Eine Woche, ein Beispiel
10.30 equivariant cohomology of $\mathbb{P}^1$

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
Other references will be add soon.

1. notations and warnings
2. result
3. computation of completion in practice
4. pt & $\mathbb{P}^1$
5 Euler class

1. notations and warnings

In this document,

$$GL_2 = GL_2(\mathbb{C}) \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset GL_2 \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2 \text{ or } SL_2$$

$$SL_2 = SL_2(\mathbb{C}) \qquad \mathbb{C}^\times = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset SL_2 \qquad \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$$

$$K_0^G(X) := K_0(\mathrm{Coh}^G(X)) \qquad\qquad H_G^*(X;\mathbb{Q}) := H^*(EG \times^G X; \mathbb{Q})$$

$$R(G) := K_0^G(pt) = \mathrm{Rep}(G) \qquad\qquad S(G) := H_G^*(pt;\mathbb{Q}) = H^*(BG;\mathbb{Q})$$

$$K_0^G(X)_I^\wedge := \varprojlim_n K_0^G(X)/I^n \qquad\qquad HP_G^0(X;\mathbb{Q}) := \prod_{n=0}^{+\infty} H_G^n(X;\mathbb{Q}) = H_G^*(X;\mathbb{Q})_I^\wedge$$

To avoid confusion, we don't consider any convolution structure in this document.
we don't consider $G \times \mathbb{C}^\times$-action either

($\mathbb{C}^\times$ is already occupied as a maximal torus of SL_2)

2. result

This time we are not so ambitious. For example, we don't fill in

$$K_0^{\mathcal{B}}(\mathcal{B} \times \mathcal{B}) \cong K_0^G(\mathcal{B} \times \mathcal{B} \times \mathcal{B}) \cong R(T) \otimes_{R(G)} R(T) \otimes_{R(G)} R(T)$$

just because the result is too long.

$K_0^-(-)$		pt	$\mathcal{B} \quad T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$
$G = SL_2$	SL_2	$\mathbb{Z}[y+y^{-1}]$	$\mathbb{Z}[z^{\pm 1}]$	$\mathbb{Z}[z_1^{\pm 1}, z_2]/((z_1-z_2)(z_1-z_2^{-1}))$
	B	$\mathbb{Z}[y^{\pm 1}]$	$\mathbb{Z}[y^{\pm 1}, z]/(z-y)(z-y^{-1})$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$
$G = GL_2$	GL_2	$\mathbb{Z}[y_1+y_2, y_1 y_2, \frac{1}{y_1 y_2}]$	$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}]$	$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}, z_1']/((z_1'-z_1)(z_1'-z_2))$
	B	$\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}]$	$\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}, z]/((z-y_1)(z-y_2))$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1', z_2']/((z_1'-1)^2, (z_2'-1)^2)$
$G = SL_n$ or GL_n	G	$R(G)$	$R(T)$	$R(T) \otimes_{R(G)} R(T)$ $\bigoplus_{w' \in W} R(G)[\overline{\Omega}_{w'}]^G$
	B	$R(T)$	$R(T) \otimes_{R(G)} R(T)$ $\bigoplus_{w \in W} R(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w' \in W} R(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Z}$	$\bigoplus_{w \in W} \mathbb{Z} \cdot [\overline{\Omega}_w]$	$\bigoplus_{w,w' \in W} \mathbb{Z}[\overline{\Omega}_{w,w'}]$

$H_-^*(-;\mathbb{Q})$		pt	$\mathcal{B} \quad T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$
$G = SL_2$	SL_2	$\mathbb{Q}[b^2]$	$\mathbb{Q}[e]$	$\mathbb{Q}[e_1, e_2]/(e_2^2 - e_1^2)$
	B	$\mathbb{Q}[b]$	$\mathbb{Q}[b,e]/(e^2-b^2)$	
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1,e_2]/(e_1^2, e_2^2)$
$G = GL_2$	GL_2	$\mathbb{Q}[b_1+b_2, b_1 b_2]$	$\mathbb{Q}[e_1, e_2]$	$\mathbb{Q}[e_1, e_2, e_1']/((e_1'-e_1)(e_1'-e_2))$
	B	$\mathbb{Q}[b_1, b_2]$	$\mathbb{Q}[b_1, b_2, e]/((e-b_1)(e-b_2))$	$e_2' = e_1 + e_2 - e_1'$
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1', e_2']/(e_1'^2, e_2'^2)$
$G = SL_n$ or GL_n	G	$S(G)$	$S(T)$	$S(T) \otimes_{S(G)} S(T)$ $\bigoplus_{w' \in W} S(G)[\overline{\Omega}_{w'}]^G$
	B	$S(T)$	$S(T) \otimes_{S(G)} S(T)$ $\bigoplus_{w \in W} S(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w' \in W} S(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Q}$	$\bigoplus_{w \in W} \mathbb{Q}[\overline{\Omega}_w]$	$\bigoplus_{w,w' \in W} \mathbb{Q}[\overline{\Omega}_{w,w'}]$

3. computation of completion in practice

Thm (cpl of Noetherian ring by power series)

R. Noetherian $\quad I := (a_1, \ldots, a_n) \lhd R$, then

$$\hat{R_I} := \varprojlim_n R/I^n$$

$$\cong R[[x_1, \ldots, x_n]]/(x_1 - a_1, \ldots, x_n - a_n)$$

$$\cong R[[a_1, \ldots, a_n]]$$
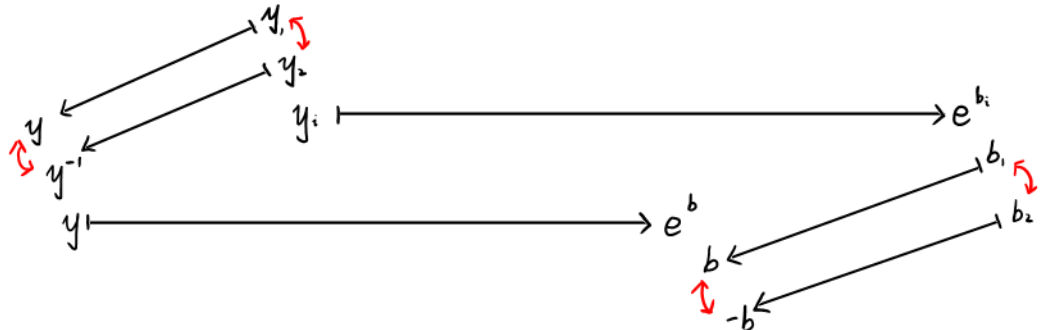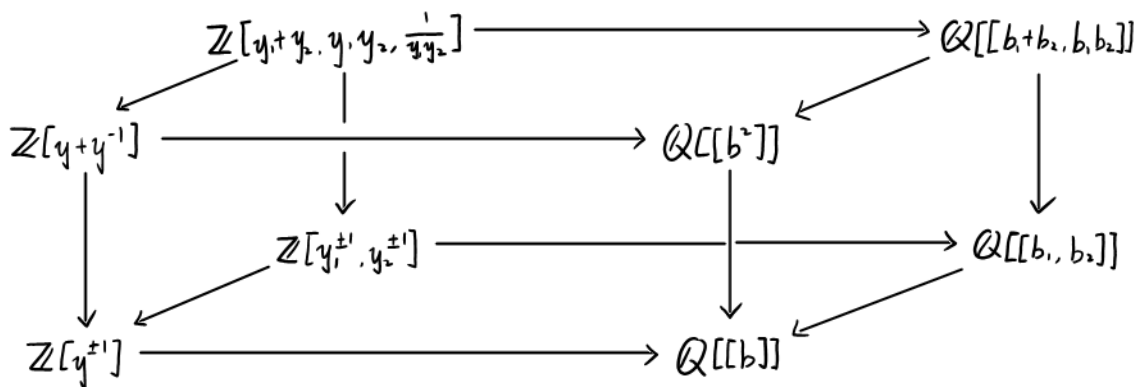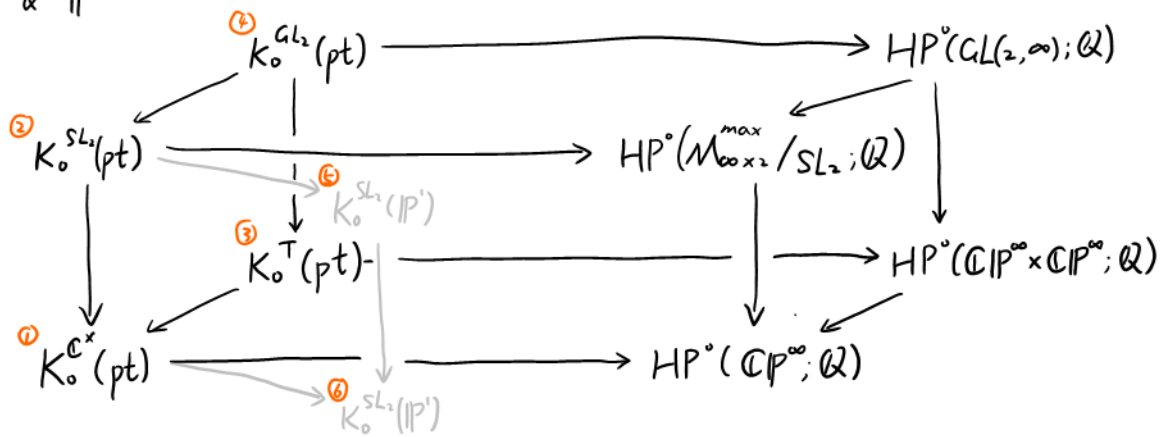
Ex. $\hat{\mathbb{Z}[x]}_{(x)} \cong \mathbb{Z}[[x]]$

$$\hat{\mathbb{Z}}_{(p)} \cong \mathbb{Z}[[x]]/(x-p) \xrightarrow{\sim} \mathbb{Z}_p$$

$$x \longmapsto p$$

$$\hat{\mathbb{Z}}_{(p^2)} \cong \mathbb{Z}_p$$

$$\hat{\mathbb{Z}}_{(n)} \cong \prod_{\substack{p \mid n \\ prime}} \mathbb{Z}_p$$

4. pt & $\mathbb{P}^1$

First diagram (K-theory and HP):

④ $K_0^{GL_2}(pt) \longrightarrow HP^0(GL(2,\infty);\mathbb{Q})$

② $K_0^{SL_2}(pt) \longrightarrow HP^0(\mathcal{M}_{\infty\times2}^{max}/SL_2;\mathbb{Q})$

⑤ $K_0^{SL_2}(\mathbb{P}^1)$

③ $K_0^{T}(pt) \longrightarrow HP^0(\mathbb{CP}^\infty\times\mathbb{CP}^\infty;\mathbb{Q})$

① $K_0^{\mathbb{C}^\times}(pt) \longrightarrow HP^0(\mathbb{CP}^\infty;\mathbb{Q})$

⑥ $K_0^{SL_2}(\mathbb{P}^1)$

Second diagram:

$\mathbb{Z}[y_1+y_2, y_1\cdot y_2, \frac{1}{y_1 y_2}] \longrightarrow \mathbb{Q}[[b_1+b_2, b_1 b_2]]$

$\mathbb{Z}[y+y^{-1}] \longrightarrow \mathbb{Q}[[b^2]]$

$\mathbb{Z}[y_1^{\pm1}, y_2^{\pm1}] \longrightarrow \mathbb{Q}[[b_1, b_2]]$

$\mathbb{Z}[y^{\pm1}] \longrightarrow \mathbb{Q}[[b]]$

Third diagram (element maps):

$y_1, \circlearrowright$
y_2
$y_i \longmapsto e^{b_i}$
y
y^{-1}
$y \longmapsto e^b$
$b_1, \circlearrowright$
b_2
b
$-b$

$\leftrightarrow$: Weyl group action

Later, $c_i = c_i^G$ is a temporate notation.

ch* is iso after <u>tensored over $\mathbb{Q}$</u>.

$$(ch^*)^{-1}: HP^0(BG;\mathbb{Q}) \xrightarrow{\sim} K_0(BG)\otimes_{\mathbb{Z}}\mathbb{Q}$$

$$HP_G^0(X;\mathbb{Q}) \xrightarrow{\sim} K_0^G(X)\otimes_{\mathbb{Z}}\mathbb{Q}$$

When I write the inverse map $(ch^*)^{-1}$, remember that the image usually has coefficient in $\mathbb{Q}$.

① $K_0^{\mathbb{C}^\times}(pt) \xrightarrow{\;cpl\;} K_0^{\mathbb{C}^\times}(pt)_I^\wedge \xrightarrow{\;AS\ map\;} K_0(B\mathbb{C}^\times) \xrightarrow{\;ch^*\;} HP^\circ(B\mathbb{C}^\times;\mathbb{Q}) \overset{cpl}{\supseteq} H^*(B\mathbb{C}^\times;\mathbb{Q})$

$\mathbb{Z}[y^{\pm 1}] \longrightarrow \mathbb{Z}[[y-1]] \longrightarrow \mathbb{Z}[[c_\cdot^{\mathbb{C}^\times}]] \longrightarrow \mathbb{Q}[[b]] \supset \mathbb{Q}[b]$

$$c_\cdot^{\mathbb{C}^\times} \longmapsto e^b - 1$$

$$\mathbb{Q}[[c_\cdot^{\mathbb{C}}]] \ni \log(1 + c_\cdot^{\mathbb{C}^\times}) \longleftarrow\!\shortmid \quad b$$

② $K_0^{SL_2}(pt) \xrightarrow{\;cpl\;} K_0^{SL_2}(pt)_I^\wedge \xrightarrow{\;AS\ map\;} K_0(BSL_2) \xrightarrow{\;ch^*\;} HP^\circ(BSL_2;\mathbb{Q}) \overset{cpl}{\supseteq} H^*(BSL_2;\mathbb{Q})$

$\mathbb{Z}[y+y^{-1}] \longrightarrow \mathbb{Z}[[y+y^{-1}-2]] \longrightarrow \mathbb{Z}[[c_\cdot^{SL_2}]] \longrightarrow \mathbb{Q}[[b^2]] \supset \mathbb{Q}[b^2]$

$$c_\cdot^{SL_2} \longmapsto e^b + e^{-b} - 1 = 4\sinh^2\frac{b}{2}$$
$$= 2\cosh b - 2$$

$$4\left(\operatorname{arcsinh}\frac{\sqrt{c_\cdot}}{2}\right)^2 \longleftarrow\!\shortmid \quad b^2$$
$$= 4\left(\ln\left(\frac{\sqrt{c_\cdot}}{2} + \sqrt{\frac{c_\cdot}{4}+1}\right)\right)^2$$
$$= \left(\ln\left(1 + \frac{c_\cdot}{2} + \sqrt{\frac{c_\cdot^2}{4}+c_\cdot}\right)\right)^2$$

③ $K_0^{T}(pt) \xrightarrow{\;cpl\;} K_0^{T}(pt)_I^\wedge \xrightarrow{\;AS\ map\;} K_0(BT) \xrightarrow{\;ch^*\;} HP^\circ(BT;\mathbb{Q}) \overset{cpl}{\supseteq} H^*(BT;\mathbb{Q})$

$\mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}] \longrightarrow \mathbb{Z}[[y_1-1, y_2-1]] \longrightarrow \mathbb{Z}[[c_\cdot^T, c_\cdot^T]] \longrightarrow \mathbb{Q}[[b_\cdot, b_\cdot]] \supset \mathbb{Q}[b_1, b_2]$

$$c_i^{\mathbb{C}^\times} \longmapsto e^{b_i} - 1$$

$$\log(1 + c_i^{\mathbb{C}^\times}) \longleftarrow\!\shortmid \quad b_i$$

④ $K_0^{GL_2}(pt) \xrightarrow{\;cpl\;} K_0^{GL_2}(pt)_I^\wedge \xrightarrow{\;AS\ map\;} K_0(BGL_2) \xrightarrow{\;ch^*\;} HP^\circ(BGL_2;\mathbb{Q}) \overset{cpl}{\supseteq} H^*(BGL_2;\mathbb{Q})$

$\mathbb{Z}[y_1+y_2, y_1 y_2, \frac{1}{y_1 y_2}] \longrightarrow \mathbb{Z}[[y_1+y_2-2, y_1 y_2-1]] \longrightarrow \mathbb{Z}[[c_1^{GL_2}, c_2^{GL_2}]] \longrightarrow \mathbb{Q}[[b_1+b_2, b_1 b_2]] \supset \mathbb{Q}[b_1+b_2, b_1 b_2]$

$$c_1^{GL_2} \longmapsto e^{b_1} + e^{b_2} - 1$$
$$c_2^{GL_2} \longmapsto e^{b_1+b_2} - 1$$

$$\log(1 + c_2^{GL_2}) \longleftarrow\!\shortmid \quad b_1 + b_2$$

$$\log(1+y_1-1)\log(1+y_2-1) \longleftarrow\!\shortmid \quad b_1 b_2$$
$$= \sum_{k=2}^{+\infty} \sum_{\substack{n+m=k \\ n,m\geq 1}} \frac{(-1)^k}{nm}(y_1-1)^n(y_2-1)^m$$
$$= \cdots$$

To facilitate the computation, use the notation
$$c_3^{GL_2} = (y_1-1)(y_2-1)$$
$$= (y_1 y_2 - 1) - (y_1+y_2-2)$$
$$= c_2^{GL_2} - c_1^{GL_2}$$

⑤

$$K_0^{SL_2}(\mathbb{P}') \xrightarrow{\ cpl\ } K_0^{SL_2}(\mathbb{P}')_I^{\wedge} \xrightarrow{\ AS\ } K_0(ESL_2 \times^{SL_2} \mathbb{P}') \xrightarrow{\ ch^*\ } HP_{SL_2}^{\circ}(\mathbb{P}';\mathbb{Q}) \supseteq H_{SL_2}^*(\mathbb{P}';\mathbb{Q})$$

$$\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[[z-1]] \longrightarrow \mathbb{Z}[[c_1]] \longrightarrow \mathbb{Q}[[e]] \supset \mathbb{Q}[e]$$

$$c_1 \longmapsto e^e - 1$$

$$\log(1 + c_1) \longleftarrow\!\shortmid\ e$$

⑥

$$K_0^{\mathbb{C}^\times}(\mathbb{P}') \xrightarrow{\ cpl\ } K_0^{\mathbb{C}^\times}(\mathbb{P}')_I^{\wedge} \xrightarrow{\ AS\ } K_0(E\mathbb{C}^\times \times^{\mathbb{C}^\times} \mathbb{P}') \xrightarrow{\ ch^*\ } HP_{\mathbb{C}^\times}^{\circ}(\mathbb{P}';\mathbb{Q}) \supseteq H_{\mathbb{C}^\times}^*(\mathbb{P}';\mathbb{Q})$$

$$\mathbb{Z}[y^{\pm 1}, z]/(z-y)(z-y^{-1}) \to \mathbb{Z}[[y-1, z-1]]/\ldots \to \mathbb{Z}[[c_1, c_2]]/(c_2-c_1)(c_1c_2+c_1+c_2) \longrightarrow \mathbb{Q}[[b,e]]/(e^2-b^2) \supset \mathbb{Q}[b,e]/(e^2-b^2)$$

$$c_1 \longmapsto e^b - 1$$

$$c_2 \longmapsto e^e - 1$$

$$\log(1 + c_1) \longleftarrow\!\shortmid\ b$$

$$\log(1 + c_2) \longleftarrow\!\shortmid\ e$$

5. Euler class

At first glance, chern class seems to be an exponential map.
    Actually, chern class induces ring isomorphism $(+ \to +, \times \to \times)$

At first glance, Euler class seems to be a termwise -log map. $(\times \to +)$
    Actually,
        in one monomial $\qquad\qquad 1 + eu(\mathcal{L}_1 \otimes \mathcal{L}_2) = (1 + eu\,\mathcal{L}_1)(1 + eu\,\mathcal{L}_2)$ $\quad x \to (1 + \cdot)^x$
        for sum among monomials, $\qquad eu(E_1 \oplus E_2) = eu(E_1)\,eu(E_2)$ $\qquad + \to \times$

Let us see some examples of Euler class.

