

fiber of $\pi_*, \pi_!, \pi^{-1}, \pi^*$ $\mathcal{G} \mathcal{F}$
 $\pi: Y \hookrightarrow X$

$$\pi_*: \mathcal{U} \rightarrow X \quad \pi_*: Z \xrightarrow{\text{close}} X$$

$$\pi_* \begin{cases} \mathcal{G}_x & x \in \mathcal{U} \\ 0 & x \notin \mathcal{U} \\ \varinjlim_{x \in V} \mathcal{G}(U \cap V) & x \in \bar{\mathcal{U}} - \mathcal{U} \end{cases} \quad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi_! \begin{cases} \mathcal{G}_x & x \in \mathcal{U} \\ 0 & x \notin \mathcal{U} \end{cases} \quad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi^{-1} \mathcal{F}_y \quad \mathcal{F}_y$$

$$\pi^* \mathcal{F}_y \otimes_{\pi^{-1} \mathcal{O}_{X,Y}} \mathcal{O}_{Y,Y} \quad \mathcal{F}_y \otimes_{\pi^{-1} \mathcal{O}_{X,Y}} \mathcal{O}_{Y,Y}$$

For étale: $f: X \xrightarrow{\mathcal{G}} Y \xrightarrow{\mathcal{F}}$ $\bar{x} \mapsto \bar{y}$
 (sheaf)

$$\bar{x} \xrightarrow{u_{\bar{x}}} x \xrightarrow{f} y$$

$$(f^* \mathcal{F})_{\bar{x}} = u_{\bar{x}}^* f^* \mathcal{F}(y) = \mathcal{F}_y$$

If f q.c. $(f^* \mathcal{F})_{\bar{x}} \cong \mathcal{F}_y$
 $(f_* \mathcal{G})_{\bar{y}} \cong \Gamma(\tilde{X}, \tilde{g}^* \mathcal{G})$ where $\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{g}} & X \\ \downarrow \wr & & \downarrow f \\ \text{Spec } \mathcal{O}_{Y,\bar{y}}^d & \xrightarrow{g} & Y \end{array}$
 e.g. for f finite, we have explicit description of f_* .

左伴随 \dashv 右伴随

$$f_! \dashv f^* \dashv f_* \dashv f'$$

自由

忘却

$$-\otimes_A N$$

$$\text{Hom}_A(N, -)$$

$$\Delta$$

$$\varprojlim$$

$$(\)^\sim$$

$$\prod_*$$

$$f_{p_i}$$

$$f^*$$

$$\text{Ind}$$

$$\text{Res}$$

$$-\otimes_k A^e$$

$$\text{Res}$$

$$\text{sh}(-)$$

$$\text{Res}_{\text{sh} \rightarrow \text{Psh}}$$

$$\pi_0$$

$$T \mapsto \bigsqcup_{t \in T} \text{Spec } \mathbb{Z}$$

$$G^{\text{gp}}$$

$$\text{Lie}$$

ad \Rightarrow 右正合

左正合

ad \Rightarrow 与余极限交换 \varinjlim 与极限交换 \varprojlim

$$\text{coker } f$$

$$\ker f$$

$$\prod A \oplus A$$

$$\prod A \quad A \times_c B$$

$$\bar{K} = \varinjlim_{L/K} L$$

$$\text{Spec } \bar{K} = \varprojlim_{L/K} \text{Spec } L \quad \text{Gal}(\bar{K}/K) = \varprojlim_{\substack{L \text{ finite} \\ \text{Galois}}} \text{Gal}(L/K)$$

$$\text{Spec } \mathbb{Z}_p = \varinjlim_{\text{completion}} \text{Spec } \mathbb{Z}/p^n \mathbb{Z}$$

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \mathbb{Z}$$

ring point of view

$$\mathcal{F}_p = \varinjlim_{\mathcal{U}} \mathcal{F}(U)$$

stalk

co limit

direct
inductive
injective } limit.

Lemma 6.21.3. Let $f : X \rightarrow Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \rightarrow PSh(X)$ which is left adjoint to f_* . For a presheaf \mathcal{G} it is determined by the rule

$$f_p \mathcal{G}(U) = \operatorname{colim}_{f(U) \subset V} \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y . The colimits are over directed partially ordered sets. (The restriction mappings of $f_p \mathcal{G}$ are explained in the proof.)

