Eine Woche, ein Beispiel
10.23 equivariant K-theory of Steinberg variety : q-coefficient.

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

This time we focus on $G\times\mathbb{C}^*$-action.

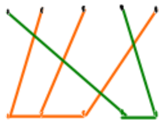
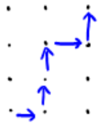
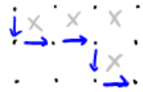
Task.
1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. change of basis
   - §6.1 two basis
   - §6.2 tangent space
   - §6.3 Euler class
   - §6.4 transition matrix, localization formula
   - §6.5 generators
7. convolution product
   - §7.1 clean intersection formula
   - §7.2. convolution for canonical basis
   - §7.3. expression of D_K.

We may use two examples for the convenience of presentation.
Readers can easily distinguish them by the dim vectors.

1. dimension vector

$|d| = 5$

$d = (3,2)$

$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \\ 0,0 \end{pmatrix} =$  $=$  $=$  $\in W_d \backslash \mathbb{W}_d$ or $Min(\mathbb{W}_{|d|}, W_d)$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$

$(1\ 4\ 5\ 3\ 2)$

Young Tableax

$r_\varpi = \pi_{\underline{d}}^{-1}(F_\varpi)$

2. Weyl group

Set	element	special element	others
$\mathbb{W}_{\lvert d\rvert} = S_5$	ϖ, x	$\varpi_{max} =$	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	ω	$\omega_{max} =$	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_d \backslash \mathbb{W}_{\lvert d\rvert} = S_3 \times S_2 \backslash S_5$	$\varpi, \underline{d}$		($Comp_d$)
$Min(\mathbb{W}_{\lvert d\rvert}, W_d) = \{ \ldots \}$	u		(Shuffled)

$$\begin{tikzcd} & & & Min(\mathbb{W}_{|d|}, W_d) \arrow[dl] \arrow[d, "\cong"] & & u \arrow[d, mapsto] \\ 0 \arrow[r] & W_d \arrow[r] & \mathbb{W}_{|d|} \arrow[r] & \mathbb{W}_{|d|} \backslash W_d \arrow[r] & 0 & \varpi = \omega u \arrow[r, mapsto] & \underline{d} \end{tikzcd}$$

$\varpi =$

$u =$

$\omega =$

Another example: $d = (1,2)$ $\quad a \longrightarrow b$, $\langle v_1 \rangle \longrightarrow \langle v_2, v_3 \rangle$

$\varpi = \omega u$				ω	$\underline{d}$, u	order of basis	$l(\varpi)$	$l(\omega)$	$\mathbb{B}_\varpi$	B_ϖ	$\varpi B \varpi^{-1}$
Id	Id	$\binom{1\,2\,3}{1\,2\,3}$	$\begin{bmatrix} 1&&\\&1&\\&&1 \end{bmatrix}$		abb	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} *&*&*\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$
t	(23)	$\binom{1\,2\,3}{1\,3\,2}$	$\begin{bmatrix} 1&&\\&&1\\&1& \end{bmatrix}$		abb	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} *&*&*\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$
s	(12)	$\binom{1\,2\,3}{2\,1\,3}$	$\begin{bmatrix} &1&\\1&&\\&&1 \end{bmatrix}$		bab	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} *&&*\\*&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&*\\&*&\\&&* \end{bmatrix}$
ts	(132)	$\binom{1\,2\,3}{3\,1\,2}$	$\begin{bmatrix} &1&\\&&1\\1&& \end{bmatrix}$		bab	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} *&*&\\&*&\\*&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&*&\\&*&\\&&* \end{bmatrix}$
st	(123)	$\binom{1\,2\,3}{2\,3\,1}$	$\begin{bmatrix} &&1\\1&&\\&1& \end{bmatrix}$		bba	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} *&&\\*&*&*\\*&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&*\\&&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\*&&* \end{bmatrix}$
sts	(13)	$\binom{1\,2\,3}{3\,2\,1}$	$\begin{bmatrix} &&1\\&1&\\1&& \end{bmatrix}$		bba	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} *&&\\*&*&\\*&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\&*&\\&*&* \end{bmatrix}$	$\begin{bmatrix} *&&\\*&*&\\&&* \end{bmatrix}$

3. alg group & Lie algebra

$\mathbb{G}_{|d|}, \mathbb{B}_{|d|}, \mathbb{T}_{|d|}, \mathbb{N}_{|d|}$ $\quad W_{|d|} = N_{\mathbb{G}_{|d|}}(\mathbb{T}_{|d|})/\mathbb{T}_{|d|}$ $\quad GL_5(\mathbb{C}) = \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$

G_d, B_d, T_d, N_d $\quad W_d = N_{G_d}(T_d)/T_d$ $\quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right)$

$$\mathbb{B}_\varpi = \varpi \mathbb{B}_{|d|} \varpi^{-1} = \mathrm{Stab}_{\mathbb{G}_{|d|}}(F_\varpi)$$

$$B_\varpi \overset{\varpi = wu}{=\!=} w B_d w^{-1} = \mathrm{Stab}_{G_d}(F_\varpi) \qquad N_\varpi = R_u(B_\varpi)$$

For $s \in \Pi$ s.t. $\varpi s \varpi^{-1} \in W_d$ (i.e. $W_d \varpi = W_d \varpi s$), define

$$P_{\varpi, \varpi s} :\overset{\varpi = wu}{=\!=} w(B_d u s u^{-1} B_d \cup B_d) w^{-1} \qquad N_{\varpi,\varpi s} = R_u(B_{\varpi,\varpi s})$$

$$= B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi \qquad = N_\varpi \cap N_{\varpi s}$$

$$M_{\varpi,\varpi s} = N_\varpi / N_{\varpi,\varpi s}$$

Ex. Show that $\qquad = B_\varpi / B_\varpi \cap B_{\varpi s}$

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{u(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\varpi,\varpi''} := N_\varpi \cap N_{\varpi''}$$

$$M_{\varpi,\varpi''} := N_\varpi / N_{\varpi,\varpi''}$$

$$= B_\varpi / B_\varpi \cap B_{\varpi''}$$

Their Lie algebras are collected here.

$\mathfrak{g}_{|d|}, \mathbb{b}_{|d|}, \mathbb{t}_{|d|}, \mathfrak{n}_{|d|}$

$\mathfrak{g}_d \quad \mathfrak{b}_d \quad \mathfrak{t}_d \quad \mathfrak{n}_d$

$\mathbb{b}_\varpi \qquad\qquad \mathbb{b}^-_\varpi \qquad\qquad \mathbb{b}^-_\varpi = \mathbb{b}_{\varpi_{max}\varpi}$

$\mathfrak{b}_\varpi \quad \mathfrak{n}_\varpi \qquad \mathfrak{b}^-_\varpi \quad \mathfrak{n}^-_\varpi \qquad \mathfrak{b}^-_\varpi = \mathfrak{b}_{w_{max}\varpi}$ ...

$\mathfrak{p}_{\varpi,\varpi s} \quad \mathfrak{n}_{\varpi,\varpi''} \qquad \mathfrak{p}^-_{\varpi,\varpi s} \quad \mathfrak{n}^-_{\varpi,\varpi''} \qquad \mathfrak{p}^-_{\varpi,\varpi s} = \mathfrak{p}_{w_{max}\varpi,\, w_{max}\varpi s}$ ...

$\mathfrak{m}_{\varpi,\varpi''} \qquad \mathfrak{m}^-_{\varpi,\varpi''}$ ...

in general case, lies in $\mathfrak{g}_{|d|}^{\oplus k}$

$$\mathrm{Rep}_d(Q) := \prod_{e \in Q_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{|d|}$$

$$r_\varpi = \{ f \in \mathrm{Rep}_d(Q) \mid f.F_{\varpi,i} \subseteq F_{\varpi,i} \} = \mu_{\underline{d}} \pi_{\underline{d}}^{-1}(F_\varpi)$$

$$= \begin{array}{c c} & \begin{array}{ccc} v_3 & v_1 & v_2 \end{array} \\ \begin{array}{c} v_5 \\ v_4 \end{array} & \begin{pmatrix} * & * & * \\ & & * \end{pmatrix} \end{array} = \begin{array}{c c} & \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{array}{c} v_4 \\ v_5 \end{array} & \begin{pmatrix} & * & \\ * & * & * \end{pmatrix} \end{array}$$

$v_{\omega(i)}$

$$r_{\varpi,\varpi''} = r_\varpi \cap r_{\varpi''}$$

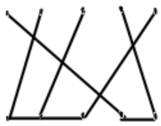
$$\partial_{\varpi,\varpi''} = r_\varpi / r_{\varpi,\varpi''}$$

Later we may twist the group actions.

E.g. $\underline{r}_{\varpi,\varpi'} := r_{\varpi,\varpi\varpi'} \qquad r_{\varpi,\varpi''} = \underline{r}_{\varpi,\varpi^{-1}\varpi''}$

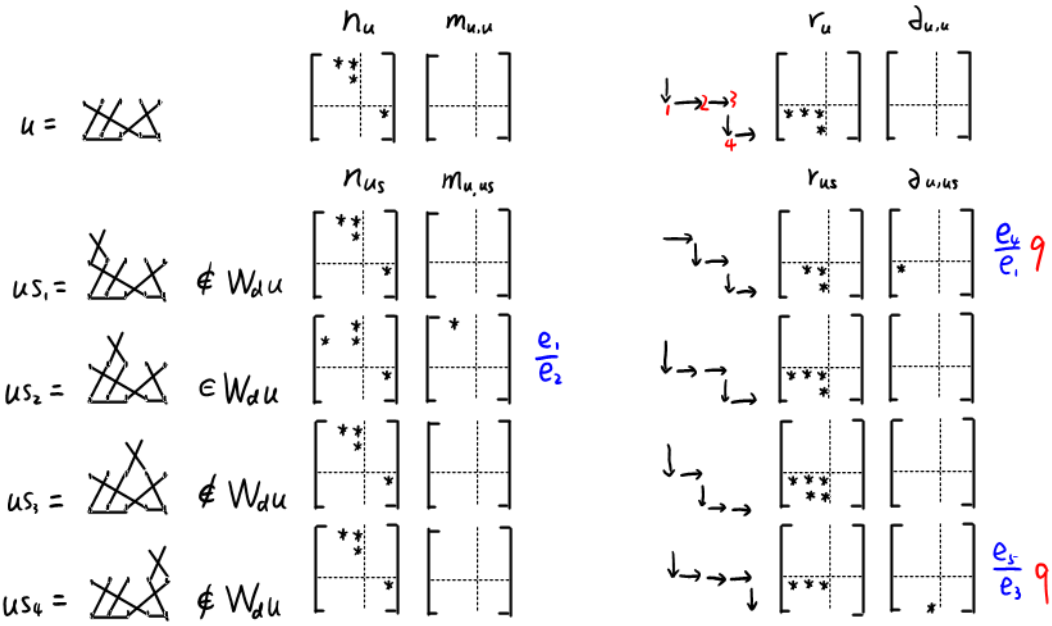
The following example may let you get familiar with those Lie algs.

$$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \\ 0,0 \end{pmatrix} =$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$$

$$u = (1\ 4\ 5\ 3\ 2)$$



4. typical variety

	Id corres to
$\mathcal{F}_{\lvert d\rvert} \cong G_{\lvert d\rvert}/B_{\lvert d\rvert}$	F_{Id}
$\mathcal{F}_{\underline{d}} \cong G_d/B_{\underline{d}}$	F_u
$\mathcal{F}_{\varpi} \cong G_d/B_{\varpi}$	F_{ϖ}
$\mathcal{F}_d := \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$	—
$\mathcal{F}_{g\varpi} \cong G_d/gB_{\varpi}g^{-1}$	$F_{g\varpi}$

$$F_{\varpi} := \varpi(F_{Id}) = F_{\{v_{\varpi(1)},\, v_{\varpi(2)},\, \dots,\, v_{\varpi(\lvert d\rvert)}\}}$$
$$= F_{\{v_5, v_3, v_1, v_4, v_2\}}$$

⚠ The action on $\mathcal{F}lag$ is not the same as in http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$\mathcal{F}_{\lvert d\rvert} \neq \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$

$\mathcal{F}_{\varpi} \cong \mathcal{F}_{\underline{d}}$ with different base pt. Base pt makes difference!

$\mathcal{F}_{\lvert d\rvert} \times \mathcal{F}_{\lvert d\rvert}$	$F_{Id, Id}$
$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_d \times \mathcal{F}_d := \coprod_{\underline{d},\underline{d}'} (\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'})$	—

$$F_{\varpi,\varpi'} := (F_{\varpi}, F_{\varpi'})$$

$\widetilde{Rep}_{\underline{d}}(Q) \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}}$

$$\begin{tikzcd} & \widetilde{Rep}_{\underline{d}}(Q) \arrow[dl, "\mu_{\underline{d}}"'] \arrow[dr, "\pi_{\underline{d}}"] & \\ Rep_d(Q) & & \mathcal{F}_{\underline{d}} \end{tikzcd}$$

$\widetilde{Rep}_d(Q) \subset Rep_d(Q) \times \mathcal{F}_d$

$$\begin{tikzcd} & \widetilde{Rep}_d(Q) \arrow[dl, "\mu_d"'] \arrow[dr, "\pi_d"] & \\ Rep_d(Q) & & \mathcal{F}_d \end{tikzcd}$$

$\mu_{\underline{d}}^{-1}(M) \cong \mathcal{F}lag_{\underline{d}}(M) \subseteq \mathcal{F}_{\underline{d}}$ is the Springer fiber.

$Z_{\underline{d},\underline{d}'} \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$

$$\begin{tikzcd} & Z_{\underline{d},\underline{d}'} \arrow[dl, "\mu_{\underline{d},\underline{d}'}"'] \arrow[dr, "\pi_{\underline{d},\underline{d}'}"] & \\ Rep_d(Q) & & \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \end{tikzcd}$$

$Z_d \subset Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d$

$$\begin{tikzcd} & Z_d \arrow[dl, "\mu_{d,d}"'] \arrow[dr, "\pi_{d,d}"] & \\ Rep_d(Q) & & \mathcal{F}_d \times \mathcal{F}_d \end{tikzcd}$$

$$\widetilde{Rep}_{\underline{d}}(Q) \subseteq Rep_d(Q) \times \mathcal{F}_{\underline{d}}$$
$$\widetilde{Rep}_d(Q) := \coprod_{\underline{d}} \widetilde{Rep}_{\underline{d}}(Q)$$

$$\widetilde{Rep}_{\varpi}(Q) \cong G_d \times^{B_{\varpi}} r_{\varpi}$$

$$Z_{\underline{d},\underline{d}'} := \widetilde{Rep}_{\underline{d}}(Q) \times_{Rep_d(Q)} \widetilde{Rep}_{\underline{d}'}(Q)$$
$$Z_d = \coprod_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$$
$$= \widetilde{Rep}_d(Q) \times_{Rep_d(Q)} \widetilde{Rep}_d(Q)$$

$$Z_{\varpi,\varpi'} := Z_{u,u'}$$

5. (equivariant) stratifications.

In the following tables, $u\omega' = \widetilde{\omega}'\widetilde{u}$.

$F_\varpi \in \widetilde{Rep}_{\underline{d}}(Q)$ means (ρ_0, F_ϖ); $(F_\varpi, F_{\varpi'}) \in \mathcal{Z}_d$ means $(\rho_0, F_\varpi, F_{\varpi'})$.

⚠ $G\times G$ acts on $\mathcal{F}\times\mathcal{F}$ in a twisted way

e.g. $(g_1,g_2)F_{\varpi,\varpi'} = F_{g_1\varpi,\, g_1\varpi g_2\varpi^{-1}\varpi'}$

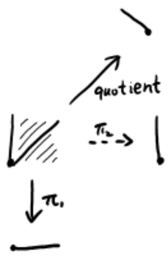
$(g_1,g_2)E_{\varpi,\varpi'} = E_{g_1\varpi,\, g_2\varpi'}$

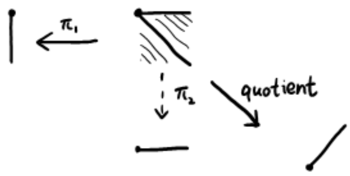
stratification / stabilizer / type variety base point		B-orbit	B×B-orbit stabilizer are twisted	B×G-orbit	G×B-orbit =	Remark G×{*}-orbit
$\mathcal{B}$ F_g	$\mathcal{B}\times\mathcal{B}$ $(F_g, F_{gg'})$	Ω_g $B\cap gBg^{-1}$	$\Omega_{g,g'}$ $B\cap gBg^{-1}\times B\cap(gg')B(gg')^{-1}$	$pr_1^{-1}(\Omega_g)$	$\Omega_{g'}$	$gBg^{-1}\cap gg'B(gg')^{-1}$
$\mathcal{F}_{\lvert d\rvert}$ F_ϖ	$\mathcal{F}_{\lvert d\rvert}\times\mathcal{F}_{\lvert d\rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_\varpi$	$\mathcal{V}_{\varpi,\varpi'}$ $\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_\varpi\times\mathbb{B}_{\lvert d\rvert}\cap\mathbb{B}_{\varpi\varpi'}$	$pr_1^{-1}(\mathcal{V}_\varpi)$	$\mathcal{V}_{\varpi'}$	$\mathbb{B}_\varpi\cap\mathbb{B}_{\varpi\varpi'}$
$\mathcal{F}_u$ $F_{\omega u}$	$\mathcal{F}_u\times\mathcal{F}_{u'}$ $(F_{\omega u}, F_{\omega\omega' u'})$	Ω^u_ω $B_d\cap B_\omega$	$\Omega^{u,u'}_{\omega,\omega'}$ $B_d\cap B_\omega\times B_d\cap B_{\omega\omega'}$	$pr_{1,u'}^{-1}(\Omega^u_\omega)$	$\Omega^{u,u'}_{\omega'}$	$B_\omega\cap B_{\omega\omega'}$
$\mathcal{F}_d$ F_ϖ $\shortparallel$ $F_{\omega u}$	$\mathcal{F}_d\times\mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$ $\shortparallel$ $(F_{\omega u}, F_{\omega\widetilde{\omega}'\widetilde{u}u'})$	Ω^u_ω $B_d\cap B_\omega$	$\Omega^{u,\widetilde{u}u'}_{\omega,\widetilde{\omega}'}$ $B_d\cap B_\omega\times B_d\cap B_{\omega\widetilde{\omega}'}$	$pr_{1,\widetilde{u}u'}^{-1}(\Omega^u_\omega)$	$\mathcal{O}^u_{\varpi'} := \Omega^{u,\widetilde{u}u'}_{\widetilde{\omega}'}$	compatability $B_\omega\cap B_{\omega\widetilde{\omega}'}$
The following may not be single orbit, but derived from the above definition.						
$\mathcal{F}_d$ F_ϖ	$\mathcal{F}_d\times\mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$ $\shortparallel$ Ω^u_ω	$\mathcal{O}_{\varpi,\varpi'}$ $\shortparallel$ $\Omega^{u,\widetilde{u}u'}_{\omega,\widetilde{\omega}'}$	$pr_1^{-1}(\mathcal{O}_\varpi)$ $\shortparallel$ $\bigsqcup_{u'} pr_{1,u'}^{-1}(\Omega^u_\omega)$	$\mathcal{O}_{\varpi'}$ $\shortparallel$ $\bigsqcup_u \mathcal{O}^u_{\varpi'}$	preimage of $\mathcal{F}_d\times\mathcal{F}_d \hookrightarrow \mathcal{F}_{\lvert d\rvert}\times\mathcal{F}_{\lvert d\rvert}$
$\widetilde{Rep}_{\underline{d}}(Q)$ $F_{\omega u}$	$\mathcal{Z}_{\underline{d},\underline{d}'}$ $(F_{\omega u}, F_{\omega' u'})$	$\widetilde{\Omega}^u_\omega$	$\widetilde{\Omega}^{u,u'}_{\omega,\omega'}$	$pr_{1,u'}^{-1}(\widetilde{\Omega}^u_\omega)$	$\widetilde{\Omega}^{u,u'}_{\omega'}$	preimage of $\mathcal{Z}_{\underline{d},\underline{d}'} \longrightarrow \mathcal{F}_{\underline{d}}\times\mathcal{F}_{\underline{d}'}$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi\varpi'})$				$\widetilde{\mathcal{O}}^u_{\varpi'}$ $\shortparallel$ $\widetilde{\Omega}^{u,\widetilde{u}u'}_{\widetilde{\omega}'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d\times\mathcal{F}_d$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$ $\shortparallel$ $\widetilde{\Omega}^u_\omega$	$\widetilde{\mathcal{O}}_{\varpi,\varpi'}$ $\shortparallel$ $\widetilde{\Omega}^{u,\widetilde{u}u'}_{\omega,\widetilde{\omega}'}$	$pr_1^{-1}(\widetilde{\mathcal{O}}_\varpi)$ $\shortparallel$ $\bigsqcup_{u'} pr_{1,u'}^{-1}(\widetilde{\Omega}^u_\omega)$	$\widetilde{\mathcal{O}}_{\varpi'}$ $\shortparallel$ $\bigsqcup_u \widetilde{\mathcal{O}}^u_{\varpi'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d\times\mathcal{F}_d$

