

# Eine Woche, ein Beispiel

## 10.2 equivariant K-theory of Steinberg variety : notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:  
[http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis\\_Tomasz%20Przezdziecki.pdf](http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf)

We changed some notation for the convenience of writing.

Task.

1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. change of basis
  - § 6.1 two basis
  - § 6.2 tangent space, Euler class
  - § 6.3 transition matrix, localizat
  - § 6.4 generators
7. convolution product
  - § 7.1 clean intersection formula
  - § 7.2. convolution for canonical basis
  - § 7.3. expression of  $D_k$ .

We may use two examples for the convenience of presentation.  
Readers can easily distinguish them by the dim vectors.

## 1. dimension vector

$$|\underline{d}| = 5$$

$$\underline{d} = (3, 2)$$

$$\underline{d} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 0 \\ 0, 0 \end{pmatrix} = \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array} = \begin{array}{c} \uparrow \uparrow \uparrow \\ \downarrow \downarrow \downarrow \end{array} = \begin{array}{c} \downarrow \times \times \times \\ \downarrow \times \end{array} \in W_d \backslash W_{\text{id}} \text{ or } \text{Min}(W_{\text{id}}, W_d)$$

$v_{\text{id}} = \pi_{\underline{d}}^{-1}(F_{\text{id}})$

## 2. Weyl group

Set	element	special element	others
$W_{\text{id}} = S_5$	$w, x$	$w_{\text{max}} = \begin{array}{ c c c c c } \hline & * & * & * & * \\ \hline * & & & & \\ \hline \end{array}$	$T = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	$w$	$w_{\text{max}} = \begin{array}{ c c c } \hline * & * & * \\ \hline * & & \\ \hline * & & \\ \hline \end{array}$	$T_d = \{s_1, s_2, s_4\}$
$W_d \backslash W_{\text{id}} = S_3 \times S_2 \backslash S_5$	$w, \underline{d}$	$\begin{array}{ c c c c c } \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$	(Comp <sub>d</sub> )

$$\text{Min}(W_{\text{id}}, W_d) = \left\{ \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}, \dots \right\} u \quad \begin{array}{c} \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array} \quad (\text{Shuffled})$$

$$0 \longrightarrow W_d \longrightarrow W_{\text{id}} \longrightarrow W_{\text{id}} \backslash W_d \longrightarrow 0 \quad w = wu \mapsto \underline{d}$$

$\xrightarrow{\text{Min}(W_{\text{id}}, W_d)} \xrightarrow{\cong} u \quad \downarrow \quad \underline{d}$

$w = \begin{array}{|c|c|c|c|c|} \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$   
 $u = \begin{array}{|c|c|c|c|c|} \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$   
 $w = \begin{array}{|c|c|c|c|c|} \hline * & * & * & * & * \\ \hline * & & & & \\ \hline * & & & & \\ \hline \end{array}$

Another example:  $\underline{d} = (1, 2)$   $\xrightarrow{a} b$   $\xrightarrow{\langle v_1 \rangle \rightarrow \langle v_2, v_3 \rangle}$

	$w = wu$	$w$	$\underline{d}, u$	order of basis	$(w)$	$l(w)$	$B_w$	$B_{wu}$	$wB_{wu}^{-1}$
Id	$(1^2 3)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
t	$(23)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
s	$(12)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
ts	$(132)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
st	$(123)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$
sts	$(13)$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\begin{array}{ c c } \hline \diagup \diagdown \diagup \diagdown \diagup \diagdown \\ \text{Young Tableaux} \end{array}$	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$

### 3. alg group & Lie algebra

$$G_{\text{Idl}}, B_{\text{Idl}}, T_{\text{Idl}}, N_{\text{Idl}} \quad W_{\text{Idl}} = N_{G_{\text{Idl}}}(\Pi_{\text{Idl}}) / \Pi_{\text{Idl}} \quad GL_5(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$G_d, B_d, T_d, N_d \quad W_d = N_{G_d}(T_d) / T_d \quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$B_\infty = B_{\text{Idl}} \omega^{-1} = \text{Stab}_{G_{\text{Idl}}}(F_\infty)$$

$$B_\infty = \omega B_d \omega^{-1} = \text{Stab}_{G_d}(F_\infty) \quad N_\infty = R_u(B_\infty)$$

For  $s \in \Pi$  s.t.  $\omega s \omega^{-1} \in W_d$  (i.e.  $W_d \omega = W_d \omega s$ ), define

$$P_{\infty, \omega s} = \omega (B_d s s^{-1} B_d \cup B_d) \omega^{-1} \quad N_{\infty, \omega s} = R_u(B_{\infty, \omega s})$$

$$= B_\infty \omega s \omega^{-1} B_\infty \cup B_\infty \quad = N_\infty \cap N_{\infty, \omega s}$$

$$M_{\infty, \omega s} = N_\infty / N_{\infty, \omega s}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{\sigma(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\infty, \omega''} := N_\infty \cap N_\infty$$

$$M_{\infty, \omega''} := N_\infty / N_{\infty, \omega''}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega''}$$

Their Lie algebras are collected here.

$$\mathfrak{g}_{\text{Idl}}, \mathfrak{b}_{\text{Idl}}, \mathfrak{t}_{\text{Idl}}, \mathfrak{n}_{\text{Idl}}$$

$$g_d \quad b_d \quad t_d \quad n_d$$

$$\mathfrak{b}_\infty \quad \bar{\mathfrak{b}}_\infty$$

$$b_\infty \quad n_\infty$$

$$P_{\infty, \omega s} \quad N_{\infty, \omega''}$$

$$m_{\infty, \omega''}$$

$$\bar{b}_\infty = b_{\omega \max \infty}$$

$$\bar{b}_\infty = b_{w \max \infty}$$

$$\bar{P}_{\infty, \omega s} = P_{w \max \infty, w \max \infty s}$$

$$m_{\infty, \omega''}$$

$$\text{Rep}_d(Q) := \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{\text{Idl}}^{\oplus k}$$

$$V_\infty = \{f \in \text{Rep}_d(Q) \mid f: F_{\infty, i} \subseteq F_{\infty, i}\} = \mu_d \pi_d^{-1}(F_\infty)$$

$$= \begin{pmatrix} v_3 & v_1 & v_2 \\ v_4 & * & * \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & * & * \end{pmatrix}$$

$$V_{\omega(i)}$$

$$V_{\infty, \omega''} = V_\infty \cap V_{\infty''}$$

$$J_{\infty, \omega''} = V_\infty / V_{\infty, \omega''}$$

Later we may twist the group actions.

