Eine Woche, ein Beispiel

10.20 Schur functor: basic formulas

Main reference:
[FH]: Willian Fulton and Joe Harris. Representation Theory. A First Course.

[Hall]: Brian Hall. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. Graduate Texts in Mathematics. Cham: Springer International Publishing, 2015.

In this document, $\operatorname{char} k = 0$, $V \in \mathrm{Vect}_k$.

Schur fctor helps us to decompose $V^{\otimes k}$ by S_k gp action.
$\mathbb{S}^\lambda V$ generalize $\mathrm{Sym}^k V$ & $\Lambda^k V$. Moreover,

$$\begin{array}{ccc} \mathrm{Rep}(GL(V)) & = & \mathrm{Rep}(A_{n-1}) \\ \cup\!\!\!\!\!\!\;\text{\rotatebox{90}{$\in$}} & & \text{\rotatebox{90}{$\in$}} \\ \mathbb{S}^\lambda V & = & \mathcal{L}(\lambda) \end{array}$$

$n = \dim V$

Here, λ has many expressions,

e.g.   partitions   weights

$$\lambda = \ydiagram{3,1} = (3,1) = 2\varpi_1 + \varpi_2$$

$$\lambda = \cdots = (\lambda_1, \lambda_2, \cdots) = \sum m_i \varpi_i$$

$$\{h_{ij}\} = \begin{ytableau} 4 & 2 & 1 \\ 1 \end{ytableau} \qquad \text{hook length}$$

1. dimension
2. $\mathbb{S}^\lambda(V \oplus W)$ and ...

1. dimension

It can be computed from the Weyl dimension formula:

$$\dim_k \mathcal{L}(\lambda) = \frac{\prod_{\alpha\in\Delta^+} (\lambda+\rho, \alpha)}{\prod_{\alpha\in\Delta^+} (\rho, \alpha)}$$

$$\overset{\substack{\mathcal{L}(\lambda)\in rep(A_{n-1}) \\ \lambda = \sum m_i \varpi_i}}{=} \frac{(m_1-1)\cdots(m_{n-1}+1)\,(m_1+m_2+2)(m_2+m_3+2)\cdots\cdots(\sum m_i + n-1)}{\underbrace{1\cdots 1}_{n-1 \text{ many}}\quad \underbrace{2\cdot 2\cdots}_{n-2 \text{ many}}\cdots\cdots n-1}$$

$$= \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i} \qquad \text{[FH, Thm 6.3 (1)]}$$

$$= \prod_{1\le i<j\le n} \frac{n-i+j}{h_{ij}} \qquad \text{[FH, Ex 6.4]}$$

Following [Hall Example 10.23],

$$\Delta^+ = \left\{ \begin{array}{ccccc} & & \sum \alpha_i & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & \\ \alpha_1+\alpha_2 & & \cdots & & \alpha_{n-2}+\alpha_{n-1} \\ \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1} \end{array} \right\}$$

$$\rho = \frac{1}{2} \sum_{\alpha\in\Delta^+} \alpha$$

$$= \frac{1}{2} \sum_{i=1}^{n-1} i(n-i)\alpha_i$$

$$= \sum_{i=1}^{n-1} \varpi_i \qquad \chi = \sum_{i=1}^{n-1} \langle \chi, \alpha_i \rangle \varpi_i$$

These would be enough to explain the first equality above.

E.g. When $\lambda = (3,1) = \ydiagram{3,1} = 2\varpi_1 + \varpi_2$,

$$\dim_k L(\lambda) = \frac{(2+1)(1+1)(0+1)\cdots(2+1+2)(1+0+2)(0+0+2)\cdots(2+1+(n-1))}{1\cdot 1\cdot 1\cdots 2\cdot 2\cdot 2\cdots n-1}$$

$$= \frac{(2+1)(1+1)\quad(2+1+2)(1+0+2)\quad\cdots(2+1+(n-1))}{1\cdot 1\quad 2\cdot 2\quad\cdots n-1}$$

$$= \left(\frac{2+1}{1}\cdot\frac{3+2}{2}\cdot\frac{3+3}{3}\cdots\frac{3+n-1}{n-1}\right)\cdot\left(\frac{1+1}{1}\cdot\frac{1+2}{2}\cdot\frac{1+3}{3}\cdots\frac{1+n-2}{n-2}\right)$$

$$= \frac{(n-1)\,n\,(n+1)(n+2)}{1\cdot 4\cdot 2\cdot 1}$$

$$= \frac{\begin{ytableau} n & n+1 & n+2 \\ n-1 \end{ytableau}}{\begin{ytableau} 4 & 2 & 1 \\ 1 \end{ytableau}}$$

2. $\mathbb{S}^{\lambda}(V \oplus W)$ and ...

E.g.
$$(V \oplus W)^{\otimes 2} = V^{\otimes 2} \oplus (V \otimes W)^{\oplus 2} \oplus W^{\otimes 2}$$
$$\mathrm{Sym}^2(V \oplus W) = \mathrm{Sym}^2 V \oplus V \otimes W \oplus \mathrm{Sym}^2 W$$
$$\Lambda^2(V \oplus W) = \Lambda^2 V \oplus V \otimes W \oplus W^{\otimes 2}$$

$$(V \oplus W)^{\otimes 3} = V^{\otimes 3} \oplus (V^{\otimes 2} \otimes W)^{\oplus 3} \oplus (V \otimes W^{\otimes 2})^{\oplus 3} \oplus W^{\otimes 3}$$
$$\mathrm{Sym}^3(V \oplus W) = \mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V \otimes W \oplus V \otimes \mathrm{Sym}^2 W \oplus \mathrm{Sym}^3 W$$
$$\mathbb{S}^{(2,1)}(V \oplus W) = \mathbb{S}^{(2,1)} V \oplus V^{\otimes 2} \otimes W \oplus V \otimes W^{\otimes 2} \oplus \mathbb{S}^{(2,1)} W$$
$$\Lambda^3(V \oplus W) = \Lambda^3 V \oplus \Lambda^2 V \otimes W \oplus V \otimes \Lambda^2 W \oplus \Lambda^3 W$$

$$\frac{\begin{array}{|c|c|}\hline m+n & m+n+1 \\ \hline m+n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} = \frac{\begin{array}{|c|c|}\hline m & m+1 \\ \hline m-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} + m^2 n + n m^2 + \frac{\begin{array}{|c|c|}\hline n & n+1 \\ \hline n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}}$$

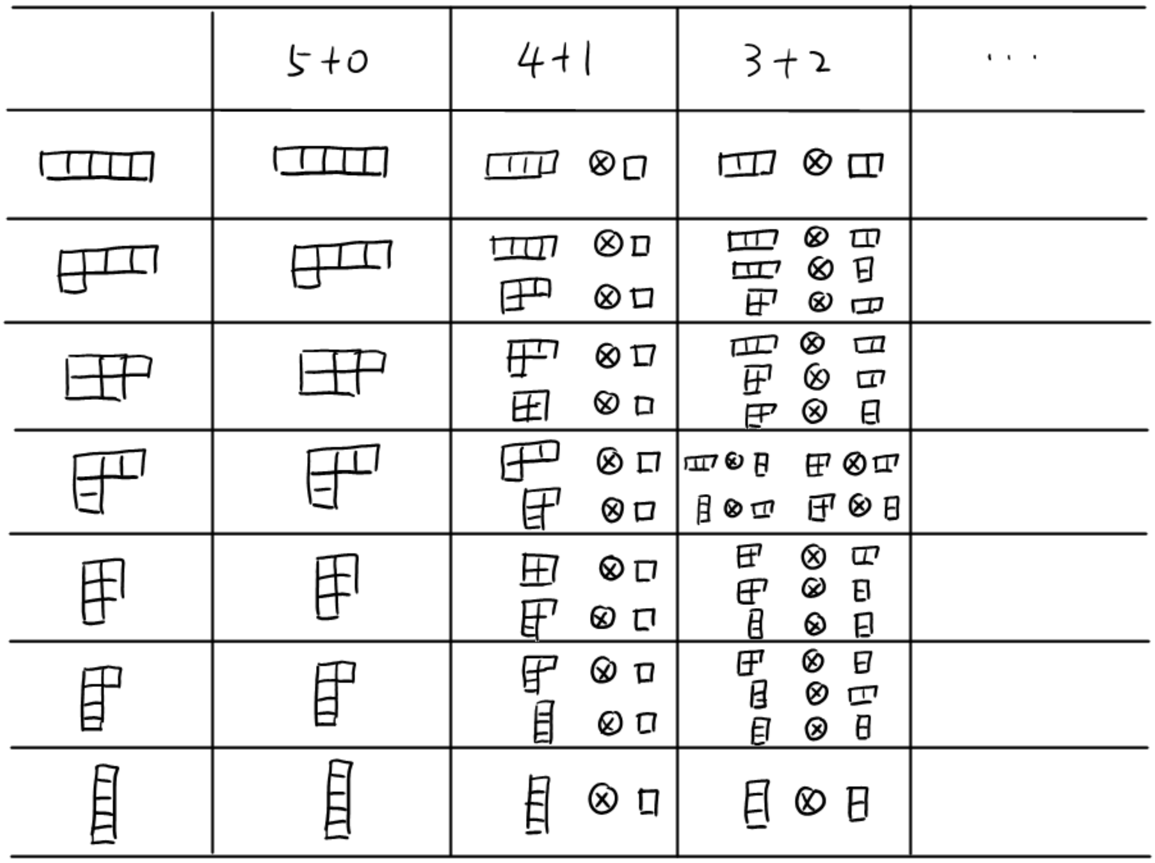
With the help of [FH,Ex 6.11], we can get the following tables:

	$2+0$	$1+1$	$0+2$
$\square\square$	$\square\square$	$\square \otimes \square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square \otimes \square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$

	3+0	2+1	1+2	0+3
$\ydiagram{3}$	$\ydiagram{3}$	$\ydiagram{2} \otimes \ydiagram{1}$	$\ydiagram{1} \otimes \ydiagram{2}$	$\ydiagram{3}$
$\ydiagram{2,1}$	$\ydiagram{2,1}$	$\ydiagram{2} \otimes \ydiagram{1}$ $\ydiagram{1,1} \otimes \ydiagram{1}$	$\ydiagram{1} \otimes \ydiagram{2}$ $\ydiagram{1} \otimes \ydiagram{1,1}$	$\ydiagram{2,1}$
$\ydiagram{1,1,1}$	$\ydiagram{1,1,1}$	$\ydiagram{1,1} \otimes \ydiagram{1}$	$\ydiagram{1} \otimes \ydiagram{1,1}$	$\ydiagram{1,1,1}$

	4+0	3+1	2+2
$\ydiagram{4}$	$\ydiagram{4}$	$\ydiagram{3} \otimes \ydiagram{1}$	$\ydiagram{2} \otimes \ydiagram{2}$
$\ydiagram{3,1}$	$\ydiagram{3,1}$	$\ydiagram{3} \otimes \ydiagram{1}$ $\ydiagram{2,1} \otimes \ydiagram{1}$	$\ydiagram{2} \otimes \ydiagram{2}$ $\ydiagram{2} \otimes \ydiagram{1,1}$ $\ydiagram{1,1} \otimes \ydiagram{2}$
$\ydiagram{2,2}$	$\ydiagram{2,2}$	$\ydiagram{2,1} \otimes \ydiagram{1}$	$\ydiagram{2} \otimes \ydiagram{2}$ $\ydiagram{1,1} \otimes \ydiagram{1,1}$
$\ydiagram{2,1,1}$	$\ydiagram{2,1,1}$	$\ydiagram{2,1} \otimes \ydiagram{1}$ $\ydiagram{1,1,1} \otimes \ydiagram{1}$	$\ydiagram{2} \otimes \ydiagram{1,1}$ $\ydiagram{1,1} \otimes \ydiagram{2}$ $\ydiagram{1,1} \otimes \ydiagram{1,1}$
$\ydiagram{1,1,1,1}$	$\ydiagram{1,1,1,1}$	$\ydiagram{1,1,1} \otimes \ydiagram{1}$	$\ydiagram{1,1} \otimes \ydiagram{1,1}$

...



https://math.stackexchange.com/questions/84103/characters-of-symmetric-and-antisymmetric-powers