ein Woche, eine Beispiel

April 16th. examples in algebraic topology

Examples:

Past	Today	Future
closed surface dim 2	S^n S^∞	Lens space
Hopf surface dim 4	$\mathbb{RP}^n$ $\mathbb{RP}^\infty$	Lie group
K3 surface	$\mathbb{CP}^n$ $\mathbb{CP}^\infty$	Grassmannian mfld, e.g. $Gr(2,4)$
	...	Moore space
		Eilenberg-Maclane space
		...

Goal.

- compute $H_n(X,\mathbb{Z})$, $H^*(X,\mathbb{Z})$, $\pi_n(X,\mathbb{Z})$
- compute characteristic class and applies the results.
- optional question. is X
  * oriented?
  * a mfld? of dim n
  * a cplx mfld? ← proj | kähler | complex
  * a Lie group?
  * ......

Today: S^n S^∞; $\mathbb{RP}^n$, $\mathbb{RP}^\infty$; $\mathbb{CP}^n$, $\mathbb{CP}^\infty$; ...

$S^\infty := \bigcup_{n\geq 1} S^n$ $\quad S^n \hookrightarrow S^m$ by $(x_0,\cdots,x_n) \mapsto (x_0,\cdots,x_n,0,\cdots,\underset{\uparrow m}{0})$

1. relations: fiber bundle

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} \to S^n & S^1 \to S^{2n+1} & \mathbb{Z}/k\mathbb{Z} \to S^{2n+1} \\ \downarrow & \downarrow & \downarrow \\ \mathbb{RP}^n & \mathbb{CP}^n & S^{2n+1}/_{\mathbb{Z}/k\mathbb{Z}} \end{array}$$

[n=1: Hopf fibration] $\quad k\in\mathbb{N}^+,\ k>1$

$$\begin{array}{ccc} \mathbb{Z}/2\mathbb{Z} \to S^\infty & S^1 \to S^\infty & \mathbb{Z}/k\mathbb{Z} \to S^\infty \\ \downarrow & \downarrow & \downarrow \\ \mathbb{RP}^\infty & \mathbb{CP}^\infty & S^\infty/_{\mathbb{Z}/k\mathbb{Z}} \end{array}$$

2. (canonical) CW structure.

e.g.

# m-cell	0	1	2	3	4	5	$m>5$
S^5	2	2	2	2	2	2	0
$\mathbb{RP}^5$	1	1	1	1	1	1	0
$\mathbb{CP}^2$	1	0	1	0	1	0	0

$$\Rightarrow \begin{cases} \chi(S^n) = \begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases} \\ \chi(\mathbb{RP}^n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases} \\ \chi(\mathbb{CP}^n) = n+1 \end{cases}$$

3. Homology & Cohomology

homology

$H_i(X,\mathbb{Z})$	0	1	2	3	4	5	$i>5$
S^5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0
$\mathbb{RP}^5$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}$	0
$\mathbb{CP}^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0
$\mathbb{RP}^4$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0

Cor. $\mathbb{RP}^{2n}$ is nonoriented; $\mathbb{RP}^{2n+1}$, S^n, $\mathbb{CP}^n$ are oriented.

S^5: $0 \to \mathbb{Z}e_1^5 \oplus \mathbb{Z}e_2^5 \to \mathbb{Z}e_1^4 \oplus \mathbb{Z}e_2^4 \to \mathbb{Z}e_1^3 \oplus \mathbb{Z}e_2^3 \to \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2^2 \to \mathbb{Z}e_1^1 \oplus \mathbb{Z}e_2^1 \to \mathbb{Z}e_1^0 \oplus \mathbb{Z}e_2^1 \to 0$

$e_1^5 \longrightarrow e_1^4 - e_2^4 \qquad e_1^3 \longrightarrow e_1^2 - e_2^2 \qquad e_1^1 \mapsto e_1^0 - e_2^0$

$e_2^5 \longrightarrow -e_1^4 + e_2^4 \qquad e_2^3 \longrightarrow -e_1^2 + e_2^2 \qquad e_2^1 \mapsto -e_1^0 + e_2^0$

$e_1^4 \longmapsto e_1^3 + e_2^3 \qquad e_1^2 \longmapsto e_1^1 + e_2^1$

$e_2^4 \longmapsto e_1^3 + e_2^3 \qquad e_2^2 \longmapsto e_1^1 + e_2^1$

$\mathbb{Z}$ $\quad 0 \quad 0 \quad 0 \quad 0 \quad$ $\mathbb{Z}$

[Rmk. The definition of cellular homology uses the (singular) homology of S^0, so seriously we can't compute $H_i(S^n,\mathbb{Z})$ by cellular homology.]

