Eine Woche, ein Beispiel

12.15 Young diagram with vectors

You can check [2024.12.01, 2024.12.08] for the reference, notations(for different weights) and the choice of the coordinates.

Motivation: Write $T \cong \mathbb{G}_m^6$ as the maximal torus of $G(E_6)$,
and $W(E_6) = N(T)/T$ as the Weyl group.
Choose $\beta_1, \dots, \beta_4$ as 4 orthogonal roots in $X_*(T)$.

We constructed a function

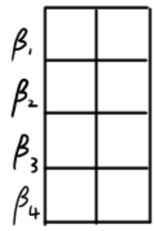
$$f: X^*(T) \longrightarrow \mathbb{R}$$

given by

$$f(\chi) = \sum_{\sigma \in W(E_6)} \langle \sigma(\beta_1), \chi \rangle^2 \langle \sigma(\beta_2), \chi \rangle^2 \langle \sigma(\beta_3), \chi \rangle^2 \langle \sigma(\beta_4), \chi \rangle^2$$

i.e.,

$$f = \sum_{\sigma \in W(E_6)} \sigma\left(\beta_1^2 \beta_2^2 \beta_3^2 \beta_4^2\right)$$



This looks like a "monomial symmetric function of type E_6".

Q: Can we generalize Young diagram to other representations (rather than S_n)?

A: Yes, but we lost some nice properties.

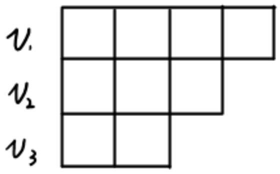
Maybe this generalization is not the "correct" one. I'm glad to hear any new ideas about the question.

1. definition & symmetric function
2. classical results for Weyl group
3. orthogonal roots
4. volume of lattices

1. definition & symmetric function

In this section, let G be a finite group.

Def For $(\rho, V) \in \mathrm{Rep}_{\mathbb{C}}(G)$, the Young diagram is some boxes with decoration $\{v_1, v_2, \cdots\} \subseteq V$.



The associated monomial sym fct (on V^*) is given by

$$M_\lambda = \sum_{\sigma \in G} \sigma\left(\prod_i v_i^{k_i}\right) \in \left(\mathrm{Sym}^{|\lambda|} V\right)^G$$

E.g. For $G = S_n$, (ρ, V) as the standard rep. and take $v_i = e_i$.
Then, the Young diagram is the usual one,
and the associated monomial sym fct is given by

$$M_\lambda = \sum_{\sigma \in S_n} \sigma\left(m_1^{k_1} \cdots m_t^{k_t}\right) \in \left(\mathrm{Sym}^{|\lambda|} V\right)^{S_n}$$

These M_λ's form a basis of $(\mathrm{Sym}^{\cdot} V)^{S_n}$,
and the multiplication is given by

https://math.stackexchange.com/questions/395842/decomposition-of-products-of-monomial-symmetric-polynomials-into-sums-of-them

Q. 1. Can we find a basis of $(\mathrm{Sym}^{\cdot} V)^{G}$?

2. Can we define

- H_j: j-th complete sym poly
- M_λ: monomial sym poly
- E_λ: elementary sym poly
- S_λ: Schur poly

and find some algorithm to get coefficients for multiplication?

3. Is this related with the cohomology ring of Grassmannians outside type A?
https://mathoverflow.net/questions/326749/reference-request-grassmannian-and-plucker-coordinates-in-type-b-c-d

4. Can we make these v_i canonical?
One possible way is to require {v_i}_i are orthonormal basis. Do we lost some sym polynomials?
Is it better to choose other bases in A_n and E_6?

2. classical results for Weyl group

It turns out that $(\mathrm{Sym}^{\cdot}(X_*(T)_{\mathbb{C}}))^W \hat{=} S(\mathfrak{h}^*)^W$ has already been seriously studied, see

https://mathoverflow.net/questions/37602/polynomial-invariants-of-the-exceptional-weyl-groups

I am summarizing the results to gain insight into what is currently known.

Prop. [Hum92]

1. $$S(\mathfrak{h}^*)^W \cong \mathbb{R}[g_1, \dots, g_n]$$
   g_i : basic invariants (homogeneous polynomial of a given degree)
2. One can give these g_i explicitly in the case of type A-D.
3. If $f_1, \dots, f_n$ are alg indep homo sym polys. and $\prod_{i=1}^{n}(\deg f_i) = |W|$, then $\{f_i\}$ are basic invarients.
4. For $f_1, \dots, f_n \in \mathbb{R}[x_1, \dots, x_n]$,
   $f_1, \dots, f_n$ are alg indep $\Leftrightarrow$ $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{i,j} \neq 0$

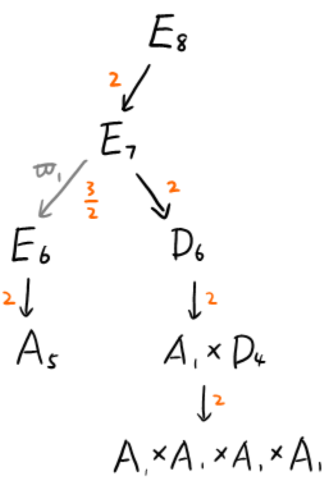
E.g.

