Eine Woche, ein Beispiel

9.3. field extension with RS

Goal: construct an equivalence between two categories:

$$\underset{\substack{\uparrow \\ \text{cpt conn}}}{RS^{cc}} = \left\{ \begin{array}{l} \text{Obj: cpt conn RS} \\ \text{Mor: non-const holo morphisms} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Obj: } F/\mathbb{C} \text{ field ext s.t.} \\ \qquad \mathrm{trdeg}_{\mathbb{C}} F = 1 \\ \qquad F/\mathbb{C} \text{ f.g as a field} \\ \text{Mor: morphism as fields}/\mathbb{C} \end{array} \right\}^{op} = \mathrm{field}^{op}_{\mathbb{C}(t)/\mathbb{C}}$$

$$\begin{array}{ccc} Y & & \mathcal{M}(Y) \\ \downarrow f & \Longrightarrow & \uparrow f^* \\ X & & \mathcal{M}(X) \end{array}$$

which obeys the following slogan:

$$\boxed{\text{(ramified) covering} \quad \approx \quad \text{(function) field extension}}$$

Rmk. 1. For requiring $F/\mathbb{C}$ f.g. as a field, we avoid examples like $\overline{\mathbb{C}(t)}$.
Do they corresponds to some non-cpt Riemann surface?
If so, how to enlarge the category RS^{cc}?

2. $\mathrm{field}_{\mathbb{C}(t)/\mathbb{C}}$ means fields over $\mathbb{C}$ which are fin ext of $\mathbb{C}(t)$ abstractly; morphisms don't need to fix $\mathbb{C}(t)$.
Do you have a better name for RS^{cc} and $\mathrm{field}_{\mathbb{C}(t)/\mathbb{C}}$?

https://math.stackexchange.com/questions/633628/threefold-category-equivalence-algebraic-curves-riemann-surfaces-and-fields-of
https://math.stackexchange.com/questions/1286286/link-between-riemann-surfaces-and-galois-theory

1. field of meromorphic functions
2. Galois covering
3. valuations
4. quadratic extension of $\mathbb{C}(x)$: hyperelliptic curve
5. miscellaneous.

1. field of meromorphic functions

Def. For X RS,

$$\begin{aligned}\mathcal{M}(X) &:= \{\text{meromorphic fcts on } X\} \\ &= \{f: X \longrightarrow \mathbb{P}^1 \text{ holomorphic}\} - \{\mathbb{1}_\infty\} \\ &\underset{\text{conn}}{\overset{X \text{ cpt}}{=}} \{\text{rational fcts on } X\}\end{aligned}$$

Ex. Verify that

$$\mathcal{M}(\mathbb{CP}^1) \cong \mathbb{C}(z)$$
$$\mathcal{M}(\mathbb{C}/\mathbb{Z}[i]) \cong \operatorname{Frac}\left(\mathbb{C}[x,y]/(y^2 - x(x+1)(x-1))\right)$$

Later we will show that, for $X \in RS^{cc}$,

$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$

Ex. For

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3,$$

compute

1) $f^*: \mathbb{C}(T) \hookrightarrow \mathbb{C}(S)$ $\quad [\mathbb{C}(S) : \mathbb{C}(T)]$ & a $\mathbb{C}(T)$-basis
2) $\operatorname{Gal}(\mathbb{C}(S)/\mathbb{C}(T))$
3) $\mathbb{C}(S)^{\mathbb{Z}/3\mathbb{Z}}$
4) $\operatorname{Aut}_f(\mathbb{CP}^1)$

Ex. For

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z + \frac{1}{z},$$

do the same work.

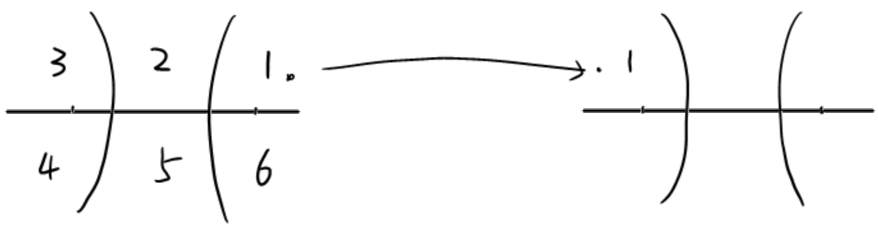
Ex. For

$$f: \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z,$$

compute the same stuff.

Why isn't $\mathbb{C}(S)/\mathbb{C}(T)$ Galois this time?

[Hint.



]

Prop. For $d \in \mathbb{N}_{>0}$, $f: Y \to X$ proper holo morphism between conn RSs,
$$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] = d.$$

Cor. For X cpt conn,
$$\exists\ \mathbb{C}(x) \hookrightarrow \mathcal{M}(X) \quad \text{s.t.} \quad [\mathcal{M}(X) : \mathbb{C}(x)] < +\infty$$
In ptc, $F/\mathbb{C}$ f.g as a field, $\operatorname{trdeg}_{\mathbb{C}} F = 1$.

To show the proposition, one need the following black box to find a basis.
Black box (meromorphic fcts seperate points)
$$X: RS, \quad x, y \in X \quad x \neq y, \text{ then}$$
$$\exists\, g \in \mathcal{M}(X) \quad \text{s.t.} \quad g(x) \neq g(y) \quad g(x), g(y) \in \mathbb{C}.$$
(stronger) $\exists\, g \in \mathcal{M}(X)$ s.t. $\operatorname{ord}_x g = -1$, $g(y) = 0$.

I prefer using Riemann-Roch when X is cpt, and Stein manifold when X is not.

Ex. Using the black box, show that,
for X: RS, $\{x_1, \dots, x_n\} \subseteq X$, $\exists\, g \in \mathcal{M}(X)$ s.t.
$$\operatorname{ord}_{x_1} g = -1, \quad g(x_i) \in \mathbb{C} \quad \forall i \in \{2, \dots, n\}$$
$$g(x_i) \neq g(x_j) \quad \forall i \neq j, \ i, j \in \{2, \dots, n\}$$

Proof of prop

$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \geq d$: Fix $x_0 \in X$ s.t. $\# f^{-1}(x_0) = d$. Denote $f^{-1}(x_0) = \{y_1, \dots, y_d\}$.
For each i, let $g_i \in \mathcal{M}(Y)$ be a meromorphic fct s.t.
$$\operatorname{ord}_{x_i} g_i = -1 \qquad g_i(y_j) \in \mathbb{C} \quad \forall j \neq i,$$
then $\{g_1, \dots, g_d\} \subseteq \mathcal{M}(Y)$ are $f^*\mathcal{M}(X)$-linear independent.
Check: $\operatorname{ord}_{y_j}(\sum_i f_i g_i) \approx \operatorname{ord}_{y_j} f_j$

$[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leq d$:
$\forall g \in \mathcal{M}(Y)$, need to find $a_i \in f^*\mathcal{M}(X)$ s.t.
$$g^d + a_{d-1} g^{d-1} + \cdots + a_0 = 0 \quad \text{in } \mathcal{M}(Y)$$
The fcts
$$a_i(z) = (-1)^i \sum_{\{k_1, \dots, k_i\} \subseteq \{1, \dots, d\}} g(z_{k_1}) \cdots g(z_{k_d})$$
$$f^{-1}(f(z)) := \{z_1, \dots, z_d\}, \text{ multiplicity is counted}$$
satisfy the conditions.
Use Riemann extension theorem to show $a_i(z) \in f^*\mathcal{M}(X)$, see [Donaldson, p148].
By primitive element theorem, $[\mathcal{M}(Y) : f^*\mathcal{M}(X)] \leq d$.

2. Galois covering

Def. Let $f: Y \longrightarrow X$ be a proper hdo map between two conn RSs.

f is Galois, if $\mathcal{M}(Y)/f^*\mathcal{M}(X)$ is a Galois extension.

normal normal

Prop. $f: Y \longrightarrow X$ is Galois/normal

$\Leftrightarrow$ $\deg f = \# \mathrm{Aut}_f(Y)$

$\Leftrightarrow$ $f^{-1}(x_0)$ is an $\mathrm{Aut}_f(Y)$-torsor, $\forall x_0 \in X \setminus f(\mathrm{Ram}(f))$

$\Leftrightarrow$ $\mathrm{Aut}_f(Y) \circlearrowright f^{-1}(x_0)$ transitively, $\forall x_0 \in X$

$\Leftrightarrow$ $Y/\mathrm{Aut}_f(Y) \cong X$, i.e. f can be written as

$$Y \twoheadrightarrow Y/G$$

Ex. For $f: Y \longrightarrow X$, suppose that

ramification index

$[\forall y_1, y_2 \in Y \text{ s.t. } f(y_1) = f(y_2),] \Rightarrow e(y_1) = e(y_2)$

Show that f is Galois by computing $\# \mathrm{Aut}_f(Y)$.

