

Eine Woche, ein Beispiel

5.15 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory. In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categories here: <http://katmat.math.uni-bremen.de/acc/acc.pdf>

For Chinese, the theory of category has been summed up in detail in [<https://wwli.asia/downloads/books/Al-jabr-1.pdf>], Chapter 2-3.

Process

0. Well-known concepts
1. Individual category
 - Complete/Cocomplete/Bicomplete category
 - Cartesian closed category / Closed category
 - Monoidal category = Tensor category
2. Functors between categories
 - Exactness
 - Adjoints
3. Examples of categories
 - Well-known examples
 - Cat
 - Hausdorff and compactness
 - Categories in condensed mathematics

Appendix

Just for the convenience of writing, I would use the following terminologies.

A **class** is a collection of sets.

A **metaclass** is a collection of classes.

A **metaclass** is a collection of metaclasses.

...

For the most time it can be turned to normal sentences without metaclass.

e.g. Consider a category \mathcal{C} .

I want to write it shorter, so I write: Consider $\mathcal{C} \in \text{Cat}_{\text{big}}$.

But "a proper class can not be written on the left of the symbol ' \in '", or

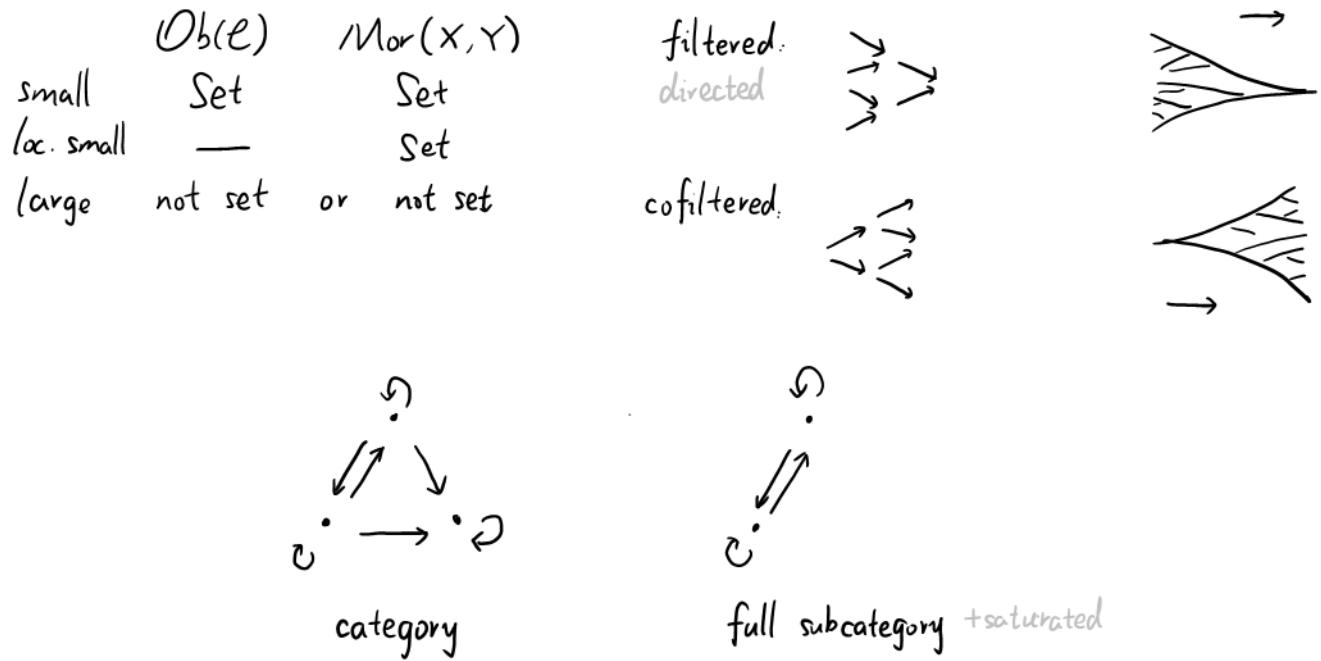
"a class is a collection of sets, so no proper class can be contained in a class".

For these silly set-theory problem, I just call Cat_{big} as the metacategory.

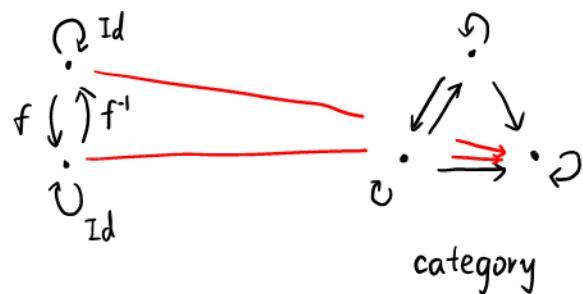
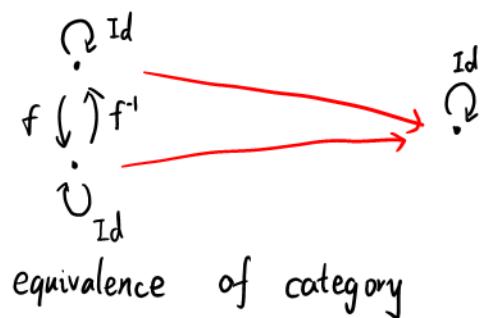
For me, it is always quicker to write "consider $M \in \text{Mod}(A)$ " than "consider an A -module M ".

0. Well-known concepts

\mathcal{C} is always a category.



<https://math.stackexchange.com/questions/2147377/are-fully-faithful-functors-injective>



fully faithful

<https://blog.juliosong.com/linguistics/mathematics/category-theory-notes-8/>

1. Individual category

Complete/Cocomplete/Bicomplete category

Def. \mathcal{C} is **complete** if

\forall small category Δ , \forall functor $F: \Delta \rightarrow \mathcal{C}$ $i \mapsto F_i$,
 $\varprojlim_{i \in \Delta} F_i$ exists $(\varprojlim_{i \in \Delta} F_i \text{ is called the small limit})$

\mathcal{C} is **cocomplete** if

\forall small category Δ , \forall functor $F: \Delta \rightarrow \mathcal{C}$ $i \mapsto F_i$,
 $\varinjlim_{i \in \Delta} F_i$ exists $(\varinjlim_{i \in \Delta} F_i \text{ is called the small colimit})$

bicomplete = complete + cocomplete

\mathcal{C} is **finitely complete** if \forall finite limit exists

\mathcal{C} is **finitely cocomplete** if \forall finite colimit exists.

Thm.

\mathcal{C} is complete $\Leftrightarrow \mathcal{C}$ has equalizers & products

$\Leftrightarrow \mathcal{C}$ has pullbacks & products

\mathcal{C} is cocomplete $\Leftrightarrow \mathcal{C}$ has coequalizers & coproducts

$\Leftrightarrow \mathcal{C}$ has pushouts & coproducts

\mathcal{C} is finitely complete $\Leftrightarrow \mathcal{C}$ has equalizers & finite products

$\Leftrightarrow \mathcal{C}$ has equalizers, binary products & terminal obj

$\Leftrightarrow \mathcal{C}$ has pullbacks & terminal obj

For small category \mathcal{C} ,

complete \Leftrightarrow cocomplete

\Downarrow \Uparrow

thin ($\#\text{Mor}(X, Y) \leq 1$)

The "closedness" of the category is that taking the morphisms between two objects gives another morphism in the same category, rather than the category of sets or some other category.

from: <https://math.stackexchange.com/questions/3486846/definition-of-cartesian-closed-category-why-do-we-need-exponential-objects>

Cartesian closed category / Closed category

Def. \mathcal{C} is **Cartesian closed** if

\mathcal{C} has terminal obj, binary product and exponential, where

$$\begin{aligned} - \times Y &\vdash (-)^Y & \text{a bifunctor } F: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \\ \text{i.e. } \text{Mor}(X \times Y, Z) &\cong \text{Mor}(X, Z^Y) & \text{which is functorial in } Y \end{aligned}$$

\mathcal{C} is **loc. Cartesian closed** if all its slice category is Cartesian closed.

<https://ncatlab.org/nlab/show/over+category>

Rmk. When \mathcal{C} is loc. Cartesian closed,

\mathcal{C} is Cartesian closed $\Leftrightarrow \mathcal{C}$ has a terminal object.

But \mathcal{C} is Cartesian closed $\nRightarrow \mathcal{C}$ is loc. Cartesian closed

For the closed category, we use the definition in <https://ncatlab.org/nlab/show/closed+category>.

Def A **closed category** is a category \mathcal{C} together with the following data.

- bifunctor $[-, -]: \mathcal{C}^{op} \times \mathcal{C} \rightarrow \mathcal{C}$ called internal hom-functor

- $I \in \text{Ob}(\mathcal{C})$ called unit object

- $i: \text{Id}_{\mathcal{C}} \xrightarrow{\cong} [I, -] \rightsquigarrow i_A: A \xrightarrow{\cong} [I, A]$

- $j_X: I \rightarrow [X, X]$ extranatural in X

- $L_{Y,Z}^X: [Y, Z] \rightarrow [[X, Y], [X, Z]]$ functorial in Y and Z

extranatural in X .

- Compatibilities

$$\begin{array}{ccc} I & \xrightarrow{j_Y} & [Y, Y] \\ & \searrow j_{[X,Y]} & \downarrow L_{Y,Y}^X \\ & & [[X, Y], [X, Y]] \end{array} \quad \begin{array}{ccc} [X, Y] & \xrightarrow{L_{XY}^X} & [[X, X], [X, Y]] \\ & \searrow i_{[X,Y]} & \downarrow [j_X, 1] \\ & & [I, [X, Y]] \end{array} \quad \begin{array}{ccc} [Y, Z] & \xrightarrow{L_{YZ}^I} & [[I, Y], [I, Z]] \\ & \searrow [1, i_Z] & \downarrow [i_Y, 1] \\ & & [Y, [I, Z]] \end{array}$$

$$\begin{array}{ccc} & [U, V] & \\ & \swarrow L_{UV}^X & \searrow L_{UV}^Y \\ [[X, U], [X, V]] & & [[Y, U], [Y, V]] \\ \\ L_{[[X, Y], [X, U]], [X, V]}^{[X, Y]} & \searrow & \swarrow L_{[[Y, U], [Y, V]]}^{[Y, Y]} \\ & \left[[[X, Y], [X, U]], [[X, Y], [X, V]] \right] & \longrightarrow \left[[[Y, U], [Y, V]], [[Y, U], [X, V]] \right] \\ & & \swarrow [1, L_{YV}^X] \end{array}$$

$$\begin{array}{ccc} \gamma: \text{Mor}(X, Y) & \longrightarrow & \text{Mor}(I, [X, Y]) \\ f & \longmapsto & [1, f] \circ j_X \end{array} \quad \text{is an iso.}$$

Monoidal category = Tensor category 公[#]定理

Def A **monoidal category** is a category \mathcal{C} together with the following data.

- bifactor $-\otimes- : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- $I \in Ob(\mathcal{C})$ called unit object
- $\alpha_{A,B,C} : A \otimes (B \otimes C) \xrightarrow{\cong} (A \otimes B) \otimes C$
- $\lambda_A : I \otimes A \xrightarrow{\cong} A$ Lambda: left
- $\rho_A : A \otimes I \xrightarrow{\cong} A$ rho: right
- Compatabilities

$$\begin{array}{ccc}
 & A \otimes (B \otimes (C \otimes D)) & \\
 1_A \otimes \alpha_{B,C,D} \swarrow & & \searrow \alpha_{A,B,C \otimes D} \\
 A \otimes ((B \otimes C) \otimes D) & \Downarrow & (A \otimes B) \otimes (C \otimes D) \\
 \downarrow \alpha_{A,B \otimes C,D} & & \downarrow \alpha_{A \otimes B,C,D} \\
 (A \otimes (B \otimes C)) \otimes D & \xrightarrow{\alpha_{A,B,C} \otimes 1_D} & ((A \otimes B) \otimes C) \otimes D \\
 \\
 A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I \otimes B}} & (A \otimes I) \otimes B \\
 \downarrow 1_A \otimes \lambda_B & & \downarrow \rho_A \otimes 1_B \\
 A \otimes B & &
 \end{array}$$

For strict monoidal category, we require in addition that $\alpha_{A,B,C}$, λ_A , ρ_A are identities.

E.g. **Cartesian monoidal category** \mathcal{C} : category with finite products

$$\otimes = \prod \quad I = \text{terminal object}$$

e.g. Set, Cat.

Cocartesian monoidal category \mathcal{C} : category with finite coproducts

$$\otimes = \coprod \quad I = \text{initial object}$$

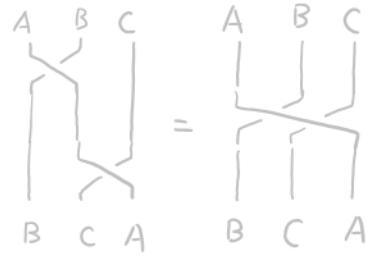
Abelian category is monoidal.

Def (Specializations)

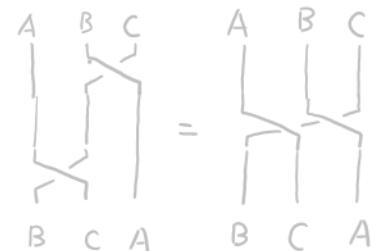
Let \mathcal{C} be a monoidal category.

If in addition we have $\gamma_{A,B} : A \otimes B \rightarrow B \otimes A$,
then \mathcal{C} is **braided monoidal category** if

$$\begin{array}{ccc}
 & (A \otimes B) \otimes C & \\
 \gamma_{A,B} \otimes 1_C \swarrow & & \searrow \alpha_{A,B,C} \\
 (B \otimes A) \otimes C & & A \otimes (B \otimes C) \\
 \downarrow \alpha_{B,A,C} & & \downarrow \gamma_{A,B \otimes C} \\
 B \otimes (A \otimes C) & & (B \otimes C) \otimes A \\
 1_B \otimes \gamma_{A,C} \searrow & & \swarrow \alpha_{B,C,A} \\
 & B \otimes (C \otimes A) &
 \end{array}$$



$$\begin{array}{ccc}
 & A \otimes (B \otimes C) & \\
 1_A \otimes \gamma_{B,C} \swarrow & & \searrow \alpha_{A,B,C}^{-1} \\
 A \otimes (C \otimes B) & & (A \otimes B) \otimes C \\
 \downarrow \alpha_{A,C,B}^{-1} & & \downarrow \gamma_{A \otimes B,C} \\
 (A \otimes C) \otimes B & & C \otimes (A \otimes B) \\
 \gamma_{C,A} \otimes 1_B \searrow & & \swarrow \alpha_{C,A,B}^{-1} \\
 & (C \otimes A) \otimes B &
 \end{array}$$



$$I \otimes A \xrightarrow{\gamma_{I,A}} A \otimes I$$

$$\lambda_A \swarrow \quad \downarrow p_A \quad \nearrow A$$

$$A \otimes I \xrightarrow{\gamma_{A,I}} I \otimes A$$

$$p_A \swarrow \quad \downarrow A \quad \nearrow \lambda_A$$

\mathcal{C} is **symmetric monoidal category** if

$$\gamma_{B,A} \circ \gamma_{A,B} = 1_{A \otimes B}. \quad + \mathcal{C} \text{ is braided.}$$

closed monoidal category = closed category + monoidal category
+ compatibility $- \otimes A \dashv [A, -]$

2. Functors between categories

Exactness

Ref:<https://ncatlab.org/nlab/show/exact+functor>

Prop/Def For a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between finitely complete categories, TFAE.

- F preserves finite limits,
- F preserves equalizers & finite products,
- F preserves equalizers, binary products & terminal objects,
- F preserves pullbacks & terminal objects,
- $\forall d \in \text{Ob}(\mathcal{D})$, the comma category F/d is filtered.

If so, we call F is a **left exact functor**.

↑ We require also that \mathcal{C} & \mathcal{D} are finitely complete categories

When \mathcal{C}, \mathcal{D} are abelian categories, this is equivalent to

- F preserves kernels, i.e.
- F sends left exact sequences to left exact sequences.

See [<https://stacks.math.columbia.edu/tag/01oM>]. You may get the following results from the argument:

|| F is a left exact functor between abelian categories $\Rightarrow F$ preserves binary products
↓ [oDLP]
|| F is additive

Def. A contravariant functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is called **left exact**, if

$F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ is left exact (In ptc. \mathcal{C} is finitely cocomplete,
 \mathcal{D} is finitely complete)

Similarly, we can define right exactness, and

exact $\stackrel{\mathcal{C}, \mathcal{D} \text{ abelian}}{=} \text{left exact} + \text{right exact}$
 $\stackrel{=}{\equiv} \text{sends SES to SES.}$

