

Modular form

3. discriminant



Def. A holo fct $f: \mathcal{H} \rightarrow \mathbb{C}$ is called a modular form of weight $k \in \mathbb{Z}$, level $\Gamma = \text{SL}_2(\mathbb{Z})$, if.

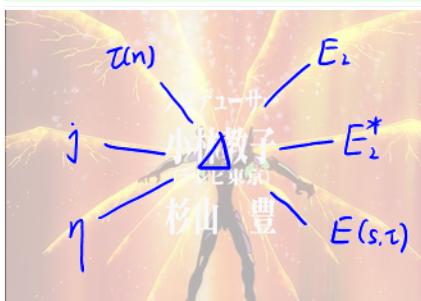
- 1) $f(\gamma\tau) = \frac{\chi(\tau)}{(c\tau + d)^k} f(\tau)$ ② quasi-modular form
e.p. $f(\tau+1) = f(\tau)$ ④ MF with character $\chi: \Gamma \rightarrow \mathbb{C}^\times$
- 2) Write $f(\tau) = \sum_{n \in \mathbb{Z}} a_n (e^{2\pi i \tau})^n$, then $a_n = 0$ for $n < 0$ ① non-entire MF
③ p-adic MF

① The order I plan to talk about

(For me they become more and more difficult)

Today: Δ , $\tau(n)$, j , E_2 , E_2^* , η

Delta的性质不可避免地用到了模形式的各类推广，这是今天讨论班的两条主线。



1. Def of Δ

2. relation with EC

3. q-exponential & Ramanujan τ -fct

4. Δ & j -invariant

- as meromorphic MF
- as modular invariant
- q-exponential and moonshine
- special values and complex multiplication

5. product formula & E_2

- automorphy condition for E_2
- product formula

· quasimodular form and almost holomorphic MF

6. Δ & Dedekind η -fct

7. Δ & $E(s, \tau)$ ← possibly to be added here

Conclusion

1. Def of Δ

Ex. find $\Delta \in S_{12}(SL_2(\mathbb{Z}))$ s.t. $\Delta \in q + q^2 \mathbb{Q}[[q]]$

$$\Delta(\tau) = \frac{1}{1728} (E_4(\tau)^3 - E_6(\tau)^2)$$

$$= q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 \\ - 16744q^7 + 84480q^8 - 113643q^9 - 115920q^{10} + O(q^{11})$$

$$\text{Cor. } S_r(SL_2(\mathbb{Z})) \cong \Delta \mathbb{C}[E_4, E_6] \triangleleft M_r(SL_2(\mathbb{Z}))$$

2. relation with EC

Rmk. $\Delta(\tau) = \frac{1}{(2\pi)^{12}} \Delta(y^2 = 4x^3 - g_2(\tau)x - g_3(\tau), \frac{dx}{y})$

which is closely related to the disc of EC.

slogan: any "fct" on moduli space reflects properties of EC.

▽ The disc depends on "the choice of equations" (in reality, the choice of differential) and can change after the change of variables.

A possible explanation may be found here:

<https://math.stackexchange.com/questions/3487698/discriminant-of-elliptic-curve-y2-ax3bx2cx0>
(the reason is, Δ is a function of lattices, and differential gives us a lattice.)

For results in char 2 & 3, see [ECII, Appendix A].

$$w = \frac{dx}{2y+a_1x+a_3}, \quad y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6$$

$$b_2 = a_1^2 + 4a_2$$

$$b_4 = 2a_4 + a_1 a_3$$

$$b_6 = a_3^2 + 4a_6$$

$$b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2$$

$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6$$

$$4b_8 = b_2 b_6 - b_4^2$$

$$\Delta = g_2^3 - 27g_3^2 \quad \Delta_{\text{wiki}} = 16\Delta$$

$$w = \alpha^{\frac{1}{3}} \frac{dx}{y} \quad y^2 = ax^3 + bx^2 + cx + d$$

$$\Delta = 16(b^2c^2 - 4ac^3 - 4b^3d - 27a^2d^2 + 18abcd)$$

$$\Delta_{\text{wiki}} = \frac{1}{16}\Delta$$

$$w = \frac{dx}{2y} \quad y^2 = x^3 + px + q$$

$$\Delta = 16(-4p^3 - 27q^2)$$

$$y^2 = (x - e_1)(x - e_2)(x - e_3)$$

$$\Delta = 16 \prod_{i < j} (e_i - e_j)^2$$

$$y^2 = x(x-1)(x-\lambda)$$

$$\Delta = 16 \lambda^2(\lambda-1)^2$$

$$\Delta_{\text{my moduli}} = \Delta_{\text{WWL}} = \Delta_{\text{silverman}} = \Delta$$

3. q -exponential & Ramanujan τ -fact

$$\Delta(\tau) = \sum_{n=1}^{+\infty} \tau(n) q^n$$

Observation

- o) $\tau(1) = 1 \quad \tau(n) \in \mathbb{Z}$
- 1) $\tau(nn') = \tau(n)\tau(n')$ for $\gcd(n, n') = 1$
- $\tau(p^{e+1}) = \tau(p)\tau(p^e) - p''\tau(p^{e-1}) \quad \forall p \text{ prime}, e \in \mathbb{Z}_{\geq 1}$
- e.p. $\tau(p^2) = \tau(p)^2 - p''$
- 2) $|\tau(p)| \leq 2p^{\frac{1}{2}} \quad \forall p \text{ prime}$
- 3) (Lehmer conj) $\forall n \in \mathbb{N}_{\geq 1}, \tau(n) \neq 0$

Rmk. o) the following exercise or $\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{-1} \leftarrow$ will be proved soon
 1). by Mordell $\xrightarrow{\text{generalize}}$ Hecke operators
 2). by Deligne, use the Weil conj.
 A much weaker version, $|\tau(p)| \leq C p^6$ can be seen in [Za, Prop 8].

Ex. Show that

$$\begin{aligned} \tau(n) &\in \mathbb{Z} \\ \tau(n) &\equiv \begin{cases} 1 \pmod{2} & \text{if } n=m^2, m \text{ odd} \\ 0 \pmod{2} & \text{otherwise} \end{cases} \\ \tau(n) &\equiv \sigma_{12}(n) \pmod{691} \end{aligned}$$

Hint. ① Write $A = \sum \sigma_3(n) q^n, B = \sum \sigma_5(n) q^n,$
 $\Delta = 5 \frac{A-B}{2} + B + 100A^2 - 147B^2 + 8000A^3$
 \Rightarrow first two equalities
 ② Write $G_{12} - \Delta$ as a combination of E_4 & E_6 , i.e.

$$G_{12} = \Delta + \frac{691}{156} \left(\frac{E_4^3}{720} + \frac{E_6^3}{1008} \right)$$

The $\tau(n)$ enjoy various congruences modulo $2^{12}, 3^6, 5^3, 7, 23, 691$. We quote some special cases (without proof):

$$(55) \quad \tau(n) \equiv n^2 \sigma_7(n) \pmod{3^3}$$

$$(56) \quad \tau(n) \equiv n \sigma_3(n) \pmod{7}$$

$$(57) \quad \tau(n) \equiv \sigma_{11}(n) \pmod{691}.$$

For other examples, and their interpretation in terms of “ l -adic representations” see Séminaire Delange-Pisot-Poitou 1967/68, exposé 14, Séminaire Bourbaki 1968/69, exposé 355 and Swinnerton-Dyer’s lecture at Antwerp (Lecture Notes, n° 350, Springer, 1973).

