

# Eine Woche, ein Beispiel

## 5.19. Weierstrass point

references:

[https://en.wikipedia.org/wiki/Weierstrass\\_point](https://en.wikipedia.org/wiki/Weierstrass_point)

[https://en.wikipedia.org/wiki/Inflection\\_point](https://en.wikipedia.org/wiki/Inflection_point)

Klein quartic has 24 inflection points:

[https://www.uio.no/studier/emner/matnat/math/MAT2000/v24/projects/2023\\_the\\_klein\\_quartic\\_and\\_its\\_n\\_weierstrass\\_points.pdf](https://www.uio.no/studier/emner/matnat/math/MAT2000/v24/projects/2023_the_klein_quartic_and_its_n_weierstrass_points.pdf)

curve of genus  $>0$  don't have single simple pole:

<https://math.stackexchange.com/questions/2841459/finding-a-meromorphic-function-on-a-compact-riemann-surface-with-prescribed-zero>

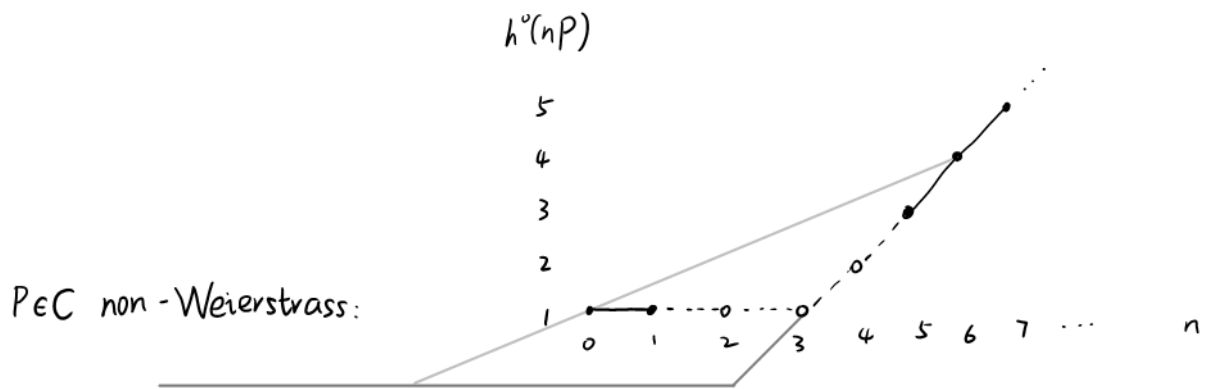
Setting:  $C$ : proj sm curve /  $\mathbb{C}$   $\bar{\mathbb{C}} = \mathbb{C}$ ,  $\text{char } \mathbb{C} = 0$

$h^0(\mathcal{O}(nP)) \backslash n$		0	1	2	3	4	5	6	7	8	$g(g^2-1)$
$g(C)$											
$g=3$ :	0	1	2	3	4	5	6	7	8	9	0
	1	1	1	2	3	4	5	6	7	8	0
	2	1	1	?	2	3	4	5	6	7	6
	3	1	1	?	?	?	3	4	5	6	24
	4	1	1	?	?	?	?	?	4	5	60
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	non-Weierstrass	1	1	1	1	2	3	4	5	6	$\emptyset$
	general quartic	1	1	1	2	2	3	4	5	6	$1 \times 24$
	W: e.g. Klein quartic	1	1	1	2	3	3	4	5	6	$2 \times 12$
	W: Fermat quartic	1	1	2	2	3	3	4	5	6	$3 \times 8$
	W: hyperelliptic case										

by Clifford's thm,  $h^0(\mathcal{O}(nP)) \leq \frac{n}{2} + 1$ ,  
so the hyperelliptic case reaches the limit.

Finiteness of Weierstrass point:

<https://math.stackexchange.com/questions/4719889/is-this-proof-that-the-number-of-weierstrass-points-on-a-compact-riemann-surface>



$g=3$  case

$$\Leftrightarrow h^0(\mathcal{O}(gP)) = 1$$

$P \in C$  Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(gP)) \geq 2$$

$\Leftrightarrow \exists f \in K(C)$ ,  $f$  has a single pole at  $P$ ,  
with  $\text{ord}_P(f) \geq -g$

$$\Leftrightarrow h^0(K - gP) \geq 1$$

e.p.

$g=2$ :  $P \in C$  Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(2P)) = 2$$

$\Leftrightarrow \exists f \in K(C)$ ,  $f$  has a single double pole at  $P$

$g=3$ :  $P \in C$  Weierstrass

$$\Leftrightarrow h^0(\mathcal{O}(3P)) \geq 2$$

$\Leftrightarrow \exists f \in K(C)$ ,  $f$  has a single triple pole at  $P$

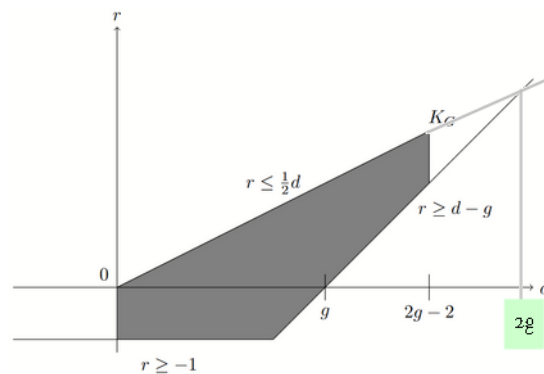


FIGURE 1. Possibilities for the degree and rank of a divisor.

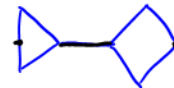
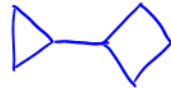
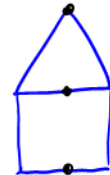
A tropical version for analog. I could draw it also for the classical AG cases, but I'm lazy.  
Photo comes from [MA 764: Chip Firing, Lec 9]:  
<https://www.ms.uky.edu/~dhje223/MA%20764%20Spring%202019.html>

A case in tropical algebraic curve where the "Weierstrass points are not finite":

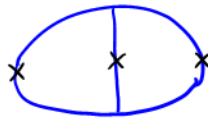
$g = 2$



"Weierstrass pts":  
 $\text{rk}(z_p) = 1$



comparison between tropical & classical.



See this article for more examples of Weierstrass points on tropical alg curves:

<https://arxiv.org/pdf/2303.07729>

See [Theorem 1.7] which computes the total weight of the Weierstrass locus:  $d - r + rg$ .

When  $D=K$ ,  $d=2g-2$ ,  $r=g-1$ , the total weight is  $g^2-1$ .

Notice that the definition of weight is slightly changed.

The Dhar's burning algorithm is mainly used for eliminating negative divisors.

Step1: blow (burn negative divisors)

Step2: suck (attract positive divisors)

This process looks like the process when I suck the river snail, therefore, I call it as "嗦田螺算法". It's an effective algorithm in determining if a divisor is effective.

