Eine Woche, ein Beispiel

8.6. Kottwitz set

This document is a continuation of [23.08.06].
Reorganized from Luozi Shi (and his partners)'s talk.

Recall that $\widehat{\mathbb{Q}_p^{ur}} = \mathrm{Frac}(\widehat{\mathbb{Z}_p^{ur}})$, $\widehat{\mathbb{Z}_p^{ur}} = W(\mathbb{F}_p)$. Here "$\widehat{\ }$" is completion w.r.t. valuation.

Setting. In this document, F is a NA local field,

$$L := \widehat{F^{ur}} \overset{\text{if } F:\text{p-adic}}{=\!=} \mathrm{Frac}(W(\bar{k}_F)\otimes_{W(k_F)} \mathcal{O}_F).$$

Def For G/F conn reductive, the Kottwitz set $B(G)$ is defined as

$$\begin{aligned} B(G) &:= H^1(W_F, G_{\bar{F}}) \\ &\cong H^1(\langle\sigma\rangle, G_L) \qquad \text{by Inf-Res seq \& } H^1(I_F, G_{\bar{F}})=0 \\ &\cong G(L)/_{\sigma\text{- twisted } G(L)\text{-conj}} \\ &\overset{\text{when } G=GL_n}{=\!=\!=} \mathrm{Isoc}/_{\sim} \end{aligned}$$

Isocrystals.

Rmk. By Hilbert 90, $H^1(\Gamma_F, GL_{n,\bar{F}}) = \{1\}$. In most cases,

$$H^1(\Gamma_F, G_{\bar{F}}) \not\cong H^1(W_F, G_{\bar{F}})$$

[even though $H^1(\Gamma_F, G_{\bar{F}}) \cong G(F) \cong H^0(W_F, G_{\bar{F}})$, we take different resolutions.]

E.g. $B(\mathbb{G}_m) \cong \mathbb{Z}$

Proof. The map

$$\beta: \mathcal{O}_L^\times \longrightarrow \mathcal{O}_L^\times \qquad x \longmapsto x\cdot\sigma(x)^{-1}$$

is surjective, since

$$\beta_1: k_L^\times \longrightarrow k_L^\times \qquad x \longmapsto x^{1-q}$$

$$\beta_2: k_L \longrightarrow k_L \qquad x \longmapsto x - x^q$$

are surjective. ($k_L \cong \overline{\mathbb{F}_p}$ is alg closed)

Then the well-defined morphism

$$v: L^\times/\mathrm{Im}\beta \longrightarrow \mathbb{Z} \qquad x \longmapsto v(x)$$

is injective, thus an iso.

Ex. Check that the SES

$$1 \longrightarrow \underline{\mathbb{Z}/2\mathbb{Z}} \longrightarrow \mathbb{G}_m \xrightarrow{(-)^2} \mathbb{G}_m \longrightarrow 1$$

induce LES in gp cohomology:

$$B(\underline{\mathbb{Z}/2\mathbb{Z}}) \xrightarrow{0} B(\mathbb{G}_m)^{\mathbb{Z}} \xrightarrow{\times 2} B(\mathbb{G}_m)^{\mathbb{Z}}$$

$$0 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow F^\times \xrightarrow{(-)^2} F^\times$$

where

$$\begin{aligned} B(\underline{\mathbb{Z}/2\mathbb{Z}}) &\overset{\text{def}}{=} H^1(W_F, \mathbb{Z}/2\mathbb{Z}) \qquad W_F \curvearrowright \mathbb{Z}/2\mathbb{Z} \text{ trivially} \\ &= \mathrm{Hom}(W_F, \mathbb{Z}/2\mathbb{Z}) \\ &\cong \{ H \lhd W_F \text{ closed with index } 2 \} \cup \{0\} \\ &\cong \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p \neq 2 \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & p = 2 \end{cases} \end{aligned}$$

This LES gives us to compute number of finite extensions. e.g. with prime degree.

