Eine Woche, ein Beispiel

3.16 Schubert calculus: subvariety with vb

This is a follow up of [2025.02.23].

Goal: relate subvarieties to some vector bundles, so that we can compute their homology class in terms of Chern class (when the dimension is correct).

The Chern class will be dealt with in the next document.

Concretely, we will write subvarieties as:

- the zero set of a section in a v.b.
- the degeneracy loci of a morphism $\mathcal{E} \longrightarrow \mathcal{F}$ among v.bs
- the preimage of known cycles in Grassmannian
- the subvariety of $Gr(r,n)$ induced by a rk r bundle (very ample)

1. Known subvarieties and known vector bundles
2. Subvariety as section
3. Subvariety as degeneracy loci
4. Subvariety given by very ample bundle

1. Known subvarieties and known vector bundles

Schubert variety

Recall that the Schubert variety has the expression $\quad \omega \overset{\text{cohom}}{\longleftrightarrow} (\lambda_1,\dots,\lambda_r)$

$$\begin{aligned}
\Sigma_{\lambda_1,\dots,\lambda_r}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+i-\lambda_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{\omega_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{\omega_i} \leq n-\lambda_i \quad \forall i\}
\end{aligned}$$

Especially,

$$\begin{aligned}
\Sigma_{k^s}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+i-k} \leq n-k \quad \forall i \leq s\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s\}
\end{aligned}$$

For special k, s, one can further simplify the formulas:

	k	1	k	$n-r$
s	$Gr(r,n)$			
1		$\Lambda + \mathcal{V}_{n-r} \subseteq H$ or $\Lambda \cap \mathcal{V}_{n-r} \neq \{0\}$	$\Lambda \cap \mathcal{V}_{n-r+1-k} \neq \{0\}$	$\mathcal{V}_1 \subset \Lambda$
s		$\Lambda + \mathcal{V}_{n-r+s-1} \subseteq H$	$\dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k$ or $\dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s$	$\mathcal{V}_s \subset \Lambda$
r		$\Lambda \subset \mathcal{V}_{n-1}$	$\Lambda \subset \mathcal{V}_{n-k}$	$\{\mathcal{V}_r\}$

Vector bundles on Grassmannian

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}^{\oplus n} \xrightarrow{\pi_{\mathcal{S}^\vee}} \mathcal{S}^\vee \longrightarrow 0$$

$\mathcal{S}$: Subspace = tautological bundle
$\mathcal{Q}$: Quotient = quotient bundle

When $r=1$, $Gr(r,n) = \mathbb{P}^{n-1}$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

With these basic v.bs, we can construct more bundles on $Gr(r,n)$:

$$T_{Gr} = Hom(\mathcal{S}, \mathcal{Q}) = \mathcal{S}^\vee \otimes \mathcal{Q} \qquad \omega_{Gr}^\vee = \det \mathcal{S}^\vee \otimes \mathcal{Q}$$
$$\Omega_{Gr} = T_{Gr}^\vee = Hom(\mathcal{Q}, \mathcal{S}) = \mathcal{Q}^\vee \otimes \mathcal{S} \qquad \omega_{Gr} = \det \mathcal{Q}^\vee \otimes \mathcal{S}$$

2. Subvariety as section

Hypersurface and its Fano variety of $(r-1)$-planes

Let $F \in k[z_1, \dots, z_n]$ be a homo poly of deg d. The hypersurface

$$Y_d := \{F = 0\} \subseteq \mathbb{P}^{n-1}$$

is given as a section of

$$\mathcal{O}(d) = \mathrm{Sym}^d \mathcal{O}(1)$$

In general, the Fano variety of $(r-1)$-planes $(\cong \mathbb{P}^{r-1})$

$$F_{r-1}(Y_d) := \{W \in Gr(r,n) \mid F|_W = 0\} \subseteq Gr(r,n)$$

is given as a section of $\mathrm{Sym}^d \mathcal{S}^\vee$, through the map

$$\mathrm{Sym}^d \pi_{\mathcal{S}^\vee} : \mathrm{Sym}^d(\mathcal{O}^{\oplus n}) \twoheadrightarrow \mathrm{Sym}^d(\mathcal{S}^\vee)$$
$$\mathrm{Sym}^d(\mathcal{O}^{\oplus n}) = (\mathrm{Sym}^d V^*) \otimes \mathcal{O}$$

Map of section: $\quad F \otimes 1 \longmapsto s_F = \mathrm{Sym}^d \pi_{\mathcal{S}^\vee}(F \otimes 1)$

Fiberwise, $(\mathrm{Sym}^d \pi_{\mathcal{S}^\vee})_W : \mathrm{Sym}^d V^* \longrightarrow \mathrm{Sym}^d W^*$

We know that

$$F|_W \equiv 0$$
$$\Leftrightarrow (\mathrm{Sym}^d \pi_{\mathcal{S}^\vee})_W (F) = 0$$
$$\Leftrightarrow s_F = 0, \text{ i.e., } \quad [W] \text{ lies in the zero set of } s_F.$$

E.g. $F_0(Y_d) = Y_d$

$F_1(Y_d) \subseteq Gr(2,n)$ — Fano variety of lines

$F_m(Y_2) \subseteq Gr(m+1, 2m+2)$ or $Gr(m+1, 2m+3)$ — Last $\}$ Grassmannian; orthogonal …

Cor. $F_{r-1}(Y_d)$ has codimension $\leq \binom{d+r-1}{d}$ (when non-empty)

Tangent line of hypersurfaces [3264, Chap 11]

Let $F = \sum a_I z^I \in k[z_1, \dots, z_n]$ be a homo poly of deg d, which is a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(d)$, and assume that $Y_d = \{F = 0\}$ is smooth. We want to describe the locus

$$\begin{aligned} \Gamma := & \{(p, l) \in \mathbb{P}^{n-1} \times Gr(2,n) \mid l \text{ is tangent to } Y_d \text{ at } p\} \\ = & \{(p, l) \in \mathbb{P}^{n-1} \times Gr(2,n) \mid F|_l \text{ has multiplicity} \geq 2 \text{ at } p\} \end{aligned}$$

as a section in the v.b. $\mathcal{E}$ over Φ, where

$$\Phi := \{(p, l) \in \mathbb{P}^{n-1} \times Gr(2,n) \mid p \in l\}$$

$$\begin{tikzcd} & \Phi \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ \mathbb{P}^{n-1} & & Gr(2,n) \end{tikzcd}$$

After that, one can describe the locus of tangent lines:

$$\alpha_*[\Gamma] = \{l \in Gr(2,n) \mid l \text{ is tangent to } Y_d\}$$

Idea: Fix $p = [1:0:0:\cdots:0]$, $l = [*:*:0:\cdots:0]$ in Φ. Consider the map

$$\begin{aligned} \varphi_{(l,p)}: H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) & \twoheadrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(d)) \longrightarrow E_{(p,l)} := H^0\left(\mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathcal{O}_{\mathbb{P}^1}/I_p^2\right) \\ \sum a_I z^I & \longmapsto \sum_k a_k z_0^k z_1^{d-k} \longmapsto \sum_{k \leq 1} a_k z_0^k z_1^{d-k} \end{aligned}$$

$$a_k = a_{(k, d-k, 0, \dots, 0)}$$

Then

$$\begin{aligned} \varphi_{(l,p)}\left(\sum a_I z^I\right) = 0 & \iff \sum_{k \leq 1} a_k z_0^k z_1^{d-k} \equiv 0 \\ & \iff F|_l \text{ has multiplicity} \geq 2 \text{ at } p. \end{aligned}$$

We want to globalize $\varphi_{(l,p)}$ to maps between v.b.s.