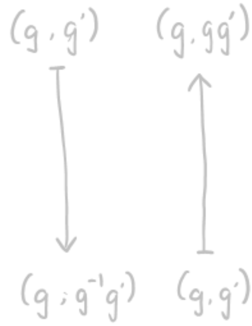
$\mathcal{Z}_{\varpi'} := \overline{\widetilde{\mathcal{O}}_{\varpi'}} \subseteq \overline{\widetilde{\mathcal{O}}}_{\varpi'}$ $\qquad \mathcal{Z}^{\varpi'}_d$

Zar - loc sub v.b. ?

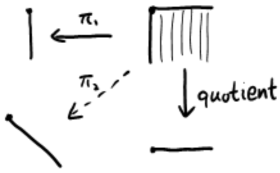
$\mathcal{Z}^{u,u'}_{\omega'} := \overline{\widetilde{\Omega}^{u,u'}_{\omega'}} \subseteq \overline{\widetilde{\Omega}}^{u,u'}_{\omega'}$ $\qquad \mathcal{Z}^{\varpi'}_{\underline{d},\underline{d}'} = \mathcal{Z}_{\varpi'}\cap\mathcal{Z}_{\underline{d},\underline{d}'}$

real case:


in table:




We want gp action to be compatible with π_1 and the quotient map. Therefore, we would do a twist.



Rmk. The stabilizer is not trivial to determine because of this twist!

$$\underline{E}_{g_1,g_2} = \underline{E}_{g_3,g_4} \not\Leftrightarrow g_1B = g_3B,\quad g_2B = g_4B$$
$$\Leftrightarrow g_1B = g_3B,\quad g_1g_2B = g_3g_4B$$
$$\underline{E}_{g_1\varpi,\, g_2\varpi'} = E_{\varpi,\varpi'} \Leftrightarrow g_1\varpi B = \varpi B,\quad g_1\varpi g_2\varpi' B = g_2\varpi' B$$
$$\Leftrightarrow g_1 = \varpi b_1 \varpi^{-1} \in B_\varpi \qquad b_1 g_2 \in B_{\varpi'}$$
$$\Leftrightarrow g_1 \in B_\varpi, \qquad g_2 \in \varpi^{-1} g_1 \varpi \cdot B_{\varpi'}$$

The following tables may help you to understand the notations.

dim / $\mathbb{B}_{\lvert d\rvert}\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$ / $\mathbb{B}_{\lvert d\rvert}\times\mathbb{B}_{\lvert d\rvert}\cdot(F_{\varpi},F_{\varpi\varpi'})$ / $\mathbb{B}_{\lvert d\rvert}\cdot F_{\omega}$ ϖ	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

shape / $B_d\cdot F_{\varpi\varpi'}$ $\varpi\varpi'$ / $B_d\times B_d\cdot(F_{\varpi},F_{\varpi\varpi'})$ / $B_d\cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{Id}$	$\mathcal{O}_{t}$	$\mathcal{O}_{s}$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	$\mathcal{O}_{t}$	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	$\mathcal{O}_{s}$	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	$\mathcal{O}_{ts}$	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	$\mathcal{O}_{sts}$	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

The following tables may help you to understand the notations.

$\varpi = ts$, $\varpi' = s$

dim; $\mathbb{B}_{\lvert d \rvert} \cdot F_{\varpi'}$; $\mathbb{B}_{\lvert d \rvert} \times \mathbb{B}_{\lvert d \rvert} \cdot (F_{\varpi}, F_{\varpi'})$ $\varpi\varpi'$; $\mathbb{B}_{\lvert d \rvert} \cdot F_{\varpi}$ ϖ	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,ts}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

$pr_1^{-1}(\vartheta_{ts})$

ϑ_s

shape; $B_d \cdot F_{\varpi'}$; $B_d \times B_d \cdot (F_{\varpi}, F_{\varpi'})$ $\varpi\varpi'$; $B_d \cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		$\mathcal{O}_{Id}$	$\mathcal{O}_{t}$	$\mathcal{O}_{s}$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega^{Id,Id}_{Id,Id}$	$\Omega^{Id,Id}_{Id,t}$	$\Omega^{Id,s}_{Id,Id}$	$\Omega^{Id,s}_{Id,t}$	$\Omega^{Id,st}_{Id,Id}$	$\Omega^{Id,st}_{Id,t}$
	$\mathcal{O}_{t}$	$\Omega^{Id,Id}_{t,t}$	$\Omega^{Id,Id}_{t,Id}$	$\Omega^{Id,s}_{t,t}$	$\Omega^{Id,s}_{t,Id}$	$\Omega^{Id,st}_{t,t}$	$\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	$\mathcal{O}_{s}$	$\Omega^{s,Id}_{Id,Id}$	$\Omega^{s,Id}_{Id,t}$	$\Omega^{s,s}_{Id,Id}$	$\Omega^{s,s}_{Id,t}$	$\Omega^{s,st}_{Id,Id}$	$\Omega^{s,st}_{Id,t}$
	$\mathcal{O}_{ts}$	$\Omega^{s,Id}_{t,t}$	$\Omega^{s,Id}_{t,Id}$	$\Omega^{s,s}_{t,t}$	$\Omega^{s,s}_{t,Id}$	$\Omega^{s,st}_{t,t}$	$\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega^{st,Id}_{Id,Id}$	$\Omega^{st,Id}_{Id,t}$	$\Omega^{st,s}_{Id,Id}$	$\Omega^{st,s}_{Id,t}$	$\Omega^{st,st}_{Id,Id}$	$\Omega^{st,st}_{Id,t}$
	$\mathcal{O}_{sts}$	$\Omega^{st,Id}_{t,t}$	$\Omega^{st,Id}_{t,Id}$	$\Omega^{st,s}_{t,t}$	$\Omega^{st,s}_{t,Id}$	$\Omega^{st,st}_{t,t}$	$\Omega^{st,st}_{t,Id}$

$pr_1^{-1}(\mathcal{O}_{ts})$ $\quad pr^{-1}_{1,Id}(\Omega^{s}_{t})$ $\quad \Omega^{s,Id}_{t,Id} = \mathcal{O}_{ts,s}$

$\mathcal{O}_s$ $\quad \Omega^{s,Id}_{Id}$

6. change of basis

§6.1 two basis

Def Let $Y \subset X$ be G-equiv closed subvariety, X proj.

$$[Y]^G := (\iota_Y)_* (\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

with same notation,

$$[Y]^G := (\iota_Y)_* (\pi_Y)^* 1_{S(G)} \in H_G^*(X; \mathbb{Q})$$

$$\begin{tikzcd} Y \arrow[r, hook, "\iota_Y"] \arrow[d, "\pi_Y"] & X \\ pt & \end{tikzcd}$$

By cellular fibration lemma,

$$\begin{tikzcd}
K_0^{T_d \times \mathbb{C}^\times}(\mathcal{F}_d) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\mathbin{/\!/}"] & K_0^{G_d \times \mathbb{C}^\times}(\mathcal{F}_d \times \mathcal{F}_d) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\mathbin{/\!/}"] & K_0^{G_d \times \mathbb{C}^\times}(Z_d) \arrow[d, phantom, "\mathbin{/\!/}"] \arrow[dd] \\
\bigoplus_{\varpi \in W_{|d|}} R(T_d \times \mathbb{C}^\times)[\overline{O}_{\varpi'}]^{T_d \times \mathbb{C}^\times} \arrow[r, phantom, "\cong"] & \bigoplus_{\varpi' \in W_{|d|}} R(T_d \times \mathbb{C}^\times)[\overline{O}_{\varpi'}]^{G_d \times \mathbb{C}^\times} \arrow[r, phantom, "\cong"] & \bigoplus_{\varpi' \in W_{|d|}} R(T_d \times \mathbb{C}^\times)[Z_{\varpi'}]^{G_d \times \mathbb{C}^\times} \\
 & \cong & K_0^{T_d \times \mathbb{C}^\times}(Z_d) \arrow[d, phantom, "\mathbin{/\!/}"] \\
 & & \bigoplus_{\varpi, \varpi' \in W_{|d|}} R(T_d \times \mathbb{C}^\times)[\overline{\widetilde{O}}_{\varpi,\varpi'}]^{T_d \times \mathbb{C}^\times}
\end{tikzcd}$$

as $R(T_d \times \mathbb{C}^\times)$-modules.

⚠ There is no evidence if $[Z_{\varpi'}]^{G_d \times \mathbb{C}^\times}$ will be maped to $\bigoplus_{\varpi \in W_{|d|}} [\overline{\widetilde{O}}_{\varpi,\varpi'}]^{T_d \times \mathbb{C}^\times}$.
Luckily, the horizontal line sends generators to generators.

