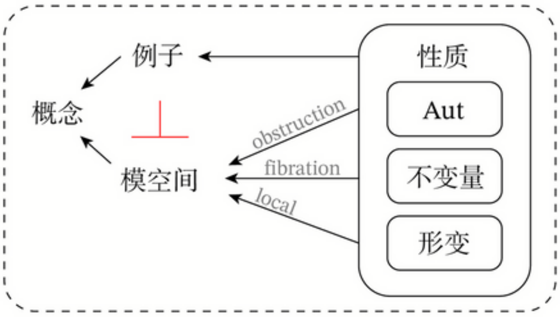
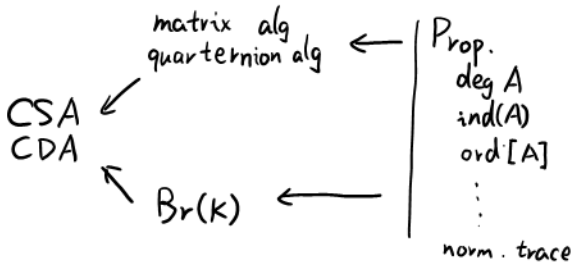
Eine Woche, ein Beispiel

10.24. central simple algebra (CSA) & central division algebra (CDA)



分类 ≌ 对概念的理解



application in CSA/CDA

Q: 在代数几何中我们把域看成一个点，那么over K的中心可除代数可以看成啥呢？

A: 可除代数 base change 到 algebraic closure上是矩阵代数；n^2 维的矩阵代数是 n 个位置暂未被观测确定的纠缠粒子.交换代表确定;非交换代表不确定，测不准 (by 李奇芮)

ref: https://en.wikipedia.org/wiki/Noncommutative_algebraic_geometry

Remark: I learned most contents from the wiki of CSA, Quaternion algebra and Brauer group. Here I just present results without proof(since I'm too lazy to read the proof). For complete discussions of these contents, you can refer to https://www2.math.ou.edu/~kmartin/quaint/.

Def (CSA) The central simple algebra over K is
A = f.d. ass K-alg
+ simple   no non-trivial two-sided ideal
+ $C(A) = K$

E.g. $M_n(k)$ is CSA over k.
E.g. Suppose A is CSA/k. Then
A is comm $\Leftrightarrow A = k$

Rmk. simple $\xrightarrow{f.d}$ semisimple.
$A = \mathbb{C}[x, \partial]/(\partial x - x\partial - 1)$ is simple but not semisimple.

For more informations, see
https://math.stackexchange.com/questions/3809479/finite-dimensional-simple-algebras-are-semisimple
https://mathoverflow.net/questions/4591/proof-a-weyl-algebra-isnt-isomorphic-to-a-matrix-ring-over-a-division-ring

Cor. When $k = \bar{k}$, the only CSAs are $M_n(k)$.
Cor. By Artin-Wedderburn thm $A \simeq M_n(S)$ where
S is a (fd) central division alg (CDA) over K.

E.g. CSA/$\mathbb{R}$: $M_n(\mathbb{H})$ & $M_n(\mathbb{R})$ no others
E.g. CSA/$\mathbb{F}_q$: $M_n(\mathbb{F}_q)$

Def (Brauer equivalent) $M_n(S) \sim M_m(T) \Leftrightarrow S \simeq T$
Def (Brauer group) $Br(K) = \{CSA/K\}/\sim \overset{\text{as set}}{=} \{CDA/K\}$
Verify $Br(K)$ is indeed group where $S \cdot T := S \otimes_k T$ $S^{-1} := S^{op}$

E.g. $Br(K) = \{*\}$ when 1) $K = \bar{K}$ $M_m(k) \otimes M_n(k) \simeq M_{mn}(k)$
2) $K = \mathbb{F}_q$
3) $K = k(C)$ C: alg curve over $k = \bar{k}$
4) alg ext of $\bigcup_n \mathbb{Q}(\zeta_n)$

E.g. $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq M_2(\mathbb{C})$
E.g. $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$ $p > 2$?
Rmk $Br(K)$ is always torsion group.
Rmk. S: CDA/k $\Rightarrow$ $\dim_k S$ is square

Def. - degree $\deg A := \sqrt{\dim A}$
- Schur index $\operatorname{ind}(A) = \deg S$ where $A \simeq M_n(S)$
- period $\operatorname{ord}[A]$: order of $[A] \in Br(K)$
exponent

Rmk $\operatorname{ord}[A] \mid \operatorname{ind}(A)$. they have same prime factors.

Def (quaternion alg) = CSA + dim 4
Cor. Suppose A is quaternion alg. A is $M_2(k)$ or CDA.
(split) (non-split)
When $Br(k) = \{*\}$, then $A = M_2(k)$.

Cor. When A is non-split quaternion alg, then $\operatorname{ind}(A) = 2 \Rightarrow \operatorname{ord}[A] = 2$
Rmk. $\{$elements of order 2 in $Br(K)\} = \langle A : A$ non-split quat alg$\rangle$

Ref: Lam, Tsit-Yuen (2005). Introduction to Quadratic Forms over Fields. Graduate Studies in Mathematics. 67. American Mathematical Society. ISBN 0-8218-1095-2. MR 2104929. Zbl 1068.11023.