Eine Woche, ein Beispiel

6.18 diagram chasing

Goal: Let's play the game of diagram chasing!

basic: five lemma, snake lemma, SES of complex $\Rightarrow$ LES of homology

[Vakil] "where there is universal property, there is diagram chasing"
e.p. Chap 1 Category + Adjoints + Spectral sequences
Chap 2 Sheaf on topology space
Please convert everything to Grothendieck topo!
Chap 23 Derived functors
Chap 28 Base change

[J. S. Milne, étale cohomology] Prop II.1.5. one description of the étale sheaf

References for the spectral sequence:

Best naive introduction by Prof. Vakil:
https://www.3blue1brown.com/content/blog/exact-sequence-picturebook/PuzzlingThroughExactSequences.pdf

applications of snake lemma and five lemma: [The rising sea2016, 1.7]
applications for algebraic geometry: [The rising sea2016, 23.3]
applications for cohomology group and homotopy group: [GTM82], [Hatcher], [2021.12.12]

ppt with nice pictures: https://github.com/CubicBear/SpectralSequences/tree/main

Now: "Fancy objects require a lot of diagram-chasing technique"
- Infinite category
- Stack related
- Condensed objects
- Triangular category, derived category and six-fctors formalism

Extension group

1. Def of $Ext_A^n(M,N)$

$$E_A(M,N) = \{0 \to N \xrightarrow{f} E \xrightarrow{g} M \to 0\}/\text{equivalence}$$
$$= \{\text{proj resolution } P_\cdot,\ H^n(Hom_A(P_\cdot, N))\}/\text{resolution}$$
$$= \{\text{inj resolution } I^\cdot,\ H^n(Hom_A(M, I^\cdot))\}/\text{resolution}$$
$$= \{\text{derivation}\}/\text{inner derivation}$$
$$= Hom_{D(A\text{-mod})}(M, N[1])$$

deri = SES = derived

G proj = SES = inj

2. Special module/ring interact with Ext?

P proj $\Leftrightarrow Ext_A^n(P,-)=0\ \forall n\geq 1 \Leftrightarrow Ext_A^1(P,-)=0$
$\Leftrightarrow$ proj dim $P=0$

I proj $\Leftrightarrow Ext_A^n(-,I)=0\ \forall n\geq 1 \Leftrightarrow Ext_A^1(-,I)=0$

A f.d alg $\quad \dim_k Ext_A^1(S(i), S(j)) = \dim_k Hom_A(rad(P(i)), S(j))$
$\underset{A=kQ/I}{=} |\{a\in Q_1 \mid s(a)=i,\ t(a)=j\}|$

Second level of detail.

equivalent of SES
$\Downarrow$
isomorphic

$\not\Uparrow \times$:

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, equal] & \mathbb{Z}/p\mathbb{Z} \arrow[r] \arrow[d, "\times 2"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot p"] & \mathbb{Z} \arrow[r, "2\cdot\pi"] & \mathbb{Z}/p\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

pushout α_* $\qquad$ pullback β^*

① ② ③$^\#$

+ com ass 0 $\quad$ k-linear $(\lambda+\mu)a$ $(\lambda\mu)a$ $1a$ $\quad \lambda(a+b)$

$\Rightarrow E_A(M,N)$: ① Def, ② bifunctor and ③$^\#$ k-linear space structure $\quad$ ① $\Rightarrow$ ② $\Rightarrow$ ③

$f_\cdot \sim g_\cdot \Rightarrow H_n(f_\cdot) = H_n(g_\cdot)$

$g_\cdot f_\cdot \sim Id \quad f_\cdot g_\cdot \sim Id \quad \Rightarrow H_n(C_\cdot) = H_n(C'_\cdot)$ ② ③

$\Rightarrow Ext_A^n(M,N)$: ① Def, bifunctor and k-linear space structure $\quad$ ① $\Rightarrow$ ③, ① $\Rightarrow$ ②

$\Rightarrow E_A(M,N) \to Ext_A^1(M,N)$ ① well-defined by resolution & lift & equiv
② bifunctor
③ k-linear map

Schanuel's lemma $\quad \left.\begin{array}{l} 0\to U \to P \to M \to 0 \\ 0 \to U' \to P' \to M \to 0 \\ P, P' \text{ proj}\end{array}\right\} \Rightarrow U\oplus P' \cong U'\oplus P$

$0\to U \to X \to V \to 0$ $\left\{\begin{array}{l}\text{non-split} \\ \text{f.d. } A\text{-mod}\end{array}\right.$ $\Rightarrow \dim_k End_A(X) < \dim_k End_A(U\oplus V)$

for f.d. A-mod $0 \to U \to X \to V \to 0$ split $\Leftrightarrow X \cong U\oplus V$ as A-module

Derived category

slogan: complex good, homology bad

Motivated:	https://arxiv.org/pdf/math/0001045.pdf
Standard reference:	S. Gelfand and Yu. Manin, Methods of Homological Algebra, Springer, 1996

we refer this without mention!

Newly recommanded references:
https://www.math.uni-hamburg.de/home/sosna/triangcat-lect.pdf
This notes shows all the details of the definition of triangular categories and derived categories, and shows even more informations of derived categories.

https://guests.mpim-bonn.mpg.de/gallauer/docs/m6ff.pdf
It introduce the six-functors formalism in detail, even though the exercises are pretty hard.

$$\begin{tikzcd} Kom(\mathcal{A}) \arrow[d, "/homotopy"] \arrow[dd, bend right, dashed, "/quasi\text{-}iso"'] & \text{category of chain complexes} \\ K(\mathcal{A}) \arrow[d, "/quasi\text{-}iso"] & \text{homotopy category} \\ D(\mathcal{A}) & \text{derived category} \end{tikzcd}$$

Remark. 1. For most time we view the category equivalence as "equal". However, the category defined by universal property is unique under isomorphism.

$$Ob(Kom(\mathcal{A})) = Ob(K(\mathcal{A})) = Ob(D(\mathcal{A}))$$

2. localizing category $B[S^{-1}]$ does not always have a good description

e.g. $D(\mathcal{A}) := Kom(\mathcal{A})[\text{quasi-iso}^{-1}]$

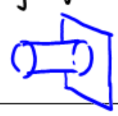
However, when S is a localizing class, then we have a good description Lemma III 2.8

e.g. $D(\mathcal{A}) := K(\mathcal{A})[\text{quasi-iso}^{-1}]$

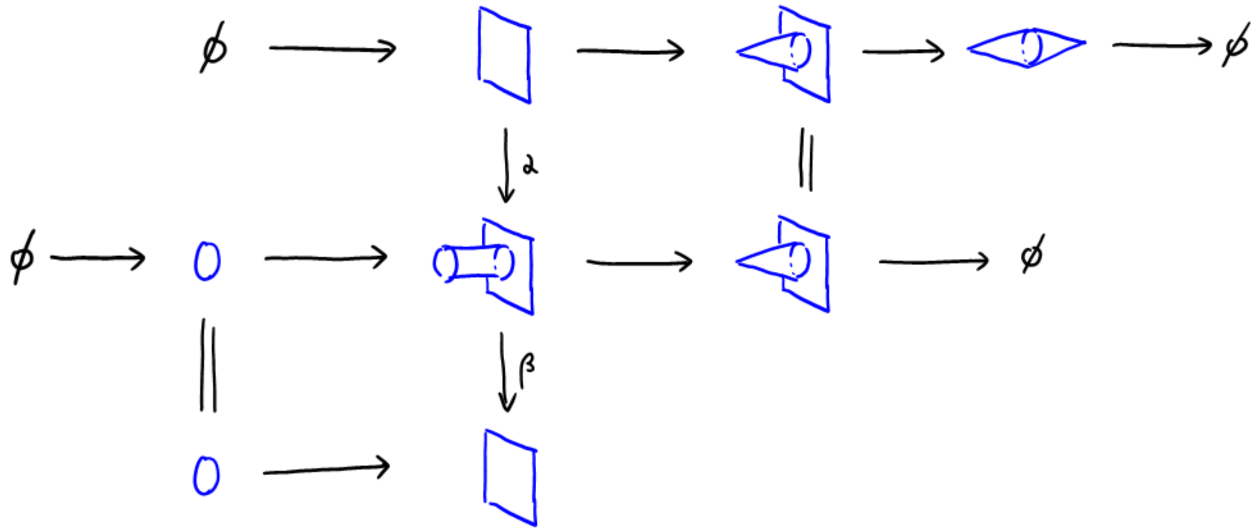
These two definitions define the same category $D(\mathcal{A})$.

3. $D(\mathcal{A})$ is a triangulated category.

To define a distinguished triangle, we denote

$f: K^{\cdot} \longrightarrow L^{\cdot}$ 	$K^{\cdot}, L^{\cdot}$: complexes	$K^0 \xrightarrow{d_K^0} K^1$, $L^0 \xrightarrow{d_L^0} L^1$; $d_K^0 = d_K = d$ to be short
$Cyl(f) := K^{\cdot} \oplus K[1]^{\cdot} \oplus L^{\cdot}$ 	$d_{Cyl(f)} = \begin{bmatrix} d & -1 & \\ & -d & \\ & f & d \end{bmatrix}$	$K^0 \oplus K^1 \oplus L^0 \xrightarrow{\begin{bmatrix} d_K^0 & -1 & \\ & -d_K^1 & \\ & f^1 & d_L^0 \end{bmatrix}} K^1 \oplus K^2 \oplus L^1$
$C(f) := K[1]^{\cdot} \oplus L^{\cdot}$ 	$d_{C(f)} = \begin{bmatrix} -d & \\ f & d \end{bmatrix}$	$K^1 \oplus L^0 \xrightarrow{\begin{bmatrix} -d_K^1 & \\ f^1 & d_L^0 \end{bmatrix}} K^2 \oplus L^1$

Then we have (Lemma III 3.3)
$\begin{cases} ⓪ \text{ well-defined} \\ ① \text{ SES on row} \\ ② \ \alpha, \beta: \text{quasi-iso} \end{cases}$



$$\begin{tikzcd}
 & 0 \arrow[r] & L^\bullet \arrow[r, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"] \arrow[d, "{\left[\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right]}"] & K[1]^\bullet \oplus L \arrow[r, "{[1\ 0]}"] \arrow[d, equal] & K[1]^\bullet \arrow[r] & 0 \\
0 \arrow[r] & K^\bullet \arrow[r, "{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]}"] \arrow[d, equal] & K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \arrow[r, "{\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]}"] \arrow[d, "{[f\ 0\ 1]}"] & K[1]^\bullet \oplus L \arrow[r] & 0 & \\
 & K^\bullet \arrow[r, "f"] & L^\bullet & & &
\end{tikzcd}$$

distinguished triangle:

$$K^\bullet \xrightarrow{\left[\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right]} K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \xrightarrow{\left[\begin{smallmatrix}0&1&0\\0&0&1\end{smallmatrix}\right]} K[1]^\bullet \oplus L \xrightarrow{[1\ 0]} K[1]^\bullet$$

SES: What's your favorate SES?

$$0 \to I \to A \to A/I \to 0 \qquad \text{as } A\text{-mod}$$

$$0 \to A \to A\oplus B \to B \to 0$$

$$0 \to I\cap J \to I \oplus J \to I+J \to 0$$

$$0 \to R/(I\cap J) \to R/I \oplus R/J \to R/(I+J) \to 0$$

$$0 \to \mathcal{O}_X \to K_X \to \bigoplus_{x\in X \text{ closed}} I_x \to 0$$

https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer12.pd

$$0 \to \underset{\Delta_*\Omega_X}{\underset{\parallel}{I/I^2}} \to \mathcal{O}_{X\times X}/I^2 \to \underset{\Delta_*\mathcal{O}_X}{\underset{\parallel}{\mathcal{O}_{X\times X}/I}} \to 0$$

https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer8.pdf

$$0 \to I_q \to D_q \to Gal(k_q/k_p) \to 0$$

$$0 \to \mathcal{O}_K^\times \to K^\times \to \bigoplus_{\mathfrak{p}\in M_K^0} \mathbb{Z} \to Cl(K) \to 0$$

$$1 \to Z(G) \to G \xrightarrow{conj} Aut(G) \to Out(G) \to 1$$

exponential $0 \to \underline{\mathbb{Z}} \to \mathcal{O}_M \to \mathcal{O}_M^\times \to 1$

generalization:https://ncatlab.org/nlab/show/exponential+exact+sequence

M: cplx mfld

Euler $0 \to \Omega_{\mathbb{P}^n_A/A} \to \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbb{P}^n_A} \to 0$

$$1 \to \mathbb{G}_m \to u_{\eta *}\mathbb{G}_{m,\eta} \to Div(X) \to 1$$

$u_\eta : \eta \to X$, $\eta = Spec(k(X))$

$$0 \dashrightarrow f^*\Omega_{X/k} \to \Omega_{Y/k} \to \Omega_{Y/X} \to 0$$

$f: Y \to X$

$$0 \dashrightarrow I/I^2 \to i^*\Omega_{X/k} \to \Omega_{Z/k} \to 0$$

https://mathoverflow.net/questions/270762/the-mittag-leffler-condition-as-necessary-and-sufficient?rq=1
https://math.stackexchange.com/questions/4260668/mittag-leffler-condition-and-exact-sequence-of-inverse-systems

$$0 \to \varprojlim_n A_n \to \prod_{n=1}^{\infty} A_n \xrightarrow{Id\text{-}shift} \prod_{n=1}^{\infty} A_n \to \varprojlim_n{}^1 A_n \to 0$$

$$Z \xrightarrow[\text{close}]{i} X \xleftarrow{j} U \quad \rightsquigarrow$$

$$\begin{tikzcd} D(Sh(Z_{\acute{e}t})) \arrow[r, "\text{pi} \quad i_* \quad \text{f.f}"] \arrow[loop left, "\mathrm{Id}"] & D(Sh(X_{\acute{e}t})) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "\text{pi} \quad i^! \quad L"] \arrow[r, "\text{pi} \quad j^*"] & D(Sh(U_{\acute{e}t})) \arrow[l, bend right, "j_! \quad \text{f.f.}"'] \arrow[l, bend left, "\text{pi} \quad Rj_* \quad L \quad \text{f.f}"] \arrow[loop right, "\mathrm{Id}"] \end{tikzcd}$$

$$\xleftarrow[0]{i^* \quad j_!} \qquad \xrightarrow[0]{i_* \quad j^*} \qquad \xleftarrow[0]{i^! \quad Rj_*}$$

L: left exact (others are exact)

f.f. fully faithful

pi: preserve injectives. (内射)

ie. inj sheaf $\rightsquigarrow$ inj sheaf

https://math.stackexchange.com/questions/2208354/faithful-direct-image-functor

$$\mathcal{F}_1 \xleftarrow{i^*} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(\mathcal{F}_1, \mathcal{F}_2, \alpha) \xmapsto{j^*} \mathcal{F}_2$$

$$\ker\alpha \xleftarrow{i^!} (\mathcal{F}_1, \mathcal{F}_2, \alpha)$$

$$(0, \mathcal{F}_2, 0) \xleftarrow{j_!} \mathcal{F}_2$$

$$\mathcal{F}_1 \xmapsto{i_*} (\mathcal{F}_1, 0, 0)$$

$$(i^*j_*\mathcal{F}_2, \mathcal{F}_2, \mathrm{Id}) \xleftarrow{Rj_*} \mathcal{F}_2$$

$$0 \to I \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 1$$

$\rightsquigarrow 0 \to \tilde{I} \to \mathcal{O}_{X\times X} \to \Delta_*\mathcal{O}_X \to 1$

$\rightsquigarrow 0 \to \Omega_{X/k} \to \Delta^*\mathcal{O}_{X\times X} \to \Delta^*\Delta_*\mathcal{O}_X \to 1$

$$i_! i^! \mathcal{F} \to \mathcal{F} \to Rj_* j^* \mathcal{F} \xrightarrow{+1}$$

~~$\mathcal{F}$ is supported on Z $\Leftrightarrow$ $\mathcal{H}^0_Z(\mathcal{F}) = \mathcal{F}$ $\Leftrightarrow$ $j_*j^*\mathcal{F} = 0$~~ ? $i^!$ is only defined on derived category?

$$0 \to j_! j^! \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

$j_!$: https://www.math.uni-bonn.de/people/gmartin/UebungenAGWS20/AGExer3.pdf

Check on stalks.

For Zariski: $j^* = j^{-1}$, $i^* \mapsto i^{-1}$

https://mathoverflow.net/questions/38168/is-the-category-of-commutative-group-schemes-abelian

Kummer sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^n-1) \longleftarrow k[x,x^{-1}] \longleftarrow k[x,x^{-1}]$$

$k/\mathbb{F}_p$

$$1 \longrightarrow \alpha_p \longrightarrow \mathbb{G}_a \xrightarrow{F:=(-)^p} \mathbb{G}_a \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^p) \longleftarrow k[x] \longleftarrow k[x]$$

Artin-Schreier sequence

$$1 \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \mathbb{G}_a \xrightarrow{F-\mathrm{Id}} \mathbb{G}_a \longrightarrow 1$$
$$0 \longleftarrow k[x]/(x^p-x) \longleftarrow k[x] \longleftarrow k[x]$$

	Zariski	étale	fppf
μ_n	×	✓ when $n \in \Gamma(X,\mathcal{O}_X)^\times$ × in general	✓
α_p	×	× in general	✓
$\underline{\mathbb{Z}/p\mathbb{Z}}$	×	✓	✓