Eine Woche, ein Beispiel
12.3. cheating sheet for six functors

$$\begin{array}{ccc} \mathcal{G} & \mathcal{F} \; \mathcal{F}' & \\ \vert & \diagdown \diagup & \\ Y & \xrightarrow{f} X & \end{array} \qquad\qquad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g'\downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

$$f^* \dashv f_*$$
$$- \otimes \mathcal{F} \dashv \underline{\mathrm{Hom}}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$$f^*(- \otimes -)$$
$$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$$
$$\color{red}{f_* \underline{\mathrm{Hom}}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{\mathrm{Hom}}(\mathcal{F}, f_*\mathcal{G})}$$

$$\otimes$$

proj formula
$$\color{blue}{f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_!\mathcal{G}}$$
$$\color{red}{f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{\mathrm{Hom}}(f_!\mathcal{G}, \mathcal{F})}$$
$$\color{gray}{f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') \cong \underline{\mathrm{Hom}}(f^*\mathcal{F}, f^!\mathcal{F}')}$$

$$f^* \underline{\qquad} f_!$$

$$\text{bc: } \color{blue}{f^*g_! \cong g'_! f'^{\,*}}$$
$$\color{gray}{g'^* f_* \cong f'_* g'^!}$$

$$\left\Vert \begin{array}{l} I: f^* = f^! \\ P: f_* = f_! \end{array}\right.$$

$$p: X \longrightarrow pt$$

$$H^\cdot(X; \underline{\mathbb{Z}}) := p_* p^* \mathbf{1} \qquad\qquad H^\cdot(X; \mathcal{F}) := p_* \mathcal{F} \qquad = R\Gamma(X; \mathcal{F})$$
$$H_c^\cdot(X; \underline{\mathbb{Z}}) := p_! p^* \mathbf{1} \qquad\qquad H_c^\cdot(X; \mathcal{F}) := p_! \mathcal{F}$$
$$H_\cdot(X; \underline{\mathbb{Z}}) := p_! p^! \mathbf{1} \qquad\qquad H_\cdot(X; \mathcal{F}) := p_!(p^! \mathbf{1} \otimes \mathcal{F}) \qquad = H_c^\cdot(X; p^! \mathbf{1} \otimes \mathcal{F})$$
$$H_\cdot^{BM}(X; \underline{\mathbb{Z}}) := p_* p^! \mathbf{1} \qquad\qquad H_\cdot^{BM}(X; \mathcal{F}) := p_*(p^! \mathbf{1} \otimes \mathcal{F}) \qquad = H^\cdot(X; p^! \mathbf{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)
$$H_c^\cdot(X; \mathcal{F}) \otimes H_c^\cdot(Y; \mathcal{G}) \cong H_c^\cdot(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_1 \downarrow & \searrow^{p} & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

reduced to: $\quad p_{X!}\mathcal{F} \otimes p_{Y!}\mathcal{G} \cong p_!(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$

App 2. (Poincaré duality)
$$X: \text{ a cpt } \underline{\text{oriented}} \text{ mfld of dim } d, \text{ then} \qquad -^\vee = \underline{\mathrm{Hom}}_{D(\mathbb{Z})}(-, \mathbb{Z})$$

proper

$$p^! \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}[d] \text{ locally (Verdier duality)}$$
$$p^! \underline{\mathbb{Z}} \cong \underline{\mathbb{Z}}[d] \text{ globally}$$

$$H^\cdot(X; \underline{\mathbb{Z}})[d] \cong H^\cdot(X; \underline{\mathbb{Z}})^\vee$$

reduced to: $\quad p_* \underline{\mathrm{Hom}}(A, p^*B \otimes p^! \mathbf{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$

$$Z \xrightarrow[\text{close}]{i} X \xleftarrow{\dot{\jmath}} U \quad \leadsto$$

$$\mathcal{D}(Z) \xrightarrow[\substack{\text{Id} \\ \text{pi } i_* = i_! \text{ f.f.} \\ \text{pi } i^!}]{i^*} \mathcal{D}(X) \xrightarrow[\substack{j_! \text{ f.f.} \\ \text{pi } j^* = j^! \\ \text{pi } Rj_*}]{} \mathcal{D}(U)$$

f.f. fully faithful

pi: preserve injectives. (内射)

ie. inj sheaf $\leadsto$ inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Z} = Or_X[n] \xequal{+\text{orientation}} \underline{\mathbb{Z}}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$0 \longrightarrow j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_! i^* \mathcal{F} \longrightarrow 0$$

$$i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^! \mathcal{F} \xrightarrow{+1}$$

