

Eine Woche, ein Beispiel

10.23 equivariant K-theory of Steinberg variety : q -coefficient.

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

This time we focus on $G \times \mathbb{C}^*$ -action.

Task.

1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. change of basis
 - § 6.1 two basis
 - § 6.2 tangent space
 - § 6.3 Euler class
 - § 6.4 transition matrix, localization formula
 - § 6.5 generators
7. convolution product
 - § 7.1 clean intersection formula
 - § 7.2 convolution for canonical basis
 - § 7.3 expression of D_k .

We may use two examples for the convenience of presentation.
Readers can easily distinguish them by the dim vectors.

1. dimension vector

$$|\underline{d}| = 5$$

$$\underline{d} = (3, 2)$$

$$\underline{d} = \begin{pmatrix} 3, 2 \\ 2, 2 \\ 2, 1 \\ 1, 1 \\ 0, 1 \\ 0, 0 \end{pmatrix} = \begin{array}{c} \text{Young Tableaux} \\ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \\ 1 & 4 & 5 & 3 & 2 \end{pmatrix} \end{array} = \begin{array}{c} \text{Young Tableaux} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \downarrow & \uparrow & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array} = \begin{array}{c} \text{Young Tableaux} \\ \begin{pmatrix} \cdot & \cdot & \cdot \\ \downarrow & \times & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} \end{array} \in W_d \setminus W_{\text{id}} \text{ or } \text{Min}(W_{\text{id}}, W_d)$$

$\nu_{\text{id}} = \pi_{\underline{d}}^{-1}(F_{\text{id}})$

2. Weyl group

Set	element	special element	others
$W_{\text{id}} = S_5$	w, x	$w_{\text{max}} = \begin{array}{ c c c c c } \hline & \times & \times & \times & \times \\ \hline \times & & & & \\ \hline & \times & \times & \times & \times \\ \hline & & & & \\ \hline \end{array}$	$T = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{\text{max}} = \begin{array}{ c c c } \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array}$	$T_d = \{s_1, s_2, s_4\}$
$W_d \setminus W_{\text{id}} = S_3 \times S_2 \setminus S_5$	w, \underline{d}	$\begin{array}{ c c c } \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array}$	(Comp _d)

$$\text{Min}(W_{\text{id}}, W_d) = \left\{ \begin{array}{|c|c|c|} \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array}, \dots \right\} \quad u \quad \begin{array}{|c|c|c|} \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array} \quad (\text{Shuffled})$$

$$0 \longrightarrow W_d \longrightarrow W_{\text{id}} \longrightarrow W_{\text{id}} \setminus W_d \longrightarrow 0 \quad w = wu \mapsto \underline{d}$$

$\xrightarrow{\text{Min}(W_{\text{id}}, W_d)}$

$w = \begin{array}{|c|c|c|} \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array}$
 $u = \begin{array}{|c|c|c|} \hline \times & \times & \times \\ \hline \times & & \\ \hline \end{array}$
 $w = \begin{array}{|c|c|c|} \hline \times & \times & \times \\ \hline & & \\ \hline \end{array}$

Another example: $\underline{d} = (1, 2)$ $\begin{array}{c} a \longrightarrow b \\ \langle v_1 \rangle \rightarrow \langle v_2, v_1 \rangle \end{array}$

	$w = uu$	w	\underline{d}, u	order of basis	(w)	$l(w)$	B_w	$B_{\underline{d}w}$	$wB_{\underline{d}w}^{-1}$
Id	$(1^2 3)$	$\begin{array}{ c c } \hline \top & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \end{array}$	$\{v_1, v_2, v_3\}$	0	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$	$\begin{bmatrix} * & * \\ * & * \end{bmatrix}$
t	(23)	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\{v_1, v_3, v_2\}$	1	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
s	(12)	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \top & \times \\ \hline \end{array}$	$\{v_2, v_1, v_3\}$	1	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
ts	(132)	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\{v_3, v_1, v_2\}$	2	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
st	(123)	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \top & \times \\ \hline \end{array}$	$\{v_2, v_3, v_1\}$	2	0	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$
sts	(13)	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\begin{array}{ c c } \hline \top & \top \\ \hline \times & \top \\ \hline \end{array}$	$\{v_3, v_2, v_1\}$	3	1	$\begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$	$\begin{bmatrix} * \\ * \end{bmatrix}$

3. alg group & Lie algebra

$$G_{\text{Idl}}, B_{\text{Idl}}, T_{\text{Idl}}, N_{\text{Idl}} \quad W_{\text{Idl}} = N_{G_{\text{Idl}}}(\Pi_{\text{Idl}}) / \Pi_{\text{Idl}} \quad GL_5(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$G_d, B_d, T_d, N_d \quad W_d = N_{G_d}(T_d) / T_d \quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$$

$$B_\infty = B_{\text{Idl}} \xrightarrow{\omega} B_{\text{Idl}} \omega^{-1} = \text{Stab}_{G_{\text{Idl}}}(F_\infty)$$

$$B_\infty \xrightarrow{\omega = w u} w B_d w^{-1} = \text{Stab}_{G_d}(F_\infty) \quad N_\infty = R_u(B_\infty)$$

For $s \in \Pi$ s.t. $\omega s \omega^{-1} \in W_d$ (i.e. $W_d \omega = W_d \omega s$), define

$$P_{\infty, \omega s} \xrightarrow{\omega = w u} w (B_d u s u^{-1} B_d \cup B_d) w^{-1} \quad N_{\infty, \omega s} = R_u(B_{\infty, \omega s})$$

$$= B_\infty \omega s \omega^{-1} B_\infty \cup B_\infty \quad = N_\infty \cap N_{\infty, \omega s}$$

$$M_{\infty, \omega s} = N_\infty / N_{\infty, \omega s}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega s}$$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{\sigma(i)} \in \Pi_d$$

We can generalize the unipotent part.

$$N_{\infty, \omega''} := N_\infty \cap N_\infty$$

$$M_{\infty, \omega''} := N_\infty / N_{\infty, \omega''}$$

$$= B_\infty / B_\infty \cap B_{\infty, \omega''}$$

Their Lie algebras are collected here.

$$\mathfrak{g}_{\text{Idl}}, \mathfrak{b}_{\text{Idl}}, \mathfrak{t}_{\text{Idl}}, \mathfrak{n}_{\text{Idl}}$$

$$g_d \quad b_d \quad t_d \quad n_d$$

$$\mathfrak{b}_\infty \quad \bar{\mathfrak{b}}_\infty$$

$$b_\infty \quad n_\infty$$

$$P_{\infty, \omega s} \quad N_{\infty, \omega''}$$

$$m_{\infty, \omega''}$$

$$\bar{\mathfrak{b}}_\infty = \mathfrak{b}_{\omega \max \infty}$$

$$\bar{b}_\infty = b_{w \max \infty}$$

$$\bar{P}_{\infty, \omega s} = P_{w \max \infty, w \max \infty s}$$

$$m_{\infty, \omega''}$$

$$\text{Rep}_d(Q) := \bigoplus_{e \in Q_1} \text{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{\text{Idl}}^{\oplus k}$$

$$V_\infty = \{f \in \text{Rep}_d(Q) \mid f: F_{\infty, i} \subseteq F_{\infty, i}\} = \underline{\mu_d} \pi_d^{-1}(F_\infty)$$

$$= \begin{smallmatrix} v_3 & v_1 & v_2 \\ v_5 & * & * \\ v_4 & * & * \end{smallmatrix} = \begin{smallmatrix} v_1 & v_2 & v_3 \\ v_4 & * & * \\ v_5 & * & * \end{smallmatrix}$$

$$V_{\omega(i)}$$

$$V_{\infty, \omega''} = V_\infty \cap V_{\infty''}$$

$$J_{\infty, \omega''} = V_\infty / V_{\infty, \omega''}$$

Later we may twist the group actions.

$$\text{E.g. } \underline{r}_{\infty, \omega'} := r_{\infty, \omega \omega'} \quad r_{\infty, \omega''} = \underline{r}_{\infty, \omega^{-1} \omega''}$$

$$\begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}$$

The following example may let you get familiar with those Lie algs.

$$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,1 \\ 0,0 \end{pmatrix} = \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagup \diagdown \diagup \diagdown \end{array}$$

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 2 & 5 & 3 \end{pmatrix}$$

\$u\$	\$n_u\$	\$m_{u,u}\$	\$r_u\$	\$\partial_{u,u}\$
	$\begin{bmatrix} * & * \\ * & * \\ \hline \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$
$us_1 = \begin{array}{c} \diagup \\ \diagup \diagdown \diagup \diagdown \end{array} \notin W_{du}$	$\begin{bmatrix} * & * \\ * & * \\ \hline \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$
$us_2 = \begin{array}{c} \diagup \\ \diagup \diagdown \diagup \diagdown \end{array} \in W_{du}$	$\begin{bmatrix} * & * \\ * & * \\ \hline \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$
$us_3 = \begin{array}{c} \diagup \\ \diagup \diagdown \diagup \diagdown \end{array} \notin W_{du}$	$\begin{bmatrix} * & * \\ * & * \\ \hline \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$
$us_4 = \begin{array}{c} \diagup \\ \diagup \diagdown \diagup \diagdown \end{array} \notin W_{du}$	$\begin{bmatrix} * & * \\ * & * \\ \hline \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$	$\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$
			$\downarrow \xrightarrow{1 \rightarrow 2 \rightarrow 3} \xrightarrow{4}$	
			$\rightarrow \downarrow \rightarrow \downarrow \rightarrow$	
			$\downarrow \rightarrow \rightarrow \downarrow \rightarrow$	
			$\downarrow \rightarrow \downarrow \rightarrow$	
			$\downarrow \rightarrow \rightarrow \rightarrow$	
				$\frac{e_4}{e_1} 9$
				$\frac{e_5}{e_3} 9$

