SL_2 - case

This is some reflections about the LLC seminar Talk 3 &4. I claim no originality. As the lecture note (of them) is already well written, we won't care about the completeness of the whole theory, and we specify to the concrete examples for the most of time.

Fix F NA local field. Then

$$L_F = W_F \times SU_2(\mathbb{R}) \qquad (F \text{ non-Archi}: L_F = W_F)$$

$$\Phi_{temp}(G) = \mathrm{Hom}_{Grp, sec}(L_F, {}^L G)/\hat{G}\text{-conj}$$

$$\overset{\text{if } SU_2(\mathbb{R})\to 1}{\supseteq} H^1(W_F, \hat{G})/\hat{G}\text{-twisted conj}$$

1. statement of conjectures.

Conj 1. $\exists$ map

$$\begin{array}{cccc} LL: & \Pi_{temp}(G) & \longrightarrow & \Phi_{temp}(G) \\ & \cup & & \Psi \\ & \Pi_\phi(G) & \longrightarrow & \phi \end{array}$$

For $\phi \in \Phi_{temp}(G)$, define

$$S_\phi := C_{\hat{G}}(\phi(L_F)) \qquad \overline{S_\phi} := S_\phi / Z(\hat{G})^\Gamma$$

Conj 2. We have a bijection $\quad$ (F non-Archi: just an injective map)

$$\iota_m : \Pi_\phi(G) \longrightarrow \mathrm{Irr}_{\mathbb{C}}(\pi_0(\overline{S_\phi}))$$

Here, $m = (\mathcal{U}, \theta)$ is a Whittaker datum introduced in [LLC, Talk3].
we require that θ is generic.

https://en.wikipedia.org/wiki/Whittaker_model

In the case where $G = SL_2/F$, $F = \mathbb{Q}_p$, one can take

$$\mathcal{U} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \qquad \theta: \mathcal{U}(F) \cong F \longrightarrow \mathbb{C}^\times$$
$$\mathbb{Z}_p \longmapsto 1$$
$$\frac{1}{p^k} \longmapsto \zeta_{p^k}$$

Def $(\pi, V) \in \mathrm{Irr}(G(F))$ is m-generic, if

$$\mathrm{Hom}_{\mathcal{U}(F)}(\pi|_{\mathcal{U}(F)}, \theta) = \mathrm{Hom}_{G(F)}(\pi, \mathrm{Ind}_{\mathcal{U}(F)}^{G(F)} \theta) \neq 0$$

Conj 3. $\exists!$ m-generic rep in $\Pi_\phi(G)$, which should be $\iota_m^{-1}(\chi_0)$.

2. discussion of $\mathrm{Irr}_{\mathbb{C}}(\pi_0(\bar{S}_\phi))$ in SL_2 case.

Let $G = SL_2/F$, then

$$\hat{G} = PGL_2(\mathbb{C}), \quad Z(\hat{G}) = \{Id\} \Rightarrow Z(\hat{G})^\Gamma = \{Id\}$$

E.g. Let E/F be biquadratic Galois, i.e., $Gal(E/F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.

Let

$$\begin{aligned} \phi : W_F \times SU_2(\mathbb{R}) \twoheadrightarrow \; & Gal(E/F) \longrightarrow PGL_2(\mathbb{C}) \\ & (1,0) \longmapsto \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \\ & (0,1) \longmapsto \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \end{aligned}$$

In this case,

$$\begin{aligned} \phi(L_F) &= \left\{Id, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right\} \subseteq PGL_2(\mathbb{C}) \\ S_\phi &= C_{\hat{G}}(\phi(L_F)) = \phi(L_F) \\ \overline{S_\phi} &= S_\phi / Z(\hat{G})^\Gamma = \phi(L_F) \\ \pi_0(\overline{S}_\phi) &= \phi(L_F) \\ \mathrm{Irr}(\pi_0(\overline{S}_\phi)) &= \{\chi_0, \chi_1, \chi_2, \chi_3\} \end{aligned}$$

Q: Can we write down $\Phi_\phi(G)$ & ι_m explicitly?

E.g. (unramified case) For $x \in \mathbb{C}^\times$, let

$$\phi_x : W_F \times SU_2(\mathbb{R}) \twoheadrightarrow \underset{\underset{Gal(F^{ur}/F)}{\cup}}{\mathbb{Z}} \longrightarrow PGL_2(\mathbb{C})$$

$$\mathrm{Frob} \longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix}$$

In this case,

$$\phi(L_F) = \left\langle \begin{pmatrix} x & \\ & 1 \end{pmatrix} \right\rangle \subseteq PGL_2(\mathbb{C})$$

$$\overline{S_\phi} = S_\phi = C_{\hat{G}}(\phi(L_F)) = \begin{cases} \hat{T} & x \neq 1, -1 \\ \hat{G} & x = 1 \\ \hat{T} \cup \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \hat{T} & x = -1 \end{cases}$$

$$\pi_0(\overline{S}_\phi) = \begin{cases} \{Id\} & x \neq -1 \\ \left\{Id, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right\} & x = -1 \end{cases}$$

$$\mathrm{Irr}(\pi_0(\overline{S}_\phi)) = \begin{cases} \{\chi_0\} & x \neq -1 \\ \{\chi_0, \chi_1\} & x = -1 \end{cases}$$

Q: Can we write down $\Phi_\phi(G)$ & ι_m explicitly?