as

Notation: - A: associate ring allowed to be non-commutative, contains 1 - There are two systems to write category of A-modules.

$$Mod_A = A - Mod$$
 $(Mod_A)^{\circ p} \neq Mod_{A^{\circ p}} = Mod - A = A^{\circ p} - Mod \Rightarrow M_A$
 $Mod_{A \otimes B^{\circ p}} = A - Mod - B \Rightarrow AM_B$

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

$$\nabla$$
 Even though you can identify $Ob(Ring) \cong Ob(Ring^{op})$, A^{op} is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

In this case, it is desirable to translate algebraic results into geometrical results. Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for
$$\otimes$$
 & Hom
2. adjunction
3. comparison between \otimes -1 Hom & f^*-1f_*
4. definition recall for \otimes & Hom , derived version
5. adjunction , derived version
6. comparison between \otimes -1 Hom & f^*-1f_* , derived version

1 definition recall for ⊗ & Hom

$$\otimes_A: \operatorname{Mod}_{A^{\circ P}} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$$

 $\operatorname{Hom}_A(-,-): (\operatorname{Mod}_A)^{\circ P} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$

$$\otimes_{B}$$
: $A - Mod - B \times B - Mod - C \longrightarrow A - Mod - C$
 $Hom_{B}(-,-)$: $(A - Mod - B)^{\overline{0}} \times B - Mod - C \longrightarrow A - Mod - C$

$$Hom_{B}^{A}(-,-)$$
: $(A-Mod-B)^{\overline{op}} \times B-Mod-A \longrightarrow \mathbb{Z}-Mod$

$$Hom_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) (\mathbb{Z}-Mod-B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}-Mod-\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}-Mod-\mathbb{Z}$$

$$(X \otimes_{B} Y) \otimes_{C} Z \cong X \otimes_{B} (Y \otimes_{C} Z)$$

$$X \otimes_{B} Y \cong Y \otimes_{B^{op}} X$$

$$A \otimes_{A} X \cong X \cong X \otimes_{B} B$$

$$Hom_{A}(A, X) \cong X$$

in
$$A-Mod-C = C^{op}-Mod-A^{op}$$

2 adjunction BXA, cYB, cZD, we get

 $Homc(Y \otimes_{B} X, Z) \cong Hom_{B}(X, Homc(Y, Z))$ in A-Mod-D.

Reason: both sides equal to the set $f: Y \times X \longrightarrow Z \mid f(cyb,x) = cf(y,bx) \quad \forall b,c$

For A = D = Z, fix $Y \in C$ -Mod-B, one gets adjunction fctors.

slogan: adjunction & associativity

$$(A-Mod-B)^{\overline{op}} \times (B-Mod-C)^{\overline{op}} \times C-Mod-D \xrightarrow{(Id, Home)} (A-Mod-B)^{\overline{op}} \times B-Mod-D$$

$$(A-Mod-B) \times B-Mod-C)^{\overline{op}} \times C-Mod-D$$

$$(A-Mod-C)^{\overline{op}} \times C-Mod-D \xrightarrow{Hom_{C}} A-Mod-D$$

$$f^*-If_*.$$

$$Hom (f^*F, G) \cong Hom (F, f_*G)$$

$$Sh(X)^{\overline{op}} \times Mor(Y, X) \times Sh(Y) \xrightarrow{(Id, pushforward)} Sh(X)^{\overline{op}} \times Sh(X)$$

$$(pullback, Id) \downarrow \qquad \qquad Hom_{Sh(Y)}(-, -)$$

$$Sh(Y)^{\overline{op}} \times Sh(Y) \xrightarrow{Hom_{Sh(Y)}(-, -)} Abel$$

$$(F, f, G) \longmapsto Hom_{Sh(Y)}(f^*F, G) \cong Hom_{Sh(X)}(F, f_*G)$$

$$f_*-If_* : similar.$$

3. comparison between ⊗-1 Hom & f*-1 f*

Forgetful fctor

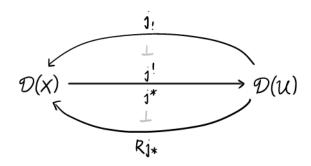
Prop. For ring homo
$$\begin{picture}(1,0) \put(0,0){\line(1,0){150}} \put($$

one has adjunction fctors

djunction fctors
$$S_{R} \otimes_{R} - \frac{\sum_{S_{R} \otimes_{R} - 1}^{S_{R} \otimes_{R}} \otimes_{S_{R} - 1}}{\sum_{S_{R} \otimes_{S} - 1}^{S_{R} \otimes_{S}} \otimes_{S_{R} - 1}} R-Mod \qquad (3.1)$$

Compare with j

Now, we compare (3.1) with part of the recollement diagram:



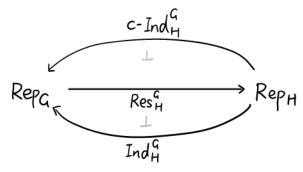
Vague slogan: $u \approx$ "forget the information of Z".

In applications, $U \longrightarrow X$ is a covering map. This change the feeling of the size between U & X.

E.g. For finite gps
$$H \leq G$$
, one has Res-Ind adjunction.
 $Res_{H}^{G} \dashv Ind_{H}^{G}$
 $c-Ind_{H}^{G} \dashv Res_{H}^{G}$

It can be generalized for
$$G: loc$$
 profinite gp , $H \leq G$ open If one only has $H \leq G$ closed, then it's possible that $j' \neq j^*$. e.g. $G = GL_1(\mathbb{Q}_p)$ $H = GL_2(\mathbb{Z}_p)$

In the diagram,



Ex Compare it with the recollement diagram & (3.1).

$$\mathcal{U}$$
 [*/H]
$$\downarrow j$$
 "cover with fiber G/H"
$$X$$
 [*/G]

translate the following geometrical results into algebraic statements.

1. One has natural fctor
$$j_! \longrightarrow j_*$$
. When $\# G/H < +\infty$, $j_! = j_*$
 $c - Ind_H^G \longrightarrow Ind_H^G$

2. Even though Sho.v.([*/G]) ≈ Repa = Q[G]-Mod. the "structure sheaf" of [*/G] is Q. not Q[G].

Res_{f*1}
$$Q = Q$$
, Res_{f*1} $Q[G] = Q[G] \neq Q$

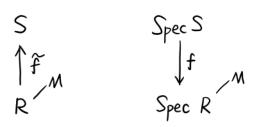
√ In this example, j*Rj* ≠ Id, j'j! ≠ Id.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

Compare with i

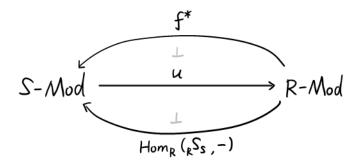
Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

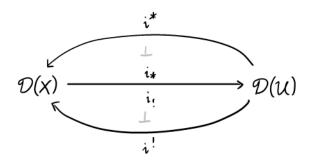


$$\exists$$
 "pullback fctor"
$$f^*: R\text{-}Mod \longrightarrow S\text{-}Mod \qquad f^*M = sS_R \otimes_R M$$
 This is also called the base change.

Now, (31) can be rewritten as



compare it with another part of the recollement diagram.



Rmk. u is usually not f faithful, unless S = R/I. (In fact, only need S is R-idempotent, i.e. $S \cong S \otimes_R S$.) which croppeds to closed embedding. In that case, $i^*i^* = Id : SS \otimes (SS \otimes_S M) \cong M$

$$i^*i_* = Id$$
: ${}_{S}S_R \otimes_R ({}_{R}S_S \otimes_S M) \cong M$
 $i^!i_* = Id$: $Hom_R ({}_{R}S_S, Hom({}_{S}S_R, M)) \cong M$

Slogan: in the comm alg., Spec $R/I \longrightarrow Spec R$ is closed embedding. In general, if S is an R-idempotent algebra. $S \cong S \otimes_R S$ then i. Spec $S \longrightarrow Spec R$ can be viewed as "closed subset".

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

E.g. R_p , R/I are idempotent R-algs. $Z[\frac{1}{6}]$, F_p , Z/p^2Z , Q, Z_p , are idem Z-algs. Usually R/1 is not an derived idem R-alg!

Rmk Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"? A: (possibly) the Verdier quotient.
We will come back to this after we derive everything.

4. LO -1 RHom

F	RF or LF	RiF or LiF	exact fctor
f* f* πx,* f: πx,: πx,:	F & LF Rf* P(X,F) Rf! Pc(X,F) f!	- Rif* H'(X;F) Rif! H'c(X;F) H'c(X;F)	f*-acyclic \(\alpha - acyclic \(\forall - acyclic \(\forall c - acyclic \)
- & - Hom _R (-, -) MG MG MG MS M/[AM] MA A/[AA] Z(A)	- branched - branched - branched - control - c	Tork(-,-) Extr(-,-) Hi(G;M) Hi(g;M) Hi(g;M) HHi(A,M) HHi(A,M) HHi(A) HHi(A)	flat injective/projective

e.g. group coh

e.g. Lie alg coh

g/x: Lie alg

e.g. Hochschild coh

For calculations, see:

[23.04.09]: gp coh [wiki]: Lie algebra coh

[21.05.21]: Hochschild coh

[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.

For Hom(-,A), see https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m.

