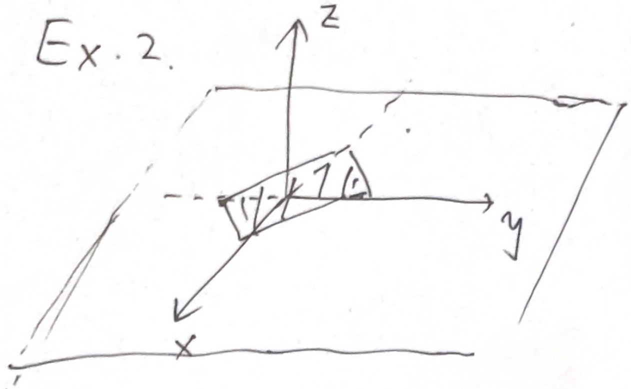
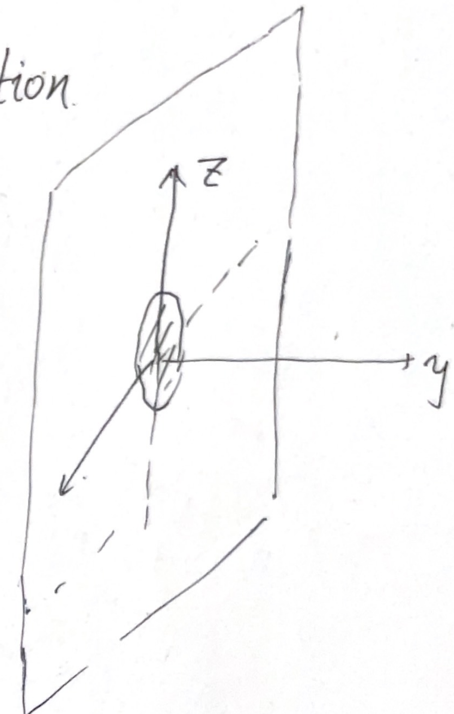
Tutorial 8 & Ex 7.

Today we do more complicated integrations.

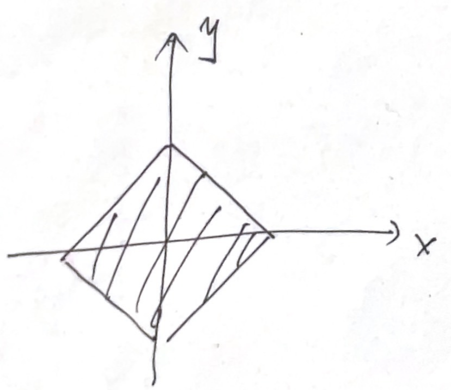
1. Measure & simple integration

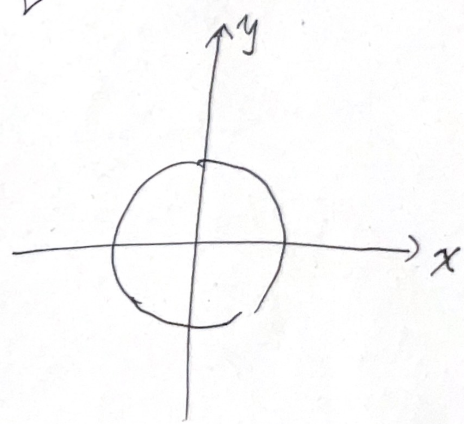
Ex. 2.





has (Jordan) measure 0.





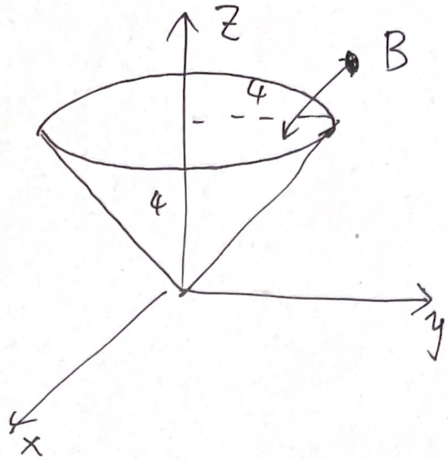
measure 2

measure 0

Rmk.

	Jordan measure	$\supsetneq$ Lebesgue measure
$[0,1]$	1	1
$\mathbb{Q} \cap [0,1]$	1	0
$\mathbb{Z}$	$+\infty$	0

Ex. 1. (b)



$$Vol(B) = \frac{1}{3}(\pi \cdot 4^2) \cdot 4 = \frac{64}{3}\pi.$$

Rigorously, $Vol(B) = \int_B 1 \, dxdydz = \cdots$

2. Indefinite integrals.

Method 1. By change of variables.

$$\int \frac{1}{y^2+1} dy \overset{y=\tan\theta}{=\!=} \int \frac{1}{\frac{1}{(\cos\theta)^2}} \frac{1}{(\cos\theta)^2} d\theta$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arctan y + C$$

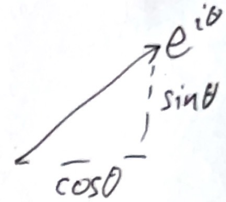
Method 2. By integration by parts.

$$\int \ln x \, dx = x \ln x - \int x \, d(\ln x)$$

$$= x \ln x - \int 1 \, dx$$

$$= x \ln x - x$$

Ex. Compute $\int x \ln x \, dx$.



Method 3. By $e^{i\theta} = \cos\theta + i\sin\theta$

$$\Rightarrow \begin{cases} \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} \\ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} \end{cases}$$

(Either: Integration by parts)

e.g. For $y > 0$, compute

$$\int \sin x \, e^{-xy} \, dx$$

$$= \int \frac{e^{ix} - e^{-ix}}{2i} e^{-xy} \, dx$$

$$= \frac{1}{2i} \int \frac{e^{ix} - e^{-ix}}{2i} e^{-xy} \, dx$$

$$= \frac{1}{2i} \left(\frac{e^{(i-y)x}}{i-y} + \frac{e^{-(i+y)x}}{i+y} \right) + C$$

$$= \frac{1}{2i} e^{-xy} \cdot \frac{e^{ix}(i+y) + e^{-ix}(i-y)}{-1-y^2} + C$$

$$= -\frac{e^{-xy}}{y^2+1} \left(\frac{e^{ix} + e^{-ix}}{2} + \frac{e^{ix} - e^{-ix}}{2i} y \right) + C$$

$$= -\frac{e^{-xy}}{y^2+1} (\cos x + y \sin x) + C$$

Method 4. By partial fraction decomposition.

e.g. $$\int \frac{1}{y^2+1}\,dy = \int \frac{1}{(y+i)(y-i)}\,dy$$

$$= \frac{1}{2i}\left(\int \frac{1}{y-i}\,dy - \int \frac{1}{y+i}\,dy\right)$$

$$= \frac{1}{2i}\ln\frac{y-i}{y+i} + C$$

$$\overset{(\Delta)}{=} \arctan y + C$$

(Δ): Let $x = \arctan y$, then

$$y = \tan x = \frac{\sin x}{\cos x}$$

$$= \frac{\frac{e^{ix}-e^{-ix}}{2i}}{\frac{e^{ix}+e^{-ix}}{2}}$$

$$\Rightarrow \quad iy\left(e^{ix}+e^{-ix}\right) = e^{ix} - e^{-ix}$$

$$\Rightarrow \quad (iy+1)\,e^{-ix} = (1-iy)\,e^{ix}$$

$$\Rightarrow \quad e^{2ix} = \frac{1+iy}{1-iy} = \frac{y-i}{y+i}$$

$$\Rightarrow \quad x = \frac{1}{2i}\ln\frac{y-i}{y+i}$$

In the exam: since $(\arctan y)' = \frac{1}{y^2+1}$,

$$\int \frac{1}{y^2+1}\,dy = \arctan y + C.$$

Ex. 4

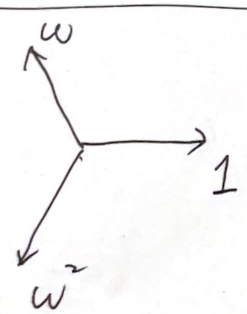
$$\int \frac{1}{y^3+1}\,dy = \int \frac{1}{(y+1)(y+\omega)(y+\omega^2)}\,dy$$

$$= \frac{1}{3}\int \frac{1}{y+1} + \frac{\omega}{y+\omega} + \frac{\omega^2}{y+\omega^2}\,dy$$

$$= \frac{1}{3}\left(\ln(y+1) + \omega\ln(y+\omega) + \omega^2\ln(y+\omega^2)\right) + C$$

$$= \frac{1}{3}\left(\ln(y+1) - \frac{1}{2}\ln(y+\omega)(y+\omega^2) + \frac{\sqrt{3}}{2}i\ln\frac{y+\omega}{y+\omega^2}\right) + C$$

$$= \frac{1}{3}\left(\ln(y+1) - \frac{1}{2}\ln(y^2-y+1) + \sqrt{3}\arctan\left(\frac{2y-1}{\sqrt{3}}\right)\right) + C$$



$$\omega^3 = 1$$

$$1 + \omega + \omega^2 = 0$$

$$\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

Ex. Shows that $\cancel{(-1\le x\le 1)}$

$$\arcsin x = \frac{1}{i}\ln\left(ix + \sqrt{1-x^2}\right) \qquad -1 \le x \le 1$$

$$\arccos x = \frac{1}{i}\ln\left(x + \sqrt{x^2-1}\right) \qquad -1 \le x \le 1$$

$$\arctan x = \frac{i}{2}\ln\frac{x+i}{x-i} \qquad x \in \mathbb{R}$$

Task 2 (a) For

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2} \qquad (x,y) \in (0,1] \times (0,1].$$

compute $\int_0^1 \int_0^1 f(x,y)\,dx\,dy$ and $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$

A: Fix y.

$$\int \frac{x^2-y^2}{(x^2+y^2)^2}\,dx$$

$$= \int \left(\frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} \right) dx$$

$$= \int \frac{x}{(x^2+y^2)^2}\,dx^2 - \int \frac{1}{x^2+y^2}\,dx$$

$$= -\int x\, d\left(\frac{1}{x^2+y^2}\right) - \int \frac{1}{x^2+y^2}\,dx$$

$$= -\frac{x}{x^2+y^2} + \int \frac{1}{x^2+y^2}\,dx - \int \frac{1}{x^2+y^2}\,dx$$

$$= -\frac{x}{x^2+y^2} + C$$

$$\Rightarrow \int_0^1 f(x,y)\,dx = -\frac{x}{x^2+y^2}\Big|_{x=0}^{1} = -\frac{1}{1+y^2}$$

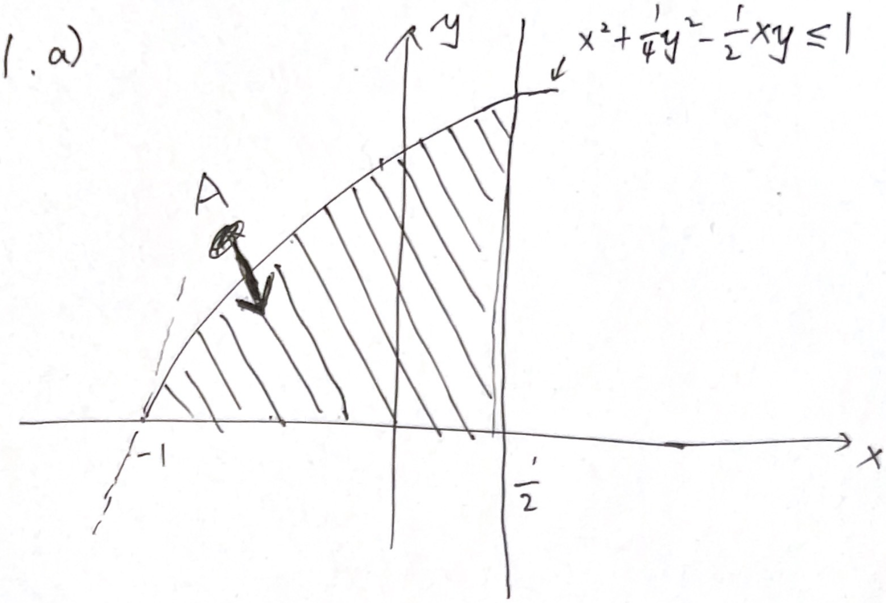
$$\Rightarrow \int_0^1 \int_0^1 f(x,y)\,dx\,dy = \int_0^1 -\frac{1}{1+y^2}\,dy$$

$$= -\arctan y \Big|_0^1$$

$$= -\arctan 1.$$

Similarly, $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \arctan 1.$

Ex. 1. a)

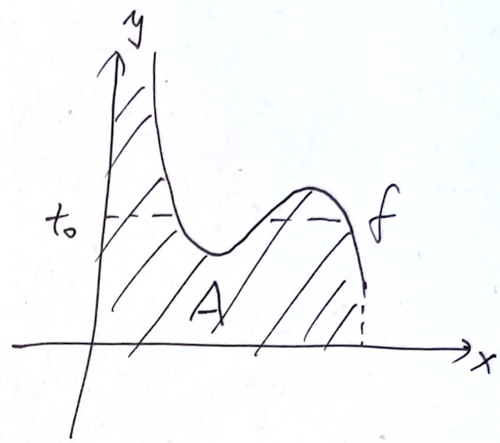


$$
\begin{aligned}
\operatorname{area}(A) &= \int_A 1 \, dx\,dy \\
&= \int_{-1}^{\frac{1}{2}} \left(\int_0^{\sqrt{4-3x^2}+x} 1 \, dy \right) dx \\
&= \int_{-1}^{\frac{1}{2}} \left(\sqrt{4-3x^2} + x \right) dx \\
&= \left(\frac{1}{2} x \sqrt{4-3x^2} + \frac{2}{\sqrt{3}} \arcsin\left(\frac{\sqrt{3}}{2}x\right) + \frac{1}{2}x^2 \right) \Bigg|_{-1}^{\frac{1}{2}} \\
&= \cdots
\end{aligned}
$$

Hint: $\int \sqrt{1-x^2}\, dx \overset{x=\sin\theta}{=\!=} \int \cos^2\theta \, d\theta = \cdots$

3. Lebesgue integration

Suppose $\Omega \subseteq \mathbb{R}^n$.



Def. (Lebesgue integration $\int_\Omega f dx$)

For $f \in L^+(\Omega)$, define
positive measurable function

$$\int_\Omega f dx = \int_A 1 d\sigma$$

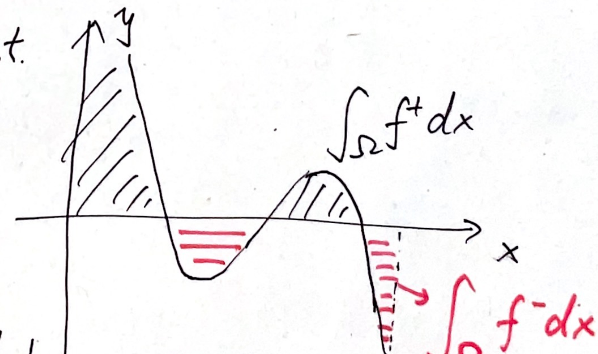
$$:= \int_0^{+\infty} m(\{x \in \Omega \mid f(x) \geq t\}) dt$$

Improper Riemann integral, but in fact ~~don't~~ can be defined directly.

When $\int_\Omega f dx < +\infty$, we say that $f \in L^1(\Omega)$
Lebesgue integrable

For $f \in L(\Omega)$, let $f^+, f^- \in L^+(\Omega)$ s.t.

$$\begin{cases} f = f^+ - f^- \\ |f| = f^+ + f^- \end{cases}$$



When $\int_\Omega |f| dx < +\infty$, we say that

$$f \in L^1(\Omega), \text{ and } \int_\Omega f dx := \int_\Omega f^+ dx - \int_\Omega f^- dx$$

Rmk. For $f \in C(\Omega)$,

f is Lebesgue-integrable $\Leftrightarrow$ f is absolute generalized Riemann-integrable

$\Leftrightarrow$ $\int_\Omega |f| dx$ converges

Riemann / Lebesgue

Thm (Fubini) If $f \in L^1(\mathbb{R}^{m+n})$ $\quad \mathbb{R}^m \rightsquigarrow \Omega \quad \mathbb{R}^n \rightsquigarrow \Omega'$

$$\Rightarrow \int_{\mathbb{R}^{n+m}} f(x,y)\,dx\,dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\,dy \right) dx$$

e.g. $$\int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^n} f(x,y)\,dx \right) dy = \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^m} f(x,y)\,dy \right) dx$$

E.g. (Task 2(b))

$$f(x,y) := \frac{x^2 - y^2}{(x^2+y^2)^2} \notin L^1((0,1]^2), \text{ since}$$

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy \neq \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx$$

Q: Show it directly?

Thm. (Lebesgue's dominated convergence theorem)

Let $f_n \in L(\Omega)$, $g \in L^1(\Omega)$, $|f_n(x)| \leq g(x) \quad \forall x \in \Omega$.

If $\lim_{n \to \infty} f_n(x) = f(x)$ a.e. for $x \in \Omega$, then

almost everywhere

$$\lim_{n\to\infty} \int_\Omega f_n(x)\,dx = \int_\Omega \lim_{n\to\infty} f_n(x)\,dx.$$

Task 1. Show that

$$\int_0^{+\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Previous exercise: ① draw $\frac{\sin x}{x}$

② show that $\int_0^{+\infty} \frac{\sin x}{x}\, dx$ converges conditionally.

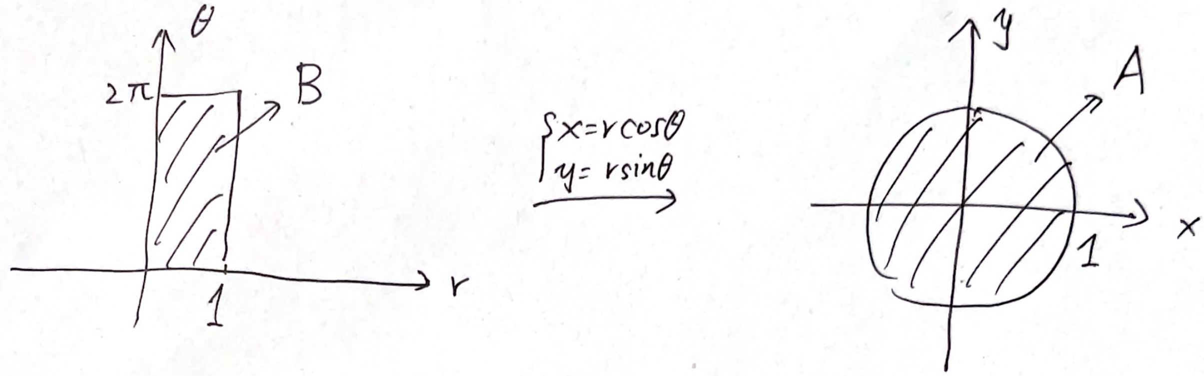
A: We compute

$$\begin{aligned}
\lim_{b \to +\infty} \int_0^b \frac{\sin x}{x}\, dx &= \lim_{b \to +\infty} \int_0^b \int_0^{+\infty} \sin x\, e^{-xy}\, dy\, dx \\
&\overset{\text{Fubini}}{=} \lim_{b \to +\infty} \int_0^{+\infty} \int_0^b \sin x\, e^{-xy}\, dx\, dy \\
&\overset{\substack{\text{Indefinite} \\ \text{integral}}}{=} \lim_{b \to +\infty} \int_0^{+\infty} -\frac{e^{-xy}}{y^2+1} (\cos x + y \sin x) \Big|_{x=0}^{b}\, dy \\
&\overset{\substack{\text{dominated} \\ \text{convergence}}}{=} \int_0^{+\infty} -\frac{e^{-xy}}{y^2+1} (\cos x + y \sin x) \Big|_{x=0}^{+\infty}\, dy \\
&= \int_0^{+\infty} \frac{1}{y^2+1}\, dy \\
&\overset{\substack{\text{Indefinite} \\ \text{integral}}}{=} \arctan y \Big|_0^{+\infty} \\
&= \frac{\pi}{2}
\end{aligned}$$

□

Task 3. For $\alpha \in \mathbb{R}$, compute

$$\int_A |x|^\alpha \, d\mu_2(x)$$



A:
$$\int_A |x|^\alpha d\mu_2(x) = \int_A (x^2+y^2)^{\frac{\alpha}{2}} dx\,dy$$

$$\underset{y = r\sin\theta}{\overset{x = r\cos\theta}{=\!=}} \int_B r^\alpha \cdot r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^1 r^{\alpha+1} dr$$

$$= \begin{cases} 2\pi \ln r \Big|_0^1 & \alpha = -2 \\ 2\pi \dfrac{r^{\alpha+2}}{\alpha+2} \Big|_0^1 & \alpha \neq -2 \end{cases}$$

$$= \begin{cases} +\infty & \alpha \leq -2 \\ \dfrac{2\pi}{\alpha+2} & \alpha > -2 \end{cases} \qquad \square$$