Eine Woche, ein Beispiel
2.13. outer automorphism

We do something very elementary but tricky, and will later find out its connection to the advanced topic, like Teichmüller space.

1. outer automorphism group Out(G) / automorphism group Aut(G)

Def. Let G be a group. We have a LES

$$1 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\text{conj}} Aut(G) \longrightarrow Out(G) \longrightarrow 1$$

where $Z(G)$ is the center of G
  $Aut(G)$ is the automorphism of G
  $Inn(G) := Im(conj)$ is the inner automorphism of G
  $Out(G) := Aut(G)/Inn(G)$ is the outer automorphism of G.

E.g. When G is commutative, $Inn(G) = Id$, $Out(G) = Aut(G)$.
  $G = \mathbb{Z}$, $Aut(\mathbb{Z}) = \{\pm 1\}$,
  $G = \mathbb{Z}/m\mathbb{Z}$, see https://zhuanlan.zhihu.com/p/97195375 ← typo. @ ⇒ 2 $(m \geq 2)$
        an easy result is that $\#Out(\mathbb{Z}/m\mathbb{Z}) = \varphi(m)$.
  e.p. $Aut(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\times} \cong \mathbb{Z}/(p-1)\mathbb{Z}$

2. Reduced to indecomposable group

Def. A group G is indecomposable if
$$G \cong A \times B \implies A \cong Id \text{ or } B \cong Id.$$

Let H be an indecomposable finite group, and let $G = H \times \cdots \times H \triangleq H_1 \times H_2 \times \cdots H_n \triangleq H^n$

Case 1. [Thm 3.1] H is non-abelian, then $\text{Aut } G = \mathcal{A} \rtimes S_n$, where

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} : \alpha_{ij} \in \begin{cases} \text{Hom}(H_j, Z(H_i)) & i \neq j \\ \text{Aut } H_i & i = j \end{cases} \right\}.$$

$$\begin{array}{ccc} & S_n & S_n \to \text{Aut}(\mathcal{A}) \\ \nearrow & \downarrow \text{Id} & \text{by matrix conjugation} \\ 0 \to \mathcal{A} \longrightarrow \text{Aut } G \longrightarrow S_n \longrightarrow 0 \end{array}$$

Case 2. H is abelian, then $H \cong \mathbb{Z}/p^r\mathbb{Z}$, $\text{Aut } G \cong GL(n, \mathbb{Z}/p^r\mathbb{Z})$

Thm. If H_i are all abelian, then

$$\text{Aut}\left(\bigoplus_{i=1}^{n} H_i \right) = \left\{ A = (a_{ij})_{i,j=1}^{n} \; \middle| \; \begin{array}{l} a_{ij} \in \text{Hom}(H_i, H_j) \\ A \text{ is invertible} \end{array} \right\}$$

Thm Let H & N be two finite group with no common direct factor
(i.e., $H \cong A \times B$ $N \cong A \times C \Rightarrow A \cong \text{Id}$), then

$$\text{Aut}(H \times N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \; \middle| \; \begin{array}{ll} \alpha \in \text{Aut}(H) & \beta \in \text{Hom}(H, Z(N)) \\ \gamma \in \text{Hom}(N, Z(H)) & \delta \in \text{Aut}(N) \end{array} \right\}$$

Cor. **Theorem 2.2.** *Let $G = H_1 \times \ldots \times H_n$ where no pair of the H_i $(1 \leq i \leq n)$ have a common direct factor. Then $\text{Aut } G \cong \mathcal{A}$ where*

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} : \alpha_{ij} \in \begin{cases} \text{Aut } H_i & i = j \\ \text{Hom}(H_j, Z(H_i)) & i \neq j \end{cases} \right\}.$$

Thus, the computation of $\text{Aut } G$ reduced to the case where G is indecomposable.

