

# Eine Woche, ein Beispiel

## 10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:  
[http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis\\_Tomasz%20Przezdziecki.pdf](http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf)

We changed some notation for the convenience of writing.

Task:

1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. change of basis
7. Convolution product

We may use two examples for the convenience of presentation.

Readers can easily distinguish them by the dim vectors.

## 1. dimension vector

$$|\underline{d}| = 5$$

$$\underline{d} = (3, 2)$$

$$\underline{d} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 0 \\ 0, 0 \end{pmatrix} = \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \end{array} = \begin{array}{c} \downarrow \times \times \times \\ \downarrow \times \end{array} \in W_d \backslash W_{\text{id}} \text{ or } \text{Min}(W_{\text{id}}, W_d)$$

$v_{\text{id}} = \pi_{\underline{d}}^{-1}(F_{\text{id}})$

## 2. Weyl group

Set	element	special element	others
$W_{\text{id}} = S_5$	$w, x$	$w_{\text{max}} = \begin{array}{ c c c c c } \hline & * & * & * & * \\ \hline * & & & & \\ \hline \end{array}$	$T = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	$w$	$w_{\text{max}} = \begin{array}{ c c c } \hline * & * & * \\ \hline * & & \\ \hline * & & \\ \hline \end{array}$	$T_d = \{s_1, s_2, s_4\}$
$W_d \backslash W_{\text{id}} = S_3 \times S_2 \backslash S_5$	$w, \underline{d}$	$\begin{array}{ c c c c c } \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$	(Comp <sub>d</sub> )

$$\text{Min}(W_{\text{id}}, W_d) = \left\{ \begin{array}{|c|c|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}, \dots \right\} u \quad \begin{array}{|c|c|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array} \quad (\text{Shuffled})$$

$$0 \longrightarrow W_d \longrightarrow W_{\text{id}} \longrightarrow W_{\text{id}} \backslash W_d \longrightarrow 0 \quad w = wu \mapsto \underline{d}$$

$\xrightarrow{\cong} \text{Min}(W_{\text{id}}, W_d)$

$w = \begin{array}{|c|c|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}$   
 $u = \begin{array}{|c|c|c|c|c|} \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \hline \end{array}$   
 $w = \begin{array}{|c|c|c|c|c|} \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$

Another example:  $\underline{d} = (1, 2)$        $a \xrightarrow{\alpha} b$        $\langle v_1 \rangle \rightarrow \langle v_2, v_3 \rangle$

	$w = wu$	$w$	$\underline{d}, u$	order of basis	$(w)$	$l(w)$	$B_w$	$B_{wu}$	$wB_{wu}^{-1}$
Id	$(1^2 3)$	$  \sqcup \sqcup$	$[1, 1]$	$  \sqcup \sqcup$	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
t	$(23)$	$  \times \sqcup$	$[1, 1]$	$  \times \sqcup$	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
s	$(12)$	$X   \sqcup$	$[1, 1]$	$  \sqcup$	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
ts	$(132)$	$X   \times$	$[1, 1]$	$  \times$	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
st	$(123)$	$X   \times$	$[1, 1]$	$  \sqcup$	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
sts	$(13)$	$*   \times$	$[1, 1]$	$  \times$	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

### 3. alg group & Lie algebra

$$G_{\text{Idl}}, B_{\text{Idl}}, T_{\text{Idl}}, N_{\text{Idl}} \quad W_{\text{Idl}} = N_{G_{\text{Idl}}}(\Pi_{\text{Idl}}) / \Pi_{\text{Idl}} \quad GL_5(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$G_d, B_d, T_d, N_d \quad W_d = N_{G_d}(T_d) / T_d \quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$B_\infty = B_{\text{Idl}} \omega^{-1} = \text{Stab}_{G_{\text{Idl}}}(F_\infty)$$

$$B_\infty = \omega B_d \omega^{-1} = \text{Stab}_{G_d}(F_\infty) \quad N_\infty = R_u(B_\infty)$$

For  $s \in \Pi$  s.t.  $\omega s \omega^{-1} \in W_d$  (i.e.  $W_d \omega = W_d \omega s$ ), define

$$P_{\infty, \omega s} = \omega (B_d s s^{-1} B_d \cup B_d) \omega^{-1} \quad N_{\infty, \omega s} = R_u(B_{\infty, \omega s})$$

$$= B_\infty \omega s \omega^{-1} B_\infty \cup B_\infty \quad = N_\infty \cap N_{\infty, \omega s}$$

$$M_{\infty, \omega s} = N_\infty / N_{\infty, \omega s}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{\sigma(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\infty, \omega''} := N_\infty \cap N_\infty$$

$$M_{\infty, \omega''} := N_\infty / N_{\infty, \omega''}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega''}$$

Their Lie algebras are collected here.

$$\mathfrak{g}_{\text{Idl}}, \mathfrak{b}_{\text{Idl}}, \mathfrak{t}_{\text{Idl}}, \mathfrak{n}_{\text{Idl}}$$

$$g_d \quad b_d \quad t_d \quad n_d$$

$$\mathfrak{b}_\infty \quad \bar{\mathfrak{b}}_\infty$$

$$b_\infty \quad n_\infty$$

$$P_{\infty, \omega s} \quad N_{\infty, \omega''}$$

$$m_{\infty, \omega''}$$

$$\bar{b}_\infty = b_{\omega \max \infty}$$

$$\bar{b}_\infty = b_{w \max \infty}$$

$$\bar{P}_{\infty, \omega s} = P_{w \max \infty, w \max \infty}$$

$$m_{\infty, \omega''}$$

$$\text{Rep}_d(Q) := \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{\text{Idl}}^{\oplus k}$$

$$V_\infty = \{f \in \text{Rep}_d(Q) \mid f: F_{\infty, i} \subseteq F_{\infty, i}\} = \mu_d \pi_d^{-1}(F_\infty)$$

$$= \begin{pmatrix} v_3 & v_1 & v_2 \\ v_4 & * & * \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & * & * \end{pmatrix}$$

$$V_{\omega(i)}$$

$$V_{\infty, \omega''} = V_\infty \cap V_{\infty''}$$

$$J_{\infty, \omega''} = V_\infty / V_{\infty, \omega''}$$

Later we may twist the group actions.

$$\text{E.g. } Y_{\infty, \omega'} = V_{\infty, \omega \omega'} \quad Y_{\infty, \omega''} = V_{\infty, \omega^{-1} \omega''}$$

#### 4. typical variety

Id corres to

$$\begin{aligned}
 F_{\text{Id}\text{Id}} &\cong G_{\text{Id}\text{Id}} / B_{\text{Id}\text{Id}} & F_{\text{Id}} \\
 F_{\underline{d}} &\cong G_{\underline{d}} / B_{\underline{d}} & F_u \\
 F_{\infty} &\cong G_{\underline{d}} / B_{\infty} & F_{\omega} \\
 F_{\underline{d}} &= \coprod_{\underline{d}'} F_{\underline{d}'} & - \\
 F_{g\infty} &\cong G_{\underline{d}} / gB_{\infty}g^{-1} & F_{g\infty} \\
 F_{\infty, \cdot} &= \text{Flag}_{\underline{d}}(F_{\text{Id}}) = F_{\{v_{\infty(1)}, v_{\infty(2)}, \dots, v_{\infty(\text{Id})}\}} & \\
 &= F_{\{u_5, u_3, u_1, u_6, u_2\}}
 \end{aligned}$$

✓ The action on Flag is not the same as in

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$$F_{\text{Id}\text{Id}} \neq \coprod_{\underline{d}} F_{\underline{d}}$$

$F_{\infty} \cong F_{\underline{d}}$  with different base pt. Base pt makes difference!

$$\begin{aligned}
 F_{\text{Id}\text{Id}} \times F_{\text{Id}\text{Id}} && F_{\text{Id}, \text{Id}} \\
 F_{\underline{d}} \times F_{\underline{d}'} && F_{u, u'} \\
 F_{\infty} \times F_{\infty'} && F_{\infty, \infty'} \\
 F_{\underline{d}} \times F_{\underline{d}'} &= \coprod_{\underline{d}, \underline{d}'} (F_{\underline{d}} \times F_{\underline{d}'}) & -
 \end{aligned}$$

$$F_{\infty, \infty'} := (F_{\infty}, F_{\infty'})$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$\mu_{\underline{d}}^{-1}(M) \cong \text{Flag}_{\underline{d}}(M) \subseteq F_{\underline{d}}$  is the Springer fiber.

$$\begin{array}{ccc}
 Z_{\underline{d}, \underline{d}'} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}'} \\
 \downarrow M_{\underline{d}, \underline{d}'} & & \downarrow \pi_{\underline{d}, \underline{d}'} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}'}
 \end{array}$$

$$\begin{array}{ccc}
 Z_{\underline{d}} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}, \underline{d}} & & \downarrow \pi_{\underline{d}, \underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}}
 \end{array}$$

$$\begin{array}{c}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \subseteq \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) := \bigsqcup_{\underline{d}} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{array}$$

$$\widetilde{\text{Rep}}_{\infty}(\mathbb{Q}) \cong G_{\underline{d}} \times^{B_{\infty}} r_{\infty}$$

$$\begin{aligned}
 Z_{\underline{d}, \underline{d}'} &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}'}(\mathbb{Q}) \\
 Z_{\underline{d}} &= \bigsqcup_{\underline{d}', \underline{d}''} Z_{\underline{d}, \underline{d}'} \\
 &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{aligned}$$

$$Z_{\infty, \infty'} = Z_{u, u'}$$

## 5. (equivariant) stratifications.

In the following tables,  $uw' = \tilde{w}'\tilde{u}$ .

$F_\infty \in \widetilde{\text{Rep}}_d(Q)$  means  $(p_0, F_\infty); (F_\infty, F_{\infty'}) \in Z_d$  means  $(p_0, F_\infty, F_{\infty'})$ .

▽  $G \times G$  acts on  $\mathcal{F} \times \mathcal{F}$  in a twisted way

$$\text{e.g. } (g_1, g_2) F_{\infty, \infty'} = F_{g_1 \infty, g_1 \tilde{w} g_2 \infty'^{-1}}$$

