

Eine Woche, ein Beispiel

3.16 Schubert calculus: subvariety with vb

This is a follow up of [2025.02.23].

Goal: relate subvarieties to some vector bundles, so that we can compute their homology class in terms of Chern class (when the dimension is correct).

The Chern class will be dealt with in the next document.

Concretely, we will write subvarieties as:

- the zero set of a section in a v.b.
- the degeneracy loci of a morphism $\mathcal{E} \rightarrow \mathcal{F}$ among v.b.s
- the preimage of known cycles in Grassmannian
- the subvariety of $\text{Gr}(r, n)$ induced by a $r \times r$ bundle (very ample)

1. Known subvarieties and known vector bundles
2. Subvariety as section
3. Subvariety as degeneracy loci
4. Subvariety given by very ample bundle

1. Known subvarieties and known vector bundles

Schubert variety

Recall that the Schubert variety has the expression $\omega \leftrightarrow (\lambda_1, \dots, \lambda_r)$ ^{cohom}

$$\begin{aligned}\Sigma_{\lambda_1, \dots, \lambda_r}(\mathcal{V}) &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+i-\lambda_i} \geq i \quad \forall i \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{w_i} \geq i \quad \forall i \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{w_i} \leq n - \lambda_i \quad \forall i \}\end{aligned}$$

Especially,

$$\begin{aligned}\Sigma_k^s(\mathcal{V}) &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{n-r+i-k} \leq n - k \quad \forall i \leq s \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n - k \} \\ &= \{ \Lambda \in \text{Gr}(r, n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s \}\end{aligned}$$

For special k, s , one can further simplify the formulas:

	k	1	k	$n-r$
s	$\text{Gr}(r, n)$			
1		$\Lambda + \mathcal{V}_{n-r} \subseteq H$ or $\Lambda \cap \mathcal{V}_{n-r} \neq \{0\}$	$\Lambda \cap \mathcal{V}_{n-r+1-k} \neq \{0\}$	$\mathcal{V}_1 \subset \Lambda$
s		$\Lambda + \mathcal{V}_{n-r+s-1} \subseteq H$	$\dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n - k$ or $\dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s$	$\mathcal{V}_s \subset \Lambda$
r		$\Lambda \subset \mathcal{V}_{n-1}$	$\Lambda \subset \mathcal{V}_{n-k}$	$\{\mathcal{V}_r\}$

Vector bundles on Grassmannian

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{O}^{\oplus n} & \longrightarrow & \mathcal{Q} \longrightarrow 0 \\ 0 & \longrightarrow & \mathcal{Q}^\vee & \longrightarrow & \mathcal{O}^{\oplus n} & \xrightarrow{\pi_{\mathcal{S}^\vee}} & \mathcal{S}^\vee \longrightarrow 0 \end{array}$$

\mathcal{S} : Subspace = tautological bundle
 \mathcal{Q} : Quotient = quotient bundle

When $r = 1$, $\text{Gr}(r, n) = \mathbb{P}^{n-1}$.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & \mathcal{O}^{\oplus n} & \longrightarrow & \mathcal{Q} \longrightarrow 0 \\ 0 & \longrightarrow & \mathcal{Q}^\vee & \longrightarrow & \mathcal{O}^{\oplus n} & \longrightarrow & \mathcal{O}(1) \longrightarrow 0 \end{array}$$

With these basic v.bs, we can construct more bundles on $\text{Gr}(r, n)$:

$$\begin{array}{ll} \mathcal{T}_{\text{Gr}} = \text{Hom}(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^\vee \otimes \mathcal{Q} & \omega_{\text{Gr}}^\vee = \det \mathcal{S}^\vee \otimes \mathcal{Q} \\ \Omega_{\text{Gr}} = \mathcal{T}_{\text{Gr}}^\vee = \text{Hom}(\mathcal{Q}, \mathcal{S}) = \mathcal{Q}^\vee \otimes \mathcal{S} & \omega_{\text{Gr}} = \det \mathcal{Q}^\vee \otimes \mathcal{S} \end{array}$$

