Modular form
5. moduli interpretation

1 level structure
2. moduli interpretation of $\mathcal{H}/\Gamma$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S,T \mid S^4 = (ST)^6 = Id \rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}^-$	$\{\pm Id\}$	$\langle S,T, \begin{pmatrix}1&0\\0&-1\end{pmatrix} \rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S,T \mid S^2 = (ST)^3 = Id \rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H} \sqcup \mathcal{H}^-$	Id	$\langle S,T, \begin{pmatrix}1&0\\0&-1\end{pmatrix} \rangle$

can't define SL_2/G_m

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A \leq B \leq C$ gp $\quad A \lhd C \Rightarrow A \lhd B$

no other restrictions. i.e. the following cases may happen:

$A \lhd B \lhd C$	$A \lhd B \leq C$	$A \lhd B \lhd C$	$A \lhd B \leq C$	$A \leq B \lhd C$	$A \leq B \leq C$
✓	✓	$C_2 \lhd A_4 \lhd S_4$	✓	✓	$S_2 \leq S_3 \leq S_4$

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$
\begin{array}{ccc}
\Gamma(N) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\Gamma_1(N) & \twoheadrightarrow & N(\mathbb{Z}/N\mathbb{Z}) = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \\
\cap & & \cap \\
\Gamma_0(N) & \twoheadrightarrow & B(\mathbb{Z}/N\mathbb{Z}) = \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \\
\cap & & \cap \\
\Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{\text{[WWL, Prop 1.4.4]}} & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\cup & & \cup \\
\Gamma^0(N) & \twoheadrightarrow & \left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right) \\
\cup & & \cup \\
\Gamma'(N) & \twoheadrightarrow & \left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)
\end{array}
$$

⚠ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$, but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid \begin{array}{l} a,b,c,d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

Ex. Verify the following tables (left comes from right)

A \ B	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	–	✓	✓	✓
$\Gamma_1(N)$	–	–	✓	×
$\Gamma_0(N)$	–	–	–	X
$\Gamma(1)$	–	–	–	–