Eine Woche, ein Beispiel

5.28. dual spaces of ∞-dim v.s.

Ref: http://staff.ustc.edu.cn/~wangzuoq/Courses/15F-FA/index.html

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. What would happen if $\mathbb{F} = \mathbb{C}_p$?

1. def
2. examples

1. def

Def. For any topo v.s. X, Y, define

$$\mathcal{L}(X,Y) := \{L : X \longrightarrow Y \mid L \text{ is linear and cont}\}$$

The dual space of X is defined as

$$X' := \mathcal{L}(X, \mathbb{F}) = \{L : X \longrightarrow \mathbb{F} \mid L \text{ is linear and cont}\}$$

! We follow the notation of analysis in this document. Other possibilities for the dual space: $X^*, X^\vee, \check{X}, \ldots$

Rmk. When X, Y are normed v.s., $\mathcal{L}(X,Y)$ is a normed v.s. with

$$\|L\| = \sup_{\|x\|_X = 1} \|L(x)\|_Y$$

On the other hand, we have the weak $*$-topology on $\mathcal{L}(X,Y)$: the weakest topo s.t.

$$ev_x : \mathcal{L}(X,Y) \longrightarrow Y \qquad L \longmapsto L(x)$$

is cont for any $x \in X$.

These two structures are not compatible with each other.

Rmk. By Klein-Milman theorem, we can show that some Banach spaces are not dual space.

2. examples.

For a bounded domain Ω, we have

$$L^\infty(\Omega) \subset \cdots \subset L^p(\Omega) \subset \cdots \subset L^1(\Omega)$$
$$\Downarrow \text{dual}$$
$$(L^\infty(\Omega))' \supset \cdots \supset L^q(\Omega) \supset \cdots \supset L^\infty(\Omega)$$

For arbitrary domain Ω, we don't have inclusion.

inclusion: cont inj map

https://math.stackexchange.com/questions/405357/when-exactly-is-the-dual-of-l1-isomorphic-to-l-infty-via-the-natural-map
https://math.stackexchange.com/questions/137677/what-is-the-predual-of-l1

Ex. Show that $(c_0)' = l^1$, $(l^p)' = l^q$, $(l^1)' = l^\infty$ by direct argument.
Show that $(l^\infty)' \supsetneq l^1$.

$$c_0 \overset{\text{not dense}}{\hookrightarrow} l^\infty \qquad l^p \qquad l^1$$
$$\Downarrow \text{dual}$$
$$l^1 \twoheadleftarrow (l^\infty)' \qquad l^q \qquad l^\infty$$

For $\Omega = \mathbb{R}^n$, we have ($\mathcal{S}(\Omega)$ is not defined for $\Omega \overset{\text{open}}{\subset} \mathbb{R}^n$, traditionally)

$$\mathcal{D}(\Omega) \subset \mathcal{S}(\Omega) \subset \mathcal{E}(\Omega)$$
$$\Downarrow \text{dual}$$
$$\mathcal{D}'(\Omega) \supset \mathcal{S}'(\Omega) \supset \mathcal{E}'(\Omega)$$

https://math.stackexchange.com/questions/4730104/is-schwartz-space-canonical-in-any-sense
Schwartz Functions on Open Subsets of Rn: https://www.math.princeton.edu/events/schwartz-functions-open-subsets-rn-2022-02-28t213000
Schwartz functions on real algebraic varieties: https://arxiv.org/abs/1701.07334

Rmk. For Hilbert space, $H' \cong H$. e.p. $(H^s(\Omega))' \cong H^s(\Omega)$

For X: cpt Hausdorff space,

$$C(X)' \subset \{\text{signed regular Borel measures}\}$$

⚠ The following illusion is common and confusing:

The dual space of bigger space is bigger/smaller.

Actually, such illusions comes from $f^*: W^* \longrightarrow V^*$ being injective/surjective.

In fin dim case, $\dim V^* = \dim V < \dim W = \dim W^*$;

In dense subspace case, it comes from the uniqueness of cont extension.