

# Eine Woche, ein Beispiel

## 5.1 Extension of NA local field

F: NA local field

### 1 List of well-known results

- in general
- unramified / totally ramified

### 2. $\hat{\mathbb{Z}}$ = profinite completion (review)

### 3. Big picture

### 4. Henselian ring

### 5. Cohomological dimension.

} not complete, I need time to check the proof

Q: Is there any subfield of  $\mathbb{Q}_p$  with finite index?

Can we classify all subfield of  $\mathbb{F}_p((t))$  with finite index?

<https://math.stackexchange.com/questions/211582/is-there-a-proper-subfield-k-subset-mathbb-r-such-that-mathbb-rk-is-fin>

Ref:

Initial motivation comes from

[AY] <https://alex-youcis.github.io/localglobalgalois.pdf>

which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:

[NSW2e] <https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/>

[JPS96] Galois cohomology by Jean-Pierre Serre

<http://p-adic.com/Local%20Fields.pdf>

<https://people.clas.ufl.edu/rcrow/files/LCFT.pdf>

<http://www.mcm.ac.cn/faculty/tianyichao/201409/W020140919372982540194.pdf>

# 1. List of well-known results

In general

$F$ : NA local field  $E/F$ : finite extension

Rmk 1.  $E$  is also a NA local field with uniquely extended norm

$$\|x\|_E = \|N_{E/F}(x)\|_F^{\frac{1}{n}} \quad \text{resp.} \quad v(x) := \frac{1}{n} v_F(N_{E/F}(x))$$

E.g.  $\|1 - \zeta_n\| = 1$  in  $\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p$   $p \nmid n$   $v(1 - \zeta_n) = 0$

$$\|1 - \zeta_p\| = \frac{1}{p} \text{ in } \mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p \quad v(1 - \zeta_p) = \frac{1}{p}$$

$$\|1 - \zeta_5\| = \|(1 - \zeta_5)(1 - \zeta_5^2)(1 - \zeta_5^3)(1 - \zeta_5^4)\|^{\frac{1}{4}}_{\mathbb{Q}_5} = \|5\|^{\frac{1}{4}}_{\mathbb{Q}_5} = \frac{1}{4} \text{ in } \mathbb{Q}_5(\zeta_5)$$

$$\|1 - \zeta_{p^n}\| = p^{-\frac{1}{p^n}} \text{ in } \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p \quad v(1 - \zeta_{p^n}) = \frac{1}{p^n}$$

$\Rightarrow 1 - \zeta_{p^n}$  is a uniformizer of  $\mathbb{Q}_p(\zeta_{p^n})$

Rmk 2. [AY, Thm 1.9]

$\mathcal{O}_E$  is monogenic, i.e.  $\mathcal{O}_E = \mathcal{O}_F[\alpha] \quad \exists \alpha \in \mathcal{O}_E$

Cor. (primitive element thm for NA local field)

$$E = F[x]/(g(x)) \quad \exists x \in \mathcal{O}_E, \quad g(x) \text{ min poly of } x.$$

Rmk: Every separable finite field extension has a primitive element, see wiki:

[https://en.wikipedia.org/wiki/Primitive\\_element\\_theorem](https://en.wikipedia.org/wiki/Primitive_element_theorem)

Separable condition is necessary, see

<https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element>

Rmk 3. Any finite extension of  $\mathbb{Q}_p$  is of form  $\mathbb{Q}_p[x]/(g(x))$ ,

where  $g(x) \in \mathbb{Q}[x]$  is an irr poly.

Any finite extension of  $\mathbb{F}_q((t))$  is of form  $\mathbb{F}_q((t))[x]/(g(x))$

where  $g(x) \in \mathbb{F}_q((t))[x]$  is an irr poly.

Both are achieved by Krasner's lemma.

<https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete>

$$\begin{array}{ccccc}
 v = v_F = \frac{1}{e} v_E & \|\cdot\| = \|\cdot\|_F = \|\cdot\|_E^{\frac{1}{e}} & \mathfrak{p}_F \mathcal{O}_E = \mathfrak{p}_E^e & & \\
 E & v_E = e v & \|\cdot\|_E = \|\cdot\|^e & \pi_E = \pi_F^{\frac{1}{e}} & v(\pi_E) = \frac{1}{e} \\
 | \deg n & & & & \\
 F & v_F & \|\cdot\|_F & \pi_F & v(\pi_F) = 1
 \end{array}$$

## Unramified/totally ramified

Good ref: [https://en.wikipedia.org/wiki/Finite\\_extensions\\_of\\_local\\_fields](https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields)  
 It collects the equivalent conditions of unramified/totally ramified field extensions.

| tot ram | wild ram  
 | tame ram  
 | field ext  
 | ~~split~~ in local case

When  $E/F$  is tot ramified,

$$e = n \quad v(\pi_E) = \frac{1}{n}$$

$$\mathcal{O}_E = \mathcal{O}_F[\pi_E] \quad \min(\pi_E) \in \mathcal{O}_F[x] \text{ is Eisenstein poly.}$$

2.  $\hat{\mathbb{Z}}$  = profinite completion of  $\mathbb{Z}$  (Recall 2022.2.13 outer auto...)

$$\hat{\mathbb{Z}} := \prod_l \mathbb{Z}_l$$

$$\hat{\mathbb{Z}}^x = \prod_l \mathbb{Z}_l^x$$

$$\hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$(\hat{\mathbb{Z}}^x)^{(p)} := \prod_{l \neq p} \mathbb{Z}_l^x = (\hat{\mathbb{Z}}^{(p)})^x$$

Prop. ①  $\text{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l=m \\ 0 & l \neq m \end{cases} \quad l, m \text{ prime.}$

$$\textcircled{2} \text{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^x$$

$$\text{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^x$$

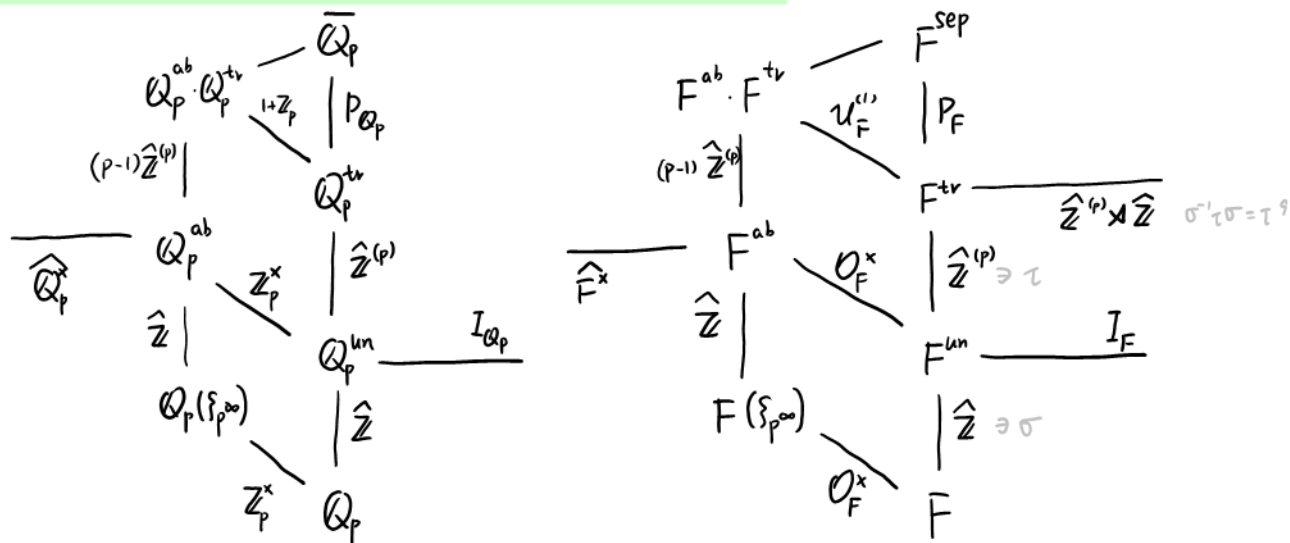
$$\text{Aut}(\hat{\mathbb{Z}}^{(p)}) = \hat{\mathbb{Z}}^{x(p)}$$

in the category of profinite gps.

$$\textcircled{3} \mathcal{O}_F, \mathcal{O}_F^x \text{ are profinite groups, so } \hat{\mathcal{O}}_F = \mathcal{O}_F \quad \hat{\mathcal{O}}_F^x = \mathcal{O}_F^x.$$

