Eine Woche, ein Beispiel

5.19. Weierstrass point

references:

https://en.wikipedia.org/wiki/Weierstrass_point

https://en.wikipedia.org/wiki/Inflection_point

Klein quartic has 24 inflection points:
https://www.uio.no/studier/emner/matnat/math/MAT2000/v24/projects/2023_the_klein_quartic_and_its_n_weierstrass_points.pdf

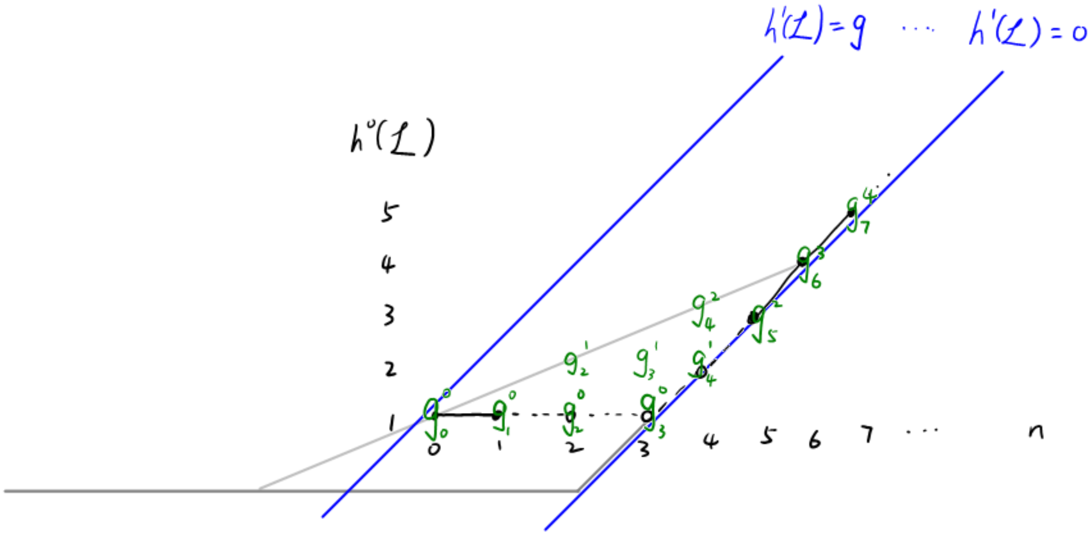
curve of genus >0 don't have single simple pole:
https://math.stackexchange.com/questions/2841459/finding-a-meromorphic-function-on-a-compact-riemann-surface-with-prescribed-zero

Setting: C: proj sm curve $/k$ $\quad \bar{k} = k$, char $k = 0$

$h^0(\mathcal{O}(nP))$ \ n ; $g(C)$	0	1	2	3	4	5	6	7	8	$g(g^2-1)$
0	1	2	3	4	5	6	7	8	9	0
1	1	1	2	3	4	5	6	7	8	0
2	1	1	?	2	3	4	5	6	7	6
3	1	1	?	?	?	3	4	5	6	24
4	1	1	?	?	?	?	?	4	5	60
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$g=3$: non-Weierstrass	1	1	1	1	2	3	4	5	6	$\emptyset$
W: general quartic e.g. Klein quartic	1	1	1	2	2	3	4	5	6	1×24
W: Fermat quartic	1	1	1	2	3	3	4	5	6	2×12
W: hyperelliptic case	1	1	2	2	3	3	4	5	6	3×8

by Clifford's thm, $h^0(\mathcal{O}(nP)) \leq \frac{n}{2} + 1$,
so the hyperelliptic case reaches the limit.

