Eine Woche, ein Beispiel

4.28 naive $\otimes$ - Hom adjunction

Notation. - A: associate ring allowed to be non-commutative, contains 1

- There are two systems to write category of A-modules.

$$\begin{array}{rcll} & \mathrm{Mod}_A & = A\text{-}\mathrm{Mod} & \ni {}_AM \\ (\mathrm{Mod}_A)^{op} \neq & \mathrm{Mod}_{A^{op}} & = \mathrm{Mod}\text{-}A = A^{op}\text{-}\mathrm{Mod} & \ni M_A \\ & \mathrm{Mod}_{A\otimes B^{op}} & = A\text{-}\mathrm{Mod}\text{-}B & \ni {}_AM_B \end{array}$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write

$$(\mathrm{Mod}_{B\otimes A^{op}})^{op} = (B\text{-}\mathrm{Mod}\text{-}A)^{op} = (A^{op}\text{-}\mathrm{Mod}\text{-}B^{op})^{op} \qquad \ni {}_BM_A$$

as

$$(\mathrm{Mod}_{A\otimes B^{op}})^{\overline{op}} = (A\text{-}\mathrm{Mod}\text{-}B)^{\overline{op}}$$

⚠ Even though you can identify $Ob(Ring) \cong Ob(Ring^{op})$,
$A^{op} \notin Ob(Ring^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\mathrm{Mod}_A = \mathrm{Mod}_{A^{op}} \subset Sh(\mathrm{Spec}\, A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1. definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$
4. definition recall for $\otimes$ & Hom , derived version
5. adjunction , derived version
6. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A : \mathrm{Mod}_{A^{op}} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$
$$\mathrm{Hom}_A(-,-) : (\mathrm{Mod}_A)^{op} \times \mathrm{Mod}_A \longrightarrow \mathrm{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B : A\text{-Mod-}B \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$
$$\mathrm{Hom}_B(-,-) : (A\text{-Mod-}B)^{\overline{op}} \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$

$$\begin{array}{ccccc} \mathrm{Hom}^A_B(-,-) : & (A\text{-Mod-}B)^{\overline{op}} & \times\ B\text{-Mod-}A & \longrightarrow & \mathbb{Z}\text{-Mod} \\ \| & \| & \| & & \| \\ \mathrm{Hom}_{B\otimes_{\mathbb{Z}} A^{op}}(-,-) & (\mathbb{Z}\text{-Mod-}B\otimes_{\mathbb{Z}} A^{op})^{\overline{op}} & \times\ (B\otimes_{\mathbb{Z}} A^{op}\text{-Mod-}\mathbb{Z})^{\overline{op}} & \longrightarrow & \mathbb{Z}\text{-Mod-}\mathbb{Z} \end{array}$$

$${}_A X_B, \quad {}_B Y_C, \quad {}_C Z_D$$

associativity: $(X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $X \otimes_B Y \cong Y \otimes_{B^{op}} X$ in $A\text{-Mod-}C = C^{op}\text{-Mod-}A^{op}$

"unit": $A \otimes_A X \cong X \cong X \otimes_B B$

$\mathrm{Hom}_A(A, X) \cong X$

2. adjunction ${}_BX_A$, ${}_CY_B$, ${}_CZ_D$, we get

$$\mathrm{Hom}_C(Y \otimes_B X,\ Z) \cong \mathrm{Hom}_B(X, \mathrm{Hom}_C(Y, Z)) \quad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$\{ f: Y \times X \longrightarrow Z \mid f(cyb, x) = c f(y, bx) \quad \forall b, c \}$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$\begin{tikzcd} B\text{-Mod} \arrow[r, bend left, "Y \otimes_B -"] \arrow[r, phantom, "\perp"] & C\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_C(Y, -)"] \end{tikzcd}$$

slogan: adjunction $\approx$ associativity

$\otimes \dashv \mathrm{Hom}$:

$$\begin{tikzcd}
(A\text{-Mod-}B)^{op} \times (B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "{(Id,\ Hom_C)}"] \arrow[d, equal] & (A\text{-Mod-}B)^{op} \times B\text{-Mod-}D \arrow[dd, "Hom_B"] \\
(A\text{-Mod-}B \times B\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[d, "{(\otimes_B,\ Id)}"'] & \\
(A\text{-Mod-}C)^{op} \times C\text{-Mod-}D \arrow[r, "Hom_C"] & A\text{-Mod-}D
\end{tikzcd}$$

