Eine Woche, ein Beispiel

11.7. Berkovich space

Ref: Spectral theory and analytic geometry over non-Archimedean fields by Vladimir G. Berkovich(we mainly follow this article)
+courses from Junyi Xie
+An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves

Goal: understand the beautiful picture copied from "An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves"

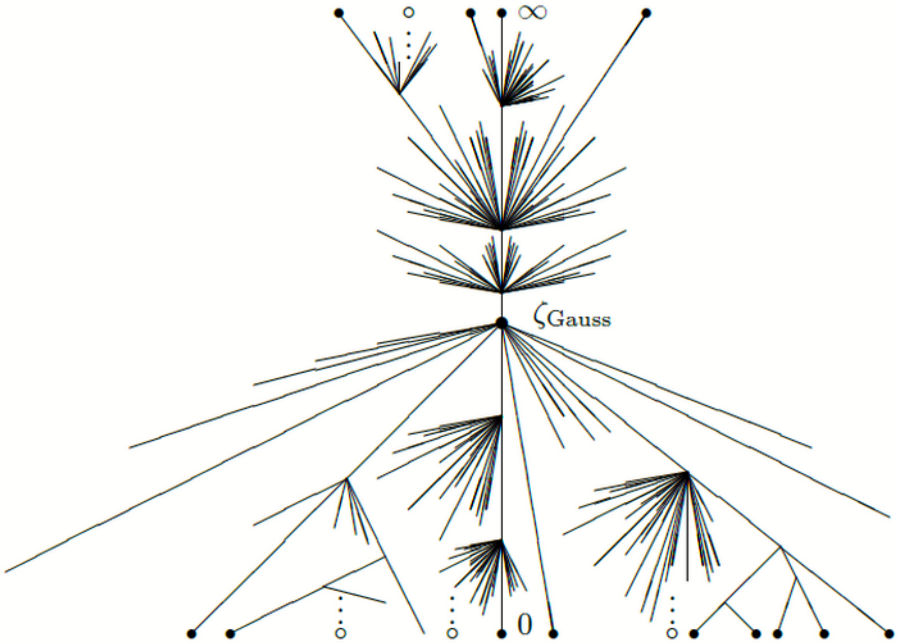


FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

A: comm with 1 (for convenience)

extra condition	local		global	closed unit disc	open unit disc
—	Spec A	affine scheme	scheme		Spec $\mathbb{Z}[[T]]$
A: adic ring with f.g. ideal of def	Spf A	affine formal scheme	formal scheme		Spf $\mathbb{Z}_p[[T]]$
A: k-affinoid alg, i.e. $A = K\langle T_1, \dots, T_n\rangle / I$	MaxSpec A	k-affinoid space	rigid-analytic space over K	MaxSpec $\mathbb{Q}_p\langle T\rangle$	$\mathcal{U} = \{ \lvert\cdot\rvert \in \mathrm{MaxSpec}\, \mathbb{Q}_p\langle T\rangle \mid \lvert T\rvert < 1\}$
(A, A^+): Huber pair	$\mathrm{Spa}(A, A^+)$	affinoid adic space	adic space	$\mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$	
A: Banach ring	$\mathcal{M}(A)$	spectrum	Berkovich space		

Ref of table: Berkeley notes

Rmk. MaxSpec A has only a Grothendieck topology.
K (in K-affinoid space) is a NA field, but can also be generalized to k-Banach alg.
↑ non-archimedean k: NA field

1. Seminorm

1.1. Def (seminorm of abelian group) $\|\cdot\|: M \longrightarrow \mathbb{R}_{\geq 0}$ s.t

$\|0\| = 0$ — norm: $\|m\| = 0 \Rightarrow m = 0$

$\|f-g\| \leq \|f\| + \|g\|$ — non-Archimedean: $\|f-g\| \leq \max(\|f\|, \|g\|)$

- seminorm $\Rightarrow$ topology

  Prop. $(M, \|\cdot\|)$ is Hausdorff $\Leftrightarrow$ $\|\cdot\|$ is norm

  Def (equivalence of norm)

- sub, quotient, homomorphism

  Def (restricted seminorm)

  Def. (residue seminorm) $\pi: (M, \|\cdot\|_M) \longrightarrow M/N$ induce the seminorm on M/N.

  $$\|\bar{m}\|_{M/N} := \inf_{\pi(m')=\bar{m}} \|m'\|_M$$

  Def (bounded / admissible) $\varphi: (M, \|\cdot\|_M) \longrightarrow (N, \|\cdot\|_N)$

  - bounded: $\exists C > 0$, $\|\varphi(m)\|_N \leq C\|m\|_M$
  - admissible: $\bar{\varphi}: (M/\ker\varphi, \|\cdot\|_{quo}) \longrightarrow (\mathrm{Im}\,\varphi, \|\cdot\|_{res})$ induces equivalence of norm.

