Eine Woche, ein Beispiel

9.5 vector bundle v.s. local system

Key objects in Geometry & Algebra:

– vector bundle	over	manifold
– module	over	ring

There are hundreds of different versions of it:

– vector bundle over manifold 几何/几何分析

diffe mfld
- (real) differential v.b. over (real) differential mfld

Riemann surface
- cplx (analytic) line bundle over Riemann surface

– sheaf over space 代数几何

scheme theory
- locally free sheaf on scheme
- coherent sheaf on scheme

geo rep theory
- local system over (real/cplx) mfld
- perverse sheaf over Riemann surface (derived)

– module over ring 代数

comm alg
- f.g module over Noetherian commutative ring (with 1)

rep of grp
- group representation over group ($\rightsquigarrow$ group algebra)

quiver theory
- quiver representation over quiver ($\rightsquigarrow$ path algebra, bound quiver algebra)

Lie algebra
- Lie alg representation over Lie alg ($\rightsquigarrow$ universal enveloping algebra)

– Arithmetic Geometry 完整 $\rightsquigarrow$ p进分析

over	Berkovich analytic space	X^{an}
over	formal scheme	$Spf A$
over	rigid-analytic space	k-affinoid space
over	adic space	$Spa(A, A^+)$

Goal
- structures & invariants
- classifications of
  - special v.b, mfld, subv.b, submfld
- symmetry & quotient
- special functors
- homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n.

(by homotopy group! $\rightsquigarrow$ generalized Picard group?)

Q: Is this group structure natural?

ref: https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres

Thm. $\{\text{rank } m \ K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $\quad K = \mathbb{R}, \mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ (n \ rank m)	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{RP}^\infty \cong K(\mathbb{Z}/2\mathbb{Z}, 1)$

$\pi_{n-1}(GL_m(\mathbb{C}))$ (n \ rank m)	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{CP}^\infty \cong K(\mathbb{Z}, 2)$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes, \Lambda^k, \mathrm{Sym}^k$, Res, Ind

For the other spaces:
https://math.stackexchange.com/questions/383838/classifying-vector-bundles
http://www.ms.uky.edu/~guillou/F18/751Notes.pdf
It's still not so explicit.

$\{\text{rank } m \ K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_k(m, \infty)]$ $\quad K = \mathbb{R}, \mathbb{C}$

M: paracompact