Modular form
5. moduli interpretation

1 level structure
2. moduli interpretation of $\mathcal{H}/\Gamma$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S,T \mid S^4=(ST)^6=Id\rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	$\{\pm Id\}$	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S,T \mid S^2=(ST)^3=Id\rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	Id	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$

can't define $SL_2/\mathbb{G}_m$

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A\leq B\leq C$ gp $\quad A\triangleleft C\Rightarrow A\triangleleft B$

no other restrictions. i.e. the following cases may happen:

$A\triangleleft B\triangleleft C$ ($A\triangleleft C$)	$A\triangleleft B\leq C$ ($A\triangleleft C$)	$A\triangleleft B\triangleleft C$	$A\triangleleft B\leq C$	$A\leq B\triangleleft C$	$A\leq B\leq C$
✓	✓	$C_2\triangleleft A_4\triangleleft S_4$	✓	✓	$S_2\leq S_3\leq S_4$

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$
\begin{array}{ccc}
\Gamma(N) & \twoheadrightarrow & \{Id\} \\
\cap & & \cap \\
\Gamma_1(N) & \twoheadrightarrow & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\
\cap & & \cap \\
\Gamma_0(N) & \twoheadrightarrow & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\
\cap & & \cap \\
\Gamma(1) = SL_2(\mathbb{Z}) & \xrightarrow{\text{[WWL, Prop 1.4.4]}} & SL_2(\mathbb{Z}/N\mathbb{Z}) \\
\cup & & \cup \\
\Gamma^0(N) & \twoheadrightarrow & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \\
\cup & & \cup \\
\Gamma^1(N) & \twoheadrightarrow & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}
\end{array}
$$

⚠ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$, but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \begin{array}{l} a, b, c, d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

Ex. Verify the following tables (left comes from right)

$A \triangleleft B$ (A \ B)	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	—	✓	✓	✓
$\Gamma_1(N)$	—	—	✓	×
$\Gamma_0(N)$	—	—	—	×
$\Gamma(1)$	—	—	—	—

$A \triangleleft B$ (A \ B)	N	B	G
N	—	✓	×
B	—	—	×
G	—	—	—

Ex. show [WWL, 练习1.4.14]

练习 1.4.14 对所有正整数 N, 证明

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma(N)) = N^3 \prod_{\substack{p:\text{素数} \\ p|N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(N)) = N^2 \prod_{\substack{p:\text{素数} \\ p|N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}) : \Gamma_0(N)) = |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1} \cdot (\mathrm{SL}(2,\mathbb{Z}) : \Gamma_1(N)) = N\prod_{p|N}\left(1+\frac{1}{p}\right).$$

A. Reduced to computation of $|SL_2(\mathbb{Z}/N\mathbb{Z})|$, $|B(\mathbb{Z}/N\mathbb{Z})|$, $|N(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5,4,6$ if you don't understand the process.

Notation: $\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z}^{\oplus 2})_{prim} / (\mathbb{Z}/N\mathbb{Z})^* \overset{[6.3\,M]}{=\!=} \mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: https://arxiv.org/pdf/2010.15543v2.pdf

∇ $\mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$[3:2] \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathbb{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different piece of $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|SL_2(\mathbb{F}_p)| = p^3 - p$

$|B(\mathbb{F}_p)| = p^2 - p$

$|N(\mathbb{F}_p)| = p$

$\# \mathbb{F}_p^\times = p-1$

② $|SL_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e} - p^{3e-2}$

$|B(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e} - p^{2e-1}$

$|N(\mathbb{Z}/p^e\mathbb{Z})| = p^e$

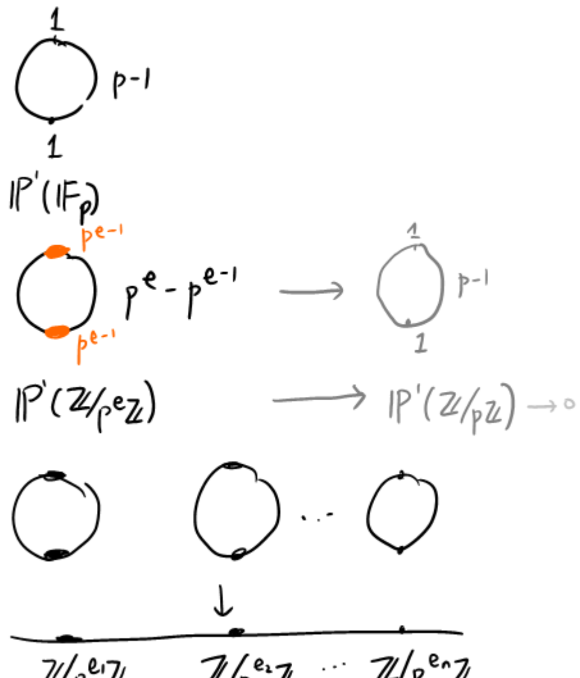
$\# (\mathbb{Z}/p^e\mathbb{Z})^\times = p^e - p^{e-1}$

