Eine Woche, ein Beispiel

6.4. Grothendieck topology, site and topos

should be read after 2022.6.5 category

A dictionary for myself:

$\{U_i \to U\}_{i\in\Delta}$ ← may be not jointly surj	sieve
topology	Grothendieck topology
topological space	site
$Sh(X)$	topos
sheaf	sheaf
irr closed set/pts	points

Discrete fibration

Ref:[https://www.illc.uva.nl/Research/Publications/Dissertations/DS-2021-09.text.pdf], begin from 3.1.8

Def A fctor $F : \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete fibration if
$\forall c \in \mathcal{C},\ b \in \mathcal{B},\ g \in Mor(b, F(c))$,
$\exists!\ c' \in \mathcal{C},\ h \in Mor(c', c)$ s.t. $F(h) = g$.

$$\begin{tikzcd} c' \arrow[r, "h", "\exists!"'] & c \\ b \arrow[r, "g"] & F(c) \end{tikzcd}$$

A fctor $F : \mathcal{C} \longrightarrow \mathcal{B}$ is a discrete opfibration if
$F^{op} : \mathcal{C}^{op} \longrightarrow \mathcal{B}^{op}$ is a discrete fibration.

From[https://arxiv.org/abs/1806.06129]: The left-handed version, now opfibrations, was originally called cofibrations, though this name was rejected to avoid confusing topologists.

E.g. For any category $\mathcal{C}$ & $x \in Ob(\mathcal{C})$, the forgetful fctor
$$\mathcal{C}/x \longrightarrow \mathcal{C}$$
is a discrete fibration (not fully faithful)

We will later see that this discrete fibration corresponds to the presheaf h_x.

Prop. (Equivalent def of discrete fibration) Let $\mathcal{C}, \mathcal{B}$ be categories. $F : \mathcal{C} \longrightarrow \mathcal{B}$ be a fctor.
F is discrete fibration $\Leftrightarrow \forall c \in \mathcal{C},\ F/c : \mathcal{C}/c \longrightarrow \mathcal{B}/F(c)$ is iso.

Let $DFib_{\mathcal{B}}$ be the metacategory of discrete fibrations. To be exact,
$$Ob(DFib_{\mathcal{B}}) = \left\{ (\mathcal{C}, p : \mathcal{C} \longrightarrow \mathcal{B}) \;\middle|\; \begin{array}{l} \mathcal{C} \in \mathcal{C}at_{big} \\ p \text{ is a discrete fibration} \end{array} \right\}$$
$$Mor(p, p') = \left\{ f : \mathcal{C} \longrightarrow \mathcal{C}' \;\middle|\; \begin{tikzcd}[column sep=small] \mathcal{C} \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{C}' \arrow[dl, "p'"] \\ & \mathcal{B} & \end{tikzcd} \text{ commutes} \right\}$$

i.e. $DFib_{\mathcal{B}}$ is a full submetacategory of $\mathcal{C}at_{big}/\mathcal{B}$.

When restrict everything to small categories, one can define $DFib_{\mathcal{B}}$ as a full subcategory of $\mathcal{C}at/\mathcal{B}$

Let $PSh(\mathcal{C}) = Fun(\mathcal{C}^{op}, Set)$,
$h_c := Mor_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \longrightarrow Set$ be a presheaf on $\mathcal{C}$. for $c \in \mathcal{C}$
$\qquad c' \longmapsto Mor_{\mathcal{C}}(c', c)$
$h_{\mathcal{C}} : \mathcal{C} \longrightarrow PSh(\mathcal{C})$
$\qquad c \longmapsto h_c$

