

# Eine Woche, ein Beispiel

## 2.6. six functors

Ref: <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>

A preparation of exams.

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{F}} & \mathcal{F}' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \quad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

Goal:

$$\begin{aligned} f^* &\dashv f_* \\ - \otimes \mathcal{F} &\dashv \underline{\mathrm{Hom}}(\mathcal{F}, -) \\ f_! &\dashv f^! \end{aligned}$$

$$\begin{aligned} f^*(- \otimes -) & \\ f^*(\mathcal{F} \otimes \mathcal{F}') &\cong f^* \mathcal{F} \otimes f^* \mathcal{F}' \\ f_* \underline{\mathrm{Hom}}(f^* \mathcal{F}, \mathcal{G}) &\cong \underline{\mathrm{Hom}}(\mathcal{F}, f_* \mathcal{G}) \end{aligned}$$

$$\begin{array}{ccc} & \otimes & \\ f^* & & f_! \\ \text{bc: } f^* g_! & \cong & g'_! f^{*'} \\ f_* g'_! & \cong & g'_! f_* \end{array}$$

proj formula

$$\begin{aligned} f_!(f^* \mathcal{F} \otimes \mathcal{G}) &\cong \mathcal{F} \otimes f_! \mathcal{G} \\ f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^* \mathcal{F}) &\cong \underline{\mathrm{Hom}}(f_! \mathcal{G}, \mathcal{F}) \\ f^! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') &\cong \underline{\mathrm{Hom}}(f^* \mathcal{F}, f^! \mathcal{F}') \end{aligned}$$

$f^*$ : stalk

$f_* / f_!$ : cohomology

$$\begin{aligned} I: f^* &= f^! \\ P: f_* &= f_! \end{aligned}$$

$$p: X \rightarrow \text{pt}$$

$$H^i(X; \mathbb{Z}) := p_* p^* \mathbb{1}$$

$$H_c^i(X; \mathbb{Z}) := p_! p^* \mathbb{1}$$

$$H^i(X; \mathbb{Z}) := p_! p^! \mathbb{1}$$

$$H^{BM}_i(X; \mathbb{Z}) := p_* p^! \mathbb{1}$$

$$H^i(X; \mathcal{F}) := p_* \mathcal{F}$$

$$H_c^i(X; \mathcal{F}) := p_! \mathcal{F}$$

$$H^i(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$$

$$H^{BM}_i(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$$

$$= R\Gamma(X; \mathcal{F})$$

$$= H_c^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

$$= H^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)

$$H_c^i(X; \mathcal{F}) \otimes H_c^j(Y; \mathcal{G}) \cong H_c^{i+j}(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\text{reduced to: } p_{X!} \mathcal{F} \otimes p_{Y!} \mathcal{G} \cong p_!(p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_! \downarrow & \searrow p & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

App 2. (Poincaré duality)

$X$ : a cpt oriented mfd of dim  $d$ , then

$$-^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

$$\begin{aligned} \text{proper} & \quad p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ locally (Verdier duality)} \\ & \quad p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ globally} \end{aligned}$$

$$H^i(X; \mathbb{Z})[d] \cong H^i(X; \mathbb{Z})^\vee$$

$$\text{reduced to: } p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$$

Upgrade:  $\infty$ -categories & sym monoidal structure

Idea:  $\mathcal{D}_\bullet: \mathcal{C}^{op} \longrightarrow \text{Cat}_\infty$

$$\begin{array}{ccc} X & \longmapsto & \mathcal{D}(X) \\ f \downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & \mathcal{D}(Y) \end{array}$$

e.g.  $X :=$  nice top space,  
 $\mathcal{D}(X) :=$  derived category of  
 abelian sheaves over  $X$ .

extends to  $\hookrightarrow$  compatibility is encoded!

$$\mathcal{D}: \text{Corr}(\mathcal{C}, \mathcal{E}) \longrightarrow \text{Mon}(\text{Cat}_\infty)$$

$$[Y \xleftarrow{f} X = X] \longmapsto f^*$$

$$[X = X \xrightarrow{f \in \mathcal{E}} X] \longmapsto f_!$$

$$[X \times X \xleftarrow{\epsilon} X = X] \longmapsto \otimes$$

Moreover, It factor through

$$\begin{array}{ccccc} \text{Corr}(\mathcal{C}, \mathcal{E}) & \longrightarrow & \text{LZ}_{\mathcal{D}} & \longrightarrow & \text{Mon}(\text{Cat}_\infty) \\ \text{Obj: } X & \longmapsto & X & \longmapsto & \mathcal{D}(X) \end{array}$$

Mor:  $\left[ \begin{array}{c} Y \\ X \xleftarrow{f} \quad \searrow g \\ Z \end{array} \right] \longmapsto \text{kernel} \longmapsto \text{FM-transformation}$

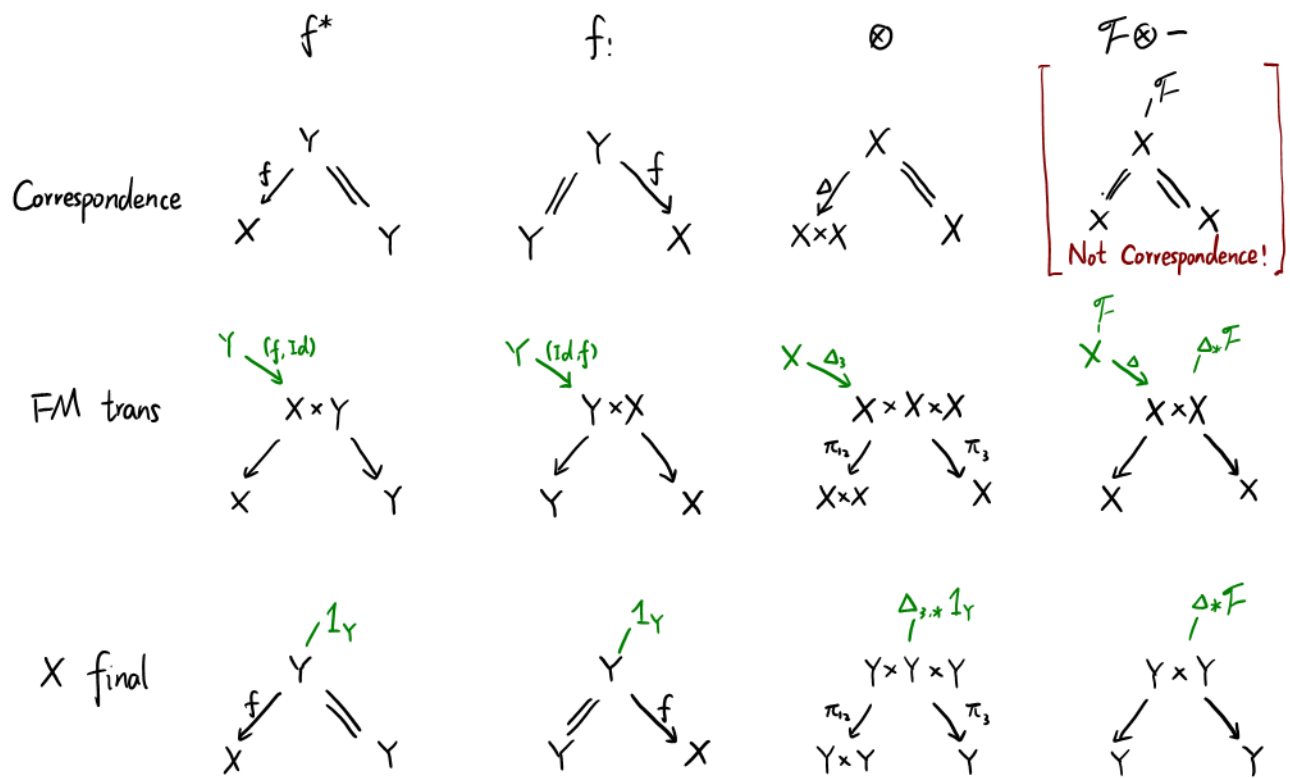
composition = convolution

2-Mor:  $\mathcal{E} \rightarrow \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$

$$\left[ \begin{array}{c} \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \\ \cong \downarrow F \\ \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \end{array} \right] \mapsto \left[ \begin{array}{c} \mathcal{E}_{12} * \mathcal{E}_{23} \\ \downarrow \\ \mathcal{E}_{13} \end{array} \right]$$

Goal: framework of  $\infty$ -category &  $\otimes$

$$\leadsto \text{Corr}(\mathcal{C}, \mathcal{E}) \text{ \& } \text{Corr}(\mathcal{C}, \mathcal{E})^{\otimes}$$



$\infty$ -category

$$\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \text{Set}) \stackrel{\Delta}{=} \text{sSet} \supseteq \text{Cat}_{\infty} \supseteq \text{An}$$

$$\begin{array}{ccc} \Delta^n & \xrightarrow{h} & X \\ \downarrow & \nearrow \exists K_{n,i}(h) & \\ \Delta^n & & \end{array} \quad \begin{array}{ll} \forall 0 \leq i < n & \text{in } \text{Cat}_{\infty} \\ \forall 0 \leq i \leq n & \text{in } \text{An} \end{array}$$

Notation

Set (0,0)-category set  
 Cat (1,1)-category category  
 An ( $\infty$ ,0)-category anima / Kan cplx /  $\infty$ -groupoid  
 Cat $_{\infty}$  ( $\infty$ ,1)-category

Ex. Realize  $\text{Corr}(C, E)$  as an  $\infty$ -category.

Monoidal structure

In (1,1)-category:

Monoidal structure on  $\mathcal{C}$ :

$$\begin{array}{ll} m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} & u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C} \\ (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G} & * \longmapsto 1_{\mathcal{C}} \end{array}$$

Monoidal object in  $(\mathcal{C}, \otimes)$ :  $X \in \text{Ob}(\mathcal{C})$  with

$$m_X: X \times X \longrightarrow X \quad u_X: 1_{\mathcal{C}} \longrightarrow X$$

In ( $\infty$ ,1)-category:

$$(C, \otimes) \stackrel{\text{def}}{\hookrightarrow} \left[ \begin{array}{ccc} X: \text{Fin}^{\text{part}} & \longrightarrow & \text{Cat}_{\infty} \\ I & \longmapsto & X(I) \\ \text{comm monoid} & & \end{array} \right] \stackrel{\text{"Straightening"}}{\longleftrightarrow} \left[ \begin{array}{ccc} \pi^{\otimes}: Y^{\otimes} & \longrightarrow & \text{Fin}^{\text{part}} \\ \text{coCartesian fibration} & & \\ Y_I^{\otimes} \xrightarrow{\sim} \prod_i Y_i^{\otimes} & & \end{array} \right]$$

$\rightsquigarrow$  See next page for details

where  $\text{Ob}(\text{Fin}^{\text{part}}) = \text{Ob}(\text{Fin})$

$$\text{Mor}_{\text{Fin}^{\text{part}}}(I, J) = \{\alpha: I \dashrightarrow J\}$$

$$\text{commutative monoid: } X(I) \xrightarrow{\sim} \prod_i X(i)$$

$$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow (\mathcal{F}, \mathcal{G}) \quad |I|=2$$

coCartesian fibration: see [Def 3.5]

$$\left[ \begin{array}{c} (C, \otimes) \\ \text{monoidal} \\ \text{co-cat} \end{array} \right] \xleftrightarrow{\text{def}} \left[ \begin{array}{c} \mathbb{X}: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty} \\ I \longmapsto \mathbb{X}(I) \\ \text{comm monoid} \end{array} \right] \xleftrightarrow{\text{"straightening"}} \left[ \begin{array}{c} \pi^{\otimes}: \mathcal{Y}^{\otimes} \longrightarrow \text{Fin}^{\text{part}} \\ \text{coCartesian fibration} \\ \mathcal{Y}_I^{\otimes} \xrightarrow{\sim} \prod_i \mathcal{Y}_i^{\otimes} \end{array} \right]$$

$$\begin{array}{ccc} (C, \otimes) & \longmapsto & \mathcal{C}^{(-)}: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty} \\ & & I \longmapsto \mathcal{C}^I := \prod_{i \in I} \mathcal{C} \\ & & \downarrow \alpha \quad \Rightarrow \quad \downarrow \\ & & J \quad \quad \quad \mathcal{C}^J \end{array} \quad \begin{array}{c} (X_i)_{i \in I} \\ \downarrow \\ (\bigotimes_{i \in \alpha^{-1}(j)} X_i)_{j \in J} \end{array}$$

$$\xrightarrow{\quad \quad \quad} \pi^{\otimes}: \mathcal{C}^{\otimes} \longrightarrow \text{Fin}^{\text{part}} \\ \mathcal{C}^{\otimes} = \{ (I, (X_i)_i) \mid I \in \text{Fin}^{\text{part}}, X_i \in \mathcal{C} \} \quad (I, (X_i)_i) \longmapsto I$$

$$\text{Mor}^{\otimes}((I, X_i), (J, Y_j)) = \{ \alpha: I \dashrightarrow J, \{ \bigotimes_{i \in \alpha^{-1}(j)} X_i \rightarrow Y_j \}_j \}$$

$$\begin{array}{ccc} (\mathbb{X}(*), \mathbb{X}(\{1,2\} \rightarrow \{*\})) & \longleftarrow & \mathbb{X} \longmapsto \mathbb{X}^{\otimes} = \{ (I, A) \mid I \in \text{Fin}^{\text{part}}, A \in \mathbb{X}(I) \} \\ & & \text{Mor}((I, A), (J, B)) = \left\{ \begin{array}{l} \alpha: I \dashrightarrow J \\ f: (\mathbb{X}(\alpha)A) \rightarrow B \text{ in } \mathbb{X}(J) \end{array} \right\} \end{array}$$

$$\begin{array}{ccc} \mathcal{Y}: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty} & \longleftarrow & \mathcal{Y}^{\otimes} \\ I & \longmapsto & \mathcal{Y}^I := \mathcal{Y}_I^{\otimes} \\ \downarrow \alpha & \Rightarrow & \downarrow \text{loc. coCartesian lift} \\ J & & \mathcal{Y}^J \end{array}$$

$$(\pi^{\otimes^{-1}}(*), \text{loc. coCartesian lift of } \{1,2\} \rightarrow \{*\}) \longleftarrow \mathcal{Y}^{\otimes}$$

$\mathbb{X}$  encodes the monoidal structure

$$\begin{array}{ccc} & \nearrow \{1,2,3\} & \\ \{1,2,3\} & & \{1,2,3\} \\ & \searrow \{1,2,3\} & \\ & \rightarrow * & \end{array}$$

$$\begin{array}{ccccc} & & \mathcal{C} \times \mathcal{C} & & \\ & \nearrow & & \searrow & \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} & & & & \mathcal{C} \\ & \searrow & \mathcal{C} \times \mathcal{C} & \nearrow & \\ & & \mathcal{C} & & \end{array}$$

$$\begin{array}{ccc} & \nearrow (X \otimes Y, Z) & \\ (X, Y, Z) & & (X \otimes Y) \otimes Z \\ & \searrow (X, Y \otimes Z) & \\ & \rightarrow X \otimes (Y \otimes Z) & \end{array}$$

$$\begin{array}{ccc} \{1,2\} & \searrow & \\ \downarrow & & \{1,2\} \\ \{1,2\} & \nearrow & * \end{array}$$

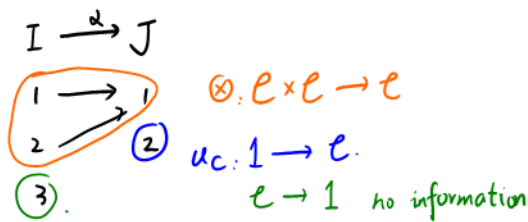
$$\begin{array}{ccc} \{1\} & \searrow & \\ \downarrow & & \{1,2\} \\ \{1,2\} & \nearrow & * \end{array}$$

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \searrow & \\ \downarrow & & \mathcal{C} \\ \mathcal{C} \times \mathcal{C} & \nearrow & \end{array}$$

$$\begin{array}{ccc} \mathcal{C} & \searrow & \\ \downarrow & & \mathcal{C} \\ \mathcal{C} \times \mathcal{C} & \nearrow & \end{array}$$

$$\begin{array}{ccc} (X, Y) & \longmapsto & X \otimes Y \\ \downarrow & & \parallel \\ (Y, X) & \longmapsto & Y \otimes X \end{array}$$

$$\begin{array}{ccc} X & \longmapsto & X \\ \downarrow & & \parallel \\ (X, 1_{\mathcal{C}}) & \longmapsto & X \otimes 1_{\mathcal{C}} \end{array}$$



Fctor. (lax) sym monoidal fctors

Special case:  $[F: (\mathcal{C}, \otimes) \longrightarrow (\mathcal{D}, \times)] \iff [F: \mathcal{C}^{\otimes} \longrightarrow \mathcal{D} \text{ with conditions}]$

Ex. Realize  $\text{Corr}(\mathcal{C}, \mathcal{E})^{\otimes}$ , and show  $f^*(- \otimes -)$ , bc & proj formula.

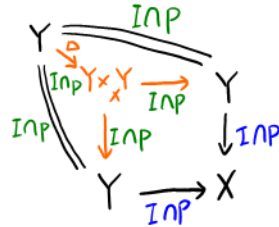
Why is  $f: \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$   $\mathcal{D}(Y)$ -linear?

Category	Object	Morphism
$\mathcal{C}$	$X \quad Y$	$X \rightarrow Y$
$\mathcal{C}^{op}$	$X \quad Y$	$X \leftarrow Y$ in $\mathcal{C}$
$(\mathcal{C}^{op})^L$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha: I \rightarrow J, \{ \bigotimes_{i \in \mathcal{A}^*(j)}^{op} X_i \rightarrow Y_j \}_j$ in $\mathcal{C}^{op}$
		$\parallel$
		$\{ \prod_{i \in \mathcal{A}^*(j)} X_i \leftarrow Y_j \}_j$ in $\mathcal{C}$
$((\mathcal{C}^{op})^L)^{op}$	$(I, (X_i)_i), (J, (Y_j)_j)$	$\alpha: I \leftarrow J, \{ \prod_{i \in \mathcal{A}^*(j)} X_i \rightarrow Y_j \}_j$ in $\mathcal{C}$

Construction "Uniqueness of  $f_!$ "

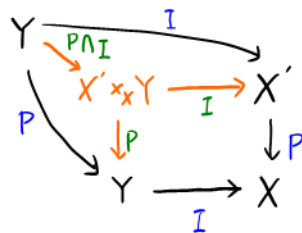
Const 1.  $f: Y \rightarrow X \quad f \in \text{INP} \quad \Rightarrow f_! \cong f_*$

By induction.

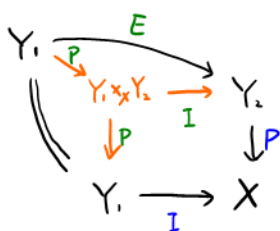
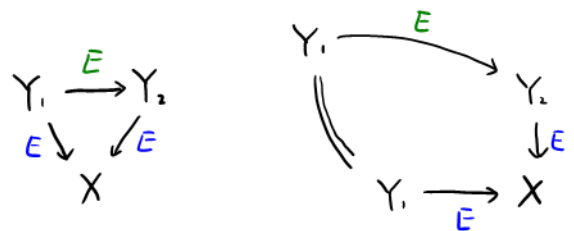


— Initial case  
— Deduced case

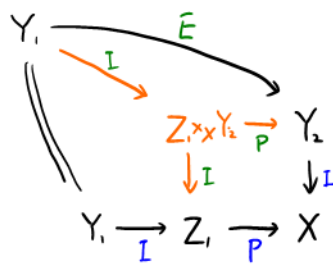
Const 2



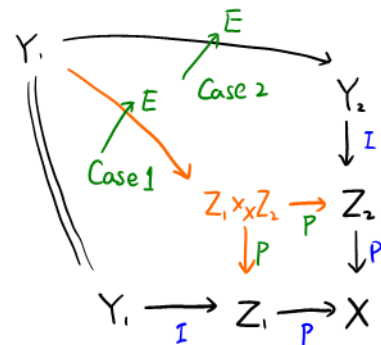
Const 3.  $E$  satisfies 2-out-of-3, i.e.