### 3. Big picture

Main ref: [AY] <https://alex-youcis.github.io/localglobalgalois.pdf>



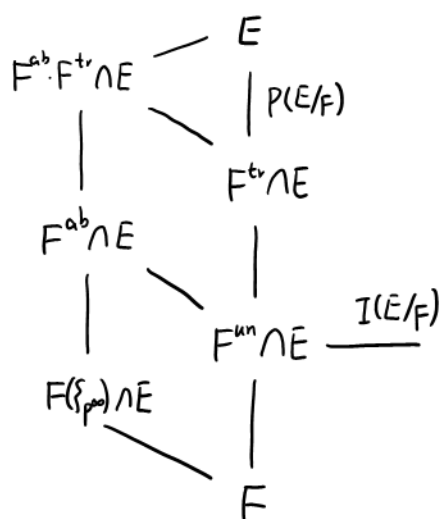
$$\text{unramified} \quad F^{un} = \bigcup_{n \geq 1} F(\zeta_{p^n-1}) \xrightarrow{\text{Fermat's little thm}} \bigcup_{\substack{n \geq 1 \\ p \nmid n}} F(\zeta_n)$$

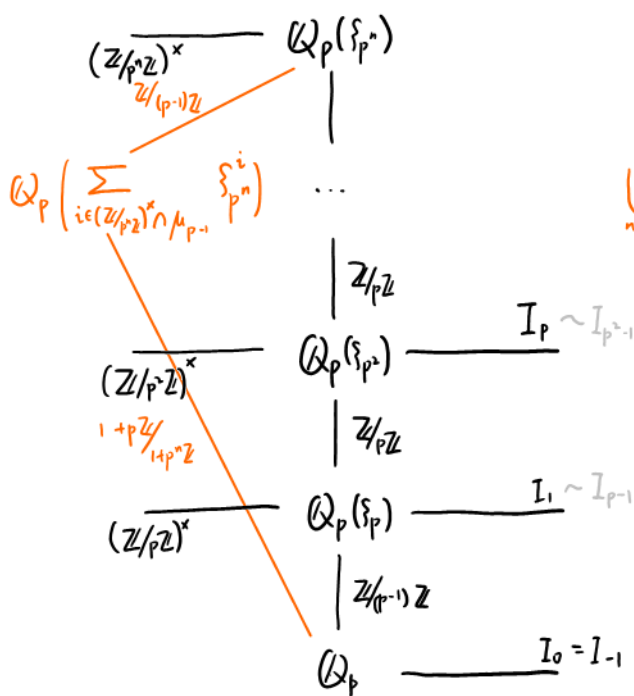
$$\text{tame ramified} \quad F^{tr} = F^{un}(\pi_F^{\frac{1}{n}} |_{(n,p)=1})$$

$$\text{abelian} \quad F^{ab} = F(\zeta_\infty) := \bigcup_{n \geq 1} F(\zeta_n)$$

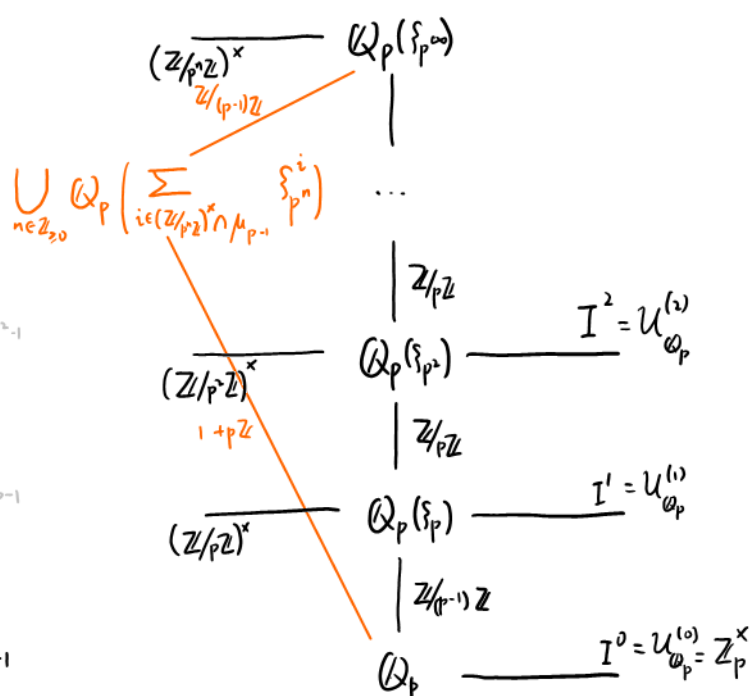
$$F^{ab} \cdot F^{tr} = F(\pi_F^{\frac{1}{n}}, \zeta_\infty |_{(n,p)=1})$$

<https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions>

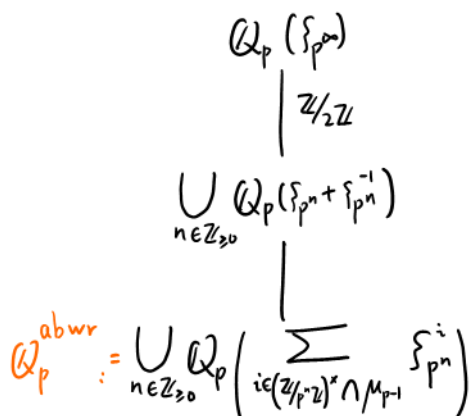
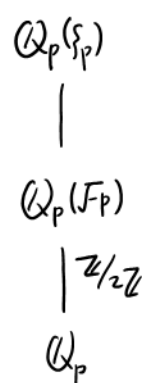
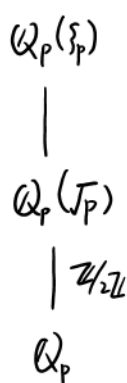
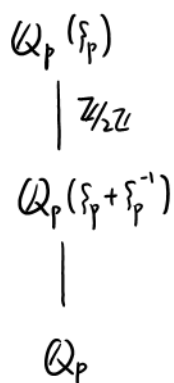




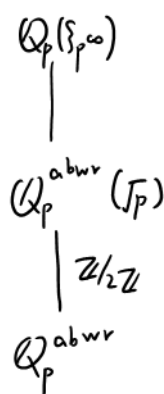
$$E/F = \mathbb{Q}_p(\xi_{p^\infty})/\mathbb{Q}_p$$



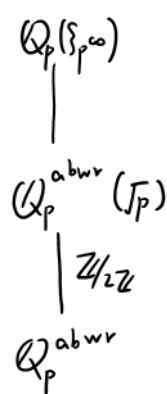
$$E/F = \mathbb{Q}_2(\xi_{2^\infty})/\mathbb{Q}_2$$



$p$  odd



$p \equiv 1 \pmod{4}$



$p \equiv 3 \pmod{4}$

#### 4. Henselian ring.

Main ref: [https://en.wikipedia.org/wiki/Henselian\\_ring](https://en.wikipedia.org/wiki/Henselian_ring)

$R$  comm with 1 (local in this section)

Def. A local ring  $(R, \mathfrak{m})$  is Henselian if Hensel's lemma holds, i.e.

for  $P \in R[x]$



$\bar{P} = \bar{g}_1 \dots \bar{g}_n \in R/\mathfrak{m}[x]$

$\exists f_i \in P[x]$



$g_i \in R/\mathfrak{m}[x]$

$\textcircled{+} P = f_1 \dots f_n$

$(R, \mathfrak{m})$  is strictly Henselian if additionally  $(R/\mathfrak{m})^{\text{sep}} = R/\mathfrak{m}$ .

E.g. Fields/Complete Hausdorff local rings are Henselian.

e.p.  $F, \mathbb{Q}_p$  are Henselian

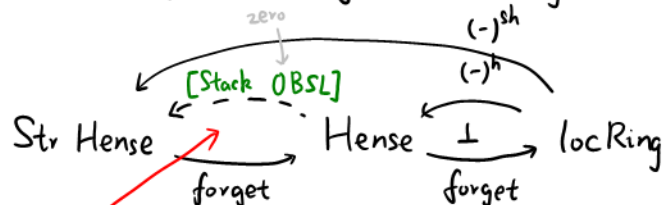
$R$  is Henselian  $\Leftrightarrow R/\text{Nil}(R)$  is Henselian

$\Leftrightarrow R/\mathfrak{I}$  is Henselian for  $\forall \mathfrak{I} \triangleleft R$

e.p. when  $\text{Spec } R = \{*\}$ ,  $R$  is Henselian.

Denote  $\text{StrHense} \subset \text{Hense} \subset \text{locRing} \subset \text{CommRing}$

full subcategories



Sadly not adjoint?

E.g.  $F^h = F$   $F^{sh} = F^{\text{un}}$

Geometrically, Henselian means  $\text{Spec } R/\mathfrak{m} \rightarrow \text{Spec } R$  has a section.

