Eine Woche, ein Beispiel

3.2 lines on cubics

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.
[Kr16, cubic threefold]: Krämer, Thomas. Cubic Threefolds, Fano Surfaces and the Monodromy of the Gauss Map. Manuscripta Mathematica 149,

This document is basically [Huy23, II.2]. Mathematicians have magic playing with this geometrical objects.

Notation in [Huy23]:

affine	proj
V	$\mathbb{P}^{n+1}$
X^{cone}	X
$\mathcal{U}$	P_L
W	L

$X \subseteq \mathbb{P}^4$: cubic threefold

$F(X) \subseteq Gr(5,2)$: moduli space of lines in X

$F_2(X) \subseteq Gr(5,2)$: moduli space of lines of second type in X.

dim & smoothness

$\dim F(X) = 2$:

- incidence variety
- $F(X) = \mathrm{Hilb}_X^{P_{\mathbb{P}^1}(t)} \rightsquigarrow T_L F(X) \cong \mathrm{Hom}(\mathcal{I}_L, i_{L,*}\mathcal{O}_L)$

$$\begin{aligned} &\cong \mathrm{Hom}(i_L^* \mathcal{I}_L, \mathcal{O}_L) \\ &\cong \mathrm{Hom}(\mathcal{I}_L/\mathcal{I}_L^2, \mathcal{O}_L) \\ &\cong \mathrm{Hom}(\mathcal{O}_L, N_{L/X}) \\ &\cong H^0(L, N_{L/X}) \end{aligned}$$

https://math.stackexchange.com/questions/239959/conormal-sheaf-morphisms-of-schemes-stacks-project
https://math.stackexchange.com/questions/4899527/why-pullback-of-ideal-sheaf-should-be-the-conormal-sheaf

$\dim F_2(X) = 1$:

- incidence variety to get an upper bound

$$\begin{array}{ccc} F_2(X) \times X & & X \\ \cup & & \cup \end{array}$$

use Gauss map to show $\mathbb{L}_2 \longrightarrow q(\mathbb{L}_2)$ is generically finite

- determinance variety to get a lower bound
- work on moduli space of cubic threefolds.

Type of lines

X : sm cubic hypersurface, $L \subseteq X$ a line.
$N_{L/X}$ is a v.b. over $L = \mathbb{P}^1$.
By classification of v.b. over $\mathbb{P}^1$, one can distinguish type of L.

$L = \{z_2 = \cdots = z_n = 0\} = \{[* : * : 0 \cdots 0]\}$

$$0 \longrightarrow N_{L/X} \longrightarrow \underset{\substack{\| \\ \mathcal{O}_L(1) \otimes_{\mathbb{C}} V/W}}{N_{L/\mathbb{P}^{n+1}}} \longrightarrow \underset{\substack{\| \\ \mathcal{O}_L(3)}}{N_{X/\mathbb{P}^{n+1}}|_L} \longrightarrow 0$$

$$\rightsquigarrow \quad 0 \longrightarrow N_{L/X}(-1) \longrightarrow \mathcal{O}_L \otimes_{\mathbb{C}} V/W \xrightarrow{\left(\frac{\partial F}{\partial z_2}, \cdots, \frac{\partial F}{\partial z_{n+1}}\right)} \mathcal{O}_L(2) \longrightarrow 0$$

$$\rightsquigarrow \quad N_{L/X}(-1) = \underbrace{\mathcal{O}_L^{\oplus n-3} \oplus \mathcal{O}_L(-1)^{\oplus 2}}_{\text{first type}} \text{ or } \underbrace{\mathcal{O}_L^{\oplus n-2} \oplus \mathcal{O}_L(-2)}_{\text{second type}}, \text{ and}$$

first type: $\mathcal{O}_L(-1)^{\oplus 2}$ second type: $\mathcal{O}_L \oplus \mathcal{O}_L(-2)$

c.f. [Huy23, Ex. II.2.1, Rmk II.1.16]

first type
second type

$$\begin{array}{ccccccccc}
0 \longrightarrow & P_L^{\text{cone}} & \longrightarrow & V & & & & \\
& {\scriptstyle n-1,\ n} & & {\scriptstyle n+2,\ n+2} & & & & \\
& \downarrow & & \downarrow & & & & \\
0 \to & H^0(L, N_{L/X}(-1)) & \longrightarrow & V/W & \xrightarrow{\left(\frac{\partial F}{\partial z_2}, \cdots, \frac{\partial F}{\partial z_{n+1}}\right)} & S^2(W^*) & \longrightarrow & H^1(L, N_{L/X}(-1)) & \to 0 \\
& {\scriptstyle n-3,\ n-2} & & {\scriptstyle n,\ n} & & {\scriptstyle 3,\ 3} & & {\scriptstyle 0,\ 1} &
\end{array}$$

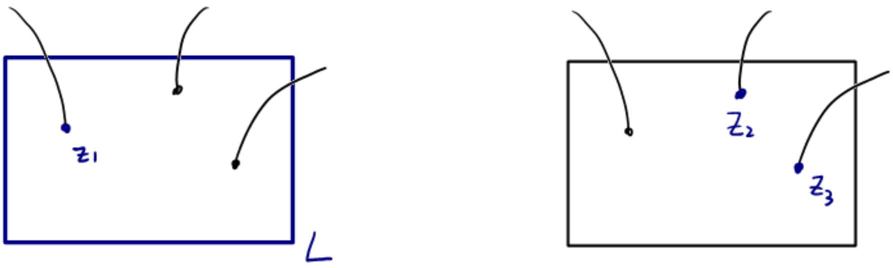
One can identify the type of L throughout