Construction: Define

$$
\begin{aligned}
\widetilde{\Phi} &:= \Phi \times_{Gr(2,n)} \Phi \\
&= \{(p_1, p_2, l) \mid p_1, p_2 \in l\}
\end{aligned}
$$

$$
\begin{tikzcd}
\Phi \arrow[dr, "\Delta"] & & & & \\
& \widetilde{\Phi} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Phi \arrow[r, "\beta"] \arrow[d, "\alpha"] \arrow[dr, "\pi_{\Phi}"] & \mathbb{P}^{n-1} \arrow[d, "\pi_{\mathbb{P}^{n-1}}"] \\
& \Phi \arrow[r, "\alpha"] & Gr(2,n) \arrow[r, "\pi_{Gr}"] & \{*\}
\end{tikzcd}
$$

Then we get

$$
\begin{tikzcd}
\varphi: \pi_{\Phi}^{*} \pi_{\mathbb{P}^{n-1},*} \mathcal{O}_{\mathbb{P}^{n-1}}(d) \arrow[r] \arrow[d, equal] & \pi_{2,*}\pi_1^{*}\beta^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(d) \arrow[r] & \pi_{2,*}\left(\pi_1^{*}\beta^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \mathcal{O}_{\widetilde{\Phi}}/I_{\Delta}^{2}\right) \arrow[d, equal, "def"] \\
\mathcal{O}_{\Phi} \otimes_k H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)) \arrow[rr] & & \mathcal{E}
\end{tikzcd}
$$

Reality check:

① $\left(\pi_{\Phi}^{*} \pi_{\mathbb{P}^{n-1},*} \mathcal{O}_{\mathbb{P}^{n-1}}(d)\right)_{(p_0,l_0)} = H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$

② $\left(\pi_{2,*}\pi_1^{*}\beta^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(d)\right)_{(p_0,l_0)} = H^0(\mathcal{O}_{\mathbb{P}^1}(d))$

③ $\left(\pi_{2,*}\left(\pi_1^{*}\beta^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \mathcal{O}_{\widetilde{\Phi}}/I_{\Delta}^{2}\right)\right)_{(p_0,l_0)} = E_{(p_0,l_0)}$

④ Construct the map $\pi_{\Phi}^{*} \pi_{\mathbb{P}^{n-1},*} \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow \pi_{2,*}\pi_1^{*}\beta^{*}\mathcal{O}_{\mathbb{P}^{n-1}}(d)$

$$
\begin{tikzcd}[column sep=small]
& & & & \mathbb{P}^{n-1} \arrow[dl] \arrow[dr] & & & \\
& & & \mathbb{P}^{n-1}\times\Phi \arrow[dl] \arrow[dr] & & \{(p_0,l_0)\} \arrow[dl] \arrow[dr, equal] & & \\
\mathcal{O}_{\mathbb{P}^{n-1}}(d) \arrow[dr, dash] & & \mathbb{P}^{n-1} \arrow[dl, equal] \arrow[dr] & & \Phi \arrow[dl] \arrow[dr, equal] & & \{(p_0,l_0)\} \arrow[dl] \arrow[dr, equal] & \\
& \mathbb{P}^{n-1} & & \{*\} & & \Phi & & \{(p_0,l_0)\}
\end{tikzcd}
$$

①

$$
\begin{tikzcd}
 & & \mathcal{O}_{\mathbb{P}^1}/I_p^2 \arrow[dr, dash] & \mathcal{O}_{\mathbb{P}^1}(d) \arrow[d, dash] & & \\
 & \mathcal{O}_{\widetilde{\Phi}}/I_\Delta^2 \arrow[dr, dash] & & \{(p, p_0, l_0) \mid p \in l_0\} \simeq \mathbb{P}^1 \arrow[dl] \arrow[ddr] & & \\
 & & \widetilde{\Phi} \arrow[dl, "\pi_1"'] \arrow[ddr, "\pi_2"] & & & \\
\mathcal{O}_{\mathbb{P}^n}(d) \arrow[d, dash] & \Phi \arrow[dl, "\beta"'] & & & \{(p_0, l_0)\} \arrow[dl] \arrow[dr] & \\
\mathbb{P}^{n-1} & & & \Phi & & \{(p_0, l_0)\}
\end{tikzcd}
$$

② & ③

④:

$$
\begin{tikzcd}
\Phi \arrow[dr, hook, "\iota_\Phi"] \arrow[drr, bend left, "\beta"] \arrow[ddr, "\alpha"'] & & \\
 & \mathbb{P}^{n-1} \times Gr(2,n) \arrow[r, "\widetilde{\beta}"] \arrow[d, "\widetilde{\alpha}"] & \mathbb{P}^{n-1} \arrow[d, "\pi_{\mathbb{P}^{n-1}}"] \\
 & Gr(2,n) \arrow[r, "\pi_{Gr}"] & \{*\}
\end{tikzcd}
$$

$$\pi_\Phi^* \, \pi_{\mathbb{P}^{n-1},*} = \alpha^* \pi_{Gr}^* \, \pi_{\mathbb{P}^{n-1},*} \longrightarrow \alpha^* \alpha_* \beta^* = \pi_{2,*} \pi_1^* \beta^*$$

where the arrow comes from

$$\pi_{Gr}^* \, \pi_{\mathbb{P}^{n-1},*} = \widetilde{\alpha}_* \widetilde{\beta}^* \longrightarrow \widetilde{\alpha}_* \, \iota_{\Phi,*} \, \iota_\Phi^* \widetilde{\beta}^* = \alpha_* \beta^*$$

Rmk. The SES

$$0 \longrightarrow I_\Delta / I_\Delta^2 \longrightarrow \mathcal{O}_{\widetilde{\Phi}} / I_\Delta^2 \longrightarrow \underset{\iota_{\Delta,*}\mathcal{O}_\Phi}{\underset{\parallel}{\mathcal{O}_{\widetilde{\Phi}} / I_\Delta}} \longrightarrow 0$$

induces the SES through $\pi_{2,*}(\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes -)$:

$$0 \longrightarrow \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \Omega_{\Phi/Gr} \longrightarrow \mathcal{E} \longrightarrow \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \longrightarrow 0$$

Here,

$$\begin{aligned}
& \pi_{2,*} \left(\pi_1^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes I_\Delta / I_\Delta^2 \right) \\
= {} & \pi_{2,*} \left(\pi_2^* \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes I_\Delta / I_\Delta^2 \right) \qquad \text{Since } I_\Delta/I_\Delta^2 \text{ supported on } \Delta \\
= {} & \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \pi_{2,*} I_\Delta / I_\Delta^2 \\
= {} & \beta^* \mathcal{O}_{\mathbb{P}^{n-1}}(d) \otimes \Omega_{\Phi/Gr}
\end{aligned}$$

3. Subvariety as degeneracy loci

Def. (degeneracy loci)

Let $X/\mathbb{C}$ sm $\quad k \in \mathbb{Z}_{\geq 0}$,

$\mathcal{E}, \mathcal{F}$: v.b. over X of rank e, f,

$\varphi: \mathcal{E} \longrightarrow \mathcal{F}$ map of v.b. (fiberwise linear).

We define the degeneracy loci

$$M_k(\varphi) := \{ x \in X \mid \operatorname{rank} \varphi_x \leq k \}$$

remember multiplicity

$\varphi_x : \mathcal{E}|_x \longrightarrow \mathcal{F}|_x$

The expected codimension is $(e-k)(f-k)$.

E.g. When $\mathcal{E} = \mathcal{O}_X$, we know $e = 1$,

$$\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F}) \qquad \varphi \longleftrightarrow s$$

$$M_1(\varphi) = X, \quad M_0(\varphi) = V(s)$$

vanishing set in X

Therefore, the degeneracy loci generalizes the section of v.b..