Cor. One can compute the automorphism of any finite abelian group G, and also $\#\text{Aut}(G)$.

$$\text{Aut}(\mathbb{Z}^n) \cong \text{Out}(\mathbb{Z}^n) \cong GL(n, \mathbb{Z}) \quad \text{is known}.$$

2. D_n, D_∞ and Q_8

$$D_n = \langle a, b \mid a^n = b^2 = 1, (ab)^2 = 1 \rangle$$
$$\text{Aut}(D_n) \cong \mathbb{Z}/n\mathbb{Z} \rtimes \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})^\times \qquad \text{where}$$

$$
\begin{array}{c}
 & & & \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \\
 & & & \| \\
0 \longrightarrow \mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z} \rtimes \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \longrightarrow \text{Aut}(\mathbb{Z}/n\mathbb{Z}) \longrightarrow 0
\end{array}
$$

$$D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$
$$\text{Aut}(D_\infty) \cong D_\infty$$

n	2	3	4	5	6	7		$\cdots$		∞
$\text{Aut}(D_n)$	S_3	S_3	D_4	F_5	D_6	F_7		$\cdots$		D_∞
$\text{Out}(D_n)$	S_3	Id	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$		$\cdots$		$\mathbb{Z}/2\mathbb{Z}$

$$Q_8 = \{-1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\}$$
$$\text{Aut}(Q_8) \cong S_4$$

3. S_n & A_n.

E.g. $G = S_n$, $(n \in \mathbb{N}_{>0})$
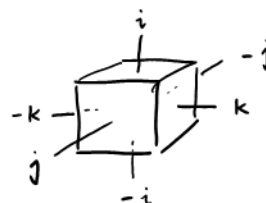
$$\text{Aut}(S_n) = \begin{cases} S_n & n \neq 2, 6 \\ \{*\} & n = 2 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\text{Out}(S_n) = \begin{cases} \{*\} & n \neq 6 \\ \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$G = A_n$.

$$\text{Aut}(A_n) = \begin{cases} S_n & n \neq 2, 3, 6 \\ \{*\} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 3 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\text{Out}(A_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \neq 2, 3, 6 \\ \{*\} & n = 2 \text{ or } 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

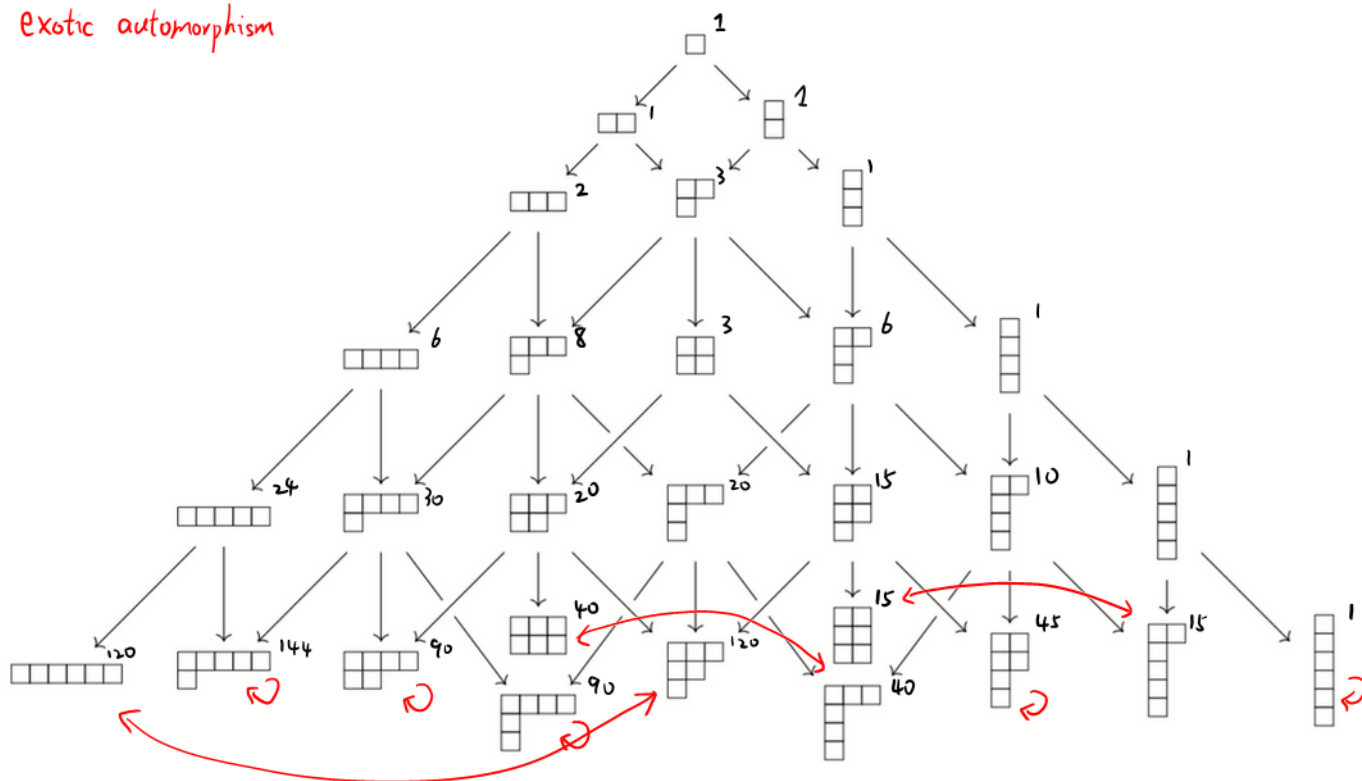For a reference of the proof and constructions of the exotic outer automorphism of S_6, see wiki and here:
https://wordpress.nmsu.edu/pamorand/files/2018/10/AutGroups.pdf
For Chinese you can also see here: https://zhuanlan.zhihu.com/p/24764617
They are elementary and everybody who have learned something about Sylow's theorem should be able to follow the proofs.