$$(g_1, g_2) E_{\infty, \infty'} = E_{g_1 \infty, g_2 \infty'}$$

variety base point	stratification stabilizer	type	B-orbit	$B \times B$ -orbit stabilizer are twisted	$B \times G$ -orbit	$G \times B$ -orbit	Remark $G \times \{*\}$ -orbit
$\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$		$\Omega_g$	$\Omega_{g, g'}$	$\text{pr}_i^{-1}(\Omega_g)$	$\Omega_{g'}$	
$F_g$ ( $F_g, F_{gg'}$ )	$B \cap gBg^{-1}$		$B \cap gBg^{-1} \times B \cap (gg')B(gg')^{-1}$				$gBg^{-1} \cap gg'B(gg')^{-1}$
$\mathcal{F}_{\text{id}}$	$\mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$		$\mathcal{V}_\infty$	$\mathcal{V}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{V}_\infty)$	$\mathcal{V}_\infty$	
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )	$B_{\text{id}} \cap B_\infty$		$B_{\text{id}} \cap B_\infty \times B_{\text{id}} \cap B_{\infty'}$				$B_\infty \cap B_{\infty\infty'}$
$\mathcal{F}_u$	$\mathcal{F}_u \times \mathcal{F}_u$		$\Omega_w^u$	$\Omega_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{w'}^{u, u'}$	
$F_{wu}$ ( $F_{wu}, F_{wuwu'}$ )	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{ww'}$				$B_w \cap B_{ww'}$
$\mathcal{F}_d$	$\mathcal{F}_d \times \mathcal{F}_d$		$\Omega_w^u$	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{\infty'}^u = \Omega_{\tilde{w}}^{u, \tilde{u} u'}$	compatibility
$F_\infty$ "	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{w\tilde{w}}$				$B_w \cap B_{w\tilde{w}}$
$F_{wu}$ ( $F_{wu}, F_{wuwu'}$ )							

The following may not be single orbit, but derived from the above definition.

$\mathcal{F}_d$	$\mathcal{F}_d \times \mathcal{F}_d$	$\mathcal{O}_\infty$	$\mathcal{O}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{O}_\infty)$	$\mathcal{O}_\infty$		preimage of $\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )		$\Omega_w^u$	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\bigsqcup_u \Omega_{\tilde{w}}^{u, \tilde{u} u'}$		preimage of $\mathcal{Z}_{d, d'} \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{\text{Rep}}_d(Q)$	$Z_d$	$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\widetilde{\Omega}_w^{u, u'}$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )							
$\widetilde{\text{Rep}}_d(Q)$	$Z_d$	$\widetilde{\mathcal{O}}_\infty$	$\widetilde{\mathcal{O}}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\widetilde{\mathcal{O}}_\infty)$	$\widetilde{\mathcal{O}}_\infty$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )		$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\bigsqcup_u \widetilde{\Omega}_{\tilde{w}}^{u, \tilde{u} u'}$		

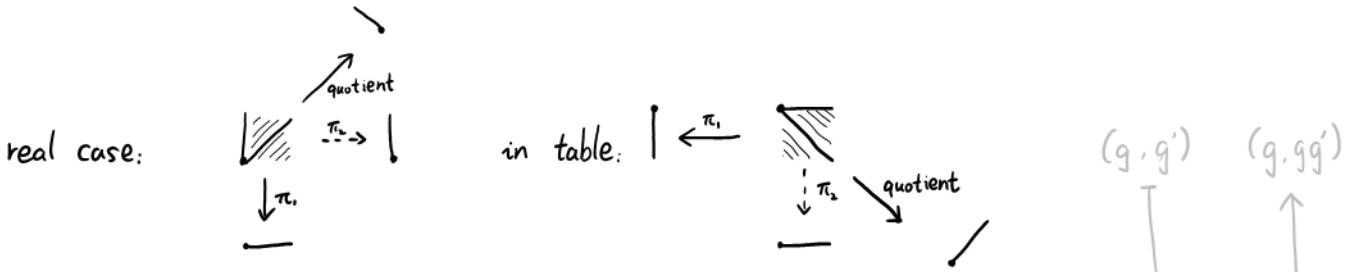
$$\mathcal{Z}_{\infty'} := \overline{\widetilde{\mathcal{O}}_\infty} \subseteq \overline{\widetilde{\mathcal{O}}_{\infty'}}$$

$$\mathcal{Z}_{w'}^{u, u'} := \overline{\widetilde{\Omega}_{w'}^{u, u'}} \subseteq \overline{\widetilde{\Omega}_w^{u, u'}}$$

Zar-loc sub v.b.?

$$\mathcal{Z}_d^\infty$$

$$\mathcal{Z}_{d, d'}^\infty = \mathcal{Z}_\infty \cap \mathcal{Z}_{d, d'}$$



We want gp action to be compatible with  $\pi_i$  and the quotient map.  
Therefore, we would do a twist.



Rmk. The stabilizer is not trivial to determine because of this twist!

$$\begin{aligned}
 E_{g_1, g_2} = E_{g_3, g_4} &\Leftrightarrow g_1 B = g_3 B, g_2 B = g_4 B \\
 &\Leftrightarrow g_1 B = g_3 B, g_1 g_2 B = g_3 g_4 B \\
 E_{g_1\omega, g_2\omega'} = E_{\omega, \omega'} &\Leftrightarrow g_1 \omega B = \omega B, g_1 \omega g_2 \omega' B = g_2 \omega' B \\
 &\Leftrightarrow g_1 = \omega b, \omega^{-1} \in B_\omega \quad b, g_2 \in B_{\omega'} \\
 &\Leftrightarrow g_1 \in B_\omega, \quad g_2 \in \omega^{-1} g_1 \omega \cdot B_{\omega'}
 \end{aligned}$$

The following tables may help you to understand the notations.

$\dim$	$B_{\text{Id}} \cdot F_{\text{var}}$	0	1	1	2	2	3
	$B_{\text{Id}} \times B_{\text{Id}} \cdot (F_{\text{var}}, F_{\text{var}})$	$\mathcal{V}_{\text{Id}}$	$\mathcal{V}_t$	$\mathcal{V}_s$	$\mathcal{V}_{ts}$	$\mathcal{V}_{st}$	$\mathcal{V}_{sts}$
	$B_{\text{Id}} \cdot F_w$	$\mathcal{V}_{\text{Id}}$	$\mathcal{V}_{\text{Id},\text{Id}}$	$\mathcal{V}_{\text{Id},t}$	$\mathcal{V}_{\text{Id},s}$	$\mathcal{V}_{\text{Id},ts}$	$\mathcal{V}_{\text{Id},st}$
0							
1							
2							
3							

$\dim$	$B_d \cdot F_{\text{var}}$	0	1	2	3	4	5
	$B_d \times B_d \cdot (F_{\text{var}}, F_{\text{var}})$	$\mathcal{O}_{\text{Id}}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
	$B_d \cdot F_w$	$\mathcal{O}_{\text{Id}}$	$\mathcal{O}_{\text{Id},\text{Id}}$	$\mathcal{O}_{\text{Id},t}$	$\mathcal{O}_{\text{Id},s}$	$\mathcal{O}_{\text{Id},ts}$	$\mathcal{O}_{\text{Id},st}$
0							
1							
2							
3							
4							
5							

The following tables may help you to understand the notations.

$$\omega = ts, \omega' = s$$

$\dim$	$B_{Id} \cdot F_{ts}$	0	1	1	2	2	3	$\text{pr}_i^{-1}(\mathcal{V}_{ts})$
	$B_{Id} \times B_{Id} \cdot (F_{ts}, F_{ts})$	$\mathcal{V}_{Id}$	$\mathcal{V}_t$	$\mathcal{V}_s$	$\mathcal{V}_{ts}$	$\mathcal{V}_{st}$	$\mathcal{V}_{sts}$	
	$B_{Id} \cdot F_{ts}$	$\mathcal{V}_{Id}$	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
0		$\mathcal{V}_{Id}$	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
1		$\mathcal{V}_t$	$\mathcal{V}_{t,t}$	$\mathcal{V}_{t,Id}$	$\mathcal{V}_{t,ts}$	$\mathcal{V}_{t,s}$	$\mathcal{V}_{t,sts}$	$\mathcal{V}_{t,st}$
1		$\mathcal{V}_s$	$\mathcal{V}_{s,s}$	$\mathcal{V}_{s,st}$	$\mathcal{V}_{s,Id}$	$\mathcal{V}_{s,sts}$	$\mathcal{V}_{s,t}$	$\mathcal{V}_{s,ts}$
2		$\mathcal{V}_{ts}$	$\mathcal{V}_{ts,st}$	$\mathcal{V}_{ts,s}$	$\mathcal{V}_{ts,sts}$	$\mathcal{V}_{ts,Id}$	$\mathcal{V}_{ts,ts}$	$\mathcal{V}_{ts,t}$
2		$\mathcal{V}_{st}$	$\mathcal{V}_{st,ts}$	$\mathcal{V}_{st,sts}$	$\mathcal{V}_{st,t}$	$\mathcal{V}_{st,st}$	$\mathcal{V}_{st,Id}$	$\mathcal{V}_{st,s}$
3		$\mathcal{V}_{sts}$	$\mathcal{V}_{sts,sts}$	$\mathcal{V}_{sts,ts}$	$\mathcal{V}_{sts,st}$	$\mathcal{V}_{sts,t}$	$\mathcal{V}_{sts,s}$	$\mathcal{V}_{sts,Id}$

$\text{shape}$	$B_d \cdot F_{ts}$	$\mathcal{F}_{Id}$	$\mathcal{F}_s$	$\mathcal{F}_{st}$	$\text{pr}_i^{-1}(\mathcal{O}_{ts})$	$\text{pr}_{i,Id}^{-1}(\Omega_t^s)$	$\Omega_{t,Id}^{s,Id} = \mathcal{O}_{ts,s}$
	$B_d \times B_d \cdot (F_{ts}, F_{ts})$	$\mathcal{O}_{Id}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
	$B_d \cdot F_{ts}$						
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega_{Id,Id}^{Id,Id}$	$\Omega_{Id,t}^{Id,Id}$	$\Omega_{Id,s}^{Id,s}$	$\Omega_{Id,t}^{Id,s}$	$\Omega_{Id,ts}^{Id,st}$	$\Omega_{Id,t}^{Id,st}$
	$\mathcal{O}_t$	$\Omega_{t,t}^{Id,Id}$	$\Omega_{t,Id}^{Id,Id}$	$\Omega_{t,t}^{Id,s}$	$\Omega_{t,Id}^{Id,s}$	$\Omega_{t,ts}^{Id,st}$	$\Omega_{t,Id}^{Id,st}$
$\mathcal{F}_s$	$\mathcal{O}_s$	$\Omega_{Id,Id}^{s,Id}$	$\Omega_{Id,t}^{s,Id}$	$\Omega_{Id,Id}^{s,s}$	$\Omega_{Id,t}^{s,s}$	$\Omega_{Id,ts}^{s,st}$	$\Omega_{Id,t}^{s,st}$
	$\mathcal{O}_{ts}$	$\Omega_{t,t}^{s,Id}$	$\Omega_{t,Id}^{s,Id}$	$\Omega_{t,t}^{s,s}$	$\Omega_{t,Id}^{s,s}$	$\Omega_{t,ts}^{s,st}$	$\Omega_{t,Id}^{s,st}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega_{Id,Id}^{st,Id}$	$\Omega_{Id,t}^{st,Id}$	$\Omega_{Id,Id}^{st,s}$	$\Omega_{Id,t}^{st,s}$	$\Omega_{Id,ts}^{st,st}$	$\Omega_{Id,t}^{st,st}$
	$\mathcal{O}_{sts}$	$\Omega_{t,t}^{st,Id}$	$\Omega_{t,Id}^{st,Id}$	$\Omega_{t,t}^{st,s}$	$\Omega_{t,Id}^{st,s}$	$\Omega_{t,ts}^{st,st}$	$\Omega_{t,Id}^{st,st}$

## 6. Change of basis

### §6.1. Two basis

Def Let  $Y \subset X$  be  $G$ -equiv closed subvariety,  $X$  proj.

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

with same notation,

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{S(G)} \in H_q^*(X; \mathbb{Q})$$

$$\begin{array}{ccc} Y & \xhookrightarrow{\iota_Y} & X \\ & \downarrow \pi_Y & \\ & pt & \end{array}$$

By cellular fibration lemma,

$$\begin{array}{ccccccc} K_0^{Td}(F_d) & \cong & K_0^{Cd}(F_d \times F_d) & \cong & K_0^{Cd}(Z_d) \\ \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Td} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Cd} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[Z_\omega]^{Cd} \\ & & & & & & \downarrow \\ & & & & & \parallel S & K_0^{Td}(Z_d) \\ & & & & & & \\ & & & & & & \oplus_{\omega, \omega' \in W_{td}} R(T_d)[\bar{\mathcal{O}}_{\omega, \omega'}]^{Td} \end{array}$$

as  $R(T_d)$ -modules.

⚠ There is no evidence if  $[Z_\omega]^{Cd}$  will be mapped to  $\oplus_{\omega \in W_{td}} [\bar{\mathcal{O}}_{\omega, \omega'}]^{Td}$ . Luckily, the horizontal line sends generators to generators.

Hint: Consider the following commutative diagram:

$$\begin{array}{ccccc} F_d & \xrightarrow{(F_d, Id)} & F_d \times F_d & \xrightarrow{(p_0, Id)} & Z_d \\ \bar{\mathcal{O}}_\omega & \nearrow & \nearrow & \nearrow & \\ pt & = & pt & = & pt \end{array}$$

To do linear alg, we take

$$\begin{aligned} \mathcal{R}(G) &:= \text{Frac } (R(G)) \\ \mathcal{S}(G) &:= \text{Frac } (S(G)) \end{aligned}$$

$$\begin{aligned} \mathcal{K}_o^G(X) &= K_o^G(X) \otimes_{R(T_d)} \mathcal{R}(T_d) \\ \mathcal{H}_G^*(X; \mathbb{Q}) &= H_G^*(X; \mathbb{Q}) \otimes_{S(T_d)} S(T_d) \end{aligned}$$

For  $R(T_d)$ -mod  $K_o^G(X)$ ,  $S(T_d)$ -mod  $H_G^*(X; \mathbb{Q})$

$$\begin{aligned} \text{Define } \psi_\infty &= [\{F_\infty\}]^{T_d} = (i_\infty)_* 1_{R(T_d)} \in K_o^{T_d}(\mathcal{F}_d) \\ \psi_{\infty, \infty'} &= [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d} = (i_{\infty, \infty'})_* 1_{R(T_d)} \in K_o^{T_d}(\mathbb{Z}_d) \\ \psi_{\infty, \infty'} &= [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d} \end{aligned}$$

We get two  $\mathcal{R}(T_d)$ -basis. ( $\psi_\infty$  is  $\mathcal{R}(T_d)$ -basis, by Localization theorem.)

$$\begin{array}{ccc} K_o^{T_d}(\mathcal{F}_d) & \longrightarrow & K_o^{T_d}(\mathbb{Z}_d) \\ [\overline{\mathcal{O}_\infty}]^{T_d} & & [\overline{\mathcal{O}_{\infty, \infty'}}]^{T_d} \\ \psi_\infty & & \psi_{\infty, \infty'} \end{array} \quad \begin{array}{l} \text{standard basis for stratification} \\ \text{canonical basis for convolution} \end{array}$$

Localization thm [Thm 10.1]

Let  $i: X^{T_d} \hookrightarrow X$ ,  $X$  is smooth.

$$\begin{array}{ccccc} \mathcal{K}_o^{T_d}(X^{T_d}) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X^{T_d}) \\ \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) \end{array}$$

are isos as  $\mathcal{R}(T_d)$  or  $\mathcal{S}(T_d)$ -module.

Q: The Steinberg variety  $Z_d$  is usually not smooth.

How to show that  $\{\psi_{\infty, \infty'}\}$  forms a basis?

Guess: apply localization thm to  $T_d \times T_d$  first.

## § 6.2. tangent space, Euler class.

Def (tangent space of fixed pts. in  $R(T_d)$ )

$$\underline{\Lambda}_{\infty} := T_{F_{\infty}} F_d \cong T_{Id}(G_d/B_{\infty}) \cong \mathfrak{g}_d/b_{\infty} = n_{\infty}$$

$$\widetilde{\Lambda}_{\infty} := T_{(p_0, F_{\infty})} \widetilde{Rep_d}(\mathbb{Q}) \cong T_{r_{\infty}} \oplus T_{F_{\infty}} F_d = r_{\infty} \oplus n_{\infty}$$

$$\underline{\Lambda}_{\infty, \omega'}^x := T_{(p_0, F_{\infty}, F_{\omega'})} \overline{\mathcal{O}}_x$$

$$\widetilde{\Lambda}_{\infty, \omega'}^x := T_{(p_0, F_{\infty}, F_{\omega'})} \mathcal{Z}_x \cong T_{r_{\infty, \omega'}} \oplus T_{(F_{\infty}, F_{\omega'})} \overline{\mathcal{O}}_x = r_{\infty, \omega'} \oplus \underline{\Lambda}_{\infty, \omega'}^x$$

