

Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety

Ref:

[Ginz] Ginzburg's book "Representation Theory and Complex Geometry"

[LCBE] Langlands correspondence and Bezrukavnikov's equivalence

[LW-BWB] The notes by Liao Wang: The Borel-Weil-Bott theorem in examples (can not be found on the internet)

<https://people.math.harvard.edu/~gross/preprints/sat.pdf>

Task. Complete the following tables.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
G	$\mathbb{Z}[X^*(T)]^W$		$\mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$	$\mathbb{Z}[W_{ext}]$
B	$\mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$		$\mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)] \otimes_{\mathbb{Z}[X^*(T)]^W} \mathbb{Z}[X^*(T)]$	
Id	\mathbb{Z}					$\mathbb{Z}[X^*(T)] /_{I_T} \rtimes \mathbb{Z}[W_f]$
$G \times \mathbb{C}^*$	$\mathbb{Z}[X^*(T)]^W[t^{\pm 1}]$					\mathcal{H}_{ext}
$B \times \mathbb{C}^*$	$\mathbb{Z}[X^*(T)][t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

We use the shorthand.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
G	$R(T)^W$	$R(T)$		$R(T) \otimes_{R(G)} R(T)$		$\mathbb{Z}[W_{ext}]$
B	$R(T)$	$R(T) \otimes_{R(G)} R(T)$		$R(T) \otimes_{R(G)} R(T) \otimes_{R(G)} R(T)$		
Id	\mathbb{Z}					$R(T) /_{I_T} \rtimes \mathbb{Z}[W_f]$
$G \times \mathbb{C}^*$	$R(G)[t^{\pm 1}]$					\mathcal{H}_{ext}
$B \times \mathbb{C}^*$	$R(T)[t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

$$\begin{aligned}
 R(B) &= \mathbb{Z}[X^*(T)] &= \mathcal{H}(\hat{\tau}(F), \hat{\tau}(\mathcal{O}_F)) \\
 R(G) &= \mathbb{Z}[X^*(T)]^W &\neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
 R(G)[q^{\pm \frac{1}{2}}] &= \mathbb{Z}[X^*(T)]^W[q^{\pm \frac{1}{2}}] &= \mathcal{H}_{sph}[q^{\pm \frac{1}{2}}] \\
 R(G \times \mathbb{C}^*) &= \mathbb{Z}[X^*(T)]^W[t^{\pm 1}] \\
 K^{G \times \mathbb{C}^*}(St) &= \mathcal{H}_{ext} &\neq \mathcal{H}(\hat{G}(F), I)
 \end{aligned}$$

Here is an initial example.

$K^{-}(-)$	pt	\mathcal{B}	$T^*\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$	$T^*(\mathcal{B} \times \mathcal{B})$	St
SL_2	$\mathbb{Z}[x]$	$\mathbb{Z}[z^{\pm 1}]$		$\mathbb{Z}[z_1^{\pm 1}, z_2^{\pm 1}] / (z_1 - z_2)(z_1 - z_2^{-1})$		$\mathbb{Z}[W_{ext}] = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}]$
B	$\mathbb{Z}[y^{\pm 1}]$	$\mathbb{Z}[y^{\pm 1}, z] / (z - y)(z - y^{-1})$		$\mathbb{Z}[y_1^{\pm 1}, z_1, z_2] / ((z_1 - y_1)(z_1 - y_1^{-1}), (z_2 - y_1)(z_2 - y_1^{-1}))$		
Id	\mathbb{Z}	$\mathbb{Z}[z] / (z - 1)^2$		$\mathbb{Z}[z_1, z_2] / (z_1 - 1)^2, (z_2 - 1)^2$		$R(T) /_{I_T} \rtimes \mathbb{Z}[W_f] = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}] / (\bar{z}_w - 1)^2$
$SL_2 \times \mathbb{C}^*$	$\mathbb{Z}[x, t^{\pm 1}]$					$\mathcal{H}_{ext} = \bigoplus_{w \in W} \mathbb{Z}[\bar{z}_w^{\pm 1}, t^{\pm 1}]$
$B \times \mathbb{C}^*$	$\mathbb{Z}[y^{\pm 1}, t^{\pm 1}]$					
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$					

This is our final task. Most of the notations are still not fixed.

$K^-(-)$	pt	$\mathcal{F}_d \quad \widetilde{\text{Rep}}_d(\mathcal{Q})$	$\mathcal{F}_d \times \mathcal{F}_{d'}$	$\mathcal{Z}_{d,d'}$
G_d	$R(T_d)^{W_d}$	$R(T_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$	
B_d	$R(T_d)$ $\oplus_{w \in W_d} R(G_d)$	$R(T_d) \otimes_{R(G_d)} R(T_d)$ $\oplus_{w \in W_d} R(T_d) [\overline{\Omega_w}]^{T_d}$	$R(T_d) \otimes_{R(G_d)} R(T_d) \otimes_{R(G_d)} R(T_d)$ $\oplus_{w, w' \in W_d} R(T_d) [\overline{\Omega_{w, w'}}]^{T_d}$	$\oplus_{w, w' \in W_d} R(T_d) [\overline{\Omega_{w, w'}}]^{T_d}$
Id	\mathbb{Z}	$\oplus_{w \in W_d} \mathbb{Z} [\overline{\Omega_w}]$	$\oplus_{w, w' \in W_d} \mathbb{Z} [\overline{\Omega_{w, w'}}]$	$\oplus_{w, w' \in W_d} \mathbb{Z} [\overline{\Omega_{w, w'}}]$
$G_d \times \mathbb{C}^*$	$R(G_d)[t^{\pm 1}]$			
$B_d \times \mathbb{C}^*$	$R(T_d)[t^{\pm 1}]$ $\oplus_{w \in W_d} R(G_d \times \mathbb{C}^*)$	$\oplus_{w \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_w}]^{T_d \times \mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{T_d \times \mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{T_d \times \mathbb{C}^*}$
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$	$\oplus_{w \in W_d} R(\mathbb{C}^*) [\overline{\Omega_w}]^{\mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(\mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{\mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(\mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{\mathbb{C}^*}$

$K^-(-)$	pt	$\mathcal{F}_d \quad \widetilde{\text{Rep}}_d(\mathcal{Q})$	$\mathcal{F}_d \times \mathcal{F}_d$	$\mathcal{Z}_d = \coprod_{d, d'} \mathcal{Z}_{d, d'}$
G_d	$R(T_d)^{W_d}$	$\oplus_d R(T_d)$	$\oplus_d R(T_d) \otimes_{R(G_d)} R(T_d)$	
B_d	$R(T_d)$ $\oplus_{w \in W_d} R(G_d)$	$\oplus_d R(T_d) \otimes_{R(G_d)} R(T_d)$ $\oplus_{w \in W_d} R(T_d) [\overline{\Omega_w}]^{T_d}$	$\oplus_d R(T_d) \otimes_{R(G_d)} R(T_d) \otimes_{R(G_d)} R(T_d)$ $\oplus_{w, w' \in W_d} R(T_d) [\overline{\Omega_{w, w'}}]^{T_d}$	$\oplus_{w, w' \in W_d} R(T_d) [\overline{\Omega_{w, w'}}]^{T_d}$
Id	\mathbb{Z}	$\oplus_{w \in W_d} \mathbb{Z} [\overline{\Omega_w}]$	$\oplus_{w, w' \in W_d} \mathbb{Z} [\overline{\Omega_{w, w'}}]$	$\oplus_{w, w' \in W_d} \mathbb{Z} [\overline{\Omega_{w, w'}}]$
$G_d \times \mathbb{C}^*$	$R(G_d)[t^{\pm 1}]$			
$B_d \times \mathbb{C}^*$	$R(T_d)[t^{\pm 1}]$ $\oplus_{w \in W_d} R(G_d \times \mathbb{C}^*)$	$\oplus_{w \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_w}]^{T_d \times \mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{T_d \times \mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(T_d \times \mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{T_d \times \mathbb{C}^*}$
\mathbb{C}^*	$\mathbb{Z}[t^{\pm 1}]$	$\oplus_{w \in W_d} R(\mathbb{C}^*) [\overline{\Omega_w}]^{\mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(\mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{\mathbb{C}^*}$	$\oplus_{w, w' \in W_d} R(\mathbb{C}^*) [\overline{\Omega_{w, w'}}]^{\mathbb{C}^*}$

$$H_*^{G_d}(\mathcal{Z}_{d, d}) \cong \bigotimes_{d_i} N H_{d_i}$$

$$K^{G_d}(\mathcal{Z}_{d, d}) \cong R(T_d) \otimes_{R(G_d)} R(T_d) \cong \bigotimes_{d_i} R(T_{d_i}) \otimes_{R(G_{d_i})} R(T_{d_i})$$

Black: know the alg structure under tensor prod (not convolution prod)

Orange: only know the $R(\text{Grp})$ -module structure, and the alg structure is yet not known

light yellow: $R(G_d)$ -module + W_d -equiv iso

$$d = (1, 2) \quad \begin{array}{l} a \rightarrow b \\ \langle v_1 \rangle \rightarrow \langle v_1, v_3 \rangle \end{array}$$

⚠ The action on Flag is not the same as in

http://www.math.uni-bonn.de/ag/stroppel/Master%20Thesis_Tomasz%20Przezdziecki.pdf

	$tu = wu$		w	$d = u$	order of basis	$l(w)$	$l(u)$	B_w	B_u	wB_u^{-1}
Id	Id $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	$\begin{array}{ c c c } \hline \square & \square & \square \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \square & \square & \square \\ \hline \end{array}$	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
t	$(23) \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	$\begin{array}{ c c c } \hline \square & \times & \square \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \times & \square & \square \\ \hline \end{array}$	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
s	$(12) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	$\begin{array}{ c c c } \hline \times & \square & \square \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \square & \times & \square \\ \hline \end{array}$	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
ts	$(132) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	$\begin{array}{ c c c } \hline \times & \times & \square \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \times & \times & \square \\ \hline \end{array}$	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
st	$(123) \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	$\begin{array}{ c c c } \hline \times & \times & \square \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \square & \times & \times \\ \hline \end{array}$	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
sts	$(13) \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	$\begin{array}{ c c c } \hline \times & \times & \times \\ \hline \end{array}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{array}{ c c c } \hline \times & \times & \times \\ \hline \end{array}$	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

Conclusions on the summer vacation.

I guess that most part of my tasks coincide with this paper:

http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

Sadly, I only found it in the last week of the vacation.

Some possible tasks to work on:

1. Work out what $K_0^{\text{Id}}(\mathcal{B})$ is.

ref:

In [3264], the author computes the Chow group of $G(2,4)$.

<https://pbelmans.ncag.info/blog/2018/08/22/rank-flag-varieties/>

http://www.math.tau.ac.il/~bernstei/Publication_list/publication_texts/BGG-SchubCells-Usp.pdf

The module structure is easy, see

[<https://math.stackexchange.com/questions/1012699/when-does-a-smooth-projective-variety-x-have-a-free-grothendieck-group>]

2. Work out what $\mathcal{H}(G(F), I)$ is, i.e.

- Bernstein presentation

- try to understand the center of $\mathcal{H}(G(F), I)$

- How does $\mathcal{H}(G(F), I)$ reflect informations on the rep theory

- How can the Hecke algebra be realized as a Hecke algebra?

ref. [Hecke, Sec 10-17], [Williamson 11.4-12.2]

3. Try to understand what the Hall algebra / Quantum group is.

ref: [Lec 1-4, Appendix 4, <https://arxiv.org/pdf/math/0611617.pdf>]

- understand $\mathcal{H}_{\text{Rep}_k(Q)}^{\text{nil}}(Q)$ where $Q = \begin{smallmatrix} & & & & \\ & \bullet & & \bullet & \\ & & \bullet & & \\ & & & \bullet & \\ & & & & \bullet \end{smallmatrix} \rightarrow \begin{smallmatrix} & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{smallmatrix}$

[Lec 2-3]

- understand $\mathcal{H}_{\text{IP}'} \cong \bigcup \mathcal{U}_\nu(\widehat{\mathfrak{sl}}_2)$

[Lec 4]

$$\mathcal{H}_{\text{Tor}(\text{IP}')} \cong \bigotimes_{x \in \text{IP}'} \mathcal{H}_{\text{Tor } x}$$

- define (Quantum) Kac-Moody / loop algs

[Appendix 4]

- Why is that

$$K_0(\text{Rep}^{\mathbb{Z}}(R)) = \mathcal{U}_q(n(Q))$$

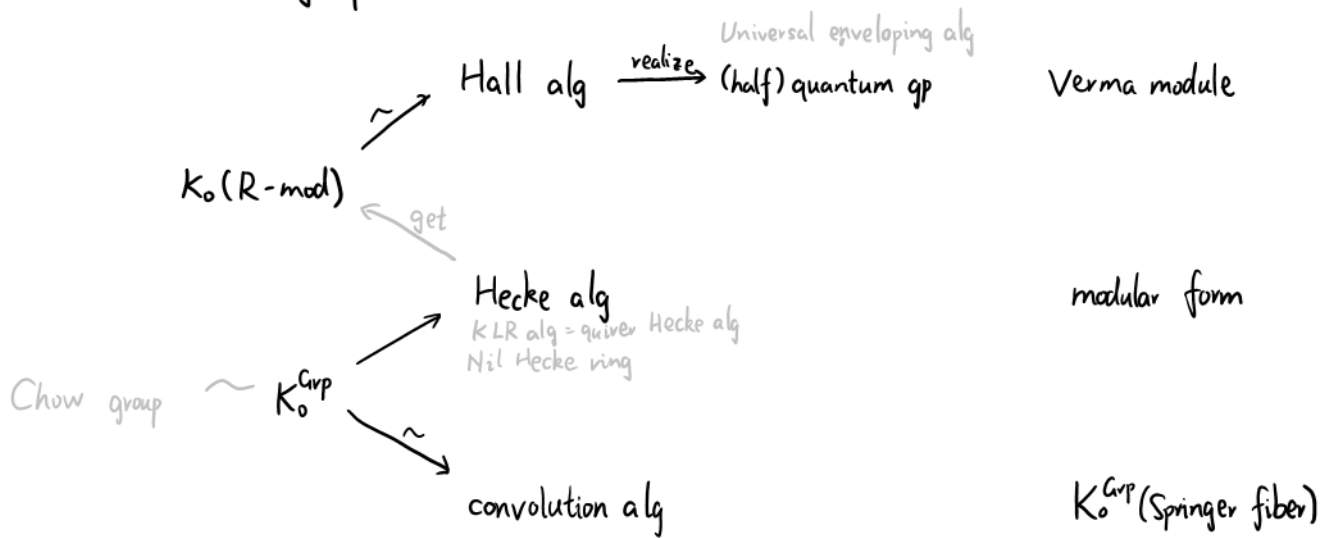
where

$$R = \bigoplus_d H^{G \times \mathbb{C}^\times, BM}(\mathbb{Z}_d)$$

and what is

$$K_0(\text{Rep}^{\mathbb{Z}}(\bigoplus_d K_0^{G \times \mathbb{C}^\times}(\mathbb{Z}_d))) ?$$

4. Work out the big picture



5. A closer check of Satake iso

$$\begin{aligned}
 & K_0 \quad \text{combinations} \quad \text{Hecke alg} \\
 & R(B) = \mathbb{Z}[x^*(T)] = \mathcal{H}(\hat{T}(F), \hat{T}(\mathcal{O}_F)) \\
 & R(G) = \mathbb{Z}[x^*(T)]^W \neq \mathcal{H}(\hat{G}(F), \hat{G}(\mathcal{O}_F)) \\
 & R(G)[q^{\pm \frac{1}{2}}] = \mathbb{Z}[x^*(T)]^W [q^{\pm \frac{1}{2}}] = \mathcal{H}_{sph}[q^{\pm \frac{1}{2}}] \\
 & R(G \times \mathbb{C}^*) = \mathbb{Z}[x^*(T)]^W [t^{\pm 1}] \\
 & R(T) \otimes_{R(G)} R(T) = NH_n \subset \text{End}_{\mathbb{Z}}(\mathbb{Z}[x^*(T)]) \\
 & K^{G \times \mathbb{C}^*}(St) = \mathcal{H}_{ext} \neq \mathcal{H}(\hat{G}(F), I)
 \end{aligned}$$

$$G = GL_n$$

It's claimed by my schoolmate that

$$K_0(\text{Perv}_B(G/B)) \cong \mathcal{H}(G, B)$$

↑
sym monoidal structure
induced from the convolution

then, what is

$$\begin{array}{ccc}
 K_0^B(\mathcal{B}) & \cong & ? \\
 K_0^{Id}(\mathcal{B}) & \cong & ? \\
 ? & \cong & \mathcal{H}(S_{m+n}, S_m \times S_n)
 \end{array}$$

Now, about Steinberg varieties.

6. Draw a picture, indicating the shape/generalization of the following spaces:
(e.g. in the case of \cdot , $\cdot \circ$, $\cdot \rightarrow \cdot$)

G, B, T

B, T^*B, St

$\mathfrak{g}, \hat{\mathfrak{g}}, \mathfrak{g}^{sr}, \hat{\mathfrak{g}}^{sr}, N, \tilde{N}, h, n$

$\hat{\mathfrak{g}}^h, \mathcal{O}_h, \Delta_w^h$

7. Try to understand what Kazhdan-Lusztig polynomials are [KL2], and

- Compute the transformation matrix between

$\{[T_w^*], w \in W_f\}$ and $\{[\Delta_w^h], w \in W_f\}$? [Ka Sai]?

- understand what standard/crystal basis is

- understand the relationship between KL poly and crystal basis

- see if it is related to two basis in $\text{Rep}(G)$ (irr reps & multiplicative basis)

Answer from my schoolmate:

<p>standard basis</p> <p>$\{[T_w^*], w \in W_f\}$</p> <p>irr reps</p>	$\xleftrightarrow{\text{KL-poly}}$	<p>canonical basis</p> <p>$\{[\Delta_w^h], w \in W_f\}$</p> <p>multiplicative basis</p>	$\xrightarrow{\text{fix } q}$	<p>crystal basis</p>
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8. Try to understand the module part, i.e.,

- numbers of components of the Springer fiber

- how does $K_0^{\text{Grp}}(St)$ act on $K_0^{\text{Grp}}(\text{Springer fiber})$ also act on $K_0^{\text{Grp}}(\text{Rep}_d(\mathbb{Q}))$

- does that occupy "all rep" of $K_0^{\text{Grp}}(St)$

9. Ways of finding multiplication structure

1. By direct computation (with techniques)

Hecke algebra

double coset calculus

2. By formulas as alg-isos

$K_0^G(B)$

induction formula

3. By geometrical computation

cohomology

cup product ? de Rham calculus

index theorem



intersection theory

Chow group

4. By deformation (indirect)

$H_{top}^{BM}(St)$

$K_0^{G \times G}(St)$

How to get a clear intersection formula of Grothendieck? It's claimed in the introduction of [https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf], but I can not find any other references about this excess intersection formula. How is it related to the Riemann-Roch theorem?

10. Different views on the double coset

$$B \backslash G / B = (* / B) \times_{* / G} (* / B)$$

- as a set
- as flag variety quotient B -action
- as a stack
- groupoid structure

Some excuses for not working a lot on the project:

Preparation for summer school	2 weeks
Summer school of the modular form	1 week
Tourism in Paris	1 week
Conference in Antwerp	1 week
Reading [Ginz, Chap 5]	2 weeks
Computing $H(G, B)$, H_{sph} , (Haff)	1 week
Applying for tutorials, extend the residence permit, preparation for TOEFL exam, Klein AG....	2 weeks
Summer school on Langlands & ICM watch (part)	1 week
In total	11 weeks

tough new semester:

- 3 Seminars (+ Master Thesis Seminar)
- Tutorial
- TOEFL exam on 15th Oct.
- The seminar handout and other materials are not completed.
 - L -parameters
 - moduli in AG
 - some following developments of the modular form (different type of q -ps, Hecke operators...)
 - reps of $GL_2(\mathbb{Q}_p)$
- applying for the PhD program.