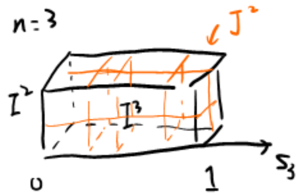
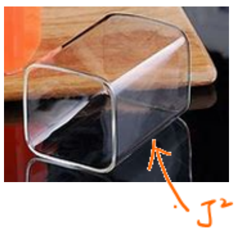
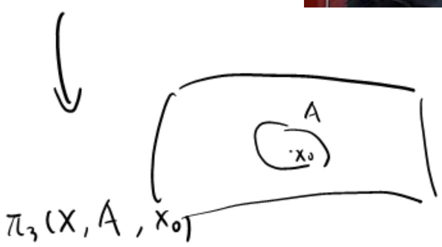
(2) **Theorem 4.41.** *Suppose $p: E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence if B is path-connected, there is a long exact sequence*

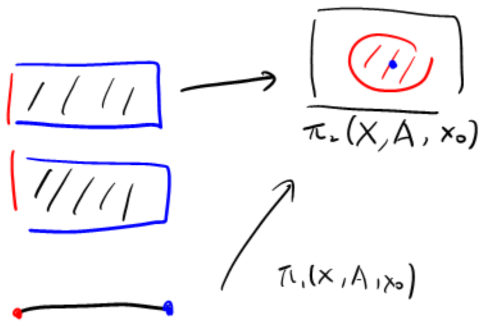
$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

Very useful generalizations of the homotopy groups $\pi_n(X, x_0)$ are the **relative homotopy groups** $\pi_n(X, A, x_0)$ for a pair (X, A) with a basepoint $x_0 \in A$. To define these, regard I^{n-1} as the face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$, the union of the remaining faces of I^n. Then $\pi_n(X, A, x_0)$ for $n \geq 1$ is defined to be the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, with homotopies through maps of the same form. There does not seem to be a completely satisfactory way of defining $\pi_0(X, A, x_0)$, so we shall leave this undefined (but see the exercises for one possible definition). Note that $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$, so absolute homotopy groups are a special case of relative homotopy groups.









$$\pi_3(A) \to \pi_3(X) \to \pi_3(X,A) \quad (=0,\ =\mathbb{Z},\ \mathbb{Z})$$
$$\to \pi_2(A) \to \pi_2(X) \to \pi_2(X,A) \quad (=0,\ =\mathbb{Z},\ \mathbb{Z}^2)$$
$$\to \pi_1(A) \to \pi_1(X) \to \pi_1(X,A) \quad (=\mathbb{Z},\ =0,\ =0)$$
$$\to \pi_0(A) \to \pi_0(X) \quad (=\{*\},\ =\{*\})$$

e.g. $X = S^2$ $A = S^1$

Probably the most useful feature of the relative groups $\pi_n(X, A, x_0)$ is that they fit into a long exact sequence

(1) $$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

Here i and j are the inclusions $(A, x_0) \hookrightarrow (X, x_0)$ and $(X, x_0, x_0) \hookrightarrow (X, A, x_0)$. The map ∂ comes from restricting maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}, or by restricting maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$ to S^{n-1}. The map ∂, called the *boundary map*, is a homomorphism when $n > 1$.

Theorem 4.3. *This sequence is exact.*

Just as elements of $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, there is an alternative definition of $\pi_n(X, A, x_0)$ as the set of homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$, since collapsing J^{n-1} to a point converts $(I^n, \partial I^n, J^{n-1})$ into (D^n, S^{n-1}, s_0). From this viewpoint, addition is done via the map $c: D^n \to D^n \vee D^n$ collapsing $D^{n-1} \subset D^n$ to a point.

$\pi_i(A)$ exact by defn

$\Rightarrow$ $\pi_i(X)$ exact

$\pi_i(X, A)$ exact by defn

$\boxed{\leftarrow}$

A useful and conceptually enlightening reformulation of what it means for an element of $\pi_n(X, A, x_0)$ to be trivial is given by the following *compression criterion*:

- A map $f: (D^n, S^{n-1}, s_0) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ iff it is homotopic rel S^{n-1} to a map with image contained in A.

For if we have such a homotopy to a map g, then $[f] = [g]$ in $\pi_n(X, A, x_0)$, and $[g] = 0$ via the homotopy obtained by composing g with a deformation retraction of D^n onto s_0. Conversely, if $[f] = 0$ via a homotopy $F: D^n \times I \to X$, then by restricting F to a family of n-disks in $D^n \times I$ starting with $D^n \times \{0\}$ and ending with the disk $D^n \times \{1\} \cup S^{n-1} \times I$, all the disks in the family having the same boundary, then we get a homotopy from f to a map into A, stationary on S^{n-1}.

