Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

Category + Groth cover	space open sets	continuous map	Covering of open sets	Sh	cohomology
site	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I},\ \bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	full sub of Sch/X	—		H
Sch_{Zar}	$Ob(Sch)$	$Mor(Sch)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		$H_{ét}$
$Sch_{ét}$	$Ob(Sch)$	$Mor(Sch)$	ét + l.f.p		
Sch_{sm}	$Ob(Sch)$	$Mor(Sch)$	smooth + l.f.p		
Sch_{fppf}	$Ob(Sch)$	$Mor(Sch)$	f.flat + l.f.p		
Sch_{fpqc}	$Ob(Sch)$	$Mor(Sch)$	f.flat + f_i^{-1}(q.c) locally q.c		
X/k $W_n := W_n(k)$ $Cris(X/W_n)$	$\left\{(U,V,i,\delta) \,\middle\vert\, \begin{array}{l} U \subseteq X \text{ open} \\ \cdots \\ \delta: \text{PD-thickening of } U \end{array}\right\}$	$\left\{(\iota,f) \,\middle\vert\, \begin{array}{l} \iota: U \xrightarrow{\text{open}} U' \\ f: V \to V' \\ \text{compatible with PD} \end{array}\right\}$	$\left\{\begin{array}{c}(U_i,V_i,i_i,\delta_i) \\ \downarrow \\ (U,V,i,\delta)\end{array} \,\middle\vert\, \{U_i\} \text{ cover of } U\right\}$		$H^i_{cris}(X/W_n, -)$

(recommended)https://sites.math.washington.edu/~jarod/moduli.pdf
https://pbelmans.ncag.info/notes/etale-cohomology.pdf
http://homepage.sns.it/vistoli/descent.pdf
(crystalline site)http://page.mi.fu-berlin.de/castillejo/docs/crystalline_cohomology.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

it tells us why we don't have small site for most condition:
https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
Here you can find some informations about comparison between fppf and fpqc topologys:
https://mathoverflow.net/questions/361664/some-basic-questions-on-quotient-of-group-schemes

Thm. ① equiv. of categories

$$\mathrm{Sets}((\mathrm{Spec}\ K)_{\acute{e}t}) \longleftrightarrow \mathrm{Disc}\ G_K\text{-Set}$$

$$\mathrm{Ab}((\mathrm{Spec}\ K)_{\acute{e}t}) \longleftrightarrow \mathrm{Disc}\ \mathrm{Mod}_{G_K} \qquad (☆)$$

② (☆) preserve cohomology

$$H^1((\mathrm{Spec}(k))_{\acute{e}t}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

$G_K := \mathrm{Gal}(K^{sep}/K)$

$(\mathrm{Spec}\ k)_{\acute{e}t} \overset{\text{Site}}{\longleftrightarrow} \overset{\text{finite}}{G_K\text{-Set}}$

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{C})_{\acute{e}t}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\mathrm{Spec}\ \mathbb{C})$)

Ex. describe sheaf on $(\mathrm{Spec}\ \mathbb{R})_{\acute{e}t}$

$$\begin{tikzcd}
\mathrm{Spec}\ \mathbb{C} \arrow[dr, "\iota^*"'] & \mathrm{Spec}\ \mathbb{C} \arrow[l, "\sigma^*"'] \arrow[d, "\iota^*"] \\
& \mathrm{Spec}\ \mathbb{R}
\end{tikzcd}
\quad \overset{\mathrm{Spec}}{\rightsquigarrow} \quad
\begin{tikzcd}
\mathbb{C} \arrow[r, "\sigma"] & \mathbb{C} \\
\mathbb{R} \arrow[u, "\iota"] \arrow[ur, "\iota"']
\end{tikzcd}$$

σ = conjugation, ι = embedding

$$\Big\downarrow \mathcal{F}$$

$$\begin{tikzcd}
\mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \arrow[r, "\mathcal{F}(\sigma^*)"] & \mathcal{F}(\mathrm{Spec}\ \mathbb{C}) \\
\mathcal{F}(\mathrm{Spec}\ \mathbb{R}) \arrow[u, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"']
\end{tikzcd}
\quad \overset{\text{Abuse of notation}}{\Longrightarrow} \quad
\begin{tikzcd}
\mathcal{F}(\mathbb{C}) \arrow[r, "\sigma"] & \mathcal{F}(\mathbb{C}) \\
\mathcal{F}(\mathbb{R}) \arrow[u, "\iota"] \arrow[ur, "\iota"']
\end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\quad Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed.

in this case $0 \to \mathcal{F}(\operatorname{Spec}\mathbb{R}) \to \mathcal{F}(\operatorname{Spec}\mathbb{C}) \underset{\iota_2}{\overset{\iota_1}{\rightrightarrows}} \mathcal{F}(\operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C})$

$$\begin{array}{ccc}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}) \\
 & \iota_1: x \longmapsto x\otimes 1 \qquad x\otimes y \longmapsto (xy, x\bar{y}) & \\
 & \iota_2: x \longmapsto 1\otimes x & \\
\| & & \wr\| \\
 & & \mathcal{F}(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\operatorname{Spec}\mathbb{C}) \\
 & & \wr\| \\
\mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})
\end{array}$$

ι_2: $\operatorname{Spec}\mathbb{C} \xleftarrow{(Id,\sigma)} \operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\operatorname{Spec}\mathbb{C}) \xrightarrow{(\mathcal{F}(Id), \mathcal{F}(\sigma))} \mathcal{F}(\operatorname{Spec}\mathbb{C} \amalg \operatorname{Spec}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})$

$\overset{\text{Abuse}}{\underset{\text{of notation}}{\rightsquigarrow}}$ $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\operatorname{Spec} k, \mathcal{F}) = \mathcal{F}(\operatorname{Spec} k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)} \qquad \mathcal{F}_{k^{sep}} := \varinjlim_{L/k \text{ finite}} \mathcal{F}(\operatorname{Spec} L)$$

Ex. descibe the stalk & fiber at $p \in \operatorname{Spec} k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\operatorname{Spec} k, p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts