Eine Woche, ein Beispiel

8.29. affine paving of quiver flag variety

Here is some personal reflection of the articles:
https://arxiv.org/abs/1804.07736
https://arxiv.org/abs/1909.04907

Plan:

① affine paving

	Grassmannian	partial flag variety	strict partial variety
D_4	✓	✓	
D_5	✓	✓	
D_6	✓	✓	
E_6	✓		
E_7	✓		
E_8	✓		

② smooth problem, dimension problem
③ explicit expressions
④ closure & intersection theories, Hasse diagram

The induction process of Grassmannian (affine paving + cellular dec)

M': indecomposable

$\mathrm{supp}\,\tilde{\underline{f}} < \mathrm{supp}\,\underline{f}$

$$\begin{tikzcd} Gr_{\underline{f}}(M) \arrow[r, "\text{weak dec}"] & Gr_{\underline{f}'}(M') \arrow[r, "\text{proj dec}"] \arrow[loop below, "\text{strong dec}"] & Gr_{\tilde{\underline{f}}}(\tilde{M}) \arrow[d, dashed] \\ & & \text{Gr of smaller quiver} \end{tikzcd}$$

$M' = \begin{smallmatrix} & & & & 3 & & \\ 2 & 3 & 4 & 5 & 6 & 4 & 2 \end{smallmatrix}$

Remark. 1. This decomposition is not canonical
depend on: order of indecomposable modules; (weak)
choose of projective module. (proj)
2. The amount of calculations grows exponentially.
Doing case-by-case is nearly impossible!
fix a dynkin quiver, you have to choose the directions of arrows,
make the AR-quivers (for all the subquiver) and choose the dim vector.
3. If we can do these three steps for partial flag variety,
then we can get cellular decomposition of part flag var.
4. We know how to compute quotients, but we want to do it easier.

E.g. E_8

$$\begin{array}{ccccccccccccc} & & & & & & & & 8 & & & & \\ & & & & & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 3 & \to & 4 & \to & 5 & \leftarrow & 6 & \leftarrow & 7 \end{array}$$

$$\eta:\ 0 \to 1223\overset{1}{3}21 \longrightarrow 2345\overset{3}{6}42 \longrightarrow 1122\overset{2}{3}21 \longrightarrow 0$$

$$\mathrm{Gr}_{\underline{1112\overset{1}{3}21}}(2345\overset{3}{6}42)$$

$$\Big\downarrow \text{strong}$$

$$\mathrm{Gr}_{\underline{1112\overset{1}{3}21}}(1223\overset{1}{3}21) \times \mathrm{Gr}_{\underline{0000\overset{0}{0}00}}(1122\overset{2}{3}21)$$

$$\coprod \quad \vdots$$

$$\coprod\ \mathrm{Gr}_{\underline{0000\overset{0}{0}00}}(1223\overset{1}{3}21) \times \mathrm{Gr}_{\underline{1112\overset{1}{3}21}}(1122\overset{2}{3}21)$$

fiber
$\{*\}$
$\vdots$
$\mathbb{C}^{26}$

$$\begin{aligned} \mathrm{Gr}_{\underline{1112\overset{1}{3}21}}(1223\overset{1}{3}21) &\overset{\text{proj}}{=\!=} \mathrm{Gr}_{\underline{0001\overset{1}{2}21}}(0112\overset{1}{2}21) \\ &= \mathrm{Gr}_{\underline{001\overset{1}{2}21}}(112\overset{1}{2}21) \\ &= \mathrm{Gr}_{\underline{01\overset{1}{2}21}}(12\overset{1}{2}21) \\ &= \mathrm{Gr}_{\underline{1\overset{1}{2}21}}(1\overset{1}{1}10 \oplus 1\overset{0}{1}11) \\ &= \mathrm{Gr}_{\underline{1\overset{1}{2}21}}(0\overset{1}{1}11 \oplus 1\overset{0}{1}11) \\ &\overset{\substack{\text{lazy} \\ \text{proj}}}{=\!=} \mathrm{Gr}_{\underline{01\overset{1}{1}1}}(01\overset{1}{1}1 \oplus 00\overset{0}{0}1) \\ &= \mathrm{Gr}_{\underline{1\overset{1}{1}1}}(1\overset{1}{1}1 \oplus 0\overset{0}{0}1) \\ &\overset{\text{proj}}{=\!=} \mathrm{Gr}_{\underline{0\overset{1}{0}1}}(0\overset{1}{0}1 \oplus 0\overset{0}{0}1) = \mathbb{P}^1 \end{aligned}$$

