Eine Woche, ein Beispiel
4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.
[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.
[Mum74]: David Mumford. Prym Varieties. I, 1974.

[Ar85]: Arbarello, E., M. Cornalba, P. A. Griffiths and J. Harris. Geometry of Algebraic Curves Volume I. Grundlehren Math. Wiss. Springer, Cham, 1985.
[JKLM23]: Ariyan Javanpeykar, Thomas Krämer, Christian Lehn and Marco Maculan. The monodromy of families of subvarieties on abelian varieties, 2023

https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve
https://mathoverflow.net/questions/402049/induced-action-on-prym-variety
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Goal: Describe some curves (maybe singular) C in A, and describe their degree and the monodromy group.

E.g. 1

Covers

$$C_9 = \{ y^2 = \prod_{j=1}^{10} (x^2 - j) \}$$ has the following covers:

genus

$$\mathrm{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

ramification degree

$$\begin{tikzcd}
 & C_9 \arrow[dl, "20"'] \arrow[d, "4"', "\tilde{f}"] \arrow[dr, "0"', "f"] & \\
\mathbb{P}^1 \arrow[dr, "2"'] & C_4 \arrow[d, "10"'] & C_5 \arrow[dl, "12"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

where

$$C_4 = \{ y^2 = \prod_{j=1}^{10} (t - j) \}$$

$$C_5 = \{ \tilde{y}^2 = t \prod_{j=1}^{10} (t - j) \}$$

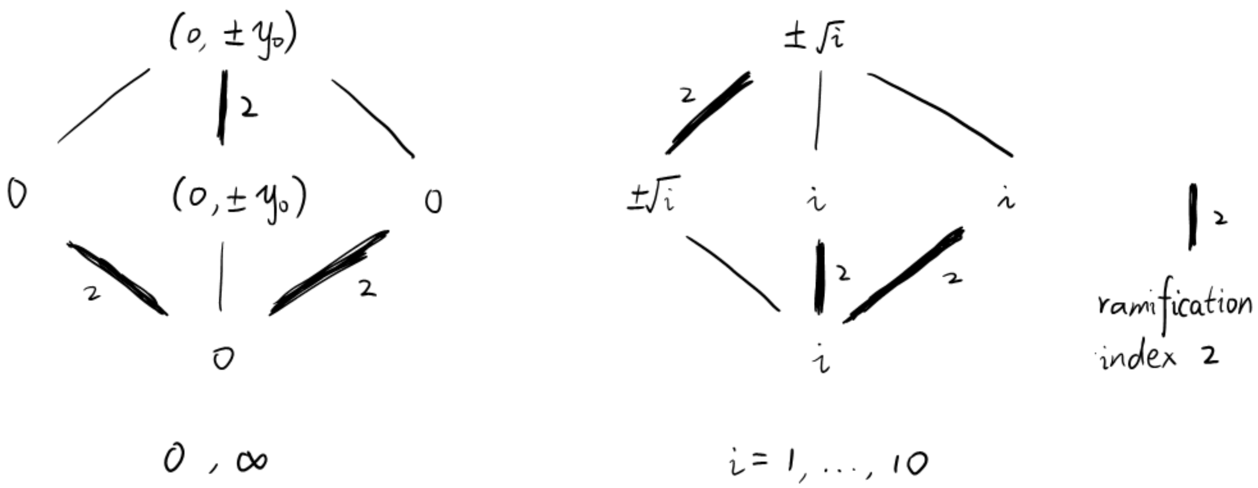
$t = x^2$

$\tilde{y} = xy$

The crspd field extension:

$$\begin{tikzcd}
 & \mathbb{C}(x)[y]/(y^2 - \prod_{j=1}^{10}(x^2-j)) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{C}(x) \arrow[dr, dash] & \mathbb{C}(t)[y]/(y^2 - \prod_{j=1}^{10}(t-j)) \arrow[d, dash] & \mathbb{C}(t)[\tilde{y}]/(\tilde{y}^2 - t\prod_{j=1}^{10}(t-j)) \arrow[dl, dash] \\
 & \mathbb{C}(t) &
\end{tikzcd}$$

