Eine Woche, ein Beispiel

5.26. 6-fctor formalism toolkit

I. before the toolkit

- I.1. $R(F\circ G) = RF \circ RG$?
- I.2. construct maps
- I.3. roadmap for proofs of topo 6-fctor formalism

II. toolkit

- II.1. conditions in topo case
- II.2. minimal toolkit
- II.3. add-ons (components)
  - recollement diagram
  - sm & flat
  - the hidden fctor $\underline{\mathrm{Hom}}(-, \mathcal{F})$
  - Künneth fctor $\boxtimes$
  - invertible sheaf
  - descent (inverse problem)

III. after the toolkit (in next document?)

- III.1. intepretation for memory
  - commute
  - FM transformation
  - ...
- III.2. cohomology intepretation — topo
  - multiplication structure
  - UCT
  - cal of coh ring — intersection theory
    - — Schubert calculus
    - — Chern class
- III.3. Weil conj intepretation — étale
- III.4. equiv intepretation — stack
- III.5. monodromy, nearby cycles — in family

I. before the toolkit

In this part, we try to prove formulas in the minimal toolkit (for topological spaces case). To make the toolkit clean and concise, we put most technicality in this part, e.p. the relationship with derived formulas and underived formulas. Readers can skip this part if they don't want to see details.

I.1. $R(F \circ G) = RF \circ RG$? [Achar 21, Table 1.4.1, Def A.6.2]

f_* : flasque $\rightsquigarrow$ flasque $\qquad R(f \circ g)_* = Rf_* \circ Rg_*$

$f_!$: c-soft $\rightsquigarrow$ c-soft $\qquad R(f \circ g)_! = Rf_! \circ Rg_!$

$\otimes$: flat $\not\rightsquigarrow$ flat $\qquad L(\mathcal{F} \otimes \mathcal{G} \otimes -) \neq (\mathcal{F} \overset{L}{\otimes} -) \circ (\mathcal{G} \overset{L}{\otimes} -)$

$$\begin{array}{ccc} L(\mathcal{F} \otimes \mathcal{G} \otimes -) & \neq & (\mathcal{F} \overset{L}{\otimes} -) \circ (\mathcal{G} \overset{L}{\otimes} -) \\ \| & & \| \\ (\mathcal{F} \otimes \mathcal{G}) \overset{L}{\otimes} - & \neq & (\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \overset{L}{\otimes} - \end{array}$$

$\underline{Hom}(\mathcal{F}, -)$: inj $\not\rightsquigarrow$ inj

$$\begin{array}{ccc} R(\underline{Hom}(\mathcal{F}, \underline{Hom}(\mathcal{G}, -))) & \neq & R\underline{Hom}(\mathcal{F}, -) \circ R\underline{Hom}(\mathcal{G}, -) \\ \| & & \| \\ R\underline{Hom}(\mathcal{F} \otimes \mathcal{G}, -) & \neq & R\underline{Hom}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}, -) \end{array}$$

//A: In needed case $R(F \circ G) = RF \circ RG$, so don't worry about that too much.

I.2. construct maps

These exercises practice your ability to use adjunctions.
In this section, we ignore the derived symbol. I would remind you if you forget.
The reference will be collected in I.3.

Base change (BC): $f^* g_! \cong g'_! f'^{,*}$

① When $g_! = g_*$, $g'_! = g'_*$, construct the map

$$f^* g_! \longrightarrow g'_! f'^{,*}$$

② When $f^* = f^!$, $f'^{,*} = f'^{,!}$, construct the map

$$f^* g_! \longleftarrow g'_! f'^{,*}$$

③ we require that the following diagram commutes:

$$\begin{tikzcd} f^* g_! \arrow[r, dashed] \arrow[d] & g'_! f'^{,*} \arrow[d] \\ f^* g_! \arrow[r] & g'_! f'^{,*} \end{tikzcd}$$

$f^*(- \otimes -)$: $f^*(\mathcal{F} \otimes \mathcal{F}') \cong f^*\mathcal{F} \otimes f^*\mathcal{F}'$

① construct the map (one side is enough)

$$f^*(\mathcal{F} \otimes \mathcal{F}') \longleftrightarrow f^*\mathcal{F} \otimes f^*\mathcal{F}'$$

by def of sheaf $\otimes$.

② show the iso by checking stalks.

③ If you did the underived version of ① & ②, try to work out derived version.

Projection formula: $f_!(f^*\mathcal{F}\otimes\mathcal{G})\cong\mathcal{F}\otimes f_!\mathcal{G}$

① When $f_!=f_*$, construct the map

$$f_!(f^*\mathcal{F}\otimes\mathcal{G})\longleftarrow\mathcal{F}\otimes f_!\mathcal{G}$$

② When $f^*=f^!$, construct the map

$$f_!(f^*\mathcal{F}\otimes\mathcal{G})\longrightarrow\mathcal{F}\otimes f_!\mathcal{G}$$

③ we require that the following diagram commutes:

$$\begin{tikzcd}
f_!(f^*\mathcal{F}\otimes\mathcal{G}) \arrow[d] & \mathcal{F}\otimes f_!\mathcal{G} \arrow[l, dashed] \arrow[d] \\
f_!(f^*\mathcal{F}\otimes\mathcal{G}) & \mathcal{F}\otimes f_!\mathcal{G} \arrow[l]
\end{tikzcd}$$

I.3 roadmap for proofs of topo 6-fctor formalism

stalkwise true

$\otimes \dashv \mathrm{Hom}$ for abelian gp

underived version

$\otimes \dashv \underline{\mathrm{Hom}}$ $\begin{cases} \mathrm{Hom} \\ \underline{\mathrm{Hom}} \end{cases}$

derived version

$\overset{L}{\otimes} \dashv R\underline{\mathrm{Hom}}$ $\begin{cases} \mathrm{Hom}_{\mathcal{D}^+} \text{ (2.6.8)} \\ R\mathrm{Hom} \\ R\underline{\mathrm{Hom}} \text{ (2.6.7)} \end{cases}$

$\otimes$

$\perp$

Hom

$f^* \dashv f_*$

$f^! \dashv f_!$

---

by def

$\mathcal{F}_p \otimes \mathcal{F}'_p \xrightarrow{\cong} (\mathcal{F} \otimes \mathcal{F}')_p$

[Prop 2.3.5]

$f^*\mathcal{F} \otimes f^*\mathcal{F}' \xrightarrow{\cong} f^*(\mathcal{F} \otimes \mathcal{F}')$

$f^*(\mathcal{F} \overset{L}{\otimes} \mathcal{F}') \cong f^*\mathcal{F} \overset{L}{\otimes} f^*\mathcal{F}'$

$Rf_* R\underline{\mathrm{Hom}}(f^*\mathcal{F}, \mathcal{G}) \cong R\underline{\mathrm{Hom}}(\mathcal{F}, Rf_*\mathcal{G})$ (2.6.15)

[Lemma 2.5.12]

$\Gamma_c(X; \mathcal{F} \otimes \mathcal{G}) \xleftarrow{\cong} \mathcal{F} \otimes \Gamma_c(X; \mathcal{G})$

for $\mathcal{F}$ flat

[Prop 2.5.13]

$f_!(f^*\mathcal{F} \otimes \mathcal{G}) \xleftarrow{\cong} \mathcal{F} \otimes f_!\mathcal{G}$

for $\mathcal{F}$ flat

$Rf_!(f^*\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \cong \mathcal{F} \overset{L}{\otimes} Rf_!\mathcal{G}$

$f^! R\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') \cong R\underline{\mathrm{Hom}}(f^*\mathcal{F}, f^!\mathcal{F}')$

$Rf_* R\underline{\mathrm{Hom}}(\mathcal{G}, f^!\mathcal{F}) \cong R\underline{\mathrm{Hom}}(Rf_!\mathcal{G}, \mathcal{F})$

$f^* \dashv Rf_*$ $\begin{cases} R\mathrm{Hom} \text{ (2.6.13)} \\ \mathrm{Hom}_{\mathcal{D}^+} \text{ (2.6.14)} \end{cases}$

[Prop 2.6.4]

$f^* \dashv f_*$ $\begin{cases} \underline{\mathrm{Hom}} \text{ (2.3.7)} \\ \mathrm{Hom} \text{ (2.3.6)} \end{cases}$

[Prop 2.3.3, Cor 2.3.4]

$\emptyset$

[Prop 3.1.10] $R\mathrm{Hom}$

[Prop 3.1.10] $\mathrm{Hom}_{\mathcal{D}^+}$

[Thm 3.1.5]

$Rf_! \dashv f^!$

$\emptyset$

$\emptyset$

$Rf_* g^! \cong g'_! f'^{,*}$

$f^* Rg_! \cong Rg'_! f'^{,*}$

[Prop 2.6.7]

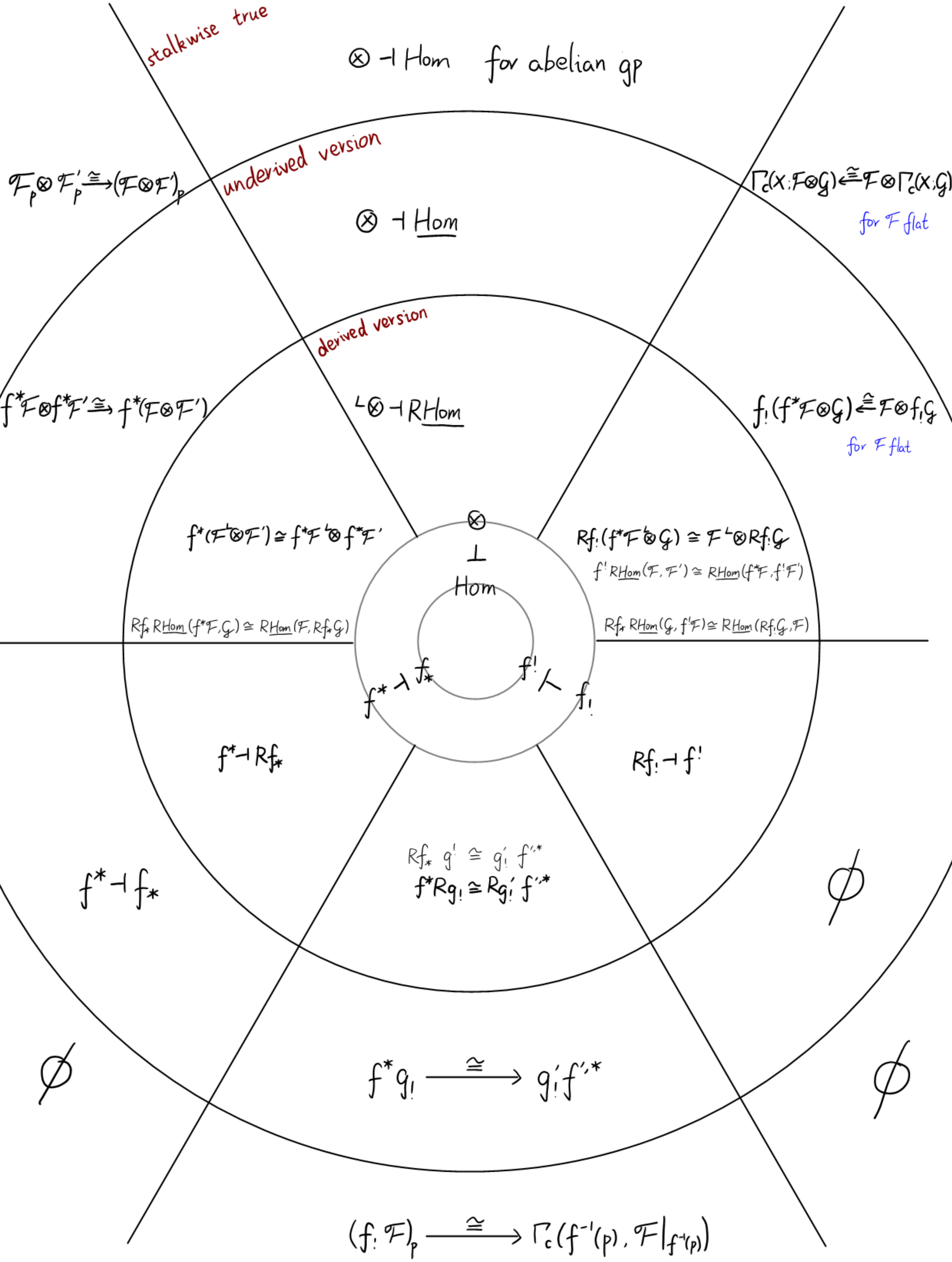
$f^* g_! \xrightarrow{\cong} g'_! f'^{,*}$

