Eine Woche, ein Beispiel

10.20 Schur functor: basic formulas

Main reference:
[FH]: Willian Fulton and Joe Harris. Representation Theory. A First Course.

[Hall]: Brian Hall. Lie Groups, Lie Algebras, and Representations: An Elementary Introduction. Graduate Texts in Mathematics. Cham: Springer International Publishing, 2015.

In this document, char $k = 0$, $V \in \mathrm{Vect}_k$.

Schur fctor helps us to decompose $V^{\otimes k}$ by S_k gp action.
$\mathbb{S}^\lambda V$ generalize $\mathrm{Sym}^k V$ & $\Lambda^k V$. Moreover,

$$\begin{array}{ccc} \mathrm{Rep}(GL(V)) & = & \mathrm{Rep}(A_{n-1}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathbb{S}^\lambda V & = & \mathcal{L}(\lambda) \end{array}$$

$n = \dim V$

Here, λ has many expressions,

e.g. partitions weights

λ =  $= (3,1) = 2\varpi_1 + \varpi_2$

$\lambda = \cdots = (\lambda_1, \lambda_2, \cdots) = \sum m_i \varpi_i$

$\{h_{ij}\} = \begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 \\ \cline{1-1} \end{array}$ hook length

1. dimension
2. $\mathbb{S}^\lambda(V \oplus W)$ and ...

1. dimension

It can be computed from the Weyl dimension formula:

$$\dim_{\mathbb{k}} L(\lambda) = \frac{\prod_{\alpha\in\Delta^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta^+}(\rho,\alpha)}$$

$L(\lambda)\in \mathrm{rep}(A_{n-1})$
$\lambda=\sum m_i\varpi_i$

$$= \frac{(m_1-1)\cdots(m_{n-1}+1)\,(m_1+m_2+2)(m_2+m_3+2)\cdots\cdots(\sum m_i+n-1)}{\underbrace{1\cdots 1}_{n-1 \text{ many}}\ \underbrace{2\cdot 2\cdot \cdots}_{n-2 \text{ many}}\cdots\ n-1}$$

$$= \prod_{1\le i<j\le n}\frac{\lambda_i-\lambda_j+j-i}{j-i} \qquad \text{[FH, Thm 6.3 (1)]}$$

$$= \prod_{1\le i<j\le n}\frac{n-i+j}{h_{ij}} \qquad \text{[FH, Ex 6.4]}$$

Following [Hall Example 10.23],

$$\Delta^+ = \left\{\begin{array}{ccccc} & & \sum\alpha_i & & \\ & \alpha_1+\alpha_2 & \cdots & \alpha_{n-2}+\alpha_{n-1} & \\ \alpha_1 & \alpha_2 & \cdots & & \alpha_{n-1}\end{array}\right\}$$

$$\begin{aligned}\rho &= \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha \\ &= \frac{1}{2}\sum_{i=1}^{n-1} i(n-i)\alpha_i \\ &= \sum_{i=1}^{n-1}\varpi_i \end{aligned} \qquad \chi=\sum_{i=1}^{n-1}\langle\chi,\alpha_i\rangle\varpi_i$$

These would be enough to explain the first equality above.

E.g. When $\lambda = (3,1) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} = 2\varpi_1 + \varpi_2$,

$$\dim_k L(\lambda) = \frac{(2+1)(1+1)(0+1)\cdots(2+1+2)(1+0+2)(0+0+2)\cdots(2+1+(n-1))}{1\cdot 1\cdot 1\cdots 2\cdot 2\cdot 2\cdots n-1}$$

$$= \frac{(2+1)(1+1)\quad(2+1+2)(1+0+2)\quad\cdots(2+1+(n-1))}{1\cdot 1\quad 2\cdot 2\quad\cdots n-1}$$

$$= \left(\frac{2+1}{1}\cdot\frac{3+2}{2}\cdot\frac{3+3}{3}\cdots\frac{3+n-1}{n-1}\right)\cdot\left(\frac{1+1}{1}\cdot\frac{1+2}{2}\cdot\frac{1+3}{3}\cdots\frac{1+n-2}{n-2}\right)$$

$$= \frac{(n-1)\,n\,(n+1)(n+2)}{1\cdot 4\cdot 2\cdot 1}$$

$$= \frac{\begin{array}{|c|c|c|}\hline n & n+1 & n+2 \\ \hline n-1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|c|}\hline 4 & 2 & 1 \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array}}$$

2. $\mathbb{S}^{\lambda}(V\oplus W)$ and ...

