Eine Woche, ein Beispiel

9.4 Hecke algebra for matrix groups

This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]
[XiongHecke][https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf]
[Ginzburg]: Representation Theory and Complex Geometry [https://link.springer.com/book/10.1007/978-0-8176-4938-8]
[Willians]: Langlands correspondence and Bezrukavnikov's equivalence
[KalethaTaïbi][http://www-personal.umich.edu/~kaletha/llcnqs.pdf]
+ The local Langlands conjecture [in sciebo document]
[BS17]: Bump, Daniel, and Anne Schilling. Crystal Bases. Representations and Combinatorics. Hackensack, NJ: World Scientific, 2017. https://doi.org/10.1142/9876.

All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

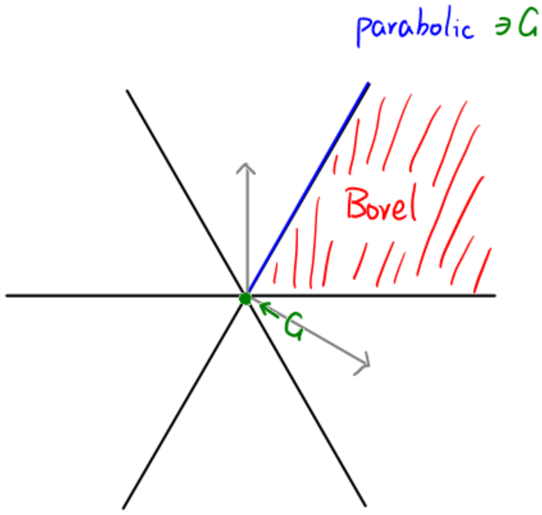
Task. For each double coset decomposition, we want to do:

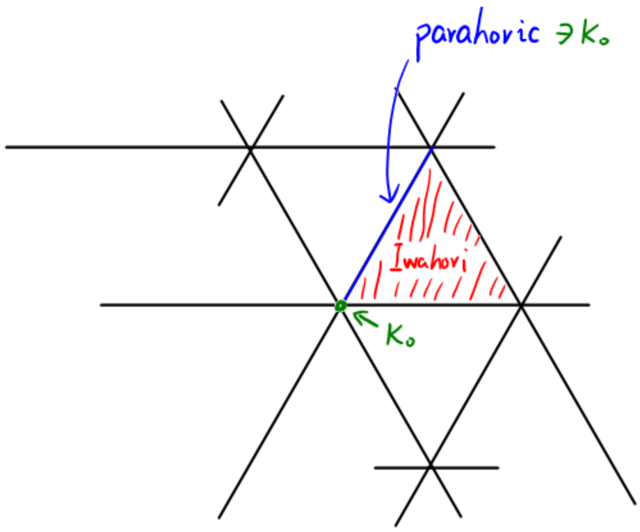
1. decomposition ($\&\Gamma t\Gamma/\Gamma$ is finite & definition of Hecke alg)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form
F finite	$G = \bigsqcup_{w\in W} BwB$		
F local	$G = \bigsqcup_{w\in W} BwB$	$G = \bigsqcup_{w\in W_{ext}} IwI$	$G = \bigsqcup_{\lambda\in T^-} K_0\lambda K_0$
F global	$G = \bigsqcup_{w\in W} BwB$		$GL_2^+(\mathbb{Q}) = \bigsqcup_{\lambda\in T^-} \Gamma\lambda\Gamma$
adèle?			

https://mathoverflow.net/questions/4547/definitions-of-hecke-alg
https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebraebras





$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$\begin{pmatrix} & & 1 \\ & 1 & \\ p & & \end{pmatrix} \in GL_2(\mathcal{O}) \Rightarrow \begin{pmatrix} & 1 \\ p & 1 \end{pmatrix} \notin I$$

mirabolic: (miracle parabolic)
parahoric (containing an Iwahori subgroup)
https://mathoverflow.net/questions/24960/why-are-parabolic-subgroups-called-parabolic-subgroups

For the classical group case, see: https://math.stackexchange.com/questions/3068424/iwahori-versus-bruhat-decompositions

$\triangledown$ $GL_2(\mathbb{Z}_p) \neq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p) \;\middle|\; a,b,c,d \in \mathbb{Z}_p \right\}$. Instead,

$$GL_2(\mathbb{Z}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Q}_p) \;\middle|\; \begin{array}{l} a,b,c,d \in \mathbb{Z}_p \\ \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}_p^\times \end{array} \right\}$$
$$\mathrel{\hat=} \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$$

Similarly,
$$I \mathrel{\hat=} \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$$
where $\det \in \mathcal{O}^\times$ is required implicitly.

e.g. $\begin{pmatrix} \pi & \\ & 1 \end{pmatrix} \notin I$.

https://mathoverflow.net/questions/15438/a-slick-proof-of-the-bruhat-decomposition-for-gl-nk

Tip: Those matrix decomposition theorems may seem quite frightening in the beginning. In fact, they are just fancy special cases of Gaussian elimination.
Remember,

> 左乘行变换，右乘列变换
> multiply a matrix on the left hand side is equiv. to do row operations.
> multiply a matrix on the right hand side is equiv. to do column operations.

The canonical form usually has entries 0 almost everywhere.
To compute the canonical form, we use allowed row/column operations.

E.g. $G = \bigsqcup_{w\in W} BwB \qquad g\sim g' \Leftrightarrow \exists\, b_1, b_2 \in B$ s.t. $g = b_1 g' b_2$

e.g. $q=7$, $G = GL_3(\mathbb{F}_7)$

$$\begin{pmatrix}5&1&6\\6&2&4\\0&4&3\end{pmatrix} \xrightarrow{\begin{pmatrix}1&&\\&-1&\\&&1\end{pmatrix}\times} \begin{pmatrix}5&1&6\\1&5&3\\0&4&3\end{pmatrix} \xrightarrow{\begin{pmatrix}1&2&\\&1&\\&&1\end{pmatrix}\times} \begin{pmatrix}0&4&5\\1&5&3\\0&4&3\end{pmatrix} \xrightarrow{\times\begin{pmatrix}1&-5&-3\\&1&\\&&1\end{pmatrix}} \begin{pmatrix}0&4&5\\1&0&0\\0&4&3\end{pmatrix}$$

$$\xrightarrow{\begin{pmatrix}1&&-1\\&1&\\&&1\end{pmatrix}\times} \begin{pmatrix}0&0&2\\1&0&0\\0&4&3\end{pmatrix} \xrightarrow{\begin{pmatrix}1&&\\&1&\\&&2\end{pmatrix}\times} \begin{pmatrix}0&0&2\\1&0&0\\0&1&6\end{pmatrix} \xrightarrow{\times\begin{pmatrix}1&&\\&1&-6\\&&1\end{pmatrix}} \begin{pmatrix}0&0&2\\1&0&0\\0&1&0\end{pmatrix}$$

$$\xrightarrow{\begin{pmatrix}4&&\\&1&\\&&1\end{pmatrix}\times} \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$$

E.g. $G = \bigsqcup_{w\in W_{ext}} IwI \qquad g\sim g' \Leftrightarrow \exists\, x_1, x_2 \in I$ s.t. $g = x_1 g' x_2$

e.g. $q=3 \quad G = GL_2(\mathbb{Q}_3) \quad I = \begin{pmatrix}\mathbb{Z}_3 & \mathbb{Z}_3\\ 3\mathbb{Z}_3 & \mathbb{Z}_3\end{pmatrix} \subset GL_2(\mathbb{Z}_3)$

$$\begin{pmatrix}1&\frac{1}{3}\\0&1\end{pmatrix} \xrightarrow{\begin{pmatrix}1&\\-3&1\end{pmatrix}\times} \begin{pmatrix}1&\frac{1}{3}\\-3&0\end{pmatrix} \xrightarrow{\times\begin{pmatrix}1&\\-3&1\end{pmatrix}} \begin{pmatrix}0&\frac{1}{3}\\-3&0\end{pmatrix} \xrightarrow{\begin{pmatrix}1&\\&-1\end{pmatrix}\times} \begin{pmatrix}&3^{-1}\\3&\end{pmatrix}$$

e.g. $q=3 \quad G = GL_3(\mathbb{Q}_3) \quad I = \begin{pmatrix}\mathbb{Z}_3&\mathbb{Z}_3&\mathbb{Z}_3\\3\mathbb{Z}_3&\mathbb{Z}_3&\mathbb{Z}_3\\3\mathbb{Z}_3&3\mathbb{Z}_3&\mathbb{Z}_3\end{pmatrix} \subset GL_3(\mathbb{Z}_3)$

$$\begin{pmatrix}2&1&3\\3&1&2\\3&2&1\end{pmatrix} \sim \begin{pmatrix}1&\frac{1}{2}&\frac{3}{2}\\3&1&2\\3&2&1\end{pmatrix} \sim \begin{pmatrix}1&\frac{1}{2}&\frac{3}{2}\\0&-\frac{1}{2}&-\frac{5}{2}\\0&\frac{1}{2}&-\frac{7}{2}\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&-\frac{1}{2}&-\frac{5}{2}\\0&\frac{1}{2}&-\frac{7}{2}\end{pmatrix}$$

$$\sim \begin{pmatrix}1&0&0\\0&0&-6\\0&\frac{1}{2}&-\frac{7}{2}\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&-6\\0&1&-7\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&-6\\0&1&0\end{pmatrix}$$

$$\sim \begin{pmatrix}1&0&0\\0&0&3\\0&1&0\end{pmatrix}$$

$$\begin{pmatrix}2&1&3\\3&1&2\\3&2&1\end{pmatrix} \sim \begin{pmatrix}2&\frac{1}{2}&3\\3&\frac{1}{2}&2\\3&1&1\end{pmatrix} \sim \begin{pmatrix}\frac{1}{2}&\frac{1}{2}&\frac{5}{2}\\\frac{3}{2}&\frac{1}{2}&\frac{3}{2}\\0&1&0\end{pmatrix} \sim \begin{pmatrix}\frac{1}{2}&0&\frac{5}{2}\\\frac{3}{2}&0&\frac{3}{2}\\0&1&0\end{pmatrix}$$

$$\sim \begin{pmatrix}1&0&5\\\frac{3}{2}&0&\frac{3}{2}\\0&1&0\end{pmatrix} \sim \begin{pmatrix}1&0&5\\0&0&-6\\0&1&0\end{pmatrix} \sim \begin{pmatrix}1&0&0\\0&0&-6\\0&1&0\end{pmatrix}$$

$$\sim \begin{pmatrix}1&0&0\\0&0&3\\0&1&0\end{pmatrix}$$

To show the disjointness (i.e. canonical form does not depend on the process), we observe that some properties of $k\times k$-minors are preserved under the restricted row/column operations. For $k=1$ these invariants can be seen easier:

E.g. $G=\bigsqcup_{w\in W} BwB$

$$\begin{pmatrix} * & & \\ \vdots & & \\ * & * & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix} \sim \begin{pmatrix} 0 & & * & & \\ \vdots & & & & \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & & & & \\ \vdots & & * & & \\ 0 & & & & \end{pmatrix}$$

not 0

$G=\bigsqcup_{\lambda\in T^-} K_0\lambda K_0$

$$A=(a_{ij})_{i,j=1}^{n} \underset{e=\min_{i,j} v(a_{ij})}{\sim} \begin{pmatrix} \pi^e & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix}$$

$G=\bigsqcup_{w\in W_{ext}} IwI$

$$A=(a_{ij})_{i,j=1}^{n} \underset{e=\min_{i,j} v(a_{ij})}{\sim} \begin{array}{cc} & j_0 \\ i_0 & \begin{pmatrix} * & 0 & * \\ 0 \cdots & \pi^e & \cdots 0 \\ * & 0 & * \end{pmatrix} \end{array}$$

$I=\{(i,j)\mid v(a_{ij})=e\}$

$(i_0,j_0)\in I$ is in the lower left corner

i.e. $\forall (i,j)\in I,\ \left.\begin{matrix} i\geq i_0 \\ j\leq j_0\end{matrix}\right\} \Rightarrow (i,j)=(i_0,j_0)$

If you have no clue on the properties of $k\times k$-minors for $k>1$, you can see [Bump, Section 9] for the case of p-adic Cartan decomposition.

