

Belyi's Theorem.

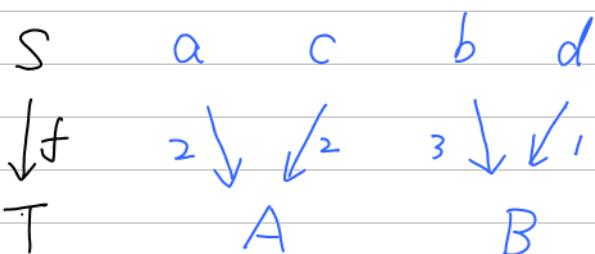
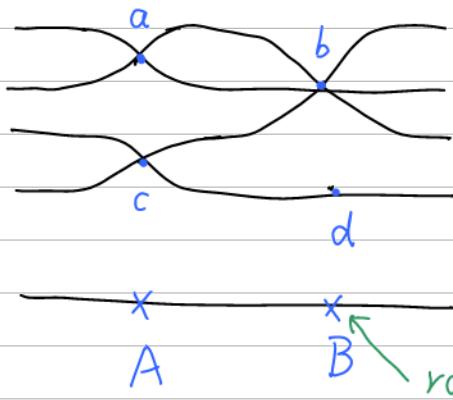
1. Ramification

2. Irreducible algebraic curves

3. Statement of Belyi's theorem

4. A one-side proof: (a) \Rightarrow (b)

1. Ramification



ramification points

$$\text{Ram}(f) = \{a, b, c\}$$

$$\text{Branch}(f) = \{A, B\}$$

$$f(\text{Ram}(f)) = \text{Branch}(f)$$

$$f^{-1}(\text{Branch}(f)) \subseteq \text{Ram}(f)$$

$$\begin{array}{c} S - \text{Ram}(f) \\ \downarrow \text{covering map} \\ T - \text{Branch}(f) \end{array}$$

定义 B.3.1 设 $e \in \mathbb{Z}_{\geq 1} \sqcup \{\infty\}$.

◦ 若 e 有限, 按 $z \mapsto z^e$ 定义单位开圆盘到自身的连续满射 $f_e : \mathcal{D} \rightarrow \mathcal{D}$;

◦ 若 $e = \infty$, 按 $\tau \mapsto \exp(2\pi i\tau)$ 定义 $\mathcal{H} \sqcup \{\infty\}$ 到 \mathcal{D} 的连续满射 f_e , 映 ∞ 为 0.

两种情形下都称 f_e 为 e 次标准分歧复叠.

定义 B.3.3 连续满射 $f : S \rightarrow T$ 具备以下性质时称为分歧复叠: 对每个 $t \in T$ 存在开邻域 $V \ni t$ 和 S 的一族无交开子集 $\{U_i\}_{i \in I}$ 使得 $f^{-1}(V) = \bigsqcup_{i \in I} U_i$, 而且对每个 $i \in I$, 皆存在 $e \in \mathbb{Z} \sqcup \{\infty\}$ 和从 $U_i \xrightarrow{f} V$ 到标准分歧复叠 f_e 的同胚, 使 t 对应到 $0 \in \mathcal{D}$.

定义-定理 B.3.4 设 $f : S \rightarrow T$ 是分歧复叠, 则对任意 $s \in S$ 及其邻域 U_1 , 总存在开邻域 $U \ni s$, $U \subset U_1$ 使得 $f^{-1}(f(s)) \cap U = \{s\}$ 而 $U \setminus \{s\} \xrightarrow{f} f(U \setminus \{s\})$ 是复叠映射, 其次数 $e(s)$ 称为 f 在 s 处的分歧指数, 它只和 s 与 f 相关.

定义 B.3.5 设 $f : S \rightarrow T$ 为分歧复叠. 满足 $e(s) > 1$ 的点 $s \in S$ 称为分歧点. 全体分歧点构成 S 的子集 $\text{Ram}(f)$.

Ex. 1) Calculate $\text{Branch}(f)$, where

$$f : \mathbb{P}\mathbb{C}^1 \longrightarrow \mathbb{P}\mathbb{C}^1 \quad z \mapsto z^3 + \frac{1}{z^3}$$

2) Suppose $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ are maps between R.S.s,
proof $\text{Branch}(g \circ f) = \text{Branch}(g) \cup g(\text{Branch}(f))$

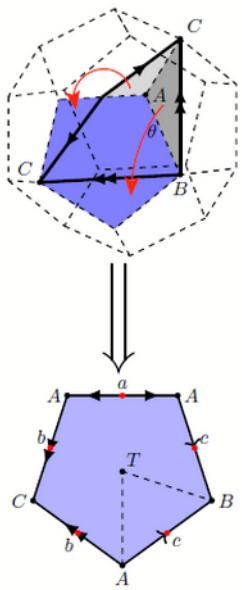


图 1.7

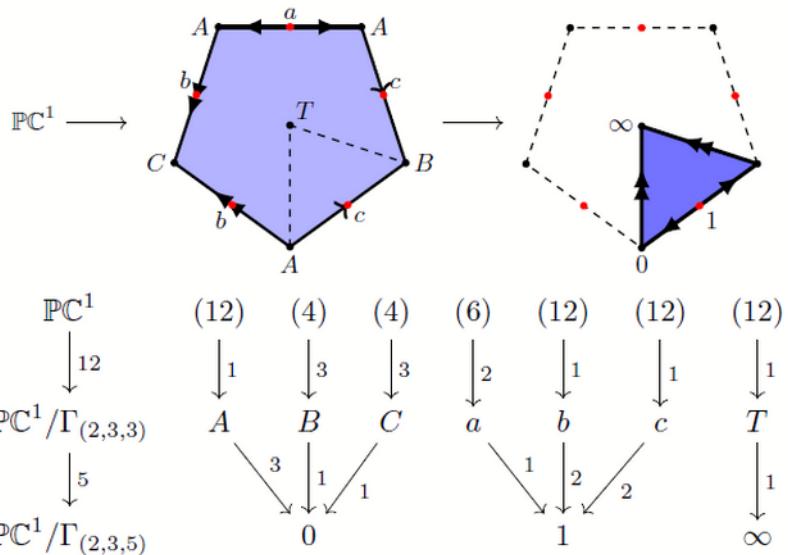


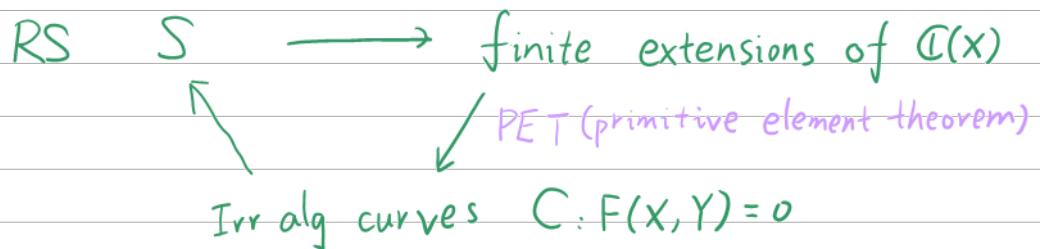
图 1.8 分歧点与分歧指标

(useless)
figure

2 Irreducible algebraic curves.

Remark 1.94 We have shown the equivalence between the following classes of objects:

- (1) Compact Riemann surfaces S .
- (2) Function fields in one variable (i.e. finite extensions of $\mathbb{C}(X)$).
- (3) Irreducible algebraic curves $C : F(X, Y) = 0$.



Theorem 1.86 Let

$$\begin{aligned} F(X, Y) &= p_0(X)Y^n + p_1(X)Y^{n-1} + \cdots + p_n(X) \\ &= q_0(Y)X^m + q_1(Y)X^{m-1} + \cdots + q_m(Y) \end{aligned}$$

be an irreducible polynomial. If $n \geq 1$ define

$$S_F^X = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0, F_Y(x, y) \neq 0, p_0(x) \neq 0\}$$

and, similarly, if $m \geq 1$ set

$$S_F^Y = \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0, F_X(x, y) \neq 0, q_0(y) \neq 0\}$$

Then:

- (i) S_F^X and S_F^Y are connected Riemann surfaces on which the coordinate functions \mathbf{x} and \mathbf{y} are holomorphic functions.
- (ii) There exists a unique compact and connected Riemann surface $S = S_F$ that contains S_F^X and S_F^Y .
- (iii) The coordinate functions \mathbf{x} and \mathbf{y} extend to meromorphic functions on S .
- (iv) The branching points of \mathbf{x} (resp. \mathbf{y}) lie in the finite set $S \setminus S_F^X$ (resp. $S \setminus S_F^Y$).

