Eine Woche, ein Beispiel
10.2 equivariant K-theory of Steinberg variety : goal

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
https://people.math.harvard.edu/~gross/preprints/sat.pdf

Task. Complete the following tables:

G here is SL_n but not GL_n (to make sure the correctness of K(St))

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]\otimes_{\mathbb{Z}[X^*(T)]^W}\mathbb{Z}[X^*(T)]$	$\mathbb{Z}[X^*(T)]/I_T \rtimes \mathbb{Z}[W_f]$
Id	$\mathbb{Z}$			$\mathcal{H}_{ext}$
$G\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)]^W[t^{\pm1}]$			
$B\times\mathbb{C}^\times$	$\mathbb{Z}[X^*(T)][t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

We use the shorthand.

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
G	$R(T)^W$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$	$R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$	
Id	$\mathbb{Z}$			$R(T)/I_T \rtimes \mathbb{Z}[W_f]$
				$\mathcal{H}_{ext}$
$G\times\mathbb{C}^\times$	$R(G)[t^{\pm1}]$			
$B\times\mathbb{C}^\times$	$R(T)[t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$$R(B) = \mathbb{Z}[X^*(T)] \qquad = \mathcal{H}(\hat{T}(F),\hat{T}(\mathcal{O}_F))$$
$$R(G) = \mathbb{Z}[X^*(T)]^W \qquad \neq \mathcal{H}(\hat{G}(F),\hat{G}(\mathcal{O}_F))$$
$$R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[X^*(T)]^W[q^{\pm\frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}]$$
$$R(G\times\mathbb{C}^\times) = \mathbb{Z}[X^*(T)]^W[t^{\pm1}]$$
$$K^{G\times\mathbb{C}^\times}(St) = \mathcal{H}_{ext} \qquad \neq \mathcal{H}(\hat{G}(F), I)$$

Here is an initial example.

$K^-(-)$	pt	$\mathcal{B}\quad T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}\quad T^*(\mathcal{B}\times\mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1}]/((z_1-z_2)(z_1-z_2^{-1}))$	$\mathbb{Z}[W_{ext}] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]$
B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1},z]/(z-y)(z-y^{-1})$	$\mathbb{Z}[y^{\pm1},z_1,z_2]/((z_1-y)(z_1-y^{-1}),(z_2-y)(z_2-y^{-1}))$	
Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$	$R(T)/I_T \rtimes \mathbb{Z}[W_f] = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1}]/(z_w-1)^2$
$SL_2\times\mathbb{C}^\times$	$\mathbb{Z}[x,t^{\pm1}]$			$\mathcal{H}_{ext} = \bigoplus_{w\in W}\mathbb{Z}[z_w^{\pm1},t^{\pm1}]$
$B\times\mathbb{C}^\times$	$\mathbb{Z}[y^{\pm1},t^{\pm1}]$			
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm1}]$			

$K^-(-)$	pt	$\mathcal{F}_{\underline{d}} \quad \widetilde{Rep}_{\underline{d}}(Q)$	$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$Z_{\underline{d},\underline{d}'}$
G_d	$R(T_d)^{W_d}$	$R(T_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$	
B_d	$R(T_d)$ $\bigoplus_{w \in W_d} R(G_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$ $\bigoplus_{w \in W_d} R(T_d)[\bar{\Omega}_w]^{T_d}$	$R(T_d) \otimes_{R(G_d)} R(T_d) \otimes_{R(G_d)} R(T_d)$ $\bigoplus_{w,w' \in W_d} R(T_d)[\bar{\Omega}_{w,w'}]^{T_d}$	$\bigoplus_{w,w' \in W_d} R(T_d)[\widetilde{\bar{\Omega}}_{w,w'}]^{T_d}$
Id	$\mathbb{Z}$	$\bigoplus_{w \in W_d} \mathbb{Z}[\bar{\Omega}_w]$	$\bigoplus_{w,w' \in W_d} \mathbb{Z}[\overline{\Omega_{w,w'}}]$	$\bigoplus_{w,w' \in W_d} \mathbb{Z}[\widetilde{\bar{\Omega}}_{w,w'}]$
$G_d \times \mathbb{C}^\times$	$R(G_d)[t^{\pm 1}]$			
$B_d \times \mathbb{C}^\times$	$R(T_d)[t^{\pm 1}]$ $\bigoplus_{w \in W_d} R(G_d \times \mathbb{C}^\times)$	$\bigoplus_{w \in W_d} R(T_d \times \mathbb{C}^\times)[\bar{\Omega}_w]^{T_d \times \mathbb{C}^\times}$	$\bigoplus_{w,w' \in W_d} R(T_d \times \mathbb{C})[\overline{\Omega_{w,w'}}]^{T_d \times \mathbb{C}^\times}$	$\bigoplus_{w,w' \in W_d} R(T_d \times \mathbb{C})[\widetilde{\bar{\Omega}}_{w,w'}]^{T_d \times \mathbb{C}^\times}$
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$	$\bigoplus_{w \in W_d} R(\mathbb{C}^\times)[\bar{\Omega}_w]^{\mathbb{C}^\times}$	$\bigoplus_{w,w' \in W_d} R(\mathbb{C}^\times)[\overline{\Omega_{w,w'}}]^{\mathbb{C}^\times}$	$\bigoplus_{w,w' \in W_d} R(\mathbb{C}^\times)[\widetilde{\bar{\Omega}}_{w,w'}]^{\mathbb{C}^\times}$

