Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

Task.

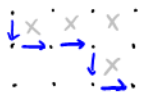
1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. tangent space, Euler class
7. basis of Hecke alg

We may use two examples for the convenience of presentation.
Readers can easily distinguish them by the dim vectors.

1. dimension vector

$|d| = 5$

$d = (3,2)$

$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,0 \\ 0,0 \end{pmatrix} =$  $=$  $=$  $\in W_d \backslash \mathbb{W}_d$ or $Min(\mathbb{W}_{|d|}, W_d)$

Young Tableax $\quad r_\varpi = \pi_{\underline{d}}^{-1}(F_\varpi)$

2. Weyl group

Set	element	special element	others
$\mathbb{W}_{\|d\|} = S_5$	ϖ	$\varpi_{max} =$	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{max} =$	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_d \backslash \mathbb{W}_{\|d\|} = S_3 \times S_2 \backslash S_5$	$\varpi, \underline{d}$		$(Comp_d)$
$Min(\mathbb{W}_{\|d\|}, W_d) = \{$  $, \cdots\}$	u		(Shuffled)

$$\begin{tikzcd}
 & & Min(\mathbb{W}_{|d|}, W_d) \arrow[dl] \arrow[d, "\cong"] & & u \arrow[d, mapsto] \\
0 \arrow[r] & W_d \arrow[r] & \mathbb{W}_{|d|} \arrow[r] & \mathbb{W}_{|d|} \backslash W_d \arrow[r] & 0 \qquad \varpi = wu \arrow[r, mapsto] & \underline{d}
\end{tikzcd}$$

$\varpi =$ 

$u =$

$w =$

Another example: $\quad d = (1,2) \qquad \begin{matrix} a \longrightarrow b \\ \langle v_1 \rangle \longrightarrow \langle v_2, v_3 \rangle \end{matrix}$

$\varpi = wu$					w	$\underline{d}$, u	order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	B_ϖ	ϖB_ϖ^{-1}
Id	Id	$\binom{1\,2\,3}{1\,2\,3}$				abb	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & * \\ & & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & * \\ & & * \end{bmatrix}$
t	(23)	$\binom{1\,2\,3}{1\,3\,2}$				abb	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ & * & \\ & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ & * & * \end{bmatrix}$
s	(12)	$\binom{1\,2\,3}{2\,1\,3}$				bab	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & & * \\ * & * & * \\ & & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & * \\ & & * \end{bmatrix}$	$\begin{bmatrix} * & & * \\ & * & \\ & & * \end{bmatrix}$
ts	(132)	$\binom{1\,2\,3}{3\,1\,2}$				bab	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & \\ & * & \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & \\ & * & \\ & & * \end{bmatrix}$
st	(123)	$\binom{1\,2\,3}{2\,3\,1}$				bba	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & & \\ * & * & * \\ * & & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & * \\ & & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ * & & * \end{bmatrix}$
sts	(13)	$\binom{1\,2\,3}{3\,2\,1}$				bba	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & & \\ * & * & \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ & * & \\ & * & * \end{bmatrix}$	$\begin{bmatrix} * & & \\ * & * & \\ & & * \end{bmatrix}$

3. alg group & Lie algebra

$\mathbb{G}_{|d|}, \mathbb{B}_{|d|}, \mathbb{T}_{|d|}, \mathbb{N}_{|d|}$ $\qquad \mathbb{G}_{|d|} = \mathbb{N}_{|d|}/\mathbb{T}_{|d|}$ $\qquad GL_5(\mathbb{C})$ $= \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$

G_d, B_d, T_d, N_d $\qquad G_d = N_d/T_d$ $\qquad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right)$

$$\mathbb{B}_\varpi = \varpi \mathbb{B}_{|d|} \varpi^{-1} = \mathrm{Stab}_{\mathbb{G}_{|d|}}(F_\varpi)$$

$$B_\varpi \overset{\varpi = wu}{=\!=} w B_d w^{-1} = \mathrm{Stab}_{G_d}(F_\varpi) \qquad N_\varpi = R_u(B_\varpi)$$

For $s \in \Pi$ s.t. $\varpi s \varpi^{-1} \in W_d$ (i.e. $W_d \varpi = W_d \varpi s$), define

$$P_{\varpi, \varpi s} :\overset{\varpi = wu}{=\!=} w(B_d u s u^{-1} B_d \cup B_d) w^{-1} \qquad N_{\varpi,\varpi s} = R_u(B_{\varpi,\varpi s})$$

$$= B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi \qquad = N_\varpi \cap N_{\varpi s}$$

$$M_{\varpi,\varpi s} = N_\varpi / N_{\varpi,\varpi s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{u(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\varpi,\varpi'} := N_\varpi \cap N_{\varpi'}$$

$$M_{\varpi,\varpi'} := N_\varpi / N_{\varpi,\varpi'}$$

Their Lie algebras are collected here.

