Eine Woche, ein Beispiel

12.17 calculation of NMD

Goal: compute normal Morse data (NMD)

$$\{f\geq 0\} \xhookrightarrow{\tilde{i}} X \xhookleftarrow{\tilde{j}} \{f<0\}$$

$$\begin{aligned}
NMD(\mathcal{F}^\cdot, S) &= (R\Gamma_{\{f|_{N\cap X} \geq f(x)\}}(\mathcal{F}^\cdot|_{N\cap X}))_x \\
&\overset{\substack{S=\{*\} \\ X \text{ is cone} \\ f(x)=0}}{\underset{\text{compatible}}{=}} (R\Gamma_{\{f\geq 0\}}(\mathcal{F}^\cdot))_x \\
&= \iota_x^* \tilde{i}^! \mathcal{F}^\cdot \\
&= R\Gamma(X, \{f<0\}, \mathcal{F}^\cdot) \\
&= \mathrm{Fiber}\left(R\Gamma(X,\mathcal{F}^\cdot) \longrightarrow R\Gamma(\{f<0\},\mathcal{F}^\cdot)\right) \\
&= \mathrm{Fiber}\left(\mathcal{F}_x \longrightarrow R\Gamma(\iota_x, \mathcal{F}^\cdot)\right)
\end{aligned}$$

1. low dimensional case
2. quadratic hypersurface
3. du val singularity
4. other quantities

Ref:

https://bastian.rieck.me/blog/posts/2019/morse_theory/

https://oldbookstonew.blogspot.com/
Contains the following books:

[MilnorMT]: Morse Theory by Milnor
[MilnorCC]: Characteristic Classes by Stasheff and Milnor

[MilnorSing]: singular points of complex hypersurfaces by Milnor

[Maxim20]:notes on vanishing cycles and applications
https://people.math.wisc.edu/~lmaxim/vanishing.pdf

1. low dimensional case

E.g. $X = \mathbb{CP}^1$ $\quad f: \mathbb{CP}^1 \dashrightarrow \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$, $\infty \longmapsto 0$ $\quad L_x = \{*\}$ $\quad S = \{\infty\}$

$\mathcal{F}$	$NMD(\mathcal{F}^\cdot, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^\cdot)$
$i_* \underline{\mathbb{Q}}_{\{\infty\}}$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_{\mathbb{CP}^1}[1]$	0	$\mathbb{Q}[1]$	$\mathbb{Q}[1]$
$Rj_* \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}[1]$
$j_! \underline{\mathbb{Q}}_{\mathbb{C}}[1]$	$\mathbb{Q}$	0	$\mathbb{Q}[1]$
$P(\phi)$	$\mathbb{Q}^2$	$\mathbb{Q}$	$\mathbb{Q}[1]$

E.g. $X = \{z_2^2 = z_1^3\}$ $\quad f: X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1} \mathbb{C} \xrightarrow{\mathrm{Re}\, z} \mathbb{R}$ $\quad L_x = \{a, b\}$ $\quad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^\cdot, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^\cdot)$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[1]$	$\mathbb{Q}$	$\mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$Rj_* \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$j_! \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	0	$\mathbb{Q}^2[1]$
$P(\phi)$	$\mathbb{Q}^3$	$\mathbb{Q}$	$\mathbb{Q}^2[1]$

E.g. $X = \mathbb{C} \cup_{\{0\}} \mathbb{C} = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_1 z_2 = 0\}$

$f : X \hookrightarrow \mathbb{C}^2 \xrightarrow{z_1 + z_2} \mathbb{C} \xrightarrow{Re\, z} \mathbb{R}$ $\quad L_x = \{a, b\}$ $\quad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^{\cdot}, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^{\cdot})$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[1]$	$\mathbb{Q}$	$\mathbb{Q}[1]$	$\mathbb{Q}^2[1]$
$Rj_* \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	$\mathbb{Q}^2 \oplus \mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$
$j_! \underline{\mathbb{Q}}_U[1]$	$\mathbb{Q}^2$	0	$\mathbb{Q}^2[1]$
$\pi^! \mathbb{Q}[-1]$	$\mathbb{Q}$	$\mathbb{Q} \oplus \mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$
$IC(\underline{\mathbb{Q}}_U[1])$	0	$\mathbb{Q}^2[1]$	$\mathbb{Q}^2[1]$

E.g. $X = X_3$ $\quad f : X \hookrightarrow \mathbb{C}^3 \xrightarrow{z_3} \mathbb{C} \xrightarrow{Re\, z} \mathbb{R}$ $\quad L_x = \mathbb{C}^\times$ $\quad S = \{0\}$

$\mathcal{F}$	$NMD(\mathcal{F}^{\cdot}, S)$	$\mathcal{F}_x$	$R\Gamma(L_x, \mathcal{F}^{\cdot})$
$i_* \underline{\mathbb{Q}}_Z$	$\mathbb{Q}$	$\mathbb{Q}$	0
$\underline{\mathbb{Q}}_X[2] = \pi^! \mathbb{Q}[-2]$	$\mathbb{Q}$	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$Rj_* \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[2] \oplus \mathbb{Q}[-1]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$j_! \underline{\mathbb{Q}}_U[2]$	$\mathbb{Q} \oplus \mathbb{Q}[1]$	0	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$
$IC(\underline{\mathbb{Q}}_U[2])$	$\mathbb{Q}$	$\mathbb{Q}[2]$	$\mathbb{Q}[1] \oplus \mathbb{Q}[2]$

2. quadratic hypersurface

Cohomology list

$$X_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \cdots z_n^2 = 0\}$$

$$M_n := \{[z_1 : \cdots : z_n] \in \mathbb{CP}^{n-1} \mid z_1^2 + z_2^2 + \cdots z_n^2 = 0\}$$

$H^i(X;\mathbb{Z})$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$M_2 \cong \{a, b\}$	$\mathbb{Z}^2$												
$M_3 \cong \mathbb{CP}^1$	$\mathbb{Z}$		$\mathbb{Z}$										
M_4	$\mathbb{Z}$		$\mathbb{Z}^2$		$\mathbb{Z}$								
M_5	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$						
M_6	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}^2$		$\mathbb{Z}$		$\mathbb{Z}$				
M_7	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		

This table is computed by Lefschetz hyperplane theorem and Chern class.

$H^i(X;\mathbb{Z})$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$M_2 - M_1 \cong \{a, b\}$	$\mathbb{Z}^2$												
$M_3 - M_2 \cong \mathbb{C}^\times$	$\mathbb{Z}$	$\mathbb{Z}$											
$M_4 - M_3$	$\mathbb{Z}$		$\mathbb{Z}$										
$M_5 - M_4$	$\mathbb{Z}$			$\mathbb{Z}$									
$M_6 - M_5$	$\mathbb{Z}$				$\mathbb{Z}$								
$M_7 - M_6$	$\mathbb{Z}$					$\mathbb{Z}$							

This table is computed by open-closed formalism. (Q-coefficient)
Using the Morse theory, one can show that (A variant of [Maxim20, Example 2.18])

$$M_n - M_{n-1} \sim S^{n-2}$$

homotopy equivalence

$H^i(X;\mathbb{Q})$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$X_2 - O \cong \mathbb{C}^\times \sqcup \mathbb{C}^\times$	$\mathbb{Q}^2$	$\mathbb{Q}^2$											
$X_3 - O$	$\mathbb{Q}$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Q}$									
$X_4 - O$	$\mathbb{Q}$		$\mathbb{Q}$	$\mathbb{Q}$		$\mathbb{Q}$							
$X_5 - O$	$\mathbb{Q}$				$\mathbb{Z}/2\mathbb{Z}$			$\mathbb{Q}$					
$X_6 - O$	$\mathbb{Q}$				$\mathbb{Q}$	$\mathbb{Q}$				$\mathbb{Q}$			
$X_7 - O$	$\mathbb{Q}$						$\mathbb{Z}/2\mathbb{Z}$					$\mathbb{Q}$	

$(n-3)$-connected truncation

This table is computed by spectral sequence and Euler class.
Using the Morse theory, one can show that

$$X_n - O \text{ is } (n-3)\text{-connected}.$$

To compute the stalk of IC sheaf, one truncates in the middle.
Z-coefficient cohomology need more work on Euler class.

