Eine Woche, ein Beispiel

6.19. idempotent algebras.

This document want to discuss some basic contents of the course "https://people.mpim-bonn.mpg.de/scholze/Complex.pdf", Lecture 5.
For me I've never noticed about this special structure before. Hope that you enjoy this small magic.
This can be a perfect series of exam questions for the Algebra III in USTC. (better if they have learned computations on tensor products)

Q: Find all (reduced) $\mathbb{Z}$-algebra A s.t.

$$A \otimes_{\mathbb{Z}} A \cong A$$

as a $\mathbb{Z}$-alg iso.

A crash recap on [Vakil 9.2] Skip if you know fiber product of schemes!

Ex. $C \in CRing$, $A, B \in C\text{-}Alg \Rightarrow A \otimes_C B$ is C-Alg. and

$$\begin{tikzcd} A\otimes_C B & A \arrow[l] \\ B \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd}$$

is a pushout.

Let $\phi: B \to A$ be a ring homomorphism. $I \lhd B$ S multiplicative set

9.2.A. (Adding an extra variable) $A \otimes_B B[t] \cong A[t]$

9.2.B (Quotient) $A \otimes_B B/I \cong A/\phi(I)$

9.2.F (Localization) $A \otimes_B S^{-1}B \cong [\phi(S)]^{-1}A$

Ex. Compute $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$

Compute fibers of $\operatorname{Spec} \mathbb{Z}[i] \longrightarrow \operatorname{Spec} \mathbb{Z}$ $\quad$ $\operatorname{Spec} \mathbb{C}[x] \longrightarrow \operatorname{Spec} \mathbb{C}[y]$

$x^2 \longleftarrow\!\shortmid y$

Definition and some cases

Def. Let $R \in \mathrm{Ring}$. $A \in R\text{-}\mathrm{Alg}$ is called **idempotent R-algebra** if

$$A \otimes_R A \cong A \qquad \text{induced by } A \cong R \otimes_R A \longrightarrow A \otimes_R A$$

as an R-alg iso.

Ex. Verify that $\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Q}$ are idempotent $\mathbb{Z}$-algebras.
Is $\mathbb{F}_{p^2}$ idempotent? Is $\mathbb{Z}/p^2\mathbb{Z}$ idempotent? Is $\mathbb{Z}_p$ idempotent?

A new topology on Spec A

Def. (Constructable topology)

$X \subseteq \mathrm{Spec}\, A$ is called constructable closed if

$$\exists\, f: \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A \qquad \mathrm{Im} f = X$$

Ex. Find all constructable closed subset of $\mathrm{Spec}\,\mathbb{Z}$

Ex. Find all constructable closed subset of $\mathrm{Spec}\,\mathbb{C}[X]$

Ex. $\{\text{Zariski closed / open subset}\} \subseteq \{\text{constructable closed set}\}$

Central result I want to prove

Fact. [Condensed, Lec 5 Ex 2] Suppose $R \in \mathrm{CRing}$ is Noetherian. Then

$$\begin{array}{ccc} \{\text{(reduced) idem } R\text{-algs}\} & \xleftrightarrow{1:1} & \{\text{constructable closed subset of } \operatorname{Spec} R\} \\ A & \longmapsto & \operatorname{Im}(\operatorname{Spec} A \longrightarrow \operatorname{Spec} R) \end{array}$$

Ex. Verify this for $\operatorname{Spec} \mathbb{Z}$.

They may be not idem $\mathbb{C}[x]$-alg?

Ex. Verify that $\mathbb{C}[x]/(x-a)$, $\mathbb{C}(x)$, $\mathbb{C}[x, \frac{1}{x}]$, $\mathbb{C}[[x]]$, $\mathcal{O}(D)$, $\mathcal{O}(\bar{D})$ are idem $\mathbb{C}[x]$-algs.

What constructable closed subset do they correspond?

$\mathbb{C}((x))$ is not $\mathbb{C}[x]$-idem algs

$$\mathcal{O}(D) = \left\{ \sum_{i=0}^{+\infty} a_i T^i \;\middle|\; a_i r^i \to 0 \quad \forall r < 1 \right\} \subseteq \mathbb{C}[[x]]$$

$$\mathcal{O}(\bar{D}) = \bigcap_{r>1} \left\{ \sum_{i=0}^{+\infty} a_i T^i \;\middle|\; a_i r^i \to 0 \right\} \subseteq \mathbb{C}[[x]]$$

Lem. A, A' are idem R-algs. Then $\# \mathrm{Mor}_{R\text{-alg}}(A, A') \leq 1$.

Proof.

$$\begin{array}{lcl} & & R \longrightarrow A \xrightarrow{f} A' \cong R \otimes_R A' \longrightarrow A \otimes_R A' \\ -\otimes_R A': & & R \otimes_R A' \longrightarrow A \otimes_R A' \xrightarrow{f \otimes_R A'} A' \otimes_R A' \cong R \otimes_R A' \otimes_R A' \longrightarrow A \otimes_R A' \otimes_R A' \\ & & \quad \| \qquad\qquad\qquad\qquad\qquad\qquad \| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ & & \quad A' \qquad\qquad\qquad\qquad\qquad\quad\;\; A' \qquad\qquad\qquad\qquad\qquad\qquad\qquad A \otimes_R A' \end{array}$$

$\Rightarrow$ $A \otimes_R A' \cong A'$ doesn't depend on f, so f given by

$$A \longrightarrow A \otimes_R A' \cong A'$$

is unique.

Cor. $\{\text{idem } R\text{-algs}\}$ is a poset.

Fact. This order is compatible with constructable topology

(Only consider reduced algs. R is Noetherian.)

Ex.

$$\begin{array}{ccc} \{\text{(reduced) idem } R\text{-algs}\} & \xleftrightarrow{1:1} & \{\text{constructable closed subset of } \operatorname{Spec} R\} \\ A & \longleftrightarrow & Z \\ A \otimes_R A' & \longleftrightarrow & Z \cap Z' \\ \operatorname{Ker}[A \oplus A' \longrightarrow A \otimes_R A'] & \longleftrightarrow & Z \cup Z' \\ \varinjlim_i A_i & \longleftrightarrow & \bigcap_i Z_i \end{array}$$

e.g. $Z \subset Z'$ iff $A \otimes_R A' \cong A$.