[Hint: Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{F}_d \arrow[r, "(F_e, Id)"] & \mathcal{F}_d \times \mathcal{F}_d \arrow[r, "(p_0, Id)"] & Z_d \\
\overline{O}_{\varpi'} \arrow[ur] \arrow[r, hook] \arrow[d] & \overline{O}_{\varpi'} \arrow[ur] \arrow[r, hook] \arrow[d] & \overline{Z}_{\varpi'} \arrow[ur] \arrow[d] & \\
pt \arrow[r, equal] & pt \arrow[r, equal] & pt &
\end{tikzcd}$$
]

To do linear alg, we take

$$\mathcal{R}(G) := \mathrm{Frac}(R(G)) \qquad \mathcal{K}_0^G(X) := K_0^G(X) \otimes_{R(T_d\times\mathbb{C}^\times)} \mathcal{R}(T_d\times\mathbb{C}^\times)$$
$$\mathcal{S}(G) := \mathrm{Frac}(S(G)) \qquad \mathcal{H}_G^*(X;\mathbb{Q}) := H_G^*(X;\mathbb{Q}) \otimes_{S(T_d\times\mathbb{C}^\times)} \mathcal{S}(T_d\times\mathbb{C}^\times)$$

For $R(T_d\times\mathbb{C}^\times)$-mod $K_0^G(X)$, $S(T_d\times\mathbb{C}^\times)$-mod $H_G^*(X;\mathbb{Q})$

Define
$$\psi_\varpi := [\{F_\varpi\}]^{T_d\times\mathbb{C}^\times} = (i_\varpi)_* 1_{R(T_d\times\mathbb{C}^\times)} \in K_0^{T_d\times\mathbb{C}^\times}(\mathcal{F}_d)$$
$$\psi_{\varpi,\varpi'} := [\{(\rho_0, F_\varpi, F_{\varpi'})\}]^{T_d\times\mathbb{C}^\times} = (i_{\varpi,\varpi'})_* 1_{R(T_d\times\mathbb{C}^\times)} \in K_0^{T_d\times\mathbb{C}^\times}(Z_d)$$
$$\psi_{\varpi,\varpi'} := [\{(\rho_0, F_\varpi, F_{\varpi\varpi'})\}]^{T_d\times\mathbb{C}^\times}$$

We get two $\mathcal{R}(T_d\times\mathbb{C}^\times)$-basis: ($\psi_\varpi$ is $\mathcal{R}(T_d\times\mathbb{C}^\times)$-basis, by Localization theorem.)

$$K_0^{T_d\times\mathbb{C}^\times}(\mathcal{F}_d) \longrightarrow K_0^{T_d\times\mathbb{C}^\times}(Z_d)$$
$$[\overline{\mathcal{O}_{\varpi'}}]^{T_d\times\mathbb{C}^\times} \qquad [\overline{\mathcal{O}_{\varpi,\varpi'}}]^{T_d\times\mathbb{C}^\times} \qquad \text{standard basis: for stratification}$$
$$\psi_{\varpi'} \qquad \psi_{\varpi,\varpi'} \qquad \text{canonical basis: for convolution}$$

Localization thm [Thm 10.1]

Let $i: X^{T_d\times\mathbb{C}^\times} \hookrightarrow X$, X is smooth.

$$\mathcal{K}_0^{T_d\times\mathbb{C}^\times}(X^{T_d\times\mathbb{C}^\times}) \xrightarrow{i_*} \mathcal{K}_0^{T_d\times\mathbb{C}^\times}(X) \xrightarrow{i^*} \mathcal{K}_0^{T_d\times\mathbb{C}^\times}(X^{T_d\times\mathbb{C}^\times})$$
$$\mathcal{H}^*_{T_d\times\mathbb{C}^\times}(X^{T_d\times\mathbb{C}^\times};\mathbb{Q}) \xrightarrow{i_*} \mathcal{H}^*_{T_d\times\mathbb{C}^\times}(X;\mathbb{Q}) \xrightarrow{i^*} \mathcal{H}^*_{T_d\times\mathbb{C}^\times}(X^{T_d\times\mathbb{C}^\times};\mathbb{Q})$$

are isos as $\mathcal{R}(T_d\times\mathbb{C}^\times)$ or $\mathcal{S}(T_d\times\mathbb{C}^\times)$-module.

Q: The Steinberg variety Z_d is usually not smooth.
How to show that $\{\psi_{\varpi,\varpi'}\}$ forms a basis?

Guess: apply localization thm to $\mathcal{F}_d\times\mathcal{F}_d$ first.

§6.2. tangent space

Def (tangent space of fixed pts. in $R(T_d)$)

$$\mathcal{T}_\varpi := T_{F_\varpi}\mathcal{F}_d \cong T_{Id}(G_d/B_\varpi) \cong \mathfrak{g}_d/\mathfrak{b}_\varpi = \mathfrak{n}_\varpi^-$$

$$\tilde{\mathcal{T}}_\varpi := T_{(\rho_0,F_\varpi)}\widetilde{Rep}_d(Q) \cong T_0 r_\varpi \oplus T_{F_\varpi}\mathcal{F}_d = r_\varpi \oplus \mathfrak{n}_\varpi^-$$

$$\mathcal{T}^x_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})}\overline{\mathcal{O}}_x$$

$$\tilde{\mathcal{T}}^x_{\varpi,\varpi'} := T_{(\rho_0,F_\varpi,F_{\varpi'})} Z_x \not\cong \cancel{T_0 r_{\varpi,\varpi'} \oplus T_{(F_\varpi,F_{\varpi'})}\overline{\mathcal{O}}_x = r_{\varpi,\varpi'} \oplus \mathcal{T}^x_{\varpi,\varpi'}}$$

↳ Notice that $Z_x \neq \widetilde{\overline{\mathcal{O}}}_x$

$$\begin{tikzcd}
\overline{\mathcal{O}}_x \arrow[r, "(\rho_0, Id)"] \arrow[dr, equal] & Z_x \arrow[d] \\
& \overline{\mathcal{O}}_x
\end{tikzcd}
\qquad
T_{x_0}(\times) = T_{x_0}(/) \oplus T_{x_0}(\backslash)$$

x_0

$$\underline{\mathcal{T}}^x_{\varpi,\varpi'} := T_{(F_\varpi, F_{\varpi\varpi'})}\overline{\mathcal{O}}_x$$

Rmk. It is still not easy to express $\tilde{\mathcal{T}}^x_{\varpi,\varpi'}$ as Lie alg.

However, we still know some special cases:

$$\begin{aligned}
\underline{\mathcal{T}}^x_{\varpi,x} &:= T_{(F_\varpi, F_{\varpi x})}\overline{\mathcal{O}}_x \\
&= T_{(F_\varpi, F_{\varpi x})}\mathcal{O}_x \\
&= T_{(F_\varpi, F_{\varpi x})}\mathcal{O}^u_x \\
&= T_{Id}\, G_d/B_\varpi \cap B_{\varpi x} \\
&= \mathfrak{g}_d - \mathfrak{b}_\varpi \cap \mathfrak{b}_{\varpi x} \\
&= \mathfrak{g}_d - \mathfrak{b}_\varpi + \mathfrak{b}_\varpi/(\mathfrak{b}_\varpi \cap \mathfrak{b}_{\varpi x}) \\
&= \mathfrak{n}_\varpi^- + \underline{\mathfrak{m}}_{\varpi,x}
\end{aligned}$$

($\underline{\mathfrak{m}}_{\varpi,Id} = 0$. For $s \in \Pi$, $\varpi s \varpi^{-1} \notin W_d$, we have $\underline{\mathfrak{m}}_{\varpi,s} = 0$)

Now suppose $\varpi s \varpi^{-1} \in W_d$.

$$\mathcal{T}^s_{\varpi,\varpi s} = \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi,\varpi s}$$

$$\begin{aligned}
\mathcal{T}^s_{\varpi,\varpi} &:= T_{(F_\varpi, F_\varpi)}\overline{\mathcal{O}}_s \\
&= T_{(Id, Id)}\, G_d/B_\varpi \times P_{\varpi,\varpi s}/B_\varpi \\
&= \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi s,\varpi}
\end{aligned}$$

$$\tilde{\mathcal{T}}^s_{\varpi,\varpi s} = r_{\varpi,\varpi s} \oplus \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi,\varpi s}$$

$$\tilde{\mathcal{T}}^s_{\varpi,\varpi} = r_{\varpi,\varpi s} \oplus \mathfrak{n}_\varpi^- \oplus \mathfrak{m}_{\varpi s,\varpi}$$

§6.3. Euler class.