Here, $H_{-1}(S^1; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$R\Gamma(X, Z; \mathcal{F}) \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$
$$R\Gamma(X, U; \mathcal{F}) \longrightarrow R\Gamma(X; \mathcal{F}) \longrightarrow R\Gamma(U; \mathcal{F}|_U) \xrightarrow{+1}$$
$$R\Gamma_Z''(X; \mathcal{F})$$

When $\mathcal{F} = \underline{\mathbb{Q}}_X$,
$$H^\cdot(X, Z) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(Z) \xrightarrow{+1}$$
$$H^\cdot(X, U) \longrightarrow H^\cdot(X) \longrightarrow H^\cdot(U) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{D}_X$,
$$R\Gamma(X, j_! \underline{\mathbb{Q}}_U) \longrightarrow H_\cdot^{BM}(X) \longrightarrow R\Gamma(Z; \mathbb{D}_X|_Z) \xrightarrow{+1}$$
$$H_\cdot^{BM}(Z) \longrightarrow H_\cdot^{BM}(X) \longrightarrow H_\cdot^{BM}(U) \xrightarrow{+1}$$

Taking $R\pi_{X,!}$

$$R\Gamma_c(U, \mathcal{F}|_U) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(Z; \mathcal{F}|_Z) \xrightarrow{+1}$$
$$R\Gamma_c(Z, i^! \mathcal{F}) \longrightarrow R\Gamma_c(X; \mathcal{F}) \longrightarrow R\Gamma_c(X, Rj_*(\mathcal{F}|_U)) \xrightarrow{+1}$$

When $\mathcal{F} = \underline{\mathbb{Q}}_X$,
$$H_c^\cdot(U) \longrightarrow H_c^\cdot(X) \longrightarrow H_c^\cdot(Z) \xrightarrow{+1}$$
$$H_c^\cdot(Z, i^! \underline{\mathbb{Q}}_X) \longrightarrow H_c^\cdot(X) \longrightarrow H_c^\cdot(X, Rj_* \underline{\mathbb{Q}}_U) \xrightarrow{+1}$$

When $\mathcal{F} = \mathbb{D}_X$,
$$H_\cdot(U) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, U) \xrightarrow{+1}$$
$$H_\cdot(Z) \longrightarrow H_\cdot(X) \longrightarrow H_\cdot(X, Z) \xrightarrow{+1}$$

$H^\cdot$ & $H_\cdot$ are invariant under homotopy equivalence, while $H_c^\cdot$ & $H_\cdot^{BM}$ are better for assembly.

$$j_! F \longrightarrow Rj_* F \longrightarrow i_! \, i^* Rj_* F \xrightarrow{+1}$$

$$i^! F \longrightarrow i^* F \longrightarrow i^* Rj_* j^! F \xrightarrow{+1}$$

$$i^! j_! j^* F \longrightarrow i^! F \longrightarrow i^* F \xrightarrow{+1}$$

$$i_* i^! j_! F \longrightarrow j_! F \longrightarrow Rj_* F \xrightarrow{+1}$$

Cor. $\quad i^* Rj_* \cong i^! j_! [1].$

$$i_* R\underline{\mathrm{Hom}}(i^* F, i^! G) \longrightarrow R\underline{\mathrm{Hom}}(F, G) \longrightarrow Rj_* R\underline{\mathrm{Hom}}(j^* F, j^! G) \xrightarrow{+1}$$

Quick reminder.

	f^*	$f^!$	$Rf_!$	Rf_*
$\pi: Y \longrightarrow \{*\}$	$\underline{\mathbb{Q}}_Y$	$\mathcal{O}r_Y[\dim_{\mathbb{R}} Y]$	$H_c^{\cdot}(Y; \mathcal{G})$	$H^{\cdot}(Y, \mathcal{G})$
$\iota_p: \{p\} \hookrightarrow X$	$\mathcal{F}_p$	$\mathcal{F}_{p, costalk}$	$sky_p(\mathbb{Q})$	
$i: Z \hookrightarrow X$	$\mathcal{F}|_Z$	$i^! \mathcal{F} = R\Gamma_Z^{\cdot}(\mathcal{F})$	$sky_Z(\mathcal{G})$	
$j: U \hookrightarrow X$	$\mathcal{F}|_U$		$j_! \mathcal{G}$	$Rj_* \mathcal{G}$
$f: X \sqcup X \longrightarrow X$	$(\mathcal{F}, \mathcal{F})$		$\mathcal{F}_1 \oplus \mathcal{F}_2$	

$$f: \mathbb{C} \xrightarrow{z^2} \mathbb{C}$$
$$\mathcal{F} = \mathcal{G} = \underline{\mathbb{Q}}_{\mathbb{C}}$$

