

Eine Woche, ein Beispiel

1.26 Numerical Chern class

Ref:
wiki: Chern class

[BWB21]: Wang, Liao. The Borel-Weil-Bott Theorem in Examples

[GK20]: Frank Gounelas and Alexis Kouvidakis. On Some Invariants of Cubic Fourfolds. European Journal of Mathematics

Nearly all the results are sourced from Wikipedia. I made this document because I tend to mix up the Chern class and the Chern character.

We omit E in notation.

All the results can be check via Macaulay2. You can find my code at

https://github.com/ramified/codelearning/raw/main/Macaulay/numerical_chern_class.txt

$$\begin{aligned} c(E) &= 1 + c_1 + \dots + c_r \in H^*(X; \mathbb{C}) \\ &= \prod_{i=1}^r (1 + a_i) \quad a_i(E) \in H^*(F(E); \mathbb{C}) \end{aligned}$$

$$\begin{aligned} c_t(E) &= 1 + c_1 t + \dots + c_r t^r \in H^*(X; \mathbb{C})[t] \\ &= \prod_{i=1}^r (1 + a_i t) \end{aligned}$$

$$\begin{aligned} ch(E) &= e^{a_1} + \dots + e^{a_r} \in H^*(X; \mathbb{C}) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} (a_1^k + \dots + a_r^k) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} s_k(c_1, \dots, c_r) \\ &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_2c_1 + 3c_3) \\ &\quad + \frac{1}{24}(c_1^4 - 4c_2c_1^2 + 4c_3c_1 + 2c_2^2 - 4c_4) + \dots \end{aligned}$$

$$\begin{aligned} td(E) &= \prod_{i=1}^r \frac{a_i}{1 - e^{-a_i}} \in H^*(X; \mathbb{C}) \\ &= \prod_{i=1}^r \left(1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} a_i^{2k} \right) \\ &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\quad + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \dots \end{aligned}$$

$$\begin{aligned} s(E) &= \prod_{i=1}^r \frac{1}{1 + a_i} \in H^*(X; \mathbb{C}) \\ &\hat{=} 1 + s_1 + \dots + s_n \\ &= 1 - c_1 + (-c_2 + c_1^2) + (-c_3 + 2c_1c_2 - c_1^3) \\ &\quad + (-c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + c_1^4) + \dots \end{aligned}$$

$$\begin{aligned} \sqrt{td(E)} &= 1 + \frac{1}{4}c_1 + \frac{1}{24}(c_1^2 + c_2) + \left(\frac{1}{2}\right)\left(\frac{1}{2}c_1\right)^2 + \dots \\ &= 1 + \frac{1}{4}c_1 + \frac{1}{96}(c_1^2 + 4c_2) + \dots \end{aligned}$$

$$c(E \oplus E') = c(E) \cup c(E')$$

$$c_t(E \oplus E') = c_t(E) \cup c_t(E')$$

$$ch(E \oplus E') = ch(E) + ch(E')$$

$$td(E \oplus E') = td(E) \cup td(E')$$

$$s(E \oplus E') = s(E) \cup s(E')$$

$$ch(E \otimes E') = ch(E) \cdot ch(E')$$

E.g. $X = \mathbb{P}^1$ $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and $H \in H^2(\mathbb{P}^1; \mathbb{C})$ as the generator

$$\begin{aligned} c(E) &= 1 + aH \\ c_t(E) &= 1 + aHt \\ ch(E) &= 1 + aH \\ td(E) &= 1 + \frac{1}{2}aH \\ s(E) &= 1 - aH \end{aligned}$$

E.g. $\Omega_{\mathbb{P}^1} = \omega_{\mathbb{P}^1} = \mathcal{O}(-2)_{a=-2}$ $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}(2)_{a=2}$

For $E = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, one gets

$$\begin{aligned} c(E) &= (1 + a_1H) \cup (1 + a_2H) &= 1 + (a_1 + a_2)H \\ c_t(E) &= (1 + a_1Ht) (1 + a_2Ht) &= 1 + (a_1 + a_2)Ht \\ ch(E) &= 1 + a_1H + 1 + a_2H &= 2 + (a_1 + a_2)H \\ td(E) &= (1 + \frac{1}{2}a_1H) \cup (1 + \frac{1}{2}a_2H) &= 1 + \frac{1}{2}(a_1 + a_2)H \\ s(E) &= (1 - a_1H) \cup (1 - a_2H) &= 1 - (a_1 + a_2)H \end{aligned}$$

Therefore, these characteristic classes can not distinguish $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

E.g. $X = C$ is of genus g , $E = \mathcal{T}_C$, then

$$\begin{aligned} c(C) &= 1 + (2-2g)[p] \\ c_t(C) &= 1 + (2-2g)t \\ ch(C) &= 1 + (2-2g)[p] \\ td(C) &= 1 + (1-g)[p] \\ s(C) &= 1 - (2-2g)[p] \end{aligned}$$

E.g. $X = \mathbb{P}^2$ $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and
 $H \in H^2(\mathbb{P}^2; \mathbb{C})$ as the generator

$$\begin{aligned} c(E) &= 1 + aH \\ c_t(E) &= 1 + aHt \\ ch(E) &= 1 + aH + \frac{1}{2}a^2H^2 \\ td(E) &= 1 + \frac{1}{2}aH + \frac{1}{12}a^2H^2 \\ s(E) &= 1 - aH + a^2H^2 \end{aligned}$$

E.g. $X = \mathbb{P}^n$ $E = \mathcal{T}_{\mathbb{P}^n}$, then the Euler sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus (n+1)} & \longrightarrow & \mathcal{T}_{\mathbb{P}^n} \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \text{Hom}(\mathcal{S}, \mathcal{S}) & & \text{Hom}(\mathcal{S}, \mathcal{O}_{\mathbb{P}^n}) & & \text{Hom}(\mathcal{S}, \mathcal{Q}) \end{array}$$

tells us that

$$\begin{aligned} c(\mathbb{P}^n) &= (1+H)^{n+1} \\ &= 1 + (n+1)H + \binom{n+1}{2}H^2 + \dots \\ c_t(\mathbb{P}^n) &= 1 + (n+1)Ht + \binom{n+1}{2}H^2t^2 + \dots \\ ch(\mathbb{P}^n) &= n + (n+1)H + \frac{n+1}{2}H^2 + \frac{n+1}{3!}H^3 + \dots \\ &= -1 + (n+1)e^H \\ td(\mathbb{P}^n) &= 1 + \frac{1}{2}(n+1)H + \frac{1}{12}((n+1)^2 + \binom{n+1}{2})H^2 + \dots \\ &= \begin{cases} 1 + \frac{3}{2}H + H^2, & n=2 \\ 1 + 2H + \frac{11}{6}H^2 + H^3, & n=3 \\ 1 + \frac{5}{2}H + \frac{35}{12}H^2 + \frac{25}{12}H^3 + H^4, & n=4 \end{cases} \\ s(\mathbb{P}^n) &= (1+H)^{-n-1} \\ &= 1 - (n+1)H + \binom{n+2}{2}H^2 - \binom{n+3}{3}H^3 + \dots \end{aligned}$$

e.p.

