Eine Woche, ein Beispiel

4.6. Curves in $\mathbb{P}^r$

Ref:

[Ar85]: Arbarello, E., M. Cornalba, P. A. Griffiths and J. Harris. Geometry of Algebraic Curves Volume I. Grundlehren Math. Wiss. Springer, Cham, 1985.

Here, we try to recollect the results in [Ar85, Chap III]. Since I learn it for the first time, the goal is to know what kind of theorems there are, but not about their proofs.

Thm [Ar85, p116] (Castelnuovo's bound)

Let $\mathcal{C}/\mathbb{C}$: smooth curve

$\phi: \mathcal{C} \longrightarrow \mathbb{P}^r$ birational to the image

$\mathrm{Im}\, f \subset \mathbb{P}^r$ non-degenerate with degree d.

Denote

$$d-1 = m(r-1)+\varepsilon \qquad m \in \mathbb{Z}_{\geq 0},\ 0 \leq \varepsilon < r-1$$

then

$$g(C) \leq \binom{m}{2}(r-1)+\varepsilon \quad = md - \binom{m+1}{2}(r-1) - m$$
$$= -\frac{r-1}{2}m^2 + \left(d - \frac{r+1}{2}\right)m$$

When "=" holds, (C,ϕ) is called the extremal curves.

Thm [Ar85, p117] (Max Noether's Theorem)

For $\mathcal{C}/\mathbb{C}$ non-hyperelliptic, $l \geq 1$, the map

$$\mathrm{Sym}^l H^0(\mathcal{C}, \omega_C) \longrightarrow H^0(\mathcal{C}, \omega_C^{\otimes l}) \text{ is surjective.}$$

Thm [Ar85, p122]

Let $r \geqslant 3$, $m \geqslant 2$. $\exists\, \phi : C \longrightarrow \mathbb{P}^r$ extremal curve, and it is one of the following cases:

① $\mathcal{C} \overset{\deg \frac{d}{2}}{\subset} \mathbb{P}^2 \longrightarrow \mathbb{P}^5$

② $\mathcal{C} = V(s) \longrightarrow S \subset \mathbb{P}^{n+1}$

where S is a rational normal scroll, i.e.,

rational: $\overset{bir}{\sim} \mathbb{P}^2$ normal: projective normality scroll: ruled surface

a ruled surface in $\mathbb{P}^{n+1}$ of degree n.

wiki: rational normal scroll

$$\begin{array}{c|cc} & H & L \\ \hline H & r-1 & 1 \\ L & 1 & 0 \end{array}$$

$\mathrm{Pic}(S) \cong \mathbb{Z}H \oplus \mathbb{Z}L$

H: hyperplane intersection

L: a line of ruling

$\mathcal{L} = \mathcal{O}_S(mH+L)$ or $\mathcal{O}_S((m+1)H-(r-\varepsilon-2)L)$, $s \in H^0(S, \mathcal{L})$

Thm [Ar85, p123]

Suppose $\mathcal{C} \subset \mathbb{P}^r$ is an integral non-degenerate curve of

degree $d \geqslant 2r+3$

genus $g > \pi_1(d, r)$

where

$$d-1 = m_1 r + \varepsilon_1 \qquad m_1 \in \mathbb{Z}_{\geq 0},\ 0 \leq \varepsilon_1 < r$$

$$\mu_1 = \begin{cases} 1, & \varepsilon_1 = r-1, \\ 0, & \varepsilon_1 \neq r-1, \end{cases}$$

$$\pi_1(d, r) = \binom{m_1}{2} r + m_1(\varepsilon_1 + 1) + \mu_1$$

Then $\mathcal{C}$ lies on a surface of degree $r-1$.

Thm [Ar85, p124] (Enriques - Babbage Theorem)

Let $\phi: C \longrightarrow \mathbb{P}^{g-1}$ be a canonical curve, then either

① C is set-theoretically cut out by quadrics, or

② C is trigonal, i.e., C has g^1_3, or

i.e., $\exists$ 3 : 1 ramified cover $C \to \mathbb{P}^1$

③ $C \cong$ smooth plane quintic.

Thm [Ar85, p126] (Base-point-free pencil trick)

Let

$C/\mathbb{C}$: sm curve

$\mathcal{L}/C$: l.b. $\quad \mathcal{F}/C$: torsion-free $\mathcal{O}_C$-module

$s_1, s_2 \in \Gamma(\mathcal{L})$: linearly independent,

$V := \langle s_1, s_2 \rangle \subset \Gamma(\mathcal{L})$

$B := V(s_1) \cap V(s_2)$: base locus of V.

Then we have a SES

$$0 \longrightarrow H^0(C, \mathcal{F} \otimes \mathcal{L}^{-1}(B)) \longrightarrow V \otimes H^0(C, \mathcal{F}) \longrightarrow H^0(C, \mathcal{F} \otimes \mathcal{L})$$

A simplified version:

Prop & Proof

Let $(\mathcal{L}, V)$ be a linear series of dim 2 s.t.

$$V \otimes \mathcal{O}_C \twoheadrightarrow \mathcal{L}$$

is surj. Denote $V = \langle s_1, s_2 \rangle$, then the Koszul cplx

$$0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \underset{\overset{\|}{V \otimes \mathcal{L}^{-1}}}{\mathcal{L}^{-1} \oplus \mathcal{L}^{-1}} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

is exact,

$- \otimes \mathcal{L}$:

$$0 \longrightarrow \mathcal{L}^{-1} \longrightarrow V \otimes \mathcal{O}_C \longrightarrow \mathcal{L} \longrightarrow 0$$

$- \otimes \mathcal{F}$: $\mathcal{F}$ l.b.

$$0 \longrightarrow \mathcal{L}^{-1} \otimes \mathcal{F} \longrightarrow V \otimes \mathcal{F} \longrightarrow \mathcal{L} \otimes \mathcal{F} \longrightarrow 0$$

$H^\cdot(-)$:

$$\begin{array}{l} \longrightarrow H^1(\mathcal{L}^{-1} \otimes \mathcal{F}) \longrightarrow V \otimes H^1(\mathcal{F}) \longrightarrow H^1(\mathcal{L} \otimes \mathcal{F}) \\ 0 \longrightarrow H^0(\mathcal{L}^{-1} \otimes \mathcal{F}) \longrightarrow V \otimes H^0(\mathcal{F}) \longrightarrow H^0(\mathcal{L} \otimes \mathcal{F}) \end{array}$$

In ptc, when $\mathcal{L} = \mathcal{O}(D) \in g^1_d$ with no base pt, $\mathcal{F} = \mathcal{O}(D')$, we get SES

$$0 \longrightarrow H^0(D' - D) \longrightarrow H^0(D) \otimes H^0(D') \longrightarrow H^0(D' + D)$$

Thm [Ar85, p131] (Petri's Theorem)

It describes the ideal of a canonical curve (of genus ≥ 4).