Eine Woche, ein Beispiel

7.13. stability manifold of $\mathbb{P}^1$

Ref:

[Okada05]: So Okada, Stability Manifold of P^1

[GKR03]: A. Gorodentscev, S. Kuleshov, A. Rudakov, t-stabilities and t-structures on triangulated categories, https://arxiv.org/abs/math/0312442

[Huy23]: Huybrechts, Daniel. The Geometry of Cubic Hypersurfaces.

[Huy06]: Huybrechts, D. Fourier-Mukai Transforms in Algebraic Geometry. Oxford Math. Monogr. Oxford: Clarendon Press, 2006

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Goal: understand the Bridgeland stability and wall crossing in this toy example.

1. equivalent definitions of stability conditions

2. structure of $\mathrm{Coh}(\mathbb{P}^1)$

1. equivalent definitions of stability conditions

Def (locally finite stability condition)

Fix a triangular category $\mathcal{T}$, and denote $K(\mathcal{T})$ as the Grothendieck gp of $\mathcal{T}$.
The set of locally finite stability conditions is defined as

$$\mathrm{Stab}(\mathcal{T}) = \left\{ (Z, \mathcal{P}) \;\middle|\; \begin{array}{l} Z : K(\mathcal{T}) \longrightarrow \mathbb{C} \qquad \text{(central charge)} \\ \mathcal{P} : \mathbb{R} \longrightarrow \{\text{full additive subcategories of } \mathcal{T}\} \\ \qquad\quad \phi \longmapsto \mathcal{P}(\phi) \qquad \text{(slicing)} \\ \text{s.t. } (a)(b)(c)(d) + (e) \end{array} \right\}$$

(a) (slicing compatible with central charge)
if $E \in \mathcal{P}(\phi)$ then $\frac{Z(E)}{e^{i\pi\phi}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)
$$\mathcal{P}(\phi+1) = \mathcal{P}(\phi)[1]$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for } A_j \in \mathcal{P}(\phi_j),\ \phi_1 > \phi_2$$

(d) (HN filtration) HN = Harder-Narashimhan
$\forall E \in \mathcal{T}$, $\exists$ finite seq of real numbers
$$\phi_1 > \phi_2 > \cdots > \phi_n$$
and a filtration

$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[dl] \\ A_1 \arrow[u, dashed, "+1"] & & & A_n \arrow[u, dashed, "+1"] & \end{tikzcd}$$

s.t. $A_j \in \mathcal{P}(\phi_j)$ $\forall j$.
we define $\phi(E) = \{\phi_1, \ldots, \phi_n\}$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{P}(I)$, $\exists$ a Jordan-Hölder filtration with finite length.
$\mathcal{P}(I) := \langle \mathcal{P}(\phi) \mid \phi \in I \rangle_{\text{extension-closed}}$

Rmk. For $E \in \mathcal{T}$, $E \neq 0$,
$E \in \mathcal{P}(\phi)$ for some $\phi \in \mathbb{R}$
$\Leftrightarrow$ the HN filtration of E has length 1
$\overset{\text{def}}{\Leftrightarrow}$ E is semistable

When E is semistable, define $\phi(E) = \phi$ when $E \in \mathcal{P}(\phi)$.

Lemma 1.1. $\mathcal{P}(\phi)$ is closed under extension.

Proof. Suppose one has one triangle

$$A_1 \longrightarrow E \longrightarrow A_2 \xrightarrow{+1} \qquad (1.1)$$

where $A_1, A_2 \in \mathcal{P}(\phi_0)$, we want to show $E \in \mathcal{P}(\phi_0)$.

Suppose $\phi(E) = \{\phi_1, \ldots, \phi_n\}$, $\phi_1 > \cdots > \phi_n$, $n \geq 1$, then

$$\phi_0 > \phi_n \quad \text{or} \quad \phi_1 > \phi_0 \quad \text{or} \quad (\underbrace{\phi_1 = \phi_0,\ n = 1}_{\Downarrow\ E \in \mathcal{P}(\phi_0)\ \checkmark})$$

w.l.o.g. assume $\phi_0 > \phi_n$, then $\exists$ triangle

$$B_1 \longrightarrow E \xrightarrow{u} B_2 \xrightarrow{+1}$$

where $u \neq 0$, $B_2 \in \mathcal{P}(\phi_n)$.

Apply $\mathrm{Hom}(-, B_2)$ to (1.1), we get

$$\cdots \longleftarrow \mathrm{Hom}(A_1[-1], B_2) \longleftarrow \mathrm{Hom}(E[-1], B_2) \longleftarrow \mathrm{Hom}(A_2[-1], B_2) \longleftarrow$$

$$\longleftarrow \underset{\substack{\| \\ 0}}{\mathrm{Hom}(A_1, B_2)} \longleftarrow \underset{\substack{\Downarrow \\ u \neq 0}}{\mathrm{Hom}(E, B_2)} \longleftarrow \underset{\substack{\| \\ 0}}{\mathrm{Hom}(A_2, B_2)} \longleftarrow \cdots$$

Contradiction!

The next lemma conclude the behavior of triangles with stability conditions.

Lemma 1.2.

Suppose

$$A_1 \xrightarrow{u_1} E \xrightarrow{u_2} A_2 \xrightarrow[+1]{u_3} \qquad (1.2)$$

is a triangle, where $\phi(A_1) = \phi_0$, $\phi(A_2) = \phi_0'$.

(1) If $\phi_0 > \phi_0'$, then

(1.2) is the HN-filtration, so E is not semistable;

(2) If $\phi_0 = \phi_0'$, then $E \in \mathcal{P}(\phi_0)$ by Lemma 1.1;

(3) If $u_3 \neq 0$, then $\widetilde{\phi_0} \leq \phi_0 + 1$.

Rmk.

$$\mathrm{Stab}(\mathcal{T}) \triangleq \left\{ (Z,\phi) \;\middle|\; \begin{array}{l} Z: K(\mathcal{T}) \longrightarrow \mathbb{C} \quad \text{(central charge)} \\ \phi: \mathcal{T} \longrightarrow \{\text{finite subsets of } \mathbb{R}\} \\ \quad\;\; E \longmapsto \{\phi_1,\dots,\phi_n\} \quad \text{(slicing)} \\ \text{s.t. } \text{(a)(b)(c)(d)} + \text{(e)} \end{array} \right\}$$

$E \in \mathcal{T}$ is semistable $\overset{\text{def}}{\Longleftrightarrow}$ $\# \phi(E) = 1$

(a) (slicing compatible with central charge)
For E semistable, $\dfrac{Z(E)}{e^{i\pi\phi(E)}} \in \mathbb{R}_{>0}$;

(b) (slicing with shift)
$$\phi(E[1]) = \phi(E) + 1$$

(c) (inverse order vanishing)
$$\mathrm{Hom}_{\mathcal{T}}(A_1, A_2) = 0 \quad \text{for} \quad \phi(A_1) > \phi(A_2), \; A_1, A_2 \text{ semistable}$$

(d) (HN filtration)
$\forall E \in \mathcal{T}$, denote $\phi(E) = \{\phi_1, \dots, \phi_n\}$, $\phi_1 < \dots < \phi_n$,

$\exists!$ filtration

$$\begin{tikzcd} 0 = E_0 \arrow[r] & E_1 \arrow[d] \arrow[r] & \cdots \arrow[r] & E_{n-1} \arrow[r] & E_n = E \arrow[d] \\ & A_1 \arrow[ul, dashed, "+1"] & & & A_n \arrow[ul, dashed, "+1"] \end{tikzcd}$$

s.t. $\phi(A_i) = \phi_i \quad \forall i$.

(e) (loc finite) $\forall t \in \mathbb{R}$, $\exists$ $I = (t-\varepsilon, t+\varepsilon) \subseteq \mathbb{R}$ s.t.
$\forall E \in \mathcal{T}$ with $\phi(E) \subset I$,
$\exists$ a Jordan-Hölder filtration with finite length.

Prop [Okada 05, Prop 2.3]

$$\mathrm{Stab}(\mathcal{T}) \cong \left\{ (\mathcal{A}, Z) \;\middle|\; \begin{array}{l} \mathcal{A} : \text{heart of } \mathcal{T} \\ Z : K(\mathcal{A}) \longrightarrow \mathbb{C} \\ \quad \text{centered slope-function} \\ \quad \text{with HN property} \end{array} \right\}$$

$$(Z, \mathcal{P}) \longmapsto (\mathcal{P}((0,1]), Z)$$
$$(Z, \mathcal{P}) \longleftarrow\!\shortmid \ (\mathcal{A}, Z)$$

where $\mathcal{P}(\phi) = \{ E \in \mathcal{A} \text{ semistable} \mid \widetilde{\phi}(E) = \phi \} \quad \forall \phi \in (0,1]$
$\widetilde{\phi}(E) = \frac{1}{\pi} \arg Z(E) \in (0,1]$

$E \in \mathcal{A}$ semistable: $\nexists$ dec $0 \longrightarrow A_1 \longrightarrow E \longrightarrow A_2 \longrightarrow 0$ s.t. $\phi(A_1) > \phi(E) > \phi(A_2)$

2. structure of $Coh(\mathbb{P}^1)$

Lemma 2.1. On $\mathbb{P}^1$, we have SESs

$$0 \longrightarrow \mathcal{O} \xrightarrow{\times x} \mathcal{O}(1) \longrightarrow \mathcal{O}_x \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(n)^{\oplus n+1} \longrightarrow \mathcal{O}(n+1)^{\oplus n} \longrightarrow 0 \qquad n \geq 0 \quad (2.1)$$

$$0 \longrightarrow \mathcal{O}(-1)^{\oplus n} \longrightarrow \mathcal{O}^{\oplus n+1} \longrightarrow \mathcal{O}(n) \longrightarrow 0 \qquad n \geq 0$$

which induces triangles

$$\mathcal{O}(k+1) \longrightarrow \mathcal{O}_x \longrightarrow \mathcal{O}(k)[1] \xrightarrow{+1}$$

$$\mathcal{O}(k+1)^{\oplus k-n}[-1] \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus k-n+1} \xrightarrow{+1} \qquad n \leq k \quad (2.2)$$

$$\mathcal{O}(k+1)^{\oplus n-k} \longrightarrow \mathcal{O}(n) \longrightarrow \mathcal{O}(k)^{\oplus n-k+1}[1] \xrightarrow{+1} \qquad n \geq k$$

Lemma 2.2. On $\mathbb{P}^1$, we have

$$RHom(\mathcal{O}, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

$$RHom(\mathcal{O}, k_p) = \mathbb{C}$$

$$RHom(k_p, \mathcal{O}) = \mathbb{C}[-1]$$

$$RHom(k_p, k_q) = \begin{cases} \mathbb{C} \oplus \mathbb{C}[-1], & p = q \\ 0, & p \neq q \end{cases}$$

Sketch of proof

$$RHom(\mathcal{O}, \mathcal{O}(n)) = H^\bullet(\mathbb{P}^1, \mathcal{O}(n)) = \begin{cases} \mathbb{C}^{n+1}, & n \geq -1 \\ \mathbb{C}^{-n-1}[-1], & n \leq -1 \end{cases}$$

Then apply $RHom(\mathcal{O}, -)$, $RHom(-, \mathcal{O})$, $RHom(-, k_q)$ to

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow k_p \longrightarrow 0$$

$\square$

The next two lemmas tells us the structure of Coh(P^1).

Lemma 2.3. [GKR03, last line in p16]

$$\forall \mathcal{F} \in \mathrm{Coh}(\mathbb{P}^1), \quad \mathcal{F} = \Big(\bigoplus_p k_p\Big) \oplus \Big(\bigoplus_i \mathcal{O}(n_i)\Big)$$

finite many

Lemma 2.4. [GKR03, Prop 6.3]

$$\forall \mathcal{F}^\cdot \in \mathcal{D}^b(\mathrm{Coh}(\mathbb{P}^1)), \quad \mathcal{F}^\cdot = \bigoplus_i A_i[-i] \quad A_i \in \mathrm{Coh}(\mathbb{P}^1)$$

It also works for $\mathcal{D}^b(\mathcal{A})$ where $\mathrm{gldim}\,\mathcal{A} = 1$.

E.g. Since $\mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n)) \cong \mathrm{Ext}^1(k_p,\mathcal{O}) \oplus \mathrm{Ext}^1(k_p,\mathcal{O}(n)) \cong \mathbb{C}^2$, let us describe the extension

$$0 \longrightarrow \mathcal{O}\oplus\mathcal{O}(n) \longrightarrow E \longrightarrow k_p \longrightarrow 0$$

crspd to $(k_1,k_2) \in \mathrm{Ext}^1(k_p, \mathcal{O}\oplus\mathcal{O}(n))$.

For simplicity, assume that $n>0$ & $k_1, k_2 \neq 0$.

It is defined as pulling back SES:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r] \arrow[d, equal] & E \arrow[r] \arrow[d] & k_p \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \mathcal{O}\oplus\mathcal{O}(n) \arrow[r, "\binom{k_1 z}{k_2 z}"] & \mathcal{O}(1)\oplus\mathcal{O}(n+1) \arrow[r] & k_p\oplus k_p \arrow[r] & 0
\end{tikzcd} \qquad (2.3)$$

Since $\deg E = n+1$, $\operatorname{rank} E = 2$, by Lemma 2.4. we get

$$E = \mathcal{O}\oplus\mathcal{O}(n+1), \quad \mathcal{O}(1)\oplus\mathcal{O}(n) \quad \text{or} \quad \mathcal{O}\oplus\mathcal{O}(n)\oplus k_p$$

but which?

We apply $R\mathrm{Hom}(-,\mathcal{O})$ to (2.3):

$$\begin{tikzcd}
0 & \underset{\mathbb{C}^{n-1}}{\mathrm{Ext}^1(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})} \arrow[l] & \mathrm{Ext}^1(E,\mathcal{O}) \arrow[l] & \underset{\mathbb{C}}{\mathrm{Ext}^1(k_p,\mathcal{O})} \arrow[l] \\
& \underset{\mathbb{C}}{\mathrm{Hom}(\mathcal{O}\oplus\mathcal{O}(n),\mathcal{O})} \arrow[urr, "k_1"'] & \mathrm{Hom}(E,\mathcal{O}) \arrow[l] & \underset{0}{\mathrm{Hom}(k_p,\mathcal{O})} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$\Rightarrow R\mathrm{Hom}(E,\mathcal{O}) = \mathbb{C}^{n-1}[-1]$

$\Rightarrow E \cong \mathcal{O}(1)\oplus\mathcal{O}(n)$.