

# Eine Woche, ein Beispiel

## 2.4 exercise sheet for six functors

Ref. from [23.11.19]

Review. For  $X$  sm mfld,  $f: X \rightarrow pt$ . compute

$$\begin{array}{ll} Rf_* \mathcal{F} = & f^* \mathcal{Q} = \\ Rf_! \mathcal{F} = & f'_! \mathcal{Q} = \end{array}$$

$f_! \dashv f'$   $\approx$  Poincaré duality  
six functor formalism  $\approx$  cohomology theory in other categories

Proper base change formula:

$$f^* Rg_* \cong Rg'_* f'^*$$

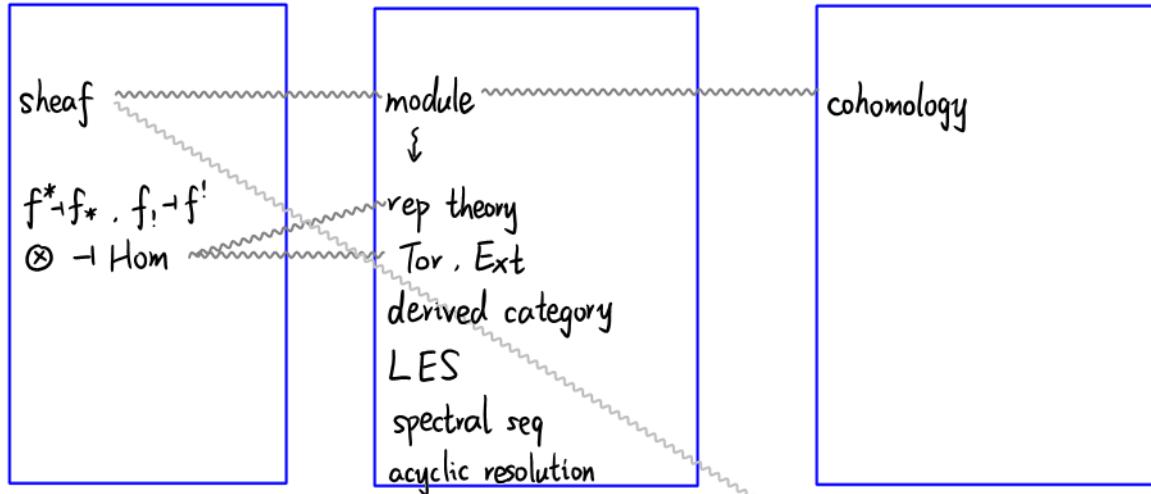
$$\begin{array}{ccc} Y & \xrightarrow{f'} & X' \\ g'_! \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

$$g'^* Rf_* \cong Rf'_* g'^! \quad [\text{Hint: Yoneda + adjunction}]$$

$X \rightarrow Y$   
between spaces      over  $A$   
Today: six functor    +    homological algebra       $\Rightarrow$  algebraic topology

over one space

classical

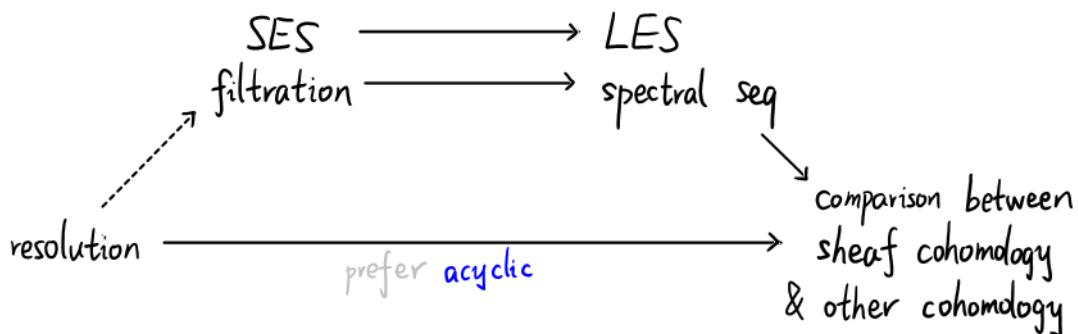


characteristic class  
vector bundle  
 $\pi_n(X, \cdot)$   
spectra  
cobordism

slogan:

SES induces LES,  
filtration induces spectral sequence.

To expand a little bit,



Even though "filtration  $\Rightarrow$  spectral seq" is the most general statement, people start with " $\text{SES} \Rightarrow \text{LES}$ " and " $\text{acyclic resolution} \Rightarrow \text{other coh} \approx \text{hyper coh}$ ".  
Let us leave spectral seq in other people's notes.

1. open-closed formalism
2. open cover
3. filtrations from chain complex
4. filtration by  $H^i(F)$
5. filtration by  $F$
6. Hodge related filtration

We will assume the Poincaré lemma:

$$H^i(\mathbb{R}^n; \underline{\mathbb{Q}}) \cong \underline{\mathbb{Q}}$$

Since  $\mathbb{R}^n$  is contractible, you know how to prove it.

Methods to construct SES:  $\left\{ \begin{array}{l} \text{check by stalks} \\ \text{filtration by } H^i(F) \\ \text{filtration by } F^i \end{array} \right.$

method	spectral seq	LES	cohomology/resolution
check by stalks	... for stratifications	relative coh seq	simplicial/cellular
	Čech-to-derived functor	MV	Čech open cover
	coefficient		closed cover
filtration by $H^i(F)$	Grothendieck		
	Leray-Serre	Cysin	Euler class Hodge-Tate
			de Rham
filtration by $F^i$ need resolution to get "another" complex	Hodge-de Rham		Hodge-de Rham $H^q(X, \Omega^p) \Rightarrow H^{p+q}(X)$
	Frölicher		Dolbeault $H^q(X, \Omega^p) = H^{p,q}(X)$
			$H^{p,q}(X) \Rightarrow H^{p+q}(X)$ "composition" singular
spectral sequences which I don't know	Adams		for stable homotopy gp
	Atiyah-Hirzebruch		for top K-theory
	Bar		for group
	Bockstein		for group homology
	Cartan-Leray		
	Eilenberg-Moore		
	Green		
	:		
			for Koszul cohomology :

For more spectral sequences, see:

[https://en.wikipedia.org/wiki/Spectral\\_sequence](https://en.wikipedia.org/wiki/Spectral_sequence)

<https://github.com/CubicBear/SpectralSequences/blob/main/SpectralSequences.pdf>

mixed Hodge theory = using two filtrations of  $Rj_* \mathbb{Q}_U$  in the same time  
to get two filtrations of  $H^i(U; \mathbb{Q})$

# 1. open-closed formalism

// related: comparison of  $j_!$  &  $j^*$   
one-point compactification.

Observe the following pictures. (called recollement diagram)

$$\begin{array}{ccccc}
 Z & \xrightarrow{i} & X & \xleftarrow{j} & U \\
 & \xleftarrow{i^*} & & \xleftarrow{j_!} & \\
 D(Z) & \xrightarrow[i_! = i_*]{i^!} & D(X) & \xrightarrow[j^* = j^!]{Rj_*} & D(U) \\
 & \xleftarrow{i^!} & & \xleftarrow{Rj_*} &
 \end{array}$$

Black box:

0. We assume some nice conditions.

e.g. in the category  $\text{Haus}^{\text{loc. cpt}}$ , and  $Z \subset X$  is loc. contractible.

Under these conditions,

1.  $i_* = i_!$ ,  $j^* = j^!$  [Achar 21, 1.3.4]

2.  $j_!$ ,  $i^*$ ,  $j^*$ ,  $i_*$  are exact.

Ex. 1. Shows that [Achar 21, Prop 1.3.9]

$$\underline{i^* i_*} = \underline{i^! i_*} = \text{Id}_{D(Z)} \quad \underline{j^* j_!} = \underline{j^* Rj_*} = \text{Id}_{D(U)}$$

$$\underline{i^* j_!} = 0, \quad \underline{j^* i_*} = 0, \quad \underline{i^! Rj_*} = 0$$

$$i^* Rj_* \cong i^! j_! [1] \quad \text{check this after (1) \& (2)}$$

— base change

~~~~ check stalkwise.

2. (for category fans)

$i_*$ ,  $j_*$ ,  $j_!$  are fully faithful, and

$i_*$ ,  $j^*$ ,  $j^*$  preserve injectives.

$i^!$ ,  $Rj_*$  are not defined for  
the abelian category

<https://mathoverflow.net/questions/100808/properties-of-functors-and-their-adjoints>

3. One has SES

$$0 \longrightarrow j_! j^* F \longrightarrow F \longrightarrow i_! i^* F \longrightarrow 0 \quad (1)$$

You might prefer

$$0 \longrightarrow j_! j^! F \longrightarrow F \longrightarrow i_* i^* F \longrightarrow 0$$

Well, writing every term by left adjoint factors is better for me.

Ex for (1).

1. Apply the  $R\pi_{X,*}$  to (1), take  $\mathcal{F} = \underline{\mathbb{Q}}_X$ , what do we get?

Define  $R\Gamma(X, \mathbb{Z}; \underline{\mathbb{Q}}) := R\Gamma(X; j_! \underline{\mathbb{Q}})$

In general, what do we get when applying  $R\pi_{X,*}$  &  $R\pi_{X,!}$ ?

Discuss 2 special cases  $\mathcal{F} = \underline{\mathbb{Q}}_X$   $|D_X| = \pi_X^! \underline{\mathbb{Q}}_{\mathbb{R}^{n+1}} = |D_X(\underline{\mathbb{Q}}_X)|$

2. Derive from (1) the triangle

$$j_! \mathcal{F} \longrightarrow Rj_* \mathcal{F} \longrightarrow i_* i^* Rj_* \mathcal{F} \xrightarrow{+1}$$

which measures the difference between  $j_! \mathcal{F}$  &  $j_* \mathcal{F}$ .

3. Shows that (require:  $\{\infty\} \hookrightarrow \bar{X}$  is loc. contractable)

$$H_c(X) \cong H(\bar{X}, \{\infty\})$$

for one pt compactification  $i: X \hookrightarrow \bar{X}$ .

Try to compute  $H_c(\mathbb{R}^n)$  in this way.

4. [Achar 21, Ex 1.3.7]

Derive from (1) the triangle

$$\xleftarrow{+1} Rj_* \underline{R\text{Hom}}(j^* \mathcal{F}, j^! G) \leftarrow \underline{R\text{Hom}}(\mathcal{F}, G) \leftarrow i^* \underline{R\text{Hom}}(i^* \mathcal{F}, i^! G)$$

which decompose the morphisms by two parts.

It seems that we get only half of the results.

### Verdier dual

Def. The Verdier dual / dualizing functor is defined as

$$\mathrm{ID}_X : \mathrm{D}^b(X; \mathbb{Q}) \longrightarrow \mathrm{D}^b(X; \mathbb{Q}) \quad \mathrm{ID}_X \mathcal{F}^\cdot := \underline{\mathrm{Hom}}_{\mathrm{D}^b(X; \mathbb{Q})}(\mathcal{F}^\cdot, \pi_X^! \underline{\mathbb{Q}}_{\mathbb{P}^1})$$

$$R\pi_{X*} \mathrm{ID}_X \mathcal{F}^\cdot = \underline{\mathrm{Hom}}_{\mathrm{D}^b(\mathbb{P}^1; \mathbb{Q})}(\pi_{X*} \mathcal{F}^\cdot, \mathbb{Q})$$

$$= H_c^*(X; \mathcal{F}^\cdot)^*$$

We know that

$$\mathrm{ID}_X \underline{\mathbb{Q}}_X = \pi_X^! \underline{\mathbb{Q}}_{\mathbb{P}^1}$$

$$\mathrm{ID}_X(\mathcal{F}[n]) = (\mathrm{ID}_X \mathcal{F})[-n]$$

$$\mathcal{F}^\cdot \longrightarrow \mathcal{G}^\cdot \longrightarrow \mathcal{H}^\cdot \xrightarrow{+1} \rightsquigarrow \mathrm{ID}\mathcal{H}^\cdot \longrightarrow \mathrm{ID}\mathcal{G}^\cdot \longrightarrow \mathrm{ID}\mathcal{F}^\cdot \xrightarrow{+1}$$

$$f^! \mathrm{ID}_X = \mathrm{ID}_Y f^*$$

$$Rf_* \mathrm{ID}_Y = \mathrm{ID}_X Rf_!$$

"left adjoint can be pulled out"

$$f: Y \longrightarrow X$$

When  $\mathcal{F}^\cdot \in \mathrm{D}^b(X; \mathbb{Q})$  is constructible /  $\mathcal{F}$  is  $\pi_X$ -smooth, then

$$\mathrm{ID}_X^2 \mathcal{F}^\cdot \cong \mathcal{F}^\cdot$$

Therefore, in the constructible setting,

$$f^* \mathrm{ID}_X = \mathrm{ID}_Y f^*$$

$$Rf_* \mathrm{ID}_Y = \mathrm{ID}_X Rf_*$$

For exact statements about  $\mathrm{ID}_X$ , see [MS21, Cor 2.11] [IHPS, Thm 5.3.9]

Ex. Derive from (1) the triangle

$$i_! i^* \mathcal{F}^\cdot \longrightarrow \mathcal{F}^\cdot \longrightarrow Rj_* j^* \mathcal{F}^\cdot \xrightarrow{+1} \tag{2}$$

for  $\mathcal{F}^\cdot \in \mathrm{D}^b(X; \mathbb{Q})$  constructible.

Rmk. It seems that in [Achar 21, Thm 13.10], (2) is true without assuming  $\mathcal{F}$  is constructible. Do I miss something?

Ex for (2). Do the same arguments in "Ex for (1)", and verifies that  $i^* Rj_* \cong i^! j_![1]$ .

E.g. For a finite graph (as a topo space)  $X$ ,

$$\begin{array}{ccc} \vdots & \text{blue graph} & \vdots \\ \text{sk}_0 X & \xleftarrow{i} & X & \xleftarrow{j} & X - \text{sk}_0 X & \text{1-cells} \end{array}$$

$$0 \longrightarrow j_! j^* \underline{\mathbb{Q}}_X \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_* i^* \underline{\mathbb{Q}}_X \longrightarrow 0$$

$$0 \longrightarrow j_! \underline{\mathbb{Q}}_{X - \text{sk}_0 X} \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow i_! \underline{\mathbb{Q}}_{\text{sk}_0 X} \longrightarrow 0$$

Take  $R\pi_{X,!}$ :

$$\underbrace{j_* H_c(X - \text{sk}_0 X) \xrightarrow{\cong \oplus \mathbb{Q}} H_c(X) \longrightarrow H_c(\text{sk}_0 X) \xrightarrow{\cong 0} 0}_{0 \longrightarrow H_c(X - \text{sk}_0 X) \xrightarrow{\cong 0} H_c(X) \longrightarrow H_c(\text{sk}_0 X) \xrightarrow{\cong \oplus \mathbb{Q}}$$

This calculates the sheaf cohomology as simplicial cohomology.

E.x. Shows that

$$H_c^i(\mathbb{R}) = \begin{cases} \mathbb{Q} & i=1 \\ 0 & \text{otherwise} \end{cases}$$

in a similar way.

Generalizing this argument, one can relate sheaf cohomology with simplicial / cellular cohomology, using the following stratification:

$$0 \subset \text{sk}^0 X \subset \text{sk}^1 X \subset \cdots \subset \text{sk}^n X = X \quad \downarrow \text{filtration}$$

$$0 \subset i_* \underline{\mathbb{Q}}_{\text{sk}^0 X} \subset i_* \underline{\mathbb{Q}}_{\text{sk}^1 X} \subset \cdots \subset \underline{\mathbb{Q}}_{\text{sk}^n X} = \underline{\mathbb{Q}}_X$$

Ex. derive the Wang LES for the cpt supp version over  $S^1$

## 2. open cover

Ex. For an open cover  $X = U_1 \cup U_2$ , deduce the SES

$$0 \leftarrow \underline{\mathbb{Q}}_X \xleftarrow{\quad} j_! \underline{\mathbb{Q}}_{U_1} \oplus j_! \underline{\mathbb{Q}}_{U_2} \xleftarrow{\quad} j_! \underline{\mathbb{Q}}_{U_1 \cap U_2} \xleftarrow{\quad} 0$$

$$\underline{\mathbb{Q}}_X \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_1} \oplus Rj_* \underline{\mathbb{Q}}_{U_2} \longrightarrow Rj_* \underline{\mathbb{Q}}_{U_1 \cap U_2} \xrightarrow{+1} 0 \quad (3)$$

▽ We omit the derived symbol and some subscripts in this section.  $U_{12} = U_1 \cap U_2$

(3) works for general sheaf

and, induce from (3) the MV sequence:

$$\begin{array}{ccccccc} & \xleftarrow{+1} & H_c^k(X) & \xleftarrow{\quad} & H_c^k(U_1) \oplus H_c^k(U_2) & \xleftarrow{\quad} & H_c^k(U_1 \cap U_2) \\ & & H^k(X) & \longrightarrow & H^k(U_1) \oplus H^k(U_2) & \longrightarrow & H^k(U_1 \cap U_2) \xrightarrow{+1} \end{array}$$

Hint: Apply  $R\pi_{X,!}$  &  $R\pi_{X,*}$ , see [StackProject, 01E9]

Ex. Derived the Wang LES. over  $S^1$

Ex. For an open cover  $X = \bigcup_{i \in \Lambda} U_i$ ,  $\Lambda$  finite, deduce the exact seq

$$0 \leftarrow \underline{\mathbb{Q}}_X \leftarrow \bigoplus_i j_! \underline{\mathbb{Q}}_{U_i} \leftarrow \bigoplus_{i < j} j_! \underline{\mathbb{Q}}_{U_i \cap U_j} \leftarrow \cdots j_! \underline{\mathbb{Q}}_{\cap U_i} \leftarrow 0$$

and t-exact seq

$$0 \longrightarrow \underline{\mathbb{Q}}_X \longrightarrow \bigoplus_i Rj_* \underline{\mathbb{Q}}_{U_i} \longrightarrow \bigoplus_{i < j} Rj_* \underline{\mathbb{Q}}_{U_i \cap U_j} \longrightarrow \cdots Rj_* \underline{\mathbb{Q}}_{\cap U_i} \longrightarrow 0$$

When  $\{U_i\}_{i \in \Lambda}$  is a good cover,  $H^i(U_{i_1, \dots, i_k}) = H^0(U_{i_1, \dots, i_k})$ ,  
 $\uparrow$  acyclic in AG

one can compute  $H^i(X)$  by applying  $R\pi_{X,*}$ :

$$\begin{array}{ccccccc} 0 \longrightarrow & \bigoplus_i \Gamma(U_i) & \xrightarrow{d^1} & \bigoplus_{i < j} \Gamma(U_i \cap U_j) & \xrightarrow{d^2} & \cdots & \Gamma(\bigcap U_i) \longrightarrow 0 \\ & H^0(X) & & H^1(X) & & \cdots & H^{|\Lambda|-1}(X) \\ & & & & \downarrow \text{Ker/Im} & & \end{array}$$

Rmk. When  $X$  is paracompact & Hausdorff, "limited" Čech = sheaf  
 $\uparrow$  e.g. loc cpt Haus + second-countable, or CW cplx

compare the first step:

$$F \longrightarrow \bigoplus_i Rj_* F|_{U_i} \qquad F \longrightarrow \bigoplus_{x \in X} F_x$$

If you haven't seen the acyclic resolution before, the following example may provide some intuition.

#  $\Delta = 3$  case:

$$\begin{array}{ccccccc}
 & & \circ & & \circ & & \\
 & & \searrow & & \swarrow & & \\
 & & F_0 & & & & \\
 & \nearrow & & \searrow & & & \\
 0 \rightarrow & Q_X \rightarrow & \bigoplus_i Rj_* \mathbb{Q}_{U_i} \xrightarrow{d'} & \bigoplus_i Rj_* \mathbb{Q}_{U_i \cap U_j} \xrightarrow{d''} & Rj_* \mathbb{Q}_{\cap U_i} \xrightarrow{d'''} & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & F_1 & & F_2 & & \\
 & & \circ & & \circ & & \\
 & & \searrow & & \swarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

$$F_2 = Rj_* \mathbb{Q}_{\cap U_i} \Rightarrow H^*(F_2) = \ker d'''$$

$$\begin{array}{ccccc}
 H^*(F_1) & \longrightarrow & 0 & \longrightarrow & H^*(F_2) \xrightarrow{\text{+1}} \\
 \underbrace{\hspace{10em}} & & & & \\
 0 \rightarrow H^0(F_1) \rightarrow \bigoplus_{i<j} \Gamma(U_i \cap U_j) \xrightarrow{d''} & & H^0(F_2) & & 
 \end{array}$$

$$\Rightarrow H^i(F_1) = \begin{cases} \ker d'' / \text{Im } d', & i=1 \\ \ker d'', & i=0 \\ 0, & \text{otherwise} \end{cases}$$

$$\begin{array}{ccccc}
 H^*(F_0) & \longrightarrow & 0 & \longrightarrow & H^*(F_1) \xrightarrow{\text{+1}} \\
 \underbrace{\hspace{10em}} & & & & \\
 H^0(F_0) & \longrightarrow & 0 & \longrightarrow & H^0(F_1) \\
 \underbrace{\hspace{10em}} & & & & \\
 0 \rightarrow H^0(F_0) \rightarrow \bigoplus_i \Gamma(U_i) \xrightarrow{d'} & & H^0(F_1) & & 
 \end{array}$$

$$\Rightarrow H^i(X) = H^i(F_0) = \begin{cases} \ker d'' / \text{Im } d', & i=2 \\ \ker d' / \text{Im } d', & i=1 \\ \ker d', & i=0 \\ 0, & \text{otherwise} \end{cases}$$

Rmk. When  $\{U_i\}_{i \in \Delta}$  is not a good cover,  
one needs Čech-to-derived functor spectral seq to compute  $H^*(X)$ .

Rmk. Closed cover can be computed in a similar way.

Ex. For an closed cover  $X = Z_1 \cup Z_2$ , deduce the SES

