# Eine Woche, ein Beispiel

## 12.21. Hodge structure

Ref:

[Car17]: James Carlson, Stefan Müller-Stach, and Chris Peters. Period Mappings and Period Domains. 2nd ed. Cambridge University Press, 2017. https://doi.org/10.1017/9781316995846.

$X$: cpt Kähler mfld of dim $n$.

Slogan: Hodge structure collects (nearly) all linear algebraic structures of $X$.

These linear algebraic structures are: $\quad k=p+q$

① $H^k(X;\mathbb{Z})$

② $H^q(X;\Omega_X^p) \cong \{\text{harmonic } (p,q)\text{-forms}\}$

③ A comparison iso as $\mathbb{C}$-v.s. (J-action)

$$H^k(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} \cong \bigoplus_{p+q=k} H^q(X;\Omega_X^p)$$

②′ A Hodge filtration

$$H^k(X;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} \cong \underset{H^{0,k}}{H^k(\Omega_X^{\bullet})} \supset \underset{H^{1,k-1}}{H^{k-1}(\Omega_X^{\geq 1})} \supset \underset{H^{2,k-2}}{H^{k-2}(\Omega_X^{\geq 2})} \supset \cdots \supset \underset{H^{k-1,1}}{H^0(\Omega_X^{\geq k})} \supset \underset{H^{k,0}}{0}$$

③′ Compatability with $\mathrm{conj}_J$:

$$H^k(X,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C} = H^{k-p}(\Omega_X^{\geq p}) \oplus \overline{H^{p-1}(\Omega_X^{\geq k-p+1})}$$

Here, $\Omega_X^{\geq p} := \sigma_{\geq p}\Omega_X^{\bullet}$. We have

$$\begin{tikzcd}
 & & & 0 \arrow[dr] & & 0 & & & & \\
 & & & & \Omega_X^{\geq 1} \arrow[ur] \arrow[dr] & & & & & \\
0 \arrow[r] & \mathbb{C} \arrow[rr] \arrow[dr] & & \Omega_X^0 \arrow[ur] \arrow[rr, "d^1"] & & \Omega_X^1 \arrow[rr, "d^2"] \arrow[dr] & & \Omega_X^2 \arrow[rr, "d^3"] \arrow[ur] & & \cdots \\
 & & \Omega_X^{\bullet}=\Omega_X^{\geq 0} \arrow[ur] & & & & \Omega_X^{\geq 2} \arrow[ur] \arrow[dr] & & & \\
 & 0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

④ (polarization)

A bilinear form

$$\Lambda: H^k(X;\mathbb{C}) \otimes H^k(X;\mathbb{C}) \xrightarrow{\Lambda} H^{2k}(X;\mathbb{C}) \xrightarrow{\wedge\omega^{n-k}} H^{2n}(X;\mathbb{C}) \cong \mathbb{C}$$

$$\Lambda(\xi,\eta) := \int_X \xi\wedge\eta\wedge\omega^{n-k}$$

with two compatability conditions:

Assume $\omega$ is integral, i.e., $X$ is proj

$$\begin{array}{ccccccccc}
\Lambda: H^k(X;\mathbb{Z}) & \otimes & H^k(X;\mathbb{Z}) & \xrightarrow{\Lambda} & H^{2k}(X;\mathbb{Z}) & \xrightarrow{\wedge\omega^{n-k}} & H^{2n}(X;\mathbb{Z})\cong\mathbb{Z} \\
 & \cap & & & \cap & & \cap \\
\Lambda: H^k(X;\mathbb{C}) & \otimes & H^k(X;\mathbb{C}) & \xrightarrow{\Lambda} & H^{2k}(X;\mathbb{C}) & \xrightarrow{\wedge\omega^{n-k}} & H^{2n}(X;\mathbb{C})\cong\mathbb{C} \\
 & \cup & & & \cup & & \cong \\
\wedge: H^q(X;\Omega_X^p) & \otimes & H^p(X;\Omega_X^q) & \xrightarrow{\wedge} & H^k(X;\Omega_X^k) & \xrightarrow{\wedge\omega^{n-k}} & H^n(X,\omega_X)\cong\mathbb{C}
\end{array}$$

with two positivity conditions:

$$\Lambda(\xi,\bar\xi)\in\mathbb{R} \quad \text{for} \quad \xi\in H^k(X;\mathbb{R}) \qquad \xi=\bar\xi \ \text{here}$$

$$i^{p-q+k(k-1)}\Lambda(\xi,\bar\xi)\geq 0 \quad \text{for} \quad \xi\in H^k_{prim}(X;\mathbb{C})\cap H^{p,q}$$

And get "=" $\Leftrightarrow$ $\xi=0$.

⚠ These positivities are not contradiction to each other.
If $\xi\in H^k(X;\mathbb{R}) \cap (H^k_{prim}(X;\mathbb{C})\cap H^{p,q})$, then $\xi\in\overline{H^{p,q}}=H^{q,p}$,
then $p=q$, i.e.

$$\xi\in H^{2p}_{prim}(X;\mathbb{R})\cap H^{p,p}, \qquad i^{p-q+k(k-1)}=(-1)^p.$$

However, when $p$ is odd,

$$H^{2p}_{prim}(X;\mathbb{R})\cap H^{p,p}=\{0\}.$$