

Eine Woche, ein Beispiel. 5.7. Lie group

Temporary roadmap of Lie Group

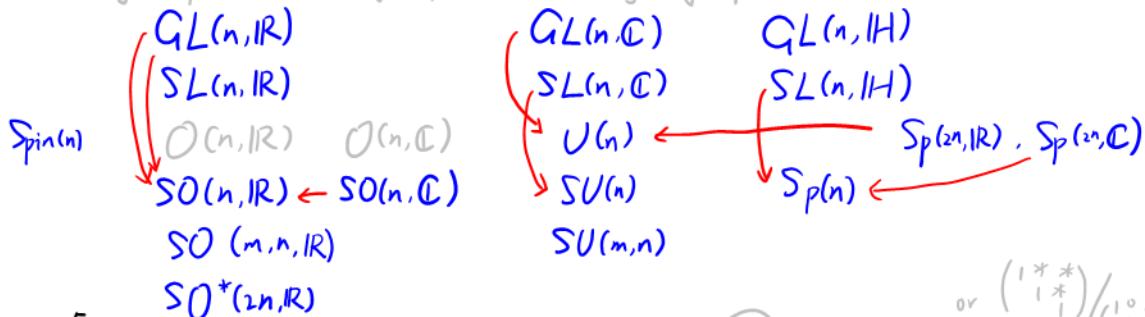
Why is it so difficult to write examples for Lie group?

- Plenty of examples, even restricted to matrix Lie groups:

- List of examples. [https://en.wikipedia.org/wiki/Table\\_of\\_Lie\\_groups](https://en.wikipedia.org/wiki/Table_of_Lie_groups)

[Some important subgroups are not written in the list,  
e.g. parabolic subgroups, Heisenberg group, etc.]

$G \rightarrow K$  maximal cpt connected subgroup



[For examples of non matrix Lie group  $(\widetilde{SL(2, IR)}, M_p(2n, IR))$ , see  
<https://math.stackexchange.com/questions/206117/is-there-a-non-matrix-lie-group>]

[On the contrary, every f.d. Lie alg can be viewed as a matrix Lie algebra by Ado's theorem, see <http://www.math.ubc.ca/~reichst/Ado's-Theorem.pdf>.]

- These groups are closely connected to each other.

e.g. the Whitehead tower:

$$\rightarrow \text{Fivebrane}(n) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow \text{O}(n)$$

- Plenty of special phenomenon on low-dimensional case:

e.g.  $GL(1, IR) \cong IR^*$  is not connected

e.g.  $SU(2) \cong S^3$

e.g.  $SO(4, IR)$  is not simple

$\downarrow$   
 $SU(2) \cong S^3$   
 $\downarrow$   
 $SO(3, IR) \cong RP^3$   
diff not as variety

e.g. <https://www.zhihu.com/question/47264301/answer/320472431>

- Fruitful structures on examples

- important subgroups

e.g. maximal torus, unipotent subgroup

Borel subgroup, parabolic subgroup,

discrete subgroup,

radical, unipotent radical, derived group

\* centralizer & normalizer of subgroups

$$G = [G, G] R(G)$$

# One page for defining these groups by pasting from

Anthony W. Knapp - Representation Theory of Semisimple Groups An Overview Based on Examples

*Complex groups:*

$$\mathrm{GL}(n, \mathbb{C}) = \{\text{nonsingular } n\text{-by-}n \text{ complex matrices}\}$$

$$\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \det g = 1\}$$

$$\mathrm{SO}(n, \mathbb{C}) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid gg^{\mathrm{tr}} = 1\}$$

$$\mathrm{Sp}(n, \mathbb{C}) = \left\{ g \in \mathrm{SL}(2n, \mathbb{C}) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

*Compact groups:*

~~$$\mathrm{SO}(n, \mathbb{R}) = \mathrm{SO}(n) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g^{\mathrm{tr}} g = 1 \text{ and } g \text{ has real entries}\}$$~~

~~$$\mathrm{U}(n) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{g}^{\mathrm{tr}} g = 1\}$$~~

~~$$\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\}$$~~

~~$$\mathrm{Sp}(n) = \left\{ g \in \mathrm{U}(2n) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$~~

*Real noncompact groups:* We list  $G$ ,  $\mathfrak{g}$ ,  $K$ , and  $\mathfrak{k}$  for the noncomplex non-compact classical groups.