§6.4. transition matrix, localization formula

Thm. (Localization formula) [Thm 10.2, Cor 5.11.3 in Ginzburg]

Suppose $Y \subset X$ is T-equivariant, $\alpha \in \mathcal{K}_0^T(X)$, X smooth.
$X^T = \{x_1, \cdots, x_m\}$, $i_k : \{x_k\} \hookrightarrow X$, then

$$\alpha = \sum_{k=1}^{m} \varepsilon_k (i_k)_* (i_k)^*(\alpha) \qquad \varepsilon_k = (eu(T_{x_k} X))^{-1} \in \mathcal{R}(T)$$

e.p. $[Y]^T = \sum_{k=1}^{m} \varepsilon_k (i_k)_* ((i_k)^* [Y]^T \cdot 1_{R(T)})$

$= \sum_{k=1}^{m} \varepsilon_k ((i_k)^* [Y]^T)(i_k)_* 1_{R(T)}$

$= \sum_{k=1}^{m} \varepsilon_k ((i_k)^* [Y]^T) [x_k]^T$

$[X]^T = \sum_{k=1}^{m} \varepsilon_k [x_k]^T$

Suppose $Y^T = \{x_1, \dots, x_n\}$, $\iota_k : \{x_k\} \hookrightarrow Y$, then

$[Y]^T = \sum_{k=1}^{n} \beta_k [x_k]^T$ $\qquad \beta_k = \varepsilon_k \cdot (i_k)^*[Y]^T$

When Y is sm at x_k,

$$\begin{cases} \beta_k = (eu(T_{x_k} Y))^{-1} \\ (i_k)^*[Y]^T = eu(T_{x_k} X) \cdot (eu(T_{x_k} Y))^{-1} \end{cases}$$

I believe that this theorem corresponds to the coherent trace formula in this article:
https://www.sciencedirect.com/science/article/pii/0022404994900884

Ex 1. $X = \widetilde{Rep}_d(Q)$, $Y = \overline{\mathcal{O}}_x$ $\qquad T = T_d \times \mathbb{C}^\times$

$i_\varpi : \{(p_0, F_\varpi)\} \hookrightarrow \widetilde{Rep}_d(Q)$

$\widetilde{Rep}_d(Q)^T = \{(p_0, F_\varpi) \mid \varpi \in \mathbb{W}_{|d|}\}$

$[\widetilde{Rep}_d(Q)]^T = \sum_{\varpi \in \mathbb{W}_{|d|}} \widetilde{\Lambda}_\varpi^{-1} (i_\varpi)_* (i_\varpi)^* 1_{K_0^T(\widetilde{Rep}_d(Q))}$

$= \sum_{\varpi \in \mathbb{W}_{|d|}} \widetilde{\Lambda}_\varpi^{-1} (i_\varpi)_* 1_{R(T)}$

$= \sum_{\varpi \in \mathbb{W}_{|d|}} \widetilde{\Lambda}_\varpi^{-1} \psi_\varpi$

$\overline{\mathcal{O}}_x^T = \{(p_0, F_\varpi) \mid \varpi \leq x\}$

$[\overline{\mathcal{O}}_x]^T = \sum_{\varpi \leq x} \widetilde{\Lambda}_\varpi^{-1} \underbrace{((i_\varpi)^* [\overline{\mathcal{O}}_x]^T)}_{f_{\varpi,x}} \psi_\varpi$

when $\overline{\mathcal{O}}_x$ is sm at (p_0, F_ϖ), $f_{\varpi,x} = \widehat{\Lambda}_\varpi (T_{(p_0,F_\varpi)} \overline{\mathcal{O}}_x)^{-1}$

Ex 2. $X = Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d$, $\quad Y = Z_x$, $\quad T = T_d \times \mathbb{C}^\times$

$(Z_s)^T = \widetilde{\mathcal{O}}_s{}^T \sqcup (Z_s - \widetilde{\mathcal{O}}_s)^T$

$= \{(p_0, F_\varpi, F_{\varpi s}) \mid \varpi \in \mathbb{W}_{|d|}\} \sqcup \{(p_0, F_\varpi, F_\varpi) \mid \varpi \in \mathbb{W}_{|d|}, \varpi s \varpi^{-1} \in W_d\}$

$\Rightarrow [Z_s]^T = \sum_{\varpi \in \mathbb{W}_{|d|}} (\widehat{\Lambda}^s_{\varpi,\varpi s})^{-1} \psi_{\varpi,\varpi s} + \sum_{\substack{\varpi \in \mathbb{W}_{|d|} \text{ s.t.} \\ \varpi s \varpi^{-1} \in W_d}} (\widehat{\Lambda}^s_{\varpi,\varpi})^{-1} \psi_{\varpi,\varpi}$ $\quad$ in $\mathcal{K}_0^T(X)$

$\Rightarrow$ in $\mathcal{K}_0^T(Z_d)$

In general, $[Z_x]^T = \sum_{\varpi,\varpi'} \beta^x_{\varpi,\varpi'} \psi_{\varpi,\varpi\varpi'}$

When Z_x is sm at $(p_0, F_\varpi, F_{\varpi\varpi'})$, $\beta^x_{\varpi,\varpi'} = (\widetilde{\Lambda}^x_{\varpi,\varpi'})^{-1}$.

Ex 1'. $X = \widetilde{\mathrm{Rep}}_{\underline{d}}(Q)$, $Y = \overline{\mathcal{O}}_x$, $T = T$

$i_{wu}: \{(\rho_0, F_{wu})\} \hookrightarrow \widetilde{\mathrm{Rep}}_d(Q)$

$$\widetilde{\mathrm{Rep}}_{\underline{d}}(Q)^T = \{(\rho_0, F_{wu}) \mid w \in W_d\}$$

$$\begin{aligned}[\widetilde{\mathrm{Rep}}_{\underline{d}}(Q)]^T &= \sum_{w \in W_d} \widehat{\Lambda}_{wu}^{-1} (i_{wu})_* (i_{wu})^* 1_{K_0^T(\widetilde{\mathrm{Rep}}_{\underline{d}}(Q))} \\ &= \sum_{w \in W_d} \widehat{\Lambda}_{wu}^{-1} (i_{wu})_* 1_{R(T)} \\ &= \sum_{w \in W_d} \widehat{\Lambda}_{wu}^{-1} \psi_{wu}\end{aligned}$$

$$f^u := f[\widetilde{\mathrm{Rep}}_{\underline{d}}(Q)]^T = \sum_{w \in W_d} (w f^u) \widehat{\Lambda}_{wu}^{-1} \psi_{wu}$$

Ex 2'. $X = \mathrm{Rep}_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$, $Y = Z_x^{u,u'}$, $T = T_d \times \mathbb{C}^\times$ $x \in W_d$

$$\begin{aligned}(Z_s^{u,u'})^T &= (\widetilde{\Omega}_s^{u,u'})^T \sqcup (Z_s^{u,u'} - \widetilde{\Omega}_s^{u,u'})^T \\ &= \begin{cases} \{(\rho_0, F_{wu}, F_{wus}) \mid w \in W_d\} \sqcup \{(\rho_0, F_{wu}, F_{wu}) \mid w \in W_d\} & u = u' \\ \{(\rho_0, F_{wu}, F_{wus}) \mid w \in W_d\} & u \neq u' \end{cases}\end{aligned}$$

$$\Rightarrow [Z_s^{u,u'}]^T = \sum_{w \in W_d} (\widehat{\Lambda}^s_{wu,wus})^{-1} \psi_{wu,wus} + \delta_{usu^{-1} \in W_d} \sum_{w \in W_d} (\widehat{\Lambda}^s_{wu,wu})^{-1} \psi_{wu,wu}$$

§ 6.5. generators. Define e_i and D_i.

In this page, $W_{|d|} := W_d$, $\widetilde{Rep}_d(Q) = \mathcal{F}_d$, $Z_d = \mathcal{F}_d \times \mathcal{F}_d$. (Otherwise $K_0^{G_d\times\mathbb{C}^\times}(\mathcal{F}_d) \cong \bigoplus_{\underline{d}} K_0^{T_d\times\mathbb{C}^\times}(pt)$)

$$K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) \cong K_0^{G_d\times\mathbb{C}^\times}(\mathcal{F}_d) \cong K_0^{T_d\times\mathbb{C}^\times}(pt)$$

$$\mathcal{R}(T_d\times\mathbb{C}^\times)[\widetilde{Rep}_d(Q)]^{G_d\times\mathbb{C}^\times} \cong \mathcal{R}(T_d\times\mathbb{C}^\times)[\mathcal{F}_d]^{G_d\times\mathbb{C}^\times} \cong \mathcal{R}(T_d\times\mathbb{C}^\times) = \mathbb{Z}[q^{\pm1}][x_1^{\pm1},\dots,x_{|d|}^{\pm1}]$$

$$\pi_T^G : K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) \longrightarrow K_0^{T_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) = \bigoplus_{\varpi\in W_{|d|}} \mathcal{R}(T_d\times\mathbb{C}^\times)\,\psi_\varpi$$

$$f[\widetilde{Rep}_d(Q)]^{G_d\times\mathbb{C}^\times} \longmapsto \sum_{\varpi\in W_{|d|}} f^{\varpi}\,\widetilde{\Lambda}_\varpi^{-1}\,\psi_\varpi$$

Let $e_i := x_i\,[\widetilde{Rep}_d(Q)]^{G_d\times\mathbb{C}^\times} \in K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q))$, then

$$\pi_T^G(e_i) = \sum_{\varpi\in W_{|d|}} x_{\varpi(i)}\,\widetilde{\Lambda}_\varpi^{-1}\,\psi_\varpi \in K_0^{T_d\times\mathbb{C}^\times}(\widetilde{Rep}(Q))$$

$$K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) \cong \mathbb{Z}[q^{\pm1}, e_1^{\pm1},\dots, e_{|d|}^{\pm1}]$$

$K_0^{G_d\times\mathbb{C}^\times}(Z_d)$ is a $\mathbb{Z}[q^{\pm1}][e_1^{\pm1},\dots,e_{|d|}^{\pm1}]$-module.

[Reason: $K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) \cong K_0^{G_d\times\mathbb{C}^\times}(Z_{Id}) \longleftrightarrow K_0^{G_d\times\mathbb{C}^\times}(Z_d)$

$$\begin{array}{ccc} K_0^{G_d\times\mathbb{C}^\times}(Z_{Id}) \times K_0^{G_d\times\mathbb{C}^\times}(Z_d) & \xrightarrow{\text{convo}} & K_0^{G_d\times\mathbb{C}^\times}(Z_d) \\ \downarrow & & \downarrow \\ K_0^{G_d\times\mathbb{C}^\times}(Z_{Id}) \times K_0^{G_d\times\mathbb{C}^\times}(Z_d) & \xrightarrow{\text{convo}} & K_0^{G_d\times\mathbb{C}^\times}(Z_d) \end{array}$$

We will mention about the convolution in the next section.]

Denote

$$D_i = [Z_{s_i}]^{G_d\times\mathbb{C}^\times} \in K_0^{G_d\times\mathbb{C}^\times}(Z_d)$$

$$\pi_T^G(D_i) = \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^{s}_{\varpi,\varpi s})^{-1}\psi_{\varpi,\varpi s} + \sum_{\substack{\varpi\in W_{|d|} \text{ s.t.} \\ \varpi s\varpi^{-1}\in W_d}} (\widetilde{\Lambda}^{s}_{\varpi,\varpi})^{-1}\psi_{\varpi,\varpi}$$

we will show that, in the case $Z_d \cong \mathcal{F}_d\times\mathcal{F}_d$,

$$K_0^{G_d\times\mathbb{C}^\times}(Z_d) = \langle q^{\pm1}, e_1^{\pm1}, e_2^{\pm1},\dots, e_{|d|}^{\pm1}, D_1,\dots D_{|d|-1}\rangle_{\mathbb{Z}[\cancel{q^{\pm1}}]\text{-alg}} \subseteq \mathrm{End}_{\mathbb{Z}[\cancel{q^{\pm1}}]\text{-mod}}(\mathbb{Z}[q^{\pm1}][e_1^{\pm1},\dots,e_{|d|}^{\pm1}])$$

Before that, the more interesting question is to compute

$$D_i \in \mathrm{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[q^{\pm1}][e_1^{\pm1},\dots,e_{|d|}^{\pm1}]).$$

§ 6.5. generators. Define e_i^u and $D_i^{u,u'}$

Now we do the general case.

$$\begin{tikzcd}
K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_{\underline{d}}(Q)) \arrow[r, phantom, "\cong"] \arrow[d, equal] \arrow[dd, "\pi_T^G"'] & K_0^{G_d\times\mathbb{C}^*}(\mathcal{F}_{\underline{d}}) \arrow[r, phantom, "\cong"] \arrow[d, equal] & K_0^{T_d\times\mathbb{C}^*}(pt) \arrow[d, equal] \\
\mathcal{R}(T_d\times\mathbb{C}^*)[\widetilde{Rep}_{\underline{d}}(Q)]^{G_d\times\mathbb{C}^*} \arrow[r, phantom, "\cong"] \arrow[d] & \mathcal{R}(T_d\times\mathbb{C}^*)[\mathcal{F}_d]^{G_d\times\mathbb{C}^*} \arrow[r, phantom, "\cong"] & \mathcal{R}(T_d\times\mathbb{C}^*) = \mathbb{Z}[q^{\pm1}][x_1^{\pm1},\dots,x_{|d|}^{\pm1}] \\
K_0^{T_d\times\mathbb{C}^*}(\widetilde{Rep}_{\underline{d}}(Q)) = \bigoplus_{w\in W_d}\mathcal{R}(T_d\times\mathbb{C}^*)\psi_{wu} & &
\end{tikzcd}$$

$$f[\widetilde{Rep}_{\underline{d}}(Q)]^{G_d\times\mathbb{C}^*} \longmapsto \sum_{w\in W_d} wuf\,\widetilde{\Lambda}_{wu}^{-1}\psi_{wu}$$

$$\begin{aligned}
\text{Let } 1^u &= [\widetilde{Rep}_{\underline{d}}(Q)]^{G_d\times\mathbb{C}^*} \in K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_{\underline{d}}(Q)) \subseteq K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \\
e_i^u &= x_i[\widetilde{Rep}_{\underline{d}}(Q)]^{G_d\times\mathbb{C}^*} \in K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_{\underline{d}}(Q)) \subseteq K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \\
e_i &= x_i[\widetilde{Rep}(Q)]^{G_d\times\mathbb{C}^*} \in K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q))
\end{aligned}$$

then $e_i = \sum_{u\in Min(W_{|d|},W_d)} e_i^u \qquad e_i^u = e_i\cdot 1^u$

$$K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_{\underline{d}}(Q)) \cong \mathbb{Z}[q^{\pm1}][e_1^{u,\pm1},\dots,e_{|d|}^{u,\pm1}]$$
$$K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \cong \bigoplus_{u\in Min(W_{|d|},W_d)}\mathbb{Z}[q^{\pm1}][e_1^{u,\pm1},\dots,e_{|d|}^{u,\pm1}]$$

Maybe it is better to write $q^{u,\pm1}$ so $q=\sum_u q^u$, $q^u = q\cdot 1^u$

$K_0^{G_d\times\mathbb{C}^*}(Z_{\underline{d},\underline{d}})$ is a $\mathbb{Z}[q^{\pm1}][e_1^{u,\pm1},\dots,e_{|d|}^{u,\pm1}]$-module

$K_0^{G_d\times\mathbb{C}^*}(Z_d)$ is a $\bigoplus_{u\in Min(W_{|d|},W_d)}\mathbb{Z}[q^{\pm1}][e_1^{u,\pm1},\dots,e_{|d|}^{u,\pm1}]$-module.

$$\left[\begin{array}{l} Z_{Id}^{u,u} \cong \widetilde{Rep}_{\underline{d}}(Q) \\ Z_{Id} \cong \widetilde{Rep}_d(Q)\end{array}\right]$$

Denote

$$\begin{aligned}
D_i^{u,u'} &= [Z_{s_i}^{u,u'}]^{G_d\times\mathbb{C}^*} \in K_0^{G_d\times\mathbb{C}^*}(Z_{\underline{d},\underline{d}'}) \subseteq K_0^{G_d\times\mathbb{C}^*}(Z_d) \\
D_i &= [Z_{s_i}]^{G_d\times\mathbb{C}^*} \in K_0^{G_d\times\mathbb{C}^*}(Z_d)
\end{aligned}$$

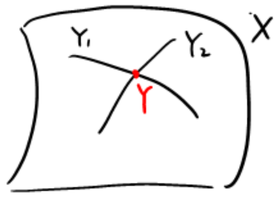
We will show that

$$K_0^{G_d}(Z_d) = \langle q^{\pm1}, e_i^{u,\pm1}, D_i^{u,u'}\rangle_{\mathbb{Z}\text{-alg}} \subseteq End_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{u\in Min(W_{|d|},W_d)}\mathbb{Z}[q^{\pm1}][e_1^{u,\pm1},\dots,e_{|d|}^{u,\pm1}]\Big)$$

$= \mathbb{Z}[q^{\pm1}]$-coefficient combinations of

7. convolution product

§7.1. clean intersection formula



$\pi_{Y,*}: K_0^G(Y) \to K_0^G(pt) = R(G)$

Thm. Suppose X sm G-equiv proj variety,

$Y_1, Y_2 \subset X$ are G-equiv subvariety.

$Y := Y_1 \cap Y_2$ $\qquad \pi_Y : Y \longrightarrow pt$

$\mathcal{T} = TX|_Y / (TY_1|_Y + TY_2|_Y)$

Assume that

$$TY_1|_Y \cap TY_2|_Y = TY,$$

then

$$[Y_1]^G \otimes [Y_2]^G = eu(\pi_{Y,*}(\mathcal{T})) \cdot [Y]^G$$

Need a reference.

I believe that the first page of this document write the same thing:
https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf
However, there is no proof.

§7.2. convolution for canonical basis.

Thm

$$\psi_{\varpi''',\varpi''} * \psi_{\varpi',\varpi} = \delta_{\varpi'',\varpi'}\, \widetilde{\Lambda}_{\varpi'}\, \psi_{\varpi''',\varpi}$$
$$\psi_{\varpi'',\varpi'} \diamond \psi_{\varpi} = \delta_{\varpi',\varpi}\, \widetilde{\Lambda}_{\varpi}\, \psi_{\varpi''}$$

Proof. It reduce to the case

$$\psi_{\varpi'',\varpi'} * \psi_{\varpi',\varpi} = \widetilde{\Lambda}_{\varpi'}\, \psi_{\varpi'',\varpi}$$
$$\psi_{\varpi',\varpi} \diamond \psi_{\varpi} = \widetilde{\Lambda}_{\varpi}\, \psi_{\varpi'}$$

$$\begin{tikzcd}[column sep=small]
 & & & \mathcal{M}_{123} \arrow[dl] \arrow[dr] & & \\
 \left[Y_{12}\right] & Z_d \arrow[r, hook] & \mathcal{M}_{12} \arrow[dl] \arrow[dr] & \left[Y_{23}\right] & Z_d \arrow[r, hook] & \mathcal{M}_{23} \arrow[dl] \arrow[dr] & \\
 & \widetilde{Rep}_d(Q) & & \widetilde{Rep}_d(Q) & & \widetilde{Rep}_d(Q)
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & & & \mathcal{M}_{123} \arrow[dl] \arrow[dr] & & \\
 \left[Y_{12}\right] & Z_d \arrow[r, hook] & \mathcal{M}_{12} \arrow[dl] \arrow[dr] & \left[\widetilde{Y_{23}}\right] & \widetilde{Rep}_d(Q) = \mathcal{M}_{23} \arrow[dl] \arrow[dr] & \\
 & \widetilde{Rep}_d(Q) & & \widetilde{Rep}_d(Q) & & pt
\end{tikzcd}$$

$$*:\ K_0^{G_d\times\mathbb{C}^\times}(Z_d) \times K_0^{G_d\times\mathbb{C}^\times}(Z_d) \longrightarrow K_0^{G_d\times\mathbb{C}^\times}(Z_d)$$

$$\diamond:\ K_0^{G_d\times\mathbb{C}^\times}(Z_d) \times K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q)) \longrightarrow K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep}_d(Q))$$

$\{(\rho_0, F_{\varpi''}), (\rho_0, F_{\varpi'})\}$
$\parallel$

$Y_{12} = \{(\rho_0, F_{\varpi''}, F_{\varpi'})\}$ $\quad Y_{23} = \{(\rho_0, F_{\varpi'}, F_{\varpi})\}$

$$\{y\} \subset Y_{12} \times \widetilde{Rep}_d(Q) \subset \mathcal{M}_{123}, \qquad \{y\} \subset \widetilde{Rep}_d \times Y_{23} \subset \mathcal{M}_{123}$$

where

$$y = ((\rho_0, F_{\varpi''}), (\rho_0, F_{\varpi'}), (\rho_0, F_{\varpi})) \in \mathcal{M}_{123}$$
$$y_{13} = ((\rho_0, F_{\varpi''}), (\rho_0, F_{\varpi})) \in \mathcal{M}_{13}$$
$$\mathcal{T} = \frac{\widetilde{\mathcal{T}}_{\varpi''} \oplus \widetilde{\mathcal{T}}_{\varpi'} \oplus \widetilde{\mathcal{T}}_{\varpi}}{\widetilde{\mathcal{T}}_{\varpi''} \oplus \widetilde{\mathcal{T}}_{\varpi}} = \widetilde{\mathcal{T}}_{\varpi'}$$