$\#\{$ elements in conj class $\boxed{\begin{smallmatrix}1\end{smallmatrix}} = (123)\}$

exotic automorphism

The diagram shows Young diagrams with numbers labeling conjugacy class sizes:

1 · 1 · 1 · 2 · 3 · 1 · 6 · 8 · 3 · 6 · 1 · 24 · 30 · 20 · 20 · 15 · 10 · 1 · 120 · 144 · 90 · 40 · 90 · 120 · 15 · 45 · 15 · 90 · 40 · 1

E.g. $\quad G = PSL(2, \mathbb{F}_7) \cong GL(3, \mathbb{F}_2)$

$\quad\quad Aut(PSL(2, \mathbb{F}_7)) \cong PGL(2, \mathbb{F}_7) \quad\quad Out(PSL(2, \mathbb{F}_7)) \cong \{\pm 1\}$

E.g. $\quad F_n :=$ free group generated by $a_1, \ldots, a_n$

$\quad\quad F_n \longrightarrow F_n/_{[F_n, F_n]} \cong \mathbb{Z}^{\oplus n} \quad\quad \rightsquigarrow \quad Out(F_n) \longrightarrow GL(n, \mathbb{Z})$

$\quad\quad$ It's claimed that $Out(F_2) \cong GL(2, \mathbb{Z})$.

Left: f.g. abelian group, like $\mathbb{Z}^n$. $(Aut(\mathbb{Z}^n) \cong Out(\mathbb{Z}^n) \cong GL(n, \mathbb{Z}))$

4. Profinite group

Now we consider automorphism in the category of profinite gp.

Lemma.
$$\mathrm{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l = m \\ 0 & l \neq m \end{cases} \qquad l, m \text{ prime.}$$

Cor.
$$\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$$
$$\mathrm{Aut}(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}^\times := \prod_p (\mathbb{Z}_p^\times)$$
$$\mathrm{Aut}(\widehat{\mathbb{Z}}^{(p)}) = \widehat{\mathbb{Z}}^{\times(p)}$$

$$\widehat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l \qquad \widehat{\mathbb{Z}}^{\times(p)} := \prod_{l \neq p} \mathbb{Z}_l^\times$$