§ 1.1. Structure of finite / local / global field

Road map

	finite field		local field: Archi		local field: NA		global field		Adèle
base field F	[7] $\mathbb{F}_1$	[1] $\mathbb{F}_p$	[2] $\mathbb{R}$ or	$\mathbb{C}$	[3] $\mathbb{Q}_p$	$\mathbb{F}_p((t))$	[4] $\mathbb{Q}$	$\mathbb{F}_p(t)$	[6] $\mathbb{A}_K$
$F^\times$	For $\mathbb{F}_{1^r}$ $\varepsilon\cdot\mu_r$	μ_{p-1}	$\mathbb{R}^+\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}^\times$	$\mathbb{Z}_p^\times\times\mathbb{Z}$	$\mathbb{F}_p[[t]]^\times\times\mathbb{Z}$	$\mathbb{Q}^\times$	$\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	—	—	—		$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	—	—	—		$\mathbb{Z}_p^\times$	$\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$\mathrm{Gal}(F^{sep}/F)$	$\hat{\mathbb{Z}}$?	$\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$	Id	most known		dream		
ari Frob	?	can	total order?	—	choose a lift		?		?
# ext of deg n	1?	1	1/0		finite		$n\neq1$ inf		countable
Spec $\mathcal{O}_F$	Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}},1)$ [étale, 2.2.4]		—						—
topology	?	discrete	Euclidean		profinite		—		restricted
topo of $\mathcal{O}_F$	—		—		cpt, not discrete		—		K is a lattice in $\mathbb{A}_K$
measure	?	discrete	Lebesgue		$\mu(\mathcal{O}_F)=1$		—		can be computed

[5] Also, discuss

- field extension, norm, trace, ...
- their connection to geometry, ramification theory
- analog with knot theory

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ as top gps

$\mathrm{Frob}_p \leftarrow 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $\mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \mathrm{Id}$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?

  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

We will discuss section 4 in [NAext] together. Some questions:

- Define I_F, P_F
- Construct $I_F/P_F \xrightarrow{\sim} \hat{\mathbb{Z}}^{(p)}$
- Explain why we have $\mathrm{Fr}\,\tau\,\mathrm{Fr}^{-1} = \tau^q$.

Task. Read [NAval], answer the following questions: (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\underbrace{\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep} / \mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right)}_{G_{\mathbb{F}_p((t))}}, \quad \underbrace{\mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}_p} / \widehat{\mathbb{Q}_p^{ur}}\right)}_{I_{\mathbb{Q}_p}}, \quad \underbrace{\mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} / \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)}_{G_{\mathbb{F}_p((t))}}$$

4. global field

$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is quite complicated.
$\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$ is less complicated, since by [Vakil, 6.5.D],
we have the equiv of cat

$$\{\text{fin ext of } \mathbb{F}_p(t)\} \longleftrightarrow \{\text{alg curve over } \mathbb{F}_p\}/\text{birational}$$

$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t))$ is even simpler. by [GalFun, Thm 3.4.8],

$$\mathrm{Gal}(\overline{\mathbb{C}(t)}/\mathbb{C}(t)) \cong \hat{F}(\mathbb{C})$$

Free profinite gp on $\mathbb{C}$

Q: Does $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ also have any natural acted object / geo realizations?

Dessin d'enfants

By [GalFun, Prop 4.7.1 - Rmk 4.7.9], we have an including

$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longhookrightarrow \mathrm{Out}(\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}))$$

induced by $\pi_{1,\overline{\mathbb{Q}}}^{\text{ét}} := \pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\})$, $\pi_{1,\mathbb{Q}}^{\text{ét}} := \pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}} - \{0,1,\infty\})$

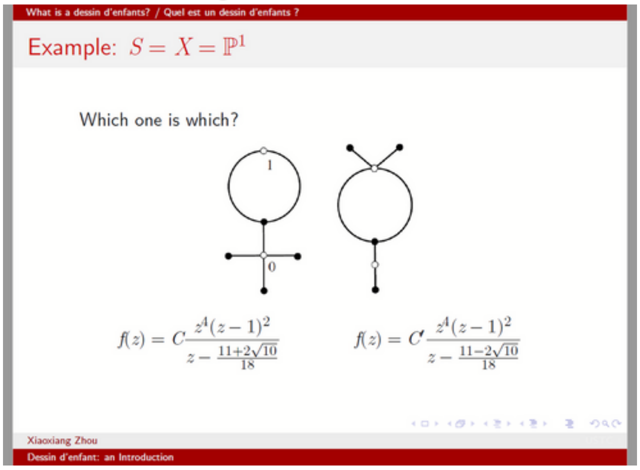
$$\begin{tikzcd}
 & 1 \arrow[r] \arrow[d, phantom, "\overset{?}{\|}\ 1"] & \pi_{1,\overline{\mathbb{Q}}}^{\text{ét}} \arrow[r] \arrow[d, equal] & \pi_{1,\mathbb{Q}}^{\text{ét}} \arrow[r] \arrow[d, "\text{conj}", "g \mapsto g\,\cdot\, g^{-1}"'] & \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \arrow[r] \arrow[d, dashed, "\exists!"] & 1 \\
1 \arrow[r] & Z(\pi_{1,\overline{\mathbb{Q}}}^{\text{ét}}) \arrow[r] & \pi_{1,\overline{\mathbb{Q}}}^{\text{ét}} \arrow[r] & \mathrm{Aut}(\pi_{1,\overline{\mathbb{Q}}}^{\text{ét}}) \arrow[r] & \mathrm{Out}(\pi_{1,\overline{\mathbb{Q}}}^{\text{ét}}) \arrow[r] & 1
\end{tikzcd}$$

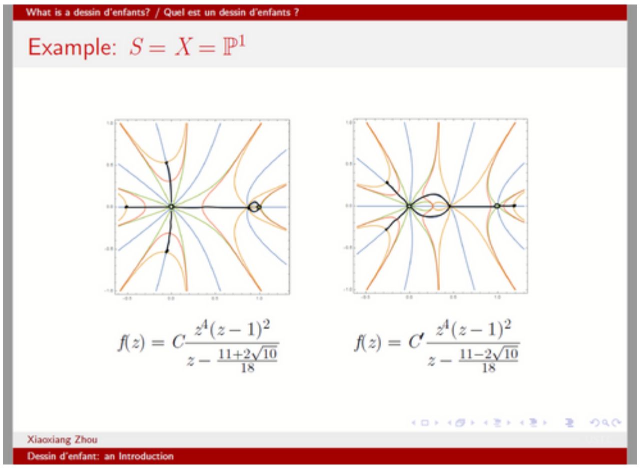
The space $\mathbb{P}^1_{\overline{\mathbb{Q}}} - \{0,1,\infty\}$ is designed for guaranteeing that
$$\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Out}(\pi_{1,\overline{\mathbb{Q}}}^{\text{ét}})$$
is inclusion.

Task. Read [Dessin d'enfant] or [Collins],

understand the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-action on the dessin d'enfants.

- Def of Dessin d'enfact
- Connections with $\mathrm{Out}(\pi_{1,\bar{\mathbb{Q}}}^{\text{ét}})$ via Belyi theorem
- Is this action faithful? Yes, in [Collins, Thm 7.1]
- Can we describe this action? Hard.





- Can we generalize this to $\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$?

  I don't know how to make a "dessin d'enfant" on alg curves over $\mathbb{F}_p$.

The global field has close connections with local fields. We will discuss these connections in next 2 sections in detail.