$T = T_d \times \mathbb{C}^\times$

$$\begin{aligned}
\psi_{\varpi'',\varpi'} * \psi_{\varpi',\varpi} &= [Y_{12}]^T * [Y_{23}]^T \\
&= \pi_{13,*}\left([Y_{12} \times \widetilde{Rep}_d(Q)]^T \otimes [\widetilde{Rep}_d(Q) \times Y_{23}]^T\right) \\
&= \pi_{13,*}\left(\widetilde{\Lambda}_{\varpi'} \cdot [y]^T\right) \\
&= \widetilde{\Lambda}_{\varpi'} \cdot [y_{13}]^T \\
&= \widetilde{\Lambda}_{\varpi'}\, \psi_{\varpi'',\varpi}
\end{aligned}$$

$Y_{12} = \{(\rho_0, F_{\varpi'}, F_{\varpi})\}$ $\quad Y_{23} = \{(\rho_0, F_{\varpi})\}$

$$\{y\} \subset Y_{12} \times pt \subset \mathcal{M}_{123}, \qquad \{y\} \subset \widetilde{Rep}_d \times Y_{23} \subset \mathcal{M}_{123}$$

where

$$y = ((\rho_0, F_{\varpi'}), (\rho_0, F_{\varpi})) \in \mathcal{M}_{123}$$
$$y_{13} = (\rho_0, F_{\varpi}) \in \mathcal{M}_{13}$$
$$\mathcal{T} = \frac{\widetilde{\mathcal{T}}_{\varpi'} \oplus \widetilde{\mathcal{T}}_{\varpi} \oplus 0}{\widetilde{\mathcal{T}}_{\varpi'} \oplus 0} = \widetilde{\mathcal{T}}_{\varpi}$$

$T = T_d \times \mathbb{C}^\times$

$$\begin{aligned}
\psi_{\varpi',\varpi} * \psi_{\varpi} &= [Y_{12}]^T * [Y_{23}]^T \\
&= \pi_{13,*}\left([Y_{12} \times pt]^T \otimes [\widetilde{Rep}_d(Q) \times Y_{23}]^T\right) \\
&= \pi_{13,*}\left(\widetilde{\Lambda}_{\varpi} \cdot [y]^T\right) \\
&= \widetilde{\Lambda}_{\varpi} \cdot [y_{13}]^T \\
&= \widetilde{\Lambda}_{\varpi}\, \psi_{\varpi'}
\end{aligned}$$

§ 7.3. expression of D_k.

In this subsection, $W_{|d|} = W_d$, $\widetilde{Rep_d}(Q) = \mathcal{F}_d$, $Z_d = \mathcal{F}_d \times \mathcal{F}_d$.

In the example, $|d| = 3$, $i = 1$

The convolution is compatible with forget map π_T^G:

$$\begin{array}{ccccc}
K_0^{G_d\times\mathbb{C}^\times}(Z_d) & \times & K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep_d}(Q)) & \longrightarrow & K_0^{G_d\times\mathbb{C}^\times}(\widetilde{Rep_d}(Q)) \\
\downarrow \pi_B^G & & \downarrow \pi_B^G & & \downarrow \pi_B^G \\
K_0^{T_d\times\mathbb{C}^\times}(Z_d) & \times & K_0^{T_d\times\mathbb{C}^\times}(\widetilde{Rep_d}(Q)) & \longrightarrow & K_0^{T_d\times\mathbb{C}^\times}(\widetilde{Rep_d}(Q))
\end{array}$$

So we do our computation in $K_0^{T_d\times\mathbb{C}^\times}$. (View $K_0^{G_d\times\mathbb{C}^\times}$ as subalg of $K_0^{T_d\times\mathbb{C}^\times}$)

Recall that

$$D_i = \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi s})^{-1}\psi_{\varpi,\varpi s} + \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi})^{-1}\psi_{\varpi,\varpi}$$

$\varpi s \varpi^{-1} \in W_d \leftarrow$ automatically satisfied

$$f = \sum_{\varpi\in W_{|d|}} (\varpi f)\, \widetilde{\Lambda}_\varpi^{-1}\psi_\varpi \qquad \text{e.g.} \quad e_i = \sum_{\varpi\in W_{|d|}} x_{\varpi(i)}\, \widetilde{\Lambda}_\varpi^{-1}\psi_\varpi$$

Therefore,

$$D_i \diamond f = \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi s})^{-1}\psi_{\varpi,\varpi s} \sum_{\varpi\in W_{|d|}} (\varpi s f)\widetilde{\Lambda}_{\varpi s}^{-1}\psi_{\varpi s} + \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi})^{-1}\psi_{\varpi,\varpi}\cdot \sum_{\varpi\in W_{|d|}} (\varpi f)\widetilde{\Lambda}_\varpi^{-1}\psi_\varpi$$

$$= \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi s})^{-1}(\varpi s f)\widetilde{\Lambda}_{\varpi s}^{-1}\widetilde{\Lambda}_{\varpi s}\psi_\varpi + \sum_{\varpi\in W_{|d|}} (\widetilde{\Lambda}^s_{\varpi,\varpi})^{-1}(\varpi f)\widetilde{\Lambda}_\varpi^{-1}\widetilde{\Lambda}_\varpi\psi_\varpi$$

$$= \sum_{\varpi\in W_{|d|}} \left[(\widetilde{\Lambda}^s_{\varpi,\varpi s})^{-1}\varpi s f + (\widetilde{\Lambda}^s_{\varpi,\varpi})^{-1}(\varpi f) \right]\psi_\varpi$$

$$= \sum_{\varpi\in W_{|d|}} \varpi\left[\left(\frac{sf}{\widetilde{\Lambda}^s_{Id,s}} + \frac{f}{\widetilde{\Lambda}^s_{Id,Id}} \right)\cdot \widetilde{\Lambda}_{Id} \right] \widetilde{\Lambda}_\varpi^{-1}\psi_\varpi$$

$$\therefore \quad D_i f = \left(\frac{sf}{\widetilde{\Lambda}^s_{Id,s}} + \frac{f}{\widetilde{\Lambda}^s_{Id,Id}} \right)\cdot \widetilde{\Lambda}_{Id}$$

In our case,

$$\widetilde{T}_{Id} = T_{Id} = n^-_{Id} = \begin{pmatrix} 0&0&0\\ *&0&0\\ *&*&0 \end{pmatrix} = A$$

$$\widetilde{T}^s_{Id,s} = T^s_{Id,s} = n^-_{Id}\oplus m_{Id,s} = \begin{pmatrix} 0&0&0\\ *&0&0\\ *&*&0 \end{pmatrix} \oplus \begin{pmatrix} 0&*&0\\ 0&0&0\\ 0&0&0 \end{pmatrix} = A + \frac{1}{B}$$

$$\widetilde{T}^s_{Id,Id} = T^{Id}_{Id,Id} = n^-_{Id}\oplus m_{s,Id} = \begin{pmatrix} 0&0&0\\ *&0&0\\ *&*&0 \end{pmatrix} \oplus \begin{pmatrix} 0&0&0\\ *&0&0\\ 0&0&0 \end{pmatrix} = A + B$$

where

$$A = \sum_{j>k}\frac{e_j}{e_k} = \frac{e_2}{e_1} + \frac{e_3}{e_1} + \frac{e_3}{e_2}$$

$$B = \frac{e_{i+1}}{e_i} = \frac{e_2}{e_1}$$

	$\mathcal{K}_0^{T_d}(Z_d)$ $\longrightarrow$	$\mathcal{H}^*_{T_d}(Z_d)$
$eu(A)$	$(1-\frac{e_1}{e_2})(1-\frac{e_1}{e_3})(1-\frac{e_2}{e_3})$	$(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)$
$eu(B)$	$1-\frac{e_1}{e_2}$	$\lambda_1-\lambda_2$
$eu(\frac{1}{B})$	$1-\frac{e_2}{e_1}$	$\lambda_2-\lambda_1$

$\therefore$

$$D_i f = \left(\frac{sf}{eu(A+\frac{1}{B})} + \frac{f}{eu(A+B)}\right) eu(A) \qquad f \in K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}(Q))$$

$$= \frac{sf}{eu(\frac{1}{B})} + \frac{f}{eu(B)}$$

$$= \frac{sf}{1-\frac{e_{i+1}}{e_i}} + \frac{f}{1-\frac{e_i}{e_{i+1}}}$$

$$= \frac{e_{i+1}f - e_i sf}{e_{i+1}-e_i}$$

$$D_i\left(\frac{e_i}{e_{i+1}} f\right) = -\frac{e_i f - e_{i+1} sf}{e_i - e_{i+1}}$$

$$D_i fg = \frac{sf\, sg}{eu(\frac{1}{B})} + \frac{sf\cdot g}{eu(B)} + \frac{f\cdot g}{eu(B)} - \frac{sf\cdot g}{eu(B)} \qquad \text{Here, } f \in K_0^{G_d\times\mathbb{C}^*}(Z_{Id}),\ g\in K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}(Q))$$

$$= sf\cdot D_i g + \frac{f-sf}{eu(B)}\, g$$

$$\Rightarrow D_i f = sf\, D_i + \frac{f-sf}{1-\frac{e_i}{e_{i+1}}}$$

$$\left(\frac{e_i}{e_{i+1}} D_i\right) f = sf\left(\frac{e_i}{e_{i+1}} D_i\right) - \frac{f-sf}{1-\frac{e_{i+1}}{e_i}}$$

$$D_i\left(\frac{e_i}{e_{i+1}} fg\right) = sf\, D_i\left(\frac{e_i}{e_{i+1}} g\right) - \frac{f-sf}{1-\frac{e_{i+1}}{e_i}}\, g$$

In the case of equivariant cohomology, the computation is similar:

$$\partial_i f = \left(\frac{sf}{eu(\widehat{\Lambda}^s_{Id\,s})} + \frac{f}{eu(\widehat{\Lambda}^s_{Id,Id})}\right)\cdot eu(\widehat{\Lambda}_{Id}) \qquad f\in K_0^{G_d}(\widetilde{Rep}(Q))$$

$$= \frac{sf}{eu(\frac{1}{B})} + \frac{f}{eu(B)}$$

$$= \frac{sf}{\lambda_{i+1}-\lambda_i} + \frac{f}{\lambda_i - \lambda_{i+1}}$$

$$\partial_i f = sf\,\partial_i + \frac{f-sf}{\lambda_i-\lambda_{i+1}} \qquad f\in K_0^{G_d}(Z_{Id})$$

§ 7.3. expression of D_k.

The convolution is compatible with forget map π_T^G:

$$\begin{tikzcd} K_0^{G_d\times\mathbb{C}^*}(Z_d) \arrow[d, "\pi_B^G"] & \times & K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \arrow[r] \arrow[d, "\pi_B^G"] & K_0^{G_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \arrow[d, "\pi_B^G"] \\ K_0^{T_d\times\mathbb{C}^*}(Z_d) & \times & K_0^{T_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \arrow[r] & K_0^{T_d\times\mathbb{C}^*}(\widetilde{Rep}_d(Q)) \end{tikzcd}$$

So we do our computation in $K_0^{T_d\times\mathbb{C}^*}$. (View $K_0^{G_d\times\mathbb{C}^*}$ as subalg of $K_0^{T_d\times\mathbb{C}^*}$)

Recall that

$$D_i^{u,u'} = \sum_{w\in W_d} (\widetilde{\Lambda}^s_{wu,wus})^{-1}\psi_{wu,wus} + \delta_{u=u'}\sum_{w\in W_d}(\widetilde{\Lambda}^s_{wu,wu})^{-1}\psi_{wu,wu}$$

$$f^u = \sum_{w\in W_d}(wuf)\,\widetilde{\Lambda}^{-1}_{wu}\,\psi_{wu}$$

$w'u' = us$

$$\begin{aligned} D_i^{u,u'} f^{u'} &= \sum_{w\in W_d}(\widetilde{\Lambda}^s_{wu,wus})^{-1}\psi_{wu,wus}\sum_{w\in W_d}(wusf)\,\widetilde{\Lambda}^{-1}_{wus}\,\psi_{wus} \\ &\quad + \delta_{u=u'}\sum_{w\in W_d}(\widetilde{\Lambda}^s_{wu,wu})^{-1}\psi_{wu,wu}\sum_{w\in W_d}(wu'f)\,\widetilde{\Lambda}^{-1}_{wu'}\,\psi_{wu'} \\ &= \sum_{w\in W_d}(\widetilde{\Lambda}^s_{wu,wus})^{-1}(wusf)\,\psi_{wu} \\ &\quad + \delta_{u=u'}\sum_{w\in W_d}(\widetilde{\Lambda}^s_{wu,wu})^{-1}(wuf)\,\psi_{wu} \end{aligned}$$

When $u=u'$,

$$\begin{aligned} D_i^{u,u} f^u &= \sum_{w\in W_d}\left(\frac{wusf}{\widetilde{\Lambda}^s_{wu,wus}} + \frac{wuf}{\widetilde{\Lambda}^s_{wu,wu}}\right)\widetilde{\Lambda}_{wu}\,(\widetilde{\Lambda}_{wu})^{-1}\psi_{wu} \\ &= \sum_{w\in W_d} w\left(\left(\frac{(sf)^u}{\widetilde{\Lambda}^s_{u,us}} + \frac{f^u}{\widetilde{\Lambda}^s_{u,u}}\right)\widetilde{\Lambda}_u\right)(\widetilde{\Lambda}_{wu})^{-1}\psi_{wu} \end{aligned}$$

$$\Rightarrow D_i^{u,u} f^u = \left(\frac{(sf)^u}{\widetilde{\Lambda}^s_{u,us}} + \frac{f^u}{\widetilde{\Lambda}^s_{u,u}}\right)\widetilde{\Lambda}_u \quad = g^u$$

When $u\neq u'$, $u'=us$,

$$\begin{aligned} D_i^{u,u'} f^{u'} &= \sum_{w\in W_d}\frac{wusf}{\widetilde{\Lambda}^s_{wu,wu'}}\,\widetilde{\Lambda}_{wu}\cdot(\widetilde{\Lambda}_{wu})^{-1}\psi_{wu} \\ &= \sum_{w\in W_d} w\left(\frac{(sf)^u}{\widetilde{\Lambda}^s_{u,us}}\,\widetilde{\Lambda}_u\right)(\widetilde{\Lambda}_{wu})^{-1}\psi_{wu} \end{aligned}$$

$$\Rightarrow D_i^{u,u'} f^{u'} = \frac{sf^u}{\widetilde{\Lambda}^s_{u,us}}\,\widetilde{\Lambda}_u \quad = g^u$$

In our case,

$$
\begin{aligned}
\widetilde{\mathcal{T}}_{Id} &= r_{Id} \oplus \mathcal{T}_{Id} & \widetilde{\mathcal{T}}_{u} &= r_{u} \oplus \mathcal{T}_{u} \\
&= r_{Id} \oplus n^-_{Id} & &= r_{u} \oplus n^-_{u} \\
\widetilde{\mathcal{T}}^s_{Id,s} &= r_{Id,s} \oplus \mathcal{T}^s_{Id,s} & \widetilde{\mathcal{T}}^s_{u,us} &= r_{u,us} \oplus \mathcal{T}^s_{u,us} \\
&= r_{Id,s} \oplus n^-_{Id} \oplus m_{Id,s} & &= r_{u,us} \oplus n^-_{u} \oplus m_{u,us} \\
\widetilde{\mathcal{T}}^s_{Id,Id} &= r_{Id,s} \oplus \mathcal{T}^s_{Id,Id} & \widetilde{\mathcal{T}}^s_{u,u} &= r_{u,us} \oplus \mathcal{T}^s_{u,u} \\
&= r_{Id,s} \oplus n^-_{Id} \oplus m_{s,Id} & &= r_{u,us} \oplus n^-_{u} \oplus m_{us,u}
\end{aligned}
$$

$$\therefore \quad D_i^{u,u} f^u = \left(\frac{sf}{eu(m_{u,us})} + \frac{f}{eu(m_{us,u})} \right) eu(\partial_{u,us})$$

$$D_i^{u,u'} f^{u'} = \frac{sf}{eu(m_{u,us})} \; eu(\partial_{u,us})$$

Name	Lie alg	$eu \in \mathcal{K}_0^{T_d}(Z_d)$	$eu \in \mathcal{H}^*_{T_d}(Z_d)$	
$m_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$1 - \frac{e_{u(i+1)}}{e_{u(i)}}$	$\lambda_{u(i+1)} - \lambda_{u(i)}$	$u = u'$
	0	1	1	$u \neq u'$
$m_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$1 - \frac{e_{u(i)}}{e_{u(i+1)}}$	$\lambda_{u(i)} - \lambda_{u(i+1)}$	$u = u'$
	0	1	1	$u \neq u'$
$\partial_{u,us}$	$k \frac{e_{u(i)} q}{e_{u(i+1)}}$	$\left(1 - \frac{e_{u(i+1)}}{e_{u(i)} q}\right)^k$	$\left(\lambda_{u(i+1)} - \lambda_{u(i)} - t\right)^k$	

$$
\begin{tikzcd}
\underset{e_{u(i+1)}}{\bullet} \arrow[r, shift left=2, "k\text{-many arrows}"] \arrow[r] \arrow[r, shift right=2] & \underset{e_{u(i)}}{\bullet}
\end{tikzcd}
$$

E.g. $\bullet \to \bullet$ u: 

Substitute everything, we get

① $\partial_i^{u,u} f^u = \dfrac{(s_i f)^u}{\lambda_{u(i+1)}-\lambda_{u(i)}} + \dfrac{f^u}{\lambda_{u(i)}-\lambda_{u(i+1)}} = \left(\dfrac{f-s_i f}{\lambda_i-\lambda_{i+1}}\right)^u$

s_2 u 

② $\partial_i^{u,u'} f^{u'} = (s_i f)^u(\lambda_{u(i+1)}-\lambda_{u(i)}-t) = \left(s_i f(\lambda_{i+1}-\lambda_i-t)\right)^u$

s_1 u 

③ $\partial_i^{u,u'} f^{u'} = (s_i f)^u$

s_3 u 

① $D_i^{u,u} f^u = \dfrac{(s_i f)^u}{1-\frac{e_{u(i+1)}}{e_{u(i)}}} + \dfrac{f^u}{1-\frac{e_{u(i)}}{e_{u(i+1)}}} = \left(\dfrac{s_i f}{1-\frac{e_{i+1}}{e_i}} + \dfrac{f}{1-\frac{e_i}{e_{i+1}}}\right)^u$

s_2 u 

② $D_i^{u,u'} f^{u'} = (s_i f)^u\left(1-\dfrac{e_{u(i+1)}}{e_{u(i)}q}\right) = \left(s_i f\left(1-\dfrac{e_{i+1}}{e_i q}\right)\right)^u$

s_1 u 

③ $D_i^{u,u'} f^{u'} = (s_i f)^u$

s_3 u 

E.g. •5

Substitute everything, we get

$$\partial_i f = \left(\frac{s_i f}{\lambda_{i+1} - \lambda_i} + \frac{f}{\lambda_i - \lambda_{i+1}} \right) (\lambda_{i+1} - \lambda_i - t)^k$$

$$\overset{t=0,k=1}{=\!=} s_i f - f$$

$$D_i f = \left(\frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \right) \left(1 - \frac{e_{i+1}}{e_i q}\right)^k$$

$$\overset{q=1,k=1}{=\!=} s_i f - \frac{e_{i+1}}{e_i} f$$