§ 1.1. Structure of finite / local / global field

Road map

	finite field		local field: Archi	local field: NA		global field		adéle
base field F	7 $\mathbb{F}_1$ For $\mathbb{F}_{1^r}$	1 $\mathbb{F}_p$	2 $\mathbb{R}$ or $\mathbb{C}$	3 $\mathbb{Q}_p$	$\mathbb{F}_p((t))$	4 $\mathbb{Q}$	$\mathbb{F}_p(t)$	6 $\mathbb{A}_K$
$F^\times$	$\varepsilon \cdot \mu_r$	μ_{p-1}	$\mathbb{R}^+ \times \mathbb{Z}/2\mathbb{Z}$ $\mathbb{C}^\times$	$\mathbb{Z}_p^\times \times \mathbb{Z}$	$\mathbb{F}_p[[t]]^\times \times \mathbb{Z}$	$\mathbb{Q}^\times$	$\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	—	—	—	$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	—	—	—	$\mathbb{Z}_p^\times$	$\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$	$\hat{\mathbb{Z}}$?	$\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$ Id	most known		dream		
ari Frob	?	can	total order? —	choose a lift		unramified $\xrightarrow{}$ $\begin{cases} \mathrm{Frob}_q \\ \mathrm{Frob}_\mathfrak{p} \text{ conj class} \end{cases}$ $\xrightarrow{\text{abelian}}$ $\mathrm{Frob}_\mathfrak{p}$		
# ext of deg n	1?	1	1/0	finite		$n \neq 1$ inf countable		
Spec $\mathcal{O}_F$	$\mathrm{Spec}\, \mathbb{F}_q = K(\hat{\mathbb{Z}}, 1)$ [étale, 2.2.4]		—					—
topology	?	discrete	Euclidean	profinite		—		restricted
topo of $\mathcal{O}_F$	—		—	cpt, not discrete		—		K is a lattice in $\mathbb{A}_K$
measure	?	discrete	Lebesgue	$\mu(\mathcal{O}_F) = 1$		—		can be computed

5 Also, discuss

- field extension, norm, trace, ...
- their connection to geometry, ramification theory
- analog with knot theory

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$\mathrm{Gal}(\overline{\mathbb{F}}_q / \mathbb{F}_q) \cong \hat{\mathbb{Z}}$ as top gps

$\mathrm{Frob}_p \longleftarrow 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $\mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \mathrm{Id}$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?
  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions.

- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $Fr\, \tau\, Fr^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions: (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\underbrace{\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep}/\mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right)}_{G_{\mathbb{F}_p((t))}},\quad \underbrace{\mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}_p}/\widehat{\mathbb{Q}_p^{ur}}\right)}_{I_{\mathbb{Q}_p}},\quad \underbrace{\mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}/\mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)}_{G_{\mathbb{F}_p((t))}}$$

4. global field

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.
$\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D],
we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\}/\text{birational}$$

$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler. by [GalFun, Thm 3.4.8],

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \hat{F}(\mathbb{C})$$

↑Free profinite gp on $\mathbb{C}$

Shafarevich's conj: See wiki: Absolute Galois group

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}^{ab})$ is a free profinite gp

$\mathbb{Q}$: Does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object/geo realizations?

Dessin d'enfants

By [GalFun, Prop 4.7.1 - Rmk 4.7.9], we have an including

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \mathrm{Out}(\pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}))$$

induced by $\pi^{ét}_{1,\overline{\mathbb{Q}}} := \pi_1^{ét}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, $\pi^{ét}_{1,\mathbb{Q}} := \pi_1^{ét}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$

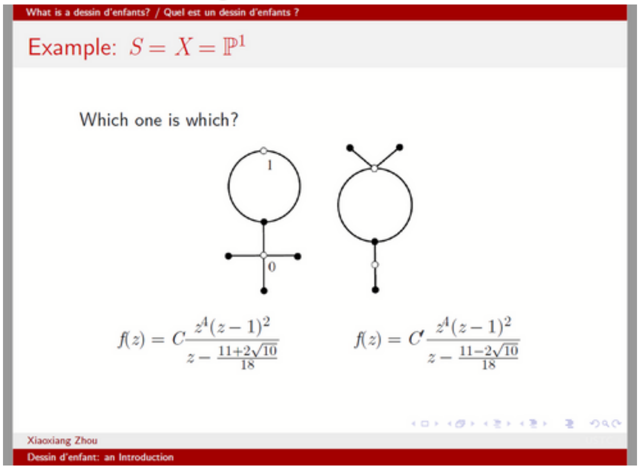
$$\begin{tikzcd}
 & 1 \arrow[r] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] \arrow[d, equal] & \pi^{ét}_{1,\mathbb{Q}} \arrow[r] \arrow[d, "\substack{\text{conj} \\ g \mapsto g\cdot g^{-1}}"] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & Z(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] \arrow[u, phantom, "\overset{1}{\|?}"] & \pi^{ét}_{1,\overline{\mathbb{Q}}} \arrow[r] & \mathrm{Aut}(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & \mathrm{Out}(\pi^{ét}_{1,\overline{\mathbb{Q}}}) \arrow[r] & 1
\end{tikzcd}$$