$G$	$\mathfrak{g}$	$K$	$\mathfrak{k}$
$\mathrm{SL}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R})$	$\mathrm{SO}(n)$	$\mathfrak{so}(n)$
$\mathrm{SL}(n, \mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H})$	$\mathrm{Sp}(n)$	$\mathfrak{sp}(n)$
$\mathrm{SO}_0(m, n)$	$\mathfrak{so}(m, n)$	$\mathrm{SO}(m) \times \mathrm{SO}(n)$	$\mathfrak{so}(m) \oplus \mathfrak{so}(n)$
$\mathrm{SU}(m, n)$	$\mathfrak{su}(m, n)$	$\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$	$\mathfrak{s}(\mathrm{u}(m) \oplus \mathrm{u}(n))$
$\mathrm{Sp}(m, n)$	$\mathfrak{sp}(m, n)$	$\mathrm{Sp}(m) \times \mathrm{Sp}(n)$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(n)$
$\mathrm{Sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$
$\mathrm{SO}^*(2n)$	$\mathfrak{so}^*(2n)$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$ .

$\mathrm{SL}(n, \mathbb{R})$  and  $\mathrm{SL}(n, \mathbb{H})$  refer to matrices of determinant one with real and quaternion entries, respectively.  $\mathrm{SO}_0(m, n)$ ,  $\mathrm{SU}(m, n)$ , and  $\mathrm{Sp}(m, n)$  are the linear isometry groups for the Hermitian form

$$|z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2$$

defined over  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ , respectively, with the subscript “0” referring to the identity component. The group  $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$  for  $\mathrm{SU}(m, n)$  is the subgroup of  $\mathrm{U}(m) \times \mathrm{U}(n)$  of matrices of determinant one.

The group  $\mathrm{Sp}(n, \mathbb{R})$  is the subgroup of real matrices in  $\mathrm{Sp}(n, \mathbb{C})$  and can be conjugated by a unitary matrix so as to become

$$\{g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} J g = J\};$$

then  $\mathrm{SO}^*(2n)$  is the analogous group

$$\left\{ g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}.$$

- decomposition: not easy to prove, but still of great significant.

[https://en.wikipedia.org/wiki/Lie\\_group\\_decomposition](https://en.wikipedia.org/wiki/Lie_group_decomposition)

e.g. Cartan decomposition

Jordan - Chevalley decomposition

Bruhat decomposition

:

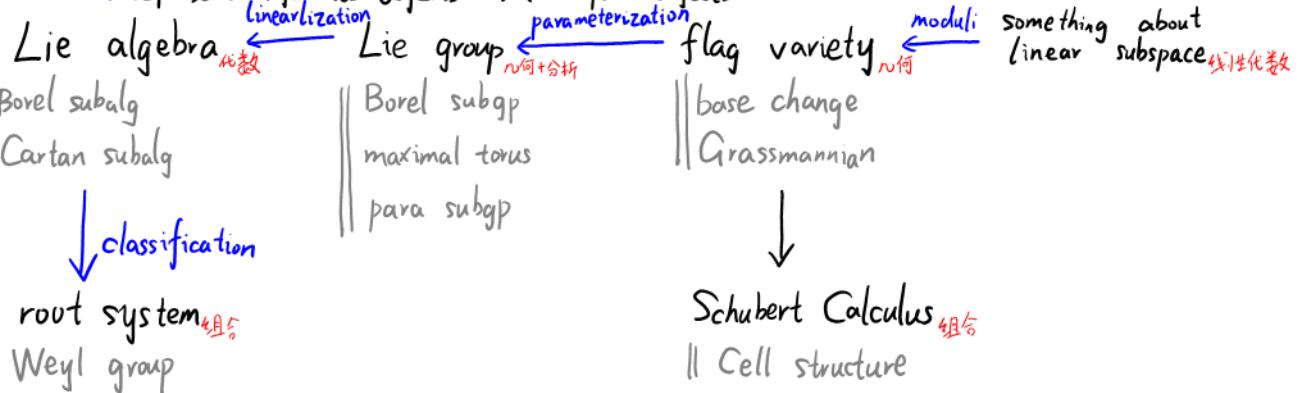
- group, mfld, variety, etc. Common questions.

- Hopf algebra structure of homology & cohomology group under mild requirements.

- A large amount of related theories.

- structure theory of Lie group & Lie algebras.

" $\rightarrow$ ": represent unfamiliar objects with simpler objects.



- representation theory

- \* group / alg, f.d / inf.d

- \* highest weight theory

- \* Schul - Weyl duality

- \* Borel - Weil - Bott theorem

- \* geometric representation theory

- \* Langlands program

- algebraic topology, Bott periodicity

- algebra groups.