$f^* \dashv f_*$:

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$\begin{tikzcd}
\mathcal{G} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\
Y \arrow[r, "f"] & X
\end{tikzcd}$$

$$\begin{tikzcd}
Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \arrow[r, "{(Id,\ pushforward)}"] \arrow[d, "{(pullback,\ Id)}"'] & Sh(X)^{op} \times Sh(X) \arrow[d, "{Hom_{Sh(X)}(-,-)}"] \\
Sh(Y)^{op} \times Sh(Y) \arrow[r, "{Hom_{Sh(Y)}(-,-)}"] & Abel
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathcal{F},\ f,\ \mathcal{G}) \arrow[r, mapsto] \arrow[d, mapsto] & (\mathcal{F},\ f_*\mathcal{G}) \arrow[d, mapsto] \\
(f^*\mathcal{F}, \mathcal{G}) \arrow[r, mapsto] & Hom_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong Hom_{Sh(X)}(\mathcal{F}, f_*\mathcal{G})
\end{tikzcd}$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$

Forgetful fctor

Prop. For ring homo $\begin{matrix} S \\ \uparrow f \\ R \end{matrix}$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \mathrm{Hom}_S({}_S S_R, M)$$

one has adjunction fctors

$$\begin{tikzcd}[column sep=huge] S\text{-Mod} \arrow[r, "\mathrm{Hom}_S({}_S S_R, -)" above, "{\color{red}u=}\ {}_R S_S \otimes_S -" below] & R\text{-Mod} \arrow[l, bend right=50, "{}_S S_R \otimes_R -" above] \arrow[l, bend left=50, "\mathrm{Hom}_R({}_R S_S, -)" below] \end{tikzcd} \qquad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

$$\begin{tikzcd}[column sep=huge] \mathcal{D}(X) \arrow[r, "j^!" above, "j^*" below] & \mathcal{D}(\mathcal{U}) \arrow[l, bend right=50, "j_!" above] \arrow[l, bend left=50, "Rj_*" below] \end{tikzcd}$$

Vague slogan: $u \approx$ "forget the information of Z".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & X.

E.g. For finite gps $H \le G$, one has Res - Ind adjunction:

$$\mathrm{Res}^G_H \dashv \mathrm{Ind}^G_H$$
$$c\text{-}\mathrm{Ind}^G_H \dashv \mathrm{Res}^G_H$$

It can be generalized for $\begin{cases} G: \text{ loc profinite gp}, \\ H \le G \text{ open} \end{cases}$

If one only has $H \le G$ closed, then it's possible that $j^! \ne j^*$.

e.g. $G = GL_2(\mathbb{Q}_p) \quad H = GL_2(\mathbb{Z}_p)$

In the diagram,

$$\begin{tikzcd}
\mathrm{Rep}_G \arrow[r, "\mathrm{Res}^G_H"'] & \mathrm{Rep}_H \arrow[l, bend right=50, "c\text{-}\mathrm{Ind}^G_H"'] \arrow[l, bend left=50, "\mathrm{Ind}^G_H"]
\end{tikzcd}$$

Ex. Compare it with the recollement diagram & (3.1).

$$\begin{tikzcd}
\mathcal{U} \arrow[d, "j"] & {[*/H]} \arrow[d, "\text{“cover with fiber } G/H\text{”}"] \\
X & {[*/G]}
\end{tikzcd}$$

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$
   $c\text{-}\mathrm{Ind}^G_H \longrightarrow \mathrm{Ind}^G_H$ $\qquad$ $c\text{-}\mathrm{Ind}^G_H = \mathrm{Ind}^G_H$

2. Even though
$$\mathrm{Sh}_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \mathrm{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$
the “structure sheaf” of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\mathrm{Res}^G_{\{*\}} \mathbb{Q} = \mathbb{Q}, \qquad \mathrm{Res}^G_{\{*\}} \mathbb{Q}[G] = \mathbb{Q}[G] \ne \mathbb{Q}$$

⚠ In this example, $j^* R j_* \ne \mathrm{Id}$, $j^! j_! \ne \mathrm{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Compare with i

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$\begin{tikzcd} S & & \operatorname{Spec} S \arrow[d, "f"] \\ R \arrow[u, "\tilde{f}"] & M & \operatorname{Spec} R & M \end{tikzcd}$$

$\exists$ "pullback fctor"

$f^*: R\text{-Mod} \longrightarrow S\text{-Mod}$ $\qquad f^*M = {}_S S_R \otimes_R M$

This is also called the base change.

Now, (3.1) can be rewritten as

$$\begin{tikzcd} S\text{-Mod} \arrow[rr, "u"] & & R\text{-Mod} \arrow[ll, bend right, "f^*"'] \arrow[ll, bend left, "\mathrm{Hom}_R({}_R S_S, -)"] \end{tikzcd}$$

Compare it with another part of the recollement diagram:

$$\begin{tikzcd} \mathcal{D}(Z) \arrow[rr, "i_*", "i_!"'] & & \mathcal{D}(X) \arrow[ll, bend right, "i^*"'] \arrow[ll, bend left, "i^!"] \end{tikzcd}$$

Rmk. u is usually not f. faithful, unless $S = R/I$.

(In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.)

which crspds to closed embedding.

In that case,

$$i^* i_* = \mathrm{Id}: \ {}_S S_R \otimes_R ({}_R S_S \otimes_S M) \cong M$$

$$i^! i_* = \mathrm{Id}: \ \mathrm{Hom}_R({}_R S_S, \mathrm{Hom}({}_S S_R, M)) \cong M$$

Slogan: in the comm alg. $\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ is closed embedding.
In general, if
S is an R-idempotent algebra: $S \cong S \otimes_R S$
then $i: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ can be viewed as "closed subset".

E.g. $R_{\mathfrak{p}}$, R/I are idempotent R-algs.
$\mathbb{Z}[\frac{1}{6}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.

⚠ Usually R/I is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?
A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. definition recall for $\otimes$ & Hom , derived version

	F	RF or LF	R^iF or L^iF	exact fctor
	f^*	f^*	—	
	f_*	Rf_*	R^if_*	f_*-acyclic
	$\pi_{X,*}\mathcal{F}$	$\Gamma(X;\mathcal{F})$	$H^i(X;\mathcal{F})$	Γ-acyclic
	$f_!$	$Rf_!$	$R^if_!$	$f_!$-acyclic
	$\pi_{X,!}\mathcal{F}$	$\Gamma_c(X;\mathcal{F})$	$H^i_c(X;\mathcal{F})$	Γ_c-acyclic
	—	$f^!$	$\mathcal{H}^i(f^!-)$	
	$-\otimes_R-$	$-\otimes^L_R-$	$Tor^i_R(-,-)$	flat
	$Hom_R(-,-)$	$RHom_R(-,-)$	$Ext^i_R(-,-)$	injective/projective
e.g. group coh	M_G	$\mathbb{Z}\otimes^L_{\mathbb{Z}[G]}M$	$H_i(G;M)$	
	M^G	$RHom_{\mathbb{Z}[G]}(\mathbb{Z},M)$	$H^i(G;M)$	
e.g. Lie alg coh	$M_{\mathfrak{g}}$	$k\otimes^L_{U\mathfrak{g}}M$	$H_i(\mathfrak{g};M)$	
$\mathfrak{g}/k$: Lie alg	$M^{\mathfrak{g}}$	$RHom_{U\mathfrak{g}}(k,M)$	$H^i(\mathfrak{g};M)$	
e.g. Hochschild coh	$M/[AM]$	$A\otimes^L_{A^e}M$	$HH_i(A,M)$	
	M^A	$RHom_{A^e}(A,M)$	$HH^i(A,M)$	
	$A/[AA]$	$A\otimes^L_{A^e}A$	$HH_i(A)$	
	$Z(A)$	$RHom_{A^e}(A,A)$	$HH^i(A)$	

For calculations, see:
[23.04.09]: gp coh
[wiki]: Lie algebra coh
[21.05.21]: Hochschild coh
[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.
For Hom(-,A), see https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m.
Chenji Fu claimed that Hom(-,A) always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.
Related:
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod
https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod

https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution

4. definition recall for $\otimes$ & Hom , derived version

To define $^L\otimes$ & RHom, one needs to extend fctors

$$\begin{aligned}
\otimes_A : \quad & A\text{-Mod} \times A\text{-Mod} \longrightarrow A\text{-Mod} \\
\mathrm{Hom}_A(-,-): \quad & (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod}
\end{aligned}$$

to fctors on double cplxes.

$\mathcal{C}(A) := $ complex of A-modules, temperate notation

$$\begin{aligned}
\otimes_{\mathcal{C}(A)} : \quad & \mathcal{C}(A) \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A) \\
\mathrm{Hom}_{\mathcal{C}(A)}(-,-): \quad & (\mathcal{C}(A))^{op} \times \mathcal{C}(A) \longrightarrow \mathcal{C}(A)
\end{aligned}$$

But how?

Wishes:

$$\begin{aligned}
(M^\cdot[i]) \otimes_{\mathcal{C}(A)} (N^\cdot[j]) &= (M^\cdot \otimes N^\cdot)[i+j] \\
\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot[-i], N^\cdot[j]) &= \mathrm{Hom}(M^\cdot, N^\cdot)[i+j]
\end{aligned}$$

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\arrow[r] & M^{-1}\otimes N^{1} \arrow[u] \arrow[r] & M^{0}\otimes N^{1} \arrow[u] \arrow[r] & M^{1}\otimes N^{1} \arrow[u] \arrow[r] & M^{2}\otimes N^{1} \arrow[u] \arrow[r] & \cdots \\
\arrow[r] & M^{-1}\otimes N^{0} \arrow[u] \arrow[r] & M^{0}\otimes N^{0} \arrow[u] \arrow[r] & M^{1}\otimes N^{0} \arrow[u] \arrow[r] & M^{2}\otimes N^{0} \arrow[u] \arrow[r] & \cdots \\
\arrow[r] & M^{-1}\otimes N^{-1} \arrow[u] \arrow[r] & M^{0}\otimes N^{-1} \arrow[u] \arrow[r] & M^{1}\otimes N^{-1} \arrow[u] \arrow[r] & M^{2}\otimes N^{-1} \arrow[u] \arrow[r] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

$\mathrm{Tot}(M^{\cdot}\otimes N^{\cdot})$, the double complex of $M^{\cdot}\otimes N^{\cdot}$.

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \vdots & \\
\arrow[r] & \mathrm{Hom}(M^{1}, N^{1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{0}, N^{1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-1}, N^{1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-2}, N^{1}) \arrow[u] \arrow[r] & \cdots \\
\arrow[r] & \mathrm{Hom}(M^{1}, N^{0}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{0}, N^{0}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-1}, N^{0}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-2}, N^{0}) \arrow[u] \arrow[r] & \cdots \\
\arrow[r] & \mathrm{Hom}(M^{1}, N^{-1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{0}, N^{-1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-1}, N^{-1}) \arrow[u] \arrow[r] & \mathrm{Hom}(M^{-2}, N^{-1}) \arrow[u] \arrow[r] & \cdots \\
 & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] & {} \arrow[u] &
\end{tikzcd}
$$

$\mathrm{Tot}(\mathrm{Hom}(M^{\cdot}, N^{\cdot}))$, the double complex of $\mathrm{Hom}(M^{\cdot}, N^{\cdot})$.

Def. For $M^\cdot, N^\cdot \in \mathcal{C}(A)$, define

$$M^\cdot \otimes N^\cdot, \quad \mathrm{Hom}_A(M^\cdot, N^\cdot) \in \mathcal{C}(A)$$

by

$$(M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot)^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$(\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot))^n = \bigoplus_{i+j=n} \mathrm{Hom}_A(M^{-i}, N^j)$$

and morphisms given by $d + (-1)^j \delta$.

⚠ $\mathcal{F}^\cdot \otimes^L \mathcal{G}^\cdot \cong \mathcal{G}^\cdot \otimes^L \mathcal{F}^\cdot$. However, for the crspd of elements, $a \otimes b \not\leftrightarrow b \otimes a$ because of this $(-1)^j$.

Ex. Let $M^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 3} \underset{0}{\mathbb{Z}}]$, $N^\cdot = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{0}{\mathbb{Z}}]$

compute $M^\cdot \otimes_{\mathcal{C}(\mathbb{Z})} N^\cdot$ & $\mathrm{Hom}_{\mathcal{C}(\mathbb{Z})}(M^\cdot, N^\cdot)$, and verify that they're complexes.

A.

$$\begin{array}{c|cc}
0 & \mathbb{Z} \xrightarrow{\times 3} & \mathbb{Z} \\
 & {\scriptstyle\times 2}\uparrow & \uparrow{\scriptstyle\times 2} \\
-1 & \mathbb{Z} \xrightarrow{\times 3} & \mathbb{Z} \\
\hline
 & -1 & 0
\end{array}
\quad \rightsquigarrow \quad
[\underset{-2}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{-1}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{0}{\mathbb{Z}}]$$

$\mathrm{Tot}(M^\cdot \otimes N^\cdot)$ $\qquad\qquad$ $M^\cdot \otimes_{\mathcal{C}(A)} N^\cdot$

$$\begin{array}{c|cc}
0 & \mathbb{Z} \xrightarrow{\times 3} & \mathbb{Z} \\
 & {\scriptstyle\times 2}\uparrow & \uparrow{\scriptstyle\times 2} \\
-1 & \mathbb{Z} \xrightarrow{\times 3} & \mathbb{Z} \\
\hline
 & 0 & 1
\end{array}
\quad \rightsquigarrow \quad
[\underset{-1}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{0}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{1}{\mathbb{Z}}]$$

$\mathrm{Tot}(\mathrm{Hom}(M^\cdot, N^\cdot))$ $\qquad\qquad$ $\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot, N^\cdot)$

Now, we can define $\overset{L}{\otimes}$ & RHom:

Def. For $M, N \in A\text{-Mod}$, one can define

$$M \overset{L}{\otimes}_A N := M \otimes_{\mathcal{C}(A)} P^{\cdot} \quad \text{when } N \xleftarrow{\simeq} P^{\cdot} \quad \text{flat resolution}$$

in general, $M^{\cdot}, N^{\cdot} \in \mathcal{D}^-(A\text{-Mod})$

$$\begin{aligned} RHom_A(M, N) &:= Hom_{\mathcal{C}(A)}(M, I^{\cdot}) \quad \text{when } N \xrightarrow{\simeq} I^{\cdot} \quad \text{inj resolution} \\ &:= Hom_{\mathcal{C}(A)}(P^{\cdot}, N) \quad \text{when } M \xleftarrow{\simeq} P^{\cdot} \quad \text{proj resolution} \end{aligned}$$

in general, $M^{\cdot} \in \mathcal{D}^-(A\text{-Mod})$, $N^{\cdot} \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat

Ex Compute $\mathbb{F}_2 \overset{L}{\otimes}_{\mathbb{Z}} \mathbb{F}_2$ & $RHom_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $Tor^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$ & $Ext^i_{\mathbb{Z}}(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that

$$Hom_{\mathcal{D}^+(A)}(M^{\cdot}, N^{\cdot}) = R^0 Hom_{\mathcal{C}(A)}(M^{\cdot}, N^{\cdot})$$

$$Hom_A(M, N) = Hom_{\mathcal{D}^+(A)}(M^{\cdot}, N^{\cdot}) = R^0 Hom_{\mathcal{C}(A)}(M, N).$$

⚠ [KS 90, Def 2.6.2]
To switch from $\mathcal{D}^-(X)$ to $\mathcal{D}^+(X)$, we need to require that

$$w.gldim(A) < +\infty. \quad \text{w.gldim: shortest flat resolution}$$

In our case, $w.gldim(\mathbb{Q}) = 0 < +\infty$. So don't worry.

A wrong proof for "flat $\longrightarrow$ proj"

"Proof": when P is flat,

$$\begin{array}{ccc} P \otimes_A - & \dashv & \mathrm{Hom}_A(P,-) \\ \| & & \\ P \overset{L}{\otimes}_A - & \dashv & R\mathrm{Hom}_A(P,-) \end{array}$$

by the uniqueness of the adjunction, $\mathrm{Hom}_A(P,-) = R\mathrm{Hom}_A(P,-)$, so P is flat.

This is wrong. $\mathbb{Q} \in \mathbb{Z}$-Mod is flat but not proj.
In the proof, we only have

$$\begin{tikzcd} A\text{-Mod} \arrow[r, bend left, "P \otimes_A -"] \arrow[d, "\iota_A"'] & A\text{-Mod} \arrow[l, bend left, "\mathrm{Hom}_A(P,-)"] \arrow[d, "\iota_A"] \\ \mathcal{D}(A) \arrow[r, bend left, "P \overset{L}{\otimes}_A -"] & \mathcal{D}(A) \arrow[l, bend left, "R\mathrm{Hom}_A(P,-)"] \end{tikzcd}$$

$$\iota_A \circ (P \otimes_A -) = (P \overset{L}{\otimes}_A -) \circ \iota_A .$$

Ex. Compute $R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-)$, and shows that

$$\iota_{\mathbb{Z}} \circ \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-) \neq R\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},-) \circ \iota_{\mathbb{Z}} .$$

5. adjunction , derived version

Prop. For A comm ring, $L, M, N \in A\text{-Mod}$, we get
$L^\bullet, M^\bullet \in \mathcal{D}^-(A)$, $N^\bullet \in \mathcal{D}^+(A)$ in general

$$R\mathrm{Hom}_A(M \otimes^L_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

Proof.

$$\mathrm{Hom}_A(M \otimes_A N, L) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

$\Downarrow$ take $(-)^n$

$$\mathrm{Hom}_{\mathcal{C}(A)}(M \otimes_{\mathcal{C}(A)} N, L) \cong \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(M, L))$$

$\Downarrow$ $\mathrm{Hom}_A(M, -)$ preserves inj modules for M flat

wiki: injective module

$$R\mathrm{Hom}_A(M \otimes^L_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

$$
\begin{aligned}
R\mathrm{Hom}_A(M \otimes^L_A N, L) &= R\mathrm{Hom}_A(P^\bullet \otimes_{\mathcal{C}(A)} N, L) && M^\bullet \xleftarrow{\cong} P^\bullet \text{ flat} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(P^\bullet \otimes_{\mathcal{C}(A)} N, I^\bullet) && L^\bullet \xrightarrow{\cong} I^\bullet \text{ inj} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^\bullet, I^\bullet)) && \text{adj in } \mathcal{C}(A) \\
&= R\mathrm{Hom}_A(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^\bullet, I^\bullet)) && \mathrm{Hom}_{\mathcal{C}(A)}(P^\bullet, I^\bullet) \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(P^\bullet, I^\bullet)) && I^\bullet \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))
\end{aligned}
$$
$\square$

∇ We don't have

$$R\mathrm{Hom}_A(M \otimes_A N, L) \cong R\mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

Find a counterexample?

