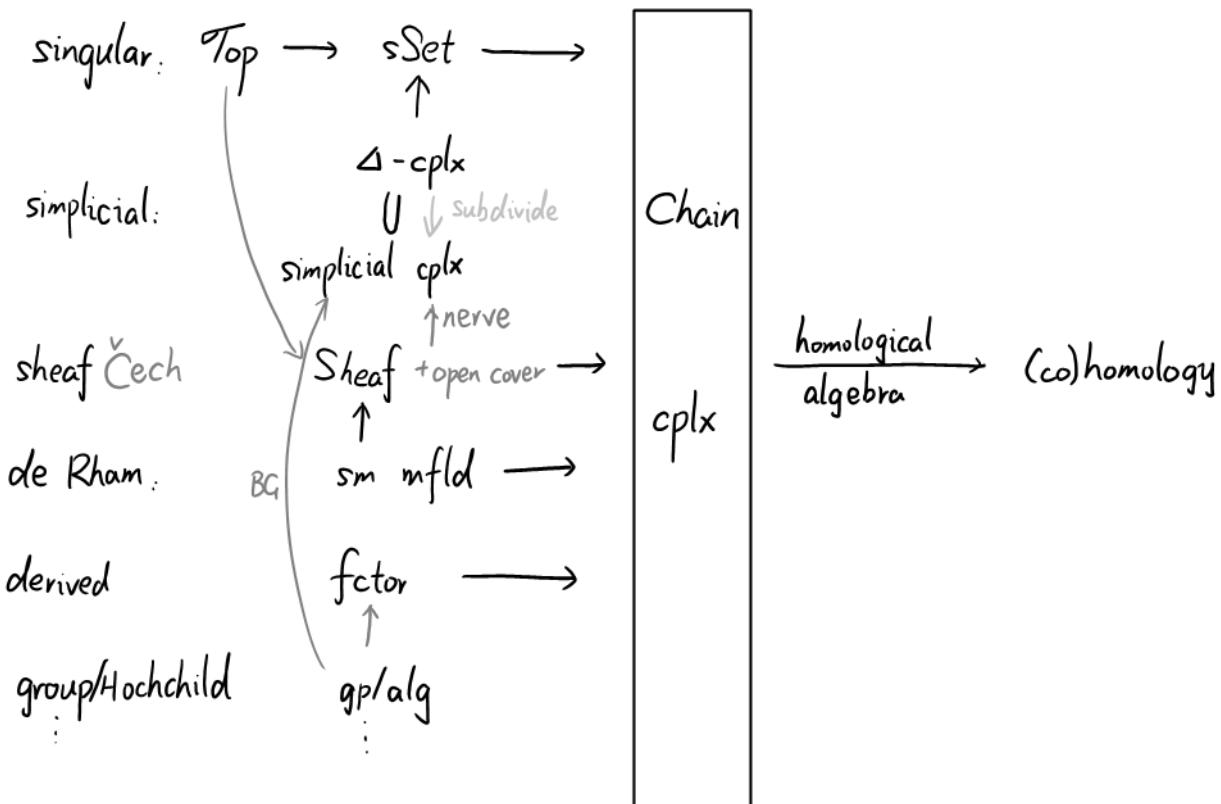


Eine Woche, ein Beispiel

6.25 (co)homology of simplicial set

This document is a continuation of [23.01.09].
<https://ncatlab.org/nlab/show/simplicial+complex>
<https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets>



Today: $sSet \longrightarrow \text{chain cplx} \dashrightarrow (\text{co})\text{homology}$

1. definition and basic examples
2. relative (co)homology
3. contractible sset has trivial cohomology
4. connection with simplicial complexes
5. more structures
6. connection with sheaf cohomology + derived category

Realize Hochschild homology as simplicial homology:
<https://arxiv.org/pdf/1802.03076.pdf>

1. definition and basic examples

Def. For $X \in s\text{Set}$, $G \in \text{Mod}(\mathbb{Z})$, define

We use Δ here because
we are considering $X = \Delta^n$ case.
May change to x in the future.

$$C_n(X; G) = \bigoplus_{x \in X_n} G \quad 0 \leftarrow \bigoplus_{x \in X_0} G \xleftarrow{(d'_0 - d'_1)^*} \bigoplus_{x \in X_1} G \xleftarrow{(d'_0 - d'_1 + d'_2)^*} \bigoplus_{x \in X_2} G \dots$$

$$C^n(X; G) = \prod_{x \in X_n} G \quad 0 \longrightarrow \prod_{x \in X_0} G \xrightarrow{\text{dual}} \prod_{x \in X_1} G \longrightarrow \prod_{x \in X_2} G \dots$$

$$C_n^{BM}(X; G) =$$

$$C_c^n(X; G) =$$

$$\text{Hom}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_{x \in X_n} \mathbb{Z}, G\right) \cong \prod_{x \in X_n} \text{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{x \in X_n} G$$

<https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%C3%b6bius-strip?rq=1>
<https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree>

