# $SL_2(\mathbb{R})$: Principal Series, Discrete Series and Modular Forms

<u>Last time</u>

| | | |
|---|---|---|
| | $SL_2(\mathbb{R}) \circlearrowleft \mathbb{P}^1$ | group action |
| $\rightsquigarrow$ | $SL_2(\mathbb{R}) \circlearrowleft L^2(\mathbb{P}^1)$ | group representation |
| $\rightsquigarrow$ | $(\mathfrak{g}, K) \circlearrowleft L^2(\mathbb{P}^1)_{(K)}$ | $(\mathfrak{g}, K)$ - module |

We have also talked about the classification of admissible irreducible $(\mathfrak{g}, K)$-modules $U$:

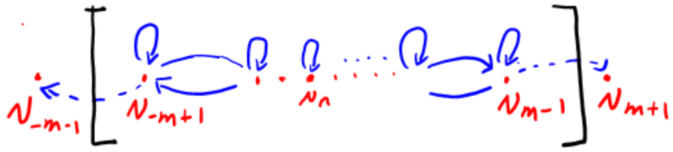
Suppose $K v_n = \varepsilon^n v_n$, $\Omega . v \equiv \gamma v$ $\quad (\exists v_n \in U, \forall v \in U)$

(1) $\gamma \neq \frac{m^2-1}{4}$ for any $m \in \mathbb{Z}_{\geq 0}$ of opposite parity to $n$:

$PS_{\gamma, n}$:


(2) $\gamma = \frac{m^2-1}{4}$, $m \in \mathbb{Z}_{\geq 0}$, $m \equiv n+1 \pmod 2$,

$|n| < m$: $FD_{m-1}$ $[-(m-1), m-1]$




$n > m$: $DS^+_{m+1}$ $[m+1, +\infty)$


$n < -m$: $DS^-_{m+1}$ $(-\infty, -(m+1)]$

We've also concluded which of the representations are unitary:

$PS_{\gamma, n}$: $\begin{cases} ① \ n \text{ is odd } \& \ \gamma \in \mathbb{R}, \gamma < \frac{1}{4} \\ ② \ n \text{ is even } \& \ \gamma \in \mathbb{R}, \gamma < 0 \end{cases}$

$FD_{m-1}$: $m = 1$

$DS^+_{m+1}$ & $DS^-_{m+1}$: always unitary

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Notations:

the split basis $\quad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad v_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad v_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

the compact basis of $sl_2(\mathbb{C})$ $\quad k = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad x_+ = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \quad x_- = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$

the Casimir element $\quad \Omega = -\frac{k^2}{4} + \frac{x_+ x_-}{2} + \frac{x_- x_+}{2}$

Today: principal series & modular forms.

Q: Which $\overset{\text{admissible, irr}}{\wedge}$ $(\mathfrak{g}, K)$-module comes from representations of $G$?

A: For $G = SL_2(\mathbb{R})$, EVERY $(\mathfrak{g}, K)$-module

Idea: by direct construction.

$$SL_2(\mathbb{R}) \circlearrowright \mathbb{P}^1 = P\backslash G$$

$$\rightsquigarrow SL_2(\mathbb{R}) \circlearrowright \Gamma(\mathbb{P}^1, E)$$ where $E$ is a $G$-equivariant line bundle of $\mathbb{P}^1$

| | | | |
|---|---|---|---|
| $E =$ | $\mathbb{P}^1 \times \mathbb{R}$ | $T\mathbb{P}^1$ | $T^*\mathbb{P}^1$ |
| $\Gamma(\mathbb{P}^1, E) =$ | $C^\infty(\mathbb{P}^1)$ | v. field | 1-form |

We will realize $\Gamma(\mathbb{P}^1, E)$ as subspace of $C^\infty(G)$.

