Eine Woche, ein Beispiel

11.13 Hecke algebra for finite groups

This document is a continuation of the document [2022.09.04_Hecke_algebra_for_matrix_groups].

main reference:
[Bump][http://sporadic.stanford.edu/bump/math263/hecke.pdf]
[XiongHecke][https://github.com/CubicBear/self-driving/blob/main/HeckeAlgebra.pdf]
All the references in https://github.com/ramified/personal_handwritten_collection/blob/main/modular_form/README.md

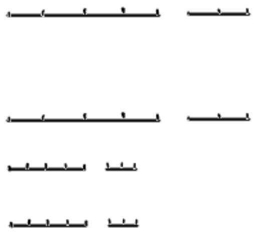
Task. For each double coset decomposition, we want to do:

1. decomposition (& $\Gamma t \Gamma / \Gamma$ is finite & definition of Hecke alg)
2. $\mathbb{Z}$-mod structure, notation
3. alg structure
4. conclusion

Today: $\mathcal{H}(S_{m+n}, S_m \times S_n)$
$\mathcal{H}(G \times G, G)$

Variation: $\mathcal{H}(S_{m+n}, S_m \times S_n)$ $\qquad m, n \in \mathbb{Z}_{>0}$

E.g. $m+n=8,\ m=5,\ n=3$; $m+n=8,\ m=7,\ n=1$.



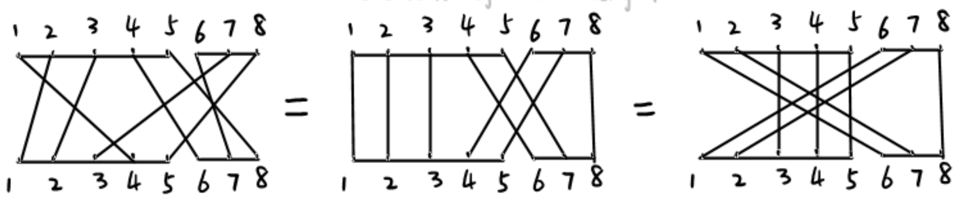
∇ For the convenience of the writing, we denote

$$S_{m,n} := S_m \times S_n$$

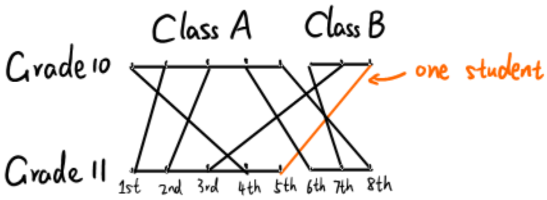
and suppose $m \geq n$.

1. decomposition

E.g. random element $\qquad$ canonical form for draw by hand (element of minimal length) $\qquad$ canonical form for computation

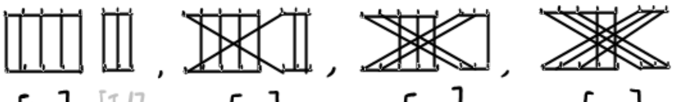


$$\begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 4&1&2&6&8&7&3&5 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 1&2&3&6&7&4&5&8 \end{pmatrix} = \begin{pmatrix} 1&2&3&4&5&6&7&8 \\ 6&7&3&4&5&1&2&8 \end{pmatrix} \text{ in } S_{5,3}\backslash S_8 / S_{5,3}$$



A vivid explanation: students are distributed into different classes on the basis of their order.
The teacher only care about the stability of the class, i.e. how many people move from Class A to Class B as time went by.

