Eine Woche, ein Beispiel
3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \curvearrowright X$

discrete	$\Rightarrow$ foundamental domain	$\Lambda \curvearrowright \mathbb{C}$	$SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$
non discrete	$\Rightarrow$ stratification by G/G_x	$S^1 \curvearrowright S^2$	$\mathbb{C}^\times \curvearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g. $\mathrm{Flag}(V) \cong GL(V)/P$ e.g. Grassmannian, $\mathbb{P}^n$

$$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$$

$O(n) := O(n,\mathbb{R})$ $\rightsquigarrow$ Stiefel mfld [21.11.14]

$SO(n) := SO(n,\mathbb{R})$

$$\mathbb{A}^n = \mathbb{A}^n$$

$$\mathcal{H}^n \cong O^+(1,n)/O(n)$$

$$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$$

$\rightsquigarrow$ Hermitian symmetric space

where $\mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1,\ v_{n+1} > 0\}$

$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ $\quad \langle v, w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$

$$O(n,1) := \mathrm{Aut}(\mathbb{R}^{n+1}, \langle\ ,\ \rangle) \subseteq GL_{n+1}(\mathbb{R})$$

$$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

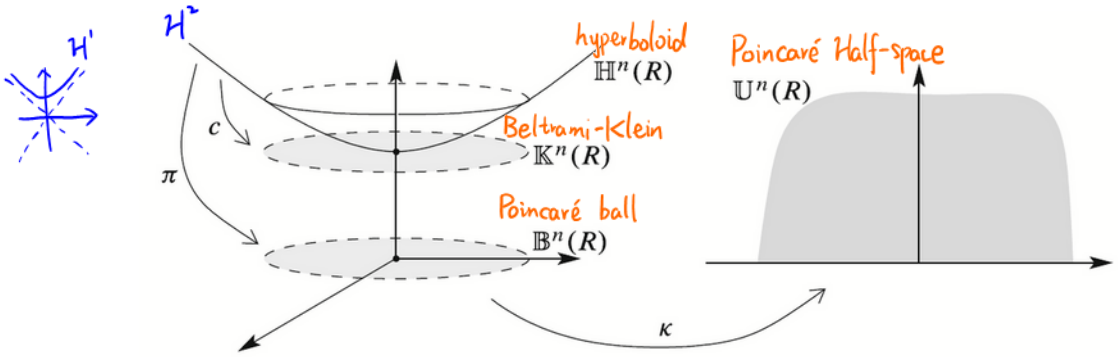


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-01-2

Terminology.

$X + G \curvearrowright X$

↓

G-space

↓ transitive

homogeneous space.

[JL18. Cor 3.18]:
Nice hom space can have
compatible Rie mfld structures.

Riemannian mfld
G preserve g

Hermitian mfld
G preserve g ← metric

Riemannian homogeneous space. $\xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}}$ Hermitian homogeneous space.

geodesic symmetry
globally defined, preserve g ↓

Symmetric space $\xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}}$ Hermitian Symmetric space

transitive on frames ↓

frame homogeneous space

$\mathbb{CP}^n$ $n \geq 1$

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

2. double coset decomposition: schedule

$$G = \coprod_{\alpha \in I} H\alpha K$$

usually, H, K are easier than G.
- comes from (usually) Gauss elimination
- I is the "foundamental domain"
- produces stratifications on G/K and ${}_H\backslash G$ indexed by I.

To be exact,

$$G/K = \coprod_{\alpha \in I} H\alpha K/K \cong \coprod_{\alpha \in I} H/H_{[\alpha K]} = \coprod_{\alpha \in I} H/(H \cap \alpha K \alpha^{-1})$$

$${}_H\backslash G = \coprod_{\alpha \in I} {}_H\backslash H\alpha K \cong \coprod_{\alpha \in I} K_{[H\alpha]}\backslash K = \coprod_{\alpha \in I} (K \cap \alpha^{-1} H \alpha)\backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$
$K_{[H\alpha]}$: stabilizer of K on $[H\alpha] \in {}_H\backslash G$

$$\#\, H/H \cap \alpha K \alpha^{-1} = \#\left\{ \begin{array}{l} \text{single cosets } [gK] \\ \text{in one double coset } H\alpha K \end{array} \right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad {}_H\backslash G \qquad \text{individually}$$

- can be viewed as stack quotient.
$[*/G]$: groupoid

$${}_H\backslash G/K \overset{\text{def}}{=\!=} [*/H] \times_{[*/G]} [*/K] \quad \text{with groupoid structure}$$

$$H^*_H(G/K) \cong H^*({}_H\backslash G/K) \cong H^*_K({}_H\backslash G)$$

slogan: the (equiv) cohomology of G/K and ${}_H\backslash G$ are connected.

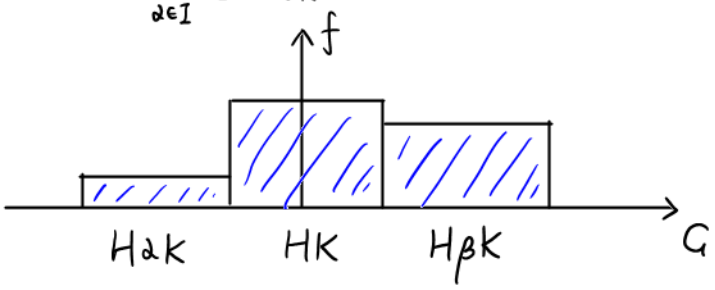
- Hecke algebra $\mathcal{H}(H\backslash G/K)$

↑ for $H=K$. You can also do $\mathcal{H}(H_1\backslash G/H_2) \hookrightarrow \bigoplus_{i,j=1}^{2} \mathcal{H}(H_i\backslash G/H_j)$

$\mathcal{H}(H\backslash G/K)$: reasonable subspaces of

$$\mathbb{C}[H\backslash G/K] \underset{\text{"0-dim"}}{=} \{ f: G \to \mathbb{C} \mid f(hgk) = f(g) \quad \forall h\in H, g\in G, k\in K \}$$

$$\Rightarrow \bigoplus_{\alpha\in I} \mathbb{C}\, \mathbb{1}_{H\alpha K}$$



with reasonable convolution structure

$$*: \mathcal{H}(H_1\backslash G/H_2) \times \mathcal{H}(H_2\backslash G/H_3) \longrightarrow \mathcal{H}(H_1\backslash G/H_3)$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

Vague: $\mathcal{H}(H\backslash G/K) \sim H^*(H\backslash G/K)$ should be a type of cohomology

$\mathcal{H}(G) \overset{G \text{ fin}}{=\!=} \mathbb{C}[G]$

$\mathcal{H}(K\backslash G/K) \cong (\mathrm{End}\,(c\text{-}\mathrm{Ind}_K^G \mathbb{1}_K))^{op}$ should be a type of base ring

Generalize: $\mathrm{Ind}_H^G \chi \approx \mathcal{H}_\chi(H\backslash G/K) \subseteq \{ f: G\to\mathbb{C} \mid f(hgk) = \chi(h) f(g) \}$

↑ depth of χ

3. examples (after [22.09.04])

Works over:

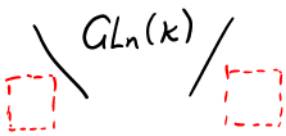
- list of possibilities
- moduli intepretation
- typical examples

finite field, $GL_n(\mathbb{F}_q)$ (Applies to any field κ, actually)

- subgps can be

Borel	max split torus	unipotent
B	T	N
parabolic	Levi	unipotent
P	L	M

nonsplit torus: T'



- moduli intepretation $V = \kappa^{\oplus n}$

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization