§2.1. Character of Galois gp

This document is originally prepared for the class field theory, but we don't have time. And also, the ref [LCFT] is fantastic(with typos).

Since we discuss §2.1 and §3.1 in the same time, it is preferable to discuss general Galois rep and then restrict to the 1-dim case.

The only speciality of 1-dim case is, the char factor through

$$\mathrm{Gal}(F^{sep}/F) \twoheadrightarrow \mathrm{Gal}(F^{ab}/F) \longrightarrow GL_1(\Lambda),$$

Therefore, the max abel ext F^{ab} plays a role.

fin	✓		
local	local	Kronecker - Weber	$F^{ab} = F(\zeta_\infty)$
global		Kronecker - Weber	$\mathbb{Q}^{ab} = \mathbb{Q}(\zeta_\infty)$

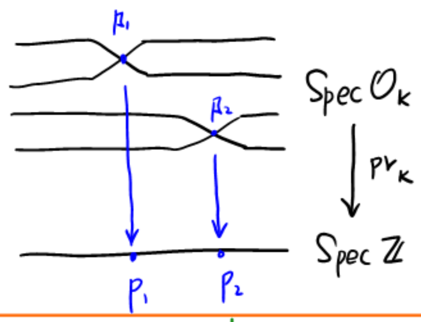
Local Kronecker - Weber

- for $\mathbb{Q}_p$: [LCFT, Thm 1.3.4] — use Kummer theory
- for F: [Allen, Thm 18.3] — use Hasse-Arf thm [Allen, Thm 17.16]

Kronecker - Weber

- for $\mathbb{Q}$: [LCFT, Thm 1.1.2] — use Minkowski's thm
- for $\mathbb{Q}(i)$: [Cox x^2+ny^2] — use CM Theory
- for $\mathbb{F}(t)$: [VS], [Hayes]

Thm $K/\mathbb{Q}$ fin abelian $\Rightarrow K \subseteq \mathbb{Q}(\zeta_n)$ $\exists n$



Proof.

Step 1. The choice of n:

Denote $\{p_1,\dots,p_r\}$ as primes over which K ramifies, pick $\mathfrak{p}_i \in pr_K^{-1}(p_i)$.

$Gal(K_{\mathfrak{p}_i}/\mathbb{Q}_{p_i}) \le Gal(K/\mathbb{Q}) \xRightarrow{\text{local KW}} \exists\, n_{p_i} \in \mathbb{N}_{\ge 1}$ s.t. $K_{\mathfrak{p}_i} \subseteq \mathbb{Q}_{p_i}(\zeta_{n_{p_i}})$

Suppose $n_{p_i} = p_i^{e_i} \cdot a_i$, $p_i \nmid a_i$. take $n := \prod_i p_i^{e_i} \in \mathbb{N}_{>0}$.

Step 2 Take $L = K(\zeta_n)$, we will show that $L = \mathbb{Q}(\zeta_n)$. Pick $\mathfrak{q}_i \in pr_{L/K}^{-1}(\mathfrak{p}_i)$.

$$\begin{tikzcd}
L_{\mathfrak{q}} \subseteq \mathbb{Q}_p(\zeta_{n_p}, \zeta_n) \arrow[d, dash, "I_{\mathfrak{q}}"] & L \supseteq \mathbb{Q}(\zeta_n) \arrow[d, dash, "I := \langle I_{\mathfrak{q}_i} \rangle_i"] \\
U_{\mathfrak{q}} := \mathbb{Q}_p^{ur} \cap L_{\mathfrak{q}} \arrow[d, dash] & U := L^I \arrow[d, dash] \\
\mathbb{Q}_p & \mathbb{Q}
\end{tikzcd}$$

$$|I| \overset{\text{Minko}}{=} [L:\mathbb{Q}] \ge [\mathbb{Q}(\zeta_n):\mathbb{Q}] = \phi(n)$$
$$|I| \le \prod_i |I_{\mathfrak{q}}| \le \prod_i \phi(p_i^{e_i}) = \phi(n)$$

$\Rightarrow [L:\mathbb{Q}] = [\mathbb{Q}(\zeta_n):\mathbb{Q}]$, $L = \mathbb{Q}(\zeta_n)$.

Rmk. This argument can not be extended to fct field K, since the residue fields of vals in K may be same (up to iso)

Left: LCFT, Galois cohomology.