§3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections
4. L-fct
5. density theorem

1. Galois rep

<u>Setting</u> G: arbitrary gp e.g. G any Galois gp
If G profinite $\Rightarrow$ open subgps are finite index subgps.
Λ: top field e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{cont}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ cont

⚠ $\rho(G)$ can be infinite! for Gal gp
E.g. When char $F \neq p$, we have p-adic cyclotomic character

$$\varepsilon_p: \mathrm{Gal}(F^{\mathrm{sep}}/F) \longrightarrow \mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p^\times \qquad \sigma \longmapsto \varepsilon_p(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_p(\sigma)} \qquad \forall \zeta \in \mu_{p^\infty}$$

This is cont by def. (Take usual topo.)

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{fi}}(G)$ fi: finite image / finite index
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with finite image

Rmk. $\mathrm{rep}_{\Lambda,\mathrm{cont}}(G) \longleftarrow \mathrm{rep}_{\Lambda,\mathrm{fi}}(G) \longleftarrow \mathrm{rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) = \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$

$$\begin{array}{ccccccc}
\mathrm{rep}_{\Lambda,\mathrm{sm}}(G) & = & \mathrm{rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) & \rightleftarrows & \mathrm{rep}_{\Lambda,\mathrm{fi}}(G) & \rightleftarrows & \mathrm{rep}_{\Lambda,\mathrm{cont}}(G) \\
\cap & & \cap & & \cap & & \cap \\
\mathrm{Rep}_{\Lambda,\mathrm{sm}}(G) & \longleftarrow & \mathrm{Rep}_{\Lambda_{\mathrm{disc}},\mathrm{cont}}(G) & \rightleftarrows & \mathrm{Rep}_{\Lambda,\mathrm{fi}}(G) & \longrightarrow & \mathrm{Rep}_{\Lambda,\mathrm{cont}}(G)
\end{array}$$

$\rightarrow$: if fin index subgps are open

$\rightarrow$: if G: profinite gp (Only need: open $\Rightarrow$ fin index)

$\rightarrow$: Artin rep (of profinite gp)

Artin rep. $\Lambda = (\mathbb{C}$, euclidean topo$)$ G profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{\mathrm{Id}\}$.

"Proof". Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\|_{max} < \frac{1}{3}\}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq \mathrm{Id}$, $\exists n \in \mathbb{N}$, s.t $A^n \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unipotent.

Case 2. A not unipotent.

Problem: $\|gAg^{-1}\|_{max} \neq \|A\|_{max}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},\mathrm{cont}}(G)$, $\rho(G)$ is finite. G profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq G_F$ finite index. $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = \mathrm{Id}$

$\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp. so ρ is semisimple.
Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

Finite field

Task. For Λ: NA local field with char $k_\Lambda = l$, compare

$$\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}}) \longleftrightarrow \mathrm{rep}_{\Lambda,\mathrm{sm}}(\mathbb{Z}) \quad + \text{ extra informations/conditions}$$