Eine Woche, ein Beispiel
1. 9. simplicial set

Today: The category $sSet$ and $\partial\Delta^n, \Lambda^n_i, sk^m X, \Delta^n/\partial\Delta^n, Hom(X,Y) \in Ob(sSet)$

Def $[n] = \{0, 1, \ldots, n\}$ $n \geq 0$
The simplex category Δ is defined by
$$Ob(\Delta) = \{[n] \mid n \geq 0\}$$
$$Mor_\Delta([m],[n]) = \{f : [m] \longrightarrow [n] \text{ weakly increasing}\}$$
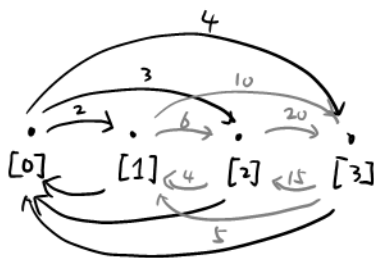The category of simplicial sets $sSet$ is defined by
$$sSet = Fun(\Delta^{op}, Set)$$

Notation   in $Mor(\Delta)$: $d_i^n : [n-1] \longrightarrow [n]$ miss i $0 \leq i \leq n$
$s_i^n : [n] \longrightarrow [n-1]$ contracts i

$\Delta \hookrightarrow sSet$ $[n] \longmapsto \Delta^n := Mor_\Delta(-,[n])$

e.p. $\Delta^n_k = Mor_\Delta([k],[n])$
↑ read from down to top

#Δ^n_k \ n	0	1	2	3
0	1	2	3	4
1	1	3	6	10
2	1	4	10	20
3	1	5	15	35

$$\#\Delta^n_k = \binom{n+k+1}{n}$$

not confuse with $\lfloor n]$

element	picture	list	count	other notations
$\alpha: [5] \to [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	
$d_1^3: [2] \to [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		$(0,2,3)$	$[1,0,1,1]$	$d_i^n: [n-1] \to [n]$ δ^n
$s_1^3: [3] \to [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		$(0,1,1,2)$	$[1,2,1]$	$s_i^n: [n] \to [n-1]$
$d_{3,2}: [3] \to [5]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$		$(0,1,2,3)$	$[1,1,1,1,0,0]$	$d_{i,j}: [i] \to [i+j]$ δ_i^f $f=front$
$d'_{3,2}: [2] \to [5]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$		$(3,4,5)$	$[0,0,0,1,1,1]$	$d'_{i,j}: [j] \to [i+j]$ δ_i^b $b=back$
$\delta_{3,(5,4)}^{out}: [5] \to [8]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$ $4 \mapsto 7$ $5 \mapsto 8$		$(0,1,2,3,7,8)$	$[1,1,1,1,0,0,0,1,1]$	$\delta_{i,(p,q)}^{out}: [p] \to [p+q-1]$ δ_i^{out}
$\delta_{3,(5,4)}^{in}: [4] \to [8]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$ $3 \mapsto 6$ $4 \mapsto 7$		$(3,4,5,6,7)$	$[0,0,0,1,1,1,1,1,0]$	$\delta_{i,(p,q)}^{in}: [q] \to [p+q-1]$ δ_i^{in}

Table 1. Morphisms in Δ.

How to compute the composition?

e.g.
$$[2,1,0,4] \circ [2,5,3,4,1,6,0] = [7,3,0,11]$$

$$[20] \to [6] \to [3]$$
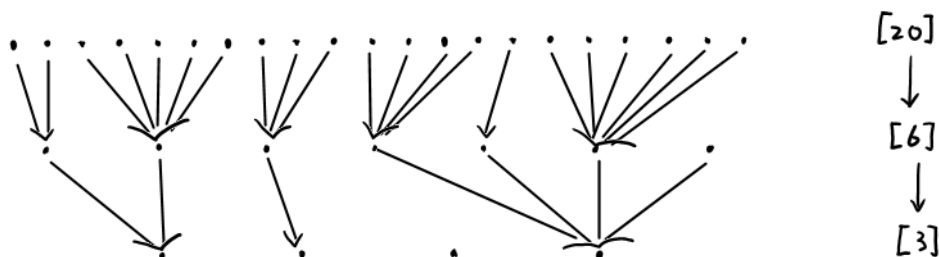
$$[1,2,1] \circ [2,1,0,3] = [2,1,3]$$
$$[4] \circ [2,1,0,3] = [6]$$
$$s_1^3 \circ d_1^3 = [1,2,1] \circ [1,0,1,1] = [1,1,1]$$
$$d_1^3 \circ s_1^3 = [1,0,1,1] \circ [1,2,1] = [1,0,2,1]$$

Rmk. In Δ we don't have finite colimit, while in $s\mathrm{Set} = \mathrm{Fun}(\Delta^{op}, \mathrm{Set})$ we have finite colimit because Set is (complete +) cocomplete.

