Eine Woche, ein Beispiel

8.27. ramified covering: RS case

$$\begin{tikzcd}[column sep=large]
 & \text{char } 0 & \text{char } p & \text{mixed char} \\
\text{global} & RS, \mathbb{P}^1_{\mathbb{R}} \arrow[r] & \mathbb{P}^1_{\mathbb{F}_p} \arrow[r] & \text{"Spec } \mathbb{Q}\text{"} \arrow[dll, bend left] \\
\text{local} & \mathbb{C}((t)), \mathbb{R}((t)) \arrow[r] & \mathbb{F}_p((t)) \arrow[r] & \mathbb{Q}_p
\end{tikzcd}$$

$\rightarrow$: route in this series

For having the best geometrical intuition, we design this route. People may prefer working with local objects first(and then global objects), since global objects are glued by local objects.
However, you don't have to sharpen your tools before playing the puzzles.

play ——	tools ——	play again
global ——	local ——	local-global principle

Today: We work on Riemann Surface (RS), the most intuitive case.
The relationship with field extension is left to next time.

1. standard ramified covering
2. definition
3. examples
   - morphisms with explicit expressions
   - RS defined by equations
   - infinite pt case
   - morphisms defined by quotients.

1. standard ramified covering

For practice, we only consider ramified covering with finite ramification index.

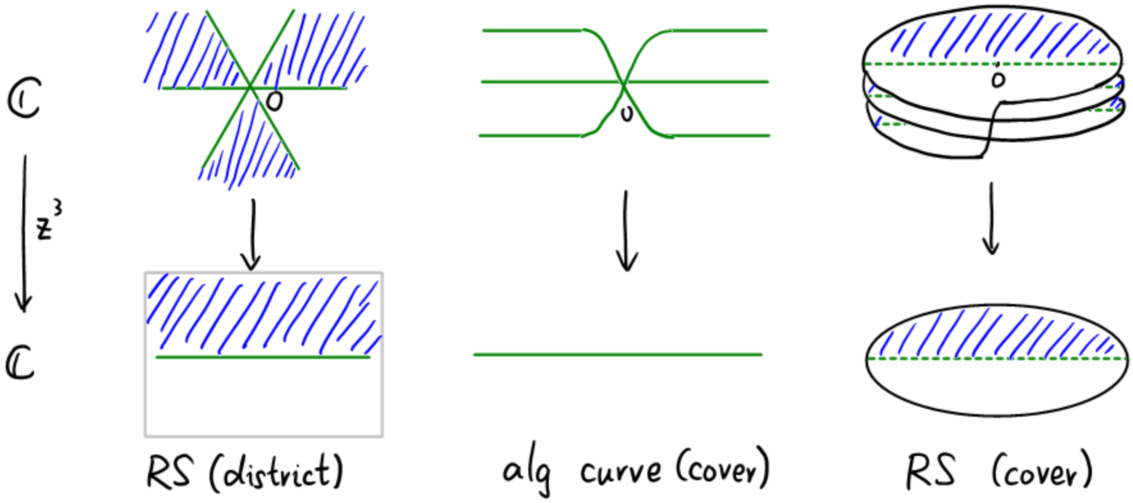
<u>Observation</u>. Consider

$$f: \mathbb{C} \longrightarrow \mathbb{C} \qquad z \longmapsto z^e$$

how to understand this fct?

- f is holomorphic ; $f \in \mathbb{Z}[z]$
- $f^*: \mathcal{M}(\mathbb{C}) \longrightarrow \mathcal{M}(\mathbb{C})$ field extension of deg e
- f is "roughly a cover":
  - $f^{-1}(z) = \begin{cases} e \text{ pts}, & z \neq 0 \\ \{0\}, & z = 0 \end{cases}$
  - $f|_{\mathbb{C}-\{0\}}: \mathbb{C}-\{0\} \longrightarrow \mathbb{C}-\{0\}$ is a cover
  - Once we divide C(=Im f) by several districts, with o lying in the boundary, we can divide domain by several districts, and see the movement of pts easily (as long as they don't pass o).

e.g. $e=3$



We will only draw the first two pictures later on, since the last one is too difficult to draw.

Fact (show in next document)

For a ramified covering $f: X \longrightarrow Y$ of deg e,
$f^*: \mathcal{M}(Y) \longrightarrow \mathcal{M}(X)$ is a field extension of deg e.
Notice that we don't assume X, Y to be cpt.

https://mathoverflow.net/questions/25085/the-riemann-correspondence-for-riemann-surfaces-made-explicit-and-its-generaliza
https://math.uchicago.edu/~may/REU2022/REUPapers/Marks.pdf

Def. For $e \in \mathbb{N}_{>0}$, we call

$$f_e : \mathcal{D} \longrightarrow \mathcal{D} \qquad z \longmapsto z^e$$

as the standard ramified covering of deg e.

2. definition

Def (Ramified covering / Branched covering)

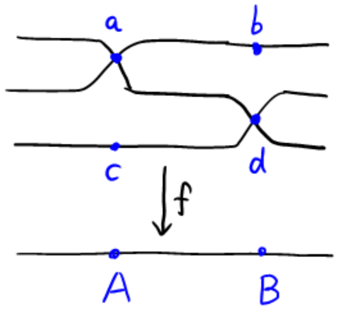
Let Y, X be oriented conn 2-dim topo mflds, $f: Y \to X$ be cont surj.
We say that f is a ramified covering, if

$$\forall x_0 \in X,\ \exists\, \mathcal{U} \overset{\text{open}}{\subset} X \text{ nbhd of } x_0 \text{ s.t.}$$

① $f^{-1}(\mathcal{U}) \cong \bigsqcup_{i \in I} V_i$ as topo spaces $\qquad V_i \overset{\text{open}}{\subset} X$

② $f|_{V_i} : V_i \longrightarrow \mathcal{U}$ is the standard ramified covering, i.e.,

$$\begin{tikzcd} V_i \arrow[r, "f|_{V_i}"] \arrow[d, "\cong"'] & \mathcal{U} \arrow[d, "\cong"] \\ \mathcal{D} \arrow[r, "f_e"] & \mathcal{D} \end{tikzcd}$$



$\mathrm{Ram}(f) = \{a, d\} \qquad \deg f = 3$

$\mathrm{Branch}(f) = \{A, B\} \qquad e_a = 2,\ e_c = 1$

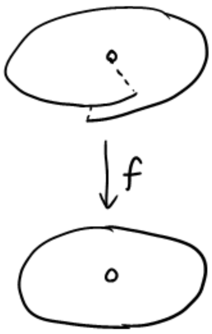
⚠ We don't consider $\mathcal{D} \subset \mathbb{C}$ as a cover, since ① does not work.

Rmk. For $f \in \mathcal{O}(X)$, $a \in X$,

$f'(a) \neq 0 \iff f$ is a local homeomorphism near a.

$n \neq -\infty, 0 \qquad \deg_a (f(x) - f(a)) = n \iff f$ is a ramified covering near a, with ramification index n.

Cor. For $f: X \to Y$ proper morphism of RS, f is a ramified covering



conn surj, not proper

not ramified covering

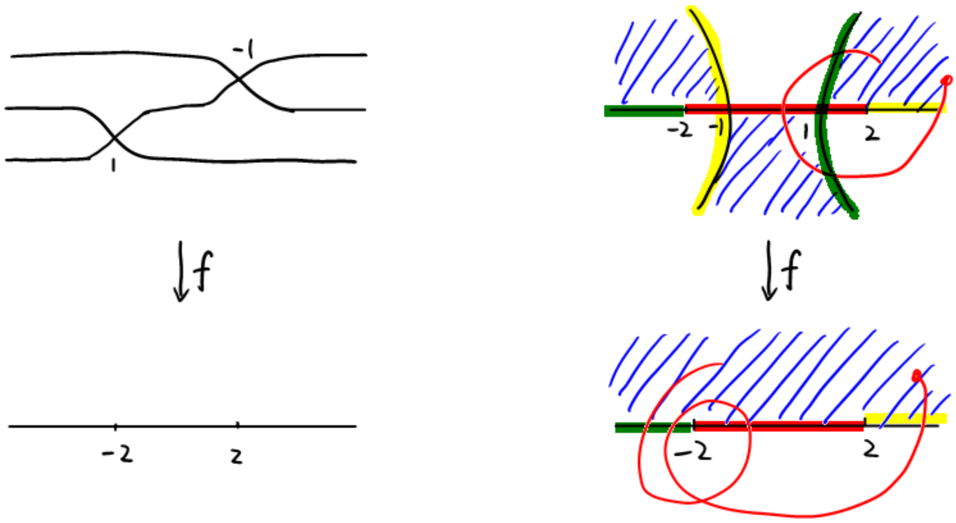
3. examples

morphisms with explicit expressions

Ex. For

$$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad f(z) = z^3 - 3z.$$

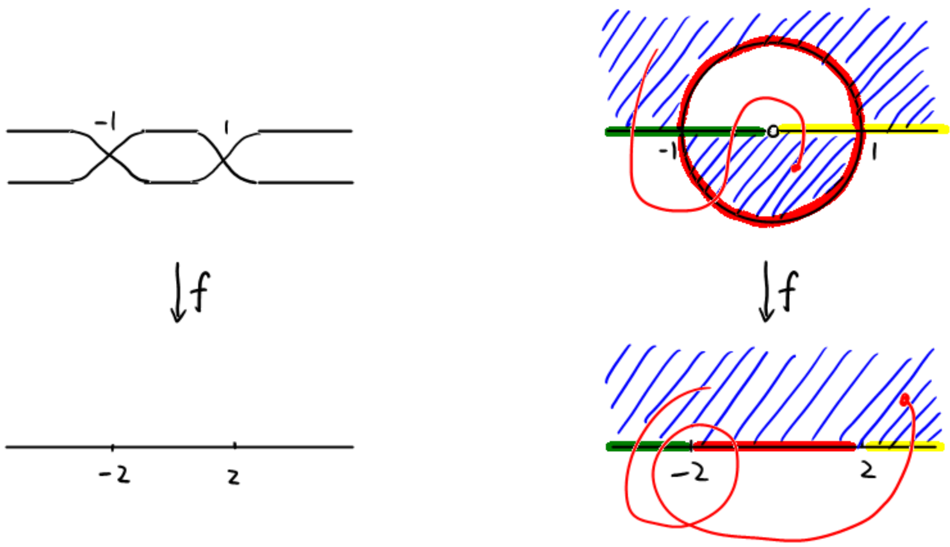
draw the picture.



Ex. For

$$f : \mathbb{C}^\times \longrightarrow \mathbb{C} \qquad f(z) = z + \frac{1}{z},$$

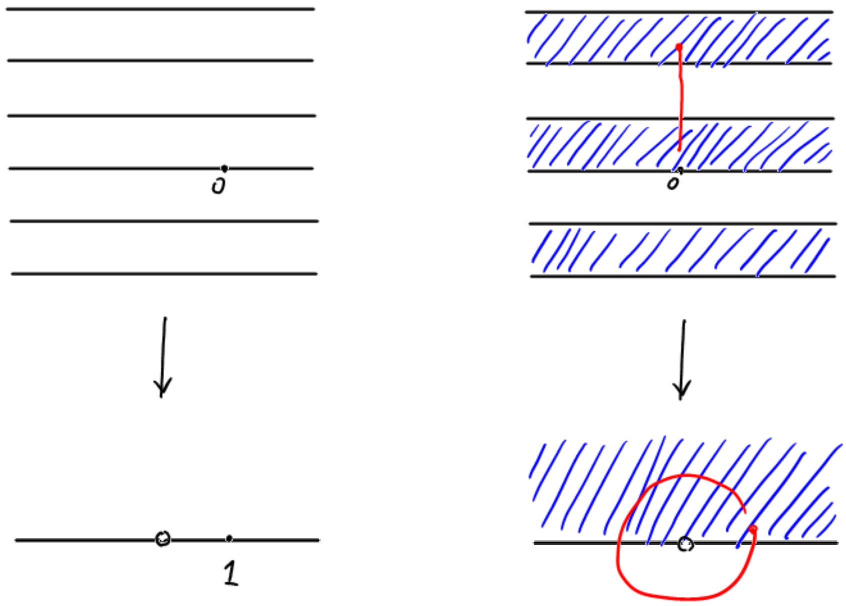
draw the picture.



Ex. For

$f : \mathbb{C} \longrightarrow \mathbb{C}^\times \qquad f(z) = e^z$

draw the picture.

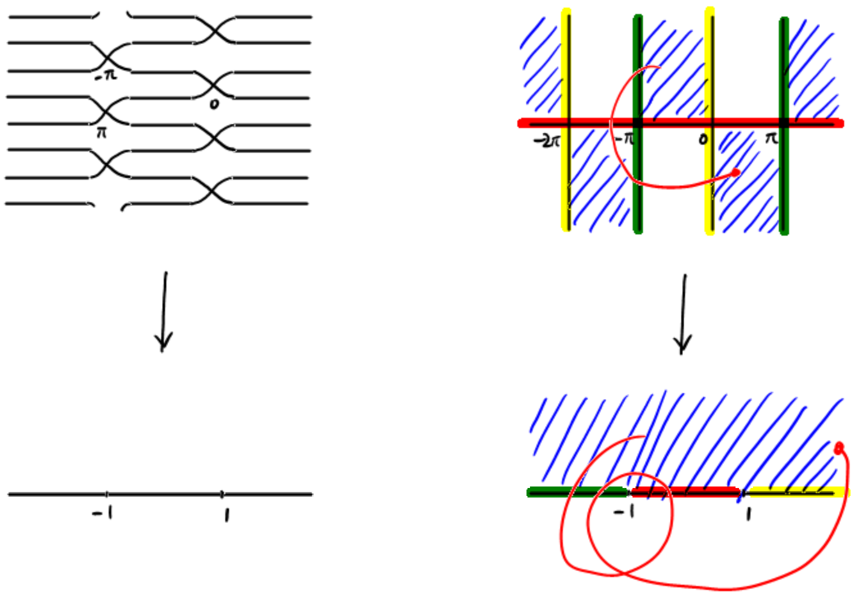


This is an infinite covering.

Ex. For

$f : \mathbb{C} \longrightarrow \mathbb{C} \qquad f(z) = \cos z$

draw the picture.



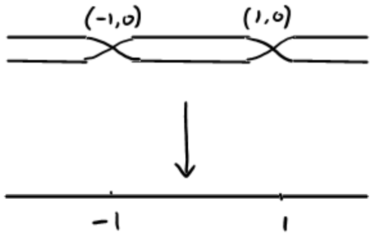
Q. How is the ramified index related with the order of zeros?

RS defined by equations

Ex. For $X := \{(x,y) \in \mathbb{C}^2 \mid x^2+y^2=1\} \subseteq \mathbb{C}^2$

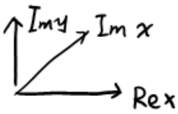
$$\begin{tikzcd} & \mathbb{C}^2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{C} & & \mathbb{C} \end{tikzcd}$$

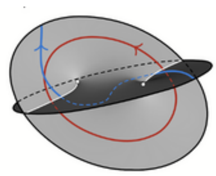
1) Shows that X is a RS
2) Shows that π_1 is a ramified cover, and determine
   - degree
   - ramified pt
   - ramification index



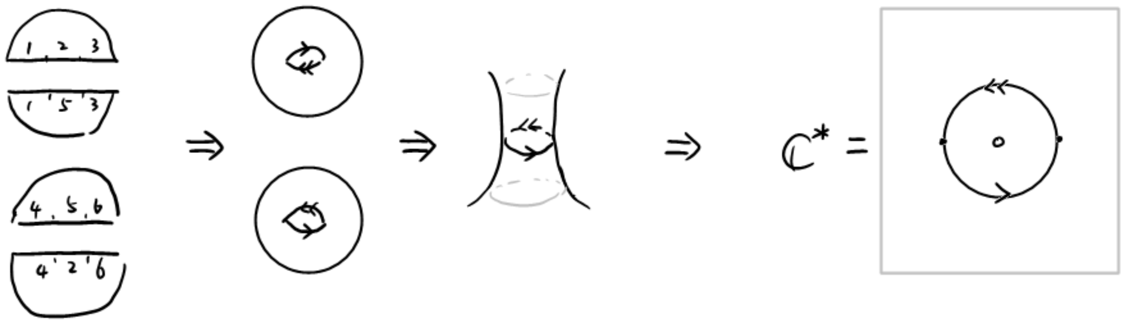
3) draw the picture of X

https://mathoverflow.net/questions/228622/intuition-for-picard-lefschetz-formula





4) Compute $H_i(X;\mathbb{R})$



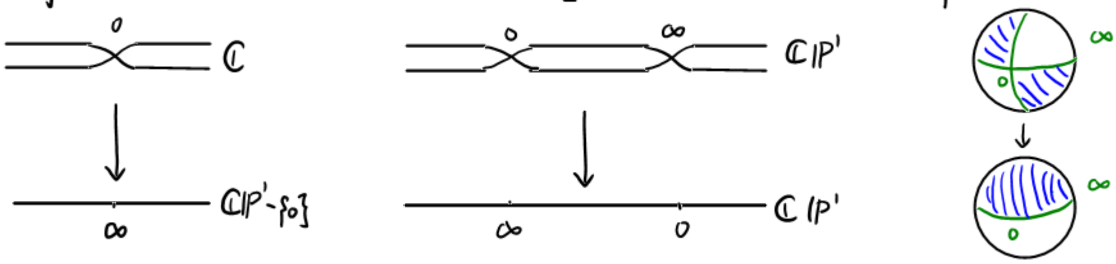
5) By identifying X by $\mathbb{C}^*$, draw the cover : $\mathbb{C}^* \longrightarrow \mathbb{C}$

$$\begin{tikzcd} X \arrow[r, "(x,y)\mapsto x+iy", "\cong"'] \arrow[d, "\pi_1"'] & \mathbb{C}^* \arrow[d, "z+\frac{1}{z}"] \\ \mathbb{C} \arrow[r, "\times 2", "\cong"'] & \mathbb{C} \end{tikzcd}$$

infinite pt case

Ex. $f: \mathbb{C} \longrightarrow \mathbb{CP}^1 - \{0\}$ $\quad z \longmapsto \frac{1}{z^2} = [1 : z^2]$ is a ramified covering

$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$ $\quad z \longmapsto \frac{1}{z^2}$ is a ramified covering



Ex. Work out the cases

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z$$
$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z + \frac{1}{z}$$

Ex. For $\widetilde{X} := \{[x:y:z] \in \mathbb{CP}^2 \mid x^2 + y^2 = z^2\} \subseteq \mathbb{CP}^2$,

$$\widetilde{f}: \widetilde{X} \longrightarrow \mathbb{CP}^1 \qquad [x:y:z] \longmapsto [x:z]$$

draw the picture.

[Hint. Consider $\widetilde{f}|_{\widetilde{f}^{-1}(\mathbb{C})}$ first.

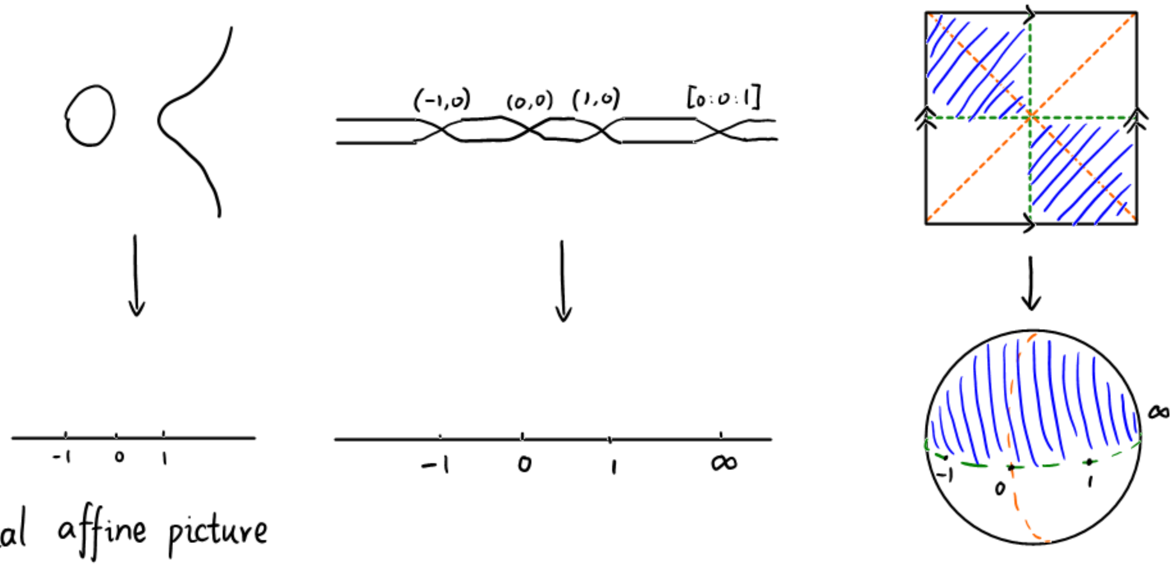
$$\begin{tikzcd} \widetilde{X} \arrow[r, "{[x:y:z] \mapsto [x+iy:z]}", "\cong"'] \arrow[d, "\widetilde{f}"'] & \mathbb{CP}^1 \arrow[d, "z \mapsto z + \frac{1}{z}"] \\ \mathbb{CP}^1 \arrow[r, "\times 2", "\cong"'] & \mathbb{CP}^1 \end{tikzcd}$$

]

Ex. For $E := \{[x:y:z] \in \mathbb{CP}^2 \mid y^2 z = x(x-z)(x+z)\} \subseteq \mathbb{CP}^2$,

$$\pi: E \longrightarrow \mathbb{CP}^1 \qquad [x:y:z] \longmapsto [x:z],$$

draw the picture.



real affine picture

Thm (Riemann-Hurwitz formula)

Let $f: X \to Y$ be non-constant morphism between cpt RS, then

$$2g(X) - 2 = (2g(Y) - 2)\deg f + \sum_{x \in \mathrm{Ram}(f)} (e(x) - 1)$$

Hint: Use triangulation on Y, which induces a triangulation on X. (may refine triangulation, if needed)

Ex. Verify RH formula for those above examples.

Ex. Compute the genus of Klein quartic:

$$\mathcal{C} = \{[x:y:z] \in \mathbb{CP}^1 \mid x^3 y + y^3 z + z^3 x = 0\}$$

morphism defined by quotients

See my bachelor thesis: https://github.com/ramified/personal_tex_collection/blob/main/bachelor_thesis/thesis/main.pdf
See more: search the keyword "Dedekind tessellation".

Try to draw $\mathcal{H} \longrightarrow SL_2(\mathbb{Z})\backslash\mathcal{H}$

$\Gamma(N)\backslash\mathcal{H} \longrightarrow SL_2(\mathbb{Z})\backslash\mathcal{H}$ finite cover

Rmk. In the next section we will see, all Galois coverings are of form

$X \longrightarrow X/G$ where $G \subset \mathrm{Aut}_{RS}(X)$.

e.g. $X = \mathbb{CP}^1$ $\qquad \mathrm{Aut}_{RS}(\mathbb{CP}^1) = PGL_2(\mathbb{C})$

$X = E$ non CM EC, then $\qquad \mathrm{Aut}_{EC}(E) = \mathbb{Z}/2\mathbb{Z}$

$\mathrm{Aut}_{RS}(E) \cong E \rtimes \mathbb{Z}/2\mathbb{Z}$

In page 10 the automorphism group of EC is listed: https://twma.files.wordpress.com/2016/10/slides2.pdf

X: RS with genus $g \geq 2$ $\qquad \# \mathrm{Aut}_{RS}(X) \leq 84(g-1)$