Eine Woche, ein Beispiel
6.12 Condensed

Main Ref: https://people.mpim-bonn.mpg.de/scholze/Course%20Summer%2022.html

That's already so well written. I collect some notations here purely for self-study, and I believe this document is useless for other people.

Condensed set

Def (pro-étale site $*_{\text{proét}}$)

Category: Prof

Cover: for $S \in \text{Prof}$,

$$\text{Cov}(S) = \left\{ \{S_i \xrightarrow{f_i} S\}_{i\in I} \text{ in Prof} \;\middle|\; \begin{array}{l} I \text{ finite} \\ S = \bigcup_{i\in I} f_i(S_i) \end{array} \right\}$$

Naive def. ⚠ Caveat: Prof is large. Need minor modification.

$$\begin{aligned} \text{CondSet} &= Sh(*_{\text{proét}}) \\ &= \left\{ X : \text{Prof}^{op} \longrightarrow \text{Set} \;\middle|\; \begin{array}{l} X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1) \times X(S_2) \\ 0 \to X(S) \to X(T) \rightrightarrows X(T\times_S T) \quad T \twoheadrightarrow S \end{array} \right\} \\ &= \{ X : \text{qcProj}^{op} \longrightarrow \text{Set} \mid X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \} \\ \text{CondAb} &= Sh(*_{\text{proét}}, \text{Ab}) \\ \text{Cond}(\mathcal{C}) &= Sh(*_{\text{proét}}, \mathcal{C}) \\ \text{Cond}(R) &= \text{Cond}(\text{Mod}(R)) = Sh(*_{\text{proét}}, \text{Mod}(R)) \quad R \in \text{Ring} \end{aligned}$$

when $R \in \text{Cond}(\text{Ring})$, require compatability.

Analytic ring and complete condensed A-module

Def. ⚠ Preliminary

An analytic ring is $\mathcal{A} = (A, \mathcal{M}_A(-), \delta)$, where

- $A \in \text{Cond}(\text{Ring})$
- $\mathcal{M}_A : \text{Prof} \longrightarrow \text{Cond}(A) \qquad S \longmapsto \mathcal{M}_A(S)$
- $\delta_S : S \longrightarrow \mathcal{M}_A(S) \qquad s \longmapsto \delta_s$
-
$$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\tilde{f}"] \\ & A \end{tikzcd} \qquad \tilde{f} : \mu \mapsto \int f \mu$$

- $M \in \text{Cond}(A)$ is complete if

$$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\tilde{f}"] \\ & M \end{tikzcd}$$

We require that the full subcategory

$\text{Cond}_{\text{cpl}}(A) := \{\text{complete condensed } A\text{-modules}\} \subseteq \text{Cond}(A)$

should be abelian category.

Liquid vector spaces. $S \in \mathrm{Prof}$.

$$\mathcal{M}(S) = \{f : C(S;\mathbb{R}) \to \mathbb{R} \mid f \text{ cont}\} = \mathbb{R}[S]^{\blacksquare}$$
$$\mathcal{M}(S)_{\ell^p \le c} = \varprojlim_i \mathcal{M}(S_i)_{\ell^p \le c} \subseteq \varprojlim_i \mathbb{R}^{\oplus S_i}$$
$$\mathcal{M}_p(S) = \bigcup_{c>0} \mathcal{M}(S)_{\ell^p \le c}$$
$$\mathcal{M}_{<p}(S) = \bigcup_{0<q<p} \mathcal{M}_q(S)$$

Def. Let $V \in \mathrm{CondAb}$ and $0 < p \le 1$. V is *p-liquid* if

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_{<p}(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd}$$

equiv: $\forall q<p$

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_q(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd}$$

equiv:
$$\bigoplus_j \mathcal{M}_{<p}(T_j) \to \bigoplus_i \mathcal{M}_{<p}(S_i) \to V \to 0$$

Relations

$$\begin{tikzcd} \mathrm{Solid} \arrow[dr] & & \\ \mathrm{Liq}_p \arrow[r, shift right] & \mathrm{CondAb} \arrow[ul, bend right, "(\ )^{\blacksquare}"', "\perp"] \arrow[l, shift right, "(-)^{\mathrm{liq}}"', "\perp"] \arrow[r, shift right] & \mathrm{CondSet} \arrow[l, shift right, "\mathbb{Z}[-]"', "\perp"] \end{tikzcd}$$

Liq_p:
- Abelian
- $- \otimes_{\mathbb{R}_{<p}} -$, $- \otimes^{\mathbb{L}}_{\mathbb{R}_{<p}} -$, $\underline{\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$, $\underline{R\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$
- flatness
- projective objects

CondAb:
- Abelian
- $- \otimes -$, $- \otimes^{\mathbb{L}} -$, $\underline{\mathrm{Hom}}(-,-)$, $\underline{R\mathrm{Hom}}(-,-)$
- flatness
- projective objects

Prop 3.4. $M \in \mathrm{CondAb}$ is flat $\Leftrightarrow M(S)$ is torsion free $\forall S \in \mathrm{qcProj}$
$\Leftrightarrow M(S)$ is torsion free $\forall S \in \mathrm{Prof}$

Prop 3.5. $M \in \mathrm{CondAb}$ is cpt proj $\Leftrightarrow M$ is a retract of $\mathbb{Z}[S]$ for some $S \in \mathrm{Extr}$
$\Leftrightarrow M$ is a retract of $\mathbb{Z}[\beta I]$ for some infinite set I with discrete topology
$\Rightarrow M \oplus \mathbb{Z}[\beta I] \simeq \mathbb{Z}[\beta I]$

Recall that for $A, B \in \mathrm{Ob}(\mathcal{C})$, A is a retract of B if $\exists (r,i)$ s.t

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[d, "\mathrm{Id}_A"] \\ B \arrow[r, "r"] & A \end{tikzcd}$$

commutes

Thm 3.14. $\mathcal{M}_{<p}(S) \in Liq_p$ is flat $\forall\, S \in Prof.$

Moreover, $\forall\, V \in Liq_p$ qs, we have an iso

$$\mathcal{M}_{<p}(S) \otimes_{\mathbb{R}_{<p}} V \cong \bigcup_{q<p} \bigcup_{\substack{K \subset V \\ q\text{-convex}}} \mathcal{M}_q(S, K)$$

Here, for S finite,

$$\mathcal{M}_q(S, K) = \langle s \otimes K \mid_{s \in S} \rangle_{q\text{-convex hull}} \subseteq \mathbb{R}[S] \otimes_{Cond(\mathbb{R})} V \;;$$

for $S = \varprojlim S_i$ profinite,

$$\mathcal{M}_q(S, K) = \varprojlim_i \mathcal{M}_q(S_i, K) \subseteq \mathbb{R}[S] \otimes_{Cond(\mathbb{R})} V .$$