

Eine Woche, ein Beispiel

9.4 Hecke algebra

This document is not finished. I need some time to digest and restate them.

I saw Hecke algebras in many different fields(modular form/p-adic group representation/K-group/...), and I want to see the difference among those Hecke algebras.

main reference:

[Bump][<http://sporadic.stanford.edu/bump/math263/hecke.pdf>]

[XiongHecke][<https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf>]

All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

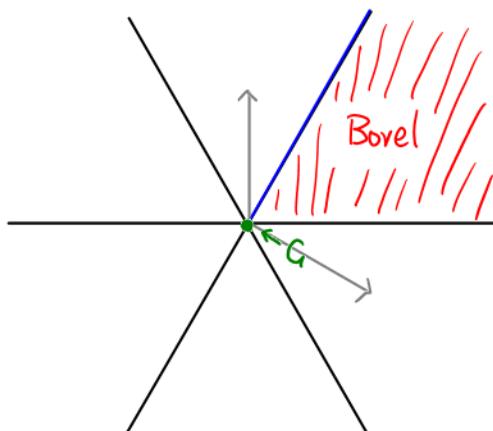
Task. For each double coset decomposition, we want to do.

1. decomposition ($\Gamma \backslash G / \Gamma$ is finite & definition of Hecke alg)
2. \mathbb{Z} -mod structure, notation
3. alg structure
4. conclusion

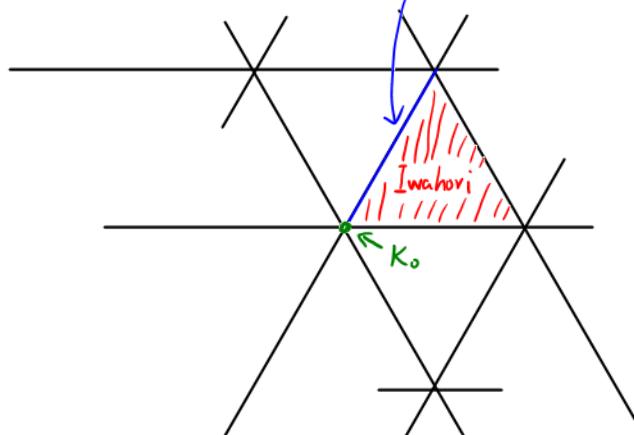
<https://math.stackexchange.com/questions/4480285/what-is-the-kak-cartan-decomposition-in-textsld-mathbb-r-in-terms-of>

	Bruhat	Iwahori	Cartan
F finite	$G = \bigsqcup_{w \in W} B_w B$	affine Bruhat	Smith normal form
F local	$G = \bigsqcup_{w \in W_{\text{ext}}} B_w B$	$G = \bigsqcup_{w \in W_{\text{ext}}} I_w I$	$G = \bigsqcup_{\alpha \in T^-} K_\alpha \alpha K_\alpha$
F global	$G = \bigsqcup_{w \in W} B_w B$		$GL_2(\mathbb{Q}) = \bigsqcup_{\alpha \in T} \Gamma \alpha \Gamma$
adèle?			

parabolic $\rightarrow G$



parahoric $\rightarrow K_0$



$$B = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \cap \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$P = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}$$

$$I = \begin{pmatrix} 0 & 0 & 0 \\ p & 0 & 0 \\ p & p & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & p & p \\ p & 0 & 0 \\ p & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} 0 & 0 & p^{-1} \\ 0 & 0 & p \\ 0 & p & 0 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p & p & 0 \end{pmatrix}$$

<https://mathoverflow.net/questions/4547/definitions-of-hecke-alg>

<https://mathoverflow.net/questions/14683/can-the-quantum-torus-be-realized-as-a-hall-algebra>

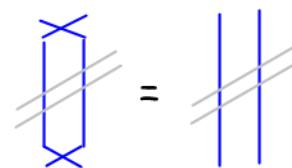
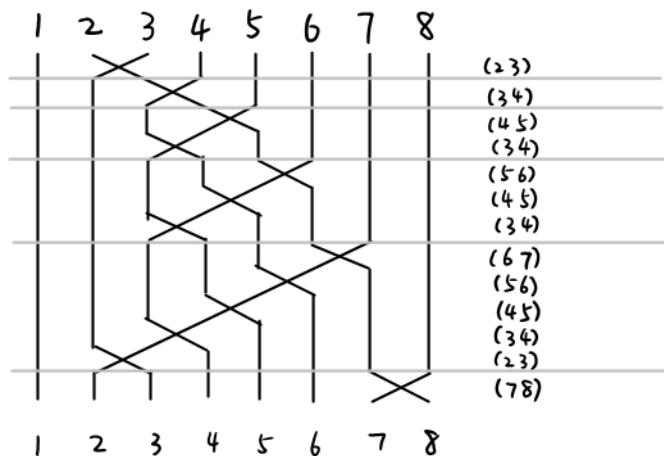
S_n and Tits system

A brief preparation for computations in Bruhat decomposition. $s_i = (i \ i+1)$, $1 \leq i \leq n-1$

$$\text{E.g. } n=8, w_0 = (287)(46) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 8 & 3 & 6 & 5 & 4 & 2 & 7 \end{pmatrix} \in S_8.$$

Ex. Compute $l(w_0)$, $l(s_i w_0)$ and $l(w_0 s_i)$.

Solution.



$$w_0 = (78)(23)(34)(45)(56)(67)(34)(45)(56)(34)(45)(34)(23)$$

$l(w_0) = 13$ = "inversion number"

$$l(s_1 w_0) = 14 \quad l(w_0 s_1) = 14$$

$$l(s_2 w_0) = 12 \quad l(w_0 s_2) = 12$$

$$l(s_3 w_0) = 14 \quad l(w_0 s_3) = 14$$

$$l(s_4 w_0) = 12 \quad l(w_0 s_4) = 12$$

$$l(s_5 w_0) = 12 \quad l(w_0 s_5) = 12$$

$$l(s_6 w_0) = 12 \quad l(w_0 s_6) = 14$$

$$l(s_7 w_0) = 14 \quad l(w_0 s_7) = 12$$

Ex. Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & \cdots & 0 \\ 0 & \ddots & * \\ 0 & \cdots & * \end{pmatrix} \leq B$,
 $w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

Shows that

$$Bs_iB \cdot Bw_0B = \begin{cases} Bs_iw_0B & l(s_iw_0) = l(w_0) + 1 \\ Bs_iw_0B \cup Bw_0B & l(s_iw_0) = l(w_0) - 1 \end{cases}$$

Solution

$$\begin{bmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix} \quad w_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iBw_0$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad s_iw_0B$$

The following computation will be also computed later on.

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad Bw_0 \cap w_0B$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad w_0Bw_0^{-1}$$

$$\begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix} \quad B \cap w_0Bw_0^{-1}$$

finite Bruhat decomposition

Let $G = GL_n(\mathbb{F}_q)$, $B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \leq G$, $T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \leq B$,
 $w_0, s_i \in N(T)$ a lift from $w_0, s_i \in S_n = N(T)/T$.
(usually take the permutation matrix)

1. decomposition $G = \bigsqcup_{w \in W} B w B$

$$\text{Ex. } (B w B)^{-1} = B w^{-1} B$$

Ex. Compute $|B w B / B|$ ↗ B w B may not be a group!

Hint: Consider the map

$$\phi: B \longrightarrow B w B / B$$

$$b \longmapsto b w B$$

$$\phi(b_1) = \phi(b_2) \Leftrightarrow b_1 w B = b_2 w B$$

$$\Leftrightarrow w^{-1} b_2^{-1} b_1 w \in B$$

$$\Leftrightarrow b_2^{-1} b_1 \in w B w^{-1}$$

$$\therefore |B w B / B| = |B| / |w B w^{-1} \cap B| = q^{\ell(w)}$$

We take Haar measure μ on G s.t. $\mu(B) = 1$, $\mu(pt) = \frac{1}{|B|}$.

Recall that $\mathcal{H}(G, B) = \{f: G \rightarrow \mathbb{Z} \mid f(b_1 g b_2) = f(g) \quad \forall b_1, b_2 \in B, g \in G\}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1} g) d\mu(x)$$

$$= \frac{1}{|B|} \sum_{x \in G} f_1(x) f_2(x^{-1} g)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(G, B) = \bigoplus_{w \in W} \mathbb{Z} \cdot \mathbf{1}_{B w B} \stackrel{\oplus n!}{\sim} \mathbb{Z}$$

Denote $T_w := \mathbf{1}_{B w B}$, $T_{s_i} := T_{s_i}$ ($T_{Id} = \mathbf{1}_B$ is the unit of $\mathcal{H}(G, B)$)

then $\{T_w\}_{w \in W}$ is a "basis" of $\mathcal{H}(G, B)$.

3. alg structure.

$$T_u * T_v = \sum_{w \in W} (T_u * T_v)(w) T_w$$

$$\begin{aligned} (T_u * T_v)(w) &= \frac{1}{|B|} \sum_{y, z \in w} T_u(y) T_v(z) \\ &= \frac{1}{|B|} \left| \{(y, z) \in B u B \times B v B \mid yz = w\} \right| \underbrace{\text{if } w \notin B u B v B}_0 \\ &= \frac{1}{|B|} |B u B \cap v B^{-1} B| \end{aligned}$$

$$B_{S_i}B \cdot B_{W_0}B = \begin{cases} B_{S_i}w_0B & l(S_iw) = l(w) + 1 \\ B_{S_i}w_0B \cup B_{W_0}B & l(S_iw) = l(w) - 1 \end{cases}$$

$$\Rightarrow T_i * T_w = \begin{cases} \mathbb{Z} \cdot T_{S_i w} & l(S_i w) = l(w) + 1 \\ \mathbb{Z} \cdot T_{S_i w} + \mathbb{Z} \cdot T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Computation of coefficient.

$$|BwB| = |BwB/B| \times |B| = q^{l(w)} \cdot |B|$$

when $l(S_iw) = l(w) + 1$,

$$(T_i * T_w)(S_iw) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz = S_iw \right\}$$

$$= \frac{1}{|B| |B_{S_i}wB|} \left\{ (y, z) \in B_{S_i}B \times B_{W_0}B \mid yz \in B_{S_i}wB \right\}$$

$$= \frac{|B_{S_i}B| |BwB|}{|B| \cdot |B_{S_i}wB|} = \frac{q^{l(S_i)} q^{l(w)}}{q^{l(S_iw)}} = 1$$

$$(T_i * T_i)(Id) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = Id \right\}$$

$$= \frac{1}{|B|} |B_{S_i}B| = q$$

$$(T_i * T_i)(S_i) = \frac{1}{|B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz = S_i \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B_{S_i}B \right\}$$

$$= \frac{1}{|B| |B_{S_i}B|} \left(|B_{S_i}B \times B_{S_i}B| - \left| \left\{ (y, z) \in B_{S_i}B \times B_{S_i}B \mid yz \in B \right\} \right| \right)$$

$$= \frac{1}{|B| |B_{S_i}B|} \left(|B_{S_i}B| |B_{S_i}B| - |B| \cdot |B_{S_i}B| \right)$$

$$= q - 1$$

when $l(S_iw) = l(w) - 1$, we get $l(S_i \cdot S_iw) = l(S_iw) + 1$,

$$T_i * T_w = T_i * T_i * T_{S_i w}$$

$$= (qT_{Id} + (q-1)T_i) * T_{S_i w}$$

$$= qT_{S_i w} + (q-1)T_w$$

$$\Rightarrow T_i * T_w = \begin{cases} T_{S_i w} & l(S_i w) = l(w) + 1 \\ qT_{S_i w} + (q-1)T_w & l(S_i w) = l(w) - 1 \end{cases}$$

Ex. Verify that

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

4. Conclusion.

$$H(G, B) = \mathbb{Z}\langle T_1, \dots, T_{n-1} \rangle_{alg} \text{ with relations } (H(G, B) \subseteq H_q(W))$$

$$T_i * T_i = q + (q-1)T_i$$

$$T_i * T_{i+1} * T_i = T_{i+1} * T_i * T_{i+1}$$

$$T_i * T_j = T_j * T_i \quad \text{for } |i-j| \geq 2$$

Q: How to show that there are no further relations?

