

Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases

Setting. We work over conn red gp over F . (G/F conn red)

\bar{F} : the separable closure of F mainly care about \mathbb{R} & p -adic field case.

$$\Gamma_F := \text{Gal}(\bar{F}/F)$$

$$Z^1(W, A) := \left\{ \begin{array}{l} L\varphi: W \rightarrow A \rtimes W \\ \gamma \mapsto (\gamma_0, \gamma) := L\gamma \end{array} \middle| \begin{array}{l} L\varphi: \text{continuous group homo} \\ L\varphi \text{ is a section} \end{array} \right\}$$

$$\sigma \in \Gamma_F$$

$$\varphi \in H^1(W, A)$$

$$= \{ \varphi: W \rightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)\gamma(\varphi(\gamma')) \}$$

$$H^1(W, A) = Z^1(W, A)/A.$$

Def. (Split)

G is split if

G has a maximal torus T over F which is split.

Def. (Quasi-split)

G is quasi-split if

G has a Borel B over F .

Borel = maximal (Zar-closed) conn sd alg subgp

= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t G/H is projective

= closed subgp containing a Borel.

Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.

https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat-Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability

<https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf>

<https://mathoverflow.net/questions/121959/classification-of-tori-of-gl2-up-to-conjugation>

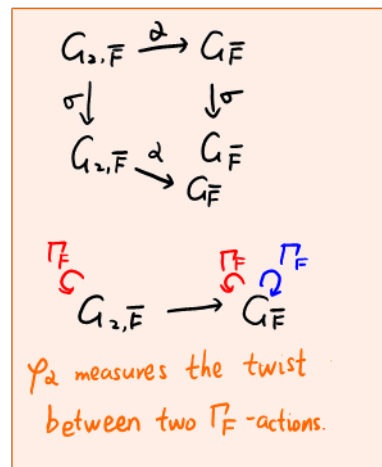
I make no claim to originality.

1. forms.

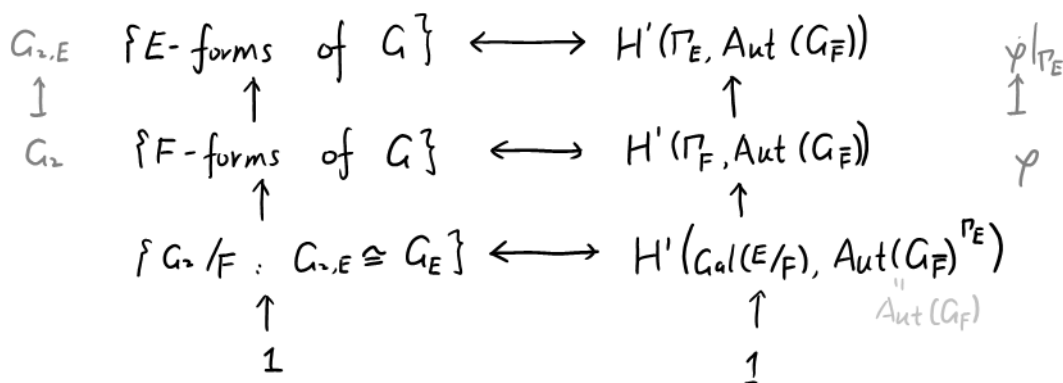
Def. $G_1, G_2/F$ are called forms, if
 $\exists \alpha: G_{2,\bar{F}} \xrightarrow{\sim} G_{1,\bar{F}}$ as qps not as Γ_F -qps!
 α is considered as the information of forms.

Thm. $\{F\text{-forms of } G\} \longleftrightarrow H'(\Gamma_F, \text{Aut}(G_{\bar{F}}))$
 $[G_2, \alpha: G_{2,\bar{F}} \rightarrow G_{\bar{F}}] \longmapsto \gamma_\alpha := \alpha \sigma \alpha^{-1} \sigma^{-1} \xrightarrow{\text{induced action}} \gamma$

$G_2(F) := \{g \in G(\bar{F}) \mid (\gamma(\sigma) \circ \sigma)g = g \quad \forall \sigma \in \Gamma_F\}$
 In general, $G_2(R) := \{g \in G(\bar{F} \otimes_F R) \mid (\gamma(\sigma) \circ \sigma)g = g \quad \forall \sigma \in \Gamma_F\}$
 e.p. $G_2(\bar{F}) = \{(\gamma(\sigma)^{-1}g)_{\sigma \in \Gamma_F} \in \prod_{\sigma \in \Gamma_F} G(\bar{F}) \mid g \in G(\bar{F})\} \cong G(\bar{F})$
 $G(\bar{F} \otimes_F \bar{F}) \cong G(\bigoplus_{\sigma \in \Gamma_F} \bar{F}) \cong \prod_{\sigma \in \Gamma_F} G(\bar{F})$



Functorial on F : (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])
 Let E/F be finite Galois.



Rmk. We have the classification of connected reductive gps:

$$\begin{array}{lcl}
 \{ \text{split red gp}/F \} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) \cong \Psi(G, B, T) \\
 \{ \text{qs red gp}/F \} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) + \Gamma_F\text{-action} \\
 & & = (X^*, \Delta, X_*, \Delta^\vee) + H'(\Gamma_F, \text{Out}(G_{\bar{F}})) \\
 \{ \text{red gp}/F \} & \longleftrightarrow & (X^*, \Delta, X_*, \Delta^\vee) + H'(\Gamma_F, \text{Aut}(G_{\bar{F}}))
 \end{array}$$

To understand the result, the following isos are needed:

$$\text{Aut}(G_{\bar{F}}) \cong \text{Inn}(G_{\bar{F}}) \rtimes \text{Aut}(G, B, T, \{u_\alpha\})$$

$$\begin{array}{ll}
 \text{Out}(G_{\bar{F}}) \cong \text{Aut}(G, B, T, \{u_\alpha\}) & \text{for embedding} \\
 \cong \text{Aut}(\Psi(G, B, T)) & \text{for combinatorics}
 \end{array}$$

Also, by the Hilbert 90, one has

$$H'(\Gamma_F, \text{Aut}(G, B, T, \{u_\alpha\})) \cong H'(\Gamma_F, \text{Aut}(G, B, T))$$

2. torus case

Let us try to find all the forms of the split torus G_m^n .

They're called (non-split) torus.

We know

$$\text{Aut}(G_m^n) \subseteq \text{End}(G_m^n)$$

$$\text{Hom}(G_m, G_m) \cong \mathbb{Z}$$

$$\begin{array}{ccc} \parallel & & \parallel \\ GL_n(\mathbb{Z}) & \subseteq & M^{n \times n}(\mathbb{Z}) \end{array}$$

$$(-)^n \leftrightarrow n$$

Therefore,

$$H^1(\Gamma_F, \text{Aut}(G_{m,\bar{F}})) = H^1(\Gamma_F, GL_n(\mathbb{Z}))$$

$$\begin{aligned} &= \text{Hom}_{\text{grp}}(\Gamma_F, GL_n(\mathbb{Z})) / GL_n(\mathbb{Z})\text{-conj} \\ &\stackrel{\text{when } F=\mathbb{R}}{=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = \text{Id}\} / GL_n(\mathbb{Z})\text{-conj} \end{aligned}$$

$$\left[\begin{array}{l} \Gamma_F \text{ acts on } \text{Aut}(G_{m,\bar{F}}) \subseteq \text{End}(G_{m,\bar{F}}) \text{ trivially.} \\ \text{see } \bar{F}\text{-pts, } n=1: \\ \begin{array}{ccc} \bar{F}^\times & \xrightarrow{\alpha} & \bar{F}^\times \\ \sigma \downarrow & & \downarrow \sigma \\ \bar{F}^\times & \xrightarrow{\sigma_\alpha} & \bar{F}^\times \end{array} \end{array} \right] \quad \left[\begin{array}{ccc} x & \mapsto & x^n \\ \downarrow & & \downarrow \\ \sigma(x) & & \sigma(x^n) = \sigma(x)^n \\ \Rightarrow \sigma_\alpha = \alpha \end{array} \right]$$

E.g. $n=1, F=\mathbb{R}$

$$\begin{array}{ccc} H^1(\Gamma_F, \text{Aut}(G_{m,\bar{F}})) \cong \{1, -1\} & \xrightarrow{\varphi(\sigma) = (-)^{-1}} & \\ \downarrow & & \downarrow \\ G_m & G = ? & SO_{2,\mathbb{R}} \end{array}$$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in G_m(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\} \\ &= \{g \in \mathbb{C}^\times \mid |g| = 1\} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = G_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = \text{Spec } \mathbb{R}[x, y] / (x^2 + y^2 - 1) = SO_{2,\mathbb{R}}$$

Check: $G(\mathbb{C}) = \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid x^2 + y^2 - 1 = 0\}$

$$\begin{aligned} &= \{(x, y) \in \mathbb{C} \times \mathbb{C} \mid (x+iy)(x-iy) = 1\} \\ &= \{(x, y, t) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^\times \mid \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array}\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mid x, y \in K, x^2 + y^2 = 1 \right\}$$

E.g. $n=2, F=\mathbb{R}$

$$H^1(\Gamma_F, \text{Aut}(\mathbb{G}_{m,\bar{F}})) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \right\}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\mathbb{G}_m \quad \mathbb{G}_m \times \text{SO}_{2,\mathbb{R}} \quad (\text{SO}_{2,\mathbb{R}})^2 \quad G = ? \quad \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \mid (\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$\parallel$$

$$\varphi(\sigma) \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

$$\begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \mid z_1 = \bar{z}_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow G = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Fact. Any (conn) \mathbb{R} -torus is product of $\mathbb{G}_m, \text{SO}_{2,\mathbb{R}}, \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$

$$\begin{array}{ccc} \updownarrow & \updownarrow & \updownarrow \\ 1 & -1 & \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \\ \updownarrow & \updownarrow & \updownarrow \end{array}$$

Fact^{dual}: $\text{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{\text{triv}}, \mathbb{Z}_{\text{sign}}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$
 i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

Rmk. Using the same argument, one can show that $\{ T/\mathbb{F}_p \text{ s.t. } T_{\mathbb{F}_p} \cong \mathbb{G}_{m,\mathbb{F}_p} \} = \text{products of } \mathbb{G}_m, \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}, \text{Res}_{\mathbb{F}_p/\mathbb{F}_p} \mathbb{G}_m$

The torus G crspd to -1 : Assume $\{ \in \mathbb{F}_p^\times \setminus \mathbb{F}_p, \{^2 = \varepsilon \in \mathbb{F}_p, \begin{pmatrix} \varepsilon \\ p \end{pmatrix} = -1$

$$G(\mathbb{F}_p) = \{ g \in \mathbb{G}_m(\mathbb{F}_p) \mid (\varphi(\sigma) \circ \sigma) g = g \quad \forall \sigma \in \Gamma_K \}$$

$$= \{ a+b\{ \in \mathbb{F}_p^\times \mid \varphi(\sigma)(a-b\{) = a+b\{ \}$$

$$\parallel$$

$$(a-b\{)^{-1}$$

$$= \{ a+b\{ \in \mathbb{F}_p^\times \mid a^2 - b^2 \varepsilon = 1 \}$$

$$\cong \left\{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \in \text{GL}_2(\mathbb{F}_p) \right\}$$

3. other cases.

By using the same methods introduced in last section,
we can compute the (inner) forms of reductive gps.

	inner forms	outer forms
G_m $(G_m)^2$ $(G_m)^n$	\emptyset	SO_2 $SO_2 \times G_m, (SO_2)^2, \text{Res}_{\mathbb{C}/\mathbb{R}} G_m$ product of lower rank torus
$GL_{2,\mathbb{R}}$ $SL_{2,\mathbb{R}}$ $PG_{2,\mathbb{R}}$	$H^x = GL_1(H \otimes_{\mathbb{R}} -)$ $H^{N_m=1} = SU_{2,\mathbb{C}/\mathbb{R}}$?	$(U_{2,\mathbb{C}/\mathbb{R},\omega} = U(1,1) \ U(2,0))$ \emptyset \emptyset
$GL_{n,\mathbb{R}}$ $SL_{n,\mathbb{R}}$ $PG_{n,\mathbb{R}}$? $GL_{n/2}^{N_m=1}(H \otimes_{\mathbb{R}} -)$ when n even ?	? $SU(a, n-a)$ ep. $SU(2,1)$? ← need clarification
$(SL_2)^2 / \mathbb{R}$ $(SL_2)^3 / \mathbb{R}$	$SL_2 \times SU_2, (SU_2)^2, \dots$ (8-1) possibilities	$\text{Res}_{\mathbb{C}/\mathbb{R}} SL_2$?

? : I have no time to compute / don't know any symbol to represent

 : quasi-split gp

Compute $\text{Aut}(G_{\mathbb{F}})$

Lemma: We understand $\text{Aut}(G_{\mathbb{F}})$ quite well.

	$G(\mathbb{F})/Z(G(\mathbb{F})) = G^{\text{ad}}(\mathbb{F})$	$\text{Aut}(\mathbb{F}_0)$	
G	$\xrightarrow{1} \text{Inn}(G_{\mathbb{F}}) \longrightarrow \text{Aut}(G_{\mathbb{F}}) \xrightarrow{1} \text{Out}(G_{\mathbb{F}}) \longrightarrow 1$		
$T \text{ rk } n$	1	$GL_n(\mathbb{Z})$	$GL_n(\mathbb{Z})$
$GL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$
$SL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1
$PGL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1
$n \geq 3$			
$GL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$	$\{\pm 1\}^{\oplus 2}$
$SL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$
$PGL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$
$(SL_2)^2/\mathbb{R}$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$
$\text{Res}_{\mathbb{C}/\mathbb{R}} SL_2$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$
$(SL_2)^n/\mathbb{R}$	$PGL_n(\mathbb{C})^n$	$PGL_n(\mathbb{C})^n \rtimes S^n$	S^n

with different $\Gamma_{\mathbb{R}}$ -action

Compute $H'(\Gamma_F, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP]

[DR09]

Theorem 12.7.7 *There is a functorial isomorphism $H^1(k, G) \rightarrow \pi_1(G)_{\Theta, \text{tor}}$.*

COROLLARY 2.4.3. *The composition*

$$[\bar{X}/(1 - \vartheta)\bar{X}]_{\text{tor}} \xrightarrow{\sim} H^1(F, \Omega_C) \xrightarrow{a_*^{-1}} H^1(F, N_C) \xrightarrow{b_*} H^1(F, G)$$

is a bijection.

$$[\bar{X}/(1 - \vartheta)\bar{X}]_{\text{tor}} = \text{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field:

Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
<https://arxiv.org/abs/1401.5913>

3.1. Theorem. *Let G , T_0 , T , and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \rightarrow H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \rightarrow H^1(\mathbb{R}, G)$ is a bijection.

global field:

Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
<https://arxiv.org/pdf/2303.04120.pdf>

E.g. $G = SL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$G \quad 1 \rightarrow \text{Inn}(G_{\mathbb{F}}) \rightarrow \text{Aut}(G_{\mathbb{F}}) \rightarrow \text{Out}(G_{\mathbb{F}}) \rightarrow 1$$

$$SL_{2,\mathbb{R}} \quad PGL_2(\mathbb{C}) \quad PGL_2(\mathbb{C}) \quad 1$$

$$\begin{aligned} H'(\Gamma_{\mathbb{R}}, \text{Aut}(SL_{2,\mathbb{C}})) &= H'(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C})) \\ &= \{1, \omega(\cdot)\omega^{-1}\} \quad \omega = \begin{pmatrix} -1 & 1 \end{pmatrix} \\ &\quad \downarrow \quad \downarrow \\ &SL_{2,\mathbb{C}} \quad G = ? \quad \mathbb{H}^{N_M=1} \end{aligned}$$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in SL_2(\mathbb{C}) \mid \omega \bar{g} \omega^{-1} = g\} \\ &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \mid \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}_{\begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{a} \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SL_2(\mathbb{C}) \right\} \\ &= \mathbb{H}^{N_M=1} \end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$G \quad 1 \rightarrow \text{Inn}(G_{\mathbb{F}}) \rightarrow \text{Aut}(G_{\mathbb{F}}) \rightarrow \text{Out}(G_{\mathbb{F}}) \rightarrow 1$$

$$GL_{2,\mathbb{R}} \quad PGL_2(\mathbb{C}) \quad PGL_2(\mathbb{C}) \rtimes \{\pm 1\} \quad \{\pm 1\}$$

$$\begin{aligned} H'(\Gamma_{\mathbb{R}}, \text{Aut}(GL_{2,\mathbb{C}})) &= \{1, \omega(\cdot)\omega^{-1}, (\cdot)^{H,-1}, \omega(\cdot)^{H,-1}\omega^{-1}\} \quad \omega = \begin{pmatrix} -1 & 1 \end{pmatrix} \\ &\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\ &GL_{2,\mathbb{C}} \quad \mathbb{H}^* \quad (\mathcal{U}(1,1)) \quad \mathcal{U}_{2,\mathbb{C}/\mathbb{R},\omega} \end{aligned}$$

$$\{F\text{-forms of } G\} \longleftrightarrow H'(\Gamma_K, \text{Aut}(G_{\bar{K}}))$$

$$\{F\text{-inner forms of } G\} \longleftrightarrow H'(\Gamma_F, \text{Inn}(G_{\bar{F}}))$$

$$G \text{ split: } \left\{ \begin{array}{l} F\text{-forms of } G \\ \text{which are quasi-split} \end{array} \right\} \longleftrightarrow H'(\Gamma_F, \underset{\text{"}}{\text{Aut}(G_{\bar{F}}, B, T)}) \cong H'(\Gamma_F, \text{Out}(G_{\bar{F}}))$$

Γ_F -actions on $(X^*, \Delta, X_*, \Delta^\vee)$

Q: Do we have

$$\begin{array}{c} \hookrightarrow H'(\Gamma_F, \text{Inn}(G_{\bar{F}})) \hookrightarrow H'(\Gamma_F, \text{Aut}(G_{\bar{F}})) \rightarrow H'(\Gamma_F, \text{Out}(G_{\bar{F}})) \\ \hline 1 \rightarrow \underset{\text{Inn}(G_F)}{\text{Inn}(G_{\bar{F}})^{\Gamma_F}} \rightarrow \underset{\text{Aut}(G_F)}{\text{Aut}(G_{\bar{F}})^{\Gamma_F}} \rightarrow \underset{\text{Out}(G_F)}{\text{Out}(G_{\bar{F}})^{\Gamma_F}} \end{array}$$

Give one example s.t. $H'(\Gamma_F, \text{Inn}(G_{\bar{F}})) \rightarrow H'(\Gamma_F, \text{Aut}(G_{\bar{F}}))$ is not inj?