Un exemple par jour

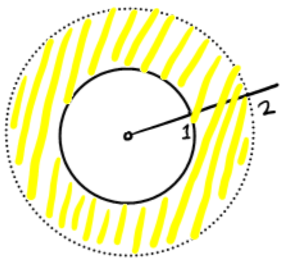
4.1. the complex torus of form $\mathbb{C}^\times/\mathbb{Z}\gamma$

$$\mathcal{C} := \mathbb{C}^\times/\mathbb{Z}\gamma \overset{\text{topo}}{=} \mathbb{T}^2 \text{ is a cpt Riemannian surface of genus } 1.$$

$\gamma \in \mathrm{Aut}(\mathbb{C}^\times)$ $\gamma(z)=az$ $a\in\mathbb{C}^\times$ $|a|>1$

Today: $a=2$

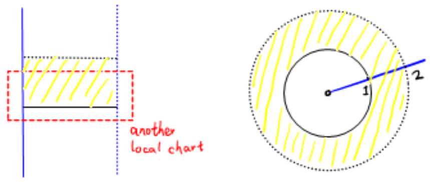
1. fundamental set:



$\Rightarrow$ only need 2 local chart

2.

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, hook] \arrow[d, "+\frac{1}{2\pi i}\ln 2"] & \mathbb{C} \arrow[r, "f:\, z\mapsto e^{2\pi i z}"] \arrow[d, "+\frac{1}{2\pi i}\ln 2"] & \mathbb{C}^\times \arrow[r] \arrow[d, "\times 2"] & 1 \\
0 \arrow[r] & \mathbb{Z}+\frac{1}{2\pi i}\ln 2 \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{C}^\times \arrow[r] & 1
\end{tikzcd}$$



$$\mathbb{C} \xrightarrow{f} \mathbb{C}^\times$$

$$\mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \Rightarrow \mathbb{C}^\times/\mathbb{Z}\gamma = \mathbb{C}/\left(\mathbb{Z}\oplus\tfrac{1}{2\pi i}\ln 2\,\mathbb{Z}\right)$$

better: $a = e^{2\pi} \approx 535.49$

$a = e^{-2\pi i w} \approx -230.765$

3. line bundle on $\mathcal{C}$

$b\in\mathbb{C}^\times$ $\quad \mathcal{L}_b := \mathbb{C}^\times\times\mathbb{C}\,/\,(z,\xi)\sim(2z,b\xi)$ $\Rightarrow$ ① $\mathcal{L}_b\in \mathrm{Pic}_0(\mathcal{C})$; $\left(\mathcal{L}_b \overset{\text{cont. deformation}}{\sim} \mathcal{L}_1 \cong \mathcal{O}_{\mathcal{C}}\right)$

$\downarrow$

$\mathcal{C} = \mathbb{C}^\times/z\sim 2z$

② $\mathrm{Pic}_0(\mathcal{C}) \cong \mathcal{C} = \mathbb{C}^\times/\mathbb{Z}\gamma$ (naive, base pt $1\in\mathbb{C}^\times/\mathbb{Z}\gamma$)

$\mathcal{L}_b \longmapsto b$

Reduced to: find a section s on $\mathcal{L}_b$ st $\mathrm{div}\, s = [b]-[1]$

Reduced to: find a meromorphic functions g on $\mathbb{C}^\times$ s.t

① $g(2z) = b\,g(z)$ $\quad b\in\mathbb{C}^\times$, $b\neq 2^k$; e.g. $b=3$

② g has simple poles on 2^n, and simple zeros on $2^n b$ $\quad n\in\mathbb{Z}$

$b = e^{2\pi i c}$, $c\in\mathbb{C}$

$\tau := \frac{1}{2\pi i}\ln 2$

$w(z) = \frac{1}{2\pi i}\ln z$

$$g(z) := \frac{\theta\begin{bmatrix}1\\1-2c\end{bmatrix}(w(z),\tau)}{\theta\begin{bmatrix}1\\1\end{bmatrix}(w(z),\tau)} \text{ is the required one.}$$

Blue — example
Orange — more than this example
Red — important results
Purple — I don't know the answer/proof
Green — sketsch of proof : in a minimal way
Grey — some supplementary explanation
Hell grey — explanation on well-known notations.