Eine Woche, ein Beispiel

2.23 Schubert calculus: coh of Grassmannian

Ref:
[3264] and [Fulton]

We will attempt to tackle Schubert calculus in a concise manner. The term "Schubert calculus" is often associated with intersection theory, enumerative geometry, combinatorics, Grassmannians, and more, making it a vast topic. However, I believe its core ideas can be clearly explained in just six hours. I will break the material into several parts:

1. $H^\cdot(Gr(n,r);\mathbb{Z})$ and its combinatorics
2. (inside Grassmannian)
   cycles in Grassmannian, including:
   - cycle class map: $CH^\cdot(Gr(n,r)) \xrightarrow{\sim} H^\cdot(Gr(n,r);\mathbb{Z})$
   - incidence variety $\begin{cases} \text{(partial) flag variety} \\ \text{Fano variety of planes} \\ \cdots \end{cases}$
   - a reintepretation of cycles

$$\begin{tikzcd} \mathcal{L} \arrow[d, dash] & \mathcal{S}^\vee \arrow[d, dash] \\ X \arrow[r, "f_{\mathcal{L}}"] & Gr(\infty,r) \end{tikzcd}$$

3. (outside Grassmannian + v.b.)

Chern class: $\quad c: VB(X) \longrightarrow H^\cdot(X;\mathbb{Z})$

$$f_{\mathcal{L}}^*: H^\cdot(Gr(\infty,r);\mathbb{Z}) \longrightarrow H^\cdot(X;\mathbb{Z})$$

e.g.,
$$\begin{array}{ccc}
VB(Gr(n,r)) & \longrightarrow & H^\cdot(Gr(n,r);\mathbb{Z}) \\
\mathcal{S}^* & \longmapsto & 1+\sigma_1+\cdots \\
\mathcal{Q} & \longmapsto & 1+\sigma_1+\cdots \\
\mathcal{T}_{Gr} & \longmapsto & 1+n\cdot\sigma_1+\cdots \\
\mathcal{S} & \longmapsto & 1-\sigma_1+\sigma_{1,1}-\sigma_{1,1,1}+\cdots+(-1)^r\sigma_{(1)^r}
\end{array}$$

4. Applications

   tangent space argument

1. Group structure of $H^{\cdot}(Gr(n,r);\mathbb{Z})$
2. Cup product

1. Group structure of $H^{\cdot}(Gr(n,r);\mathbb{Z})$

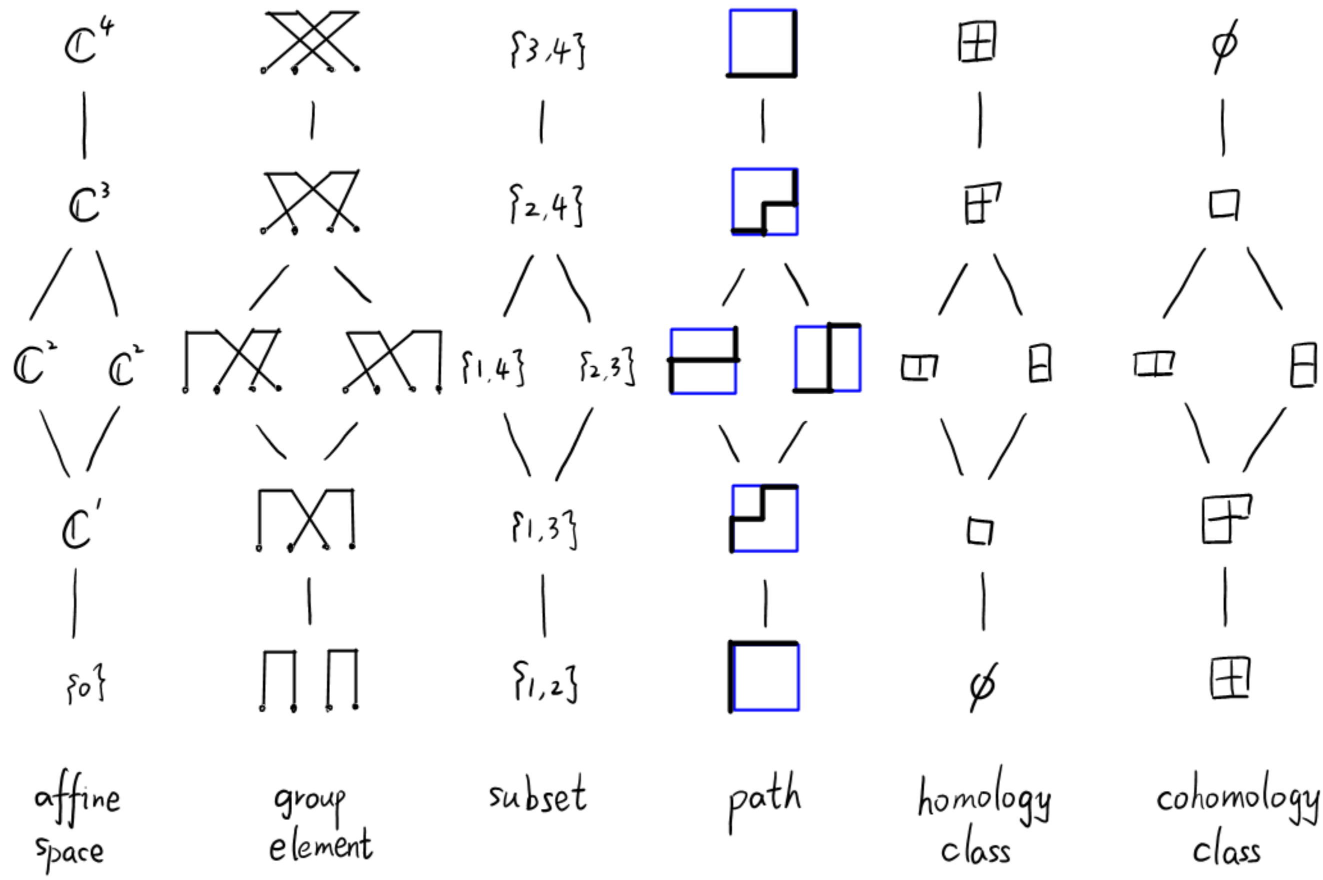
It's well-known that $Gr(n,r) \cong GL_n(\mathbb{C})/P$ has an affine paving w.r.t. $S_n/S_r \times S_{n-r}$:

$$Gr(n,r) = \coprod_{w \in S_n/S_r \times S_{n-r}} BwP/P \cong \coprod_{w \in S_n/S_r\times S_{n-r}} \mathbb{C}^{l(w)}$$

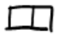
$$\# S_n/S_r\times S_{n-r} = \binom{n}{r}$$

We read the diagram from top to bottom, the map from right to left.

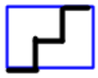
E.g. $n=4$ $r=2$

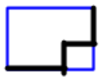


Hint from gp element to homology class:

$\rightsquigarrow$ $(2,0) =$ 

E.g. $n = 5$, $r = 2$

 $\sim \{2,4\} \sim$  $\sim$  $\sim$ 

 $\sim \{3,5\} \sim$  $\sim$  $\sim$ 

Ex. compute w_0-action (left mult) on $S_n / S_r \times S_{n-r}$, where $w_0 =$ .

2. Cup product

We want to compute intersection number by moving one cycle(so that they intersect transversally)

Lemma 1. $[B^- w P/P] = [B w_0 w P/P]$ in $H^\cdot(Gr(r,n); \mathbb{Z})$.

Proof. $B^- w P/P = w_0 B w_0 w P/P \sim B w_0 w P/P$.

Lemma 2.

$$\#\left(BwP/P \cap B^-\eta P/P\right) = \begin{cases} 0 & \eta > w \\ 1 & \eta = w \\ 0 & \eta \not> w \ \ \& \ \ l(\eta) = l(w) \\ ? & \text{otherwise} \end{cases}$$

Moreover, when $\eta = w$, BwP/P and $B^-\eta P/P$ intersect transversally.

Idea: Find a set of representative elements $C_w^+ \cong \mathbb{C}^{l(w)}$ in B, s.t.

$$BwP/P \xleftarrow{\cong} C_w^+ w P/P \cong C_w^+.$$

Similarly, find a set of representative elements $C_\eta^- \cong \mathbb{C}^{l(w_0\eta)}$ in B^-, s.t.

$$B^-\eta P/P \xleftarrow{\cong} C_\eta^- \eta P/P \cong C_\eta^-.$$

After that,

$$\begin{aligned} BwP/P \cap B^-\eta P/P &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_+ wP = c_- \eta P\} \\ &= \{(c_+, c_-) \in C_w^+ \times C_\eta^- \mid c_-^{-1} c_+ \in \eta P w^{-1}\} \end{aligned}$$

can be written as the zero sets of polynomials (of deg ≤ 2) in $C_w^+ \times C_\eta^- \cong \mathbb{C}^{l(w)+l(w_0\eta)}$.

E.g. $n=5$, $r=2$,

$$w = $$  $$= \left(\begin{array}{cc:ccc} & & 1 & & \\ & & & 1 & \\ 1 & & & & \\ & & & & 1 \\ & 1 & & & \end{array}\right) = \{35\,|\,124\} \sim$$ hom  $\sim$ cohom 

$$\eta_0 = $$  $$= \left(\begin{array}{cc:ccc} 1 & & & & \\ & & 1 & & \\ & 1 & & & \\ & & & 1 & \\ & & & & 1 \end{array}\right) = \{13\,|\,245\} \sim$$  $\sim$ 

Let $\eta = \eta_0$, we want to describe $BwP/P \cap B^-\eta P/P \subset C_w^+ \times C_\eta^-$.
By direct calculation,

$$P = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ & & * & * & * \\ & & * & * & * \\ & & * & * & * \end{pmatrix} \qquad \eta P w^{-1} = \begin{array}{c} \\ 1 \\ \\ 3 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array} \qquad \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{pmatrix}$$

for copy

$$wPw^{-1} = \begin{array}{c} \\ \\ \\ 3 \\ \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & & 4 & \end{array} \\ \begin{pmatrix} * & * & & * & \\ * & * & & * & \\ * & * & * & * & * \\ * & * & & * & \\ * & * & * & * & * \end{pmatrix} \end{array} \qquad \eta P \eta^{-1} = \begin{array}{c} \\ 1 \\ \\ 3 \\ \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & 2 & & 4 & 5 \end{array} \\ \begin{pmatrix} * & * & * & * & * \\ & * & & * & * \\ * & * & * & * & * \\ & * & & * & * \\ & * & & * & * \end{pmatrix} \end{array}$$

$$C_w^+ = \begin{pmatrix} 1 & & * & & * \\ & 1 & * & & * \\ & & 1 & & \\ & & & 1 & * \\ & & & & 1 \end{pmatrix} \qquad C_\eta^- = \begin{pmatrix} 1 & & & & \\ * & 1 & & & \\ & & 1 & & \\ * & & * & 1 & \\ * & & * & & 1 \end{pmatrix}$$

Now, suppose

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ b_{21} & 1 & & & \\ & & 1 & & \\ b_{41} & & b_{43} & 1 & \\ b_{51} & & b_{53} & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ b_{21} & 1 & b_{21}a_{13} + a_{23} & & b_{21}a_{15} + a_{25} \\ & & 1 & & \\ b_{41} & & b_{41}a_{13} + b_{43} & 1 & b_{41}a_{15} + a_{45} \\ b_{51} & & b_{51}a_{13} + b_{53} & & b_{51}a_{15} + 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in \eta P w^{-1} \Longleftrightarrow \begin{cases} b_{21}a_{13} + a_{23} = 0 \\ b_{21}a_{15} + a_{25} = 0 \\ b_{41}a_{13} + b_{43} = 0 \\ b_{41}a_{15} + a_{45} = 0 \\ b_{51}a_{13} + b_{53} = 0 \\ b_{51}a_{15} + 1 = 0 \end{cases}$$

In this case, $BwP/P \cap B^-\eta P/P \cong \mathbb{C}^3 \times \mathbb{C}^\times$.

Now, take $\eta = w$, one suppose that

$$C_-^{-1} = \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & b_{43} & 1 & \\ & & & & 1 \end{pmatrix} \qquad C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & & 1 & a_{45} \\ & & & & 1 \end{pmatrix}$$

then

$$C_-^{-1} C_+ = \begin{pmatrix} 1 & & a_{13} & & a_{15} \\ & 1 & a_{23} & & a_{25} \\ & & 1 & & \\ & & b_{43} & 1 & a_{45} \\ & & & & 1 \end{pmatrix}.$$

Therefore,

$$C_-^{-1} C_+ \in w P w^{-1} \Longleftrightarrow a_{13} = a_{15} = a_{23} = a_{25} = a_{45} = b_{43} = 0.$$

In this case $BwP/P \cap B^- wP/P = \{*\}$.

Furthermore, one can show the transversality through the tangent argument.

Ex. When $\eta = w_0$, verify that

$$BwP/P \cap B^- w_0 P/P = \phi$$

Generalize this example to prove Lemma 2.

$\Leftrightarrow l(w_0 w) + l(w_0 w') = r(n-r)$

Cor of Lemma 2. When $l(w) + l(w') = r(n-r)$,

$$\deg\left([BwP/P] \cup [Bw'P/P]\right) = \begin{cases} 1 & w = w_0 w' \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, denote

$$\sigma_w := [BwP/P] \in H^{\cdot}(Gr(r,n);\mathbb{Z})$$

then $\sigma_w \sigma_{w_0 w} = \sigma_{Id}$

$\sigma_w \sigma_\eta = 0$ when $l(w) + l(\eta) = r(n-r)$.

⚠ When we view $w = a = (a_1, \dots, a_r)$ as the Young diagram in the cohom class,

$$l(w) = r(n-r) - |a|$$

$$\sigma_w \stackrel{\Delta}{=} \sigma_a \in H_{l(w)}(Gr(r,n);\mathbb{Z}) \cong H^{|a|}(Gr(r,n);\mathbb{Z}).$$