Eine Woche, ein Beispiel
6.4. basics of fields

This document is aimed for people who have enough mathematical maturity, but miss the chance and time to study Galois theory.
For a (relative) complete study of Galois theory which takes time, please see [GTM167].

1. classical motivation
2. common confusion
3. field extension
4. examples of algebraic closed field

1. classical motivation

	ruler-and-compass construction 尺规作图		solving higher degree equations 求根公式	
possible	⬠ 17-gon	$\cos\frac{2\pi}{5}$ ζ_5 $\cos\frac{2\pi}{17}$ ζ_{17}	$\deg F \leq 4$	$x: F(x)=0$
impossible	Squaring the circle Doubling the cube Angle trisection	π 化圆为方 $\sqrt[3]{2}$ 倍立方 $x: 4x^3-3x-a=0$ 三等分角	$\deg F \geq 5$	$x: F(x)=0$

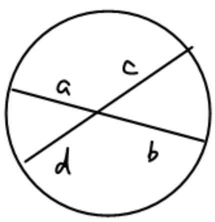
Ex. Denote

$$F_R := \{z\in\mathbb{C} \mid z \text{ can be drawn by ruler-and-compass, given } 0,1\}$$

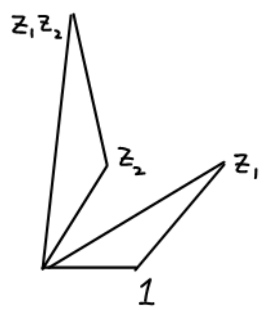
$$F_{开根} := \{z\in\mathbb{C} \mid z \text{ can be expressed by } +,-,\times,\div, \text{radicals}\}$$

Verify that F_R, $F_{开根}$ are fields.

Hint. Verify that $\mathbb{Q}\subseteq F_R$ to get some intuition.



$ab=cd$



Ex. Given $1, a \in \mathbb{R}^+$, try to draw $\sqrt{a}$ by ruler-and-compass.
Argue that why we can draw ⬠ and 17-gons.

Hint. $\cos\frac{2\pi}{5} = \frac{\sqrt{5}}{4} - 1$

$\cos\frac{2\pi}{17} = \frac{1}{16}\left(-1+\sqrt{17}+\sqrt{2(17-\sqrt{17})}+2\sqrt{17+3\sqrt{17}-\sqrt{2(17-\sqrt{17})}-2\sqrt{2(17+\sqrt{17})}}\right)$

Slogan: consider

element	$\rightsquigarrow$	set
object	$\rightsquigarrow$	moduli spaces
if x can be realized	$\rightsquigarrow$	$\{x \mid x$ can be realized$\}$

2. common confusion

	Abstract field	Subfield of $\bar{K}$ or $\mathbb{C}$
name of category	Field	$\text{Subfield}_{\bar{K}}$
Ob	$\{F : \text{field}\}$	$\{(F,\iota) \mid \iota : F \hookrightarrow \bar{K}\}$
Mor	$\text{Mor}_{\text{Field}}(F,E) = \{\alpha : F \hookrightarrow E\}$ usually: finitely many elements	$\text{Mor}_{\text{Subfield}}(F,E) = \{\alpha : F \hookrightarrow E \text{ s.t. } F \searrow \bar{K} \swarrow E\}$ at most 1 element
Examples	$\mathbb{Q}[x]/(x^2+1)$ $\mathbb{Q}[x]/(x^3-2)$ $\mathbb{Q}(x)$	$\mathbb{Q}(i)$ $\mathbb{Q}(\sqrt[3]{2})$ $\mathbb{Q}(\pi)$

Common questions: (Which category are we considering for these questions?)
- $\#\{$extensions of K of deg $3\}$
- Automorphism gp of the field.

Abstract fields are not as hard as you may think!

Ex 1). Write down the definition of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$, as well as $\mathbb{Q}(i)$, $\mathbb{Q}(\pi)$
2). Find a $\mathbb{Q}$-basis of $\mathbb{Q}[x]/(x^2+1)$, $\mathbb{Q}(x)$. Compute the dim.

