Eine Woche, ein Beispiel

7.10 Non-Archimedean valued field

See [https://math.stackexchange.com/questions/186326/non-archimedean-fields] for definition and examples.
However, in this document, we only care about field extensions of NA local fields.

In this document, E, F are field extensions over $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$ with extended valuation.
E/F is usually an alg field extension.
Some results can be generalized to NA valued field where the valuation is of rank 1. For the "Completion" section, the assumption here is essential.

Goal.

1. Basic informations
2. Perfection
3. Completion
4. Tilting

$\left.\begin{matrix}\text{2. Perfection}\\ \text{3. Completion}\\ \text{4. Tilting}\end{matrix}\right\}$ three operators which do not change the Galois group

	effective structure	main tool
Perfection	field	Galois theory
Completion	topology	Krasner's lemma
Tilting	(mixed) character	almost mathematics

1. Basic informations

(F, v): NA valued field

$\rightsquigarrow \mathcal{O} := \{x \in F \mid v(x) \geq 0\}$

$\mathfrak{p} := \{x \in F \mid v(x) > 0\}$

$k := \mathcal{O}/\mathfrak{p} \qquad p = \operatorname{char} k$

$\mathcal{U} := \mathcal{U}^{(0)} := \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\}$

Prop. (still true)

- $(\mathcal{O}, \mathfrak{p})$ is still a local ring. $\mathcal{O}$ is integral closed.
- F is totally disconnected. $\lhd$ 
- Every open ball $B_x(<r)$ is closed and every closed ball $B_x(r)$ is open || $\{0\}$ is closed but not open in $\mathbb{Q}_p$. $\mathbb{Q}_p - \{0\}$ is open but not closed in $\mathbb{Q}_p$.

  ⚠ Open ball may be not closed ball! Vice versa. (We never define "ball" alone)

Prop. (New Phenomenon) compared with NA local field

- It's possible that $\mathfrak{p}^2 = \mathfrak{p}$, so the uniformizer π may be not picked.
  Luckily have topological uniformizer $\pi \in \mathfrak{p}$.
  e.g. $K = \mathbb{Q}_p(p^{\frac{1}{p^\infty}})$, $\mathcal{O} = \mathbb{Z}_p(p^{\frac{1}{p^\infty}})$, $\pi = p \in \mathfrak{p} = \mathfrak{p}^2$
- k may be not finite
- $\mathcal{O}$ may be not DVR (Noetherian $\Leftrightarrow$ F local field, not dim 1)
  https://math.stackexchange.com/questions/363166/examples-of-non-noetherian-valuation-rings
- $\mathcal{O}$ may be not cpt. $\mathcal{O}^\times$ neither.
- No classification and good enough understanding of the structure (for me)!

2. Perfection

In this section F is a field (can be with no valuation)

Ref: wiki:perfect field

Def. A field is perfect if every fin ext is sep.

Thm [Thm 4.13 in GTM167] When char $F = p$,

$$F \text{ is perfect} \Leftrightarrow F \xrightarrow{(-)^p} F \text{ is surjective.}$$

E.g. char $F = 0$ / finite field $\Rightarrow$ F perfect

$\mathbb{F}_p(t)$, $\mathbb{F}_p((t))$ are not perfect

$\mathbb{F}_p((t^{\frac{1}{p^\infty}}))$ is perfect.

Warning. Don't mix perfectoid field with perfect field!

e.g. $\mathbb{Q}_p$ is not a perfectoid field, but it is perfect;

$\mathbb{Q}_p(p^{\frac{1}{p^\infty}})$ is both a perfectoid field and a perfect field.

A perfectoid field is always perfect by
[Remark 17.1.8, http://math.stanford.edu/~conrad/Perfseminar/Notes/L17.pdf].

