Eine Woche, ein Beispiel

5.14. modular representation of $\mathbb{Z}/p\mathbb{Z}$

Let $\mathcal{C} = \mathrm{rep}_\Lambda(\mathbb{Z}/p\mathbb{Z}) = \mathrm{mod}(\Lambda[\mathbb{Z}/p\mathbb{Z}])$, where
$\Lambda = \overline{\Lambda}$ is a field with $\mathrm{char}\,\Lambda = p$.

Goal: understand $\mathcal{C}$ in detail.

1. indecomposable representations
2. tensor category structure
3. semisimplification

1. indecomposable representations

We have

$$\Lambda[\mathbb{Z}/p\mathbb{Z}] \cong \Lambda[x]/(x^p-1) \cong \Lambda[x]/(x-1)^p \cong \Lambda[T]/T^p$$

$$\begin{array}{cc}
N(p) & \left(\Lambda^p, \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
\vdots & \\
\uparrow\downarrow & \\
N(2) & \left(\Lambda^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \\
\uparrow\downarrow & \\
N(1) & (\Lambda, 0)
\end{array}$$

AR-quiver of $\quad \bullet \circlearrowleft T \,/\, T^p = 0 \quad = \Lambda[T]/T^p$

https://math.stackexchange.com/questions/368722/what-does-the-group-ring-mathbbzg-of-a-finite-group-know-about-g

2. tensor category structure.

For general ring A/Λ, there is no tensor structure on mod(A).
However, for a Hopf algebra A/Λ, we can construct a natural tensor structure on mod(A).

Construction.

$$c^\#: A \longrightarrow A \otimes_\Lambda A \quad \rightsquigarrow \quad \otimes: \mathrm{mod}(A) \times \mathrm{mod}(A) \longrightarrow \mathrm{mod}(A \otimes_\Lambda A) \xrightarrow{c^{\#,*}} \mathrm{mod}(A)$$

$$(M, N) \longmapsto M \otimes_\Lambda N \longmapsto M \otimes_\Lambda N$$

where A acts on $M \otimes_\Lambda N$ by

$$A \times M \otimes_\Lambda N \longrightarrow M \otimes_\Lambda N \qquad a \cdot (m \otimes n) := c^\#(a)(m \otimes n) = \sum_i b_i m \otimes c_i n$$

when $c^\#(a) = \sum_i b_i \otimes c_i$

$$e^\#: A \longrightarrow \Lambda \quad \rightsquigarrow \quad e^{\#,*}: \mathrm{mod}(\Lambda) \longrightarrow \mathrm{mod}(A)$$

$$\Lambda \longmapsto \Lambda$$

$$A \times \Lambda \longrightarrow \Lambda \qquad (a, t) \longmapsto e^\#(a) \cdot t$$

$$i^\#: A \longrightarrow A^{op} \quad \rightsquigarrow \quad (-)^\vee: \mathrm{mod}(A) \xrightarrow{\mathrm{Hom}_\Lambda(-,\Lambda)} \mathrm{mod}(A^{op}) \xrightarrow{i^{\#,*}} \mathrm{mod}(A)$$

$$M \longmapsto M^\vee \longmapsto M^\vee$$

$$A \times M^\vee \longrightarrow M^\vee \qquad (a, f) \longmapsto f(i^\#(a) -)$$

Q: Let A be a Λ-alg.
Given a tensor category structure on mod(A), can we recover the Hopf algebra on A?
I.e., is the map

$$\left\{ \text{Hopf algebra structures on } A \right\} \longrightarrow \left\{ \text{tensor category structures on } \mathrm{mod}(A) \right\}$$

inj or surj?

E.g. (tensor category structure of mod$(\Lambda[G])$) $\quad G$: finite gp

$\mathrm{rep}_\Lambda(G)$ is naturally endowed with $\otimes$-structure:

$$G \circlearrowright M\otimes N \qquad g\cdot(m\otimes n) := gm\otimes gn$$
$$\rightsquigarrow \Lambda[G]\circlearrowright M\otimes N \qquad (\sum_i t_i g_i)(m\otimes n) = \sum_i t_i\, g_i\cdot(m\otimes n) = \sum_i t_i (g_i m\otimes g_i n) = (\sum_i t_i(g_i\otimes g_i))(m\otimes n)$$

so the Hopf algebra structure on $\Lambda[G]$ should be

$$c^\#: \Lambda[G]\longrightarrow \Lambda[G]\otimes_\Lambda\Lambda[G] \qquad \sum_i t_i g_i \longmapsto \sum_i t_i g_i\otimes g_i$$
$$e^\#: \Lambda[G]\longrightarrow \Lambda \qquad \sum_i t_i g_i \longmapsto \sum_i t_i$$
$$i^\#: \Lambda[G]\longrightarrow \Lambda[G]^{op} \qquad \sum_i t_i g_i \longmapsto \sum_i t_i g_i^{-1}$$

[Verify:

$$G\circlearrowright\Lambda \qquad g.t := t$$
$$\rightsquigarrow \Lambda[G]\circlearrowright\Lambda \qquad (\sum_i t_i g_i)\, t = \sum_i t_i(g_i\cdot t) = \sum_i t_i t$$

$$G\circlearrowright M^\vee \qquad g\cdot f := f(g^{-1}\cdot -)$$
$$\rightsquigarrow \Lambda[G]\circlearrowright M^\vee \qquad (\sum_i t_i g_i) f = \sum_i t_i(g_i\cdot f) = \sum_i t_i f(g_i^{-1}\cdot -) = f(\sum_i t_i g_i^{-1} -)$$
]

e.p. $\mathrm{Spec}\,\Lambda[\mathbb{Z}/n\mathbb{Z}] \cong \mu_{n,\Lambda}$ as a finite gp scheme.

E.g. (tensor category structure of mod$(\mathcal{U}(\mathfrak{g}))$) $\quad \mathfrak{g}$: f.d. Lie alg over $\mathbb{C}$

$\mathrm{rep}_\mathbb{C}(\mathfrak{g})$ is naturally endowed with $\otimes$-structure:

$$\mathfrak{g}\circlearrowright M\otimes N \qquad X\cdot(m\otimes n) := X.m\otimes n + m\otimes X.n$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M\otimes N \qquad X_1X_2\cdots X_n(m\otimes n) = \sum_{\{1,\cdots,k\}=I\sqcup J} (X_I m)\otimes(X_J n) \quad \text{shuffle!}$$

e.p. $[X,Y](m\otimes n) = [X,Y]m\otimes n + m\otimes[X,Y]n$

(For $I=\{i_1,\cdots,i_l\}$ fix an order $i_1<i_2<\cdots<i_l$, $X_I := X_{i_1}X_{i_2}\cdots X_{i_n}$)

so the Hopf algebra structure on $\mathcal{U}(\mathfrak{g})$ should be

$$c^\#: \mathcal{U}(\mathfrak{g})\longrightarrow \mathcal{U}(\mathfrak{g})\otimes_\mathbb{C}\mathcal{U}(\mathfrak{g}) \qquad X_{\{1,\cdots,k\}}\longmapsto \sum_{\{1,\cdots,k\}=I\sqcup J} X_I\otimes X_J$$
$$e^\#: \mathcal{U}(\mathfrak{g})\longrightarrow\mathbb{C} \qquad \sum_\alpha t_\alpha X_\alpha \longmapsto t_\emptyset$$
$$i^\#: \mathcal{U}(\mathfrak{g})\longrightarrow\mathcal{U}(\mathfrak{g})^{op} \qquad \sum_\alpha t_\alpha X_\alpha \longmapsto \sum_\alpha (-1)^{|\alpha|} t_\alpha X_\alpha$$

[Verify:

$$\mathfrak{g}\circlearrowright\mathbb{C} \qquad X.t := 0$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright\mathbb{C} \qquad (\sum_\alpha t_\alpha X_\alpha).t = t_\emptyset t$$

$$\mathfrak{g}\circlearrowright M^\vee \qquad X.f := \overset{\text{minus!}}{-} f(X.-)$$
$$\rightsquigarrow \mathcal{U}(\mathfrak{g})\circlearrowright M^\vee \qquad (\sum_\alpha t_\alpha X_\alpha).t = \sum_\alpha t_\alpha(-1)^{|\alpha|} f(X_\alpha.-) = f(\sum_\alpha (-1)^{|\alpha|} t_\alpha X_\alpha.-)$$
]

For more examples of Hopf algebras, see wiki: Hopf algebras.

