Eine Woche, ein Beispiel

2.23 Schubert calculus: coh of Grassmannian

Ref:
[3264] and [Fulton]
[LW21]: https://www.math.uni-bonn.de/ag/stroppel/Masterarbeit_Wang.pdf

We will attempt to tackle Schubert calculus in a concise manner. The term "Schubert calculus" is often associated with intersection theory, enumerative geometry, combinatorics, Grassmannians, and more, making it a vast topic. However, I believe its core ideas can be clearly explained in just six hours. I will break the material into several parts:

1. $H^\cdot(Gr(r,n);\mathbb{Z})$ and its combinatorics
2. (inside Grassmannian)
   cycles in Grassmannian, including:
   - cycle class map: $CH^\cdot(Gr(r,n)) \xrightarrow{\sim} H^\cdot(Gr(r,n);\mathbb{Z})$
   - incidence variety $\begin{cases} \text{(partial) flag variety} \\ \text{Fano variety of planes} \\ \cdots \end{cases}$
   - a reintepretation of cycles
3. (outside Grassmannian + v.b.)

$$\begin{array}{ccc} \mathcal{L} & & \mathcal{S}^\vee \\ | & & | \\ X & \xrightarrow{f_{\mathcal{L}}} & Gr(r,\infty) \end{array}$$

Chern class: $\quad c: VB(X) \longrightarrow H^\cdot(X;\mathbb{Z})$

$$f_{\mathcal{L}}^*: H^\cdot(Gr(r,\infty);\mathbb{Z}) \longrightarrow H^\cdot(X;\mathbb{Z})$$

e.g.,

$$\begin{array}{ccc} VB(Gr(r,n)) & \longrightarrow & H^\cdot(Gr(r,n);\mathbb{Z}) \\ \mathcal{S}^* & \longmapsto & 1+\sigma_1+\cdots \\ Q & \longmapsto & 1+\sigma_1+\cdots \\ T_{Gr} & \longmapsto & 1+n\cdot\sigma_1+\cdots \\ \mathcal{S} & \longmapsto & 1-\sigma_1+\sigma_{1,1}-\sigma_{1,1,1}+\cdots+(-1)^r\sigma_{(1)^r} \end{array}$$

4. Applications

   tangent space argument

1. Group structure of $H^{\cdot}(Gr(r,n);\mathbb{Z})$
2. Cup product

1. Group structure of $H^{\cdot}(Gr(r,n);\mathbb{Z})$

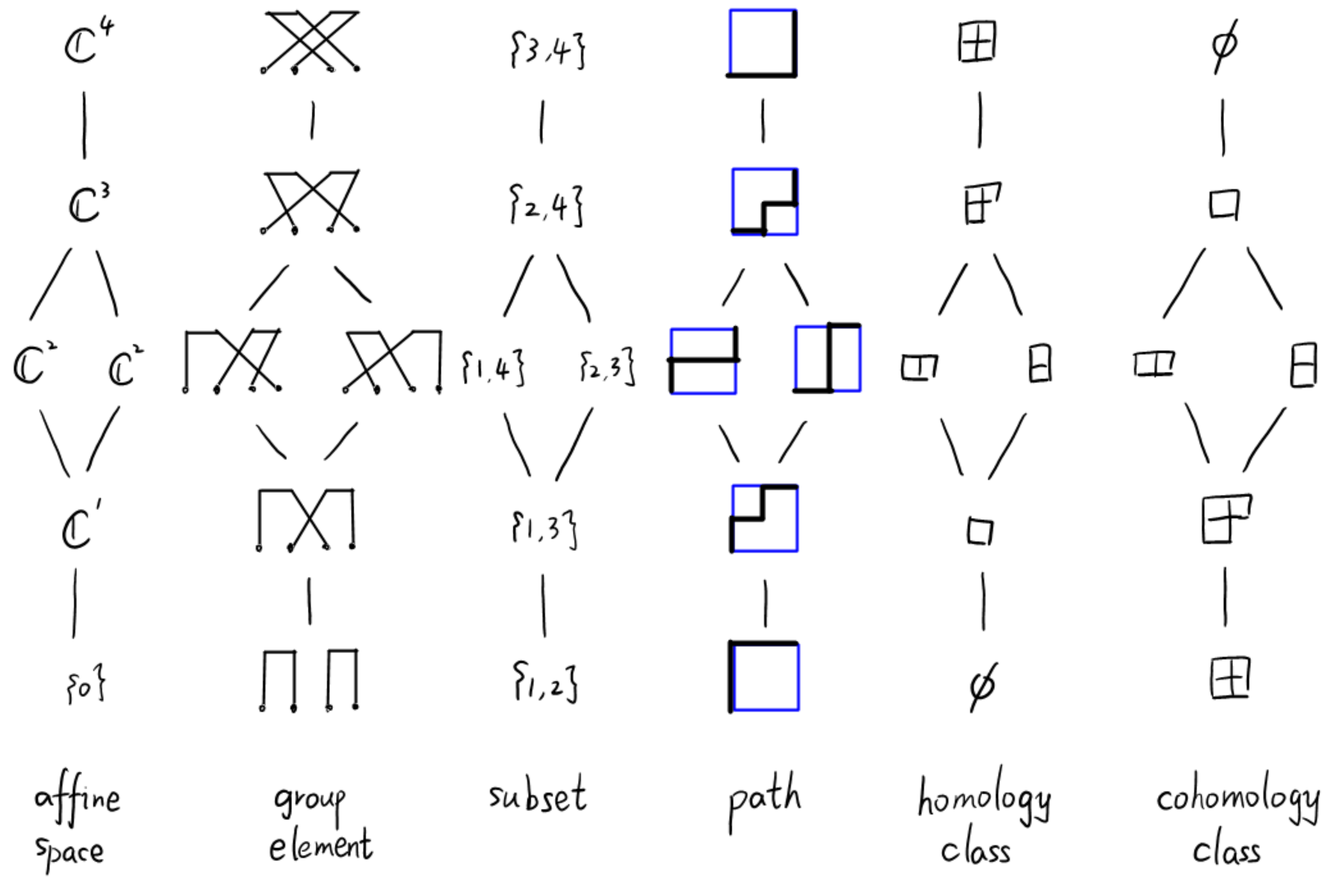
It's well-known that $Gr(r,n) \cong GL_n(\mathbb{C})/P$ has an affine paving w.r.t. $S_n/S_r \times S_{n-r}$:

$$Gr(r,n) = \bigsqcup_{w\in S_n/S_r\times S_{n-r}} BwP/P \cong \bigsqcup_{w\in S_n/S_r\times S_{n-r}} \mathbb{C}^{l(w)}$$

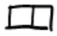
$$\# S_n/S_r\times S_{n-r} = \binom{n}{r}$$

We read the diagram from top to bottom, the map from right to left.

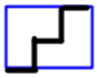
E.g. $n=4$ $r=2$

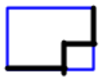


Hint from gp element to homology class:

 $\rightsquigarrow$ $(2,0) =$ 

E.g. $n = 5$, $r = 2$

 $\sim \{2,4\} \sim$  $\sim$  $\sim$ 

 $\sim \{3,5\} \sim$  $\sim$  $\sim$ 

Ex. compute w_0-action (left mult) on $S_n/S_r \times S_{n-r}$, where $w_0 =$ .

2. Cup product

We want to compute intersection number by moving one cycle(so that they intersect transversally)

Lemma 1. $[B^- wP/P] = [Bw_0wP/P]$ in $H^\cdot(Gr(r,n);\mathbb{Z})$.

Proof. $B^- wP/P = w_0Bw_0wP/P \sim Bw_0wP/P$.

Lemma 2.

$$\#(BwP/P \cap B^-\eta P/P) = \begin{cases} 0 & \eta > w \\ 1 & \eta = w \\ 0 & \eta \not\geq w \quad \& \quad l(\eta) = l(w) \\ ? & \text{otherwise} \end{cases}$$

Moreover, when $\eta = w$, BwP/P and $B^-\eta P/P$ intersect transversally.

Idea: Find a set of representative elements $C_w^+ \cong \mathbb{C}^{l(w)}$ in B, s.t.

$$BwP/P \xleftarrow{\cong} C_w^+ wP/P \cong C_w^+.$$

Similarly, find a set of representative elements $C_\eta^- \cong \mathbb{C}^{l(w_0\eta)}$ in B^-, s.t.

$$B^-\eta P/P \xleftarrow{\cong} C_\eta^- \eta P/P \cong C_\eta^-.$$

After that,

$$\begin{aligned} BwP/P \cap B^-\eta P/P &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_+ wP = c_- \eta P\} \\ &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_-^{-1} c_+ \in \eta P w^{-1}\} \end{aligned}$$

can be written as the zero sets of polynomials (of deg ≤ 2) in $C_w^+ \times C_\eta^- \cong \mathbb{C}^{l(w)+l(w_0\eta)}$.

E.g. $n=5$, $r=2$,

$$w = $$  $$= \left(\begin{array}{cc:ccc} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{35 \mid 124\} \sim$$ hom  $\sim$ cohom 

$$\eta_0 = $$  $$= \left(\begin{array}{cc:ccc} 1 & & & & \\ & & 1 & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) = \{13 \mid 245\} \sim$$  $\sim$ 

Let $\eta = \eta_0$, we want to describe $BwP/P \cap B^{-}\eta P/P \subset C_w^{+} \times C_\eta^{-}$.
By direct calculation,

$$P = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & * & * & * \end{pmatrix} \qquad \eta P w^{-1} = \begin{array}{c} \\ 1 \\ \\ 3 \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array}$$

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$
for copy

$$w P w^{-1} = \begin{array}{c} \\ \\ \\ 3 \\ \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{pmatrix} * & * & & * & \\ * & * & & * & \\ * & * & * & * & * \\ * & * & & * & \\ * & * & * & * & * \end{pmatrix} \end{array} \qquad \eta P \eta^{-1} = \begin{array}{c} \\ 1 \\ \\ 3 \\ \\ \\ \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} & 2 & & 4 & 5 \end{array} \\ \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array}$$

$$C_w^{+} = \begin{pmatrix} 1 & & * & & * \\ & 1 & * & & * \\ & & 1 & & \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \qquad C_\eta^{-} = \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ & & 1 & & \\ * & & * & 1 & \\ * & & * & & 1 \end{pmatrix}$$

Now, suppose

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ b_{21} & 1 & & & \\ & & 1 & & \\ b_{41} & & b_{43} & 1 & \\ b_{51} & & b_{53} & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$c_-^{-1} c_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ b_{21} & 1 & b_{21}a_{13} + a_{23} & & b_{21}a_{15} + a_{25} \\ & & 1 & & \\ b_{41} & & b_{41}a_{13} + b_{43} & 1 & b_{41}a_{15} + a_{45} \\ b_{51} & & b_{51}a_{13} + b_{53} & & b_{51}a_{15} + 1 \end{pmatrix}.$$

Therefore,

$$c_-^{-1} c_+ \in \eta P w^{-1} \Leftrightarrow \begin{cases} b_{21}a_{13} + a_{23} = 0 \\ b_{21}a_{15} + a_{25} = 0 \\ b_{41}a_{13} + b_{43} = 0 \\ b_{41}a_{15} + a_{45} = 0 \\ b_{51}a_{13} + b_{53} = 0 \\ b_{51}a_{15} + 1 = 0 \end{cases}$$

In this case, $BwP/P \cap B^-\eta P/P \cong \mathbb{C}^3 \times \mathbb{C}^\times$.

Now, take $\eta = w$, one suppose that

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & b_{43} & 1 & \\ & & & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$c_-^{-1} c_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & b_{43} & 1 & a_{45} \\ & & & & 1 \end{pmatrix}.$$

Therefore,

$$c_-^{-1} c_+ \in w P w^{-1} \Leftrightarrow a_{13} = a_{15} = a_{23} = a_{25} = a_{45} = b_{43} = 0.$$

In this case $BwP/P \cap B^- wP/P = \{*\}$.

Furthermore, one can show the transversality through the tangent argument.

Ex. When $\eta = w_0$, verify that

$$BwP/P \cap B^- w_0 P/P = \phi$$

Generalize this example to prove Lemma 2.

$\Leftrightarrow l(w_0 w) + l(w_0 w') = r(n-r)$

Cor of Lemma 2. When $l(w) + l(w') = r(n-r)$,

$$\deg\left([BwP/P] \cup [Bw'P/P]\right) = \begin{cases} 1 & w = w_0 w' \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, denote

$$\sigma_w := [BwP/P] \in H^{\cdot}(Gr(r,n); \mathbb{Z})$$

then
$$\sigma_w \sigma_{w_0 w} = \sigma_{Id}$$
$$\sigma_w \sigma_\eta = 0 \quad \text{when} \quad l(w) + l(\eta) = r(n-r).$$

⚠ When we view $w = a = (a_1, \dots, a_r)$ as the Young diagram in the cohom class,

$$l(w) = r(n-r) - |a|$$
$$\sigma_w \overset{!}{=} \sigma_a \in H_{l(w)}(Gr(r,n); \mathbb{Z}) \cong H^{|a|}(Gr(r,n); \mathbb{Z}).$$

For simplicity, we write $\sigma_k = \sigma_{(k,0,\dots,0)}$ and $\sigma_{1^k} = \sigma_{(\underbrace{1,\dots,1}_{k \text{ many}},0,\dots,0)}$.

The moduli interpolation of Schubert variety

To prove the Pieri rule, the method in the proof of Lemma 2 need to be modified. Working with the moduli interpolation of Schubert varieties can help understanding.

E.g. $n=5,\ r=2,$

$w =$  $= \left(\begin{array}{cc|ccc} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{35 \mid 124\} \sim$ hom  $\sim$ cohom $\square$

$$wP/P \in G/P \quad \longleftrightarrow \quad w\langle e_1, e_2\rangle = \langle e_3, e_5\rangle \in Gr(2,5)$$

standard

$$\begin{array}{cccccccccccc} \mathcal{V}^{st}: & 0 & \subset & \langle e_1\rangle & \subset & \langle e_1,e_2\rangle & \subset & \langle e_1,e_2,e_3\rangle & \subset & \langle e_1,\cdots,e_4\rangle & \subset & \langle e_1,\cdots,e_5\rangle \\ & \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ \langle e_3,e_5\rangle\cap\mathcal{V}^{st}: & 0 & = & 0 & = & 0 & \subset & \langle e_3\rangle & = & \langle e_3\rangle & \subset & \langle e_3,e_5\rangle \end{array}$$

$$\begin{aligned} \Sigma_w(\mathcal{V}_0): \ &\triangleq \overline{BwP/P} \\ &= \left\{ \Lambda \in Gr(2,5) \,\middle|\, \begin{array}{l} \dim \Lambda\cap \mathcal{V}^{st}_3 \geq 1 \\ \dim \Lambda\cap \mathcal{V}^{st}_5 \geq 2 \end{array} \right\} \end{aligned}$$

$$BwP/P = \left\{ \Lambda \in Gr(2,5) \,\middle|\, \begin{array}{l} \dim \Lambda\cap \mathcal{V}^{st}_3 = 1 \\ \dim \Lambda\cap \mathcal{V}^{st}_5 = 2 \\ \dim \Lambda\cap \mathcal{V}^{st}_2 = 0 \\ \dim \Lambda\cap \mathcal{V}^{st}_4 = 1 \end{array} \right\}$$

Def. For the flag $\mathcal{V} = g\mathcal{V}^{st}$, define

$$\begin{aligned} \Sigma_w(\mathcal{V}) &= g\,\overline{BwP/P} \\ &= \left\{ \Lambda \in Gr(2,5) \,\middle|\, \begin{array}{l} \dim \Lambda\cap \mathcal{V}_3 \geq 1 \\ \dim \Lambda\cap \mathcal{V}_5 \geq 2 \end{array} \right\} \end{aligned}$$

General case:

$$\Sigma_w(\mathcal{V}) = \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{w(i)} \geq i\}$$

Easy to see that $\Sigma_w(w_0\mathcal{V}^{st}) = \overline{B^- w_0 w P/P}$.

Lemma 3. Let a, c be Young diagrams which crspd to w, w' s.t.

$$\begin{cases} |c| = |a| + k \\ a_i \leq c_i \leq a_{i-1} \quad \forall i \end{cases}$$

Then $$\Sigma_a(\mathcal{V}^{st}) \cap \Sigma_c(w_0 \mathcal{V}^{st}) = \underbrace{\mathbb{P}^{w(1)+w'(r)-n-1} \times \cdots \times \mathbb{P}^{w(r)+w'(1)-n-1}}_{r \text{ many}}$$

E.g. $n=5$, $r=2$, write $\mathcal{V} := \mathcal{V}_{st}$, $\mathcal{W} = w_0 \mathcal{V}_{st}$,

$w =$  $= \left(\begin{array}{cc:ccc} & & 1 & & \\ 1 & & & & \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{25 \mid 134\} \overset{\text{cohom}}{\sim}$  $\leftarrow a = (2,0)$

$w' =$  $= \left(\begin{array}{cc:ccc} & & 1 & & \\ 1 & & & & \\ & & & 1 & \\ & 1 & & & \\ & & & & 1 \end{array}\right) = \{24 \mid 135\} \overset{\text{cohom}}{\sim}$  $\leftarrow c = (2,1)$

We want to show $\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) \cong \mathbb{P}^0 \times \mathbb{P}^1$.

We write

$$A_1 := \mathcal{V}_2 \cap \mathcal{W}_4 = \langle v_2 \rangle$$
$$A_2 := \mathcal{V}_5 \cap \mathcal{W}_2 = \langle v_4, v_5 \rangle$$

then

$$\begin{aligned} \Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) &= Gr(1, A_1) \times Gr(1, A_2) \\ \Lambda \quad &\longmapsto (\Lambda \cap A_1, \ \Lambda \cap A_2) \\ W_1 \oplus W_2 \quad &\longleftarrow (W_1, \ W_2) \end{aligned}$$

Hint: ① $A \triangleq A_1 + A_2 = A_1 \oplus A_2$

② $\dim \Lambda \cap A_i \geq 1$

$$\begin{aligned} 2 = \dim \Lambda &= \dim \Lambda \cap (\mathcal{V}_2 + \mathcal{W}_4) \\ &= \dim \Lambda \cap \mathcal{V}_2 + \dim \Lambda \cap \mathcal{W}_4 - \dim \Lambda \cap A_1 \\ &\geq 1 + 2 - \dim \Lambda \cap A_1 \end{aligned}$$

③ $\dim \Lambda \cap A_i = 1$, $\quad \Lambda = \oplus \Lambda \cap A_i \qquad \Lambda \subset A$

$$\begin{aligned} 2 = \dim \Lambda &\geq \dim \Lambda \cap A \\ &\geq \dim \Lambda \cap A_1 + \dim \Lambda \cap A_2 \\ &\geq 1 + 1 = 2 \end{aligned}$$

Lemma 4. Let a, c be Young diagrams which crspd to w, w' s.t.

$$\begin{cases} |c| = |a| + k \\ a_i \le c_i \le a_{i-1} \quad \forall i \end{cases}$$

Let $(k, \dots, 0)$ be Young diagram which crspds to w''.
Let $\mathcal{V}, \mathcal{W}, \mathcal{U}$ be general complete flags in $\mathbb{C}^n$, then

$$\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) \cap \Sigma_k(\mathcal{U}) = \{*\}.$$

Proof. W.l.o.g. let $\mathcal{V} = \mathcal{V}^{st}$, $\mathcal{W} = w_0 \mathcal{V}^{st}$. [3264, Def 4.4]
We know

$$\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) = \prod_{i=1}^{r} Gr(1, A_i)$$
$$\Sigma_k(\mathcal{U}) = \left\{ \Lambda \in Gr(r,n) \;\middle|\; \dim \Lambda \cap \mathcal{U}_{n-k+1} \geq 1 \right\}$$

By transversality, $\dim A \cap \mathcal{U}_{n-k+1} = 1$. $\Rightarrow \Lambda \supset A \cap \mathcal{U}_{n-k+1}$
Define

$$\psi_i : A \cap \mathcal{U}_{n-k+1} \subset A \twoheadrightarrow A_i$$

Claim: $\Lambda \cap A_i = \mathrm{Im}\, \psi_i$

$$\left[\begin{array}{ll} \quad \Lambda \cap \mathcal{U}_{n-k+1} \subset A \cap \mathcal{U}_{n-k+1} & \text{with equal dim} \\ \Rightarrow \Lambda \cap \mathcal{U}_{n-k+1} = A \cap \mathcal{U}_{n-k+1} & \\ \Rightarrow \mathrm{Im}\, \psi_i \subseteq \Lambda \cap A_i & \text{with equal dim} \\ \Rightarrow \mathrm{Im}\, \psi_i = \Lambda \cap A_i & \end{array}\right]$$

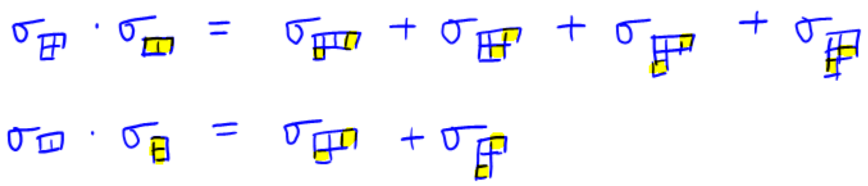
Therefore, $\Lambda = \bigoplus_i \Lambda \cap A_i = \bigoplus_i \mathrm{Im}\, \psi_i$ is uniquely determined. $\square$

Write Lemma 4 in terms of cohomology class, we get Pieri's formula: [3264, Prop 4.9, Thm 4.14]

$$\sigma_a \cdot \sigma_{(k,\dots,0)} = \sum_{\substack{|c| = |a|+k \\ a_i \le c_i \le a_{i-1}}} \sigma_c$$

$$\sigma_a \cdot \sigma_{(\underbrace{1,\dots,1}_{k\text{-many}},\dots 0)} = \sum_{\substack{|c| = |a|+k \\ a_i \le c_i \le a_i+1}} \sigma_c$$

E.g.



We will play with Young diagrams in the next section.