Eine Woche, ein Beispiel

9.10. ramified covering: alg curve case

Today we are going to move out of the world of RS, trying to switch from cplx alg geo to number theory. The pictures become less intuitive; on the other hand, more interesting phenomenons will appear during the journey.

1. alg curve viewed as stack quotient
2. ramified covering for alg curve/$\mathbb{R}$
3. Frobenius for alg curve/$\mathbb{R}$
4. complexify is a ramified covering by non geometrical connected spaces
5. alg curves and function fields
   - Correspondence
   - Valuations
6. alg curve over $\mathbb{F}_p$: miscellaneous.

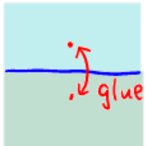
1. alg curve viewed as stack quotient

	$\operatorname{Spec}\mathbb{R}$	base change $\operatorname{Spec}\mathbb{C}/_{\mathbb{C}}$	$\operatorname{Spec}\mathbb{C}/_{\mathbb{R}}$
$\mathbb{R}$-pts	$\{*\}$	—	$\emptyset$
$\mathbb{C}$-pts	$\{*\}$	$\{*\}$	$\{Id, \tau\}$
$\Gamma_{\mathbb{R}} = Gal(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	$Id \rightleftarrows \tau$

This table can clarify many confusions during the study of varieties over non alg close fields.

Rmk. $\operatorname{Spec}\mathbb{C}$ over $\mathbb{R}$ is not geo connected!

When we take the base change, there are no difference for C-pts.
However, when we try to count C-pts on the fiber of X/R of form Spec C, then we see a pair of C-pts.



E.g. Let's work on $\mathbb{A}^1_{\mathbb{R}} = \operatorname{Spec}\mathbb{R}[x]$. As a set,

$$\begin{aligned}
\operatorname{Spec}\mathbb{R}[x] &= \{(x-a) \mid a\in\mathbb{R}\} \cup \{(x^2+bx+c) \mid {b,c\in\mathbb{R} \atop b^2-4c<0}\} \cup \{(0)\} \\
&= \qquad \mathbb{R} \qquad \cup \qquad \mathcal{H} \qquad \cup \{(0)\} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{R}) = \mathbb{R} \\
\mathbb{A}^1_{\mathbb{R}}(\mathbb{C}) &= Mor_{\mathbb{R}\text{-alg}}(\mathbb{R}[x], \mathbb{C}) = \mathbb{C} = \mathbb{A}^1_{\mathbb{C}}(\mathbb{C})
\end{aligned}$$

One gets a $\Gamma_{\mathbb{R}}$-action on $\mathbb{A}^1_{\mathbb{R}}(\mathbb{C})$ by $x \longmapsto \tau\circ x$. Observe that

$$\operatorname{MaxSpec}\mathbb{R}[x] = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})/_{\Gamma_{\mathbb{R}}} \qquad \mathbb{A}^1_{\mathbb{R}}(\mathbb{R}) = \mathbb{A}^1_{\mathbb{R}}(\mathbb{C})^{\Gamma_{\mathbb{R}}}$$

as a set, so we can view $\mathbb{A}^1_{\mathbb{R}}$ as the quotient stack of $\mathbb{A}^1_{\mathbb{C}}/_{\mathbb{R}}$ quotienting out $\Gamma_{\mathbb{R}}$-action.

Ex. Work out the same results for $\mathbb{A}^1_{\mathbb{F}_p}$. E.p., shows that

$$\mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{F}_p \qquad \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p = \mathbb{A}^1_{\overline{\mathbb{F}}_p}(\overline{\mathbb{F}}_p)$$

$$\operatorname{MaxSpec}\mathbb{F}_p[x] = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)/_{\Gamma_{\mathbb{F}_p}} \qquad \mathbb{A}^1_{\mathbb{F}_p}(\mathbb{F}_p) = \mathbb{A}^1_{\mathbb{F}_p}(\overline{\mathbb{F}}_p)^{\Gamma_{\mathbb{F}_p}}$$

Ex. For an (sm) alg curve X over κ (In general, X: f.t. over a field κ), try to show that

$$\{\text{closed pts of } X\} = X(\kappa^{sep})/_{\Gamma_\kappa} \qquad X(\kappa) = X(\kappa^{sep})^{\Gamma_\kappa}$$

by Hilbert's Nullstellensatz.

