Eine Woche, ein Beispiel

7.18 irreducible representation of semisimple Lie alg

today: $sl_2(\mathbb{C})$ & $sl_3(\mathbb{C})$

Goal. 1 Get some informations of irr rep
- dim
- weight space + dim
- realization (e.g. $\mathrm{Sym}^n V$, $\wedge^n V, \dots$)

2. Understand why "each irr rep corresponds to each highest weight vector".

1. $sl_2(\mathbb{C})$

Notations:

the split basis $\quad h=\begin{bmatrix}1&0\\0&-1\end{bmatrix} \quad v_+=\begin{bmatrix}0&1\\0&0\end{bmatrix} \quad v_-=\begin{bmatrix}0&0\\1&0\end{bmatrix}$

the compact basis of $sl_2(\mathbb{C})$ $\quad k=\begin{bmatrix}0&-1\\1&0\end{bmatrix} \quad x_+=\frac{1}{2}\begin{bmatrix}1&-i\\-i&-1\end{bmatrix} \quad x_-=\frac{1}{2}\begin{bmatrix}1&i\\i&-1\end{bmatrix}$

the Casimir element $\quad \Omega=-\frac{k^2}{4}+\frac{x_+x_-}{2}+\frac{x_-x_+}{2}$

$$h.v_+=2v_+ \qquad h.v_-=-2v_- \qquad [v_+,v_-]=h$$

$$\operatorname{ad} h=\begin{bmatrix}0&0&0\\0&2&0\\0&0&-2\end{bmatrix} \quad \operatorname{ad} v_+=\begin{bmatrix}0&0&1\\-2&0&0\\0&0&0\end{bmatrix} \quad \operatorname{ad} v_-=\begin{bmatrix}0&-1&0\\0&0&0\\2&0&0\end{bmatrix}$$

Lie bracket structure on $\Lambda^{\cdot}sl_2(\mathbb{C})=\mathbb{C}\oplus sl_2(\mathbb{C})\oplus\Lambda^2 sl_2(\mathbb{C})\oplus\Lambda^3 sl_2(\mathbb{C})$:
of degree -1

$[\downarrow,\rightarrow]$	1	h	e	f	$e\wedge f$	$h\wedge f$	$h\wedge e$	$h\wedge e\wedge f$
1	0	0	0	0	0	0	0	0
h	0	0	$2e$	$2f$	0	$-2h\wedge f$	$2h\wedge e$	0
e	0	$-2e$	0	h	$-h\wedge e$	$-2e\wedge f$	0	0
f	0	$-2f$	h	0	$-h\wedge f$	0	$-2e\wedge f$	0
$e\wedge f$	0	0	$h\wedge e$	$h\wedge f$	0	0	0	0
$h\wedge f$	0	$2h\wedge f$	$2e\wedge f$	0	0	0	$-4h\wedge e\wedge f$	0
$h\wedge e$	0	$-2h\wedge e$	0	$2e\wedge f$	0	$-4h\wedge e\wedge f$	0	0
$h\wedge e\wedge f$	0	0	0	0	0	0	0	0

Representations: $\mathrm{Sym}^n V$ $\qquad V\cong\mathbb{C}^2$: standard representation

e.g. $n=3$

$$h\longmapsto\begin{bmatrix}3&&&\\&1&&\\&&-1&\\&&&-3\end{bmatrix} \qquad v_+\longmapsto\begin{bmatrix}0&1&&\\&0&2&\\&&0&3\\&&&0\end{bmatrix} \qquad v_-\longmapsto\begin{bmatrix}0&&&\\3&0&&\\&2&0&\\&&1&0\end{bmatrix}$$

e.p. the adjoint representation is $\mathrm{Sym}^2 V$

$$\begin{tikzcd} 2v_- \arrow[r, bend left, "/2"] & h \arrow[l, bend left] \arrow[r, bend left] & -2v_+ \arrow[l, bend left, "/2"] \end{tikzcd} \qquad \begin{tikzcd} y^2 \arrow[r, bend left, "/2"] & xy \arrow[l, bend left] \arrow[r, bend left] & x^2 \arrow[l, bend left, "/2"] \end{tikzcd}$$

2. $sl_3(\mathbb{C})$

Ref: I would recommend the book "Representation Theory -- a First Course" by Fulton.[Lecture 11-13].

Actually, if you just want to find the answer, then the website "https://www.jgibson.id.au/lievis/" can satisfy most of your requirement. And also if you want to draw the rank 2 root diagrams, then "https://ctan.org/pkg/rank-2-roots" may be a good choice.