A: By comparing the dimensions.

$$\text{E.g. For } n=2, \quad \mathcal{H}(G, B) \cong \mathbb{Z}[T_1] / (T_1^2 - (q-1)T_1 - q)$$

$$\cong \mathbb{Z}[T_1] / (T_1 - q)(T_1 + 1)$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1$$

$$\text{For } n=3, \quad \mathcal{H}(G, B) \cong \mathbb{Z}\langle T_1, T_2 \rangle / ((T_1 - q)(T_1 + 1), (T_2 - q)(T_2 + 1), T_1 T_2 T_1 = T_2 T_1 T_2)$$

$$\stackrel{\mathbb{Z}-\text{mod}}{=} \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_1 T_2 \oplus \mathbb{Z} T_2 T_1 \oplus \mathbb{Z} T_1 T_2 T_1$$

$$= \mathbb{Z} \oplus \mathbb{Z} T_1 \oplus \mathbb{Z} T_2 \oplus \mathbb{Z} T_{(12)} \oplus \mathbb{Z} T_{(13)} \oplus \mathbb{Z} T_{(123)}$$

Variation: $H(S_{m+n}, S_m \times S_n)$

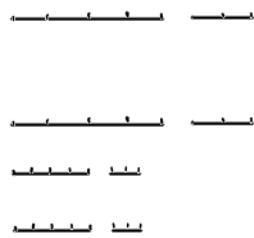
$m, n \in \mathbb{Z}_{>0}$

E.g. $m+n=8, m=5, n=3$; $m+n=8, m=7, n=1$.

For the convenience of the writing, we denote

$$S_{m,n} := S_m \times S_n$$

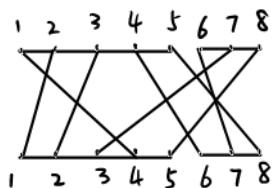
and suppose $m \geq n$.



1. decomposition

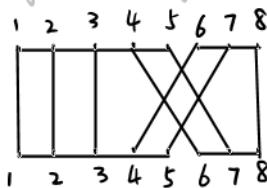
E.g.

random element



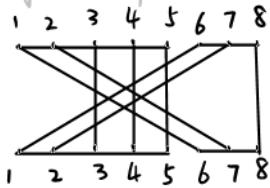
canonical form

for draw by hand

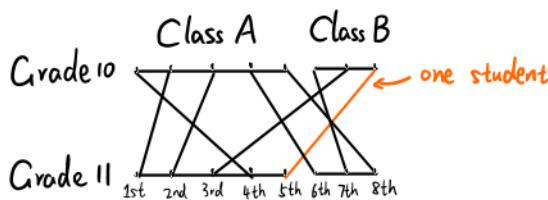


canonical form

for computation



$$\left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 2 & 6 & 8 & 7 & 3 & 5 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 4 & 5 & 8 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 7 & 3 & 4 & 5 & 1 & 2 & 8 \end{smallmatrix} \right) \text{ in } S_8 \setminus S_8 / S_{5,3}$$



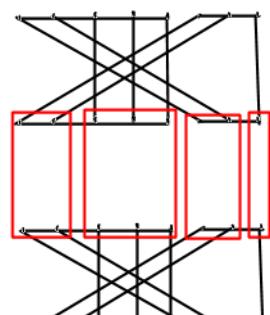
A vivid explanation: students are distributed into different classes on the basis of their order.
The teacher only care about the stability of the class, i.e. how many people move from Class A to Class B as time went by.

$$\therefore S_{5,3} \setminus S_8 / S_{5,3} = \left\{ \begin{array}{c} \text{[W}_0\text{]} = [\text{Id}] \\ \text{[W}_1\text{]} \\ \text{[W}_2\text{]} \\ \text{[W}_3\text{]} \end{array} \right\} \quad \begin{array}{l} \text{e.g. } w_1 = (16) \in S_8 \\ [w] \in S_{5,3} \setminus S_8 / S_{5,3} \end{array}$$

$$\text{In general, } S_{m+n} = \bigsqcup_{i=0}^n S_{m,n} w_i S_{m,n}$$

Ex. Compute $|S_{m,n} w_i S_{m,n} / S_{m,n}|$

$$\begin{aligned} A. |S_{m,n} w_i S_{m,n} / S_{m,n}| &= |S_{m,n} / w_i S_{m,n} w_i^{-1} \cap S_{m,n}| \\ &= |S_{m,n} / S_{i,m-i} \times S_{i,n-i}| \\ &= \frac{m! n!}{i! (m-i)! i! (n-i)!} \\ &= \binom{m}{i} \binom{n}{i} \end{aligned}$$



$$w_i S_{m,n} w_i^{-1} \cap S_{m,n}$$

E.g. (canonical form)

$$\begin{array}{c}
 \begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
 \end{array} =
 \begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
 \end{array} \\
 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 5 & 7 & 8 & 6 \end{smallmatrix} \right) \quad \text{in } S_{5,3}/S_{2,3} \times S_{2,1}
 \end{array}$$

$$\begin{array}{c}
 \begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
 \end{array} =
 \begin{array}{ccccccccc}
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
 \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown & \diagup & \diagdown \\
 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8
 \end{array} \\
 \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 4 & 5 & 8 \end{smallmatrix} \right) \quad \text{in } S_{2,3} \times S_{2,1} \setminus S_{5,3}
 \end{array}$$

Recall that $\mathcal{H}(G, H) = \{f: G \rightarrow \mathbb{Z} \mid f(h_1gh_2) = f(g) \quad \forall h_1, h_2 \in H, g \in G\}$ where
 $(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$

$$\begin{aligned}
 &= \frac{1}{|H|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)
 \end{aligned}$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(S_{m+n}, S_{m,n}) = \bigoplus_{i=0}^n \mathbb{Z} \cdot \mathbb{1}_{S_{m,n} w_i S_{m,n}} = \mathbb{Z}^{\oplus(n+1)}$$

denote $T_i := \mathbb{1}_{S_{m,n} w_i S_{m,n}}$ $(T_0 = \mathbb{1}_{S_{m,n}} \text{ is the unit of } \mathcal{H}(S_{m+n}, S_{m,n}))$

