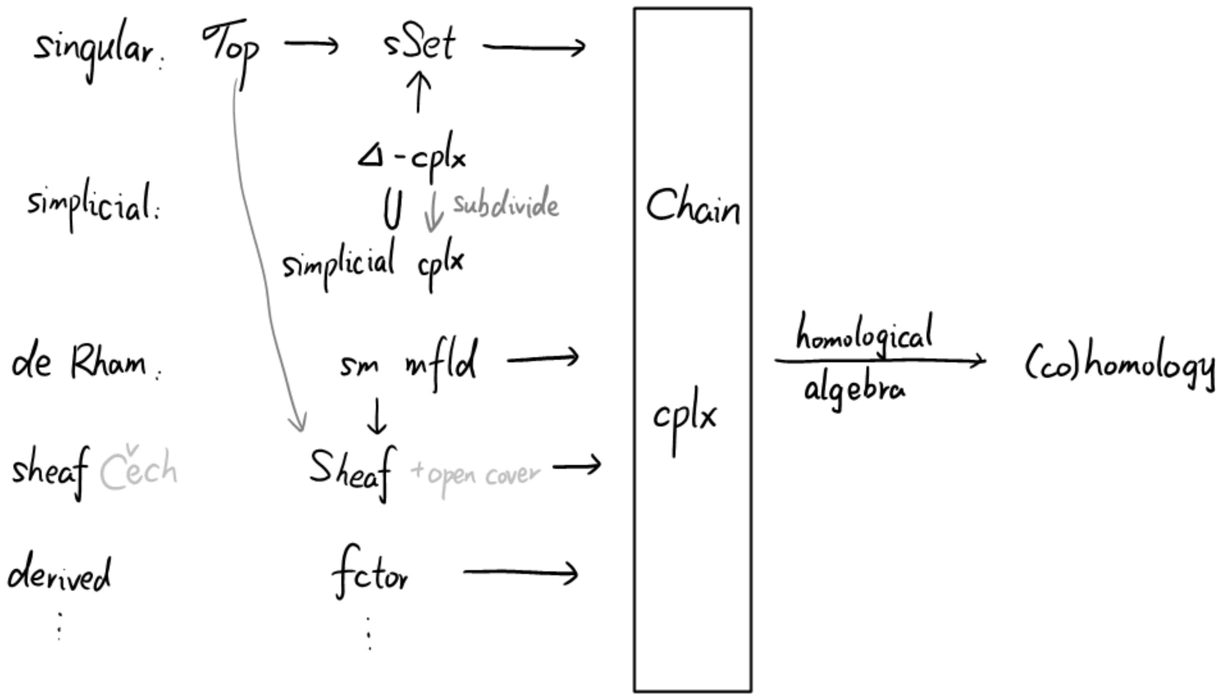
Eine Woche, ein Beispiel

6.24 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. more structures
3. connection with sheaf cohomology + derived category

1. definition and basic examples

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha\in X_n} G \qquad 0 \longleftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1-d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2-d_0^1+d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C_n^{BM}(X;G) =$$

$$C_c^n(X;G) =$$

$$\mathrm{Hom}_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n} \mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n} \mathrm{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

E.g. 1 For $A \in Top$, $X_\cdot = \mathcal{S}(A) \in sSet$, one can compute

$$C_\cdot(X;G): \quad 0 \longleftarrow \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C^\cdot(X;G): \quad 0 \longrightarrow \prod_{\alpha\in A} G \xrightarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \cdots$$

$$C_\cdot^{BM}(X;G): \quad 0 \longleftarrow \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C_c^\cdot(X;G): \quad 0 \longrightarrow \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \cdots$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H_n^{BM}(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H_c^n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.

Notice that $\#\Delta'_k = k+2$, so

$$C_\cdot(\Delta';G):\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0&1&0\\ 0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1&1&0&0\\ 0&0&0&0\\ 0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0&1&0&0&0\\ 0&-1&0&0&0\\ 0&0&0&1&0\\ 0&0&0&-1&0\end{pmatrix}} G^{\oplus 5}$$

basis: $d'_0 \triangleq x_0$, $d'_1 \triangleq x_1$; x_0, x_1, x_2; x_0, x_1, x_2, x_3; x_0, x_1, x_2, x_3, x_4

remember indexes.

$$\begin{aligned}
0 = x_0 - x_0 &\longleftarrow\!\shortmid\ x_0\\
x_0 - x_1 = x_0 - x_1 &\longleftarrow\!\shortmid\ x_1\\
0 = x_1 - x_1 &\longleftarrow\!\shortmid\ x_2
\end{aligned}$$

$$\begin{aligned}
0 = x_0 - x_0 + x_0 - x_0 &\longleftarrow\!\shortmid\ x_0\\
x_0 - x_1 = x_0 - x_1 + x_1 - x_1 &\longleftarrow\!\shortmid\ x_1\\
0 = x_1 - x_1 + x_2 - x_2 &\longleftarrow\!\shortmid\ x_2\\
x_2 - x_3 = x_2 - x_2 + x_2 - x_3 &\longleftarrow\!\shortmid\ x_3\\
0 = x_3 - x_3 + x_3 - x_3 &\longleftarrow\!\shortmid\ x_4
\end{aligned}$$

$$\begin{aligned}
x_0 = x_0 - x_0 + x_0 &\longleftarrow\!\shortmid\ x_0\\
x_0 = x_0 - x_1 + x_1 &\longleftarrow\!\shortmid\ x_1\\
x_2 = x_1 - x_1 + x_2 &\longleftarrow\!\shortmid\ x_2\\
x_2 = x_2 - x_2 + x_2 &\longleftarrow\!\shortmid\ x_3
\end{aligned}$$