Eine Woche, ein Beispiel

8.3. wall examples

Ref:

[MS19]: Emanuele Macrì, Benjamin Schmidt, Lectures on Bridgeland Stability, https://arxiv.org/abs/1607.01262
[GR87]: A. L. Gorodentsev, A. N. Rudakov, Exceptional vector bundles on projective spaces

Well, the main part of this document aims to solve some results in [MS19].

Recall that [MS19]

$$
\begin{aligned}
ch^B(E) &= ch(E)\cdot e^{-B} \qquad [p14] \\
&= ch_0(E) + (ch_1(E) - B\cdot ch_0(E)) \\
&\quad + (ch_2(E) - B\cdot ch_1(E) + \tfrac{B^2}{2} ch_0(E)) + \cdots
\end{aligned}
$$

in p31 there is a typo

$$
\begin{aligned}
Z_{\omega,B}(E) &= -\int_X e^{-B-i\omega}\, ch(E) \qquad [p31] \\
&= \left(-ch_2^B(E) + \tfrac{\omega^2}{2} ch_0^B(E)\right) + i\omega\cdot ch_1^B(E)
\end{aligned}
$$

$$
\begin{aligned}
\nu_{\omega,B}(E) &= -\frac{\operatorname{Re} Z_{\omega,B}(E)}{\operatorname{Im} Z_{\omega,B}(E)} \\
&= \frac{ch_2^B(E) - \frac{\omega^2}{2}\cdot ch_0^B(E)}{\omega\cdot ch_1^B(E)}
\end{aligned}
$$

$$
\begin{aligned}
Z_{\alpha,\beta}(E) &= Z_{\alpha H, B_0+\beta H}(E) \qquad [p37] \\
\nu_{\alpha,\beta}(E) &= \nu_{\alpha H, B_0+\beta H}(E)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
Z_{\alpha,\beta}(E) &= Z_{\alpha H, B_0+\beta H}(E) \\
&= -\int_X e^{-B_0-\beta H - i\alpha H}\, ch(E) \\
&= -\int_X e^{-B_0-(\beta+i\alpha)H}\, ch(E) \\
&= -ch_2^{B_0}(E) + (\beta+i\alpha)H ch_1^{B_0}(E) - \tfrac{1}{2}(\beta+i\alpha)^2 H^2 ch_0^{B_0}(E) \\
&\overset{\substack{B_0=0 \\ z=\beta+i\alpha}}{\underset{\text{in } \mathbb{P}^2}{=\!=}} -ch_2(E) + z\, ch_1(E) - \tfrac{z^2}{2} ch_0(E)
\end{aligned}
$$

[MS19, Ex 7.3]

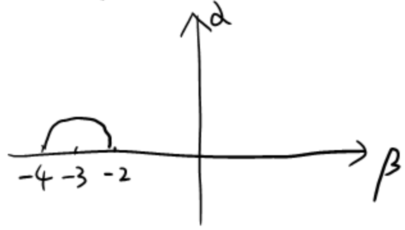
$$\mathrm{ch}(I_Z) = 1 - 4H^2 = (1, 0, -4)$$
$$\mathrm{ch}(\mathcal{O}(-2)) = 1 - 2H + 2H^2 = (1, -2, 2)$$

Therefore,

$$Z_{\alpha,\beta}(I_Z) = 4 - \frac{z^2}{2} = -\frac{1}{2}(z^2 - 8)$$
$$Z_{\alpha,\beta}(\mathcal{O}(-2)) = -2 - 2z - \frac{z^2}{2} = -\frac{1}{2}(z+2)^2$$

$$0 = \operatorname{Im} \frac{z^2-8}{(z+2)^2}$$

$$\Leftrightarrow (z^2-8)(\bar{z}^2+4\bar{z}+4) - (\bar{z}^2-8)(z^2+4z+4) = 0$$
$$\Leftrightarrow (\bar{z}-z)(\quad z\bar{z} + 3(z+\bar{z}) + 8 \quad) = 0$$
$$\Leftarrow \qquad |z|^2 + 6\operatorname{Re} z + 8 = 0$$
$$\Leftrightarrow \qquad \alpha^2 + (\beta+3)^2 = 1$$



Or:
$$(\beta + i\alpha)^2 - 8 \sim (\beta + i\alpha + 2)^2$$
$$\| \qquad\qquad \|$$
$$\beta^2 - \alpha^2 - 8 + 2\alpha\beta i \qquad (\beta+2)^2 - \alpha^2 + 2\alpha(\beta+2) i$$

$$\Leftrightarrow (\beta^2 - \alpha^2 - 8)(\cancel{2\alpha}(\beta+2)) = ((\beta+2)^2 - \alpha^2)\cancel{2\alpha}\beta$$
$$\Leftarrow 2(\beta^2 - \alpha^2 - 8) - 4\beta(\beta+3) = 0$$
$$\Leftrightarrow \qquad \alpha^2 + (\beta+3)^2 = 1$$

In general, in $\mathbb{P}^2$, $(NS(X) = \mathbb{Z} \cdot H,\ H^2 = [p])$

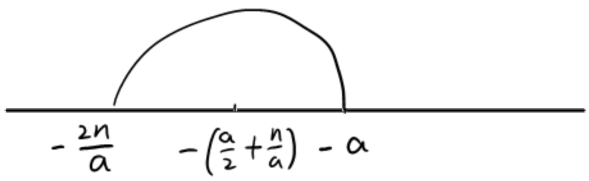
$$ch(\mathcal{O}(-a)) = 1 - aH + \frac{1}{2}a^2H^2 = (1, -a, \frac{1}{2}a^2)$$

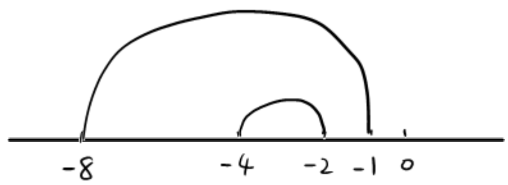
$$Z_{\alpha,\beta}(\mathcal{O}(-a)) = -\frac{1}{2}a^2 - az - \frac{z^2}{2} = -\frac{1}{2}(z+a)^2$$

and the equation

$$\nu_{\alpha,\beta}((1, 0, -n)) = \nu_{\alpha,\beta}(\mathcal{O}(-a))$$

$$\overset{\sim}{\Longleftrightarrow} \quad \alpha^2 + \left(\beta + \frac{a}{2} + \frac{n}{a}\right)^2 = \left(\frac{n}{a} - \frac{a}{2}\right)^2$$





Lemma. [GR87, 4.2]

Let $\mathcal{A}$ be an abelian category, and $F, E, G \in \mathcal{A}$. Assume that we have a SES

$$0 \longrightarrow F \longrightarrow E \longrightarrow G \longrightarrow 0$$

with $[F,G]^0 = [G,F]^2 = 0$. Then

$$[E,E]' \geq [F,F]' + [G,G]'.$$

Proof

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
0 & {[F,F]'} \arrow[l] \arrow[d] & {[E,F]'} \arrow[l] \arrow[d] \arrow[dl, phantom, "\mathrm{Im}\, g"] & {[G,F]'} \arrow[l] \arrow[d] & \\
 & {[F,E]'} \arrow[d] & {[E,E]'} \arrow[l, "g"'] \arrow[d] & {[G,E]'} \arrow[l, "g"'] \arrow[d] \arrow[dl, phantom, "\mathrm{Im} f"] & \\
 & {[F,G]'} & {[E,G]'} \arrow[l] & {[G,G]'} \arrow[l] \arrow[d] & 0 \arrow[l] \\
 & & & 0 &
\end{tikzcd}$$

$$\begin{aligned}
[E,E]' &= \dim \mathrm{Im}\, g + \dim \mathrm{Im} f \\
&\geq [F,F]' + [G,G]'
\end{aligned}$$