$\mathbb{R}P^5$: $0 \longrightarrow \mathbb{Z}e^5 \longrightarrow \mathbb{Z}e^4 \longrightarrow \mathbb{Z}e^3 \longrightarrow \mathbb{Z}e^2 \longrightarrow \mathbb{Z}e^1 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$e^5 \longmapsto 0$ $\quad e^3 \longmapsto 0$ $\quad e^1 \longmapsto 0$

$e^4 \longmapsto 2e^3$ $\quad e^2 \longmapsto 2e^1$

$\mathbb{Z} \quad 0 \quad \mathbb{Z}/2\mathbb{Z} \quad 0 \quad \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}$

$\mathbb{R}P^4$: $0 \longrightarrow \mathbb{Z}e^4 \longrightarrow \mathbb{Z}e^3 \longrightarrow \mathbb{Z}e^2 \longrightarrow \mathbb{Z}e^1 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$e^3 \longmapsto 0$ $\quad e^1 \longmapsto 0$

$e^4 \longmapsto 2e^3$ $\quad e^2 \longmapsto 2e^1$

$0 \quad \mathbb{Z}/2\mathbb{Z} \quad 0 \quad \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}$

$\mathbb{C}P^2$ $\quad 0 \longrightarrow \mathbb{Z}e^4 \longrightarrow 0 \longrightarrow \mathbb{Z}e^2 \longrightarrow 0 \longrightarrow \mathbb{Z}e^0 \longrightarrow 0$

$\mathbb{Z} \quad 0 \quad \mathbb{Z} \quad 0 \quad \mathbb{Z}$

Similarly, $H_n(S^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{otherwise} \end{cases}$

$$H_n(\mathbb{R}P^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}/2\mathbb{Z} & n \text{ odd} \\ 0 & \text{otherwise} \end{cases} \qquad H_n(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$$

$$H_n(\mathbb{C}P^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & n \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

cohomology

$H^i(X,\mathbb{Z})$	0	1	2	3	4	5	$i>5$
S^5	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}$	0
$\mathbb{R}P^5$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	0
$\mathbb{C}P^2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0
$\mathbb{R}P^4$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0	0

$$\Rightarrow \begin{cases} H^*(\mathbb{R}P^{2n}) = \mathbb{Z}[x]/(2x, x^{n+1}) \\ H^*(\mathbb{R}P^{2n+1}) = \mathbb{Z}[x]/(2x, x^{n+1}) \oplus \mathbb{Z}y \\ H^*(\mathbb{C}P^n) = \mathbb{Z}[x]/(x^{n+1}) \end{cases}$$

$\boxed{\begin{array}{l} \deg x = 2 \\ \deg y = 5 \\ \deg t = 1 \end{array}}$

$$\Rightarrow \begin{cases} H^*(\mathbb{R}P^\infty) = \mathbb{Z}[x]/(2x) \\ H^*(\mathbb{C}P^\infty) = \mathbb{Z}[x] \\ H^*(\mathbb{R}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[t] \\ H^*(\mathbb{C}P^\infty, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[x] \end{cases}$$

prod structure: use Poincaré duality & cellular cohomology. see [May, P153].

$H^{2q}(\mathbb{C}P^n) \xrightarrow{\sim} H^{2q}(\mathbb{C}P^{n-1})$ for $q<n$

https://math.stackexchange.com/questions/1128712
/integral-cohomology-ring-of-real-projective-space

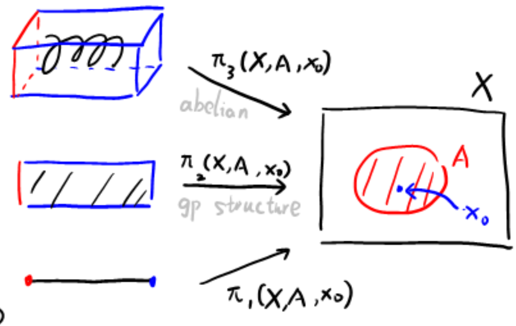
By spectral sequence: GTM 82 Example 14.22, 14.32,
Ex 18.4, 18.10

Interlude: LES of homotopy groups
$x_0 \in B \subset A \subset X$, X top space

Def. relative homotopy group

$$\pi_n(X, A, x_0) = [f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)] \quad n \geq 1 \quad J^{n-1} := \partial I^n \setminus I^{n-1}$$

Relations: $f \sim g \iff \exists\, F_t: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ s.t.
(denote by $[f]=[g]$) $\quad F_0 = f \quad F_1 = g$



Lemma: Suppose $f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, $f(I^n) \subseteq A$, then $[f] = [0]$

Thm. we have LES $\leftarrow$ ker = Im

$$\cdots \to \pi_2(A,B,x_0) \to \pi_2(X,B,x_0) \to \pi_2(X,A,x_0) \to \pi_1(A,B,x_0) \to \pi_1(X,B,x_0) \to \pi_1(X,A,x_0)$$

Cor. we have LES

$$\pi_2(A,x_0) \to \pi_2(X,x_0) \to \pi_2(X,A,x_0) \to \pi_1(A,x_0) \to \pi_1(X,x_0) \to \pi_1(X,A,x_0) \to \pi_0(A,x_0) \to \pi_0(X,x_0)$$

Thm. when $p: E \to B$ has the homotopy lifting property w.r.t. I^k ($\forall k \geq 0$) ↗ called Serre fibration
then (denote $b_0 \in B$, $x_0 \in F := p^{-1}(B)$)

$$p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0) \quad n \geq 1$$

is an isomorphism.

$$\begin{array}{ccc} I^k \times [0,1] & \longrightarrow & E \ni x_0 \\ \downarrow & \overset{\exists}{\nearrow} & \downarrow p \\ I^k & \longrightarrow & B \ni b_0 \end{array}$$

Rmk. 1 The proof mainly uses the HLP: homotopy lifting property.
2. Any fiber bundle $p: E \to B$ is a Serre fibration.

Cor Suppose $p: E \to B$ is the fiber bundle map, then we have LES
(denote $b_0 \in B$, $x_0 \in F := p^{-1}(B)$)

$$\pi_2(F,x_0) \to \pi_2(E,x_0) \to \pi_2(B,b_0) \to \pi_1(F,x_0) \to \pi_1(E,x_0) \to \pi_1(B,b_0) \to \pi_0(F,x_0) \to \pi_0(E,x_0)$$

4. Homotopy: by LES of fibration, we obtain $\quad n \geq 2$

$$\pi_m(\mathbb{RP}^n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & m=1 \\ \pi_m(S^n) & m>1 \end{cases} \qquad \pi_m(\mathbb{CP}^n) = \begin{cases} 0 & m=1 \\ \mathbb{Z} & m=2 \\ \pi_m(S^{2n+1}) & m>2 \end{cases}$$

$$S^1 \to S^{2n+1} \to \mathbb{CP}^n$$

Rmk. S^∞ is contractable by the argument in https://mathoverflow.net/questions/198

Cor. $\mathbb{RP}^\infty$ is of type $K(\mathbb{Z}/2\mathbb{Z}, 1)$
$\mathbb{CP}^\infty$ is of type $K(\mathbb{Z}, 2)$

in GTM 82 What I can prove now

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}
S^0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_{15}$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_{24}\times\mathbb{Z}_3$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

by Hopf fibration

$\pi_7(S^7)$

5. Characteristic class

We have both tautological vector bundle and tangent bundle for $S^n, \mathbb{RP}^n, \mathbb{CP}^n$.

$\mathbb{CP}^n$: by https://en.wikipedia.org/wiki/Chern_class ,

$$c(\mathbb{CP}^n) \overset{\text{def}}{=} c(T\mathbb{CP}^n) = c(\mathcal{O}_{\mathbb{CP}^n}(1))^{n+1} = (1+a)^{n+1},$$

where a is the canonical generator of the cohomology group $H^2(\mathbb{CP}^n, \mathbb{Z})$;

tautological bundle $\mathcal{O}_{\mathbb{CP}^n}(-1)$: $c(\mathcal{O}_{\mathbb{CP}^n}(-1)) = 1 - a$

Cor. $T\mathbb{CP}^n$, $\mathcal{O}_{\mathbb{CP}^n}(-1)$ are not spin; $\mathbb{CP}^n$ is not a boundary.

$\mathbb{RP}^n$: similarly, $w(\gamma_n^1) = 1+t$ $\quad w(\mathbb{RP}^n) = w(\gamma_n^1)^{n+1} = (1+t)^{n+1}$

Cor. γ_n^1 is not orientable;

$T\mathbb{RP}^n$ is orientable only when $n \equiv 1 \mod 2$;

$T\mathbb{RP}^n$ is spin only when $n \equiv 3 \mod 4$ or $n = 1$.

S^n: Lemma $\pi^*: H^n(\mathbb{RP}^n, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^n(S^n, \mathbb{Z}/2\mathbb{Z})$ is zero.

Proof by computation.

$$C_\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \quad 0 \longrightarrow e^5 \longrightarrow e^4 \longrightarrow e^3 \longrightarrow e^2 \longrightarrow e^1 \longrightarrow e^0 \longrightarrow 0$$

$$e_1^5 \mapsto e^5, \quad e_2^5 \mapsto e^5$$

$$C_\cdot(S^5, \mathbb{Z}/2\mathbb{Z}) \quad 0 \longrightarrow e_1^5, e_2^5 \longrightarrow e_1^4, e_2^4 \longrightarrow e_1^3, e_2^3 \to e_1^2, e_2^2 \to e_1^1, e_2^1 \to e_1^0, e_2^0 \to 0$$

$$e_1^5 \longmapsto e_1^4 - e_2^4$$
$$e_2^5 \longmapsto -e_1^4 + e_2^4$$

$$C^\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \quad 0 \longleftarrow e^{5*} \longleftarrow e^{4*} \longleftarrow e^{3*} \longleftarrow e^{2*} \longleftarrow e^{1*} \longleftarrow e^{0*} \longleftarrow 0$$

$$e^{5*} \mapsto e_1^{5*} + e_2^{5*}$$

$$C^\cdot(\mathbb{RP}^5, \mathbb{Z}/2\mathbb{Z}) \quad 0 \longleftarrow e_1^{5*}, e_2^{5*} \longleftarrow e_1^{4*}, e_2^{4*} \leftarrow e_1^{3*}, e_2^{3*} \leftarrow e_1^{2*}, e_2^{2*} \leftarrow e_1^{1*}, e_2^{1*} \leftarrow e_1^{0*}, e_2^{0*} \leftarrow 0$$

$$e_1^{5*} - e_2^{5*} \longleftarrow e_1^{4*}$$
$$-e_1^{5*} + e_2^{5*} \longleftarrow e_2^{4*}$$

btw. when n is odd,
$$\begin{array}{ccc} H^n(\mathbb{RP}^n, \mathbb{Z}) & \longrightarrow & H^n(S^n, \mathbb{Z}) \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} \end{array}$$

Cor. $w(\gamma_{n,S^n}^1) = \pi^* w(\gamma_{n,\mathbb{RP}^n}^1) = 1$

$w(TS^n) = \pi^* w(T\mathbb{RP}^n) = 1$

γ_{n,S^n}^1, TS^n are spin, $S^n = \partial D^n$.

6. Cplx mfld

$\mathbb{CP}^n$ is undoubtedly proj cplx mfld.

$\mathbb{RP}^{2n-1}$, S^{2n-1} are not cplx mflds since they're of odd dim.

$\mathbb{RP}^{2n}$ is not cplx mfld since it's not orientable.

$S^n (n>6)$, S^4 are not cplx mflds. see https://mathoverflow.net/questions/11664/complex-structure-on-sn

whether S^6 is a cplx mfld is still an open problem, see https://mathoverflow.net/questions/1973/is-there-a-complex-structure-on-the-6-sphere

related problems: is the cplx structure of $\mathbb{CP}^n$ unique? Still open, see https://mathoverflow.net/questions/382442/is-the-complex-structure-of-mathbb-cpn-unique

7. Lie group: S^1, S^3, $\mathbb{RP}^1$, $\mathbb{RP}^3$, we have $\mathbb{RP}^1 \cong S^1$ and

$$\begin{tikzcd} S^3 \arrow[d, "\pi"] \arrow[r, phantom, "\cong"] & SU_2 \arrow[d, "\pi'"] \arrow[r, phantom, "\cong"] & \{q \in \mathbb{H} \mid |q| = 1\} \\ \mathbb{RP}^3 \arrow[r, phantom, "\cong"] & SO_3 & \end{tikzcd}$$

https://math.stackexchange.com/questions/4065801/collecting-proofs-so3-cong-bbb-r-p3

for S^n: https://math.stackexchange.com/questions/12453/is-there-an-easy-way-to-show-which-spheres-can-be-lie-groups

for $\mathbb{RP}^n$: lemma: a Lie/topological group structure lifts to a covering space

proof: see https://math.stackexchange.com/questions/5391/covering-of-a-topological-group-is-a-topological-group

Cor: $\mathbb{RP}^n$ $(n>3)$ is not a Lie group

for $\mathbb{CP}^n$: lemma. for the connected Lie group G, $\pi_2(G) = 0$ $\quad$ $\pi_3(G)$ has no torsion!

proof: see https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups

Cor. $\mathbb{CP}^n$ is not a Lie group.

different proof of this cor: https://math.stackexchange.com/questions/3043483/lie-group-structure-on-the-complex-projective-space

Interesting results during the ways of searching

Lemma: a cpt Lie group is either abelian $\Rightarrow$ torus

or nonabelian & have nonzero H^3.

See https://math.stackexchange.com/questions/3421788/topological-lie-group-structure-on-projective-spaces

Lemma every compact Lie group has zero Euler characteristic since it is parallelizable

See https://math.stackexchange.com/questions/829928/can-s2-be-turned-into-a-topological-group/