Lemma 6.31.4. Let X be a topological space. Let $j : U \rightarrow X$ be the inclusion of an open subset.

- (1) The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).
- (2) The functor $j_!$ is a left adjoint to restriction, in a formula

$$\operatorname{Mor}_{Sh(X)}(j_! \mathcal{F}, \mathcal{G}) = \operatorname{Mor}_{Sh(U)}(\mathcal{F}, j^{-1} \mathcal{G}) = \operatorname{Mor}_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in \mathcal{F} and \mathcal{G} .

- (3) Let \mathcal{F} be a sheaf of sets on U . The stalks of the sheaf $j_! \mathcal{F}$ are described as follows

$$j_! \mathcal{F}_x = \begin{cases} \emptyset & \text{if } x \notin U \\ \mathcal{F}_x & \text{if } x \in U \end{cases}$$

- (4) On the category of presheaves of U we have $j_p j_{p!} = \operatorname{id}$.
- (5) On the category of sheaves of U we have $j^{-1} j_! = \operatorname{id}$.

situation	category \mathcal{A}	category \mathcal{B}	left adjoint $F: \mathcal{A} \rightarrow \mathcal{B}$	right adjoint $G: \mathcal{B} \rightarrow \mathcal{A}$
A-modules (Ex. 1.5.D)	Mod_A	Mod_A	$(\cdot) \otimes_A N$	$Hom_A(N, \cdot)$
ring maps $B \rightarrow A$ (Ex. 1.5.E)	Mod_B	Mod_A	$(\cdot) \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(pre)sheaves on a topological space X (Ex. 2.4.L)	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups (§1.5.3)	semigroups	groups	groupification	forgetful
sheaves, $\pi: X \rightarrow Y$ (Ex. 2.7.B)	sheaves on Y	sheaves on X	π^{-1}	π_*
sheaves of abelian groups or \mathcal{O} -modules, open embeddings $\pi: U \hookrightarrow Y$ (Ex. 2.7.G)	sheaves on U	sheaves on Y	$\pi_!$	π^{-1}
quasicoherent sheaves, $\pi: X \rightarrow Y$ (Prop. 16.3.6)	$QCoh_Y$	$QCoh_X$	π^*	π_*
ring maps $B \rightarrow A$ (Ex. 30.3.A)	Mod_A	Mod_B	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto$ $Hom_B(A, N)$
quasicoherent sheaves, affine $\pi: X \rightarrow Y$ (Ex. 30.3.B(b))	$QCoh_X$	$QCoh_Y$	π_*	$\pi_{sh}^!$

Other examples will also come up, such as the adjoint pair (\sim, Γ_\bullet) between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.4).

E.g. $\text{Spec } \mathbb{C} \longrightarrow \text{Spec } \mathbb{R}$ ring space but not scheme

$$(f^*: \mathbb{R}\text{-mod} \longrightarrow (\text{Spec } \mathbb{C}, \widehat{\mathbb{R}})^{\text{-mod}})$$

quasi-coherent: $f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{C}\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C}$
 $f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \quad \text{forget}$

$$f^* f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{C}\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C} \leftarrow \mathbb{C}\text{-mod structure}$$

$$f_* f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C}$$

étale: $f^*: \mathbb{Z}[\frac{1}{2}\mathbb{Z}]\text{-mod} \longrightarrow \mathcal{A}b \quad \text{forget}$
 $f_*: \mathcal{A}b \longrightarrow \mathbb{Z}[\frac{1}{2}\mathbb{Z}]\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C}$
 $f^* f_*: \mathcal{A}b \longrightarrow \mathcal{A}b \quad - \otimes_{\mathbb{R}} \mathbb{C}$
 $f_* f^*: \mathbb{Z}[\frac{1}{2}\mathbb{Z}]\text{-mod} \longrightarrow \mathbb{Z}[\frac{1}{2}\mathbb{Z}]\text{-mod} \quad - \otimes_{\mathbb{R}} \mathbb{C} \leftarrow \text{Gal}(\mathbb{C}/\mathbb{R})\text{-action}$

$$\pi: X_{\text{ét}} \longrightarrow X_{\text{Zar}} \quad \left[\begin{array}{l} \text{Zariski sheafification} \\ \mathcal{F} \mapsto \left[\pi^* \mathcal{F} : \bigcup_x U \rightarrow \Gamma(U, \pi^* \mathcal{F}) \right] \\ \text{is not enough...} \end{array} \right]$$

$$\pi_*: \text{Sh}(X_{\text{ét}}) \longrightarrow \text{Sh}(X_{\text{Zar}}) \quad \text{forget}$$

$$\pi^* \dashv \pi_* \quad \pi_* \pi^* = \text{Id}_{\text{Sh}(X_{\text{Zar}})} \Rightarrow \pi^* \text{ fully faithful}$$