The proof will use a relative form of the homotopy lifting property. The map $p:E\to B$ is said to have the **homotopy lifting property for a pair (X,A)** if each homotopy $f_t:X\to B$ lifts to a homotopy $\widetilde{g}_t:X\to E$ starting with a given lift $\widetilde{g}_0$ and extending a given lift $\widetilde{g}_t:A\to E$. In other words, the homotopy lifting property for (X,A) is the lift extension property for $(X\times I, X\times\{0\}\cup A\times I)$.

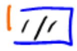
$$\begin{tikzcd} A\times I\cup X\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ X\times I \arrow[r, "f_t"'] \arrow[ur, dashed, "\tilde{g}_t"] & B \end{tikzcd}$$

The homotopy lifting property for D^k is equivalent to the homotopy lifting property for $(D^k,\partial D^k)$ since the pairs $(D^k\times I, D^k\times\{0\})$ and $(D^k\times I, D^k\times\{0\}\cup\partial D^k\times I)$ are homeomorphic. This implies that the homotopy lifting property for disks is equivalent to the homotopy lifting property for all CW pairs (X,A). For by induction over the skeleta of X it suffices to construct a lifting $\widetilde{g}_t$ one cell of $X-A$ at a time. Composing with the characteristic map $\Phi:D^k\to X$ of a cell then gives a reduction to the case $(X,A)=(D^k,\partial D^k)$. A map $p:E\to B$ satisfying the homotopy lifting property for disks is sometimes called a *Serre fibration*.

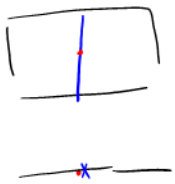
$$\begin{tikzcd} D^k\times\{0\} \arrow[r] \arrow[d, hook] & E \arrow[d, "p"] \\ D^k\times I \arrow[r, "f_t"'] \arrow[ur, dashed] & B \end{tikzcd}$$

Proof: First we show that p_* is onto. Represent an element of $\pi_n(B,b_0)$ by a map $f:(I^n,\partial I^n)\to(B,b_0)$. The constant map to x_0 provides a lift of f to E over the subspace $J^{n-1}\subset I^n$, so the relative homotopy lifting property for $(I^{n-1},\partial I^{n-1})$ extends this to a lift $\widetilde{f}:I^n\to E$, and this lift satisfies $\widetilde{f}(\partial I^n)\subset F$ since $f(\partial I^n)=b_0$. Then $\widetilde{f}$ represents an element of $\pi_n(E,F,x_0)$ with $p_*([\widetilde{f}])=[f]$ since $p\widetilde{f}=f$.

$$\begin{tikzcd} & (E,x_0) \arrow[d] \\ (I^n,\partial I^n) \arrow[r, "f"] & (B,b_0) \end{tikzcd}$$



Injectivity of p_* is similar. Given $\widetilde{f}_0,\widetilde{f}_1:(I^n,\partial I^n,J^{n-1})\to(E,F,x_0)$ such that $p_*([\widetilde{f}_0])=p_*([\widetilde{f}_1])$, let $G:(I^n\times I,\partial I^n\times I)\to(B,b_0)$ be a homotopy from $p\widetilde{f}_0$ to $p\widetilde{f}_1$. We have a partial lift $\widetilde{G}$ given by $\widetilde{f}_0$ on $I^n\times\{0\}$, $\widetilde{f}_1$ on $I^n\times\{1\}$, and the constant map to x_0 on $J^{n-1}\times I$. After permuting the last two coordinates of $I^n\times I$, the relative homotopy lifting property gives an extension of this partial lift to a full lift $\widetilde{G}:I^n\times I\to E$. This is a homotopy $\widetilde{f}_t:(I^n,\partial I^n,J^{n-1})\to(E,F,x_0)$ from $\widetilde{f}_0$ to $\widetilde{f}_1$. So p_* is injective.



For the last statement of the theorem we plug $\pi_n(B,b_0)$ in for $\pi_n(E,F,x_0)$ in the long exact sequence for the pair (E,F). The map $\pi_n(E,x_0)\to\pi_n(E,F,x_0)$ in the exact sequence then becomes the composition $\pi_n(E,x_0)\to\pi_n(E,F,x_0)\xrightarrow{p_*}\pi_n(B,b_0)$, which is just $p_*:\pi_n(E,x_0)\to\pi_n(B,b_0)$. The 0 at the end of the sequence, surjectivity of $\pi_0(F,x_0)\to\pi_0(E,x_0)$, comes from the hypothesis that B is path-connected since a path in E from an arbitrary point $x\in E$ to F can be obtained by lifting a path in B from $p(x)$ to b_0. $\square$