Modular form
5. moduli interpretation

1 level structure
2. moduli interpretation of $\Gamma\backslash\mathcal{H}$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S,T \mid S^4=(ST)^6=Id\rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	$\{\pm Id\}$	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S,T \mid S^2=(ST)^3=Id\rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	Id	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$

can't define $SL_2/\mathbb{G}_m$

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A\leq B\leq C$ gp $\quad A\lhd C \not\Rightarrow A\lhd B$

no other restrictions. i.e. the following cases may happen:

$A\lhd B\lhd C$	$A\lhd B\leq C$	$A\lhd B\lhd C$	$A\lhd B\leq C$	$A\leq B\lhd C$	$A\leq B\leq C$
$\vdash \lhd \dashv$	$\vdash \lhd \dashv$				
✓	✓	$C_2\lhd A_4\lhd S_4$	✓	✓	$S_2\leq S_3\leq S_4$

wiki: congruence subgroup

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$\begin{array}{ccc}
\Gamma(N) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\Gamma_1(N) & \twoheadrightarrow & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(N) & \twoheadrightarrow & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \cap \\
\Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{\text{[WWL, Prop 1.4.4]}} & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\cup & & \cup \\
\Gamma^0(N) & \twoheadrightarrow & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \\
\cup & & \cup \\
\Gamma^1(N) & \twoheadrightarrow & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}
\end{array}$$

$\triangledown$ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$,
but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

Ex. Verify the following tables (left comes from right)

$A \triangleleft B$	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	–	✓	✓	✓
$\Gamma_1(N)$	–	–	✓	×
$\Gamma_0(N)$	–	–	–	×
$\Gamma(1)$	–	–	–	–

$A \triangleleft B$	N	B	G
N	–	✓	×
B	–	–	×
G	–	–	–

Ex. show [WWL, 练习 1.4.14]

练习 1.4.14 对所有正整数 N, 证明

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(N)) = N^2 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)) = |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1} \cdot (\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(N)) = N \prod_{p\mid N} \left(1+\frac{1}{p}\right).$$

A. Reduced to computation of $|SL_2(\mathbb{Z}/N\mathbb{Z})|$, $|B(\mathbb{Z}/N\mathbb{Z})|$, $|N(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5, 4, 6$ if you don't understand the process.

Notation: $\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z}^{\oplus 2})_{prim} / (\mathbb{Z}/N\mathbb{Z})^* \overset{[6.3\,M]}{=\!=} \mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: https://arxiv.org/pdf/2010.15543v2.pdf

$\triangledown$ $\mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$\left[\begin{smallmatrix}3\\2\end{smallmatrix}\right] \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathbb{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different piece of $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|SL_2(\mathbb{F}_p)| = p^3 - p$

$|B(\mathbb{F}_p)| = p^2 - p$

$|N(\mathbb{F}_p)| = p$

$\# \mathbb{F}_p^\times = p-1$

② $|SL_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e} - p^{3e-2}$

$|B(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e} - p^{2e-1}$

$|N(\mathbb{Z}/p^e\mathbb{Z})| = p^e$

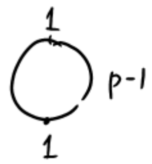
$\# (\mathbb{Z}/p^e\mathbb{Z})^\times = p^e - p^{e-1}$

③ $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p \text{ prime}\\ p\mid N}} (1 - \frac{1}{p^2})$

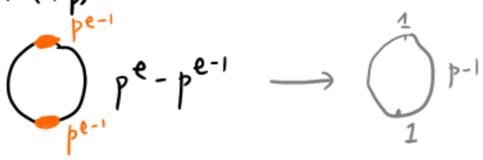
$|B(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p \text{ prime}\\ p\mid N}} (1 - \frac{1}{p})$

$|N(\mathbb{Z}/N\mathbb{Z})| = N$

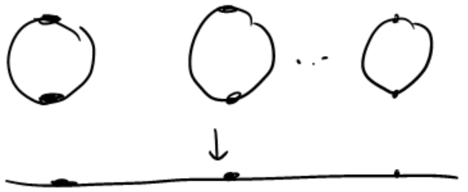
$\#(\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N \prod_{\substack{p \text{ prime}\\ p\mid N}} (1 - \frac{1}{p})$



