

# Eine Woche, ein Beispiel

## 11.19. Basic sheaf calculation

Goal: Motivate  $f_*$ ,  $f^*$ ,  $f_!$ ,  $f'$ , by connecting them with (co)homology theory

- After story:
- ~> calculation of  $\text{Perv}_{\Delta}(\text{CIP}')$
  - ~> generalize Morse theory
  - ~> Characteristic classes / cycles
  - ~> index theorem

Minor advantages from my talk:

- offers examples for derived category.  
(more geometrical compared with examples about quiver reps)
- the first step toward 6-factor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)  
~> understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...  
covering: (étale) descent, ramification, ...

Three types: closed immersion, submersion, covering.

Usual setting:  $X \in \text{Top}$

$\text{Ob}(\text{Sh}(X)) = \{\text{sheaves of abelian qps}\}$

e.g.  $\text{Sh}(f_*) = \text{Abel}$

$\mathbb{Q}_{\text{perf}} \longleftrightarrow \mathbb{Q}$

0. sheaf

- 1.  $f_*$ , skyscraper sheaf & global sections
- 2.  $f^*$ , constant sheaf & stalks
- 3.  $Rf_*$  & cohomology
- 4.  $f_!$  & global sections with cpt supp
- 5.  $Rf_!$  & cohomology with cpt supp
- 6.  $f'$  & homology
- $\otimes$  - & product structure on cohomology
- $\text{Hom}(-, -)$  & Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016

[IHPS] Laurențiu G. Maxim, Intersection Homology & Perverse Sheaves with Applications to Singularities, 2019

## 0. Sheaf

Recall the definition of

- |                  |   |
|------------------|---|
| • presheaf       | $\mathcal{F}$                                     |
| • sheaf          | $\mathcal{F}$                                     |
| • stalk          | $\mathcal{F}_x$                                   |
| • global section | $\Gamma(X; \mathcal{F}) = H^0(X; \mathcal{F})$    |
| • cohomology     | $R^n\Gamma(X; \mathcal{F}) = H^n(X; \mathcal{F})$ |

<https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology>  
If  $X$  is paracompact and Hausdorff, Čech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

<https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology>

<https://math.stackexchange.com/questions/3305512/cech-cohomology-and-the-simplicial-cohomology-of-the-nerve-of-an-open-cover>

Recall examples of sheaves:

- complicated
- |   |
|---|
| • $\mathcal{E}_X$ : sheaf of cont fcts on $X$                                   |
| • $\mathcal{O}_X$ : structure sheaf on $X$ e.g., $X$ : (cplx) mfld, scheme, ... |
| • $\underline{\mathbb{Q}}_X$ : constant sheaf on $X$                            |
| • $\text{sky}_p(\mathbb{Q})$ : skyscraper sheaf of $p \in X$ on $X$ .           |

Ex. For  $X = \mathbb{C}$  as cplx mfld,  $x=0$ , compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{E}_X)_x \quad \& \quad (\text{sky}_p(\mathbb{Q}))_x.$$

1.  $f_*$ , skyscraper sheaf & global sections

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

Def.  $f_* \mathcal{F} \in \text{Sh}(X)$  is given by

$$f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

This defines a factor

$$f_*: \text{Sh}(Y) \longrightarrow \text{Sh}(X)$$

$$\begin{array}{ccc} \mathcal{F} & & f_* \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ | & & | \\ U & & U \end{array}$$

E.g. For  $p \in X$ ,  $i_p: \{p\} \hookrightarrow X$ ,  $i_{p*} \underline{\mathbb{Q}}_{\{p\}} = \text{sky}_p(\mathbb{Q})$   
 For  $\pi: Y \rightarrow \{*\}$ ,  $\pi_* \mathcal{F} = \mathcal{F}(Y) = \Gamma(Y; \mathcal{F})$

Ex (hard?)

For  $j: \mathbb{C} \rightarrow \mathbb{CP}^1$ , compute  $j_* \underline{\mathbb{Q}}_{\mathbb{C}}$ .