Ramanujan’s congruences [WWL, 例4.4.8]



p-adic modular form

4. Δ & j -invariant

as mere MF

$$\text{Ex. } \left\{ \begin{array}{l} \text{entire/non-entire MF} \\ \text{with level } SL_2(\mathbb{Z}) \end{array} \right\} \cong \mathbb{C}[E_4, E_6, \Delta^{-1}]$$

$$\left\{ \begin{array}{l} \text{meromorphic MF} \\ \text{with level } SL_2(\mathbb{Z}) \end{array} \right\} \cong \mathbb{C}(E_4, E_6)$$

$$\left\{ \begin{array}{l} \text{meromorphic modular fct} \\ \text{with level } SL_2(\mathbb{Z}) \end{array} \right\} \cong \mathbb{C}(j) \quad \begin{matrix} \text{modular fct} \\ = \text{MF of weight 0} \end{matrix}$$

where $j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)^2} = 1728 \frac{E_4^3}{E_4^3 - E_6^2}$

as modular invariant

Ex. $j(p)=0$ $j(i)=1728$ $j(\infty)=\infty$
 $j : (H/SL_2(\mathbb{Z}))^* \rightarrow \mathbb{C}\text{P}^1$ is an iso of RS.
 another road to this result. [WWL, P233]

Cor. [WWL, 定理8.2.4] X_1, X_2 : conn cpt RS of genus 1, then
 $X_1 \cong X_2 \Leftrightarrow j(X_1) = j(X_2)$

Ex. (maybe hard) Let $X_\lambda := V(y^2 = x(x-1)(x-\lambda))$ $\lambda \neq 0, 1, \infty$, compute $j(X_\lambda)$.
 A. $j(X_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}$ [WWL, (8.2.6)]

$$j(X_\lambda) = j(X_{\frac{1}{\lambda}}) = j(X_{1-\lambda}) = \dots$$

Rmk. we will see that λ is also a modular form (of level $\Gamma(2)$)

q -exponential and moonshine [WWL, p 65]

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + \dots$$

$$j(\tau) \xleftarrow[\text{VOA: vertex operator alg}]{\text{∞-dim Lie alg}} M \xrightarrow{\text{zhihu: 313723878}}$$

special values and cplx multiplication

Def An EC $E = \mathbb{C}/\Lambda$ admits cplx multiplication (CM) if
 $\exists \lambda \in \mathbb{C} - \mathbb{R}, \quad \lambda \Lambda \subset \Lambda.$

$\tau \in \mathcal{H}$ is a CM point if $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ admits CM

E.g. $i, \rho, \sqrt{2}i$ are CM points.

Prop. [Za Prop 22, 23] For any CM pt τ , $j(\tau)$ is an alg number.

E.g. $j(\rho) = 0 \quad j(i) = 1728 \quad j\left(\frac{1+\sqrt{-5}}{2}\right) = -3375$

$$j(\sqrt{2}i) = 8000$$

$$j(2i) = 287496$$

$$j(3i) = 76771008 + 44330490\sqrt{3}$$

We will see from [Za Prop 25] that $j(\tau_0) \in \mathbb{Z}$ for $\tau_0 = \frac{i}{2}(1 + \sqrt{163}i)$

$$\Rightarrow q \approx -4 \times 10^{-18}$$

$$\Rightarrow j(\tau_0) = q^{-1} + 744 + O(q) \approx q^{-1} + 744$$

$$\Rightarrow e^{\frac{\pi i}{\sqrt{163}}} = -q^{-1} \approx 744 - j(\tau_0)$$

$$e^{\frac{\pi i}{\sqrt{163}}} = 262537412640768743.999999999999250072597\dots$$

5. product formula & E_2

In this section, we will introduce E_2 , and prove the central identity of today's talk.