Hint:

$$\begin{array}{ccc} \overline{\mathcal{O}}_x & \xrightarrow{(p_0, Id)} & \mathcal{Z}_x \\ & \searrow & \downarrow \\ & & \overline{\mathcal{O}}_x \end{array} \quad T_{x_0} (\cancel{\times}) = T_{x_0} (\cancel{\vee}) \oplus T_{x_0} (\cancel{\wedge})$$

$$\underline{\Lambda}_{\infty, \omega'}^x := T_{(F_{\infty}, F_{\infty, \omega'})} \overline{\mathcal{O}}_x$$

Rmk. It is still not easy to express  $\widetilde{\Lambda}_{\infty, \omega'}$  as Lie alg.  
However, we still know some special cases:

$$\begin{aligned} \underline{\Lambda}_{\infty, x}^x &:= T_{(F_{\infty}, F_{\infty x})} \overline{\mathcal{O}}_x \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{Id} G_d/B_{\infty} \cap B_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} \cap b_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} + b_{\infty}/(b_{\infty} \cap b_{\infty x}) \\ &= n_{\infty} + \underline{m}_{\infty, x} \end{aligned}$$

( $\underline{m}_{\infty, Id} = 0$ . For  $s \in \Pi$ ,  $\omega s \omega^{-1} \notin W_d$ , we have  $\underline{m}_{\infty, s} = 0$ )

Now suppose  $\omega s \omega^{-1} \in W_d$ .

$$\begin{aligned} \underline{\Lambda}_{\infty, \omega s}^s &= n_{\infty} \oplus \underline{m}_{\infty, \omega s} \\ \widetilde{\Lambda}_{\infty, \omega}^s &:= T_{(F_{\infty}, F_{\infty})} \overline{\mathcal{O}}_s \\ &= T_{(Id, Id)} G_d/B_{\infty} \times P_{\infty, \omega s}/B_{\infty} \\ &= n_{\infty} \oplus \underline{m}_{\infty s, \omega} \end{aligned}$$

$$\widetilde{\Lambda}_{\infty, \omega s}^s = r_{\infty, \omega s} \oplus n_{\infty}^- \oplus \underline{m}_{\infty, \omega s}$$

$$\widetilde{\Lambda}_{\infty, \omega}^s = r_{\infty, \omega} \oplus n_{\infty}^- \oplus \underline{m}_{\infty s, \omega}$$

### 6.3. transition matrix, localization formula

Thm. (Localization formula) [Thm 10.2]

Suppose  $Y \subset X$  is  $T$ -equivariant,  $\alpha \in \mathcal{K}_o^T(X)$ ,  $X$  smooth.

$X^T = \{x_1, \dots, x_m\}$ ,  $i_k : \{x_k\} \hookrightarrow X$ , then

$$\alpha = \sum_{k=1}^m \varepsilon_k (i_k)_* (i_k)^*(\alpha)$$

$$\varepsilon_k = (T_{x_k} X)^{-1} \in R(T)$$

$$\begin{aligned} \text{e.p. } [Y]^T &= \sum_{k=1}^m \varepsilon_k (i_k)_* (i_k)^* [Y]^T \cdot 1_{R(T)} \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^* [Y]^T) (i_k)_* 1_{R(T)} \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^* [Y]^T) [x_k]^T \\ [X]^T &= \sum_{k=1}^m \varepsilon_k [x_k]^T \end{aligned}$$

Suppose  $Y^T = \{x_1, \dots, x_n\}$ ,  $i_k : \{x_k\} \hookrightarrow Y$ , then

$$[Y]^T = \sum_{k=1}^n \beta_k [x_k]^T \quad \beta_k = \varepsilon_k \cdot (i_k)^* [Y]^T$$

When  $Y$  is sm at  $x_k$ ,

$$\begin{cases} \beta_k &= (T_{x_k} Y)^{-1} \\ (i_k)^* [Y]^T &= T_{x_k} X \cdot (T_{x_k} Y)^{-1} \end{cases}$$

Ex 1.  $X = \widetilde{\text{Rep}}_d(Q)$ ,  $Y = \overline{\mathcal{O}}_x$ ,  $T = T_d$

$$i_\infty : \{(p_0, F_\infty)\} \hookrightarrow \widetilde{\text{Rep}}_d(Q)$$

$$\widetilde{\text{Rep}}_d(Q)^{T_d} = \{(p_0, F_\infty) \mid \infty \in W_{ldl}\}$$

$$\begin{aligned} [\widetilde{\text{Rep}}_d(Q)]^{T_d} &= \sum_{\omega \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* (i_\infty)^* 1_{K_o^{T_d}(\widetilde{\text{Rep}}_d(Q))} \\ &= \sum_{\omega \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* 1_{R(T_d)} \\ &= \sum_{\omega \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} \psi_\omega \end{aligned}$$

$$\overline{\mathcal{O}}_x^{T_d} = \{(p_0, F_\infty) \mid \infty \leq x\}$$

$$[\overline{\mathcal{O}}_x]^{T_d} = \sum_{\omega \leq x} \widetilde{\Delta}_\infty^{-1} \underbrace{(i_\infty)^* [\overline{\mathcal{O}}_x]^{T_d}}_{f_{\infty, x}} \psi_\omega$$

when  $\overline{\mathcal{O}}_x$  is sm at  $(p_0, F_\infty)$ ,  $f_{\infty, x} = \widetilde{\Delta}_\infty (T_{(p_0, F_\infty)} \overline{\mathcal{O}}_x)^{-1}$

