Eine Woche, ein Beispiel

6.29 cotangent bundle and ideal sheaf

For many years I was confused for the cotangent bundle. It has many expressions, many explanations, but they look like far away from each other. For example, I always wonder why the cotangent bundle is related with the ideal sheaf of the diagonal.

Recently I tried to understand the tangent space of the Hilbert space (of points, see[2025.06.22]). This has a similar flavor with the tangent space of Grassmannian(which means, more combinatorial). To my surprise, these special cases recover the general tangent spaces. This document concludes this "reverse engineering" and helps me to remember tangent space in a more general way.

Notation: $\mathcal{F}^\vee = \underline{RHom}_{\mathcal{O}}(\mathcal{F}, \mathcal{O})$

$V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$

Recall that for Grassmannian, one has SES

$$0 \longrightarrow S \longrightarrow \mathcal{O}_{Gr(k,n)}^{\oplus n} \longrightarrow Q \longrightarrow 0$$

and $T_{Gr(k,n)} = \underline{Hom}(S, Q) = S^\vee \otimes Q$.

Similarly, for the Hilbert scheme $Hilb^P(X)$, one has SES

$$0 \longrightarrow \mathcal{I}_Z \longrightarrow \mathcal{O}_X \longrightarrow i_{Z,*}\mathcal{O}_Z \longrightarrow 0$$

and $T_{[Z]} Hilb^P(X) = R^0 Hom_{\mathcal{O}_X}(\mathcal{I}_Z, i_{Z,*}\mathcal{O}_Z) = H^0(\mathcal{I}_Z^\vee \otimes i_{Z,*}\mathcal{O}_Z)$

Now, let $P \equiv 1$, $Z = \{p\}$, then we get SES

$$0 \longrightarrow \mathcal{I}_p \longrightarrow \mathcal{O}_X \longrightarrow i_{p,*}\mathcal{O}_p \longrightarrow 0 \qquad (\star)$$

and
$$\begin{aligned} T_p X &= R^0 Hom_{\mathcal{O}_X}(\mathcal{I}_p, i_{p,*}\mathcal{O}_p) \\ &= R^0 Hom_{\mathbb{C}}(Li_p^* \mathcal{I}_p, \mathbb{C}) \\ &= H^0\left(\left(\mathcal{I}_p \otimes^L_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_p\right)^*\right) \\ &= H^0\left(\left(\mathcal{I}_p/\mathcal{I}_p \otimes^L_{\mathcal{O}_X} \mathcal{I}_p\right)^*\right) \\ &= (\mathcal{I}_p/\mathcal{I}_p^2)^* \end{aligned}$$

$$T_p^* X = \mathcal{I}_p/\mathcal{I}_p^2$$

Globalize $(\star)$ gives us a SES of diagonals

$$0 \longrightarrow \mathcal{I}_\Delta \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_\Delta \longrightarrow 0$$

and $\Omega_X = T^*X = L^0 i_\Delta^* \mathcal{I}_\Delta = \mathcal{I}_\Delta \otimes^{L^0}_{\mathcal{O}_X} \mathcal{O}_X/\mathcal{I}_\Delta = \mathcal{I}_\Delta/\mathcal{I}_\Delta \otimes^{L^0} \mathcal{I}_\Delta = \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$

check: $\Omega_{X,p} = i_p^*(\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) = i_p^*\mathcal{I}_\Delta/i_p^*\mathcal{I}_\Delta^2 = \mathcal{I}_p/\mathcal{I}_p^2$

Divisor ideal sheaf $\mathcal{I}_D = \mathcal{O}(-D)$ $\qquad$ X smooth

When $Z = D$ is a divisor, $\mathcal{I}_D = \mathcal{O}(-D)$ is a l.b., so we can do more magic.

