Eine Woche, ein Beispiel

10.20 Schur functor: basic formulas

Main reference:
[FH]: Willian Fulton and Joe Harris. Representation Theory. A First Course.

[Hall]: Brian Hall. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. Graduate Texts in Mathematics. Cham: Springer International Publishing, 2015.
https://ocw.mit.edu/courses/18-755-lie-groups-and-lie-algebras-ii-spring-2024/resources/mit18_755_s24_leco2_pdf/

In this document, char $\Bbbk = 0$, $V \in \mathrm{Vect}_{\Bbbk}$.

Schur fctor helps us to decompose $V^{\otimes k}$ by S_k gp action. $\mathbb{S}^\lambda V$ generalize $\mathrm{Sym}^k V$ & $\Lambda^k V$. Moreover,

$$\begin{array}{ccc} \mathrm{Rep}(GL(V)) & = & \mathrm{Rep}(A_{n-1}) \\ \Psi & & \Psi \\ \mathbb{S}^\lambda V & = & L(\lambda) \end{array}$$

$n = \dim V$

Here, λ has many expressions,

e.g. partitions weights

$$\lambda = \begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ \\ \cline{1-1}\end{array} = (3,1) = 2\varpi_1 + \varpi_2$$

$$\lambda = \quad \cdots \quad = (\lambda_1, \lambda_2, \cdots) = \sum m_i \varpi_i$$

$$\{h_{ij}\} = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1}\end{array}$$ hook length

1. dimension
2. $\mathbb{S}^\lambda(V \oplus W)$ and ...

1. dimension

It can be computed from the Weyl dimension formula:

$$\dim_{\mathbb{C}} \mathcal{L}(\lambda) = \frac{\prod_{\alpha\in\Delta^+} (\lambda+\rho, \alpha)}{\prod_{\alpha\in\Delta^+} (\rho, \alpha)}$$

$\mathcal{L}(\lambda)\in rep(A_{n-1})$
$\lambda = \sum m_i \varpi_i$

$$= \frac{(m_1-1)\cdots(m_{n-1}+1)\,(m_1+m_2+2)(m_2+m_3+2)\cdots\quad\cdots\,(\sum m_i + n-1)}{\underbrace{1 \cdots 1}_{n-1 \text{ many}}\qquad \underbrace{2 \cdot 2 \cdots}_{n-2 \text{ many}}\cdots\cdots\quad n-1}$$

$$= \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \qquad \text{[FH, Thm 6.3 (1)]}$$

$$= \prod_{1\le i<j\le n} \frac{n-i+j}{h_{ij}} \qquad \text{[FH, Ex 6.4]}$$

Following [Hall Example 10.23],

$$\Delta^+ = \left\{ \begin{array}{ccccc} & & \sum\alpha_i & & \\ & \alpha_1+\alpha_2 & \cdots & \alpha_{n-2}+\alpha_{n-1} & \\ \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1} \end{array} \right\}$$

$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$$
$$= \frac{1}{2}\sum_{i=1}^{n-1} i(n-i)\alpha_i$$
$$= \sum_{i=1}^{n-1}\varpi_i \qquad \chi = \sum_{i=1}^{n-1}\langle \chi, \alpha_i\rangle \varpi_i$$

These would be enough to explain the first equality above.

E.g. When $\lambda = (3,1) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} = 2\varpi_1 + \varpi_2$,

$$\dim_k \mathcal{L}(\lambda) = \frac{(2+1)(1+1)(0+1)\cdots(2+1+2)(1+0+2)(0+0+2)\cdots(2+1+(n-1))}{1\cdot 1\cdot 1\cdots 2\cdot 2\cdot 2\cdots n-1}$$

$$= \frac{(2+1)(1+1)\quad(2+1+2)(1+0+2)\quad\cdots(2+1+(n-1))}{1\cdot 1\quad 2\cdot 2\quad\cdots n-1}$$

$$= \left(\frac{2+1}{1}\cdot\frac{3+2}{2}\cdot\frac{3+3}{3}\cdots\frac{3+n-1}{n-1}\right)\cdot\left(\frac{1+1}{1}\cdot\frac{1+2}{2}\cdot\frac{1+3}{3}\cdots\frac{1+n-2}{n-2}\right)$$

$$= \frac{(n-1)\,n\,(n+1)(n+2)}{1\cdot 4\cdot 2\cdot 1}$$

$$= \frac{\begin{array}{|c|c|c|}\hline n & n+1 & n+2 \\ \hline n-1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}$$