$$\text{E.g. } Y_{\infty, \omega'} = V_{\infty, \omega \omega'} \quad Y_{\infty, \omega''} = V_{\infty, \omega^{-1} \omega''}$$

#### 4. typical variety

Id corres to

$$\begin{aligned}
 F_{\text{Id}\text{Id}} &\cong G_{\text{Id}\text{Id}} / B_{\text{Id}\text{Id}} & F_{\text{Id}} \\
 F_{\underline{d}} &\cong G_{\underline{d}} / B_{\underline{d}} & F_u \\
 F_{\infty} &\cong G_{\underline{d}} / B_{\infty} & F_{\omega} \\
 F_{\underline{d}} &= \coprod_{\underline{d}'} F_{\underline{d}'} & - \\
 F_{g\infty} &\cong G_{\underline{d}} / gB_{\infty}g^{-1} & F_{g\infty} \\
 F_{\infty, \cdot} &= \text{Flag}_{\underline{d}}(F_{\text{Id}}) = F_{\{v_{\infty(1)}, v_{\infty(2)}, \dots, v_{\infty(\text{Id})}\}} & \\
 &= F_{\{u_5, u_3, u_1, u_6, u_2\}}
 \end{aligned}$$

✓ The action on Flag is not the same as in

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$$F_{\text{Id}\text{Id}} \neq \coprod_{\underline{d}} F_{\underline{d}}$$

$F_{\infty} \cong F_{\underline{d}}$  with different base pt. Base pt makes difference!

$$\begin{aligned}
 F_{\text{Id}\text{Id}} \times F_{\text{Id}\text{Id}} && F_{\text{Id}, \text{Id}} \\
 F_{\underline{d}} \times F_{\underline{d}'} && F_{u, u'} \\
 F_{\infty} \times F_{\infty'} && F_{\infty, \infty'} \\
 F_{\underline{d}} \times F_{\underline{d}'} &= \coprod_{\underline{d}, \underline{d}'} (F_{\underline{d}} \times F_{\underline{d}'}) & -
 \end{aligned}$$

$$F_{\infty, \infty'} := (F_{\infty}, F_{\infty'})$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}} & & \downarrow \pi_{\underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}}
 \end{array}$$

$\mu_{\underline{d}}^{-1}(M) \cong \text{Flag}_{\underline{d}}(M) \subseteq F_{\underline{d}}$  is the Springer fiber.

$$\begin{array}{ccc}
 Z_{\underline{d}, \underline{d}'} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}'} \\
 \downarrow M_{\underline{d}, \underline{d}'} & & \downarrow \pi_{\underline{d}, \underline{d}'} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}'}
 \end{array}$$

$$\begin{array}{ccc}
 Z_{\underline{d}} & \subset & \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \times F_{\underline{d}} \\
 \downarrow M_{\underline{d}, \underline{d}} & & \downarrow \pi_{\underline{d}, \underline{d}} \\
 \text{Rep}_{\underline{d}}(\mathbb{Q}) & & F_{\underline{d}} \times F_{\underline{d}}
 \end{array}$$

$$\begin{array}{c}
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \subseteq \text{Rep}_{\underline{d}}(\mathbb{Q}) \times F_{\underline{d}} \\
 \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) := \bigsqcup_{\underline{d}} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{array}$$

$$\widetilde{\text{Rep}}_{\infty}(\mathbb{Q}) \cong G_{\underline{d}} \times^{B_{\infty}} r_{\infty}$$

$$\begin{aligned}
 Z_{\underline{d}, \underline{d}'} &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}'}(\mathbb{Q}) \\
 Z_{\underline{d}} &= \bigsqcup_{\underline{d}', \underline{d}''} Z_{\underline{d}, \underline{d}'} \\
 &= \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q}) \times_{\text{Rep}_{\underline{d}}(\mathbb{Q})} \widetilde{\text{Rep}}_{\underline{d}}(\mathbb{Q})
 \end{aligned}$$

$$Z_{\infty, \infty'} = Z_{u, u'}$$

## 5. (equivariant) stratifications.

In the following tables,  $uw' = \tilde{w}'\tilde{u}$ .

$F_\infty \in \widetilde{\text{Rep}}_d(Q)$  means  $(p_0, F_\infty); (F_\infty, F_{\infty'}) \in Z_d$  means  $(p_0, F_\infty, F_{\infty'})$ .

▽  $G \times G$  acts on  $\mathcal{F} \times \mathcal{F}$  in a twisted way

$$\text{e.g. } (g_1, g_2) F_{\infty, \infty'} = F_{g_1 \infty, g_1 \tilde{w} g_2 \infty'^{-1}}$$

$$(g_1, g_2) E_{\infty, \infty'} = E_{g_1 \infty, g_2 \infty'}$$

variety base point	stratification stabilizer	type	B-orbit	$B \times B$ -orbit stabilizer are twisted	$B \times G$ -orbit	$G \times B$ -orbit	Remark $G \times \{*\}$ -orbit
$\mathcal{B}$	$\mathcal{B} \times \mathcal{B}$		$\Omega_g$	$\Omega_{g, g'}$	$\text{pr}_i^{-1}(\Omega_g)$	$\Omega_{g'}$	
$F_g$ ( $F_g, F_{gg'}$ )	$B \cap gBg^{-1}$		$B \cap gBg^{-1} \times B \cap (gg')B(gg')^{-1}$				$gBg^{-1} \cap gg'B(gg')^{-1}$
$\mathcal{F}_{\text{id}}$	$\mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$		$\mathcal{V}_\infty$	$\mathcal{V}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{V}_\infty)$	$\mathcal{V}_\infty$	
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )	$B_{\text{id}} \cap B_\infty$		$B_{\text{id}} \cap B_\infty \times B_{\text{id}} \cap B_{\infty'}$				$B_\infty \cap B_{\infty\infty'}$
$\mathcal{F}_u$	$\mathcal{F}_u \times \mathcal{F}_u$		$\Omega_w^u$	$\Omega_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{w'}^{u, u'}$	
$F_{wu}$ ( $F_{wu}, F_{wuwu'}$ )	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{ww'}$				$B_w \cap B_{ww'}$
$\mathcal{F}_d$	$\mathcal{F}_d \times \mathcal{F}_d$		$\Omega_w^u$	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{\infty'}^u = \Omega_{\tilde{w}}^{u, \tilde{u} u'}$	compatibility
$F_\infty$ "	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{w\tilde{w}}$				$B_w \cap B_{w\tilde{w}}$
$F_{wu}$ ( $F_{wu}, F_{wuwu'}$ )							

The following may not be single orbit, but derived from the above definition.

