Eine Woche, ein Beispiel

12.1 weights of type E

There are already much information in wiki and other references about the exceptional Lie algebra. It is nice, but I always have to check the compatability among different references. In this document, I try to fix a standard coordinate, and state all the combinatorical results without proofs.

We will make a list of the following objects, for E_6, E_7 and E_8.

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2023. https://doi.org/10.1017/9781009280020.

- Weights nearest to the origin
  - some graphs
  - weight lattice
- Simple roots
- Fundamental wts
- Weyl group action

Remark: There is another coordinate system which is written in wiki: del Pezzo surface. We don't use them. There, the different weight spaces are identified, while in our coordinate system, we identify the root lattices.

1. E_6.

– Weights nearest to the origin

There are two minuscule representations of E 6. So we just fix one.

affine version

#	typical coordinates	symbol
6	$(1, 0, 0, 0, 0, 0, 1, 0)^T$	v_i
6	$(1, 0, 0, 0, 0, 0, 0, 1)^T$	$\tilde{v}_i$
15	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{0, 1, 2\}$$

could be $\tilde{v}_i$ or v_{ij} edge

weight lattice version

#	typical coordinates	symbol
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, 3, -3)^T$	v_i
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, -3, 3)^T$	$\tilde{v}_i$
15	$\frac{1}{3}(-2, -2, 1, 1, 1, 1, 0, 0)^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{\tfrac{4}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\}$$

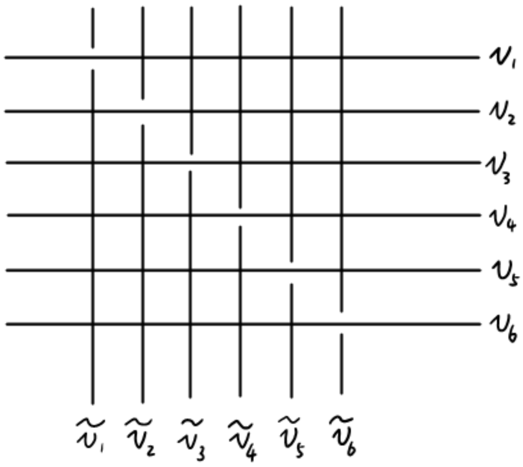
could be $\tilde{v}_i$ or v_{ij} edge

in $$\left\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \right\} \cong \mathbb{R}^6$$

The graph constructed is called the Schläfli graph, which has 27 vertices and 216 edges (with HoG Id 1300).
This graph is also the configuration graph of 27 lines.

vertices $\rightsquigarrow$ lines
edges $\rightsquigarrow$ intersection points
triangle $\rightsquigarrow$ triangle cut by H
only in E_6

Here are some typical subgraphs:

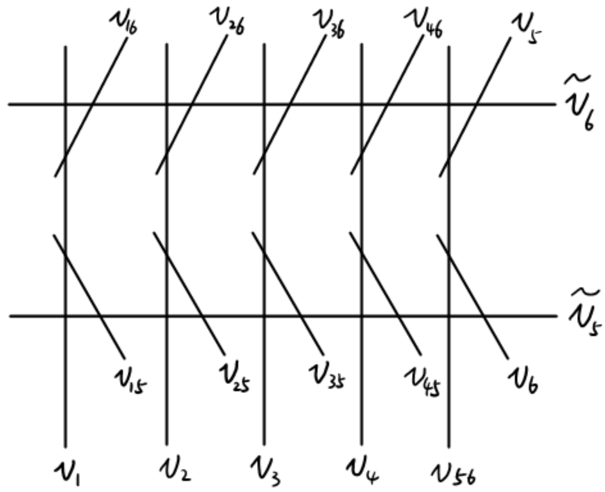


Schläfli double six configuration

$\# V = 12 \qquad \# E = 30$

HoG Id = 32794

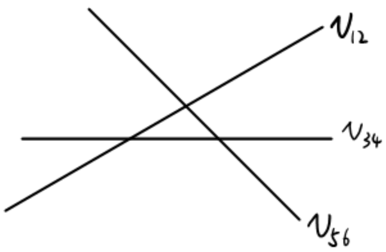
36 many, from [Huy24, Ex 3.6]



(v_{16} intersect with v_{25}, v_{35}, v_{45}, v_6. $\cdots$)

$\# V = 17 \qquad \# E = 30 + 20 = 50$

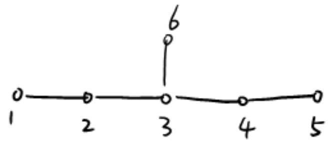
HoG Id = none.



triangle
720 many

Q: For each type of subgraph, how many are they in the Schläfli graph?
I don't know if there are any simple answer for general subgraphs, and I don't know if there are any efficient algorithm for doing this. But this already produces many mysterious combinatorical numbers.

- Simple roots



$$
\begin{aligned}
&\{\ \alpha_1,\quad \alpha_2,\quad \alpha_3,\quad \alpha_4,\quad \alpha_5,\quad \alpha_6\ \}\\
=&\{\ v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6,\ v_4 - v_{56}\ \}\\
=&\left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}-\frac{1}{2}\\-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix} \right\}
\end{aligned}
$$

Ex. Verify that all the 72 roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	v_{ij}
$40 = \binom{6}{3} \cdot 2$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$	$v_{456,7}$
2	$(0, 0, 0, 0, 0, 0, 1, -1)^T$	v_7

- Fundamental wts

denote by $A = (a_{ij})$ the Cartan matrix, then

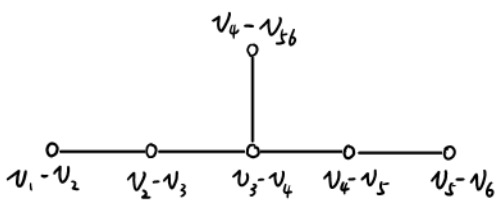
$$(\alpha_1, \dots, \alpha_r) = (\varpi_1, \dots, \varpi_r)\, A \qquad \langle \alpha_i, \alpha_j \rangle = A$$
$$(\varpi_1, \dots, \varpi_r) = (\alpha_1, \dots, \alpha_r)\, A^{-1} \qquad \langle \varpi_i, \varpi_j \rangle = A^{-1}$$

As a result,

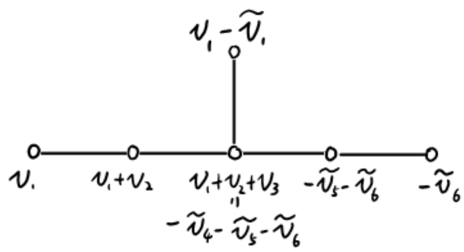
$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$$
$$= \{ v_1, \ v_1 + v_2, \ v_1+v_2+v_3, \ -\tilde{v}_5 - \tilde{v}_6, \ -\tilde{v}_6, \ v_1 - \tilde{v}_1 \} \qquad -\tilde{v}_6 \neq \sum_{i=1}^{5} v_i$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ -2 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \\ -3 \\ -3 \\ 9 \\ -9 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ -4 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



α_i



ϖ_i