2. $\mathbb{S}^{\lambda}(V \oplus W)$ and ...

E.g.

$$(V \oplus W)^{\otimes 2} = V^{\otimes 2} \oplus (V \otimes W)^{\oplus 2} \oplus W^{\otimes 2}$$

$$\mathrm{Sym}^2(V \oplus W) = \mathrm{Sym}^2 V \oplus V \otimes W \oplus \mathrm{Sym}^2 W$$

$$\Lambda^2 (V \oplus W) = \Lambda^2 V \oplus V \otimes W \oplus W^{\otimes 2}$$

$$(V \oplus W)^{\otimes 3} = V^{\otimes 3} \oplus (V^{\otimes 2} \otimes W)^{\oplus 3} \oplus (V \otimes W^{\otimes 2})^{\oplus 3} \oplus W^{\otimes 3}$$

$$\mathrm{Sym}^3(V \oplus W) = \mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V \otimes W \oplus V \otimes \mathrm{Sym}^2 W \oplus \mathrm{Sym}^3 W$$

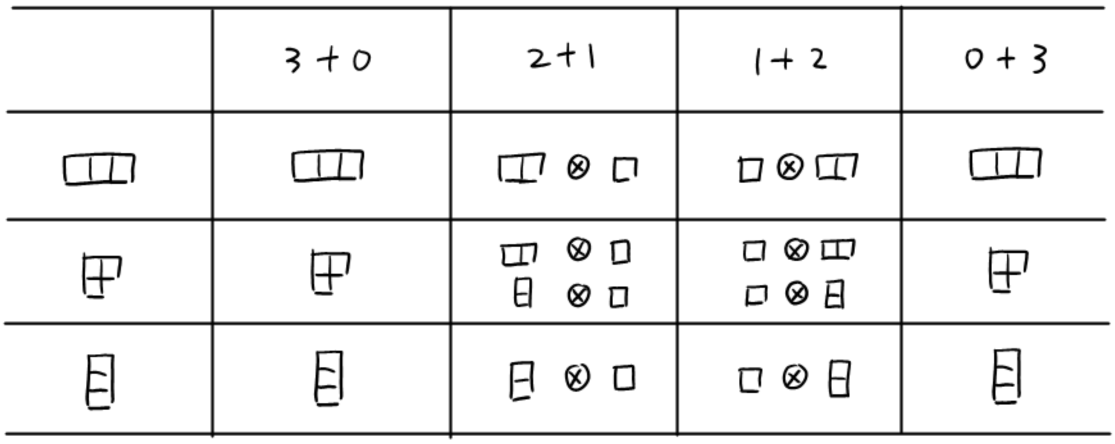
$$\mathbb{S}^{\boxplus}(V \oplus W) = \mathbb{S}^{\boxplus} V \oplus V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2} \oplus \mathbb{S}^{\boxplus} W$$

