fiber of $\pi_*, \pi_!, \pi^{-1}, \pi^*$

$\mathcal{G} \quad \mathcal{F}$

$\pi : Y \hookrightarrow X$

$\pi : U \hookrightarrow X$ $\qquad\qquad$ $\pi : Z \xrightarrow{\text{close}} X$

$$\pi_* \quad \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin \bar{U} \\ \varinjlim_{x \in V} \mathcal{G}(U \cap V) & x \in \bar{U} \setminus U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi_! \quad \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi^{-1} \quad \mathcal{F}_y \qquad\qquad \mathcal{F}_y$$

$$\pi^* \quad \mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y} \qquad\qquad \mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y}$$

$\bar{x} \xrightarrow{u_{\bar{x}}} X \xrightarrow{f} Y$

$(f^*\mathcal{F})_{\bar{x}} = u_{\bar{x}}^* f^* \mathcal{F}(Y) = \mathcal{F}_y$

For étale (sheaf): $f : \underset{\mathcal{G}}{X} \longrightarrow \underset{\mathcal{F}}{Y}$ $\qquad \bar{x} \longmapsto \bar{y}$

$(f^*\mathcal{F})_{\bar{x}} \cong \mathcal{F}_y$

If f q.c. $\quad (f_*\mathcal{G})_{\bar{y}} \cong \Gamma(\tilde{X}, \tilde{g}^*\mathcal{G})$ where
$$\begin{tikzcd} \tilde{X} \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ \operatorname{Spec} \mathcal{O}^{sh}_{Y,\bar{y}} \arrow[r, "g"] & Y \end{tikzcd}$$

e.g. for f finite, we have explicit description of f_*.

Lemma 6.21.3. *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to f_*. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \mathrm{colim}_{f(U) \subset V}\, \mathcal{G}(V)$$

where the colimit is over the collection of open neighbourhoods V of $f(U)$ in Y. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)

Lemma 6.31.4. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor j_p (see Lemma 6.31.1).*

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.

(3) *Let $\mathcal{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of presheaves of U we have $j_p j_{p!} = \mathrm{id}$.*

(5) *On the category of sheaves of U we have $j^{-1} j_! = \mathrm{id}$.*

situation	category $\mathscr{A}$	category $\mathscr{B}$	left adjoint $F\colon \mathscr{A} \to \mathscr{B}$	right adjoint $G\colon \mathscr{B} \to \mathscr{A}$
A-modules (Ex. 1.5.D)	Mod_A	Mod_A	$(\cdot) \otimes_A N$	$\mathrm{Hom}_A(N, \cdot)$
ring maps $B \to A$ (Ex. 1.5.E)	Mod_B	Mod_A	$(\cdot) \otimes_B A$ (extension of scalars)	$M \mapsto M_B$ (restriction of scalars)
(pre)sheaves on a topological space X (Ex. 2.4.L)	presheaves on X	sheaves on X	sheafification	forgetful
(semi)groups (§1.5.3)	semigroups	groups	groupification	forgetful
sheaves, $\pi\colon X \to Y$ (Ex. 2.7.B)	sheaves on Y	sheaves on X	π^{-1}	π_*
sheaves of abelian groups or $\mathscr{O}$-modules, open embeddings $\pi\colon U \hookrightarrow Y$ (Ex. 2.7.G)	sheaves on U	sheaves on Y	$\pi_!$	π^{-1}
quasicoherent sheaves, $\pi\colon X \to Y$ (Prop. 16.3.6)	$QCoh_Y$	$QCoh_X$	π^*	π_*
ring maps $B \to A$ (Ex. 30.3.A)	Mod_A	Mod_B	$M \mapsto M_B$ (restriction of scalars)	$N \mapsto \mathrm{Hom}_B(A, N)$
quasicoherent sheaves, affine $\pi\colon X \to Y$ (Ex. 30.3.B(b))	$QCoh_X$	$QCoh_Y$	π_*	$\pi^!_{sh}$

Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.4).

E.g. $\operatorname{Spec}\mathbb{C} \longrightarrow \operatorname{Spec}\mathbb{R}$ ring space but not scheme

$\left(f^{-1}: \mathbb{R}\text{-mod} \longrightarrow (\operatorname{Spec}\mathbb{C}, \widehat{\mathbb{R}})\text{-mod}\right)$

quasi-coherent: $f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{C}\text{-mod}$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$

$f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{R}\text{-mod}$ $\quad$ forget

$f^*f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{C}\text{-mod}$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$ ← $\mathbb{C}$-mod structure

$f_*f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{R}\text{-mod}$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$

étale: $f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow \mathcal{A}b$ $\quad$ forget

$f_*: \mathcal{A}b \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod}$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$

$f^*f_*: \mathcal{A}b \longrightarrow \mathcal{A}b$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$

$f_*f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod}$ $\quad - \otimes_{\mathbb{R}} \mathbb{C}$ ← $\operatorname{Gal}(\mathbb{C}/\mathbb{R})$-action

$\pi: X_{ét} \longrightarrow X_{Zar}$

$\pi^{-1}: Sh(X_{Zar}) \longrightarrow Sh(X_{ét})$ $\quad$ [Zariski sheafification $\mathcal{F} \longmapsto \left[\pi^{-1}\mathcal{F}: \begin{smallmatrix} U \\ \downarrow i \\ X \end{smallmatrix} \longmapsto \Gamma(U, i^{-1}\mathcal{F})\right]$ is not enough...]

$\pi_*: Sh(X_{ét}) \longrightarrow Sh(X_{Zar})$ $\quad$ forget

$\pi^{-1} \vdash \pi_*$ $\quad \pi_*\pi^{-1} = Id_{Sh(X_{Zar})} \Rightarrow \pi^{-1}$ fully faithful

https://math.stackexchange.com/questions/321633/when-is-a-fully-faithful-functor-an-equivalant-functor

$$\begin{tikzcd}
 & QCoh(X_{Zar}) \arrow[dl, "W"'] \arrow[d, "\iota"] \arrow[dl, phantom, shift left=4, "\times"] \\
Sh(X_{ét}) \arrow[r, shift left, "\pi_*"] & Sh(X_{Zar}) \arrow[l, shift left, "\pi^*"]
\end{tikzcd}$$

full subcategory

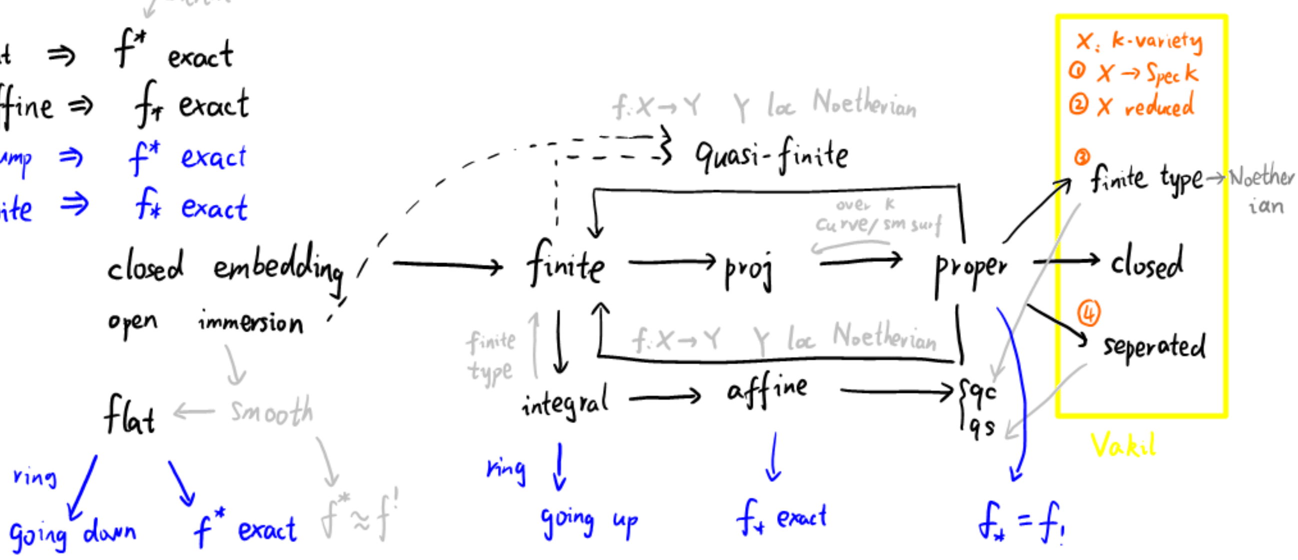
18.1.7

f flat $\Rightarrow$ f^* exact (coherent)

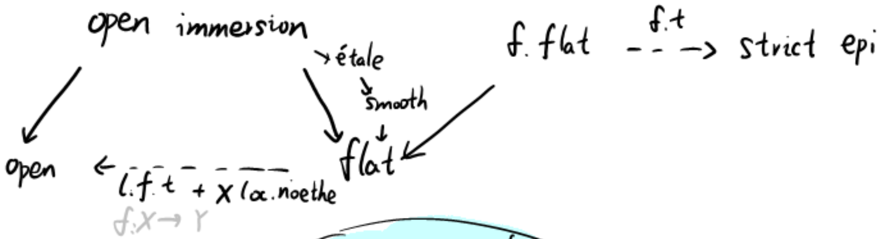
f affine $\Rightarrow$ f_* exact

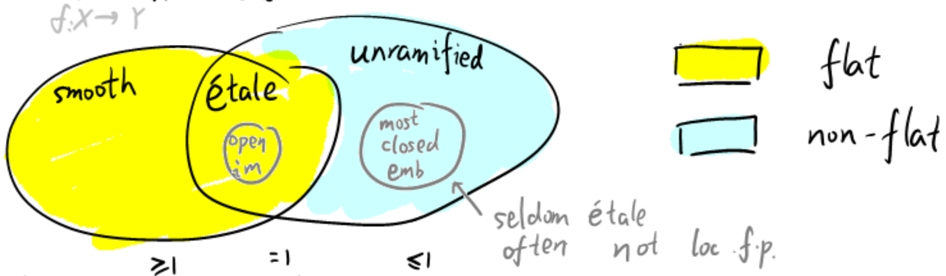
étale: mild assump $\Rightarrow$ f^* exact

f finite $\Rightarrow$ f_* exact



Various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercises 7.3.B(b), 7.3.D, and 10.1.H resp.) are quasiseparated, f.t.=> Noetherian: https://math.stackexchange.com/questions/707877/noetherian-schemes-and-varieties





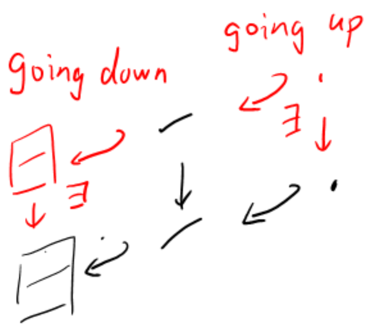
Rmk locally f.p is crucial.

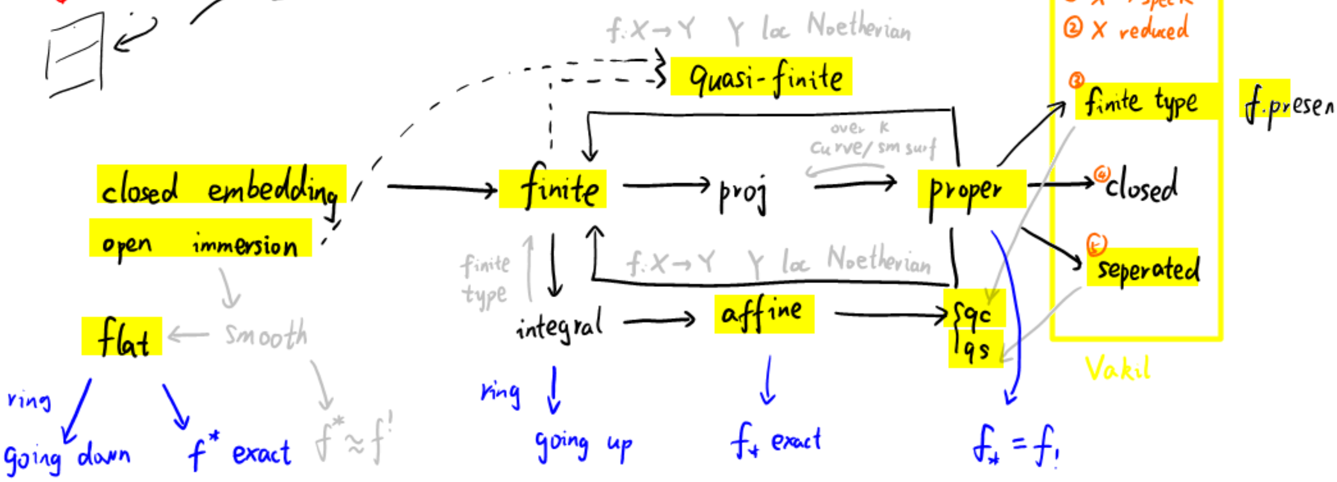
lift

e.g. $\mathbb{C}[t^\lambda : \lambda\in\mathbb{Q}_{>0}] \longrightarrow \mathbb{C}$, $t^\lambda \longmapsto 0$ is formally étale but not flat

https://rankeya.people.uic.edu/formallyunramifiedetale.pdf
https://mathoverflow.net/questions/288466/idea-behind-grothendiecks-proof-that-formally-smooth-implies-flat

$\mathbb{F}_p[T] \longrightarrow (\mathbb{F}_p[T])^{sep}$ is formally unramified, not formally étale but flat.





: properties which satisfy the f.flat descent.