Eine Woche, ein Beispiel

8.6. Kottwitz set

This document is a continuation of [23.08.06].
Reorganized from Luozi Shi (and his partners)'s talk.

Recall that $\widehat{\mathbb{Q}_p^{ur}} = \mathrm{Frac}(\widehat{\mathbb{Z}_p^{ur}})$, $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$. Here "$\wedge$" is completion w.r.t. valuation.

Setting. In this document, F is a NA local field,

$$L := \widehat{F^{ur}} \overset{\text{if } F \text{: p-adic}}{=\!=} \mathrm{Frac}(W(\overline{k_F}) \otimes_{W(k_F)} \mathcal{O}_F).$$

Def For G/F conn reductive, the Kottwitz set $B(G)$ is defined as

$$\begin{aligned} B(G) := \; & H^1(W_F, G_{\overline{F}}) \\ \cong \; & H^1(\langle\sigma\rangle, G_L) \qquad \text{by Inf-Res seq \& } H^1(I_F, G_{\overline{F}}) = 0 \\ \cong \; & G(L)/\sigma\text{-twisted } G(L)\text{-conj} \\ \overset{\text{when } G = GL_n}{\cong} \; & \mathrm{Isoc}/\sim \end{aligned}$$

$\uparrow$ Isocrystals.

Rmk. By Hilbert 90, $H^1(\Gamma_F, GL_{n,\overline{F}}) = \{1\}$. In most cases,

$$H^1(\Gamma_F, G_{\overline{F}}) \not\cong H^1(W_F, G_{\overline{F}})$$

[even though $H^1(\Gamma_F, G_{\overline{F}}) \cong G(F) \cong H^0(W_F, G_{\overline{F}})$, we take different resolutions.]

E.g. $B(\mathbb{G}_m) \cong \mathbb{Z}$

Proof. The map

$$\beta: \mathcal{O}_L^\times \longrightarrow \mathcal{O}_L^\times \qquad x \longmapsto x \cdot \sigma(x)^{-1}$$

is surjective, since

$$\beta_1: k_L^\times \longrightarrow k_L^\times \qquad x \longmapsto x^{1-q}$$

$$\beta_2: k_L \longrightarrow k_L \qquad x \longmapsto x - x^q$$

are surjective. ($k_L \cong \overline{\mathbb{F}_p}$ is alg closed)

Then the well-defined morphism

$$v: L^\times/\mathrm{Im}\beta \longrightarrow \mathbb{Z} \qquad x \longmapsto v(x)$$

is injective, thus an iso.

Ex. Check that the SES

$$1 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \longrightarrow 1$$

induce LES in gp cohomology:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & F^\times \arrow[r, "(-)^2"] & F^\times \arrow[dll, out=-10, in=170] \\
 & B(\underline{\mathbb{Z}/2\mathbb{Z}}) \arrow[r, "0"] & \underset{\mathbb{Z}}{B(\mathbb{G}_m)} \arrow[r, "\times 2"] & \underset{\mathbb{Z}}{B(\mathbb{G}_m)}
\end{tikzcd}$$

where

$$\begin{aligned}
B(\underline{\mathbb{Z}/2\mathbb{Z}}) &\overset{def}{:=} H^1(W_F, \mathbb{Z}/2\mathbb{Z}) \qquad W_F \curvearrowright \mathbb{Z}/2\mathbb{Z} \text{ trivially} \\
&= \mathrm{Hom}(W_F, \mathbb{Z}/2\mathbb{Z}) \\
&\cong \{H \triangleleft W_F \text{ closed with index } 2\} \cup \{0\} \\
&\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 2 \end{cases}
\end{aligned}$$

This LES gives us to compute number of finite extensions. e.p with prime degree.
One gets

$$F^\times \xrightarrow{(-)^n} F^\times \rightsquigarrow B(\underline{\mathbb{Z}/n\mathbb{Z}}) \xrightarrow{0} \cdots$$

E.g. $B(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{Z} \times \mathbb{Z} = X_*(\mathbb{G}_m \times \mathbb{G}_m)$

Proof. $B(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{G}_m(L) \times \mathbb{G}_m(L) /\sigma\text{-twisted } \mathbb{G}_m(L) \times \mathbb{G}_m(L)\text{-conj}$

$$\begin{aligned}
&\cong \mathbb{G}_m(L)/\sim \times \mathbb{G}_m(L)/\sim \\
&\cong \mathbb{Z} \times \mathbb{Z}
\end{aligned}$$

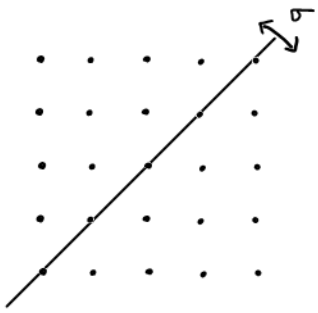
Rmk. In general, for a torus T/F,

$$B(T) \cong X_*(T)_{\Gamma_F} \cong X^*(\hat{T}^{\Gamma_F})$$

E.g. For E/F any Galois extension of deg 2, $\mathrm{Gal}(E/F) := \{1, \sigma\}$.
When $T = \mathrm{Res}_{E/F}\mathbb{G}_{m,E}$, $B(T) \cong \mathbb{Z}$.

$$X_*(T) \cong \mathbb{Z} \times \mathbb{Z} \qquad \sigma : \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} b \\ a \end{pmatrix}$$

$$X_*(T)_{\Gamma_F} \cong \mathbb{Z} \times \mathbb{Z} / \langle \sigma(v) - v \rangle \xrightarrow[\sim]{+} \mathbb{Z}$$



$$\begin{array}{ccc}
\hat{T} & \cong & \mathbb{C}^\times \times \mathbb{C}^\times \\
\cup & & \uparrow \Delta \\
\hat{T}^{\Gamma_F} & \cong & \mathbb{C}^\times
\end{array} \qquad \sigma : \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} b \\ a \end{pmatrix}$$

$$X^*(\hat{T}^{\Gamma_F}) \cong \mathbb{Z}$$

When $T = \mathrm{Res}'_{E/F}\mathbb{G}_{m,E}$, $B(T) \cong \mathbb{Z}/2\mathbb{Z}$

$$X_*(T) \cong \mathbb{Z} \qquad \sigma : a \longmapsto -a$$
$$X_*(T)_{\Gamma_F} \cong \mathbb{Z}/\langle \sigma(a) - a \rangle \cong \mathbb{Z}/2\mathbb{Z}$$

$$\hat{T} \cong \mathbb{C}^\times \qquad \sigma : a \longmapsto a^{-1}$$
$$\cup \qquad \cup$$
$$\hat{T}^{\Gamma_F} \cong \mu_2(\mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$$
$$X^*(\hat{T}^{\Gamma_F}) \cong \mathbb{Z}/2\mathbb{Z}$$

Ex. Check that the SES

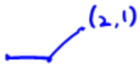
$$1 \longrightarrow \mathrm{Res}'_{E/F}\mathbb{G}_{m,E} \longrightarrow \mathrm{Res}_{E/F}\mathbb{G}_{m,E} \longrightarrow \mathbb{G}_{m,F} \longrightarrow 1$$

induce LES in gp cohomology:

$$\begin{array}{ccccccc}
 & & \mathbb{Z}/2\mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
 & & \| & & \| & & \| \\
 & & B(\mathrm{Res}'_{E/F}\mathbb{G}_{m,E}) & \xrightarrow{0} & B(\mathrm{Res}_{E/F}\mathbb{G}_{m,E}) & \xrightarrow{\times 2} & B(\mathbb{G}_{m,F}) \\
0 & \longrightarrow & H^0(W_F, (\mathrm{Res}'_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) & \longrightarrow & H^0(W_F, (\mathrm{Res}_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) & \xrightarrow{\text{Norm}} & H^0(W_F, \mathbb{G}_{m,\bar{F}}) \\
 & & \| & & \| & & \| \\
 & & (\mathrm{Res}'_{E/F}\mathbb{G}_{m,E})(F) & & E^\times & & F^\times
\end{array}$$