E.g. When $\mathcal{E} = \mathcal{O}_X^{\oplus e}$,

$$\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \cong \Gamma(X; \mathcal{F})^{\oplus e} \qquad \varphi \longleftrightarrow (s_1, \dots, s_e)$$

$M_e(\varphi) = X$

$M_{e-1}(\varphi) = \{ x \in X \mid s_1(x), \dots, s_e(x) \text{ are linear dependent} \}$

$M_k(\varphi) = \{ x \in X \mid \dim \langle s_i(x) \rangle_i \leq k \}$

$M_0(\varphi) = V(s_1, \dots, s_e)$

Flag variety

E.g.

$$\begin{aligned}\Sigma_{k'}^{union} :&= \{(V,V') \in Gr(r,n) \times Gr(r',n) \mid \dim V\cap V' \geq k'\} \\ &= \{(V,V') \in Gr(r,n)\times Gr(r',n) \mid \dim V+V' \leq r+r'-k'\} \\ &= \{(V,V') \mid V\oplus V' \longrightarrow \mathbb{C}^n \text{ is of rank} \leq r+r'-k'\} \\ &= M_{r+r'-k'}(\varphi : \pi_1^{-1}S \oplus \pi_2^{-1}S' \longrightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

The expected dimension is

$$(r+r'-(r+r'-k'))(n-(r+r'-k')) = k'(n+k'-r-r')$$

When $\begin{cases} k' \leq \min(r,r') \\ n+k'-r-r' \geq 0 \end{cases}$, $\Sigma_{k'}^{union}$ has the expected codimension.

In general, one can define

$$\begin{aligned}\Sigma_k^{sum} :&= \{(V_i)_i \in \prod_i Gr(r_i,n) \mid \dim \sum_i V_i \leq k\} \\ &= M_k(\varphi : \bigoplus_i \pi_i^{-1} S_i \longrightarrow \mathcal{O}^{\oplus n})\end{aligned}$$

with the expected dimension $(\sum_i r_i - k)(n-k)$.

When $\begin{cases} k \geq \max\{r_i\}_i \\ k \leq n \end{cases}$, Σ_k^{sum} has expected codimension.

A more general case (also generalize [3264, Ex 12.11, Ex 12.9]).

Let X : sm proj, $\mathcal{F}_i \subset \mathcal{E}$ are v.b.s

rank $\rightarrow$ r_i n

$$\begin{aligned}\Sigma_k^{sum} :&= \{p\in X \mid \dim \sum_i \mathcal{F}_i|_p \leq k\} \\ &= \{p \in X \mid \oplus \mathcal{F}_i|_p \longrightarrow \mathcal{E}|_p \text{ is of rank} \leq k\} \\ &= M_k(\varphi : \bigoplus_i \mathcal{F}_i \longrightarrow \mathcal{E})\end{aligned}$$

The general partial flag variety can be express as the degenaracy loci.

E.g. $\mathcal{F}lag_{r_1,r_2,r_3}(\mathbb{C}^n) := \{ 0 \subset V_1 \subset V_2 \subset V_3 \subset \mathbb{C}^n \mid \dim V_i = r_i \}$

$$= \{ (V_1, V_2, V_3) \in \prod_i Gr(r_i, n) \mid \dim V_i + V_{i+1} \leq r_{i+1} \}$$

$$= M_{r_2+r_3}\left(\varphi : \begin{matrix} \pi_1^{-1} S_1 \oplus \pi_2^{-1} S_2 \\ \oplus \\ \pi_2^{-1} S_2 \oplus \pi_3^{-1} S_3 \end{matrix} \longrightarrow \begin{matrix} \mathcal{O}^{\oplus n} \\ \oplus \\ \mathcal{O}^{\oplus n} \end{matrix} \right)$$

$$= \pi_{12}^{-1} \Sigma_{r_2}^{sum} \cap \pi_{23}^{-1} \Sigma_{r_3}^{sum}$$

Ramification locus [Barth 04 I.16]

