$\mathcal{G}$ $\mathcal{F}$

fiber of $\pi_*, \pi_!, \pi^{-1}, \pi^*$

$\pi : Y \hookrightarrow X$

$\pi : U \hookrightarrow X$ $\qquad\qquad$ $\pi : Z \xrightarrow{\text{close}} X$

$$\pi_* \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin \bar{U} \\ \varinjlim_{x \in V} \mathcal{G}(U \cap V) & x \in \bar{U} \setminus U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi_! \begin{cases} \mathcal{G}_x & x \in U \\ 0 & x \notin U \end{cases} \qquad \begin{cases} \mathcal{G}_x & x \in Z \\ 0 & x \notin Z \end{cases}$$

$$\pi^{-1} \quad \mathcal{F}_y \qquad\qquad \mathcal{F}_y$$

$$\pi^* \quad \mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y} \qquad\qquad \mathcal{F}_y \otimes_{\pi^{-1}\mathcal{O}_{X,y}} \mathcal{O}_{Y,y}$$

$$\bar{x} \xrightarrow{u_{\bar{x}}} X \xrightarrow{f} Y$$

$$(f^*\mathcal{F})_{\bar{x}} = u_{\bar{x}}^* f^* \mathcal{F}(Y) = \mathcal{F}_y$$

For étale: $f : \overset{\mathcal{G}}{X} \longrightarrow \overset{\mathcal{F}}{Y}$ $\qquad$ $\bar{x} \longmapsto \bar{y}$
(sheaf)

$$(f^*\mathcal{F})_{\bar{x}} \cong \mathcal{F}_y$$

If $f$ q.c. $\quad (f_*\mathcal{G})_{\bar{y}} \cong \Gamma(\widetilde{X}, \tilde{g}^*\mathcal{G})$ where

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{g}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\ \operatorname{Spec}\mathcal{O}^{sh}_{Y,\bar{y}} \arrow[r, "g"] & Y \end{tikzcd}$$

e.g. for $f$ finite, we have explicit description of $f_*$.

左伴随 $\dashv$ 右伴随 $\qquad f_! \dashv f^* \dashv f_* \dashv f^!$

| | |
|---|---|
| 自由 | 忘却 |
| $-\otimes_A N$ | $\mathrm{Hom}_A(N,-)$ |
| $\Delta$ | $\varprojlim$ |
| $(\ )^\sim$ | $\Gamma_*$ |
| $f_p$ | $f^*$ |
| Ind | Res |
| $-\otimes_k A^e$ | Res |
| sh $(-)$ | $\mathrm{Res}_{sh\to Psh}$ |
| $\pi_0$ | $T \mapsto \coprod_{t\in T} \mathrm{Spec}\,\mathbb{Z}$ |
| $G^{[p]}$ | Lie |

ad $\Rightarrow$ 右正合 $\qquad$ 左正合

ad $\Rightarrow$ 与余极限交换 $\varinjlim$ $\qquad$ 与极限交换 $\varprojlim$

| | |
|---|---|
| coker $f$ | ker $f$ |
| $\coprod A \quad \bigoplus A$ | $\prod A \quad A\times_C B$ |
| $\mathcal{F}_p = \varinjlim_{U} \mathcal{F}(U)$ | $\mathbb{Z}_p = \varprojlim_{n} \mathbb{Z}/p^n\mathbb{Z}$ |
| stalk | completion |

$$\mathrm{Gal}(\bar{k}/k) = \varprojlim_{L \text{ finite}} \mathrm{Gal}(L/k)$$

co limit

$\left.\begin{array}{l}\text{direct}\\ \text{inductive}\\ \text{injective}\end{array}\right\}$ limit.

**Lemma 6.21.3.** *Let $f : X \to Y$ be a continuous map. There exists a functor $f_p : PSh(Y) \to PSh(X)$ which is left adjoint to $f_*$. For a presheaf $\mathcal{G}$ it is determined by the rule*

$$f_p\mathcal{G}(U) = \operatorname{colim}_{f(U)\subset V} \mathcal{G}(V)$$

*where the colimit is over the collection of open neighbourhoods $V$ of $f(U)$ in $Y$. The colimits are over directed partially ordered sets. (The restriction mappings of $f_p\mathcal{G}$ are explained in the proof.)*

**Lemma 6.31.4.** *Let $X$ be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(1) *The functor $j_{p!}$ is a left adjoint to the restriction functor $j_p$ (see Lemma 6.31.1).*

(2) *The functor $j_!$ is a left adjoint to restriction, in a formula*

$$Mor_{Sh(X)}(j_!\mathcal{F}, \mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, j^{-1}\mathcal{G}) = Mor_{Sh(U)}(\mathcal{F}, \mathcal{G}|_U)$$