- ... see [https://en.wikipedia.org/wiki/List\\_of\\_Lie\\_groups\\_topics](https://en.wikipedia.org/wiki/List_of_Lie_groups_topics)

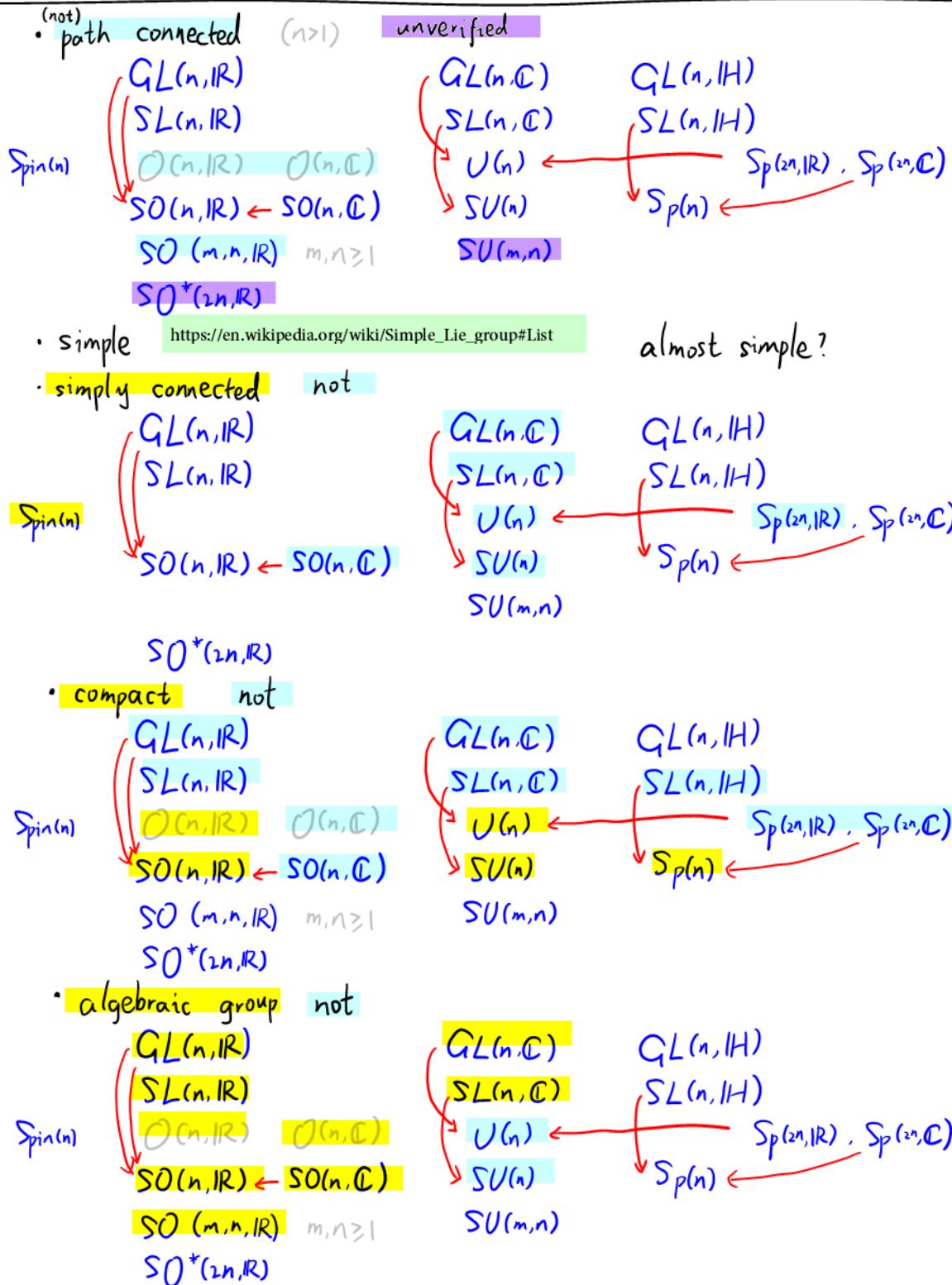
<http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html>

<https://mathoverflow.net/questions/235917/roadmap-to-geometric-representation-theory-leading-to-langlands>

Basic knowledge which has been well-written in [Naive Lie Theory].

- canonical maximal tori
- center

- (1)  $Z(\mathrm{SO}(2m)) = \{\pm 1\}$ .
- (2)  $Z(\mathrm{SO}(2m+1)) = \{1\}$ .
- (3)  $Z(\mathrm{U}(n)) = \{\omega \mathbf{1} : |\omega| = 1\}$ .
- (4)  $Z(\mathrm{SU}(n)) = \{\omega \mathbf{1} : \omega^n = 1\}$ .
- (5)  $Z(\mathrm{Sp}(n)) = \{\pm 1\}$ .