3. alg structure

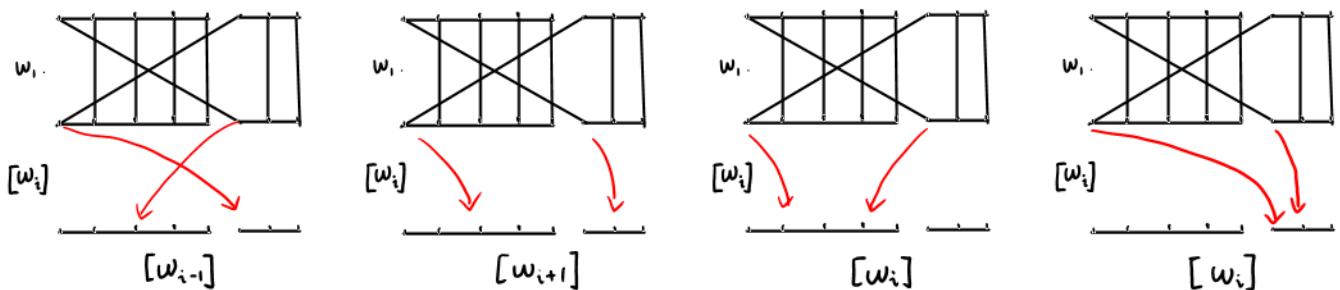
$$\begin{aligned}
 \text{E.g. } \mathcal{H}(S_8, S_7) &= \mathbb{Z} \oplus \mathbb{Z} T = \mathbb{Z}[T]/(T-7)(T+1) \\
 g_{w_1, w_2}^{\text{Id}} &= \frac{1}{|S_7|} \{ (y, z) \in S_7 w_1 S_7 \times S_7 w_2 S_7 \mid yz = 1 \} \\
 &= \frac{|S_7 w_2 S_7|}{|S_7|} \\
 &= \frac{8! - 7!}{7!} = 7
 \end{aligned}$$

$$\begin{aligned}
g_{w_i, w_i}^{w_i} &= \frac{1}{|S_7|} \# \{(y, z) \in S_7 w_i S_7 \times S_7 w_i S_7 \mid yz = w_i\} \\
&= \frac{1}{|S_7| |S_7 w_i S_7|} \# \{(y, z) \in S_7 w_i S_7 \times S_7 w_i S_7 \mid yz \in S_7 w_i S_7\} \\
&= \frac{1}{|S_7| |S_7 w_i S_7|} \# \{(y, z) \in S_7 w_i S_7 \times S_7 w_i S_7 \mid yz \notin S_7\} \\
&= \frac{|S_7 w_i S_7| (|S_7 w_i S_7| - |S_7 w_i S_7|)}{|S_7| |S_7 w_i S_7|} \\
&= (7)(1) - 1 \\
&= 6
\end{aligned}$$

$$\text{In general, } H(S_{m+1}, S_m) = \mathbb{Z}[T]/(T-m)(T+1).$$

Direct argument shows that

$$T_i * T_i \in \left\{ \begin{array}{ll} \mathbb{Z} \cdot T_{i-1} + \mathbb{Z} \cdot T_i + \mathbb{Z} \cdot T_{i+1} & 0 < i < n \\ \mathbb{Z} \cdot T_0 + \mathbb{Z} \cdot T_i & i=0 \\ \mathbb{Z} \cdot T_{n-1} + \mathbb{Z} \cdot T_n & i=n \end{array} \right.$$



Computation of the coefficient.

$$\begin{aligned}
g_{w_i, w_i}^{w_{i-1}} &= \frac{1}{|S_{m,n}|} \# \{(y, z) \in S_{m,n} w_i S_{m,n} \times S_{m,n} w_i S_{m,n} \mid yz = w_{i-1}\} \quad (0 < i \leq n) \\
&= \frac{1}{|S_{m,n}| |S_{m,n} w_i S_{m,n}|} \# \{(y, z) \in S_{m,n} w_i S_{m,n} \times S_{m,n} w_i S_{m,n} \mid yz \in S_{m,n} w_{i-1} S_{m,n}\} \\
&= \frac{|S_{m,n} w_i S_{m,n}|}{|S_{m,n}| |S_{m,n} w_i S_{m,n}|} \# \{z \in S_{m,n} w_i S_{m,n} \mid w_i z \in S_{m,n} w_{i-1} S_{m,n}\} \\
&= \frac{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,i)} w_i S_{m,n}|}{|S_{m,n}| / |S_{m,n} w_{i-1} S_{m,n}|} \\
g_{w_i, w_i}^{w_{i+1}} &= \frac{|S_{m,n}| / |S_{m,n} w_{i+1} S_{m,n}|}{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,i+1)} w_i S_{m,n}|} \quad (0 \leq i < n) \\
g_{w_i, w_i}^{w_i} &= \frac{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,0)} w_i S_{m,n}|}{|S_{m,n}| / |S_{m,n} w_i S_{m,n}|}
\end{aligned}$$

where

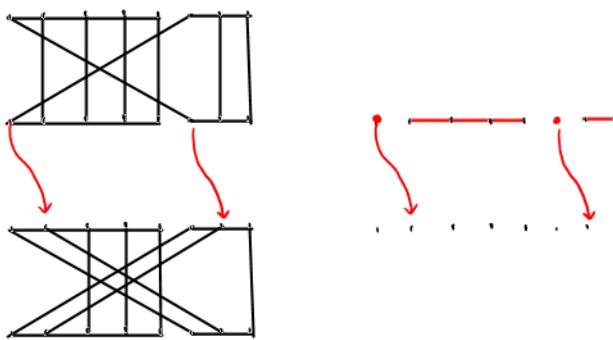
$$\begin{aligned} S_{m,n}^{(i,-)} &:= \left\{ g \in S_{m,n} \mid w_i g w_i^{-1} \in S_{m,n} \cap S_{m,n}^{(i,-)} \right\} \\ S_{m,n}^{(i,+)} &:= \left\{ g \in S_{m,n} \mid w_i g w_i^{-1} \in S_{m,n} \cap S_{m,n}^{(i,+)} \right\} \\ S_{m,n}^{(i,0)} &:= \left\{ g \in S_{m,n} \mid w_i g w_i^{-1} \in S_{m,n} \cap S_{m,n}^{(i,0)} \right\} \end{aligned}$$

Recall that

$$S_{m,n} w_i S_{m,n} / S_{m,n} \subset S_{m,n} w_i S_{m,n} / S_{m,n} \text{ as left coset}$$

By the following picture, for $g \in S_{m,n}$,

$$g \in S_{m,n} \Leftrightarrow \begin{cases} g(1) \in \{1, 2, \dots, i\} \\ g(m+1) \in \{m+1, m+2, \dots, m+i\} \end{cases}$$



$$\therefore |S_{m,n}^{(i,-)} w_i S_{m,n} / S_{m,n}| = i^2$$

$$\text{Similarly, } |S_{m,n}^{(i,+)} w_i S_{m,n} / S_{m,n}| = (m-i)(n-i)$$

$$\begin{aligned} |S_{m,n}^{(i,0)} w_i S_{m,n} / S_{m,n}| &= i(n-i) + (m-i)i \\ &= i(m+n-2i) \end{aligned}$$

$$\therefore g_{w_i w_i}^{w_{i-1}} = \frac{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,-)} w_i S_{m,n}|}{|S_{m,n}| / |S_{m,n} w_{i-1} S_{m,n}|} \quad (0 < i \leq n)$$

$$= \frac{\binom{m}{i} \binom{n}{i} i^2}{\binom{m}{i-1} \binom{n}{i-1}} \quad \binom{m}{n} = \frac{m-i+1}{i} \binom{m}{i-1}$$

$$= (m-i+1)(n-i+1)$$

$$g_{w_i w_i}^{w_{i+1}} = \frac{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,+)} w_i S_{m,n}|}{|S_{m,n}| / |S_{m,n} w_{i+1} S_{m,n}|} \quad (0 \leq i < n)$$

$$= \frac{\binom{m}{i} \binom{n}{i} (m-i)(n-i)}{\binom{m}{i+1} \binom{n}{i+1}}$$

$$= (i+1)^2$$

$$g_{w_i w_i}^{w_i} = \frac{|S_{m,n} w_i S_{m,n}| / |S_{m,n}^{(i,0)} w_i S_{m,n}|}{|S_{m,n}| / |S_{m,n} w_i S_{m,n}|}$$

$$= |S_{m,n}^{(i,0)} w_i S_{m,n} / S_{m,n}|$$

$$= i(m+n-2i)$$

Therefore,

$$T_i * T_1 = \begin{cases} (m-i+1)(n-i+1) T_{i-1} + i(m+n-2i) T_i + (i+1)^n T_{i+1}, & 0 < i < n \\ T_1, & i=0 \\ (m-n+1) T_{n-1} + n(m-n) T_n, & i=n \end{cases}$$

4. Conclusion

By [Hecke, Prop 6], $S_{m,n}$ is a Gelfand subgp of S_{m+n} .
thus $\mathcal{H}(S_{m+n}, S_{m,n})$ is commutative.

Gelfand involution: $\sigma \mapsto \sigma^\tau$ $(S_{m+n} \hookrightarrow GL_{m+n}(\mathbb{K}))$

Possible extension: compute $\mathcal{H}(S_{m+n+l}, S_m \times S_n \times S_l)$.

The rest of the section is devoted to compute $F_{m,n} \in \mathbb{Z}[T]$ s.t
 $\mathcal{H}(S_{m+n}, S_{m,n}) \cong \mathbb{Z}[T]/(F_{m,n})$

global Cartan decomposition
1. decomposition

Thm (Elementary divisor thm) R : PID (In naive proof R should be ED)

$$M_{2 \times 2}(R) = \coprod_{(b) \subseteq (a)} GL_2(R) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(R)$$

$$\text{Cor } M_{2 \times 2}(\mathbb{Z}) = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 \leq a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det \neq 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} GL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} GL_2(\mathbb{Z})$$

$$M_{2 \times 2}(\mathbb{Z})_{\det > 0} = \coprod_{\substack{a, b \in \mathbb{Z} \\ 0 < a \leq b}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) = \coprod_{\substack{a, b \in \mathbb{Q}_{>0}^\times \\ v_p(a) \leq v_p(b) \quad \forall p}} SL_2(\mathbb{Z}) \begin{pmatrix} a & \\ & b \end{pmatrix} SL_2(\mathbb{Z})$$

$$GL_2^+(\mathbb{Q}) := GL_2(\mathbb{Q})_{\det > 0}$$

Denote $\Gamma = SL_2(\mathbb{Z})$,

$$\Gamma^- = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \in GL_2^+(\mathbb{Q}) \mid \begin{array}{l} a, b > 0 \\ v_p(a) \leq v_p(b) \quad \forall p \text{ prime} \end{array} \right\} \stackrel{\text{Grp}}{\cong} \mathbb{Q}_{>0}^\times \times (\mathbb{Z}_{>0}, \times)$$

then

$$GL_2^+(\mathbb{Q}) = \coprod_{\alpha \in \Gamma^-} \Gamma \alpha \Gamma$$

Ex. Verify that $\Gamma \alpha \Gamma / \Gamma$ is finite, and compute the order. $\alpha = \begin{pmatrix} \alpha_1 & \\ & \alpha_2 \end{pmatrix} \in \Gamma^-$

Hint. See [WWL, 引理 5.1.4].

$$\# \Gamma \alpha \Gamma / \Gamma = \# \Gamma / \Gamma \cap \alpha \Gamma \alpha^{-1} = \# \Gamma / \Gamma_0 \left(\frac{\alpha_1}{\alpha_2} \right) = \# \text{Irr} \left(\frac{\alpha_1}{\alpha_2} \right) = \frac{\alpha_2}{\alpha_1} \prod_{p \mid \frac{\alpha_1}{\alpha_2}} \left(1 + \frac{1}{p} \right)$$

$$\left[\alpha \begin{pmatrix} a & \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} a & \frac{a_1}{a_2} b \\ \frac{c}{a_2} c & d \end{pmatrix} \right] \Rightarrow \Gamma \cap \alpha \Gamma \alpha^{-1} = \left(\frac{\mathbb{Z}}{\alpha_1 \mathbb{Z}}, \frac{\mathbb{Z}}{\alpha_2 \mathbb{Z}} \right)_{\det=1} = \Gamma_0 \left(\frac{\alpha_2}{\alpha_1} \right)$$

$$\text{e.g. } \# \Gamma \begin{pmatrix} \alpha_1 & \\ 0 & \alpha_2 \end{pmatrix} \Gamma / \Gamma = 1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma / \Gamma = p+1, \quad \# \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^e \end{pmatrix} \Gamma / \Gamma = p^e + p^{e-1}$$

The desired measure can not be realized here, i.e.,

a Haar measure μ on $GL_2^+(\mathbb{Q})$ s.t. $\mu(\Gamma) = 1$.

Reason: measure satisfies countable additivity, and Γ is a countable set.

Q: How to remedy the problem?

short A: replace countable by finite. (measure \rightsquigarrow semimeasure)

To e.g.: There is no way to define a Haar measure μ on \mathbb{Q} s.t. $\mu(\mathbb{Z}) = 1$.

However, if we only require finite additivity, we can do it.

Def (Semimeasure on \mathbb{Q})

For any periodic set $X \subseteq \mathbb{Q}$ (i.e., $\exists m \in \mathbb{Q}_{>0}$ s.t. $m + X = X$)
we set

$$\text{Rmk. 1. } \mu(X) = \frac{1}{m} |X/m\mathbb{Z}| = \frac{1}{m} |X \cap [0, m]|$$

$$\mathbb{Z} \supset \frac{X}{m\mathbb{Z}} \quad |X/m\mathbb{Z}|, |m\mathbb{Z}/m\mathbb{Z}| < +\infty$$

" $m\mathbb{Z}$ are all commensurable gps of \mathbb{Z} "

2. $X = \bigsqcup_{\alpha \in \Delta} \alpha + m\mathbb{Z}$ for some $\Delta \subseteq \mathbb{Q}/m\mathbb{Z}$

" X is a commensurable set of \mathbb{Z} (when $\mu(X) < +\infty$)"

Long A: Def. (Semimeasure on $GL_2^+(\mathbb{Q})$)

For any gp $H \leq GL_2^+(\mathbb{Q})$ which is commensurable with Γ

(i.e., $\#H/\mathbb{H} \cap \Gamma, \#\Gamma/\mathbb{H} \cap \Gamma$ are finite), set

$$\mu(H) = \frac{|H/\mathbb{H} \cap \Gamma|}{|\Gamma/\mathbb{H} \cap \Gamma|} \stackrel{\text{if } H \leq \Gamma}{=} \frac{1}{|\Gamma/H|} \in \mathbb{Q}_{>0}$$

Similarly we can specify μ to any commensurable set $X \subseteq GL_2^+(\mathbb{Q})$.

$$\left(\begin{array}{l} \text{i.e., } X = \bigsqcup_{\alpha \in \Delta} \alpha H \text{ for some } H, H' \leq GL_2^+(\mathbb{Q}) \text{ commensurable with } \Gamma, \\ X = \bigsqcup_{\alpha \in \Delta'} H' \alpha' \quad \Delta \subseteq GL_2^+(\mathbb{Q})/H, \Delta' \subseteq H' \backslash GL_2^+(\mathbb{Q}) \\ \Delta, \Delta' \text{ finite} \end{array} \right)$$

Rmk: In the most of references the terminology (semi)measure
is avoid by the double coset calculus.

If you don't like semimeasure, just view it as intuition and
take the second line as a def of the convolution.

Def. (Hecke alg $\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)$)

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) := \left\{ f: GL_2^+(\mathbb{Q}) \rightarrow \mathbb{Z} \mid \begin{array}{l} f(\gamma_1 \alpha \gamma_2) = f(\alpha) \quad \forall \gamma_1, \gamma_2 \in \Gamma, \alpha \in GL_2^+(\mathbb{Q}) \\ \#(\text{supp } f)/\Gamma < +\infty \end{array} \right\}$$

$$(f_1 * f_2)(g) := \int_{GL_2^+(\mathbb{Q})} f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} f_1(x) f_2(x^{-1}g) = \sum_{y \in \Gamma \backslash GL_2^+(\mathbb{Q})} f_1(gy^{-1}) f_2(y)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma) = \bigoplus_{\alpha \in \Gamma} \mathbb{Z} \cdot \mathbf{1}_{\Gamma \alpha \Gamma}$$

denote $T_\alpha := \mathbf{1}_{\Gamma \alpha \Gamma}$

$$\begin{aligned} \lambda \in \mathbb{Q}^\times & \quad R_\lambda := T_{(\lambda)} = \mathbf{1}_{\Gamma(\lambda)} \Gamma = \mathbf{1}_{\lambda \Gamma} \quad (R_1 = \mathbf{1}_\Gamma \text{ is the unit of } \mathcal{H}(GL_2^+(\mathbb{Q}), \Gamma)) \\ p \text{ prime, } e \geq 1 & \quad T_{p^e} := T_{(p^e)} = \mathbf{1}_{\Gamma(p^e)} \Gamma \quad T_p := T_{(p)} = \mathbf{1}_{\Gamma(p)} \Gamma \end{aligned}$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in \Gamma} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha \beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in GL_2^+(\mathbb{Q})/\Gamma} T_\alpha(x) T_\beta(x^{-1} \gamma) \\ &= \# \left\{ x \in GL_2^+(\mathbb{Q})/\Gamma \mid \begin{array}{l} x \in \Gamma \alpha \Gamma \\ x^{-1} \gamma \in \Gamma \beta \Gamma \end{array} \right\} \\ &= |\Gamma \alpha \Gamma \cap \gamma \Gamma \beta^{-1} \Gamma / \Gamma| \end{aligned}$$

$$\text{e.p. } \mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1} g) = f(g \lambda^{-1}) = (f * R_\lambda)(g)$$

$$R_\lambda * R_\mu = R_{\lambda \mu}$$

$$\text{E.g. } g_{\alpha \beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad \text{where } |\alpha|_+ = \det \alpha$$

The formula above is still not feasible for effective calculation.
We will derived the easiest way to compute $g_{\alpha \beta}^\gamma$ in the next page.

$$\text{Suppose } \Gamma_\alpha \Gamma / \Gamma = \{x_1 \Gamma, \dots, x_i \Gamma, \dots\}$$

$$\Gamma_\beta \Gamma / \Gamma = \{y_1 \Gamma, \dots, y_j \Gamma, \dots\}$$

then

$$\begin{aligned} g_{\alpha\beta} &= \sum_{x \in \Gamma_\alpha \cap \Gamma_\beta} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \sum_i T_\beta(x_i^{-1}\gamma) \\ &= \sum_i \mathbf{1}_{x_i^{-1}\gamma \in \Gamma_\beta \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i^{-1}\gamma \in y_j \Gamma} \\ &= \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_i \sum_j \mathbf{1}_{\Gamma_\alpha y_j \Gamma = \gamma \Gamma} \\ &= \frac{1}{|\Gamma_\alpha \Gamma / \Gamma|} \sum_{y \in \Gamma_\beta \Gamma / \Gamma} \mathbf{1}_{\Gamma_\alpha y \Gamma = \gamma \Gamma} \\ &= \frac{|\Gamma_\alpha \Gamma / \Gamma|}{|\Gamma_\beta \Gamma / \Gamma|} \end{aligned}$$

where

$$\Gamma' := \{\gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_\beta \Gamma\} = \alpha^{-1} \Gamma_\beta \Gamma \beta^{-1} \cap \Gamma$$

$$\Rightarrow \Gamma'_\beta \Gamma / \Gamma = \alpha^{-1} \Gamma_\beta \Gamma \cap \Gamma \beta^{-1} / \Gamma$$

depends on α, β, γ .

The rest is a routine work.

$$\text{Ex. } \Gamma('m) \Gamma \cdot \Gamma('n) \Gamma = \Gamma('_{mn}) \Gamma \quad (m, n) = 1$$

$$\Gamma('_{p^e}) \Gamma \cdot \Gamma('_p) \Gamma = \Gamma('_{p^{e+1}}) \Gamma \sqcup \Gamma('_{p^e}) \Gamma \quad p \text{ prime, } e \geq 1$$

[Hint. $('m)(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})('n) = (\begin{smallmatrix} a & nb \\ mc & mnd \end{smallmatrix}) \in \Gamma('_{\frac{m}{l}}) \Gamma \quad \text{for } (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in SL_2(\mathbb{Z})$]

$$\Rightarrow \begin{cases} T_m * T_n \in \mathbb{Z} \cdot T_{mn} \\ T_{p^e} * T_p \in \mathbb{Z} \cdot T_{p^{e+1}} + \mathbb{Z} T_{p^{e-1}} R_p \end{cases} \quad (m, n) = 1$$

$$p \text{ prime, } e \geq 1$$

Computation of coefficient:

when $(m, n) = 1$, $\alpha = (1_m)$, $\beta = (1_n)$, $\gamma = (1_{mn})$,

$$\begin{aligned} g_{\alpha\beta}^{\gamma} &= \frac{1}{|\Gamma_{\gamma}\Gamma/\Gamma|} \sum_i \sum_j \mathbf{1}_{x_i y_j \in \Gamma_{\gamma}\Gamma} \\ &= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma_{\beta}\Gamma/\Gamma|}{|\Gamma_{\gamma}\Gamma/\Gamma|} \\ &= 1 \end{aligned}$$

when p is prime, $e \geq 1$, $\alpha = (1_{p^e})$, $\beta = (1_p)$, $\gamma_2 = (p_{p^e})$, $\gamma_1 = (1_{p^{e+1}})$,

$$\begin{aligned} \Gamma'_2 &\triangleq \left\{ \gamma' \in \Gamma \mid \alpha \gamma' \beta \in \Gamma_{\gamma_2} \Gamma \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \gcd(a, pb, p^e c, p^{e+1} d) = p \right\} \\ &= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Gamma \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \equiv \left(\begin{smallmatrix} 0 & * \\ * & * \end{smallmatrix} \right) \pmod{p} \right\} \\ &= \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ |\Gamma'_2 \beta \Gamma/\Gamma| &= \left| \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma/\Gamma \right| \\ &= \left| \Gamma^0(p) \right| / \left| \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \Gamma^0(p) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \cap \Gamma^0(p) \right| \\ &\stackrel{\text{def}}{=} \left| \Gamma^0(p) \right| / \left| \Gamma^0(p) \right| \\ &= 1 \end{aligned}$$

$$\begin{aligned} \left[\begin{aligned} \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix} \right)^{-1} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} a & b \\ pc & pd \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right)^{-1} \\ &= \left(\begin{smallmatrix} a & b \\ -pc & -pd \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \\ &= \left(\begin{smallmatrix} a & b \\ -pc & -pd \end{smallmatrix} \right) \end{aligned} \right] \end{aligned}$$

$$\therefore g_{\alpha\beta}^{\gamma_2} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_2 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_2} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot 1}{p^{e-1} - p^{e-2}}$$

$$g_{\alpha\beta}^{\gamma_1} = \frac{|\Gamma_{\alpha}\Gamma/\Gamma| |\Gamma'_1 \beta \Gamma/\Gamma|}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{|\Gamma_{\alpha}\Gamma/\Gamma| (|\Gamma_{\beta}\Gamma/\Gamma| - |\Gamma'_1 \beta \Gamma/\Gamma|)}{|\Gamma_{\gamma_1} \Gamma/\Gamma|}$$

$$= \frac{(p^e - p^{e-1}) \cdot (p+1 - 1)}{p^{e+1} - p^e}$$

$$= 1$$

$$4. \text{ Conclusion. } H(GL_2^+(\mathbb{Q}), \Gamma) = \mathbb{Z}[R_p^{\pm 1}, T_p \mid p \text{ prime}]$$

with

$$\begin{cases} T_m T_n = T_{mn} \\ T_p^e T_p = T_{p^{e+1}} + p T_{p^{e-1}} R_p \end{cases}$$

$$(m, n) = 1$$

$$p \text{ prime, } e \geq 1$$

Task: generalize it to other congruence subgps.

p -adic Cartan decomposition / not Grothendieck group!
Set $G = GL_2(F)$, $K_0 = GL_2(\mathcal{O}_F)$.

