

Eine Woche, ein Beispiel

11.7. Berkovich space

Ref: Spectral theory and analytic geometry over non-Archimedean fields by Vladimir G. Berkovich (we mainly follow this article)

+courses from Junyi Xie

+An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves

Goal: understand the beautiful picture copied from "An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves"

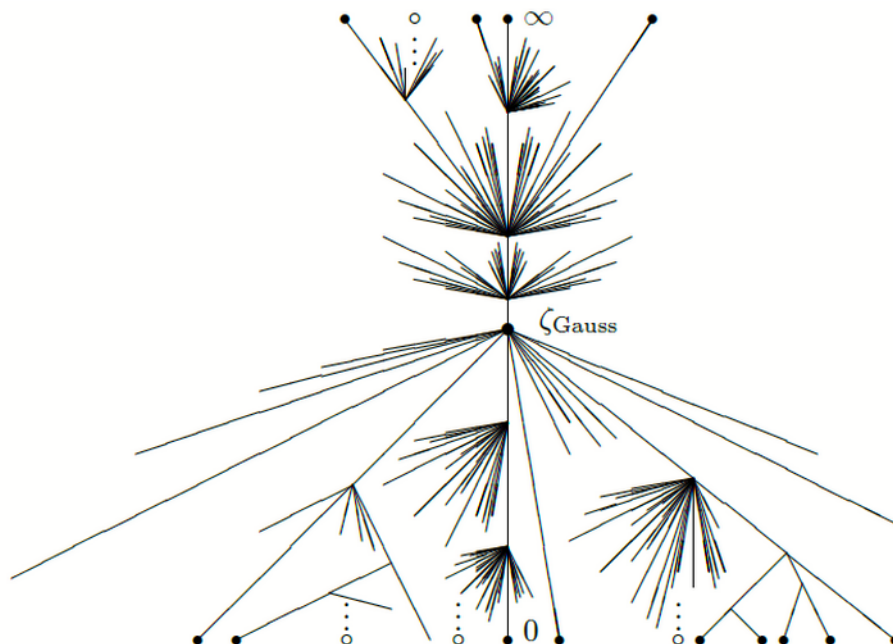


FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

A : comm with 1 (for convenience)

extra condition	local		global	closed unit disc	open unit disc
—	$\text{Spec } A$	affine scheme	scheme		$\text{Spec } \mathbb{Z}[[T]]$
A : adic ring with f.g. ideal of def	$\text{Spf } A$	affine formal scheme	formal scheme		$\text{Spf } \mathbb{Z}_p[[T]]$
A : K -affinoid alg. i.e. $A = K\langle T_1, \dots, T_n \rangle$	$\text{MaxSpec } A$	K -affinoid space	rigid-analytic space over K	$\text{MaxSpec } \mathbb{Q}_p\langle T \rangle$	$\mathcal{U} = \{t \in \text{MaxSpec } \mathbb{Q}_p\langle T \rangle \mid t < 1\}$
(A, A^+) : Huber pair	$\text{Spa}(A, A^+)$	affinoid adic space	adic space	$\text{Spa}(K\langle T \rangle, \mathcal{O}_K\langle T \rangle)$	
A : Banach ring	$\mathcal{M}(A)$	spectrum	Berkovich space		

Ref of table: Berkeley notes

Rmk. $\text{MaxSpec } A$ has only a Grothendieck topology.

K (in K -affinoid space) is a NA field, but can also be generalized to K -Banach alg.
 \uparrow non-archimedean K : NA field

1. Seminorm

1.1. Def (seminorm of abelian group) $\|\cdot\|: M \rightarrow \mathbb{R}_{\geq 0}$ s.t

$$\|0\| = 0$$

$$\text{norm: } \|m\| = 0 \Rightarrow m = 0$$

$$\|f+g\| \leq \|f\| + \|g\|$$

$$\text{non-Archimedean: } \|f+g\| \leq \max(\|f\|, \|g\|)$$

- seminorm \Rightarrow topology

Prop. $(M, \|\cdot\|)$ is Hausdorff $\Leftrightarrow \|\cdot\|$ is norm

Def (equivalence of norm)

- sub, quotient, homomorphism

Def (restricted seminorm)

Def. (residue seminorm) $\pi: (M, \|\cdot\|_M) \rightarrow M/N$ induce the seminorm on M/N :

$$\|\bar{m}\|_{M/N} := \inf_{\pi(m') = \bar{m}} \|m'\|_M$$

Def (bounded / admissible) $\varphi: (M, \|\cdot\|_M) \rightarrow (N, \|\cdot\|_N)$

- bounded: $\exists C > 0, \|\varphi(m)\|_N \leq C \|m\|_M$

- admissible: $\bar{\varphi}: (M/\ker \varphi, \|\cdot\|_{\text{quo}}) \rightarrow (\text{Im } \varphi, \|\cdot\|_{\text{res}})$
induces equivalence of norm.

