Eine Woche, ein Beispiel
3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \curvearrowright X$

discrete $\Rightarrow$ foundamental domain $\quad \Lambda \curvearrowright \mathbb{C} \quad SL_2(\mathbb{Z}) \curvearrowright \mathcal{H}$

non discrete $\Rightarrow$ stratification by $G/G_x \quad S^1 \curvearrowright S^2 \quad \mathbb{C}^\times \curvearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g.
$$\mathrm{Flag}(V) \cong GL(V)/P \quad \text{e.p. Grassmannian, } \mathbb{P}^n$$
$$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$$
$O(n) := O(n,\mathbb{R})$ $\rightsquigarrow$ Stiefel mfld [21.11.14]
$SO(n) := SO(n,\mathbb{R})$
$$\mathbb{A}^n = \mathbb{A}^n$$
$$\mathcal{H}^n \cong O^+(1,n)/O(n)$$
$$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$$
$\rightsquigarrow$ Hermitian symmetric space

where
$$\mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1,\ v_{n+1} > 0\}$$
$$\langle\, ,\, \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad \langle v, w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$$

$$O(n,1) := \mathrm{Aut}\,(\mathbb{R}^{n+1}, \langle\, ,\, \rangle) \subseteq GL_{n+1}(\mathbb{R})$$
$$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

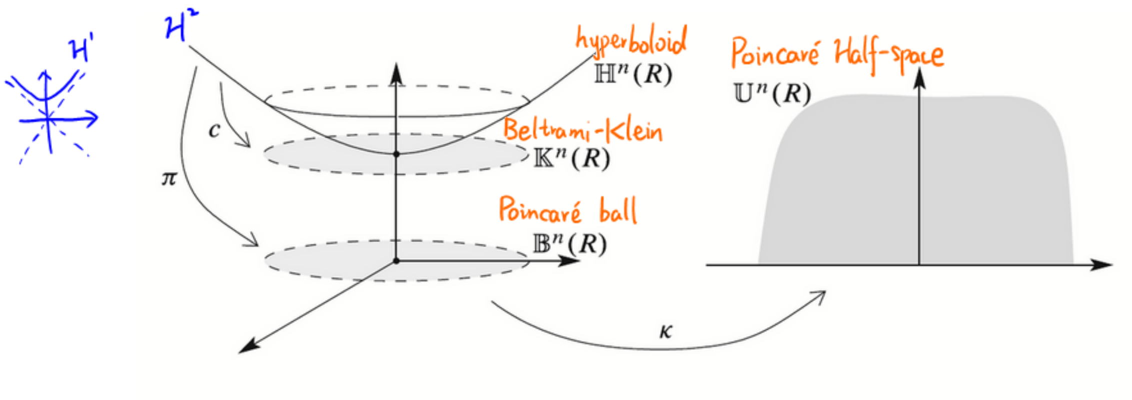


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-01-2

Terminology.

$$\downarrow\ X + G \curvearrowright X$$

G-space

$$\downarrow \text{transitive}$$

homogeneous space.

[JL18. Cor 3.18]: Nice hom space can have compatible Rie mfld structures.

$$\begin{array}{ccc}
 & \text{homogeneous space} & \\
\swarrow{\scriptstyle \text{Riemannian mfld, } G \text{ preserve } g} & & \searrow{\scriptstyle \text{Hermitian mfld, } G \text{ preserve } g \text{ (metric)}} \\
\text{Riemannian homogeneous space} & \xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}} & \text{Hermitian homogeneous space} \\
\downarrow{\scriptstyle \text{geodesic symmetry globally defined, preserve } g} & & \downarrow{\scriptstyle \text{geodesic symmetry globally defined, preserve } g} \\
\text{Symmetric space} & \xrightarrow{\mathbb{R} \rightsquigarrow \mathbb{C}} & \text{Hermitian symmetric space} \\
\downarrow{\scriptstyle \text{transitive on frames}} & & \\
\text{frame homogeneous space} & &
\end{array}$$

$\mathbb{C}P^n$ $n \geq 1$

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

2. double coset decomposition: schedule

$$G = \bigsqcup_{\alpha\in I} H\alpha K$$

usually, H, K are easier than G.

- comes from (usually) Gauss elimination
- I is the "foundamental domain"
- produces stratifications on G/K and $_H\backslash G$ indexed by I.

To be exact,

$$G/K = \bigsqcup_{\alpha\in I} H\alpha K/K \cong \bigsqcup_{\alpha\in I} H/H_{[\alpha K]} = \bigsqcup_{\alpha\in I} H/(H\cap\alpha K\alpha^{-1})$$

$$_H\backslash G = \bigsqcup_{\alpha\in I} {}_H\backslash H\alpha K \cong \bigsqcup_{\alpha\in I} K_{[H\alpha]}\backslash K = \bigsqcup_{\alpha\in I} (K\cap\alpha^{-1}H\alpha)\backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$

$K_{[H\alpha]}$: stabilizer of K on $[H\alpha] \in {}_H\backslash G$

$$\# \, H/H\cap\alpha K\alpha^{-1} = \#\left\{\begin{matrix}\text{single cosets } [gK] \\ \text{in one double coset } H\alpha K\end{matrix}\right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad {}_H\backslash G$$ individually

- can be viewed as stack quotient.
  $[*/G]$: groupoid

$$_H\backslash G/K \overset{def}{=} [*/H] \times_{[*/G]} [*/K] \quad \text{with groupoid structure}$$

$$H_H^*(G/K) \cong H^*(_H\backslash G/K) \cong H_K^*(_H\backslash G)$$

slogan: the (equiv) cohomology of G/K and $_H\backslash G$ are connected.

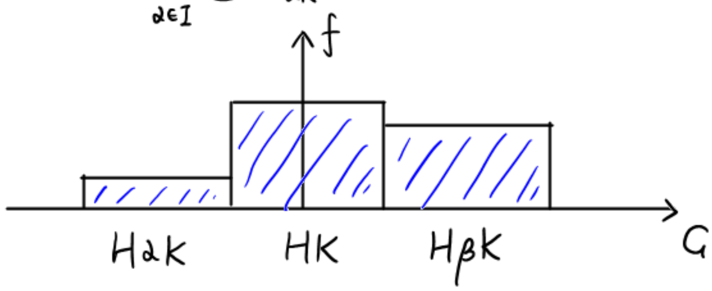
- Hecke algebra $\mathcal{H}({}_H\backslash G/_K)$

for $H=K$. You can also do $\mathcal{H}({}_{H_1}\backslash G/_{H_2}) \hookrightarrow \bigoplus_{i,j=1}^{2} \mathcal{H}({}_{H_i}\backslash G/_{H_j})$

$\mathcal{H}({}_H\backslash G/_K)$: reasonable subspaces of

$$\mathbb{C}[{}_H\backslash G/_K] = \{f : G \to \mathbb{C} \mid f(hgk) = f(g) \quad \forall h\in H, g\in G, k\in K\}$$

$$\underset{\text{"o-dim"}}{\supseteq} \bigoplus_{\alpha\in I} \mathbb{C}\, \mathbb{1}_{H\alpha K}$$



with reasonable convolution structure

$$* : \mathcal{H}({}_{H_1}\backslash G/_{H_2}) \times \mathcal{H}({}_{H_2}\backslash G/_{H_3}) \longrightarrow \mathcal{H}({}_{H_1}\backslash G/_{H_3})$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization