Eine Woche, ein Beispiel

11.12 algebraic de Rham cohomology

Ref:

[BK23] Notes of p-adic Hodge theory by Bruno Klingler.

[VPIH] notes on intesection homology, by Vishwambhar Pati
https://www.isibang.ac.in/~adean/infsys/database/notes/homology.pdf

[HM16]: Periods and Nori Motives, by Annette Huber , Stefan Müller-Stach
https://link.springer.com/content/pdf/10.1007/978-3-319-50926-6

[GTM281]: Intersection Homology & Perverse Sheaves with Applications to Singularities, by Laurențiu G. Maxim
https://link.springer.com/book/10.1007/978-3-030-27644-7

[KS90] Masaki Kashiwara, Pierre Schapira. Sheaves on Manifolds. With a Short History "Les Débuts de La Théorie Des Faisceaux" by Christian Houzel. Grundlehren Math. Wiss. Berlin etc.: Springer-Verlag, 1990.

1. definition
2. period

1. definition

Def. Let $F \subseteq \mathbb{C}$ field, X/F sm variety. (integral)
we define the <u>algebraic de Rham complex</u>

$$\Omega^{\cdot}_{X/F} = (\mathcal{O}_X \xrightarrow{d} \Omega^1_{X/F} \xrightarrow{d} \Omega^2_{X/F} \xrightarrow{d} \cdots)$$

and the <u>algebraic de Rham cohomology</u>

$$H^i_{dR}(X/F) := R^i\Gamma(X; \Omega^{\cdot}_{X/F})$$

For the def of relative de Rham complex $H^i_{dR}(X,Z)$, see [HM16, Definition 3.2.6].

In ptc, when X, Z are sm, $Z \xhookrightarrow{i} X$ closed subscheme.

$$\Omega^{\cdot}_{(X,Z)} = \ker(\Omega^{\cdot}_{X/F} \longrightarrow i_* \Omega^{\cdot}_{Z/F})$$

$$H^i_{dR}(X,Z) := R^i\Gamma(X; \Omega^{\cdot}_{(X,Z)}).$$

https://mathoverflow.net/questions/298838/why-use-hypercohomology-when-defining-the-de-rham-cohomology-of-a-smooth-scheme

Ordinary cohomology: $\ker/\mathrm{Im}$

hyper cohomology: resolution + $\ker/\mathrm{Im}$

https://math.stackexchange.com/questions/2429835/injective-mathcalo-x-module-is-flasque
https://mathoverflow.net/questions/55725/interesting-examples-of-flasque-sheaves
https://math.stackexchange.com/questions/3508441/derived-functors-can-be-computed-by-an-acyclic-resolution
https://math.stackexchange.com/questions/1038292/why-can-we-use-flabby-sheaves-to-define-cohomology/1038346#1038346
every sheaf has a canonical flabby resolution, called the Godement resolution.
I think, we can not compute the higher direct image by flasque resultion, since flasque sheave are usually not pi_*-acyclic.

$$\begin{tikzcd}
\text{injective} \arrow[r, Rightarrow] & \text{flasque} \arrow[r, Rightarrow] \arrow[rr, Rightarrow, bend right] \arrow[dr, Rightarrow] & \text{soft} \arrow[r, Rightarrow, "\text{paracompact}"] & \Gamma\text{-acyclic} \\
& & c\text{-soft} \arrow[r, Rightarrow] \arrow[ur, Rightarrow, "\text{if covered by countable cpt sets}"'] & \Gamma_c\text{-acyclic}
\end{tikzcd}$$

blue: assume loc cpt spaces. See [KS90, Def 2.5.5-2.5.10].

Rmk. 1). When $F=\mathbb{R}$, $\Omega^i_{X^{an}}$ is soft, thus acyclic,

From [VPIH, Example 1.3.7], softness follows by the fact that smooth Urysohn functions exist in $\Gamma(X^{an}, \Omega^0_{X^{an}})$

and $\Omega^i_{X^{an}}$ are modules over $\Omega^0_{X^{an}}$.

$$H^i_{dR}(X^{an};\mathbb{R}) = \frac{\mathrm{Ker}\ [\Omega^i_{X^{an}}(X) \xrightarrow{d} \Omega^{i+1}_{X^{an}}(X)]}{\mathrm{Im}\ [\Omega^{i-1}_{X^{an}}(X) \xrightarrow{d} \Omega^i_{X^{an}}(X)]}$$

2). When $F=\mathbb{C}$ & X is a Stein mfld, $\Omega^i_{X^{an}}$ is acyclic.

See [GTM 281, Example 4.3.17]

Moreover, for $X/\mathbb{C}$ sm variety,

$$H^i_{dR}(X/\mathbb{C}) \cong H^i_{dR}(X^{an}/\mathbb{C}),$$

which is a non-trivial Corollary of GAGA.

3) From the remark of
https://mathoverflow.net/questions/298838/why-use-hypercohomology-when-defining-the-de-rham-cohomology-of-a-smooth-scheme

When X/F is affine,

$\Omega^i_{X/F}$ is coherent $\Rightarrow$ $\Omega^i_{X/F}$ is acyclic.

E.g. For $X = \mathbb{A}^1_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x]$,

$$\Gamma(X;\Omega^\bullet_{X/\mathbb{Q}}) = \left(\mathbb{Q}[x] \xrightarrow{d} \mathbb{Q}[x]dx \longrightarrow 0 \cdots\right)$$

$$R^i\Gamma(X;\Omega^\bullet_{X/\mathbb{Q}}) = \begin{cases} \mathbb{Q}\cdot 1 & i=0 \\ 0 & i\neq 0 \end{cases}$$

E.g. For $X = \mathbb{G}_{m,\mathbb{Q}} = \operatorname{Spec} \mathbb{Q}[x,x^{-1}]$,

$$\Gamma(X;\Omega^\bullet_{X/\mathbb{Q}}) = \left(\mathbb{Q}[x,x^{-1}] \xrightarrow{d} \mathbb{Q}[x,x^{-1}]dx \longrightarrow 0 \cdots\right)$$

$$R^i\Gamma(X;\Omega^\bullet_{X/\mathbb{Q}}) = \begin{cases} \mathbb{Q}\cdot 1 & i=0 \\ \mathbb{Q}\cdot \frac{dx}{x} & i=1 \\ 0 & \text{otherwise} \end{cases} \quad \Leftarrow \text{residue}$$

E.g. For $X = \mathbb{P}^1_{\mathbb{C}}$, see

https://math.stackexchange.com/questions/3156041/algebraic-de-rham-cohomology-of-projective-space-over-mathbbc

or [HM16, Example 3.1.3].

2. Period

Ref:
https://en.wikipedia.org/wiki/Period_(algebraic_geometry)
https://math.stackexchange.com/questions/2959421/is-pi-e-a-period
https://math.stackexchange.com/questions/2574608/do-numbers-get-worse-than-transcendental

Def (complex period)

For F: # field, X/F: variety, $Z \overset{i}{\hookrightarrow} X$ closed subscheme over F, one has a pairing

$$\begin{aligned} \langle -, - \rangle : H^i_{dR}(X/F) \times H_i(X^{an}_{\mathbb{C}}; \mathbb{Z}) &\longrightarrow \mathbb{C} \\ (\omega \quad , \quad \gamma) &\longmapsto \int_\gamma \omega \\ \langle -, - \rangle : H^i_{dR}(X, Z) \times H_i(X^{an}_{\mathbb{C}}, Z^{an}_{\mathbb{C}}; \mathbb{Z}) &\longrightarrow \mathbb{C} \\ (\omega \quad , \quad \gamma) &\longmapsto \int_\gamma \omega \end{aligned}$$

$\int_\gamma \omega$ is called the period of ω over γ.

Q: What kind of number can be a period?

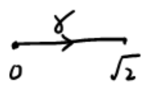
A: True: $\overline{\mathbb{Q}}$, π, $\ln 2$, $\zeta(n)$, $\Gamma(\frac{p}{q})^q$, ...

Conjectured false: e, $\frac{1}{\pi}$, γ, ...

$\{a \in \mathbb{C} \mid a \text{ is a period}\}$ is a ring.

E.g. Let $F = \mathbb{Q}$, $X = \mathbb{A}^1_{\mathbb{Q}}$, $Z = V(x^3 - 2x) = \{-\sqrt{2}, 0, \sqrt{2}\}$ over $\mathbb{Q}(\sqrt{2})$, then

$$\int_\gamma dx = \int_0^{\sqrt{2}} dx = \sqrt{2}.$$



E.g. Let $F = \mathbb{Q}$, $X = \mathbb{G}_{m\mathbb{Q}}$, $Z = \{1, 2\}$, then

$$\int_{\gamma_1} \frac{dx}{x} = 2\pi i \qquad \int_{\gamma_2} \frac{dx}{x} = \ln 2$$