Eine Woche, ein Beispiel

11.19. Basic sheaf calculation

Goal: Motivate $f_*, f^*, f_!, f^!$, by connecting them with (co)homology theory

After story: $\rightsquigarrow$ calculation of $\mathrm{Perv}_\Delta(\mathbb{C}\mathbb{P}^1)$
$\rightsquigarrow$ generalize Morse theory
$\rightsquigarrow$ Characteristic classes / cycles
$\rightsquigarrow$ index theorem

Minor advantages from my talk:

- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    $\rightsquigarrow$ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale) descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \mathrm{Top}$
$\mathrm{Obj}(\mathrm{Sh}(X)) = \{\text{sheaves of abelian gps}\}$
e.p. $\mathrm{Sh}(\{*\}) = \mathrm{Abel}$ $\qquad \mathbb{Q}_{\{*\}} \longleftrightarrow \mathbb{Q}$

0. sheaf
1. f_*, skyscraper sheaf & global sections
2. f^*, constant sheaf & stalks
3. Rf_* & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology

$- \otimes -$ & product structure on cohomology
$\mathrm{Hom}(-,-)$ & Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016

0. Sheaf

Recall the definition of

- presheaf $\mathcal{F}$
- sheaf $\mathcal{F}$
- stalk $\mathcal{F}_x$
- global section $\mathcal{F}(X) = \Gamma(X;\mathcal{F}) = H^0(X;\mathcal{F})$
- cohomology $R^n\Gamma(X;\mathcal{F}) = H^n(X;\mathcal{F})$

https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology
If X is paracompact and Hausdorff, Cech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

Recall examples of sheaves:

complicated $\left\{\begin{array}{l} \cdot\ \mathcal{C}_X:\ \text{sheaf of cont fcts on } X \\ \cdot\ \mathcal{O}_X:\ \text{structure sheaf on } X \end{array}\right.$ e.g., X: (cplx) mfld, scheme, $\cdots$

- $\underline{\mathbb{Q}}_X$: constant sheaf on X

- $\mathrm{sky}_p(\mathbb{Q})$: skyscraper sheaf of $p \in X$ on X.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{C}_X)_x \qquad \& \quad (\mathrm{sky}_p(\mathbb{Q}))_x .$$

1. f_*, skyscraper sheaf & global sections

Setting $X, Y \in Top$, $\mathcal{F} \in Sh(Y)$, $f : Y \longrightarrow X$ cont

Def. $f_*\mathcal{F} \in Sh(X)$ is given by
$$f_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(f^{-1}(\mathcal{U}))$$

$$\begin{array}{ccc} \mathcal{F} & & f_*\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

This defines a fctor
$$f_* : Sh(Y) \longrightarrow Sh(X)$$

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_{p*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$

For $\pi : Y \longrightarrow \{*\}$, $\pi_*\mathcal{F} = \mathcal{F}(Y) = \Gamma(Y;\mathcal{F})$

Ex (hard?) For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, compute $j_*\underline{\mathbb{Q}}_{\mathbb{C}}$.

- ○ It is a constant sheaf on $\mathbb{CP}^1$.
- ○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$.
- ○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$.
- ○ All the above is wrong.
- ○ I don't know, but I don't want to make a wrong choice.

2. f^*, constant sheaf & stalks

In [Vakil, Chapter 2], it is f^{-1}, the inverse image functor.

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(X)$, $f : Y \longrightarrow X$ cont

Def. $f^*\mathcal{F} \in Sh(Y)$ is given by sheafification of

$$f^{*,\mathrm{pre}}\mathcal{F}(\mathcal{U}) = \varinjlim_{f(\mathcal{U}) \subseteq V} \mathcal{F}(V)$$

$$\begin{tikzcd} f^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ \mathcal{U} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

This defines a fctor

$$f^* : Sh(X) \longrightarrow Sh(Y)$$

Recall:

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (x_p)_p \in \prod_{p \in \mathcal{U}} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall\, x_0 \in \mathcal{U},\ \exists\, \mathcal{U}_{x_0} \subseteq \mathcal{U} \text{ nbhd of } x_0, \\ s \in \mathcal{F}(\mathcal{U}) \text{ s.t.} \\ s_p = x_p \qquad \forall\, p \in \mathcal{U}_{x_0} \end{array} \right\}$$

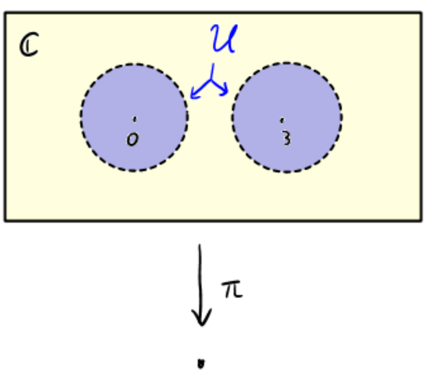
By definition, $(\mathcal{F}^{sh})_p = \mathcal{F}_p$.

Universal property:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f \in Mor_{PSh}"] \arrow[d, "sh"'] & \mathcal{G} \\ \mathcal{F}^{sh} \arrow[ur, dashed, "\exists!\, f^{sh} \in Mor_{sh}"'] & \end{tikzcd}$$

$\mathcal{G}$: sheaf

Ex. For $\pi : \mathbb{C} \longrightarrow \{*\}, U = B_1(0) \cup B_1(3)$, which one is correct:



- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ All the above is wrong.

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_p^* \mathcal{F} = \mathcal{F}_p$

For $\pi : Y \longrightarrow \{*\}$, $\pi^* \underline{\mathbb{Q}}_{\{*\}} = \underline{\mathbb{Q}}_Y$

For $\mathcal{U} \subset X$ open, $j : \mathcal{U} \longrightarrow X$, $j^* \mathcal{F} = \mathcal{F}|_{\mathcal{U}}$

People generalize the last notation to arbitrary subset:

For $Y \subset X$, $\iota_Y : Y \hookrightarrow X$, $\iota_Y^* \mathcal{F} \triangleq \mathcal{F}|_Y$

Q: For $\mathcal{U} \subset X$ open, how to express $\mathcal{F}(\mathcal{U})$ by fctors?

A:

$$\begin{tikzcd} \mathcal{U} \arrow[r, hook, "\iota_{\mathcal{U}}"] \arrow[dr, "\pi_{\mathcal{U}}"'] & X \arrow[d, "\pi_X"] \\ & \{*\} \end{tikzcd}$$

$$\mathcal{F}(\mathcal{U}) = \pi_{\mathcal{U},*} \underbrace{\iota_{\mathcal{U}}^* \mathcal{F}}_{\mathcal{F}|_{\mathcal{U}}}$$

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

Prop. One has the adjunction $f^* \dashv f_*$, i.e.,

$$\mathrm{Mor}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G}) \qquad + \text{naturality}$$

Hint. [Vakil, 2.7.B] Show that both side give the same information, i.e.,

$$\phi_{UV} \in \mathrm{Mor}_{Ab}(\mathcal{F}(U), \mathcal{G}(V)) \qquad \text{for each pair } (V, U) \text{ s.t. } f(V) \subset U$$

+ compatability

Cor. f^* is right adjoint, f_* is left adjoint.

Rmk. f^* is an exact functor.

Hint: exactness can be checked on stalks!

⚠ After "polished" (because of the structure sheaf), f^* is again only right adjoint.

3. Rf_* & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input $\mathcal{F}$, output a complex (before Ker/Im procedure)
- SES induces LES: for
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
one has
$$\begin{tikzcd} & H^0(X;\mathcal{F}) \arrow[r] \arrow[d, equal] & H^0(X;\mathcal{G}) \arrow[r] \arrow[d, equal] & H^0(X;\mathcal{H}) \arrow[dll, out=-10, in=170] \\ 0 \arrow[ur] & \pi_*\mathcal{F} & \pi_*\mathcal{G} & \pi_*\mathcal{H} \end{tikzcd}$$
$$H^0(X;\mathcal{F}) \to H^0(X;\mathcal{G}) \to H^0(X;\mathcal{H}) \to H^1(X;\mathcal{F}) \to H^1(X;\mathcal{G}) \to H^1(X;\mathcal{H}) \to H^2(X;\mathcal{F}) \to \cdots$$
$$\pi : X \longrightarrow \{*\}$$
- can be viewed as right derived fctor of
$$H^0(X,-) = \Gamma(X,-) = \pi_*$$
one gets
$$H^n(X,-) = R^n\Gamma(X,-) = R^n\pi_*$$
We denote the complex (before the Ker/Im procedure) as
$$R\Gamma(X,-) = R\pi_*$$
up to homotopy equiv & quasi-iso, i.e., in the derived category of $\{*\}$.

$$\begin{aligned} \mathcal{D}(X) = \mathcal{D}(Sh(X)) &= \text{“derived category of sheaves over } X\text{”} \\ &= \text{“ complexes of sheaves over } X\text{, up to } \cdots \text{”} \\ &= \{\cdots \to \mathcal{F}^{-2} \to \mathcal{F}^{-1} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots\} \triangleq \{\mathcal{F}^\bullet\} \end{aligned}$$

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

Def. $Rf_* \mathcal{F}$ = "derived pushforward of $\mathcal{F}$"
$= f_* \mathcal{I}^\cdot$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & Rf_*\mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ & \cup \\ & \mathcal{U} \end{tikzcd}$$

[Here, $\mathcal{I}^\cdot$ is the injective resolution of $\mathcal{F}$:
$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$
$(\Rightarrow \mathcal{F} \overset{\text{quasi-iso}}{\sim} \mathcal{I}^\cdot)$]

This defines a fctor
$$Rf_*: \mathcal{D}^b(Y) \longrightarrow \mathcal{D}(X)$$

The derived pushforward is hard to compute.
just like cohomology, and even worse, since we need more information
Luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]
$R^n f_* \mathcal{F}$ is given by the sheafification of
$$(R^n f_*^{pre} \mathcal{F})(\mathcal{U}) = H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$
sometimes omit

e.p. one can compute the stalk
$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in \mathcal{U}} H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$

Cor For $\pi: X \to \{*\}$,
$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For $\pi: \mathbb{CP}^1 \to \{*\}$,
$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in $\mathcal{D}(*)$ are proj, we work over $\mathbb{Q}$]
$$R\pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$
$$= [0 \to \cdots \underset{-1}{\to} \underset{0}{\mathbb{Q}} \to \underset{1}{0} \to \underset{2}{\mathbb{Q}} \to \underset{3}{0} \to \underset{4}{\cdots}]$$

Ex.

For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, what is true about $Rj_*\underline{\mathbb{Q}}_{\mathbb{C}}$?

- ○ $(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- ○ $(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- ○ $(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- ○ $(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- ○ What the hell is that?

In fact, $(Rj_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1]$.

$i : \{\infty\} \to \mathbb{CP}^1$ is exact, so $Ri_* = i_*$.