Eine Woche, ein Beispiel

4.13 lattices defining abelian variety

Ref:

[Deb99]: Complex tori and abelian varieties
[Mum74]: Mumford, David, Abelian varieties. Oxford university press Oxford, 1974.

This document try to work out [Deb99, p28, Ex (3)].

Claim 1. [Mum74, p35, (1)⇔(4)]
Let $\Lambda \subseteq \mathbb{C}^g$ be a full lattice. Then

$\mathbb{C}^g/\Lambda$ is an abelian variety
$\Leftrightarrow$ $\exists$ an $\mathbb{R}$-bilinear alternating form $\omega: \mathbb{C}^g \times \mathbb{C}^g \longrightarrow \mathbb{R}$ s.t.

$$\begin{cases} \omega(\Lambda \times \Lambda) \subseteq \mathbb{Z} \\ \omega(x, ix) > 0 \quad \forall x \neq 0 \\ \omega(ix, iy) = \omega(x, y) \end{cases} \qquad (*)$$

i.e. an integral Kähler form
$$\omega(u,v) = \mathrm{Re}\, h(iu,v) = \mathrm{Im}\, h(u,v) \qquad h(au,v) = \bar{a} h(u,v)$$
$$= -\mathrm{Re}\, h(u,iv)$$

From now on. suppose $\Lambda = \langle v_1, \dots, v_{2g} \rangle_{\mathbb{Z}}$, we denote

$$A := (a_{ij})_{i,j=1}^{2g} := (v_1^*, \dots, v_{2g}^*)^T \qquad \Rightarrow v_i^* = \sum a_{ij} e_j^*$$

The matrix A encodes all information of the lattice (add a basis)

Q: For what kind of conditions of A, can we find ω satisfying $(*)$?

Let $\omega = \sum_{i<j} c_{ij}\, v_i^* \wedge v_j^*$, then

$$\omega(\Lambda \times \Lambda) \subset \mathbb{Z} \iff c_{ij} \in \mathbb{Z} \qquad \forall i,j.$$

Write $x = \sum x_i e_i$, $y = \sum y_i e_i$, we get

$$\begin{aligned}
\omega(x,y) &= \sum_{k,l} x_k y_l\, \omega(e_k, e_l) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\; v_i^* \wedge v_j^* \,(e_k, e_l) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\, (a_{ik} a_{jl} - a_{jk} a_{il}) \\
\omega(x,iy) &= \sum_{k,l} x_k y_{l-g} \sum_{i<j} c_{ij}\, (a_{ik} a_{jl} - a_{jk} a_{il}) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\, (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \\
\omega(ix,iy) &= \sum_{k,l} x_{k-g} y_{l-g} \sum_{i<j} c_{ij}\, (a_{ik} a_{jl} - a_{jk} a_{il}) \\
&= \sum_{k,l} x_k y_l \sum_{i<j} c_{ij}\, (a_{i(k+g)} a_{j(l+g)} - a_{j(k+g)} a_{i(l+g)})
\end{aligned}$$

Therefore,

$\omega(x, ix) > 0$ is an open condition on A.

$$\omega(ix,iy) = \omega(x,y) \iff \sum_{i<j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0$$

$$\forall\ k,l \in \{1, \dots, 2g\}$$

Claim 2. $\mathbb{C}^g/\Lambda$ is an abelian variety

$\Lambda = \langle v_1, \dots, v_{2g} \rangle_{\mathbb{Z}}$

$A = (a_{ij})_{i,j=1}^{2g} = (v_1^*, \dots, v_{2g}^*)^T$

$\iff \exists\ c_{ij} \in \mathbb{Z}$ for all $i,j \in \{1, \dots, 2g\}$ s.t.

① $\sum_{i<j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0 \quad \forall k,l \in \{1, \dots, 2g\}$

② $\left(\sum_{i<j} c_{ij}\, (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \right)_{k,l=1}^{2g}$ is def positive.

Rmk. The equations in ① are not linear independent.
There are at most $\binom{g}{2}\cdot\frac{4\cdot 2}{4} = g(g-1)$ equations.

Reason: When $l=k$ or $l=k+g$, we get 0.
e.p. when $g=1$, we get no condition.

When $l\neq k$ & $l\neq k+g$, denote $\{k,l,k+g,l+g\}=\{k_1,k_2,k_3,k_4\}$,

$$a_{ik_1}a_{jk_2}-a_{ik_2}a_{jk_1}-a_{ik_3}a_{jk_4}+a_{ik_4}a_{jk_3}$$

$k_1\longleftrightarrow k_2 \quad k_3\longleftrightarrow k_4$ neg
$k_1\longleftrightarrow k_3 \quad k_2\longleftrightarrow k_4$ neg
$k_1\longleftrightarrow k_4 \quad k_2\longleftrightarrow k_3$ same

Cor. Since

$$\mathcal{A}_g \subseteq \{\mathbb{C}^n/\Lambda \text{ a.v}\} \cong \{A\in\mathbb{R}^{2g\times 2g} \mid A \text{ satisfies ①② for some } (c_{ij})\}/GL_g(\mathbb{C})$$

We know

$$\dim \mathcal{A}_g = \frac{1}{2}\left((2g)^2-g(g-1)\right)-g^2 = \frac{1}{2}g(g+1).$$