Rmk. Prof. Scholze told me that we cannot define

Borel-Moore homology or cpt supported cohomology, not to say six factors for sset.
If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \text{Top}$ discrete, $X := \mathcal{S}(A) \in \text{Set}$, one can compute

$$\text{wished} \left\{ \begin{array}{l} C(X; G) : 0 \xleftarrow{\oplus_{a \in A} G} \xleftarrow{o} \oplus_{a \in A} G \xleftarrow{\text{Id}} \oplus_{a \in A} G \xleftarrow{o} \oplus_{a \in A} G \xleftarrow{\text{Id}} \dots \\ C^*(X; G) : 0 \rightarrow \prod_{a \in A} G \xrightarrow{o} \prod_{a \in A} G \xrightarrow{\text{Id}} \prod_{a \in A} G \xleftarrow{o} \prod_{a \in A} G \xrightarrow{\text{Id}} \dots \\ C_c^{\text{BM}}(X; G) : 0 \leftarrow \prod_{a \in A} G \xleftarrow{o} \prod_{a \in A} G \xrightarrow{\text{Id}} \prod_{a \in A} G \xleftarrow{o} \prod_{a \in A} G \xleftarrow{\text{Id}} \dots \\ C_c(X; G) : 0 \rightarrow \bigoplus_{a \in A} G \xrightarrow{o} \bigoplus_{a \in A} G \xrightarrow{\text{Id}} \bigoplus_{a \in A} G \xrightarrow{o} \bigoplus_{a \in A} G \xrightarrow{\text{Id}} \dots \end{array} \right.$$

Therefore,

$$H_n(X; G) = \begin{cases} \bigoplus_{a \in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X; G) = \begin{cases} \prod_{a \in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H_n^{\text{BM}}(X; G) = \begin{cases} \prod_{a \in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H_c^n(X; G) = \begin{cases} \bigoplus_{a \in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta'; G)$ & $H^n(\Delta'; G)$.

Notice that $\#\Delta'_k = k+2$, so

$$C_*(\Delta'; G) : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

basis:

$$d'_0 = x_0 \rightarrow \vdots$$

$$x_0 \rightarrow \vdots$$

$$x_0 \rightarrow \vdots$$

$$x_0 \rightarrow \vdots$$

remember indexes:

$$d'_1 = x_1 \rightarrow \vdots$$

$$x_1 \rightarrow \vdots$$

$$x_1 \rightarrow \vdots$$

$$x_2 \rightarrow \vdots$$

$$x_2 \rightarrow \vdots$$

$$x_2 \rightarrow \vdots$$

$$x_3 \rightarrow \vdots$$

$$x_3 \rightarrow \vdots$$

$$x_3 \rightarrow \vdots$$

$$x_4 \rightarrow \vdots$$

$$x_4 \rightarrow \vdots$$

$$0 = x_0 - x_0 \longleftarrow x_0$$

$$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow x_0$$

$$x_0 - x_1 = x_0 - x_1 \longleftarrow x_1$$

$$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow x_1$$

$$0 = x_1 - x_1 \longleftarrow x_2$$

$$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow x_2$$

$$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow x_3$$

$$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow x_4$$

$$x_0 = x_0 - x_0 + x_0 \longleftarrow x_0$$

$$x_0 = x_0 - x_1 + x_1 \longleftarrow x_1$$

$$x_2 = x_1 - x_1 + x_2 \longleftarrow x_2$$

$$x_2 = x_2 - x_2 + x_2 \longleftarrow x_3$$

By taking the transpose, one get

$$C^*(\Delta'; G) : 0 \rightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

Therefore,

$$H_n(\Delta'; G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta'; G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

E.g. 3. When we want to compute $H_n(\Delta^m; G)$ and $H^n(\Delta^m; G)$, we'd better to give elements in $\Delta_n^m \approx \{\text{basis of } C_n(\Delta^m; G)\}$ a better notation, see [23.01.09]
 The following table shows some typical element in

$$C_n(\Delta^m; G) = \langle d: [n] \rightarrow [m] \rangle_{d \in \Delta_n^m}.$$

not confuse with $[n]$

element	picture	list	count	degenerate degree
$d: [5] \rightarrow [3]$ 0 → 0 1 → 0 2 → 1 3 → 3 4 → 3 5 → 3		(0,0,1,3,3,3)	[2,1,0,3]	$\Delta_5^{3,43}$
$d^3: [2] \rightarrow [3]$ 0 → 0 1 → 2 2 → 3		(0,2,3)	[1,0,1,1]	$\Delta_2^{3,4}$
$s^3: [3] \rightarrow [2]$ 0 → 0 1 → 1 2 → 1 3 → 2		(0,1,1,2)	[1,2,1]	$\Delta_3^{2,41}$
∂_2	—	(0,0,3,3,3) -(0,0,1,3,3)	[2,0,0,3] -[2,1,0,2]	$\Delta_4^{3,43}$ $\Delta_4^{3,42}$

e.g. $\partial[2,5,3,4,1,6,0]$

$$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$$

In this case, $d: C^n(\Delta^m; G) \rightarrow C^{n+1}(\Delta^m; G)$ is also not hard to describe.

e.g. $\partial[2,1,0,3] = [3,1,0,3] - [2,1,1,3]$

$$\partial[2,5,3,4,1,6,0]$$

$$= [3,5,3,4,1,6,0] + [2,5,3,5,1,6,0]$$

$$- [2,5,3,4,1,7,0] - [2,5,3,4,1,6,1]$$

Rmk. The computation of ∂^* and d actually comes from the computation of $d_i^{n,*}$ and $d_{i,*}^n = \text{the dual of } d_i^{n,*}$ (not $s_i^{n,*}$!). In general, one can derive the formula of α^* & α_* .