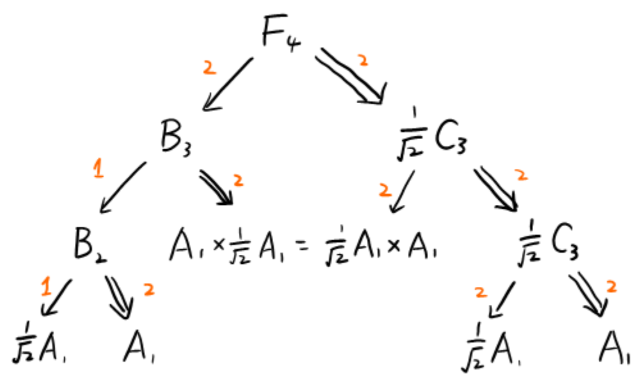
Type	V^*	λ
A_5	fw	(2), (3), (4), (5), (6)
B_5	root	(2), (4), (6), (8), (10)
C_5	fw	
D_6	fw	(2), (4), (6), (8), (10), (1,1,1,1,1,1)

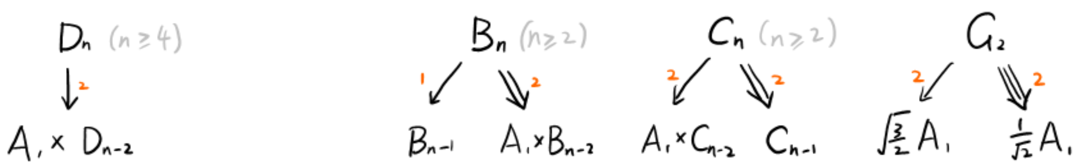
3. orthogonal roots

Φ	$\lvert\Phi\rvert$	$\lvert\Phi^<\rvert$	I	D (= det Cartan = $\#\Delta(\Phi)$)	$\lvert W\rvert$	h (Coxeter number)	volume $=\sqrt{\det A}$
A_n $(n \geq 1)$	$n(n+1)$			$n+1$	$(n+1)!$	$n+1$	$\sqrt{n+1}$
B_n $(n \geq 2)$	$2n^2$	$2n$	2	2	$2^n\,n!$	$2n$	1
C_n $(n \geq 3)$	$2n^2$	$2n(n-1)$	2^{n-1}	2	$2^n\,n!$	$2n$	2
D_n $(n \geq 4)$	$2n(n-1)$			4	$2^{n-1}\,n!$	$2(n-1)$	2
E_6	72			3	51840	12	$\sqrt{3}$
E_7	126			2	2903040	18	$\sqrt{2}$
E_8	240			1	696729600	30	1
F_4	48	24	4	1	1152	12	$\frac{1}{2}$
G_2	12	6	3	1	12	6	$\frac{\sqrt{3}}{2}$

Notice that, $\langle \varpi_i \rangle^\perp = \langle \alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_r \rangle$ is the root lattice associated with the Dynkin diagram obtained by deleting vertice i.
Luckily, for every wt lattice outside type A, $\exists!$ fundamental wt ϖ_k which is also the short root. (also $\exists!$ for the long root outside type C_n) $\varpi_1 = \frac{1}{2}\alpha_{long}$
Using this strategy, we can find the numbers of max ortho roots.







$D_2 = A_1 \times A_1$
$B_1 = \frac{1}{\sqrt{2}} A_1$
$C_1 = \sqrt{2}\, A_1$

$\xrightarrow{\varpi_1}$ other wts
$\longrightarrow$ (short) root
$\Longrightarrow$ long root
$\Rrightarrow$ very long root
2 relative volume $= [Q_2 : Q_1 \times \langle\alpha\rangle]$

In type ADE, for Dynkin diagrams outside A_n & E_6, the root lattices always contain a sublattice iso to A_1^n.

4. volume of lattices = volume of $\mathbb{R}^r/\Lambda$

$$\begin{tikzcd} Q_2 \arrow[d, "{[Q_2 : Q_1 \times \langle\alpha\rangle]}"] \\ Q_1 \end{tikzcd} \qquad \begin{tikzcd} & Q_1 \times \langle\alpha\rangle \arrow[dl, no head, shift left] \arrow[dl, no head, shift right] \arrow[dr, no head, shift left] \arrow[dr, no head, shift right, "\#(\mathbb{Z}\varpi_k/\mathbb{Z}\alpha) = 1"] & \\ Q_2 & & Q_1 \times \langle\varpi_k\rangle \end{tikzcd}$$

not a map of root lattice

usually

$$Q_1 := \langle\alpha\rangle^\perp \subseteq Q_2$$

$$\mathrm{vol}(Q_2) = \mathrm{vol}(Q_1) \cdot \frac{|\alpha|}{[Q_2 : Q_1 \times \langle\alpha\rangle]}$$

E.g. E_6, E_7, E_8, D_n

$$\begin{tikzcd} & \overset{\sqrt{6} \times \sqrt{2}}{A_5 \times \langle\alpha\rangle} \arrow[dl, no head, "2"'] \arrow[dr, no head] & \\ \underset{\sqrt{3}}{E_6} & & A_5 \times \langle\varpi_6\rangle \end{tikzcd} \qquad \begin{tikzcd} & \overset{2 \times \sqrt{2}}{D_6 \times \langle\alpha\rangle} \arrow[dl, no head, "2"'] \arrow[dr, no head] & \\ \underset{\sqrt{2}}{E_7} & & D_6 \times \langle\varpi_6\rangle \end{tikzcd}$$

$$\begin{tikzcd} & \overset{\sqrt{2} \times \sqrt{2}}{E_7 \times \langle\alpha\rangle} \arrow[dl, no head, "2"'] \arrow[dr, no head] & \\ \underset{1}{E_8} & & E_7 \times \langle\varpi_6\rangle \end{tikzcd} \qquad \begin{tikzcd} & \overset{\sqrt{2} \cdot 2 \cdot \sqrt{2}}{A_1 \times D_{n-2} \times \langle\alpha\rangle} \arrow[dl, no head, "2"'] \arrow[dr, no head] & \\ \underset{2}{D_n} & & A_1 \times D_{n-2} \times \langle\varpi_2\rangle \end{tikzcd}$$

E.g. B_n, C_n

$$\begin{tikzcd}[column sep=small] & \overset{1}{B_{n-1}} \times \overset{1}{\langle \alpha_{short} \rangle} \arrow[dl, Rightarrow, no head, "1"'] \arrow[dr, Rightarrow, no head] & \\ \underset{1}{B_n} & & B_{n-1} \times \langle \varpi_1 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{\sqrt{2}}{A_1} \times \overset{1}{B_{n-2}} \times \overset{\sqrt{2}}{\langle \alpha_{long} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{1}{B_n} & & A_1 \times B_{n-2} \times \langle \varpi_2 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{\sqrt{2}}{A_1} \times \overset{2}{C_{n-2}} \times \overset{\sqrt{2}}{\langle \alpha_{short} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{2}{C_n} & & A_1 \times C_{n-2} \times \langle \varpi_2 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{2}{C_{n-1}} \times \overset{2}{\langle \alpha_{long} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head, "2"] & \\ \underset{2}{C_n} & & \underset{2}{C_{n-1}} \times \underset{1}{\langle \varpi_1 \rangle} \end{tikzcd}$$

E.g. F_4, G_2

$$\begin{tikzcd}[column sep=small] & \overset{1}{B_3} \times \overset{1}{\langle \alpha_{short} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{\frac{1}{2}}{F_4} & & B_3 \times \langle \varpi_4 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{\frac{1}{\sqrt{2}}}{\tfrac{1}{\sqrt{2}} C_3} \times \overset{\sqrt{2}}{\langle \alpha_{long} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{\frac{1}{2}}{F_4} & & \tfrac{1}{\sqrt{2}} C_3 \times \langle \varpi_1 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{\sqrt{3}}{\sqrt{\tfrac{3}{2}} A_1} \times \overset{1}{\langle \alpha_{short} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{\frac{\sqrt{3}}{2}}{G_2} & & \sqrt{\tfrac{3}{2}} A_1 \times \langle \varpi_1 \rangle \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & \overset{1}{\tfrac{1}{\sqrt{2}} A_1} \times \overset{\sqrt{3}}{\langle \alpha_{long} \rangle} \arrow[dl, Rightarrow, no head, "2"'] \arrow[dr, Rightarrow, no head] & \\ \underset{\frac{\sqrt{3}}{2}}{G_2} & & \tfrac{1}{\sqrt{2}} A_1 \times \langle \varpi_2 \rangle \end{tikzcd}$$