Ramification information



$\sigma: \ x \longmapsto -x$, $\ y \longmapsto -y$ — Prym involution

$\tau: \ x \longmapsto -x$, $\ y \longmapsto y$ — C_4 involution

$\sigma\tau: \ x \longmapsto x$, $\ y \longmapsto -y$ — hyperelliptic involution

$$\begin{tikzcd}
 & Id \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle \sigma\tau \rangle \arrow[dr] & \langle \tau \rangle \arrow[d] & \langle \sigma \rangle \arrow[dl] \\
 & \langle \sigma, \tau \rangle &
\end{tikzcd}$$

H

The geometric signature of $C_g \longrightarrow \mathbb{P}^1$ is

$$\begin{array}{ccccccc}
(0\ ; & \langle\tau\rangle, & \langle\tau\rangle, & \langle\sigma\tau\rangle, & \dots & , \langle\sigma\tau\rangle) \\
\text{genus} & 0 & \infty & 1 & \dots & 10
\end{array}$$

Therefore,

$$g_{C_H} = [G:H](g-1) + 1 + \frac{1}{2} \sum_{j \in I} \left([G:H] - |H \backslash G / G_j| \right)$$

$$\overset{\text{when } |H|=2}{\underset{g=0,\ G=\langle\sigma,\tau\rangle}{=\!=}} -1 + \frac{1}{2} \sum_{j \in I} \left(2 - |G/(G_j + H)| \right)$$

$$= \begin{cases} 0, & H = \langle \sigma\tau \rangle \\ 4, & H = \langle \tau \rangle \\ 5, & H = \langle \sigma \rangle \end{cases}$$

Local charts for $f: C_9 \longrightarrow C_5$

$$\begin{tikzcd}
C_9 \arrow[d, "f"] & \overset{R_{xy}}{\mathbb{C}[x][y]/(y^2-\prod_{j=1}^{10}(x^2-j))} \arrow[r, dashed, leftrightarrow] & \overset{R_{uv}}{\mathbb{C}[u][v]/(v^2-\prod_{j=1}^{10}(1-ju^2))} \\
C_5 & \mathbb{C}[t][\tilde{y}]/(\tilde{y}^2-t\prod_{j=1}^{10}(t-j)) \arrow[u, "R_{t\tilde{y}}"'] \arrow[r, dashed, leftrightarrow] & \mathbb{C}[s][\tilde{v}]/(\tilde{v}^2-s\prod_{j=1}^{10}(1-js)) \arrow[u, "R_{s\tilde{v}}"']
\end{tikzcd}$$

$U_{xy} := \operatorname{Spec} R_{xy}$: the name of the local chart

$$\begin{tikzcd}
x^2 & xy \\
t \arrow[u, mapsto] & \tilde{y} \arrow[u, mapsto]
\end{tikzcd}
\qquad
\begin{cases} x = \frac{1}{u} \\ y = \frac{v}{u^{10}} \end{cases}
\qquad
\begin{cases} u = \frac{1}{x} \\ v = \frac{y}{x^{10}} \end{cases}
\qquad
\begin{tikzcd}
u^2 & uv \\
s \arrow[u, mapsto] & \tilde{v} \arrow[u, mapsto]
\end{tikzcd}$$

$$\begin{cases} t = \frac{1}{s} \\ \tilde{y} = \frac{\tilde{v}}{s^6} \end{cases}
\qquad
\begin{cases} s = \frac{1}{t} \\ \tilde{v} = \frac{\tilde{y}}{t^6} \end{cases}$$

Q: I saw from [Mum74, p 326] that

$$f_* \mathcal{O}_{C_9} = \mathcal{O}_{C_5} \oplus \eta$$

for some line bundle $\eta \in \operatorname{Pic}(C_5)$.

