# Eine Woche, ein Beispiel

## 8.21 equivariant K-theory of $\mathbb{P}^1$

Let us do a simple case over $\mathbb{P}^1$. It can be generlized "easily" to flag variety, but $\mathbb{P}^1$ is the beginning case of study.

Ref:
[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"
[LCBE] Langlands correspondence and Bezrukavnikov's equivalence
[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand

$$\begin{tikzcd}
K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[d] \\
K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2}(pt) \arrow[d] \\
K^{B}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{B}(pt) \arrow[d] \\
K(\mathbb{P}^1) \arrow[r] & K(pt)
\end{tikzcd}$$

where $SL_2 = SL_{2,\mathbb{C}}$, $\quad B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subseteq SL_{2,\mathbb{C}}$,

$\mathbb{P}^1 = \mathbb{P}^1_{\mathbb{C}} \cong G/B$. $\quad G \underset{mult}{\overset{left}{\curvearrowright}} \mathbb{P}^1 \quad \mathbb{C}^\times \overset{trivial}{\curvearrowright} \mathbb{P}^1$

Notation. For linear alg gp $G$ [Ginz, 5.1],

$K_i^G(X) := K_i(Coh^G(X)) \qquad K^G(X) := K_0^G(X) \qquad K(X) := K^{\{Id\}}(X)$

$R(G) := K^G(pt) = K_0(Coh^G(pt)) = K_0(Rep\, G)$

e.g. $R(\{Id\}) = \mathbb{Z}$, $R(B) \cong R(T) \cong \mathbb{Z}[y^{\pm 1}]$, $R(G) \cong \mathbb{Z}[x]$, $R(G\times\mathbb{C}^\times) \cong \mathbb{Z}[x, t^{\pm 1}]$

We only care about addition structure here. Everything is only $\mathbb{Z}$-module.

[Ginz, (5.2.4)] $\quad \begin{matrix} G_1 \curvearrowright X \\ G_2 \underset{trivial}{\curvearrowright} X \end{matrix} \Rightarrow K^{G_1\times G_2}(X) \cong K^{G_1}(X)\otimes_{\mathbb{Z}} R(G_2)$

e.g. $K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \cong K^{SL_2}(\mathbb{P}^1)\otimes_{\mathbb{Z}} R(\mathbb{C}^\times) \cong K^{SL_2}(\mathbb{P}^1)\otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$

$K^B(\mathbb{P}^1) \cong K(\mathbb{P}^1)\otimes_{\mathbb{Z}} R(B) \cong K(\mathbb{P}^1)\otimes_{\mathbb{Z}} \mathbb{Z}[y^{\pm 1}]$

[Ginz, (5.2.17)]

$$K_i^H(X) \underset{Ind_H^G}{\overset{Res_H^G}{\rightleftarrows}} K_i^G(G\times_H X)$$

e.g. $K^{SL_2}(\mathbb{P}^1) \cong K^{SL_2}(SL_2\times_B pt) \cong K^B(pt) = R(B) = \mathbb{Z}[z^{\pm 1}]$

[LCBE, 2.1.1] $\quad K(\mathbb{P}^1) \cong \mathbb{Z}\mathcal{O}_{\mathbb{P}^1} \oplus \mathbb{Z}\mathcal{O}_{\mathbb{P}^1}(1) = \mathbb{Z}[w]/(w-1)^2 \quad = \mathbb{Z}[w^{\pm 1}]/(w-1)^2$

[Ginz, 5.2.13] gives def of pushforward.

