

# Eine Woche, ein Beispiel

## 5.15 Category

Everybody knows a little about category theory, but nobody can conclude all the terms emerged in the category theory. In this document I try to collect the notations and basic examples used in the course "Condensed Mathematics and Complex Geometry". I'm sure that it won't be better than the wikipedia, I just collect results I'm happy with.

I have to divide it into two parts which interact with each other, but you can always jump through examples which you're not familiar. You can also find a "complete" list of categorys here: <http://katmat.math.uni-bremen.de/acc/acc.pdf>

$\mathcal{C}$  is always a category.

	$Ob(\mathcal{C})$	$Mor(X, Y)$
small	Set	Set
loc. small	—	Set
large	not set	or not set

filtered:



cofiltered:



## Complete/Cocomplete/Bicomplete category

Def.  $\mathcal{C}$  is **complete** if

$\forall$  small category  $\Delta$ ,  $\forall$  factor  $F: \Delta \rightarrow \mathcal{C} \quad i \mapsto F_i$ ,  
 $\varprojlim_{i \in \Delta} F_i$  exists  $\left( \varprojlim_{i \in \Delta} F_i \text{ is called the small limit} \right)$

$\mathcal{C}$  is **cocomplete** if

$\forall$  small category  $\Delta$ ,  $\forall$  factor  $F: \Delta \rightarrow \mathcal{C} \quad i \mapsto F_i$ ,  
 $\varinjlim_{i \in \Delta} F_i$  exists  $\left( \varinjlim_{i \in \Delta} F_i \text{ is called the small colimit} \right)$

**bicomplete** = complete + cocomplete

$\mathcal{C}$  is **finitely complete** if  $\forall$  finite limit exists

$\mathcal{C}$  is **finitely cocomplete** if  $\forall$  finite colimit exists.

Thm.

$\mathcal{C}$  is complete  $\Leftrightarrow \mathcal{C}$  has equalizers & products

$\Leftrightarrow \mathcal{C}$  has pullbacks & products

$\mathcal{C}$  is cocomplete  $\Leftrightarrow \mathcal{C}$  has coequalizers & coproducts

$\Leftrightarrow \mathcal{C}$  has pushouts & coproducts

$\mathcal{C}$  is finitely complete  $\Leftrightarrow \mathcal{C}$  has equalizers & finite products

$\Leftrightarrow \mathcal{C}$  has equalizers, binary products & terminal obj

$\Leftrightarrow \mathcal{C}$  has pullbacks & terminal obj

For small category  $\mathcal{C}$ ,

complete  $\Leftrightarrow$  cocomplete

$\Rightarrow$

$\Leftarrow$

**thin**  $(\# \text{Mor}(X, Y) \leq 1)$

Cartesian closed category

Monoidal category = Tensor category

A list of categories which I'm interested:

Set Top Grp Ab Vect(k) Mod(R)

Ring: identity + preserve identity

CRing Rng

Field: full subcategory of CRing

$$0: Ob(0) = \emptyset$$

$$1: Ob(1) = \{*\} \quad Mor(*, *) = \{1_*\}$$

$$K(2): Ob(K(2)) = \{V, E\} \quad Mor(V, V) = \{1_V\} \quad Mor(E, E) = \{1_E\} \\ Mor(V, E) = \emptyset \quad Mor(E, V) = \{s, t\}$$

$$\begin{array}{ccc} 1_E & s & \\ \textcircled{Q} E & \xrightarrow{\quad} & V \textcircled{S} 1_V \\ & t & \end{array}$$

$$\Delta: Ob(\Delta) = \{[n] := \{0, 1, 2, \dots, n\} \mid n \geq 0\}$$

$$Mor([m], [n]) = \{\text{weakly monotone maps}\}$$

$$sSet: Ob(sSet) = \left\{ X: \Delta^{op} \rightarrow Set \atop [n] \mapsto X_n \right\} \quad Mor(X, Y) = \left\{ \alpha: \Delta^{op} \begin{array}{c} \xrightarrow{X} \\ \Downarrow \alpha \\ Y \end{array} Set \right\}$$

$$CHaus: Ob(CHaus) = \left\{ \underbrace{\text{cpt Hausdorff space}}_{\text{cptum/cpta}} X \right\}$$

<https://ncatlab.org/nlab/show/compactum>

$$Mor(X, Y) = \{f: X \rightarrow Y \mid f \text{ cont}\}$$

Met: full subcategory of CHaus whose objects are metric spaces.

! For the category of Graph, there're different realizations.

$$Quiv(e): Ob(Quiv(e)) = \{fctor \Gamma: K(2) \rightarrow e\}$$

$$Mor(\Gamma_1, \Gamma_2) = \left\{ \alpha: K(2) \begin{array}{c} \xrightarrow{\Gamma_1} \\ \Downarrow \alpha \\ \Gamma_2 \end{array} e \right\}$$

$$Quiv = Quiv(Set)$$

= Category of presheaves on  $\Delta^{op}$ .

$\mathbf{Cat}$  = {the category of small categories} is a 2-category.

$\text{Ob}(\mathbf{Cat}) = \{\text{small category } \mathcal{C}\}$

$\text{Mor}(\mathcal{C}, \mathcal{D})$  is a category by

$\text{Ob}(\text{Mor}(\mathcal{C}, \mathcal{D})) = \{F: \mathcal{C} \rightarrow \mathcal{D}\}$

$\text{Mor}(F, G) = \left\{ \alpha: \mathcal{C} \xrightarrow[\alpha]{F} \mathcal{D} \right\}$

Basic properties of  $\mathbf{Cat}$ :

1. Initial object  $0$ , Terminal object  $1$ .

2.  $\mathbf{Cat}$  is loc. small but not small

3.  $\mathbf{Cat}$  is bicomplete

4.  $\mathbf{Cat}$  is Cartesian closed but not loc. Cartesian closed

5.  $\mathbf{Cat}$  is **loc. finitely presentable**

<https://ncatlab.org/nlab/show/locally+finitely+presentable+category>

6.  $\mathbf{Cat} \xleftarrow[\text{forget}]{\text{free}} \mathbf{Quiv}$

e.g of "free"

$$f: \mathcal{C} \rightarrow \mathcal{D} \Leftarrow \cdot \mathcal{D} f$$

$$\begin{array}{ccc} 1_a \mathcal{C} & \begin{array}{c} \xrightarrow{efef} \\ \xrightarrow{efe} \\ \xrightarrow{e} \\ \xleftarrow{f} \\ \xleftarrow{efe} \end{array} & \cdot \mathcal{D} 1_b \\ a & & b \end{array} \Leftarrow \begin{array}{ccc} & e & \\ a & \xrightarrow{\quad} & b \\ & f & \end{array}$$

$$\begin{array}{ccc} 1_a \mathcal{C} & \begin{array}{ccc} \xrightarrow{f} & \cdot \mathcal{D} & \xrightarrow{g} \\ a & b & c \end{array} & \cdot \mathcal{D} 1_c \\ & \xrightarrow{gf} & \end{array} \Leftarrow \begin{array}{ccccc} & & & & \\ a & \xrightarrow{f} & b & \xrightarrow{g} & c \end{array}$$