Notation Perfection = maximal purely inseparable ext

= purely inseparable closure

E.g. The perfection of $\mathbb{F}_p((t))$ is $\mathbb{F}_p((t^{\frac{1}{p^\infty}})) := \bigcup_n \mathbb{F}_p((t^{\frac{1}{p^n}}))$

Thm [A special case of Thm 4.23 in GTM167]

$F^{insep} :=$ perfection of F, then

$$\mathrm{Gal}(F^{sep}/F) = \mathrm{Gal}((F^{insep})^{sep}/F^{insep})$$

Rmk. Perfect fields admit Witt vectors.

i.e. $\forall$ perfect field F, we can define $W(F)$.

e.p. $W_{\infty,p}(\mathbb{F}_p) = \mathbb{Z}_p$ $\quad W_{\infty,p}(\mathbb{F}_{p^k}) = \mathbb{Z}_{p^k}$ $\quad W_{\infty,p}(\overline{\mathbb{F}_p}) = \mathcal{O}_{\widehat{\mathbb{Q}_p^{ur}}}$

3. Completion

Ref: https://math.mit.edu/classes/18.785/2017fa/LectureNotes8.pdf

https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete

A lot of NA valued fields are not complete:

Lemma. E/F an alg extension, F NA local field. Then

$$E \text{ is complete} \iff [E:F] < +\infty$$

Proof: "$\Leftarrow$": $[E:F] < +\infty \Rightarrow E$ NA local field $\Rightarrow E$ is complete

"$\Rightarrow$": If not,

$$E = \bigcup_{\substack{F'/F \text{ finite} \\ F' \subseteq E}} F' \xrightarrow{[E:F] = +\infty \Rightarrow F' \neq E} E \subset E \text{ is of second category}$$

$$E \text{ is complete} \xrightarrow{\text{Baire}} E \subset E \text{ is of first category}$$

$\Rightarrow$ Contradiction!

We usually have 3 ways to complete $\mathcal{O} = \mathcal{O}_F$:

$$\check{\mathcal{O}}_\pi := \varprojlim_n \mathcal{O}/(\pi^n) \qquad \pi\text{-adic completion}$$

$$\check{\mathcal{O}}_\mathfrak{p} := \varprojlim_n \mathcal{O}/(\mathfrak{p}^n) \qquad \mathfrak{p}\text{-adic completion}$$

$$\hat{\mathcal{O}} := \text{completion w.r.t. } \|\cdot\|_F.$$

[Prop 8.11, https://math.mit.edu/classes/18.785/2017fa/LectureNotes8.pdf] tells us, when F is a NA local field, these three completions are equivalent.

Universal property:

Define.

$\mathrm{Ob}(\mathrm{Field}_{NAv}) = \{(F, v: F \to \Gamma \cup \{\infty\}) \text{ is a NA valued field}\}/\sim$

$\mathrm{Mor}(F, E) = \{f: F \to E \mid f \text{ cont field embedding}\}$

$\mathrm{CplField}_{NAv}$: full subcategory consisting of complete objects.

We get adjoint fctors

$$\begin{tikzcd} \mathrm{CplField}_{NAv} \arrow[r, bend right, "\text{forget}"'] \arrow[r, phantom, "\perp"] & \mathrm{Field}_{NAv} \arrow[l, bend right, "\text{cpl w.r.t. } \|\cdot\|"'] \end{tikzcd}$$

i.e. $\forall f: F \to E$ cont field embedding, E cpl,

$\exists!\ \hat{f}: \hat{F} \to E$ s.t. $f = \hat{f} \circ \iota$.

$$\begin{tikzcd} F \arrow[r, hook, "\iota"] \arrow[dr, "f"'] & \hat{F} \arrow[d, dashed, "\exists!\hat{f}"] \\ & E \end{tikzcd}$$

Cor. $\hat{\hat{F}} = \hat{F}$.

Krasner's lemma

We would like to recall the Krasner's lemma, which is a key lemma in the theory of NA completed field.

Thm (Krasner's lemma)

K: NA complete field

$\alpha \in K^{sep}/K$, $\quad Gal(K^{sep}/K)\alpha = \{\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n\}$ $\quad n \geq 2$

$\beta \in K^{sep}$.

- If $\alpha \notin K(\beta)$, then $\quad |\alpha - \beta| \geq \min_{2\leq i\leq n} |\alpha - \alpha_i|$

  Two useful cases:
  - $\quad \mathrm{dist}(\alpha, K) \geq \min_{2\leq i\leq n} |\alpha - \alpha_i| > 0$
  - For F/K sep ext, $\alpha \notin F$, we have
    $$\mathrm{dist}(\alpha, F) \geq \min_{2\leq i\leq n} |\alpha - \alpha_i| > 0 \quad \Rightarrow \alpha \notin \hat{F}$$
    i.e. $\hat{F} \cap F^{sep} = F$

- If $|\alpha - \beta| < \min_{2\leq i\leq n} |\alpha - \alpha_i|$, then $\alpha \in K(\beta)$

  Combined with Lemma 1, this version is usually used for approximation.

  $\varepsilon := \min_{2\leq i\leq n} |\alpha - \alpha_i|$ when applied

Lemma 1. K: NA complete field.

Let $f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i \in K[x]$ irr sep, $\alpha \in K^{sep}$ be a root of f.

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall g(x) = x^n + \sum_{i=0}^{n-1} b_i x^i \in K[x]$ with
$$\|f - g\| := \max_{0\leq i\leq n-1} |a_i - b_i| < \delta,$$
$\exists \beta \in K^{sep}$ be a root of g, with
$$|\alpha - \beta| < \varepsilon.$$

Proof of Lemma 1. Let $C_0 = \left(\max_{0\leq i\leq n-1} |a_i|^{\frac{1}{n-i}}\right) + 2$.

$$\alpha^n = \sum_{i=0}^{n-1} -a_i \alpha^i \Rightarrow |\alpha|^n \leq \max_{0\leq i\leq n-1} |a_i||\alpha|^i$$
$$\Rightarrow |\alpha| \leq \max_{0\leq i\leq n-1} |a_i|^{\frac{1}{n-i}} < C_0$$

$\forall \varepsilon > 0$, $\exists \delta := \frac{\varepsilon^n}{C_0^n} > 0$ s.t. $\forall g(x) = x^n + \sum_{i=0}^{n-1} b_i x^i \in K[x]$ with $\|f - g\| < \delta$,

(β_j: roots of g) $\quad (\min_j |\alpha - \beta_j|)^n \leq \prod_j |\alpha - \beta_j| = |g(\alpha)| = |f(\alpha) - g(\alpha)|$
$$\leq \max_{0\leq i\leq n-1} |a_i - b_i||\alpha|^i$$
$$< \delta\, C_0^n = \varepsilon^n$$
$$\Rightarrow \min_j |\alpha - \beta_j| \leq \varepsilon$$

Rmk. Since $\alpha_i \neq \alpha_j$ for $i \neq j$, we can set δ small enough s.t. $\beta_i \neq \beta_j$ for $i \neq j$.
In this case, we can require that $\beta \in K^{sep}$.

We can enhance Lemma 1 to stronger version by Krasner's Lemma.

Lemma 2. K: NA complete field.

Let $f(x) = x^n + \sum_{i=0}^{n-1} a_i x^i \in K[x]$ irr sep, $\{\alpha_i\}_{i=1}^{n} \subseteq K^{sep}$ be roots of f.

$\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall\ g(x) = x^n + \sum_{i=0}^{n-1} b_i x^i \in K[x]$ with

$$\|f-g\| := \max_{0 \le i \le n-1} |a_i - b_i| < \delta,$$

$\exists$ a ordering $\{b_1, \cdots, b_n\}$ of roots of g, s.t

1. $|\alpha_i - \beta_i| < \varepsilon$
2. $K(\alpha_i) = K(\beta_i)$
3. g is irreducible.

[Idea of proof. Reset $\varepsilon' = \min_{2 \le i \le n} \{\varepsilon, |\alpha - \alpha_i|\}$ so that we can apply Krasner's lemma.

$$\overbrace{\underset{\deg n}{K \subseteq} K(\alpha_i)}^{\text{deg most } n} \subseteq K(\beta_i) \quad \Rightarrow \quad \begin{cases} K(\alpha_i) = K(\beta_i) \\ g \text{ is irr} \end{cases}$$

$\square$]

Galois with completion

All the arguments work if you replace $\mathbb{Q}_p$ by $\mathbb{F}_p((t))$;
however, some technical conditions (sep) can be removed if you focus on $\mathbb{Q}_p$.

In this section, F: alg sep ext of $\mathbb{Q}_p$

$C := \widehat{F^{sep}} = \widehat{\mathbb{Q}_p^{sep}}$ is alg closed by S29722/Krasner

Every field is considered in a fixed $\mathbb{C}$.

<u>Cor 1 from Krasner's lemma</u>. $\hat{F} \cap F^{sep} = F$.

When F is perfect (all fin ext are sep), this is equivalent to

$\hat{F}/F$ is purely transcendental.

Q: If $F/\mathbb{F}_p((t))$ is not perfect, is $\hat{F}/F$ still purely transcendental?

Q: How much do we know about the transcendental degree?

Fun fact: $\mathbb{Q}_p(\zeta_\infty, p^{\frac{1}{p^\infty}})$ is not dense in $\mathbb{C}_p$ since

$\mathbb{Q}_p(\zeta_\infty, p^{\frac{1}{p^\infty}})$ is not alg closed (irr poly. $x^p - x + p^{-1}$)

<u>Main theorem</u>. We have the iso of Galois gp

$$\mathrm{Gal}(F^{sep}/F) \cong \mathrm{Gal}(\hat{F}^{sep}/\hat{F})$$

Equivalently, we have the canonical one-to-one correspondense

$$\{E/F \text{ fin sep ext}\} \longleftrightarrow \{\mathbb{E}/\hat{F} \text{ fin sep ext}\}$$
$$E \longmapsto \hat{E}$$
$$\overline{F}^{\mathbb{E}} := F^{sep} \cap \mathbb{E} \longleftarrow\!\shortmid \mathbb{E}$$

Proof. — $\overline{F}^{\hat{E}} = \overline{E}^{\hat{E}} \overset{\text{Cor 1}}{=\!=} E$

— For $\mathbb{E}/\hat{F}$ fin sep ext, let $E := \overline{F}^{\mathbb{E}}$. Want: $\hat{E} = \mathbb{E}$.

- $\mathbb{E}/\hat{F}$ fin sep $\Rightarrow$ $\mathbb{E}$ is complete $\Rightarrow$ $\hat{E} \subseteq \mathbb{E}$
- $\forall x \in \mathbb{E}$, $\forall \varepsilon > 0$, want to find $y \in E$ s.t. $|x-y| < \varepsilon$. (Thus $\mathbb{E} \subseteq \hat{E}$)

$\exists a_i \in \hat{F}$, $x^n + \sum_{i=0}^{n-1} a_i x^i = 0$

$\xrightarrow{\text{Lemma 2}}$ $\exists y \in F^{sep}$ $\exists b_i \in F$, $y^n + \sum_{i=0}^{n-1} b_i y^i = 0$ s.t.

- $|x-y| < \varepsilon$
- $\hat{F}(y) = \hat{F}(x) \subseteq \mathbb{E}$ $\Rightarrow$ $y \in F^{sep} \cap \mathbb{E} = E$ $\square$

Rmk. Finiteness is essential. (Otherwise $\mathbb{E}$ may be not complete).

E.g. $\widehat{\mathbb{Q}_p^{un}} \neq \mathbb{C}_p$ since $\overline{\mathbb{Q}_p}/\mathbb{Q}_p^{un}$ is inf field ext.

4. Tilting

There is no need to write anything new for the Prof. Peter Scholze's work. I cannot do better, of course :->
So here I just collect everything I think worthwhile to cite:

https://www.youtube.com/watch?v=SA1lkTuESco&list=PLx5f8IelFRgEZ-Qk_SGo3n5jE-ykcAfZX
https://www.math.uni-bonn.de/people/scholze/PerfectoidSpaces.pdf
https://mathoverflow.net/questions/65729/what-are-perfectoid-spaces

Maybe here is also a good place to remind me of some mathematical videos to see?