In conclusion, we get

$$
\begin{tikzcd}
K^{SL_2\times\mathbb{C}^\times}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2\times\mathbb{C}^\times}(pt) \arrow[d] \\
K^{SL_2}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{SL_2}(pt) \arrow[d] \\
K^{B}(\mathbb{P}^1) \arrow[r] \arrow[d] & K^{B}(pt) \arrow[d] \\
K(\mathbb{P}^1) \arrow[r] & K(pt)
\end{tikzcd}
\qquad
\begin{tikzcd}
t \arrow[d, mapsto] & \mathbb{Z}[z^{\pm 1}, t^{\pm 1}] \arrow[r] \arrow[d] & \mathbb{Z}[x, t^{\pm 1}] \arrow[d] & t \arrow[d, mapsto] \\
1 & \mathbb{Z}[z^{\pm 1}] \arrow[r] \arrow[d] & \mathbb{Z}[x] \arrow[d] & 1 \\
y \arrow[d, mapsto] & \mathbb{Z}[w, y^{\pm 1}]/(w-1)^2 \arrow[r] \arrow[d] & \mathbb{Z}[y^{\pm 1}] \arrow[d] & y \arrow[d, mapsto] \\
1 & \mathbb{Z}[w]/(w-1)^2 \arrow[r] & \mathbb{Z} & 1 \\
 & w \arrow[r, mapsto] & 1 &
\end{tikzcd}
$$

The difficult part is the middle square.

Down: $\mathbb{Z}[w, y^{\pm 1}]/(w-1)^2 \longrightarrow \mathbb{Z}[y^{\pm 1}]$

$$
\begin{aligned}
w &\longmapsto 1 \\
y &\longmapsto y \\
y^{-1} &\longmapsto y^{-1}
\end{aligned}
$$

Right: by rep theory,

$\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm 1}]$

$$
\begin{aligned}
1 &\longmapsto 1 \\
x &\longmapsto y + y^{-1} \\
x^2 &\longmapsto y^2 + 1 + y^{-2} \\
x^3 &\longmapsto y^3 + y + y^{-1} + y^{-3} \\
&\ \ \vdots
\end{aligned}
$$

Up: by Borel-Weil-Bott theorem,

$\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[x]$

$$
\begin{aligned}
1 &\longmapsto 1 & \qquad z &\longmapsto 0 \\
z^{-1} &\longmapsto x & z^2 &\longmapsto -1 \\
z^{-2} &\longmapsto x^2 & z^3 &\longmapsto -x \\
z^{-3} &\longmapsto x^3 & z^4 &\longmapsto -x^2 \\
&\ \ \vdots & z^5 &\longmapsto -x^3 \\
& & &\ \ \vdots
\end{aligned}
$$

Left: by [LW-BWB, Ex 2.6], $\mathcal{L}_n \cong \mathcal{O}(-n)$, combined with "Up", we get

$\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[w, y^{\pm 1}]/(w-1)^2$

e.g. $z^3 \longmapsto -w^{-3}(y + y^{-1})$ (see table below)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $z$ | $z^{-2}$ | $z^{-1}$ | $1$ | $z$ | $z^2$ | $z^3$ | $z^4$ | $z^m$ |
| $x$ | $x^2$ | $x$ | $1$ | $0$ | $-1$ | $-x$ | $-x^2$ | $\begin{cases} -x^{m-2} & m \geq 2 \\ 0 & m = 1 \\ x^{-m} & m \leq 0 \end{cases}$ |
| $y$ | $y^2+1+y^{-2}$ | $y+y^{-1}$ | $1$ | $0$ | $-1$ | $-y-y^{-1}$ | $-y^2-1-y^{-2}$ | $-\frac{y^m - y^{-m+2}}{y^2-1}$ |
| $w$ | $w^2$ | $w$ | $1$ | $w^{-1}$ | $w^{-2}$ | $w^{-3}$ | $w^{-4}$ | $w^{-m}$ |
| $w, y$ | $w^2(y^2+1+y^{-2})$ | $w(y+y^{-1})$ | $1$ | $0$ | $-w^{-2}$ | $-w^{-3}(y+y^{-1})$ | $-w^{-4}(y^2+1+y^2)$ | $-w^{-m}\frac{y^m - y^{-m+2}}{y^2-1}$ |

Ex. Generalize to

- $SL_2 \rightsquigarrow SL_n$, $\mathbb{P}^1 \rightsquigarrow \mathcal{F}lag(\mathbb{C}^n)$
- $SL_2 \rightsquigarrow GL_2$
- $\mathbb{C} \rightsquigarrow \mathbb{F}_p$ $\quad \mathbb{C}^\times \rightsquigarrow \mathbb{F}_p^\times$

Q: How to compute $K_i^{SL_2 \times \mathbb{C}^\times}(\mathbb{P}^1)$ for $i \geq 1$?