

# Eine Woche, ein Beispiel

## 8.28 global field

This note mainly follows [现代数学基础12-数论I: Fermat的梦想和类域论-日 加藤和也 & 黑川信重-胥鸣伟 & 印林生(译)].  
 Another reference for complement (and also for non-Chinese reader):  
 [MIT] <https://math.mit.edu/classes/18.785/2015fa/lectures.html>

I should have done this in 2021.06.27\_adèles\_and\_idèles. However, I was not familiar with local field at that time.

### 1. definition

- 2. adèle ring and idèle group
- 3. topological properties of  $\mathbb{A}_K$  &  $\mathbb{I}_K$
- 4. Tate's thesis

$\parallel$  def  
measure  
topo      fundamental domain      dense  
cpt  
discrete

### 1. definition

Def A global field is

- a finite extension of  $\mathbb{Q}$  (number field), or
- a finite extension of  $\mathbb{F}_p(T)$  (function field)

For an axiomatic definition, see  
<https://math.stackexchange.com/questions/873666/definition-of-global-field>

In this note we denote  $K$  for the global field;  
 when  $K$  is used as a cpt open subgp, we denote  $\mathbb{E}$  or  $\mathbb{F}$  for the global field.

Rmk 1. Ostrowski's thm states that

every non-trivial norm on  $\mathbb{Q}$  is equiv to  $|\cdot|_p$  or  $|\cdot|_\infty$ .

In [Thm3, Cor4, [https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF\(T\).pdf](https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF(T).pdf)],

every non-trivial norm on  $\mathbb{F}_p(T)$  equiv to  $|\cdot|_\pi$  or  $|\cdot|_\infty$

where

$$\left| \frac{a}{b} \pi^k \right|_\pi = p^{-\deg \pi \cdot k}$$

$$\left| \frac{a}{b} \right|_\infty = p^{\deg a - \deg b}$$

for some monic irr  $\pi(T) \in \mathbb{F}_p[T]$   
 $a, b \in \mathbb{F}_p[T], \pi \nmid ab \quad a, b \neq 0$   
 $a, b \in \mathbb{F}_p[T] \quad a, b \neq 0$

Ex. Compute  $K_v, \mathcal{O}_v$  for  $v = |\cdot|_\infty, |\cdot|_T, |\cdot|_{T-1}, |\cdot|_{T^2+1}$   $K = \mathbb{F}_p(T), p=7$

$$\text{A. } \mathcal{O}_{|\cdot|_\infty} = \mathbb{F}_p\left[\left(\frac{1}{T}\right)\right] \quad \mathcal{O}_{|\cdot|_T} = \mathbb{F}_p[[T]] \quad \mathcal{O}_{|\cdot|_{T-1}} = \mathbb{F}_p[[T-1]] \\ K_{|\cdot|_\infty} = \mathbb{F}_p\left(\left(\frac{1}{T}\right)\right) \quad K_{|\cdot|_T} = \mathbb{F}_p((T)) \quad K_{|\cdot|_{T-1}} = \mathbb{F}_p((T-1))$$

$\mathcal{O}_K = \mathbb{F}_p[T]$  can not embed in  $\mathcal{O}_{|\cdot|_\infty}$ , since  $\mathbb{F}_p[T] = \mathcal{O}_{\mathbb{F}_p((1))}$ .

The prod formula also prohibit  $\mathcal{O}_K$  embed to all  $\mathcal{O}_v$ .

Show that  $\mathbb{F}_p\left(\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(T-\frac{1}{\alpha}\right)\right)$  for  $\alpha \in \mathbb{F}_p^\times$ :

$$\mathbb{F}_p\left(\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(\frac{1-\alpha T}{T}\right)\right) = \mathbb{F}_p\left(\left(-\frac{\alpha}{T}(T-\frac{1}{\alpha})\right)\right) \\ \mathbb{F}_p\left(\left(-\frac{(T^2-\alpha+\alpha)}{\alpha}\right)^{-1}\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(-\frac{T}{\alpha}\left(\frac{1}{T}-\alpha\right)\right)\right) = \mathbb{F}_p\left(\left(T-\frac{1}{\alpha}\right)\right)$$

$$\mathcal{O}_{|\cdot|_{T^2+1}} = \mathbb{F}_p(2)[[T^2+1]] \quad \alpha^2 + 1 = 0$$

$$K_{|\cdot|_{T^2+1}} = \mathbb{F}_p(2)((T^2+1))$$

$$\begin{aligned} \mathbb{F}_p[T] &\hookrightarrow \mathbb{F}_p(2)[[T^2+1]] \\ T &\mapsto 2 - \frac{5}{2}(T^2+1) - \frac{25}{8}(T^2+1)^2 - \frac{25}{16}(T^2+1)^3 - \frac{52}{128}(T^2+1)^4 - \dots \\ T^2 &\mapsto -1 + T^2+1 \end{aligned}$$

Rmk 2. Product formula is still true; that is, for  $K = \mathbb{F}_p(T)$

$$\prod_{\pi \text{ fin}} \|f\|_\pi = 1 \quad \forall f \in \mathbb{F}_p(T)^\times$$

Ex. Verify the product formula for other  $K$ .

