Modular form
3. discriminant

① meromorphic MF
③ almost holomorphic MF (殆全纯)
⑤ smooth automorphic form

⑥ Siegel MF
⑦ Hilbert MF

⓪ MF of congruence subgp (同余子群)
④ half-integral weight MF (半整权)

Def. A holo fct $f: \mathcal{H} \longrightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, level $\Gamma := SL_2(\mathbb{Z})$ if.

1) $f(\gamma\tau) = (c\tau+d)^k f(\tau)$ ⟶ ② quasimodular form

$\forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$

e.p. $f(\tau+1) = f(\tau)$

2) Write $f(\tau) = \sum_{n\in\mathbb{Z}} a_n (e^{2\pi i\tau})^n$, then $\underbrace{a_n = 0 \text{ for } n<0}$

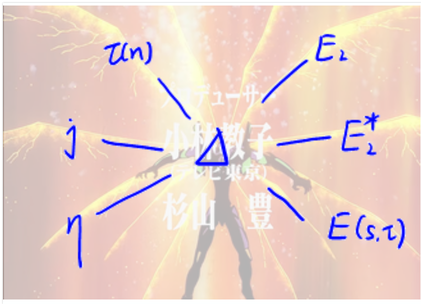
↳ ⑧ p-adic MF

↳ ⓪ non-entire MF

ⓘ The order I plan to talk about
(For me they become more and more difficult)

Today: Δ, $\tau(n)$, ① j, ② E_2, ③ E_2^*, ④ η

\Delta的性质不可避免地用到了模形式的各类推广，这是今天讨论班的两条主线。



1. Def of Δ
2. relation with EC
3. q-exponential & Ramanujan τ-fct
4. Δ & j-invariant
   - as mero MF
   - as modular invariant
   - q-exponential and moonshine
   - special values and cplx multiplication
5. product formula & E_2
   - automorphy condition for E_2
   - product formula
   - quasimodular form and almost holomorphic MF
6. Δ & Dedekind η-fct
7. Δ & $E(s,\tau)$ ← possiblely to be added here

conclusion
bonus

1. Def of Δ

Ex. find $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ s.t. $\Delta \in q + q^2\mathbb{Q}[[q]]$

A. $\Delta(\tau) = \frac{1}{1728}\left(E_4(\tau)^3 - E_6(\tau)^2\right)$

$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6$$
$$- 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + O(q^{11})$$

Cor. $S_*(SL_2(\mathbb{Z})) \cong \Delta\,\mathbb{C}[E_4, E_6] \triangleleft M_*(SL_2(\mathbb{Z}))$

2. relation with EC

Rmk. $\Delta(\tau) = \frac{1}{(2\pi)^{12}}\,\Delta\left(y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \frac{dx}{y}\right)$

which is closely related to the disc of EC.

slogan: any "fct" on moduli space reflects properties of EC.

⚠ The disc depends on "the choice of equations" (in reality, the choice of differential) and can change after the change of variables.

A possible explanation may be found here:
https://math.stackexchange.com/questions/3487698/discriminant-of-elliptic-curve-y2-ax3bx2cxd
(the reason is, \Delta is a function of lattices, and differential gives us a lattice.)

For results in char 2 & 3, see [ECII. Appendix A].

$$w = \frac{dx}{2y+a_1x+a_3} \qquad y^2+a_1xy+a_3y = x^3+a_2x^2+a_4x+a_6 \qquad \Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$$

$$b_2 = a_1^2+4a_2$$

$$b_4 = 2a_4+a_1a_3$$

$$b_6 = a_3^2+4a_6$$

$$b_8 = a_1^2a_6+4a_2a_6-a_1a_3a_4+a_2a_3^2-a_4^2$$

$$w = \frac{dx}{y} \qquad y^2 = 4x^3+b_2x^2+2b_4x+b_6 \qquad \Delta = -b_2^2b_8-8b_4^3-27b_6^2+9b_2b_4b_6$$

$$4b_8 = b_2b_6-b_4^2$$

$$y^2 = 4x^3-g_2x-g_3 \qquad \Delta = g_2^3-27g_3^2 \qquad \Delta_{wiki} = 16\Delta$$

$$y^2 = 4(x-e_1)(x-e_2)(x-e_3) \qquad \Delta = 16\prod_{i<j}(e_i-e_j)^2$$

$$y^2 = 4x(x-1)(x-\lambda) \qquad \Delta = 16\lambda^2(\lambda-1)^2$$

$$w = a^{\frac{1}{6}}\frac{dx}{y} \qquad y^2 = ax^3+bx^2+cx+d \qquad \Delta = 16(b^2c^2-4ac^3-4b^3d-27a^2d^2+18abcd)$$

$$\Delta_{wiki} = \frac{1}{16}\Delta$$

$$w = \frac{dx}{2y} \qquad y^2 = x^3+px+q \qquad \Delta = 16(-4p^3-27q^2)$$

$$y^2 = (x-e_1)(x-e_2)(x-e_3) \qquad \Delta = 16\prod_{i<j}(e_i-e_j)^2$$

$$y^2 = x(x-1)(x-\lambda) \qquad \Delta = 16\lambda^2(\lambda-1)^2$$

$$\Delta_{my\ moduli} = \Delta_{WWL} = \Delta_{silverman} = \Delta$$

3. q-exponential & Ramanujan τ-fct

$$\Delta(\tau) = \sum_{n=1}^{+\infty} \tau(n) q^n$$

Observation

0) $\tau(1) = 1 \quad \tau(n) \in \mathbb{Z}$

1) $\tau(nn') = \tau(n)\tau(n')$ for $\gcd(n,n') = 1$

$\tau(p^{e+1}) = \tau(p)\tau(p^e) - p^{11}\tau(p^{e-1})$ $\quad \forall\, p$ prime, $e \in \mathbb{Z}_{\geq 1}$

e.p. $\tau(p^2) = \tau(p)^2 - p^{11}$

2) $|\tau(p)| \leq 2p^{\frac{11}{2}}$ $\quad \forall\, p$ prime

3) (Lehmer conj) $\forall n \in \mathbb{N}_{\geq 1},\ \tau(n) \neq 0$

Rmk. 0). the following exercise or $\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{24}$ ← will be proved soon

1). by Mordell $\underset{\text{generalize}}{\rightsquigarrow}$ Hecke operators

2). by Deligne, use the Weil conj.

A much weaker version, $|\tau(p)| \leq Cp^6$ can be seen in [Za, Prop 8].

Ex. Show that

$$\tau(n) \in \mathbb{Z}$$

$$\tau(n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = m^2, m \text{ odd} \\ 0 \pmod 2 & \text{otherwise} \end{cases}$$

$$\tau(n) \equiv \sigma_{11}(n) \pmod{691}$$

Hint. ① Write $A = \sum \sigma_3(n) q^n$, $B = \sum \sigma_5(n) q^n$,

$$\Delta = 5\frac{A-B}{2} + B + 100A^2 - 147B^2 + 8000A^3$$

$\Rightarrow$ first two equalities

② Write $G_{12} - \Delta$ as a combination of E_4 & E_6, i.e.

$$G_{12} = \Delta + \frac{691}{156}\left(\frac{E_4^3}{720} + \frac{E_6^3}{1008}\right)$$

The $\tau(n)$ enjoy various congruences modulo 2^{12}, 3^6, 5^3, 7, 23, 691. We quote some special cases (without proof):

(55) $\quad \tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}$

(56) $\quad \tau(n) \equiv n\sigma_3(n) \pmod 7$

(57) $\quad \tau(n) \equiv \sigma_{11}(n) \pmod{691}$.

Ramanujan's congruences [WWL, 例4.4.8]

⇑

p-adic modular form

For other examples, and their interpretation in terms of "l-adic representations" see *Sém. Delange-Pisot-Poitou* 1967/68, exposé 14, *Sém. Bourbaki* 1968/69, exposé 355 and Swinnerton-Dyer's lecture at Antwerp (*Lecture Notes*, n° 350, Springer, 1973).