

Eine Woche, ein Beispiel

8.28 global field

This note mainly follows [现代数学基础12-数论I: Fermat的梦想和类域论-日 加藤和也 & 黑川信重-胥鸣伟 & 印林生(译)].
 Another reference for complement (and also for non-Chinese reader):
 [MIT] <https://math.mit.edu/classes/18.785/2015fa/lectures.html>

I should have done this in 2021.06.27_adèles_and_idèles. However, I was not familiar with local field at that time.

1. definition

- 2. adèle ring and idèle group
- 3. topological properties of \mathbb{A}_K & \mathbb{I}_K
- 4. Tate's thesis

\parallel def
measure
topo fundamental domain dense
cpt
discrete

1. definition

Def A global field is

- a finite extension of \mathbb{Q} (number field), or
- a finite extension of $\mathbb{F}_p(T)$ (function field)

For an axiomatic definition, see
<https://math.stackexchange.com/questions/873666/definition-of-global-field>

In this note we denote K for the global field;
 when K is used as a cpt open subgp, we denote \mathbb{E} or \mathbb{F} for the global field.

Rmk 1. Ostrowski's thm states that

every non-trivial norm on \mathbb{Q} is equiv to $|\cdot|_p$ or $|\cdot|_\infty$.

In [Thm3, Cor4, [https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF\(T\).pdf](https://kconrad.math.uconn.edu/blurbs/gradnumthy/ostrowskiF(T).pdf)],

every non-trivial norm on $\mathbb{F}_p(T)$ equiv to $|\cdot|_\pi$ or $|\cdot|_\infty$

where

$$\left| \frac{a}{b} \pi^k \right|_\pi = p^{-\deg \pi \cdot k}$$

$$\left| \frac{a}{b} \right|_\infty = p^{\deg a - \deg b}$$

for some monic irr $\pi(T) \in \mathbb{F}_p[T]$
 $a, b \in \mathbb{F}_p[T], \pi \nmid ab \quad a, b \neq 0$
 $a, b \in \mathbb{F}_p[T] \quad a, b \neq 0$

Ex. Compute K_v, \mathcal{O}_v for $v = |\cdot|_\infty, |\cdot|_T, |\cdot|_{T-1}, |\cdot|_{T^2+1}$ $K = \mathbb{F}_p(T), p=7$

$$\text{A. } \mathcal{O}_{|\cdot|_\infty} = \mathbb{F}_p\left[\left(\frac{1}{T}\right)\right] \quad \mathcal{O}_{|\cdot|_T} = \mathbb{F}_p[[T]] \quad \mathcal{O}_{|\cdot|_{T-1}} = \mathbb{F}_p[[T-1]] \\ K_{|\cdot|_\infty} = \mathbb{F}_p\left(\left(\frac{1}{T}\right)\right) \quad K_{|\cdot|_T} = \mathbb{F}_p((T)) \quad K_{|\cdot|_{T-1}} = \mathbb{F}_p((T-1))$$

$\mathcal{O}_K = \mathbb{F}_p[T]$ can not embed in $\mathcal{O}_{|\cdot|_\infty}$, since $\mathbb{F}_p[T] = \mathcal{O}_{\mathbb{F}_p((1))}$.

The prod formula also prohibit \mathcal{O}_K embed to all \mathcal{O}_v .

Show that $\mathbb{F}_p\left(\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(T-\frac{1}{\alpha}\right)\right)$ for $\alpha \in \mathbb{F}_p^\times$:

$$\mathbb{F}_p\left(\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(\frac{1-\alpha T}{T}\right)\right) = \mathbb{F}_p\left(\left(-\frac{\alpha}{T}(T-\frac{1}{\alpha})\right)\right) \\ \mathbb{F}_p\left(\left(-\frac{(T^2-\alpha+\alpha)}{\alpha}\right)^{-1}\left(\frac{1}{T}-\alpha\right)\right) = \mathbb{F}_p\left(\left(-\frac{T}{\alpha}\left(\frac{1}{T}-\alpha\right)\right)\right) = \mathbb{F}_p\left(\left(T-\frac{1}{\alpha}\right)\right)$$

$$\mathcal{O}_{|\cdot|_{T^2+1}} = \mathbb{F}_p(2)[[T^2+1]] \quad \alpha^2 + 1 = 0$$

$$K_{|\cdot|_{T^2+1}} = \mathbb{F}_p(2)((T^2+1))$$

$$\begin{aligned} \mathbb{F}_p[T] &\hookrightarrow \mathbb{F}_p(2)[[T^2+1]] \\ T &\mapsto 2 - \frac{5}{2}(T^2+1) - \frac{25}{8}(T^2+1)^2 - \frac{25}{16}(T^2+1)^3 - \frac{52}{128}(T^2+1)^4 - \dots \\ T^2 &\mapsto -1 + T^2+1 \end{aligned}$$

Rmk 2. Product formula is still true; that is, for $K = \mathbb{F}_p(T)$

$$\prod_{\pi \text{ fin}} \|f\|_\pi = 1 \quad \forall f \in \mathbb{F}_p(T)^\times$$

Ex. Verify the product formula for other K .

For relationships between local fields and global fields, see: <https://alex-yuclis.github.io/localglobalgalois.pdf>
We only list two results which will be used later:

Let L/K be fin ext of global field. We get two isos as topo ring

$$\begin{array}{ccc} L \otimes_K K_v & \xrightarrow{\cong} & \prod_{i=1}^g L_{w_i} \\ \uparrow & & \cup \\ \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_v & \xrightarrow{\cong} & \prod_{i=1}^g \mathcal{O}_{w_i} \end{array} \quad \begin{array}{c} w_1 \cdots w_g \\ \backslash \cdots / \\ v \end{array} \quad \begin{array}{c} L_{w_1} \cdots L_{w_g} \\ \backslash \cdots / \\ K_v \end{array}$$

[MIT, Cor 11.7]

2. adèle ring and idèle group

Every book begins this topic by restricted product, which is totally correct but a little boring/confusing. Let us derive the restricted product naturally.

$$\begin{array}{cccc}
 \text{global} & A_K & \mathbb{I}_K & \mathbb{I}_K^\times \\
 \text{local} & F & F^\times & \mathcal{O}_F^\times \\
 \\
 (\mathbb{Q}, +) & G & (\mathbb{Q}^\times, \times) & G \\
 \prod_p \mathbb{Q}_p \times \mathbb{R} & & \prod_p \mathbb{Q}_p^\times \times \mathbb{R}^\times & \\
 \bigcup & & \bigcup & \\
 \\
 A_{\mathbb{Q}} = \underbrace{\prod_p' \mathbb{Q}_p \times \mathbb{R}}_{A_{\mathbb{Q}, \text{fin}}} \bigcup & A_{\mathbb{Q}}^\times = \underbrace{\prod_p' \mathbb{Q}_p^\times \times \mathbb{R}^\times}_{A_{\mathbb{Q}, \text{fin}}^\times} = \mathbb{I}_{\mathbb{Q}} & & \\
 \\
 \mathbb{Q} \sqrt{A_{\mathbb{Q}}} = \underbrace{\prod_p \mathbb{Z}_p \times [0, 1)}_{\widehat{\mathbb{Z}}} & \mathbb{Q}^\times \setminus A_{\mathbb{Q}}^\times = \underbrace{\prod_p \mathbb{Z}_p^\times \times \mathbb{R}_{>0}}_{\widehat{\mathbb{Z}}^\times} = C_{\mathbb{Q}} & &
 \end{array}$$

= : as a set, topo may be different

∇ A_F^\times contains the F^\times -orbit of $\prod_{p \text{ fin}} \mathcal{O}_p^\times \times \prod_{p \text{ inf}} F_p^\times$,
but not equal to it in general.

When $\#\text{Cl}(\mathcal{O}_F) = 1$, we are lucky,
as the UFD property provides us with such an element.

Q: Do we always have

$$A_F = F \cdot \left(\prod_{p \text{ fin}} \mathcal{O}_p \times \prod_{p \text{ inf}} F_p \right) ?$$

adèle ring

Def (adèle ring $A_{\mathbb{Q}}$) We know that

$$\left(\prod_{p \text{ prime}} \mathbb{Z}_p \right) \times [0,1) \subseteq \left(\prod_{p \text{ prime}} \mathbb{Q}_p \right) \times \mathbb{R}$$

where \mathbb{Q} acts diagonally on $\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$.

$$+ : \mathbb{Q} \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R} \right) \longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p \times \mathbb{R}$$

$$(t, (a_p, a_\infty)) \mapsto (t + a_p, t + a_\infty)$$

The adèle ring $A_{\mathbb{Q}}$ is defined as the orbit of $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$, i.e.

$$A_{\mathbb{Q}} := \mathbb{Q} + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1) \right)$$

$$= \{ (a_v)_v \in \prod_v F_v \mid a_v \in O_v \text{ for almost all } v \} \stackrel{\triangle}{=} \prod_v' F_v$$

\exists we don't define O_v for $v = 1/\infty$,
but that doesn't matter.

Rmk. You can also replace $[0,1)$ by \mathbb{R} in the definition ($A_{\mathbb{Z}} := \prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R}$),
then it may happen that

$$t + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R} \right) = t' + \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times \mathbb{R} \right) \quad \text{for } t \neq t' \in \mathbb{Q}.$$

Rmk. The measure is easy to define while the topo is a bit tricky.

By letting $\mu_p(\mathbb{Z}_p) = 1$, $\mu_\infty([0,1)) = 1$ and give $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0,1)$ with the prod measure,
the **measures** on $A_{\mathbb{Q}/\mathbb{Q}}$ and $A_{\mathbb{Q}}$ are defined.

For the **topology** on $A_{\mathbb{F}}$, we take the weakest topo s.t. all the subspaces

$$\prod_{v \in S} F_v \times \prod_{v \notin S} O_v = \left(\prod_{\substack{p \in S \\ p \text{ prime}}} \mathbb{Q}_p \times \mathbb{R} \times \prod_{p \notin S} \mathbb{Z}_p \right)$$

(for any S : set of finite places containing all infinite places)

are open, and the subspace topo of $\prod_{v \in S} F_v \times \prod_{v \notin S} O_v$ coincides with the prod topo.

This topology is a little stronger than the subspace topo of $A_{\mathbb{F}} \subset \prod_v F_v$,
since $\prod_{v \in S} F_v \times \prod_{v \notin S} O_v$ are not open in this subspace topo.

The same method can be applied to defining the topo of any restricted product.

Ex. Verify that

$\prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1]$ is the fundamental domain of $A_{\mathbb{Q}/\mathbb{Q}}$, so

$$\mu \left(\prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1] \right) = 1 \Rightarrow \mu (A_{\mathbb{Q}/\mathbb{Q}}) = 1$$

Ex. How do they glue with each other?

- $\mathbb{Q} \hookrightarrow A_{\mathbb{Q}}$ is discrete. (by considering the preimage of $\prod_{p \text{ prime}} \mathbb{Z}_p \times (-\frac{1}{2}, \frac{1}{2})$)
- $\prod_{p \text{ prime}} \mathbb{Z}_p \times [0, 1] \hookrightarrow A_{\mathbb{Q}} \rightarrow A_{\mathbb{Q}/\mathbb{Q}}$ is cont
- $\Rightarrow A_{\mathbb{Q}/\mathbb{Q}}$ is cpt. $A_{\mathbb{Q}}$ is loc. cpt.
- $\mathbb{Q} \hookrightarrow \prod'_{p \text{ prime}} \mathbb{Q}_p$, $\mathbb{Q} \hookrightarrow \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$ are dense;
- $\mathbb{Z}[\frac{1}{p}] \hookrightarrow \mathbb{Q}_p \times \mathbb{R}$, $\{\frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b\} \hookrightarrow \prod'_{\substack{p \text{ prime} \\ p \neq 7}} \mathbb{Q}_p \times \mathbb{R}$ are lattices

discrete & quotient is cpt

Ex. define A_F in general, apply it to $F = \mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{F}_p(T)$, and compute their measures and fundamental domains.

From [MIT, #22, p5], $\mu_v(U) = 2\mu_c(U)$ for $F_v \cong \mathbb{C}$

Hint. $\mathbb{F}_p[T] \subset \mathbb{F}_p((\frac{1}{T}))$ is a lattice. $\mathbb{F}_p((\frac{1}{T})) = \mathbb{F}_p[T] \oplus \frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]$.

$$\begin{aligned} \text{Set } \mu(\mathbb{F}_p[[\frac{1}{T}]]) &= 1, \text{ then } \mu\left(\frac{1}{T}\mathbb{F}_p[[\frac{1}{T}]]\right) = \frac{1}{p} \\ &\Rightarrow \mu(A_{\mathbb{F}_p(T)} / \mathbb{F}_p(T)) = \frac{1}{p}. \end{aligned}$$

For convenience, we will define

$$A_{F, \text{fin}} = \prod'_{v \text{ fin}} F_v = \widehat{\bigcap_{\substack{\uparrow \text{ in some article} \\ \text{not in our notes}}} F} \quad A_{F, \text{inf}} = \prod'_{v \text{ inf}} F_v \quad (A_F = A_{F, \text{fin}} \times A_{F, \text{inf}})$$

$$\widehat{O}_F = \prod'_{v \text{ fin}} O_v$$

S denotes for any finite set of places containing all infinite places, and T denotes for any set of places containing all infinite places.

$S, T \neq \emptyset$

idèle group

Def (idèle group \mathbb{I}_α) We know that

$$\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) \subseteq \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times\right)$$

where \mathbb{Q}^\times acts diagonally on $\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$.

$$\begin{aligned} \cdot : \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times\right) &\longrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times \\ (t, (a_p, a_\infty)) &\longmapsto (ta_p, ta_\infty) \end{aligned}$$

The idèle group \mathbb{I}_α is defined as the orbit of $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$, i.e.

$$\begin{aligned} \mathbb{I}_\alpha &:= \mathbb{Q}^\times \times \left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) \\ &= \{(a_v)_v \in \prod_v F_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \prod_v' F_v^\times \\ &= (\prod_v' F_v)^\times = A_\alpha^\times \end{aligned}$$

In general,

$$\begin{aligned} \mathbb{I}_F &\stackrel{\text{not unique expression}}{=} F^\times \times \left(\prod_{v \text{ fin}} \mathcal{O}_v^\times \times \prod_{v \text{ inf}} F_v^\times\right) \\ &\stackrel{\text{def}}{=} \{(a_v)_v \in \prod_v F_v^\times \mid a_v \in \mathcal{O}_v^\times \text{ for almost all } v\} \triangleq \prod_v' F_v^\times \\ &= (\prod_v' F_v)^\times = A_F^\times \\ &= F^\times \times \left(\prod_{v \notin S} \mathcal{O}_v^\times \times \prod_{v \in S} F_v^\times\right) \text{ for } S \text{ big enough} \end{aligned}$$

Rmk. The definition of measure and topology are similar.

The topo defined is stronger than the subspace topo $A_F^\times \subset A_F$,

since $\prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$ (for any S) is not open in the subspace topology.

Ex. Verify that

• $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$ is the fundamental domain of $\mathbb{I}_\alpha/\alpha^\times$, so

- $\mu\left(\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}\right) = +\infty \Rightarrow \mu(\mathbb{I}_\alpha/\alpha^\times) = +\infty$
- $\mathbb{Q}^\times \hookrightarrow \mathbb{I}_\alpha$ is discrete. (by considering the preimage of $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \mathbb{R}_{>0}$)
- $\mathbb{I}_\alpha/\alpha^\times$ is not cpt. \mathbb{I}_α is loc. cpt.
- $\mathbb{Q}^\times \hookrightarrow \prod_{p \text{ prime}} \mathbb{Q}_p^\times \times \mathbb{R}^\times$ is discrete (by considering the preimage of $\prod_{p \neq 7} \mathbb{Z}_p^\times \times (1+7\mathbb{Z}_7)$)
- $\mathbb{Q} \hookrightarrow \prod_{p \neq 7} \mathbb{Q}_p^\times \times \mathbb{R}^\times$ is dense;
- $\mathbb{Z}[\frac{1}{p}] = \pm p^{\mathbb{Z}} \hookrightarrow \mathbb{Q}_p^\times \times \mathbb{R}^\times$, $\left\{ \frac{a}{b} \in \mathbb{Q} \mid 7 \nmid b \right\}^\times = \mathbb{Q}^\times \cap \mathbb{Z}_7^\times \hookrightarrow \prod_{p \neq 7} \mathbb{Q}_p^\times \times \mathbb{R}^\times$ are discrete.

To remedy the cptness, we introduce the group of 1-idèles.

Def (1-idèles group)

$$\mathbb{I}_{\mathbb{Q}} := \mathbb{Q}^{\times} \times \left(\prod_{p \text{ prime}} \mathbb{Z}_p^{\times} \times \{1\} \right) \\ = \{ (a_v)_v \in \prod_v K_v^{\times} \mid \prod_v |a_v|_v = 1 \} = \left(\prod_v F_v^{\times} \right)^1 = A_{\mathbb{Q}}^{\times, 1}$$

In general,

$$\mathbb{I}'_F \stackrel{\text{not unique expression}}{\cong} F^{\times} \times \left(\prod_{v \text{ fin}} \mathcal{O}_v^{\times} \times \left(\prod_{v \text{ inf}} F_v^{\times} \right)^1 \right) \\ \cong \{ (a_v)_v \in \prod_v F_v^{\times} \mid \prod_v |a_v|_v = 1 \} = \left(\prod_v F_v^{\times} \right)^1 = A_F^{\times, 1}$$

where

$$\left(\prod_{v \text{ inf}} F_v^{\times} \right)^1 := \{ (a_v)_v \in \prod_{v \text{ inf}} F_v^{\times} \mid \prod_v |a_v|_v = 1 \}$$

We have SESs:

$$0 \rightarrow \mathbb{I}'_F \rightarrow \mathbb{I}_F \xrightarrow{||.||} \mathbb{R}_{>0}^{\times} \rightarrow 0 \quad \text{for } F \text{ number field}$$

$$0 \rightarrow \mathbb{I}'_F \rightarrow \mathbb{I}_F \xrightarrow{||.||} \mathbb{P}^1 \rightarrow 0 \quad \text{for } F \text{ function field}$$

Rmk [引理6.106] [MIT, Lemma 23.8, 23.9]

For measures, I set $\mu(S) = 2\pi$, $\mu(\mathbb{Z}_p^{\times}) = 1$, $\mu(pt) = 1$. I hope they're fine.

The subspace topologies $\mathcal{O}_F^{\times} \subseteq F^{\times}$, $\mathcal{O}_F^{\times} \subseteq F$ coincide. $\mathcal{O}_F^{\times} \subseteq F$ is closed.

Observation. It's clear if you see

$$\mathbb{I}_F \cong \{ (x, x^{-1}) \in A_F^{\times} \} \subseteq GL_2(A_F)$$

Ex. Verify that

• $\prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \{1\}$ is the fundamental domain of $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$, so

$$\cdot \mathbb{Q}^\times \triangleleft \prod_{p \text{ prime}} \mathbb{Z}_p^\times \times \{1\} = 1 \Rightarrow \mu(\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times) = 1$$

• \mathbb{Q}^\times is discrete, $\mathbb{I}_{\mathbb{Q}}^1/\mathbb{Q}^\times$ is cpt.

Ex. Compute $\mathbb{I}_F, \mathbb{I}'_F$ for $F = \mathbb{Q}(i), \mathbb{Q}(\sqrt{3}), \mathbb{F}_p(t)$.

Notice that they're all UFD.

Ex. Try to compute it for $F = \mathbb{Q}(\sqrt{-5})$.

Here, the fundamental domain is hard to describe.

For convenience, we define

$$C_K := \mathbb{I}_K/K^\times$$

$$\mathbb{I}_{K,\text{fin}} := \prod_{v \text{ fin}} K_v^\times$$

$$C_K^1 := \mathbb{I}_K^1/K^\times$$

$$\mathbb{I}_{K,\text{inf}} := \prod_{v \text{ inf}} K_v^\times$$

$$(\mathbb{I}_K = \mathbb{I}_{K,\text{fin}} \times \mathbb{I}_{K,\text{inf}})$$

so C_K is cpt, while C_K^1 is loc cpt.

(We've shown this for $K = \mathbb{Q}$.)

⚠ I may use the symbols

$$\mathbb{A}_{K,T} = \prod_{v \in T} K_v \times \prod_{v \notin T} \mathcal{O}_v$$

$$\mathbb{I}_{K,T} = \prod_{v \in T} K_v^\times \times \prod_{v \notin T} \mathcal{O}_v^\times$$

$$\mathbb{I}'_{K,T} = \left(\prod_{v \in T} K_v^\times \right)^1 \times \prod_{v \notin T} \mathcal{O}_v^\times$$

to make the result simpler and more symmetrical.

I don't do it now just because I'm lazy.

3. topological properties of A_F & I_F .

All the properties in this section have been checked for $F=\mathbb{Q}$, $I_F(t)$ in the last section (for results concerning S , we checked some examples also). To make everything rigorous and easy to cite (and get some important applications), we make this section.

The roadmap of this section:

$$\begin{array}{c} F \subseteq A_F \text{ for } F = \mathbb{Q}, I_F(t) \Rightarrow F \subseteq A_F \Rightarrow O_T \subseteq \prod'_{v \in T} F_v \\ \Downarrow \\ F^\times \subseteq I_F' \Rightarrow O_T^\times \subseteq (\prod'_{v \in T} F_v)^\times \\ \Downarrow \qquad \Downarrow \\ \text{class number} \qquad \text{Dirichlet unit} \\ \text{dense comes from the theory of duality. not from lattice.} \end{array}$$

topo results needed

Def (iso up to cpt gp, Iso_{cpt})

$f: G_1 \rightarrow G_2 \in \text{Mor}(\text{Abel}_{\text{Top}})$ is called iso up to cpt gp (Iso_{cpt}) if

(1) $G_1/\text{kerf} \cong \text{Im}f$ in Abel_{Top} ;

(2) $\text{kerf}, \text{cokerf}$ are cpt.

Def (lattice)

$L \subseteq G$ in Abel_{Top} is called a lattice, if

(1) L is discrete;

(2) G/L is cpt.

When $G = (\mathbb{R}^n, +)$, this is equiv to a full lattice.

Cor: for $G_1 \xrightarrow{f} G_2 \in \text{Iso}_{\text{cpt}}$, if G_1 is discrete, then

$\text{Im}f$ is a lattice in G_2 .

Lemma 1. (1) $G_1 \xrightarrow{f} G_2, G_2 \xrightarrow{g} G_3 \in \text{Iso}_{\text{cpt}}$

$\Rightarrow G_1 \xrightarrow{f} G_2 \xrightarrow{g} G_3 \in \text{Iso}_{\text{cpt}}$

(2) $G_1 \xrightarrow[f]{\text{V/open}} G_2 \in \text{Iso}_{\text{cpt}}$

H_2

$G_1 \xrightarrow{f} G_2 \in \text{Iso}_{\text{cpt}}$

V/open

$f^{-1}(H_2) \rightarrow H_2 \in \text{Iso}_{\text{cpt}}$

(3) $H \subseteq G$ in Abel_{Top}

H is open $\Leftrightarrow G/H$ is discrete

\Downarrow

\Downarrow

H is closed $\Leftrightarrow G/H$ is Hausdorff.

lattice

Lemma 2 [6.10] [MIT, Prop 22.10] E/F finite ext of global field. We get an iso of topo rings

$$\Phi: E \otimes_F A_F \xrightarrow{\cong} A_F$$

$$(t, (a_v)_v) \mapsto (ta_{w_k})_w$$

In ptc, A_F is a subring of A_E , $A_E \cong A_F^{\oplus [E:F]}$, and we have an iso

$$\begin{array}{ccc} E & \hookrightarrow & A_E \\ \uparrow \cong & & \uparrow \cong \\ E \otimes_F F & \xrightarrow{\text{Id}_E \otimes_F \alpha} & E \otimes_F A_F \end{array}$$

Proof. Locally we have

$$\begin{array}{ccc} E \otimes_F F_v & \xrightarrow{\sim} & \prod_{i=1}^n E_{w_i} \\ \mathcal{O}_E \otimes_{\mathcal{O}_F} \mathcal{O}_v & \xrightarrow{\sim} & \prod_{i=1}^n \mathcal{O}_{w_i} \end{array}$$

Since $E \otimes_F -$, $\mathcal{O}_E \otimes_{\mathcal{O}_F} -$ are exact [Stackexchange, 1916457], one get

$$\begin{array}{ccc} \mathcal{O}_E \otimes_{\mathcal{O}_F} \prod_{v \in T} \mathcal{O}_v & \cong & \prod_{w \in T} \mathcal{O}_w \\ \downarrow & & \cap \\ E \otimes_F A_F & \longrightarrow & A_E \\ \downarrow & & \cap \\ E \otimes_F \prod_{v \in T} F_v & \cong & \prod_w E_w \end{array}$$

which shows the bijection.

(Φ is well-defined, since $a_w \in \mathcal{O}_w \Rightarrow a_v \in \mathcal{O}_v$;
 (" Φ & Φ^{-1} are cont" should be a routine check) (but I don't check it))

Prop 1 [6.78] F is a lattice in A_F .

Proof. We have checked for $F = \mathbb{Q}, \mathbb{F}_p(T)$. The rest comes from Lemma 1.

Prop 2 [6.80(1)] Let T be a set of places of F containing all infinite places, $T \neq \emptyset$.
 Let

$$\mathcal{O}_T = \{x \in F \mid x \in \mathcal{O}_v \text{ for } v \notin T\}$$

then \mathcal{O}_T is a lattice in $\prod'_{v \in T} F_v$.

Rmk. For F/\mathbb{Q} of degree n

When $T = \{\text{all places of } F\}$, $\mathcal{O}_T = F$; Application 1.

When $T = \{\text{all inf places of } F\}$, $\mathcal{O}_T = \mathcal{O}_F \Rightarrow \mathcal{O}_F$ is a free \mathbb{Z} -module of rank n .

Proof.

$$F \hookrightarrow A_F$$

$$\vee \quad \vee \text{ open}$$

$$\mathcal{O}_T \rightarrow \prod'_{v \in T} F_v \times \prod_{v \notin T} \mathcal{O}_v \xrightarrow{\pi} \prod'_{v \in T} F_v$$

By Lemma 1, $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v \in T} F_v$ is iso up to cpt gp,

and $\mathcal{O}_T \xrightarrow{\Delta} \prod'_{v \in T} F_v$ is obviously injective.