Principal Series
- Characters
- Definition of principal series
- Basis of principal series
- Actions on the basis
- Properties: irr & unitary

Discrete Series

Modular forms

# Character

<u>Def</u>: A character of a locally cpt group $H$ is a 1-dim representation of $H$:

$$\chi: H \longrightarrow \mathbb{C}^\times$$

$\chi$ is called a unitary character when $\operatorname{Im}\chi \subseteq \{z \mid |z|=1\}$

E.g. 1 the character of $\mathbb{R}^\times_{>0} \cong \mathbb{R}$ is of the form

$$\chi_s: x \longmapsto x^s \qquad s \in \mathbb{C}$$

and the character of $\mathbb{R}^\times$ is of the form

$$\chi_s \operatorname{sgn}^n: x \longmapsto |x|^s \operatorname{sgn}^n(x) \qquad s \in \mathbb{C},\ n \in \mathbb{Z}/2\mathbb{Z}$$

E.g. 2. the characters of $K = SO_2(\mathbb{R}) \cong S^1$ is of the form

$$\varepsilon^n: \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto (c+si)^n \qquad n \in \mathbb{Z}$$

E.g. 3. the character of $P = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \in SL_2(\mathbb{R}) \right\}$ is of the form

$$\chi_s \operatorname{sgn}^n: \begin{bmatrix} t & x \\ 0 & \frac{1}{t} \end{bmatrix} \longmapsto |t|^s \operatorname{sgn}^n(t) \qquad s \in \mathbb{C} \quad n \in \mathbb{Z}/2\mathbb{Z}$$

[These are all char of $P$, since $[PP] = N$
$\Rightarrow \chi$ is trivial on $N$
$\Rightarrow \chi$ is lifted from $P/N \cong \mathbb{R}^\times$]

We're specially interested in the modulus character

$$\delta = \chi_2 \operatorname{sgn}^0: \begin{bmatrix} t & x \\ 0 & \frac{1}{t} \end{bmatrix} \longmapsto t^2$$

[ Check: $R_p: \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \quad [x:y] \mapsto [x:y]p$
$\rightsquigarrow (R_p)^*_0: T^*_0\mathbb{P}^1 \longrightarrow T^*_0\mathbb{P}^1$ is multiplication by $\delta(p)$ ]

and we denote

$$\delta^{\frac{1}{2}} = \chi_1 \operatorname{sgn}^0: \begin{bmatrix} t & x \\ 0 & \frac{1}{t} \end{bmatrix} \longmapsto |t|$$

"Principal": Characters give us information about the degree of twist of a line bundle.

E.g.4 For the mobius strip over $S^1$, the sections can be viewed as a fct $f$ on $\mathbb{R}$ s.t $f(z+n) = (-1)^n f(z)$. $\forall z \in \mathbb{R}$

E.g.5 we know that $P\backslash G \cong \mathbb{P}^1$, then we have the correspondence. $(a=[0:1])$

$$\Gamma(T^*\mathbb{P}^1) \longleftrightarrow \{ f: G \to T_a^*\mathbb{P}^1 \mid f(pg) = (R_p)_a^*(f(g)) \} \quad \forall p \in P, g \in G$$

$$X \rightsquigarrow \Theta_X : g \longmapsto (R_g)^* X_{R_g(a)}$$

after calculation, we get

$$(R_p)_a^* : T_a^*\mathbb{P}^1 \longrightarrow T_a^*\mathbb{P}^1$$

is just multiplication by $\delta(p)$, so

$$\Gamma(T^*\mathbb{P}^1) \longleftrightarrow \{ f \in C^\infty(G) \mid f(px) = \delta(p) f(x) \} \quad \forall p \in P \ \ x \in G$$

In general, we can construct the isomorphism between

$$\Gamma(E) \longleftrightarrow \{ f \in C^\infty(G) \mid f(px) = \chi(p) f(x) \ \ \forall p \in P, x \in X \}$$

where $E$ : a $G$-equivariant line bundle

$\chi$ : a char of $P$

## Definition of $\mathrm{Ind}^\infty(\chi)$

Def. the principal series $\mathrm{Ind}^\infty(\chi)$ for a char $\chi : P \to \mathbb{C}^\times$ is defined as

$$\mathrm{Ind}^\infty(\chi) := \{ f \in C^\infty(G) \mid f(pg) = \chi(p) \underbrace{\delta^{\frac{1}{2}}(p)}_{\text{for the bilinear pairing}} f(g) \quad \forall p \in P, g \in G \}$$

e.g. $\mathrm{Ind}^\infty(\delta^{-\frac{1}{2}}) = \{ f \in C^\infty(G) \mid f(pg) = f(g) \quad \forall p \in P, g \in G \}$

$= C^\infty(P\backslash G) \cong C^\infty(\mathbb{P}^1)$

$\mathrm{Ind}^\infty(\delta^{\frac{1}{2}}) = \{ f \in C^\infty(G) \mid f(pg) = \delta(p) f(g) \quad \forall p \in P, g \in G \}$

$= \Omega^1(\mathbb{P}^1)$

for $f \in \mathrm{Ind}^\infty(\delta^{\frac{1}{2}})$, we can define the integral on $P\backslash G$

$$\int_{P\backslash G} f(x)\, dx := \int_K f(k)\, dk$$

For $f \in \mathrm{Ind}^\infty(\chi)$, $\varphi \in \mathrm{Ind}^\infty(\chi^{-1})$, we can define the bilinear pairing

$$\langle -,- \rangle : \mathrm{Ind}^\infty(\chi) \times \mathrm{Ind}^\infty(\chi^{-1}) \longrightarrow \mathbb{R}$$

$$\langle f, \varphi \rangle := \int_{P\backslash G} f\varphi(x)\, dx$$

$$G \curvearrowright C^\infty(G) \qquad f(-) \rightsquigarrow f(-g)$$

$\rightsquigarrow G \curvearrowright \mathrm{Ind}^\infty(\chi)$

$\rightsquigarrow (\mathfrak{g}, K) \curvearrowright \mathrm{Ind}(\chi)$ , where

$\mathrm{Ind}(\chi)$ = the Harish-Chandra modules of $\mathrm{Ind}^\infty(\chi)$

= subspace of $K$-finite vectors.

## The basis of $\mathrm{Ind}(\chi)$

Now suppose $f \in \mathrm{Ind}(\chi)$ is an eigenvector of $K(\kappa)$ with $f(I)=1$

$k.f = \varepsilon^m(k) f$

$$f(k) = (k.f)(I) = \varepsilon^m(k) f(I) = \varepsilon^m(k)$$

$G=PK$

$P \cap K = \{\pm I\}$

$$f(pk) = \chi(p)\delta^{\frac{1}{2}}(p) f(k) = \chi(p)\delta^{\frac{1}{2}}(p)\varepsilon^m(k)$$

Condition: $f((-I)(-I)) = f(I \cdot I) = 1$

$\Leftrightarrow \chi(-I)\, \mathrm{sgn}^m(-I) = 1$

$\Leftrightarrow$ (when $\chi = \chi_s \mathrm{sgn}^n$) $m \equiv n \pmod 2$

Conclusion: the basis of $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ are

$$\varepsilon_s^m(g) = (\chi_s \mathrm{sgn}^n)(p)\, \delta^{\frac{1}{2}}(p)\, \varepsilon^m(k) \qquad g = pk,\ m \equiv n \pmod 2$$

## Actions on the basis: $K$, $\mathfrak{g}$ & $\Omega$

$K$:
$$k'.\varepsilon_s^m(g) = \varepsilon_s^m(pkk')$$
$$= (\chi_s \mathrm{sgn}^n)(p)\delta^{\frac{1}{2}}(p)\varepsilon^m(k)\varepsilon^m(k')$$
$$= \varepsilon^m(k')\, \varepsilon_s^m(g)$$

