Eine Woche, ein Beispiel

11.14. Stiefel manifold

Goal: · understand some homotopy gp of $V_k(\mathbb{R}^n)$, $V_k(\mathbb{C}^n)$

· metric on Stiefel mfld $\rightsquigarrow$ geodesic, volumn, ...

· cellular structure

· dim, some fiber bundle, with Lie gp.

· measure

Ref: https://en.wikipedia.org/wiki/Stiefel_manifold

For the description of metric, see https://math.stackexchange.com/questions/1371410/geodesic-of-stiefel-manifold
For the cellular structure, see https://math.stackexchange.com/questions/58041/cell-structure-on-stiefel-manifolds

Orthogonal basis = 么正基.

1. fiber bundle

$$\begin{array}{ccc} O(k,\mathbb{R}) \to V_k(\mathbb{R}^n) & V_{k-1}(\mathbb{R}^{n-1}) \to V_k(\mathbb{R}^n) & S^{n-k} \to V_k(\mathbb{R}^n) \\ \downarrow & \downarrow & \downarrow \\ Gr_k(\mathbb{R}^n) & V_1(\mathbb{R}^n) & V_{k-1}(\mathbb{R}^n) \end{array}$$

$$\begin{array}{ccc} U(k) \to V_k(\mathbb{C}^n) & V_{k-1}(\mathbb{C}^{n-1}) \to V_k(\mathbb{C}^n) & S^{2n-2k+1} \to V_k(\mathbb{C}^n) \\ \downarrow & \downarrow & \downarrow \\ Gr_k(\mathbb{C}^n) & V_1(\mathbb{C}^n) & V_{k-1}(\mathbb{C}^n) \end{array}$$

2. homotopy gp

Ref: https://people.math.ethz.ch/~jagnaw/Seminar_Notes/Obstruction_theory_Stiefel_Whitney_classes.pdf

Lemma 5 The homotopy groups of the Stiefel manifold $V_k(\mathbb{R}^n)$ for $l \leq n-k$ are

$$\pi_l(V_k(\mathbb{R}^n)) = \begin{cases} 0 & \text{if } l < n-k \\ \mathbb{Z} & \text{if } l = n-k \text{ and } k = 1 \\ \mathbb{Z} & \text{if } l = n-k \text{ is even} \\ \mathbb{Z}_2 & \text{if } l = n-k \text{ is odd and } k \neq 1. \end{cases}$$

ref:http://math.uchicago.edu/~may/REU2012/REUPapers/Fung.pdf
in[Lemma 1.10], the author proved this result by the elementary argument. Really nice!

For the references on
https://projecteuclid.org/journals/journal-of-the-institute-of-polytechnics-osaka-city-university-series-a-mathematics/volume-6/issue-1/On-the-homotopy-groups-of-Stiefel-manifolds/ojm/1353054734.pdf
https://projecteuclid.org/journals/bulletin-of-the-american-mathematical-society-new-series/volume-71/issue-4/Some-homotopy-groups-of-Stiefel-manifolds/bams/1183527242.full
https://www.maths.ed.ac.uk/~v1ranick/papers/paechter5.pdf
they all concern only with the stable homotopy group. So in general it's quite difficult to compute the other homotopy groups.

E.g. $n=5$ $\pi_l(V_k(\mathbb{R}^5))$

l \ k	0 $\{*\}$	1 S^4	2	3	4 $SO(5)$	5 $O(5)$
1	0	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
2	0	0	0	$\mathbb{Z}$	0	0
3	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	0	$\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
5	0	$\mathbb{Z}/2\mathbb{Z}$		$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
6	0	$\mathbb{Z}/2\mathbb{Z}$		0	0	0
7	0	$\mathbb{Z}\times\mathbb{Z}/12\mathbb{Z}$		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
8	0	$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$		0	0	0

E.g. $n=6$ $\pi_l(V_k(\mathbb{R}^6))$

l \ k	0 $\{*\}$	1 S^5	2	3	4	5 $SO(6)$	6 $O(6)$
1	0	0	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
2	0	0	0	0	$\mathbb{Z}$	0	0
3	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
4	0	0	$\mathbb{Z}$		0	0	0
5	0	$\mathbb{Z}$			$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
6	0	$\mathbb{Z}/2\mathbb{Z}$			0	0	0
7	0	$\mathbb{Z}/2\mathbb{Z}$			$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
8	0	$\mathbb{Z}/24\mathbb{Z}$			$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$	$\mathbb{Z}/24\mathbb{Z}$

3. cohomology group.

we compute by spectral sequence.

E.g. $n=5$ $H^i(V_k(\mathbb{R}^5),\mathbb{Z})$

$i \backslash k$	0 $\{*\}$	1 S^4	2	3	4 $SO(5)$	5 $O(5)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$
1	0	0	0	0	0	0
2	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
3	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}$
4	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
5	0	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
6	0	0	0	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
7	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$
8	0	0	0	0	0	0
9	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$
10	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}^{\oplus 2}$

E.g. $n=6$ $H^i(V_k(\mathbb{R}^6),\mathbb{Z})$

$i \backslash k$	0 $\{*\}$	1 S^5	2	3	4	5 $SO(6)$	6 $O(6)$
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	
1							
2					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
3						$\mathbb{Z}$	
4			$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	E$^{\oplus 2}$
5		$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	V
6					$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	E
7				$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$	R
8						$\mathbb{Z}$	Y
9			$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}^{\oplus 2}$	T
10						$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	H
11					$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	I
12				$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}/2\mathbb{Z}$	N
13							G
14					$\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	
15						$\mathbb{Z}$	