$\therefore\ S_{5,3}\backslash S_8 / S_{5,3} = \{$



$[\varpi_0] = [\mathrm{Id}]$, $[\varpi_1]$, $[\varpi_2]$, $[\varpi_3]$ $\}$ $\qquad$ e.g. $\varpi_1 = (16) \in S_8$, $[\varpi_1] \in S_{5,3}\backslash S_8 / S_{5,3}$

In general, $\quad S_{m+n} = \coprod_{i=0}^{n} S_{m,n}\, \varpi_i\, S_{m,n}$

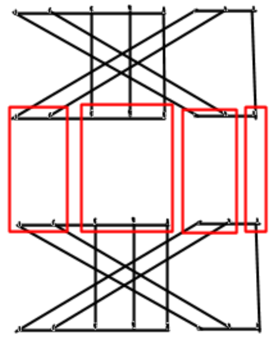
Ex. Compute $|S_{m,n}\varpi_i S_{m,n} / S_{m,n}|$

A. $|S_{m,n}\varpi_i S_{m,n} / S_{m,n}| = |S_{m,n} / \varpi_i S_{m,n} \varpi_i^{-1} \cap S_{m,n}|$

$= |S_{m,n} / S_{i,m-i} \times S_{i,n-i}|$

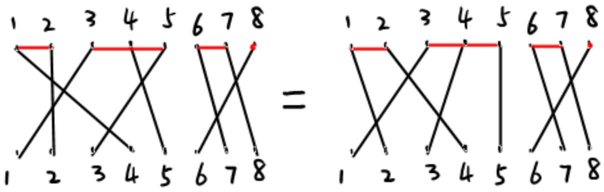
$= \dfrac{m!\, n!}{i!\,(m-i)!\, i!\,(n-i)!}$

$= \binom{m}{i}\binom{n}{i}$

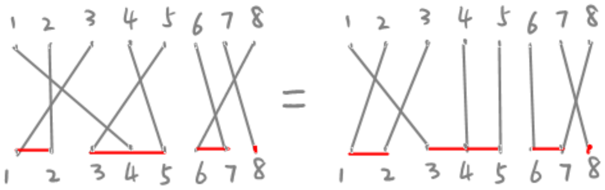


$\varpi_i S_{m,n} \varpi_i^{-1} \cap S_{m,n}$

E.g. (canonical form)



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 5 & 7 & 8 & 6 \end{pmatrix} \quad \text{in } S_{5,3} / S_{2,3} \times S_{2,1}$$



$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 5 & 3 & 7 & 8 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 6 & 7 & 4 & 5 & 8 \end{pmatrix} \quad \text{in } S_{2,3} \times S_{2,1} \backslash S_{5,3}$$

Recall that $\mathcal{H}(G,H) := \{ f : G \to \mathbb{Z} \mid f(h_1 g h_2) = f(g) \quad \forall h_1, h_2 \in H, g \in G \}$ where

$$(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g)\, d\mu(x)$$

$$= \frac{1}{|H|} \sum_{x \in G} f_1(x) f_2(x^{-1}g)$$

2. $\mathbb{Z}$-mod structure. notation

$$\mathcal{H}(S_{m+n}, S_{m,n}) = \bigoplus_{i=0}^{n} \mathbb{Z} \cdot \mathbb{1}_{S_{m,n}\varpi_i S_{m,n}} = \mathbb{Z}^{\oplus(n+1)}$$

denote $T_i := \mathbb{1}_{S_{m,n}\varpi_i S_{m,n}}$ ($T_0 = \mathbb{1}_{S_{m,n}}$ is the unit of $\mathcal{H}(S_{m+n}, S_{m,n})$)

3. alg structure

E.g. $\mathcal{H}(S_8, S_7) = \mathbb{Z} \oplus \mathbb{Z}T = \mathbb{Z}[T]/(T-7)(T+1)$

$$g^{Id}_{\varpi_1,\varpi_1} = \frac{1}{|S_7|} \{ (y,z) \in S_7 \varpi_1 S_7 \times S_7 \varpi_1 S_7 \mid yz = 1 \}$$

$$= \frac{|S_7 \varpi_1 S_7|}{|S_7|}$$

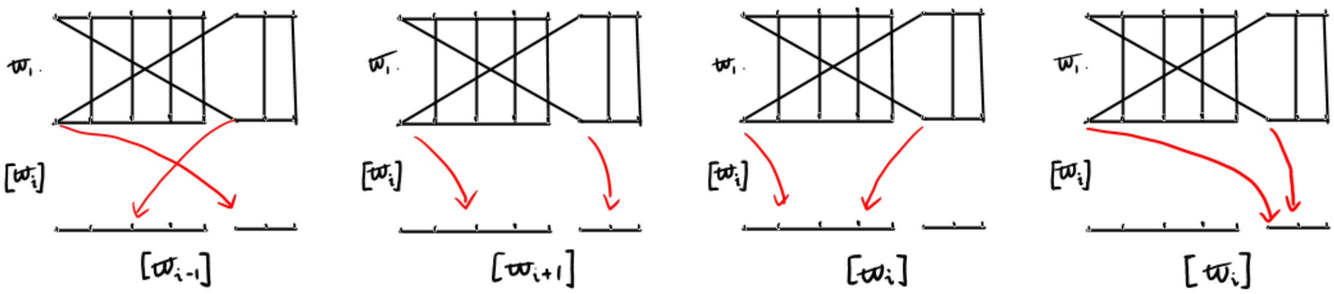
$$= \frac{8! - 7!}{7!} = 7$$

$$
\begin{aligned}
g^{\varpi_1}_{\varpi_1,\varpi_1} &= \frac{1}{|S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz=\varpi_1\} \\
&= \frac{1}{|S_7|\,|S_7\varpi_1 S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz\in S_7\varpi_1 S_7\} \\
&= \frac{1}{|S_7|\,|S_7\varpi_1 S_7|}\{(y,z)\in S_7\varpi_1 S_7 \times S_7\varpi_1 S_7 \mid yz\notin S_7\} \\
&= \frac{\cancel{|S_7\varpi_1 S_7|}\,|S_7\varpi_1 S_7| - \cancel{|S_7\varpi_1 S_7|}\,|S_7|}{|S_7|\,\cancel{|S_7\varpi_1 S_7|}} \\
&= \binom{7}{1}\binom{1}{1}-1 \\
&= 6
\end{aligned}
$$

In general, $\mathcal{H}(S_{m+1}, S_m) = \mathbb{Z}[T]/(T-m)(T+1)$.

Direct argument shows that

$$
T_i * T_1 \in \begin{cases} \mathbb{Z}\cdot T_{i-1} + \mathbb{Z}\cdot T_i + \mathbb{Z}\cdot T_{i+1} & 0<i<n \\ \cancel{\mathbb{Z}\cdot T_0} + \mathbb{Z}\cdot T_1 & i=0 \\ \mathbb{Z}\cdot T_{n-1} + \mathbb{Z}\cdot T_n & i=n \end{cases}
$$



Computation of the coefficient.

$$
\begin{aligned}
g^{\varpi_{i-1}}_{\varpi_i,\varpi_1} &= \frac{1}{|S_{m,n}|}\#\{(y,z)\in S_{m,n}\varpi_i S_{m,n} \times S_{m,n}\varpi_1 S_{m,n} \mid yz=\varpi_{i-1}\} \qquad (0<i\le n) \\
&= \frac{1}{|S_{m,n}|\,|S_{m,n}\varpi_{i-1}S_{m,n}|}\#\{(y,z)\in S_{m,n}\varpi_i S_{m,n} \times S_{m,n}\varpi_1 S_{m,n} \mid yz\in S_{m,n}\varpi_{i-1}S_{m,n}\} \\
&= \frac{|S_{m,n}\varpi_i S_{m,n}|}{|S_{m,n}|\,|S_{m,n}\varpi_{i-1}S_{m,n}|}\#\{z\in S_{m,n}\varpi_1 S_{m,n} \mid \varpi_i z\in S_{m,n}\varpi_{i-1}S_{m,n}\} \\
&= \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,-)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}|\,|S_{m,n}\varpi_{i-1}S_{m,n}|}
\end{aligned}
$$