$$\begin{aligned} c(\mathbb{P}^2) &= 1 + 3H + 3H^2 \\ c_t(\mathbb{P}^2) &= 1 + 3Ht + 3H^2t^2 \\ ch(\mathbb{P}^2) &= 2 + 3H + \frac{3}{2}H^2 \\ td(\mathbb{P}^2) &= 1 + \frac{3}{2}H + H^2 \\ s(\mathbb{P}^2) &= 1 - 3H + 6H^2 \end{aligned}$$

$$\begin{aligned} c(\mathbb{P}^3) &= 1 + 4H + 6H^2 + 4H^3 \\ c_t(\mathbb{P}^3) &= 1 + 4Ht + 6H^2t^2 + 4H^3t^3 \\ ch(\mathbb{P}^3) &= 3 + 4H + 2H^2 + \frac{2}{3}H^3 \\ td(\mathbb{P}^3) &= 1 + 2H + \frac{11}{6}H^2 + H^3 \\ s(\mathbb{P}^3) &= 1 - 4H + 10H^2 - 20H^3 \end{aligned}$$

<https://math.stackexchange.com/questions/998797/the-second-and-third-chern-classes-of-calabi-yau-threefolds>

E.g. $X = \text{alg K3 surface}$, $E \in \mathcal{T}_X$, then

$$c_1(\Omega_X) = c_1(\omega_X) = c_1(\mathcal{O}_X) = 0 \Rightarrow c_1(\mathcal{T}_X) = 0$$

$[p] \in H^4(X; \mathbb{C})$ generator

$$\begin{aligned} c(X) &= 1 + 24[p] \\ c_t(X) &= 1 + 24t^2 \\ ch(X) &= 2 - 24[p] \\ td(X) &= 1 - 2[p] \\ s(X) &= 1 - 24[p] \end{aligned}$$

What's the Schur functor for Chern class? Give me formulas for $c(S^\lambda(E))$, for some vector bundle E over X .

Answer from chatgpt:

If E has Chern roots x_1, \dots, x_r and $W(\lambda)$ is the set of weights of the irreducible $GL(r)$ -representation of highest weight λ , then

$$c(S^\lambda(E)) = \prod_{(w \in W(\lambda))} (1 + a_1 x_1 + \dots + a_r x_r) \quad (\text{for } w = (a_1, \dots, a_r)),$$

or equivalently,

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1}, \dots, e^{x_r}).$$

In particular,

$$c(S^k E) = \prod_{(a_1 + \dots + a_r = k)} (1 + a_1 x_1 + \dots + a_r x_r)$$

and

$$c(\wedge^k E) = \prod_{(1 \leq i_1 < \dots < i_k \leq r)} (1 + x_{i_1} + \dots + x_{i_k}).$$

Explanation: One very standard answer is to use the splitting principle. Suppose that the vector bundle E has rank r and splits (formally) as a sum of line bundles, so that

$$E = L_1 \oplus L_2 \oplus \dots \oplus L_r$$

with first Chern classes x_1, x_2, \dots, x_r (so that $c(L_i) = 1 + x_i$). Then any Schur functor S^λ applied to E has, by representation theory, "Chern roots" given by the weights of the irreducible representation of $GL(r)$ with highest weight λ . In other words, if $W(\lambda)$ is the (multi)set of weights – that is, if every weight $w \in W(\lambda)$ is given by an r -tuple (a_1, \dots, a_r) , then the corresponding "root" is

$$\langle w, x \rangle = a_1 x_1 + \dots + a_r x_r.$$

Thus the total Chern class of $S^\lambda(E)$ is

$$c(S^\lambda(E)) = \prod_{(w \in W(\lambda))} (1 + \langle w, x \rangle).$$

An equivalent formulation is to note that the Chern character of $S^\lambda(E)$ is given by

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1}, \dots, e^{x_r}),$$

where s_λ is the Schur polynomial corresponding to the partition λ . (From ch one may recover the Chern classes via the usual formal relations.)