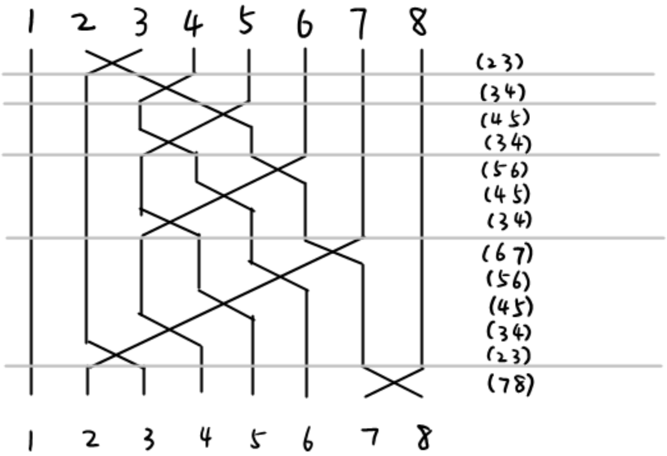
S_n and Tits system

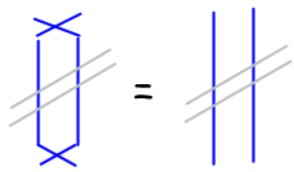
A brief preparation for computations in Bruhat decomposition. $s_i = (i\ i{+}1),\ 1 \le i \le n-1$

E.g. $n=8$, $w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8$.

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.

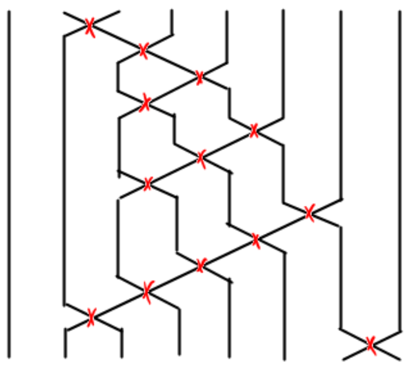




$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0) = 13$ = "inversion number"

$l(s_1 w_0) = 14$	$l(w_0 s_1) = 14$
$l(s_2 w_0) = 12$	$l(w_0 s_2) = 12$
$l(s_3 w_0) = 14$	$l(w_0 s_3) = 14$
$l(s_4 w_0) = 12$	$l(w_0 s_4) = 12$
$l(s_5 w_0) = 12$	$l(w_0 s_5) = 12$
$l(s_6 w_0) = 12$	$l(w_0 s_6) = 14$
$l(s_7 w_0) = 14$	$l(w_0 s_7) = 12$



$l(w_0) = 13$

How to see the length:
count the intersection number

Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

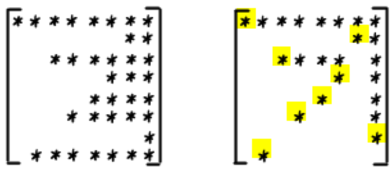
Solution

w_0 $\quad$ Bw_0 $\quad$ w_0B

s_1Bw_0 $\quad$ s_1w_0B

s_2Bw_0 $\quad$ s_2w_0B

The following computation will be also computed later on.

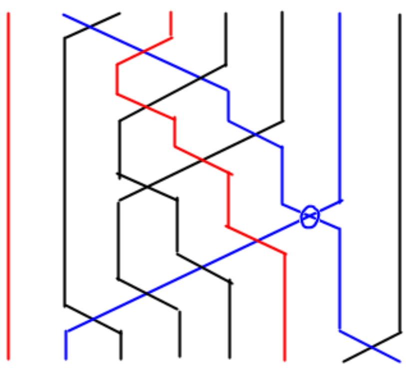


w_0B $\quad$ $Bw_0 \cap w_0B$

$w_0Bw_0^{-1}$ $\quad$ $B \cap w_0Bw_0^{-1}$

How to see $w_0Bw_0^{-1}$:

$w_0Bw_0^{-1}$



red: no intersection
blue: have intersection

finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$ but $BwB \cdot Bw^{-1}B \neq B$ is possible

Ex. Compute $|BwB/B|$ $\triangledown$ BwB may not be a group!

Hint: Consider the map

$$\phi : B \longrightarrow BwB/B$$
$$b \longmapsto bwB$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 wB = b_2 wB$$
$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$
$$\Leftrightarrow b_2^{-1} b_1 \in wBw^{-1}$$
$$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{l(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) = \{ f : G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \ \forall b_1, b_2 \in B, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) \, d\mu(x)$$
$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G,B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbb{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote $T_w := \mathbb{1}_{BwB}$, $T_i := T_{s_i}$ ($T_{Id} = \mathbb{1}_B$ is the unit of $\mathcal{H}(G,B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G,B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) \, T_w$$

$$(T_u * T_v)(w) = \frac{1}{|B|} \sum_{yz = w} T_u(y) T_v(z)$$
$$= \frac{1}{|B|} \{ (y,z) \in BuB \times BvB \mid yz = w \} \quad \underline{\text{if } w \notin BuBBvB} \ 0$$
$$= \frac{1}{|B|} | BuB \cap wBv^{-1}B |$$
$$= \frac{1}{|B|} | \cup \{ wbv^{-1}B \mid b \in B, wbv^{-1} \in BuB \} |$$

$$Bs_iB\cdot Bw_oB = \begin{cases} Bs_iw_oB & l(s_iw)=l(w)+1 \\ Bs_iw_oB\cup Bw_oB & l(s_iw)=l(w)-1 \end{cases}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} \mathbb{Z}\cdot T_{s_iw} & l(s_iw)=l(w)+1 \\ \mathbb{Z}\cdot T_{s_iw} + \mathbb{Z}\cdot T_w & l(s_iw)=l(w)-1 \end{cases}$$

Computation of coefficient:

$$|BwB| = |BwB/B|\times|B| = q^{l(w)}\cdot|B|$$

when $l(s_iw)=l(w)+1$,

$$(T_i * T_w)(s_iw) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times BwB \mid yz=s_iw\}$$

$$= \frac{1}{|B|\,|Bs_iwB|}\{(y,z)\in Bs_iB\times BwB\mid yz\in Bs_iwB\}$$

$$= \frac{|Bs_iB|\,|BwB|}{|B|\cdot|Bs_iwB|} = \frac{q^{l(s_i)}q^{l(w)}}{q^{l(s_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times Bs_iB\mid yz=Id\}$$

$$= \frac{1}{|B|}|Bs_iB| = q$$

$$(T_i * T_i)(s_i) = \frac{1}{|B|}\{(y,z)\in Bs_iB\times Bs_iB\mid yz=s_i\}$$

$$= \frac{1}{|B|\,|Bs_iB|}\{(y,z)\in Bs_iB\times Bs_iB\mid yz\in Bs_iB\}$$

$$= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB\times Bs_iB| - \{(y,z)\in Bs_iB\times Bs_iB\mid yz\in B\}\right)$$

$$= \frac{1}{|B|\,|Bs_iB|}\left(|Bs_iB|\,|Bs_iB| - |B|\cdot|Bs_iB|\right)$$

$$= q-1$$

when $l(s_iw)=l(w)-1$, we get $l(s_i\cdot s_iw)=l(s_iw)+1$,

$$T_i * T_w = T_i * T_i * T_{s_iw}$$

$$=(qT_{Id} + (q-1)T_i) * T_{s_iw}$$

$$= qT_{s_iw} + (q-1)T_w$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} T_{s_iw} & l(s_iw)=l(w)+1 \\ qT_{s_iw} + (q-1)T_w & l(s_iw)=l(w)-1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$\mathcal{H}(G,B) = \mathbb{Z}\langle T_1,\dots,T_{n-1}\rangle_{alg}$ with relations $\quad (\mathcal{H}(G,B)\simeq\mathcal{H}_q(W))$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \qquad \text{for } |i-j|\geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

E.g. For $n=2$, $\mathcal{H}(G,B) \cong \mathbb{Z}[T_1] / (T_1^2-(q-1)T_1-q)$

$$\cong \mathbb{Z}[T_1] / (T_1-q)(T_1+1)$$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1$$

For $n=3$, $\mathcal{H}(G,B) \cong \mathbb{Z}\langle T_1, T_2\rangle \Big/ \big((T_1-q)(T_1+1), (T_2-q)(T_2+1), T_1T_2T_1 = T_2T_1T_2\big)$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_1T_2 \oplus \mathbb{Z}T_2T_1 \oplus \mathbb{Z}T_1T_2T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_{(123)} \oplus \mathbb{Z}T_{(132)} \oplus \mathbb{Z}T_{(13)}$$

global Cartan decomposition

1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2\times 2}(R) = \coprod_{(b)\subseteq(a)} GL_2(R)\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(R)$$

Cor
$$M_{2\times 2}(\mathbb{Z}) = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0\le a\le b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det\neq 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\le b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det> 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\le b}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a,b\in\mathbb{Q}^\times_{>0}\\ v_p(a)\le v_p(b)\ \forall p}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} SL_2(\mathbb{Z})$$ $GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det>0}$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$T^- = \left\{ \begin{pmatrix} a & \\ & b\end{pmatrix} \in GL_2^+(\mathbb{Q}) \;\middle|\; \begin{array}{l} a,b>0 \\ v_p(a)\le v_p(b) \quad \forall p \text{ prime}\end{array} \right\} \overset{\text{Grp}}{\cong} \mathbb{Q}^\times_{>0}\times(\mathbb{Z}_{>0},\times)$$

then
$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha\in T^-} \Gamma\alpha\Gamma$$

Ex. Verify that $\Gamma\alpha\Gamma/\Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix}\alpha_1 & \\ & \alpha_2\end{pmatrix}\in T^-$

Hint. See [WWL, 引理5.1.4].

$$\#\Gamma\alpha\Gamma/\Gamma = \#\Gamma/\Gamma\cap\alpha\Gamma\alpha^{-1} = \#\Gamma/\Gamma_0\!\left(\tfrac{\alpha_2}{\alpha_1}\right) = \#\,\mathbb{P}^1\!\left(\tfrac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_2}{\alpha_1}\prod_{p\mid\frac{\alpha_2}{\alpha_1}}\left(1+\frac{1}{p}\right)$$

$$\left[\alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}\alpha^{-1} = \begin{pmatrix} a & \frac{\alpha_1}{\alpha_2}b\\ \frac{\alpha_2}{\alpha_1}c & d\end{pmatrix} \quad\Rightarrow\quad \Gamma\cap\alpha\Gamma\alpha^{-1} = \begin{pmatrix} \mathbb{Z} & \mathbb{Z}\\ \frac{\alpha_2}{\alpha_1}\mathbb{Z} & \mathbb{Z}\end{pmatrix}_{\det=1} = \Gamma_0\!\left(\tfrac{\alpha_2}{\alpha_1}\right)\right]$$

e.g. $\#\Gamma\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}\Gamma/\Gamma = p+1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p^e\end{pmatrix}\Gamma/\Gamma = p^e+p^{e-1}$

The desired measure can not be realized here, i.e.,

a Haar measure μ on $GL_2^+(\mathbb{Q})$ s.t. $\mu(\Gamma)=1$.

