Eine Woche, ein Beispiel

8.6. Kottwitz set

This document is a continuation of [23.08.06].
Reorganized from Luozi Shi (and his partners)'s talk.

Recall that $\widehat{\mathbb{Q}_p^{ur}} = \mathrm{Frac}(\widehat{\mathbb{Z}_p^{ur}})$, $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$. Here "$\wedge$" is completion w.r.t. valuation.

Setting. In this document, F is a NA local field,

$$L := \widehat{F^{ur}} \overset{\text{if } F:\ p\text{-adic}}{=\!=\!=} \mathrm{Frac}(W(\overline{k}_F) \otimes_{W(\kappa_F)} \mathcal{O}_F).$$

Def For G/F reductive, the Kottwitz set $B(G)$ is defined as

$$\begin{aligned} B(G) :&= H^1(W_F, G_{\overline{F}}) \\ &\cong H^1(\langle\sigma\rangle, G_L) \qquad \text{by Inf-Res seq \& } H^1(I_F, G_{\overline{F}}) = 0 \\ &\cong G(L)/\sigma\text{-twisted } G(L)\text{-conj} \\ &\overset{\text{when } G = GL_n}{\cong} \mathrm{Isoc}/\sim \end{aligned}$$

Isocrystals.

Rmk. By Hilbert 90, $H^1(\Gamma_F, GL_{n,\overline{F}}) = \{1\}$. In most cases,

$$H^1(\Gamma_F, G_{\overline{F}}) \not\cong H^1(W_F, G_{\overline{F}})$$

[even though $H^1(\Gamma_F, G_{\overline{F}}) \cong G(F) \cong H^0(W_F, G_{\overline{F}})$, we take different resolutions.]

E.g. $B(\mathbb{G}_m) \cong \mathbb{Z}$

Proof. The map

$$\beta: \mathcal{O}_L^\times \longrightarrow \mathcal{O}_L^\times \qquad x \longmapsto x \cdot \sigma(x)^{-1}$$

is surjective, since

$$\beta_1: \kappa_L^\times \longrightarrow \kappa_L^\times \qquad x \longmapsto x^{1-q}$$

$$\beta_2: \kappa_L \longrightarrow \kappa_L \qquad x \longmapsto x - x^q$$

are surjective. ($\kappa_L \cong \overline{\mathbb{F}_p}$ is alg closed)

Then the well-defined morphism

$$v: L^\times/\mathrm{Im}\beta \twoheadrightarrow \mathbb{Z} \qquad x \longmapsto v(x)$$

is injective, thus an iso.

Ex. Check that the SES

$$1 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \longrightarrow 1$$

induce LES in gp cohomology:

$$\begin{tikzcd}
 & & & B(\underline{\mathbb{Z}/2\mathbb{Z}}) \arrow[r, "0"] & B(\mathbb{G}_m) \arrow[r, "\times 2"] & B(\mathbb{G}_m) \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & F^\times \arrow[r, "(-)^2"] & F^\times \arrow[u]
\end{tikzcd}$$

$B(\mathbb{G}_m) = \mathbb{Z}$, $B(\mathbb{G}_m) = \mathbb{Z}$

where

$$\begin{aligned}
B(\underline{\mathbb{Z}/2\mathbb{Z}}) &\overset{def}{:=} H^1(W_F, \mathbb{Z}/2\mathbb{Z}) \qquad W_F \curvearrowright \mathbb{Z}/2\mathbb{Z} \text{ trivially} \\
&= \mathrm{Hom}(W_F, \mathbb{Z}/2\mathbb{Z}) \\
&\cong \{H \triangleleft W_F \text{ closed with index } 2\} \cup \{0\} \\
&\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 2 \end{cases}
\end{aligned}$$

This LES gives us to compute number of finite extensions. e.p with prime degree.
One gets

$$F^\times \xrightarrow{(-)^n} F^\times \rightsquigarrow B(\underline{\mathbb{Z}/n\mathbb{Z}}) \xrightarrow{0} \cdots$$