Suspect: $X_5 - O$:

$$\begin{tikzcd} \mathbb{Z} \arrow[dr, "1"] & \mathbb{Z} \arrow[dr, "2"] & \mathbb{Z} \arrow[dr, "1"] & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

$X_6 - O$:

$$\begin{tikzcd} \mathbb{Z} \arrow[dr, "1"] & \mathbb{Z} \arrow[dr, "{(1,1)}"] & \mathbb{Z}^2 \arrow[dr, "\binom{1}{1}"] & \mathbb{Z} \arrow[dr, "1"] & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z} \end{tikzcd}$$

"reason":

$$\begin{tikzcd} \mathbb{Z} \cong H^2(\mathbb{CP}^4) \arrow[r] \arrow[d] & H^2(M_5) \cong \mathbb{Z} \arrow[d] \\ \mathbb{Z} \cong H^6(\mathbb{CP}^4) \arrow[r] & H^6(M_5) \cong \mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} 1 = ev_{\mathcal{O}(-1)} \arrow[r, mapsto] \arrow[d, mapsto] & ev_{\mathcal{O}(-1)} = 1 \arrow[d, mapsto] \\ 1 \arrow[r, mapsto] & 2 \end{tikzcd}$$

3 hyperplane intersect at 2 pts.

$H^i(X_n - 0, M_n - M_{n-1}; \mathbb{Q})$ $n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
2	0	$\mathbb{Q}^2$											
3			$\mathbb{Q}$	$\mathbb{Q}$									
4				$\mathbb{Q}$		$\mathbb{Q}$							
5					$\mathbb{Q}$			$\mathbb{Q}$					
6						$\mathbb{Q}$				$\mathbb{Q}$			
7							$\mathbb{Q}$					$\mathbb{Q}$	

After truncation, only the red one remains.

The $\mathbb{Z}/2\mathbb{Z} \hat{=} \mathbb{F}_2$ - coefficient is as follows. (Assuming $NMD(X, IC(\mathbb{Z}_U))$ support at index 0)

$H^i(X_n - 0, M_n - M_{n-1}; \mathbb{F}_2)$ $n \backslash i$	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
2		$\mathbb{F}_2^2$											
3			$\mathbb{F}_2$	$\mathbb{F}_2$									
4				$\mathbb{F}_2$		$\mathbb{F}_2$							
5					$\mathbb{F}_2$			$\mathbb{F}_2$					
6						$\mathbb{F}_2$				$\mathbb{F}_2$			
7							$\mathbb{F}_2$					$\mathbb{F}_2$	

After truncation, nothing remains.

Possible direct calculation:

blow up

$$H^i(X_n - 0, M_n - M_{n-1}; \mathbb{Z}) \cong H^{BM}_{2n-2-i}\left((M_n - M_{n-1}) \times \mathbb{R}_{>0} \cup (X_{n-1} - 0) \times \{0\} ; \mathbb{Z}\right)$$

$$\cong H^{BM}_{2n-3-i}\left(S^{n-2} ; \mathbb{Q}\right)$$

Conclusion: for $X_n := \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_1^2 + z_2^2 + \cdots z_n^2 = 0\}$,

$$NMD\left(X_n ; IC(\mathbb{Q}_U[n-1])\right) = \begin{cases} \mathbb{Q}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

3. du val singularity

https://math.stackexchange.com/questions/40351/what-are-the-finite-subgroups-of-su-2c