③ $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p^2}\right)$

$|B(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$

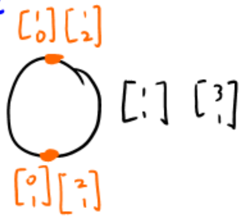
$|N(\mathbb{Z}/N\mathbb{Z})| = N$

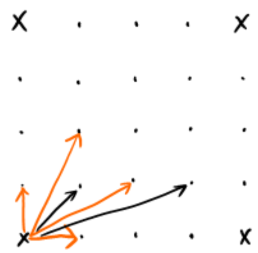
$\#(\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N \prod_{\substack{p \text{ prime} \\ p|N}} \left(1 - \frac{1}{p}\right)$



$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_{k=1}^{n} \mathbb{P}'(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$

E.g. $\mathbb{Z}/4\mathbb{Z}$





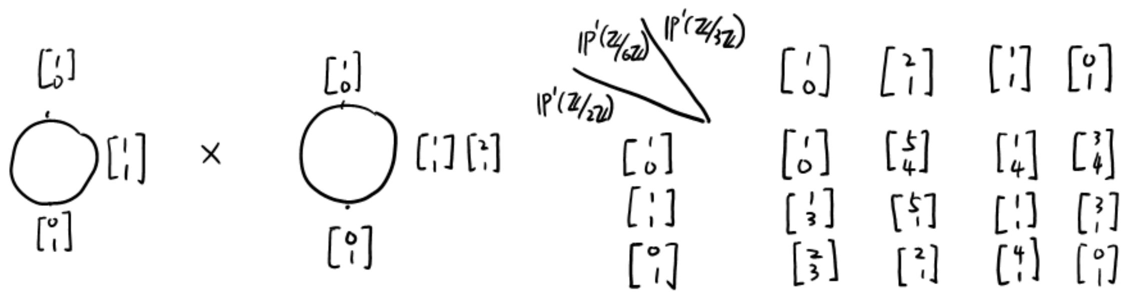
E.g. $\mathbb{Z}/6\mathbb{Z}$

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f \in S_+ \\ f \text{ homogeneous}}} \operatorname{Spec}\left(\mathbb{Z}/6\mathbb{Z}[x,y]_f\right)_0$$

e.g. $(x-2, y-3) \triangleleft \mathbb{Z}/6\mathbb{Z}[x,y]$ is not prime.

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3)$$
$$\cong \mathbb{P}^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3)$$

Ex. Use [Vakil, 6.3.M] to compute $\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$. Enjoy it!



Rmk. The original proof is also good, but less geometrically obvious:
(Now you should understand the geometry in every step)

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & SL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[d] & SL_2(\mathbb{F}_p) \arrow[d] & \\
0 \arrow[r] & 1+pM_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[r] & GL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[r] \arrow[d] & GL_2(\mathbb{F}_p) \arrow[r] \arrow[d] & 0 \\
 & & (\mathbb{Z}/p^e\mathbb{Z})^\times \arrow[d] & \mathbb{F}_p^\times \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

p^{4e-4} $\quad (p^2-1)(p^2-p)$ $\quad p^e - p^{e-1}$ $\quad p-1$

Finally, use Chinese remainder theorem to get
$$SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^{n} SL_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z}) \qquad \square$$

Ex. do the exactly same thing with SL_2 replaced by GL_2 and PGL_2.

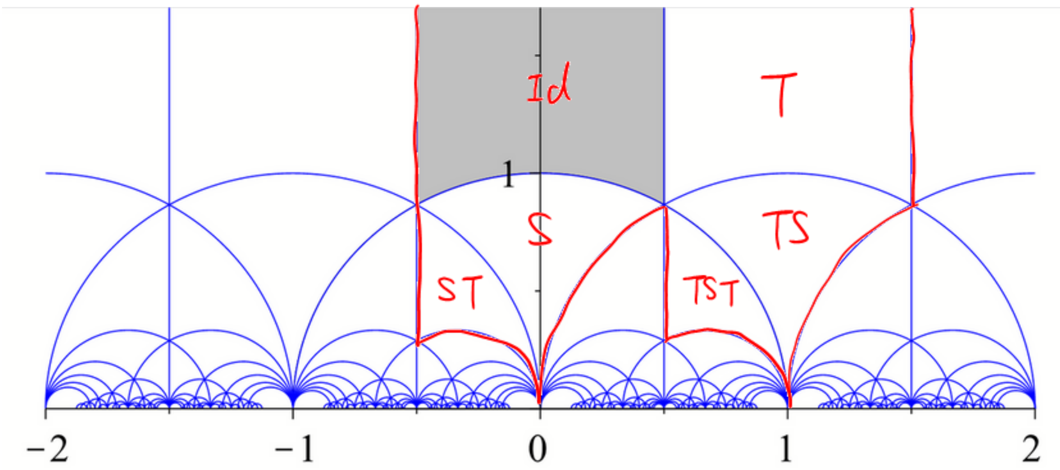
Ex. (hard) explore the Tits building & rep theory of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

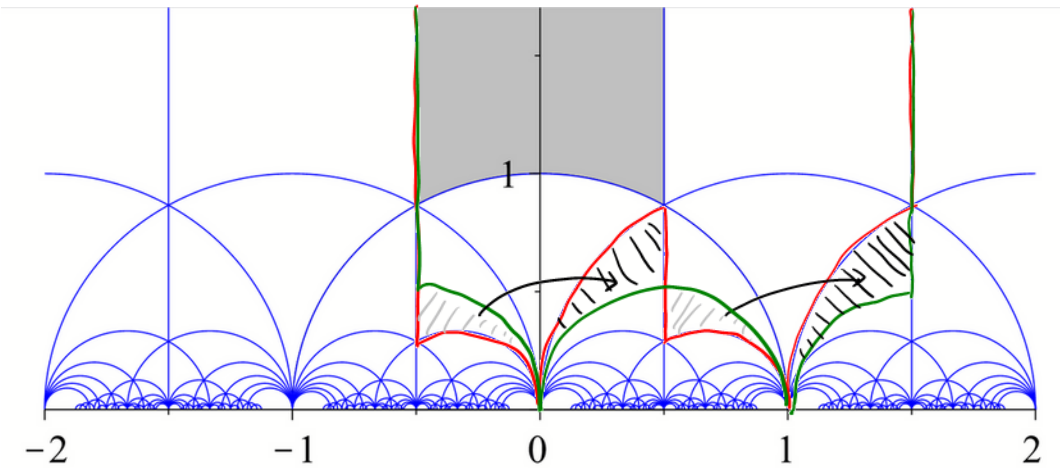
It will be used later on (I believe)

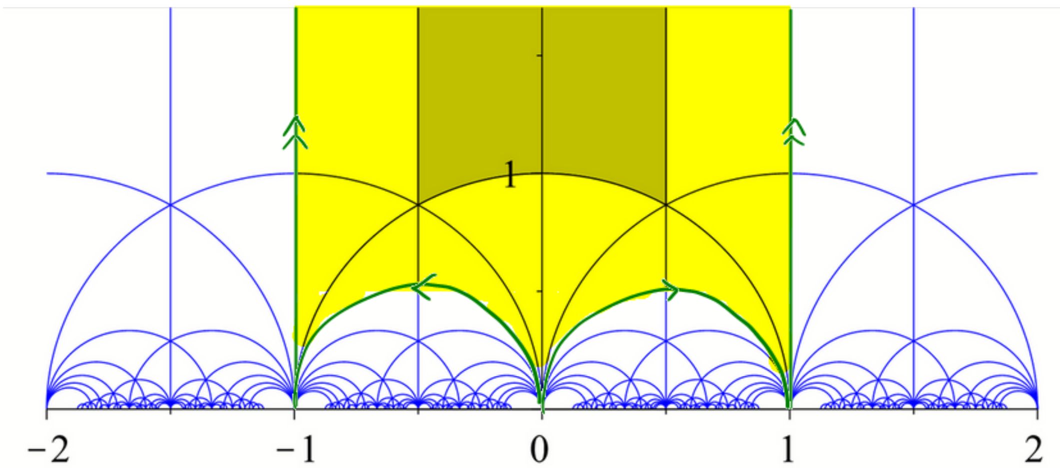
Is the Tits building of $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ functorial?

Ex. Draw the fundamental domain of $\mathcal{H}/\Gamma(2)$.

Hint. $\Gamma(1)/\Gamma(2) = \{Id, T, S, TS, ST, TST\}$







Cor. $\Gamma(2)/\{\pm Id\} = \mathbb{Z} * \mathbb{Z} = \langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \rangle$

2. moduli interpretation of $\mathcal{H}/\Gamma$

Def. A basis (v_1,v_2) of a lattice $\Lambda\subseteq\mathbb{C}$ is called oriented if $\mathrm{Im}\frac{v_1}{v_2}>0$.

Def (Weil pairing) [WWL, 注记8.5.9, 定义3.8.9, 练习3.8.10]

For $N\in\mathbb{Z}_{\geq 1}$, $E=\mathbb{C}/\Lambda$, $\Lambda=\mathbb{Z}u\oplus\mathbb{Z}v$, $\mathrm{Im}\frac{u}{v}>0$, we define the Weil pairing e_N:

$$\begin{tikzcd}
E[N]\times E[N] \arrow[d, "\wr\!\shortparallel"] \arrow[ddr, "e_N"] & \\
\frac{1}{n}\Lambda/\Lambda\times\frac{1}{n}\Lambda/\Lambda \arrow[d, "\wr\!\shortparallel"] & \\
(\mathbb{Z}/N\mathbb{Z})^{\oplus 2}\times(\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \arrow[r] & \mu_N \cong (\mathbb{Z}/N\mathbb{Z},+) \\
\left(\begin{pmatrix}a\\c\end{pmatrix},\begin{pmatrix}b\\d\end{pmatrix}\right) \arrow[r, mapsto] & \zeta_N^{\begin{vmatrix}a&b\\c&d\end{vmatrix}} \longmapsto \begin{vmatrix}a&b\\c&d\end{vmatrix}
\end{tikzcd}$$

$$a\frac{u}{N}+c\frac{v}{N} \longmapsto \begin{pmatrix}a\\c\end{pmatrix}$$

Ex. Let $e_1,e_2\in E[n]$.

1. e_N is antisymmetric and bilinear.

   $e_N(\gamma(e_1,e_2)) = \zeta_N^{\det\gamma}\, e_N(e_1,e_2)$ $\forall\gamma\in GL_2(\mathbb{Z}/N\mathbb{Z})$

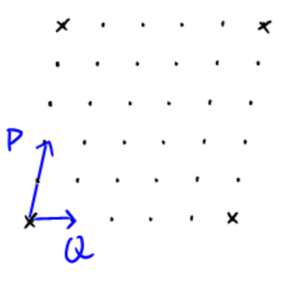
   e.p. e_N only depends on E and N (does not depend on Λ and u,v)

2. $e_N(e_1,e_2)\in\mu_N^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \iff E[N]=\langle e_1,e_2\rangle_{\mathbb{Z}}$

   $e_N(e_1,e_2)=\zeta_N \longmapsto 1 \iff \exists P,Q\in\frac{1}{N}\Lambda$, $\bar{P}=e_1$, $\bar{Q}=e_2$, (NP,NQ) is an oriented basis of Λ.

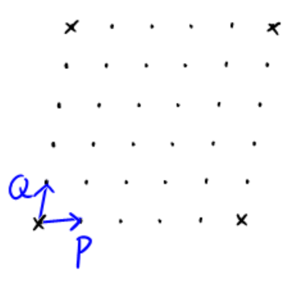
Def. (e_1,e_2) is called a pretty oriented basis of $E[N]$, if $e_N(e_1,e_2)=\zeta_N$.

Ex. $N=5$



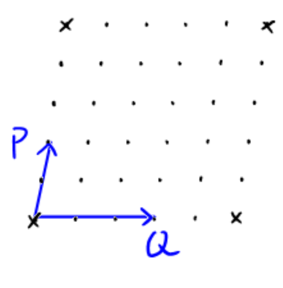
$e_N(\bar{P},\bar{Q})=\zeta_5^2$

$(5P,5Q)$ is not a basis of Λ.



$e_N(\bar{P},\bar{Q})=\zeta_5^4=\zeta_5^{-1}$

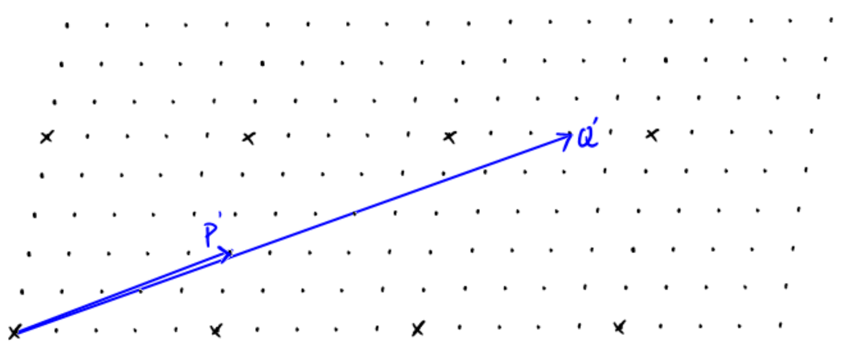
$(5P,5Q)$ is a basis of Λ, but not an oriented basis.



$e_N(\bar{P},\bar{Q})=\zeta_5^6=\zeta_5$

$(5P,5Q)$ is not a basis of Λ but $(5P',5Q')$ is an oriented basis.

$$\underset{SL_2(\mathbb{Z})}{\underset{\cap}{\begin{pmatrix}2&5\\5&13\end{pmatrix}}}\overset{\bmod 5}{\equiv}\begin{pmatrix}2&0\\0&3\end{pmatrix}$$



Recall: For $E = \mathbb{C}/\Lambda$, $E[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$

Main Thm. We have the following moduli interpolations (E: any cplx EC curve)