As an example, note that when $\lambda = (k)$ the Schur functor is the k -th symmetric power, and its total Chern class may be written as

$$c(S^k(E)) = \prod_{(a_1, \dots, a_r) \in \mathbb{N}^r, a_1 + \dots + a_r = k} (1 + a_1 x_1 + \dots + a_r x_r),$$

while when $\lambda = (1^k)$ (that is, a column of k boxes) one has $S^\lambda(E) = \wedge^k E$ and

$$c(\wedge^k E) = \prod_{(1 \leq i_1 < \dots < i_k \leq r)} (1 + x_{i_1} + \dots + x_{i_k}).$$

Riemann - Roch

Roughly speaking, Riemann-Roch computes chern class of pushforward.

$$\begin{array}{c} G \\ / \\ f: Y \longrightarrow X \end{array}$$

$$\text{GRR: } \text{ch}(f_* G) \text{td}(X) = f_* (\text{ch}(G) \text{td}(Y))$$

$$\text{HRR: } \chi(Y, G) = \int_Y \text{ch}(G) \text{td}(Y) = (\text{ch}(G) \text{td}(Y))_{\deg Y}$$

$$\begin{aligned} \text{RR for surface: } \mathcal{L} = \mathcal{O}(D) \quad \chi(Y, \mathcal{L}) &= \left[(1 + c_1(\mathcal{L}) + \frac{1}{2} c_1(\mathcal{L})^2) (1 + \frac{1}{2} c_1(Y) + \frac{1}{12} (c_1(Y)^2 + c_2(Y))) \right]_2 \\ &= \frac{1}{2} c_1(\mathcal{L})^2 + \frac{1}{2} c_1(\mathcal{L}) c_1(Y) + \frac{1}{12} (c_1(Y)^2 + c_2(Y)) \\ &= \frac{1}{2} D(D-K) + \frac{1}{12} (K^2 + e) \\ \Rightarrow \begin{cases} \chi(\mathcal{O}) = \frac{1}{12} (K^2 + e) \\ \chi(D) = \chi(\mathcal{O}) + \frac{1}{2} D(D-K) \end{cases} \end{aligned}$$

$$\begin{aligned} \text{E.g. In case } D = -C, \\ 0 \longrightarrow \mathcal{O}(-C) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_C \longrightarrow 0 \\ \Rightarrow \chi(\mathcal{O}_C) = \chi(\mathcal{O}) - \chi(D) = -\frac{1}{2} C(C+K) \\ 1 - g(C) = -\frac{1}{2} D(D-K) \\ \Rightarrow 2g(C) - 2 = C(C+K) = c_1(\mathcal{L}) (c_1(\mathcal{L}) + c_1(w_X)) \\ \text{where } \mathcal{L} = \mathcal{O}(C) \end{aligned}$$

⚠ $\mathcal{O}_C \otimes \mathcal{O}_X(C) \neq \mathcal{O}_C$, so we only have

$$\begin{aligned} 0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0 \\ \Rightarrow \chi(\mathcal{O}_C(C)) = \chi(\mathcal{O}(C)) - \chi(\mathcal{O}) = \frac{1}{2} C(C-K) \end{aligned}$$

<https://math.stackexchange.com/questions/3446039/adjunction-formula-and-riemann-roch-for-surfaces>

$$\begin{aligned} \text{RR for curve: } \mathcal{L} = \mathcal{O}(D) \quad \chi(Y, \mathcal{L}) &= \left[(1 + c_1(\mathcal{L})) (1 + \frac{1}{2} c_1(Y)) \right]_1 \\ &= c_1(\mathcal{L}) + \frac{1}{2} c_1(Y) \\ &= \deg D + 1 - g \end{aligned}$$

RR for Flag or Grassmannian: Borel-Weil-Bott theorem.

BWB is stronger, because it tells $H^k(\mathrm{Gr}(r,n); \mathbb{C})$ for specific k ,
and it constructs an explicit isomorphism.

