

Eine Woche, ein Beispiel. 5.7. Lie group

Temporary roadmap of Lie Group

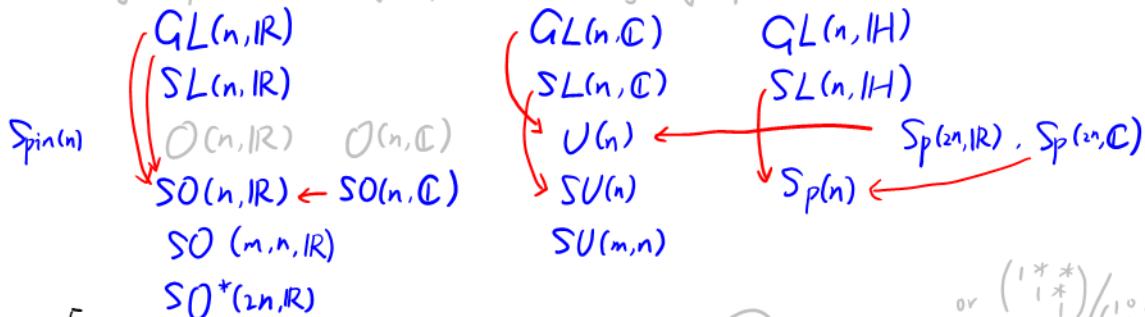
Why is it so difficult to write examples for Lie group?

- Plenty of examples, even restricted to matrix Lie groups:

- List of examples. [https://en.wikipedia.org/wiki/Table\\_of\\_Lie\\_groups](https://en.wikipedia.org/wiki/Table_of_Lie_groups)

[Some important subgroups are not written in the list,  
e.g. parabolic subgroups, Heisenberg group, etc.]

$G \rightarrow K$  maximal cpt connected subgroup



[For examples of non matrix Lie group  $(\widetilde{SL(2, IR)}, M_p(2n, IR))$ , see

<https://math.stackexchange.com/questions/206117/is-there-a-non-matrix-lie-group>

$$\text{or } \begin{pmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{pmatrix} / \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

On the contrary, every f.d. Lie alg can be viewed as a matrix Lie algebra by Ado's theorem, see <http://www.math.ubc.ca/~reichst/Ado's-Theorem.pdf>.

- These groups are closely connected to each other.

e.g. the Whitehead tower:

$$\rightarrow \text{Fivebrane}(n) \rightarrow \text{String}(n) \rightarrow \text{Spin}(n) \rightarrow \text{SO}(n) \rightarrow \text{O}(n)$$

- Plenty of special phenomena on low-dimensional case:

e.g.  $GL(1, IR) \cong IR^*$  is not connected

e.g.  $SU(2) \cong S^3$

e.g.  $SO(4, IR)$  is not simple

$\downarrow$   
 $SO(3, IR) \cong RP^3$   
diff  
not as variety

e.g. <https://www.zhihu.com/question/47264301/answer/320472431>

- Fruitful structures on examples

- important subgroups

e.g. maximal torus, unipotent subgroup

Borel subgroup, parabolic subgroup.

discrete subgroup.

radical, unipotent radical, derived group

$$G = [G, G] R(G)$$

\* centralizer & normalizer of subgroups

This article may give a classification of the cocompact subgroup:  
<https://people.uleth.ca/~dave.morris/papers/cocompact.pdf>

# One page for defining these groups by pasting from

Anthony W. Knapp - Representation Theory of Semisimple Groups An Overview Based on Examples

*Complex groups:*

$$\mathrm{GL}(n, \mathbb{C}) = \{\text{nonsingular } n\text{-by-}n \text{ complex matrices}\}$$

$$\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \det g = 1\}$$

$$\mathrm{SO}(n, \mathbb{C}) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid gg^{\mathrm{tr}} = 1\}$$

$$\mathrm{Sp}(n, \mathbb{C}) = \left\{ g \in \mathrm{SL}(2n, \mathbb{C}) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$

*Compact groups:*

~~$$\mathrm{SO}(n, \mathbb{R}) = \mathrm{SO}(n) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g^{\mathrm{tr}} g = 1 \text{ and } g \text{ has real entries}\}$$~~

~~$$\mathrm{U}(n) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{g}^{\mathrm{tr}} g = 1\}$$~~

~~$$\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\}$$~~

~~$$\mathrm{Sp}(n) = \left\{ g \in \mathrm{U}(2n) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}.$$~~

*Real noncompact groups:* We list  $G$ ,  $\mathfrak{g}$ ,  $K$ , and  $\mathfrak{k}$  for the noncomplex non-compact classical groups.

$G$	$\mathfrak{g}$	$K$	$\mathfrak{k}$
$\mathrm{SL}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R})$	$\mathrm{SO}(n)$	$\mathfrak{so}(n)$
$\mathrm{SL}(n, \mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H})$	$\mathrm{Sp}(n)$	$\mathfrak{sp}(n)$
$\mathrm{SO}_0(m, n)$	$\mathfrak{so}(m, n)$	$\mathrm{SO}(m) \times \mathrm{SO}(n)$	$\mathfrak{so}(m) \oplus \mathfrak{so}(n)$
$\mathrm{SU}(m, n)$	$\mathfrak{su}(m, n)$	$\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$	$\mathfrak{s}(\mathrm{u}(m) \oplus \mathrm{u}(n))$
$\mathrm{Sp}(m, n)$	$\mathfrak{sp}(m, n)$	$\mathrm{Sp}(m) \times \mathrm{Sp}(n)$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(n)$
$\mathrm{Sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$
$\mathrm{SO}^*(2n)$	$\mathfrak{so}^*(2n)$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$ .

$\mathrm{SL}(n, \mathbb{R})$  and  $\mathrm{SL}(n, \mathbb{H})$  refer to matrices of determinant one with real and quaternion entries, respectively.  $\mathrm{SO}_0(m, n)$ ,  $\mathrm{SU}(m, n)$ , and  $\mathrm{Sp}(m, n)$  are the linear isometry groups for the Hermitian form

$$|z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2$$

defined over  $\mathbb{R}$ ,  $\mathbb{C}$ , and  $\mathbb{H}$ , respectively, with the subscript “0” referring to the identity component. The group  $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$  for  $\mathrm{SU}(m, n)$  is the subgroup of  $\mathrm{U}(m) \times \mathrm{U}(n)$  of matrices of determinant one.

The group  $\mathrm{Sp}(n, \mathbb{R})$  is the subgroup of real matrices in  $\mathrm{Sp}(n, \mathbb{C})$  and can be conjugated by a unitary matrix so as to become

$$\{g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} J g = J\};$$

then  $\mathrm{SO}^*(2n)$  is the analogous group

$$\left\{ g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \right\}.$$

- decomposition: not easy to prove, but still of great significance.  
[https://en.wikipedia.org/wiki/Lie\\_group\\_decomposition](https://en.wikipedia.org/wiki/Lie_group_decomposition)
  - e.g. Jordan - Chevalley decomposition
  - group, mfld, variety, etc. Common questions.

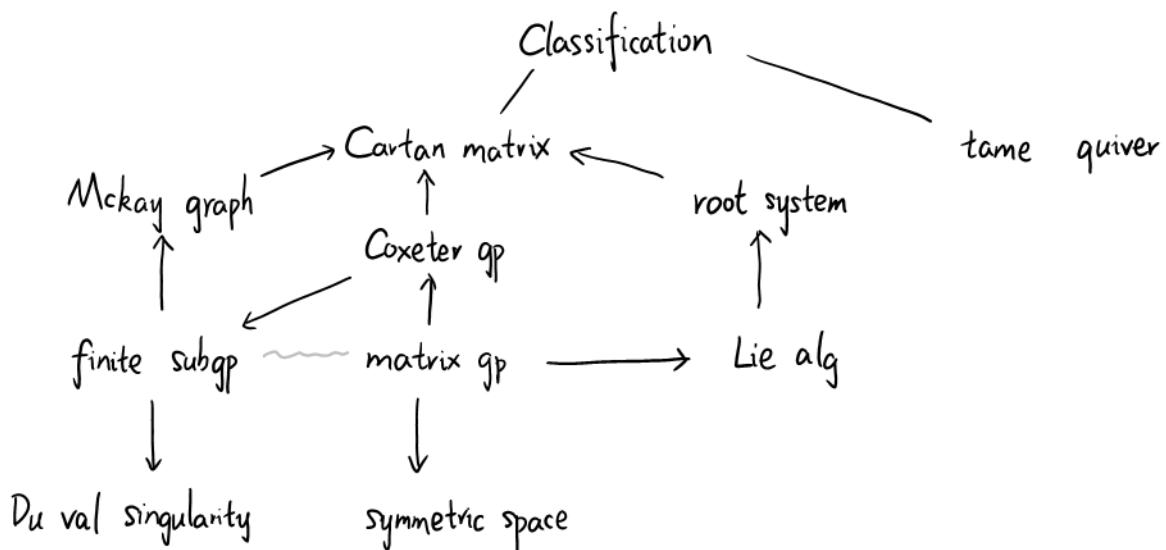
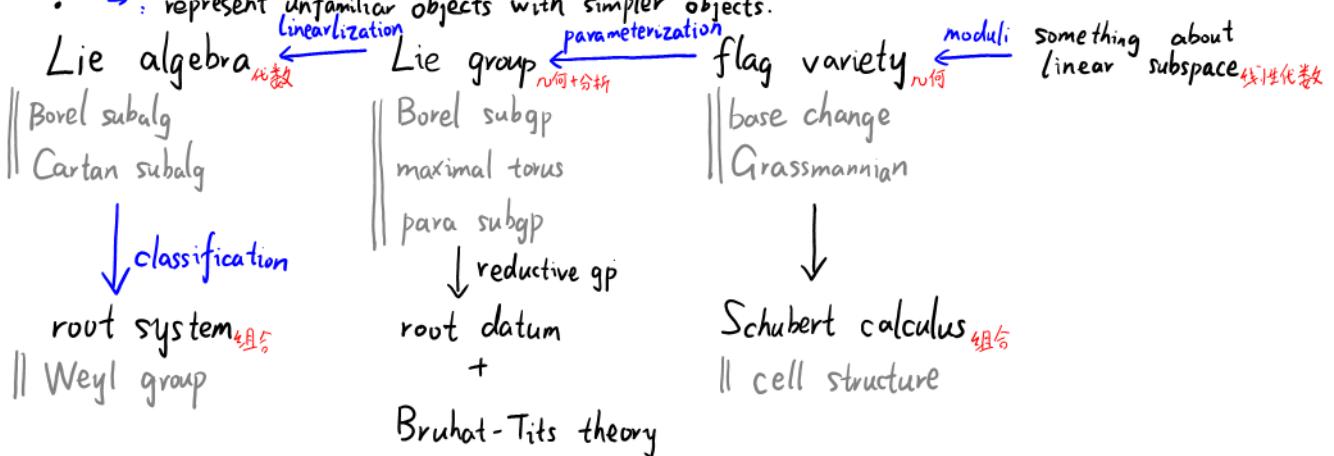
} See a thorough discussion in [23.03.26]

- Hopf algebra structure of homology & cohomology group under mild requirements.

- A large amount of related theories.

- structure theory of Lie group & Lie algebras.

- “ $\rightarrow$ ”: represent unfamiliar objects with simpler objects.



- representation theory
  - \* group /alg , f.d / inf.d
  - \* highest weight theory
  - \* Schul - Weyl duality
  - \* Borel - Weil - Bott theorem
  - \* geometric representation theory
  - \* Langlands program
- algebraic topology, Bott periodicity
- algebraic groups.

<https://math.stackexchange.com/questions/2748772/what-is-the-real-representation-theory-of-so3>

<https://mathoverflow.net/questions/235917/roadmap-to-geometric-representation-theory-leading-to-langlands>

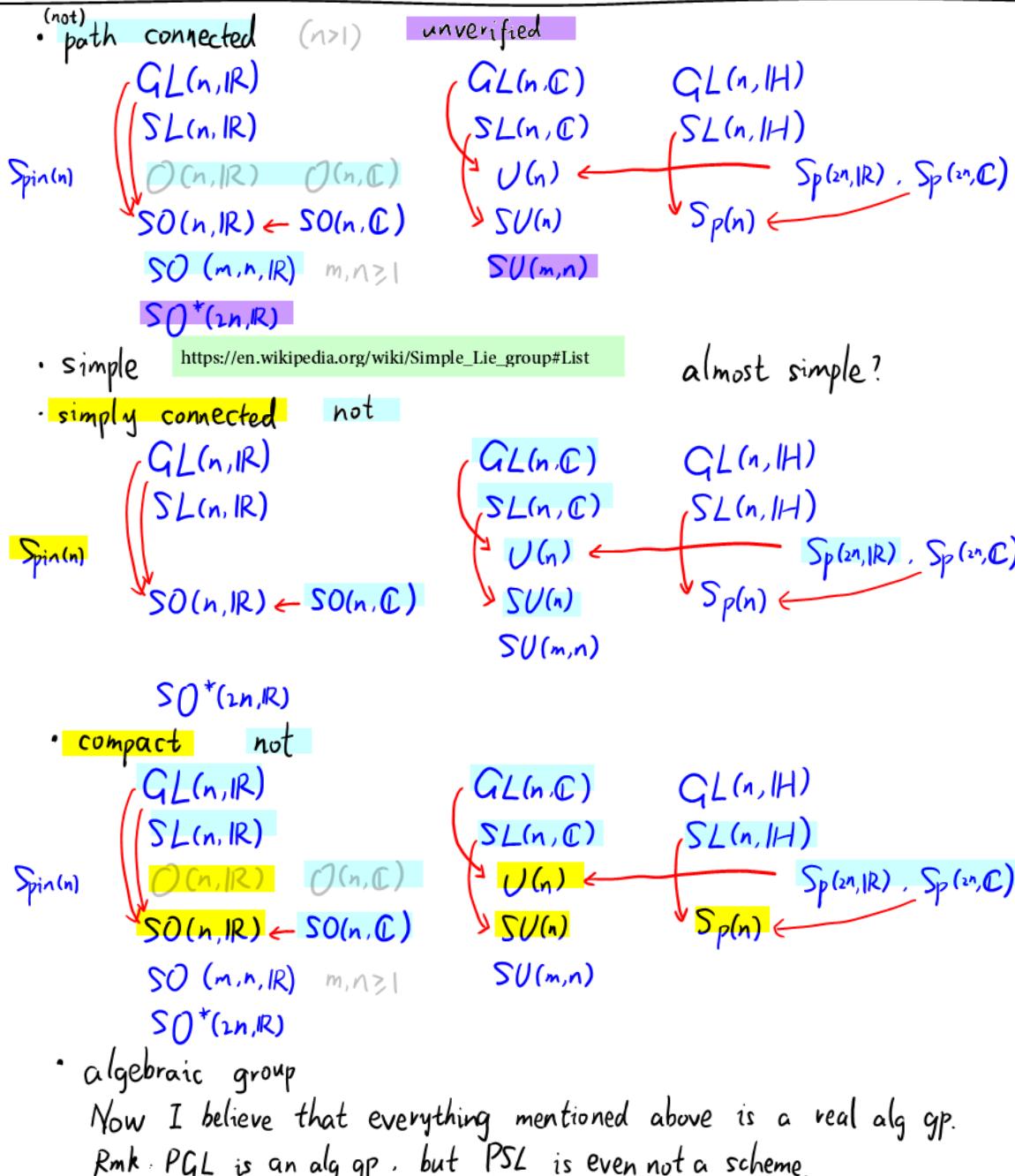
• ... see [https://en.wikipedia.org/wiki/List\\_of\\_Lie\\_groups\\_topics](https://en.wikipedia.org/wiki/List_of_Lie_groups_topics)

<http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html>

Basic knowledge which has been well-written in [Naive Lie Theory].

- canonical maximal tori
- center

- (1)  $Z(\mathrm{SO}(2m)) = \{\pm 1\}$ .
- (2)  $Z(\mathrm{SO}(2m+1)) = \{1\}$ .
- (3)  $Z(\mathrm{U}(n)) = \{\omega \mathbf{1} : |\omega| = 1\}$ .
- (4)  $Z(\mathrm{SU}(n)) = \{\omega \mathbf{1} : \omega^n = 1\}$ .
- (5)  $Z(\mathrm{Sp}(n)) = \{\pm 1\}$ .



In alg top: homotopy groups:

$\text{typo: } \pi_*(U(n)) = \mathbb{Z}$

see

<http://felix.physics.sunysb.edu/~abanov/Teaching/Spring2009/Notes/abanov-cpA1-upload.pdf>

more about  $\pi_i(SO(n, \mathbb{R}))$ :

<https://ncatlab.org/nlab/show/orthogonal+group#HomotopyGroups>

more about  $\pi_i(G)$  (in Chinese):

<https://www.zhihu.com/question/443652195>

more about  $\pi_n(G)$ :

<https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups>

$\pi_i(SO(m, n))$ :

<https://math.stackexchange.com/questions/4126129/homotopy-group-of-op-q>

my try: Cartan decomposition  $\Rightarrow$  only consider cpt Lie group.

fibration + low-dimensional understanding, + Bott periodicity

e.g. •  $O(1) = S^0$ , a two-point discrete space

$U(n) \cong SU(n) \times S^1$

•  $SO(1) = \{1\}$

•  $SO(2) \cong S^1$

•  $SO(3) \cong RP^3$  [3]

•  $SO(4)$  is doubly covered by  $SU(2) \times SU(2) = S^3 \times S^3$ .

[https://en.wikipedia.org/wiki/Orthogonal\\_group](https://en.wikipedia.org/wiki/Orthogonal_group)

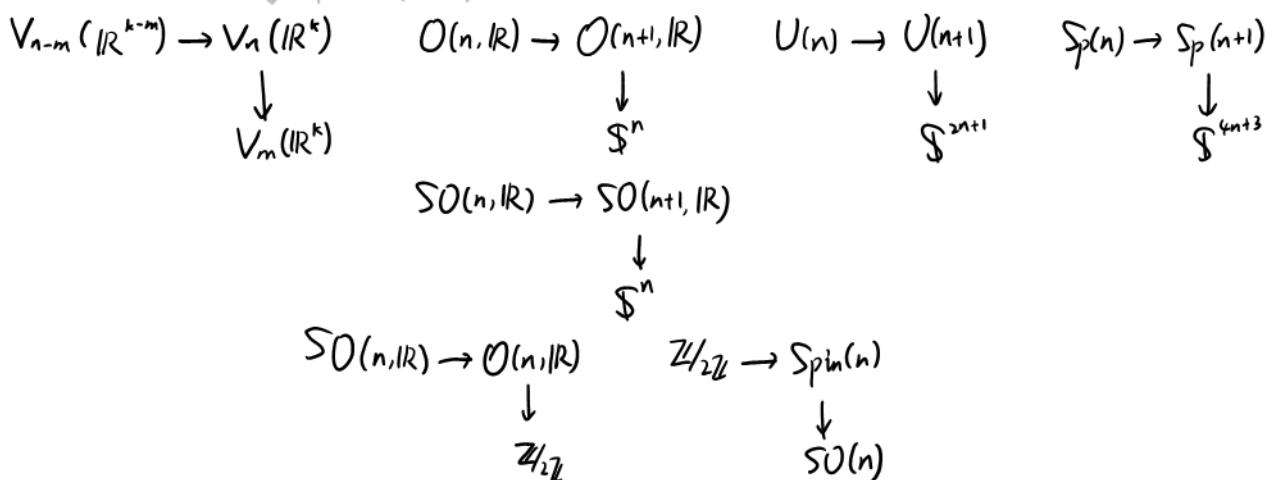
Homotopy groups of orthogonal groups								
$\pi_1$	$\pi_2$	$\pi_3$	$\pi_4$	$\pi_5$	$\pi_6$	$\pi_7$	$\pi_8$	$\pi_9$
$SO(2)$	$Z$	0	0	0	0	0	0	0
$SO(3)$	$[Z_2]$	0	$Z$	$Z_2$	$Z_2$	$Z_{12}$	$Z_2$	$Z_2$
$SO(4)$	$Z_2$	$[0]$	$(Z)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_{12})^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$
$SO(5)$	$Z_2$	0	$[Z]$	$Z_2$	$Z_2$	0	$Z$	0
$SO(6)$	$Z_2$	0	$Z$	$[0]$	$Z$	0	$Z$	$Z_{24}$
$SO(n), n > 6$	$Z_2$	0	$Z$	0	0	0	0	0

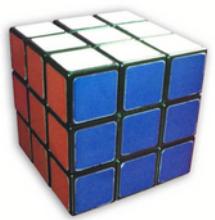
Activate Windows

my understanding

fibration: [Hatcher Example 4.54, 4.55]

space of  $n$ -frames in  $\mathbb{R}^k$





reductive group  
 $G$



Borel subgroup  
 $B$



maximal torus  
 $T$

<https://users.math.msu.edu/users/ruiterj2/math/Documents/Notes%20and%20talks/General%20linear%20group%20butterfly.pdf>

# Structure theory of Lie group

Ref:

<http://www.math.columbia.edu/~maki/sumi/cld/reductivegroups.pdf>

← notations, results, examples, pictures.

Today:

$$G = GL(2, \mathbb{C})$$

center

$$Z(G) = \mathbb{C}^{\times} \quad \pi_1(G) = \mathbb{Z}$$

radical

$$\text{rad}(G) := R(G) = \mathbb{C}^{\times}$$

reductive

$$\text{Unipotent rad} \quad \text{rad}_u(G) := R_u(G) = \{I\}$$

$$\text{rank } G = 2$$

canonical maximal torus  $T = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$$C_G(T) = T$$

$$N_G(T) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \cup \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

$$W(G) = S_2 \quad \text{Weyl group} \\ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

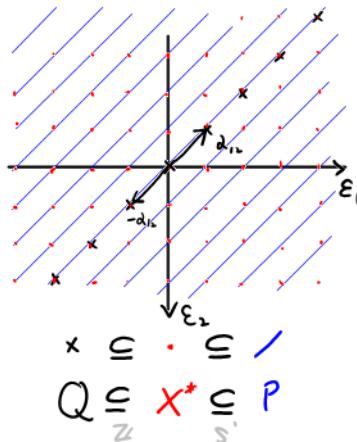
$$X^* := X^*(T) = \langle \varepsilon_i : (t_1, t_2) \mapsto t_i \rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \alpha \mapsto \alpha(g^{-1}g)$$

$$X_{\neq} := X_{\neq}(T) = \left\langle \begin{array}{l} \varepsilon_{1,*} : t \mapsto (t_1, 1) \\ \varepsilon_{2,*} : t \mapsto (1, t_2) \end{array} \right\rangle_{\mathbb{Z}} \quad \hookrightarrow W(G) \quad \lambda \mapsto g \lambda(-)g^{-1}$$

$$\Phi = \{\varepsilon_1 - \varepsilon_2, -(\varepsilon_1 + \varepsilon_2)\} := \{\alpha_{12}, -\alpha_{12}\}$$

$$\Phi^V = \{\alpha_{12}^V, -\alpha_{12}^V\} = \{\varepsilon_{1,*} - \varepsilon_{2,*}, -(\varepsilon_{1,*} + \varepsilon_{2,*})\}$$

$$\boxed{r_? = \text{action of } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \begin{array}{c} X^* \longrightarrow X^* \\ \varepsilon_1 \longmapsto \varepsilon_2 \\ \varepsilon_2 \longmapsto \varepsilon_1 \end{array} \Rightarrow r_? = r_{\alpha_{12}} \quad \text{r}_? \text{ is reflection and} \\ X - r_?(X) = x_1 \varepsilon_1 + x_2 \varepsilon_2 - x_1 \varepsilon_2 - x_2 \varepsilon_1 = (x_1 - x_2) \alpha_{12} \quad x_? = x_1 \varepsilon_1 + x_2 \varepsilon_2 \\ \langle -, \alpha_{12}^V \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^V = \varepsilon_{1,*} - \varepsilon_{2,*}}$$



$$G = SL(2, \mathbb{C})$$

$$Z(G) = \{\pm 1\} \quad \pi_1(G) = \{*\}$$

$$\text{rad}(G) = \{I\} \Rightarrow \text{semisimple}$$

$$\text{rad}_u(G) = \{I\}$$

$$\text{rank } G = 1$$

canonical maximal torus  $T = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$

$$C_G(T) = T$$

$$N_G(T) = \{t' \mid t' \in T\}$$

$$W(G) = S_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$$

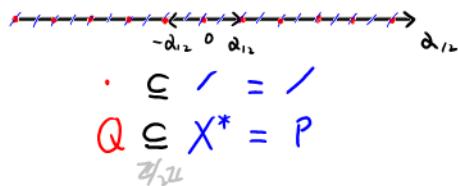
$$X^* = \langle \varepsilon_i : (t_{-t}) \mapsto t \rangle_{\mathbb{Z}} \quad \text{with } \alpha \mapsto \alpha(g^{-1} - g) \quad \alpha_1 = 2\varepsilon_1!$$

$$X_t = \langle \varepsilon_i^t : t \mapsto (t_{-t}) \rangle_{\mathbb{Z}} \quad \text{with } \lambda \mapsto g \cdot \lambda(-) \cdot g^{-1}$$

$$\Phi = \{d_{12}, -d_{12}\} = \{2\varepsilon_1, -2\varepsilon_1\}$$

$$\vec{\Phi}^V = \{d_{12}^V, -d_{12}^V\} = \{E_{1,+}, -E_{1,+}\}$$

$$\boxed{\begin{aligned} r_? &= \text{action of } (\downarrow^1): X^* \longrightarrow X^* \quad \varepsilon_i \mapsto -\varepsilon_i \\ x - r_?(x) &= x_0 \varepsilon_i - x_0 (-\varepsilon_i) = x_0 \alpha_{12} \stackrel{r_? \text{ ref}}{\uparrow} + r_2 = r_{122} \quad x := x_0 \varepsilon_i \\ \langle -, \alpha_{12}^\vee \rangle: x_0 \varepsilon_i &\mapsto x_0 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_{1,2} \end{aligned}}$$



$G = GL(3, \mathbb{C})$

$$\begin{aligned} Z(G) &= \mathbb{C}^\times & \pi_1(G) &= \mathbb{Z} \\ \text{rad}(G) &= \mathbb{C}^\times & \left. \begin{aligned} \text{rad}(G) &= \{I\} \end{aligned} \right\} &\Rightarrow \text{reductive} \\ \text{rank } G &= 3 & &\text{but not semi} \end{aligned}$$

canonical maximal torus  $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$   
 $C_G(T) = T$   
 $N_G(T) = \text{monomial matrixs}$   
 $W(G) = S_3$

$$X^* = \langle \varepsilon_i, \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}}$$

$$X_* = \langle \varepsilon_i^*, t \mapsto (t, t, t) \rangle_{\mathbb{Z}}$$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$$\left[ \begin{array}{l} r_{\alpha_{12}} = \text{action of } \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, X^* \rightarrow X^* \\ \begin{array}{l} \varepsilon_1 \mapsto \varepsilon_2 \\ \varepsilon_2 \mapsto \varepsilon_1 \\ \varepsilon_3 \mapsto \varepsilon_3 \end{array} \end{array} \right]$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 \\ = (x_1 - x_2) \alpha_{12}$$

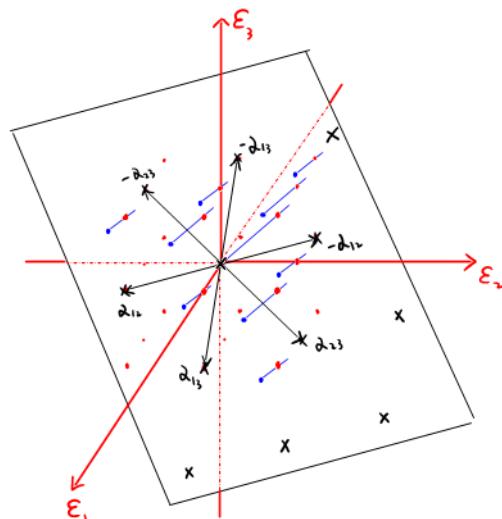
$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \Rightarrow \alpha_{12}^\vee = \varepsilon_{1,*} - \varepsilon_{2,*}$$

$$\text{Similarly, } \alpha_{23}^\vee = \varepsilon_{2,*} - \varepsilon_{3,*}$$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$\text{where } \alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$$

$$\begin{aligned} \varepsilon &\mapsto \varepsilon, \left( \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) \\ &= \varepsilon, \begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix} = \varepsilon_2 \end{aligned}$$



$$x \subseteq \cdot \subseteq /$$

$$Q \subseteq X^* \subseteq P$$

$G = SL(3, \mathbb{C})$

$$Z(G) = \mathbb{Z}/3\mathbb{Z} \quad \pi_1(G) = \langle \gamma \rangle$$

$\text{rad}(G) = \langle I \rangle \Rightarrow \text{semisimple}$

$\text{rad}_u(G) = \langle I \rangle$

rank  $G = 2$

canonical maximal torus  $T = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \text{monomial matrixs}$

$W(G) = S_3$

$$X^* = \langle \varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}} / \mathbb{Z}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$$

$$X_* = \langle \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^* : \varepsilon_1^* - \varepsilon_2^*, \varepsilon_2^* - \varepsilon_3^* \rangle_{\mathbb{Z}}$$

$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ : dual basis of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

$$\left[ \begin{array}{l} r_{\alpha_{12}} = \text{action of } \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, X^* \rightarrow X^* \\ \varepsilon_1 \mapsto \varepsilon_2 \\ \varepsilon_2 \mapsto \varepsilon_1 \\ \varepsilon_3 \mapsto \varepsilon_3 \end{array} \right]$$

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

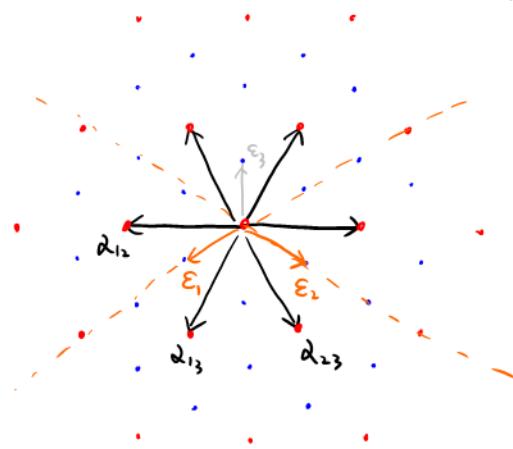
$$\text{where } \alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$$

$$\varepsilon_1 \mapsto \varepsilon_1 \left( \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \right) = \varepsilon_2$$

$$x - r_{\alpha_{12}}(x) = x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 - x_1 \varepsilon_2 - x_2 \varepsilon_1 - x_3 \varepsilon_3 = (x_1 - x_2) \alpha_{12}$$

$$\langle -, \alpha_{12}^\vee \rangle : x_1 \varepsilon_1 + x_2 \varepsilon_2 + x_3 \varepsilon_3 \mapsto x_1 - x_2 \Rightarrow \alpha_{12}^\vee = \varepsilon_{1,*} - \varepsilon_{2,*}$$

$$\text{Similarly, } \alpha_{23}^\vee = \varepsilon_{2,*} - \varepsilon_{3,*}$$



$$\begin{matrix} & \subseteq & & = & \\ Q & \subseteq & X^* & = & P \\ & \mathbb{Z}/3\mathbb{Z} & & & \end{matrix}$$

$$G = \mathrm{PGL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C}) / \mu_3$$

$$Z(G) = \{\mathbf{I}\} \quad \pi_1(G) = \mathbb{Z}/3\mathbb{Z}$$

$\mathrm{rad}(G) = \{\mathbf{I}\} \Rightarrow$  semisimple

$$\mathrm{rad}_u(G) = \{\mathbf{I}\}$$

rank  $G = 2$

$$\mathbb{C}^* \cong \mathbb{C}_{\mu_3}^* \times \mathbb{C}^*$$

$$t^3 \leftrightarrow \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \times \begin{pmatrix} 1 & & \\ & t_1 & \\ & & t_2 \end{pmatrix}$$

canonical maximal torus  $T = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} / \mu_3$

$$C_G(T) = T$$

$N_G(T)$  = monomial matrixs /  $\mu_3$

$$W(G) = S_3$$

$$X^* = \{n_1 \varepsilon_1 + n_2 \varepsilon_2 + n_3 \varepsilon_3 \mid n_i \in \mathbb{Z}, 3 \mid \sum n_i\} /_{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \mathbb{Z}}$$

$$= \langle \alpha_{12}, \alpha_{23} \rangle_{\mathbb{Z}}$$

$\varepsilon_i : \begin{pmatrix} t_1 & t_2 & t_3 \end{pmatrix} \rightarrow t_i$  is no longer well-defined

$$\alpha_{ij} := \varepsilon_i - \varepsilon_j$$

$$X_* = \{n_1 \varepsilon_{1,*} + n_2 \varepsilon_{2,*} + n_3 \varepsilon_{3,*} \mid n_i \in \frac{1}{3}\mathbb{Z}, \sum n_i = 0\}$$

$$= \left\langle \frac{1}{3} \varepsilon_{1,*} + \frac{1}{3} \varepsilon_{2,*} - \frac{2}{3} \varepsilon_{3,*}, \varepsilon_{3,*} - \varepsilon_{2,*} \right\rangle_{\mathbb{Z}}$$

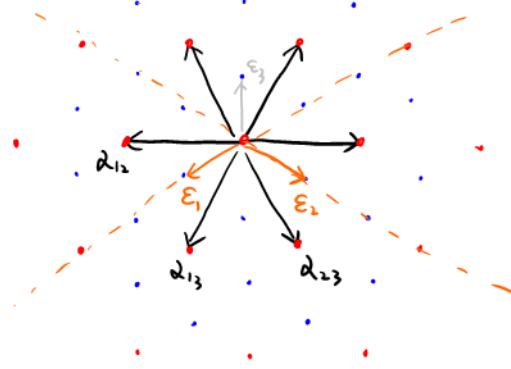
$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ : dual basis of  $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$$

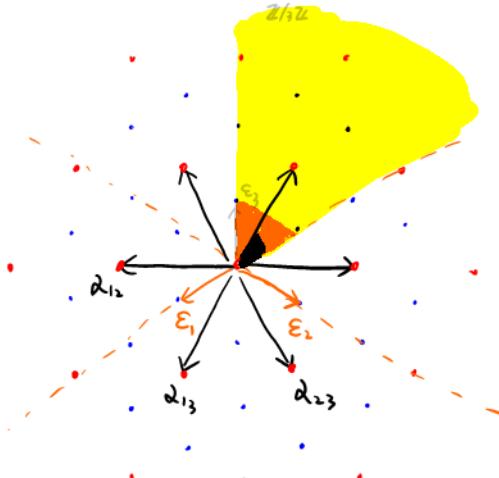
$$\alpha_{ij,*} = \varepsilon_i - \varepsilon_j$$

$$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$$

where  $\alpha_{ij}^\vee = \varepsilon_{i,*} - \varepsilon_{j,*}$



$$Q = X^* \subseteq P$$



fundamental domain of

Weyl group [ ]  
affine Weyl group [ ]  
extended affine Weyl group [ ]

$$G = \mathrm{Sp}(4, \mathbb{C}) = \{ A \in GL(4, \mathbb{C}) \mid A^T M A = M \}$$

$Z(G) = \{\pm I\}$     $\pi_1(G) = \mathbb{Z}_2$   
 $\mathrm{rad}(G) = \{I\} \Rightarrow \text{semisimple}$   
 $\mathrm{rad}_u(G) = \{I\}$   
 $\mathrm{rank } G = 2$

$M = \begin{pmatrix} & -1 & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$   
 canonical maximal torus    $T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_3^{-1} & \\ & & & t_4^{-1} \end{pmatrix}$   
 $C_G(T) = T$   
 $N_G(T) = \text{monomial matrixs}$   
 $W(G) = D_4 \subseteq S_4$

$$\mathrm{sp}(4, \mathbb{C}) = \{ A \in M^{4 \times 4}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

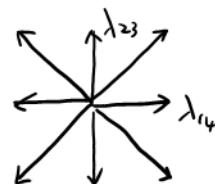
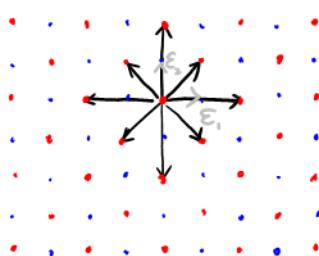
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & -a_{13} & & \\ & & a_{22} & a_{12} \\ & & -a_{31} & -a_{11} \end{pmatrix} \in M^{4 \times 4}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{14} \oplus \mathbb{Z}\lambda_{23}$$

$$\Phi = \{\pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23}\} = \{\pm (\varepsilon_1 - \varepsilon_2), \pm (\varepsilon_1 + \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$$

$$\Phi^\vee = \{\pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee\} = \{\pm (\lambda_{14} - \lambda_{23}), \pm (\lambda_{14} + \lambda_{23}), \pm \lambda_{14}, \pm \lambda_{23}\}$$



$$\Phi \subseteq Q \subseteq X^* = P$$

$$\Phi^\vee$$

$\mathbb{Z}/2\mathbb{Z}$

$$G = SO(5, \mathbb{C}) \cong \{ A \in SL(5, \mathbb{C}) \mid A^T M A = M \}$$

$$Z(G) = \{ \text{Id} \} \quad \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$$