4. typical variety

Id corres to

$$\begin{aligned}
 F_{\text{Id}} &\cong G_{\text{Id}} / B_{\text{Id}} & F_{\text{Id}} \\
 F_d &\cong G_d / B_d & F_u \\
 F_\infty &\cong G_d / B_\infty & F_\infty \\
 F_d &= \coprod_d F_d & - \\
 F_{g\infty} &\cong G_d / gB_\infty g^{-1} & F_{g\infty} \\
 F_\infty := &_{\infty}(F_{\text{Id}}) = F_{\{v_{\infty(1)}, v_{\infty(2)}, \dots, v_{\infty(\text{Id})}\}} & \\
 &= F_{\{u_5, u_3, v_1, v_4, v_2\}}
 \end{aligned}$$

✓ The action on Flag is not the same as in

http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$$F_{\text{Id}} \neq \coprod_d F_d$$

$F_\infty \cong F_d$ with different base pt. Base pt makes difference!

$$\begin{aligned}
 F_{\text{Id}} \times F_{\text{Id}} && F_{\text{Id}, \text{Id}} \\
 F_d \times F_{d'} && F_{u, u'} \\
 F_\infty \times F_\infty && F_{\infty, \infty} \\
 F_d \times F_d := \coprod_{d, d'} (F_d \times F_{d'}) && -
 \end{aligned}$$

$$F_{\infty, \infty'} := (F_\infty, F_{\infty'})$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_d(Q) & \subset & \text{Rep}_d(Q) \times F_d \\
 \downarrow M_d & & \downarrow \pi_d \\
 \text{Rep}_d(Q) & & F_d
 \end{array}$$

$$\begin{array}{ccc}
 \widetilde{\text{Rep}}_d(Q) & \subset & \text{Rep}_d(Q) \times F_d \\
 \downarrow M_d & & \downarrow \pi_d \\
 \text{Rep}_d(Q) & & F_d
 \end{array}$$

$\mu_d^*(M) \cong \text{Flag}_d(M) \subseteq F_d$ is the Springer fiber.

$$\begin{array}{ccc}
 Z_{d, d'} & \subset & \text{Rep}_d(Q) \times F_d \times F_{d'} \\
 \downarrow M_{d, d'} & & \downarrow \pi_{d, d'} \\
 \text{Rep}_d(Q) & & F_d \times F_{d'}
 \end{array}$$

$$\begin{array}{ccc}
 Z_d & \subset & \text{Rep}_d(Q) \times F_d \times F_d \\
 \downarrow M_{dd} & & \downarrow \pi_{d,d} \\
 \text{Rep}_d(Q) & & F_d \times F_d
 \end{array}$$

$$\begin{array}{c}
 \widetilde{\text{Rep}}_d(Q) \subseteq \text{Rep}_d(Q) \times F_d \\
 \widetilde{\text{Rep}}_d(Q) := \bigsqcup_d \widetilde{\text{Rep}}_d(Q)
 \end{array}$$

$$\widetilde{\text{Rep}}_\infty(Q) \cong G_d \times^{B_\infty} r_\infty$$

$$\begin{aligned}
 Z_{d, d'} &= \widetilde{\text{Rep}}_d(Q) \times_{\text{Rep}_d(Q)} \widetilde{\text{Rep}}_{d'}(Q) \\
 Z_d &= \bigsqcup_{d, d'} Z_{d, d'} \\
 &= \widetilde{\text{Rep}}_d(Q) \times_{\text{Rep}_d(Q)} \widetilde{\text{Rep}}_d(Q)
 \end{aligned}$$

$$Z_{\infty, \infty'} = Z_{u, u'}$$

5. (equivariant) stratifications.

In the following tables, $uw' = \tilde{w}'\tilde{u}$.

$F_\infty \in \widetilde{\text{Rep}}_d(Q)$ means $(p_0, F_\infty); (F_\infty, F_{\infty'}) \in Z_d$ means $(p_0, F_\infty, F_{\infty'})$.

▽ $G \times G$ acts on $\mathcal{F} \times \mathcal{F}$ in a twisted way

$$\text{e.g. } (g_1, g_2) F_{\infty, \infty'} = F_{g_1 \infty, g_1 \tilde{w} g_2 \infty'^{-1}}$$

$$(g_1, g_2) E_{\infty, \infty'} = E_{g_1 \infty, g_2 \infty'}$$

variety base point	stratification stabilizer	type	B-orbit	$B \times B$ -orbit stabilizer are twisted	$B \times G$ -orbit	$G \times B$ -orbit	Remark $G \times \{*\}$ -orbit
\mathcal{B}	$\mathcal{B} \times \mathcal{B}$		Ω_g	$\Omega_{g, g'}$	$\text{pr}_i^{-1}(\Omega_g)$	$\Omega_{g'}$	
F_g ($F_g, F_{gg'}$)	$B \cap gBg^{-1}$		$B \cap gBg^{-1} \times B \cap (gg')B(gg')^{-1}$				$gBg^{-1} \cap gg'B(gg')^{-1}$
\mathcal{F}_{id}	$\mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$		\mathcal{V}_∞	$\mathcal{V}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{V}_\infty)$	\mathcal{V}_∞	
F_∞ ($F_\infty, F_{\infty\infty'}$)	$B_{\text{id}} \cap B_\infty$		$B_{\text{id}} \cap B_\infty \times B_{\text{id}} \cap B_{\infty'}$				$B_\infty \cap B_{\infty\infty'}$
\mathcal{F}_u	$\mathcal{F}_u \times \mathcal{F}_u$		Ω_w^u	$\Omega_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{w'}^{u, u'}$	
F_{wu} ($F_{wu}, F_{wuwu'}$)	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{ww'}$				$B_w \cap B_{ww'}$
\mathcal{F}_d	$\mathcal{F}_d \times \mathcal{F}_d$		Ω_w^u	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\Omega_{\infty'}^u = \Omega_{\tilde{w}}^{u, \tilde{u} u'}$	compatibility
F_∞ "	$B_d \cap B_w$		$B_d \cap B_w \times B_d \cap B_{w\tilde{w}}$				$B_w \cap B_{w\tilde{w}}$
F_{wu} ($F_{wu}, F_{wuwu'}$)							

The following may not be single orbit, but derived from the above definition.

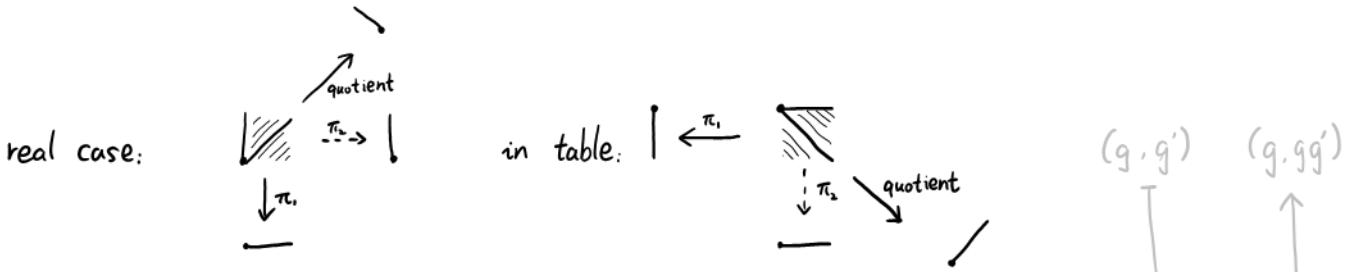
\mathcal{F}_d	$\mathcal{F}_d \times \mathcal{F}_d$	\mathcal{O}_∞	$\mathcal{O}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\mathcal{O}_\infty)$	\mathcal{O}_∞		preimage of $\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\text{id}} \times \mathcal{F}_{\text{id}}$
F_∞ ($F_\infty, F_{\infty\infty'}$)		Ω_w^u	$\Omega_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\Omega_w^u)$	$\bigsqcup_u \Omega_{\tilde{w}}^{u, \tilde{u} u'}$		preimage of $\mathcal{Z}_{d, d'} \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{\text{Rep}}_d(Q)$	Z_d	$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, w'}^{u, u'}$	$\text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\widetilde{\Omega}_w^{u, u'}$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
F_∞ ($F_\infty, F_{\infty\infty'}$)							
$\widetilde{\text{Rep}}_d(Q)$	Z_d	$\widetilde{\mathcal{O}}_\infty$	$\widetilde{\mathcal{O}}_{\infty, \infty'}$	$\text{pr}_i^{-1}(\widetilde{\mathcal{O}}_\infty)$	$\widetilde{\mathcal{O}}_\infty$		preimage of $\mathcal{Z}_d \rightarrow \mathcal{F}_d \times \mathcal{F}_d$
F_∞ ($F_\infty, F_{\infty\infty'}$)		$\widetilde{\Omega}_w^u$	$\widetilde{\Omega}_{w, \tilde{w}}^{u, \tilde{u} u'}$	$\bigsqcup_u \text{pr}_{i,u}^{-1}(\widetilde{\Omega}_w^u)$	$\bigsqcup_u \widetilde{\Omega}_{\tilde{w}}^{u, \tilde{u} u'}$		

$$\mathcal{Z}_{\infty'} := \overline{\widetilde{\mathcal{O}}_\infty} \subseteq \overline{\widetilde{\mathcal{O}}_{\infty'}}$$

$$\mathcal{Z}_{w'}^{u, u'} := \overline{\widetilde{\Omega}_{w'}^{u, u'}} \subseteq \overline{\widetilde{\Omega}_w^{u, u'}}$$

$$\mathcal{Z}_{d, d'}^\infty := \mathcal{Z}_\infty \cap \mathcal{Z}_{d, d'}$$

Zar-loc sub v.b.?



We want gp action to be compatible with π_i and the quotient map.
Therefore, we would do a twist.



Rmk. The stabilizer is not trivial to determine because of this twist!

$$\begin{aligned}
 E_{g_1, g_2} = E_{g_3, g_4} &\Leftrightarrow g_1 B = g_3 B, g_2 B = g_4 B \\
 &\Leftrightarrow g_1 B = g_3 B, g_1 g_2 B = g_3 g_4 B \\
 E_{g_1\omega, g_2\omega'} = E_{\omega, \omega'} &\Leftrightarrow g_1\omega B = \omega B, g_1\omega g_2\omega' B = g_2\omega' B \\
 &\Leftrightarrow g_1 = \omega b, \omega^{-1} \in B_\omega \quad b, g_2 \in B_{\omega'} \\
 &\Leftrightarrow g_1 \in B_\omega, \quad g_2 \in \omega^{-1} g_1 \omega \cdot B_{\omega'}
 \end{aligned}$$

The following tables may help you to understand the notations.

\dim	$B_{\text{Id}} \cdot F_{\text{var}}$	0	1	1	2	2	3
	$B_{\text{Id}} \times B_{\text{Id}} \cdot (F_{\text{var}}, F_{\text{var}})$	\mathcal{V}_{Id}	\mathcal{V}_t	\mathcal{V}_s	\mathcal{V}_{ts}	\mathcal{V}_{st}	\mathcal{V}_{sts}
	$B_{\text{Id}} \cdot F_w$	\mathcal{V}_{Id}	$\mathcal{V}_{\text{Id},\text{Id}}$	$\mathcal{V}_{\text{Id},t}$	$\mathcal{V}_{\text{Id},s}$	$\mathcal{V}_{\text{Id},ts}$	$\mathcal{V}_{\text{Id},st}$
0							
1							
2							
3							

\dim	$B_d \cdot F_{\text{var}}$	0	1	1	2	2	3
	$B_d \times B_d \cdot (F_{\text{var}}, F_{\text{var}})$	\mathcal{F}_{Id}	\mathcal{F}_s	\mathcal{F}_{st}			
	$B_d \cdot F_w$	\mathcal{O}_{Id}	\mathcal{O}_t	\mathcal{O}_s	\mathcal{O}_{ts}	\mathcal{O}_{st}	\mathcal{O}_{sts}
0							
1							
2							
3							

The following tables may help you to understand the notations.

$$\omega = ts, \omega' = s$$

\dim	$B_{Id} \cdot F_{ts}$	0	1	1	2	2	3	$\text{pr}_i^{-1}(\mathcal{V}_{ts})$
	$B_{ts} \times B_{st} \cdot (F_s, F_{st})$	\mathcal{V}_{Id}	\mathcal{V}_t	\mathcal{V}_s	\mathcal{V}_{ts}	\mathcal{V}_{st}	\mathcal{V}_{sts}	
	$B_{st} \cdot F_{st}$	\mathcal{V}_{Id}	$\mathcal{V}_{Id,Id}$	$\mathcal{V}_{Id,t}$	$\mathcal{V}_{Id,s}$	$\mathcal{V}_{Id,ts}$	$\mathcal{V}_{Id,st}$	$\mathcal{V}_{Id,sts}$
0								\mathcal{V}_s
1								\mathcal{V}_t
2								\mathcal{V}_{ts}
3								\mathcal{V}_{st}
4								\mathcal{V}_{sts}

shape	$B_d \cdot F_{ts}$	\mathcal{F}_{Id}	\mathcal{F}_s	\mathcal{F}_{st}	$\text{pr}_i^{-1}(\mathcal{O}_{ts})$	$\text{pr}_{i,Id}^{-1}(\Omega_t^s)$	$\Omega_{t,Id}^{s,Id} = \mathcal{O}_{ts,s}$
	$B_d \times B_d \cdot (F_{ts}, F_{st})$	\mathcal{O}_{Id}	\mathcal{O}_t	\mathcal{O}_s	\mathcal{O}_{ts}	\mathcal{O}_{st}	\mathcal{O}_{sts}
	$B_{ts} \cdot F_{st}$	$\Omega_{Id,Id}$	$\Omega_{Id,t}$	$\Omega_{Id,s}$	$\Omega_{Id,ts}$	$\Omega_{Id,st}$	$\Omega_{Id,sts}$
\mathcal{F}_{Id}	\mathcal{O}_{Id}	$\Omega_{Id,Id}$	$\Omega_{Id,t}$	$\Omega_{Id,s}$	$\Omega_{Id,ts}$	$\Omega_{Id,st}$	$\Omega_{Id,sts}$
	\mathcal{O}_t	$\Omega_{t,Id}$	$\Omega_{t,t}$	$\Omega_{t,s}$	$\Omega_{t,ts}$	$\Omega_{t,st}$	$\Omega_{t,sts}$
\mathcal{F}_s	\mathcal{O}_s	$\Omega_{s,Id}$	$\Omega_{s,t}$	$\Omega_{s,s}$	$\Omega_{s,ts}$	$\Omega_{s,st}$	$\Omega_{s,sts}$
	\mathcal{O}_{ts}	$\Omega_{ts,Id}$	$\Omega_{ts,t}$	$\Omega_{ts,s}$	$\Omega_{ts,ts}$	$\Omega_{ts,st}$	$\Omega_{ts,sts}$
\mathcal{F}_{st}	\mathcal{O}_{st}	$\Omega_{st,Id}$	$\Omega_{st,t}$	$\Omega_{st,s}$	$\Omega_{st,ts}$	$\Omega_{st,st}$	$\Omega_{st,sts}$
	\mathcal{O}_{sts}	$\Omega_{sts,Id}$	$\Omega_{sts,t}$	$\Omega_{sts,s}$	$\Omega_{sts,ts}$	$\Omega_{sts,st}$	$\Omega_{sts,sts}$

6. change of basis

§6.1 two basis

Def Let $Y \subset X$ be G -equiv closed subvariety, X proj.

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{R(G)} \in K_0^G(X)$$

with same notation,

$$[Y]^G := (\iota_Y)_*(\pi_Y)^* 1_{S(G)} \in H_q^*(X; \mathbb{Q})$$

$$\begin{array}{ccc} Y & \xhookrightarrow{\iota_Y} & X \\ & \downarrow \pi_Y & \\ & pt & \end{array}$$

By cellular fibration lemma,

$$\begin{array}{ccccccc} K_0^{T_d \times \mathbb{C}^\times}(F_d) & \cong & K_0^{G_d \times \mathbb{C}^\times}(F_d \times F_d) & \cong & K_0^{G_d \times \mathbb{C}^\times}(Z_d) \\ \oplus_{\infty, \infty' \in W_{\text{id}}} R(T_d \times \mathbb{C}) [\bar{\mathcal{O}}_{\infty}]^{T_d \times \mathbb{C}^\times} & \cong & \oplus_{\infty, \infty' \in W_{\text{id}}} R(T_d \times \mathbb{C}) [\bar{\mathcal{O}}_{\infty}]^{G_d \times \mathbb{C}^\times} & \cong & \oplus_{\infty, \infty' \in W_{\text{id}}} R(T_d \times \mathbb{C}) [Z_{\infty}]^{G_d \times \mathbb{C}^\times} \\ & \parallel & \parallel & \parallel & \parallel & & \downarrow \\ & & & & & & K_0^{T_d \times \mathbb{C}^\times}(Z_d) \\ & & & & & \parallel & \\ & & & & & & \oplus_{\infty, \infty' \in W_{\text{id}}} R(T_d \times \mathbb{C}) [\bar{\mathcal{O}}_{\infty, \infty'}]^{T_d \times \mathbb{C}^\times} \end{array}$$

as $R(T_d \times \mathbb{C})$ -modules.

⚠ There is no evidence if $[Z_{\infty'}]^{G_d \times \mathbb{C}^\times}$ will be mapped to $\oplus_{\infty, \infty' \in W_{\text{id}}} [\bar{\mathcal{O}}_{\infty, \infty'}]^{T_d \times \mathbb{C}^\times}$.
 Luckily, the horizontal line sends generators to generators.

Hint: Consider the following commutative diagram:

$$\begin{array}{ccccc} F_d & \xrightarrow{(F_d, \text{Id})} & F_d \times F_d & \xrightarrow{(P_d, \text{Id})} & Z_d \\ \bar{\mathcal{O}}_{\infty} \nearrow & \nearrow & \nearrow & \nearrow & \\ \bar{\mathcal{O}}_{\infty} & \longrightarrow & \bar{\mathcal{O}}_{\infty} & \longrightarrow & \bar{Z}_{\infty} \\ \downarrow & = & \downarrow & = & \downarrow \\ pt & = & pt & = & pt \end{array}$$

To do linear alg, we take

$$\begin{aligned} R(G) &:= \text{Frac}(R(G)) & \mathcal{K}_G^G(X) &:= K_0^G(X) \otimes_{R(T_d \times \mathbb{C}^\times)} R(T_d \times \mathbb{C}^\times) \\ S(G) &:= \text{Frac}(S(G)) & \mathcal{H}_G^*(X; \mathbb{Q}) &:= H_G^*(X; \mathbb{Q}) \otimes_{S(T_d \times \mathbb{C}^\times)} S(T_d \times \mathbb{C}^\times) \end{aligned}$$

For $R(T_d \times \mathbb{C}^\times)$ -mod $K_0^G(X)$, $S(T_d \times \mathbb{C}^\times)$ -mod $H_G^*(X; \mathbb{Q})$

Define $\psi_\infty := [\{F_\infty\}]^{T_d \times \mathbb{C}^\times} = (\iota_\infty)_* 1_{R(T_d \times \mathbb{C}^\times)} \in K_0^{T_d \times \mathbb{C}^\times}(F_d)$
 $\psi_{\infty, \infty'} := [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d \times \mathbb{C}^\times} = (\iota_{\infty, \infty'})_* 1_{R(T_d \times \mathbb{C}^\times)} \in K_0^{T_d \times \mathbb{C}^\times}(\mathbb{Z}_d)$
 $\psi_{\infty, \infty'} := [\{(\rho_0, F_\infty, F_{\infty'})\}]^{T_d \times \mathbb{C}^\times}$

We get two $R(T_d \times \mathbb{C}^\times)$ -basis. (ψ_∞ is $R(T_d \times \mathbb{C}^\times)$ -basis, by Localization theorem.)

$$\begin{array}{ccc} K_0^{T_d \times \mathbb{C}^\times}(F_d) & \longrightarrow & K_0^{T_d \times \mathbb{C}^\times}(\mathbb{Z}_d) \\ [\overline{\mathcal{O}_\infty}]^{T_d \times \mathbb{C}^\times} & & [\overline{\mathcal{O}_{\infty, \infty'}}]^{T_d \times \mathbb{C}^\times} \\ \psi_\infty & & \psi_{\infty, \infty'} \end{array} \quad \begin{array}{l} \text{standard basis for stratification} \\ \text{canonical basis for convolution} \end{array}$$

Localization thm [Thm 10.1]

Let $i: X^{T_d \times \mathbb{C}^\times} \hookrightarrow X$, X is smooth.

$$\begin{array}{ccccc} \mathcal{K}_0^{T_d \times \mathbb{C}^\times}(X^{T_d \times \mathbb{C}^\times}) & \xrightarrow{i_*} & \mathcal{K}_0^{T_d \times \mathbb{C}^\times}(X) & \xrightarrow{i^*} & \mathcal{K}_0^{T_d \times \mathbb{C}^\times}(X^{T_d \times \mathbb{C}^\times}) \\ \mathcal{H}_{T_d \times \mathbb{C}^\times}^*(X^{T_d \times \mathbb{C}^\times}; \mathbb{Q}) & \xrightarrow{i_*} & \mathcal{H}_{T_d \times \mathbb{C}^\times}^*(X; \mathbb{Q}) & \xrightarrow{i^*} & \mathcal{H}_{T_d \times \mathbb{C}^\times}^*(X^{T_d \times \mathbb{C}^\times}; \mathbb{Q}) \end{array}$$

are isos as $R(T_d \times \mathbb{C}^\times)$ or $S(T_d \times \mathbb{C}^\times)$ -module.

Q: The Steinberg variety \mathbb{Z}_d is usually not smooth.

How to show that $\{\psi_{\infty, \infty'}\}$ forms a basis?

Guess: apply localization thm to $F_d \times F_d$ first.

§ 6.2. tangent space

Def (tangent space of fixed pts. in $R(T_d)$)

$$\widetilde{T}_{\infty} := T_{F_{\infty}} F_d \cong T_{Id}(G_d/B_{\infty}) \cong \mathfrak{g}_d/b_{\infty} = n_{\infty}$$

$$\widetilde{T}_{\infty} := T_{(p_0, F_{\infty})} \widetilde{Rep_d}(\mathbb{Q}) \cong T_{r_{\infty}} \oplus T_{F_{\infty}} F_d = r_{\infty} \oplus n_{\infty}$$

$$T_{\infty, \infty'}^x := T_{(p_0, F_{\infty}, F_{\infty'})} \overline{\mathcal{O}}_x$$

$$\widetilde{T}_{\infty, \infty'}^x := T_{(p_0, F_{\infty}, F_{\infty'})} Z_x \not\cong T_{r_{\infty, \infty'}} \oplus T_{(F_{\infty}, F_{\infty'})} \overline{\mathcal{O}}_x = r_{\infty, \infty'} \oplus T_{\infty, \infty'}^x$$

Notice that $Z_x \neq \overline{\mathcal{O}}_x$

$$\begin{array}{ccc} \overline{\mathcal{O}}_x & \xrightarrow{(p_0, Id)} & Z_x \\ & \searrow & \downarrow \\ & & \overline{\mathcal{O}}_x \end{array} \quad T_{x_0} (\cancel{\times}) = T_{x_0} (\cancel{\nearrow}) \oplus T_{x_0} (\cancel{\searrow})$$

$$\underline{T}_{\infty, \infty'}^x := T_{(F_{\infty}, F_{\infty \infty'})} \overline{\mathcal{O}}_x$$

Rmk. It is still not easy to express $\widetilde{T}_{\infty, \infty'}^x$ as Lie alg.
However, we still know some special cases:

$$\begin{aligned} T_{\infty, x}^x &:= T_{(F_{\infty}, F_{\infty x})} \overline{\mathcal{O}}_x \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{(F_{\infty}, F_{\infty x})} \mathcal{O}_x^u \\ &= T_{Id} G_d/B_{\infty} \cap B_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} \cap b_{\infty x} \\ &= \mathfrak{g}_d - b_{\infty} + b_{\infty}/(b_{\infty} \cap b_{\infty x}) \\ &= n_{\infty} + m_{\infty, x} \end{aligned}$$

($m_{\infty, Id} = 0$. For $s \in \Pi$, $\infty s \in W_d$, we have $m_{\infty, s} = 0$)

Now suppose $\infty s \in W_d$.

$$\begin{aligned} T_{\infty, \infty s}^s &= n_{\infty} \oplus m_{\infty, \infty s} \\ T_{\infty, \infty}^s &:= T_{(F_{\infty}, F_{\infty})} \overline{\mathcal{O}}_s \\ &= T_{(Id, Id)} G_d/B_{\infty} \times P_{\infty, \infty s}/B_{\infty} \\ &= n_{\infty} \oplus m_{\infty s, \infty} \end{aligned}$$

$$\widetilde{T}_{\infty, \infty s}^s = r_{\infty, \infty s} \oplus n_{\infty} \oplus m_{\infty, \infty s}$$

$$\widetilde{T}_{\infty, \infty}^s = r_{\infty, \infty s} \oplus n_{\infty} \oplus m_{\infty s, \infty}$$

§6.3. Euler class.

§6.4. transition matrix, localization formula

Thm. (Localization formula) [Thm 10.2, Cor 5.11.3 in Ginzburg]

Suppose $Y \subset X$ is T -equivariant, $\alpha \in \mathcal{K}_o^T(X)$, X smooth.

$X^T = \{x_1, \dots, x_m\}$, $i_k : \{x_k\} \hookrightarrow X$, then

$$\alpha = \sum_{k=1}^m \varepsilon_k (i_k)_* (i_k)^*(\alpha) \quad \varepsilon_k = (\text{eu}(T_{x_k} X))^{-1} \in \mathcal{R}(T)$$

$$\begin{aligned} \text{e.p. } [Y]^T &= \sum_{k=1}^m \varepsilon_k (i_k)_* ((i_k)^*[Y]^T \cdot 1_{R(T)}) \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^*[Y]^T) (i_k)_* 1_{R(T)} \\ &= \sum_{k=1}^m \varepsilon_k ((i_k)^*[Y]^T) [x_k]^T \\ [X]^T &= \sum_{k=1}^m \varepsilon_k [x_k]^T \end{aligned}$$

Suppose $Y^T = \{x_1, \dots, x_n\}$, $i_k : \{x_k\} \hookrightarrow Y$, then

$$[Y]^T = \sum_{k=1}^n \beta_k [x_k]^T \quad \beta_k = \varepsilon_k \cdot (i_k)^*[Y]^T$$

When Y is sm at x_k ,

$$\begin{cases} \beta_k &= (\text{eu}(T_{x_k} Y))^{-1} \\ (i_k)^*[Y]^T &= \text{eu}(T_{x_k} X) \cdot (\text{eu}(T_{x_k} Y))^{-1} \end{cases}$$

I believe that this theorem corresponds to the coherent trace formula in this article:
<https://www.sciencedirect.com/science/article/pii/0022404994900884>

Ex 1. $X = \widetilde{\text{Rep}}_d(Q)$, $Y = \overline{\mathcal{O}}_x$, $T = T_d \times \mathbb{C}^\times$

$$\begin{aligned} i_\infty : \{(p_0, F_\infty)\} &\hookrightarrow \widetilde{\text{Rep}}_d(Q) \\ \widetilde{\text{Rep}}_d(Q)^T &= \{(p_0, F_\infty) \mid \infty \in W_{\text{ldl}}\} \\ [\widetilde{\text{Rep}}_d(Q)]^T &= \sum_{\infty \in W_{\text{ldl}}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* (i_\infty)^* 1_{K_o^T(\widetilde{\text{Rep}}_d(Q))} \\ &= \sum_{\infty \in W_{\text{ldl}}} \widetilde{\Delta}_\infty^{-1} (i_\infty)_* 1_{R(T)} \\ &= \sum_{\infty \in W_{\text{ldl}}} \widetilde{\Delta}_\infty^{-1} \psi_\infty \end{aligned}$$

$$\begin{aligned} \overline{\mathcal{O}}_x^T &= \{(p_0, F_\infty) \mid \infty \leq x\} \\ [\overline{\mathcal{O}}_x]^T &= \sum_{\infty \leq x} \widetilde{\Delta}_\infty^{-1} \underbrace{((i_\infty)^* [\overline{\mathcal{O}}_x]^T)}_{f_{\infty, x}} \psi_\infty \end{aligned}$$

when $\overline{\mathcal{O}}_x$ is sm at (p_0, F_∞) , $f_{\infty, x} = \widetilde{\Delta}_\infty (T_{(p_0, F_\infty)} \overline{\mathcal{O}}_x)^{-1}$

Ex 2. $X = \text{Rep}_d(Q) \times \mathbb{F}_d \times \mathbb{F}_d$, $Y = \mathbb{Z}_x$, $T = T_d \times \mathbb{C}^\times$

$$\begin{aligned} (\mathbb{Z}_s)^T &= \overline{\mathcal{O}}_s^T \sqcup (\mathbb{Z}_s - \overline{\mathcal{O}}_s)^T \\ &= \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{\text{ldl}}\} \sqcup \{(p_0, F_\infty, F_{\infty s}) \mid \infty \in W_{\text{ldl}}, \infty s \in W_d\} \\ \Rightarrow [\mathbb{Z}_s]^T &= \sum_{\infty \in W_{\text{ldl}}} (\widetilde{\Delta}_{\infty, \infty s})^{-1} \psi_{\infty, \infty s} + \sum_{\substack{\infty \in W_{\text{ldl}} \text{ s.t.} \\ \infty s \in W_d}} (\widetilde{\Delta}_{\infty, \infty s})^{-1} \psi_{\infty, \infty s} \quad \text{in } \mathcal{K}_o^T(X) \\ &\Rightarrow \text{in } \mathcal{K}_o^T(\mathbb{Z}_d) \end{aligned}$$

In general, $[\mathbb{Z}_x]^T = \sum_{\infty, \infty'} \beta_{\infty, \infty'}^x \psi_{\infty, \infty \infty'}$

When \mathbb{Z}_x is sm at $(p_0, F_\infty, F_{\infty \infty'})$, $\beta_{\infty, \infty'}^x = (\widetilde{\Delta}_{\infty, \infty'})^{-1}$.

$$Ex\ 1': X = \widetilde{\text{Rep}_d}(\mathcal{Q}), \quad Y = \overline{\mathbb{O}}_x \quad T = T$$

$$i_{wu}: \{(p_0, F_{wu})\} \hookrightarrow \widetilde{\text{Rep}_d}(\mathcal{Q})$$

$$\widetilde{\text{Rep}_d}(\mathcal{Q})^T = \{(p_0, F_{wu}) \mid w \in W_d\}$$

$$[\widetilde{\text{Rep}_d}(\mathcal{Q})]^T = \sum_{w \in W_d} \widetilde{\Delta}_{wu}^{-1} (i_{wu})_* (i_{wu})^* 1_{K_0^T(\widetilde{\text{Rep}_d}(\mathcal{Q}))}$$

$$= \sum_{w \in W_d} \widetilde{\Delta}_{wu}^{-1} (i_{wu})_* 1_{R(T)}$$

$$= \sum_{w \in W_d} \widetilde{\Delta}_{wu}^{-1} \psi_{wu}$$

$$f'' := f[\widetilde{\text{Rep}_d}(\mathcal{Q})]^T = \sum_{w \in W_d} (wf) \widetilde{\Delta}_{wu}^{-1} \psi_{wu}$$

$$Ex\ 2': X = \text{Rep}_d(\mathcal{Q}) \times \mathbb{F}_d \times \mathbb{F}'_d, \quad Y = Z_x^{u,u'}, \quad T = T_d \times \mathbb{C}^\times$$

$$(Z_s^{u,u'})^T = (\widetilde{\Omega}_s^{u,u'})^T \sqcup (\widetilde{Z}_s^{u,u'} - \widetilde{\Omega}_s^{u,u'})^T$$

$$= \begin{cases} \{(p_0, F_{wu}, F_{wus}) \mid w \in W_d\} \sqcup \{(p_0, F_{wu}, F_{wu}) \mid w \in W_d\} & u = u' \\ \{(p_0, F_{wu}, F_{wus}) \mid w \in W_d\} & u \neq u' \end{cases}$$

$$\Rightarrow [Z_s^{u,u'}]^T = \sum_{w \in W_d} (\widetilde{\Delta}_{wu,wus}^s)^{-1} \psi_{wu,wus} + \sum_{w \in W_d} \sum_{w' \in W_d} (\widetilde{\Delta}_{wu,wu}^s)^{-1} \psi_{wu,wu}$$

§ 6.5. generators. Define e_i and D_i .

$$K_0^{G_d \times C^\times}(\widetilde{\text{Rep}}_d(Q)) \cong K_0^{G_d \times C^\times}(F_d) \cong K_0^{T_d \times C^\times}(\text{pt})$$

$$\begin{array}{ccccc} & \pi_T^G & & & \\ & \downarrow & & & \\ K_0^{T_d \times C^\times}(\widetilde{\text{Rep}}_d(Q)) & \cong & R(T_d \times C)[\widetilde{\text{Rep}}_d(Q)]^{G_d \times C^\times} & \cong & R(T_d \times C)[F_d]^{G_d \times C^\times} \cong R(T_d \times C) = \mathbb{Z}[q^{\pm 1}][x_1^{\pm 1}, \dots, x_{|Id|}^{\pm 1}] \\ & \cong & \downarrow f[\widetilde{\text{Rep}}_d(Q)]^{G_d \times C^\times} & \cong & \downarrow \\ & & \sum_{\infty \in W_{Id}} f \widetilde{\Delta}_{\infty}^{-1} \psi_\infty & & \end{array}$$

Let $e_i := x_i [\widetilde{\text{Rep}}_d(Q)]^{G_d \times C^\times} \in K_0^{G_d \times C^\times}(\widetilde{\text{Rep}}_d(Q))$, then

$$\pi_T^G(e_i) = \sum_{\infty \in W_{Id}} x_{i\infty} (\widetilde{\Delta}_{\infty}^{-1}) \psi_\infty \in K_0^{T_d \times C^\times}(\widetilde{\text{Rep}}_d(Q))$$

$$K_0^{G_d \times C^\times}(\widetilde{\text{Rep}}_d(Q)) \cong \mathbb{Z}[q^{\pm 1}, e_1^{\pm 1}, \dots, e_{|Id|}^{\pm 1}]$$

$K_0^{G_d \times C^\times}(Z_d)$ is a $\mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \dots, e_{|Id|}^{\pm 1}]$ -module.

$$\left[\begin{array}{l} \text{Reason: } K_0^{G_d \times C^\times}(\widetilde{\text{Rep}}_d(Q)) \cong K_0^{G_d \times C^\times}(Z_{Id}) \hookrightarrow K_0^{G_d \times C^\times}(Z_d) \\ K_0^{G_d \times C^\times}(Z_{Id}) \times K_0^{G_d \times C^\times}(Z_d) \xrightarrow{\text{convolution}} K_0^{G_d \times C^\times}(Z_d) \\ \downarrow \\ K_0^{G_d \times C^\times}(Z_{Id}) \times K_0^{G_d \times C^\times}(Z_d) \xrightarrow{\text{convolution}} K_0^{G_d \times C^\times}(Z_d) \end{array} \right]$$

We will mention about the convolution in the next section.