$\mathbb{P}'(\mathbb{F}_p)$



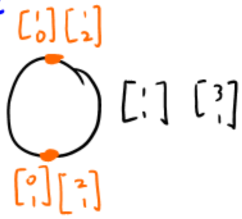
$0 \to p\mathbb{Z}/p^e\mathbb{Z} \to \mathbb{P}'(\mathbb{Z}/p^e\mathbb{Z}) \longrightarrow \mathbb{P}'(\mathbb{Z}/p\mathbb{Z}) \to 0$

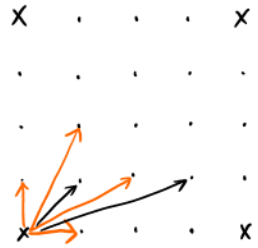


$\mathbb{Z}/p_1^{e_1}\mathbb{Z} \quad \mathbb{Z}/p_2^{e_2}\mathbb{Z} \quad \cdots \quad \mathbb{Z}/p_n^{e_n}\mathbb{Z}$

$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_{k=1}^{n} \mathbb{P}'(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$

E.g. $\mathbb{Z}/4\mathbb{Z}$





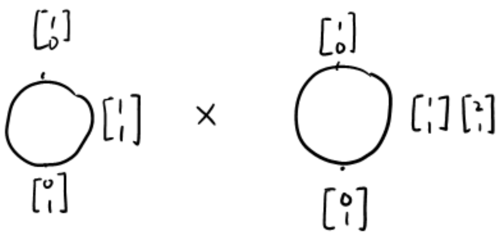
E.g. $\mathbb{Z}/6\mathbb{Z}$

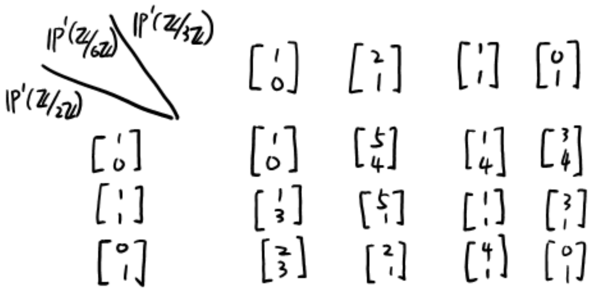
$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f\in S_+ \\ f \text{ homogeneous}}} \operatorname{Spec}(\mathbb{Z}/6\mathbb{Z}[x,y]_f)_0$$

e.g. $(x-2, y-3) \trianglelefteq \mathbb{Z}/6\mathbb{Z}[x,y]$ is not prime.

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3)$$
$$\cong \mathbb{P}^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3)$$

Ex. Use [Vakil, 6.3.M] to compute $\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$. Enjoy it!





Rmk. The original proof is also good, but less geometrically obvious.
(Now you should understand the geometry in every step)

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & SL_2(\mathbb{Z}/p^e\mathbb{Z}) & & SL_2(\mathbb{F}_p) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underbrace{1+pM_2(\mathbb{Z}/p^e\mathbb{Z})}_{p^{4e-4}} & \longrightarrow & GL_2(\mathbb{Z}/p^e\mathbb{Z}) & \longrightarrow & \underbrace{GL_2(\mathbb{F}_p)}_{(p^2-1)(p^2-p)} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \underbrace{(\mathbb{Z}/p^e\mathbb{Z})^\times}_{p^e-p^{e-1}} & & \underbrace{\mathbb{F}_p^\times}_{p-1} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Finally, use Chinese remainder theorem to get

$$SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^{n} SL_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z}) \qquad \square$$

Ex. do the exactly same thing with SL_2 replaced by GL_2 and PGL_2.

Ex. (hard) explore the Tits building & rep theory of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