Reason: measure satisfies countable additivity, and Γ is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure $\rightsquigarrow$ semimeasure)

Toy e.g.: There is no way to define a Haar measure μ on $\mathbb{Q}$ s.t. $\mu(\mathbb{Z})=1$.

However, if we only require finite additivity, we can do it.

Def (Semimeasure on $\mathbb{Q}$)

For any periodic set $X \subseteq \mathbb{Q}$ (i.e., $\exists m \in \mathbb{Q}_{>0}$ s.t. $m+X=X$)

we set

$$\mu(X) := \tfrac{1}{m} \left|X/m\mathbb{Z}\right| = \tfrac{1}{m}\left|X \cap [0,m)\right|$$

Rmk. 1. $\mathbb{Z} \supset \mathbb{Z}\cap m\mathbb{Z} \subset m\mathbb{Z}$ $\quad \left|\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}\right|,\ \left|m\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}\right| < +\infty$

"$m\mathbb{Z}$ are all commensurable gps of $\mathbb{Z}$"

2. $X = \bigsqcup_{\alpha \in \Lambda} \alpha + m\mathbb{Z}$ for some $\Lambda \subseteq \mathbb{Q}/m\mathbb{Z}$

"X is a commensurable set of $\mathbb{Z}$ (when $\mu(X)<+\infty$)"

Long A: Def. (Semimeasure on $GL_2^+(\mathbb{Q})$)

For any gp $H \leq GL_2^+(\mathbb{Q})$ which is commensurable with Γ

(i.e., $\# H/H\cap\Gamma$, $\# \Gamma/H\cap\Gamma$ are finite), set

$$\mu(H) = \frac{\left|H/H\cap\Gamma\right|}{\left|\Gamma/H\cap\Gamma\right|} \overset{\text{if } H\leq\Gamma}{=\!=} \frac{1}{\left|\Gamma/H\right|} \in \mathbb{Q}_{>0}$$

Similarly we can specify μ to any commensurable set $X \subseteq GL_2^+(\mathbb{Q})$.

$\left(\begin{array}{lll} \text{i.e.,} \quad X = \bigsqcup_{\alpha\in\Lambda} \alpha H & \text{for some} & H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ \quad\quad\ X = \bigsqcup_{\alpha\in\Lambda'} H'\alpha & & \Lambda \subseteq GL_2^+(\mathbb{Q})/H,\ \Lambda' \subseteq H'\backslash GL_2^+(\mathbb{Q}) \\ & & \Lambda, \Lambda' \text{ finite} \end{array}\right)$

Rmk: In the most of references the terminology (semi)measure is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and take the second line as a def of the convolution.

Def. (Hecke alg $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma)$)

$$\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \to \mathbb{Z} \;\middle|\; \begin{array}{l} f(\gamma_1\alpha\gamma_2) = f(\alpha) \quad \forall \gamma_1,\gamma_2\in\Gamma,\ \alpha\in GL_2^+(\mathbb{Q}) \\ \#\left(\operatorname{supp} f/\Gamma\right) < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \sum_{x\in GL_2^+(\mathbb{Q})/\Gamma} f_1(x) f_2(x^{-1}g) = \sum_{y\in\Gamma\backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1}) f_2(y)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha\in T^-} \mathbb{Z}\cdot \mathbb{1}_{\Gamma\alpha\Gamma}$$

denote $T_\alpha := \mathbb{1}_{\Gamma\alpha\Gamma}$

$\lambda\in\mathbb{Q}^\times$ $\quad R_\lambda := T_{\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)\Gamma} = \mathbb{1}_{\lambda\Gamma}$ $\quad$ ($R_1 = \mathbb{1}_\Gamma$ is the unit of $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma)$)

p prime, $e\geq 1$ $\quad T_{p^e} := T_{\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)\Gamma}$ $\quad$ $T_p := T_{\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)\Gamma}$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma\in T^-} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) = \sum_{x\in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma)$$

$$= \#\left\{ x\in GL_2^+(\mathbb{Q})/\Gamma \;\middle|\; \begin{matrix} x\in\Gamma\alpha\Gamma \\ x^{-1}\gamma\in\Gamma\beta\Gamma \end{matrix} \right\}$$

$$= \left| \Gamma\alpha\Gamma \cap \gamma\Gamma\beta^{-1}\Gamma/\Gamma \right|$$

e.p. $\quad \mathbb{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$

$R_\lambda * R_\mu = R_{\lambda\mu}$

E.g. $\quad g_{\alpha\beta}^{\gamma} \neq 0 \Rightarrow |\gamma| = |\alpha||\beta|$ $\quad$ where $|\alpha| := \det\alpha$

The formula above is still not feasible for effective calculation. We will derived the easiest way to comute $g_{\alpha\beta}^{\gamma}$ in the next page.

Suppose $\Gamma\alpha\Gamma/\Gamma = \{x_1\Gamma, \cdots, x_i\Gamma, \cdots\}$

$\Gamma\beta\Gamma/\Gamma = \{y_1\Gamma, \cdots, y_j\Gamma, \cdots\}$

then

$$
\begin{aligned}
g_{\alpha\beta}^{\gamma} &= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma) \\
&= \sum_i T_\beta(x_i^{-1}\gamma) \\
&= \sum_i \mathbb{1}_{x_i^{-1}\gamma \in \Gamma\beta\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i^{-1}\gamma \in y_j\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i y_j \Gamma = \gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{x_i y_j \in \Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma x_i y_j \Gamma = \Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma\alpha y_j \Gamma = \Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|} \sum_{y \in \Gamma\beta\Gamma/\Gamma} \mathbb{1}_{\Gamma\alpha y \Gamma = \Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma'\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}
\end{aligned}
$$

where

$$\Gamma' := \{\gamma' \in \Gamma \mid \alpha\gamma'\beta \in \Gamma\gamma\Gamma\} = \alpha^{-1}\Gamma\gamma\Gamma\beta^{-1} \cap \Gamma$$

$$\Rightarrow \Gamma'\beta\Gamma/\Gamma = \alpha^{-1}\Gamma\gamma\Gamma \cap \Gamma\beta\Gamma/\Gamma$$

depends on α, β, γ.

The rest is a routine work.

Ex. $\Gamma\begin{pmatrix}1 & \\ & m\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1 & \\ & n\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1 & \\ & mn\end{pmatrix}\Gamma$ $\quad (m,n)=1$

$\Gamma\begin{pmatrix}1 & \\ & p^e\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1 & \\ & p\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1 & \\ & p^{e+1}\end{pmatrix}\Gamma \sqcup \Gamma\begin{pmatrix}p & \\ & p^e\end{pmatrix}\Gamma$ $\quad p$ prime, $e \geq 1$

[Hint. $\begin{pmatrix}1 & \\ & m\end{pmatrix}\begin{pmatrix}a & b \\ c & d\end{pmatrix}\begin{pmatrix}1 & \\ & n\end{pmatrix} = \begin{pmatrix}a & nb \\ mc & mnd\end{pmatrix} \in \Gamma\begin{pmatrix}l & \\ & \frac{mn}{l}\end{pmatrix}\Gamma$ $\quad$ for $\begin{pmatrix}a & b \\ c & d\end{pmatrix} \in SL_2(\mathbb{Z})$

$l = \gcd(a, nb, mc, mnd)$]

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z} \cdot T_{mn} & (m,n)=1 \\ T_{p^e} * T_p \in \mathbb{Z} \cdot T_{p^{e+1}} + \mathbb{Z}\, T_{p^{e-1}} R_p & p \text{ prime},\ e \geq 1 \end{cases}$$

Computation of coefficient:

when $(m,n)=1$, $\alpha=\begin{pmatrix}1&\\&m\end{pmatrix}$, $\beta=\begin{pmatrix}1&\\&n\end{pmatrix}$, $\gamma=\begin{pmatrix}1&\\&mn\end{pmatrix}$.

$$g_{\alpha\beta}^{\gamma}=\frac{1}{|\Gamma\gamma\Gamma/\Gamma|}\sum_i\sum_j \mathbb{1}_{x_iy_j\in\Gamma\gamma\Gamma}$$

$$=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}$$

$$=1$$

when p is prime, $e\geq 1$, $\alpha=\begin{pmatrix}1&\\&p^e\end{pmatrix}$, $\beta=\begin{pmatrix}1&\\&p\end{pmatrix}$, $\gamma_2=\begin{pmatrix}p&\\&p^e\end{pmatrix}$, $\gamma_1=\begin{pmatrix}1&\\&p^{e+1}\end{pmatrix}$,

$$\Gamma_2'\triangleq\{\gamma'\in\Gamma \mid \alpha\gamma'\beta\in\Gamma\gamma_2\Gamma\}$$

$$=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma \;\middle|\; \gcd(a,\,pb,\,p^ec,\,p^{e+1}d)=p\right\}$$

$$=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma \;\middle|\; \begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}0&*\\ *&*\end{pmatrix} \bmod p\right\}$$

$$=\Gamma^0(p)\begin{pmatrix}&1\\-1&\end{pmatrix}$$

$$|\Gamma_2'\beta\Gamma/\Gamma| = \left|\Gamma^0(p)\begin{pmatrix}&p\\-1&\end{pmatrix}\Gamma/\Gamma\right|$$

$$=|\Gamma^0(p)|\Big/\left|\begin{pmatrix}&p\\-1&\end{pmatrix}\Gamma^0(p)\begin{pmatrix}&p\\-1&\end{pmatrix}^{-1}\cap\Gamma^0(p)\right|$$

$$=|\Gamma^0(p)|/|\Gamma^0(p)|$$

$$=1$$

$$\begin{pmatrix}&p\\-1&\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}&p\\-1&\end{pmatrix}^{-1}=\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}^{-1}\begin{pmatrix}&1\\-1&\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}a&\frac{1}{p}b\\pc&d\end{pmatrix}\begin{pmatrix}&1\\-1&\end{pmatrix}^{-1}$$

$$=\begin{pmatrix}pc&d\\-a&-\frac{1}{p}b\end{pmatrix}\begin{pmatrix}&-1\\1&\end{pmatrix}$$

$$=\begin{pmatrix}d&-pc\\-\frac{1}{p}b&a\end{pmatrix}$$

$\therefore$

$$g_{\alpha\beta}^{\gamma_2}=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_2'\beta\Gamma/\Gamma|}{|\Gamma\gamma_2\Gamma/\Gamma|}$$

$$=\frac{(p^e-p^{e-1})\cdot 1}{p^{e-1}-p^{e-2}}$$

$$=p$$

$$g_{\alpha\beta}^{\gamma_1}=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_1'\beta\Gamma/\Gamma|}{|\Gamma\gamma_1\Gamma/\Gamma|}$$

$$=\frac{|\Gamma\alpha\Gamma/\Gamma|\left(|\Gamma\beta\Gamma/\Gamma|-|\Gamma_2'\beta\Gamma/\Gamma|\right)}{|\Gamma\gamma_1\Gamma/\Gamma|}$$

$$=\frac{(p^e-p^{e-1})\cdot(p+1-1)}{p^{e+1}-p^e}$$

$$=1$$

4. Conclusion. $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$

with $\begin{cases} T_m T_n = T_{mn} & (m,n)=1 \\ T_{p^e} T_p = T_{p^{e+1}} + p T_{p^{e-1}} R_p & p \text{ prime}, e \geq 1 \end{cases}$

By [Hecke, Thm 12], Γ is a Gelfand subgp of $GL_2^+(\mathbb{Q})$, thus $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma)$ is commutative.

Gelfand involution: $\sigma \mapsto \sigma^T$

Task: generalize it to other congruence subgps.

p-adic Cartan decomposition $\swarrow$ not Grothendieck group!

Set $G = GL_2(F)$, $K_0 = GL_2(\mathcal{O}_F)$.

1. decomposition [Bump Prop 3.5]

$$M_{2\times 2}(\mathcal{O}_F) = \coprod_{0\le e_1\le e_2\le +\infty} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

$$M_{2\times 2}(\mathcal{O}_F)_{\det\neq 0} = \coprod_{0\le e_1\le e_2<+\infty} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

$$G = \coprod_{e_1\le e_2} K_0 \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} K_0$$

Denote $T^- = \left\{\begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix} \in G \,\middle|\, e_1 \le e_2,\ e_1, e_2 \in \mathbb{Z}\right\} \overset{\text{semi gp}}{\cong} \mathbb{Z}\oplus\mathbb{Z}_{\ge 0}$, then

$$G = \coprod_{\alpha\in T^-} K_0 \alpha K_0$$

Ex. Verify that $K_0\alpha K_0/K_0$ is finite, and compute the order. $\alpha = \begin{pmatrix}\pi^{e_1} & \\ & \pi^{e_2}\end{pmatrix}\in T^-$

Hint.

$$\# K_0\alpha K_0/K_0 = \# K_0/K_0\cap \alpha K_0\alpha^{-1} = \# K_0/\Gamma_0(\mathfrak{p}^{e_2-e_1}) = \# \mathbb{P}^1(\mathcal{O}_F/\mathfrak{p}_F^{e_2-e_1}) = \begin{cases} q^{e_2-e_1}+q^{e_2-e_1-1} & e_1<e_2 \\ 1 & e_1=e_2\end{cases}$$

$$\left[\alpha\begin{pmatrix}a & b\\ c & d\end{pmatrix}\alpha^{-1} = \begin{pmatrix}\mathcal{O} & \mathfrak{p}^{e_1-e_2}\\ \mathfrak{p}^{e_2-e_1} & \mathcal{O}\end{pmatrix} \Rightarrow K_0\cap\alpha K_0\alpha^{-1} = \begin{pmatrix}\mathcal{O} & \mathcal{O}\\ \mathfrak{p}^{e_2-e_1} & \mathcal{O}\end{pmatrix} = \Gamma_0(\mathfrak{p}^{e_2-e_1})\right]$$

e.g. $\#\Gamma\begin{pmatrix}\pi & 0\\ 0 & \pi\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\Gamma/\Gamma = q+1$, $\#\Gamma\begin{pmatrix}1 & 0\\ 0 & \pi^e\end{pmatrix}\Gamma/\Gamma = q^e+q^{e-1}$

Here we use the similar notation in modular form [https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/5.moduli_interpretation.pdf]:

$$\begin{array}{ccc}
\Gamma(\mathfrak{p}^e) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\mathrm{bal}.\Gamma_1(\mathfrak{p}^e) & \twoheadrightarrow & N(\mathcal{O}_F/\mathfrak{p}_F^e) = \begin{pmatrix}1 & *\\ 0 & 1\end{pmatrix} \\
\cap & & \cap \\
\Gamma_1(\mathfrak{p}^e) & \twoheadrightarrow & \begin{pmatrix}1 & *\\ & *\end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(\mathfrak{p}^e) & \twoheadrightarrow & B(\mathcal{O}_F/\mathfrak{p}_F^e) = \begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \\
\cap & & \cap \\
\Gamma(\mathcal{O}) = K^0 = GL_2(\mathcal{O}_F) & \twoheadrightarrow & GL_2(\mathcal{O}_F/\mathfrak{p}_F^e)
\end{array}$$

Take the unique Haar measure on G s.t. $\mu(K^0)=1$, then

$$\mu(K^0 t K^0) = \# K^0 t K^0/K^0$$

μ is induced from the measure on coset G/K^0.

The Hecke algebra has been defined here:
https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.04.17_preliminary_facts_of_reps_of_p-adic_groups.pdf
We still recall the convolution here:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \sum_{x\in G/K_0} f_1(x) f_2(x^{-1}g) = \sum_{y\in K_0\backslash G} f_1(gy^{-1}) f_2(y)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G, K_0) = \bigoplus_{\alpha\in T^-} \mathbb{Z}\cdot \mathbb{1}_{K_0\alpha K_0}$$

denote $T_\alpha := \mathbb{1}_{K_0\alpha K_0}$

$\lambda\in F^\times$ $\quad R_\lambda := T_{\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)} = \mathbb{1}_{K_0\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)K_0} = \mathbb{1}_{\lambda K_0}$ $\quad$ ($R_1 = \mathbb{1}_{K_0}$ is the unit of $\mathcal{H}(G, K_0)$)

$e\geq 1$ $\quad T_{\pi^e} := T_{\left(\begin{smallmatrix}1 & \\ & \pi^e\end{smallmatrix}\right)} = \mathbb{1}_{K_0\left(\begin{smallmatrix}1 & \\ & \pi^e\end{smallmatrix}\right)K_0}$ $\quad$ $T_\pi := T_{\left(\begin{smallmatrix}1 & \\ & \pi\end{smallmatrix}\right)} = \mathbb{1}_{K_0\left(\begin{smallmatrix}1 & \\ & \pi\end{smallmatrix}\right)K_0}$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma\in T^-} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) = \sum_{x\in G/K_0} T_\alpha(x) T_\beta(x^{-1}\gamma)$$

$$= \#\left\{ x\in G/K_0 \;\middle|\; \begin{array}{l} x\in K_0\alpha K_0 \\ x^{-1}\gamma\in K_0\beta K_0\end{array}\right\}$$

$$= \left| K_0\alpha K_0 \cap \gamma K_0\beta^{-1}K_0 / K_0 \right|$$

e.p. $\quad \mathbb{1}_\Gamma * f = f \qquad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$

$$R_\lambda * R_\mu = R_{\lambda\mu}$$

E.g. $\quad g_{\alpha\beta}^{\gamma}\neq 0 \Rightarrow |\gamma| = |\alpha||\beta|$ $\quad$ where $|\alpha| := \det\alpha$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^{\gamma} = \frac{|K_0\alpha K_0/K_0|\,|K_0'\beta K_0/K_0|}{|K_0\gamma K_0/K_0|}$$

where

$$K_0' := \{\gamma'\in K_0 \mid \alpha\gamma'\beta\in K_0\gamma K_0\} = \alpha^{-1}K_0\gamma K_0\beta^{-1}\cap K_0$$

$$\Rightarrow K_0'\beta K_0/K_0 = \alpha^{-1}K_0\gamma K_0 \cap K_0\beta K_0/K_0$$

depends on α, β, γ.

$$\Rightarrow T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q\, T_{\pi^{e-1}} R_\pi$$

4. Conclusion (Tamagawa, Satake)

$$\mathcal{H}(G, K^\circ) = \mathbb{Z}[R_\pi^{\pm 1}, T_\pi] \quad \text{with}$$

$$T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q\, T_{\pi^{e-1}} \cdot R_\pi$$

e.p. $\mathcal{H}(GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z})) = \bigotimes_{\mathbb{Z}} \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$

p-adic Iwahori decomposition

We only consider GL_n, SL_n, PGL_n in this section. SL_2 case is special focused.

E.g. $G = SL_2(F)$ $\quad T(F) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $\quad I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}_{\det = 1}$

$\mathcal{U}(F) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\quad \mathcal{U}^-(F) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$

$\mathcal{U}(\mathcal{O}_F) = \begin{pmatrix} 1 & \mathcal{O}_F \\ 0 & 1 \end{pmatrix}$ $\quad \mathcal{U}^-(\mathcal{O}_F) = \begin{pmatrix} 1 & 0 \\ \mathcal{O}_F & 1 \end{pmatrix}$

$$N_G(T(F))/T(F) := W_f$$
$$N_G(T(\mathcal{O}_F))/T(\mathcal{O}_F) := W_{ext} \cong \langle s_1, s_0 \rangle / (s_1^2 = s_0^2 = 1) \qquad s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi \\ -\pi^{-1} & \end{pmatrix}$$

	basis roots / root / char / weight	basis coroots / coroot / cochar / coweight		
My notation	$\Delta \subseteq \Phi \subseteq Q \subseteq X^* \subseteq P$	$\Delta^\vee \subseteq \Phi^\vee \subseteq Q^\vee \subseteq X_* \subseteq P^\vee$	$W_f \quad W_{aff} \quad W_{ext}$	in dual
[Ginzburg]	$R \subseteq Q \subseteq P$	$R^\vee \subseteq Q^\vee \subseteq P^\vee$	$W \quad W_{aff}$	not in dual
[Willians]	$R \subseteq X$	$R^\vee \subseteq X^\vee$	$W_f \quad W \quad W_{ext}$	
[Bump]	$\Phi \subseteq Q \subseteq P$	$\hat{\Phi} \subseteq Q^\vee \subseteq P^\vee$	$W \quad W_{aff}$	$\tilde{W}_{aff}$ not in dual

[Bump]: Iwahori Hecke algebra to $P^\vee$, not the standard $\mathcal{H}(G, I)$

[Kaletha Taibi]	$\Delta \subseteq R \subseteq X$	$\subseteq R^\vee \subseteq X^\vee$		
[BS17]	$\Phi \subseteq \Lambda_{root} \subseteq \Lambda \subseteq \Lambda_{sc}$	$\Phi^\vee \subseteq$		

Λ called wt lattice in this book

$$\begin{tikzcd}
0 \arrow[r] & Q^\vee \arrow[r] \arrow[d, phantom, "\cap"] & W_{aff} \arrow[r] \arrow[d, phantom, "\triangle"] & W_f \arrow[r] \arrow[l, bend right] \arrow[d, phantom, "\|"] & 0 \\
0 \arrow[r] & X_*(T) \arrow[r] & W_{ext} \arrow[r, "p^\vee"'] & W_f \arrow[r] \arrow[l, bend right, "s"'] & 0
\end{tikzcd}$$

$\Rightarrow$ Standard isomorphism

$$W_{aff} \cong Q^\vee \rtimes W_f$$
$$W_{ext} \cong X_*(T) \rtimes W_f$$

In case GL_n and PGL_n, people tend to have different definitions for Iwahori, N and the extended Weyl group.
In this document, I=I(SL_n) \times T^o , N_G(T) is the usual normalizer, then the extended Weyl group in the GL_n case is not a Coxeter group.
In [Kaletha and Prasad _BruhatnTits theory_ a new approach], I is the same, while N and W are different.

e.g. $I(GL_n) = I(SL_n) \cdot \begin{pmatrix} \mathcal{O}^\times & & \\ & \ddots & \\ & & \mathcal{O}^\times \end{pmatrix} = I(SL_n) \cdot \begin{pmatrix} \mathcal{O}^\times & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$

$\underset{\text{image of}}{I(PGL_n) =} I(SL_n) \cdot \begin{pmatrix} \mathcal{O}^\times & & \\ & \ddots & \\ & & \mathcal{O}^\times \end{pmatrix} = I(SL_n) \cdot \begin{pmatrix} \mathcal{O}^\times & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix}$

In ptc, $\begin{pmatrix} & 1 \\ p & \end{pmatrix} \in GL_2(\mathbb{Q}_p) - I$, $\begin{pmatrix} & 1 \\ p & \end{pmatrix} \in PGL_2(\mathbb{Q}_p) - I$

Q: How does W_{ext} act on

$$\pi_1(T) = X_*(T) \cong T(F)/T(\mathcal{O}_F) \cong \mathbb{Z}^{\operatorname{rank} G} \subseteq \mathbb{Z}^n \ ?$$

$$\left[\lambda : t \mapsto \begin{pmatrix} t^{e_1} & & \\ & \ddots & \\ & & t^{e_n}\end{pmatrix}\right] \longleftrightarrow \begin{pmatrix} \pi^{e_1} & & \\ & \ddots & \\ & & \pi^{e_n}\end{pmatrix} \longleftrightarrow \begin{pmatrix} e_1 \\ \vdots \\ e_n\end{pmatrix}$$

A: Denote

$$pr: W_{ext} \longrightarrow W_f \qquad \begin{pmatrix} & * & \\ & & * \\ * & & \\ & * & \end{pmatrix} \longmapsto \begin{pmatrix} & 1 & & \\ & & & 1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

$$s: W_f \hookrightarrow W_{ext} \qquad \begin{pmatrix} & 1 & & \\ & & & 1 \\ -1 & & & \\ & 1 & & \end{pmatrix} \longmapsto \begin{pmatrix} & 1 & & \\ & & & 1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

This is canonical embedding! We will omit s to simplify the formula.

Then, W_{ext} acts on $X_*(T)$ by

$$\begin{array}{ccccc} W_{ext} & \times & X_*(T) & \longrightarrow & X_*(T) \\ N_G(T(F))/T(\mathcal{O}_F) & \times & T(F)/T(\mathcal{O}_F) & \longrightarrow & T(F)/T(\mathcal{O}_F) \\ (w, & & \lambda) & \longmapsto & w.\lambda := w\lambda\, s(pr(w))^{-1} \end{array}$$

$$\underset{w=\mu\times u}{=\!=} \mu u \lambda u^{-1} \qquad \mu \in X_*(T),\ u \in W_f$$

$$\underset{\times \to +}{\overset{\text{in } \mathbb{Z}^n}{=\!=}} \mu + u\lambda u^{-1}$$

So roughly speaking, W_{ext} acts on $X_*(T)$ by a rotation/flip followed with a translation.

We will denote $w = t_\mu u$ when we view $w \in W_{ext}$ as operators on $X_*(T)$.

∇ $w \neq r_{\alpha,k}$ in many cases.

Recall that for $\alpha \in X^*(T)$, $k \in \mathbb{Z}$, we can define

$$\psi = \alpha + k : X_*(T) \longrightarrow \mathbb{R} \qquad \lambda \longmapsto \langle \lambda, \alpha + k\rangle := \langle \lambda, \alpha\rangle + k$$

$$r_\psi = r_{\alpha,k} : X_*(T) \longrightarrow X_*(T) \qquad \lambda \longmapsto \lambda - \langle \lambda, \alpha + k\rangle \alpha^\vee$$

$$= \lambda - \langle \lambda, \alpha\rangle \alpha^\vee - k\alpha^\vee$$

$$H_\psi = H_{\alpha,k} = \{\lambda \in X_*(T) \mid r_{\alpha,k}(\lambda) = \lambda\}$$

$$= \{\lambda \in X_*(T) \mid \langle \lambda, \psi\rangle = 0\} \subseteq X_*(T)$$

The set

$$\Psi := \{\psi : X_*(T) \longrightarrow \mathbb{R} \mid \psi = \alpha + k \text{ for some } \alpha \in \Phi,\ k \in \mathbb{Z}\}$$

is an affine root system, and W_{ext} acts on Ψ by

$$W_{ext} \times \Psi \longrightarrow \Psi \qquad (w, \psi) \longmapsto \psi \circ w^{-1}$$

E.g. For $G = SL_2$,

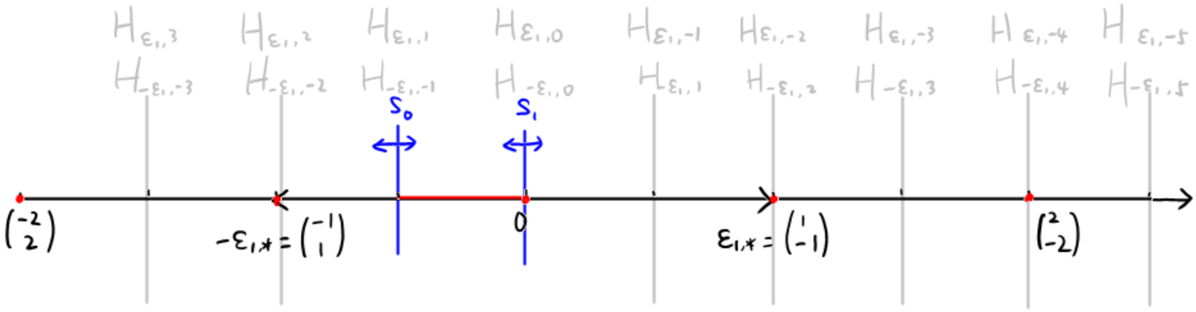
$$\begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ -\lambda_1 - e \end{pmatrix} \qquad \begin{pmatrix} & \pi^e \\ -\pi^{-e} & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 + e \\ \lambda_1 - e \end{pmatrix}$$

Sign is useless

e.p. $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 \\ \lambda_1 \end{pmatrix} \qquad \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 - 1 \\ \lambda_1 + 1 \end{pmatrix}$

Let $s_1 := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $s_0 := \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$, then $s_1 s_0 = \begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix} = \varepsilon_{1,*} \cdot \mathrm{Id}$

Never mind symbol! $\begin{pmatrix} -1 & \\ & -1 \end{pmatrix} \in T(\mathcal{O}_F)$



shortest	$s_0 s_1 s_0$	$s_0 s_1$	s_0	Id	s_1	$s_1 s_0$	$s_1 s_0 s_1$	$s_1 s_0 s_1 s_0$	$s_1 s_0 s_1 s_0 s_1$
matrix	$\begin{pmatrix} & \pi^{-2} \\ -\pi^2 & \end{pmatrix}$	$\begin{pmatrix} \pi^{-1} & \\ & \pi \end{pmatrix}$	$\begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} \pi & \\ & \pi^{-1} \end{pmatrix}$	$\begin{pmatrix} & \pi \\ -\pi^{-1} & \end{pmatrix}$	$\begin{pmatrix} \pi^2 & \\ & \pi^{-2} \end{pmatrix}$	$\begin{pmatrix} & \pi^3 \\ -\pi^{-2} & \end{pmatrix}$
	$(s_1 s_0)^{-2} s_1$	$(s_1 s_0)^{-1}$	$(s_1 s_0)^{-1} s_1$	Id	s_1	$s_1 s_0$	$(s_1 s_0) s_1$	$(s_1 s_0)^2$	$(s_1 s_0)^2 s_1$
operator	$t_{-2\varepsilon_{1,*}} s_1$	$t_{-\varepsilon_{1,*}}$	$t_{-\varepsilon_{1,*}} s_1$	Id	s_1	$t_{\varepsilon_{1,*}}$	$t_{\varepsilon_{1,*}} s_1$	$t_{2\varepsilon_{1,*}}$	$t_{2\varepsilon_{1,*}} s_1$
reflections	$r_{\varepsilon_1,2}$		$r_{\varepsilon_1,1}$		$r_{\varepsilon_1,0}$		$r_{\varepsilon_1,-1}$		$r_{\varepsilon_1,-2}$
	$r_{-\varepsilon_1,2}$		$r_{-\varepsilon_1,-1}$		$r_{-\varepsilon_1,0}$		$r_{-\varepsilon_1,1}$		$r_{-\varepsilon_1,2}$
length	3	2	1	0	1	2	3	4	5
Iwahori	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-3} \\ \mathfrak{p}^4 & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-2} \\ \mathfrak{p}^3 & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^2 \\ \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^3 \\ \mathfrak{p}^{-2} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^4 \\ \mathfrak{p}^{-3} & \mathcal{O} \end{pmatrix}$	$\begin{pmatrix} \mathcal{O} & \mathfrak{p}^5 \\ \mathfrak{p}^{-4} & \mathcal{O} \end{pmatrix}$

parahoric $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-3} \\ \mathfrak{p}^3 & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-2} \\ \mathfrak{p}^2 & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p}^1 & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^2 \\ \mathfrak{p}^{-2} & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^3 \\ \mathfrak{p}^{-3} & \mathcal{O} \end{pmatrix}$ $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^4 \\ \mathfrak{p}^{-4} & \mathcal{O} \end{pmatrix}$

$\bullet = \bullet \subseteq ,$

$Q^\vee = X_* \subseteq P^\vee$

E.g. For $G = PGL_2$,

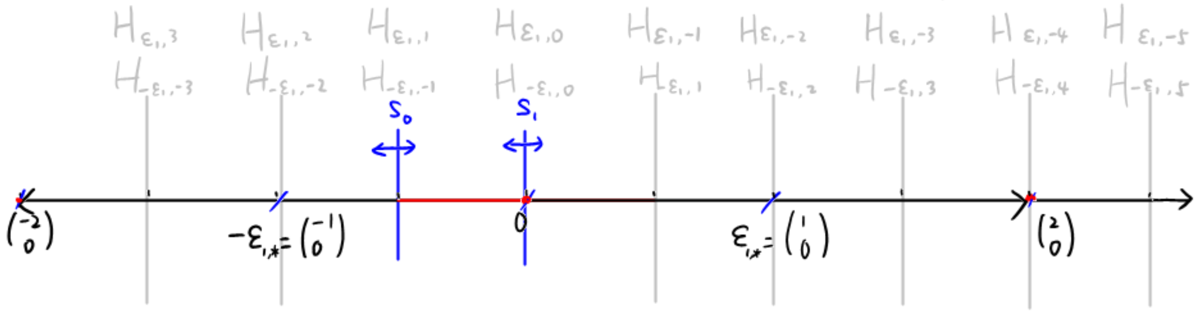
$$\begin{pmatrix} \pi^e & \\ & 1 \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ \lambda_2 \end{pmatrix} \qquad \begin{pmatrix} & 1 \\ -\pi^e & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + e \end{pmatrix}$$

