Eine Woche, ein Beispiel

6.2. Roof structure for moduli of pairs

Setting $\quad \mathcal{C}$: category $\quad$ e.g. Top

Def A roof / correspondence in $\mathcal{C}$ is a diagram

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

$X, Y, Z \in \mathcal{C}$
$f \in \mathrm{Mor}_{\mathcal{C}}(Z, X)$, $g \in \mathrm{Mor}_{\mathcal{C}}(Z, Y)$

"Equivalently", this can be written as
when $X \times Y$ $\exists$

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, bend right, "f"'] \arrow[drrr, bend right, "g"] & & X \times Y \arrow[dl] \arrow[dr] & \\ & X & & Y \end{tikzcd}$$

Roofs are used in many different areas.

E.g. - construct derived category by "quotienting out quasi-isos"
- define $\mathrm{Corr}(C, E)$ in abstract 6-fctor formalism
- define Fourier-Mukai transformation

$$\Phi_{\mathcal{F}} = g_! \circ (\mathcal{F} \otimes -) \circ f^*$$

$$\begin{tikzcd} \mathcal{F} \arrow[dr, dash] & & \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

- ???

In most cases, roofs are used in understanding the moduli of pairs.

E.g. $X = \{x\text{'s}\}$
$Y = \{y\text{'s}\}$
$Z = \{(x,y) \in X \times Y \mid \phi(x,y) = \mathrm{True}\}$ $\qquad \phi : X \times Y \longrightarrow \{0, \underset{\mathrm{True}}{1}\}$

Then the figure

$$\begin{tikzcd} \{y \mid \phi(x_0, y)\} \arrow[dr, dash] & & \{x \mid \phi(x, y_0)\} \arrow[dl, dash] \\ & \{(x,y) \mid \phi(x,y)\} \arrow[dl] \arrow[dr] & \\ \{x\} & & \{y\} \end{tikzcd} \qquad \begin{tikzcd} f^{-1}(x_0) \arrow[dr, dash] & & g^{-1}(y_0) \arrow[dl, dash] \\ & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y \end{tikzcd}$$

presents many moduli spaces in a clear way.

E.p., one can describe Z by stratifications through f and g.