Un example par jour

4.5. nonorientable closed surfaces without boundary $\Rightarrow \begin{cases} \text{lens space} \\ \mathbb{RP}^n, \mathbb{CP}^n \end{cases}$

$$\tilde{\Sigma}_l := \underbrace{\mathbb{RP}^2 \# \cdots \mathbb{RP}^2}_{l\ \mathbb{RP}^2}$$

For a complete statement and proof for four versions of universal coefficiennt theorem, see Section 2.6 in Lecture Notes in Algebraic Topology: https://www.maths.ed.ac.uk/~v1ranick/papers/davkir.pdf

Today: $X = \mathbb{RP}^2$

nonorientable $\Rightarrow \begin{cases} \text{cannot be embedded in } \mathbb{R}^3 \\ \text{can't be realized as a Lie group.} \end{cases}$ embedded in $\mathbb{R}^4$.

universal cover of degree 2 $\pi: S^2 \to \mathbb{RP}^2$

$\Rightarrow$

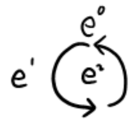
n	1	2	3	4	5	6	$n>1$
$\pi_n(\mathbb{RP}^2)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/n\mathbb{Z}$	$\pi_n(S^2)$

cellular homology

$$0 \to \underset{\mathbb{Z}e^2}{C_2} \to \underset{\mathbb{Z}e^1}{C_1} \to \underset{\mathbb{Z}e^0}{C_0} \to 0$$

$e^2 \mapsto 2e^1$

$e^1 \mapsto 0$



$\chi(\mathbb{RP}^2) = 1$

$\Rightarrow$

n	0	1	2	$n>2$
$H_n(\mathbb{RP}^2)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0

$$0 \leftarrow \underset{\mathbb{Z}e^{2*}}{\mathrm{Hom}_{\mathbb{Z}}(C_2, \mathbb{Z})} \leftarrow \underset{\mathbb{Z}e^{1*}}{\mathrm{Hom}_{\mathbb{Z}}(C_1, \mathbb{Z})} \leftarrow \underset{\mathbb{Z}e^{0*}}{\mathrm{Hom}_{\mathbb{Z}}(C_0, \mathbb{Z})} \leftarrow 0$$

$2e^{2*} \leftarrow\!\shortmid e^{1*}$

$0 \leftarrow\!\shortmid e^{0*}$

$\Rightarrow$

n	0	1	2	$n>2$
$H^n(\mathbb{RP}^2)$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0

$\Rightarrow H^*(\mathbb{RP}^2) = \mathbb{Z}[x]/(2x, x^2)$ deg $x = 2$

Let X be a topo space.

Prop. Universal coefficient thm for cohomology ($\mathbb{Z}$-coefficient)

natural SES

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \mathbb{Z}) \to H^n(X) \xrightarrow{h} \mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z}) \to 0$$

(unnatural) splits

$$\Rightarrow H^n(X) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(X), \mathbb{Z}) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X), \mathbb{Z})$$

Prop. **Lemma 3.8.** *Let A be a K-algebra, and let $(M_i)_{i\in I}$ be a family of A-modules. There are natural isomorphisms*

$$\mathrm{Ext}^m_A\left(\bigoplus_{i\in I} M_i, -\right) \to \prod_{i\in I} \mathrm{Ext}^m_A(M_i, -)$$

and

$$\mathrm{Ext}^m_A\left(-, \prod_{i\in I} M_i\right) \to \prod_{i\in I} \mathrm{Ext}^m_A(-, M_i)$$

for each $m \geq 0$.