For a construction, see
https://math.stackexchange.com/questions/3837844/limits-and-colimits-are-computed-pointwise-in-functor-categories

Notice that
$$\partial \Delta^n, \Lambda_i^n, sk^m \Delta^n, \Delta^n / \partial \Delta^n \in s\mathrm{Set} - \Delta$$

Conclusion: $s\mathrm{Set}$ is a Grothendieck topos.
It is Cartesian closed, complete and cocomplete.
In $s\mathrm{Set}$, we can glue objects ($\approx$ pushforward), which is impossible in Δ.

Slogan: $s\mathrm{Set} \sim$ simplicial complex
$X_n \sim$ the index set of n-dim cells

Rmk. ([sSet]) If you have strong enough geometrical background, you will find out the adjoint pair

$$\text{sSet} \underset{S}{\overset{|-|}{\rightleftarrows}} \text{Top}$$

quite useful, where

$$|X| := \left(\coprod_{n \geq 0} X_n \times \nabla^n \right) / \sim$$
$$S(A)_n := \text{Mor}_{\text{Top}}(\nabla^n, A)$$
$$\alpha^* : S(A)_n \longrightarrow S(A)_m \qquad x \longmapsto x \circ S(\alpha)$$

$$\alpha : [m] \to [n]$$
$$S(\alpha) : \nabla^m \to \nabla^n$$

Moreover, we have eqvlc of categories

$$\text{sSet}[\text{weq}^{-1}] \underset{S}{\overset{|-|}{\rightleftarrows}} \overset{\text{Ho}(\text{Top}_{cw})}{\underset{\text{Top}[\text{weq}^{-1}]}{\downarrow \uparrow}}$$

where

Top$_{cw}$ is the full subcategory of Top with objects the top spaces admitting a CW-cplx structure, and

Ho(Top$_{cw}$) is the homotopy category of CW cplxes.

Q: Do we have the following comm diag. (as eqvlc of categories)

$$\text{sSet}[\text{weq}^{-1}] \xrightarrow{|-|} \text{Ho}(\text{Top}_{cw})$$
$$\uparrow \qquad \overset{S}{\leftarrow} \qquad \downarrow$$
$$\text{An} \qquad \text{Top}[\text{weq}^{-1}]$$

Q: For $\mathcal{C} \in \text{Cat}_\infty \subseteq \text{sSet}$, how to view $\mathcal{C}$ as a topo space?
e.p. compute $\pi_n(\mathcal{C})$?

Roughly, we have three ways to define/determine a simplicial set:
  1. By writing down their def directly;   brutal force
  2. By universal property (pullback, pushforward,...)   abstract construction
  3. By its geometrical realization   name

Let us see how they're compatible with each other.

E.g.1 For $A \in \mathcal{T}op$ discrete, define $X = S(A)$, i.e.,

$$X_n = A \qquad \alpha^* = Id_A \qquad \forall \alpha : [m] \longrightarrow [n]$$

$$|S(A)| = (\coprod_k X_k \times \nabla^k)/\sim$$
$$\sim A \times \nabla^0$$
$$\sim A$$

Eg. 2. $\Delta_k^n = Mor_\Delta([k], [n]) = \{ \, x : [k] \longrightarrow [n] \text{ weakly increasing} \}$

$$|\Delta^n| = (\coprod_k \Delta_k^n \times \nabla^k)/\sim$$
$$\sim (\Delta_n^n \times \nabla^n)/\sim$$
$$\sim \nabla^n$$