1.2. Def (seminorm of ring non-comm, with 1): seminorm group +

$\|1\| = 1$ — norm $\xrightarrow{+\text{completed}}$ Banach ring

$\|fg\| \leq \|f\|\,\|g\|$

power-multi: $\|f^n\| = \|f\|^n$ — multiplicative: $\|fg\| = \|f\|\|g\|$ $\Big\}$ $\Rightarrow$ absolute value

- quotient, $\prod_{\text{infinite}}$, $\mathcal{A}\langle r^{-1}T\rangle$, ...
- comparison among norms: bounded.
- Def related to valuation field

https://math.stackexchange.com/questions/2151779/normed-vector-spaces-over-finite-fields

1.3. Def (seminorm of $\mathcal{A}$-module, where $\mathcal{A}$: normed ring)

seminorm group + $\exists C > 0$, $\|fm\| \leq C\|f\|\,\|m\|$

- $\hat{\otimes}_{\mathcal{A}}$

seminormed ring

$\uparrow$ — $(\mathbb{Z}, |\ |_p)$, $(\mathbb{Q}, |\ |_\infty)$, $(\mathbb{Q}, |\ |_p)$

Banach ring

$\uparrow$ — (R, triv) R: not field, $(\mathbb{Z}, |\ |_\infty)$, $(\mathbb{C}\langle r^{-1}T\rangle, \|\ \|_{ind})$

valuation field

Arch field	non-Arch field
$(\mathbb{R}, |\ |_\infty)$ $(\mathbb{R}/\mathbb{C}, |\ |_\infty^{\varepsilon})$ $0 < \varepsilon \leq 1$	$(\mathbb{F}_q, \text{triv})$ (K, triv)
$(\mathbb{C}, |\ |_\infty)$	$(\mathbb{Q}_p, |\cdot|_p)$

⚠ In analysis, the word "seminorm" is defined in a "totally" different way:

Definition 1.1.3. A **seminorm** on a $\mathbb{K}$-vector space E is a function $p : E \to \mathbb{R}$ such that

(1) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$.

(2) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{K}, x \in E$.

In analysis, the ring usually has no unit (e.g. $\mathcal{L}^1(\mathbb{R})$), and (semi)norms are absolute homogeneous.

Moreover, we don't require semimultiplicative.

e.g. in $\mathcal{L}^p(\mathbb{R})$, one don't have $\|fg\|_p \leq \|f\|_p \|g\|_p$

Apart from analysis, the terminology is concluded as follows.

<table>
<tr><td colspan="3">seminorm</td></tr>
<tr><td rowspan="2"></td><td colspan="2">(multiplicative) norm = absolute value = places</td></tr>
<tr><td>valuation (Bourbaki)
exponential valuation
NA absolute value
ultrametric absolute value</td><td>Archi absolute value</td></tr>
</table>

I prefer Bourbaki's terminology, because valuations are always written additive, and the natural triangular inequality is the ultrametric inequality, i.e.,

$$v(a+b) \geq \min(v(a), v(b)), \quad \text{with equality if } v(a) \neq v(b)$$

Many people don't use "absolute value" for high rank valuations.

In the main ref (as well as this document, e.g. no example found yet) the norm can be not multiplicative, but I assume norm to be multiplicative in other documents.

2. Affine case

suppose $\mathcal{A}$: Banach ring comm + 1

$\mathcal{M}(\mathcal{A}) := \{$bounded mult seminorms on $\mathcal{A}\}$

with top basis generated by $U_{m,(a,b)} := \{\|\cdot\| \in \mathcal{M}(\mathcal{A}) \mid \|m\| \in (a,b)\}$

$m \in \mathcal{A}$, $(a,b) \subseteq \mathbb{R}$

$\mathcal{M}(\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)) := \{$mult seminorms on $\mathcal{A}\}$

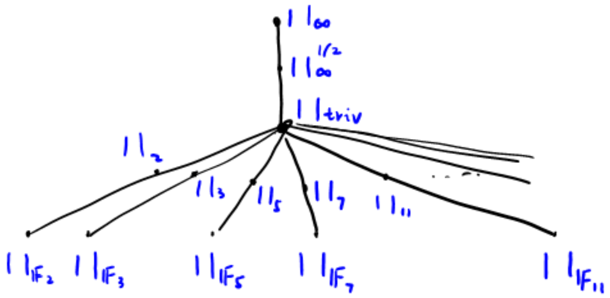
E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$ Don't confuse with $|\cdot|_p = |\cdot|_{L^p}$ in functional analysis!

