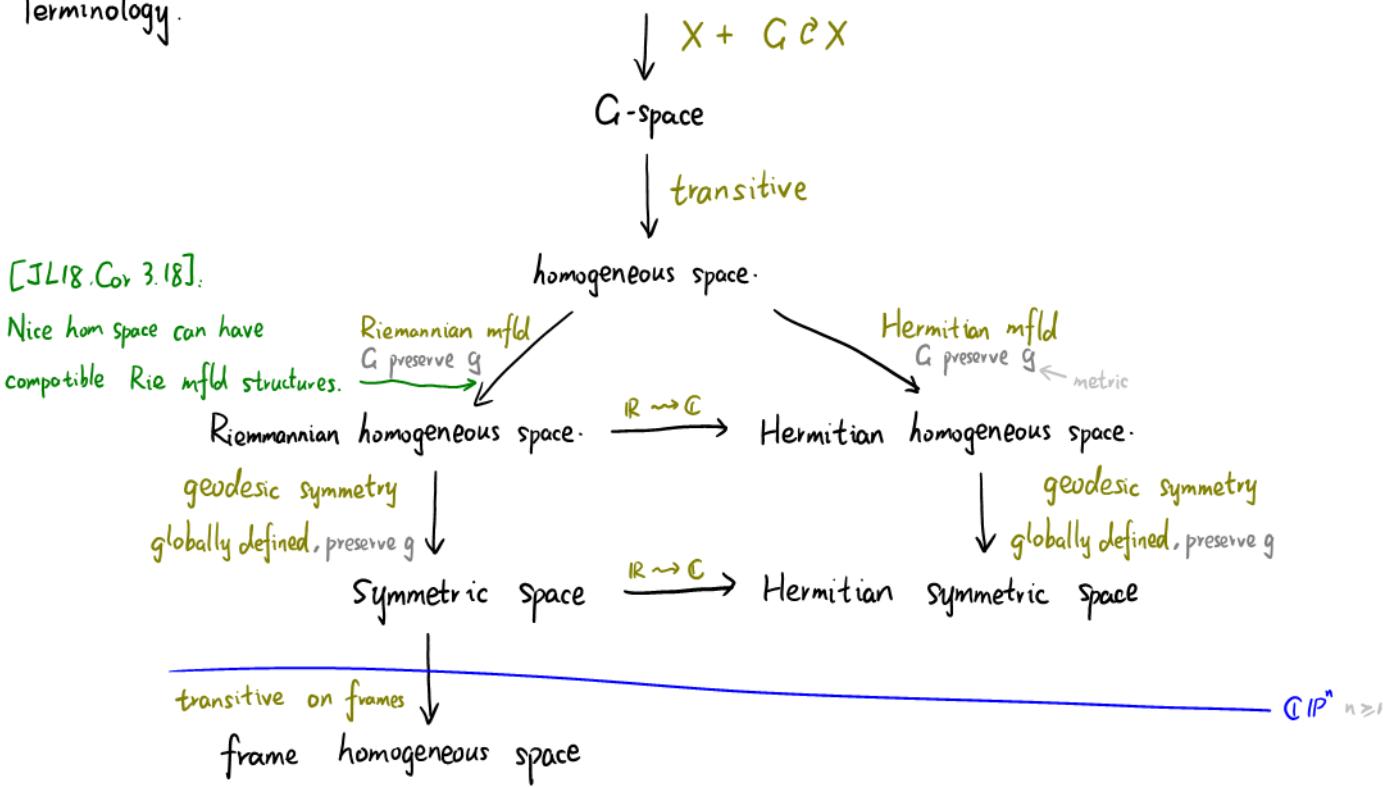


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

<https://math.stackexchange.com/questions/3340992/sl2-mathbb{R}-as-a-lorentz-group-0-1-2>

Terminology.



Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

Task: try to read materials about Drinfeld p-adic half-space.

double coset decomposition is the key technique to understand homogeneous spaces.

2. double coset decomposition: schedule

$$G = \bigsqcup_{\alpha \in I} H_\alpha K$$

usually, H, K are easier than G .

- comes from (usually) Gauss elimination
- I is the "fundamental domain"
- produces stratifications on G/K and $H\backslash G$ indexed by I .
each piece is again a homogeneous space.

To be exact,

$$G/K = \bigsqcup_{\alpha \in I} H_\alpha K / K \cong \bigsqcup_{\alpha \in I} H / H_{[\alpha K]} = \bigsqcup_{\alpha \in I} H / (H \cap \alpha K \alpha^{-1})$$

$$H\backslash G = \bigsqcup_{\alpha \in I} H \backslash H_\alpha K \cong \bigsqcup_{\alpha \in I} K_{[H_\alpha]} \backslash K = \bigsqcup_{\alpha \in I} (K \cap \alpha^{-1} H_\alpha) \backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$

$K_{[H_\alpha]}$: stabilizer of K on $[H_\alpha] \in H\backslash G$

$$\# H / (H \cap \alpha K \alpha^{-1}) = \# \left\{ \begin{array}{l} \text{single cosets } [gK] \\ \text{in one double coset } H_\alpha K \end{array} \right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad H\backslash G \quad \text{individually}$$

- can be viewed as stack quotient.

$[\ast/G]$: groupoid

$$H\backslash G / K \stackrel{\text{def}}{=} [\ast/H] \times_{[\ast/G]} [\ast/K] \text{ with groupoid structure}$$

Analog: $H\backslash G / K \approx \text{Spec } E \otimes_F E'$



$$H^*_H(G/K) \cong H^*(H\backslash G / K) \cong H^*_K(H\backslash G)$$

slogan: the (equiv) cohomology of G/K and $H\backslash G$ are connected.

- Hecke algebra $\mathcal{H}(H \backslash G / K)$

\uparrow for $H = K$. You can also do $\mathcal{H}(H_i \backslash G / H_j) \hookrightarrow \bigoplus_{i,j=1}^r \mathcal{H}(H_i \backslash G / H_j)$

$\mathcal{H}(H \backslash G / K)$: reasonable subspaces of

$$\mathbb{C}[H \backslash G / K] \stackrel{\text{"o-dim."}}{=} \left\{ f: G \rightarrow \mathbb{C} \mid f(hgk) = f(g) \quad \forall h \in H, g \in G, k \in K \right\}$$

$$\cong \bigoplus_{\alpha \in I} \mathbb{C} \mathbf{1}_{H\alpha K}$$

with reasonable convolution structure

$$*: \mathcal{H}(H_1 \backslash G / H_2) \times \mathcal{H}(H_2 \backslash G / H_3) \longrightarrow \mathcal{H}(H_1 \backslash G / H_3)$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

Vague: $\mathcal{H}(H \backslash G / K) \sim H^*(H \backslash G / K)$ should be a type of cohomology

$$H(G) \xrightarrow{G \text{ fin}} \mathbb{C}[G]$$

$\mathcal{H}(K \backslash G / K) \cong (\text{End}(c\text{-Ind}_K^G \mathbf{1}_K))^{\text{op}}$ should be a type of base ring

Generalize: $\text{Ind}_H^G X \approx \mathcal{H}_X(H \backslash G / K) \subseteq \left\{ f: G \rightarrow \mathbb{C} \mid f(hgk) = \underset{\sim \text{depth of } X}{X(h)f(g)} \right\}$

3. examples (after [22.09.04])

Works over:

- list of possibilities
- moduli interpretation
- typical examples

finite field, $GL_n(\mathbb{F}_q)$ (Applies to any field κ , actually)

- subgps can be

Borel	max split torus	unipotent	
B	T	N	
parabolic	Levi	unipotent	
P	L	M	
	nonsplit torus		
	T'		
+ SL_n, Sp_n, \dots	(as subgps)		$GL_n(\kappa)$