Finiteness of Weierestrass point:
https://math.stackexchange.com/questions/4719889/is-this-proof-that-the-number-of-weierstrass-points-on-a-compact-riemann-surface

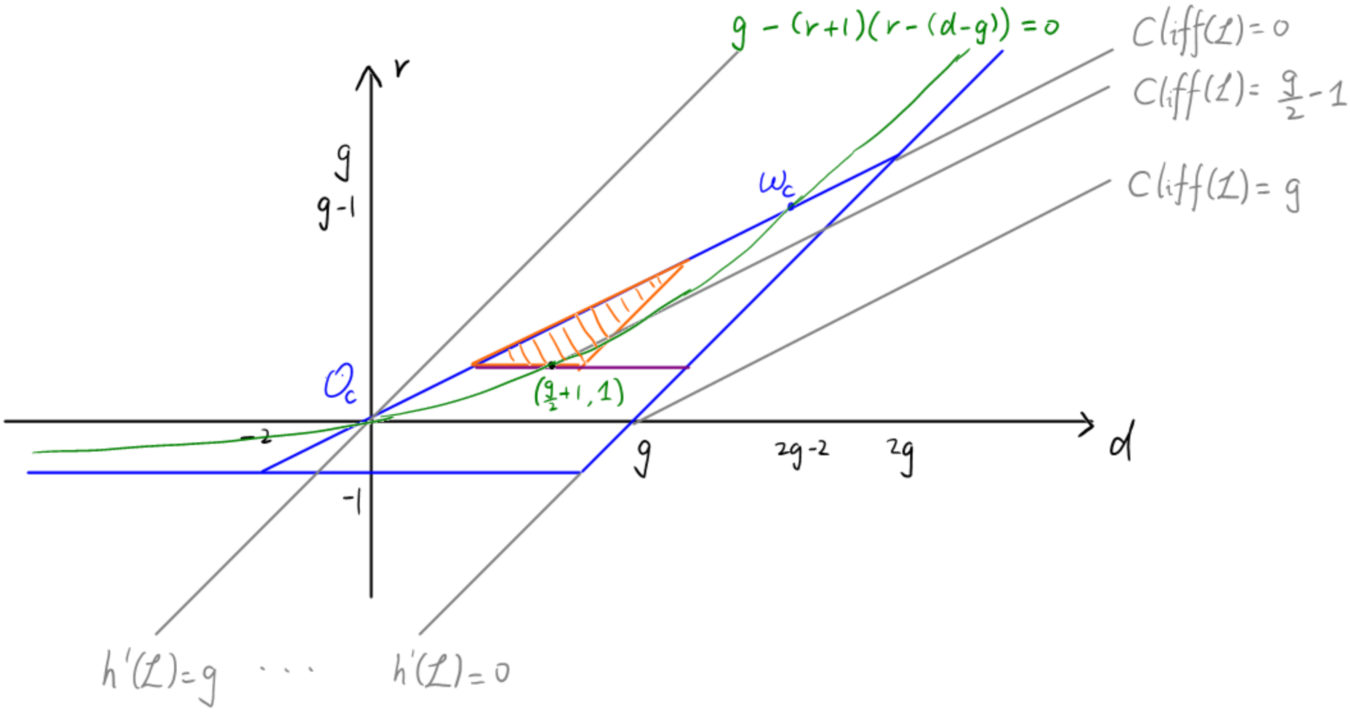


$(V, \mathcal{L}) \in g^{\dim V}_{\deg \mathcal{L}}$, where $\mathcal{O}_X \otimes_k V \subset \mathcal{O}_X \otimes_k H^0(X;\mathcal{L}) \longrightarrow \mathcal{L}$

$$g^{\dim V}_{\deg \mathcal{L}} \cong \mathbb{P}^{\dim V - 1} \hookrightarrow C^{\deg \mathcal{L}}$$

First $g^{\dim V}_{\deg \mathcal{L}}$: moduli of linear systems

Second $g^{\dim V}_{\deg \mathcal{L}}$: one special linear system = moduli of divisors



