Eine Woche, ein Beispiel

8.21 equivariant cohomology of $\mathbb{P}^1$

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)
Other references will be add soon.

1. notations and warnings
2. result
3. computation of completion in practice
4. pt & $\mathcal{B}$
5 Euler class

1. notations and warnings

In this document,

$$GL_2 = GL_2(\mathbb{C}) \qquad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset GL_2 \qquad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset GL_2 \text{ or } SL_2$$

$$SL_2 = SL_2(\mathbb{C}) \qquad \mathbb{C}^\times = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset SL_2 \qquad \mathbb{P}^1 = \mathbb{P}^1(\mathbb{C})$$

$$K_0^G(X) := K_0(\mathrm{Coh}^G(X)) \qquad H_G^*(X;\mathbb{Q}) := H^*(EG \times^G X;\mathbb{Q})$$

$$R(G) := K_0^G(pt) = \mathrm{Rep}(G) \qquad S(G) := H_G^*(pt;\mathbb{Q}) = H^*(BG;\mathbb{Q})$$

$$K_0^G(X)^\wedge_I := \varprojlim_n K_0^G(X)/I^n \qquad HP_G^0(X;\mathbb{Q}) := \prod_{n=0}^{+\infty} H_G^n(X;\mathbb{Q}) = H_G^*(X;\mathbb{Q})^\wedge_{I'}$$

To avoid confusion, we don't consider any convolution structure in this document.

we don't consider $G \times \mathbb{C}^\times$-action either

($\mathbb{C}^\times$ is already occupied as a maximal torus of SL_2)

2. result

This time we are not so ambitious. For example, we don't fill in

$$K_0^B(\mathcal{B}\times\mathcal{B}) \cong K_0^G(\mathcal{B}\times\mathcal{B}\times\mathcal{B}) \cong R(T)\otimes_{R(G)}R(T)\otimes_{R(G)}R(T)$$

just because the result is too long.

We don't want to use these symbols(like x,y,z) in later documents either. If you want to fix a notation, please use the notations in https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.10.23_notation_K%5EG(St).pdf

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Z}[y+y^{-1}]$	$\mathbb{Z}[z^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2]/((z_1-z_2)(z_1-z_2^{-1}))$
	B	$\mathbb{Z}[y^{\pm1}]$	$\mathbb{Z}[y^{\pm1},z]/(z-y)(z-y^{-1})$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1,z_2]/((z_1-1)^2,(z_2-1)^2)$
$G=GL_2$	GL_2	$\mathbb{Z}[y_1+y_2,y_1y_2,\frac{1}{y_1y_2}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1}]$	$\mathbb{Z}[z_1^{\pm1},z_2^{\pm1},z_1']/((z_1'-z_1)(z_1'-z_2))$
	B	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1}]$	$\mathbb{Z}[y_1^{\pm1},y_2^{\pm1},z_1]/((z_1-y_1)(z_1-y_2))$	
	Id	$\mathbb{Z}$	$\mathbb{Z}[z]/(z-1)^2$	$\mathbb{Z}[z_1',z_2']/((z_1'-1)^2,(z_2'-1)^2)$
$G=SL_n$ or GL_n	G	$R(G)$	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w'\in W} R(G)[\overline{\Omega}_{w'}]^G$
	B	$R(T)$	$R(T)\otimes_{R(G)}R(T)$ $\bigoplus_{w\in W} R(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W} R(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Z}$	$\bigoplus_{w\in W} \mathbb{Z}\cdot[\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W} \mathbb{Z}\ [\overline{\Omega}_{w,w'}]$

$K_0^-(-)$		pt	$\mathcal{B}$ $T^*\mathcal{B}$	$\mathcal{B}\times\mathcal{B}$
$G=SL_2$	SL_2	$\mathbb{Q}[b^2]$	$\mathbb{Q}[e]$	$\mathbb{Q}[e_1,e_2]/(e_2^2-e_1^2)$
	B	$\mathbb{Q}[b]$	$\mathbb{Q}[b,e]/(e^2-b^2)$	
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1,e_2]/(e_1^2,e_2^2)$
$G=GL_2$	GL_2	$\mathbb{Q}[b_1+b_2,b_1b_2]$	$\mathbb{Q}[e_1,e_2]$	$\mathbb{Q}[e_1,e_2,e_1']/((e_1'-e_1)(e_1'-e_2))$
	B	$\mathbb{Q}[b_1,b_2]$	$\mathbb{Q}[b_1,b_2,e_1]/((e_1-b_1)(e_1-b_2))$	$e_2'=e_1+e_2-e_1'$
	Id	$\mathbb{Q}$	$\mathbb{Q}[e]/(e^2)$	$\mathbb{Q}[e_1',e_2']/(e_1'^2,e_2'^2)$
$G=SL_n$ or GL_n	G	$S(G)$	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w'\in W} S(G)[\overline{\Omega}_{w'}]^G$
	B	$S(T)$	$S(T)\otimes_{S(G)}S(T)$ $\bigoplus_{w\in W} S(T)[\overline{\Omega}_w]^T$	$\bigoplus_{w,w'\in W} S(T)[\overline{\Omega}_{w,w'}]^T$
	Id	$\mathbb{Q}$	$\bigoplus_{w\in W} \mathbb{Q}\ [\overline{\Omega}_w]$	$\bigoplus_{w,w'\in W} \mathbb{Q}\ [\overline{\Omega}_{w,w'}]$