Cor. For Hn(X) is finitely generated for all n, e.p. if X has the homotopy type of a CW-complex with finitely many cells in each degree.

we have $H_n(X) \xleftrightarrow{\text{torsion shift}} H^n(X)$

e.g. $H_n(X) \cong \mathbb{Z}^{b_n} \oplus T_n \Rightarrow H^n(X) \cong \mathbb{Z}^{b_n} \oplus T_{n-1}$

$\mathbb{Z}/2\mathbb{Z}$-coefficient (co)homology:

$$0 \longrightarrow C_2' \longrightarrow C_1' \longrightarrow C_0' \longrightarrow 0$$

$$C_2' = \mathbb{Z}/2\mathbb{Z}\, e^2, \quad C_1' = \mathbb{Z}/2\mathbb{Z}\, e^1, \quad C_0' = \mathbb{Z}/2\mathbb{Z}\, e^0$$

$$e^2 \longmapsto 0, \qquad e^1 \longmapsto 0$$

$\Rightarrow$

n	0	1	2	$n>2$
$H_n(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

$$0 \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(C_2', \mathbb{Z}/2\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(C_1', \mathbb{Z}/2\mathbb{Z}) \longleftarrow \mathrm{Hom}_{\mathbb{Z}}(C_0', \mathbb{Z}/2\mathbb{Z}) \longleftarrow 0$$

$$\mathrm{Hom}_{\mathbb{Z}}(C_2', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\, e^{2*}, \quad \mathrm{Hom}_{\mathbb{Z}}(C_1', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\, e^{1*}, \quad \mathrm{Hom}_{\mathbb{Z}}(C_0', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}\, e^{0*}$$

$$0 \longleftarrow\!\shortmid e^{1*}, \qquad 0 \longleftarrow\!\shortmid e^{0*}$$

$\Rightarrow$

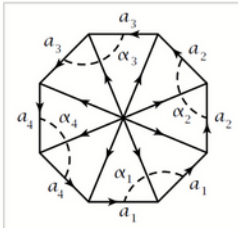
n	0	1	2	$n>2$
$H_n(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0

$\Rightarrow H^*(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}[a]/(a^3)$

$\deg a = 1$

Verify. $a^2 \neq 0$ [Hatcher Ex 3.8]

Example 3.8. The closed nonorientable surface N of genus g can be treated in similar fashion if we use $\mathbb{Z}_2$ coefficients. Using the Δ-complex structure shown, the edges a_i give a basis for $H_1(N;\mathbb{Z}_2)$, and the dual basis elements $\alpha_i \in H^1(N;\mathbb{Z}_2)$ can be represented by cocycles with values given by counting intersections with the arcs labeled α_i in the figure. Then one computes that $\alpha_i \smile \alpha_i$ is the nonzero element of $H^2(N;\mathbb{Z}_2) \approx \mathbb{Z}_2$ and $\alpha_i \smile \alpha_j = 0$ for $i \neq j$. In particular, when $g = 1$ we have $N = \mathbb{RP}^2$, and the cup product of a generator of $H^1(\mathbb{RP}^2;\mathbb{Z}_2)$ with itself is a generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2)$.



X a topo space, R: Abelian group.

Prop. Universal coefficient thm for homology

natural SES

$$0 \longrightarrow H_n(X) \otimes_{\mathbb{Z}} R \xrightarrow{\mu} H_n(X, R) \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), R) \longrightarrow 0$$

(unnatural) splits

$\mathrm{Tor}_n^A(M, N) := H_n(M \otimes_A P_\cdot)$

$$\Rightarrow H_n(X, R) \cong H_n(X) \otimes_{\mathbb{Z}} R \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), R)$$

Ex.

n	0	1	2	$n>2$
$H_n(\mathbb{RP}^2)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
$H_n(\mathbb{RP}^2, \mathbb{R})$	$\mathbb{R}$	0	0	0
$H_n(\mathbb{RP}^2, \mathbb{C})$	$\mathbb{C}$	0	0	0
$H_n(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H_n(\mathbb{RP}^2, \mathbb{Z}/2^3\mathbb{Z})$	$\mathbb{Z}/2^3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H_n(\mathbb{RP}^2, (\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	0

Remark. $S^2 \longrightarrow \mathbb{RP}^2$ is cover, but $H_n(S^2, \mathbb{R}) \ncong H_n(\mathbb{RP}^2, \mathbb{R})$, so for every cover we need to recompute its (co)homology group.