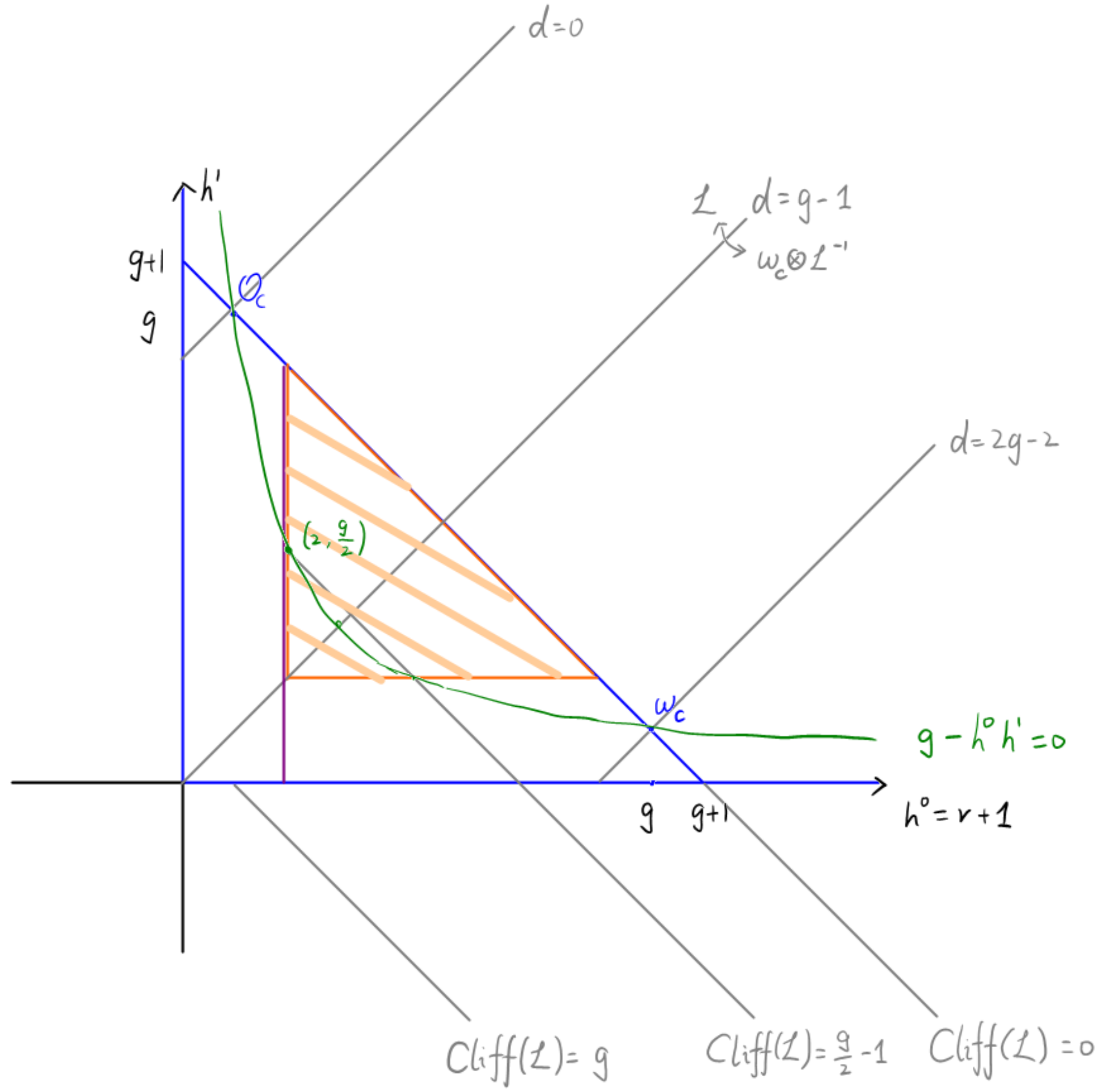
$\mathrm{Cliff}(C) := \min\{\mathrm{Cliff}(\mathcal{L}) \mid h^0(\mathcal{L}), h^1(\mathcal{L}) \geq 2\}$ when $g \geq 4$

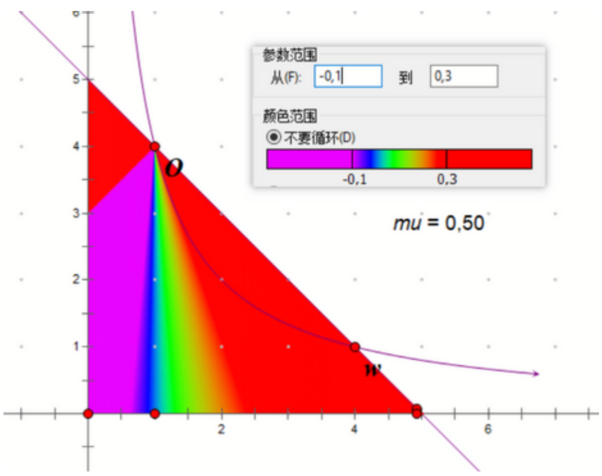
$\mathrm{gon}(C) := \min\{\deg(\mathcal{L}) \mid r(\mathcal{L}) = 1\}$

