Eine Woche, ein Beispiel

6.26. adic space

Ref: Berkeley notes by Peter Scholze.
Section 4, http://www.math.uni-bonn.de/people/ja/lubintate/lecture_notes_lubin_tate.pdf
I just feel the need to record these results, so that I won't check them again as time went by.
Also I just learned a little about the discrete version. It is still not obvious for me to find out all valuations up to equivalence.

1. Discrete Huber pairs.

Set level

(A, A^+): $A \in CRing$ $\quad$ $A^+ \leq$ integrally closed subring (containing 1)

$$\mathrm{Spa}(A, A^+) = \left\{ v: A \longrightarrow \Gamma \cup \{+\infty\} \;\middle|\; \begin{matrix} v(f+g) \geq \min\{v(f), v(g)\} & v(0) = +\infty \\ v(fg) = v(f) + v(g) & v(1) = 0 \\ v(A^+) \geq 0 & \end{matrix} \right\} \Big/ \sim$$

$$= \left\{ |\cdot|: A \longrightarrow \Gamma' \cup \{0\} \;\middle|\; \begin{matrix} |f+g| \leq \max\{|f|, |g|\} & |0| = 0 \\ |fg| = |f||g| & |1| = 1 \\ |A^+| \leq 1 & \end{matrix} \right\} \Big/ \sim$$

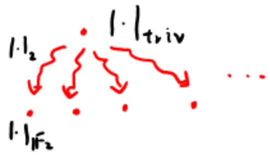
$v \sim v' \Leftrightarrow [v(a) \geq v(b) \Leftrightarrow v'(a) \geq v'(b)]$

$|\cdot| \sim |\cdot|' \Leftrightarrow [|a| \leq |b| \Leftrightarrow |a|' \leq |b|']$

$(\Gamma, +, 0)$, $(\Gamma', \cdot, 1)$: totally ordered abelian group

We use the some notations as in the document [Berkovich space, 2021.11.7].

E.g. $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) = \{|\cdot|_{triv}, |\cdot|_p, |\cdot|_{\mathbb{F}_p}\}$



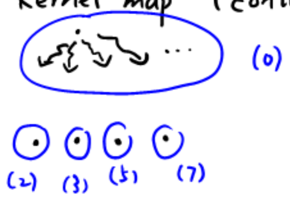
$\mathrm{Spa}(\mathbb{Q}, \mathbb{Z}) = \{|\cdot|_{triv}, |\cdot|_p\}$

$\mathrm{Spa}(\mathbb{Q}, \mathbb{Q}) = \{|\cdot|_{triv}\}$

$\bullet\ |\cdot|_{triv}$

supp: $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Spec}\,\mathbb{Z}$

k = supp: kernel map (contraction)



Topology level.

Def (Rational open subsets of $\mathrm{Spa}(A, A^+)$) For $f, g_1, \cdots, g_n \in A$,

$$\begin{aligned}
\mathcal{U}\left(\frac{g_1,\dots,g_n}{f}\right) &= \{ v \mid v(g_i) \geq v(f) \neq +\infty \quad \forall i \} \\
&= \{ |\cdot| \mid |g_i| \leq |f| \neq 0 \quad \forall i \} \\
&= \mathcal{U}\left(\tfrac{g_1}{f}\right) \cap \mathcal{U}\left(\tfrac{g_2}{f}\right) \cap \cdots \mathcal{U}\left(\tfrac{g_n}{f}\right)
\end{aligned}$$

E.g. For $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, Yellow: remove

$$\mathcal{U}\left(\tfrac{0}{1}\right) = \mathcal{U}\left(\tfrac{9}{1}\right) = \mathcal{U}\left(\tfrac{3}{1}\right) = \mathcal{U}\left(\tfrac{1}{1}\right) = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$$

$$\mathcal{U}\left(\tfrac{0}{3}\right) = \mathcal{U}\left(\tfrac{9}{3}\right) = \mathcal{U}\left(\tfrac{3}{3}\right) = \{ \cdots \} \qquad \mathcal{U}\left(\tfrac{1}{3}\right) = \{ \cdots \} = \mathcal{U}\left(\tfrac{1}{9}\right)$$

$$\mathcal{U}\left(\tfrac{45}{36}\right) = \{ \cdots \} \qquad \mathcal{U}\left(\tfrac{a}{0}\right) = \emptyset$$

For $\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$,

$$\mathcal{U}\left(\tfrac{0}{1}\right) = \mathcal{U}\left(\tfrac{9}{1}\right) = \mathcal{U}\left(\tfrac{3}{1}\right) = \mathcal{U}\left(\tfrac{1}{1}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$$

$$\mathcal{U}\left(\tfrac{0}{3}\right) = \mathcal{U}\left(\tfrac{9}{3}\right) = \mathcal{U}\left(\tfrac{3}{3}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z}) \qquad \mathcal{U}\left(\tfrac{1}{3}\right) = \{ \cdots \} = \mathcal{U}\left(\tfrac{1}{9}\right)$$

$$\mathcal{U}\left(\tfrac{45}{36}\right) = \{ \cdots \} \qquad \mathcal{U}\left(\tfrac{a}{0}\right) = \emptyset$$

$$\mathcal{U}\left(\frac{\frac{1}{9}}{1}\right) = \{ \cdots \} \qquad \mathcal{U}\left(\frac{1}{\frac{1}{9}}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z}) \qquad \mathcal{U}\left(\frac{\frac{1}{9}}{\frac{1}{9}}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$$

It is actually easier in this case, we just take the reduction of a fraction, and remove valuations which correspond to primes on the denominators.

Rmk. Rational open subsets of $\mathrm{Spa}(A, A^+)$ is the basis of a topo, as

$$\mathcal{U}\left(\frac{g_1,\dots,g_n}{f}\right) \cap \mathcal{U}\left(\frac{g_1',\dots,g_m'}{f'}\right) = \mathcal{U}\left(\frac{g_1f',\dots,g_nf',\ g_1'f,\dots g_m'f}{ff'}\right)$$

Now we can do the same formalism as two years ago.

- Topology
  - Closed subset of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$: $\bigcup_{p \in \Lambda_1} \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \} \cup \bigcup_{p \in \Lambda_2} \{ |\cdot|_{\mathbb{F}_p} \}$, or $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$

    $\Lambda_1, \Lambda_2 \subset \{\text{primes } p\}$ finite, $\Lambda_1 \cap \Lambda_2 = \emptyset$
  - Open subset of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$: $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) - \bigcup_{p \in \Lambda_1} \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \} \cup \bigcup_{p \in \Lambda_2} \{ |\cdot|_{\mathbb{F}_p} \}$, or $\emptyset$.
  - closure of a pt:

    $\overline{\{|\cdot|_{triv}\}} = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) \qquad \overline{\{|\cdot|_p\}} = \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \} \qquad \overline{\{|\cdot|_{\mathbb{F}_p}\}} = \{ |\cdot|_{\mathbb{F}_p} \}$
  - Topo properties: connected ✓ Hausdorff ✗ quasi-compact ✓

    irreducible ✓

$$\begin{tikzcd}
A^{\circ\circ} \arrow[rr, phantom, "\subset"] \arrow[dr, phantom, "\subset" sloped] & & A^{\circ} \arrow[r, phantom, "\subset"] & A \\
& A^{+} \arrow[ur, phantom, "\subset" sloped] & & \\
I \arrow[u, phantom, "\subset" sloped] \arrow[rr, phantom, "\subset"] & & A_{0} \arrow[uu, phantom, "\subset" sloped] &
\end{tikzcd}$$