Eine Woche, ein Beispiel

9.5. vector bundle v.s. local system

Key objects in Geometry & Algebra:

- vector bundle over manifold
- module over ring

There are hundreds of different versions of it:

- vector bundle over manifold 几何/几何分析
  - diffe mfld · (real) differential v.b. over (real) differential mfld
  - Riemann surface · cplx (analytic) line bundle over Riemann surface
- sheaf over space 代数几何
  - scheme theory · locally free sheaf on scheme
  - · coherent sheaf on scheme
  - geo rep theory · local system over (real/cplx) mfld
  - · perverse sheaf over Riemann surface (derived)
- module over ring 代数
  - comm alg · f.g module over Noetherian commutative ring (with 1)
  - rep of grp · group representation over group ($\rightsquigarrow$ group algebra)
  - p-adic rep · smooth representation over unimodular gp ($\rightsquigarrow$ Hecke algebra $\mathcal{H}(G)$) smooth module
  - quiver theory · quiver representation over quiver ($\rightsquigarrow$ path algebra, bound quiver algebra)
  - Lie algebra · Lie alg representation over Lie alg ($\rightsquigarrow$ universal enveloping algebra)
- Arithmetic Geometry 范数 $\rightsquigarrow$ p进分析
  - · hermitian line bundle over projective arithmetic variety $\mathcal{X}$
  - · adelic line bundle over essentially quasi-proj scheme
  - · over Berkovich analytic space X^{an}
  - · over formal scheme $\mathrm{Spf}\,A$
  - · over rigid-analytic space K-affinoid space
  - · over adic space $\mathrm{Spa}(A,A^{+})$

Picture:

$$\begin{array}{cccc} \text{v.b.} & \text{sheaf} & \text{mod} & \text{rep} \\ | & | & | & | \\ \text{mfld} & \text{scheme} & \text{ring} & \text{alg} \end{array}$$

① variation (e.g. v.b → f.b, mfld → CW cplx, sheaf → fctor, scheme → stack/adic space, ...)

② vertical relation: ↓: v.b as mfld, representable fct, Spec/Proj construction, ...

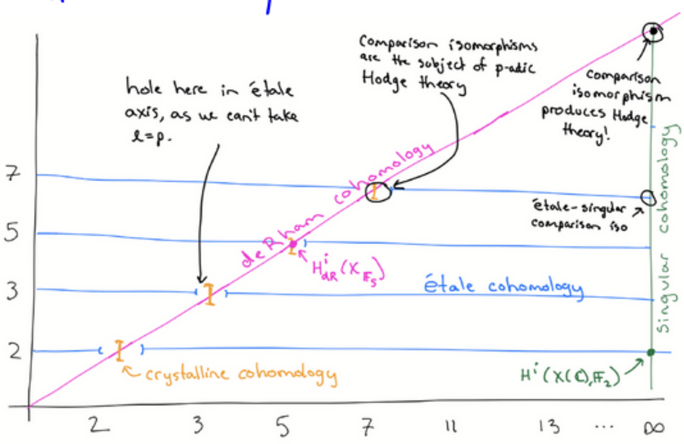
↑: tangent/trivial v.b, structure sheaf, R as R-mod, regular rep, ...

③ horizontal relation:

$$\begin{tikzcd} \text{v.b.} \arrow[r, shift left, "\Gamma(-,\mathcal{U})"'] \arrow[d, no head] & \text{sheaf} \arrow[l, dashed, shift left, "\text{Spec}"'] \arrow[d, no head] & \text{mod} \arrow[l, "M \mapsto \widetilde{M}"'] \arrow[d, no head] & \text{rep} \arrow[l] \arrow[d, no head] \\ \text{mfld} \arrow[r, dashed, shift right, "\text{model}"'] & \text{scheme} \arrow[l, shift right, "\otimes_{\mathbb{Z}}\mathbb{C}"'] & \text{ring} \arrow[l, dashed, "\text{Spec}"'] & \text{alg} \arrow[l] \end{tikzcd}$$

④ homology and cohomology:

$$\begin{array}{cccccccc} \text{v.b.} & & \text{sheaf} & \text{Sheaf, Čech} & \text{mod} & \text{Ext}^n,\ \text{Tor}^n,\ \text{Hochchild} & \text{rep} & \begin{cases}\text{gp/Galois} \\ \text{Lie-alg}\end{cases} \\ | & & | & & | & & | & \\ \text{mfld} & \begin{cases} \text{singular} \\ \text{simplicial} \\ \text{cellular} \\ \text{Borel-Moore} \\ \text{de Rham} \\ \text{cpt supported} \end{cases} & \text{scheme} & & \text{ring} & & \text{alg} & \end{array}$$



when axes meet: comparison isomorphisms (the "glue" of the "sheaf")

Prof. Scholze's ICM picture

Remaining (co)homology:

l-adic cohomology
intersection (co)homology
elliptic cohomology
flat cohomology
infinitesimal cohomology

Objects in upper row can be already viewed as element in (co)homology.
eg. v.b. $\leftrightarrow$ transition fct $\leftrightarrow H^1(X, -)$

$$\begin{array}{cccc}
\text{v.b.}\ \begin{smallmatrix}\text{differential form}\\ \text{metrics}\\ \text{curvature}\end{smallmatrix} & \text{sheaf} & \text{mod}\ ^{\text{Ext}\ \ \text{Tor}} & \text{rep}\ ^{L\text{-fct}} \\
| & | & | & | \\
\text{mfld}\ \begin{smallmatrix}\text{measure}\\ \text{submfld}\\ [-, K(n,A)]\end{smallmatrix} & \text{scheme}\ \begin{smallmatrix}\text{divisor}\\ \text{subvariety}\end{smallmatrix} & \text{ring} & \text{alg}
\end{array}$$

⑤ Chern class : from cohomology to cohomology (also for the other Char class)
There are several ways of defining/viewing Chern class.

i) $\mathcal{L} \in \mathrm{Pic}_{\mathbb{C}}(X) \longmapsto c_1(\mathcal{L}) \in H^2(X;\mathbb{Z})$

ii) $H^1(X, \mathcal{O}_X^\times) \longrightarrow H^2(X;\mathbb{Z})$ by LES

iii) As the coefficient of equation ($CH^*(\mathbb{P}E)$ is a free $CH^*(B)$-module)
Euler class

iv) As the pull back of the universal Chern class in Grassmannian

v) From curvature ; Chern-Weil theory

vi) From Chow group

Goal
- structures & invariants
- classifications of
  special v.b, mfld, sub v.b, submfld
- symmetry & quotient
- special functors
- homological algebra, derived version

Today we will focus on the comparison between v.b. and local system.

1. classifications of real/cplx v.b. on S^n.

(by homotopy group! $\rightsquigarrow$ generalized Picard group?)

Q: Is this group structure natural?

ref: https://math.stackexchange.com/questions/1923402/understanding-vector-bundles-over-spheres

Thm. $\{\text{rank } m \ K\text{-v.b. over } S^n\} \longleftrightarrow \pi_{n-1}(GL_m(K))$ $\quad K=\mathbb{R},\mathbb{C}$

$\pi_{n-1}(GL_m(\mathbb{R}))$ (n \ rank m)	1	2	3	4	5	6	>6
S^1 1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^2 2	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	0	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$	$\mathbb{Z}/2\mathbb{Z}$	0	0
S^6 6	0	0	$\mathbb{Z}/12\mathbb{Z}$	$(\mathbb{Z}/12\mathbb{Z})^{\oplus 2}$	0	0	0

$\mathbb{RP}^\infty \cong K(\mathbb{Z}/2\mathbb{Z},1)$

$\pi_{n-1}(GL_m(\mathbb{C}))$ (n \ rank m)	1	2	3	4	5	6	>6
S^1 1	0	0	0	0	0	0	0
S^2 2	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^3 3	0	0	0	0	0	0	0
S^4 4	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
S^5 5	0	$\mathbb{Z}/2\mathbb{Z}$	0	0	0	0	0
S^6 6	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

$\mathbb{CP}^\infty \cong K(\mathbb{Z},2)$

Problems. Describe the special bundles, e.g. TS^n

Describe the operations, e.g. dual, $\oplus, \otimes$, Λ^k, Sym^k, Res, Ind

For the other spaces:
https://math.stackexchange.com/questions/383838/classifying-vector-bundles
http://www.ms.uky.edu/~guillou/F18/751Notes.pdf
It's still not so explicit.

$\{\text{rank } m \ K\text{-v.b. over } M\} \longleftrightarrow [M, Gr_K(m,\infty)]$ $\quad K=\mathbb{R},\mathbb{C}$

M: paracompact