[Hint. Use geodesics to divide X into several smaller triangles.
If geodesics are hard, take $g: X \longrightarrow \mathbb{CP}^1$ non-constant,
and reduce the problem to $g \circ f$.]

This proof is not completely rigorous, and you are encouraged to find a reference to rigorously prove it, or read this dicsussion on stackexchange:
https://math.stackexchange.com/questions/1952655/ramification-index-and-inertia-degree-same-for-all-the-primes-then-is-the-exten

You may need the following materials for completing the proof, relating with questions about geodesic triangulations.

google: geodesic triangulations
https://math.stackexchange.com/questions/1661331/proof-of-equivalence-of-conformal-and-complex-structures-on-a-riemann-surface?rq=1
https://arxiv.org/pdf/2103.16702.pdf
(If a non geodesic triangulation is given, in a sufficiently fine subdivision one can replace all edges by geodesics, which leaves the Euler characteristic unchanged.)
copied from p2, in https://www.mathematik.uni-muenchen.de/~forster/eprints/gaussbonnet.pdf

http://czamfirescu.tricube.de/CTZamfirescu-08.pdf
[Thm 2] http://www-fourier.univ-grenoble-alpes.fr/~ycolver/All-Articles/91c.pdf

https://mathoverflow.net/questions/138267/what-prevents-a-cover-to-be-galois

E.g. Consider the covering

$$f : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1 \qquad z \longmapsto z^3 - 3z$$

This is not a Galois covering. Consider the Galois closure

$$\begin{array}{ccccc}
\mathbb{CP}^1 & & \mathbb{C}(u) = \mathbb{C}(S)[R]/(R^2+S^2-4) & & u + \frac{1}{u} \\
\big\downarrow z+\frac{1}{z} & & \uparrow & & \uparrow \\
\mathbb{CP}^1 & & \mathbb{C}(S) = \mathbb{C}(T)[S]/(S^3-3S-T) & & S \quad S^3-3S \\
\big\downarrow z^3-3z & & \uparrow & & \uparrow \\
\mathbb{CP}^1 & & \mathbb{C}(T) & & T
\end{array}$$

Determination of the Galois closure

$$\begin{aligned}
\min(S, \mathbb{C}(T)) &= x^3 - 3x - T && \text{in } \mathbb{C}(T)[x] \\
&= x^3 - 3x - (S^3 - 3S) \\
&= (x - S)(x^2 + Sx + S^2 - 3) && \text{in } \mathbb{C}(S)[x]
\end{aligned}$$

To decompose the polynomial $x^2 + Sx + S^2 - 3$,
we have to add root of discriminant:

$$\sqrt{\Delta} := \sqrt{S^2 - 4(S^2-3)} = \sqrt{3}\sqrt{-S^2+4}.$$

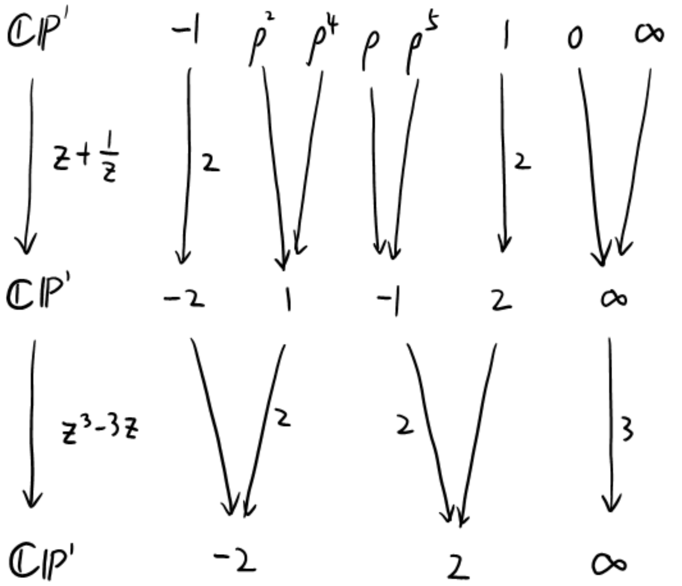
Therefore, the Galois closure of $\mathbb{C}(S)/\mathbb{C}(T)$ is

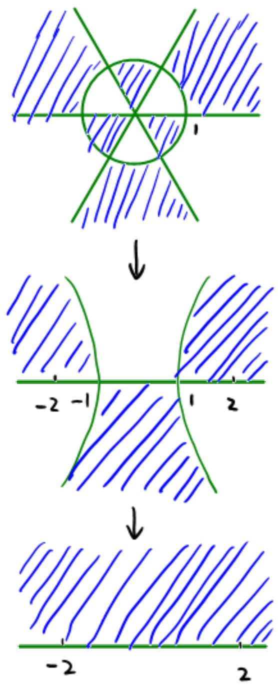
$$\mathbb{C}(S)[R]/(R^2+S^2-4) \cong \mathbb{C}\left(\frac{S+iR}{2}\right) \triangleq \mathbb{C}(u)$$

where

$$S = \frac{S+iR}{2} + \frac{S-iR}{2} = u + \frac{1}{u}$$

The picture from the RS side is as follows:





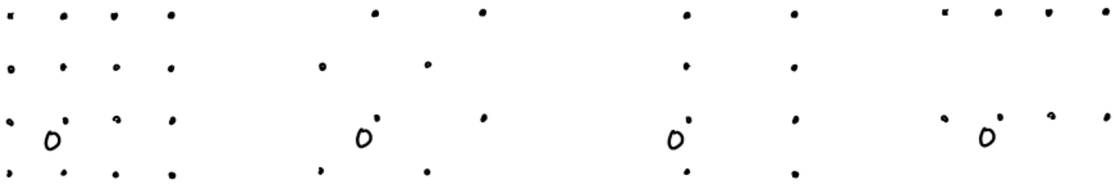
only ramified pts are drawn

affine version

Question:
How to know the genus of the RS corresponding to the Galois closure?
Answer:
https://mathoverflow.net/questions/152/how-do-you-see-the-genus-of-a-curve-just-looking-at-its-function-field
Question:
Do we have any Galois closure whose ramification information is not minimal as we expected?

E.g. 2. For $E = \mathbb{C}/\Lambda$, since $\pi_1(E, 0) \cong \mathbb{Z} \oplus \mathbb{Z}$,
E has three unramified coverings of deg 2.
When $\Lambda = \mathbb{Z}[i]$, what are the crspd field extensions?

There are more deg 2 ramified coverings from the higher genus RS, but we don't discuss them here.



normalized equation

$$y^2 = x(x+1)(x-1) \qquad y^2 = x(x+1)(x-1)$$

$$\begin{aligned} y^2 &= 4x^3 - 11x - 7 \qquad \Delta = 8 \\ &= (x+1)(4x^2 - 4x - 7) \\ &= 4(x+1)\left(x - \tfrac{1}{2} + \sqrt{2}\right)\left(x - \tfrac{1}{2} - \sqrt{2}\right) \end{aligned}$$

$$j(2i) = \left(\frac{11}{2}\right)^3 \cdot 1728 = 66^3 \qquad g_2(2i) = \frac{11\,\Gamma(\frac{1}{4})^8}{2^8 \pi^2} \qquad g_3(2i) = \frac{7\,\Gamma(\frac{1}{4})^{12}}{2^{12}\pi^3}$$

equation is given by $\quad y^2 = 4x^3 - g_2 x - g_3$

Ex. 1) Show that

$$\mathrm{Aut}_{RS}(\mathbb{C}/\Lambda)[2] \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

no matter $\mathbb{C}/\Lambda$ has CM or not.