Adjoints For more adjoints, see §3 on examples.

left adjoint \dashv right adjoint $f_! \dashv f^* \dashv f_* \dashv f^!$

free	forget
$- \otimes_A N$	$\text{Hom}_A(N, -)$
Δ	\varprojlim
$(\)^\sim$	Γ^*
f_p	f^*
$c\text{-Ind}$	Res
Res	Ind
$- \otimes_k A^e$	Res
$\text{sh}(-)$	$\text{Res}_{\text{sh} \rightarrow \text{Psh}}$
π_0	$T \mapsto \coprod_{t \in T} \text{Spec } \mathbb{Z}$
$G^{[n]}$	Lie
$I\text{-I}$	S
$\tau_1 = H_0(-)$	$N \cdot \text{nerve}$

ad \Rightarrow preserve colimits preserve limits

\Rightarrow right exacts left exacts
in (co)complete category

coker f $\ker f$
 $\amalg A \oplus A$ $\prod A$ $A \times_B$

pushforward pullback
coequalizer equalizer

$$\bar{k} = \varinjlim_{L/k} L \quad \text{Spec } \bar{k} = \varprojlim_{L/k} \text{Spec } L \quad \text{Gal}(\bar{k}/k) = \varprojlim_{\substack{L \text{ finite} \\ \text{Galois}}} \text{Gal}(L/k)$$

$$\text{Spec } \mathbb{Z}_p = \varprojlim_n \text{Spec } \mathbb{Z}_{p^n} \quad \mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \mathbb{Z}$$

completion ring point of view

$$F_p = \varinjlim_U F(U)$$

stalk

co limit direct inductive injective	limit inverse projective
\bigcup limit.	\bigcap limit.

How to memorize:

$$\xleftarrow{\text{limit}} \quad \xrightarrow{\text{colimit}} \quad .$$

$$0 \longrightarrow \text{ker} \longrightarrow M \longrightarrow N \longrightarrow \text{coker} \longrightarrow 0$$

3. Examples of categories

Well-known examples

Set Top Grp Ab $\text{Vect}(k)$ $\text{Mod}(R)$

Ring : identity + preserve identity + $\circ \in \text{Ring}$

CRing Rng

Field : full subcategory of CRing

$\text{O}:$ $\text{Ob}(\text{O}) = \emptyset$

$\text{L}:$ $\text{Ob}(\text{L}) = \{\ast\}$ $\text{Mor}(\ast, \ast) = \{1_\ast\}$

K(2) : $\text{Ob}(\text{K(2)}) = \{V, E\}$ $\text{Mor}(V, V) = \{1_V\}$ $\text{Mor}(E, E) = \{1_E\}$
^{Another name: Arr} $\text{Mor}(V, E) = \emptyset$ $\text{Mor}(E, V) = \{s, t\}$

$$\begin{array}{c} \text{I}_E \\ \text{Q} \\ \text{E} \end{array} \xrightarrow[s]{t} V \xleftarrow{s} V$$

$\Delta:$ $\text{Ob}(\Delta) = \{[n] := \{0, 1, 2, \dots, n\} \mid n \geq 0\}$

$\text{Mor}([m], [n]) = \{\text{weakly monotone maps}\}$

$s\text{Set}:$ $\text{Ob}(s\text{Set}) = \left\{ X : \Delta^{\text{op}} \rightarrow \text{Set} \right\}$ $\text{Mor}(X, Y) = \left\{ \alpha : \Delta^{\text{op}} \xrightarrow[X]{Y} \text{Set} \right\}$

$\text{CHaus}:$ $\text{Ob}(\text{CHaus}) = \left\{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} X \right\}$

<https://ncatlab.org/nlab/show/compactum>

$\text{Mor}(X, Y) = \{f : X \rightarrow Y \mid f \text{ cont}\}$

$\text{Met}:$ full subcategory of CHaus whose objects are metric spaces.

For the category of Graph, there're different realizations.

$\text{Quiv}(\mathcal{C}):$ $\text{Ob}(\text{Quiv}(\mathcal{C})) = \{\text{factor } \Gamma, K(2) \rightarrow \mathcal{C}\}$

$\text{Mor}(\Gamma_1, \Gamma_2) = \left\{ \alpha : K(2) \xrightarrow[\Gamma_2]{\Gamma_1} \mathcal{C} \right\}$

$\text{Quiv} = \text{Quiv}(\text{Set})$

= Category of presheaves on \mathcal{Q}^{op} .