Sign is useless

e.p. $\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$ $\qquad \begin{pmatrix} & 1 \\ -\pi & \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + 1 \end{pmatrix}$

Let $s_1 := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, $s_0 := \begin{pmatrix} & 1 \\ -\pi & \end{pmatrix}$, then $s_1 s_0 = \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} = \varepsilon_1^* \cdot Id$

Never mind symbol! $\begin{pmatrix} & -1 \\ -1 & \end{pmatrix} \in T(\mathcal{O}_F)$



shortest	$s_0 s_1 s_0$	$s_0 s_1$	s_0	Id	s_1	$s_1 s_0$	$s_1 s_0 s_1$	$s_1 s_0 s_1 s_0$	$s_1 s_0 s_1 s_0 s_1$
matrix	$\begin{pmatrix} & 1 \\ -\pi^2 & \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -\pi & \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi^{-1} \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -\pi^{-1} & \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi^{-2} \end{pmatrix}$	$\begin{pmatrix} & 1 \\ -\pi^{-2} & \end{pmatrix}$
	$(s_1 s_0)^{-2} s_1$	$(s_1 s_0)^{-1}$	$(s_1 s_0)^{-1} s_1$	Id	s_1	$s_1 s_0$	$(s_1 s_0) s_1$	$(s_1 s_0)^2$	$(s_1 s_0)^2 s_1$
operator	$t_{-2\varepsilon_{1,*}} s_1$	$t_{-\varepsilon_{1,*}}$	$t_{\varepsilon_{1,*}} s_1$	Id	s_1	$t_{\varepsilon_{1,*}}$	$t_{\varepsilon_{1,*}} s_1$	$t_{2\varepsilon_{1,*}}$	$t_{2\varepsilon_{1,*}} s_1$
reflections	$r_{\varepsilon_1,2}$		$r_{\varepsilon_1,1}$		$r_{\varepsilon_1,0}$		$r_{\varepsilon_1,-1}$		$r_{\varepsilon_1,-2}$
	$r_{-\varepsilon_1,-2}$		$r_{-\varepsilon_1,-1}$		$r_{-\varepsilon_1,0}$		$r_{-\varepsilon_1,1}$		$r_{-\varepsilon_1,2}$
length	3	2	1	0	1	2	3	4	5

Iwahori: $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p}^2 & \mathcal{O} \end{pmatrix}$ | $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$ | $\begin{pmatrix} \mathcal{O} & \mathfrak{p} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$ | $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^2 \\ \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix}$ | $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^3 \\ \mathfrak{p}^{-2} & \mathcal{O} \end{pmatrix}$

parahoric: $\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p}^1 & \mathcal{O} \end{pmatrix}$ $\quad \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$ $\quad \begin{pmatrix} \mathcal{O} & \mathfrak{p} \\ \mathfrak{p}^{-1} & \mathcal{O} \end{pmatrix}$ $\quad \begin{pmatrix} \mathcal{O} & \mathfrak{p}^2 \\ \mathfrak{p}^{-2} & \mathcal{O} \end{pmatrix}$

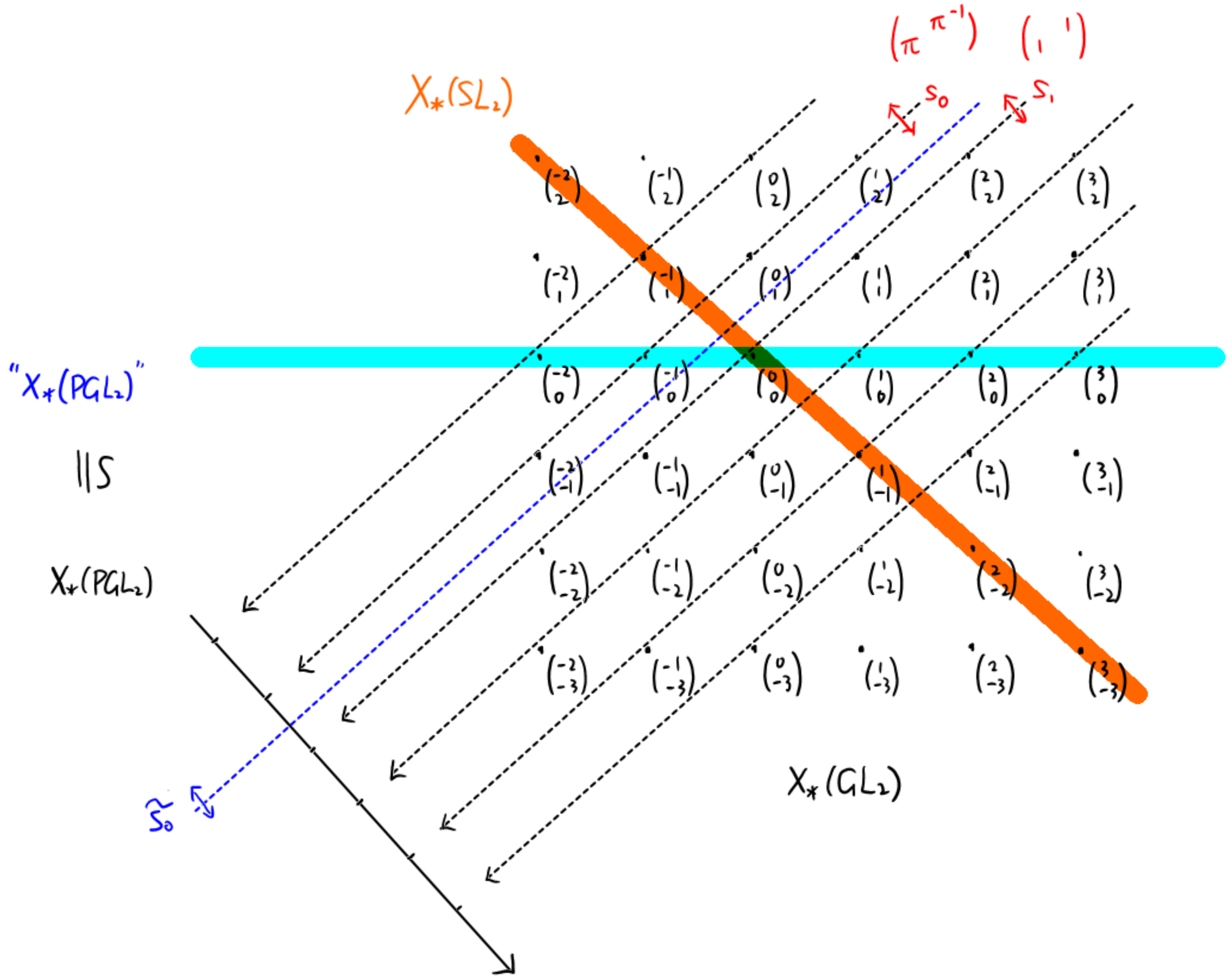
$\cdot \subseteq / = /$

$Q^\vee \subseteq X_* = P^\vee$

⚠ Here, we still require $x \in \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$ satisfies $\det x \in \mathcal{O}^\times$, so $\begin{pmatrix} & 1 \\ p & \end{pmatrix} \notin \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}$

You see also that $\begin{pmatrix} & 1 \\ p & \end{pmatrix} \in N_G(T(\mathcal{O}_F)) - T(\mathcal{O}_F)$.

A vivid picture describing relationships among $X_*(GL_2), X_*(SL_2), X_*(PSL_2)$



$$\begin{array}{ccc} X_*(SL_2) & \hookrightarrow & X_*(GL_2) \\ & & \downarrow \\ & & X_*(PGL_2) \end{array} \qquad \begin{array}{ccc} W_{ext}(SL_2) & \hookrightarrow & W_{ext}(GL_2) \\ & & \downarrow \\ & & W_{ext}(PGL_2) \end{array}$$

$$\begin{pmatrix} & 1 \\ \pi & \end{pmatrix} = s_1 \circ \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cdots \right)$$

$$= \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} + \tilde{s}_0 \circ \cdots$$

In GL_n case, W_{ext} is not the usual affine Weyl gp.

In both cases (SL_2 & PGL_2), $W_{ext} = X_* \sqcup X_* \cdot s_1$ as set.

We will focus on the case $G = SL_2$.

1. decomposition.
$$\begin{aligned} G &= \bigsqcup_{w \in W_{ext}} IwI \\ &= \bigsqcup_{e \in \mathbb{Z}} I \begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix} I \sqcup \bigsqcup_{e \in \mathbb{Z}} I \begin{pmatrix} & \pi^e \\ -\pi^{-e} & \end{pmatrix} I \\ &= \bigsqcup_{e \in \mathbb{Z}} I (s_1 s_0)^e I \sqcup \bigsqcup_{e \in \mathbb{Z}} I (s_1 s_0)^e s_1 I \end{aligned}$$

Ex. Verify that IwI/I is finite, and compute the order.

Hint. $\# IwI/I = \# I/I \cap wIw^{-1}$ $\qquad \# I/I\cap wIw^{-1}$

When $w = \begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix}$,
$$wIw^{-1} = \begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix} \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \begin{pmatrix} \pi^{-e} & \\ & \pi^e \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{2e} \\ \mathfrak{p}^{1-2e} & \mathcal{O} \end{pmatrix}$$
$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{2e} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} & e \geq 0 \qquad q^{2e} \\ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^{1-2e} & \mathcal{O} \end{pmatrix} & e < 0 \qquad q^{-2e} \end{cases}$$

When $w = \begin{pmatrix} & \pi^e \\ -\pi^{-e} & \end{pmatrix}$,
$$wIw^{-1} = \begin{pmatrix} & \pi^e \\ -\pi^{-e} & \end{pmatrix} \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} \begin{pmatrix} & \pi^e \\ -\pi^{-e} & \end{pmatrix} = \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{2e+1} \\ \mathfrak{p}^{-2e} & \mathcal{O} \end{pmatrix}$$
$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} \mathcal{O} & \mathfrak{p}^{2e+1} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix} & e \geq 0 \qquad q^{2e+1} \\ \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}^{-2e} & \mathcal{O} \end{pmatrix} & e < 0 \qquad q^{-2e-1} \end{cases}$$

In conclusion,
$$\# IwI/I = \# I/I\cap wIw^{-1} = q^{l(w)}$$
$\square$

Take the unique Haar measure on G s.t. $\mu(I) = 1$, then
$$\mu(IwI) = \# IwI/I = q^{l(w)}$$
μ is induced from the measure on coset G/I.