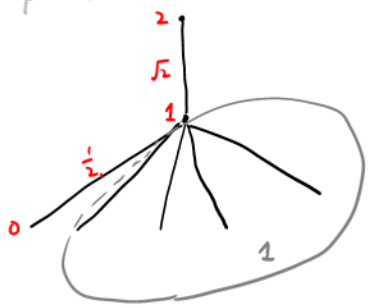
We have

$$\mathcal{M}(\mathbb{Z}, |\ |_\infty) = \left\{ \begin{array}{ll} |\ |_{triv} := \text{trivial norm} & \\ |\ |_p^t \ : \ t \in (0, +\infty] & |m|_{\mathbb{F}_p} := |m|_p^\infty = \begin{cases} 0 & p \mid m \\ 1 & p \nmid m \end{cases} \\ |\ |_\infty^\varepsilon \ : \ \varepsilon \in (0,1] & \end{array} \right\}$$

$|\ |_{triv} = |\ |_p^0 = |\ |_\infty^0$

Picture:

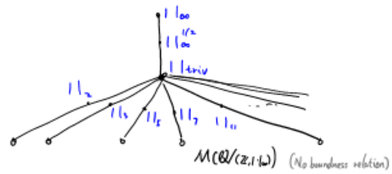




value of 2.

From this picture, we want to get:
Bound relations among seminorms
Topology properties: Hausdorff? compact?
Residue field, injection and contraction
... See next page

Rmk. When we do not identify the norm, we mean $\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)$.



E.g. $\mathcal{A} = (\mathbb{Q}, \|\cdot\|_{any})$, $\mathcal{M}(\mathcal{A}) = \{*\}$

E.g. $\mathcal{A} = (\mathbb{F}_q, \|\cdot\|_{triv})$ $\mathcal{M}(\mathbb{F}_q) = \{*\}$

E.g. $\mathcal{A} = \mathbb{R}/\mathbb{C}$ continuous seminorms are $\|\cdot\|_\infty^\varepsilon$, $\varepsilon \in [0,1]$.

Do we have any other cont seminorms? No.

E.g. $\mathcal{A} = \mathbb{Z}_p$ continuous seminorms are $\|\cdot\|_p^t$, $t \in [0, +\infty]$. ($\mathcal{A} = \mathbb{Q}_p$ is also interesting)

Do we have any other cont seminorms?

E.g. $\mathcal{A} = \mathbb{C}_p$

E.g. $\mathcal{A} = \mathbb{C}[x]$ If we only consider the norm which restricted to $\mathbb{C}$ is $|\ |_\infty$, we would get $\mathbb{C}$. Need to verify...

What would happen in the other cases?

If we only consider the norm which restricted to $\mathbb{C}$ is $|\ |_{triv}$, we would get $\mathbb{C}\mathbb{P}^1$.

E.g. $\mathcal{A} = \mathbb{C}_p\langle r^{-1}T\rangle$ or $\mathbb{P}^1_{\mathbb{C}_p}$

E.g. $\mathcal{A} = (\mathbb{Z}[i], \|\ \|_\infty)$

I'm very happy to do the homework one years ago.

E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

Try to answer the following questions:

- Set
  - $\mathcal{M}(\mathbb{Z})$ = ✓
  - partial order ⇝ bound order
  - Picture ✓
  - maximal / minimal seminorm — max: $\|\cdot\|_{\mathbb{F}_p}$, min $\|\cdot\|_\infty$
  - Berkovich structure of $\|\cdot\| \in \mathcal{M}(\mathbb{Z})$?
  - Archi or non Archi?

