Eine Woche, ein Beispiel

12.1 weights of type E

There are already much information in wiki and other references about the exceptional Lie algebra. It is nice, but I always have to check the compatability among different references. In this document, I try to fix a standard coordinate, and state all the combinatorical results without proofs.

We will make a list of the following objects, for E_6, E_7 and E_8.

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2023. https://doi.org/10.1017/9781009280020.
[Hum92]: Humphreys, James E. Reflection groups and Coxeter groups. 29. Cambridge university press, 1992.

- Weights nearest to the origin
  - some graphs
  - weight lattice
- Simple roots
- Fundamental weights
- Weyl group action

Remark: There is another coordinate system which is written in wiki: del Pezzo surface. We don't use them. There, the different weight spaces are identified, while in our coordinate system, we identify the root lattices.

The order we present:
The order we compute:

$$\begin{tikzcd} ② \text{ simple roots} \arrow[r] \arrow[d] & ④ \text{ fundamental wts} \arrow[d] \\ ③ \text{ roots} & ① \text{ wt orbits} \end{tikzcd}$$

We present in this way, only because we want to express everything in terms of weight orbits.

1. E_6.

– Weights nearest to the origin

There are two minuscule representations of E 6. So we just fix one.

affine version

#	typical coordinates	symbol
6	$(1, 0, 0, 0, 0, 0, 1, 0)^T$	v_i
6	$(1, 0, 0, 0, 0, 0, 0, 1)^T$	$\tilde{v}_i$
15	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{0, 1, 2\}$$

could be $\tilde{v}_i$ or v_{ij} edge

weight lattice version

#	typical coordinates	symbol
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, 3, -3)^T$	v_i
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, -3, 3)^T$	$\tilde{v}_i$
15	$\frac{1}{3}(-2, -2, 1, 1, 1, 1, 0, 0)^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{\tfrac{4}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\}$$

could be $\tilde{v}_i$ or v_{ij} edge

in $\left\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \right\} \cong \mathbb{R}^6$

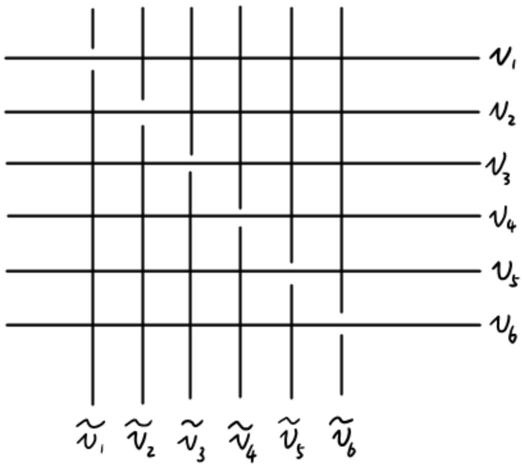
The graph constructed is called the Schläfli graph, which has 27 vertices and 216 edges (with HoG Id 1300).
This graph is also the configuration graph of 27 lines.

vertices $\cdot$ $\rightsquigarrow$ lines

empty edges $\cdot\ \cdot$ $\rightsquigarrow$ intersection points

empty triangle $\therefore$ $\rightsquigarrow$ triangle cut by H ← only in E_6

Here are some typical subgraphs:

