

Eine Woche, ein Beispiel

1.26 Numerical Chern class

Ref:
wiki: Chern class

Nearly all the results are sourced from Wikipedia. I made this document because I tend to mix up the Chern class and the Chern character.
We omit E in notation.

$$\begin{aligned} c(E) &= 1 + c_1 + \dots + c_r \in H^*(X; \mathbb{C}) \\ &= \prod_{i=1}^r (1 + a_i) \quad a_i(E) \in H^*(F(E); \mathbb{C}) \end{aligned}$$

$$\begin{aligned} c_t(E) &= 1 + c_1 t + \dots + c_r t^r \in H^*(X; \mathbb{C})[t] \\ &= \prod_{i=1}^r (1 + a_i t) \end{aligned}$$

$$\begin{aligned} ch(E) &= e^{a_1} + \dots + e^{a_r} \in H^*(X; \mathbb{C}) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} (a_1^k + \dots + a_r^k) \\ &= \sum_{k=0}^{+\infty} \frac{1}{k!} s_k(c_1, \dots, c_r) \\ &= r + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_2c_1 + 3c_3) \\ &\quad + \frac{1}{24}(c_1^4 - 4c_2c_1^2 + 4c_3c_1 + 2c_2^2 - 4c_4) + \dots \end{aligned}$$

$$\begin{aligned} td(E) &= \prod_{i=1}^r \frac{a_i}{1 - e^{-a_i}} \in H^*(X; \mathbb{C}) \\ &= \prod_{i=1}^r \left(1 + \frac{a_i}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} a_i^{2k} \right) \\ &= 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2 \\ &\quad + \frac{1}{720}(-c_1^4 + 4c_1^2c_2 + c_1c_3 + 3c_2^2 - c_4) + \dots \end{aligned}$$

$$\begin{aligned} s(E) &= \prod_{i=1}^r \frac{1}{1 + a_i} \in H^*(X; \mathbb{C}) \\ &\hat{=} 1 + s_1 + \dots + s_n \\ &= 1 - c_1 + (-c_2 + c_1^2) + (-c_3 + 2c_1c_2 - c_1^3) \\ &\quad + (-c_4 + c_2^2 + 2c_1c_3 - 3c_1^2c_2 + c_1^4) + \dots \end{aligned}$$

$$c(E \oplus E') = c(E) \cup c(E')$$

$$c_t(E \oplus E') = c_t(E) \cup c_t(E')$$

$$ch(E \oplus E') = ch(E) + ch(E')$$

$$td(E \oplus E') = td(E) \cup td(E')$$

$$s(E \oplus E') = s(E) \cup s(E')$$

$$ch(E \otimes E') = ch(E) \cdot ch(E')$$

E.g. $X = \mathbb{P}^1$ $E = \mathcal{O}(a)$, then $c_1(E) = aH$, and $H \in H^2(\mathbb{P}^1; \mathbb{C})$ as the generator

$$\begin{aligned} c(E) &= 1 + aH \\ c_t(E) &= 1 + aHt \\ ch(E) &= 1 + aH \\ td(E) &= 1 + \frac{1}{2}aH \\ s(E) &= 1 - aH \end{aligned}$$

For $E = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$, one gets

$$\begin{aligned} c(E) &= (1 + a_1H) \cup (1 + a_2H) &= 1 + (a_1 + a_2)H \\ c_t(E) &= (1 + a_1Ht) (1 + a_2Ht) &= 1 + (a_1 + a_2)Ht \\ ch(E) &= 1 + a_1H + 1 + a_2H &= 2 + (a_1 + a_2)H \\ td(E) &= (1 + \frac{1}{2}a_1H) \cup (1 + \frac{1}{2}a_2H) &= 1 + \frac{1}{2}(a_1 + a_2)H \\ s(E) &= (1 - a_1H) \cup (1 - a_2H) &= 1 - (a_1 + a_2)H \end{aligned}$$

Therefore, these characteristic classes can not distinguish $\mathcal{O}^{\oplus 2}$ and $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

What's the Schur functor for Chern class? Give me formulas for $c(S^\lambda(E))$, for some vector bundle E over X .

Answer from chatgpt:

If E has Chern roots x_1, \dots, x_r and $W(\lambda)$ is the set of weights of the irreducible $GL(r)$ -representation of highest weight λ , then

$$c(S^\lambda(E)) = \prod_{(w \in W(\lambda))} (1 + a_1 x_1 + \dots + a_r x_r) \quad (\text{for } w = (a_1, \dots, a_r)),$$

or equivalently,

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1}, \dots, e^{x_r}).$$

In particular,

$$c(S^k E) = \prod_{(a_1 + \dots + a_r = k)} (1 + a_1 x_1 + \dots + a_r x_r)$$

and

$$c(\wedge^k E) = \prod_{(1 \leq i_1 < \dots < i_k \leq r)} (1 + x_{i_1} + \dots + x_{i_k}).$$

Explanation: One very standard answer is to use the splitting principle. Suppose that the vector bundle E has rank r and splits (formally) as a sum of line bundles, so that

$$E = L_1 \oplus L_2 \oplus \dots \oplus L_r$$

with first Chern classes x_1, x_2, \dots, x_r (so that $c(L_i) = 1 + x_i$). Then any Schur functor S^λ applied to E has, by representation theory, "Chern roots" given by the weights of the irreducible representation of $GL(r)$ with highest weight λ . In other words, if $W(\lambda)$ is the (multi)set of weights – that is, if every weight $w \in W(\lambda)$ is given by an r -tuple (a_1, \dots, a_r) , then the corresponding "root" is

$$\langle w, x \rangle = a_1 x_1 + \dots + a_r x_r.$$

Thus the total Chern class of $S^\lambda(E)$ is

$$c(S^\lambda(E)) = \prod_{(w \in W(\lambda))} (1 + \langle w, x \rangle).$$

An equivalent formulation is to note that the Chern character of $S^\lambda(E)$ is given by

$$ch(S^\lambda(E)) = s_\lambda(e^{x_1}, \dots, e^{x_r}),$$

where s_λ is the Schur polynomial corresponding to the partition λ . (From ch one may recover the Chern classes via the usual formal relations.)

As an example, note that when $\lambda = (k)$ the Schur functor is the k -th symmetric power, and its total Chern class may be written as

$$c(S^k(E)) = \prod_{(a_1, \dots, a_r) \in \mathbb{N}^r, a_1 + \dots + a_r = k} (1 + a_1 x_1 + \dots + a_r x_r),$$

while when $\lambda = (1^k)$ (that is, a column of k boxes) one has $S^{(1^k)}(E) = \wedge^k E$ and

$$c(\wedge^k E) = \prod_{(1 \leq i_1 < \dots < i_k \leq r)} (1 + x_{i_1} + \dots + x_{i_k}).$$