Eine Woche, ein Beispiel

9.4 Hecke algebra

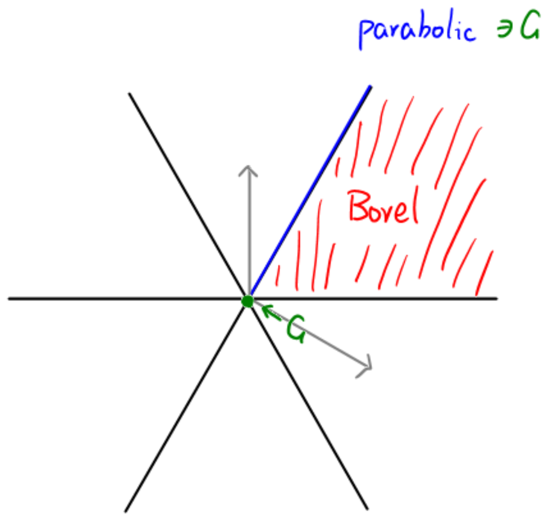
This document is not finished. I need some time to digest and restate them.
I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.
main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]
[XiongHecke][https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf]
All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

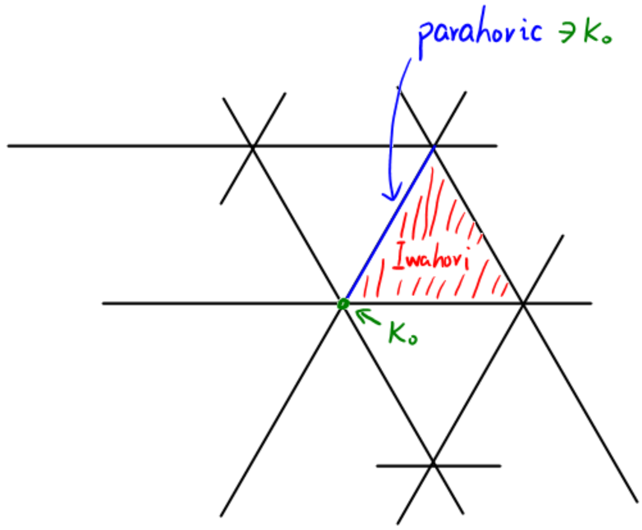
Task. For each double coset decomposition, we want to do:

1. decomposition ($\#\Gamma t\Gamma/\Gamma$ is finite & definition of Hecke alg)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of

	Bruhat	Iwahori affine Bruhat	Cartan Smith normal form
F finite	$G=\bigsqcup_{w\in W} BwB$		
F local	$G=\bigsqcup_{w\in W} BwB$	$G=\bigsqcup_{w\in W_{ext}} IwI$	$G=\bigsqcup_{\alpha\in T^-} K_0\alpha K_0$
F global	$G=\bigsqcup_{w\in W} BwB$		$GL_2^+(\mathbb{Q})=\bigsqcup_{\alpha\in T^-}\Gamma\alpha\Gamma$
adèle?			





$$B=\begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}=\begin{pmatrix} * & * & * \\ & * & * \\ & * & * \end{pmatrix}\cap\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$P=\begin{pmatrix} * & * & * \\ * & * & * \\ & & * \end{pmatrix}$$

$$I=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{pmatrix}\cap\begin{pmatrix} \mathcal{O} & \mathfrak{p}^{-1} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \end{pmatrix}\cap\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathcal{O} & \mathcal{O} & \mathfrak{p}^{-1} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

$$P=\begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

https://mathoverflow.net/questions/4547/definitions-of-hecke-alg
https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebraebras

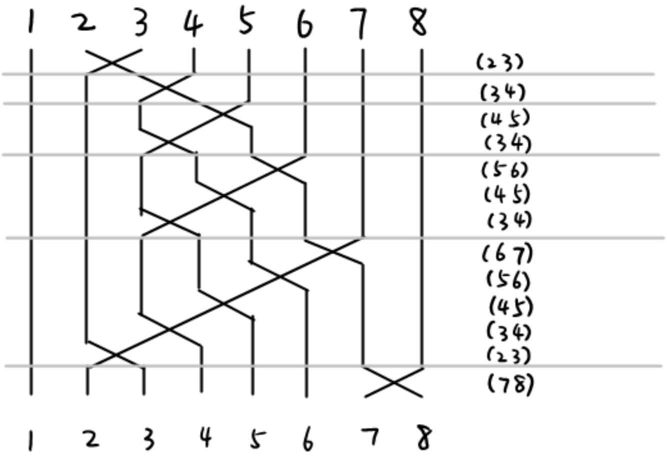
S_n and Tits system

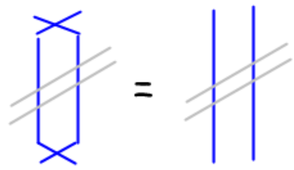
A brief preparation for computations in Bruhat decomposition. $s_i=(i\ i{+}1)$, $1\le i\le n-1$

E.g. $n=8$, $w_0=(287)(46)=\begin{pmatrix}1&2&3&4&5&6&7&8\\1&8&3&6&5&4&2&7\end{pmatrix}\in S_8$.

Ex. Compute $l(w_0)$, $l(s_iw_0)$ and $l(w_0s_i)$.

Solution.





$w_0=(78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$

$l(w_0)=13=$ "inversion number"

$l(s_1w_0)=14$ $\quad l(w_0s_1)=14$

$l(s_2w_0)=12$ $\quad l(w_0s_2)=12$

$l(s_3w_0)=14$ $\quad l(w_0s_3)=14$

$l(s_4w_0)=12$ $\quad l(w_0s_4)=12$

$l(s_5w_0)=12$ $\quad l(w_0s_5)=12$

$l(s_6w_0)=12$ $\quad l(w_0s_6)=14$

$l(s_7w_0)=14$ $\quad l(w_0s_7)=12$

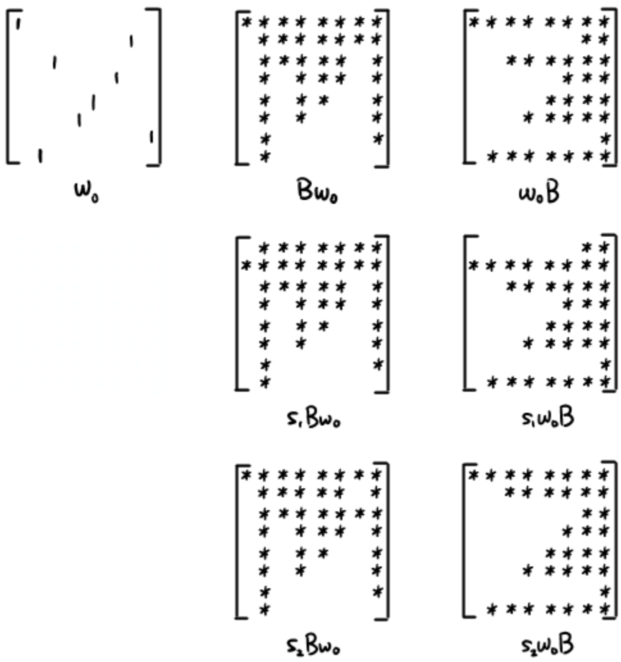
Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & \ddots & 0 \\ 0 & \ddots & * \end{pmatrix} \leq B$,

$w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

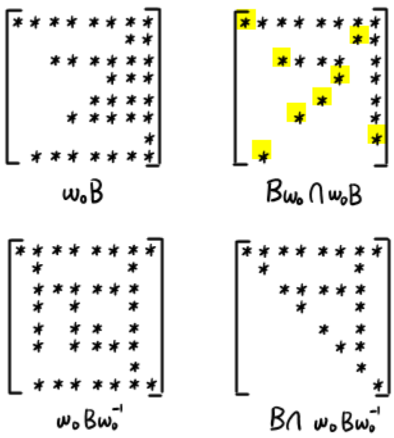
Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0)+1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0)-1 \end{cases}$$

Solution



The following computation will be also computed later on.



finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix} \le G$, $T = \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \le B$,

w_0, $s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$

Ex. Compute $|BwB/B|$ ⚠ BwB may not be a group!

