# Eine Woche, ein Beispiel

## 12.1 weights of type E

It feels incomplete to discuss only the type E case without addressing the other classical cases.

Hence, this document serves as a complement to [2024.12.01].

⚠ There are some new phenomenons outside type E (which are not essential).

1. The formula becomes

$$2\frac{\langle \varpi_i, \alpha_j\rangle}{\langle \alpha_j, \alpha_j\rangle} = \delta_{ij} \qquad \text{i.e.,} \quad \langle \varpi_i \tfrac{2}{\langle\alpha_i,\alpha_i\rangle}, \alpha_j\rangle = \delta_{ij}$$

when $i\neq j$, $\frac{2}{\langle\alpha_i,\alpha_i\rangle}$ won't impact, as $\delta_{ij}=0$

$$s_k v = v - 2\frac{\langle \alpha_k, v\rangle}{\langle \alpha_k, \alpha_k\rangle}\alpha_k$$

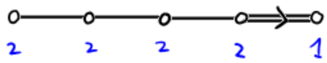
2. $A = (\langle \alpha_i, \alpha_j\rangle)_{i,j}$ is not Cartan matrix. It is $\left(2\frac{\langle \alpha_i,\alpha_j\rangle}{\langle\alpha_i,\alpha_i\rangle}\right)_{i,j}$.

3. The minuscule weight orbit may not generate the whole lattice in type A, B, D

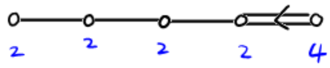
4. The minuscule weight orbit may not be the wts nearest to the origin in type A, B, C, D

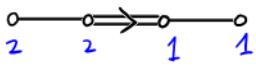
Since the coordinate itself already offers good symmetry(compared with type E case), we will omit many details.

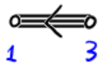
5. In type B, C, F_4, G_2, the coordinate slightly affact the inner product structures, therefore the volume. We choose the coordinate such that most simple roots have length \sqrt(2).

$B_n$ 


$\langle \alpha_i, \alpha_i\rangle = |\alpha_i|^2$

$C_n$ 


$F_4$ 

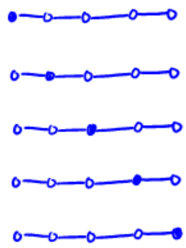

$G_2$ 

## 1. $A_n$ E.g. $n=5$

### – Weights nearest to the origin

There are n many minuscule representations of A_n:

| # | typical coordinates | symbol |
|---|---|---|
| $\binom{6}{1}$ | $\frac{1}{6}(5,-1,-1,-1,-1,-1)^T$ | $v_i$ |
| $\binom{6}{2}$ | $\frac{1}{6}(4,4,-2,-2,-2,-2)^T$ | $v_{ij}$ |
| $\binom{6}{3}$ | $\frac{1}{6}(3,3,3,-3,-3,-3)^T$ | $v_{ijk}$ |
| $\binom{6}{4}$ | $\frac{1}{6}(2,2,2,2,-4,-4)^T$ | $v_{ijkl}$ |
| $\binom{6}{5}$ | $\frac{1}{6}(1,1,1,1,1,-5)^T$ | $v_{ijklm}$ |



$$|v_i|^2 = \langle v_i, v_i \rangle \in \left\{ \frac{5}{6}, \frac{4}{3}, \frac{3}{2} \right\}$$

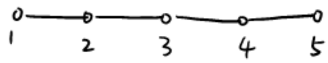
in general, in $\binom{n+1}{k}$, $\langle v_i, v_i \rangle = \frac{k(n+1-k)}{n+1}$ in $\left\{ \sum_{i=1}^{n+1} z_i = 0 \right\} \cong \mathbb{R}^n$

Restrict to the standard rep case,

$$\langle v_i, v_j \rangle \in \left\{ \frac{n}{n+1}, -1 \right\}.$$

The graph has no edges.

