§ 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections
4. L-fct
5. density theorem

1. Galois rep

Setting G: arbitrary topo gp e.g. G any Galois gp
If G profinite $\Rightarrow$ open subgps are finite index subgps.
Λ: top field e.g. $\overline{\mathbb{F}_p}$, $\overline{\mathbb{Q}_p}$, $\mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{cont}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ cont

⚠ $\rho(G)$ can be infinite! for Gal gp
E.g. When char $F \neq l$, we have l-adic cyclotomic character

$$\varepsilon_l: \mathrm{Gal}(F^{sep}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \qquad \sigma \longmapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \qquad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute ε_l for $F = \mathbb{F}_p$.
A: $\varepsilon_l: \hat{\mathbb{Z}} \cong \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \qquad 1 \longmapsto p$
↑ lift from $\mathbb{Z} \longrightarrow \mathbb{Z}_l^\times$

Notice the following two definitions don't depend on the topo of Λ.

Def (sm Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{sm}}(G)$
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with open stabilizer.

Def (fin image Galois rep) $(\rho, V) \in \mathrm{rep}_{\Lambda,\mathrm{fi}}(G)$ fi: finite image / finite index
$V \in \mathrm{vect}_\Lambda$ + $\rho: G \longrightarrow GL(V)$ with finite image

Rmk.

$$\begin{tikzcd}
\mathrm{rep}_{\Lambda,sm}(G) \arrow[r, phantom, "="] \arrow[d, phantom, "\bigcap"] & \mathrm{rep}_{\Lambda_{disc},cont}(G) \arrow[rr] \arrow[d, phantom, "\bigcap"] \arrow[dr, shift left] & & \mathrm{rep}_{\Lambda,cont}(G) \arrow[dl] \arrow[d, phantom, "\bigcap"] \\
& & \mathrm{rep}_{\Lambda,fi}(G) \arrow[ul, shift left] & \\
\mathrm{Rep}_{\Lambda,sm}(G) & \mathrm{Rep}_{\Lambda_{disc},cont}(G) \arrow[l] \arrow[rr] \arrow[dr, shift left] & & \mathrm{Rep}_{\Lambda,cont}(G) \\
& & \mathrm{Rep}_{\Lambda,fi}(G) \arrow[ul, shift left] \arrow[uu, phantom, "\bigcap"] &
\end{tikzcd}$$

$\rightarrow$: if fin index subgps are open

$\rightarrow$: if G: profinite gp (Only need: open $\Rightarrow$ fin index)

$\rightarrow$: Artin rep (of profinite gp)

Artin rep. $\Lambda = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)

$\exists\ \mathcal{U} \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.

$\forall H \leq GL_n(\mathbb{C})$, $H \subseteq \mathcal{U} \Rightarrow H = \{Id\}$.

Proof. Take $\mathcal{U} = \{A \in GL_n(\mathbb{C}) \mid \|A - I\| < \frac{1}{3n}\}$ $\quad \|\cdot\| = \|\cdot\|_{max}$, $|\cdot| = |\cdot|_{max}$

Only need to show, $\forall A \in GL_n(\mathbb{C})$, $A \neq Id$, $\exists m \in \mathbb{N}$, s.t. $A^m \notin \mathcal{U}$.

Consider the Jordan form of A.

Case 1. A unipotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1$, $v \in \mathbb{C}^n - \{0\}$ s.t. $Av = \lambda v$. Take $m \in \mathbb{N}$ s.t. $|\lambda^m - 1| > \frac{1}{3}$.

$\frac{1}{3}|v| < |\lambda^m - 1||v| = \|(A^m - Id)v\| \leq n\|A^m - Id\||v| \Rightarrow \|A^m - Id\| \geq \frac{1}{3n}$.

Prop. For $(\rho, V) \in \mathrm{rep}_{\mathbb{C},cont}(G)$, $\rho(G)$ is finite. G profinite

Proof. Take $\mathcal{U}$ in Lemma 1, then

$\rho^{-1}(\mathcal{U})$ is open $\Rightarrow \exists I \leq G_F$ finite index, $\rho(I) \subseteq \mathcal{U}$

$\xRightarrow{\text{Lemma 1}} \rho(I) = Id$

$\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp, so ρ is semisimple.
Since classifications of irr reps for $\mathbb{C}$ & $\overline{\mathbb{Q}}$ are the same,
every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$.

2. #{ fin subgps in GL_n(C) of "exponent m" } is bounded, see:
https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-nc

2. Weil-Deligne rep

Now we work over "the skeloton of the Galois gp" in general.

Finite field

Task. For Λ: NA local field with char $k_\Lambda = l$, understand $\mathrm{rep}_{\Lambda,\mathrm{cont}}(\hat{\mathbb{Z}})$.