Take $N = A$, reduce to:

$$R\mathrm{Hom}_A(M, L) \cong \mathrm{Hom}_A(M, L)$$

then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between $\otimes \dashv \mathrm{Hom}$ & $f^* \dashv f_*$, derived version

Now it's time to extend Section 3 to derived version!
Notice that most fctors in (3.1) are already exact:

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(S\text{-Mod}) \arrow[r, "\mathrm{Hom}_S({}_S S_R\,,-)" above, "u = {}_R S_S \otimes_S -" below] & \mathcal{D}(R\text{-Mod}) \arrow[l, bend right=40, "{}_S S_R \overset{L}{\otimes}_R -" above] \arrow[l, bend left=40, "R\mathrm{Hom}_R({}_R S_S\,,-)" below] \arrow[l, phantom, bend right=20, "\perp"] \arrow[l, phantom, bend left=20, "\perp"]
\end{tikzcd} \qquad (6.1)$$

Compare with j
In the extreme cases where S is a free R-Mod,
e.g. $S = k[G]$, $R = k[H]$
all the fctors in (3.1) are exact.

Compare with i
(6.1) can be rewritten as

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(S\text{-Mod}) \arrow[r, "u"] & \mathcal{D}(R\text{-Mod}) \arrow[l, bend right=40, "Lf^*" above] \arrow[l, bend left=40, "R\mathrm{Hom}_R({}_R S_S\,,-)" below] \arrow[l, phantom, bend right=20, "\perp"] \arrow[l, phantom, bend left=20, "\perp"]
\end{tikzcd} \qquad (6.2)$$

where the "pullback fctor"
$Lf^* : \mathcal{D}(R\text{-Mod}) \longrightarrow \mathcal{D}(S\text{-Mod})$ $\qquad Lf^*M = {}_S S_R \overset{L}{\otimes}_R M$
is now derived.

compare it with i-part of the recollement diagram:

$$\begin{tikzcd}[column sep=huge]
\mathcal{D}(Z) \arrow[r, "i_*", "i_!"'] & \mathcal{D}(X) \arrow[l, bend right=40, "i^*"'] \arrow[l, bend left=40, "i^!"] \arrow[l, phantom, bend right=20, "\perp"] \arrow[l, phantom, bend left=20, "\perp"]
\end{tikzcd}$$

To guarantee that

$$\begin{cases} u \text{ is f.faithful} \\ i^* i_* = i^! i_* = \mathrm{Id} \end{cases}$$

one has to require that $S \cong S \overset{L}{\otimes}_R S$.

e.g. $S = R_\mathfrak{p}$ or $\mathcal{S}^{-1}R$

Slogan: Zariski open subsets are closed!

To extend (6.1) to the whole recollement diagram, one has to define the Verdier quotient.

Verdier quotient

Def. For $\mathcal{C} \subset \mathcal{D}$ triangulated categories, the <u>Verdier quotient</u> $\mathcal{D}/\mathcal{C}$ is the triangulated category s.t.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[rrd, bend right, "0"'] & \mathcal{D} \arrow[r] \arrow[rd] & \mathcal{D}/\mathcal{C} \arrow[d, dashed, "\exists!"] \\
 & & \mathcal{E}
\end{tikzcd}$$

full subcategory:
https://math.stackexchange.com/questions/1285761/why-is-it-an-equivalent-definition-of-a-triangulated-full-subcategory

Rmk: Verdier quotient always exist.
For an explicit construction, see
https://stacks.math.columbia.edu/tag/05RA

Ex. Show the following isomorphism:

$$Z \xrightarrow{i} X \xleftarrow{j} \mathcal{U}$$

$$\begin{tikzcd}
\mathcal{D}(Z) \arrow[r, "i_! = i_*"] & \mathcal{D}(X) \arrow[l, shift right=3, "i^*"'] \arrow[l, shift left=3, "i^!"] \arrow[r, "j^* = j^!"] & \mathcal{D}(\mathcal{U}) \arrow[l, shift right=3, "j_!"'] \arrow[l, shift left=3, "Rj_*"]
\end{tikzcd}$$

$$\mathcal{D}(X)/_{j_!\mathcal{D}(\mathcal{U})} \cong \mathcal{D}(Z)$$
$$\mathcal{D}(X)/_{i_!\mathcal{D}(Z)} \cong \mathcal{D}(\mathcal{U})$$
restricted to π_X-sm sheaves: $$\mathcal{D}(X)/_{Rj_*\mathcal{D}(\mathcal{U})} \cong \mathcal{D}(Z)$$

[Hint: take the middle one as an example:

$$\begin{tikzcd}
\mathcal{D}(Z) \arrow[r, "i_!"] \arrow[rrd, bend right, "0"'] & \mathcal{D}(X) \arrow[r, "j^*"] \arrow[rd, "F"] & \mathcal{D}(\mathcal{U}) \arrow[d, dashed, "\exists! G"] \\
 & & \mathcal{E}
\end{tikzcd}$$

Uniqueness: $G \circ j^* = F \Rightarrow G = F \circ j_! = F \circ Rj_*$

Existence: Let $G = F \circ j_!$. Apply F to

$j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_! i^* \mathcal{F} \xrightarrow{+1}$

get $F \circ j_! \circ j^* = F$.]

Now, we can extend (6.2) to a full recollement:
(use stable ∞-category to guarentee that cocone is a fctor)

$$
\begin{tikzcd}
Z \arrow[r, "i"] & X & \mathcal{U} \arrow[l, hook', "j"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
\mathcal{D}(S\text{-Mod}) \arrow[r, "i_! = i_*" description] & \mathcal{D}(R\text{-Mod}) \arrow[l, bend right, "Li^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^* = j^!" description] & \mathcal{D}(R\text{-Mod}) / \mathcal{D}(S\text{-Mod}) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}
$$

$$
\begin{aligned}
Li^* &= S \overset{L}{\otimes}_R - & j_! &= \operatorname{cocone}\left[\underset{0}{-} \longrightarrow \underset{1}{S \overset{L}{\otimes}_R -} \right] \\
i_! &= u(-) & j^* &= \text{projection} \\
Ri^! &= R\mathrm{Hom}_R(S,-) & Rj_* &= R\mathrm{Hom}_R\left(\operatorname{cocone}\left[\underset{0}{R} \to \underset{1}{S}\right], -\right)
\end{aligned}
$$

Rmk: For any half recollement diagram (satisfying nec conditions), we can extend it to a full recollement diagram through the Verdier quotient. We will prove that rigorously in the 6-fctor formalism toolkit.

geometrical realizations of representations

representations	realized as sheaves	on some space	formulas
gp	equiv sheaves	stack	$\mathrm{Rep}(G) \longrightarrow \mathrm{Sh}([*/G]; \underline{\mathbb{Q}})$
Lie alg	D-modules	flag variety	$\mathrm{Mod}(\mathfrak{g}, \chi_\lambda) \cong \mathrm{Mod}^e_{qc}(D_\lambda)$
Hochschild	Differential, like	Hodge-Tate	$A\text{-Mod-}A \longrightarrow\ ?$
some quiver	perverse sheaves	stratified space	$\mathrm{Rep}(Q) \longrightarrow \mathrm{Perv}_S(X)$

Remark: for quiver, there's no "structure sheaf/rep" of a given quiver. Therefore, we don't discuss about quiver cohomology.

Realize homogeneous vector bundle on Hermitian symmetric variety as quiver representations:
https://arxiv.org/abs/math/0403307

The reference for Lie algebra cohomology is [HT07, Corollary 11.2.6(p274), p298 and Chapter 11]. Aaron told me about this story, and wrote this equivalence in detail in his master thesis.