## – Simple roots



$$
\begin{aligned}
&\{\ \alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ ,\ \alpha_5\ \}\\
&= \{\ v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6\ \}\\
&= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \right\}
\end{aligned}
$$

Ex. Verify that all the $2\binom{n+1}{2} = 30$ roots are given by

| # | typical coordinates | symbol |
|---|---|---|
| 30 | $(1,\ -1, 0, 0,\ 0,\ 0)^T$ | $\alpha_{1-2}$ |

## – Fundamental weights

$$
\begin{aligned}
&\{\ \varpi_1\ ,\ \varpi_2\ ,\ \varpi_3\ ,\ \varpi_4\ ,\ \varpi_5\ \}\\
&= \{\ v_1\ ,\ v_{12}\ ,\ v_{123}\ ,\ v_{1234}\ ,\ v_{12345}\ \}\\
&= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{6}\begin{pmatrix}4\\4\\-2\\-2\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}3\\3\\3\\-3\\-3\\-3\end{pmatrix}, \frac{1}{6}\begin{pmatrix}2\\2\\2\\2\\-4\\-4\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\}\\
&= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\2\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\1\\1\\1\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\}
\end{aligned}
$$

## – Weyl group action

$S_{n+1}$ acts by permutations. Nothing special.

## 2. $D_n$ E.g. $n=6$ $n>4$ for avoiding special cases.

### – Weights nearest to the origin

There are 3 minuscule representations of D_n:

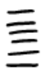
| | # | typical coordinates | | symbol | |
|---|---|---|---|---|---|
| vector | $2\cdot 6$ | $(1, 0, 0, 0, 0, 0)^T$ | | $v_i$ & $-v_i$ |  |
| half spin | $2^5$ | $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)^T$ | odd (neg) sign | $v_{\pm\pm\pm\pm\pm\pm}$ |  |
| | | $\frac{1}{2}(1, 1, 1, 1, 1, -1)^T$ | | $v_-$ | |
| | $2^5$ | $\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)^T$ | even (neg) sign | $v_{\pm\pm\pm\pm\pm\pm}$ |  |
| | | $\frac{1}{2}(1, 1, 1, 1, 1, 1)^T$ | | $v_+$ | |

in general, $\langle v_i, v_i \rangle \in \{1, \frac{n}{4}\}$ in $\mathbb{R}^n$

Restrict to the standard rep case,

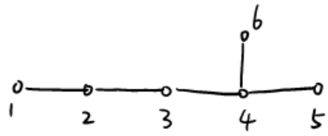
$$\langle v_i, v_j \rangle \in \{1, 0, -1\}.$$

edge

The graph is .

Here, the weights corresponding to standard reps does not generate all other weights.

## - Simple roots




$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$
$$= \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4 - v_5, v_5 - v_6, v_5 + v_6\}$$
$$= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \right\}$$

Ex. Verify that all the $4\binom{n}{2} = 60$ roots are given by

| # | typical coordinates | symbol |
|---|---|---|
| 60 | $(\pm1, \pm1, 0, 0, 0, 0)^T$ | $\alpha_{\pm i \pm j}$ |
| | $(-1, 1, 0, 0, 0, 0)^T$ | $\alpha_{-1+2}$ |

## - Fundamental weights

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$$
$$= \{v_1, v_1 + v_2, \sum_{i=1}^{3} v_i, \sum_{i=1}^{4} v_i, v_-, v_+\}$$
$$= \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \right\}$$

## - Weyl group action

$s_k = s_{(k,k+1)}$ for $k = 1, \dots, n-1$.

$$s_n = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & & -1 \\ & & & -1 & \end{pmatrix}$$

$$W(D_n) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n \subseteq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

## 3. $D_4$

### – Weights nearest to the origin

D_4 is more symmetric.

| # | typical coordinates | symbol | |
|---|---|---|---|
| $8 = 2\cdot 4$ | $(1,0,0,0)^T$ | $v_i$ & $-v_i$ |  |
| $8 = 2^3$ | $\frac{1}{2}(\pm1,\pm1,\pm1,\pm1)^T$ odd sign<br>$\frac{1}{2}(1,1,1,1)^T$ | $v_{\pm\pm\pm\pm}$<br>$v_-$ |  |
| $8 = 2^3$ | $\frac{1}{2}(\pm1,\pm1,\pm1,\pm1)^T$ even sign<br>$\frac{1}{2}(1,1,1,1)^T$ | $v_{\pm\pm\pm\pm}$<br>$v_+$ |  |

If not restricted to the standard representation case,

$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1\}.$$

## 4. $B_n$ E.g. $n=5$

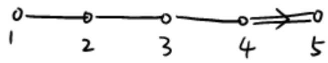
– minuscule weights

The minuscule weight of B_n is usually not the nearest weight orbit:

| | # | typical coordinates | symbol |
|---|---|---|---|
| spin | $32 = 2^5$ | $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1)^T$ | $v_{\pm\pm\pm\pm\pm}$ |
| | | $\frac{1}{2}(1, -1, 1, 1, -1)^T$ | $v_{+-++-}$ |

$$\langle v_i, v_j \rangle \in \left\{ \frac{n}{4}, \frac{n-2}{4}, \dots, \frac{-n}{4} \right\}. \quad \text{in } \mathbb{R}^n$$

## - Simple roots




$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$
$$= \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5\}$$
$$= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\}$$

Ex. Verify that all the $2 \cdot n^2 = 50$ roots are given by

| # | typical coordinates | symbol |
|---|---|---|
| $20 = 2 \cdot \binom{5}{2}$ | $(1, -1, 0, 0, 0)^T$ | $v_{1-2}$ |
| $10 = 2 \cdot 5$ | $(1, 0, 0, 0, 0)^T$ | $v_1$ |
| $20 = 2 \cdot \binom{5}{2}$ | $(1, 1, 0, 0, 0)^T$ | $v_{1+2}$ & $v_{-1-2}$ |

## - Fundamental weights

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$$
$$= \{v_1,\ v_1 + v_2,\ \sum_{i=1}^{3} v_i,\ \sum_{i=1}^{4} v_i,\ v_{+++++}\}$$
$$= \left\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \right\}$$

## - Weyl group action

$s_k = s_{(k,k+1)}$ for $k = 1, \dots, n-1$.

$$s_n = \begin{pmatrix}1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1\end{pmatrix}$$

$$W(B_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

5. $C_n$ E.g. $n=5$

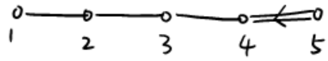
- minuscule weights

The minuscule representation of C_n is the standard representation:

| | # | typical coordinates | symbol |
|---|---|---|---|
| vector | $10 = 2 \cdot 5$ | $(1, 0, 0, 0, 0)^T$ | $v_i$ & $-v_i$ |

$\langle v_i, v_j \rangle \in \{1, 0\}$. in $\mathbb{R}^n$

- Simple roots




$$
\begin{aligned}
&\{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5\ \} \\
=\ &\{\ v_1-v_2,\ v_2-v_3,\ v_3-v_4,\ v_4-v_5,\ v_5\ \} \\
=\ &\left\{ \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\2\end{pmatrix}, \right\}
\end{aligned}
$$

Ex. Verify that all the $2\cdot n^2 = 50$ roots are given by

| # | typical coordinates | symbol |
|---|---|---|
| $20 = 2\cdot\binom{5}{2}$ | $(1, -1, 0, 0, 0)^T$ | $v_{1-2}$ |
| $10 = 2\cdot 5$ | $(2, 0, 0, 0, 0)^T$ | $2v_1$ |
| $20 = 2\cdot\binom{5}{2}$ | $(1, 1, 0, 0, 0)^T$ | $v_{1+2}$ & $v_{-1-2}$ |

- Fundamental weights

$$
\begin{aligned}
&\{\ \varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4,\ \varpi_5\ \} \\
=\ &\{\ v_1,\ v_1+v_2,\ \sum_{i=1}^{3} v_i,\ \sum_{i=1}^{4} v_i,\ 2v_{+++++}\ \} \\
=\ &\left\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \right\}
\end{aligned}
$$

