Invariance [edit]

Homotopy equivalence is important because in algebraic topology many concepts are **homotopy invariant**, that is, they respect the relation of homotopy equivalence. For example, if X and Y are homotopy equivalent spaces, then:

- X is path-connected if and only if Y is.
- X is simply connected if and only if Y is.
- The (singular) homology and cohomology groups of X and Y are isomorphic.
- If X and Y are path-connected, then the fundamental groups of X and Y are isomorphic, and so are the higher homotopy groups. (Without the path-connectedness assumption, one has $\pi_1(X, x_0)$ isomorphic to $\pi_1(Y, f(x_0))$ where $f : X \to Y$ is a homotopy equivalence and $x_0 \in X$.)

An example of an algebraic invariant of topological spaces which is not homotopy-invariant is compactly supported homology (which is, roughly speaking, the homology of the compactification, and compactification is not homotopy-invariant).

Homotopy equivalence [edit]

Given two topological spaces X and Y, a **homotopy equivalence** between X and Y is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$, such that $g \circ f$ is homotopic to the identity map id_X and $f \circ g$ is homotopic to id_Y. If such a pair exists, then X and Y are said to be **homotopy equivalent**, or of the same **homotopy type**. Intuitively, two spaces X and Y are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. Spaces that are homotopy-equivalent to a point are called contractible.

$f \circ g : Y \longrightarrow Y$

$\exists F : Y \times [0,1] \longrightarrow Y$

$F_0 = f \circ g \quad F_1 = Id_Y$

- A deformation retraction is a homotopy equivalence.

Let X be a topological space and A a subspace of X. Then a continuous map

$$r: X \to A$$

is a **retraction** if the restriction of r to A is the identity map on A;

Proof. $r : X \to A$ $\quad \iota : A \hookrightarrow X$ $\quad r \circ \iota = Id$

$\iota \circ r : X \longrightarrow X$

A continuous map

$$F: X \times [0, 1] \to X$$

is a *deformation retraction* of a space X onto a subspace A if, for every x in X and a in A,

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad \text{and} \quad F(a, 1) = a.$$

In other words, a deformation retraction is a homotopy between a retraction and the identity map on X.

过程中A也可以移开