## 5. Cohomological dimension

main reference for cohomological dimension:

[NSW2e] <https://www.math.uni-heidelberg.de/~schmidt/NSW2e/>

<https://mathoverflow.net/questions/349484/what-is-known-about-the-cohomological-dimension-of-algebraic-number-fields>

This section is initially devoted to the following result:

Prop. [(7.5.1)] The wild inertia gp  $P_F$  is free pro- $p$ -group of countably infinite rank.

See [Galois Theory of  $p$ -Extensions, Chap 4] for the definition and construction of free pro- $p$ -groups.

Q: Do we have the adjoint

$$\begin{array}{ccc} & \xleftarrow{(\ )^{\text{free}}} & \\ \text{Pro-}p\text{-gp} & \xrightleftharpoons[\text{forget}]{\perp} & \text{Set} \end{array} \quad ?$$

Now let

$G$ : profinite gp  
 $\text{Mod}(G)$ : category of discrete  $G$ -modules  
 full subcategory of  $\text{Mod}(G)$   $\left\{ \begin{array}{l} \text{Mod}_t(G): \text{torsion} \\ \text{Mod}_p(G): p\text{-torsion} \\ \text{Mod}_f(G): \text{finite} \end{array} \right\}$  viewed as abelian gp

Lemma For abelian torsion gp  $X$ , denote

$$X(p) := \{x \in X \mid x^{p^k} = 1 \quad \exists k \in \mathbb{N}_{>0}\}$$

we have  $X = \bigoplus_p X(p)$ .

This is trivial when  $X$  is finite, but I don't know how to prove this in the general case. It should be not too hard.

Def [(3.3.1)] (cohomological dimension)  $p$  prime

$$\text{cd } G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A) \neq 0\}$$

$$\text{tcd } G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A) \neq 0\}$$

$$\text{cd}_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A)(p) \neq 0\}$$

$$\text{tcd}_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A)(p) \neq 0\}$$

$$\text{Prop. (local to global)} \quad \text{cd } G = \sup_p \text{cd}_p G \quad \text{scd } G = \sup_p \text{scd}_p G$$

$$\text{Prop. [(3.3.2)]} \quad \text{cd}_p G \leq n \Leftrightarrow H^{n+1}(G, A) = 0 \quad \forall \text{ simple } G\text{-mod } A \text{ with } pA = 0$$

e.p. for  $G$ : pro- $p$ -gp,

$$\text{cd}_p G \leq n \Leftrightarrow H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

$$\text{E.g. } \text{cd}_p \hat{\mathbb{Z}} = 1 \quad \text{scd}_p \hat{\mathbb{Z}} = 2$$

Prop. [(3.3.5)] For  $H \leq G$  closed,

$$\text{cd}_p H \leq \text{cd}_p G \quad \text{scd}_p H \leq \text{scd}_p G$$

When  $p \nmid [G:H]$  or  $[H \text{ open} + \text{cd}_p G < +\infty]$ , the equality holds.

Weaker condition: see [(3.3.5, Serre)]

Cor.  $G$ : profinite gp, then

$$\text{cd}_p G = 0 \Leftrightarrow p \nmid \#G$$

Prop. [(3.5.17)] A pro- $p$ -gp  $G$  is free iff  $\text{cd } G \leq 1$ .

Prop [7.1.8] (i)  $F$  NA local field with  $\text{char } k = p$ .

$$\text{cd}_l(F) = \begin{cases} 2 & \text{if } l \neq \text{char } F, \\ 1 & \text{if } l = \text{char } F. \end{cases}$$

For any  $E/F$  field extension s.t.  $l^\infty \mid \deg E/F$ ,  $\text{cd}_l(E) \leq 1$ .

(ii) Fix  $n \in \mathbb{N}_{>0}$  s.t.  $\text{char } F \nmid n$ .

$$H^i(F, \mu_n) = \begin{cases} F^\times / (F^\times)^n & i=1 \\ \frac{1}{n} \mathbb{Z} / \mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases}$$

[ Proof for Prop (7.5.1) □

Now  $l^\infty \mid \deg F^{\text{tr}}/F$

(7.1.8)

$\Rightarrow \text{cd}_l(F^{\text{tr}}) \leq 1 \quad \forall \text{ prime } l$

$\Leftrightarrow \text{cd}(F^{\text{tr}}) \leq 1$

$\Leftrightarrow P_F$  is free pro- $p$ -group.

□