Eine Woche, ein Beispiel
4.28 naive $\otimes$- Hom adjunction

Notation. - A: associate ring   *allowed to be non-commutative, contains 1*
  - There are two systems to write category of A-modules.

$$\text{Mod}_A = A - \text{Mod} \qquad\qquad \ni {}_A M$$
$$(\text{Mod}_A)^{op} \neq \text{Mod}_{A^{op}} = \text{Mod} - A = A^{op} - \text{Mod} \ni M_A$$
$$\text{Mod}_{A \otimes B^{op}} = A - \text{Mod} - B \qquad\qquad \ni {}_A M_B$$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

For convenience, we write
$$(\text{Mod}_{B \otimes A^{op}})^{op} = (B - \text{Mod} - A)^{op} = (A^{op} - \text{Mod} - B^{op})^{op} \qquad \ni {}_B M_A$$

as
$$(\text{Mod}_{A \otimes B^{op}})^{\overline{op}} = (A - \text{Mod} - B)^{\overline{op}}$$

$\triangledown$ Even though you can identify $\text{Ob}(\text{Ring}) \cong \text{Ob}(\text{Ring}^{op})$,
$A^{op} \notin \text{Ob}(\text{Ring}^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

- For A comm, $\text{Mod}_A = \text{Mod}_{A^{op}} \subset \text{Sh}(\text{Spec } A)$.

In this case, it is desirable to translate algebraic results into geometrical results.
Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for $\otimes$ & Hom
2. adjunction
3. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$
4. definition recall for $\otimes$ & Hom   , derived version
5. adjunction   , derived version
6. comparison between $\otimes \dashv$ Hom & $f^* \dashv f_*$, derived version

1. definition recall for $\otimes$ & Hom

$$\otimes_A: \quad \text{Mod}_{A^{op}} \times \text{Mod}_A \longrightarrow \text{Mod}_{\mathbb{Z}}$$

$$\text{Hom}_A(-,-): (\text{Mod}_A)^{op} \times \text{Mod}_A \longrightarrow \text{Mod}_{\mathbb{Z}}$$

In general,

$$\otimes_B: \quad A\text{-Mod-}B \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$

$$\text{Hom}_B(-,-): (A\text{-Mod-}B)^{\overline{op}} \times B\text{-Mod-}C \longrightarrow A\text{-Mod-}C$$

$$\text{Hom}_B^A(-,-): (A\text{-Mod-}B)^{\overline{op}} \times B\text{-Mod-}A \longrightarrow \mathbb{Z}\text{-Mod}$$

$$\parallel \qquad\qquad \parallel \qquad\qquad \parallel \qquad\qquad\qquad \parallel$$

$$\text{Hom}_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) \;\; (\mathbb{Z}\text{-Mod-}B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}\text{-Mod-}\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}\text{-Mod-}\mathbb{Z}$$

$$_A X_B, \quad _B Y_C, \quad _C Z_D$$

associativity: $\quad (X \otimes_B Y) \otimes_C Z \cong X \otimes_B (Y \otimes_C Z)$

"commutativity": $\quad X \otimes_B Y \cong Y \otimes_{B^{op}} X \qquad$ in $A\text{-Mod-}C = C^{op}\text{-Mod-}A^{op}$

"unit": $\quad A \otimes_A X \cong X \cong X \otimes_B B$

$\qquad\qquad \text{Hom}_A(A, X) \cong X$

2. adjunction $_BX_A$, $_CY_B$, $_CZ_D$.   we get

$$\text{Hom}_C(Y \otimes_B X, Z) \cong \text{Hom}_B(X, \text{Hom}_C(Y, Z)) \qquad \text{in } A\text{-Mod-}D.$$

Reason: both sides equal to the set
$$\{f: Y \times X \longrightarrow Z \mid f(cyb, x) = c\, f(y, bx) \quad \forall b, c\}$$

For $A = D = \mathbb{Z}$, fix $Y \in C\text{-Mod-}B$, one gets adjunction fctors:

$$B\text{-Mod} \underset{\text{Hom}_C(Y, -)}{\overset{Y \otimes_B -}{\rightleftarrows}} C\text{-Mod}$$

(with $\perp$ between the arrows)

slogan: adjunction $\approx$ associativity

$\otimes \dashv$ Hom:

$$(A\text{-}Mod\text{-}B)^{\overline{op}} \times (B\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D \xrightarrow{(Id,\, Hom_C)} (A\text{-}Mod\text{-}B)^{\overline{op}} \times B\text{-}Mod\text{-}D$$

$$\|$$

$$(A\text{-}Mod\text{-}B \times B\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D$$

$(\otimes_B,\, Id) \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow Hom_B$

$$(A\text{-}Mod\text{-}C)^{\overline{op}} \times C\text{-}Mod\text{-}D \xrightarrow{\;Hom_C\;} A\text{-}Mod\text{-}D$$

$$\mathcal{G} \qquad \mathcal{F}$$
$$| \qquad |$$
$$Y \xrightarrow{\;f\;} X$$

$f^* \dashv f_*$:

$$Hom(f^*\mathcal{F},\, \mathcal{G}) \cong Hom(\mathcal{F},\, f_*\mathcal{G})$$

$$Sh(X)^{op} \times Mor(Y,X) \times Sh(Y) \xrightarrow{(Id,\, pushforward)} Sh(X)^{op} \times Sh(X)$$

$(pullback,\, Id) \downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow Hom_{Sh(X)}(-,-)$

$$Sh(Y)^{op} \times Sh(Y) \xrightarrow{\;Hom_{Sh(Y)}(-,-)\;} Abel$$

$$(\mathcal{F},\quad f,\quad \mathcal{G}) \longmapsto (\mathcal{F},\, f_*\mathcal{G})$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$(f^*\mathcal{F},\, \mathcal{G}) \longmapsto Hom_{Sh(Y)}(f^*\mathcal{F},\, \mathcal{G}) \cong Hom_{Sh(X)}(\mathcal{F},\, f_*\mathcal{G})$$

$f_! \dashv f^!$: similar.

3. comparison between $\otimes \dashv \text{Hom}$ & $f^* \dashv f_*$

Prop. For ring homo $\begin{smallmatrix} S \\ \uparrow f \\ R \end{smallmatrix}$, $\exists$ "forgetful fctor"

$$u: S\text{-Mod} \longrightarrow R\text{-Mod} \qquad M \longmapsto u(M)$$
$$u(M) = {}_R S_S \otimes_S M = \text{Hom}_S ({}_S S_R, M)$$

one has adjunction fctors

$$(3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:

Vague slogan: $u \approx$ "forget the information of Z".

In applications, $\mathcal{U} \longrightarrow X$ is a covering map.
This change the feeling of the size between $\mathcal{U}$ & X.

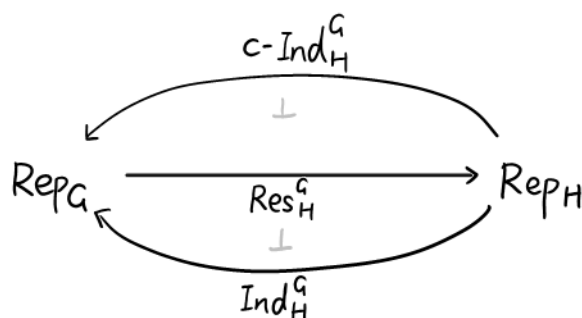**E.g.** For finite gps $H \leq G$, one has Res - Ind adjunction:
$$\text{Res}_H^G \dashv \text{Ind}_H^G$$
$$c\text{-}\text{Ind}_H^G \dashv \text{Res}_H^G$$

It can be generalized for $\begin{cases} G: \text{loc profinite gp}, \\ H \leq G \text{ open} \end{cases}$

If one only has $H \leq G$ closed, then it's possible that $j^! \neq j^*$.

e.g. $G = GL_2(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,

Ex. Compare it with the recollement diagram & (3.1).

"cover with fiber G/H"

translate the following geometrical results into algebraic statements.

1. One has natural fctor $j_! \longrightarrow j_*$. When $\# G/H < +\infty$, $j_! = j_*$
   $\quad\quad\quad\quad\quad c\text{-}\text{Ind}_H^G \longrightarrow \text{Ind}_H^G \quad\quad\quad\quad\quad c\text{-}\text{Ind}_H^G = \text{Ind}_H^G$

2. Even though
$$\text{Sh}_{\mathbb{Q}\text{-v.s}}([*/G]) \approx \text{Rep}_G = \mathbb{Q}[G]\text{-Mod},$$
   the "structure sheaf" of $[*/G]$ is $\mathbb{Q}$, not $\mathbb{Q}[G]$.

$$\text{Res}_{\{*\}}^G \mathbb{Q} = \mathbb{Q}, \quad\quad \text{Res}_{\{*\}}^G \mathbb{Q}[G] = \mathbb{Q}[G] \neq \mathbb{Q}$$

⚠ In this example, $j^* R j_* \neq \text{Id}$, $j^! j_! \neq \text{Id}$.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

$$
\begin{array}{ccc}
S & \qquad & \operatorname{Spec} S \\
\uparrow \tilde{f} & \quad M & \downarrow f \quad M \\
R & \nearrow & \operatorname{Spec} R \nearrow
\end{array}
$$

$\exists$ "pullback fctor"
$$f^* : R\text{-Mod} \longrightarrow S\text{-Mod} \qquad f^*M = {}_SS_R \otimes_R M$$
This is also called the base change.

Now, (3.1) can be rewritten as

$$
S\text{-Mod} \xrightarrow{\quad u \quad} R\text{-Mod}
$$

with f^* on top and $\operatorname{Hom}_R({}_RS_S, -)$ on bottom.

Compare it with another part of the recollement diagram:

$$
\mathcal{D}(X) \xrightarrow{\quad i_* \quad} \mathcal{D}(\mathcal{U})
$$

with i^* on top, $i_!$ and $i^!$ below.

Rmk. u is usually not f. faithful, unless $S = R/I$.
(In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.)
which crspds to closed embedding.
In that case,
$$i^* i_* = \operatorname{Id}: \ {}_SS_R \otimes_R ({}_RS_S \otimes_S M) \cong M$$
$$i^! i_* = \operatorname{Id}: \ \operatorname{Hom}_R({}_RS_S, \operatorname{Hom}({}_SS_R, M)) \cong M$$

Slogan: in the comm alg, $\operatorname{Spec} R/I \longrightarrow \operatorname{Spec} R$ is closed embedding.
In general, if
S is an R-idempotent algebra: $S \cong S \otimes_R S$
then $i: \operatorname{Spec} S \longrightarrow \operatorname{Spec} R$ can be viewed as "closed subset".

E.g. $R_{\mathfrak{p}}$, R/I are idempotent R-algs.
$\mathbb{Z}[\frac{1}{b}]$, $\mathbb{F}_p$, $\mathbb{Z}/p^2\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\cdots$ are idem $\mathbb{Z}$-algs.
$\triangledown$ Usually R/I is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk. Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"?
A. (possibly) the Verdier quotient.

We will come back to this after we derive everything.

4. $\,{}^L\otimes \dashv R\mathrm{Hom}$

F	RF or LF	R^iF or L^iF	exact fctor
f^*	f^*	—	
f_*	Rf_*	R^if_*	f_*-acyclic
$\pi_{x,*}\mathscr{F}$	$\Gamma(X;\mathscr{F})$	$H^i(X;\mathscr{F})$	Γ - acyclic
$f_!$	$Rf_!$	$R^if_!$	$f_!$ - acyclic
$\pi_{x,!}\mathscr{F}$	$\Gamma_c(X;\mathscr{F})$	$H^i_c(X;\mathscr{F})$	Γ_c -acyclic
—	$f^!$	$H^i(f^!-)$	
$-\otimes_R-$	$-\,{}^L\!\otimes_R-$	$\mathrm{Tor}^R_i(-,-)$	flat
$\mathrm{Hom}_R(-,-)$	$R\mathrm{Hom}_R(-,-)$	$\mathrm{Ext}^i_R(-,-)$	injective/projective
M_G	$\mathbb{Z}\,{}^L\!\otimes_{\mathbb{Z}[G]}M$	$H_i(G;M)$	
M^G	$R\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M)$	$H^i(G;M)$	
$M_{\mathfrak{g}}$	$x\,{}^L\!\otimes_{u\mathfrak{g}}M$	$H_i(\mathfrak{g};M)$	
$M^{\mathfrak{g}}$	$R\mathrm{Hom}_{u\mathfrak{g}}(x,M)$	$H^i(\mathfrak{g};M)$	
$M/[A,M]$	$A\,{}^L\!\otimes_{A^e}M$	$HH_i(A,M)$	
M^A	$R\mathrm{Hom}_{A^e}(A,M)$	$HH^i(A,M)$	
$A/[A,A]$	$A\,{}^L\!\otimes_{A^e}A$	$HH_i(A)$	
$Z(A)$	$R\mathrm{Hom}_{A^e}(A,A)$	$HH^i(A)$	