One gets

$$F^\times \xrightarrow{(-)^n} F^\times \rightsquigarrow B(\underline{\mathbb{Z}/n\mathbb{Z}}) \xrightarrow{0} \cdots$$

E.g. $B(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{Z} \times \mathbb{Z} = X_*(\mathbb{G}_m \times \mathbb{G}_m)$

Proof. $B(\mathbb{G}_m \times \mathbb{G}_m) \cong \mathbb{G}_m(L) \times \mathbb{G}_m(L)/_{\sigma\text{-twisted } \mathbb{G}_m(L) \times \mathbb{G}_m(L)\text{-conj}}$

$$\begin{aligned} &\cong \mathbb{G}_m(L)/_\sim \times \mathbb{G}_m(L)/_\sim \\ &\cong \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

Rmk. In general, for a torus T/F,

$$B(T) \cong X_*(T)_{\Gamma_F} \cong X^*(\hat{T}^{\Gamma_F})$$

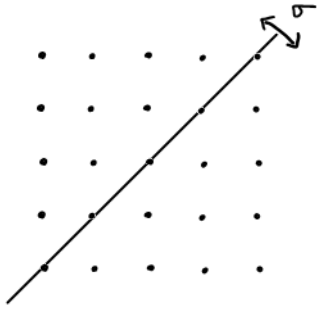
E.g. For E/F any Galois extension of deg 2, $\mathrm{Gal}(E/F) = \{1, \sigma\}$.

When $T = \mathrm{Res}_{E/F}\mathbb{G}_{m,E}$, $B(T) \cong \mathbb{Z}$.

$$X_*(T) \cong \mathbb{Z} \times \mathbb{Z} \qquad \sigma : \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} b \\ a \end{pmatrix}$$

$$X_*(T)_{\Gamma_F} \cong \mathbb{Z} \times \mathbb{Z}/\langle \sigma(v) - v \rangle \xrightarrow[\sim]{+} \mathbb{Z}$$

$$\begin{array}{ccc} \hat{T} & \cong & \mathbb{C}^\times \times \mathbb{C}^\times \qquad \sigma : \begin{pmatrix} a \\ b \end{pmatrix} \longmapsto \begin{pmatrix} b \\ a \end{pmatrix} \\ \cup & & \uparrow \Delta \\ \hat{T}^{\Gamma_F} & \cong & \mathbb{C}^\times \\ X^*(\hat{T}^{\Gamma_F}) & \cong & \mathbb{Z} \end{array}$$



When $T = \mathrm{Res}^1_{E/F}\mathbb{G}_{m,E}$, $B(T) \cong \mathbb{Z}/2\mathbb{Z}$

$X_*(T) \cong \mathbb{Z}$ $\qquad \sigma : a \mapsto -a$

$X_*(T)_{\Gamma_F} \cong \mathbb{Z}/\langle \sigma(a) - a \rangle \cong \mathbb{Z}/2\mathbb{Z}$

$\hat{T} \cong \mathbb{C}^\times$ $\qquad \sigma : a \mapsto a^{-1}$

$\cup \qquad \cup$

$\hat{T}^{\Gamma_F} \cong \mu_2(\mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$

$X^*(\hat{T}^{\Gamma_F}) \cong \mathbb{Z}/2\mathbb{Z}$

Ex. Check that the SES

$$1 \longrightarrow \mathrm{Res}^1_{E/F}\mathbb{G}_{m,E} \longrightarrow \mathrm{Res}_{E/F}\mathbb{G}_{m,E} \longrightarrow \mathbb{G}_{m,F} \longrightarrow 1$$

induce LES in gp cohomology:

$$B(\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E}) \xrightarrow{0} B(\mathrm{Res}_{E/F}\mathbb{G}_{m,E}) \xrightarrow{\times 2} B(\mathbb{G}_{m,F})$$

($B(\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E}) = \mathbb{Z}/2\mathbb{Z}$, $B(\mathrm{Res}_{E/F}\mathbb{G}_{m,E}) = \mathbb{Z}$, $B(\mathbb{G}_{m,F}) = \mathbb{Z}$)

$$0 \longrightarrow H^0(W_F, (\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) \longrightarrow H^0(W_F, (\mathrm{Res}_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) \xrightarrow{\text{Norm}} H^0(W_F, \mathbb{G}_{m,\bar{F}})$$

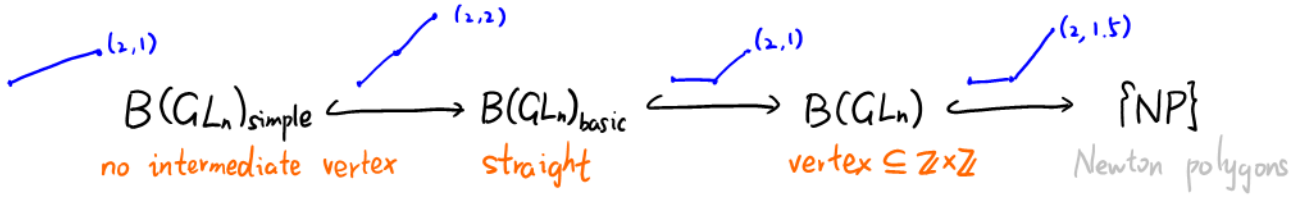
($H^0(W_F, (\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) = (\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E})(F)$, $H^0(W_F, (\mathrm{Res}_{E/F}\mathbb{G}_{m,E})_{\bar{F}}) = E^\times$, $H^0(W_F, \mathbb{G}_{m,\bar{F}}) = F^\times$)

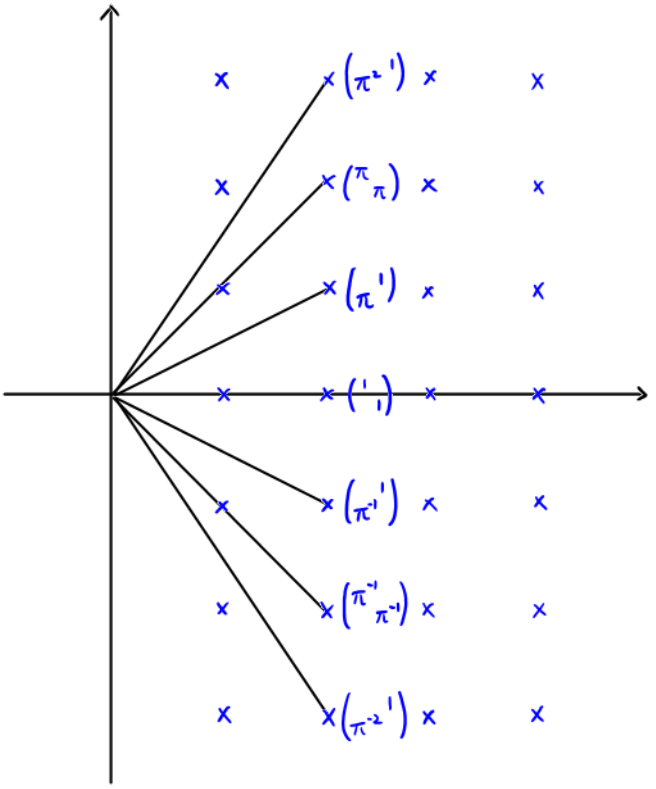
[Aside: $(\mathrm{Res}^1_{E/F}\mathbb{G}_{m,E})(F) = \ker\left[(\mathrm{Res}_{E/F}\mathbb{G}_{m,E})(F) \xrightarrow{\text{Norm}} \mathbb{G}_m(F)\right]$

$= \ker\left[E^\times \xrightarrow{\text{Norm}} F^\times\right]$

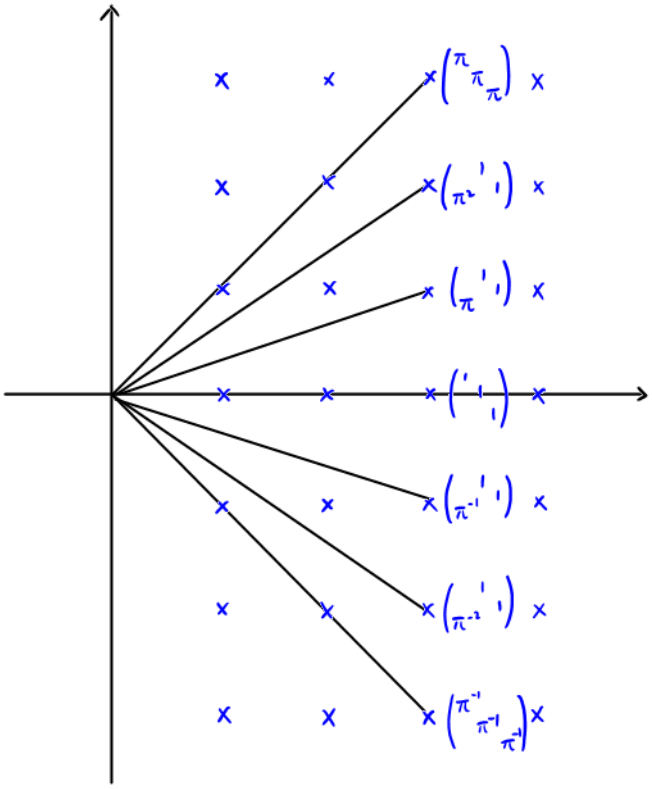
$= \{x \in E^\times \mid x\sigma(x) = 1\}$]

E.g. $G = GL_n$





$B(GL_2)_{basic}$



$B(GL_3)_{basic}$