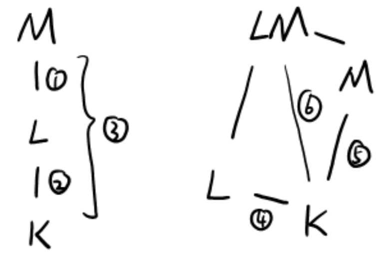


① $\xrightarrow{\text{③ pure ins}}$ ③

normal: ③ ⇒ ① ③ ⇏ ② ① + ② ⇏ ③ ⑥ ⇏ ④ ④ + ⑤ ⇒ ⑥

\+

seperable: ① + ② = ③ ④ + ⑤ = ⑥

\=

Galois: ③ ⇒ ① ③ ⇏ ② ① + ② ⇏ ③ ⑥ ⇏ ④ ④ + ⑤ ⇒ ⑥

purely inseparable ① + ② = ③ ④ + ⑤ = ⑥

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then

F perfect $\Leftrightarrow F^p = F$

$\overline{K}$		$\overline{\mathbb{F}_p}$	$\overline{\mathbb{F}_p}$	$\overline{\mathbb{F}_p}$
|	closed subgroup	| $\mathbb{Z}_l$	| $\prod_{p \nmid l} \mathbb{Z}_p$	| $d\hat{\mathbb{Z}}$
L		$\bigcup_{l^a} \mathbb{F}_{p^{l^a}}$	$\bigcup_n \mathbb{F}_{p^{l^n}}$	$\mathbb{F}_q$
(finite) |	quotient group	| $\prod_{p \nmid l} \mathbb{Z}_p$	| $\mathbb{Z}_l$	| $\mathbb{Z}/d\mathbb{Z}$
K		$\mathbb{F}_p$	$\mathbb{F}_p$	$\mathbb{F}_p$

$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ $(q = p^d)$

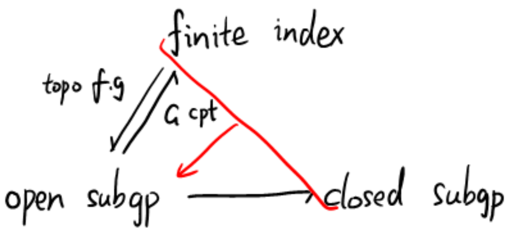
open subgroup $\overset{\{1\} \subseteq \mathbb{Z}_p}{\subsetneq}$ closed subgroup $= \{Gal(\overline{K}/L) \mid L/k \text{ ext}\} \overset{\mathbb{Z} \subseteq \mathbb{Z}_p}{\subsetneq}$ subgroup

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Q: Do we have any finite index gp of $Gal(\overline{k}/k)$ which is not open?

In general,



https://groupprops.subwiki.org/wiki/Closed_subgroup_of_finite_index_implies_open
In a topological group, any closed subgroup of finite index must be an open subgroup.
https://groupprops.subwiki.org/wiki/Open_subgroup_implies_closed
Any open subgroup of a topological group is closed.

https://math.stackexchange.com/questions/4350043/open-subgroup-and-finite-index-subgroup-of-topological-group
https://math.stackexchange.com/questions/1092974/finite-index-subgroup-g-of-mathbbz-p-is-open

Some wonderful exercises for Galois correspondence:

Let E/F be Galois field ext of deg n, $m \mid n$. prove: $\exists$ subfield ext of deg m.

(Sylow thm & $Z(G) \neq \{1\}$ for a p-gp & classification of f.g. abelian gp)

Cor. For p prime, F field, one can define ${}^{p}\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^{p}\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}} \cap \mathbb{R}$)?