

Eine Woche, ein Beispiel

12.21. Hodge structure

Ref:

[Car17]: James Carlson, Stefan Müller-Stach, and Chris Peters. Period Mappings and Period Domains. 2nd ed. Cambridge University Press, 2017. <https://doi.org/10.1017/9781316995846>.

X : cpt Kähler mfld of dim n .

Slogan: Hodge structure collects (nearly) all linear algebraic structures of X .

These linear algebraic structures are: $k = p+q$

$$\textcircled{1} \quad H^k(X; \mathbb{Z})$$

$$\textcircled{2} \quad H^k(X; \Omega_X^p) \cong \{\text{harmonic } (p,q)\text{-forms}\}$$

\textcircled{3} A comparison iso as C -v.s. (J -action)

$$H^k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \bigoplus_{p+q=k} H^q(X; \Omega_X^p)$$

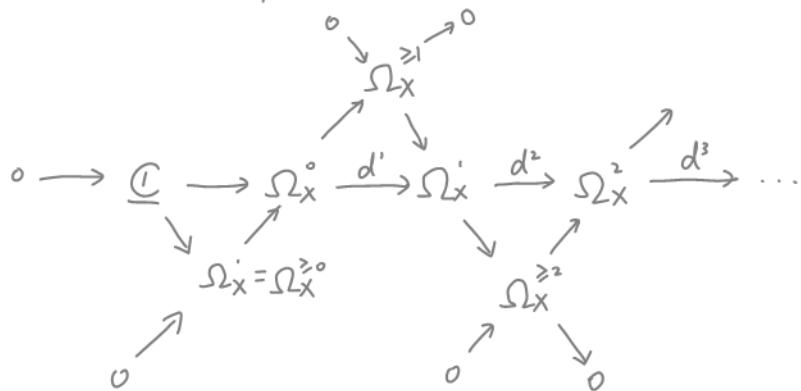
\textcircled{2}' A Hodge filtration

$$H^k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \cong H^k(\Omega_X^{\geq 0}) \supset H^{k-1}(\Omega_X^{\geq 1}) \supset H^{k-2}(\Omega_X^{\geq 2}) \supset \dots \supset H^0(\Omega_X^{\geq k}) \supset 0$$

\textcircled{3}' Compatibility with conj $_J$:

$$H^k(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} = H^{k-p}(\Omega_X^{\geq p}) \oplus \overline{H^{p-1}(\Omega_X^{\geq k-p+1})}$$

Here, $\Omega_X^{\geq p} := \bigcup_{q \geq p} \Omega_X^q$. We have



④ (polarization)
A bilinear form

$$\Delta: H^k(X; \mathbb{C}) \otimes H^k(X; \mathbb{C}) \xrightarrow{\Delta} H^{2k}(X; \mathbb{C}) \xrightarrow{\wedge \omega^{n-k}} H^{2n}(X; \mathbb{C}) \cong \mathbb{C}$$

$$\langle \xi, \eta \rangle := \int_X \xi \wedge \eta \wedge \omega^{n-k}$$

with two compatibility conditions:

$$\Delta: H^k(X; \mathbb{Z}) \otimes H^k(X; \mathbb{Z}) \xrightarrow[\wedge]{\Delta} H^{2k}(X; \mathbb{Z}) \xrightarrow{\wedge \omega^{n-k}} H^{2n}(X; \mathbb{Z}) \cong \mathbb{Z}$$

$$\Delta: H^k(X; \mathbb{C}) \otimes H^k(X; \mathbb{C}) \xrightarrow[\wedge]{\Delta} H^{2k}(X; \mathbb{C}) \xrightarrow{\wedge \omega^{n-k}} H^{2n}(X; \mathbb{C}) \cong \mathbb{C}$$

$$\Lambda: H^q(X; \Omega_X^p) \otimes H^p(X; \Omega_X^q) \xrightarrow{\wedge} H^k(X; \Omega_X^k) \xrightarrow{\wedge \omega^{n-k}} H^n(X, \omega_X) \cong \mathbb{C}$$

Assume ω is integral, i.e., X is proj

$$H^{2n}(X; \mathbb{Z}) \cong \mathbb{Z}$$

$$H^{2n}(X; \mathbb{C}) \cong \mathbb{C}$$

$$H^n(X, \omega_X) \cong \mathbb{C}$$

with two positivity conditions:

$$\begin{aligned} \Delta(\xi, \bar{\xi}) &\in \mathbb{R} & \text{for } \xi \in H^k(X; \mathbb{R}) & \quad \xi = \bar{\xi} \text{ here} \\ i^{p-q+k(k-1)} \Delta(\xi, \bar{\xi}) &\geq 0 & \text{for } \xi \in H_{\text{prim}}^k(X; \mathbb{C}) \cap H^{p,q} \end{aligned}$$

And get " $=$ " $\Leftrightarrow \xi = 0$.

These positivities are not contradiction to each other.

If $\xi \in H^k(X; \mathbb{R}) \cap (H_{\text{prim}}^k(X; \mathbb{C}) \cap H^{p,q})$, then $\xi \in \overline{H^{p,q}} = H^{q,p}$,
then $p=q$, i.e.

$$\xi \in H_{\text{prim}}^{2p}(X; \mathbb{R}) \cap H^{p,p}, \quad i^{p-q+k(k-1)} = (-1)^p.$$

However, when p is odd,

$$H_{\text{prim}}^{2p}(X; \mathbb{R}) \cap H^{p,p} = \{0\}.$$