

Modular form

1. origin of definition of modular form

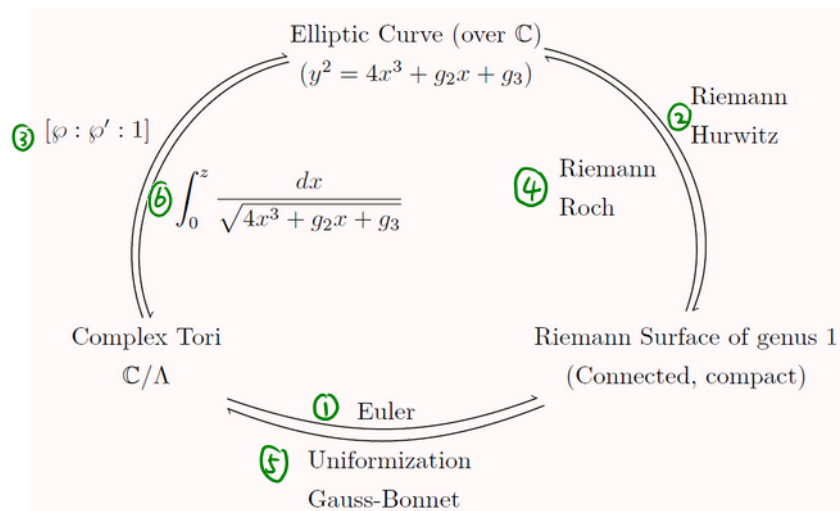
1. EC

2. moduli space (from cplx points of view)

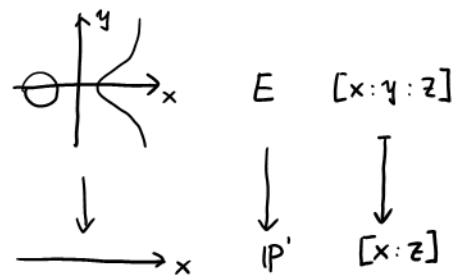
3. modular form

<https://www.math1.uni-heidelberg.de/~otmar/diplom/williams.pdf>

1. EC



② RH formula. $f: X \rightarrow Y$ $\chi = 2 - 2g$
 $\chi(X) = \chi(Y) \deg f - \sum_{x \in \text{Ran}(f)} (e(x) - 1)$



③④ $\iota: \mathbb{C}/\Lambda \longrightarrow \mathbb{P}^2$
 $z \longmapsto \begin{cases} [\wp(z): \wp'(z): 1] & z \neq 0 \\ [0: 1: 0] & z = 0 \end{cases}$

- well-define: \wp, \wp' + holomorphic
- equation (computation in [WWL, 命题 8.3.2])
- closed embedding [Vakil, 19.2.7, 19.2.10]

⑤ $\tilde{X} \quad \tilde{X} \text{ RS} + \Gamma \subset \text{Aut}_{\text{RS}, \text{tran}}(\tilde{X})$
 $\downarrow \Rightarrow \tilde{X} \cong \mathbb{CP}^1, \mathbb{C} \text{ or } \mathbb{H} \quad \Gamma \subset \text{Aut}_{\text{RS}, \text{tran}}(\tilde{X}) \subset \text{Isom}^+(\tilde{X})$
 $X \Rightarrow \text{Riemannian metric with constant curvature on } X$
 $\xrightarrow{\text{GB}} \tilde{X} \cong \mathbb{C}, X = \mathbb{C}/\Lambda$

⑥ $E \xrightarrow{\cong} H^0(\tilde{E}, \Omega_E)^\vee / \pi_1(E, e)$
 $z \longmapsto [w \longmapsto \int_{\gamma: 0 \rightarrow z} \omega]$

Idea: $\tilde{E} \xrightarrow{\sim} H^0(\tilde{E}, \Omega_E)^\vee$
 $\downarrow \quad \downarrow$
 $E \xrightarrow{\sim} H^0(E, \Omega_E)^\vee / \pi_1(E, e)$

- Ex. 1. Discuss \mathcal{O} . Discuss addition structure and their compatibilities.
 2. Some computations of $\mathcal{O}, \mathcal{O}'$.
 3. Describe rational fct field on EC.

2. moduli space (from cplx points of view)

Origin of $\mathcal{H}/SL_2(\mathbb{Z})$

Lemma. $\mathbb{C}/\Lambda \cong \mathbb{C}/\Lambda' \Leftrightarrow \Lambda' = z_0 \cdot \Lambda \quad \exists z_0 \in \mathbb{C}^\times$

Proof. [WWL, 命题 3.8.3, 练习 3.8.4]

Reduced to: Classify lattices (up to cplx scalar)

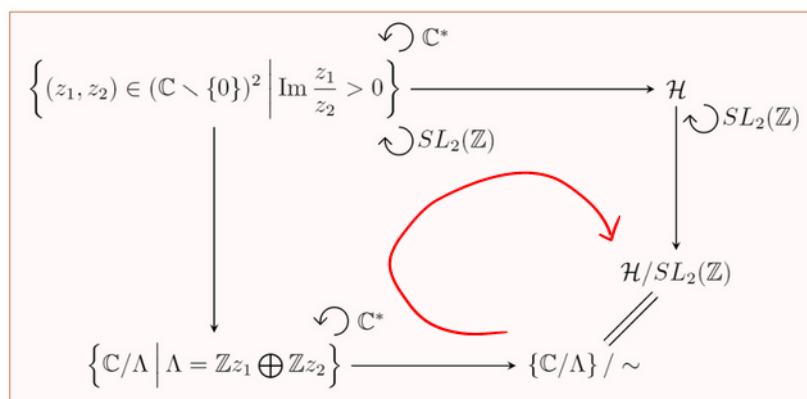
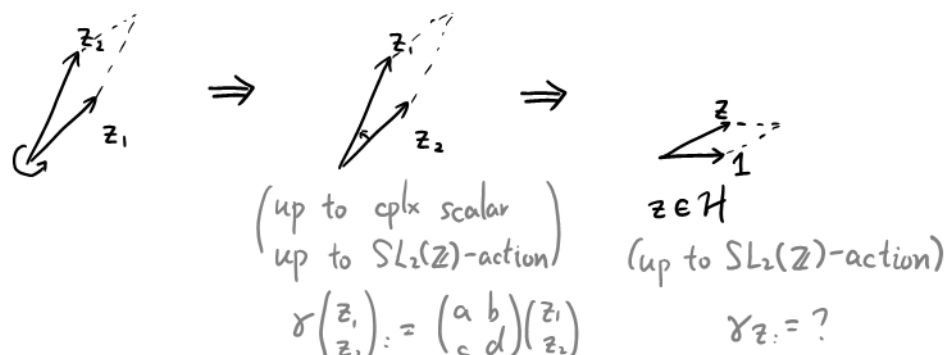



图 2.1 构造模空间/模形式的过程

Description of $\mathcal{H}/SL_2(\mathbb{Z})$

Ex. 1. Special items of $SL_2(\mathbb{Z})$ $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2. (difficult)  & $SL_2(\mathbb{Z}) = \langle T, S \rangle$ [Za, Prop 1] [JP, P.78, Thm 1.2]
 $= \langle T, S | S^2, (ST)^3 \rangle$ [NT II §9.6 p354]

3.  Describe glue, elliptic pts and cusp pt, volume
 \uparrow
the corresponding lattices

$$\begin{aligned} i: E_{\mathbb{Z}[i]}: y^2 &= 4x^3 + x & \phi(x, y) &= (-x, iy) \\ \rho: E_{\mathbb{Z}[\rho]}: y^2 &= 4x^3 - 1 & \phi(x, y) &= (\rho x, -y) \end{aligned}$$

