Eine Woche, ein Beispiel

5.11 genus of generalized Fermat curve

Goal. 1. Find a basis of $H^{p,q}(X)$ by harmonic forms.
2. Compute the geometric genus of curves

$$C := \{y^n = x^m - 1\} \subseteq \mathbb{P}^2$$

Rmk: [2024.11.03] try to compute a special case in detail. In this document, more advanced methods are applied, so we don't need to blow up explicitly.
The reference also follows [2024.11.03].

Extra Ref:

Generalised Fermat equation: a survey of solved cases
https://arxiv.org/abs/2412.11933

Connection between Fermat curve and hyperelliptic curve:
https://math.stackexchange.com/questions/3493593/transformation-which-takes-fermat-curve-xnyn-1-to-a-hyperelliptic-curve

1. Harmonic forms

- Affine plane curve
- Plane curve
  Fermat curve
- Hyperelliptic curve
- generalized Fermat curve
- $\mathbb{P}^n$
- Hypersurface

2. Riemann-Hurwitz
3. Milnor formula

1. Harmonic forms

Almost all the results in this section come from the answer here:
https://mathoverflow.net/questions/324812/the-construction-of-a-basis-of-holomorphic-differential-1-forms-for-a-given-plan

Affine plane curve

Prop. Suppose $C = \{f(x,y) = 0\} \subseteq \mathbb{A}^2$ is a sm curve, then

$$\omega \triangleq \frac{dx}{f_2(x,y)} = -\frac{dy}{f_1(x,y)}$$

is a global generator of $H^0(C, \Omega^1)$.

i.e., $\forall\, \omega' \in H^0(C,\Omega^1)$, $\omega' = f\omega$ for some $f \in \mathcal{O}_{hol}(C)$.

Proof. Notice that

$$f_1(x,y)\,dx + f_2(x,y)\,dy = 0.$$

When $f_1(x_0,y_0) \neq 0$,

$$\begin{aligned} y: C &\longrightarrow \mathbb{A}^1 \quad \text{is a local chart,} \\ (x,y) &\longmapsto y \end{aligned}$$

$\Rightarrow$ dy is a global generator near (x_0,y_0).

$\Rightarrow$ $\dfrac{dy}{f_1(x,y)}$ is a global generator near (x_0,y_0).

When $f_2(x_0,y_0) \neq 0$,

$$\begin{aligned} x: C &\longrightarrow \mathbb{A}^1 \quad \text{is a local chart,} \\ (x,y) &\longmapsto x \end{aligned}$$

$\Rightarrow$ dx is a global generator near (x_0,y_0).

$\Rightarrow$ $-\dfrac{dx}{f_2(x,y)}$ is a global generator near (x_0,y_0).