## Eine Woche, ein Beispiel 8.21 equivariant K-theory of IP'

Let us do a simple case over IP'. It can be generlized "easily" to flag variety, but IP' is the beginning case of study.

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

## Task. Understand

where 
$$SL_{1} = SL_{1}, C$$
,  $B = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \subseteq SL_{1}, C$ ,  $C = C/B$ .  $C = C/B$ .  $C = C/B$ .  $C = C/B$ .  $C = C/B$ .

Notation. For linear alg gp G [
$$G_{inz}$$
,  $\pm 1$ ],
$$K_i^G(X) = K_i(C_0h(X)) \qquad K^G(X) := K_0^G(X) \qquad K(X) := K^{fid}(X)$$

$$R(G) := K^G(pt) = K_0(C_0h^G(pt)) = K_0(Rep G)$$
e.g.  $R(fid) = \mathbb{Z}$ ,  $R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}]$ ,  $R(SL) \cong \mathbb{Z}[x]$ ,  $R(SL_1 \times \mathbb{C}^x) \cong \mathbb{Z}[x, t^{\pm 1}]$ 

Some further discussion of R(SL2).

 $R(SL_1) = \bigoplus_{i \in N_{20}} C \times_i$  where  $\times_i$  represents the (i+1)-dim irr rep of  $SL_2$ . As an algebra,  $R(SL_2) = C[\times]$  where

$$1 = X_{0}$$

$$X = X_{1}$$

$$X^{2} = X_{2} + 1$$

$$X^{3} = X_{3} + 2X,$$

$$X^{4} = X_{11} + 3X_{2} + 2$$

$$X_{0} = 1$$

$$X_{1} = X$$

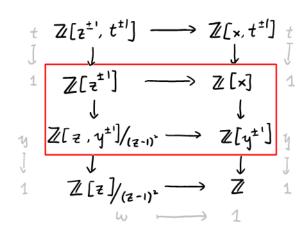
$$X_{2} = X^{2} - 1$$

$$X_{3} = X^{3} - 2X$$

$$X_{4} = X^{4} - 3X^{2} + 1$$

[LCBE, 2.1.1] 
$$K(P') \cong \mathbb{Z}\mathcal{O}_{P'} \oplus \mathbb{Z}\mathcal{O}_{P'}(1) = \mathbb{Z}[\mathbb{Z}]/(\mathbb{Z}-1)^2 = \mathbb{Z}[\mathbb{Z}^{\pm 1}/(\mathbb{Z}-1)^2]$$
 $\mathbb{Z}$  corresponds to  $\mathbb{Z}$ [ $\mathbb{Z}$ ]/ $\mathbb{Z}$  gives def of pushforward.

In conclusion, we get



The difficult part is the middle square.  $\mathbb{Z}[z,y^{\pm i}]/_{(w-i)^{2}} \longrightarrow \mathbb{Z}[y^{\pm i}]$ 

Right: by rep theory,

$$Z[x] \longrightarrow Z[y^{\pm 1}] \qquad homo \quad as \quad Z-alg$$

$$x_0 \longmapsto 1$$

$$x_1 \longmapsto y+y^{-1}$$

$$x_2 \longmapsto y^2+1+y^{-2}$$

$$x_3 \longmapsto y^3+y+y^{-1}+y^{-3}$$

Up by Borel-Weil-Bott theorem.

Left: by [LW-BWB, Ex 2.6], 
$$L_n \supseteq O(-n)$$
, combined with "Up", we get

 $\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[z, y^{\pm 1}]/(z-1)^2$ 

e.g.  $z^3 \longmapsto -z^3(y+y^{-1})$  (see table below)

 $z \mapsto z^{-1} \quad z^{-1} \quad 1 \quad z \quad z^{2} \quad z^{3} \quad z^{4} \quad \sum_{x=-\infty}^{\infty} \frac{z^{x}}{x^{2}} \quad z^{2} \quad z^{2}$ 

Under these (natural) ring structure, 
$$\mathbb{Z}[x,t^{\pm i}] \longrightarrow \mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}] \longrightarrow \mathbb{Z}$$
 are homo of rings.

Ex. Generalize to 
$$SL_1 \longrightarrow SL_n$$
,  $P' \longrightarrow Flag(C')$   
 $SL_2 \longrightarrow GL_2$   
 $C \longrightarrow FP$   $C^* \longrightarrow FP$   
 $Q:$  How to compute  $K_i^{SL_1 \times C^*}(P')$  for  $i \ge 1$ ?