Coxeter gp ⟵ Braid gp ⟶ Fundamental gp ⟶ Aut

Weyl gp   Galois gp

↑ core   ↑

matrix gp / red gp   Gal($\overline{\mathbb{Q}}_p/\mathbb{Q}_p$)

↑   ⌇

"Rep of $G(\mathbb{Q}_p)$" ⟷ "Rep of $W_{\mathbb{Q}_p}$"

Fact: any f.g. gp can be
realized as $\pi_1(X)$
for some 4-dim mfld X.

Local Langlands correspondence   ⤳ FLC, GLC ?

Roadmap

1.   Group ⤳ Representation ⤳ Geometrical object.

structure   character   construction
· set   classification   geo rep
· topology   L-fct   cohomology, intersection, ζ-fct
· Tits system,
  BT-theory.

2.

Shimura variety ⟶ Motive

Automorphic   FM conj ⋮ ↓ étale cohomology
world ⟵ --- modularity --- Gal rep

3.

	finite field		local field			global field		Adéle	
			Archi	NA						
base field	$\mathbb{F}_r$	$\mathbb{F}_p$	$\mathbb{R}$ or $\mathbb{C}$	$\mathbb{Q}_p$	$\mathbb{F}_p((t))$	$\mathbb{Q}$	$\mathbb{F}_p(t)$	$\mathbb{A}_K$	
integral ring	—	—	—	$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K	

only analog

4. ($G(F)$-case) G $\mathbb{A}^1$ $\mathbb{G}_m$ GL_n red gp (G, B, Unipotent, ...)

$G(F)$ F $F^\times$ $GL_n(F)$ $G(F)$

$G(\mathbb{A}_K)$ $\mathbb{A}_K$ I_K $GL_n(\mathbb{A}_K)$ $G(\mathbb{A}_K)$

(both $G(F)$ & Galois)

coefficient ring Λ: $\mathbb{C}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{F}}_p$, $\overline{\mathbb{Z}}_p$, ...

Roughly, need to solve $3 \times 2 \times 8 \times 4 = 192$ cases
+ much more connections

Usual route: fix 3 & 4,
$$\boxed{\begin{array}{c} 2 \\ G \text{ or } A \end{array}} \longrightarrow 1$$

Anna's route: fix 4 (GL₁),
$$\boxed{\begin{array}{c} 2 \\ G, A, M \end{array}} \longrightarrow 3 \longrightarrow 1$$

Our route:
$$1 \longrightarrow \boxed{\begin{array}{c} 2 \\ G, A \end{array}} \longrightarrow 3 + 4$$

connections in 2 are delayed.

The arrow roughly means:
2 |
1
1

2 |
1
1

Program

§ 1. No rep
- ① [1. Structure of finite/local/global field
- ② [2. Structure of reductive gp (GLₙ)

§ 2. 1-dim rep
- [1. Character of Galois gp
- [2. Character of red gp

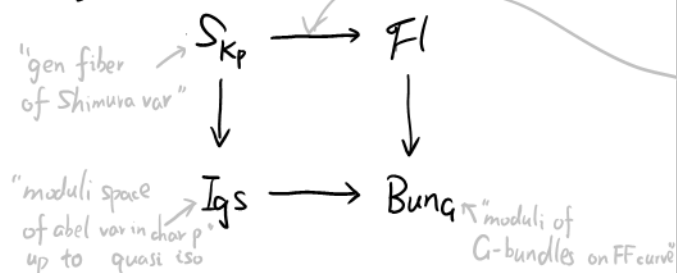§ 3. Rep
- [1. Galois rep
- [2. Rep theory of red gp

§ 4. Geometrical rep
- ④ [1. EC
- ③ [2. MF (Moduli space / Shimura variety / Modular curve)
- 3. Flag variety

§ 5. Connections
- [1. MF $\xrightarrow{ES\ iso}$ Gal rep
- ⑤ [2. MF $\xleftarrow{modularity}$ Gal rep
- 3.

$S_{K_p} \longrightarrow Fl$

"gen fiber of Shimura var" →

$\downarrow \qquad \downarrow$

"moduli space of abel var in char p up to quasi iso" → $Igs \longrightarrow Bun_G$ ↖ "moduli of G-bundles on FF curve"

§ 6. Non-classical Langlands
- 1. Geometrical Langlands
- 2. Categorical geometrical Langlands

(right column)

Galois gp, Frob, Weil gp
Tits system.   BT-theory

local class field theory
$\widehat{F^*}$ & $\widehat{F}^*$, $\widehat{O_F^*}$ & $\widehat{O}_F^*$, Hecke character

WD-rep
l≠p: l-adic monodromy thm
l=p: Hierarchy of p-adic Galois rep
global: Chebychev density thm
        fin /NA/IR /𝔸_K
preliminary
Hecke alg
classification (Hierarchy) { principal series / cuspidal / ...

étale cohomology, Fontaine-Mazur conj
Shimura data
equiv def of MF

Rep Ⅱ

ES iso , ES relation
Deligne-Serre thm
Modularity

Mingjia's work:
    HT period map
    Torelli theorem

– Farques-Scholze
– Chenji's work

Also, in each part :
  · Describe L-fction
  · Describe connections   in section / among sections / with last part.