Eine Woche, ein Beispiel

8.21 equivariant cohomology of $\mathbb{P}^1$

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
Other references will be add soon.

1. notations and warnings
2. result
3. computation of completion in practice
4. pt & $\mathbb{P}^1$
5 Euler class

1. notations and warnings

In this document,

$$GL_2 = GL_2(\mathbb{C}) \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset GL_2 \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2 \text{ or } SL_2$$

$$SL_2 = SL_2(\mathbb{C}) \qquad \mathbb{C}^\times = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset SL_2 \qquad \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$$

$$K_0^G(X) := K_0(Coh^G(X)) \qquad H_G^*(X;\mathbb{Q}) := H^*(EG \times^G X;\mathbb{Q})$$

$$R(G) := K_0^G(pt) = Rep(G) \qquad S(G) := H_G^*(pt;\mathbb{Q}) = H^*(BG;\mathbb{Q})$$

$$K_0^G(X)_I^\wedge := \varprojlim_n K_0^G(X)/I^n \qquad HP_G^0(X;\mathbb{Q}) := \prod_{n=0}^{+\infty} H_G^n(X;\mathbb{Q}) = H_G^*(X;\mathbb{Q})_{I'}^\wedge$$

To avoid confusion, we don't consider any convolution structure in this document.

we don't consider $G \times \mathbb{C}^\times$-action either

($\mathbb{C}^\times$ is already occupied as a maximal torus of SL_2)

2. result

This time we are not so ambitious. For example, we don't fill in

$$K_0^{B}(\mathcal{B}\times\mathcal{B}) \cong K_0^{G}(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) \cong R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$$

just because the result is too long.

We don't want to use these symbols(like x,y,z) in later documents either. If you want to fix a notation, please use the notations in https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.10.23_notation_K%5EG(St).pdf

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Z}[y+y^{-1}]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2]/((z_1-z_2)(z_1-z_2^{-1}))$
	B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1},z]/(z-y)(z-y^{-1})$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$
$G=GL_2$	GL_2	$\mathbb{Z}[y_1+y_2,y_1y_2,\frac{1}{y_1y_2}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},z_1']/((z_1'-z_1)(z_1'-z_2))$
	B	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1}]$	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1},z_1]/((z_1-y_1)(z_1-y_2))$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1',z_2']/((z_1'-1)^2,(z_2'-1)^2)$
$G=SL_n$ or GL_n	G	$R(G)$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w'\in W}R(G)[\overline{\Omega}_{w'}]^G$
	B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w\in W}R(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W}R(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Z}$	$\bigoplus_{w\in W}\mathbb{Z}\cdot[\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W}\mathbb{Z}[\overline{\Omega}_{w,w'}]$

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Q}[b^2]$	$\mathbb{Q}[e]$	$\mathbb{Q}[e_1,e_2]/(e_2^2-e_1^2)$
	B	$\mathbb{Q}[b]$	$\mathbb{Q}[b,e]/(e^2-b^2)$	
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1,e_2]/(e_1^2,e_2^2)$
$G=GL_2$	GL_2	$\mathbb{Q}[b_1+b_2,b_1b_2]$	$\mathbb{Q}[e_1,e_2]$	$\mathbb{Q}[e_1,e_2,e_1']/((e_1'-e_1)(e_1'-e_2))$
	B	$\mathbb{Q}[b_1,b_2]$	$\mathbb{Q}[b_1,b_2,e_1]/((e_1-b_1)(e_1-b_2))$	$e_2'=e_1+e_2-e_1'$
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1',e_2']/(e_1'^2,e_2'^2)$
$G=SL_n$ or GL_n	G	$S(G)$	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w'\in W}S(G)[\overline{\Omega}_{w'}]^G$
	B	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w\in W}S(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W}S(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Q}$	$\bigoplus_{w\in W}\mathbb{Q}[\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W}\mathbb{Q}[\overline{\Omega}_{w,w'}]$

3. computation of completion in practice

Thm (cpl of Noetherian ring by power series)

R: Noetherian $I := (a_1, \dots, a_n) \triangleleft R$, then

$$\hat{R}_I := \varprojlim_n R/I^n$$
$$\cong R[[x_1, \dots, x_n]] / (x_1 - a_1, \dots, x_n - a_n)$$
$$\cong R[[a_1, \dots, a_n]]$$

Ex. $\widehat{\mathbb{Z}[x]}_{(x)} \cong \mathbb{Z}[[x]]$

$$\hat{\mathbb{Z}}_{(p)} \cong \mathbb{Z}[[x]]/(x-p) \xrightarrow{\sim} \mathbb{Z}_p$$
$$x \longmapsto p$$

$$\hat{\mathbb{Z}}_{(p^2)} \cong \mathbb{Z}_p$$

$$\hat{\mathbb{Z}}_{(n)} \cong \prod_{\substack{p \mid n \\ \text{prime}}} \mathbb{Z}_p$$

4. pt & $\mathbb{P}^1$

$$\begin{tikzcd}[column sep=small]
 & ④\,K_0^{GL_2}(pt) \arrow[dl] \arrow[dd] \arrow[rr] & & HP^0(GL(2,\infty);\mathbb{Q}) \arrow[dl] \arrow[dd] \\
②\,K_0^{SL_2}(pt) \arrow[r] \arrow[dd] \arrow[rr] & ⑤\,K_0^{SL_2}(\mathbb{P}^1) & HP^0(\mathcal{M}^{max}_{\infty\times 2}/SL_2;\mathbb{Q}) \arrow[dd] & \\
 & ③\,K_0^T(pt) \arrow[dl] \arrow[d] \arrow[rr] & & HP^0(\mathbb{C}P^\infty\times\mathbb{C}P^\infty;\mathbb{Q}) \arrow[dl] \\
①\,K_0^{\mathbb{C}^\times}(pt) \arrow[r] \arrow[rr] & ⑥\,K_0^{SL_2}(\mathbb{P}^1) & HP^0(\mathbb{C}P^\infty;\mathbb{Q}) &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z}[y_1+y_2,\, y_1y_2,\, \frac{1}{y_1y_2}] \arrow[dl] \arrow[dd] \arrow[rr] & & \mathbb{Q}[[b_1+b_2,\, b_1b_2]] \arrow[dl] \arrow[dd] \\
\mathbb{Z}[y+y^{-1}] \arrow[dd] \arrow[rr] & & \mathbb{Q}[[b^2]] \arrow[dd] & \\
 & \mathbb{Z}[y_1^{\pm1},\, y_2^{\pm1}] \arrow[dl] \arrow[rr] & & \mathbb{Q}[[b_1,\, b_2]] \arrow[dl] \\
\mathbb{Z}[y^{\pm1}] \arrow[rr] & & \mathbb{Q}[[b]] &
\end{tikzcd}$$

$$\begin{tikzcd}
y & y_1 \arrow[l, mapsto] \\
y^{-1} & y_2 \arrow[l, mapsto]
\end{tikzcd}
\qquad y_i \longmapsto e^{b_i}$$

$$y \longmapsto e^b \qquad
\begin{tikzcd}
b & b_1 \arrow[l, mapsto] \\
-b & b_2 \arrow[l, mapsto]
\end{tikzcd}$$

$\leftrightarrow$: Weyl group action

Later, $\mathbb{C}_i := \mathbb{C}_i^G$ is a temporate notation.

ch^* is iso after <u>tensored over $\mathbb{Q}$</u>.

$$(ch^*)^{-1} : HP^0(BG;\mathbb{Q}) \xrightarrow{\sim} K_0(BG)\otimes_{\mathbb{Z}}\mathbb{Q}$$
$$HP^0_G(X;\mathbb{Q}) \xrightarrow{\sim} K_0^G(X)\otimes_{\mathbb{Z}}\mathbb{Q}$$

When I write the inverse map $(ch^*)^{-1}$, remember that the image usually has coefficient in $\mathbb{Q}$.

