Eine Woche, ein Beispiel

3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

[svd-notes]: Notes on singular value decomposition for Math 54
https://math.berkeley.edu/~hutching/teach/54-2017/svd-notes.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \curvearrowright X$

discrete $\Rightarrow$ foundamental domain $\qquad \Lambda \curvearrowright \mathbb{C} \qquad SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$

non discrete $\Rightarrow$ stratification by G/G_x $\qquad S^1 \curvearrowright S^2 \qquad \mathbb{C}^\times \curvearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g. $\mathcal{F}lag(V) \cong GL(V)/P$ $\qquad$ e.p. Grassmannian, $\mathbb{P}^n$

$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$

$O(n) := O(n,\mathbb{R})$ $\qquad \rightsquigarrow$ Stiefel mfld [21.11.14]

$SO(n) := SO(n,\mathbb{R})$

$\mathbb{A}^n = \mathbb{A}^n$

$\mathcal{H}^n \cong O^+(1,n)/O(n)$

$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$

$\rightsquigarrow$ Hermitian symmetric space

where $\mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v,v\rangle = -1,\ v_{n+1} > 0\}$

$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad \langle v,w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$

$O(n,1) := \mathrm{Aut}\,(\mathbb{R}^{n+1}, \langle\ ,\ \rangle) \subseteq GL_{n+1}(\mathbb{R})$

$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

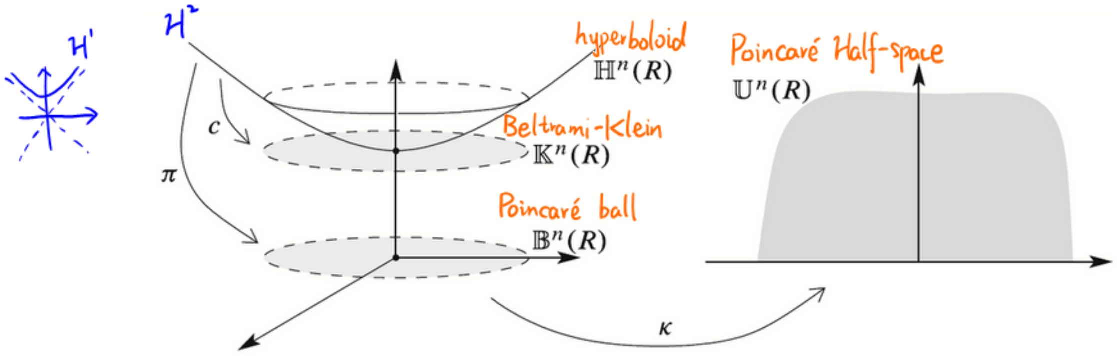


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-01-2

Terminology.

$$\begin{tikzcd}
{} \arrow[d, "X + G \curvearrowright X"] & & \\
G\text{-space} \arrow[d, "\text{transitive}"] & & \\
\text{homogeneous space} \arrow[d, "\substack{\text{Riemannian mfld} \\ G \text{ preserve } g}"'] \arrow[dr, "\substack{\text{Hermitian mfld} \\ G \text{ preserve } g \leftarrow \text{metric}}"] & & \\
\text{Riemannian homogeneous space} \arrow[r, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\substack{\text{geodesic symmetry} \\ \text{globally defined, preserve } g}"'] & \text{Hermitian homogeneous space} \arrow[d, "\substack{\text{geodesic symmetry} \\ \text{globally defined, preserve } g}"] \\
\text{Symmetric space} \arrow[r, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\text{transitive on frames}"'] & \text{Hermitian symmetric space} \\
\text{frame homogeneous space} &
\end{tikzcd}$$

[JL18, Cor 3.18]: Nice hom space can have compatible Rie mfld structures.

$\mathbb{C}\mathbb{P}^n$ $n \geq 1$

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

2. double coset decomposition: schedule

$$G = \bigsqcup_{\alpha\in I} H\alpha K$$

usually, H, K are easier than G.
- comes from (usually) Gauss elimination
- I is the "foundamental domain"
- produces stratifications on G/K and $_H\backslash G$ indexed by I.

To be exact,

$$G/K = \bigsqcup_{\alpha\in I} H\alpha K/K \cong \bigsqcup_{\alpha\in I} H/H_{[\alpha K]} = \bigsqcup_{\alpha\in I} H/(H\cap\alpha K\alpha^{-1})$$

$$_H\backslash G = \bigsqcup_{\alpha\in I} {}_H\backslash H\alpha K \cong \bigsqcup_{\alpha\in I} K_{[H\alpha]}\backslash K = \bigsqcup_{\alpha\in I} (K\cap\alpha^{-1}H\alpha)\backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$
$K_{[H\alpha]}$: stabilizer of K on $[H\alpha] \in {}_H\backslash G$

$$\# \; H/H\cap\alpha K\alpha^{-1} = \# \left\{ \begin{array}{l} \text{single cosets } [gK] \\ \text{in one double coset } H\alpha K \end{array} \right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad {}_H\backslash G \qquad \text{individually}$$

- can be viewed as stack quotient.
$[*/G]$: groupoid

$${}_H\backslash G/K \overset{\text{def}}{=} [*/H] \times_{[*/G]} [*/K] \quad \text{with groupoid structure}$$

$$H^*_H(G/K) \cong H^*({}_H\backslash G/K) \cong H^*_K({}_H\backslash G)$$

slogan: the (equiv) cohomology of G/K and $_H\backslash G$ are connected.

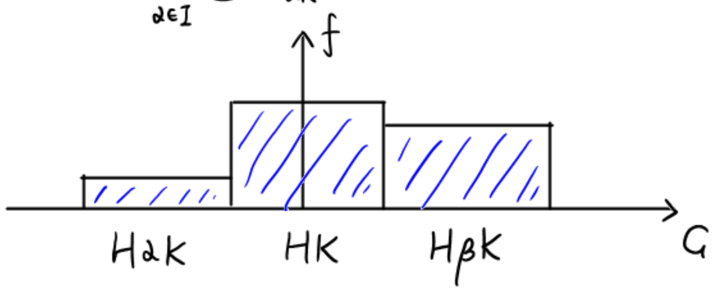
- Hecke algebra $\mathcal{H}(H\backslash G/K)$

for $H=K$. You can also do $\mathcal{H}(H_1\backslash G/H_2) \hookrightarrow \bigoplus_{i,j=1}^{2} \mathcal{H}(H_i\backslash G/H_j)$

$\mathcal{H}(H\backslash G/K)$: reasonable subspaces of

$$\mathbb{C}[H\backslash G/K] = \{f: G \to \mathbb{C} \mid f(hgk) = f(g) \quad \forall h\in H, g\in G, k\in K\}$$

$$\overset{\text{"o-dim"}}{\supseteq} \bigoplus_{\alpha\in I} \mathbb{C}\, \mathbb{1}_{H\alpha K}$$



with reasonable convolution structure

$$*: \mathcal{H}(H_1\backslash G/H_2) \times \mathcal{H}(H_2\backslash G/H_3) \longrightarrow \mathcal{H}(H_1\backslash G/H_3)$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

Vague: $\mathcal{H}(H\backslash G/K) \sim H^*(H\backslash G/K)$ should be a type of cohomology

$\mathcal{H}(G) \overset{G \text{ fin}}{=\!=} \mathbb{C}[G]$

$\mathcal{H}(K\backslash G/K) \cong (\mathrm{End}(c\text{-}\mathrm{Ind}_K^G \mathbb{1}_K))^{op}$ should be a type of base ring

Generalize: $\mathrm{Ind}_H^G \chi \approx \mathcal{H}_\chi(H\backslash G/K) \subseteq \{f: G \to \mathbb{C} \mid f(hgk) = \chi(h) f(g)\}$

depth of χ

3. examples (after [22.09.04])

Works over:

- list of possibilities
- moduli intepretation
- typical examples

finite field, $GL_n(\mathbb{F}_q)$ (Applies to any field κ, actually)

- subgps can be

Borel	max split torus	unipotent
B	T	N
parabolic	Levi	unipotent
P	L	M
	nonsplit torus	
	T'	

$$\square \backslash GL_n(\kappa) / \square$$

- moduli intepretation $V = \kappa^{\oplus n}$

$$G/B = \{ \text{cpl flags in } V \}$$

$$G/T = \{ (V_i)_{i=1}^n \mid V = \bigoplus_{i=1}^n V_i,\ \dim V_i = 1 \}$$

$$G/N = \left\{ (\mathcal{F}, m_i) \,\middle|\, \begin{array}{l} \mathcal{F}: 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = V \text{ cpl} \\ 0 \neq m_i \in M_i / M_{i-1} \end{array} \right\}$$

$$G/P = \{ \text{flags in } V \}$$

$$G/L = \{ (V_i)_{i \in I} \mid V = \bigoplus_{i \in I} V_i \}$$

$$G/M = \left\{ (\mathcal{F}, \mathcal{B}_i) \,\middle|\, \begin{array}{l} \mathcal{F}: 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = V \\ \mathcal{B}_i: \text{ a basis of } M_i / M_{i-1} \end{array} \right\}$$

Rmk. We have a fiber bundle

$$\begin{array}{ccc} \mathbb{A}^{\oplus\binom{n}{2}} \cong B/N & \longrightarrow & G/N \\ & & \downarrow \\ & & G/B \end{array}$$

which makes G/N a $\mathbb{A}^{\oplus\binom{n}{2}}$-torsor over G/B

⚠ G/N is not a $\underbrace{\text{rk } \binom{n}{2} \text{ v.b}}_{\text{i.e. } GL_{\binom{n}{2}}(\kappa)\text{-torsor}}$ over G/B, so G/N can be affine space.

- E.g. Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

  - Gauss elimination gives "$\subseteq$",
    while the observation of process gives "$\sqcup$" (Something is invariant)
  - the "fundamental domain" W has a gp structure, and crsp to B-orbits of G/B.
    gp structure comes from Tits system
  - produces an affine paving of G/B, and the Zariski topo gives Bruhat order
    works also for Euclidean topo, $\kappa = \mathbb{R}$ or $\mathbb{C}$.
  - $B\backslash{}^{G}/_{B} = [{}^{*}/_{B}] \times_{[{}^{*}/_{G}]} [{}^{*}/_{B}]$, with
    $$H_B^*(G/B) \cong H_T^*(G/B) \cong \bigoplus_{w} H_T^*(pt)$$ [my master thesis]
  - $\mathcal{H}(G,B)$: see [22.09.04]

- possible exercise:
  - Work out

$$T\backslash{}^{G}/_{B}$$

$$P_1\backslash{}^{G}/_{P_2} \quad GL_m\times GL_n\backslash{}^{GL_{m+n}}/_{GL_m\times GL_n} \qquad S_m\times S_n\backslash{}^{S_{m+n}}/_{S_m\times S_n}$$ [22.11.13]

$$\mathbb{F}_{q^n}^{\times}\backslash{}^{GL_n(\mathbb{F}_q)}/_{B}, \quad \ldots$$

  $\kappa = \mathbb{F}_1$; $GL_n \rightsquigarrow$ other gps
  - Computation of cardinals.

Archi field, $\mathbb{R}$ or $\mathbb{C}$

– subgps can be

nearly affine			cpt
Borel	max split torus	unipotent	
B	T	N	$O(n)$
parabolic	Levi	unipotent	or $SO(n)$
P	L	M	
	nonsplit torus		$\mathcal{U}(n) := \mathcal{U}_{\mathbb{C}/\mathbb{R}}(n)$
	T'		or $SU(n)$

$GL_n(\mathbb{R})$

$GL_n(\mathbb{C})$

+ real & cplx

https://mathoverflow.net/questions/249313/real-orbits-on-flag-varieties

∇ $M^{sym}_{n\times n}(\mathbb{R})$, $M^{sym,>0}_{n\times n}(\mathbb{R})$ are not gps!