[BWB21, Thm 2.4] For a GL_n -regular and dominant (resp. P) weight $\lambda \in X^*(T(GL_n))$,

$$H^{l(\omega)}(\mathrm{Gr}(r,n), \mathcal{U}(\lambda)) \cong V_{GL_n}(\omega \cdot \lambda) \quad \omega \cdot \lambda := \omega(\lambda + \rho) - \rho$$

\uparrow Verma module

[GK20, Sec 3]

$$H^{l(\omega)}(\mathrm{Gr}(r,n), \Sigma_{\lambda'} \mathcal{S}^{\vee} \otimes \Sigma_{\lambda''} \mathcal{Q}^{\vee}) \cong \sum_{\omega \cdot \lambda} \mathbb{C}^n$$

Compare HRR with BWB:

$$\begin{aligned} \mathrm{ch}(\mathcal{U}(\lambda) \mathrm{td}(\mathrm{Gr}(r,n))) &= \mathrm{ch}(\Sigma_{\omega'} \mathcal{S}^{\vee} \otimes \Sigma_{\omega''} \mathcal{Q}^{\vee}) \mathrm{td}(\mathcal{S}^{\vee} \otimes \mathcal{Q}) \\ &\stackrel{?}{=} (-1)^{l(\omega)} \prod_{1 \leq i < j \leq n} \frac{(\omega \cdot \lambda)_i - (\omega \cdot \lambda)_j + j - i}{j - i} \\ &= (-1)^{l(\omega)} \dim V_{GL_n}(\omega \cdot \lambda). \end{aligned}$$

E.g.

$$\chi(P^1; G) = ch_1(G) + H ch_0(G)$$

$$\chi(P^2; G) = ch_2(G) + \frac{3}{2} H ch_1(G) + H^3 ch_0(G)$$

$$\chi(P^3; G) = ch_3(G) + 2 H ch_2(G) + \frac{11}{6} H^2 ch_1(G) + H^3 ch_0(G).$$

Mukai vector & Mukai pairing

Def Let X/\mathbb{C} be a sm proj variety, and $E \in \mathcal{D}^b(X)$.

Define

$$\begin{aligned} \nu(E) &= \text{ch}(E) \sqrt{\text{td}(X)} \in H^*(X; \mathbb{C}) \\ &= \left(\text{ch}_0(E), \text{ch}_1(E) + \frac{1}{4} c_1(X) \text{ch}_0(E), \right. \\ &\quad \left. \text{ch}_2(E) + \frac{1}{4} c_1(X) \text{ch}_1(E) + \frac{1}{96} (c_1^2(X) + 4c_2(X)) \text{ch}_0(E), \dots \right) \end{aligned}$$

as the Mukai vector, and the Mukai pairing

also called Euler pairing

$$\begin{aligned} \langle -, - \rangle &: H^*(X; \mathbb{C}) \times H^*(X; \mathbb{C}) \longrightarrow \mathbb{C} \\ \langle \nu, \nu' \rangle &= \int_X e^{\frac{c_1(X)}{2}} \cdot (\nu^\vee \cdot \nu') \end{aligned}$$

where

$$\nu^\vee = \sum_j (\sqrt{-1})^{2j} \nu_j \quad \nu_j \in H^{2j}(X; \mathbb{C})$$

Check [Huy06, p133] for this definition.

<https://math.stackexchange.com/questions/4346782/the-mukai-pairing>

$$\begin{aligned} \text{So } \langle \nu(E), \nu(F) \rangle &= \int_X \text{ch}(E^\vee) \text{ch}(F) \text{td}(X) \\ &= \chi(X, E^\vee \otimes F) = \sum_j (-1)^j \dim \text{Hom}(E, F[j]) \\ &= \chi(E, F) \end{aligned}$$

Rmk. The factor $e^{\frac{c_1(X)}{2}}$ comes from [Huy06, p133].