1.2. Def (seminorm of ring non-comm, with 1): seminorm group +

$$\|1\| = 1$$

$$\|fg\| \leq \|f\| \|g\|$$

power-multi:

$$\|f^n\| = \|f\|^n$$

multiplicative:

$$\|fg\| = \|f\| \|g\|$$

+completed \rightarrow Banach ring

\Rightarrow valuation

- quotient, \prod_{infinite} , $\mathbb{A}\langle r^{-1}T \rangle, \dots$

- comparison among norms: bounded.

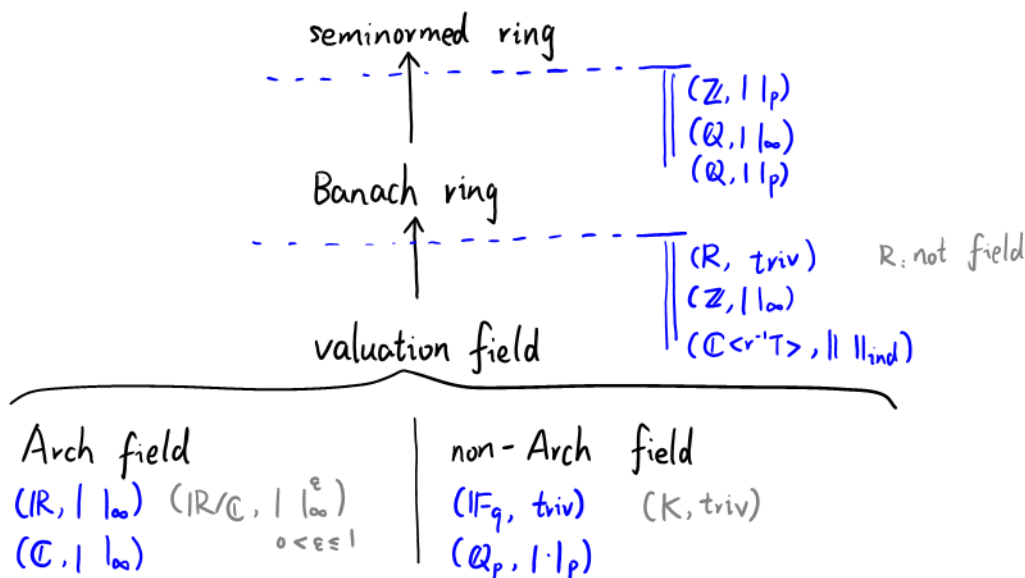
- Def related to valuation field.

<https://math.stackexchange.com/questions/2151779/normed-vector-spaces-over-finite-fields>

1.3. Def (seminorm of \mathbb{A} -module, where \mathbb{A} : normed ring)

seminorm group + $\exists C > 0, \|fm\| \leq C \|f\| \|m\|$

- $\hat{\otimes}_{\mathbb{A}}$



2. Affine case

suppose \mathcal{A} : Banach ring comm + 1

$\mathcal{M}(\mathcal{A}) := \{\text{bounded mult seminorms on } \mathcal{A}\}$

with top basis generated by $U_{m,(a,b)} := \{\| \cdot \| \in \mathcal{M}(\mathcal{A}) \mid \|m\| \in (a,b)\}$
 $m \in \mathcal{A}, (a,b) \in \mathbb{R}$

$\mathcal{M}(\mathcal{A}/(\mathbb{Z}, \|\cdot\|_\infty)) := \{\text{mult seminorms on } \mathcal{A}\}$

E.g. $\mathcal{A} = (\mathbb{Z}, \|\cdot\|_\infty)$

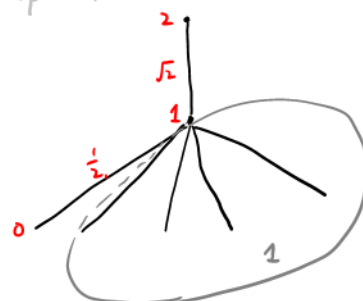
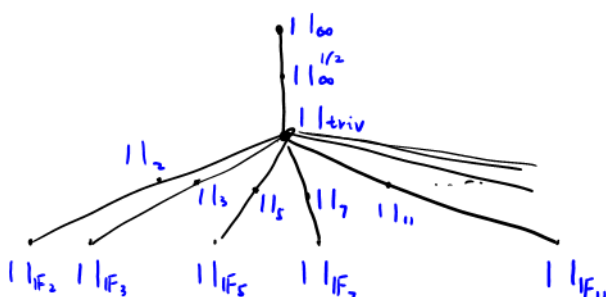
We have

