Eine Woche, ein Beispiel
3.13 dual variety

Dual variety is useful is the research of subvarieties of P^n (and symplectic geometry). We emphasize the embedding here.

Main reference:
https://arxiv.org/abs/math/0112028v1
other ref:
Discriminants, Resultants, and Multidimensional Determinants by Israel M. GelfandMikhail M. KapranovAndrei V. Zelevinsky.
https://en.wikipedia.org/wiki/Dual_curve

A vivid animation: https://www.youtube.com/watch?v=HTXpf4jDgYE
Some pictures: https://www.ima.umn.edu/materials/2006-2007/W9.18-22.06/2203/Piene_190906.pdf

Goal.

1. Definition
2. Basic properties
   - Reflexivity theorem
   - dimension and defect
   - d, g, b, f, δ, k
3. Basic examples
   - Smooth proj plane curve of deg 2,3,4.
   - Fermat curve
   - Veronese curve/variety
   - K3 surface
   - Other examples

Let $K = \bar{K}$ be a field, V a v.s. of dim $n+1$.

1. Definition

Def (Dual variety)

Let $X \subset \mathbb{P}V$: irr proj variety

X_{sm} : smooth locus

$I_X^\circ := \{(z, H) \mid z \in X_{sm},\ H \in \mathbb{P}V^*,\ T_zX \subset H\}$

$I_X := \overline{I_X^\circ}$.

Then $X^* := pr_2(I_X)$ is called the dual variety of X.

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd}$$

$\mathbb{P}V^* = \mathbb{P}(V^*)$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\operatorname{def}X}{X^*} \end{tikzcd}$$

$\dim V = n+1$

$\dim X = m$

$\operatorname{def} X = \operatorname{codim}_{\mathbb{P}V^*} X^* - 1$

Relation with symplectic geometry

Def (Lagrangian construction)

Let M be a sm proj irr variety, $Y \subset M$ be any irr subvariety.
We define

$$\mathrm{Lag}(Y) := \overline{N^*_{Y_{sm}} M} \qquad \text{(closure in } T^*M\text{)}$$

Def. Any set $S \subset T^*M$ is called conical if S is closed under scalar multiplication.

Rmk. [Thm 1.9] $\mathrm{Lag}(Y)$ is a conical Lagrangian subvariety,
and every conical Lagrangian subvariety S is of this form, i.e.

$$S = \mathrm{Lag}(\pi(S)) \qquad \pi : T^*M \to M$$

Rmk. $\mathrm{Lag}(Y)$ is an analog of I_X. see the following picture:

$$\begin{tikzcd} & \mathbb{P}V \times \mathbb{P}V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ \mathbb{P}V & & \mathbb{P}V^* \end{tikzcd} \qquad \begin{tikzcd} & V \times V^* \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ V & & V^* \end{tikzcd}$$

$$\begin{tikzcd} & \overset{n-1}{I_X} \arrow[dl, "n-m-1"'] \arrow[dr, "\text{typically } 0"] & \\ \overset{m}{X} & & \overset{n-1-\operatorname{def} X}{X^*} \end{tikzcd} \qquad \begin{tikzcd} & \overset{n+1}{\mathrm{Lag}(Y)} \arrow[dl, "n-m"'] \arrow[dr, "\text{typically } 1"] & \\ \overset{m+1}{Y} & & \overset{n-\operatorname{def} X}{Y^*} \end{tikzcd}$$

$$Y := \mathrm{Cone}(X) \Rightarrow Y^* = \mathrm{Cone}(X^*)$$

2. Basic properties

2.1. Thm (Reflexivity thm) $X^{**} = X$

Sketch of proof.

$$X \xrightarrow{\cong} X^{**}$$

$$\Leftrightarrow \quad [(z,H) \in I_X^\circ \Leftrightarrow (H,z) \in I_{X^*}^\circ]$$

$$\Leftrightarrow \quad I_X \cong I_{X^*} \quad \text{under the iso } \mathbb{P}V \times \mathbb{P}V^* \xrightarrow{\sim} \mathbb{P}V^* \times \mathbb{P}V^{**}$$

$$\Leftrightarrow \quad \mathrm{Lag}(Y) \cong \mathrm{Lag}(Y^*) \quad \text{where } Y := \mathrm{Cone}(X) \quad Y^* := \mathrm{Cone}(X^*)$$

under the iso $T^*V \cong V \times V^* \cong V^* \times V \cong T^*V^*$

Under this iso, $\mathrm{Lag}(Y)$ is a conical Lagrangian subvariety of T^*V^*, so

$$\mathrm{Lag}(Y) \cong \mathrm{Lag}(pr_2(\mathrm{Lag}(Y))) \cong \mathrm{Lag}(Y^*)$$

2.2. Dimension and defect

Def (Defect) $\Big\|$ $\operatorname{def} X := \operatorname{codim}_{\mathbb{P}V^*} X^* - 1$. $\Rightarrow \dim X^* = n-1-\operatorname{def} X$

Typically, $\operatorname{def} X = 0$.

Def (Ruled space) X is ruled in proj subspaces of dim r if

$\forall x \in X \quad \exists L$: proj subspace of dim r s.t $x \in L \subseteq X$.

Rmk. Sufficient to check $x \in X_{sm}$.

E.g. $X = V(xw - yz)$ is ruled in proj subspaces of dim 1,

$X = V(x^3+y^3+z^3+w^3)$ is not ruled. (Strictly speaking, it's ruled in dim 0)

Prop. [Thm 1.12]

$$\operatorname{def} X = r \iff X \text{ is (maximal) ruled in proj subspaces of dim } r.$$

[Proof. Since $X = X^{**}$, the statement is equivalent to

$\dim X = n-r-1 \Rightarrow X^*$ is ruled in proj subspaces of dim r.

For any $(z, H) \in I_X^\circ$, $pr_1^{-1}(z) \cap I_X^\circ \simeq \{z\} \times \mathbb{P}^r$ is maped by pr_2 to a proj subspace L of $\mathbb{P}V^*$, s.t. $\dim L = r$ & $H \in L \subseteq X^*$.]

Rmk. Now we know that

X is smooth $\overset{\text{Thm 1.10}}{\Rightarrow}$ I_X is smooth $\Big\}$ $\iff pr_2$ is a resolution

X is not ruled $\iff \operatorname{def} X = 0 \iff X^*$ hypersurface $\iff pr_2$ is birational $\Big\}$

E.g. When $X = V(xw - yz)$, $\dim X^* = 3-1-1 = 1$;

when $X = V(x^3+y^3+z^3+w^3)$, $\dim X^* = 3-1-0 = 2$, $pr_2 : I_X \longrightarrow X^*$ is birational.

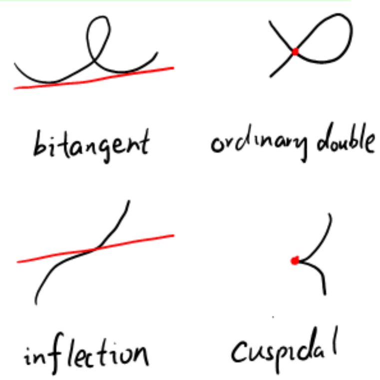
Def. When X is not ruled, Δ_X is the polynomial defining X^*, which is unique up to scaling. Δ_X is called the discriminant of X.

We now assume $k = \mathbb{C}$.

By doing so, some potential problems for the genus formula and other formula will be solved. Moreover, we don't need to do case by case analysis in those specific examples.

2.3. $d, g, b, f, \delta, \kappa$

Here, we need to assume $C \subset \mathbb{P}^2$ is a generic curve, i.e., both C and C^* have only double points and cusps as their singularities.



Def. $\Big\|$ d: degrees

g: geo genus

b: #bitangents $\qquad \delta$: #double points

f: #flexs $\qquad \kappa$: #cusps

Formulas:

$$\begin{cases} d^* \\ g^* \\ b^* \quad \delta^* \\ f^* \quad \kappa^* \end{cases} = \begin{cases} d(d-1) - 2\delta - 3\kappa & \text{(called Plücker-Clebsch formula)} \\ g = \frac{1}{2}(d-1)(d-2) - \delta - \kappa & \text{by genus formula} \\ \delta \quad b \\ \kappa \quad b \end{cases}$$

Rmk. If d, δ, k is known, then b, f can be computed.

E.p. when $\delta, k = 0$, $\begin{cases} b = \frac{1}{2}d^4 - d^3 - \frac{9}{2}d^2 + 9d \\ f = 3d^2 - 6d \end{cases}$

d	2	3	4	5	6	7	8	9
b	0	0	28	120	324	700	1320	2268
f	0	9	24	45	72	105	144	189

3. Basic examples

3.1. Smooth proj plane curve [Eg 1.19 - 1.22]

Degree 2

Let $C = V(\sum_{i,j=1}^{3} a_{ij} x_i x_j)$ be a sm conic, where

$A = (a_{ij})_{i,j=1}^{3}$ is a non-deg sym matrix, then

$C^* = V(\sum_{i,j=1}^{3} b_{ij} p_i p_j)$ is also a sm conic, where $B = (b_{ij})_{i,j=1}^{3} := A^{-1}$

e.g. $A = \begin{pmatrix} a_1 & & \\ & a_2 & \\ & & a_3 \end{pmatrix}$. The dual curve of $C: a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0$ is the curve $C^*: \frac{p_1^2}{a_1} + \frac{p_2^2}{a_2} + \frac{p_3^2}{a_3} = 0$.

Degree 3

Let $C = V(f) \subseteq \mathbb{P}^2$ be a sm cubic, then

$d = 3$		$d^* = 6$	
$g = 1$		$g^* = 1$	
$b = 0$	$\delta = 0$	$b^* = 0$	$\delta^* = 0$
$f = 9$	$k = 0$	$f^* = 0$	$f^* = 9$

and Δ_C is computed by the Schläffli's formula:

$$V(p, x) = \begin{vmatrix} 0 & p_1 \quad p_2 \quad p_3 \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \dfrac{\partial^2 f}{\partial x_i \partial x_j} \end{vmatrix} \qquad \Delta_C(p) = \begin{vmatrix} 0 & p_1 \quad p_2 \quad p_3 \\ \begin{matrix} p_1 \\ p_2 \\ p_3 \end{matrix} & \dfrac{\partial^2 V}{\partial x_i \partial x_j} \end{vmatrix}$$

e.g. $C: a_1x_1^3 + a_2x_2^3 + a_3x_3^3 = 0$, then

$$V(p,x) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 6a_1x_1 & & \\ p_2 & & 6a_2x_2 & \\ p_3 & & & 6a_3x_3 \end{vmatrix} = -36 \sum_{cyc} a_2 a_3 x_2 x_3 p_1^2$$

$$\Delta_C(p) = \begin{vmatrix} 0 & p_1 & p_2 & p_3 \\ p_1 & 0 & -36a_1a_2p_3^2 & -36a_1a_3p_2^2 \\ p_2 & -36a_1a_2p_3^2 & 0 & -36a_2a_3p_1^2 \\ p_3 & -36a_1a_3p_2^2 & -36a_2a_3p_1^2 & 0 \end{vmatrix} = 6^4 \sum_{cyc} (a_2^2a_3^2p_1^6 - 2a_1^2a_2a_3p_2^3p_3^3)$$

e.p. when $a_1 = a_2 = a_3 = 1$, $\Delta_C = 6^4 \sum_{cyc} (p_1^6 - p_2^3 p_3^3)$

when $a_1 = a_2 = 1$, $a_3 = -a^{-3}$, $\Delta_C = 6^4 \left(\begin{array}{l} p_3^6 + a^{-6} p_1^6 + a^{-6} p_2^6 \\ -2\, a^{-6} p_1^3 p_2^3 + 2a^{-3} p_1^3 p_3^3 + 2a^{-3} p_2^3 p_3^3 \end{array} \right)$

it corresponds to curve defined by

$$p_1^{\frac{3}{2}} + p_2^{\frac{3}{2}} = a^{\frac{3}{2}} p_3^{\frac{3}{2}}$$

This is not rigorously defined equation, and has no difference with

$$p_1^{\frac{3}{2}} + p_2^{\frac{3}{2}} = -a^{\frac{3}{2}} p_3^{\frac{3}{2}}$$

Degree 4.

Let $C = V(f) \subseteq \mathbb{P}^2$ be a generic sm quartic curve, then

$b = 28$ is explained in [Eg 1.22]

$d=4$		$d^* = 12$		
$g=3$		3		
$b=28$	$\delta = 0$	0	28	
$f=24$	$\kappa = 0$	0	24	

e.g. Let $C = V(x_1 x_2^3 + x_2 x_3^3 + x_3 x_1^3)$ be the Fermat quartic curve, then the result comes from the article:

Computation of the Dual of a Plane Projective Curve

$$\Delta_C = \sum_{cyc} \left(-27 p_1^{10} p_2^2 + 4 p_1^3 p_2^9 - 42 p_1^5 p_2^6 p_3 + 282 p_1^7 p_2^3 p_3^2 \right) - 651\, p_1^4 p_2^4 p_3^4$$

3.2. Fermat curve [Eg 1.15]

The dual curve of

$$C: \quad x_1^p + x_2^p = x_3^p \qquad p > 1,\ p \in \mathbb{Q}$$

is

$$C^*: \quad p_1^q + p_2^q = p_3^q \qquad \frac{1}{p} + \frac{1}{q} = 1$$

This is not rigorously defined, since it is computed by not-rigorous formula

$$\begin{cases} p_1(t) = \dfrac{-\dot{x}_2}{\dot{x}_1 x_2 - x_1 \dot{x}_2} \\[2ex] p_2(t) = \dfrac{\dot{x}_1}{\dot{x}_1 x_2 - x_1 \dot{x}_2} \end{cases} \qquad x_1 = x_1(t) \quad x_2 = x_2(t)$$

3.3 Veronese curve/variety [Eg 2.1]

Let $C \subset \mathbb{P}^d$ be the curve given by the image of Veronese embedding

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^d \qquad [x:y] \longmapsto [x^d : x^{d-1}y : \dots : y^d]$$

then $C^* \subset \mathbb{P}^d$ is a hypersurface cut by

$$\Delta_C = \text{discriminant of } f(x,y) := \sum_{i=0}^{d} p_i x^{d-i} y^i$$

See wiki for the definition of discriminant: https://en.wikipedia.org/wiki/Discriminant
In general, see here: https://mathoverflow.net/questions/304957/definition-of-a-discriminant-in-three-variables

e.g. $d=2 \quad \Delta_C = p_1^2 - 4p_0p_2$

$d=3 \quad \Delta_C = p_1^2p_2^2 - 4p_0p_2^3 - 4p_1^3p_3 - 27p_0^2p_3^2 + 18\,p_0p_1p_2p_3$

In general, when $C = Im\left(\mathbb{P}^m \longrightarrow \mathbb{P}^{\binom{d+1}{m}-1}\right)$, then

$C^* \subset \mathbb{P}^d$ is a hypersurface cut by

$$\Delta_C = \text{discriminant of } f(x) := \sum_I p_I x^I .$$

3.4. K3 surface

See http://www-personal.umich.edu/~jakubw/masterthesis.pdf. Until now, I still don't know the equation of the dual variety of the Fermat cubic.

3.5. Other examples.

I'm not so interested now, but maybe I'll add it here when I need it.
[Eg 2.1]Grassmannians
[Eg 2.2]Spinor varieties
[Eg 2.3]Severi varieties
[Eg 2.4]Adjoint varieties