Prop. For $\mathcal{B} \in \mathcal{C}at$, we have an equivalence of categories:
$$\int_{\mathcal{B}} : PSh(\mathcal{B}) \longrightarrow DFib_{\mathcal{B}}$$
$$F \longmapsto \int_{\mathcal{B}} F \text{ be the strict pullback}$$

$$\begin{tikzcd} \int_{\mathcal{B}} F \arrow[r] \arrow[d] & PSh(\mathcal{B})/F \arrow[d] \\ \mathcal{B} \arrow[r, "h_{\mathcal{B}}"] & PSh(\mathcal{B}) \end{tikzcd}$$

$$\left[b \in \mathcal{B} \longmapsto Mor_{DFib_{\mathcal{B}}}\left(\begin{array}{c}\mathcal{B}/b \\ \downarrow \\ \mathcal{B}\end{array}, \begin{array}{c}\mathcal{C} \\ \downarrow p \\ \mathcal{B}\end{array}\right) \right] \longleftarrow\!\shortmid\ (\mathcal{C},\ p : \mathcal{C} \longrightarrow \mathcal{B})$$

Sieve

Def (sieve in small category)

Let $\mathcal{C}$ be a small category. $S \in \mathrm{Cat}/\mathcal{C}$.

S is a sieve in $\mathcal{C}$ if the fctor

$$S \longrightarrow \mathcal{C}$$

is fully faithful and a discrete fibration.

For $c \in \mathcal{C}$, $T \in \mathrm{Cat}/(\mathcal{C}/c)$.

T is a sieve on c if the fctor

$$T \longrightarrow \mathcal{C}/c$$

is fully faithful and a discrete fibration.

Viewing T as a fullsubcategory of $\mathcal{C}/c$, this is equivalent to

A sieve on c is a subset $T \subseteq \mathrm{Ob}(\mathcal{C}/c)$ s.t.

$(f \circ g : e \to c) \in T$ for any $e, d \in \mathcal{C}, (f : d \to c) \in T, g \in \mathrm{Mor}(e, d)$.

Def. Now $\mathcal{C}$ can be any category, $c \in \mathcal{C}$.

A sieve on c is a subclass $T \subseteq \mathrm{Ob}(\mathcal{C}/c)$ s.t.

$(f \circ g : e \to c) \in T$ for any $e, d \in \mathcal{C}, (f : d \to c) \in T, g \in \mathrm{Mor}(e, d)$.

$$\begin{tikzcd} e \arrow[r, "g"] \arrow[dr, dashed, "f\circ g \in T"'] & d \arrow[d, "f \in T"] \\ & c \end{tikzcd}$$

Let $h_c := \mathrm{Mor}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$ be a presheaf on $\mathcal{C}$.

$$c' \longmapsto \mathrm{Mor}_{\mathcal{C}}(c', c)$$

Thm. When $\mathcal{C}$ is small, There is a bijection between Sets

$$\{\text{sieves on } c \in \mathcal{C}\} \longleftrightarrow \{\text{subfctors of } h_c\}$$

$$T \longmapsto F_T : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$$

$$\begin{array}{ccc} d & & \{(d \to c) \in T\} \\ \scriptstyle{\alpha \in \mathrm{Mor}_{\mathcal{C}}(d,d')} \quad \alpha \downarrow & \Rightarrow & \uparrow \alpha\circ - \\ d' & & \{(d' \to c) \in T\} \end{array}$$

$$T_F := \coprod_{d \in \mathrm{Ob}(\mathcal{C})} F(d) \longleftarrow F \subset h_c$$

Q: How to get a correct statement for this theorem when $\mathcal{C}$ is large?

Grothendieck topology, site and topos

On set theoretic issues: https://stacks.math.columbia.edu/tag/ooVI
Ironnically, even though what I can actually understand is the Grothendieck topology over a small category, nearly all the applications I need is the Grothendieck topology over a large category.

Def. A **Grothendieck topology** $\mathcal{T}$ on a category $\mathcal{C}$ is an assignment

$$\mathcal{T}(-): \mathcal{C} \longrightarrow \mathcal{P}(\{\text{sieves on } c \in \mathcal{C} \text{ for some } c\})$$
$$c \longmapsto \mathcal{T}(c) \subseteq \{\text{sieves on } c\}$$

s.t.

1). (Base change) $\forall g \in Mor_{\mathcal{C}}(d, c)$, $T \in \mathcal{T}(c) \Rightarrow g^* T \in \mathcal{T}(d)$
2) (Local character) Let T be a sieve on $c \in \mathcal{C}$. If $[\exists S \in \mathcal{T}(c) \text{ st } \forall (g: d \to c) \in S,\ g^* T \in \mathcal{T}(d)]$ then $T \in \mathcal{T}(c)$
3) $h_c \in \mathcal{T}(c)$

Def. A **site** $\mathcal{C} = (\mathcal{C}, \mathcal{T})$ is a category equipped with a Grothendieck topology.
A **topos** is a category equivalent to $Sh(\mathcal{C})$, where $\mathcal{C}$ is a site.