$$\mathcal{M}(\mathbb{Z}, \|\cdot\|_\infty) = \left\{ \begin{array}{l} \|\cdot\|_{\text{triv}} := \text{trivial norm} \\ \|\cdot\|_p^t : t \in (0, +\infty] \\ \|\cdot\|_\infty^\varepsilon : \varepsilon \in (0, 1] \end{array} \right.$$

$$\|m\|_{\mathbb{F}_p} := \|m\|_p^\infty = \begin{cases} 0 & p \mid m \\ 1 & p \nmid m \end{cases}$$

$$\|\cdot\|_{\text{triv}} = \|\cdot\|_p^0 = \|\cdot\|_\infty^0$$

Picture:



value of 2.

From this picture, we want to get:
 Bound relations among seminorms
 Topology properties: Hausdorff? compact?
 Residue field, injection and contraction
 ... See next page

Rmk. When we do not identify the norm, we mean $\mathcal{A}/(\mathbb{Z}, \|\cdot\|_\infty)$.

E.g. $\mathcal{A} = (\mathbb{Q}, \|\cdot\|_{\text{any}})$, $\mathcal{M}(\mathcal{A}) = \{*\}$

E.g. $\mathcal{A} = (\mathbb{F}_q, \|\cdot\|_{\text{triv}})$, $\mathcal{M}(\mathbb{F}_q) = \{*\}$

E.g. $\mathcal{A} = \mathbb{R}/\mathbb{C}$ reasonable seminorms are $\|\cdot\|_\infty^\varepsilon$, $\varepsilon \in [0, 1]$.

Do we have any other seminorms?

E.g. $\mathcal{A} = \mathbb{Z}_p$ reasonable seminorms are $\|\cdot\|_p^t$, $t \in [0, +\infty]$. ($\mathcal{A} = \mathbb{Q}_p$ is also interesting)

Do we have any other seminorms?

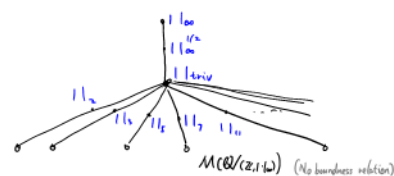
E.g. $\mathcal{A} = \mathbb{C}_p$

E.g. $\mathcal{A} = \mathbb{C}[X]$ If we only consider the norm which restricted to \mathbb{C} is $\|\cdot\|_\infty$, we would get \mathbb{C} .

What would happen in the other cases?

E.g. $\mathcal{A} = \mathbb{C}_p \langle r^{-1}T \rangle$ or $\mathbb{P}_{\mathbb{C}_p}^1$

E.g. $\mathcal{A} = (\mathbb{Z}[i], \|\cdot\|_\infty)$



I'm very happy to do the homework one years ago.

E.g. $\mathcal{A} = (\mathbb{Z}, \|\cdot\|_\infty)$

Try to answer the following questions:

- Set

• $\mathcal{M}(\mathbb{Z}) = \checkmark$

• partial order \rightsquigarrow bound order

• Picture \checkmark

• maximal/minimal seminorm $\max: \|\cdot\|_{1p}$
 $\min: \|\cdot\|_{\infty}$

• Berkovich structure of $\|\cdot\| \in \mathcal{M}(\mathbb{Z})$?

• Archi or non Archi ?

!



- Topo

• Close set

• Open set

not contain $\|\cdot\|_{triv}$: normal way + contain only finite $\|\cdot\|_p^+$

contain $\|\cdot\|_{triv}$: normal way

not contain $\|\cdot\|_{triv}$: normal way

contain $\|\cdot\|_{triv}$: normal way + contain all $\|\cdot\|_p^+$ except finite p



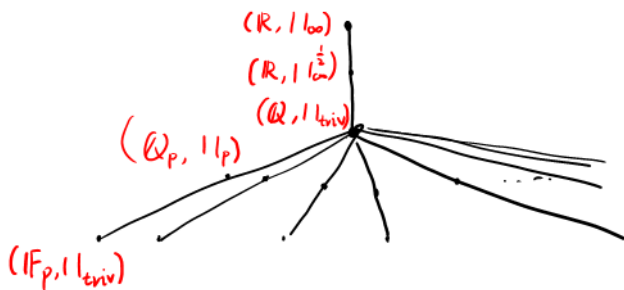
• Topo properties: connected? \checkmark Hausdorff? \checkmark (quasi)compact? \checkmark

irreducible? \times

$X = Y \sqcup Z$

Def. $p \in X$ is a closed pt
iff $\{p\}$ is closed
Then every pt is closed pt

The definitions of Residue field, injection and contraction follows from [3.1.1, <https://arxiv.org/abs/2105.13587v3>]



Residue field

