Eine Woche, ein Beispiel
1.14. reminder on Morse theory

Ref:

https://bastian.rieck.me/blog/posts/2019/morse_theory/

https://oldbookstonew.blogspot.com/
Contains the following books:

[MilnorMT]: Morse Theory by Milnor
[MilnorCC]: Characteristic Classes by Stasheff and Milnor

[MilnorSing]: singular points of complex hypersurfaces by Milnor

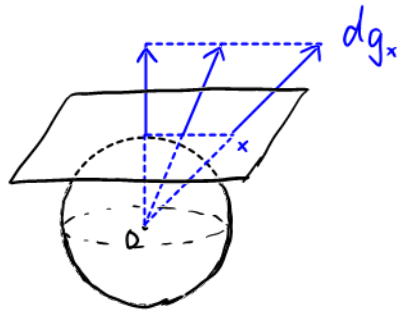
[Maxim20]:notes on vanishing cycles and applications
https://people.math.wisc.edu/~lmaxim/vanishing.pdf

1. Calculations of index

E.g.

$$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^3 \xrightarrow{g = \|\cdot\|^2} \mathbb{R}$$
$$(u_1, u_2) \longmapsto (u_1, u_2, 1)$$
$$x = (x_1, x_2, x_3) \longmapsto \langle x, x \rangle = x_1^2 + x_2^2 + x_3^2$$

$$x = (u_1, u_2, 1)$$
$$\frac{\partial x}{\partial u_1} = (1, 0, 0) \qquad g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\frac{\partial x}{\partial u_2} = (0, 1, 0)$$
$$\frac{\partial^2 x}{\partial u_i \partial u_j} = (0, 0, 0) \qquad \forall i, j$$



$$g(x) = \langle x, x \rangle$$
$$dg_x = 2x\, dx = \sum_i 2 x_i\, dx_i$$
$$dg_x(\vec{v}) = 2 \langle x, \vec{v} \rangle \qquad \vec{v} \in T_x \mathbb{R}^3$$

$$f(u) = \langle u, u \rangle + 1$$
$$df_u = 2u\, du = \sum_i 2 u_i\, du_i$$
$$df_u(\vec{v}) = 2 \langle u, \vec{v} \rangle \qquad \vec{v} \in T_u \mathbb{R}^2$$

$$f(u_1, u_2) = u_1^2 + u_2^2 + 1$$
$$\frac{\partial f}{\partial u_1} = 2u_1$$
$$\frac{\partial f}{\partial u_2} = 2u_2$$
$$\left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right)_{i,j} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\frac{\partial f}{\partial u_i} = df_u(\vec{e}_i) = 2 \langle u, \vec{e}_i \rangle = 2u_i$$
$$\frac{\partial f}{\partial u_i} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u_i} = \sum_j \frac{\partial g}{\partial x_j} \cdot \frac{\partial x_j}{\partial u_i}$$
$$= \sum_j 2 x_j \cdot \delta_{ij}$$
$$= 2u_i$$

have one critical pt $(0,0,1) \in \mathbb{R}^3$, with Morse index 0.
(attach one 0-cell.)

E.g.

$$f: \mathbb{R}^2 \xrightarrow{x} \mathbb{R}^3 \xrightarrow{g = \|\cdot\|^2} \mathbb{R}$$

$$(u_1, u_2) \longmapsto (u_1, u_2, u_1^2+u_2^2+t)$$

$$x = (x_1, x_2, x_3) \longmapsto \langle x, x\rangle = x_1^2+x_2^2+x_3^2$$

$$x = (u_1, u_2, u_1^2+u_2^2+t)$$

$$\frac{\partial x}{\partial u_1} = (1, 0, 2u_1) \qquad g_{ij} = \begin{pmatrix} 1+4u_1^2 & 4u_1u_2 \\ 4u_1u_2 & 1+4u_2^2 \end{pmatrix}$$

$$\frac{\partial x}{\partial u_2} = (0, 1, 2u_2)$$

$$\frac{\partial^2 x}{\partial u_i \partial u_j} = \begin{pmatrix} (0,0,2), & (0,0,0) \\ (0,0,0), & (0,0,2) \end{pmatrix}$$

function

vector $\rightarrow$ $$x \cdot \frac{\partial^2 x}{\partial u_i \partial u_j} = \begin{pmatrix} 2x_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$t=1$ case

$$g(x) = \langle x, x\rangle$$

$$dg_x = 2x\,dx = \sum_i 2x_i\,dx_i$$

$$dg_x(\vec{v}) = 2\langle x, \vec{v}\rangle$$

$\vec{v} \in T_x\mathbb{R}^3$

$$f(u) = \langle u, u\rangle + (u_1^2+u_2^2+t)^2$$

$$df_u = 2u\,du + 2(u_1^2+u_2^2+t)(2u_1du_1+2u_2du_2)$$

$$= (2u_1^2+2u_2^2+t+1)(2u\,du)$$

$$df_u(\vec{v}) = (2u_1^2+2u_2^2+t+1)\,2\langle u, \vec{v}\rangle$$

$\vec{v} \in T_u\mathbb{R}^2$

$$f(u_1,u_2) = u_1^2+u_2^2+(u_1^2+u_2^2+t)^2$$

$$\frac{\partial f}{\partial u_1} = 2u_1 + 2(u_1^2+u_2^2+t)\cdot 2u_1 \qquad = df_u(\vec{e}_1)$$