e.p., for x: closed pt of X,

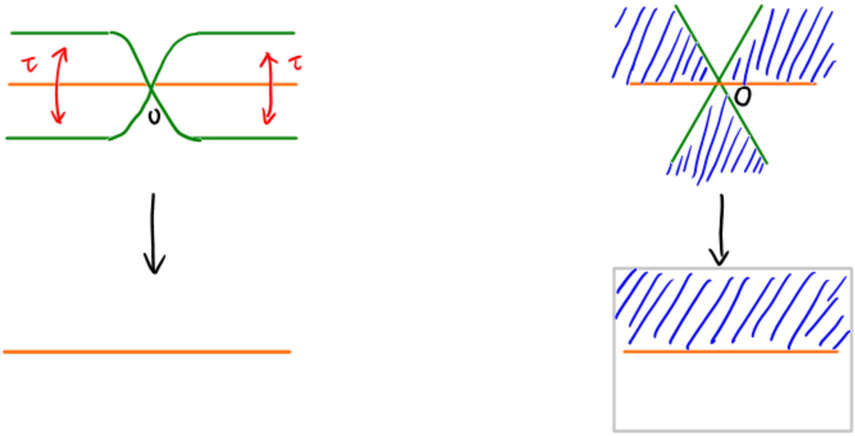
$$Stab_x(\Gamma_\kappa) = \Gamma_{\kappa'} \iff \text{fiber at } x = \operatorname{Spec}\kappa'.$$

	$\mathbb{A}^1_{\mathbb{R}}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{C}$	$\mathbb{A}^1_{\mathbb{C}}/\mathbb{R}$
MaxSpec	$\mathbb{R}\cup\mathcal{H}$	$\mathbb{C}$	$\mathbb{C}$ $\circlearrowleft$ cplx conj
$\mathbb{R}$ - pts	$\mathbb{R}$	—	$\emptyset$
$\mathbb{C}$ - pts	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}\sqcup\mathbb{C}\tau$
$\Gamma_{\mathbb{R}}=\mathrm{Gal}(\mathbb{C}/\mathbb{R})$	trivial on pts & fcts	no action	see orange arrows

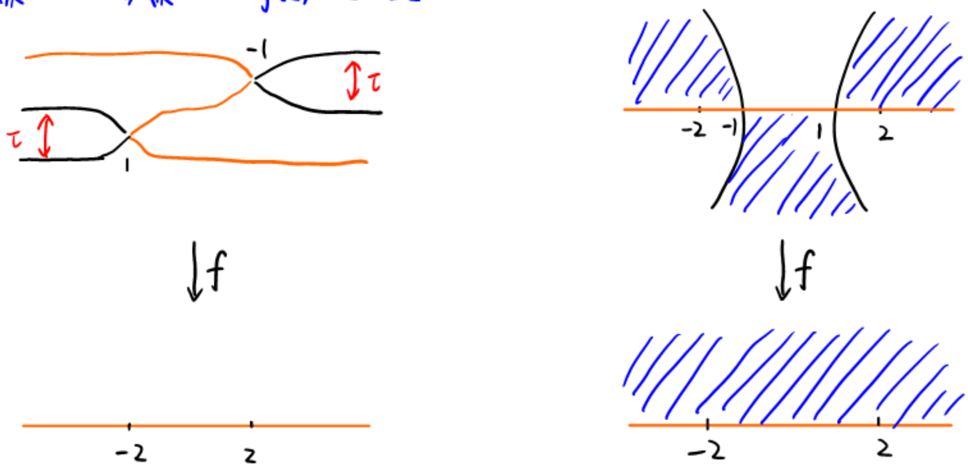
2. ramified covering for alg curve/$\mathbb{R}$

Many examples we worked on RS can be reused in this setting.

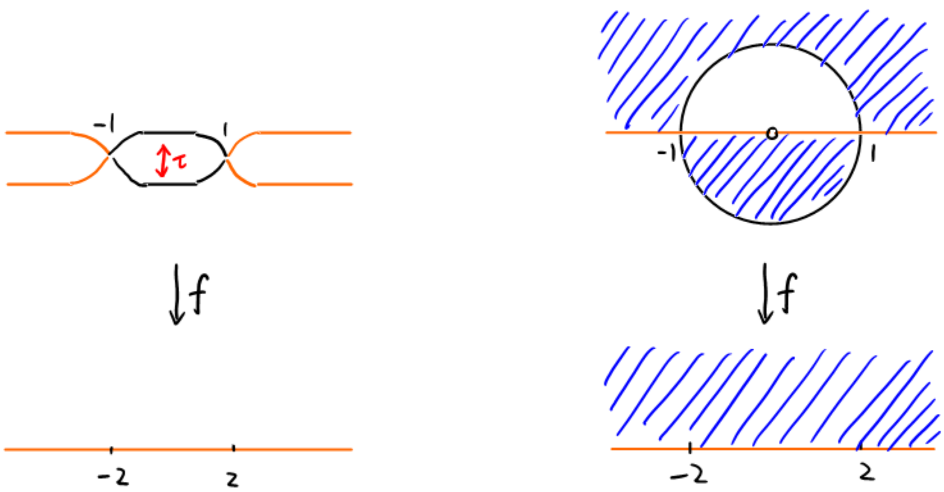
E.g. $f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3$



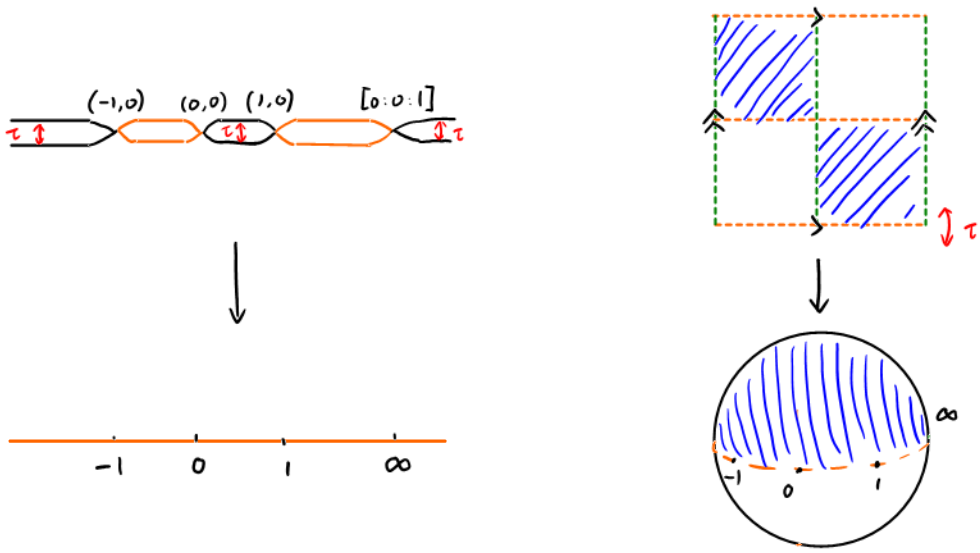
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z^3 - 3z$



$f: \mathbb{G}_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = z + \frac{1}{z}$

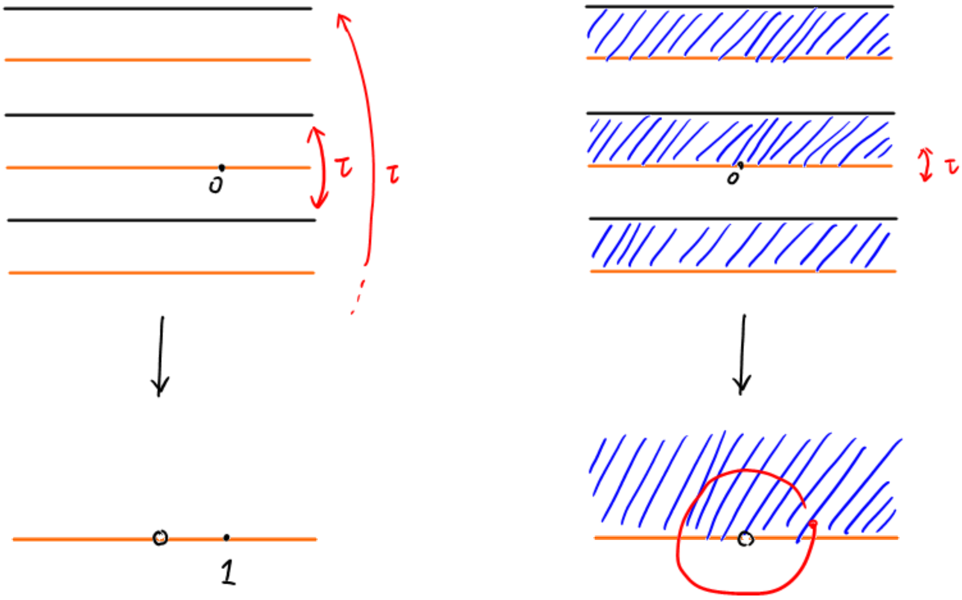


$f: E_{\mathbb{R}} \longrightarrow \mathbb{P}^1_{\mathbb{R}}$ $\quad [x:y:z] \longmapsto [x:z]$ $\quad E_{\mathbb{R}} = \mathrm{Proj}\ \mathbb{R}[x,y,z]/(y^2z - x(x-z)(x+z))$

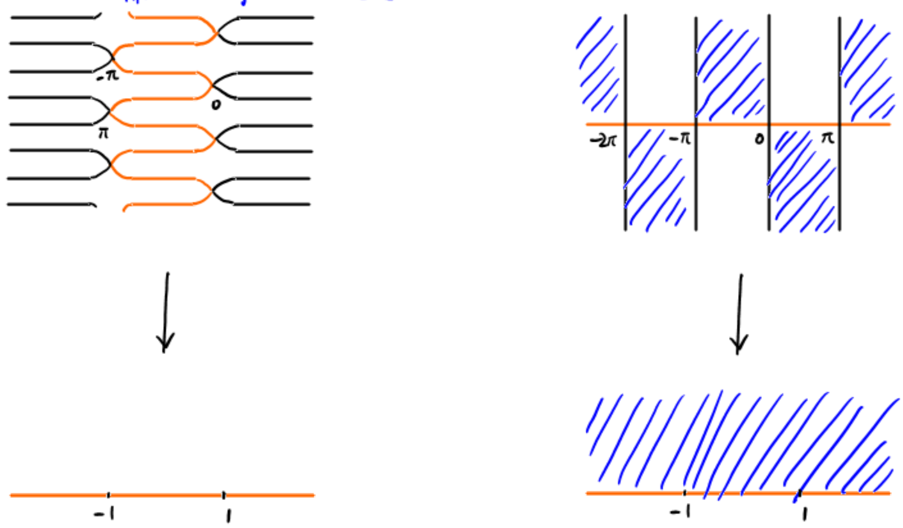


⚠ The following are not alg morphisms!

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = e^z$



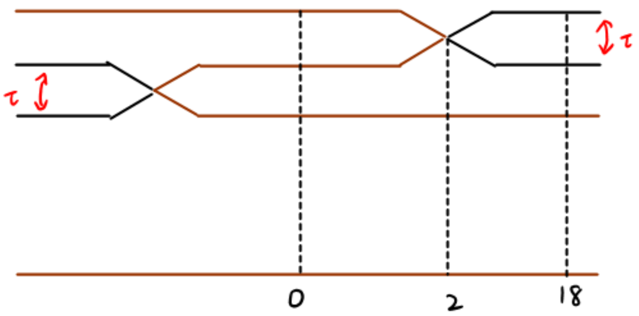
$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\quad f(z) = \cos z$



Let's focus on the case

$f: \mathbb{A}^1_{\mathbb{R}} \longrightarrow \mathbb{A}^1_{\mathbb{R}}$ $\qquad f(z) = z^3 - 3z$

classical picture



split: $f^{-1}(0) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$ $\qquad f^{-1}(z_0) := f^{-1}((z-z_0))$

$f^{-1}((z^2+1)) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{C}$

(partially) inert: $f^{-1}(18) = \operatorname{Spec}\mathbb{C} \sqcup \operatorname{Spec}\mathbb{R}$

generic point: $f^{-1}((0)) = \operatorname{Spec}\mathbb{R}(z')$

ramified: $f^{-1}(2) = \operatorname{Spec}\mathbb{R} \sqcup \operatorname{Spec}\mathbb{R}$

algebraic picture