

§2.1. Character of Galois gp

This document is originally prepared for the class field theory, but we don't have time. And also, the ref [LCFT] is fantastic (with typos).

Since we discuss §2.1 and §3.1 in the same time, it is preferable to discuss general Galois rep and then restrict to the 1-dim case.

The only speciality of 1-dim case is, the char factor through

$$\mathrm{Gal}(F^{\mathrm{sep}}/F) \rightarrow \mathrm{Gal}(F^{\mathrm{ab}}/F) \rightarrow \mathrm{GL}_1(\Delta),$$

Therefore, the max abel ext F^{ab} plays a role.

fin	✓		
local	local	Kronecker - Weber	$F^{\mathrm{ab}} = F(\zeta_{\infty})$
global		Kronecker - Weber	$\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}(\zeta_{\infty})$

Local Kronecker - Weber

for \mathbb{Q}_p : [LCFT, Thm 1.3.4]

for F : [Allen, Thm 18.3]

use Kummer theory

use Hasse-Arf thm [Allen, Thm 17.16]

Kronecker - Weber

for \mathbb{Q} : [LCFT, Thm 1.1.2]

for $\mathbb{Q}(i)$: [Cox x^2+ny^2]

for $\mathbb{F}(t)$: [VS], [Hayes]

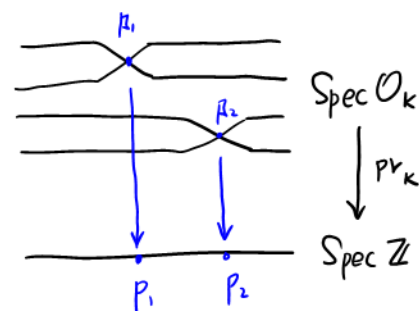
use Minkowski's thm

use CM Theory

<https://math.stackexchange.com/questions/2125609/classical-version-and-idelic-version-of-class-field-theory>

<https://math.stackexchange.com/questions/132006/the-kernel-of-the-reciprocity-map-in-global-class-field-theory>

Thm K/\mathbb{Q} fin abelian $\Rightarrow K \subseteq \mathbb{Q}(\zeta_n) \quad \exists n$



Proof.

Step 1. The choice of n :

Denote $\{p_1, \dots, p_r\}$ as primes over which K ramifies, pick $\mu_i \in p_{r_K}^{-1}(p_i)$.

$\text{Gal}(K_{\mu_i}/\mathbb{Q}_{p_i}) \leq \text{Gal}(K/\mathbb{Q}) \xrightarrow{\text{local KW}} \exists n_{p_i} \in \mathbb{N}_{>0}$ s.t. $K_{\mu_i} \subseteq \mathbb{Q}(\zeta_{n_{p_i}})$

Suppose $n_{p_i} = p_i^{e_i} \cdot a_i$, $p_i \nmid a_i$, take $n := \prod_i p_i^{e_i} \in \mathbb{N}_{>0}$.

Step 2 Take $L = K(\zeta_n)$, we will show that $L = \mathbb{Q}(\zeta_n)$. Pick $q_i \in p_{r_{L/K}}^{-1}(p_i)$.

$$|I| \stackrel{\text{Minko}}{=} [L:\mathbb{Q}] \geq [\mathbb{Q}(\zeta_n):\mathbb{Q}] = \phi(n)$$

$$|I| \leq \prod_i |I_{q_i}| \leq \prod_i \phi(p_i^{e_i}) = \phi(n)$$

$$\Rightarrow [L:\mathbb{Q}] = [\mathbb{Q}(\zeta_n):\mathbb{Q}], \quad L = \mathbb{Q}(\zeta_n).$$

$L_{q_i} \subseteq \mathbb{Q}_{p_i}(\zeta_{n_{p_i}}, \zeta_n)$	$L \subseteq \mathbb{Q}(\zeta_n)$
$ I_{q_i} $	$ I = \langle I_{q_i} \rangle_i$
$U_{q_i} = \mathbb{Q}_{p_i}^{\text{ur}} \cap L_{q_i}$	$U_i = L^I$
\mathbb{Q}_{p_i}	\mathbb{Q}

Rmk. This argument can not be extended to fct field K , since the residue fields of vals in K may be same (up to iso)

Left: LCFT, Galois cohomology.

Global class field theory

Observe that

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0} \cong \widehat{\mathbb{Z}}^\times \cong \text{Gal}(\mathbb{Q}^{\text{ab}}/\mathbb{Q}) \hat{=} \Gamma_\mathbb{Q}^{\text{ab}}$$

In fact, we have Artin reciprocity:

$$\text{Art}: (\mathbb{F}^\times \backslash \mathbb{A}_\mathbb{F}^\times) / (\overline{\mathbb{F}_\infty^\times})^\circ \cong \Gamma_\mathbb{F}^{\text{ab}}$$

Ex. What does $-1 \in \widehat{\mathbb{Z}}^\times$ corresponds to in $\Gamma_\mathbb{Q}^{\text{ab}}$?