

Eine Woche, ein Beispiel

9.4 Hecke algebra for matrix groups

This document is not finished. I need some time to digest and restate them.

I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.

main reference:

[Bump][<http://sporadic.stanford.edu/bump/math263/hecke.pdf>]

[XiongHecke][<https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf>]

All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

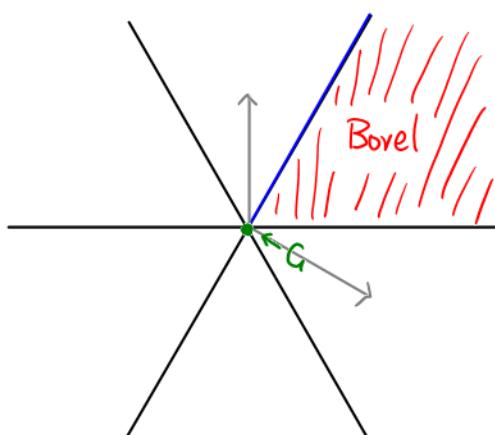
Task. For each double coset decomposition, we want to do.

1. decomposition ($\Gamma \backslash G / \Gamma$ is finite & definition of Hecke alg)
2. \mathbb{Z} -mod structure, notation
3. alg structure
4. conclusion

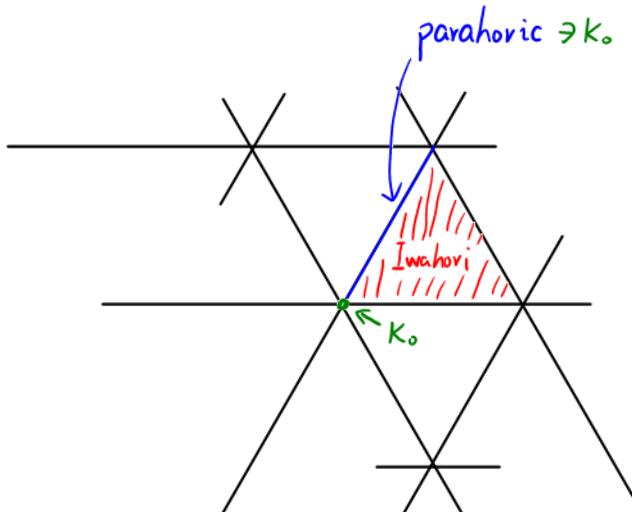
<https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of>

	Bruhat	Iwahori	Cartan
F finite	$G = \bigsqcup_{w \in W} B_w B$	affine Bruhat	Smith normal form
F local	$G = \bigsqcup_{w \in W_{\text{ext}}} B_w B$	$G = \bigsqcup_{w \in W_{\text{ext}}} I_w I$	$G = \bigsqcup_{\alpha \in T^-} K_\alpha \alpha K_\alpha$
F global	$G = \bigsqcup_{w \in W} B_w B$		$GL_2(\mathbb{Q}) = \bigsqcup_{\alpha \in T} \Gamma \alpha \Gamma$
adèle?			

parabolic $\ni G$



parahoric $\ni K_0$



$$B = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ p & p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & p & p \\ p & 0 & 0 \\ p & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & p^{-1} \\ 0 & p & 0 \\ p & p & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & p & 0 \end{pmatrix}$$

$$\begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{Q}) \Rightarrow \begin{pmatrix} p & 1 \\ 0 & 1 \end{pmatrix} \notin I$$

<https://mathoverflow.net/questions/4547/definitions-of-hecke-alg>

<https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebra>

For the classical group case, see: <https://math.stackexchange.com/questions/3068424/iwahori-versus-bruhat-decompositions>

Tip: Those matrix decomposition theorems may seem quite frightening in the beginning.
 In fact, they are just fancy special cases of Gaussian elimination.
 Remember,

左乘行变换，右乘列变换

multiply a matrix on the left hand side is equiv. to do row operations,
 multiply a matrix on the right hand side is equiv. to do column operations.

The canonical form usually has entries 0 almost everywhere.
 To compute the canonical form, we use allowed row/column operations.

E.g. $G = \bigcup_{w \in W} B w B$ $g \sim g' \Leftrightarrow \exists b_1, b_2 \in B \text{ s.t. } g = b_1 g' b_2$
 e.g. $q = 7, G = GL_3(\mathbb{F}_7)$

$$\begin{pmatrix} 5 & 1 & 6 \\ 0 & 2 & 4 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} -1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 5 & 1 & 6 \\ 1 & 5 & 3 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 5 & 3 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & -5 & -3 \\ & 1 & 1 \\ & & 1 \end{smallmatrix} \right)} \begin{pmatrix} 0 & 4 & 5 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix}$$

$$\xrightarrow{\left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 4 & 3 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 6 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right)} \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\left(\begin{smallmatrix} 4 & & \\ & 1 & \\ & & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

E.g. $G = \bigcup_{w \in W_{\text{ext}}} I w I$ $g \sim g' \Leftrightarrow \exists x_1, x_2 \in I \text{ s.t. } g = x_1 g' x_2$

e.g. $q = 3$ $G = GL_2(\mathbb{Q}_3)$ $I = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \subset GL_2(\mathbb{Z}_3)$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} -3 & 1 \\ & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 1 & \frac{1}{3} \\ -3 & 0 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & \\ & -3 \end{smallmatrix} \right)} \begin{pmatrix} 0 & \frac{1}{3} \\ -3 & 0 \end{pmatrix} \xrightarrow{\left(\begin{smallmatrix} 1 & -1 \\ & 1 \end{smallmatrix} \right) \times} \begin{pmatrix} 3 & 3^{-1} \end{pmatrix}$$

e.g. $q = 3$ $G = GL_3(\mathbb{Q}_3)$ $I = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & \mathbb{Z}_3 & \mathbb{Z}_3 \\ 3\mathbb{Z}_3 & 3\mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \subset GL_3(\mathbb{Z}_3)$

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{7}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & \frac{1}{2} & -\frac{7}{2} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 2 & \frac{1}{2} & 3 \\ 3 & \frac{1}{2} & 2 \\ 3 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} \frac{1}{2} & 0 & \frac{5}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 \\ 0 & 0 & -6 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -6 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 1 & 0 \end{pmatrix}$$

To show the disjointness (i.e. canonical form does not depend on the process), we observe that some properties of $k \times k$ -minors are preserved under the restricted row/column operations. For $k=1$ these invariants can be seen easier.

$$\text{E.g. } G = \bigsqcup_{w \in W} BwB$$

$$\xrightarrow{\text{not } 0} \begin{pmatrix} * & & \\ \vdots & & \\ * & * & \\ \vdots & & \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix} \sim \begin{pmatrix} 0 & & * & & \\ \vdots & 0 & \cdots & \cdots & 0 \\ 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & * & & \\ 0 & & & & \end{pmatrix}$$

$$G = \bigsqcup_{\lambda \in T} K_\lambda K_\lambda$$

$$A = (a_{ij})_{i,j=1}^n \quad e = \underbrace{\min_{i,j} v(a_{ij})}_{\sim} \quad \begin{pmatrix} \pi^e & & 0 \\ & \ddots & \\ 0 & \ddots & \ddots \end{pmatrix}$$

$$G = \bigsqcup_{w \in W_{\text{ext}}} I_w I$$

$$A = (a_{ij})_{i,j=1}^n \quad e = \underbrace{\min_{i,j} v(a_{ij})}_{\sim} \quad i_0 \begin{pmatrix} * & & * & & \\ & 0 & & & \\ & \vdots & \pi^e & \cdots & 0 \\ * & & 0 & & * \\ & & & & \end{pmatrix}$$

$$I = \{(i,j) \mid v(a_{ij}) = e\}$$

$(i_0, j_0) \in I$ is in the lower left corner

$$\text{i.e. } \forall (i,j) \in I, \quad \left. \begin{array}{l} i \geq i_0 \\ j \leq j_0 \end{array} \right\} \Rightarrow (i,j) = (i_0, j_0)$$

If you have no clue on the properties of $k \times k$ -minors for $k > 1$, you can see [Bump, Section 9] for the case of p -adic Cartan decomposition.

