Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases
4. conclusions on various forms

Setting. We work over conn red gp over F. ($G/_F$ conn red)

$\bar{F}$: the seperable closure of F   mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_F := Gal(\bar{F}/F)$ $\sigma \in \Gamma_F$

$$Z^1(W,A) := \left\{ \begin{array}{c} {}^L\varphi : W \longrightarrow A \rtimes W \\ \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{array} \middle| \begin{array}{l} {}^L\varphi : \textit{continuous group homo} \\ {}^L\varphi \ \textit{is a section} \end{array} \right\}$$

$$= \{ \varphi : W \longrightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)^{\gamma}(\varphi(\gamma')) \}$$

$\varphi \in H^1(W, A)$

$$H^1(W,A) = Z^1(W,A)/A.$$

Def. (Split)   G is split if
G has a maximal torus T over F which is split.

Def. (Quasi-split) G is quasi-split if
G has a Borel B over F.

Borel = maximal (Zar-closed) conn sol alg subgp
= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t G/H is projective
= closed subgp containing a Borel.

Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat{Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability
https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf

https://mathoverflow.net/questions/121959/classification-of-tori-of-gl2-up-to-conjugation

I make no claim to originality.

1. forms.

Def. $G_1, G_2/F$ are called forms, if

$$\exists\ \alpha : G_{2,\bar F} \xrightarrow{\sim} G_{1,\bar F} \quad \text{as gps} \qquad \text{not as } \Gamma_F\text{-gps!}$$

α is considered as the information of forms.

Thm.

$$\{F\text{-forms of } G\}/\sim \ \longleftrightarrow\ H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$$

$$[G_2,\ \alpha : G_{2,\bar F} \longrightarrow G_{\bar F}] \longmapsto \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1}$$

$$G_2 \longleftarrow\!\shortmid\ \varphi$$

$$G_2(F) := \{ g \in G(\bar F) \mid \underbrace{(\varphi(\sigma)\circ\sigma)\, g}_{\text{induced action}} = g \quad \forall \sigma \in \Gamma_F \}$$

In general, $G_2(R) := \{ g \in G(\bar F \otimes_F R) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_F \}$

e.p. $G_2(\bar F) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_F} \in \prod_{\sigma\in\Gamma_F} G(\bar F) \mid g \in G(\bar F) \} \cong G(\bar F)$

$$G(\bar F \otimes_F \bar F) \cong G\Big(\bigoplus_{\sigma\in\Gamma_F} \bar F\Big) \cong \prod_{\sigma\in\Gamma_F} G(\bar F)$$

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} \end{tikzcd}$$

$$\Gamma_F \circlearrowright G_{2,\bar F} \longrightarrow G_{\bar F} \circlearrowleft \Gamma_F,\ \Gamma_F$$

φ_α measures the twist between two Γ_F-actions.

$(G_2, \alpha) \sim (G_2', \alpha')$, if
$\exists\ \beta : G_2 \longrightarrow G_2'$ as an iso.

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

Functorial on F. (Inflation - Restriction seq. [EC II, Appendix B, Prop 1.3])
Let E/F be finite Galois.

$$\begin{tikzcd}
G_{2,E} & \{E\text{-forms of } G\} \arrow[r, leftrightarrow] & H^1(\Gamma_E, \mathrm{Aut}(G_{\bar F})) & \varphi|_{\Gamma_E} \\
G_2 \arrow[u, mapsto] & \{F\text{-forms of } G\} \arrow[r, leftrightarrow] \arrow[u] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[u] & \varphi \arrow[u, mapsto] \\
& \{G_2/F : G_{2,E} \cong G_E\} \arrow[r, leftrightarrow] \arrow[u] & H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(G_{\bar F})^{\Gamma_E}) \arrow[u] & \\
& 1 \arrow[u] & 1 \arrow[u] &
\end{tikzcd}$$

$\mathrm{Aut}(G_{\bar F})^{\Gamma_E} = \mathrm{Aut}(G_E)$

Rmk. We have the classification of connected reductive gps:

$$\begin{aligned}
\{\text{split red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) \quad \simeq \Psi(G,B,T) \\
\text{G split} \quad \{\text{qs red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + \Gamma_F\text{-action} \\
&\qquad = (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, Out(G_{\bar F})) \\
\{\text{red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, Aut(G_{\bar F}))
\end{aligned}$$

To understand the result, the following isos are needed:

$$Aut(G_{\bar F}) \cong Inn(G_{\bar F}) \rtimes Aut(G, B, T, \{u_\alpha\})$$

$$\begin{aligned}
Out(G_{\bar F}) &\cong Aut(G, B, T, \{u_\alpha\}) && \text{for embedding} \\
&\cong Aut(\Psi(G,B,T)) && \text{for combinatorics}
\end{aligned}$$

Also, by the Hilbert 90, one has

$$H^1(\Gamma_F, Aut(G,B,T,\{u_\alpha\})) \cong H^1(\Gamma_F, Aut(G,B,T))$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} Aut(\mathbb{G}_m^n) & \subseteq & End(\mathbb{G}_m^n) \\ \parallel & & \parallel \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$Hom(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow n$

Therefore,

$$H^1(\Gamma_F, Aut(\mathbb{G}^n_{m,\bar{F}})) = H^1(\Gamma_F, GL_n(\mathbb{Z}))$$
$$= Hom_{Grp}(\Gamma_F, GL_n(\mathbb{Z}))/_{GL_n(\mathbb{Z})\text{-conj}}$$
$$\overset{\text{when } F=\mathbb{R}}{=\!=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = Id\}/_{GL_n(\mathbb{Z})\text{-conj}}$$

[Γ_F acts on $Aut(\mathbb{G}^n_{m,\bar{F}}) \subseteq End(\mathbb{G}^n_{m,\bar{F}})$ trivially:

see $\bar{F}$-pts, $n=1$:

$$\begin{array}{ccc} \bar{F}^\times & \xrightarrow{\alpha} & \bar{F}^\times \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma} \\ \bar{F}^\times & \xrightarrow{{}^\sigma\alpha} & \bar{F}^\times \end{array} \qquad \begin{array}{ccc} x & \longmapsto & x^n \\ \downarrow & & \downarrow \\ \sigma(x) & & \sigma(x^n) = \sigma(x)^n \end{array}$$

$\Rightarrow {}^\sigma\alpha = \alpha$]

E.g. $n=1$, $F=\mathbb{R}$

$$H^1(\Gamma_F, Aut(\mathbb{G}_{m,\bar{F}})) \cong \{1, -1\}$$

$\varphi(\sigma) = (-)^{-1}$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ \mathbb{G}_m & G = \,?\ SO_{2,\mathbb{R}} \end{array}$$

$$G(\mathbb{R}) = \{g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\}$$
$$= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\}$$
$$= \{g \in \mathbb{C}^\times \mid |g| = 1\}$$
$$= S^1$$
$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$
$$\Rightarrow G = Spec\ \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$

Check: $G(\mathbb{C}) = \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1=0\}$

$$= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1\}$$
$$= \left\{ (x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \;\middle|\; \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array} \right\}$$
$$\cong \mathbb{C}^\times$$

$$SO_2(K) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; x,y \in K,\ x^2+y^2 = 1 \right\}$$

E.g. $n=2$, $F=\mathbb{R}$

$$H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}^2_{m,\bar F})) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$\mathbb{G}_m^2 \quad \mathbb{G}_m \times SO_{2,\mathbb{R}} \quad (SO_{2,\mathbb{R}})^2 \quad G = ? \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; (\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$(\varphi(\sigma)\circ\sigma)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \varphi(\sigma) \begin{pmatrix} \bar z_1 \\ \bar z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar z_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Rmk. As a scheme,

$$\begin{aligned} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m &= \mathrm{Spec}\ \mathbb{C}[\mathbb{Z} \times \mathbb{Z}]^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\left(\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[s,s^{-1}] \right)^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\ \mathbb{C}[t,s,t^{-1},s^{-1}]^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\ \mathbb{R}\left[\tfrac{t+s}{2}, \tfrac{t-s}{2i}, t^{-1}s^{-1}\right] \\ &\cong \mathrm{Spec}\ \mathbb{R}[x,y,u]/((x^2+y^2)u-1) \end{aligned}$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\begin{array}{ccc} \mathbb{G}_m, & SO_{2,\mathbb{R}}, & \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \\ \updownarrow & \updownarrow & \updownarrow \\ 1 & -1 & \left(\begin{smallmatrix} & 1 \\ 1 & \end{smallmatrix}\right) \\ \updownarrow & \updownarrow & \updownarrow \end{array}$$

