Eine Woche, ein Beispiel
5.21 Hochschild (co)homology

1 Def. $C_\bullet^{bar}(A)$: $\quad \to A\otimes_k A\otimes_k A\otimes_k A \xrightarrow{\partial_2} A\otimes_k A\otimes_k A \xrightarrow{\partial_1} A\otimes_k A \to \cancel{A} \to 0$

a free $A^e := A\otimes_k A^{op}$-module resolution

$HH_\bullet(A,M) := Tor_\bullet^{A^e}(A,M) = H_\bullet(M\otimes_{A^e} C_\bullet^{bar}(A)) \qquad HH_\bullet(A) = HH_\bullet(A,A)$

$HH^\bullet(A,M) := Ext_{A^e}^\bullet(A,M) = H^\bullet(Hom_{A^e}(C_\bullet^{bar}(A),M)) \qquad HH^\bullet(A) = HH^\bullet(A,A)$

$C_n(A,M) := \qquad\qquad C_\bullet^{bar}(A)\otimes_{A^e} M$

$C^n(A,M) := Hom_{A^e}(C_\bullet^{bar}(A),M)$

$\bullet$: covariant $\Big\}\leadsto$ $HH_\bullet(-)$ is a functor

$-$: contravariant $\Big\}$ $HH^\bullet(-)$ is not a functor, e.p. $Z(ring)$ is not a functor.

$\partial_n: A^{\otimes n+2} \longrightarrow A^{\otimes n+1} \qquad a_0\otimes\cdots\otimes a_{n+1} \longmapsto \sum_{i=0}^n (-1)^i a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_{n+1}$

$\partial_n: C_n(A,M) \longrightarrow C_{n-1}(A,M)$

$A^{\otimes n+2}\otimes_{A^e} M \longrightarrow A^{\otimes n+1}\otimes_{A^e} M \qquad a_0\otimes\cdots\otimes a_{n+1}\otimes m \longmapsto \sum_{i=0}^n (-1)^i a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_{n+1}\otimes m$

$A^{\otimes n+1}\otimes_A M \longrightarrow A^{\otimes n}\otimes_A M \qquad a_0\otimes\cdots\otimes a_n\otimes m \longmapsto \sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_n\otimes m$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n a_0\otimes\cdots\otimes a_{n-1}\otimes a_n m$

$A^{\otimes n}\otimes_k M \longrightarrow A^{\otimes n-1}\otimes_k M \qquad a_1\otimes\cdots\otimes a_n\otimes m \longmapsto \sum_{i=1}^{n-1}(-1)^i a_1\otimes\cdots a_i a_{i+1}\cdots\otimes a_n\otimes m$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n a_0\otimes\cdots\otimes a_{n-1}\otimes a_n m$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_2\otimes\cdots\otimes a_n\otimes m a_1$

$d^n: C^{n-1}(A,M) \longrightarrow C^n(A,M)$

$Hom_{A^e}(A^{\otimes n+1},M) \longrightarrow Hom_{A^e}(A^{\otimes n+2},M) \qquad d^n f(a_0\otimes\cdots\otimes a_{n+1}) = \sum_{i=0}^n (-1)^i f(a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_{n+1})$

$Hom_A(A^{\otimes n},M) \longrightarrow Hom_A(A^{\otimes n+1},M) \qquad d^n f(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i f(a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_n)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n f(a_0\otimes\cdots\otimes a_{n-1})a_n$

$Hom_k(A^{\otimes n-1},M) \longrightarrow Hom_k(A^{\otimes n},M) \qquad d^n f(a_1\otimes\cdots\otimes a_n) = \sum_{i=1}^{n-1}(-1)^i f(a_1\otimes\cdots a_i a_{i+1}\cdots\otimes a_n)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n f(a_1\otimes\cdots\otimes a_{n-1})a_n$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + a_1 f(a_2\otimes\cdots\otimes a_n)$

$\|$ when $M = A^\vee := Hom_k(A_A, k)$

$Hom_k(A^{\otimes n},k) \longrightarrow Hom_k(A^{\otimes n+1},k) \qquad d^n f(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i f(a_0\otimes\cdots a_i a_{i+1}\cdots\otimes a_n)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (-1)^n f(a_n a_0\otimes a_1\otimes\cdots\otimes a_{n-1})$

