§ 1.1. Structure of finite / local / global field

Road map

	finite field		local field: Archi	local field: NA	global field	adèle
base field F	7 $\mathbb{F}_1$ (For $\mathbb{F}_{1^r}$)	1 $\mathbb{F}_p$	2 $\mathbb{R}$ or $\mathbb{C}$	3 $\mathbb{Q}_p$ $\mathbb{F}_p((t))$	4 $\mathbb{Q}$ $\mathbb{F}_p(t)$	6 $\mathbb{A}_K$
$F^\times$	$\varepsilon\cdot\mu_r$	μ_{p-1}	$\mathbb{R}^+\times\mathbb{Z}/2\mathbb{Z}$ $\mathbb{C}^\times$	$\mathbb{Z}_p^\times\times\mathbb{Z}$ $\mathbb{F}_p[[t]]^\times\times\mathbb{Z}$	$\mathbb{Q}^\times$ $\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	—	—	—	$\mathbb{Z}_p$ $\mathbb{F}_p[[t]]$	$\mathbb{Z}$ $\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	—	—	—	$\mathbb{Z}_p^\times$ $\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$ $\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$	$\hat{\mathbb{Z}}$?	$\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$ Id	most known	dream	
ari Frob	?	can	total order? —	choose a lift	unramified $\begin{cases}\mathrm{Frob}_q\\ \mathrm{Frob}_p \text{ conj class}\end{cases}$ $\xrightarrow{\text{abelian}}$ Frob_p	
# ext of deg n	1?	1	1/0	finite	$n\neq1$ inf countable	
Spec $\mathcal{O}_F$	Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}},1)$ [étale, 2.2.4]		—			—
topology	?	discrete	Euclidean	profinite	—	restricted
topo of $\mathcal{O}_F$	—		—	cpt, not discrete	—	K is a lattice in $\mathbb{A}_K$
measure	?	discrete	Lebesgue	$\mu(\mathcal{O}_F)=1$	—	can be computed

5 Also, discuss

- field extension, norm, trace, …
- their connection to geometry, ramification theory
- analog with knot theory

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$Gal(\overline{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}}$ as top gps

$Frob_p \leftarrow 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $Gal(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $Gal(\mathbb{C}/\mathbb{C}) = Id$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?
  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions.

- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $Fr\, \tau\, Fr^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions: (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep}/\mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right)},\quad \underset{I_{\mathbb{Q}_p}}{\mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}_p}/\widehat{\mathbb{Q}_p^{ur}}\right)},\quad \underset{G_{\mathbb{F}_p((t))}}{\mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}/\mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)}$$

4. global field

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.

$\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D], we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\}/\text{birational}$$

$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler. by [GalFun, Thm 3.4.8],

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \hat{F}(\mathbb{C})$$

Free profinite gp on $\mathbb{C}$

Shafarevich's conj: See wiki: Absolute Galois group

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite gp

$\mathbb{Q}$: Does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object/geo realizations?

Dessin d'enfants

By [GalFun, Prop 4.7.1 - Rmk 4.7.9], we have an including

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\pi_1^{\acute{e}t}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}))$$

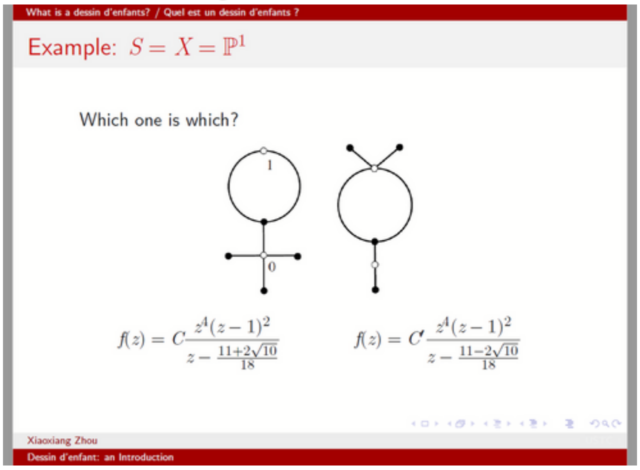
induced by $\pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}} := \pi_1^{\acute{e}t}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, $\pi^{\acute{e}t}_{1,\mathbb{Q}} := \pi_1^{\acute{e}t}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$

