

Eine Woche, ein Beispiel

11.6 equivariant K-theory of Steinberg variety: abstract nil Hecke alg

Recall that we have an alg homo

$$\mathbb{Z}[e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_d^{\pm 1}] \xrightarrow{\quad} \mathbb{Q}[[\lambda_1, \lambda_2, \dots, \lambda_d]] \supseteq \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

$$e_i \xrightarrow{\quad} e^{\lambda_i}$$

Set $s_i = (i, i+1) \in S_d$, $i \in \{1, \dots, d-1\}$ for e_i, λ_i , $i \in \{1, \dots, d\}$

Ex 1. define $\partial_i \in \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$ by

$$\partial_i f = \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \quad f \in \mathbb{Q}[\lambda_1, \dots, \lambda_d]$$

compute $\partial_i \lambda_i$, $\partial_i \lambda_{i+1}$, $\partial_i (\lambda_1^3 \lambda_2 - 3 \lambda_2 \lambda_4 \lambda_5)$.

Ex 2. derive that

$$\partial_i f g = (s_i f) \partial_i g + \frac{f - s_i f}{\lambda_i - \lambda_{i+1}} g \quad f \in \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])$$

$$f: g \mapsto f \cdot g$$

as operators.

Ex 3. verify that

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ \partial_i \lambda_i = \lambda_{i+1} \partial_i + 1 \end{array} \quad \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ \partial_i \lambda_{i+1} = \lambda_i \partial_i - 1 \end{array}$$

$$\begin{array}{c} \text{Diagram: two crossings sharing a strand} \\ \partial_i \partial_{i+1} \partial_i = \partial_{i+1} \partial_i \partial_{i+1} \end{array} \quad \begin{array}{c} \text{Diagram: square crossing} \\ \partial_i^2 = 0 \end{array}$$

Ex 1'. define $D_i \in \text{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_i^{\pm}, \dots, e_d^{\pm}])$ by

$$D_i f = \frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}}$$

$$= \frac{e_{i+1}f - e_i s_i f}{e_{i+1} - e_i}$$

compute

$$D_i 1 = 1$$

$$D_i e_i = 0$$

$$D_i e_{i+1} = e_i + e_{i+1}$$

$$D_i e_i^{-1} = e_i^{-1} + e_{i+1}^{-1}$$

$$D_i e_{i+1}^{-1} = 0$$

Ex 2'. derive that

$$D_i f g = (s_i f) D_i g + \frac{f - s_i f}{1 - \frac{e_i}{e_{i+1}}} g$$

as operators.

Ex 3'. verify that

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ D_i e_i \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ e_{i+1} D_i \end{array} - \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the right one} \\ e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-right strand} \\ D_i e_{i+1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-left strand} \\ e_i D_i \end{array} + \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the right one} \\ e_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-left strand} \\ D_i e_i^{-1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-right strand} \\ e_{i+1}^{-1} D_i \end{array} + \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the left one} \\ e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: crossing with dot on top-right strand} \\ D_i e_{i+1}^{-1} \end{array} = \begin{array}{c} \text{Diagram: crossing with dot on bottom-left strand} \\ e_i^{-1} D_i \end{array} - \begin{array}{c} \text{Diagram: two parallel vertical strands, dot on the left one} \\ e_i^{-1} \end{array}$$

$$\begin{array}{c} \text{Diagram: braid relation for } D_i, D_{i+1} \\ D_i D_{i+1} D_i \end{array} = \begin{array}{c} \text{Diagram: braid relation for } D_{i+1}, D_i, D_{i+1} \\ D_{i+1} D_i D_{i+1} \end{array}$$

$$\begin{array}{c} \text{Diagram: square formed by two crossings} \\ D_i^2 \end{array} = \begin{array}{c} \text{Diagram: single crossing} \\ D_i \end{array}$$

Ex 4. Verify that

$$\begin{array}{ccc}
 \text{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[e_i^{\pm 1}, \dots, e_d^{\pm 1}])_{\text{cpl}} & & \text{End}_{\mathbb{Q}\text{-v.s.}}(\mathbb{Q}[\lambda_1, \dots, \lambda_d])_{\text{cpl}} \\
 \cup & & \cup \\
 \langle e_i^{\pm 1}, \dots, e_d^{\pm 1}, D_1, \dots, D_{d-1} \rangle_{\mathbb{Z}\text{-alg, cpl}} & \longrightarrow & \langle \lambda_1, \dots, \lambda_d, \partial_1, \dots, \partial_d \rangle_{\mathbb{Q}\text{-alg, cpl}} \\
 e_i & \xrightarrow{\quad} & e^{\lambda_i} \\
 D_i & \xrightarrow{\quad} & \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} = \frac{\lambda_{i+1} - \lambda_i}{1 - e^{\lambda_{i+1} - \lambda_i}} \partial_i + 1 \\
 \\
 \log e_i & \longleftarrow & \lambda_i \\
 \\
 D_i \frac{1 - \frac{e_i}{e_{i+1}}}{\log \frac{e_i}{e_{i+1}}} = \frac{1 - \frac{e_{i+1}}{e_i}}{\log \frac{e_{i+1}}{e_i}} (D_i - 1) & \longleftarrow & \partial_i
 \end{array}$$

is an alg iso.

Hint.

$$\begin{array}{lcl}
 D_i e_k & = & s_i(e_k) D_i \\
 \Leftrightarrow \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} e^{\lambda_k} & = & s_i(e^{\lambda_k}) \partial_i \frac{\lambda_i - \lambda_{i+1}}{1 - e^{\lambda_i - \lambda_{i+1}}} + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{1 - e^{\lambda_i - \lambda_{i+1}}} \\
 \Leftrightarrow \partial_i e^{\lambda_k} & = & s_i(e^{\lambda_k}) \partial_i + \frac{e^{\lambda_k} - s_i(e^{\lambda_k})}{\lambda_i - \lambda_{i+1}}
 \end{array}$$