Eine Woche, ein Beispiel
9.3. field extension with RS

Goal: construct an equivalence between two categories:

cpt conn
$$RS^{cc} = \left\{ \begin{array}{l} Obj: \text{ cpt conn RS} \\ Mor: \text{ non-const holo morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} Obj: F/\mathbb{C} \text{ field ext s.t.} \\ \quad trdeg_{\mathbb{C}} F = 1 \\ \quad F/\mathbb{C} \text{ f.g as a field} \\ Mor: \text{ morphism as fields}/\mathbb{C} \end{array} \right\}^{op} = field^{op}_{\mathbb{C}(t)/\mathbb{C}}$$

$$\begin{array}{ccc} Y & & M(Y) \\ \downarrow f & \Longrightarrow & \uparrow f^* \\ X & & M(X) \end{array}$$

which obeys the following slogan:

> (ramified) covering $\approx$ (function) field extension

Rmk. 1. For requiring $F/\mathbb{C}$ f.g. as a field, we avoid examples like $\overline{\mathbb{C}(t)}$.
   Do they corresponds to some non-cpt Riemann surface?
   If so, how to enlarge the category RS^{cc}?

2. $field_{\mathbb{C}(t)/\mathbb{C}}$ means fields over $\mathbb{C}$ which are fin ext of $\mathbb{C}(t)$ abstractly;
   morphisms don't need to fix $\mathbb{C}(t)$.
   Do you have a better name for RS^{cc} and $field_{\mathbb{C}(t)/\mathbb{C}}$?

1. field of meromorphic functions
2. Galois covering
3. valuations
4. quadratic extension of $\mathbb{C}(x)$: hyperelliptic curve
5. miscellaneous.

1. field of meromorphic functions

Def. For X RS,
$$\mathcal{M}(X) := \{\text{meromorphic fcts on } X\}$$
$$= \{f: X \longrightarrow \mathbb{P}^1 \text{ holomorphic}\} \sim \{\mathbb{1}_\infty\}$$
$$\underset{\substack{X \text{ cpt} \\ \text{conn}}}{=} \{\text{rational fcts on } X\}$$

Ex. Verify that
$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$
$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[i]) \cong \mathrm{Frac}\left(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1))\right)$$

Later we will show that, for $X \in RS^{cc}$,
$$\exists\, \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For
$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3,$$
compute
1) $f^*: \mathbb{C}(T) \hookrightarrow \mathbb{C}(S) \qquad [\mathbb{C}(S):\mathbb{C}(T)]$ & a $\mathbb{C}(T)$-basis
2) $\mathrm{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$
3) $\mathbb{C}(S)^{\mathbb{Z}/3\mathbb{Z}}$
4) $\mathrm{Aut}_f(\mathbb{CP}^1)$

Ex. For
$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z + \tfrac{1}{z},$$
do the same work.

Ex. For
$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z,$$
compute the same stuff.
Why isn't $\mathbb{C}(S)/\mathbb{C}(T)$ Galois this time?

Prop. For $d \in \mathbb{N}_{>0}$, $f: Y \longrightarrow X$ proper holo morphism between conn RSs,
$$[\mathcal{M}(Y) : f^* \mathcal{M}(X)] = d.$$

Cor. For X cpt conn,
$$\exists \; \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$
In ptc, $F/\mathbb{C}$ f.g as a field, $\mathrm{trdeg}_{\mathbb{C}} F = 1$.

To show the proposition, one need the following black box to find a basis.
Black box (meromorphic fcts seperate points)
$$X : RS, \quad x, y \in X \quad x \neq y \quad , \text{ then}$$
$$\exists \; g \in \mathcal{M}(X) \quad \text{s.t.} \quad g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$
(stronger) $\quad \exists \; g \in \mathcal{M}(X) \quad \text{s.t.} \quad \mathrm{ord}_x g = -1 , \quad g(y) = 0.$

I prefer using Riemann-Roch when X is cpt, and Stein manifold when X is not.

Ex. Using the black box, show that,
$$\text{for } X : RS, \quad \{x_1, \cdots, x_n\} \subseteq X, \quad \exists \; g \in \mathcal{M}(X) \text{ s.t.}$$
$$\mathrm{ord}_{x_1} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \cdots, n\}$$
$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \quad i, j \in \{2, \cdots, n\}$$

Proof of prop
$\quad [\mathcal{M}(Y) : f^* \mathcal{M}(X)] \geq d$: Fix $x_0 \in X$ s.t. $\# f^{-1}(x_0) = d$. Denote $f^{-1}(x_0) = \{y_1, \cdots, y_d\}$.
$\quad$ For each i, let $g_i \in \mathcal{M}(Y)$ be a meromorphic fct s.t.
$$\mathrm{ord}_{x_i} g_i = -1 \quad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$
$\quad$ then $\{g_1, \cdots, g_d\} \subseteq \mathcal{M}(Y)$ are $f^* \mathcal{M}(X)$ -linear independent.
$\quad$ Check: $\mathrm{ord}_{y_j}(\sum_i f_i g_i) \approx \mathrm{ord}_{y_j} f_j$

$[\mathcal{M}(Y) : f^* \mathcal{M}(X)] \leq d$:
$\quad \forall g \in \mathcal{M}(Y)$, need to find $a_i \in f^* \mathcal{M}(X)$ s.t.
$$g^d + a_{d-1} g^{d-1} + \cdots + a_0 = 0 \quad \text{in } \mathcal{M}(Y)$$
$\quad$ The fcts
$$a_i(z) = (-1)^i \sum_{\{k_1, \cdots, k_i\} \subseteq \{1, \cdots, d\}} g(z_{k_1}) \cdots g(z_{k_d})$$
$\quad f^{-1}(f(z)) := \{z_1, \cdots, z_d\}$, multiplicity is counted
$\quad$ satisfy the conditions.
$\quad$ Use Riemann extension theorem to show $a_i(z) \in f^* \mathcal{M}(X)$, see [Donaldson, p148].
$\quad$ By primitive element theorem, $[\mathcal{M}(Y) : f^* \mathcal{M}(X)] \leq d.$

2. Galois covering

Def. Let $f: Y \to X$ be a proper holo map between two conn RSs.
f is Galois, if $M(Y)/f^*M(X)$ is a Galois extension.
(normal) ... (normal)

Prop. $f: Y \to X$ is Galois/normal
$\Leftrightarrow \deg f = \# \operatorname{Aut}_f(Y)$
$\Leftrightarrow f^{-1}(x_0)$ is an $\operatorname{Aut}_f(Y)$-torsor, $\quad \forall x_0 \in X - f(\operatorname{Ram}(f))$
$\Leftrightarrow \operatorname{Aut}_f(Y) \circlearrowright f^{-1}(x_0)$ transitively, $\quad \forall x_0 \in X$
$\Leftrightarrow Y/\operatorname{Aut}_f(Y) \cong X$, i.e. f can be written as
$$Y \longrightarrow\!\!\!\!\rightarrow Y/G$$

Ex. For $f: Y \to X$, suppose that
$$[\forall y_1, y_2 \in Y \text{ s.t. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2)$$
($\swarrow$ ramification index)
Show that f is Galois by computing $\# \operatorname{Aut}_f(Y)$.

[Hint. Use geodesics to divide X into several smaller triangles.
If geodesics are hard, take $g: X \to \mathbb{CP}^1$ non-constant,
and reduce the problem to $g \circ f$.]

E.g. Consider the covering
$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z$$
This is not a Galois covering. Consider the Galois closure

$$
\begin{array}{ccc}
\mathbb{CP}^1 & \mathbb{C}(\mathcal{U}) = \mathbb{C}(S)[R]/(R^2 + S^2 - 4) & \mathcal{U} + \frac{1}{\mathcal{U}} \\
\downarrow z + \frac{1}{z} & \uparrow & \uparrow \\
\mathbb{CP}^1 & \mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3 - 3S - T) & S \qquad S^3 - 3S \\
\downarrow z^3 - 3z & \uparrow & \updownarrow \\
\mathbb{CP}^1 & \mathbb{C}(T) & T
\end{array}
$$

Determination of the Galois closure
$$\min(S, \mathbb{C}(T)) = x^3 - 3x - T \qquad \text{in } \mathbb{C}(T)[x]$$
$$= x^3 - 3x - (S^3 - 3S)$$
$$= (x - S)(x^2 + Sx + S^2 - 3) \qquad \text{in } \mathbb{C}(S)[x]$$

To decompose the polynomial $x^2 + Sx + S^2 - 3$,
we have to add root of discriminant:
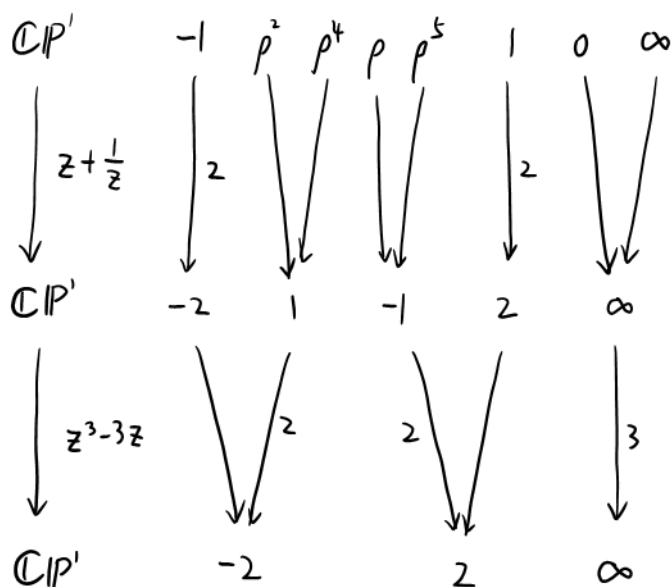$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2 - 3)} = \sqrt{3}\sqrt{-S^2 + 4}.$$
Therefore, the Galois closure of $\mathbb{C}(S)/\mathbb{C}(T)$ is
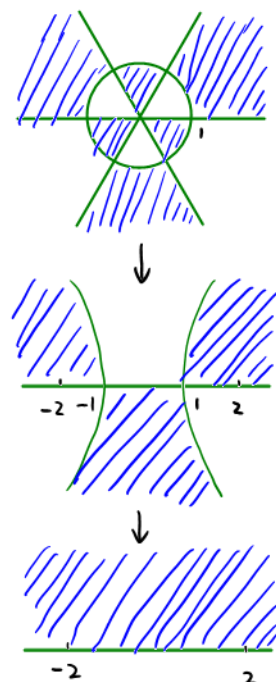$$\mathbb{C}(S)[R]/(R^2 + S^2 - 4) \cong \mathbb{C}\left(\frac{S + iR}{2}\right) \overset{\wedge}{=} \mathbb{C}(\mathcal{U})$$
where
$$S = \frac{S + iR}{2} + \frac{S - iR}{2} = \mathcal{U} + \frac{1}{\mathcal{U}}$$

The picture from the RS side is as follows:

$$
\begin{array}{c}
\mathbb{CP}^1 \qquad -1 \quad \rho^2 \; \rho^4 \; \rho \; \rho^5 \quad 1 \quad 0 \quad \infty \\
\downarrow z + \frac{1}{z} \quad \Big\downarrow 2 \quad \bigvee\bigvee \quad \bigvee \quad \Big\downarrow 2 \quad \bigvee \\
\mathbb{CP}^1 \qquad -2 \quad 1 \quad -1 \quad 2 \quad \infty \\
\downarrow z^3 - 3z \quad \bigvee 2 \quad \bigvee 2 \quad \Big\downarrow 3 \\
\mathbb{CP}^1 \qquad -2 \qquad 2 \qquad \infty
\end{array}
$$

only ramified pts are drawn

affine version

E.g. 2. For $E = \mathbb{C}/\Lambda$, since $\pi_1(E, 0) \cong \mathbb{Z} \oplus \mathbb{Z}$,
E has three unramified coverings of deg 2.
When $\Lambda = \mathbb{Z}[i]$, what are the crspd field extensions?