2. Subvariety as section

Hypersurface and its Fano variety of $(r-1)$ -planes

Let $F \in K[z_1, \dots, z_n]$ be a homo poly of deg d . The hypersurface

$$Y_d := \{F = 0\} \subseteq \mathbb{P}^{n-1}$$

is given as a section of

$$\mathcal{O}(d) = \text{Sym}^d \mathcal{O}(1)$$

In general, the Fano variety of $(r-1)$ -planes ($\cong \mathbb{P}^{r-1}$)

$$F_{r-1}(Y_d) := \{W \in \text{Gr}(r, n) \mid F|_W = 0\} \subseteq \text{Gr}(r, n)$$

is given as a section of $\text{Sym}^d \mathcal{S}^\vee$, through the map

$$\begin{aligned} \text{Sym}^d \pi_{\mathcal{S}^\vee}: \text{Sym}^d(\mathcal{O}^{\oplus n}) &\longrightarrow \text{Sym}^d(\mathcal{S}^\vee) \\ &\parallel \\ &(\text{Sym}^d V^*) \otimes \mathcal{O} \end{aligned}$$

Map of section: $F \otimes 1 \longmapsto s_F = \text{Sym}^d \pi_{\mathcal{S}^\vee}(F \otimes 1)$

Fiberwise, $(\text{Sym}^d \pi_{\mathcal{S}^\vee})_W: \text{Sym}^d V^* \longrightarrow \text{Sym}^d W^*$

We know that

$$\begin{aligned} &F|_W \equiv 0 \\ \Leftrightarrow &(\text{Sym}^d \pi_{\mathcal{S}^\vee})_W(F) = 0 \\ \Leftrightarrow &s_F = 0, \text{ i.e., } [W] \text{ lies in the zero set of } s_F. \end{aligned}$$

E.g.

$$\begin{aligned} F_0(Y_d) &= Y_d \\ F_1(Y_d) &\subseteq \text{Gr}(2, n) \\ F_m(Y_2) &\subseteq \text{Gr}(m+1, 2m+2) \\ &\text{or} \\ &\text{Gr}(m+1, 2m+3) \end{aligned}$$

Fano variety of lines
Last } Grassmannian
orthogonal }
...

Cor. $F_{r-1}(Y_d)$ has codimension $\leq \binom{d+r-1}{d}$ (when non-empty)

3. Subvariety as degeneracy loci

Def. (degeneracy loci)

Let X/\mathbb{C} sm $k \in \mathbb{Z}_{\geq 0}$,

\mathcal{E}, \mathcal{F} : v.b. over X of rank e, f ,

$\varphi: \mathcal{E} \rightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

We define the degeneracy loci

$$M_k(\varphi) := \{x \in X \mid \text{rank } \varphi_x \leq k\}$$

remember multiplicity
 $\varphi_x: \mathcal{E}|_x \rightarrow \mathcal{F}|_x$

The expected codimension is $(e-k)(f-k)$.

E.g. When $\mathcal{E} = \mathcal{O}_X$, we know $e=1$,

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F}) \quad \varphi \longleftrightarrow s$$

$$M_1(\varphi) = X, \quad M_0(\varphi) = V(s)$$

\uparrow vanishing set in X

Therefore, the degeneracy loci generalizes the section of v.b..

E.g. When $\mathcal{E} = \mathcal{O}_X^{\oplus e}$,

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F})^{\oplus e} \quad \varphi \longleftrightarrow (s_1, \dots, s_e)$$

$$M_e(\varphi) = X$$

$$M_{e-1}(\varphi) = \{x \in X \mid s_1(x), \dots, s_e(x) \text{ are linear dependent}\}$$

$$M_k(\varphi) = \{x \in X \mid \dim \langle s_i(x) \rangle_i \leq k\}$$

$$M_0(\varphi) = V(s_1, \dots, s_e)$$

Flag variety

E.g.

$$\begin{aligned}\Sigma_{k'}^{\text{union}} &= \{ (V, V') \in \text{Gr}(r, n) \times \text{Gr}(r', n) \mid \dim V \cap V' \geq k' \} \\ &= \{ (V, V') \in \text{Gr}(r, n) \times \text{Gr}(r', n) \mid \dim V + V' \leq r + r' - k' \} \\ &= \{ (V, V') \mid V \oplus V' \longrightarrow \mathbb{C}^n \text{ is of rank } \leq r + r' - k' \} \\ &= \mathcal{M}_{r+r'-k'}(\gamma: \pi_1^* \mathcal{S} \oplus \pi_2^* \mathcal{S}' \longrightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

The expected dimension is

$$(r + r' - (r + r' - k'))(n - (r + r' - k')) = k'(n + k' - r - r')$$

When $\begin{cases} k' \leq \min(r, r') \\ n + k' - r - r' \geq 0 \end{cases}$, $\Sigma_{k'}^{\text{union}}$ has the expected codimension.

