Un example par jour

4.8   K3 surface.   cpt cplx surf s.t $\omega_x \cong \mathcal{O}_x$ $\boxed{H^1(x, \mathcal{O}_x) = 0}$ can be changed by $b_1 = 0$ or $\pi_1(x) = \{*\}$

Today: Fermat quartic surface $X: z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$

$i: X \hookrightarrow \mathbb{P}^3_{\mathbb{C}}$

1. proj smooth alg surf ✓

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to i_* \mathcal{O}_x \to 0$$

Lemma VIII.9 *Let $V \subset \mathbb{P}^n$ be a d-dimensional complete intersection; then $H^i(V, \mathcal{O}_V) = 0$ for $0 < i < d$.* [Beauville]

H^3 $\rightarrow \mathbb{C} \to 0$

H^2 $\rightarrow 0 \to 0 \to H^2(X, \mathcal{O}_x) \cong \mathbb{C}$

H^1 $\rightarrow 0 \to 0 \to H^1(X, \mathcal{O}_x) \cong 0$

H^0 $0 \to 0 \to \mathbb{C} \to H^0(X, \mathcal{O}_x) \cong \mathbb{C}$

$\omega_{\mathbb{P}^3} = \mathcal{O}_{\mathbb{P}^3}(-4) \xRightarrow{21.5.B \text{ in [Vakil]}} \omega_x \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(4))|_x = \mathcal{O}_x$.

$\dfrac{z_0 \, dz_1 \wedge dz_2}{4 z_3^3}$ $\begin{matrix} z_0 \neq 0 \\ z_3 \neq 0 \end{matrix}$

∴ X is a K3-surface.

2. $\Big[$ Prop. X: K3 surfaces $\Rightarrow$ X is minimal.

$P_n = 1$ $n \geq 1$ $\Rightarrow k(X) = 0$

$\begin{matrix} 2 & -20 & 2 & & 24 \\ & & 1 & & 1 \\ & 0 & 0 & & 0 \\ 1 & 20 & 1 & & 22 \\ & 0 & 0 & & 0 \\ & & 1 & & 1 \end{matrix}$

$(b_+, b_-) = (3, 19)$

$c_2 = 24$

$c_1^2 = K^2 = 0$

Some conclusions on alg topo:

↑cpt M: closed connected mfld of dim n

	orientable	nonorientable
$H_n(M)$	$\mathbb{Z}$	0
$H^n(M)$	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H_n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
$H^n(M, \mathbb{Z}/2\mathbb{Z})$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

Moreover, when M is oriented +cpt $\Rightarrow$ $T^p \cong T^{n-p+1}$ $(H^i(X) \cong \mathbb{Z}^{\oplus b_i} \oplus T^i)$

Cor $T_i := H_i(X)$ ↙def $\sqsubseteq$ by universal coefficient thm & Poincaré duality

n	0	1	2	3	4	≥ 5
$H_n(X)$	$\mathbb{Z}$	T	$\mathbb{Z}^{22} \oplus T$	0	$\mathbb{Z}$	0
$H^n(X)$	$\mathbb{Z}$	0	$\mathbb{Z}^{22} \oplus T$	T	$\mathbb{Z}$	0

Claim. T = 0, i.e (co)homology has no torsion!

https://math.stackexchange.com/questions/2882059/

$\Big[$ Proof. by LES induced by $0 \to \mathbb{Z} \to \mathcal{O}_x \to \mathcal{O}_x^* \to 1$,

only need to prove $Pic(X)$ is torsion free.

Suppose $D \in Div(X)$ s.t $nD \equiv 0$ $\Rightarrow \chi(D) = 2$

$\Rightarrow h^0(D) \geq 1$ or $h^0(-D + K_x) = h^2(D) \geq 1$ w.l.o.g suppose $h^0(D) \geq 1$

$\Rightarrow D \sim D'$ where D' is effective

$\underset{nD \equiv 0}{\Rightarrow} D' = 0$

Another proof. If $H_1(X)$ has torsion, then $\exists\, G \triangleleft H_1(X)$ s.t $H_1(X)/G \cong \mathbb{Z}/_{m\mathbb{Z}}$, $m \in \mathbb{N}^+$, $m > 1$
denote $p: \pi_1(X) \longrightarrow H_1(X)$, then $\pi_1(X)/_{p^{-1}(G)} \cong H_1(X)/G \cong \mathbb{Z}/_{m\mathbb{Z}}$
$\therefore \exists$ a nontrivial unramified covering of degree m
$$\Phi: \widehat{X} \longrightarrow X$$
$\Rightarrow \begin{cases} K_{\widehat{X}} = 0 \\ \chi_{top}(\widehat{X}) = 24m \end{cases} \Rightarrow \chi(\mathcal{O}_{\widehat{X}}) = \frac{1}{12}(K_{\widehat{X}}^2 + \chi_{top}(\widehat{X})) = 2m > 2 - h^1(\mathcal{O}_{\widehat{X}}) = \chi(\mathcal{O}_{\widehat{X}})$

Contradiction!

12.2. **K3 lattice.** We deduce that a K3 surface has second Betti number $b_2 = 22$. Cup-product equips $H^2(X, \mathbb{Z})$ with the structure of an integral lattice of rank 22. Often this lattice is called **K3 lattice** and denoted by Λ. The following properties of Λ are well-known:

- Λ is unimodular by Poincaré-duality;
- Λ has signature $(3, 19)$ by the topological index theorem;
- Λ is even by Wu's formula since the first Chern class is even. ← one signature:

$C^2 = 2g(C) - 2$ is even

Hence the classification of even unimodular lattices implies that

$$\Lambda \cong U^{\oplus 3} \oplus \widetilde{H}(-E_8)^{\oplus 2}$$

$U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$

↖ $\widetilde{b}(x,y) := -b(x,y)$

Now let us consider the fundamental group of X.

Thm (Lefschetz hyperplane thm) [In wiki there are several proofs].

Let Y: n-dim cplx proj variety
X: hyperplane section of Y
U: $Y - X$ is smooth

then $H_k(Y,X,\mathbb{Z})$, $H^k(Y,X,\mathbb{Z})$, $\pi_k(Y,X) = 0$ for $k \le n-1$. $k \in \mathbb{N}^+$

Cor.

	$k < n-1$	$k = n$
$H_k(X) \longrightarrow H_k(Y)$ is	iso	or surj
$H^k(Y) \longrightarrow H^k(X)$ is	iso	or inj
$\pi_k(X) \longrightarrow \pi_k(Y)$ is	iso	or surj.

For X Fermat quartic, $X \overset{i}{\hookrightarrow} \mathbb{P}^3_{\mathbb{C}} \overset{\psi}{\hookrightarrow} \mathbb{P}^{34}_{\mathbb{C}}$ $Y = \phi(\mathbb{P}^3_{\mathbb{C}})$,
$H_1(X) = H_1(Y) = H_1(\mathbb{P}^3_{\mathbb{C}}) = 0$ $H^1(X) = 0$ $\pi_1(X) = 0$
$\Rightarrow X$ is simply connected 4-dim mfld.

Rmk. By [GTM82, Thm 17.21], $\pi_2(X) = H_2(X) = \mathbb{Z}^{22}$;
By

$\pi_n(X) = \pi_n(\#^{20} S^2 \times S^3)$ $n \ge 3$, and

n	1	2	3	4	5	6	7
$\pi_n(X)$	0	$\mathbb{Z}^{22}$	$\mathbb{Z}^{252}$	$\mathbb{Z}^{3520} \oplus (\mathbb{Z}/_{2\mathbb{Z}})^{42}$	$\mathbb{Z}^{57960} \oplus tor$	$\mathbb{Z}^{1020096} \oplus tor$	$\mathbb{Z}^{22397760} \oplus tor$

Picard group:
Again, by $0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^\times \longrightarrow 0$

$$\nearrow H^2(X, \mathbb{Z}) \underset{\cong \mathbb{Z} \cdots}{\longrightarrow} H^2(\mathcal{O}_X) \underset{\cong \mathbb{C}}{\longrightarrow} H^2(\mathcal{O}_X^\times)$$

$$\nearrow H^1(X, \mathbb{Z}) \underset{\cong 0}{\longrightarrow} H^1(\mathcal{O}_X) \underset{\cong 0}{\longrightarrow} H^1(\mathcal{O}_X^\times) = \mathrm{Pic}(X)$$

$$0 \to H^0(X, \mathbb{Z}) \longrightarrow H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_X^\times)$$
$$\underset{\cong \mathbb{Z}}{\qquad} \underset{\cong \mathbb{C}}{\qquad} \underset{\cong \mathbb{C}^\times}{\qquad}$$

$\Rightarrow \mathrm{Pic}(X) = NS(X) \subseteq H^2(X, \mathbb{Z}) = U^{\oplus 3} \oplus (-E_8)^{\oplus 2}$

$\Rightarrow \mathrm{Pic}(X) \cong \mathbb{Z}^{\rho(X)}, 1 \le \rho(X) \le 20$

<u>Rmk.</u> for X Fermat quartic, $\rho(X) = 20$, $\mathrm{Pic}(X) \cong (-E_8)^{\oplus 2} \oplus U \oplus (-8)^{\oplus 2}$ $\mathrm{Pic}(X)^\dagger \cong (8)^{\oplus 2}$

[Schütt, Shioda and van Luijk]

Q. find a metric s.t $\mathrm{Ric}(\rho) \equiv 0$?