Chenji Fu claimed that Hom(-,A) always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.

https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod

https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution

4. definition recall for ⊗ & Hom

, derived version

To define 6 & RHom, one needs to extend fctors

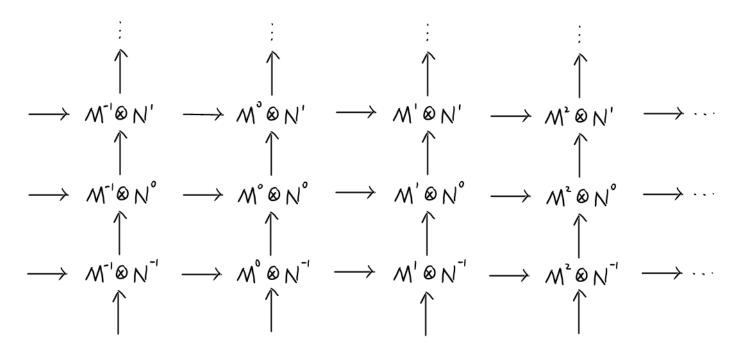
$$\otimes_{A}: A-Mod \times A-Mod \longrightarrow A-Mod$$
 $Hom_{A}(-,-): (A-Mod)^{op} \times A-Mod \longrightarrow A-Mod$

to fctors on double cplxes. C(A): = complex of A-modules, temporate notation

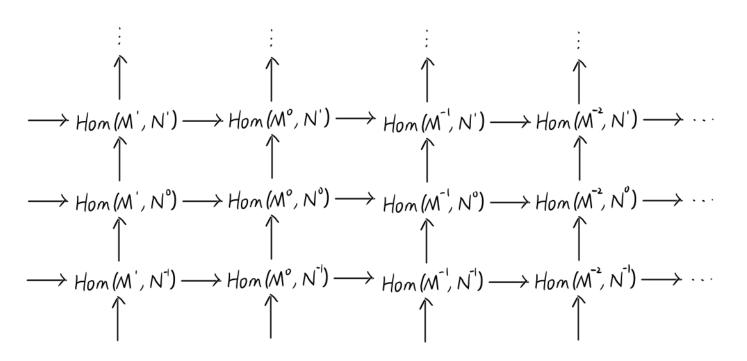
$$\begin{array}{cccc}
\otimes & \mathcal{L}(A) & \times & \mathcal{L}(A) & \longrightarrow & \mathcal{L}(A) \\
\text{Hom } (-,-): & (& \mathcal{L}(A) &)^{op} & \times & \mathcal{L}(A) & \longrightarrow & \mathcal{L}(A)
\end{array}$$

But how?

Wishes:



Tot $(M \otimes N)$, the double complex of $M \otimes N$.



Tot (Hom(M', N')), the double complex of Hom(M', N').

Def For M, N'
$$\in$$
 $\mathcal{L}(A)$, define $M' \otimes N'$, $Hom_A(M', N') \in \mathcal{L}(A)$ by $(M' \otimes_{\mathcal{L}(A)} N')^n = \bigoplus_{i \neq j = n} M^i \otimes_A N^j$ $(Hom_{\mathcal{L}(A)}(M', N'))^n = \bigoplus_{i \neq j = n} Hom_A(M^{-i}, N^j)$ and morphisms given by $d + (-1)^j S$.

$$E_{x}$$
. Let $M' = \begin{bmatrix} \mathbb{Z} \xrightarrow{\times^{3}} \mathbb{Z} \end{bmatrix}$, $N' = \begin{bmatrix} \mathbb{Z} \xrightarrow{\times^{2}} \mathbb{Z} \end{bmatrix}$
compute $M' \otimes_{e(\mathbb{Z})} N' \otimes_{e(\mathbb{Z})} M' \otimes_{e(\mathbb{Z$

Tot (Hom (M', N'))

Now, we can define L⊗ & RHom.

Def. For M, N ∈ A-Mod, one can define

 $M^L \otimes_A N := M \otimes_{e(A)} P'$ when $N \stackrel{\cong}{\leftarrow} P'$ flat resolution in general, $M', N' \in \mathcal{D}^-(A-Mod)$

 $\begin{array}{lll} \text{RHom}_{A}\left(M,\,N\right):=\, \text{Hom}_{e(A)}(M,\,I') & \text{when} & N\stackrel{\cong}{\longrightarrow} I' & \text{inj} & \text{resolution} \\ & :=\, \text{Hom}_{e(A)}\left(P',\,N\right) & \text{when} & M\stackrel{\cong}{\longleftarrow} P' & \text{proj} & \text{resolution} \\ & \text{in general}, & M' \in \mathcal{D}^{-}(A\text{-Mod}), & N' \in \mathcal{D}^{+}(A\text{-Mod}) \end{array}$

Side Rmk. Proj module is flat. Since free module is flat https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat

Ex Compute F. & F. & RHomz (F., F.), and get Torz (F., F.) & Extz (F., F.)

Ex. Shows that $Hom_{\mathcal{D}(A)}(M', N') = R^{\circ} Hom_{\mathcal{D}(A)}(M', N') + R^{\circ} Hom_{\mathcal{D}(A)}(M', N') = R^{\circ} Hom_{\mathcal{D}(A)}(M, N') = R^{\circ} Hom_{\mathcal{D}(A)}(M, N')$

[KS 90, Def 2.6.2]
To switch from D(X) to $D^{\dagger}(X)$, we need to require that

w. gldim $(A) < +\infty$. w. gldim: shortest flat resolution

A wrong proof for "flat -> proj"

"Proof" when P is flat,

$$P \otimes_A - Hom_A(P, -)$$
 $P \stackrel{\square}{\otimes}_A - Hom_A(P, -)$

by the uniqueness of the adjunction, $Hom_A(P, -) = RHom_A(P, -)$, so P is flat.

This is wrong. $Q \in \mathbb{Z}$ -Mod is flat but not proj. In the proof, we only have

Ex. Compute $RHom_Z(Q,-)$, and shows that

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5. adjunction
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, derived version

Prop. For A comm ring, $L,M,N\in A-Mod$, we get $L',M'\in \mathcal{D}^-(A)$, $N'\in \mathcal{D}^+(A)$ in general

RHoma (M & N, L) = RHoma (N, RHoma (M, L))

Proof.

 $Hom_A(M \otimes_A N, L) \cong Hom_A(N, Hom_A(M, L))$ $\downarrow take (-)^n$

 $Hom_{e(A)}(M \otimes_{e(A)}N, L) \cong Hom_{e(A)}(N, Hom_{e(A)}(M, L))$ $\downarrow Hom_{A}(M, -)$ preserves in modules for M flat wiki: injective module $RHom_{A}(M \otimes_{A}N, L) \cong RHom_{A}(N, RHom_{A}(M, L))$

 $\begin{array}{lll} \mathsf{R}\mathsf{Hom}_{\mathsf{A}}(\mathsf{M} & \otimes_{\mathsf{A}}\mathsf{N}, \mathsf{L}) &= \mathsf{R}\mathsf{Hom}_{\mathsf{A}}(\mathsf{P} \otimes_{\mathsf{C}(\mathsf{A})}\mathsf{N}, \mathsf{L}) & \mathsf{M} & \stackrel{\cong}{=} \mathsf{P}' \; \mathsf{flat} \\ &= \mathsf{Hom}_{\mathsf{C}(\mathsf{A})}(\mathsf{P} \otimes_{\mathsf{C}(\mathsf{A})}\mathsf{N}, \mathsf{I}') & \mathsf{L}' & \stackrel{\cong}{=} \mathsf{I}' \; \mathsf{inj} \\ &= \mathsf{Hom}_{\mathsf{C}(\mathsf{A})}(\mathsf{N}, \; \mathsf{Hom}_{\mathsf{C}(\mathsf{A})}(\mathsf{P}, \mathsf{I}')) & \mathsf{adj} \; \mathsf{in} \; \mathsf{C}(\mathsf{A}) \\ &= \mathsf{R}\mathsf{Hom}_{\mathsf{A}}\left(\mathsf{N}, \; \mathsf{Hom}_{\mathsf{C}(\mathsf{A})}(\mathsf{P}, \mathsf{I}')\right) & \mathsf{Hom}_{\mathsf{C}(\mathsf{A})}(\mathsf{P}, \mathsf{I}') \; \mathsf{is} \; \mathsf{inj} \\ &= \mathsf{R}\mathsf{Hom}_{\mathsf{A}}\left(\mathsf{N}, \; \mathsf{R}\mathsf{Hom}_{\mathsf{A}}\left(\mathsf{P}', \mathsf{I}'\right)\right) & \mathsf{I}' \; \mathsf{is} \; \mathsf{inj} \end{array}$

= RHOMA (N, RHOMA (M, L))

口

√ We don't have

RHoma (M & N, L) = RHoma (N, Homa (M, L))

Find a counterexample? Take N = A, reduce to: $R Hom_A (M, L) \cong Hom_A(M, L)$ then take $A = \mathbb{Z}$, $M = L = \mathbb{Z}/2\mathbb{Z}$.

6. comparison between \otimes -1 Hom & $f^* - f_*$, derived version

Will write: verdier category interpretation of 3 cohomology ($gp+Lie\,alg+Hochschild$)