- It is a constant sheaf on  $\mathbb{CP}^1$ .
- It is not a constant sheaf on  $\mathbb{CP}^1$ , and  $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$ .
- It is not a constant sheaf on  $\mathbb{CP}^1$ , and  $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$ .
- All the above is wrong.
- I don't know, but I don't want to make a wrong choice.

2.  $f^*$ , constant sheaf & stalks

In [Vakil, Chapter 2], it is  $f^{-1}$ , the inverse image functor.

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(X)$ ,  $f: Y \rightarrow X$  cont

Def.  $f^*\mathcal{F} \in \text{Sh}(Y)$  is given by sheafification of

$$f^{*,\text{pre}}\mathcal{F}(U) = \varinjlim_{f(u) \in V} \mathcal{F}(V)$$

This defines a factor

$$f^*: \text{Sh}(X) \longrightarrow \text{Sh}(Y)$$

$$\begin{array}{ccc} f^*\mathcal{F} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ \cup & & \cup \\ U & & \mathcal{U} \end{array}$$

Recall:

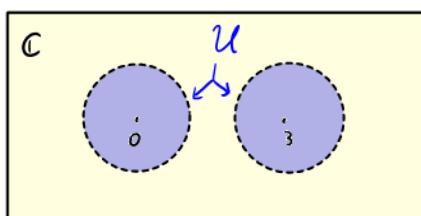
$$\mathcal{F}^{\text{sh}}(U) = \left\{ (x_p)_p \in \prod_{p \in U} \mathcal{F}_p \mid \begin{array}{l} \forall x_0 \in U, \exists U_{x_0} \subseteq U \text{ nbhd of } x_0, \\ s \in \mathcal{F}(U) \text{ st. } \\ s_p = x_p \quad \forall p \in U_{x_0} \end{array} \right\}$$

By definition,  $(\mathcal{F}^{\text{sh}})_p = \mathcal{F}_p$ .

Universal property:

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{fcMor}_{\text{PSH}}} & \mathcal{G} \\ \text{sh} \downarrow & \text{G} \nearrow & \text{G: sheaf} \\ \mathcal{F}^{\text{sh}} & \dashv \exists! f^{\text{sh}} \in \text{Mor}_{\text{Sh}} & \end{array}$$

Ex. For  $\pi: \mathbb{C} \rightarrow \{\ast\}$ ,  $U = B_1(0) \cup B_1(3)$ , which one is correct:



$$\downarrow \pi$$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2.$

$(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2, \quad (\pi^*\underline{\mathbb{Q}}_{\{\ast\}})(U) = \mathbb{Q}^2.$

All the above is wrong.

E.g. For  $p \in X$ ,  $\{p\} \hookrightarrow X$ ,  $\{p\}^* F = F_p$   
 For  $\pi: Y \rightarrow \{\ast\}$ ,  $\pi^* \underline{Q}_{\{\ast\}} = Q_Y$   
 For  $U \subset X$  open,  $j: U \hookrightarrow X$ ,  $j^* F = F_U$

People generalize the last notation to arbitrary subset:  
 For  $Y \subset X$ ,  $\downarrow Y: Y \hookrightarrow X$ ,  $\downarrow Y^* F = F_Y$

Q: For  $U \subset X$  open, how to express  $\mathcal{F}(U)$  by factors?

A:

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\text{inj}} & X \\ \pi_{\mathcal{U}} \downarrow & \swarrow \pi_X & \\ \{*\} & & \mathcal{F}(\mathcal{U}) = \pi_{\mathcal{U}, *} \underbrace{\iota^* \mathcal{F}}_{\mathcal{F}_{\mathcal{U}}} \end{array}$$

Prop. One has the adjunction  $f^* \dashv f_*$ , i.e.,

$$\begin{array}{ccc} G & & F \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

$$\text{Mor}_{\text{Sh}(Y)}(f^*F, G) \cong \text{Mor}_{\text{Sh}(X)}(F, f_*G) \quad + \text{naturality}$$

Hint. [Vakil, 2.7.B] Show that both sides give the same information, i.e.,

$$\phi_{uv} \in \text{Mor}_{\text{Ab}}(F(U), G(V)) \quad \begin{matrix} \text{for each pair } (V, U) \\ \text{s.t. } f(V) \subset U \\ + \text{compatibility} \end{matrix}$$

Cor.  $f^*$  is right adjoint,  $f_*$  is left adjoint.

Rmk.  $f^*$  is an exact functor.

Hint: exactness can be checked on stalks!

⚠ After "polished" (because of the structure sheaf),  $f^*$  is again only right adjoint.

Application.

$$\text{Hom}_{\text{Sh}(X; \mathbb{Q})}(\underline{\mathbb{Q}}_X, F) \cong \Gamma(X; F)$$

$$\text{Hom}_{\text{Sh}(X; \mathbb{Q})}(F, l_{p,*}\mathbb{Q}) \cong \text{Hom}_{\mathbb{Q}\text{-v.s.}}(F_p, \mathbb{Q}) \cong F_p^*$$

$$\text{Hom}_{\text{Sh}(U; \mathbb{Q})}(F|_U, G) \cong \text{Hom}_{\text{Sh}(X; \mathbb{Q})}(F, l_{U,*}G)$$

not f faithful

where  $\text{Sh}(X; \mathbb{Q}) \subset \text{Sh}(X)$ : sheaves of  $\mathbb{Q}$ -v.s.

$$\begin{array}{ccc} G & & F \\ | & & | \\ U & \xrightarrow{l_U} & X \end{array}$$

### 3. $Rf_*$ & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input  $\mathcal{F}$ , output a complex (before Ker/Im procedure)
- SES induces LES: for

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

one has

$$\rightarrow H^2(X; \mathcal{F}) \longrightarrow \dots$$

$$\rightarrow H^1(X; \mathcal{F}) \longrightarrow H^1(X; \mathcal{G}) \longrightarrow H^1(X; \mathcal{H}) \rightarrow$$

$$0 \rightarrow H^0(X; \mathcal{F}) \longrightarrow H^0(X; \mathcal{G}) \longrightarrow H^0(X; \mathcal{H}) \rightarrow$$

$$\pi''_{*\mathcal{F}}$$

$$\pi''_{*\mathcal{G}}$$

$$\pi''_{*\mathcal{H}}$$

$$\pi: X \rightarrow \{*\}$$

- can be viewed as right derived factor of

$$H^0(X, -) = \Gamma(X, -) = \pi_*$$

one gets

$$H^n(X, -) = R^n \Gamma(X, -) = R^n \pi_*$$

We denote the complex (before the Ker/Im procedure) as

$$R\Gamma(X, -) = R\pi_*$$

up to homotopy equiv & quasi-iso, i.e., in the derived category of  $\{*\}$ .

$$\begin{aligned} \mathcal{D}(X) = \mathcal{D}(\text{Sh}(X)) &= \text{"derived category of sheaves over } X \text{"} \\ &= \text{"complexes of sheaves over } X, \text{ up to ...}" \\ &= \{ \dots \rightarrow \mathcal{F}^{-2} \rightarrow \mathcal{F}^{-1} \rightarrow \mathcal{F}^0 \rightarrow \mathcal{F}^1 \rightarrow \dots \} \hat{=} \{\mathcal{F}^i\} \end{aligned}$$

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

Def.  $Rf_* \mathcal{F}$  = "derived pushforward of  $\mathcal{F}$ "  
 $= f_* \mathcal{I}'$

[Here,  $\mathcal{I}'$  is the injective resolution of  $\mathcal{F}$ .  
 $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}' \rightarrow \mathcal{I}^2 \rightarrow \dots$   
 $(\Rightarrow \mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}')$ ]

$$\begin{array}{ccc} \mathcal{F} & Rf_* \mathcal{F} \\ | & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & U \end{array}$$

This defines a functor

$$Rf_*: \mathcal{D}^+(Y) \rightarrow \mathcal{D}^+(X)$$

The derived pushforward is hard to compute.

just like cohomology, and even worse, since we need more information  
 Luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]

$$R^n f_* \mathcal{F} \text{ is given by the sheafification of } (R^n f_*^{\text{pre}} \mathcal{F})(U) = H^n(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

$\uparrow$  sometimes omit

e.p. one can compute the stalk

$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in U} H^n(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)})$$

$\mathcal{F}$

|

Cor For  $\pi: X \rightarrow \mathbb{P}^*$ ,

$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For  $\pi: \mathbb{CP}^1 \rightarrow \mathbb{P}^1$ ,

$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in  $\mathcal{D}^*(*)$  are proj, we work over  $\mathbb{Q}$ ]

$$R \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= \left[ \underset{-1}{0} \rightarrow \dots \rightarrow \underset{0}{\mathbb{Q}} \rightarrow \underset{1}{0} \rightarrow \underset{2}{\mathbb{Q}} \rightarrow \underset{3}{0} \rightarrow \underset{4}{\dots} \right]$$

*Ex.* For  $j : \mathbb{C} \rightarrow \mathbb{CP}^1$ , what is true about  $Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}$ ?

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$

$(R^1 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \quad (R^2 j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$

What the hell is that?

In fact,  $(Rj_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1]$ .

$i : \mathbb{P}^1 \rightarrow \mathbb{CP}^1$  is exact, so  $Ri_* = i_*$ .

Upgrade formulas to derived version  
 $f^* g_! \cong g_! f'^*$   $\xrightarrow{f^*, f'^* \text{ exact}}$   $f^* Rg_! \cong Rg'_! f'^*$

$$\begin{aligned} \text{Hom}(f^* \mathcal{F}, \mathcal{G}) &\cong \text{Hom}(\mathcal{F}, f_* \mathcal{G}) \\ \rightsquigarrow \text{Hom}(f^* \mathcal{F}, \mathcal{G}) &\cong \text{Hom}(f^* \mathcal{F}, \mathcal{I}) \\ &\cong \text{Hom}(\mathcal{F}, f_* \mathcal{I}) \\ &\cong \text{Hom}(\mathcal{F}, Rf_* \mathcal{G}) \end{aligned}$$

Is this argument correct?

4.  $f_!$ , extension by zeros & global sections with cpt supp

$$\begin{array}{ccc} \mathcal{F} & & f_*\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \downarrow \\ & & U \\ & & \downarrow \\ & & \mathcal{U} \end{array}$$

Setting  $X, Y \in \text{Top}$ ,  $\mathcal{F} \in \text{Sh}(Y)$ ,  $f: Y \rightarrow X$  cont

Def.  $f_!\mathcal{F} \in \text{Sh}(X)$  is given by

$$f_!\mathcal{F}(U) = \left\{ s \in \mathcal{F}(f^{-1}(U)) \mid \begin{array}{l} f|_{\text{supp}(s)}: \text{supp}(s) \rightarrow U \text{ is proper} \\ (f_*\mathcal{F})(U) \end{array} \right\}$$

This defines a factor

$$f_!: \text{Sh}(Y) \rightarrow \text{Sh}(X)$$

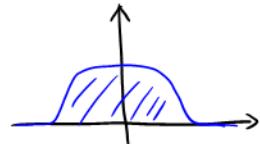
$$\text{Recall: } \text{supp}(s) = \overline{\{x \in f^{-1}(U) \mid s_x \neq 0\}}$$

proper: preimage of cpt set is cpt.

Rmk. By def.  $(f_!\mathcal{F})(U) \subseteq (f_*\mathcal{F})(U)$ , one has natural transformation  $f_! \rightarrow f_*$ .  
When  $f$  is proper,  $f_! = f_*$ .

E.g. For  $p \in X$ ,  $\iota_p: \{p\} \hookrightarrow X$ ,  $\iota_{p,!} \underline{\mathbb{Q}}_{\{p\}} = \iota_{p,*} \underline{\mathbb{Q}}_{\{p\}} = \text{sky}_p(\mathbb{Q})$   
For  $\pi: Y \rightarrow \{*\}$ ,  $\pi_* \mathcal{F} = \Gamma_c(Y, \mathcal{F}) = H^0_c(Y, \mathcal{F})$

cpt supp facts on  $Y$



Ex.

Do you know what is  $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}})$  and  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1})$ ?

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$ .

$\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$ ,  $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$ .

Could you explain the notation again?

E.g. 4.3. For  $U \xrightarrow{j} X$  open,  $j_! F$  is the classical "extension by zero":

$$(j_! F)^{\text{pre}}(V) = \begin{cases} F(U) & V \subseteq U \\ 0 & \text{otherwise} \end{cases}$$

e.p.  $(j_! F)_p = \begin{cases} F_p & p \in U \\ 0 & p \notin U \end{cases}$

In general, [IHPS, p82]

$$(f_! F)_p = \Gamma_c(f^{-1}(p); F|_{f^{-1}(p)})$$

This comes from the proper base change formula:

$$l_p^* f_! F \cong \pi_* \tilde{l}_p^* F$$

Prove it?

$$\begin{array}{ccc} f^{-1}(p) & \xrightarrow{\tilde{l}_p} & Y^F \\ \pi \downarrow & & \downarrow f \\ \{p\} & \xrightarrow{l_p} & X \end{array}$$

Rmk. In E.g. 4.3,  $j_!$  is exact. (Check the stalks!)

In general,  $f_!$  is only left adjoint.

e.p. when  $f: Y \rightarrow X$  is proper, then  $f_! = f_*$  is usually not right adjoint.  
 Notice that  $Rf_! \dashv f^!$ , and we don't have  $f_! \dashv f^!$ .

<https://math.stackexchange.com/questions/3132036/direct-image-functor-f-left-exact>  
 the same method here argues why  $f_!$  is left exact.

Sidemark:

<https://math.stackexchange.com/questions/4671873/compare-two-definition-of-rf-derived-pushforward-with-proper-support>  
 it gives another definition of  $f_!$  in étale case.

## 5. $Rf_!$ & cohomology with cpt supp

Just like  $Rf_*$ , we derive the factor

$$H_c^0(X, -) = \Gamma_c(X, -) = \pi_!$$

to get

$$H_c^n(X, -) = R^n \Gamma_c(X, -) = R^n \pi_!$$

$X$   
 $\downarrow \pi$   
 $\{*\}$

Def.  $Rf_! \mathcal{F} =$  "derived proper pushforward of  $\mathcal{F}$ "

$$= f_! \mathcal{I}^*$$

Here,  $\mathcal{I}^*$  is the injective resolution of  $\mathcal{F}$ .

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow \mathcal{I}^2 \rightarrow \dots$$

(⇒  $\mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}^*$ )

$$\begin{array}{ccc} \mathcal{F} & Rf_! \mathcal{F} \\ | & | \\ Y & \xrightarrow{f} & X \\ \cup & & \cup \\ U & & U \end{array}$$

This defines a factor

$$Rf_! : \mathcal{D}^+(Y) \longrightarrow \mathcal{D}^+(X)$$

$\mathcal{F}$

|

Cor For  $\pi: X \rightarrow \{*\}$ ,

$$R^n \pi_! \mathcal{F} = H_c^n(X; \mathcal{F})$$

E.g. For  $\pi: \mathbb{C}\mathbb{P}^1 \rightarrow \{*\}$ ,

$$R^n \pi_! \underline{\mathbb{Q}}_{\mathbb{C}\mathbb{P}^1} = H_c^n(\mathbb{C}\mathbb{P}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in  $\mathcal{D}(*)$  are proj, we work over  $\mathbb{Q}$ ]

$$R \pi_! \underline{\mathbb{Q}}_{\mathbb{C}\mathbb{P}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= \left[ \underset{-1}{0} \rightarrow \dots \rightarrow \underset{0}{\mathbb{Q}} \rightarrow \underset{1}{0} \rightarrow \underset{2}{\mathbb{Q}} \rightarrow \underset{3}{0} \rightarrow \underset{4}{\mathbb{Q}} \rightarrow \dots \right]$$

$\mathbb{C}\mathbb{P}^1 \rightsquigarrow \mathbb{C}$ , what would happen?

For  $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$ , what is true about  $Rj_! \underline{\mathbb{Q}}_{\mathbb{C}}$ ?