1. decomposition [Bump Prop 35]

$$M_{2 \times 2}(\mathcal{O}_F) = \coprod_{0 \leq e_i \leq e_1 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$M_{2 \times 2}(\mathcal{O}_F)_{\text{det} \neq 0} = \coprod_{0 \leq e_1 \leq e_2 \leq +\infty} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

$$G = \coprod_{e_1 \leq e_2} K_0 \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} K_0$$

Denote $T^- = \left\{ \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in G \mid e_1 \leq e_2, e_1, e_2 \in \mathbb{Z} \right\} \stackrel{\text{semi gp}}{\cong} \mathbb{Z} \oplus \mathbb{Z}_{\geq 0}$, then
 $G = \coprod_{\alpha \in T^-} K_0 \alpha K_0$

Ex. Verify that $K_0 \alpha K_0 / K_0$ is finite, and compute the order. $\alpha = \begin{pmatrix} \pi^{e_1} & \\ & \pi^{e_2} \end{pmatrix} \in T^-$

Hint.

$$\# K_0 \alpha K_0 / K_0 = \# K_0 / K_0 \cap \alpha K_0 \alpha^{-1} = \# K_0 / \Gamma_0(\mathfrak{p}_F^{e_1 - e_2}) = \# \text{IP}'(\mathcal{O}_F / \mathfrak{p}_F^{e_1 - e_2}) = \begin{cases} q^{e_2 - e_1} + q^{e_2 - e_1 - 1} & e_1 < e_2 \\ 1 & e_1 = e_2 \end{cases}$$

$$\left[\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O}^{e_1 - e_2} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} \Rightarrow K_0 \cap \alpha K_0 \alpha^{-1} = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathfrak{p}_F^{e_2 - e_1} & \mathcal{O} \end{pmatrix} = \Gamma_0(\mathfrak{p}_F^{e_2 - e_1}) \right]$$

$$\text{e.g. } \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = 1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q+1, \# \Gamma_0(\mathfrak{p}_F^e) \Gamma_0 / \Gamma_0 = q^e + q^{e-1}$$

Here we use the similar notation in modular form

[https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/5.moduli_interpretation.pdf]:

$$\begin{array}{ccc} \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \{ \text{Id} \} \\ \cap & & \cap \\ \text{bal. } \Gamma(\mathfrak{p}_F^e) & \xrightarrow{\quad} & N(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & \begin{pmatrix} 1 & * \\ * & 1 \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathfrak{p}_F^e) & \xrightarrow{\quad} & B(\mathcal{O}_F / \mathfrak{p}_F^e) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \\ \cap & & \cap \\ \Gamma_0(\mathcal{O}) = K^0 = GL_2(\mathcal{O}_F) & \xrightarrow{\quad} & GL_2(\mathcal{O}_F / \mathfrak{p}_F^e) \end{array}$$

Take the unique Haar measure on G s.t. $\mu(K^0) = 1$, then

$$\mu(K^0 \alpha K^0) = \# K^0 \alpha K^0 / K^0$$

μ is induced from the measure on coset G / K^0 .

The Hecke algebra has been defined here:

https://github.com/ramified/personal_handwritten_collection/blob/main/weeklyupdate/2022.04.17_preliminary_facts_of_reps_of_p-adic_groups.pdf

We still recall the convolution here:

$$(f_1 * f_2)(g) := \int_G f_1(x) f_2(x^{-1}g) d\mu(x)$$

$$= \sum_{x \in G/K_0} f_1(x) f_2(x^{-1}g) = \sum_{y \in K_0 \backslash G} f_1(gy^{-1}) f_2(y)$$

2. \mathbb{Z} -mod structure, notation

$$\mathcal{H}(G, K_0) = \bigoplus_{\alpha \in T} \mathbb{Z} \cdot \mathbf{1}_{K_0 \alpha K_0}$$

denote $T_\alpha := \mathbf{1}_{K_0 \alpha K_0}$

$$\begin{array}{lll} \lambda \in F^\times & R_\lambda := T_{(\lambda)} = \mathbf{1}_{K_0(\lambda) K_0} = \mathbf{1}_{\lambda K_0} & (R_1 = \mathbf{1}_{K_0} \text{ is the unit of } \mathcal{H}(G, K_0)) \\ e \geq 1 & T_{\pi^e} := T_{(\pi^e)} = \mathbf{1}_{K_0(\pi^e) K_0} & T_\pi := T_{(\pi)} = \mathbf{1}_{K_0(\pi) K_0} \end{array}$$

3. alg structure

$$T_\alpha * T_\beta = \sum_{\gamma \in T} (T_\alpha * T_\beta)(\gamma) T_\gamma$$

$$\begin{aligned} g_{\alpha\beta}^\gamma &:= (T_\alpha * T_\beta)(\gamma) = \sum_{x \in G/K_0} T_\alpha(x) T_\beta(x^{-1}\gamma) \\ &= \# \left\{ x \in G/K_0 \mid \begin{array}{l} x \in K_0 \alpha K_0 \\ x^{-1}\gamma \in K_0 \beta K_0 \end{array} \right\} \\ &= |K_0 \alpha K_0 \cap \gamma K_0 \beta^{-1} K_0| / |K_0| \end{aligned}$$

$$\text{e.p. } \mathbf{1}_\Gamma * f = f \quad (R_\lambda * f)(g) = f(\lambda^{-1}g) = f(g\lambda^{-1}) = (f * R_\lambda)(g)$$

$$R_\lambda * R_\mu = R_{\lambda\mu}$$

$$\text{E.g. } g_{\alpha\beta}^\gamma \neq 0 \Rightarrow |\gamma| = |\alpha||\beta| \quad \text{where } |\alpha| := \det \alpha$$

By the exactly same argument as in the global Cartan decomposition, one can show

$$g_{\alpha\beta}^\gamma = \frac{|K_0 \alpha K_0| |K_0 \beta K_0|}{|K_0 \gamma K_0|}$$

where

$$K_0' := \{\gamma' \in K_0 \mid \alpha \gamma' \beta \in K_0 \gamma K_0\} = \alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0$$

$$\Rightarrow |K_0' \beta K_0| = |\alpha^{-1} K_0 \gamma K_0 \beta^{-1} \cap K_0| / |\alpha|$$

depends on α, β, γ .

$$\Rightarrow T_{\pi^e} T_\pi = T_{\pi^{e+1}} + q T_{\pi^{e-1}} R_\pi$$

4. Conclusion (Tamagawa, Satake)

$$\mathcal{H}(G, K^\circ) = \mathbb{Z} [R_\pi^{\pm 1}, T_\pi] \quad \text{with}$$
$$T_\pi e T_\pi = T_\pi^{e+1} + q T_\pi^{e-1} \cdot R_\pi$$

$$\text{e.p. } \mathcal{H}(GL_2^+(\mathbb{Q}), SL_2(\mathbb{Z})) = \bigotimes_{\mathbb{Z}} \mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p))$$