Why is that true? How to describe η?

https://math.stackexchange.com/questions/527194/picard-group-of-a-affine-scheme
When A is an integral domain, the Picard group is is known as the class group.

(!) $R_{xy} = \mathbb{C}[x][y]/(y^2-\prod_{j=1}^{10}(x^2-j))$ is not a PID, as

$$y^2 = \prod_{j=1}^{10}(x^2-j)$$

How to compute $\operatorname{Pic}(\operatorname{Spec} R_{xy})$?

Global differential forms

Pulling back differential forms give the following maps:

$$\begin{tikzcd}
 & \langle x^k \frac{dx}{y} \mid k=0,\dots,8\rangle & \\
 & \langle 2x^{2k+1}\frac{dx}{y} \mid k=0,\dots,3\rangle \arrow[u, "\tilde{f}^*"] \arrow[d, equal] & \langle 2x^{2k}\frac{dx}{y} \mid k=0,\dots,4\rangle \arrow[ul, "f^*"'] \arrow[d, equal] \\
0 \arrow[uur] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3\rangle & \langle 2t^k \frac{dt}{\tilde{y}} \mid k=0,\dots,4\rangle \\
 & 0 \arrow[ul, bend left] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Therefore,

$$H^0(C_9;\omega_{C_9}) \cong \tilde{f}^* H^0(C_4;\omega_{C_4}) \oplus f^* H^0(C_5;\omega_{C_5}) \qquad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which crspds to pulling back of divisors)

$$\begin{tikzcd}
 & \langle x^k \frac{dx}{y} \mid k=0,\dots,8\rangle \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] \arrow[ddl] & \\
 & \langle 2x^{2k+1}\frac{dx}{y} \mid k=0,\dots,3\rangle \arrow[d, equal] & \langle 2x^{2k}\frac{dx}{y} \mid k=0,\dots,4\rangle \arrow[d, equal] \\
0 \arrow[dr, bend right] & \langle t^k \frac{dt}{y} \mid k=0,\dots,3\rangle \arrow[d] & \langle 2t^k \frac{dt}{\tilde{y}} \mid k=0,\dots,4\rangle \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

However, since $\mathrm{Jac}(C) = H^0(C;\omega_C)^* / H_1(C;\mathbb{Z})$,
we are working on the dual spaces. The notations are again switched:

$$f^* \rightsquigarrow Nm_f$$
$$f_* \rightsquigarrow f^*$$

One may get

different meaning compared with (1)!

$$H^0(C_9;\omega_{C_9})^* \cong \tilde{f}^* H^0(C_4;\omega_{C_4})^* \oplus f^* H^0(C_5;\omega_{C_5})^* \qquad (2)$$

(co)homology class

This page may be easier to understand, and it helps to understand the previous page.

$$\begin{tikzcd}
 & H^1(C_9;\mathbb{Z}) & \\
0 \arrow[ur] & H^1(C_4;\mathbb{Z}) \arrow[u, "\tilde{f}^*"] & H^1(C_5;\mathbb{Z}) \arrow[ul, "f^*"'] \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Q: Do we have

$$H^1(C_9;\mathbb{Z}) \cong \tilde{f}^* H^1(C_4;\mathbb{Z}) \oplus f^* H^1(C_5;\mathbb{Z})\ ?$$

$$\begin{tikzcd}
 & H_1(C_9;\mathbb{Z}) \arrow[dl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\
0 \arrow[dr] & H_1(C_4;\mathbb{Z}) \arrow[d] & H_1(C_5;\mathbb{Z}) \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

Q: Do we have

$$H_1(C_9;\mathbb{Z})^* \cong \tilde{f}^* H_1(C_4;\mathbb{Z})^* \oplus f^* H_1(C_5;\mathbb{Z})^*\ ?$$

