Eine Woche, ein Beispiel

1.16 Whitehead bracket

The story begins with the following naive question: what's the homotopy group of

$X =$  ?

Answer: the universal covering of X is

$\tilde{X} =$  $\cong$  $\cong$ 

$\overset{\text{homotopy}}{\sim} S^2 \vee S^2$

so the question reduces to the computation of $\pi_n(S^2 \vee S^2)$.

What is relative easy to do:

$$\pi_n(S^2 \vee S^2, *) \cong \begin{cases} 0 & n=0,1 \\ \mathbb{Z} & n=2 \quad \text{by Hurewicz} \\ \mathbb{Z}^3 & n=3 \quad \text{by [Hatcher Example 4.52]} \end{cases}$$

Idea for $\pi_3(S^2 \vee S^2, *) \cong \mathbb{Z}^3$:

By the LES induced by the CW-pair, we get a split SES:

$$0 \longrightarrow \pi_{n+1}(S^2 \times S^2, S^2 \vee S^2, *) \longrightarrow \pi_n(S^2 \vee S^2, *) \longrightarrow \pi_n(S^2 \times S^2, *) \longrightarrow 0 \quad \forall n \geq 1$$

By repeatly applying [Hatcher, Prop 4.28], we get

- $(S^2 \times S^2, S^2 \vee S^2)$ is 3-connected
- $\pi_4(S^2 \times S^2, S^2 \vee S^2, *) \cong \pi_4(S^4, *)$

$$\therefore \quad \pi_3(S^2 \vee S^2, *) \cong \pi_3(S^2, *) \times \pi_3(S^2, *) \times \pi_4(S^4, *) \cong \mathbb{Z}^3$$

This problem has been fully solved in some sense, see the first two pages for the description and also the rest for the proof(I'm too lazy to see the proof): http://nlab-pages.s3.us-east-2.amazonaws.com/nlab/files/Hilton55.pdf

Finally we get

$$\pi_n(S^2 \vee S^2, *) \cong \pi_n(S^2)^{\oplus 2} \oplus \pi_n(S^3)^{\oplus 1} \oplus \pi_n(S^4)^{\oplus 2} \oplus \pi_n(S^5)^{\oplus 3} \oplus \pi_n(S^6)^{\oplus 6} \oplus \cdots$$

Some computations for the future to check: abbreviate $[\iota_1,[\iota_1,\iota_2]]$ as $[1[12]]$

2 weight 1 $\quad 1 \quad 2$

1 weight 2 $\quad [12]$

2 weight 3 $\quad [1[12]] \quad [2[12]]$

3 weight 4 $\quad [1[1[12]_3 \quad [2[1[12]_3 \quad [2[2[12]_3$

6 weight 5 $\begin{cases} 1+4 & [1[1[1[12]_4 \quad [2[1[1[12]_4 \quad [2[2[1[12]_4 \quad [2[2[2[12]_4 \\ 2+3 & [[12][1[12]_3 \quad [[12][2[12]_3 \end{cases}$

9 weight 6 $\begin{cases} 1+5 & [1[1[1[1[12]_5 \quad [2[1[1[1[12]_5 \quad [2[2[1[1[12]_5 \quad [2[2[2[1[12]_5 \quad [2[2[2[2[12]_5 \\ 2+4 & [[12][1[1[12]_4 \quad [[12][2[1[12]_4 \quad [[12][2[2[12]_4 \\ 3+3 & [[1[12]][2[12]_3 \end{cases}$

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It is more interesting that we have the extra structure for the homotopy group, which gives us a simple way to construct nontrivial element in the higher homotopy groups(maybe very difficult to prove, though): the Whitehead bracket.

See wiki for its definition: https://en.wikipedia.org/wiki/Whitehead_product
results about Whitehead bracket of spheres: https://mathoverflow.net/questions/315255/whitehead-products-in-homotopy-groups-of-spheres

Some exercises for myself:
prove that Whitehead bracket is a graded quasi-Lie algebra;
verify if the action of the fundamental group on homotopy groups compatible with the Whitehead bracket;

Q: Suppose $[f]=[g]\in\pi_2(S^2,*)$. Do we have homotopy equivalent between $S^2\cup_{\varphi_f}D^2$ and $S^2\cup_{\varphi_g}D^2$?

$$\begin{tikzcd} \partial D^2 \arrow[r, hook] \arrow[d, "f"'] & D^2 \arrow[d] \\ S^2 \arrow[r] & S^2\cup_{\varphi_f}D^2 \end{tikzcd}$$

pushout

A: Yes, see [Hatcher, Prop 0.18]

Q: Let $f: S^2\to S^2$ be the map of degree 3, how to compute $\pi_i(S^2\cup_{\varphi_f}D^2,*)$?