Denote

$$D_i = [Z_{S_i}]^{G_d \times C^\times} \in K_0^{G_d \times C^\times}(Z_d)$$

$$\pi_T^G(D_i) = \sum_{\infty \in W_{Id}} (\widetilde{\Delta}_{\infty, \infty S_i}^{-1}) \psi_{\infty, \infty S_i} + \sum_{\substack{\infty \in W_{Id} \\ \infty S_i^{-1} \in W_d}} (\widetilde{\Delta}_{\infty, \infty}^{-1}) \psi_{\infty, \infty}$$

we will show that, in the case $Z_d \cong F_d \times F_d$,

$$K_0^{G_d \times C^\times}(Z_d) = \langle q^{\pm 1}, e_1^{\pm 1}, e_2^{\pm 1}, \dots, e_{|Id|}^{\pm 1}, D_1, \dots, D_{|Id|-1} \rangle_{\mathbb{Z}[q^{\pm 1}]\text{-alg}} \subseteq \text{End}_{\mathbb{Z}[q^{\pm 1}]\text{-mod}}(\mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \dots, e_{|Id|}^{\pm 1}])$$

Before that, the more interesting question is to compute

$$D_i \in \text{End}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}[q^{\pm 1}][e_1^{\pm 1}, \dots, e_{|Id|}^{\pm 1}]).$$

$$G \rightarrow G \times \mathbb{C}^\times$$

§ 6.5. generators. Define e_i^u and $D_i^{u,u'}$

Now we do the general case.

$$\begin{array}{c} K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) \cong K_0^{G_d \times \mathbb{C}^\times}(F_d) \cong K_0^{T_d \times \mathbb{C}^\times}(\text{pt}) \\ \downarrow \pi_T^G \quad \downarrow \quad \downarrow f[\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \\ K_0^{T_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) \cong \bigoplus_{w \in W_d} R(T_d \times \mathbb{C}^\times)[\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \\ \downarrow \quad \downarrow \quad \downarrow \\ \bigoplus_{w \in W_d} R(T_d \times \mathbb{C}^\times)\psi_{wu} \cong \sum_{w \in W_d} f[\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \psi_{wu} \end{array}$$

$$\text{Let } 1^u = [\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \in K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) \subseteq K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q))$$

$$e_i^u := x_i [\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \in K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) \subseteq K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q))$$

$$e_i = x_i [\widetilde{\text{Rep}}_d(Q)]^{G_d \times \mathbb{C}^\times} \in K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q))$$

|| | | |
| t | | |

$$\text{then } e_i = \sum_{u \in \text{Min}(W_{Id}, W_d)} e_i^u \quad e_i^u = e_i \cdot 1^u$$

$$\begin{aligned} K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) &\cong \mathbb{Z}[q^{\pm 1}][e_i^{u,\pm 1}, \dots, e_{Id}^{u,\pm 1}] \\ K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}_d(Q)) &\cong \bigoplus_{u \in \text{Min}(W_{Id}, W_d)} \mathbb{Z}[q^{\pm 1}][e_i^{u,\pm 1}, \dots, e_{Id}^{u,\pm 1}] \end{aligned} \quad \begin{array}{l} \text{Maybe it is better to write } q^{u,\pm 1} \\ \text{so } q = \sum_u q^u, \quad q^u = q \cdot 1^u \end{array}$$

$$\begin{array}{ll} K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_{d,d}) \text{ is a } & \mathbb{Z}[q^{\pm 1}][e_i^{u,\pm 1}, \dots, e_{Id}^{u,\pm 1}] - \text{module} \\ K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_d) \text{ is a } & \bigoplus_{u \in \text{Min}(W_{Id}, W_d)} \mathbb{Z}[q^{\pm 1}][e_i^{u,\pm 1}, \dots, e_{Id}^{u,\pm 1}] - \text{module.} \end{array} \quad \begin{bmatrix} \mathbb{Z}_{Id}^{u,u} \cong \widetilde{\text{Rep}}_d(Q) \\ \mathbb{Z}_{Id} \cong \widetilde{\text{Rep}}_d(Q) \end{bmatrix}$$

Denote

$$\begin{array}{ll} D_i^{u,u'} = [\mathbb{Z}_{S_i}]^{G_d \times \mathbb{C}^\times} \in K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_{d,d}) \subseteq K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_d) & \text{X | | |} \\ D_i = [\mathbb{Z}_{S_i}]^{G_d \times \mathbb{C}^\times} \in K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_d) & \text{X | | |} \end{array}$$

We will show that

$$\begin{aligned} K_0^{G_d}(\mathbb{Z}_d) &= \langle q^{\pm 1}, e_i^{u,\pm 1}, D_i^{u,u'} \rangle_{\mathbb{Z}\text{-alg}} \subseteq \text{End}_{\mathbb{Z}\text{-mod}}\left(\bigoplus_{u \in \text{Min}(W_{Id}, W_d)} \mathbb{Z}[q^{\pm 1}][e_i^{u,\pm 1}, \dots, e_{Id}^{u,\pm 1}]\right) \\ &= \mathbb{Z}[q^{\pm 1}] \text{-coefficient combinations of} \end{aligned}$$



7. convolution product

§7.1. clean intersection formula

Thm. Suppose X sm G -equiv proj variety,
 $Y_1, Y_2 \subset X$ are G -equiv subvariety,
 $Y = Y_1 \cap Y_2$ $\pi_Y: Y \rightarrow \text{pt}$
 $T = TX|_Y / (TY_1|_Y + TY_2|_Y)$

Assume that

$$TY_1|_Y \wedge TY_2|_Y = TY,$$

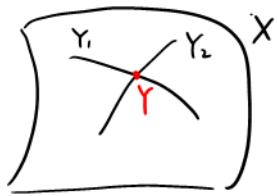
then

$$[Y_1]^G \otimes [Y_2]^G = \text{eu}(\pi_{Y,*}(T)) \cdot [Y]^G$$

Need a reference.

I believe that the first page of this document write the same thing:

https://www.uni-due.de/~adc301m/staff.uni-duisburg-essen.de/Publications_files/excessgw.pdf
 However, there is no proof.



$$\pi_{Y,*}: K^G_0(Y) \rightarrow K^G_0(\text{pt}) = R(G)$$

§7.2. convolution for canonical basis.

Thm $\psi_{\infty'', \infty''} * \psi_{\infty', \infty} = \delta_{\infty'', \infty'} \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty}$

 $\psi_{\infty'', \infty'} \diamond \psi_{\infty} = \delta_{\infty', \infty} \widetilde{\Delta}_{\infty} \psi_{\infty''}$

Proof. It reduce to the case

$$\begin{aligned} \psi_{\infty'', \infty''} * \psi_{\infty', \infty} &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty} \\ \psi_{\infty', \infty} \diamond \psi_{\infty} &= \widetilde{\Delta}_{\infty} \psi_{\infty'} \end{aligned}$$

$$\begin{array}{ccc} [Y_{12}] & & M_{123} \\ \downarrow & \searrow & \downarrow \\ Z_d \hookrightarrow M_{12} & & Z_d \hookrightarrow M_{23} \\ \downarrow & \downarrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & \widetilde{\text{Rep}}_d(Q) & \widetilde{\text{Rep}}_d(Q) \end{array}$$

$$*: K_0^{G_d \times C}(Z_d) \times K_0^{G_d \times C}(Z_d) \rightarrow K_0^{G_d \times C}(Z_d)$$

$$\{(p_0, F_{\infty''}), (p_0, F_{\infty'})\}$$

$$Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} \quad Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\}$$

$$\begin{array}{ccc} Y_{12} \times \widetilde{\text{Rep}}_d(Q) & \subset & M_{123} \\ \{y\} \subset \widetilde{\text{Rep}}_d \times Y_{23} & \subset & \end{array}$$

where

$$\begin{aligned} y &= ((p_0, F_{\infty''}), (p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123} \\ y_{13} &= ((p_0, F_{\infty''}), (p_0, F_{\infty})) \in M_{13} \\ T &= \frac{\widetilde{T}_{\infty''} \oplus \widetilde{T}_{\infty'} \oplus \widetilde{T}_{\infty}}{\widetilde{T}_{\infty''} \oplus \widetilde{T}_{\infty}} = \widetilde{T}_{\infty'} \end{aligned}$$

$$T = T_d \times \mathbb{C}^\times$$

$$\begin{aligned} \psi_{\infty'', \infty'} * \psi_{\infty, \infty} &= [Y_{12}]^T * [Y_{23}]^T \\ &= \pi_{13,*}([Y_{12} \times \widetilde{\text{Rep}}_d(Q)]^T \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^T) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty'} \cdot [y]^T) \\ &= \widetilde{\Delta}_{\infty'} \cdot [y_{13}]^T \\ &= \widetilde{\Delta}_{\infty'} \psi_{\infty'', \infty} \end{aligned}$$

$$\begin{array}{ccc} [Y_{12}] & & M_{123} \\ \downarrow & \searrow & \downarrow \\ Z_d \hookrightarrow M_{12} & & \overset{\sim}{\text{Rep}}_d(Q) = M_{23} \\ \downarrow & \downarrow & \downarrow \\ \widetilde{\text{Rep}}_d(\omega) & \widetilde{\text{Rep}}_d(Q) & pt \end{array}$$

$$\diamond: K_0^{G_d \times C}(Z_d) \times K_0^{G_d \times C}(\widetilde{\text{Rep}}_d(Q)) \rightarrow K_0^{G_d \times C}(\widetilde{\text{Rep}}_d(Q))$$

$$\begin{array}{ccc} Y_{12} = \{(p_0, F_{\infty''}, F_{\infty'})\} & & Y_{23} = \{(p_0, F_{\infty'}, F_{\infty})\} \\ \{y\} \subset Y_{12} \times pt & & \{y\} \subset M_{123} \\ \subset \widetilde{\text{Rep}}_d \times Y_{23} & & \subset M_{123} \end{array}$$

where

$$\begin{aligned} y &= ((p_0, F_{\infty'}), (p_0, F_{\infty})) \in M_{123} \\ y_{13} &= (p_0, F_{\infty'}) \in M_{13} \\ T &= \frac{\widetilde{T}_{\infty'} \oplus \widetilde{T}_{\infty} \oplus 0}{\widetilde{T}_{\infty'} \oplus 0} = \widetilde{T}_{\infty} \end{aligned}$$

$$T = T_d \times \mathbb{C}^\times$$

$$\begin{aligned} \psi_{\infty', \infty} * \psi_{\infty} &= [Y_{12}]^T * [Y_{23}]^T \\ &= \pi_{13,*}([Y_{12} \times pt]^T \otimes [\widetilde{\text{Rep}}_d(Q) \times Y_{23}]^T) \\ &= \pi_{13,*}(\widetilde{\Delta}_{\infty} \cdot [y_{13}]^T) \\ &= \widetilde{\Delta}_{\infty} \cdot [y_{13}]^T \\ &= \widetilde{\Delta}_{\infty} \psi_{\infty'} \end{aligned}$$

§ 7.3. expression of D_k .

In this subsection, $W_{Id} = W_d$, $\widetilde{Rep}_d(Q) = F_d$. $Z_d = F_d \times F_d$.

In the example, $|Id| = 3$, $i = 1$

The convolution is compatible with forget map π_T^G .

$$\begin{array}{ccc} K_0^{G_d \times C^\times}(Z_d) \times K_0^{G_d \times C^\times}(\widetilde{Rep}_d(Q)) & \longrightarrow & K_0^{G_d \times C^\times}(\widetilde{Rep}_d(Q)) \\ \downarrow \pi_B^G & \downarrow \pi_B^G & \downarrow \pi_B^G \\ K_0^{T_d \times C^\times}(Z_d) \times K_0^{T_d \times C^\times}(\widetilde{Rep}_d(Q)) & \longrightarrow & K_0^{T_d \times C^\times}(\widetilde{Rep}_d(Q)) \end{array}$$

So we do our computation in $K_0^{T_d \times C^\times}$. (View $K_0^{G_d \times C^\times}$ as subalg of $K_0^{T_d \times C^\times}$)

Recall that

$$D_i = \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s})^{-1} \psi_{\infty, \omega s} + \sum_{\substack{\omega \in W_{Id} \\ \omega s \omega^{-1} \in W_d}} (\widetilde{\Delta}_{\infty, \omega}^{-1})^{-1} \psi_{\infty, \omega}$$

$\omega s \omega^{-1} \in W_d \leftarrow \text{automatically satisfied}$

$$f = \sum_{\omega \in W_{Id}} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \quad \text{e.p. } e_i = \sum_{\omega \in W_{Id}} x_{\omega(i)} \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty}$$

Therefore,

$$\begin{aligned} D_i \diamond f &= \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s})^{-1} \psi_{\infty, \omega s} \sum_{\omega \in W_{Id}} (\omega s f) \widetilde{\Delta}_{\infty s}^{-1} \psi_{\infty s} + \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega}^{-1})^{-1} \psi_{\infty, \omega} \sum_{\omega \in W_{Id}} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega s})^{-1} (\omega s f) \widetilde{\Delta}_{\infty s}^{-1} \widetilde{\Delta}_{\infty s} \psi_{\infty} + \sum_{\omega \in W_{Id}} (\widetilde{\Delta}_{\infty, \omega}^{-1})^{-1} (\omega f) \widetilde{\Delta}_{\infty}^{-1} \widetilde{\Delta}_{\infty} \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} \left[(\widetilde{\Delta}_{\infty, \omega s}^{-1}) \omega s f + (\widetilde{\Delta}_{\infty, \omega}^{-1})^{-1} (\omega f) \right] \psi_{\infty} \\ &= \sum_{\omega \in W_{Id}} \omega \left[\left(\frac{sf}{\widetilde{\Delta}_{Id,s}^s} + \frac{f}{\widetilde{\Delta}_{Id,Id}^s} \right) \cdot \widetilde{\Delta}_{Id} \right] \widetilde{\Delta}_{\infty}^{-1} \psi_{\infty} \\ \therefore D_i f &= \left(\frac{sf}{\widetilde{\Delta}_{Id,s}^s} + \frac{f}{\widetilde{\Delta}_{Id,Id}^s} \right) \cdot \widetilde{\Delta}_{Id} \end{aligned}$$

$$\begin{aligned} \text{In our case, } \widetilde{T}_{Id} &= T_{Id} = n_{Id}^- & = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} & = A \\ \widetilde{T}_{Id,s}^s &= T_{Id,s}^s = n_{Id}^- \oplus m_{Id,s} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & = A + \frac{1}{3} \\ \widetilde{T}_{Id,Id}^s &= T_{Id,Id}^s = n_{Id}^- \oplus m_{s,Id} = \begin{pmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & = A + B \end{aligned}$$

$$\begin{aligned} \text{where } A &= \sum_{j>k} \frac{e_j}{e_k} = \frac{e_2}{e_1} + \frac{e_3}{e_1} + \frac{e_3}{e_2} \\ B &= \frac{e_{i+1}}{e_i} = \frac{e_3}{e_1} \end{aligned}$$

$$\begin{array}{ccc} K_0^{T_d}(Z_d) & \xrightarrow{\hspace{2cm}} & H_{T_d}^*(Z_d) \\ eu(A) & (1 - \frac{e_1}{e_2})(1 - \frac{e_1}{e_3})(1 - \frac{e_2}{e_3}) & \cdot (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3) \\ eu(B) & 1 - \frac{e_3}{e_1} & \lambda_1 - \lambda_2 \\ eu(\frac{1}{B}) & 1 - \frac{e_2}{e_1} & \lambda_2 - \lambda_1 \end{array}$$

$$\begin{aligned}
D_i f &= \left(\frac{sf}{eu(A + \frac{1}{B})} + \frac{f}{eu(A+B)} \right) eu(A) \quad f \in K_0^{G_d \times \mathbb{C}^\times}(\widehat{\text{Rep}}(\mathcal{Q})) \\
&= \frac{sf}{eu(\frac{1}{B})} + \frac{f}{eu(B)} \\
&= \frac{sf}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \\
&= \frac{e_{i+1}f - e_i sf}{e_{i+1} - e_i}
\end{aligned}$$

$$D_i \left(\frac{e_i}{e_{i+1}} f \right) = - \frac{e_i f - e_{i+1} sf}{e_i - e_{i+1}}$$

$$\begin{aligned}
D_i fg &= \frac{sf \cdot g}{eu(\frac{1}{B})} + \frac{sf \cdot g}{eu(B)} + \frac{f \cdot g}{eu(B)} - \frac{sf \cdot g}{eu(B)} \quad \text{Here, } f \in K_0^{G_d \times \mathbb{C}^\times}(\mathbb{Z}_{Id}), \quad g \in K_0^{G_d \times \mathbb{C}^\times}(\widetilde{\text{Rep}}(\mathcal{Q})) \\
&= sf \cdot D_i g + \frac{f - sf}{eu(B)} g \\
\Rightarrow D_i f &= sf D_i + \frac{f - sf}{1 - \frac{e_i}{e_{i+1}}} \quad \left(\frac{e_i}{e_{i+1}} D_i \right) f = sf \left(\frac{e_i}{e_{i+1}} D_i \right) - \frac{f - sf}{1 - \frac{e_i}{e_{i+1}}} \\
D_i \left(\frac{e_i}{e_{i+1}} f g \right) &= sf D_i \left(\frac{e_i}{e_{i+1}} g \right) - \frac{f - sf}{1 - \frac{e_i}{e_{i+1}}} g
\end{aligned}$$

In the case of equivariant cohomology, the computation is similar:

$$\begin{aligned}
\partial_i f &= \left(\frac{sf}{eu(\widehat{\Delta}_{Id,S}^s)} + \frac{f}{eu(\widehat{\Delta}_{Id,Id}^s)} \right) \cdot eu(\widehat{\Delta}_{Id}) \quad f \in K_0^{G_d}(\widehat{\text{Rep}}(\mathcal{Q})) \\
&= \frac{sf}{eu(\frac{1}{B})} + \frac{f}{eu(B)} \\
&= \frac{sf}{\lambda_{i+1} - \lambda_i} + \frac{f}{\lambda_i - \lambda_{i+1}} \\
\partial_i f &= sf \partial_i + \frac{f - sf}{\lambda_i - \lambda_{i+1}} \quad f \in K_0^{G_d}(\mathbb{Z}_{Id})
\end{aligned}$$

§ 7.3. expression of D_k .

The convolution is compatible with forget map π_T^G .

$$K_0^{G_d \times C^\times}(\mathbb{Z}_d) \times K_0^{G_d \times C^\times}(\widetilde{\text{Rep}_d(Q)}) \longrightarrow K_0^{G_d \times C^\times}(\widetilde{\text{Rep}_d(Q)})$$

$$\downarrow \pi_B^G \qquad \downarrow \pi_B^G \qquad \downarrow \pi_B^G$$

$$K_0^{T_d \times C^\times}(\mathbb{Z}_d) \times K_0^{T_d \times C^\times}(\widetilde{\text{Rep}_d(Q)}) \longrightarrow K_0^{T_d \times C^\times}(\widetilde{\text{Rep}_d(Q)})$$

So we do our computation in $K_0^{T_d \times C^\times}$. (View $K_0^{G_d \times C^\times}$ as subalg of $K_0^{T_d \times C^\times}$)

Recall that

$$D_i^{u,u} = \sum_{w \in W_d} (\widetilde{\prod}_{wu,wus}^s)^{-1} \psi_{wu,wus} + \delta_{u=u'} \sum_{w \in W_d} (\widetilde{\prod}_{wu,wu}^s)^{-1} \psi_{wu,wu}$$

$$f^u = \sum_{w \in W_d} (wuf) \widetilde{\prod}_{wu}^{-1} \psi_{wu} \quad wu' = us$$

$$\begin{aligned} D_i^{u,u'} f^{u'} &= \sum_{w \in W_d} (\widetilde{\prod}_{wu,wus}^s)^{-1} \psi_{wu,wus} \sum_{w \in W_d} (wusf) \widetilde{\prod}_{wus}^{-1} \psi_{wus} \\ &\quad + \delta_{u=u'} \sum_{w \in W_d} (\widetilde{\prod}_{wu,wu}^s)^{-1} \psi_{wu,wu} \sum_{w \in W_d} (wu'f) \widetilde{\prod}_{wu'}^{-1} \psi_{wu'} \\ &= \sum_{w \in W_d} (\widetilde{\prod}_{wu,wus}^s)^{-1} (wusf) \psi_{wu} \\ &\quad + \delta_{u=u'} \sum_{w \in W_d} (\widetilde{\prod}_{wu,wu}^s)^{-1} (wuf) \psi_{wu} \end{aligned}$$

When $u=u'$,

$$D_i^{u,u} f^u = \sum_{w \in W_d} \left(\frac{wusf}{\widetilde{\prod}_{wu,wus}^s} + \frac{wuf}{\widetilde{\prod}_{wu,wu}^s} \right) \widetilde{\prod}_{wu} (\widetilde{\prod}_{wu})^{-1} \psi_{wu}$$

$$= \sum_{w \in W_d} w \left(\left(\frac{(sf)^u}{\widetilde{\prod}_{u,us}^s} + \frac{f^u}{\widetilde{\prod}_{u,u}^s} \right) \widetilde{\prod}_u \right) (\widetilde{\prod}_{wu})^{-1} \psi_{wu}$$

$$\Rightarrow D_i^{u,u} f^u = \left(\frac{(sf)^u}{\widetilde{\prod}_{u,us}^s} + \frac{f^u}{\widetilde{\prod}_{u,u}^s} \right) \widetilde{\prod}_u = g^u$$

When $u \neq u'$, $u'=us$,

$$D_i^{u,u'} f^{u'} = \sum_{w \in W_d} \frac{wusf}{\widetilde{\prod}_{wu,wu}^s} \widetilde{\prod}_{wu} (\widetilde{\prod}_{wu})^{-1} \psi_{wu}$$

$$= \sum_{w \in W_d} w \left(\frac{(sf)^u}{\widetilde{\prod}_{u,us}^s} \widetilde{\prod}_u \right) (\widetilde{\prod}_{wu})^{-1} \psi_{wu}$$

$$\Rightarrow D_i^{u,u'} f^{u'} = \frac{sf^u}{\widetilde{\prod}_{u,us}^s} \widetilde{\prod}_u = g^u$$

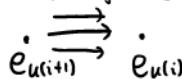
In our case,

$$\begin{aligned}
 \widetilde{T}_{Id} &= r_{Id} \oplus T_{Id} & \widetilde{T}_u &= r_u \oplus T_u \\
 \widetilde{T}_{Id,s}^s &= r_{Id} \oplus n_{Id}^- & \widetilde{T}_{u,us}^s &= r_u \oplus n_u^- \\
 \widetilde{T}_{Id,s} &= r_{Id,s} \oplus T_{Id,s}^s & \widetilde{T}_{u,us} &= r_{u,us} \oplus T_{u,us}^s \\
 &= r_{Id,s} \oplus n_{Id}^- \oplus m_{Id,s} & &= r_{u,us} \oplus n_u^- \oplus m_{u,us} \\
 \widetilde{T}_{Id,Id}^s &= r_{Id,s} \oplus T_{Id,Id}^s & \widetilde{T}_{Id,Id}^s &= r_{u,us} \oplus T_{u,u}^s \\
 &= r_{Id,s} \oplus n_{Id}^- \oplus m_{s,Id} & &= r_{u,us} \oplus n_u^- \oplus m_{u,us}
 \end{aligned}$$

$$D_i^{u,u} f^u = \left(\frac{sf}{eu(m_{u,us})} + \frac{f}{eu(m_{us,u})} \right) eu(\partial_{u,us})$$

$$D_i^{u,u'} f^{u'} = \frac{sf}{eu(m_{u,us})} eu(\partial_{u,us})$$

Name	Lie alg	$eu \in \mathcal{K}_{\text{Lie}}^{T_d}(Z_d)$	$eu \in \mathcal{U}_{T_d}^*(Z_d)$	
$m_{u,us}$	$\frac{e_{u(i)}}{e_{u(i+1)}}$	$1 - \frac{e_{u(i+1)}}{e_{u(i)}}$	$\lambda_{u,us} - \lambda_{u(i)}$	$u = u'$
	0	1	1	$u \neq u'$
$m_{us,u}$	$\frac{e_{u(i+1)}}{e_{u(i)}}$	$1 - \frac{e_{u(i)}}{e_{u(i+1)}}$	$\lambda_{u(i)} - \lambda_{u(i+1)}$	$u = u'$
	0	1	1	$u \neq u'$
$\partial_{u,us}$	$k \frac{e_{u(i)}^q}{e_{u(i+1)}}$	$\left(1 - \frac{e_{u(i+1)}}{e_{u(i)}^q}\right)^k$	$(\lambda_{u(i+1)} - \lambda_{u(i)} - t)^k$	

k-mony arrows


E.g. $\bullet \rightarrow \bullet$ u: ~~$\times \times$~~

Substitute everything, we get

$$\textcircled{1} \quad \partial_i^{u,u} f^u = \frac{(s_i f)^u}{\lambda_{u(i+1)} - \lambda_{u(i)}} + \frac{f^u}{\lambda_{u(i)} - \lambda_{u(i+1)}} = \left(\frac{f - s_i f}{\lambda_i - \lambda_{i+1}} \right)^u$$

s_2
u ~~$\times \times$~~

$$\textcircled{2} \quad \partial_i^{u,u'} f^{u'} = (s_i f)^u (\lambda_{u(i+1)} - \lambda_{u(i)} - t) = (s_i f (\lambda_{i+1} - \lambda_i - t))^u$$

s_1
u ~~$\times \times$~~

$$\textcircled{3} \quad \partial_i^{u,u'} f^{u'} = (s_i f)^u$$

s_3
u ~~$\times \times$~~

$$\textcircled{1} \quad D_i^{u,u} f^u = \frac{(s_i f)^u}{1 - \frac{e_{u(i+1)}}{e_{u(i)}}} + \frac{f^u}{1 - \frac{e_{u(i)}}{e_{u(i+1)}}} = \left(\frac{s_i f}{1 - \frac{e_{i+1}}{e_i}} + \frac{f}{1 - \frac{e_i}{e_{i+1}}} \right)^u$$

s_2
u ~~$\times \times$~~

$$\textcircled{2} \quad D_i^{u,u'} f^{u'} = (s_i f)^u \left(1 - \frac{e_{u(i+1)}}{e_{u(i)} q} \right) = (s_i f \left(1 - \frac{e_{i+1}}{e_i q} \right))^u$$

s_1
u ~~$\times \times$~~

$$\textcircled{3} \quad D_i^{u,u'} f^{u'} = (s_i f)^u$$

s_3
u ~~$\times \times$~~