Curve in Prym variety

Define A as the quotient of Jacobians, i.e.,

$$A := Jac(C_9)/_{f^* Jac(C_5)} \cong Prym(C_9/C_5)$$

$$\begin{tikzcd} & C_9 \arrow[r] \arrow[d, "AJ_{C_9}"] & C_4 \arrow[d] \arrow[r] & Jac(C_4) \arrow[dl, dashed, "\exists! \text{ isogeny}"] & \\ Jac(C_5) \arrow[r, "f^*"] & Jac(C_9) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \quad (3)$$

Prop
0. A is isogenous to $Jac(C_4)$;
1. $f^*: Jac(C_5) \longrightarrow Jac(C_9)$ is an isogeny to its image;
2. $\pi \circ AJ_{C_9}$ is not injective, it factors through C_4;
3. $C_4 \longrightarrow A$ is generically injective;
4. $C_4 \longrightarrow A$ produces a sm image of A, outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof. 0. Taking the tangent space of (3), one gets

$$0 \longrightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df_0^*} H^0(C_9, \omega_{C_9})^* \longrightarrow T_0A \longrightarrow 0$$

Combined with (2),

$$T_0A \cong H^0(C_4; \omega_{C_4})^*.$$

Late we will find a natural isogeny $Jac(C_4) \longrightarrow A$.
What's the degree of this isogeny?

1. Since

$$Nm_f \circ f^* = 2\, Id_{Jac(C_5)}$$

$\# Ker f^* < +\infty$ Is f^* inj in this case?

2. For $p_1=(x_0,y_0)$, $p_2=(-x_0,y_0)$, we want to show that

$$\int_{\gamma_1: p\sim p_1} x^{2k+1}\frac{dx}{y} = \int_{\gamma_2: p\sim p_2} x^{2k+1}\frac{dx}{y}$$

$$\mathrm{LHS} = \int_{\tilde{\gamma}_1: p\sim p_2} (-x)^{2k+1}\frac{d(-x)}{y} = \mathrm{RHS}.$$

3.

https://mathoverflow.net/questions/68503/has-anyone-studied-the-prym-map-for-double-covers-with-two-ramification-points
https://arxiv.org/abs/1010.4483: It proves that many Prym maps (C->Prym) are generically finite.

Notice: $C_4 \subset \mathrm{Jac}(C_4)$ is only invariant under $p \mapsto -p$,
not invariant under $p \mapsto p + a_0$.

Otherwise, the Gauss map would be cover of deg >2.
Therefore, after isogeny $C_4 \longrightarrow A$ is still gen inj.
Q: Is this map really inj?

(4) $C_4 \hookrightarrow \mathrm{Jac}(C_4)$ is sm, so after isogeny it is still sm outside of non-injective locus.

Rmk. Suppose $f: \widetilde{C} \longrightarrow C$ is a deg 2 (ramified) covering,
$\sigma: \widetilde{C} \longrightarrow \widetilde{C}$ the crspd involution,
define A as the quotient

$$\begin{tikzcd} & \widetilde{C} \arrow[d, "AJ_{\widetilde{C}}"] & & \\ \mathrm{Jac}(C) \arrow[r, "f^*"] & \mathrm{Jac}(\widetilde{C}) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

one can identify A with $\mathrm{Prym}(\widetilde{C}/C) \subset \mathrm{Jac}(\widetilde{C})$, why?
and the Abel-Prym map is given by

$$\begin{aligned} AP_{\widetilde{C}} = \pi \circ AJ_{\widetilde{C}}: \widetilde{C} &\longrightarrow \mathrm{Jac}(\widetilde{C}) \longrightarrow A \\ p &\longmapsto \mathcal{O}_{\widetilde{C}}(p - p_0) \longmapsto \mathcal{O}_{\widetilde{C}}(p - \sigma(p)) \end{aligned}$$

