Eine Woche, ein Beispiel

11.19. Basic sheaf calculation

Goal: Motivate $f_*, f^*, f_!, f^!$, by connecting them with (co)homology theory

After story: $\rightsquigarrow$ calculation of $\mathrm{Perv}_\Delta(\mathbb{CP}^1)$
$\rightsquigarrow$ generalize Morse theory
$\rightsquigarrow$ Characteristic classes / cycles
$\rightsquigarrow$ index theorem

Minor advantages from my talk:

- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism.
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    $\rightsquigarrow$ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale) descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \mathrm{Top}$
$\mathrm{Obj}(Sh(X)) = \{\text{sheaves of abelian gps}\}$
e.p. $Sh(\{*\}) = \mathrm{Abel}$ $\qquad \mathbb{Q}_{\{*\}} \longleftrightarrow \mathbb{Q}$

0. sheaf
1. f_*, skyscraper sheaf & global sections
2. f^*, constant sheaf & stalks
3. Rf_* & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology

$- \otimes -$ & product structure on cohomology
$\mathrm{Hom}(-,-)$ & Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016
[IHPS] Laurențiu G. Maxim, Intersection Homology & Perverse Sheaves with Applications to Singularities, 2019
[BI86] Birger Iversen, Cohomology of Sheaves, 1986
https://link.springer.com/book/10.1007/978-3-642-82783-9

0. Sheaf

Recall the definition of

- presheaf $\mathcal{F}$
- sheaf $\mathcal{F}$
- stalk $\mathcal{F}_x$
- global section $\mathcal{F}(X) = \Gamma(X;\mathcal{F}) = H^0(X;\mathcal{F})$
- cohomology $R^n\Gamma(X;\mathcal{F}) = H^n(X;\mathcal{F})$

https://mathoverflow.net/questions/4214/equivalence-of-grothendieck-style-versus-cech-style-sheaf-cohomology
If X is paracompact and Hausdorff, Cech cohomology coincides with Grothendieck cohomology for ALL SHEAVES

https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology
https://math.stackexchange.com/questions/3305512/cech-cohomology-and-the-simplicial-cohomology-of-the-nerve-of-an-open-cover

Recall examples of sheaves:

complicated $\Big\{$
- $\mathcal{C}_X$: sheaf of cont fcts on X
- $\mathcal{O}_X$: structure sheaf on X e.g., X: (cplx) mfld, scheme, ⋯
- $\underline{\mathbb{Q}}_X$: constant sheaf on X

- $\mathrm{sky}_p(\mathbb{Q})$: skyscraper sheaf of $p \in X$ on X.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_X)_x \subseteq (\mathcal{O}_X)_x \subseteq (\mathcal{C}_X)_x \quad \& \quad (\mathrm{sky}_p(\mathbb{Q}))_x .$$

1. f_*, skyscraper sheaf & global sections

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(Y)$, $f: Y \longrightarrow X$ cont

Def. $f_*\mathcal{F} \in Sh(X)$ is given by

$$f_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(f^{-1}(\mathcal{U}))$$

This defines a fctor

$$f_*: Sh(Y) \longrightarrow Sh(X)$$

$$\begin{array}{ccc} \mathcal{F} & & f_*\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_{p,*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$

For $\pi: Y \longrightarrow \{*\}$, $\pi_*\mathcal{F} = \mathcal{F}(Y) = \Gamma(Y;\mathcal{F})$

Ex (hard?) For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, compute $j_*\underline{\mathbb{Q}}_{\mathbb{C}}$.

- ○ It is a constant sheaf on $\mathbb{CP}^1$.
- ○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$.
- ○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$.
- ○ All the above is wrong.
- ○ I don't know, but I don't want to make a wrong choice.

2. f^*, constant sheaf & stalks

In [Vakil, Chapter 2], it is f^{-1}, the inverse image functor.

Setting $X, Y \in \mathcal{T}op$, $\mathcal{F} \in Sh(X)$, $f: Y \longrightarrow X$ cont

Def. $f^*\mathcal{F} \in Sh(Y)$ is given by sheafification of

$$f^{*,pre}\mathcal{F}(\mathcal{U}) = \varinjlim_{f(\mathcal{U}) \subseteq V} \mathcal{F}(V)$$

This defines a fctor

$$f^*: Sh(X) \longrightarrow Sh(Y)$$

$$\begin{tikzcd} f^*\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ \mathcal{U} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

Recall:

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (x_p)_p \in \prod_{p \in \mathcal{U}} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall\, x_0 \in \mathcal{U},\ \exists\, \mathcal{U}_{x_0} \subseteq \mathcal{U} \text{ nbhd of } x_0, \\ s \in \mathcal{F}(\mathcal{U}) \text{ s.t.} \\ s_p = x_p \qquad \forall p \in \mathcal{U}_{x_0} \end{array} \right\}$$

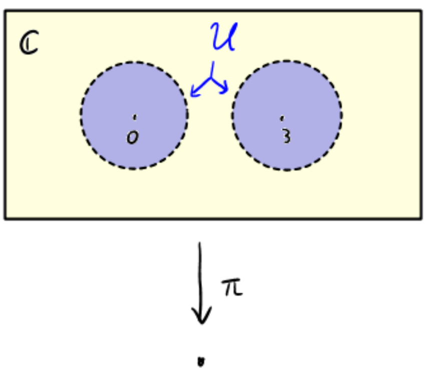
By definition, $(\mathcal{F}^{sh})_p = \mathcal{F}_p$.

Universal property:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "f \in Mor_{PSh}"] \arrow[d, "sh"'] & \mathcal{G} \\ \mathcal{F}^{sh} \arrow[ur, dashed, "\exists !\, f^{sh} \in Mor_{Sh}"'] \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

$\mathcal{G}$: sheaf

Ex. For $\pi : \mathbb{C} \longrightarrow \{*\}, U = B_1(0) \cup B_1(3)$, which one is correct:



- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ $(\pi^{*,\mathrm{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.
- ○ All the above is wrong.

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_p^* \mathcal{F} = \mathcal{F}_p$

For $\pi : Y \longrightarrow \{*\}$, $\pi^* \underline{\mathbb{Q}}_{\{*\}} = \underline{\mathbb{Q}}_Y$

For $U \subset X$ open, $j : U \longrightarrow X$, $j^* \mathcal{F} = \mathcal{F}|_U$

People generalize the last notation to arbitrary subset:

For $Y \subset X$, $\iota_Y : Y \hookrightarrow X$, $\iota_Y^* \mathcal{F} \triangleq \mathcal{F}|_Y$

Q: For $U \subset X$ open, how to express $\mathcal{F}(U)$ by fctors?

A:

$$\begin{tikzcd} U \arrow[rr, hook, "\iota_U"] \arrow[dr, "\pi_U"'] & & X \arrow[dl, "\pi_X"] \\ & \{*\} & \end{tikzcd}$$

$$\mathcal{F}(U) = \pi_{U,*} \underbrace{\iota_U^* \mathcal{F}}_{\mathcal{F}|_U}$$

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

Prop. One has the adjunction $f^* \dashv f_*$, i.e.,

$$\mathrm{Mor}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G}) \qquad + \text{naturality}$$

[Hint. [Vakil, 2.7.B] Show that both side give the same information, i.e.,

$$\phi_{UV} \in \mathrm{Mor}_{Ab}(\mathcal{F}(U), \mathcal{G}(V)) \qquad \text{for each pair } (V, U) \text{ s.t. } f(V) \subset U$$

+ compatability]

Cor. f^* is right exact, f_* is left exact.

Rmk. f^* is an exact functor.

Hint: exactness can be checked on stalks!

⚠ After "polished" (because of the structure sheaf), f^* is again only right adjoint.

Application.

$$\mathrm{Hom}_{Sh(X;\mathbb{Q})}(\underline{\mathbb{Q}}_X, \mathcal{F}) \cong \Gamma(X;\mathcal{F})$$

$$\mathrm{Hom}_{Sh(X;\mathbb{Q})}(\mathcal{F}, i_{p,*}\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}\text{-v.s.}}(\mathcal{F}_p, \mathbb{Q}) \cong \mathcal{F}_p^*$$

$$\mathrm{Hom}_{Sh(U;\mathbb{Q})}(\mathcal{F}|_U, \mathcal{G}) \cong \mathrm{Hom}_{Sh(X;\mathbb{Q})}(\mathcal{F}, i_{U,*}\mathcal{G})$$

where $Sh(X;\mathbb{Q}) \overset{\text{f.faithful}}{\subset} Sh(X)$: sheaves of $\mathbb{Q}$-v.s.

$Sh(X;\mathbb{F}_4) \subset Sh(X;\mathbb{F}_2)$ is not f.faithful

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F} \\ | & & | \\ U & \xhookrightarrow{i_U} & X \end{array}$$

3. Rf_* & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input $\mathcal{F}$, output a complex (before Ker/Im procedure)
- SES induces LES: for

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

  one has

$$\begin{tikzcd}
 & H^2(X;\mathcal{F}) \arrow[r] & \cdots & \\
 & H^1(X;\mathcal{F}) \arrow[r] & H^1(X;\mathcal{G}) \arrow[r] & H^1(X;\mathcal{H}) \arrow[lll, out=-10, in=170, looseness=0.5] \\
0 \arrow[r] & H^0(X;\mathcal{F}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^0(X;\mathcal{G}) \arrow[r] \arrow[d, phantom, "\shortparallel"] & H^0(X;\mathcal{H}) \arrow[d, phantom, "\shortparallel"] \arrow[lll, out=-10, in=170, looseness=0.5] \\
 & \pi_*\mathcal{F} & \pi_*\mathcal{G} & \pi_*\mathcal{H}
\end{tikzcd}$$

$\pi : X \longrightarrow \{*\}$

- can be viewed as right derived fctor of

$$H^0(X,-) = \Gamma(X,-) = \pi_*$$

  one gets

$$H^n(X,-) = R^n\Gamma(X,-) = R^n\pi_*$$

  We denote the complex (before the Ker/Im procedure) as

$$R\Gamma(X,-) = R\pi_*$$

  up to homotopy equiv & quasi-iso, i.e., in the derived category of $\{*\}$.

$$\begin{aligned}
\mathcal{D}(X) = \mathcal{D}(Sh(X)) &= \text{“derived category of sheaves over } X\text{”} \\
&= \text{“ complexes of sheaves over } X\text{, up to } \ldots \text{ ”} \\
&= \{ \cdots \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^{0} \longrightarrow \mathcal{F}^{1} \longrightarrow \cdots \}/\sim \;\triangleq\; \{\mathcal{F}^\bullet\}
\end{aligned}$$

Setting $X, Y \in Top$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

Def. $Rf_* \mathcal{F}$ = "derived pushforward of $\mathcal{F}$"
$= f_* \mathcal{I}^\cdot$

$$\begin{tikzcd} \mathcal{F} \arrow[d, dash] & Rf_*\mathcal{F} \arrow[d, dash] \\ Y \arrow[r, "f"] & X \\ & \mathcal{U} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

[Here, $\mathcal{I}^\cdot$ is the injective resolution of $\mathcal{F}$:
$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$
$(\Rightarrow \mathcal{F} \overset{\text{quasi-iso}}{\sim} \mathcal{I}^\cdot)$]

This defines a fctor
$$Rf_*: \mathcal{D}^+(Y) \longrightarrow \mathcal{D}^+(X)$$

in fact, can be $\mathcal{D}(Y) \to \mathcal{D}(X)$ (nontrivial)

The derived pushforward is hard to compute.
just like cohomology, and even worse, since we need more information
Luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]
$R^n f_* \mathcal{F}$ is given by the sheafification of
$$(R^n f_* \mathcal{F})^{pre}(\mathcal{U}) = H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$
↖ sometimes omit

e.p. one can compute the stalk
$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in \mathcal{U}} H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$

$\mathcal{F}$
|

Cor For $\pi: X \to \{*\}$,
$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For $\pi: \mathbb{CP}^1 \to \{*\}$,
$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in $\mathcal{D}(*)$ "split", we work over $\mathbb{Q}$]

$$R\pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$
$$= [0 \to \cdots \underset{-1}{\to} \underset{0}{\mathbb{Q}} \to \underset{1}{0} \to \underset{2}{\mathbb{Q}} \to \underset{3}{0} \to \underset{4}{\cdots}]$$

Ex. For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, what is true about $Rj_*\underline{\mathbb{Q}}_{\mathbb{C}}$?

- $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- $(R^1 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^2 j_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- What the hell is that?

In fact, $(Rj_*\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1]$.

$i : \{\infty\} \to \mathbb{CP}^1$ is exact, so $Ri_* = i_*$.

Upgrade formulas to derived version

$$f^* g_! \cong g'_! f'^{,*} \xrightarrow{f^*,\, f'^{,*} \text{ exact}} f^* Rg_! \cong Rg'_! f'^{,*}$$

$$\mathrm{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, f_*\mathcal{G})$$

$$\rightsquigarrow \mathrm{Hom}(f^*\mathcal{F}^\cdot, \mathcal{G}^\cdot) \cong \mathrm{Hom}(f^*\mathcal{F}^\cdot, \mathcal{I}^\cdot)$$
$$\cong \mathrm{Hom}(\mathcal{F}^\cdot, f_*\mathcal{I}^\cdot)$$
$$\cong \mathrm{Hom}(\mathcal{F}^\cdot, Rf_*\mathcal{G})$$

Is this argument correct?

4. $f_!$, extension by zeros & global sections with cpt supp

$$\begin{array}{ccc} \mathcal{F} & & f_!\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

Setting $X, Y \in Top$, $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont
X, Y loc. cpt of fin coh dim [IHPS, P81] $\rightsquigarrow f_!\mathcal{F}$ is a sheaf

Def. $f_!\mathcal{F} \in Sh(X)$ is given by

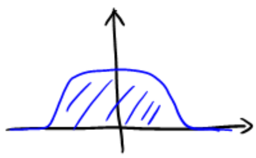
$$f_!\mathcal{F}(\mathcal{U}) = \{ s \in \underset{\underset{(f_*\mathcal{F})(\mathcal{U})}{\parallel}}{\mathcal{F}(f^{-1}(\mathcal{U}))} \mid f|_{supp(s)} : supp(s) \longrightarrow \mathcal{U} \text{ is proper} \}$$

This defines a fctor
$$f_! : Sh(Y) \longrightarrow Sh(X)$$

Recall: $supp(s) = \overline{\{x \in f^{-1}(\mathcal{U}) \mid s_x \neq 0\}} = \{x \in f^{-1}(\mathcal{U}) \mid s_x \neq 0\}$
proper: preimage of cpt set is cpt.

Rmk. By def, $(f_!\mathcal{F})(\mathcal{U}) \subseteq (f_*\mathcal{F})(\mathcal{U})$, one has natural transformation $f_! \longrightarrow f_*$.
When f is proper, $f_! = f_*$.

E.g. For $p \in X$, $\iota_p : \{p\} \hookrightarrow X$, $\iota_{p,!}\underline{\mathbb{Q}}_{\{p\}} = \iota_{p,*}\underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$
For $\pi: Y \longrightarrow \{*\}$, $\pi_!\mathcal{F} = \Gamma_c(Y;\mathcal{F}) = H^0_c(Y;\mathcal{F})$
cpt supp fcts on Y



Ex. Do you know what is $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}})$ and $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1})$?

- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$, $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$.
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = \mathbb{Q}$, $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$.
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$, $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = \mathbb{Q}$.
- ○ $\Gamma_c(\mathbb{C}, \underline{\mathbb{Q}}_{\mathbb{C}}) = 0$, $\Gamma_c(\mathbb{CP}^1, \underline{\mathbb{Q}}_{\mathbb{CP}^1}) = 0$.
- ○ Could you explain the notation again?