completion = cpl, Atiyah-Segal = AS map

①

$$\begin{array}{ccccccccc}
K_0^{\mathbb{C}^\times}(pt) & \xrightarrow{cpl} & K_0^{\mathbb{C}^\times}(pt)^\wedge_I & \xrightarrow{AS\ map} & K_0(B\mathbb{C}^\times) & \xrightarrow{ch^*} & HP^\circ(B\mathbb{C}^\times;\mathbb{Q}) & \overset{cpl}{\supset} & H^*(B\mathbb{C}^\times;\mathbb{Q}) \\
\mathbb{Z}[y^{\pm1}] & \longrightarrow & \mathbb{Z}[[y-1]] & \longrightarrow & \mathbb{Z}[[c_1^{\mathbb{C}^\times}]] & \longrightarrow & \mathbb{Q}[[b]] & \supset & \mathbb{Q}[b] \\
 & & & & c_1^{\mathbb{C}^\times} & \longmapsto & e^b-1 & & \\
 & & & \mathbb{Q}[[c_1^{\mathbb{C}^\times}]] \ni & \log(1+c_1^{\mathbb{C}^\times}) & \longleftarrow\!\shortmid & b & &
\end{array}$$

②

$$\begin{array}{ccccccccc}
K_0^{SL_2}(pt) & \xrightarrow{cpl} & K_0^{SL_2}(pt)^\wedge_I & \xrightarrow{AS\ map} & K_0(BSL_2) & \xrightarrow{ch^*} & HP^\circ(BSL_2;\mathbb{Q}) & \overset{cpl}{\supset} & H^*(BSL_2;\mathbb{Q}) \\
\mathbb{Z}[y+y^{-1}] & \longrightarrow & \mathbb{Z}[[y+y^{-1}-2]] & \longrightarrow & \mathbb{Z}[[c_1^{SL_2}]] & \longrightarrow & \mathbb{Q}[[b^2]] & \supset & \mathbb{Q}[b^2] \\
 & & & & c_1^{SL_2} & \longmapsto & e^b+e^{-b}-1 & = & 4\sinh^2\frac{b}{2} \\
 & & & & & & & = & 2\cosh b-2
\end{array}$$

$$\begin{aligned}
4\left(\operatorname{arcsinh}\frac{\sqrt{c_1}}{2}\right)^2 &\longleftarrow\!\shortmid \quad b^2 \\
=4\left(\ln\left(\frac{\sqrt{c_1}}{2}+\sqrt{\frac{c_1}{4}+1}\right)\right)^2 & \\
=\left(\ln\left(1+\frac{c_1}{2}+\sqrt{\frac{c_1^2}{4}+c_1}\right)\right)^2 &
\end{aligned}$$

③

$$\begin{array}{ccccccccc}
K_0^{T}(pt) & \xrightarrow{cpl} & K_0^{T}(pt)^\wedge_I & \xrightarrow{AS\ map} & K_0(BT) & \xrightarrow{ch^*} & HP^\circ(BT;\mathbb{Q}) & \overset{cpl}{\supset} & H^*(BT;\mathbb{Q}) \\
\mathbb{Z}[y_1^{\pm1},y_2^{\pm1}] & \longrightarrow & \mathbb{Z}[[y_1-1,y_2-1]] & \longrightarrow & \mathbb{Z}[[c_1^T,c_2^T]] & \longrightarrow & \mathbb{Q}[[b_1,b_2]] & \supset & \mathbb{Q}[b_1,b_2] \\
 & & & & c_i^{\mathbb{C}^\times} & \longmapsto & e^{b_i}-1 & & \\
 & & & & \log(1+c_i^{\mathbb{C}^\times}) & \longleftarrow\!\shortmid & b_i & &
\end{array}$$

④

$$\begin{array}{ccccccccc}
K_0^{GL_2}(pt) & \xrightarrow{cpl} & K_0^{GL_2}(pt)^\wedge_I & \xrightarrow{AS\ map} & K_0(BGL_2) & \xrightarrow{ch^*} & HP^\circ(BGL_2;\mathbb{Q}) & \overset{cpl}{\supset} & H^*(BGL_2;\mathbb{Q}) \\
\mathbb{Z}[y_1+y_2,y_1y_2,\frac{1}{y_1y_2}] & \longrightarrow & \mathbb{Z}[[y_1+y_2-2,y_1y_2-1]] & \longrightarrow & \mathbb{Z}[[c_1^{GL_2},c_2^{GL_2}]] & \longrightarrow & \mathbb{Q}[[b_1+b_2,b_1b_2]] & \supset & \mathbb{Q}[b_1+b_2,b_1b_2] \\
 & & & & c_1^{GL_2} & \longmapsto & e^{b_1}+e^{b_2}-1 & & \\
 & & & & c_2^{GL_2} & \longmapsto & e^{b_1+b_2}-1 & & \\
 & & & & \log(1+c_2^{GL_2}) & \longleftarrow\!\shortmid & b_1+b_2 & & \\
 & & & & \log(1+y_1-1)\log(1+y_2-1) & \longleftarrow\!\shortmid & b_1b_2 & &
\end{array}$$

$$\begin{aligned}
&\log(1+y_1-1)\log(1+y_2-1) \\
&=\sum_{k=2}^{+\infty}\sum_{\substack{n+m=k\\ n,m\geq1}}\frac{(-1)^k}{nm}(y_1-1)^n(y_2-1)^m \\
&=\cdots
\end{aligned}$$

To facilitate the computation, use the notation

$$\begin{aligned}
c_3^{GL_2} &= (y_1-1)(y_2-1) \\
&= (y_1y_2-1)-(y_1+y_2-2) \\
&= c_2^{GL_2}-c_1^{GL_2}
\end{aligned}$$

5

$$K_0^{SL_2}(\mathbb{P}^1) \xrightarrow{\text{cpl}} K_0^{SL_2}(\mathbb{P}^1)^{\wedge}_{I} \xrightarrow{AS} K_0(ESL_2 \times^{SL_2} \mathbb{P}^1) \xrightarrow{ch^*} HP^{\circ}_{SL_2}(\mathbb{P}^1;\mathbb{Q}) \overset{\text{cpl}}{\supset} H^{*}_{SL_2}(\mathbb{P}^1;\mathbb{Q})$$

$$\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[[z-1]] \longrightarrow \mathbb{Z}[[c_1]] \longrightarrow \mathbb{Q}[[\mathfrak{e}]] \supset \mathbb{Q}[\mathfrak{e}]$$

$$c_1 \longmapsto e^{\mathfrak{e}} - 1$$

$$\log(1+c_1) \longleftarrow\!\shortmid\ \mathfrak{e}$$

6

$$K_0^{\mathbb{C}^\times}(\mathbb{P}^1) \xrightarrow{\text{cpl}} K_0^{\mathbb{C}^\times}(\mathbb{P}^1)^{\wedge}_{I} \xrightarrow{AS} K_0(E\mathbb{C}^\times \times^{\mathbb{C}^\times} \mathbb{P}^1) \xrightarrow{ch^*} HP^{\circ}_{\mathbb{C}^\times}(\mathbb{P}^1;\mathbb{Q}) \overset{\text{cpl}}{\supset} H^{*}_{\mathbb{C}^\times}(\mathbb{P}^1;\mathbb{Q})$$

$$\mathbb{Z}[y^{\pm 1}, z]/((z-y)(z-y^{-1})) \longrightarrow \mathbb{Z}[[y-1, z-1]]/\ldots \longrightarrow \mathbb{Z}[[c_1, c_2]]/((c_1-c_2)(c_1c_2+c_1+c_2)) \longrightarrow \mathbb{Q}[[b,\mathfrak{e}]]/(\mathfrak{e}^2-b^2) \supset \mathbb{Q}[b,\mathfrak{e}]/(\mathfrak{e}^2-b^2)$$

$$c_1 \longmapsto e^{b} - 1$$

$$c_2 \longmapsto e^{\mathfrak{e}} - 1$$

$$\log(1+c_1) \longleftarrow\!\shortmid\ b$$

$$\log(1+c_2) \longleftarrow\!\shortmid\ \mathfrak{e}$$