Fact^{dual}: $\mathrm{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{triv}, \mathbb{Z}_{sign}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

More details can be found here: https://personal.math.ubc.ca/~cass/research/pdf/realtori.pdf

The classification is much more general than we may think of. When I really understand this I would make a new document for this.
See: Dieudonné–Manin classification theorem in wiki
http://faculty.bicmr.pku.edu.cn/~dingyiwen/dm.pdf

Rmk. Using the same argument, one can show that

$\{ T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}^n_{m,\mathbb{F}_{p^2}} \}$ = products of $\mathbb{G}_m$, $\begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}$, $\mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\mathbb{G}_m$

The torus G crspd to -1: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$\begin{aligned} G(\mathbb{F}_p) &= \{ g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k \} \\ &= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid \underbrace{\varphi(\sigma)(a-b\xi)}_{= (a-b\xi)^{-1}} = a+b\xi \} \\ &= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2\varepsilon = 1 \} \\ &\cong \{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p) \} \end{aligned}$$

3. other cases.

By using the same methods introduced in last section, we can compute the (inner) forms of reductive gps.

	inner forms	outer forms
$\mathbb{G}_m$ $(\mathbb{G}_m)^2$ $(\mathbb{G}_m)^n$	ϕ	SO_2 $SO_2 \times \mathbb{G}_m$, $(SO_2)^2$, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ product of lower rank torus
$GL_{2,\mathbb{R}}$ $SL_{2,\mathbb{R}}$ $PGL_{2,\mathbb{R}}$	$\mathbb{H}^\times = GL_1(\mathbb{H}\otimes_{\mathbb{R}} -)$ $\mathbb{H}^{Nm=1} = SU_{2,\mathbb{C}/\mathbb{R}}$?	$(U_{2,\mathbb{C}/\mathbb{R},\omega} = U(1,1)\ \ U(2,0))$ ϕ ϕ
$GL_{n,\mathbb{R}}$ $SL_{n,\mathbb{R}}$ $PGL_{n,\mathbb{R}}$	? $GL_{n/2}^{Nm=1}(\mathbb{H}\otimes_{\mathbb{R}} -)$ when n even ?	$\left(U_{n,\mathbb{C}/\mathbb{R},\omega} = \begin{cases} U(\frac{n}{2},\frac{n}{2}) & n \text{ even} \\ U(\frac{n+1}{2},\frac{n-1}{2}) & n \text{ odd} \end{cases}\ \ U(p,q) \ldots\right)$ $SU(a,n-a)$ e.p. $SU(2,1)$ ← need clarification ?
$(SL_2)^2/\mathbb{R}$ $(SL_2)^3/\mathbb{R}$	$SL_2\times SU_2$, $(SU_2)^2$, ϕ? $(8-1)$ possibilities	$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$?

? : I have no time to compute / don't know any symbol to represent

: quasi-split gp

Compute $\mathrm{Aut}(G_{\bar{F}})$

Lemma. We understand $\mathrm{Aut}(G_{\bar F})$ quite well.

$$G(\bar F)/Z(G(\bar F)) = G^{ad}(\bar F) \qquad\qquad \mathrm{Aut}(\Psi_0)$$

$$1 \longrightarrow \mathrm{Inn}(G_{\bar F}) \longrightarrow \mathrm{Aut}(G_{\bar F}) \longrightarrow \mathrm{Out}(G_{\bar F}) \longrightarrow 1$$

G	$\mathrm{Inn}(G_{\bar F})$	$\mathrm{Aut}(G_{\bar F})$	$\mathrm{Out}(G_{\bar F})$	
T rk n	1	$GL_n(\mathbb{Z})$	$GL_n(\mathbb{Z})$	
$GL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$SL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$PGL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$n \geq 3$				
$GL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$	$\{\pm 1\}^{\oplus 2}$	
$SL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$PGL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$(SL_2)^2/\mathbb{R}$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	with different $\Gamma_{\mathbb{R}}$-action
$(SL_2)^n/\mathbb{R}$	$PGL_n(\mathbb{C})^n$	$PGL_n(\mathbb{C})^n \rtimes S^n$	S^n	

Compute $H^1(\Gamma_F, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP] **Theorem 12.7.7** *There is a functorial isomorphism* $\mathrm{H}^1(k, G) \to \pi_1(G)_{\Theta,\mathrm{tor}}$.

