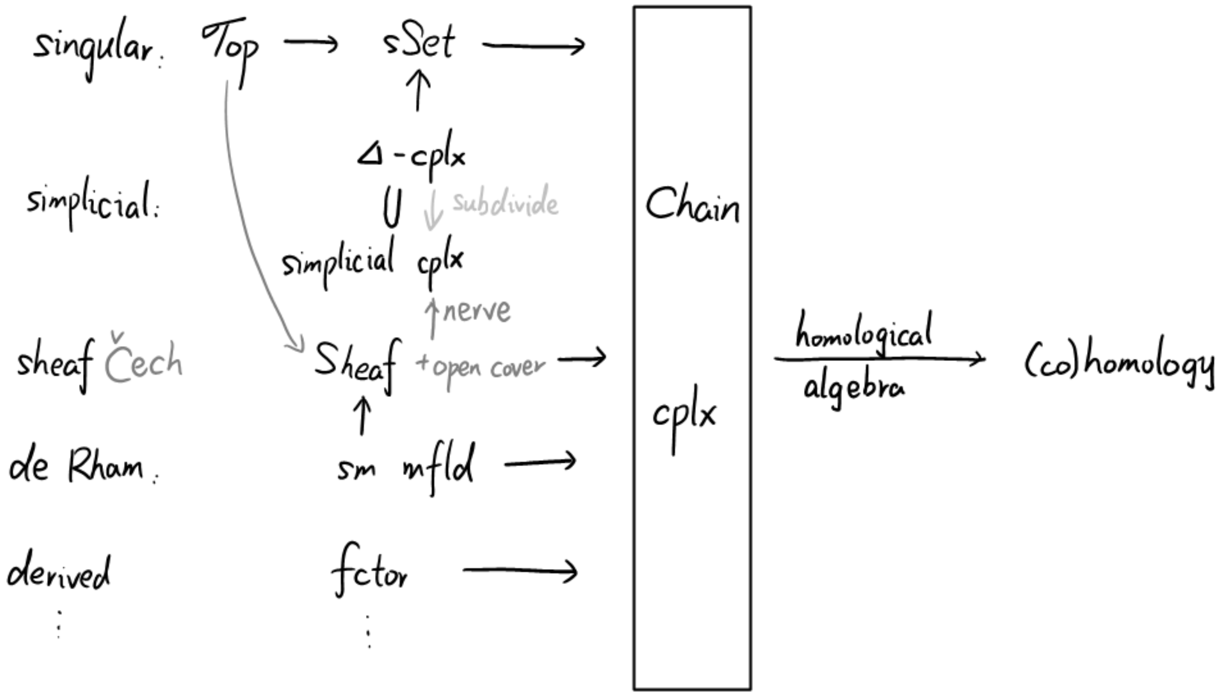
Eine Woche, ein Beispiel
6.25 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



$$\begin{tikzcd} \text{Today: } \mathrm{sSet} \arrow[r] & \text{chain cplx} \arrow[r, dashed] & \text{(co)homology} \end{tikzcd}$$

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

We use α here because
we are considering $X = \Delta^n$ case.
May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha \in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha \in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{dual} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$Hom_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n} \mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n} Hom_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

Rmk. Prof. Scholze told me that we cannot define
Borel-Moore homology or cpt supported cohomology, not to say six fctors for sset.
If there were any sheaf on sset, it should behave like perverse sheaf.

E.g. 1 For $A \in \mathcal{T}op$ discrete, $X_\cdot = \mathcal{S}(A) \in sSet$, one can compute

$$\begin{array}{ll}
C_\cdot(X;G): & 0 \longleftarrow \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \cdots \\
C^\cdot(X;G): & 0 \longrightarrow \prod_{\alpha\in A} G \xrightarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \cdots
\end{array}$$

$$\text{wished}\left\{\begin{array}{ll}
C^{BM}_\cdot(X;G): & 0 \longleftarrow \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \cdots \\
C^\cdot_c(X;G): & 0 \longrightarrow \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \cdots
\end{array}\right.$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.
Notice that $\#\Delta'_k = k+2$, so

$$\begin{tikzcd} C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0&1&0 \\ 0&-1&0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1&1&0&0 \\ 0&0&0&0 \\ 0&0&1&1 \end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0&1&0&0&0 \\ 0&-1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&0&-1&0 \end{pmatrix}"'] & \cdots \end{tikzcd}$$

basis:

remember indexes:

$0 = x_0 - x_0 \longmapsfrom x_0$

$x_0 - x_1 = x_0 - x_1 \longmapsfrom x_1$

$0 = x_1 - x_1 \longmapsfrom x_2$

$0 = x_0 - x_0 + x_0 - x_0 \longmapsfrom x_0$

$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longmapsfrom x_1$

$0 = x_1 - x_1 + x_2 - x_2 \longmapsfrom x_2$

$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longmapsfrom x_3$

$0 = x_3 - x_3 + x_3 - x_3 \longmapsfrom x_4$

$x_0 = x_0 - x_0 + x_0 \longmapsfrom x_0$

$x_0 = x_0 - x_1 + x_1 \longmapsfrom x_1$

$x_2 = x_1 - x_1 + x_2 \longmapsfrom x_2$

$x_2 = x_2 - x_2 + x_2 \longmapsfrom x_3$

By taking the transpose, one get

$$\begin{tikzcd} C^\bullet(\Delta';G): & 0 \arrow[r] & G^{\oplus 2} \arrow[r, "\begin{pmatrix} 0&0 \\ 1&-1 \\ 0&0 \end{pmatrix}"] & G^{\oplus 3} \arrow[r, "\begin{pmatrix} 1&0&0 \\ 1&0&0 \\ 0&0&1 \\ 0&0&1 \end{pmatrix}"] & G^{\oplus 4} \arrow[r, "\begin{pmatrix} 0&0&0&0 \\ 1&-1&0&0 \\ 0&0&0&0 \\ 0&0&1&-1 \\ 0&0&0&0 \end{pmatrix}"] & G^{\oplus 5} \cdots \end{tikzcd}$$

Therefore,

$$H_n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta':G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\cdot(\Delta^1;G)$ and $C_\cdot(\Delta^0;G)$.

$$\begin{tikzcd}
\Delta^1 \arrow[d, "s_0^1"] & C_\cdot(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "(1\,1)"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1)"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1\,1\,1)"] \\
\Delta^0 & C_\cdot(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "Id"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Delta^0 \arrow[d, "d_0^1"] & C_\cdot(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "Id"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] \\
\Delta^1 & C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}$$

s.t. $s^1_{0,*} \circ d^1_{0,*} = Id_{C_\cdot(\Delta^1;G)}$, $\quad d^1_{0,*} \circ s^1_{0,*} \sim Id_{C_\cdot(\Delta^1;G)}$.

In fact, we have

$$\begin{tikzcd}
C_\cdot(\Delta^1;G): \arrow[d, shift right, "Id"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}0&0&0\\0&0&1\\0&0&-1\\0&0&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dr, dashed, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \\
C_\cdot(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}$$

$x_0 \mapsto x_0$

$x_1 \mapsto x_1$

$x_0 \mapsto x_0 - x_0 + x_0 = x_0$

$x_1 \mapsto x_1 - x_1 + x_1 = x_1$

$x_2 \mapsto x_1 - x_2 + x_2 = x_1$

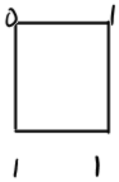
$x_3 \mapsto x_1 - x_2 + x_3$

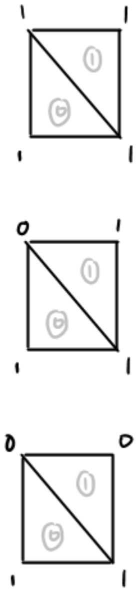
$x_0 \mapsto x_0 - x_0 = 0$

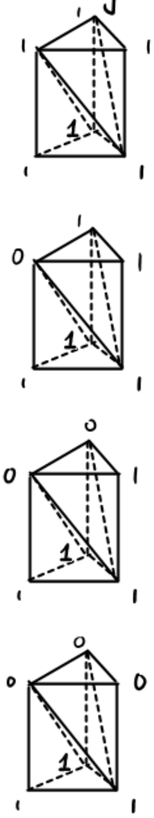
$x_1 \mapsto x_1 - x_1 = 0$

$x_2 \mapsto x_1 - x_2$

Ex. Observe the picture, try to translate the calculation in geometrical language.

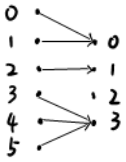
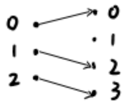
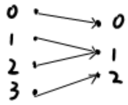






E.g. 3. When we want to compute $H_n(\Delta^m;G)$ and $H^n(\Delta^m;G)$, we'd better to give elements in $\Delta_n^m \approx \{\text{basis of } C_n(\Delta^m;G)\}$ a better notation. The following table shows some typical element in

$$C_n(\Delta^m;G) = \langle \alpha : [n] \longrightarrow [m] \rangle_{\alpha \in \Delta_n^m} .$$

element	picture	list	count	degenerate degree
$\alpha : [5] \longrightarrow [3]$ $0 \longmapsto 0$ $1 \longmapsto 0$ $2 \longmapsto 1$ $3 \longmapsto 3$ $4 \longmapsto 3$ $5 \longmapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	$\Delta_5^{3,\lightning 3}$
$d_1^3 : [2] \longrightarrow [3]$ $0 \longmapsto 0$ $1 \longmapsto 2$ $2 \longmapsto 3$		$(0,2,3)$	$[1,0,1,1]$	$\Delta_2^{3,\theta}$
$s_1^3 : [3] \longrightarrow [2]$ $0 \longmapsto 0$ $1 \longmapsto 1$ $2 \longmapsto 1$ $3 \longmapsto 2$		$(0,1,1,2)$	$[1,2,1]$	$\Delta_3^{2,\lightning 1}$
$\partial\alpha$	——	$(0,0,3,3,3)$ $-(0,0,1,3,3)$	$[2,0,0,3]$ $-[2,1,0,2]$	$\Delta_4^{3,\lightning 3}$ $\Delta_4^{3,\lightning 2}$

e.g. $\partial[2,5,3,4,1,6,0]$

$= [2,4,3,4,1,6,0] - [2,5,2,4,1,6,0] + [2,5,3,4,0,6,0]$

2. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\overset{\text{non-degenerate}}{\overset{\downarrow}{\diamond}}}$:

$$\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5}\cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "0"] \\
C_\cdot(\Delta';G)^{\diamond}: & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0\cdots \arrow[l, "0"']
\end{tikzcd}$$

$$\begin{tikzcd}
C_\cdot(\Delta';G)^{\diamond}: \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}0\\1\\0\end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0\cdots \arrow[l, "0"'] \arrow[d, "0"] \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5}\cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}$$

In fact, we have

$$\begin{tikzcd}
C_\cdot(\Delta';G): \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5}\cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \\
C_\cdot(\Delta';G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5}\cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)
For $X \in sSet$, define

$\Diamond$: non-degenerate
$\lightning$: degenerate

$$X_k^{\Diamond} := \{x \in X_k \mid x \text{ non-degerate}\} = X_k \smallsetminus (sk^{k-1}X)_k$$
$$X_k^{\lightning} := \{x \in X_k \mid x \text{ degenerate}\} = (sk^{k-1}X)_k$$
$$X_k^{\lightning i} := \left\{x \in X_k \;\middle|\; x = \alpha^*(y) \text{ for some } \begin{matrix} y \in X_{k-i}^{\Diamond} \\ \alpha : [k-i] \to [k] \end{matrix}\right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$0 = \underbrace{(sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}}_{X_k^{\lightning}} \underbrace{X)_k \overset{X_k^{\lightning 0}}{\subset} (sk^k}_{X_k^{\Diamond}} X) = X_k$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\Diamond} = \bigoplus_{\alpha \in X_n^{\Diamond}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\Diamond}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\Diamond}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\Diamond}} G \cdots$$

$$C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots$$

and $H_\bullet(X;G)^{\Diamond}$, $H_\bullet(X;G)^{\lightning}$ as crspd homology.

By definition, $$C_\bullet(X;G) \cong C_\bullet(X;G)^{\Diamond} \oplus C_\bullet(X;G)^{\lightning}$$

Claim 1. $H_\bullet(X;G)^{\lightning} = 0$, so
$$H_\bullet(X;G) \cong H_\bullet(X;G)^{\Diamond}. \qquad (*)$$

Rmk, Roughly, $(*)$ says that
singular homology $\approx$ simplicial homology.
Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expend $C_\bullet(X;G)$ by double complex.

Def (Double complex of $C_\cdot(X;G)$) $\swarrow\leftarrow + \leftarrow\swarrow = 0$

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0^{\delta 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{\delta 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{\delta 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{\delta 3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\delta 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_1^{\delta 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_2^{\delta 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_3^{\delta 2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\delta 1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_1^{\delta 1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_2^{\delta 1}} G \arrow[l] \arrow[dl, "0"'] & \bigoplus_{\alpha\in X_3^{\delta 1}} G \arrow[l] \arrow[dl, "0"'] \\
0 & \bigoplus_{\alpha\in X_0^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_1^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_2^{\emptyset}} G \arrow[l] & \bigoplus_{\alpha\in X_3^{\emptyset}} G \arrow[l]
\end{tikzcd}$$

$0 \Leftarrow$

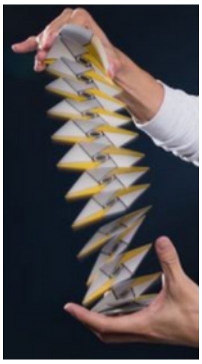
dim 0

dim 1

dim 2

fold $\Downarrow\Uparrow$ expand

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0} G \arrow[l] & \bigoplus_{\alpha\in X_1} G \arrow[l] & \bigoplus_{\alpha\in X_2} G \arrow[l] & \bigoplus_{\alpha\in X_3} G \arrow[l]
\end{tikzcd}$$



fold / expend

Eg. For $X = \Delta^1$, we have double complex

$0 \Leftarrow$

$$\begin{tikzcd}
 & & & & & \cdots \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_4\rangle \arrow[l] \arrow[dl, "0"'] & \cdots \arrow[l, "\begin{pmatrix}1&0&0&0\\0&0&0&1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}0&0&0&0\\0&1&1&0\\0&0&0&0\end{pmatrix}"'] \\
0 & 0 \arrow[l] \arrow[dl] & 0 \arrow[l] \arrow[dl] & \langle x_0, x_3\rangle \arrow[l] \arrow[dl, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] & \langle x_1, x_2, x_3\rangle \arrow[l, "\begin{pmatrix}1&0&0\\0&0&-1\end{pmatrix}"'] \arrow[dl, "\begin{pmatrix}-1&0&0\\0&0&1\end{pmatrix}"'] & \\
0 & 0 \arrow[l] \arrow[dl] & \langle x_0, x_2\rangle \arrow[l] \arrow[dl, "0"'] & \langle x_1, x_2\rangle \arrow[l, "\begin{pmatrix}1&0\\0&1\end{pmatrix}"'] \arrow[dl, "0"] & 0 \arrow[l] \arrow[dl] & \\
0 & \langle x_0, x_1\rangle \arrow[l] & \langle x_1\rangle \arrow[l, "\begin{pmatrix}1\\-1\end{pmatrix}"'] & 0 \arrow[l] & 0 \arrow[l] &
\end{tikzcd}$$

fold $\Downarrow\Uparrow$ expand

$$\begin{tikzcd}
0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] & \cdots
\end{tikzcd}$$

Claim 2. We have chain homotopy equivalence between the following two cplx.

$$\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_n^{\Diamond}} G \arrow[l] \arrow[d, equal] & \bigoplus_{\alpha\in X_{n+1}^{\leq 1}} G \arrow[l, "0"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+2}^{\leq 2}} G \arrow[l, "\partial^{\swarrow}"'] \arrow[d, shift right, "0"'] & \bigoplus_{\alpha\in X_{n+3}^{\leq 3}} G \arrow[l, "\partial^{\swarrow}"'] \arrow[d, shift right, "0"'] & (**) \\
0 & \bigoplus_{\alpha\in X_n^{\Diamond}} G \arrow[l] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] & 0 \arrow[l] \arrow[u, shift right, "0"'] &
\end{tikzcd}$$

i.e. (**) is exact on all terms except $\bigoplus_{\alpha\in X_n^{\Diamond}} G$.

Proof idea for $X=\Delta^m$. (can be generalized to arbitrary X).

$$\begin{tikzcd}
\cdots \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k-1}^{\leq k-1}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k}^{\leq k}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \bigoplus_{\alpha\in X_{n+k+1}^{\leq k+1}} G \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "0"] \arrow[dr, "s"] & \cdots \arrow[l] \\
\cdots & \bigoplus_{\alpha\in X_{n+k-1}^{\leq k-1}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k}^{\leq k}} G \arrow[l] & \bigoplus_{\alpha\in X_{n+k+1}^{\leq k+1}} G \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Define

$$s\,[\underbrace{a_1,\cdots a_l}_{\in\{0,1\}},\ \underset{\notin\{0,1\}}{a_{l+1}},\cdots,a_m] = \begin{cases} (-1)^{i}\,[a_1,\cdots,a_l,\ a_{l+1}+1,\cdots,a_m], & a_{k+1} \text{ even} \\ 0, & a_{k+1} \text{ odd} \end{cases}$$

$i=\sum_{j=1}^{l} a_j$

Ex. Check that s is a homotopy.

e.g. $X=\Delta^3$, $n=2$, $k=3$ $\Rightarrow m=3$, $n+k=5$

$$\begin{tikzcd}
-[2,1,0,2] \arrow[dr, mapsto] & {[2,1,0,3]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& -[3,1,0,2] & \\
& \begin{array}{c}[2,1,0,3]\\ +[3,1,0,2]\end{array} & {[3,1,0,3]} \arrow[l, mapsto]
\end{tikzcd}$$

$X=\Delta^6$, $n=5$, $k=15$ $\Rightarrow m=6$, $n+k=20$

$$\begin{tikzcd}
\begin{array}{c}[2,4,3,4,1,6,0]\\ -[2,5,2,4,1,6,0]\end{array} \arrow[dr, mapsto] & {[2,5,3,4,1,6,0]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& \begin{array}{c}[3,4,3,4,1,6,0]\\ -[3,5,2,4,1,6,0]\end{array} & \\
& \begin{array}{c}[2,5,3,4,1,6,0]\\ -[3,4,3,4,1,6,0]\\ +[3,5,2,4,1,6,0]\end{array} & {[3,5,3,4,1,6,0]} \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
{[4,4]} \arrow[dr, mapsto] & {[4,5]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& {[5,4]} & \\
& {\begin{matrix}[4,5]\\ -[5,4]\end{matrix}} & {[5,5]} \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
{\begin{matrix}[2,5]\\ -[3,4]\end{matrix}} \arrow[dr, mapsto] & {[3,5]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& {[3,5]} & \\
& 0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
{-[1,4,4]} \arrow[dr, mapsto] & {[1,4,5]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& {[1,5,4]} & \\
& {\begin{matrix}[1,4,5]\\ -[1,5,4]\end{matrix}} & {-[1,5,5]} \arrow[l, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
{\begin{matrix}-[1,2,5]\\ +[1,3,4]\end{matrix}} \arrow[dr, mapsto] & {[1,3,5]} \arrow[l, mapsto] \arrow[ddr, mapsto] & \\
& {[1,3,5]} & \\
& 0 & 0 \arrow[l, mapsto]
\end{tikzcd}$$