Eine Woche, ein Beispiel

1.13 structure thms of f.p. modules

Ref:

wiki:
Structure theorem for finitely generated modules over a principal ideal domain (check the proof!)
Coherent ring
Dedekind domain (see section "Finitely generated modules over a Dedekind domain")

stackexchange & mathoverflow:

(classifies finitely generated modules over a Dedekind domain)
https://math.stackexchange.com/questions/3684112/structure-theorem-for-modules-over-dedekind-domains
(finitely generated modules over general rings, recent result)
https://mathoverflow.net/questions/67772/structure-theorem-for-modules
https://mathoverflow.net/questions/473249/steinitz-isomorphism-theorem-for-non-dedekind-domains

I just realized that the proof (of f.g. modules over PID) uses the Gaussian elimination. The method can be used to "classify" all f.p. modules, or f.g. modules over coherent ring. The only problem is that, we can no longer get nice canonical form like the PID case.

$$\text{PID} \xRightarrow{\mathbb{Z}[\sqrt{-5}]} \text{left Noetherian} \xRightarrow{\mathbb{C}[z_1, \dots, z_n]} \text{left coherent}$$

1. PID case
2. Dedekind ring case

1. PID case

Proofs [edit]

One proof proceeds as follows:

- Every finitely generated module over a PID is also finitely presented because a PID is Noetherian, an even stronger condition than coherence.
- Take a presentation, which is a map $R^r \to R^g$ (relations to generators), and put it in Smith normal form.

Gaussian elimination

This yields the invariant factor decomposition, and the diagonal entries of Smith normal form are the invariant factors.

E.g. $R = \mathbb{Z}$ $\rightsquigarrow$ f.g. abelian gp

E.g. $R = k[T]$ $\rightsquigarrow$ $(V,\ A : V \longrightarrow V)$ $\dim_k V < +\infty$

i.e. get the torsion part

$$\eta:\ k[T] \otimes_k V \xrightarrow{T\cdot Id - A} k[T] \otimes_k V \longrightarrow V \longrightarrow 0$$

$$f(T) \otimes v \longmapsto f(A).v$$

$$f(T) \otimes v \longmapsto Tf(T) \otimes v - f(T) \otimes A.v$$

It can be directly checked that η is exact.
After the Gaussian elimination, one can write down $\ker(T \cdot Id - A)$ as free $k[T]$-modules, where $k[T] \otimes_k V / \ker(T \cdot Id - A)$ is also free.

Here, the char poly $T \cdot Id - A$ works as the matrix, determing uniquely the $k[T]$-module V.

2. Dedekind ring case

Rmk: Steinitz's theorem tells us the structure theorems of f.g. modules over a Dedekind ring.
https://math.stackexchange.com/questions/3684112/structure-theorem-for-modules-over-dedekind-domains

Let R be a Dedekind domain. For every element $\alpha \in C(R)$, let a representative I_α in the group of fractional ideals be chosen. Then, to a fin. gen. R-Mod M there are unique natural numbers r and s, $\alpha \in C(R)$ with $\alpha = 0$ if $s = 0$, and proper nonzero ideals $I_r \subset \ldots \subset I_1$ such that

$M \cong R/I_1 \oplus \ldots \oplus R/I_r$, if $s = 0$

$M \cong R/I_1 \oplus \ldots \oplus R/I_r \oplus R^{s-1} \oplus I_\alpha$, if $s > 0$

Q: Can we proof Steinitz's thm through the Gauss elimination?

E.g. $R = \mathbb{Z}[\sqrt{-5}]$

$$\begin{pmatrix} \sqrt{5}-1 & 2 & 2 & 2 \\ 2 & \sqrt{5}-1 & 2 & 2 \\ 2 & 2 & \sqrt{5}-1 & 2 \\ 2 & 2 & 2 & \sqrt{5}-1 \end{pmatrix} \sim \begin{pmatrix} \sqrt{5}-1 & 2 & 0 & 0 \\ 2 & \sqrt{5}-1 & 3-\sqrt{5} & 3-\sqrt{5} \\ 2 & 2 & \sqrt{5}-3 & 0 \\ 2 & 2 & 0 & \sqrt{5}-3 \end{pmatrix} \sim \begin{pmatrix} \sqrt{5}-1 & 2 & 0 & 0 \\ 6 & \sqrt{5}+5 & 0 & 0 \\ 2 & 2 & \sqrt{5}-3 & 0 \\ 2 & 2 & 0 & \sqrt{5}-3 \end{pmatrix}$$

Ziyang Zhu provides me with some references where I can find the proof:

[p228, (4.3) Proposition]
J. Neukirch. Algebraic Number Theory, Springer-Verlag, 2010.

[p210, 81.2]
O. T. O'Meara. Introduction to Quadratic Forms, Springer-Verlag, 1973.

I will check if we can really find a Gaussian elimination argument for this.