Eine Woche, ein Beispiel

1.21. complex multilinear algebra

The title comes from
http://staff.ustc.edu.cn/~wangzuoq/Courses/16F-Manifolds/Notes/Lec16.pdf

We also take the reference from "Introduction to complex geometry", written by Yalong Shi:
http://maths.nju.edu.cn/~yshi/BICMR_ComplexGeometry.pdf

M: cplx mfld, $p \in M$ $\qquad$ $M_{\mathbb{R}}$: M viewed as smooth mfld, not base change

better: M_{sm}

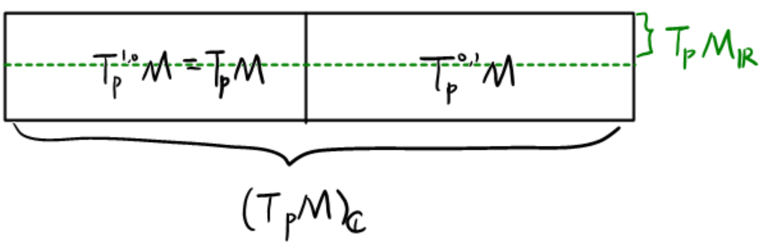
e.g. $M = \mathbb{C}^3$ $\quad p = 0$

Notation	base field	dim	basis	name	[YS20]
T_pM	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_pM_{\mathbb{R}}$	$\mathbb{R}$	6	$\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$	real tangent vector	$T_p^{\mathbb{R}}M$
$(T_pM)_{\mathbb{C}} := T_pM_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar z_i}$ or $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$	complexified tangent vector	$T_p^{\mathbb{C}}M$
$T_p^{1,0}M = T_pM$	$\mathbb{C}$	3	$\frac{\partial}{\partial z_i}$	holomorphic tangent vector	
$T_p^{0,1}M$	$\mathbb{C}$	3	$\frac{\partial}{\partial \bar z_i}$	anti-holomorphic tangent vector	
T_p^*M	$\mathbb{C}$	3	dz_i	holomorphic 1-form	Ω_p^1
$T_p^*M_{\mathbb{R}} \triangleq \Omega^1_{\mathbb{R},p}$	$\mathbb{R}$	6	dx_i, dy_i	real 1-form	
$(T_p^*M)_{\mathbb{C}} := T_p^*M_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$	$\mathbb{C}$	6	$dz_i, d\bar z_i$ or dx, dy	complexified 1-form	$T_p^{*\mathbb{C}}M = A_p^1$
$T_p^{1,0,*}M \triangleq \Omega_p^{1,0} = T_p^*M$	$\mathbb{C}$	3	dz_i	$(1,0)$-form	$T_p^{*1,0}M = A_p^{1,0}$
$T_p^{0,1,*}M \triangleq \Omega_p^{1,0}$	$\mathbb{C}$	3	$d\bar z_i$	$(0,1)$-form	$T_p^{*0,1}M = A_p^{0,1}$

$\Omega^i, \Omega^{i,j}$: sheaves on M

Rmk. We don't have any natural identification between T_pM & $T_pM_{\mathbb{R}}$.
Notice that $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $-\frac{1}{2}i$ is not real, so $\frac{\partial}{\partial z} \notin T_pM_{\mathbb{R}}$.

although our geometrical intuition of T_pM is often $T_pM_{\mathbb{R}}$,
$T_pM \cap T_pM_{\mathbb{R}} = \emptyset$ in $(T_pM)_{\mathbb{C}}$.



Reminder: the (induced) almost complex structure is defined as

$$J : T_pM_{\mathbb{R}} \longrightarrow T_pM_{\mathbb{R}}$$
$$\frac{\partial}{\partial x_i} \longmapsto \frac{\partial}{\partial y_i}$$
$$\frac{\partial}{\partial y_i} \longmapsto -\frac{\partial}{\partial x_i}$$

$$\rightsquigarrow \quad J : T_pM \longrightarrow T_pM$$

$$J\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right) = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\right)\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$$

$$J\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right) = \left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_i}\right)\begin{pmatrix} i & \\ & -i \end{pmatrix}$$

real basis of $(T_pM)_{\mathbb{C}}$:

$$\mathcal{B}_1 = \{\underbrace{\partial_x, \partial_y}, \underbrace{i\partial_x, i\partial_y}\}$$

$$\begin{tikzcd} \circlearrowleft T_pM_{\mathbb{R}} \arrow[r, leftrightarrow] \arrow[loop below, "\mathrm{Id}"] & iT_pM_{\mathbb{R}} \circlearrowright \arrow[loop below, "-\mathrm{Id}"] \end{tikzcd}$$

$$\begin{cases} \partial_x = \partial_z + \partial_{\bar{z}} \\ \partial_y = \frac{1}{i}(\partial_z - \partial_{\bar{z}}) \end{cases} \qquad dx = \tfrac{1}{2}(dz + d\bar{z}) \qquad dy = \tfrac{1}{2i}(dz - d\bar{z})$$

$$\mathcal{B}_2 = \{\underbrace{\partial_z}, \underbrace{\partial_{\bar{z}}}, \underbrace{i\partial_z}, \underbrace{i\partial_{\bar{z}}}\}$$

$$\begin{tikzcd} \circlearrowleft T_pM \arrow[r, leftrightarrow] \arrow[loop below, "\times i"] & T^{0,1}M \circlearrowright \arrow[loop below, "\times i"] \end{tikzcd}$$

$$\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases} \qquad dz = dx + i\,dy \qquad d\bar{z} = dx - i\,dy$$

$\circlearrowright$: J, conj_J

$\circlearrowright$: conj_i

$\circlearrowright$: $\times\, i$

$$
\begin{aligned}
J\,(f\partial_x + g\partial_y) &= f\partial_y - g\partial_x \\
J\,(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_y, -\partial_x, i\partial_y, -i\partial_x) \\
&= (\partial_x, \partial_y, i\partial x, i\partial y) \begin{pmatrix} & -1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
J\,(f\partial_z + g\partial_{\bar z}) &= if\partial_z - ig\partial_{\bar z} \\
J\,(\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) &= (i\partial_z, -i\partial_{\bar z}, -\partial_z, \partial_{\bar z}) \\
&= (\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) \begin{pmatrix} & & -1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{conj}_J\,(f\partial_x + g\partial_y) &= \bar f\partial_x - \bar g\partial_y \\
\mathrm{conj}_J\,(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, -\partial_y, -i\partial_x, i\partial_y) \\
&= (\partial_x, \partial_y, i\partial x, i\partial y) \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{conj}_J\,(f\partial_z + g\partial_{\bar z}) &= \bar f\partial_z + \bar g\partial_{\bar z} \\
\mathrm{conj}_J\,(\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) &= (\partial_z, \partial_{\bar z}, -i\partial_z, -i\partial_{\bar z}) \\
&= (\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{conj}_i\,(f\partial_x + g\partial_y) &= \bar f\partial_x + \bar g\partial_y \\
\mathrm{conj}_i\,(\partial_x, \partial_y, i\partial_x, i\partial_y) &= (\partial_x, \partial_y, -i\partial_x, -i\partial_y) \\
&= (\partial_x, \partial_y, i\partial x, i\partial y) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{conj}_i\,(f\partial_z + g\partial_{\bar z}) &= \bar f\partial_{\bar z} + \bar g\partial_z \\
\mathrm{conj}_i\,(\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) &= (\partial_{\bar z}, \partial_z, -i\partial_{\bar z}, -i\partial_z) \\
&= (\partial_z, \partial_{\bar z}, i\partial_z, i\partial_{\bar z}) \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & -1 & \end{pmatrix}
\end{aligned}
$$

Hermitian metric

$$\begin{aligned}
H &= h_{\alpha\beta}\, dz^\alpha \otimes d\bar{z}^\beta \quad (= ds^2) && \text{Hermitian metric} \quad (h_{\alpha\beta}) \in \mathbb{R}^{n\times n} \text{ pos def} \\
g &= \tfrac{1}{2}(H + \overline{H}) && \text{Riemannian metric} \\
\omega &= \tfrac{i}{2}(H - \overline{H}) && \text{Hermitian form}
\end{aligned}$$

e.g.

$$\begin{aligned}
H &= dz \otimes d\bar{z} \\
&= (dx \otimes dx + dy \otimes dy) - i(dx \otimes dy - dy \otimes dx) = g - i\omega \\
g &= \tfrac{1}{2}(dz \otimes d\bar{z} + d\bar{z} \otimes dz) \\
&= dx \otimes dx + dy \otimes dy \\
\omega &= \tfrac{i}{2}(dz \otimes d\bar{z} - d\bar{z} \otimes dz) && = i\, dz \wedge d\bar{z} \\
&= dx \otimes dy - dy \otimes dx && = 2\, dx \wedge dy
\end{aligned}$$

$$\kappa = -\frac{1}{h}\partial_z \partial_{\bar{z}} \ln h = -\frac{1}{h}\frac{1}{4}\Delta(\ln h) \qquad \Delta = \partial_x^2 + \partial_y^2$$

Two methods to show $\mathrm{II}(\partial_z, \partial_{\bar{z}}) = 0$

Method 1.

$$i\,\mathrm{II}(\partial_z, \partial_{\bar{z}}) = \mathrm{II}(J\partial_z, \partial_{\bar{z}}) = \mathrm{II}(\partial_z, J\partial_{\bar{z}}) = -i\,\mathrm{II}(\partial_z, \partial_{\bar{z}})$$

Method 2.

$$\begin{aligned}
\mathrm{II}(\partial_z, \partial_{\bar{z}}) &= \mathrm{II}\left(\tfrac{1}{2}(\partial_x - i\partial_y),\ \tfrac{1}{2}(\partial_x + i\partial_y)\right) \\
&= \mathrm{II}\left(\tfrac{1}{2}(1 - iJ)\partial_x,\ \tfrac{1}{2}(1+iJ)\partial_x\right) \\
&= \tfrac{1}{2}(1 - iJ)\cdot\tfrac{1}{2}(1+iJ)\,\mathrm{II}(\partial_x, \partial_x) \\
&= \tfrac{1}{4}(1 - i^2J^2)\,\mathrm{II}(\partial_x, \partial_x) \\
&= 0.
\end{aligned}$$