Eine Woche, ein Beispiel

5.12 sheaf version of ⊗ & Hom

1. def of sheaf Hom
2. def of sheaf ⊗

sheaf version of Tensor-Hom adjunction is left in the next document.

Compared with ⊗, Hom is more delicated, and it is harder than you expected.

1. def of sheaf Hom

$$\mathrm{Hom}_A(-,-) : (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod}$$

A : comm ring

$$\rightsquigarrow$$

$$\mathrm{Hom}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow \mathbb{Z}\text{-Mod}$$

$$\rightsquigarrow$$

$$\underline{\mathrm{Hom}}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$

$$\rightsquigarrow$$

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^+(X)}(-,-) : \mathcal{D}^-(X)^{op} \times \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

non-derived sheaf Hom

Def [Vakil, 2.3.1] For $\mathcal{F}, \mathcal{G} \in Sh(X)$, a morphism of sheave

$$\phi : \mathcal{F} \longrightarrow \mathcal{G}$$

is the data of maps

$$\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \quad \text{for all } U \subseteq X \text{ open.}$$

which is compatible with restriction.

We write

$$\phi \in \mathrm{Hom}_{Sh(X)}(\mathcal{F}, \mathcal{G})$$

Similarly, one can define

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$$

as the set of morphisms in $\mathcal{D}(X)$.

Def [Vakil, 2.3.C] (Sheaf Hom / Internal Hom)
For $\mathcal{F}, \mathcal{G} \in Sh(X)$, one gets a sheaf $\underline{Hom}(\mathcal{F}, \mathcal{G}) \in Sh(X)$ given by

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)(\mathcal{U}) = Hom_{Sh(\mathcal{U})}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}})$$

Cor.
$$Hom = \Gamma \circ \underline{Hom} : Sh(X)^{op} \times Sh(X) \xrightarrow{\underline{Hom}} Sh(X) \xrightarrow{\Gamma} Abel$$

⚠ Even though $(\mathcal{F} \otimes \mathcal{G})_p \cong \mathcal{F}_p \otimes \mathcal{G}_p$.
$\underline{Hom}$ does not commute with taking stalks:

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)_p \xrightarrow{\ncong} Hom(\mathcal{F}_p, \mathcal{G}_p)$$

It's neither inj nor surj.
[Left adj comm with limit, $\otimes \dashv \underline{Hom}$].

Ex. Try to compute coefficient $\mathbb{Q}$.

$$\underline{Hom}_{Sh(X)}(\underline{\mathbb{Q}}_X, \mathcal{F}) \cong \mathcal{F}$$
$$\underline{Hom}_{Sh(X)}(j_! \underline{\mathbb{Q}}_{\mathcal{U}}, \mathcal{F}) \cong j_*(\mathcal{F}|_{\mathcal{U}})$$
$$\underline{Hom}_{Sh(\mathbb{C})}(sky_0(\mathbb{Q}), \underline{\mathbb{Q}}_{\mathbb{C}}) \cong 0$$

derived sheaf Hom

Def. For $\mathcal{F}, \mathcal{G} \in Sh(X)$, the derived internal Hom

in general, $\mathcal{F} \in \mathcal{D}(X)^-$, $\mathcal{G} \in \mathcal{D}^+(X)$

$$R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^+(X)$$

is given by

$$\begin{array}{lll} \underline{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \mathcal{I}^\cdot) & \text{when } \mathcal{G} \xrightarrow{\simeq} \mathcal{I}^\cdot & \text{inj resolution} \\ \underline{Hom}_{\mathcal{C}(X)}(\mathcal{P}^\cdot, \mathcal{G}) & \text{when } \mathcal{F} \xleftarrow{\simeq} \mathcal{P}^\cdot & \text{proj resolution} \end{array}$$

Here,

$$\underline{Hom} : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$

is extended to the double complex

$\mathcal{C}(X) :=$ complex of sheaves on X, temperate notation

$$\underline{Hom}_{\mathcal{C}(X)} : \mathcal{C}(X)^{op} \times \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$$

Other versions of sheaf Hom

$$\begin{array}{ccc} Hom_A(-,-) & \rightsquigarrow & RHom_A(-,-) \\ \wr\downarrow & & \wr\downarrow \\ Hom_{Sh(X)}(-,-) & \rightsquigarrow & RHom_{\mathcal{D}^+(X)}(-,-) \\ \wr\downarrow & & \wr\downarrow \\ \underline{Hom}_{Sh(X)}(-,-) & \rightsquigarrow & R\underline{Hom}_{\mathcal{D}^+(X)}(-,-) \end{array}$$

$$Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

$$Hom_{Sh(X)}(\mathcal{F}, \mathcal{G}) = Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}).$$

$$RHom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R\Gamma \circ R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G})$$