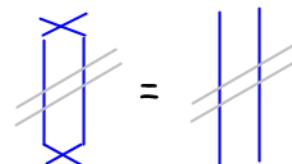
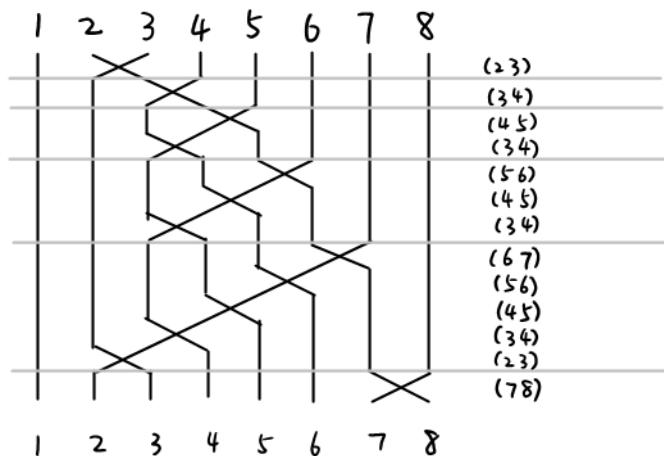
S_n and Tits system

A brief preparation for computations in Bruhat decomposition. $s_i = (i \ i+1)$, $1 \leq i \leq n-1$

$$\text{E.g. } n=8, w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8.$$

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.



$$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$$

$l(w_0) = 13$ = "inversion number"

$$l(s_1 w_0) = 14 \quad l(w_0 s_1) = 14$$

$$l(s_2 w_0) = 12 \quad l(w_0 s_2) = 12$$

$$l(s_3 w_0) = 14 \quad l(w_0 s_3) = 14$$

$$l(s_4 w_0) = 12 \quad l(w_0 s_4) = 12$$

$$l(s_5 w_0) = 12 \quad l(w_0 s_5) = 12$$

$$l(s_6 w_0) = 12 \quad l(w_0 s_6) = 14$$

$$l(s_7 w_0) = 14 \quad l(w_0 s_7) = 12$$

Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & 0 \\ 0 & 0 & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & \cdots & 0 \\ 0 & \ddots & * \\ 0 & 0 & * \end{pmatrix} \leq B$,
 $w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0) + 1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0) - 1 \end{cases}$$

Solution

$$\begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix} \quad w_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

The following computation will be also computed later on.

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0 \cap w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0Bw_0^{-1}$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad B \cap w_0Bw_0^{-1}$$

finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \leq B$,
 $w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} BwB$

Ex. $(BwB)^{-1} = Bw^{-1}B$ but $BwB \cdot Bw^{-1}B \neq B$ is possible

Ex. Compute $|BwB/B|$ ∇BwB may not be a group!

Hint: Consider the map

$$\phi: B \longrightarrow BwB/B$$

$$b \mapsto b w B$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 w B = b_2 w B$$

$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$

$$\Leftrightarrow b_2^{-1} b_1 \in w B w^{-1}$$

$$\therefore |BwB/B| = |B| / |wBw^{-1} \cap B| = q^{\ell(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G, B) = \{f: G \rightarrow \mathbb{Z} \mid f(b_1 g b_2) = f(g) \quad \forall b_1, b_2 \in B, g \in G\}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(G, B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbf{1}_{BwB} \stackrel{\oplus n!}{\sim} \mathbb{Z}$$

Denote $T_w := \mathbf{1}_{BwB}$, $T_{s_i} := T_{s_i}$ ($T_{Id} = \mathbf{1}_B$ is the unit of $\mathcal{H}(G, B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G, B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) T_w$$

$$\begin{aligned} (T_u * T_v)(w) &= \frac{1}{|B|} \sum_{y, z \in w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} \left| \{(y, z) \in B_u B \times B_v B \mid yz = w\} \right| \text{ if } w \in B_u B \cap B_v B \\ &= \frac{1}{|B|} |B_u B \cap u B_v^{-1} B| \end{aligned}$$

$$B_{S_i}B \cdot B_{W_0}B = \begin{cases} B_{S_i}w_0B & l(S_iw) = l(w) + 1 \\ B_{S_i}w_0B \cup B_{W_0}B & l(S_iw) = l(w) - 1 \end{cases}$$

$$\Rightarrow T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{S_i w} & l(S_i w) = l(w) + 1 \\ \mathbb{Z} \cdot T_{S_i w} + \mathbb{Z} \cdot T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Computation of coefficient.

$$|B_{W_0}B| = |B_{W_0}B/B| \times |B| = q^{l(w)} \cdot |B|$$

when $l(S_i w) = l(w) + 1$,

$$(T_i * T_w)(S_i w) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz = S_i w \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz \in B_{S_i}wB \right\}$$

$$= \frac{|B_{S_i}B| |B_{W_0}B|}{|B| |B_{S_i}wB|} = \frac{q^{l(S_i)} q^{l(w)}}{q^{l(S_i w)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = Id \right\}$$

$$= \frac{1}{|B|} |B_{S_i}B| = q$$

$$(T_i * T_i)(S_i) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = S_i \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B_{S_i}B \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left(|B_{S_i}B \times B_{S_i}B| - \left| \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B \right\} \right| \right)$$

$$= \frac{1}{|B| |B_{S_i}B|} \left(|B_{S_i}B| |B_{S_i}B| - |B| \cdot |B_{S_i}B| \right)$$

$$= q - 1$$

when $l(S_i w) = l(w) - 1$, we get $l(S_i \cdot S_i w) = l(S_i w) + 1$,

$$T_i * T_w = T_i * T_i * T_{S_i w}$$

$$= (qT_{Id} + (q-1)T_i) * T_{S_i w}$$

$$= qT_{S_i w} + (q-1)T_w$$

$$\Rightarrow T_i * T_w = \begin{cases} T_{S_i w} & l(S_i w) = l(w) + 1 \\ qT_{S_i w} + (q-1)T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$$H(G, B) = \mathbb{Z}\langle T_1, \dots, T_{n-1} \rangle_{alg} \text{ with relations } (H(G, B) \subseteq H_q(W))$$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j| \geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

$$\text{E.g. For } n=2, \quad \mathcal{H}(G, B) \cong \mathbb{Z}[T_1] / (T_1^2 - (q-1)T_1 - q)$$

$$\cong \mathbb{Z}[T_1] / (T_1 - q)(T_1 + 1)$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1$$

$$\text{For } n=3, \quad \mathcal{H}(G, B) \cong \mathbb{Z}\langle T_1, T_2 \rangle / ((T_1 - q)(T_1 + 1), (T_2 - q)(T_2 + 1), T_1 T_2 T_1 = T_2 T_1 T_2)$$

$$\stackrel{\mathbb{Z}\text{-mod}}{=} \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_1 T_2 \oplus \mathbb{Z} T_2 T_1 \oplus \mathbb{Z} T_1 T_2 T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_{(12)} \oplus \mathbb{Z} T_{(13)} \oplus \mathbb{Z} T_{(123)}$$

global Cartan decomposition
1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2 \times 2}(R) = \coprod_{(b) \subseteq (a)} GL_2(R) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(R)$$

$$\text{Cor } M_{2 \times 2}(\mathbb{Z}) = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 \leq a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det \neq 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det > 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a, b \in \mathbb{Q}_{>0}^\times \\ v_p(a) \leq v_p(b) \quad \forall p}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det > 0}$$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$\Gamma^- = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \in GL_2^+(\mathbb{Q}) \mid \begin{array}{l} a, b > 0 \\ v_p(a) \leq v_p(b) \quad \forall p \text{ prime} \end{array} \right\} \stackrel{\text{Grp}}{\cong} \mathbb{Q}_{>0}^\times \times (\mathbb{Z}_{>0}, \times)$$

then

$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha \in \Gamma^-} \Gamma \alpha \Gamma$$

Ex. Verify that $\Gamma \alpha \Gamma / \Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \in \Gamma^-$

Hint. See [WWL, 引理 5.1.4].

$$\# \Gamma \alpha \Gamma / \Gamma = \# \Gamma / \Gamma \cap \alpha \Gamma \alpha^{-1} = \# \Gamma / \Gamma_0 \left(\frac{\alpha_1}{\alpha_2} \right) = \# \text{Irr} \left(\frac{\alpha_1}{\alpha_2} \right) = \frac{\alpha_2}{\alpha_1} \prod_{p \mid \frac{\alpha_1}{\alpha_2}} \left(1 + \frac{1}{p} \right)$$

$$\left[\alpha \begin{pmatrix} a & \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} a & \frac{a_1}{a_2} b \\ \frac{c}{a_2} c & d \end{pmatrix} \right] \Rightarrow \Gamma \cap \alpha \Gamma \alpha^{-1} = \left(\frac{\mathbb{Z}}{\alpha_1 \mathbb{Z}}, \frac{\mathbb{Z}}{\alpha_2 \mathbb{Z}} \right)_{\det=1} = \Gamma_0 \left(\frac{\alpha_2}{\alpha_1} \right)$$

$$\text{e.g. } \# \Gamma \begin{pmatrix} \alpha_1 & \\ 0 & \alpha_2 \end{pmatrix} \Gamma / \Gamma = 1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma / \Gamma = p+1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix} \Gamma / \Gamma = p^e + p^{e-1}$$

The desired measure can not be realized here, i.e.,

a Haar measure μ on $GL_2^+(\mathbb{Q})$ s.t. $\mu(\Gamma) = 1$.

Reason: measure satisfies countable additivity, and Γ is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure \rightsquigarrow semimeasure)

To e.g.: There is no way to define a Haar measure μ on \mathbb{Q} s.t. $\mu(\mathbb{Z}) = 1$.

However, if we only require finite additivity, we can do it.

Def (Semimeasure on \mathbb{Q})

For any periodic set $X \subseteq \mathbb{Q}$ (i.e., $\exists m \in \mathbb{Q}_{>0}$ s.t. $m + X = X$)
we set

$$\text{Rmk. 1. } \mu(X) = \frac{1}{m} |X/m\mathbb{Z}| = \frac{1}{m} |X \cap [0, m]|$$

$$\mathbb{Z} \supset \frac{X}{m\mathbb{Z}} \quad |X/m\mathbb{Z}|, |m\mathbb{Z}/m\mathbb{Z}| < +\infty$$

" $m\mathbb{Z}$ are all commensurable gps of \mathbb{Z} "

2. $X = \bigsqcup_{\alpha \in \Delta} \alpha + m\mathbb{Z}$ for some $\Delta \subseteq \mathbb{Q}/m\mathbb{Z}$

" X is a commensurable set of \mathbb{Z} (when $\mu(X) < +\infty$)"

Long A: Def. (Semimeasure on $GL_2^+(\mathbb{Q})$)

For any gp $H \leq GL_2^+(\mathbb{Q})$ which is commensurable with Γ

(i.e., $\#H/\mathbb{H}\cap\Gamma, \#\Gamma/\mathbb{H}\cap\Gamma$ are finite), set

$$\mu(H) = \frac{|H/\mathbb{H}\cap\Gamma|}{|\Gamma/\mathbb{H}\cap\Gamma|} \stackrel{\text{if } H \leq \Gamma}{=} \frac{1}{|\Gamma/H|} \in \mathbb{Q}_{>0}$$

Similarly we can specify μ to any commensurable set $X \subseteq GL_2^+(\mathbb{Q})$.

$$\left(\begin{array}{l} \text{i.e., } X = \bigsqcup_{\alpha \in \Delta} \alpha H \text{ for some } H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ X = \bigsqcup_{\alpha \in \Delta'} H' \alpha' \quad \Delta \subseteq GL_2^+(\mathbb{Q})/H, \Delta' \subseteq H' \backslash GL_2^+(\mathbb{Q}) \\ \Delta, \Delta' \text{ finite} \end{array} \right)$$

Rmk: In the most of references the terminology (semi)measure
is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and
take the second line as a def of the convolution.

Def. (Hecke alg $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$)

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \rightarrow \mathbb{Z} \mid \begin{array}{l} f(\gamma_1 \alpha \gamma_2) = f(\alpha) \quad \forall \gamma_1, \gamma_2 \in \Gamma, \alpha \in GL_2^+(\mathbb{Q}) \\ \#(\text{supp } f)/\Gamma < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} f_1(x) f_2(x^{-1}g) = \sum_{y \in \Gamma \backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1}) f_2(y)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha \in \Gamma} \mathbb{Z} \cdot \mathbf{1}_{\Gamma \alpha \Gamma}$$

denote $T_\alpha := \mathbf{1}_{\Gamma \alpha \Gamma}$

$$\begin{aligned} \lambda \in \mathbb{Q}^\times & \quad R_\lambda := T_{(\lambda)} = \mathbf{1}_{\Gamma(\lambda)} \Gamma = \mathbf{1}_{\lambda \Gamma} \quad (R_1 = \mathbf{1}_\Gamma \text{ is the unit of } \mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)) \\ p \text{ prime, } e \geq 1 & \quad T_{p^e} := T_{(p^e)} = \mathbf{1}_{\Gamma(p^e)} \Gamma \quad T_p := T_{(p)} = \mathbf{1}_{\Gamma(p)} \Gamma \end{aligned}$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in \Gamma} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha \beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x) T_\beta(x^{-1} \gamma) \\ &= \# \left\{ x \in GL_2^+(\mathbb{Q})/\Gamma \mid \begin{array}{l} x \in \Gamma \alpha \Gamma \\ x^{-1} \gamma \in \Gamma \beta \Gamma \end{array} \right\} \\ &= |\Gamma \alpha \Gamma \cap \gamma \Gamma \beta^{-1} \Gamma / \Gamma| \end{aligned}$$

$$\text{e.p. } \mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1} g) = f(g \lambda^{-1}) = (f * R_\lambda)(g)$$

$$R_\lambda * R_\mu = R_{\lambda \mu}$$

$$\text{E.g. } g_{\alpha \beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad \text{where } |\alpha|_+ = \det \alpha$$

The formula above is still not feasible for effective calculation.
We will derived the easiest way to compute $g_{\alpha \beta}^\gamma$ in the next page.

$$\text{Suppose } \Gamma_\alpha \Gamma / \Gamma = \{x_1 \Gamma, \dots, x_i \Gamma, \dots\}$$

$$\Gamma_\beta \Gamma / \Gamma = \{y_1 \Gamma, \dots, y_j \Gamma, \dots\}$$

then

$$\begin{aligned} g_{\alpha\beta} &= \sum_{x \in \Gamma_\alpha \cap \Gamma_\beta} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \sum_i T_\beta(x_i^{-1}\gamma) \\ &= \sum_i \mathbf{1}_{x_i^{-1}\gamma \in \Gamma_\beta \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i^{-1}\gamma \in y_j \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{\Gamma_\alpha y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_{y \in \Gamma_\beta \Gamma / \Gamma} \mathbf{1}_{\Gamma_\alpha y \Gamma = \gamma \Gamma} \\ &= \frac{|\Gamma_\alpha \Gamma / \Gamma|}{|\Gamma_\beta \Gamma / \Gamma|} \end{aligned}$$

where

$$\Gamma' := \{\gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_\beta \Gamma\} = \alpha^{-1} \Gamma_\beta \Gamma \beta^{-1} \cap \Gamma$$

$$\Rightarrow \Gamma'_\beta \Gamma / \Gamma = \alpha^{-1} \Gamma_\beta \Gamma \cap \Gamma \beta^{-1} / \Gamma$$

depends on α, β, γ .

The rest is a routine work.

$$\text{Ex. } \Gamma('m) \Gamma \cdot \Gamma('n) \Gamma = \Gamma('_{mn}) \Gamma \quad (m, n) = 1$$

$$\Gamma('_{p^e}) \Gamma \cdot \Gamma('_p) \Gamma = \Gamma('_{p^{e+1}}) \Gamma \sqcup \Gamma('_{p^e}) \Gamma \quad p \text{ prime, } e \geq 1$$

[Hint. $('m)(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})('n) = (\begin{smallmatrix} a & nb \\ mc & mnd \end{smallmatrix}) \in \Gamma('_{\frac{m}{l}}) \Gamma \quad \text{for } (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z})$]

$$l = \gcd(a, nb, mc, mnd)$$

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z} \cdot T_{mn} \\ T_{p^e} * T_p \in \mathbb{Z} \cdot T_{p^{e+1}} + \mathbb{Z} T_{p^{e-1}} R_p \end{cases} \quad (m, n) = 1$$

$$p \text{ prime, } e \geq 1$$

Computation of coefficient:

when $(m, n) = 1$, $\alpha = (1_m)$, $\beta = (1_n)$, $\gamma = (1_{mn})$,

$$\begin{aligned} g_{\alpha\beta}^{\gamma} &= \frac{1}{|\Gamma_{\gamma}\Gamma/\Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \in \Gamma_{\gamma}\Gamma} \\ &= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma_{\beta}\Gamma/\Gamma|}{|\Gamma_{\gamma}\Gamma/\Gamma|} \\ &= 1 \end{aligned}$$

when p is prime, $e \geq 1$, $\alpha = (1_{p^e})$, $\beta = (1_p)$, $\gamma_2 = (p_{p^e})$, $\gamma_1 = (1_{p^{e+1}})$,

$$\begin{aligned} \Gamma'_2 &\triangleq \left\{ \gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_{\gamma_2} \Gamma \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \gcd(a, pb, p^e c, p^{e+1} d) = p \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix} \right) \pmod{p} \right\} \\ &= \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ |\Gamma'_2 \beta \Gamma/\Gamma| &= \left| \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma/\Gamma \right| \\ &= \left| \Gamma^0(p) \right| / \left| \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \cap \Gamma^0(p) \right| \\ &= \left| \Gamma^0(p) \right| / \left| \Gamma^0(p) \right| \end{aligned}$$

$$\begin{aligned} \left[\begin{aligned} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left(\begin{smallmatrix} pc & b \\ a & pd \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ &= \left(\begin{smallmatrix} d & -pc \\ -pb & a \end{smallmatrix} \right) \end{aligned} \right] \end{aligned}$$

$$\therefore g_{\alpha\beta}^{\gamma_2} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_2 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_2} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot 1}{p^{e-1} - p^{e-2}}$$

$$g_{\alpha\beta}^{\gamma_1} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_1 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| (|\Gamma_{\beta}\Gamma/\Gamma| - |\Gamma'_1 \beta \Gamma/\Gamma|)}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot (p+1 - 1)}{p^{e+1} - p^e}$$

$$= 1$$

$$4. \text{ Conclusion.} \quad H(GL_2^+(\mathbb{Q}), \Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$$

with $\begin{cases} T_m T_n = T_{mn} \\ T_p^e T_p = T_{p^{e+1}} + p T_{p^{e-1}} R_p \end{cases}$

$(m, n) = 1$
 $p \text{ prime, } e \geq 1$

Task: generalize it to other congruence subgps.

p -adic Cartan decomposition / not Grothendieck group!
Set $G = GL_2(F)$, $K_0 = GL_2(\mathcal{O}_F)$.

1. decomposition [Bump Prop 35]

$$M_{2 \times 2}(\mathcal{O}_F) = \coprod_{0 \leq e_i \leq e_2 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$M_{2 \times 2}(\mathcal{O}_F)_{\text{det} \neq 0} = \coprod_{0 \leq e_1 \leq e_2 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$G = \coprod_{e_1 \leq e_2} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

Denote $T^- = \left\{ \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in G \mid e_1 \leq e_2, e_1, e_2 \in \mathbb{Z} \right\} \stackrel{\text{semi gp}}{\cong} \mathbb{Z} \oplus \mathbb{Z}_{\geq 0}$, then
 $G = \coprod_{\alpha \in T^-} K_0 \alpha K_0$

Ex. Verify that $K_0 \alpha K_0 / K_0$ is finite, and compute the order. $\alpha = \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in T^-$

Hint.

$$\# K_0 \alpha K_0 / K_0 = \# K_0 / K_0 \cap \alpha K_0 \alpha^{-1} = \# K_0 / \Gamma_0(\mathfrak{p}_F^{e_2 - e_1}) = \# \text{IP}'(\mathcal{O}_F / \mathfrak{p}_F^{e_2 - e_1}) = \begin{cases} q^{e_2 - e_1} + q^{e_2 - e_1 - 1} & e_1 < e_2 \\ 1 & e_1 = e_2 \end{cases}$$

$$\left[\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O}^{e_1, e_2} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} \Rightarrow K_0 \cap \alpha K_0 \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} = \Gamma_0(\mathfrak{p}_F^{e_2 - e_1}) \right]$$

$$\text{e.g. } \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = 1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q+1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q^e + q^{e-1}$$

Here we use the similar notation in modular form

[https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/5.moduli_interpretation.pdf]:

$$\begin{array}{ccc} \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \{ \text{Id} \} \\ \cap & & \cap \\ \text{bal. } \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & N(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & B(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathcal{O}) = K^0 = GL_2(\mathcal{O}_F) & \xrightarrow{\quad} & GL_2(\mathcal{O}_F / \mathfrak{p}_F^e) \end{array}$$

Take the unique Haar measure on G s.t. $\mu(K^0) = 1$, then

$$\mu(K^0 \alpha K^0) = \# K^0 \alpha K^0 / K^0$$

μ is induced from the measure on coset G / K^0 .

The Hecke algebra has been defined here:

https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.04.17_preliminary_facts_of_reps_of_p-adic_groups.pdf

We still recall the convolution here:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in G/K_0} f_1(x) f_2(x^{-1}g) = \sum_{y \in K_0 \backslash G} f_1(gy^{-1}) f_2(y)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(G, K_0) = \bigoplus_{\alpha \in T} \mathbb{Z} \cdot \mathbf{1}_{K_0 \alpha K_0}$$

denote $T_\alpha := \mathbf{1}_{K_0 \alpha K_0}$

$$\begin{array}{lll} \lambda \in F^\times & R_\lambda := T_{(\lambda)} = \mathbf{1}_{K_0(\lambda) K_0} = \mathbf{1}_{\lambda K_0} & (R_1 = \mathbf{1}_{K_0} \text{ is the unit of } \mathcal{H}(G, K_0)) \\ \epsilon \geq 1 & T_\pi^\epsilon := T_{(\pi^\epsilon)} = \mathbf{1}_{K_0(\pi^\epsilon) K_0} & T_\pi := T_{(\pi)} = \mathbf{1}_{K_0(\pi) K_0} \end{array}$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/K_0} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \# \left\{ x \in G/K_0 \mid \begin{array}{l} x \in K_0 \alpha K_0 \\ x^{-1}\gamma \in K_0 \beta K_0 \end{array} \right\} \\ &= |K_0 \alpha K_0 \cap K_0 \beta^{-1} K_0| / |K_0| \end{aligned}$$

$$\text{e.p. } \mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$$

$$R_\lambda * R_\mu = R_{\lambda\mu}$$

$$\text{E.g. } g_{\alpha\beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad \text{where } |\alpha| := \det \alpha$$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^\gamma = \frac{|K_0 \alpha K_0| / |K_0 \beta K_0|}{|K_0 \gamma K_0| / |K_0|}$$

where

$$K_0' := \{\gamma' \in K_0 \mid \alpha \gamma' \beta \in K_0 \gamma K_0\} = \alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0$$

$$\Rightarrow |K_0' \beta K_0| / |K_0| = |\alpha^{-1} K_0 \gamma K_0 \beta^{-1}| / |K_0|$$

depends on α, β, γ .

$$\Rightarrow T_{\pi^\epsilon} T_\pi = T_{\pi^{\epsilon+1}} + q T_{\pi^{\epsilon+1}} R_\pi$$

4. Conclusion (Tamagawa, Satake)

$$\mathcal{H}(G, K^\circ) = \mathbb{Z} [R_\pi^{\pm 1}, T_\pi] \quad \text{with}$$
$$T_\pi e T_\pi = T_\pi^{e+1} + q T_\pi^{e-1} \cdot R_\pi$$

$$\text{e.p. } \mathcal{H}(GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z})) = \bigotimes_{\mathbb{Z}} \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$$

p-adic Iwahori decomposition

We only consider GL_n, SL_n, PGL_n in this section. SL_2 case is special focused.

$$\text{E.g. } G = SL_2(F) \quad T(F) = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad I = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}_{\det=1}$$

$$U(F) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \quad U(F) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$$

$$U(O_F) = \begin{pmatrix} 1 & 0_F \\ 0 & 1 \end{pmatrix} \quad U(O_F) = \begin{pmatrix} 1 & 0 \\ 0_F & 1 \end{pmatrix}$$

$$N_{T(F)}(G)/T(F)_{\circ} = W_f$$

$$N_{T(F)}(G)/T(F)_{\circ} \cong \langle s_1, s_0 \rangle / (s_1^2 = s_0^2 = 1) \quad s_1 = \begin{pmatrix} 1 & \\ -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & \pi \\ -\pi^{-1} & \pi \end{pmatrix}$$

	\triangle root char weight	\triangle^\vee coroot cochar coweight	
My notation	$\emptyset \subseteq Q \subseteq X^* \subseteq P$	$\emptyset^\vee \subseteq Q^\vee \subseteq X_r \subseteq P^\vee$	$W_f \quad W_{aff} \quad W_{ext} \quad \text{in dual}$
[Ginzburg]	$R \subseteq Q \subseteq P$	$R^\vee \subseteq Q^\vee \subseteq P^\vee$	$W \quad W_{aff} \quad \text{not in dual}$
[Williams]	$R \subseteq \subseteq X$	$R^\vee \subseteq \subseteq X^\vee$	$W_f \quad W \quad W_{ext}$
[Bump] ₇	$\emptyset \subseteq Q \subseteq \subseteq P$	$\emptyset^\vee \subseteq Q^\vee \subseteq \subseteq P^\vee$	$W \quad W_{aff} \quad \tilde{W}_{aff} \text{ not in dual}$

Iwahori Hecke algebra to P^\vee , not the standard $H(G, I)$

$$\begin{array}{ccccccc} 0 & \longrightarrow & Q^\vee & \longrightarrow & W_{aff} & \xleftarrow{\quad} & W_f \longrightarrow 0 \\ & & \cap & & \Delta & & \\ 0 & \longrightarrow & X_r(T) & \longrightarrow & W_{ext} & \xleftarrow[\text{pr}]{} & W_f \longrightarrow 0 \end{array}$$

\Rightarrow Standard isomorphism

$$W_{aff} \cong Q^\vee \rtimes W_f$$

$$W_{ext} \cong X_r(T) \rtimes W_f$$

Q: How does W_{ext} act on

$$\pi_*(T) = X_*(T) \cong T(F)/T(O_F) \cong \mathbb{Z}^{\text{rank } G} \subseteq \mathbb{Z}^n . ?$$

$$\left[\lambda : t \mapsto \begin{pmatrix} t^{e_1} \\ \vdots \\ t^{e_n} \end{pmatrix} \right] \leftrightarrow \begin{pmatrix} \pi^{e_1} & & \\ & \ddots & \\ & & \pi^{e_n} \end{pmatrix} \leftrightarrow \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$$

A: Denote

$$\text{pr}: W_{\text{ext}} \longrightarrow W_f \quad \begin{pmatrix} * & * & * \\ * & * & * \\ -1 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$s: W_f \longleftrightarrow W_{\text{ext}} \quad \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} \mapsto \begin{pmatrix} * & * & * \\ * & * & * \\ -1 & 1 & 1 \end{pmatrix}$$

This is canonical embedding! We will omit s to simplify the formula.

Then, W_{ext} acts on $X_*(T)$ by

$$\begin{aligned} W_{\text{ext}} \times X_*(T) &\longrightarrow X_*(T) \\ N_G(T(F)) / T(O_F) \times T(F) / T(O_F) &\longrightarrow T(F) / T(O_F) \\ (w, \lambda) &\mapsto w \cdot \lambda = \underbrace{w \lambda s(\text{pr}(w))}_{\substack{w = \mu x u \\ \text{in } \mathbb{Z}^n \\ x \rightarrow +}} \underbrace{\mu u \lambda u^{-1}}_{\mu + u \lambda u^{-1}} \end{aligned} \quad \mu \in X_*(T), u \in W_f$$

So roughly speaking, W_{ext} acts on $X_*(T)$ by a rotation/flip followed with a translation.

We will denote $w = t_\mu u$ when we view $w \in W_{\text{ext}}$ as operators on $X_*(T)$.

$\checkmark w \neq r_{\alpha, k}$ in many cases.

Recall that for $\alpha \in X^*(T)$, $k \in \mathbb{Z}$, we can define

$$\begin{aligned} \psi &= \alpha + k : X_*(T) \rightarrow \mathbb{R} & \lambda &\mapsto \langle \lambda, \alpha + k \rangle = \langle \lambda, \alpha \rangle + k \\ r_\psi &= r_{\alpha, k} : X_*(T) \rightarrow X_*(T) & \lambda &\mapsto \lambda - \langle \lambda, \alpha + k \rangle \alpha^\vee \\ &&&= \lambda - \langle \lambda, \alpha \rangle \alpha^\vee - k \alpha^\vee \end{aligned}$$

$$\begin{aligned} H_\psi &= H_{\alpha, k} = \{ \lambda \in X_*(T) \mid r_{\alpha, k}(\lambda) = \lambda \} \\ &= \{ \lambda \in X_*(T) \mid \langle \lambda, \psi \rangle = 0 \} \subseteq X_*(T) \end{aligned}$$

The set

$$\Psi := \{ \psi : X_*(T) \rightarrow \mathbb{R} \mid \psi = \alpha + k \text{ for some } \alpha \in \Phi, k \in \mathbb{Z} \}$$

is an affine root system, and W_{ext} acts on Ψ by

$$W_{\text{ext}} \times \Psi \rightarrow \Psi \quad (w, \psi) \mapsto \psi \circ w^{-1}$$

E.g. For $G = SL_2$,

$$\text{e.p. } \begin{pmatrix} \pi^e & \\ -\pi^{-e} & \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ -\lambda_1 - e \end{pmatrix}$$

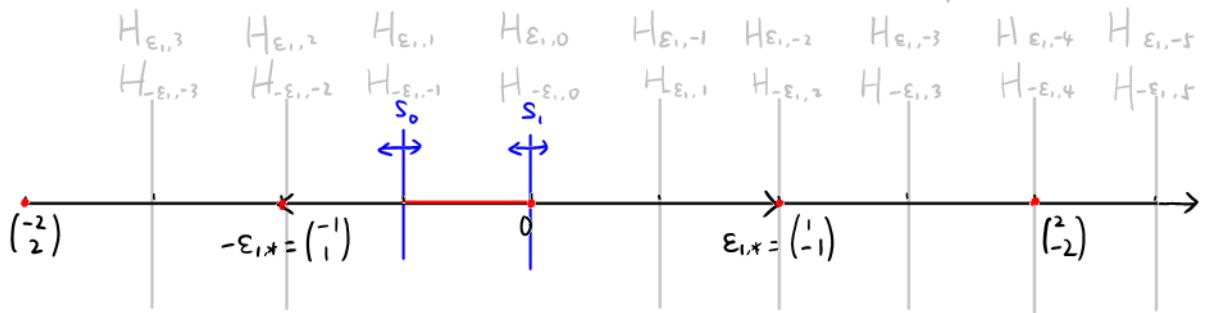
Sign is useless

$$\begin{pmatrix} -\pi^{-e} & \pi^e \\ -\pi & \pi^{-e} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ -\lambda_1 \end{pmatrix} = \begin{pmatrix} -\lambda_1 + e \\ \lambda_1 - e \end{pmatrix}$$

Let $s_i = \begin{pmatrix} \lambda_1 & \\ -\lambda_1 & \end{pmatrix}$, $s_o = \begin{pmatrix} \pi^e & \\ -\pi^{-e} & \end{pmatrix}$, then

$$s_i s_o = \begin{pmatrix} \pi^e & \\ -\pi^{-e} & \end{pmatrix} = \varepsilon_{i,*} \cdot \text{Id}$$

Never mind symbol! $\begin{pmatrix} \lambda_1 & \\ -\lambda_1 & \end{pmatrix} \in T(\mathcal{O}_F)$



shortest matrix	$s_o s_i s_o$	$s_o s_i$	s_o	Id	s_i	$s_i s_o$	$s_i s_o s_i$	$s_i s_o s_i s_o$	$s_i s_o s_s s_i$
	$(-\pi^e \pi^{-e})$	$(\pi^{-e} \pi)$	$(-\pi \pi^{-e})$	$(1 \ 1)$	$(-1 \ 1)$	$(\pi \pi^{-e})$	$(-\pi^{-e} \pi)$	$(\pi^e \pi^{-e})$	$(-\pi^{-2} \pi^2)$
	$(s_i s_o) s_i$	$(s_i s_o)^{-1}$	$(s_i s_o)^{-1} s_i$	Id	s_i	$s_i s_o$	$(s_i s_o) s_i$	$(s_i s_o)^2$	$(s_i s_o)^2 s_i$
operator	$t_{-2\varepsilon_1, s_i}$	$t_{-\varepsilon_1, *}$	$t_{-\varepsilon_1, s_i}$	Id	s_i	$t_{\varepsilon_1, *}$	$t_{\varepsilon_1, *} s_i$	$t_{2\varepsilon_1, *}$	$t_{2\varepsilon_1, *} s_i$
reflections	$r_{\varepsilon_1, 2}$		$r_{\varepsilon_1, 1}$		$r_{\varepsilon_1, 0}$		$r_{\varepsilon_1, -1}$		$r_{\varepsilon_1, -2}$
	$r_{-2\varepsilon_1}$		$r_{-\varepsilon_1, -1}$		$r_{-\varepsilon_1, 0}$		$r_{-\varepsilon_1, 1}$		$r_{-\varepsilon_1, 2}$
length	3	2	1	0	1	2	3	4	5
								$\bullet = \bullet \subseteq \text{I}$	
								$Q^\vee = X_* \subseteq P^\vee$	

E.g. For $G = \mathrm{PGL}_2$,

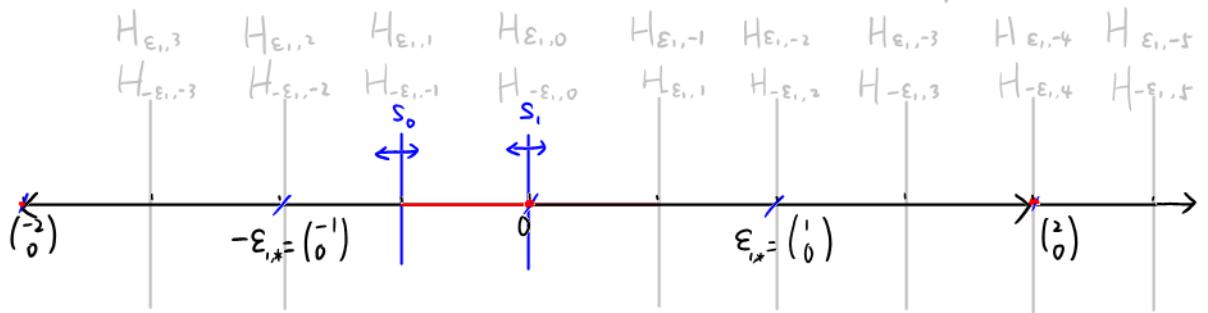
$$\begin{pmatrix} \pi^e & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + e \\ \lambda_2 \end{pmatrix}$$

e.p. $\begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix}$

$$\begin{pmatrix} -\pi^e & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + e \end{pmatrix}$$

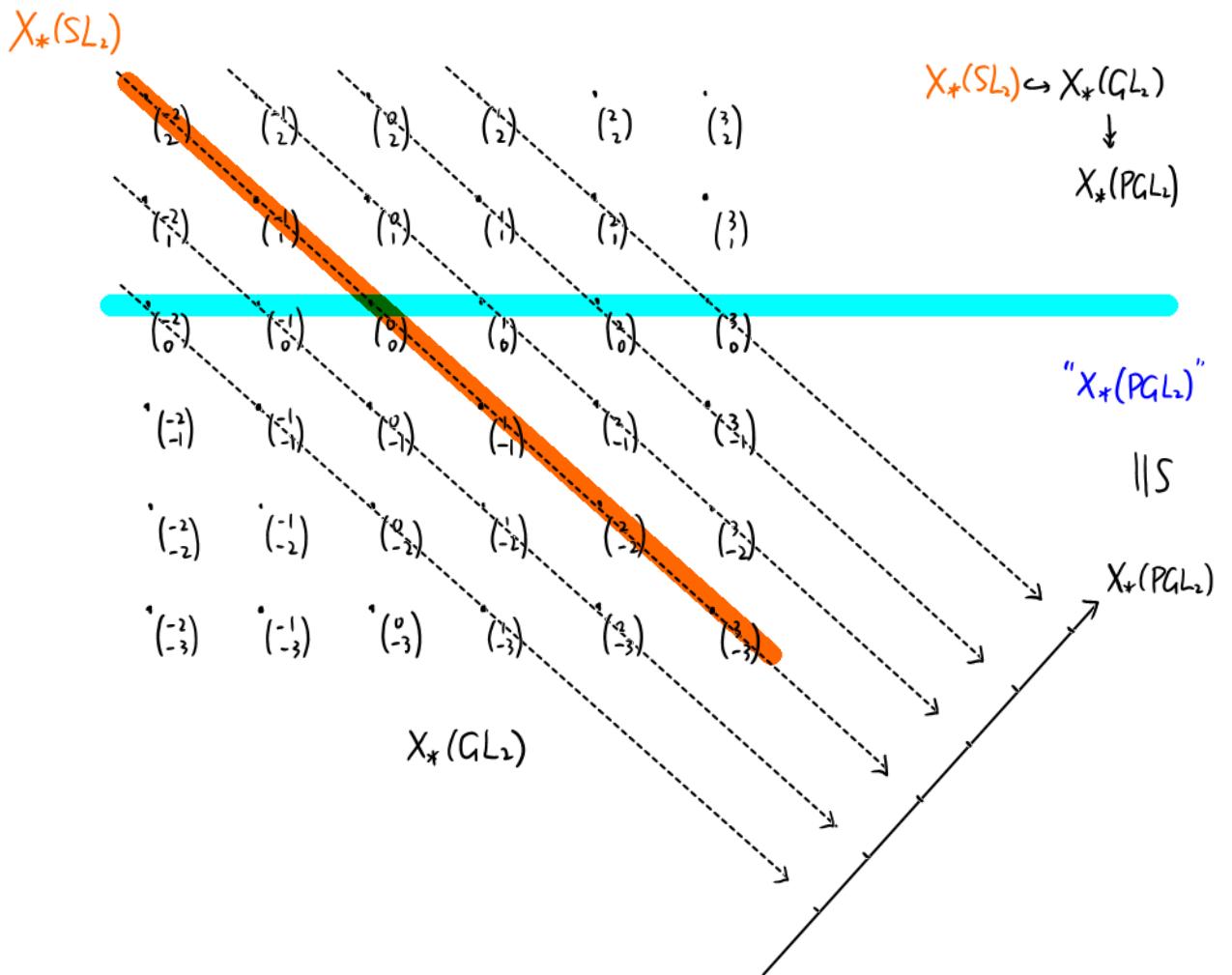
$$\begin{pmatrix} -\pi & \\ & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 + 1 \end{pmatrix}$$

Let $s_i = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$, $s_o = \begin{pmatrix} -\pi & \\ & 1 \end{pmatrix}$, then $s_i s_o = \begin{pmatrix} \pi & \\ & 1 \end{pmatrix} = \varepsilon_i^* \cdot \mathrm{Id}$
 Never mind symbol! $(^{-1} -1) \in T(\mathcal{O}_F)$



shortest matrix	$s_o s_i s_o$	$s_o s_i$	s_o	Id	s_i	$s_i s_o$	$s_i s_o s_i$	$s_i s_o s_i s_o$	$s_i s_o s_s s_i$
	$\begin{pmatrix} -\pi^e & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi^{-1} \end{pmatrix}$	$\begin{pmatrix} -\pi^{-1} & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & \pi^{-2} \end{pmatrix}$	$\begin{pmatrix} -\pi^{-2} & \\ & 1 \end{pmatrix}$
	$(s_i s_o)^2 s_i$	$(s_i s_o)^{-1}$	$(s_i s_o)^1 s_i$	Id	s_i	$s_i s_o$	$(s_i s_o) s_i$	$(s_i s_o)^2$	$(s_i s_o)^{-1} s_i$
operator	$t_{-\varepsilon_{i,*}} s_i$	$t_{-\varepsilon_{i,*}}$	$t_{-\varepsilon_{i,*}} s_i$	Id	s_i	$t_{\varepsilon_{i,*}}$	$t_{\varepsilon_{i,*}} s_i$	$t_{2\varepsilon_{i,*}}$	$t_{2\varepsilon_{i,*}} s_i$
reflections	$r_{\varepsilon_{i,*} 2}$		$r_{\varepsilon_{i,*} 1}$		$r_{\varepsilon_{i,*} 0}$		$r_{\varepsilon_{i,*} -1}$		$r_{\varepsilon_{i,*} -2}$
length	3	2	1	0	1	2	3	4	5
								$\bullet \subseteq / = /$	
								$Q^\vee \subseteq X_* = P^\vee$	

A vivid picture describing relationships among $X_*(GL_2)$, $X_*(SL_2)$, $X_*(PSL_2)$



In both cases (SL_2 & PGL_2), $W_{\text{ext}} = X_* \sqcup X_*$'s, as set.
We will focus on the case $G = SL_2$.

$$\begin{aligned} 1. \text{ decomposition. } G &= \bigsqcup_{w \in W_{\text{ext}}} IwI \\ &= \bigsqcup_{e \in \mathbb{Z}} I(\pi^e \pi^{-e})I \sqcup \bigsqcup_{e \in \mathbb{Z}} I(-\pi^{-e} \pi^e)I \\ &= \bigsqcup_{e \in \mathbb{Z}} I(s, s_0)^e I \sqcup \bigsqcup_{e \in \mathbb{Z}} I(s, s_0)^e s, I \end{aligned}$$

Ex. Verify that IwI/I is finite, and compute the order.

Hint. $\# IwI/I = \# I/I \cap wIw^{-1}$ $\# I/I \cap wIw^{-1}$

When $w = (\pi^e \pi^{-e})$, $wIw^{-1} = \begin{pmatrix} \pi^e & \pi^{-e} \\ \pi^{-e} & \pi^e \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \begin{pmatrix} \pi^{-e} & \pi^e \\ \pi^e & \pi^{-e} \end{pmatrix}$

$$= \begin{pmatrix} 0 & p^{-e} \\ p^{e-1} & 0 \end{pmatrix}$$

$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} & e \geq 0 \\ \begin{pmatrix} 0 & 0 \\ p^{1-e} & 0 \end{pmatrix} & e < 0 \end{cases}$$
 q^{2e}
 q^{-2e}

When $w = (-\pi^{-e} \pi^e)$, $wIw^{-1} = \begin{pmatrix} -\pi^{-e} \pi^e & 0 \\ 0 & \pi^{-e} \pi^e \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \begin{pmatrix} -\pi^{-e} \pi^e & 0 \\ 0 & \pi^{-e} \pi^e \end{pmatrix}$

$$= \begin{pmatrix} 0 & p^{2e+1} \\ p^{-e} & 0 \end{pmatrix}$$

$$I \cap wIw^{-1} = \begin{cases} \begin{pmatrix} 0 & p^{2e+1} \\ p^{-e} & 0 \end{pmatrix} & e \geq 0 \\ \begin{pmatrix} 0 & 0 \\ p^{-e} & 0 \end{pmatrix} & e < 0 \end{cases}$$
 q^{2e+1}
 q^{-2e-1}
 q

In conclusion,

$$\# IwI/I = \# I/I \cap wIw^{-1} = q^{l(w)}$$

□

Take the unique Haar measure on G s.t. $\mu(I) = 1$, then
 $\mu(IwI) = \# IwI/I = q^{l(w)}$
 μ is induced from the measure on coset G/I .

2. \mathbb{Z} -mod structure, notation.

$$\mathcal{H}(G, I) = \bigoplus_{w \in W_{\text{ext}}} \mathbb{Z} \cdot \mathbf{1}_{IwI}$$

denote $T_w = \mathbf{1}_{IwI}$ $T_{Id} = \mathbf{1}_I$ is the unit of $\mathcal{H}(G, I)$

$$\forall e \in \mathbb{Z} \quad T_{s_e} := T_s, \quad T_{s_0} := T_{s_0}$$

$$S_{(s_e)} = \overline{T_{(\pi^e \pi^{-e})}} = T_{(s, s_0)^e} = T_{(s, s_0)^e} \quad \text{The general notation is } S_\lambda, \text{ for } \lambda \in X_*(T)$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in W_{\text{ext}}} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/I} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \# \left\{ x \in G/I \mid \begin{array}{l} x \in I\alpha I \\ x^{-1}\gamma \in I\beta I \end{array} \right\} \\ &= |I\alpha I \cap \gamma I\beta^{-1} I|_I \end{aligned}$$

e.p. $\mathbf{1}_I * f = f * \mathbf{1}_I = f$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^\gamma = \frac{|I\alpha I|_I |I\alpha\beta^{-1} I|_I}{|I\gamma I|_I}$$

where

$$\begin{aligned} I_{\alpha\beta}^\gamma &:= \{ w' \in I \mid \alpha w' \beta \in I\gamma I \} = \alpha^{-1} I \gamma I \beta^{-1} \cap I \\ &\Rightarrow I_{\alpha\beta}^\gamma I/I = \alpha^{-1} I \gamma I \cap I \beta^{-1} I/I \end{aligned}$$

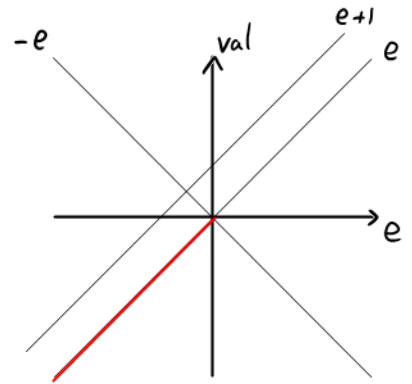
depends on α, β, γ .

In the following computation, the minus sign is not important, so we ignore it.

E.g. When we write $(1')$, we actually mean $(-1')$.

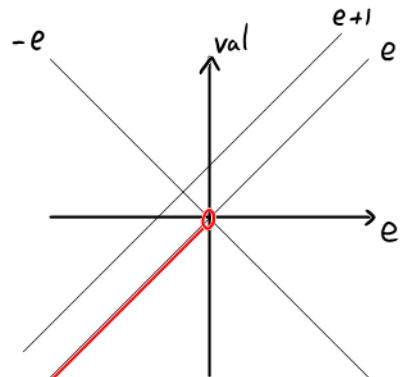
$$(1') \left(\begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} \pi^e & \\ & \pi^{-e} \end{smallmatrix} \right) = \left(\begin{smallmatrix} \pi^{e+1} c & \pi^e d \\ \pi^e a & \pi^{-e} b \end{smallmatrix} \right)$$

$\left(\begin{smallmatrix} \pi^e & \pi^{-e} \\ \pi^e & \end{smallmatrix} \right)$	$e \leq 0$
$\left(\begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^e \end{smallmatrix} \right)$	$e > 0, \text{val}(b) = 0$
$\left(\begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^e \end{smallmatrix} \right)$	$e > 0, \text{val}(b) > 0$



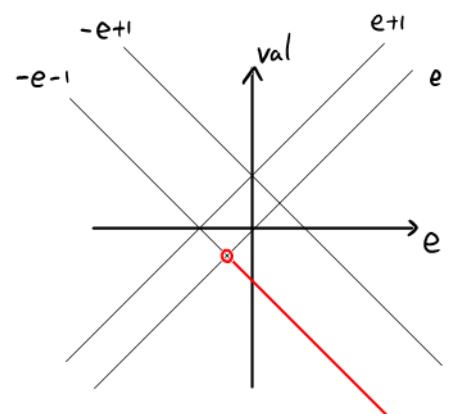
$$(1') \left(\begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} & \pi^e \\ \pi^{-e} & \end{smallmatrix} \right) = \left(\begin{smallmatrix} \pi^e d & \pi^{e+1} c \\ \pi^{-e} b & \pi^e a \end{smallmatrix} \right)$$

$\left(\begin{smallmatrix} \pi^{-e} & \pi^e \\ \pi^e & \end{smallmatrix} \right)$	$e < 0$
$\left(\begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^e \end{smallmatrix} \right)$	$e \geq 0, \text{val}(b) = 0$
$\left(\begin{smallmatrix} \pi^{-e} & \pi^e \\ & \pi^e \end{smallmatrix} \right)$	$e \geq 0, \text{val}(b) > 0$



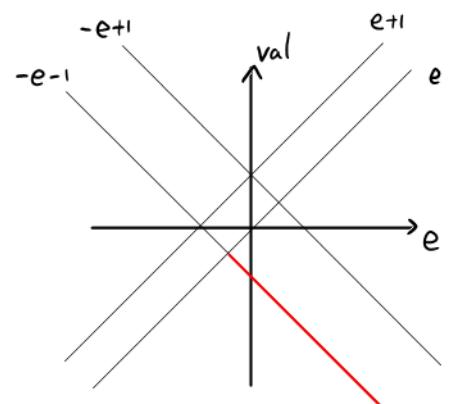
$$(\pi^{-1}) \left(\begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} \pi^e & \\ & \pi^{-e} \end{smallmatrix} \right) = \left(\begin{smallmatrix} \pi^e c & \pi^{e-1} d \\ \pi^{-e+1} a & \pi^{-e+1} b \end{smallmatrix} \right)$$

$\left(\begin{smallmatrix} \pi^{e+1} & \pi^{-e-1} \\ \pi^e & \end{smallmatrix} \right)$	$e \geq 0$
$\left(\begin{smallmatrix} \pi^e & \pi^{-e} \\ & \pi^{-e-1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) = 0$
$\left(\begin{smallmatrix} \pi^e & \pi^{-e-1} \\ & \pi^{e+1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) > 0$



$$(\pi^{-1}) \left(\begin{smallmatrix} a & b \\ \pi^c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} & \pi^e \\ \pi^{-e} & \end{smallmatrix} \right) = \left(\begin{smallmatrix} \pi^{e-1} d & \pi^e c \\ \pi^{-e+1} b & \pi^{e+1} a \end{smallmatrix} \right)$$

$\left(\begin{smallmatrix} \pi^{-e-1} & \pi^{e+1} \\ \pi^{-e} & \end{smallmatrix} \right)$	$e \geq 0$
$\left(\begin{smallmatrix} \pi^{-e-1} & \pi^{e+1} \\ & \pi^e \end{smallmatrix} \right)$	$e < 0, \text{val}(c) = 0$
$\left(\begin{smallmatrix} \pi^{-e-1} & \pi^{e+1} \\ & \pi^{e+1} \end{smallmatrix} \right)$	$e < 0, \text{val}(c) > 0$



In conclusion,

$$T_i * T_\omega \in \left\{ \begin{array}{l} \mathbb{Z} T_{s_i \omega} \\ \mathbb{Z} T_{s_i \omega} + \mathbb{Z} T_\omega \end{array} \right.$$

$$l(s_i \omega) = l(\omega) + 1$$

$$l(s_i \omega) = l(\omega) - 1$$

In the following computation, $\alpha = s_i$, $\beta = w$, $\gamma = s_i w$ or $\gamma = w$.

When $l(s_i w) = l(w) + 1$,

$$g_{s_i, w}^{s_i w} = \frac{|I s_i I/I| |I w I/I|}{|I s_i w I/I|}$$

$$= \frac{q^{l(s_i)} q^{l(w)}}{q^{l(s_i w)}}$$

$$= 1$$

When $l(s_i w) = l(w) - 1$ and $i = 1$, $w = (\pi^e \pi^{-e})$, $e > 0$ or $w = (\pi^{-e} \pi^e)$, $e \geq 0$

$$I_{s_i, w}^{s_i w} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I_{s_i, w}^w = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$\begin{aligned} \# I_{s_i, w}^{s_i w} w I/I &= \# I_{\alpha \beta}^\gamma / (I_{\alpha \beta}^\gamma \cap w I w^{-1}) & \gamma = s_i w \\ &= \# I_{\alpha \beta}^\gamma / (I \cap w I w^{-1}) \\ &= \frac{\# I/I \cap w I w^{-1}}{\# I/I_{\alpha \beta}^\gamma} \\ &= q^{l(w)-1} \\ g_{s_i, w}^{s_i w} &= \frac{|I s_i I/I| |I_{s_i, w}^{s_i w} w I/I|}{|I s_i w I/I|} \\ &= \frac{q^{l(s_i)} q^{l(w)-1}}{q^{l(s_i w)}} \\ &= q \\ g_{s_i, w}^w &= \frac{|I s_i I/I| |I_{s_i, w}^w w I/I|}{|I w I/I|} \\ &= \frac{|I s_i I/I| (|I w I/I| - |I_{s_i, w}^{s_i w} w I/I|)}{|I w I/I|} \\ &= \frac{q^{l(s_i)} (q^{l(w)} - q^{l(w)-1})}{q^{l(w)}} \\ &= q - 1 \end{aligned}$$

When $l(s_i w) = l(w) - 1$ and $i=0$, $w = (\pi^e \pi^{-e})$, $e < 0$ or $w = (\pi^{-e} \pi^e)$, $e < 0$

$$I_{s_i w}^{s_i w} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p^2 & \mathcal{O} \end{pmatrix}, \quad I_{s_i, w}^w = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ p-p^2 & \mathcal{O} \end{pmatrix}.$$

$$\begin{aligned} \# I_{s_i w}^{s_i w} wI/I &= \# I_{\alpha\beta}^\gamma / (I_{\alpha\beta}^\gamma \cap wIw^{-1}) & \gamma = s_i w \\ &= \# I_{\alpha\beta}^\gamma / (I \cap wIw^{-1}) \\ &= \frac{\# I/I \cap wIw^{-1}}{\# I/I_{\alpha\beta}^\gamma} \\ &= q^{l(w)-1} \\ g_{s_i w}^{s_i w} &= \frac{|I_{s_i} I/I| |I_{s_i w}^{s_i w} wI/I|}{|I_{s_i w} I/I|} \\ &= \frac{q^{l(s_i)} q^{l(w)-1}}{q^{l(s_i w)}} \\ &= q \\ g_{s_i w}^w &= \frac{|I_{s_i} I/I| |I_{s_i, w}^w wI/I|}{|I_w I/I|} \\ &= \frac{|I_{s_i} I/I| (|I_w I/I| - |I_{s_i, w}^{s_i w} wI/I|)}{|I_w I/I|} \\ &= \frac{q^{l(s_i)} (q^{l(w)} - q^{l(w)-1})}{q^{l(w)}} \\ &= q-1 \end{aligned}$$

In conclusion,

$$T_i * T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w) + 1 \\ q T_{s_i w} + (q-1) T_w & l(s_i w) = l(w) - 1 \end{cases}$$

4. Conclusion.

$$\mathcal{H}(G, I) = \mathbb{Z}\{T_0, T_1\}$$

with

$$T_i * T_w = \begin{cases} T_{s_i w} & l(s_i w) = l(w) + 1 \\ q T_{s_i w} + (q-1) T_w & l(s_i w) = l(w) - 1 \end{cases}$$