E.g.

$$K_0^{Gl_n}(\mathcal{B}) \longrightarrow HP_{GL_n}^{\circ}(\mathcal{B};\mathbb{Q}) \supset H_{GL_n}^*(\mathcal{B};\mathbb{Q})$$
$$\mathbb{Z}[y_1^{\pm 1}, \ldots, y_n^{\pm 1}] \longrightarrow \mathbb{Q}[[b_1, \ldots, b_n]] \supset \mathbb{Q}[b_1, \ldots, b_n]$$

$$y_i \longmapsto e^{b_i} \quad \overset{\log(\ )^{-1}}{\dashleftarrow}$$
$$\downarrow$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow -b_i$$

$$y_i^{-1}(1 - y_i) \quad \text{or} \quad 1 - y_i$$
$$\log y_i^{-1} = \log(1 + (y_i^{-1} - 1)) \approx y_i^{-1} - 1 \quad \text{or} \quad -\log y_i = -\log(1 + (y_i - 1)) \approx 1 - y_i$$
$$\approx y_i^{-1}(1 - y_i)$$

$$\prod_i y_i^{k_i} \longmapsto e^{\sum_i k_i b_i}$$
$$\downarrow$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow -\sum_i k_i b_i$$

$$\left(\prod_i y_i^{-k_i}\right)\left(1 - \left(\prod_i y_i^{k_i}\right)\right) \quad \text{or} \quad 1 - \left(\prod_i y_i^{k_i}\right)$$

$$\frac{y_2}{y_1} + \frac{y_3}{y_1} + \frac{y_3}{y_2} \longmapsto e^{b_2 - b_1} + e^{b_3 - b_1} + e^{b_3 - b_2}$$
$$\downarrow$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \longrightarrow$$

$$\left(1 - \frac{y_2}{y_1}\right)\left(1 - \frac{y_3}{y_1}\right)\left(1 - \frac{y_3}{y_2}\right) \qquad\qquad (b_1 - b_2)(b_1 - b_3)(b_2 - b_3)$$

Q: What is right definition of $eu(T)$?

$$eu(T) = \sum_{i=0}^{+\infty} (-1)^i [\Lambda^i T^*]$$

$$eu\left(\frac{y_2}{y_1}\right) = 1 - \frac{y_1}{y_2}$$

or

$$eu(T) = \sum_{i=0}^{+\infty} (-1)^{i+1} [\Lambda^i T] \ ?$$

$$eu\left(\frac{y_2}{y_1}\right) = \frac{y_2}{y_1} - 1 = \frac{y_2}{y_1}\left(1 - \frac{y_1}{y_2}\right)$$

compatible with Euler characteristic.

$$e(X) = \sum_{i=0}^{+\infty} (-1)^i H^i(X; \mathbb{Q})$$

will induce

$$D_i f = sf \, D_i + \frac{f - sf}{1 - \frac{e_i}{e_{i+1}}}$$

will induce

$$D_i f = sf \, D_i - \frac{f - sf}{1 - \frac{e_{i+1}}{e_i}}$$

$$\left(\frac{e_i}{e_{i+1}} D_i\right) f = sf \left(\frac{e_i}{e_{i+1}} D_i\right) - \frac{f - sf}{1 - \frac{e_{i+1}}{e_i}}$$

$$D_i\left(\frac{e_i}{e_{i+1}} fg\right) = sf \, D_i\left(\frac{e_i}{e_{i+1}} g\right) - \frac{f - sf}{1 - \frac{e_{i+1}}{e_i}} g$$

In 22.13, Another definition is mentioned:
https://pages.uoregon.edu/ddugger/kgeom.pdf
p75:
https://www.sciencedirect.com/science/article/pii/0022404994900884
It's also the definition in [Ginzburg, Cor 5.11.3]

reasons for each possibility

1.15: https://arxiv.org/pdf/math/0309168.pdf
p50: https://link.springer.com/content/pdf/10.1007/b10326.pdf
p93: http://sporadic.stanford.edu/bump/math263/hecke.pdf
p3: https://arxiv.org/pdf/math/0405333.pdf is not correct