Name	$R(x,y,z)$	gp G	$\#G$	G/G'	det(Cartan)	
A_n	$x^2+y^2+z^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$	
D_n	$x^2+y^2z+z^{n-1}$	$\mathbb{BD}_{2(n-2)} = Dic_{n-2}$	$4(n-2)$	$\begin{cases} \mathbb{Z}/4\mathbb{Z}, & n \text{ odd} \\ (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}, & n \text{ even} \end{cases}$	4	dicyclic
E_6	$x^2+y^3+z^4$	$\mathbb{BT} \cong SL_2(\mathbb{F}_3)$	24	$\mathbb{Z}/3\mathbb{Z}$	3	
E_7	$x^2+y^3+yz^3$	$\mathbb{BO} \cong 2.S_4^-$	48	$\mathbb{Z}/2\mathbb{Z}$	2	(48,28)
E_8	$x^2+y^3+z^5$	$\mathbb{BD} \cong SL_2(\mathbb{F}_5)$	120	1	1	

$\mathcal{U}$ = link	0	1	2	3
$H^\cdot(\mathcal{U};\mathbb{Z})$	$\mathbb{Z}$	0	G/G'	$\mathbb{Z}$
$H_\cdot(\mathcal{U};\mathbb{Z})$	$\mathbb{Z}$	G/G'	0	$\mathbb{Z}$

L_X homotopic equiv to $\begin{cases} S^1 & A_n \,\&\, D_n \\ S^1 \vee S^1 & E_6, E_7, E_8 \end{cases}$

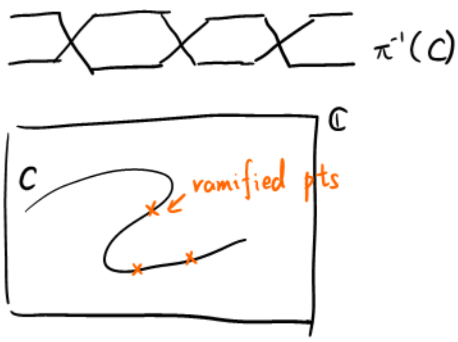
$\Rightarrow H^\cdot(L_X;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}[-1] & A_n \,\&\, D_n \\ \mathbb{Z} \oplus \mathbb{Z}^2[-1] & E_6, E_7, E_8 \end{cases}$

$\Rightarrow H^\cdot(\mathcal{U}, L_X;\mathbb{Q}) = \begin{cases} \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] & A_n \,\&\, D_n \\ \mathbb{Q}[-2] \oplus \mathbb{Q}[-3] & E_6, E_7, E_8 \end{cases}$

$\Rightarrow NMD(X; IC(\mathbb{Q}_{\mathcal{U}}[2])) = \begin{cases} \mathbb{Q} & A_n \,\&\, D_n \\ \mathbb{Q}^2 & E_6, E_7, E_8 \end{cases}$

Three different arguments for $L_X \sim S^1$ or $S^1 \vee S^1$:

1. Morse index [MilnorSing, Theorem 6.5 & 5.11]
2. Riemann surface, contract to $\pi^{-1}(C)$
3. Join construction, see [Maxim 20, Example 2.18]



These singularities can be used to understand 2-dimensional weighted projective spaces.
For weighted projective spaces, the local charts are of form C^n quotient cyclic group.

The topology of cone is still easy to compute by spectral sequence. For the result, see:
http://www.map.mpim-bonn.mpg.de/Fake_lens_spaces
However, the equations become harder to get, and we don't know the topology of the link.
I just believe that there should be a answer for all these singularities.

4. other quantities

Setting M : analytic mfld e.g. $M = \mathbb{C}^n$ or $\mathbb{CP}^n$

$X \subset M$ analytic variety of $\dim_{\mathbb{C}} X = m$

$\mathcal{S}$: $\phi \subset \{o\} \subset X$ where o is the only singularity

$x_0 \in X - \{o\}$

$\mathcal{F} \in \mathrm{Perv}_{\mathcal{S}}(X)$

$\mathcal{L} := \mathcal{F}|_{X-\{o\}}[-m]$ local system on $X - \{o\}$ with rank r

Special case: $\mathcal{F} = IC(\mathcal{L}[m])$

Task: Compute the following quantities.

$$
\begin{tikzcd}
(1)\ \mathrm{NMD}(\mathcal{F}, \{o\}) \arrow[d] & & (3)\ \mathrm{NMD}(\mathcal{F}, X-\{o\}) \arrow[d] & & & \\
(2)\ m(\{o\}) \arrow[dr] & & (4)\ m(X-\{o\}) \arrow[dl] & & (6)\ Eu_X \arrow[d] & Eu_{\{o\}} \arrow[d] \\
& (5)\ CC(\mathcal{F}) \arrow[drr] & & & (7)\ \deg(\Lambda_X) \arrow[dl] & \deg(\Lambda_{\{o\}}) \arrow[dll] \\
& & & (8)\ \chi_{IC}(X) & &
\end{tikzcd}
$$

Here we use notations in https://arxiv.org/abs/2105.13069v2.
6-8 comes from my supervisor's notation, and you may find the definition of Euler obstruction here:
Jiang, Yunfeng. Note on MacPherson's local Euler obstruction

① See the examples before

② $m(\{0\}) = \chi(NMD(\mathcal{F},\{0\}))$

③ $NMD(\mathcal{F}, X-\{0\}) \cong \mathcal{F}_{x_0} \cong \mathbb{Q}^r[m]$

④ $m(X-\{0\}) = (-1)^{\dim_{\mathbb{C}}(X-\{0\})}\chi(NMD(\mathcal{F}, X-\{0\}))$

$= (-1)^m \cdot (-1)^m \cdot r$

$= r$

⑤ $CC(\mathcal{F}) = m(X-\{0\})\,[\overline{T^*_{X-\{0\}}M}] + m(\{0\})\,[\overline{T^*_{\{0\}}M}]$

$= r\,[T^*_X M] + m(\{0\})\,[T^*_{\{0\}}M]$

$= r\,\Lambda_X + m(\{0\})\,\Lambda_{\{0\}}$

recall: $[T^*_X M] = [\overline{T^*_{X-\{0\}}M}]$ $\quad \Lambda_{\overline{S}} := [\overline{T^*_S M}]$

⑥ Need to check the definition.

For $X \subset \mathbb{C}^2$ cuspidal cubic, $Sing(X) = \{p_0\}$,

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X - \{p_0\} \\ 2 & p = p_0 \end{cases}$$

In general. from my memory it looks like:

$$Eu_X(p) = \begin{cases} 0 & p \notin X \\ 1 & p \in X_{sm} \\ \cancel{\geq 1} & \cancel{p \in X - X_{sm}} \end{cases}$$

This is wrong. If you consider the cone on a nonsingular plane curve of degree d, then you get 2d-d^2, which is negative when d>3!

⑦ $\deg(\Lambda_X) := \#(\Lambda_X \cdot \Lambda_M)$ in T^*M

$= (-1)^m \chi(X, Eu_X)$

$= (-1)^m \left(\chi(X-\{0\}) \cdot Eu_X(x_0) + \chi(\{0\}) \cdot Eu_X(0)\right)$

$= (-1)^m \left(\chi(X-\{0\}) + Eu_X(0)\right)$ $= -2$ for $X = M = \mathbb{CP}^1$

$\deg(\Lambda_{\{0\}}) := \#(\Lambda_{\{0\}} \cdot \Lambda_M)$

$= \chi(\{0\}, Eu_{\{0\}})$

$= \chi(\{0\}) \cdot Eu_{\{0\}}(0)$

$= 1$

⑧ $(-1)^m \chi_{IC}(X) = \deg(CC(\mathcal{F}))$ Here, $\mathcal{F} := IC(\mathbb{Q}_{X-\{0\}}[m])$, $r = 1$

$= \deg(r\Lambda_X + m(\{0\})\Lambda_{\{0\}})$

$= r \cdot \deg \Lambda_X + m(\{0\}) \deg \Lambda_0$

$= \deg \Lambda_X + m(\{0\})$

$\Rightarrow \quad \chi_{IC}(X) = \chi(X-\{0\}) + Eu_X(0) + (-1)^m m(\{0\})$

X	$\chi(X-\{0\})$	$Eu_X(0)$	$m(\{0\})$	$\chi(X)$
$\mathbb{C}$	0	1	0	1
$\{y^2 = x^3\}$	0	2	1	1