Eine Woche, ein Beispiel

2.13. outer automorphism

We do something very elementary but tricky, and will later find out its connection to the advanced topic, like Teichmüller space.

1. outer automorphism group Out(G) / automorphism group Aut(G)

Ref:
https://en.wikipedia.org/wiki/Outer_automorphism_group
https://en.wikipedia.org/wiki/Automorphisms_of_the_symmetric_and_alternating_groups

Def. Let G be a group. We have a LES

$$1 \longrightarrow Z(G) \longrightarrow G \xrightarrow{\text{conj}} \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1$$

where $Z(G)$ is the center of G
$\mathrm{Aut}(G)$ is the automorphism of G
$\mathrm{Inn}(G) := \mathrm{Im}(\text{conj})$ is the inner automorphism of G
$\mathrm{Out}(G) := \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is the outer automorphism of G.

E.g. When G is commutative, $\mathrm{Inn}(G) = \mathrm{Id}$, $\mathrm{Out}(G) = \mathrm{Aut}(G)$.
$G = \mathbb{Z}$, $\mathrm{Aut}(\mathbb{Z}) = \{\pm 1\}$,
$G = \mathbb{Z}/m\mathbb{Z}$, see https://zhuanlan.zhihu.com/p/97195375 ← typo: ② ⇒ 2 $(m \geq 2)$
an easy result is that $\#\mathrm{Out}(\mathbb{Z}/m\mathbb{Z}) = \varphi(m)$.
e.p. $\mathrm{Aut}(\mathbb{Z}/p\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$

2. Reduced to indecomposable group

Main reference in this section:
http://www.math.hawaii.edu/~williamdemeo/latticetheory/Bidwell-AutomorphismsOfDirectProductsII-2008.pdf
There are also quite a lot of concrete examples. Examples are also fruitful here:
http://www.math.hawaii.edu/~williamdemeo/latticetheory/Bidwell-thesis-2006.pdf

In this section we suppose every group is finite. I doubt that it's also true for infinite group, but I didn't check the proof.

Def. A group G is indecomposable if

$$G \cong A \times B \Rightarrow A \cong \mathrm{Id} \text{ or } B \cong \mathrm{Id}.$$

Let H be an indecomposable finite group, and let $G = H \times \cdots \times H \hat{=} H_1 \times H_2 \times \cdots H_n \hat{=} H^n$

Case 1. [Thm 3.1] H is non-abelian, then $\mathrm{Aut}\, G = \mathcal{A} \rtimes S_n$, where

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} : \alpha_{ij} \in \begin{cases} \mathrm{Hom}\,(H_j, Z(H_i)) & i \neq j \\ \mathrm{Aut}\ H_i & i = j \end{cases} \right\}.$$

$$\begin{tikzcd} & & & S_n \arrow[dl, dashed] \arrow[d, "Id"] & \\ 0 \arrow[r] & \mathcal{A} \arrow[r] & \mathrm{Aut}\, G \arrow[r] & S_n \arrow[r] & 0 \end{tikzcd}$$

$S_n \to \mathrm{Aut}(\mathcal{A})$ by matrix conjugation

Case 2. H is abelian, then $H \cong \mathbb{Z}/p^r\mathbb{Z}$, $\mathrm{Aut}\, G \cong GL(n, \mathbb{Z}/p^r\mathbb{Z})$

See https://math.stackexchange.com/questions/34449/automorphism-group-of-an-abelian-group.
This is actually the special case of a theorem:
(from: https://math.stackexchange.com/questions/55262/the-automorphism-group-of-a-direct-product-of-abelian-groups-is-isomorphic-to-a)

Thm. If H_i are all abelian, then

$$\mathrm{Aut}\left(\bigoplus_{i=1}^{n} H_i\right) = \left\{ A = (a_{ij})_{i,j=1}^{n} \;\middle|\; \begin{array}{l} a_{ij} \in \mathrm{Hom}\,(H_i, H_j) \\ A \text{ is invertible} \end{array} \right\}$$

Thm Let H & N be two finite group with no common direct factor
(i.e., $H \cong A \times B$ $N \cong A \times C \Rightarrow A \cong Id$), then

$$\mathrm{Aut}\,(H \times N) = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \;\middle|\; \begin{array}{ll} \alpha \in \mathrm{Aut}(H) & \beta \in \mathrm{Hom}(H, Z(N)) \\ \gamma \in \mathrm{Hom}(N, Z(H)) & \delta \in \mathrm{Aut}(N) \end{array} \right\}$$

For a proof, see "Automorphisms of direct products of finite groups".
See also here: https://math.stackexchange.com/questions/1236571/automorphism-group-of-direct-product-of-groups

Cor. **Theorem 2.2.** *Let $G = H_1 \times \ldots \times H_n$ where no pair of the H_i $(1 \leq i \leq n)$ have a common direct factor. Then* $\mathrm{Aut}\ G \cong \mathcal{A}$ *where*

$$\mathcal{A} = \left\{ \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} : \ \alpha_{ij} \in \begin{cases} \mathrm{Aut}\ H_i & i = j \\ \mathrm{Hom}\ (H_j, Z(H_i)) & i \neq j \end{cases} \right\}.$$

Thus, the computation of $\mathrm{Aut}\, G$ reduced to the case where G is indecomposable.

Cor. One can compute the automorphism of any finite abelian group G, and also $\#\mathrm{Aut}(G)$.

Task: check if we can compute the automorphism group of f.g. abelian group in this way.

$\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{Out}(\mathbb{Z}^n) \cong GL(n, \mathbb{Z})$ is known.

2. D_n, D_∞ and Q_8

For a concrete proof in this section, see here: http://home.ustc.edu.cn/~yx3x/USTC/anonymousnotes.zip

$$D_n = \langle a, b \mid a^n = b^2 = 1,\ (ab)^2 = 1 \rangle$$

$$\mathrm{Aut}(D_n) \cong \mathbb{Z}/n\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})^\times \quad \text{where}$$

$$\begin{tikzcd}
 & & & \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \arrow[dl, dashed] \arrow[d, equal] & \\
0 \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & \mathbb{Z}/n\mathbb{Z} \rtimes \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \arrow[r] & \mathrm{Aut}(\mathbb{Z}/n\mathbb{Z}) \arrow[r] & 0
\end{tikzcd}$$

$$D_\infty = \langle a, b \mid a^2 = b^2 = 1 \rangle = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$$

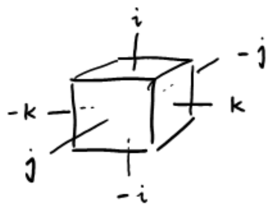
$$\mathrm{Aut}(D_\infty) \cong D_\infty$$

n	2	3	4	5	6	7	$\cdots$	∞
$\mathrm{Aut}(D_n)$	S_3	S_3	D_4	F_5	D_6	F_7	$\cdots$	D_∞
$\mathrm{Out}(D_n)$	S_3	Id	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\cdots$	$\mathbb{Z}/2\mathbb{Z}$

The notation F_5, F_6 come from the website GroupNames.

$$Q_8 = \{-1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1\}$$

$$\mathrm{Aut}(Q_8) \cong S_4$$



3. S_n & A_n.

E.g. $G = S_n$, $(n \in \mathbb{N}_{>0})$

$$\mathrm{Aut}(S_n) = \begin{cases} S_n & n \neq 2, 6 \\ \{*\} & n = 2 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\mathrm{Out}(S_n) = \begin{cases} \{*\} & n \neq 6 \\ \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

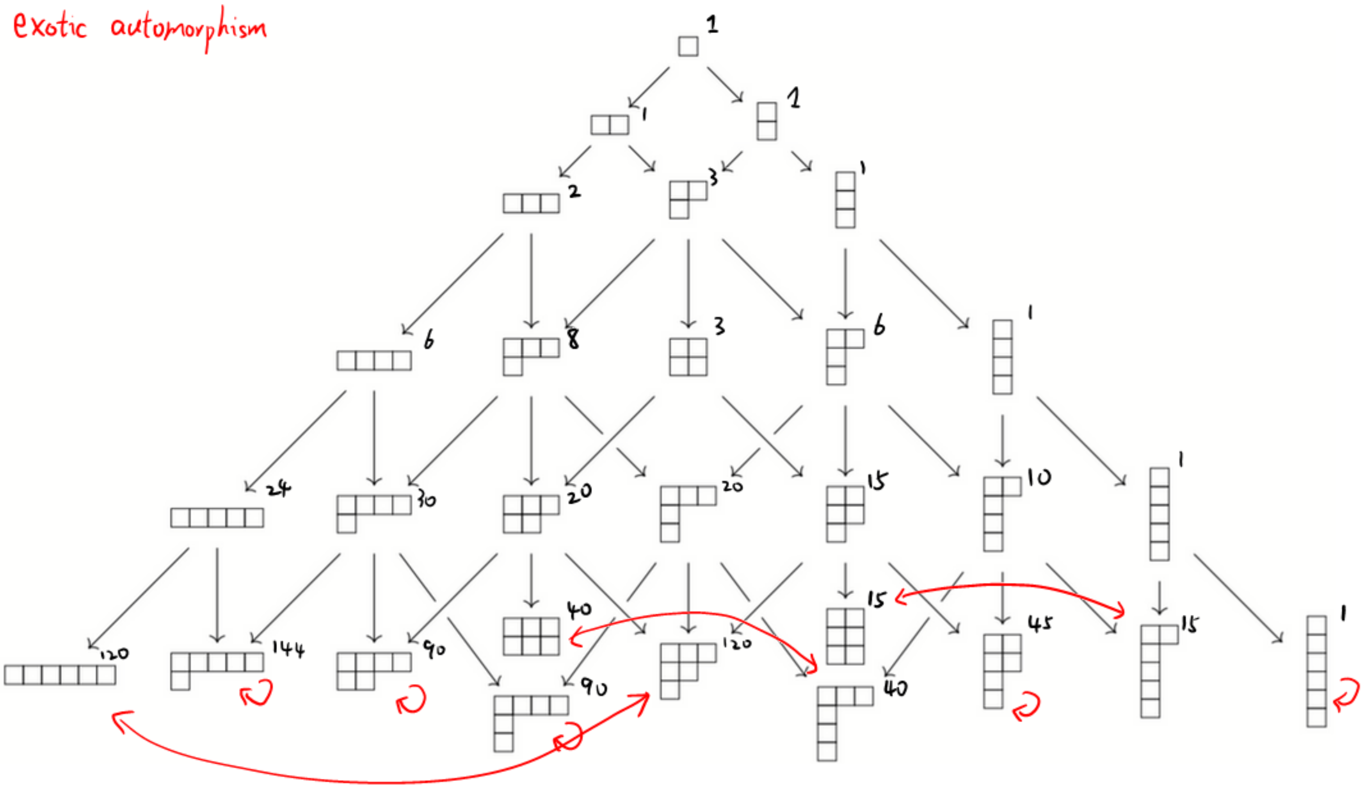
$G = A_n$.