Prop 3 [6.82] F^\times is a lattice in \mathbb{I}_F'

For a proof, see [Theorem 3.3.6, <https://bicmr.pku.edu.cn/~dingyiwen/nt1.pdf>]

Prop 4 [6.83] Let T be a set of places containing all infinite places. $T \neq \emptyset$, let

$$\left(\prod'_{v \in T} \mathbb{R} \right)^\circ = \left\{ (c_v)_{v \in T} \in \prod'_{v \in T} \mathbb{R} \mid \begin{array}{l} c_v = 0 \text{ for almost all } v \\ \sum_{v \in T} c_v = 0 \end{array} \right\}$$

$$R_T : \mathcal{O}_T^\times \longrightarrow \left(\prod'_{v \in T} \mathbb{R} \right)^\circ \quad x \mapsto (\ln |x|_v)_{v \in T}$$

then R_T makes $R_T(\mathcal{O}_T^\times)$ as a lattice in $\left(\prod'_{v \in T} \mathbb{R} \right)^\circ$, and $\ker R_T$ is finite.

Proof. $F^\times \xrightarrow{\Delta} \mathbb{I}_F'$

\forall open v open

$$\mathcal{O}_T^\times \xrightarrow{\cong} \left(\prod'_{v \in T} F_v^\times \right) \times \prod'_{v \notin T} \mathcal{O}_v^\times \xrightarrow{\pi} \left(\prod'_{v \in T} F_v^\times \right)' \longrightarrow \left(\prod'_{v \in T} \mathbb{R} \right)^\circ$$

$$(a_v)_{v \in T} \longmapsto (\ln |a_v|_v)_{v \in T}$$

Rmk. R_T is in general not injective.

Application 2.

$$\begin{aligned} \ker R_T &= \{\text{root of unity in } F\} \leftarrow \text{the unity root gp of } F \\ &= \{x \in F \mid x^n = 1, \exists n \in \mathbb{N}_{>1}\} \end{aligned}$$

where " \subseteq " comes from the finiteness of $\ker R_T$.

As a corollary, the unity root gp of F is finite.

Application 3. Suppose F/\mathbb{Q} is a number field, $\#T < +\infty$. We get SES

$$1 \longrightarrow \ker R_T \longrightarrow \mathcal{O}_T^\times \longrightarrow \mathbb{Z}^{\oplus(\#T-1)} \longrightarrow 1$$

When $T = \{\text{all inf places}\}$, we get Dirichlet unit theorem.

Application 4. For F/α , define

$$\begin{aligned}\mathbb{I}_F &\longrightarrow I(F) \\ (a_v)_v &\longmapsto \prod_{\substack{v \text{ fin} \\ v = v_p}} \beta^{v(a_v)}\end{aligned}$$

we get SES's.

$$\begin{array}{ccccccc} 1 & & 1 & & 1 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow \mathcal{O}_F^x & \longrightarrow & \prod_{v \text{ fin}} \mathcal{O}_v^x \times \prod_{v \text{ inf}} F_v^x & \xrightarrow{\cong \mathcal{U}} & \bar{u} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow F^x & \longrightarrow & \mathbb{I}_F & \longrightarrow & C_F & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow P(F) & \longrightarrow & I(F) & \longrightarrow & Cl(F) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & \end{array}$$

$\Rightarrow 1 \rightarrow \mathcal{O}_F^x \rightarrow F^x \rightarrow I(F) \rightarrow Cl(F) \rightarrow 1$

By replacing \mathbb{I}_F by \mathbb{I}'_F , we get

$$\begin{array}{ccccccc} 1 & & 1 & & 1 & & \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow \mathcal{O}_F^x & \longrightarrow & \prod_{v \text{ fin}} \mathcal{O}_v^x \times (\prod_{v \text{ inf}} F_v^x)' & \xrightarrow{\cong \mathcal{U}'} & \bar{u}' & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow F^x & \longrightarrow & \mathbb{I}'_F & \longrightarrow & C'_F & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 \rightarrow P(F) & \longrightarrow & I(F) & \longrightarrow & Cl(F) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & \end{array}$$

C'_F cpt $\Rightarrow Cl(F)$ is cpt

$\prod_{v \text{ fin}} \mathcal{O}_v^x \times (\prod_{v \text{ inf}} F_v^x)' \subseteq \mathbb{I}'_F$ is open $\Rightarrow I(F)$ is discrete $\Rightarrow Cl(F)$ is discrete

$\Rightarrow Cl(F)$ is finite. □

dense.

Work over the category of loc. cpt Abelian gp Abel_{lc} .

$G \in \text{Abel}_{\text{lc}}$

$$\widehat{G}^{*,u} = \{x: G \rightarrow S' \text{ cont}\}$$

unitary

$$\widehat{G}^* = \{x: G \rightarrow \mathbb{C}^\times \text{ cont}\}$$

\widehat{K} : "profinite completion of K "

V^* : "dual space of V "

Thm (Pontrjagin dual) For $G \in \text{Abel}_{\text{lc}}$,

$$\begin{aligned} \phi: G &\xrightarrow{\sim} \widehat{G}^{*,u} \\ g &\mapsto [\phi_g: \widehat{G}^{*,u} \rightarrow S' \\ &\quad x \mapsto x(g)] \end{aligned}$$

is an iso (as topo abelian gp)

$$x^k \neq \text{Id} \quad \forall k \in \mathbb{N}_{\geq 1}$$

Thm. 1. For F local field, take $x \in \widehat{F}^{*,u}$ not root of unit,

$$\begin{aligned} F &\xrightarrow{\sim} \widehat{F}^{*,u} \\ x &\mapsto x(x-) \end{aligned}$$

is an iso

2. For F global field, take $x \in \widehat{\mathbb{A}_F/F}^{*,u}$ not root of unit.

$$\begin{aligned} F &\xrightarrow{\sim} \widehat{\mathbb{A}_F/F}^{*,u} \\ x &\mapsto x(x-) \\ \mathbb{A}_F &\xrightarrow{\sim} \widehat{\mathbb{A}_F}^{*,u} \\ x &\mapsto x(x-) \end{aligned}$$

are isos.

3. For F global field, T , take $x \in \widehat{(\prod'_{v \in T} F_v)/\mathcal{O}_T}^{*,u}$ not root of unit.

$$\mathcal{O}_T \xrightarrow{\sim} \widehat{(\prod'_{v \in T} F_v)/\mathcal{O}_T}^{*,u}$$

$$\prod'_{v \in T} F_v \xrightarrow{\sim} \widehat{\prod'_{v \in T} F_v}^{*,u}$$

are isos.

Lemma. For $f \in \text{Mor}_{\text{Abel}_{lc}}(G_1, G_2)$,

$$\text{Im } f \subseteq G_2 \text{ is dense} \Leftrightarrow \widehat{f}^{*,u}: \widehat{G}_2^{*,u} \xrightarrow{\quad} \widehat{G}_1^{*,u} \text{ is inj.}$$

$$x \mapsto \chi(f(x))$$

Proof. Let $H = \overline{\text{Im } f}$.

$$\begin{aligned} & \text{Im } f \subseteq G_2 \text{ is dense} \\ \Leftrightarrow & H = G_2 \\ \Leftrightarrow & G_2/H = \{\text{Id}\} \\ \Leftrightarrow & \widehat{G}_2/H^{*,u} = \{\text{Id}\} \\ \Leftrightarrow & \widehat{f}^{*,u} \text{ is inj.} \end{aligned}$$

Prop 5 [6.79] Let $T \subseteq \{\text{places of } F\}$, then the image of $F \xrightarrow{\quad} \prod'_{v \in T} F_v$ is dense.

Proof. Reduce to show, for any w ,

$$\begin{aligned} & \text{the image of } F \xrightarrow{\quad} \prod'_{v \neq w} F_v \text{ is dense} \\ \Leftrightarrow & \underbrace{\prod'_{v \neq w} F_v^{*,u}}_{\text{HS}} \xrightarrow{\quad} \widehat{F}^{*,u} \xrightarrow{\quad} \prod'_{v \neq w} F_v \text{ is inj} \\ \Leftrightarrow & \prod'_{v \neq w} F_v \xrightarrow{\quad} A_F/F \text{ is inj} \end{aligned}$$

□

Prop 6 [6.80(2)] Suppose $T' \subseteq T$. then the image of $\mathcal{O}_T \xrightarrow{\quad} \prod'_{v \in T'} F_v$ is dense.

Proof. Reduce to show, for any $w \in T$,

the image of $\mathcal{O}_T \xrightarrow{\quad} \prod'_{v \in T-w} F_v$ is dense.

$$\Leftrightarrow \underbrace{\prod'_{v \in T-w} F_v^{*,u}}_{\text{HS}} \xrightarrow{\quad} \widehat{\mathcal{O}}_T^{*,u} \text{ is inj}$$

$$\Leftrightarrow \prod'_{v \in T-w} F_v \xrightarrow{\quad} \prod'_{v \in T-w} F_v/\mathcal{O}_T \text{ is inj}$$

□