§2.2. Character of torus

Global case

Hecke charater

Notation
$$
\begin{aligned}
T &= \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{m,F} \\
T(\mathbb{Q}) &= F^\times \\
T(F) &= \mathbb{G}_m(F\otimes_{\mathbb{Q}} F) \cong \prod_{n \text{ many}} F^\times \\
T(\mathbb{R}) &= \prod_{\tau: F\hookrightarrow \mathbb{R}} \mathbb{R}^\times \prod_{\substack{(\tau, c\tau):\\ F\hookrightarrow \mathbb{C}}} \mathbb{C}^\times \qquad \hat{=}\ F_\infty^\times \\
T(\mathbb{Q}_p) &= \prod_{v\mid p} F_v^\times \qquad \hat{=}\ F_p^\times \\
T(\mathbb{A}_{\mathbb{Q}}) &= \mathbb{A}_F^\times
\end{aligned}
$$

For $F^{nc} := $ normal closure of F in $\overline{\mathbb{Q}}/\mathbb{Q}$, (normal closure)

$$T_{F^{nc}} = \prod_{\tau: F\hookrightarrow F^{nc}} \mathbb{G}_{m,F^{nc}} \qquad T(F^{nc}) = \prod_{\tau: F\hookrightarrow F^{nc}} F^{nc,\times}$$

$$X^*(T) = \mathrm{Hom}(T_{F^{nc}}, \mathbb{G}_{m,F^{nc}}) \cong \bigoplus_{\tau: F\hookrightarrow F^{nc}} \mathbb{Z}[\tau] \circlearrowleft \Gamma_F$$

$$\sigma.[\tau] = [\sigma\circ\tau]$$

We can rewrite

$$F^\times \backslash \mathbb{A}_F^\times / \overline{(F_\infty^\times)^\circ} \cong T(\mathbb{Q}) \backslash T(\mathbb{A}_{\mathbb{Q}}) / \overline{T(\mathbb{R})^\circ}$$

<u>Notation</u> $T = \mathrm{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, $\rho \in X^*(T)$

When $\rho: T \longrightarrow \mathbb{G}_m$ is defined over $\mathbb{Q}$,

$\rho_\infty: T(\mathbb{R}) \longrightarrow \mathbb{R}^\times$ $\qquad$ $\rho_p: T(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$;

When $\rho: T_{F'} \longrightarrow \mathbb{G}_{m,F'}$ is defined over F',

$\rho_\infty: T(\mathbb{C}) \longrightarrow \mathbb{C}^\times$ $\qquad$ $\rho_p: T(\overline{F}_p) \longrightarrow \overline{F}_p^\times$

Prop 2. One has bijection

$$
\begin{tikzcd}
\mathrm{Char}_{\mathbb{C},\mathrm{alg,wt}\,0}\left(F^\times\backslash\mathbb{A}_F^\times\right) \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{Char}_{\mathbb{C}}(\Gamma_F) \arrow[d, "\text{twist}"] \\
\mathrm{Char}_{\mathbb{C},\mathrm{alg}}\left(F^\times\backslash\mathbb{A}_F^\times\right) \arrow[r, leftrightarrow, "\text{twisted by}", "\frac{\rho_\infty(x_\infty)}{\rho_p(x_p)} \underline{\underline{\text{when } x_\infty = x_p}}\ 1"'] & \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) + \text{de Rham} \\
\|\cdot\| \arrow[r, leftrightarrow] & \varepsilon_p
\end{tikzcd}
$$

We will explain Prop 2 in the following pages.

$$\begin{tikzcd}
\chi_\infty & F_\infty^\times \arrow[dr, hook] & & \\
\chi_p : & F_p^\times \arrow[r, hook] & \mathbb{A}_F^\times \arrow[r, "\chi"] & \mathbb{C}^\times \\
\chi_{fin} : & \mathbb{A}_{F,fin}^\times \arrow[ur, hook] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
T(\mathbb{R}) \arrow[dr, hook] & & \\
T(\mathbb{Q}_p) \arrow[r, hook] & T(\mathbb{A}_{\mathbb{Q}}) \arrow[r, "\chi"] & \mathbb{C}^\times \\
T(\mathbb{A}_{\mathbb{Q},fin}) \arrow[ur, hook] & &
\end{tikzcd}$$

Def. Let $\rho \in X^*(T)$. $\chi \in \widehat{\mathbb{A}_F^\times}$ is alg of wt ρ, if

$$\chi_\infty|_{F_\infty^{\times,0}} : F_\infty^{\times,0} = T(\mathbb{R})^0 \hookrightarrow T(\mathbb{C}) \xrightarrow{\frac{1}{\rho_\infty(-)}} \mathbb{C}^\times$$

Lemma. Let $\chi \in \mathrm{Char}_{\mathbb{C},alg}(F^\times \backslash \mathbb{A}_F^\times)$, then
$\exists$ E # field, s.t. $\mathrm{Im}\ \chi_{fin} \subset E^\times$.

Proof. Step 1 $\quad \chi_{fin} : \mathbb{A}_{F,fin}^\times \longrightarrow \mathbb{C}^\times$

$$\chi_{fin}(F^\times) \subseteq F^{nc}$$
$$\parallel$$
comes from $T(\mathbb{R})^0$
$$\chi_\infty(F^\times)^{-1} = \pm \rho(F^\times)$$

Step 2.

$$\begin{tikzcd}
& F^\times \backslash \mathbb{A}_{F,fin}^\times / \prod_v \mathcal{O}_{F,v}^\times \cap \ker \chi_{fin} \arrow[dl] \arrow[dr, "\text{finite}"] & \\
F^\times \backslash \mathbb{A}_{F,fin}^\times / \ker \chi_{fin} & & F^\times \backslash \mathbb{A}_F^\times / \prod_v \mathcal{O}_{F,v}^\times \cong Cl(F)
\end{tikzcd}$$

since χ_{fin} is cont

$$\Rightarrow \quad m := \# F^\times \backslash \mathbb{A}_{F,fin}^\times / \ker \chi_{fin} < +\infty$$

Step 3. Denote $F^\times \backslash \mathbb{A}_{F,fin}^\times / \ker \chi_{fin} = \{\bar{g}_1, \ldots, \bar{g}_m\}$. then
$$(\chi_{fin}(g_i))^m = \chi_{fin}(g_i^m) \in F^{nc},$$
then
$$E := F^{nc}(\chi_{fin}(g_i))$$
satisfies the conditions.

Proof of Prop 2.

Goal: twist $\chi \in \mathrm{Char}_{\mathbb{C},\mathrm{alg}}(F^\times \backslash \mathbb{A}_F^\times)$ to $\psi_\chi \in \mathrm{Char}_{\overline{\mathbb{Q}}_p,\mathrm{cont}}(\Gamma_F^{ab})$.

In fact, we construct

$$\psi_\chi : \mathbb{A}_F^\times = F_\infty^\times \times F_p^\times \times \mathbb{A}_{F,fin}^{p,\times} \longrightarrow \overline{\mathbb{Q}}_p$$

$$T(\mathbb{A}_{\mathbb{Q}}) = T(\mathbb{R}) \times T(\mathbb{Q}_p) \times T(\mathbb{A}^p_{\mathbb{Q},fin})$$

$$x = (x_\infty, \underbrace{x_p, x^p_{fin}}_{x_{fin}}) \longmapsto \chi(x) \cdot \frac{\rho_\infty(x_\infty)}{\rho_p(x_p)}$$

$$= \underbrace{(\chi_\infty \rho_\infty)(x_\infty)}_{\in \{\pm 1\}} \cdot \underbrace{\frac{1}{\rho_p(x_p)}}_{\in F_p^\times} \cdot \underbrace{\chi_{fin}(x_{fin})}_{\in E^\times}$$

Rmk. When $\rho = \mathrm{Id}$, $\chi_{fin}(F^\times) = \pm 1$, $\rho_\infty = \mathrm{Id}$, $\rho_p = \mathrm{Id}$, there is no twist.

E.g.

1) $\chi \in \widehat{\mathbb{A}_F^\times}$ is alg of wt 0 $\longleftrightarrow$ $\chi \in \mathrm{Char}_{\mathbb{C}}(\Gamma_F)$ is the Artin character

2) $\|\cdot\| \in \widehat{\mathbb{A}_F^\times}$ is alg of wt $-\sum_{\tau: F \hookrightarrow F^{nc}} [\tau]$

$\|\cdot\| \longleftrightarrow \varepsilon_p$

3) For $t \in \mathbb{C}$,

$\|\cdot\|^t$ is alg $\Leftrightarrow$ $t \in \mathbb{Z}$.