Therefore, for $p_1 \neq p_2$,

$$\begin{aligned} & AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2) \\ \Leftrightarrow\ & \mathcal{O}_{\widetilde{C}}(p_1 - \sigma(p_1)) = \mathcal{O}_{\widetilde{C}}(p_2 - \sigma(p_2)) \\ \Leftrightarrow\ & \mathcal{O}_{\widetilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\widetilde{C}}(p_2 + \sigma(p_1)) \end{aligned}$$

① When $p_1, p_2 \in \widetilde{C}$ are ramification pts of f,
i.e., $p_1 = \sigma(p_1)$, $p_2 = \sigma(p_2)$

$$AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2).$$

As a result, when f is ramified, $AP_{\widetilde{C}}$ is never injective.

② Now assume $AP_{\widetilde{C}}$ is not inj. $AP_{\widetilde{C}}(p_1) = AP_{\widetilde{C}}(p_2)$.
When $p_1 \neq \sigma(p_1)$ or $p_2 \neq \sigma(p_2)$,

$$\mathcal{L} := \mathcal{O}_{\widetilde{C}}(p_1 + \sigma(p_2)) = \mathcal{O}_{\widetilde{C}}(p_2 + \sigma(p_1))$$

is a l.b. with deg 2 & rk ≥ 1, so $\widetilde{C}$ must be hyperelliptic,
and $\mathcal{L}$ induces a degree 2 map

$$\widetilde{C} \longrightarrow \mathbb{P}^1.$$

③ In the example,

$$\widetilde{C} = C_9, \quad C = C_5,$$

$\sigma: x \longmapsto -x$, $y \longmapsto -y$ — Prym involution

$\tau: x \longmapsto -x$, $y \longmapsto y$ — C_4 involution

$\sigma\tau: x \longmapsto x$, $y \longmapsto -y$ — hyperelliptic involution

we can show directly that

$$AP_{C_9}(\tau(p)) = AP_{C_9}(p).$$

This gives a second proof for Prop (2).

Reason:

$$\mathcal{O}_{C_9}(\tau(p)+\sigma(p)) = \mathcal{O}_{C_9}(\tau(p+\sigma\tau(p))) \qquad \sigma\tau=\tau\sigma$$

$$= \mathcal{O}_{C_9}(p+\sigma\tau(p)) \qquad \sigma\tau \text{ is an hyperelliptic involution}$$

Gauss map

Taking the Gauss map of (3), one gets

$$\begin{tikzcd} C_9 \arrow[r, "2:1", "4"'] \arrow[d, "2:1"', "20"] & C_4 \arrow[d, "2:1", "10"'] \\ R_1 \arrow[r, "2:1", "\text{ram at } 0,\infty"'] \arrow[d] & R_2 \arrow[d] \\ \mathbb{P}^8 \arrow[r, dashed] & \mathbb{P}^3 \end{tikzcd}$$

$$[a_0, \cdots, a_8] \longmapsto [a_1 : a_3 : a_5 : a_7]$$

$\Rightarrow \deg_A C_4 = 6, \quad Gal(\gamma) = S_6 = W(C_3).$

E.g. 2. $C_{11} = \{ y^2 = \prod_{j=1}^{12} (x^2 - j) \}$ has the following covering:

$$\begin{tikzcd}
 & C_{11} \arrow[dl, "24"'] \arrow[d, "4"] \arrow[dr, "0"] & \\
\mathbb{P}^1 \arrow[dr, "2"'] & C_5 \arrow[d, "12"] & C_6 \arrow[dl, "14"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

Curves in Jacobians: ($A = \mathrm{Prym}(C_{11}/C_6)$?)

$$\begin{tikzcd}
 & C_{11} \arrow[r] \arrow[d] & C_5 \arrow[d] \arrow[r] & \mathrm{Jac}(C_5) \arrow[dl, dashed, "\exists!\ \text{isogeny}"] & \\
\mathrm{Jac}(C_6) \arrow[r] & \mathrm{Jac}(C_{11}) \arrow[r] & A \arrow[r] & 0
\end{tikzcd}$$

