Eine Woche, ein Beispiel

12.26 "Average Resistance" $\tau(\Gamma)$

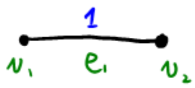
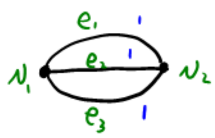
Goal: compute parameters in

[Cin]:https://arxiv.org/pdf/0901.3945.pdf
[BF]:https://arxiv.org/pdf/math/0407428.pdf
[BR]:https://arxiv.org/pdf/math/0407427.pdf

and think of their physical meaning (I need your help!)
If possible, find a way to explain the Cinkir's bound [Cin, Thm 5.21].

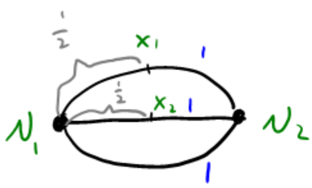
We begin with an undirected weighted connected graph Γ.
(weight is always positive, and can be thought as the length ; Γ have at lease 1 edge)

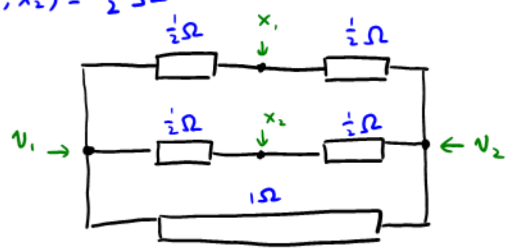
E.g.

			
Vertices $V = V(\Gamma)$	$\{v_1, v_2\}$	$\{v\}$	$\{v_1, v_2\}$
Edges $E = E(\Gamma)$	$\{e_1\}$	$\{e\}$	$\{e_1, e_2, e_3\}$
total length $l = l(\Gamma)$	1	l	3
genus $g = g(\Gamma)$	0	1	2

You can think a graph Γ as some electrical wires with given length and constant resistivity $1\Omega/m$. Then we can compute the resistance between two points $p, q \in \Gamma$, and denote it by $r(p,q)$.
↖ can be points on edges

E.g. In Fig 1, $r(v_1, v_2) = \frac{1}{3}\Omega$, $r(x_1, x_2) = \frac{1}{2}\Omega$





Thm. There exists a unique real signal Borel measure μ_{can} on Γ, satisfying:

(i) $\mu_{can}(\Gamma) = 1$, $|\mu_{can}|(\Gamma) < \infty$

(ii) The expression $(x, y \in \Gamma)$

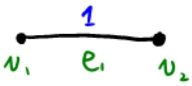
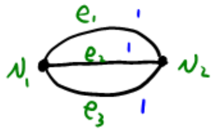
$$\frac{1}{2}\int_\Gamma r(x,y)\, d\mu_{can}(y)$$

is independent of the variant x.

We denote $\tau = \tau(\Gamma) = \frac{1}{2}\int_\Gamma r(x,y)\, d\mu_{can}(y)$, and call it the "average resistance".

Actually this quantity is more weird then what I thought.

E.g.

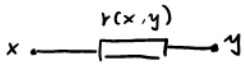
			
μ_{can}	$\frac{1}{2}\delta_{v_1} + \frac{1}{2}\delta_{v_2}$	$\frac{1}{l}\,dx$	$-\frac{1}{2}\delta_{v_1} - \frac{1}{2}\delta_{v_2} + \frac{2}{3}dx$
$\tau = \tau(\Gamma)$	$\frac{1}{4}\,\Omega$	$\frac{1}{12}l\,\Omega$	$\frac{7}{36}\,\Omega$

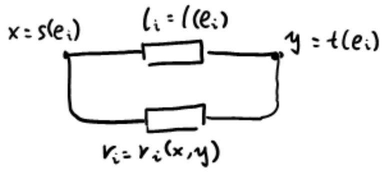
Ex. Verify the value of $\tau(\Gamma)$ in the tables. (assuming that μ_{can} is already known)

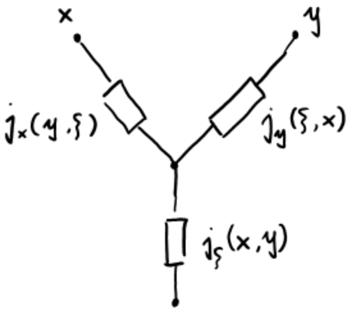
Q: Do we have any physical explanation for $\tau(\Gamma)$?

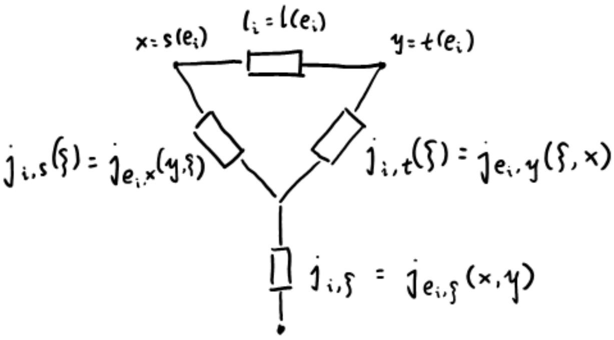
Actually we can write down μ_{can} explicitly. For doing so we have to introduce some new concepts.

Def. $\left(l_i, r_i, j_\xi(x,y), j_{e_i,\xi}, j_{e_i,s}(\xi), j_{e_i,t}(\xi)\right)$









Reality check: ⓪ $\dfrac{1}{r(x,y)} = \dfrac{1}{l_i} + \dfrac{1}{r_i}$

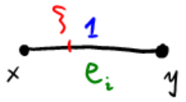
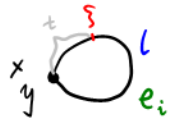
① $r(x,y) = j_x(y,\xi) + j_y(\xi,x)$ $\qquad$ $r_i(x,y) = j_{i,s}(\xi) + j_{i,t}(\xi)$

② $j_x(y,\xi) = \dfrac{l_i \cdot j_{i,s}(\xi)}{l_i + r_i}$ $\qquad$ $j_y(\xi,x) = \dfrac{l_i \cdot j_{i,t}(\xi)}{l_i + r_i}$

$$j_\xi(x,y) = j_{i,\xi} + \frac{j_{i,s}(\xi)\cdot j_{i,t}(\xi)}{l_i + r_i}$$

③ $r(x,y) \leq r_i$ $\qquad$ $j_\xi(x,y) \geq j_{i,\xi}$

E.g.

			
$r(x,y)$	1	0	$\frac{1}{3}$
r_i	∞	0	$\frac{1}{2}$
$j_\xi(x,y)$	0	$\frac{1}{\ell}t(1-t)$	$\frac{2}{3}t(1-t)$
$j_{i,\xi}$	error $(\xi \in e_i)$	error	$\frac{1}{2}t(1-t)$
$j_{i,s}(\xi)$	error	error	$\frac{1}{2}t$
$j_{i,t}(\xi)$	error	error	$\frac{1}{2}(1-t)$

Def Let $p \in |\Gamma|$. $v(p) = \#\{\text{half edges end at } p\}$

e.g.

$v(p) = 5$

Thm [(14.1), BR] $\mu_{can}(x) = \sum_{p \in |\Gamma|} \left(1 - \frac{1}{2}v(p)\right)\delta_p(x) + \sum_{e_i \in E} \frac{dx}{l_i + r_i}$

Thm. $$\tau(\Gamma) = \frac{1}{2}\int_\Gamma r(x,y)\, d\mu_{can}(x)$$
$$= \frac{1}{4}\int_\Gamma \left(\frac{dr}{dx}(x,y)\right)^2 dx$$
$$= \frac{1}{12}\sum_{e_i \in E} \frac{l_i^3 + 3l_i\left(j_{i,s}(\xi) - j_{i,t}(\xi)\right)^2}{(l_i + r_i)^2}$$
$$= \frac{l(\Gamma)}{12} - \frac{1}{6}\sum_{q \in |\Gamma|}(v(q) - 2)\, r(\xi, q) + \frac{1}{3}\sum_{e_i \in E}\frac{l_i}{l_i + r_i}\, j_{i,\xi}$$

Left: δ_K θ

μ_{ad} ε

μ_{path} a Cinkir's bound

φ

λ

Δ, g on graphs.