Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

should be read after 2022.6.5 category

A dictionary for myself:

$\{U_i \to U\}_{i\in\Delta}$	may be not jointly surj	sieve
topology		Grothendieck topology
topological space		site
$Sh(X)$		topos
sheaf		sheaf
irr closed set/pts		points

Discrete fibration

Ref:[https://www.illc.uva.nl/Research/Publications/Dissertations/DS-2021-09.text.pdf], begin from 3.1.8

Def A fctor $F: \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete fibration if
$\forall c \in \mathcal{C}, b \in \mathcal{B}, g \in Mor(b, F(c))$,
$\exists!\ c' \in \mathcal{C}, h \in Mor(c', c)$ s.t. $F(h) = g$.

$$\begin{array}{ccc} c' & \overset{\exists!\ h}{\longrightarrow} & c \\ & \Downarrow & \\ b & \xrightarrow{g} & F(c) \end{array}$$

A fctor $F: \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete opfibration if
$F^{op}: \mathcal{C}^{op} \longrightarrow \mathcal{B}^{op}$ is a discrete fibration.

From[https://arxiv.org/abs/1806.06129]: The left-handed version, now opfibrations, was originally called cofibrations, though this name was rejected to avoid confusing topologists.

E.g. For any category $\mathcal{C}$ & $x \in Ob(\mathcal{C})$, the forgetful fctor
$$\mathcal{C}/_x \longrightarrow \mathcal{C}$$
is a discrete fibration (not fully faithful)

We will later see that this discrete fibration corresponds to the presheaf h_x.

Prop. (Equivalent def of discrete fibration) Let $\mathcal{C}, \mathcal{B}$ be categories. $F: \mathcal{C} \longrightarrow \mathcal{B}$ be a fctor.
$$F \text{ is discrete fibration} \Leftrightarrow \forall c \in \mathcal{C},\ F/c: \mathcal{C}/c \longrightarrow \mathcal{B}/_{F(c)} \text{ is iso.}$$

Let $DFib_{\mathcal{B}}$ be the metacategory of discrete fibrations. To be exact,
$$Ob(DFib_{\mathcal{B}}) = \left\{ (\mathcal{C}, p: \mathcal{C} \longrightarrow \mathcal{B}) \;\middle|\; \begin{array}{l} \mathcal{C} \in \mathcal{C}at_{big} \\ p \text{ is a discrete fibration} \end{array} \right\}$$
$$Mor(p, p') = \left\{ f: \mathcal{C} \to \mathcal{C}' \;\middle|\; \begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{C}' \\ & {\scriptstyle p}\searrow \ \mathcal{B} \ \swarrow{\scriptstyle p'} & \end{array} \text{ commutes} \right\}$$

i.e. $DFib_{\mathcal{B}}$ is a full submetacategory of $Cat_{big}/\mathcal{B}$.

When restrict everything to small categories, one can define $DFib_{\mathcal{B}}$ as a full subcategory of $Cat/\mathcal{B}$

Let $PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$,
$h_c := Mor_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \longrightarrow Set$ be a presheaf on $\mathcal{C}$. for $c \in \mathcal{C}$
$\qquad c' \longmapsto Mor_{\mathcal{C}}(c', c)$
$h_{\mathcal{C}}: \mathcal{C} \longrightarrow PSh(\mathcal{C}) \qquad c \longmapsto h_c$

Prop. For $\mathcal{B} \in Cat$, we have an equivalence of categories:
$$\int_{\mathcal{B}} : PSh(\mathcal{B}) \longrightarrow DFib_{\mathcal{B}}$$
$$F \longmapsto \int_{\mathcal{B}} F \quad \text{be the strict pullback}$$

$$\begin{array}{ccc} \int_{\mathcal{B}} F & \longrightarrow & PSh(\mathcal{B})/F \\ \downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{h_{\mathcal{B}}} & PSh(\mathcal{B}) \end{array}$$

$$\left[b \in \mathcal{B} \longmapsto Mor_{DFib_{\mathcal{B}}}\left(\begin{array}{c}\mathcal{B}/b \\ \downarrow \\ \mathcal{B}\end{array}, \begin{array}{c}\mathcal{C} \\ \downarrow p \\ \mathcal{B}\end{array}\right) \right] \longleftarrow\!\shortmid \ (\mathcal{C}, p: \mathcal{C} \longrightarrow \mathcal{B})$$

Sieve

Def (sieve in small category)

Let $\mathcal{C}$ be a small category. $S \in \mathrm{Cat}/\mathcal{C}$.

S is a sieve in $\mathcal{C}$ if the fctor

$$S \longrightarrow \mathcal{C}$$

is fully faithful and a discrete fibration.

For $c \in \mathcal{C}$, $T \in \mathrm{Cat}/(\mathcal{C}/c)$.

T is a sieve on c if the fctor

$$T \longrightarrow \mathcal{C}/c$$

is fully faithful and a discrete fibration.

Viewing T as a fullsubcategory of $\mathcal{C}/c$, this is equivalent to

A sieve on c is a subset $T \subseteq \mathcal{O}b(\mathcal{C}/c)$ s.t.

$(f \circ g : e \to c) \in T$ for any $e, d \in \mathcal{C}$, $(f : d \to c) \in T$, $g \in \mathrm{Mor}(e, d)$.

Def. Now $\mathcal{C}$ can be any category, $c \in \mathcal{C}$.

A sieve on c is a subclass $T \subseteq \mathcal{O}b(\mathcal{C}/c)$ s.t.

$(f \circ g : e \to c) \in T$ for any $e, d \in \mathcal{C}$, $(f : d \to c) \in T$, $g \in \mathrm{Mor}(e, d)$.

$$\begin{tikzcd} e \arrow[r, "g"] \arrow[dr, dashed, "f\circ g \in T"'] & d \arrow[d, "f \in T"] \\ & c \end{tikzcd}$$

Let $h_c := \mathrm{Mor}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ be a presheaf on $\mathcal{C}$.

$$c' \longmapsto \mathrm{Mor}_{\mathcal{C}}(c', c)$$

Thm. When $\mathcal{C}$ is small, There is a bijection between Sets

$$\{\text{sieves on } c \in \mathcal{C}\} \longleftrightarrow \{\text{subfctors of } h_c\}$$

$$T \longmapsto F_T : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

$$\begin{matrix} d & & \{(d \to c) \in T\} \\ a \downarrow & \Rightarrow & \uparrow a\circ - \\ d' & & \{(d' \to c) \in T\} \end{matrix} \quad a \in \mathrm{Mor}_{\mathcal{C}}(d, d')$$

$$T_F := \coprod_{d \in \mathcal{O}b(\mathcal{C})} F(d) \longleftarrow F \subset h_c$$

Q: How to get a correct statement for this theorem when $\mathcal{C}$ is large?

Grothendieck topology, site and topos

On set theoretic issues: https://stacks.math.columbia.edu/tag/ooVI
Ironnically, even though what I can actually understand is the Grothendieck topology over a small category, nearly all the applications I need is the Grothendieck topology over a large category.

Def. A Grothendieck topology $\mathcal{T}$ on a category $\mathcal{C}$ is an assignment

$$\begin{aligned} \mathcal{T}(-): \mathcal{C} &\longrightarrow \mathcal{P}(\{\text{sieves on } c \in \mathcal{C} \text{ for some } c\}) \\ c &\longmapsto \mathcal{T}(c) \subseteq \{\text{sieves on } c\} \end{aligned}$$

s.t.

1). (Base change) $\forall g \in Mor_{\mathcal{C}}(d, c)$, $T \in \mathcal{T}(c) \Rightarrow g^* T \in \mathcal{T}(d)$
2) (Local character) Let T be a sieve on $c \in \mathcal{C}$. If $[\exists S \in \mathcal{T}(c) \text{ st } \forall (g: d \to c) \in S,\ g^* T \in \mathcal{T}(d)]$ then $T \in \mathcal{T}(c)$
3) $h_c \in \mathcal{T}(c)$

Def. A site $\mathcal{C} = (\mathcal{C}, \mathcal{T})$ is a category equipped with a Grothendieck topology.
A topos is a category equivalent to $Sh(\mathcal{C})$, where $\mathcal{C}$ is a site.

Category + Groth cover	space (open sets)	continuous map	Covering of (open sets)	Sh	cohomology
site	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I}, \bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	full sub of Sch/X	—		H
Sch_{Zar}	$Ob(Sch)$	$Mor(Sch)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		$H_{ét}$
$Sch_{ét}$	$Ob(Sch)$	$Mor(Sch)$	ét + l.f.p		
Sch_{sm}	$Ob(Sch)$	$Mor(Sch)$	smooth + l.f.p		
Sch_{fppf}	$Ob(Sch)$	$Mor(Sch)$	f. flat + l.f.p		
Sch_{fpqc}	$Ob(Sch)$	$Mor(Sch)$	f. flat + f_i^{-1}(q.c) locally q.c		
X/k, $W_n := W_n(k)$: $Cris(X/W_n)$	$\left\{(U, V, i, \delta) \,\middle\vert\, \begin{array}{l} U \subseteq X \text{ open} \\ \cdots \\ \delta: \text{PD-thickening of } U \end{array}\right\}$	$\left\{(\iota, f) \,\middle\vert\, \begin{array}{l} \iota: U \xrightarrow{\text{open}} U' \\ f: V \to V' \\ \text{compatible with PD} \end{array}\right\}$	$\left\{\begin{array}{c}(U_i, V_i, i_i, \delta_i) \\ \downarrow \\ (U, V, i, \delta)\end{array} \,\middle\vert\, \{U_i\} \text{ cover of } U\right\}$		$H^i_{cris}(X/W_n, -)$

(recommended)https://sites.math.washington.edu/~jarod/moduli.pdf
https://pbelmans.ncag.info/notes/etale-cohomology.pdf
http://homepage.sns.it/vistoli/descent.pdf
(crystalline site)http://page.mi.fu-berlin.de/castillejo/docs/crystalline_cohomology.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

it tells us why we don't have small site for most condition:
https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
Here you can find some informations about comparison between fppf and fpqc topologys:
https://mathoverflow.net/questions/361664/some-basic-questions-on-quotient-of-group-schemes

Thm. ① equiv. of categories $G_k := \mathrm{Gal}(k^{sep}/k)$

$$\mathrm{Sets}((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \mathrm{Disc}\ G_k\text{-Set} \qquad (\mathrm{Spec}\ k)_{ét} \overset{\text{Site}}{\longleftrightarrow} \text{finite } G_k\text{-Set}$$

$$\mathrm{Ab}((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \mathrm{Disc}\ \mathrm{Mod}_{G_k} \qquad (☆)$$

② (☆) preserve cohomology

$$H^1((\mathrm{Spec}(k))_{ét}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{C})_{ét}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\mathrm{Spec}\,\mathbb{C})$)

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{R})_{ét}$

$$\begin{tikzcd} \mathrm{Spec}\ \mathbb{C} \arrow[dr, "\iota^*"'] & \mathrm{Spec}\ \mathbb{C} \arrow[l, "\sigma^*"'] \arrow[dl, "\iota^*"] \\ & \mathrm{Spec}\ \mathbb{R} \end{tikzcd} \overset{\mathrm{Spec}}{\rightsquigarrow} \begin{tikzcd} \mathbb{C} \arrow[rr, "\sigma\ (= \text{conjugation})"] & & \mathbb{C} \\ & \mathbb{R} \arrow[ul, "\iota"] \arrow[ur, "\iota = \text{embedding}"'] & \end{tikzcd}$$

$$\Big\downarrow \mathcal{F}$$

$$\begin{tikzcd} \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \arrow[rr, "\mathcal{F}(\sigma^*)"] & & \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \\ & \mathcal{F}(\mathrm{Spec}\,\mathbb{R}) \arrow[ul, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"'] & \end{tikzcd} \overset{\text{Abuse of notation}}{\Longrightarrow} \begin{tikzcd} \mathcal{F}(\mathbb{C}) \arrow[rr, "\sigma"] & & \mathcal{F}(\mathbb{C}) \\ & \mathcal{F}(\mathbb{R}) \arrow[ul, "\iota"] \arrow[ur, "\iota"'] & \end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\qquad Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

[$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed.

in this case $0 \to \mathcal{F}(\operatorname{Spec}\mathbb{R}) \to \mathcal{F}(\operatorname{Spec}\mathbb{C}) \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C})$

$$\begin{tikzcd}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[r] \arrow[dd, equal] & \mathcal{F}(\operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}) \arrow[d, phantom, "\cong"] \\
 & \mathcal{F}(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\operatorname{Spec}\mathbb{C}) \arrow[d, phantom, "\cong"] \\
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[r] & \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})
\end{tikzcd}$$

$\iota_1: x \longmapsto x \otimes 1$ $\qquad x\otimes y \longmapsto (xy, x\bar{y})$

$\iota_2: x \longmapsto 1 \otimes x$

ι_2: $\operatorname{Spec}\mathbb{C} \xleftarrow{(Id,\sigma)} \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\operatorname{Spec}\mathbb{C}) \xrightarrow{(\mathcal{F}(Id),\mathcal{F}(\sigma))} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})$

$\overset{\text{Abuse}}{\underset{\text{of notation}}{\rightsquigarrow}}$ $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$]

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\operatorname{Spec} k, \mathcal{F}) = \mathcal{F}(\operatorname{Spec} k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)} \qquad \mathcal{F}_{k^{sep}} := \varinjlim_{\substack{L/k \\ \text{finite}}} \mathcal{F}(\operatorname{Spec} L)$$

Ex. describe the stalk & fiber at $p \in \operatorname{Spec} k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\operatorname{Spec} k, p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts