Eine Woche, ein Beispiel

4.24 irreducible representation of the Mirabolic group

Main reference: The Local Langlands Conjecture for GL(2) by Colin J. Bushnell and Guy Henniart.
[https://link.springer.com/book/10.1007/3-540-31511-X]

Process

1. Notations
2. Constructions
3. Classification
4. Applications
   - Computation of $V(N)$, V_N, $V(\vartheta)$, V_ϑ.
   - Dual, Sym^m, Λ^m, ...
   - Decompose $\mathrm{Res}^B_P\ \mathrm{Rep}^G_B\ \mathrm{Ind}^G_B\chi$ (not today, need knowledge of G & B)
   - Trace formula
5. Irr rep of B?

1. Notations

F: non-archi local field.

https://math.stackexchange.com/questions/299626/the-center-of-operatornamegln-k

$A = M_{2\times 2}(F)$ $\quad G = GL_2(F)$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $\quad T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ $\quad N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ $\quad Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = Z(G)$ $\quad S = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$

$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad T^0 = \begin{pmatrix} \mathcal{O}^\times & 0 \\ 0 & \mathcal{O}^\times \end{pmatrix}$ $\quad N_j = \begin{pmatrix} 1 & \mathfrak{p}^j \\ 0 & 1 \end{pmatrix}$ $\quad N_j' = \begin{pmatrix} 1 & 0 \\ \mathfrak{p}^j & 1 \end{pmatrix}$

Temporarily, $P := \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in GL_2(F) \right\} = F \rtimes F^\times := N \rtimes S$ $\qquad 0 \longrightarrow \underset{N}{(F,+)} \longrightarrow P \longrightarrow \underset{S}{F^\times} \longrightarrow 0$

parabolic subgp

to be short, $\mathrm{Ind} = \mathrm{Ind}_N^P$, $\quad c\text{-}\mathrm{Ind} = c\text{-}\mathrm{Ind}_N^P$.

2. Constructions

E.g. 1 (Irr rep from quotient gp)

When $(\rho, V) \in \widehat{P}^*$, ρ is the inflation of some $\chi \in \widehat{P/N}^* = \widehat{F^\times}^*$, i.e.,

$$\rho: P \longrightarrow \mathbb{C}^\times \qquad \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longmapsto \chi(a)$$

E.g. 2. (Irr rep from subgp)

For every $\vartheta \in \widehat{N}^* \setminus \{\mathbb{1}_N\}$, we claim that $\text{c-Ind}\, \vartheta \in \mathrm{Irr}(P)$.

Rmk. For $\vartheta, \vartheta' \in \widehat{N}^* \setminus \{\mathbb{1}_N\}$, we have an iso $(\exists s \in S,\ \vartheta' = \vartheta(s^{-1} \cdot s))$

$$\text{c-Ind}\, \vartheta \longrightarrow \text{c-Ind}\, \vartheta' \qquad f \longmapsto f(s^{-1} \cdot s) \qquad \text{in } \mathrm{Rep}(P).$$

So those irr reps in E.g. 2 are iso to each other.

The rest of this section is organized to prove E.g. 2. Step 1, 2 are also used in the next section.

Step 1. If $(\sigma, W) \in \mathrm{Rep}(N)$ is restricted too much, then $W=0$ (in Cor)

Prop. For $(\sigma, W) \in \mathrm{Rep}(N)$, $\bigcap_{\vartheta \in \widehat{N}} W(\vartheta) = \{0\}$.

Cor. (1) For $(\sigma, W) \in \mathrm{Rep}(N)$,

$$W_\vartheta = 0 \quad \forall \vartheta \in \widehat{N}^* \quad \Rightarrow W = 0$$

(2) When $(\sigma, W) \in \mathrm{Rep}(P)$, since $W_\vartheta \cong W_{\vartheta'}$ for $\vartheta, \vartheta' \in \widehat{N}^* - \{\mathbb{1}_N\}$, we can further reduce (1) to

$$W_N = 0 \quad W_\vartheta = 0 \quad \exists \vartheta \in \widehat{N}^* - \{\mathbb{1}_N\} \Rightarrow W = 0$$

Proof of Prop. $N = F$ here. Let $w \in W$, $w \neq 0$, we would find $\vartheta_0 \in \widehat{F}^*$ s.t. $w \notin W(\vartheta_0)$.

