Preview of $\mathrm{Ext}_A^n(M,N)$

How do you think of the importance of Homological Algebra?

1. Def of $\mathrm{Ext}_A^n(M,N)$

$$E_A(M,N) = \{0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0\}/\text{equivalence}$$

$$= \{\text{proj resolution } P_\cdot,\ H^n(\mathrm{Hom}_A(P_\cdot,N))\}/\text{resolution}$$

$$= \{\text{inj resolution } I^\cdot,\ H^n(\mathrm{Hom}_A(M,I^\cdot))\}/\text{resolution}$$

$$= \{\text{derivation}\}/\text{inner derivation}$$

devi
=
SES
G proj = = inj

2. Special module/ring interact with Ext?

$$P \text{ proj} \Leftrightarrow \mathrm{Ext}_A^n(P,-)=0 \quad \forall n\geq 1 \Leftrightarrow \mathrm{Ext}_A^1(P,-)=0$$

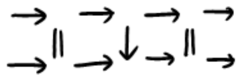
$$\Leftrightarrow \text{proj} \dim P = 0$$

$$I \text{ proj} \Leftrightarrow \mathrm{Ext}_A^n(-,I)=0 \quad \forall n\geq 1 \Leftrightarrow \mathrm{Ext}_A^1(-,I)=0$$

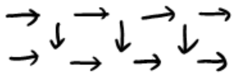
A f.d alg $\quad \dim_k \mathrm{Ext}_A^1(S(i),S(j)) = \dim_k \mathrm{Hom}_A(\mathrm{rad}(P(i)),S(j))$

$$\overset{A=kQ/I}{=\!=} |\{a\in Q_1 \mid s(a)=i,\ t(a)=j\}|$$

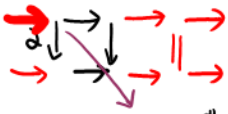
Second level of detail.

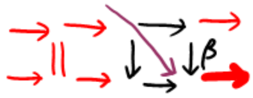
equivalent of SES 

$\Downarrow$

isomorphic 

$$\Uparrow \times: \quad \begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, equal] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\times 2"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] & \mathbb{Z} \arrow[r, "2\cdot\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

pushout α_* 

pullback β^* 

① ② ③* 

+ com ass 0   k-linear $(\lambda+\mu)a$ $(\lambda\mu)a$ $1a$ $\lambda(a+b)$

$\Rightarrow E_A(M,N)$: ① Def, ② bifunctor and ③* k-linear space structure ① ⇒ ② ⇒ ③

$f_\cdot \sim g_\cdot \quad \Rightarrow H_n(f_\cdot) = H_n(g_\cdot)$

$g_\cdot f_\cdot \sim Id \quad f_\cdot g_\cdot \sim Id \quad \Rightarrow \quad H_n(C_\cdot) = H_n(C'_\cdot)$

$\Rightarrow Ext_A^n(M,N)$: ① Def, ② bifunctor and ③ k-linear space structure ① ⇒ ③, ① ⇒ ②

$\Rightarrow E_A(M,N) \to Ext_A^1(M,N)$ ① well-defined by resolution & lift & equiv
② bifunctor
③ k-linear map

Schanuel's lemma
$$\left.\begin{array}{l} 0 \to U \to P \to M \to 0 \\ 0 \to U' \to P' \to M \to 0 \\ \quad P, P' \text{ proj} \end{array}\right\} \Rightarrow U \oplus P' \cong U' \oplus P$$

$0 \to U \to X \to V \to 0$ $\begin{cases} \text{non-split} \\ \text{f.d. } A\text{-mod} \end{cases}$ $\Rightarrow \dim_k End_A(X) < \dim_k End_A(U \oplus V)$

for f.d. A-mod $0 \to U \to X \to V \to 0$ split $\Leftrightarrow X \cong U \oplus V$ as A-module

Third level of details: purely for difficulty?

- submodules in Snake lemma
- realize J-mod complex: $C(\mathrm{Mod}(F(J))) \hookrightarrow \mathcal{M}\mathrm{od}(F(J'))$ similar to $\mathrm{Mod}(kQ) \cong \mathrm{Rep}(Q)$
- factor category C/I, stable module category, bounded homotopy category
- relations between $S(i)$, $P(i)$, $I(i)$, rad, soc, top, semisimple, minimal proj resolution, duality
- $0 \to \mathrm{Hom}_A(V,U) \to \mathrm{Hom}_k(V,U) \xrightarrow{d_{V,U}} Z_A(V,U) \xrightarrow{\phi_{V,U}} E_A(V,U) \to 0$ & res rep embedding by orthogonal bricks.

The most important A-modules are:

$$\begin{array}{rll} {}_AA \rightsquigarrow & P(1), \dots, P(n) & \text{indecomposable projective } A\text{-modules} \\ D(A_A) \rightsquigarrow & I(1), \dots, I(n) & \text{indecomposable injective } A\text{-modules} \\ A/J(A) \rightsquigarrow & S(1), \dots, S(n) & \text{simple } A\text{-modules} \end{array}$$

We can label these modules such that

$$\mathrm{top}(P(i)) \cong S(i) \cong \mathrm{soc}(I(i))$$

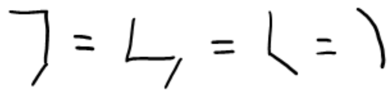
for $1 \le i \le n$.

Fourth level of details: (partly) 人类驯服野生正合列的珍贵图像

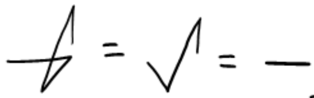
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "h_1"] & B \arrow[r, "g"] \arrow[d, "h_2"] \arrow[ddr, bend left, "g"] & C \arrow[r] \arrow[d, dashed, "\Phi_C"] & 0 \\
0 \arrow[r] & D \arrow[r, "f'"] \arrow[drr, bend right, "0"'] & E \arrow[r, "g'"] \arrow[dr, dashed, "\Phi_E"'] & \mathrm{coker} f' \arrow[r] \arrow[d, dashed, "\Phi_F"] & 0 \\
 & & & C &
\end{tikzcd}$$

① $g' = \Phi_C \circ \Phi_E$?

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h_1"'] & B \arrow[d, "h_2"] \arrow[ddr, bend left, "\Phi_C \circ g"] & \\
D \arrow[r, "f'"] \arrow[drr, bend right, "0"'] & E \arrow[dr, shift left, "\Phi_C \circ \Phi_E"] \arrow[dr, shift right, "g'"'] & \\
 & & \mathrm{coker} f
\end{tikzcd}$$

② 

$g \circ \Phi_C \Phi_F = g \Rightarrow \Phi_C \Phi_F = Id_C$

③ 

$g' \circ \Phi_F \Phi_C = g' \Rightarrow \Phi_F \Phi_C = Id_{\mathrm{coker} f'}$

$$\begin{tikzcd}
{} \arrow[r] & P_2 \arrow[rrrr, "p_2"] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "f_2"] & & & & P_1 \arrow[rrrr, "p_1"] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "f_1"] \arrow[dllll, dashed, "s_1"'] \arrow[ddll, "f_1 - Id - s_0 p_1"] & & & & P_0 \arrow[r, "p_0"] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "f_0"] \arrow[dllll, dashed, "s_0"'] \arrow[ddll, "f_0 - Id"] & M \arrow[r] \arrow[d, equal] \arrow[dl, dashed, "0"'] & 0 \\
{} \arrow[r] & P_2' \arrow[rrrr, "p_2'"] \arrow[drr] & & & & P_1' \arrow[rrrr, "p_1'"] \arrow[drr] & & & & P_0' \arrow[r, "p_0'"] & M \arrow[r] & 0 \\
 & & & \Omega^2(M) \arrow[urr] \arrow[dr] & & & & \Omega^1(M) \arrow[urr] \arrow[dr] & & & & \\
 & & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$