

§ 3.1. Galois representation

1. Galois rep
2. Weil-Deligne rep
3. connections (Characters)
4. L-fct
5. density theorem

Just for convenience, we allow

element \in_c class class \subset_c class $\{\dots | \dots\}_c$ be a class

We may add c to emphasize that the family can be a class, instead of set.

1. Galois rep

Setting G : arbitrary topo gp e.g. G any Galois gp

If G profinite \Rightarrow open subgps are finite index subgps.

Δ : top field e.g. $\overline{\mathbb{F}_p}, \overline{\mathbb{Q}_p}, \mathbb{C}$, don't want to mention $\overline{\mathbb{Z}_p}$ now.

Def (cont Galois rep) $(\rho, V) \in \text{rep}_{\Delta, \text{cont}}(G)$
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{cont}$

∇ $\rho(G)$ can be infinite! for Gal gp

E.g. When $\text{char } F \neq l$, we have l -adic cyclotomic character

$$\varepsilon_l: \text{Gal}(\overline{F}/F) \longrightarrow \mathbb{Z}_l^\times \hookrightarrow \mathbb{Q}_l^\times \quad \sigma \mapsto \varepsilon_l(\sigma) \text{ satisfying}$$

$$\sigma(\zeta) = \zeta^{\varepsilon_l(\sigma)} \quad \forall \zeta \in \mu_{l^\infty}$$

This is cont by def. (Take usual topo.)

Ex: Compute ε_l for $F = \mathbb{F}_p$.

$$\mathbb{A}: \quad \varepsilon_l: \widehat{\mathbb{Z}} \cong \text{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \longrightarrow \mathbb{Z}_l^\times \quad 1 \mapsto p$$

\uparrow lift from $\mathbb{Z} \rightarrow \mathbb{Z}_l^\times$

Ex. Compute ε_l for $F = \mathbb{Q}_p$.

$$\mathbb{A}: \quad \varepsilon_l: \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p) \longrightarrow \text{Gal}(\mathbb{Q}_p^{\text{ur}}/\mathbb{Q}_p) \longrightarrow \text{Gal}(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p)$$

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\text{IIS}} & \mathbb{Z}_l^\times \\ \text{Frob} \mapsto 1 & \xrightarrow{\quad} & p \end{array}$$

Notice that

$$\begin{aligned} \text{Gal}(\mathbb{Q}_p(\zeta_{l^\infty})/\mathbb{Q}_p) &\cong \text{Gal}(\mathbb{F}_p(\zeta_{l^\infty})/\mathbb{F}_p) \cong \varprojlim_k (\mathbb{Z}/l^k \mathbb{Z})^\times \cong \mathbb{Z}_l^\times \\ x \in \widehat{\mathbb{Z}} \text{ fix } \zeta_{l^k}: &\Leftrightarrow \zeta_{l^k}^{p^x} = \zeta_{l^k} \\ &\Leftrightarrow p^x \equiv 1 \pmod{l^k} \end{aligned}$$

Ex. Compute ε_l for $F = \mathbb{Q}_l$.

Ex: Compute ε_i for $i = \infty$.

A: $\varepsilon_i: \text{Gal}(\bar{\mathbb{Q}}_i/\mathbb{Q}_i) \longrightarrow \text{Gal}(\mathbb{Q}_i^{\text{ab}}/\mathbb{Q}_i) \longrightarrow \text{Gal}(\mathbb{Q}(\zeta_i^\infty)/\mathbb{Q}_i)$

$$\widehat{\mathbb{Q}_i^*} \cong \widehat{\mathbb{Z}} \times \mathbb{Z}_i^* \xrightarrow{\pi_{\mathbb{Z}_i^*}} \mathbb{Z}_i^*$$

Rmk. Usually we denote $Z_l(1)$ as Z_l with twisted G_F -action by ε_l , i.e.,
 $(\varepsilon_l, Z_l(1)) \in \text{rep}_{Z_l, \text{cont}}(G_F)$

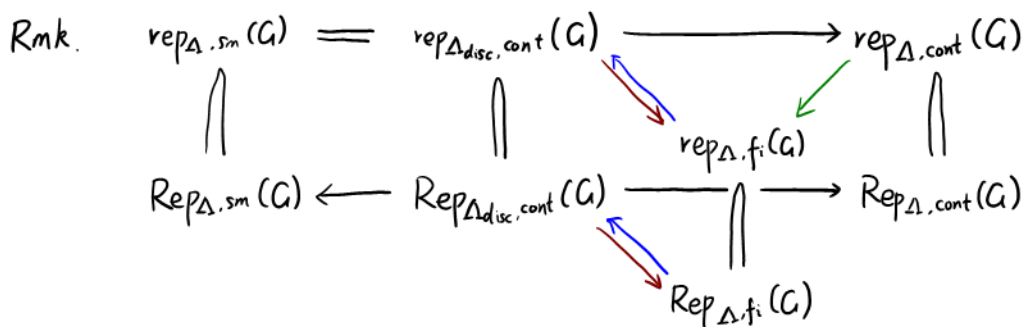
We use ε_i to twist reps.

$$V \in \text{Rep}_{\mathbb{Z}_\ell, \text{cont}}(G_F) \rightsquigarrow V(j) = V \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)^{\otimes j} \in \text{Rep}_{\mathbb{Z}_\ell, \text{cont}}(G_F)$$

Notice the following two definitions don't depend on the topo of Δ .

Def (sm Galois rep) $(\rho, V) \in \text{rep}_{\Delta, \text{sm}}(G)$
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V) \quad \text{with open stabilizer.}$

Def (fin image Galois rep) $(\rho, V) \in \text{rep}_{\Delta, \text{fi}}(G)$ with fi : finite image / finite index
 $V \in \text{vect}_{\Delta} \quad + \quad \rho: G \longrightarrow GL(V)$ with finite image



- : if fin index subgps are open
- : if G : profinite gp (Only need: open \Rightarrow fin index)
- : Artin rep (of profinite gp)

Artin rep: $\Delta = (\mathbb{C}, \text{euclidean topo})$ G profinite

Lemma 1 (No small gp argument)

$\exists U \subset GL_n(\mathbb{C})$ open nbhd of 1 s.t.
 $\forall H \leq GL_n(\mathbb{C}), H \subseteq U \Rightarrow H = \{\text{Id}\}.$

Proof. Take $U = \{A \in GL_n(\mathbb{C}) \mid \|A - \text{Id}\| < \frac{1}{3n}\}$ $\|\cdot\| = \|\cdot\|_{\max}, \|\cdot\| = \|\cdot\|_{\max}$

Only need to show, $\forall A \in GL_n(\mathbb{C}), A \neq \text{Id}, \exists m \in \mathbb{N}, \text{ s.t. } A^m \notin U.$

Consider the Jordan form of A .

Case 1. A unipotent.

Case 2. A not unipotent.

$\exists \lambda \neq 1, v \in \mathbb{C}^n \setminus \{0\} \text{ s.t. } Av = \lambda v.$ Take $m \in \mathbb{N}$ s.t. $|\lambda^m - 1| > \frac{1}{3}.$

$\frac{1}{3} \|v\| < |\lambda^m - 1| \|v\| = \|(A^m - \text{Id})v\| \leq n \|A^m - \text{Id}\| \|v\| \Rightarrow \|A^m - \text{Id}\| \geq \frac{1}{3n}.$

Prop. For $(\rho, V) \in \text{rep}_{\mathbb{C}, \text{cont}}(G), \rho(G)$ is finite.

G profinite

Proof. Take U in Lemma 1, then

$\rho^{-1}(U)$ is open $\Rightarrow \exists I \leq G_F$ finite index, $\rho(I) \subseteq U$
 $\xRightarrow{\text{Lemma 1}} \rho(I) = \text{Id}$
 $\Rightarrow \rho(G_F)$ is finite

Rmk. For Artin rep we can speak more:

1. ρ is conj to a rep valued in $GL_n(\overline{\mathbb{Q}})$

ρ can be viewed as cplx rep of fin gp, so ρ is semisimple.
 Since classifications of irr reps for \mathbb{C} & $\overline{\mathbb{Q}}$ are the same,
 every irr rep is conj to a rep valued in $GL_n(\overline{\mathbb{Q}}).$

2. $\#\{\text{fin subgps in } GL_n(\mathbb{C}) \text{ of "exponent } m"\}$ is bounded, see:
<https://mathoverflow.net/questions/24764/finite-subgroups-of-gl-n-c>

2. Weil-Deligne rep

Now we work over "the skeleton of the Galois gp" in general.

Setting: Δ : NA local field with char $k_\Delta = l$

Q: What would happen if Δ is only a NA local field?

Finite field

Goal: For Δ : NA local field with char $k_\Delta = l$, understand $\text{rep}_{\Delta, \text{cont}}(\hat{\mathbb{Z}})$.

Def/Prop. Let $A \in GL_n(\Delta)$, TFAE:

① $\hat{\mathbb{Z}} \rightarrow GL_n(\Delta)$ is a well-defined cont gp homo
 $1 \mapsto A$

② $\exists g \in GL_n(\Delta)$, $gAg^{-1} \in GL_n(\mathcal{O}_\Delta)$

③ $\det(\lambda I - A) \in \mathcal{O}_\Delta[\lambda]$, with $\det A \in \mathcal{O}_\Delta^\times$

A is called bounded in these cases.

Proof

$$\textcircled{1} \xrightleftharpoons[\text{local}]{\text{local}} \textcircled{2} \xrightleftharpoons[\text{local}]{} \textcircled{3}$$

$\textcircled{1} \Rightarrow \textcircled{2}$: $\hat{\mathbb{Z}}$ is cpt, so image lies in a max cpt subgp of $GL_n(\Delta)$, which conjugates to $GL_n(\mathcal{O}_\Delta)$

https://math.stackexchange.com/questions/4461815/maximal-compact-subgroups-of-mathrmgl_2-mathbb-q-p

Another method:

Lemma 1: ρ, μ : two ways of expressions of gp action

$\rho: \hat{\mathbb{Z}} \rightarrow GL_n(\mathbb{Z})$ is cont $\Leftrightarrow \mu: \hat{\mathbb{Z}} \times \Delta^n \rightarrow \Delta^n$ is cont

$$\Rightarrow: \mu: \hat{\mathbb{Z}} \times \Delta^n \xrightarrow{\rho \times \text{Id}_{\Delta^n}} GL_n(\Delta) \times \Delta^n \xrightarrow{\quad} \Delta^n \text{ is cont.}$$

$\Delta^n \uparrow$ is Haus loc cpt.

See [Theorem III.3, III.4]:

https://github.com/lrnmhl/AT1/blob/main/Algebraic_Topology_I_Stefan_Schwede_Bonn_Winter_2021.pdf

\Leftarrow : $\rho: \hat{\mathbb{Z}} \rightarrow GL_n(\Delta)$ is cont

$\Leftrightarrow \rho: \hat{\mathbb{Z}} \rightarrow M_{n \times n}(\Delta)$ is cont

$\Leftrightarrow \rho_{ij}: \hat{\mathbb{Z}} \rightarrow \Delta$ is cont $\forall i, j \in \{1, \dots, n\}$

We know that

$$\rho_{ij}: \hat{\mathbb{Z}} \xrightarrow{(\text{Id}, e_i)} \hat{\mathbb{Z}} \times \Delta^n \xrightarrow{\mu} \Delta^n \xrightarrow{e_i^*} \Delta$$

is cont

linear map between f.d v.s is cont

In this case, e_i^* is projection.

Another \Leftarrow : (suggested by Longke Tang)

$$\Leftrightarrow \begin{array}{ccc} \mu: \widehat{\mathbb{Z}} \times \widehat{\Lambda}^n & \longrightarrow & \Lambda^n \text{ is cont} \\ \widehat{\mathbb{Z}} & \xrightarrow{\exists!} & \text{Mor}_{\text{Top}}(\Lambda^n, \Lambda^n) \end{array} \begin{array}{l} \swarrow \text{open cpt topo} \\ \text{is cont} \end{array}$$

$GL_n(\Lambda)$

Only need: $GL_n(\Lambda) \subseteq M_{n \times n}(\Lambda)$, $GL_n(\Lambda) \subset \text{Mor}_{\text{Top}}(\Lambda^n, \Lambda^n)$
define the same topo on $GL_n(\Lambda)$.

This is hard. Assuming Lemma 1, this can be proved,
but then this method can't be a real proof for Lemma 1.