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0.$

$(R^0 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}, \quad (R^1 j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = \mathbb{Q}.$

This question is too easy for me. Ask more difficult questions next time!

In fact,  $j_!$  is exact, so  $(Rj_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = (j_! \underline{\mathbb{Q}}_{\mathbb{C}})_{\infty} = 0$ .

## 6. $f^!$ & homology

The correct title should actually be:  
 $f^!$ , orientation sheaf & costalks  
 homology is just some special (sheaf) cohomology.

Now we come to the hard part, since  $f^!$  can only be defined over the derived category. That's why there's no derived symbol for  $f^!$ .

"Def"  $f^!$ , if exists, should be given by  
 the right adjoint of  $Rf_!$ :

$$Rf_! \dashv f^!$$

Remember:

$$\begin{aligned} f^* &\dashv Rf_* \\ Rf_! &\dashv f^! \\ M^L \otimes - &\dashv R\text{Hom}(M, -) \end{aligned}$$

Hard exercise: for  $\pi: \mathbb{R}^n \rightarrow \{\ast\}$ , shows that  
 $\pi^! \underline{\mathbb{Q}} \cong \underline{\mathbb{Q}}_{\mathbb{R}^n}[n]$

by using the def of  $\pi^!$ .

Black box: (reduced from the Hard exercise)

For  $M$  n-dim mfld (without boundary),  $\pi: M \rightarrow \{\ast\}$ ,

$$\pi^! \underline{\mathbb{Q}} \cong \mathcal{O}_M[n] \quad \mathcal{O}_M: \underline{\mathbb{Q}}\text{-orientation sheaf of } M$$

e.g. when  $M$  is oriented, fixing an orientation  $\mathcal{O}_M \cong \underline{\mathbb{Q}}_M$ , one gets  
 $\pi^! \underline{\mathbb{Q}} \cong \underline{\mathbb{Q}}_M[n]$ .

What does the adjunction tell us?

<https://mathoverflow.net/questions/404706/how-duality-follows-from-a-six-functor-formalism>

Poincaré duality, version 1

$$\mathrm{Hom}_{\mathcal{D}^b(Y, \mathbb{Q})}(Rf_! \mathcal{F}, G) \cong \mathrm{Hom}_{\mathcal{D}^b(X, \mathbb{Q})}(\mathcal{F}, f^! G)$$

$$\begin{array}{ccc} \mathcal{F} & & G \\ | & & | \\ X & \xrightarrow{f} & Y \end{array}$$

Now, take  $Y = \{\ast\}$ ,  $G = \mathbb{Q}$ ,  $X$ : n-dim mfld, one gets

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Q}\text{-vs.}}(R\Gamma_c(X; \mathcal{F}), \mathbb{Q}) &\cong \mathrm{Hom}_{\mathcal{D}^b(X, \mathbb{Q})}(\mathcal{F}, \mathcal{O}_X[n]) \\ R\Gamma_c(X; \mathcal{F})^* &\cong \mathrm{Hom}_{\mathrm{Sh}(X, \mathbb{Q})}(\mathcal{F}, \mathcal{O}_X)[n] \end{aligned}$$

Take  $\mathcal{F} = \underline{\mathbb{Q}}_X$ , one gets

$$R\Gamma_c(X; \mathbb{Q}) \cong R\Gamma(X; \mathcal{O}_X)[n]$$

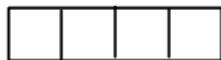
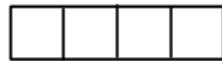
Take  $(-i)$ 's cohomology, one gets

$$H_c^i(X; \mathbb{Q}) \cong H^{n-i}(X; \mathcal{O}_X)$$

E.g.  $n=3$   $i=1$   $X=S^2 \times \mathbb{R}$



$$R\Gamma_c(X; \mathcal{O}_X)^*$$



[https://en.wikipedia.org/wiki/Borel-Moore\\_homology](https://en.wikipedia.org/wiki/Borel-Moore_homology)  
<https://mathoverflow.net/questions/249342/two-points-of-view-about-borel-moore-homology>