$$
\begin{array}{c}
M \\
| \; ① \\
L \\
| \; ② \\
K
\end{array}
\Bigg\} ③
\qquad\qquad
\begin{tikzcd}
 & LM \arrow[dl, dash] \arrow[ddr, dash, "⑥"] \arrow[dr, dash] & \\
L \arrow[rr, dash, "④"'] & & M \arrow[d, dash, "⑤"] \\
 & & K
\end{tikzcd}
$$

$① \xrightarrow{③\ \text{pure ins}} ③$

normal:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
+ seperable:		① + ② = ③			④ + ⑤ = ⑥
= Galois:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
purely inseparable		① + ② = ③			④ + ⑤ = ⑥

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then
F perfect $\Leftrightarrow F^p = F$

$$
\begin{array}{cl}
\overline{K} & \\
| & \text{closed subgroup} \\
L & \\
\text{(finite)}\ | & \text{quotient group} \\
K &
\end{array}
\qquad
\begin{array}{c}
\overline{\mathbb{F}}_p \\
| \ \mathbb{Z}_l \\
\bigcup_{l \nmid a} \mathbb{F}_{p^a} \\
| \ \prod_{p \nmid l} \mathbb{Z}_p \\
\mathbb{F}_p
\end{array}
\qquad
\begin{array}{c}
\overline{\mathbb{F}}_p \\
| \ \prod_{p \nmid l} \mathbb{Z}_p \\
\bigcup_n \mathbb{F}_{p^{l^n}} \\
| \ \mathbb{Z}_l \\
\mathbb{F}_p
\end{array}
\qquad
\begin{array}{c}
\overline{\mathbb{F}}_p \\
| \ d\hat{\mathbb{Z}} \\
\mathbb{F}_q \\
| \ \mathbb{Z}/d\mathbb{Z} \\
\mathbb{F}_p
\end{array}
\Bigg]
\quad \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \quad (q = p^d)
$$

open subgroup $\overset{\{1\} \subseteq \mathbb{Z}_p}{\subsetneq}$ closed subgroup $= \{Gal(\overline{K}/L) \mid L/K \text{ ext}\} \overset{\mathbb{Z} \subseteq \mathbb{Z}_p}{\subsetneq}$ subgroup

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Some wonderful exercises for Galois correspondence:

Let E/F be Galois field ext of deg n, $m \mid n$. prove: $\exists$ subfield ext of deg m.
(Sylow thm & $Z(G) \neq \{1\}$ for a p-gp & classification of f.g. abelian gp)
Cor. For p prime, F field, one can define ${}^p\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^p\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}} \cap \mathbb{R}$)?