Eine Woche, ein Beispiel

7.9. Steenrod algebra

The Steenrod algebra has been studied for decades. As usual, I make no claim to originality.

For calculating the Steenrod algebra in compute, you can get it from this link:
https://math.berkeley.edu/~kruckman/adem/

The original article by José Adem: The Iteration of the Steenrod Squares in Algebraic Topology
https://www.pnas.org/doi/10.1073/pnas.38.8.720

The survey talk(recommend):
http://www.math.uni-bonn.de/people/daniel/2022/algtop/Steenrod_Squares.pdf

A combinatorial method for computing Steenrod squares:
https://www.sciencedirect.com/science/article/pii/S0022404999000067

Chinese collections on Steenrod algebra:
https://www.zhihu.com/question/265308226

Problems in the Steenrod Algebra:
https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=3ba0259a7d1afc849fb1796d5002bc9c7eab1b5a

1. binomial coefficient mod p
2. Adem relations
3. Steenrod algebra

https://en.wikipedia.org/wiki/Adams_operation

1. binomial coefficient mod p

$\binom{m+n}{n}$ mod 2, m \ n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0
3	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
4	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0
5	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0
6	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0
7	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
8	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0
9	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0
10	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0
11	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
12	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
17	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	1	0	0	1	1	0	0	1	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20	1	1	1	1	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21	1	0	1	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
22	1	1	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

period

$$\binom{m+n}{r}=\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$$

We conclude from table (or Chu-Vandermonde identity) the following practical formula:

Prop. Let $a=\sum_{n\geq 0} a_n 2^n$, $b=\sum_{n\geq 0} b_n 2^n$, $a_n, b_n \in \{0,1\}$. We get

$$\binom{a+b}{a}\equiv 0 \mod 2 \iff \exists\, n\in\mathbb{N}_{\geq 0} \text{ s.t. } a_n=b_n=1$$

Eg. $a=(1101101010100)_2$, $b=(1000000110)_2$, then

$$\binom{a+b}{a}\equiv 0 \mod 2 \quad \text{since} \quad \begin{array}{r} 110110101\boxed{1}00 \\ 100000\boxed{1}10 \end{array}$$

Rmk. Similarly, one can show:
for $a=\sum_{n\geq 0} a_n p^n$, $b=\sum_{n\geq 0} b_n p^n$, $a_n, b_n \in \{0,1,\cdots,p-1\}$,

$$\binom{a+b}{a}\equiv \prod_{n\geq 0}\binom{a_n+b_n}{a_n} \mod p$$

Rmk. It is possible to define $\binom{a+b}{a}\in\mathbb{F}_p$ for $a,b\in\mathbb{Z}[\frac{1}{p}]$.
One may want to:
① Verify if the usual formulas in https://en.wikipedia.org/wiki/Binomial_coefficient work;
② Find a combinatorical explanation of it.

https://en.wikipedia.org/wiki/%CE%9B-ring

$\lambda^n: \mathbb{Z}\longrightarrow\mathbb{Z}$ $\quad x\longmapsto\binom{x}{n}$
is the unique λ-ring on $\mathbb{Z}$.

2. Adem relations

Def (Steenrod squares) see [wiki: Steenrod algebra] for detail.

$$Sq^k : H^*(- ; \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^{*+k}(- ; \mathbb{Z}/2\mathbb{Z})$$
$$Sq = Sq^0 + Sq^1 + Sq^2 + \cdots \qquad Sq^0 = Id_{H^*(- ; \mathbb{Z}/2\mathbb{Z})}$$

⚠ $\quad Sq^3 \neq Sq^1 Sq^1 Sq^1 \qquad Sq \neq Sq^1$

Prop (Adem relations)

For $0 < a < 2b$, we have a formula

$$Sq^a Sq^b = \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{b-j-1}{a-2j} Sq^{a+b-j} Sq^j$$
$$= \sum_{j=0}^{\lfloor a/2 \rfloor} \binom{(b-a+j-1)+(a-2j)}{a-2j} Sq^{a+b-j} Sq^j$$

Here we list first several terms: $\left(b > \frac{a}{2} \right)$