Eine Woche, ein Beispiel

12.1 weights of type E

There are already much information in wiki and other references about the exceptional Lie algebra. It is nice, but I always have to check the compatability among different references. In this document, I try to fix a standard coordinate, and state all the combinatorical results without proofs.

We will make a list of the following objects, for E_6, E_7 and E_8.

Ref:
[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2023. https://doi.org/10.1017/9781009280020.

- Weights nearest to the origin
  - some graphs
  - weight lattice
- Simple roots
- Fundamental weights
- Weyl group action

Remark: There is another coordinate system which is written in wiki: del Pezzo surface. We don't use them. There, the different weight spaces are identified, while in our coordinate system, we identify the root lattices.

1. E_6.

– Weights nearest to the origin

There are two minuscule representations of E 6. So we just fix one.

affine version

#	typical coordinates	symbol
6	$(1, 0, 0, 0, 0, 0, 1, 0)^T$	v_i
6	$(1, 0, 0, 0, 0, 0, 0, 1)^T$	$\tilde{v}_i$
15	$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{0, 1, 2\}$$

could be $\tilde{v}_i$ or v_{ij} edge

weight lattice version

#	typical coordinates	symbol
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, 3, -3)^T$	v_i
6	$\frac{1}{6}(5, -1, -1, -1, -1, -1, -3, 3)^T$	$\tilde{v}_i$
15	$\frac{1}{3}(-2, -2, 1, 1, 1, 1, 0, 0)^T$	v_{ij}

$$\langle v_i, v_j \rangle \in \{\tfrac{4}{3}, \tfrac{1}{3}, -\tfrac{2}{3}\}$$

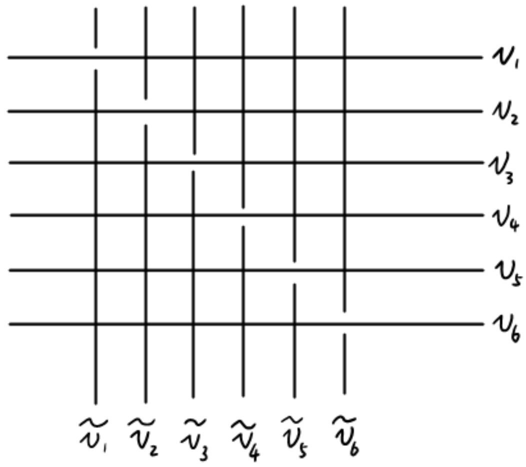
could be $\tilde{v}_i$ or v_{ij} edge

in $\left\{ \sum_{i=1}^{6} z_i = z_7 + z_8 = 0 \right\} \cong \mathbb{R}^6$

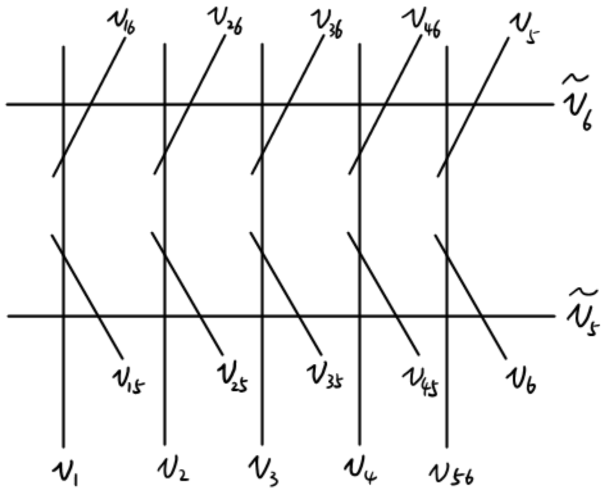
The graph constructed is called the Schläfli graph, which has 27 vertices and 216 edges (with HoG Id 1300).
This graph is also the configuration graph of 27 lines.

vertices $\rightsquigarrow$ lines
edges $\rightsquigarrow$ intersection points
triangle $\rightsquigarrow$ triangle cut by H only in E_6

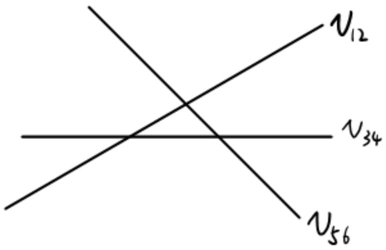
Here are some typical subgraphs:



Schläfli double six configuration
$\# V = 12 \quad \# E = 30$
HoG Id = 32794
36 many, from [Huy24, Ex 3.6]



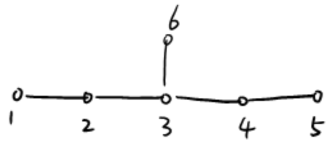
(v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6$. $\cdots$)
$\# V = 17 \quad \# E = 30 + 20 = 50$
HoG Id = none.



triangle
720 many

Q: For each type of subgraph, how many are they in the Schläfli graph?
I don't know if there are any simple answer for general subgraphs, and I don't know if there are any efficient algorithm for doing this. But this already produces many mysterious combinatorical numbers.

– Simple roots



$$
\begin{aligned}
&\{\ \alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\ \} \\
&= \{\ v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6,\ v_4 - v_{56}\ \} \\
&= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \right\}
\end{aligned}
$$

Ex. Verify that all the 72 roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	α_{1-2}
$40 = \binom{6}{3} \cdot 2$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})^T$	$\alpha_{456,7}$
2	$(0, 0, 0, 0, 0, 0, 1, -1)^T$	α_7

- Fundamental weights

denote by $A = (a_{ij})$ the Cartan matrix, then

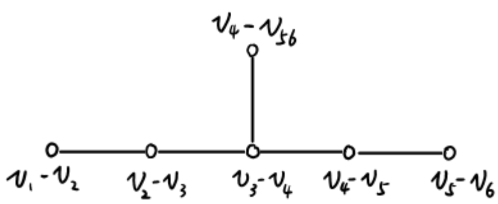
$$(\alpha_1, \dots, \alpha_r) = (\varpi_1, \dots, \varpi_r) A \qquad \langle \alpha_i, \alpha_j \rangle = A$$
$$(\varpi_1, \dots, \varpi_r) = (\alpha_1, \dots, \alpha_r) A^{-1} \qquad \langle \varpi_i, \varpi_j \rangle = A^{-1}$$

As a result,

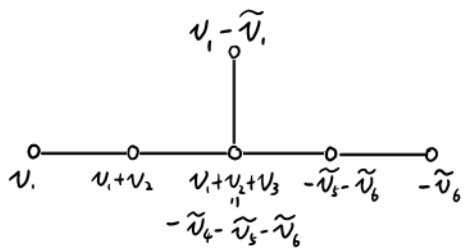
$$\{ \varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6 \}$$
$$= \{ v_1, v_1 + v_2, v_1+v_2+v_3, -\tilde{v}_5 - \tilde{v}_6, -\tilde{v}_6, v_1 - \tilde{v}_1 \} \qquad -\tilde{v}_6 \neq \sum_{i=1}^{5} v_i$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 4 \\ 4 \\ -2 \\ -2 \\ -2 \\ -2 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 3 \\ 3 \\ 3 \\ -3 \\ -3 \\ -3 \\ 9 \\ -9 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \\ -4 \\ -4 \\ 6 \\ -6 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{6}\begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 2 \\ 2 \\ -1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ -1 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -2 \\ -2 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -5 \\ 3 \\ -3 \end{pmatrix}, \frac{1}{6}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$



α_i



ϖ_i

- Weyl group action

We know that

$$s_k \alpha_i = \alpha_i - \langle \alpha_k, \alpha_i \rangle \alpha_k$$
$$= \alpha_i - a_{ki} \alpha_k$$

$$\rightsquigarrow \quad s_k(\alpha_1, \dots, \alpha_r) = (\alpha_1, \dots, \alpha_r) \underbrace{\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ -a_{k1} & \cdots & 1-a_{kk} & \cdots & -a_{kr} \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}}_{\left(\delta_{ij} - \delta_{ik} a_{ij}\right)_{i,j}}$$

In practice, we want to compute s_k-action on coordinates, it's easier to use the formula

$$s_k e_i = e_i - \langle \alpha_k, e_i \rangle \alpha_k$$

E.g. In E_6-case, when $k=1$, $\alpha = (1,-1,0,\dots,0)^T = e_1 - e_2$,

$$s_1 e_1 = e_1 - (e_1 - e_2) = e_2$$
$$s_1 e_2 = e_2 - (-1)(e_1 - e_2) = e_1$$

$$\rightsquigarrow \quad s_1 = \begin{pmatrix} & 1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

Similarly, $s_k = s_{(k,k+1)}$ for $i = 1, \dots, 5$.

When $k=6$, $\alpha_k = \frac{1}{2}(-1,-1,-1,1,1,1,1,-1)^T$,

$$s_6 e_1 = e_1 - (-\tfrac{1}{2})\alpha_6 = e_1 + \tfrac{1}{2}\alpha_6$$
$$= \tfrac{1}{4}(3,-1,-1,1,1,1,1,-1)^T$$
$$s_6 e_4 = e_4 - \tfrac{1}{2}\alpha_6$$
$$= \tfrac{1}{4}(1,1,1,-3,-1,-1,-1,1)^T$$

$$\rightsquigarrow \quad s_6 = \frac{1}{4}\left(\begin{array}{c|c|c} \begin{smallmatrix} 3 & & -1 \\ & 3 & \\ -1 & & 3 \end{smallmatrix} & 1 & -1 \\ \hline 1 & \begin{smallmatrix} 3 & & & -1 \\ & 3 & & \\ & & 3 & \\ -1 & & & 3 \end{smallmatrix} & 1 \\ \hline -1 & 1 & 3 \end{array}\right)$$

The action of $s_1, \dots, s_5$ on the Schläfli graph is easy. s_6 is hard.

E.g. $s_6 v_1 = s_6(e_1+e_7) = e_1+e_7 = v_1$ $\quad s_6 v_2 = v_2 \quad s_6 v_3 = v_3$

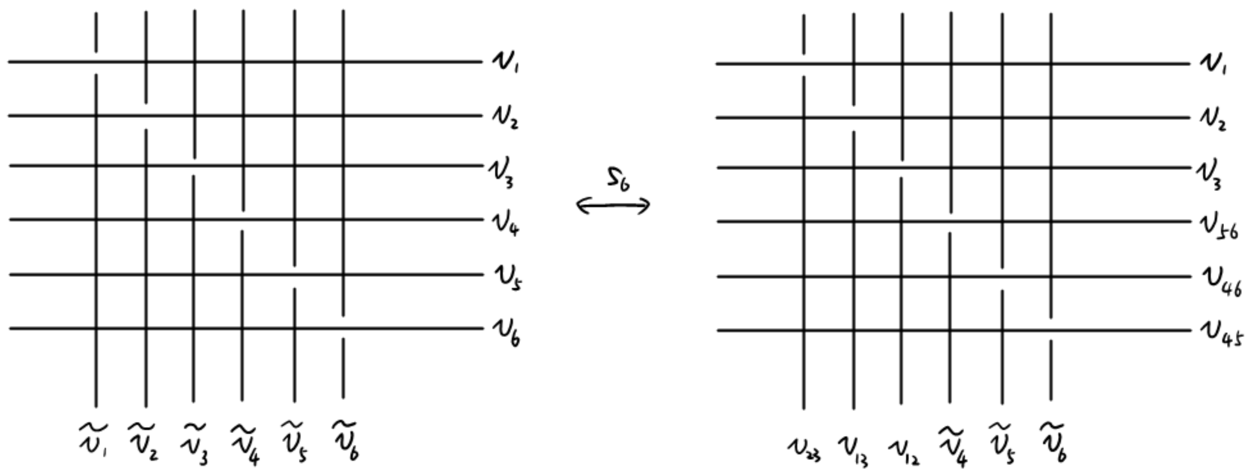
$s_6 v_4 = s_6(e_4+e_7) = \frac{1}{2}(1,1,1,1,-1,-1,1,1)^T = v_{56}$ $\quad s_6 v_5 = v_{46} \quad s_6 v_6 = v_{45}$

$s_6 \tilde{v}_1 = s_6(e_1+e_8) = \frac{1}{2}(1,-1,-1,1,1,1,1,1)^T = v_{23}$ $\quad s_6 \tilde{v}_2 = v_{13} \quad s_6 \tilde{v}_3 = v_{12}$

$s_6 \tilde{v}_4 = s_6(e_4+e_8) = e_4+e_8 = \tilde{v}_4$ $\quad s_6 \tilde{v}_5 = \tilde{v}_5 \quad s_6 \tilde{v}_6 = \tilde{v}_6$

The rest are easy to detemine through the Schläfli double six configuration.

e.g. $s_6 v_{14} = v_{14}$



2. E_7.

– Weights nearest to the origin

There is just one minuscule representations of E_7.

<u>integer version</u>

#	typical coordinates	symbol
28	$(3, 3, -1, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$(-3, -3, 1, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij}$ $= -v_{ij}$

$$\langle v_i, v_j \rangle \in \{24, 8, \underset{\text{edge}}{-8}, -24\}$$

<u>weight lattice version</u>

#	typical coordinates	symbol
28	$\frac{1}{4}(3, 3, -1, -1, -1, -1, -1, -1)^T$	v_{ij}
28	$\frac{1}{4}(-3, -3, 1, 1, 1, 1, 1, 1)^T$	$\tilde{v}_{ij}$ $= -v_{ij}$

$$\langle v_i, v_j \rangle \in \{\tfrac{3}{2}, \tfrac{1}{2}, \underset{\text{edge}}{-\tfrac{1}{2}}, -\tfrac{3}{2}\}$$

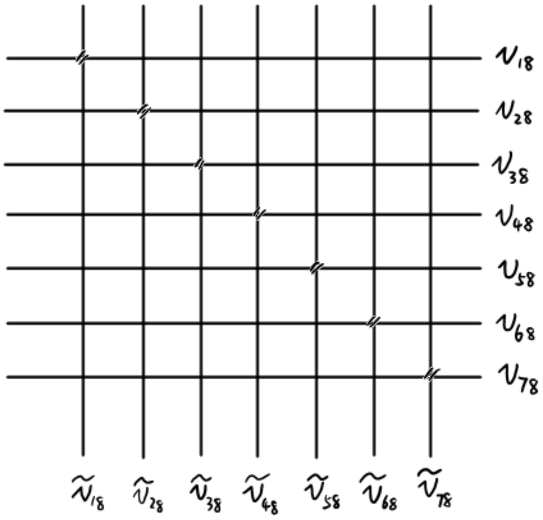
in $\quad \left\{ \sum_{i=1}^{8} z_i = 0 \right\} \cong \mathbb{R}^7$

The graph constructed is called the Gosset graph, which has 56 vertices and 756 edges (with HoG Id 1114).
This graph is also the configuration graph of 56 (-1)-curves on P^2 blowing up 7 points.

$56 = 7 + \binom{7}{2} + \binom{7}{5} + 7$

vertices	$\rightsquigarrow$	lines
edges	$\rightsquigarrow$	intersection points
triangle	$\rightsquigarrow$	triangle

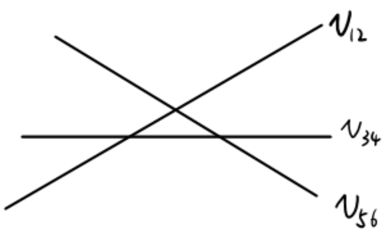
Here are some typical subgraphs:



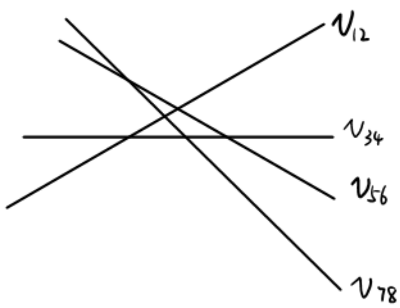
$\{ v_{ij} \}_{i,j}$

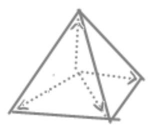
"double seven configuration"
$\# V = 14$ $\quad \# E = 42$
HoG Id = 50584

v_{16} intersect with $v_{25}, v_{35}, v_{45}, v_6 \cdots$
$\# V = 28$ $\quad \# E = 210$
HoG Id = 50698.



triangle
4032 many





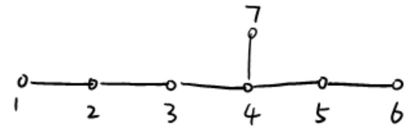
1008 many

in (-1)-curves setting,

$$\langle v_i, v_j \rangle \in \left\{ \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2} \right\}$$

intersection number: $\quad -1 \quad 0 \quad 1 \quad 2$

– Simple roots



$$
\begin{aligned}
&\{ \alpha_1 , \alpha_2 , \alpha_3 , \alpha_4 , \alpha_5 , \alpha_6 , \alpha_7 \} \\
&= \{ v_{18} - v_{28},\ v_{28} - v_{38},\ v_{38} - v_{48},\ v_{48} - v_{58},\ v_{58} - v_{68},\ v_{68} - v_{78},\ -v_{12} - v_{34} \} \\
&= \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} \right\}
\end{aligned}
$$

Ex. Verify that all the 126 roots are given by

#	typical coordinates	symbol
$56 = 8 \cdot 7$	$(1, -1, 0, 0, 0, 0, 0, 0)^T$	$\alpha_{1,-2}$
$70 = \binom{8}{4}$	$(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$	$\alpha_{5,6,7,8}$

- Fundamental weights

For convenient, denote

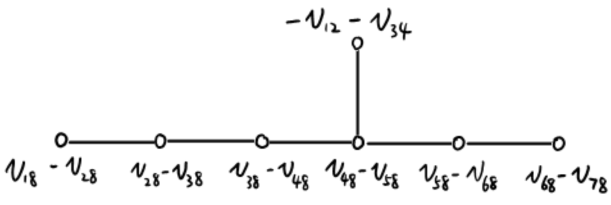
$$v_{j-k} = v_{ij} - v_{ik} = e_j - e_k \quad \text{for some } i = j, k.$$

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6, \varpi_7\}$$

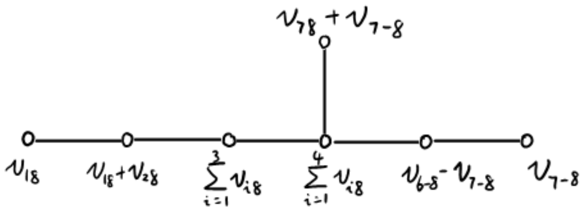
$$= \left\{ v_{18},\ v_{18}+v_{28},\ v_{18}+v_{28}+v_{38},\ \sum_{i=0}^{4} v_{i8},\ v_{6-8}+v_{7-8},\ v_{7-8},\ v_{78}+v_{7-8} \right\}$$

$$= \left\{ \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{4}\begin{pmatrix}2\\2\\-2\\-2\\-2\\-2\\-2\\6\end{pmatrix}, \frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\-4\\-4\\-4\\12\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\-4\\-4\\8\end{pmatrix}, \frac{1}{4}\begin{pmatrix}0\\0\\0\\0\\0\\0\\-4\\4\end{pmatrix}, \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$

$$= \left\{ \frac{1}{4}\begin{pmatrix}3\\-1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\\-1\\-1\\-1\\3\end{pmatrix}, \frac{1}{4}\begin{pmatrix}1\\1\\1\\-3\\-3\\-3\\-3\\9\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\-1\\-1\\-1\\3\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\-1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\0\\0\\-1\\1\end{pmatrix}, \frac{1}{4}\begin{pmatrix}-1\\-1\\-1\\-1\\-1\\-1\\-1\\7\end{pmatrix} \right\}$$



α_i



ϖ_i

- Weyl group action

Using the similar methods like E_6, we get

$$s_k = s_{(k,\, k+1)} \qquad \text{for} \quad i = 1, \dots, 6$$

$$s_7 = \frac{1}{4} \left(\begin{array}{cccc|cccc} 3 & & & -1 & & & & \\ & 3 & & & & 1 & & \\ & & 3 & & & & & \\ -1 & & & 3 & & & & \\ \hline & & & & 3 & & & \\ & 1 & & & & 3 & & -1 \\ & & & & & & 3 & \\ & & & & -1 & & & 3 \end{array} \right)$$

$$s_7\, v_{ij} = \begin{cases} v_{ij} & \text{if} \quad i \in \{1,2,3,4\}\ ,\ \ j = \{5,6,7,8\} \\ \tilde{v}_{kl} & \text{if} \quad \{i,j,k,l\} = \{1,2,3,4\} \text{ or } \{5,6,7,8\} \end{cases}$$