Cat

$\text{Cat} = \{\text{the category of small categories}\}$ is a 2-category.

$\text{Ob}(\text{Cat}) = \{\text{small category } \mathcal{C}\}$

$\text{Mor}(\mathcal{C}, \mathcal{D})$ is a category by

$\text{Ob}(\text{Mor}(\mathcal{C}, \mathcal{D})) = \{F: \mathcal{C} \rightarrow \mathcal{D}\}$

$\text{Mor}(F, G) = \{a: \mathcal{C} \xrightarrow[F]{\Downarrow a} \mathcal{D}\}$

Basic properties of Cat:

1. Initial object $\mathbf{0}$, Terminal object $\mathbf{1}$.

2. Cat is loc. small but not small

3. Cat is bicomplete

4. Cat is Cartesian closed but not loc. Cartesian closed

5. Cat is loc. finitely presentable <https://ncatlab.org/nlab/show/locally+finitely+presentable+category>

6. Cat $\begin{array}{c} \xleftarrow{\text{free}} \\[-1ex] T \\[-1ex] \xrightarrow{\text{forget}} \end{array}$ Quiv

e.g. of "free"

$$\begin{matrix} f \\ \circ \\ fG \\ \circ \\ g \\ \circ \\ g_1 \\ \dots \end{matrix}$$

$$\Leftarrow \quad \cdot \circ f$$

$$\begin{matrix} & \xrightarrow{\circ f \circ e} \\ 1_a & \xrightarrow{f} & G_1 & \xrightarrow{e} & 1_b \\ a & \xrightarrow{\circ f} & \circ & \xrightarrow{\circ e} & b \\ & \xrightarrow{\circ f \circ e} & & & \end{matrix}$$

$$\Leftarrow \quad \begin{matrix} & \xrightarrow{e} \\ a & \xrightarrow{f} & b \end{matrix}$$

$$\begin{matrix} & & \xrightarrow{1_b} \\ 1_a & \xrightarrow{f} & G_1 & \xrightarrow{g} & 1_c \\ a & \xrightarrow{f} & \circ & \xrightarrow{g} & c \\ & & \xrightarrow{gf} & & \end{matrix}$$

$$\Leftarrow \quad \begin{matrix} & \xrightarrow{f} & \xrightarrow{g} \\ a & \xrightarrow{f} & b & \xrightarrow{g} & c \end{matrix}$$

<https://math.stackexchange.com/questions/750731/is-there-a-category-of-categories>

Inspired by this question and answer, I will use metacategory to indicate Cat_{big} , where

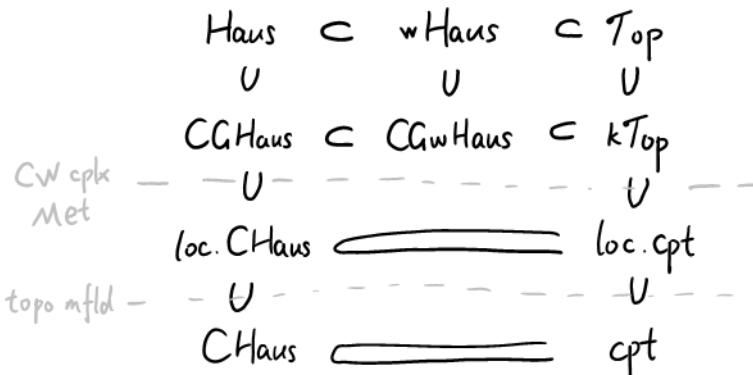
$\text{Ob}(\text{Cat}_{\text{big}}) = \{\text{all categories}\}$ as a metaclass

$\text{Mor}(\mathcal{C}, \mathcal{D})$ is a metacategory by

$\text{Ob}(\text{Mor}(\mathcal{C}, \mathcal{D})) = \{F: \mathcal{C} \rightarrow \mathcal{D}\}$ as a metaclass

$\text{Mor}(F, G) = \{a: \mathcal{C} \xrightarrow[F]{\Downarrow a} \mathcal{D}\}$ as a metaclass

Hausdorff and compactness $\leftarrow \text{cpt} \approx (\text{quasi})\text{cpt}$



Def. $X \in \text{Top}$ is a **weak Hausdorff space** (in $w\text{Haus}$) if
 $\forall K \in \text{CHaus}, \forall g: K \rightarrow X$ cont, $g(K) \subset X$ is closed.