$\text{rad}(G) = \{ I \} \Rightarrow \text{semisimple}$

$$\text{rad}_u(G) = \{ I \}$$

$$\text{rank } G = 2$$

$$M = \begin{pmatrix} & & & & \\ & t_1^{-1} & & & \\ & & t_2^{-1} & & \\ & & & t_3^{-1} & \\ & & & & t_4^{-1} \end{pmatrix}$$

canonical maximal torus

$$C_G(T) = T$$

$N_G(T) = \text{monomial matrixs}$

$$W(G) = D_4$$

$$SO(5, \mathbb{C}) = \{ A \in M^{5 \times 5}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

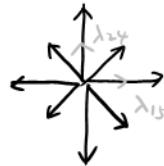
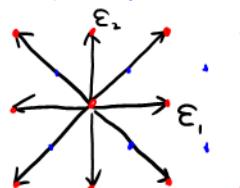
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ 0 & a_{22} & a_{23} & a_{24} & 0 \\ 0 & a_{32} & a_{33} - a_{22} & a_{34} & 0 \\ 0 & a_{42} - a_{31} & a_{43} - a_{21} & a_{44} - a_{11} & 0 \end{pmatrix} \in M^{5 \times 5}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_5, \varepsilon_2 + \varepsilon_4, \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$

$$X_* = \mathbb{Z}\lambda_{15} \oplus \mathbb{Z}\lambda_{24}$$

$$\Phi = \{ \pm \alpha_{12}, \pm \alpha_{13}, \pm \alpha_{14}, \pm \alpha_{23} \} = \{ \pm (\varepsilon_1 - \varepsilon_2), \pm \varepsilon_1, \pm (\varepsilon_1 + \varepsilon_2), \pm \varepsilon_2 \}$$

$$\Phi^\vee = \{ \pm \alpha_{12}^\vee, \pm \alpha_{13}^\vee, \pm \alpha_{14}^\vee, \pm \alpha_{23}^\vee \} = \{ \pm (\lambda_{15} - \lambda_{24}), \pm 2\lambda_{15}, \pm (\lambda_{15} + \lambda_{24}), \pm 2\lambda_{24} \}$$

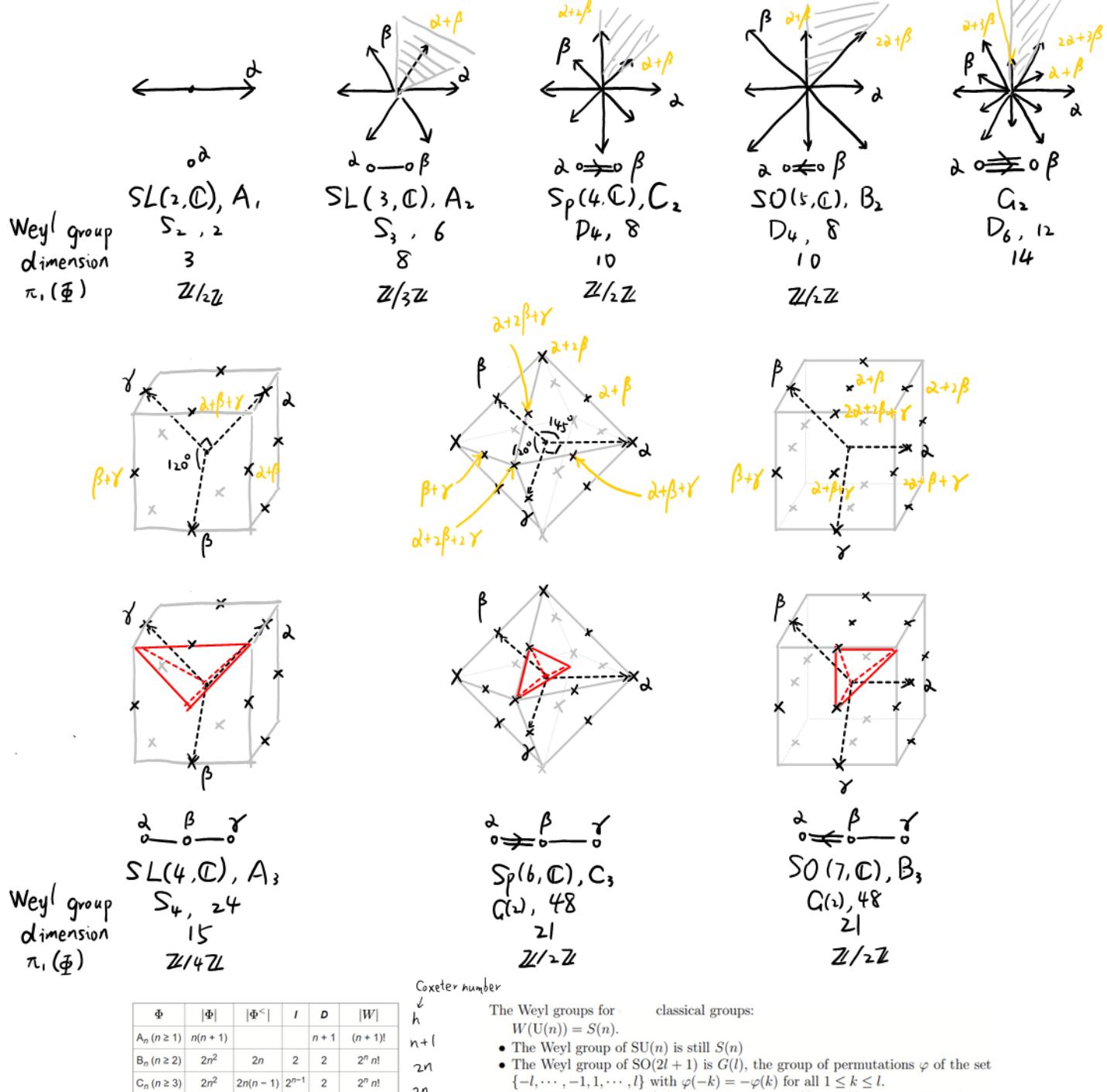


$$\Phi \subseteq Q = X^* \subseteq P$$

$\mathbb{Z}/2\mathbb{Z}$

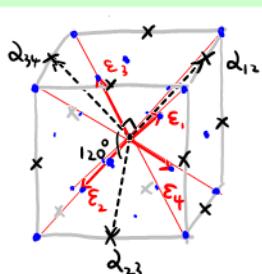
$$\Phi^\vee$$

Now let's enjoy the beauty of low-dim root system!

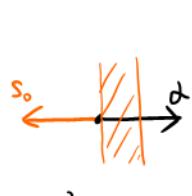


[https://en.wikipedia.org/wiki/Root\\_system](https://en.wikipedia.org/wiki/Root_system)

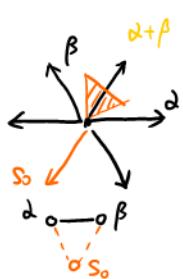
A better table can be found here: [https://www.jgibson.id.au/lievis/tables\\_fin\\_aff/](https://www.jgibson.id.au/lievis/tables_fin_aff/)



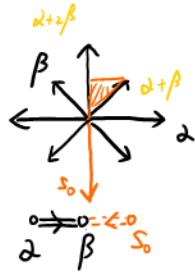
Now let's construct some associated extended Dynkin diagrams!



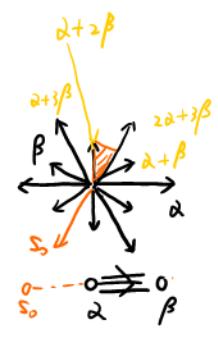
$\tilde{A}_1$



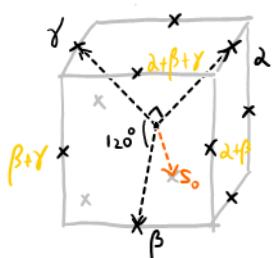
$\tilde{A}_2$



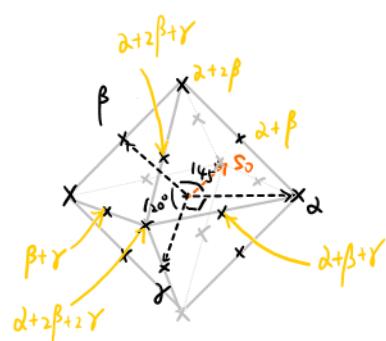
$\tilde{C}_2$



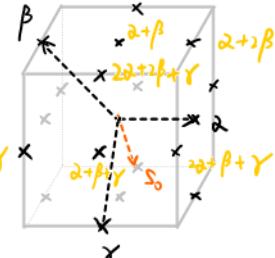
$\tilde{G}_2$



$\tilde{A}_3$

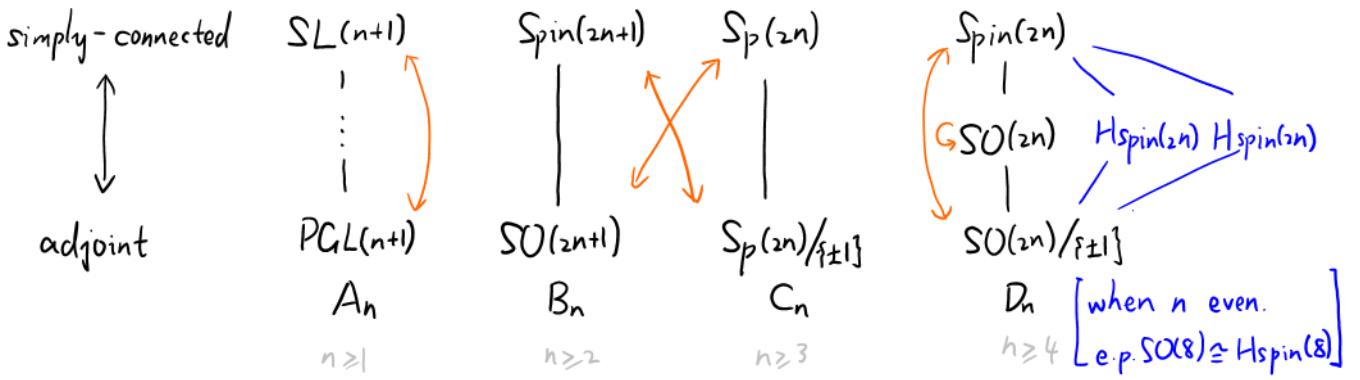


$\tilde{C}_3$



$\tilde{B}_3$

$k = \overline{k}$ ,  $\text{char } k = 0$ .



The following table comes from [linear algebraic groups and finite groups of Lie type, p72].  
Errata for this book: [https://www.mathematik.uni-kl.de/~malle/download/MT\\_errata.pdf](https://www.mathematik.uni-kl.de/~malle/download/MT_errata.pdf)

Table 9.2 *Isogeny types of simple algebraic groups*

$\Phi$	$\Lambda(\Phi)$	$G_{\text{sc}}$	$G_{\text{ad}}$	in between
$A_{n-1}, n \geq 2$	$Z_n$	$\text{SL}_n$	$\text{PGL}_n$	$\text{SL}_n/Z_d \ (d n)$
$B_n, n \geq 2$	$Z_2$	$\text{Spin}_{2n+1}$	$\text{SO}_{2n+1}$	—
$C_n, n \geq 2$	$Z_2$	$\text{Sp}_{2n}$	$\text{PCSp}_{2n}$	—
$D_n, n \geq 3 \text{ odd}$	$Z_4$	$\text{Spin}_{2n}$	$\text{PCO}_{2n}^\circ$	$\text{SO}_{2n}$
$D_n, n \geq 4 \text{ even}$	$Z_2 \times Z_2$	$\text{Spin}_{2n}$	$\text{PCO}_{2n}^\circ$	$\text{SO}_{2n}, \text{HSpin}_{2n}$
$G_2$	1	$G_2$		—
$F_4$	1	$F_4$		—
$E_6$	$Z_3$	$(E_6)_{\text{sc}}$	$(E_6)_{\text{ad}}$	—
$E_7$	$Z_2$	$(E_7)_{\text{sc}}$	$(E_7)_{\text{ad}}$	—
$E_8$	1	$E_8$		—