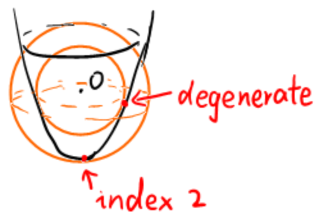
$$\frac{\partial f}{\partial u_2} = 2u_2 + 2(u_1^2+u_2^2+t)\cdot 2u_2 \qquad = df_u(\vec{e}_2)$$

$$\left(\frac{\partial^2 f}{\partial u_i \partial u_j}\right)_{i,j} = \begin{pmatrix} 2+12u_1^2+4(u_2^2+t) & 8u_1u_2 \\ 8u_1u_2 & 2+12u_2^2+4(u_1^2+t) \end{pmatrix}$$

$$\overset{u_1=u_2=0}{=\!=} \begin{pmatrix} 2+4t & 0 \\ 0 & 2+4t \end{pmatrix}$$

At the critical pt $(0,0,t) \in \mathbb{R}^3$, one has

$$\begin{cases} \text{index } 0 & t > -\frac{1}{2} \\ \text{degenerated critical pt} & t = -\frac{1}{2} \\ \text{index } 2 & t < -\frac{1}{2} \end{cases}$$



$t<-\frac{1}{2}$ case

E.g. Fix $a \in \mathbb{R}^r$.

For

$$f : \mathbb{R}^k \xrightarrow{\ x\ } \mathbb{R}^r \xrightarrow{\ g = \|\cdot - a\|^2\ } \mathbb{R}$$

$$u \longmapsto x(u)$$

$$x \longmapsto \langle x-a, x-a \rangle$$

one gets

$$\frac{\partial f}{\partial u_i} = \left\langle \frac{\partial}{\partial u_i}(x-a), x-a \right\rangle + \left\langle x-a, \frac{\partial}{\partial u_i}(x-a) \right\rangle$$

$$= 2 \left\langle x-a, \frac{\partial x}{\partial u_i} \right\rangle$$

coordinate calculation:

$$\frac{\partial f}{\partial u_i} = \sum_j \frac{\partial g}{\partial x_j} \frac{\partial x_j}{\partial u_i}$$

$$= \sum_j 2(x_j - a_j) \frac{\partial x_j}{\partial u_i}$$

$$= 2 \left\langle x-a, \frac{\partial x}{\partial u_i} \right\rangle$$

$$\frac{\partial^2 f}{\partial u_i \partial u_j} = 2 \left\langle \frac{\partial x}{\partial u_j}, \frac{\partial x}{\partial u_i} \right\rangle + 2 \left\langle x-a, \frac{\partial x}{\partial u_i \partial u_j} \right\rangle$$

$$= 2 \left(g_{ij} - (a-x) \cdot \underset{\Phi(u_i, u_j)}{\underset{\uparrow}{\vec{l}_{ij}}} \right)$$

Q: What happens in the cplx case?

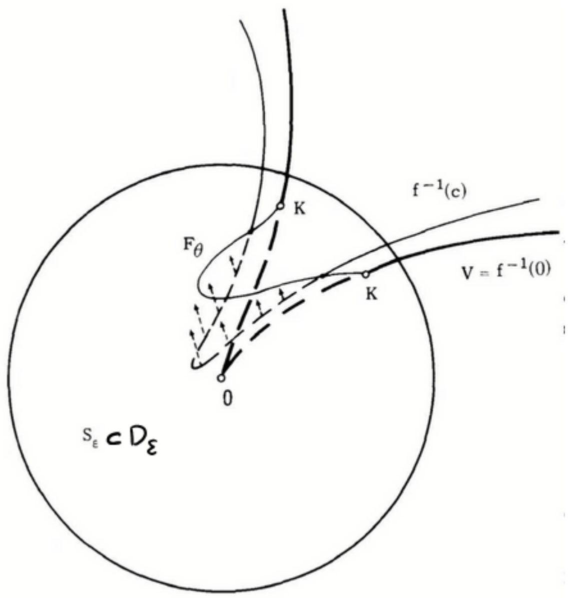


Figure 4.

from [MilnorSing, p54]

$$f^{-1}(c) \cap D_\varepsilon \overset{\text{diff}}{\cong} \{ z \in F_\theta \mid |f(z)| > |c| \} \xrightarrow[|c| \ll 1]{\text{diff}} F_\theta$$