For relationships between local fields and global fields, see: <https://alex-yuclis.github.io/localglobalgalois.pdf>  
We only list two results which will be used later:

Let  $L/K$  be fin ext of global field. We get two isos as topo ring

$$\begin{array}{ccc} L \otimes_K K_v & \xrightarrow{\cong} & \prod_{i=1}^g L_{w_i} \\ \uparrow & & \cup \\ \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_v & \xrightarrow{\cong} & \prod_{i=1}^g \mathcal{O}_{w_i} \end{array} \quad \begin{array}{c} w_1 \cdots w_g \\ \backslash \cdots / \\ v \end{array} \quad \begin{array}{c} L_{w_1} \cdots L_{w_g} \\ \backslash \cdots / \\ K_v \end{array}$$

[MIT, Cor 11.7]

## 2. adèle ring and idèle group

Every book begins this topic by restricted product, which is totally correct but a little boring/confusing. Let us derive the restricted product naturally.

global (local)	$A_k$ $F$	$\mathbb{I}_k$ $F^\times$	$\mathbb{I}_k^\times$ $\mathcal{O}_F^\times$
$(\mathbb{Q}, +)_G$	$\prod_p \mathbb{Q}_p \times \mathbb{R}$	$(\mathbb{Q}^\times, \times)_G$	$\prod_p \mathbb{Q}_p^\times \times \mathbb{R}^\times$
$\bigcup$	$\bigcup$	$\bigcup$	$\bigcup$
$A_\mathbb{Q} = \underbrace{\prod_p \mathbb{Q}_p \times \mathbb{R}}_{A_{\mathbb{Q}, \text{fin}}} \bigcup$	$A_\mathbb{Q}^\times = \underbrace{\prod_p \mathbb{Q}_p^\times \times \mathbb{R}^\times}_{A_{\mathbb{Q}, \text{fin}}^\times} = \mathbb{I}_\mathbb{Q}$		
$\mathbb{Q} \sqrt{A_\mathbb{Q}} = \underbrace{\prod_p \mathbb{Z}_p \times [0, 1)}_{\widehat{\mathbb{Z}}}$	$\mathbb{Q}^\times \backslash A_\mathbb{Q}^\times = \underbrace{\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}}_{\widehat{\mathbb{Z}}^\times} = C_\mathbb{Q}$		

= : as a set, topo may be different

▽  $A_F^\times$  contains the  $F^\times$ -orbit of  $\prod_{p \text{ fin}} \mathcal{O}_p^\times \times \prod_{p \text{ inf}} F_p^\times$ ,  
but not equal to it in general.

When  $\#\text{Cl}(\mathcal{O}_F) = 1$ , we are lucky,  
as the UFD property provides us with such an element.

Q: Do we always have

$$A_F = F \cdot \left( \prod_{p \text{ fin}} \mathcal{O}_p \times \prod_{p \text{ inf}} F_p \right) ?$$

Answer: Yes, by strong approximation theorem.  
<https://math.mit.edu/classes/18.785/2015fa/LectureNotes22.pdf>

## adèle ring

Def (adèle ring  $A_{\mathbb{Q}}$ ) We know that

$$\left( \prod_{p \text{ prime}} \mathbb{Z}_p \right) \times [0,1) \subseteq \left( \prod_{p \text{ prime}} \mathbb{Q}_p \right) \times \mathbb{R}$$

where  $\mathbb{Q}$  acts diagonally on  $\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$ .

$$+ : \mathbb{Q} \times \left( \prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R} \right) \longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$$

$$(t, (a_p, a_\infty)) \mapsto (t + a_p, t + a_\infty)$$

The adèle ring  $A_{\mathbb{Q}}$  is defined as the orbit of  $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$ , i.e.

$$A_{\mathbb{Q}} := \mathbb{Q} + \left( \prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1) \right)$$

$$= \{ (a_v)_v \in \prod_v F_v \mid a_v \in O_v \text{ for almost all } v \} \stackrel{\triangle}{=} \prod_v' F_v$$

<sup>we don't define  $O_v$  for  $v = 1/\infty$ ,  
but that doesn't matter.</sup>

Rmk. You can also replace  $[0,1)$  by  $\mathbb{R}$  in the definition ( $A_{\mathbb{Z}} := \prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R}$ ), then it may happen that

$$t + \left( \prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R} \right) = t' + \left( \prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R} \right) \quad \text{for } t \neq t' \in \mathbb{Q}.$$

Rmk. The measure is easy to define while the topo is a bit tricky.

By letting  $\mu_p(\mathbb{Z}_p) = 1$ ,  $\mu_\infty([0,1]) = 1$  and give  $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$  with the prod measure, the **measures** on  $A_{\mathbb{Q}/\mathbb{Q}}$  and  $A_{\mathbb{Q}}$  are defined.

For the **topology** on  $A_{\mathbb{F}}$ , we take the weakest topo s.t. all the subspaces

$$\prod_{v \in S} F_v \times \prod_{v \notin S} O_v = \left( \prod_{\substack{p \in S \\ p \text{ prime}}} \mathbb{Q}_p \times \mathbb{R} \times \prod_{p \notin S} \mathbb{Z}_p \right)$$

(for any  $S$ : set of finite places containing all infinite places)

are open, and the subspace topo of  $\prod_{v \in S} F_v \times \prod_{v \notin S} O_v$  coincides with the prod topo.

This topology is a little stronger than the subspace topo of  $A_{\mathbb{F}} \subset \prod_v F_v$ , since  $\prod_{v \in S} F_v \times \prod_{v \notin S} O_v$  are not open in this subspace topo.

The same method can be applied to defining the topo of any restricted product.

Ex. Verify that

$\prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1]$  is the fundamental domain of  $A_{\mathbb{Q}/\mathbb{Q}}$ , so

$$\mu \left( \prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1] \right) = 1 \Rightarrow \mu (A_{\mathbb{Q}/\mathbb{Q}}) = 1$$

Ex. How do they glue with each other?

- $\mathbb{Q} \hookrightarrow A_{\mathbb{Q}}$  is discrete. (by considering the preimage of  $\prod_{p \text{ prime}} \mathbb{Z}_p \times (-\frac{1}{2}, \frac{1}{2})$ )
- $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1] \hookrightarrow A_{\mathbb{Q}} \rightarrow A_{\mathbb{Q}/\mathbb{Q}}$  is cont
- $\Rightarrow A_{\mathbb{Q}/\mathbb{Q}}$  is cpt.  $A_{\mathbb{Q}}$  is loc. cpt.
- $\mathbb{Q} \hookrightarrow \prod'_{p \text{ prime}} \mathbb{Q}_p$ ,  $\mathbb{Q} \hookrightarrow \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$  are dense;
- $\mathbb{Z}[\frac{1}{p}] \hookrightarrow \mathbb{Q}_p \times \mathbb{R}$ ,  $\{\frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b\} \hookrightarrow \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$  are lattices  
discrete & quotient is cpt

Ex. define  $A_F$  in general, apply it to  $F = \mathbb{Q}(i)$ ,  $\mathbb{Q}(\sqrt{3})$ ,  $\mathbb{F}_p(T)$ , and compute their measures and fundamental domains.

From [MIT, #22, p5],  $\mu_v(U) = 2\mu_c(U)$  for  $F_v \cong \mathbb{C}$

Hint.  $\mathbb{F}_p[T] \subset \mathbb{F}_p((\frac{1}{T}))$  is a lattice.  $\mathbb{F}_p((\frac{1}{T})) = \mathbb{F}_p[T] \oplus \frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]$ .

$$\begin{aligned} \text{Set } \mu(\mathbb{F}_p[[\frac{1}{T}]]) &= 1, \text{ then } \mu\left(\frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]\right) = \frac{1}{p} \\ &\Rightarrow \mu(A_{\mathbb{F}_p(T)} / \mathbb{F}_p(T)) = \frac{1}{p}. \end{aligned}$$

For convenience, we will define

$$A_{F, \text{fin}} = \prod'_{v \text{ fin}} F_v = \widehat{\bigcap_{\substack{\uparrow \text{ in some article} \\ \text{not in our notes}}} F} \quad A_{F, \text{inf}} = \prod'_{v \text{ inf}} F_v \quad (A_F = A_{F, \text{fin}} \times A_{F, \text{inf}})$$

$$\widehat{O}_F = \prod'_{v \text{ fin}} O_v$$

$S$  denotes for any finite set of places containing all infinite places, and  $T$  denotes for any set of places containing all infinite places.

$$S, T \neq \emptyset$$

## idèle group

Def (idèle group  $\mathbb{I}_\alpha$ ) We know that

$$\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) \subseteq \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times\right)$$

where  $\mathbb{Q}^\times$  acts diagonally on  $\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$ .

