Un exemple par jour

4.3. the (primary) Hopf surface $X := \mathbb{C}^2 - \{0\} / \mathbb{Z}\gamma$

$$\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^n, \beta z_2) \quad \text{where } \begin{cases} \alpha, \beta \in \mathbb{C} \quad n \in \mathbb{N}^+ \\ 0 < |\alpha| \leq |\beta| < 1 \\ \lambda = 0 \text{ or } \alpha = \beta^n \end{cases}$$

Today. $\alpha = \frac{1}{4}, \beta = \frac{1}{2}, \lambda = 3, n = 2$ applies well for $\lambda \neq 0$

$$\gamma(z_1, z_2) = (\tfrac{1}{4} z_1 + 3 z_2^2, \tfrac{1}{2} z_2)$$

Notice that we have the group action

$$\mathbb{R} \times X \longrightarrow X$$
$$(t, [z_1, z_2]) \mapsto [\alpha^t z_1 + \lambda t \alpha^{t-1} z_2, \beta^t z_2^n]$$

So $X \overset{\text{diffeo}}{\cong} S^3 \times S^1$, and everything about topo & Hodge numbers are the same,

$$\mathrm{Pic}(X) = \mathbb{C}^\times.$$

2. Compute K_X. $\quad \psi = \frac{1}{z_2^{n+1}} dz_1 \wedge dz_2 \in H^0_{\mathcal{M}}(X, \omega_X)$ ← meromorphic

$$C := [z_2 = 0] = \mathbb{C}^* / \mathbb{Z}\gamma_1 \cong \mathbb{C} / \mathbb{Z} \oplus (\tfrac{1}{2\pi i} \ln \alpha)\mathbb{Z} \qquad (\gamma_1 z_1 = \alpha z_1)$$

$$K_X = -(n+1)C \implies P_k = h^0(kK_X) = 0 \text{ for } k \geq 1 \implies k(X) = -\infty$$

3. X is not an elliptic surface. || THEOREM 31. *$W/\{f\}$ is an elliptic surface if and only if $\lambda = 0$ and $\alpha_1^k = \alpha_2^l$ for certain positive integers k, l.*

If not, then $\exists\ \Phi : X \to \Delta$, fibers are elliptic curves (may degenerate)

then $H^2(X, \mathbb{Z}) = 0 \Rightarrow C$ is fiber of Φ. $\Phi(C) = u$ is a pt

choose one non-constant fct $x \in H^0(\Delta, \mathcal{O}_\Delta(ku))$,

pullback to $x \in H^0(X, \mathcal{O}_X(kC))$, then

$$\phi := z_2^k x \in \Gamma_{hol}(\mathbb{C}^2 - \{0\}) \overset{\text{Hartog}}{=\!=} \Gamma_{hol}(\mathbb{C}^2)$$

$$\phi(\alpha z_1 + \lambda z_2^n, \beta z_2) = \beta^k \phi(z_1, z_2) \qquad (1)$$

$$\begin{array}{c} \mathbb{C}^2 - \{0\} \hookrightarrow \mathbb{C}^2 \\ \downarrow \\ X \\ \downarrow \\ \Delta \end{array}$$

let $\phi_\nu := \frac{\partial^\nu \phi}{\partial z_1^\nu}$, then $(\nu \in \mathbb{N}^+)$

$$\alpha^\nu \phi_\nu(\alpha z_1 + \lambda z_2^n, \beta z_2) = \beta^k \phi_\nu(z_1, z_2) \qquad (2)$$

$$\Rightarrow \phi_\nu(z_1, z_2) = \lim_{N \to +\infty} \left(\frac{\alpha^\nu}{\beta^k}\right)^N \phi_\nu(\alpha^N z_1 + \lambda N \alpha^{N-1} z_2, \beta^N z_2) \equiv 0 \text{ when } \alpha^\nu < \beta^k \qquad (\nu \gg 1)$$

let $l := \min\{l' \in \mathbb{N}_{\geq 0} \mid \phi_{l'+1} \equiv 0\}$

① $l = 0 \overset{(1)}{\Rightarrow} \phi(0, \beta z_2) = \beta^k \phi(0, z_2) \Rightarrow \phi(z_1, z_2) = c z_2^k$ $(c \in \mathbb{C})$, contradiction!

② $l > 0 \overset{(2)}{\Rightarrow} \phi_l(0, \beta z_2) = \frac{\beta^k}{\alpha^l} \phi_l(0, z_2) = \beta^{k-ln} \phi_l(0, z_2)$

$$\Rightarrow \phi_l(z_1, z_2) = c\, z_2^{k-ln} \quad (c \in \mathbb{C})$$

$$\Rightarrow \phi_{l-1}(z_1, z_2) = c\, z_1 z_2^{k-ln} + \sum a_h z_2^h$$

$$\overset{(2)}{\Rightarrow} \beta^{k-n} c \lambda = 0 \Rightarrow c = 0, \text{ contradiction!}$$

Cor. $\mathrm{tr.dim}\, \mathcal{M}(X) = 0 \Rightarrow \mathcal{M}(X) = \mathbb{C}$

suppose [Kodaira I, Thm 4].

Q: Do we have other curves except C in X?

https://mathoverflow.net/questions/360082/fun-examples-relating-to-hopf-surfaces