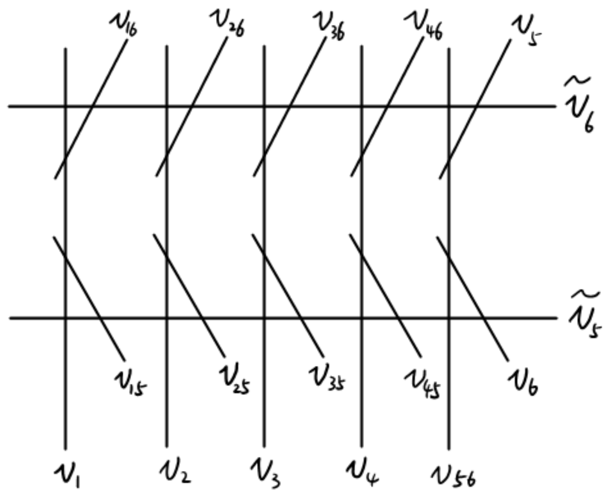


Schläfli double six configuration

$\# V = 12 \qquad \# E = 30$

HoG Id = 32794

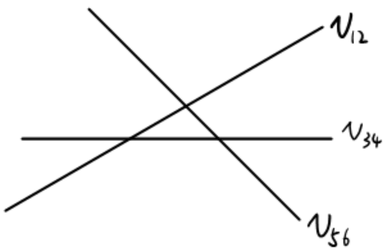
36 many, from [Huy24, Ex 3.6]



(v_{16} intersect with v_{25}, v_{35}, v_{45}, v_6. $\cdots$)

$\# V = 17 \qquad \# E = 30 + 20 = 50$

HoG Id = none.

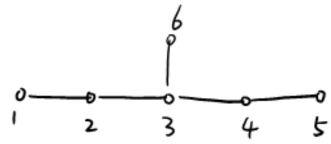


triangle (not for graph)

45 many

Q: For each type of subgraph, how many are they in the Schläfli graph?
I don't know if there are any simple answer for general subgraphs, and I don't know if there are any efficient algorithm for doing this. But this already produces many mysterious combinatorical numbers.

– Simple roots



$$\{ \alpha_1 , \quad \alpha_2 , \quad \alpha_3 , \quad \alpha_4 , \quad \alpha_5 , \quad \alpha_6 \}$$

$$= \{ v_1 - v_2 , \ v_2 - v_3 , \ v_3 - v_4 , \ v_4 - v_5 , \ v_5 - v_6 , \ v_4 - v_{56} \}$$

$$= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}$$

Ex. Verify that all the 72 roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}
$40 = \binom{6}{3} \cdot 2$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$	$\alpha_{456,7}$
2	$(0, 0, 0, 0, 0, 0, 1, -1)^T$	α_7

- Fundamental weights

denote by $A = (a_{ij})$ the Cartan matrix, then

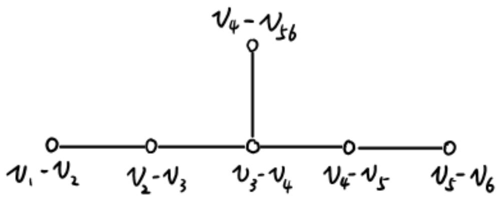
$$(\alpha_1, \dots, \alpha_r) = (\varpi_1, \dots, \varpi_r)\, A \qquad \langle \alpha_i, \alpha_j \rangle = A$$
$$(\varpi_1, \dots, \varpi_r) = (\alpha_1, \dots, \alpha_r)\, A^{-1} \qquad \langle \varpi_i, \varpi_j \rangle = A^{-1}$$

As a result,

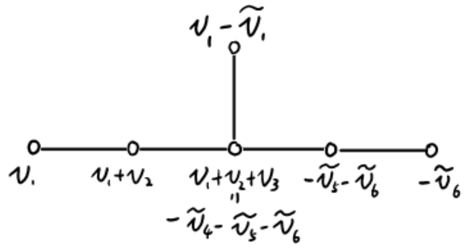
$$\{ \varpi_1, \ \varpi_2, \ \varpi_3, \ \varpi_4, \ \varpi_5, \ \varpi_6 \}$$
$$= \{ v_1, \ v_1 + v_2, \ v_1+v_2+v_3, \ -\tilde{v}_5 - \tilde{v}_6, \ -\tilde{v}_6, \ v_1 - \tilde{v}_1 \} \qquad -\tilde{v}_6 \neq \sum_{i=1}^{5} v_i$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ -2 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \\ -3 \\ -3 \\ 9 \\ -9 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ -4 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ -6 \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



