§ 1.1. Structure of finite / local / global field

Road map

	finite field		local field				global field		adèle
			Archi		NA				
base field F	7 $\mathbb{F}_1$ (For $\mathbb{F}_{1^r}$)	1 $\mathbb{F}_p$	2 $\mathbb{R}$ or $\mathbb{C}$		3 $\mathbb{Q}_p$	$\mathbb{F}_p((t))$	4 $\mathbb{Q}$	$\mathbb{F}_p(t)$	6 $\mathbb{A}_K$
$F^\times$	$\varepsilon \cdot \mu_r$	μ_{p-1}	$\mathbb{R}^+ \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}^\times$	$\mathbb{Z}_p^\times \times \mathbb{Z}$	$\mathbb{F}_p[[t]]^\times \times \mathbb{Z}$	$\mathbb{Q}^\times$	$\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	—	—	—		$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	—	—	—		$\mathbb{Z}_p^\times$	$\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$	$\hat{\mathbb{Z}}$?	$\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$	Id	most known		dream		
ari Frob	?	can	total order?	—	choose a lift		$\xrightarrow{\text{unramified}} \begin{cases} \mathrm{Frob}_q \\ \mathrm{Frob}_p \end{cases}$ conj class		$\xrightarrow{\text{abelian}}$ Frob_p
# ext of deg n	1?	1	1/0		finite		$n \neq 1$ inf	countable	
Spec $\mathcal{O}_F$	Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}}, 1)$ [étale, 2.2.4]		—						—
topology	?	discrete	Euclidean		profinite		—		restricted
topo of $\mathcal{O}_F$	—		—		cpt, not discrete		—		K is a lattice in $\mathbb{A}_K$
measure	?	discrete	Lebesgue		$\mu(\mathcal{O}_F) = 1$		—		can be computed

5 Also, discuss

- field extension, norm, trace, …
- their connection to geometry, ramification theory
- analog with knot theory

Process

1. finite field $\mathbb{F}_q$
2. Archi local field $\mathbb{R}$ or $\mathbb{C}$
3. NA local field
    - Individual structure
    - Field extension
4. global field

    - Dessin d'enfants
5. local and global: connections
    - Basics
    - Traditional point of view
        - Frobenius
        - Application: Quadratic reciprocity
    - Étale point of view
6. local to global: adèle

    - Base field with automorphism
    - Galois extension
7. $\mathbb{F}_1$

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}} \quad \text{as top gps}$$
$$\mathrm{Frob}_q \leftarrow 1$$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $\quad \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \mathrm{Id}$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt? Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions:

- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $\mathrm{Fr}\,\tau\,\mathrm{Fr}^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions. (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep} / \mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right)}, \quad \underset{I_{\mathbb{Q}_p}}{\mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}}_p / \widehat{\mathbb{Q}_p^{ur}}\right)}, \quad \underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} / \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)}$$

4. global field

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.

$\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D],
we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\}/\text{birational}$$

$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler. by [GalFun, Thm 3.4.8],

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \underbrace{\hat{F}}_{\text{Free profinite gp on } \mathbb{C}}(\mathbb{C})$$

Shafarevich's conj: See wiki: Absolute Galois group

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite gp

Q: Does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object/geo realizations?

Dessin d'enfants

By [GalFun, Prop 4.7.1 - Rmk 4.7.9], we have an including

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}))$$

induced by $\pi^{ét}_{1,\overline{\mathbb{Q}}} := \pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, $\pi^{ét}_{1,\mathbb{Q}} := \pi_1^{ét}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$

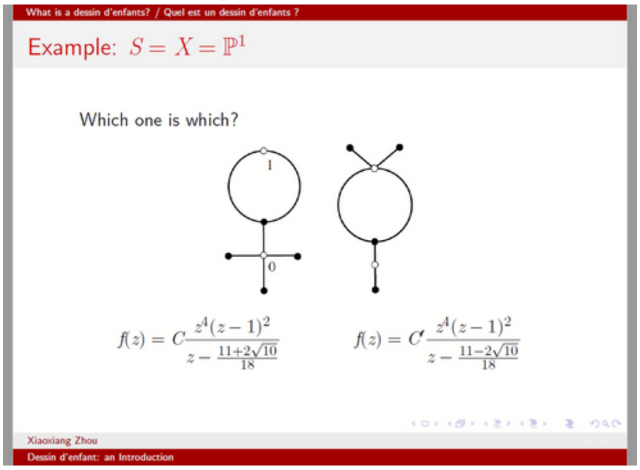
$$\begin{tikzcd}
 & 1 \arrow[r] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] \arrow[d, equal] & \pi^{ét}_{1,\mathbb{Q}} \arrow[r] \arrow[d, "\substack{\text{conj} \\ g \mapsto g \cdot g^{-1}}"] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & Z(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] \arrow[u, phantom, "1 \overset{?}{=}"] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] & \mathrm{Aut}(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & \mathrm{Out}(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & 1
\end{tikzcd}$$

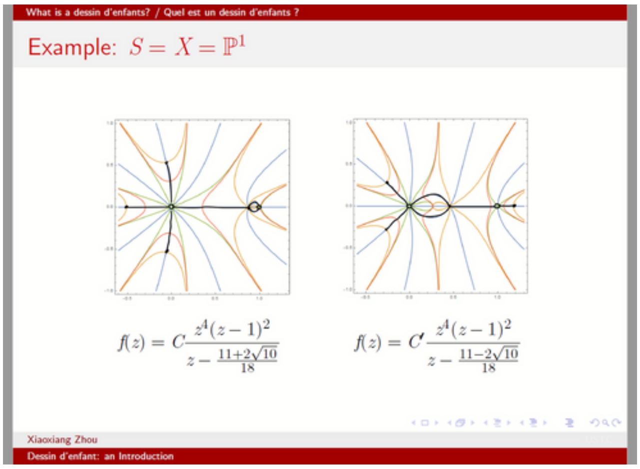
The space $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ is designed for guaranteeing that
$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\pi^{ét}_{1,\overline{\mathbb{Q}}})$$
is inclusion.

Task. Read [Dessin d'enfant] or [Collins],

understand the $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the dessin d'enfants.

- Def of Dessin d'enfact
- Connections with $Out(\pi_{1,\bar{\mathbb{Q}}}^{\acute{e}t})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.





- Can we generalize this to $Gal(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$?

  I don't know how to make a "dessin d'enfant" on alg curves over $\mathbb{F}_p$.

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.

5. local and global connections (Wait: $K \rightsquigarrow F$)

Basics

$$\begin{tikzcd} F \arrow[r, hook] & F_v \\ \mathcal{O}_F \arrow[u, phantom, "\cup"] \arrow[r, hook] \arrow[rrd] & \mathcal{O}_v \arrow[u, phantom, "\cup"] \arrow[rd] & \\ & & k_v \end{tikzcd}$$

for any valuation v $(\rightsquigarrow K \hookrightarrow \mathbb{A}_K)$

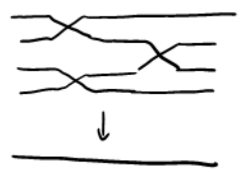
and $\mathcal{O}_v$ is the integral closure of $\mathcal{O}_F$ in F_v (for v fin)

and $\mathcal{O}_F/\mathfrak{p}^r\mathcal{O}_F \cong (\mathcal{O}_F/\mathfrak{p}^r\mathcal{O}_F)_v \cong \mathcal{O}_v/\mathfrak{p}^r\mathcal{O}_v \quad \forall r \in \mathbb{N}_{\geq 0}$

The connections are also compatible with field exts. (E/F Galois)

E.g.

$$\begin{array}{ccc} E & \mathcal{O}_E & \mathfrak{q} \sim w \\ | & \uparrow & | \\ F & \mathcal{O}_F & \mathfrak{p} \sim v \end{array}$$



$$\mathfrak{p}\mathcal{O}_E = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_g^{e_g} \qquad\qquad E \otimes_F F_v \cong \prod_w E_w$$

$\alpha \in E$, $\iota_w : E \hookrightarrow E_w$

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d, "N_{L/K}"'] & \prod_w E_w \arrow[d, "\prod_w N_{E_w/F_v}^{e_w}"] \\ F \arrow[r, hook] & F_v \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, hook] \arrow[d, "Tr_{L/K}"'] & \prod_w E_w \arrow[d, "\sum_w e_w Tr_{L_w/K_v}"] \\ K \arrow[r, hook] & K_v \end{tikzcd}$$

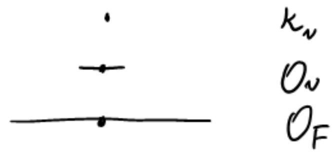
$$N_{E/F}(\alpha) = \prod_{w|v} N_{E_w/F_v}(\iota_w(\alpha))^{e_w} \qquad Tr_{E/F}(\alpha) = \prod_{w|v} e_w Tr_{E_w/F_v}(\iota_w(\alpha))$$

Traditional point of view

$k_v \longrightarrow$ pt

$\mathcal{O}_v \longrightarrow$ "loc" curve

$\mathcal{O}_F \longrightarrow$ curve



Task. Read [Algfungp, 0.2], answer the following questions:

- Understand ramified, inert, split.
- Understand (from geo meaning)
$$\mathcal{O}_E/\mathfrak{p}\mathcal{O}_E \cong \bigoplus_{i=1} \mathcal{O}_E/\mathfrak{q}^{e_i}\mathcal{O}_E$$
- Compute some classical examples
- Compare it with ramified covering in RS.

Frobenius (E/F Galois)

$$\begin{tikzcd}
 & 1 \arrow[r] & \mathrm{Gal}(E/F)_w \arrow[r, "\leq"] \arrow[d, phantom, "\cong"] & \mathrm{Gal}(E/F) & \\
1 \arrow[r] & \mathrm{Gal}(E_w/F_v^{ur}\cap E_w) \arrow[r] \arrow[d, phantom, "\cong"] & \mathrm{Gal}(E_w/F_v) \arrow[r, two heads] \arrow[d, phantom, "\cong"] & \mathrm{Gal}(F_v^{ur}\cap E_w/F_v) \arrow[r] \arrow[d, phantom, "\cong"] & 1 \\
1 \arrow[r] & I(E_w/F_v) \arrow[r] & \mathrm{Aut}_{\mathcal{O}_v\text{-alg}}(\mathcal{O}_w) \arrow[r, two heads] & \mathrm{Gal}(k_w/k_v) \arrow[r] & 1
\end{tikzcd}$$

where

$$\mathrm{Gal}(E/F)_w = \{\sigma \in \mathrm{Gal}(E/F) \mid \sigma(q) = q\} \leq \mathrm{Gal}(E/F)$$

By https://math.stackexchange.com/questions/4131855/frobenius-elements,

more conditions $\Rightarrow$ better props of Frob

① E/F is unramified at w (True for all except finite w) $\Rightarrow$ $\begin{cases} I(E_w/F_v) = \mathrm{Id} \\ \mathrm{Fr}_w \in \mathrm{Gal}(E/F)_w \text{ well-defined} \end{cases}$

② E/F Galois $\Rightarrow$ $\begin{cases} \text{all symbols are meaningful} \\ \forall q, q' \in \mathrm{Spec}\, \mathcal{O}_E,\ \exists \sigma \in \mathrm{Gal}(E/F),\ \sigma(q) = q' \\ \mathrm{Frob}_v = [\mathrm{Frob}_w] \text{ is a conj class in } \mathrm{Gal}(E/F) \end{cases}$

③ $E \subseteq K^{ab}$ $\Rightarrow$ $\mathrm{Frob}_v \in \mathrm{Gal}(E/F)$

E.g. When $E/F = \mathbb{Q}^{ab}/\mathbb{Q}$,

$$\begin{tikzcd}
 & 1 \arrow[r] & \widehat{\mathbb{Q}}_p^\times \arrow[r, hook] \arrow[d, phantom, "\cong"] & \widehat{\mathbb{Z}}^\times & \\
1 \arrow[r] & \mathbb{Z}_p^\times \arrow[r] & \widehat{\mathbb{Q}}_p^\times \arrow[r, two heads] & \widehat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

where $\widehat{\mathbb{Q}}_p^\times \longrightarrow \widehat{\mathbb{Z}}^\times$ is given by

$$\begin{aligned} \mathbb{Z}_p^\times \times \widehat{\mathbb{Z}} &\longrightarrow \mathbb{Z}_p^\times \times \prod_{l \neq p} \mathbb{Z}_l^\times \\ (a, b) &\longmapsto (a, p^b, \cdots, p^b) \end{aligned}$$

$$p^b = \left(p^{b \bmod \varphi(l^k)}\right)_k \in \varprojlim_k (\mathbb{Z}/l^k\mathbb{Z})^\times = \mathbb{Z}_l^\times$$

$\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \widehat{\mathbb{Z}}^\times$: see

https://math.stackexchange.com/questions/142236/what-is-a-maximal-abelian-extension-of-a-number-field-and-what-does-its-galois-g

Application: Quadratic reciprocity

Thm. p, l odd primes, $p \neq l$, then

$$\left(\frac{p}{l}\right)\left(\frac{l}{p}\right) = (-1)^{\frac{p-1}{2}\frac{l-1}{2}}$$

$$\left(\frac{-1}{l}\right) = (-1)^{\frac{l-1}{2}} = \begin{cases} 1 & l \equiv 1 \mod 4 \\ -1 & l \equiv 3 \mod 4 \end{cases}$$

$$\left(\frac{2}{l}\right) = (-1)^{\frac{l^2-1}{8}} = \begin{cases} 1 & l \equiv 1, 7 \mod 8 \\ -1 & l \equiv 3, 5 \mod 8 \end{cases}$$

Proof Assume $p \equiv 1 \bmod 4$, then $\mathbb{Q}_\ell(\sqrt{p}) \hookrightarrow \mathbb{Q}_\ell(\zeta_p)$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_p)/\mathbb{Q}_\ell) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{p})/\mathbb{Q}_\ell) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_\ell \arrow[r, mapsto] \arrow[d, mapsto] & \ell \arrow[d, mapsto] \\ \mathrm{Frob}_\ell \arrow[r, mapsto] & \left(\frac{p}{\ell}\right) = \left(\frac{\ell}{p}\right) \end{tikzcd}$$

$$\begin{aligned} \chi(\mathrm{Frob}_\ell) = 1 &\iff \mathrm{Frob}_\ell(\sqrt{p}) = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_\ell} \\ &\iff (\sqrt{p})^\ell = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_\ell} \\ &\iff \sqrt{p} \in \mathbb{F}_\ell \\ &\iff \left(\tfrac{p}{\ell}\right) = 1 \end{aligned}$$

Assume $p \equiv 3 \bmod 4$, then $\mathbb{Q}_\ell(\sqrt{-p}) \hookrightarrow \mathbb{Q}_\ell(\zeta_p)$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_p)/\mathbb{Q}_\ell) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{-p})/\mathbb{Q}_\ell) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_\ell \arrow[r, mapsto] \arrow[d, mapsto] & \ell \arrow[d, mapsto] \\ \mathrm{Frob}_\ell \arrow[r, mapsto] & \left(\frac{-p}{\ell}\right) = \left(\frac{\ell}{p}\right) \end{tikzcd}$$

$$\begin{aligned} \chi(\mathrm{Frob}_\ell) = 1 &\iff \mathrm{Frob}_\ell(\sqrt{-p}) = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_\ell} \\ &\iff (\sqrt{-p})^\ell = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_\ell} \\ &\iff \sqrt{-p} \in \mathbb{F}_\ell \\ &\iff \left(\tfrac{-p}{\ell}\right) = 1 \end{aligned}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_4)/\mathbb{Q}_\ell) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/4\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{-1})/\mathbb{Q}_\ell) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_\ell \arrow[r, mapsto] \arrow[d, mapsto] & \ell \arrow[d, mapsto] \\ \mathrm{Frob}_\ell \arrow[r, mapsto] & \left(\frac{-1}{\ell}\right) = (-1)^{\frac{\ell-1}{2}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_8)/\mathbb{Q}_\ell) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{2})/\mathbb{Q}_\ell) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_\ell \arrow[r, mapsto] \arrow[d, mapsto] & \ell \arrow[d, mapsto] \\ \mathrm{Frob}_\ell \arrow[r, mapsto] & \left(\frac{2}{\ell}\right) = (-1)^{\frac{\ell^2-1}{8}} \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Gal}(\mathbb{Q}_\ell(\zeta_8)/\mathbb{Q}_\ell) \arrow[r, phantom, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\ \mathrm{Gal}(\mathbb{Q}_\ell(\sqrt{-2})/\mathbb{Q}_\ell) \arrow[r, "\chi", "\sim"'] & \{\pm 1\} \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Frob}_\ell \arrow[r, mapsto] \arrow[d, mapsto] & \ell \arrow[d, mapsto] \\ \mathrm{Frob}_\ell \arrow[r, mapsto] & \left(\frac{-2}{\ell}\right) = (-1)^{\frac{(\ell+1)(\ell+3)}{8}} \end{tikzcd}$$

Q: Quadratic reciprocity for $\mathbb{F}_p(t)$?

Cubic reciprocity ?
[Cox x^2+ny^2, § 4.A, 4.B]

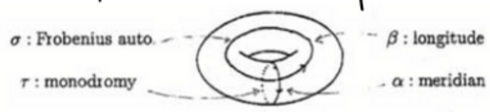
$\begin{cases} \text{primes over } \mathbb{Q}(\zeta_3),\ \left(\frac{\alpha}{\beta}\right)_3 \quad (\alpha),(\beta)\in \operatorname{Spec}\mathbb{Z}[\zeta_3] \\ \deg 3,\ \text{like } \operatorname{Gal}(F/\mathbb{Q}(\sqrt{-23})) : \text{More difficult} \end{cases}$

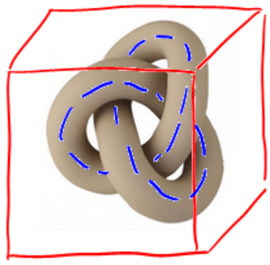
Étale point of view

$k \longrightarrow S^1$

$\mathcal{O}_v \longrightarrow$ tubular nbhd of S^1

$\mathcal{O}_F \longrightarrow$ 3-dim spaces ($\mathbb{R}^3$ when $K=\mathbb{Q}$)





k

$\mathcal{O}_v$

$\mathcal{O}_F$

See [Knotprime Table 1] for more informations.
Also, see this:
http://www.neverendingbooks.org/mazurs-dictionary
算术拓扑的初步理论: https://zhuanlan.zhihu.com/p/563347112

6. local to global: adèle

Recall: Ostrowski's thm & Product formula.

Task. Read [Adèle] and answer the following questions:

- Give a def of $\mathbb{A}_F$ & $\mathbb{I}_F$ (set, topo and measure)
- Verify that

$$F \subseteq \mathbb{A}_F \qquad \mathcal{O}_T \subseteq \prod_{v\in T}' F_v$$
$$F^\times \subseteq \mathbb{I}_F^\times \qquad \mathcal{O}_T^\times \subseteq \left(\prod_{v\in T}' F_v\right)'$$

are lattices. Give fundamental domain in easy cases.

- Deduce the finiteness of class number and Dirichlet unit theorem.

Base field with automorphism

We know that

	finite field	local field: Archi	local field: NA	global field	Adèle
base field F	$\mathbb{F}_p$	$\mathbb{R}$	$\mathbb{Q}_p$ $\quad \mathbb{F}_p((t))$	$\mathbb{Q}$ $\quad \mathbb{F}_p(t)$	$\mathbb{A}_K$
	$\mathrm{Aut}_{ring}(\mathbb{F}_p)=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{R})=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{Q}_p)=1$ $\mathrm{Aut}_{top\ ring}(\mathbb{F}_p((t)))\neq 1$	$\mathrm{Aut}_{ring}(\mathbb{Q})=1$ $\mathrm{Aut}_{ring}(\mathbb{F}_p(t))\neq 1$	 $\mathrm{Aut}_{ring}(\mathbb{A}_{\mathbb{F}_p(t)})\neq 1$

Q: Do we have $\mathrm{Aut}_{ring}(\mathbb{A}_\mathbb{Q}) = 1$?

A: Yes. See [LCFT, Ex 6.3.6]. I don't understand this proof.

Galois extension

Setting: E/F fin ext of global field

Recall that we have an iso

$$E\otimes_F \mathbb{A}_F \xrightarrow{\cong} \mathbb{A}_E \text{ of topo rings with compatible embedding of } E$$

$\rightsquigarrow$ $\mathbb{A}_F \subseteq \mathbb{A}_E$ subring, $\mathbb{A}_E \cong \mathbb{A}_F^{\oplus[E:F]}$ as $\mathbb{A}_E$-module.

Lemma. [LCFT, Ex 6.3.2]

$$\text{integral closure of } F \text{ in } \mathbb{A}_E = E$$

Proof. Reduce to

$$\text{integral closure of } E \text{ in } \mathbb{A}_E \subseteq E$$

If $\exists\, x\in\mathbb{A}_E - E$ which is integral over E, then $E(x)/E$ is a fin field ext in $\mathbb{A}_E$, and

$$\#\{\mathfrak{q}\in\operatorname{Spec}\mathcal{O}_E \mid \mathfrak{q} \text{ do not split completely}\}$$
$$\overset{?}{\leq} \#\{\mathfrak{q}\in\operatorname{Spec}\mathcal{O}_E \mid x_\mathfrak{q}\notin\mathcal{O}_\mathfrak{q}\} < +\infty.$$

But fin nontrivial field ext have inf many non split primes. $\lightning$

7. $\mathbb{F}_1$

This is better explained in §1.2. Anyhow, it is still a "field".

⚠ It is always better to think $\# \mathbb{F}_1 = 1+\varepsilon$, where $\varepsilon \ll 1$.

In that way you can "see object at different level", like

$\# \mathbb{F}_1^\times = \varepsilon$ $\qquad$ $\mathbb{F}_1^\times$ is not empty!

Slogan: "Infinitesimal is only visible when constant level is zero"

This phenomenon already happens when we learn integrals:

$$\int x^n = \begin{cases} \frac{1}{n+1} x^{n+1} + C & n \neq -1 \\ \log x + C & n = -1 \end{cases}$$

Here, $\log x$ is that "infinitesimal".

See
wiki: https://en.wikipedia.org/wiki/Field_with_one_element
nlab: https://ncatlab.org/nlab/show/field+with+one+element

It is desirable to define

- The "field extension" $\mathbb{F}_{1^n}/\mathbb{F}_1$ of deg n $\quad$ + $\mathbb{F}_{1^n}^\times \cong \varepsilon\mu_n$
- The "Galois group" $\mathrm{Gal}(\overline{\mathbb{F}}_1/\mathbb{F}_1) \cong \widehat{\mathbb{Z}}$
- The "v.s. over $\mathbb{F}_1$"

Q: Do we have $\mathbb{Q}_1$?

A: See https://mathoverflow.net/questions/309664/what-is-mathbbq-1-the-field-of-1-adic-numbers