Eine Woche, ein Beispiel

12.3. cheating sheet for six functors

Ref: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

$$\begin{tikzcd} \mathcal{G} \arrow[d, dash] & \mathcal{F}\ \mathcal{F}' \arrow[d, dash] \\ Y \arrow[r, "f"] & X \end{tikzcd} \qquad \begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "g'"'] & X' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

$f^* \dashv f_*$

$- \otimes \mathcal{F} \dashv \underline{Hom}(\mathcal{F}, -)$

$f_! \dashv f^!$

$f^*(- \otimes -)$

$f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

$f_* \underline{Hom}(f^*\mathcal{F}, \mathcal{G}) \cong \underline{Hom}(\mathcal{F}, f_*\mathcal{G})$

$$\begin{tikzcd} & \otimes \arrow[dl, dash] \arrow[dr, dash] & \\ f^* \arrow[rr, dash] & & f_! \end{tikzcd}$$

proj formula

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \cong \mathcal{F} \otimes f_! \mathcal{G}$

$f_* \underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong \underline{Hom}(f_! \mathcal{G}, \mathcal{F})$

$f^! \underline{Hom}(\mathcal{F}, \mathcal{F}') \cong \underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$

bc: $f^* g_! \cong g'_! f'^{,*}$

$f_* g^! \cong g'^{,!} f'_*$

These extra formulas (compatabilities) come from the upgrade of adjunction formula to internal Hom.
To upgrade the adjunction between tensor product and internal Hom, one don't need extra formula, except the association law of tensor product.

I: $f^* = f^!$

P: $f_* = f_!$

$p: X \to pt$

$H^\cdot(X; \underline{\mathbb{Z}}) := p_* p^* \mathbb{1}$ $\qquad H^\cdot(X; \mathcal{F}) := p_* \mathcal{F}$ $\qquad = R\Gamma(X; \mathcal{F})$

$H_c^\cdot(X; \underline{\mathbb{Z}}) := p_! p^* \mathbb{1}$ $\qquad H_c^\cdot(X; \mathcal{F}) := p_! \mathcal{F}$

$H_\cdot(X; \underline{\mathbb{Z}}) := p_! p^! \mathbb{1}$ $\qquad H_\cdot(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad = H_c^\cdot(X; p^! \mathbb{1} \otimes \mathcal{F})$

$H_\cdot^{BM}(X; \underline{\mathbb{Z}}) := p_* p^! \mathbb{1}$ $\qquad H_\cdot^{BM}(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$ $\qquad = H^\cdot(X; p^! \mathbb{1} \otimes \mathcal{F})$

App 1. (Künneth formula)

$$H_c(X; \mathcal{F}) \otimes H_c(Y; \mathcal{G}) \cong H_c(X \times Y; \mathcal{F} \otimes \mathcal{G})$$

$$\begin{tikzcd} X \times Y \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "p"] & Y \arrow[d, "p_Y"] \\ X \arrow[r, "p_X"] & * \end{tikzcd}$$

reduced to: $p_{X,!} \mathcal{F} \otimes p_{Y,!} \mathcal{G} \cong p_!(p_1^* \mathcal{F} \otimes p_2^* \mathcal{G})$

App 2. (Poincaré duality)

X: a cpt oriented mfld of dim d, then $\qquad -^\vee = \underline{Hom}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$

proper

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ locally (Verdier duality)

$p^! \mathbb{Z} \cong \mathbb{Z}[d]$ globally

$$H^\cdot(X; \underline{\mathbb{Z}})[d] \cong H^\cdot(X; \underline{\mathbb{Z}})^\vee$$

reduced to: $p_* \underline{Hom}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{Hom}(p_! A, B)$

$$Z \xrightarrow[\text{close}]{i} X \xleftarrow{j} U \quad \rightsquigarrow$$

$$\begin{tikzcd}
\mathcal{D}(Z) \arrow[r, "p_i\ i_* = i_!\ f.f"] & \mathcal{D}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "p_i\ i^!"] \arrow[r, "p_i\ j^* = j^!"] & \mathcal{D}(U) \arrow[l, bend right, "j_!\ f.f."'] \arrow[l, bend left, "p_i\ Rj_*\ f.f"]
\end{tikzcd}$$