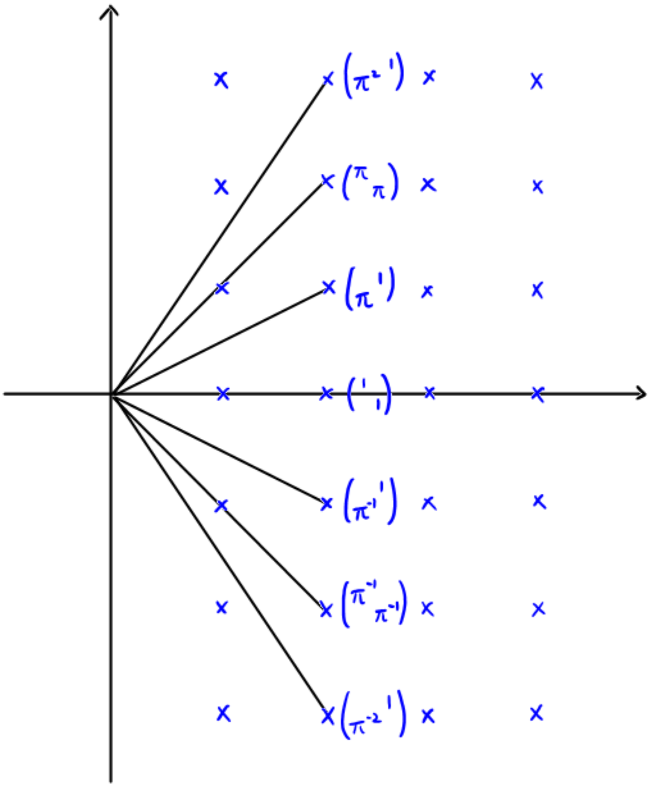
[Aside:
$$\begin{aligned}
(\mathrm{Res}'_{E/F}\mathbb{G}_{m,E})(F) &= \ker\left[(\mathrm{Res}_{E/F}\mathbb{G}_{m,E})(F) \xrightarrow{\text{Norm}} \mathbb{G}_m(F)\right] \\
&= \ker\left[E^\times \xrightarrow{\text{Norm}} F^\times\right] \\
&= \{x \in E^\times \mid x\sigma(x) = 1\}
\end{aligned}$$
]

E.g. $G = GL_n$

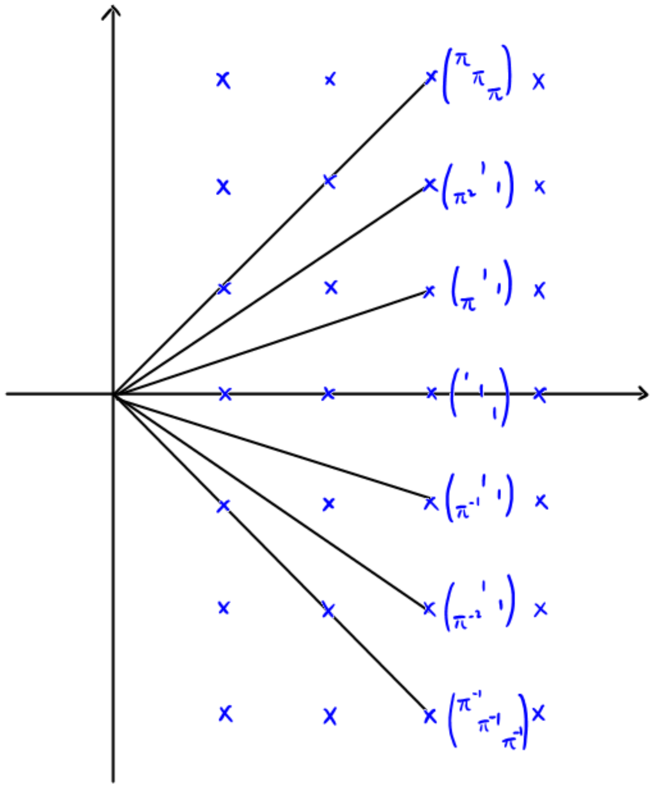


$$B(GL_n)_{simple} \hookrightarrow B(GL_n)_{basic} \hookrightarrow B(GL_n) \hookrightarrow \{NP\}$$

no intermediate vertex — straight — vertex $\subseteq \mathbb{Z}\times\mathbb{Z}$ — Newton polygons



$B(GL_2)_{basic}$



$B(GL_3)_{basic}$