Eine Woche, ein Beispiel

1.30 homotopy addition theorem

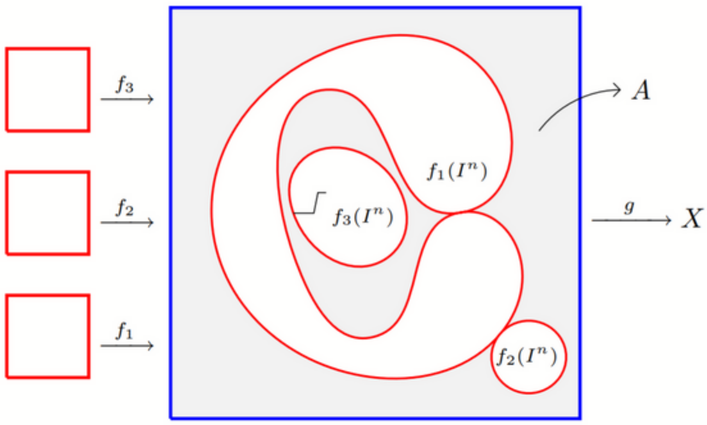
ref:https://github.com/lrnmhl/AT1
Some pictures are also copied there. Here we just want to draw some figures, and explain it in 15 minutes(for oral exam!)

(For convenience, we take singular homology, and abbreviate $H_n(X,A;\mathbb{Z})$ as $H_n(X,A)$.)

I.7. Theorem (Homotopy Addition Theorem). — *Assume we have $f_1, \ldots, f_k : I^n \to I^n$ such that $f_i|_{\mathring{I}^n}$ is an open embedding and the sets $f_i(\mathring{I}^n)$ are pairwise disjoint. Furthermore, let $g : (I^n, \partial I^n) \to (X, A)$ such that $g(I^n \setminus \bigcup_{i=1}^k f_i(\mathring{I}^n)) \subset A$. Then*

$$[g] = \sum_{i=1}^{k} (\deg f_i)[g \circ f_i]$$

in $\pi_n(X, A)^{\#}$.



For this we need the definition of deg f_i.

Def 1 Recall the notion of local degree. Let $f : \mathring{I}^n \to \mathring{I}^n$ be an open embedding, $p \in \mathring{I}^n$. We have that f induces a commutative diagram:

$$\begin{tikzcd}
H_n(\mathring{I}^n, \mathring{I}^n \setminus \{p\}) \arrow[r, "i_*"] \arrow[d, "f_*"'] \arrow[dr, "f_*"] & H_n(I^n, I^n \setminus \{p\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"'] \arrow[d, dashed, "d\cdot-"] \\
H_n(f(\mathring{I}^n), f(\mathring{I}^n) \setminus \{f(p)\}) \arrow[r, "i_*"'] & H_n(I^n, I^n \setminus \{f(p)\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

where the maps are all isomorphisms by homotopies and excision, hence they induce the dashed arrow. This is an automorphism of $H_n(I^n, \partial I^n) \cong \mathbb{Z}$ and thus $d = \pm 1$. One can show this is independent of p, hence we call it the **local degree** of f, and write $\deg(f) = d$.

After proving that every arrow is an iso, the useful part of comm diag reduced to

$$\begin{tikzcd}
 & H_n(I^n, \partial I^n) \arrow[dl, "f_*"'] \arrow[d, dashed, "d\cdot-"] \\
H_n(I^n, I^n \setminus \{f(p)\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"]
\end{tikzcd}$$

Def 2. When $f : I^n \to I^n$ sends ∂I^n to ∂I^n, then deg f is defined by

$f_* : H_n(I^n, \partial I^n) \longrightarrow H_n(I^n, \partial I^n)$

Rmk. These two degree coincide when f satisfies both conditions ($f|_{\mathring{I}^n}$ open embedding, $f(\partial I^n) \subset \partial I^n$)

Lemma 1 (Obvious)

Suppose $g, g' : (I^n, \partial I^n) \longrightarrow (X, A)$ are homotopic in $\pi_n(X,A)^{\#}$, then $[g] = [g']$, $[g \circ f_i] = [g' \circ f_i]$ in $\pi_n(X,A)^{\#}$. $(f_i : I^n \longrightarrow I^n)$

Lemma 2 (Not obvious)

Assume $H : I^n \times [0,1] \longrightarrow I^n$ is an homotopy between f and f', $H_t|_{\mathring{I}^n}$ is an open embedding for any $t \in [0,1]$. Then $\deg f = \deg f'$.

(This lemma is still not so perfect anyhow, so we have Lemma 3 as a compliment)

Proof. Fix $p \in \mathring{I}^n$. For $\forall\, t \in [0,1]$, $\exists$ open nbhd I_t of t s.t. $Im(H|_{\partial I^n \times \bar{I}_t}) \cap Im(H|_{\{t\} \times \bar{I}_t}) = \phi$.

Take open subset $\mathcal{U}_t$ s.t. $Im(H|_{\{t\} \times \bar{I}_t}) \subset \mathcal{U}_t \subset \bar{\mathcal{U}}_t \subset I^n - Im(H|_{\partial I^n \times I_t})$,

then for any $t', t'' \in I_t$, we have

$$\begin{tikzcd}
 & & H_n(I^n, \partial I^n) \arrow[dl, shift right, "H_{t'*}"'] \arrow[dl, shift left, "H_{t''*}"] \arrow[d, dashed, "\text{induce same degree map}"] \arrow[dll, dashed, bend right, "H_{t'*}"'] \arrow[dll, dashed, "H_{t''*}"] \\
H_n(I^n, I^n - \{H(p,t')\}) \\
H_n(I^n, I^n - \{H(p,t'')\}) & H_n(I^n, I^n - \mathcal{U}_t) \arrow[ul, dashed] \arrow[l, dashed] & H_n(I^n, \partial I^n) \arrow[l, "i_*"']
\end{tikzcd}$$

homotopy eq $\sim$

$\therefore \deg H_{t'} = \deg H_{t''}$ for $t', t'' \in I_t$

Finally, by using the Heine-Borel theorem we get $\deg f = \deg f'$.

Lemma 3. Assume that $H : I^n \times [0,1] \longrightarrow I^n$ is an homotopy between f and f'.

- $f|_{\mathring{I}^n}$ is an open embedding
- $f'(\partial I^n) \subseteq \partial I^n$
- $\exists\, \mathcal{U} \overset{\text{open}}{\subset} f(I^n)$ s.t. $H(\partial I^n \times [0,1]) \subset I^n - \mathcal{U}$

  better: $\exists\, p \in \mathring{I}^n$ s.t. $H(\partial I^n \times [0,1]) \subset I^n - \{f(p)\}$

Then $\deg f = \deg f'$.

Proof. Pick $p \in f^{-1}(\mathcal{U})$. Then

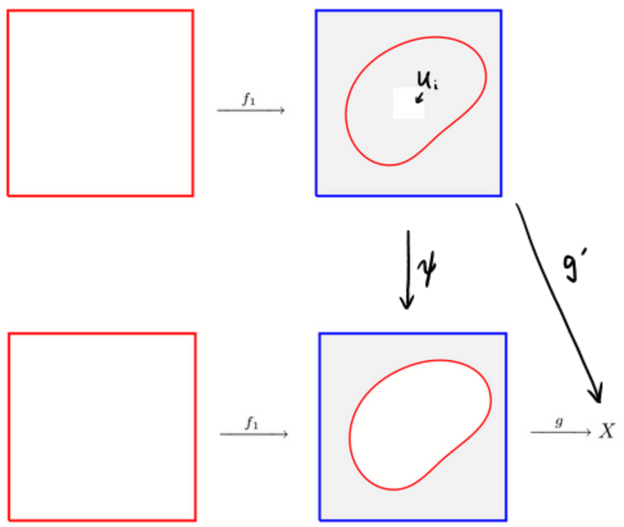
$$\begin{tikzcd}
 & H_n(I^n, \partial I^n) \arrow[dl, "f_*"'] \arrow[dl, shift left=2, red, "i_* \circ f'_*"] \arrow[d, "f'_*"] \arrow[d, shift right=3, dashed, red, "(i_*)^{-1} \circ f_*"'] \\
H_n(I^n, I^n - \{f(p)\}) & H_n(I^n, \partial I^n) \arrow[l, "i_*"']
\end{tikzcd}$$

homotopy eq $\sim$

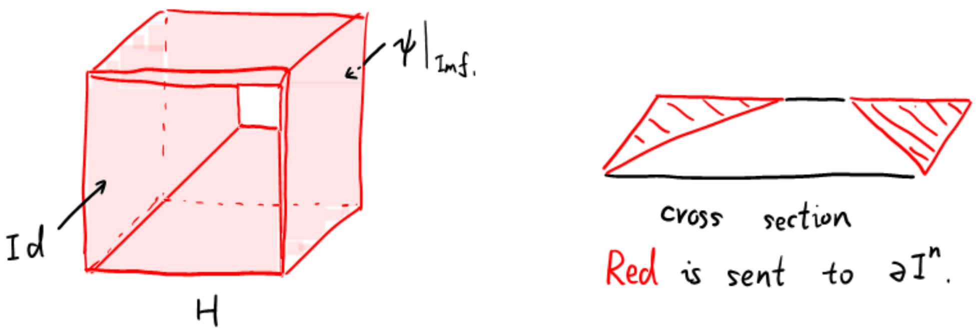
Now we can squeeze / contort fcts by homotopy without too much worries!