$\mathfrak{g}$:
$$\kappa.\varepsilon_s^m(g) = \frac{d}{dt}\Big|_{t=0} \exp(t\kappa).\varepsilon_s^m(g)$$
$$= \frac{d}{dt}\Big|_{t=0} \varepsilon^m(\exp(t\kappa)).\varepsilon_s^m(g)$$
$$= \frac{d}{dt}\Big|_{t=0} (\cos t + i \sin t)^m\, \varepsilon_s^m(g)$$
$$= mi\, \varepsilon_s^m(g)$$

$$(\kappa . x_+ . \varepsilon_s^m)(g) = (2i+mi)(x_+.\varepsilon_s^m(g)) = (m+2)i\,(x_+.\varepsilon_s^m(g))$$

$\Rightarrow (x_+.\varepsilon_s^m)(g) = C\, \varepsilon_s^{m+2}(g)$ $\qquad C$ to be determined

$\Rightarrow (x_+.\varepsilon_s^m)(1) = C$ $\qquad 2X_+ = h - i\kappa - 2iv_+$

$$C = (x_+.\varepsilon_s^m)(1) = \tfrac{1}{2}\left[(h.\varepsilon_s^m)(1) - i(\kappa.\varepsilon_s^m)(1) - 2i(v_+.\varepsilon_s^m)(1)\right]$$
$$= \tfrac{1}{2}\left[(s+1) - i(mi) - 2i \times 0\right]$$
$$= \tfrac{1}{2}(s+1+m)$$

$\therefore\ X_+.\varepsilon_s^m(g) = \frac{1}{2}(s+1+m)\,\varepsilon_s^{m+2}(g)$

Similarly, $X_-.\varepsilon_s^m(g) = \frac{1}{2}(s+1-m)\,\varepsilon_s^{m-2}(g)$.

$\Omega$: By direct calculation, we get $\quad \Omega = -\frac{k^2}{4} + \frac{X_- X_+}{2} + \frac{X_+ X_-}{2}$

$$\Omega . \varepsilon_s^m(g) = \Big[ -\tfrac{1}{4}(mi)^2 + \tfrac{1}{2}\cdot\tfrac{1}{2}(s+1+m)\tfrac{1}{2}(s+1-m-2) + \tfrac{1}{2}\cdot\tfrac{1}{2}(s+1-m)\tfrac{1}{2}(s+1+m-2) \Big] \varepsilon_s^m(g)$$
$$= \tfrac{1}{4}m^2 + \tfrac{1}{4}\left[(s+1)(s+1-2) - m^2\right] \varepsilon_s^m(g)$$
$$= \tfrac{1}{4}(s^2-1)\,\varepsilon_s^m(g)$$

As a conclusion:

$K$: $\quad k . \varepsilon_s^m(g) = \varepsilon^m(k') \; \varepsilon_s^m(g)$

$\Omega$: $\quad \Omega . \varepsilon_s^m(g) = \tfrac{1}{4}(s^2-1) \; \varepsilon_s^m(g)$

$\mathfrak{g}$: $\quad k . \varepsilon_s^m(g) = mi \; \varepsilon_s^m(g)$

$\quad X_+ . \varepsilon_s^m(g) = \tfrac{1}{2}(s+1+m) \; \varepsilon_s^{m+2}(g)$

$\quad X_- . \varepsilon_s^m(g) = \tfrac{1}{2}(s+1-m) \; \varepsilon_s^{m-2}(g)$

Properties: irr & unitary

(Ir)reducibility of $\mathrm{Ind}(\chi_s \, \mathrm{sgn}^n) = \langle \varepsilon_s^m \rangle_{\mathbb{C}\text{-basis}}$ $\quad m \equiv n \pmod 2$

Normally: irreducible $\quad s \notin \mathbb{Z}$ or $s \equiv n \pmod 2$

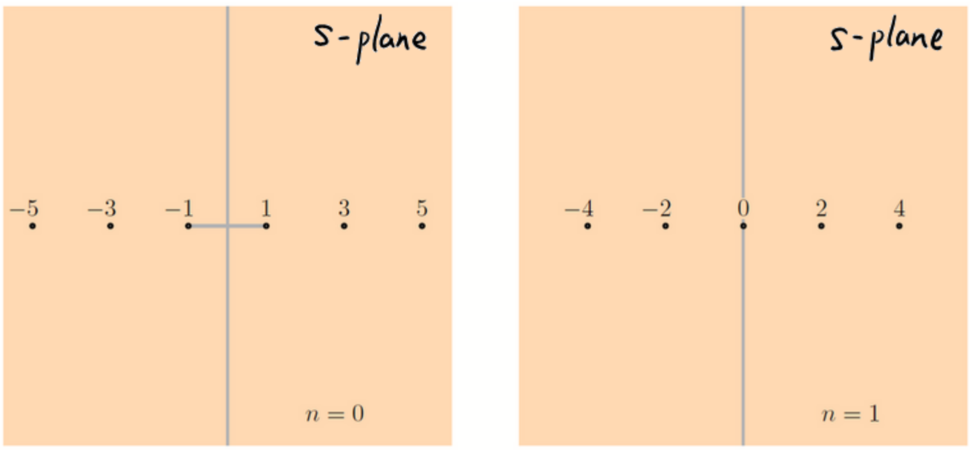
$$\mathrm{Ind}(\chi_s \, \mathrm{sgn}^n) \cong PS_{\frac{1}{4}(s^2-1), n}$$

Special situation: reducible $\quad s \in \mathbb{Z}$, $s \equiv n+1 \pmod 2$

| | submodule | quotient module |
|---|---|---|
| i) $s > 0$ | $DS_{s+1}^+ \oplus DS_{s+1}^-$ | $FD_{s-1}$ |
| ii) $s = 0$ | $DS_1^+ \oplus DS_1^-$ | $0$ |
| iii) $s < 0$ | $FD_{-s-1}$ | $DS_{-s+1}^+ \oplus DS_{-s+1}^-$ |

Unitary: The irr rep $PS_{\frac{1}{4}(s^2-1), n}$ is unitary iff

$$\begin{cases} n=0: & \tfrac{1}{4}(s^2-1) \in \mathbb{R},\ \tfrac{1}{4}(s^2-1) < 0 & \Leftrightarrow\ s^2 \in \mathbb{R},\ s^2 < 1 \\ n=1: & \tfrac{1}{4}(s^2-1) \in \mathbb{R},\ \tfrac{1}{4}(s^2-1) < -\tfrac{1}{4} & \Leftrightarrow\ s^2 \in \mathbb{R},\ s^2 < 0 \end{cases}$$




• reducible

— unitary irr

*Points of unitarity and reducibility of the principal series*

The duality between $Ind(\chi)$ & $Ind(\chi^{-1})$ can explain the situations where $\operatorname{Re} s = 0$

$\operatorname{Re} s = 0 \Rightarrow \chi = \chi_s \operatorname{sgn}^n$ is unitary, $\bar{\chi} = \chi^{-1}$

induce an Hermite inner product of $Ind(\chi)$:

$$\langle f, \phi \rangle := \int_{P \backslash G} f(x) \bar{\phi}(x) \, dx \qquad \bar{\phi} \in Ind(\bar{\chi}) = Ind(\chi^{-1})$$

## Discrete Series

$$SL_2(\mathbb{R}) \circlearrowright \mathbb{RP}^1$$

$$SL_2(\mathbb{R}) \circlearrowright \mathbb{CP}^1 = \mathbb{RP}^1 \sqcup \mathcal{H}^+ \sqcup \mathcal{H}^-$$

$$\rightsquigarrow SL_2(\mathbb{R}) \circlearrowright \Gamma(\mathcal{O}(m)) = \{\text{homogeneous poly of degree } m\}$$

