Review + Tutorial 11 & Ex 10

+: not required

~~Sa~~ Space
- $\langle -,- \rangle \Rightarrow \|-\| \Rightarrow d(-,-) \Rightarrow \mathcal{U}$
- open/closed, $A^\circ, \bar{A}, A', \partial A, \cdots$
- connectedness:
  connected, path connected,
  simply connected $\rightsquigarrow \pi_1$
- compactness cpt $\Leftrightarrow$ net cpt $\Rightarrow$ sequence cpt
  https://math.stackexchange.com/questions/324842/nets-and-compactness-in-topological-spaces
+ separation axioms
+ countability
- convergence, completeness Cauchy sequence are only defined for metric space / TVS
  topological v.s.
  https://math.stackexchange.com/questions/1407695/is-it-possible-to-define-cauchy-sequences-in-a-topological-space

Differential
- derivation: one variable
- directional differential } Task 1
- total differential } Task 1
- mean value thm
- Taylor expansion
- criterion of Max/Min } Task 3
- Lagrange multiplier } Task 3

Integration:

Calculation: - indefinite integral
- multiple variable integral
- change of variables
- path integration
- integrate v.f. on surfaces } Task 2
+ Stokes' formula

Theory: - measure
- Riemann integral
- Lebesgue integral
+ de Rham cohomology

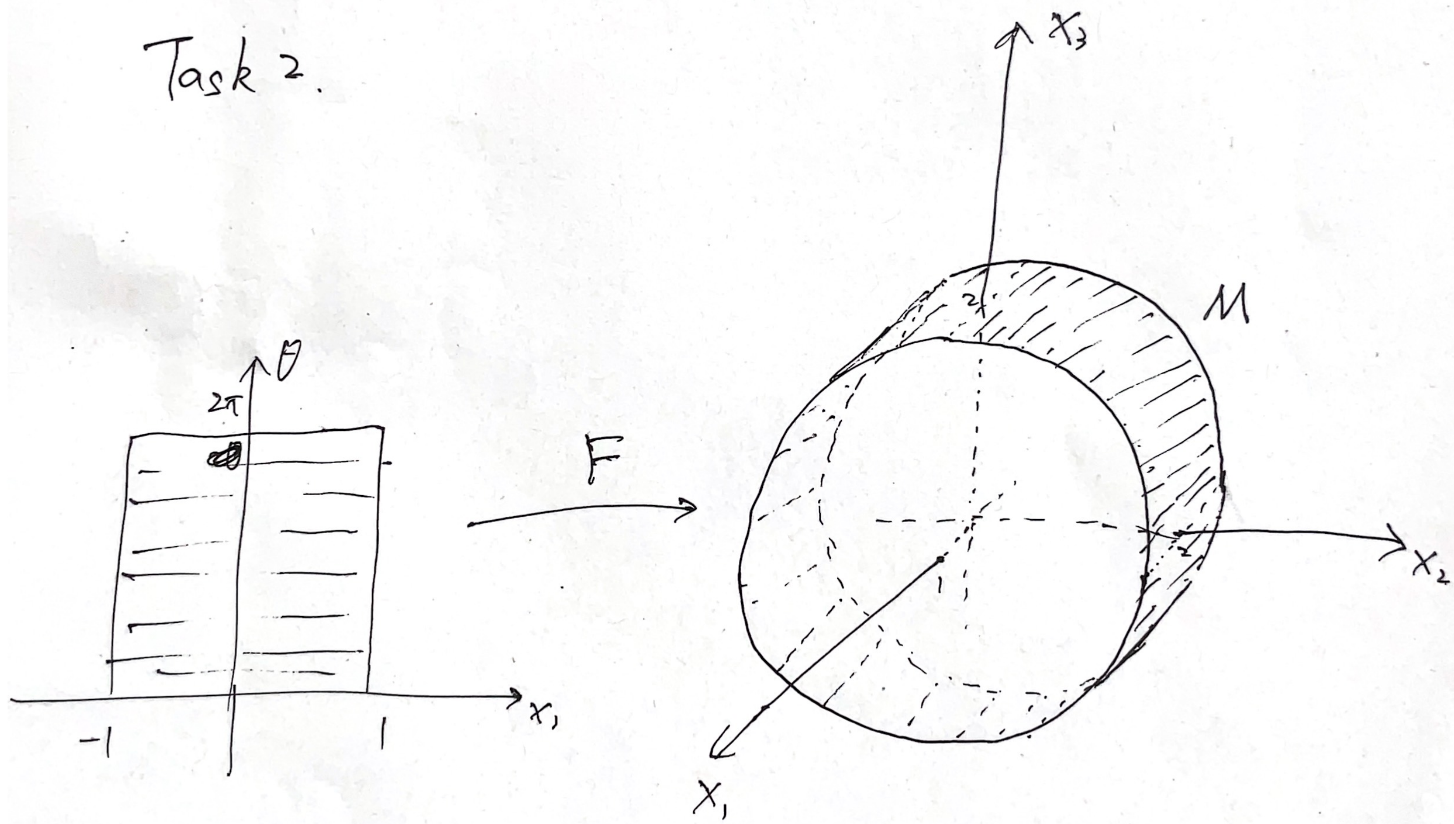
This time, the answer of the "PräsentzBlatt" is also available for you.