α_i



ϖ_i

- Weyl group action

We know that

$$s_k \alpha_i = \alpha_i - \langle \alpha_k, \alpha_i \rangle \alpha_k$$
$$= \alpha_i - a_{ki}\alpha_k$$

$$\rightsquigarrow \quad s_k(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r) \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ -a_{k1} & \cdots & 1-a_{kk} & \cdots & -a_{kr} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$(\delta_{ij} - \delta_{ik}a_{ij})_{i,j}$$

In practice, we want to compute s_k-action on coordinates, it's easier to use the formula

$$s_k e_i = e_i - \langle \alpha_k, e_i \rangle \alpha_k$$

E.g. In E_6-case, when $k=1$, $\alpha = (1,-1,0,\dots,0)^T = e_1 - e_2$,

$$s_1 e_1 = e_1 - (e_1 - e_2) = e_2$$
$$s_1 e_2 = e_2 - (-1)(e_1 - e_2) = e_1$$

$$\rightsquigarrow \quad s_1 = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Similarly. $s_k = s_{(k,k+1)}$ for $i = 1, \dots, 5$.

When $k=6$, $\alpha_k = \frac{1}{2}(-1,-1,-1,1,1,1,1,-1)^T$,

$$s_6 e_1 = e_1 - (-\tfrac{1}{2})\alpha_6 = e_1 + \tfrac{1}{2}\alpha_6$$
$$= \tfrac{1}{4}(3,-1,-1,1,1,1,1,-1)^T$$
$$s_6 e_4 = e_4 - \tfrac{1}{2}\alpha_6$$
$$= \tfrac{1}{4}(1,1,1,-3,-1,-1,-1,1)^T$$

$$\rightsquigarrow \quad s_6 = \frac{1}{4}\left(\begin{array}{c|c|c} \begin{smallmatrix} 3 & & -1 \\ & 3 & \\ -1 & & 3 \end{smallmatrix} & 1 & -1 \\ \hline 1 & \begin{smallmatrix} 3 & & & -1 \\ & 3 & & \\ & & 3 & \\ -1 & & & 3 \end{smallmatrix} & 1 \\ \hline -1 & 1 & 3 \end{array}\right)$$

The action of $s_1, \dots, s_5$ on the Schläfli graph is easy. s_6 is hard.

E.g.

$$s_6 v_1 = s_6(e_1+e_7) = e_1+e_7 = v_1 \qquad s_6 v_2 = v_2 \qquad s_6 v_3 = v_3$$

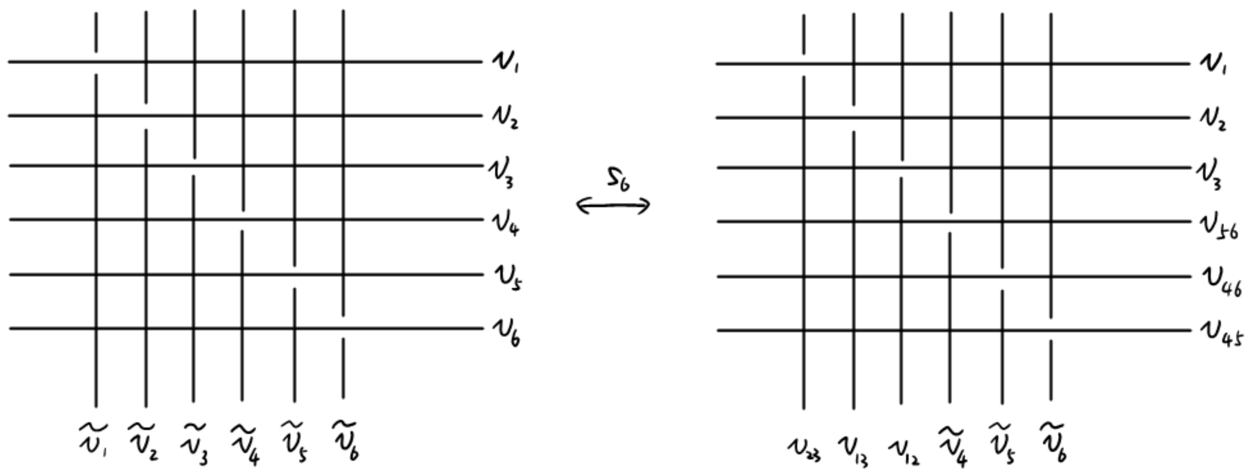
$$s_6 v_4 = s_6(e_4+e_7) = \tfrac{1}{2}(1,1,1,1,-1,-1,1,1)^T = v_{56} \qquad s_6 v_5 = v_{46} \qquad s_6 v_6 = v_{45}$$

