Eine Woche, ein Beispiel

11.26 calculation of $\mathrm{Perv}_\Delta(\mathbb{CP}^1)$

Final goal: Fill in the tables in the next page.

(for first time, remove the $i^!$ column)

We won't show the following fact in this document:

<u>Fact</u> There are exactly 5 indec reps in $\mathrm{Perv}_\Delta(\mathbb{CP}^1)$.

Ref:

[Willians]: Langlands correspondence and Bezrukavnikov's equivalence

calculations from Lukas Bonfert's note (don't forward this to anyone else).

$\psi \underset{\text{var}}{\overset{\text{can}}{\rightleftarrows}} \phi$ alias

$\text{var} \circ \text{can} + 1 = 1$

$i_* \underline{\mathbb{Q}}_{\{\infty\}}$
$(0,1,1,1)$

	n	-2	-1	0	1
$\mathbb{C}$	j^*	0	0	0	0
$\{\infty\}$	i^*	0	0	$\mathbb{Q}$	0
	$i^!$	0	0	$\mathbb{Q}$	0
	$R^n\Gamma$	0	0	$\mathbb{Q}$	0

$0 \underset{0}{\overset{0}{\rightleftarrows}} \mathbb{Q}$ IC_0

In [Willians], $\{0\}$ is digged out

$\underline{\mathbb{Q}}_{\mathbb{CP}^1}[1]$
$(-1,-1,-1,-2)$

	n	-2	-1	0	1
$\mathbb{C}$	j^*	0	$\underline{\mathbb{Q}}$	0	0
$\{\infty\}$	i^*	0	$\mathbb{Q}$	0	0
	$i^!$	0	0	0	$\mathbb{Q}$
	$R^n\Gamma$	0	$\mathbb{Q}$	0	$\mathbb{Q}$

$\mathbb{Q} \underset{0}{\overset{0}{\rightleftarrows}} 0$ IC_∞
$IC(\mathbb{CP}^1, \underline{\mathbb{C}}_{\mathbb{C}})$

$Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}[1]$
$(-1,0,0,-1)$

	n	-2	-1	0	1
$\mathbb{C}$	j^*	0	$\underline{\mathbb{Q}}$	0	0
$\{\infty\}$	i^*	0	$\mathbb{Q}$	$\mathbb{Q}$	0
	$i^!$	0	0	0	0
	$R^n\Gamma$	0	$\mathbb{Q}$	0	0
	Γ	0	$\mathbb{Q}$	$\mathbb{Q}$	0

$\mathbb{Q} \underset{1}{\overset{0}{\rightleftarrows}} \mathbb{Q}$ $I(\psi)$

$j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1]$
$(-1,0,0,-1)$

	n	-2	-1	0	1
$\mathbb{C}$	j^*	0	$\underline{\mathbb{Q}}$	0	0
$\{\infty\}$	i^*	0	0	0	0
	$i^!$	0	0	$\mathbb{Q}$	$\mathbb{Q}$
	$R^n\Gamma$	0	0	0	$\mathbb{Q}$

$\mathbb{Q} \underset{0}{\overset{1}{\rightleftarrows}} \mathbb{Q}$ $P(\psi)$

???
$(-1,1,1,0)$

	n	-2	-1	0	1
$\mathbb{C}$	j^*	0	$\underline{\mathbb{Q}}$	0	0
$\{\infty\}$	i^*	0	0	$\mathbb{Q}$	0
	$i^!$	0	0	$\mathbb{Q}$	0
	$R^n\Gamma$	0	0	0	0

$\mathbb{Q} \underset{(1\ 0)}{\overset{\binom{0}{1}}{\rightleftarrows}} \mathbb{Q}^2$ big tilting sheaf
$P(\phi) = I(\phi)$

Hint for calculation

1. Stalk are usually easy to compute, while global sections are collections of compatible stalks.
2. Use some triangles can facilitate calculations.

e.g.

$$\begin{array}{ccccc} R^0 j_* \underline{\mathbb{Q}}_{\mathbb{C}} & \longrightarrow & Rj_*\underline{\mathbb{Q}}_{\mathbb{C}} & \longrightarrow & R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}}[-1] \xrightarrow{+1} \\ \| & & & & \| \\ \underline{\mathbb{Q}}_{\mathbb{CP}^1} & & & & i_*\underline{\mathbb{Q}}_{\{\infty\}}[-1] \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & j_! j^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & i_* i^* \mathcal{F} & \longrightarrow & 0 \\ 0 & \longrightarrow & j_! \underline{\mathbb{Q}}_{\mathbb{C}} & \longrightarrow & \underline{\mathbb{Q}}_{\mathbb{CP}^1} & \longrightarrow & i_*\underline{\mathbb{Q}}_{\{\infty\}} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccc} i_! i^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & Rj_* j^* \mathcal{F} & \xrightarrow{+1} \\ i_! \underline{\mathbb{Q}}_{\{\infty\}}[-2] & \longrightarrow & \underline{\mathbb{Q}}_{\mathbb{CP}^1} & \longrightarrow & Rj_* \underline{\mathbb{Q}}_{\mathbb{C}} & \xrightarrow{+1} \end{array}$$

3. Open-closed formalism also save some time.

e.g. $(j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = i^* j_! \underline{\mathbb{Q}}_{\mathbb{C}} = 0$

Ex. Check the following table for $\mathcal{F} \in \mathcal{D}_\Lambda(\mathbb{CP}^1)$ or $PSh_\Lambda(\mathbb{CP}^1)$:

$\Lambda: \emptyset \subseteq \{\infty\} \subseteq \mathbb{CP}^1$

$\mathcal{F}$	$\mathcal{F}_0 \cong \mathcal{F}(\mathbb{C})$	$\mathcal{F}_\infty \cong \mathcal{F}(\mathbb{CP}^1 - \{0\})$	$\mathcal{F}(\mathbb{CP}^1)$	$R\Gamma(\mathcal{F})$	
$i_*\underline{\mathbb{Q}}_{\{\infty\}}$	0	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	
$\underline{\mathbb{Q}}_{\mathbb{CP}^1}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}[-2]$	
$Rj_*\underline{\mathbb{Q}}_{\mathbb{C}}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}[-1]$	—	$\mathbb{Q}$	
$j_!\underline{\mathbb{Q}}_{\mathbb{C}}$	$\mathbb{Q}$	0	$\mathbb{Q}$	$\mathbb{Q}[-2]$	
$(R^0 j_*\underline{\mathbb{Q}}_{\mathbb{C}})^{pre} = R^0 j_*\underline{\mathbb{Q}}_{\mathbb{C}}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}[-2]$	$= \underline{\mathbb{Q}}_{\mathbb{CP}^1}$
$(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})^{pre}$	0	$\mathbb{Q}$	0	—	
$R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}}$	0	$\mathbb{Q}$	$\mathbb{Q}$	$\mathbb{Q}$	$= i_*\underline{\mathbb{Q}}_{\{\infty\}}$
$(j_!\underline{\mathbb{Q}}_{\mathbb{C}})^{pre}$	$\mathbb{Q}$	0	0	—	

How to compute $f^!$?

https://math.stackexchange.com/questions/2167554/how-to-calculate-ri

We do it by cases.

① When $f = j : \mathcal{U} \xhookrightarrow{\text{open}} X$, $j^! = j^*$;

② When $f = i : Z \xhookrightarrow{\text{closed}} X$, use the triangle

$$\begin{array}{ccccccc} i_! i^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & Rj_* j^* \mathcal{F} & \xrightarrow{+1} & \\ i^! \mathcal{F} & \longrightarrow & i^* \mathcal{F} & \longrightarrow & i^* Rj_* j^* \mathcal{F} & \xrightarrow{+1} & \end{array}$$

So $\quad i^! \mathcal{F} = \mathrm{Fiber}\left(i^* \mathcal{F} \longrightarrow i^* Rj_* j^* \mathcal{F} \right)$.

e.p. costalks can be computed in this way, and

$$i_x^! \mathcal{F} = \varinjlim_{x \in V} H^{\cdot}(V, U \cap V ; \mathcal{F}|_V)$$

③ When $f = \pi : X \longrightarrow \{*\}$, X mfld of dim n, one gets

$$\pi^! \mathbb{Q} = Or_X[n]$$

④ Other cases: try to write f as composition of maps we're familiar with.

e.g. for $X \subseteq \mathbb{C}^n$ hypersurface, want $\pi_X^! \mathbb{Q}$.

use the composition $\pi_X : X \hookrightarrow \mathbb{C}^n \longrightarrow \{*\}$

Surprising: $f^!$ does not depend on the composition we choose!

reason: adjunction is unique

⑤ ramified covering and blow up

Ex. Compute $f^! \mathbb{Q}$ for the following cases.

1. $f : [0,1) \longrightarrow \{*\}$
2. $f : M \longrightarrow \{*\}$ $\quad M$: mfld with boundary
3. $f : \mathbb{C} \cup_{\{0\}} \mathbb{C} \longrightarrow \{*\}$ $\quad \left\{ \begin{array}{l} \{x^2 + y^2 = 0\} \subseteq \mathbb{C}^2 \\ \quad \subseteq \mathbb{R}^3 \end{array} \right.$

4. $f : \{z_1^2 + \cdots + z_n^2 = 0\} \longrightarrow \{*\}$
5. $f :$  $\longrightarrow \{*\}$

"hard" Ex. Take the ramified covering $f: \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = z^2$, we have $f^*\mathcal{F} \cong f^!\mathcal{F}$ for which $\mathcal{F} \in Sh(\mathbb{C})$?

A. $\mathcal{F} = sky_p(\mathbb{Q})$ for $p \in \mathbb{C}^\times$
B. $\mathcal{F} = sky_0(\mathbb{Q})$
C. $\mathcal{F} = \underline{\mathbb{Q}}_{\mathbb{C}}$
D. $\mathcal{F} = \iota_* \underline{\mathbb{Q}}_{\mathbb{R}}$ for $\iota: \mathbb{R} \hookrightarrow \mathbb{C}$