$$0 \rightarrow \underline{\mathbb{Q}}_X \xrightarrow{+1} i_! \underline{\mathbb{Q}}_{Z_1} \oplus i_! \underline{\mathbb{Q}}_{Z_2} \xrightarrow{+1} i_! \underline{\mathbb{Q}}_{Z_1 \cup Z_2} \rightarrow 0$$

$$\xleftarrow{+1} \underline{\mathbb{Q}}_X \xleftarrow{i_* i^!} \underline{\mathbb{Q}}_X \oplus i_* i^! \underline{\mathbb{Q}}_X \xleftarrow{i_* i^!} \underline{\mathbb{Q}}_X \quad (3')$$

In general, one would get

$$0 \rightarrow F \xrightarrow{+1} i_! i^* F \oplus i_* i^* F \xrightarrow{+1} i_! i^* F \rightarrow 0$$

$$\xleftarrow{+1} DF \xleftarrow{i_* i^!} DF \oplus i_* i^! DF \xleftarrow{i_* i^!} DF$$

$\downarrow$  different  $i$  here...

Ex. Use the closed cover, try to compute

$$H_{\text{BM}}^*(Z; \mathbb{Q}) \quad Z := \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 z_2 z_3 = 0\} = \begin{array}{c} \text{a cube} \\ \text{with a cut} \end{array}$$

or  $Z = \{z_1, z_2, z_3 \mid (z_1 + z_2 + z_3) = 0\}$

In the first case,  $\mathbb{C}^3 - Z \cong \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  You can use this fact to double check results.

closed cover of  $Z \approx$  "excessive cover" of  $\mathbb{C}^3 - Z$   
 $\neq$  open cover of  $\mathbb{C}^3 - Z$   
no relation

Rmk. stratification & open cover are two main tools to extract topological information.  
 They appear with different names in different fields.  
 Once you realize them, you can apply the six-functor machine to analyze structures.

stratification with extra properties

|   |                        |
|---|------------------------|
| { | CW cplx                |
|   | triangulization        |
|   | dessin d'enfant        |
|   | affine paving          |
|   | Whitney stratification |
|   | :                      |

Q. How to construct stratifications?

A. For me, there are two efficient methods:

|   |                    |
|---|--------------------|
| { | orbit of gp action |
|   | Morse theory       |

That's why some geometrical problems are finally reduced to combinatorical / analytic problems.  
 Other fields come to geometry by providing stratifications.

In fact, there is only one method:

find a fct  $f: X \rightarrow Y$ , and get stratification of  $X$  from  $Y$ .  
 get better stratification by analyzing  $f$

|      |                        |                                                                 |
|------|------------------------|-----------------------------------------------------------------|
| E.g. | 1. Morse theory        | $f: X \rightarrow \mathbb{R}$                                   |
|      | 2. tessellation        | $f: H \rightarrow H/\Gamma$                                     |
|      | 3. semi-continuous fct | $f: X \rightarrow \mathbb{N}_{\geq 0}$ e.g. $f(p) = \dim T_p X$ |
|      | 4. my master thesis    | $f: Gr(X) \rightarrow Gr(S) \times Gr(X/S)$                     |
|      | 5. orbit of gp action  | $f: X \rightarrow [X/G] \rightarrow X/G$                        |

### 3. filtrations from chain complex [Stack Project, 0118]

Lots of filtrations are obtained just from the naive complex.

"chopping pork ribs"

Consider a chain complex  $C$ :

$$\dots \xrightarrow{d^{-2}} C^{-2} \xrightarrow{d^{-1}} C^{-1} \xrightarrow{d^0} C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots$$

One can make a "stupid" truncation

$$\begin{array}{ccccccccc} \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \end{array}$$

which is denoted by

$$0 \longrightarrow \sigma_{\geq 0} C \longrightarrow C \longrightarrow \sigma_{\leq -1} C \longrightarrow 0$$

One can also make a canonical truncation

$$\begin{array}{ccccccccc} \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & \ker d' & \longrightarrow & 0 \longrightarrow 0 \longrightarrow \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{d^{-2}} & C^{-2} & \xrightarrow{d^{-1}} & C^{-1} & \xrightarrow{d^0} & C^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \text{coker } d^0 & \xrightarrow{d^1} & C^1 \xrightarrow{d^2} C^2 \xrightarrow{d^3} \dots \end{array}$$

which is denoted by

$$0 \longrightarrow \tau_{\leq 0} C \longrightarrow C \longrightarrow \tau_{\geq 1} C \longrightarrow 0$$

Using these truncations, one can easily construct filtrations:

$$0 \subset \cdots \subset \tau_{\geq 1} C \overset{C'[-1]}{\subset} \tau_{\geq 0} C \overset{C'}{\subset} \tau_{\geq -1} C \subset \cdots \subset C$$

$$0 \subset \cdots \subset \tau_{\leq -1} C \overset{H^0(C)}{\subset} \tau_{\leq 0} C \overset{H^0(C)[1]}{\subset} \tau_{\leq 1} C \subset \cdots \subset C$$

Rmk. 1. These two filtrations have opposite directions!

(a striking feature for me)

2. The "stupid" truncation extracts pieces of the chain cplx, while the canonical truncation extracts cohomology. ( $\text{Ker}/\text{Im}$ )  
Therefore, only the canonical truncation can be defined on the derived category.

This information is culmulated in the standard/natural t-structure  $(D_{\leq 0}, D_{\geq 0})$ .

One has adjoint factors:

$$\begin{array}{ccccc} & l_{\leq 0} & & T_{\geq 1} & \\ D_{\leq 0} & \begin{array}{c} \xrightarrow{\quad} \\ \perp \\ \xleftarrow{\quad} \end{array} & D & \begin{array}{c} \xleftarrow{\quad} \\ \perp \\ \xrightarrow{\quad} \end{array} & D_{\geq 1} \\ \end{array}$$

where  $l_{\leq 0}, l_{\geq 1}$  are natural inclusions.

The following notations are from: <https://ncatlab.org/nlab/show/t-structure>

$D_{\leq 0}$ : t-co-connective objects

$D_{\geq 0}$ : t-connective objects

$\tau_{\geq 0}$ : connective cover

Let's apply these filtrations!

#### 4. filtration by $H^i(F)$

Assume that  $B$  is loc. contractable.  $\Leftarrow$  Can we remove this condition?

(+  $B$  simply connected to simplify the argument)

Ex. Suppose that  $\pi: E \rightarrow B$  is an oriented  $S^k$ -bundle.

Analyze  $R\pi_* \underline{\mathbb{Q}}_E$ , and apply  $R\pi_{B,*}$  to get the Gysin sequence.

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[+1]{ev_\pi \wedge}$$

Hint. consider the easy case ( $B = \{*\}$ ,  $E \cong S^k \times B$ ) to see the shape of  $R\pi_* \underline{\mathbb{Q}}_E$ .

Q. Why does  $\pi_*: D^b(E) \rightarrow D^b(B)$  takes injective objects to  $\pi_{B,*}$ -acyclic objects?

A. [Vo02, p195] By definition,  $\pi_*$  takes flasque sheaves to flasque sheaves.

Similarly,  $\pi_!$  takes c-soft sheaves to c-soft sheaves.

Rmk. 1. Here we can't use the "stupid" truncation,

because  $R\pi_* \underline{\mathbb{Q}}_E$  lies in the derived category.

2. You can generalize it to fiber bundle. then

you will get the Leray-Serre spectral sequence.

Think how the following conditions simplify the final results.

①  $\pi$  is oriented  $S^k$ -bundle

②  $B$  is simply-connected

③ (Leray-Hirsch)  $H^*(E; \mathbb{Q}) \rightarrow H^*(F; \mathbb{Q})$  is surjection

④  $\pi_!(B)$  acts on  $H^*(F)$  trivially.

3. This is also a special case of Grothendieck-Serre spectral sequence.

▽  $R^i \pi_* \underline{\mathbb{Q}}_E = H^i(R\pi_* \underline{\mathbb{Q}}_E)$

$\neq$  the  $i$ -th term in the "chain cplx"  $R\pi_* \underline{\mathbb{Q}}_E$  (undefined)

## 5. filtration by $\mathcal{F}$

For a single sheaf  $\mathcal{F}$ , we have no way to produce filtrations.

Since we only care about  $H^i(\mathcal{F})$ , we may replace  $\mathcal{F}$  by another complex  $C^\bullet$ , which is usually achieved by resolution:

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varepsilon} C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \dots \quad \text{exact}$$

i.e.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{F} & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots \\ & & \downarrow \varepsilon & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^0 & \xrightarrow{d^0} & C^1 & \xrightarrow{d^1} & C^2 \xrightarrow{d^2} \dots \end{array} \quad \text{iso in } \mathcal{D}(X)$$

Then, one can use "stupid" truncation to get filtrations, and finally spectral sequences.  
(The canonical truncation won't give you anything new)

E.p. When  $C^\bullet$  is  $\Gamma$ -acyclic, the spectral seq deg. and  
 $H^i(X; \mathcal{F}) \cong H^i(\Gamma(C^\bullet))$

E.g. The spectral seqs in Section 1,2 are special cases.

## Coefficient spectral seq

E.g. For  $r \in \mathbb{Z}_{>0}$ , consider the SES

$$0 \longrightarrow \underline{\mathbb{Z}} \xrightarrow{x^r} \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow 0$$

which induces the iso

$$\underline{\mathbb{Z}/r\mathbb{Z}} \cong [\rightarrow \underline{\mathbb{Z}} \xrightarrow{x^r} \underline{\mathbb{Z}} \longrightarrow]$$

the triangle

$$\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}/r\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}[1] \xrightarrow{+1}$$

and the LES

$$H^k(X; \mathbb{Z}) \longrightarrow H^k(X, \underline{\mathbb{Z}/r\mathbb{Z}}) \longrightarrow H^{k+1}(X, \underline{\mathbb{Z}}) \xrightarrow{+1}$$

Ex. Compute  $H^k(\mathbb{R}\mathbb{P}^2; \underline{\mathbb{Z}/8\mathbb{Z}})$  in this way.

$$A: H^k(\mathbb{R}\mathbb{P}^2; \underline{\mathbb{Z}/8\mathbb{Z}}) = \begin{cases} \underline{\mathbb{Z}/8\mathbb{Z}}, & k=0 \\ \underline{\mathbb{Z}/2\mathbb{Z}}, & k=1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Rmk. In general, one can get coefficient spectral seq [FF16, 20.5.B, p318] through the coefficient seq.  $G$  abelian

$$\dots \xrightarrow{\partial_2} \underline{G}_1 \xrightarrow{\partial_1} \underline{G}_0 \xrightarrow{\epsilon} \underline{G} \longrightarrow 0$$

Q. Can we recover the inflation - restriction seq in this way?

$$0 \longrightarrow H^*(G/H, M^H) \xrightarrow{\text{Inf}} H^*(G, M) \xrightarrow{\text{Res}} H^*(H, M)$$

A: I think no. We haven't introduce  ${}^L\otimes$  &  $R\text{Hom}$ .

## Singular cohomology

We present a defective proof in [Voo2, Theorem 4.47].

The bug has been discussed in the stack exchange:

<https://math.stackexchange.com/questions/1794725/detail-in-the-proof-that-sheaf-cohomology-singular-cohomology>

|                 |                     |                             |                                      |
|-----------------|---------------------|-----------------------------|--------------------------------------|
| [Stackexchange] | $\mathcal{S}^k$     | $\widetilde{\mathcal{S}}^k$ | $\mathcal{S}^k(X)_0$                 |
| [Voo2]          | $C_{\text{sing}}^k$ | $C_{\text{sing}}^k$         | $C_{\text{sing}}^k(U, \mathbb{Z})_0$ |

$X \in \mathbf{Top}$ .

Def The presheaf  $\mathcal{S}^k \in \text{Psh}(X)$  is defined as

$$\begin{aligned}\mathcal{S}^k(U) &= \text{singular } k\text{-cochains on } U \\ &= \text{Hom}_{\text{Abel}}(\text{Sing}_k(U), \mathbb{Z})\end{aligned}$$

where

$$\begin{aligned}\text{Sing}_k(U) &= \text{singular } k\text{-chains on } U \\ &= \langle \sigma : \Delta^k \rightarrow U \rangle_{\text{free, Abel}}\end{aligned}$$

and  $\widetilde{\mathcal{S}}^k := (\mathcal{S}^k)^{\text{sh}}$  is called the sheaf of singular cochains.

Def

$$H_{\text{sing}}^k(X, \mathbb{Z}) := H^k(\mathcal{S}(X))$$

Thm For  $X$  loc contractible,

$$H^k(X, \mathbb{Z}) \cong H_{\text{sing}}^k(X, \mathbb{Z})$$

Sketch of the wrong proof: Still hard. I don't check details.

1. Construct a complex of sheaves

$$0 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathcal{S}}^0 \longrightarrow \widetilde{\mathcal{S}}^1 \longrightarrow \widetilde{\mathcal{S}}^2 \longrightarrow \dots \quad (\star)$$

2. Show that  $(\star)$  is exact when  $X$  is loc contractible.  $\Rightarrow \mathbb{Z} \cong \widetilde{\mathcal{S}}^0$

3. Show that  $\widetilde{\mathcal{S}}^k$  is flabby, then

$$H^k(X, \mathbb{Z}) \cong H^k(\widetilde{\mathcal{S}}^k(X))$$

Reduced to show:  $H^k(\widetilde{\mathcal{S}}^k(X)) \cong H^k(\mathcal{S}^k(X))$

4. Consider the exact sequence of chain complexes

$$0 \longrightarrow \mathcal{S}^k(X)_0 \longrightarrow \mathcal{S}^k(X) \longrightarrow \widetilde{\mathcal{S}}^k(X) \longrightarrow \mathcal{S}^k(X)^0 \longrightarrow 0$$

shows that:

$$\textcircled{1} \quad \mathcal{S}^k(X)^0 = 0 \quad \text{This is wrong. We ignore this bug...}$$

$$\textcircled{2} \quad \mathcal{S}^k(X)_0 = \left\{ \alpha \in \mathcal{S}^k(U) \mid \begin{array}{l} \exists \text{ open covering } \{U_i\}_i \text{ of } U \text{ s.t.} \\ \alpha|_{U_i} = 0 \quad \forall i \end{array} \right\}$$

$$\textcircled{3} \quad H^k(\mathcal{S}^k(X)_0) = 0$$

Finally,  $H^k(X, \mathbb{Z}) \cong H^k(\widetilde{\mathcal{S}}^k(X)) \cong H^k(\mathcal{S}^k(X)) \stackrel{\text{def}}{=} H_{\text{sing}}^k(X, \mathbb{Z})$ .

Q. Can one shows  $H^k(\mathcal{S}^k(X)^0) = 0$  to replace the bug in 4.①?

## 6. Hodge related filtration

### Smooth de Rham resolution

The classical story is the real de Rham cohomology.

Thm For  $X$  mfld, one has iso

$$H^k(X; \mathbb{R}) \cong H_{dR}^k(X; \mathbb{R}) := H^k(\Omega_X(X))$$

Here,

$\Omega_X^k$  = sheaf of smooth k-forms with coefficient  $\mathbb{R}$

[https://en.wikipedia.org/wiki/De\\_Rham\\_cohomology#Proof](https://en.wikipedia.org/wiki/De_Rham_cohomology#Proof)

#### Sketch of proof

1. Construct a cplx of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_X^0 \xrightarrow{d^0} \Omega_X^1 \xrightarrow{d^1} \Omega_X^2 \xrightarrow{d^2} \dots (*)$$

2. Show that  $(*)$  is exact by Poincaré lemma. [Cl17, Lec1, Lemma 5]

3. Show that  $\Omega_X^i$  is  $\Gamma$ -acyclic [Cl17, Lec1, Lemma 4], then

$$H^k(X; \mathbb{R}) \cong H^k(\Omega_X(X)) \stackrel{\text{def}}{=} H_{dR}^k(X; \mathbb{R})$$

POU  $\Rightarrow$  fine  $\xrightarrow{\text{paracpt}}$  acyclic

Rmk. You can change coefficient from  $\mathbb{R}$  to  $\mathbb{C}$ , where

$$\begin{aligned} \Omega_{X,C}^k &= \text{sheaf of smooth k-forms with coefficient } \mathbb{C} \\ &= \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} = \Omega_X^k \otimes_{\mathbb{R}} \mathbb{C} \\ &\quad \uparrow \text{it can be viewed as } \mathbb{R}\text{-module} \end{aligned}$$

then

$$H^k(X; \mathbb{C}) \cong H_{dR}^k(X; \mathbb{C}) := H^k(\Omega_{X,C}(X)).$$

Now we step into the world of cplx mfld.

▽ Hereafter,  $X$ : cplx mfld,

$$\Omega_X^k = \text{sheaf of holomorphic } k\text{-forms} = \Omega_X^{k,0}$$

$$= \{ f dz_1 \wedge \dots \wedge dz_k \mid f: X \rightarrow \mathbb{C} \text{ holomorphic} \}$$

$\uparrow_{\text{no } d\bar{z}}$

while

$$A_{sm}^k = \Omega_{X_{sm}}^k = \text{sheaf of smooth } k\text{-forms}$$

$$= \{ f dx_1 \wedge \dots \wedge dx_k \mid f: X \rightarrow \mathbb{R} \text{ smooth} \}$$

$\uparrow d\bar{z} \text{ is allowed}$

$$A^k := \Omega_{X_{sm}, \mathbb{C}}^k = \{ f dx_1 \wedge \dots \wedge dx_k \mid f: X \rightarrow \mathbb{C} \text{ smooth} \} = \bigoplus_{p+q=k} A^{p,q}$$

$$\begin{array}{ll} \Omega: \text{holomorphic} & \Omega^k = \Omega^{k,0} \\ A: C^\infty & A^k = \bigoplus_{p+q=k} A^{p,q} \end{array}$$

This abuse of notation confuses me a lot.

For example, we still have the resolution (called the holomorphic de Rham resolution)

$$0 \longrightarrow \underline{\mathbb{C}} \longrightarrow \Omega_X^0 \xrightarrow{d^1} \Omega_X^1 \xrightarrow{d^2} \Omega_X^2 \xrightarrow{d^3} \dots \quad (**)$$

but  $\Omega_X^k$  is no longer  $\Gamma$ -acyclic, and

$$\dim_{\mathbb{C}} \Gamma(X; \Omega_X^k) < +\infty \quad \text{for } X \text{ proj.}$$

$$H^q(X; \Omega_X^p) \longrightarrow H^q(X; \Omega_X^{p+1}) \text{ is 0-map} \quad \text{for } X \text{ proj.}$$

Find reference for the 0-map. ( $E_1$ -degeneration)

Do we get the 0-map by the dim argument + Hodge decomposition?

Do we have any elegant proof to replace it?

[Vo 02, p214]

1. Let  $S$  be a compact complex surface.

- (a) Show that the Frölicher spectral sequence of  $S$  degenerates at  $E_3$  for degree reasons. What are the possibly non-zero differentials  $d_2$ ?

## Holomorphic de Rham resolution & Hodge-Tate

For simplicity, let's consider a proj curve  $X/\mathbb{C}$ .

cpt kähler is enough

Where do we use kähler condition?

### Hodge de-Rham

$$\underline{\mathcal{C}} \cong [\Omega^0 \xrightarrow{0} \Omega^1]_1$$

"stupid" filtration degenerates to the triangle

$$\begin{array}{ccccccc} \Omega^1[-1] & \longrightarrow & \underline{\mathcal{C}} & \longrightarrow & \Omega^0 & \xrightarrow{+1} & \\ & & \curvearrowright H^1(X; \Omega^1) & \longrightarrow & H^2(X; \mathbb{C}) & \longrightarrow & H^2(X; \Omega^0) \\ & & \curvearrowright H^0(X; \Omega^1) & \longrightarrow & H^1(X; \mathbb{C}) & \longrightarrow & H^1(X; \Omega^0) \\ & & \curvearrowright H^1(X; \Omega^1) & \xrightarrow{=0} & H^0(X; \mathbb{C}) & \longrightarrow & H^0(X; \Omega^0) \end{array}$$

$\rightsquigarrow$  Hodge filtration:

$$0 \longrightarrow H^0(X; \Omega^1) \longrightarrow H^1(X; \mathbb{C}) \longrightarrow H^1(X; \Omega^0) \longrightarrow 0$$

Hodge Tate similar to Euler class (I'm foolish in this field)

$$\nu: X_{\text{proét}} \xrightarrow{\widehat{\mathcal{O}_X}} X_{\text{ét}}$$

$$\begin{array}{ccccccc} R^0 \nu_* \widehat{\mathcal{O}_X} & \longrightarrow & R \nu_* \widehat{\mathcal{O}_X} & \longrightarrow & R^1 \nu_* \widehat{\mathcal{O}_X}[-1] & \xrightarrow{+1} & \\ \parallel & & \parallel & & \parallel & & \\ \Omega_X^0 & & & & \Omega_X^1[-1] & & \\ & & & & & & \\ & & \curvearrowright H^1(X; \Omega^0) & \xrightarrow{=0} & H^2(X; \mathbb{C}) & \longrightarrow & H^2(X; \Omega^1) \\ & & \curvearrowright H^1(X; \Omega^0) & \longrightarrow & H^1(X; \mathbb{C}) & \longrightarrow & H^0(X; \Omega^1) \\ & & & & & & \xrightarrow{0} \\ & & & & H^0(X; \Omega^0) & \longrightarrow & H^1(X; \Omega^1) \end{array}$$

$$\rightsquigarrow 0 \longrightarrow H^0(X; \Omega^0) \longrightarrow H^1(X; \mathbb{C}) \longrightarrow H^0(X; \Omega^1) \longrightarrow 0$$

"Euler class of  $\nu$ " is trivial?

Q. Combining these two filtrations, does it give Hodge decomposition? in the p-adic sense

Hodge decomposition in the algebraic surface case.

$$\begin{array}{ccc}
 \chi(\mathcal{O}_X) \chi(\Omega_X) \chi(\omega_X) & & e := \chi_{\text{top}}(X) \\
 \downarrow & & \downarrow \\
 \begin{matrix} 1 \\ q \\ Pg \\ Pg \end{matrix} & \begin{matrix} 1 \\ q \\ h' \\ 1 \end{matrix} & \begin{matrix} b_4 \\ b_3 \\ b_2 \\ b_1 \\ b_0 \end{matrix} \\
 \begin{matrix} c_2 = e \\ c_1 = K^2 = 12\chi(\mathcal{O}_X) - e \end{matrix} & \begin{matrix} q \\ 1 \end{matrix} & \begin{matrix} h^{2,2} \\ h^{1,2} \quad h^{2,1} \\ h^{0,2} \quad h^{1,1} \quad h^{2,0} \\ h^{0,1} \quad h^{1,0} \\ h^{0,0} \end{matrix}
 \end{array}$$

## Dolbeault resolution [Vo02, p94]

For  $\pi_E: E \rightarrow X$  holo v.b., let

$\mathcal{A}_E^{p,q} = \text{sheaf of smooth } (p,q)\text{-forms with coefficient as sections in } E$

$\mathcal{E} = \text{sheaf of holo sections of } E \triangleq \Omega_E^{0,0}$

One can construct a cplx of sheaves

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{A}_E^{0,0} \xrightarrow{\bar{\partial}} \mathcal{A}_E^{0,1} \xrightarrow{\bar{\partial}} \mathcal{A}_E^{0,2} \xrightarrow{\bar{\partial}} \dots \quad (***)$$

Show that  $(***)$  is exact, and  $\mathcal{A}_E^{p,q}$  is  $\pi_*$ -acyclic.

$$\Rightarrow H^k(X; \mathcal{E}) \cong H^k(\mathcal{A}_E^{0,\cdot}(X))$$

E.g. When  $\mathcal{E} = \Omega_X^p$ ,  $\mathcal{A}_E^{p,q} = \mathcal{A}^{p,q}$ , one gets

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots \quad (***)'$$

$$\Rightarrow H^q(X; \Omega_X^p) \cong H^q(\mathcal{A}_E^{p,\cdot}(X)) \stackrel{\text{def}}{=} H_{\bar{\partial}}^{p,q}(X)$$

Rmk.  $(*)$ ,  $(**)$  &  $(***)'$  are related by the following double complex:

$$\begin{array}{ccccccc}
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \uparrow \bar{\partial} \\
 0 & \longrightarrow & \mathcal{A}^{0,1} & \xrightarrow{\partial} & \mathcal{A}^{1,1} & \xrightarrow{\partial} & \mathcal{A}^{2,1} \xrightarrow{\partial} \mathcal{A}^{3,1} \xrightarrow{\partial} \dots \\
 & \uparrow \bar{\partial} \\
 0 & \longrightarrow & \mathcal{A}^{0,0} & \xrightarrow{\partial} & \mathcal{A}^{1,0} & \xrightarrow{\partial} & \mathcal{A}^{2,0} \xrightarrow{\partial} \mathcal{A}^{3,0} \xrightarrow{\partial} \dots \\
 & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array}$$