Eine Woche, ein Beispiel

5.15 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory.
In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categorys here: http://katmat.math.uni-bremen.de/acc/acc.pdf

$\mathcal{C}$ is always a category.

	$Ob(\mathcal{C})$		$Mor(X,Y)$
small	Set		Set
loc. small	—		Set
large	not set	or	not set

filtered:

cofiltered:

Complete/Cocomplete/Bicomplete category

Def. $\mathcal{C}$ is complete if

$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow \mathcal{C}$ $\quad i \longmapsto F_i$,

$\varprojlim_{i \in \Lambda} F_i$ exists $\quad$ ($\varprojlim_{i\in\Lambda} F_i$ is called the small limit)

$\mathcal{C}$ is cocomplete if

$\forall$ small category Λ, $\forall$ fctor $F: \Lambda \longrightarrow \mathcal{C}$ $\quad i \longmapsto F_i$,

$\varinjlim_{i \in \Lambda} F_i$ exists $\quad$ ($\varinjlim_{i\in\Lambda} F_i$ is called the small colimit)

bicomplete = complete + cocomplete

$\mathcal{C}$ is finitely complete if $\forall$ finite limit exists

$\mathcal{C}$ is finitely cocomplete if $\forall$ finite colimit exists.

Thm. $\mathcal{C}$ is complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & products

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & products

$\mathcal{C}$ is cocomplete $\Leftrightarrow$ $\mathcal{C}$ has coequalizers & coproducts

$\Leftrightarrow$ $\mathcal{C}$ has pushouts & coproducts

$\mathcal{C}$ is finitely complete $\Leftrightarrow$ $\mathcal{C}$ has equalizers & finite products

$\Leftrightarrow$ $\mathcal{C}$ has equalizers, binary products & terminal obj

$\Leftrightarrow$ $\mathcal{C}$ has pullbacks & terminal obj

For small category $\mathcal{C}$,

$$\begin{array}{ccc} \text{complete} & \Leftrightarrow & \text{cocomplete} \\ & \Downarrow & \\ & \text{thin} & (\#\mathrm{Mor}(X,Y) \leq 1) \end{array}$$

Cartesian closed category / Closed category

Def. $\mathcal{C}$ is Cartesian closed if
$\mathcal{C}$ has terminal obj, binary product and exponential, where

$$- \times Y \dashv (-)^Y$$

a bifctor $F: \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which is functorial in Y

i.e. $\mathrm{Mor}(X\times Y, Z) \cong \mathrm{Mor}(X, Z^Y)$

$\mathcal{C}$ is loc. Cartesian closed if all its slice category is Cartesian closed.

Rmk. When $\mathcal{C}$ is loc. Cartesian closed,
$\mathcal{C}$ is Cartesian closed $\Leftrightarrow$ $\mathcal{C}$ has a terminal object.
But $\mathcal{C}$ is Cartesian closed $\not\Rightarrow$ $\mathcal{C}$ is loc. Cartesian closed

For the closed category, we use the definition in https://ncatlab.org/nlab/show/closed+category.

Def A closed category is a category $\mathcal{C}$ together with the following data.

- bifctor $[-,-]: \mathcal{C}^{op}\times \mathcal{C} \to \mathcal{C}$ — called internal hom-fctor
- $I \in \mathrm{Ob}(\mathcal{C})$ — called unit object
- $i: \mathrm{Id}_{\mathcal{C}} \xrightarrow{\cong} [I,-]$ $\rightsquigarrow$ $i_A: A \xrightarrow{\cong} [I,A]$
- $j_X: I \longrightarrow [X,X]$ — extranatural in X
- $L^X_{Y,Z}: [Y,Z] \to [[X,Y],[X,Z]]$ — functorial in Y and Z, extranatural in X.
- Compatabilities

$$\begin{tikzcd}
I \arrow[r, "j_Y"] \arrow[dr, "j_{[X,Y]}"'] & {[Y,Y]} \arrow[d, "L^X_{Y,Y}"] \\
& {[[X,Y],[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[X,Y]} \arrow[r, "L^X_{XY}"] \arrow[dr, "i_{[X,Y]}"'] & {[[X,X],[X,Y]]} \arrow[d, "{[j_X,1]}"] \\
& {[I,[X,Y]]}
\end{tikzcd}
\qquad
\begin{tikzcd}
{[Y,Z]} \arrow[r, "L^I_{YZ}"] \arrow[dr, "{[1,i_Z]}"'] & {[[I,Y],[I,Z]]} \arrow[d, "{[i_Y,1]}"] \\
& {[Y,[I,Z]]}
\end{tikzcd}$$

$$\begin{tikzcd}
& {[U,V]} \arrow[dl, "L^X_{UV}"'] \arrow[dr, "L^Y_{UV}"] & \\
{[[X,U],[X,V]]} \arrow[d, "L^{[X,Y]}_{[X,U][X,V]}"'] & & {[[Y,U],[Y,V]]} \arrow[d, "{[1,L^X_{YV}]}"] \\
{[[[X,Y],[X,U]],[[X,Y],[X,V]]]} \arrow[rr, "{[L^X_{YU},1]}"] & & {[[Y,U],[[X,Y],[X,V]]]}
\end{tikzcd}$$

$$\begin{aligned}
\gamma: \mathrm{Mor}(X,Y) &\longrightarrow \mathrm{Mor}(I,[X,Y]) \\
f &\longmapsto [1,f]\circ j_X
\end{aligned}$$

is an iso.