*bifunctorially in $\mathcal{F}$ and $\mathcal{G}$.*

(3) *Let $\mathcal{F}$ be a sheaf of sets on $U$. The stalks of the sheaf $j_!\mathcal{F}$ are described as follows*

$$j_!\mathcal{F}_x = \begin{cases} \emptyset & \text{if} \quad x \notin U \\ \mathcal{F}_x & \text{if} \quad x \in U \end{cases}$$

(4) *On the category of presheaves of $U$ we have $j_p j_{p!} = \mathrm{id}$.*

(5) *On the category of sheaves of $U$ we have $j^{-1} j_! = \mathrm{id}$.*

| situation | category $\mathscr{A}$ | category $\mathscr{B}$ | left adjoint $F\colon \mathscr{A} \to \mathscr{B}$ | right adjoint $G\colon \mathscr{B} \to \mathscr{A}$ |
|---|---|---|---|---|
| A-modules (Ex. 1.5.D) | $Mod_A$ | $Mod_A$ | $(\cdot) \otimes_A N$ | $\mathrm{Hom}_A(N, \cdot)$ |
| ring maps $B \to A$ (Ex. 1.5.E) | $Mod_B$ | $Mod_A$ | $(\cdot) \otimes_B A$ (extension of scalars) | $M \mapsto M_B$ (restriction of scalars) |
| (pre)sheaves on a topological space X (Ex. 2.4.L) | presheaves on X | sheaves on X | sheafification | forgetful |
| (semi)groups (§1.5.3) | semigroups | groups | groupification | forgetful |
| sheaves, $\pi\colon X \to Y$ (Ex. 2.7.B) | sheaves on Y | sheaves on X | $\pi^{-1}$ | $\pi_*$ |
| sheaves of abelian groups or $\mathscr{O}$-modules, open embeddings $\pi\colon U \hookrightarrow Y$ (Ex. 2.7.G) | sheaves on U | sheaves on Y | $\pi_!$ | $\pi^{-1}$ |
| quasicoherent sheaves, $\pi\colon X \to Y$ (Prop. 16.3.6) | $QCoh_Y$ | $QCoh_X$ | $\pi^*$ | $\pi_*$ |
| ring maps $B \to A$ (Ex. 30.3.A) | $Mod_A$ | $Mod_B$ | $M \mapsto M_B$ (restriction of scalars) | $N \mapsto \mathrm{Hom}_B(A, N)$ |
| quasicoherent sheaves, affine $\pi\colon X \to Y$ (Ex. 30.3.B(b)) | $QCoh_X$ | $QCoh_Y$ | $\pi_*$ | $\pi^!_{sh}$ |

Other examples will also come up, such as the adjoint pair $(\sim, \Gamma_\bullet)$ between graded modules over a graded ring, and quasicoherent sheaves on the corresponding projective scheme (§15.4).

E.g. $\operatorname{Spec}\mathbb{C} \longrightarrow \operatorname{Spec}\mathbb{R}$  ring space but not scheme

$(f^{-1}: \mathbb{R}\text{-mod} \longrightarrow (\operatorname{Spec}\mathbb{C}, \widetilde{\mathbb{R}})\text{-mod} \quad )$

quasi-coherent: $f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{C}\text{-mod} \qquad -\otimes_{\mathbb{R}}\mathbb{C}$

$f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \qquad$ forget

$f^*f_*: \mathbb{C}\text{-mod} \longrightarrow \mathbb{C}\text{-mod} \qquad -\otimes_{\mathbb{R}}\mathbb{C}$ ← $\mathbb{C}$-mod structure

$f_*f^*: \mathbb{R}\text{-mod} \longrightarrow \mathbb{R}\text{-mod} \qquad -\otimes_{\mathbb{R}}\mathbb{C}$

étale: $f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow Ab \qquad$ forget

$f_*: Ab \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \qquad -\otimes_{\mathbb{R}}\mathbb{C}$

$f^*f_*: Ab \longrightarrow Ab \qquad -\otimes_{\mathbb{R}}\mathbb{C}$

$f_*f^*: \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \longrightarrow \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\text{-mod} \qquad -\otimes_{\mathbb{R}}\mathbb{C}$ ← $Gal(\mathbb{C}/\mathbb{R})$-action

$\pi: X_{ét} \longrightarrow X_{Zar}$

$\pi^{-1}: Sh(X_{Zar}) \longrightarrow Sh(X_{ét})$  [Zariski sheafification $\mathcal{F} \mapsto \left[\pi^{-1}\mathcal{F}: \; U \xrightarrow{i} X \mapsto \Gamma(U, i^{-1}\mathcal{F})\right]$ is not enough...]