E.g. Klein quartic $F(X, Y) = X^3Y + Y^3 + X$

$$\begin{aligned} S_F^X &= \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0, x^3 + 3y^2 \neq 0, 1 \neq 0\} \\ S_F^Y &= \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0, 3x^2 + 1 \neq 0, y \neq 0\} \\ S_F^X \cup S_F^Y &= \{(x, y) \in \mathbb{C}^2 \mid F(x, y) = 0\} - \{(0, 0)\} \end{aligned}$$

$$\begin{array}{ccc} S_F^X & & S_F \\ \downarrow & \Rightarrow & \downarrow \\ \mathbb{CP}^1 - \{\text{finite pts}\} & & \mathbb{CP}^1 \end{array} \quad \text{projective} \rightarrow$$

Rmk. 1. S_F is generally not the corresponding proj variety
 2. If the corresponding proj variety is RS (no singularity)
 \Rightarrow it's the required one. Riemann Surface

Def (defined over $K \subseteq \mathbb{C}$)

We shall say that a Riemann surface S is defined over a field $K \subset \mathbb{C}$ (or that K is a field of definition of S) if $S \simeq S_F$ for some irreducible polynomial $F(X, Y) = \sum a_{ij}X^iY^j$ with coefficients $a_{ij} \in K$.

$$S \simeq S_F \quad F(X, Y) = \sum a_{ij}X^iY^j \quad a_{ij} \in K.$$

E.g. Elliptic curve over \mathbb{Q}

$$(1) \quad S_{F_1}, \quad F_1(X, Y) = Y^2 - [X^3 - 1]$$

$$(2) \quad S_{F_2}, \quad F_2(X, Y) = Y^2 - [X^3 - \pi^3] \quad (\mathbf{x}, \mathbf{y})$$

Reason:

$$S_{F_1}, \quad F_1(W, Z) = Z^2 - [W^3 - 1]$$

$$\left(\frac{y}{\pi\sqrt{\pi}}\right)^2 - \left[\left(\frac{x}{\pi}\right)^3 - 1\right] = 0 \quad \text{well-defined } \left(\frac{x}{\pi}, \frac{y}{\pi\sqrt{\pi}}\right)$$

3. Claim of Belyi's theorem

Philosophy: functions reflect properties of space.

- topology
- geometry
- arithmetic

Theorem 3.1 (Belyi's Theorem) Let S be a compact Riemann surface. The following statements are equivalent:

- S is defined over $\bar{\mathbb{Q}}$.
- S admits a morphism $f : S \rightarrow \mathbb{P}^1$ with at most three branching values.

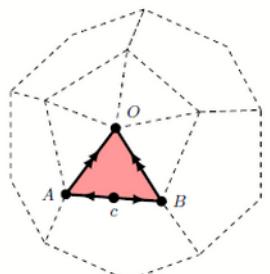
defined over $\bar{\mathbb{Q}} \Leftrightarrow$ admits a Belyi fct.

Rmk $n := \# \text{Branch}(f)$

- when $n=0, 1, 2$, $S \cong \mathbb{P}^1$;
- when $n=3$, we can assume $\text{Branch}(f) = \{\infty, 0, 1\}$

Eg. of (b).

$$1) \quad \mathbb{P}\mathbb{C}^1 \xrightarrow{\pi} \mathbb{P}\mathbb{C}^1/\Gamma \cong \mathbb{P}\mathbb{C}^1 \quad (\text{Of course } \mathbb{P}\mathbb{C}^1 \text{ is defined over } \bar{\mathbb{Q}})$$

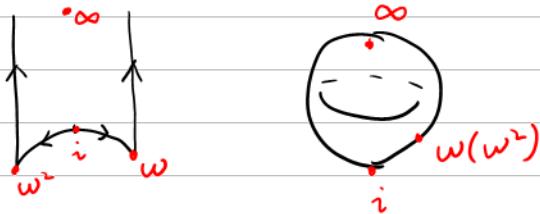


$$\begin{array}{ccccccc} \mathbb{P}\mathbb{C}^1 & & A, B, \dots (12) & & c, \dots (20) & & O, \dots (30) \\ \pi \downarrow 60 & & \downarrow 5 & & \downarrow 3 & & \downarrow 2 \\ \mathbb{P}\mathbb{C}^1/\Gamma_{(2,3,5)} & & 0 & & 1 & & \infty \end{array}$$

