Eine Woche, ein Beispiel

1.30 Tits system

For many time I want to understand Tits system, I always see the reference to Bourbaki's work. But I believe that the proof can be shown more elegant.

Ref: [Building] Buildings, by Peter Abramenko and Kenneth S. Brown

Def. (Tits system, BN-pair)

(G, B, N, S) gp + (subgp, subgp) + gen of $W := N/_{B\cap N}$

(TS0) $G = \langle B, N \rangle_{Grp}$

(TS1) $T \triangleleft N$ where $T := B \cap N$

(TS2) $W = \langle s \in S \rangle_{Grp}$ + s of order 2 $\forall s \in S$ where $W := N/T$

(TS3) $BwsB \subseteq BwB \cdot BsB \subseteq BwsB \cup BwB$

(TS4) $BsB \cdot BsB = B \cup BsB$

Saturated Tits system: e.g. of non-Satuated?

(TS5) $T = \bigcap_{g \in N} gBg^{-1}$ $\subseteq$ is obvious, need $\supseteq$

Notation.

W : Weyl gp of the Tits system

$s \in S$: simple reflections

$l(w) := \min\{r \mid w = s_1 \cdots s_r,\ s_i \in S\}$: length of $w \in W$

For example of Tits system, see wiki or [Prasad, Eg 1.4.3].

Basic results of Tits system

Prop.
① $(BwB)^{-1} = Bw^{-1}B$
② (TS3') $BswB \subseteq BsB \cdot BwB \subseteq BswB \cup BwB$
③ (TS3'') $Bww'B \subseteq BwB \cdot Bw'B \subseteq \bigcup_{l(x) \leq l(w)+l(w')} BxB$
④ (TS4') $BwB \cdot BsB \cdot BsB = BwB \cup BwsB$

Proof. ① : by def.

②. by (TS3),

$$Bw^{-1}B \cdot BsB \subseteq Bw^{-1}sB \cup Bw^{-1}B$$

Apply $(-)^{-1}$ to both sides.

③. by (TS3).

④ by (TS3)+(TS4),

$$BwB \cup BwsB \subseteq BwB(B \cup BsB) = BwB \cdot BsB \cdot BsB \subseteq BwB \cup BwsB.$$

Thm (Bruhat decomposition)

$$G = \bigsqcup_{w \in W} BwB \qquad (5)$$

left...

(5) is proved by the following two lemmas.

Lemma: $G = BNB$ by (TS0) + (TS3'')

Lemma: $BwB = Bw'B \Rightarrow w = w'$

Idea. If $w, w' \in S$, then

$$\begin{aligned} B = Bw'^2B &\subseteq Bw'B \cdot Bw'B \\ &= BwB \cdot Bw'B \\ &\subseteq Bww'B \cup BwB \end{aligned}$$

$\Rightarrow ww' = Id$ or $\cancel{w = Id}$

$\Rightarrow w = w'$

Proof. Induction on $\min(l(w), l(w'))$. w.l.o.g. $l(w) \geq l(w')$

$l(w') = 0$: $BwB = B \Rightarrow \tilde{w} \in B$, $w = Id$

$l(w') = k+1$: write $w' = ys$ for some $y, s \in W$, $l(y) = k$, $l(s) = 1$.

$$\begin{aligned} ByB = Bw's B &\subseteq Bw'B \cdot BsB \\ &= BwB \cdot BsB \\ &\subseteq BwsB \cup BwB \end{aligned}$$

$\Rightarrow ByB = BwsB$ or $ByB = BwB$

$\Rightarrow y = ws$ or $\cancel{y = w}$ (but $l(y) < l(w)$)

$\Rightarrow w' = w$ $\square$

Prop. (TS3 + length)

$$BwB \cdot BsB = \begin{cases} BwsB & l(ws) \geq l(w) \quad (6) \\ BwsB \cup BwB & l(ws) < l(w) \quad (7) \end{cases}$$

Idea for (6): If $w = t \in S$, then

$$BtB \cdot BsB \overset{(TS3)}{\subseteq} BtsB \cup BtB$$

$$BtB \cdot BsB \overset{(TS3')}{\supseteq} BtsB \cup BsB$$

Since $l(ts) \geq l(t)$, $t \neq s \Rightarrow BtB \cdot BsB \subseteq BtsB$.

Proof for (6)+(6)': induction on $l(w)$.

(6)': $BsB \cdot BwB = BswB$ when $l(sw) \geq l(w)$

$l(w) = 0$: ✓

$l(w) = k+1$: just show (6). Write $w = ty$ for some $y, t \in W$. $l(y) = k$, $l(t) = 1$

$\Rightarrow l(ys) \cancel{= k+1} \geq l(y)$

$$BtB \cdot ByB \cdot BsB \overset{(6)' \text{ ind}}{=} BtyB \cdot BsB \overset{(TS3)}{\subseteq} BtysB \cup BtyB$$

$$BtB \cdot ByB \cdot BsB \overset{(6) \text{ ind}}{=} BtB \cdot BysB \overset{(TS3')}{\subseteq} BtysB \cup BysB$$

Since $ty \neq ys$ (Otherwise $l(tys) = l(ys^2) = l(y) = k \Rightarrow l(ws) < l(w)$, contradiction!),

$BtB \cdot ByB \cdot BsB \subseteq BtysB \Rightarrow BwBBsB = BwsB$. $\square$

Cor. Either $l(ws) > l(w)$ or $l(ws) < l(w)$.

[Proof. Otherwise. $Bw \cdot s \cdot s \cdot B \overset{⑥}{=} BwsB \cdot BsB \overset{⑥}{=} BwB \cdot BsB \cdot BsB \overset{(TS4)}{=} BwB \cdot (B \cup BsB) \ni ws$ ↯]

Cor. For $l(ww') = l(w) + l(w')$, we have

$$BwB\, Bw'B = Bww'B \qquad ⑧$$

Prop. S is unique, i.e.,

$$(G, B, N, S),\ (G, B, N, S') \text{ are Tits systems} \Rightarrow S = S' \qquad ⑨$$

Proof. W.l.o.g. suppose $S \subseteq S'$. $(G, B, N, S \cup S')$ is also a Tits system.

Only need to show: In (G, B, N, S), $\forall\, w \in W$, $w^2 = 1$, $l(w) > 1$,

$$BwB \cdot BwB \not\subseteq B \cup BwB.$$

Write $w = ys$ $\quad l(y) = l(w) - 1$, $l(s) = 1$.

$$BysB \cdot BsB = ByBBsBBsB = ByB \cup BysB$$

$\Rightarrow \exists\, b \in B$ s.t. $wbs \in BwB$

$\Rightarrow s \in BwB \cdot BwB$ but $s \notin B \cup BwB$. $l(s) \neq l(w)$ $\square$

Rmk. See [buildings, Lemma 6.4.1] for a generalization of ⑨. In ptc, one can see that, write $t = s_1 \cdots s_r$ with $l(t) = r$,

$$\langle BtB \rangle = \langle B, tBt^{-1} \rangle = \langle Bs_iB \rangle_i$$

We may come back to this remark when we mention about parabolic subgps.

Prop. (W, S) is a Coxeter gp.

Proof. Only need to show the folding condition.

For $w \in W$, $s, t \in S$ s.t. $l(tw) = l(w) + 1$, $l(ws) = l(w) + 1$, $l(tws) \neq l(w) + 2$,

$$BtB \cdot BwB \cdot BsB \begin{cases} = BtwB \cdot BsB = BtwsB \cup BtwB \\ = BtB \cdot BwsB = BtwsB \cup BwsB \end{cases}$$

$\Rightarrow tw = ws \Rightarrow tws = w$ $\square$

Remaining:

- Saturated Tits system, see 2421047.
- Is N the normalizer of T?

- Tits's simplicity theorem

  (G, B, N, S) B: solvable

  $\bigcap_{g\in G} gBg^{-1} = Id$

  The Coxeter graph of (W, S) is connected

Then

$$G \text{ is simple} \iff G \text{ is perfect, i.e., } G = [GG]$$

https://en.wikipedia.org/wiki/(B,_N)_pair