Eine Woche, ein Beispiel

5.1 Extension of NA local field

F: NA local field

1. List of well-known results
   - in general
   - unramified / totally ramified
2. $\hat{\mathbb{Z}}$ = profinite completion (review)
3. Big picture
4. Henselian ring
5. Cohomological dimension

} not completed, appendixes

Ref:
Initial motivation comes from
[AY]https://alex-youcis.github.io/localglobalgalois.pdf
which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

[JPS96] Galois cohomology by Jean-Pierre Serre
http://p-adic.com/Local%20Fields.pdf
https://people.clas.ufl.edu/rcrew/files/LCFT.pdf
http://www.mcm.ac.cn/faculty/tianyichao/201409/W020140919372982540194.pdf

1. List of well-known results

In general

F: NA local field E/F: finite extension

Rmk 1. E is also a NA local field with uniquely extended norm

$$\|x\|_: = \|N_{E/F}(x)\|_F^{\frac{1}{n}} \qquad \text{resp. } v(x) := \tfrac{1}{n} v_F(N_{E/F}(x))$$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$ is monogenic, i.e. $\mathcal{O}_E = \mathcal{O}_F[\alpha]$ $\exists\, \alpha \in \mathcal{O}_E$

Cor. (primitive element thm for NA local field)

$E = F[x]/(g(x))$ $\exists\, x \in \mathcal{O}_E$, $g(x)$ min poly of x.

Rmk: Every separable finite field extension has a primitive element, see wiki:
https://en.wikipedia.org/wiki/Primitive_element_theorem
Separable condition is necessary, see
https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element

Rmk 3. Any finite extension of $\mathbb{Q}_p$ is of form $\mathbb{Q}_p[x]/(g(x))$, where $g(x) \in \mathbb{Q}[x]$ is an irr poly.
Any finite extension of $\mathbb{F}_q((t))$ is of form $\mathbb{F}_q((t))[x]/(g(x))$ where $g(x) \in \mathbb{F}_q(t)[x]$ is an irr poly.
Both are achieved by Krasner's lemma.

$v = v_F = \frac{1}{e} v_E$ $\qquad \|\cdot\| = \|\cdot\|_F = \|\cdot\|_E^{\frac{1}{e}}$ $\qquad \mathfrak{p}_F \mathcal{O}_E = \mathfrak{p}_E^e$

E	$v_E = ev$	$\|\cdot\|_E = \|\cdot\|^e$	$\pi_E = \pi_F^{\frac{1}{e}}$	$v(\pi_E) = \frac{1}{e}$
$\mid$ deg n				
F	v_F	$\|\cdot\|_F$	π_F	$v(\pi_F) = 1$

Unramified / totally ramified

Good ref: https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields
It collects the equivalent conditions of unramified/totally ramified field extensions.

| tot ram | wild ram
| | tame ram
| field ext
| split

When E/F is tot ramified,

$e = n \qquad v(\pi_E) = \frac{1}{n}$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E] \qquad \min(\pi_E) \in \mathcal{O}_F[x]$ is Eisenstein poly.

2. $\hat{\mathbb{Z}}$ = profinite completion of $\mathbb{Z}$ (Recall 2022.2.13 outer auto...)

$$\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l \qquad \hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$\hat{\mathbb{Z}}^\times = \prod_l \mathbb{Z}_l^\times \qquad (\hat{\mathbb{Z}}^\times)^{(p)} := \prod_{l \neq p} \mathbb{Z}_l = (\hat{\mathbb{Z}}^{(p)})^\times$$

Prop. ① $\mathrm{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l = m \\ 0 & l \neq m \end{cases}$ $\quad l, m$ prime.

② $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$

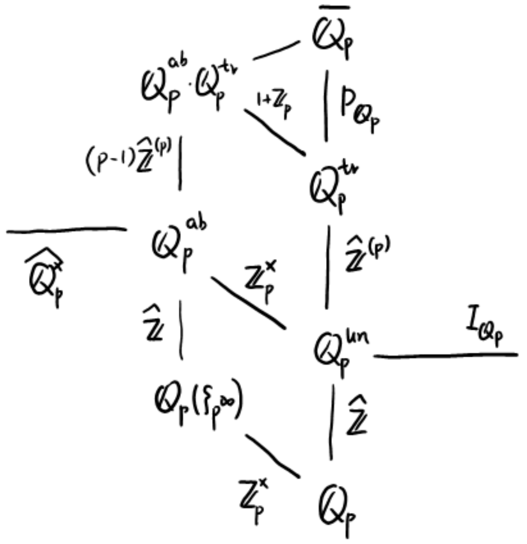
$\mathrm{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^\times$ $\qquad$ in the category of profinite gps.