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

Def.

$$G_2(\tau) = \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau+n)^2} \quad \forall \text{ in } [\mathbb{Z}] \quad G_2(\tau) = \frac{1}{2} \dots$$

Ex. show that

$$\frac{1}{2} G_2(\tau) = \frac{(2\pi i)^2}{(2-1)!} \left(-\frac{B_2}{2 \cdot 2} + \sum_{n=1}^{\infty} \sigma_1(n) q^n \right)$$

Similarly, define

$$E_2(\tau) = \left(-\frac{2 \cdot 2}{B_2} \right) \left(-\frac{B_2}{2 \cdot 2} + \sum_{n=1}^{\infty} \sigma_1(n) q^n \right)$$

$$G_2(\tau) = E_2(\tau) = -\frac{B_2}{2 \cdot 2} + \sum_{n=1}^{\infty} \sigma_1(n) q^n$$

for convenience,

$$E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n = 1 - 24 \sum_{n=1}^{\infty} \frac{n q^n}{1 - q^n}$$

Sadly, $G_2(\tau)$ is no longer a modular form. (quasi MF instead)

automorphy condition for E_2

Prop [Za, Prop 6]

$$G_2(\gamma\tau) = (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d)$$

$$E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) - \frac{6ic}{\pi}(c\tau+d)$$

Proof. We use deformation. for $\varepsilon > 0$, let

$$G_{2,\varepsilon}(\Lambda) = \sum_{z_0 \in \Lambda} \frac{1}{z_0^2 |z_0|^{2\varepsilon}}$$

then

$$G_{2,\varepsilon}(\lambda\Lambda) = \lambda^{-2} |\lambda|^{-2\varepsilon} G_{2,\varepsilon}(\Lambda)$$

$$\Rightarrow G_{2,\varepsilon}(\gamma\tau) = (c\tau+d)^2 |c\tau+d|^{2\varepsilon} G_{2,\varepsilon}(\tau)$$

$$\text{Claim: } \lim G_{2,\varepsilon}(\tau) = G_2(\tau) - \frac{\pi}{\text{Im } \tau} \triangleq G_2^*(\tau)$$

$$\begin{aligned} &\text{almost holo MF} \\ &G_2^*(\tau) = G_2(\tau) - \frac{\pi}{\text{Im } \tau} \\ &E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi \text{Im } \tau} \\ &G_2^*(\tau) = G_2(\tau) + \frac{1}{8\pi \text{Im } \tau} \end{aligned}$$

$$\begin{aligned} \text{If so, } G_2(\gamma\tau) - \frac{\pi}{2\text{Im } \gamma\tau} &= (c\tau+d)^2 \left(G_2(\tau) - \frac{\pi}{\text{Im } \tau} \right) \\ \Rightarrow G_2(\gamma\tau) &= (c\tau+d)^2 G_2(\tau) - \pi \left(\frac{(c\tau+d)^2}{\text{Im } \tau} - \frac{1}{\text{Im } \gamma\tau} \right) \\ &= (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d) \end{aligned}$$

Proof of claim Idea. 配湊法

$$\begin{aligned} \frac{1}{2} G_{2,\varepsilon}(\tau) &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau+n)^2 |m\tau+n|^{2\varepsilon}} \\ \frac{1}{2} G_{2,\varepsilon} - \sum_{m=1}^{\infty} I_{\varepsilon}(m\tau) &= \sum_{n=1}^{\infty} \frac{1}{n^{2+2\varepsilon}} + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \left(\frac{1}{(m\tau+n)^2 |m\tau+n|^{2\varepsilon}} - \int_n^{n+1} \frac{dt}{(m\tau+t)^2 |m\tau+t|^{2\varepsilon}} \right) (*) \end{aligned}$$

where

$$I_{\varepsilon}(\tau) = \int_{-\infty}^{\infty} \frac{dt}{(\tau+t)^2 |t+\tau|^{2\varepsilon}} \quad (\text{defined for all } \varepsilon > -\frac{1}{2}) \quad I(\varepsilon) = \int_{-\infty}^{\infty} \frac{ds}{(s+i)^2 |s+i|^{2\varepsilon}}$$

$$I_{\varepsilon}(x+iy) = \int_{-\infty}^{\infty} \frac{dt}{(x+t+iy)^2 ((x+t)^2 + y^2)^{\varepsilon}} = \frac{1}{y^{1+2\varepsilon}} I(\varepsilon)$$

$$\Rightarrow \sum_{m=1}^{\infty} I_{\varepsilon}(m\tau) = \frac{1}{y^{1+2\varepsilon}} \int (1+2\varepsilon) I(\varepsilon) \\ = \frac{1}{y^{1+2\varepsilon}} \left(\frac{1}{2\varepsilon} + O(1) \right) (-\pi\varepsilon + O(\varepsilon^2)) \xrightarrow{\varepsilon \rightarrow 0} -\frac{\pi}{2y}$$