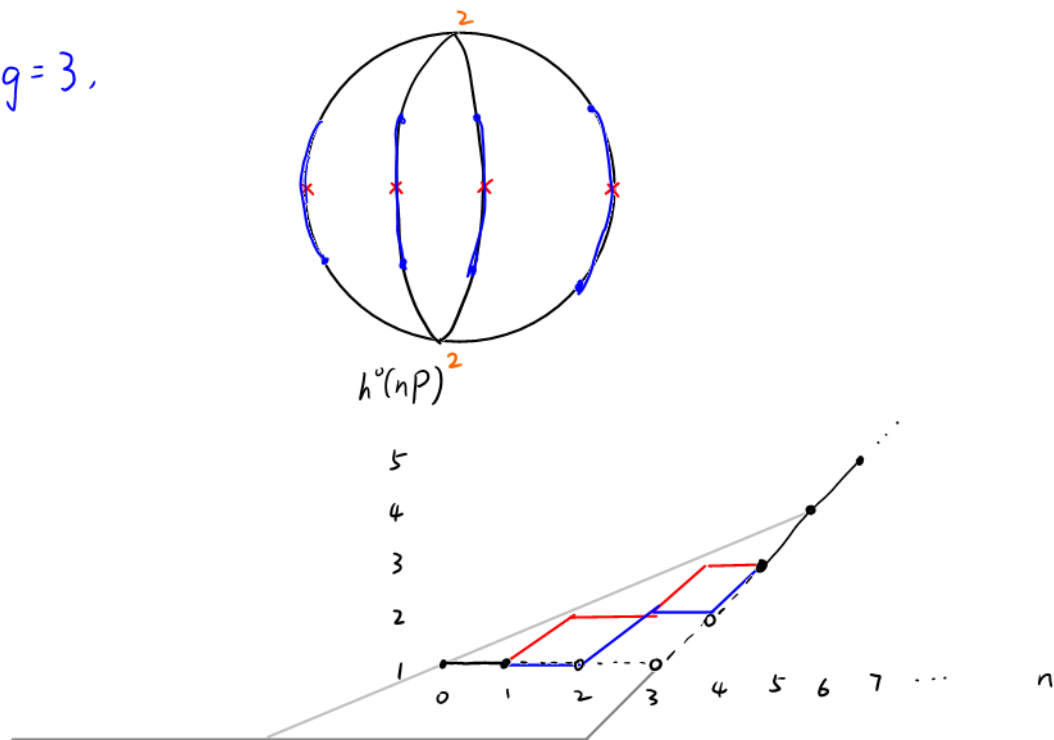
Some differences between classical algebraic curves and tropical algebraic curves:

We have Dhar's burning algorithm for tropical algebraic curves, which is not so explicit in classical case. (Maybe I'm wrong: the hyperelliptic curves can be seen in [Theorem 4.1.6]: <https://algant.eu/documents/theses/dipiazza.pdf>)

We can also divide  $K$  into two canonical parts.

In classical algebraic curves, the Weierstrass point is finite, which is not true in tropical algebraic curves.

E.g.  $g=3$ ,

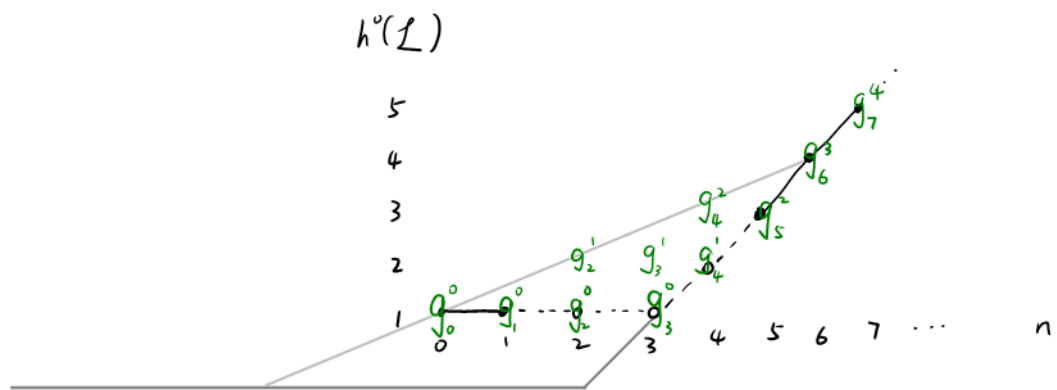


$g=3$  case

By Riemann-Roch,

$$r(kP) = r(K - kP) + k - g + 1$$

$$\underline{\underline{g=3}} \quad \begin{cases} r(K - 2P) & k=2 \\ r(K - 3P) + 1 & k=3 \\ r(K - 4P) + 2 & k=4 \end{cases}$$



$$(V, \mathcal{L}) \in g_{\deg \mathcal{L}}^{\dim V}, \text{ where } \mathcal{O}_X \otimes_k V \subset \mathcal{O}_X \otimes_k H^0(X, \mathcal{L}) \rightarrow \mathcal{L}$$