- $\dim \langle \partial_i F|_L \rangle$ where $\partial_i F|_L \in S^2(W^*)$ — Rmk II.2.2
- $P_L^{\text{cone}} = \bigcap_{y \in W} T_y X^{\text{cone}} \cong \mathbb{C}^{n-1}$ or $\mathbb{C}^n$ — Cor II.2.6
- Gauss map γ_X: — Ex II.2.10

$$\begin{array}{ccc}
\mathbb{P}^{n+1} & \xrightarrow[\text{2:1}]{\text{gen}} & (\mathbb{P}^{n+1})^* \\
\cup & & \cup \\
X & \xrightarrow[\text{1:1}]{\text{gen}} & X^* \\
\cup & & \cup \\
L & \longrightarrow & \gamma_X(L)
\end{array}$$

first type: 1:1
second type: 2:1

Motive



$$\begin{array}{ccc} (L, z_1) & \longleftrightarrow & (z_2, z_3) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ |\mathbb{L}_G|_X - \mathbb{L} & & X^{[2]} - \mathbb{L}^{[2]} \end{array}$$

Special case:

Copy from [Huy23, (4.7)]:

$$\begin{tikzcd}[column sep=tiny]
& \overline{\{(L, x, Z) \mid \{x\}, Z \subset L \cap X, x \notin Z\}} = Bl_{\mathbb{L}}(|\mathbb{L}_G|_X) \arrow[dl] \arrow[dr] & \\
|\mathbb{L}_G|_X = \{(L, x) \mid x \in L \cap X\} \arrow[dr] \arrow[d, "\mathbb{P}^n"'] & & \{Z \mid Z \subset X\} = X^{[2]} \arrow[dl] \\
X & \{L\} &
\end{tikzcd}$$

When $L \subset X$, the diagram reduces to:

$$\begin{tikzcd}[column sep=small]
 & \{(L,x,Z) \mid \{x\} \sqcup Z \subset L \subset X\} = E \arrow[dl, "\mathbb{P}^2"'] \arrow[dr, "\mathbb{P}^1"] & \\
\mathbb{L} = \{(L,x) \mid x \in L \subset X\} \arrow[dr, "\mathbb{P}^1"'] & & \{Z \mid \langle z \in Z \rangle \subset X\} = \mathbb{L}^{[2]} \arrow[dl, "\mathbb{P}^2"] \\
 & \{L \mid L \subset X\} = F(X) &
\end{tikzcd}$$

When $L \not\subset X$, the diagram reduces to:

$$\begin{tikzcd}
 & Bl_{\mathbb{L}}(\mathbb{L}_G|_X) - E \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
\mathbb{L}_G|_X - \mathbb{L} \arrow[dr, "3:1"'] & & X^{[2]} - \mathbb{L}^{[2]} \arrow[dl, "3:1"] \\
 & \{L \mid L \not\subset X\} &
\end{tikzcd}$$

ramified when $L \cap X$ is not reduced

Therefore, from

$$[\mathbb{L}_G|_X] - [\mathbb{L}] = [X^{[2]}] - [\mathbb{L}^{[2]}],$$

one gets

$$[\mathbb{P}^n][X] - [\mathbb{P}^1][F(X)] = [X^{[2]}] - [\mathbb{P}^2][F(X)],$$

i.e.,

$$[X^{[2]}] = [\mathbb{P}^n]\cdot[X] + l^2[F(X)]$$
$$[X^{(2)}] = (1+l^n)[X] + l^2[F(X)]$$

from

$$[X^{[2]}] - [\mathbb{P}^{n-1}][X] = [X^{(2)}] - [X],$$

one gets

$$[X^{[2]}] = [X^{(2)}] + (l+\cdots+l^{n-1})[X]$$

Assuming the cancellation,

$$\mathfrak{h}(\mathbb{L}_G|_X) - \mathfrak{h}(\mathbb{L})(-3) \cong \mathfrak{h}(X^{[2]}) - \mathfrak{h}(\mathbb{L}^{[2]})(-2)$$

(combined with the standard formula for proj bds) one gets

$$\mathfrak{h}(X)\oplus\cdots\mathfrak{h}(X)(-n) - \mathfrak{h}(F(X))(-3)\oplus\mathfrak{h}(F(X))(-4) \cong \mathfrak{h}(X^{[2]}) - \mathfrak{h}(F(X))(-2)\oplus\cdots\oplus\mathfrak{h}(F(X))(-4)$$

i.e.,

$$\mathfrak{h}(X^{[2]}) \cong \mathfrak{h}(X)\oplus\cdots\oplus\mathfrak{h}(X)(-n)\oplus\mathfrak{h}(F(X))(-2)$$
$$\mathfrak{h}(X^{(2)}) \cong \mathfrak{h}(X)\oplus\mathfrak{h}(X)(-n)\oplus\mathfrak{h}(F(X))(-2)$$

fiber of $X^{[2]}\longrightarrow X^{(2)}$ over X

from

$$\mathfrak{h}(X^{[2]}) - \mathfrak{h}(F)(-1) \cong \mathfrak{h}(X^{(2)}) - \mathfrak{h}(X)(-n)$$

i.e.,

$$\mathfrak{h}(X^{[2]}) - \mathfrak{h}(X)(-1)\oplus\cdots\oplus\mathfrak{h}(X)(-n) \cong S^2\mathfrak{h}(X) - \mathfrak{h}(X)(-n)$$

one gets

$$\mathfrak{h}(X^{[2]}) \cong S^2\mathfrak{h}(X)\oplus\mathfrak{h}(X)(-1)\oplus\cdots\oplus\mathfrak{h}(X)(-n+1)$$