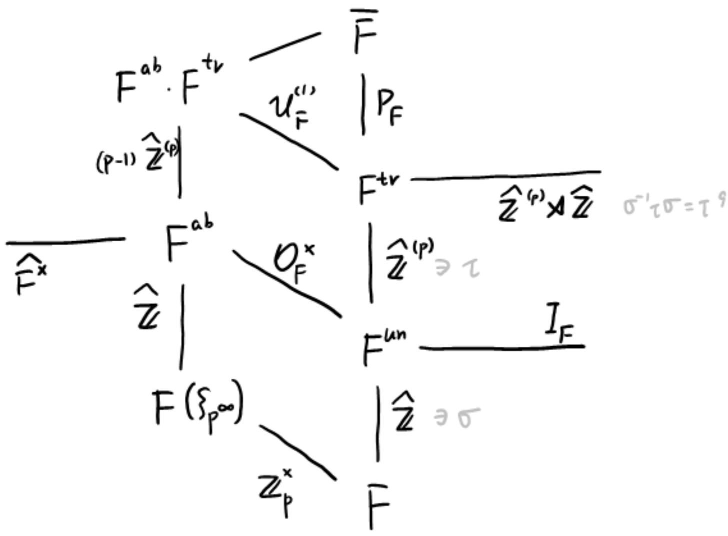
$\mathrm{Aut}(\hat{\mathbb{Z}}^{(p)}) = \hat{\mathbb{Z}}^{\times (p)}$

③ $\mathcal{O}_F, \mathcal{O}_F^\times$ are profinite groups, so $\hat{\mathcal{O}}_F = \mathcal{O}_F \quad \hat{\mathcal{O}}_F^\times = \mathcal{O}_F^\times$.

3. Big picture

Main ref: [AY]https://alex-youcis.github.io/localglobalgalois.pdf





F: NA local field with char $k = p$

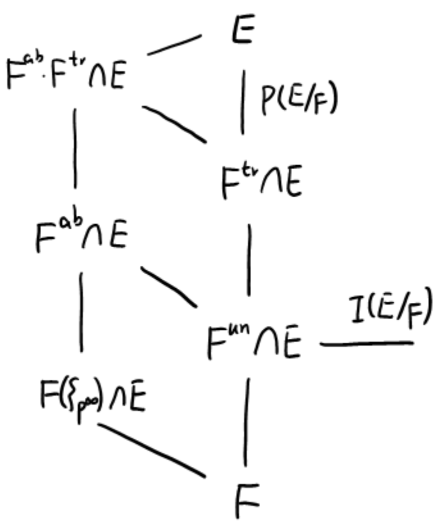
unramified $\qquad F^{un} = \bigcup_{n\geq 1} F(\zeta_{p^n-1}) \overset{\text{Fermat's little thm}}{=\!=} \bigcup_{n\geq 1,\ p\nmid n} F(\zeta_n)$

tame ramified $\qquad F^{tr} = F^{un}\left(\pi_F^{\frac{1}{n}}\big|_{(n,p)=1}\right) = F\left(\pi_F^{\frac{1}{n}}, \zeta_n\big|_{(n,p)=1}\right)$

abelian $\qquad F^{ab} = F(\zeta_\infty) := \bigcup_{n\geq 1} F(\zeta_n)$

$$F^{ab}\cdot F^{tr} = F\left(\pi_F^{\frac{1}{n}}, \zeta_\infty\big|_{(n,p)=1}\right)$$

https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions



$$\begin{tikzcd}[column sep=small]
(\mathbb{Z}/p^n\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_{p^n}) \arrow[d, dash] & \\
\mathbb{Q}_p\left(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\right) \arrow[urrr, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \arrow[ddddrrr, dash, "1+p\mathbb{Z}/1+p^n\mathbb{Z}"'] & & & \cdots \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] & \\
(\mathbb{Z}/p^2\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \arrow[r, dash] & I_p \sim I_{p^2-1} \\
(\mathbb{Z}/p\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \arrow[r, dash] & I_1 \sim I_{p-1} \\
 & & & \mathbb{Q}_p \arrow[r, dash] & I_0 = I_{-1}
\end{tikzcd}$$