$$s_6 \tilde{v}_1 = s_6(e_1+e_8) = \tfrac{1}{2}(1,-1,-1,1,1,1,1,1)^T = v_{23} \qquad s_6 \tilde{v}_2 = v_{13} \qquad s_6 \tilde{v}_3 = v_{12}$$

$$s_6 \tilde{v}_4 = s_6(e_4+e_8) = e_4+e_8 = \tilde{v}_4 \qquad s_6 \tilde{v}_5 = \tilde{v}_5 \qquad s_6 \tilde{v}_6 = \tilde{v}_6$$

The rest are easy to detemine through the Schläfli double six configuration.

e.g. $s_6 v_{14} = v_{14}$



2. E_7.

– Weights nearest to the origin

There is just one minuscule representations of E_7.

<u>integer version</u>

#	typical coordinates	symbol
28	$(3, 3, -1, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$(-3, -3, 1, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij}$ $= -v_{ij}$

$$\langle v_i, v_j \rangle \in \{24, 8, -8, -24\}$$

edge

<u>weight lattice version</u>

#	typical coordinates	symbol
28	$\frac{1}{4}(3, 3, -1, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$\frac{1}{4}(-3, -3, 1, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij}$ $= -v_{ij}$

$$\langle v_i, v_j \rangle \in \left\{\frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right\}$$

edge

in $\quad \left\{ \sum_{i=1}^{8} z_i = 0 \right\} \cong \mathbb{R}^7$

The graph constructed is called the Gosset graph, which has 56 vertices and 756 edges (with HoG Id 1114).
This graph is also the configuration graph of 56 (-1)-curves on P^2 blowing up 7 points.

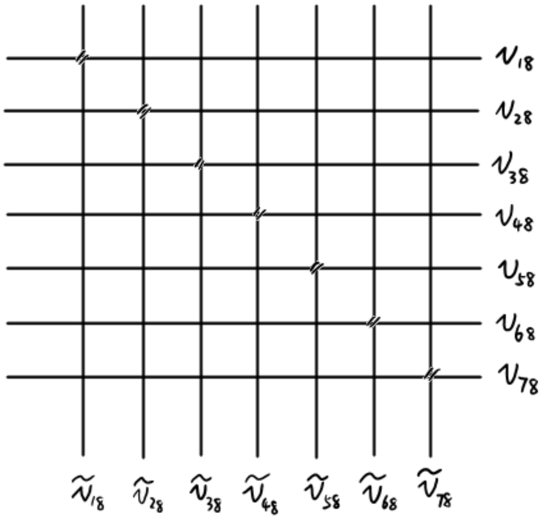
$56 = 7 + \binom{7}{2} + \binom{7}{5} + 7$

vertices $\rightsquigarrow$ lines

empty edges $\rightsquigarrow$ intersection points

empty triangle $\rightsquigarrow$ triangle

Here are some typical subgraphs:



$\{ v_{ij} \}_{i,j}$

"double seven configuration"

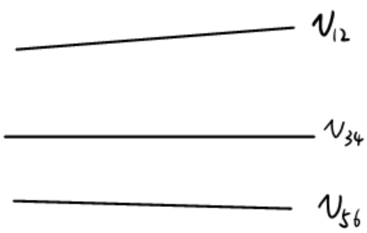
$\# V = 14 \qquad \# E = 42$

HoG Id = 50584

v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6, \cdots$

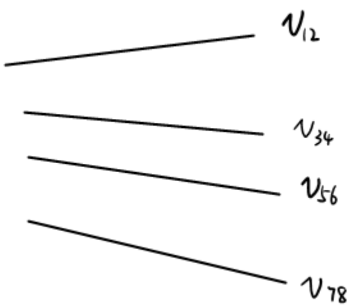
$\# V = 28 \qquad \# E = 210$

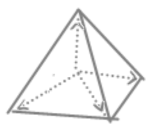
HoG Id = 50698.



triangle

4032 many





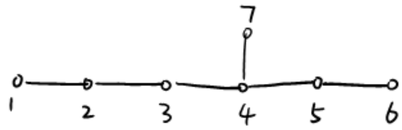
1008 many

in (-1)-curves setting,

$$\langle v_i, v_j \rangle \in \left\{ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\}$$

intersection number: $\quad -1 \quad 0 \quad 1 \quad 2$

– Simple roots



$$
\begin{aligned}
&\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\} \\
&= \{v_{18}-v_{28},\ v_{28}-v_{38},\ v_{38}-v_{48},\ v_{48}-v_{58},\ v_{58}-v_{68},\ v_{68}-v_{78},\ -v_{12}-v_{34}\} \\
&= \left\{ \begin{pmatrix} 1\\-1\\0\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\1\\-1\\0\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\1\\-1\\0\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\1\\-1\\0\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\1\\-1\\0\\0 \end{pmatrix}, \begin{pmatrix} 0\\0\\0\\0\\0\\1\\-1\\0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2} \end{pmatrix} \right\}
\end{aligned}
$$

Ex. Verify that all the 126 roots are given by

#	typical coordinates	symbol
$56 = 8 \cdot 7$	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	$\alpha_{1,-2}$
$70 = \binom{8}{4}$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	$\alpha_{5,6,7,8}$

- Fundamental weights

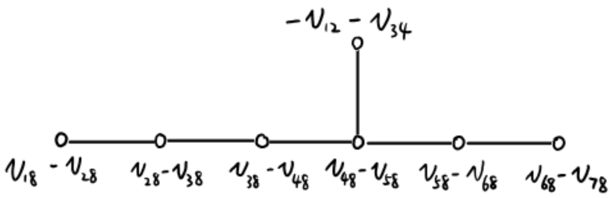
For convenient, denote

$$v_{j-k} = v_{ij} - v_{ik} = e_j - e_k \quad \text{for some } i = j,k.$$

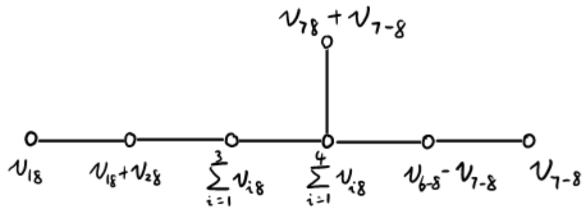
$$\{ \varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7 \}$$
$$= \{ v_{18},\ v_{18}+v_{28},\ v_{18}+v_{28}+v_{38},\ \sum_{i=0}^{4} v_{i8},\ v_{6-8}+v_{7-8},\ v_{7-8},\ v_{78}+v_{7-8} \}$$

$$= \left\{ \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{4}\begin{pmatrix}2\\2\\-2\\-2\\-2\\-2\\-2\\6\end{pmatrix}, \frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\-4\\-4\\-4\\12\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\-4\\-4\\8\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\0\\-4\\4\end{pmatrix}, \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\-1\\-1\\-1\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\-1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{pmatrix}, \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$



