Local Langlands Correspondence for GL_n

As modifying files in the sciebo folder is prohibited, the corrected version of my portion (with the typo rectified) will be available in the Github directories:

Talk1:
https://github.com/ramified/personal_handwritten_collection/raw/main/weeklyupdate/2023.04.23_(non-split)_reductive_group.pdf
Talk2(this one):
https://github.com/ramified/personal_handwritten_collection/raw/main/Langlands/GL_case.pdf

F : local field   NA local field + $\mathbb{R}$ & $\mathbb{C}$ case

$\Gamma_F := \mathrm{Gal}(F^{sep}/F)$

$W_F :=$ Weil group of F

NA case: $W_F = \Gamma_F \times_{\hat{\mathbb{Z}}} \mathbb{Z}$

$\mathbb{C}$ case: $W_{\mathbb{C}} = \mathbb{C}^\times$

$\mathbb{R}$ case: $W_{\mathbb{R}} := \mathbb{C}^\times \sqcup j\mathbb{C}^\times \subseteq \mathbb{H}^\times$

Rep = sm rep   For Archimedean case, we only have continuous condition.

Irr = irr sm rep

Π = adm irr sm rep $\overset{\text{p-adic}}{=\!=}$ $\mathrm{Irr}_{\mathbb{C}}$

WDrep = Weil-Deligne rep

1. $GL_n(F)$ for F NA local
2. $GL_n(F)$ for $F = \mathbb{C}$ or $\mathbb{R}$
3. G nonsplit torus over $\mathbb{R}$
4. unramified case
5. injectivity

1. $GL_n(F)$ for F NA local

Let us first state the GL_n case for a NA local field F.

Thm (LLC for $GL_n(F)$, Harris-Taylor, Henniart, Scholze)
We have a natural bijection (called the reciprocity map)

$$\mathrm{rec}_{F,GL_n} : \mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \longleftrightarrow \mathrm{WDrep}_{\substack{n\text{-dim} \\ \mathrm{Frob\ s.s}}}(W_F)$$

$$\mathrm{WDrep}_{\substack{n\text{-dim} \\ \mathrm{Frob\ s.s}}}(W_F) = \left\{ \begin{array}{ll} \phi : W_F \longrightarrow GL_n(\mathbb{C}) & \phi(\mathrm{Frob})\ \text{s.s.} \\ + \ N \in \mathrm{End}(\mathbb{C}^n) & \\ + \ \text{compatability} & \end{array} \right\}$$

$n=1$

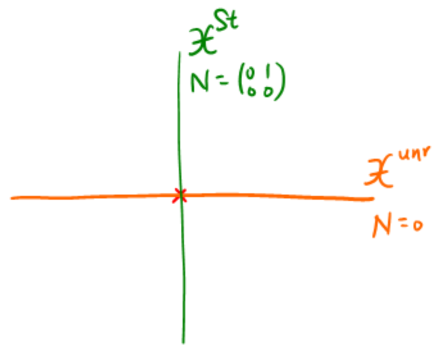
$$\chi : F^\times \longrightarrow \mathbb{C}^\times \quad \longleftrightarrow \quad \chi : W_F \twoheadrightarrow W_F^{ab} \cong F^\times \xrightarrow{\chi} \mathbb{C}^\times$$

$n=2$.

$$\begin{array}{lccc}
1) & \chi \circ \det & \longleftrightarrow & \left(\begin{pmatrix} \chi\cdot|\cdot|_F^{-\frac{1}{2}} & \\ & \chi\cdot|\cdot|_F^{\frac{1}{2}} \end{pmatrix}, 0 \right) \\
2) & n\text{-}\mathrm{Ind}_B^{GL_2}(\chi_1,\chi_2) & \longleftrightarrow & \left(\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}, 0 \right) \\
3) & \mathrm{St} \otimes (\chi\circ\det) & \longleftrightarrow & \left(\begin{pmatrix} \chi\cdot|\cdot|_F^{-\frac{1}{2}} & \\ & \chi\cdot|\cdot|_F^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \\
4) & c\text{-}\mathrm{Ind}_{KZ}^{GL_2} \rho & \longleftrightarrow & \text{don't know how to describe}
\end{array}$$

Rmk.

$$\begin{array}{lccc}
\text{action by } \chi & -\otimes(\chi\circ\det) & \longleftrightarrow & -\otimes\chi \\
\text{unramified} & V^K \neq \{0\} & \longleftrightarrow & \phi|_{I_F} = \mathbb{1}_{I_F},\ N=0 \\
 & \text{disc series} & \longleftrightarrow & \text{indec rep} \\
 & \text{cuspidal} & \longleftrightarrow & \text{irr rep} \\
 & \text{“}n\text{-Ind”} & \longleftrightarrow & \oplus
\end{array}$$



Let us try to work out $n=1$ case. In that case,

$$\begin{aligned} \text{RHS} &= \{\phi : W_F \longrightarrow \mathbb{C}^\times\} \\ &= \{\phi : W_F^{ab} \longrightarrow \mathbb{C}^\times\} \\ &\overset{\text{Artin}}{=\!=} \{\rho : F^\times \longrightarrow \mathbb{C}^\times\} = \text{LHS} \end{aligned}$$

Rem. The key argument is the Artin map

$$W_F^{ab} \cong F^\times$$

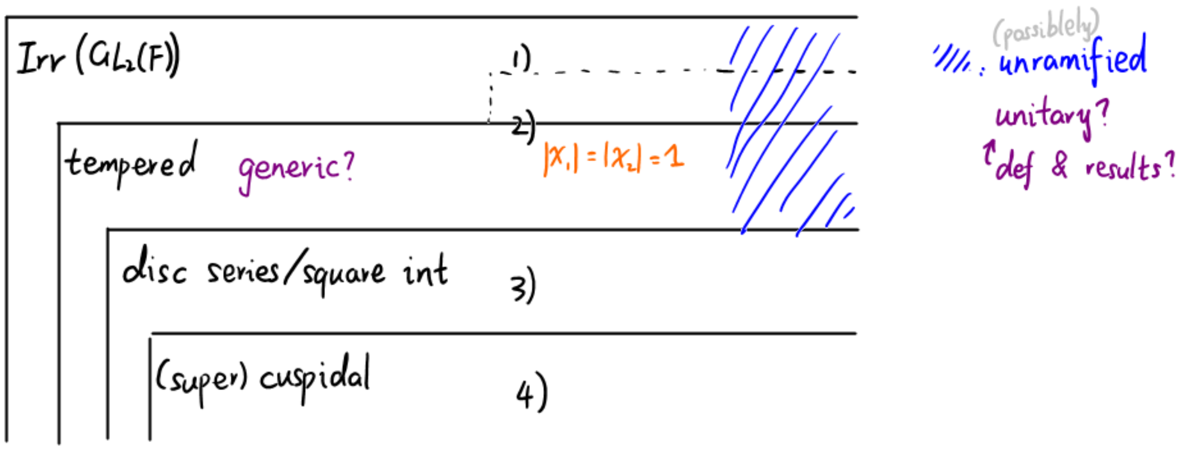
For $n=2$ case, we still have nice descriptions on both side. However, it would already take the content of a whole book for us to comprehend the details of this case.

Thm (Langlands classification for $\mathrm{Irr}_{\mathbb{C}}(GL_2(F))$)

We have a classification of $\mathrm{Irr}_{\mathbb{C}}(GL_2(F))$. $\qquad \chi : K^\times \to \mathbb{C}$

	1) 1-dim	$\chi \circ \det$	
$\chi\lvert\cdot\rvert_F^{-\frac{1}{2}} \boxplus \chi\lvert\cdot\rvert_F^{\frac{1}{2}}$	2) principal series	$n\text{-}\mathrm{Ind}_B^{GL_2}(\chi_1,\chi_2)$	$\chi_1\chi_2^{-1} \neq \lVert\cdot\rVert^{\pm 1}$
$\chi_1 \boxplus \chi_2$	3) a twist of St by χ	$St \otimes (\chi \circ \det)$	
$Sp_2(\chi)$	4) supercuspidal rep	$c\text{-}\mathrm{Ind}_{KZ}^{GL_2} \rho$	for some $\rho \in \mathrm{Irr}_{\mathbb{C}}(KZ)$

in [Gee]



For $\mathbb{F}_p(t)$-case, see wiki: Drinfeld module.

Rmk. $c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G(K)}\rho \in \mathrm{Irr}(G) \rightsquigarrow c\text{-}\mathrm{Ind}_{\mathcal{U}}^{G(K)}\rho$ sc

ref: Prof. Can's lecture note in Lec 9, p74

Conj. all sc reps arise in this way (true for GL_n)

$\tilde{\rho} \in \mathrm{Rep}(GL_n(\mathcal{O}_F)\cdot F^\times)$ s.t. $\rho \in \mathrm{Cusp}(GL_n(\kappa)) \rightsquigarrow c\text{-}\mathrm{Ind}_{GL_n(\mathcal{O}_F)\cdot F^\times}^{GL_n(F)}\rho \in \mathrm{Cusp}(GL_n(F))$

$\tilde{\rho}|_{1+\varpi M_n(\mathcal{O}_F)} = \mathrm{Id}$

[Thm 1.7] https://nms.kcl.ac.uk/james.newton/week4.pdf

https://mathoverflow.net/questions/101067/are-all-irreducible-supercuspidal-representation-induced-from-compact-mod-center
https://math.stackexchange.com/questions/821417/is-supercuspidal-representation-the-same-as-cuspidal-representation

2. $GL_n(F)$ for $F = \mathbb{C}$ or $\mathbb{R}$

For the Archimedean case, we also want to construct such a correspondence. In this case, we have a relatively explicit description on both sides, since the structure of the Weyl gp is easier. Also, we don't need to worry about cuspidal reps here.

For avoiding technical conditions, we only state the LLC for $GL_n(F)$.

$F = \mathbb{R}$ or $\mathbb{C}$.

Thm (LLC for $GL_n(F)$)

We have a 1-to-1 correspondence

$$\begin{tikzcd} \Pi(GL_n(F))/\sim \arrow[r, leftrightarrow] \arrow[d, "\cong"] & \Phi(GL_{n,F})/\sim \arrow[d, equal, "def"] \\ \{\text{Irr adm } (\mathfrak{gl}_n, K)\text{-modules}\} & \left\{ \begin{array}{l} \phi : W_F \longrightarrow GL_n(\mathbb{C}) \\ \text{semisimple as reps} \end{array} \right\} / GL_n(\mathbb{C})\text{-conj} \end{tikzcd}$$

where

$K := O(n)$ or $U(n)$

$\sim$: up to infinitesimally equivalence

i.e. induce the same $(\mathfrak{g}, K)$-modules

For letting $n=1$ case to be true, we have to ask at least

$$W_F^{ab} \cong F^\times$$

Also, W_K should be related to Γ_F.

Def (Weil gp for $F = \mathbb{R}, \mathbb{C}$)

$$W_{\mathbb{C}} := \mathbb{C}^\times$$
$$W_{\mathbb{R}} := \mathbb{C}^\times \sqcup j\mathbb{C}^\times \subset \mathbb{H}^\times$$

Ex.
$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \Gamma_{\mathbb{R}} \longrightarrow 1$$
$$j^2 = -1 \qquad jzj^{-1} = \bar{z} \qquad \forall z \in \mathbb{C}^\times$$

$$\Rightarrow \frac{\bar{z}}{z} = jzj^{-1}z^{-1} \in [W_{\mathbb{R}}, W_{\mathbb{R}}]$$
$$\Rightarrow [W_{\mathbb{R}}, W_{\mathbb{R}}] = S^1$$
$$\Rightarrow W_{\mathbb{R}}^{ab} \cong (\mathbb{C}^\times \sqcup j\mathbb{C}^\times)/S^1 \cong \mathbb{R}_{>0} \sqcup j\mathbb{R}_{>0} \cong \mathbb{R}^\times$$

By this iso ($W_F^{ab} \cong F^\times$), we have shown the LLC for $n=1$ case abstractly. To understand more, we must discuss this case in more detail.

$GL_n(F)$	$\mathbb{R}$	$\mathbb{C}$
$n=1$	$\mathbb{C} \times \{\pm 1\}$ $i\mathbb{R} \times \{\pm 1\}$	$\mathbb{C} \times \mathbb{Z}$ $i\mathbb{R} \times \mathbb{Z}$
$n=2$	$\cdots$ $\mathbb{C} \times \mathbb{N}_{>0}$ $i\mathbb{R} \times \mathbb{N}_{>0}$	$\cdots$ ϕ
$n>2$	$\cdots$ ϕ	

$\cdots$: written as direct sum of lower dim reps.

orange: unitary representations. for L-parameters side.

E.g. $n=1$, $F=\mathbb{R}$

$$\{\rho : \underset{\mathbb{R}_{>0} \times \{\pm 1\}}{\overset{\wr\wr}{\mathbb{R}^\times}} \longrightarrow \mathbb{C}^\times\} \cong \underset{t}{\overset{\in}{\mathbb{C}}} \times \{\pm 1\}$$

$$x \longmapsto x^t \qquad\qquad \rightsquigarrow \begin{cases} \chi_{triv} \otimes |\cdot|^t \\ \chi_{sign} \otimes |\cdot|^t \end{cases}$$

$$-1 \longmapsto \pm 1$$

The characters of $W_\mathbb{R}$ are given by

$$\begin{tikzcd} W_\mathbb{R} \arrow[r] \arrow[d] & \mathbb{C}^\times \\ \mathbb{R}^\times \cong W_\mathbb{R}^{ab} \arrow[ur, dashed, "\exists !"'] & \end{tikzcd} \qquad \begin{tikzcd} z \arrow[d, mapsto] & |z|^t \\ |z| \arrow[ur, mapsto] & \end{tikzcd} \qquad \begin{tikzcd} j \arrow[d, mapsto] & \pm 1 \\ -1 \arrow[ur, mapsto] & \end{tikzcd}$$

e.p. the unitary reps are parameterized by $i\mathbb{R} \times \{\pm 1\}$.

E.g. $n=1$, $F=\mathbb{C}$

$$\{\rho : \underset{\mathbb{R}_{>0} \times S^1}{\overset{\wr\wr}{\mathbb{C}^\times}} \longrightarrow \mathbb{C}^\times\} \cong \underset{(t,l)}{\overset{\in}{\mathbb{C}}} \times \mathbb{Z}$$

$$z = re^{i\theta} \longmapsto r^t e^{il\theta} = z^{\frac{t+l}{2}} \bar{z}^{\frac{t-l}{2}}$$

$\Big\downarrow$ reparameterization

$$z \longmapsto z^\mu \bar{z}^\nu \qquad \{(\mu,\nu) \in \mathbb{C} \times \mathbb{C} \mid \mu - \nu \in \mathbb{Z}\}$$

e.p. the unitary reps are parameterized by $i\mathbb{R} \times \mathbb{Z}$.

E.g. $n=2$, $F=\mathbb{R}$

$$\left\{\phi: \underset{\substack{\cup\!\shortmid \\ \mathbb{C}^\times \\ z}}{W_{\mathbb{R}}} \longrightarrow GL_2(\mathbb{C})\right\}/\sim$$

$$z \longmapsto \begin{pmatrix} z^{\mu}\bar{z}^{\nu} & \\ & z^{\mu'}\bar{z}^{\nu'} \end{pmatrix}$$

①: $\phi = \chi_1 \oplus \chi_2$ $\quad \dim \chi_i = 1$

$\rightsquigarrow$ subquotient of $n\text{-}\mathrm{Ind}_B^G(\chi_1, \chi_2)$
quotient, when $\operatorname{Re} t_1 \geq \operatorname{Re} t_2$
FD & principal series
finite dim reps.

②: ϕ irreducible.
By linear algebra arguments, i.e. choose a good basis

$$\left\{\phi: W_{\mathbb{R}} \longrightarrow GL_2(\mathbb{C}) \text{ irr}\right\}/\sim \;\; \cong \;\; \mathbb{C} \times \mathbb{N}_{>0}$$

$$z \longmapsto \begin{pmatrix} z^{\mu}\bar{z}^{\nu} & \\ & z^{\nu}\bar{z}^{\mu} \end{pmatrix} \qquad \left(\tfrac{\mu+\nu}{2}, \mu-\nu\right)$$

$$j \longmapsto \begin{pmatrix} & (-1)^{\mu-\nu} \\ 1 & \end{pmatrix} \qquad (t, l)$$

$\rightsquigarrow DS_l \otimes |\det(-)|_{\mathbb{R}}^t$

discrete series $\quad \chi \circ \det$

Rem. In Prof. Caraiani's course, we did the classification of irr adm $(\mathfrak{gl}_{2,\mathbb{R}}, O(2))$-modules.
We reproduce it by the LLC!

Details about linear algebras should be put in this page.

Ref here: [Knapp91, Sec 3]: https://www.math.stonybrook.edu/~aknapp/pdf-files/motives.pdf

Step 1. Analyze $\phi|_{\mathbb{C}^\times}$

$$\left.\begin{array}{r}\phi(z) \text{ is diagonalizable} \\ \mathbb{C}^\times \text{ is commutative}\end{array}\right\} \Rightarrow \phi|_{\mathbb{C}^\times} \cong \chi_1 \oplus \chi_2$$

i.e. under some basis $\{u, v\}$,

$$\phi: z \longmapsto \begin{pmatrix} z^{\mu}\bar{z}^{\nu} & \\ & z^{\mu'}\bar{z}^{\nu'} \end{pmatrix}$$

$\phi(z)\cdot u = z^{\mu}\bar{z}^{\nu} u$
$\phi(z)\cdot v = z^{\mu'}\bar{z}^{\nu'} v$

Step 2. Remove decomposable cases.

When $\mu = \mu'$, $\nu = \nu'$: (same eigenvalues)

$$\left.\begin{array}{r}\phi(j) \text{ is diagonalizable} \\ \phi(\mathbb{C}^\times) \subset Z(GL_2(\mathbb{C}))\end{array}\right\} \Rightarrow \phi \cong \chi_1 \oplus \chi_2$$

Assume $\mu \neq \mu'$ or $\nu \neq \nu'$ now.

$$\phi(z)\phi(j)u = \phi(j)\phi(\bar{z})u = z^{\nu}\bar{z}^{\mu}\phi(j)u$$

$\Rightarrow \phi(j)u$ is an eigenvector with eigenvalue $z^{\nu}\bar{z}^{\mu}$

When $\mu = \nu$, then

$\mathbb{C}u$ is irr subrep $\rightsquigarrow \phi \cong \chi_1 \oplus \chi_2$;

When $\mu \neq \nu$, then $\mu' = \nu$, $\nu' = \mu$.

under the basis $\{u, \phi(j)u\}$,

$$\phi: z \longmapsto \begin{pmatrix} z^{\mu}\bar{z}^{\nu} & \\ & z^{\nu}\bar{z}^{\mu} \end{pmatrix}$$

$$j \longmapsto \begin{pmatrix} & ? \\ 1 & ? \end{pmatrix} \overset{j^2=-1}{=\!=} \begin{pmatrix} & (-1)^{\mu-\nu} \\ 1 & \end{pmatrix}$$

Step 3. By the symmetry, we can assume that $\mu - \nu > 0$.

under the basis $\{\phi(j)u, (-1)^{\mu-\nu}u\}$,

$$\phi: z \longmapsto \begin{pmatrix} z^{\nu}\bar{z}^{\mu} & \\ & z^{\mu}\bar{z}^{\nu} \end{pmatrix}$$

$$j \longmapsto \begin{pmatrix} & (-1)^{\nu-\mu} \\ 1 & \end{pmatrix}$$

$$\left.\begin{array}{r}(\det\phi)(z) = |z|^{\mu+\nu} \\ (\det\phi)(j) = (-1)^{\mu-\nu+1} \\ \phi|_{\mathbb{C}^\times} \cong \chi_{\mu,\nu}\oplus\chi_{\nu,\mu}\end{array}\right\} \Rightarrow \left(\tfrac{\mu+\nu}{2}, |\mu-\nu|\right) \in \mathbb{C}\times\mathbb{N}_{>0} \text{ are determined by the rep } \phi.$$

$\square$

Rmk. By the similar linear algebra argument, one can show

$$\phi \in \mathrm{Irr}_{\mathbb{C}}(W_{\mathbb{R}}) \rightsquigarrow \dim_{\mathbb{C}} \phi = 1 \text{ or } 2$$

$$\phi \in \mathrm{Irr}_{\mathbb{C}}(W_{\mathbb{C}}) \rightsquigarrow \dim_{\mathbb{C}} \phi = 1$$

By the correspondence, we get classifications of $GL_n(F)$-reps explicitly:

[Knapp91, p400]: https://www.math.stonybrook.edu/~aknapp/pdf-files/motives.pdf

Theorem 1. For $G = GL_n(\mathbb{R})$,

(a) if the parameters $n_j^{-1} t_j$ of $(\sigma_1, \ldots, \sigma_r)$ satisfy

$$n_1^{-1}\,\mathrm{Re}\,t_1 \geq n_2^{-1}\,\mathrm{Re}\,t_2 \geq \cdots \geq n_r^{-1}\,\mathrm{Re}\,t_r, \tag{2.5}$$

$n\text{-}\mathrm{Ind}_P^{GL_n}(\sigma_1, \cdots, \sigma_r)$

then $I(\sigma_1, \ldots, \sigma_r)$ has a unique irreducible quotient $J(\sigma_1, \ldots, \sigma_r)$,

(b) the representations $J(\sigma_1, \ldots, \sigma_r)$ exhaust the irreducible admissible representations of G, up to infinitesimal equivalence,

(c) two such representations $J(\sigma_1, \ldots, \sigma_r)$ and $J(\sigma_1', \ldots, \sigma_r')$ are infinitesimally equivalent if and only if $r' = r$ and there exists a permutation $j(i)$ of $\{1, \ldots, r\}$ such that $\sigma_i' = \sigma_{j(i)}$ for $1 \leq i \leq r$.

Q: Find a reference for the statement of $GL_n(\mathbb{C})$.

3. G nonsplit torus over $\mathbb{R}$

We can also state (and even prove) LLC for nonsplit torus over $\mathbb{R}$.

I saw the result here [Part V, section 30]: Bill Casselman, Representations of SL2(R)
https://personal.math.ubc.ca/~cass/research/pdf/Irr.pdf

For the examples, I try to do computations in a more natural way.

Thm (LLC for $G/\mathbb{R}$ torus) e.g. $G = \mathbb{G}_{m,\mathbb{R}}$, $SO_{2,\mathbb{R}}$, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$

We have a 1-to-1 correspondence

$$\Pi(G(\mathbb{R}))/\sim \quad \longleftrightarrow \quad \Phi(G_{\mathbb{R}})/\sim$$

$$\Phi(G_{\mathbb{R}})/\sim \;\overset{\text{def}}{=}\; \left\{ {}^L\phi : W_{\mathbb{R}} \longrightarrow {}^LG \;\; \text{cont gp homo "sec"} \right\} / \hat{G}(\mathbb{C})\text{-conj}$$

where "sec" means, the following diagram commutes:

$$\begin{tikzcd} W_{\mathbb{R}} \arrow[rr, "{}^L\phi"] \arrow[dr] & & {}^LG = \hat{G}(\mathbb{C}) \rtimes \Gamma_{\mathbb{R}} \arrow[dl] \\ & \Gamma_{\mathbb{R}} & \end{tikzcd}$$

Q: How could we state LLC for reductive gp over $\mathbb{C}$ or $\mathbb{R}$ rigorously?

See [BVA92, Theorem 1.18]

E.g. For $G = \mathbb{G}_{m,\mathbb{R}}$, $\rtimes$ becomes $\times$, and

$$\mathrm{RHS} = \{\phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times\} = \{\rho : \mathbb{R}^\times \longrightarrow \mathbb{C}^\times\} = \mathrm{LHS} \qquad \square$$

E.g. For $G = SO_{2,\mathbb{R}}$, we get

$$\widehat{G}(\mathbb{C}) = \mathbb{C}^\times \qquad {}^LG = \mathbb{C}^\times \rtimes \Gamma_{\mathbb{R}}$$

where $\Gamma_{\mathbb{R}}$ acts on $\mathbb{C}^\times$ by

$$\Gamma_{\mathbb{R}} \times \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \qquad (\sigma, z) \longmapsto {}^\sigma z := z^{-1}$$

$$\begin{aligned}\mathrm{RHS} &= \{{}^L\phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times \rtimes \Gamma_{\mathbb{R}} \text{ cont ``sec''}\}/\mathbb{C}^\times\text{-conj} \\ &= \{\phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times \text{ cocycle}\}/\text{twisted } \mathbb{C}^\times\text{-conj}\end{aligned}$$

By mimicking the proof in split torus case, one computes

$$\phi(\bar{z}) = \phi(jzj^{-1}) = \phi(j)\,{}^\sigma\phi(z)\,\phi(j)^{-1} = {}^\sigma\phi(z) = \phi(z)^{-1}$$

$$\Rightarrow \phi(|z|^2) = \phi(jzj^{-1}z) = 1$$

$$\Rightarrow \begin{tikzcd} {}^L\phi : W_{\mathbb{R}} \arrow[r] \arrow[d] & \mathbb{C}^\times \rtimes \Gamma_{\mathbb{R}} \\ S' \sqcup jS' \cong W_{\mathbb{R}}/\langle jzj^{-1}z\rangle_{z \in \mathbb{C}^\times} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Therefore,

$$\begin{aligned}\mathrm{RHS} &= \{{}^L\phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times \rtimes \Gamma_{\mathbb{R}} \text{ cont ``sec''}\}/\mathbb{C}^\times\text{-conj} \\ &= \{{}^L\phi : S' \sqcup jS' \longrightarrow \mathbb{C}^\times \rtimes \Gamma_{\mathbb{R}} \text{ cont ``sec''}\}/\mathbb{C}^\times\text{-conj} \\ &= \{\rho : S' \longrightarrow \mathbb{C}^\times \text{ cont}\} = \mathrm{LHS} \qquad \square\end{aligned}$$

For the effect of $\mathbb{C}^\times$-conj,

$$\begin{aligned}(z, Id)(\phi(j), \sigma)(z^{-1}, Id) &= (z\phi(j)\,{}^\sigma(z^{-1}), \sigma) \\ &= (z\phi(j)z, \sigma) \\ &= (z^2\phi(j), \sigma)\end{aligned}$$

So we can assume $\phi(j) = 1$, and remove $\mathbb{C}^\times$-conj action.

E.g. For $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$, we get

$$\widehat{G}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \qquad {}^L G = (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_{\mathbb{R}}$$

where $\Gamma_{\mathbb{R}}$ acts on $\mathbb{C}^\times \times \mathbb{C}^\times$ by

$$\Gamma_{\mathbb{R}} \times (\mathbb{C}^\times \times \mathbb{C}^\times) \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \qquad (\sigma, (z_1, z_2)) \longmapsto {}^\sigma(z_1, z_2) := (z_2, z_1)$$

$$\begin{aligned} \mathrm{RHS} &= \{ {}^L\phi : W_{\mathbb{R}} \longrightarrow (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_{\mathbb{R}} \text{ cont "sec"} \} / (\mathbb{C}^\times \times \mathbb{C}^\times)\text{-conj} \\ &= \{ \phi : W_{\mathbb{R}} \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \text{ cocycle } \} / \text{twisted } (\mathbb{C}^\times \times \mathbb{C}^\times)\text{-conj} \end{aligned}$$

By the formula

$$\phi(\bar{z}) = \phi(j z j^{-1}) = {}^\sigma(\phi(z))$$

we get

$$\phi(z) = (z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu).$$

By $(\mathbb{C}^\times \times \mathbb{C}^\times)$-conj, we can assume

$$\phi(j) = (1, 1)$$

Therefore,

$$\begin{aligned} \mathrm{RHS} &= \left\{ \rho : \begin{array}{l} \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \times \mathbb{C}^\times \\ z \longmapsto (z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu) \end{array} \right\} \\ &= \{ \rho : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times \text{ cont } \} \\ &= \{ \rho : (\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}})(\mathbb{R}) \longrightarrow \mathbb{C}^\times \text{ cont} \} = \mathrm{LHS}. \qquad \square \end{aligned}$$

One may compute

$$\begin{aligned} &((z_1, z_2), \mathrm{Id})((a, b), \sigma)((z_1^{-1}, z_2^{-1}), \mathrm{Id}) \\ &= ((z_1 a \, {}^\sigma(z_1^{-1}), z_2 b \, {}^\sigma(z_2^{-1})), \sigma) \\ &= ((z_1^2 a, z_2^2 b), \sigma) \end{aligned}$$

Fun game: you have already some examples of LLC. (GL_n + torus) Try to make some comparisons and find some functoriality results!

e.g.

$$\begin{array}{ccc} \mathbb{G}_{m,\mathbb{R}} \hookrightarrow GL_{2,\mathbb{R}} & \Rightarrow & \Pi(\mathbb{R}^\times) \leftarrow \Pi(GL_2(\mathbb{R})) \\ SO_{2,\mathbb{R}} \nearrow & & \Pi(S^1) \swarrow \end{array}$$

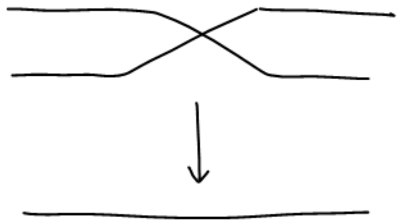
e.g.

$$\Pi(\mathbb{G}_m(\mathbb{C})) = \Pi((\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}})(\mathbb{R}))$$

Task left: work out some examples of torus over F, where F: local NA field

4. unramified case.

⚠ I didn't find any ref for ramified L-packets for real red gps, so use at your own risk!



ramified

$I_F \subseteq \Gamma_F$ Inertia subgp

$$f(x) = \begin{cases} \ln|x| & n=2 \\ |x|^{2-n} & n>2 \end{cases}$$

spherical

$S^1 \subseteq \mathbb{C}^\times$ max cpt subgp

Compare: for F' local field,

$$\begin{tikzcd} 1 \arrow[r] & I_{F'} \arrow[r] \arrow[d] & W_{F'} \arrow[r] \arrow[d, two heads] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \mathcal{O}_{F'}^\times \arrow[r] & F'^{,\times} \arrow[r] & \mathbb{Z} \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & S^1 \sqcup j S^1 \arrow[r] \arrow[d] & W_{\mathbb{R}} \arrow[r] \arrow[d, two heads] & \mathbb{R}_{>0} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathbb{R}^\times \arrow[r] & \mathbb{R}_{>0} \arrow[r] & 1 \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & S^1 \arrow[r] \arrow[d, equal] & W_{\mathbb{C}} \arrow[r] \arrow[d, equal] & \mathbb{R}_{>0} \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & S^1 \arrow[r] & \mathbb{C}^\times \arrow[r] & \mathbb{R}_{>0} \arrow[r] & 1 \end{tikzcd}$$

where $\mathcal{O}_{F'}^\times \subseteq F'^{,\times}$, $\{\pm 1\} \subseteq \mathbb{R}^\times$, $S^1 \subseteq \mathbb{C}^\times$ are maximal compact subgps.

Slogan: Inertia gp can be viewed as the preimage of maximal compact subgp.

Def. Let $F=\mathbb{R}$ or $\mathbb{C}$. We say that

$$\phi \in \Phi(GL_{n,F}) \text{ is unramified, if } \begin{cases} \phi(S' \sqcup jS') = 1_{n\times n} & F=\mathbb{R} \\ \phi(S') = 1_{n\times n} & F=\mathbb{C} \end{cases}$$

$\rho \in \Pi(GL_n(F))$ is unramified, if the crspd $\phi \in \Phi(GL_{n,F})$ is unramified.

One get

$$\begin{array}{ccc} \Pi(GL_n(F))/\sim & \longleftrightarrow & \Phi(GL_{n,F})/\sim \\ \cup & & \cup \\ \Pi_{ur}(GL_n(F))/\sim & \longleftrightarrow & \Phi_{ur}(GL_{n,F})/\sim \end{array}$$

Later, we will see that $K = O(n)$ or $\mathcal{U}(n)$

$$\Pi_{ur}(GL_n(F)) = \{(\rho, V) \in \Pi(GL_n(F)) \mid V^K \neq 0\}$$

In general, G: conn red gp

for $G/\mathbb{R}$, $\phi \in \Phi(G)$ is unramifed $\Leftrightarrow \phi'(S' \sqcup jS') = 1_{\hat{G}}$ for some $\phi' \sim \phi$

for $G/\mathbb{C}$, $\phi \in \Phi(G)$ is unramifed $\Leftrightarrow \phi(S') = 1_{\hat{G}}$

Q: Do we have

$$\Pi_{ur}(G(F)) = \{(\rho, V) \in \Pi(G(F)) \mid V^K \neq 0 \text{ for some } K \text{ max cpt subgp}\} ?$$

$GL_n(F)$	$\mathbb{R}$	$\mathbb{C}$
$n=1$	$\mathbb{C} \times \{\pm 1\}$ $i\mathbb{R} \times \{\pm 1\}$ $\mathbb{C} \times \{1\}$	$\mathbb{C} \times \mathbb{Z}$ $i\mathbb{R} \times \mathbb{Z}$ $\mathbb{C} \times \{1\}$
$n=2$	$\mathbb{C} \times \mathbb{N}_{>0}$ $i\mathbb{R} \times \mathbb{N}_{>0}$	$\emptyset$
$n>2$	$\emptyset$	

$\mathbb{C}^{\oplus n}/S_n$-action

orange: unitary representations. for L-parameters side.

blue: unramified L-parameters

E.g. $n=1$, $F=\mathbb{R}$

All the unramified L-parameters are given by

$$\begin{tikzcd} W_{\mathbb{R}} \arrow[r] \arrow[d] & \mathbb{C}^\times \\ \mathbb{R}_{>0} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} z \arrow[d, mapsto] & {|z|^t} \\ {|z|} \arrow[ur, mapsto] & \end{tikzcd} \qquad \begin{tikzcd} j \arrow[d, mapsto] & 1 \\ 1 \arrow[ur, mapsto] & \end{tikzcd}$$

i.e. $\quad \Phi_{ur}(\mathbb{G}_{m,\mathbb{R}})/\sim \;\cong\; \mathbb{C} \times \{1\}$

$\rightsquigarrow \quad \Pi_{ur}(\mathbb{G}_m(\mathbb{R}))/\sim \;\cong\; \{|\cdot|^t \mid t \in \mathbb{C}\}$

E.g. $n=1$, $F=\mathbb{C}$

All the unramified L-parameters are given by

$$\begin{tikzcd} W_{\mathbb{C}} \arrow[r] \arrow[d] & \mathbb{C}^\times \\ \mathbb{R}_{>0} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} z \arrow[d, mapsto] & {|z|^t} \\ {|z|} \arrow[ur, mapsto] & \end{tikzcd}$$

i.e. $\quad \Phi_{ur}(\mathbb{G}_{m,\mathbb{C}})/\sim \;\cong\; \mathbb{C} \times \{1\}$

$\rightsquigarrow \quad \Pi_{ur}(\mathbb{G}_m(\mathbb{C}))/\sim \;\cong\; \{|\cdot|^t \mid t \in \mathbb{C}\}$

E.g. $n=2$, $F=\mathbb{R}$

All the unramified L-parameters are given by

$$\begin{tikzcd} W_{\mathbb{R}} \arrow[r] \arrow[d] & GL_2(\mathbb{C}) \\ \mathbb{R}_{>0} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} z \arrow[d, mapsto] & {\begin{pmatrix} |z|^{t_1} & \\ & |z|^{t_2} \end{pmatrix}} \\ {|z|} \arrow[ur, mapsto] & \end{tikzcd} \qquad \begin{tikzcd} j \arrow[d, mapsto] & {\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}} \\ 1 \arrow[ur, mapsto] & \end{tikzcd}$$

i.e. $\quad \Phi_{ur}(GL_{2,\mathbb{R}})/\sim \;\cong\; \mathbb{C} \times \mathbb{C}/S_2\text{-action}$

$\rightsquigarrow \quad \Pi_{ur}(\mathbb{G}_m(\mathbb{R}))$ are FD or principal series.

In ptc, they're all the subquotients of $n\text{-}\mathrm{Ind}_B^G(|\cdot|^{t_1}, |\cdot|^{t_2})$.

Rmk. We know that, for $\chi_1, \chi_2 \in \widehat{\mathbb{R}^{\times}}$,

$n\text{-}\mathrm{Ind}_B^G(\chi_1,\chi_2)$ has an $O_2(\mathbb{R})$-fixed vector

$$\Leftrightarrow \quad f: GL_2(\mathbb{R}) \longrightarrow \mathbb{C}^\times, \quad g = bk \longmapsto \chi_1(b_1)\chi_2(b_2)\left|\frac{b_1}{b_2}\right|^{\frac{1}{2}}$$

$k \in O_2(\mathbb{R})$, $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B(\mathbb{R})$

is well-defined

$$\Leftrightarrow \quad \chi_1(-1) = \chi_2(-1) = 1$$

Ex. Check that in NA local field case, $\chi_1, \chi_2 \in \widehat{F'^{\times}}$

$n\text{-}\mathrm{Ind}_B^G(\chi_1,\chi_2)$ has an $GL_2(\mathcal{O}_{F'})$-fixed vector

$$\Leftrightarrow \quad f: GL_2(F') \longrightarrow \mathbb{C}^\times, \quad g = bk \longmapsto \chi_1(b_1)\chi_2(b_2)\left|\frac{b_1}{b_2}\right|_{F'}^{\frac{1}{2}}$$

$k \in GL_2(\mathcal{O}_{F'})$, $b = \begin{pmatrix} b_1 & * \\ 0 & b_2 \end{pmatrix} \in B(F')$

is well-defined

$$\Leftrightarrow \quad \chi_1(b_1)\chi_2(b_2) = 1 \qquad \forall\, b_1, b_2 \in \mathcal{O}_{F'}^{\times}$$

$$\Leftrightarrow \quad \chi_1, \chi_2 \text{ are unramified.}$$

E.g. For $G = SO_{2,\mathbb{R}}$, all the unramified L-parameters are given by

$$\begin{array}{ccc}
\begin{array}{ccc} W_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{C}^\times \\ \downarrow & \nearrow & \\ \mathbb{R}_{>0} & & \end{array}
&
\begin{array}{ccc} z & & |z|^t \\ \updownarrow & \swarrow\!\!\nearrow & \\ |z| & & \end{array}
&
\begin{array}{ccc} j & & 1 \\ \updownarrow & \swarrow\!\!\nearrow & \\ 1 & & \end{array}
\end{array}$$

with $\phi(|z|^2) = \phi(jzj^{-1}z) = 1 \quad \Rightarrow \quad t = 0$

i.e. $\quad \Phi_{ur}(SO_{2,\mathbb{R}})/\sim \;\cong\; \{1_{W_{\mathbb{R}}}\}$

$\rightsquigarrow \quad \Pi_{ur}(SO_2(\mathbb{R}))/\sim \;\cong\; \{1_{SO_2(\mathbb{R})}\}$ $\square$

E.g. For $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}}$, all the unramified L-parameters are given by

$$\begin{array}{ccc}
\begin{array}{ccc} W_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{C}^\times \times \mathbb{C}^\times \\ \downarrow & \nearrow & \\ \mathbb{R}_{>0} & & \end{array}
&
\begin{array}{ccc} z & & (|z|^{t_1}, |z|^{t_2}) \\ \updownarrow & \swarrow\!\!\nearrow & \\ |z| & & \end{array}
&
\begin{array}{ccc} j & & (1,1) \\ \updownarrow & \swarrow\!\!\nearrow & \\ 1 & & \end{array}
\end{array}$$

with $\phi(\bar{z}) = \phi(jzj^{-1}) = {}^{\sigma}(\phi(z)) \quad \Rightarrow \quad t_1 = t_2$

i.e. $\quad \Phi_{ur}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{R}})/\sim \;\cong\; \mathbb{C}$

$\rightsquigarrow \quad \Pi_{ur}(\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m(\mathbb{R}))/\sim \;\cong\; \{|\cdot|^t \mid t \in \mathbb{C}\}$ $\square$

5. injectivity $\qquad \hat{G} := \hat{G}(\mathbb{C}) \qquad \hat{T} := \hat{T}(\mathbb{C})$

In this section, we show that

$$\pi_0(\bar{S}_\phi) = \{*\} \quad \text{for} \quad \phi \in \Phi(G), \quad G = GL_{n,\mathbb{R}} \text{ or torus}/\mathbb{R}$$

where $\qquad\qquad$ $GL_{n,\mathbb{C}}$ or torus$/\mathbb{C}$ are easier.

$$S_\phi := C_{{}^LG}({}^L\phi(W_\mathbb{R})) \cap \hat{G} \qquad\qquad \bar{S}_\phi := S_\phi / Z(\hat{G})^\Gamma$$

$$Z(\hat{G})^\Gamma := C_{{}^LG}({}^LG) \cap \hat{G} \overset{\text{if } G \text{ split}}{=\!=\!=} Z(\hat{G})$$

E.g. $G = \mathbb{G}_{m,\mathbb{R}}$. We get

$$\hat{G} = \mathbb{C}^\times \qquad Z(\hat{G})^\Gamma = Z(\hat{G}) = \mathbb{C}^\times.$$

For $\phi \in \Phi(\mathbb{G}_{m,\mathbb{R}})$,

$$S_\phi = \mathbb{C}^\times \qquad \bar{S}_\phi = \{*\} \qquad \pi_0(\bar{S}_\phi) = \{*\}.$$

Lemma. In $GL_2(\mathbb{C})$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & \\ & y \end{pmatrix} = \begin{pmatrix} x & \\ & y \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff b = c = 0 \text{ or } x = y$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} & x \\ y & \end{pmatrix} = \begin{pmatrix} & x \\ y & \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \iff a = d,\ by = cx$$

i.e.

$$C_{GL_2(\mathbb{C})}\begin{pmatrix} x & \\ & y \end{pmatrix} = \begin{cases} T(\mathbb{C}) & x \neq y \\ GL_2(\mathbb{C}) & x = y \end{cases}$$

$$C_{GL_2(\mathbb{C})}\begin{pmatrix} & x \\ y & \end{pmatrix} = \left\{ \begin{pmatrix} a & tx \\ ty & a \end{pmatrix} \,\middle|\, a, t \in \mathbb{C},\ a^2 \neq txy \right\}$$

E.g. $G = GL_{2,\mathbb{R}}$. We get

$$\hat{G} = GL_2(\mathbb{C}) \qquad Z(\hat{G})^\Gamma = Z(\hat{G}) = \begin{pmatrix} t & \\ & t \end{pmatrix} \cong \mathbb{C}^\times$$

For $\phi = \chi_1 \oplus \chi_2$ corresponding to $(t_1, \varepsilon_1), (t_2, \varepsilon_2)$, $\qquad t_i \in \mathbb{C},\ \varepsilon \in \{\pm 1\}$

we write

$$\begin{aligned} {}^L\phi : W_\mathbb{R} &\longrightarrow GL_2(\mathbb{C}) \times \Gamma_\mathbb{R} \\ z &\longmapsto \left(\begin{pmatrix} |z|^{t_1} & \\ & |z|^{t_2} \end{pmatrix}, 1 \right) \\ j &\longmapsto \left(\begin{pmatrix} \varepsilon_1 & \\ & \varepsilon_2 \end{pmatrix}, \sigma \right) \end{aligned}$$

$$\Rightarrow \quad S_\phi = \begin{cases} \hat{G} & t_1 = t_2,\ \varepsilon_1 = \varepsilon_2, \\ \hat{T} & \text{otherwise}, \end{cases}$$

$$\Rightarrow \quad \bar{S}_\phi = \begin{cases} \hat{G}/\mathbb{C}^\times & t_1 = t_2,\ \varepsilon_1 = \varepsilon_2, \\ \hat{T}/\mathbb{C}^\times & \text{otherwise}, \end{cases}$$

$$\Rightarrow \quad \pi_0(\bar{S}_\phi) = \{*\}$$

For ϕ simple, we write $\qquad\qquad \mu-\nu>0$

$$
\begin{aligned}
{}^L\phi :\ W_{\mathbb{R}} &\longrightarrow GL_2(\mathbb{C})\times\Gamma_{\mathbb{R}}\\
z &\longmapsto \left(\begin{pmatrix} z^{\mu}\bar z^{\nu} & \\ & z^{\nu}\bar z^{\mu}\end{pmatrix},\ 1\right)\\
j &\longmapsto \left(\begin{pmatrix} & (-1)^{\mu-\nu}\\ 1 & \end{pmatrix},\ \sigma\right)
\end{aligned}
$$

$\Rightarrow S_\phi \cong \mathbb{C}^\times$, $\bar S_\phi \cong \mathbb{C}^\times/\mathbb{C}^\times = \{*\}$, $\pi_0(\bar S_\phi) = \{*\}$

To compute the $GL_{n,\mathbb{R}}$ case, one may want to compute

$$C_{GL_n(\mathbb{C})}\left(\mathrm{diag}\left(\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\dots,\begin{pmatrix} & 1\\ 1 & \end{pmatrix},\begin{pmatrix} & -1\\ 1 & \end{pmatrix},\dots,\begin{pmatrix} & -1\\ 1 & \end{pmatrix},1,\dots,1\right)\right)$$

Luckily, by observation one can conclude that $\pi_0(S_\phi)=\{*\}$, $\pi_0(\bar S_\phi)=\{*\}$ without really computing the centralizer.

If you really struggle to compute it, use

$$
\text{Centralizer of}
\left(\begin{array}{cccc|cccc|ccc}
1 &&&&&&&&&& \\
& -1 &&&&&&&&& \\
&& 1 &&&&&&&& \\
&&& -1 &&&&&&& \\ \hline
&&&& i &&&&&& \\
&&&&& -i &&&&& \\
&&&&&& i &&&& \\
&&&&&&& -i &&& \\ \hline
&&&&&&&& 1 && \\
&&&&&&&&& \ddots & \\
&&&&&&&&&& 1
\end{array}\right)
=
\left(\begin{array}{cccc|cccc|cccc}
* & & * & & & & & & * & * & * & * \\
& * & & * & & & & & & & & \\
* & & * & & & & & & * & * & * & * \\
& * & & * & & & & & & & & \\ \hline
& & & & * & & * & & & & & \\
& & & & & * & & * & & & & \\
& & & & * & & * & & & & & \\
& & & & & * & & * & & & & \\ \hline
* & & * & & & & & & * & * & * & * \\
* & & * & & & & & & * & * & * & * \\
* & & * & & & & & & * & * & * & * \\
* & & * & & & & & & * & * & * & *
\end{array}\right)
$$

and

$$\begin{pmatrix} & 1\\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & 1\\ 1 & -1\end{pmatrix}^{-1}\begin{pmatrix} 1 & \\ & -1\end{pmatrix}\begin{pmatrix} 1 & 1\\ 1 & -1\end{pmatrix} \qquad \begin{pmatrix} & -1\\ 1 & \end{pmatrix} = \begin{pmatrix} 1 & 1\\ -i & i\end{pmatrix}^{-1}\begin{pmatrix} i & \\ & -i\end{pmatrix}\begin{pmatrix} 1 & 1\\ -i & i\end{pmatrix}$$

E.g. $G = GL_{n,\mathbb{R}}$. We get

$\hat G = GL_n(\mathbb{C}) \qquad Z(\hat G)^\Gamma = Z(\hat G) = \begin{pmatrix} t & & \\ & \ddots & \\ & & t\end{pmatrix} \cong \mathbb{C}^\times$

$\Rightarrow \pi_0(S_\phi) = \{*\} \qquad \pi_0(\bar S_\phi) = \{*\}.$

Eg. $G = SO_{2,\mathbb{R}}$. We get

$$\widehat{G} = \mathbb{C}^\times \qquad Z(\widehat{G}) = \mathbb{C}^\times \qquad Z(\widehat{G})^\Gamma \cong \{z \in \mathbb{C}^\times \mid z^{-1} = z\} \cong \{\pm 1\}$$

For

$$\begin{aligned} {}^L\phi : W_\mathbb{R} &\longrightarrow \mathbb{C}^\times \rtimes \Gamma_\mathbb{R} \\ z &\longmapsto (|z|^t, 1) \\ j &\longmapsto (1, \sigma) \end{aligned}$$

we can compute

$$\begin{aligned} S_\phi &= \left\{ \omega \in \mathbb{C}^\times \;\middle|\; \begin{matrix} (|z|^t, 1)(\omega, 1) = (\omega, 1)(|z|^t, 1) \\ (1, \sigma)(\omega, 1) = (\omega, 1)(1, \sigma) \end{matrix} \right\} \\ &= \{\omega \in \mathbb{C}^\times \mid \omega^{-1} = \omega\} \\ &= \{\pm 1\} \end{aligned}$$

$$\Rightarrow \overline{S}_\phi = \{*\}, \quad \pi_0(\overline{S}_\phi) = \{*\}.$$

⚠ Here $\pi_0(S_\phi) \neq \{*\}$.

Eg. $G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{C}}$. We get

$$\widehat{G} = \mathbb{C}^\times \times \mathbb{C}^\times \qquad Z(\widehat{G}) = \widehat{G} \qquad Z(\widehat{G})^\Gamma \cong \{(z_1, z_2) \in \mathbb{C}^\times \times \mathbb{C}^\times \mid z_1 = z_2\} \cong \{(z, z) \in \mathbb{C}^\times \times \mathbb{C}^\times\} \cong \mathbb{C}^\times \leftarrow \text{diagonal.}$$

For

$$\begin{aligned} {}^L\phi : W_\mathbb{R} &\longrightarrow (\mathbb{C}^\times \times \mathbb{C}^\times) \rtimes \Gamma_\mathbb{R} \\ z &\longmapsto ((z^\mu \bar{z}^\nu, z^\nu \bar{z}^\mu), 1) \\ j &\longmapsto ((1, 1), \sigma) \end{aligned}$$

we get finally

$$S_\phi = Z(\widehat{G})^\Gamma \cong \mathbb{C}^\times \qquad \overline{S}_\phi = \{*\} \qquad \pi_0(\overline{S}_\phi) = \{*\}. \qquad \square$$