In general, one can define

$$\begin{aligned}\Sigma_k^{\text{sum}} &= \{ (V_i)_i \in \prod_i \text{Gr}(r_i, n) \mid \dim \sum_i V_i \leq k \} \\ &= \mathcal{M}_k(\gamma: \bigoplus_i \pi_i^* \mathcal{S}_i \longrightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

with the expected dimension $(\sum_i r_i - k)(n - k)$.

When $\begin{cases} k \geq \max\{r_i\}_i \\ k \leq n \end{cases}$, Σ_k^{sum} has expected codimension.

A more general case (also generalize [3264, Ex 12.11, Ex 12.9]):

Let $X: \text{sm proj}$, $\mathcal{F}_i \subset \mathcal{E}$ are v.b.s
rank $\rightarrow r_i$ n

$$\begin{aligned}\Sigma_k^{\text{sum}} &= \{ p \in X \mid \dim \sum_i \mathcal{F}_i|_p \leq k \} \\ &= \{ p \in X \mid \bigoplus_i \mathcal{F}_i|_p \longrightarrow \mathcal{E}|_p \text{ is of rank } \leq k \} \\ &= \mathcal{M}_k(\gamma: \bigoplus_i \mathcal{F}_i \longrightarrow \mathcal{E})\end{aligned}$$

The general partial flag variety can be expressed as the degeneracy loci.

E.g.

$$\begin{aligned}
 \text{Flag}_{r_1, r_2, r_3}(\mathbb{C}^n) &= \{ 0 \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^n \mid \dim V_i = r_i \} \\
 &= \{ (V_1, V_2, V_3) \in \prod_{i=1}^3 \text{Gr}(r_i, n) \mid \dim V_i + V_{i+1} \leq r_{i+1} \} \\
 &= M_{r_2+r_3} \left(\begin{array}{ccc} \pi_1^* S_1 \oplus \pi_2^* S_2 & & \mathcal{O}^{\oplus n} \\ \varphi: \oplus & \longrightarrow & \oplus \\ \pi_2^* S_2 \oplus \pi_3^* S_3 & & \mathcal{O}^{\oplus n} \end{array} \right) \\
 &= \pi_{12}^{-1} \sum_{r_2}^{\text{sum}} \cap \pi_{23}^{-1} \sum_{r_3}^{\text{sum}}
 \end{aligned}$$

Ramification locus [Barth 04 I.16]

Let $Y, X/\mathbb{C}$: sm of $\dim n$, $f: Y \rightarrow X$ finite.
The ramification divisor of f is defined as

$$\begin{aligned} R &:= \{y \in Y \mid T_y f: T_y Y \rightarrow T_{f(y)} X \text{ is not surj}\} \\ &= \{y \in Y \mid f^*: T_{f(y)}^* X \rightarrow T_y^* Y \text{ is not surj}\} \\ &= \left\{ y \in Y \mid \begin{array}{l} \text{rank } \varphi_y \leq n-1, \text{ where } \\ \varphi: f^* \Omega_X \rightarrow \Omega_Y \end{array} \right\} \\ &= M_{n-1}(f^* \Omega_X \rightarrow \Omega_Y) \end{aligned}$$

with the expected $\text{codim } (n - (n-1))(n - (n-1)) = 1$

Rmks. 1. R may have multiplicity, which is also counted in the degeneracy loci.
Recall that, for the zero set of section, we also count the multiplicity

2. Since

$\mathbb{C}^n \rightarrow \mathbb{C}^n$ is of $\text{rk} \leq n-1 \Leftrightarrow \det \mathbb{C}^n \rightarrow \det \mathbb{C}^n$ is zero,
we get

$$\begin{aligned} R &= M_0(f^* \omega_X \rightarrow \omega_Y) \\ \omega_Y &= f^* \omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R) \quad \leadsto \text{Hurwitz formula} \end{aligned}$$

$$0 \rightarrow f^* \omega_X \rightarrow \omega_Y \rightarrow L_{R,*} \mathcal{O}_R \rightarrow 0$$

3. I guess that we can generalize to f generic finite,
the we can get ramification locus + special fiber part.

How to distinguish these two locus?

Guess: for those special fibers, the pushforward will give us zero cycle. Can we use that?

4. For Y, X sm variety of $\dim Y, \dim X$,
when $f: Y \rightarrow X$ is a closed embedding, we get:
 $0 \rightarrow N_{Y/X}^\vee \rightarrow f^* \Omega_X \rightarrow \Omega_Y \rightarrow 0$
In this case, $\varphi: f^* \Omega_X \rightarrow \Omega_Y$ is always surj,
so the degeneracy loci is meaningless.

4. Subvariety given by very ample bundle

For X/\mathbb{C} sm proj, \mathcal{F}/X v.b. of rank r , assume that $(\mathcal{F}, s_1, \dots, s_n)$ provides an embedding

$$\phi_{\mathcal{F}}: X \longrightarrow \mathrm{Gr}(r, n),$$

we want to compute $[X] \in H_{2\dim X} \mathrm{Gr}(r, n)$.

Rmk 1.

This is different from the previous construction. Here, the cycle we want to compute is the pushforward, not the pullback. By Poincaré duality in $\mathrm{Gr}(r, n)$, the pushforward can be computed using the pullback of special cycles in $\mathrm{Gr}(r, n)$.

Rmk 2. We know

$$\begin{aligned} \mathcal{L} \text{ is very ample} &\iff (\mathcal{L}, \Gamma(X; \mathcal{L})) \text{ induces embedding} \\ \mathcal{F} \text{ is very ample} &\implies (\mathcal{F}, \Gamma(X; \mathcal{F})) \text{ induces embedding} \end{aligned}$$

See p73 in
R Hartshorne, Ample vector bundles
https://www.numdam.org/item/PMIHES_1966__29__63_0.pdf

E.g. When \mathcal{L}/X induces an embedding $\phi_{\mathcal{L}}: X \longrightarrow \mathbb{P}^{n-1}$,

$$\begin{aligned} [X] &= \phi_{\mathcal{L}}^* ([H]^{\dim X}) [IP^{\dim X}] \\ &= \deg \mathcal{L} \cdot [IP^{\dim X}]. \end{aligned}$$

Therefore,

$$\text{compute } X \iff \text{compute } \deg \mathcal{L}.$$

Special embeddings

Let us include several special embeddings here.

E.g. For $l_X: X \hookrightarrow \text{Gr}(r, n)$, l_X is induced by $l_X^* \mathcal{S}^\vee$.

E.g. (Veronese embedding)
For $d \geq 1$, $\mathcal{O}_{\mathbb{P}^n}(d)$ induces the Veronese embedding $\text{Sym}^d \mathcal{O}^{\oplus n} \rightarrow \text{Sym}^d \mathcal{O}(1)$

$$\begin{aligned} \nu_d: \mathbb{P}^n &\longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \\ [z_0: \dots: z_n] &\longmapsto [z_0^d: \dots: z_n^d]_{\sum i_k = d} \end{aligned}$$

of degree d^n .

It describes when the deg d hypersurfaces in \mathbb{P}^n degenerates as a (non-reduced) hyperplane.

E.g. (Segre embedding)
 $\mathcal{L} := \mathcal{O}_{\mathbb{P}^m}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(1)$ induces the Segre embedding $\mathcal{O}^{\oplus(m+1)} \boxtimes \mathcal{O}^{\oplus(n+1)} \rightarrow \mathcal{L}$

$$\begin{aligned} \phi_{\mathcal{L}}: \mathbb{P}^m \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{(m+1)(n+1)-1} \\ ([x_0: \dots: x_m], [y_0: \dots: y_n]) &\longmapsto [x_i y_j]_{i,j} \end{aligned}$$

of degree $\binom{m+n}{m}$.

E.g. (Plücker embedding)
In $\text{Gr}(r, n)$, $\det \mathcal{S}^\vee$ induces the Plücker embedding $\Lambda^r \mathcal{O}^{\oplus n} \rightarrow \Lambda^r \mathcal{S}^\vee$

$$\begin{aligned} l_{\text{Gr}}: \text{Gr}(r, n) &\longrightarrow \mathbb{P}(\Lambda^r \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{r}-1} \\ W = \langle w_1, \dots, w_r \rangle &\longmapsto w_1 \wedge \dots \wedge w_r \end{aligned}$$

of degree $(r(n-r))! \prod_{i=0}^{r-1} \frac{i!}{(n-r+i)!}$ [3264, Prop 4.12]

Not very ample case

There are cases where L is not very ample.

E.g.
$$\begin{array}{ccccc} \mathrm{Sym}^2 \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \parallel & & \parallel & & \parallel \\ \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 6} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}^{\oplus 9} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array}$$
 induces the map

$$\begin{aligned} \phi: \mathbb{P}^2 \times \mathbb{P}^2 &\longrightarrow \mathbb{P}^8 \dashrightarrow \mathbb{P}^5 \\ ([x_0:x_1:x_2], [y_0:y_1:y_2]) &\mapsto \begin{bmatrix} x_0 y_0 & \cdots & x_0 y_2 \\ \vdots & & \vdots \\ x_2 y_0 & \cdots & x_2 y_2 \end{bmatrix} \mapsto \begin{bmatrix} x_0 y_0 : x_0 y_1 + x_1 y_0 : x_0 y_2 + x_2 y_0 \\ \vdots : x_1 y_1 : x_1 y_2 + x_2 y_1 \\ \vdots : x_2 y_2 \end{bmatrix} \end{aligned}$$

$\mathrm{Im} \phi$ describes when the quadric curve degenerates as union of two lines.
Since $\phi: \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathrm{Im} \phi$ is a 2:1 ramified cover,

$$\phi^*([H]^4) = 6 \Rightarrow \mathrm{Im} \phi \subset \mathbb{P}^5 \text{ is of degree 3.}$$

E.g. [3264, 2.2.1]

$$\begin{array}{ccccc} \mathrm{Sym}^3 \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathrm{Sym}^2 \mathcal{O}_{\mathbb{P}^5}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \\ \parallel & & \parallel & & \parallel \\ \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 10} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^5}^{\oplus 18} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \end{array}$$

induces the map

$$\begin{aligned} \phi: \mathbb{P}^2 \times \mathbb{P}^5 &\longrightarrow \mathbb{P}^{17} \dashrightarrow \mathbb{P}^9 \\ (F_1, F_2) &\longmapsto F_1 F_2 \\ \sum_{i,j,k} x_i z_j z_k &\quad \sum_{i,j,k} x_i y_{jk} z_i z_j z_k \end{aligned}$$

$\mathrm{Im} \phi$ describes when the cubic curve contains a line.

Since $\phi: \mathbb{P}^2 \times \mathbb{P}^5 \longrightarrow \mathrm{Im} \phi$ is generically injective,

$$\phi^*([H]^7) = \binom{7}{2} = 21 \Rightarrow \mathrm{Im} \phi \subset \mathbb{P}^9 \text{ is of degree 21.}$$

Eg. [3264, 2.2.2]

$$\begin{array}{ccccc} \mathrm{Sym}^3 \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2(1)} \boxtimes \mathcal{O}_{\mathbb{P}^2(1)} \boxtimes \mathcal{O}_{\mathbb{P}^2(1)} \\ \parallel^{\oplus 10} & & \parallel^{\oplus 27} & & \parallel \\ \mathcal{O}_{(\mathbb{P}^2)^3} & \longrightarrow & \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 27} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2(1)} \boxtimes \mathcal{O}_{\mathbb{P}^2(1)} \boxtimes \mathcal{O}_{\mathbb{P}^2(1)} \end{array}$$

induces the map

$$\phi: \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^{26} \dashrightarrow \mathbb{P}^9 \quad (F_1, F_2, F_3) \longmapsto F_1 F_2 F_3$$

$\mathrm{Im} \phi$ describes when the cubic curve degenerates as union of three lines.
 Since $\phi: \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathrm{Im} \phi$ is a 6:1 ramified cover,

$$\phi^*([H]^6) = \binom{6}{4} \binom{4}{2} = \frac{6!}{2!2!2!} = 90 \Rightarrow \mathrm{Im} \phi \subset \mathbb{P}^9 \text{ is of degree } 15.$$