$K^-(-)$	pt	$\mathcal{F}_{\underline{d}} \quad \widetilde{Rep}_{\underline{d}}(Q)$	$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}}$	$Z_d = \coprod_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$
G_d	$R(T_d)^{W_d}$	$\bigoplus_{\underline{d}} R(T_d)$	$\bigoplus_{\underline{d}} R(T_d) \otimes_{R(G_d)} R(T_d)$	
B_d	$R(T_d)$ $\bigoplus_{w \in W_d} R(G_d)$	$\bigoplus_{\underline{d}} R(T_d) \otimes_{R(G_d)} R(T_d)$ $\bigoplus_{\varpi \in W_{|d|}} R(T_d)[\bar{\mathcal{O}}_\varpi]^{T_d}$	$\bigoplus_{\underline{d}} R(T_d) \otimes_{R(G_d)} R(T_d) \otimes_{R(G_d)} R(T_d)$ $\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(T_d)[\overline{\mathcal{O}_{\varpi,\varpi'}}]^{T_d}$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(T_d)[\widehat{\bar{\mathcal{O}}_{\varpi,\varpi'}}]^{T_d}$
Id	$\mathbb{Z}$	$\bigoplus_{\varpi \in W_{|d|}} \mathbb{Z}[\bar{\mathcal{O}}_\varpi]$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} \mathbb{Z}[\overline{\mathcal{O}_{\varpi,\varpi'}}]$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} \mathbb{Z}[\widehat{\bar{\mathcal{O}}_{\varpi,\varpi'}}]$
$G_d \times \mathbb{C}^\times$	$R(G_d)[t^{\pm 1}]$			
$B_d \times \mathbb{C}^\times$	$R(T_d)[t^{\pm 1}]$ $\bigoplus_{w \in W_d} R(G_d \times \mathbb{C})$	$\bigoplus_{\varpi \in W_{|d|}} R(T_d \times \mathbb{C}^\times)[\bar{\mathcal{O}}_\varpi]^{T_d \times \mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(T_d \times \mathbb{C})[\overline{\mathcal{O}_{\varpi,\varpi'}}]^{T_d \times \mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(T_d \times \mathbb{C})[\widehat{\bar{\mathcal{O}}_{\varpi,\varpi'}}]^{T_d \times \mathbb{C}^\times}$
$\mathbb{C}^\times$	$\mathbb{Z}[t^{\pm 1}]$	$\bigoplus_{\varpi \in W_{|d|}} R(\mathbb{C}^\times)[\bar{\mathcal{O}}_\varpi]^{\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(\mathbb{C}^\times)[\overline{\mathcal{O}_{\varpi,\varpi'}}]^{\mathbb{C}^\times}$	$\bigoplus_{\varpi,\varpi' \in W_{|d|}} R(\mathbb{C}^\times)[\widehat{\bar{\mathcal{O}}_{\varpi,\varpi'}}]^{\mathbb{C}^\times}$

$$H_*^{G_d}(Z_{\underline{d},\underline{d}}) \cong \bigotimes_{d_i} NH_{d_i}$$

$$K^{G_d}(Z_{\underline{d},\underline{d}}) \cong R(T_d) \otimes_{R(G_d)} R(T_d) \cong \bigotimes_{d_i} R(T_{d_i}) \otimes_{R(G_{d_i})} R(T_{d_i})$$

Black: know the alg structure under tensor prod
Grey: know the alg structure under tensor prod, which is not prefered
red: know the alg structure under convolution prod
Orange: only know the $R(Grp)$-module structure, and the alg structure is yet not known
Light yellow: $R(G_d)$-module + W_d-equiv iso

$d = (1,2)$

$a \longrightarrow b$

$\langle v_1 \rangle \longrightarrow \langle v_2, v_3 \rangle$

⚠ The action on Flag is not the same as in

		$ww = wu$			w	$d = u$	order of basis	$l(\varpi)$	$l(w)$	B_w	B_w	wB_w^{-1}
Id	Id	$\begin{pmatrix}123\\123\end{pmatrix}$				abb	$\{v_1, v_2, v_3\}$	0	0			
t	(23)	$\begin{pmatrix}123\\132\end{pmatrix}$				abb	$\{v_1, v_3, v_2\}$	1	1			
s	(12)	$\begin{pmatrix}123\\213\end{pmatrix}$				bab	$\{v_2, v_1, v_3\}$	1	0			
ts	(132)	$\begin{pmatrix}123\\312\end{pmatrix}$				bab	$\{v_3, v_1, v_2\}$	2	1			
st	(123)	$\begin{pmatrix}123\\231\end{pmatrix}$				bba	$\{v_2, v_3, v_1\}$	2	0			
sts	(13)	$\begin{pmatrix}123\\321\end{pmatrix}$				bba	$\{v_3, v_2, v_1\}$	3	1			

Conclusions on the summer vacation.

Some possible tasks to work on.

1. Work out what $K_0^{Id}(\mathcal{B})$ is.

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.
   - Bernstein presentation
   - try to understand the center of $\mathcal{H}(G(F), I)$
   - How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory
   - How can the Hecke algebra be realized as a Hecke algebra?

   ref. [Hecke, Sec 10-17], [Williamson 11.4 - 12.2]

3. Try to understand what the Hall algebra / Quantum group is.

   - understand $\mathcal{H}_{Rep_k^{nil}(Q)}$ where $Q = \cdot \longrightarrow \cdot \circlearrowleft$ [Lec 2-3]
   - understand $\mathcal{H}_{\mathbb{P}^1} \cong \mathcal{U}_v(\widehat{\mathfrak{sl}_2})$ [Lec 4]

   $$\cup \qquad\qquad \cup$$

   $$\mathcal{H}_{Tor(\mathbb{P}^1)} \cong \bigotimes_{x \in \mathbb{P}^1} \mathcal{H}_{Tor \, x}$$

   - define (Quantum) Kac-Moody / loop algs   [Appendix 4]
   - Why is that
   $$K_0(Rep^{\mathbb{Z}}(R)) = \mathcal{U}_q(n(Q))$$
   (graded)

   where
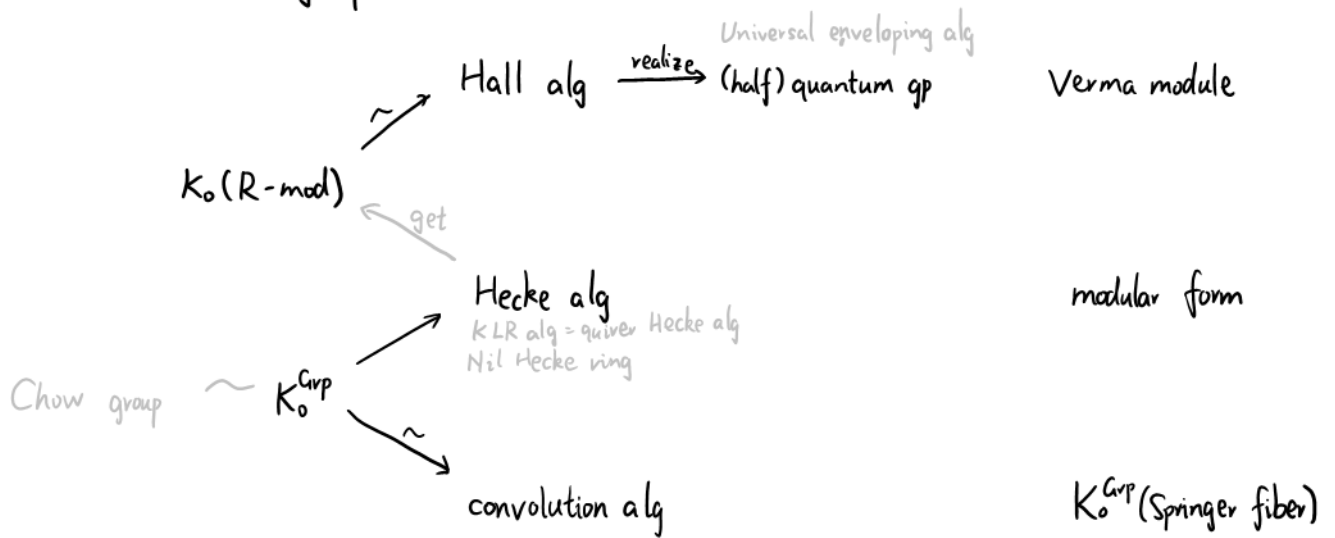 $$R = \bigoplus_{\underline{d}} H_{\bullet}^{G \times \mathbb{C}^\times, BM}(Z_{\underline{d}})$$

   and what is
   $$K_0\left(Rep^{\mathbb{Z}}\left(\bigoplus_{\underline{d}} K_0^{G \times \mathbb{C}^\times}(Z_{\underline{d}})\right)\right) ?$$

4. Work out the big picture

Hall alg $\xrightarrow{\text{realize}}$ (half) quantum gp   Verma module

~ Universal enveloping alg

$K_0(R\text{-mod})$

get

Hecke alg   modular form

KLR alg = quiver Hecke alg
Nil Hecke ring

Chow group ~ K_0^{Grp}

~

convolution alg $K_0^{Grp}(\text{Springer fiber})$

5. A closer check of Satake iso

$$
\begin{array}{lll}
K_0 & \text{combinations} & \text{Hecke alg} \\
R(B) = \mathbb{Z}[x^*(T)] & & = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F)) \\
R(G) = \mathbb{Z}[x^*(T)]^W & & \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
R(G)[q^{\pm\frac{1}{2}}] = \mathbb{Z}[x^*(T)]^W[q^{\pm\frac{1}{2}}] & & = \mathcal{H}_{sph}[q^{\pm\frac{1}{2}}] \\
R(G \times \mathbb{C}^*) = \mathbb{Z}[x^*(T)]^W[t^{\pm 1}] & & \\
R(T) \otimes_{R(G)} R(T) = NH_n \subset End_{\mathbb{Z}}(\mathbb{Z}[x^*(T)]) & & \\
K^{G \times \mathbb{C}^*}(St) = \mathcal{H}_{ext} & & \overset{?}{\neq} \mathcal{H}(\hat{G}(F), I)
\end{array}
$$

It's claimed by my schoolmate that

$$K_0(Perv_B(G/B)) \cong \mathcal{H}(G, B)$$

↑

sym monoidal structure
induced from the convolution

then, what is

$$
\begin{array}{lll}
K_0^B(\mathcal{B}) & \cong & ? \\
K_0^{Id}(\mathcal{B}) & \cong & ? \\
? & \cong & \mathcal{H}(S_{m+n}, S_m \times S_n)
\end{array}
$$

$G = GL_n$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces.
   (e.p. in the case of $\cdot$, $\cdot\circlearrowleft$, $\cdot\to\cdot$)

$$G, B, T$$

$$\mathcal{B}, T^*\mathcal{B}, St$$

$$\mathfrak{g}, \widetilde{\mathfrak{g}}, \mathfrak{g}^{sr}, \widetilde{\mathfrak{g}}^{sr}, \mathcal{N}, \widetilde{\mathcal{N}}, \mathcal{L}, n$$

$$\widetilde{\mathfrak{g}}^h, \mathcal{O}_h, \Lambda^h_\omega$$

7. Try to understand what Kazhdan–Lusztig polynomials are [KL2], and
   - Compute the transformation matrix between
     $$\{[T^*_\omega], \omega \in W_f\} \quad \text{and} \quad \{[\Lambda^h_\omega], \omega \in W_f\} \; ? \qquad \text{[KaSai]?}$$
   - understand what standard/crystal basis is
   - understand the relationship between KL poly and crystal basis
   - see if it is related to two basis in Rep (G)   (irr reps & multiplicative basis)

Answer from my schoolmate:

standard basis $\xleftrightarrow{\text{KL-poly}}$ canonical basis $\xrightarrow{\text{fix } q}$ crystal basis

$$\{[T^*_\omega], \omega \in W_f\} \qquad \{[\Lambda^h_\omega], \omega \in W_f\}$$

irr reps   multiplicative basis

8. Try to understand the module part, i.e.,
   - numbers of components of the Springer fiber
   - how does $K_0^{Grp}(St)$ act on $K_0^{Grp}(\text{Springer fiber})$ also act on $K_0^{Grp}(\text{Rep}_{pol}(G))$
   - does that occupy "all rep" of $K_0^{Grp}(St)$

9. Ways of finding multiplication structure
   1. By direct computation (with techniques)
      Hecke algebra   double coset calculus
   2. By formulas as alg-isos
      $K_0^G(\mathcal{B})$ induction formula
   3. By geometrical computation
      cohomology   cup product? de Rham calculus
                                                      index theorem

      Chow group   intersection theory
   4. By deformation (indirect)
      $H_{top}^{BM}(St)$
      $K_0^{G\times\mathbb{C}^\times}(St)$

10. Different views on the double coset
$$B\backslash G/B = (*/B)\times_{*/G}(*/B)$$
    - as a set
    - as flag variety quotient B-action
    - as a stack
    - groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $\mathcal{H}(G, B)$, $\mathcal{H}_{sph}$, ($\mathcal{H}_{aff}$)	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG,....	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:
- 3 Seminars   (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
  - L-parameters
  - moduli in AG
  - some following developments of the modular form (different type of gps, Hecke operators...)
  - reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.