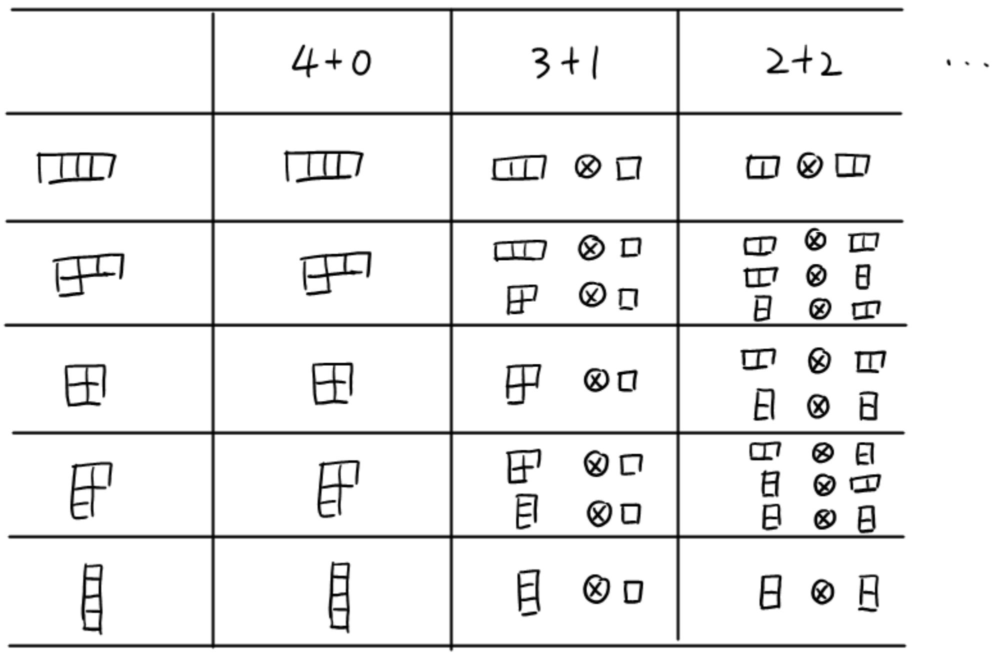
$$\Lambda^3 (V \oplus W) = \Lambda^3 V \oplus \Lambda^2 V \otimes W \oplus V \otimes \Lambda^2 W \oplus \Lambda^3 W$$

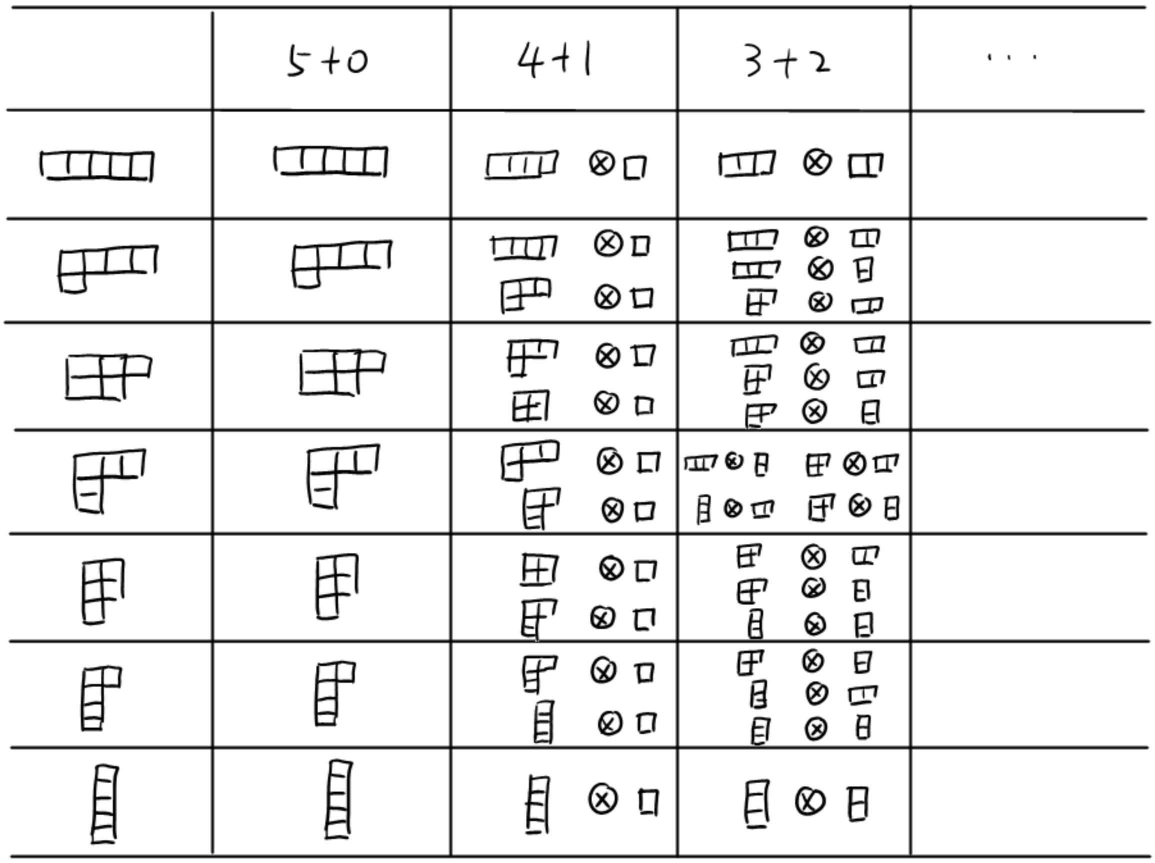
$$\frac{\begin{array}{|c|c|}\hline m+n & m+n+1 \\ \hline m+n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} = \frac{\begin{array}{|c|c|}\hline m & m+1 \\ \hline m-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} + m^2 n + n m^2 + \frac{\begin{array}{|c|c|}\hline n & n+1 \\ \hline n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}}$$

With the help of [FH,Ex 6.11], we can get the following tables:

	$2+0$	$1+1$	$0+2$
$\square\square$	$\square\square$	$\square \otimes \square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square \otimes \square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$


3 + 0
2+1
1 + 2
0 + 3


4+0
3+1
2+2
. . .



https://math.stackexchange.com/questions/84103/characters-of-symmetric-and-antisymmetric-powers

We have collected many remarkable coefficients from combinatorics, presented in the following formulas:

$N_{\lambda\mu\nu}$: Littlewood - Richardson number

$$\mathbb{S}^{\nu}(V \oplus W) \cong \bigoplus_{\mu,\lambda} N_{\lambda\mu\nu} \left(\mathbb{S}^{\lambda} V \otimes \mathbb{S}^{\mu} W\right) \qquad [Ex\ 6.11(a)]$$

$$\mathbb{S}^{\lambda} V \otimes \mathbb{S}^{\mu} V \cong \bigoplus_{\nu} N_{\lambda\mu\nu}\ \mathbb{S}^{\nu} V \qquad [Ex\ 6.19]$$

$$\mathbb{S}^{\nu/\lambda} V \cong \bigoplus_{\mu} N_{\lambda\mu\nu}\ \mathbb{S}^{\mu} V \qquad [p79, (6.7)]$$

As a Corollery,

$$\mathrm{Res}^{GL_{d_1+d_2}}_{GL_{d_1}\times GL_{d_2}} \mathbb{S}^{\nu}(\mathbb{C}^{d_1+d_2}) \cong \bigoplus_{\mu,\lambda} N_{\lambda\mu\nu}\ \mathbb{S}^{\mu}(\mathbb{C}^{d_1}) \otimes \mathbb{S}^{\lambda}(\mathbb{C}^{d_2})$$

$$\mathrm{Ind}^{S_{d_1+d_2}}_{S_{d_1}\times S_{d_2}} S^{\lambda} \otimes S^{\mu} \cong \bigoplus_{\nu} N_{\lambda\mu\nu}\ S^{\nu}$$

$$S^{\nu/\lambda} \cong \bigoplus_{\mu} N_{\lambda\mu\nu}\ S^{\mu}$$

$M_{\lambda\mu\nu}$: Plethysm coefficient

$$\mathbb{S}^{\mu}(\mathbb{S}^{\lambda} V) \cong \bigoplus_{\nu} M_{\lambda\mu\nu}\ \mathbb{S}^{\nu} V \qquad [Ex\ 6.17]$$

Later, $\lambda, \mu, \nu \vdash d$.