$\underline{\mathbb{Q}}_{\mathbb{C}}$

$$\mathbb{Q}^2 \underset{\binom{2}{-2}}{\overset{(1,-1)}{\rightleftarrows}} \mathbb{Q}$$

$$f: \mathbb{C} \xrightarrow{z^2} \mathbb{C}$$
$$\mathcal{F} = \mathcal{G} = \underline{\mathbb{Q}}_{\mathbb{C}}$$

$\underline{\mathbb{Q}}_{\mathbb{C}}$

$$\mathbb{Q}^3 \overset{\left(\begin{smallmatrix}1 & -1 & 0 \\ 0 & 1 & -1\end{smallmatrix}\right)}{\rightleftarrows} \mathbb{Q}^2$$

Application
One point compactification

$$X \overset{\iota}{\hookrightarrow} \bar{X}$$
$$\pi \searrow \swarrow \bar{\pi}$$
$$\{*\}$$

$$
\begin{aligned}
H_c^{\cdot}(X) &= R\pi_! \, \pi^* \mathbb{Z} \\
&= R\bar{\pi}_! \, \iota_! \, \iota^* \pi^* \mathbb{Z} \\
&= R\bar{\pi}_* \, (\iota_! \, \iota^! \, \mathbb{Z}_{\bar{X}}) \\
&= H^{\cdot}(\bar{X}, \{\infty\}; \mathbb{Z})
\end{aligned}
$$

$$
\begin{aligned}
H_{\cdot}^{BM}(X) &= R\pi_* \, \pi^! \mathbb{Z} \\
&= R\bar{\pi}_* \, \iota_* \, \iota^! \bar{\pi}^! \mathbb{Z} \\
&= R\bar{\pi}_* \, (\iota_* \, \iota^* \bar{\pi}^! \mathbb{Z}) \\
&= \mathrm{cone} \, (R\bar{\pi}_* \, i_! \, i^! \bar{\pi}^! \mathbb{Z} \longrightarrow R\bar{\pi}_* \, \bar{\pi}^! \mathbb{Z}) \\
&= H_{\cdot}(\bar{X}, \{\infty\}; \mathbb{Z})
\end{aligned}
$$

Originally, this is another def of cpt supp coh & BM homology.

Vector bundle with 6-functors.

Goal: Define Thom class & Euler class as in [GTM82, §6]

$$H_c^\cdot(F) \longleftarrow H_{cv}^\cdot(E) \xrightarrow{\ s^* \ } H^\cdot(B)$$

$$\downarrow \cong$$

$$H^{\cdot-r}(B)$$

$$\text{generator} \in H_c^r(F) \longleftarrow \Phi \in H_{cv}^r(E)$$

$$\downarrow \qquad \qquad \searrow eu_B \in H^r(B)$$

$$1 \in H^0(B)$$

Setting

$\pi : E \longrightarrow B$ oriented v.b. with fiber $F \cong \mathbb{R}^r$

β_F : one point compactification $\quad (\mathbb{R}^n \subset S^n)$

β_E : fiberwise compactification

$\pi_X : X \longrightarrow \{*\}$

$$\bar{F} \xrightarrow{\ \iota_{\bar{F}} \ } \bar{E}$$
$$\beta_F \nearrow \qquad \nearrow \beta_E$$
$$F \xhookrightarrow{\ \iota_F \ } E \qquad$$
$$\pi_F \downarrow \qquad \pi \downarrow \qquad \searrow \bar{\pi}$$
$$\{p\} \xrightarrow{\ \iota_p \ } B$$

Def

$$H_{cv}^\cdot(E) \overset{\wedge}{=} H^\cdot(\bar{E}, \bar{E} - E)$$

(compact vertical)

$$= R\pi_{\bar{E},*} \, \beta_{E,!} \, \beta_E^! \, \underline{\mathbb{Z}}_{\bar{E}}$$

$$= R\pi_{B,*} \, (R\bar{\pi}_! \, \beta_{E,!})(\beta_E^* \, \underline{\mathbb{Z}}_{\bar{E}})$$

$$= R\pi_{B,*} \, R\pi_! \, \underline{\mathbb{Z}}_E$$

Ex. Construct the following canonical maps by six functors.