在 $0, 1, \infty$ 处分歧, π is a Belyi fct

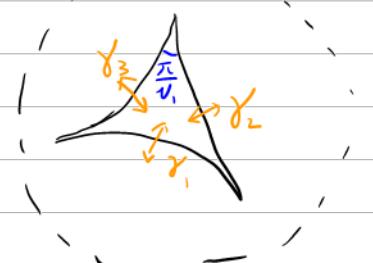
图 1.2 例: 正十二面体

$$2) \quad H^*/\Gamma(N) \xrightarrow{\pi} H^*/SL_2(\mathbb{Z}) \xrightarrow[\sim]{j} \mathbb{P}\mathbb{C}^1$$



The ramification points here are called elliptic points.
"Why modular form / modular space is an arithmetic object"

3) $\text{ID}/K \longrightarrow \text{ID}/\Gamma$
 \downarrow
 to make ID/K opt



$$\Gamma := \langle \gamma_1, \gamma_2, \gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_2 \gamma_3 \rangle$$

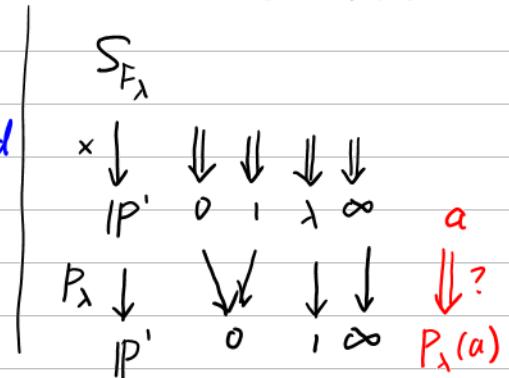
4. a one-side proof: $(a) \Rightarrow (b)$

(1) an example

e.g. for S_{F_λ} , $F_\lambda: Y^2 = X(X-1)(X-\lambda)$, $\lambda \in \mathbb{Q} \cap (0, 1)$
 ① F_λ is defined over $\bar{\mathbb{Q}}$
 ② X : ramified at $\infty, 0, 1, \lambda$.
 ③ adjustment: $P_\lambda \circ X$ ramified at 3 points, where

$$\lambda = \frac{m}{m+n}, \quad P_\lambda(x) = \frac{(m+n)^{m+n}}{m^m n^n} x^m (1-x)^n$$

$$0 = P'_\lambda(a) = \frac{(m+n)^{m+n}}{m^m n^n} [ma + n(1-a)] a^{m-1} (1-a)^{n-1} \Rightarrow a = 0, 1, \lambda$$



Thm. (Belyi's Theorem) Suppose S is a cpt RS, then TFAE:

(a) S is defined over $\bar{\mathbb{Q}}$.

(b0) $\exists f: S \rightarrow \bar{\mathbb{P}}'$ with $\text{Branch}(f) = \{\infty, 0, 1\}$.

(b1) $\exists f: S \rightarrow \bar{\mathbb{P}}'$ with $\#\text{Branch}(f) \leq 3$

(b2) $\exists f: S \rightarrow \bar{\mathbb{P}}'$ with $\text{Branch}(f) \subseteq \bar{\mathbb{Q}} \cup \{\infty\}$.

(b3) $\exists f: S \rightarrow \bar{\mathbb{P}}'$ with $\text{Branch}(f) \subseteq \bar{\mathbb{Q}} \cup \{\infty\}$

Sketch

(b0)

Rmk $\uparrow \downarrow$

(b1)

(2)(5) $\uparrow \downarrow$

(b2)

(4) $\uparrow \downarrow$

(b3)

(a)

Verify

(3)

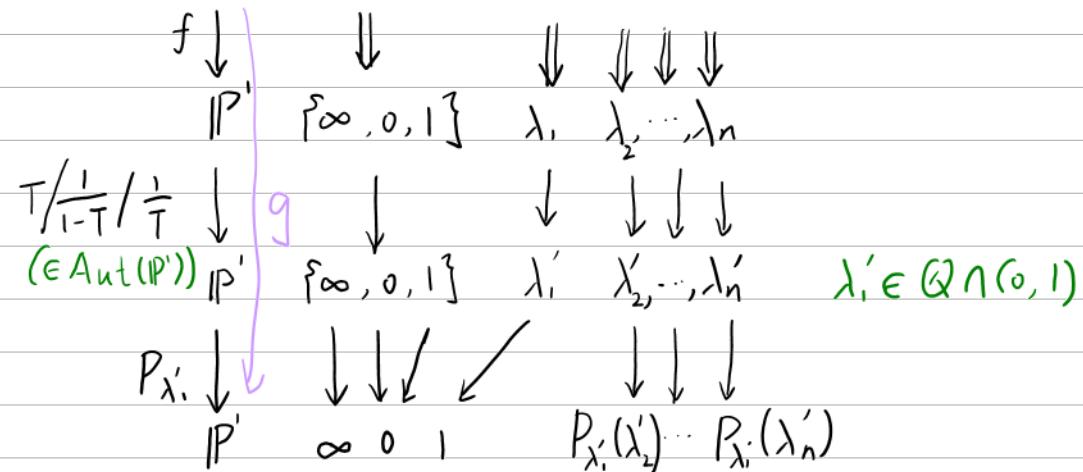
(1)

(2) : $(b_2) \Rightarrow (b_1)$

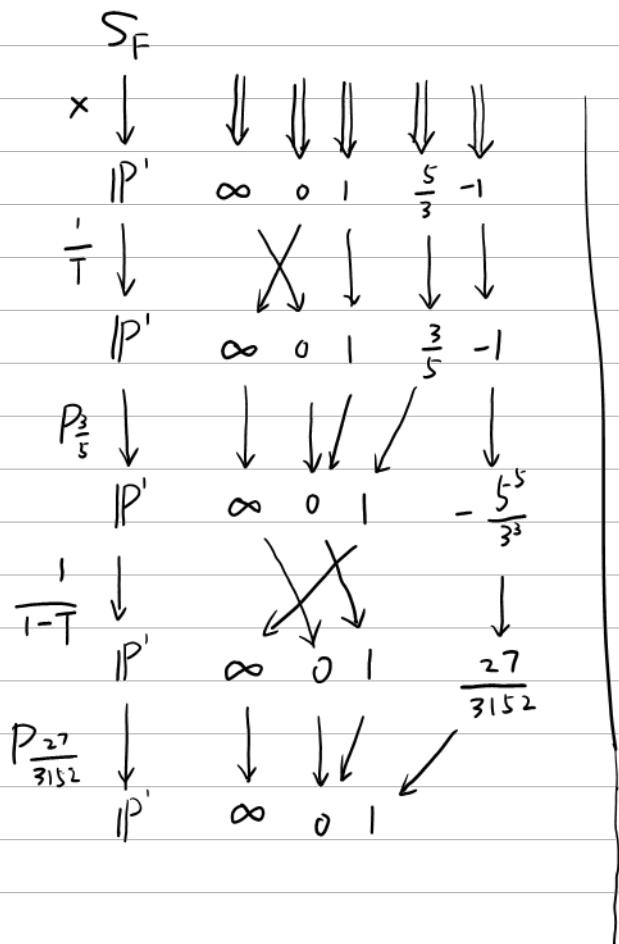
Use induction.

[Suppose $f: S \rightarrow \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \lambda_1, \dots, \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$
 need construct $g: S \rightarrow \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \mu_1, \dots, \mu_{n+1}\} \subset \mathbb{Q} \cup \{\infty\}$]

S



Eg. of (2) F. $y^2 = x(x-1)(x+1)(x - \frac{5}{3})$



$$\lambda = \frac{3}{5} = \frac{3}{3+2} \quad m=3 \quad n=2$$

$$P_{\frac{3}{5}}(T) = \frac{5^5}{3^3 2^2} T^3 (1-T)^2$$

$$P_{\frac{27}{3152}}(T) = \frac{3152^{3152}}{27^{27} 3125^{3125}} T^{27} (1-T)^{3125}$$

$$f(x, y) = P_{\frac{27}{3152}} \left(\frac{1}{1 - P_{\frac{3}{5}} \left(\frac{1}{x} \right)} \right)$$

(3), (a) \Rightarrow (b 3)

$$\left[\begin{array}{l} S \cong S_F \quad F(X, Y) = \sum a_{ij} X^i Y^j \in \overline{\mathbb{Q}}[X, Y] \text{ irr} \\ \text{Verify } x: S_F \rightarrow \mathbb{P}^1 \text{ only ramified in } \overline{\mathbb{Q}} \cup \{\infty\}. \end{array} \right]$$