[Prop 2.5.11]

$(f_! \mathcal{F})_p \xrightarrow{\cong} \Gamma_c(f^{-1}(p), \mathcal{F}|_{f^{-1}(p)})$

[Prop 2.5.2]

Ref: [KS90]



stalkwise true
$\otimes \dashv Hom$ for abelian gp
underived version
$\mathcal{F}_p \otimes \mathcal{F}'_p \xrightarrow{\cong} (\mathcal{F}\otimes\mathcal{F}')_p$
$\underline{\otimes} \dashv \underline{Hom}$
$\Gamma_c(X;\mathcal{F}\otimes\mathcal{G}) \xleftarrow{\cong} \mathcal{F}\otimes\Gamma_c(X;\mathcal{G})$
for $\mathcal{F}$ flat
derived version
$f^*\mathcal{F}\otimes f^*\mathcal{F}' \xrightarrow{\cong} f^*(\mathcal{F}\otimes\mathcal{F}')$
$\overset{L}{\otimes} \dashv R\underline{Hom}$
$f_!(f^*\mathcal{F}\otimes\mathcal{G}) \xleftarrow{\cong} \mathcal{F}\otimes f_!\mathcal{G}$
for $\mathcal{F}$ flat
$f^*(\mathcal{F}\overset{L}{\otimes}\mathcal{F}') \cong f^*\mathcal{F}\overset{L}{\otimes} f^*\mathcal{F}'$
$\otimes \perp Hom$
$Rf_!(f^*\mathcal{F}\overset{L}{\otimes}\mathcal{G}) \cong \mathcal{F}\overset{L}{\otimes} Rf_!\mathcal{G}$
$f^! R\underline{Hom}(\mathcal{F},\mathcal{F}') \cong R\underline{Hom}(f^*\mathcal{F}, f^!\mathcal{F}')$
$Rf_* R\underline{Hom}(f^*\mathcal{F},\mathcal{G}) \cong R\underline{Hom}(\mathcal{F}, Rf_*\mathcal{G})$
$Rf_* R\underline{Hom}(\mathcal{G}, f^!\mathcal{F}) \cong R\underline{Hom}(Rf_!\mathcal{G}, \mathcal{F})$
$f^* \dashv f_*$
$f^! \vdash f_!$
$f^* \dashv Rf_*$
$Rf_! \dashv f^!$
$Rf_* g^! \cong g'_! f'^{*}$
$f^* Rg_! \cong Rg'_! f'^{*}$
$f^* \dashv f_*$
$\emptyset$
$\emptyset$
$\emptyset$
$f^* g_! \xrightarrow{\cong} g'_! f'^{*}$
$(f_!\mathcal{F})_p \xrightarrow{\cong} \Gamma_c(f^{-1}(p), \mathcal{F}|_{f^{-1}(p)})$

The figure shows the routes for upgrading isomorphisms, while you can also downgrade isomorphisms.

To move from derived to underived categories, take the 0-th cohomology. To go from underived categories to basic formulas, take the stalks.