E.g. 4.3. For $\mathcal{U} \overset{j}{\hookrightarrow} X$ open, $j_!\mathcal{F}$ is the classical "extension by zero":

$$(j_!\mathcal{F})^{pre}(V) = \begin{cases} \mathcal{F}(\mathcal{U}) & V \subseteq \mathcal{U} \\ 0 & \text{otherwise} \end{cases}$$

e.p.
$$(j_!\mathcal{F})_p = \begin{cases} \mathcal{F}_p & p \in \mathcal{U} \\ 0 & p \notin \mathcal{U} \end{cases}$$

In general, [IHPS, p82]

$$(f_!\mathcal{F})_p = \Gamma_c(f^{-1}(p); \mathcal{F}|_{f^{-1}(p)})$$

This comes from the proper base change formula:

$$\iota_p^* f_! \mathcal{F} \cong \pi_! \tilde{\iota}_p^* \mathcal{F}$$

$$\begin{tikzcd} f^{-1}(p) \arrow[r, hook, "\tilde{\iota}_p"] \arrow[d, "\pi"'] & Y \arrow[d, "f"] \\ \{p\} \arrow[r, "\iota_p"] & X \end{tikzcd}$$

Prove it?

Rmk. In E.g. 4.3, $j_!$ is exact. (Check the stalks!)
In general, $f_!$ is only left exact.

e.p. when $f: Y \to X$ is proper, then $f_! = f_*$ is usually not right adjoint.
Notice that $Rf_! \dashv f^!$, and we don't have $f_! \dashv f^!$.

https://math.stackexchange.com/questions/3132036/direct-image-functor-f-left-exact
the same method here argues why f_! is left exact.

Sidemark:
https://math.stackexchange.com/questions/4671873/compare-two-definition-of-rf-derived-pushforward-with-proper-support
it gives another definition of f_! in étale case.

5. $Rf_!$ & cohomology with cpt supp

Just like Rf_*, we derive the fctor

$$H^0_c(X,-) = \Gamma_c(X,-) = \pi_!$$

to get

$$H^n_c(X,-) = R^n\Gamma_c(X,-) = R^n\pi_!$$

$$\begin{array}{c} X \\ \downarrow \pi \\ \{*\} \end{array}$$

Def. $Rf_!\mathcal{F}$ = "derived proper pushforward of $\mathcal{F}$"

$$= f_!\mathcal{I}^\cdot$$

[Here, $\mathcal{I}^\cdot$ is the injective resolution of $\mathcal{F}$:

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

$(\Rightarrow \mathcal{F} \overset{\text{quasi-iso}}{\rightsquigarrow} \mathcal{I}^\cdot)$]

$$\begin{array}{ccc} \mathcal{F} & & Rf_!\mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

This defines a fctor

$$Rf_! : \mathcal{D}^+(Y) \longrightarrow \mathcal{D}^+(X)$$

Cor For $\pi: X \longrightarrow \{*\}$,

$$R^n\pi_!\mathcal{F} = H^n_c(X;\mathcal{F})$$

E.g. For $\pi: \mathbb{CP}^1 \longrightarrow \{*\}$,

$$R^n\pi_!\underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n_c(\mathbb{CP}^1;\mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0,2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, [all objects in $\mathcal{D}(*)$ are proj, we work over $\mathbb{Q}$]

$$R\pi_!\underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= [\underset{}{0} \to \cdots \to \underset{0}{\mathbb{Q}} \to \underset{1}{0} \to \underset{2}{\mathbb{Q}} \to \underset{3}{0} \to \underset{4}{\cdots}]$$

(degrees: -1, 0, 1, 2, 3, 4)

$\mathbb{CP}^1 \rightsquigarrow \mathbb{C}$, what would happen?

For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, what is true about $Rj_!\underline{\mathbb{Q}}_{\mathbb{C}}$?

- $(R^0 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^1 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- $(R^0 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^1 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- $(R^0 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0, \qquad (R^1 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0.$
- $(R^0 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}, \qquad (R^1 j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}.$
- This question is too easy for me. Ask more difficult questions next time!

In fact, $j_!$ is exact, so $(Rj_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = (j_!\underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$.

6. $f^!$ & homology

[The correct title should actually be:
$f^!$, orientation sheaf & costalks
homology is just some special (sheaf) cohomology.]

Now we come to the hard part, since $f^!$ can only be defined over the derived category. That's why there's no derived symbol for $f^!$.

"Def" $f^!$, if exists, should be given by the right adjoint of $Rf_!$:

$$Rf_! \dashv f^!$$

Remember:

$$f^* \dashv Rf_*$$
$$Rf_! \dashv f^!$$
$$M \overset{L}{\otimes} - \dashv R\mathcal{H}om(M, -)$$

Hard exercise: for $\pi: \mathbb{R}^n \longrightarrow \{*\}$, shows that

$$\pi^! \mathbb{Q} \cong \underline{\mathbb{Q}}_{\mathbb{R}^n}[n]$$

by using the def of $\pi^!$.

Black box: (reduced from the Hard exercise)

For M n-dim mfld (without boundary), $\pi: M \longrightarrow \{*\}$,

$$\pi^! \mathbb{Q} \cong \mathcal{O}r_M [n]$$

$\mathcal{O}r_M$: $\mathbb{Q}$-orientation sheaf of M

e.g. when M is oriented, fixing an orientation $\mathcal{O}r_M \cong \underline{\mathbb{Q}}_M$, one gets

$$\pi^! \mathbb{Q} \cong \underline{\mathbb{Q}}_M [n].$$

What does the adjunction tell us?

https://mathoverflow.net/questions/404706/how-duality-follows-from-a-six-functor-formalism

Poincaré duality, version 1

$$\underline{Hom}_{D^+(Y;\mathbb{Q})}(Rf_!\mathcal{F},\mathcal{G}) \cong {\scriptstyle Rf_*}\,\underline{Hom}_{D^+(X;\mathbb{Q})}(\mathcal{F}, f^!\mathcal{G})$$

$$\begin{array}{ccc} \mathcal{F} & & \mathcal{G} \\ | & & | \\ X & \xrightarrow{f} & Y \end{array}$$

Now, take $Y=\{*\}$, $\mathcal{G}=\mathbb{Q}$, X: n-dim mfld, one gets

$$\underline{Hom}_{D^+(*;\mathbb{Q})}(R\Gamma_c(X;\mathcal{F}),\mathbb{Q}) \cong {\scriptstyle Rf_*}\,\underline{Hom}_{D^+(X;\mathbb{Q})}(\mathcal{F}, Or_X[n])$$

$$R\Gamma_c(X;\mathcal{F})^* \cong {\scriptstyle Rf_*}\,\underline{Hom}_{D^+(X;\mathbb{Q})}(\mathcal{F}, Or_X)[n]$$

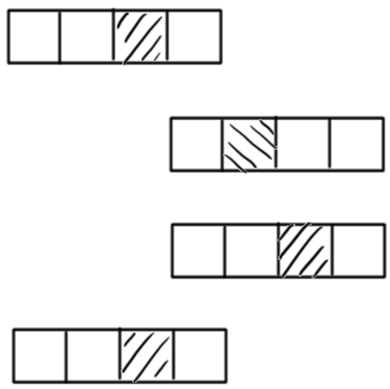
Take $\mathcal{F}=\underline{\mathbb{Q}}_X$, one gets

$$R\Gamma_c(X;\mathbb{Q})^* \cong R\Gamma(X;Or_X)[n]$$

Take $(-i)$'s cohomology, one gets

$$H^i_c(X;\mathbb{Q})^* \cong H^{n-i}(X;Or_X)$$

E.g. $n=3$ $i=1$ $X=S^2\times\mathbb{R}$



$R\Gamma_c(X;Or_X)^*$	$H^{-i}(R\Gamma_c(X;Or_X)^*)$
	$\text{\rotatebox{90}{$\cong$}}$
$R\Gamma_c(X;Or_X)$	$H^i_c(X;Or_X)^*$
	$\cong$ PD
$R\Gamma(X;\mathbb{Q})$	$H^{n-i}(X;\mathbb{Q})$
	$\cong$
$R\Gamma(X;\mathbb{Q})[n]$	$H^{-i}(R\Gamma(X;\mathbb{Q})[n])$

