Eine Woche, ein Beispiel

8.10 toric variety

Ref:

[2021.04.09]
[BP15]: Taras E. Panov and Victor Buchstaber, Toric topology
[ACM25]: Omid Amini, Daniel Corey, Leonid Monin. Tropical Abel-Jacobi theory
https://arxiv.org/abs/2504.14415

I learned toric variety before, but I forget the notation right after I learn it. Anyhow, next month I need these information to study tropical geometry.

1. affine chart

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Def (affine toric variety V_σ) [BP15, p180]

Fix a lattice $N \cong \mathbb{Z}^n$, and a cone $\sigma \subset N_{\mathbb{R}}$. Define

$\mathbb{R}_{\geq 0}$-module

dual space $\quad \sigma^\vee := \{u \in N^*_{\mathbb{R}} \mid \langle u, v\rangle \geq 0 \quad \forall v \in N\}$

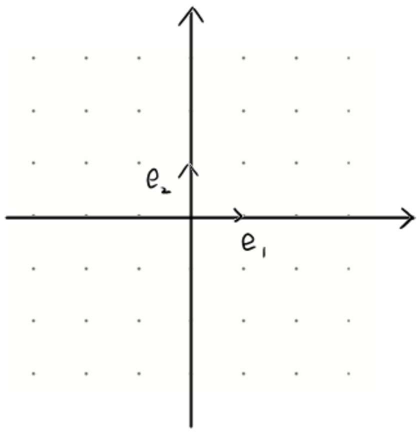
lattice pts of dual cone $\quad S_\sigma := \sigma^\vee \cap N^* = \{u \in N^* \mid \langle u, v\rangle \geq 0 \quad \forall v \in N\}$

$$A_\sigma := \mathbb{C}[S_\sigma] = \mathbb{C}[\chi^u \mid u \in S_\sigma] \big/ (\chi^u \cdot \chi^{u'} - \chi^{u+u'}, \chi^0 - 1) = \bigoplus_{u \in S_\sigma} \mathbb{C} \cdot \chi^u$$

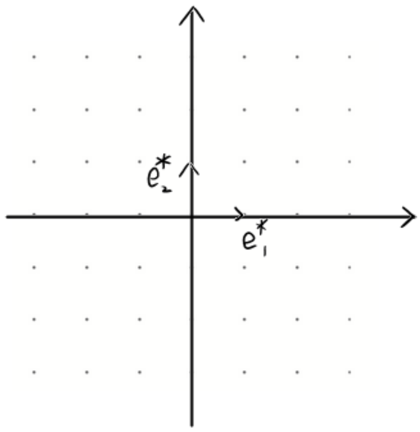
affine toric variety $\quad V_\sigma := \operatorname{Spec} A_\sigma$

$$V_\sigma(\mathbb{C}) = \operatorname{Hom}_{ring}(\mathbb{C}[S_\sigma], \mathbb{C}) = \operatorname{Hom}_{sg}(S_\sigma, \mathbb{C}_m)$$

