Eine Woche, ein Beispiel

11.3 normalization of a plane curve

In this document, we try to solve the exercise in [Ar85, I. Ex A-6]. After the discussion with Vincent Ariksoy, I want to record the proof here.

[Ar85]: Arbarello, E., M. Cornalba, P. A. Griffiths and J. Harris. Geometry of Algebraic Curves Volume I. Grundlehren Math. Wiss. Springer, Cham, 1985.
[Vakil]: Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016

For the general theory please read [2025.05.11].

Ex. For the plane curve

$$y^3 = x^5 - 1 \quad \text{in } \mathbb{C}^2$$

1) compactify it by blowing up, get C.
2) compute $g(C)$.
3) write a basis of $H^0(C, \omega_C)$.
4) describe the canonical map

$$\phi : C \hookrightarrow \mathbb{P}H^0(C, \omega_C)^* \cong \mathbb{P}^3$$

5) verify that ϕ is an embedding, and C is non-hyperelliptic.
6) determine the equations of $\phi(C)$.

1). Projection.

$$X: x_1^5 - x_3^5 - x_2^3 x_3^2 = 0$$

$[x_1 : x_2 : x_3]$

$\mathcal{U}_1 = \{[1 : x_2 : x_3]\}$

$\mathcal{U}_2 = \{[x_1 : 1 : x_3]\}$

$\mathcal{U}_3 = \{[x_1 : x_2 : 1]\}$

On $\mathcal{U}_1$: $1 - x_3^5 - x_2^3 x_3^2 = 0$ smooth

On $\mathcal{U}_2$: $x_1^5 - x_3^5 - x_3^2 = 0$ singular at $[0:1:0]$

On $\mathcal{U}_3$: $x_1^5 - 1 - x_2^3 = 0$ smooth

Blow up $\mathcal{U}_2$ at $[0:1:0]$:

$[y_1 : y_3],\ (x_1, x_3)$

$B_1 = \{[1 : y_3],\ (x_1, x_1 y_3)\}$

$B_2 = \{[y_1 : 1],\ (x_3 y_1, x_3)\}$

On B_1: $x_1^2 (x_1^3 - x_1^3 y_3^5 - y_3^2) = 0$

$\widetilde{X}|_{B_1} = Z(x_1^3 - x_1^3 y_3^5 - y_3^2)$ singular at $([1:0],(0,0))$

On B_2: $x_3^2 (x_3^3 y_1^5 - x_3^3 - 1) = 0$

$\widetilde{X}|_{B_2} = Z(x_3^3 y_1^5 - x_3^3 - 1)$ smooth

$\widetilde{X}|_{\widetilde{\mathcal{U}}_2} = Z(y_1^2 x_1^3 - y_3^2 x_3^3 - y_3^2)$ is still not smooth.

Blow up B_1 at $([1:0],(0,0))$.

$[z_1 : z_3],\ (x_1, y_3)$

$B_1' = \{[1 : z_3],\ (x_1, x_1 z_3)\}$

$B_2' = \{[z_1 : 1],\ (y_3 z_1, y_3)\}$

On B_1': $x_1^2 (x_1 - x_1^6 z_3^5 - z_3^2) = 0$

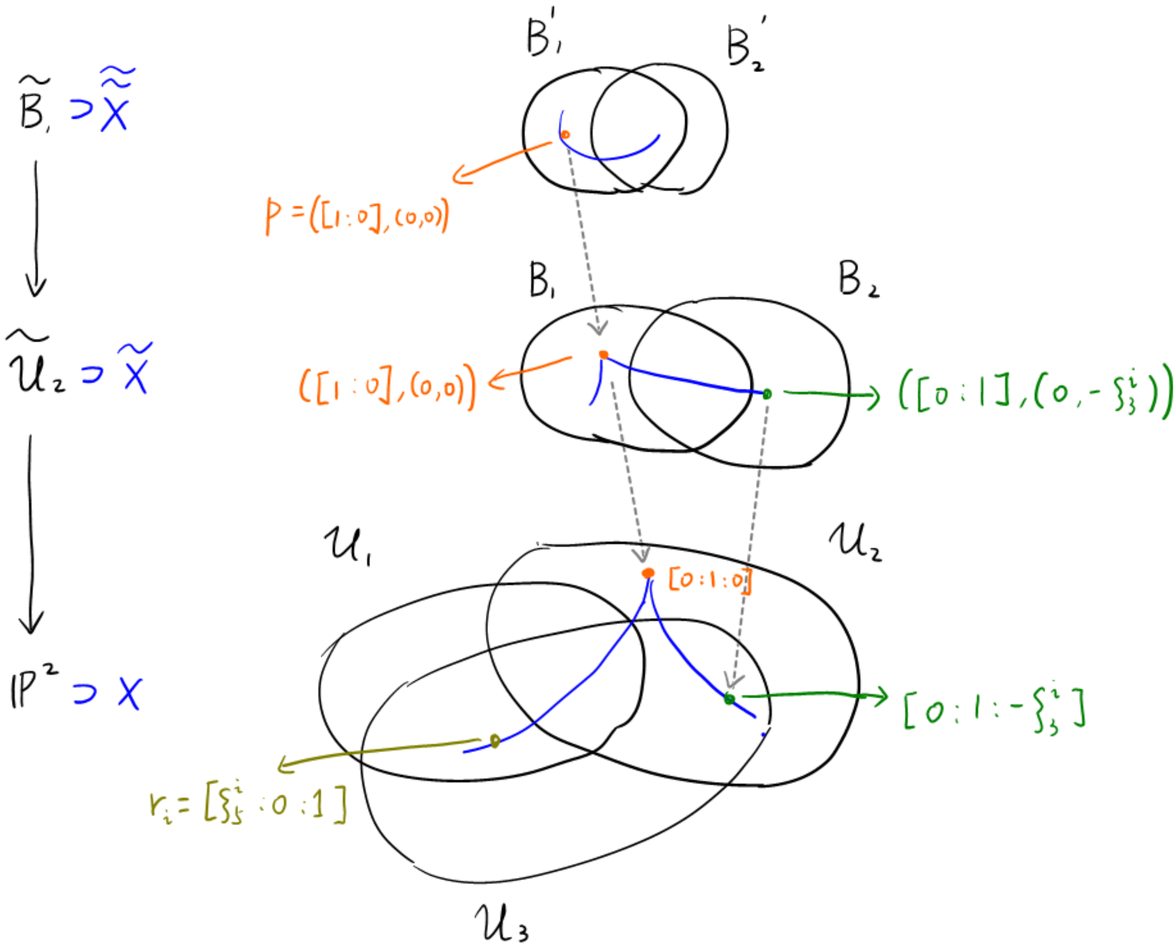
$\widetilde{\widetilde{X}}|_{B_1'} = Z(x_1 - x_1^6 z_3^5 - z_3^2)$ smooth

On B_2': $y_3^2 (y_3 z_1^3 - y_3^6 z_1^3 - 1) = 0$

$\widetilde{\widetilde{X}}|_{B_2'} = Z(y_3 z_1^3 - y_3^6 z_1^3 - 1)$ smooth

$\widetilde{\widetilde{X}}|_{\widetilde{B}_1} = Z(z_1^2 x_1 - z_1^2 x_1 y_3^5 - z_3^2)$ is finally smooth.

Apart from finite points (marked below), the curves always lie in the intersections.

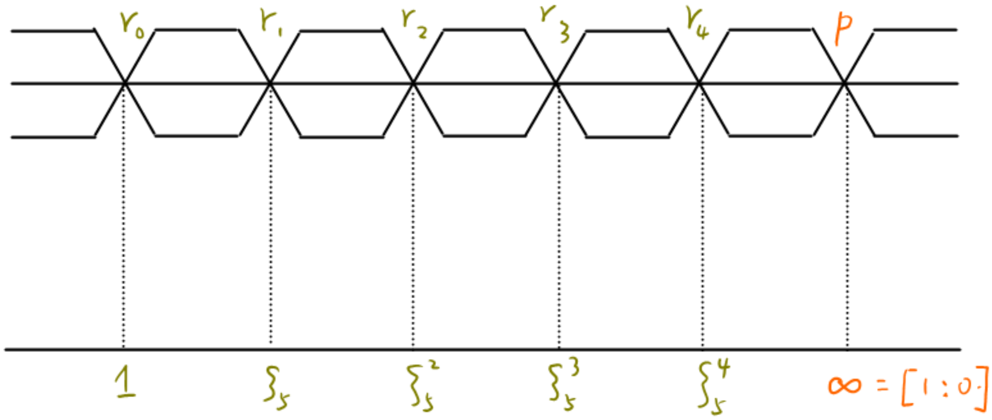


2). Consider

$$\begin{tikzcd} C \arrow[d] \\ X \subseteq \mathbb{P}^2 \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

$$[x_1 : x_2 : x_3] \longmapsto [x_1 : x_3] \qquad \text{when } x_2 \neq 0$$

$$(x, y) \longmapsto x$$



$$2g(C) - 2 = 3 \cdot (-2) + 6 \cdot (3-1) \Rightarrow g(C) = 4$$

3) Lemma. View dx as a meromorphic section on ω_C, one has

$$\mathrm{div}(dx) = \sum_{i=0}^{4} 2r_i - 4p$$

combining with the fact that

$$\mathrm{div}(x-\xi_5) = 3r_1 - 3p$$
$$\mathrm{div}(y) = \sum_{i=0}^{4} r_i - 5p$$

one gets

$$H^0(C, \omega_C) = \left\langle \frac{dx}{y}, \frac{dx}{y^2}, \frac{(x-\xi_5)dx}{y^2}, \frac{(x-\xi_5)^2 dx}{y^2} \right\rangle$$

By checking the order at p, these differential forms are linear independent.

Proof of Lemma

https://math.stackexchange.com/questions/2981416/a-basis-for-the-holomorphic-differentials-of-a-hyper-elliptic-riemann-surface
https://math.stackexchange.com/questions/820927/holomorphic-differentials-on-a-non-singular-curve

dx is holomorphic and has no zero pt in $C \cap (\mathcal{U}_2 - \{[0:1:0]\})$.

In $\mathcal{U}_3$, $dx = d\left(\frac{x_1}{x_3}\right) = dx_1$, and

$$5x_1^4 dx_1 = 3x_2^2 dx_2$$

basis near r_i

$$\Rightarrow dx_1 = \frac{3x_2^2}{5x_1^4} dx_2$$

$$\Rightarrow \mathrm{ord}_{r_i}(dx) = 2 \qquad \forall i \in \{0, \dots, 4\}$$

In B_1', $dx = d\left(\frac{x_1}{x_3}\right) = d\left(\frac{y_1}{y_3}\right) = d\left(\frac{1}{y_3}\right) = d\left(\frac{1}{x_1 z_3}\right)$, and

$$(1 - 6x_1^5 z_3^5)\, dx_1 = (5x_1^6 z_3^4 - 2z_3)\, dz_3$$

$\uparrow$ basis near p

$$\Rightarrow d\left(\frac{1}{x_1 z_3}\right) = -\frac{1}{x_1^2 z_3^2}\,(x_1 dz_3 + z_3 dx_1)$$

$$= -\frac{1}{x_1^2 z_3^2}\left(x_1 dz_3 + \frac{z_3(5x_1^6 z_3^4 - 2z_3)}{1 - 6x_1^5 z_3^5}\, dz_3\right)$$

$$= -\frac{1}{x_1^2 z_3^2}\,\frac{x_1 - 6x_1^6 z_3^5 + 5x_1^6 z_3^5 - 2z_3^2}{1 - 6x_1^5 z_3^5}\, dz_3$$

$$\sim -\frac{1}{x_1 z_3^2}\, dz_3$$

$$\sim -\frac{1}{z_3^4}\, dz_3$$

$\Rightarrow \operatorname{ord}_p(dx) = -4.$ $\square$

4) The embedding is given by

$$\begin{aligned}
\varphi:\quad & C \longrightarrow \mathbb{P}^3 \\
& (x, y) \longmapsto [y : 1 : (x - \zeta_5) : (x - \zeta_5)^2] \\
& [x_1 : x_2 : x_3] \longmapsto [x_2 x_3 : x_3^2 : (x_1 - \zeta_5 x_3) x_3 : (x_1 - \zeta_5 x_3)^2] \\
\text{on } \mathcal{U}_2:\quad & [x_1 : 1 : x_3] \longmapsto [x_3 : x_3^2 : (x_1 - \zeta_5 x_3) x_3 : (x_1 - \zeta_5 x_3)^2] \\
\text{on } B_1:\quad & ([1 : y_3], (x_1, x_1 y_3)) \longmapsto [y_3 : x_1 y_3^2 : x_1 y_3 (1 - \zeta_5 y_3) : x_1 (1 - \zeta_5 y_3)^2] \\
\text{on } B_1':\quad & ([1 : z_3], (x_1, x_1 z_3)) \longmapsto [x_1^2 z_3 : x_1^4 z_3^2 : x_1^3 z_3 (1 - \zeta_5 x_1 z_3) : x_1^2 (1 - \zeta_5 x_1 z_3)^2] \\
& x_3 = x_1^2 z_3 \qquad\qquad = [z_3 : x_1^2 z_3^2 : x_1 z_3 (1 - \zeta_5 x_1 z_3) : (1 - \zeta_5 x_1 z_3)^2] \\
\text{e.p.}\quad & p \longmapsto [0 : 0 : 0 : 1]
\end{aligned}$$

5). If $\varphi(p_0) = [t_1 : t_2 : t_3 : t_4]$, $t_2 \neq 0$, then

$$p_0 = (x, y) = \left(\frac{t_3 + \zeta_5 t_2}{t_2}, \frac{t_1}{t_2}\right)$$

is uniquely determined.

6). Equation

$$\begin{cases} t_3^2 = t_2 t_4 \\ t_1^3 t_2^2 = (t_3 + \zeta_5 t_2)^5 - t_2^5 \end{cases} \qquad (*)$$

multiplicity 2

cut out $\phi(C)$ and $\{[t_1 : 0 : 0 : t_4]\} \cong \mathbb{CP}^1$.
To remove $\mathbb{CP}^1$, one should add another equation.

https://math.stackexchange.com/questions/3387244/decomposition-of-ideal-into-intersection-of-prime-ideals

code:

```
K.<zeta> = CyclotomicField(5)
R.<x,y,z,w> = PolynomialRing(K)
J = R*[ z^2 - y*w,    x^2*y^2 - (z+zeta*y)^5-y^5]
for Q in J.primary_decomposition():
    print("Ideal generated by", Q.gens())
```

answer:

Ideal generated by [z^2 - y*w, y^2]
Ideal generated by [z^2 - y*w, 2*y^3 + (-5*zeta^3 - 5*zeta^2 - 5*zeta - 5)*y^2*z + (10*zeta^3)*y^2*w + (10*zeta^2)*y*z*w + (5*zeta)*y*w^2 + z*w^2 - x^2]