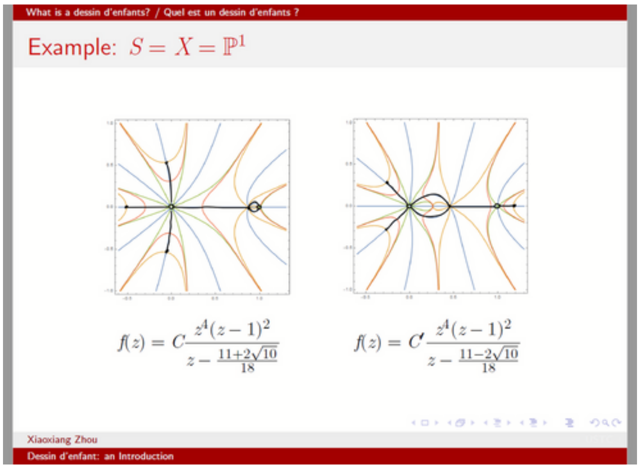
The space $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ is designed for guaranteeing that
$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\pi^{ét}_{1,\overline{\mathbb{Q}}})$
is inclusion.

Task. Read [Dessin d'enfant] or [Collins],

understand the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-action on the dessin d'enfants.

- Def of Dessin d'enfact
- Connections with $Out(\pi_{1,\overline{\mathbb{Q}}}^{ét})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.





- Can we generalize this to $Gal(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$?

  I don't know how to make a "dessin d'enfant" on alg curves over $\mathbb{F}_p$.

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.

5. local and global: connections

6. local to global: adèle

Recall: Ostrowski's thm & Product formula.

Task. Read [Adèle] and answer the following questions:

- Give a def of $\mathbb{A}_K$ & $\mathbb{I}_K$ (set, topo and measure)
- Verify that

$$K \subseteq \mathbb{A}_K \qquad \mathcal{O}_T \subseteq \prod_{v\in T}' K_v$$
$$K^\times \subseteq \mathbb{I}_K^\times \qquad \mathcal{O}_T^\times \subseteq \Big(\prod_{v\in T}' K_v\Big)'$$

  are lattices. Give fundamental domain in easy cases.
- Deduce the finiteness of class number and Dirichlet unit theorem.

Base field with automorphism

We know that

	finite field	local field: Archi	local field: NA	global field	Adèle
base field F	$\mathbb{F}_p$	$\mathbb{R}$	$\mathbb{Q}_p$ $\quad \mathbb{F}_p((t))$	$\mathbb{Q}$ $\quad \mathbb{F}_p(t)$	$\mathbb{A}_K$
	$\mathrm{Aut}_{ring}(\mathbb{F}_p)=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{R})=1$	$\mathrm{Aut}_{top\ ring}(\mathbb{Q}_p)=1$	$\mathrm{Aut}_{ring}(\mathbb{Q})=1$	
			$\mathrm{Aut}_{top\ ring}(\mathbb{F}_p((t)))\neq 1$	$\mathrm{Aut}_{ring}(\mathbb{F}_p(t))\neq 1$	$\mathrm{Aut}_{ring}(\mathbb{A}_{\mathbb{F}_p(t)})\neq 1$

Q: Do we have $\mathrm{Aut}_{ring}(\mathbb{A}_\mathbb{Q})=\{\mathrm{Id}\}$?

A: Yes. See [LCFT, Ex 6.3.6]. I don't understand this proof.

Galois extension

Setting: L/K fin ext of global field

Recall that we have an iso

$$L\otimes_K \mathbb{A}_K \xrightarrow{\cong} \mathbb{A}_L \quad \text{of topo rings with compatible embedding of } L$$

$\rightsquigarrow$ $\mathbb{A}_K \subseteq \mathbb{A}_L$ subring, $\mathbb{A}_L \cong \mathbb{A}_K^{\oplus[L:K]}$ as $\mathbb{A}_K$-module.

Lemma. [LCFT, Ex 6.3.2]

$$\text{integral closure of } K \text{ in } \mathbb{A}_L = L$$

Proof. Reduce to

$$\text{integral closure of } L \text{ in } \mathbb{A}_L \subseteq L$$

If $\exists\, x\in \mathbb{A}_L - L$ which is integral over L, then

$L(x)/L$ is a fin field ext in $\mathbb{A}_L$, and

$$\#\{\mathfrak{q}\in \operatorname{Spec}\mathcal{O}_L \mid \mathfrak{q} \text{ do not split completely}\}$$
$$\overset{?}{\leq} \#\{\mathfrak{q}\in \operatorname{Spec}\mathcal{O}_L \mid x_\mathfrak{q}\notin \mathcal{O}_\mathfrak{q}\} < +\infty.$$

But fin nontrivial field ext have inf many non split primes. ↯