$\mathcal{F}_d$	$\mathcal{F}_d \times \mathcal{F}_d$	$\mathcal{O}_\infty$	$\mathcal{O}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{O}_\infty)$	$\mathcal{O}_\infty$		preimage of $\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )		$\Omega_w^u$	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\bigsqcup_u \Omega_{\tilde{w}}^{u, \tilde{u} u'}$		preimage of $\mathcal{Z}_{d, d'} \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{\text{Rep}}_d(Q)$	$Z_d$	$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\widetilde{\Omega}_w^{u, u'}$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )							
$\widetilde{\text{Rep}}_d(Q)$	$Z_d$	$\widetilde{\mathcal{O}}_\infty$	$\widetilde{\mathcal{O}}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\widetilde{\mathcal{O}}_\infty)$	$\widetilde{\mathcal{O}}_\infty$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$F_\infty$ ( $F_\infty, F_{\infty\infty'}$ )		$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\bigsqcup_u \widetilde{\Omega}_{\tilde{w}}^{u, \tilde{u} u'}$		

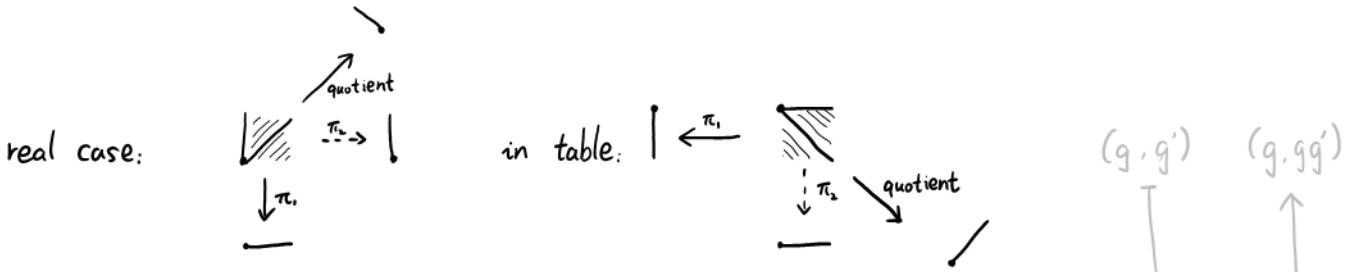
$$\mathcal{Z}_{\infty'} := \overline{\widetilde{\mathcal{O}}_\infty} \subseteq \overline{\widetilde{\mathcal{O}}_{\infty'}}$$

$$\mathcal{Z}_{w'}^{u, u'} := \overline{\widetilde{\Omega}_{w'}^{u, u'}} \subseteq \overline{\widetilde{\Omega}_w^{u, u'}}$$

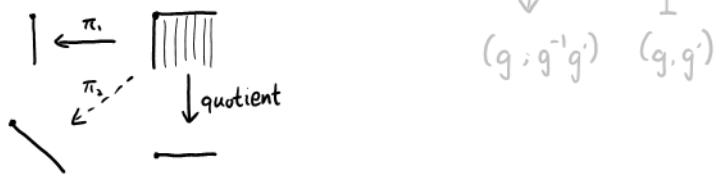
Zar-loc sub v.b.?

$$\mathcal{Z}_d^\infty$$

$$\mathcal{Z}_{d, d'}^\infty = \mathcal{Z}_\infty \cap \mathcal{Z}_{d, d'}$$



We want gp action to be compatible with  $\pi_i$  and the quotient map.  
Therefore, we would do a twist.



Rmk. The stabilizer is not trivial to determine because of this twist!

$$\begin{aligned}
 E_{g_1, g_2} = E_{g_3, g_4} &\Leftrightarrow g_1 B = g_3 B, g_2 B = g_4 B \\
 &\Leftrightarrow g_1 B = g_3 B, g_1 g_2 B = g_3 g_4 B \\
 E_{g_1\omega, g_2\omega'} = E_{\omega, \omega'} &\Leftrightarrow g_1 \omega B = \omega B, g_1 \omega g_2 \omega' B = g_2 \omega' B \\
 &\Leftrightarrow g_1 = \omega b, \omega^{-1} \in B_\omega \quad b, g_2 \in B_{\omega'} \\
 &\Leftrightarrow g_1 \in B_\omega, \quad g_2 \in \omega^{-1} g_1 \omega \cdot B_{\omega'}
 \end{aligned}$$

The following tables may help you to understand the notations.

$\dim$	$B_{\text{Id}} \cdot F_{\text{var}}$	0	1	1	2	2	3
	$B_{\text{Id}} \times B_{\text{Id}} \cdot (F_{\text{var}}, F_{\text{var}})$	$\mathcal{V}_{\text{Id}}$	$\mathcal{V}_t$	$\mathcal{V}_s$	$\mathcal{V}_{ts}$	$\mathcal{V}_{st}$	$\mathcal{V}_{sts}$
	$B_{\text{Id}} \cdot F_w$	$\mathcal{V}_{\text{Id}}$	$\mathcal{V}_{\text{Id},\text{Id}}$	$\mathcal{V}_{\text{Id},t}$	$\mathcal{V}_{\text{Id},s}$	$\mathcal{V}_{\text{Id},ts}$	$\mathcal{V}_{\text{Id},st}$
0							
1							
2							
3							

$\dim$	$B_d \cdot F_{\text{var}}$	0	1	2	3	4	5
	$B_d \times B_d \cdot (F_{\text{var}}, F_{\text{var}})$	$\mathcal{O}_{\text{Id}}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
	$B_d \cdot F_w$	$\mathcal{O}_{\text{Id}}$	$\mathcal{O}_{\text{Id},\text{Id}}$	$\mathcal{O}_{\text{Id},t}$	$\mathcal{O}_{\text{Id},s}$	$\mathcal{O}_{\text{Id},ts}$	$\mathcal{O}_{\text{Id},st}$
0							
1							
2							
3							
4							
5							

The following tables may help you to understand the notations.