Cor $\mathrm{gon}(C) \geq \mathrm{Cliff}(C) + 2$

"=" $\Leftrightarrow$ Clifford dim of C is 1.

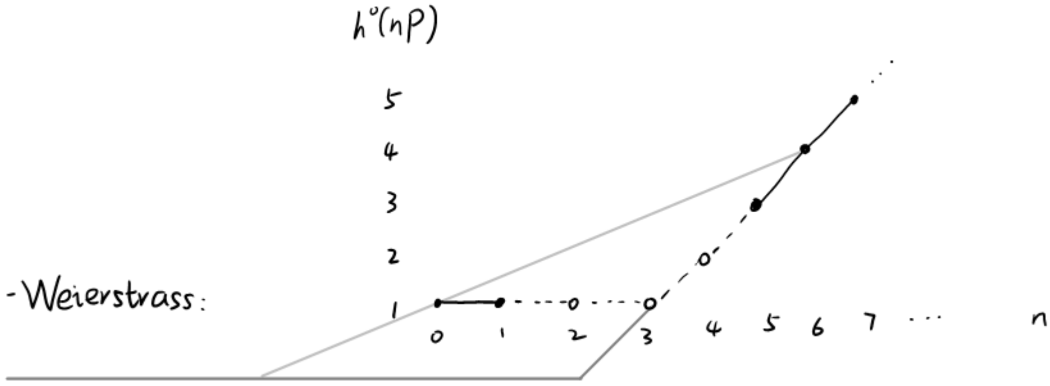
https://mathoverflow.net/questions/9734/on-the-clifford-index-of-a-curve
https://mathoverflow.net/questions/333072/clifford-index-and-clifford-dimension





$$\mu := \frac{r(\mathcal{L})}{\deg(\mathcal{L})} = \frac{h^0 - 1}{h^0 - h^1 + g - 1}$$

P∈C non-Weierstrass:



$g=3$ case

$\Leftrightarrow h^0(\mathcal{O}(gP)) = 1$

P∈C Weierstrass $\Leftrightarrow h^0(\mathcal{O}(gP)) \geq 2$

$\Leftrightarrow \exists\ f \in K(C)$, f has a single pole at P, with $\mathrm{ord}_P(f) \geq -g$

$\Leftrightarrow h^0(K - gP) \geq 1$

e.p.

$g=2$: P∈C Weierstrass $\Leftrightarrow h^0(\mathcal{O}(2P)) = 2$

$\Leftrightarrow \exists\ f \in K(C)$, f has a single double pole at P

$g=3$: P∈C Weierstrass $\Leftrightarrow h^0(\mathcal{O}(3P)) \geq 2$

$\Leftarrow \exists\ f \in K(C)$, f has a single triple pole at P

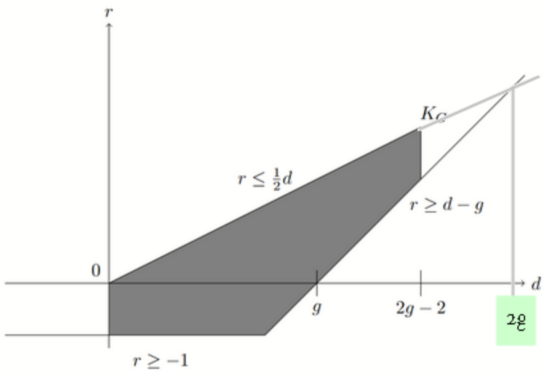


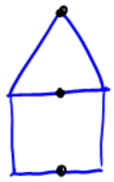
FIGURE 1. Possibilities for the degree and rank of a divisor.