- moduli interpretation $V = \kappa^{\oplus n}$

$$G/B = \{ \text{cpl flags in } V \}$$

$$G/T = \{ (V_i)_{i=1}^n \mid V = \bigoplus_{i=1}^n V_i, \dim V_i = 1 \}$$

$$G/N = \left\{ (\mathcal{F}, m_i) \mid \begin{array}{l} \mathcal{F}: 0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_n = V \text{ cpl} \\ 0 \neq m_i \in M_i/M_{i-1} \end{array} \right\}$$

$$G/P = \{ \text{flags in } V \}$$

$$G/L = \{ (V_i)_{i \in I} \mid V = \bigoplus_{i \in I} V_i \}$$

$$G/M = \left\{ (\mathcal{F}, \mathcal{B}_i) \mid \begin{array}{l} \mathcal{F}: 0 = M_0 \subseteq M_1 \subseteq \dots \subseteq M_d = V \\ \mathcal{B}_i: \text{a basis of } M_i/M_{i-1} \end{array} \right\}$$

Rmk. After the help of Bruhat dec & transitivity, one can show

$$G/B = \{ \text{Borel subgps of } GL_n(\kappa) \}$$

$$G/N(T) = \{ \text{Torus of } GL_n(\kappa) \}$$

$$G/T = \{ \text{Borel pairs } (B, T) \text{ of } GL_n(\kappa) \}$$

Rmk. We have a fiber bundle

$$A^{\oplus \binom{n}{2}} \cong B/T \longrightarrow G/T$$
$$\downarrow$$
$$G/B$$

which makes G/T a $A^{\oplus \binom{n}{2}}$ -torsor over G/B .
◻ G/T is not a $\underbrace{\text{rk } \binom{n}{2} \text{ v.b.}}$ over G/B , so G/T can be affine space.
i.e. $GL(\binom{n}{2})(k)$ -torsor

- E.g. Bruhat decomposition

$$G = \bigsqcup_{w \in W} B w B$$

- Gauss elimination gives " \leq ", while the observation of process gives " \sqcup " (Something is invariant)
- the "fundamental domain" W has a gp structure, and crsp to B -orbits of G/B .
gp structure comes from Tits system
- produces an affine paving of G/B , and the Zariski topo gives Bruhat order
works also for Euclidean topo, $K = \mathbb{R}$ or \mathbb{C} .
- $B \backslash G/B = [\ast/B] \times_{[\ast/G]} [\ast/B]$, with $H_B^*(G/B) \cong H_T^*(G/B) \cong \bigoplus_w H_T^*(pt)$ [my master thesis]
- $H(G, B)$, see [22.09.04]
- More: Schubert calculus
 G -equiv v.b.
Borel - Weil - Bott theorem

- possible exercise:

- Work out

$$\begin{array}{cccc} P \backslash G / P_2 & GL_m \times GL_n \backslash GL_{m+n} / GL_m \times GL_n & T \backslash G / B \\ \mathbb{F}_q^\times \backslash GL_n(\mathbb{F}_q) / B, & \dots & S_m \times S_n \backslash S_{m+n} / S_m \times S_n \quad [22.11.13] \end{array}$$

$\kappa = \mathbb{F}_q$; $GL_n \rightsquigarrow$ other gps

- Computation of cardinals.

Ex. For $g \in GL_n(\kappa)$, show that

g admits an LU factorization \Leftrightarrow leading principal minors are all nonzero

$$g = \begin{bmatrix} \cdot & * \\ 0 & \cdot \end{bmatrix} \begin{bmatrix} \cdot & 0 \\ * & \cdot \end{bmatrix}$$

Archi field, \mathbb{R} or \mathbb{C}

Borel max split torus unipotent

B T N

R T N $(\cdot)_n$

11. Law

parabolic Levi unipotent

P L M

nonsplit torus

T'

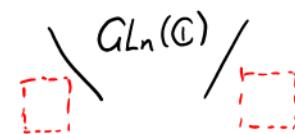
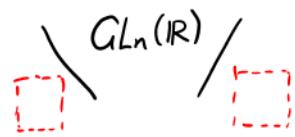
cpt

$O(n)$

or $SO(n)$

$$\mathcal{U}(n) := \mathcal{U}_{\mathbb{C}/\mathbb{R}}(n)$$

or $SU(n)$



+ real & cplx

<https://mathoverflow.net/questions/249313/real-orbits-on-flag-varieties>

+ discrete $\mathbb{Z}, \mathbb{Q}, \dots$

$\forall M_{n \times n}^{\text{sym}}(\mathbb{R}), M_{n \times n}^{\text{sym},>0}(\mathbb{R})$ are not subgps!

