

Eine Woche, ein Beispiel

9.5. vector bundle v.s. local system

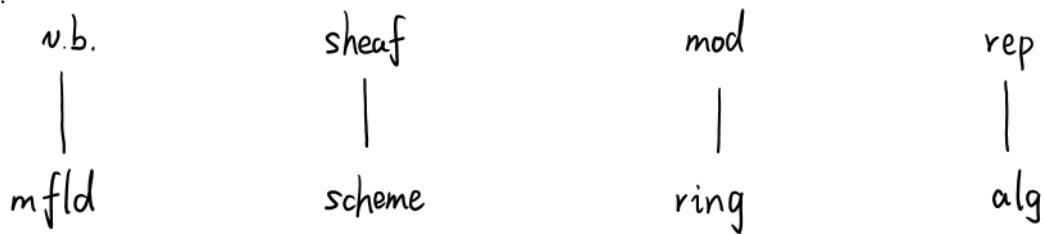
Key objects in Geometry & Algebra:

- vector bundle over manifold
- module over ring

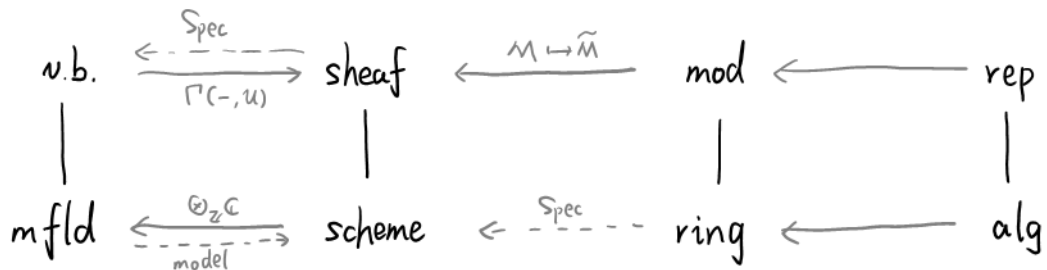
There are hundreds of different versions of it:

- vector bundle over manifold 几何/几何分析
 - diffe mfld • (real) differential v.b. over (real) differential mfld
 - Riemann surface • cplx (analytic) line bundle over Riemann surface
- sheaf over space 代数几何
 - scheme theory • locally free sheaf on scheme
 - coherent sheaf on scheme
 - geo rep theory • local system over (real/cplx) mfld
 - perverse sheaf over Riemann surface (derived)
 - simplicial set over category Δ
- module over ring 代数
 - comm alg • f.g module over Noetherian commutative ring (with 1)
 - rep of grp • group representation over group (\leadsto group algebra)
 - p-adic rep • smooth representation over unimodular gp (\leadsto Hecke algebra $\mathcal{H}(G)$) smooth module
 - quiver theory • quiver representation over quiver (\leadsto path algebra, bound quiver algebra)
 - Lie algebra • Lie alg representation over Lie alg (\leadsto universal enveloping algebra)
- Arithmetic Geometry 代数 \leadsto p-分析
 - hermitian line bundle over projective arithmetic variety \mathcal{X}
 - adelic line bundle over essentially quasi-proj scheme
 - over Berkovich analytic space X^{an}
 - over formal scheme $\mathrm{Spf} A$
 - over rigid-analytic space $k\text{-affinoid space}$
 - over adic space $\mathrm{Spa}(A, A^+)$

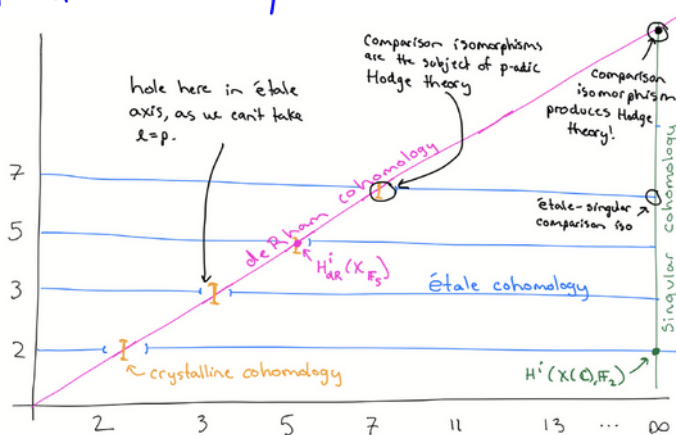
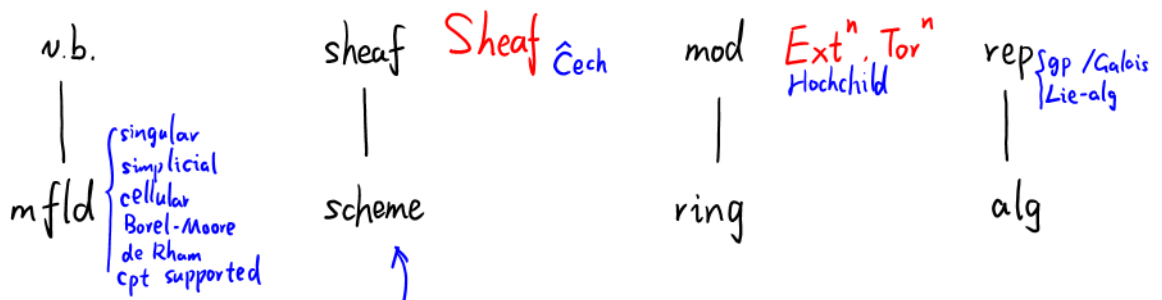
Picture:



