Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve
https://mathoverflow.net/questions/402049/induced-action-on-prym-variety
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Goal: Describe some curves (maybe singular) C in A, and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10}(x^2 - j)\}$ has the following covers:

genus

$\mathrm{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

ramification degree

$$\begin{tikzcd}
 & C_9 \arrow[dl, "20"'] \arrow[d, "4"', "\tilde{f}"] \arrow[dr, "0"', "f"] & \\
\mathbb{P}^1 \arrow[dr, "2"'] & C_4 \arrow[d, "10"'] & C_5 \arrow[dl, "12"] \\
 & \mathbb{P}^1 &
\end{tikzcd}$$

where

$$C_4 = \{y^2 = \prod_{j=1}^{10}(t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t\prod_{j=1}^{10}(t - j)\}$$

$t = x^2$

$\tilde{y} = xy$

The crspd field extension:

$$\begin{tikzcd}
 & \mathbb{C}(x)[y]/(y^2 - \prod_{j=1}^{10}(x^2 - j)) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{C}(x) \arrow[dr, dash] & \mathbb{C}(t)[y]/(y^2 - \prod_{j=1}^{10}(t - j)) \arrow[d, dash] & \mathbb{C}(t)[\tilde{y}]/(\tilde{y}^2 - t\prod_{j=1}^{10}(t - j)) \arrow[dl, dash] \\
 & \mathbb{C}(t) &
\end{tikzcd}$$

Global differential forms

Pulling back differential forms give the following maps:

$$\begin{tikzcd}
 & \left\langle x^k \frac{dx}{y} \;\middle|\; k=0,\dots,8 \right\rangle & \\
 & \left\langle 2x^{2k+1} \frac{dx}{y} \;\middle|\; k=0,\dots,3 \right\rangle \arrow[u, "\tilde{f}^*"] \arrow[d, equal] & \left\langle 2x^{2k} \frac{dx}{y} \;\middle|\; k=0,\dots,4 \right\rangle \arrow[ul, "f^*"'] \arrow[d, equal] \\
0 \arrow[uur] & \left\langle t^k \frac{dt}{y} \;\middle|\; k=0,\dots,3 \right\rangle & \left\langle 2t^k \frac{dt}{y} \;\middle|\; k=0,\dots,4 \right\rangle \\
 & 0 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

Therefore,

$$H^0(C_9;\omega_{C_9}) \cong \tilde{f}^* H^0(C_4;\omega_{C_4}) \oplus f^* H^0(C_5;\omega_{C_5}) \qquad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which crspds to pulling back of divisors)

$$\begin{tikzcd}
 & \left\langle x^k \frac{dx}{y} \;\middle|\; k=0,\dots,8 \right\rangle \arrow[ddl] \arrow[d, "\tilde{f}_*"] \arrow[dr, "f_*"] & \\
 & \left\langle 2x^{2k+1} \frac{dx}{y} \;\middle|\; k=0,\dots,3 \right\rangle \arrow[d, equal] & \left\langle 2x^{2k} \frac{dx}{y} \;\middle|\; k=0,\dots,4 \right\rangle \arrow[d, equal] \\
0 \arrow[dr] & \left\langle t^k \frac{dt}{y} \;\middle|\; k=0,\dots,3 \right\rangle \arrow[d] & \left\langle 2t^k \frac{dt}{y} \;\middle|\; k=0,\dots,4 \right\rangle \arrow[dl] \\
 & 0 &
\end{tikzcd}$$

However, since $\mathrm{Jac}(C) = H^0(C;\omega_C)^* / H_1(C;\mathbb{Z})$,
we are working on the dual spaces. The notations are again switched:

$$f^* \rightsquigarrow Nm_f$$
$$f_* \rightsquigarrow f^*$$

One may get

different meaning compared with (1)!

$$H^0(C_9;\omega_{C_9})^* \cong \tilde{f}^* H^0(C_4;\omega_{C_4})^* \oplus f^* H^0(C_5;\omega_{C_5})^* \qquad (2)$$

Curve in Prym variety

Define A as the quotient of Jacobians, i.e.,

$$A := \mathrm{Jac}(C_9) / f^* \mathrm{Jac}(C_5) \cong \mathrm{Prym}(C_9/C_5)$$

$$\begin{tikzcd}
 & & C_9 \arrow[r] \arrow[d, "AJ_{C_9}"] & C_4 \arrow[d] \arrow[r] & \mathrm{Jac}(C_4) \arrow[dl, dashed, "\exists! \text{ isogeny}"] & \\
0 \arrow[r] & \mathrm{Jac}(C_5) \arrow[r, "f^*"] & \mathrm{Jac}(C_9) \arrow[r, "\pi"] & A \arrow[r] & 0
\end{tikzcd} \tag{3}$$

Prop.
0. A is isogenous to $\mathrm{Jac}(C_4)$;
1. $f^* : \mathrm{Jac}(C_5) \longrightarrow \mathrm{Jac}(C_9)$ is injective;
2. $\pi \circ AJ_{C_9}$ is not injective, it factors through C_4;
3. $C_4 \longrightarrow A$ is generically injective;
4. $C_4 \longrightarrow A$ produces a sm image of A, outside of non-injective locus.