- ## - moduli interpretation

$V = \mathbb{R}^{n}$ In \mathbb{C}^{n} case, replace inner product by Hermitian prod.

$$G/O(n) \cong \{ \text{inner products on } V \} \cong M_{n \times n}^{\text{sym}, >0}(\mathbb{R})$$

$g = (v_1, \dots, v_n) \mapsto \langle \cdot, \cdot \rangle \text{ s.t. } \{v_1, \dots, v_n\} \text{ is an ortho basis} \mapsto (\langle e_i, e_j \rangle)_{i,j=1}^n$

$v_i = g e_i \quad \text{i.e. } \langle x, y \rangle := x^T (g^{-1})^T g^{-1} y$

$\underbrace{g}_{\longleftarrow} \qquad \qquad \qquad (g^{-1})^T g^{-1}$

as G -sets, where

$$g \cdot x := gx \quad g \cdot \langle \cdot, \cdot \rangle := \langle g^{-1} \cdot \cdot, g^{-1} \cdot \cdot \rangle \quad g \cdot A := (g^{-1})^T A g^{-1}$$

i.e. $\langle gx, gy \rangle_g = \langle x, y \rangle$

action on inner product

Rmk. We actually get the polar decomposition here: not hard, but not obvious

$$GL_n(\mathbb{R}) = M_{n \times n}^{\text{sym}, >0}(\mathbb{R}) O(n) \quad GL_n(\mathbb{C}) = M_{n \times n}^{\text{herm}, >0}(\mathbb{C}) U(n)$$

$$\begin{aligned} \text{Eq. } H &\cong GL_2(\mathbb{R}) / O(2) \cdot \mathbb{R}_{\geq 0} \cong SL_2(\mathbb{R}) / SO(2) \\ &\cong PGL_2(\mathbb{R}) / PO(2) \cong PSL_2(\mathbb{R}) / PSO(2) \end{aligned}$$

$$\begin{aligned} &\cong \{ \text{inner products on } V \} / \text{scalars} \\ &\cong M_{n \times n}^{\text{sym}, \geq 0}(\mathbb{R}) / \text{scalars} \\ &\cong \{ \text{max cpt subgps of } GL_2(\mathbb{R}) \} \\ &\stackrel{\text{Lemma 1, 2}}{\hookrightarrow} \end{aligned}$$

Lemma 1. cpt subgps are conj to a subgp of $O(2)$.

Idea of proof. $K \subset GL_2(\mathbb{R}) \subset H$ maps bounded set to bounded set
 $\Rightarrow K$ preserves one pt in H

Lemma 2. $gO(2)g^{-1} = O(2) \Leftrightarrow g \in O(2) \cdot \mathbb{R}_{\geq 0}$

Idea of proof. use G $\subset H$ or SVD \leftarrow shown later

- E.g. singular value decomposition (SVD) [svd-notes]

$$GL_n(\mathbb{R}) = \bigsqcup_{a_1 > a_2 > \dots > a_n > 0} O(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} O(n)$$

$$GL_n(\mathbb{C}) = \bigsqcup_{a_1 > a_2 > \dots > a_n > 0} U(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} U(n)$$

- " \subseteq ", lazy proof:

When $A \in GL_n(\mathbb{R})$ is symmetric, $A \xrightarrow{O(n)\text{-conj}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \in \mathbb{R}^\times$.

When $A \in GL_n(\mathbb{C})$ is normal matrix, $A \xrightarrow{U(n)\text{-conj}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \in \mathbb{C}^\times$

One can then use polar dec to show SVD.

- " \sqcup ", algorithm.

Suppose $A = U \Sigma V^T \in O(n) \Sigma O(n)$ $\Sigma = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. $a_i \in \mathbb{R}_{>0}$
Observe that

$$A^T A = V \Sigma^T \Sigma V^T = V \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} V^{-1}$$

\Rightarrow eigenvalues of $A^T A$ tell us Σ .

- " \subseteq ", algorithm. [svd-notes, Thm 3.2]

$$A^T A = V \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} V^{-1} \quad a_i \in \mathbb{R}_{>0} \quad A^T A(v_1, \dots, v_n) = (v_1, \dots, v_n) \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix}$$

Take $\Sigma := \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$, $U = AV\Sigma^{-1}$, then $U \in O(n)$, $A = U\Sigma V^T$.

- " \sqcup ", geometry:

$$a_1 = \max_{v \neq 0} \frac{\|Av\|}{\|v\|} \quad \|\cdot\| \text{, 2-norm}$$

$$a_k = \min_{\substack{V \subseteq \mathbb{C}^n \\ \dim k=1}} \max_{\substack{v \perp V \\ v \neq 0}} \frac{\|Av\|}{\|v\|}$$

Compare with: https://en.wikipedia.org/wiki/Min-max_theorem
(Courant-Fischer-Weyl min-max principle)

- the "fundamental domain"

$$I = \{(a_1, \dots, a_n) \in \mathbb{R}_{>0}^{\oplus n} \mid a_1 \geq a_2 \geq \dots \geq a_n\} = \bigsqcup_{\substack{(k, (n_1, \dots, n_k)) \\ \sum n_i = n}} I_{n_1, \dots, n_k}$$

$$I_{n_1, \dots, n_k} = \left\{ \underbrace{(a_1, \dots, a_1)}_{n_1}, \dots, \underbrace{(a_k, \dots, a_k)}_{n_k} \in \mathbb{R}_{>0}^{\oplus n} \mid a_1 > a_2 > \dots > a_k \right\}$$

is an n -dim real mfld, with boundary $I - I_{1, \dots, 1}$.

- produces a foliation of $GL_n(\mathbb{R})/\mathcal{O}(n)$ or $GL_n(\mathbb{C})/\mathcal{U}(n)$ indexed by I , with each piece iso to

$$\begin{aligned}\mathcal{O}(n)/\sum \mathcal{O}(n) \Sigma^{-1} \cap \mathcal{O}(n) &\cong \mathcal{O}(n)/\mathcal{O}(n_1) \times \dots \times \mathcal{O}(n_k) \cong GL_n(\mathbb{R})/L \\ \mathcal{U}(n)/\sum \mathcal{U}(n) \Sigma^{-1} \cap \mathcal{U}(n) &\cong \mathcal{U}(n)/\mathcal{U}(n_1) \times \dots \times \mathcal{U}(n_k) \cong GL_n(\mathbb{C})/L\end{aligned}$$

↑
QR dec

Space	$\dim_{\mathbb{R}}$	Space	$\dim_{\mathbb{R}}$
$GL_n(\mathbb{R})$	n^2	$GL_n(\mathbb{C})$	$2n^2$
$\mathcal{O}(n)$	$\frac{n(n-1)}{2}$	$\mathcal{U}(n)$	n^2
$GL_n(\mathbb{R})/\mathcal{O}(n)$	$\frac{n(n+1)}{2}$	$GL_n(\mathbb{C})/\mathcal{U}(n)$	n^2
$GL_n(\mathbb{R})/L$	$\frac{n(n+1)}{2}$	$GL_n(\mathbb{C})/L$	$\frac{n(n+1)}{2} \times 2$
I_{n_1, \dots, n_k}	k	I_{n_1, \dots, n_k}	k

E.g. The $SO(2)$ -orbit on $\mathcal{H} = SL_2(\mathbb{R})/SO(2)$ is as follows.



- stack quotient: not discussed yet

- [Getz, 3.3] <https://mathoverflow.net/questions/301410/what-is-the-archimedean-hecke-algebra>

$$\begin{aligned}\mathcal{H}(GL_n(\mathbb{R}), \mathcal{O}(n)) &= \left\{ f: GL_n(\mathbb{R}) \rightarrow \mathbb{C} \middle| \begin{array}{l} f \text{ distributions} \\ \text{supp } f \subseteq \mathcal{O}(n) \\ f \text{ bi } \mathcal{O}(n) \text{-finite} \end{array} \right\} \\ &\neq \left\{ f: GL_n(\mathbb{R}) \rightarrow \mathbb{C} \middle| \begin{array}{l} f \text{ sm, supp } f \text{ cpt,} \\ f(k_1 g k_2) = f(g) \quad \forall k_1, k_2 \in \mathcal{O}(n) \end{array} \right\}\end{aligned}$$

▼

bi $\mathcal{O}(n)$ -finite: $\langle f \rangle_{(\mathcal{O}(n), \mathcal{O}(n))\text{-module}} \subseteq \{\text{Distributions on } GL_n(\mathbb{R})\}$
 is of fin dim.