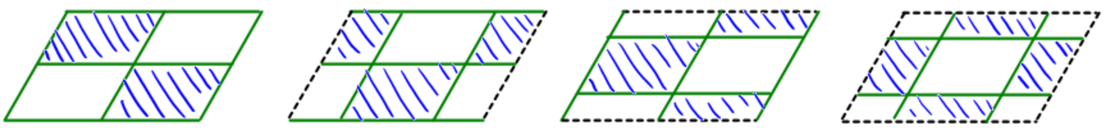
2) We get 7 ramified coverings of deg 2:

$$\pi_\tau : E \longrightarrow E/\langle \tau \rangle \qquad \forall\, \tau \in \mathrm{Aut}_{RS}(\mathbb{C}/\Lambda)[2] - \{Id\}$$

Which are ramified coverings? Compute the genus & ramification information.

3 unramified , $g(E/\langle\tau\rangle) = 1$

4 ramified at 4 pts , $g(E/\langle\tau\rangle) = 0$



3) Find all index 2 subfields of $\mathcal{M}(\mathbb{C}/\Lambda)$. hard!

3. valuations.

Q: How to reconstruct the RS of the crspd fct field extension?
i.e., how to give an inverse fctor of the fctor

$$\mathcal{M}(-): RS^{cc} \longrightarrow \mathrm{field}^{op}_{\mathbb{C}(t)/\mathbb{C}} \qquad X \longmapsto \mathcal{M}(X) \quad ?$$

Observation: $\forall X \in RS^{cc}$, $x \in X$, one can define a valuation

$$\nu^X_x: \mathcal{M}(X) \longrightarrow \mathbb{Z} \sqcup \{\infty\} \qquad f \longmapsto \deg_x f$$

indicating the order of fcts on x.

If we collect all the valuations on $\mathcal{M}(X)$, we may recover the RS X.

Ref: [Perfseminar, L2]

See:
The Zariski-Riemann Space of Valuation Rings by Bruce Olberding
https://link.springer.com/chapter/10.1007/978-3-030-89694-2_21

https://math.stackexchange.com/questions/188652/finite-extensions-of-rational-functions
https://mathoverflow.net/questions/75923/the-space-of-valuations-of-a-function-field

Def (valuation (Bourbaki) / NA absolute value*) Γ: some tot ordered gp.

For $A \in \mathrm{CRing}$, a valuation of A is a map

$$\nu: A \longrightarrow \Gamma \cup \{\infty\}$$

s.t.

- $\nu(0) = \infty$
- $\nu(ab) = \nu(a) + \nu(b)$, $\qquad \nu(1) = 0$
- $\nu(a+b) \geq \min(\nu(a), \nu(b))$, $\qquad$ with equality if $\nu(a) \neq \nu(b)$

↳ which makes ν more algebraic rather than analytic

rigid $\qquad$ flexible

If $A \in \mathrm{Field}_{\mathbb{C}}$, we require additionally that $\nu|_{\mathbb{C}^\times} \equiv 0$.

Rmk. The additional assumption on $\mathbb{C}^\times$ is natural, as we want

$$\nu|_{\mathbb{C}^\times}: \mathbb{C}^\times \longrightarrow \Gamma \cup \{\infty\}$$

↑ order top

to be a cont gp homo. One gets

$$\nu(z) = \nu(|z|) = \nu(e)^{\ln|z|} \qquad \forall z \in \mathbb{C}^\times.$$

Moreover, we want ν to be an NA absolute value, so $\nu(z) = 0$.

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* Many people don't use "absolute value" for high rank valuations.

Def (continue) Denote

$$\mathrm{Spv}(A) = \mathrm{NAval}(A) = \{\text{valuations of } A\}/\sim$$

where

$$v \sim v' \iff \exists\, v_0 : A \longrightarrow \Gamma_0 \cup \{\infty\},\ \Gamma_0 \longrightarrow \Gamma,\ \Gamma_0 \longrightarrow \Gamma' \text{ s.t.}$$

$$\begin{tikzcd}
 & & \Gamma \cup \{\infty\} \\
A \arrow[r, "v_0"] \arrow[urr, bend left, "v"] \arrow[drr, bend right, "v'"'] & \Gamma_0 \cup \{\infty\} \arrow[ur] \arrow[dr] & \\
 & & \Gamma' \cup \{\infty\}
\end{tikzcd}$$

commutes

$$\iff \forall x, y \in A,\ [\, v(x) \geq v(y) \iff v'(x) \geq v'(y) \,]$$

$$\iff \bar{\mathfrak{p}}_v = \bar{\mathfrak{p}}_{v'},\ \mathcal{O}_v = \mathcal{O}_{v'}.$$