$$\omega = ts, \omega' = s$$

$\dim$	$B_{Id} \cdot F_{ts}$	0	1	1	2	2	3	$\text{pr}_i^{-1}(\mathcal{V}_{ts})$
	$B_{Id} \times B_{Id} \cdot (F_{ts}, F_{ts})$	$\mathcal{V}_{Id}$	$\mathcal{V}_t$	$\mathcal{V}_s$	$\mathcal{V}_{ts}$	$\mathcal{V}_{st}$	$\mathcal{V}_{sts}$	
	$B_{Id} \cdot F_{ts}$	$\mathcal{V}_{Id}$	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
0		$\mathcal{V}_{Id}$	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
1		$\mathcal{V}_t$	$\mathcal{V}_{t,t}$	$\mathcal{V}_{t,Id}$	$\mathcal{V}_{t,ts}$	$\mathcal{V}_{t,s}$	$\mathcal{V}_{t,sts}$	$\mathcal{V}_{t,st}$
1		$\mathcal{V}_s$	$\mathcal{V}_{s,s}$	$\mathcal{V}_{s,st}$	$\mathcal{V}_{s,Id}$	$\mathcal{V}_{s,sts}$	$\mathcal{V}_{s,t}$	$\mathcal{V}_{s,ts}$
2		$\mathcal{V}_{ts}$	$\mathcal{V}_{ts,st}$	$\mathcal{V}_{ts,s}$	$\mathcal{V}_{ts,sts}$	$\mathcal{V}_{ts,Id}$	$\mathcal{V}_{ts,ts}$	$\mathcal{V}_{ts,t}$
2		$\mathcal{V}_{st}$	$\mathcal{V}_{st,ts}$	$\mathcal{V}_{st,sts}$	$\mathcal{V}_{st,t}$	$\mathcal{V}_{st,st}$	$\mathcal{V}_{st,Id}$	$\mathcal{V}_{st,s}$
3		$\mathcal{V}_{sts}$	$\mathcal{V}_{sts,sts}$	$\mathcal{V}_{sts,ts}$	$\mathcal{V}_{sts,st}$	$\mathcal{V}_{sts,t}$	$\mathcal{V}_{sts,s}$	$\mathcal{V}_{sts,Id}$

$\text{shape}$	$B_d \cdot F_{ts}$	$\mathcal{F}_{Id}$	$\mathcal{F}_s$	$\mathcal{F}_{st}$	$\text{pr}_i^{-1}(\mathcal{O}_{ts})$	$\text{pr}_{i,Id}^{-1}(\Omega_t^s)$	$\Omega_{t,Id}^{s,Id} = \mathcal{O}_{ts,s}$
	$B_d \times B_d \cdot (F_{ts}, F_{ts})$	$\mathcal{O}_{Id}$	$\mathcal{O}_t$	$\mathcal{O}_s$	$\mathcal{O}_{ts}$	$\mathcal{O}_{st}$	$\mathcal{O}_{sts}$
	$B_d \cdot F_{ts}$						
$\mathcal{F}_{Id}$	$\mathcal{O}_{Id}$	$\Omega_{Id,Id}^{Id,Id}$	$\Omega_{Id,t}^{Id,Id}$	$\Omega_{Id,s}^{Id,s}$	$\Omega_{Id,t}^{Id,s}$	$\Omega_{Id,ts}^{Id,st}$	$\Omega_{Id,t}^{Id,st}$
	$\mathcal{O}_t$	$\Omega_{t,t}^{Id,Id}$	$\Omega_{t,Id}^{Id,Id}$	$\Omega_{t,t}^{Id,s}$	$\Omega_{t,Id}^{Id,s}$	$\Omega_{t,ts}^{Id,st}$	$\Omega_{t,Id}^{Id,st}$
$\mathcal{F}_s$	$\mathcal{O}_s$	$\Omega_{Id,Id}^{s,Id}$	$\Omega_{Id,t}^{s,Id}$	$\Omega_{Id,Id}^{s,s}$	$\Omega_{Id,t}^{s,s}$	$\Omega_{Id,ts}^{s,st}$	$\Omega_{Id,t}^{s,st}$
	$\mathcal{O}_{ts}$	$\Omega_{t,t}^{s,Id}$	$\Omega_{t,Id}^{s,Id}$	$\Omega_{t,t}^{s,s}$	$\Omega_{t,Id}^{s,s}$	$\Omega_{t,ts}^{s,st}$	$\Omega_{t,Id}^{s,st}$
$\mathcal{F}_{st}$	$\mathcal{O}_{ts}$	$\Omega_{Id,Id}^{st,Id}$	$\Omega_{Id,t}^{st,Id}$	$\Omega_{Id,Id}^{st,s}$	$\Omega_{Id,t}^{st,s}$	$\Omega_{Id,ts}^{st,st}$	$\Omega_{Id,t}^{st,st}$
	$\mathcal{O}_{sts}$	$\Omega_{t,t}^{st,Id}$	$\Omega_{t,Id}^{st,Id}$	$\Omega_{t,t}^{st,s}$	$\Omega_{t,Id}^{st,s}$	$\Omega_{t,ts}^{st,st}$	$\Omega_{t,Id}^{st,st}$

## 6. change of basis

### §6.1 two basis

Def Let  $Y \subset X$  be  $G$ -equiv closed subvariety,  $X$  proj.

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

with same notation,

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{S(G)} \in H_q^*(X; \mathbb{Q})$$

$$\begin{array}{ccc} Y & \xhookrightarrow{\iota_Y} & X \\ & \downarrow \pi_Y & \\ & pt & \end{array}$$

By cellular fibration lemma,

$$\begin{array}{ccccccc} K_0^{Td}(F_d) & \cong & K_0^{Cd}(F_d \times F_d) & \cong & K_0^{Cd}(Z_d) \\ \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Td} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[\bar{\mathcal{O}}_\omega]^{Cd} & \cong & \oplus_{\omega \in W_{td}} R(T_d)[Z_\omega]^{Cd} \\ & & & & & & \downarrow \\ & & & & & \parallel S & K_0^{Td}(Z_d) \\ & & & & & & \\ & & & & & & \oplus_{\omega, \omega' \in W_{td}} R(T_d)[\bar{\mathcal{O}}_{\omega, \omega'}]^{Td} \end{array}$$

as  $R(T_d)$ -modules.

⚠ There is no evidence if  $[Z_\omega]^{Cd}$  will be mapped to  $\oplus_{\omega \in W_{td}} [\bar{\mathcal{O}}_{\omega, \omega'}]^{Td}$ . Luckily, the horizontal line sends generators to generators.

Hint: Consider the following commutative diagram:

$$\begin{array}{ccccc} F_d & \xrightarrow{(F_d, Id)} & F_d \times F_d & \xrightarrow{(p_0, Id)} & Z_d \\ \bar{\mathcal{O}}_\omega & \nearrow & \nearrow & \nearrow & \\ pt & = & pt & = & pt \end{array}$$

To do linear alg, we take

$$\begin{aligned} \mathcal{R}(G) &:= \text{Frac } (R(G)) \\ \mathcal{S}(G) &:= \text{Frac } (S(G)) \end{aligned}$$

$$\begin{aligned} \mathcal{K}_o^G(X) &= K_o^G(X) \otimes_{R(T_d)} \mathcal{R}(T_d) \\ \mathcal{H}_G^*(X; \mathbb{Q}) &= H_G^*(X; \mathbb{Q}) \otimes_{S(T_d)} S(T_d) \end{aligned}$$

For  $R(T_d)$ -mod  $K_o^G(X)$ ,  $S(T_d)$ -mod  $H_G^*(X; \mathbb{Q})$

$$\begin{aligned} \text{Define } \psi_\infty &= [\{F_\infty\}]^{T_d} = (i_\infty)_* 1_{R(T_d)} \in K_o^{T_d}(\mathcal{F}_d) \\ \psi_{\infty, \infty'} &= [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d} = (i_{\infty, \infty'})_* 1_{R(T_d)} \in K_o^{T_d}(\mathbb{Z}_d) \\ \psi_{\infty, \infty'} &= [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d} \end{aligned}$$

We get two  $\mathcal{R}(T_d)$ -basis. ( $\psi_\infty$  is  $\mathcal{R}(T_d)$ -basis, by Localization theorem.)

$$\begin{array}{ccc} K_o^{T_d}(\mathcal{F}_d) & \longrightarrow & K_o^{T_d}(\mathbb{Z}_d) \\ [\overline{O_\infty}]^{T_d} & & [\overline{O_{\infty, \infty'}}]^{T_d} \\ \psi_\infty & & \psi_{\infty, \infty'} \end{array} \quad \begin{array}{l} \text{standard basis for stratification} \\ \text{canonical basis for convolution} \end{array}$$

Localization thm [Thm 10.1]

Let  $i: X^{T_d} \hookrightarrow X$ ,  $X$  is smooth.

$$\begin{array}{ccccc} \mathcal{K}_o^{T_d}(X^{T_d}) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X) & \xrightarrow{i^*} & \mathcal{K}_o^{T_d}(X^{T_d}) \\ \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d}^*(X^{T_d}; \mathbb{Q}) \end{array}$$

are isos as  $\mathcal{R}(T_d)$  or  $\mathcal{S}(T_d)$ -module.

Q: The Steinberg variety  $Z_d$  is usually not smooth.

How to show that  $\{\psi_{\infty, \infty'}\}$  forms a basis?

Guess: apply localization thm to  $T_d \times T_d$  first.

## § 6.2. tangent space, Euler class.

Def (tangent space of fixed pts. in  $R(T_d)$ )

$$\underline{\Lambda}_{\infty} := T_{F_{\infty}} F_d \cong T_{Id}(G_d/B_{\infty}) \cong \mathfrak{g}_d/b_{\infty} = n_{\infty}$$

$$\widetilde{\Lambda}_{\infty} := T_{(p_0, F_{\infty})} \widetilde{Rep_d}(\mathbb{Q}) \cong T_{r_{\infty}} \oplus T_{F_{\infty}} F_d = r_{\infty} \oplus n_{\infty}$$

$$\underline{\Lambda}_{\infty, \omega'}^x := T_{(p_0, F_{\infty}, F_{\omega'})} \overline{\mathcal{O}}_x$$

$$\widetilde{\Lambda}_{\infty, \omega'}^x := T_{(p_0, F_{\infty}, F_{\omega'})} \mathcal{Z}_x \cong T_{r_{\infty, \omega'}} \oplus T_{(F_{\infty}, F_{\omega'})} \overline{\mathcal{O}}_x = r_{\infty, \omega'} \oplus \underline{\Lambda}_{\infty, \omega'}^x$$

Hint:

$$\begin{array}{ccc} \overline{\mathcal{O}}_x & \xrightarrow{(p_0, Id)} & \mathcal{Z}_x \\ & \searrow & \downarrow \\ & & \overline{\mathcal{O}}_x \end{array} \quad T_{x_0} (\cancel{\times}) = T_{x_0} (\cancel{\vee}) \oplus T_{x_0} (\cancel{\wedge})$$

$$\underline{\Lambda}_{\infty, \omega'}^x := T_{(F_{\infty}, F_{\infty, \omega'})} \overline{\mathcal{O}}_x$$

Rmk. It is still not easy to express  $\widetilde{\Lambda}_{\infty, \omega'}$  as Lie alg.  
However, we still know some special cases:

$$\begin{aligned} \underline{\Lambda}_{\infty, x}^x &:= T_{(F_{\infty}, F_{\infty x})} \overline{\mathcal{O}}_x \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{Id} G_d/B_{\infty} \cap B_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} \cap b_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} + b_{\infty}/(b_{\infty} \cap b_{\infty x}) \\ &= n_{\infty} + \underline{m}_{\infty, x} \end{aligned}$$

( $\underline{m}_{\infty, Id} = 0$ . For  $s \in \Pi$ ,  $\omega s \omega^{-1} \notin W_d$ , we have  $\underline{m}_{\infty, s} = 0$ )

Now suppose  $\omega s \omega^{-1} \in W_d$ .

$$\begin{aligned} \underline{\Lambda}_{\infty, \omega s}^s &= n_{\infty} \oplus \underline{m}_{\infty, \omega s} \\ \underline{\Lambda}_{\infty, \omega}^s &:= T_{(F_{\infty}, F_{\infty})} \overline{\mathcal{O}}_s \\ &= T_{(Id, Id)} G_d/B_{\infty} \times P_{\infty, \omega s}/B_{\infty} \\ &= n_{\infty} \oplus \underline{m}_{\infty s, \omega} \end{aligned}$$

$$\widetilde{\Lambda}_{\infty, \omega s}^s = r_{\infty, \omega s} \oplus n_{\infty}^- \oplus \underline{m}_{\infty, \omega s}$$

$$\widetilde{\Lambda}_{\infty, \omega}^s = r_{\infty, \omega} \oplus n_{\infty}^- \oplus \underline{m}_{\infty s, \omega}$$

### §6.3. transition matrix, localization formula

Thm. (Localization formula) [Thm 10.2]

Suppose  $Y \subset X$  is  $T$ -equivariant,  $\alpha \in \mathcal{K}_o^T(X)$ ,  $X$  smooth.

$X^T = \{x_1, \dots, x_m\}$ ,  $i_k : \{x_k\} \hookrightarrow X$ , then

$$\alpha = \sum_{k=1}^m \varepsilon_k (i_k)_* (i_k)^*(\alpha) \quad \varepsilon_k = (T_{x_k} X)^{-1} \in \mathcal{R}(T)$$

$$\begin{aligned} \text{e.p. } [Y]^T &= \sum_{k=1}^m \varepsilon_k (i_k)_* ((i_k)^*[Y]^T \cdot 1_{R(T)}) \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^*[Y]^T) (i_k)_* 1_{R(T)} \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^*[Y]^T) [x_k]^T \\ [X]^T &= \sum_{k=1}^m \varepsilon_k [x_k]^T \end{aligned}$$

Suppose  $Y^T = \{x_1, \dots, x_n\}$ ,  $i_k : \{x_k\} \hookrightarrow Y$ , then

$$[Y]^T = \sum_{k=1}^n \beta_k [x_k]^T \quad \beta_k = \varepsilon_k \cdot (i_k)^*[Y]^T$$

When  $Y$  is sm at  $x_k$ ,

$$\begin{cases} \beta_k = (T_{x_k} Y)^{-1} \\ (i_k)^*[Y]^T = T_{x_k} X \cdot (T_{x_k} Y)^{-1} \end{cases}$$

Ex 1.  $X = \widetilde{\text{Rep}}_d(Q)$ ,  $Y = \overline{\mathcal{O}}_x$ ,  $T = T_d$

$$i_\infty : \{(p_0, F_\infty)\} \hookrightarrow \widetilde{\text{Rep}}_d(Q)$$

$$\widetilde{\text{Rep}}_d(Q)^{T_d} = \{(p_0, F_\infty) \mid \infty \in W_{ldl}\}$$

$$\begin{aligned} [\widetilde{\text{Rep}}_d(Q)]^{T_d} &= \sum_{\infty \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* (i_\infty)^* 1_{K_o^{T_d}(\widetilde{\text{Rep}}_d(Q))} \\ &= \sum_{\infty \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* 1_{R(T_d)} \\ &= \sum_{\infty \in W_{ldl}} \widetilde{\Delta}_\infty^{-1} \psi_\infty \end{aligned}$$

$$\overline{\mathcal{O}}_x^{T_d} = \{(p_0, F_\infty) \mid \infty \leq x\}$$

$$[\overline{\mathcal{O}}_x]^{T_d} = \sum_{\infty \leq x} \widetilde{\Delta}_\infty^{-1} \underbrace{(i_\infty)^* [\overline{\mathcal{O}}_x]^{T_d}}_{f_{\infty, x}} \psi_\infty$$

when  $\overline{\mathcal{O}}_x$  is sm at  $(p_0, F_\infty)$ ,  $f_{\infty, x} = \widetilde{\Delta}_\infty (T_{(p_0, F_\infty)} \overline{\mathcal{O}}_x)^{-1}$

Ex 2.  $X = \text{Rep}_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d$ ,  $Y = \mathcal{Z}_x$ ,  $T = T_d$

$$(\mathcal{Z}_s)^{T_d} = \overline{\mathcal{O}}_s^{T_d} \sqcup (\mathcal{Z}_s - \overline{\mathcal{O}}_s)^{T_d}$$

$$= \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{ldl}\} \sqcup \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{ldl}, \infty s \in W_d\}$$

$$\Rightarrow [\mathcal{Z}_s]^{T_d} = \sum_{\infty \in W_{ldl}} (\widetilde{\Delta}_{\infty, \infty s})^{-1} \psi_{\infty, \infty s} + \sum_{\substack{\infty \in W_{ldl} \text{ s.t.} \\ \infty s \in W_d}} (\widetilde{\Delta}_{\infty, \infty s})^{-1} \psi_{\infty, \infty s} \quad \text{in } \mathcal{K}_o^{T_d}(X)$$

$\Rightarrow$  in  $\mathcal{K}_o^{T_d}(\mathcal{Z}_d)$

In general,  $[\mathcal{Z}_x]^{T_d} = \sum_{\infty, \infty'} \beta_{\infty, \infty'}^\times \psi_{\infty, \infty \infty'}$

When  $\mathcal{Z}_x$  is sm at  $(p_0, F_\infty, F_{\infty \infty'})$ ,  $\beta_{\infty, \infty'}^\times = (\widetilde{\Delta}_{\infty, \infty'})^{-1}$ .

§ 6.4. generators. Define  $e_i$  and  $D_i$ .

In this subsection,  $W_{\text{id}} = W_d$ ,  $\widetilde{\text{Rep}_d(Q)} = F_d$ ,  $Z_d = F_d \times F_d$ . (Otherwise  $K_0^{G_d}(F_d) \cong \bigoplus_{\text{id}} K_0^{T_d}(\text{pt})$ )

$$K_0^{G_d}(\widetilde{\text{Rep}_d(Q)}) \cong K_0^{G_d}(F_d) \cong K_0^{T_d}(\text{pt})$$

$$\begin{array}{ccccc} \pi_T^G & \downarrow & R(T_d) [\widetilde{\text{Rep}_d(Q)}]^{G_d} & \cong & R(T_d) [F_d]^{G_d} \cong R(T_d) = \mathbb{Z}[x_1^{\pm 1}, \dots, x_{|W_d|}^{\pm 1}] \\ \downarrow & & f [\widetilde{\text{Rep}_d(Q)}]^{G_d} & & \\ K_0^{T_d}(\widetilde{\text{Rep}_d(Q)}) & \cong & \sum_{w \in W_d} f \widetilde{\Delta}_w^{-1} \psi_w & & \\ \downarrow & & & & \\ \bigoplus_{w \in W_d} R(T_d) \psi_w & & & & \\ & & \downarrow & & \\ & & \sum_{w \in W_d} f \widetilde{\Delta}_w^{-1} \psi_w & & \end{array}$$

Q: Does  $f [\widetilde{\text{Rep}_d(Q)}]^{T_d} = \sum_{w \in W_d} f \widetilde{\Delta}_w^{-1} \psi_w$ ?

This is needed  
but I can't get it.

Let  $e_i := x_i [\widetilde{\text{Rep}_d(Q)}]^{G_d} \in K_0^{G_d}(\widetilde{\text{Rep}_d(Q)})$ , then

$$\pi_T^G(e_i) = \sum_{w \in W_d} x_{w(i)} \widetilde{\Delta}_w^{-1} \psi_w$$

$$\in K_0^{T_d}(\widetilde{\text{Rep}_d(Q)})$$

$$K_0^{G_d}(\widetilde{\text{Rep}_d(Q)}) \cong \mathbb{Z}[e_1, \dots, e_{|W_d|}]$$

$K_0^{G_d}(Z_d)$  is a  $\mathbb{Z}[e_1, \dots, e_{|W_d|}]$ -module.

Reason: 
$$\begin{aligned} K_0^{G_d}(\widetilde{\text{Rep}_d(Q)}) &\cong K_0^{G_d}(Z_d) \hookrightarrow K_0^{G_d}(Z_d) \\ K_0^{G_d}(Z_d) \times K_0^{G_d}(Z_d) &\xrightarrow{\text{convolution}} K_0^{G_d}(Z_d) \\ K_0^{G_d}(Z_d) \times K_0^{G_d}(Z_d) &\xrightarrow{\text{convolution}} K_0^{G_d}(Z_d) \end{aligned}$$

We will mention about the convolution in the next section.

Denote

$$\begin{aligned} D_i &= [Z_{s_i}]^{G_d} \in K_0^{G_d}(Z_d) \\ \pi_T^G(D_i) &= \sum_{w \in W_d} (\widetilde{\Delta}_{w,w}^s)^{-1} \psi_{w,w} + \sum_{\substack{w \in W_d \text{ s.t.} \\ wsw^{-1} \in W_d}} (\widetilde{\Delta}_{w,w}^s)^{-1} \psi_{w,w} \end{aligned}$$

we will show that, in the case  $Z_d \cong F_d \times F_d$ ,

$$K_0^{G_d}(Z_d) = \langle e_1, e_2, \dots, e_{|W_d|}, D_1, \dots, D_{|W_d|-1} \rangle_{\mathbb{Z}\text{-alg}} \subseteq \text{End}_{\mathbb{C}\text{-v.s.}}(\mathbb{Z}[e_1, \dots, e_{|W_d|}])$$

Before that, the more interesting question is to compute

$$D_i \in \text{End}_{\mathbb{C}\text{-v.s.}}(\mathbb{Z}[e_1, \dots, e_{|W_d|}]).$$

## 7. convolution product

### §7.1. clean intersection formula

Thm. Suppose  $X$  sm  $G$ -equiv proj variety,  
 $Y_1, Y_2 \subset X$  are  $G$ -equiv subvariety,  
 $Y = Y_1 \cap Y_2$        $\pi_Y: Y \rightarrow \text{pt}$   
 $T = TX|_Y / (TY_1|_Y + TY_2|_Y)$

Assume that

$$TY_1|_Y \wedge TY_2|_Y = TY,$$

then

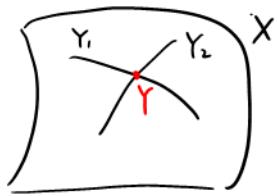
$$[Y_1]^G \otimes [Y_2]^G = \pi_{Y,*}(T) \cdot [Y]^G$$

Need a reference.

I believe that the first page of this document write the same thing:

[https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications\\_files/excessgw.pdf](https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf)

However, there is no proof.



$$\pi_{Y,*}: K^G_0(Y) \rightarrow K^G_0(\text{pt}) = R(G)$$

## §7.2. convolution for canonical basis.

Thm  $\psi_{\infty'', \infty''} * \psi_{\infty', \infty} = \delta_{\infty'', \infty'} \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty}$

 $\psi_{\infty'', \infty'} \diamond \psi_{\infty} = \delta_{\infty', \infty} \widetilde{\Delta}_{\infty} \psi_{\infty''}$

Proof. It reduce to the case

$$\begin{aligned} \psi_{\infty'', \infty''} * \psi_{\infty', \infty} &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty} \\ \psi_{\infty', \infty} \diamond \psi_{\infty} &= \widetilde{\Delta}_{\infty} \psi_{\infty'} \end{aligned}$$

$*: K_0^{Cd}(Z_d) \times K_0^{Cd}(Z_d) \rightarrow K_0^{Cd}(Z_d)$

$\{(p_0, F_{\infty''}), (p_0, F_{\infty'})\}$

$Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} \quad Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\}$

$\begin{matrix} Y_{12} \times \widetilde{\text{Rep}}_d(Q) \\ \hookrightarrow \\ \{y\} \subset \widetilde{\text{Rep}}_d \times Y_{23} \end{matrix} \quad \begin{matrix} \hookleftarrow \\ Y_{12} \times \widetilde{\text{Rep}}_d(Q) \\ \hookrightarrow \\ M_{123} \end{matrix}$

where

$y = ((p_0, F_{\infty''}), (p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123}$

$y_{13} = ((p_0, F_{\infty''}), (p_0, F_{\infty})) \in M_{13}$

$T = \frac{\widetilde{\Delta}_{\infty''} \oplus \widetilde{\Delta}_{\infty'} \oplus \widetilde{\Delta}_{\infty}}{\widetilde{\Delta}_{\infty''} \oplus \widetilde{\Delta}_{\infty'}} = \widetilde{\Delta}_{\infty}$

$$\begin{aligned} \psi_{\infty'', \infty''} * \psi_{\infty', \infty} &= [Y_{12}]^{Td} * [Y_{23}]^{Td} \\ &= \pi_{13,*}([Y_{12} \times \widetilde{\text{Rep}}_d(Q)]^{Td} \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^{Td}) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty'} \cdot [y]^{Td}) \\ &= \widetilde{\Delta}_{\infty'} \cdot [y_{13}]^{Td} \\ &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty} \end{aligned}$$

$$\begin{matrix} [Y_{12}] & & M_{123} \\ \downarrow & & \downarrow \\ Z_d \hookrightarrow M_{12} & \xrightarrow{Z_d \hookrightarrow M_{23}} & \text{Rep}_d(Q) = M_{23} \\ \widetilde{\text{Rep}}_d(Q) & \xrightarrow{\widetilde{\text{Rep}}_d(Q)} & \widetilde{\text{Rep}}_d(Q) \end{matrix} \quad \begin{matrix} pt \\ \downarrow \\ \widetilde{\text{Rep}}_d(Q) \end{matrix}$$

