Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

Top. space	space	continuous map	Covering of open sets	Sh	cohomology
site = Category + Groth cover	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, $\bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	$U_i \to U_i$ over X	—		
$(Sch/X)_{Zar}$	$Ob(Sch/X)$	$Mor(Sch/X)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		
$(Sch/X)_{ét}$	$Ob(Sch/X)$	$Mor(Sch/X)$	ét + l.f.p		
$(Sch/X)_{sm}$	$Ob(Sch/X)$	$Mor(Sch/X)$	smooth + l.f.p		
$(Sch/X)_{fppf}$	$Ob(Sch/X)$	$Mor(Sch/X)$	f.flat + l.f.p		
$(Sch/X)_{fpqc}$	$Ob(Sch/X)$	$Mor(Sch/X)$	f.flat + f_i^{-1}(q.c) locally q.c		

https://pbelmans.ncag.info/notes/etale-cohomology.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
it tells us why we don't have small site for most condition.

Thm. ① equiv. of categories

$G_k := Gal(k^{sep}/k)$

$$Sets((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \text{Disc } G_k\text{-Set}$$
$$Ab((\mathrm{Spec}\ k)_{ét}) \longleftrightarrow \text{Disc } Mod_{G_k} \quad (*)$$

$(\mathrm{Spec}\ k)_{ét} \overset{\text{Site}}{\longleftrightarrow} \text{finite } G_k\text{-Set}$

② $(*)$ preserve cohomology
$$H^1((\mathrm{Spec}(k))_{ét}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{C})_{ét}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\mathrm{Spec}\ \mathbb{C})$)

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{R})_{ét}$

$$\begin{tikzcd}
\mathrm{Spec}\ \mathbb{C} \arrow[dr, "\iota^*"'] & \mathrm{Spec}\ \mathbb{C} \arrow[l, "\sigma^*"'] \arrow[d, "\iota^*"] \\
& \mathrm{Spec}\ \mathbb{R}
\end{tikzcd}
\quad \overset{\mathrm{Spec}}{\rightsquigarrow} \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\sigma"] & \mathbb{C} \\
\mathbb{R} \arrow[u, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

σ = conjugation, ι = embedding

$\Big\downarrow \mathcal{F}$

$$\begin{tikzcd}
\mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \arrow[r, "\mathcal{F}(\sigma^*)"] & \mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \\
\mathcal{F}(\mathrm{Spec}\ \mathbb{R}) \arrow[u, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"'] &
\end{tikzcd}
\quad \xrightarrow{\text{Abuse of notation}} \quad
\begin{tikzcd}
\mathcal{F}(\mathbb{C}) \arrow[r, "\sigma"] & \mathcal{F}(\mathbb{C}) \\
\mathcal{F}(\mathbb{R}) \arrow[u, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\quad Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

[$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed.

in this case $0 \to \mathcal{F}(\mathrm{Spec}\,\mathbb{R}) \to \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{F}(\mathrm{Spec}\,\mathbb{C} \amalg \mathrm{Spec}\,\mathbb{C})$

$$\begin{array}{ccc}
\mathcal{F}(\mathrm{Spec}\,\mathbb{C}) & \longrightarrow & \mathcal{F}(\mathrm{Spec}\,\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}\left(\mathrm{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}\right) \\
& \iota_1: x \longmapsto x\otimes 1 \qquad x\otimes y \longmapsto (xy, x\bar{y}) & \cong \\
\| & \iota_2: x \longmapsto 1\otimes x & \mathcal{F}\left(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathrm{Spec}\,\mathbb{C}\right) \\
& & \cong \\
\mathcal{F}(\mathrm{Spec}\,\mathbb{C}) & \longrightarrow & \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \times \mathcal{F}(\mathrm{Spec}\,\mathbb{C})
\end{array}$$

ι_2: $\mathrm{Spec}\,\mathbb{C} \xleftarrow{(Id,\sigma)} \mathrm{Spec}\,\mathbb{C} \amalg \mathrm{Spec}\,\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \xrightarrow{(\mathcal{F}(Id),\mathcal{F}(\sigma))} \mathcal{F}(\mathrm{Spec}\,\mathbb{C} \amalg \mathrm{Spec}\,\mathbb{C}) \cong \mathcal{F}(\mathrm{Spec}\,\mathbb{C}) \times \mathcal{F}(\mathrm{Spec}\,\mathbb{C})$

$\overset{\text{Abuse}}{\underset{\text{of notation}}{\rightsquigarrow}}$ $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$]

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\mathrm{Spec}\,k, \mathcal{F}) = \mathcal{F}(\mathrm{Spec}\,k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)} \qquad \mathcal{F}_{k^{sep}} := \varinjlim_{\substack{L/k \\ \text{finite}}} \mathcal{F}(\mathrm{Spec}\,L)$$

Ex. describe the stalk & fiber at $p \in \mathrm{Spec}\,k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\mathrm{Spec}\,k,p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts

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