- Topo
  - Close set — not contain $|\ |_{triv}$: normal way + contain only finite $\|\cdot\|_p^t$; contain $|\ |_{triv}$: normal way
  - Open set — not contain $|\ |_{triv}$: normal way; contain $|\ |_{triv}$: normal way + contain all $\|\cdot\|_p^t$ except finite p
  - Topo properties: connected? ✓ ($X = Y \sqcup Z$) Hausdorff? ✓ (quasi)compact? ✓
  - irreducible? ✗ ($X = Y \cup Z$)

finite

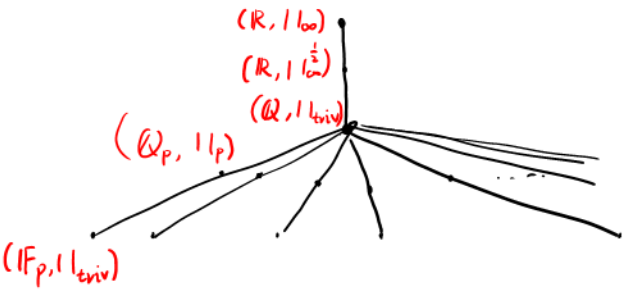
$(\mathcal{M}(\mathbb{Z}), \text{graph})$ ↓ $(\mathcal{M}(\mathbb{Z}), \text{weak})$ is continuous

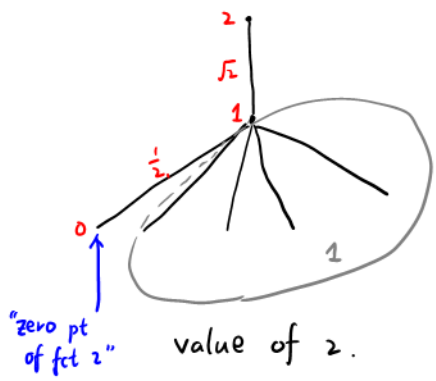
weak top is a little weaker then graph top

Def. $p \in X$ is a closed pt iff $\{p\}$ is closed

Then every pt is closed pt

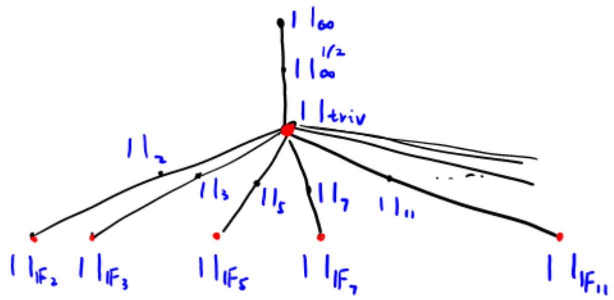
The definitions of Residue field, injection and contraction follows from [3.1.1, https://arxiv.org/abs/2105.13587v3]

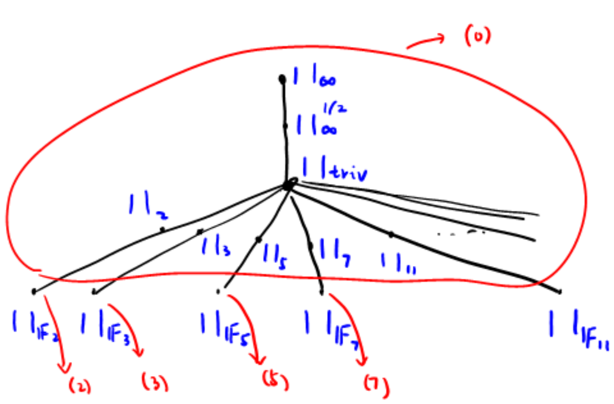




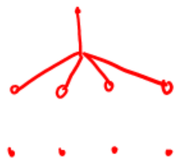
value of 2.

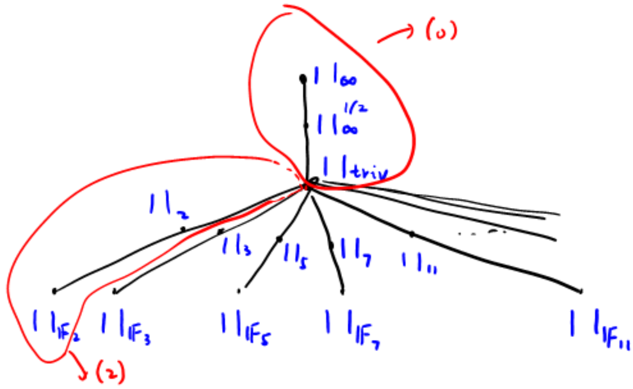
Residue field





k: kernel map (contraction)





r: reduction map (specialization)