

There are more examples in Ahlfors' book or Jihuai Shi's book, but I think it's enough for one-time show.

https://math.stackexchange.com/questions/585182/why-is-the-riemann-mapping-theorem-important

with sphere packing: https://scholarworks.calstate.edu/downloads/rn301358k By 2.2.1, every simply connected proper open subset of C is not biholomorphic to C.

Liouville's theorem: every bounded entire function must be constant.

Cor1. For every entire function f, if Re f is bounded, then f is constant.

Cor2. For every entire function f, if $f^{-1}([0,\inf y])$ is empty, then f is constant.

Cor3. For every entire function f, if f is injective, then f is surjective.

Little Picard Theorem is the strongest version of this type of results. For a statement, see wiki: Picard_theorem; for a proof, see [WWL, $\{\emptyset\}_3,3.6$].