Id Id Id Id

$$i^* j_! = 0, \quad j^* i_* = 0, \quad i^! Rj_* = 0$$

f.f. fully faithful
pi: preserve injectives. (内射)
ie. inj sheaf $\rightsquigarrow$ inj sheaf

For X mfld, $\dim_{\mathbb{R}} X = n$, $\pi_X^! \mathbb{Z} = Or_X[n] \overset{+\text{orientation}}{=\!=\!=} \mathbb{Z}_X[n]$

Just by checking the stalk & taking the dual, one gets

$$0 \longrightarrow j_! j^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow 0$$

$$i_! i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow Rj_* j^* \mathcal{F} \xrightarrow{+1}$$

Here, $H_{-1}(S^1; \mathbb{Q}) = \mathbb{Q}$ for convenience of index.

Taking $R\pi_{X,*}$

$$\begin{array}{lllll}
 & R\Gamma(X, Z; \mathcal{F}) \longrightarrow & R\Gamma(X; \mathcal{F}) \longrightarrow & R\Gamma(Z; \mathcal{F}|_Z) & \xrightarrow{+1} \\
 & R\Gamma(X, U; \mathcal{F}) \longrightarrow & R\Gamma(X; \mathcal{F}) \longrightarrow & R\Gamma(U; \mathcal{F}|_U) & \xrightarrow{+1} \\
 & \quad \| & & & \\
 & R\Gamma_Z(X; \mathcal{F}) & & & \\
\text{When } \mathcal{F} = \mathbb{Q}_X, & H^\cdot(X, Z) \longrightarrow & H^\cdot(X) \longrightarrow & H^\cdot(Z) & \xrightarrow{+1} \\
 & H^\cdot(X, U) \longrightarrow & H^\cdot(X) \longrightarrow & H^\cdot(U) & \xrightarrow{+1} \\
\text{When } \mathcal{F} = \mathbb{D}_X, & R\Gamma(X, j_! \mathbb{Q}_U) \longrightarrow & H^{BM}_\cdot(X) \longrightarrow & R\Gamma(Z; \mathbb{D}_X|_Z) & \xrightarrow{+1} \\
 & H^{BM}_\cdot(Z) \longrightarrow & H^{BM}_\cdot(X) \longrightarrow & H^{BM}_\cdot(U) & \xrightarrow{+1}
\end{array}$$

Taking $R\pi_{X,!}$

$$\begin{array}{lllll}
 & R\Gamma_c(U, \mathcal{F}|_U) \longrightarrow & R\Gamma_c(X; \mathcal{F}) \longrightarrow & R\Gamma_c(Z; \mathcal{F}|_Z) & \xrightarrow{+1} \\
 & R\Gamma_c(Z, i^! \mathcal{F}) \longrightarrow & R\Gamma_c(X; \mathcal{F}) \longrightarrow & R\Gamma_c(X, Rj_*(\mathcal{F}|_U)) & \xrightarrow{+1} \\
\text{When } \mathcal{F} = \mathbb{Q}_X, & H^\cdot_c(U) \longrightarrow & H^\cdot_c(X) \longrightarrow & H^\cdot_c(Z) & \xrightarrow{+1} \\
 & H^\cdot_c(Z, i^! \mathbb{Q}_X) \longrightarrow & H^\cdot_c(X) \longrightarrow & H^\cdot_c(X, Rj_* \mathbb{Q}_U) & \xrightarrow{+1} \\
\text{When } \mathcal{F} = \mathbb{D}_X, & H_\cdot(U) \longrightarrow & H_\cdot(X) \longrightarrow & H_\cdot(X, U) & \xrightarrow{+1} \\
 & H_\cdot(Z) \longrightarrow & H_\cdot(X) \longrightarrow & H_\cdot(X, Z) & \xrightarrow{+1}
\end{array}$$

$$i^!\mathcal{F} \longrightarrow i^*\mathcal{F} \longrightarrow i^*Rj_*j^*\mathcal{F} \xrightarrow{+1}$$

local cohomology compares the difference between stalks and costalks.

Application

One point compactification

$$\begin{tikzcd} X \arrow[r, hook, "\iota"] \arrow[dr, "\pi"'] & \overline{X} \arrow[d, "\overline{\pi}"] \\ & \{*\} \end{tikzcd}$$

$$\begin{aligned} H_c^{\bullet}(X) &= R\pi_! \pi^* \mathbb{Z} \\ &= R\overline{\pi}_! \iota_! \iota^* \pi^* \mathbb{Z} \\ &= R\overline{\pi}_* (\iota_! \iota^! \mathbb{Z}_{\overline{X}}) \\ &= H^{\bullet}(\overline{X}, \{\infty\}; \mathbb{Z}) \end{aligned}$$

$$\begin{aligned} H_{\bullet}^{BM}(X) &= R\pi_* \pi^! \mathbb{Z} \\ &= R\overline{\pi}_* \iota_* \iota^! \overline{\pi}^! \mathbb{Z} \\ &= R\overline{\pi}_* (\iota_* \iota^* \overline{\pi}^! \mathbb{Z}) \\ &= \mathrm{cone}\, (R\overline{\pi}_* i_! i^! \overline{\pi}^! \mathbb{Z} \longrightarrow R\overline{\pi}_* \overline{\pi}^! \mathbb{Z}) \\ &= H_{\bullet}(\overline{X}, \{\infty\}; \mathbb{Z}) \end{aligned}$$

Originally, this is another def of cpt supp coh & BM homology.

Vector bundle with 6-functors.

Goal: Define Thom class & Euler class as in [GTM82, §6]

[GTM82]: Raoul Bott, Loring W. Tu, Differential Forms in Algebraic Topology, 1982
https://link.springer.com/book/10.1007/978-1-4757-3951-0

$$\begin{tikzcd} H_c^\bullet(F) & H_{cv}^\bullet(E) \arrow[l] \arrow[d, "\cong"] \arrow[dr, bend left, "s^*"] & \\ & H^{\bullet-r}(B) & H^\bullet(B) \end{tikzcd}
\qquad
\begin{tikzcd} \text{generator} \in H_c^r(F) & \Phi \in H_{cv}^r(E) \arrow[l, mapsto] \arrow[d, mapsto] \arrow[dr, bend left, mapsto] & \\ & 1 \in H^0(B) & eu_B \in H^r(B) \end{tikzcd}$$

Setting

$\pi: E \longrightarrow B$ oriented v.b. with fiber $F \cong \mathbb{R}^r$

β_F: one point compactification $(\mathbb{R}^n \subset S^n)$

β_E: fiberwise compactification

$\pi_X: X \longrightarrow \{*\}$

$$\begin{tikzcd} & \bar{F} \arrow[r, "\iota_{\bar{F}}"] & \bar{E} \arrow[ddr, "\bar{\pi}"] & \\ F \arrow[ur, "\beta_F"] \arrow[rr, hook, "\iota_F"] \arrow[d, "\pi_F"'] & & E \arrow[u, "\beta_E"'] \arrow[d, "\pi"] & \\ \{p\} \arrow[rr, "\iota_p"] & & B & \end{tikzcd}$$

Def $H_{cv}^\bullet(E) :\hat{=} H^\bullet(\bar{E}, \bar{E} - E)$ (compact vertical)

$$\begin{aligned} &= R\pi_{\bar{E},*}\, \beta_{E,!}\, \beta_E^!\, \mathbb{Z}_{\bar{E}} \\ &= R\pi_{B,*}\, (R\bar{\pi}_!\, \beta_{E,!})(\beta_E^*\, \mathbb{Z}_{\bar{E}}) \\ &= R\pi_{B,*}\, R\pi_!\, \mathbb{Z}_E \end{aligned}$$

Ex. Construct the following canonical maps by six functors.

$$\begin{tikzcd} H^\bullet(\bar{F}, \bar{F} - F) \arrow[d, phantom, "\wr\|"] & H^\bullet(\bar{E}, \bar{E} - E) \arrow[l] \arrow[d, phantom, "\wr\|"] \\ H_c^\bullet(F) & H_{cv}^\bullet(E) \arrow[l] \arrow[d, "\sim"] \\ & H^{\bullet-r}(B) \end{tikzcd}
\qquad
\begin{tikzcd} R\pi_{\bar{F},*}\, \beta_{F,!}\, \mathbb{Z}_F \arrow[d, phantom, "\wr\|"] & R\pi_{\bar{E},*}\, \beta_{E,!}\, \mathbb{Z}_E \arrow[l] \arrow[d, phantom, "\wr\|"] \\ R\pi_{F,!}\, \mathbb{Z}_F & R\pi_{B,*}\, R\pi_!\, \mathbb{Z}_E \arrow[l] \arrow[d, "\sim"] \\ & R\pi_{B,*}\, \mathbb{Z}_B[-r] \end{tikzcd}$$

This page works on the details of the last exercise. If you did that exercise or don't care the details, then skip this page.
Notice that we ignore the difference between the complex and cohomology, which is just for the convenience of presentation.

Lemma. $R\pi_! \mathbb{Z}_E \cong \mathbb{Z}_B[-r]$. As a result, $H^{\cdot}_{cv}(E) \cong H^{\cdot-r}(B)$.

Proof.

$$
\begin{aligned}
R\pi_! \mathbb{Z}_E &= R\pi_! \pi^* \mathbb{Z}_B && \text{expand} \\
&\cong R\pi_! \pi^! \mathbb{Z}_B[-r] && \text{Verdier duality, } \pi \text{ is a v.b.} \\
&\xrightarrow{\sim} \mathbb{Z}_B[-r] && \text{adjunction, iso comes from} \\
& && H_\cdot(\mathbb{R}^r;\mathbb{Z}) \cong \mathbb{Z} \text{ (check locally)}
\end{aligned}
$$

strictly speaking, $H^i_{cv}(E) = \mathcal{H}^i(\pi_{B,*} R\pi_! \mathbb{Q}_E)$, but we're lazy to write $\mathcal{H}$

$$
\begin{aligned}
H^{\cdot}_c(F) &\overset{\substack{\text{trick!} \\ bc}}{=\!=} (R\pi_! \mathbb{Z}_E)_p \\
&=\!= \iota_p^* R\pi_! \mathbb{Z}_E \\
&=\!= \pi_{B,*} \iota_{p,*} \iota_p^* R\pi_! \mathbb{Z}_E \longleftarrow \pi_{B,*} R\pi_! \mathbb{Z}_E = H^{\cdot}_{cv}(E)
\end{aligned}
$$

Compare with the first row:

$$
\begin{aligned}
R\pi_{\bar{F},*} \beta_{F,!} \mathbb{Z}_F &=\!= R\pi_{\bar{E},*} \iota_{\bar{F},*} \beta_{F,!} \iota_F^* \mathbb{Z}_E \\
&\overset{bc}{=\!=} R\pi_{\bar{E},*} \iota_{\bar{F},*} \iota_{\bar{F}}^* \beta_{E,!} \mathbb{Z}_E \longleftarrow R\pi_{\bar{E},*} \beta_{E,!} \mathbb{Z}_E
\end{aligned}
$$

Q: How to show that $H^0(B) \longrightarrow H^r_c(F)$ $\quad 1 \longmapsto$ generator?

A:

$$
\begin{aligned}
H^{\cdot-r}(B) = R\pi_{B,*} \mathbb{Z}_B[-r] \longrightarrow\ & R\pi_{B,*} \iota_{p,*} \iota_p^* \mathbb{Z}_B[-r] \\
&= \iota_p^* \mathbb{Z}_B[-r] \qquad = \mathbb{Z}_{\{p\}}[-r]
\end{aligned}
$$

$$1 \longmapsto \text{generator}$$

$$
\begin{aligned}
&\cong \iota_p^* R\pi_! \mathbb{Z}_E \\
&\overset{bc}{\cong} R\pi_{F,!} \iota_F^* \mathbb{Z}_E \\
&= R\pi_{F,!} \mathbb{Z}_F \qquad = H^{\cdot}_c(F)
\end{aligned}
$$

Euler class of v.b.

People are lazy to write all contravarient functors induced by s as s^*. It may be not the pullback in the 6-functor formalism.
These may cause confusion, and I hope it is fine for you.

Ex construct

$$\begin{tikzcd} s^*: H^\cdot_{cv}(E) \arrow[r] \arrow[d, equal] & H^\cdot(B) \arrow[d, equal] \\ R\pi_{B,*}R\pi_! \mathbb{Z}_E & R\pi_{B,*}\mathbb{Z}_B \end{tikzcd} \qquad \Phi \longmapsto eu_E$$

After this exercise, we get euler class $eu_E \in H^r(B)$.

/A for Ex: $\pi \circ s = Id$, so

$$R\pi_{B,*} R\pi_! \mathbb{Z}_E \longrightarrow R\pi_{B,*} (R\pi_! s_*) s^* \mathbb{Z}_E = R\pi_{B,*} \mathbb{Z}_B . \qquad \square$$

Ex. Find a way to define eu_E without using Φ, i.e. construct

$$\begin{tikzcd} H^{\cdot-r}(B) \arrow[r] \arrow[d, equal] & H^\cdot(B) \arrow[d, equal] \\ R\pi_{B,*}\mathbb{Z}_B[-r] & R\pi_{B,*}\mathbb{Z}_B \end{tikzcd} \qquad 1 \longmapsto eu_E$$

[Hint. Induced by

$$\mathbb{Z}_B[-r] \cong R\pi_! \mathbb{Z}_E \longrightarrow (R\pi_! s_*) s^* \mathbb{Z}_E \cong \mathbb{Z}_B$$

in fact, use $\mathbb{Z}_E \longrightarrow s_* \mathbb{Z}_B$]

Relation with tubular nbhd

Let $j: E - B \hookrightarrow E$, T: tubular nbhd of B in E,

s(B), if you prefer rigorous notation

by applying $R\pi_{B,*} R\pi_!$ to

$$0 \longrightarrow j_! j^! \mathbb{Z}_E \longrightarrow \mathbb{Z}_E \longrightarrow s_* s^* \mathbb{Z}_E \longrightarrow 0$$

we get LES:

$$\begin{tikzcd} R\pi_{B,*} R\pi|_{E\setminus B,!} \mathbb{Z}_{E\setminus B} \arrow[r] \arrow[dd, equal] & H^\cdot_{cv}(E) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^\cdot(B) \arrow[r, "+1"] \arrow[dd, equal] & {} \\ & H^{\cdot-r}(B) \arrow[ur, "eu_E \wedge"] \arrow[d, phantom, "\cong" rotate=90] & & \\ H^{\cdot-1}(\partial T) \arrow[r] & H^\cdot(T, \partial T) \arrow[r] & H^\cdot(T) \arrow[r, "+1"] & {} \end{tikzcd}$$

Explanation

Exactness & derived

$j_!$ is exact
i^*, j^* are exact
i_* is exact

by checking on stalks
in the category Top
when $Z \subset X$ is (strongly) loc. contractable.

j_* is not exact $\rightsquigarrow R j_*$
$i^!$ is already derived.

Rmk: strongly loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p s.t. $U_n \cap Z$ is contractable

loc. contractable: $\forall p \in X$, $\exists$ a nbhd basis $\{U_n\}_n$ of p s.t. $U_n \cap Z \subset U_n$ is contractable

E.g. $\{$strongly loc. contractable$\} \subseteq \{$loc. contractable$\} \not\subseteq$ Top

CW-cplx, topo mflds & algebraic varieties (Check?)

Cantor set

See the answer in
https://math.stackexchange.com/questions/1082601/anr-is-locally-contractible
for the subtlety of these two definitions.

I don't care. In both cases, the local cohomology vanishes in higher degree, and that's what I want.

For the non-exact functors, there maybe some problems in the composition of derived functors.

https://mathoverflow.net/questions/108734/theorem-on-composition-of-derived-functors-question-about-proof
https://mathoverflow.net/questions/435310/what-can-be-said-about-the-derived-functor-of-a-composition-between-unbounded-de

E.g. we need to check if $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.
Luckily, in the open-closed formalism, we won't meet these problems.

Prop1. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}'$ ($F \dashv G$), assume F is exact. Then

① G preserves injective sheaves;
② $RG(f) \circ RG(g) = RG(f \circ g)$

Proof. ①: by universal property.
②: by adjunction

Prop 2. Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{C}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathcal{C}''$. Suppose F or F' is exact, then

$$RG \circ RG'(f) = (R(G \circ G'))(f)$$

Proof. By adjunction & Grothendieck-Serre sequence. $(LF' \circ LF = L(F' \circ F))$

When F' is exact, can use Prop 1 ①.

Cor. $R\pi_{X,*} \circ Rj_* = R\pi_{U,*}$.

Rf_* & $Rf_!$ are nice in general.

Reason: f_! sends skyscraper sheaf to skyscraper sheaf, and in general preserve injective sheaves. (need double check)