Take $\lim_{\varepsilon \rightarrow 0}$ in (*), the claim is proved. \square

product formula

Now set $\Delta(\tau) = q \prod_{n=1}^{\infty} (1-q^n)^{-1}$

We want $\Delta = \Delta$. Reduce to $\Delta(\gamma\tau) = (c\tau+d)^{12} \Delta(\tau)$.

Lemma. $\frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta(\tau) = E_2(\tau)$ $E_2(\gamma\tau) = (c\tau+d)^2 E_2(\tau) - \frac{bi\zeta}{\pi} (c\tau+d)$

$$\frac{1}{2\pi i} \frac{d}{ds} \Big|_{s=\tau} \log \Delta(s)$$

Proof

$$\frac{1}{2\pi i} \frac{d}{d\tau} \log \left(\frac{\Delta(\gamma\tau)}{(c\tau+d)^2 \Delta(\tau)} \right) = \frac{1}{(c\tau+d)^2} E_2(\gamma\tau) - E_2(\tau) - \frac{12}{2\pi i} \frac{c}{c\tau+d} = 0$$

$$\Rightarrow \Delta(\gamma\tau) = C(\gamma) (c\tau+d)^{12} \Delta(\tau) \quad \forall \gamma \in SL_2(\mathbb{Z})$$

Reduced to $C(T) = C(S) = 1$. \square

Rmk. [NT II §9.2, p312] lists 5 methods to prove $\Delta \in S_{12}(SL_2(\mathbb{Z}))$.

We adopt Method 5, which is also used in [Serre, Za, WWL, 228].

Method	name	make use of	fct equation of
1	η -fct	pentagonal number thm	Poisson sum of χ_x
2	Kronecker limit formula	$\frac{\partial E}{\partial s}(0, \tau) = \frac{1}{6} \log((Im\tau)^6 \Delta(\tau))$	$E(s, \tau) = \frac{1}{2} \sum_{\substack{gcd(c,d)=1 \\ c\tau+d=s}} \frac{(Im\tau)^3}{ c\tau+d ^{2s}}$ convergent for $Re s > 1$
3	Siegel	η -fct	contour integral of $f_V(z) = \frac{1}{2} \cot(\nu z) \cot\left(\frac{\nu z}{it}\right)$
4	Weil	$\sum_{n=1}^{\infty} \log(1-q^n) = \sum_{n=1}^{\infty} c_n q^n$ $\sum_{n=1}^{\infty} c_n n^{-s} = -\zeta(s) \zeta(s+1)$	$\zeta(s)$
5	Hurwitz method	$E_2 = \frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta(\tau)$	E_2

quasimodular form and almost holomorphic MF

We will return to this topic of quasi/almost holo MF in [Za, Sec 5, e.p. 5.3].
We only list results here:

quasi MF	$\widetilde{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_2, E_4, E_6]$
almost holo MF	$\widehat{M}_*(SL_2(\mathbb{Z})) = \mathbb{C}[E_2^*, E_4, E_6]$

Rmk. $\widetilde{M}_*(SL_2(\mathbb{Z}))$ is closed under differential.

e.g. $E_2' = \frac{1}{12} (E_2^2 - E_4)$
 $\Rightarrow \sum_{m=1}^{n-1} \sigma_3(m) \sigma_3(n-m) = \frac{1}{12} (5\sigma_3(n) - (6n-1)\sigma_3(n))$

b. Δ & Dedekind η -fct

Def.