In alg top: homotopy groups:

*typo:  $\pi_*(U(n)) = \mathbb{Z}$*

see

<http://felix.physics.sunysb.edu/~abanov/Teaching/Spring2009/Notes/abanov-cpA1-upload.pdf>

more about  $\pi_i(SO(n, \mathbb{R}))$ : <https://ncatlab.org/nlab/show/orthogonal+group#HomotopyGroups>

more about  $\pi_i(G)$  (in Chinese): <https://www.zhihu.com/question/443652195>

$\pi_i(SO_{\theta}(m, n))$ : <https://math.stackexchange.com/questions/4126129/homotopy-group-of-op-q>

my try: Cartan decomposition  $\Rightarrow$  only consider cpt Lie group.

fibration + low-dimensional understanding, + Bott periodicity

e.g. •  $O(1) = S^0$ , a two-point discrete space

$$U(n) \cong SU(n) \times S^1$$

[https://en.wikipedia.org/wiki/Othogonal\\_group](https://en.wikipedia.org/wiki/Othogonal_group)

- $SO(1) = \{1\}$
- $SO(2)$  is  $S^1$
- $SO(3)$  is  $RP^3$  [3]
- $SO(4)$  is doubly covered by  $SU(2) \times SU(2) = S^3 \times S^3$ .

Homotopy groups of orthogonal groups									
	$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$	$\pi_7$	$\pi_8$	$\pi_9$
$SO(2)$	$Z$	0	0	0	0	0	0	0	0
$SO(3)$	$[Z_2]$	0	$Z$	$Z_2$	$Z_2$	$Z_{12}$	$Z_2$	$Z_2$	$Z_3$
$SO(4)$	$Z_2$	$[0]$	$(Z)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_{12})^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_3)^{\times 2}$
$SO(5)$	$Z_2$	0	$[Z]$	$Z_2$	$Z_2$	0	$Z$	0	0
$SO(6)$	$Z_2$	0	$Z$	$[0]$	$Z$	0	$Z$	$Z_{24}$	$Z_2$
$SO(n), n > 6$	$Z_2$	0	$Z$	0	0	0			

Activate Windows

fibration: [Hatcher Example 4.54, 4.55]

space of  $n$ -frames in  $\mathbb{R}^k$

$$V_{n-m}(\mathbb{R}^{k-m}) \rightarrow V_n(\mathbb{R}^k) \quad O(n, \mathbb{R}) \rightarrow O(n+1, \mathbb{R}) \quad U(n) \rightarrow U(n+1) \quad Sp(n) \rightarrow Sp(n+1)$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$V_m(\mathbb{R}^k) \quad \quad \quad \mathbb{S}^n \quad \quad \quad \mathbb{S}^{2n+1} \quad \quad \quad \mathbb{S}^{4n+3}$$

$$SO(n, \mathbb{R}) \rightarrow SO(n+1, \mathbb{R})$$

$$\downarrow \\ \mathbb{S}^n$$

$$SO(n, \mathbb{R}) \rightarrow O(n, \mathbb{R}) \quad \mathbb{Z}/2\mathbb{Z} \rightarrow Spin(n)$$

$$\downarrow \quad \quad \quad \downarrow$$

$$\mathbb{Z}/2\mathbb{Z} \quad \quad \quad SO(n)$$

# Structure theory of Lie group

Ref:

<http://www.math.columbia.edu/~maki/sumi/cld/reductivegroups.pdf>

← notations, results, examples, pictures.

Today:

$G = GL(2, \mathbb{C})$

center  $Z(G) = \mathbb{C}^\times \quad \pi_1(G) = \mathbb{Z}$

radical  $\text{rad}(G) := R(G) = \mathbb{C}^\times \quad \Rightarrow$  reductive

Unipotent  $\text{rad}_{\text{un}}(G) := R_{\text{u}}(G) = \{I\} \quad \text{not semisimple}$

rank  $G = 2$

canonical maximal torus  $T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cup \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$W(G) = S_2 \quad \text{Weyl group}$   
 $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$X^* := X^*(T) = \langle \varepsilon_i : (t_1, t_2) \mapsto t_i \rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \alpha \mapsto \alpha(g^{-1}g)$$

$$X_{\neq} := X_{\neq}(T) = \left\langle \begin{array}{c} \varepsilon_1^* : t \mapsto (t_1) \\ \varepsilon_2^* : t \mapsto (1, t) \end{array} \right\rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \lambda \mapsto g \lambda(-)g^{-1}$$

$$\Phi = \{\varepsilon_1 - \varepsilon_2, -(\varepsilon_1 + \varepsilon_2)\} := \{\alpha_{12}, -\alpha_{12}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, -\alpha_{12}^\vee\} = \{\varepsilon_1^* - \varepsilon_2^*, -(\varepsilon_1^* + \varepsilon_2^*)\}$$

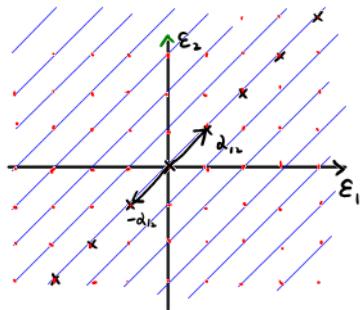
$r_?$  = action of  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ :  $X^* \longrightarrow X^*$