Def. $X \in \text{Top}$ is **locally compact** (in loc.cpt) if
 $\forall p \in X, \exists \text{ cpt nbhd } V$ (i.e. $p \in U \subseteq V \subseteq X$ $U \subseteq X$ open, V cpt)
 $\text{loc.CHaus} = \text{loc.cpt} \cap \text{Haus}$

see https://en.wikipedia.org/wiki/Locally_compact_space for other common definitions which are not equivalent in general.

Def. $X \in \text{Top}$ is a **compactly generated/a k-space** (in $k\text{Top}$) if

cpt gen in Condensed Math

Hausdorff-cpt gen/k-space in wiki

cpt gen/k-space in nlab

k-space in ATII

\forall map $f: X \rightarrow Y$.

f is cont $\Leftrightarrow \forall K \xrightarrow{g} X \xrightarrow{f} Y$ is cont

$\forall K \in \text{CHaus}, g: K \rightarrow X$ cont

$$\begin{bmatrix} \text{CGwHaus} = k\text{Top} \cap w\text{Haus} \\ \text{CGHaus} = k\text{Top} \cap \text{Haus} \end{bmatrix}$$

<https://mathoverflow.net/questions/47702/why-the-w-in-cgwh-compactly-generated-weakly-hausdorff-spaces>

equivalently,

$\forall A \subseteq X$ subspace,

$A \subseteq X$ is closed $\Leftrightarrow g^{-1}(A) \subseteq K$ is closed

$\forall K \in \text{CHaus}, g: K \rightarrow X$ cont

When X is Hausdorff, this is equivalent to

$\forall A \subseteq X$ subspace,

$A \subseteq X$ is closed $\Leftrightarrow A \cap K \subseteq K$ is closed

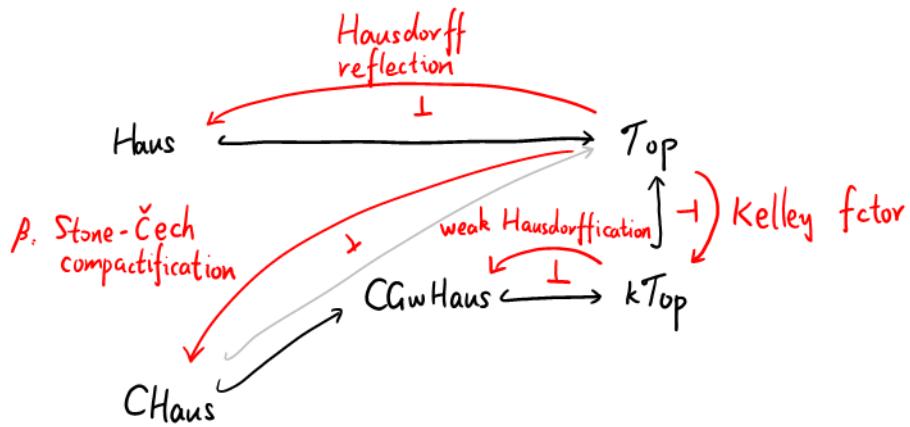
$\forall K \in \text{CHaus}$:

Prop. $X \in \text{Top}$, then

X is a k-space $\Leftrightarrow X \cong \bigsqcup_{i \in I} S_i / \sim$ $S_i \in \text{CHaus}$

Rmk. In the def/prop of $k\text{Top}$, CHaus can be replaced by Prof.

Adjoints



Kelley factor $(-)^{cg}$: $\text{Top} \longrightarrow k\text{Top}$
 $X \mapsto X^{cg}$ compactly generated

Set: $X^{cg} = X$

Topo: $A \subseteq X^{cg}$ is closed if $g^{-1}(A) \subseteq K$ is closed
 $\forall K \in \text{CHaus}, g: K \rightarrow X$ cont

Definition and properties of Hausdorff reflection:

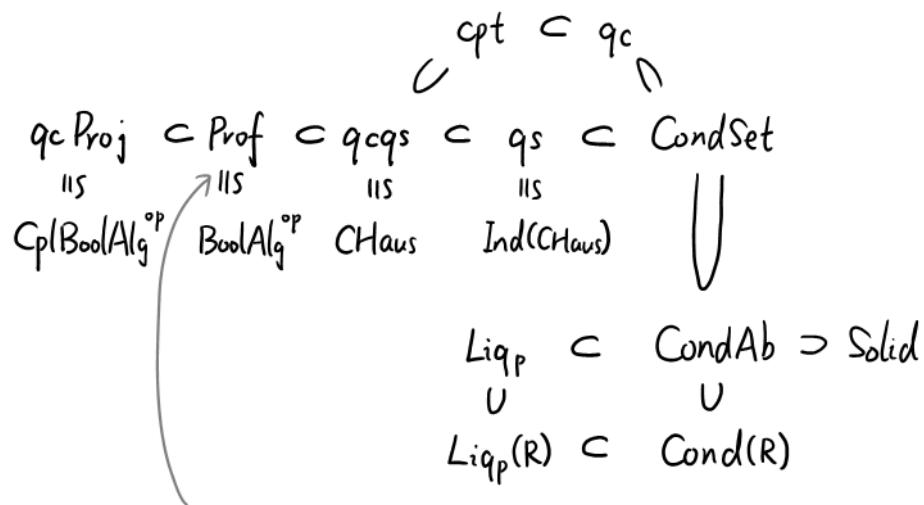
<https://ncatlab.org/nlab/show/Hausdorff+space#HausdorffReflections>

Detailed verification of the Hausdorff reflection:

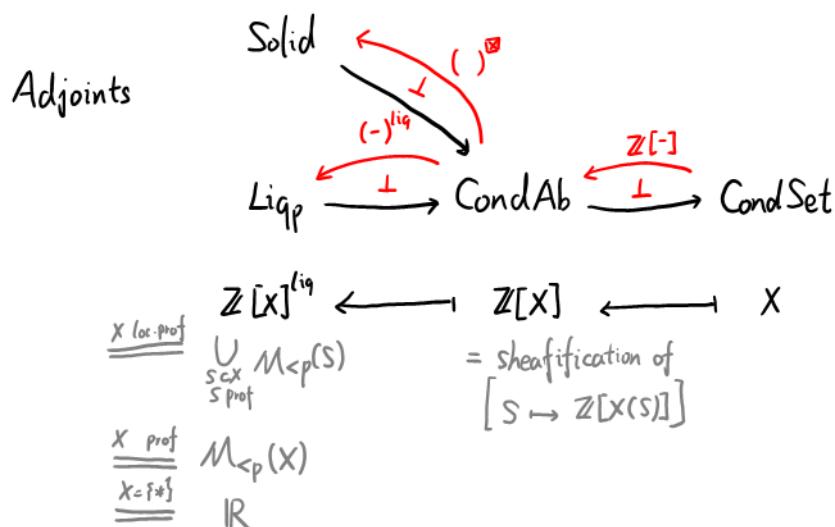
<https://math.stackexchange.com/questions/471238/construction-of-a-hausdorff-space-from-a-topological-space>

More beautiful pictures of adjoint functors: <https://arxiv.org/abs/2112.13654>

Categories in condensed mathematics



<https://ncatlab.org/nlab/show/Stone+duality>
<https://math.unice.fr/tacl/assets/2019/contributed/s3/1/6-nurakunov-stronkowski.pdf>



Appendix

I'm just too lazy to fill in this table. If you know more, tell me and I will fill in, thanks!

Category	cpl	fin cpl	cocpl	fin cocpl	Cartesian closed	closed	monoidal
Set	✓		✓		✓		✓
Top	✓		✓		✗		✓
Grp	✓		✓		✗		✓
Ab	✓		✓		✗		✓
Vect(k)	✓		✓		✗		✓
Mod(R)	✓		✓		✗		✓
Ring	✓		✓				
CRing	✓		✓				
Rng	✓		✓				
Field	✗	✗	✗	✗			
o							
l	✓		✓		✓	✓	✓
K(2)							
Δ	✗	✗	✗	✗			
sSet	✓		✓				
CHaus	✓		✓				
Met	✗	✓	✗	✗			
Quiv(e)							
Quiv							
Cat	✓		✓		✓		✓
kTop							✓
CGHaus					✓		
CGwHaus	✓		✓		✓		
Prof	✓						