Eine Woche, ein Beispiel
5.28. dual spaces of ∞-dim v.s.

Ref: http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-FA/index.html

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What would happen if $\mathbb{F} = \mathbb{C}_p$?

1. def
2. examples

1. def

Def. For any topo v.s. X, Y, define
$$\mathcal{L}(X,Y) := \{L: X \longrightarrow Y \mid L \text{ is linear and cont}\}$$
The dual space of X is defined as
$$X' := \mathcal{L}(X, \mathbb{F}) = \{L: X \longrightarrow \mathbb{F} \mid L \text{ is linear and cont}\}$$

⚠ We follow the notation of analysis in this document.
Other possibilities for the dual space: $X^*, X^\vee, \check{X}, \dots$

Rmk. When X, Y are normed v.s., $\mathcal{L}(X,Y)$ is a normed v.s. with
$$\|L\| = \sup_{\|x\|_X = 1} \|L(x)\|_Y$$

On the other hand, we have the weak $*$-topology on $\mathcal{L}(X,Y)$: the weakest topo s.t.
$$ev_x: \mathcal{L}(X,Y) \longrightarrow Y \qquad L \longmapsto L(x)$$
is cont for any $x \in X$.

These two structures are not compatible with each other.

Rmk. By Klein-Milman theorem, we can show that some Banach spaces are not dual space.

2. initial examples.

For a bounded domain Ω, we have

$$\begin{array}{c} L^\infty(\Omega) \subset \cdots \subset L^p(\Omega) \subset \cdots \subset L^1(\Omega) \\ \Downarrow \text{dual} \\ (L^\infty(\Omega))' \supset \cdots \supset L^q(\Omega) \supset \cdots \supset L^\infty(\Omega) \end{array}$$

For arbitrary domain Ω, we don't have inclusion.

inclusion: cont inj map

https://math.stackexchange.com/questions/405357/when-exactly-is-the-dual-of-l1-isomorphic-to-l-infty-via-the-natural-map
https://math.stackexchange.com/questions/137677/what-is-the-predual-of-l1

Ex. Show that $(c_0)' = l^1$, $(l^p)' = l^q$, $(l^1)' = l^\infty$ by direct argument.

Show that $(l^\infty)' \supsetneq l^1$.

$$\begin{array}{cccc} c_0 \overset{\text{not dense}}{\hookrightarrow} & l^\infty & l^p & l^1 \\ & & \Downarrow \text{dual} & \\ l^1 \twoheadleftarrow & (l^\infty)' & l^q & l^\infty \end{array}$$

For $\Omega = \mathbb{R}^n$, we have ($\mathcal{S}(\Omega)$ is not defined for $\Omega \overset{\text{open}}{\subset} \mathbb{R}^n$, traditionally)

$$\begin{array}{c} \mathcal{D}(\Omega) \subset \mathcal{S}(\Omega) \subset \mathcal{E}(\Omega) \\ \Downarrow \text{dual} \\ \mathcal{D}'(\Omega) \supset \mathcal{S}'(\Omega) \supset \mathcal{E}'(\Omega) \end{array}$$

https://math.stackexchange.com/questions/4730104/is-schwartz-space-canonical-in-any-sense
Schwartz Functions on Open Subsets of Rn: https://www.math.princeton.edu/events/schwartz-functions-open-subsets-rn-2022-02-28t213000
Schwartz functions on real algebraic varieties: https://arxiv.org/abs/1701.07334

Rmk. For Hilbert space, $H' \cong H$. e.p. $(H^s(\Omega))' \cong H^s(\Omega)$

For X: cpt Hausdorff space,

$$C(X)' \subset \{\text{signed regular Borel measures}\}$$

⚠ The following illusion is common and confusing:

The dual space of bigger space is bigger/smaller.

Actually, such illusions comes from $f^*: W^* \to V^*$ being injective/surjective.

In fin dim case, $\dim V^* = \dim V < \dim W = \dim W^*$;

In dense subspace case, it comes from the uniqueness of cont extension.

3. an overview of spaces in analysis

$1 < p < +\infty, \quad \frac{1}{p} + \frac{1}{q} = 1$

$\mathbb{1}_{\mathbb{R}}, e^{i\langle a, x\rangle}$

$$
\begin{array}{lccccc}
\text{distribution} & \mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
\text{measure} & C'(\mathbb{R}^n) & \subset & (C_0(\mathbb{R}^n) \cap L^p)' & \subset & C_c'(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
 & (L^q_{loc}(\mathbb{R}^n))' & \subsetneqq & (L^q(\mathbb{R}^n))' & \subsetneqq & (L^q_c(\mathbb{R}^n))' \\
 & \cup\, \text{bij} & & \cup\, \text{bij} & & \cup\, \text{bij} \\
 & L^p_c(\mathbb{R}^n) & \subsetneqq & L^p(\mathbb{R}^n) & \subsetneqq & L^p_{loc}(\mathbb{R}^n) \\
\text{functions} & \cup & & \cup & & \cup \\
 & C_c(\mathbb{R}^n) & \subset & C_0(\mathbb{R}^n) \cap L^p & \subset & C(\mathbb{R}^n) \\
 & \cup & & \cup & & \cup \\
 & \mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
\end{array}
$$