Another example: E_6

$$\begin{array}{ccccccccc} & & & & 6 & & & & \\ & & & & \downarrow & & & & \\ 1 & \to & 2 & \to & 3 & \leftarrow & 4 & \leftarrow & 5 \end{array}$$

$$\eta:\ 0 \to 11\overset{1}{2}21 \longrightarrow 11\overset{1}{2}21 \oplus 12\overset{1}{2}11 \longrightarrow 12\overset{1}{2}11 \longrightarrow 0$$

$$\mathrm{Gr}_{\underline{11\overset{1}{2}11}}(11\overset{1}{2}21 \oplus 12\overset{1}{2}11) \cong \mathbb{C}^{10} \sqcup \{*\}$$

$$\Big\downarrow \text{weak}$$

$$\mathrm{Gr}_{\underline{11\overset{1}{2}11}}(11\overset{1}{2}21) \times \mathrm{Gr}_{\underline{00\overset{0}{0}00}}(12\overset{1}{2}11)$$

$$\coprod \quad \vdots$$

$$\coprod\ \mathrm{Gr}_{\underline{00\overset{0}{0}00}}(11\overset{1}{2}21) \times \mathrm{Gr}_{\underline{11\overset{1}{2}11}}(12\overset{1}{2}11)$$

fiber
$\{*\}$
$\vdots$
$\mathbb{C}^{10}$

Some unsuccessful tries:

1. For each $N \in Gr_f(M)$, by Krull-Remak-Schmidt Theorem,

$\exists$ indecomposable modules N_i & $t_i \in \mathbb{N}_{>0}$ s.t. $N \cong \bigoplus_i N_i^{\oplus t_i}$
($\Rightarrow \sum_i t_i \underline{\dim} N_i = f$)

So it's natural to consider

$$Gr_{(N_1^{t_1}, \dots, N_r^{t_r})}(M) = \{ N \leq M \mid N \cong \bigoplus_i N_i^{\oplus t_i} \}$$

and we have "explicit expression"

$$\boxed{Gr_{(N_1^{t_1}, \dots, N_r^{t_r})}(M) = \left(Hom\left(\bigoplus_i N_i^{\oplus t_i}, M\right) - \{\text{not inj}\}\right) \Big/ Aut\left(\bigoplus_i N_i^{\oplus t_i}\right)}$$

$Hom\left(\bigoplus_i N_i^{\oplus t_i}, M\right)$:

$$Hom\left(\bigoplus_i N_i^{\oplus t_i}, M\right) \cong \bigoplus_i Hom(N_i, M)^{\oplus t_i}$$

where the basis of $Hom(N_i, M)$ can be read off from the AR-quiver
(though not easy! any technique for it?)

Injectivity: If $f: \bigoplus_i N_i^{\oplus t_i} \longrightarrow M$ is not inj, then $\ker f \neq 0$,

f factors through $\bigoplus_i N_i^{\oplus t_i} / \ker f \longrightarrow M$

$$\therefore \{\text{not inj}\} \cong \{ Hom(\bigoplus_i N_i^{\oplus t_i}/T, M) \mid 0 < T \leq \bigoplus_i N_i^{\oplus t_i} \}$$

Difficulty: It's doable (but not easy!) to compute the quotient (using SES)
You need to understand all submodules of $\bigoplus_i N_i^{\oplus t_i}$, (by induction...)
and there may be infinite many submodules!

$Aut(\bigoplus_i N_i^{\oplus t_i})$:

Lemma. $Aut(N_i^{\oplus t_i}) \cong GL_{t_i}(\mathbb{C})$

In general, we can imagine $Aut(\bigoplus_i N_i^{\oplus t_i})$ as "quasi upper triangler matrix".

Advantages:

① This decomposition is natural, and doesn't depend on our choices;

② It's easier to get dimensions of $GL_f(M)$ (If $\exists$ inj map) (inj map is open)
e.g. when $g: N \hookrightarrow M$ is a sectional morphism, then $GL_N(M) = \mathbb{CP}^{[N,M]-1}$

Conj: $\dim GL_N(M) = \begin{cases} -\infty & [N,M]=0 \\ [N,M]-1 & \text{otherwise} \end{cases}$ for N, M indecomposable

③ It can be easily generalized to (strict) partial flag variety.

③ It's possible: further decompositions according to the shape of quotients.

Disadventages:

① It's not affine paving

② Computations are too ugly to write down.

③ Even though it's possible to "fill in the holes", relations among these pieces are still unclear.

2. proj. dec for partial flag variety does not work.

E.g. η: $0 \longrightarrow 11\overset{0}{1}00 \longrightarrow 12\overset{2}{3}21 \longrightarrow 01\overset{1}{1}10 \oplus 00\overset{1}{1}11 \longrightarrow 0$

$$\begin{array}{ccc} 11\overset{0}{1}00 & 12\overset{1}{3}21 & 01\overset{1}{2}21 \\ \cup & \cup & \cup \\ 00\overset{0}{0}00 & 01\overset{1}{2}11 & 01\overset{1}{2}11 \\ \cup & \cup & \cup \\ 00\overset{0}{0}00 & 01\overset{1}{2}10 & 01\overset{1}{2}10 \end{array}$$

$$\mathrm{Ext}^1(01\overset{1}{1}10 \oplus 00\overset{1}{1}11,\ 11\overset{0}{1}00) = \mathrm{Hom}(11\overset{0}{1}00,\ 11\overset{1}{2}11 \oplus 11\overset{0}{1}10) = \mathbb{C}^2$$