Ex 6 & 7

Today we discuss some exercises before. You can ask me anything about the exercises.

If there are no enough questions, I'd like to say something about ~~pointwise~~ convergenge. It is important for math students to distinguish the concept of pointwise converg and uniformly converg.

Since you have some exercises about this ~~distingu~~ key point, I have to emphasize it.

$U \subseteq \mathbb{R}^n$

Def. $f_n : U \longrightarrow \mathbb{R}$ converges pointwise, uniformly to f, if

$\forall x \in U$ $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}_{>0}$ s.t. $\forall n > N$, $\forall x \in U$ we have

$$|f_n(x) - f(x)| < \varepsilon$$

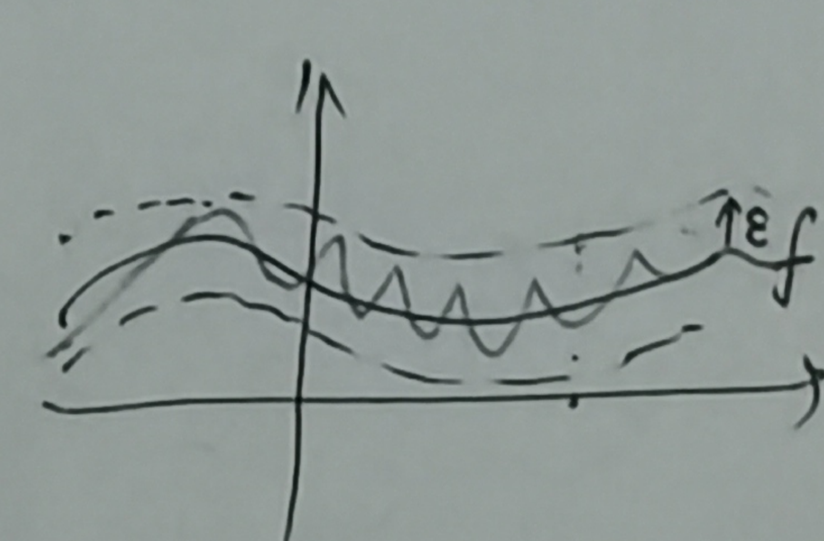
$\Longleftrightarrow$ $\forall x \in U$, $\lim\limits_{n \to +\infty} f_n(x) = f(x)$ (pointwise, $f_n \xrightarrow{\text{pointwise}} f$)

$\Longleftrightarrow$ $\lim\limits_{n \to +\infty} \sup\limits_{x \in U} |f_n(x) - f(x)| = 0$ (uniformly, $f_n \overset{un}{\rightrightarrows} f$)

Ex. Write down the definition of ~~$\{f_n\}_{n\in\mathbb{N}}$~~ "does not converge".

f_n $U \longrightarrow \mathbb{R}$ doesn't converge pointwise, uniformly to f, if.

Ex. uniformly $\Rightarrow$ pointwise.



$\forall \varepsilon > 0$

Task (1) $\quad f_n(x) = \frac{1}{n}\sin(nx) \qquad x \in \mathbb{R}$.

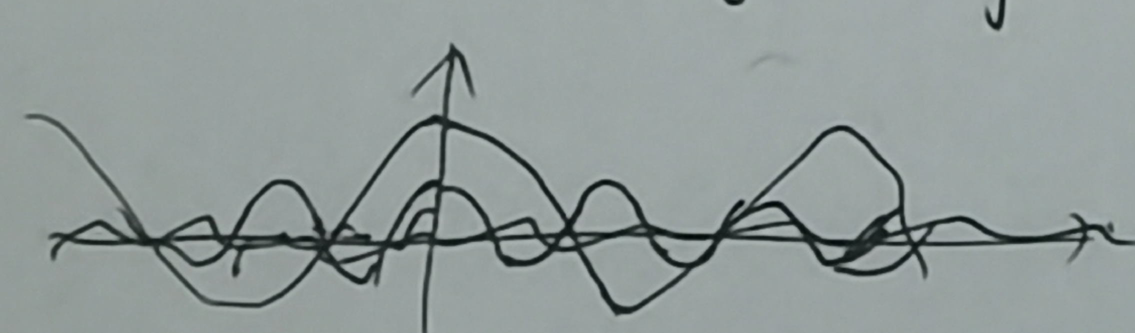
Fix $x \in \mathbb{R}$, $\quad \lim_{n\to+\infty} \frac{1}{n} \sin(nx) = 0 \qquad \left|\frac{1}{n}\sin(nx)\right| \le \left|\frac{1}{n}\right| \xrightarrow{n\to+\infty} 0$

