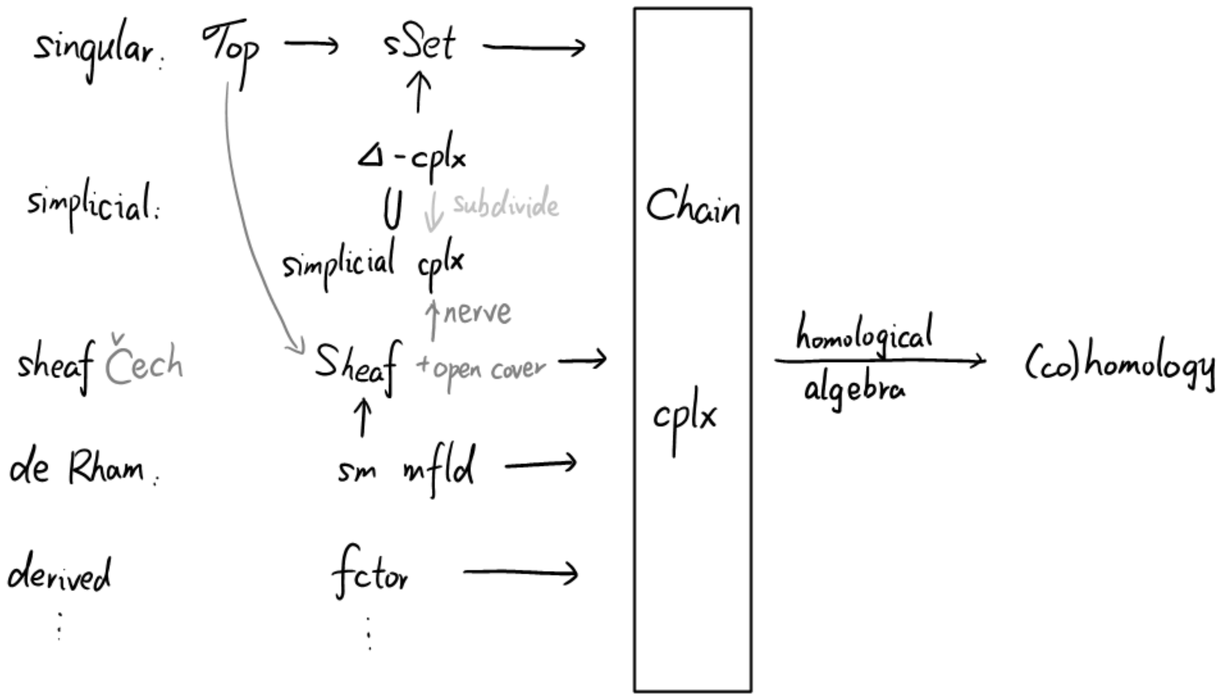
Eine Woche, ein Beispiel
6.25 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

We use α here because we are considering $X=\Delta^n$ case. May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha \in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n} \mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n} \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define
Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset.
If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \mathcal{T}op$ discrete, $X_\cdot = \mathcal{S}(A) \in sSet$, one can compute

$$
\begin{array}{ll}
C_\cdot(X;G): & 0 \leftarrow \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \cdots \\
C^\cdot(X;G): & 0 \rightarrow \prod_{\alpha\in A} G \xrightarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \cdots \\
\text{wished} \left\{ C^{BM}_\cdot(X;G): \right. & 0 \leftarrow \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \cdots \\
\phantom{\text{wished} \{} C^\cdot_c(X;G): & 0 \rightarrow \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \cdots
\end{array}
$$

Therefore,

$$
H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}
$$

$$
H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}
$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.

Notice that $\#\Delta'_k = k+2$, so

$$C_\cdot(\Delta';G):\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \cdots$$

basis:

remember indexes:

$d'_0 \triangleq x_0$: x_0: x_0: x_0:

$d'_1 \triangleq x_1$: x_1: x_1: x_1:

x_2: x_2: x_2:

x_3: x_3:

x_4:

$$\begin{aligned}
0 = x_0 - x_0 &\longleftarrow\!\!\shortmid\; x_0 \\
x_0 - x_1 = x_0 - x_1 &\longleftarrow\!\!\shortmid\; x_1 \\
0 = x_1 - x_1 &\longleftarrow\!\!\shortmid\; x_2
\end{aligned}
\qquad
\begin{aligned}
0 = x_0 - x_0 + x_0 - x_0 &\longleftarrow\!\!\shortmid\; x_0 \\
x_0 - x_1 = x_0 - x_1 + x_1 - x_1 &\longleftarrow\!\!\shortmid\; x_1 \\
0 = x_1 - x_1 + x_2 - x_2 &\longleftarrow\!\!\shortmid\; x_2 \\
x_2 - x_3 = x_2 - x_2 + x_2 - x_3 &\longleftarrow\!\!\shortmid\; x_3 \\
0 = x_3 - x_3 + x_3 - x_3 &\longleftarrow\!\!\shortmid\; x_4
\end{aligned}$$

$$\begin{aligned}
x_0 = x_0 - x_0 + x_0 &\longleftarrow\!\!\shortmid\; x_0 \\
x_0 = x_0 - x_1 + x_1 &\longleftarrow\!\!\shortmid\; x_1 \\
x_2 = x_1 - x_1 + x_2 &\longleftarrow\!\!\shortmid\; x_2 \\
x_2 = x_2 - x_2 + x_2 &\longleftarrow\!\!\shortmid\; x_3
\end{aligned}$$

By taking the transpose, one get

$$C^\cdot(\Delta';G):\quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$
\begin{tikzcd}
\Delta^1 \arrow[d, "s_0^1"] & C_\cdot(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\,1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1)"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1\,1)"] \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "Id"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
\Delta^0 \arrow[d, "d_0^1"] & C_\cdot(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "Id"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

s_0^1 d_0^1

s.t. $s^1_{0,*} \circ d^1_{0,*} = Id_{C_\cdot(\Delta^0;G)}$, $d^1_{0,*} \circ s^1_{0,*} \sim Id_{C_\cdot(\Delta^1;G)}$.

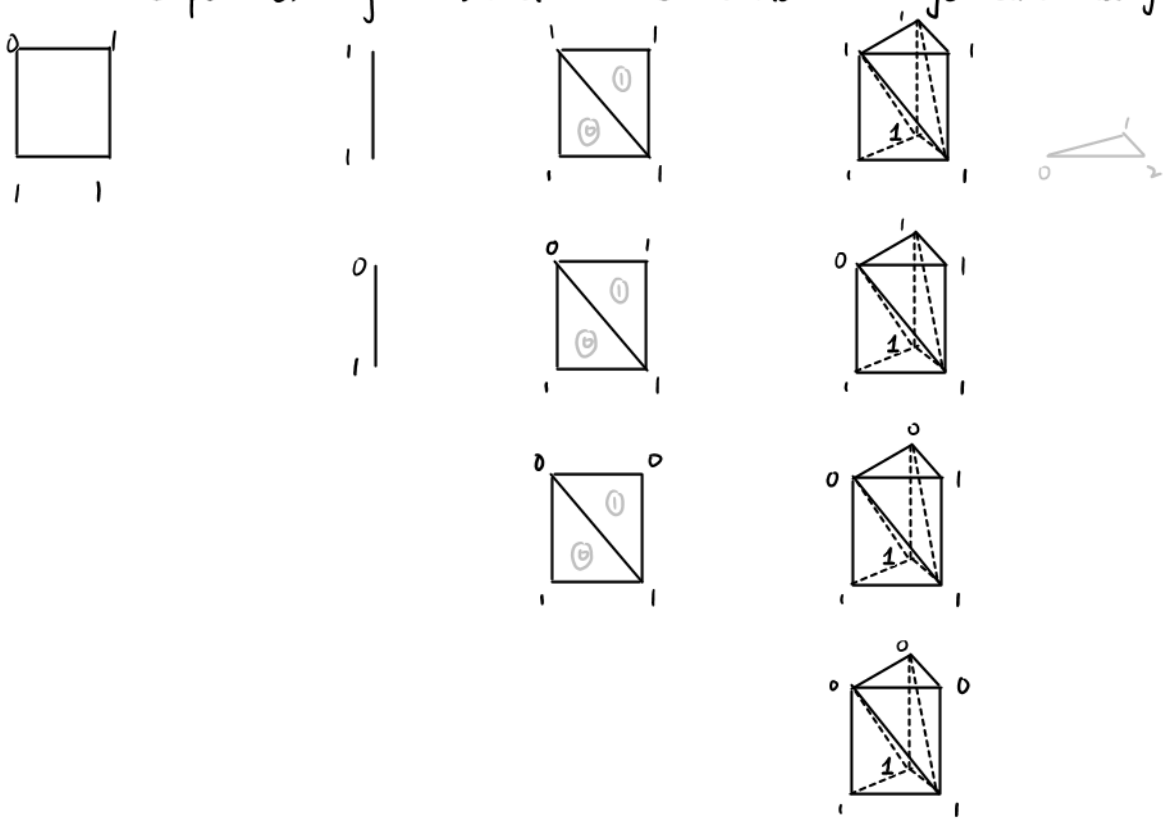
In fact, we have

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
C_\cdot(\Delta^1;G): \arrow[d, shift right, "Id"'] \arrow[d, shift left, "(d^1_0 \circ s^1_0)_*"] & 0 \arrow[dr, dashed, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}0&0&0\\0&0&1\\0&0&-1\\0&0&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"]
\end{tikzcd}
$$

$$
\begin{aligned}
x_0 &\mapsto x_0 \\
x_1 &\mapsto x_1
\end{aligned}
\qquad\qquad
\begin{aligned}
x_0 &\mapsto x_0 - x_0 + x_0 = x_0 \\
x_1 &\mapsto x_1 - x_1 + x_1 = x_1 \\
x_2 &\mapsto x_1 - x_2 + x_2 = x_1 \\
x_3 &\mapsto x_1 - x_2 + x_3
\end{aligned}
$$

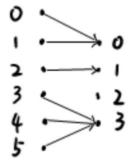
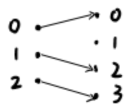
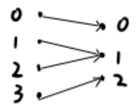
$$
\begin{aligned}
x_0 &\mapsto x_0 - x_0 = 0 \\
x_1 &\mapsto x_1 - x_1 = 0 \\
x_2 &\mapsto x_1 - x_2
\end{aligned}
$$

Ex. Observe the picture, try to translate the calculation in geometrical language.



E.g. 3. When we want to compute $H_n(\Delta^m;G)$ and $H^n(\Delta^m;G)$, we'd better to give elements in $\Delta_n^m \approx \{\text{basis of } C_n(\Delta^m;G)\}$ a better notation. The following table shows some typical element in

$$C_n(\Delta^m;G) = \langle \alpha : [n] \longrightarrow [m] \rangle_{\alpha \in \Delta_n^m}.$$

element	picture	list	count	degenerate degree
$\alpha : [5] \longrightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	$\Delta_5^{3,\leq 3}$
$d_1^3 : [2] \longrightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		$(0,2,3)$	$[1,0,1,1]$	$\Delta_2^{3,\emptyset}$
$s_1^3 : [3] \longrightarrow [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		$(0,1,1,2)$	$[1,2,1]$	$\Delta_3^{2,\leq 1}$
$\partial\alpha$	—	$(0,0,3,3,3)$ $-(0,0,1,3,3)$	$[2,0,0,3]$ $-[2,1,0,2]$	$\Delta_4^{3,\leq 3}$ $\Delta_4^{3,\leq 2}$

e.g. $\partial[2,5,3,4,1,6,0]$

$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$

2. connection with simplicial complexes.

Continuation of Eq. 2.

non-degenerate

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\Diamond}$:

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G)^{\Diamond}: & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_\cdot(\Delta';G)^{\Diamond}: \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}0\\1\\0\end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

In fact, we have

$$
\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)

For $X \in sSet$, define

$\diamondsuit$: non-degenerate
$\lightning$: degenerate

$$X_k^{\diamondsuit} := \{x \in X_k \mid x \text{ non-degerate}\} = X_k \setminus (sk^{k-1}X)_k$$
$$X_k^{\lightning} := \{x \in X_k \mid x \text{ degenerate}\} = (sk^{k-1}X)_k$$
$$X_k^{\lightning i} := \left\{x \in X_k \;\middle|\; x = \alpha^*(y) \text{ for some } \begin{array}{l} y \in X_{k-i}^{\diamondsuit} \\ \alpha: [k-i] \to [k] \end{array}\right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$0 = \underbrace{(sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}X)_k}_{X_k^{\lightning}} \underbrace{\overset{X_k^{\lightning 0}}{\subset} (sk^kX)}_{X_k^{\diamondsuit}} = X_k$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\diamondsuit} = \bigoplus_{\alpha \in X_n^{\diamondsuit}} G \qquad 0 \leftarrow \bigoplus_{\alpha \in X_0^{\diamondsuit}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\diamondsuit}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\diamondsuit}} G \cdots$$

$$C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \leftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots$$

and $H_\cdot(X;G)^{\diamondsuit}$, $H_\cdot(X;G)^{\lightning}$ as crspd homology.

By definition, $$C_\cdot(X;G) \cong C_\cdot(X;G)^{\diamondsuit} \oplus C_\cdot(X;G)^{\lightning}$$

Claim 1. $H_\cdot(X;G)^{\lightning} = 0$, so
$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\diamondsuit}. \qquad (*)$$

Rmk 1. Roughly, $(*)$ says that

singular homology $\approx$ simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expend $C_\cdot(X;G)$ by double complex.

Def (Double complex of $C_\cdot(X;G)$) $\swarrow\!\leftharpoonup + \leftarrow\!\swarrow = 0$

$$
\begin{tikzcd}
0 & \bigoplus\limits_{\alpha\in X_0^{S3}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_1^{S3}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_2^{S3}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_3^{S3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus\limits_{\alpha\in X_0^{S2}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_1^{S2}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_2^{S2}} G \arrow[l] \arrow[dl] & \bigoplus\limits_{\alpha\in X_3^{S2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus\limits_{\alpha\in X_0^{S1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus\limits_{\alpha\in X_1^{S1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus\limits_{\alpha\in X_2^{S1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus\limits_{\alpha\in X_3^{S1}} G \arrow[l] \arrow[dl, "0"'] \\
0 & \bigoplus\limits_{\alpha\in X_0^{\emptyset}} G \arrow[l] & \bigoplus\limits_{\alpha\in X_1^{\emptyset}} G \arrow[l] & \bigoplus\limits_{\alpha\in X_2^{\emptyset}} G \arrow[l] & \bigoplus\limits_{\alpha\in X_3^{\emptyset}} G \arrow[l]
\end{tikzcd}
$$

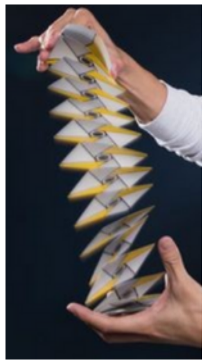
dim 0

dim 1

dim 2

$$\text{fold} \Downarrow \; \Uparrow \text{expand}$$

$$
\begin{tikzcd}
0 & \bigoplus\limits_{\alpha\in X_0} G \arrow[l] & \bigoplus\limits_{\alpha\in X_1} G \arrow[l] & \bigoplus\limits_{\alpha\in X_2} G \arrow[l] & \bigoplus\limits_{\alpha\in X_3} G \arrow[l]
\end{tikzcd}
$$



fold / expend

Eg. For $X = \Delta^1$, we have double complex

$$
\begin{tikzcd}
 & & & & & \cdots \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_4\rangle \arrow[l] \arrow[dl, "0"'] & \cdots \arrow[l, "\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] & \langle x_1, x_2, x_3\rangle \arrow[l, "\begin{pmatrix}1&0&0\\0&0&-1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}"'] & \\
0 & 0 \arrow[l] \arrow[dl] & \langle x_0, x_2\rangle \arrow[l] \arrow[dl, "0"'] & \langle x_1, x_2\rangle \arrow[l, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \arrow[dl, "0"'] & 0 \arrow[l] \arrow[dl] & \\
0 & \langle x_0, x_1\rangle \arrow[l] & \langle x_1\rangle \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l] & 0 \arrow[l] &
\end{tikzcd}
$$

$$\text{fold} \Downarrow \; \Uparrow \text{expand}$$

$$
0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots
$$

Claim 2. We have chain homotopy equivalence between the following two cplx:

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_n^{\phi}} G \arrow[l] \arrow[d, equal] & \bigoplus_{\alpha\in X_{n+1}^{s_1}} G \arrow[l, "0"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+2}^{s_2}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+3}^{s_3}} G \arrow[l, "\partial"'] \arrow[d, shift right, "0"'] & (**) \\
0 & \bigoplus_{\alpha\in X_n^{\phi}} G \arrow[l] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] &
\end{tikzcd}$$

i.e. (**) is exact on all terms except $\bigoplus_{\alpha\in X_n^{\phi}} G$.

