Eine Woche, ein Beispiel

1.23 Coxeter group

1. def & realizations
   - def
   - geometrical representation
   - root system
   - polytopes
   - as subgp of S_n
   - as Weyl gp of some Tits system

2. combinatorical results
3. Bruhat order
4. geometrical realization (faithfulness)

Roodmap

gen & relations $\longleftrightarrow$ characteristic properties $\rightarrow$ realizations.

wide river

In the first section, we omit technical details, which will be filled in later on. (Mainly: injectivity)

1. def & realizations

def

Def (Coxeter system) (W,S) gp + gen, $m(s,t)\in\mathbb{Z}_{>0}\cup\{+\infty\}$, $m(s,s)=1$

$$W=\langle s\in S\rangle/(s^2=(st)^{m(s,t)}=1,\ \forall s,t\in S)$$

W is a Coxeter gp if $\exists\ S\subseteq W$, (W,S) is a Coxeter system.

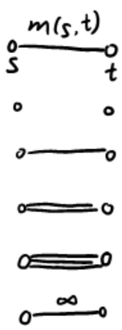
E.g.

$$S_n\cong\langle s_i\rangle_i/(s_i^2=(s_is_j)^2=(s_is_{i+1})^3=1)$$

$|i-j|\geq 2$, and undefined relations (eg. $(s_{n-1}s_n)^3$) should be removed.

Coxeter graph

$m(s,t)$
2
3
4
6
$+\infty$



Notation

S — simple reflections/transpositions

$l(\omega) = \min\{r \mid w = s_1 \cdots s_r,\ s_i \in S\}$ — length of $\omega \in W$

$T = \{wsw^{-1} \mid w \in W,\ s \in S\}$ — reflections /transpositions

geometrical representation

⚠ We suppose $|S| < \infty$, which is not necessary (but helpful for concentrating mind)

$$(W, S) \rightsquigarrow (\rho_{geo}, V_{geo}, \langle -,- \rangle) \in \mathrm{Rep}_{\mathbb{R},\mathrm{ortho}}(W)$$

$$V_{geo} := \bigoplus_{s \in S} \mathbb{R}\alpha_s$$

$$\langle -,- \rangle : V_{geo} \otimes V_{geo} \longrightarrow \mathbb{R}$$

$$(\alpha_s, \alpha_t) \longmapsto \begin{cases} -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq +\infty \\ -1 & m(s,t) = +\infty \end{cases}$$

$m(s,t)$	1	2	3	4	5	6	$\cdots$	∞
(α_s, α_t)	1	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{5}+1}{4}$	$-\frac{\sqrt{3}}{2}$	$\cdots$	-1

$$\rho_{geo} : W \longrightarrow GL(V_{geo})$$
$$s \longmapsto r_{\alpha_s}$$

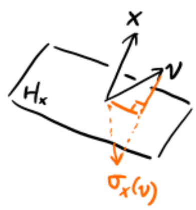
For $x, v \in V_{geo}$, $\langle x, x \rangle = 1$, define

$$r_x(v) = v - 2\langle v, x \rangle x$$

Check: $\sigma_x(x) = -x$

$\sigma_x(v) = v \iff v \in H_x$, where

$H_x = \{v \in V_{geo} \mid \langle v, x \rangle = 0\}$



Ex Verify the well-definess.

- $\rho_{geo}(s)$ is linear & orthogonal;
- $\rho_{geo}(\text{relations}) = Id$

Also, $\langle wv, wv' \rangle = \langle v, v' \rangle$.

Thm. ρ_{geo} is faithful (sketch of proof later on)