Eine Woche, ein Beispiel

5.29 Unitary group

Ref: [L-group, 4-5]https://personal.math.ubc.ca/~cass/research/pdf/miyake.pdf
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/unitary_groups_basic_definitions.pdf

Notation. F NA local field (not necessary)
E/F Galois $\deg = 2$ $\mathrm{Gal}(E/F) = \{1, \sigma\}$

$$w = \begin{bmatrix} & & 1 \\ & -1 & \\ 1 & & \end{bmatrix} \in GL_3(F) \hookrightarrow GL_3(E \otimes R) \qquad A^H := \sigma(A^T)$$

Def. $G = U_w(3, E/F)$ is an alg gp over F defined by

$$G(R) = \left\{ A = (a_{ij})_{i,j=1}^{3} \;\middle|\; \begin{array}{l} a_{ij} \in E \otimes_F R \\ A w A^H = w \end{array} \right\} \circlearrowleft \mathrm{Gal}(E/F)$$

(G has a $\mathrm{Gal}(E/F)$-action)

E.x. 1.

$$\begin{array}{ccc} E \otimes_F E & \longrightarrow & E \oplus E \\ x_1 \otimes x_2 & \longmapsto & (x_1 x_2, \sigma(x_1) x_2) \\ \downarrow \sigma & & \downarrow \sigma \\ \sigma(x_1) \otimes x_2 & \longmapsto & (\sigma(x_1) x_2, x_1 x_2) \end{array}$$

is a $\mathrm{Gal}(E/F)$-equiv E-alg homomorphism, where (on $E \oplus E$)

$$\sigma(y_1, y_2) := (y_2, y_1)$$
$$k(y_1, y_2) := (k y_1, \sigma(k) y_2) \quad \forall k \in E$$
$$(y_1, y_2)(y_1', y_2') := (y_1 y_1', y_2 y_2')$$

E.x. 2.

$$G(E) = \left\{ (A, B) \;\middle|\; \begin{array}{l} (a_{ij}, b_{ij}) \in E \oplus E \quad \forall i,j \\ (A, B)(w, \sigma(w))(A, B)^H = (w, \sigma(w)) \end{array} \right\}$$

$$= \left\{ (A, B) \;\middle|\; \begin{array}{l} (a_{ij}, b_{ij}) \in E \oplus E \quad \forall i,j \\ A w B^H = w \end{array} \right\} \qquad B = (w^{-1} A^{-1} w)^H$$

$$\cong GL_n(E)$$

with the action of $\mathrm{Gal}(E/F)$ by

$$\sigma : G(E) \longrightarrow G(E) \qquad A \longmapsto (w^{-1} A^{-1} w)^H$$

e.p. $\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \mapsto \begin{pmatrix} \sigma(t_3)^{-1} & & \\ & \sigma(t_2)^{-1} & \\ & & \sigma(t_1)^{-1} \end{pmatrix}$

So G is an inner form / outer twisted form of GL_n.

Torus $\quad T(R) = \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \in G(R) \right\}$

$$= \left\{ \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \in GL_3(E \otimes_F R) \;\middle|\; \begin{matrix} t_1\sigma(t_3) = 1 \\ t_2\sigma(t_2) = 1 \end{matrix} \right\}$$

$$\cong (\mathrm{Res}_{E/F}\, GL_1)(R) \times \mathcal{U}(1, E/F)(R)$$

Action on the root datum:

$$X^*(T_E) = \left\langle \varepsilon_i : \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \longmapsto t_i \right\rangle_{\mathbb{Z}}$$

$$X_*(T_E) = \left\langle \varepsilon_i^* : t \longmapsto \begin{pmatrix} 1 & & \\ & \ddots\, t\, \ddots & \\ & & 1 \end{pmatrix} \right\rangle_{\mathbb{Z}}$$

$\sigma:\ \varepsilon_1 \longmapsto -\varepsilon_3$

$\varepsilon_2 \longmapsto -\varepsilon_2$

$\varepsilon_3 \longmapsto -\varepsilon_1$

$\varepsilon_1^* \longmapsto -\varepsilon_3^*$

$\varepsilon_2^* \longmapsto -\varepsilon_2^*$

$\varepsilon_3^* \longmapsto -\varepsilon_1^*$

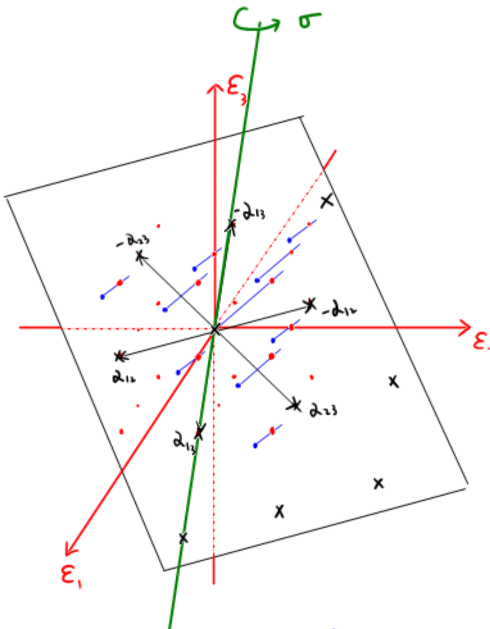
E.g. $\quad \mathrm{Gal}(E/F) \overset{\phi}{\circlearrowright} T_E \qquad \mathrm{Gal}(E/F) \overset{\phi'}{\circlearrowright} \mathbb{G}_{m,E}$

$\rightsquigarrow \mathrm{Gal}(E/F) \overset{\psi}{\circlearrowright} X^*(T_E)$ by

$$\psi(\sigma):\ \mathrm{Hom}(T_E, \mathbb{G}_{m,E}) \longrightarrow \mathrm{Hom}(T_E, \mathbb{G}_{m,E})$$
$$f \longmapsto \phi'(\sigma) \circ f \circ (\phi(\sigma))^{-1}$$

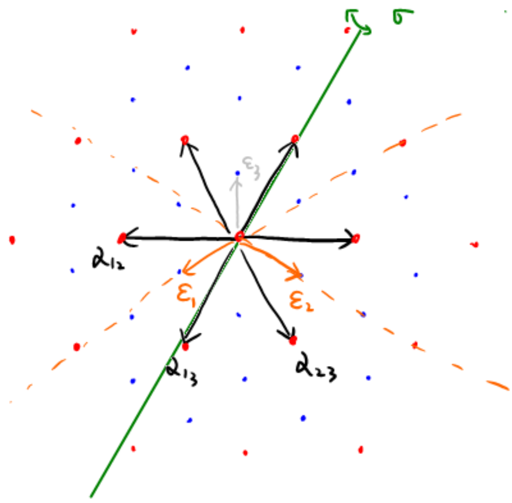
e.g.
$$\begin{tikzcd} T_E \arrow[r, "\varepsilon_1"] \arrow[d, "\phi(\sigma)"'] & \mathbb{G}_{m,E} \arrow[d, "\phi'(\sigma)"] \\ T_E \arrow[r, "\sigma.\varepsilon_1"] & \mathbb{G}_{m,E} \end{tikzcd}$$

$$\begin{tikzcd} \left(\begin{pmatrix} \sigma(t_3)^{-1} & & \\ & \sigma(t_2)^{-1} & \\ & & \sigma(t_1)^{-1} \end{pmatrix}, \begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix} \right) \arrow[r, mapsto, "\varepsilon_1"] \arrow[d, mapsto, "\sigma"] & \sigma(t_3)^{-1} \arrow[d, mapsto, "\sigma"] \\ \left(\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}, \begin{pmatrix} \sigma(t_3)^{-1} & & \\ & \sigma(t_2)^{-1} & \\ & & \sigma(t_1)^{-1} \end{pmatrix} \right) \arrow[r, mapsto, "-\varepsilon_3"] & t_3^{-1} \end{tikzcd}$$



$\times \ \subseteq\ \cdot\ \subseteq\ /$

$Q \underset{\mathbb{Z}}{\subseteq} X^* \underset{S'}{\subseteq} P$



$\cdot\ \subseteq\ \cdot\ =\ \cdot$

$Q \underset{\mathbb{Z}/3\mathbb{Z}}{\subseteq} X^* = P$

▽ In this case, the action of σ on $X^*(T_E)$ does not coincide with any element in Weyl group.

Action on the dual group $\hat{G} = GL_3/\mathbb{Z}$

$$\sigma:\ GL_3 \longrightarrow GL_3 \qquad A \longmapsto (w^{-1}A^{-1}w)^T$$

σ fixes $\hat{B} = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$ & $\hat{T} = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$, and induces the same action on

$$(X^*(\hat{T}),\ \Delta(\hat{B}),\ X_*(\hat{T}),\ \check{\Delta}(\hat{B})) \cong (X_*(T_E),\ \check{\Delta}(B_E),\ X^*(T_E),\ \Delta(B_E))$$