– moduli intepretation

$V := \mathbb{R}^{\oplus n}$ In $\mathbb{C}^{\oplus n}$ case, replace inner product by Hermittian prod.

$$G/O(n) \cong \{\text{inner products on } V\} \cong M^{sym,>0}_{n\times n}(\mathbb{R})$$

$$g = (v_1, \cdots, v_n) \longmapsto \langle\, ,\, \rangle \text{ s.t. } \{v_1, \cdots, v_n\} \text{ is an ortho basis} \longmapsto (\langle e_i, e_j\rangle)^n_{i,j=1}$$

$v_i = ge_i$ i.e. $\langle x, y\rangle := x^T (g^{-1})^T g^{-1} y$

$$g \longmapsto (g^{-1})^T g^{-1}$$

as G-sets, where

$g.x := gx$ $\quad g.\langle\, ,\, \rangle := \langle g^{-1}-, g^{-1}-\rangle$ $\quad g.A := (g^{-1})^T A g^{-1}$

i.e. $\langle gx, gy\rangle_g = \langle x, y\rangle$

action on inner product

Rmk. We actually get the polar decomposition here. not hard, but not obvious

$$GL_n(\mathbb{R}) = M^{sym,>0}_{n\times n}(\mathbb{R})\, O(n) \qquad GL_n(\mathbb{C}) = M^{herm,>0}_{n\times n}(\mathbb{C})\, \mathcal{U}(n)$$

Ex.
$$\begin{aligned}
\mathcal{H} &\cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{\geq 0} \cong SL_2(\mathbb{R})/SO(2) \\
&\cong \{\text{inner products on } V\}/\text{scalars} \\
&\cong M^{sym,>0}_{n\times n}(\mathbb{R})/\text{scalars} \\
&\cong \{\text{max cpt subgps of } GL_2(\mathbb{R})\}
\end{aligned}$$

Lemma 1,2

Lemma 1. cpt subgps are conj to a subgp of $O(2)$.

[Idea of proof. $K \hookrightarrow GL_2(\mathbb{R}) \circlearrowright \mathcal{H}$ maps bounded set to bounded set
$\Rightarrow K$ preserves one pt in $\mathcal{H}$]

Lemma 2. $gO(2)g^{-1} = O(2) \iff g \in O(2)\cdot\mathbb{R}_{\geq 0}$

[Idea of proof. use $G \circlearrowright \mathcal{H}$ or SVD ← shown later]

– singular value decomposition (SVD) [svd-notes]

$$GL_n(\mathbb{R}) = \bigsqcup_{a_1 \geq a_2 \cdots a_n > 0} \mathcal{O}(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mathcal{O}(n)$$

$$GL_n(\mathbb{C}) = \bigsqcup_{a_1 \geq a_2 \cdots a_n > 0} \mathcal{U}(n) \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \mathcal{U}(n)$$

- "$\subseteq$", lazy proof.
  When $A \in GL_n(\mathbb{R})$ is symmetric, $A \underset{\sim}{\overset{\mathcal{O}(n)\text{-conj}}{}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \in \mathbb{R}^\times$.
  When $A \in GL_n(\mathbb{C})$ is normal matrix, $A \underset{\sim}{\overset{\mathcal{U}(n)\text{-conj}}{}} \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \in \mathbb{C}^\times$
  One can then use polar dec to show SVD.
- "$\sqcup$", algorithm.
  Suppose $A = U \Sigma V^T \in \mathcal{O}(n) \Sigma \mathcal{O}(n)$ $\Sigma = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$. $a_i \in \mathbb{R}_{>0}$
  Observe that

$$A^T A = V \Sigma^T \Sigma V^T = V \begin{pmatrix} a_1^2 & & \\ & \ddots & \\ & & a_n^2 \end{pmatrix} V^{-1}$$

$\Rightarrow$ eigenvalues of $A^T A$ tells us Σ.

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization