Eine Woche, ein Beispiel

6.15 Betti diagram

Main Ref:

Eisenbud, David. The Geometry of Syzygies. Graduate Texts in Mathematics. Springer New York,

This book suits me best. It begins with the Betti diagram of points on P^n, while other references may begin with examples of curves.

Koszul cohomology is a nice replacement for the equations, because it is uniquely determined, while it also describe some properties of exterior geometry. You can consider the stratification on the moduli space given by the Koszul cohomology invariant.

$$F_\bullet: \quad 0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$$

$$F_i = \bigoplus_j S(-j)^{\beta_{i,j}}$$

$j-i$ \ i	0	1	2	3	$\cdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\beta_{3,3}$	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\beta_{3,4}$	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\beta_{3,5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

1. Examples

Complete intersections

E.g. $\mathbb{P}^{n-k} \subset \mathbb{P}^n$ [p4]

hyperplane $\mathbf{K}(x_0) : 0 \longrightarrow S(-1) \xrightarrow{(x_0)} S$ 1 1

codim 2 plane $\mathbf{K}(x_0,x_1) : 0 \longrightarrow S(-2) \xrightarrow{\begin{pmatrix} x_1 \\ -x_0 \end{pmatrix}} S^2(-1) \xrightarrow{(x_0\ x_1)} S$ 1 2 1

codim 3 plane $\mathbf{K}(x_0,x_1,x_2) : 0 \longrightarrow S(-3) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}} S^3(-1) \xrightarrow{(x_0\ x_1\ x_2)} S$ 1 3 3 1

E.g.

deg d hypersurface

$$K(f) : 0 \longrightarrow S(-d) \xrightarrow{(f)} S$$

	0	1
0	1	–
⋮	⋮	⋮
$d-1$	–	1

cpl intersection of two hyperserface [Ex 1C.1]

$$0 \longrightarrow S(-d_1-d_2) \xrightarrow{\binom{g}{-f}} S(-d_1) \oplus S(-d_2) \xrightarrow{(f\ g)} S$$

	0	1	2
0	1	–	–
⋮	⋮	⋮	⋮
d_1-1	–	1	–
⋮	⋮	⋮	⋮
d_2-1	–	1	–
⋮	⋮	⋮	⋮
d_1+d_2-1	–	–	1

Finite points on $\mathbb{P}^n$

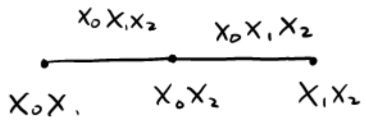
E.g. 3 pts on $\mathbb{P}^2$ [Eg. 2.1]

$$0 \longrightarrow \underset{\substack{\parallel \\ \langle x_0x_1x_2,\, x_0x_1x_2\rangle}}{S^2(-3)} \xrightarrow{\begin{pmatrix} -x_2 & 0 \\ x_1 & -x_1 \\ 0 & x_0 \end{pmatrix}} \underset{\substack{\parallel \\ \langle x_0x_1,\, x_0x_2,\, x_1x_2\rangle}}{S^3(-2)} \xrightarrow{(x_0x_1 \quad x_0x_2 \quad x_1x_2)} S$$

1	–	–
–	3	2

Other expressions:

	3	3
2	1	1
2	1	1
2	1	1



E.g. 8 general pts in $\mathbb{P}^2$ [p37]

$$0 \longrightarrow S^2(-5) \longrightarrow S^2(-4) \oplus S(-3) \longrightarrow S$$

1	–	–
–	–	–
–	2	–
–	1	2

	5	5
3	2	2
3	2	2
4	1	1

E.g. $\exists$ X: 10 pts in $\mathbb{P}^2$ s.t. S_X has free resolution [p7]

$$0 \longrightarrow S(-6) \oplus S(-5) \longrightarrow S(-4) \oplus S(-4) \oplus S(-3) \longrightarrow S$$

1	–	–
–	–	–
–	1	–
–	2	1
–	–	1

	6	5
4	2	1
4	2	1
3	3	2

E.g. $X = \operatorname{Spec} k[x,y]/(x^2, y^3)$ in $\mathbb{P}^2$ [Ex 2D.15]

$$\begin{array}{cccc} 1 & - & - & - \\ - & - & - & - \\ - & 4 & 3 & 1 \\ - & - & 1 & - \end{array}$$

E.g. X: 7 pts in $\mathbb{P}^3$ (in linear general position) [Thm 2.8]

$$\begin{array}{cccc} 1 & - & - & - \\ - & 3 & - & - \\ - & 1 & 6 & 3 \end{array} \qquad\qquad \begin{array}{cccc} 1 & - & - & - \\ - & 3 & 2 & - \\ - & 3 & 6 & 3 \end{array}$$

don't lie in curve of deg 3   lie in curve of deg 3

Rational normal curve & Elliptic normal curve

E.g. $R \subset \mathbb{P}^3$ RNC [Ex 2D.8]

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \\ x_2 & x_3 \end{pmatrix}} S^3(-2) \xrightarrow{(x_1x_3 - x_2^2,\ -x_0x_3 + x_1x_2,\ x_0x_2 - x_1^2)} S$$

1	–	–
–	3	2

In general, the Betti table of a RNC $R \subset \mathbb{P}^r$ is given by [Cor 2.6]

	0	1	2	$\cdots$	$r-1$
0	1	–	–	$\cdots$	–
1	–	$\binom{r}{2}$	$2\binom{r}{3}$	$\cdots$	$(r-1)\binom{r}{r} = r-1$

E.g. $E \subset \mathbb{P}^r$ ENC $\quad \deg E = r+1$ [6D, Thm 6.26]

$r=2$, $d=3$:

1	–
–	–
–	1

$r=3$, $d=4$:

1	–	–
–	2	–
–	–	1

$r \geq 4$, $d \geq 5$:

1	–	–	$\cdots$	–	–
–	b_1	b_2	$\cdots$	b_{r-2}	–
–	–	–	$\cdots$	–	1

$$b_i = i\binom{r-1}{i+1} + (r-i-1)\binom{r-1}{i-1}$$

Canonical curve in $\mathbb{P}^{g-1}$ non-hyperelliptic curve [9B, p 180-185]

1	–
–	–
–	–
–	1

$g=3$
deg 4 plane curve

1	–	–
–	1	–
–	1	–
–	–	1

$g=4$
cpl intersection of deg 2 & deg 3 hypersurfaces

1	–	–	–
–	3	–	–
–	–	3	–
–	–	–	1

$g=5$
$\mathrm{Cliff}(C)=2$
$\mathrm{gon}(C)=4$

1	–	–	–
–	3	2	–
–	2	3	–
–	–	–	1

$g=5$
$\mathrm{Cliff}(C)=1$
$\mathrm{gon}(C)=3$

1	–	–	–	–
–	6	5	–	–
–	–	5	6	–
–	–	–	–	1

$g=6$
general curve

https://www.mathematik.hu-berlin.de/~farkas/progr_syz.pdf
https://www.math.uni-sb.de/ag/schreyer/images/PDFs/talks/schreyerICM2010.pdf (the screenshot comes from here)

Example: Canonical curves of genus 7 [S 1986]

The Betti table of a smooth canonically embedded curve $C \subset \mathbb{P}^6$ of genus $g = 7$ is one of the following:

1	–	–	–	–	–
–	10	20	15	4	–
–	4	15	20	10	–
–	–	–	–	–	1

trigonal

1	–	–	–	–	–
–	10	16	9	–	–
–	–	9	16	10	–
–	–	–	–	–	1

$\exists\, g^2_6$

1	–	–	–	–	–
–	10	16	3	–	–
–	–	3	16	10	–
–	–	–	–	–	1

4-gonal

1	–	–	–	–	–
–	10	16	–	–	–
–	–	–	16	10	–
–	–	–	–	–	1

general case, $char(\mathbb{K}) \neq 2$

Canonical curve in $\mathbb{P}^{g-1}$ non-hyperelliptic curve   [9B, p 180-185]

1	–
–	–
–	–
–	1

$g=3$
deg 4 plane curve

1	–	–
–	1	–
–	1	–
–	–	1

$g=4$
cpl intersection of deg 2 & deg 3 hypersurfaces

1	–	–	–
–	3	–	–
–	–	3	–
–	–	–	1

$g=5$
$\mathrm{Cliff}(C)=2$
$\mathrm{gon}(C)=4$

1	–	–	–
–	3	2	–
–	2	3	–
–	–	–	1

$g=5$
$\mathrm{Cliff}(C)=1$
$\mathrm{gon}(C)=3$

1	–	–	–	–
–	6		–	–
–	–		6	–
–	–	–	–	1

$g=6$
general curve

Setting $\beta_i = \beta_{i,i+1}$ the Betti diagram of S_X has the form

	0	1	$\cdots$	a	$a+1$	$\cdots$	$b-1$	b	$\cdots$	$g-3$	$g-2$
0	1	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–	–
1	–	β_1	$\cdots$	β_a	β_{a+1}	$\cdots$	β_{g-3-a}	–	$\cdots$	–	–
2	–	–	$\cdots$	–	β_{g-3-a}	$\cdots$	β_{a+1}	β_a	$\cdots$	β_1	–
3	–	–	$\cdots$	–	–	$\cdots$	–	–	$\cdots$	–	1

$a+b=g-2$

where the terms marked "–" are zero, the β_i are nonzero, and $\beta_1 = \binom{g-2}{2}$.

$a \leq \mathrm{Cliff}\, C - 1$

Green conj: $a = \mathrm{Cliff}\, C - 1$