$x|_{S_F^X}: S_F^X \rightarrow \mathbb{C}$ is unramified, so

the possible ramified points:
 ① ∞

$$② x: p_0(x) = 0$$

$$F(X, Y) = p_0(X)Y^n + \dots + p_n(X)$$

$$③ x \in \{(x, y) \mid F(x, y) = F_Y(x, y) = 0\}$$

(4). (b 3) \Rightarrow (b 2)

$\left[\begin{array}{l} \text{Suppose } f: S \rightarrow \mathbb{P}^1 \text{ ramified at } \{\infty, o, 1, \lambda_1, \dots, \lambda_m\} \subset \overline{\mathbb{Q}} \cup \{\infty\} \\ \text{need construct } g: S \rightarrow \mathbb{P}^1 \text{ ramified at } \{\infty, o, 1, \mu_1, \dots, \mu_m\} \subset \overline{\mathbb{Q}} \cup \{\infty\} \end{array} \right]$

Denote $B_r(f) = \text{Branch}(f) \cap (\mathbb{Q} \cup \{\infty\}) = \{\lambda_1, \dots, \lambda_m\}$

$$B_r(f) = \text{Branch}(f) \cap (\mathbb{Q} \cup \{\infty\})$$

Let $m_f(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $B_r(f)$

Induction on $\deg m_f(T)$:

construct $g: S \rightarrow \mathbb{P}^1$ s.t. $\deg m_g(T) < \deg m_f(T)$

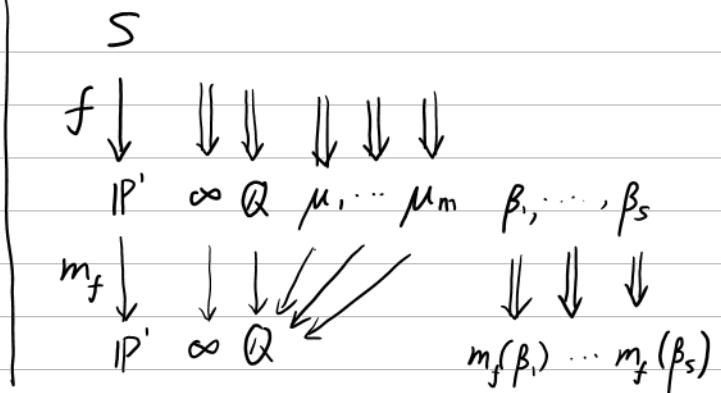
or $B_r(g) = \emptyset$

By convention $m_g(T) = 0$

Claim: $\deg m_{m_f \circ f} < \deg m_f$

Let $\Phi = \{\beta_1, \dots, \beta_s\}$ be the roots of m'_f
 then $B_r(m_f \circ f) = \{m_f(\beta_1), \dots, m_f(\beta_s)\} \cap \mathbb{Q}$

Notation Let $p(T) \in \mathbb{Q}[T]$ be the minimal polynomial of $\Phi - \mathbb{Q}$



We have

$$\deg m_{m_f \circ f} \leq \deg p \leq \deg m'_f < \deg m_f$$

def of min poly

$$\left\{ \begin{array}{l} \deg \min(m_f(\beta_i)) \leq \deg \min(\beta_i) \\ \beta_i = \sigma(\beta_j) \Rightarrow m_f(\beta_i) = \sigma(m_f(\beta_j)) \end{array} \right.$$

E.g. of (4)

$$S_F: \quad y^2 = x(x - \sqrt{2})(x - 3\sqrt{11})$$

$$\begin{array}{ccccccc} S_F & & & & & & \\ x \downarrow & \Downarrow & \Downarrow & \Downarrow & \Downarrow & & \\ \mathbb{P}^1 & \infty & 0 & \sqrt{2} & 3\sqrt{11} & \beta_3 & \beta_4 \\ (\tau^2 - 2)(\tau^3 - 11) \downarrow & \downarrow & \downarrow & \downarrow & \swarrow & \downarrow & \downarrow \\ \mathbb{P}^1 & \infty & b & 0 & m_x(\beta_3) & m_x(\beta_4) & \frac{2733}{64} \\ \downarrow & \downarrow & \downarrow & \downarrow & \searrow & \downarrow & \downarrow \\ \mathbb{P}^1 & \infty & 373248 & 388494 & 0 & \frac{42257943}{128} & \end{array}$$

$$B_1(x) = \{\sqrt{2}, 3\sqrt{11}\}$$

$$m_x(\tau) = (\tau^2 - 2)(\tau^3 - 11)$$

$$m'_x(\tau) = \tau(\tau - 2)(2\tau^2 + 7\tau + 14)$$

$$\Phi = \{\beta_1, \beta_2, \beta_3, \beta_4\}$$

$$= \{0, 2, \frac{1}{4}(-7 \pm 3\sqrt{7}i)\}$$

$$f := m_x \circ x$$

$$B_1(f) = \{\beta_3, \beta_4\}$$

$$m_f(\tau) = 32\tau^2 - 2733\tau + 388494$$

$$m'_f(\tau) = 64\tau - 2733$$

$$\Phi' = \{\frac{2733}{64}\}$$

(5). (b2) \Rightarrow (b1)

