# Eine Woche, ein Beispiel

## 6.18 Diagram chasing

Goal: Let's play the game of diagram chasing!

basic: five lemma, snake lemma, SES of complex $\Rightarrow$ LES of homology

[Vakil] "where there is universal property, there is diagram chasing"

e.p. Chap 1 Category + Adjoints + Spectral sequences

Chap 2 Sheaf on topology space

Please convert everything to Grothendieck topo!

Chap 23 Derived functors

Chap 28 Base change

[J. S. Milne, étale cohomology] Prop II.1.5. one description of the étale sheaf

Extension group

1. Def of $\mathrm{Ext}_A^n(M,N)$

$$E_A(M,N) = \{0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0\}/\text{equivalence}$$
$$= \{\text{proj resolution } P_\cdot,\ H^n(\mathrm{Hom}_A(P_\cdot, N))\}/\text{resolution}$$
$$= \{\text{inj resolution } I^\cdot,\ H^n(\mathrm{Hom}_A(M, I^\cdot))\}/\text{resolution}$$
$$= \{\text{derivation}\}/\text{inner derivation}$$
$$= \mathrm{Hom}_{D(A\text{-mod})}(M, N[1])$$

deri = SES = derived

proj = SES = inj

2. Special module/ring interact with Ext?

$P$ proj $\Leftrightarrow \mathrm{Ext}_A^n(P,-)=0\ \forall n\geq 1 \Leftrightarrow \mathrm{Ext}_A^1(P,-)=0$

$\Leftrightarrow$ proj dim $P=0$

$I$ proj $\Leftrightarrow \mathrm{Ext}_A^n(-,I)=0\ \forall n\geq 1 \Leftrightarrow \mathrm{Ext}_A^1(-,I)=0$

$A$ f.d alg $\quad \dim_k \mathrm{Ext}_A^1(S(i), S(j)) = \dim_k \mathrm{Hom}_A(\mathrm{rad}(P(i)), S(j)) \underset{A=kQ/I}{=} |\{a\in Q_1 \mid s(a)=i,\ t(a)=j\}|$

Second level of detail.

equivalent of SES
$$\begin{array}{ccccccccc} \to & & \to & & \to & & \to & \\ & \| & & \downarrow & & \| & & \\ \to & & \to & & \to & & \to & \end{array}$$
$\Downarrow$

isomorphic
$$\begin{array}{ccccccccc} \to & & \to & & \to & & \to & \\ & \downarrow & & \downarrow & & \downarrow & & \\ \to & & \to & & \to & & \to & \end{array}$$

$\Uparrow \times$:

$$\begin{array}{ccccccccc} 0 \to & \mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/p\mathbb{Z} & \to 0 \\ & \| & & \| & & \downarrow{\times 2} & \\ 0 \to & \mathbb{Z} & \xrightarrow{\cdot p} & \mathbb{Z} & \xrightarrow{2\cdot\pi} & \mathbb{Z}/p\mathbb{Z} & \to 0 \end{array}$$

pushout $\alpha_*$

pullback $\beta^*$

① ② ③$^*$

$+$ com ass $0$   $k$-linear $(\lambda+\mu)a$ $(\lambda\mu)a$ $1a$   $\lambda(a+b)$

$\Rightarrow E_A(M,N)$: ① Def, ② bifunctor and ③$^*$ $k$-linear space structure   ① $\Rightarrow$ ② $\Rightarrow$ ③

$f_\cdot \sim g_\cdot \quad \Rightarrow H_n(f_\cdot) = H_n(g_\cdot)$

$g_\cdot f_\cdot \sim \mathrm{Id} \quad f_\cdot g_\cdot \sim \mathrm{Id} \quad \Rightarrow \quad H_n(C_\cdot) = H_n(C'_\cdot)$

$\Rightarrow \mathrm{Ext}_A^n(M,N)$: ① Def, ② bifunctor and ③ $k$-linear space structure   ① $\Rightarrow$ ③, ① $\Rightarrow$ ②

$\Rightarrow E_A(M,N) \to \mathrm{Ext}_A^1(M,N)$ ① well-defined by resolution & lift & equiv
② bifunctor
③ $k$-linear map