$$H^\cdot(\bar{F}, \bar{F} - F) \longleftarrow H^\cdot(\bar{E}, \bar{E} - E)$$
$$\| S \qquad \qquad \| S$$
$$H_c^\cdot(F) \longleftarrow H_{cv}^\cdot(E)$$
$$\downarrow \sim$$
$$H^{\cdot-r}(B)$$

$$R\pi_{\bar{F},*} \, \beta_{F,!} \, \underline{\mathbb{Z}}_F \longleftarrow R\pi_{\bar{E},*} \, \beta_{E,!} \, \underline{\mathbb{Z}}_E$$
$$\| S \qquad \qquad \| S$$
$$R\pi_{F,!} \, \underline{\mathbb{Z}}_F \longleftarrow R\pi_{B,*} \, R\pi_! \, \underline{\mathbb{Z}}_E$$
$$\downarrow \sim$$
$$R\pi_{B,*} \, \underline{\mathbb{Z}}_B[-r]$$

Lemma. $R\pi_! \, \underline{\mathbb{Z}}_E \cong \underline{\mathbb{Z}}_B[-r]$. As a result, $H^\cdot_{cv}(E) \cong H^{\cdot-r}(B)$.

Proof. $R\pi_! \, \underline{\mathbb{Z}}_E = R\pi_! \, \pi^* \underline{\mathbb{Z}}_B$ *expand*

$\qquad\qquad \cong R\pi_! \, \pi^! \, \underline{\mathbb{Z}}_B[-r]$ *Verdier duality, π is a v.b.*

$\qquad\qquad \xrightarrow{\sim} \underline{\mathbb{Z}}_B[-r]$ *adjunction, iso comes from*

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad H_\cdot(\mathbb{R}^r; \mathbb{Z}) \cong \mathbb{Z}$ *(check locally)*

strictly speaking, $H^\cdot_{cv}(E) = H^i(\pi_{B,*} \, R\pi_! \, \underline{\mathbb{Q}}_E)$, but we're lazy to write $\mathcal{H}$

$\overset{\text{trick!}}{}$

$H^\cdot_c(F) \overset{bc}{=\!=\!=} (R\pi_! \, \underline{\mathbb{Z}}_E)_p$

$\qquad =\!=\!= l_p^* R\pi_! \, \underline{\mathbb{Z}}_E$

$\qquad =\!=\!= \pi_{B,*} \, l_{p,*} \, l_p^* R\pi_! \, \underline{\mathbb{Z}}_E \longleftarrow \pi_{B,*} \, R\pi_! \, \underline{\mathbb{Z}}_E = H^\cdot_{cv}(E)$

Compare with the first row:

$R\pi_{\bar{F},*} \, \beta_{F,!} \, \underline{\mathbb{Z}}_F =\!=\!= R\pi_{\bar{E},*} \, l_{\bar{F},*} \, \beta_{F,!} \, l_F^* \, \underline{\mathbb{Z}}_E$

$\qquad\qquad \overset{bc}{=\!=\!=} R\pi_{\bar{E},*} \, l_{\bar{F},*} \, l_F^* \, \beta_{E,!} \, \underline{\mathbb{Z}}_E \longleftarrow R\pi_{\bar{E},*} \, \beta_{E,!} \, \underline{\mathbb{Z}}_E$

Q: How to show that $\qquad H^0(B) \longrightarrow H^r_c(F) \qquad 1 \longmapsto$ generator?

A: $\qquad\qquad H^{\cdot-r}(B) = R\pi_{B,*} \, \underline{\mathbb{Z}}_B[-r] \longrightarrow R\pi_{B,*} \, l_{p,*} \, l_p^* \, \underline{\mathbb{Z}}_B[-r]$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = l_p^* \, \underline{\mathbb{Z}}_B[-r] \qquad\qquad = \underline{\mathbb{Z}}_{\{p\}}[-r]$

$\qquad\qquad\qquad\qquad 1 \longmapsto \qquad\qquad\qquad\qquad\qquad\qquad\qquad$ generator

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cong l_p^* \, R\pi_! \, \underline{\mathbb{Z}}_E$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{bc}{\cong} R\pi_{F,!} \, l_F^* \, \underline{\mathbb{Z}}_E$

$\qquad\qquad\qquad\qquad\qquad\qquad = R\pi_{F,!} \, \underline{\mathbb{Z}}_F \qquad\qquad = H^\cdot_c(F)$

Euler class of v.b.