Hint: Consider the map

$$\phi: B \longrightarrow BwB/B$$
$$b \longmapsto bwB$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 wB = b_2 wB$$
$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$
$$\Leftrightarrow b_2^{-1} b_1 \in wBw^{-1}$$
$$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{l(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G,B) := \{ f: G \to \mathbb{Z} \mid f(b_1 g b_2) = f(g) \quad \forall b_1, b_2 \in B, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$
$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(G,B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbb{1}_{BwB} = \mathbb{Z}^{\oplus n!}$$

Denote $T_w := \mathbb{1}_{BwB}$, $T_i := T_{s_i}$ ($T_{Id} = \mathbb{1}_B$ is the unit of $\mathcal{H}(G,B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G,B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w)\, T_w$$

$$(T_u * T_v)(w) = \frac{1}{|B|} \sum_{yz = w} T_u(y) T_v(z)$$
$$= \frac{1}{|B|} \{ (y,z) \in BuB \times BvB \mid yz = w \} \quad \underline{\text{if } w \notin BuB\,BvB}\ 0$$
$$= \frac{1}{|B|} |BuB \cap uBv^{-1}B|$$

$$Bs_iB \cdot Bw_oB = \begin{cases} Bs_iw_oB & l(s_iw) = l(w)+1 \\ Bs_iw_oB \cup Bw_oB & l(s_iw) = l(w)-1 \end{cases}$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{s_iw} & l(s_iw) = l(w)+1 \\ \mathbb{Z} \cdot T_{s_iw} + \mathbb{Z} \cdot T_w & l(s_iw) = l(w)-1 \end{cases}$$

Computation of coefficient:

$$|BwB| = |BwB/B| \times |B| = q^{l(w)} \cdot |B|$$

when $l(s_iw) = l(w)+1$,

$$(T_i * T_w)(s_iw) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times BwB \mid yz = s_iw\}$$

$$= \frac{1}{|B|\,|Bs_iwB|} \{(y,z) \in Bs_iB \times BwB \mid yz \in Bs_iwB\}$$

$$= \frac{|Bs_iB|\,|BwB|}{|B| \cdot |Bs_iwB|} = \frac{q^{l(s_i)}\, q^{l(w)}}{q^{l(s_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz = Id\}$$

$$= \frac{1}{|B|} |Bs_iB| = q$$

$$(T_i * T_i)(s_i) = \frac{1}{|B|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz = s_i\}$$

$$= \frac{1}{|B|\,|Bs_iB|} \{(y,z) \in Bs_iB \times Bs_iB \mid yz \in Bs_iB\}$$

$$= \frac{1}{|B|\,|Bs_iB|} \left(|Bs_iB \times Bs_iB| - \{(y,z) \in Bs_iB \times Bs_iB \mid yz \in B\}\right)$$

$$= \frac{1}{|B|\,|Bs_iB|} \left(|Bs_iB|\,|Bs_iB| - |B| \cdot |Bs_iB|\right)$$

$$= q-1$$

when $l(s_iw) = l(w)-1$, we get $l(s_i \cdot s_iw) = l(s_iw)+1$,

$$T_i * T_w = T_i * T_i * T_{s_iw}$$

$$= (qT_{Id} + (q-1)T_i) * T_{s_iw}$$

$$= qT_{s_iw} + (q-1)T_w$$

$$\Rightarrow \quad T_i * T_w = \begin{cases} T_{s_iw} & l(s_iw) = l(w)+1 \\ qT_{s_iw} + (q-1)T_w & l(s_iw) = l(w)-1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$\mathcal{H}(G,B) = \mathbb{Z}\langle T_1, \dots, T_{n-1}\rangle_{alg}$ with relations $\quad (\mathcal{H}(G,B) \cong \mathcal{H}_q(W))$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \qquad \text{for } |i-j| \geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

E.g. For $n=2$, $\mathcal{H}(G,B) \cong \mathbb{Z}[T_1] / (T_1^2-(q-1)T_1-q)$

$$\cong \mathbb{Z}[T_1] / (T_1-q)(T_1+1)$$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1$$