$\pi_*: Sh(X_{ét}) \longrightarrow Sh(X_{Zar}) \qquad$ forget

$\pi^{-1} \dashv \pi_* \qquad \pi_*\pi^{-1} = Id_{Sh(X_{Zar})} \Rightarrow \pi^{-1}$ fully faithful

https://math.stackexchange.com/questions/321633/when-is-a-fully-faithful-functor-an-equivalant-functor

$$\begin{tikzcd} & QCoh(X_{Zar}) \arrow[dl, "W"'] \arrow[d, "\iota"] \\ Sh(X_{ét}) \arrow[r, shift left, "\pi_*"] & Sh(X_{Zar}) \arrow[l, shift left, "\pi^*"] \end{tikzcd}$$

full subcategory

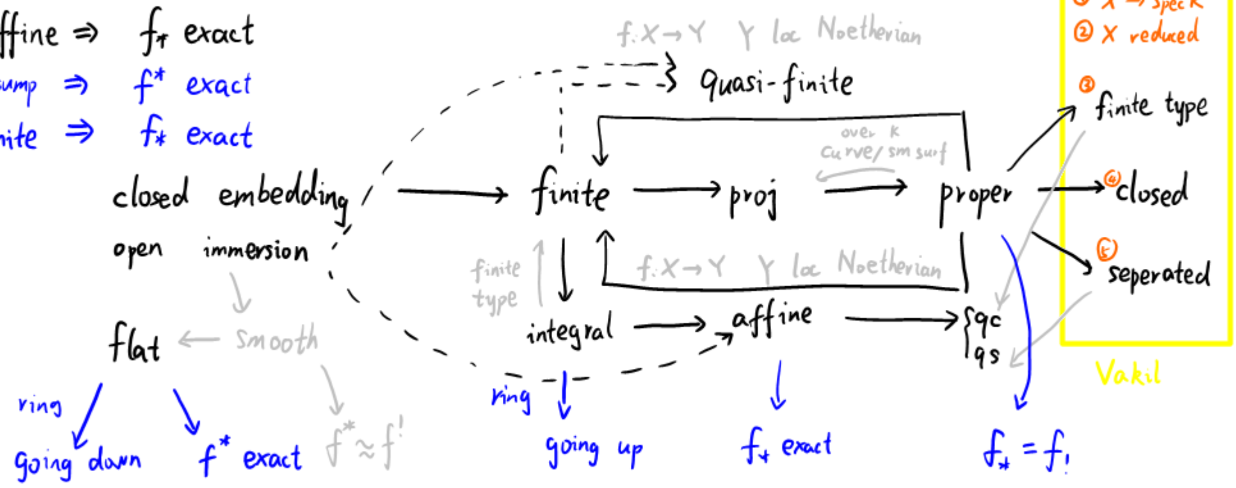
18.1.7

$f$ flat $\Rightarrow f^*$ exact (coherent)

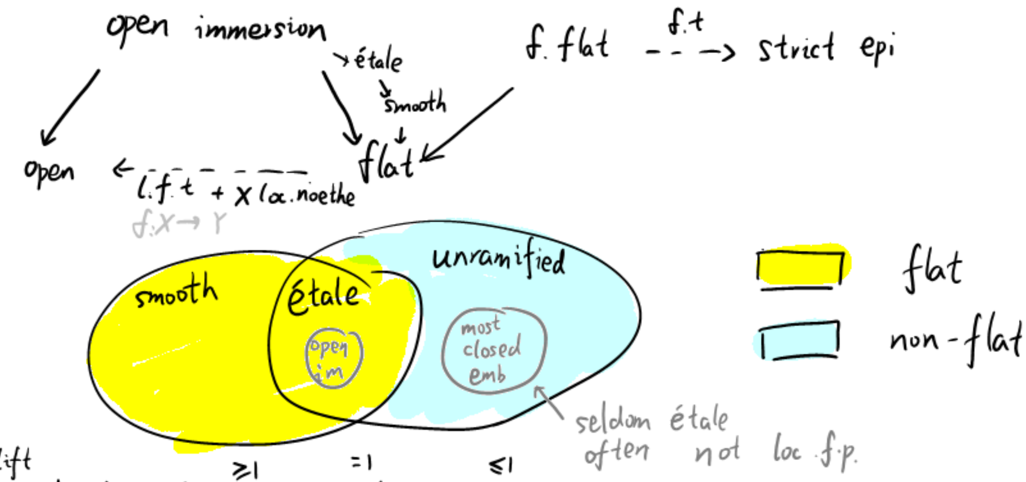
$f$ affine $\Rightarrow f_*$ exact

étale: mild assump $\Rightarrow f^*$ exact

$f$ finite $\Rightarrow f_*$ exact




Various interesting kinds of morphisms (locally Noetherian source, affine, separated, see Exercises 7.3.B(b), 7.3.D, and 10.1.H resp.) are quasiseparated,

