Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases

Setting We work over conn red gp over K.

$\bar{K}$: the seperable closure of K mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_K := Gal(\bar{K}/K)$ $\sigma \in \Gamma_K$

$H^1(W, A) := Hom_{Grp}(W, A \rtimes W)/A\text{-conj}$ $\varphi \in H^1(W, A)$

Ref:
[ECII] Silverman, The Arithmetic of Elliptic Curves

1. forms.

Def. $G_1, G_2/k$ are called forms, if

$\exists\ \alpha : G_{2,\bar{k}} \xrightarrow{\sim} G_{1,\bar{k}}$ as gps not as Γ_k-gps!

α is considered as the information of forms.

Thm. $\{k\text{-forms of } G\} \longleftrightarrow H^1(\Gamma_k, \mathrm{Aut}(G_{\bar{k}}))$

$[G_2, \alpha : G_{2,\bar{k}} \longrightarrow G_{\bar{k}}] \longmapsto \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1}$

$G_2 \longleftarrow \varphi$

$$G_2(k) := \{ g \in G(\bar{k}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k \}$$

In general, $G_2(R) := \{ g \in G(\bar{k}\otimes_k R) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k \}$

e.p. $G_2(\bar{k}) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_k} \in \prod_{\sigma\in\Gamma_k} G(\bar{k}) \mid g \in G(\bar{k}) \} \cong G(\bar{k})$

$G(\bar{k}\otimes_k\bar{k}) \cong G(\bigoplus_{\sigma\in\Gamma_k}\bar{k}) \cong \prod_{\sigma\in\Gamma_k} G(\bar{k})$

$$\begin{array}{ccc} G_{2,\bar{k}} & \xrightarrow{\alpha} & G_{\bar{k}} \\ \sigma\downarrow & & \downarrow\sigma \\ G_{2,\bar{k}} & \xrightarrow{\alpha} & G_{\bar{k}} \end{array}$$

$\Gamma_k \circlearrowright G_{2,\bar{k}} \longrightarrow G_{\bar{k}} \circlearrowleft \Gamma_k, \Gamma_k$

φ_α measures the commutativity between two Γ_k-actions.

Functorial on k. (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])

Let L/k be finite Galois.

$$\begin{array}{ccccc} G_{2,L} & \{L\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_L, \mathrm{Aut}(G_{\bar{k}})) & \varphi|_{\Gamma_L} \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ G_2 & \{k\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_k, \mathrm{Aut}(G_{\bar{k}})) & \varphi \\ & \uparrow & & \uparrow & \\ & \{G_2/k : G_{2,L} \cong G_L\} & \longleftrightarrow & H^1(\mathrm{Gal}(L/k), \mathrm{Aut}(G_{\bar{k}})^{\Gamma_L}) & \\ & & & \text{``} = \mathrm{Aut}(G_L)\text{''} & \\ & \uparrow & & \uparrow & \\ & 1 & & 1 & \end{array}$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{G}_m^n) & \subseteq & \mathrm{End}(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow\!\shortmid\ n$

Therefore,

$$\begin{aligned} H^1(\Gamma_K, \mathrm{Aut}(\mathbb{G}_{m,\bar{K}}^n)) &= H^1(\Gamma_K, GL_n(\mathbb{Z})) \\ &= \mathrm{Hom}_{Grp}(\Gamma_K, GL_n(\mathbb{Z})) / GL_n(\mathbb{Z})\text{-conj} \\ &\overset{\text{when } K=\mathbb{R}}{=\!=} \{ g \in GL_n(\mathbb{Z}) \mid g^2 = Id \} / GL_n(\mathbb{Z})\text{-conj} \end{aligned}$$

[Γ_K acts on $\mathrm{Aut}(\mathbb{G}_{m,\bar{K}}^n) \subseteq \mathrm{End}(\mathbb{G}_{m,\bar{K}}^n)$ trivially:

see $\bar{K}$-pts, $n=1$:

$$\begin{tikzcd} \bar{K}^\times \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & \bar{K}^\times \arrow[d, "\sigma"] \\ \bar{K}^\times \arrow[r, "{}^\sigma\alpha"] & \bar{K}^\times \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & x^n \arrow[d, mapsto] \\ \sigma(x) & \sigma(x^n) = \sigma(x)^n \end{tikzcd}$$

$\Rightarrow {}^\sigma\alpha = \alpha$]

E.g. $n=1$, $K=\mathbb{R}$

$\varphi(\sigma) = (-)^{-1}$

$$H^1(\Gamma_K, \mathrm{Aut}(\mathbb{G}_{m,\bar{K}})) \cong \{1, -1\}$$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ \mathbb{G}_m & G = \ ?\ SO_{2,\mathbb{R}} \end{array}$$

$$\begin{aligned} G(\mathbb{R}) &= \{ g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}} \} \\ &= \{ g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g \} \\ &= \{ g \in \mathbb{C}^\times \mid |g| = 1 \} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = \mathrm{Spec}\ \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$

Check:

$$\begin{aligned} G(\mathbb{C}) &= \{ (x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1 = 0 \} \\ &= \{ (x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1 \} \\ &= \left\{ (x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \;\middle|\; \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array} \right\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; x,y \in K,\ x^2+y^2 = 1 \right\}$$

E.g. $n=2$, $K=\mathbb{R}$

$$H^1(\Gamma_K, Aut(\mathbb{G}^2_{m,\bar{K}})) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$\mathbb{G}_m^2 \quad \mathbb{G}_m \times SO_{2,\mathbb{R}} \quad (SO_{2,\mathbb{R}})^2 \quad G = ? \quad Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; \underbrace{(\varphi(\sigma)\circ\sigma)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{\substack{\| \\ \varphi(\sigma)\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} \\ \| \\ \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \end{pmatrix}}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar{z}_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\mathbb{G}_m, \quad SO_{2,\mathbb{R}}, \quad Res_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$1 \qquad -1 \qquad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

Fact^{dual}: $Ind_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{triv}, \mathbb{Z}_{sign}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

Rmk. Using the same argument, one can show that

$$\{ T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}^n_{m,\mathbb{F}_{p^2}} \} = \text{products of } \mathbb{G}_m, \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}, Res_{\mathbb{F}_{p^2}/\mathbb{F}_p}\mathbb{G}_m$$

The torus G crspd to -1: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$G(\mathbb{F}_p) = \{ g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_K \}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid \underbrace{\varphi(\sigma)(a-b\xi)}_{\substack{\| \\ (a-b\xi)^{-1}}} = a+b\xi \}$$

$$= \{ a+b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2\varepsilon = 1 \}$$

$$\cong \left\{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p) \right\}$$