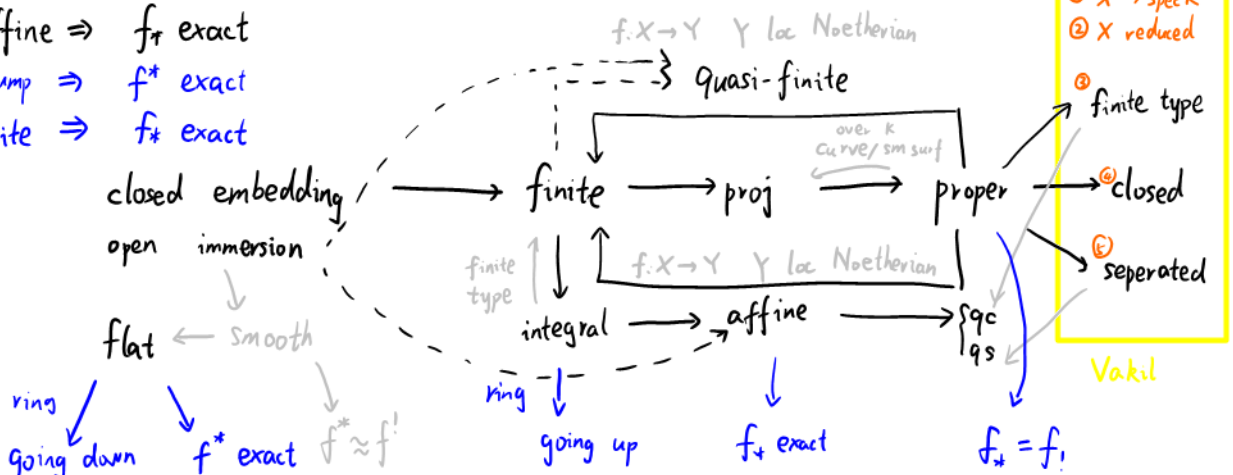
<https://math.stackexchange.com/questions/321633/when-is-a-fully-faithful-functor-a-n-equivalent-functor>

$$\begin{array}{ccc} & \text{QCoh}(X_{\text{Zar}}) & \\ \swarrow W & \downarrow \hookrightarrow & \text{full subcategory} \\ \text{Sh}(X_{\text{ét}}) & \xrightleftharpoons[\pi^*]{\pi_*} & \text{Sh}(X_{\text{Zar}}) \end{array}$$

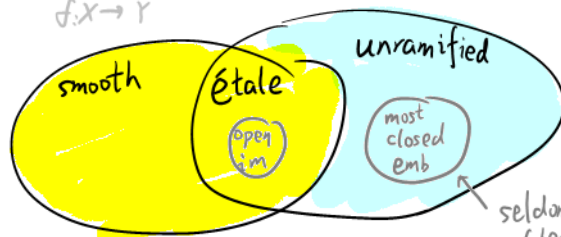
18.1.7

coherent

étale: $f \text{ flat} \Rightarrow f^* \text{ exact}$
 $f \text{ affine} \Rightarrow f_* \text{ exact}$
 $f \text{ finite} \Rightarrow f_* \text{ exact}$



Various interesting kinds of morphisms (locally Noetherian source, **affine**, separated, see Exercises 7.3.B(b), 7.3.D, and 10.1.H resp.) are quasiseparated,