Constructing new field by adding roots

$$
\begin{array}{r}
132 \\
102 \overline{\smash{)}\,13562} \\
\underline{102} \\
3362 \\
\underline{306} \\
302 \\
\underline{202} \\
100
\end{array}
\qquad\qquad
\begin{array}{r}
x^2+3x+7 \\
x^2-2 \overline{\smash{)}\, x^4+3x^3+5x^2+6x+2} \\
\underline{x^4 \qquad -2x^2 \qquad\qquad} \\
3x^3+7x^2+6x \\
\underline{3x^3 \qquad\quad -6x} \\
7x^2+12x+2 \\
\underline{7x^2 \qquad\quad -14} \\
12x+16
\end{array}
$$

$13562 \div 102 = 132 \cdots 100$ $\qquad$ $(x^4+3x^3+5x^2+6x+2) \div (x^2-2) = (x^2+3x+7) \cdots (12x+16)$

$13562 = 102 \times 132 + 100$ $\qquad$ $x^4+3x^3+5x^2+6x+2 = (x^2-2)(x^2+3x+7)+(12x+16)$

Ex. factorize $x^3+4x^2-7x-10$ in $\mathbb{Q}[x]$ or $\mathbb{F}_3[x]$.

Ex. Let $F = \mathbb{F}_7[x]/(x^3-3)$.

1) Compute $(x^2+1)\cdot(x-1)$, $\frac{1}{x}$,
2) Show that x^3-3 is irr in $\mathbb{F}_7[x]$, i.e.

   $x^3-3 = f(x)g(x) \Rightarrow \deg f = 0$ or $\deg g = 0$

   $f, g \in \mathbb{F}_7[x]$
3) Show that $(x^3-3,\ x^2+x+1) = (1)$ in $\mathbb{F}_7[x]$, by Euclidean division.

   In fact, $\mathbb{F}_7[x]$ is ED $\Rightarrow$ PID
4) Compute $(x^2+x+1)^{-1}$ in F.
5) Factorize T^3-3 in $F[T]$.

Rmk. In fact, $K[T]/(f(T))$ is a field $\Leftrightarrow$ $f(T) \in K[T]$ is irreducible

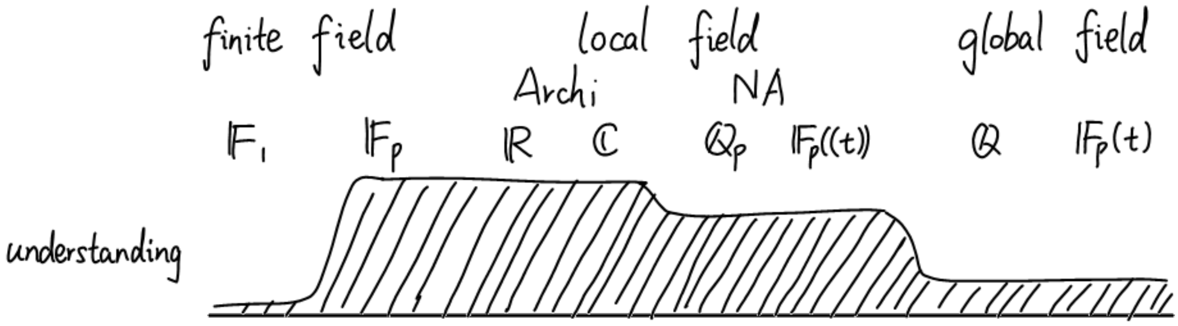
Ex. Let $F = \mathbb{Q}[x]/(x^3-2)$.

1) Compute $\mathrm{Mor}_{\mathrm{Field}}(F, \mathbb{C})$. Are all embeddings real?
2) Discussion: What is the difference between

   $\mathbb{Q}[x]/(x^3-2)$ with $\mathbb{Q}(\sqrt[3]{2})$?

3. field extension

Main examples of fields



Definitions

Def: E/F field extension: $(E, F, \iota: F \hookrightarrow E)$

Def: Base field: $\begin{cases} \mathbb{Q} & \text{char } F = 0 \\ \mathbb{F}_p & \text{char } F = p \end{cases}$

Def. (Algebraic extension)

E/F is alg, if $\forall a \in E$ is alg$/F$, i.e., the following equivalent conditions are true.

1) $\forall a \in E$, $\exists f \in F[x]$, $f \neq 0$, $f(a) = 0$.
2) $\forall a \in E$, $[F(\alpha) : F] < +\infty$.
3) $E = \bigcup_{\substack{F \subset F' \subset E \\ F'/F \text{ finite}}} F'$
4) $\forall a \in E$, $\exists$ f.d. F-v.s. $V \subseteq E$ s.t $aV \subseteq V$.

For $a \in E$, $\mathrm{Min}(a, F) :=$ minimal monic polynomial of a in F.

E.g. $\overline{\mathbb{Q}}/\mathbb{Q}$, $\mathbb{Q}(\pi)/\mathbb{Q}$, $\mathbb{C}/\mathbb{Q}$

We mainly consider alg extension, e.p. fin field extension.

Assume: E/F alg

Slogan: Galois = normal + seperable

Def. (Normal extension)

E/F normal, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \subseteq F[x] \subseteq E[x] \text{ splits}}_{a\in E \text{ is normal}}$.

E.g. $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ $\quad \mathbb{Q}(\zeta_3)/\mathbb{Q}$ $\quad \mathbb{Q}(\sqrt[3]{2},\zeta_3)/\mathbb{Q}$

Def. (Seperable extension)

E/F sep, if $\forall a \in E$, $\underbrace{\mathrm{Min}(a,F) \text{ has no repeated roots in } \bar{F}[x]}_{a\in E \text{ is sep}}$.

E.g. $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$, where $\mathbb{F}_p(T^{\frac{1}{p}}) := \mathbb{F}_p(T)[x]/(x^p-T)$

Rmk. When $\operatorname{char} F = 0$ or $\# F < +\infty$, E/F is always seperable.

Def. (Galois extension)

E/F Galois, if E/F is normal and sep. We denote

$$\begin{aligned} \mathrm{Gal}(E/F) &= \mathrm{Aut}_{F\text{-alg}}(E) \\ &= \{\sigma : E \longrightarrow E \mid \sigma|_F = \mathrm{Id}_F\} \end{aligned}$$

Rmk. When E/F finite,

$$E/F \text{ Galois} \Leftrightarrow [E:F] = \#\mathrm{Aut}_{F\text{-alg}}(E)$$

E.g. & Exercise: Compute $\#\mathrm{Aut}_{F\text{-alg}}(E)$ for $E/F = \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$, $\mathbb{F}_p(T^{\frac{1}{p}})/\mathbb{F}_p(T)$.

https://kconrad.math.uconn.edu/blurbs/galoistheory/galoiscorrexamples.pdf

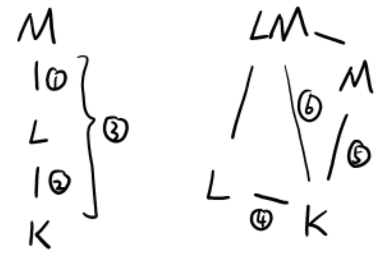
Ex. Read it, and compute

$\mathrm{Gal}(\mathbb{Q}(\sqrt[3]{2},\zeta_3)/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\sqrt[4]{2},i)/\mathbb{Q})$

$\mathrm{Gal}(\mathbb{Q}[x]/(x^4-x^2-1)\,/\mathbb{Q})$

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q})$, $\quad \mathrm{Gal}(\mathbb{Q}[T]/(T^2-2)^2-2\,/\mathbb{Q})$

$\mathrm{Gal}(\mathbb{Q}(\sqrt{2+\sqrt{2}})/\mathbb{Q})$



① $\xrightarrow{\text{③ pure ins}}$ ③

normal:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
seperable:	① + ② = ③			④ + ⑤ = ⑥	
Galois:	③ ⇒ ①	③ ⇏ ②	① + ② ⇏ ③	⑥ ⇏ ④	④ + ⑤ ⇒ ⑥
purely inseparable	① + ② = ③			④ + ⑤ = ⑥	

only 1 root for minimal poly

[GTM 167, Thm 4.13] char $F = p$. then
F perfect $\Leftrightarrow F^p = F$

$$\begin{array}{lll}
\overline{K} & & \\
| & \text{closed subgroup} & \\
L & & \\
\text{(finite)}\ | & \text{quotient group} & \\
K & &
\end{array}
\qquad
\left.\begin{array}{ccc}
\overline{\mathbb{F}_p} & \overline{\mathbb{F}_p} & \overline{\mathbb{F}_p} \\
\big|\ \mathbb{Z}_l & \big|\ \prod_{p\nmid l}\mathbb{Z}_p & \big|\ d\hat{\mathbb{Z}} \\
\bigcup_{l\nmid a}\mathbb{F}_{p^a} & \bigcup_n \mathbb{F}_{p^{l^n}} & \mathbb{F}_q \\
\big|\ \prod_{p\nmid l}\mathbb{Z}_p & \big|\ \mathbb{Z}_l & \big|\ \mathbb{Z}/d\mathbb{Z} \\
\mathbb{F}_p & \mathbb{F}_p & \mathbb{F}_p
\end{array}\right]
\quad \hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \quad (q = p^d)$$

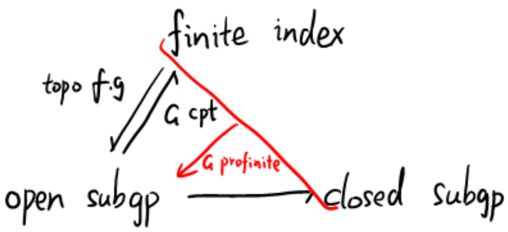
open subgroup $\overset{\{1\}\subseteq\mathbb{Z}_p}{\subsetneq}$ closed subgroup $= \{\mathrm{Gal}(\overline{K}/L) \mid L/k \text{ ext}\} \overset{\mathbb{Z}\subseteq\mathbb{Z}_p}{\subsetneq}$ subgroup

Lem. A subgroup of a profinite group is open iff it's closed and has finite index.

Ref: https://ctnt-summer.math.uconn.edu/wp-content/uploads/sites/1632/2020/06/CTNT-InfGaloisTheory.pdf

Q: Do we have any finite index gp of $\mathrm{Gal}(\overline{k}/k)$ which is not open?

In general,



https://groupprops.subwiki.org/wiki/Closed_subgroup_of_finite_index_implies_open
In a topological group, any closed subgroup of finite index must be an open subgroup.
https://groupprops.subwiki.org/wiki/Open_subgroup_implies_closed
Any open subgroup of a topological group is closed.

https://math.stackexchange.com/questions/4350043/open-subgroup-and-finite-index-subgroup-of-topological-group
https://math.stackexchange.com/questions/1092974/finite-index-subgroup-g-of-mathbbz-p-is-open
https://math.stackexchange.com/questions/3734250/are-normal-subgroups-of-finite-index-in-an-absolute-galois-groups-open

Some wonderful exercises for Galois correspondence:

Let E/F be Galois field ext of deg n, $m|n$. prove: $\exists$ subfield ext of deg m.
(Sylow thm & $Z(G)\neq\{1\}$ for a p-gp & classification of f.g. abelian gp)
Cor. For p prime, F field, one can define ${}^p\overline{F} := \bigcup_{[E:F]=p^k} E$, and

$$\overline{F} = \prod_{p \text{ prime}} {}^p\overline{F}$$

Sadly this is totally wrong. Notice that a Sylow p-subgroup may be not normal.

https://math.stackexchange.com/questions/2125547/finite-field-extension-with-no-non-trivial-subextension

https://math.stackexchange.com/questions/1068327/is-bar-mathbb-q-bar-mathbb-q-cap-mathbb-r-2

Are there any other subfield of $\overline{\mathbb{Q}}$ with finite index (except $\overline{\mathbb{Q}}$ & $\overline{\mathbb{Q}}\cap\mathbb{R}$)?

4. examples of algebraic closed field

① $\overline{\mathbb{Q}} \overset{\pi}{\underset{\overline{\mathbb{Z}}}{\subsetneq}} \mathbb{C} \overset{t}{\subsetneq} \bigcup_n \mathbb{C}((t^{\frac{1}{n}})) = \overline{\mathbb{C}((t))}$ $\mathbb{C}[[t]]$

$\sum_{n=0}^{+\infty} p^n$ $\quad \cap$ Puiseux series

$\overline{\mathbb{Q}_p} \underset{\overline{\mathbb{Z}}_p}{\subset} \mathbb{C}_p$

$\overset{\shortparallel}{\mathbb{Q}_p \cdot \overline{\mathbb{Q}}}$

② char $K = p$: $\overline{\mathbb{F}_q} = \overline{\mathbb{F}_p}$ $\quad (\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\mathbb{Z}})$

Task. ① Prove they are alg closed. $(\mathbb{C}, \bigcup_n \mathbb{C}((t^{\frac{1}{n}})), \mathbb{C}_p)$

② Find an element in each "c".

$Q_p \cap \bar{Q}:$ https://math.stackexchange.com/questions/1280053/explicit-description-of-bbb-q-p-cap-bar-bbb-q
$C_p \setminus \bar{Q_p}:$ https://math.stackexchange.com/questions/123925/is-the-algebraic-closure-of-a-p-adic-field-complete
https://math.stackexchange.com/questions/2430665/algebraic-closure-of-q-p-is-composite-of-bar-mathbbq-and-mathbbq-p
https://math.stackexchange.com/questions/2153580/transcendental-numbers-in-mathbbq-p