- E.g. QR decomposition
 ortho \uparrow upper

We write "RQ dec" instead.

$$GL_n(\mathbb{R}) = B \cdot O(n) = \bigsqcup_{t_i \in \mathbb{R}_{>0}} N \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} O(n)$$

$$GL_n(\mathbb{C}) = B \cdot U(n) = \bigsqcup_{t_i \in \mathbb{R}_{>0}} N \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} U(n)$$

- Gauss elimination by B: Gram-Schmidt process
- Gauss elimination by $O(n)$: rotation s.t. $Av_i \in \langle e_1, \dots, e_i \rangle$
- the "fundamental domain" is a single pt
- $GL_n(\mathbb{R})/O(n) \cong B/B \cap O(n) \stackrel{\text{as gp}}{\cong} \mathbb{R}_{>0}^n \oplus \mathbb{R}^{\oplus \binom{n}{2}}$
- $GL_n(\mathbb{C})/U(n) \cong B/B \cap U(n) \stackrel{\text{as gp}}{\cong} \mathbb{R}_{>0}^n \oplus \mathbb{C}^{\oplus \binom{n}{2}}$
- $B \setminus GL_n(\mathbb{R}) \cong B \cap O(n) \setminus O(n) \cong \mathbb{P}^{\oplus n} \setminus O(n)$ is cpt
- $B \setminus GL_n(\mathbb{C}) \cong B \cap U(n) \setminus U(n) \cong \mathbb{S}^{\oplus n} \setminus U(n)$ is cpt

Rmk. As a Corollary, we know the (higher) homotopy gp of $B \setminus GL_n(\mathbb{R})$.
 It's fundamental gp is still hard to construct.

e.g.

$$\pi_1(B \setminus GL_n(\mathbb{R})) \cong \begin{cases} \{\text{Id}\} & n=1 \\ \mathbb{Z} & n=2 \\ 1 \rightarrow \mathbb{Z}/\mathbb{Z} \rightarrow ? \rightarrow (\mathbb{Z}/\mathbb{Z})^{\oplus n} \rightarrow 1 & n>2 \end{cases}$$

The fundamental group of a real flag manifold
https://www.researchgate.net/publication/222792895_The_fundamental_group_of_a_real_flag_manifold

From this ref [Thm 1.1 + § 5.2], we see

$$\pi_1(B \setminus GL_n(\mathbb{R})) \cong \langle t_{21}, \dots, t_{n-1} \rangle / \left(\begin{array}{l} t_{21}t_{21+1} = t_{21+1}t_{21}^{-1}, t_{21+1}t_{21} = t_{21}t_{21+1}^{-1}, \\ t_{21}t_{2j} = t_{2j}t_{21} \quad |i-j| \geq 2 \end{array} \right)$$

e.g. $\pi_1(B \setminus GL_2(\mathbb{R})) \cong \langle t \rangle$

$$\begin{aligned} \pi_1(B \setminus GL_3(\mathbb{R})) &\cong \langle t, s \rangle / (sts^{-1}, stst^{-1}) \\ &\cong \langle t, s \rangle / (t^4 = 1, s^2 = t^2, sts^{-1} = t^{-1}) \cong Q_8 \end{aligned}$$

Cohomology rings of real flag manifolds are also well understood:

On the cohomology rings of real flag manifolds: Schubert cycles:
<https://link.springer.com/article/10.1007/s00208-021-02237-z>

- $H_{O(n)}^*(B \setminus GL_n(\mathbb{R})) \cong H_{O(n)}^*(B \cap O(n) \setminus O(n)) \cong H_{B \cap O(n)}^*(\text{pt})$
- $H_{U(n)}^*(B \setminus GL_n(\mathbb{C})) \cong H_{U(n)}^*(B \cap U(n) \setminus U(n)) \cong H_{B \cap U(n)}^*(\text{pt})$

