

# Eine Woche, ein Beispiel

## 4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

<https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve>

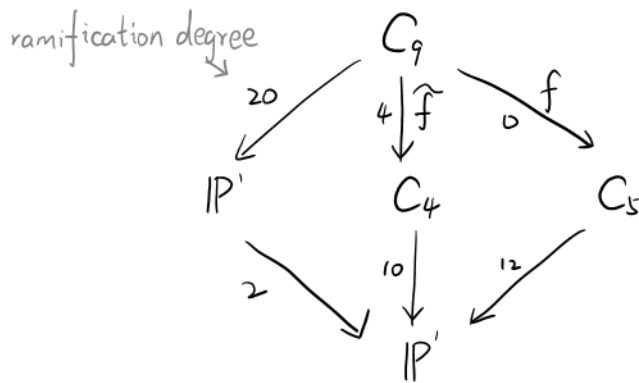
<https://mathoverflow.net/questions/402049/induced-action-on-prym-variety>

Goal: Describe some curves (maybe singular)  $C$  in  $A$ , and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$  has the following covers:  
 $\text{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



where

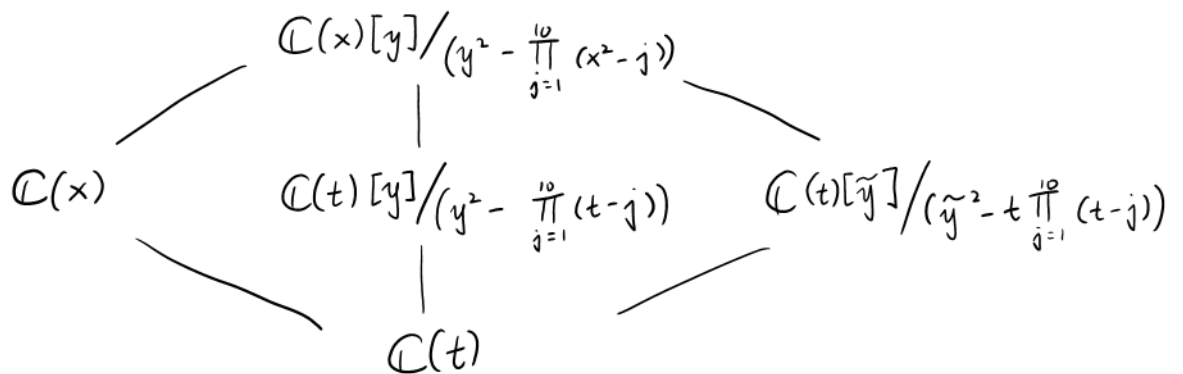
$$C_4 = \{y^2 = \prod_{j=1}^{10} (t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t \prod_{j=1}^{10} (t - j)\}$$

$$t = x^2$$

$$\tilde{y} = xy$$

The crspd field extension:



## Global differential forms

Pulling back differential forms give the following maps:

$$\begin{array}{ccccc}
 & & \langle x^k \frac{dx}{y} \mid k=0, \dots, 8 \rangle & & \\
 & \nearrow & \uparrow \tilde{f}^* & \nwarrow f^* & \\
 & & \langle 2x^{2k+1} \frac{dx}{y} \mid k=0, \dots, 3 \rangle & & \langle 2x^{2k} \frac{dx}{y} \mid k=0, \dots, 4 \rangle \\
 0 & \nwarrow & \parallel & \nearrow & \\
 & & \langle t^k \frac{dt}{y} \mid k=0, \dots, 3 \rangle & & \langle 2t^k \frac{dt}{y} \mid k=0, \dots, 4 \rangle \\
 & \nearrow & \downarrow & \nwarrow & \\
 & & 0 & & 
 \end{array}$$

Therefore,

$$H^0(C_9; \omega_{C_9}) \cong \tilde{f}^* H^0(C_4; \omega_{C_4}) \oplus f^* H^0(C_5; \omega_{C_5}) \quad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which corresponds to pulling back of divisors)

$$\begin{array}{ccccc}
 & & \langle x^k \frac{dx}{y} \mid k=0, \dots, 8 \rangle & & \\
 & \nwarrow & \downarrow \tilde{f}_* & \searrow f_* & \\
 & & \langle 2x^{2k+1} \frac{dx}{y} \mid k=0, \dots, 3 \rangle & & \langle 2x^{2k} \frac{dx}{y} \mid k=0, \dots, 4 \rangle \\
 0 & \nwarrow & \parallel & \nearrow & \\
 & & \langle t^k \frac{dt}{y} \mid k=0, \dots, 3 \rangle & & \langle 2t^k \frac{dt}{y} \mid k=0, \dots, 4 \rangle \\
 & \nearrow & \downarrow & \nwarrow & \\
 & & 0 & & 
 \end{array}$$

However, since  $\text{Jac}(C) = H^0(C; \omega_C)^* / H_1(C; \mathbb{Z})$ , we are working on the dual spaces. The notations are again switched:

$$\begin{array}{ccc}
 f^* & \rightsquigarrow & N_{mf} \\
 f_* & \rightsquigarrow & f^*
 \end{array}$$

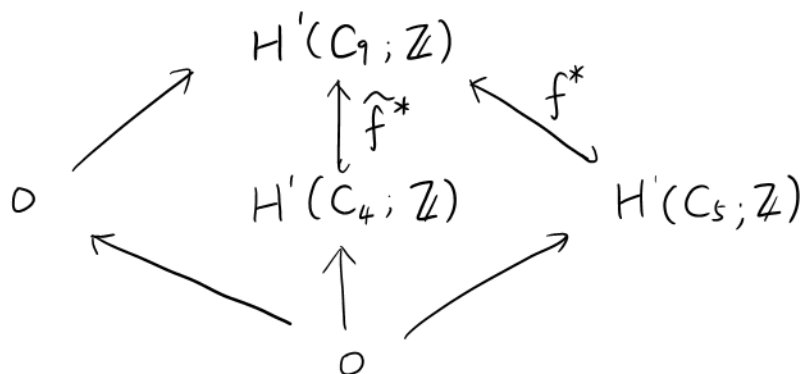
One may get

$$H^0(C_9; \omega_{C_9})^* \cong \tilde{f}^* H^0(C_4; \omega_{C_4})^* \oplus f^* H^0(C_5; \omega_{C_5})^* \quad (2)$$

different meaning compared with (1)!

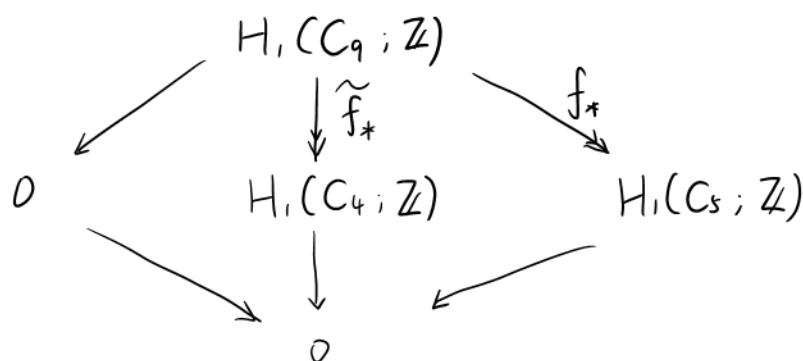
### (co)homology class

This page may be easier to understand, and it helps to understand the previous page.



Q: Do we have

$$H'(C_9; \mathbb{Z}) \cong \tilde{f}^* H'(C_4; \mathbb{Z}) \oplus f^* H'(C_5; \mathbb{Z})?$$



Q: Do we have

$$H_1(C_9; \mathbb{Z})^* \cong \tilde{f}^* H_1(C_4; \mathbb{Z})^* \oplus f^* H_1(C_5; \mathbb{Z})^*?$$

## Curve in Prym variety

Define  $A$  as the quotient of Jacobians, i.e.,

$$A := \text{Jac}(C_9) / f^* \text{Jac}(C_5) \cong \text{Prym}(C_9/C_5)$$

$$\begin{array}{ccccccc}
 & & C_9 & \longrightarrow & C_4 & \xrightarrow{\quad} & \text{Jac}(C_4) \\
 & & \downarrow A|_{C_9} & & \downarrow & \nearrow \exists! \text{ isogeny} & \\
 0 \longrightarrow & \text{Jac}(C_5) & \xrightarrow{f^*} & \text{Jac}(C_9) & \xrightarrow{\pi} & A & \longrightarrow 0
 \end{array} \quad (3)$$

- Prop.
0.  $A$  is isogenous to  $\text{Jac}(C_4)$ ;
  1.  $f^*: \text{Jac}(C_5) \rightarrow \text{Jac}(C_9)$  is injective;
  2.  $\pi \circ A|_{C_9}$  is not injective, it factors through  $C_4$ ;
  3.  $C_4 \rightarrow A$  is generically injective;
  4.  $C_4 \rightarrow A$  produces a sm image of  $A$ , outside of non-injective locus.

Idea: observe everything from the tangent space.

Proof.

0. Taking the tangent space of (3), one gets

$$0 \rightarrow H^0(C_5, \omega_{C_5})^* \xrightarrow{df^*} H^0(C_9, \omega_{C_9})^* \rightarrow T_0 A \rightarrow 0$$

Combined with (2),

$$T_0 A \cong H^0(C_4, \omega_{C_4})^*.$$

Late we will find a natural isogeny  $\text{Jac}(C_4) \rightarrow A$ .  
What's the degree of this isogeny?

1. Since

$$Nm_f \circ f^* = 2 \text{Id}_{\text{Jac}(C_5)} \quad \& \text{char } K \neq 2,$$

$f^*$  is injective.

2. For  $p_1 = (x_0, y_0)$ ,  $p_2 = (-x_0, y_0)$ , we want to show that

$$\int_{\gamma_1: p \sim p_1} x^{2k+1} \frac{dx}{y} = \int_{\gamma_2: p \sim p_2} x^{2k+1} \frac{dx}{y}$$

$$\text{LHS} = \int_{\gamma_1: p \sim p_2} (-x)^{2k+1} \frac{d(-x)}{y} = \text{RHS}.$$

3.

<https://mathoverflow.net/questions/68503/has-anyone-studied-the-ptym-map-for-double-covers-with-two-ramification-points>  
<https://arxiv.org/abs/1010.4483>: It proves that many Prym maps  $(C \rightarrow \text{Prym})$  are generically finite.

Notice:  $C_4 \subset \text{Jac}(C_4)$  is only invariant under  $p \mapsto -p$ ,  
 not invariant under  $p \mapsto p + a_0$ .

Otherwise, the Gauss map would be cover of  $\deg > 2$ .

Therefore, after isogeny  $C_4 \rightarrow A$  is still gen inj.

Q: Is this map really inj?

4.  $C_4 \hookrightarrow \text{Jac}(C_4)$  is sm, so after isogeny it is still sm  
 outside of non-injective locus.