Eine Woche, ein Beispiel

2.6. six functors

$$\begin{array}{ccc} \mathcal{G} & \mathcal{F} \; \mathcal{F}' \\ | & \vee \; \vee \\ Y & \xrightarrow{f} X \end{array} \qquad\qquad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g'\downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

Goal:

$$f^* \dashv f_*$$
$$-\otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$$
$$f_! \dashv f^!$$

$$f^*(-\otimes -)$$
$$f^*(\mathcal{F}\otimes\mathcal{F}') \cong f^*\mathcal{F}\otimes f^*\mathcal{F}'$$
$$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$$

proj formula
$$f_!(f^*\mathcal{F}\otimes\mathcal{G}) \cong \mathcal{F}\otimes f_!\mathcal{G}$$
$$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_!\mathcal{G}, \mathcal{F})$$
$$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$$

$$bc: \quad f^* g_! \cong g'_! f'^{*}$$
$$f_* g'_* \cong g_*' f'_*$$

$$f^*: \text{stalk} \qquad f_*/f_!: \text{cohomology}$$

$$\begin{array}{l} I: \; f^* = f^! \\ P: \; f_* = f_! \end{array}$$

$$p: X \longrightarrow pt$$

$$H^{\cdot}(X; \underline{\mathbb{Z}}) := p_* p^* \mathbb{1} \qquad H^{\cdot}(X;\mathcal{F}) := p_*\mathcal{F} \qquad = R\Gamma(X;\mathcal{F})$$
$$H_c^{\cdot}(X; \underline{\mathbb{Z}}) := p_! p^* \mathbb{1} \qquad H_c^{\cdot}(X;\mathcal{F}) := p_!\mathcal{F}$$
$$H_{\cdot}(X; \underline{\mathbb{Z}}) := p_! p^! \mathbb{1} \qquad H_{\cdot}(X;\mathcal{F}) := p_!(p^!\mathbb{1}\otimes\mathcal{F}) \qquad = H_c^{\cdot}(X; p^!\mathbb{1}\otimes\mathcal{F})$$
$$H_{\cdot}^{BM}(X; \underline{\mathbb{Z}}) := p_* p^! \mathbb{1} \qquad H_{\cdot}^{BM}(X;\mathcal{F}) := p_*(p^!\mathbb{1}\otimes\mathcal{F}) \qquad = H^{\cdot}(X; p^!\mathbb{1}\otimes\mathcal{F})$$

App 1. (Künneth formula)
$$H_c^{\cdot}(X;\mathcal{F}) \otimes H_c^{\cdot}(Y;\mathcal{G}) \cong H_c^{\cdot}(X\times Y; \mathcal{F}\boxtimes\mathcal{G})$$

$$\begin{array}{ccc} X\times Y & \xrightarrow{P_2} & Y \\ P_1\downarrow & \searrow^{P} & \downarrow P_Y \\ X & \xrightarrow{P_X} & * \end{array}$$

reduced to: $\quad P_{X!}\mathcal{F} \otimes P_{Y!}\mathcal{G} \cong P_!(p_1^*\mathcal{F}\otimes p_2^*\mathcal{G})$

App 2. (Poincaré duality)

$$X: \text{ a cpt } \underline{\text{oriented}} \text{ mfld of dim } d, \text{ then} \qquad -^{\vee} = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

proper $\qquad p^!\mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$\qquad\qquad\qquad\longrightarrow p^!\mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^{\cdot}(X; \underline{\mathbb{Z}})[d] \cong H^{\cdot}(X; \underline{\mathbb{Z}})^{\vee}$$

reduced to: $\quad p_* \underline{Hom}(A, p^*B \otimes p^!\mathbb{1}) \cong \underline{Hom}(p_!A, B)$

Upgrade: ∞-categories & sym monoidal structure

Idea: $\mathcal{D}_\circ : \mathcal{C}^{op} \longrightarrow Cat_\infty$

$$\begin{array}{ccc} X & \longmapsto & D(X) \\ f\downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & D(Y) \end{array}$$

e.g. X := nice top space,
$D(X)$:= derived category of
abelian sheaves over X.

extends to $\quad$ ⌐compatability is encoded!