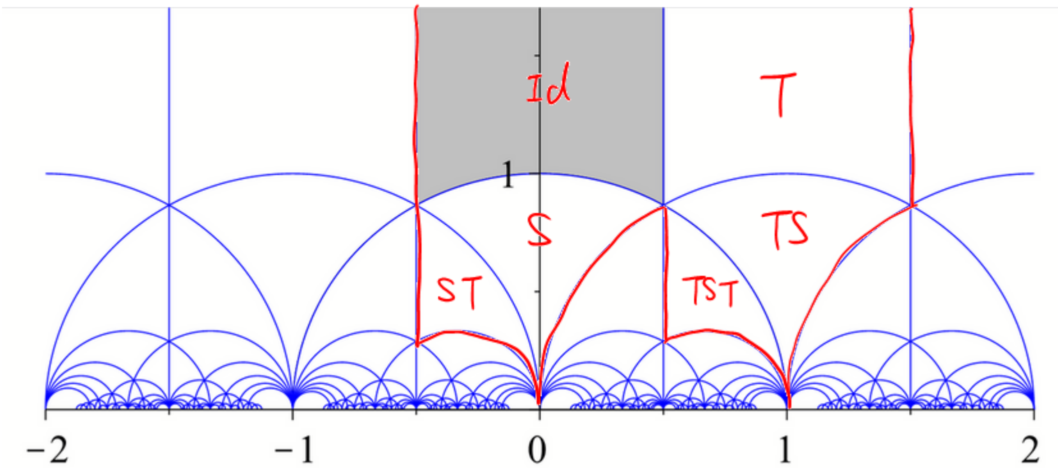
It will be used later on (I believe)

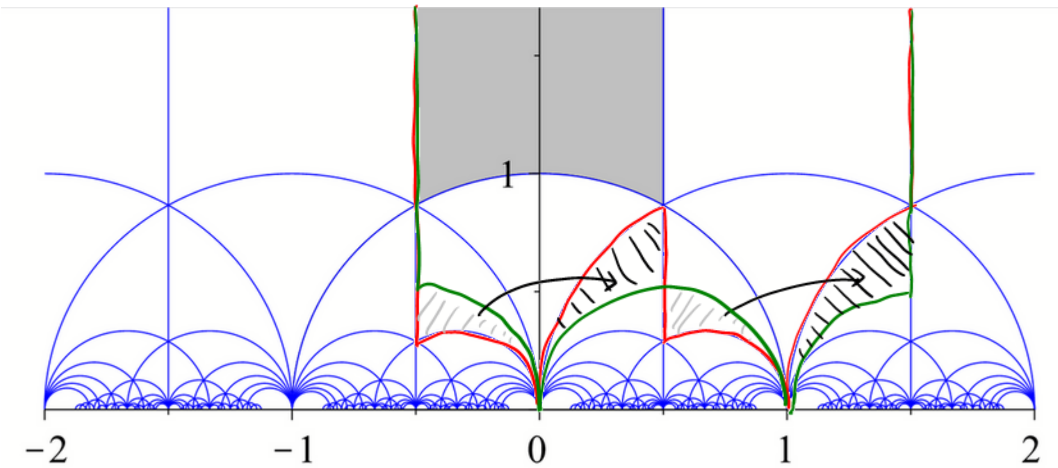
Is the Tits building of $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ functorial?

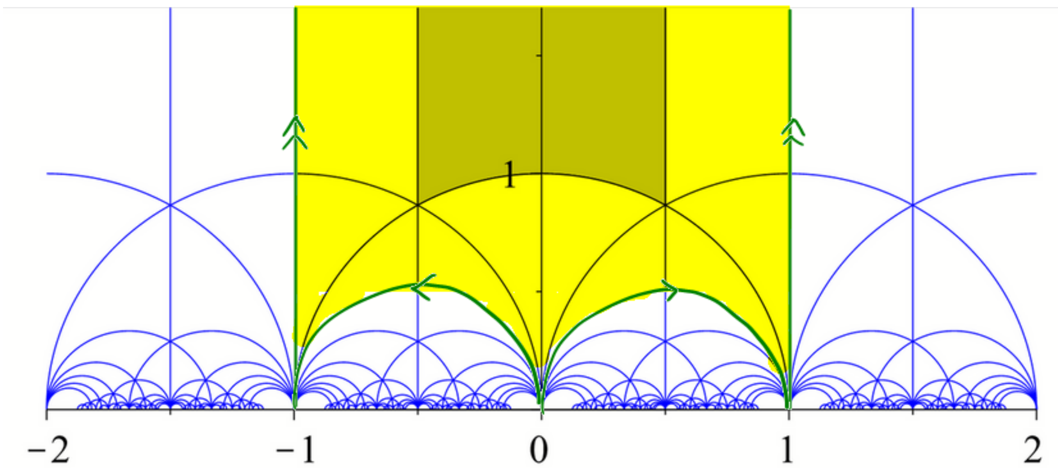
we write left quotient from now on, since it's a left action

Ex. Draw the fundamental domain of $\Gamma(2)\backslash\mathcal{H}$.

Hint. $\Gamma(1)/\Gamma(2) = \{Id, T, S, TS, ST, TST\}$





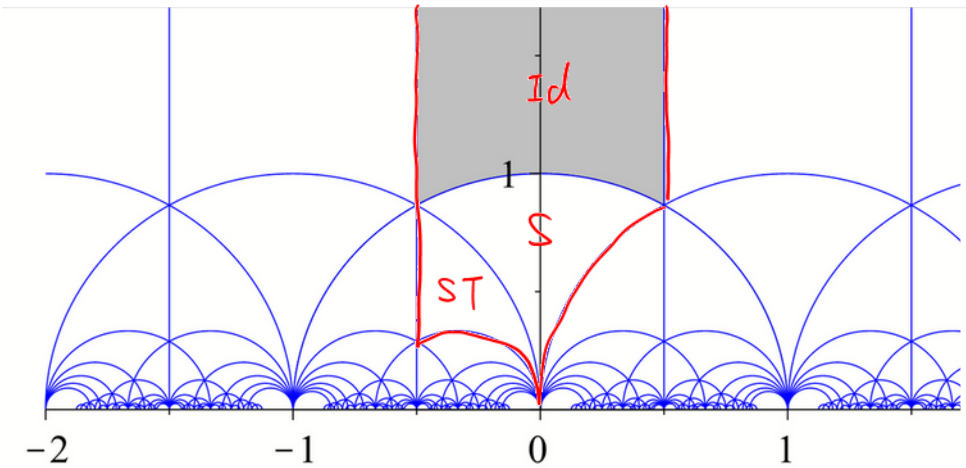


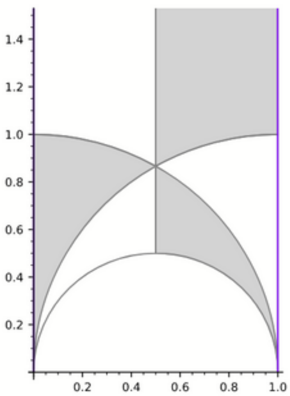
Cor. $\Gamma(2)/\{\pm Id\} = \mathbb{Z} * \mathbb{Z} = \left\langle \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \right\rangle$

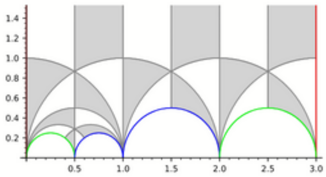
Ex. Draw the fundamental domain of $\Gamma_0(2)\backslash\mathcal{H}$. $\Gamma_0(2)=\Gamma_1(2)$

∇ $SL_2(\mathbb{Z}) \curvearrowright \Gamma_0(2)\backslash\mathcal{H}$ is not well-defined. e.g. $S\,i \neq S\,(i+1)$ in $\Gamma_0(2)\backslash\mathcal{H}$.

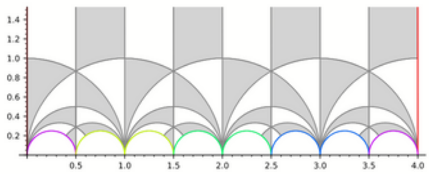
Hint. $\Gamma_0(2)\backslash\Gamma(1) = \{Id, S, ST\}$



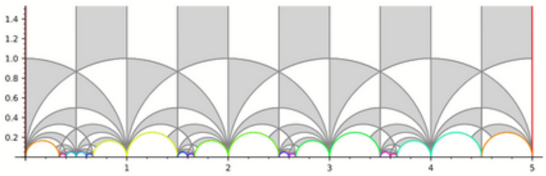




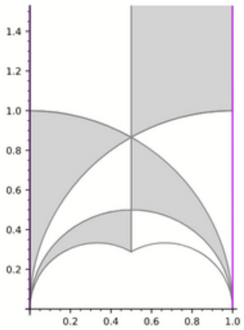
$\Gamma(3)$



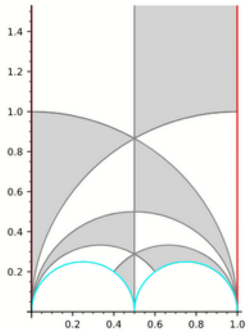
$\Gamma(4)$



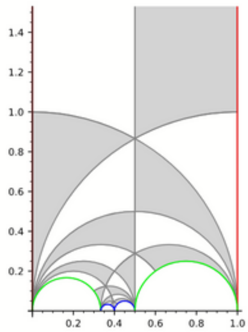
$\Gamma(5)$



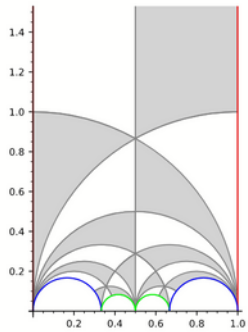
$\Gamma_1(3)=\Gamma_0(3)$



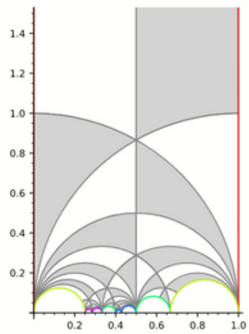
$\Gamma_1(4)=\Gamma_0(4)$



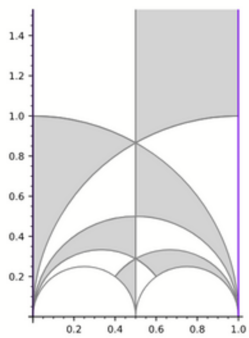
$\Gamma_1(5)$



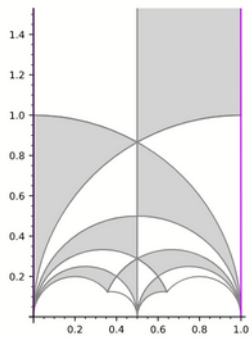
$\Gamma_1(6)=\Gamma_0(6)$



$\Gamma_1(7)$



$\Gamma_0(5)$



$\Gamma_0(7)$

2. moduli interpretation of $\Gamma\backslash\mathcal{H}$

Def. A basis (v_1, v_2) of a lattice $\Lambda \subseteq \mathbb{C}$ is called oriented if $\mathrm{Im}\,\frac{v_1}{v_2} > 0$.

Def (Weil pairing) [WWL, 注记8.5.9, 定义3.8.9, 练习3.8.10]

For $N \in \mathbb{Z}_{\geq 1}$, $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$, $\mathrm{Im}\,\frac{u}{v} > 0$, we define the Weil pairing e_N:

$$\begin{array}{ccc}
E[N] \times E[N] & \xrightarrow{\ e_N\ } & \\
\wr\wr & & \\
\frac{1}{n}\Lambda/\Lambda \times \frac{1}{n}\Lambda/\Lambda & & \\
\wr\wr & & \\
(\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \times (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} & \longrightarrow & \mu_N \cong (\mathbb{Z}/N\mathbb{Z}, +) \\
\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right) & \longmapsto & \zeta_N^{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \longmapsto \begin{vmatrix} a & b \\ c & d \end{vmatrix}
\end{array}$$

$a\frac{u}{N} + c\frac{v}{N} \mapsto \begin{pmatrix} a \\ c \end{pmatrix}$

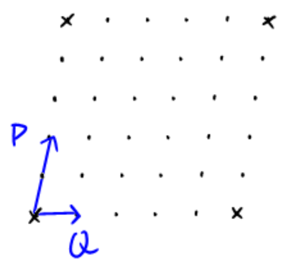
Ex. Let $e_1, e_2 \in E[n]$.

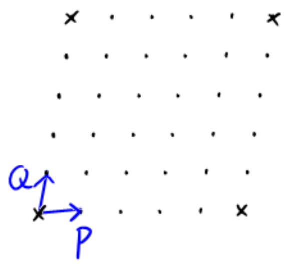
1. e_N is antisymmetric and bilinear.
   $e_N(\gamma(e_1, e_2)) = \zeta_N^{\det\gamma}\, e_N(e_1, e_2)$ $\forall \gamma \in GL_2(\mathbb{Z}/N\mathbb{Z})$
   e.g. e_N only depends on E and N (does not depend on Λ and u, v)
2. $e_N(e_1, e_2) \in \mu_N^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \iff E[N] = \langle e_1, e_2 \rangle_{\mathbb{Z}}$
   $e_N(e_1, e_2) = \zeta_N \longmapsto 1 \iff \exists P, Q \in \frac{1}{N}\Lambda$, $\bar{P} = e_1$, $\bar{Q} = e_2$, (NP, NQ) is an oriented basis of Λ.

