Eine Woche, ein Beispiel

4.27. homomorphism between Jacobians

[2025.04.20] provides us with many examples and references, and here we do things more theoretically.

Idea:

$$\mathrm{Jac}(C) = \underbrace{H^0(C;\omega_C)^*}_{\substack{\text{linear part}\\ \text{coherent}}} \Big/ \underbrace{H_1(C;\mathbb{Z})}_{\substack{\text{lattice part}\\ \text{constant}}}$$

To understand Jac(C), we need to understand these two parts separately.

For a morphism between two sm proj curves $/\mathbb{C}$:

$$f : \widehat{C} \longrightarrow C$$

$$\begin{aligned} \mathrm{Nm}_{f,a} &: H^0(\widehat{C}; \omega_{\widehat{C}})^* \to H^0(C; \omega_C)^* \\ \mathrm{Nm}_{f,r} &: H_1(\widehat{C}; \mathbb{Z}) \to H_1(C; \mathbb{Z}) \\ \mathrm{Nm}_f &: \mathrm{Jac}(\widehat{C}) \to \mathrm{Jac}(C) \end{aligned} \qquad \begin{aligned} (f^*)_a &: H^0(C; \omega_C)^* \to H^0(\widetilde{C}; \omega_{\widetilde{C}})^* \\ (f^*)_r &: H_1(C; \mathbb{Z}) \to H_1(\widetilde{C}; \mathbb{Z}) \\ f^* &: \mathrm{Jac}(C) \to \mathrm{Jac}(\widetilde{C}) \end{aligned}$$

cohom pullback

sheaf origin

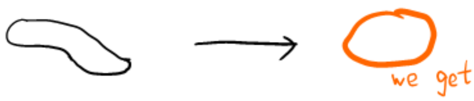
$$\begin{array}{ccc} \omega_{\widetilde{C}} & \longleftarrow & f^! \omega_C \\ f_! \pi^!_{\widetilde{C}} \mathbb{Z} & \longrightarrow & \pi^!_C \mathbb{Z} \end{array} \qquad \begin{array}{ccc} \omega_C & \longleftarrow & f_! \omega_{\widetilde{C}} \\ \underline{\mathbb{Z}}_C & \longrightarrow & f_* \underline{\mathbb{Z}}_{\widetilde{C}} \end{array}$$

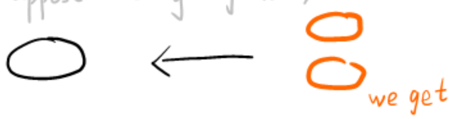
$= f_* \omega_{\widetilde{C}}$

$$g(f(\omega))\, d(f(\omega)) \longleftarrow\!\!\mid\; g(z)\,dz \qquad \sum_{f(\omega)=z} g(\omega)\,dz \longleftarrow\!\!\mid\; g(\omega)\,d\omega$$

suppose locally $f^*(dz) = d\omega$

geometric picture





$$[q] \longmapsto [f(q)] \qquad [p] \longmapsto \sum_{f(q)=p} [q]$$

Ex. Show that

$$\mathrm{Nm}_f \circ f^* = [\deg f] : \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C)$$

Also,

$$\begin{aligned} \mathrm{Nm}_{f,a} \circ (f^*)_a &= \deg f \cdot \mathrm{Id}_{H^0(C;\omega_C)^*} \\ \mathrm{Nm}_{f,r} \circ (f^*)_r &= \deg f \cdot \mathrm{Id}_{H_1(C;\mathbb{Z})} \end{aligned}$$

Hint: use Poincaré duality.