$\varepsilon_1 \longrightarrow$	$\varepsilon_1$
$\varepsilon_2 \longrightarrow$	$\varepsilon_2$

$r_?$  is reflection and  $\Rightarrow r_? = r_{\alpha_{12}}$

$x - r_?(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_1 \varepsilon_2 - x_2 \varepsilon_1 = (x_1 - x_2) \alpha_{12} \quad x := x_1 \varepsilon_1 + x_2 \varepsilon_2$

$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$



$$\times \subseteq \cdot \subseteq /$$

$$Q \subseteq X^* \subseteq P$$

$$G = SL(2, \mathbb{C})$$

$$Z(G) = \{\pm 1\} \quad \pi_1(G) = \{*\}$$

$$\text{rad}(G) = \{I\} \Rightarrow \text{semisimple}$$

$$\text{rad}_u(G) = \{I\}$$

$$\text{rank } G = 1$$

canonical maximal torus  $T = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$

$$C_G(T) = T$$

$$N_G(T) = \{t' \mid t' \sim t\}$$

$$W(G) = S_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

$$X^* = \langle \varepsilon_i : (t_{-t}) \mapsto t \rangle_{\mathbb{Z}} \quad \text{with } \alpha \mapsto \alpha(g^{-1} - g) \quad \alpha_1 = 2\varepsilon_1!$$

$$X_t = \langle \varepsilon_i^t : t \mapsto (t_{-t}) \rangle_{\mathbb{Z}} \quad \text{with } \lambda \mapsto g \cdot \lambda(-) \cdot g^{-1}$$

$$\Phi = \{\alpha_{12}, -\alpha_{12}\} = \{2\varepsilon_1, -2\varepsilon_1\}$$

$$\vec{\Phi}^V = \{d_{12}^V, -d_{12}^V\} = \{\varepsilon_1^+, -\varepsilon_1^+\}$$

$$\left[ \begin{array}{l} r_? = \text{action of } (\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}): X^* \rightarrow X^* \quad \varepsilon_i \mapsto -\varepsilon_i \\ x - r_?(x) = x_0 \varepsilon_i - x_0 (-\varepsilon_i) = x_0 \alpha_{12} \stackrel{r_? \text{ ref}}{\substack{+ r_2 = r_{12}}} x := x_0 \varepsilon_i \\ \langle -, \alpha_{12}^\vee \rangle: x_0 \varepsilon_i \mapsto x_0 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_i^* \end{array} \right]$$

$\cdot c \neq 1$

$$\textcircled{1} \quad C \subset X^* - P$$

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$$G = GL(3, \mathbb{C})$$

$$\begin{aligned} Z(G) &= \mathbb{C}^\times & \pi_1(G) &= \mathbb{Z} \\ \text{rad}(G) &= \mathbb{C}^\times & \left. \begin{aligned} \text{rad}_0(G) &= \{I\} \end{aligned} \right\} &\Rightarrow \text{reductive} \\ \text{rank } G &= 3 & &\text{but not semi} \end{aligned}$$

canonical maximal torus  $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

 $C_G(T) = T$ 
 $N_G(T) = \text{monomial matrixs}$ 
 $W(G) = S_3$

$$X^* = \langle \varepsilon_i, \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}}$$

$$X_* = \langle \varepsilon_i^*, t \mapsto (t, t, t) \rangle_{\mathbb{Z}}$$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$$\left[ \begin{array}{l} r_{\alpha_{12}} = \text{action of } \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, X^* \rightarrow X^* \\ \begin{array}{l} \varepsilon_1 \mapsto \varepsilon_2 \\ \varepsilon_2 \mapsto \varepsilon_1 \\ \varepsilon_3 \mapsto \varepsilon_3 \end{array} \end{array} \right]$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 \\ = (x_1 - x_2) \alpha_{12}$$

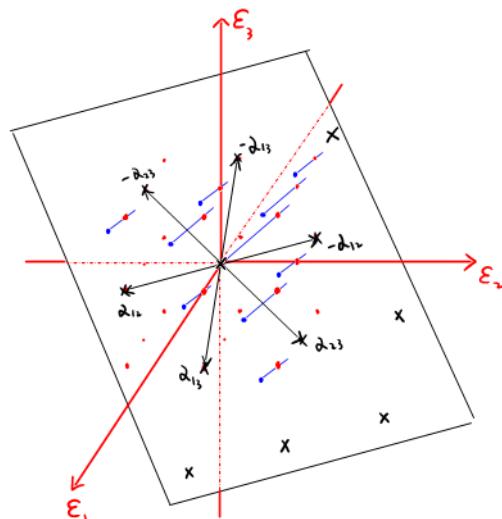
$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$$

Similarly,  $\alpha_{23}^\vee = \varepsilon_2^* - \varepsilon_3^*$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$\text{where } \alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$$

$$\begin{aligned} \varepsilon_1 &\mapsto \varepsilon_1 \left( \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \\ &= \varepsilon_1 \begin{pmatrix} t_2 & t_3 & t_3 \end{pmatrix} = \varepsilon_2 \end{aligned}$$



$$x \subseteq \cdot \subseteq /$$

$$Q \subseteq X^* \subseteq P$$

$G = SL(3, \mathbb{C})$

$$Z(G) = \mathbb{Z}/3\mathbb{Z} \quad \pi_1(G) = \langle \gamma \rangle$$

$\text{rad}(G) = \{I\} \Rightarrow \text{semisimple}$

$$\text{rad}_u(G) = \{I\}$$

rank  $G = 2$

canonical maximal torus  $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$$C_G(T) = T$$

$N_G(T) = \text{monomial matrixs}$

$$W(G) = S_3$$

$$X^* = \langle \varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}} / \mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$X_\gamma = \langle \varepsilon_1^* - \varepsilon_2^*, \varepsilon_2^* - \varepsilon_3^* \rangle_{\mathbb{Z}}$$

$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ : dual basis of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$r_{\alpha_{12}}$  = action of  $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ ,  $X^* \rightarrow X^*$

$$\begin{aligned} \varepsilon_1 &\mapsto \varepsilon_2 \\ \varepsilon_2 &\mapsto \varepsilon_1 \\ \varepsilon_3 &\mapsto \varepsilon_3 \end{aligned}$$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$\text{where } \alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$$

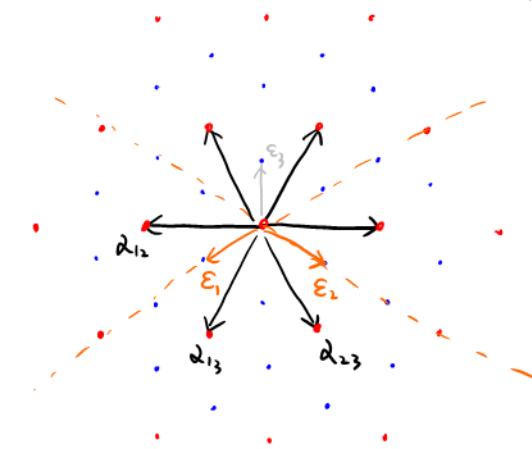
$$\begin{aligned} \varepsilon_1 &\mapsto \varepsilon_1 \left( \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \\ &= \varepsilon_1 \begin{pmatrix} t_2 & t_1 & t_3 \end{pmatrix} = \varepsilon_2 \end{aligned}$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 \quad x = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3$$

$$= (x_1 - x_2) \alpha_{12}$$

$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$$

$$\text{Similarly, } \alpha_{23}^\vee = \varepsilon_2^* - \varepsilon_3^*$$



$$\begin{array}{c} \subseteq \\ Q \subseteq X^* = P \\ \mathbb{Z}/3\mathbb{Z} \end{array}$$

$$G = \mathrm{PGL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C}) / \mu_3$$

$$Z(G) = \{\mathbf{I}\} \quad \pi_1(G) = \mathbb{Z}/3\mathbb{Z}$$

$\mathrm{rad}(G) = \{\mathbf{I}\} \Rightarrow$  semisimple

$$\mathrm{rad}_u(G) = \{\mathbf{I}\}$$

rank  $G = 2$

$$\mathbb{C}^* \cong \mathbb{C}_{\mu_3}^* \times \mathbb{C}^*$$

$$t_i^3 \leftrightarrow \begin{pmatrix} t_1 & t_2 \\ t_1 & t_2 \\ t_1 & t_2 \end{pmatrix} \times \begin{pmatrix} t_1 & t_2 \\ t_1 & t_2 \\ t_1 & t_2 \end{pmatrix}$$

canonical maximal torus  $T = \begin{pmatrix} * & * \\ * & * \end{pmatrix} / \mu_3$

$$C_G(T) = T$$

$N_G(T)$  = monomial matrixs /  $\mu_3$

$$W(G) = S_3$$

$$X^* = \{n_1 \varepsilon_1 + n_2 \varepsilon_2 + n_3 \varepsilon_3 \mid n_i \in \mathbb{Z}, 3 \mid \sum n_i\} /_{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \mathbb{Z}}$$

$$= \langle \alpha_{12}, \alpha_{23} \rangle_{\mathbb{Z}}$$

$\varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \rightarrow t_i$  is no longer well-defined

