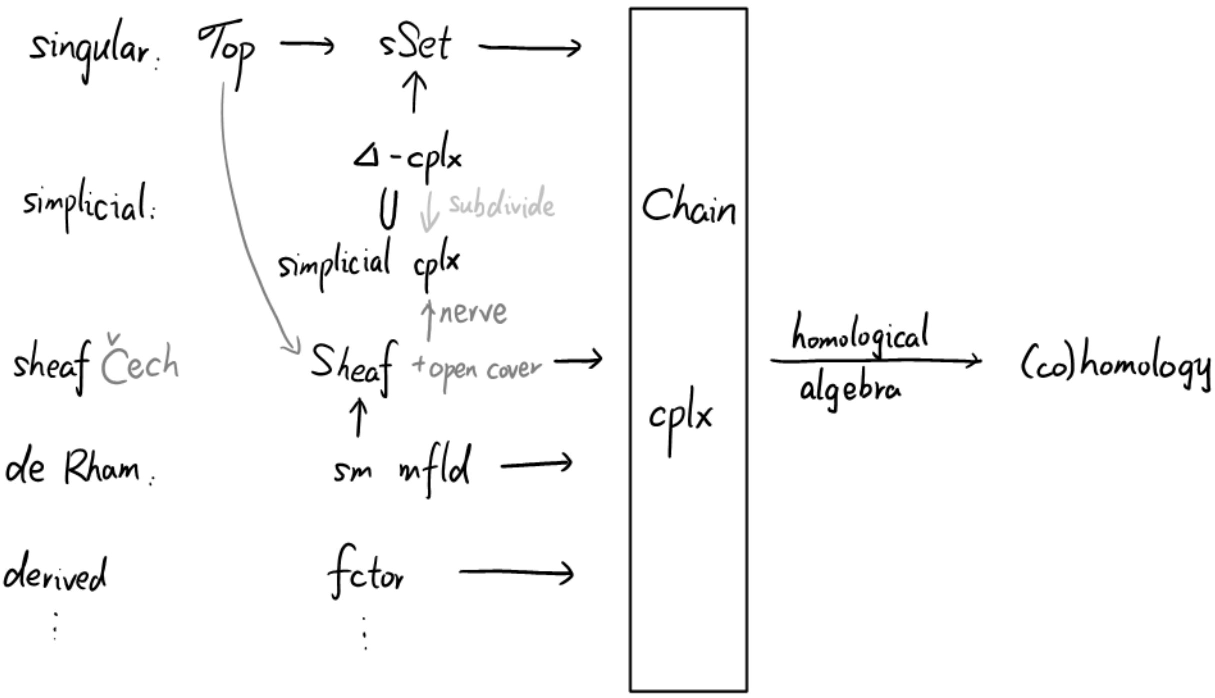
Eine Woche, ein Beispiel

6.25 (co)homology of simplicial set

https://ncatlab.org/nlab/show/simplicial+complex
https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets



Today: sSet ⟶ chain cplx - - → (co)homology

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

We use α here because we are considering $X = \Delta^n$ case. May change to x in the future.

Def. For $X \in sSet$, $G \in Mod(\mathbb{Z})$, define

$$C_n(X;G) = \bigoplus_{\alpha \in X_n} G \qquad 0 \leftarrow \bigoplus_{\alpha\in X_0} G \xleftarrow{(d_0^1-d_1^1)^*} \bigoplus_{\alpha\in X_1} G \xleftarrow{(d_0^2-d_0^1+d_2^2)^*} \bigoplus_{\alpha\in X_2} G \cdots$$

$$C^n(X;G) = \prod_{\alpha\in X_n} G \qquad 0 \longrightarrow \prod_{\alpha\in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha\in X_1} G \longrightarrow \prod_{\alpha\in X_2} G \cdots$$

$$C^{BM}_n(X;G) =$$

$$C^n_c(X;G) =$$

$$Hom_{\mathbb{Z}\text{-mod}}\Big(\bigoplus_{\alpha\in X_n} \mathbb{Z}, G\Big) \cong \prod_{\alpha\in X_n} Hom_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha\in X_n} G$$

https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-m%c3%b6bius-strip?rq=1
https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree

E.g. 1 For $A \in Top$ discrete, $X := \mathcal{S}(A) \in sSet$, one can compute

$$C_\bullet(X;G): \quad 0 \leftarrow \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \bigoplus_{\alpha\in A} G \xleftarrow{0} \bigoplus_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C^\bullet(X;G): \quad 0 \rightarrow \prod_{\alpha\in A} G \xrightarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xrightarrow{Id} \cdots$$

$$C^{BM}_\bullet(X;G): \quad 0 \leftarrow \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \prod_{\alpha\in A} G \xleftarrow{0} \prod_{\alpha\in A} G \xleftarrow{Id} \cdots$$

$$C^\bullet_c(X;G): \quad 0 \rightarrow \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \bigoplus_{\alpha\in A} G \xrightarrow{0} \bigoplus_{\alpha\in A} G \xrightarrow{Id} \cdots$$

Therefore,

$$H_n(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^{BM}_n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(X;G) = \begin{cases} \prod_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases} \qquad H^n_c(X;G) = \begin{cases} \bigoplus_{\alpha\in A} G & n=0 \\ 0 & n>0 \end{cases}$$

Eg. 2. We want to compute $H_n(\Delta';G)$ & $H^n(\Delta';G)$.

Notice that $\#\Delta'_k = k+2$, so

$$C_\cdot(\Delta';G):\quad 0 \longleftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}} G^{\oplus 5} \cdots$$

basis: $d'_0 \triangleq x_0$; $d'_1 \triangleq x_1$; x_0 ; x_1 ; x_2 ; x_0 ; x_1 ; x_2 ; x_3 ; x_0 ; x_1 ; x_2 ; x_3 ; x_4

remember indexes.

$$0 = x_0 - x_0 \longleftarrow\!\shortmid\ x_0$$
$$x_0 - x_1 = x_0 - x_1 \longleftarrow\!\shortmid\ x_1$$
$$0 = x_1 - x_1 \longleftarrow\!\shortmid\ x_2$$

$$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow\!\shortmid\ x_0$$
$$x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow\!\shortmid\ x_1$$
$$0 = x_1 - x_1 + x_2 - x_2 \longleftarrow\!\shortmid\ x_2$$
$$x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow\!\shortmid\ x_3$$
$$0 = x_3 - x_3 + x_3 - x_3 \longleftarrow\!\shortmid\ x_4$$

$$x_0 = x_0 - x_0 + x_0 \longleftarrow\!\shortmid\ x_0$$
$$x_0 = x_0 - x_1 + x_1 \longleftarrow\!\shortmid\ x_1$$
$$x_2 = x_1 - x_1 + x_2 \longleftarrow\!\shortmid\ x_2$$
$$x_2 = x_2 - x_2 + x_2 \longleftarrow\!\shortmid\ x_3$$

By taking the transpose, one get

$$C^\cdot(\Delta';G):\quad 0 \longrightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix}0&0\\1&-1\\0&0\end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix}1&0&0\\1&0&0\\0&0&1\\0&0&1\end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix}0&0&0&0\\1&-1&0&0\\0&0&0&0\\0&0&1&-1\\0&0&0&0\end{pmatrix}} G^{\oplus 5} \cdots$$

Therefore,

$$H_n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta';G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C_\bullet(\Delta^1;G)$ and $C_\bullet(\Delta^0;G)$.

$$
\begin{tikzcd}[column sep=large]
\Delta^1 \arrow[d, "s_0^1"] & C_\bullet(\Delta^1;G): \arrow[d, "s^1_{0,*}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\begin{pmatrix}1&1\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}1&1&1\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}1&1&1&1\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, "\begin{pmatrix}1&1&1&1&1\end{pmatrix}"] \\
\Delta^0 & C_\bullet(\Delta^0;G): & 0 & G \arrow[l] & G \arrow[l, "0"'] & G \arrow[l, "Id"'] & G \cdots \arrow[l, "0"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
\Delta^0 \arrow[d, "d_0^1"] & C_\bullet(\Delta^0;G): \arrow[d, "d^1_{0,*}"] & 0 & G \arrow[l] \arrow[d, "\begin{pmatrix}1\\0\end{pmatrix}"] & G \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\end{pmatrix}"] & G \arrow[l, "Id"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\end{pmatrix}"] & G \cdots \arrow[l, "0"'] \arrow[d, "\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}"] \\
\Delta^1 & C_\bullet(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

s_0^1 $\quad$ d_0^1

s.t. $\quad s^1_{0,*} \circ d^1_{0,*} = Id_{C_\bullet(\Delta^1;G)}, \qquad d^1_{0,*} \circ s^1_{0,*} \sim Id_{C_\bullet(\Delta^1;G)}.$

In fact, we have

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
C_\bullet(\Delta^1;G): \arrow[d, shift right, "Id"'] \arrow[d, shift left, "(\cdot)_*"] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&0\\0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0\\0&1\\0&0\end{pmatrix}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}0&0&0\\0&0&1\\0&0&-1\\0&0&0\end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix}1&0&0&0\\0&1&1&1\\0&0&0&-1\\0&0&0&1\\0&0&0&0\end{pmatrix}"] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"'] \arrow[d, shift right, "Id"'] \arrow[d, shift left, "\begin{pmatrix}1&1&1&1&1\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{pmatrix}"] \\
C_\bullet(\Delta^1;G): & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix}0&1&0\\0&-1&0\end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix}1&1&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix}"'] & G^{\oplus 5} \cdots \arrow[l, "\begin{pmatrix}0&1&0&0&0\\0&-1&0&0&0\\0&0&0&1&0\\0&0&0&-1&0\end{pmatrix}"']
\end{tikzcd}
$$

$x_0 \mapsto x_0$

$x_1 \mapsto x_1$

$x_0 \mapsto x_0 - x_0 + x_0 = x_0$

$x_1 \mapsto x_1 - x_1 + x_1 = x_1$

$x_2 \mapsto x_1 - x_2 + x_2 = x_1$

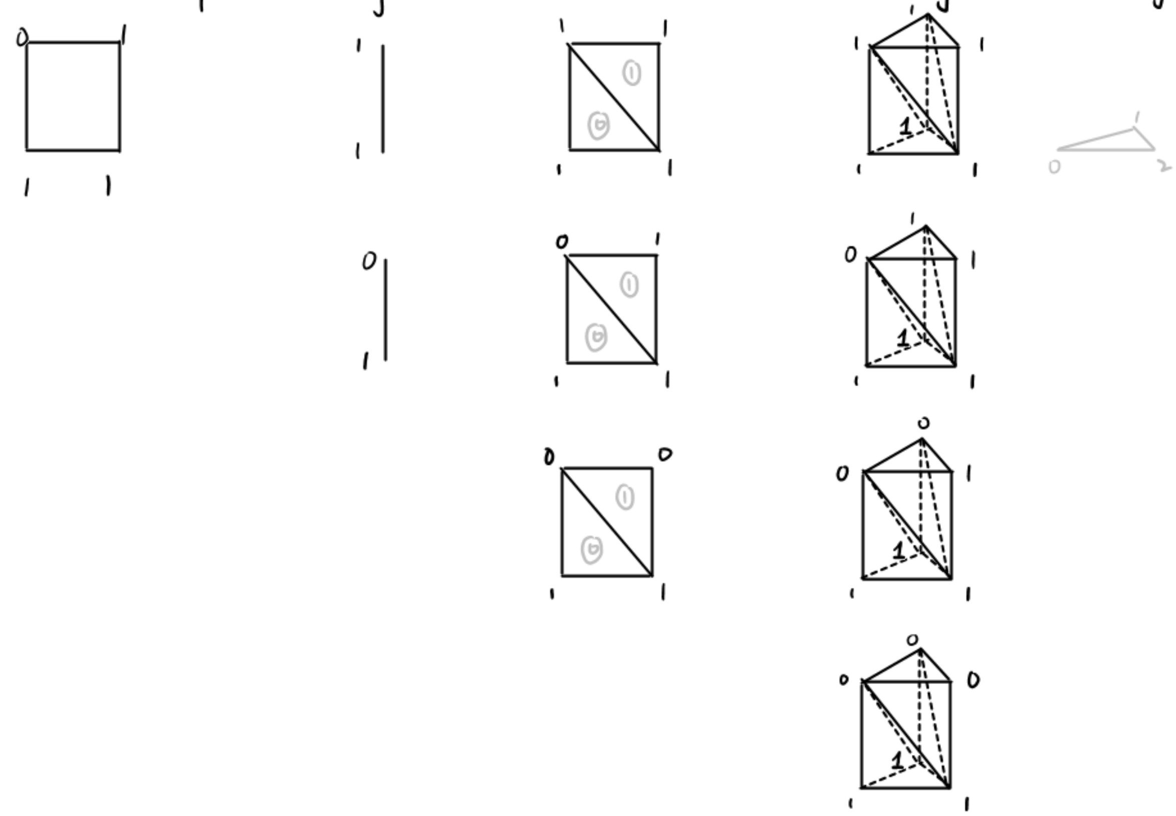
$x_3 \mapsto x_1 - x_2 + x_3$

$x_0 \mapsto x_0 - x_0 = 0$

$x_1 \mapsto x_1 - x_1 = 0$

$x_2 \mapsto x_1 - x_2$

Ex. Observe the picture, try to translate the calculation in geometrical language.



2. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_\cdot(\Delta';G)$ and $C_\cdot(\Delta';G)^{\diamond}$:

non-degenerate

$$
\begin{tikzcd}
C_\cdot(\Delta';G) : \arrow[d, "\text{projection}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, "(1\,1\,1)"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] \arrow[d, "0"] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G)^{\diamond} : & 0 & G^{\oplus 2} \arrow[l] & G \arrow[l, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"'] & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}
C_\cdot(\Delta';G)^{\diamond} : \arrow[d, "\text{inclusion}"] & 0 & G^{\oplus 2} \arrow[l] \arrow[d, "\mathrm{Id}"] & G \arrow[l, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"'] \arrow[d, "\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & 0 \arrow[l, "0"'] \arrow[d, "0"] & \cdots \\
C_\cdot(\Delta';G) : & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

In fact, we have

$$
\begin{tikzcd}
C_\cdot(\Delta';G) : \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left] & 0 \arrow[dr, dashed, red, "0"] & G^{\oplus 2} \arrow[l] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\mathrm{Id}"] \arrow[dr, dashed, red, "0"] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}"] \arrow[dr, dashed, red, "\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}"] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] \arrow[dr, dashed, red] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] \arrow[d, shift right, "\mathrm{Id}"'] \arrow[d, shift left, "0"] & \cdots \\
C_\cdot(\Delta';G) : & 0 & G^{\oplus 2} \arrow[l] & G^{\oplus 3} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}"'] & G^{\oplus 4} \arrow[l, "\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"'] & G^{\oplus 5} \arrow[l, "\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}"'] & \cdots
\end{tikzcd}
$$

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)
For $X \in sSet$, define

$\Diamond$: non-degenerate
$\lightning$: degenerate

$$X_k^{\Diamond} := \{ x \in X_k \mid x \text{ non-degerate} \} = X_k \setminus (sk^{k-1}X)_k$$
$$X_k^{\lightning} := \{ x \in X_k \mid x \text{ degenerate} \} = (sk^{k-1}X)_k$$
$$X_k^{\lightning i} := \left\{ x \in X_k \;\middle|\; x = \alpha^*(y) \text{ for some } y \in X_{k-i}^{\Diamond},\ \alpha : [k-i] \to [k] \right\} = (sk^{k-i}X)_k - (sk^{k-i-1}X)_k$$

$$0 = \underbrace{(sk^{-1}X)_k \overset{X_k^{\lightning k}}{\subset} (sk^0X)_k \overset{X_k^{\lightning k-1}}{\subset} (sk^1X)_k \overset{X_k^{\lightning k-2}}{\subset} \cdots \overset{X_k^{\lightning 1}}{\subset} (sk^{k-1}X)_k}_{X_k^{\lightning}} \underbrace{\overset{X_k^{\lightning 0}}{\subset} (sk^kX)}_{X_k^{\Diamond}} = X_k$$

Def. For $X \in sSet$, $G \in Abel$, define the chain cplx

$$C_n(X;G)^{\Diamond} = \bigoplus_{\alpha \in X_n^{\Diamond}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\Diamond}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\Diamond}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\Diamond}} G \cdots$$

$$C_n(X;G)^{\lightning} = \bigoplus_{\alpha \in X_n^{\lightning}} G \qquad 0 \longleftarrow \bigoplus_{\alpha \in X_0^{\lightning}} G \xleftarrow{(d_0^1 - d_1^1)^*} \bigoplus_{\alpha \in X_1^{\lightning}} G \xleftarrow{(d_0^2 - d_0^1 + d_2^2)^*} \bigoplus_{\alpha \in X_2^{\lightning}} G \cdots$$

and $H_\cdot(X;G)^{\Diamond}$, $H_\cdot(X;G)^{\lightning}$ as crspd homology.

By definition, $\quad C_\cdot(X;G) \cong C_\cdot(X;G)^{\Diamond} \oplus C_\cdot(X;G)^{\lightning}$

Claim 1. $H_\cdot(X;G)^{\lightning} = 0$, so

$$H_\cdot(X;G) \cong H_\cdot(X;G)^{\Diamond}. \qquad (*)$$

Rmk. Roughly, $(*)$ says that

singular homology $\approx$ simplicial homology.

Finally, one can compute the (co)homology of sSets without too much pain.

To prove Claim 1, one has to expend $C_\cdot(X;G)$ by double complex.

Def (Double complex of $C.(X;G)$)

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0^{\leq 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 3}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 3}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\leq 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 2}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 2}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\leq 1}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 1}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 1}} G \arrow[l] \arrow[dl] & \bigoplus_{\alpha\in X_0^{\leq 1}} G \arrow[l] \arrow[dl] \\
0 & \bigoplus_{\alpha\in X_0^{\varnothing}} G \arrow[l] & \bigoplus_{\alpha\in X_0^{\varnothing}} G \arrow[l] & \bigoplus_{\alpha\in X_0^{\varnothing}} G \arrow[l] & \bigoplus_{\alpha\in X_0^{\varnothing}} G \arrow[l]
\end{tikzcd}
$$

fold $\Downarrow \Uparrow$ expand

$$
\begin{tikzcd}
0 & \bigoplus_{\alpha\in X_0} G \arrow[l] & \bigoplus_{\alpha\in X_0} G \arrow[l] & \bigoplus_{\alpha\in X_0} G \arrow[l] & \bigoplus_{\alpha\in X_0} G \arrow[l]
\end{tikzcd}
$$