For $n=3$, $\mathcal{H}(G,B) \cong \mathbb{Z}\langle T_1, T_2 \rangle \Big/ \Big((T_1-q)(T_1+1), (T_2-q)(T_2+1), T_1T_2T_1 = T_2T_1T_2\Big)$

$$\overset{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_1T_2 \oplus \mathbb{Z}T_2T_1 \oplus \mathbb{Z}T_1T_2T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z}T_1 \oplus \mathbb{Z}T_2 \oplus \mathbb{Z}T_{(123)} \oplus \mathbb{Z}T_{(132)} \oplus \mathbb{Z}T_{(13)}$$

global Cartan decomposition

1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2\times 2}(R) = \coprod_{(b)\subseteq(a)} GL_2(R)\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(R)$$

Cor

$$M_{2\times 2}(\mathbb{Z}) = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0\le a\le b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det\neq 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\le b}} GL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2\times 2}(\mathbb{Z})_{\det> 0} = \coprod_{\substack{a\mid b\in\mathbb{Z}\\ 0< a\le b}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a,b\in\mathbb{Q}^\times_{>0}\\ v_p(a)\le v_p(b)\ \forall p}} SL_2(\mathbb{Z})\begin{pmatrix} a & \\ & b\end{pmatrix} SL_2(\mathbb{Z}) \qquad GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det>0}$$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$T^- = \left\{ \begin{pmatrix} a & \\ & b\end{pmatrix}\in GL_2^+(\mathbb{Q}) \;\middle|\; \begin{array}{l} a,b>0 \\ v_p(a)\le v_p(b) \quad \forall p \text{ prime}\end{array}\right\} \overset{\text{Grp}}{\cong} \mathbb{Q}^\times_{>0}\times(\mathbb{Z}_{>0},\times)$$

then

$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha\in T^-} \Gamma\alpha\Gamma$$

Ex. Verify that $\Gamma\alpha\Gamma/\Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix}\alpha_1 & \\ & \alpha_2\end{pmatrix}\in T^-$

Hint. See [WWL, 引理5.1.4].

$$\#\Gamma\alpha\Gamma/\Gamma = \#\Gamma/\Gamma\cap\alpha\Gamma\alpha^{-1} = \#\Gamma/\Gamma_0\!\left(\tfrac{\alpha_2}{\alpha_1}\right) = \#\mathbb{P}^1\!\left(\tfrac{\alpha_2}{\alpha_1}\right) = \frac{\alpha_2}{\alpha_1}\prod_{p\mid\frac{\alpha_2}{\alpha_1}}\left(1+\frac{1}{p}\right)$$

$$\left[\alpha\begin{pmatrix} a & b\\ c & d\end{pmatrix}\alpha^{-1} = \begin{pmatrix} a & \frac{\alpha_1}{\alpha_2}b\\ \frac{\alpha_2}{\alpha_1}c & d\end{pmatrix} \quad\Rightarrow\quad \Gamma\cap\alpha\Gamma\alpha^{-1} = \begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ \frac{\alpha_2}{\alpha_1}\mathbb{Z} & \mathbb{Z}\end{pmatrix}_{\det=1} = \Gamma_0\!\left(\tfrac{\alpha_2}{\alpha_1}\right)\right]$$

e.g. $\#\Gamma\begin{pmatrix} d & 0\\ 0 & d\end{pmatrix}\Gamma/\Gamma = 1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p\end{pmatrix}\Gamma/\Gamma = p+1$, $\#\Gamma\begin{pmatrix} 1 & 0\\ 0 & p^e\end{pmatrix}\Gamma/\Gamma = p^e+p^{e-1}$

The desired measure can not be realized here, i.e.,

a Haar measure μ on $GL_2^+(\mathbb{Q})$ s.t. $\mu(\Gamma)=1$.

Reason: measure satisfies countable additivity, and Γ is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure $\rightsquigarrow$ semimeasure)

Toy e.g.: There is no way to define a Haar measure μ on $\mathbb{Q}$ s.t. $\mu(\mathbb{Z})=1$.

However, if we only require finite additivity, we can do it:

Def (Semimeasure on $\mathbb{Q}$)

For any periodic set $X \subseteq \mathbb{Q}$ (i.e., $\exists m \in \mathbb{Q}_{>0}$ s.t. $m+X=X$)

we set

$$\mu(X) := \tfrac{1}{m}\,|X/m\mathbb{Z}| = \tfrac{1}{m}\,|X \cap [0,m)|$$

Rmk. 1. $\mathbb{Z} \supset \mathbb{Z}\cap m\mathbb{Z} \subset m\mathbb{Z}$ $\quad |\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}|,\ |m\mathbb{Z}/\mathbb{Z}\cap m\mathbb{Z}| < +\infty$

"$m\mathbb{Z}$ are all commensurable gps of $\mathbb{Z}$"

2. $X = \bigsqcup_{\alpha\in\Lambda} \alpha + m\mathbb{Z}$ for some $\Lambda \subseteq \mathbb{Q}/m\mathbb{Z}$

"X is a commensurable set of $\mathbb{Z}$ (when $\mu(X)<+\infty$)"

Long A: Def. (Semimeasure on $GL_2^+(\mathbb{Q})$)

For any gp $H \leq GL_2^+(\mathbb{Q})$ which is commensurable with Γ

(i.e., $\#H/H\cap\Gamma$, $\#\Gamma/H\cap\Gamma$ are finite), set

$$\mu(H) = \frac{|H/H\cap\Gamma|}{|\Gamma/H\cap\Gamma|} \overset{\text{if } H\leq\Gamma}{=\!=} \frac{1}{|\Gamma/H|} \in \mathbb{Q}_{>0}$$

Similarly we can specify μ to any commensurable set $X \subseteq GL_2^+(\mathbb{Q})$.

$$\left(\begin{array}{lll} \text{i.e.,} \quad X = \bigsqcup_{\alpha\in\Lambda} \alpha H & \text{for some} & H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ \phantom{\text{i.e.,}} \quad X = \bigsqcup_{\alpha\in\Lambda'} H'\alpha & & \Lambda \subseteq GL_2^+(\mathbb{Q})/H,\ \Lambda' \subseteq H\backslash GL_2^+(\mathbb{Q}) \\ & & \Lambda, \Lambda' \text{ finite} \end{array}\right)$$

Rmk: In the most of references the terminology (semi)measure is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and take the second line as a def of the convolution.

Def. (Hecke alg $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma)$)

$$\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \to \mathbb{Z} \;\middle|\; \begin{array}{l} f(\gamma_1\alpha\gamma_2) = f(\alpha) \quad \forall \gamma_1,\gamma_2\in\Gamma, \alpha\in GL_2^+(\mathbb{Q}) \\ \#\,(\operatorname{supp} f/\Gamma) < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x)\, f_2(x^{-1}g)\, d\mu(x)$$

$$= \sum_{x\in GL_2^+(\mathbb{Q})/\Gamma} f_1(x)\, f_2(x^{-1}g) = \sum_{y\in\Gamma\backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1})\, f_2(y)$$

2. $\mathbb{Z}$-mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha \in T^-} \mathbb{Z} \cdot \mathbb{1}_{\Gamma\alpha\Gamma}$$

denote $T_\alpha := \mathbb{1}_{\Gamma\alpha\Gamma}$

$\lambda \in \mathbb{Q}^\times$ $\quad R_\lambda := T_{\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)\Gamma} = \mathbb{1}_{\lambda\Gamma}$ $\quad$ ($R_1 = \mathbb{1}_\Gamma$ is the unit of $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$)

p prime, $e \geq 1$ $\quad T_{p^e} := T_{\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p^e\end{smallmatrix}\right)\Gamma}$ $\quad$ $T_p := T_{\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)} = \mathbb{1}_{\Gamma\left(\begin{smallmatrix}1 & \\ & p\end{smallmatrix}\right)\Gamma}$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T^-} (T_\alpha * T_\beta)(\gamma)\, T_\gamma$$

$$g_{\alpha\beta}^{\gamma} := (T_\alpha * T_\beta)(\gamma) = \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma)$$

$$= \# \left\{ x \in GL_2^+(\mathbb{Q})/\Gamma \;\middle|\; \begin{matrix} x \in \Gamma\alpha\Gamma \\ x^{-1}\gamma \in \Gamma\beta\Gamma \end{matrix} \right\}$$

$$= \left| \Gamma\alpha\Gamma \cap \gamma\Gamma\beta^{-1}\Gamma/\Gamma \right|$$

e.p. $\quad \mathbb{1}_\Gamma * f = f \qquad (R_\lambda * f)(g) = f(\lambda^{-1} g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$

$\qquad\qquad\qquad\qquad R_\lambda * R_\mu = R_{\lambda\mu}$

E.g. $\quad g_{\alpha\beta}^{\gamma} \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \qquad$ where $|\alpha| := \det \alpha$

The formula above is still not feasible for effective calculation. We will derived the easiest way to comute $g_{\alpha\beta}^{\gamma}$ in the next page.

Suppose $\Gamma\alpha\Gamma/\Gamma = \{x_1\Gamma, \cdots, x_i\Gamma, \cdots\}$
$\Gamma\beta\Gamma/\Gamma = \{y_1\Gamma, \cdots, y_j\Gamma, \cdots\}$

then

$$
\begin{aligned}
g_{\alpha\beta}^{\gamma} &= \sum_{x\in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x)\, T_\beta(x^{-1}\gamma) \\
&= \sum_i T_\beta(x_i^{-1}\gamma) \\
&= \sum_i \mathbb{1}_{x_i^{-1}\gamma\in\Gamma\beta\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i^{-1}\gamma\in y_j\Gamma} \\
&= \sum_i \sum_j \mathbb{1}_{x_i y_j\Gamma=\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{x_i y_j\in\Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma x_i y_j\Gamma=\Gamma\gamma\Gamma} \\
&= \frac{1}{|\Gamma\gamma\Gamma/\Gamma|} \sum_i \sum_j \mathbb{1}_{\Gamma\alpha y_j\Gamma=\Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|} \sum_{y\in\Gamma\beta\Gamma/\Gamma} \mathbb{1}_{\Gamma\alpha y\Gamma=\Gamma\gamma\Gamma} \\
&= \frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma'\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}
\end{aligned}
$$

where

$$\Gamma' := \{\gamma'\in\Gamma \mid \alpha\gamma'\beta\in\Gamma\gamma\Gamma\} = \alpha^{-1}\Gamma\gamma\Gamma\beta^{-1}\cap\Gamma$$
$$\Rightarrow \Gamma'\beta\Gamma/\Gamma = \alpha^{-1}\Gamma\gamma\Gamma \cap \Gamma\beta\Gamma/\Gamma$$

depends on α, β, γ.

The rest is a routine work.

Ex. $\Gamma\begin{pmatrix}1&\\&m\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1&\\&n\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1&\\&mn\end{pmatrix}\Gamma$ $\quad (m,n)=1$

$\Gamma\begin{pmatrix}1&\\&p^e\end{pmatrix}\Gamma \cdot \Gamma\begin{pmatrix}1&\\&p\end{pmatrix}\Gamma = \Gamma\begin{pmatrix}1&\\&p^{e+1}\end{pmatrix}\Gamma \sqcup \Gamma\begin{pmatrix}p&\\&p^e\end{pmatrix}\Gamma$ $\quad p$ prime, $e\geq 1$

[Hint. $\begin{pmatrix}1&\\&m\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&\\&n\end{pmatrix} = \begin{pmatrix}a&nb\\mc&mnd\end{pmatrix} \in \Gamma\begin{pmatrix}l&\\&\frac{mn}{l}\end{pmatrix}\Gamma$ for $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$, $l=\gcd(a, nb, mc, mnd)$]

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z}\cdot T_{mn} & (m,n)=1 \\ T_{p^e} * T_p \in \mathbb{Z}\cdot T_{p^{e+1}} + \mathbb{Z}\, T_{p^{e-1}} R_p & p \text{ prime}, e\geq 1 \end{cases}$$

Computation of coefficient:

when $(m,n)=1$, $\alpha=\begin{pmatrix}1&\\&m\end{pmatrix}$, $\beta=\begin{pmatrix}1&\\&n\end{pmatrix}$, $\gamma=\begin{pmatrix}1&\\&mn\end{pmatrix}$.

$$g_{\alpha\beta}^{\gamma}=\frac{1}{|\Gamma\gamma\Gamma/\Gamma|}\sum_i\sum_j \mathbb{1}_{x_iy_j\in\Gamma\gamma\Gamma}$$
$$=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma\beta\Gamma/\Gamma|}{|\Gamma\gamma\Gamma/\Gamma|}$$
$$=1$$

when p is prime, $e\geq 1$, $\alpha=\begin{pmatrix}1&\\&p^e\end{pmatrix}$, $\beta=\begin{pmatrix}1&\\&p\end{pmatrix}$, $\gamma_2=\begin{pmatrix}p&\\&p^e\end{pmatrix}$, $\gamma_1=\begin{pmatrix}1&\\&p^{e+1}\end{pmatrix}$,

$$\Gamma_2'\triangleq\{\gamma'\in\Gamma\mid \alpha\gamma'\beta\in\Gamma\gamma_2\Gamma\}$$
$$=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\;\middle|\;\gcd(a,pb,p^ec,p^{e+1}d)=p\right\}$$
$$=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma\;\middle|\;\begin{pmatrix}a&b\\c&d\end{pmatrix}\equiv\begin{pmatrix}0&*\\ *&*\end{pmatrix}\bmod p\right\}$$
$$=\Gamma^0(p)\begin{pmatrix}&1\\-1&\end{pmatrix}$$

$$|\Gamma_2'\beta\Gamma/\Gamma|=\left|\Gamma^0(p)\begin{pmatrix}&p\\-1&\end{pmatrix}\Gamma/\Gamma\right|$$
$$=|\Gamma^0(p)|\Big/\left|\begin{pmatrix}&p\\-1&\end{pmatrix}\Gamma^0(p)\begin{pmatrix}&p\\-1&\end{pmatrix}^{-1}\cap\Gamma^0(p)\right|$$
$$=|\Gamma^0(p)|/|\Gamma^0(p)|$$
$$=1$$

$$\left[\begin{aligned}\begin{pmatrix}&p\\-1&\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}&p\\-1&\end{pmatrix}^{-1}&=\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&\\&p\end{pmatrix}^{-1}\begin{pmatrix}&1\\-1&\end{pmatrix}^{-1}\\&=\begin{pmatrix}&1\\-1&\end{pmatrix}\begin{pmatrix}a&\frac{1}{p}b\\pc&d\end{pmatrix}\begin{pmatrix}&1\\-1&\end{pmatrix}^{-1}\\&=\begin{pmatrix}pc&d\\-a&-\frac{1}{p}b\end{pmatrix}\begin{pmatrix}&-1\\1&\end{pmatrix}\\&=\begin{pmatrix}d&-pc\\-\frac{1}{p}b&a\end{pmatrix}\end{aligned}\right]$$

$\therefore$

$$g_{\alpha\beta}^{\gamma_2}=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_2'\beta\Gamma/\Gamma|}{|\Gamma\gamma_2\Gamma/\Gamma|}$$
$$=\frac{(p^e-p^{e-1})\cdot 1}{p^{e-1}-p^{e-2}}$$
$$=p$$

$$g_{\alpha\beta}^{\gamma_1}=\frac{|\Gamma\alpha\Gamma/\Gamma|\,|\Gamma_1'\beta\Gamma/\Gamma|}{|\Gamma\gamma_1\Gamma/\Gamma|}$$
$$=\frac{|\Gamma\alpha\Gamma/\Gamma|\left(|\Gamma\beta\Gamma/\Gamma|-|\Gamma_2'\beta\Gamma/\Gamma|\right)}{|\Gamma\gamma_1\Gamma/\Gamma|}$$
$$=\frac{(p^e-p^{e-1})\cdot(p+1-1)}{p^{e+1}-p^e}$$
$$=1$$

4. Conclusion. $\mathcal{H}(GL_2^+(\mathbb{Q}),\Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$

with $\begin{cases} T_m T_n = T_{mn} & (m,n)=1 \\ T_{p^e} T_p = T_{p^{e+1}} + p\, T_{p^{e-1}} R_p & p \text{ prime},\ e \geq 1 \end{cases}$