$$\eta(\tau) := \Delta(\tau)^{\frac{1}{24}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = e^{\frac{2\pi i \tau}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})$$

Ex. show that

$$\begin{aligned} \eta(\tau+1) &= e^{\frac{2\pi i}{24}} \eta(\tau) \\ \eta(-\frac{1}{\tau}) &= \sqrt{-i\tau} \eta(\tau) \end{aligned}$$

$\xrightarrow{\quad \quad \quad \quad \quad}$ $\xrightarrow{\quad \quad \quad \quad \quad}$ $\xrightarrow{\quad \quad \quad \quad \quad}$

Ex. compute q -expansions of $\eta(\tau)$.

see wiki: Pentagonal number theorem if you don't work out the proof.

Another proof uses theta-function. (i.e., equivalent with Jacobi triple product)

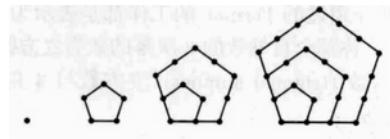
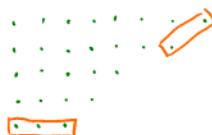


图 0.2 n 角数

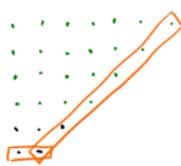
$$\begin{aligned} A: \eta(\tau) &= \sum_{n=1}^{+\infty} \chi_{12}(n) q^{\frac{n^2}{24}} = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \dots \\ &= q^{\frac{1}{24}} \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{3n^2+n}{24}} = q^{\frac{1}{24}} (1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - \dots) \end{aligned}$$

$$\begin{array}{cccccc} n & & 1 & 5 & 7 & 11 \\ \chi_{12}(n) & & 1 & -1 & -1 & 1 \end{array}$$

"Proof"

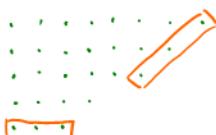


\Rightarrow



$$27 = 8 + 7 + 5 + 4 + 3$$

$$27 = 7 + 6 + 5 + 4 + 3 + 2$$



\Rightarrow



$$28 = 8 + 7 + 6 + 4 + 3$$

$$28 = 9 + 8 + 7 + 4$$

For a definition of half-integral weight modular form with character, see my bachelor thesis:
https://github.com/ramified/personal_tex_collection/blob/main/bachelor_thesis/thesis/main.pdf

Conclusion

type	symbol	EC	relation with Δ	q-expansion	further study in
-	Δ, τ	Δ .disc	$\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n) q^n = q - 24q^2 + 252q^3 - \dots$	$j(\tau) = q^{-1} + 744 + 196884q + \dots$	Sec 4
①	j	modular inv	$j = \frac{E_4^3}{\Delta}$	$E_2(\tau) = 1 - 24q - 72q^2 - 96q^3 - \dots$	Sec 6
②,③	E_2, E_2^*	-	$E_2 = \frac{1}{2\pi i} \frac{d}{d\tau} \log \Delta(\tau)$	$\eta(\tau) = q^{\frac{1}{24}} - q^{\frac{25}{24}} - q^{\frac{49}{24}} + q^{\frac{121}{24}} + \dots$	Sec 5
④	η	-	$\eta = \Delta^{\frac{1}{24}}$		Sec 3

$$\begin{array}{l}
 \Delta \quad \infty \quad \frac{i}{2^6} \frac{\omega}{\pi^{12}} \quad ? \quad P \\
 0 \quad \frac{1}{2^6} \frac{\omega}{\pi^{12}} \quad ? \\
 \\
 j \quad \infty \quad 1728 \quad 0 \\
 \\
 E_2 \quad 1 \quad \frac{3}{\pi} \quad \frac{2\sqrt{3}}{\pi} \\
 \\
 \eta \quad 0 \quad \frac{1}{j_2} \frac{\omega}{\pi} \quad ?
 \end{array}$$