

Eine Woche, ein Beispiel

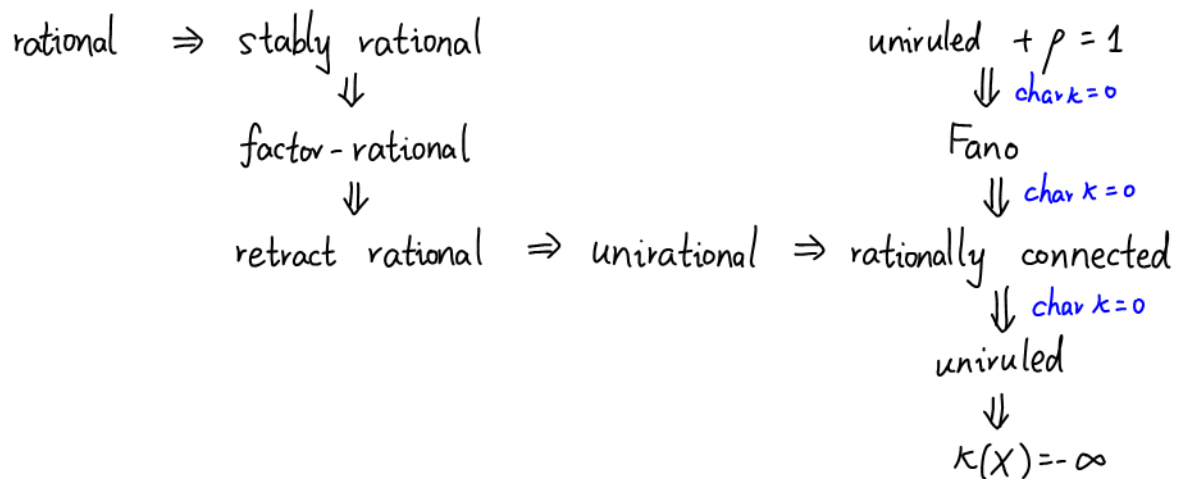
12.7 rationality in algebraic geometry

I heard these concepts in Jan Lange's talk, so I want to record them.

Ref:

[PP16]: Beauville, Arnaud, Brendan Hassett, Alexander Kuznetsov, and Alessandro Verra. Rationality Problems in Algebraic Geometry. Edited by Rita Pardini and Gian Pietro Pirola. Vol. 2172. Lecture Notes in Mathematics. Springer International Publishing, 2016. <https://doi.org/10.1007/978-3-319-46209-7>.

[Debo1]: Olivier Debarre. Higher-Dimensional Algebraic Geometry. Universitext, edited by S. Axler, F. W. Gehring, and K. A. Ribet. Springer, 2001. <https://doi.org/10.1007/978-1-4757-5406-3>.



This diagram is collected from the following resources:

[PP16, p14, p106]

[Debo1, 5.6]: mainly for Fano \Rightarrow rationally connected

<https://mathoverflow.net/questions/66569/uniruled-picard-number-1-fano>

In [PP16] everything is over \mathbb{C} , [Debo1] is a bit more relaxed. Still, most arrows are true (by checking the definition), so in char p they are still fine.

Variety	Unirational	Rational	Method	Reference
$V_6 \subset \mathbb{P}(1, 1, 1, 2, 3)$?	No	Bir(V)	[Gr]
Quartic double \mathbb{P}^3	Yes	No	JV	[V1]
$V_3 \subset \mathbb{P}^4$	"	No	JV	[C-G]
$V_{2,2} \subset \mathbb{P}^5, X_5 \subset \mathbb{P}^6$	"	Yes		
Sextic double \mathbb{P}^3	?	No	Bir(V)	[I-M]
$V_4 \subset \mathbb{P}^4$	Some	No	Bir(V)	[I-M]
$V_{2,3} \subset \mathbb{P}^5$	Yes	No (generic)	JV, Bir(V)	[B1, P]
$V_{2,2,2} \subset \mathbb{P}^6$	"	No	JV	[B1]
$X_{10} \subset \mathbb{P}^7$	"	No (generic)	JV	[B1]
$X_{12}, X_{16}, X_{18}, X_{22}$	"	Yes		
$X_{14} \subset \mathbb{P}^9$	"	No	JV	[C-G] + [F3] ¹

This comes from [PP16, p6].

Now we have better database: <https://www.fanography.info/>

Here, X : a variety, i.e., an integral scheme of f.t. over k .

Def. [PPI6, p4, p13-14]

X is rational	if \exists birational map $\mathbb{P}^n \xrightarrow{\sim} X$
X is stably rational	if \exists birational map $\mathbb{P}^n \xrightarrow{\sim} X \times \mathbb{P}^k$
X is factor-rational	if \exists birational map $\mathbb{P}^n \xrightarrow{\sim} X \times X'$
X is retract rational	if \exists rational dominant map $\mathbb{P}^n \dashrightarrow X$ + a rational section
X is unirational	if \exists rational dominant map $\mathbb{P}^n \dashrightarrow X$

Take a sm proj model V

X is rationally connected if $\forall p, q \in X, \exists$ a rational curve $C \cong \mathbb{P}^1$
passing p & q

[Deb01, Def 4.1]

X is uniruled if \exists rational dominant map $\mathbb{P}^1 \times X' \dashrightarrow X$