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Schanuel's lemma
$$\left.\begin{array}{l} 0 \to U \to P \to M \to 0 \\ 0 \to U' \to P' \to M \to 0 \\ \quad P, P' \text{ proj} \end{array}\right\} \Rightarrow U \oplus P' \cong U' \oplus P$$

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$0 \to U \to X \to V \to 0$ $\left\{\begin{array}{l}\text{non-split} \\ \text{f.d. } A\text{-mod}\end{array}\right.$ $\Rightarrow \dim_k \mathrm{End}_A(X) < \dim_k \mathrm{End}_A(U\oplus V)$

for f.d. $A$-mod $0\to U \to X \to V \to 0$ split $\Leftrightarrow X \cong U \oplus V$ as $A$-module

# Derived category

slogan: complex good, homology bad

Motivated: https://arxiv.org/pdf/math/0001045.pdf
Standard reference: S. Gelfand and Yu. Manin, Methods of Homological Algebra, Springer, 1996 ← we refer this without mention!

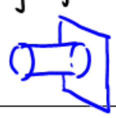
$$\begin{tikzcd}
\mathrm{Kom}(\mathcal{A}) \arrow[d, "/\text{homotopy}"] \arrow[dd, bend right, dashed, "/\text{quasi-iso}"'] & \text{category of chain complexes} \\
K(\mathcal{A}) \arrow[d, "/\text{quasi-iso}"] & \text{homotopy category} \\
D(\mathcal{A}) & \text{derived category}
\end{tikzcd}$$

**Remark.** 1. For most time we view the category equivalence as "equal".
However, the category defined by universal property is unique under isomorphism.

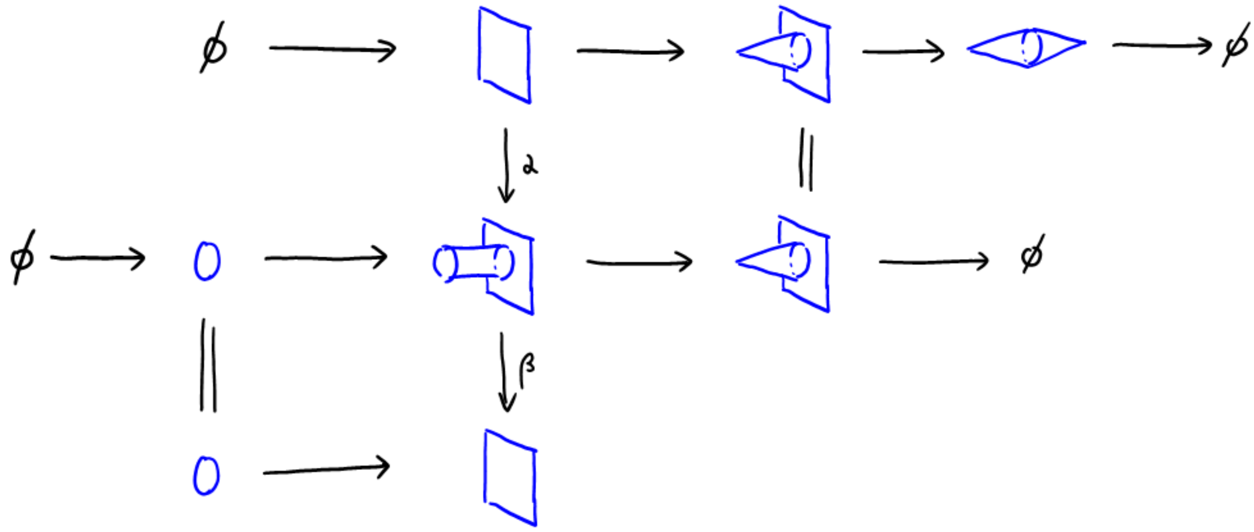
$$\mathcal{O}b(\mathrm{Kom}(\mathcal{A})) = \mathcal{O}b(K(\mathcal{A})) = \mathcal{O}b(D(\mathcal{A}))$$

2. localizing category $B[S^{-1}]$ does not always have a good description
e.g. $D(\mathcal{A}) := \mathrm{Kom}(\mathcal{A})[\text{quasi-iso}^{-1}]$
However, when $S$ is a localizing class, then we have a good description (Lemma III 2.8)
e.g. $D(\mathcal{A}) := K(\mathcal{A})[\text{quasi-iso}^{-1}]$
These two definitions define the same category $D(\mathcal{A})$.

3. $D(\mathcal{A})$ is a triangulated category.
To define a distinguished triangle, we denote

| | | |
|---|---|---|
| $f: K^\bullet \longrightarrow L^\bullet$  | $K^\bullet, L^\bullet$ : complexes | $K^0 \xrightarrow{d_K^0} K^1$, $L^0 \xrightarrow{d_L^0} L^1$; $d_K^0 = d_K = d$ to be short |
| $Cyl(f)^\bullet := K^\bullet \oplus K[1]^\bullet \oplus L^\bullet$  | $d_{Cyl(f)} = \begin{bmatrix} d & -1 & \\ & -d & \\ & f & d \end{bmatrix}$ | $K^0 \oplus K^1 \oplus L^0 \xrightarrow{\begin{bmatrix} d_K^0 & -1 & \\ & -d_K^1 & \\ & f^1 & d_L^0 \end{bmatrix}} K^1 \oplus K^2 \oplus L^1$ |
| $C(f)^\bullet := K[1]^\bullet \oplus L^\bullet$  | $d_{C(f)} = \begin{bmatrix} -d & \\ f & d \end{bmatrix}$ | $K^1 \oplus L^0 \xrightarrow{\begin{bmatrix} -d_K^1 & \\ f^1 & d_L^0 \end{bmatrix}} K^2 \oplus L^1$ |

Then we have (Lemma III 3.3) $\begin{cases} \text{⓪ well-defined} \\ \text{① SES on row} \\ \text{② } \alpha, \beta \text{: quasi-iso} \end{cases}$



$$\begin{tikzcd}
 & 0 \arrow[r] & L^\bullet \arrow[r, "{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}"] & K[1]^\bullet \oplus L \arrow[r, "{\begin{bmatrix} 1 & 0 \end{bmatrix}}"] \arrow[d, equal] & K[1]^\bullet \arrow[r] & 0 \\
0 \arrow[r] & K^\bullet \arrow[r, "{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}"] \arrow[d, equal] & K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \arrow[r, "{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}"] \arrow[d, "{\begin{bmatrix} f & 0 & 1 \end{bmatrix}}"] & K[1]^\bullet \oplus L \arrow[r] & 0 \\
 & K^\bullet \arrow[r, "f"] & L^\bullet & & &
\end{tikzcd}$$

distinguished triangle:

$$K^\bullet \xrightarrow{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \xrightarrow{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} K[1]^\bullet \oplus L \xrightarrow{\begin{bmatrix} 1 & 0 \end{bmatrix}} K[1]^\bullet$$

SES: What's your favorate SES?

$$0 \to I \to A \to A/I \to 0 \qquad \text{as } A\text{-mod}$$

$$0 \to A \to A \oplus B \to B \to 0$$

$$0 \to I \cap J \to I \oplus J \to I + J \to 0$$

$$0 \to R/(I\cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$$

$$0 \to \mathcal{O}_X \to K_X \to \bigoplus_{x \in X \text{ closed}} I_x \to 0$$

https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer12.pd

$$0 \to \underset{\Delta_* \Omega_X}{\overset{\parallel}{I/I^2}} \to \mathcal{O}_{X\times X}/I^2 \to \underset{\Delta_* \mathcal{O}_X}{\overset{\parallel}{\mathcal{O}_{X\times X}/I}} \to 0$$

https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer8.pdf

$$0 \to I_q \to D_q \to \mathrm{Gal}(k_q/k_p) \to 0$$

$$0 \to \mathcal{O}_K^\times \to K^\times \to \bigoplus_{p \in M_K^\circ} \mathbb{Z} \to Cl(K) \to 0$$

exponential $\quad 0 \to \mathbb{Z} \to \mathcal{O}_M \to \mathcal{O}_M^\times \to 1$

generalization:https://ncatlab.org/nlab/show/exponential+exact+sequence

$M$: cplx mfld

Euler $\quad 0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0$

$$1 \to \mathbb{G}_m \to u_{\eta *}\mathbb{G}_{m,\eta} \to \mathrm{Div}(X) \to 1 \qquad u_\eta : \underset{= \operatorname{Spec}(k(X))}{\eta} \to X$$

$$0 \dashrightarrow f^*\Omega_{X/k} \to \Omega_{Y/k} \to \Omega_{Y/X} \to 0 \qquad f: Y \to X$$

$$0 \dashrightarrow I/I^2 \to i^*\Omega_{X/k} \to \Omega_{Z/k} \to 0$$

$$Z \xrightarrow[\text{close}]{i} X \xleftarrow{j} U \quad \rightsquigarrow$$

$$\begin{tikzcd}[column sep=huge]
\mathrm{Sh}(Z_{\acute{e}t}) \arrow[r, "\text{pi} \quad i_* \quad \text{f.f}" description] & \mathrm{Sh}(X_{\acute{e}t}) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\text{pi} \quad i^! \quad L"] \arrow[r, "\text{pi} \quad j^*" description] & \mathrm{Sh}(U_{\acute{e}t}) \arrow[l, bend right, "j_! \quad \text{f.f.}"'] \arrow[l, bend left, "\text{pi} \quad j_* \quad L \quad \text{f.f}"]
\end{tikzcd}$$

Id Id non-zero Id Id

L: left exact (others are exact)

ff. fully faithful

pi: preserve injectives. (内射)

ie. inj sheaf $\rightsquigarrow$ inj sheaf

$$\mathcal{F}_1 \xleftarrow{i^*} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(\mathcal{F}_1, \mathcal{F}_2, \alpha) \xrightarrow{j^*} \mathcal{F}_2$$

$$\ker\alpha \xleftarrow{i^!} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(0, \mathcal{F}_2, 0) \xleftarrow{j_!} \mathcal{F}_2$$

$$\mathcal{F}_1 \xmapsto{i_*} (\mathcal{F}_1, 0, 0)$$

$$(i^* j_* \mathcal{F}_2, \mathcal{F}_2, \mathrm{Id}) \xleftarrow{j_*} \mathcal{F}_2$$

$$0 \to I \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 1 \quad \rightsquigarrow 0 \to \widetilde{I} \to \mathcal{O}_{X\times X} \to \Delta_*\mathcal{O}_X \to 1$$

$$\rightsquigarrow 0 \to \Omega_{X/k} \to \Delta^*\mathcal{O}_{X\times X} \to \Delta^*\Delta_*\mathcal{O}_X \to 1$$

$$0 \to \underset{\mathcal{H}^0_Z\mathcal{F}}{\underset{\shortparallel}{i_* i^! \mathcal{F}}} \to \mathcal{F} \to j_* j^* \mathcal{F} \to 0$$

$\mathcal{H}^0_Z$: https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer1.pdf

$\mathcal{F}$ is supported on $Z \iff \mathcal{H}^0_Z(\mathcal{F}) = \mathcal{F} \iff j_* j^* \mathcal{F} = 0$

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

$j_!$: https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer3.pdf

https://math.stackexchange.com/questions/2208354/faithful-direct-image-functor

For Zariski: $j^* = j^{-1}$, $i^* \mapsto i^{-1}$

https://mathoverflow.net/questions/38168/is-the-category-of-commutative-group-schemes-abelian

Kummer sequence
$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^n-1) \longleftarrow k[x,x^{-1}] \longleftarrow k[x,x^{-1}]$$

$k/\mathbb{F}_p$
$$1 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F:=(-)^p} \mathbb{G}_a \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^p) \longleftarrow k[x] \longleftarrow k[x]$$

Artin-Schreier sequence
$$1 \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \mathbb{G}_a \xrightarrow{F-\mathrm{Id}} \mathbb{G}_a \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^p-x) \longleftarrow k[x] \longleftarrow k[x]$$

| | Zariski | étale | fppf |
|---|---|---|---|
| $\mu_n$ | × | ✓ when $n \in \Gamma(X,\mathcal{O}_X)^\times$<br>× in general | ✓ |
| $\alpha_p$ | × | × in general | ✓ |
| $\underline{\mathbb{Z}/p\mathbb{Z}}$ | × | ✓ | ✓ |