

Eine Woche, ein Beispiel

7.11 Universal properties

Here we conclude some processes of understanding.

- | | | |
|-------------|---|---|
| obvious | [| ① Commutative diagram |
| | | ② Explicitly description |
| | | ③ Initial object or Final object |
| | | ④ Adjoint functor or Representable functors |
| non-trivial | [| ⑤ Geometry |
| | | ⑥ Functor & Derived functor |
| | | ⑦ Other properties & Corollary |

E.g. tensor product

① $M \times N \xrightarrow{A\text{-bilinear}} M \otimes_A N \in \text{Mod}(A)$
 $\searrow A\text{-bilinear} \quad \downarrow \exists! A\text{-mod morphism}$
 $T \in \text{Mod}(A)$

we have $\varphi \otimes \psi$ s.t.
 for any $\varphi \otimes \psi$,
 $\exists! \varphi \otimes \psi$.

② $M \otimes_A N = \bigoplus_{(m,n) \in M \times N} A / \sim$

e.g. when $A=k$ is a field, then $M \otimes_k N$ has a basis.

③ Initial object of $\{\alpha: M \times N \rightarrow T \mid T \in \text{Mod}(A), \alpha: A\text{-bilinear}\}$

④ $\text{Mor}_{A\text{-bil}}(M \times N, T) \cong \text{Hom}_{A\text{-mod}}(M \otimes_A N, T)$
 $\Rightarrow \text{Mor}_{A\text{-bil}}(M \times N, -)$ is represented by $M \otimes_A N$.

⑤

$M, N \in \text{Mod}(A)$: coherent sheaf of rank m \otimes coherent sheaf of rank n $\stackrel{\text{is}}{=}$ coherent sheaf of rank mn

e.g. \otimes is the multiplication in $\text{Pic}(X)$

$N: A\text{-alg}, M \in \text{Mod}(A)$: $- \otimes_A N$ would be pullback of sheaf.

e.g. when $N = A_f$, $\pi^* \mathcal{F} = \mathcal{F}|_{\text{Spec } A_f}$

when $N = A/\mathfrak{m}$, $\pi^* \mathcal{F} = \mathcal{F}_{\mathfrak{m}}$

$M, N: A\text{-alg}$: Fiber product [Vakil chap 9]

⑥ $- \otimes -$ is the $(A\text{-Mod}, A\text{-Mod})$ -bifunctor
 $- \otimes_A N \dashv \text{Hom}_A(N, -)$

$\Rightarrow - \otimes_A N$ is right exact

$\Rightarrow - \otimes_A N$ has left derived functor $\text{Tor}_i^A(-, N)$

⑦ Flat module, ...

e.g. Kähler differentials

k : comm ring with unit 1

A, k -alg

$$\textcircled{1} \quad A \xrightarrow{k\text{-derivation}} \Omega_{A/k} \in \text{Mor}(A)$$

$$\begin{array}{c} \downarrow k\text{-derivation} \\ \mathcal{M} \in \text{Mor}(A) \end{array} \quad \begin{array}{c} \downarrow \exists! A\text{-mod morphism} \end{array}$$

Hochschild cohomology

where $I := \ker(A \otimes_k A \rightarrow A)$

$$\textcircled{2} \quad \Omega_{A/k} = \bigoplus_{a \in A} A da / \sim \cong I/I^2 \cong I \otimes_{A \otimes_k A} A$$

$$\Rightarrow \Omega_{X/k} = \Delta^*(I)$$

Proof. If $A = k[x_i]/(f_j)$, then

$$\begin{aligned} A \otimes_k A &= k[x_i, y_i]/(f_j(x), f_j(y)) \\ &= k[x_i, \Delta_i]/(f_j(x), f_j(x+\Delta)) \quad \Delta_i = y_i - x_i \\ &\cong A[\Delta_i]/(f_j(x+\Delta)) \\ I &= \langle \Delta_i \rangle \quad I^2 = \langle \Delta_i \Delta_j \rangle_{i,j} \\ A \otimes_k A / I^2 &\cong A[\Delta_i]/(\sum_i \frac{\partial f_j}{\partial x_i} \Delta_i)_j \\ I/I^2 &\cong \langle \Delta_i \rangle / (\sum_i \frac{\partial f_j}{\partial x_i} \Delta_i) \cong \Omega_{A/k} \end{aligned}$$

$\textcircled{3} \quad A \rightarrow \Omega_{A/k}$ is the initial object of category $\{d: A \rightarrow \mathcal{M} \text{ is the derivation}\}$

$$\textcircled{4} \quad \text{Der}_k(A, \mathcal{M}) = \text{Mor}_{A\text{-mod}}(\Omega_{A/k}, \mathcal{M})$$

$\Rightarrow \text{Der}_k(A, -)$ is represented by $\Omega_{A/k}$. $\Rightarrow \text{Der}_k(A, A)$ can be viewed as tangent bundle

$\textcircled{5} \quad \Omega_{A/k}$ is a sheaf on $\text{Spec } A$. It represents the cotangent bundle of $\text{Spec } A$.

$$\textcircled{6} \quad A \leftarrow B \leftarrow k$$

$$\leadsto i^* \Omega_{B/k} \rightarrow \Omega_{A/k} \rightarrow \Omega_{A/B} \rightarrow 0$$

"contravariant functor $\Omega_{-/k}$ "

$\textcircled{7} \quad \text{Let } k \text{ comm alg with unit } S, S', R: k\text{-alg}$

$\mathcal{M}: S\text{-mod}, \mathcal{M}^2 = 0 \text{ (in } S')$

Prop 1 $0 \rightarrow \mathcal{M} \rightarrow S' \rightarrow S \rightarrow 0$

$$\begin{array}{ccc} \text{Spec } S' & \xrightarrow{\alpha} & \text{Spec } R \\ \uparrow & \searrow & \downarrow \\ \text{Spec } S & \xrightarrow{\quad} & \text{Spec } R \\ & \searrow & \downarrow \\ & & \text{Spec } k \end{array}$$

$$\begin{array}{ccc} \text{Spec } S & \rightarrow & \text{Spec } R \\ \downarrow & \searrow \alpha & \downarrow \\ \text{Spec } S' & \rightarrow & \text{Spec } k \end{array}$$

then the lift α is \emptyset or a torsor under $\text{Der}_k(R, \mathcal{M}) = \text{Mor}_{R\text{-mod}}(\Omega_{R/k}, \mathcal{M})$

Prop 2 set $S' = S[M]$, $\gamma \in \text{Hom}_{k\text{-alg}}(R, S)$, then

$$\text{Hom}_{S\text{-alg}}(R, S[M]) = \text{Der}_k(R, \mathcal{M}_\gamma)$$

e.p. let $S = R$, then

$$\text{Hom}_{R\text{-alg}}(R, R[M]) = \text{Der}_k(R, \mathcal{M})$$

$$\begin{array}{ccc} \text{Spec } R & = & \text{Spec } R \\ \downarrow & \searrow \alpha & \downarrow \\ \text{Spec } R[M] & \rightarrow & \text{Spec } k \end{array}$$

Cor $A \xrightarrow{k \text{ is field}} k$ formally $\xrightarrow[\textcircled{7} \text{ Prop 1}]{\textcircled{7} \text{ Prop 2}} \text{Der}_k(A, -) \equiv 0 \iff \Omega_{A/k} = 0 \xrightarrow[\textcircled{3}]{\textcircled{2} f. \text{ lift}} \Delta \text{ is open immersion}$

$A \downarrow_k$ étale