My notation	A	v	Γ_v	$\bar{\mathfrak{p}}_v$	$\bar{\kappa}_v$	$\mathcal{O}_v$	$\mathfrak{p}_v$	κ_v
[Perfseminar, L2]	A	v	Γ_v	$\mathfrak{p}_v = \mathrm{supp}(v) = v^{-1}(\infty)$	$\kappa(\mathfrak{p}_v)$	R_v	—	—
e.g.	$\mathbb{Q}_p$	p-adic	$\mathbb{Z}$	0	$\mathbb{Q}_p$	$\mathbb{Z}_p$	$p\mathbb{Z}_p$	$\mathbb{Z}/p\mathbb{Z}$
	$\mathbb{Z}$	p-adic	$\mathbb{Z}$	0	$\mathbb{Q}$	$\mathbb{Z}_{(p)}$	$p\mathbb{Z}_{(p)}$	$\mathbb{Z}/p\mathbb{Z}$
	$\mathbb{Z}$	$\lvert\cdot\rvert_{\mathbb{F}_p}$	0	$p\mathbb{Z}$	$\mathbb{F}_p$	$\mathbb{F}_p$	0	$\mathbb{F}_p$
	$\mathbb{Z}$	v_{triv}	0	0	$\mathbb{Q}$	$\mathbb{Q}$	0	$\mathbb{Q}$

$\lvert\cdot\rvert_\infty \notin \mathrm{Spv}(\mathbb{Z})$, since $\lvert\cdot\rvert_\infty$ is Archimedean.

Ex. In this exercise we want to describe $\mathrm{Spv}(\mathbb{C}(z))$.

1). For $v \in \mathrm{Spv}(\mathbb{C}(z))$, suppose $v(z-3) = 1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z+3)}\right)$.

2). For $v \in \mathrm{Spv}(\mathbb{C}(z))$, suppose $v(z-3) = -1$, compute $v\left(\frac{(z-3)^2(z-\pi)^2}{z^4(z+3)}\right)$.

3). Define

$$v_{triv}: \mathbb{C}(z) \longrightarrow 0 \cup \{\infty\} \qquad f \neq 0 \longmapsto 0$$

Show that $v_{triv} \in \mathrm{Spv}(\mathbb{C}(z))$.

4) Show that as Sets,

$$\begin{array}{ccc} \mathrm{Spv}(\mathbb{C}(z)) & \cong & \{v_{triv}\} \sqcup \mathbb{CP}^1 \\ v_{z_0}^{\mathbb{CP}^1} & \longleftarrow & z_0 \end{array}$$

Ex. Use the same method to describe $\mathrm{Spv}(\mathbb{Q})$.

Hint. 1. Use the strong triangular inequality, for some $a \neq 0$

$$v(a+a+\cdots+a) \geqslant v(a)$$
$$\|$$
$$v(na) = v(n)+v(a)$$

$\Rightarrow v(n) \geqslant 0 \qquad \forall n \in \mathbb{N}_{\geqslant 1}$.

