Eine Woche, ein Beispiel

8.6. Kottwitz set

This document is a continuation of [23.08.06].
Reorganized from Luozi Shi (and his partners)'s talk.

Recall that $\widehat{\mathbb{Q}_p^{ur}} = \mathrm{Frac}(\widehat{\mathbb{Z}_p^{ur}})$, $\widehat{\mathbb{Z}_p^{ur}} = W(\overline{\mathbb{F}_p})$. Here "$\wedge$" is completion w.r.t. valuation.

Setting. In this document, F is a NA local field,

$$L := \widehat{F^{ur}} \overset{\text{if } F:\text{ p-adic}}{=\!=} \mathrm{Frac}(W(\overline{k_F}) \otimes_{W(k_F)} \mathcal{O}_F).$$

Def For G/F reductive, the Kottwitz set $B(G)$ is defined as

$$\begin{aligned} B(G) :&= H^1(W_F, G_{\overline{F}}) \\ &\cong H^1(\langle\sigma\rangle, G_L) \qquad \text{by Inf-Res seq \& } H^1(I_F, G_{\overline{F}}) = 0 \\ &\cong G(L)/\sigma\text{- twisted } G(L)\text{-conj} \\ &\overset{\text{when } G = GL_n}{\simeq} \mathrm{Isoc}/\sim \end{aligned}$$

$\uparrow$ Isocrystals.

E.g. $B(\mathbb{G}_m) \cong \mathbb{Z}$

Proof. The map

$$\beta: \mathcal{O}_L^\times \longrightarrow \mathcal{O}_L^\times \qquad x \longmapsto x \cdot \sigma(x)^{-1}$$

is surjective, since

$$\begin{aligned} \beta_1 &: k_L^\times \longrightarrow k_L^\times & x &\longmapsto x^{1-q} \\ \beta_2 &: k_L \longrightarrow k_L & x &\longmapsto x - x^q \end{aligned}$$

are surjective. ($k_L \cong \overline{\mathbb{F}_p}$ is alg closed)

Then the well-defined morphism

$$v: L^\times / \mathrm{Im}\,\beta \twoheadrightarrow \mathbb{Z} \qquad x \longmapsto v(x)$$

is injective, thus an iso.

Ex. Check that the SES

$$1 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \longrightarrow 1$$

induce LES in gp cohomology:

$$\begin{tikzcd}
 & B(\underline{\mathbb{Z}/2\mathbb{Z}}) \arrow[r] & B(\mathbb{G}_m)^{= \mathbb{Z}} \arrow[r, "\times 1?", "\times 2?"'] & B(\mathbb{G}_m)^{= \mathbb{Z}} \\
0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & F^\times \arrow[r, "(-)^2"] & F^\times \arrow[ull, bend right]
\end{tikzcd}$$

where

$$\begin{aligned}
B(\underline{\mathbb{Z}/2\mathbb{Z}}) &:\overset{\text{def}}{=} H^1(W_F, \mathbb{Z}/2\mathbb{Z}) \qquad W_F \circlearrowright \mathbb{Z}/2\mathbb{Z} \text{ trivially} \\
&= \mathrm{Hom}(W_F, \mathbb{Z}/2\mathbb{Z}) \\
&\cong \{H \triangleleft W_F \text{ closed with index } 2\} \cup \{0\} \\
&\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 2 \end{cases}
\end{aligned}$$