sg: semigroup
$\mathbb{C}_m := (\mathbb{C}, \cdot)$

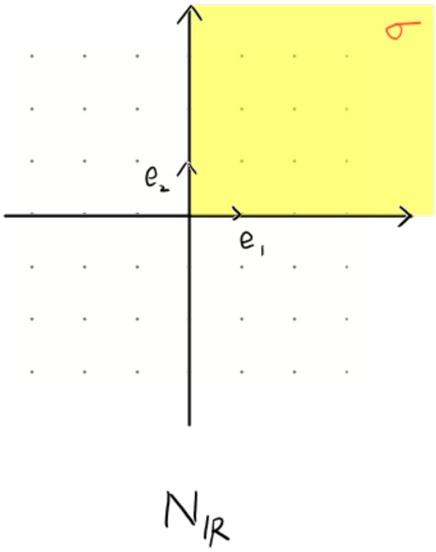


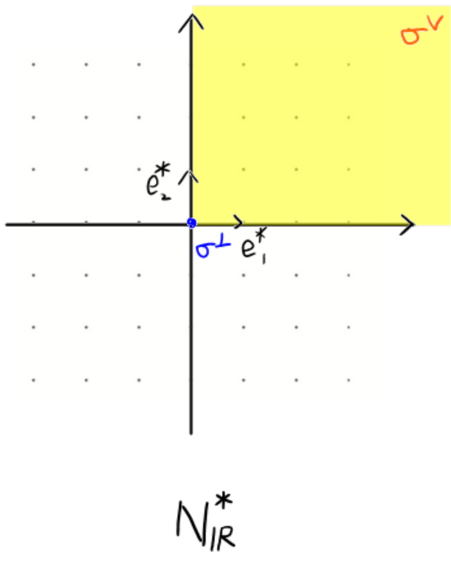
$N_{\mathbb{R}}$



$N^*_{\mathbb{R}}$

E.g. $n=2$

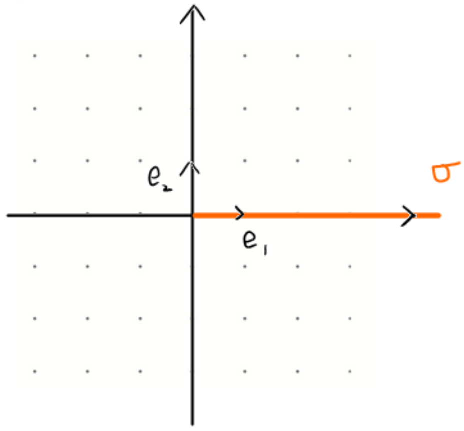




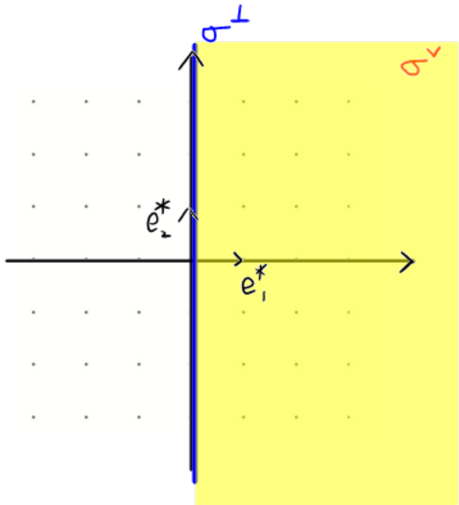
$$\sigma = \mathbb{R}_{\geq 0}\, e_1 \oplus \mathbb{R}_{\geq 0}\, e_2$$
$$\sigma^\vee = \mathbb{R}_{\geq 0}\, e_1^* \oplus \mathbb{R}_{\geq 0}\, e_2^*$$
$$S_\sigma = \mathbb{Z}_{\geq 0}\, e_1^* \oplus \mathbb{Z}_{\geq 0}\, e_2^*$$

$$A_\sigma = \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] \triangleq \mathbb{C}[x_1, x_2]$$
$$V_\sigma = \operatorname{Spec} \mathbb{C}[x_1, x_2] = \mathbb{A}^2_{\mathbb{C}}$$
$$V_\sigma(\mathbb{C}) = \operatorname{Hom}_{sg}(\mathbb{Z}_{\geq 0}\, e_1^* \oplus \mathbb{Z}_{\geq 0}\, e_2^*, \mathbb{C}_m) = \mathbb{C}^2$$

E.g. $n=2$



$N_{\mathbb{R}}$

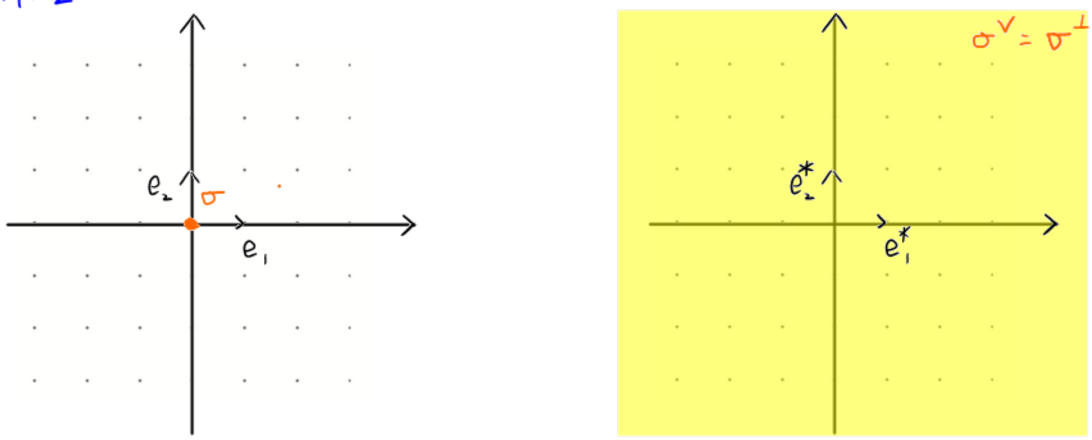


$N_{\mathbb{R}}^*$

$$\sigma = \mathbb{R}_{\geq 0}\, e_1$$
$$\sigma^{\vee} = \mathbb{R}_{\geq 0}\, e_1^* \oplus \mathbb{R}\, e_2^*$$
$$S_\sigma = \mathbb{Z}_{\geq 0}\, e_1^* \oplus \mathbb{Z}\, e_2^*$$

$$A_\sigma = \mathbb{C}[\chi^{e_1^*}, (\chi^{e_2^*})^{\pm 1}] \triangleq \mathbb{C}[x_1, x_2^{\pm 1}]$$
$$V_\sigma = \operatorname{Spec} \mathbb{C}[x_1, x_2^{\pm 1}] = \mathbb{A}_{\mathbb{C}} \oplus \mathbb{G}_{m,\mathbb{C}}$$
$$V_\sigma(\mathbb{C}) = \operatorname{Hom}_{sg}(\mathbb{Z}_{\geq 0}\, e_1^* \oplus \mathbb{Z} e_2^*, \mathbb{C}_m) = \mathbb{C} \oplus \mathbb{C}^{\times}$$

E.g. $n=2$

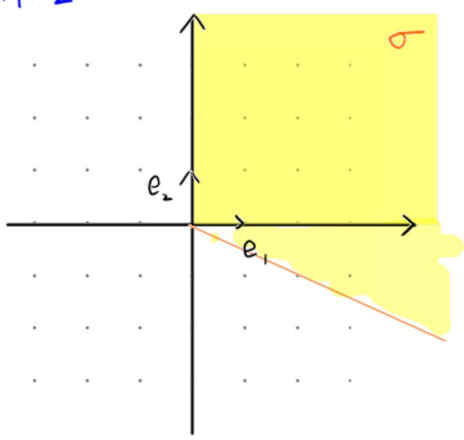


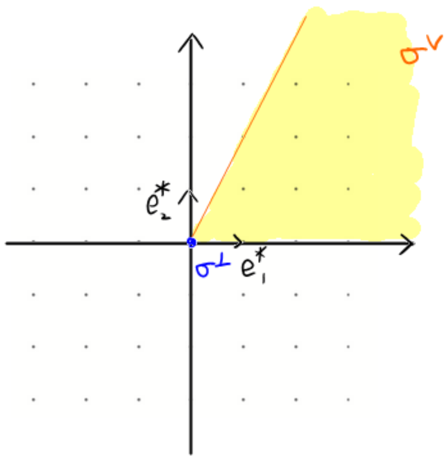
$N_{\mathbb{R}}$ $\qquad$ $N_{\mathbb{R}}^*$

$$\sigma = \{0\}$$
$$\sigma^\vee = \mathbb{R}\, e_1^* \oplus \mathbb{R}\, e_2^*$$
$$S_\sigma = \mathbb{Z}\, e_1^* \oplus \mathbb{Z}\, e_2^*$$

$$A_\sigma = \mathbb{C}[(\chi^{e_1^*})^{\pm 1}, (\chi^{e_2^*})^{\pm 1}] \triangleq \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}]$$
$$V_\sigma = \operatorname{Spec} \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}] = \mathbb{G}_{m,\mathbb{C}}^{\oplus 2}$$
$$V_\sigma(\mathbb{C}) = \operatorname{Hom}_{sg}(\mathbb{Z} e_1^* \oplus \mathbb{Z} e_2^*, \mathbb{C}_m) = (\mathbb{C}^\times)^2$$

E.g. $n=2$



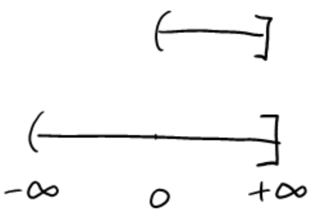


$N_{\mathbb{R}}$ $N_{\mathbb{R}}^*$

$$\sigma = \mathbb{R}_{\geq 0}\, e_2 \oplus \mathbb{R}_{\geq 0}(2e_1 - e_2)$$
$$\sigma^\vee = \mathbb{R}_{\geq 0}\, e_1^* \oplus \mathbb{R}_{\geq 0}(e_1^* + 2e_2^*)$$
$$S_\sigma = \langle e_1^*,\ e_1^* + 2e_2^*,\ e_1^* + e_2^* \rangle_{\mathbb{Z}_{\geq 0}}$$

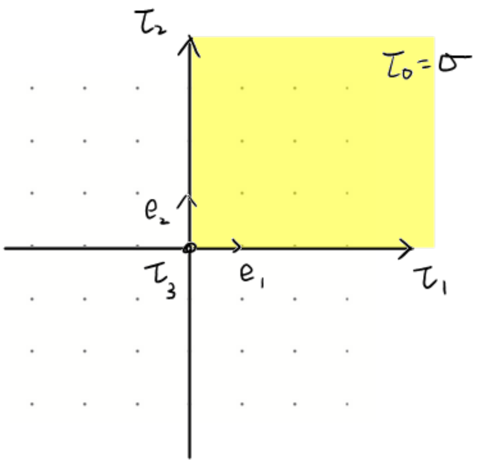
$$A_\sigma = \mathbb{C}[x, xy, xy^2] \cong \mathbb{C}[u,v,w]/(v^2 - uw) \qquad x = \chi^{e_1^*},\ y = \chi^{e_2^*}$$
$$V_\sigma = \operatorname{Spec} \mathbb{C}[u,v,w]/(v^2-uw)$$

tropical toric variety [ACM25, p7-8]



$$(0,+\infty] = \mathbb{T}^+ \overset{e^{-x}}{\approx} [0,1)$$

$$(-\infty,+\infty] = \mathbb{T} \approx [0,+\infty) = \mathbb{R}_{\geq 0}$$



$N_{\mathbb{R}}$

i	U_{τ_i}	$O(\tau_i)$	∞_{τ_i}	$\sigma_\infty^{\tau_i}$
0	$\mathbb{T}\oplus\mathbb{T}$	$\{+\infty\}\oplus\{+\infty\}$	$(+\infty,+\infty)$	$\{+\infty\}\oplus\{+\infty\}$
1	$\mathbb{T}\oplus\mathbb{R}$	$\{+\infty\}\oplus\mathbb{R}$	$(+\infty,0)$	$\{+\infty\}\oplus\mathbb{R}_{\geq 0}$
2	$\mathbb{R}\oplus\mathbb{T}$	$\mathbb{R}\oplus\{+\infty\}$	$(0,+\infty)$	$\mathbb{R}_{\geq 0}\oplus\{+\infty\}$
3	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$(0,0)$	$\mathbb{R}_{\geq 0}\oplus\mathbb{R}_{\geq 0}$

$\tau_{j,\infty}^{\tau_i}$ ($i \backslash j$)	0	1	2	3	$O(\tau_i)$	$\Sigma_\infty^{\tau_i}$
0	$\{+\infty\}\oplus\{+\infty\}$	–	–	–	$\{+\infty\}\oplus\{+\infty\}$	
1	$\{+\infty\}\oplus\mathbb{R}_{\geq 0}$	$\{+\infty\}\oplus\{0\}$	–	–	$\{+\infty\}\oplus\mathbb{R}$	
2	$\mathbb{R}_{\geq 0}\oplus\{+\infty\}$	–	$\{0\}\oplus\{+\infty\}$	–	$\mathbb{R}\oplus\{+\infty\}$	
3	$\mathbb{R}_{\geq 0}\oplus\mathbb{R}_{\geq 0}$	$\mathbb{R}_{\geq 0}\oplus\{0\}$	$\{0\}\oplus\mathbb{R}_{\geq 0}$	$\{0\}\oplus\{0\}$	$\mathbb{R}\oplus\mathbb{R}$