

Eine Woche, ein Beispiel
 6.4. Grothendieck topology, site and topos

Top. space	space	continuous map	Covering of open sets	Sh	cohomology
site	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i \in I} \cup \coprod_{i \in I} \text{Im } f_i = U$	topos	new cohomology
X_{zar} $(\text{Sch}/X)_{\text{zar}}$	open immersion over X $\text{Ob}(\text{Sch}/X)$	$U_i \rightarrow U$ $\downarrow \text{Sch}$ $\text{Mor}(\text{Sch}/X)$	— —		
$X_{\text{ét}}$ $(\text{Sch}/X)_{\text{ét}}$	étale + l.f.p over X $\text{Ob}(\text{Sch}/X)$	full sub of Sch/X $\text{Mor}(\text{Sch}/X)$	ét + l.f.p ét + l.f.p		
$(\text{Sch}/X)_{\text{sm}}$	$\text{Ob}(\text{Sch}/X)$	$\text{Mor}(\text{Sch}/X)$	smooth + l.f.p		
$(\text{Sch}/X)_{\text{fppf}}$	$\text{Ob}(\text{Sch}/X)$	$\text{Mor}(\text{Sch}/X)$	f.flat + l.f.p		
$(\text{Sch}/X)_{\text{fqc}}$	$\text{Ob}(\text{Sch}/X)$	$\text{Mor}(\text{Sch}/X)$	f.flat + $f_i^+(q.o)$ locally qc		

<https://pbelmans.ncag.info/notes/etale-cohomology.pdf>

<https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology>

<https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites>
 it tells us why we don't have small site for most condition.

\Rightarrow [Hilbert's theorem 90 \Leftrightarrow no non-trivial line bundle on $\text{Spec } k$]

Thm. ① equiv. of categories

$$\text{Sets}((\text{Spec } k)_{\text{ét}}) \longleftrightarrow \text{Disc } G_k\text{-Set}$$

$$\text{Ab}((\text{Spec } k)_{\text{ét}}) \longleftrightarrow \text{Disc } \text{Mod}_{G_k} \quad (*)$$

$$G_k = \text{Gal}(K/k)^{\text{sep}}$$

$$(\text{Spec } k)_{\text{ét}} \xleftrightarrow{\text{Site}} G_k\text{-Set} \xleftrightarrow{\text{finite}}$$

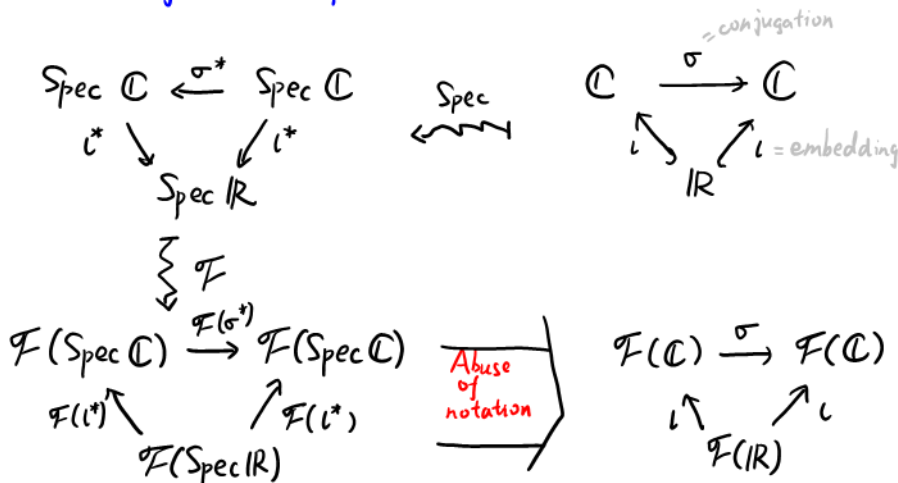
② (*) preserve cohomology

$$H^i((\text{Spec } k)_{\text{ét}}, \mathcal{F}) = H_{\text{cont}}^i(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\text{Spec } \mathbb{C})_{\text{ét}}$

