Eine Woche, ein Beispiel

8.20. diagonalizable group

Ref:

[Borel91]: Borel, Linear Algebraic Groups
https://link.springer.com/book/10.1007/978-1-4612-0941-6

[PerrinAG]:
http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/ag-chap3.pdf
http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/ag-chap4.pdf

[Eberhardt23]: lecture notes of "spaces in GRT"
https://jenseberhardt.com/teaching/W2324data/Spaces%20in%20GRT.pdf

[Vakil]: Vakil, The Rising Sea: Foundations of Algebraic Geometry

In this document, κ is a field. In [PerrinAG], $\kappa = \bar{\kappa}$; in [Eberhardt23], $\kappa = \bar{\kappa}$, char $\kappa = 0$

https://mathoverflow.net/questions/12118/what-is-an-algebraic-group-over-a-noncommutative-ring
https://mathoverflow.net/questions/448426/is-diagonalizability-a-local-property

We follow the notation of [Vakil].

gpSch : scheme + gp
AffgpSch : affine + gpSch
gpVar : variety + gp $= \mathrm{GpSch}_\kappa$ + f.t. + reduced + sep + ~~(irr)~~
Alggp : sm + gpVar
AffAlggp : affine + Alggp = linear alg gp
AbVar : (conn)proj + Alggp + geo integral

$$\begin{array}{ccc}
\mathrm{gpSch} & \supset & \mathrm{gpVar} \\
\cup & & \cup \\
\mathrm{gpSch}^{sm} & \supset & \mathrm{Alggp} \\
\cup & & \cup \\
\mathrm{AffgpSch}^{sm} & \supset & \mathrm{AffAlggp} \\
 & & \\
\mathrm{AntiaffgpSch}^{sm} & \supset & \mathrm{AbVar}
\end{array}$$

https://math.stackexchange.com/questions/3237148/how-does-an-affine-algebraic-group-become-a-group-scheme
def of anti-affine group schemes:
https://link.springer.com/chapter/10.1007/978-93-86279-58-3_5
arxiv.org/abs/0710.5211

Chevalley's structure thm: [wiki] For κ perfect,
every sm conn alggp is an extension of an abelian variety by sm conn linea alggp.

We mainly follow [Borel91, §8] in the following material.

Def. $D \in \mathrm{AffAlggp}_{\Bbbk}$ is called diagonalizable, if $X^*(D)^{\Gamma_{\Bbbk}}$ generates $\Bbbk[D]$, where

$$\Bbbk[D] = \mathcal{O}_D(D)$$
$$X^*(D)^{\Gamma_{\Bbbk}} = \mathrm{Mor}_{\mathrm{Alggp}_{\Bbbk}}(D, \mathbb{G}_m) = \mathrm{Mor}_{\mathrm{Hopf}_{\Bbbk}}(\Bbbk[t^{\pm 1}], \Bbbk[D]) \subseteq \Bbbk[D]$$

Prop. [PerrinAG, Prop 3.3.2, Thm 4.1.8] for $\Bbbk = \bar{\Bbbk}$, TFAE: