Eine Woche, ein Beispiel   5.7. Lie group

Temporary roadmap of Lie Group

Why is it so difficult to write examples for Lie group?

- Plenty of examples, even restricted to matrix Lie groups.
  - List of examples. https://en.wikipedia.org/wiki/Table_of_Lie_groups

[Some important subgroups are not written in the list, e.g. parabolic subgroups, Heisenberg group, etc.]

$G \to K$ maximal cpt connected subgroup

$GL(n,\mathbb{R})$ $GL(n,\mathbb{C})$ $GL(n,\mathbb{H})$

$SL(n,\mathbb{R})$ $SL(n,\mathbb{C})$ $SL(n,\mathbb{H})$

$Spin(n)$ $O(n,\mathbb{R})$ $O(n,\mathbb{C})$ $U(n)$ $Sp(2n,\mathbb{R})$, $Sp(2n,\mathbb{C})$

$SO(n,\mathbb{R}) \leftarrow SO(n,\mathbb{C})$ $SU(n)$ $Sp(n)$

$SO(m,n,\mathbb{R})$ $SU(m,n)$

$SO^*(2n,\mathbb{R})$

[For examples of non matrix Lie group ($\widetilde{SL(2,\mathbb{R})}$, $Mp(2n,\mathbb{R})$, $\begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} / \begin{pmatrix} 1 & 0 & \mathbb{Z} \\ & 1 & 0 \\ & & 1 \end{pmatrix}$), see https://math.stackexchange.com/questions/206117/is-there-a-non-matrix-lie-group

Every complex semisimple group and every compact lie group has a faithful rep, i.e. is matrix Lie group, see https://mathoverflow.net/questions/64195/when-is-a-finite-dimensional-real-or-complex-lie-group-not-a-matrix-group/64212#64212

On the contrary, every f.d. Lie alg can be viewed as a matrix Lie algebra by Ado's theorem, see http://www.math.ubc.ca/~reichst/Ado's-Theorem.pdf .]

- These groups are closely connected to each other.

  e.g. the Whitehead tower:

  $$\to \mathrm{Fivebrane}(n) \to \mathrm{String}(n) \to \mathrm{Spin}(n) \to SO(n) \to O(n)$$

- Plenty of special phenomenons on low-dimensional case:

  e.g. $GL(1,\mathbb{R}) \cong \mathbb{R}^\times$ is not connected

  e.g. $\mathfrak{so}(4,\mathbb{R})$ is not simple

  e.g. https://www.zhihu.com/question/47264301/answer/320472431

  e.g.
  $$\begin{array}{ccc} SU(2) & \cong & S^3 \\ \downarrow & & \downarrow \\ SO(3,\mathbb{R}) & \overset{\text{diff}}{\cong} & \mathbb{R}P^3 \end{array}$$
  not as variety

- Fruitful structures on examples
  - important subgroups

    e.g. maximal torus, unipotent subgroup

    Borel subgroup, parabolic subgroup,

    discrete subgroup,

    radical, unipotent radical, derived group $G = [G,G]R(G)$

  * centralizer & normalizer of subgroups

This article may give a classification of the cocompact subgroup: https://people.uleth.ca/~dave.morris/papers/cocompact.pdf

One page for defining these groups by pasting from

Anthony W. Knapp - Representation Theory of Semisimple Groups An Overview Based on Examples

Complex groups:

$$\mathrm{GL}(n, \mathbb{C}) = \{\text{nonsingular } n\text{-by-}n \text{ complex matrices}\}$$

$$\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \det g = 1\}$$

$$\mathrm{SO}(n, \mathbb{C}) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g g^{\mathrm{tr}} = 1\}$$

$$\mathrm{Sp}(n, \mathbb{C}) = \left\{g \in \mathrm{SL}(2n, \mathbb{C}) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\right\}.$$

Compact groups:

$$\mathrm{SO}(n,\mathbb{R}) = \mathrm{SO}(n) = \{g \in \mathrm{SL}(n, \mathbb{C}) \mid g^{\mathrm{tr}} g = 1 \text{ and } g \text{ has real entries}\}$$

$$\mathrm{U}(n) = \{g \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{g}^{\mathrm{tr}} g = 1\}$$

$$\mathrm{SU}(n) = \{g \in \mathrm{U}(n) \mid \det g = 1\}$$

$$\mathrm{Sp}(n) = \left\{g \in \mathrm{U}(2n) \mid g^{\mathrm{tr}} J g = J \text{ for } J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}\right\}.$$

Real noncompact groups: We list G, $\mathfrak{g}$, K, and $\mathfrak{k}$ for the noncomplex noncompact classical groups.

G	$\mathfrak{g}$	K	$\mathfrak{k}$
$\mathrm{SL}(n, \mathbb{R})$	$\mathfrak{sl}(n, \mathbb{R})$	$\mathrm{SO}(n)$	$\mathfrak{so}(n)$
$\mathrm{SL}(n, \mathbb{H})$	$\mathfrak{sl}(n, \mathbb{H})$	$\mathrm{Sp}(n)$	$\mathfrak{sp}(n)$
$\mathrm{SO}_0(m, n)$	$\mathfrak{so}(m, n)$	$\mathrm{SO}(m) \times \mathrm{SO}(n)$	$\mathfrak{so}(m) \oplus \mathfrak{so}(n)$
$\mathrm{SU}(m, n)$	$\mathfrak{su}(m, n)$	$\mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$	$\mathfrak{s}(\mathfrak{u}(m) \oplus \mathfrak{u}(n))$
$\mathrm{Sp}(m, n)$	$\mathfrak{sp}(m, n)$	$\mathrm{Sp}(m) \times \mathrm{Sp}(n)$	$\mathfrak{sp}(m) \oplus \mathfrak{sp}(n)$
$\mathrm{Sp}(n, \mathbb{R})$	$\mathfrak{sp}(n, \mathbb{R})$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$
$\mathrm{SO}^*(2n)$	$\mathfrak{so}^*(2n)$	$\mathrm{U}(n)$	$\mathfrak{u}(n)$.

$\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{SL}(n, \mathbb{H})$ refer to matrices of determinant one with real and quaternion entries, respectively. $\mathrm{SO}_0(m, n)$, $\mathrm{SU}(m, n)$, and $\mathrm{Sp}(m, n)$ are the linear isometry groups for the Hermitian form

$$|z_1|^2 + \cdots + |z_m|^2 - |z_{m+1}|^2 - \cdots - |z_{m+n}|^2$$

defined over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, respectively, with the subscript "0" referring to the identity component. The group $K = \mathrm{S}(\mathrm{U}(m) \times \mathrm{U}(n))$ for $\mathrm{SU}(m, n)$ is the subgroup of $\mathrm{U}(m) \times \mathrm{U}(n)$ of matrices of determinant one.

The group $\mathrm{Sp}(n, \mathbb{R})$ is the subgroup of real matrices in $\mathrm{Sp}(n, \mathbb{C})$ and can be conjugated by a unitary matrix so as to become

$$\{g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} J g = J\};$$

then $\mathrm{SO}^*(2n)$ is the analogous group

$$\left\{g \in \mathrm{SU}(n, n) \mid g^{\mathrm{tr}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right\}.$$

- decomposition: not easy to prove, but still of great significant.
  https://en.wikipedia.org/wiki/Lie_group_decomposition

  e.g. Cartan decomposition
  Jordan-Chevalley decomposition
  Bruhat decomposition
  ⋮

- group, mfld, variety, etc. Common questions.
- Hopf algebra structure of homology & cohomology group under mild requirements.

– A large amount of related theories.
- structure theory of Lie group & Lie algebras.
- "→": represent unfamiliar objects with simpler objects.

Lie algebra 代数 $\xleftarrow{\text{linearlization}}$ Lie group 几何+分析 $\xleftarrow{\text{parameterization}}$ flag variety 几何 $\xleftarrow{\text{moduli}}$ something about linear subspace 线性代数

Lie algebra	Lie group	flag variety
‖ Borel subalg Cartan subalg	‖ Borel subgp maximal torus para subgp	‖ base change Grassmannian
↓ classification	↓ reductive gp	↓
root system 组合 ‖ Weyl group	root datum + Bruhat-Tits theory	Schubert calculus 组合 ‖ cell structure

- representation theory
  - group / alg, f.d / inf.d
  - highest weight theory
  - Schul-Weyl duality
  - Borel-Weil-Bott theorem
  - geometric representation theory
  - Langlands program

  https://mathoverflow.net/questions/235917/roadmap-to-geometric-representation-theory-leading-to-langlands
- algebraic topology, Bott periodicity
- algebraic groups.
- ..., see https://en.wikipedia.org/wiki/List_of_Lie_groups_topics
  http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Lie.html

Basic knowledge which has been well-written in [Naive Lie Theory]:

- canonical maximal tori
- center

(1) $Z(SO(2m)) = \{\pm 1\}$.

(2) $Z(SO(2m+1)) = \{1\}$.

(3) $Z(U(n)) = \{\omega \mathbf{1} : |\omega| = 1\}$.

(4) $Z(SU(n)) = \{\omega \mathbf{1} : \omega^n = 1\}$.

(5) $Z(Sp(n)) = \{\pm 1\}$.

- (not) path connected $(n>1)$ unverified

$GL(n,\mathbb{R})$ $\quad$ $GL(n,\mathbb{C})$ $\quad$ $GL(n,\mathbb{H})$

$SL(n,\mathbb{R})$ $\quad$ $SL(n,\mathbb{C})$ $\quad$ $SL(n,\mathbb{H})$

$Spin(n)$ $\quad$ $O(n,\mathbb{R})$ $\quad$ $O(n,\mathbb{C})$ $\quad$ $U(n)$ $\quad$ $Sp(2n,\mathbb{R})$, $Sp(2n,\mathbb{C})$

$SO(n,\mathbb{R}) \leftarrow SO(n,\mathbb{C})$ $\quad$ $SU(n)$ $\quad$ $Sp(n)$

$SO(m,n,\mathbb{R})$ $\quad m,n\geq 1$ $\quad$ $SU(m,n)$

$SO^*(2n,\mathbb{R})$

- simple https://en.wikipedia.org/wiki/Simple_Lie_group#List almost simple?
- simply connected $\quad$ not

$GL(n,\mathbb{R})$ $\quad$ $GL(n,\mathbb{C})$ $\quad$ $GL(n,\mathbb{H})$

$SL(n,\mathbb{R})$ $\quad$ $SL(n,\mathbb{C})$ $\quad$ $SL(n,\mathbb{H})$

$Spin(n)$ $\quad$ $U(n)$ $\quad$ $Sp(2n,\mathbb{R})$, $Sp(2n,\mathbb{C})$

$SO(n,\mathbb{R}) \leftarrow SO(n,\mathbb{C})$ $\quad$ $SU(n)$ $\quad$ $Sp(n)$

$SU(m,n)$

$SO^*(2n,\mathbb{R})$

- compact $\quad$ not

$GL(n,\mathbb{R})$ $\quad$ $GL(n,\mathbb{C})$ $\quad$ $GL(n,\mathbb{H})$

$SL(n,\mathbb{R})$ $\quad$ $SL(n,\mathbb{C})$ $\quad$ $SL(n,\mathbb{H})$

$Spin(n)$ $\quad$ $O(n,\mathbb{R})$ $\quad$ $O(n,\mathbb{C})$ $\quad$ $U(n)$ $\quad$ $Sp(2n,\mathbb{R})$, $Sp(2n,\mathbb{C})$

$SO(n,\mathbb{R}) \leftarrow SO(n,\mathbb{C})$ $\quad$ $SU(n)$ $\quad$ $Sp(n)$

$SO(m,n,\mathbb{R})$ $\quad m,n\geq 1$ $\quad$ $SU(m,n)$

$SO^*(2n,\mathbb{R})$

- algebraic group

Now I believe that everything mentioned above is a real alg gp.

Rmk. PGL is an alg gp. but PSL is even not a scheme.

https://mathoverflow.net/questions/16145/what-is-the-difference-between-psl-2-and-pgl-2

In alg top: homotopy groups:

typo: $\pi_1(U(n)) = \mathbb{Z}$

see http://felix.physics.sunysb.edu/~abanov/Teaching/Spring2009/Notes/abanov-cpA1-upload.pdf

more about $\pi_i(SO(n,\mathbb{R}))$: https://ncatlab.org/nlab/show/orthogonal+group#HomotopyGroups

more about $\pi_1(G)$ (in Chinese): https://www.zhihu.com/question/443652195

more about $\pi_n(G)$: https://mathoverflow.net/questions/8957/homotopy-groups-of-lie-groups

$\pi_i(SO_0(m,n))$: https://math.stackexchange.com/questions/4126129/homotopy-group-of-op-q

my try: Cartan decomposition $\Rightarrow$ only consider cpt Lie group.

fibration + low-dimensional understanding. + Bott periodicity

$U(n) \cong SU(n) \times S^1$

e.g.
- $O(1) = S^0$, a two-point discrete space
- $SO(1) = \{1\}$
- $SO(2)$ is S^1
- $SO(3)$ is $\mathbf{RP}^3$ [3]
- $SO(4)$ is doubly covered by $SU(2) \times SU(2) = S^3 \times S^3$.

https://en.wikipedia.org/wiki/Orthogonal_group

Homotopy groups of orthogonal groups

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9
$SO(2)$	Z	0	0	0	0	0	0	0	0
$SO(3)$	$\boxed{Z_2}$	0	Z	Z_2	Z_2	Z_{12}	Z_2	Z_2	Z_3
$SO(4)$	Z_2	$\boxed{0}$	$(Z)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_{12})^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_2)^{\times 2}$	$(Z_3)^{\times 2}$
$SO(5)$	Z_2	0	$\boxed{Z}$	Z_2	Z_2	0	Z	0	0
$SO(6)$	Z_2	0	Z	$\boxed{0}$	Z	0	Z	Z_{24}	Z_2
$SO(n),\ n>6$	Z_2	0	Z	0	0	0			

my understanding

Activate Windows

fibration: [Hatcher Example 4.54, 4.55]

space of n-frames in $\mathbb{R}^k$

$$\begin{tikzcd} V_{n-m}(\mathbb{R}^{k-m}) \arrow[r] & V_n(\mathbb{R}^k) \arrow[d] \\ & V_m(\mathbb{R}^k) \end{tikzcd} \qquad \begin{tikzcd} O(n,\mathbb{R}) \arrow[r] & O(n+1,\mathbb{R}) \arrow[d] \\ & S^n \end{tikzcd} \qquad \begin{tikzcd} U(n) \arrow[r] & U(n+1) \arrow[d] \\ & S^{2n+1} \end{tikzcd} \qquad \begin{tikzcd} Sp(n) \arrow[r] & Sp(n+1) \arrow[d] \\ & S^{4n+3} \end{tikzcd}$$

$$\begin{tikzcd} SO(n,\mathbb{R}) \arrow[r] & SO(n+1,\mathbb{R}) \arrow[d] \\ & S^n \end{tikzcd}$$

$$\begin{tikzcd} SO(n,\mathbb{R}) \arrow[r] & O(n,\mathbb{R}) \arrow[d] \\ & \mathbb{Z}/2\mathbb{Z} \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}/2\mathbb{Z} \arrow[r] & Spin(n) \arrow[d] \\ & SO(n) \end{tikzcd}$$

reductive group
G

Borel subgroup
B

maximal torus
T

Structure theory of Lie group

Ref: http://www.math.columbia.edu/~makisumi/old/reductivegroups.pdf ← notations, results, examples, pictures.

Today:

$G = GL(2,\mathbb{C})$

center $Z(G) = \mathbb{C}^\times$ $\quad \pi_1(G) = \mathbb{Z}$

radical $\mathrm{rad}(G) := R(G) = \mathbb{C}^\times$

unipotent rad $\mathrm{rad}_u(G) := R_u(G) = \{1\}$

$\Rightarrow$ reductive, not semisimple

$\mathrm{rank}\, G = 2$

canonical maximal torus $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \begin{pmatrix} * & \\ & * \end{pmatrix} \cup \begin{pmatrix} & * \\ * & \end{pmatrix}$

$W(G) = S_2$ Weyl group $= \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

$X^* := X^*(T) = \left\langle \varepsilon_i : \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \mapsto t_i \right\rangle_{\mathbb{Z}} \circlearrowleft W(G) \quad \alpha \mapsto \alpha(g^{-1} - g)$

$X_* := X_*(T) = \left\langle \begin{array}{l} \varepsilon_1^* : t \mapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix} \\ \varepsilon_2^* : t \mapsto \begin{pmatrix} 1 & \\ & t \end{pmatrix} \end{array} \right\rangle_{\mathbb{Z}} \circlearrowleft W(G) \quad \lambda \mapsto g\, \lambda(-) g^{-1}$

$\Phi = \{\varepsilon_1 - \varepsilon_2, -(\varepsilon_1 - \varepsilon_2)\} =: \{\alpha_{12}, -\alpha_{12}\}$

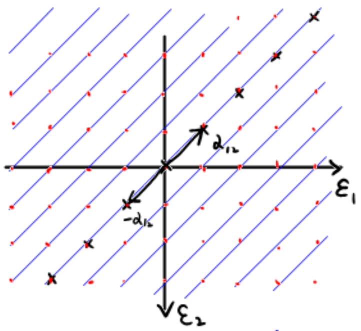
$\Phi^\vee = \{\alpha_{12}^\vee, -\alpha_{12}^\vee\} = \{\varepsilon_1^* - \varepsilon_2^*, -(\varepsilon_1^* - \varepsilon_2^*)\}$

$r_? =$ action of $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$: $X^* \longrightarrow X^*$, $\varepsilon_1 \longmapsto \varepsilon_2$, $\varepsilon_2 \longmapsto \varepsilon_1$

$\Rightarrow r_?$ is reflection and $r_? = r_{\alpha_{12}}$

$x - r_?(x) = x_1\varepsilon_1 + x_2\varepsilon_2 - x_1\varepsilon_2 - x_2\varepsilon_1 = (x_1 - x_2)\alpha_{12}$ $\quad x := x_1\varepsilon_1 + x_2\varepsilon_2$

$\langle -, \alpha_{12}^\vee \rangle : x_1\varepsilon_1 + x_2\varepsilon_2 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$



$\times \subseteq \cdot \subseteq /$

$Q \underset{\mathbb{Z}}{\subseteq} X^* \underset{S}{\subseteq} P$

$G = SL(2,\mathbb{C})$

$Z(G) = \{\pm 1\}$ $\quad \pi_1(G) = \{*\}$

$\mathrm{rad}(G) = \{I\} \Rightarrow$ semisimple

$\mathrm{rad}_u(G) = \{I\}$

$\mathrm{rank}\ G = 1$

canonical maximal torus $T = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}$

$C_G(T) = T$

$N_G(T) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \cup \begin{pmatrix} & t^{-1} \\ t & \end{pmatrix}$

$W(G) = S_2 = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \cup \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$

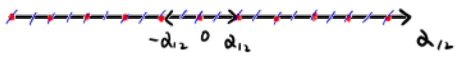
$X^* = \left\langle \varepsilon_1 : \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \mapsto t \right\rangle_{\mathbb{Z}} \circlearrowleft W(G) \quad \alpha \mapsto \alpha(g^{-1} - g)$ $\qquad \alpha_{12} = 2\varepsilon_1 !$

$X_* = \left\langle \varepsilon_1^* : t \mapsto \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right\rangle_{\mathbb{Z}} \circlearrowleft W(G) \quad \lambda \mapsto g\, \lambda(-) g^{-1}$

$\Phi = \{\alpha_{12}, -\alpha_{12}\} = \{2\varepsilon_1, -2\varepsilon_1\}$

$\Phi^\vee = \{\alpha_{12}^\vee, -\alpha_{12}^\vee\} = \{\varepsilon_1^*, -\varepsilon_1^*\}$

$$\left[\begin{array}{l} r_? = \text{action of } \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} : X^* \longrightarrow X^* \qquad \varepsilon_1 \mapsto -\varepsilon_1 \\ x - r_?(x) = x_0 \varepsilon_1 - x_0(-\varepsilon_1) = x_0 \alpha_{12} \quad r_? \text{ ref} + r_? = r_{\alpha_{12}} \quad x = x_0 \varepsilon_1 \\ \langle -, \alpha_{12}^\vee \rangle : x_0 \varepsilon_1 \mapsto x_0 \qquad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* \end{array}\right]$$



$\cdot \subseteq / = /$

$Q \underset{\mathbb{Z}/2\mathbb{Z}}{\subseteq} X^* = P$

$G = GL(3,\mathbb{C})$

$Z(G) = \mathbb{C}^\times \qquad \pi_1(G) = \mathbb{Z}$

$\left.\begin{array}{l} \mathrm{rad}(G) = \mathbb{C}^\times \\ \mathrm{rad}_u(G) := \{I\} \end{array}\right\} \Rightarrow$ reductive but not semi

$\mathrm{rank}\, G = 3$

canonical maximal torus $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) =$ monomial matrixs

$W(G) = S_3$

$X^* = \left\langle \varepsilon_i : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mapsto t_i \right\rangle_{\mathbb{Z}}$

$X_* = \left\langle \varepsilon_i^* : t \mapsto \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & t & \\ & & & \ddots \; 1 \end{pmatrix} \right\rangle_{\mathbb{Z}}$

$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\} \qquad \alpha_{ij} := \varepsilon_i - \varepsilon_j$

$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$ where $\alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$

$r_{\alpha_{12}} =$ action of $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} : X^* \longrightarrow X^*$

$\varepsilon_1 \longmapsto \varepsilon_2$

$\varepsilon_2 \longmapsto \varepsilon_1$

$\varepsilon_3 \longmapsto \varepsilon_3$

$\varepsilon_1 \mapsto \varepsilon_1\left(\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\right)$

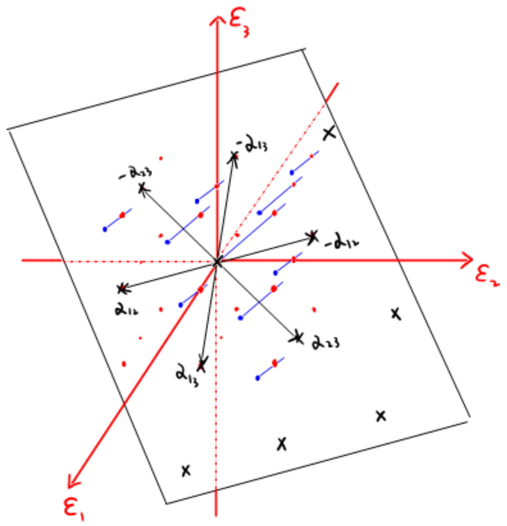
$= \varepsilon_1\begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix} = \varepsilon_2$

$x - r_{\alpha_{12}}(x) = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 - x_1\varepsilon_2 - x_2\varepsilon_1 - x_3\varepsilon_3 \qquad x = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$

$= (x_1 - x_2)\alpha_{12}$

$\langle -, \alpha_{12}^\vee \rangle : x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$

Similarly, $\alpha_{23}^\vee = \varepsilon_2^* - \varepsilon_3^*$



$\times \subseteq \cdot \subseteq /$

$Q \underset{\mathbb{Z}}{\subseteq} X^* \underset{S'}{\subseteq} P$

$G = SL(3,\mathbb{C})$

$Z(G) = \mathbb{Z}/3\mathbb{Z}$ $\quad \pi_1(G) = \{*\}$

$\mathrm{rad}(G) = \{I\} \Rightarrow$ semisimple

$\mathrm{rad}_u(G) = \{I\}$

$\mathrm{rank}\ G = 2$

canonical maximal torus $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$

$C_G(T) = T$

$N_G(T) =$ monomial matrixs

$W(G) = S_3$

$X^* = \langle \varepsilon_i : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mapsto t_i \rangle_{\mathbb{Z}} / \mathbb{Z}(\varepsilon_1+\varepsilon_2+\varepsilon_3)$

$X_* = \langle \varepsilon_1^* - \varepsilon_2^*, \varepsilon_2^* - \varepsilon_3^* \rangle_{\mathbb{Z}}$ $\qquad \varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$: dual basis of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$ $\qquad \alpha_{ij} := \varepsilon_i - \varepsilon_j$

$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$ where $\alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$

$r_{\alpha_{12}} :=$ action of $\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$: $X^* \longrightarrow X^*$

$\varepsilon_1 \longmapsto \varepsilon_2$

$\varepsilon_2 \longmapsto \varepsilon_1$

$\varepsilon_3 \longmapsto \varepsilon_3$

$\varepsilon_1 \mapsto \varepsilon_1\left(\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}^{-1} \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\right)$

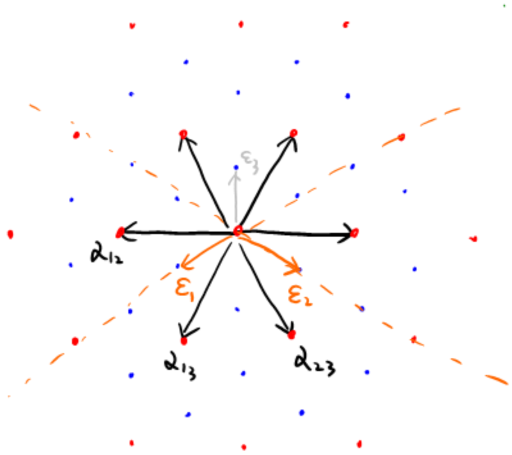
$= \varepsilon_1 \begin{pmatrix} t_2 & & \\ & t_1 & \\ & & t_3 \end{pmatrix} = \varepsilon_2$

$x - r_{\alpha_{12}}(x) = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 - x_1\varepsilon_2 - x_2\varepsilon_1 - x_3\varepsilon_3$ $\qquad x = x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3$

$= (x_1 - x_2)\alpha_{12}$

$\langle -, \alpha_{12}^\vee \rangle : x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 \mapsto x_1 - x_2 \quad \Rightarrow \alpha_{12}^\vee = \varepsilon_1^* - \varepsilon_2^*$

Similarly, $\alpha_{23}^\vee = \varepsilon_2^* - \varepsilon_3^*$



$\cdot \subseteq \cdot = \cdot$

$Q \underset{\mathbb{Z}/3\mathbb{Z}}{\subseteq} X^* = P$

$$\mathbb{C}^\times \cong \mathbb{C}^\times/\mu_3 \times \mathbb{C}^\times$$
$$t_1^3 \leftarrow\!\shortmid \begin{pmatrix} t_1 & & \\ & t_1 & \\ & & t_1^{-2}\end{pmatrix} \times \begin{pmatrix} 1 & & \\ & t_2 & \\ & & t_2^{-1}\end{pmatrix}$$

$G = PGL(3,\mathbb{C}) = SL(3,\mathbb{C})/\mu_3$

$Z(G) = \{I\}$ $\quad \pi_1(G) = \mathbb{Z}/3\mathbb{Z}$

$\mathrm{rad}(G) = \{I\} \Rightarrow$ semisimple

$\mathrm{rad}_u(G) = \{I\}$

$\mathrm{rank}\ G = 2$

canonical maximal torus $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}/\mu_3$

$C_G(T) = T$

$N_G(T) =$ monomial matrixs $/\mu_3$

$W(G) = S_3$

$X^* = \{n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 \mid n_i \in \mathbb{Z},\ 3 \mid \sum_i n_i\}/(\varepsilon_1+\varepsilon_2+\varepsilon_3)\mathbb{Z}$

$\quad = \langle \alpha_{12}, \alpha_{23} \rangle_{\mathbb{Z}}$

$\varepsilon_i : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3\end{pmatrix} \to t_i$ is no longer well-defined

$\alpha_{ij} := \varepsilon_i - \varepsilon_j$

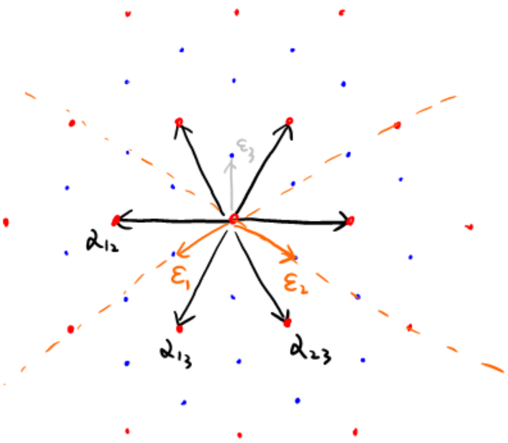
$X_* = \{n_1\varepsilon_1^* + n_2\varepsilon_2^* + n_3\varepsilon_3^* \mid n_i \in \frac{1}{3}\mathbb{Z},\ \sum_i n_i = 0\}$

$\quad = \langle \frac{1}{3}\varepsilon_1^* + \frac{1}{3}\varepsilon_2^* - \frac{2}{3}\varepsilon_3^*,\ \varepsilon_2^* - \varepsilon_3^* \rangle_{\mathbb{Z}}$

$\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$: dual basis of $\varepsilon_1, \varepsilon_2, \varepsilon_3$

$\Phi = \{\alpha_{12}, \alpha_{23}, \alpha_{13}, -\alpha_{12}, -\alpha_{23}, -\alpha_{13}\}$ $\quad \alpha_{ij} := \varepsilon_i - \varepsilon_j$

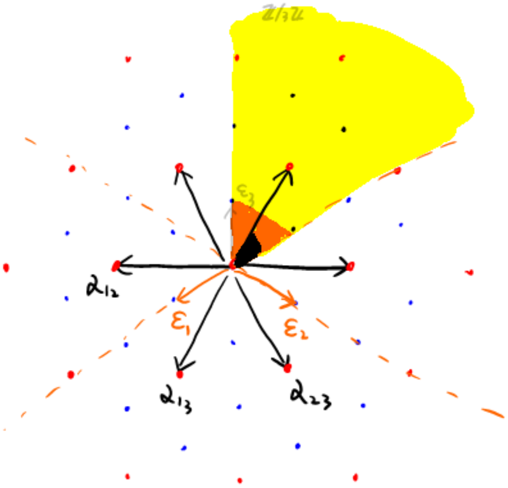
$\Phi^\vee = \{\alpha_{12}^\vee, \alpha_{23}^\vee, \alpha_{13}^\vee, -\alpha_{12}^\vee, -\alpha_{23}^\vee, -\alpha_{13}^\vee\}$ where $\alpha_{ij}^\vee = \varepsilon_i^* - \varepsilon_j^*$



$\cdot \ = \ \cdot \ \subseteq \ \cdot$

$Q = X^* \subseteq P$

$\mathbb{Z}/3\mathbb{Z}$



fundamental domain of

Weyl group

affine Weyl group

extended affine Weyl group

$\mathbb{Z}/3\mathbb{Z}$

$G = Sp(4,\mathbb{C}) = \{ A \in GL(4,\mathbb{C}) \mid A^T M A = M \}$ $\qquad M = \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$

$Z(G) = \{\pm 1\} \qquad \pi_1(G) = \{*\}$

$rad(G) = \{I\} \Rightarrow$ semisimple

$rad_u(G) = \{I\}$

$rank\ G = 2$

canonical maximal torus $T = \begin{pmatrix} t_1 & & & \\ & t_2 & & \\ & & t_2^{-1} & \\ & & & t_1^{-1} \end{pmatrix}$

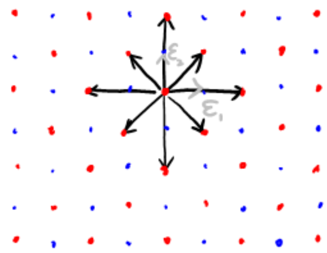
$C_G(T) = T$

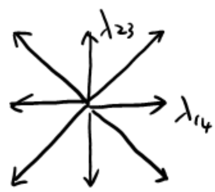
$N_G(T) =$ monomial matrixs

$W(G) = D_4 \subseteq S_4$

$$sp(4,\mathbb{C}) = \{ A \in M^{4\times 4}(\mathbb{C}) \mid A^T M + M A = 0 \}$$
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ & & & -a_{13} \\ & & -a_{22} & a_{12} \\ & -a_{31} & a_{21} & -a_{11} \end{pmatrix} \in M^{4\times 4}(\mathbb{C}) \right\}$$

$$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$$
$$X_* = \mathbb{Z}\lambda_{14} \oplus \mathbb{Z}\lambda_{23}$$
$$\Phi = \{\pm\alpha_{12}, \pm\alpha_{13}, \pm\alpha_{14}, \pm\alpha_{23}\} = \{\pm(\varepsilon_1 - \varepsilon_2), \pm(\varepsilon_1 + \varepsilon_2), \pm 2\varepsilon_1, \pm 2\varepsilon_2\}$$
$$\Phi^\vee = \{\pm\alpha_{12}^\vee, \pm\alpha_{13}^\vee, \pm\alpha_{14}^\vee, \pm\alpha_{23}^\vee\} = \{\pm(\lambda_{14} - \lambda_{23}), \pm(\lambda_{14} + \lambda_{23}), \pm\lambda_{14}, \pm\lambda_{23}\}$$





$$\underset{}{\bullet} \subseteq \bullet = \bullet$$
$$\Phi \subseteq Q \underset{\mathbb{Z}/2\mathbb{Z}}{\subseteq} X^* = P$$

$\Phi^\vee$

$G = SO(5,\mathbb{C}) \cong \{ A \in SL(5,\mathbb{C}) \mid A^T M A = M \}$ $\qquad M = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}$

$Z(G) = \{Id\} \qquad \pi_1(G) = \mathbb{Z}/2\mathbb{Z}$

$rad(G) = \{I\} \Rightarrow$ semisimple

$rad_u(G) = \{I\}$

$rank\ G = 2$

canonical maximal torus $\quad T = \begin{pmatrix} t_1 & & & & \\ & t_2 & & & \\ & & 1 & & \\ & & & t_2^{-1} & \\ & & & & t_1^{-1} \end{pmatrix}$

$C_G(T) = T$

$N_G(T) =$ monomial matrixs

$W(G) = D_4$

$$so(5,\mathbb{C}) = \{ A \in M^{5\times 5}(\mathbb{C}) \mid A^T M + M A = 0 \}$$

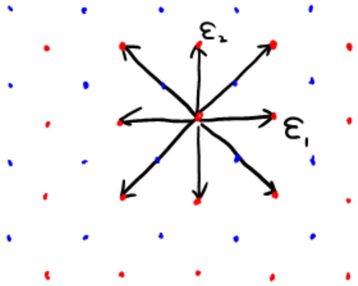
$$= \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & 0 \\ & & & 0 & a_{14} \\ & & 0 & a_{23} & -a_{13} \\ & 0 & a_{32} & -a_{22} & a_{12} \\ 0 & a_{41} & -a_{31} & a_{21} & -a_{11} \end{pmatrix} \in M^{5\times 5}(\mathbb{C}) \right\}$$

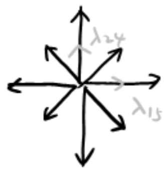
$X^* = \mathbb{Z}\varepsilon_i / (\varepsilon_1+\varepsilon_5, \varepsilon_2+\varepsilon_4, \varepsilon_3) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$

$X_* = \mathbb{Z}\lambda_{15} \oplus \mathbb{Z}\lambda_{24}$

$\Phi = \{ \pm\alpha_{12}, \pm\alpha_{13}, \pm\alpha_{14}, \pm\alpha_{23} \} = \{ \pm(\varepsilon_1-\varepsilon_2), \pm\varepsilon_1, \pm(\varepsilon_1+\varepsilon_2), \pm\varepsilon_2 \}$

$\Phi^\vee = \{ \pm\alpha_{12}^\vee, \pm\alpha_{13}^\vee, \pm\alpha_{14}^\vee, \pm\alpha_{23}^\vee \} = \{ \pm(\lambda_{15}-\lambda_{24}), \pm 2\lambda_{15}, \pm(\lambda_{15}+\lambda_{24}), \pm 2\lambda_{24} \}$

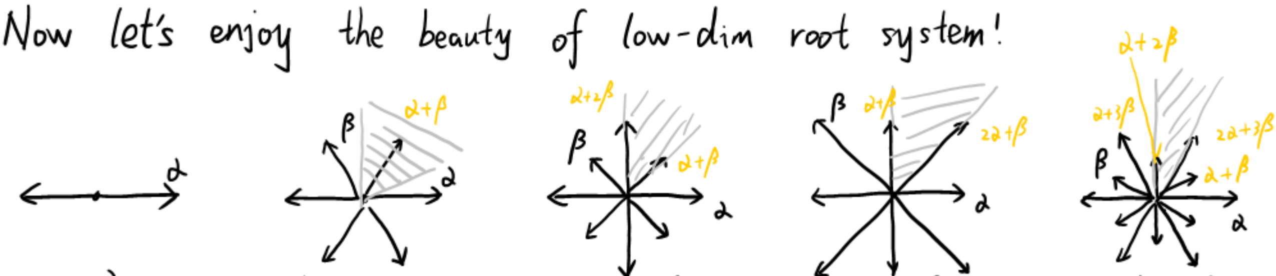




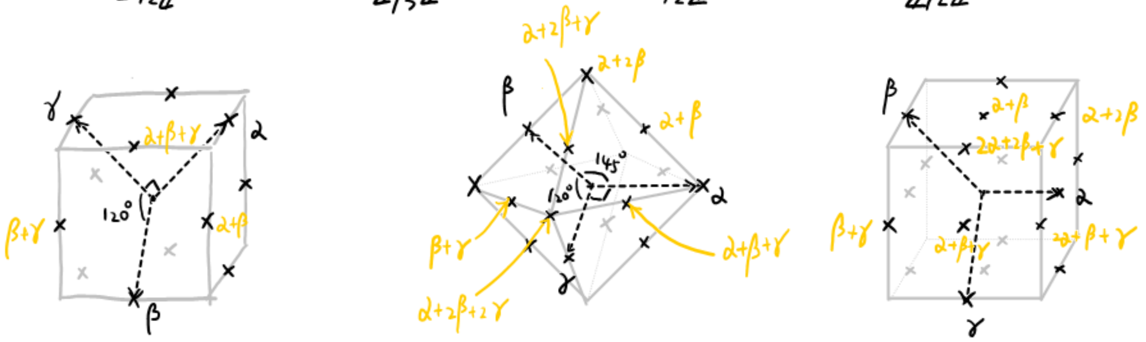
$$\Phi \subseteq \underset{\bullet}{Q} \overset{=}{=} \underset{\bullet}{X^*} \underset{\mathbb{Z}/2\mathbb{Z}}{\overset{\subseteq}{\subseteq}} \underset{\bullet}{P}$$

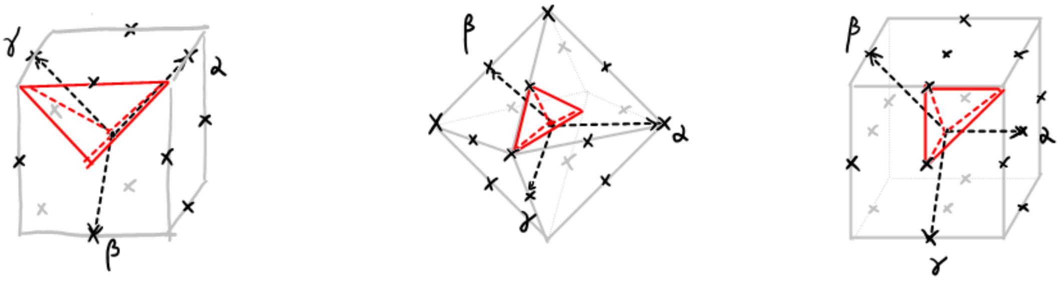
$\Phi^\vee$

Now let's enjoy the beauty of low-dim root system!



	$\circ\alpha$	$\alpha\ \circ\!-\!\circ\ \beta$	$\alpha\ \circ\!\Rightarrow\!\circ\ \beta$	$\alpha\ \circ\!\Leftarrow\!\circ\ \beta$	$\alpha\ \circ\!\Rrightarrow\!\circ\ \beta$
	$SL(2,\mathbb{C}), A_1$	$SL(3,\mathbb{C}), A_2$	$Sp(4,\mathbb{C}), C_2$	$SO(5,\mathbb{C}), B_2$	G_2
Weyl group	S_2, 2	S_3, 6	D_4, 8	D_4, 8	D_6, 12
dimension	3	8	10	10	14
$\pi_1(\Phi)$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	





	$\alpha\ \circ\!-\!\circ\ \beta\ -\!\circ\ \gamma$	$\alpha\ \circ\!\Rightarrow\!\circ\ \beta\ -\!\circ\ \gamma$	$\alpha\ \circ\!\Leftarrow\!\circ\ \beta\ -\!\circ\ \gamma$
	$SL(4,\mathbb{C}), A_3$	$Sp(6,\mathbb{C}), C_3$	$SO(7,\mathbb{C}), B_3$
Weyl group	S_4, 24	$G(2)$, 48	$G(2)$, 48
dimension	15	21	21
$\pi_1(\Phi)$	$\mathbb{Z}/4\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$

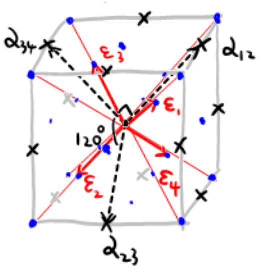
Φ	$\lvert\Phi\rvert$	$\lvert\Phi^<\rvert$	I	D	$\lvert W\rvert$	Coxeter number h
A_n ($n \geq 1$)	$n(n+1)$			$n+1$	$(n+1)!$	$n+1$
B_n ($n \geq 2$)	$2n^2$	$2n$	2	2	$2^n\, n!$	$2n$
C_n ($n \geq 3$)	$2n^2$	$2n(n-1)$	2^{n-1}	2	$2^n\, n!$	$2n$
D_n ($n \geq 4$)	$2n(n-1)$			4	$2^{n-1}\, n!$	$2(n-1)$
E_6	72			3	51840	12
E_7	126			2	2903040	18
E_8	240			1	696729600	30
F_4	48	24	4	1	1152	12
G_2	12	6	3	1	12	6

The Weyl groups for classical groups:
$W(U(n)) = S(n)$.

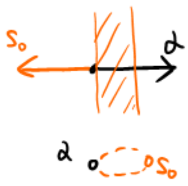
- The Weyl group of $SU(n)$ is still $S(n)$
- The Weyl group of $SO(2l+1)$ is $G(l)$, the group of permutations φ of the set $\{-l, \cdots, -1, 1, \cdots, l\}$ with $\varphi(-k) = -\varphi(k)$ for all $1 \leq k \leq l$.
- The Weyl group of $SO(2l)$ is the subgroup $SG(l)$ of $G(l)$ that consists of even permutations.
- The Weyl group of $Sp(n)$ is still $G(n)$.

http://staff.ustc.edu.cn/~wangzuoq/Courses/13F-Lie/Notes/Lec%2027.pdf

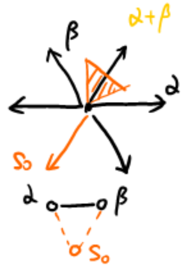
https://en.wikipedia.org/wiki/Root_system
A better table can be found here: https://www.jgibson.id.au/lievis/tables_fin_aff/



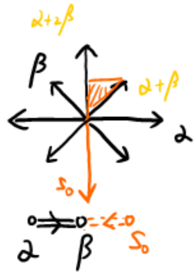
Now let's construct some associated extended Dynkin diagrams!



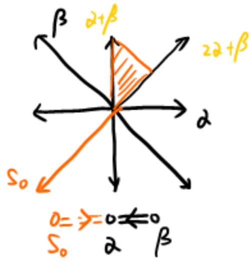
$\tilde{A}_1$



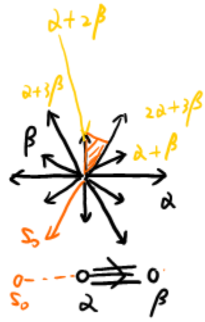
$\tilde{A}_2$



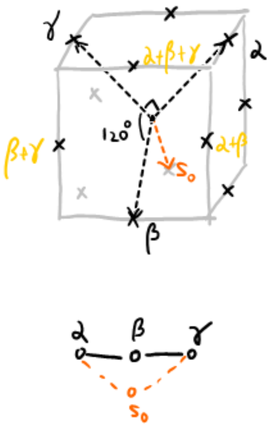
$\tilde{C}_2$



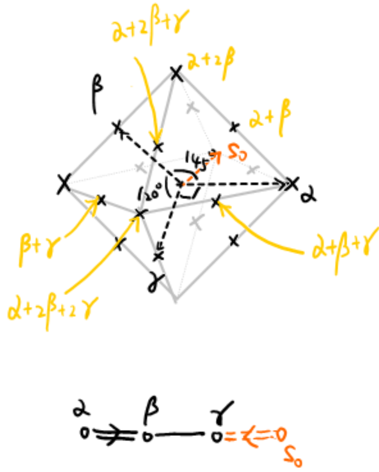
$\tilde{C}_2$



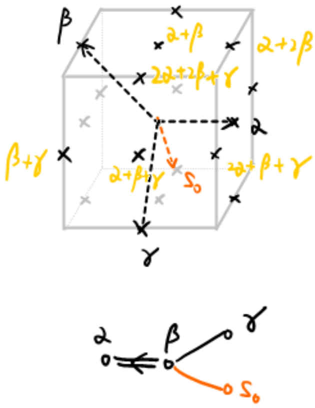
$\tilde{G}_2$



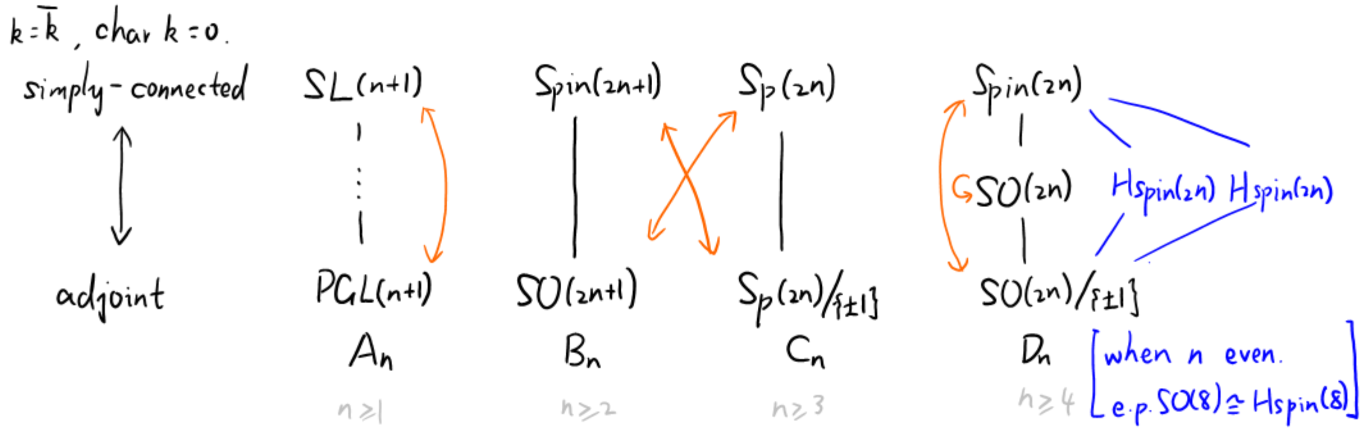
$\tilde{A}_3$



$\tilde{C}_3$



$\tilde{B}_3$



The following table comes from [linear algebraic groups and finite groups of Lie type, p72].
Errata for this book: https://www.mathematik.uni-kl.de/~malle/download/MT_errata.pdf

Table 9.2 *Isogeny types of simple algebraic groups*

Φ	$\Lambda(\Phi)$	G_{sc}	G_{ad}	in between
$A_{n-1},\ n\geq 2$	Z_n	SL_n	PGL_n	$\mathrm{SL}_n/Z_d\ (d\vert n)$
$B_n,\ n\geq 2$	Z_2	Spin_{2n+1}	SO_{2n+1}	–
$C_n,\ n\geq 2$	Z_2	Sp_{2n}	PCSp_{2n}	–
$D_n,\ n\geq 3$ odd	Z_4	Spin_{2n}	$\mathrm{PCO}^{\circ}_{2n}$	SO_{2n}
$D_n,\ n\geq 4$ even	$Z_2\times Z_2$	Spin_{2n}	$\mathrm{PCO}^{\circ}_{2n}$	$\mathrm{SO}_{2n},\mathrm{HSpin}_{2n}$
G_2	1	G_2		–
F_4	1	F_4		–
E_6	Z_3	$(E_6)_{\mathrm{sc}}$	$(E_6)_{\mathrm{ad}}$	–
E_7	Z_2	$(E_7)_{\mathrm{sc}}$	$(E_7)_{\mathrm{ad}}$	–
E_8	1	E_8		–