Eine Woche, ein Beispiel 8.21 equivariant K-theory of IP'

Let us do a simple case over IP'. It can be generlized "easily" to flag variety, but IP' is the beginning case of study.

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

Task. Understand

$$\begin{array}{cccc} K(\mathbb{b}_{i}) & \longrightarrow & K(\mathbb{b}_{f}) \\ & & & & & \\ K_{B}(\mathbb{b}_{i}) & \longrightarrow & K_{B}(\mathbb{b}_{f}) \\ & & & & & \\ K_{2\Gamma^{r}}(\mathbb{b}_{i}) & \longrightarrow & K_{2\Gamma^{r}}(\mathbb{b}_{f}) \\ & & & & & \\ & & & & \\ K_{2\Gamma^{r}\times\mathbb{C}_{x}}(\mathbb{b}_{i}) & \longrightarrow & K_{2\Gamma^{r}\times\mathbb{C}_{x}}(\mathbb{b}_{f}) \end{array}$$

where 
$$SL_{1} = SL_{2}, C$$
,  $B = \begin{pmatrix} * & * \\ * & * \end{pmatrix} \subseteq SL_{1}, C$ ,  $C \in \mathbb{R}^{n}$   $C \in \mathbb{R}^{n}$   $C \in \mathbb{R}^{n}$ 

Notation. For linear alg qp G [Ginz, 5.1], 
$$K_i^G(X) = K_i(Coh(X))$$
  $K_i^G(X) = K_i(Coh(X))$   $K$ 

$$[G_{inz}, (5.2.4)] \qquad G_{i}G_{X} \Rightarrow \qquad K^{G_{i}\times G_{i}}(X) \cong K^{G_{i}}(X) \otimes_{\mathbb{Z}} R(G_{i})$$

$$e.g. \qquad K^{SL_{i}\times C^{*}}(|P^{i}\rangle) \cong K^{SL_{i}}(|P^{i}\rangle) \otimes_{\mathbb{Z}} R(C^{*}) \cong K^{SL_{i}}(|P^{i}\rangle) \otimes_{\mathbb{Z}} \mathbb{Z}[t^{\pm 1}]$$

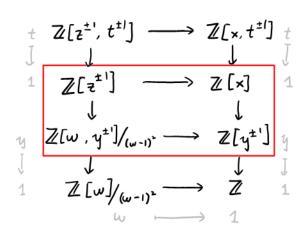
$$K^{B} \qquad (|P^{i}\rangle) \cong K \qquad (|P^{i}\rangle) \otimes_{\mathbb{Z}} R(B) \cong K \qquad (|P^{i}\rangle) \otimes_{\mathbb{Z}} \mathbb{Z}[y^{\pm 1}]$$

$$[G_{inz}, (5.2.17)] \qquad K_{i}^{H}(X) \qquad \underbrace{Res_{H}^{G}}_{Ind_{H}} \qquad K_{i}^{G}(G_{XH}X)$$

e.g. 
$$K^{SL_1}(IP') \cong K^{SL_1}(SL_1 \times_B pt) \cong K^B(pt) = R(B) = \mathbb{Z}[z^{\pm 1}]$$
  
[LCBE, 2.1.1]  $K(IP') \cong \mathbb{Z}\mathcal{O}_{IP'} \oplus \mathbb{Z}\mathcal{O}_{IP'}(I) = \mathbb{Z}[\omega]/(\omega-1)^2 = \mathbb{Z}[\omega^{\pm 1}]/(\omega-1)^2$   
[Ginz, 5.2.13] gives def of pushforward.

In conclusion, we get

$$\begin{array}{cccc} K(\mathbb{b}_{i}) & \longrightarrow & K(\mathbb{b}_{f}) \\ & \uparrow & & \uparrow \\ K_{\mathbb{R}^{r}}(\mathbb{b}_{i}) & \longrightarrow & K_{\mathbb{R}^{r}}(\mathbb{b}_{f}) \\ & \uparrow & & \uparrow \\ K_{\mathbb{R}^{r} \times \mathbb{C}_{x}}(\mathbb{b}_{i}) & \longrightarrow & K_{\mathbb{R}^{r} \times \mathbb{C}_{x}}(\mathbb{b}_{f}) \end{array}$$



The difficult part is the middle square.  $\mathbb{Z}[w,y^{\pm 1}]/(w-1)^{\pm} \longrightarrow \mathbb{Z}[y^{\pm 1}]$  $\begin{array}{ccc} y & \longmapsto y \\ y^{-1} & \longmapsto y^{-1} \end{array}$ 

Right: by rep theory,  $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[y^{\pm i}]$ 

Up by Borel-Weil-Bott theorem.

Left: by [LW-BWB, Ex 2.6], 
$$L_n \cong O(-n)$$
, combined with "Up", we get

 $\mathbb{Z}[z^{\pm 1}] \longrightarrow \mathbb{Z}[\omega, y^{\pm 1}]/(\omega-1)^2$ 

e.g.  $z^3 \longmapsto -\omega^3(y+y^{-1})$  (see table below)

 $z \mapsto z^{-1} \quad z^{-1} \quad 1 \quad z \mapsto z^3 \quad z^4 \quad \sum_{x=m}^{\infty} \sum_{x=m}^{\infty} \sum_{x=m}^{\infty} z^4 \quad \sum_{x=m}^{\infty} \sum_{x=m}^{\infty} z^4 \quad \sum_{x=m}^{\infty} \sum_{x=m}^{\infty} z^4 \quad \sum_{x=$ 

Ex. Generalize to  $SL_1 \longrightarrow SL_n$ ,  $P' \longrightarrow Flag(C')$   $SL_2 \longrightarrow GL_2$   $C \longrightarrow FP$   $C \longrightarrow FP$  $C \longrightarrow FP$