Category + Groth cover	space open sets	continuous map	Covering of open sets	Sh	cohomology
site	Object	Morphism	Grothendieck Top. $\{U_i \xrightarrow{f_i} U\}_{i\in I}$, $\bigcup_{i\in I} \mathrm{Im} f_i = U$	topos	new cohomology
X_{Zar}	open immersion over X	full sub of Sch/X	—		H
Sch_{Zar}	$Ob(Sch)$	$Mor(Sch)$	—		
$X_{ét}$	étale + l.f.p over X	full sub of Sch/X	ét + l.f.p		$H_{ét}$
$Sch_{ét}$	$Ob(Sch)$	$Mor(Sch)$	ét + l.f.p		
Sch_{sm}	$Ob(Sch)$	$Mor(Sch)$	smooth + l.f.p		
Sch_{fppf}	$Ob(Sch)$	$Mor(Sch)$	f.flat + l.f.p		
Sch_{fpqc}	$Ob(Sch)$	$Mor(Sch)$	f.flat + f_i^{-1}(q.c) locally q.c		
X/k $W_n := W_n(k)$ $Cris(X/W_n)$	$\left\{(U,V,i,\delta) \,\middle\vert\, \begin{array}{l} U \subseteq X \text{ open} \\ \cdots \\ \delta: \text{PD-thickening of } U \end{array}\right\}$	$\left\{(\iota,f) \,\middle\vert\, \begin{array}{l} \iota: U \xrightarrow{\text{open}} U' \\ f: V \to V' \\ \text{compatible with PD} \end{array}\right\}$	$\left\{\begin{array}{c} (U_i,V_i,i_i,\delta_i) \\ \downarrow \\ (U,V,i,\delta) \end{array} \,\middle\vert\, \begin{array}{l} \{U_i\} \text{ cover} \\ \text{of } U \end{array}\right\}$		$H^i_{cris}(X/W_n,-)$

(recommended)https://sites.math.washington.edu/~jarod/moduli.pdf
https://pbelmans.ncag.info/notes/etale-cohomology.pdf
http://homepage.sns.it/vistoli/descent.pdf
(crystalline site)http://page.mi.fu-berlin.de/castillejo/docs/crystalline_cohomology.pdf

[Hilbert's theorem 90 $\Leftrightarrow$ no non-trivial line bundle on Spec k]

https://math.stackexchange.com/questions/1424102/relationship-between-galois-cohomology-and-etale-cohomology

it tells us why we don't have small site for most condition:
https://mathoverflow.net/questions/247044/small-fppf-syntomic-smooth-sites
Here you can find some informations about comparison between fppf and fpqc topologys:
https://mathoverflow.net/questions/361664/some-basic-questions-on-quotient-of-group-schemes

Thm. ① equiv. of categories

$$\mathrm{Sets}((\operatorname{Spec} k)_{\text{ét}}) \longleftrightarrow \text{Disc } G_k\text{-Set}$$
$$\mathrm{Ab}((\operatorname{Spec} k)_{\text{ét}}) \longleftrightarrow \text{Disc } \mathrm{Mod}_{G_k} \qquad (☆)$$

$G_k := \mathrm{Gal}(k^{sep}/k)$

$(\operatorname{Spec} k)_{\text{ét}} \overset{\text{Site}\quad\text{finite}}{\longleftrightarrow} G_k\text{-Set}$

② (☆) preserve cohomology
$$H^1((\operatorname{Spec}(k))_{\text{ét}}, \mathcal{F}) = H^1_{cont}(G_k, \mathcal{F}_k)$$

Ex. describe sheaf on $(\operatorname{Spec} \mathbb{C})_{\text{ét}}$ (Verify: $\mathcal{F}$ is decided by $\mathcal{F}(\operatorname{Spec}\mathbb{C})$)

