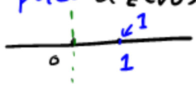
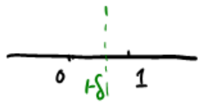
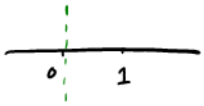
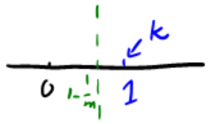
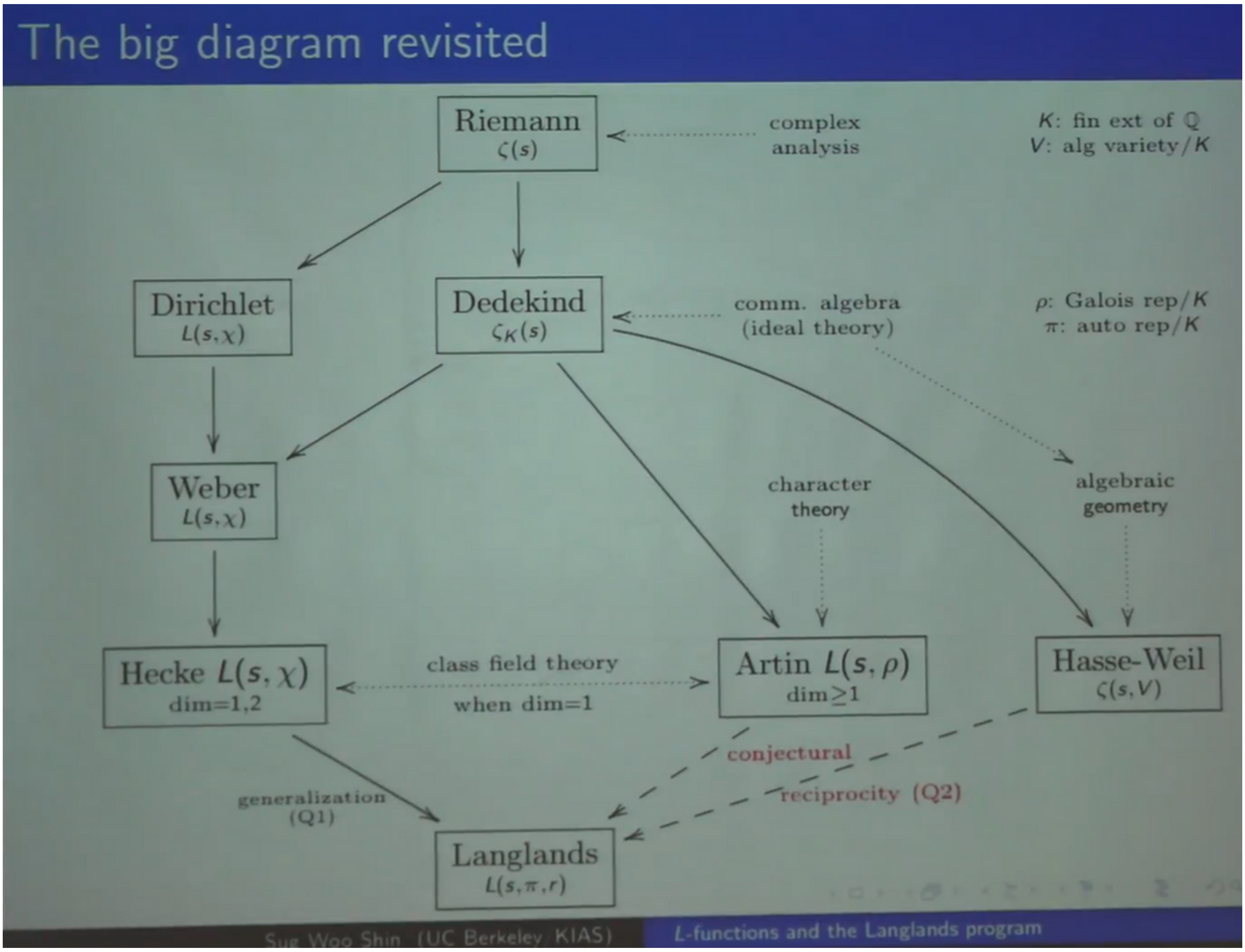
$$\begin{tikzcd} \zeta_K(s) \arrow[r, squiggly] \arrow[d, dash] \arrow[dr, "\text{dec}"] & \text{Hecke} \arrow[d, dash] \\ \zeta(s) \arrow[r, squiggly] & \text{Dirichlet} \end{tikzcd}$$

A partial conclusion on L-functions

	domain (at least) poles & zeros	equations	Conj
$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1-p^{-s}}$			RH
$f(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ where $S_t := O(t^{1-\delta})$ & basic convergence			
Dirichlet L $L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}}$ $\chi: (\mathbb{Z}/n\mathbb{Z})^\times \to S^1$			GRH
Hecke L $L(s,\chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}$.			
Artin L $L(s,\rho)$			
自守 L			
$[K:\mathbb{Q}]=m$ $\zeta_K(s) := \sum_{I \subseteq \mathcal{O}_K} N(I)^{-s} = \prod_{\mathfrak{p}\text{ prime}} \frac{1}{1-N(\mathfrak{p})^{-s}}$ $= \sum_{n=1}^{\infty} \frac{\#\{I \mid N(I)=n\}}{n^s}$			ERH

Hasse-Weil $\zeta(X,s) := \exp\left(\sum_{m\geq 1} \frac{\#X(\mathbb{F}_{q^m})}{m} q^{-ms}\right)$

$X/\mathbb{Z}$ finite type $\zeta_X(s) := \prod_{x\in|X|} \frac{1}{1-N(x)^{-s}}$



A pretty nice picture copied from: https://www.youtube.com/watch?v=oaNFaOiUEr8

https://math.stackexchange.com/questions/2501635/show-that-zeta-k2-frac-pi448-sqrt2-with-k-mathbbq-sqrt?rq=1

$$\zeta_K(s) = \prod_{\chi \in \widehat{G}} L(\chi, s) \quad \Rightarrow \quad \kappa = \prod_{\substack{\chi \in \widehat{G} \\ \chi \neq \chi_0}} L(\chi, 1)$$

e.g. $K = \mathbb{Q}(\zeta_{12})$ $\kappa \approx 0.3610515$

	0	1	2	3	4	5	6	7	8	9	10	11
$\chi_1 = \mathrm{Id}$	1	1	1	1	1	1	1	1	1	1	1	1
χ_2		1	-1		1	-1		1	-1		1	-1
χ_3		1		-1		1		-1		1		-1
χ_4		1				-1		-1				1

$$\kappa = \left(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \cdots\right)\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)\left(1 - \frac{1}{5} - \frac{1}{7} + \frac{1}{11} + \cdots\right)$$

A Dirichlet character χ is called *odd* if $\chi(-1) = 1$ and *even* if $\chi(-1) = 1$. If χ is a Dirichlet character modulo m and $m|m'$, then χ can be lifted to a Dirichlet character modulo m' by pulling back using the projection. A Dirichlet character χ is called *primitive* if it cannot be lifted from Dirichlet character character of smaller modulus. Let

$$a = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Let χ be a primitive character modulo m. Let

$$\Lambda(s, \chi) = (\pi/m)^{-(s+a)/2} \Gamma(\frac{s+a}{2}) L(s, \chi).$$

The Dirichlet L-function satisfies the following functional equation:

$$\Lambda(1 - s, \overline{\chi}) = \frac{i^a k^{1/2}}{\tau(\chi)} \Lambda(s, \chi),$$

where

$$\tau(\chi) = \sum_{n=1}^{m} \chi(n) e^{2\pi i n/m}.$$