E.g. $$(V\oplus W)^{\otimes 2} = V^{\otimes 2} \oplus (V\otimes W)^{\oplus 2} \oplus W^{\otimes 2}$$

$$\mathrm{Sym}^2(V\oplus W) = \mathrm{Sym}^2 V \oplus V\otimes W \oplus \mathrm{Sym}^2 W$$
$$\Lambda^2(V\oplus W) = \Lambda^2 V \oplus V\otimes W \oplus W^{\otimes 2}$$

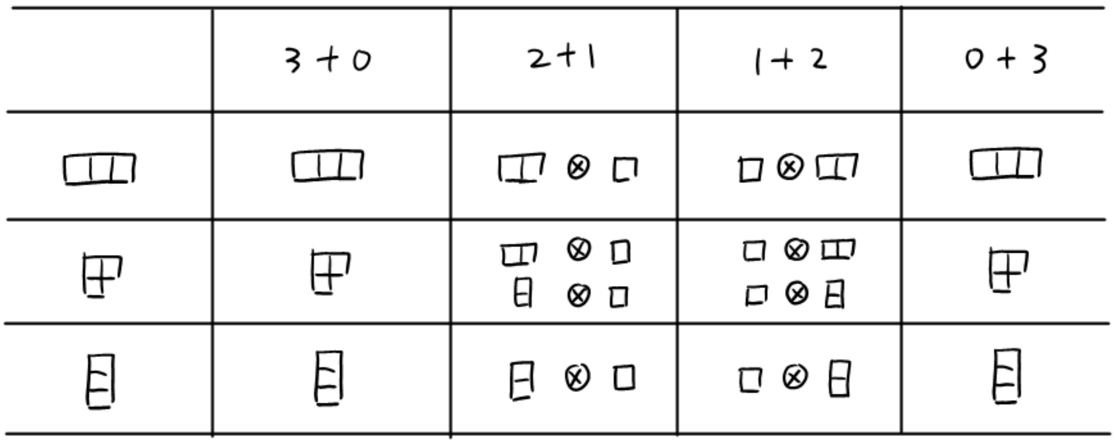
$$(V\oplus W)^{\otimes 3} = V^{\otimes 3} \oplus (V^{\otimes 2}\otimes W)^{\oplus 3} \oplus (V\otimes W^{\otimes 2})^{\oplus 3} \oplus W^{\otimes 3}$$

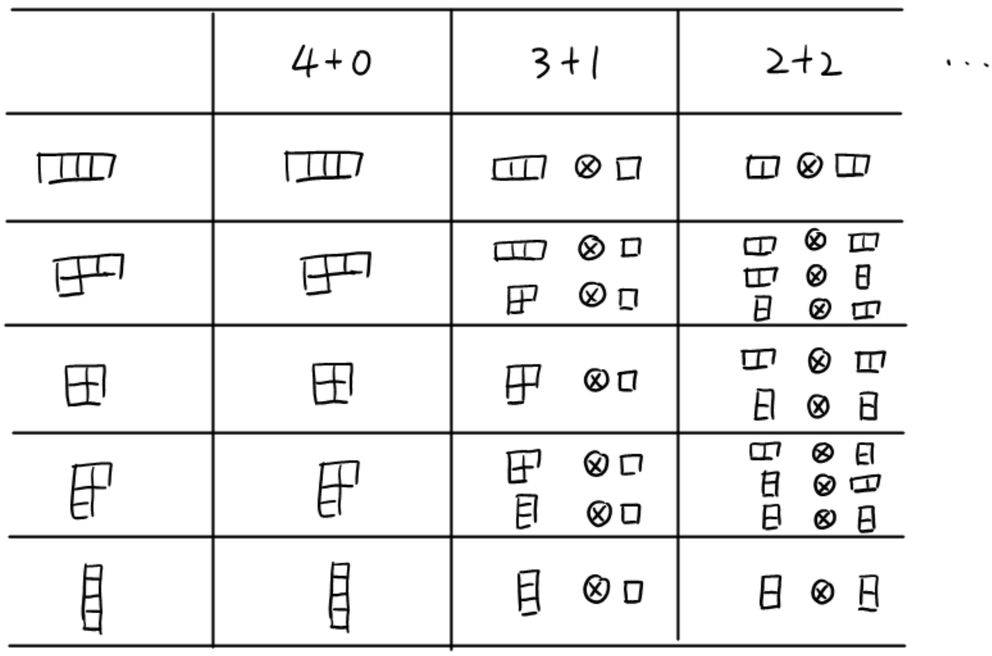
$$\mathrm{Sym}^3(V\oplus W) = \mathrm{Sym}^3 V \oplus \mathrm{Sym}^2 V\otimes W \oplus V\otimes \mathrm{Sym}^2 W \oplus \mathrm{Sym}^3 W$$
$$\mathbb{S}^{\boxplus'}(V\oplus W) = \mathbb{S}^{\boxplus'} V \oplus V^{\otimes 2}\otimes W \oplus V\otimes W^{\otimes 2} \oplus \mathbb{S}^{\boxplus'} W$$
$$\Lambda^3(V\oplus W) = \Lambda^3 V \oplus \Lambda^2 V\otimes W \oplus V\otimes \Lambda^2 W \oplus \Lambda^3 W$$