$\therefore f_n$ converges pointwise to 0.

$$\sup_{x\in U} |f_n(x) - f(x)| \le \frac{1}{n} \xrightarrow{n\to+\infty} 0$$

$$\Rightarrow \lim_{n\to+\infty} \sup_{x\in U} |f_n(x) - f(x)| = 0$$

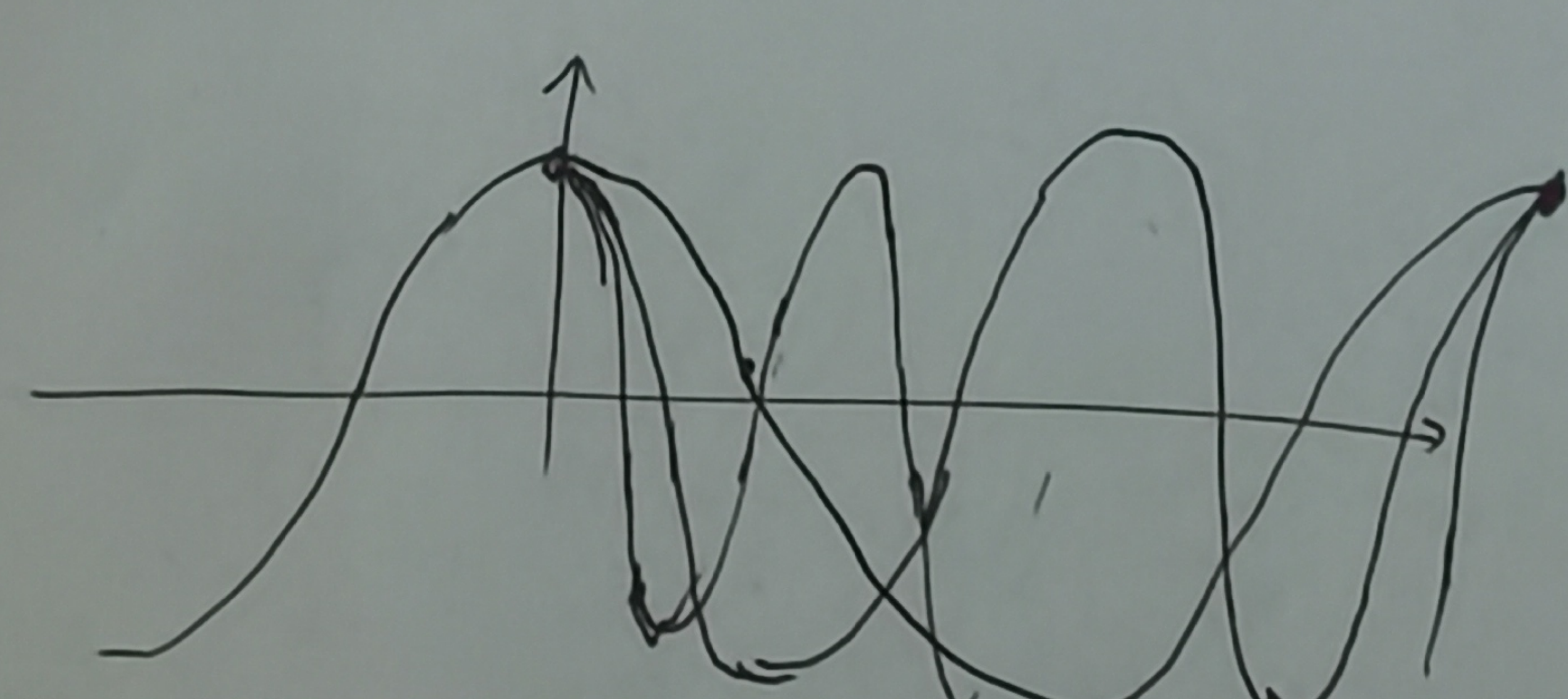
$\therefore f_n$ —— ——uniformly to 0.



(2)

$$g_n(x) = \cos(nx) \qquad x \in \mathbb{R}$$

Fix $x \in \mathbb{R}$. $\quad \lim_{n\to+\infty} \cos(nx) = \begin{cases} \nexists & x \notin \pi\mathbb{Z} \\ 1 & x \in \pi\mathbb{Z} \end{cases}$



$\therefore f_n$ doesn't converge pointwise.

(3) $h_n(x) = x^n \qquad x \in (0,1)$

Fix $x \in (0,1)$ $\qquad \lim\limits_{n\to+\infty} x^n = 0$

$h_n \xrightarrow{pt} h \equiv 0$

$$\sup_{x\in(0,1)} |h_n(x) - h(x)| = \sup_{x\in(0,1)} |x^n| = 1 \not\xrightarrow{n\to+\infty} 0$$

$\therefore h_n \overset{un}{\not\rightrightarrows} h$

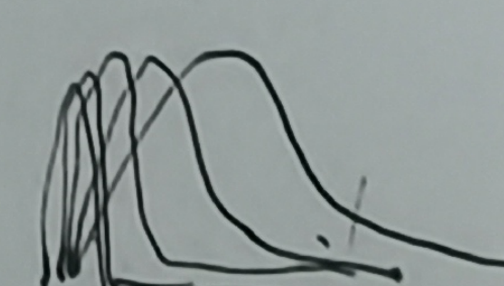
(4). $i_n(x) = x^n \qquad x \in (0, \frac{1}{2})$

$$\sup_{x\in(0,\frac{1}{2})} |i_n(x) - i(x)| = \sup_{x\in(0,\frac{1}{2})} |x^n| = \left(\frac{1}{2}\right)^n$$

$$\therefore \lim_{n\to+\infty} \left(\qquad\qquad \right) = \lim_{n\to+\infty} \left(\frac{1}{2}\right)^n = 0$$

$i_n \overset{un}{\rightrightarrows} i \equiv 0$

(5) $j_n(x) = nxe^{-nx} \qquad x \in [0, +\infty)$



Fix $x \in [0, +\infty)$.

$$\lim_{n\to+\infty} nxe^{-nx} = \begin{cases} \lim\limits_{y\to+\infty} ye^{-y} = 0 & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\therefore j_n \xrightarrow{pt} j \equiv 0$

$$\sup_{x\in[0,+\infty)} |nxe^{-nx} - 0| = \sup_{x\in[0,+\infty)} nxe^{-nx} \geq 1 \cdot e^{-1} = \frac{1}{e}$$

$\Rightarrow \lim\limits_{n\to+\infty} \sup\limits_{x} |nxe^{-nx} - 0| \nexists$

$\therefore j_n \overset{un}{\not\rightrightarrows} 0$

Uniform convergence has many nice properties

(Uniform limit teorem)

App 1. $f_n \in C(\mathcal{U})$ $\quad f_n \rightrightarrows f \quad \Rightarrow f \in C(\mathcal{U})$

App 2. $f_n \in C'([a,b])$ $\quad f_n' \rightrightarrows g \quad \Rightarrow f_n \rightrightarrows f \quad f \in C'([a,b])$

$$g = f'$$

App 3. $f_n \in C([a,b])$. $\quad f_n \rightrightarrows f \quad \Rightarrow \lim_{n\to+\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

Pf. $$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)dx \right|$$

$$= \left| \int_a^b f_n(x) - f(x)\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f(x)|\,dx$$

$$\leq \int_a^b \sup_{y\in[a,b]} |f_n(y) - f(y)|\,dx$$

$$= (b-a) \sup_{y\in[a,b]} |f_n(y) - f(y)| \xrightarrow{n\to+\infty} 0.$$

Rmk. For ~~cpx~~ hol fct we have stronger result then App 2.

Thm 1. [Ahlfors, P176]

Let $f_n \in \mathcal{O}(\mathcal{U})$, $f_n \rightrightarrows f \quad \Rightarrow f \in \mathcal{O}(\mathcal{U})$

(for $\forall$ $K \subseteq \mathcal{U}$ cpt) $\quad f_n' \rightrightarrows f'$

Cor For $f(z) = \sum_{n=0}^{+\infty} a_n z^n$, $a_n \in \mathbb{C}$ denote R as the radius of convergence. Then,

$$f(z) \in \mathcal{O}(B_0(R))$$

$\cancel{\forall z \in B_0(R)}$, $f'(z) = \sum_{n=0}^{+\infty} n a_n z^{n-1}$, $\forall z \in B_0(R)$