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$X^* = \{n_1 \varepsilon_1^* + n_2 \varepsilon_2^* + n_3 \varepsilon_3^* \mid n_i \in \frac{1}{3}\mathbb{Z}, \sum n_i = 0\}$$

$$= \left\langle \frac{1}{3} \varepsilon_1^* + \frac{1}{3} \varepsilon_2^* - \frac{2}{3} \varepsilon_3^*, \varepsilon_2^* - \varepsilon_3^* \right\rangle_{\mathbb{Z}}$$

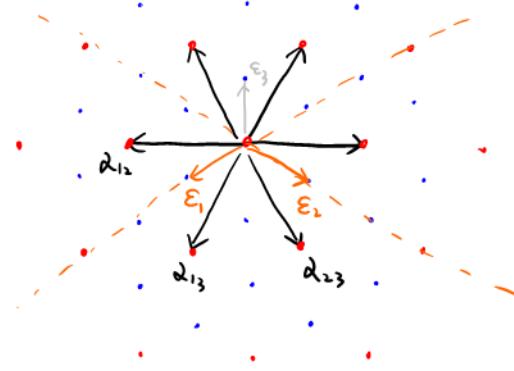
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$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

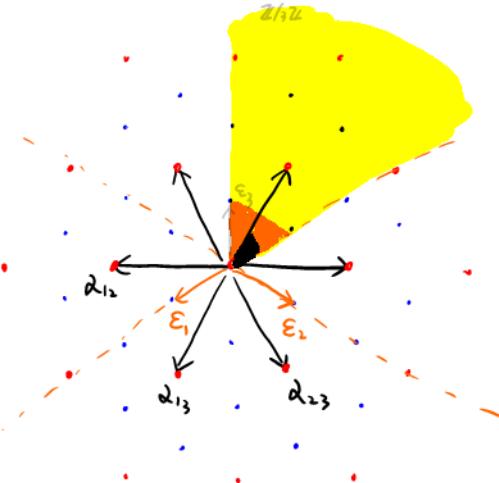
$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ : dual basis of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

where  $\alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$



$$Q = X^* \subseteq P$$



fundamental domain of

Weyl group [ ]  
affine Weyl group [ ]  
extended affine Weyl group [ ]

$$G = \mathrm{Sp}(4, \mathbb{C}) = \{ A \in GL(4, \mathbb{C}) \mid A^T M A = M \}$$

$Z(G) = \{\pm I\}$     $\pi_1(G) = \mathbb{Z}_2$   
 $\mathrm{rad}(G) = \{I\} \Rightarrow \text{semisimple}$   
 $\mathrm{rad}_u(G) = \{I\}$   
 $\mathrm{rank } G = 2$

$M = \begin{pmatrix} & & -1 & 1 \\ & -1 & & \\ -1 & & & \\ & & 1 & -1 \end{pmatrix}$   
 canonical maximal torus    $T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3^{-1} & \\ & & & t_4^{-1} \end{pmatrix}$   
 $C_G(T) = T$   
 $N_G(T) = \text{monomial matrixs}$   
 $W(G) = D_4 \subseteq S_4$

$$\mathrm{sp}(4, \mathbb{C}) = \{ A \in M^{4 \times 4}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

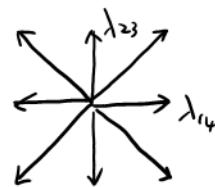
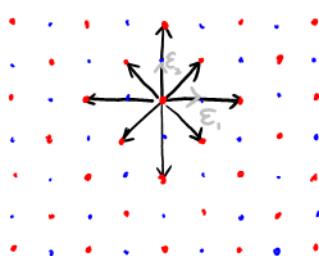
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & -a_{13} & & \\ a_{21} & a_{22} & a_{23} & \\ & & -a_{31} & a_{31} \end{pmatrix} \in M^{4 \times 4}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{14} \oplus \mathbb{Z}\lambda_{23}$$

$$\Phi = \{\pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23}\} = \{\pm (\varepsilon_1 - \varepsilon_2), \pm (\varepsilon_1 + \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$$

$$\Phi^\vee = \{\pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee\} = \{\pm (\lambda_{14} - \lambda_{23}), \pm (\lambda_{14} + \lambda_{23}), \pm \lambda_{14}, \pm \lambda_{23}\}$$



$$\Phi \subseteq Q \subseteq X^* = P$$

$\mathbb{Z}/2\mathbb{Z}$

$$\Phi^\vee$$

$$G = SO(5, \mathbb{C}) \cong \{ A \in SL(5, \mathbb{C}) \mid A^T M A = M \}$$

$$Z(G) = \{ \text{Id} \} \quad \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$$