$$\mathrm{Aut}(A_n) = \begin{cases} S_n & n \neq 2, 3, 6 \\ \{*\} & n = 2 \\ \mathbb{Z}/2\mathbb{Z} & n = 3 \\ S_6 \rtimes \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

$$\mathrm{Out}(A_n) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & n \neq 2, 3, 6 \\ \{*\} & n = 2 \text{ or } 3 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 6 \end{cases}$$

For a reference of the proof and constructions of the exotic outer automorphism of S_6, see wiki and here:
https://wordpress.nmsu.edu/pamorand/files/2018/10/AutGroups.pdf
For Chinese you can also see here: https://zhuanlan.zhihu.com/p/24764617
They are elementary and everybody who have learned something about Sylow's theorem should be able to follow the proofs.

$\#\{$ elements in conj class $= (123)\}$



E.g. $G = PSL(2, \mathbb{F}_7) \cong GL(3, \mathbb{F}_2)$

$Aut(PSL(2,\mathbb{F}_7)) \cong PGL(2,\mathbb{F}_7)$ $\quad$ $Out(PSL(2,\mathbb{F}_7)) \cong \{\pm 1\}$

Statement:
https://mathoverflow.net/questions/348440/what-is-the-outer-automorphism-group-of-operatornamesl2-mathbbf-q
For the other lie group, e.g. group in wiki: https://en.wikipedia.org/wiki/Projective_linear_group,
there is a general theory for its outer automorphism group, please see this book: (Even though I'm not so interested now)
https://www.cambridge.org/core/journals/canadian-journal-of-mathematics/article/automorphisms-of-finite-linear-groups/16c
23F257E0F21D57873B1450E9F15E4

E.g. $F_n :=$ free group generated by $a_1, \dots, a_n$

$F_n \longrightarrow F_n/[F_n,F_n] \cong \mathbb{Z}^{\oplus n}$ $\quad \rightsquigarrow \quad$ $Out(F_n) \longrightarrow GL(n,\mathbb{Z})$

It's claimed that $Out(F_2) \cong GL(2,\mathbb{Z})$.

Left: f.g. abelian group, like $\mathbb{Z}^n$. $(Aut(\mathbb{Z}^n) \cong Out(\mathbb{Z}^n) \cong GL(n,\mathbb{Z}))$

4. Profinite group

Now we consider automorphism in the category of profinite gp.

Lemma. $\mathrm{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l = m \\ 0 & l \neq m \end{cases}$ $\quad l, m$ prime.

Cor. $\mathrm{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^\times$

$\mathrm{Aut}(\hat{\mathbb{Z}}) = \hat{\mathbb{Z}}^\times := \prod_p \mathbb{Z}_p^\times$

$\mathrm{Aut}(\hat{\mathbb{Z}}^{(p)}) = \hat{\mathbb{Z}}^{\times(p)}$ $\quad \hat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l \quad \hat{\mathbb{Z}}^{\times(p)} := \prod_{l \neq p} \mathbb{Z}_l^\times$