Ex. describe sheaf on $(\operatorname{Spec} \mathbb{R})_{\text{ét}}$

$$\begin{tikzcd}
\operatorname{Spec}\mathbb{C} \arrow[dr, "\iota^*"'] & & \operatorname{Spec}\mathbb{C} \arrow[ll, "\sigma^*"'] \arrow[dl, "\iota^*"] & \arrow[l, squiggly, "\operatorname{Spec}"'] & \mathbb{C} \arrow[rr, "\sigma", "= \text{conjugation}"'] & & \mathbb{C} \\
& \operatorname{Spec}\mathbb{R} \arrow[d, squiggly, "\mathcal{F}"] & & & & \mathbb{R} \arrow[ul, "\iota"] \arrow[ur, "\iota = \text{embedding}"'] & \\
& \; & & & & &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(\operatorname{Spec}\mathbb{C}) \arrow[rr, "\mathcal{F}(\sigma^*)"] & & \mathcal{F}(\operatorname{Spec}\mathbb{C}) & \overset{\text{Abuse of notation}}{\Longrightarrow} & \mathcal{F}(\mathbb{C}) \arrow[rr, "\sigma"] & & \mathcal{F}(\mathbb{C}) \\
& \mathcal{F}(\operatorname{Spec}\mathbb{R}) \arrow[ul, "\mathcal{F}(\iota^*)"] \arrow[ur, "\mathcal{F}(\iota^*)"'] & & & & \mathcal{F}(\mathbb{R}) \arrow[ul, "\iota"] \arrow[ur, "\iota"'] &
\end{tikzcd}$$

Sub Ex. $\mathcal{F}$ is sheaf $\rightsquigarrow \mathcal{F}(\mathbb{R}) = \mathcal{F}(\mathbb{C})^{Gal}$ $\quad Gal := Gal(\mathbb{C}/\mathbb{R})$

[partial results: $\mathcal{F}$ is seperated $\rightsquigarrow \mathcal{F}(\mathbb{R}) \to \mathcal{F}(\mathbb{C})$ inj
Comm diagram $\rightsquigarrow \mathcal{F}(\mathbb{R}) \subseteq \mathcal{F}(\mathbb{C})^{Gal}$]

[$\mathcal{F}$ sheaf: $0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$ $\leftarrow$ $i=j$ is allowed.

in this case $0 \to \mathcal{F}(\operatorname{Spec}\mathbb{R}) \to \mathcal{F}(\operatorname{Spec}\mathbb{C}) \overset{\iota_1}{\underset{\iota_2}{\rightrightarrows}} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C})$

$$\begin{array}{ccc} \mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}) \cong \mathcal{F}\left(\operatorname{Spec}\prod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\mathbb{C}\right) \\ & \begin{array}{l} \iota_1 : x \longmapsto x\otimes 1 \\ \iota_2 : x \longmapsto 1\otimes x \end{array} \quad x\otimes y \longmapsto (xy, x\bar{y}) & \cong \\ \| & & \mathcal{F}\left(\coprod_{\sigma\in Gal(\mathbb{C}/\mathbb{R})}\operatorname{Spec}\mathbb{C}\right) \\ & & \cong \\ \mathcal{F}(\operatorname{Spec}\mathbb{C}) & \longrightarrow & \mathcal{F}(\operatorname{Spec}\mathbb{C})\times\mathcal{F}(\operatorname{Spec}\mathbb{C}) \end{array}$$

ι_2: $\operatorname{Spec}\mathbb{C} \xleftarrow{(Id,\sigma)} \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

$\rightsquigarrow \mathcal{F}(\operatorname{Spec}\mathbb{C}) \xrightarrow{(\mathcal{F}(Id),\mathcal{F}(\sigma))} \mathcal{F}(\operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}) \cong \mathcal{F}(\operatorname{Spec}\mathbb{C}) \times \mathcal{F}(\operatorname{Spec}\mathbb{C})$

$\underset{\text{of notation}}{\overset{\text{Abuse}}{\rightsquigarrow}}$ $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,\sigma)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$