$g_{\lambda\mu\nu}$: Kronecker coefficient

$$\mathbb{S}^\nu(V \otimes W) \cong \bigoplus_{\mu,\lambda} g_{\lambda\mu\nu} (\mathbb{S}^\lambda V \otimes \mathbb{S}^\mu W) \qquad [\text{Ex } 6.11(b)]$$

$$S^\lambda \otimes S^\mu \cong \bigoplus_\nu g_{\lambda\mu\nu} S^\nu \qquad [\text{Ex } 4.51]$$

where

$$g_{\lambda\mu\nu} = \sum_{\xi \vdash d} \frac{1}{z_\xi} \chi_\lambda(C_\xi) \chi_\mu(C_\xi) \chi_\nu(C_\xi) \qquad [\text{Ex } 4.51]$$

$$z_\xi = \frac{d!}{|C_\xi|} = \prod_j (j^{\xi_j} \xi_j !) \qquad C_\xi : \text{conj class crspd to } \xi$$

$K_{\lambda\mu}$: Kostka number

https://mathoverflow.net/questions/314594/about-relation-between-kostka-numbers-and-littlewood-richardson-coefficient

$$\Lambda^{\mu_1} V \otimes \cdots \otimes \Lambda^{\mu_r} V \cong \bigoplus_\lambda K_{\lambda\mu} \mathbb{S}^{\lambda^T} V \qquad [\text{Ex } 6.13]$$

$$S_\lambda = \sum_\mu K_{\lambda\mu} M_\mu \qquad [(A.19)]$$

Left:

$$\mathbb{S}^\lambda V \otimes \mathbb{S}^\mu V \cong \ ? \qquad \text{inner tensor product}$$

$$\mathbb{S}^\mu \mathbb{S}^\lambda V \cong \ ? \qquad \text{inner plethysm}$$

The inner action is given by

$$GL_d \overset{\Delta}{\hookrightarrow} GL_d \times GL_d \subset GL_{2d} \circlearrowright V$$

the inner plethysm of s-symmetric functions
https://www.math.ucla.edu/~pak/hidden/papers/Thibon-Plethysm.pdf

Description of $\mathbb{S}_\lambda V$. [Ex 6.14, 6.19, 6.20]

$\mu = \lambda^T$, $\tilde{\mu} = \tilde{\lambda}^T$

$$\mathbb{S}^\lambda V = \mathrm{Im}\,(c_\lambda|_{V^{\otimes d}})$$
$$= \mathrm{Im}\left(\bigotimes_i \Lambda^{\mu_i} V \longrightarrow \otimes V^{\otimes d} \longrightarrow \bigotimes_j \mathrm{Sym}^{\lambda_j} V \right)$$
$$= \mathrm{Im}\left(\bigotimes_j \mathrm{Sym}^{\lambda_j} V \longrightarrow \otimes V^{\otimes d} \longrightarrow \bigotimes_i \Lambda^{\mu_i} V \right)$$
$$= \mathrm{Ker}\left(\mathbb{S}^{(\lambda_1, \dots, \lambda_{k-1})} V \otimes \mathrm{Sym}^{\lambda_k} V \longrightarrow V^{\otimes d - \lambda_k + 1} \otimes \mathrm{Sym}^{\lambda_k - 1} V \right)$$

$$\mathbb{S}^{\lambda/\tilde{\lambda}} V = \mathrm{Im}\left(\bigotimes_i \Lambda^{\mu_i - \tilde{\mu}_i} V \longrightarrow \otimes V^{\otimes d} \longrightarrow \bigotimes_j \mathrm{Sym}^{\lambda_j - \tilde{\lambda}_j} V \right)$$
$$= \mathrm{Im}\left(\bigotimes_j \mathrm{Sym}^{\lambda_j - \tilde{\lambda}_j} V \longrightarrow \otimes V^{\otimes d} \longrightarrow \bigotimes_i \Lambda^{\mu_i - \tilde{\mu}_i} V \right)$$