By the integral criterion,

$$w \notin W(\vartheta) \Leftrightarrow \text{For any } N_0 \in \mathrm{Cos}(F), \int_{N_0} \vartheta(n)^{-1} n.w \, d\mu_N(n) \neq 0$$

$$\Leftrightarrow \text{For any } j \in \mathbb{Z}, \int_{\mathfrak{p}^j} \vartheta(n)^{-1} n.w \, d\mu_N(n) \neq 0$$

(σ, W) sm $\Rightarrow \exists j_0 \in \mathbb{Z}$ s.t. $\mathfrak{p}^{j}.w = w$.

For $j \in \mathbb{Z}$, let $W_j := \langle w \rangle_{\mathrm{Rep}(\mathfrak{p}^j)}$. We will define $\vartheta_0 \in \widehat{F}^*$ inductively, i.e.

$$\boxed{\vartheta_0|_{\mathfrak{p}^{j_0}} \rightsquigarrow \vartheta_0|_{\mathfrak{p}^j} \rightsquigarrow \vartheta_0}$$

(1) $\vartheta_0|_{\mathfrak{p}^{j_0}} := \mathbb{1}_{\mathfrak{p}^{j_0}}$, then

$$\int_{\mathfrak{p}^j} \vartheta_0(n)^{-1} n.w \, d\mu_N(n) = \mu_N(\mathfrak{p}^j).v \neq 0 \quad \text{for } j \geq j_0$$

(2) Suppose $\vartheta_0|_{\mathfrak{p}^{j+1}} \triangleq \eta_{j+1}$ is defined s.t. $W_{j+1}^{\eta_{j+1}} = 0$.

We define $\vartheta_0|_{\mathfrak{p}^j} \triangleq \eta_j$ s.t.

1. $\eta_j|_{\mathfrak{p}^{j+1}} = \eta_{j+1}$
2. $W_j^{\eta_j} \neq 0$
3. $e_{\eta_j} * w = \int_{\mathfrak{p}^j} \eta_j(n)^{-1} n.w \, d\mu_N(n) \neq 0$.

$$0 \subsetneq \langle W_{j+1}^{\eta_{j+1}} \rangle_{\mathrm{Rep}(\mathfrak{p}^j)} \subset W_j$$

$$\Rightarrow \exists \eta_j \in \widehat{\mathfrak{p}^j}^* \text{ contained in } \langle W_{j+1}^{\eta_{j+1}} \rangle_{\mathrm{Rep}(\mathfrak{p}^j)}$$

$$\Rightarrow ①, ②$$

$$\Rightarrow e_{\eta_j} * - : \begin{array}{ccc} W_j & \twoheadrightarrow & W_j^{\eta_j} \\ x & \longmapsto & \int_{N_j} \eta_j(n)^{-1} n.x \, d\mu_N(n) \end{array} \quad \text{is not } 0$$

$$\xRightarrow{W_j = \langle w \rangle_{\mathrm{Rep}(\mathfrak{p}^j)}} e_{\eta_j} * w \neq 0$$

(3) Let $\vartheta_0(n) = \eta_j(n)$ $(n \in \mathfrak{p}^j, j \in \mathbb{Z})$. Then ϑ_0 is well-defined (by ①), and satisfy

$$\int_{\mathfrak{p}^j} \vartheta(n)^{-1} n.w \, d\mu_N(n) \neq 0 \quad \forall j \in \mathbb{Z}$$

□

[$N \triangleleft P$ closed but not open.
So we only have $\varepsilon_\vartheta : \operatorname{Ind}\vartheta \to \mathbb{C}$]

Step 2. We show that $c\text{-}\operatorname{Ind}\vartheta$ is heavily restricted.

Prop. \label{prop:jacqofind} Let $\vartheta \in \widehat{N}^* - \{\mathbb{1}_N\}$.

(1) $(c\text{-}\operatorname{Ind}\vartheta)(N) = (\operatorname{Ind}\vartheta)(N) = c\text{-}\operatorname{Ind}\vartheta$

$(c\text{-}\operatorname{Ind}\vartheta)_N = 0$ $\qquad (\operatorname{Ind}\vartheta)_N \cong \operatorname{Ind}\vartheta / c\text{-}\operatorname{Ind}\vartheta$

(2) $(c\text{-}\operatorname{Ind}\vartheta)(\vartheta) \cong \ker \varepsilon_\vartheta \cap c\text{-}\operatorname{Ind}\vartheta$ $\qquad (\operatorname{Ind}\vartheta)(\vartheta) \cong \ker \varepsilon_\vartheta$

$(c\text{-}\operatorname{Ind}\vartheta)_\vartheta \cong \mathbb{C}$ $\qquad (\operatorname{Ind}\vartheta)_\vartheta \cong \mathbb{C}$

Proof. (1). $(c\text{-}\operatorname{Ind}\vartheta)(N) \subset (\operatorname{Ind}\vartheta)(N) \subset c\text{-}\operatorname{Ind}\vartheta$: by direct computation.

$c\text{-}\operatorname{Ind}\vartheta \subset (c\text{-}\operatorname{Ind}\vartheta)(N)$: find generators of $c\text{-}\operatorname{Ind}\vartheta$, and verify it.

Generators: $\{ f_{a,j} \in C^\infty(P) \mid a \in F^\times, j \geq 1 \}$, where

$$f_{a,j}(g) = \begin{cases} \vartheta\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} & g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} au & 0 \\ 0 & 1 \end{pmatrix} \\ 0 & g \notin N \cdot \begin{pmatrix} aU_F^{(j)} & 0 \\ 0 & 1 \end{pmatrix} \end{cases}$$

$\exists x \in F,\ u \in U_F^{(j)}$

Informal: think it as $F \rtimes aU_F^{(j)}$

Verify $f_{a,j} \in (c\text{-}\operatorname{Ind}\vartheta)(N)$. Let $d = \operatorname{level}(\vartheta)$. $\quad \Rightarrow \vartheta|_{\mathfrak{p}^d} = \mathbb{1}_{\mathfrak{p}^d},\ \vartheta|_{\mathfrak{p}^{d-1}} \neq \mathbb{1}_{\mathfrak{p}^{d-1}}$

Let $y_0 \in \mathfrak{p}^{d-1}$ s.t. $\vartheta\begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix} \triangleq c \neq 1$, and $x_0 := \frac{y_0}{a}$. We get

$$\vartheta\begin{pmatrix} 1 & aU_F^{(j)}x_0 \\ 0 & 1 \end{pmatrix} = \vartheta\begin{pmatrix} 1 & y_0U_F^{(j)} \\ 0 & 1 \end{pmatrix} \equiv c \neq 1$$

$$\Rightarrow f_{a,j} = \frac{1}{1-c}\left(f_{a,j} - \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix} . f_{a,j}\right) \in (c\text{-}\operatorname{Ind}\vartheta)(N).$$

Other results are then obvious.

(2)

$$\begin{tikzcd}[column sep=small]
& 0 \arrow[r] & (c\text{-Ind}\,\vartheta)(\vartheta) \arrow[dl, "①"'] \arrow[dd] \arrow[rr] & & c\text{-Ind}\,\vartheta \arrow[dl, equal] \arrow[dd] \arrow[rr] & & (c\text{-Ind}\,\vartheta)_\vartheta \arrow[dl, "②"'] \arrow[dd, "⓪"] \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}\varepsilon_\vartheta \cap c\text{-Ind}\,\vartheta \arrow[dd] \arrow[rr] & & c\text{-Ind}\,\vartheta \arrow[dd] \arrow[rr] & & \mathbb{C} \arrow[dd, equal] \arrow[r] & 0 & \\
& 0 \arrow[r] & (\text{Ind}\,\vartheta)(\vartheta) \arrow[dl, "④"'] \arrow[rr] & & \text{Ind}\,\vartheta \arrow[dl, equal] \arrow[rr] & & (\text{Ind}\,\vartheta)_\vartheta \arrow[dl, "③"'] \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ker}\varepsilon_\vartheta \arrow[rr] & & \text{Ind}\,\vartheta \arrow[rr] & & \mathbb{C} \arrow[r] & 0 &
\end{tikzcd}$$

⓪: $0 \to c\text{-Ind}\,\vartheta \to \text{Ind}\,\vartheta \to \underbrace{\text{Ind}\,\vartheta / c\text{-Ind}\,\vartheta}_{\cong (\text{Ind}\,\vartheta)_N} \to 0$

$\rightsquigarrow 0 \to (c\text{-Ind}\,\vartheta)_\vartheta \to (\text{Ind}\,\vartheta)_\vartheta \to \underbrace{(\text{Ind}\,\vartheta)_{N,\vartheta}}_{=0} \to 0$

$\rightsquigarrow$ ⓪ is iso

①: To verify that $\operatorname{Ker}\varepsilon_\vartheta \cap c\text{-Ind}\,\vartheta \subset (c\text{-Ind}\,\vartheta)(\vartheta)$,

we only need to show the generators of $\operatorname{Ker}\varepsilon_\vartheta \cap c\text{-Ind}\,\vartheta$ belong to $(c\text{-Ind}\,\vartheta)(\vartheta)$.

Generators: $\{f_{a,j} \in C^\infty(P) \mid a \in F^\times - U_F^{(j)},\ j \geq 1\}$

Verify $f_{a,j} \in (c\text{-Ind}\,\vartheta)(\vartheta)$: Let $d = \text{level}(\vartheta)$, $j_0 := v_F(a-1) \underset{a \notin U_F^{(j)}}{<} j$.

Let $y_0 \in \mathfrak{p}^{d-1}$ s.t. $\vartheta\begin{pmatrix}1 & y_0\\ 0 & 1\end{pmatrix} := c \neq 1$, and $x_0 := \frac{y_0}{a-1}$. We get

$$v_F(ax_0\mathfrak{p}^j) \geq v_F(a) + d - 1 - j_0 + j \geq \begin{cases} v_F(a) + d \geq d, & v_F(a) \geq 0 \\ v_F(a) + d - j_0 = d, & v_F(a) < 0 \end{cases}$$

$$\begin{aligned}
&\vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} - \vartheta\begin{pmatrix}1 & aU_F^{(j)}x_0\\ 0 & 1\end{pmatrix} \\
&= \vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} - \vartheta\begin{pmatrix}1 & ax_0\\ 0 & 1\end{pmatrix}\vartheta\begin{pmatrix}1 & ax_0\mathfrak{p}^j\\ & 1\end{pmatrix} \\
&\overset{v_F(ax_0\mathfrak{p}^j)\geq d}{=\!=} \vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} - \vartheta\begin{pmatrix}1 & ax_0\\ 0 & 1\end{pmatrix} \\
&= \vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} - \vartheta\begin{pmatrix}1 & (a-1)x_0\\ 0 & 1\end{pmatrix}\vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} \\
&= (1-c)\,\vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} \neq 0
\end{aligned}$$

$$\Rightarrow f_{a,j} = \frac{1}{(1-c)\,\vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix}}\left(\vartheta\begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix} f_{a,j} - \begin{pmatrix}1 & x_0\\ 0 & 1\end{pmatrix}.f_{a,j}\right) \in (c\text{-Ind}\,\vartheta)(\vartheta).$$

Finally, ① iso $\Rightarrow$ ② iso $\overset{⓪\ \text{iso}}{\Longrightarrow}$ ③ iso $\Rightarrow$ ④ iso. $\square$

Step 3 Finally we can prove that $c\text{-}\mathrm{Ind}\,\vartheta \in \mathrm{Irr}(P)$. $\forall \vartheta \in \widehat{N}^* - \{\mathbb{1}_N\}$.

Proof. Let $V \leq c\text{-}\mathrm{Ind}\,\vartheta$ in $\mathrm{Rep}(P)$, we show that $V = 0$ or $c\text{-}\mathrm{Ind}\,\vartheta/V = 0$.

$$0 \longrightarrow V \longrightarrow c\text{-}\mathrm{Ind}\,\vartheta \longrightarrow c\text{-}\mathrm{Ind}\,\vartheta/V \longrightarrow 0$$

$$\Rightarrow \begin{cases} 0 \longrightarrow \underbrace{V_N}_{=0} \longrightarrow \underbrace{(c\text{-}\mathrm{Ind}\,\vartheta)_N}_{=0} \longrightarrow \underbrace{(c\text{-}\mathrm{Ind}\,\vartheta/V)_N}_{=0} \longrightarrow 0 \\ 0 \longrightarrow V_\vartheta \longrightarrow \underbrace{(c\text{-}\mathrm{Ind}\,\vartheta)_\vartheta}_{\cong\mathbb{C}} \longrightarrow (c\text{-}\mathrm{Ind}\,\vartheta/V)_\vartheta \longrightarrow 0 \end{cases}$$

$$\Rightarrow \quad V_\vartheta = 0 \xRightarrow{V_N = 0} V = 0$$

or

$$(c\text{-}\mathrm{Ind}\,\vartheta/V)_\vartheta = 0 \xRightarrow{(c\text{-}\mathrm{Ind}\,\vartheta/V)_N = 0} c\text{-}\mathrm{Ind}\,\vartheta/V = 0$$

$\square$

3. Classification.

We will prove that the two examples in the last section are all irr reps of P.

Lemma. Let $(\rho, V) \in \mathrm{Rep}(P)$, we get

$$\begin{tikzcd} V \arrow[r, "\pi_*"] & \mathrm{Ind}_N^P V_\vartheta \\ V(N) \arrow[u, phantom, "\cup"] \arrow[r, "\pi_*|_{V(N)}", "\cong"'] & (\mathrm{Ind}_N^P V_\vartheta)(N) \arrow[u, phantom, "\cup"] \cong c\text{-}\mathrm{Ind}_N^P V_\vartheta . \end{tikzcd} \qquad \text{induced by} \quad \mathrm{Res}_N^P V \xrightarrow{\pi} V_\vartheta$$

[**Proof.** Denote $W = \mathrm{Ker}\, \pi_*|_{V(N)}$, $W' = \mathrm{Coker}\, \pi_*|_{V(N)}$. we get LES

$$0 \longrightarrow W \longrightarrow V(N) \longrightarrow (\mathrm{Ind}_N^P V_\vartheta)(N) \longrightarrow W' \longrightarrow 0$$

$$\Rightarrow \begin{cases} 0 \longrightarrow W_N \longrightarrow \underbrace{V(N)_N}_{=0} \longrightarrow \underbrace{(\mathrm{Ind}_N^P V_\vartheta)(N)_N}_{=0} \longrightarrow W'_N \longrightarrow 0 \\ 0 \longrightarrow W_\vartheta \longrightarrow \underbrace{V(N)_\vartheta}_{= V_\vartheta} \longrightarrow \underbrace{(\mathrm{Ind}_N^P V_\vartheta)(N)_\vartheta}_{\cong (\mathrm{Ind}_N^P V_\vartheta)_\vartheta \cong V_\vartheta} \longrightarrow W'_\vartheta \longrightarrow 0 \end{cases}$$

$$\Rightarrow \begin{cases} W_N = 0 & W'_N = 0 \\ W_\vartheta = 0 & W'_\vartheta = 0 \end{cases} \Rightarrow W = 0, W' = 0 \Rightarrow \pi_*|_{V(N)} \text{ is iso.}$$

$\square$]

Thm. Let $(\rho, V) \in \mathrm{Irr}(P)$. Fix $\vartheta \in \widehat{N}^* \setminus \{\mathbb{1}_N\}$

(1) When $V(N) = 0$, $\rho \in \widehat{P}^*$ is the inflation of some $\chi \in \widehat{P/N}^* = \widehat{F^{\times}}^{*}$;

(2) When $V(N) = V$, $V \cong c\text{-}\mathrm{Ind}_N^P \vartheta$.

[**Proof.** When $V(N) = 0$, $\rho|_N = \mathrm{Id}_V \Rightarrow \exists\, \chi \in \mathrm{Irr}(P/N) = \widehat{P/N}^*$, $\chi \xrightarrow{P} \rho$.

When $V(N) = V$, $V = V(N) \overset{\text{Lemma}}{\cong} c\text{-}\mathrm{Ind}\, V_\vartheta \in \mathrm{Irr}(P)$

$\Rightarrow \dim_{\mathbb{C}} V_\vartheta = 1$, i.e., $V_\vartheta \cong \vartheta$ in $\mathrm{Rep}(N)$

$\Rightarrow V \cong c\text{-}\mathrm{Ind}\, \vartheta$.

$\square$]

4. Applications.

4.1. Computation of $V(N), V_N, V(\vartheta), V_\vartheta$ $(\rho, V) \in \mathrm{Irr}(P)$

For $\rho = \text{c-Ind}\,\vartheta$ $\vartheta \in \widehat{N}^* - \{\mathbb{1}_N\}$, we have computed in [prop. jacqofind].

For $\rho_\chi : P \longrightarrow \mathbb{C}^\times$ $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \longmapsto \chi(a)$, we know that

$$V(N) = 0 \qquad V_N \cong \mathbb{C}$$

$$V(\vartheta) \cong \mathbb{C} \qquad V_\vartheta = 0 \qquad \forall \vartheta \in \widehat{N}^* - \{\mathbb{1}_N\}.$$

4.2. Dual, Sym^m, Λ^m, ...

$N \leq P$ closed, N is unimodular, while P is not. $\delta_P|_N = \mathbb{1}_N$, so

$$(\text{c-Ind}\,\vartheta)^\vee \cong \mathrm{Ind}\,(\delta_P|_N \otimes \check{\vartheta}) \cong \mathrm{Ind}\,\check{\vartheta} \cong \mathrm{Ind}\,\vartheta$$

Q: $\delta_P = ?$ (lazy to compute it.)