Eine Woche, ein Beispiel

2.23 Schubert calculus: coh of Grassmannian

Ref:
[3264] and [Fulton]
[LW21]: https://www.math.uni-bonn.de/ag/stroppel/Masterarbeit_Wang.pdf
[BWB21]: Wang, Liao. The Borel–Weil–Bott Theorem in Examples

[PV04]: https://arxiv.org/abs/math/0407170
[Barth04]: Barth, Wolf P., Klaus Hulek, Chris A. M. Peters and Antonius Van De Ven. Compact Complex Surfaces.
Igor Pak, Ernesto Vallejo, Combinatorics and geometry of Littlewood-Richardson cones
[GK20]: Frank Gounelas and Alexis Kouvidakis. On Some Invariants of Cubic Fourfolds. European Journal of Mathematics

We will attempt to tackle Schubert calculus in a concise manner. The term "Schubert calculus" is often associated with intersection theory, enumerative geometry, combinatorics, Grassmannians, and more, making it a vast topic. However, I believe its core ideas can be clearly explained in just six hours. I will break the material into several parts:

1. $H^{\cdot}(Gr(r,n);\mathbb{Z})$ and its combinatorics
2. Relate cycles with v.b.s
   cycles in Grassmannian, including:
   - cycle class map: $CH^{\cdot}(Gr(r,n)) \xrightarrow{\sim} H^{\cdot}(Gr(r,n);\mathbb{Z})$
   - incidence variety $\begin{cases} \text{(partial) flag variety} \\ \text{Fano variety of planes} \\ \cdots \end{cases}$
   - a reintepretation of cycles

$$\begin{array}{ccc} \mathcal{L} & & S^\vee \\ | & & | \\ X & \xrightarrow{f_{\mathcal{L}}} & Gr(r,n) \end{array}$$

3. Chern class calculation

Chern class: $\quad c: VB(X) \longrightarrow H^{\cdot}(X;\mathbb{Z})$

$$f_{\mathcal{L}}^*: H^{\cdot}(Gr(r,\infty);\mathbb{Z}) \longrightarrow H^{\cdot}(X;\mathbb{Z})$$

e.g., $\quad VB(Gr(r,n)) \longrightarrow H^{\cdot}(Gr(r,n);\mathbb{Z})$

$$S^\vee \longmapsto 1+\sigma_1+\sigma_1^2+\cdots\sigma_1^r$$
$$T_{Gr} \longmapsto 1+n\cdot\sigma_1+\cdots$$

4. Applications

1. Group structure of $H^{\cdot}(Gr(r,n);\mathbb{Z})$
2. Cup product
3. Young diagram formulas
4. Other combinatorial models for LR coefficients

1. Group structure of $H^{\cdot}(Gr(r,n);\mathbb{Z})$

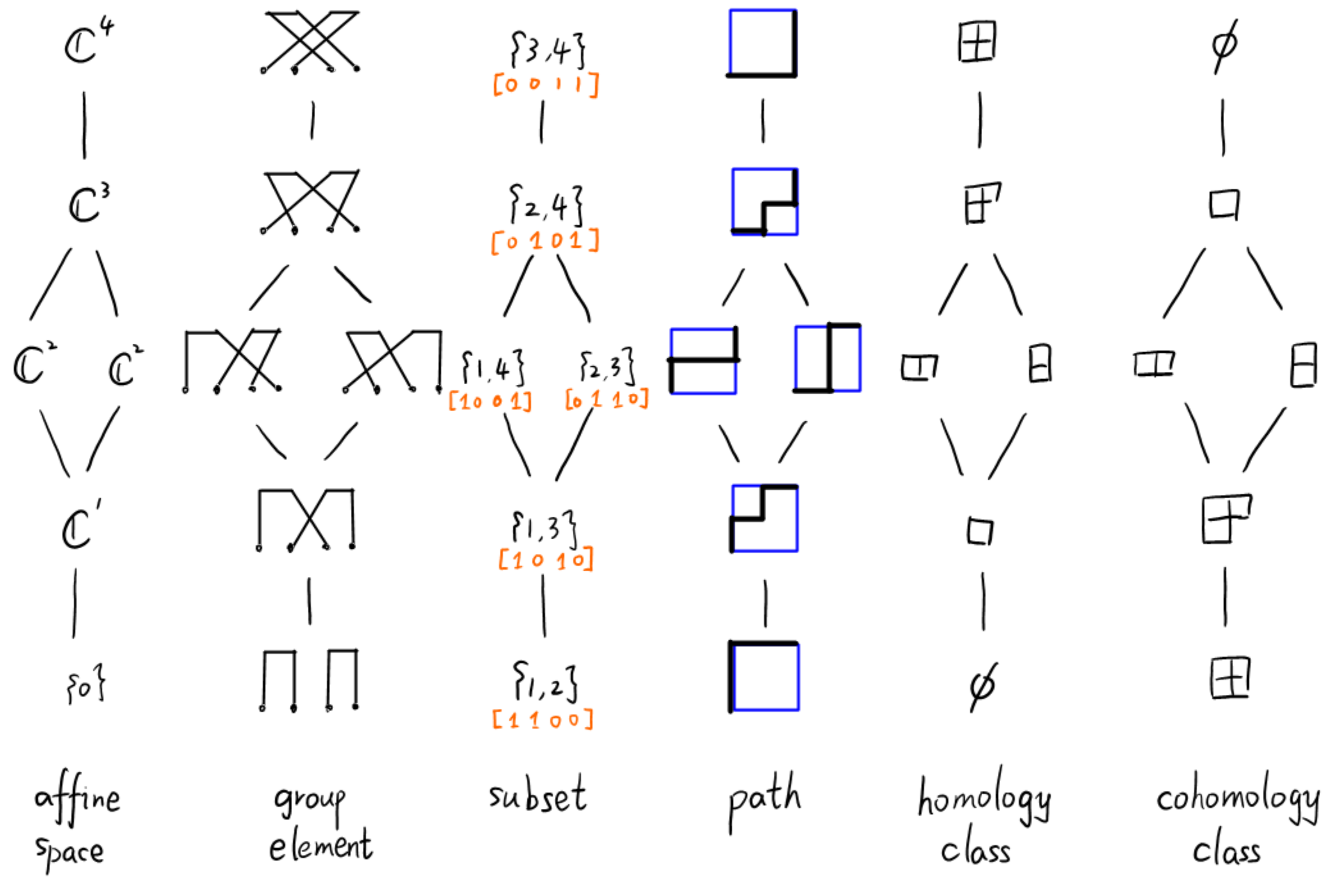
It's well-known that $Gr(r,n) \cong GL_n(\mathbb{C})/P$ has an affine paving w.r.t. $S_n/S_r\times S_{n-r}$:

$$Gr(r,n) = \coprod_{w\in S_n/S_r\times S_{n-r}} BwP/P \cong \coprod_{w\in S_n/S_r\times S_{n-r}} \mathbb{C}^{l(w)}$$

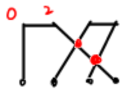
$$\# S_n/S_r\times S_{n-r} = \binom{n}{r}$$

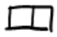
We read the diagram from top to bottom, the map from right to left.

E.g. $n=4$ $r=2$

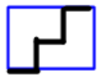


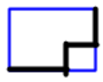
Hint from gp element to homology class:



$\rightsquigarrow$ $(2,0) =$ 

E.g. $n = 5, \ r = 2$

 $\sim \{2,4\} \sim$  $\sim$  $\sim$ 

 $\sim \{3,5\} \sim$  $\sim$  $\sim$ 

Ex. compute w_0-action (left mult) on $S_n / S_r \times S_{n-r}$, where $w_0 =$ .

2. Cup product

We want to compute intersection number by moving one cycle(so that they intersect transversally)

Lemma 1. $[B^- wP/P] = [Bw_0wP/P]$ in $H^\cdot(Gr(r,n);\mathbb{Z})$.

Proof. $B^- wP/P = w_0 Bw_0 w P/P \sim Bw_0 wP/P$.

Lemma 2.

$$\#\left(BwP/P \cap B^-\eta P/P\right) = \begin{cases} 0 & \eta > w \\ 1 & \eta = w \\ 0 & \eta \ngtr w \quad \& \quad l(\eta) = l(w) \\ ? & \text{otherwise} \end{cases}$$

Moreover, when $\eta = w$, BwP/P and $B^-\eta P/P$ intersect transversally.

Idea: Find a set of representative elements $C_w^+ \cong \mathbb{C}^{l(w)}$ in B, s.t.

$$BwP/P \xleftarrow{\cong} C_w^+ wP/P \cong C_w^+.$$

Similarly, find a set of representative elements $C_\eta^- \cong \mathbb{C}^{l(w_0\eta)}$ in B^-, s.t.

$$B^-\eta P/P \xleftarrow{\cong} C_\eta^- \eta P/P \cong C_\eta^-.$$

After that,

$$\begin{aligned} BwP/P \cap B^-\eta P/P &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_+ wP = c_- \eta P\} \\ &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_-^{-1} c_+ \in \eta P w^{-1}\} \end{aligned}$$

can be written as the zero sets of polynomials (of deg ≤ 2) in $C_w^+ \times C_\eta^- \cong \mathbb{C}^{l(w) + l(w_0\eta)}$.

E.g. $n=5$, $r=2$,

$$w = $$  $$= \left(\begin{array}{cc:ccc} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{35 \mid 124\} \sim$$  $\sim$ 

hom cohom

$$\eta_0 = $$  $$= \left(\begin{array}{cc:ccc} 1 & & & & \\ & & 1 & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) = \{13 \mid 245\} \sim$$  $\sim$ 

Let $\eta = \eta_0$, we want to describe $BwP/P \cap B^- \eta P/P \subset C_w^+ \times C_\eta^-$.
By direct calculation,

$$P = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & * & * & * \end{pmatrix} \qquad \eta P w^{-1} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{array}{c} 1 \\ \\ 3 \\ \\ \\ \end{array} & \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array}$$

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

for copy

$$wPw^{-1} = \begin{array}{c c} & \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{array}{c} \\ \\ 3 \\ \\ 5 \end{array} & \begin{pmatrix} * & * & & * & \\ * & * & & * & \\ * & * & * & * & * \\ * & * & & * & \\ * & * & * & * & * \end{pmatrix} \end{array} \qquad \eta P \eta^{-1} = \begin{array}{c c} & \begin{array}{ccccc} & 2 & & 4 & 5 \end{array} \\ \begin{array}{c} 1 \\ \\ 3 \\ \\ \\ \end{array} & \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array}$$

$$C_w^+ = \begin{pmatrix} 1 & & * & & * \\ & 1 & * & & * \\ & & 1 & & \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \qquad C_\eta^- = \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ & & 1 & & \\ * & & * & 1 & \\ * & & * & & 1 \end{pmatrix}$$

Now, suppose

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ b_{21} & 1 & & & \\ & & 1 & & \\ b_{41} & & b_{43} & 1 & \\ b_{51} & & b_{53} & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ b_{21} & 1 & b_{21}a_{13} + a_{23} & & b_{21}a_{15} + a_{25} \\ & & 1 & & \\ b_{41} & & b_{41}a_{13} + b_{43} & 1 & b_{41}a_{15} + a_{45} \\ b_{51} & & b_{51}a_{13} + b_{53} & & b_{51}a_{15} + 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in \eta P w^{-1} \iff \begin{cases} b_{21}a_{13} + a_{23} = 0 \\ b_{21}a_{15} + a_{25} = 0 \\ b_{41}a_{13} + b_{43} = 0 \\ b_{41}a_{15} + a_{45} = 0 \\ b_{51}a_{13} + b_{53} = 0 \\ b_{51}a_{15} + 1 = 0 \end{cases}$$

In this case, $BwP/P \cap B^- \eta P/P \cong \mathbb{C}^3 \times \mathbb{C}^\times$.

Now, take $\eta = w$, one suppose that

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & b_{43} & 1 & \\ & & & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & b_{43} & 1 & a_{45} \\ & & & & 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in w P w^{-1} \iff a_{13} = a_{15} = a_{23} = a_{25} = a_{45} = b_{43} = 0.$$

In this case $BwP/P \cap B^- wP/P = \{*\}$.

Furthermore, one can show the transversality through the tangent argument.

Ex. When $\eta = w_0$, verify that

$$BwP/P \cap B^- w_0 P/P = \phi$$

Generalize this example to prove Lemma 2.

$\Leftrightarrow l(w_0 w) + l(w_0 w') = r(n-r)$

Cor of Lemma 2. When $l(w) + l(w') = r(n-r)$,

$$\deg([BwP/P] \cup [Bw'P/P]) = \begin{cases} 1 & w = w_0 w' \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, denote

$$\sigma_w := [BwP/P] \in H^{\cdot}(Gr(r,n);\mathbb{Z})$$

then
$$\sigma_w \sigma_{w_0 w} = \sigma_{Id}$$
$$\sigma_w \sigma_\eta = 0 \quad \text{when} \quad l(w) + l(\eta) = r(n-r).$$

⚠ When we view $w = a = (a_1, \dots, a_r)$ as the Young diagram in the cohom class,

$$l(w) = r(n-r) - |a|$$
$$\sigma_w \stackrel{!}{=} \sigma_a \in H_{l(w)}(Gr(r,n);\mathbb{Z}) \cong H^{|a|}(Gr(r,n);\mathbb{Z}).$$

For simplicity, we write $\sigma_k = \sigma_{(k,0,\dots,0)}$ and $\sigma_{1^k} = \sigma_{(\underbrace{1,\dots,1}_{k \text{ many}},0,\dots 0)}$.

The moduli interpolation of Schubert variety

To prove the Pieri rule, the method in the proof of Lemma 2 need to be modified. Working with the moduli interpolation of Schubert varieties can help understanding.

E.g. $n=5,\ r=2,$

$$w = $$  $$= \left(\begin{array}{cc|ccc} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{35 \mid 124\} \sim$$ hom  $\sim$ cohom $\square$

$$wP/P \in G/P \quad \longleftrightarrow \quad w\langle e_1, e_2\rangle = \langle e_3, e_5\rangle \in Gr(2,5)$$

standard

$$\begin{array}{rccccccccccc} \mathcal{V}^{st}: & 0 & \subset & \langle e_1\rangle & \subset & \langle e_1,e_2\rangle & \subset & \langle e_1,e_2,e_3\rangle & \subset & \langle e_1,\cdots,e_4\rangle & \subset & \langle e_1,\dots,e_5\rangle \\ & \cup & & \cup & & \cup & & \cup & & \cup & & \cup \\ \langle e_3,e_5\rangle \cap \mathcal{V}^{st}: & 0 & = & 0 & = & 0 & \subset & \langle e_3\rangle & = & \langle e_3\rangle & \subset & \langle e_3,e_5\rangle \end{array}$$

$$\Sigma_w(\mathcal{V}^{st}) : \triangleq \overline{BwP/P}$$
$$= \left\{ \Lambda \in Gr(2,5) \;\middle|\; \begin{array}{l} \dim \Lambda \cap \mathcal{V}^{st}_3 \geq 1 \\ \dim \Lambda \cap \mathcal{V}^{st}_5 \geq 2 \end{array} \right\}$$

$$BwP/P = \left\{ \Lambda \in Gr(2,5) \;\middle|\; \begin{array}{l} \dim \Lambda \cap \mathcal{V}^{st}_3 = 1 \\ \dim \Lambda \cap \mathcal{V}^{st}_5 = 2 \\ \dim \Lambda \cap \mathcal{V}^{st}_2 = 0 \\ \dim \Lambda \cap \mathcal{V}^{st}_4 = 1 \end{array} \right\}$$

Def. For the flag $\mathcal{V} = g\mathcal{V}^{st}$, define

$$\Sigma_w(\mathcal{V}) = g\overline{BwP/P}$$
$$= \left\{ \Lambda \in Gr(2,5) \;\middle|\; \begin{array}{l} \dim \Lambda \cap \mathcal{V}_3 \geq 1 \\ \dim \Lambda \cap \mathcal{V}_5 \geq 2 \end{array} \right\}$$

General case:

$$\Sigma_w(\mathcal{V}) = \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{w(i)} \geq i\}$$

Easy to see that $\Sigma_w(w_0\mathcal{V}^{st}) = \overline{B^- w_0 w P/P}$.

Lemma 3. Let a, c be Young diagrams which crspd to w, w' s.t.

$$\begin{cases} |c| = |a| + k \\ a_i \leq c_i \leq a_{i-1} \quad \forall i \end{cases}$$

Then $$\Sigma_a(\mathcal{V}^{st}) \cap \Sigma_c(w_0 \mathcal{V}^{st}) = \underbrace{\mathbb{P}^{w(1)+w'(r)-n-1} \times \cdots \times \mathbb{P}^{w(r)+w'(1)-n-1}}_{r \text{ many}}$$

E.g. $n=5$, $r=2$, write $\mathcal{V} := \mathcal{V}_{st}$, $\mathcal{W} = w_0 \mathcal{V}_{st}$,

$w =$  $= \left(\begin{array}{cc|ccc} 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{25 \mid 134\} \overset{\text{cohom}}{\sim}$  $\leftarrow a = (2,0)$

$w' =$  $= \left(\begin{array}{cc|ccc} 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & 1 & & & \\ & & & & 1 \end{array}\right) = \{24 \mid 135\} \overset{\text{cohom}}{\sim}$  $\leftarrow c = (2,1)$

We want to show $\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) \cong \mathbb{P}^0 \times \mathbb{P}^1$.

We write

$$A_1 := \mathcal{V}_2 \cap W_4 = \langle v_2 \rangle$$
$$A_2 := \mathcal{V}_5 \cap W_2 = \langle v_4, v_5 \rangle$$

then

$$\begin{aligned} \Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) &= Gr(1, A_1) \times Gr(1, A_2) \\ \Lambda &\longmapsto (\Lambda \cap A_1, \ \Lambda \cap A_2) \\ W_1 \oplus W_2 &\longleftarrow (W_1, \ W_2) \end{aligned}$$

Hint: ① $A \triangleq A_1 + A_2 = A_1 \oplus A_2$

② $\dim \Lambda \cap A_i \geq 1$

$$\begin{aligned} 2 = \dim \Lambda &= \dim \Lambda \cap (\mathcal{V}_2 + W_4) \\ &= \dim \Lambda \cap \mathcal{V}_2 + \dim \Lambda \cap W_4 - \dim \Lambda \cap A_1 \\ &\geq 1 + 2 - \dim \Lambda \cap A_1 \end{aligned}$$

③ $\dim \Lambda \cap A_i = 1$, $\quad \Lambda = \oplus \Lambda \cap A_i \qquad \Lambda \subset A$

$$\begin{aligned} 2 = \dim \Lambda &\geq \dim \Lambda \cap A \\ &\geq \dim \Lambda \cap A_1 + \dim \Lambda \cap A_2 \\ &\geq 1 + 1 = 2 \end{aligned}$$

Lemma 4. Let a, c be Young diagrams which crspd to w, w' s.t.

$$\begin{cases} |c| = |a| + k \\ a_i \leq c_i \leq a_{i-1} \quad \forall i \end{cases}$$

Let $(k, \dots, 0)$ be Young diagram which crspds to w''.
Let $\mathcal{V}, \mathcal{W}, \mathcal{U}$ be general complete flags in $\mathbb{C}^n$, then

$$\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) \cap \Sigma_k(\mathcal{U}) = \{*\}.$$

Proof. W.l.o.g. let $\mathcal{V} = \mathcal{V}^{st}$, $\mathcal{W} = \omega_0 \mathcal{V}^{st}$. [3264, Def 4.4]
We know

$$\Sigma_a(\mathcal{V}) \cap \Sigma_c(\mathcal{W}) = \prod_{i=1}^{r} Gr(1, A_i)$$

$$\Sigma_k(\mathcal{U}) = \left\{ \Lambda \in Gr(r,n) \,\middle|\, \dim \Lambda \cap \mathcal{U}_{n-r+1-k} \geq 1 \right\}$$

By transversality, $\dim A \cap \mathcal{U}_{n-r+1-k} = 1$. $\Rightarrow \Lambda \supset A \cap \mathcal{U}_{n-r+1-k}$
Define

$$\psi_i : A \cap \mathcal{U}_{n-r+1-k} \subset A \twoheadrightarrow A_i$$

Claim: $\Lambda \cap A_i = \operatorname{Im} \psi_i$

[
$\Lambda \cap \mathcal{U}_{n-r+1-k} \subseteq A \cap \mathcal{U}_{n-r+1-k}$ with equal dim
$\Rightarrow \Lambda \cap \mathcal{U}_{n-r+1-k} = A \cap \mathcal{U}_{n-r+1-k}$
$\Rightarrow \operatorname{Im} \psi_i \subseteq \Lambda \cap A_i$ with equal dim
$\Rightarrow \operatorname{Im} \psi_i = \Lambda \cap A_i$
]

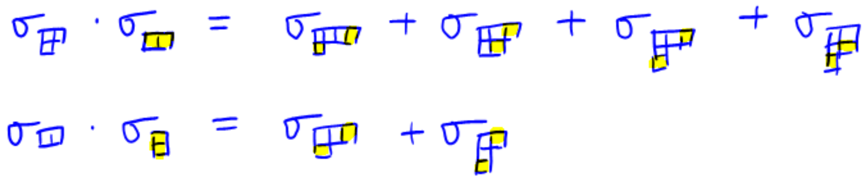
Therefore, $\Lambda = \bigoplus_i \Lambda \cap A_i = \bigoplus_i \operatorname{Im} \psi_i$ is uniquely determined. $\square$

Write Lemma 4 in terms of cohomology class, we get Pieri's formula: [3264, Prop 4.9, Thm 4.14]

$$\sigma_a \cdot \sigma_{(k,\dots,0)} = \sum_{\substack{|c|=|a|+k \\ a_i \le c_i \le a_{i-1}}} \sigma_c$$

$$\sigma_a \cdot \sigma_{(\underbrace{1,\dots,1}_{k\text{-many}},\dots 0)} = \sum_{\substack{|c|=|a|+k \\ a_i \le c_i \le a_i+1}} \sigma_c$$

E.g.



We will play with Young diagrams in the next section.

3. Young diagram formulas

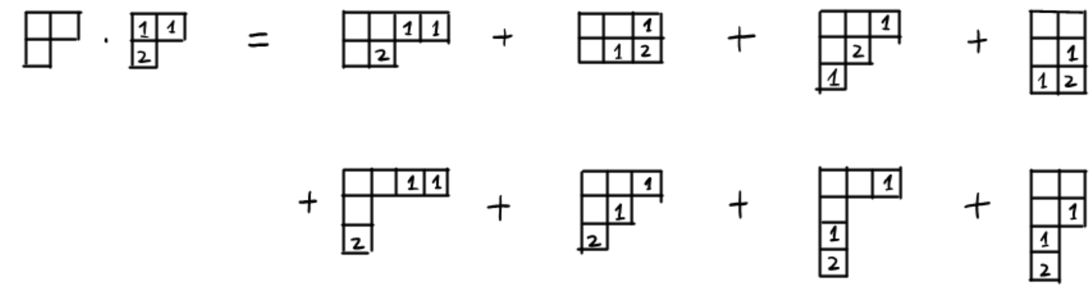
shifted Littlewood-Richardson rule:
https://arxiv.org/pdf/2503.14609

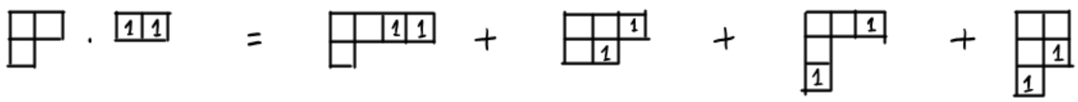
Littlewood - Richardson rule

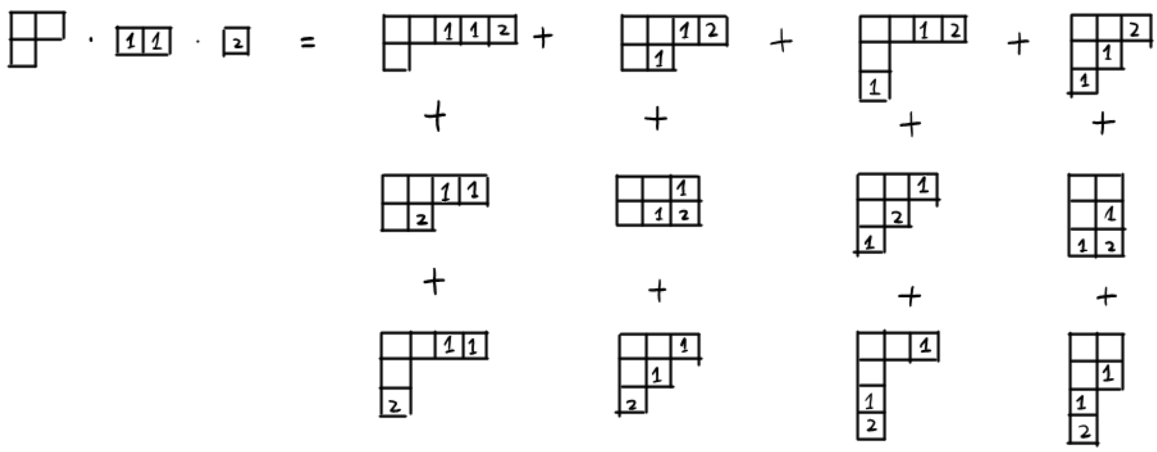
The Pieri formula can be upgraded to the Littlewood--Richardson rule:

$$\sigma_\lambda \, \sigma_\mu = \sum_\nu N_{\lambda\mu\nu} \, \sigma_\nu$$

E.g.







from: https://en.wikipedia.org/wiki/Littlewood%E2%80%93Richardson_rule

The Littlewood–Richardson rule is notorious for the number of errors that appeared prior to its complete, published proof. Several published attempts to prove it are incomplete, and it is particularly difficult to avoid errors when doing hand calculations with it: even the original example in D. E. Littlewood and A. R. Richardson (1934) contains an error.

That's why I don't want to prove it (using only Pieri formula).

Giambelli's formula

This formula expresses σ_λ as polynomials in σ_k.

Ex. [3264, Prop 4.16]
Show that (by induction)

$$\begin{array}{c} \xrightarrow{\text{index } +1} \\ \sigma_{(\lambda_1,\dots,\lambda_k)} = \begin{vmatrix} \sigma_{\lambda_1} & \cdots & \sigma_{\lambda_1+k-1} \\ & \ddots & \\ \sigma_{\lambda_k-k+1} & \cdots & \sigma_{\lambda_k} \end{vmatrix} \end{array}$$

e.p.

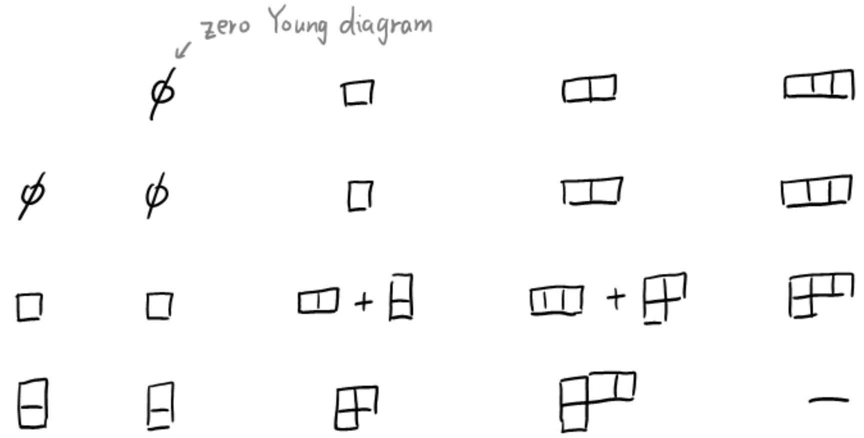
$$\sigma_{(1,\dots,1)} = \begin{vmatrix} \sigma_1 & \cdots & \sigma_k \\ 1 & \ddots & \vdots \\ 0 & 1 & \sigma_1 \end{vmatrix} = \begin{cases} \sigma_1 \\ \sigma_2 - \sigma_1^2 \\ \sigma_3 - \sigma_1\sigma_2 - (\sigma_2 - \sigma_1^2)\sigma_1 \\ \sigma_4 - \sigma_1\sigma_3 - (\sigma_2 - \sigma_1^2)\sigma_2 - (\sigma_{(1,1,1)})\sigma_1 \\ \vdots \end{cases}$$

Relations in $H^\cdot(Gr(r,n);\mathbb{Z})$

E.g. [3264, Cor 4.10]

$$(1+\sigma_1+\cdots+\sigma_{n-r})(1-\sigma_1+\cdots+(-1)^r\sigma_{1^r}) = 1$$
$$(1-\sigma_1+\cdots+(-1)^{n-r}\sigma_{n-r})(1+\sigma_1+\cdots+\sigma_{1^r}) = 1$$

In $Gr(5,2)$, we list the table of products for a hint:



Thm [3264, Thm 5.26]

$$H^\cdot(Gr(r,n);\mathbb{Z}) \cong \mathbb{Z}[c_1,\dots,c_r]/I$$

where

$$c_k = c_k(S) = (-1)^k\sigma_{1^k}$$

$$I = \left\langle \left(\frac{1}{1+c_1+\cdots+c_r}\right)^{\deg=n-r+1}, \dots, \left(\frac{1}{1+c_1+\cdots+c_r}\right)^{\deg=n} \right\rangle$$

$$= \langle \sigma_{n-r+1}, \dots, \sigma_n \rangle$$

$$\frac{1}{1+c_1+\cdots+c_r} = 1-(c_1+\cdots+c_r)+(c_1+\cdots+c_r)^2-\cdots$$

$$\overset{r\geq 5}{=\!=} 1 - c_1 + (c_1^2-c_2) + (c_1^3-2c_1c_2+c_3)$$
$$+ (c_1^4-3c_1^2c_2+c_2^2+2c_1c_3-c_4)$$
$$+ (c_1^5-4c_1^3c_2+3c_1c_2^2+3c_1^2c_3-2c_2c_3-2c_1c_4+c_5)$$
$$+ \cdots$$

$$= 1+\sigma_1+(\sigma_1^2-\sigma_{1^2})+(-\sigma_1^3+2\sigma_1\sigma_{1^2}-\sigma_{1^3})+\cdots$$

Q: How to describe I' in

$$H^{\cdot}(Gr(r,n);\mathbb{Z}) \cong \mathbb{Z}[\sigma_1,\dots,\sigma_{n-r}]/_{I'} \quad ?$$

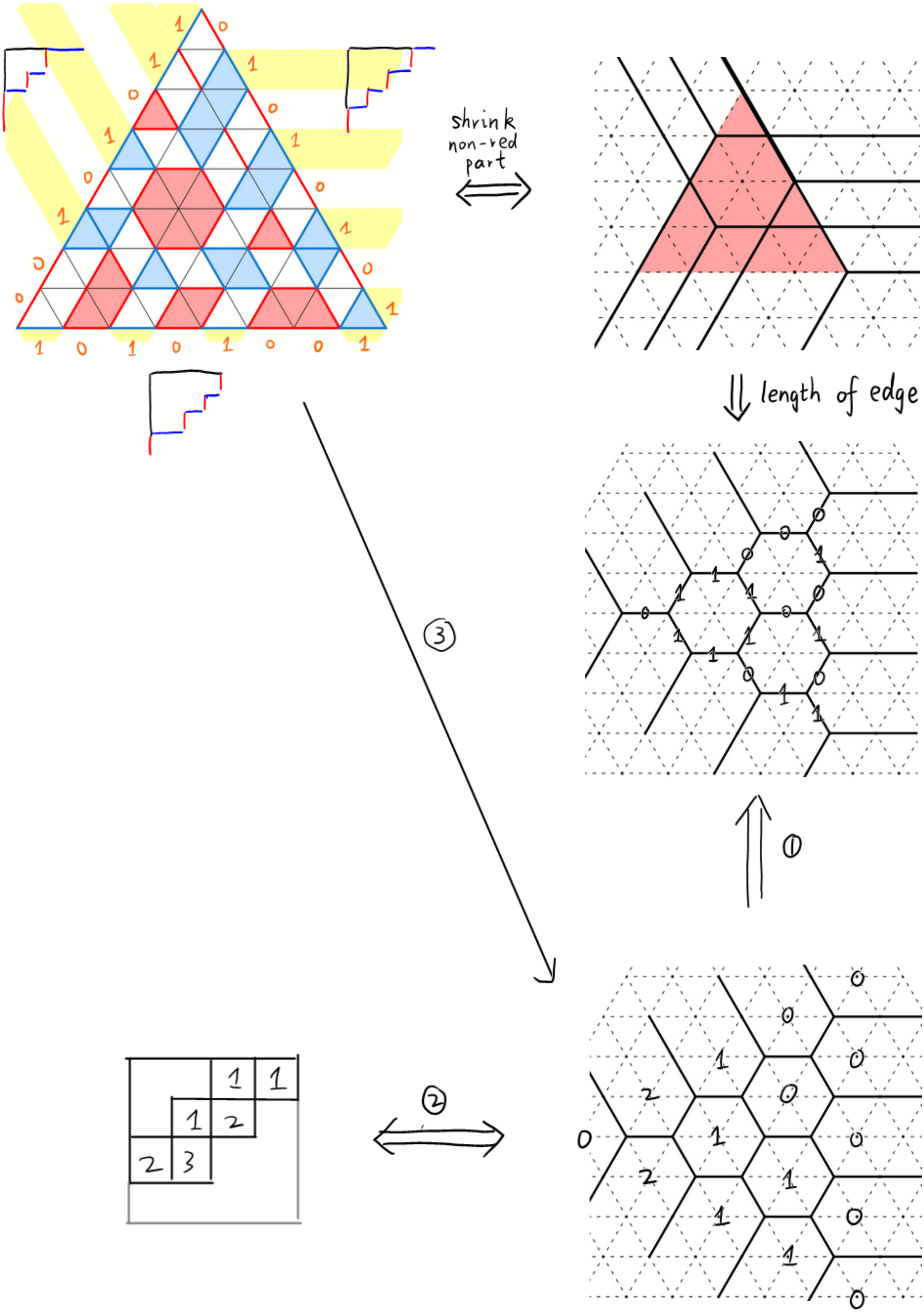
Guess: by duality [3264, Ex 4.31],

$$I' = \left\langle \left(\frac{1}{1-\sigma_1+\cdots+(-1)^{n-r}\sigma_{n-r}}\right)^{\deg = r+1}, \dots, \left(\frac{1}{1-\sigma_1+\cdots+(-1)^{n-r}\sigma_{n-r}}\right)^{\deg = n} \right\rangle$$

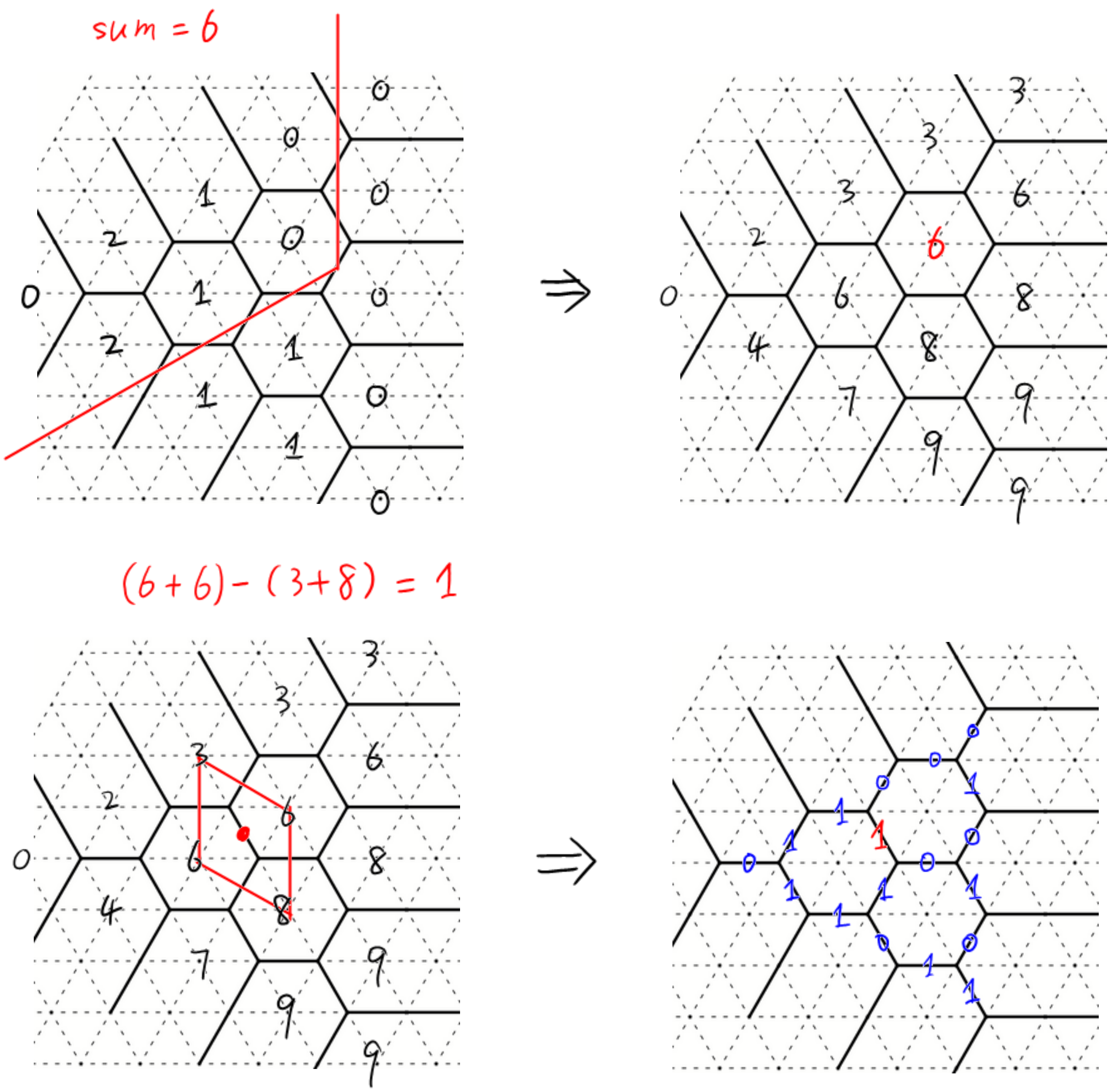
$$= \langle \sigma_1^{r+1}, \dots, \sigma_1^{n} \rangle$$

4. Other combinatorial models for LR coefficients

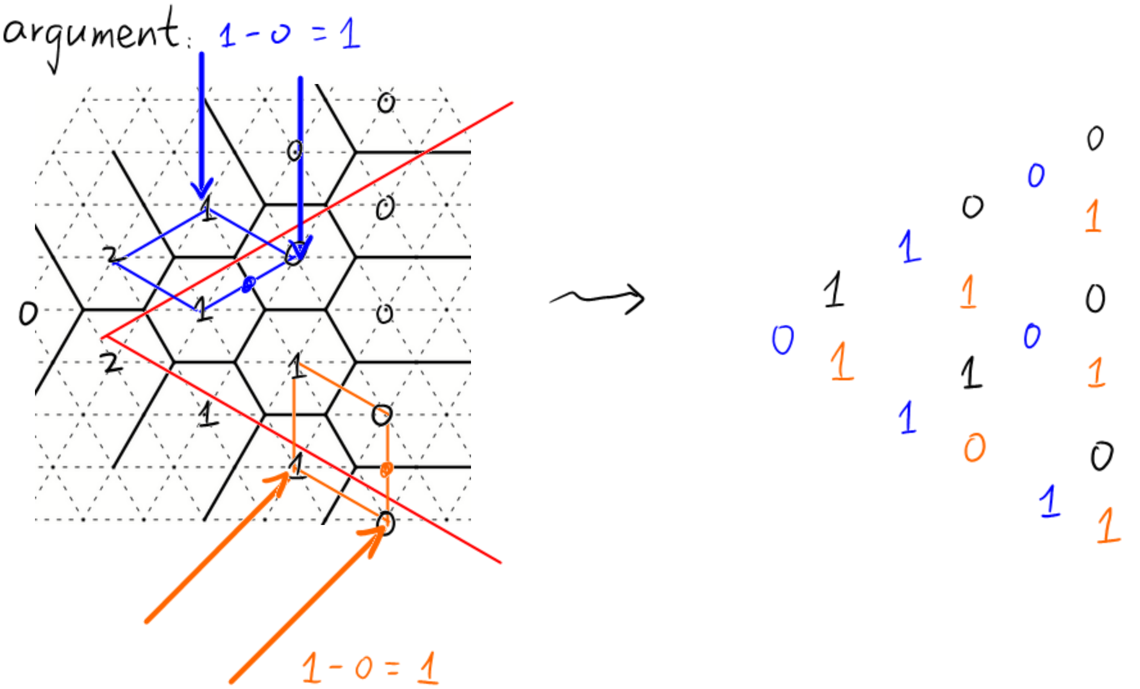
I would highly recommand to read [PV04] before this section. It clarifies the correspondence clearly.
The main goal is the correspondence among these combinatorial models, especially the Knutson--Tao puzzle and the Young diagram.



①:

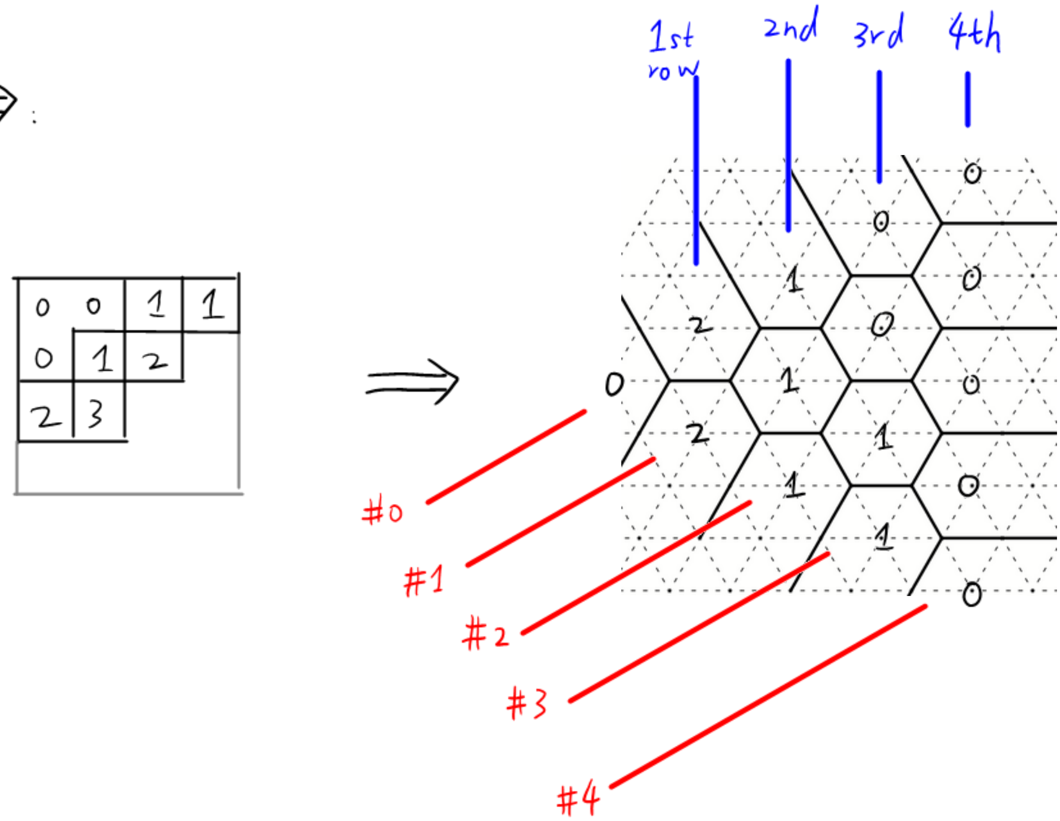


Direct argument:

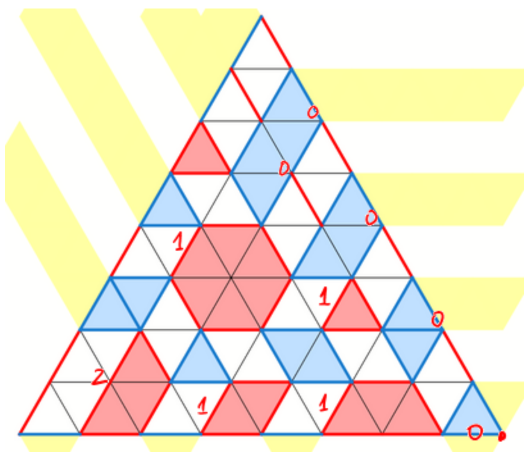


⇑: Read the black one gives 1 $\begin{smallmatrix} & & 0 \\ & 0 & \\ 1 & & 0 \\ & 1 & \\ & & 0 \end{smallmatrix}$, and the rest is easy.

② : ⇒ :



③ ⇓ :



reading the length of the NW side gives the numbers.