$\mathcal{D} : Corr(C,E) \longrightarrow Mon(Cat_\infty)$

$[Y \xleftarrow{f} X = X] \longmapsto f^*$

$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$

$[X \times X \xleftarrow{\Delta} X = X] \longmapsto \otimes$

Moreover, It factor through

$$\begin{array}{ccccc} Corr(C,E) & \longrightarrow & LZ_\mathcal{D} & \longrightarrow & Mon(Cat_\infty) \end{array}$$

Obj: $\qquad X \qquad \longmapsto \qquad X \qquad \longmapsto \qquad \mathcal{D}(X)$

Mor:
$$\left[\begin{array}{c} Y \\ {}^f\nearrow \searrow^g \\ X \qquad Z \end{array} \right] \longmapsto \begin{array}{c} kernel \\ (f,g)_* 1_Y \in \mathcal{D}(X\times Z) \end{array} \longmapsto \begin{array}{c} FM\text{-transformation} \end{array}$$

composition = convolution

2-Mor:
$\qquad\qquad\qquad \varepsilon \to \varepsilon' \longmapsto \Phi_\varepsilon \longrightarrow \Phi_{\varepsilon'}$

$$\left[\begin{array}{c} Z \\ \swarrow \downarrow \searrow \\ Y_1 \quad\quad Y_2 \\ \swarrow \searrow \swarrow \searrow \\ X_1 \quad X_2 \quad X_3 \\ \\ \cong \downarrow F \\ \\ Z \\ \swarrow \searrow \\ X_1 \quad X_3 \end{array} \right] \longmapsto \left[\begin{array}{c} \varepsilon_{12} * \varepsilon_{23} \\ \\ \cong \downarrow \\ \\ \varepsilon_{13} \end{array} \right]$$

Goal: framework of ∞-category & $\otimes$ $\qquad \rightsquigarrow Corr(C,E)$ & $Corr(C,E)^\otimes$

	f^*	$f_!$	$\otimes$	$\mathcal{F} \otimes -$

Correspondence

Columns:

f^*: Y at top, $f: Y \to X$ on left, $Y = Y$ on right (so $X \xleftarrow{f} Y = Y$).

$f_!$: Y at top, $Y = Y$ on left, $f: Y \to X$ on right.

$\otimes$: X at top, $\Delta: X \to X \times X$ on left, $X = X$ on right.

$\mathcal{F} \otimes -$: X at top with $\mathcal{F}$, X left, X right. $\boxed{\text{Not Correspondence!}}$

FM trans

f^*: $Y \xrightarrow{(f, \mathrm{Id})} X \times Y$, then projections to X and Y.

$f_!$: $Y \xrightarrow{(\mathrm{Id}, f)} Y \times X$, then projections to Y and X.

$\otimes$: $X \xrightarrow{\Delta_3} X \times X \times X$, then $\pi_{12}: \to X \times X$ and $\pi_3: \to X$.

$\mathcal{F} \otimes -$: $X \xrightarrow{\mathcal{F}} {}^{\Delta} X \times X \xrightarrow{\Delta_* \mathcal{F}}$, then projections to X and X.

X final

f^*: $Y \xrightarrow{1_Y}$, $f: Y \to X$ on left, $Y = Y$ on right.

$f_!$: $Y \xrightarrow{1_Y}$, $Y = Y$ on left, $f: Y \to X$ on right.

$\otimes$: $Y \times Y \times Y$ with $\Delta_{3,*} 1_Y$, then $\pi_{12}: \to Y \times Y$ and $\pi_3: \to Y$.

$\mathcal{F} \otimes -$: $Y \times Y$ with $\Delta_* \mathcal{F}$, then projections to Y and Y.

Ex. Explain base change, projection formula and Poincaré duality.

∞- category

$$\Delta \hookrightarrow \operatorname{Fun}(\Delta^{op}, \operatorname{Set}) \overset{\triangle}{=} \operatorname{sSet} \supseteq \operatorname{Cat}_\infty \supseteq \operatorname{An}$$

$$\begin{array}{ccc} \Lambda_i^n & \overset{h}{\longrightarrow} & X \\ \downarrow & \nearrow & \\ \Delta^n & \exists K_{n,i}(h) & \end{array} \qquad \begin{array}{ll} \forall 0 < i < n & \text{in } \operatorname{Cat}_\infty \\ \forall 0 \leq i \leq n & \text{in } \operatorname{An} \end{array}$$

Notation

Set	$(0,0)$- category	set
Cat	$(1,1)$- category	category
An	$(\infty, 0)$- category	anima / Kan cplx / ∞- groupoid
$\operatorname{Cat}_\infty$	$(\infty, 1)$- category	

Ex.   Realize $\operatorname{Corr}(C, E)$ as an ∞- category.

Monoidal structure

In $(1,1)$- category:

Monoidal structure on $\mathscr{C}$:
$$m_{\mathscr{C}}: \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C} \qquad u_{\mathscr{C}}: 1 \longrightarrow \mathscr{C}$$
$$(F, G) \longmapsto F \otimes G \qquad\qquad * \longmapsto 1_{\mathscr{C}}$$

Monoidal object in $(\mathscr{C}, \otimes)$: $X \in \operatorname{Ob}(\mathscr{C})$ with
$$m_X: X \times X \longrightarrow X \qquad u_X: 1_{\mathscr{C}} \to X$$

In $(\infty, 1)$- category:

$$(C, \otimes) \overset{def}{\longleftrightarrow} \begin{bmatrix} \mathbb{X}: \operatorname{Fin}^{part} \longrightarrow \operatorname{Cat}_\infty \\ I \longmapsto X(I) \\ \text{comm monoid} \end{bmatrix} \overset{\text{"straightening"}}{\longleftrightarrow} \begin{bmatrix} \pi^\otimes: Y^\otimes \longrightarrow \operatorname{Fin}^{part} \\ \text{coCartesian fibration} \\ Y_I^\otimes \overset{\sim}{\to} \prod_i Y_i^\otimes \end{bmatrix}$$

⤳ See next page for details

where $\operatorname{Ob}(\operatorname{Fin}^{part}) = \operatorname{Ob}(\operatorname{Fin})$

$\operatorname{Mor}_{\operatorname{Fin}^{part}}(I, J) = \{\alpha: I \dashrightarrow J\}$

commutative monoid: $X(I) \overset{\sim}{\longrightarrow} \prod_i X(i)$
$$F \boxtimes G \longleftarrow\!\shortmid (F, G) \qquad |I| = 2$$

coCartesian fibration: see [Def 3.5]

$$\begin{bmatrix} (C, \otimes) \\ \underbrace{\quad}_{\substack{\text{monoidal} \\ \infty\text{-cat}}} \end{bmatrix} \underset{\text{def}}{\longleftrightarrow} \begin{bmatrix} \mathbb{X}: \text{Fin}^{part} \longrightarrow \text{Cat}_\infty \\ I \longmapsto \mathbb{X}(I) \\ \text{comm monoid} \end{bmatrix} \underset{\text{"straightening"}}{\longleftrightarrow} \begin{bmatrix} \pi^\otimes: Y^\otimes \longrightarrow \text{Fin}^{part} \\ \text{coCartesian fibration} \\ Y_I^\otimes \xrightarrow{\sim} \prod_i Y_i^\otimes \end{bmatrix}$$

$(\mathcal{C}, \otimes) \longmapsto \mathcal{C}^{(-)}: \text{Fin}^{part} \longrightarrow \text{Cat}_\infty$

$\qquad\qquad\qquad I \longmapsto \mathcal{C}^I := \prod_{i \in I} \mathcal{C}$

$\qquad\qquad \alpha \downarrow \quad \Rightarrow \quad \downarrow \qquad\qquad (X_i)_{i \in I}$

$\qquad\qquad J \qquad\qquad \mathcal{C}^J \qquad\qquad\qquad \downarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\otimes_{i \in \alpha^{-1}(j)} X_i)_{j \in J}$

$$\xrightarrow{\hspace{6cm}} \quad \pi^\otimes: \mathcal{C}^\otimes \longrightarrow \text{Fin}^{part}$$

$\mathcal{C}^\otimes = \{(I, (X_i)_i) \mid I \in \text{Fin}^{part}, X_i \in \mathcal{C}\} \qquad (I, (X_i)_i) \longmapsto I$

$\text{Mor}_{\mathcal{C}^\otimes}((I, X_i), (J, Y_j)) = \left\{ \alpha: I \dashrightarrow J, \ \{\otimes_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j\}_j \right\}$

$(\mathbb{X}(*), \mathbb{X}(\{1,2\} \to \{*\})) \longleftarrowtail \mathbb{X} \longmapsto \mathbb{X}^\otimes = \{(I, A) \mid I \in \text{Fin}^{part}, A \in \mathbb{X}(I)\}$

$$\text{Mor}((I,A),(J,B)) = \begin{bmatrix} \alpha: I \dashrightarrow J \\ f: (\mathbb{X}(\alpha))A \longrightarrow B \quad \text{in } \mathbb{X}(J) \end{bmatrix}$$

$Y: \text{Fin}^{part} \longrightarrow \text{Cat}_\infty \quad \longleftarrowtail \quad Y^\otimes$

$\qquad I \longmapsto Y^I := Y_I^\otimes$

$\qquad \alpha \downarrow \quad \Rightarrow \quad \downarrow \text{ loc. coCartesian lift}$

$\qquad J \qquad\qquad Y^J$

$(\pi^{\otimes^{-1}}(*), \text{ loc. coCartesian lift of } \{1,2\} \to \{*\}) \longleftarrow\!\!\!\longrightarrow Y^\otimes$

$\mathbb{X}$ encodes the monoidal structure

$$I \xrightarrow{\alpha} J$$

$$\otimes : \ell \times \ell \to \ell$$

② $u_C : 1 \to \ell$

③. $\ell \to 1$ no information

Fctor. (lax) sym monoidal fctors

Special case: $\left[F : (C, \otimes) \longrightarrow (D, \times) \right] \longleftrightarrow \left[\begin{array}{c} F : C^{\otimes} \longrightarrow D \\ \text{with conditions} \end{array} \right]$

Ex. Realize $\mathrm{Corr}(C, E)^{\otimes}$, and show $f^*(- \otimes -)$, bc & proj formula.

Why is $f_! : \mathcal{D}(X) \longrightarrow \mathcal{D}(Y)$ $\mathcal{D}(Y)$-linear?

Category	Object		Morphism
ℓ	X	Y	$X \longrightarrow Y$
ℓ^{op}	X	Y	$X \longleftarrow Y$ in ℓ
$(\ell^{op})^{\sqcup}$	$(I, (X_i)_i)$,	$(J, (Y_j)_j)$	$\alpha : I \to J, \left\{ \bigotimes\limits_{i \in \alpha^{-1}(j)}^{op} X_i \longrightarrow Y_j \right\}_j$ in ℓ^{op}
			$\parallel$
			$\left\{ \prod\limits_{i \in \alpha^{-1}(j)} X_i \longleftarrow Y_j \right\}_j$ in ℓ
$((\ell^{op})^{\sqcup})^{op}$	$(I, (X_i)_i)$,	$(J, (Y_j)_j)$	$\alpha : I \leftarrow J, \left\{ \prod\limits_{i \in \alpha^{-1}(j)} X_i \longrightarrow Y_j \right\}_j$ in ℓ

Construction *"Uniqueness of $f_!$"*

Const 1. $f: Y \longrightarrow X$ $\qquad f \in I \cap P$ $\qquad \Rightarrow f_! \cong f_*$

By induction,
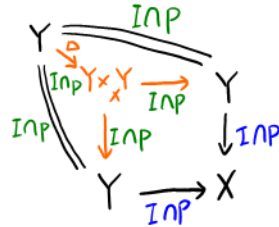
$\quad$ $\underline{\quad}$ Initial case
$\quad$ $\underline{\quad}$ Deduced case

Const 2
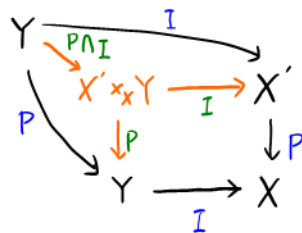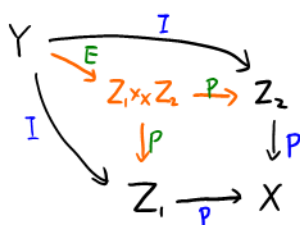

Const 3 E satisfies 2-out-of-3, i.e.
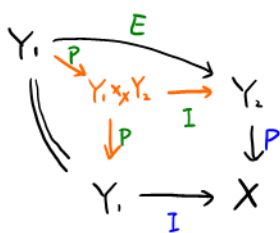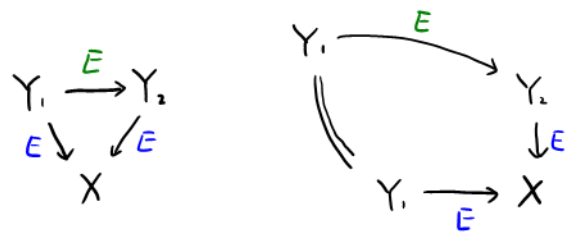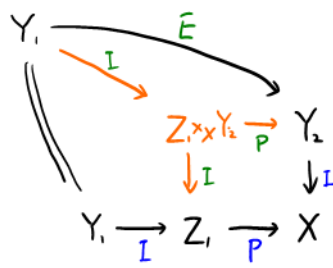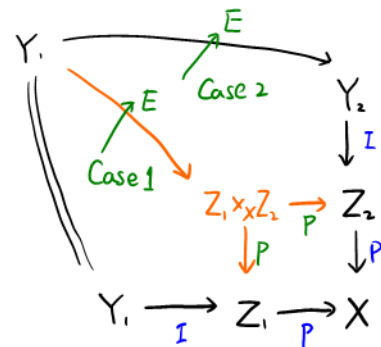
Case 1 $\qquad\qquad$ Case 2 $\qquad\qquad$ Case 3

Const 4. $\quad Y \xrightarrow{i_2} Z_2$ $\qquad$ want: $f_{1*} j_{1!} \cong f_{2*} j_{2!}$
$\qquad\qquad i_1 \downarrow I \quad P \downarrow f_2$
$\qquad\qquad Z_1 \xrightarrow{f_1} X$

sym
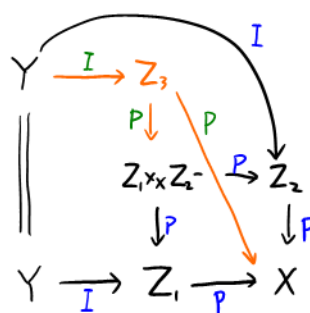
Construction

$$\text{Corr}(C, E) = \delta_+^* \text{Corr}(C,E)_{ct, co}$$
$$\overset{\text{Fun}(-,E)}{\sim} \delta^* \text{Corr}(C,E)_{ct, co}$$
$$\sim \delta^* \text{Corr}(C, I, P)_{ct, co, co}$$
$$\longleftarrow \delta^* \text{Corr}(C, I, P)_{ct, ct, ct}$$

$$\Delta^n \times \Delta^n \times \Delta^n \longrightarrow \text{Cat}_\infty$$
$$\Delta^n \times \Delta^n \longrightarrow \text{Fun}(\Delta^n, \text{Cat}_\infty)$$
$$\begin{array}{c} \\ \searrow \qquad \cup \\ \text{Fun}^L(\Delta^n, \text{Cat}_\infty) \\ \downarrow \\ \text{Fun}((\Delta^n)^{op}, \text{Cat}_\infty) \end{array}$$

f-smooth (= f-admissible) $f: Y \longrightarrow X$

$$F: f_!(A \otimes -) \quad \dashv \quad G: B \otimes f^* -$$
$$\dashv \quad \mathrm{Hom}(A, f^! -)$$

① $\quad B \otimes f^* - \;\cong\; \mathrm{Hom}(A, f^! -)$

App 1. $\quad \Delta_! 1_Y$ cpt $\;\Rightarrow\; A$ cpt

[Proof. $\mathrm{Hom}(\Delta_! 1_Y, B \otimes f^* -) \cong \mathrm{Hom}(A, -)$ preserves filtered colimit.]

② $\qquad\qquad B \cong \mathrm{Hom}(A, f^! 1_X)$
$$p_2^* B \otimes p_1^* - \;\cong\; \mathrm{Hom}(p_2^* A, p_1^! -)$$

Prop. $\quad A$ is f-smooth $\quad\Leftrightarrow\quad p_2^* B \otimes p_1^* A \cong \mathrm{Hom}(p_2^* A, p_1^! A)$ ②b

$\qquad\qquad\qquad\qquad$ where $\quad B \cong \mathrm{Hom}(A, f^! 1_X)$ ②a

    [Verdier's diagonal trick]

$\qquad \Rightarrow: \checkmark$

$\qquad \Leftarrow:$ Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = \mathrm{Id}$,

$\qquad A$ is f-smooth $\quad\Leftrightarrow\quad \mathrm{Hom}(A, 1_X) \otimes A \cong \mathrm{Hom}(A, A)$

$\qquad\qquad\qquad\qquad\qquad\Leftrightarrow\quad A$ is dualizable

App 3. When $A = 1_Y$, $B = f^! 1_X$,

$\qquad 1_Y$ is f-smooth $\underset{+ f^! 1_X \text{ inv}}{\overset{\Leftrightarrow}{\Longleftrightarrow}}$ $p_2^* f^! 1_X \cong p_1^! 1_Y$

$\qquad\qquad\qquad\qquad\qquad\qquad f$ is coh smooth

Using this, one can prove results on coh étale.

Write $B = \mathbb{D}_f(A)$. we get $\mathbb{D}_f(\mathbb{D}_f(A)) \cong A$.   (adjunction is symmetric in A & B).

$\qquad B = S\mathbb{D}_f(A) \qquad w_f := S\mathbb{D}_f(1_Y)$

$\qquad\qquad \uparrow \text{smooth dual}$

f-proper   (f-coadmissible) $f: Y \longrightarrow X$
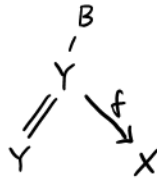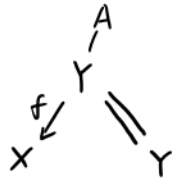
$$F: A \otimes f^* - \quad \dashv \quad G. \ f_!(B \otimes -)$$
$$\dashv \quad f_* \mathrm{Hom}(A, -)$$

① $f_!(B \otimes -) \cong f_* \mathrm{Hom}(A, -)$

App 1. 1_X cpt $\Rightarrow$ A cpt

[Proof. $\mathrm{Hom}(1_X, f_!(B \otimes -)) \cong \mathrm{Hom}(A, -)$ preserves filtered colimit.]

② $\quad p_{1,!}(p_2^* B \otimes -) \cong p_{1,*} \mathrm{Hom}(p_2^* A, -)$ [Verdier's diagonal trick]
$$B \cong p_{1,*} \mathrm{Hom}(p_2^* A, \Delta_! 1_Y)$$

Prop. A is f-proper $\quad \Leftrightarrow \quad f_!(B \otimes A) \cong f_* \mathrm{Hom}(A, A)$ ②b
$$\text{where} \quad B \cong p_{1,*} \mathrm{Hom}(p_2^* A, \Delta_! 1_Y) \quad \text{②a}$$

$\Rightarrow$. ✓

$\Leftarrow$. Writing down adjunctions in 2-category.

App 2.   When $Y = X$, $f = \mathrm{Id}$,
$$A \text{ is } f\text{-proper} \quad \Leftrightarrow \quad \mathrm{Hom}(A, 1_X) \otimes A \cong \mathrm{Hom}(A, A)$$
$$\Leftrightarrow \quad A \text{ is dualizable}$$

App 3.   When $A = 1_Y$, $B = p_{1,*} \Delta_! 1_Y$
$$1_Y \text{ is } f\text{-proper} \quad \Leftrightarrow \quad f_! p_{1,*} \Delta_! 1_Y \cong f_* 1_Y$$
Using this, one can prove results on coh proper.

Write $B = \mathbb{D}_f^{\mathrm{pro}}(A)$. we get $\mathbb{D}_f^{\mathrm{pro}}(\mathbb{D}_f^{\mathrm{pro}}(A)) \cong A$.   (adjunction is symmetric in A & B)
$B = PD_f(A) \qquad \omega_f := PD_f(1_Y)$
$\quad \uparrow$ proper dual

When $\Delta_! = \Delta_*$, $\mathbb{D}_f^{\mathrm{pro}} = \mathrm{Hom}(-, 1_Y)$ is the naive dual.

Relations

open immersion $\longrightarrow$ coh smooth $\underset{\text{if } f^! 1_X \text{ inv}}{\overset{}{\rightleftarrows}}$ 1_Y is f-sm $\underset{}{\overset{\text{if } \Delta \text{ coh étale} \atop f \text{ is n-truncated}}{\rightleftarrows}}$ coh étale

proper $\xrightarrow{\quad \text{if } \Delta_! = \Delta_* \quad}$ 1_Y is f-proper $\underset{}{\overset{\text{if } \Delta \text{ coh proper} \atop f \text{ is n-truncated}}{\rightleftarrows}}$ coh proper