e.g. group coh

e.g. Lie alg coh
 $\mathfrak{g}/\kappa$: Lie alg
e.g. Hochschild coh

4. definition recall for $\otimes$ & Hom   , derived version

To define $\overset{L}{\otimes}$ & RHom, one needs to extend fctors

$$\otimes_A : \quad A\text{-Mod} \quad \times \quad A\text{-Mod} \quad \longrightarrow \quad A\text{-Mod}$$
$$\mathrm{Hom}_A(-,-): \quad (A\text{-Mod})^{op} \quad \times \quad A\text{-Mod} \quad \longrightarrow \quad A\text{-Mod}$$

to fctors on double cplxes.

$\mathcal{C}(A) := $ complex of A-modules, temperate notation

$$\otimes_{\mathcal{C}(A)} \quad \mathcal{C}(A) \quad \times \quad \mathcal{C}(A) \quad \longrightarrow \quad \mathcal{C}(A)$$
$$\mathrm{Hom}_{\mathcal{C}(A)}(-,-): \quad (\mathcal{C}(A))^{op} \quad \times \quad \mathcal{C}(A) \quad \longrightarrow \quad \mathcal{C}(A)$$

But how ?

Wishes:

$$(M^\cdot[i]) \otimes_{\mathcal{C}(A)} (N^\cdot[j]) = (M^\cdot \otimes N^\cdot)[i+j]$$
$$\mathrm{Hom}_{\mathcal{C}(A)}(M^\cdot[-i], N^\cdot[j]) = \mathrm{Hom}(M^\cdot, N^\cdot)[i+j]$$

$$
\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & M^{-1} \otimes N^1 & \longrightarrow & M^0 \otimes N^1 & \longrightarrow & M^1 \otimes N^1 & \longrightarrow & M^2 \otimes N^1 & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & M^{-1} \otimes N^0 & \longrightarrow & M^0 \otimes N^0 & \longrightarrow & M^1 \otimes N^0 & \longrightarrow & M^2 \otimes N^0 & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & M^{-1} \otimes N^{-1} & \longrightarrow & M^0 \otimes N^{-1} & \longrightarrow & M^1 \otimes N^{-1} & \longrightarrow & M^2 \otimes N^{-1} & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\end{array}
$$

Tot ($M^\bullet \otimes N^\bullet$), the double complex of $M^\bullet \otimes N^\bullet$.

$$
\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & \mathrm{Hom}(M^1, N^1) & \longrightarrow & \mathrm{Hom}(M^0, N^1) & \longrightarrow & \mathrm{Hom}(M^{-1}, N^1) & \longrightarrow & \mathrm{Hom}(M^{-2}, N^1) & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & \mathrm{Hom}(M^1, N^0) & \longrightarrow & \mathrm{Hom}(M^0, N^0) & \longrightarrow & \mathrm{Hom}(M^{-1}, N^0) & \longrightarrow & \mathrm{Hom}(M^{-2}, N^0) & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\longrightarrow & \mathrm{Hom}(M^1, N^{-1}) & \longrightarrow & \mathrm{Hom}(M^0, N^{-1}) & \longrightarrow & \mathrm{Hom}(M^{-1}, N^{-1}) & \longrightarrow & \mathrm{Hom}(M^{-2}, N^{-1}) & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\end{array}
$$

Tot (Hom($M^\bullet, N^\bullet$)), the double complex of Hom($M^\bullet, N^\bullet$).

Def. For $M^{\cdot}, N^{\cdot} \in \mathcal{C}(A)$, define
$$M^{\cdot} \otimes N^{\cdot}, \quad \mathrm{Hom}_A(M^{\cdot}, N^{\cdot}) \in \mathcal{C}(A)$$
by
$$(M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot})^n = \bigoplus_{i+j=n} M^i \otimes_A N^j$$

$$(\mathrm{Hom}_{\mathcal{C}(A)}(M^{\cdot}, N^{\cdot}))^n = \bigoplus_{i+j=n} \mathrm{Hom}_A(M^{-i}, N^j)$$

and morphisms given by $d + (-1)^j \delta$.

Ex. Let $M^{\cdot} = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 3} \underset{0}{\mathbb{Z}}]$, $N^{\cdot} = [\underset{-1}{\mathbb{Z}} \xrightarrow{\times 2} \underset{0}{\mathbb{Z}}]$

compute $M^{\cdot} \otimes_{\mathcal{C}(\mathbb{Z})} N^{\cdot}$ & $\mathrm{Hom}_{\mathcal{C}(\mathbb{Z})}(M^{\cdot}, N^{\cdot})$, and verify that they're complexes.

A.

$$
\begin{array}{ccc}
0 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} & \\
& \times 2 \uparrow \qquad \uparrow \times 2 & \\
-1 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} & \\
& -1 \qquad 0 & \\
& \mathrm{Tot}(M^{\cdot} \otimes N^{\cdot}) &
\end{array}
\quad\rightsquigarrow\quad
\left[\underset{-2}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{-1}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{0}{\mathbb{Z}} \right]
$$
$$M^{\cdot} \otimes_{\mathcal{C}(A)} N^{\cdot}$$

$$
\begin{array}{ccc}
0 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} & \\
& \times 2 \uparrow \qquad \uparrow \times 2 & \\
-1 & \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} & \\
& 0 \qquad 1 & \\
& \mathrm{Tot}(\mathrm{Hom}(M^{\cdot}, N^{\cdot})) &
\end{array}
\quad\rightsquigarrow\quad
\left[\underset{-1}{\mathbb{Z}} \xrightarrow{\binom{-3}{2}} \underset{0}{\mathbb{Z}^2} \xrightarrow{(2\ 3)} \underset{1}{\mathbb{Z}} \right]
$$
$$\mathrm{Hom}_{\mathcal{C}(A)}(M^{\cdot}, N^{\cdot})$$

Now, we can define $\overset{L}{\otimes}$ & RHom.

Def. For $M, N \in A\text{-Mod}$, one can define

$$M \overset{L}{\otimes}_A N := M \otimes_{e(A)} P^\cdot \qquad \text{when } N \overset{\simeq}{\longleftarrow} P^\cdot \quad \text{flat resolution}$$

in general, $M^\cdot, N^\cdot \in \mathcal{D}^-(A\text{-Mod})$

$$RHom_A(M, N) := Hom_{e(A)}(M, I^\cdot) \qquad \text{when } N \overset{\simeq}{\longrightarrow} I^\cdot \quad \text{inj resolution}$$
$$:= Hom_{e(A)}(P^\cdot, N) \qquad \text{when } M \overset{\simeq}{\longleftarrow} P^\cdot \quad \text{proj resolution}$$

in general, $M^\cdot \in \mathcal{D}^-(A\text{-Mod})$, $N^\cdot \in \mathcal{D}^+(A\text{-Mod})$

Side Rmk. Proj module is flat. Since free module is flat

Ex Compute $\mathbb{F}_2 \overset{L}{\otimes}_\mathbb{Z} \mathbb{F}_2$ & $RHom_\mathbb{Z}(\mathbb{F}_2, \mathbb{F}_2)$,
and get $Tor^i_\mathbb{Z}(\mathbb{F}_2, \mathbb{F}_2)$ & $Ext^i_\mathbb{Z}(\mathbb{F}_2, \mathbb{F}_2)$

Ex. Shows that
$$Hom_{\mathcal{D}^i(A)}(M^\cdot, N^\cdot) = R^0 Hom_{e(A)}(M^\cdot, N^\cdot)$$
$$Hom_A(M, N) = Hom_{\mathcal{D}^+(A)}(M^\cdot, N^\cdot) = R^0 Hom_{e(A)}(M, N).$$

$\overset{!}{\triangledown}$ [KS 90, Def 2.6.2]
   To switch from $\mathcal{D}^-(X)$ to $\mathcal{D}^+(X)$, we need to require that

$$w.gldim(A) < +\infty. \qquad \text{w.gldim: shortest flat resolution}$$

In our case, $w.gldim(Q) = 0 < +\infty$. So don't worry.

A wrong proof for "flat ⟶ proj"

"Proof": when P is flat,

$$P \otimes_A - \quad \dashv \quad \text{Hom}_A(P, -)$$

$$\| $$

$$P \overset{L}{\otimes}_A - \quad \dashv \quad R\text{Hom}_A(P, -)$$

by the uniqueness of the adjunction, $\text{Hom}_A(P, -) = R\text{Hom}_A(P, -)$, so P is flat.

This is wrong. $\mathbb{Q} \in \mathbb{Z}\text{-Mod}$ is flat but not proj. In the proof, we only have

$$L_A \circ (P \otimes_A -) = (P \overset{L}{\otimes}_A -) \circ L_A.$$

Ex. Compute $R\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$, and shows that

$$L_{\mathbb{Z}} \circ \text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -) \neq R\text{Hom}_{\mathbb{Z}}(\mathbb{Q}, -) \circ L_{\mathbb{Z}}.$$

5. adjunction   , derived version

Prop. For A comm ring, $L, M, N \in A\text{-Mod}$, we get
$L^{\cdot}, M^{\cdot} \in \mathcal{D}^-(A)$, $N^{\cdot} \in \mathcal{D}^+(A)$ in general

$$R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

Proof.

$$\mathrm{Hom}_A(M \otimes_A N, L) \cong \mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

$\Downarrow$ take $(-)^n$

$$\mathrm{Hom}_{\mathcal{C}(A)}(M \otimes_{\mathcal{C}(A)} N, L) \cong \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(M, L))$$

$\Downarrow$ $\mathrm{Hom}_A(M, -)$ preserves inj modules
  for M flat   wiki: injective module

$$R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) \cong R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L))$$

$$
\begin{aligned}
R\mathrm{Hom}_A(M \overset{L}{\otimes}_A N, L) &= R\mathrm{Hom}_A(P^{\cdot} \otimes_{\mathcal{C}(A)} N, L) & M^{\cdot} \overset{\cong}{\Leftarrow} P^{\cdot} \text{ flat} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(P^{\cdot} \otimes_{\mathcal{C}(A)} N, I^{\cdot}) & L^{\cdot} \overset{\cong}{\to} I^{\cdot} \text{ inj} \\
&= \mathrm{Hom}_{\mathcal{C}(A)}(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^{\cdot}, I^{\cdot})) & \text{adj in } \mathcal{C}(A) \\
&= R\mathrm{Hom}_A(N, \mathrm{Hom}_{\mathcal{C}(A)}(P^{\cdot}, I^{\cdot})) & \mathrm{Hom}_{\mathcal{C}(A)}(P^{\cdot}, I^{\cdot}) \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(P^{\cdot}, I^{\cdot})) & I^{\cdot} \text{ is inj} \\
&= R\mathrm{Hom}_A(N, R\mathrm{Hom}_A(M, L)) & \square
\end{aligned}
$$

∇ We don't have

$$R\mathrm{Hom}_A(M \otimes_A N, L) \cong R\mathrm{Hom}_A(N, \mathrm{Hom}_A(M, L))$$

Find a counterexample?
|| Take $N = A$. reduce to:
|| $R\mathrm{Hom}_A(M, L) \cong \mathrm{Hom}_A(M, L)$
|| then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between $\otimes \dashv \operatorname{Hom}$ & $f^* \dashv f_*$, derived version

Will write: verdier category
interpretation of 3 cohomology (gp + Lie alg + Hochschild)