- Possible ex: work out

$$\begin{aligned} SO(n) \backslash SL_n(\mathbb{R}) / SO(n) \\ O(n) \backslash GL_n(\mathbb{R}) / N, \quad O(n) \backslash GL_n(\mathbb{R}) / T, \quad O(n) \backslash GL_n(\mathbb{R}) / P, \\ GL_n(\mathbb{R}) \backslash GL_n(\mathbb{C}) / B, \dots \\ B \backslash SO(n+1) / SO(n) \end{aligned}$$

$\xrightarrow{\text{Borel of } SO(n+1)}$

~ Q: Can we find a good stratification of S^h in this way?

<https://math.stackexchange.com/questions/466998/what-are-the-borels-parabolics-of-the-orthogonal-or-symplectic-groups>

E.g. $SL_2(\mathbb{R}) \subset IP'(\mathbb{C}) = SL_2(\mathbb{C}) / B$

has three orbits.



$$H^+ = SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

$$IP'(\mathbb{R}) = SL_2(\mathbb{R}) / B(\mathbb{R})$$

$$H^- = SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

This connects Shimura variety & flag variety in a different way.

Therefore, a discussion of

$$GL_n(\mathbb{R}) \backslash GL_n(\mathbb{C}) / P$$

is needed.

NA field: $GL_n(F)$ [K_0 is maximal cpt. by Cartan dec]

- subgps can be
nearly affine

Borel max split torus unipotent

$$B(F) \quad T(F) \quad N(F)$$

parabolic Levi unipotent

$$P(F) \quad L(F) \quad M(F)$$

nonsplit torus

$$T'(F)$$

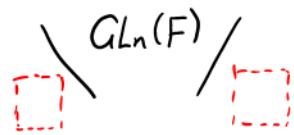
cpt

$$K_0 := GL_n(\mathcal{O}_F)$$

\cup

$$I_0$$

\vdots



+ field extension

cpt open subset

+ For SL_n , we have another coset $K'_0 = \left(\begin{smallmatrix} \mathcal{O}_F & \\ & F^\times \end{smallmatrix} \right)_{\det=1}$ not conj to K_0 .
In GL_n , $K'_0 = \left(\begin{smallmatrix} \pi & \\ & 1 \end{smallmatrix} \right) K_0 \left(\begin{smallmatrix} \pi^{-1} & \\ & 1 \end{smallmatrix} \right)$.

- moduli interpretation

$$V = F^{\oplus n}$$

Affine Grassmannian

$$\begin{aligned} G/K_0 &= \{ \mathcal{O}_F\text{-lattices in } V \} \\ &= \{ \text{NA norms in } V \} \end{aligned}$$

$$G/K_0 \cdot F^\times = \{ \text{max cpt subgps of } G \text{ conj to } K_0 \} \\ = \dots$$

Affine flag variety

$$\begin{aligned} G/I_0 &= \left\{ (L, \mathcal{F}) \mid \begin{array}{l} L: \mathcal{O}_F\text{-lattice in } V \\ \mathcal{F}: \text{cpl flag of } L/\pi L \end{array} \right\} / \sim \\ &= \{ \text{Iwahori subgps of } G \} \\ &= \dots \end{aligned}$$

Idea: Analog & comparison between Archi & NA:

$$GL_n(\mathbb{R})/\mathcal{O}(n)$$

$$GL_n(F)/K_0$$

Inner prod

\mathcal{O}_F -lattice L

orthonormal $\begin{cases} \text{normal} \\ \text{orthogonal} \end{cases}$

$$v \in L - \pi L$$

$\{ \pi_K(a_i v_i) \}$ basis of $L/\pi L$ $\left\{ \{ \pi_K(v_i) \} \text{ basis of } L/\pi L \right\}$

two many

- E.g. Cartan decomposition

$$GL_n(F) = \bigsqcup_{\lambda \in T} K_0 \lambda K_0$$

- Gauss elimination gives " \subseteq ", while the observation of process gives " \sqcup " (Something is invariant)
- the "fundamental domain" has no gp structure. They crsp to dominant weights of GL_n in $X^+(T)$
- produces a stratification of $GL_n(F)/K_0$ by K_0 -orbits, where

$$K_0 / K_0 \cap \lambda K_0 \lambda^{-1} = K_0 / \left(\begin{matrix} 0 & P^{e_i - e_j} \\ 0 & 0 \end{matrix} \right)_{\det \neq 0}$$

the NA local field topo gives the dominance order. $\lambda \geq \mu \Leftrightarrow \lambda - \mu$ is dom coroots