2. $\mathbb{Z}$-mod structure, notation.
$$\mathcal{H}(G, I) = \bigoplus_{w \in W_{ext}} \mathbb{Z} \cdot \mathbb{1}_{IwI}$$
denote $T_w = \mathbb{1}_{IwI}$ $\qquad T_{Id} = \mathbb{1}_I$ is the unit of $\mathcal{H}(G, I)$

$T_1 := T_{s_1}$, $\quad T_0 := T_{s_0}$

$e \in \mathbb{Z}$ $\qquad S_{\binom{e}{-e}} := T_{\begin{pmatrix} \pi^e & \\ & \pi^{-e} \end{pmatrix}} = T_{(s_1 s_0)^e} = T_{(s_1 s_0)^e}$ $\qquad$ The general notation is S_λ, for $\lambda \in X_*(T)$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in W_{ext}} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/I} T_\alpha(x)\, T_\beta(x^{-1}\gamma)$$

$$= \# \left\{ x \in G/I \;\middle|\; \begin{matrix} x \in I\alpha I \\ x^{-1}\gamma \in I\beta I \end{matrix} \right\}$$

$$= \left| I\alpha I \cap \gamma I \beta^{-1} I / I \right|$$

e.p. $\mathbb{1}_I * f = f * \mathbb{1}_I = f$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^{\gamma} = \frac{|I\alpha I/I|\; |I_{\alpha\beta}^{\gamma}\beta I/I|}{|I\gamma I/I|}$$

where

$$I_{\alpha\beta}^{\gamma} := \{ w' \in I \mid \alpha w' \beta \in I\gamma I \} = \alpha^{-1} I \gamma I \beta^{-1} \cap I$$

$$\Rightarrow I_{\alpha\beta}^{\gamma}\beta I/I = \alpha^{-1} I \gamma I \cap I\beta I/I$$

depends on α, β, γ.

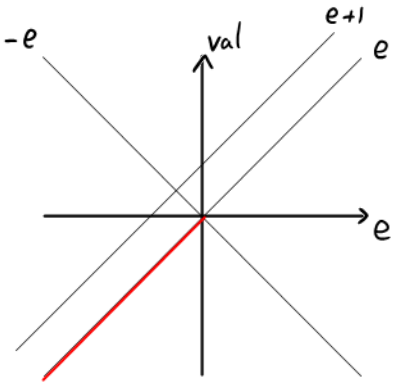
In the following computation, the minus sign is not important, so we ignore it.
E.g When we write $(1\,')$, we actually mean $(-1\,')$.

$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix}\begin{pmatrix}\pi^e & \\ & \pi^{-e}\end{pmatrix} = \begin{pmatrix}\pi^{e+1}c & \pi^{-e}d \\ \pi^e a & \pi^{-e} b\end{pmatrix}$$

$\begin{pmatrix} & \pi^{-e}\\ \pi^e & \end{pmatrix}$ $\quad e\leq 0$

$\begin{pmatrix}\pi^e & \\ & \pi^{-e}\end{pmatrix}$ $\quad e>0,\ val(b)=0$

$\begin{pmatrix} & \pi^{-e}\\ \pi^e & \end{pmatrix}$ $\quad e>0,\ val(b)>0$

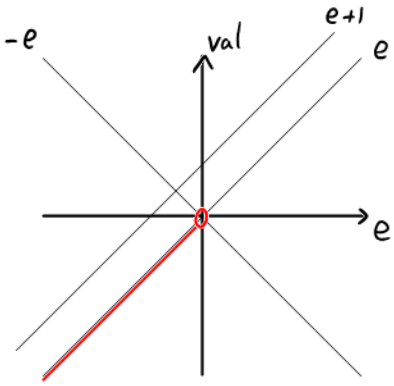


$$\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix}\begin{pmatrix} & \pi^e \\ \pi^{-e} & \end{pmatrix} = \begin{pmatrix}\pi^{-e}d & \pi^{e+1}c \\ \pi^{-e} b & \pi^e a\end{pmatrix}$$

$\begin{pmatrix}\pi^{-e} & \\ & \pi^e\end{pmatrix}$ $\quad e<0$

$\begin{pmatrix} & \pi^e\\ \pi^{-e} & \end{pmatrix}$ $\quad e\geq 0,\ val(b)=0$

$\begin{pmatrix}\pi^{-e} & \\ & \pi^e\end{pmatrix}$ $\quad e\geq 0,\ val(b)>0$

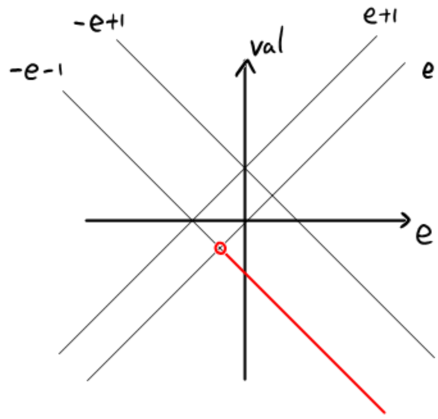


$$\begin{pmatrix} & \pi^{-1} \\ \pi & \end{pmatrix}\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix}\begin{pmatrix}\pi^e & \\ & \pi^{-e}\end{pmatrix} = \begin{pmatrix}\pi^{e}c & \pi^{-e-1}d \\ \pi^{e+1} a & \pi^{-e+1} b\end{pmatrix}$$

$\begin{pmatrix} & \pi^{-e-1}\\ \pi^{e+1} & \end{pmatrix}$ $\quad e\geq 0$

$\begin{pmatrix}\pi^e & \\ & \pi^{-e}\end{pmatrix}$ $\quad e<0,\ val(c)=0$

$\begin{pmatrix} & \pi^{-e-1}\\ \pi^{e+1} & \end{pmatrix}$ $\quad e<0,\ val(c)>0$

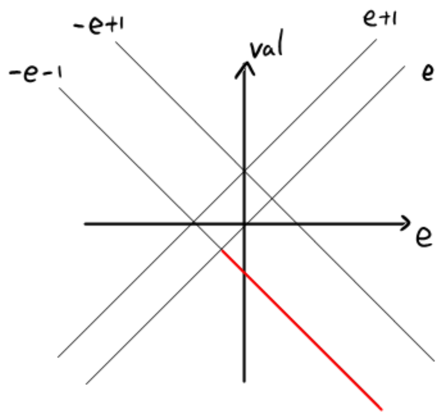


$$\begin{pmatrix} & \pi^{-1} \\ \pi & \end{pmatrix}\begin{pmatrix} a & b \\ \pi c & d\end{pmatrix}\begin{pmatrix} & \pi^e \\ \pi^{-e} & \end{pmatrix} = \begin{pmatrix}\pi^{-e-1}d & \pi^{e}c \\ \pi^{-e+1} b & \pi^{e+1} a\end{pmatrix}$$

$\begin{pmatrix}\pi^{-e-1} & \\ & \pi^{e+1}\end{pmatrix}$ $\quad e\geq 0$

$\begin{pmatrix} & \pi^e\\ \pi^{-e} & \end{pmatrix}$ $\quad e<0,\ val(c)=0$

$\begin{pmatrix}\pi^{-e-1} & \\ & \pi^{e+1}\end{pmatrix}$ $\quad e<0,\ val(c)>0$



In conclusion,

$$T_i * T_w \in \begin{cases} \mathbb{Z}\, T_{s_i w} & l(s_i w) = l(w)+1 \\ \mathbb{Z}\, T_{s_i w} + \mathbb{Z}\, T_w & l(s_i w) = l(w) - 1 \end{cases}$$

In the following computation, $\alpha = s_i$, $\beta = \omega$, $\gamma = s_i\omega$ or $\gamma = \omega$.

When $l(s_i\omega) = l(\omega) + 1$, $\qquad I^{s_i\omega}_{s_i,\omega} = I$,

$$g^{s_i\omega}_{s_i,\omega} = \frac{|Is_iI/I|\,|I\omega I/I|}{|Is_i\omega I/I|} = \frac{q^{l(s_i)}q^{l(\omega)}}{q^{l(s_i\omega)}} = 1$$

When $l(s_i\omega) = l(\omega) - 1$ and $i = 1$, $\qquad \omega = \begin{pmatrix} & \pi^{e} \\ \pi^{-e} & \end{pmatrix}$, $e > 0$ or $\omega = \begin{pmatrix} & \pi^{-e} \\ \pi^{e} & \end{pmatrix}$, $e \geq 0$

$$I^{s_i\omega}_{s_i,\omega} = \begin{pmatrix} \mathcal{O} & \mathfrak{p} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}, \quad I^{\omega}_{s_i,\omega} = \begin{pmatrix} \mathcal{O} & \mathcal{O}-\mathfrak{p} \\ \mathfrak{p} & \mathcal{O} \end{pmatrix}.$$

$$\# I^{s_i\omega}_{s_i,\omega}\omega I/I = \# I^{\gamma}_{\alpha\beta}/(I^{\gamma}_{\alpha\beta} \cap \omega I\omega^{-1})$$

$\gamma = s_i\omega$

$$= \# I^{\gamma}_{\alpha\beta}/(I \cap \omega I\omega^{-1}) = \frac{\# I/I\cap\omega I\omega^{-1}}{\# I/I^{\gamma}_{\alpha\beta}} = q^{l(\omega)-1}$$

$$g^{s_i\omega}_{s_i,\omega} = \frac{|Is_iI/I|\,|I^{s_i\omega}_{s_i,\omega}\omega I/I|}{|Is_i\omega I/I|} = \frac{q^{l(s_i)}q^{l(\omega)-1}}{q^{l(s_i\omega)}} = q$$

$$g^{\omega}_{s_i,\omega} = \frac{|Is_iI/I|\,|I^{\omega}_{s_i,\omega}\omega I/I|}{|I\omega I/I|} = \frac{|Is_iI/I|\left(|I\omega I/I| - |I^{s_i\omega}_{s_i,\omega}\omega I/I|\right)}{|I\omega I/I|} = \frac{q^{l(s_i)}\left(q^{l(\omega)} - q^{l(\omega)-1}\right)}{q^{l(\omega)}} = q-1$$

When $l(s_i\omega)=l(\omega)-1$ and $i=0$, $\omega=\begin{pmatrix} \pi^e & \\ & \pi^{-e}\end{pmatrix}$, $e<0$ or $\omega=\begin{pmatrix} & \pi^e \\ \pi^{-e} & \end{pmatrix}$, $e<0$

$$I_{s_i,\omega}^{s_i\omega}=\begin{pmatrix}\mathcal{O} & \mathcal{O}\\ \mathfrak{p}^2 & \mathcal{O}\end{pmatrix},\quad I_{s_i,\omega}^{\omega}=\begin{pmatrix}\mathcal{O} & \mathcal{O}\\ \mathfrak{p}-\mathfrak{p}^2 & \mathcal{O}\end{pmatrix}.$$

$$\begin{aligned}
\# I_{s_i,\omega}^{s_i\omega}\omega I/I &= \# I_{\alpha\beta}^{\gamma}/(I_{\alpha\beta}^{\gamma}\cap\omega I\omega^{-1}) \qquad \gamma=s_i\omega\\
&= \# I_{\alpha\beta}^{\gamma}/(I\cap\omega I\omega^{-1})\\
&= \frac{\# I/I\cap\omega I\omega^{-1}}{\# I/I_{\alpha\beta}^{\gamma}}\\
&= q^{l(\omega)-1}
\end{aligned}$$

$$\begin{aligned}
g_{s_i,\omega}^{s_i\omega} &= \frac{|Is_iI/I|\,|I_{s_i,\omega}^{s_i\omega}\omega I/I|}{|Is_i\omega I/I|}\\
&= \frac{q^{l(s_i)}q^{l(\omega)-1}}{q^{l(s_i\omega)}}\\
&= q
\end{aligned}$$

$$\begin{aligned}
g_{s_i,\omega}^{\omega} &= \frac{|Is_iI/I|\,|I_{s_i,\omega}^{\omega}\omega I/I|}{|I\omega I/I|}\\
&= \frac{|Is_iI/I|\left(|I\omega I/I|-|I_{s_i,\omega}^{s_i\omega}\omega I/I|\right)}{|I\omega I/I|}\\
&= \frac{q^{l(s_i)}\left(q^{l(\omega)}-q^{l(\omega)-1}\right)}{q^{l(\omega)}}\\
&= q-1
\end{aligned}$$

In conclusion,

$$T_i * T_\omega = \begin{cases} T_{s_i\omega} & l(s_i\omega)=l(\omega)+1\\ q\,T_{s_i\omega}+(q-1)T_\omega & l(s_i\omega)=l(\omega)-1\end{cases}$$

4. Conclusion

$$\mathcal{H}(G,I) = \mathbb{Z}\{T_0, T_1\}$$

with

$$T_i * T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w)+1 \\ q\, T_{s_i w} + (q-1) T_w & l(s_i w) = l(w) - 1 \end{cases}$$

Bernstein presentation of $\mathcal{H}_q(G,I)$

To continue, we now work on $\mathbb{Z}[q^{\pm 1}]$-coefficient Iwahori Hecke algebra

$$\mathcal{H}_q(G,I) = \mathcal{H}(G,I) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}]$$
$$= \mathcal{H}(G,I) \otimes_{\mathbb{Z}} \mathbb{Z}[x]/(qx-1)$$

⚠ The map

$$X_*(T) \longrightarrow \mathcal{H}_q(G,I)^\times \qquad \lambda \longmapsto T_\lambda$$

is not a gp homomorphism, i.e.

$$\mathbb{Z}[q^{\pm 1}][X_*(T)] \longrightarrow \mathcal{H}_q(G,I) \qquad \lambda \longmapsto T_\lambda$$

is not a $\mathbb{Z}[q^{\pm 1}]$-alg homomorphism.

Instead, if we twist it (e.p. for non-dominant part), then the map

$$\theta: \mathbb{Z}[q^{\pm 1}][X_*(T)] \longrightarrow \mathcal{H}_q(G,I) \qquad \lambda \text{ dominant} \longmapsto q^{-\frac{l(\lambda)}{2}} T_\lambda$$
$$\lambda = \lambda_1 \lambda_2^{-1} \longmapsto q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1}$$

in usual reference, $\lambda = \lambda_1 - \lambda_2$ to indicate $X_*(T)$ is commutative.

is a well-defined $\mathbb{Z}[q^{\pm 1}]$-alg homomorphism.

Prop θ is a well-defined $\mathbb{Z}[q^{\pm 1}]$-alg homomorphism.

"Proof".

Well-defined: If $\lambda = \lambda_1 \lambda_2^{-1} = \lambda_1' \lambda_2'^{-1}$, then $\qquad \lambda_1, \lambda_2, \lambda_1', \lambda_2'$ dominant

- $\lambda_1 \lambda_2' = \lambda_1' \lambda_2 \Rightarrow l(\lambda_1) + l(\lambda_2') = l(\lambda_1') + l(\lambda_2)$
$$\Rightarrow q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} = q^{-\frac{l(\lambda_1') - l(\lambda_2')}{2}}$$
- $$T_{\lambda_1} T_{\lambda_2'}^{-1} = T_{\lambda_1} T_{\lambda_2'} T_{\lambda_2'}^{-1} T_{\lambda_2}^{-1} = T_{\lambda_1 \lambda_2'} T_{\lambda_2' \lambda_2}^{-1}$$
$$\parallel$$
$$T_{\lambda_1'} T_{\lambda_2'}^{-1} = T_{\lambda_1'} T_{\lambda_2} T_{\lambda_2}^{-1} T_{\lambda_2'}^{-1} = T_{\lambda_1' \lambda_2} T_{\lambda_2 \lambda_2'}^{-1}$$
- $$q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1} = q^{-\frac{l(\lambda_1') - l(\lambda_2')}{2}} T_{\lambda_1'} T_{\lambda_2'}^{-1}$$

multiplication. If $\lambda = \lambda_1 \lambda_2^{-1}$, $\mu = \mu_1 \mu_2^{-1}$, then $\lambda_1, \lambda_2, \mu_1, \mu_2$ dominant

- $\lambda\mu = (\lambda_1 \mu_1)(\lambda_2 \mu_2)^{-1}$
- $T_{\lambda_2} T_{\mu_1} = T_{\mu_1} T_{\lambda_2} \Rightarrow T_{\mu_1} T_{\lambda_2}^{-1} = T_{\lambda_2}^{-1} T_{\mu_1}$
- $$\begin{aligned} T_{\lambda_1 \mu_1} T_{\lambda_2 \mu_2}^{-1} &= T_{\lambda_1} T_{\mu_1} T_{\mu_2}^{-1} T_{\lambda_2}^{-1} \\ &= T_{\lambda_1} T_{\mu_1} T_{\lambda_2}^{-1} T_{\mu_2}^{-1} \\ &= T_{\lambda_1} T_{\lambda_2}^{-1} T_{\mu_1} T_{\mu_2}^{-1} \end{aligned}$$
- $q^{-\frac{l(\lambda_1\mu_1) - l(\lambda_2\mu_2)}{2}} T_{\lambda_1\mu_1} T_{\lambda_2\mu_2}^{-1} = q^{-\frac{l(\lambda_1) - l(\lambda_2)}{2}} T_{\lambda_1} T_{\lambda_2}^{-1} \; q^{-\frac{l(\mu_1) - l(\mu_2)}{2}} T_{\mu_1} T_{\mu_2}^{-1}$ ▨

⚠ People prefer to write multiplication in $X_*(T)$ as addition.
e.g. $\lambda = \lambda_1 - \lambda_2$, $\lambda = 3\varepsilon_{1,*}$
Be careful that these additions are not additions in $\mathbb{Z}[q^{\pm 1}][X_*(T)]$.
to avoid the clash of terminology, use multiplication symbol;
to make it easier to digest, use addition symbol.

Prop. (Bernstein presentation) For $\lambda \in X_*(T)$, we have

$$\theta(\lambda) * T_1 - T_1 * \theta(-\lambda) = (q-1) \frac{\theta(\lambda) - \theta(-\lambda)}{1 - \theta(-\varepsilon_{1,*})} \qquad (\star)$$

Proof 1. If λ, λ' satisfy $(\star)$, then $\lambda + \lambda'$ satisfies $(\star)$.

$$\begin{aligned}
&\theta(\lambda+\lambda') * T_1 - T_1 * \theta(-\lambda-\lambda') \\
&= \theta(\lambda)\left(\theta(\lambda')T_1 - T_1\theta(-\lambda')\right) + \left(\theta(\lambda)T_1 - T_1\theta(-\lambda)\right)\theta(-\lambda') \\
&= (q-1)\frac{\theta(\lambda)\left(\theta(\lambda') - \theta(-\lambda')\right)}{1-\theta(-\varepsilon_{1,*})} + (q-1)\frac{\left(\theta(\lambda) - \theta(-\lambda)\right)\theta(-\lambda')}{1-\theta(-\varepsilon_{1,*})} \\
&= (q-1)\frac{\theta(\lambda+\lambda') - \theta(-(\lambda+\lambda'))}{1-\theta(-\varepsilon_{1,*})}
\end{aligned}$$

2. $\lambda = 0$ satisfies $(\star)$. Obviously.

3. If λ satisfies $(\star)$, then $-\lambda$ satisfies $(\star)$

$$\begin{aligned}
&\theta(-\lambda) * T_1 - T_1 * \theta(\lambda) \\
&= \theta(-\lambda)\left(T_1\theta(-\lambda) - \theta(\lambda)T_1\right)\theta(\lambda) \\
&= \theta(-\lambda)\left((q-1)\frac{-\theta(\lambda)+\theta(-\lambda)}{1-\theta(-\varepsilon_{1,*})}\right)\theta(\lambda) \\
&= (q-1)\frac{\theta(-\lambda)-\theta(\lambda)}{1-\theta(-\varepsilon_{1,*})}
\end{aligned}$$

4. $\lambda = \varepsilon_{1,*}$ satisfies $(\star)$ Therefore, $(\star)$ is true for any $\lambda \in X_*(T)$.

$$\mathrm{RHS} = (q-1)\frac{\theta(\lambda)-\theta(-\lambda)}{1-\theta(-\lambda)} = (q-1)(1+\theta(\lambda))$$

$$T_i^2 = (q-1)T_i + q \quad \Rightarrow \quad qT_i^{-1} = T_i - (q-1)$$

$$\begin{aligned}
\mathrm{LHS} &= q^{-1}T_\lambda T_1 - qT_1T_\lambda^{-1} \\
&= q^{-1}\left[T_1T_0T_1 - T_1(qT_0^{-1})(qT_1^{-1})\right] \\
&= q^{-1}\left[T_1T_0T_1 - T_1(T_0-(q-1))(T_1-(q-1))\right] \\
&= q^{-1}\left[(q-1)T_1(T_0+T_1) - (q-1)^2T_1\right] \\
&= \frac{q-1}{q}\left[T_1T_0 + T_1^2 - (q-1)T_1\right] \\
&= \frac{q-1}{q}(T_1T_0 + q) \\
&= (q-1)(1+\theta(\lambda))
\end{aligned}$$

In conclusion, denote

$$\Theta := \langle \theta(\lambda) \,|_{\lambda \in X_*(T)} \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} = \theta(\mathbb{Z}[q^{\pm 1}][X_*(T)]) \subseteq \mathcal{H}_q(G, I)$$
$$= X_*(T) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}]$$
$$= \langle \theta(\varepsilon_{1,*}),\ \theta(-\varepsilon_{1,*}) \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}}$$
$$= \mathbb{Z}[q^{\pm 1}][\lambda_0^{\pm 1}] \qquad \lambda_0 = \varepsilon_{1,*}$$
$$\mathcal{H}_q(W_f) := \mathcal{H}(W_f) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1}]$$

Then

$$\mathcal{H}_q(G, I) \cong \Theta \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathcal{H}_q(W_f) \qquad \text{as left } \Theta\text{-module}$$
$$\cong \bigoplus_{w \in W_f} \Theta \cdot T_w$$

Guess. $\mathcal{H}_q(G, I) \cong \langle \theta(\varepsilon_{1,*})^{\pm 1}, T_1 \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} \subseteq \mathrm{End}_{\mathbb{Z}[q^{\pm 1}]\text{-mod}}(\Theta)$

where

$$T_1 * \theta(\lambda) = (q-1)\left(\frac{\theta(\lambda)}{1 - \theta(-\varepsilon_{1,*})} + \frac{\theta(-\lambda)}{1 - \theta(\varepsilon_{1,*})} \right) \qquad \theta(\lambda) \in \Theta$$

Center of $\mathcal{H}_q(G, I)$

Q. What is the center of $\mathcal{H}_q(G, I)$?

A. It is

$$\Theta^{W_f} := \langle \theta(\varepsilon_{1,*}) + \theta(-\varepsilon_{1,*}) \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}}$$
$$= \mathbb{Z}[q^{\pm 1}][\lambda_0 + \lambda_0^{-1}] \qquad \lambda_0 = \varepsilon_{1,*}$$

I believe that people can get this result by direct computation.
See [Bump, Theorem 23] for a proof.