Task 1.

$$f(x_1, x_2) = \begin{cases} \dfrac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0,0) \\ 0 & (x_1, x_2) = (0,0) \end{cases}$$

1) Compute $\partial_1 f$
2) Compute $\partial_2 \partial_1 f$
3) Show that $\partial_2 \partial_1 f \neq \partial_1 \partial_2 f$
4) Show that $\partial_1 \partial_2 f$ or $\partial_2 \partial_1 f$ is not cont.

Task 2.



$$F(x, \theta) = (x, 2\cos\theta, 2\sin\theta)$$

Compute

$\cancel{\int_M e_3\, dA}$, $\int_M \vec{e}_3\, dA$ and $\int_M \vec{e}_3 \cdot d\vec{O}$.

A. First, we compute $d\mathcal{A}$ & $d\vec{\mathcal{O}}$.

$$\frac{\partial F}{\partial x} = (1, 0, 0)$$

$$\frac{\partial F}{\partial \theta} = (0, -2\sin\theta, 2\cos\theta)$$

$$\vec{\mathcal{O}} = \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \\ i & j & k \end{vmatrix}$$

$$= -2\cos\theta\, j - 2\sin\theta \cdot k$$

$$= (0, -2\cos\theta, -2\sin\theta)$$

$$d\mathcal{A} = \left|\frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta}\right| dx\, d\theta = 2\, dx\, d\theta$$

$$d\vec{\mathcal{O}} = \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta}\, dx\, d\theta = (0, -2\cos\theta, -2\sin\theta)\, dx\, d\theta$$

$$\int_M \vec{e}_3\, d\mathcal{A} = \vec{e}_3 \int_M 1\, d\mathcal{A}$$

$$= \vec{e}_3 \cdot \int_{-1}^{1} \int_0^{2\pi} 2\, d\theta\, dx,$$

$$= e_3 \cdot \int_{-1}^{1} dx \cdot \int_0^{2\pi} 2\, d\theta = 8\pi e_3$$

$$\int_M \vec{e}_3 \cdot d\vec{\mathcal{O}} = \int_0^{2\pi} \int_{-1}^{1} \langle x_3, (0, -2\cos\theta, -2\sin\theta) \rangle\, dx\, d\theta$$

$$= \int_0^{2\pi} \int_{-1}^{1} -2\sin\theta\, dx\, d\theta$$

$$= \int_0^{2\pi} -2\sin\theta\, d\theta \cdot \int_{-1}^{1} dx = 0$$

Task 3.

For $f(x,y) = x^2+y^2$, the minimum is reached at $(0,0)$.

Reason: $f(x,y) = x^2+y^2 \geq 0 = f(0,0)$

$(\Rightarrow \partial_x f(0,0) = \partial_y f(0,0) = 0)$

Q: What would happen when we restrict to the set

$$B := \{(x,y) \in \mathbb{R}^2 \mid x-y = 1\} ?$$

Bad new: We can no longer find possible max/mins by solving

$$\partial_x f = \partial_y f = 0$$

since in general, the solutions $z \notin B$

Good new: We can perturb f by

$$\tilde{f}(x,y) := f(x,y) + a(x-y-1) \text{ for some } a \in \mathbb{R}$$

so that the solutions of the equations

$$\partial_x \tilde{f} = \partial_y \tilde{f} = 0$$

lies in B. These solutions are possible max/mins of $\tilde{f}$.

Moreover, $\tilde{f}|_B = f|_B$, so

$$\min_{z\in B} \tilde{f}(z) = \min_{z\in B} f(z), \quad \max_{z\in B} \tilde{f}(z) = \max_{z\in B} f(z).$$

Ex. Check that the following three steps work.

① $\begin{cases} (\partial_x \tilde{f})(x,y) = 0 \\ (\partial_y \tilde{f})(x,y) = 0 \\ x - y = 1 \end{cases} \Leftrightarrow \begin{cases} 2x + a = 0 \\ 2y - a = 0 \\ x - y - 1 = 0 \end{cases}$

$\Leftrightarrow \begin{cases} a = 1 \\ x = -\frac{1}{2} \\ y = \frac{1}{2} \end{cases}$

② Let $a = 1$, then

$$\tilde{f}(-\tfrac{1}{2}, \tfrac{1}{2}) = \min_{z \in \mathbb{R}^2} \tilde{f}(z)$$

$$= \min_{z \in B} \tilde{f}(z)$$

$$= \min_{z \in B} f(z)$$

③ $\lim_{\substack{(x,y) \to \infty \\ (x,y) \in B}} f(x,y) = +\infty \quad \Rightarrow$ no maximum.

In conclusion, $f|_B$ has minimum $\frac{1}{2}$ at $(-\frac{1}{2}, \frac{1}{2})$,

$f|_B$ has no maximum.

Ex. Use the GeoGebra (online, free) to draw the fct

$$z = x^2 + y^2 - a(x - y - 1).$$

Ex. For $g(x,y) = 2x+y$, compute

$$\min_{(x,y)\in S^1} g(x,y) \qquad \max_{(x,y)\in S^1} g(x,y)$$

where

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$$

Hint: Let $\tilde{g}(x,y) := g(x,y) + a(x^2+y^2-1)$, then

$$\begin{cases} (\partial_x \tilde{g})(x,y) = 0 \\ (\partial_y \tilde{g})(x,y) = 0 \\ x^2+y^2 = 1 \end{cases} \iff \begin{cases} 2+2ax = 0 \\ 1+2ay = 0 \\ x^2+y^2 = 1 \end{cases}$$

$$\iff \begin{cases} a = \frac{\sqrt{5}}{2} \\ x = -\frac{2\sqrt{5}}{5} \\ y = -\frac{\sqrt{5}}{5} \end{cases} \text{ or } \begin{cases} a = -\frac{\sqrt{5}}{2} \\ x = \frac{2\sqrt{5}}{5} \\ y = \frac{\sqrt{5}}{5} \end{cases}$$

Can check that

$$g\left(-\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) = -\sqrt{5} \qquad \text{minimum}$$

$$g\left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) = \sqrt{5} \qquad \text{maximum.}$$

$$\because\ 5 = (x^2+y^2)(4+1) \geq (2x+y)^2$$

Ex. Read wiki: Lagrange multiplier, try to understand examples there.