Eine Woche, ein Beispiel
6.25 (co)homology of simplicial set

singular: Top $\longrightarrow$ sSet $\longrightarrow$

$\uparrow$

Δ-cplx

simplicial: $\cup$ $\downarrow$ subdivide

simplicial cplx

$\uparrow$ nerve

sheaf Čech Sheaf + open cover $\longrightarrow$

$\uparrow$

de Rham. sm mfld $\longrightarrow$

derived fctor $\longrightarrow$

$\vdots$ $\vdots$

Chain

cplx

$\xrightarrow{\text{homological algebra}}$ (co)homology
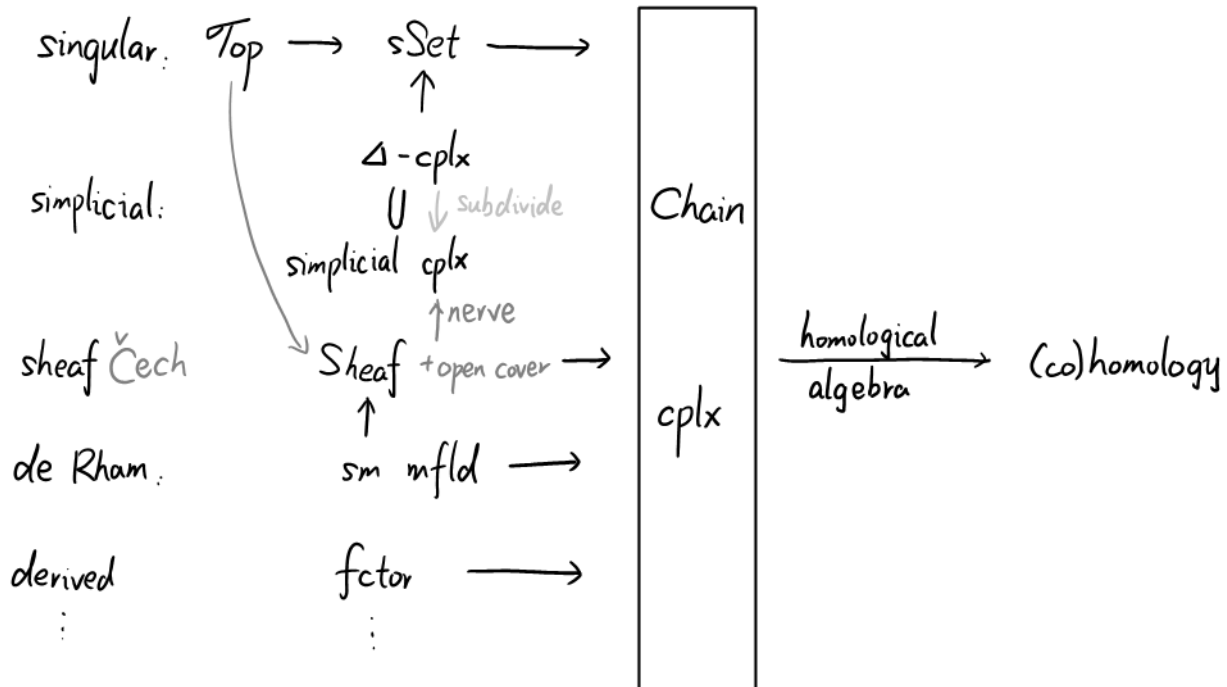
Today: sSet $\longrightarrow$ chain cplx $--\rightarrow$ (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha \in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0} G \xleftarrow{(d_0' - d_1')^*} \bigoplus_{\alpha \in X_1} G \xleftarrow{(d_0^2 - d_0' + d_2^2)^*} \bigoplus_{\alpha \in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha \in X_n} G \qquad 0 \longrightarrow \prod_{\alpha \in X_0} G \xrightarrow{dual} \prod_{\alpha \in X_1} G \longrightarrow \prod_{\alpha \in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_{\alpha \in X_n} \mathbb{Z}, G\right) \cong \prod_{\alpha \in X_n} \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha \in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset. If there were any sheaf on sset, it should behave like perverse sheaf.

