Thm (Baire) (X,d) cpl, $A_i \subset X$ dense $\Rightarrow \bigcap_{\substack{i\in\Lambda \\ \text{countable}}} A_i \subset X$ dense

Thm (Arzela-Ascoli) For $K \subseteq \mathbb{R}^N$ cpt, $f_n : K \to \mathbb{R}$,

uniformly bounded + uniformly equicontinuous $\Rightarrow$ subseq uniformly converges

$\sup_n \|f\|_\infty < +\infty$ $\quad \inf_{\delta>0} \sup_{\substack{n \\ |x-y|<\delta}} |f_n(x) - f_n(y)| = 0$ $\quad \lim_{k\to+\infty} \|f_{n_k} - f\|_\infty = 0$

$f_n \in C^\infty(K)$, $\sup_n \|f_n^{(i)}\|_\infty < +\infty \ \forall i \Rightarrow \exists f \in C^\infty(K)$ $f_{n_k}^{(i)} \rightrightarrows f^{(i)}$, $f_{n_k} \xrightarrow{C^\infty(K)} f$ (diagonal method)

Thm (Banach-Steinhaus) X Frechét, Y: TVS, $\Lambda \subseteq \mathcal{L}(X,Y)$,

"ptws bounded": $\forall x \in X$, $\{L(x) \mid L \in \Lambda\} \subset Y$ is bd $\Rightarrow$ equicont: $\bigcap_{L\in\Lambda} L^{-1}(\mathcal{U}) \subset X$ is a nbhd of 0

"Cor". X Frechét, Y: TVS, $\{L_n\} \subset \Lambda$, ptws converge $\Rightarrow L \in \mathcal{L}(X,Y)$

Thm (Uniform boundedness principal) X: Banach, Y: normed v.s. $\Lambda \subseteq \mathcal{L}(X,Y)$

$\forall x \in X$, $\sup_{L\in\Lambda} \|Lx\|_Y < +\infty \Rightarrow \exists C > 0$ s.t. $\forall L \in \Lambda$, $\|L\| < C$.

~~Thm (Open~~

Cor (Inverse mapping principle) X, Y: Frechét, $L \in \mathcal{L}(X,Y)$ bij $\Rightarrow L^{-1} \in \mathcal{L}(Y,X)$

When X, Y: Banach, norms on X, Y are equivalent

Thm (Closed graph theorem) X, Y: Frechet, Y Hausdorff, $f: X \to Y$

$\Gamma_f \subset X \times Y$ closed $\Leftrightarrow f \in \mathcal{L}(X,Y)$

Thm (Open mapping thm) X: Fréchet, Y: TVS, $L \in \mathcal{L}(X,Y)$

$L(X) \subset Y$ of 2^{nd} category $\Rightarrow L$ is open, $L(X) = Y$

Thm (Hahn-Banach thm) X: $\mathbb{R}$-v.s. $Y \subset X$

$p: X \to \mathbb{R}$ quasi-seminorm. ($\cancel{X}$ $p \geq 0$)

$l: Y \to \mathbb{R}$ s.t. $l(y) \leq p(y)$

$\Rightarrow \exists L: X \to \mathbb{R}$ s.t. $L(x) \leq p(x)$ $L|_Y = l$

Cor. X normed, $Y \subset X$, $l \in Y' \Rightarrow \exists L \in X'$ s.t. $\|L\|_{X^*} = \|l\|_{Y^*}$, $L|_Y = l$

In LCTVS $V = (V, p_\alpha)$

basis of 0: $\bigcap_{fin} \{x \mid p_\alpha(x) < r\}$

seq : $f_n \to f \Leftrightarrow p_\alpha(f_n) \to p_\alpha(f) \ \forall \alpha$

seminorm. p cont $\Leftrightarrow p(x) \leq \sum_{fin} C_\alpha p_\alpha(x)$

lin fct. L cont $\Leftrightarrow |L(x)| \leq \sum_{fin} C_\alpha p_\alpha(x)$

In X' (X: TVS), $p_x(f) = |f(x)|$ are seminorms

$\widehat{f(\lambda -)}(\xi) = \frac{1}{\lambda^n} \hat{f}\left(\frac{\xi}{\lambda}\right)$ $\quad \widehat{f(\vec{v} + -)}(\xi) = e^{i\langle \vec{v}, \xi\rangle} \hat{f}(\xi)$

$\widehat{D^\beta f}(\xi) = \xi^\beta \hat{f}(\xi)$ $\quad \widehat{f * g}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$

$\widehat{e^{-\frac{\|x\|^2}{2}}} = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}$ $\quad \|f\|_{L^2}^2 = \frac{1}{(2\pi)^n} \|\hat{f}\|_{L^2}^2$

$\widehat{\delta_a(x)} = e^{-i\langle a, \xi\rangle}$ $\quad \widehat{e^{i\langle a, x\rangle}} = (2\pi)^n \delta_a(\xi)$

$\langle f, g\rangle = \int_\Omega \bar{f}(x) g(x) dx$ $\quad \langle \partial^\beta f, g\rangle = (-1)^{|\beta|} \langle f, \partial^\beta g\rangle$

$D^\beta = i^{-|\beta|} \partial^\beta = i^{-|\beta|} \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$ $\quad \langle D^\beta f, g\rangle = \langle f, D^\beta g\rangle$

$\mathcal{F}: \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ $\quad \langle \mathcal{F}f, g\rangle = \langle f, \mathcal{F}g\rangle$

$k \in \mathcal{D}'(\Omega \times \Omega') \leftrightsquigarrow A_k: \mathcal{D}(\Omega) \to \mathcal{D}'(\Omega')$

$k \in \mathcal{E}(\Omega \times \Omega') \leftrightsquigarrow A_k: \mathcal{E}'(\Omega) \to \mathcal{E}(\Omega')$ smoothing

$\overset{\Omega = \Omega'}{\leftrightsquigarrow} A_k \in \Psi DO^{-\infty}(\Omega) \twoheadleftarrow S^{-\infty}(\Omega \times \Omega \times \mathbb{R}^n) \ni a.$

$[L^p, L^q]_\theta = L^r$ $\quad [H^s, H^t]_\theta = H^{(1-\theta)s + \theta t}$

$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x, \xi\rangle} f(x) dx$

$f(x) = \int_{\mathbb{R}^n} e^{i\langle x, \xi\rangle} \hat{f}(\xi) đ\xi = \frac{1}{(2\pi)^n} \widehat{\hat{f}(-\xi)}$

$đ\xi = \frac{1}{(2\pi)^n} d\xi$

$\mathcal{E}(\Omega)$ $\quad p_{\alpha,K}(f) = \sup_{x\in K} \|\partial^\alpha f\|_\infty$

$\mathcal{S}(\mathbb{R}^n)$ $\quad p_{\alpha,\beta}(f) = \|x^\beta \partial^\alpha f\|_\infty$

$p_{\alpha,N}(f) = \|(1+|x|)^N \partial^\alpha f\|_\infty$

$\mathcal{D}(\Omega) = \varinjlim_K \mathcal{D}_K(\Omega)$ $\quad p_{\alpha,K}(f) = \sup_{x\in K} \|\partial^\alpha f\|_\infty$

E: Banach $\quad f: \bar{\Omega} \to E$ bd cont hol

$\mathcal{H} := \mathcal{H}(E_0, E_1) = \left\{ f: \bar{\Omega} \to E_0 + E_1 \;\middle|\; \begin{array}{l} \text{bd cont hol} \\ f(it) \in E_0,\ f(1+it) \in E_1 \end{array} \right\}$

Banach Space with norm $|||\cdot|||_\infty$

$E_\theta := [E_0, E_1]_\theta := \mathcal{H}(E_0, E_1) / \{f : f(\theta) = 0\} \sim \mathrm{Im}\, ev_\theta$

$M_\theta(f) := \sup_{t\in\mathbb{R}} \|f(r+it)\| < +\infty \Rightarrow M_\theta(f) \leq (M_0(f))^\theta (M_1(f))^{1-\theta}$

$T(E_0) \subseteq F_0 \quad T(E_1) \subseteq F_1 \Rightarrow T(E_\theta) \subseteq F_\theta$, $\|T\|_\theta \leq \|T\|_0^{1-\theta} \|T\|_1^\theta$

