

## Invariance [edit]

Homotopy equivalence is important because in [algebraic topology](#) many concepts are **homotopy invariant**, that is, they respect the relation of homotopy equivalence. For example, if  $X$  and  $Y$  are homotopy equivalent spaces, then:

- $X$  is [path-connected](#) if and only if  $Y$  is.
- $X$  is [simply connected](#) if and only if  $Y$  is.
- The (singular) [homology](#) and [cohomology groups](#) of  $X$  and  $Y$  are [isomorphic](#).
- If  $X$  and  $Y$  are path-connected, then the [fundamental groups](#) of  $X$  and  $Y$  are isomorphic, and so are the higher [homotopy groups](#). (Without the path-connectedness assumption, one has  $\pi_1(X, x_0)$  isomorphic to  $\pi_1(Y, f(x_0))$  where  $f: X \rightarrow Y$  is a homotopy equivalence and  $x_0 \in X$ .)

An example of an algebraic invariant of topological spaces which is not homotopy-invariant is [compactly supported homology](#) (which is, roughly speaking, the homology of the [compactification](#), and compactification is not homotopy-invariant).

## Homotopy equivalence [edit]

Given two topological spaces  $X$  and  $Y$ , a **homotopy equivalence** between  $X$  and  $Y$  is a pair of continuous [maps](#)  $f: X \rightarrow Y$  and  $g: Y \rightarrow X$ , such that  $g \circ f$  is homotopic to the [identity map](#)  $\text{id}_X$  and  $f \circ g$  is homotopic to  $\text{id}_Y$ . If such a pair exists, then  $X$  and  $Y$  are said to be **homotopy equivalent**, or of the same **homotopy type**. Intuitively, two spaces  $X$  and  $Y$  are homotopy equivalent if they can be transformed into one another by bending, shrinking and expanding operations. Spaces that are homotopy-equivalent to a point are called [contractible](#).

- A [deformation retraction](#) is a homotopy equivalence.

Let  $X$  be a topological space and  $A$  a subspace of  $X$ . Then a continuous map

$$r: X \rightarrow A$$

is a **retraction** if the [restriction](#) of  $r$  to  $A$  is the [identity map](#) on  $A$ ;

Proof.

$$r: X \rightarrow A$$

$$\iota: A \hookrightarrow X$$

$$r \circ \iota = \text{Id}$$

A continuous map

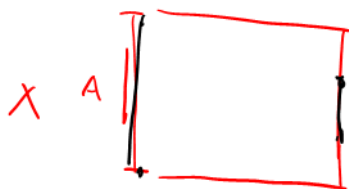
$$\iota \circ r: X \rightarrow X$$

$$F: X \times [0, 1] \rightarrow X$$

is a **deformation retraction** of a space  $X$  onto a subspace  $A$  if, for every  $x$  in  $X$  and  $a$  in  $A$ ,

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad \text{and} \quad F(a, 1) = a.$$

In other words, a deformation retraction is a [homotopy](#) between a retraction and the identity map on  $X$ .



过程中 A 也可以移动

$$f \circ g: Y \rightarrow Y$$

$$\exists F: Y \times [0, 1] \rightarrow Y$$

$$F_0 = f \circ g \quad F_1 = \text{Id}_Y$$