Ex. describe sheaf on $(\text{Spec } \mathbb{R})_{\text{ét}}$

(Verify: \mathcal{F} is decided by $\mathcal{F}(\text{Spec } \mathbb{C})$)



Sub Ex. \mathcal{F} is sheaf $\leadsto \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{\text{Gal}}$ $\text{Gal} := \text{Gal}(\mathbb{C}/\mathbb{R})$
 partial results: \mathcal{F} is separated $\leadsto \mathcal{F}(\mathbb{R}) \rightarrow \mathcal{F}(\mathbb{C})$ inj
 Comm diagram $\leadsto \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{\text{Gal}}$

\mathcal{F} sheaf: $0 \rightarrow \mathcal{F}(U) \rightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_j U_j)$
 $i, j \leftarrow i=j$ is allowed:

in this case $0 \rightarrow \mathcal{F}(\text{Spec } \mathbb{R}) \rightarrow \mathcal{F}(\text{Spec } \mathbb{C}) \xrightarrow[\hookrightarrow]{\hookrightarrow} \mathcal{F}(\text{Spec } \mathbb{C} \amalg \text{Spec } \mathbb{C})$

$$\begin{array}{ccc} \mathcal{F}(\text{Spec } \mathbb{C}) & \longrightarrow & \mathcal{F}(\text{Spec } \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathcal{F}(\text{Spec } \prod_{\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})} \mathbb{C}) \\ \downarrow \text{ } & \begin{array}{l} \hookrightarrow_1: x \mapsto x \otimes 1 \\ \hookrightarrow_2: x \mapsto 1 \otimes x \end{array} & \begin{array}{l} x \otimes y \mapsto (xy, x\bar{y}) \\ \parallel \end{array} \end{array}$$

$$\mathcal{F}(\coprod_{\sigma \in \text{Gal}(\mathbb{C}/\mathbb{R})} \text{Spec } \mathbb{C}) \parallel \mathcal{F}(\text{Spec } \mathbb{C})$$

$$\mathcal{F}(\text{Spec } \mathbb{C}) \longrightarrow \mathcal{F}(\text{Spec } \mathbb{C}) \times \mathcal{F}(\text{Spec } \mathbb{C})$$

$$\hookrightarrow_2: \text{Spec } \mathbb{C} \xleftarrow{(Id, \sigma)} \text{Spec } \mathbb{C} \amalg \text{Spec } \mathbb{C}$$

$$\begin{array}{l} \leadsto \mathcal{F}(\text{Spec } \mathbb{C}) \xrightarrow{(\mathcal{F}(Id), \mathcal{F}(\sigma))} \mathcal{F}(\text{Spec } \mathbb{C} \amalg \text{Spec } \mathbb{C}) \cong \mathcal{F}(\text{Spec } \mathbb{C}) \times \mathcal{F}(\text{Spec } \mathbb{C}) \\ \text{Abuse of notation} \quad \mathcal{F}(\mathbb{C}) \xrightarrow{(Id, \sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C}) \\ \hookrightarrow_1: \mathcal{F}(\mathbb{C}) \xrightarrow{(Id, Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C}) \end{array}$$

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\text{Spec } K, \mathcal{F}) = \mathcal{F}(\text{Spec } K) = \mathcal{F}_{K^{\text{sep}}}^{\text{Gal}(K^{\text{sep}}/K)} \quad \mathcal{F}_{K^{\text{sep}}} := \varinjlim_{\substack{L/K \\ \text{finite}}} \mathcal{F}(\text{Spec } L)$$

Ex. describe the stalk & fiber at $p \in \text{Spec } K$

$$\mathcal{F}_p := \varinjlim_{p \in U} \mathcal{F}(U) = \mathcal{F}_{K^{\text{sep}}} \quad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\text{Spec } K, p}} K(p) = \mathcal{F}_p = \mathcal{F}_{K^{\text{sep}}}$$

<https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts-peck-mu-n-and-h1-texts>

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