Eine Woche, ein Beispiel

4.27. homomorphism between Jacobians

[2025.04.20] provides us with many examples and references, and here we do things more theoretically.

Idea:

$$\mathrm{Jac}(C) = \underbrace{H^0(C;\omega_C)^*}_{\substack{\text{linear part}\\ \text{coherent}}} \Big/ \underbrace{H_1(C;\mathbb{Z})}_{\substack{\text{lattice part}\\ \text{constant}}}$$

To understand Jac(C), we need to understand these two parts separately.

For a morphism between two sm proj curves $/\mathbb{C}$:

$$f: \widehat{C} \longrightarrow C$$

$$\begin{aligned} Nm_{f,a}&: H^0(\widehat{C}; \omega_{\widehat{C}})^* \longrightarrow H^0(C; \omega_C)^* \\ Nm_{f,r}&: H_1(\widehat{C}; \mathbb{Z}) \longrightarrow H_1(C; \mathbb{Z}) \\ Nm_f&: \mathrm{Jac}(\widehat{C}) \longrightarrow \mathrm{Jac}(C) \end{aligned}$$

$$\begin{aligned} (f^*)_a&: H^0(C; \omega_C)^* \longrightarrow H^0(\widetilde{C}; \omega_{\widetilde{C}})^* \\ (f^*)_r&: H_1(C; \mathbb{Z}) \longrightarrow H_1(\widetilde{C}; \mathbb{Z}) \\ f^*&: \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(\widetilde{C}) \uparrow \end{aligned}$$

cohom pullback

sheaf origin

$$\begin{aligned} \omega_{\widetilde{C}} &\longleftarrow f^* \omega_C \\ f_! \pi^!_{\widetilde{C}} \mathbb{Z} &\longrightarrow \pi^!_C \mathbb{Z} \end{aligned}$$

$$\begin{aligned} \omega_C &\longleftarrow f_! \omega_{\widetilde{C}} \;\; (= f_* \omega_{\widetilde{C}}) \\ \mathbb{Z}_C &\longrightarrow f_* \mathbb{Z}_{\widetilde{C}} \end{aligned}$$

$$g(f(\omega))\, d(f(\omega)) \longleftarrow\!\shortmid\; g(z)\,dz$$

$$\sum_{f(\omega)=z} g(\omega)\, dz \longleftarrow\!\shortmid\; g(\omega)\, d\omega$$

suppose locally $f^*(dz) = d\omega$

geometric picture

$$[q] \longmapsto [f(q)]$$

$$[p] \longmapsto \sum_{f(q)=p} [q]$$

Ex. Show that

$$Nm_f \circ f^* = [\deg f] : \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C)$$

Also,

$$\begin{aligned} Nm_{f,a} \circ (f^*)_a &= \deg f \cdot Id_{H^0(C;\omega_C)^*} \\ Nm_{f,r} \circ (f^*)_r &= \deg f \cdot Id_{H_1(C;\mathbb{Z})} \end{aligned}$$

Hint: use Poincaré duality.