<https://math.stackexchange.com/questions/2051526/eisenstein-series-for-hexagonal-lattice?rq=1>
<http://www.fen.bilkent.edu.tr/~franz/publ/wald.pdf>
<https://math.stackexchange.com/questions/4043509/how-can-i-calculate-the-eisenstein-series-of-a-complex-lattice>
<https://mathematica.stackexchange.com/questions/89430/eisenstein-series-in-mathematica>

1.1.2. (a) Show that $\text{Im}(\gamma(\tau)) = \text{Im}(\tau)/|c\tau + d|^2$ for all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

(b) Show that $(\gamma\gamma')(\tau) = \gamma(\gamma'(\tau))$ for all $\gamma, \gamma' \in SL_2(\mathbb{Z})$ and $\tau \in \mathcal{H}$.

(c) Show that $d\gamma(\tau)/d\tau = 1/(c\tau + d)^2$ for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$.

3. Modular form

Def. A holo fct $f: \mathcal{H} \rightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, level $\Gamma := SL_2(\mathbb{Z})$ if.

$$1) \quad f(\gamma\tau) = (c\tau + d)^k f(\tau) \quad \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

$$\text{e.p.} \quad f(\tau+1) = f(\tau)$$

$$2) \text{ Write } f(\tau) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n \tau}, \text{ then } a_n = 0 \text{ for } n < 0$$

By cplx analysis, this condition is equivalent to

$\exists C > 0$ s.t. $\{ |f(\tau)| \mid \text{Im}(\tau) > C \}$ is bounded.

$$\mathcal{M}_k(\Gamma) \supseteq \Sigma_k(\Gamma) \leftarrow \text{Cusp form} = \text{Spitzenform}$$

Ex. 1. View modular form as fcts on the space of lattices

2. Eisenstein fct.

练习 2.1.1. 对于 $k \in \mathbb{Z}, k \geq 2$, 定义 Eisenstein 函数

$$G_k(\tau) := \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m\tau + n)^k} = \frac{1}{2} \sum_{(m,n) \in \mathbb{Z}^2} \frac{1}{(m\tau + n)^k}$$

其中 \sum' 表示对 $(0,0)$ 以外的点求和, 可以验证

1. 级数在 \mathcal{H} 的紧子集上一致收敛, G_k 为 \mathcal{H} 上的全纯函数;

2. k 为奇数时, $G_k \equiv 0$;

3. k 为偶数时, G_k 满足 (2.1.1), 且有 Fourier 展开

$$G_k(\tau) = \frac{(2\pi i)^k}{(k-1)!} \left(-\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \right)$$

故 G_k 为权 k , 级 $SL_2(\mathbb{Z})$ 的模形式.

为方便起见, 取 $E_k := G_k/(2k)$ 使得 Fourier 常数项为 1. 可以证明, $M_k(SL_2(\mathbb{Z})) \cong \mathbb{C}[E_4, E_6]$, 且 E_4, E_6 代数无关.

We use $G_k(\Delta) := \sum' \frac{1}{z_0^k}$ instead
(In [Za] $G_k(\Delta) := \frac{1}{2} \sum' \frac{1}{z_0^k}$)

Next time

3. Δ and j

$$4. \mathcal{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_4, E_6]$$

This will be put to Lec 3.

① mero modular fct (of weight 0)
 ② almost holomorphic MF
 ④ smooth Automorphic form

⑥ Siegel MF
 ⑦ Hilbert MF

⑤ MF of congruence subgp
 ④ half-integral weight MF

Def. A holo fct $f: \mathcal{H} \rightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, level $\Gamma := SL_2(\mathbb{Z})$, if.

1) $f(\gamma\tau) = (c\tau + d)^k f(\tau)$

② quasimodular form
 $\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

e.p. $f(\tau+1) = f(\tau)$

2) Write $f(\tau) = \sum_{n \in \mathbb{Z}} a_n (e^{2\pi i \tau})^n$, then $a_n = 0$ for $n < 0$

③ p-adic MF

① meromorphic MF

① The order I plan to talk about

(For me they become more and more difficult)