Eine Woche, ein Beispiel

4.10. non-Archimedean local field F

Classification:
- finite extension of $\mathbb{Q}_p$
- $\mathbb{F}_q((T))$ $(q = p^r)$

Process:
1. Basic structures and results.
2. Topological results.
3. Haar measure
4. Representation of $(F, +)$ and $F^\times$ (next week)

1. Basic structures and results

1.1. None of them is alg closed.

1.2. The natural valuation $v: F \longrightarrow \mathbb{Z} \cup \{+\infty\}$ is defined. Then
$$\mathcal{O}, \mathfrak{p}, k = \mathcal{O}/\mathfrak{p} \qquad p = \text{char } k, \quad q = |k| = p^r$$
$$\mathcal{U} = \mathcal{U}^{(0)} = \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) = 0\} \qquad \mathcal{U}^{(n)} = 1 + \mathfrak{p}^n \qquad n \geq 1$$
are defined, and $\pi \in \mathfrak{p}\mathcal{O}^\times \cong \mathfrak{p} - \mathfrak{p}^2$ is picked.

Moreover, $\mathcal{O}$ is DVR, k is finite,
$$\mathcal{U}^{(0)}/\mathcal{U}^{(1)} \overset{\text{split iso}}{\cong} k^\times \qquad \mathcal{U}^{(n)}/\mathcal{U}^{(n+1)} \overset{\text{non-canonical}}{\cong} k \qquad n \geq 1$$
$$\mathcal{U}^{(0)}/\mathcal{U}^{(n)} \overset{\text{non split iso}}{\cong} (\mathcal{O}/\mathfrak{p}^n)^\times \quad n \geq 1 \qquad \mathcal{U}^{(m+1)}/\mathcal{U}^{(n+1)} \overset{\text{non-canonical}}{\cong} \mathcal{O}/\mathfrak{p}^{m-n} \quad 2n+1 \geq m > n \geq 0$$

$$0 \longrightarrow \mathcal{U}^{(1)} \longrightarrow \mathcal{O}^\times \longrightarrow k^\times \longrightarrow 0$$

with μ_{q-1}, $\downarrow \cong$ the Teichmüller lift

$$\mu_{q-1} = \{\alpha \in F \mid \alpha^{q-1} = 1\}$$

$$\Rightarrow \mathcal{O}^\times \cong \mathcal{U}^{(1)} \times \mu_{q-1}$$

1.3. $F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times \cong \langle \pi \rangle \times \mu_{q-1} \times \mathcal{U}^{(1)}$

e.g. when $F = \mathbb{Q}_p$, $\mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p & p \neq 2 \\ \mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2) & p = 2 \end{cases}$

Thm. When $p \geq 3$, $(p\mathbb{Z}_p, +) \underset{\log}{\overset{\exp}{\rightleftarrows}} (1 + p\mathbb{Z}_p, \cdot)$ is an iso as topological gps.

⚠️ The topology of $F^\times$ is not the same as the subspace topology $F^\times \subset F$, but is the same as the subspace topology of $F^\times \cong GL_1(F) \cong \{(x, x') \in F^2\} \subset F^2$. (Maybe that is the origin of the topology of the Weil group).

2. Topological results.

$\mathcal{O} = \varprojlim_n \mathcal{O}/\mathfrak{p}^n$ is cpt and profinite group, while F is loc. cpt and loc. profinite group.

$\mathcal{O}^\times = \varprojlim_n \mathcal{O}^\times/U^{(n)}$ is cpt and profinite group, while $F^\times$ is loc. cpt and loc. profinite group.

Cpt open subgps of $(F, +)$ are $\{\mathfrak{p}^k\}$.

Cpt open subgps of $F^\times$ are not restricted in $\{U^{(k)}\}$,

but $\{U^{(k)}\}$ is a nbhd system of $F^\times$, i.e.,

$\{a\,U^{(k)}\}_{a \in F^\times}$ is a topological basis of $F^\times$.

$\{\text{open subgps}\} \subseteq \{\text{closed subgps}\}$ for $(F, +)$ and $F^\times$.

Q: Are there any other cpt closed subgp?

A: Yes.   eg $\{0\} \subseteq (F, +)$ $\{1\} \subseteq F^\times$

Q: Can we classify all cpt closed subgp?