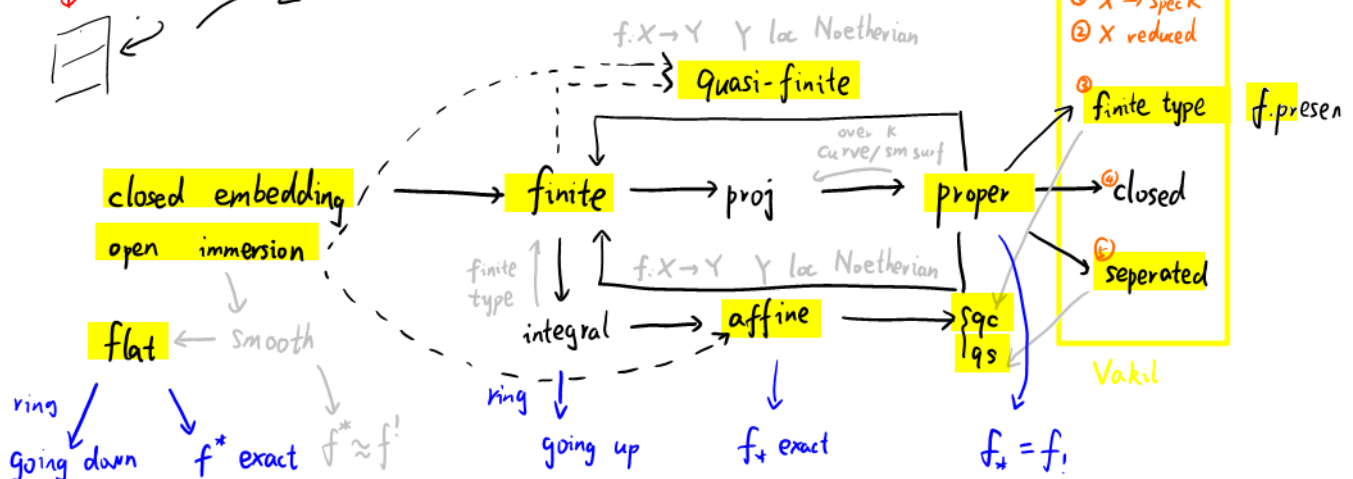
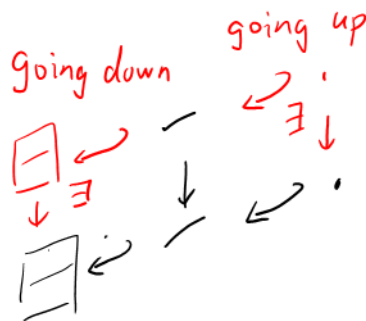
flat
 non-flat

Rmk ^{lift} locally f.p. is crucial.

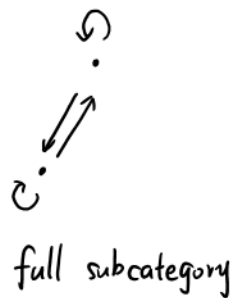
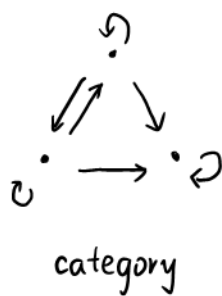
eg. $\mathbb{C}[t^{\frac{1}{n}}] \xrightarrow{t^{\frac{1}{n}} \mapsto 0} \mathbb{C}$ is formally étale but not flat

<https://rankaya.people.uic.edu/formallyunramifiedetale.pdf>

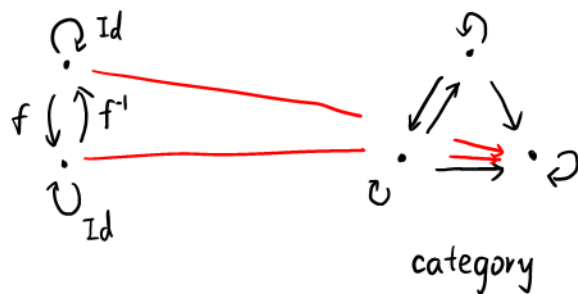
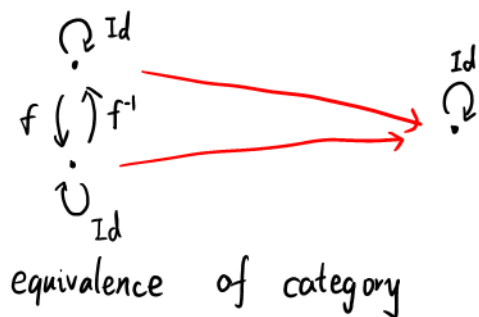
$\mathbb{F}_p[T] \rightarrow (\mathbb{F}_p[T])^{sep}$ is formally unramified, not formally étale but flat.



: properties which satisfy the f.flat descent.



<https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective>



fully faithful

<https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/>