[DR09] Corollary 2.4.3. *The composition*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

is a bijection.

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field: Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
https://arxiv.org/abs/1401.5913

3.1. Theorem. *Let G, T_0, T, and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

global field: Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
https://arxiv.org/pdf/2303.04120.pdf

E.g. $G = SL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{lccccccccc}
G & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow & 1 \\
SL_{2,\mathbb{R}} & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) & & 1 & &
\end{array}$$

$$\begin{aligned}
H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(SL_{2,\mathbb{C}})) &= H^1(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C})) \\
&= \{\ 1,\quad \omega(\ \ )\omega^{-1}\} \qquad \omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \\
& \qquad \Downarrow \qquad\quad \Downarrow \\
& \quad SL_{2,\mathbb{C}} \qquad G = {\color{red}?} \qquad \mathbb{H}^{N_m = 1}
\end{aligned}$$

$$\begin{aligned}
G(\mathbb{R}) &= \{ g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}} \} \\
&= \{ g \in SL_2(\mathbb{C}) \mid \omega \bar{g} \omega^{-1} = g \} \\
&= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \;\middle|\; \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\
&= \left\{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in SL_2(\mathbb{C}) \right\} \\
&= \mathbb{H}^{N_m = 1}
\end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{lccccccccc}
G & 1 & \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow & 1 \\
GL_{2,\mathbb{R}} & & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) \rtimes \{\pm 1\} & & \{\pm 1\} & &
\end{array}$$

$$\begin{aligned}
H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(GL_{2,\mathbb{C}})) &= \{\ 1,\quad \omega(\ \ )\omega^{-1},\quad (\ \ )^{H,-1},\quad \omega(\ \ )^{H,-1}\omega^{-1} \} \qquad \omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \\
& \quad\ \Downarrow \qquad\quad \Downarrow \qquad\qquad \Downarrow \qquad\qquad\quad \Downarrow \\
& \ (GL_{2,\mathbb{C}} \qquad \mathbb{H}^{\times}) \qquad (\mathcal{U}(2,0) \qquad\quad U_{2,\mathbb{C}/\mathbb{R},\omega})
\end{aligned}$$

4. conclusions on various forms

$H^1(\Gamma_F, -)$ as parameter space

$$
\begin{tikzcd}
 & & & 1 \arrow[dr] & & 1 & & \\
 & & & & \mathrm{Inn}(G_{\bar F}) \arrow[ur] \arrow[dr] & & & \\
1 \arrow[r] & Z(G(\bar F)) \arrow[r] & G(\bar F) \arrow[urr] & & & \mathrm{Aut}(G_{\bar F}) \arrow[r] & \mathrm{Out}(G_{\bar F}) \arrow[r] & 1
\end{tikzcd}
$$

$$
\rightsquigarrow H^1(\Gamma_F, Z(G(\bar F))) \to \underset{\text{pure inner twist}}{H^1(\Gamma_F, G(\bar F))} \to \underset{\text{inner twist}}{H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))} \to \underset{\text{form}}{H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))} \to H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))
$$

$$H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \to H^2(\Gamma_F, Z(G(\bar F)))$$

$\{F\text{-forms of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$

$\{F\text{- inner forms of } G\}/\sim \;\longleftrightarrow\; \mathrm{Im}\big(H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))\big)$

$\{F\text{- inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$

$\{F\text{-pure inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, G(\bar F))$

G split: $\left\{\begin{array}{l} F\text{- forms of } G \\ \text{which are quasi-split}\end{array}\right\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) \cong H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))$

$\qquad\qquad H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) = \Gamma_F\text{-actions on } (X^*, \Delta, X_*, \Delta^\vee)$

Q: Do we have

$$
\begin{tikzcd}
1 \arrow[r] & \underset{\mathrm{Inn}(G_F)}{\mathrm{Inn}(G_{\bar F})^{\Gamma_F}} \arrow[r] & \underset{\mathrm{Aut}(G_F)}{\mathrm{Aut}(G_{\bar F})^{\Gamma_F}} \arrow[r] & \underset{\mathrm{Out}(G_F)}{\mathrm{Out}(G_{\bar F})^{\Gamma_F}} \arrow[r, "0"] & H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \arrow[r, hook] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[r] & H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))
\end{tikzcd}
$$

Give one example s.t. $H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$ is not inj?