$$\nu(E^\vee) = \nu(E)^\vee \cdot e^{\frac{c_1(X)}{2}}$$

E.g. When $X = C$ is a curve of genus g , $c(C) = 1 + (2-2g)[p]$,
so

$$\begin{aligned} \nu(E) &= (ch_0(E), ch_1(E) + \frac{1}{2}(1-g)ch_0(E)) \\ &= (rk, \deg + \frac{1}{2}(1-g)rk) \end{aligned}$$

$$\begin{aligned} \langle \nu(E), \nu(F) \rangle &= ((1 + (1-g)[p]) (\nu_0(E) - \nu_1(E)) (\nu_0(F) + \nu_1(F)))_{deg=1} \\ &= \nu_0(E)\nu_1(F) - \nu_1(E)\nu_0(F) + (1-g)\nu_0(E)\nu_0(F) \\ &= r(E)d(F) - d(E)r(F) + (1-g)r(E)r(F) \end{aligned}$$

E.g. When $X = \mathbb{P}^2$, $c(\mathbb{P}^2) = 1 + 3H + 3H^2$, so

$$\begin{aligned} \nu(E) &= (ch_0(E), ch_1(E) + \frac{3}{4}Hch_0(E), ch_2(E) + \frac{3}{4}Hch_1(E) + \frac{7}{32}H^2ch_0(E)) \\ &= ch(E) \cdot \begin{pmatrix} 1 & \frac{3}{4}H & \frac{7}{32}H^2 \\ & 1 & \frac{3}{4}H \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \langle \nu(E), \nu(F) \rangle &= ((1 + \frac{3}{2}H + \frac{9}{8}H^2) (\nu_0(E) - \nu_1(E) + \nu_2(E)) (\nu_0(F) - \nu_1(F) + \nu_2(F)))_{deg=2} \\ &= \nu_0(E)\nu_2(F) - \nu_1(E)\nu_1(F) + \nu_2(E)\nu_0(F) \\ &\quad + \frac{3}{2}H(\nu_0(E)\nu_1(F) - \nu_1(E)\nu_0(F)) \\ &\quad + \frac{9}{8}H^2\nu_0(E)\nu_0(F) \\ &= H^2ch_0(E)ch_0(F) - \frac{3}{2}Hch_1(E)ch_0(F) + \frac{3}{2}Hch_0(E)ch_1(F) \\ &\quad + ch_2(E)ch_0(F) - ch_1(E)ch_1(F) + ch_0(E)ch_2(F) \\ &= (Hch_0(E), ch_1(E), H^{-1}ch_2(E)) \begin{pmatrix} 1 & \frac{3}{4} & \frac{1}{2} \\ -\frac{3}{4} & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} Hch_0(F) \\ ch_1(F) \\ H^{-1}ch_2(F) \end{pmatrix} \\ &= (ch_0(E), ch_1(E), ch_2(E)) \begin{pmatrix} H^2 & \frac{3}{4}H & \frac{1}{2} \\ -\frac{3}{4}H & -1 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} ch_0(F) \\ ch_1(F) \\ ch_2(F) \end{pmatrix} \end{aligned}$$

E.g. When X is a K3 surface, $c(X) = 1 + 24[p]$,
so

$$\begin{aligned} \nu(E) &= (ch_0(E), ch_1(E), ch_2(E) + ch_0(E)) \\ \langle \nu(E), \nu(F) \rangle &= ((\nu_0(E) - \nu_1(E) + \nu_2(E)) (\nu_0(F) + \nu_1(F) + \nu_2(F)))_{deg=2} \\ &= \nu_0(E)\nu_2(F) - \nu_1(E)\nu_1(F) + \nu_2(E)\nu_0(F) \\ &= ch_0(E)ch_2(F) - ch_1(E)ch_1(F) + ch_2(E)ch_0(F) + 2ch_0(E)ch_0(F) \\ &= (ch_0(E), ch_1(E), ch_2(E)) \begin{pmatrix} 2[p] & & \frac{1}{2} \\ & -1 & \\ \frac{1}{2} & & \end{pmatrix} \begin{pmatrix} ch_0(F) \\ ch_1(F) \\ ch_2(F) \end{pmatrix} \end{aligned}$$