$\Sigma \delta_a$

δ_a

$\mathbb{1}_{[0,+\infty)}$

$e^{\varepsilon x} \mathbb{1}_{[0,+\infty)}$

for Fourier transform — sheaf

cpt supported — globally bd — locally bd

seminorm of the last line:

$\varinjlim_K \mathcal{D}_K(\mathbb{R}^n)$ — $p_{\alpha,K}(f) = \sup_{x \in K} \|\partial^\alpha f\|_\infty$

$p_{\alpha,\beta}(f) = \|x^\beta \partial^\alpha f\|_\infty$

$p_{\alpha,N}(f) = \|(1+|x|)^N \partial^\alpha f\|_\infty$

$p_{\alpha,K}(f) = \sup_{x \in K} \|\partial^\alpha f\|_\infty$

measure line: $C'(\mathbb{R}^n)$: fcts of bounded variation

$C_0'(\mathbb{R}^n)$: signed regular Borel measures on $\mathbb{R}^n$.

$(C_c'(\mathbb{R}^n))^+$: Radon measure

Q: is $C_c'(\mathbb{R}^n)$ the signed Radon measure?

https://math.stackexchange.com/questions/4448590/how-to-generalize-riesz-markov-kakutani-representation-theorem-from-c-cx-to
https://math.stackexchange.com/questions/4500358/3-versions-of-riesz-markov-kakutani-theorem

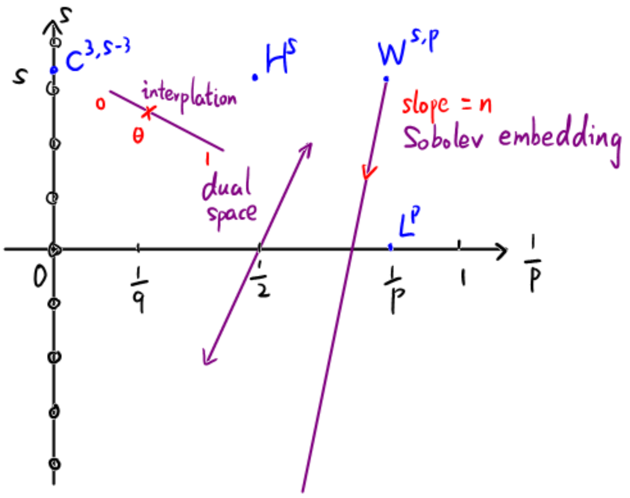
The above diagram has many variations. For example,

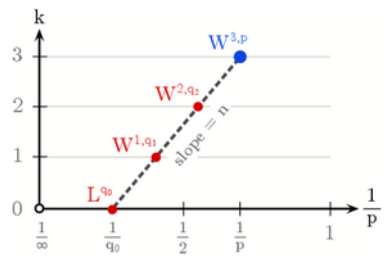
$$
\begin{array}{ccccc}
\mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
C'(\mathbb{R}^n) & \subset & C_0'(\mathbb{R}^n) & \subset & C_c'(\mathbb{R}^n) \\
\uparrow & & \uparrow & & \uparrow \\
(L^1_{loc}(\mathbb{R}^n))' & \subset & (L^1(\mathbb{R}^n))' & \subset & (L^1_c(\mathbb{R}^n))' \\
\uparrow \text{bij?} & & \cup \text{ bij} & & \uparrow \text{bij?} \\
L^\infty_c(\mathbb{R}^n) & \subset & L^\infty(\mathbb{R}^n) & \subset & L^\infty_{loc}(\mathbb{R}^n) \\
\uparrow & & \uparrow & & \uparrow \\
C_c(\mathbb{R}^n) & \subset & C_0(\mathbb{R}^n) & \subset & C(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
\mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{E}'(\mathbb{R}^n) & \subset & \mathcal{S}'(\mathbb{R}^n) & \subset & \mathcal{D}'(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
H^{-s}_{loc}(\mathbb{R}^n) & \subset & H^{-s}(\mathbb{R}^n) & \subset & H^{-s}_c(\mathbb{R}^n) \\
\cup \text{ bij?} & & \cup \text{ bij} & & \cup \text{ bij?} \\
H^s_c(\mathbb{R}^n) & \subset & H^s(\mathbb{R}^n) & \subset & H^s_{loc}(\mathbb{R}^n) \\
\cup & & \cup & & \cup \\
\mathcal{D}(\mathbb{R}^n) & \subset & \mathcal{S}(\mathbb{R}^n) & \subset & \mathcal{E}(\mathbb{R}^n)
\end{array}
$$

https://math.stackexchange.com/questions/221069/why-are-continuous-functions-not-dense-in-l-infty

In fact, in the middle, we can change various of Sobolev spaces.

https://en.wikipedia.org/wiki/Sobolev_inequality
https://en.wikipedia.org/wiki/Sobolev_space
https://arxiv.org/PS_cache/arxiv/pdf/1104/1104.4345v2.pdf





$C^{r,\alpha}(\mathbb{R}^n)$: Hölder spaces