$E/F = \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p$

$$\begin{tikzcd}[column sep=small]
(\mathbb{Z}/p^n\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] & \\
\bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\right) \arrow[urrr, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \arrow[ddddrrr, dash, "1+p\mathbb{Z}"'] & & & \cdots \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] & \\
(\mathbb{Z}/p^2\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_{p^2}) \arrow[d, dash, "\mathbb{Z}/p\mathbb{Z}"] \arrow[r, dash] & I^2 = \mathcal{U}^{(2)}_{\mathbb{Q}_p} \\
(\mathbb{Z}/p\mathbb{Z})^\times \arrow[rrr, dash] & & & \mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/(p-1)\mathbb{Z}"] \arrow[r, dash] & I^1 = \mathcal{U}^{(1)}_{\mathbb{Q}_p} \\
 & & & \mathbb{Q}_p \arrow[r, dash] & I^0 = \mathcal{U}^{(0)}_{\mathbb{Q}_p} = \mathbb{Z}_p^\times
\end{tikzcd}$$

$E/F = \mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p$

$$\begin{tikzcd}
\mathbb{Q}_p(\zeta_p) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] & \mathbb{Q}_p(\zeta_p) \arrow[d, dash] & \mathbb{Q}_p(\zeta_p) \arrow[d, dash] \\
\mathbb{Q}_p(\zeta_p+\zeta_p^{-1}) \arrow[d, dash] & \mathbb{Q}_p(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] & \mathbb{Q}_p(\sqrt{-p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p & \mathbb{Q}_p & \mathbb{Q}_p
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] & \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] & \mathbb{Q}_p(\zeta_{p^\infty}) \arrow[d, dash] \\
\bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p(\zeta_{p^n}+\zeta_{p^n}^{-1}) \arrow[d, dash] & \mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] & \mathbb{Q}_p^{abwr}(\sqrt{p}) \arrow[d, dash, "\mathbb{Z}/2\mathbb{Z}"] \\
\mathbb{Q}_p^{abwr} := \bigcup_{n\in\mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i\in(\mathbb{Z}/p^n\mathbb{Z})^\times\cap\mu_{p-1}} \zeta_{p^n}^i\right) & \mathbb{Q}_p^{abwr} & \mathbb{Q}_p^{abwr}
\end{tikzcd}$$

p odd $\qquad p \equiv 1 \pmod 4 \qquad p \equiv 3 \pmod 4$

4. Henselian ring.

Main ref: https://en.wikipedia.org/wiki/Henselian_ring

R comm with 1 (local in this section)

Def. A local ring (R, m) is Henselian if Hensel's lemma holds, i.e.

$$
\begin{tikzcd}
\text{for } P \in R[x] \arrow[d, mapsto] & & \exists f_i \in P[x] \arrow[d, mapsto, "①"] & ② \; P = f_1 \cdots f_n \\
\bar{P} = g_1 \cdots g_n \in R/m[x] & & g_i \in R/m[x] &
\end{tikzcd}
$$

(R, m) is strictly Henselian if additionally $(R/m)^{sep} = R/m$.

E.g.

- Fields / Complete Hausdorff local rings are Henselian.
  e.p. F, $\mathcal{O}_F$ are Henselian
- R is Henselian $\Leftrightarrow$ $R/Nil(R)$ is Henselian
  $\Leftrightarrow$ R/I is Henselian for $\forall I \lhd R$
  e.p. when $\operatorname{Spec} R = \{*\}$, R is Henselian.

Denote $\quad$ StrHense $\subset$ Hense $\subset$ locRing $\subset$ CommRing $\quad$ full subcategories

$$
\begin{tikzcd}
\text{StrHense} \arrow[r, bend right, "\text{forget}"'] & \text{Hense} \arrow[l, dashed, bend right, "{[Stack\_OBSL]}"'] \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \text{locRing} \arrow[l, bend right, "(-)^h"'] \arrow[ll, bend right=40, "(-)^{sh}"']
\end{tikzcd}
$$

zero

Sadly not adjoint

E.g. $F^h = F \qquad F^{sh} = F^{un}$