Monoidal category = Tensor category

Def A monoidal category is a category $\mathcal{C}$ together with the following data.

- bifctor $-\otimes- : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- $I \in Ob(\mathcal{C})$ called unit object
- $a_{A,B,C} : A\otimes(B\otimes C) \xrightarrow{\cong} (A\otimes B)\otimes C$
- $\lambda_A : I\otimes A \xrightarrow{\cong} A$ lambda: left
- $\rho_A : A\otimes I \xrightarrow{\cong} A$ rho: right
- Compatabilities

$$\begin{tikzcd}
 & A\otimes(B\otimes(C\otimes D)) \arrow[dl, "1_A\otimes a_{B,C,D}"'] \arrow[dr, "a_{A,B,C\otimes D}"] & \\
A\otimes((B\otimes C)\otimes D) \arrow[d, "a_{A,B\otimes C,D}"'] & \circlearrowright & (A\otimes B)\otimes(C\otimes D) \arrow[d, "a_{A\otimes B,C,D}"] \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "a_{A,B,C}\otimes 1_A"] & & ((A\otimes B)\otimes C)\otimes D
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(I\otimes B) \arrow[rr, "a_{A,I,B}"] \arrow[dr, "1_A\otimes\lambda_B"'] & & (A\otimes I)\otimes B \arrow[dl, "\rho_A\otimes 1_B"] \\
 & A\otimes B &
\end{tikzcd}$$

For strict monoidal category, we require in addition that $a_{A,B,C}$, λ_A, ρ_A are identities.

E.g. Cartesian monoidal category $\mathcal{C}$: category with finite products

$\otimes = \prod$ $\quad I$ = terminal object

e.g. Set, Cat.

Cocartesian monoidal category $\mathcal{C}$: category with finite coproducts

$\otimes = \coprod$ $\quad I$ = initial object

Abelian category is monoidal.

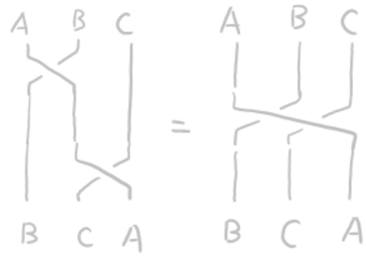
Def (Specializations)

Let $\mathcal{C}$ be a monoidal category.

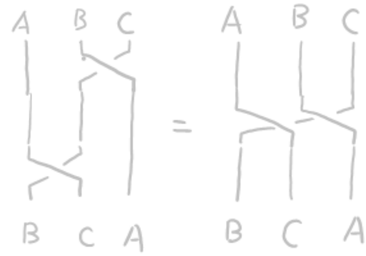
If in addition we have $\gamma_{A,B} : A \otimes B \longrightarrow B \otimes A$,

then $\mathcal{C}$ is **braided monoidal category** if

$$\begin{tikzcd}
 & (A\otimes B)\otimes C \arrow[dl, "\gamma_{A,B}\otimes 1_C"'] \arrow[dr, "\alpha_{A,B,C}"] & \\
(B\otimes A)\otimes C \arrow[d, "\alpha_{B,A,C}"] & & A\otimes(B\otimes C) \arrow[d, "\gamma_{A,B\otimes C}"] \\
B\otimes(A\otimes C) \arrow[dr, "1_B\otimes\gamma_{A,C}"'] & & (B\otimes C)\otimes A \arrow[dl, "\alpha_{B,C,A}"] \\
 & B\otimes(C\otimes A) &
\end{tikzcd}$$



$$\begin{tikzcd}
 & A\otimes(B\otimes C) \arrow[dl, "1_A\otimes\gamma_{B,C}"'] \arrow[dr, "\alpha^{-1}_{A,B,C}"] & \\
A\otimes(C\otimes B) \arrow[d, "\alpha^{-1}_{A,C,B}"] & & (A\otimes B)\otimes C \arrow[d, "\gamma_{A\otimes B,C}"] \\
(A\otimes C)\otimes B \arrow[dr, "\gamma_{C,A}\otimes 1_B"'] & & C\otimes(A\otimes B) \arrow[dl, "\alpha^{-1}_{C,A,B}"] \\
 & (C\otimes A)\otimes B &
\end{tikzcd}$$



$\mathcal{C}$ is **symmetric monoidal category** if

$$\gamma_{B,A} \circ \gamma_{A,B} = 1_{A\otimes B}. \qquad + \mathcal{C} \text{ is braided.}$$

closed monoidal category = closed category + monoidal category
+ compatabilite $\quad -\otimes A \dashv [A,-]$

A list of categories which I'm interested:

Set **Top** **Grp** **Ab** **Vect(k)** **Mod(R)**

Ring: identity + preserve identity **CRing** **Rng**

Field: full subcategory of CRing

0: $Ob(0) = \emptyset$

1: $Ob(1) = \{*\}$ $Mor(*,*) = \{1_*\}$

K(2): $Ob(K(2)) = \{V, E\}$ $Mor(V,V) = \{1_V\}$ $Mor(E,E) = \{1_E\}$ $Mor(V,E) = \emptyset$ $Mor(E,V) = \{s,t\}$

$$\begin{tikzcd} E \arrow[loop left, "1_E"] \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V \arrow[loop right, "1_V"] \end{tikzcd}$$

Δ: $Ob(\Delta) = \{[n] := \{0,1,2,\dots n\} \mid n \geq 0\}$

$Mor([m],[n]) = \{\text{weakly monotone maps}\}$

sSet: $Ob(sSet) = \left\{ X : \Delta^{op} \longrightarrow Set \atop [n] \mapsto X_n \right\}$ $Mor(X,Y) = \left\{ \alpha : \begin{tikzcd} \Delta^{op} \arrow[r, bend left, "X"] \arrow[r, bend right, "Y"'] \arrow[r, phantom, "\Downarrow\alpha"] & Set \end{tikzcd} \right\}$

CHaus: $Ob(CHaus) = \{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} \ X \}$ https://ncatlab.org/nlab/show/compactum

$Mor(X,Y) = \{ f : X \longrightarrow Y \mid f \text{ cont} \}$

Met: full subcategory of CHaus whose objects are metric spaces.

⚠ For the category of Graph, there're different realizations.

Quiv(C): $Ob(Quiv(\mathcal{C})) = \{ \text{fctor } \Gamma : K(2) \longrightarrow \mathcal{C} \}$

$Mor(\Gamma_1, \Gamma_2) = \left\{ \alpha : \begin{tikzcd} K(2) \arrow[r, bend left, "\Gamma_1"] \arrow[r, bend right, "\Gamma_2"'] & \mathcal{C} \end{tikzcd} \right\}$

Quiv $= Quiv(Set)$

$=$ Category of presheaves on $\mathcal{Q}^{op}$.

$\mathbf{Cat}$ = {the category of small categories} is a 2-category.

$Ob(\mathbf{Cat}) = \{\text{small category } \mathcal{C}\}$

$Mor(\mathcal{C}, \mathcal{D})$ is a category by

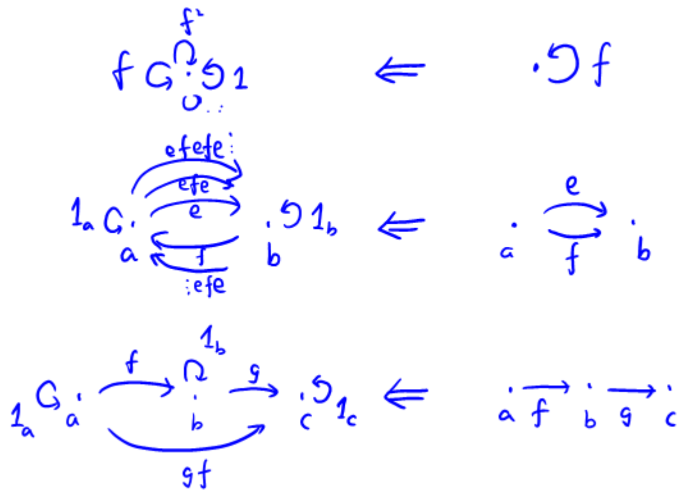
$$Ob(Mor(\mathcal{C},\mathcal{D})) = \{F : \mathcal{C} \to \mathcal{D}\}$$

$$Mor(F,G) = \left\{ \alpha : \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \end{tikzcd} \right\}$$

Basic properties of Cat.

1. Initial object 0, Terminal object 1.
2. Cat is loc. small but not small
3. Cat is bicomplete
4. Cat is Cartesian closed but not loc. Cartesian closed
5. Cat is loc. finitely presentable https://ncatlab.org/nlab/show/locally+finitely+presentable+category
6. $\begin{tikzcd} \mathbf{Cat} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\top"] & \mathbf{Quiv} \arrow[l, bend right, "\text{free}"'] \end{tikzcd}$

e.g of "free"



I'm just too lazy to fill in this table. If you know more, tell me and I will fill in, thanks!

	Category	cpl	fin cpl	cocpl	fin cocpl	Cartesian closed	monoidal
	Set	✓		✓		✓	✓
	Top	✓		✓		×	✓
	Grp	✓		✓		×	✓
	Ab	✓		✓		×	✓
	Vect(K)	✓		✓		×	✓
ReCRing	Mod(R)	✓		✓		×	✓
	Ring	✓		✓			
	CRing	✓		✓			
	Rng	✓		✓			
	Field	×	×	×	×		
	0						
	1						
	K(2)						
	Δ	×	×	×	×		
	sSet	✓		✓			
	CHaus	✓		✓			
	Met	×	✓	×	×		
	Quiv($\mathcal{C}$)						
	Quiv						
	Cat	✓		✓		✓	✓
	CGTop						✓
	CGHaus					✓	
	Prof	✓					