$\text{rad}(G) = \{ I \} \Rightarrow \text{semisimple}$

$$\text{rad}_u(G) = \{ I \}$$

$$\text{rank } G = 2$$

$$M = \begin{pmatrix} & & & & \\ & t_1^{-1} & & & \\ & & t_2^{-1} & & \\ & & & t_3^{-1} & \\ & & & & t_4^{-1} \end{pmatrix}$$

canonical maximal torus

$$C_G(T) = T$$

$N_G(T) = \text{monomial matrixs}$

$$W(G) = D_4$$

$$SO(5, \mathbb{C}) = \{ A \in M^{5 \times 5}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

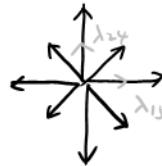
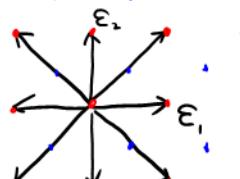
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & a_{32} & a_{33} - a_{22} & a_{34} & a_{13} \\ 0 & a_{42} - a_{31} & a_{43} - a_{21} & a_{44} - a_{11} & a_{12} \end{pmatrix} \in M^{5 \times 5}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_5, \varepsilon_2 + \varepsilon_4, \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{15} \oplus \mathbb{Z}\lambda_{24}$$

$$\Phi = \{ \pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23} \} = \{ \pm (\varepsilon_1 - \varepsilon_2), \pm \varepsilon_1, \pm (\varepsilon_1 + \varepsilon_2), \pm \varepsilon_2 \}$$

$$\Phi^\vee = \{ \pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee \} = \{ \pm (\lambda_{15} - \lambda_{24}), \pm 2\lambda_{15}, \pm (\lambda_{15} + \lambda_{24}), \pm 2\lambda_{24} \}$$



$$\Phi \subseteq Q = X^* \subseteq P$$

$\mathbb{Z}/2\mathbb{Z}$

$$\Phi^\vee$$

Now let's enjoy the beauty of low-dim root system!

Dynkin Type	Weyl group	Dimension	Order
$A_1$	$SL(2, \mathbb{C}), A_1$	3	$\mathbb{Z}/2\mathbb{Z}$
$D_2$	$SL(3, \mathbb{C}), A_2$	8	$\mathbb{Z}/3\mathbb{Z}$
$D_3$	$Sp(4, \mathbb{C}), C_2$	10	$\mathbb{Z}/2\mathbb{Z}$
$D_4$	$SO(5, \mathbb{C}), B_2$	10	$\mathbb{Z}/2\mathbb{Z}$
$E_6$	$D_4, 8$	14	$\mathbb{Z}/2\mathbb{Z}$
$E_7$	$D_4, 8$	48	$\mathbb{Z}/2\mathbb{Z}$
$E_8$	$D_4, 8$	48	$\mathbb{Z}/2\mathbb{Z}$
$A_2$	$SL(4, \mathbb{C}), A_3$	15	$\mathbb{Z}/4\mathbb{Z}$
$D_4$	$Sp(6, \mathbb{C}), C_3$	21	$\mathbb{Z}/2\mathbb{Z}$
$E_6$	$SO(7, \mathbb{C}), B_3$	48	$\mathbb{Z}/2\mathbb{Z}$
$E_7$	$C_{(2)}, 48$	21	$\mathbb{Z}/2\mathbb{Z}$
$E_8$	$C_{(2)}, 48$	21	$\mathbb{Z}/2\mathbb{Z}$

$\Phi$	$ \Phi $	$ \Phi^< $	$I$	$D$	$ W $
$A_n$ ( $n \geq 1$ )	$n(n+1)$			$n+1$	$(n+1)!$
$B_n$ ( $n \geq 2$ )	$2n^2$	$2n$	$2$	$2$	$2^n n!$
$C_n$ ( $n \geq 3$ )	$2n^2$	$2n(n-1)$	$2^{n-1}$	$2$	$2^n n!$
$D_n$ ( $n \geq 4$ )	$2n(n-1)$			$4$	$2^{n-1} n!$
$E_6$	72			3	51840
$E_7$	126			2	2903040
$E_8$	240			1	696729600
$F_4$	48	24	4	1	1152
$G_2$	12	6	3	1	12

The Weyl groups for classical groups:

- The Weyl group of  $SU(n)$  is still  $S(n)$
  - The Weyl group of  $SO(2l+1)$  is  $G(l)$ , the group of permutations  $\varphi$  of the set  $\{-l, \dots, -1, 1, \dots, l\}$  with  $\varphi(-k) = -\varphi(k)$  for all  $1 \leq k \leq l$ .
  - The Weyl group of  $SO(2l)$  is the subgroup  $SG(l)$  of  $G(l)$  that consists of even permutations.
  - The Weyl group of  $Sp(n)$  is still  $G(n)$ .

<http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2027.pdf>

[https://en.wikipedia.org/wiki/Root\\_system](https://en.wikipedia.org/wiki/Root_system)