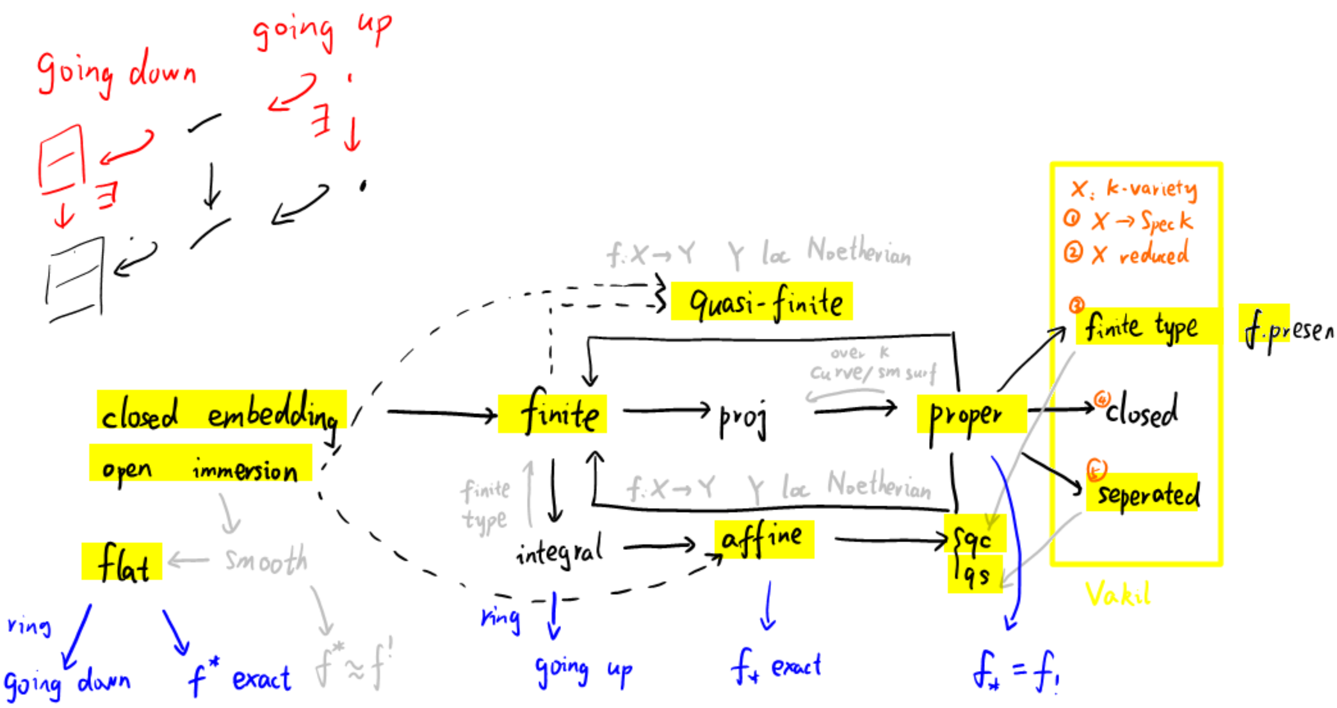



Rmk locally f.p is crucial.

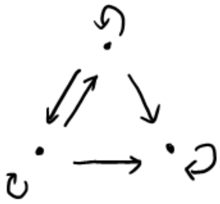
e.g. $\mathbb{C}[t^\lambda : \lambda \in \mathbb{Q}_{>0}] \longrightarrow \mathbb{C}$ is formally étale but not flat
$t^\lambda \longmapsto 0$

https://rankeya.people.uic.edu/formallyunramifiedetale.pdf

$\mathbb{F}_p[T] \longrightarrow (\mathbb{F}_p[T])^{sep}$ is formally unramified, not formally étale but flat.



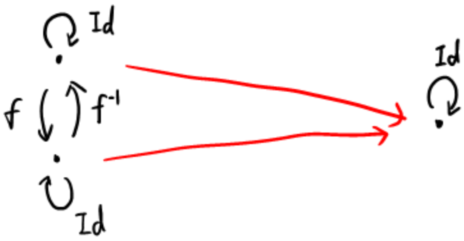

: properties which satisfy the f.flat descent.



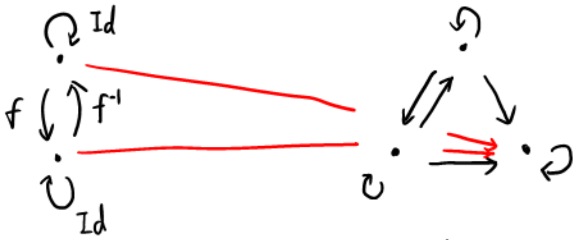
category

full subcategory

https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective




equivalence of category




category

fully faithful

https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/