ι_1: $\mathcal{F}(\mathbb{C}) \xrightarrow{(Id,Id)} \mathcal{F}(\mathbb{C}) \times \mathcal{F}(\mathbb{C})$]

Ex. describe the global section of sheaf under the equivalence

$$\Gamma(\operatorname{Spec} k, \mathcal{F}) = \mathcal{F}(\operatorname{Spec} k) = \mathcal{F}_{k^{sep}}^{Gal(k^{sep}/k)} \qquad \mathcal{F}_{k^{sep}} := \varinjlim_{\substack{L/k \\ \text{finite}}} \mathcal{F}(\operatorname{Spec} L)$$

Ex. describe the stalk & fiber at $p \in \operatorname{Spec} k$

$$\mathcal{F}_p := \varinjlim_{p\in U} \mathcal{F}(U) = \mathcal{F}_{k^{sep}} \qquad \mathcal{F}|_p := \mathcal{F}_p \otimes_{\mathcal{O}_{\operatorname{Spec} k, p}} k(p) = \mathcal{F}_p = \mathcal{F}_{k^{sep}}$$

neighborhood

Nbhd category, stalks and points

For defining pts & stalks, we need an index category which corresponds to the nbhd of a pt (realzed as a fct, e.g. the "skyscraper sheaf")

Def. (Neighborhood category $Nbhd_p$)

Let $\mathcal{C}$ be a site, $p: \mathcal{C} \longrightarrow Set$ be a covariant fctor.

$$Ob(Nbhd_p) = \{(\mathcal{U}, x) \mid \mathcal{U} \in Ob(\mathcal{C}),\ x \in p(\mathcal{U})\}$$

$$Mor((V,y),(\mathcal{U},x)) = \{\alpha: V \longrightarrow \mathcal{U} \text{ morphism s.t. } x = p(\alpha)(y)\}$$

Def. (Stalk at p)

Let $\mathcal{C}$ be a site, $p: \mathcal{C} \longrightarrow Set$ be a covariant fctor.

The set

$$\mathcal{F}_p := \varinjlim_{(\mathcal{U},x)\in Nbhd^{op}} \mathcal{F}(\mathcal{U})$$

is called the stalk at p.

Here, the concept of pt generalizes the "skyscraper sheaf".

Def. [00Y3] Let $\mathcal{C} = (\mathcal{C}, Cov(\mathcal{C}))$ be a site.

A fctor $p: \mathcal{C} \longrightarrow Set$ is called a <u>point</u> of $\mathcal{C}$, if

(1) $\forall \{\mathcal{U}_i \longrightarrow \mathcal{U}\} \in Cov(\mathcal{U})$, $\coprod_i p(\mathcal{U}_i) \longrightarrow p(\mathcal{U})$ is surj;

(2) $\forall \{\mathcal{U}_i \longrightarrow \mathcal{U}\} \in Cov(\mathcal{U})$, $\forall V \longrightarrow \mathcal{U}$ morphism in $\mathcal{C}$, $\forall i$,

$$p(\mathcal{U}_i \times_{\mathcal{U}} V) \longrightarrow p(\mathcal{U}_i) \times_{p(\mathcal{U})} p(V)$$

are bijection.

(3) The fctor

$$Sh(\mathcal{C}) \longrightarrow Set \qquad \mathcal{F} \longmapsto \mathcal{F}_p$$

is left exact.

Rmk. The usual skyscraper sheaf is

$$i_{p,*}A: Open(X)^{op} \longrightarrow Set \qquad i_{p,*}A(\mathcal{U}) = \begin{cases} A, & p \in \mathcal{U} \\ \{*\}, & p \notin \mathcal{U} \end{cases}$$

Here, the skyscraper point is

$$u_p: Open(X) \longrightarrow Set \qquad u_p(\mathcal{U}) = \begin{cases} \{*\}, & p \in \mathcal{U} \\ \emptyset, & p \notin \mathcal{U} \end{cases}$$

We use the initial/final object of the category Set, thus switch the direction of arrows.

https://stacks.math.columbia.edu/tag/03PN

https://math.stackexchange.com/questions/2856987/computing-%C3%A9tale-cohomology-group-h1-texts
peck-mu-n-and-h1-texts