$m \geq 0$: $\dim \Gamma(\mathcal{O}(m)) = m+1$

$v_{m-2k} := (z+iw)^k (z-iw)^{m-k}$ ; $\quad k \cdot v_{m-2k} = \varepsilon^{m-2k}(k) v_{m-2k}$

$k.(z \pm iw) = \varepsilon^{\mp 1}(k)(z \pm iw)$

$$[v_{-m}, \cdots\cdots, v_m]_{FD_m}$$

$m<0$: $\Gamma(\mathcal{O}(m)) = 0$

$$\Gamma(\mathcal{O}(m)|_{\mathcal{H}^+}) \supseteq \left\langle v_{2p-m} := \frac{(z-iw)^p}{(z+iw)^{p-m}} \,\middle|\, p \geq 0 \right\rangle$$

$$[v_{-m}, \cdots \cdots +\infty)_{DS^+_{|m|}}$$

$$\Gamma(\mathcal{O}(m)|_{\mathcal{H}^-}) \supseteq \left\langle v_{m-2p} := \frac{(z+iw)^p}{(z-iw)^{p-m}} \,\middle|\, p \geq 0 \right\rangle$$

$$(-\infty \cdots\cdots v_m]_{DS^-_{|m|}}$$

# Modular forms

Def. A modular form of weight $k \in \mathbb{Z}$ for the group $SL_2(\mathbb{Z})$ is a holomorphic fct on $\mathcal{H}$, satisfying the following conditions:

1) $f(\gamma\tau) = (c\tau+d)^k f(\tau)$

for $\forall\ \tau \in \mathcal{H}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

$j(\gamma,\tau) := c\tau + d$

$f(\gamma\tau) = j(\gamma,\tau)^k f(\tau)$

$j(\gamma_1\gamma_2,\tau) = j(\gamma_1,\gamma_2\tau)\,j(\gamma_2,\tau)$

$\chi(p): P \to \mathbb{C}^\times$

$f(pg) = \chi(p) f(g)$

$\chi(p_1p_2) = \chi(p_1)\chi(p_2)$

2) $f$ is bounded on $\{z \in \mathcal{H} \mid \operatorname{Im} z > 1\}$

Remark. We consider modular form as

- fct on $\mathcal{H}$
- sections in the v.b. of $SL_2(\mathbb{Z})\backslash\mathcal{H}$
- a highest weight vector of $C^\infty(SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})) \circlearrowleft G$

$$\{\text{modular forms of weight } m\} \longrightarrow C^\infty(SL_2(\mathbb{R}))$$
$$f \longmapsto [\Phi_f : g \longmapsto f(g(i))\,j(g,i)^{-m}]$$

[In general, we define
$$C^\infty(\mathcal{H}) \longrightarrow C^\infty(SL_2(\mathbb{R}))$$
$$f \longmapsto \Phi_f : g \longmapsto f(g(i))\,j(g,i)^{-m}]$$

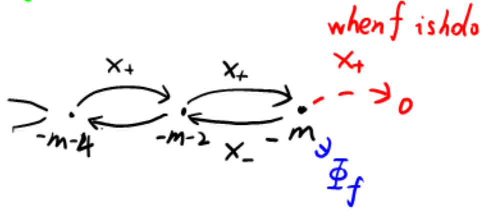
then for $\forall\ f \in C^\infty(\mathcal{H})$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$,

$K$: $\quad k.\Phi_f(g) = \varepsilon^{-m}(k)\,\Phi_f(g)$

$\mathfrak{g}$: $\quad K.\Phi_f(g) = -mi\,\Phi_f(g)$

$$x_+.\Phi_f(g) = -\frac{2i}{(-ci+d)^2}\,\Phi_{\frac{\partial f}{\partial \bar z}}(g) \overset{f \text{ is holo}}{=\!=\!=} 0$$

$$x_-.\Phi_f(g) = \frac{2i}{(ci+d)^2}\,\Phi_{\frac{\partial f}{\partial z}}(g) - m\,\frac{ci-d}{ci+d}\,\Phi_f(g)$$