Eg. 3. $\Delta_{(i)}^{n-1} := Im(d_i^n : \Delta^{n-1} \longrightarrow \Delta^n)$ in sSet

$$\Rightarrow (\Delta_{(i)}^{n-1})_k = \{ x \in \Delta_k^n \mid \exists \, y \in \Delta_k^{n-1} \text{ s.t. } x = d_i^n \circ y \}$$

$$|\Delta_{(i)}^{n-1}| = (\coprod_k (\Delta_{(i)}^{n-1})_k \times \nabla^k)/\sim$$

$$\sim ((\Delta_{(i)}^{n-1})_{n-1}^{nondeg} \times \nabla^{n-1})/\sim$$

$$\sim (Sd_i^n)(\nabla^{n-1})$$

Denote $|\Delta_{(i)}^{n-1}|$ by $\nabla_{(i)}^{n-1}$, i.e. $\nabla_{(i)}^{n-1} := Im(Sd_i^n : \nabla^{n-1} \longrightarrow \nabla^n)$

Eg. 4. $(\partial \Delta^n)_k = \{ x \in \Delta_k^n \mid x \text{ is not surj} \}$

$$\partial \Delta^n = \bigcup_{i=0}^n \Delta_{(i)}^{n-1} = \text{colimit of } \cdots$$

$$\text{e.g. } \partial \Delta^2 = \left[\text{colimit of} \begin{array}{ccc} \Delta^0 & \Delta^0 & \Delta^0 \\ \downarrow \searsearrow \swarrow \searrow \swarrow \downarrow & & \\ \Delta^1 & \Delta^1 & \Delta^1 \end{array} \right]$$

$$|\partial \Delta^n| = (\coprod_k (\partial \Delta^n)_k \times \nabla^k)/\sim$$
$$\sim (Mor_\Delta^{nondeg}([n-1], [n]) \times \nabla^{n-1})/\sim$$
$$\sim (\coprod_{i=0}^n Sd_i^n(\nabla^{n-1}))/\sim$$
$$= \bigcup_{i=0}^n \nabla_{(i)}^{n-1}$$
$$= \partial \nabla^n$$

Eg. 5. $(\Lambda_i^n)_k = \left\{ x \in \Delta_k^n \mid x = \alpha^*(y) \text{ for some } y \in \Delta_{n-1}^n \text{ and } \alpha: [k] \to [n-1] \atop y \neq d_i^n \right\}$

$\Lambda_i^n = \bigcup_{j \neq i} \Delta_{(j)}^{n-1} = \text{colimit of } \cdots$

e.g. $\Lambda_1^2 = \left[\text{colimit of} \quad \begin{array}{ccc} & & \Delta^0 \\ & & {\scriptstyle d_0^1} \downarrow \\ \Delta^0 & \xrightarrow{d_1^1} & \Delta^1 \\ {\scriptstyle d_0^1} \downarrow & & \downarrow {\scriptstyle d_0^1} \\ \Delta^0 & \dashrightarrow[d_1^1] & \Delta^1 \end{array} \right]$

$= \Delta^1 \sqcup_{\Delta^0} \Delta^1$

ex. write down $(X \sqcup_Y Z)_k$ for $X, Y, Z \in sSet$

$|\Lambda_i^n| = \left(\coprod_k (\Lambda_i^n)_k \times \nabla^k \right) / \sim$
$\sim \left((\Lambda_i^n)_{n-1}^{\text{nondeg}} \times \nabla^{n-1} \right) / \sim$
$\sim \left(\coprod_{j \neq i} (Sd_j^n)(\nabla^{n-1}) \right) / \sim$

$\sim \bigcup_{j \neq i} \nabla_{(j)}^{n-1}$

Eg. 6. $(sk^m \Delta)_k = \left\{ x \in \Delta_k^n \mid x = \alpha^*(y) \text{ for some } y \in \Delta_m^n \text{ and } \alpha: [k] \to [m] \right\}$

$sk^m \Delta^n = \bigcup_{\beta: [m] \to [n]} \beta(\Delta^m) = \text{colimit of } \cdots$

$|sk^m \Delta^n| = \left(\coprod_k (sk^m \Delta^n)_k \times \nabla^k \right) / \sim$
$\sim \left((sk^m \Delta^n)_m^{\text{nondeg}} \times \nabla^m \right) / \sim$
$\sim \left(Mor^{\text{nondeg}}([m],[n]) \times \nabla^m \right) / \sim$
$\sim \bigcup_{\beta: [m] \to [n]} (S\beta)(\nabla^m)$

Eg. 7. $(\Delta^n / \partial \Delta^n)_k = \Delta_k^n / (\partial \Delta^n)_k = \Delta_k^n / \sim$

