§ 1.1. Structure of finite / local / global field

Road map

	finite field		local field: Archi		local field: NA		global field		Adéle
base field F	6 $\mathbb{F}_q$ (For $\mathbb{F}_{q^r}$)	1 $\mathbb{F}_p$	2 $\mathbb{R}$ or $\mathbb{C}$		3 $\mathbb{Q}_p$	$\mathbb{F}_p((t))$	4 $\mathbb{Q}$	$\mathbb{F}_p(t)$	5 $\mathbb{A}_K$
$F^\times$	$\varepsilon\cdot\mu_r$	μ_{p-1}	$\mathbb{R}^+\times\mathbb{Z}/2\mathbb{Z}$	$\mathbb{C}^\times$	$\mathbb{Z}_p^\times\times\mathbb{Z}$	$\mathbb{F}_p[[t]]^\times\times\mathbb{Z}$	$\mathbb{Q}^\times$	$\mathbb{F}_p(t)^\times$	$\mathbb{I}_K$
integral ring $\mathcal{O}_F$	—	—	—		$\mathbb{Z}_p$	$\mathbb{F}_p[[t]]$	$\mathbb{Z}$	$\mathbb{F}_p[t]$	K ?
units $\mathcal{O}_F^\times$	—	—	—		$\mathbb{Z}_p^\times$	$\mathbb{F}_p[[t]]^\times$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{F}_p^\times$	$\mathbb{I}_K^\times$?
$Gal(F^{sep}/F)$	$\hat{\mathbb{Z}}$?	$\hat{\mathbb{Z}}$	$\mathbb{Z}/2\mathbb{Z}$	Id	most known		dream		
ari Frob	?	can	total order?	—	choose a lift		?		?
# ext of deg n	1?	1	1/0		finite		$n\neq 1$ inf		countable
Spec $\mathcal{O}_F$	Spec $\mathbb{F}_q = K(\hat{\mathbb{Z}},1)$ [étale, 2.2.4]		—						—
topology	?	discrete	Euclidean		profinite		—		restricted
topo of $\mathcal{O}_F$	—		—		cpt, not discrete		—		K is a lattice in $\mathbb{A}_K$
measure	?	discrete	Lebesgue		$\mu(\mathcal{O}_F)=1$		—		can be computed

Also, discuss

- field extension, norm, trace, ⋯
- their connection to geometry, ramification theory
- analog with knot theory

1. finite field $\mathbb{F}_q$

Any fin field is of form $\mathbb{F}_q$, where $q = p^r$, $r \in \mathbb{N}_{\geq 1}$.

$\mathbb{F}_q$ = the splitting field of $X^q - X$ over $\mathbb{F}_p$.

$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \widehat{\mathbb{Z}}$ as top gps

$\mathrm{Frob}_p \longleftarrow 1$

2. Archi local field $\mathbb{R}$ or $\mathbb{C}$

No difficulty: $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$ $\quad \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \mathrm{Id}$

$\mathbb{C}$ is the unique local field which is alg closed.

3. NA local field

Define NA local field as (finite ext of $\mathbb{Q}_p$) or $\mathbb{F}_q((T))$.

Individual structure

Task. Read [NAlocal], answer the following questions:

- Describe $\mathcal{O}, \mathfrak{p}, \kappa, \mathcal{U}, \mathcal{U}^{(n)}$ in terms of ν
- What is the structure of $\mathbb{Q}_p^\times$?
- For $F, F^\times, \mathcal{O}, \mathcal{O}^\times$, which are cpt?

  Can we classify open subgps of $F, F^\times$?
- Give a description of the Haar measure on F and $F^\times$.

Field extension

Task. Read [NAext], answer the following questions:

- Describe the field extension tower of F.
- Find a wild extension of $\mathbb{Q}_p$ & $\mathbb{F}_p[[t]]$
- Can we "see the geometry of $\mathbb{Q}_p$" vividly?

Something is needed here.

Task. Read [NAval], answer the following questions. (Not necessary for future discussion)

- What is the difference of NA valued field (with NA local field)?
- When is the field extension over $\mathbb{Q}_p$ complete?
- Using the result in [NAval], computes the following Galois gps:

$$\mathrm{Gal}\left(\mathbb{F}_p((t^{\frac{1}{p^\infty}}))^{sep} / \mathbb{F}_p((t^{\frac{1}{p^\infty}}))\right), \quad \mathrm{Gal}\left(\widehat{\overline{\mathbb{Q}}}_p / \widehat{\mathbb{Q}_p^{ur}}\right), \quad \mathrm{Gal}\left(\overline{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})} / \mathbb{Q}_p(p^{\frac{1}{p^\infty}})\right)$$

$$G_{\mathbb{F}_p((t))} \qquad\qquad I_{\mathbb{Q}_p} \qquad\qquad G_{\mathbb{F}_p((t))}$$