Eine Woche, ein Beispiel

11.19. Basic sheaf calculation

Goal: Motivate $f_*, f^*, f_!, f^!$ by connecting them with (co)homology theory

⇝ calculation of $\text{Perv}_\Delta(\mathbb{CP}^1)$
⇝ generalize Morse theory
⇝ Characteristic classes /cycles
⇝ index theorem

Minor advantages from my talk:

- offers examples for derived category.
  (more geometrical compared with examples about quiver reps)
- the first step toward 6-fctor formalism:
  - formal nonsense: adjointness, open-closed, SES(triangles)
  - application: Riemann-Roch, Serre duality, index theorem (guess)
    ⇝ understand cpt RS, Weil conj, ...
  - glue: open-closed, cellular fibration, Morse theory, ...
    covering: (étale)descent, ramification, ...
    Three types: closed immersion, submersion, covering.

Usual setting: $X \in \text{Top}$
$\text{Obj}(\text{Sh}(x)) = \{$sheaves of abelian gps$\}$
e.p. $\text{Sh}(\{*\}) = \text{Abel}$ $\mathbb{Q}_{\{*\}} \longleftrightarrow \mathbb{Q}$

0. sheaf
1. f_* , skyscraper sheaf & global sections
2. f^* , constant sheaf & stalks
3. Rf_* & cohomology
4. $f_!$ & global sections with cpt supp
5. $Rf_!$ & cohomology with cpt supp
6. $f^!$ & homology
   $- \otimes -$ & product structure on cohomology
   $\text{Hom}(-,-)$ & Poincaré duality.

Ref:

[Vakil] Vakil, The Rising Sea: Foundations of Algebraic Geometry, 2016

0. Sheaf

Recall the definition of
- presheaf $\qquad\qquad\qquad \mathcal{F}$
- sheaf $\qquad\qquad\qquad\quad \mathcal{F}$
- stalk $\qquad\qquad\qquad\quad\; \mathcal{F}_x$
- global section $\qquad\quad\; \mathcal{F}(X) = \Gamma(X; \mathcal{F}) = H^0(X; \mathcal{F})$
- cohomology $\qquad\qquad\qquad R^n\Gamma(X; \mathcal{F}) = H^n(X; \mathcal{F})$

Recall examples of sheaves:

complicated
- $\mathcal{C}_x$: sheaf of cont fcts on X
- $\mathcal{O}_x$: structure sheaf on X $\qquad$ e.g., X: (cplx) mfld, scheme, $\cdots$
- $\underline{\mathbb{Q}}_x$: constant sheaf on X

- $\mathrm{sky}_p(\mathbb{Q})$: skyscraper sheaf of $p \in X$ on X.

Ex. For $X = \mathbb{C}$ as cplx mfld, $x = 0$, compute

$$(\underline{\mathbb{Q}}_x)_x \subseteq (\mathcal{O}_x)_x \subseteq (\mathcal{C}_x)_x \qquad \& \quad (\mathrm{sky}_p(\mathbb{Q}))_x .$$

1. f_*, skyscraper sheaf & global sections

Setting $X, Y \in Top$. $\mathcal{F} \in Sh(Y)$, $f: Y \rightarrow X$ cont

Def. $f_* \mathcal{F} \in Sh(X)$ is given by
$$f_* \mathcal{F}(u) = \mathcal{F}(f^{-1}(u))$$
This defines a fctor
$$f_*: Sh(Y) \longrightarrow Sh(X)$$

$$\mathcal{F} \qquad f_* \mathcal{F}$$
$$\downarrow \qquad \downarrow$$
$$Y \xrightarrow{f} X$$
$$\cup$$
$$u$$

E.g. For $p \in X$, $\iota_p: \{p\} \hookrightarrow X$, $\iota_{p*} \underline{\mathbb{Q}}_{\{p\}} = sky_p(\mathbb{Q})$

For $\pi: Y \longrightarrow \{*\}$, $\pi_* \mathcal{F} = \mathcal{F}(Y) = \Gamma(Y; \mathcal{F})$

Ex (hard?) For $j: \mathbb{C} \longrightarrow \mathbb{CP}^1$, compute $j_* \underline{\mathbb{Q}}_{\mathbb{C}}$.

○ It is a constant sheaf on $\mathbb{CP}^1$.

○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = \mathbb{Q}$.

○ It is not a constant sheaf on $\mathbb{CP}^1$, and $(j_* \underline{\mathbb{Q}}_{\mathbb{C}})_\infty = 0$.

○ All the above is wrong.

○ I don't know, but I don't want to make a wrong choice.

2. f^*, constant sheaf & stalks

In [Vakil, Chapter 2], it is f^{-1}, the inverse image functor.

Setting $X, Y \in \text{Top}$, $\mathcal{F} \in \text{Sh}(X)$, $f: Y \longrightarrow X$ cont

Def. $f^*\mathcal{F} \in \text{Sh}(Y)$ is given by sheafification of

$$f^{*,\text{pre}}\mathcal{F}(\mathcal{U}) = \varinjlim_{f(\mathcal{U}) \subseteq V} \mathcal{F}(V)$$

This defines a fctor

$$f^*: \text{Sh}(X) \longrightarrow \text{Sh}(Y)$$

$$
\begin{array}{ccc}
f^*\mathcal{F} & & \mathcal{F} \\
| & & | \\
Y & \xrightarrow{f} & X \\
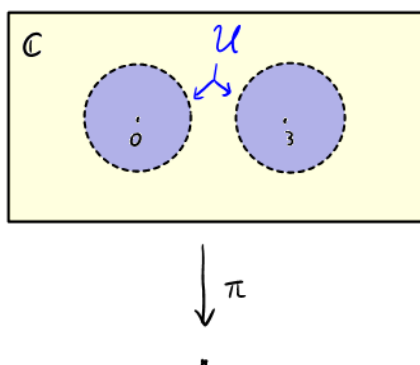\cup & & \\
\mathcal{U} & &
\end{array}
$$

Recall:

$$\mathcal{F}^{sh}(\mathcal{U}) = \left\{ (x_p)_p \in \prod_{p \in \mathcal{U}} \mathcal{F}_p \;\middle|\; \begin{array}{l} \forall x_0 \in \mathcal{U}, \exists \, \mathcal{U}_{x_0} \subseteq \mathcal{U} \text{ nbhd of } x_0, \\ s \in \mathcal{F}(\mathcal{U}) \text{ st.} \\ s_p = x_p \qquad \forall p \in \mathcal{U}_{x_0} \end{array} \right\}$$

By definition, $(\mathcal{F}^{sh})_p = \mathcal{F}_p$.

Universal property:

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{f \in \text{Mor}_{\text{PSh}}} & \mathcal{G} \qquad\qquad \mathcal{G} : \text{sheaf} \\
sh \downarrow & \nearrow{\scriptstyle G} & \\
\mathcal{F}^{sh} & \exists! \, f^{sh} \in \text{Mor}_{\text{Sh}} &
\end{array}
$$

Ex. For $\pi : \mathbb{C} \longrightarrow \{*\}, U = B_1(0) \cup B_1(3)$, which one is correct:

○ $(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.

○ $(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$.

○ $(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.

○ $(\pi^{*,\text{pre}}\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$, $\quad (\pi^*\underline{\mathbb{Q}}_{\{*\}})(U) = \mathbb{Q}^2$.

○ All the above is wrong.

E.g. For $p \in X$, $\quad \iota_p : \{p\} \hookrightarrow X, \quad \iota_p^* \mathcal{F} \;=\; \mathcal{F}_p$

For $\qquad\qquad\quad \pi : Y \longrightarrow \{*\}, \quad \pi^* \underline{\mathbb{Q}}_{\{*\}} \;=\; \underline{\mathbb{Q}}_Y$

For $\mathcal{U} \subset X$ open, $\quad j : \mathcal{U} \longrightarrow X, \quad j^* \mathcal{F} \;=\; \mathcal{F}|_{\mathcal{U}}$

People generalize the last notation to arbitrary subset:

For $Y \subset X$, $\qquad\qquad \iota_Y : Y \hookrightarrow X, \quad \iota_Y^* \mathcal{F} \;\overset{\Delta}{=}\; \mathcal{F}|_Y$

Q: For $\mathcal{U} \subset X$ open, how to express $\mathcal{F}(\mathcal{U})$ by fctors?

A:

$$\mathcal{U} \overset{\iota_{\mathcal{U}}}{\hookrightarrow} X$$
$$\pi_{\mathcal{U}} \searrow \quad \swarrow \pi_X$$
$$\{*\}$$

$$\mathcal{F}(\mathcal{U}) = \pi_{\mathcal{U},*} \underbrace{\iota_{\mathcal{U}}^* \mathcal{F}}_{\mathcal{F}|_{\mathcal{U}}}$$

$$\begin{array}{cc} \mathcal{G} & \mathcal{F} \\ | & | \\ Y & \xrightarrow{f} X \end{array}$$

Prop. One has the adjunction $f^* \dashv f_*$, i.e.,

$$\text{Mor}_{Sh(Y)}(f^*\mathcal{F}, \mathcal{G}) \cong \text{Mor}_{Sh(X)}(\mathcal{F}, f_*\mathcal{G}) \qquad + \text{naturality}$$

Hint. [Vakil, 2.7.B] Show that both side give the same information, i.e.,

$$\phi_{uv} \in \text{Mor}_{Ab}(\mathcal{F}(U), \mathcal{G}(V)) \qquad \begin{array}{l} \text{for each pair } (V, U) \\ \text{s.t. } f(V) \subset U \end{array}$$

$+ \text{compatability}$

Cor. f^* is right adjoint, f_* is left adjoint.

Rmk. f^* is an exact functor.
  Hint: exactness can be checked on stalks!
⚠ After "polished" (because of the structure sheaf), f^* is again only right adjoint.

3. Rf_* & cohomology

Recall that cohomology is usually a derived object:

- It is (often) computed by resolutions;
- Input $\mathcal{F}$, output a complex (before Ker/Im procedure)
- SES induces LES: for
$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$
one has

$$\curvearrowright H^2(X;\mathcal{F}) \longrightarrow \cdots$$

$$\curvearrowright H^1(X;\mathcal{F}) \longrightarrow H^1(X;\mathcal{G}) \longrightarrow H^1(X;\mathcal{H})$$

$$0 \longrightarrow \underset{\overset{\shortparallel}{\pi_*\mathcal{F}}}{H^0(X;\mathcal{F})} \longrightarrow \underset{\overset{\shortparallel}{\pi_*\mathcal{G}}}{H^0(X;\mathcal{G})} \longrightarrow \underset{\overset{\shortparallel}{\pi_*\mathcal{H}}}{H^0(X;\mathcal{H})} \qquad \pi: X \longrightarrow \{*\}$$

- can be viewed as right derived fctor of
$$H^0(X,-) = \Gamma(X,-) = \pi_*$$
one gets
$$H^n(X,-) = R^n\Gamma(X,-) = R^n\pi_*$$
We denote the complex (before the Ker/Im procedure) as
$$R\Gamma(X,-) = R\pi_*$$
up to homotopy equiv & quasi-iso, i.e., in the derived category of $\{*\}$.

$$\mathcal{D}(X) = \mathcal{D}(\mathrm{Sh}(X)) = \text{"derived category of sheaves over } X\text{"}$$
$$= \text{" complexes of sheaves over } X, \text{ up to } \cdots \text{"}$$
$$= \{ \cdots \longrightarrow \mathcal{F}^{-2} \longrightarrow \mathcal{F}^{-1} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots \} \overset{\wedge}{=} \{\mathcal{F}^{\cdot}\}$$

Setting $X, Y \in \text{Top}$, $\mathcal{F} \in \text{Sh}(Y)$, $f: Y \longrightarrow X$ cont

Def. $Rf_* \mathcal{F}$ = "derived pushforward of $\mathcal{F}$"
$$= f_* \mathcal{I}^\bullet$$

$$\left[\begin{array}{l} \text{Here, } \mathcal{I}^\bullet \text{ is the injective resolution of } \mathcal{F}: \\ \quad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \cdots \\ \qquad\qquad (\Rightarrow \mathcal{F} \xrightarrow{\text{quasi-iso}} \mathcal{I}^\bullet) \end{array}\right]$$

$$\begin{array}{ccc} \mathcal{F} & & Rf_* \mathcal{F} \\ | & & | \\ Y & \xrightarrow{f} & X \\ & & \cup \\ & & \mathcal{U} \end{array}$$

This defines a fctor
$$Rf_* : \mathcal{D}^b(Y) \longrightarrow \mathcal{D}(X)$$

The derived pushforward is hard to compute.
just like cohomology, and even worse, since we need more information
Luckily, the following proposition helps us to cheat a little bit.