$\diamond: K_0^{Cd}(Z_d) \times K_0^{Cd}(\widetilde{\text{Rep}}_d(Q)) \rightarrow K_0^{Cd}(\widetilde{\text{Rep}}_d(Q))$

$$\begin{matrix} Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} & Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\} \\ \hookrightarrow Y_{12} \times pt & \hookrightarrow M_{123} \\ \{y\} \subset \widetilde{\text{Rep}}_d \times Y_{23} & \end{matrix}$$

where

$y = ((p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123}$

$y_{13} = (p_0, F_{\infty'}) \in M_{13}$

$T = \frac{\widetilde{\Delta}_{\infty'} \oplus \widetilde{\Delta}_{\infty} \oplus 0}{\widetilde{\Delta}_{\infty'} \oplus 0} = \widetilde{\Delta}_{\infty}$

$$\begin{aligned} \psi_{\infty', \infty} * \psi_{\infty} &= [Y_{12}]^{Td} * [Y_{23}]^{Td} \\ &= \pi_{13,*}([Y_{12} \times pt]^{Td} \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^{Td}) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty} \cdot [y]^{Td}) \\ &= \widetilde{\Delta}_{\infty} \cdot [y_{13}]^{Td} \\ &= \widetilde{\Delta}_{\infty} \psi_{\infty'} \end{aligned}$$

### § 7.3. expression of $D_k$ .

In this subsection,  $W_{Id} = W_d$ ,  $\widetilde{Rep}_d(Q) = F_d$ .  $Z_d = F_d \times F_d$ . (Otherwise  $K_0^G(F_d) \cong \bigoplus_{\underline{d}} K_0^{T_d}(pt)$ )

In the example,  $|d| = 3$ ,  $i = 1$

The convolution is compatible with forget map  $\pi_T^G$ .

$$\begin{array}{ccc} K_0^{G_d}(Z_d) \times K_0^{G_d}(\widetilde{Rep}_d(Q)) & \longrightarrow & K_0^{G_d}(\widetilde{Rep}_d(Q)) \\ \downarrow \pi_B^G & \downarrow \pi_B^G & \downarrow \pi_B^G \\ K_0^{T_d}(Z_d) \times K_0^{T_d}(\widetilde{Rep}_d(Q)) & \longrightarrow & K_0^{T_d}(\widetilde{Rep}_d(Q)) \end{array}$$

So we do our computation in  $K_0^{T_d}$ . (View  $K_0^{G_d}$  as subalg of  $K_0^{T_d}$ )

Recall that

$$D_i = \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s}^s)^{-1} \psi_{\infty, \omega s} + \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega}^s)^{-1} \psi_{\infty, \omega}$$

$\omega s \omega^{-1} \in W_d$  ← automatically satisfied

$$f = \sum_{\omega \in W_{Id}} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \quad \text{e.p. } e_i = \sum_{\omega \in W_{Id}} x_{\omega(i)} \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty}$$

Therefore,

$$\begin{aligned} D_i \diamond f &= \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s}^s)^{-1} \psi_{\infty, \omega s} \sum_{\omega' \in W_{Id}} (\omega s f) \widetilde{\Delta}_{\omega s}^{-1} \psi_{\omega s} + \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega}^s)^{-1} \psi_{\infty, \omega} \sum_{\omega' \in W_{Id}} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s}^s)^{-1} (\omega s f) \widetilde{\Delta}_{\infty s}^{-1} \widetilde{\Delta}_{\omega s} \psi_{\infty} + \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega}^s)^{-1} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \widetilde{\Delta}_{\omega} \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} \left[ (\widetilde{\Delta}_{\infty, \omega s}^s)^{-1} \omega s f + (\widetilde{\Delta}_{\infty, \omega}^s)^{-1} (\omega f) \right] \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} \omega \left[ \left( \frac{sf}{\widetilde{\Delta}_{Id,s}^s} + \frac{f}{\widetilde{\Delta}_{Id,Id}^s} \right) \cdot \widetilde{\Delta}_{Id} \right] \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \\ \therefore D_i f &= \left( \frac{sf}{\widetilde{\Delta}_{Id,s}^s} + \frac{f}{\widetilde{\Delta}_{Id,Id}^s} \right) \cdot \widetilde{\Delta}_{Id} \end{aligned}$$

$$\begin{array}{llll} \text{In our case, } \widetilde{\Delta}_{Id} = \Delta_{Id} = n_{Id}^- & = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} & = A \\ \widetilde{\Delta}_{Id,s}^s = \Delta_{Id,s}^s = n_{Id}^- \oplus m_{Id,s} & = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & = A + \frac{1}{B} \\ \widetilde{\Delta}_{Id,Id}^s = \Delta_{Id,Id}^s = n_{Id}^- \oplus m_{s,Id} & = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & = A + B \end{array}$$

where  $A = \prod_{j>k} \frac{e_j}{e_k} = \frac{e_2}{e_1} + \frac{e_3}{e_1} + \frac{e_3}{e_2}$   
 $B = \frac{e_{ii'}}{e_i} = \frac{e_3}{e_1}$

$$\therefore D_i f = \left( \frac{sf}{A+\frac{1}{B}} + \frac{f}{A+B} \right) A = \frac{sf}{1+\frac{1}{AB}} + \frac{f}{1+\frac{B}{A}} \quad f \in K_0^G(\widetilde{Rep}(Q))$$

Here,  $f \in K_0^{G_d}(\mathbb{Z}_{Id})$ ,  $g \in K_0^{G_d}(\widetilde{\text{Rep}}(\mathbb{Q}))$

$$\begin{aligned} D_i f g &= \frac{sf \cdot sg}{1 + \frac{f}{AB}} + \frac{sf \cdot g}{1 + \frac{B}{A}} - \frac{f \cdot g}{1 + \frac{B}{A}} + \frac{sf \cdot g}{1 + \frac{B}{A}} \\ &= sf \cdot D_i g + \frac{sf - f}{1 + \frac{B}{A}} g \\ \Rightarrow D_i f &= sf \cdot D_i + \frac{sf - f}{1 + \frac{B}{A}} \end{aligned}$$

This is, however, not the correct result. The desired one:

$$D_i f = sf \cdot D_i + \frac{sf - f}{1 - B}$$