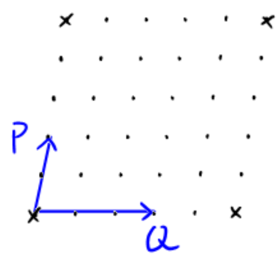
Def. (e_1, e_2) is called a pretty oriented basis of $E[N]$, if $e_N(e_1, e_2) = \zeta_N$.

In [KM85], this is called Drinfeld basis.

Ex. $N = 5$







$e_N(\bar{P}, \bar{Q}) = \zeta_5^2$

$(5P, 5Q)$ is not a basis of Λ.

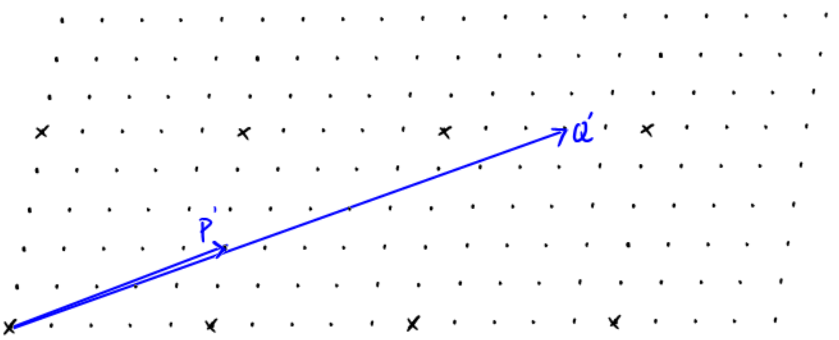
$e_N(\bar{P}, \bar{Q}) = \zeta_5^4 = \zeta_5^{-1}$

$(5P, 5Q)$ is a basis of Λ, but not an oriented basis.

$e_N(\bar{P}, \bar{Q}) = \zeta_5^6 = \zeta_5$

$(5P, 5Q)$ is not a basis of Λ but $(5P', 5Q')$ is an oriented basis.

$$\underset{\substack{\cap \\ SL_2(\mathbb{Z})}}{\begin{pmatrix} 2 & 5 \\ 5 & 13 \end{pmatrix}} \overset{\mathrm{mod}\,5}{\equiv} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$



Recall: For $E = \mathbb{C}/\Lambda$, $E[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$

Main Thm. We have the following moduli interpolations (E: any cplx EC curve)

$$\begin{array}{ccc}
\left\{(E,\alpha) \,\middle|\, \begin{array}{l}\alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[N] \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N\end{array}\right\}/\sim & \xrightarrow{\sim} & \Gamma(N)\backslash\mathcal{H} \\
\text{E with a pretty oriented basis } (e_1,e_2) & & \\
\Big\downarrow {\scriptstyle \beta = \alpha(0,-)} & & \Big\downarrow {\scriptstyle N} \\
\{(E,\beta) \mid \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]\}/\sim & \longrightarrow & \Gamma_1(N)\backslash\mathcal{H} \\
\text{E with an N-torsion pt of E} & & \\
\Big\downarrow {\scriptstyle C = \operatorname{Im}\beta} & & \Big\downarrow {\scriptstyle \begin{cases} \frac{N}{2}\prod_{p\mid N}(1-\frac{1}{p}) & N \neq 2 \\ 1 & N = 2\end{cases}} \\
\{(E,\mathcal{F}) \mid \mathcal{F}: 0 \subseteq C \subseteq E[N]\}/\sim & \longrightarrow & \Gamma_0(N)\backslash\mathcal{H} \\
\text{E with } C \subseteq E[N],\ C \cong \mathbb{Z}/N\mathbb{Z} & & \\
\Big\downarrow & & \Big\downarrow {\scriptstyle N\prod_{p\mid N}(1+\frac{1}{p})} \\
\{\text{cplx EC } E\}/\sim & \longrightarrow & \Gamma(1)\backslash\mathcal{H}
\end{array}$$

Idea. A pretty oriented basis on E[N] gives us a oriented basis on E up to Γ(N)-action;

coefficient has to be 1, so that (v_1+bv_2, v_2) is a pretty oriented basis.