Ex   construct

$$s^*: \quad \dot{H}_{cv}(E) \longrightarrow \dot{H}(B) \qquad\qquad \Phi \longmapsto eu_E$$

$$\| \qquad\qquad\qquad \|$$

$$R\pi_{B,*} R\pi_! \,\underline{\mathbb{Z}}_E \qquad R\pi_{B,*} \,\underline{\mathbb{Z}}_B$$

After this exercise, we get euler class $\quad eu_E \in H^r(B)$.

$\mathbb{A}$ for Ex: $\pi \circ s = Id$, so

$$R\pi_{B,*} R\pi_! \,\underline{\mathbb{Z}}_E \longrightarrow R\pi_{B,*} (R\pi_! \, s_*) s^* \underline{\mathbb{Z}}_E = R\pi_{B,*} \,\underline{\mathbb{Z}}_B. \qquad\qquad \square$$

Ex. Find a way to define eu_E without using Φ, i.e. construct

$$H^{\cdot -r}(B) \longrightarrow \dot{H}(B) \qquad\qquad 1 \longmapsto eu_E$$

$$\| \qquad\qquad\qquad \|$$

$$R\pi_{B,*} \,\underline{\mathbb{Z}}_B[-r] \qquad R\pi_{B,*} \,\underline{\mathbb{Z}}_B$$

$$\left[\begin{array}{l} \text{Hint.} \quad \text{Induced by} \\[4pt] \quad \underline{\mathbb{Z}}_B[-r] \cong R\pi_! \,\underline{\mathbb{Z}}_E \xrightarrow{\quad\underset{\uparrow}{\quad}\quad} (R\pi_! \, s_*) s^* \underline{\mathbb{Z}}_E \cong \underline{\mathbb{Z}}_B \\[4pt] \qquad\qquad\qquad \text{in fact, use } \underline{\mathbb{Z}}_E \longrightarrow s_* \underline{\mathbb{Z}}_B \end{array} \right]$$

Relation with tubular nbhd

Let $j: E - B \hookrightarrow E$, T: tubular nbhd of B in E,

$\qquad\qquad\qquad \underset{s(B), \text{ if you prefer rigorous notation}}{\uparrow}$

by applying $R\pi_{B,*} R\pi_!$ to

$$0 \longrightarrow j_! j^! \,\underline{\mathbb{Z}}_E \longrightarrow \underline{\mathbb{Z}}_E \longrightarrow s_* s^* \underline{\mathbb{Z}}_E \longrightarrow 0$$

we get LES:

$$R\pi_{B,*} R\pi|_{E \setminus B,!} \,\underline{\mathbb{Z}}_{E \setminus B} \longrightarrow \dot{H}_{cv}(E) \longrightarrow \dot{H}(B) \xrightarrow{+1}$$

$$\| \qquad\qquad\qquad \|S \qquad eu_E \wedge \nearrow \qquad \|$$

$$\qquad\qquad\qquad H^{\cdot -r}(B)$$

$$\qquad\qquad\qquad \|S$$

$$\dot{H}^{\cdot -1}(\partial T) \longrightarrow \dot{H}(T, \partial T) \longrightarrow \dot{H}(T) \xrightarrow{+1}$$

Euler class of sphere bundle

Ex: Use the similar method, construct the Gysin sequence:

$$H^n(B) \xrightarrow{\ \pi^*\ } H^n(E) \xrightarrow{\ \pi_*\ } H^{n-k}(B) \xrightarrow[+1]{\ eu_\pi \wedge\ }$$

$$S^k \xrightarrow{\ l_k\ } E$$
$$\downarrow \pi \quad \text{oriented}$$
$$B$$

Hint: Apply the factor $R\pi_{B,*}$ to

$$R^0\pi_* \underline{\mathbb{Q}}_E \longrightarrow R\pi_* \underline{\mathbb{Q}}_E \longrightarrow R^k\pi_* \underline{\mathbb{Q}}_E[-k] \xrightarrow{\ +1\ }$$

$$\shortparallel$$
$$\underline{\mathbb{Q}}_B \qquad\qquad\qquad\qquad \shortparallel$$
$$\underline{\mathbb{Q}}_B[-k]$$

Therefore, euler class roughly describes the connecting map
$$\underline{\mathbb{Q}}_B[-k] \longrightarrow \underline{\mathbb{Q}}_B[1] \quad \textit{after applying } R\pi_{B,*}$$
It works as an obstruction for "the triangle to be exact".