Categorification of $H^1(\Gamma_F, -)$

These categories are all groupoids.
These $H^1(\Gamma_F, -)$ are all achieved as isomorphism classes.

<table>
<tr><th></th><th>Obj</th><th>Mor $((G_2,\alpha),(G_2',\alpha'))$</th></tr>
<tr>
<td>form

$H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] \arrow[rr, bend left, "\sigma(\alpha)"] & G_{\bar F} & G_{\bar F} \arrow[l, "\sigma(\alpha)\circ\alpha^{-1}"'] \end{tikzcd}$$

commutes $\forall \sigma \in \Gamma_F$

</td>
<td>

$\beta: G_2 \longrightarrow G_2'$ iso

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes

</td>
</tr>
<tr>
<td>inner form

$\mathrm{Im}\begin{pmatrix} H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \\ \downarrow \\ H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \end{pmatrix}$

full subcategory of "form"</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & G_{\bar F} \arrow[l, "\sigma(\alpha)\circ\alpha^{-1}"'] \end{tikzcd}$$

$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.

</td>
<td>

$\beta: G_2 \longrightarrow G_2'$ iso

$$\Rightarrow \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes

</td>
</tr>
<tr>
<td>inner twist

$H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$

less isomorphisms compared with inner form</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F})$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & G_{\bar F} \arrow[l, "\sigma(\alpha)\circ\alpha^{-1}"'] \end{tikzcd}$$

$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.

</td>
<td>

$\beta: G_2 \longrightarrow G_2'$ iso

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

$\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}$ is inner auto.

</td>
</tr>
<tr>
<td>pure inner twist

$H^1(\Gamma_F, G(\bar F))$</td>
<td>

$(G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F},\ \phi)$
$\phi \in Z^1(\Gamma_F, G(\bar F))$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & G_{\bar F} \arrow[l, "\phi(\sigma)\text{-conj}"'] \end{tikzcd}$$

commutes

</td>
<td>

(β, δ)
$\beta: G_2 \longrightarrow G_2'$ iso $\quad \delta \in G(\bar F)$

$$\text{s.t. } \begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

commutes, and
$\phi_1(\sigma) = \delta^{-1}\phi_2(\sigma)\,\sigma(\delta)$

</td>
</tr>
</table>

	Obj	Mor $((G_2,\alpha),(G_2',\alpha'))$
rigid inner twist $H^1\begin{pmatrix} u(\bar F)\to\mathcal{E}^{rig}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$	$(G_2,\ \alpha: G_{2,\bar F}\to G_{\bar F},\ Z,\ z)$ $Z\subset Z(G)$ finite gp subscheme $z\in Z^1\begin{pmatrix} u(\bar F)\to\mathcal{E}^{rig}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\bar z(\sigma)\text{-conj}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$ commutes	(β,δ) $\beta: G_2\longrightarrow G_2'$ iso $\delta\in G(\bar F)$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ commutes, and $z_1(\sigma)=\delta^{-1}z_2(\sigma)\,\sigma(\delta)$
when $Z=Z(G)$ $\left\{\begin{array}{c}\text{basic Kottwitz set} \\ B(G)_{basic} \\ H^1_{basic}(\mathcal{E}^{iso},G)\end{array}\right.$ $H^1\begin{pmatrix} \mathbb{D}(\bar F)\to\mathcal{E}^{iso}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$	$(G_2,\ \alpha: G_{2,\bar F}\to G_{\bar F},\ Z,\ z)$ $Z\subset Z(G)$ finite gp subscheme $z\in Z^1\begin{pmatrix} \mathbb{D}(\bar F)\to\mathcal{E}^{iso}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\bar z(\sigma)\text{-conj}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$ commutes	(β,δ) $\beta: G_2\longrightarrow G_2'$ iso $\delta\in G(\bar F)$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ commutes, and $z_1(\sigma)=\delta^{-1}z_2(\sigma)\,\sigma(\delta)$

https://mathoverflow.net/questions/117033/center-of-the-algebraic-group-g-mathbbr-for-a-centerless-g
https://math.stackexchange.com/questions/953526/relative-center-of-relative-group-scheme