$x \sim y \iff x, y \in (\partial \Delta^n)_k \quad \text{or} \quad x = y$

Universal property:

$$\partial \Delta^n \longrightarrow \Delta^n \longrightarrow \Delta^n / \partial \Delta^n \longrightarrow 0$$

contract to one pt $\searrow \quad \downarrow \exists !$

X

$$\left| \Delta^n / \partial \Delta^n \right| = \left(\coprod_k (\Delta^n / \partial \Delta^n)_k \times \nabla^k \right) / \sim$$
$$\sim \left((\Delta^n / \partial \Delta^n)_n^{nondeg} \times \nabla^n \right) / \sim$$
$$\sim \nabla^n / \sim$$
$$\sim \nabla^n / \partial \nabla^n$$

Eg. 8. Define
$$X = \left[\text{colimit of} \quad \Delta^1 \underset{d_0^2}{\overset{d_2^2}{\underset{-d_1^2}{\rightrightarrows}}} \Delta^2 \right]$$

$$X_k = \Delta_k^2 / \sim \qquad \text{here we identify } d_2^2 x = -d_1^2 x = d_0^2 x$$
$$|X| = \left(\coprod_k X_k \times \nabla^k \right) / \sim$$
$$\sim \left(X_2^{nondeg} \times \nabla^2 \right) / \sim$$

Eg. 9 $(\text{Hom}(X,Y))_n = \text{Hom}_{sSet}(\Delta^n \times X, Y)$

$\alpha^* : \text{Hom}_{sSet}(\Delta^n \times X, Y) \longrightarrow \text{Hom}_{sSet}(\Delta^m \times X, Y)$ for $\alpha: [m] \to [n]$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \alpha: \Delta^m \to \Delta^n$

$$sSet \underset{\text{Hom}(X,-)}{\overset{-\times X}{\rightleftarrows}} \bot \quad sSet$$

$$
\begin{bmatrix}
\text{"Proof"} \quad \text{Hom}_{sSet}(Z, \text{Hom}(X,Y)) \cong \left\{ \begin{array}{c} g_m : Z_m \longrightarrow \text{Hom}_{sSet}(\Delta^n \times X, Y) \\ + \cdots \end{array} \right\} \\
\cong \left\{ \begin{array}{c} h_{m,k} : Z_m \times \Delta^m_k \times X_k \longrightarrow Y_k \\ + \cdots \end{array} \right\} \\
\cong \left\{ \begin{array}{c} h_k : Z_k \times X_k \longrightarrow Y_k \\ + \cdots \end{array} \right\} \\
\cong \text{Hom}_{sSet}(Z \times X, Y)
\end{bmatrix}
$$

Cor. $\text{Hom}(Z \times X, Y) \cong \text{Hom}(Z, \text{Hom}(X,Y))$

e.g. $\text{Hom}(\Delta^0, Y) \cong Y$ $\qquad\qquad (\text{Hom}(\Delta^n, Y))_m \cong (\text{Hom}(\Delta^m, Y))_n$
$\qquad\;\; \text{Hom}(X, \Delta^0) \cong \Delta^0$ $\qquad\quad$ e.p. $(\text{Hom}(\Delta^n, Y))_0 \cong Y_n$

$$|\text{Hom}(X,Y)| = \left(\coprod_k \text{Hom}_{sSet}(\Delta^k \times X, Y) \times \nabla^k \right) / \sim$$

$$= \;?$$

Remaining: Compute $\#(\text{Hom}(\Delta^n, \Delta^m))_k$
Compute $(\text{Hom}(\Delta^n/\partial\Delta^n, Y))_k$. How is it related to Y_{k+n} or $\pi_n(|Y|)$?
How to see the geometrical realization of $\text{Hom}(X,Y)$,
$\qquad\qquad$ e.p. in these examples?

Eg. 10. Let X be a subset of Δ whose realization is as follows. Write down X_k for $k \leq 3$.

e.g. $X_1 = \left\{ \begin{array}{l} [2,0,0,0], [0,2,0,0], [0,0,2,0], [0,0,0,2], \\ [1,1,0,0], [1,0,1,0], [1,0,0,1], [0,1,0,1], [0,0,1,1] \end{array} \right\}$

Eg. 11. BG

Eg. 12.