- $H_{K_0}^*(GL_n(F)/K_0)$. No idea yet.
- $H(GL_n(F), K_0)$: see [22.09.04]

... \rightarrow minuscule

- E.g. Iwahori decomposition

$$GL_n(F) = \bigsqcup_{w \in W_{\text{ext}}} I_0 \omega I_0$$

- Gauss elimination gives " \subseteq ", while the observation of process gives " \sqcup " (Something is invariant)
- the "fundamental domain" W_{ext} has a gp structure, called the extended affine Weyl gp.
gp structure comes from Tits system.

Cor.

$$K_0 = \bigsqcup_{w \in W_f} I_0 w I_0$$

- produces a stratification of $GL_n(F)/I_0$ by I_0 -orbits, where each piece is iso to $\kappa^{\oplus k}$ as a set.

the NA local field topo gives the Bruhat order.

- $I_0 \backslash GL_n(F) / I_0 = [\ast / I_0] \times_{[\ast / G]} [\ast / I_0]$, with $H_{I_0}^*(G/I_0) \cong \bigoplus_{w \in W_{\text{ext}}} H_{I_0}^*(\text{pt})$
- $H(GL_n(F), I_0)$: see [22.09.04]

- E.g. Iwasawa decomposition

$$GL_n(F) = B(F) \cdot K_0 = \bigsqcup_{\nu_i \in \mathbb{Z}} N(F) \begin{pmatrix} \pi^{\nu_1} & & \\ & \ddots & \\ & & \pi^{\nu_n} \end{pmatrix} K_0$$

$$= \bigsqcup_{w \in W_F} B(F) w I_0 = \bigsqcup_{\alpha \in W_{\text{ext}}} N(F) \alpha I_0$$

- Gauss elimination gives " \subseteq ",
while the observation of process gives " \sqcup " (Something is invariant)
- e.g. $F = \mathbb{Q}_3$.

Gauss elimination by $B(F)$:

$$\begin{pmatrix} 1 & & \\ \frac{1}{3} & 2 & \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ \frac{1}{3} & -6 & \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ \frac{1}{3} & 2 & \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} = \begin{pmatrix} 9 & & \\ 1 & \frac{1}{3} & \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & & \\ 6 & & \\ 1 & 1 & 3 \end{pmatrix}$$

Hint: find a basis of $(O_F/\mathfrak{p}_F)^{\oplus 3}$: $((1,1,3), (1,6,0), \dots)$

Gauss elimination by K_0 :

$$\begin{pmatrix} 1 & & \\ \frac{1}{3} & 2 & \\ \frac{1}{3} & \frac{1}{3} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ \frac{1}{3} & 2 & \\ 1 & \frac{1}{3} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ -6 & 2 & \\ 1 & \frac{1}{3} & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & & \\ -\frac{18}{5} & 1 & \\ -\frac{3}{5} & 2 & \frac{1}{3} \end{pmatrix}$$

- the "fundamental domain" is a single pt
- $GL_n(F)/K_0 \cong B(F)/B(F) \cap K_0 \cong B(F)/B(O_F)$
 $B(F) \backslash GL_n(F) \cong K_0 \cap B(F) \cong B(O_F) \backslash K_0$ is cpt
- $H_{B(F)}(GL_n(F)/K_0)$ is not clear for me.

- Possible ex: work out

$$GL_n(F) \backslash GL_n(E) / GL_n(F)$$

$$N(F) \backslash GL_n(F) / K_0, \quad B(F) \backslash GL_n(F) / I_0, \dots$$

4. special case: v, b on \mathbb{P}^1 .

https://en.wikipedia.org/wiki/Birkhoff_factorization