Case 1



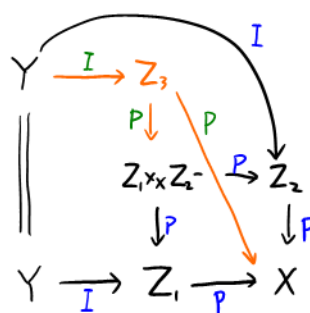
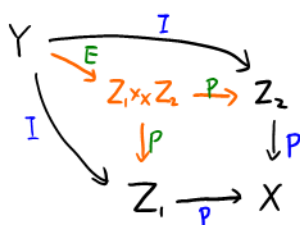
Case 2



Case 3

Const 4.  $Y \xrightarrow{i_2} Z_2$   
 $i_1 \downarrow I \quad P \downarrow f_!$   
 $Z_1 \xrightarrow{f_!} X$

want:  $f_{1*} j_{1!} \cong f_{2*} j_{2!}$



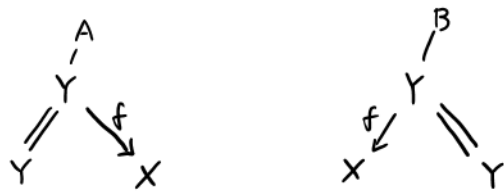
## Construction

$$\begin{aligned}
 \text{Corr}(C, E) &= \delta_+^* \text{Corr}(C, E)_{ct, co} \\
 &\underset{\text{Fun}(-, E)}{\sim} \delta^* \text{Corr}(C, E)_{ct, co} \\
 &\sim \delta^* \text{Corr}(C, I, P)_{ct, co, co} \\
 &\longleftarrow \delta^* \text{Corr}(C, I, P)_{ct, ct, ct}
 \end{aligned}$$

$$\begin{array}{ccc}
 \Delta^n \times \Delta^n \times \Delta^n & \longrightarrow & \text{Cat}_\infty \\
 \Delta^n \times \Delta^n & \longrightarrow & \text{Fun}(\Delta^n, \text{Cat}_\infty) \\
 & \searrow & \downarrow \\
 & & \text{Fun}^L(\Delta^n, \text{Cat}_\infty) \\
 & & \downarrow \\
 & & \text{Fun}(\mathbb{A}^n{}^{\text{op}}, \text{Cat}_\infty)
 \end{array}$$



$f$ -smooth  $f: Y \rightarrow X$



$$F: f_*(A \otimes -) \dashv G: B \otimes f^* - \\ \dashv \text{Hom}(A, f^! -)$$

$$① \quad B \otimes f^* - \cong \text{Hom}(A, f^! -)$$

App 1.  $\Delta: 1_Y \text{ cpt} \Rightarrow A \text{ cpt}$

[Proof.  $\text{Hom}(\Delta, 1_Y, B \otimes f^* -) \cong \text{Hom}(A, -)$  preserves filtered colimit.]

$$② \quad B \cong \text{Hom}(A, f^! 1_X)$$

$$p_2^* B \otimes p_1^* - \cong \text{Hom}(p_2^* A, p_1^! -)$$

[Verdier's diagonal trick]

$$\text{Prop. } A \text{ is } f\text{-smooth} \Leftrightarrow p_2^* B \otimes p_1^* A \cong \text{Hom}(p_2^* A, p_1^! A) \quad (2b)$$

$$\text{where } B \cong \text{Hom}(A, f^! 1_X) \quad (2a)$$

$\Rightarrow: \checkmark$

$\Leftarrow: \text{Writing down adjunctions in 2-category.}$

App 2. When  $Y = X$ ,  $f = \text{Id}$ ,

$$A \text{ is } f\text{-smooth} \Leftrightarrow \text{Hom}(A, 1_X) \otimes A \cong \text{Hom}(A, A)$$

$$\Leftrightarrow A \text{ is dualizable}$$

App 3. When  $A = 1_Y$ ,  $B = f^! 1_X$ ,

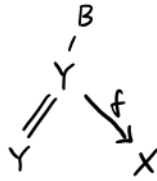
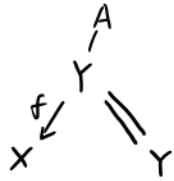
$$1_Y \text{ is } f\text{-smooth} \Leftrightarrow p_2^* f^! 1_X \cong p_1^! 1_Y$$

$$\xLeftrightarrow{+f^! 1_X \text{ inv}} f \text{ is coh smooth}$$

Using this, one can prove results on coh étale.

Write  $B = \mathbb{D}_f(A)$ . we get  $\mathbb{D}_f(\mathbb{D}_f(A)) \cong A$ . (adjunction is symmetric in  $A$  &  $B$ ).

$f$ -proper  $f: Y \rightarrow X$



$$F: A \otimes f^* - \dashv G: f_!(B \otimes -) \\ \dashv f_* \text{Hom}(A, -)$$

$$\textcircled{1} f_!(B \otimes -) \cong f_* \text{Hom}(A, -)$$

App 1.  $1_X \text{ cpt} \Rightarrow A \text{ cpt}$

[Proof.  $\text{Hom}(1_X, f_!(B \otimes -)) \cong \text{Hom}(A, -)$  preserves filtered colimit.]

$$\textcircled{2} p_{1,!}(p_2^* B \otimes -) \cong p_{1,*} \text{Hom}(p_2^* A, -) \quad [\text{Verdier's diagonal trick}] \\ B \cong p_{1,*} \text{Hom}(p_2^* A, \Delta_! 1_Y)$$

$$\text{Prop. } A \text{ is } f\text{-proper} \Leftrightarrow f_!(B \otimes A) \cong f_* \text{Hom}(A, A) \quad \textcircled{2b} \\ \text{where } B \cong p_{1,*} \text{Hom}(p_2^* A, \Delta_! 1_Y) \quad \textcircled{2a}$$

$\Rightarrow: \checkmark$

$\Leftarrow$ : Writing down adjunctions in 2-category.

App 2. When  $Y = X$ ,  $f = \text{Id}$ ,

$$A \text{ is } f\text{-proper} \Leftrightarrow \text{Hom}(A, 1_X) \otimes A \cong \text{Hom}(A, A) \\ \Leftrightarrow A \text{ is dualizable}$$

App 3. When  $A = 1_Y$ ,  $B = p_{1,*} \Delta_! 1_Y$

$$1_Y \text{ is } f\text{-proper} \Leftrightarrow f_! p_{1,*} \Delta_! 1_Y \cong f_* 1_Y$$

Using this, one can prove results on coh proper.

Write  $B = \mathbb{D}_f^{\text{pro}}(A)$ . we get  $\mathbb{D}_f^{\text{pro}}(\mathbb{D}_f^{\text{pro}}(A)) \cong A$ . (adjunction is symmetric in  $A$  &  $B$ )

When  $\Delta_! = \Delta_*$ ,  $\mathbb{D}_f^{\text{pro}} = \text{Hom}(-, 1_Y)$  is the naive dual.

## Relations

$$\text{open immersion} \longrightarrow \text{coh smooth} \xrightleftharpoons[\text{if } f^! 1_X \text{ inv}]{\text{if } \Delta \text{ coh étale, } f \text{ is } n\text{-truncated}} 1_Y \text{ is } f\text{-sm} \xrightleftharpoons[\text{if } \Delta \text{ coh étale, } f \text{ is } n\text{-truncated}]{} \text{coh étale}$$

$$\text{proper} \xrightarrow{\text{if } \Delta_! = \Delta_*} 1_Y \text{ is } f\text{-proper} \xrightleftharpoons[\text{if } \Delta \text{ coh proper, } f \text{ is } n\text{-truncated}]{} \text{coh proper}$$