⚠ $Rf \neq \oplus R^i f[-i]$. Instead,

$$\phi \overset{R^0 f}{\subset} \cdots \overset{R^1 f[-1]}{\subset} \cdots \cdots \overset{R^i f[-n]}{\subset} Rf.$$

It's constructed by truncations.

Hard Exercise

1. When the S^k-bundle $\pi: E \longrightarrow B$ has a section,
   show that $ev_E = 0$. Do we have
   $$R\pi_! \pi^! \underline{\mathbb{Q}}_B \cong \underline{\mathbb{Q}}_B \oplus \underline{\mathbb{Q}}_B[-k] \,?$$
   I think so.

2. For a rk r v.b. $\pi: E \longrightarrow B$ and the crspd S^{r-1}-bundle $\widetilde{\pi}: \widetilde{E} \longrightarrow B$,
   show that they define the same Euler class.
   How are their Gysin sequences related?

3. Discuss the maps
   $$\mathrm{Vect}_B^{\mathrm{rk}\, r} \longrightarrow \{S^{r-1}\text{-bundle over } B\} \longrightarrow H^r(B; \mathbb{Z})$$
   $$E \longmapsto \widetilde{E} \longmapsto ev_{\widetilde{E}}$$
   Are they surj? inj?
   Here, $E \cong E'$ iff $\exists$

   $\widetilde{E} \cong \widetilde{E}'$ iff $\exists$

4. https://en.wikipedia.org/wiki/Chern_class#Via_an_Euler_class
   Can we directly define Chern class by 6-fctors?
   How to define other characteristic classes by 6-fctors?

Explanation

Exactness & derived

$$j_! \quad \text{is exact}$$
$$i^*, j^* \quad \text{are exact}$$
$$i_* \quad \text{is exact}$$

by checking on stalks
in the category Top
when $Z \subset X$ is (strongly) loc. contractable.

$$j_* \quad \text{is not exact} \rightsquigarrow Rj_*$$
$$i^! \quad \text{is already derived}$$

Rmk. strongly loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z$ is contractable

loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p
s.t. $U_n \cap Z \subset U_n$ is contractable

E.g. | $\{\text{strongly loc. contractable}\} \subseteq \{\text{loc. contractable}\} \subsetneq Top$
CW-cplx, topo mflds
& algebraic varieties (Check?)

Cantor set

E.g. we need to check if $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.
Luckily, in the open-closed formalism, we won't meet these problems.

Prop1. Let $e \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} e'$, assume F is exact. Then

① G preserves injective sheaves;
② $RG(f) \circ RG(g) = RG(f \circ g)$

Proof. ①: by universal property.
② by adjunction

Prop 2. Let $\quad e \underset{G}{\overset{F}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} e' \underset{G'}{\overset{F'}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} e''$. Suppose F or F' is exact, then

$$RG \circ RG'(f) = \left(R(G \circ G')\right)(f)$$

Proof. By adjunction & Grothendieck - Serre sequence. $(LF' \circ LF = L(F' \circ F))$
When F' is exact, can use Prop 1 ①.

Cor. $R\pi_{x,*} \circ R_{j_*} = R\pi_{u,*}$.
$Rf_* \& Rf_!$ are nice in general.

Reason: f_! sends skyscraper sheaf to skyscraper sheaf, and in general preserve injective sheaves. (need double check)

Local properties

1. They can be checked locally on the target.

f^*: composition
$f^!$: composition + $j^* = j^!$
$Rf_!$: proper bc
Rf_*: adjunction of proper bc

2. Stalk:

f^*: composition
$f^!$: for $i^!$, use triangles $\leadsto R\Gamma(X, \mathcal{U}; F)$
for $j^!$, $j^! = j^*$
for $\pi^!$, $\pi^! \sim Or[n]$
$Rf_!$: proper bc $\leadsto$ local (proper) cohomology
Rf_*: local cohomology by definition

$$
\begin{array}{ccc}
Y_p & \xrightarrow{\ell_{p,Y}} & Y \; {}^{\nearrow g} \\
{\scriptstyle f|_{Y_p}} \downarrow & & \downarrow f \quad {}_{\nearrow F} \\
\{p\} & \xrightarrow{\ell_p} & X \, {}^{\nearrow}
\end{array}
$$

proper bc to compute stalk of $Rf_!$