Eg.1 For $A \in \mathcal{T}op$ discrete, $X := S(A) \in sSet$, one can compute

$$C_\bullet(X;G): \quad 0 \leftarrow \bigoplus_{\alpha \in A} G \xleftarrow{0} \bigoplus_{\alpha \in A} G \xleftarrow{Id} \bigoplus_{\alpha \in A} G \xleftarrow{0} \bigoplus_{\alpha \in A} G \xleftarrow{Id} \cdots$$

$$C^\bullet(X;G): \quad 0 \longrightarrow \prod_{\alpha \in A} G \xrightarrow{0} \prod_{\alpha \in A} G \xrightarrow{Id} \prod_{\alpha \in A} G \xleftarrow{0} \prod_{\alpha \in A} G \xrightarrow{Id} \cdots$$

wished $\Bigg\{$

$$\overset{BM}{C_\bullet}(X;G): \quad 0 \leftarrow \prod_{\alpha \in A} G \xleftarrow{0} \prod_{\alpha \in A} G \xleftarrow{Id} \prod_{\alpha \in A} G \xleftarrow{0} \prod_{\alpha \in A} G \xleftarrow{Id} \cdots$$

$$C_c^\bullet(X;G): \quad 0 \longrightarrow \bigoplus_{\alpha \in A} G \xrightarrow{0} \bigoplus_{\alpha \in A} G \xrightarrow{Id} \bigoplus_{\alpha \in A} G \xrightarrow{0} \bigoplus_{\alpha \in A} G \xrightarrow{Id} \cdots$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha \in A} G & n = 0 \\ 0 & n > 0 \end{cases} \qquad \overset{BM}{H_n}(X;G) = \begin{cases} \prod_{\alpha \in A} G & n = 0 \\ 0 & n > 0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha \in A} G & n = 0 \\ 0 & n > 0 \end{cases} \qquad H_c^n(X;G) = \begin{cases} \bigoplus_{\alpha \in A} G & n = 0 \\ 0 & n > 0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.
Notice that $\#\Delta'_k = k+2$, so

$$C_\bullet(\Delta';G): \quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

basis:

remember indexes:
$d'_0 \triangleq x_0$
$d'_1 \triangleq x_1$

$0 = x_0 - x_0 \longleftarrow x_0$

$x_0 - x_1 = x_0 - x_1 \longleftarrow x_1$

$0 = x_1 - x_1 \longleftarrow x_2$

$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow x_0$

$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow x_1$

$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow x_2$

$x_2 - x_3 = x_2 - x_2 + x_3 - x_3 \longleftarrow x_3$

$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow x_4$

$x_0 = x_0 - x_0 + x_0 \longleftarrow x_0$

$x_0 = x_0 - x_1 + x_1 \longleftarrow x_1$

$x_2 = x_1 - x_1 + x_2 \longleftarrow x_2$

$x_2 = x_2 - x_2 + x_2 \longleftarrow x_3$

By taking the transpose, one get

$$C^\bullet(\Delta';G): \quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xrightarrow{\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xrightarrow{\left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C.(\Delta';G)$ and $C.(\Delta^\circ;G)$.

Δ' $\quad C.(\Delta';G):$ $\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0\\0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 1\end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 5} \cdots$

$\Big\downarrow s'_0 \qquad \Big\downarrow s'_{0,*} \qquad\quad \Big\downarrow (1\,1) \qquad\quad \Big\downarrow (1\,1\,1) \qquad\quad \Big\downarrow (1\,1\,1\,1) \qquad\quad \Big\downarrow (1\,1\,1\,1\,1)$

Δ° $\quad C.(\Delta^\circ;G):$ $\quad 0 \longleftarrow G \xleftarrow{\;0\;} G \xleftarrow{\;Id\;} G \xleftarrow{\;0\;} G \cdots$

Δ° $\quad C.(\Delta^\circ;G):$ $\quad 0 \longleftarrow G \xleftarrow{\;0\;} G \xleftarrow{\;Id\;} G \xleftarrow{\;0\;} G \cdots$

$\Big\downarrow d'_0 \qquad \Big\downarrow d'_{0,*} \qquad\quad \Big\downarrow \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) \qquad\quad \Big\downarrow \left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right) \qquad\quad \Big\downarrow \left(\begin{smallmatrix}1\\0\\0\\0\end{smallmatrix}\right) \qquad\quad \Big\downarrow \left(\begin{smallmatrix}1\\0\\0\\0\\0\end{smallmatrix}\right)$

Δ' $\quad C.(\Delta';G):$ $\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0\\0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 1\end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 5} \cdots$

$\Big\Downarrow s'_0 \qquad\qquad \Big\downarrow d'_0$

s.t. $\quad s'_{0,*} \circ d'_{0,*} = Id_{C.(\Delta^\circ;G)}, \qquad d'_{0,*} \circ s'_{0,*} \sim Id_{C.(\Delta';G)}.$

In fact, we have

$C.(\Delta';G):$ $\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0\\0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 1\end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 5} \cdots$

$Id \Big\| \left(\begin{smallmatrix} \cdot\\ \cdot \end{smallmatrix}\right)_* \qquad Id \Big\| \left(\begin{smallmatrix}1 & 0\\0 & 0\end{smallmatrix}\right) \qquad Id \Big\| \left(\begin{smallmatrix}1 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{smallmatrix}\right) \qquad Id \Big\| \left(\begin{smallmatrix}1 & 1 & 1 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{smallmatrix}\right) \qquad Id \Big\| \left(\begin{smallmatrix}1 & 1 & 1 & 1 & 1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{smallmatrix}\right)$

red dashed arrows with matrices: $\left(\begin{smallmatrix}1 & 0\\0 & 1\\0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 0 & 0\\0 & 0 & 1\\0 & 0 & -1\\0 & 0 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 0 & 0 & 0\\0 & 1 & 1 & 1\\0 & 0 & -1\\0 & 0 & 0 & 1\\0 & 0 & 0 & 0\end{smallmatrix}\right)$

$C.(\Delta';G):$ $\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0\\0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix}1 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 1 & 1\end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix}0 & 1 & 0 & 0 & 0\\0 & -1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & -1 & 0\end{smallmatrix}\right)} G^{\oplus 5} \cdots$

$x_0 \mapsto$
$x_1 \mapsto x_0$
$\qquad\quad x_1$

$x_0 \mapsto$
$x_1 \mapsto x_0 - x_0 + x_0 = x_0$
$x_2 \mapsto x_1 - x_1 + x_1 = x_1$
$x_3 \mapsto x_1 - x_2 + x_2 = x_1$
$\qquad\quad x_1 - x_2 + x_3$

$x_0 \mapsto$
$x_1 \mapsto x_0 - x_0 = 0$
$x_2 \mapsto x_1 - x_1 = 0$
$\qquad\quad x_1 - x_2$

Ex. Observe the picture, try to translate the calculation in geometrical language.
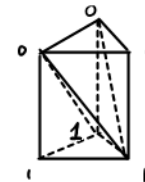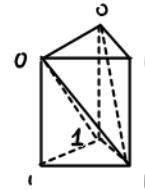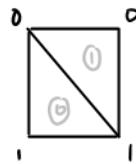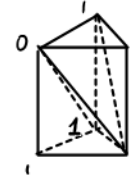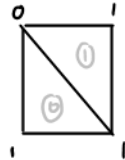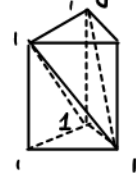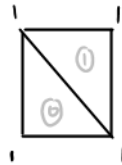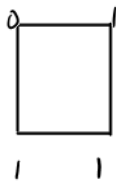
2. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta'; G)$ and $C_\cdot(\Delta'; G)^\Delta$.

non-degenerate
$\downarrow$
Δ

$C_\cdot(\Delta'; G):$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

downward maps: projection, Id, $(1\ 1\ 1)$, 0, 0

$C_\cdot(\Delta'; G)^\Delta:$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} G \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \cdots$$

$C_\cdot(\Delta'; G)^\Delta:$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} G \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \cdots$$

downward maps: inclusion, Id, $\left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$, 0, 0

$C_\cdot(\Delta'; G):$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

In fact, we have

$C_\cdot(\Delta'; G):$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

Id, 0, Id $\|$ Id, 0, Id $\left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 \\ 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, Id $\| 0$, Id $\| 0$

$C_\cdot(\Delta'; G):$
$$0 \longleftarrow G^{\oplus 2} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 3} \xleftarrow{\left(\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{smallmatrix}\right)} G^{\oplus 4} \xleftarrow{\left(\begin{smallmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{smallmatrix}\right)} G^{\oplus 5} \cdots$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)
  For $X \in sSet$, define