X: topo space A: PID R: an A-module.

Prop. Universal coefficient thm for homology

natural SES:

$$0 \to \mathrm{Ext}^1_A(H_{n-1}(X,A),R) \to H^n(X,R) \xrightarrow{h} \mathrm{Hom}_A(H_n(X,A),R) \to 0$$

(unnatural) splits

$$\Rightarrow H^n(X,R) \cong \mathrm{Hom}_A(H_n(X,A),R) \oplus \mathrm{Ext}^1_A(H_{n-1}(X,A),R)$$

e.p. when $A=\mathbb{Z}$,

$$H^n(X,R) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(X),R) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),R)$$

when $A=R$ is a field,

$$H^n(X,R) \cong \mathrm{Hom}_R(H_n(X,R),R)$$

Cor. For Hn(X) is finitely generated for all n, e.p. if X has the homotopy type of a CW-complex with finitely many cells in each degree.

we have $H_n(X,F) \cong H^n(X,F)$

Rmk. F field,

$b_i(F) := \dim_F H_i(X,F) = \dim_F H^i(X,F)$.

$b_i(\mathbb{Z}/2\mathbb{Z}) \neq b_i(\mathbb{C})$ but $\chi(\mathbb{Z}/2\mathbb{Z}) = \chi(\mathbb{C}) = v-e+f$ for surfaces...

Ex. compute it twice!

n	0	1	2	$n>2$
$H^n(\mathbb{RP}^2)$	$\mathbb{Z}$	0	$\mathbb{Z}/2\mathbb{Z}$	0
$H^n(\mathbb{RP}^2,\mathbb{R})$	$\mathbb{R}$	0	0	0
$H^n(\mathbb{RP}^2,\mathbb{C})$	$\mathbb{C}$	0	0	0
$H^n(\mathbb{RP}^2,\mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H^n(\mathbb{RP}^2,\mathbb{Z}/2^3\mathbb{Z})$	$\mathbb{Z}/2^3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	0
$H^n(\mathbb{RP}^2,(\mathbb{Z}/2\mathbb{Z})^{\oplus 3})$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 3}$	0

Characteristic class I'm new in this field, so in the beginning we just pick up props and apply them.

special vector bundle $\begin{cases} \text{tautological line bundle } \gamma_2^1 \text{ on } \mathbb{RP}^2 \\ \text{tangent bundle } T(\mathbb{RP}^2) =: TX \end{cases}$

Stiefel–Whitney class $w(\gamma_2^1) = 1 + a \qquad \in H^1(\mathbb{RP}^2, \mathbb{Z}/2\mathbb{Z})$

$w(TX) = (1+a)^3 = 1+a+a^2 \in H^1(\mathbb{RP}^2,\mathbb{Z}/2\mathbb{Z}) \Rightarrow w_1^2 = w_2 = 1 \in \mathbb{Z}/2\mathbb{Z}$

Prop. for a real v.b. ξ, ξ is orientable $\Leftrightarrow w_1(\xi)=0$

ξ is spin $\Leftrightarrow w_1(\xi)=0,\ w_2(\xi)=0$

Cor. For line bundle, orientable $\Leftrightarrow$ spin $\Leftrightarrow w_1(\xi)=0 \Leftrightarrow w(\xi)=1 \Leftrightarrow$ trivial

Cor. γ_2^1, TX is not orientable.

Thm (Pontryagin & Thom) fix a cpt smooth mfld M (without boundary), then

$\exists$ cpt smooth mfld N with boundary $\partial N \cong M \Leftrightarrow$ all SW-numbers of M are 0

Cor. $\mathbb{RP}^2$ is not a boundary.

$\mathbb{RP}^{2n}$ is not a boundary.

$\mathbb{RP}^{2n-1}$ is a boundary.

for RP^3, see https://mathoverflow.net/questions/8829/what-manifolds-are-bounded-by-rpodd