E.g. $\mathbb{Q}_{p^r}$ = the splitting field of $X^q - X$ over $\mathbb{Q}_p$ $q = p^r$

    = the unique unramified extension of $\mathbb{Q}_p$ of degree r

$\mathrm{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \mathrm{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p) \cong \mathbb{Z}/r\mathbb{Z}$

3. Haar measure

G: loc. profinite gp

$C^\infty(G) := \{f : G \to \mathbb{C} \mid f \text{ is loc. const}\}$

$C_c^\infty(G) := \{f \in C^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$

Rmk. G has topo basis $\{gK\}_{\substack{g \in G \\ K \leq G \text{ cpt open}}}$.

$\forall f \in C_c^\infty(G), \exists K \leq G$ cpt open, s.t.
$$f = \sum_{g \in G} a_g \mathbb{1}_{K_g K} \qquad a_g \in \mathbb{C} \qquad \#\{g \in G \mid a_g \neq 0\} < +\infty$$

e.g. When $G = (F, +)$, $C_c^\infty(F) = \langle a + \beta^k \rangle_{\substack{a \in F \\ k \in \mathbb{Z}}}$

when $G = F^\times$, $C_c^\infty(F^\times) = \langle a U^{(k)} \rangle_{\substack{a \in F^\times \\ k \in \mathbb{Z}_{\geq 0}}}$

Def (Left Haar integral & Left Haar measure)

integral: $I : C_c^\infty(G) \longrightarrow \mathbb{C}$ s.t

- (left invariant) $\quad I(f(g \cdot)) = I(f(\cdot)) \qquad \forall f \in C_c^\infty(G) \quad g \in G$
- (positive) $\quad I(f) \geq 0 \qquad\qquad\qquad \forall f \in C_c^\infty(G) \quad f \geq 0$

measure: $\mu_G : \mathcal{L}(G) \longrightarrow \mathbb{R} \qquad S \subset G$ cpt open $\longmapsto I(\mathbb{1}_S)$

The domain of I is not extended, so here it is not perfect.

relation/notation: $\quad I(f) = \int_G f(g) \, d\mu_G(g)$

Rmk.

Later on, Haar measure = left + right Haar measure.

E.g. Let μ be the Haar measure on F, then
$\mu^\times$ is a Haar mesure on $F^\times$, and $\quad \left(d\mu^\times(x) = \dfrac{d\mu(x)}{\|x\|}\right)$

$$\int_{F^\times} f(x) \, d\mu^\times(x) = \int_F f(x) \, \frac{d\mu(x)}{\|x\|} \qquad \forall f \in C_c^\infty(F^\times) \subset C_c^\infty(F)$$

Let μ be the Haar measure on $A := M_{n \times n}(F)$, then
$\mu^\times$ is a Haar measure on $G := GL_n(F)$, and $\left(d\mu^*(g) = \dfrac{d\mu(g)}{\|\det g\|^n}\right)$

$$\int_G f(g) \, d\mu^\times(g) = \int_A f(g) \, \frac{d\mu(g)}{\|\det g\|^n} \qquad \forall f \in C_c^\infty(G) \subset C_c^\infty(A)$$

Def   Unimodular. left Haar measure = right Haar measure

Rmk. G is cpt $\Rightarrow$ G is unimodular $\Leftrightarrow$ $\delta_G = 1$

G is abelian $\Rightarrow$ $G/z(G)$ is $^{\text{①}}$ unimodular

where $\delta_G : G \longrightarrow \mathbb{C}^\times$ is determined by

$d\mu_G(g^{-1}xg) \stackrel{\text{left inv}}{=\!=\!=} d\mu_G(xg) = \delta_G(g)\, d\mu_G(x)$.

Actually, $\forall K \leq G$ cpt open , $\delta_G|_K = \mathbb{1}_K$.

e.g. $(F, +), (\mathcal{O}, +), F^\times, \mathcal{O}^\times$ are all unimodular.

e.g. $G = GL_2(\mathbb{Q}_p)$ is unimodular, while

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ $M = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$ are not unimodular.

It's claimed that every reductive gp over non-archi local field is unimodular, but I don't know the reference.