θ: non-degenerate
φ: degenerate

$$X_k^{\theta} := \{x \in X_k \mid x \text{ non-degerate}\} \qquad = X_k - (sk^{k-1}X)_k$$
$$X_k^{\varphi} := \{x \in X_k \mid x \text{ degenerate}\} \qquad = (sk^{k-1}X)_k$$
$$X_k^{\varphi i} := \left\{x \in X_k \mid \begin{array}{l} x = \alpha^*(y) \text{ for some } y \in X_{k-i}^{\theta} \\ \quad\quad \alpha: [k-i] \to [k] \end{array}\right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$0 = (sk^{-1}X)_k \overset{X_k^{\varphi k}}{\subset} (sk^0 X)_k \overset{X_k^{\varphi k-1}}{\subset} (sk^1 X)_k \overset{X_k^{\varphi k-2}}{\subset} \cdots \overset{X_k^{\varphi 1}}{\subset} (sk^{k-1}X)_k \overset{X_k^{\varphi 0}}{\subset} (sk^k X) = X_k$$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{X_k^{\varphi}} \underbrace{\qquad\qquad}_{X_k^{\theta}}$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\theta} = \bigoplus_{\alpha \in X_n^{\theta}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\theta}} G \overset{(d_0' - d_1')^*}{\longleftarrow} \bigoplus_{\alpha \in X_1^{\theta}} G \overset{(d_0^2 - d_0^1 + d_2^2)^*}{\longleftarrow} \bigoplus_{\alpha \in X_2^{\theta}} G \cdots$$

$$C_n(X;G)^{\varphi} = \bigoplus_{\alpha \in X_n^{\varphi}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\varphi}} G \overset{(d_0' - d_1')^*}{\longleftarrow} \bigoplus_{\alpha \in X_1^{\varphi}} G \overset{(d_0^2 - d_0^1 + d_2^2)^*}{\longleftarrow} \bigoplus_{\alpha \in X_2^{\varphi}} G \cdots$$

and $H_\cdot(X;G)^{\theta}$, $H_\cdot(X;G)^{\varphi}$ as crspd homology.

By definition, $\qquad C_\cdot(X;G) \cong C_\cdot(X;G)^{\theta} \oplus C_\cdot(X;G)^{\varphi}$

Claim 1. $H_\cdot(X;G)^{\varphi} = 0$, so
$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\theta}. \qquad\qquad (*)$$
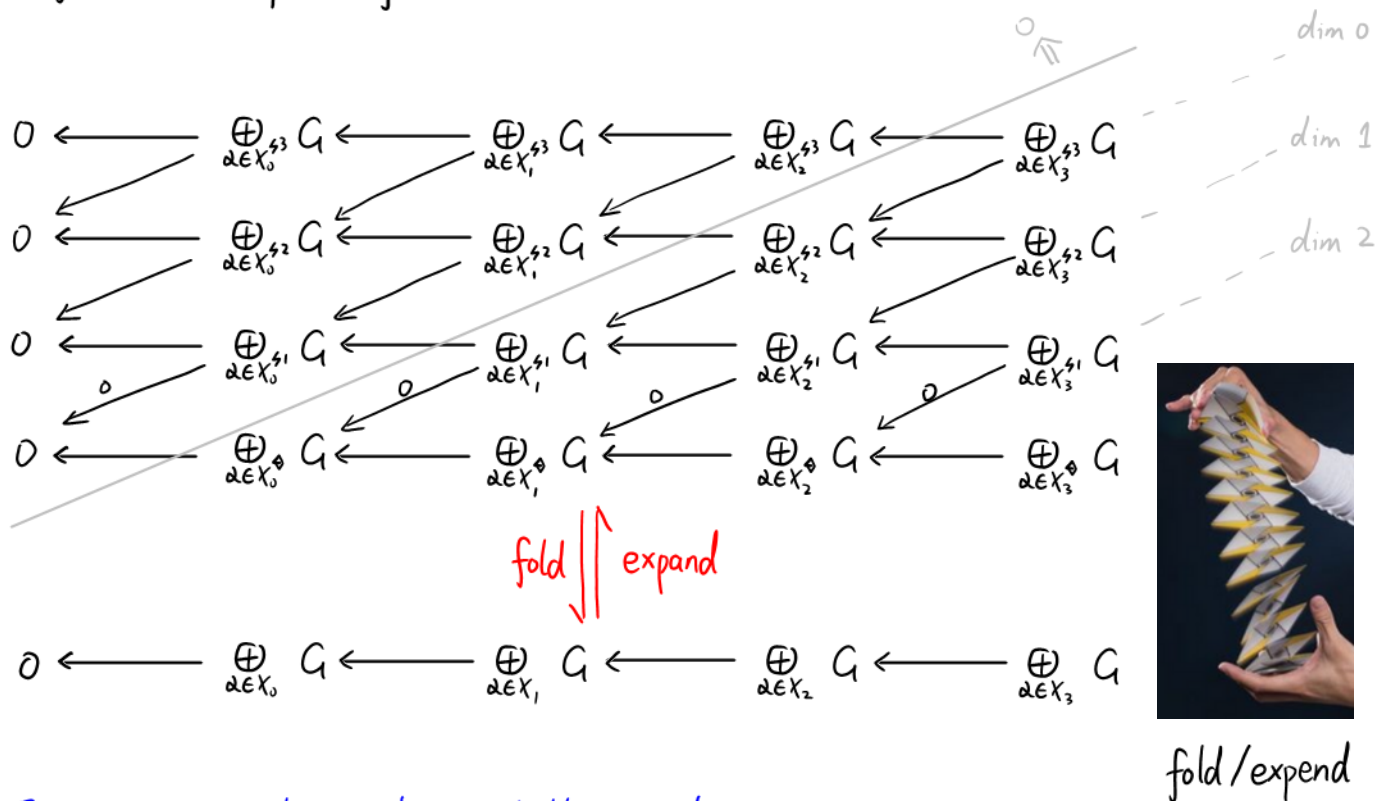
Rmk. Roughly, (*) says that
    singular homology $\approx$ simplicial homology.
  Finally, one can compute the (co)homology of sSets without too much pain.