$\mathbb{g}_{|d|}, \mathbb{b}_{|d|}, \mathbb{t}_{|d|}, \mathbb{n}_{|d|}$

$\mathfrak{g}_d \quad \mathfrak{b}_d \quad \mathfrak{t}_d \quad \mathfrak{n}_d$

$\mathbb{b}_\varpi$		$\mathbb{b}^-_\varpi$		$\mathbb{b}^-_\varpi = \mathbb{b}_{\varpi_{max}\varpi}$
$\mathfrak{b}_\varpi$	$\mathfrak{n}_\varpi$	$\mathfrak{b}^-_\varpi$	$\mathfrak{n}^-_\varpi$	$\mathfrak{b}^-_\varpi = \mathfrak{b}_{w_{max}\varpi}$ $\cdots$
$\mathfrak{p}_{\varpi,\varpi s}$	$\mathfrak{n}_{\varpi,\varpi'}$	$\mathfrak{p}^-_{\varpi,\varpi s}$	$\mathfrak{n}^-_{\varpi,\varpi'}$	$\mathfrak{p}^-_{\varpi,\varpi s} = \mathfrak{p}_{w_{max}\varpi, w_{max}\varpi s}$ $\cdots$
	$\mathfrak{m}_{\varpi,\varpi'}$		$\mathfrak{m}^-_{\varpi,\varpi'}$	$\cdots$

$$\mathrm{Rep}_d(Q) := \prod_{e \in Q_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathbb{g}_{|d|}$$

in general case, lies in $\mathbb{g}_{|d|}^{\oplus k}$

$$r_\varpi = \{ f \in \mathrm{Rep}_d(Q) \mid f.F_{\varpi,i} \subseteq F_{\varpi,i} \} = \mu_{\underline{d}} \pi_{\underline{d}}^{-1}(F_\varpi)$$

$$= \begin{array}{c|ccc} & v_3 & v_1 & v_2 \\ \hline v_5 & * & * & * \\ v_4 & & & * \end{array} = \begin{array}{c|ccc} & v_1 & v_2 & v_3 \\ \hline v_4 & & * & \\ v_5 & * & * & * \end{array}$$

$v_{w(i)}$

$$r_{\varpi,\varpi'} = r_\varpi \cap r_{\varpi'}$$

$$\mathfrak{d}_{\varpi,\varpi'} = r_\varpi / r_{\varpi,\varpi'}$$

4. typical variety

Id corres to

$\mathcal{F}_{\lvert d\rvert} \cong G_{\lvert d\rvert}/B_{\lvert d\rvert}$	F_{Id}
$\mathcal{F}_{\underline{d}} \cong G_d/B_{\underline{d}}$	F_u
$\mathcal{F}_{\varpi} \cong G_d/B_{\varpi}$	F_{ϖ}
$\mathcal{F}_d := \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$	—

$$F_{\varpi} := \varpi(F_{Id}) = F_{\{v_{\varpi(1)}, v_{\varpi(2)}, \cdots, v_{\varpi(|d|)}\}}$$
$$= F_{\{v_5, v_3, v_1, v_4, v_2\}}$$

⚠ The action on Flag is not the same as in http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$\mathcal{F}_{\lvert d\rvert} \neq \coprod_{d} \mathcal{F}_d$

$\mathcal{F}_{\varpi} \cong \mathcal{F}_{\underline{d}}$ with different base pt. Base pt makes difference!

$\mathcal{F}_{\lvert d\rvert} \times \mathcal{F}_{\lvert d\rvert}$	$F_{Id,Id}$
$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_d \times \mathcal{F}_d := \coprod_{\underline{d},\underline{d}'} (\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'})$	—

$$F_{\varpi,\varpi'} := (F_{\varpi}, F_{\varpi'})$$

$$\begin{tikzcd} \widetilde{Rep}_{\underline{d}}(Q) \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \arrow[d, "\mu_{\underline{d}}"'] \arrow[dr, "\pi_{\underline{d}}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{Rep}_d(Q) \subset Rep_d(Q) \times \mathcal{F}_d \arrow[d, "\mu_d"'] \arrow[dr, "\pi_d"] & \\ Rep_d(Q) & \mathcal{F}_d \end{tikzcd}$$

$\mu_{\underline{d}}^{-1}(M) \cong Flag_{\underline{d}}(M) \subseteq \mathcal{F}_{\underline{d}}$ is the Springer fiber.

$$\begin{tikzcd} Z_{\underline{d},\underline{d}'} \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \arrow[d, "\mu_{\underline{d},\underline{d}'}"'] \arrow[dr, "\pi_{\underline{d},\underline{d}'}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \end{tikzcd} \qquad \begin{tikzcd} Z_d \subset Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d \arrow[d, "\mu_{d,d}"'] \arrow[dr, "\pi_{d,d}"] & \\ Rep_d(Q) & \mathcal{F}_d \times \mathcal{F}_d \end{tikzcd}$$

$\widetilde{Rep}_{\underline{d}}(Q) \subseteq Rep_d(Q) \times \mathcal{F}_{\underline{d}}$

$\widetilde{Rep}_d(Q) := \coprod_{\underline{d}} \widetilde{Rep}_{\underline{d}}(Q)$

$\widetilde{Rep}_{\varpi}(Q) \cong G_d \times^{B_\varpi} r_\varpi$

$$Z_{\underline{d},\underline{d}'} := \widetilde{Rep}_{\underline{d}}(Q) \times_{Rep_d(Q)} \widetilde{Rep}_{\underline{d}'}(Q)$$
$$Z_d = \coprod_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$$
$$= \widetilde{Rep}_d(Q) \times_{Rep_d(Q)} \widetilde{Rep}_d(Q)$$

$$Z_{\varpi,\varpi'} := Z_{u,u'}$$

5. (equivariant) stratifications.