

# Eine Woche, ein Beispiel

## 4.10. non-Archimedean local field $F$

wiki: local field

See <https://mathoverflow.net/questions/17061/locally-profinite-fields> for different definition of local fields. We follow wiki instead.

### Classification:

- finite extension of  $\mathbb{Q}_p$
- $\mathbb{F}_q((T))$  ( $q = p^r$ )

### Process:

1. Basic structures and results.
2. Topological results.
3. Haar measure
4. Representation of  $(F, +)$  and  $F^\times$  (next week)

### 1. Basic structures and results

1.1. None of them is alg closed.

1.2. The natural valuation  $v: F \rightarrow \mathbb{Z}$  is defined. Then

$$\mathcal{O}, \mathfrak{p}, \kappa = \mathcal{O}/\mathfrak{p}$$

$$p = \text{char } \kappa, \quad q = |\kappa| = p^r$$

$$\mathcal{U} = \mathcal{U}^{(0)} = \mathcal{O}^\times = \mathcal{O} - \mathfrak{p} = \{x \in F \mid v(x) \geq 0\}$$

$$\mathcal{U}^{(n)} = 1 + \mathfrak{p}^n \quad n \geq 1$$

are defined, and  $\pi \in \mathfrak{p} \setminus \mathfrak{p}^2 \cong \mathfrak{p} - \mathfrak{p}^2$  is picked.

Moreover,  $\mathcal{O}$  is DVR,  $\kappa$  is finite,

$$\mathcal{U}^{(0)}/\mathcal{U}^{(1)} \xrightarrow{\text{split iso}} \kappa^\times$$

$$\mathcal{U}^{(0)}/\mathcal{U}^{(n)} \xrightarrow{\text{non-split iso}} (\mathcal{O}/\mathfrak{p}^n)^\times \quad n \geq 1$$

$$\mathcal{U}^{(n)}/\mathcal{U}^{(n+1)}$$

$$\mathcal{U}^{(m+1)}/\mathcal{U}^{(n+1)}$$

non-canonical

$$\cong \kappa$$

$$\cong \mathcal{O}/\mathfrak{p}^{m-n}$$

$n \geq 1$

$2n+1 \geq m > n \geq 0$

$$0 \rightarrow \mathcal{U}^{(n)} \rightarrow \mathcal{O}^\times \xrightarrow{\mu_{q-1}} \kappa^\times \rightarrow 0$$

$$\mu_{q-1} = \{a \in F \mid a^{q-1} = 1\}$$

$\curvearrowright$ : the Teichmüller lift

$$\Rightarrow \mathcal{O}^\times \cong \mathcal{U}^{(n)} \times \mu_{q-1}$$

$$1.3. \quad F^\times \cong \langle \pi \rangle \times \mathcal{O}^\times \cong \langle \pi \rangle \times \mu_{q-1} \times \mathcal{U}^{(n)}$$

$$\text{e.g. when } F = \mathbb{Q}_p, \quad \mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/(q-1)\mathbb{Z} \oplus \mathbb{Z}_p & p \neq 2 \\ \mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_2) & p = 2 \end{cases}$$

Thm. When  $p \geq 3$ ,  $(p\mathbb{Z}_p, +) \xrightleftharpoons[\log]{\exp} (1+p\mathbb{Z}_p, \cdot)$  is an iso as topological grps.

## 2. Topological results.

$\mathcal{O} = \varprojlim_n \mathcal{O}/\mathfrak{p}^n$  is cpt and profinite group, while  $F$  is loc. cpt and loc. profinite group

$\mathcal{O}^\times = \varprojlim_n \mathcal{O}^\times/\mathcal{U}^{(n)}$  is cpt and profinite group, while  $F^\times$  is loc. cpt and loc. profinite group

Cpt open subgps of  $(F, +)$  are  $\{\mathfrak{p}^k\}$ .

Cpt open subgps of  $F^\times$  are not restricted in  $\{\mathcal{U}^{(k)}\}$ ,

but  $\{\mathcal{U}^{(k)}\}$  is a nbhd system of  $F^\times$ , i.e.,

$\{a\mathcal{U}^{(k)}\}_{a \in F^\times}$  is a topological basis of  $F^\times$ .

$\{\text{open subgps}\} \subseteq \{\text{closed subgps}\}$  for  $(F, +)$  and  $F^\times$ .

Q: Are there any other cpt closed subgp?

A: Yes. e.g.  $\{0\} \subseteq (F, +)$   $\{1\} \subseteq F^\times$

Q: Can we classify all cpt closed subgp?

E.g.  $\mathbb{Q}_{p^r} =$  the splitting field of  $X^q - X$  over  $\mathbb{Q}_p$   $q = p^r$   
 $=$  the unique unramified extension of  $\mathbb{Q}_p$  of degree  $r$

$$\text{Gal}(\mathbb{Q}_{p^r}/\mathbb{Q}_p) \cong \text{Gal}(\mathbb{F}_{p^r}/\mathbb{F}_p) \cong \mathbb{Z}/r\mathbb{Z}$$

### 3. Haar measure

Main reference: The Local Langlands Conjecture for  $GL(2)$  by Colin J. Bushnell Guy Henniart.  
 [https://link.springer.com/book/10.1007/3-540-31511-X]  
 Ref: https://en.wikipedia.org/wiki/Haar\_measure

$G$ : loc. profinite gp

$$C_c^\infty(G) := \{f: G \rightarrow \mathbb{C} \mid f \text{ is loc. const}\}$$

$$C_c^\infty(G) := \{f \in C_c^\infty(G) \mid \text{supp } f \subset G \text{ is cpt}\}$$

Rmk.  $G$  has topo basis  $\{g_k\}_{k \in \mathbb{N}}$  cpt open.

$\forall f \in C_c^\infty(G), \exists k \in \mathbb{N}$  cpt open, s.t.

$$f = \sum_{g \in G} a_g \mathbb{1}_{g_k} \quad a_g \in \mathbb{C} \quad \#\{g \in G \mid a_g \neq 0\} < +\infty$$

e.g. When  $G = (F, +)$ ,  $C_c^\infty(F) = \langle a + \beta^k \rangle_{\substack{a \in F \\ k \in \mathbb{Z}}}$   
 when  $G = F^\times$ ,  $C_c^\infty(F^\times) = \langle a U^{(k)} \rangle_{\substack{a \in F^\times \\ k \in \mathbb{Z}_{\geq 0}}}$

Def (Left Haar integral & Left Haar measure)

integral:  $I: C_c^\infty(G) \rightarrow \mathbb{C}$  s.t

• (left invariant)  $I(f(g \cdot)) = I(f(\cdot))$

• (positive)  $I(f) \geq 0$

measure:  $\mu_G: \mathcal{L}(G) \rightarrow \mathbb{R}$

Lebesgue  $\sigma$ -algebra, see  
<https://math.stackexchange.com/question/s/3117419/lebesgue-sigma-algebra>

$\forall f \in C_c^\infty(G) \quad g \in G$

$\forall f \in C_c^\infty(G) \quad f \geq 0$

$S \subset G$  cpt open  $\mapsto I(\mathbb{1}_S)$

The domain of  $I$  is not extended, so here it is not perfect.

relation/notation:  $I(f) = \int_G f(g) d\mu_G(g)$

Rmk.

Left Haar measure exists and is unique (up to scalar) on every loc. cpt gp  $G$ , see  
<https://www.diva-portal.org/smash/get/diva2:1564300/FULLTEXT01.pdf>

Later on, Haar measure = left + right Haar measure.

E.g. Let  $\mu$  be the Haar measure on  $F$ , then

$\mu^\times$  is a Haar measure on  $F^\times$ , and  $(d\mu^\times(x) = \frac{d\mu(x)}{|x|})$

$$\int_{F^\times} f(x) d\mu^\times(x) = \int_F f(x) \frac{d\mu(x)}{|x|} \quad \forall f \in C_c^\infty(F^\times) \subset C_c^\infty(F)$$

Let  $\mu$  be the Haar measure on  $A := M_{n \times n}(F)$ , then

$\mu^\times$  is a Haar measure on  $G := GL_n(F)$ , and  $(d\mu^\times(g) = \frac{d\mu(g)}{|\det g|^n})$

$$\int_G f(g) d\mu^\times(g) = \int_A f(g) \frac{d\mu(g)}{|\det g|^n} \quad \forall f \in C_c^\infty(G) \subset C_c^\infty(A)$$

Def Unimodular: left Haar measure = right Haar measure

Rmk.  $G$  is cpt  $\Rightarrow G$  is unimodular  $\Leftrightarrow \delta_G = 1$   
 $\nearrow$   
 $G$  is abelian

where  $\delta_G : G \rightarrow \mathbb{C}^\times$  is determined by  
 $d\mu_G(g^{-1}xg) \stackrel{\text{left inv}}{=} d\mu_G(x) = \delta_G(g) d\mu_G(x).$

Actually,  $\forall K \leq G$  cpt open,  $\delta_K = \mathbb{1}_K$ .

e.g.  $(F, +), (\mathcal{O}, +), F^\times, \mathcal{O}^\times$  are all unimodular.

e.g.  $G = GL_2(\mathbb{Q}_p)$  is unimodular, while

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$   $M = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$  are not unimodular.