Prop. [Vakil, 18.8, p497]
$R^n f_* \mathcal{F}$ is given by the sheafification of
$$(R^n f_*^{pre} \mathcal{F})(\mathcal{U}) = H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$
$\qquad\qquad\qquad\qquad\qquad\qquad \hookleftarrow$ sometimes omit

e.p. one can compute the stalk
$$(R^n f_* \mathcal{F})_x = \varinjlim_{x \in \mathcal{U}} H^n(f^{-1}(\mathcal{U}), \mathcal{F}|_{f^{-1}(\mathcal{U})})$$

$$\begin{array}{c} \mathcal{F} \\ | \end{array}$$

Cor For $\pi: X \longrightarrow \{*\}$,
$$R^n \pi_* \mathcal{F} = H^n(X; \mathcal{F})$$

E.g. For $\pi: \mathbb{CP}^1 \longrightarrow \{*\}$,
$$R^n \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = H^n(\mathbb{CP}^1; \mathbb{Q}) = \begin{cases} \mathbb{Q} & n = 0, 2 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\hookleftarrow$ [all objects in $\mathcal{D}(*)$ are proj, we work over $\mathbb{Q}$]

$$R \pi_* \underline{\mathbb{Q}}_{\mathbb{CP}^1} = \mathbb{Q} \oplus \mathbb{Q}[-2]$$

$$= \left[0 \longrightarrow \cdots \longrightarrow \underset{-1}{\mathbb{Q}} \longrightarrow \underset{0}{0} \longrightarrow \underset{1}{\mathbb{Q}} \longrightarrow \underset{2}{0} \longrightarrow \underset{3}{\cdots} \right]$$

Ex.

For $j : \mathbb{C} \longrightarrow \mathbb{CP}^1$, what is true about $Rj_*\underline{\mathbb{Q}}_\mathbb{C}$?

○ $(R^1 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = 0,$ $(R^2 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = \mathbb{Q}.$

○ $(R^1 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = \mathbb{Q},$ $(R^2 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = 0.$

○ $(R^1 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = 0,$ $(R^2 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = 0.$

○ $(R^1 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = \mathbb{Q},$ $(R^2 j_*\underline{\mathbb{Q}}_\mathbb{C})_\infty = \mathbb{Q}.$

○ What the hell is that?

In fact, $\left(Rj_*\underline{\mathbb{Q}}_\mathbb{C}\right)_\infty = \mathbb{Q} \oplus \mathbb{Q}[-1].$

$i : \{\infty\} \longrightarrow \mathbb{CP}^1$ is exact, so $Ri_* = i_*.$

4. $f_!$, extension by zeros & global sections with cpt supp

$$
\begin{array}{ccc}
\mathcal{F} & & f_!\mathcal{F} \\
| & & | \\
Y & \xrightarrow{\;f\;} & X \\
& & \cup \\
& & \mathcal{U}
\end{array}
$$

Setting $X, Y \in Top.$ $\mathcal{F} \in Sh(Y)$, $f: Y \to X$ cont

Def. $f_!\mathcal{F} \in Sh(X)$ is given by

$$f_!\mathcal{F}(\mathcal{U}) = \underbrace{\{s \in \mathcal{F}(f^{-1}(\mathcal{U})) \mid}_{(f_*\mathcal{F})(\mathcal{U})} f|_{supp(s)} : supp(s) \longrightarrow \mathcal{U} \text{ is proper}\}$$

This defines a fctor
$$f_! : Sh(Y) \longrightarrow Sh(X)$$

Recall : $supp(s) = \overline{\{x \in f^{-1}(\mathcal{U}) \mid s_x \neq 0\}}$
proper : preimage of cpt set is cpt.

Rmk. By def, $(f_!\mathcal{F})(\mathcal{U}) \subseteq (f_*\mathcal{F})(\mathcal{U})$, one has natural transformation $f_! \to f_*$.
When f is proper, $f_! = f_*$.

https://en.wikipedia.org/wiki/Borel%E2%80%93Moore_homology
https://mathoverflow.net/questions/249342/two-points-of-view-about-borel-moore-homology