3. semisimplification.

$\mathrm{Ver}_p := \overline{\mathcal{C}}$ is a fusion category with simple objects

see wiki, or lecture notes.

$\overline{N(1)}, \dots, \overline{N(p-1)}$, denoted as $X_1, \dots, X_{p-1}$.

⚠ For $M, N \in \mathrm{Ver}_p$, T acts on $M \otimes N$ by

$$\begin{aligned} T(m \otimes n) &= (x-1)(m \otimes n) \\ &= xm \otimes xn - m \otimes n \\ &= (T+1)m \otimes (T+1)n - m \otimes n \\ &= Tm \otimes Tn + Tm \otimes n + m \otimes Tn \end{aligned}$$

So we don't have $T(m\otimes n) = Tm \otimes Tn$, i.e. T is not a group-like element.

Lemma. In any Ver_p,

$$X_2 \otimes X_i \cong \begin{cases} X_0 \oplus X_2 & i = 1 \\ X_{i-1} \oplus X_{i+1} & 1 < i < p-1 \\ X_{p-2} \oplus X_p & i = p-1 \end{cases}$$

virtual minus sign

If we write $X_2 \otimes X_i = X_{i-1} \oplus X_{i+1}$, we need to assume $X_0 = X_p = 0$, $X_{p+1} = -X_{p-1}, \dots$

Proof. Let $M = \begin{pmatrix} J_i & J_i \\ & J_i \end{pmatrix}$, $J_i = \mathrm{Id} + N_i$, $N_i = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$,

we need to find the Jordan normal form of M.

$$M - I = \begin{pmatrix} N_i & J_i \\ & N_i \end{pmatrix} = \begin{pmatrix} N_i & \\ & N_i \end{pmatrix} + \begin{pmatrix} & J_i \\ & \end{pmatrix}$$

Since N_i commutes with J_i, $\begin{pmatrix} N_i & \\ & N_i \end{pmatrix}$ commutes with $\begin{pmatrix} & J_i \\ & \end{pmatrix}$,

$$\begin{aligned} (M-I)^l &= \left(\begin{pmatrix} N_i & \\ & N_i \end{pmatrix} + \begin{pmatrix} & J_i \\ & \end{pmatrix} \right)^l \qquad \text{for } l \in \mathbb{N}_{>0} \\ &= \sum_{k=0}^{l} \binom{l}{k} \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^{l-k} \begin{pmatrix} & J_i \\ & \end{pmatrix}^k \\ &= \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^l + l \begin{pmatrix} N_i & \\ & N_i \end{pmatrix}^{l-1} \begin{pmatrix} & J_i \\ & \end{pmatrix} \\ &= \begin{pmatrix} N_i^l & l N_i^{l-1} J_i \\ & N_i^l \end{pmatrix} \end{aligned}$$

$$\left. \begin{aligned} \mathrm{rk}(M-I) &\geq 2p-2 \\ (M-I)^i &\neq 0 \\ (M-I)^{i+1} &= 0 \end{aligned} \right\} \Rightarrow M - I \sim \begin{array}{c} \begin{array}{cc} \scriptstyle i+1 & \scriptstyle i-1 \end{array} \\ \left(\begin{array}{c|c} \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{matrix} & \\ \hline & \begin{matrix} 0 & \ddots & \\ & \ddots & 1 \\ & & 1 \end{matrix} \end{array} \right) \end{array}$$

$\square$