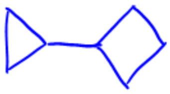
A tropical version for analog. I could draw it also for the classical AG cases, but I'm lazy.
Photo comes from [MA 764: Chip Firing, Lec 9]:
https://www.ms.uky.edu/~dhje223/MA%20764%20Spring%202019.html

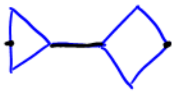
A case in tropical algebraic curve where the "Weierstrass points are not finite":

$g = 2$

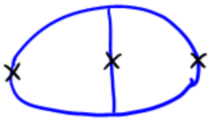
"Weierstrass pts": $rk(2p) = 1$

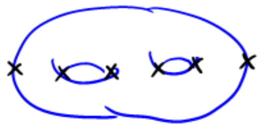






comparison between tropical & classical:





See this article for more examples of Weierestrass points on tropical alg curves:
https://arxiv.org/pdf/2303.07729
See [Theorem 1.7] which computes the total weight of the Weierstrass locus: $d - r + rg$.
When D=K, d=2g-2, r=g-1, the total weight is g^2-1.
Notice that the definition of weight is slightly changed.

The Dhar's burning algorithm is mainly used for eliminating negative divisors.

Step1: blow (burn negative divisors)
Step2: suck (attract positive divisors)

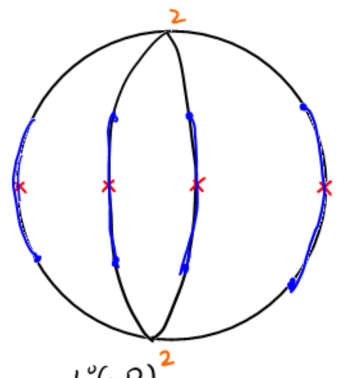
This process looks like the process when I suck the river snail, therefore, I call it as "嗦田螺算法". It's an effective algorithm in determining if a divisor is effective.

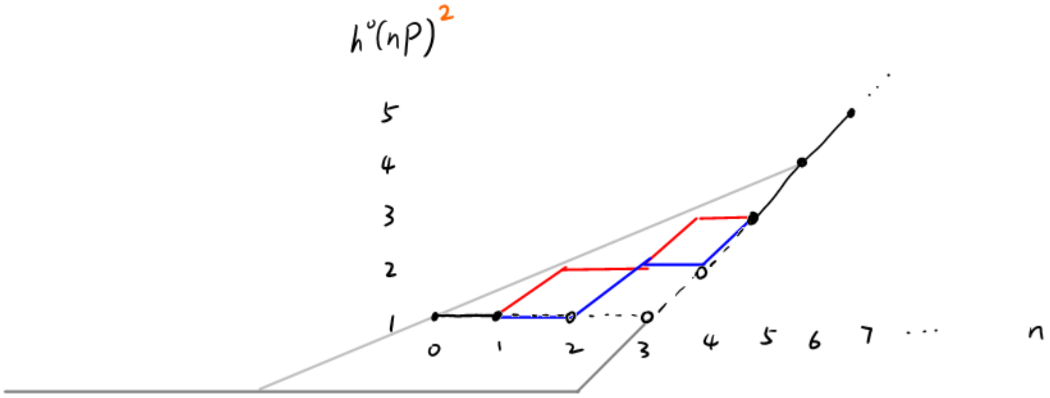
Some differences between classical algebraic curves and tropical algebraic curves:

We have Dhar's burning algorithm for tropical algebraic curves, which is not so explicit in classical case. (Maybe I'm wrong: the hyperelliptic curves can be seen in [Theorem 4.1.6]: https://algant.eu/documents/theses/dipiazza.pdf)
We can also divide K into two canonical parts.

In classical algebraic curves, the Weierstrass point is finite, which is not true in tropical algebraic curves.

E.g. $g=3$,





$g=3$ case

By Riemann-Roch,

$$r(kP) = r(K-kP) + k - g + 1$$

$$\underset{=\!=}{g=3} \begin{cases} r(K-2P) & k=2 \\ r(K-3P)+1 & k=3 \\ r(K-4P)+2 & k=4 \end{cases}$$