α_i



ϖ_i

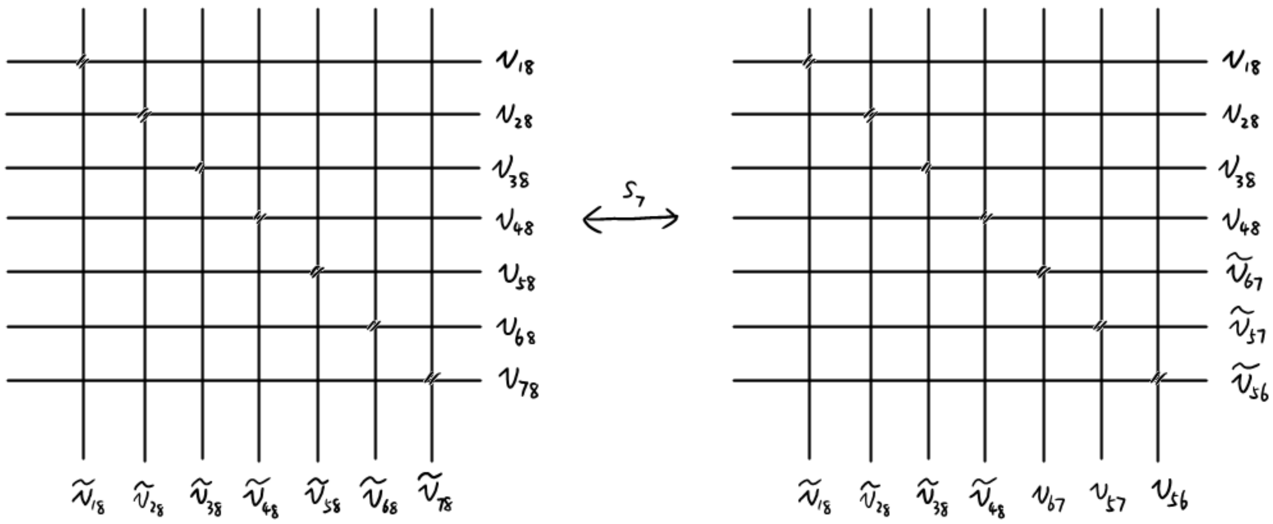
- Weyl group action

Using the similar methods like E_6, we get

$$s_k = s_{(k,\,k+1)} \qquad \text{for} \quad i = 1, \dots, 6$$

$$s_7 = \frac{1}{4}\left(\begin{array}{c|c} \begin{matrix} 3 & & & -1 \\ & 3 & & \\ & & 3 & \\ -1 & & & 3 \end{matrix} & 1 \\ \hline 1 & \begin{matrix} 3 & & & -1 \\ & 3 & & \\ & & 3 & \\ -1 & & & 3 \end{matrix} \end{array}\right)$$

$$s_7\, v_{ij} = \begin{cases} v_{ij} & \text{if} \quad i \in \{1,2,3,4\}\,,\ j = \{5,6,7,8\} \\ \tilde{v}_{kl} & \text{if} \quad \{i,j,k,l\} = \{1,2,3,4\} \text{ or } \{5,6,7,8\} \end{cases}$$



2. E_8.

– Weights nearest to the origin

There is no minuscule representations of E_8, the 240 weights are roots.

<u>weight lattice version</u> (allow negative roots)

#	typical coordinates	symbol	inverse
$56 = 2\cdot\binom{8}{2}$	$(1, 1, 0, 0, 0, 0, 0, 0)^T$	α_{12}	$\tilde{\alpha}_{12}$
$56 = 8\cdot 7$	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}	α_{2-1}
2	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	v_ϕ	$\tilde{v}_\phi$
$56 = 2\cdot\binom{8}{2}$	$(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	v_{12}	$\tilde{v}_{12}$
$70 = \binom{8}{4}$	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})^T$	v_{1234}	v_{5678}

$$\langle v_i, v_j \rangle \in \{2, 1, 0, -1, 2\} \quad \text{in } \mathbb{R}^8$$

edge

<u>shorter</u>:

#	typical coordinates		symbol
$112 = 4\cdot 28$	$(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)^T$		$\alpha_{\pm i \pm j}$
$128 = 2^7$	$(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})^T$	even sign	v_I

We call the constructed graph as the E_8-Gosset graph. It has 240 vertices and 126*240/2=15120 edges, with no HoG Id.

Q: $\mathrm{Aut}(\Gamma) = W(E_8)$?

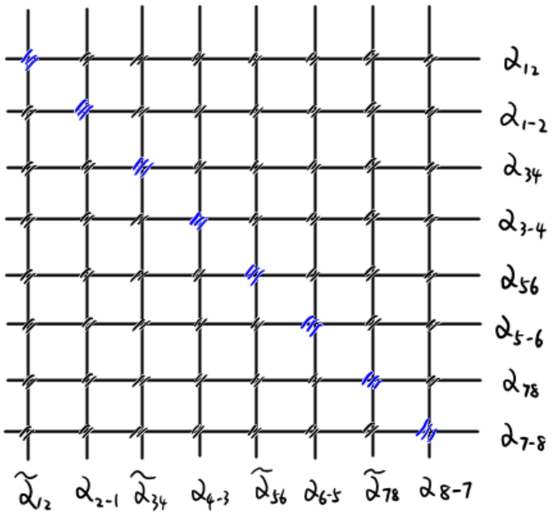
in (-1)-curves setting,

$$\langle v_i, v_j \rangle \in \{2,\ 1,\ 0,\ -1,\ -2\}$$

intersection number: $-1 \quad 0 \quad 1 \quad 2 \quad 3$

If we allow multiple edges, then I believe $\mathrm{Aut}(\Gamma_{mult}) = W(E_8)$.

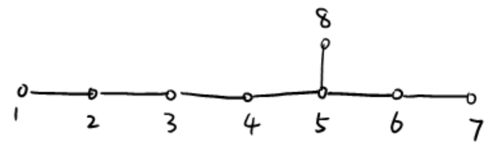
Here are some typical subgraphs:



"double eight configuration"

$\# V = 16 \qquad \# E = 0$

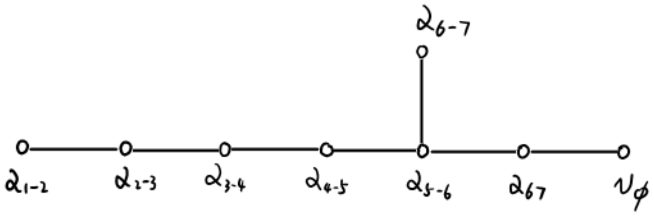
– Simple roots



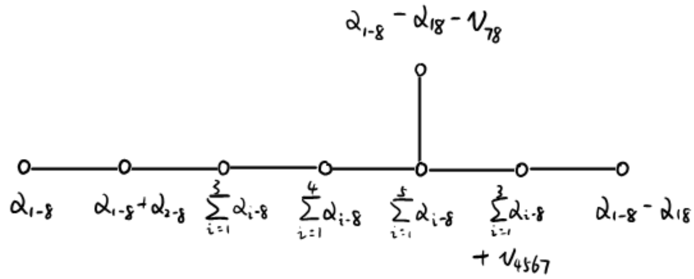
$$
\begin{aligned}
&\{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7,\ \alpha_8\ \}\\
=&\{\ \alpha_{1\text{-}2},\ \alpha_{2\text{-}3},\ \alpha_{3\text{-}4},\ \alpha_{4\text{-}5},\ \alpha_{5\text{-}6},\ \alpha_{67},\ \nu_\phi,\ \alpha_{6\text{-}7}\ \}\\
=&\left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\1\\-1\\0\\0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\1\\-1\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\\0\\1\\1\\0\end{pmatrix},\ \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\\0\\1\\-1\\0\end{pmatrix} \right\}
\end{aligned}
$$

– Fundamental weights

$$
\begin{aligned}
&\{\ \varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4,\ \varpi_5,\ \varpi_6,\ \varpi_7,\ \varpi_8\ \}\\
=&\{\ \alpha_{1\text{-}8},\ \alpha_{1\text{-}8}-\alpha_{2\text{-}8},\ \sum_{i=1}^{3}\alpha_{i\text{-}8},\ \sum_{i=1}^{4}\alpha_{i\text{-}8},\ \sum_{i=1}^{5}\alpha_{i\text{-}8},\ \alpha_{1\text{-}8}+\alpha_{2\text{-}8}+\alpha_{3\text{-}8}+\nu_{4567},\ \alpha_{1\text{-}8}-\alpha_{18},\ \alpha_{1\text{-}8}-\alpha_{18}-\nu_{78}\}\\
=&\left\{ \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\-1\end{pmatrix},\ \begin{pmatrix}1\\1\\0\\0\\0\\0\\0\\-2\end{pmatrix},\ \begin{pmatrix}1\\1\\1\\0\\0\\0\\0\\-3\end{pmatrix},\ \begin{pmatrix}1\\1\\1\\1\\0\\0\\0\\-4\end{pmatrix},\ \begin{pmatrix}1\\1\\1\\1\\1\\0\\0\\-5\end{pmatrix},\ \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{7}{2}\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\-2\end{pmatrix},\ \begin{pmatrix}\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{5}{2}\end{pmatrix} \right\}
\end{aligned}
$$