$$
g^{\varpi_{i+1}}_{\varpi_i,\varpi_1} = \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,+)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}|\,|S_{m,n}\varpi_{i+1}S_{m,n}|} \qquad (0\le i<n)
$$

$$
g^{\varpi_i}_{\varpi_i,\varpi_1} = \frac{|S_{m,n}\varpi_i S_{m,n}|\,|S^{(i,0)}_{m,n}\varpi_1 S_{m,n}|}{|S_{m,n}|\,|S_{m,n}\varpi_i S_{m,n}|}
$$

where

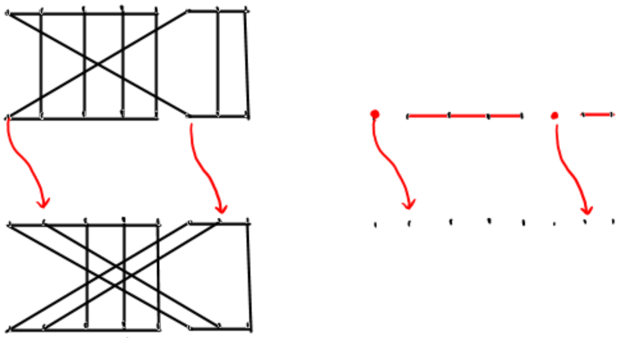
$$S_{m,n}^{(i,-)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_{i-1} S_{m,n} \}$$
$$S_{m,n}^{(i,+)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_{i+1} S_{m,n} \}$$
$$S_{m,n}^{(i,0)} := \{ g \in S_{m,n} \mid \varpi_i g \varpi_1 \in S_{m,n} \varpi_i S_{m,n} \}$$

Recall that

$$S_{m,n}^{(i,-)} \varpi_1 S_{m,n} / S_{m,n} \subset S_{m,n} \varpi_1 S_{m,n} / S_{m,n} \quad \text{as left coset}$$

By the following picture, for $g \in S_{m,n}$,

$$g \in S_{m,n}^{-} \iff \begin{cases} g(1) \in \{1, 2, \cdots, i\} \\ g(m+1) \in \{m+1, m+2, \cdots, m+i\} \end{cases}$$



$$\therefore \quad \left| S_{m,n}^{(i,-)} \varpi_1 S_{m,n} / S_{m,n} \right| = i^2$$

Similarly,

$$\left| S_{m,n}^{(i,+)} \varpi_1 S_{m,n} / S_{m,n} \right| = (m-i)(n-i)$$
$$\left| S_{m,n}^{(i,0)} \varpi_1 S_{m,n} / S_{m,n} \right| = i(n-i) + (m-i)i$$
$$= i(m+n-2i)$$

$$\therefore \quad g_{\varpi_i, \varpi_1}^{\varpi_{i-1}} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,-)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_{i-1} S_{m,n}|} \qquad (0 < i \leq n)$$

$$= \frac{\binom{m}{i}\binom{n}{i} i^2}{\binom{m}{i-1}\binom{n}{i-1}} \qquad \binom{m}{n} = \frac{m-i+1}{i}\binom{m}{i-1}$$

$$= (m-i+1)(n-i+1)$$

$$g_{\varpi_i, \varpi_1}^{\varpi_{i+1}} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,+)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_{i+1} S_{m,n}|} \qquad (0 \leq i < n)$$

$$= \frac{\binom{m}{i}\binom{n}{i} (m-i)(n-i)}{\binom{m}{i+1}\binom{n}{i+1}}$$

$$= (i+1)^2$$

$$g_{\varpi_i, \varpi_1}^{\varpi_i} = \frac{|S_{m,n} \varpi_i S_{m,n}| \, |S_{m,n}^{(i,0)} \varpi_1 S_{m,n}|}{|S_{m,n}| \, |S_{m,n} \varpi_i S_{m,n}|}$$

$$= \left| S_{m,n}^{(i,0)} \varpi_1 S_{m,n} / S_{m,n} \right|$$

$$= i(m+n-2i)$$

Therefore,

$$T_i * T_1 = \begin{cases} (m-i+1)(n-i+1)\,T_{i-1} + i(m+n-2i)\,T_i + (i+1)^2\,T_{i+1}, & 0<i<n \\ T_1, & i=0 \\ (m-n+1)\,T_{n-1} + n(m-n)\,T_n, & i=n \end{cases}$$

4. Conclusion

By [Hecke, Prop 6], $S_{m,n}$ is a Gelfand subgp of S_{m+n}.
thus $\mathcal{H}(S_{m+n}, S_{m,n})$ is commutative.
Gelfand involution: $\sigma \longmapsto \sigma^T$ $\quad (S_{m+n} \hookrightarrow GL_{m+n}(K))$

Possible extension: compute $\mathcal{H}(S_{m+n+l}, S_m \times S_n \times S_l)$.

The rest of the section is devoted to compute $F_{m,n} \in \mathbb{Z}[T]$ s.t

$$\mathcal{H}(S_{m+n}, S_{m,n}) \cong \mathbb{Z}[T]/(F_{m,n}) \qquad T = T_1$$

Appendix: "Linear algebra"

Set $v_i = T_i$, $w_i = T^i$

First cases:

$w_0 = 1 = T_0 = v_0$

$w_1 = T = T_1 = v_1$

$w_2 = T^2 = Tv_1 = mn\,v_0 + (m+n-2)\,v_1 + 4\,v_2$

$w_3 = T^3 = T(mn\,v_0 + (m+n-2)\,v_1 + 4\,v_2) = \dots$

Define
$$\mathcal{A}: \mathcal{H}(S_{m+n}, S_{m,n}) \longrightarrow \mathcal{H}(S_{m+n}, S_{m,n})$$
$$f \longmapsto f * T$$

Then $\mathcal{A}(w_i) = w_{i+1}$

$$\mathcal{A}(v_0,\dots,v_n) = (v_0,\dots,v_n)\begin{bmatrix} 0 & mn & & & & & \\ 1 & (m+n-2) & (m-1)(n-1) & & & & \\ & 4 & 2(m+n-4) & (m-2)(n-2) & & & \\ & & 9 & 3(m+n-6) & (m-3)(n-3) & & \\ & & & 16 & \ddots & \ddots & \\ & & & & \ddots & \ddots & m-n+1 \\ & & & & & n^2 & n(m-n) \end{bmatrix}_{(n+1)\times(n+1)}$$

Therefore, if $w_i = \sum_j a_{ij} v_j$, then

$$w_{i+1} = \mathcal{A}(w_i) = \sum_j a_{ij}\,\mathcal{A}(v_j)$$

$$(w_0,\cdots,w_n)=(v_0,\dots,v_n)\begin{bmatrix} 1 & 0 & mn & * & * & \cdots \\ & 1 & m+n-2 & * & * & \cdots \\ & & 4 & * & * & \cdots \\ & & & 4\cdot 9 & * & \cdots \\ & & & & 4\cdot9\cdot16 & \cdots \\ & & & & & \ddots \end{bmatrix}_{(n+1)\times(n+1)}$$

after tensored over $\mathbb{Q}$, $(w_0,\cdots,w_n)$ become a basis, and

$$\mathcal{A}(w_0,\dots,w_n)=(w_0,\dots,w_n)\begin{bmatrix} & & & & -c_0 \\ 1 & & & & \vdots \\ & 1 & & & \vdots \\ & & \ddots & & -c_{n-1} \\ & & & 1 & -c_n \end{bmatrix}_{(n+1)\times(n+1)}$$

where $F_{m,n}(T)=b_{n+1}(T^{n+1}+c_nT^n+c_{n-1}T^{n-1}+\cdots c_0)\in\mathbb{Z}[T]$ $\quad c_i\in\mathbb{Q}$

$b_{n+1}:=\operatorname{lcm}(\text{denominators of } c_i)$

least common multiple

Therefore, the problem reduces to the computation of

$$\begin{aligned} &T^{n+1}+c_nT^n+c_{n-1}T^{n-1}+\cdots+c_0 \\ &=\text{char poly of } \mathcal{A} \\ &=\text{char poly of } \begin{bmatrix} 0 & mn & & \\ 1 & m+n-2 & (m-1)(n-1) & \\ & 4 & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}_{(n+1)\times(n+1)} \end{aligned}$$

Since $c_i\in\mathbb{Z}$, we get $b_{n+1}=1$, i.e.

$$F_{m,n}(T)=\text{char poly of } \mathcal{A}.$$

Fix $m\geq n$, denote $n\geq k\geq 0$,

$$\beta_k^T=[0,\cdots,0,1]\in\mathbb{Z}^k,$$

$$A_k:=A_{m,n,k}:=\begin{bmatrix} 0 & mn & & & \\ 1 & m+n-2 & (m-1)(n-1) & & \\ & 4 & \ddots & \ddots & \\ & & \ddots & \ddots & (m-k+1)(n-k+1) \\ & & & k^2 & k(m+n-2k) \end{bmatrix}_{(k+1)\cdot(k+1)}$$

$$\underset{\text{if } k\geq1}{=}\begin{bmatrix} A_{m,n,k-1} & (m-k+1)(n-k+1)\beta_k \\ k^2\beta_k^T & k(m+n-2k) \end{bmatrix}$$

e.p. $\mathcal{A}=A_{m,n,n}$

$$\lambda I - A_k = \begin{bmatrix} \lambda I - A_{k-1} & -(m-k+1)(n-k+1)\beta_k \\ -k^2\beta_k^T & \lambda - k(m+n-2k) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ -k^2\beta_k^T(\lambda I - A_{k-1})^{-1} & 1 \end{bmatrix} \begin{bmatrix} \lambda I - A_{k-1} & -(m-k+1)(n-k+1)\beta_k \\ 0 & \begin{array}{c} -k^2(m-k+1)(n-k+1)\beta_k^T(\lambda I - A)^{-1}\beta_k \\ +\lambda - k(m+n-2k) \end{array} \end{bmatrix}$$

$$\beta_k^T(\lambda I - A_{k-1})^{-1}\beta_k = \left((\lambda I - A_{k-1})^{-1}\right)_{k,k}$$

$$= \begin{cases} \dfrac{\det(\lambda I - A_{k-2})}{\det(\lambda I - A_{k-1})} & k \geq 1 \\ \lambda^{-1} & k = 1 \end{cases}$$

Hint:

$$B^{-1} = \frac{1}{\det B} \begin{bmatrix} B_{11} & -B_{12} & B_{13} & \cdots & \\ -B_{21} & B_{22} & & & \\ B_{31} & & & & -B_{(n-1)n} \\ \vdots & & & -B_{n,(n-1)} & B_{n,n} \end{bmatrix}$$

Denote $Det_k := Det(\lambda I - A_k)$, then

$$Det_k = Det_{k-1}\left(-k^2(m-k+1)(n-k+1)\frac{Det_{k-2}}{Det_{k-1}} + \lambda - k(m+n-2k)\right)$$

$$= (\lambda - k(m+n-2k))\,Det_{k-1} - k^2(m-k+1)(n-k+1)\,Det_{k-2} \quad (\text{for } k \geq 2)$$

$Det_0 = \lambda I - A_0 = \lambda$

$Det_1 = \lambda I - A_1 = \lambda^2 - (m+n-2)\lambda - mn$

$F_{m,n}(\lambda) = Det_n$ $\square$