Ex 2.  $X = \text{Rep}_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d$ ,  $Y = \mathcal{Z}_x$ ,  $T = T_d$

$$(\mathcal{Z}_s)^{T_d} = \overline{\mathcal{O}}_s^{T_d} \sqcup (\mathcal{Z}_s - \overline{\mathcal{O}}_s)^{T_d}$$

$$= \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{ldl}\} \sqcup \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{ldl}, \infty s \in W_d\}$$

$$\Rightarrow [\mathcal{Z}_s]^{T_d} = \sum_{\omega \in W_{ldl}} (\widetilde{\Delta}_{\infty, \omega s})^{-1} \psi_{\infty, \omega s} + \sum_{\substack{\omega \in W_{ldl} \text{ s.t.} \\ \infty s \in W_d}} (\widetilde{\Delta}_{\infty, \omega})^{-1} \psi_{\infty, \omega} \quad \text{in } \mathcal{K}_o^{T_d}(X)$$

$\Rightarrow$  in  $\mathcal{K}_o^{T_d}(\mathcal{Z}_d)$

In general,  $[\mathcal{Z}_x]^{T_d} = \sum_{\omega, \omega'} \beta_{\infty, \omega, \omega'}^x \psi_{\infty, \omega \omega'}$

When  $\mathcal{Z}_x$  is sm at  $(p_0, F_\infty, F_{\infty \omega'})$ ,  $\beta_{\infty, \omega'}^x = (\widetilde{\Delta}_{\infty, \omega'})^{-1}$ .

### § 6.4. Generators. Define $e_i$ and $D_i$ .

$$\begin{array}{ccccc}
 K_0^{G_d}(\widetilde{Rep}_d(Q)) & \cong & K_0^{G_d}(F_d) & \cong & K_0^{T_d}(\text{pt}) \\
 \downarrow \pi_B^G & \Downarrow & \downarrow & \Downarrow & \downarrow \\
 R(T_d)[\widetilde{Rep}_d(Q)]^{G_d} & \cong & R(T_d)[F_d]^{G_d} & \cong & R(T_d) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{|I_d|}^{\pm 1}] \\
 \downarrow f[\widetilde{Rep}_d(Q)]^{G_d} & & & & \\
 K_0^{B_d}(\widetilde{Rep}_d(Q)) & \cong & \oplus_{\infty \in W_{|I_d|}} R(T_d) \psi_\infty & & \\
 \Downarrow & \Downarrow & & & \\
 \sum_{\infty \in W_{|I_d|}} f[\widetilde{Rep}_d(Q)]^{T_d} \psi_\infty & & & & \\
 \text{Q: Does } f[\widetilde{Rep}_d(Q)]^{T_d} = \sum_{\infty \in W_{|I_d|}} f[\widetilde{\Delta}_\infty^{-1}] \psi_\infty?
 \end{array}$$

Let  $e_i := x_i [\widetilde{\text{Rep}}_d(Q)]^{G_d} \in K_0^{G_d}(\widetilde{\text{Rep}}(Q))$ , then

$$\pi_B^G(e_i) = \sum_{\theta \in W(B)} x_{\theta(i)} \prod_{\omega}^{-1} \psi_{\omega} \in K_0^{Td}(\widetilde{Rep}(Q))$$

$$K_o^{G_d}(\widetilde{\text{Rep}_d(Q)}) \cong \mathbb{Z}[e_1, \dots, e_{|D|}]$$

$K_0^{G_d}(Z_d)$  is a  $\mathbb{Z}[e_1, \dots, e_{|d|}]$ -module.

Reason:  $K_0^{\text{Gr}}(\widehat{\text{Rep}}_f(Q)) \cong K_0^{\text{Gr}}(\mathbb{Z}_{\text{id}}) \hookrightarrow K_0^{\text{Gr}}(\mathbb{Z}_d)$

$$K_0^{Gal}(\mathbb{Z}_{Id}) \times K_0^{Gal}(\mathbb{Z}_{Id}) \xrightarrow{\text{CONVO}} K_0^{Gal}(\mathbb{Z}_{Id})$$

$$K_o^{Gal}(Z_{Id}) \times K_o^{Gal}(Z_d) \xrightarrow{\text{convolution}} K_o^{Gal}(Z_d)$$

We will mention about the convolution in the next section.]

Denote

$$\pi_B^G(D_i) = [Z_{S_i}]^{G_d} \in K_0^{G_d}(Z_d)$$

$$\pi_B^G(D_i) = \sum_{\omega \in W_{id}} (\widehat{\Delta}_{\infty, \infty}^s)^{-1} \psi_{\bar{\omega}, \omega} + \sum_{\substack{\omega \in W_{id} \\ \omega \leq \omega^{-1} \in W_d}} (\Delta_{-\infty, \infty}^s)^{-1} \psi_{\bar{\omega}, \omega}$$

we will show that

$$K_o^{G_d}(\mathbb{Z}_d) = \langle e_1, e_2, \dots, e_{|d|}, D_1, \dots, D_{|d|-1} \rangle_{\mathbb{Z}\text{-alg}} \subseteq \text{End}_{\mathbb{C}\text{-v.s.}}(\mathbb{Z}[e_1, \dots, e_{|d|}])$$

Before that, the more interesting question is to compute

$$D_i \in \text{End}_{\mathbb{C}\text{-vs.}}(\mathbb{Z}[e_1, \dots, e_{|D|}]).$$

## 7. Convolution product

### §7.1. Clean intersection formula

Thm. Suppose  $X$  sm  $G$ -equiv proj variety,  
 $Y_1, Y_2 \subset X$  are  $G$ -equiv subvariety,  
 $Y = Y_1 \cap Y_2$        $\pi_Y: Y \rightarrow \text{pt}$   
 $T = TX|_Y / (TY_1|_Y + TY_2|_Y)$

Assume that

$$TY_1|_Y \wedge TY_2|_Y = TY,$$

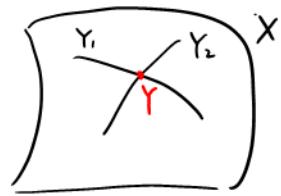
then

$$[Y_1]^G \otimes [Y_2]^G = \pi_{Y,*}(T) \cdot [Y]^G$$

Need a reference.

