# Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases

Setting We work over conn red gp over $K$.

$\bar{K}$: the seperable closure of $K$ mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_K := Gal(\bar{K}/K)$ $\sigma \in \Gamma_K$

$H^1(W,A) := Hom_{Grp}(W, A \rtimes W)/A\text{-conj}$ $\varphi \in H^1(W,A)$

Ref:
[ECII] Silverman, The Arithmetic of Elliptic Curves

## 1. forms.

**Def.** $G_1, G_2/K$ are called forms, if
$\exists\ \alpha : G_{2,\bar{K}} \xrightarrow{\sim} G_{1,\bar{K}}$ as gps — not as $\Gamma_K$-gps!

$\alpha$ is considered as the information of forms.

**Thm.** $\{K\text{-forms of } G\} \longleftrightarrow H^1(\Gamma_K, \mathrm{Aut}(G_{\bar{K}}))$

$[G_2,\ \alpha : G_{2,\bar{K}} \to G_{\bar{K}}] \longmapsto \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1}$

$G_2 \longleftarrow\!\shortmid\ \varphi$

$$\begin{tikzcd} G_{2,\bar{K}} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & G_{\bar{K}} \arrow[d, "\sigma"] \\ G_{2,\bar{K}} \arrow[r, "\alpha"] & G_{\bar{K}} \end{tikzcd} \qquad G_{2,\bar{K}} \xrightarrow{\alpha} G_{\bar{K}}$$

$\Gamma_K \circlearrowleft G_{2,\bar{K}} \longrightarrow G_{\bar{K}} \circlearrowleft \Gamma_K,\ \Gamma_K$

$\varphi_\alpha$ measures the commutativity between two $\Gamma_K$-actions.

$$G_2(K) := \{ g \in G(\bar{K}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_K \}$$

In general, $G_2(R) := \{ g \in G(\bar{K} \otimes_K R) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_K \}$

e.p. $G_2(\bar{K}) = \{ (\varphi(\sigma)^{-1} g)_{\sigma \in \Gamma_K} \in \prod_{\sigma\in\Gamma_K} G(\bar{K}) \mid g \in G(\bar{K}) \} \cong G(\bar{K})$

$G(\bar{K} \otimes_K \bar{K}) \cong G\left(\bigoplus_{\sigma\in\Gamma_K} \bar{K}\right) \cong \prod_{\sigma\in\Gamma_K} G(\bar{K})$

Functorial on $K$. (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])

Let $L/K$ be finite Galois.

$$\begin{array}{cccccc}
G_{2,L} & \{L\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_L, \mathrm{Aut}(G_{\bar{K}})) & & \varphi|_{\Gamma_L} \\
\updownarrow & \uparrow & & \uparrow & & \updownarrow \\
G_2 & \{K\text{-forms of } G\} & \longleftrightarrow & H^1(\Gamma_K, \mathrm{Aut}(G_{\bar{K}})) & & \varphi \\
 & \uparrow & & \uparrow & & \\
 & \{G_2/K : G_{2,L} \cong G_L\} & \longleftrightarrow & H^1(\mathrm{Gal}(L/K), \mathrm{Aut}(G_{\bar{K}})^{\Gamma_L}) & & \\
 & & & & \mathrm{Aut}(G_L) & \\
 & \uparrow & & \uparrow & & \\
 & 1 & & 1 & &
\end{array}$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} Aut(\mathbb{G}_m^n) & \subseteq & End(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$Hom(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow\!\shortmid n$

Therefore,

$$\begin{aligned} H^1(\Gamma_K, Aut(\mathbb{G}^n_{m,\bar{K}})) &= H^1(\Gamma_K, GL_n(\mathbb{Z})) \\ &= Hom_{Grp}(\Gamma_K, GL_n(\mathbb{Z})) / _{GL_n(\mathbb{Z})\text{-conj}} \\ &\overset{\text{when } K=\mathbb{R}}{=\!=} \{ g \in GL_n(\mathbb{Z}) \mid g^2 = Id \} / _{GL_n(\mathbb{Z})\text{-conj}} \end{aligned}$$

[ $\Gamma_K$ acts on $Aut(\mathbb{G}^n_{m,\bar{K}}) \subseteq End(\mathbb{G}^n_{m,\bar{K}})$ trivially:

see $\bar{K}$-pts, $n=1$:

$$\begin{tikzcd} \bar{K}^\times \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & \bar{K}^\times \arrow[d, "\sigma"] \\ \bar{K}^\times \arrow[r, "{}^{\sigma}\alpha"] & \bar{K}^\times \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & x^n \arrow[d, mapsto] \\ \sigma(x) & \sigma(x^n) = \sigma(x)^n \end{tikzcd}$$

$\Rightarrow {}^{\sigma}\alpha = \alpha$ ]

E.g. $n = 1$, $K = \mathbb{R}$

$$H^1(\Gamma_K, Aut(\mathbb{G}_m)) \cong \{ 1, -1 \} \qquad \varphi(\sigma) = (-)^{-1}$$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ \mathbb{G}_m & G = \ ?\ SO_{2,\mathbb{R}} \end{array}$$

$$\begin{aligned} G(\mathbb{R}) &= \{ g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}} \} \\ &= \{ g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g \} \\ &= \{ g \in \mathbb{C}^\times \mid |g| = 1 \} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = Spec\ \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$