Lemma 2. $\mathcal{L}_1, \mathcal{L}_2$ lattice in $\Lambda^n \Rightarrow \mathcal{L}_1 + \mathcal{L}_2$ lattice in Λ

$$\left[\begin{array}{l} \mathcal{L}_1 \supseteq (\mathfrak{p}^{k_1})^{\oplus n} \\ \mathcal{L}_2 \supseteq (\mathfrak{p}^{k_2})^{\oplus n} \end{array} \right] \Rightarrow \# \mathcal{L}_1 + \mathcal{L}_2 / \mathcal{L}_1 < +\infty \Rightarrow \mathcal{L}_1 + \mathcal{L}_2 \text{ is a lattice}$$

Take $\mathcal{L}_1 = \mathcal{O}_{\Lambda}^n \subseteq \Lambda^n$, then the stabilizer

$$\begin{aligned} \text{Stab}(\mathcal{L}) &= \{g \in \widehat{\mathbb{Z}} \mid g \cdot \mathcal{L} = \mathcal{L}\} \\ &= \{g \in \widehat{\mathbb{Z}} \mid g \cdot e_i \in \mathcal{L} \ \forall i\} \\ &= \bigcap_i \mu_{e_i}^{-1}(\mathcal{L}) \end{aligned}$$

is open, where

$$\mu_{e_i}: \widehat{\mathbb{Z}} \longrightarrow \Lambda^n \quad g \mapsto g \cdot e_i \quad (\text{cont by Lemma 1})$$

$\Rightarrow \mathcal{L}$ has finite orbit
 $\xRightarrow{\text{Lemma 2}} \sum_{i \in \mathbb{Z}} \mathcal{L}_i$ is a lattice stabilized by \mathbb{Z} .

After conjugation, $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

② \Rightarrow ①: w.l.o.g. $A \in GL_n(\mathcal{O}_\Delta)$. Then we get a lift

$$\begin{array}{ccc} \widehat{\mathbb{Z}} & \xrightarrow{\exists! \text{ cont}} & \widehat{GL_n(\mathcal{O}_\Delta)} \cong GL_n(\mathcal{O}_\Delta) \\ \uparrow & & \uparrow \\ \mathbb{Z} & \longrightarrow & GL_n(\mathcal{O}_\Delta) \end{array}$$

② \Rightarrow ③: Obvious

③ \Rightarrow ②: $\sum_{i \in \mathbb{Z}} A^i \mathcal{L} = \sum_{i=0}^{n-1} A^i \mathcal{L}$ is a lattice fixed by A, A^{-1} (Lemma 2)

After conjugation, $A, A^{-1} \in M^{n \times n}(\mathcal{O}_\Delta) \Rightarrow A \in GL_n(\mathcal{O}_\Delta)$

$\nabla A, B \in GL_n(\Delta)$ bounded $\not\Rightarrow AB$ bounded
 counter eg: (from Longke Tang)

Consider $A = \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} p & 1 \\ 1 & 1 \end{pmatrix}^{-1}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then $AB = \begin{pmatrix} p & p^{-1} \\ 1 & 1 \end{pmatrix}$.

Cor. $\text{rep}_{\Delta, \text{cont}}(\widehat{\mathbb{Z}}) \cong \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(\mathbb{Z})$
 \nwarrow full subcategory of $\text{rep}_{\Delta, \text{cont}}(\mathbb{Z})$.

Local field

Goal. For Δ : NA local field with $\text{char } \kappa_\Delta = l$,

F : NA local field with $\text{char } \kappa_F = p \neq l$,

realize cont Galois rep as bounded Weil-Deligne rep.
via the following diagrams:

$$\begin{array}{ccccc}
 & & \text{rep}_{\Delta, \text{sm}}(W_F) \text{ with } N & & \\
 & & \cup & & \\
 & \swarrow & & \searrow & \\
 & \text{rep}_{\Delta, \text{cont}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}(W_F) & \\
 & \cup & & \cup & \\
 \text{rep}_{\Delta, \text{cont}}(\Gamma_F) & \xrightarrow{\sim} & \text{rep}_{\Delta, \text{cont}}^{\text{bdd}}(W_F) & \xrightarrow{\sim} & \text{WDrep}_{\Delta, \text{sm}}^{\text{bdd}}(W_F)
 \end{array}$$

here, "bdd" means $\text{Im } \rho$ are bounded.

4.

References:

https://en.wikipedia.org/wiki/Dirichlet_character

在算术几何中格罗藤迪克的 l -进上同调(l -adic cohomology)可以看作对于函数域(function field)上的 L -函数(L -function)的一种范畴化:

- a) 函数方程(functional equation)对应庞加莱对偶(Poincare duality)
- b) 欧拉分解(Euler factorisation)对应迹公式(trace formula)
- c) 解析延拓(analytic continuation)对应有限性(finitude)

from <https://www.zhihu.com/question/31823394/answer/54820421>