Un exemple par jour

4.3. the (primary) Hopf surface $X := \mathbb{C}^2 - \{0\}/\mathbb{Z}\gamma$

$$\gamma(z_1, z_2) = (\alpha z_1 + \lambda z_2^{n'}, \beta z_2) \quad \text{where } \begin{cases} \alpha, \beta \in \mathbb{C} \quad n' \in \mathbb{N}^+ \\ 0 < |\alpha| \leq |\beta| < 1 \\ \lambda = 0 \text{ or } \alpha = \beta^{n'} \end{cases}$$

Today: $\alpha = \frac{1}{3}, \beta = \frac{1}{2}, \lambda = 0$ applies well for ① $\lambda = 0, \alpha^n \neq \beta^m$ $(\forall n, m \in \mathbb{N}^+)$

We also want to discuss the condition ② $\lambda = 0, \alpha^n = \beta^m$ $(\exists n, m \in \mathbb{N}^+)$

e.g. $\alpha = \frac{1}{8}$ $\beta = \frac{1}{4}$ $n = 2$ $m = 3$

Rmk. normally α^t, β^t $(t \in \mathbb{R})$ is not well-defined.
here we fix one value of $\ln\alpha, \ln\beta$ (and in ②, $n\ln\alpha = m\ln\beta$) and define
$\alpha^t := e^{t\ln\alpha}$ $\quad \beta^t := e^{t\ln\beta}$

1. We still have $T^2 \longrightarrow X$ $[\mathbb{R} \curvearrowright \mathbb{C}^2 - \{0\} : t(z_1, z_2) := (\alpha^t z_1, \beta^t z_2)]$
$$\downarrow$$
$$\mathbb{CP}^1$$

and $X \overset{\text{diffeo}}{\cong} S^3 \times S^1$

$[\alpha^t w_1, \beta^t w_2] \longleftarrow ((w_1, w_2), \bar{t} \in \mathbb{R}/\mathbb{Z})$

One sidemark for b^+, b^-: "new" numerial invarient.

Def. Suppose X is a cpt complex surface, ← assumption in this orange block
then the symmetric bilinear form
$\cup : H^2(X, \mathbb{R}) \times H^2(X, \mathbb{R}) \longrightarrow H^4(X, \mathbb{R}) \cong \mathbb{R}$ is non-degenerate
$[\cup : H^2(X, \mathbb{Z})/_{\text{Tor}} \times H^2(X, \mathbb{Z})/_{\text{Tor}} \longrightarrow H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ is a perfect pairing (iso) $]$
b^+/b^- : numbers of positive/negative eigenvalues.

result: [Kodaira I, Thm 3]

$$b_1 \text{ even} \Rightarrow \begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array} \quad \& \quad \begin{cases} b^+ = 2p_g + 1 \\ b^- = h^{1,1} - 1 \end{cases}$$

$$b_1 \text{ odd} \Rightarrow \begin{array}{ccccc} & & 1 & & \\ & q\text{-}1 & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q\text{-}1 & \\ & & 1 & & \end{array} \quad \& \quad \begin{cases} b^+ = 2p_g \\ b^- = h^{1,1} \end{cases}$$

(but $K(X)$ is not a topo invarient)

Cor. for cpt cplx surface, its Hodge diamond is totally decided by its topo properties.

Q: Are there two cpt complex mfld X_1, X_2, s.t $X_1 \overset{\text{homeo}}{\cong} X_2$ but $h^{i,j}(X_1) \neq h^{i,j}(X_2)$? $(\exists i, j \in \mathbb{N}^+)$

Sketsch of the result:

① Hirzebruch signature theorem (a special case of Atiyah-Singer index thm for the signature operater) for the surface:
$$b^+ - b^- = \frac{1}{3} p_1 = \frac{1}{3}(c_1^2 - 2c_2)$$

②
$$\begin{tikzcd}[column sep=small]
 & & 0 \arrow[r] & d\mathcal{O}_X \arrow[r] \arrow[dr] & 0 & & \\
0 \arrow[r] & \mathbb{C} \arrow[r] & \mathcal{O}_X \arrow[rr, "d"] \arrow[ur] & & \Omega_X \arrow[r] & \omega_X \arrow[r] & 0
\end{tikzcd}$$

lemma 1. $h^{1,0} = h^0(d\mathcal{O}_X)$ [Kodaira I, Thm 1]
$2h^0(d\mathcal{O}_X) \le b_1$ [Kodaira I, Thm 2] $\Big\} \Rightarrow 2h^{1,0} \le b_1 \Rightarrow 2q - b_1 \ge 0$

lemma 2. $2p_g \le b^+$ [Kodaira I, (11)] $\Rightarrow b^+ - 2p_g \ge 0$

$$\begin{cases} b^+ - b^- = \frac{1}{3}(12\chi - e - 2e) = 4\chi - e \\ b^+ + b^- = 2p_g + h^{1,1} \end{cases}$$
$$\Rightarrow 2b^+ = 4\chi + 2p_g + h^{1,1} - e$$
$$\Rightarrow 2q - b_1 + b^+ - 2p_g = 1 \overset{\text{lemma}}{\Longrightarrow} \begin{cases} 2q - b_1 = 0 \\ b^+ - 2p_g = 1 \end{cases} \text{ or } \begin{cases} 2q - b_1 = 1 \\ b^+ - 2p_g = 0 \end{cases} \quad \square$$

Cor. Again, the same Hodge diamond & topological properties applies.
$\mathrm{Pic}(X) = \mathbb{C}^\times$.

2. Compute K_X. $\psi = \frac{1}{z_1 z_2} dz_1 \wedge dz_2 \in H^0_M(X, \omega_X)$
↖ meromorphic

$$C_1 := [z_1 = 0] = \mathbb{C}^*/\mathbb{Z}\gamma_2 \cong \mathbb{C}/\mathbb{Z} \oplus (\tfrac{1}{2\pi i}\ln\beta)\mathbb{Z} \qquad (\gamma_2 z_2 = \beta z_2)$$
$$C := [z_2 = 0] = \mathbb{C}^*/\mathbb{Z}\gamma_1 \cong \mathbb{C}/\mathbb{Z} \oplus (\tfrac{1}{2\pi i}\ln\alpha)\mathbb{Z} \qquad (\gamma_1 z_1 = \alpha z_1)$$

$$K_X = -C - C_1 \Longrightarrow P_k = h^0(kK_X) = 0 \text{ for } k \ge 1 \Rightarrow \kappa(X) = -\infty$$

3. In condition ②, X is an elliptic surface by
$$\pi: X \longrightarrow \mathbb{P}^1_{\mathbb{C}}$$
$$[z_1, z_2] \longmapsto [z_1^m : z_2^n]$$

By an *elliptic surface* we shall mean an analytic fibre space of elliptic curves over a non-singular algebraic curve, i.e., a surface S together with a holomorphic map Ψ of S onto a non-singular algebraic curve Δ such that the inverse image $\Psi^{-1}(u)$ of any general point $u \in \Delta$ is an elliptic curve. [Kodaira I, P 767]

⇒ all curves are fiber, minimal.

In condition ①, X is not an elliptic surface.

If not, then $\exists\ \Phi: X \to \Delta$, fibers are elliptic curves. (may degenerate)
then $H^2(X,\mathbb{Z}) = 0 \Rightarrow C$ is fiber of Φ. $\Phi(C) = u$ is a pt
choose one non-constant fct $x \in H^0(\Delta, \mathcal{O}_\Delta(ku))$,
pullback to $x \in H^0(X, \mathcal{O}_X(kC))$, then

$$\begin{array}{c} \mathbb{C}^2 - \{0\} \hookrightarrow \mathbb{C}^2 \\ \downarrow \\ X \\ \downarrow \\ \Delta \end{array}$$

$$\phi := z_2^k x \in \Gamma_{hol}(\mathbb{C}^2 - \{0\}) \overset{\text{Hartog}}{=\!=} \Gamma_{hol}(\mathbb{C}^2)$$
$$\phi(\alpha z_1, \beta z_2) = \beta^k \phi(z_1, z_2) \qquad (1)$$
let $\phi_\nu := \frac{\partial^\nu \phi}{\partial z_1^\nu}$, then $(\nu \in \mathbb{N}^+)$
$$\alpha^\nu \phi_\nu(\alpha z_1, \beta z_2) = \beta^k \phi_\nu(z_1, z_2) \qquad (2)$$
$$\Rightarrow \phi_\nu(z_1, z_2) = \lim_{N \to +\infty} \left(\frac{\alpha^\nu}{\beta^k}\right)^N \phi_\nu(\alpha^N z_1, \beta^N z_2) \equiv 0 \quad \text{when } \alpha^\nu < \beta^k \ (\nu \gg 1)$$
let $l := \min\{l' \in \mathbb{N}_{\ge 0} \mid \phi_{l'+1} \equiv 0\}$

① $l = 0 \overset{(1)}{\Rightarrow} \phi(0, \beta z_2) = \beta^k \phi(0, z_2) \Rightarrow \phi(z_1, z_2) = c z_2^k$ $(c \in \mathbb{C})$, contradiction!

② $l > 0 \overset{(2)}{\Rightarrow} \phi_l(0, \beta z_2) = \frac{\beta^k}{\alpha^l} \phi_l(0, z_2) \Rightarrow \begin{cases} \alpha^l = \beta^{l'} \\ \phi_l(z_1, z_2) = c z_2^{l - l'} \end{cases}$ $(c \in \mathbb{C},\ l' \in \mathbb{N}^+)$, contradiction!

THEOREM 4. *If there exist on S two algebraically independent meromorphic functions, then S is an algebraic surface. If there exists on S one and only one algebraically independent meromorphic function, then S is an elliptic surface* (see Kodaira [8], Chow and Kodaira [3]).

Cor. $\mathrm{tr.dim}\, \mathcal{M}(X) = 0$ in condition ② $\Rightarrow$ $\mathcal{M}(X) = \mathbb{C}$

Q: In condition ②, can we show it directly?

Q: $\mathrm{Aut}(X)$?

4. curves in condition ②

We have only C & C'. Is it true? Why?