$\begin{pmatrix}1 & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}v_1+bv_2\\ v_2\end{pmatrix}$ $\Rightarrow$ an n-torsion pt v_2

$\begin{pmatrix}a & b\\ 0 & d\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}av_1+bv_2\\ dv_2\end{pmatrix}$ $\Rightarrow$ a flag $0 \subseteq \underset{\langle v_2\rangle}{\overset{\shortparallel}{C}} \subseteq \underset{\langle v_1,v_2\rangle}{\overset{\shortparallel}{E[N]}}$

Proof. For $\Gamma(N)\backslash\mathcal{H}$,

$$\begin{aligned}
\text{LHS} &\cong \left\{(\Lambda,\alpha) \,\middle|\, \begin{array}{l}\alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \longrightarrow \frac{1}{N}\Lambda/\Lambda \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,e_1,e_2) \,\middle|\, \begin{array}{l} e_1,e_2 \in \frac{1}{N}\Lambda/\Lambda \\ e_N(e_1,e_2) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,z_1,z_2,e_1,e_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_1 = \frac{z_1}{N} \quad e_2 = \frac{z_2}{N} \\ e_N(e_1,e_2) = \zeta_N\end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \left\{(\Lambda,z_1,z_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ (z_1,z_2) \text{ is an oriented basis of } \Lambda\end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \left\{(z_1,z_2) \in (\mathbb{C}-\{0\})^2 \,\middle|\, \operatorname{Im}\frac{z_1}{z_2} > 0\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowright \Gamma(N)\end{array} \\
&\cong \Gamma(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}$$

For $\Gamma_1(N)\backslash\mathcal{H}$,

$$
\begin{aligned}
\text{LHS} &\cong \{(\Lambda,\beta) \mid \beta : \mathbb{Z}/N\mathbb{Z} \hookrightarrow \tfrac{1}{N}\Lambda/\Lambda\} \circlearrowleft \mathbb{C}^\times \\
&\cong \{(\Lambda, e_2) \mid e_2 \in \tfrac{1}{N}\Lambda/\Lambda,\ \mathrm{order}(e_2) = N\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{ (\Lambda, z_1, z_2, e_2) \;\middle|\; \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_2 = \frac{z_2}{N} \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array} \right\} \begin{array}{l} \circlearrowleft \mathbb{C}^\times \\ \\ \circlearrowleft \Gamma_1(N) \end{array} \qquad \text{If not, set } (z_1,z_2) \mapsto (-z_1, z_2) \\
&\cong \left\{ (\Lambda, z_1, z_2) \;\middle|\; \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array} \right\} \begin{array}{l} \circlearrowleft \mathbb{C}^\times \\ \circlearrowleft \Gamma_1(N) \end{array} \\
&\cong \Gamma_1(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}
$$

For $\Gamma_0(N)\backslash\mathcal{H}$,

$$
\begin{aligned}
\text{LHS} &\cong \{(\Lambda, C) \mid C \subseteq \tfrac{1}{N}\Lambda/\Lambda,\ C \cong \mathbb{Z}/N\mathbb{Z}\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{ (\Lambda, z_1, z_2, C) \;\middle|\; \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ C = \left(\frac{z_2}{N}\right) \subseteq \frac{1}{N}\Lambda/\Lambda \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array} \right\} \begin{array}{l} \circlearrowleft \mathbb{C}^\times \\ \\ \circlearrowleft \Gamma_0(N) \end{array} \\
&\cong \left\{ (\Lambda, z_1, z_2) \;\middle|\; \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ \mathrm{Im}\,\frac{z_1}{z_2} > 0 \end{array} \right\} \begin{array}{l} \circlearrowleft \mathbb{C}^\times \\ \circlearrowleft \Gamma_0(N) \end{array} \\
&\cong \Gamma_0(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}
$$

□

Rmk. If you observe carefully, you will find out that what we prove actually is

$\mathbb{C}^\times = SO_2(\mathbb{R}) \times \mathbb{R}_{>0}$

$$
\begin{tikzcd}
\{(E,\alpha)\} \arrow[r, "\sim"] \arrow[d] & \Gamma(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong \Gamma(N)\backslash\mathcal{H} \arrow[d] \\
\{(E,\beta)\} \arrow[r, "\sim"] \arrow[d] & \Gamma_1^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong \Gamma_1(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong \Gamma_1(N)\backslash\mathcal{H} \arrow[d] \\
\{(E,\mathcal{F})\} \arrow[r, "\sim"] \arrow[d] & \Gamma_0^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong \Gamma_0(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong \Gamma_0(N)\backslash\mathcal{H} \arrow[d] \\
\{E\} \arrow[r, "\sim"] & GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{Z})\backslash\mathcal{H}
\end{tikzcd}
$$

where

$$
\begin{array}{ccl}
\Gamma_1^{\pm}(N) & \longrightarrow & \begin{pmatrix} 1 & * \\ 0 & \pm 1 \end{pmatrix} \\
\cap & & \quad\cap \\
\Gamma_0^{\pm}(N) & \longrightarrow & \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \quad\cap \\
GL_2(\mathbb{Z}) & \twoheadrightarrow & GL_2(\mathbb{Z}/N\mathbb{Z})_{\det = \pm 1} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})
\end{array}
$$

This page is by no means complete. Be skeptical about every result here.

Def [Milne. LEC, Def 6.1] Let $Y \to X$ f.flat, G finite, $G \subset \mathrm{Aut}(Y/X)$. $Y \to X$ is called a Galois covering with gp G if

$$\begin{aligned} G \times Y &\longrightarrow Y \times_X Y \\ (g, y) &\longmapsto (gy, y) \end{aligned}$$

is an iso.

I believe that

without condition that $e_E(\alpha(1,0), \alpha(0,1)) = \zeta_N$

$$\begin{aligned}
\bigsqcup_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathcal{H} \cong \{(E,\alpha)\}/\sim
&\cong \left\{ (E, Y, \alpha) \;\middle|\; \begin{array}{l} E/Y \text{ Galois étale} \\ \alpha: \mathrm{Gal}(E/Y) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} /\sim \\
&= \{ (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\text{-torsors } E \twoheadrightarrow Y \}/\sim \\
&\cong \left\{ (E, X, \alpha') \;\middle|\; \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} (\mu_N)^{\oplus 2} \end{array} \right\} /\sim \\
&= \{ (\mu_N)^{\oplus 2}\text{-torsors } X \twoheadrightarrow E \}/\sim
\end{aligned}$$

$$\begin{aligned}
\Gamma_1(N)\backslash\mathcal{H} \cong \{(E,\beta)\}/\sim
&\cong \left\{ (E, Y, \beta) \;\middle|\; \begin{array}{l} E/Y \text{ Galois étale} \\ \alpha: \mathrm{Gal}(E/Y) \xrightarrow{\sim} \mathbb{Z}/N\mathbb{Z} \end{array} \right\} /\sim \\
&= \{ \mathbb{Z}/N\mathbb{Z}\text{-torsors } E \twoheadrightarrow Y \}/\sim \\
&\cong \left\{ (E, X, \alpha') \;\middle|\; \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} \mu_N \end{array} \right\} /\sim \\
&= \{ \mu_N\text{-torsors } X \twoheadrightarrow E \}/\sim
\end{aligned}$$

$$\begin{aligned}
\Gamma_0(N)\backslash\mathcal{H} \cong \{(E,\mathcal{F})\}/\sim
&\cong \left\{ (E, Y) \;\middle|\; \begin{array}{l} E/Y \text{ Galois étale} \\ \mathrm{Gal}(E/Y) \cong \mathbb{Z}/N\mathbb{Z} \end{array} \right\} /\sim \\
&= \{ \mathbb{Z}/N\mathbb{Z}\text{-isogeny } E \twoheadrightarrow Y \} \\
&\cong \left\{ (E, X) \;\middle|\; \begin{array}{l} X/E \text{ Galois étale} \\ \mathrm{Gal}(X/E) \cong \mu_N \end{array} \right\} /\sim \\
&= \{ \mu_N\text{-isogeny } X \twoheadrightarrow E \}
\end{aligned}$$

My confusion: In [https://arxiv.org/pdf/1510.05687.pdf](and its historical version), it's claimed that

without condition that $e_E(\alpha(1,0), \alpha(0,1)) = \zeta_N$

$$\begin{aligned}
\bigsqcup_{i \in (\mathbb{Z}/N\mathbb{Z})^\times} \Gamma(N)\backslash\mathcal{H} \cong \{(E,\alpha)\}/\sim
&\cong \left\{ (E, X, \alpha') \;\middle|\; \begin{array}{l} X/E \text{ Galois étale} \\ \alpha': \mathrm{Gal}(X/E) \xrightarrow{\sim} (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \end{array} \right\} /\sim \\
&= \{ (\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\text{-torsors } X \twoheadrightarrow E \}/\sim
\end{aligned}$$