

Eine Woche, ein Beispiel

5.4. line bundles on abelian varieties

Ref: follows [2025.04.13].

Most contents in this document can be found in [BL04, Chap 2 and Appendix B].

Goal: For $A = V/\Lambda$, identify

$$\text{Pic}(A) \xrightarrow{\sim} H^1(\Lambda, H^0(\mathcal{O}_V^*)) \xrightarrow{\text{def}} \mathcal{P}(\Lambda)$$

algebraic info gp cohom info analytic info

\mathbb{Z} $a_\lambda: \Lambda \times V \rightarrow \mathbb{C}$ (H, χ)

$a_\lambda(\lambda, v) = \chi(\lambda) \exp(\pi H(\lambda, v) + \frac{\pi}{2} H(\lambda, \lambda))$

hidden sheaf argument?

<https://mathoverflow.net/questions/30611/relation-between-sheaf-and-group-cohomology>

This explains the iso between $\text{Pic}(A)$ with gp cohom by using the spectral sequences on two functors:

$$(-)^{\pi_1(A)} \circ \pi_{A*} \pi_* \pi^* = \pi_{A*}$$

I wonder if it is possible to express invariant functor as one in the six functor (like the pushforward from classifying space to one point), and argue this identity in abstract nonsense.

$$\begin{array}{c} V \\ \downarrow \pi \\ A \\ \downarrow \pi_A \\ \{*\} \end{array}$$

Thm (Appell-Humbert) [BL04, p32]

$$\begin{array}{ccccccc} \{ \chi \} & & \{ (H, \chi) \} & & \{ H \} & \text{where a polarization} \\ \parallel & & \parallel & & \parallel & \swarrow \text{lives} \\ 0 \longrightarrow \text{Hom}(\Lambda, S') \longrightarrow \mathcal{P}(\Lambda) \longrightarrow \text{NS}(A) \longrightarrow 0 \\ \cong \downarrow & & \cong \downarrow & & \parallel & \\ 0 \longrightarrow \text{Pic}^\circ(A) \longrightarrow \text{Pic}(A) \longrightarrow \text{NS}(A) \longrightarrow 0 \\ \parallel \text{def} & & & & & \\ \hat{A} & & & & & \end{array}$$

where

$$\text{NS}(A) = \left\{ H: V \times V \rightarrow \mathbb{C} \mid \begin{array}{l} H \text{ Hermitian} \\ \text{Im } H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$\mathcal{P}(\Lambda) = \left\{ (H, \chi) \mid \begin{array}{l} H \in \text{NS}(A) \\ \chi: \Lambda \rightarrow S' \text{ semicharacter w.r.t. } H, \text{ i.e.,} \\ \chi(\lambda + \mu) = \chi(\lambda) \chi(\mu) \exp(\pi i \text{Im } H(\lambda, \mu)) \\ \forall \lambda, \mu \in \Lambda \end{array} \right\}$$

1. Cohomology of abelian varieties (Betti & Hodge)

Thm. [BL04, Thm 1.4.1 b)]

We have

$$\begin{aligned} \Omega &:= \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \stackrel{V^*}{=} H^{1,0}(A) \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) & f dz \\ \bar{\Omega} &:= \text{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) = H^{0,1}(A) \cong \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) & \bar{f} d\bar{z} \\ \Omega \oplus \bar{\Omega} &= H^1(A; \mathbb{C}) = \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \end{aligned}$$

Proof.

$$\begin{aligned} \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) &\longleftrightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ \downarrow &\longmapsto \text{Re } \downarrow \\ k(-) - ik(i-) &\longleftrightarrow k \end{aligned} \quad \begin{aligned} dz &= dx + idy \mapsto dx \\ idz &= -dy + idx \mapsto -dy \end{aligned}$$

$$\begin{aligned} \text{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) &\longleftrightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ \downarrow &\longmapsto i \text{Im } \downarrow \\ -k(i-) + ik(-) &\longleftrightarrow ik \end{aligned} \quad \begin{aligned} d\bar{z} &= dx - idy \mapsto -idy \\ id\bar{z} &= dy + idx \mapsto idx \end{aligned}$$

$$\begin{aligned} \Omega \oplus \bar{\Omega} &\longleftrightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{C}) \\ (f dz, \bar{g} d\bar{z}) &\longmapsto ? \end{aligned}$$

$$\begin{aligned} (dz, -id\bar{z}) &\longleftrightarrow dz = dx + idy \\ (idz, id\bar{z}) &\longleftrightarrow idz = idx - dy \\ (dz, d\bar{z}) &\longleftrightarrow d\bar{z} = dx - idy \\ (idz, -id\bar{z}) &\longleftrightarrow id\bar{z} = idx + dy \end{aligned}$$

$$\begin{aligned} ((f_1 + if_2) dz, 0) &\mapsto f_1 dx - f_2 dy \\ (0, (g_1 - ig_2) d\bar{z}) &\mapsto -i(g_1 dy + g_2 dx) \end{aligned}$$

$$f_1, f_2, g_1, g_2 \in C^\infty(A; \mathbb{R})$$

Cor. $H^q(A; \Omega_A^p) \cong \Lambda^p \Omega \otimes \Lambda^q \bar{\Omega}$

$\Omega_A^p = \text{Alt}^p \Omega_A$

Proof Sketch

$$\begin{aligned} H^q(A; \Omega_A^p) &\cong H_{\bar{\partial}}^{p,q}(A) \\ &= \{ \bar{\partial}\text{-closed } (p,q)\text{-forms on } V/\Lambda \} / \sim \\ &= \{ \bar{\partial}\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } \Lambda \} / \sim \\ &= \{ \bar{\partial}\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } V \} \\ &= \Lambda^p \Omega \otimes \Lambda^q \bar{\Omega} \end{aligned}$$

Another proof, though essentially the same:

Step 1 Ω_A is a free \mathcal{O}_A -module with rank n , so

$$\begin{aligned} \Omega_A &\cong \mathcal{O}_A \otimes_{\mathbb{C}} V^* \\ \Rightarrow \Omega_A^p &= \Lambda^p \Omega_A = \mathcal{O}_A \otimes_{\mathbb{C}} \Lambda^p \Omega \end{aligned}$$

Step 2 By Dolbeault resolution,

$$H^q(A; \mathcal{O}_A) \cong H^q(A_{A \times \mathbb{C}}^{\circ, q}) \cong H_{\bar{\partial}}^{\circ, q}(A) \cong \Lambda^q \bar{\Omega}$$

\uparrow
trivial l.b. over A

Dual abelian variety

Rmk. We have canonical bilinear pairing

$$\langle -, - \rangle: \bar{\Omega} \times \Omega^* \xrightarrow{\vee} \mathbb{R} \quad \langle l, v \rangle = \text{Im}(lv)$$

which restricts to

$$\langle -, - \rangle: \hat{\Lambda} \times \Lambda \longrightarrow \mathbb{Z}$$

where

$$\begin{aligned} \hat{\Lambda} &:= \{ l \in \bar{\Omega} \mid \langle l, \Lambda \rangle \subseteq \mathbb{Z} \} \\ &\cong \text{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \\ &\cong H^1(A; \mathbb{Z}) \end{aligned}$$

Prop. The abelian variety A and its dual \hat{A} have following expressions:

$$\begin{aligned} A &= \Omega^* / \Lambda = H^0(A; \Omega_A)^* / H_1(A; \mathbb{Z}) \\ \hat{A} &= \bar{\Omega} / \hat{\Lambda} = H^1(A; \mathcal{O}_A) / H^1(A; \mathbb{Z}) \\ &= \text{Pic}^0(A) \cong \text{Hom}(\Lambda, S') = \{x\} \end{aligned}$$

Proof. We only need to show that $\overline{\Omega}/\hat{\Delta} \cong \text{Hom}_{\mathbb{Z}}(\Delta, S')$,
 which follows from applying $\text{Hom}_{\mathbb{Z}}(\Delta, -)$ to $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{R} \rightarrow S' \rightarrow 0$.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\Delta, \mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}}(\Delta, \mathbb{R}) & \longrightarrow & \text{Hom}(\Delta, S') \longrightarrow 0 \\
 & & \parallel & & \begin{array}{c} \text{||S} \\ \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) \\ \text{||S} \\ \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \end{array} & & \\
 & & & & \parallel & & \\
 0 & \longrightarrow & \hat{\Delta} & \longrightarrow & \hat{\overline{\Omega}} & \longrightarrow & \overline{\Omega}/\hat{\Delta} \longrightarrow 0
 \end{array}$$

$NS(A)$: more descriptions

Lemma [BL04, Prop 2.1.6]

Let

$$NS(A) := \text{Pic}(A) / \text{Pic}^{\text{red}}(A) \cong H^2(A; \mathbb{Z}) \cap H^{1,1}(A)$$

$$NS'(A) := \left\{ \omega: V \times V \rightarrow \mathbb{R} \left| \begin{array}{l} \omega \text{ } \mathbb{R}\text{-bilinear alternating form} \\ \omega(ix, iy) = \omega(x, y) \\ \omega(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right. \right\}$$

$$NS''(A) = \left\{ H: V \times V \rightarrow \mathbb{R} \left| \begin{array}{l} H \text{ Hermitian} \\ \text{Im } H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right. \right\}$$

↑ imaginary part

Then

$$NS(A) \cong NS'(A) \cong NS''(A).$$

As a reminder, H Hermitian:

$$H(av, bv) = \bar{a}b H(u, v) \quad + \text{ } \mathbb{R}\text{-linear}$$

$$H(u, v) = \overline{H(v, u)}$$

corresponds to the matrix M s.t. $M^H = M$

$$H(u, v) = \omega(u, v) + i\omega(u, v)$$

$$\omega(u, v) = \text{Im } H(u, v)$$

Rmk. A Hermitian form is equiv to a \mathbb{C} -linear map

$$\begin{aligned} \phi: \Omega^* = V &\longrightarrow \text{Hom}_{\mathbb{R}}(V, \mathbb{R}) \cong \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) = \bar{\Omega} \\ v_2 &\longmapsto \omega(-, v_2) \longmapsto H(-, v_2) \end{aligned}$$

$$\phi(iv) = H(-, iv) = iH(-, v) = i\phi(v)$$

or a cplx bilinear map $\bar{V} \times V \rightarrow \mathbb{C}$

or an element in $\Omega \otimes_{\mathbb{C}} \bar{\Omega} \cong H^{1,1}(A)$

Moreover,

$$\begin{aligned} H \text{ is non-deg} &\Leftrightarrow \phi \text{ is an iso} \\ \omega(\Lambda \times \Lambda) \subset \mathbb{Z} &\Leftrightarrow \langle \phi(\Lambda), \Lambda \rangle \subset \mathbb{Z} \Leftrightarrow \phi(\Lambda) \subseteq \hat{\Lambda} \end{aligned}$$

$$\begin{aligned} \text{Therefore, } NS'(A) &= \{ \phi \in \text{Hom}(\Omega^*, \bar{\Omega}) \mid \phi(\Delta) \subset \hat{\Delta} \} \\ &= \text{Hom}(\Omega^*/\Delta, \bar{\Omega}/\hat{\Delta}) \\ &= \text{Hom}(A, \hat{A}). \end{aligned}$$

This explains why some references would like to call a morphism

$$\phi: A \longrightarrow \hat{A}$$

(induced by an ample l.b.) as the polarization of A .

From $\phi: \Omega^* \rightarrow \bar{\Omega}$ to $H(-, -)$:

In fact, once we fixed the \mathbb{C} -linear map $\phi: \Omega^* \rightarrow \bar{\Omega}$, the Hermitian form comes from the canonical bilinear pairing:

$$\begin{array}{ccc} \langle -, - \rangle: \bar{\Omega} \times \Omega^* & \longrightarrow & \mathbb{R} & \langle l, v \rangle = \text{Im}(l(v)) \\ & \uparrow (\phi, \text{Id}) & & \end{array}$$

$$\omega(-, -): \Omega^* \times \Omega^* \longrightarrow \mathbb{R} \quad \omega(v_1, v_2) = \text{Im } H(v_1, v_2)$$

Hint for the main lemma. Consider the ambient spaces.

$$\begin{array}{ccccccc} & & NS(A) & & NS'(A) & & NS''(A) \\ & & \cap & & \cap & & \cap \\ H^1(\mathcal{O}_A^*) & \xrightarrow{c_1} & H^2(A; \mathbb{Z}) & \hookrightarrow & H^2(A; \mathbb{R}) & \hookrightarrow & H^2(A; \mathbb{C}) \end{array}$$

2. Miscellaneous

I will collect some basic results as well as its proofs here for my personal reference. They show the strategy to work with these line bundles, dual varieties stuff, but not the most important thing to put in the main goal. Maybe someday I will move them to better places.

Prop. (dual variety with SES) [BL04, Prop 2.4.2]

If

$$0 \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow 0$$

is a SES of cplx tori, then

$$0 \leftarrow \hat{X}_1 \leftarrow \hat{X}_2 \leftarrow \hat{X}_3 \leftarrow 0$$

is exact.

Proof. Let $X_i = V_i / \Delta_i$, then

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Delta_1 & \rightarrow & \Delta_2 & \rightarrow & \Delta_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & V_1 & \rightarrow & V_2 & \rightarrow & V_3 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & X_1 & \rightarrow & X_2 & \rightarrow & X_3 \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

exact

Taking $\text{Hom}_{\mathbb{Z}}(-, S')$,

$$0 \leftarrow \text{Hom}_{\mathbb{Z}}(\Delta_1, S') \leftarrow \text{Hom}_{\mathbb{Z}}(\Delta_2, S') \leftarrow \text{Hom}_{\mathbb{Z}}(\Delta_3, S') \leftarrow 0$$

is exact, i.e.,

$$0 \leftarrow \hat{X}_1 \leftarrow \hat{X}_2 \leftarrow \hat{X}_3 \leftarrow 0$$

is exact.

Prop. (dual variety with isogeny) [BL04, Prop 2.4.3]

If $f: Y \rightarrow X$ is an isogeny of cplx tori, then
 $\hat{f}: \hat{X} \rightarrow \hat{Y}$ is an isogeny of cplx tori, with

$$\text{Ker } \hat{f} = \text{Hom}(\text{Ker } f, S').$$

Proof. Let $Y = V/\Lambda_2$, $X = V/\Lambda_3$, then

$$\begin{array}{ccccccc}
 0 & \rightarrow & 0 & \rightarrow & \Lambda_2 & \rightarrow & \Lambda_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & 0 & \rightarrow & V & \rightarrow & V \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & \text{Ker } f & \rightarrow & Y & \xrightarrow{f} & X \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{Ker } f & \rightarrow & 0 & \rightarrow & 0
 \end{array}$$

exact

Taking $\text{Hom}(-, S')$,

$$\begin{array}{ccccccc}
 0 & \leftarrow & \text{Hom}(\Lambda_2, S') & \xleftarrow{\hat{f}} & \text{Hom}(\Lambda_3, S') & \leftarrow & \text{Hom}(\text{Ker } f, S') \leftarrow 0 \\
 & & \parallel & & \parallel & & \\
 & & \hat{Y} & & \hat{X} & &
 \end{array}$$

so

$$\text{Ker } \hat{f} = \text{Hom}(\text{Ker } f, S').$$

Prop (Image of f^* via isogeny) [BL04, Cor 2.4.4]

Let $f: X_1 \rightarrow X_2$ be an isogeny of cplx tori, $X_i = V/\Delta_i$.
For $\mathcal{L} = \mathcal{L}(H, x) \in \text{Pic}(X_1)$,

$$\mathcal{L} = f^* \mathcal{M} \quad \Leftrightarrow \quad \overset{\substack{\text{imaginary part} \\ \downarrow}}{\text{Im } H(\Delta_2 \times \Delta_2)} \subseteq \mathbb{Z}$$

for some $\mathcal{M} \in \text{Pic}(X_2)$

Proof. Diagram chasing. Find H' and x' .

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Pic}^\circ(X_1) & \longrightarrow & \text{Pic}(X_1) & \longrightarrow & \text{NS}(X_1) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \text{Pic}^\circ(X_2) & \longrightarrow & \text{Pic}(X_2) & \longrightarrow & \text{NS}(X_2) \longrightarrow 0 \end{array}$$

$$\begin{array}{ccccc} x_1 & \xrightarrow{\quad} & \left\{ \begin{array}{l} \mathcal{L} \\ f^* \mathcal{L}_2 \end{array} \right. & \xrightarrow{\quad} & H \\ \uparrow & & \uparrow & & \uparrow \\ x' & \xrightarrow{\quad} & \mathcal{L}_2 & \xrightarrow{\quad} & H \end{array}$$

(An orange arrow curves from x_1 to x' and another from \mathcal{L} to \mathcal{L}_2)

Prop. (line bundles under pullbacks) [BL04, Lemma 2.3.2 & Lemma 2.3.4]

For $\mathcal{L} = \mathcal{L}(H, x) \in \text{Pic}(A)$ and $[v] \in A$
 $f: \tilde{A} \rightarrow A$ homo of cplx tori,

① $t_{[v]}^* \mathcal{L}(H, x) = \mathcal{L}(H, x \exp(2\pi i \text{Im } H(-, v)))$

② $f^* \mathcal{L}(H, x) = \mathcal{L}(f_a^* H, f_r^* x)$

Cor. (equiv def of polarization)
the diagram commutes:

$$\begin{array}{ccc}
 V \xrightarrow{\phi_H} \overline{\Omega} & v \mapsto H(-, v) & \\
 \downarrow & \downarrow \pi_{\overline{\Omega}} & \\
 V/\Lambda \xrightarrow{\phi_H} \overline{\Omega}/\hat{\Lambda} & [v] \mapsto H(-, v) & \\
 \parallel & \parallel & \downarrow \mathcal{L} \\
 A \xrightarrow{\phi_{\mathcal{L}}} \hat{A} = \text{Pic}^0(A) & [v] \mapsto t_{[v]}^* \mathcal{L} \otimes \mathcal{L}^{-1} & \exp(2\pi i \text{Im}(\mathcal{L}(-)))
 \end{array}$$

Proof. Write $\mathcal{L} = \mathcal{L}(H, \chi)$, then

$$\begin{aligned}
 t_{[v]}^* \mathcal{L} \otimes \mathcal{L}^{-1} &= \mathcal{L}(H, \chi \exp(2\pi i \text{Im} H(-, v))) \otimes \mathcal{L}(H, \chi)^{-1} \\
 &= \mathcal{L}(0, \exp(2\pi i \text{Im} H(-, v))) \\
 &= \pi_{\overline{\Omega}} \circ \phi_H(v).
 \end{aligned}$$