  To prove Claim 1, one has to expend $C_\cdot(X;G)$ by double complex.

Def (Double complex of $C_\bullet(X;G)$)

dim 0

dim 1

dim 2

$$0 \longleftarrow \bigoplus_{\alpha \in X_0^{\leq 3}} G \longleftarrow \bigoplus_{\alpha \in X_1^{\leq 3}} G \longleftarrow \bigoplus_{\alpha \in X_2^{\leq 3}} G \longleftarrow \bigoplus_{\alpha \in X_3^{\leq 3}} G$$

$$0 \longleftarrow \bigoplus_{\alpha \in X_0^{\leq 2}} G \longleftarrow \bigoplus_{\alpha \in X_1^{\leq 2}} G \longleftarrow \bigoplus_{\alpha \in X_2^{\leq 2}} G \longleftarrow \bigoplus_{\alpha \in X_3^{\leq 2}} G$$

$$0 \longleftarrow \bigoplus_{\alpha \in X_0^{\leq 1}} G \overset{0}{\longleftarrow} \bigoplus_{\alpha \in X_1^{\leq 1}} G \overset{0}{\longleftarrow} \bigoplus_{\alpha \in X_2^{\leq 1}} G \overset{0}{\longleftarrow} \bigoplus_{\alpha \in X_3^{\leq 1}} G$$

$$0 \longleftarrow \bigoplus_{\alpha \in X_0^{0}} G \longleftarrow \bigoplus_{\alpha \in X_1^{0}} G \longleftarrow \bigoplus_{\alpha \in X_2^{0}} G \longleftarrow \bigoplus_{\alpha \in X_3^{0}} G$$

$$\text{fold} \Big\updownarrow \text{expand}$$

$$0 \longleftarrow \bigoplus_{\alpha \in X_0} G \longleftarrow \bigoplus_{\alpha \in X_1} G \longleftarrow \bigoplus_{\alpha \in X_2} G \longleftarrow \bigoplus_{\alpha \in X_3} G$$

fold / expend

Eg. For $X = \Delta^1$, we have double complex

$$0 \longleftarrow 0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \langle x_0, x_4 \rangle \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$0 \longleftarrow 0 \longleftarrow 0 \longleftarrow \langle x_0, x_3 \rangle \overset{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}}{\longleftarrow} \langle x_1, x_2, x_3 \rangle$$

$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$0 \longleftarrow 0 \longleftarrow \langle x_0, x_2 \rangle \overset{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}{\longleftarrow} \langle x_1, x_2 \rangle \longleftarrow 0$$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$0 \longleftarrow \langle x_0, x_1 \rangle \overset{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}{\longleftarrow} \langle x_1 \rangle \overset{0}{\longleftarrow} 0 \longleftarrow 0$$

$$\text{fold} \Big\updownarrow \text{expand}$$

$$0 \longleftarrow G^{\oplus 2} \overset{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}}{\longleftarrow} G^{\oplus 3} \overset{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}{\longleftarrow} G^{\oplus 4} \overset{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}}{\longleftarrow} G^{\oplus 5} \cdots$$