Proof idea of Claim 2 for $X=\Delta^m$. (can be generalized to arbitrary X).

$$\begin{tikzcd}
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{s_{k-1}}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k}^{s_k}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k+1}^{s_{k+1}}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \cdots \arrow[l] \\
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{s_{k-1}}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k}^{s_k}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k+1}^{s_{k+1}}} G \arrow[l] & \cdots \arrow[l]
\arrow[from=1-1, to=2-2, "s"]
\end{tikzcd}$$

Define

$$s[\underbrace{a_1,\cdots a_l}_{\in\{0,1\}}, \underbrace{a_{l+1}}_{\notin\{0,1\}},\cdots,a_m] = \begin{cases} (-1)^{i}[a_1,\cdots,a_l,a_{l+1}+1,\cdots,a_m], & a_{k+1} \text{ even} \\ 0, & a_{k+1} \text{ odd}\end{cases} \qquad i=\sum_{j=1}^{l} a_j$$

Ex. Check that s is a homotopy.

e.g. $X=\Delta^3$, $n=2$, $k=3$ $\Rightarrow$ $m=3$, $n+k=5$

$-[2,1,0,2] \longleftarrow [2,1,0,3]$

$\longrightarrow -[3,1,0,2]$

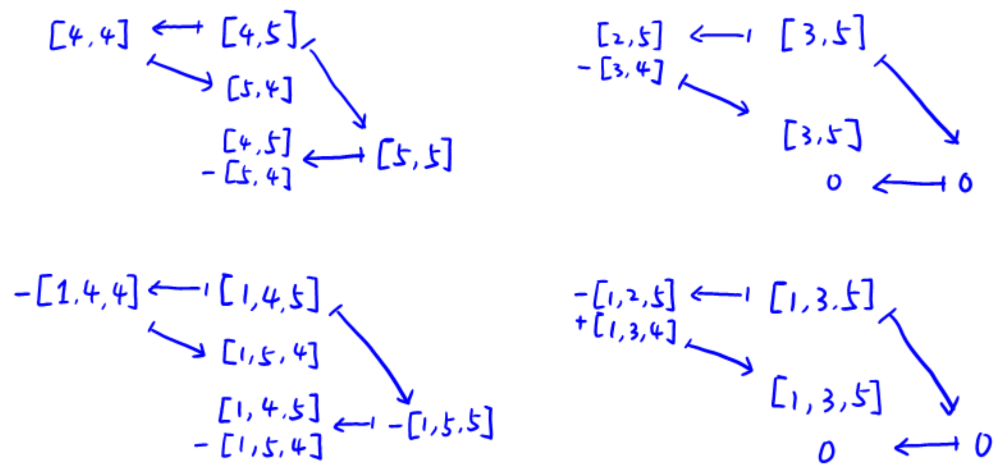
$[2,1,0,3] + [3,1,0,2] \longleftarrow [3,1,0,3]$

$X=\Delta^6$, $n=5$, $k=15$ $\Rightarrow$ $m=6$, $n+k=20$

$[2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] \longleftarrow [2,5,3,4,1,6,0]$

$\longrightarrow [3,4,3,4,1,6,0] - [3,5,2,4,1,6,0]$

$[2,5,3,4,1,6,0] - [3,4,3,4,1,6,0] + [3,5,2,4,1,6,0] \longleftarrow [3,5,3,4,1,6,0]$

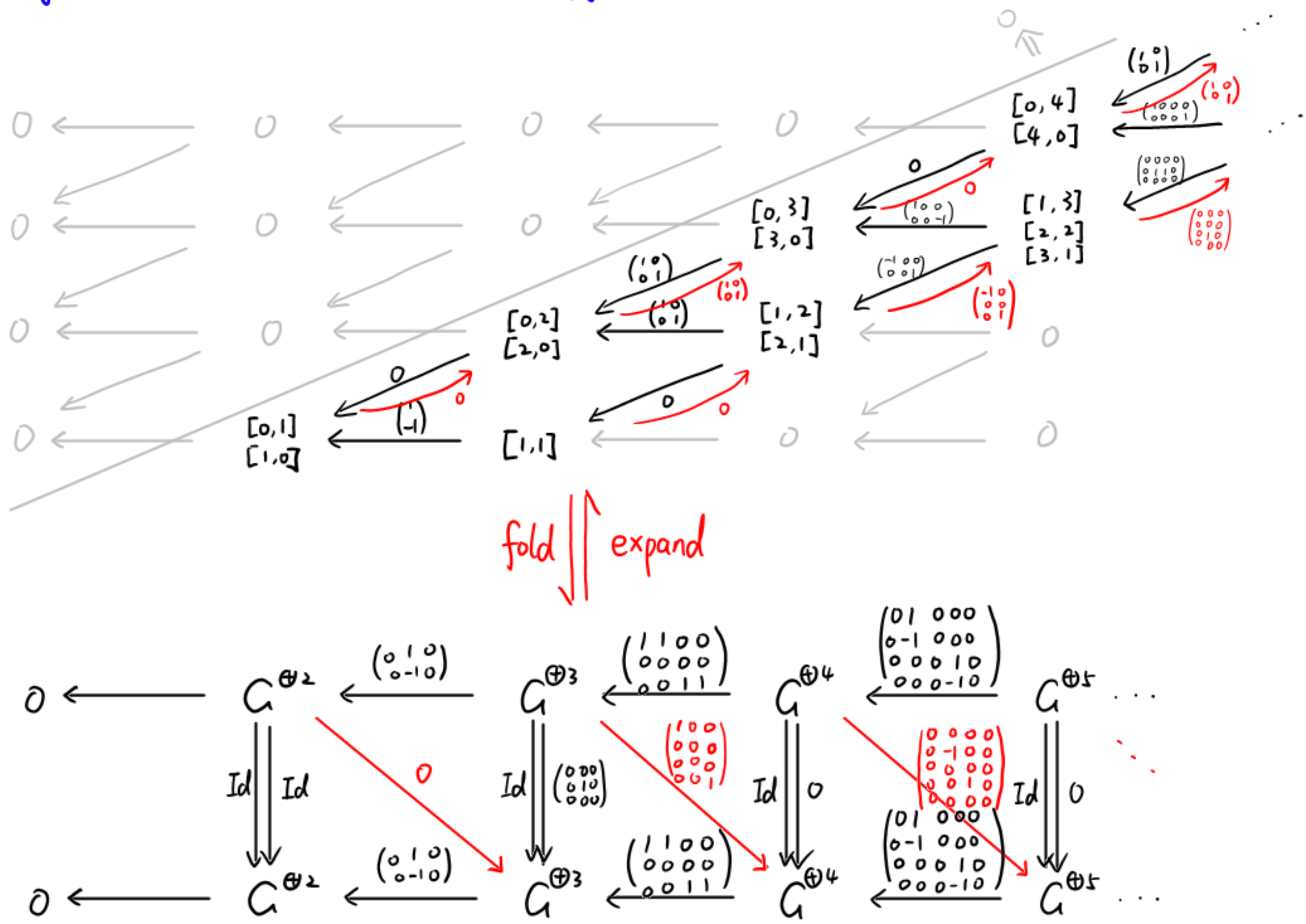


In conclusion,

$$\text{Claim 2} \Rightarrow \text{Claim 1} \Rightarrow \text{Rmk 1}$$

Coming back to Eg.2, one can now find a homotopy without guess.

Eg. For $X = \Delta^1$, we have homotopy



Ex. Check that (I believe that this argument also works for general sset X)

① $\overset{s}{\nearrow}\underset{\partial^{\leftarrow}}{\longleftarrow} + \overset{\partial^{\leftarrow}}{\longleftarrow}\overset{}{\nearrow}_{s} = 0$

② the collected s is a homotopy.

3. more structures

math.stackexchange.com/questions/2559705/cup-product-why-do-we-need-to-consider-cohomology-with-coefficients-in-a-ring

When $G = R$ is a K-alg, the product structure on $C^{\cdot}(X; R)$ is defined by