$\phi(z) = \frac{f(-)}{|f(-)|} |f(-)|^{r\left(\frac{1-z}{p} + \frac{z}{q}\right)} \mathbb{1}_{\{|f|>0\}}$ $\quad \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$

$\phi(z) = f(-) \Lambda(-)^{(\theta - z)(t-s)}$ $\quad \Lambda(\xi) = (1 + |\xi|^2)^{\frac{1}{2}}$

$$\underset{\Psi DO^m(\Omega)}{\hat{P}u(x)} = \int_\Omega \Big(\underbrace{\int_{\mathbb{R}^n} \overbrace{e^{i\langle x-y,\xi\rangle}}^{\text{phase fct}}\, \overbrace{\sigma_P(x,\xi)}^{\text{amplitude} \in CS^m(\Omega\times\mathbb{R}^N)}\, \bar{d}\xi}_{\text{kernel/oscilatory integral} \in \mathcal{D}'(\Omega\times\Omega)}\Big) u(y)\,dy$$

$$\text{"="} \int_\Omega \Big(\int_{\mathbb{R}^n} e^{-i\langle y,\xi\rangle} (\mathcal{F}_y^{-1}P)(x,\xi)\,\bar{d}\xi\Big) u(y)\,dy$$

$Op_a: \mathcal{D}(\Omega) \longrightarrow \mathcal{E}(\Omega)$ $\quad u \mapsto Op_a u$ $\quad (a \in S^m(\Omega\times\Omega\times\mathbb{R}^n))$

$$(Op_a u)(x) = \int_{\mathbb{R}^n}\Big(\int_\Omega e^{i\langle x-y,\xi\rangle} a(x,y,\xi) u(y)\,dy\Big)\bar{d}\xi$$

$$= \text{"}\int_\Omega \Big(\underbrace{\int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\,\bar{d}\xi}_{K(x,y)}\Big) u(y)\,dy\text{"}$$

$$\Psi^{Phase} = \left\{\varphi: \Omega\times\mathbb{R}^N \longrightarrow H\sqcup\mathbb{R} \;\middle|\; \begin{array}{l}\varphi(x,\lambda\xi) = \lambda\varphi(x,\xi) \quad \forall\lambda>0,\ \xi\neq 0 \\ d\varphi(x,\xi)\neq 0\end{array}\right\}$$

$$CS^{-\infty} = S^{-\infty} \subset I_m\, l_{m,m'} \text{ pt conv topo}, \qquad S^{-\infty} \to S^m \overset{l_{m,m'}}{\subset} S^{m'}$$

$m \geq m_1 > m_2 \cdots \to -\infty$, $a \in S^m(\Omega\times\mathbb{R}^N)$, $a_{m_j} \in S^{m_j} \ldots$

$a \sim \sum_{j=1}^{\infty} a_{m_j}$: $\forall k \in \mathbb{N}_{>0}$,

$$a - \sum_{j=1}^{k} a_{m_j} \in S^{m_{j+1}}(\Omega\times\mathbb{R}^N)$$

$$S^m(\Omega\times\mathbb{R}^N) = \left\{ a(x,\xi) \in C^\infty(\Omega\times\mathbb{R}^N) \;\middle|\; \begin{array}{l} \forall K \subset \Omega \text{ cpt},\ \forall \alpha,\beta \\ \exists C = C(\alpha,\beta,K) > 0 \text{ s.t.} \\ |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \leq C(1+|\xi|)^{m-|\beta|} \text{ in } K\times\mathbb{R}^N \end{array}\right\}$$