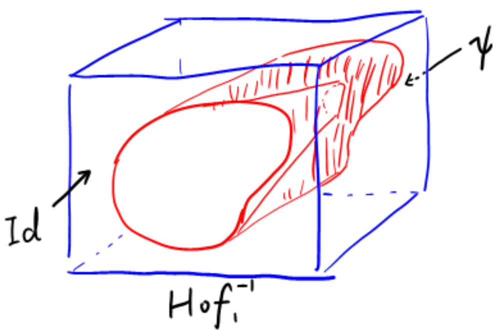
Lemma 4. (contort g).

I.8. Lemma. — *For $1 \leqslant i \leqslant k$, let $\mathcal{U}_i \subset f_i(\mathring{I}^n)$ be any non-empty open set. Then g is homotopic relative to $I^n \smallsetminus f_i(\mathring{I}^n)$ to a map g' that sends $f_i(I^n) \smallsetminus \mathcal{U}_i$ to A for all i.*



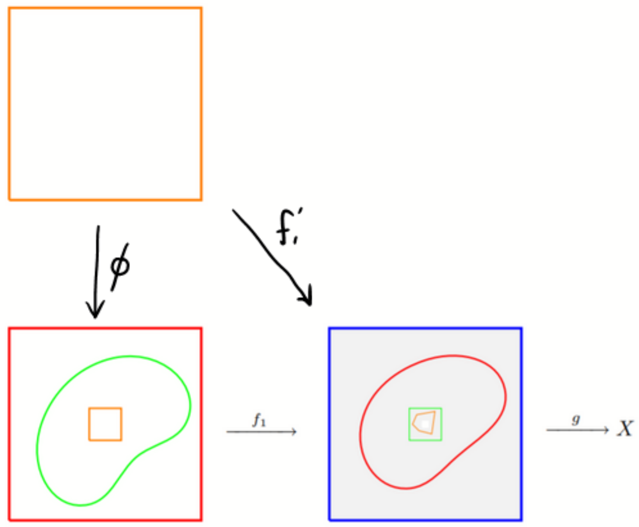
Idea: We only contort the space in $\operatorname{Im} f_i$, and fix pts outside f_i.

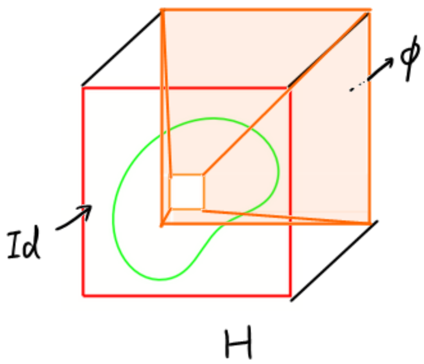




By using Lemma 1, we can assume that only a small part is not sent to A.

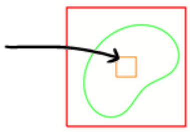
Shrink f_1: (By using Lemma 2)



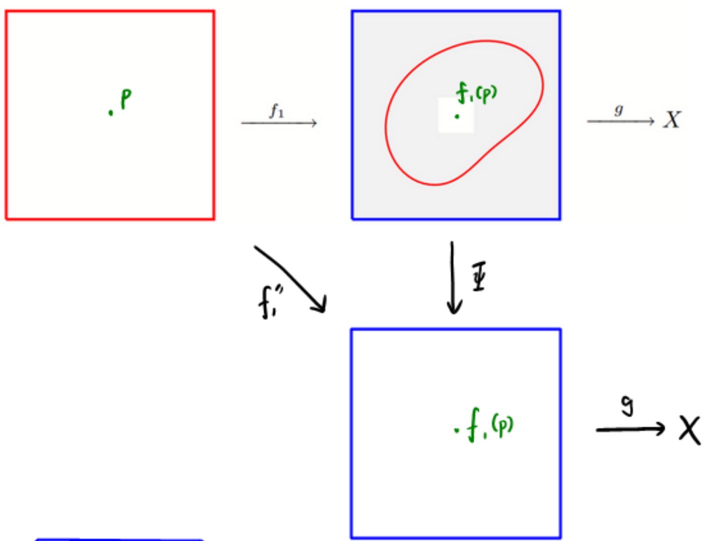


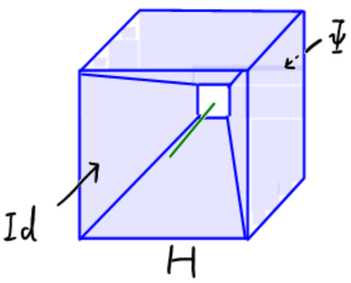
cross section

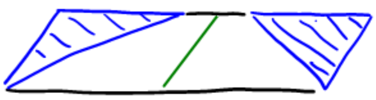
Orange is sent to



Blow image of f_1:







cross section

Blue is sent to ∂I^n.

Be careful: f_1'' may not be an open embedding when restricted to $\mathring{I}^n$,
but now we have $f_1''(\partial I^n) \subset \partial I^n$, so now we can apply Lemma 3!

The rest is easy. Apply Lemma I.5 in ref.