α_i



ϖ_i

- Weyl group action

Using the similar methods like E_6, we get

$$s_k = s_{(k,\,k+1)} \qquad \text{for } i = 1, \dots, 5$$

$$s_6 = \begin{pmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& & -1 & \\ &&&&& -1 & & \\ &&&&&&& 1 \end{pmatrix} \qquad s_7 = \frac{1}{4}\begin{pmatrix} 3 & & -1 \\ & \ddots & \\ -1 & & 3 \end{pmatrix} \qquad s_8 = s_{(6,7)}$$

Ex. Check the s_7-action on roots are given by

$$s_7(\alpha_{ij}) = v_{ij} \qquad\qquad s_7(v_\phi) = -v_\phi$$

$$s_7(\alpha_{i\text{-}j}) = \alpha_{i\text{-}j} \qquad\qquad s_7(v_{ij}) = \alpha_{ij}$$

$$s_7(v_{ijkl}) = v_{ijkl}$$

4. Comparison among different root systems.

Rmk. For the root lattice,

$$
\begin{aligned}
E_8 &= \left\{ z_i \in \mathbb{Z}^8 \cup (\mathbb{Z}+\tfrac{1}{2})^8 \;\middle|\; \sum_{i=1}^{8} z_i \equiv 0 \mod 2 \right\} \\
E_7 &= E_8 \cap \left\{ \sum_{i=1}^{8} z_i = 0 \right\} \\
E_6 &= E_8 \cap \left\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \right\}
\end{aligned}
$$

E_8 $\quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5$

$$
s_6 = \begin{pmatrix} 1 &&&&& \\ & \ddots &&&& \\ && 1 &&& \\ &&&& -1 & \\ &&& -1 && \\ &&&&& 1 \end{pmatrix}
\qquad
s_7 = \frac{1}{4}\begin{pmatrix} 3 && -1 \\ & \ddots & \\ -1 && 3 \end{pmatrix}
\qquad
s_8 = \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ &&& 1 & \\ && 1 && \\ &&&& 1 \end{pmatrix}
$$

E_7 $\quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5$

$$
s_6 = \begin{pmatrix} 1 &&&& \\ & \ddots &&& \\ &&& 1 & \\ && 1 && \\ &&&& 1 \end{pmatrix}
\qquad
s_7 = \frac{1}{4}\left(\begin{array}{c|c} \begin{matrix} 3 && -1 \\ & 3 \;\; 3 & \\ -1 && 3 \end{matrix} & 1 \\ \hline 1 & \begin{matrix} 3 && -1 \\ & 3 \;\; 3 & \\ -1 && 3 \end{matrix} \end{array}\right)
$$

E_6 $\quad s_1 \quad s_2 \quad s_3 \quad s_4 \quad s_5$

$$
s_6 = \frac{1}{4}\left(\begin{array}{c|c|c} \begin{matrix} 3 && \\ & 3 & \\ -1 && 3 \end{matrix} & 1 & -1 \\ \hline 1 & \begin{matrix} 3 &&& -1 \\ & 3 && \\ && 3 & \\ -1 &&& 3 \end{matrix} & 1 \\ \hline -1 & 1 & 3 \end{array}\right)
$$

The action of the Weyl group can also be represented as matrices with respect to the basis of either simple roots or fundamental weights, but I don't want to write it down.