E.g.	$\alpha^*: C_3(\Delta^n; G) \rightarrow C_5(\Delta^n; G)$	$\beta \mapsto \beta \circ \alpha$
	$[1, 2, 1] \mapsto [1, 2, 1] \circ [2, \underline{1}, \underline{0}, \underline{3}] = [2, 1, 3]$	
	$d_i^{3,*}: C_3(\Delta^n; G) \rightarrow C_2(\Delta^n; G)$	$\beta \mapsto \beta \circ d_i^3$
	$[1, 2, 1] \mapsto [1, 2, 1] \circ [1, \underline{0}, \underline{1}, \underline{1}] = [1, 1, 1]$	
	$s_{i,*}^{3,*}: C_2(\Delta^n; G) \rightarrow C_3(\Delta^n; G)$	$\beta \mapsto \beta \circ s_i^3$
	$[2, 1] \mapsto [2, 1] \circ [1, \underline{2}, \underline{1}] = [3, 1]$	
	$\alpha_*: C^3(\Delta^n; G) \rightarrow C^3(\Delta^n; G)$	$[2, \underline{1}, \underline{0}, \underline{3}]$
	$[2, 1, 3] \mapsto [1, 2, 1] + [1, 1, 2]$	$\sqcup \quad \sqcup \quad \sqcup$
	$[2, 2, 2] \mapsto 0$	\times
	$[3, 3] \mapsto [3, 1] + [2, 2]$	$\sqcup \quad \sqcup \quad \sqcup$
	$d_{i,*}^3: C^2(\Delta^n; G) \rightarrow C^3(\Delta^n; G)$	$[1, 0, 1, 1]$
	$[1, 1, 1] \mapsto [2, 1, 1] + [1, 2, 1]$	$\sqcup \quad \sqcup \quad \sqcup$
	$[0, 3] \mapsto [0, 4]$	$\sqcup \quad \sqcup \quad \sqcup$
	$[1, 0, 1, 1] \mapsto [2, 0, 1, 1] + [1, 1, 1, 1] + [1, 0, 2, 1]$	$\sqcup \quad \sqcup \quad \sqcup \quad \sqcup$
	$s_{i,*}^3: C^3(\Delta^n; G) \rightarrow C^2(\Delta^n; G)$	$[1, 2, 1]$
	$[3, 1] \mapsto [2, 1]$	$\sqcup \quad \sqcup$
	$[2, 2] \mapsto 0$	\times
	$[1, 0, 3] \mapsto [1, 0, 2]$	$\sqcup \quad \sqcup \quad \sqcup$

2. relative (co)homology

Def For $A \subseteq X$ ssets, denote

$$C_k(X, A; G) := C_k(X; G)/C_k(A; G)$$

$$= \bigoplus_{x \in X_k - A_k} G$$

⚠ Notice that $X - A$ is not a sset, so we can't write

$$\underline{C_k(X, A; G)} = C_k(X - A; G)$$

we have $(X/A)_k = X_k/A_k = (X_k - A_k) \cup \{\ast\} \neq X_k - A_k$, so one may write

$$C_k(X, A; G) = \widehat{C}_k(X/A; G)$$

Ex. Verify that $Y \subset A \subset X$ as ssets

$$C_k(X, \emptyset; G) = C_k(X; G)$$

$$C_k(X, \{\ast\}; G) = \widehat{C}_k(X; G)$$

$$C_k(X, X; G) = 0$$

$$0 \longrightarrow C_k(A; G) \longrightarrow C_k(X; G) \longrightarrow C_k(X, A; G) \rightarrow 0$$

$$0 \longrightarrow C_k(A, Y; G) \longrightarrow C_k(X, Y; G) \longrightarrow C_k(X, A; G) \rightarrow 0$$

Eg. 4. We want to compute $H_n(\Delta^2/\partial\Delta^2; G)$ & $H^n(\Delta^2/\partial\Delta^2; G)$.

From the SES

$$\begin{array}{ccccccc}
 0 & 0 & 0 & 0 & 0 & 0 & \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 C_*(\partial\Delta^2; G) & 0 \leftarrow G^{\oplus 3} \leftarrow G^{\oplus 6} \leftarrow G^{\oplus 9} \leftarrow G^{\oplus 12} \leftarrow G^{\oplus 15} \leftarrow \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 C_*(\Delta^2; G) & 0 \leftarrow G^{\oplus 3} \leftarrow G^{\oplus 6} \leftarrow G^{\oplus 10} \leftarrow G^{\oplus 15} \leftarrow G^{\oplus 21} \leftarrow \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 \widetilde{C}_*(\Delta^2/\partial\Delta^2; G) & 0 \leftarrow 0 \leftarrow 0 \leftarrow 0 \xleftarrow{0} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} G^{\oplus 6} \leftarrow \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 0 & 0 & 0 & 0 & 0 & 0 &
 \end{array}$$

we get

$$\widetilde{H}_k(\Delta^2/\partial\Delta^2; G) = \begin{cases} 0, & k < 2 \\ G, & k = 2 \\ 0, & k > 2 \end{cases}$$

$$\widetilde{H}^k(\Delta^2/\partial\Delta^2; G) = \begin{cases} 0, & k < 2 \\ G, & k = 2 \\ 0, & k > 2 \end{cases}$$

↑ dimension argument, or comparison with sset.

Here, the basis of $\widehat{C}(\Delta^2/\partial\Delta^2; G) \subset C(\Delta^2/\partial\Delta^2; G)$ is given by the following order:

$$\begin{array}{ccccccc}
 C.(\Delta^2/\partial\Delta^2; G) & 0 \leftarrow G \xleftarrow{o} G \xleftarrow{(11)} G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 7} \leftarrow \dots & [3,1,1] \\
 & & & & [2,1,1] & [1,3,1] \\
 \widehat{C}.(\Delta^2/\partial\Delta^2; G) : & [1,1,1] & [1,2,1] & [1,1,3] \\
 & [1,1,2] & [2,2,1] \\
 & & [2,1,2] \\
 & & [1,2,2] \\
 \text{xtra basis vector:} & [1,0,0] & [2,0,0] & [3,0,0] & [4,0,0] & [5,0,0]
 \end{array}$$

Ex. Check that we have SES of cplxes

$$\begin{array}{ccccccc}
 & 0 & 0 & 0 & 0 & 0 & 0 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 \widetilde{C}(\Delta^2/\partial\Delta^2; G) & 0 \leftarrow 0 & \leftarrow 0 & \leftarrow G & \leftarrow G^{\oplus 3} & \leftarrow G^{\oplus 6} & \leftarrow \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\
 C(\Delta^2/\partial\Delta^2; G) & 0 \leftarrow G & \leftarrow G & \leftarrow G^{\oplus 2} & \leftarrow G^{\oplus 4} & \leftarrow G^{\oplus 7} & \leftarrow \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\
 C(\partial\Delta^2/\partial\Delta^2; G) & 0 \leftarrow G & \overset{o}{\leftarrow} G & \overset{!}{\leftarrow} G & \overset{o}{\leftarrow} G & \overset{!}{\leftarrow} G & \overset{o}{\leftarrow} \dots \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \dots \\
 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$

Also, deduce that

$$H_k(\Delta^2/\alpha\Delta^2; G) = \begin{cases} G & k=0, 2 \\ 0 & \text{otherwise} \end{cases}$$

3. contractible sset has trivial cohomology

Continue of E.g. 2

Rmk. Actually, we have chain homotopy equivalence between $C_*(\Delta'; G)$ and $C_*(\Delta^o; G)$.

$$\begin{array}{c}
 \Delta' \quad C.(\Delta'; G) : \quad o \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots \\
 \downarrow s'_* \\
 S'_* \\
 \Delta' \quad C.(\Delta'; G) : \quad o \leftarrow G \xleftarrow{o} G \xleftarrow{Id} G \xleftarrow{o} G \dots \\
 \\
 \Delta^\circ \quad C.(\Delta^\circ; G) : \quad o \leftarrow G \xleftarrow{o} G \xleftarrow{Id} G \xleftarrow{o} G \dots \\
 \downarrow d'_* \\
 d'_* \\
 \Delta' \quad C.(\Delta'; G) : \quad o \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots
 \end{array}$$

$$\text{s.t. } s_{*,*}^i \circ d_{*,*}^i = \text{Id}_{C_*(\Delta^i; G)}, \quad d_{*,*}^i \circ s_{*,*}^i \sim \text{Id}_{C_*(\Delta^i; G)}.$$

In fact, we have

$$\begin{array}{ccccccc}
 & & & & & & \\
 C.(\Delta'; G) : & 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots \\
 & \downarrow \text{Id} \quad \downarrow \text{Id} \\
 & C.(\Delta'; G) : & 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots
 \end{array}$$

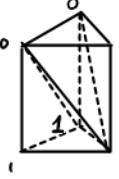
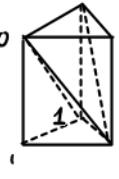
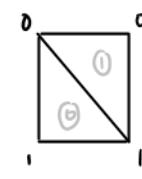
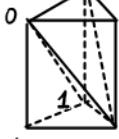
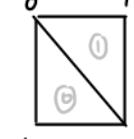
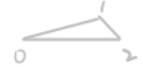
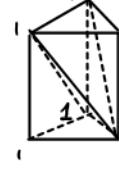
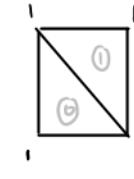
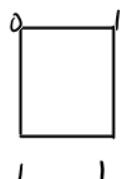
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graph LR
    x0[x₀] --> x0[x₀]
    x1[x₁] --> x0[x₀]

```

$$\begin{array}{l}
 x_0 \xrightarrow{x_1} x_0 - x_0 + x_0 = x_0 \\
 x_1 \xrightarrow{x_2} x_1 - x_1 + x_1 = x_1 \\
 x_2 \xrightarrow{x_3} x_1 - x_2 + x_2 = x_1 \\
 x_3 \xrightarrow{x_0} x_1 - x_2 + x_3 \\
 x_0 \xrightarrow{x_1} x_0 - x_0 = 0 \\
 x_1 \xrightarrow{x_2} x_1 - x_1 = 0 \\
 x_2 \xrightarrow{x_1} x_1 - x_2
 \end{array}$$

Ex. Observe the picture, try to translate the calculation in geometrical language.



4. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_*(\Delta'; G)$ and $C_*(\Delta'; G)^\diamond$.

$$C_*(\Delta'; G) : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

$$\downarrow \text{projection} \quad \downarrow \text{Id} \quad \downarrow (111) \quad \downarrow 0 \quad \downarrow 0$$

$$C_*(\Delta'; G)^\diamond : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} G \xleftarrow{0} 0 \leftarrow 0 \leftarrow 0 \dots$$

$$C_*(\Delta'; G)^\diamond : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} G \xleftarrow{0} 0 \leftarrow 0 \leftarrow 0 \dots$$

$$\downarrow \text{inclusion} \quad \downarrow \text{Id} \quad \downarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \downarrow 0 \quad \downarrow 0$$

$$C_*(\Delta'; G) : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

In fact, we have

$$C_*(\Delta'; G) : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

$$\downarrow \text{Id} \quad \downarrow \text{Id} \quad \downarrow \text{Id} \quad \downarrow \text{Id} \quad \downarrow \text{Id}$$

$$C_*(\Delta'; G) : 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)

For $X \in \text{Set}$, define

\triangleleft : non-degenerate

\triangleleft : degenerate

$$\begin{aligned} X_k^\triangleleft &= \{x \in X_k \mid x \text{ non-degenerate}\} & = X_k - (sk^{k-1}X)_k \\ X_k^\triangleleft &= \{x \in X_k \mid x \text{ degenerate}\} & = (sk^{k-1}X)_k \\ X_k^{\triangleleft i} &= \left\{ x \in X_k \mid \begin{array}{l} x = \varphi^*(y) \text{ for some } y \in X_{k-i}^\triangleleft \\ \varphi: [k-i] \rightarrow [k] \end{array} \right\} & = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k \end{aligned}$$