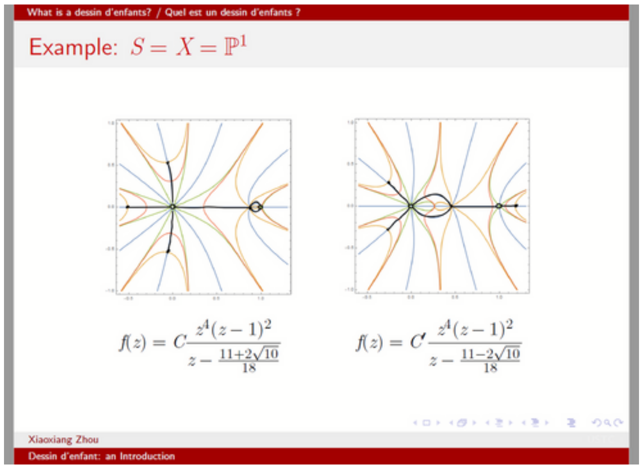
$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[d, equal, "?"'] & \pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}} \arrow[r] \arrow[d, equal] & \pi^{\acute{e}t}_{1,\mathbb{Q}} \arrow[r] \arrow[d, "\text{conj}", "g \mapsto g \cdot g^{-1}"'] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & Z(\pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}}) \arrow[r] & \pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}} \arrow[r] & \mathrm{Aut}(\pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}}) \arrow[r] & \mathrm{Out}(\pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}}) \arrow[r] & 1
\end{tikzcd}$$

The space $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ is designed for guaranteeing that
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\pi^{\acute{e}t}_{1,\overline{\mathbb{Q}}})$
is inclusion.

Task. Read [Dessin d'enfant] or [Collins],
understand the $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the dessin d'enfants.

- Def of Dessin d'enfact
- Connections with $Out(\pi_{1,\bar{\mathbb{Q}}}^{\acute{e}t})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.





- Can we generalize this to $Gal(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$?
  I don't know how to make a "dessin d'enfant" on alg curves over $\mathbb{F}_p$.

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.

5. local and global connections

Basics

$$\begin{tikzcd} K \arrow[r, hook] & K_v \\ \mathcal{O}_K \arrow[u, phantom, "\cup" rotate=90] \arrow[r, hook] \arrow[drr] & \mathcal{O}_v \arrow[u, phantom, "\cup" rotate=90] \arrow[dr] & \\ & & k_v \end{tikzcd}$$

for any valuation v $(\rightsquigarrow K \hookrightarrow \mathbb{A}_K)$

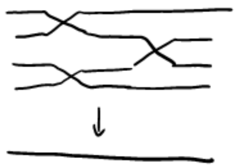
and $\mathcal{O}_v$ is the integral closure of $\mathcal{O}_K$ in K_v (for v fin)

and $\mathcal{O}_K/\mathfrak{p}^r\mathcal{O}_K \cong (\mathcal{O}_K/\mathfrak{p}^r\mathcal{O}_K)_v \cong \mathcal{O}_v/\mathfrak{p}^r\mathcal{O}_v \quad \forall r \in \mathbb{N}_{\geq 0}$

The connections are also compatible with field exts. (L/K Galois)

E.g.

$$\begin{array}{ccc} L & \mathcal{O}_L & \mathfrak{q} \sim w \\ | & \uparrow & | \\ K & \mathcal{O}_K & \mathfrak{p} \sim v \end{array}$$



$$\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_g^{e_g} \qquad\qquad L \otimes_K K_v \cong \prod_w L_w$$

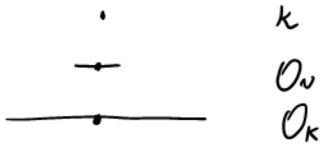
$$\begin{tikzcd} L \arrow[r, hook] \arrow[d, "N_{L/K}"'] & \prod_w L_w \arrow[d, "\prod_w N_{L_w/K_v}^{e_w}"] \\ K \arrow[r, hook] & K_v \end{tikzcd} \qquad \begin{tikzcd} L \arrow[r, hook] \arrow[d, "\mathrm{Tr}_{L/K}"'] & \prod_w L_w \arrow[d, "\sum_w e_w \mathrm{Tr}_{L_w/K_v}"] \\ K \arrow[r, hook] & K_v \end{tikzcd}$$

Traditional point of view

$k \longrightarrow$ pt

$\mathcal{O}_v \longrightarrow$ "loc" curve

$\mathcal{O}_K \longrightarrow$ curve



Task. Read [Algfungp, 0.2], answer the following questions:

- Understand ramified, inert, split.
- Understand (from geo meaning)
$$\mathcal{O}_L/\mathfrak{p}\mathcal{O}_L \cong \bigoplus_{i=1}^{g} \mathcal{O}_L/\mathfrak{q}_i^{e_i}\mathcal{O}_L$$
- Compute some classical examples
- Compare it with ramified covering in RS.

Frobenius (L/K Galois)

$$
\begin{tikzcd}
1 \arrow[r] & \mathrm{Gal}(L/K)_w \arrow[r, "\le"] \arrow[d, phantom, "\cong" rotate=90] & \mathrm{Gal}(L/K) & & \\
& \mathrm{Gal}(L_w/K_v) \arrow[d, phantom, "\cong" rotate=90] & & & \\
1 \arrow[r] & I(L_w/K_v) \arrow[r] & \mathrm{Aut}_{\mathcal{O}_v\text{-alg}}(\mathcal{O}_w) \arrow[r, two heads] & \mathrm{Gal}(k_w/k_v) \arrow[r] & 1
\end{tikzcd}
$$

where

$$\mathrm{Gal}(L/K)_w = \{\sigma \in \mathrm{Gal}(L/K) \mid \sigma(q) = q\} \le \mathrm{Gal}(L/K)$$

By https://math.stackexchange.com/questions/4131855/frobenius-elements,

more conditions		better props of Frob
① L/K is unramified at w	$\Rightarrow$	$I(L_w/K_v) = \mathrm{Id}$; $\mathrm{Fr}_w \in \mathrm{Gal}(L/K)_w$ well-defined
② L/K Galois	$\Rightarrow$	all symbols are meaningful; $\forall q, q' \in \mathrm{Spec}\, \mathcal{O}_L$, $\exists \sigma \in \mathrm{Gal}(L/K)$, $\sigma(q) = q'$; $\mathrm{Frob}_v = [\mathrm{Frob}_w]$ is a conj class in $\mathrm{Gal}(L/K)$
③ $L \subseteq K^{ab}$	$\Rightarrow$	$\mathrm{Frob}_v \in \mathrm{Gal}(L/K)$

Application: Quadratic reciprocity

Thm. p, l odd primes, $p \neq l$, then

$$\left(\frac{p}{l}\right)\left(\frac{l}{p}\right) = (-1)^{\frac{p-1}{2}\frac{l-1}{2}}$$

$$\left(\frac{-1}{l}\right) = (-1)^{\frac{l-1}{2}} = \begin{cases} 1 & l \equiv 1 \mod 4 \\ -1 & l \equiv 3 \mod 4 \end{cases}$$

$$\left(\frac{2}{l}\right) = (-1)^{\frac{l^2-1}{8}} = \begin{cases} 1 & l \equiv 1, 7 \mod 8 \\ -1 & l \equiv 3, 5 \mod 8 \end{cases}$$

Proof Assume $p \equiv 1 \bmod 4$, then $\mathbb{Q}(\sqrt{p}) \hookrightarrow \mathbb{Q}(\zeta_p)$

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\
\mathrm{Gal}(\mathbb{Q}(\sqrt{p})/\mathbb{Q}) \arrow[r, "\chi", "\sim"'] & \{\pm 1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\
\mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{p}{l}\right) \left(\frac{l}{p}\right)
\end{tikzcd}$$

$$\begin{aligned}
\chi(\mathrm{Frob}_l) = 1 &\iff \mathrm{Frob}_l(\sqrt{p}) = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l} \\
&\iff (\sqrt{p})^l = \sqrt{p} \quad \text{in } \overline{\mathbb{F}_l} \\
&\iff \sqrt{p} \in \mathbb{F}_l \\
&\iff \left(\frac{p}{l}\right) = 1
\end{aligned}$$

Assume $p \equiv 3 \bmod 4$, then $\mathbb{Q}(\sqrt{-p}) \hookrightarrow \mathbb{Q}(\zeta_p)$

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/p\mathbb{Z})^\times \arrow[d, two heads] \\
\mathrm{Gal}(\mathbb{Q}(\sqrt{-p})/\mathbb{Q}) \arrow[r, "\chi", "\sim"'] & \{\pm 1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\
\mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-p}{l}\right) \left(\frac{l}{p}\right)
\end{tikzcd}$$

$$\begin{aligned}
\chi(\mathrm{Frob}_l) = 1 &\iff \mathrm{Frob}_l(\sqrt{-p}) = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l} \\
&\iff (\sqrt{-p})^l = \sqrt{-p} \quad \text{in } \overline{\mathbb{F}_l} \\
&\iff \sqrt{-p} \in \mathbb{F}_l \\
&\iff \left(\frac{-p}{l}\right) = 1
\end{aligned}$$

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_4)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/4\mathbb{Z})^\times \arrow[d, two heads] \\
\mathrm{Gal}(\mathbb{Q}(\sqrt{-1})/\mathbb{Q}) \arrow[r, "\chi", "\sim"'] & \{\pm 1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\
\mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-1}{l}\right) (-1)^{\frac{l-1}{2}}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\
\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q}) \arrow[r, "\chi", "\sim"'] & \{\pm 1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\
\mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{2}{l}\right) (-1)^{\frac{l^2-1}{8}}
\end{tikzcd}$$

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\mathrm{Gal}(\mathbb{Q}(\zeta_8)/\mathbb{Q}) \arrow[r, "\cong"] \arrow[d] & (\mathbb{Z}/8\mathbb{Z})^\times \arrow[d, two heads] \\
\mathrm{Gal}(\mathbb{Q}(\sqrt{-2})/\mathbb{Q}) \arrow[r, "\chi", "\sim"'] & \{\pm 1\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Frob}_l \arrow[r, mapsto] \arrow[d, mapsto] & l \arrow[d, mapsto] \\
\mathrm{Frob}_l \arrow[r, mapsto] & \left(\frac{-2}{l}\right) (-1)^{\frac{(l+1)(l+3)}{8}}
\end{tikzcd}$$

$\mathbb{Q}$: Quadratic reciprocity for $\mathbb{F}_p(t)$?

Cubic reciprocity ?
[Cox x^2+ny^2, §4.A, 4.B]

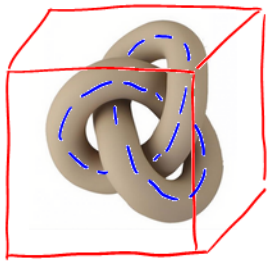
$\left\{\begin{array}{l}\text{primes over } \mathbb{Q}(\zeta_3) \\ \text{deg } 3, \text{ like } \mathrm{Gal}(K/\mathbb{Q}(\sqrt{-23})) : \text{More difficult}\end{array}\right.$

Étale point of view

$k \longrightarrow S^1$

$\mathcal{O}_v \longrightarrow$ tubular nbhd of S^1

$\mathcal{O}_K \longrightarrow$ 3-dim spaces ($\mathbb{R}^3$ when $K=\mathbb{Q}$)



k

$\mathcal{O}_v$

$\mathcal{O}_K$

See [Knotprime Table 1] for more informations.

6. local to global: adèle

Recall: Ostrowski's thm & Product formula.

Task. Read [Adèle] and answer the following questions:

- Give a def of $\mathbb{A}_K$ & $\mathbb{I}_K$ (set, topo and measure)
- Verify that

$$K \subseteq \mathbb{A}_K \qquad \mathcal{O}_T \subseteq \prod_{v \in T}' K_v$$
$$K^\times \subseteq \mathbb{I}_K^\times \qquad \mathcal{O}_T^\times \subseteq \left(\prod_{v \in T}' K_v\right)'$$

are lattices. Give fundamental domain in easy cases.
- Deduce the finiteness of class number and Dirichlet unit theorem.

Base field with automorphism

We know that

	finite field	local field: Archi	local field: NA	global field	Adèle
base field F	$\mathbb{F}_p$	$\mathbb{R}$	$\mathbb{Q}_p$ $\quad \mathbb{F}_p((t))$	$\mathbb{Q}$ $\quad \mathbb{F}_p(t)$	$\mathbb{A}_K$
	$Aut_{ring}(\mathbb{F}_p)=1$	$Aut_{top\ ring}(\mathbb{R})=1$	$Aut_{top\ ring}(\mathbb{Q}_p)=1$	$Aut_{ring}(\mathbb{Q})=1$	
			$Aut_{top\ ring}(\mathbb{F}_p((t)))\neq 1$	$Aut_{ring}(\mathbb{F}_p(t))\neq 1$	$Aut_{ring}(\mathbb{A}_{\mathbb{F}_p(t)})\neq 1$

Q: Do we have $Aut_{ring}(\mathbb{A}_{\mathbb{Q}}) = \{Id\}$?

A: Yes. See [LCFT, Ex 6.3.6]. I don't understand this proof.

Galois extension

Setting: L/K fin ext of global field

Recall that we have an iso

$$L \otimes_K \mathbb{A}_K \xrightarrow{\cong} \mathbb{A}_L$$ of topo rings with compatible embedding of L

$\rightsquigarrow$ $\mathbb{A}_K \subseteq \mathbb{A}_L$ subring, $\mathbb{A}_L \cong \mathbb{A}_K^{\oplus [L:K]}$ as $\mathbb{A}_K$-module.

Lemma. [LCFT, Ex 6.3.2]

$$\text{integral closure of } K \text{ in } \mathbb{A}_L = L$$

Proof. Reduce to

$$\text{integral closure of } L \text{ in } \mathbb{A}_L \subseteq L$$

If $\exists\, x \in \mathbb{A}_L - L$ which is integral over L, then $L(x)/L$ is a fin field ext in $\mathbb{A}_L$, and

$$\# \{\mathfrak{q} \in \operatorname{Spec} \mathcal{O}_L \mid \mathfrak{q} \text{ do not split completely}\}$$
$$\overset{?}{\leq} \# \{\mathfrak{q} \in \operatorname{Spec} \mathcal{O}_L \mid x_{\mathfrak{q}} \notin \mathcal{O}_{\mathfrak{q}}\} < +\infty.$$

But fin nontrivial field ext have inf many non split primes. $\lightning$

7. $\mathbb{F}_1$

This is better explained in §1.2. Anyhow, it is still a "field".

⚠ It is always better to think $\# \mathbb{F}_1 = 1+\varepsilon$, where $\varepsilon \ll 1$.
In that way you can "see object at different level", like
$\# \mathbb{F}_1^\times = \varepsilon$ $\qquad$ $\mathbb{F}_1^\times$ is not empty!

Slogan: "Infinitesimal is only visible when constant level is zero"

This phenomenon already happens when we learn integrals.

$$\int x^n = \begin{cases} \frac{1}{n+1} x^{n+1} + C & n \neq -1 \\ \log x + C & n = -1 \end{cases}$$

Here, $\log x$ is that "infinitesimal".

See
wiki: https://en.wikipedia.org/wiki/Field_with_one_element
nlab: https://ncatlab.org/nlab/show/field+with+one+element

It is desirable to define

- The "field extension" $\mathbb{F}_{1^n}/\mathbb{F}_1$ of deg n $\quad + \quad \mathbb{F}_{1^n}^\times \cong \varepsilon \mu_n$
- The "Galois group" $\mathrm{Gal}(\overline{\mathbb{F}}_1/\mathbb{F}_1) \cong \widehat{\mathbb{Z}}$
- The "v.s. over $\mathbb{F}_1$"

Q: Do we have $\mathbb{Q}_1$?

A: See https://mathoverflow.net/questions/309664/what-is-mathbbq-1-the-field-of-1-adic-numbers