- Weyl group action

$$s_k = s_{(k,k+1)} \quad \text{for } k = 1, \dots, n-1.$$

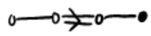
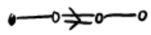
$$s_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$$W(C_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

6. $F_4$

## – Weights nearest to the origin

We make a list of the root lattices:

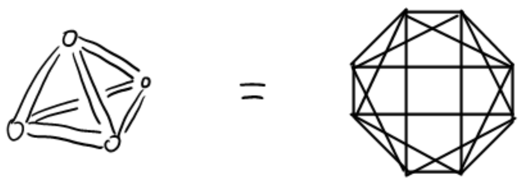
| # | typical coordinates | | symbol | | |
|---|---|---|---|---|---|
| $8 = 4 \cdot 2$ | $(\pm 1, 0, 0, 0)^T$ | | $e_1$ | short | |
| $16 = 2^4$ | $\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$ | even sign / odd sign | $\alpha_{\pm\pm\pm\pm}$ | short |  |
| $24 = 4 \cdot \binom{4}{2}$ | $(\pm 1, \pm 1, 0, 0)^T$ | | $\alpha_{\pm 1 \pm 2}$ | long |  |

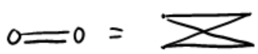
Restrict to the short roots:

$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1\} \quad \text{in } \mathbb{R}^4$$

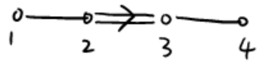
edge

Restrict to thhe short roots, the graph constructed has 24 vertices and 72 edges. It is not connected, and has 3 components. The connected component has HoG Id 176.





- Simple roots




$$\{\ \alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ \}$$
$$= \{\ \alpha_{2-3}\ ,\ \alpha_{3-4}\ ,\ e_4\ ,\ \alpha_{+---}\ \}$$
$$= \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \right\}$$

- Fundamental weights

$$\{\ \varpi_1\ ,\ \varpi_2\ ,\ \varpi_3\ ,\ \varpi_4\ \}$$
$$\{\ \alpha_{1+2}\ ,\ \alpha_{1+2} + \alpha_{1+3}\ ,\ e_1 + \alpha_{++++}\ ,\ e_1\ \}$$
$$= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \right\}$$

- Weyl group action

$$s_1 = s_{(2,3)} \qquad s_2 = s_{(3,4)} \qquad s_3 \rightsquigarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \qquad s_4 \rightsquigarrow \frac{1}{4}\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 1 & -1 & 3 & -1 \\ 1 & -1 & -1 & 3 \end{pmatrix}$$

## 7. $G_2$

### – Weights nearest to the origin

We make a list of the root lattices:

| # | typical coordinates | symbol | | |
|---|---|---|---|---|
| 6 | $\frac{1}{\sqrt{2}}(1, -1, 0)^T$ | $\alpha_{1-2}$ | short | |
| 6 | $\frac{1}{\sqrt{2}}(2, -1, -1)^T$ | $\beta_1$ & $-\beta_1$ | long | |

Restrict to the short roots:

$$\langle v_i, v_j \rangle \in \left\{1, \tfrac{1}{2}, -\tfrac{1}{2}, -1\right\}$$

$$\text{in} \quad \left\{\sum_{i=1}^{3} z_i = 0\right\} \cong \mathbb{R}^2$$

### – Simple roots

$$\begin{aligned}
&\{\alpha_1, \quad \alpha_2\} \\
=&\{\alpha_{1-2}, \quad \beta_2\} \\
=&\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\}
\end{aligned}$$

### – Fundamental weights

$$\begin{aligned}
&\{\varpi_1, \quad \varpi_2\} \\
=&\{\alpha_{3-2}, \quad -\beta_3\} \\
=&\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\}
\end{aligned}$$

### – Weyl group action

$$s_1 = S_{(1,2)} \qquad s_2 \rightsquigarrow \frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & 2 & -1\\ 2 & -1 & 2\end{pmatrix}$$