2. For $p \neq p'$ primes, use Bézout's identity to show that $[v(p) > 0 \Rightarrow v(p') = 0]$

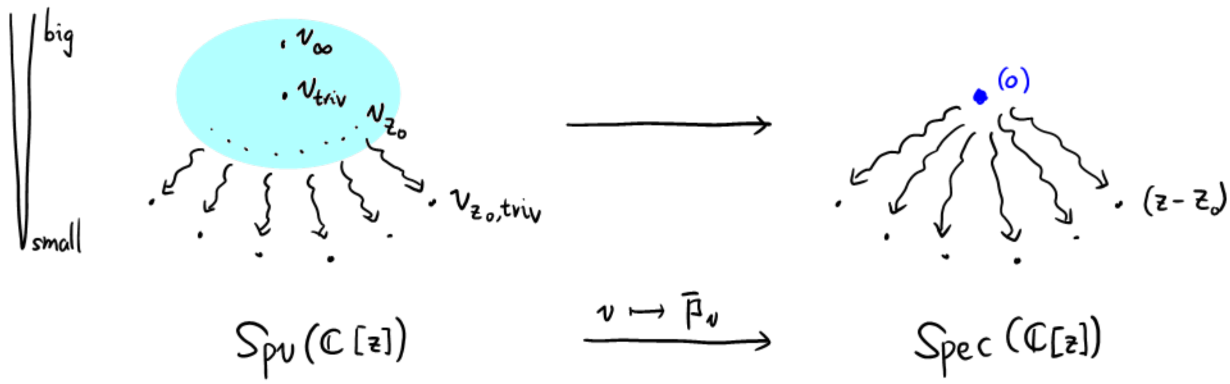
3. Conclude that as Sets,

$$\begin{array}{ccc} \mathrm{Spv}(\mathbb{Q}) & \cong & \{v_{triv}\} \sqcup \{\text{primes}\} \\ v_p & \longleftarrow & p \end{array}$$

Ex. In this exercise we want to describe $\mathrm{Spv}(\mathbb{C}[z])$.
Since $\mathfrak{p}_v$ is a prime ideal of $\mathbb{C}[z]$, we get (as Sets)

$$\mathrm{Spv}(\mathbb{C}[z]) \cong \coprod_{\mathfrak{p}\in \mathrm{Spec}\,\mathbb{C}[z]} \mathrm{Spv}\left(\mathrm{Frac}(\mathbb{C}[z]/\mathfrak{p})\right)$$
$$\cong \mathrm{Spv}(\mathbb{C}(z)) \ \sqcup \coprod_{z_0\in\mathbb{C}} \mathrm{Spv}\left(\mathbb{C}[z]/(z-z_0)\right)$$
$$\cong \{v_{triv}\} \sqcup \mathbb{CP}^1 \sqcup \mathbb{C}$$

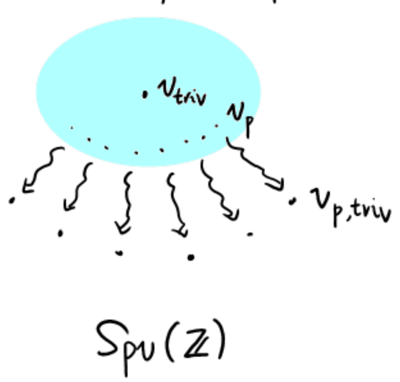
Here, $v \le v'$ iff $\exists$ $\Gamma_v \to \Gamma_0$, $\Gamma_{v'} \to \Gamma_0$, $v(a) \ge v'(a)$ $\forall a\in A$
e.p. $v_\infty \ge v_0$ in $\mathrm{Spv}(\mathbb{C}[z])$, while they are incomparable in $\mathrm{Spv}(\mathbb{C}(z))$.



	0	1	z ($z_0\neq 0$)	$z-z_0$
v_∞	∞	0	-1	-1
v_{triv}	∞	0	0	0
v_{z_0}	∞	0	0	1
$v_{z_0,triv}$	∞	0	0	∞

The generic point contains information about the curves.
This philosophy becomes clearer when working with Spv. We observe that the (preimage of the) generic point inherits all the closed points, even those outside the local affine chart.

Ex. Use the same method to compute $\mathrm{Spv}(\mathbb{Z})$.



Q: How to understand $\mathrm{Spv}(F)$, for $F = \mathbb{C}(x)[y]/(y^2 - x(x+1)(x-1))$?

Idea: Use the restriction map

$$\begin{tikzcd} \mathrm{Spv}(F) \arrow[d, "\pi"] \\ \mathrm{Spv}(\mathbb{C}(x)) \end{tikzcd}$$

we only need to understand the fiber at each pt.

5. miscellaneous.

- genus of a fct field F?
- non-cpt RS, infinite covering
- Spv for higher dimensional varieties, high rank pts
- RS/scheme structures reconstruction
- gp structures on valuations of $\mathbb{C}[x,y,z]/(y^2z - x(x-z)(x+z))$

- cyclic extension
- maximal abelian extension/unramified extension/unramified outside some places

https://math.stackexchange.com/questions/2836916/what-are-the-abelian-extensions-of-bbb-cx