Suppose $f: S \rightarrow \mathbb{P}^1$ ramified at $\{\infty, 0, 1, \lambda_1, \dots, \lambda_n\} \subset \mathbb{Q} \cup \{\infty\}$
 need construct $g: S \rightarrow \mathbb{P}^1$ ramified at $\{\infty, 0, 1\}$

Idea: find a "good ramified fact" $G(\tau) = \prod_{i=1}^n (x - \lambda_i)^{a_i}$ $a_i \in \mathbb{N} - \{0\}$

\uparrow
 $\text{Ram}(G) \subseteq \{\lambda_1, \dots, \lambda_n\}$

① Suppose $\text{Branch}(f) \subseteq \mathbb{Z} \cup \{\infty\}$

② Let $y_i := \frac{1}{\prod_{j \neq i} (\lambda_i - \lambda_j)}$ $V_i = \prod_{j > i} (\lambda_j - \lambda_i)$ $a_{ii} = V y_i$

③ Verified: i) $a_i \in \mathbb{N} - \{0\} \Rightarrow G(\lambda_i) = 0$ or ∞
 ii) $\sum a_i = 0 \Rightarrow G(\infty) = 1$

$$a_n = \prod_{n > j > i} (\lambda_j - \lambda_i) = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_{n-1} & \cdots & \lambda_1^{n-2} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix} \quad a_{n-1} = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_{n-2} & \cdots & \lambda_{n-2}^{n-2} & 0 \\ 0 & \cdots & 0 & 1 & 1 \\ 1 & \lambda_n & \cdots & \lambda_n^{n-2} & 0 \end{vmatrix}$$

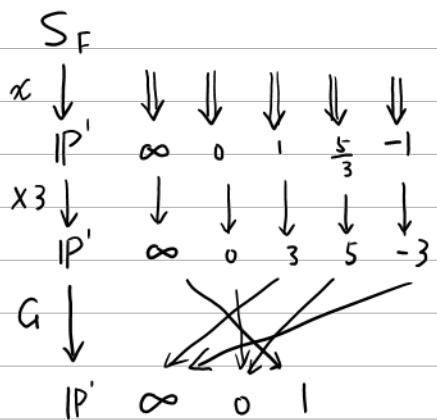
$$\Rightarrow \sum a_i = \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-2} & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-2} & 1 \end{vmatrix} = 0$$

$$\text{iii) } \text{Ram}(G) \subseteq \{\lambda_1, \dots, \lambda_n\}$$

$$\frac{G(T)}{G(T)} = (\log G(T))' = \sum_{i=1}^n \frac{a_i}{x - \lambda_i} = \sum_{i=1}^n \frac{\sqrt{y_i}}{x - \lambda_i} = \frac{\sqrt{V}}{\prod_{i=1}^n (x - \lambda_i)}$$

$$\Rightarrow G'(T) = \sqrt{V} \prod_{i=1}^n (x - \lambda_i)^{a_i - 1}$$

E.g. of (5) F: $y^2 = x(x-1)(x+1)(x - \frac{5}{3})$



$$\{\lambda_1, \dots, \lambda_4, \infty\} = \{-3, 0, 3, 5, \infty\}$$

$$V = \prod_{j>i} (\lambda_j - \lambda_i) = 4320$$

$$\{y_1, y_2, y_3, y_4\} = \{-\frac{1}{144}, \frac{1}{45}, -\frac{1}{36}, \frac{1}{80}\}$$

$$\{a_1, a_2, a_3, a_4\} = \{-30, 96, -120, 54\}$$

$$G(x) = \frac{x^{96}(x-5)^{54}}{(x+3)^{30}(x-3)^{120}}$$

$$G'(x) = 4320 \frac{x^{95}(x-5)^{53}}{(x+3)^{29}(x-3)^{119}}$$