- ① variation (e.g. v.b. \rightarrow f.b., mfld \rightarrow CW cplx, sheaf \rightarrow fctor, scheme \rightarrow stack/adic space,...)
- ② vertical relation: \downarrow : v.b. as mfld, representable fct, Spec/Proj construction, ...
 \uparrow : tangent/trivial v.b, structure sheaf, R as R-mod, regular rep, ...
- ③ horizontal relation:



④ homology and cohomology:



when axes meet: comparison isomorphisms (the "glue" of the "sheaf")

Prof. Scholze's ICM picture

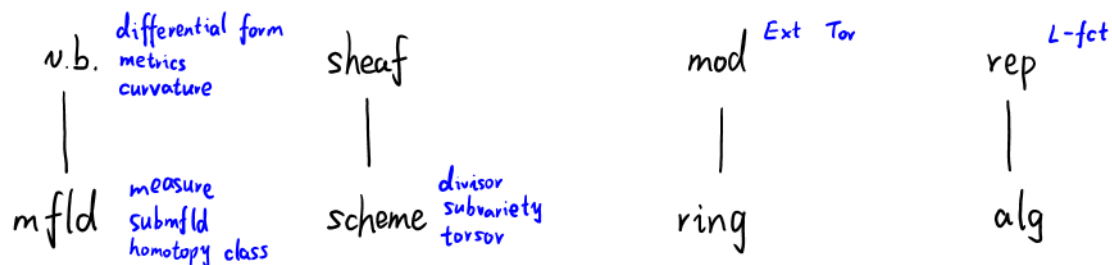
Remaining (co)homology:

l-adic cohomology
 intersection (co)homology
 elliptic cohomology
 flat cohomology
 infinitesimal cohomology

Objects in upper row can be already viewed as element in (co)homology.

eg. v.b. \leftrightarrow transition fct $\leftrightarrow H^1(X, -)$

One motivation for ∞ -category: make a generalization from H^1 to H^i

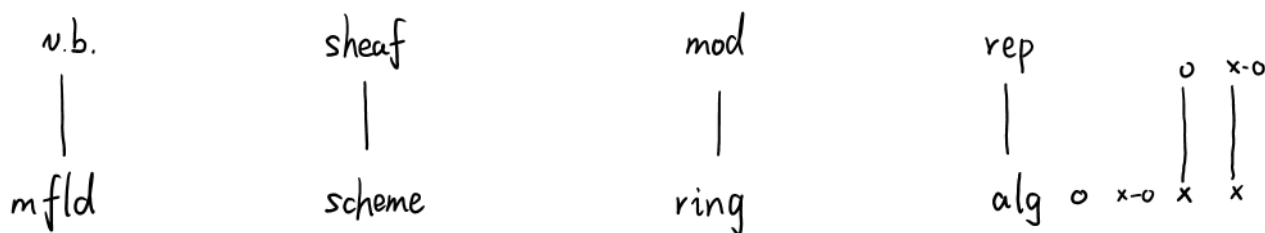


⑤ Chern class: from cohomology to cohomology (also for the other Char class)

There are several ways of defining/viewing Chern class.

- i) $\mathcal{L} \in \text{Pic}_G(X) \mapsto c_1(\mathcal{L}) \in H^2(X; \mathbb{Z})$
- ii) $H^1(X, \mathcal{O}_X^\times) \rightarrow H^2(X; \mathbb{Z})$ by LES
- iii) As the coefficient of equation ($CH^*(PE)$ is a free $CH^*(B)$ -module)
Euler class
- iv) As the pull back of the universal Chern class in Grassmannian
- v) From curvature; Chern-Weil theory
- vi) From Chow group
- vii) $\partial \bar{\partial}, \Delta$

⑥ moduli problems



Three type of geometry:

PDE	elliptic	parabolic	hyperbolic
curvature	+	0	-
genus	0	1	≥ 2
Euler number	-2	0	≥ 2
Kodaira dim	$-\infty$	0	dim X
variety	Fano	Calabi-Yau	general type
filtration	unramified	tame	wild
quiver	Dynkin	affine	strictly wild
condensed	solid	liquid	gaseous

- Goal
- structures & invariants
 - classifications of
 - special v.b, mfld, subv.b, submfld
 - symmetry & quotient
 - special functors
 - homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n .

(by homotopy group! \leadsto generalized Picard group?)

Q: Is this group structure natural?

ref: <https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres>

Thm. $\{\text{rank } m \text{ } K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$

$K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(K))$ \ rank n \ m	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	\mathbb{Z}	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	\mathbb{Z}	$\mathbb{Z}^{\oplus 2}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{R}P^\infty \cong K(\mathbb{Z}/2\mathbb{Z}, 1)$

$\pi_{n-1}(GL_m(K))$ \ rank n \ m	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
S^3 3	0	0	0	0	0	0	0
S^4 4	0	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}	\mathbb{Z}

$\mathbb{C}P^\infty \cong K(\mathbb{Z}, 2)$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes, \wedge^k, \text{Sym}^k, \text{Res}, \text{Ind}$

For the other spaces:

<https://math.stackexchange.com/questions/383838/classifying-vector-bundles>

<http://www.ms.uky.edu/~guillou/F18/751Notes.pdf>

It's still not so explicit.

$\{\text{rank } m \text{ } K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_K(m, \infty)]$

$K = \mathbb{R}, \mathbb{C}$

M : paracompact

Unfinished task: introduce the concept of local systems and compute examples in [<https://arxiv.org/pdf/2103.02329.pdf>], 16.3.