Eine Woche, ein Beispiel

5.1 Extension of NA local field

1. List of well-known results
   - in general
   - unramified / totally ramified
2. $\hat{\mathbb{Z}}$ = profinite completion (review)
3. Big picture
4. Henselian ring
5. Cohomological dimension.

Ref:
Initial motivation comes from
[AY]https://alex-youcis.github.io/localglobalgalois.pdf

main reference for cohomological dimension:
[NSW2e]https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/

[JPS96] Galois cohomology by Jean-Pierre Serre
http://p-adic.com/Local%20Fields.pdf
https://people.clas.ufl.edu/rcrew/files/LCFT.pdf
http://www.mcm.ac.cn/faculty/tianyichao/201409/W020140919372982540194.pdf

1. List of well-known results

In general

F: NA local field $\quad E/F$: finite extension

Rmk 1. E is also a NA local field with uniquely extended norm

$$\|x\| := \|N_{E/F}(x)\|_F^{\frac{1}{n}} \quad \text{resp. } v(x) := \tfrac{1}{n} v_F(N_{E/F}(x))$$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$ is monogenic, i.e. $\mathcal{O}_E = \mathcal{O}_F[\alpha]$ $\quad \exists \alpha \in \mathcal{O}_E$

Cor. (primitive element thm for NA local field)

$E = F[x]/(g(x))$ $\quad \exists x \in \mathcal{O}_E$, $g(x)$ min poly of x.

Rmk: Every separable finite field extension has a primitive element, see wiki:
https://en.wikipedia.org/wiki/Primitive_element_theorem
Separable condition is necessary, see
https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element

Rmk 3. Any finite extension of $\mathbb{Q}_p$ is of form $\mathbb{Q}_p[x]/(g(x))$,
where $g(x) \in \mathbb{Q}[x]$ is an irr poly.
Any finite extension of $\mathbb{F}_q((t))$ is of form $\mathbb{F}_q((t))[x]/(g(x))$
where $g(x) \in \mathbb{F}_q(t)[x]$ is an irr poly..
Both are achieved by Krasner's lemma.

$$v = v_F = \frac{1}{e} v_E \qquad \|\cdot\| = \|\cdot\|_F = \|\cdot\|_E^{\frac{1}{e}} \qquad \mathfrak{p}_F \mathcal{O}_E = \mathfrak{p}_E^e$$

$$\begin{array}{llllll} E & v_E = ev & \|\cdot\|_E = \|\cdot\|^e & \pi_E = \pi_F^{\frac{1}{e}} & v(\pi_E) = \frac{1}{e} \\ \mid \deg n & & & & \\ F & v_F & \|\cdot\|_F & \pi_F & v(\pi_F) = 1 \end{array}$$

Unramified / totally ramified

Good ref: https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields
It collects the equivalent conditions of unramified/totally ramified field extensions.

$$\begin{array}{lll} \mid & \text{tot ram} & \mid \text{wild ram} \\ \mid & & \mid \text{tame ram} \\ \mid \text{field ext} & & \\ \mid \text{split} & & \end{array}$$

When E/F is tot ramified,

$e = n \qquad v(\pi_E) = \frac{1}{n}$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E] \qquad \min(\pi_E) \in \mathcal{O}_F[x]$ is Eisenstein poly.