Rmk. It works similarly for group cohomology.
    In gp coh case, one even has identification

$$G^{\otimes n} \longrightarrow G^{\otimes n, homo} \quad \leftarrow homogeneous$$

$$a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \longmapsto a_0 \otimes a_0 a_1 \otimes \cdots \otimes a_0 a_1 \cdots a_{n-1}$$
$$b_0 \otimes b_0^{-1} b_1 \otimes \cdots \otimes b_{n-2}^{-1} b_{n-1} \qquad b_0 \otimes b_1 \otimes \cdots \otimes b_{n-1}$$

and therefore more expressions for ∂_n & d^n.

Rmk. $H^n(G, M) = Ext^n_{\mathbb{Z}[G]}(\mathbb{Z}, M)$ $RHom_{\mathbb{Z}[G]}(\mathbb{Z}, M)$
        $H_n(G, M) = Tor_n^{\mathbb{Z}[G]}(\mathbb{Z}, M)$ $\mathbb{Z} \otimes^L_{\mathbb{Z}[G]} M$

        $HH^n(A, M) = Ext^n_{A^e}(A, M)$ $RHom_{A^e}(A, M)$
        $HH_n(A, M) = Tor_n^{A^e}(A, M)$ $A \otimes^L_{A^e} M$

2. Cyclic cohomology

$$C_\lambda^n(A, A^\vee) := \{f \in C^n(A, A^\vee) \mid f(a_0 \otimes \cdots \otimes a_n) = (-1)^n f(a_1 \otimes \cdots \otimes a_n)\}$$

$$HC^*(A) := H^*(C_\lambda^\cdot(A, A^\vee))$$

We have a LES:

$$HC^{n+1}(A) \longrightarrow HH^{n+1}(A, A^\vee) \longrightarrow H^{n+1}(C^\cdot/C_\lambda^\cdot) \underset{S}{=} HC^n(A)$$

$$HC^n(A) \xrightarrow{\ I\ } HH^n(A, A^\vee) \xrightarrow{\ B\ } H^n(C^\cdot/C_\lambda^\cdot) = HC^{n-1}(A)$$

2. In low-dimension

n	$HH_n(A, M)$	$HH_n(A)$	$HH^n(A, M)$	$HH^n(A)$
0	$A \otimes_{A^e} M$ $= M/[A M]$	$A \otimes_{A^e} A$ $= A/[A A]$	$Hom_{A^e}(A, M)$ $= M^A = \{m \in M \mid am = ma\}$	$Hom_{A^e}(A, A)$ $= Z(A)$
1	—	$\Omega_{A/k}$ when A is comm (no alg struc)	OutDer(A, M) (Lie alg)	OutDer(A)
2	—	—	AlgExt(A, M)	Square-zero deformation $=$ flat $k[t]/(t^2)$-alg E s.t $E \otimes_{k[t]/(t^2)} k \cong A$
3	—	—	crossed bimodules	obstruction space

3. Examples

in $k[t]/(t^n)$, k

A	$HH_n(A,M)$	$HH^n(A,M)$	proj resolution
k	$\begin{cases} M & n=0 \\ 0 & n\geq 1 \end{cases}$	$\begin{cases} M & n=0 \\ 0 & n\geq 1 \end{cases}$	$0 \longrightarrow K \longrightarrow K \longrightarrow 0$ $1 \longmapsto 1$
$k[x]$	$\begin{cases} k[x] & n=0,1 \\ 0 & n\geq 2 \end{cases}$	$\begin{cases} k[x] & n=0,1 \\ 0 & n=2 \end{cases}$	$0 \longrightarrow A^e \longrightarrow A^e \longrightarrow A \longrightarrow 0$ $1 \longmapsto 1$ $1 \longmapsto x_2-x_1$
$k[x,y]$	$\begin{cases} k[x,y] & n=0 \\ k[x,y]^{\oplus 2} & n=1 \\ k[x,y] & n=2 \\ 0 & n\geq 3 \end{cases}$	$\begin{cases} k[x,y] & n=0 \\ k[x,y]^{\oplus 2} & n=1 \\ k[x,y] & n=2 \\ 0 & n\geq 3 \end{cases}$	$0 \to A^e \longrightarrow A^{e\oplus 2} \longrightarrow A^e \to A \longrightarrow 0$ $1 \mapsto (-(y_1-y_2), x_1-x_2)$ $(1,0) \longmapsto x_1-x_2$ $(0,1) \longmapsto y_1-y_2$
$k[t]/(t^k)$ char $k\nmid n$	$\begin{cases} M & n=0 \\ tM & 2\nmid n \\ M/t^{k-1}M & 2\mid n,\ n>0 \end{cases}$	$\begin{cases} M & n=0 \\ tM & 2\nmid n \\ M/t^{k-1}M & 2\mid n,\ n>0 \end{cases}$	$\cdots \xrightarrow{u} A^e \xrightarrow{v} A^e \xrightarrow{u} A^e \longrightarrow A \longrightarrow 0$ $1 \longmapsto \sum_{i+j=k-1} r^i s^j \qquad 1 \longmapsto 1$ $1 \longmapsto r-s$
$k[t]/(t^k)$ char $k\mid n$	M	M	
$k\langle x,y\rangle/(yx-xy-1)$ char $k=0$	$\begin{cases} 0 & n=0,1 \\ K & n=2 \\ 0 & n\geq 3 \end{cases}$	$\begin{cases} K & n=0 \\ 0 & n\geq 1 \end{cases}$	$0 \longrightarrow A^e \longrightarrow A^{e\oplus 2} \longrightarrow A^e \longrightarrow A \longrightarrow 0$ $1 \mapsto (-(y_1-y_2), x_1-x_1)$ $(1,0) \longmapsto x_2-x_1$ $(0,1) \longmapsto y_2-y_1$
kQ Q: connected acyclic	$\begin{cases} K^{\oplus r} & n=0 \\ 0 & n\geq 1 \end{cases}$	—	$0 \to \bigoplus_{\alpha\in Q_1} A\, e_{t(\alpha)}\otimes e_{s(\alpha)} \to \bigoplus_{v\in Q_0} A\, e_v\otimes e_v \to A \longrightarrow 0$ $e_v\otimes e_v \mapsto e_v$ $e_{t(\alpha)}\otimes e_{s(\alpha)} \longmapsto \alpha\otimes 1(e_{s(\alpha)}\otimes e_{s(\alpha)}) - 1\otimes\alpha(e_{t(\alpha)}\otimes e_{t(\alpha)})$
$K[G]$ G: finite gp	$\begin{cases} K^{\oplus \#\text{conj class}} & n=0 \\ 0 & n\geq 1 \end{cases}$	—	

Morita equivalence

$$A \overset{\text{morita}}{\sim} B \overset{\text{def}}{\iff} \text{Mod}_A \overset{\text{equiv}}{\sim} \text{Mod}_B$$

Thm. Each f.d. alg over $k = \bar{k}$ is Morita equiv to a basic alg.

[Rep notes 1, Cor 24.5]

Theorem 24.4. *Let A be a finite-dimensional K-algebra. Let P be a projective generator of $\mathrm{mod}(A)$, and set $B := \mathrm{End}_A(P)^{\mathrm{op}}$. Then*

$$F := P \otimes_B - : \mathrm{mod}(B) \to \mathrm{mod}(A)$$

and

$$G := \mathrm{Hom}_A(P, -) : \mathrm{mod}(A) \to \mathrm{mod}(B)$$

are equivalences of categories which are quasi-inverses of each other.

e.g.
$$M^{n \times n}(\mathbb{C}) \sim \mathbb{C} \qquad\qquad \mathbb{C}[S_3] \sim \mathbb{C}^{\oplus 3}$$

$$X^{\oplus n} := X \otimes_{\mathbb{C}} \mathbb{C}^{\oplus n} \longleftarrow\!\!| \; X$$

$$Y \longmapsto \mathrm{Hom}_{M^{n \times n}(\mathbb{C})\text{-mod}}(\mathbb{C}^{\oplus n}, Y)$$

or

$$\mathrm{Hom}_{\mathbb{C}\text{-mod}}(\mathbb{C}^{\oplus n}, X) \longleftarrow\!\!| \; X$$

$$Y \longmapsto Y \otimes_{M^n(\mathbb{C})} \mathbb{C}^{\oplus n}$$