Eine Woche, ein Beispiel
12.21. Hodge structure

Ref:

[Car17]: James Carlson, Stefan Müller-Stach, and Chris Peters. Period Mappings and Period Domains. 2nd ed. Cambridge University Press, 2017. https://doi.org/10.1017/9781316995846.

X: cpt Kähler mfld of dim n.

Slogan: Hodge structure collects (nearly) all linear algebraic structures of X.

These linear algebraic structures are: $k = p+q$

① $H^k(X; \mathbb{Z})$

② $H^q(X; \Omega_X^p) \cong \{$harmonic (p,q)-forms$\}$

③ A comparison iso as $\mathbb{C}$-v.s.   (J-action)
$$H^k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{p+q=k} H^q(X; \Omega_X^p)$$

②′ A Hodge filtration

$$H^k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^k(\Omega_X^{\cdot}) \supset H^{k-1}(\Omega_X^{\geq 1}) \supset H^{k-2}(\Omega_X^{\geq 2}) \supset \cdots \supset H^0(\Omega_X^{\geq k}) \supset 0$$

$ H^{0,k} H^{1,k-1} H^{2,k-2} H^{k-1,1} H^{k,0}$

③′ Compatability with conj$_J$:

$$H^k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{k-p}(\Omega_X^{\geq p}) \oplus \overline{H^{p-1}(\Omega_X^{\geq k-p+1})}$$

Here, $\Omega_X^{\geq p} := \sigma_{\geq p} \Omega_X^{\cdot}$. We have