Proof. Recall the definition

$$\Phi_f(g) = f(g(i))\, j(g,i) \qquad \Phi_f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = f\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-m}$$

We have

$$\begin{aligned} k.\,\Phi_f(g) &= \Phi_f(gk) \\ &= f(gk(i))\, j(gk,i)^{-m} \\ &= f(g(i))\, j(g,k(i))^{-m} j(k,i)^{-m} \\ &= \Phi_f(g)\cdot(i\sin t + \cos t)^{-m} \\ &= \varepsilon^{-m}(k)\,\Phi_f(g) \end{aligned}$$

$$k.\,\Phi_f(g) = -mi\,\Phi_f(g)$$

$$\begin{aligned} X_+.\,\Phi\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \frac{d}{dt}\Big|_{t=0}\Phi\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}\exp(tX_+)\right) \\ &= \frac{d}{dt}\Big|_{t=0}\Phi\begin{pmatrix} a+\frac{1}{2}at-\frac{i}{2}bt & b-\frac{1}{2}bt-\frac{i}{2}at \\ c+\frac{1}{2}ct-\frac{i}{2}dt & d-\frac{1}{2}dt-\frac{i}{2}ct\end{pmatrix} \\ &= \frac{1}{2}\left[(a-bi)\Phi_1-(-b-ai)\Phi_2+(c-di)\Phi_3+(-d-ci)\Phi_4\right] \\ &= \frac{1}{2i}\left[(ai+b)(\Phi_1-i\Phi_2)+(ci+d)(\Phi_3-i\Phi_4)\right] \end{aligned}$$

$$\begin{aligned} \Phi_{f,1} := \frac{\partial}{\partial a}\Phi_f &= \frac{\partial}{\partial a}\left[f\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-m}\right] \\ &= (ci+d)^{-m}\frac{\partial f}{\partial z}\cdot\frac{i}{ci+d} + (ci+d)^{-m}\frac{\partial f}{\partial \bar z}\,\frac{-i}{-ci+d} \\ &= \frac{i}{ci+d}\Phi_{f_z} - \frac{i}{-ci+d}\Phi_{f_{\bar z}} \end{aligned} \qquad f_z := \frac{\partial f}{\partial z} \quad f_{\bar z} := \frac{\partial f}{\partial \bar z}$$

Similarly, $\Phi_{f,2} = \frac{1}{ci+d}\Phi_{f_z} + \frac{1}{-ci+d}\Phi_{f_{\bar z}}$

$$\Phi_{f,3} = -i\frac{ai+b}{(ci+d)^2}\Phi_{f_z} + i\frac{-ai+b}{(-ci+d)^2}\Phi_{f_{\bar z}} - \frac{mi}{ci+d}\Phi_f$$

$$\Phi_{f,4} = -\frac{ai+b}{(ci+d)^2}\Phi_{f_z} - \frac{-ai+b}{(-ci+d)^2}\Phi_{f_{\bar z}} - \frac{m}{ci+d}\Phi_f$$

then
$$\begin{aligned} X_+.\,\Phi_f\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \frac{1}{2i}\left[(ai+b)(\Phi_{f,1}-i\Phi_{f,2})+(ci+d)(\Phi_{f,3}-i\Phi_{f,4})\right] \\ &= -\frac{2i}{(-ci+d)^2}\Phi_{f_{\bar z}} \end{aligned}$$

similarly,
$$\begin{aligned} X_-.\,\Phi_f\begin{pmatrix} a & b \\ c & d\end{pmatrix} &= \frac{1}{2i}\left[(ai-b)(\Phi_{f,1}+i\Phi_{f,2})+(ci-d)(\Phi_{f,3}+i\Phi_{f,4})\right] \\ &= \frac{2i}{(ci+d)^2}\Phi_{f_z} - m\frac{ci-d}{ci+d}\Phi_f \end{aligned}$$