$$\begin{aligned} \cdot : \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times\right) &\longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times \\ (t, (a_p, a_\infty)) &\longmapsto (ta_p, ta_\infty) \end{aligned}$$

The idèle group  $\mathbb{I}_\alpha$  is defined as the orbit of  $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ , i.e.

$$\begin{aligned} \mathbb{I}_\alpha &:= \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) \\ &= \{(a_v)_v \in \prod_v F_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \prod_v' F_v^\times \\ &= (\prod_v' F_v)^\times = A_\alpha^\times \end{aligned}$$

In general,

$$\begin{aligned} \mathbb{I}_F &\stackrel{\text{not unique expression}}{=} F^\times \times \left(\prod_{v \text{ fin}} \mathcal{O}_v^\times \times \prod_{v \text{ inf}} F_v^\times\right) \\ &\stackrel{\text{def}}{=} \{(a_v)_v \in \prod_v F_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \prod_v' F_v^\times \\ &= (\prod_v' F_v)^\times = A_F^\times \\ &= F^\times \times \left(\prod_{v \notin S} \mathcal{O}_v^\times \times \prod_{v \in S} F_v^\times\right) \text{ for } S \text{ big enough} \end{aligned}$$

Rmk. The definition of measure and topology are similar.

The topo defined is stronger than the subspace topo  $A_F^\times \subset A_F$ ,

since  $\prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$  (for any  $S$ ) is not open in the subspace topology.

Ex. Verify that

•  $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$  is the fundamental domain of  $\mathbb{I}_\alpha/\alpha^\times$ , so

- $\mu\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) = +\infty \Rightarrow \mu(\mathbb{I}_\alpha/\alpha^\times) = +\infty$
- $\mathbb{Q}^\times \hookrightarrow \mathbb{I}_\alpha$  is discrete. (by considering the preimage of  $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ )
- $\mathbb{I}_\alpha/\alpha^\times$  is not cpt.  $\mathbb{I}_\alpha$  is loc. cpt.
- $\mathbb{Q}^\times \hookrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$  is discrete (by considering the preimage of  $\prod_{p \neq 7} \mathbb{Z}_p^\times \times (1+7\mathbb{Z}_7)$ )
- $\mathbb{Q} \hookrightarrow \prod_{p \neq 7} \mathbb{Q}_p^\times \times \mathbb{R}^\times$  is dense;
- $\mathbb{Z}[\frac{1}{p}] = \pm p^{\mathbb{Z}} \hookrightarrow \mathbb{Q}_p^\times \times \mathbb{R}^\times$ ,  $\left\{ \frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b \right\}^\times = \mathbb{Q}^\times \cap \mathbb{Z}_7^\times \hookrightarrow \prod_{p \neq 7} \mathbb{Q}_p^\times \times \mathbb{R}^\times$  are discrete.

To remedy the cptness, we introduce the group of 1-idèles.

Def (1-idèles group)

$$\mathbb{I}_{\mathbb{Q}} := \mathbb{Q}^{\times} \times \left( \prod_{p \text{ prime}} \mathbb{Z}_p^{\times} \times \{1\} \right) \\ = \{ (a_v)_v \in \prod_v K_v^{\times} \mid \prod_v |a_v|_v = 1 \} = \left( \prod_v F_v^{\times} \right)^1 = A_{\mathbb{Q}}^{\times, 1}$$

In general,

$$\mathbb{I}'_F \stackrel{\text{not unique expression}}{\cong} F^{\times} \times \left( \prod_{v \text{ fin}} \mathcal{O}_v^{\times} \times \left( \prod_{v \text{ inf}} F_v^{\times} \right)^1 \right) \\ = \{ (a_v)_v \in \prod_v F_v^{\times} \mid \prod_v |a_v|_v = 1 \} = \left( \prod_v F_v^{\times} \right)^1 = A_F^{\times, 1}$$

where

$$\left( \prod_{v \text{ inf}} F_v^{\times} \right)^1 := \{ (a_v)_v \in \prod_{v \text{ inf}} F_v^{\times} \mid \prod_v |a_v|_v = 1 \}$$

We have SESs:

$$0 \rightarrow \mathbb{I}'_F \rightarrow \mathbb{I}_F \xrightarrow{|| \cdot ||} \mathbb{R}_{>0}^{\times} \rightarrow 0 \quad \text{for } F \text{ number field}$$

$$0 \rightarrow \mathbb{I}'_F \rightarrow \mathbb{I}_F \xrightarrow{|| \cdot ||} \mathbb{P}^1 \rightarrow 0 \quad \text{for } F \text{ function field}$$

Rmk [引理6.106] [MIT, Lemma 23.8, 23.9]

For measures, I set  $\mu(S) = 2\pi$ ,  $\mu(\mathbb{Z}_p^{\times}) = 1$ ,  $\mu(pt) = 1$ . I hope they're fine.

The subspace topologies  $\mathcal{O}_F^{\times} \subseteq F^{\times}$ ,  $\mathcal{O}_F^{\times} \subseteq F$  coincide.  $\mathcal{O}_F^{\times} \subseteq F$  is closed.

Observation. It's clear if you see

$$\mathbb{I}_F \cong \{ (x, x^{-1}) \in A_F^{\times} \} \subseteq GL_2(A_F)$$

Ex. Verify that

$\{\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \{1\}\}$  is the fundamental domain of  $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$ , so

$$\cdot \mathbb{Q}^\times \triangleleft \prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \{1\} = 1 \Rightarrow \mu(\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times) = 1$$

$\mathbb{Q}^\times$  is discrete,  $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$  is cpt.

Ex. Compute  $\mathbb{I}_F, \mathbb{I}'_F$  for  $F = \mathbb{Q}(i), \mathbb{Q}(\sqrt{3}), \mathbb{F}_p(t)$ .

Notice that they're all UFD.

Ex. Try to compute it for  $F = \mathbb{Q}(\sqrt{-5})$ .

Here, the fundamental domain is hard to describe.

<http://faculty.bicmr.pku.edu.cn/~lxiao/2024fall/2024fall.htm>

In Exercise 1.8, we see that the volume has an explicit expression (as long as we know the regulator, discriminant, ...)

For convenience, we define

$$C_K := \mathbb{I}_K/K^\times$$

$$\mathbb{I}_{K,\text{fin}} := \prod_{v \text{ fin}} K_v^\times$$

$$C_K^1 := \mathbb{I}_K^1/K^\times$$

$$\mathbb{I}_{K,\text{inf}} := \prod_{v \text{ inf}} K_v^\times$$

$$(\mathbb{I}_K = \mathbb{I}_{K,\text{fin}} \times \mathbb{I}_{K,\text{inf}})$$

so  $C_K$  is cpt, while  $C_K^1$  is loc cpt.

(We've shown this for  $K = \mathbb{Q}$ .)

⚠ I may use the symbols

$$\mathbb{A}_{K,T} = \prod_{v \in T} K_v \times \prod_{v \notin T} \mathcal{O}_v$$

$$\mathbb{I}_{K,T} = \prod_{v \in T} K_v^\times \times \prod_{v \notin T} \mathcal{O}_v^\times$$

$$\mathbb{I}'_{K,T} = \left( \prod_{v \in T} K_v^\times \right)^1 \times \prod_{v \notin T} \mathcal{O}_v^\times$$

to make the result simpler and more symmetrical.

I don't do it now just because I'm lazy.

### 3. topological properties of $A_F$ & $I_F$ .

All the properties in this section have been checked for  $F=\mathbb{Q}$ ,  $I_F(t)$  in the last section (for results concerning  $S$ , we checked some examples also). To make everything rigorous and easy to cite (and get some important applications), we make this section.

The roadmap of this section:

$$\begin{array}{c} F \subseteq A_F \text{ for } F = \mathbb{Q}, I_F(t) \Rightarrow F \subseteq A_F \Rightarrow \mathcal{O}_T \subseteq \prod'_{v \in T} F_v \\ \Downarrow \\ F^\times \subseteq I_F' \Rightarrow \mathcal{O}_T^\times \subseteq (\prod'_{v \in T} F_v)^1 \\ \Downarrow \qquad \Downarrow \\ \text{class number} \qquad \text{Dirichlet unit} \\ \text{dense comes from the theory of duality. not from lattice.} \end{array}$$

topo results needed

Def (iso up to cpt gp,  $\text{Iso}_{\text{cpt}}$ )

$f: G_1 \rightarrow G_2 \in \text{Mor}(\text{Abel}_{\text{Top}})$  is called iso up to cpt gp ( $\text{Iso}_{\text{cpt}}$ ) if

(1)  $G_1/\text{ker}f \cong \text{Im}f$  in  $\text{Abel}_{\text{Top}}$ ;

(2)  $\text{ker}f, \text{coker}f$  are cpt.

Def (lattice)

$L \subseteq G$  in  $\text{Abel}_{\text{Top}}$  is called a lattice, if

(1)  $L$  is discrete;

(2)  $G/L$  is cpt.

When  $G = (\mathbb{R}^n, +)$ , this is equiv to a full lattice.

Cor: for  $G_1 \xrightarrow{f} G_2 \in \text{Iso}_{\text{cpt}}$ , if  $G_1$  is discrete, then

$\text{Im}f$  is a lattice in  $G_2$ .

Lemma 1. (1)  $G_1 \xrightarrow{f} G_2, G_2 \xrightarrow{g} G_3 \in \text{Iso}_{\text{cpt}}$

$\Rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \in \text{Iso}_{\text{cpt}}$

(2)  $G_1 \xrightarrow[f]{\text{V/open}} G_2 \in \text{Iso}_{\text{cpt}}$

$H_2$

$G_1 \xrightarrow{f} G_2 \in \text{Iso}_{\text{cpt}}$

$\text{V/open}$

$f^{-1}(H_2) \rightarrow H_2 \in \text{Iso}_{\text{cpt}}$

(3)  $H \subseteq G$  in  $\text{Abel}_{\text{Top}}$

$H$  is open  $\Leftrightarrow G/H$  is discrete

$\Downarrow$

$\Downarrow$

$H$  is closed  $\Leftrightarrow G/H$  is Hausdorff.

lattice

Lemma 2 [6.10] [MIT, Prop 22.10]  $E/F$  finite ext of global field. We get an iso of topo rings

$$\Phi: E \otimes_F A_F \xrightarrow{\cong} A_F$$

$$(t, (a_v)_v) \mapsto (ta_{w_k})_w$$

In ptc,  $A_F$  is a subring of  $A_E$ ,  $A_E \cong A_F^{\oplus [E:F]}$ , and we have an iso

$$\begin{array}{ccc} E & \hookrightarrow & A_E \\ \uparrow \cong & & \uparrow \cong \\ E \otimes_F F & \xrightarrow{\text{Id}_E \otimes_F a} & E \otimes_F A_F \end{array}$$

Proof. Locally we have

$$\begin{array}{ccc} E \otimes_F F_v & \xrightarrow{\sim} & \prod_{i=1}^n E_{w_i} \\ \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_v & \xrightarrow{\sim} & \prod_{i=1}^n \mathcal{O}_{w_i} \end{array}$$

Since  $E \otimes_F -$ ,  $\mathcal{O}_E \otimes_{\mathcal{O}_F} -$  are exact [Stackexchange, 1916457], one get

$$\begin{array}{ccc} \mathcal{O}_E \otimes_{\mathcal{O}_F} \prod_{v \in T} \mathcal{O}_v & \cong & \prod_{w \in T} \mathcal{O}_w \\ \downarrow & & \cap \\ E \otimes_F A_F & \longrightarrow & A_E \\ \downarrow & & \cap \\ E \otimes_F \prod_{v \in T} F_v & \cong & \prod_w E_w \end{array}$$

which shows the bijection.

( $\Phi$  is well-defined, since  $a_w \in \mathcal{O}_w \Rightarrow a_v \in \mathcal{O}_v$ ;  
 (" $\Phi$  &  $\Phi^{-1}$  are cont" should be a routine check) (but I don't check it))

Prop 1 [6.78]  $F$  is a lattice in  $A_F$ .

Proof. We have checked for  $F = \mathbb{Q}, \mathbb{F}_p(T)$ . The rest comes from Lemma 1.

Prop 2 [6.80(1)] Let  $T$  be a set of places of  $F$  containing all infinite places,  $T \neq \emptyset$ .  
 Let

$$\mathcal{O}_T = \{x \in F \mid x \in \mathcal{O}_v \text{ for } v \notin T\}$$

then  $\mathcal{O}_T$  is a lattice in  $\prod'_{v \in T} F_v$ .

Rmk. For  $F/\mathbb{Q}$  of degree  $n$

When  $T = \{\text{all places of } F\}$ ,  $\mathcal{O}_T = F$ ; Application 1.

When  $T = \{\text{all inf places of } F\}$ ,  $\mathcal{O}_T = \mathcal{O}_F \Rightarrow \mathcal{O}_F$  is a free  $\mathbb{Z}$ -module of rank  $n$ .

Proof.

$$F \hookrightarrow A_F$$

$$\vee \quad \vee \text{ open}$$

$$\mathcal{O}_T \rightarrow \prod'_{v \in T} F_v \times \prod_{v \notin T} \mathcal{O}_v \xrightarrow{\pi} \prod'_{v \in T} F_v$$

By Lemma 1,  $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v \in T} F_v$  is iso up to cpt gp,

and  $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v \in T} F_v$  is obviously injective.

Prop 3 [6.82]  $F^\times$  is a lattice in  $\mathbb{I}_F'$

For a proof, see [Theorem 3.3.6, <https://bicmr.pku.edu.cn/~dingyiwen/nt1.pdf>]

Prop 4 [6.83] Let  $T$  be a set of places containing all infinite places.  $T \neq \emptyset$ , let

$$\left( \prod'_{v \in T} \mathbb{R} \right)^\circ = \left\{ (c_v)_{v \in T} \in \prod_{v \in T} \mathbb{R} \mid \begin{array}{l} c_v = 0 \text{ for almost all } v \\ \sum_{v \in T} c_v = 0 \end{array} \right\}$$

$$R_T : \mathcal{O}_T^\times \longrightarrow \left( \prod'_{v \in T} \mathbb{R} \right)^\circ \quad x \mapsto (\ln |x|_v)_{v \in T}$$

then  $R_T$  makes  $R_T(\mathcal{O}_T^\times)$  as a lattice in  $\left( \prod'_{v \in T} \mathbb{R} \right)^\circ$ , and  $\ker R_T$  is finite.

Proof.  $F^\times \xrightarrow{\Delta} \mathbb{I}_F'$

$\forall$  open  $\mathfrak{w}$  open

$$\mathcal{O}_T^\times \xrightarrow{\Delta} \left( \prod'_{v \in T} F_v^\times \right) \times \prod_{v \notin T} \mathcal{O}_v^\times \xrightarrow{\pi} \left( \prod'_{v \in T} F_v^\times \right)' \longrightarrow \left( \prod'_{v \in T} \mathbb{R} \right)^\circ$$

$$(a_v)_{v \in T} \mapsto (\ln |a_v|_v)_{v \in T}$$

Rmk.  $R_T$  is in general not injective.

Application 2.

$$\begin{aligned} \ker R_T &= \{\text{root of unity in } F\} \leftarrow \text{the unity root gp of } F \\ &= \{x \in F \mid x^n = 1, \exists n \in \mathbb{N}_{>1}\} \end{aligned}$$

where " $\subseteq$ " comes from the finiteness of  $\ker R_T$ .

As a corollary, the unity root gp of  $F$  is finite.

Application 3. Suppose  $F/\mathbb{Q}$  is a number field,  $\#T < +\infty$ . We get SES

$$1 \longrightarrow \ker R_T \longrightarrow \mathcal{O}_T^\times \longrightarrow \mathbb{Z}^{\oplus(\#T-1)} \longrightarrow 1$$

When  $T = \{\text{all inf places}\}$ , we get Dirichlet unit theorem.

Application 4. For  $F/\alpha$ , define

$$\begin{aligned}\mathbb{I}_F &\longrightarrow I(F) \\ (a_v)_v &\longmapsto \prod_{\substack{v \text{ fin} \\ v = v_p}} \beta^{v(a_v)}\end{aligned}$$

we get SES's.

$$\begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \rightarrow \mathcal{O}_F^x & \longrightarrow & \prod_{v \text{ fin}} \mathcal{O}_v^x \times \prod_{v \text{ inf}} F_v^x & \xrightarrow{\cong \mathcal{U}} & \bar{U} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow F^x & \longrightarrow & \mathbb{I}_F & \longrightarrow & C_F & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow P(F) & \longrightarrow & I(F) & \longrightarrow & Cl(F) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ & 1 & & 1 & & 1 & \end{array}$$

$\Rightarrow 1 \rightarrow \mathcal{O}_F^x \rightarrow F^x \rightarrow I(F) \rightarrow Cl(F) \rightarrow 1$

By replacing  $\mathbb{I}_F$  by  $\mathbb{I}'_F$ , we get

$$\begin{array}{ccccccc} & 1 & & 1 & & 1 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 1 \rightarrow \mathcal{O}_F^x & \longrightarrow & \prod_{v \text{ fin}} \mathcal{O}_v^x \times (\prod_{v \text{ inf}} F_v^x)' & \xrightarrow{\cong \mathcal{U}'} & \bar{U}' & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow F^x & \longrightarrow & \mathbb{I}'_F & \longrightarrow & C'_F & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow P(F) & \longrightarrow & I(\bar{F}) & \longrightarrow & Cl(F) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ & 1 & & 1 & & 1 & \end{array}$$

$C'_F$  cpt  $\Rightarrow Cl(F)$  is cpt

$\prod_{v \text{ fin}} \mathcal{O}_v^x \times (\prod_{v \text{ inf}} F_v^x)' \subseteq \mathbb{I}'_F$  is open  $\Rightarrow I(\bar{F})$  is discrete  $\Rightarrow Cl(F)$  is discrete

$\Rightarrow Cl(F)$  is finite. □

dense.

Work over the category of loc. cpt Abelian gp  $\text{Abel}_{\text{lc}}$ .

$G \in \text{Abel}_{\text{lc}}$

$$\widehat{G}^{*,u} = \{x: G \rightarrow S' \text{ cont}\}$$

unitary

$$\widehat{G}^* = \{x: G \rightarrow \mathbb{C}^\times \text{ cont}\}$$

$\widehat{K}$  : "profinite completion of  $K$ "

$V^*$  : "dual space of  $V$ "

Thm (Pontrjagin dual) For  $G \in \text{Abel}_{\text{lc}}$ ,

$$\begin{aligned} \phi: G &\xrightarrow{\sim} \widehat{G}^{*,u} \\ g &\mapsto [\phi_g: \widehat{G}^{*,u} \rightarrow S' \\ &\quad x \mapsto x(g)] \end{aligned}$$

is an iso (as topo abelian gp)

$$x^k \neq \text{Id} \quad \forall k \in \mathbb{N}_{\geq 1}$$

Thm. 1. For  $F$  local field, take  $x \in \widehat{F}^{*,u}$  not root of unit,

$$\begin{aligned} F &\xrightarrow{\sim} \widehat{F}^{*,u} \\ x &\mapsto x(x-) \end{aligned}$$

is an iso

2. For  $F$  global field, take  $x \in \widehat{\mathbb{A}_F/F}^{*,u}$  not root of unit.

$$\begin{aligned} F &\xrightarrow{\sim} \widehat{\mathbb{A}_F/F}^{*,u} \\ x &\mapsto x(x-) \\ \mathbb{A}_F &\xrightarrow{\sim} \widehat{\mathbb{A}_F}^{*,u} \\ x &\mapsto x(x-) \end{aligned}$$

are isos.

3. For  $F$  global field,  $T$ , take  $x \in \widehat{(\prod'_{v \in T} F_v)/\mathcal{O}_T}^{*,u}$  not root of unit.

$$\mathcal{O}_T \xrightarrow{\sim} \widehat{(\prod'_{v \in T} F_v)/\mathcal{O}_T}^{*,u}$$

$$\prod'_{v \in T} F_v \xrightarrow{\sim} \widehat{\prod'_{v \in T} F_v}^{*,u}$$

are isos.

Lemma. For  $f \in \text{Mor}_{\text{Abel}_{lc}}(G_1, G_2)$ ,

$$\text{Im } f \subseteq G_2 \text{ is dense} \Leftrightarrow \widehat{f}^{*,u}: \widehat{G}_2^{*,u} \xrightarrow{\quad} \widehat{G}_1^{*,u} \text{ is inj.}$$

$$x \mapsto \chi(f(x))$$

Proof. Let  $H = \overline{\text{Im } f}$ .

$$\begin{aligned} & \text{Im } f \subseteq G_2 \text{ is dense} \\ \Leftrightarrow & H = G_2 \\ \Leftrightarrow & G_2/H = \{\text{Id}\} \\ \Leftrightarrow & \widehat{G}_2/H^{*,u} = \{\text{Id}\} \\ \Leftrightarrow & \widehat{f}^{*,u} \text{ is inj.} \end{aligned}$$

Prop 5 [6.79] Let  $T \subseteq \{\text{places of } F\}$ , then the image of  $F \xrightarrow{\quad} \prod'_{v \in T} F_v$  is dense.

Proof. Reduce to show, for any  $w$ ,

$$\begin{aligned} & \text{the image of } F \xrightarrow{\quad} \prod'_{v \neq w} F_v \text{ is dense} \\ \Leftrightarrow & \underbrace{\prod'_{v \neq w} F_v^{*,u}}_{\text{HS}} \xrightarrow{\quad} \widehat{F}^{*,u} \xrightarrow{\quad} \prod'_{v \neq w} F_v \text{ is inj} \\ \Leftrightarrow & \prod'_{v \neq w} F_v \xrightarrow{\quad} A_F/F \text{ is inj} \end{aligned}$$

□

Prop 6 [6.80(2)] Suppose  $T' \subseteq T$ . then the image of  $\mathcal{O}_T \xrightarrow{\quad} \prod'_{v \in T'} F_v$  is dense.

Proof. Reduce to show, for any  $w \in T$ ,

the image of  $\mathcal{O}_T \xrightarrow{\quad} \prod'_{v \in T-w} F_v$  is dense.

$$\Leftrightarrow \underbrace{\prod'_{v \in T-w} F_v^{*,u}}_{\text{HS}} \xrightarrow{\quad} \widehat{\mathcal{O}}_T^{*,u} \text{ is inj}$$

$$\Leftrightarrow \prod'_{v \in T-w} F_v \xrightarrow{\quad} \prod'_{v \in T-w} F_v/\mathcal{O}_T \text{ is inj}$$

□