$$0 = (sk^{-1}X)_k \subset \underbrace{(sk^0X)_k \subset (sk^1X)_k \subset \dots \subset (sk^{k-1}X)_k}_{X_k^\triangleleft} \subset (sk^kX)_k = X_k$$

Def. For $X \in \text{Set}$, $G \in \text{Abel}$, define the chain cplx

$$C_n(X; G)^\triangleleft = \bigoplus_{x \in X_n^\triangleleft} G \quad 0 \leftarrow \bigoplus_{x \in X_0^\triangleleft} G \xleftarrow{(d_0 - d_1)^*} \bigoplus_{x \in X_1^\triangleleft} G \xleftarrow{(d_0^+ - d_0^- + d_1^+)^*} \bigoplus_{x \in X_2^\triangleleft} G \dots$$

$$C_n(X; G)^\triangleleft = \bigoplus_{x \in X_n^\triangleleft} G \quad 0 \leftarrow \bigoplus_{x \in X_0^\triangleleft} G \xleftarrow{(d_0 - d_1)^*} \bigoplus_{x \in X_1^\triangleleft} G \xleftarrow{(d_0^+ - d_0^- + d_1^+)^*} \bigoplus_{x \in X_2^\triangleleft} G \dots$$

and $H_*(X; G)^\triangleleft$, $H_*(X; G)^\triangleleft$ as crspd homology.

By definition,

$$C_*(X; G) \cong C_*(X; G)^\triangleleft \oplus C_*(X; G)^\triangleleft$$

Claim 1. $H_*(X; G)^\triangleleft = 0$, so

$$H_*(X; G) \cong H_*(X; G)^\triangleleft. \quad (\#)$$

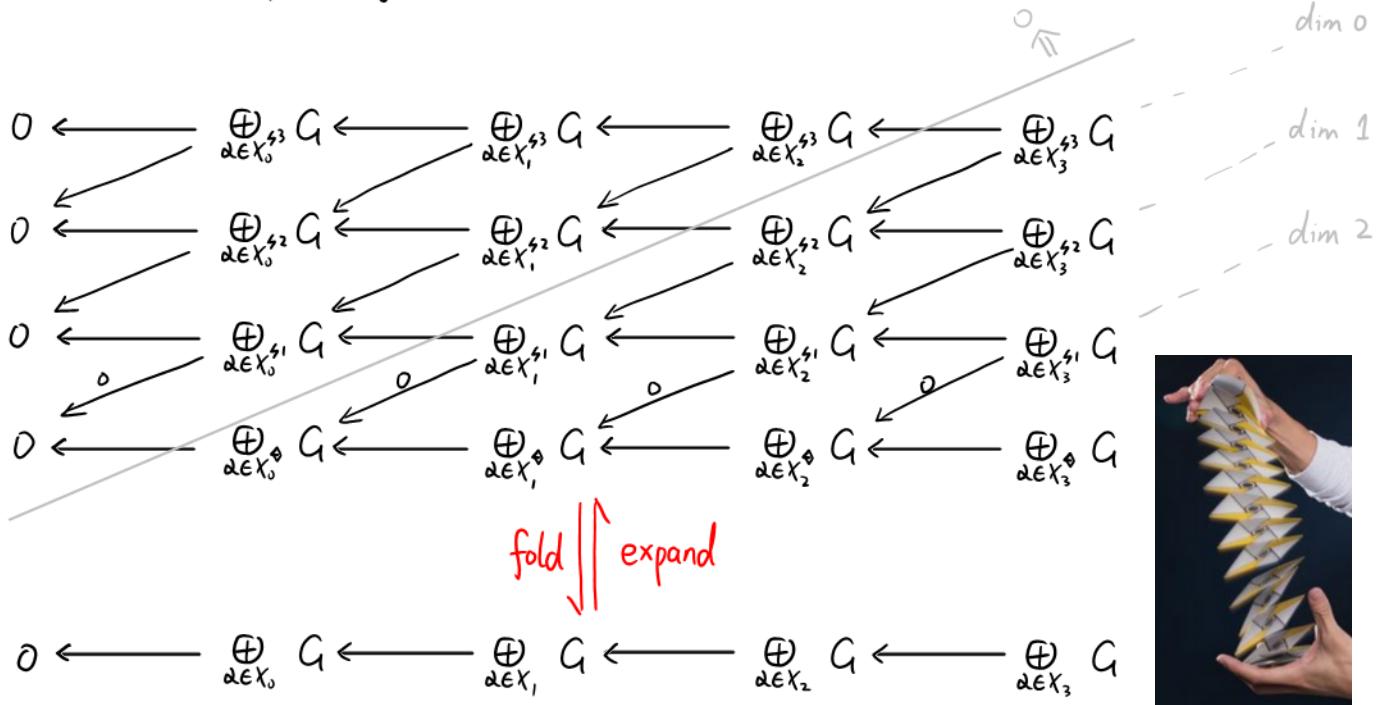
Rmk 1. Roughly, $(\#)$ says that

singular homology \approx simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

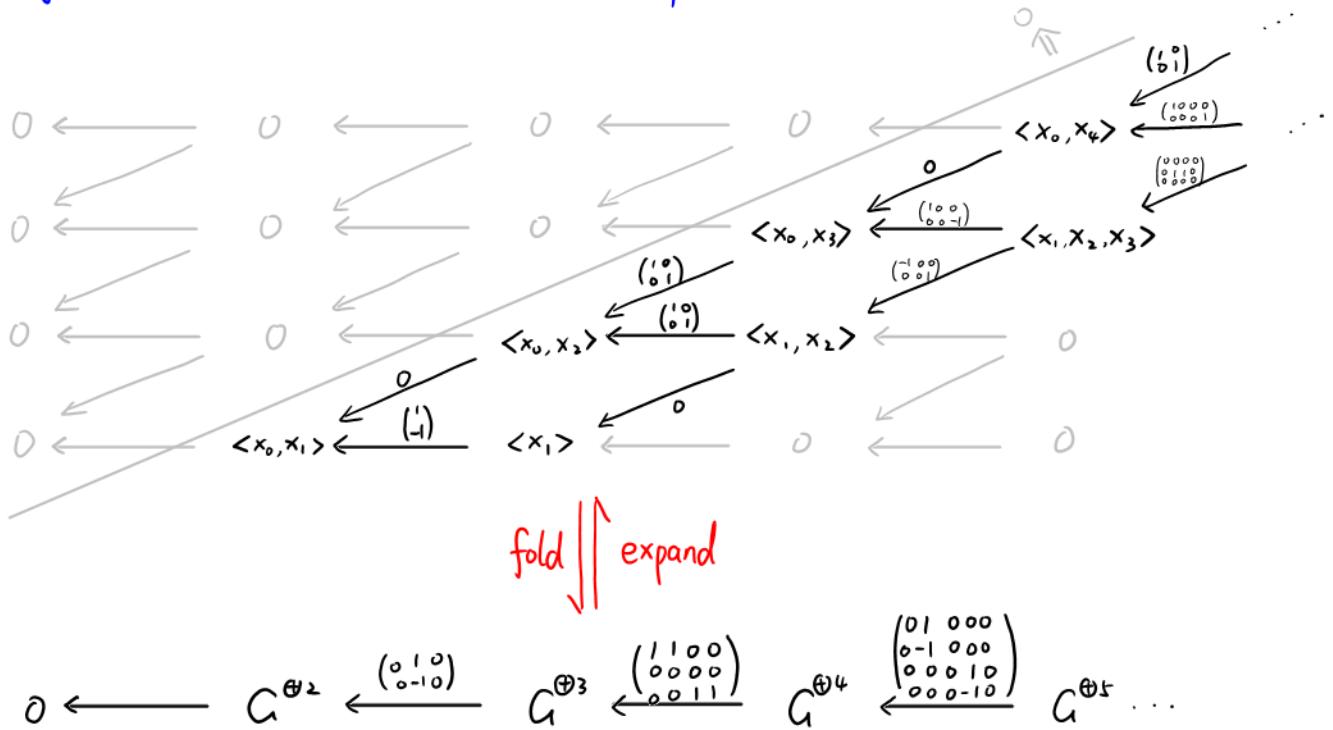
To prove Claim 1, one has to expand $C_*(X; G)$ by double complex.

Def (Double complex of $C_*(X; G)$) $\swarrow + \searrow = 0$



Eg. For $X = \Delta'$, we have double complex

fold / expend



Claim 2. We have chain homotopy equivalence between the following two cplx.

$$\begin{array}{ccccccc}
 0 & \longleftarrow & \bigoplus_{\alpha \in X_n^{s_k}} G & \xleftarrow{\circ} & \bigoplus_{\alpha \in X_{n+1}^{s_k}} G & \xleftarrow{\partial'} & \bigoplus_{\alpha \in X_{n+2}^{s_k}} G & \xleftarrow{\partial''} & \bigoplus_{\alpha \in X_{n+3}^{s_k}} G \\
 & & \parallel & & \circ(\uparrow) \circ & & \circ(\uparrow) \circ & & \circ(\uparrow) \circ \\
 0 & \longleftarrow & \bigoplus_{\alpha \in X_n^{s_k}} G & \longleftarrow & 0 & \longleftarrow & 0 & \longleftarrow & 0
 \end{array} \quad (**)$$

i.e. $(**)$ is exact on all terms except $\bigoplus_{\alpha \in X_n^{s_k}} G$.

Proof idea of Claim 2 for $X = \Delta^m$.

(can be generalized to arbitrary subset of Δ^m , e.g. for any triangulation)

Q: How far can we weaken our conditions on X ?

Does this proof work for $\partial\Delta^m$? (No, I think).

$$\begin{array}{ccccccc}
 \cdots & \longleftarrow & \bigoplus_{\alpha \in X_{n+k-1}^{s_k}} G & \longleftarrow & \bigoplus_{\alpha \in X_{n+k}^{s_k}} G & \longleftarrow & \bigoplus_{\alpha \in X_{n+k+1}^{s_k}} G & \longleftarrow & \cdots \\