An illusion for index shift

Take $\mathcal{F}=Or_X$, one gets

$$R\Gamma_c(X;Or_X)^* \cong \underline{Hom}_{D(X;\mathbb{Q})}(Or_X,Or_X)[n]$$

$$\overset{Or_X \text{ is l.b.}}{\cong} \underline{Hom}_{D(X;\mathbb{Q})}(\underline{\mathbb{Q}}, Or_X^\vee\otimes Or_X)[n]$$

$$\overset{\pi^*\dashv R\pi_*}{\cong} R\Gamma(X;\mathbb{Q})[n]$$

Take $(-i)$'s cohomology, one gets

$$(H^i_c(X;Or_X))^* \cong H^{n-i}(X;\mathbb{Q})$$

These are the Poincaré duality.

Rmk. 1. When X is cpt, one gets $R\pi_! = R\pi_*$, $H^i = H^i_c$,

$$(H^i(X;\mathbb{Q}))^{**} \cong (H^{n-i}(X;\mathcal{O}r_X))^* \cong H^i(X;\mathbb{Q})$$

So $\dim_{\mathbb{Q}} H^i(X;\mathbb{Q}) < +\infty$.

https://math.stackexchange.com/questions/35779/what-can-be-said-about-the-dual-space-of-an-infinite-dimensional-real-vector-spa

$H^i(X;\mathbb{Q})$ has an upper bound given by the triangulation of X.

2. In fact, these isos come from the non-degenerate bilinear pairing maps (why?)

$$\cup: H^i_c(X;\mathbb{Q}) \times H^{n-i}(X;\mathcal{O}r_X) \longrightarrow H^n_c(X;\mathcal{O}r_X) \cong \bigoplus_{\text{comp}} \mathbb{Q} \xrightarrow{\text{sum}} \mathbb{Q}$$

$$\cup: H^i(X;\mathbb{Q}) \times H^{n-i}_c(X;\mathcal{O}r_X) \longrightarrow H^n_c(X;\mathcal{O}r_X) \cong \bigoplus_{\text{comp}} \mathbb{Q} \xrightarrow{\text{sum}} \mathbb{Q}$$

Poincaré duality, version 2 & 3

Def. Let X be a mfld of dim n, the homology of X are defined as

$$H^{BM}_{-i}(X;\mathbb{Q}) = (R\pi_* \pi^! \mathbb{Q})_i = H^i(X; Or_X[n]) = H^{n+i}(X; Or_X)$$
$$H_{-i}(X;\mathbb{Q}) = (R\pi_! \pi^! \mathbb{Q})_i = H^i_c(X; Or_X[n]) = H^{n+i}_c(X; Or_X)$$

This is Poincaré duality, version 2.

https://en.wikipedia.org/wiki/Borel%E2%80%93Moore_homology
Definitions are not equivalent when X=\mathbb{Z}, but fine for good spaces

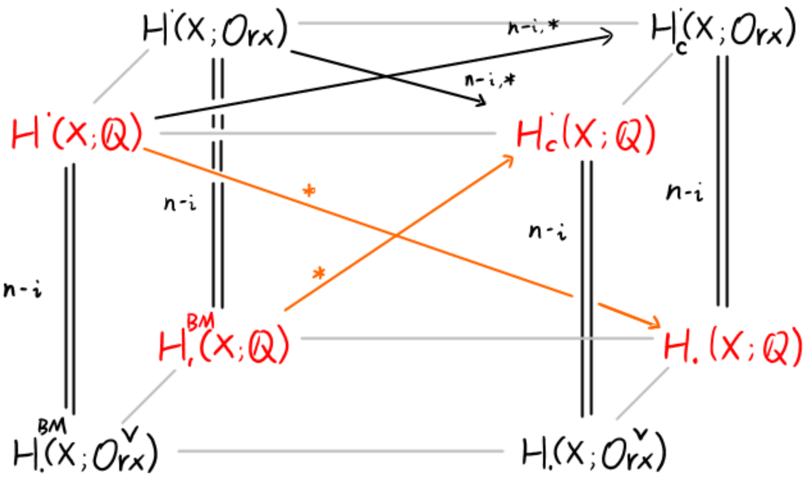
https://mathoverflow.net/questions/277069/what-is-homology-anyway
https://mathoverflow.net/questions/249342/two-points-of-view-about-borel-moore-homology

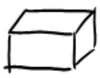
Ex. Do you know what is $H_\cdot(\mathbb{C};\mathbb{Q})$ & $H^{BM}_\cdot(\mathbb{C};\mathbb{Q})$?

	-3	-2	-1	0	1	2	3	
A	0	$\mathbb{Q}$	0	0	0	0	0	$R\pi_* \pi^! \mathbb{Q}$
B	0	0	0	$\mathbb{Q}$	0	0	0	$H_\cdot(\mathbb{C};\mathbb{Q})$
C	0	0	0	0	0	$\mathbb{Q}$	0	$H^{BM}_\cdot(\mathbb{C};\mathbb{Q})$

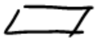
Combining two versions, one gets Poincaré duality, version 3: (UCT)

$$(H^i_c(X;\mathbb{Q}))^* \cong H^{n-i}(X;Or_X) = H_i^{BM}(X;\mathbb{Q})$$
$$(H_{n-i}(X;\mathbb{Q}))^* = (H^i_c(X;Or_X))^* \cong H^{n-i}(X;\mathbb{Q})$$





$\leftrightarrow$: compact
$\swarrow\!\!\nearrow$: oriented
$\updownarrow$: def (n-i shift)

 : cohomological guy (six functor formalism)

 : topology guy (no sheaf cohomology)

e.g. in oriented mfld case, it becomes

$$\begin{tikzcd} H^\cdot(X;\mathbb{Q}) \arrow[r, "n-i,*"] \arrow[d, equal, "n-i"'] & H^\cdot_c(X;\mathbb{Q}) \arrow[d, equal, "n-i"'] \\ H^{BM}_\cdot(X;\mathbb{Q}) & H_\cdot(X;\mathbb{Q}) \end{tikzcd}$$

in cpt oriented mfld case, it becomes

$$\begin{tikzcd} H^\cdot(X;\mathbb{Q}) \arrow[loop right, "n-i,*"] \arrow[d, equal, "n-i"] \\ H_\cdot(X;\mathbb{Q}) \end{tikzcd}$$

Poincaré duality for non-mflds

Def. For $X \in \mathcal{T}op$, $\pi : X \to \{*\}$, if $\pi^! \mathbb{Q}$ is well-defined, then the homology of X are defined as

$$H^{BM}_{-\bullet}(X;\mathbb{Q}) = R\pi_* \pi^! \mathbb{Q} = H^\bullet(X; \pi^! \mathbb{Q})$$
$$H_{-\bullet}(X;\mathbb{Q}) = R\pi_! \pi^! \mathbb{Q} = H^\bullet_c(X; \pi^! \mathbb{Q})$$

We will compute f^! for many cases next time.
Remember, when we compute new f^!'s, we discover new versions of Poincaré duality.

E.g. In [BI86, V.2.8, Example V.2.9., VI 3.2], it claims that

For $(X, \partial X)$ a mfld with boundary, $\mathcal{U} := X - \partial X$, one has

$$\pi_X^! \mathbb{Q} = \iota_{\mathcal{U},!} \pi_{\mathcal{U}}^! \mathbb{Q}$$

$$\begin{tikzcd} \partial X \arrow[r, "\iota_{\partial X}"] & X \arrow[d, "\pi_X"] & \mathcal{U} \arrow[l, hook', "\iota_{\mathcal{U}}"'] \arrow[dl, "\pi_{\mathcal{U}}"] \\ & \{*\} & \end{tikzcd}$$

With this in hand, one derives the Poincaré-Lefschetz duality:

$$H^i_c(X;\mathbb{Q})^* \simeq H^{n-i}(X, \partial X; \iota_{\mathcal{U},*} Or_{\mathcal{U}}) \overset{\text{when } X \text{ is oriented}}{=\!=\!=} H^{n-i}(X, \partial X; \mathbb{Q})$$

https://math.stackexchange.com/questions/2212857/poincare-lefschetz-duality-from-poincare-lefschetz-alexander-duality
I include this link just for remembering the name of these dualities.