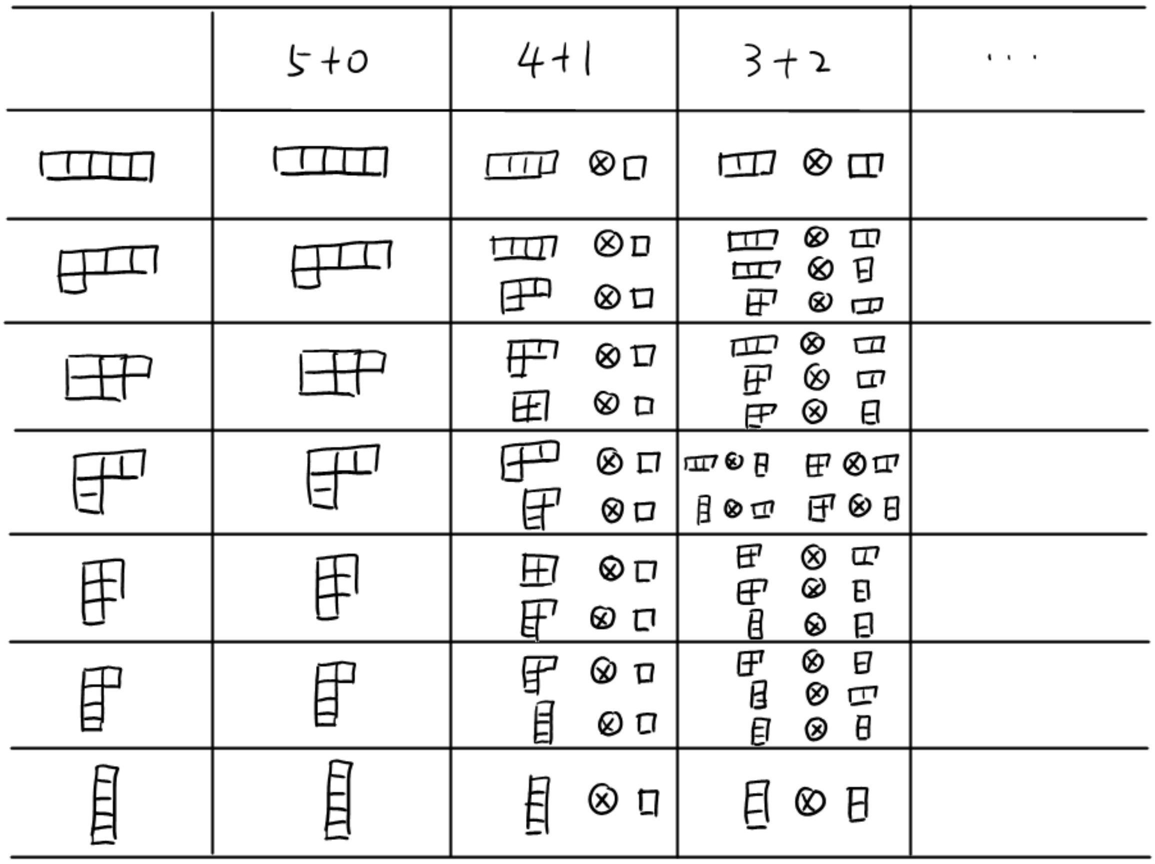
$$\frac{\begin{array}{|c|c|}\hline m+n & m+n+1 \\ \hline m+n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} = \frac{\begin{array}{|c|c|}\hline m & m+1 \\ \hline m-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}} + m^2 n + n m^2 + \frac{\begin{array}{|c|c|}\hline n & n+1 \\ \hline n-1 & \\ \cline{1-1}\end{array}}{\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 1 & \\ \cline{1-1}\end{array}}$$

With the help of [FH,Ex 6.11], we can get the following tables:

	$2+0$	$1+1$	$0+2$
$\square\square$	$\square\square$	$\square\otimes\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\square\otimes\square$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$


3 + 0
2+1
1 + 2
0 + 3


4+0
3+1
2+2
. . .


5+0
4+1
3+2
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