 & \searrow s & \downarrow \text{Id} & \searrow s & \downarrow \text{Id} & \searrow s & \downarrow \text{Id} & \searrow s & \\
 \cdots & \longleftarrow & \bigoplus_{\alpha \in X_{n+k-1}^{s_k}} G & \longleftarrow & \bigoplus_{\alpha \in X_{n+k}^{s_k}} G & \longleftarrow & \bigoplus_{\alpha \in X_{n+k+1}^{s_k}} G & \longleftarrow & \cdots
 \end{array}$$

Define

$$s [a_1, \dots, \underbrace{a_l}_{\{0,1\}}, a_{l+1}, \dots, a_m] = \begin{cases} (-1)^i [a_1, \dots, a_l, a_{l+1}+1, \dots, a_m], & a_{k+1} \text{ even} \\ 0 & a_{k+1} \text{ odd} \end{cases} \quad i = \sum_{j=1}^l a_j$$

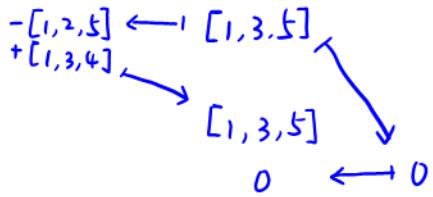
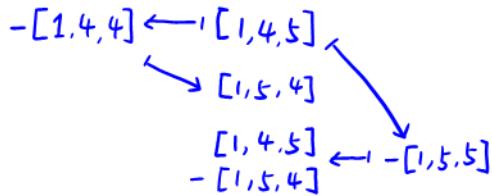
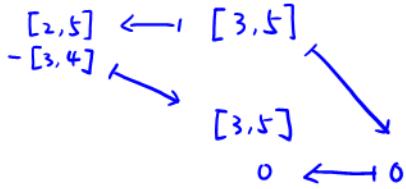
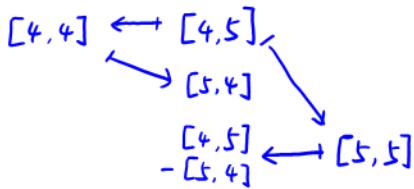
Ex. Check that s is a homotopy.

$$\text{e.g. } X = \Delta^3, n=2, k=3 \Rightarrow m=3, n+k=5$$

$$\begin{array}{ccccc}
 -[2,1,0,2] & \longleftrightarrow & [2,1,0,3] & & \\
 & \swarrow & \searrow & & \\
 & -[3,1,0,2] & & & \\
 & \uparrow & & & \\
 & [2,1,0,3] & \longleftrightarrow & [3,1,0,3] & \\
 & +[3,1,0,2] & & &
 \end{array}$$

$$X = \Delta^6, n=5, k=15 \Rightarrow m=6, n+k=20$$

$$\begin{array}{ccccc}
 [2,4,3,4,1,6,0] & \longleftrightarrow & [2,5,3,4,1,6,0] & & \\
 -[2,5,2,4,1,6,0] & \swarrow & \uparrow & \searrow & \\
 & [3,4,3,4,1,6,0] & & & \\
 & -[3,5,2,4,1,6,0] & & & \\
 & [2,5,3,4,1,6,0] & & & \\
 & -[3,4,3,4,1,6,0] & \longleftrightarrow & [3,5,3,4,1,6,0] & \\
 & +[3,5,2,4,1,6,0] & & &
 \end{array}$$

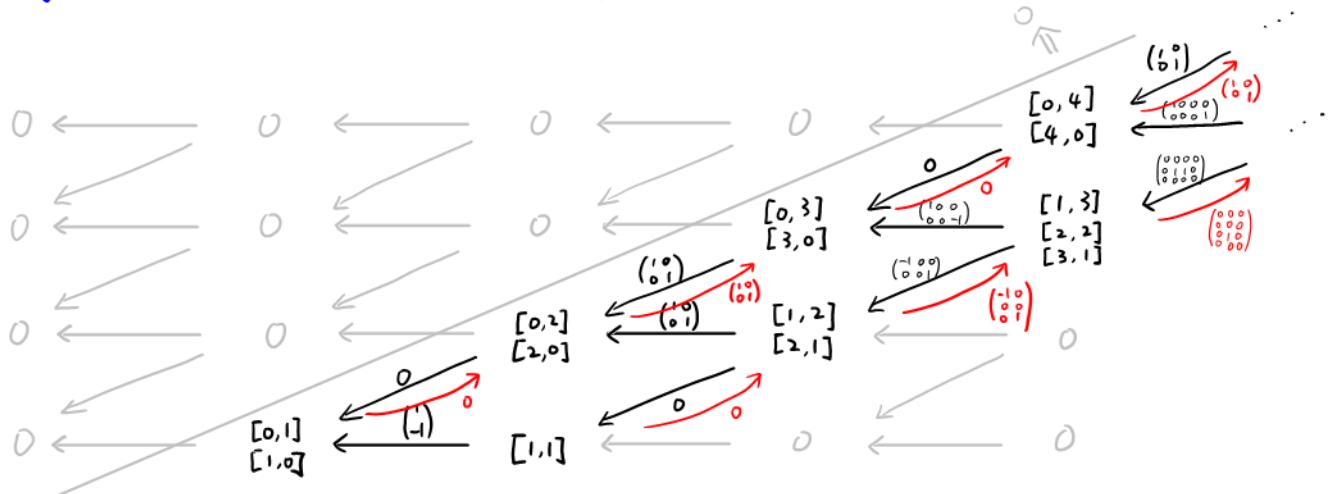


In conclusion,

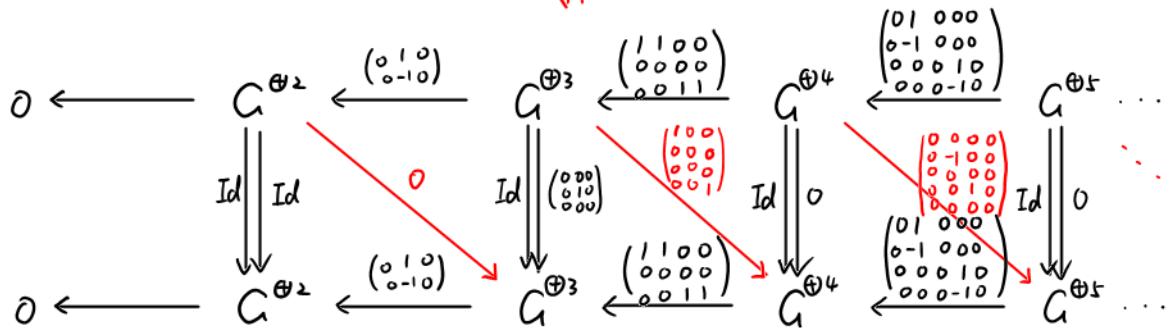
$$\text{Claim 2} \Rightarrow \text{Claim 1} \Rightarrow \text{Rmk 1}$$

Coming back to E.g. 2, one can now find a homotopy without guess.

Eg. For $X = \Delta'$, we have homotopy



fold // expand



Ex. Check that (I believe that this argument also works for general sset X)

$$\textcircled{1} \quad \begin{array}{c} \nearrow s \\ \searrow s \end{array} + \begin{array}{c} \nearrow s \\ \searrow s \end{array} = 0$$

\textcircled{2} the collected s is a homotopy.

5. more structures

math.stackexchange.com/questions/2559705/cup-product-why-do-we-need-to-consider-cohomology-with-coefficients-in-a-ring
<https://arxiv.org/pdf/1105.0802v5.pdf>
<https://users.math.msu.edu/users/rulterj2/math/Documents/Spring%202019/Galois%20Cohomology%20Seminar%20Week%203.pdf>

When $G = R$ is a k -alg, the product structure on $C^*(X; R)$ is defined by
 (cup product)

$$\begin{array}{ccc}
 \text{wedge in de Rham} & & \\
 \cup \approx \wedge & d_{i,j,*} \otimes d'_{i,j,*} & C^{i+j}(X; R) \otimes C^{i+j}(X; R) \\
 \text{bad symbol compatibility...} & \nearrow & \downarrow \\
 \cup: C^i(X; R) \otimes C^j(X; R) & f_1 \otimes f_2 & \xrightarrow{\text{multiply}} C^{i+j}(X; R) \\
 & \longleftarrow & (f_1 \circ d_{i,j,*}) \otimes (f_2 \circ d'_{i,j,*}) \\
 & & f \otimes g \xrightarrow{\quad} fg
 \end{array}$$

the $C^*(X; R)$ -module structure on $C_*(X; G)$ is defined by
 (cap product)

$$\begin{array}{ccc}
 \text{Id} \otimes d_{i,j}^*(-) \otimes d'^{*}_{i,j}(-) & C^i(X; R) \otimes C_i(X; R) \otimes C_j(X; R) \\
 \nearrow & \downarrow & \\
 \cap: C^i(X; R) \otimes C_{i+j}(X; R) & f \otimes \alpha & \xrightarrow{\text{multiply}} C_j(X; R) \\
 & \longleftarrow & r(d_{i,j}^*(\alpha)) \otimes d'^{*}_{i,j}(\alpha) \\
 & & r \otimes \beta \xrightarrow{\quad} r\beta
 \end{array}$$

where ($i=3, j=2$)

$$\begin{array}{ccc}
 & \cdot \quad \cdot \quad \cdot \quad \cdot & \\
 & \searrow \quad \searrow \quad \searrow \quad \searrow & \\
 \cdot \quad \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
 & \downarrow & \\
 d_{i,j}: [i] & \longrightarrow & [i+j] \\
 d_{i,j} = [\underbrace{1, \dots, 1}_{i+1 \text{ many}}, \underbrace{0, \dots, 0}_j \text{ many}] & & \\
 & & \\
 & \cdot \quad \cdot \quad \cdot & \\
 & \swarrow \quad \swarrow \quad \swarrow & \\
 \cdot \quad \cdot \quad \cdot & & \cdot \quad \cdot \quad \cdot \\
 & \downarrow & \\
 d'_{i,j}: [j] & \longrightarrow & [i+j] \\
 d'_{i,j} = [\underbrace{0, \dots, 0}_i \text{ many}, \underbrace{1, \dots, 1}_{j+1 \text{ many}]}
 \end{array}$$

\otimes are over k . Notice that

$$\begin{cases} C_i(X; R) = \bigoplus_{x \in X_i} R \\ C^i(X; R) = \prod_{x \in X_i} R \end{cases}$$

are bi R -modules.

<https://math.stackexchange.com/questions/4439483/applications-of-the-cup-product-before-descending-to-cohomology>

Ex: Show that:

- $C^*(X; R)$ is a dga,
- $H^*(X; R)$ is a graded R -alg. (graded commutative)
- $H_*(X; R)$ is a graded $H^*(X; R)$ -module.

Hint. Once the following key formulas are verified, we are done.

For $a \in C^p(X; R)$, $b \in C^q(X; R)$,

$$d^{p+q}(a \cup b) = d^p(a) \cup b + (-1)^p a \cup d^q(b)$$

$$a \cup b - (-1)^{pq} b \cup a = (-1)^{p+q-1} (d^{p+q-1}(a \cup b) - d^p(a) \cup b - (-1)^p a \cup d^q(b))$$

where

$$a \cup b \in C^{p+q-1}(X; R)$$

$$a \cup b(x) = \sum_{i=0}^{p-1} (-1)^{(p-i)(q+1)} a(\delta_{i,(p,q)}^{\text{out},*}(x)) \cdot b(\delta_{i,(p,q)}^{\text{in},*}(x))$$

For the definition of $\delta_{i,(p,q)}^{\text{out}}$, $\delta_{i,(p,q)}^{\text{in}}$, see [23.01.09, Table 1].