Eine Woche, ein Beispiel

7.13. stability manifold of $\mathbb{P}^1$

Ref:

[Okada05]: So Okada, Stability Manifold of P^1

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Huy06]: Huybrechts, D. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr. Oxford: Clarendon Press, 2006
.

Goal: understand the Bridgeland stability and wall crossing in this toy example.

Def (locally finite stability condition)

Fix a triangular category $\mathcal{T}$, and denote $K(\mathcal{T})$ as the Grothendieck gp of $\mathcal{T}$.
The set of locally finite stability conditions is defined as

$$\mathrm{Stab}(\mathcal{T}) = \left\{ (Z, \mathcal{P}) \;\middle|\; \begin{array}{l} Z : K(\mathcal{T}) \longrightarrow \mathbb{C} \qquad \text{(central charge)} \\ \mathcal{P} : \mathbb{R} \longrightarrow \{\text{full additive subcategories of } \mathcal{T}\} \\ \qquad \phi \longmapsto \mathcal{P}(\phi) \qquad \text{(slicing)} \\ \text{s.t. } (a)(b)(c)(d) + (e) \end{array} \right\}$$

(a) (slicing compatible with central charge)
if $E \in \mathcal{P}(\phi)$ then $\dfrac{Z(E)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)
$$\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for } A_j \in \mathcal{P}(\phi_j),\ \phi_1 > \phi_2$$

(d) (HN filtration) HN = Harder-Narashimhan
$\forall E \in \mathcal{T}$, $\exists$ finite seq of real numbers
$$\phi_1 > \phi_2 > \cdots > \phi_n$$
and a filtration
$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
A_1 \arrow[u, dashed, "+1"] & & & A_n \arrow[u, dashed, "+1"] &
\end{tikzcd}$$
s.t. $A_j \in \mathcal{P}(\phi_j)$ $\forall j$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{P}(I)$, $\exists$ a Jordan-Hölder filtration with finite length.
$\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\text{extension-closed}}$

Rmk. For $E \in \mathcal{T}$, $E \neq 0$,
$E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$
$\Leftrightarrow$ the HN filtration of E has length 1
$\overset{\text{def}}{\Longleftrightarrow}$ E is semistable

When E is semistable, define $\phi(E) = \phi$ when $E \in \mathcal{P}(\phi)$.

Rmk.

$$\mathrm{Stab}(\mathcal{T}) \triangleq \left\{ (Z,\phi) \;\middle|\; \begin{array}{l} Z: K(\mathcal{T}) \longrightarrow \mathbb{C} \qquad \text{(central charge)} \\ \phi: \mathcal{T} \longrightarrow \{\text{finite subsets of } \mathbb{R}\} \\ \qquad E \longmapsto \{\phi_1, \dots, \phi_n\} \quad \text{(slicing)} \\ \text{s.t. } \ (a)(b)(c)(d) + (e) \end{array} \right\}$$

$$E \in \mathcal{T} \text{ is semistable} \overset{\text{def}}{\iff} \# \phi(E) = 1$$

(a) (slicing compatible with central charge)

For E semistable, $\dfrac{Z(E)}{e^{i\pi\phi(E)}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)

$$\phi(E[1]) = \phi(E) + 1$$

(c) (inverse order vanishing)

$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for} \quad \phi(A_1) > \phi(A_2), \ A_1, A_2 \text{ semistable}$$

(d) (HN filtration)

$\forall E \in \mathcal{T}$, denote $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 < \dots < \phi_n$,

$\exists!$ filtration

$$\begin{tikzcd}
0 = E_0 \arrow[r] & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\
A_1 \arrow[u, dashed, "+1"] & & & A_n \arrow[u, dashed, "+1"] &
\end{tikzcd}$$

s.t. $\phi(A_i) = \phi_i \quad \forall i$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists \ I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.

$\forall E \in \mathcal{T}$ with $\phi(E) \subset I$,

$\exists$ a Jordan-Hölder filtration with finite length.

Prop [Okada 05, Prop 2.3]

$$\mathrm{Stab}(\mathcal{T}) \cong \left\{ (\mathcal{A}, Z) \;\middle|\; \begin{array}{l} \mathcal{A} : \text{heart of } \mathcal{T} \\ Z : K(\mathcal{A}) \longrightarrow \mathbb{C} \\ \quad \text{centered slope-function} \\ \quad \text{with HN property} \end{array} \right\}$$

$$\begin{array}{ccc} (Z, \mathcal{P}) & \longmapsto & (\mathcal{P}((0,1]), Z) \\ (Z, \mathcal{P}) & \longleftarrow\!\shortmid & (\mathcal{A}, Z) \end{array}$$

where $\mathcal{P}(\phi) = \{ E \in \mathcal{A} \text{ semistable} \mid \tilde{\phi}(E) = \phi \} \quad \forall \phi \in (0,1]$

$\tilde{\phi}(E) = \frac{1}{\pi} \arg Z(E) \in (0,1]$

$E \in \mathcal{A}$ semistable: $\nexists$ dec $0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$ s.t. $\phi(A_1) > \phi(E) > \phi(A_2)$

Lemma. On $\mathbb{P}^1$, we have SESs

$$0 \longrightarrow \mathcal{O} \xrightarrow{\times x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(n)^{\oplus n+1} \longrightarrow \mathcal{O}(n+1)^{\oplus n} \longrightarrow 0 \qquad n \geq 0 \qquad (a)$$

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{O}(n) \longrightarrow 0 \qquad n \geq 0$$

which induces triangles

$$\mathcal{O}(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}(k)[1] \xrightarrow{+1}$$

$$\mathcal{O}(k+1)^{\oplus k-n}[-1] \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus k-n+1} \xrightarrow{+1} \qquad n \leq k \qquad (b)$$

$$\mathcal{O}(k+1)^{\oplus n-k} \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus n-k+1}[1] \xrightarrow{+1} \qquad n \geq k$$