

Eine Woche, ein Beispiel  
1.28 conormal bundle

1. conceptions describing the singularity
2. smooth mfd case

# 1. conceptions describing the singularity

tangent cone  $\subseteq \bigcup$  limit of tangent spaces  $\subseteq$  tangent space  
in Grassmannian, for affine variety

e.g

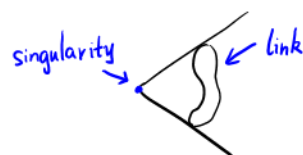
$$\begin{aligned} \{w^2 = z^3\} \quad \{w^2 = 0\} &= \{v \in \mathbb{C}^2 \mid \langle v, dw \rangle = 0\} \subseteq \mathbb{C}^2 \\ \{z_1^2 + z_2^3 + z_3^4 = 0\} \quad \{z_1^2 = 0\} &= \left\{ v \in \mathbb{C}^3 \mid \langle v, a dz_1 + b dz_2 \rangle = 0 \right. \\ &\quad \left. \text{for some } (a,b) \in \mathbb{C}^2 - 0 \right\} \subseteq \mathbb{C}^3 \end{aligned}$$

Apart from these conceptions, we have topological cone:



$$\{w^2 = z^3\}$$

with link  $S' \subseteq S^3$



$$\{z_1^2 + z_2^3 + z_3^4 = 0\}$$

with link  $S^3 / \mathbb{Z}_3 \mathbb{Z} \subseteq S^5$

## 2. smooth mfld case

Def For  $X \subseteq V$  an immersion of **smooth** mflds,  $p \in X$ , the normal space  $N_p X$  at  $p$  is defined as

$$N_p X := \frac{T_p V}{T_p X}$$

(also denoted as  $\underbrace{(T_p V / T_p X)}_{\text{Riemannian metric}} = (T_p X)^\perp$ )

and the conormal space  $(T_x^* V)_p$  at  $p$  is defined as

$$\begin{aligned} (T_x^* V)_p &:= \ker [T_p^* V \rightarrow T_p^* X] \\ &= \{ \alpha \in T_p^* V \mid \alpha(\vec{v}) = 0 \quad \forall \vec{v} \in T_p X \} \\ &= (T_p X)^\perp \end{aligned}$$

(also denoted as  $(T_{V/X}^*)_p, N_p^* X, \dots$ )

One has SESs

$$0 \longrightarrow T_p X \longrightarrow T_p V \longrightarrow N_p X \longrightarrow 0$$

$$0 \longrightarrow (T_x^* V)_p \longrightarrow T_p^* V \longrightarrow T_p^* X \longrightarrow 0$$

As v.b.,

$$0 \longrightarrow T X \longrightarrow T V|_X \longrightarrow N X \longrightarrow 0$$

$$0 \longrightarrow T_x^* V \longrightarrow T^* V|_X \longrightarrow T^* X \longrightarrow 0$$

Rmk. When  $X$  is defined by equations, then  $(T_X^*V)_p$  is easier to compute than other spaces  
 e.g. tangent space  $T_pX$

E.g. remove  $0 \in V$  to avoid singularity

For  $V = \mathbb{C}^3$ ,

$$X = x^2 + y^3 + z^5 = 0, \text{ at } p = (x_0, y_0, z_0) \in X,$$

$$\begin{aligned} (T_X^*V)_p &= \langle 2x_0 dx + 3y_0^2 dy + 5z_0^4 dz \rangle_{\mathbb{C}} \\ &= \langle (df)_p \rangle_{\mathbb{C}} \end{aligned}$$

$$\left[ \begin{array}{l} \text{where } f: V \longrightarrow \mathbb{C} \\ (x, y, z) \longmapsto x^2 + y^3 + z^5 \end{array} \right. \quad \left. \begin{array}{ccc} X & \longrightarrow & V \\ \downarrow & \lrcorner & \downarrow f \\ \{0\} & \hookrightarrow & \mathbb{C} \end{array} \right]$$

$$T_p^*X \cong \mathbb{C}^3 / \langle (df)_p \rangle_{\mathbb{C}}$$

$$\begin{aligned} T_p X &= \{ \vec{v} \in T_p V \mid \alpha(\vec{v}) = 0 \quad \forall \alpha \in (T_X^*V)_p \} \\ &= \{ v_x \partial_x + v_y \partial_y + v_z \partial_z \mid 2x_0 v_x + 3y_0^2 v_y + 5z_0^4 v_z = 0 \} \\ &\stackrel{p \neq 0}{\cong} \mathbb{C}^2 \end{aligned}$$

$$N_p X \cong \mathbb{C}^3 / T_p X$$

In conclusion, conormal space is more natural when spaces are defined by equations.