I believe that the first page of this document write the same thing:

[https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications\\_files/excessgw.pdf](https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf)  
 However, there is no proof.



$$\pi_{Y,*}: K^G_0(Y) \rightarrow K^G_0(\text{pt}) = R(G)$$

## 2. Convolution for canonical basis.

Thm  $\psi_{\infty'', \infty''} * \psi_{\infty', \infty} = \delta_{\infty'', \infty'} \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty}$   
 $\psi_{\infty'', \infty'} \diamond \psi_{\infty} = \delta_{\infty', \infty} \widetilde{\Delta}_{\infty} \psi_{\infty''}$

Proof. It reduce to the case

$$\begin{aligned}\psi_{\infty'', \infty''} * \psi_{\infty', \infty} &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty} \\ \psi_{\infty', \infty} \diamond \psi_{\infty} &= \widetilde{\Delta}_{\infty} \psi_{\infty'}\end{aligned}$$

$$\begin{array}{ccc}[Y_{12}] & & M_{123} \\ \downarrow & Z_d \hookrightarrow M_{12} & \downarrow \\ Y_{12} & \xrightarrow{\quad} & M_{12} \\ \downarrow & \searrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & & \widetilde{\text{Rep}}_d(Q)\end{array}$$

$$\begin{array}{ccc}[Y_{23}] & & M_{123} \\ \downarrow & Z_d \hookrightarrow M_{23} & \downarrow \\ Y_{23} & \xrightarrow{\quad} & M_{23} \\ \downarrow & \searrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & & \widetilde{\text{Rep}}_d(Q)\end{array}$$

$*: K_o^{Cd}(Z_d) \times K_o^{Cd}(Z_d) \rightarrow K_o^{Cd}(Z_d)$

$\{(p_0, F_{\infty''}), (p_0, F_{\infty'})\}$

$Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} \quad Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\}$

$\begin{array}{ccc}Y_{12} \times \widetilde{\text{Rep}}_d(\omega) & \subset & M_{123} \\ \{y\} \subset \widetilde{\text{Rep}}_d \times Y_{23} & \subset & M_{123}\end{array}$

where

$y = ((p_0, F_{\infty''}), (p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123}$

$y_{13} = ((p_0, F_{\infty''}), (p_0, F_{\infty})) \in M_{13}$

$T = \frac{\widetilde{\Delta}_{\infty''} \oplus \widetilde{\Delta}_{\infty'} \oplus \widetilde{\Delta}_{\infty}}{\widetilde{\Delta}_{\infty''} \oplus \widetilde{\Delta}_{\infty'}} = \widetilde{\Delta}_{\infty}$

$$\begin{aligned}\psi_{\infty'', \infty''} * \psi_{\infty', \infty} &= [Y_{12}]^{Td} * [Y_{23}]^{Td} \\ &= \pi_{13,*}([Y_{12} \times \widetilde{\text{Rep}}_d(Q)]^{Td} \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^{Td}) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty'} \cdot [y]^{Td}) \\ &= \widetilde{\Delta}_{\infty'} \cdot [y_{13}]^{Td} \\ &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty}\end{aligned}$$

$$\begin{array}{ccc}[Y_{12}] & & M_{123} \\ \downarrow & Z_d \hookrightarrow M_{12} & \downarrow \\ Y_{12} & \xrightarrow{\quad} & M_{12} \\ \downarrow & \searrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & & \widetilde{\text{Rep}}_d(Q)\end{array}$$

$$\begin{array}{ccc}[Y_{23}] & & M_{123} \\ \downarrow & Z_d \hookrightarrow M_{23} & \downarrow \\ Y_{23} & \xrightarrow{\quad} & M_{23} \\ \downarrow & \searrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & & \widetilde{\text{Rep}}_d(Q) \\ & & pt\end{array}$$

$\diamond: K_o^{Cd}(Z_d) \times K_o^{Cd}(\widetilde{\text{Rep}}_d(Q)) \rightarrow K_o^{Cd}(\widetilde{\text{Rep}}_d(Q))$

$$\begin{array}{ccc}Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} & & Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\} \\ \{y\} \subset Y_{12} \times pt & \subset & M_{123} \\ \{y\} \subset \widetilde{\text{Rep}}_d \times Y_{23} & \subset & M_{123}\end{array}$$

where

$y = ((p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123}$

$y_{13} = (p_0, F_{\infty'}) \in M_{13}$

$T = \frac{\widetilde{\Delta}_{\infty'} \oplus \widetilde{\Delta}_{\infty} \oplus 0}{\widetilde{\Delta}_{\infty'} \oplus 0} = \widetilde{\Delta}_{\infty}$

$$\begin{aligned}\psi_{\infty', \infty} * \psi_{\infty} &= [Y_{12}]^{Td} * [Y_{23}]^{Td} \\ &= \pi_{13,*}([Y_{12} \times pt]^{Td} \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^{Td}) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty} \cdot [y]^{Td}) \\ &= \widetilde{\Delta}_{\infty} \cdot [y_{13}]^{Td} \\ &= \widetilde{\Delta}_{\infty} \psi_{\infty'}.\end{aligned}$$

§ 7.3. expression of  $D_k$ .