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0$$
$$\Uparrow \text{twist}$$
$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_D \longrightarrow 0$$
$$\Downarrow \text{dual}$$
$$\mathcal{O}_D^\vee \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(D) \xrightarrow{+1}$$
$$\Downarrow \text{shift}$$
$$\mathcal{O} \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_D^\vee[1] \xrightarrow{+1}$$

$$\Rightarrow \mathcal{O}_D^\vee[1] = \mathcal{O}_D(D)$$
$$\begin{aligned}\Rightarrow H^{n-(k+1)}(\omega_X|_D)^* &= H^{n-(k+1)}(\omega_X \otimes \mathcal{O}_D)^* \\ &= H^{k+1}(\mathcal{O}_D^\vee) \qquad \text{Serre duality} \\ &= H^k(\mathcal{O}_D(D))\end{aligned}$$

Check. When $X = C$ is a curve, $D \in \mathrm{Div}(C)$ effective of deg n.

$$T_{[D]} C^{[n]} = \mathrm{Hom}(\mathcal{O}(-D), \mathcal{O}_D) = \mathrm{Hom}(\mathcal{O}, \mathcal{O}_D(D)) = H^0(\mathcal{O}_D(D)).$$

When $X = C$, $D = p$, $n = 1$, $k = 0$,

$$T_pC = H^0(\omega_C|_p)^* = H^0(k_p(p))$$
$$T_p^*C = H^0(\omega_C|_p) = H^0(k_p(p))$$

In contrast, for general ideal sheaf, we only get a comparison of two triangles:

$$
\begin{tikzcd}
 & \mathcal{I}\otimes^L_{\mathcal{O}}\mathcal{I}^\vee \arrow[r] & \mathcal{I}^\vee \arrow[r] & \mathcal{O}_Z\otimes^L_{\mathcal{O}}\mathcal{I}^\vee \arrow[r, "+1"] & {} \\
0 \arrow[r] & \mathcal{I} \arrow[r] & \mathcal{O} \arrow[r] \arrow[u, Rightarrow, "\text{twist}\quad -\otimes^L_{\mathcal{O}}\mathcal{I}^\vee"'] \arrow[d, Rightarrow, "\text{dual}"] & \mathcal{O}_Z \arrow[r] & 0 \\
 & \mathcal{O}_Z^\vee \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, Rightarrow, "\text{shift}"] & \mathcal{I}^\vee \arrow[r, "+1"] & {} \\
 & \mathcal{O} \arrow[r] & \mathcal{I}^\vee \arrow[r] & \mathcal{O}_Z^\vee[1] \arrow[r, "+1"] & {}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{I}\otimes^L_{\mathcal{O}}\mathcal{I}^\vee \arrow[r] \arrow[d] & \mathcal{I}^\vee \arrow[r] \arrow[d, equal] & \mathcal{O}_Z\otimes^L_{\mathcal{O}}\mathcal{I}^\vee \arrow[r, "+1"] \arrow[d] & {} \\
\mathcal{O} \arrow[r] & \mathcal{I}^\vee \arrow[r] & \mathcal{O}_Z^\vee[1] \arrow[r, "+1"] & {}
\end{tikzcd}
$$

E.g.

$$
\begin{aligned}
T_{[Z]}\mathrm{Hilb}^P(X) &= R^0\mathcal{H}om_{\mathcal{O}_X}(\mathcal{I},\ \iota_{Z,*}\mathcal{O}_Z) \\
&= R^0\mathcal{H}om_{\mathcal{O}_Z}(L\iota_{Z}^{*}\mathcal{I},\ \mathcal{O}_Z) \\
&= R^0\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}\otimes^L_{\mathcal{O}_X}\mathcal{O}_Z,\ \mathcal{O}_Z) \\
&= R^0\mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}\otimes^L_{\mathcal{O}_X}\mathcal{I},\ \mathcal{O}_Z) \\
&= \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{I}/\mathcal{I}^2,\ \mathcal{O}_Z)
\end{aligned}
$$