

Eine Woche, ein Beispiel

4.20 hyperelliptic curves in abelian varieties

Ref:

[LR22]: Herbert Lange and Rubí E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

<https://math.stackexchange.com/questions/710899/prym-variety-associated-to-an-%c3%a9tale-cover-of-degree-2-of-an-hyperelliptic-curve>

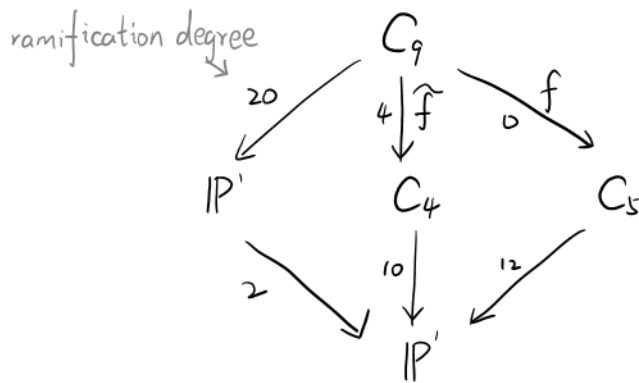
<https://mathoverflow.net/questions/402049/induced-action-on-prym-variety>

Goal: Describe some curves (maybe singular) C in A , and describe their degree and the monodromy group.

E.g. 1

Covers

$C_9 = \{y^2 = \prod_{j=1}^{10} (x^2 - j)\}$ has the following covers:
 $\text{Aut}(C_9) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$



where

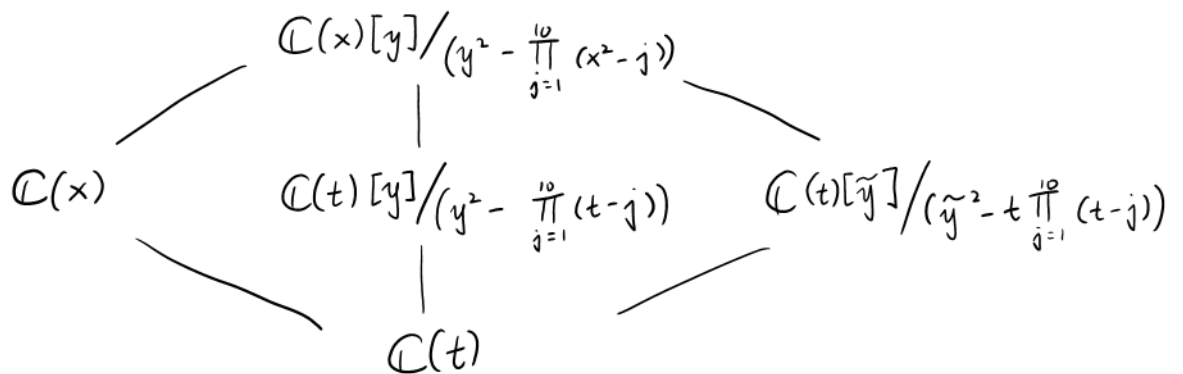
$$C_4 = \{y^2 = \prod_{j=1}^{10} (t - j)\}$$

$$C_5 = \{\tilde{y}^2 = t \prod_{j=1}^{10} (t - j)\}$$

$$t = x^2$$

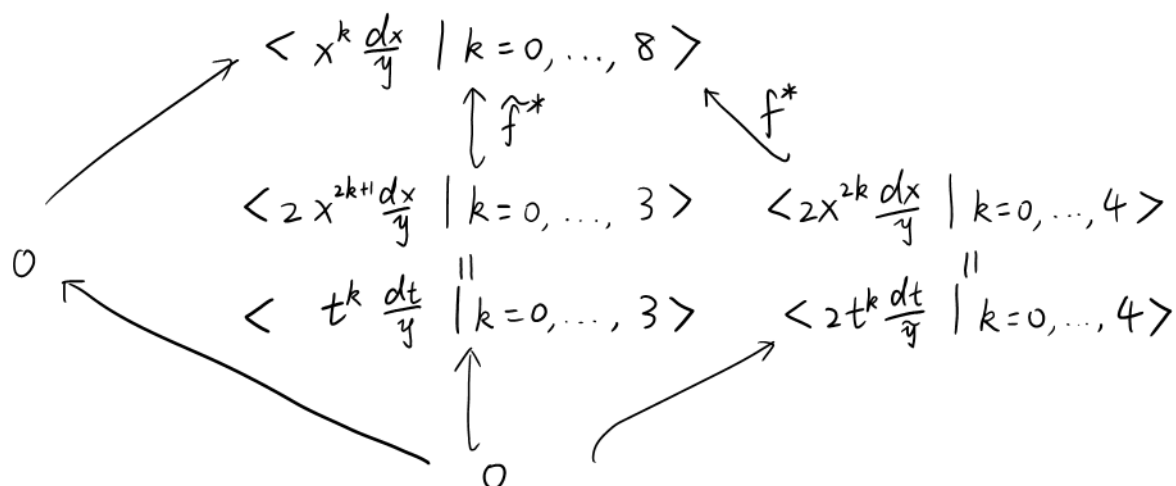
$$\tilde{y} = xy$$

The crspd field extension:



Global differential forms

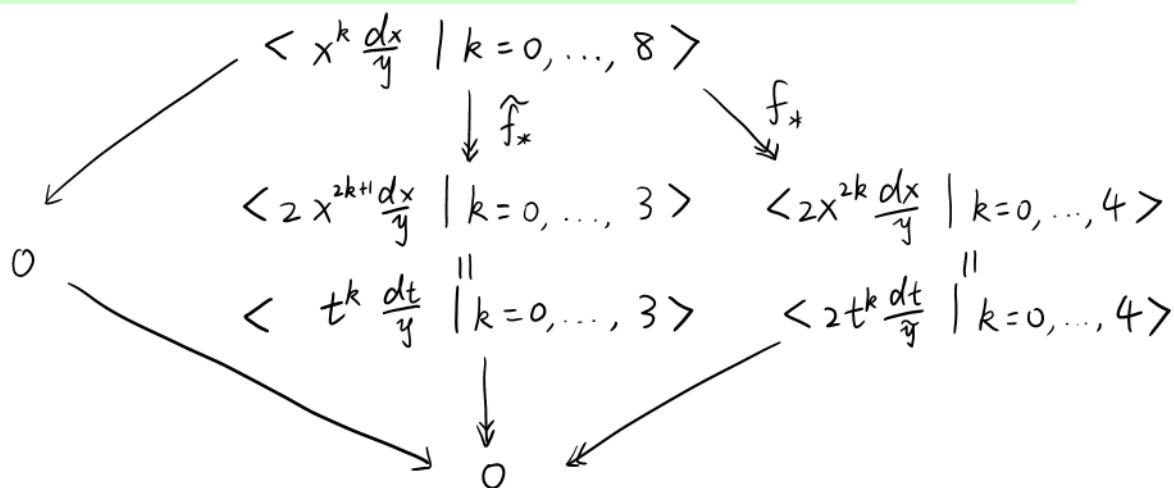
Pulling back differential forms give the following maps:



Therefore,

$$H^0(C_9; \omega_{C_9}) \cong \tilde{f}^* H^0(C_4; \omega_{C_4}) \oplus f^* H^0(C_5; \omega_{C_5}) \quad (1)$$

Since the maps are (ramified) covering, we have the maps in opposite direction: (which corresponds to pulling back of divisors)



However, since $\text{Jac}(C) = H^0(C; \omega_C)^* / H_1(C; \mathbb{Z})$, we are working on the dual spaces. The notations are again switched:

$$\begin{aligned} f^* &\rightsquigarrow N_{mf} \\ f_* &\rightsquigarrow f^* \end{aligned}$$

One may get

$$H^0(C_9; \omega_{C_9})^* \cong \tilde{f}^* H^0(C_4; \omega_{C_4})^* \oplus f^* H^0(C_5; \omega_{C_5})^* \quad (2)$$

different meaning compared with (1)!

Curve in Prym variety

Define A as the quotient of Jacobians, i.e.,

$$A := \text{Jac}(C_9) / f^* \text{Jac}(C_5) \cong \text{Prym}(C_9/C_5)$$

$$\begin{array}{ccccccc}
 & & C_9 & \longrightarrow & C_4 & \xrightarrow{\quad} & \text{Jac}(C_4) \\
 & & \downarrow A|_{C_9} & & \downarrow & \swarrow \exists! \text{ isogeny} & \\
 0 \longrightarrow & \text{Jac}(C_5) & \xrightarrow{f^*} & \text{Jac}(C_9) & \xrightarrow{\pi} & A & \longrightarrow 0
 \end{array} \quad (3)$$

- Prop.
0. A is isogenous to $\text{Jac}(C_4)$;
 1. $f^*: \text{Jac}(C_5) \rightarrow \text{Jac}(C_9)$ is injective;
 2. $\pi \circ A|_{C_9}$ is not injective, it factors through C_4 ;
 3. $C_4 \rightarrow A$ is generically injective;
 4. $C_4 \rightarrow A$ produces a sm image of A , outside of non-injective locus.