Fréchet with seminorm $p_{\alpha,\beta,K} = \min C$

$$\underset{\text{subspace topo}}{CS^m(\Omega\times\mathbb{R}^N)} = \left\{ a(x,\xi) \in S^m(\Omega\times\mathbb{R}^N) \;\middle|\; \begin{array}{l} \exists a_{m-j} \in C^\infty(\Omega\times\mathbb{R}^N) \text{ for all } j \in \mathbb{N}_{>0} \text{ s.t.} \\ a_{m-j}(x,\lambda\xi) = \lambda^{m-j} a_{m-j}(x,\xi) \quad \forall \lambda \geq 1,\ |\xi|\geq 1 \\ a \sim \sum_{j=0}^{\infty} a_{m-j}, \text{ i.e. } a - \sum_{j\geq 0}^{k} a_{m-j} \in S^{m-k-1}(\Omega\times\mathbb{R}^N) \end{array}\right\}$$

$$\Psi DO^m(\Omega) = \{A: \mathcal{D}(\Omega) \longrightarrow \mathcal{E}(\Omega) \mid A = Op_a \text{ for some } a \in S^m(\Omega\times\Omega\times\mathbb{R}^n)\}$$

$$L^m(\Omega) = \{A \in \Psi DO^m(\Omega) \mid \operatorname{supp} k_A \subset \Omega\times\Omega \text{ is } \pi\text{-proper}\}$$

$$= \{A = Op_a \mid \operatorname{supp}_{\Omega\times\Omega} a \text{ is } \pi\text{-proper}\}$$

$$CL^m(\Omega) = \{A \in L^m(\Omega) \mid A = Op_a \text{ for some } a \in CS^m(\Omega\times\Omega\times\mathbb{R}^n)\}$$

$$\begin{tikzcd} R \subset X\times Y \arrow[d, "\pi_{R,X}"'] \arrow[dr, "\pi_{R,Y}"] & \\ X & Y \end{tikzcd}$$

R: π-proper $\Leftrightarrow \pi_{R,X}, \pi_{R,Y}$ proper

$$\begin{array}{ccc} \mathcal{D}(\Omega) & \dashrightarrow & \mathcal{D}(\Omega) \\ \cap & & \cap \\ \mathcal{E}(\Omega) & \dashrightarrow & \mathcal{E}(\Omega) \\ \cap & & \cap \\ \mathcal{D}'(\Omega) & \dashrightarrow & \mathcal{D}'(\Omega) \end{array}$$

for $L^m(\Omega)$

$$\text{"}I(a,\varphi)(x) := \int_{\mathbb{R}^N} e^{i\varphi(x,\xi)} a(x,\xi)\,\bar{d}\xi\text{"}$$

$$\langle I(a,\varphi), u\rangle = \int_{\mathbb{R}^N}\Big(\int_\Omega e^{i\varphi(x,\xi)} a(x,\xi) u(x)\,dx\Big)\bar{d}\xi$$

$$\underset{\varphi(x,\xi) = \langle x,\xi\rangle}{\overset{a \in S^m_{cpt\,\Omega}}{=\!=}} \int_{\mathbb{R}^N}\Big(\int_\Omega e^{i\langle x,\xi\rangle} \sigma(\xi) u(x)\,dx\Big)\bar{d}\xi$$

(Workhorse)

$$\sigma(\xi) = \widehat{I(a,\varphi)}(\xi) = \underbrace{\sum_\alpha \frac{i^{|\alpha|}}{\alpha!} \partial_x^\alpha \partial_\xi^\alpha a(x,\xi)\Big|_{x=0}}_{\in S^{m-|\alpha|}(\{0\}\times\mathbb{R}^N)}$$

$$S^m(\Omega\times\mathbb{R}^N) \times \Psi^{Phase} \longrightarrow \mathcal{D}'(\Omega)$$

$$\cup \qquad\qquad\qquad \cup$$

$$S^m_{cpt\,\Omega}(\Omega\times\mathbb{R}^N) \times \Psi^{Phase} \longrightarrow \mathcal{E}'(\Omega)$$

$$(a, \varphi) \longmapsto I(a,\varphi)$$

Def. $a \in S^m(\Omega\times\mathbb{R}^N)$ is <u>elliptic</u>, if

$\forall K \subset \Omega$ cpt, $\exists R > 0$, $C = C(K)$ s.t.

$\forall |\xi| > R$, $|a(x,\xi)|^{-1} \leq C(1+|\xi|)^{-m}$

$A \in L^m(\Omega)$ is <u>elliptic</u>, if σ_A is elliptic

$\boxed{Op_a = A_k}$ $\overset{POV}{\Longrightarrow} \exists b \in S^{-m}(\Omega\times\mathbb{R}^n)$, $b(x,\xi) = a(x,\xi)^{-1}$ outside

Prop. $a \in S^{-\infty}(\Omega\times\Omega\times\mathbb{R}^n) \overset{\exists}{\longrightarrow} \mathcal{D}'(\Omega\times\Omega) \ni k(x,y)$

$$a \longmapsto \int_{\mathbb{R}^n} e^{i\langle x-y,\xi\rangle} a(x,y,\xi)\,\bar{d}\xi$$

$$k(x,y) e^{-i\langle x-y,\xi\rangle}\chi(\xi) \longleftarrow k$$

$\chi \in C_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\chi(\xi)\,\bar{d}\xi = 1$

Thm. $\exists B \in S^{-m}(\Omega\times\mathbb{R}^n)$ s.t. $AB - I \in L^{-\infty}(\Omega)$

$BA - I \in L^{-\infty}(\Omega)$

Cor $\operatorname{singsupp} Au \subseteq \operatorname{singsupp} u$ "$\subseteq$" $\to$ "$=$"

Def (Complete symbol of A)

$\sigma: L^m(\Omega) \longrightarrow C^\infty(\Omega\times\mathbb{R}^n)$ $\quad A \longmapsto \sigma_A$

$$\sigma_A(x,\xi) = e^{-i\langle x,\xi\rangle}(Ae^{i\langle -,\xi\rangle})(x)$$

Thm. $\sigma_A \in S^m(\Omega\times\mathbb{R}^n)$, $A = Op_{\sigma_A}$, $\forall a$ with $A = Op_a$,

$$\sigma_A \sim \underbrace{\sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha \partial_y^\alpha a(x,y,\xi))\Big|_{y=x}}_{\in S^{m-|\alpha|}(\Omega\times\mathbb{R}^n)}$$

$$q_a(\theta,\xi) = \int_\Omega e^{i\langle x,\theta\rangle} a(x,\xi)\,\bar{d}x$$

$$k_a(\tau,\xi) = q_a(\xi-\tau,\xi)$$

$$\tilde{k}_a(\tau,\xi) = (1+|\tau|)^{s-m} q_a(\xi-\tau,\xi)(1+|\xi|)^{-s}$$

$\forall N$, $\exists C_N$ s.t. $|q_a(\theta,\xi)| \leq C_N(1+|\theta|)^{-N}(1+|\xi|)^m$

$$\Rightarrow |\tilde{k}_a(\tau,\xi)| \leq C_N(1+|\xi-\tau|)^{|m-s|-N}$$

$$\begin{tikzcd} H^s(\mathbb{R}^n) \arrow[r, "A = Op(a)"] \arrow[d, "\mathcal{F}"] & H^{s-m}(\mathbb{R}^n) \arrow[d, "\mathcal{F}"] \\ L^2(\mathbb{R}^n, (1+|\xi|^2)^s) \arrow[r, "k_a"] \arrow[d, "\times(1+|\xi|^2)^{\frac{s}{2}}"] & L^2(\mathbb{R}^n, (1+|\xi|^2)^{s-m}) \arrow[d, "\times(1+|\xi|^2)^{\frac{s-m}{2}}"] \\ L^2(\mathbb{R}^n) \arrow[r, "\tilde{k}_a"] & L^2(\mathbb{R}^n) \end{tikzcd}$$

$$\sigma_{A^*}(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha \partial_x^\alpha \overline{\sigma_A(x,\xi)} \qquad k_{A^*}(x,y) = \overline{k_A(y,x)}$$

$$\sigma_{A\circ B}(x,\xi) \sim \sum_\alpha \frac{i^{-|\alpha|}}{\alpha!}(\partial_\xi^\alpha \sigma_A(x,\xi))\cdot(\partial_x^\alpha \sigma_B(x,\xi)) \quad (k\text{; integral})$$

$$\sigma_{k^*A}(x,\xi) \equiv \sigma_A(k(x), (T^*_{k(x)}k)(\xi)) \bmod S^{m-1} \qquad X \xrightarrow{k} Y \quad A \in L^m(Y)$$