Gauss map:

$$\begin{tikzcd}
C_{11} \arrow[r, "2:1", "4"'] \arrow[d, "2:1"', "24"] & C_6 \arrow[d, "2:1", "14"'] \\
R_1 \arrow[r, "2:1", "\text{ram at } 0,\infty"'] \arrow[d, hook] & R_2 \arrow[d, hook] \\
\mathbb{P}^{10} \arrow[r, dashed] & \mathbb{P}^5 \\[-20pt]
{[a_0 : \cdots : a_{10}]} \arrow[r, mapsto] & {[a_1 : a_3 : a_5 : a_7 : a_9]}
\end{tikzcd}$$

$\Rightarrow \deg_A C_6 = 14, \quad \mathrm{Gal}(\gamma) = S_{14} = W(C_7)$.

E.g. 3.

$$\begin{tikzcd} C_5 = \{ y^2 = (x^3+x+2)^4 + 1 \} \arrow[d, "f"', "\substack{3:1 \\ 8}"] \\ E = \{ y^2 = t^4 + 1 \} \end{tikzcd}$$

Let us write down the change of variables for f:

$$\begin{tikzcd} C_5 \arrow[d] & \mathbb{C}(x)[y] / \left(y^2 - ((x^3+x+2)^4+1)\right) \arrow[r, phantom, "\cong"] & \mathbb{C}(u)[v'] / \left(v'^2 - ((1+u^2+2u^3)^4 + u^{12})\right) \\ E & \mathbb{C}(t)[y] / \left(y^2 - (t^4+1)\right) \arrow[u, hook] \arrow[r, phantom, "\cong"] & \mathbb{C}(s)[v] / \left(v^2 - (1+s^4)\right) \arrow[u, hook] \end{tikzcd}$$

$$\begin{array}{cccccc} x^3+x+2 & y & \left\{\begin{array}{l} x = \frac{1}{u} \\ y = \frac{v'}{u^6} \end{array}\right. & \left\{\begin{array}{l} u = \frac{1}{x} \\ v' = \frac{y}{x^6} \end{array}\right. & \frac{u^3}{1+u^2+2u^3} & \frac{v'}{(1+u^2+2u^3)^2} \\ \uparrow & \uparrow & & & \uparrow & \uparrow \\ t & y & \left\{\begin{array}{l} t = \frac{1}{s} \\ y = \frac{v}{s^2} \end{array}\right. & \left\{\begin{array}{l} s = \frac{1}{t} \\ v = \frac{y}{t^2} \end{array}\right. & s & v \end{array}$$

Also, we write down the local coordinate charts of f:

$$\begin{tikzcd} C_5 \arrow[d] & \mathbb{C}[x][y] / \left(y^2 - ((x^3+x+2)^4+1)\right) \arrow[r, dashed, leftrightarrow] & \mathbb{C}[u][v'] / \left(v'^2 - ((1+u^2+2u^3)^4 + u^{12})\right) \\ E & \mathbb{C}[t][y] / \left(y^2 - (t^4+1)\right) \arrow[u, hook] \arrow[r, dashed, leftrightarrow] & \mathbb{C}[s][v] / \left(v^2 - (1+s^4)\right) \arrow[u, hook] \end{tikzcd}$$

Curves in Jacobians: $A = \mathrm{Prym}(C_s/E)$?

$$\begin{tikzcd} & C_s \arrow[d, "AJ_{C_s}"] \arrow[dr, dashed] & & \\ E \arrow[r, "f^*"] & \mathrm{Jac}(C_s) \arrow[r, "\pi"] & A \arrow[r] & 0 \end{tikzcd}$$

Prop. 1. $\# \ker f^* < +\infty$;

2. $\pi \circ AJ_{C_s} : C_s \longrightarrow A$ is gen injective ;

3. The image of $C_s \longrightarrow A$ is smooth outside the non-injective locus.