§2.2. Character of torus

Global case

Hecke charater

Notation
$$T = \mathrm{Res}_{F/\mathbb{Q}}\, \mathbb{G}_{m,F}$$
$$T(\mathbb{Q}) = F^\times$$
$$T(F) = \mathbb{G}_m(F\otimes_{\mathbb{Q}} F) \cong \prod_{n \text{ many}} F^\times$$
$$T(\mathbb{R}) = \prod_{\tau: F\hookrightarrow\mathbb{R}} \mathbb{R}^\times \prod_{\substack{(\tau,c\tau):\\ F\hookrightarrow\mathbb{C}}} \mathbb{C}^\times \quad \hat{=}\; F_\infty^\times$$
$$T(\mathbb{Q}_p) = \prod_{v|p} F_v^\times \quad \hat{=}\; F_p^\times$$
$$T(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_F^\times$$

For F^{nc} := normal closure of F in $\overline{\mathbb{Q}}/\mathbb{Q}$, (normal closure)
$$T_{F^{nc}} = \prod_{\tau: F\hookrightarrow F^{nc}} \mathbb{G}_{m,F^{nc}} \qquad T(F^{nc}) = \prod_{\tau: F\hookrightarrow F^{nc}} F^{nc,\times}$$

$$X^*(T) = \mathrm{Hom}(T_{F^{nc}}, \mathbb{G}_{m,F^{nc}}) \cong \bigoplus_{\tau: F\hookrightarrow F^{nc}} \mathbb{Z}[\tau] \circlearrowleft \Gamma_F$$
$$\sigma.[\tau] = [\sigma\circ\tau]$$

We can rewrite
$$F^\times \backslash \mathbb{A}_F^\times / (F_\infty^\times)^\circ \cong T(\mathbb{Q})\backslash T(\mathbb{A}_{\mathbb{Q}}) / T(\mathbb{R})^\circ$$

<u>Notation</u> $T = \mathrm{Res}_{F/\mathbb{Q}}\mathbb{G}_m$, $\rho \in X^*(T)$

When $\rho: T \longrightarrow \mathbb{G}_m$ is defined over $\mathbb{Q}$,

$\rho_\infty: T(\mathbb{R}) \longrightarrow \mathbb{R}^\times$ $\qquad$ $\rho_p: T(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$;

When $\rho: T_{F'} \longrightarrow \mathbb{G}_{m,F'}$ is defined over F',

$\rho_\infty: T(\mathbb{C}) \longrightarrow \mathbb{C}^\times$ $\qquad$ $\rho_p: T(\overline{F}_p) \longrightarrow \overline{F}_p^\times$

Prop 2. One has bijection

$$\begin{array}{ccc}
\mathrm{Char}_{\mathbb{C},\mathrm{alg},\mathrm{wt}\,0}\left(F^\times\backslash\mathbb{A}_F^\times\right) & \longleftrightarrow & \mathrm{Char}_{\mathbb{C}}(\Gamma_F) \\
\downarrow & & \downarrow \text{twist} \\
\mathrm{Char}_{\mathbb{C},\mathrm{alg}}\left(F^\times\backslash\mathbb{A}_F^\times\right) & \xleftrightarrow[\frac{\rho_\infty(x_\infty)}{\rho_p(x_p)} \;\underline{\underline{\text{when } x_\infty = x_p}}\; 1]{\text{twisted by}} & \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) + \text{de Rham} \\
\|\cdot\| & \longleftrightarrow & \varepsilon_p
\end{array}$$

We will explain Prop 2 in the following pages.

Def. Let $\rho \in X^*(T)$. $\chi \in \widehat{\mathbb{A}_F^*}$ is alg of wt ρ, if

$$\chi_\infty|_{F_\infty^{\times,0}} : F_\infty^{\times,0} = T(\mathbb{R})^0 \hookrightarrow T(\mathbb{C}) \xrightarrow{\frac{1}{\rho_\infty(-)}} \mathbb{C}^\times$$

E.g. 1). $\chi \in \widehat{\mathbb{A}_F^*}$ is alg of wt 0 $\leftrightsquigarrow$ $\chi \in \mathrm{Char}_{\mathbb{C}}(\Gamma_F)$ is the Artin character

2) $\|\cdot\| \in \widehat{\mathbb{A}_F^*}$ is alg of wt $-\sum_{\tau : F \hookrightarrow F^{nc}} [\tau]$

3) For $t \in \mathbb{C}$,

$$\|\cdot\|^t \text{ is alg } \Leftrightarrow t \in \mathbb{Z}.$$