Let $Y, X/\mathbb{C}$: sm of dim n, $f: Y \longrightarrow X$ finite.
The ramification divisor of f is defined as

$$\begin{aligned} R := & \{ y \in Y \mid T_y f : T_y Y \longrightarrow T_{f(y)} X \text{ is not surj} \} \\ = & \{ y \in Y \mid f^* : T^*_{f(y)} X \longrightarrow T^*_y Y \text{ is not surj} \} \\ = & \left\{ y \in Y \;\middle|\; \begin{array}{l} \text{rank } \varphi_y \leq n-1, \text{ where} \\ \varphi : f^*\Omega_X \longrightarrow \Omega_Y \end{array} \right\} \\ = & M_{n-1} (f^*\Omega_X \xrightarrow{\varphi} \Omega_Y) \end{aligned}$$

with the expected codim $(n-(n-1))(n-(n-1)) = 1$

Rmks. 1. R may have multiplicity, which is also counted in the degeneracy loci.
Recall that, for the zero set of section, we also count the multiplicity

2. Since

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n \text{ is of } rk \leq n-1 \Leftrightarrow \det \mathbb{C}^n \longrightarrow \det \mathbb{C}^n \text{ is zero,}$$

we get

$$R = M_0 (f^*\omega_X \longrightarrow \omega_Y)$$

$$\omega_Y = f^*\omega_X \otimes_{\mathcal{O}_Y} \mathcal{O}_Y(R) \quad \rightsquigarrow \text{Hurwitz formula}$$

$$0 \longrightarrow f^*\omega_X \longrightarrow \omega_Y \longrightarrow \iota_{R,*}\mathcal{O}_R \longrightarrow 0$$

3. I guess that we can generalize to f generic finite,
the we can get ramification locus + special fiber part.

How to distinguish these two locus?
Guess: for those special fibers, the pushforward will give us zero cycle. Can we use that?

4. For Y, X sm variety of dim Y, dim X,
when $f: Y \longrightarrow X$ is a closed embedding, we get:

$$0 \longrightarrow N^\vee_{Y/X} \longrightarrow f^*\Omega_X \longrightarrow \Omega_Y \longrightarrow 0$$

In this case, $\varphi: f^*\Omega_X \longrightarrow \Omega_Y$ is always surj,
so the degeneracy loci is meaningless.

4. Subvariety given by very ample bundle

For $X/\mathbb{C}$ sm proj, $\mathcal{F}/X$ v.b. of rank r, assume that $(\mathcal{F}, s_1, \dots, s_n)$ provides an embedding

$$\phi_{\mathcal{F}}: X \longrightarrow Gr(r,n),$$

we want to compute $[X] \in H_{2\dim_{\mathbb{C}} X}\, Gr(r,n)$.

Rmk 1. This is different from the previous construction. Here, the cycle we want to compute is the pushforward, not the pullback. By Poincaré dauility in Gr(r,n), the pushforward can be computed using the pullback of special cycles ir Gr(r,n).

Rmk 2. We know

$$\mathcal{L} \text{ is very ample} \iff (\mathcal{L}, \Gamma(X;\mathcal{L})) \text{ induces embedding}$$
$$\mathcal{F} \text{ is very ample} \implies (\mathcal{F}, \Gamma(X;\mathcal{F})) \text{ induces embedding}$$

See p73 in
R Hartshorne, Ample vector bundles
https://www.numdam.org/item/PMIHES_1966__29__63_0.pdf

E.g. When $\mathcal{L}/X$ induces an embedding $\phi_{\mathcal{L}}: X \longrightarrow \mathbb{P}^{n-1}$,

$$\begin{aligned}[X] &= \phi_{\mathcal{L}}^*([H]^{\dim X})\,[\mathbb{P}^{\dim X}] \\ &= \deg \mathcal{L} \cdot [\mathbb{P}^{\dim X}].\end{aligned}$$

Therefore,

$$\text{compute } X \iff \text{compute } \deg \mathcal{L}.$$

Special embeddings

Let us include several special embeddings here.

E.g. For $\iota_X : X \hookrightarrow Gr(r,n)$, ι_X is induced by $\iota_X^* \mathcal{S}^\vee$.

E.g. (Veronese embedding)
For $d \geq 1$, $\mathcal{O}_{\mathbb{P}^n}(d)$ induces the Veronese embedding $\quad Sym^d \mathcal{O}^{\oplus n} \twoheadrightarrow Sym^d \mathcal{O}(1)$

$$\begin{aligned} \nu_d : \mathbb{P}^n &\longrightarrow \mathbb{P}^{\binom{n+d}{d}-1} \\ [z_0 : \cdots : z_n] &\longmapsto [z_0^{i_0} \cdots z_n^{i_n}]_{\sum i_k = d} \end{aligned}$$

of degree d^n.

It describes when the deg d hypersurfaces in P^n degenerates as a (non-reduced) hyperplane.

E.g. (Segre embedding)
$\mathcal{L} := \mathcal{O}_{\mathbb{P}^m}(1) \boxtimes \mathcal{O}_{\mathbb{P}^n}(1)$ induces the Segre embedding $\quad \mathcal{O}^{\oplus(m+1)} \boxtimes \mathcal{O}^{\oplus(n+1)} \twoheadrightarrow \mathcal{L}$

$$\begin{aligned} \phi_{\mathcal{L}} : \mathbb{P}^m \times \mathbb{P}^n &\longrightarrow \mathbb{P}^{(m+1)(n+1)-1} \\ ([x_0 : \cdots : x_m], [y_0 : \cdots : y_n]) &\longmapsto [x_i y_j]_{i,j} \end{aligned}$$

of degree $\binom{m+n}{m}$.

E.g. (Plücker embedding) $\quad \Lambda^r \mathcal{O}^{\oplus n} \twoheadrightarrow \Lambda^r \mathcal{S}^\vee$
In $Gr(r,n)$, $\det \mathcal{S}^\vee$ induces the Plücker embedding

$$\begin{aligned} \iota_{Gr} : Gr(r,n) &\longrightarrow \mathbb{P}(\Lambda^r \mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{r}-1} \\ W = \langle w_1, \dots, w_r \rangle &\longmapsto w_1 \wedge \cdots \wedge w_r \end{aligned}$$

of degree $(r(n-r))! \prod_{i=0}^{r-1} \frac{i!}{(n-r+i)!}$ [3264, Prop 4.12]

Not very ample case

There are cases where L is not very ample.

E.g.

$$\begin{array}{ccccc} \mathrm{Sym}^2 \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \\ \| & & \| & & \| \\ \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}^{\oplus 6} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}^{\oplus 9} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1) \end{array}$$

induces the map

$$\phi : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathbb{P}^8 \dashrightarrow \mathbb{P}^5$$

$$([x_0:x_1:x_2],[y_0:y_1:y_2]) \mapsto \begin{bmatrix} x_0y_0 & \cdots & x_0y_2 \\ \vdots & & \vdots \\ x_2y_0 & \cdots & x_2y_2 \end{bmatrix} \longmapsto \begin{bmatrix} x_0y_0 : x_0y_1+x_1y_0 : x_0y_2+x_2y_0 \\ : x_1y_1 : x_1y_2+x_2y_1 \\ : x_2y_2 \end{bmatrix}$$

$\mathrm{Im}\,\phi$ describes when the quadric curve degenerates as union of two lines.
Since $\phi : \mathbb{P}^2 \times \mathbb{P}^2 \longrightarrow \mathrm{Im}\,\phi$ is a 2:1 ramified cover,

$$\phi^*([H]^4) = 6 \quad \Rightarrow \quad \mathrm{Im}\,\phi \subset \mathbb{P}^5 \text{ is of degree } 3.$$

E.g. [3264, 2.2.1]

$$\begin{array}{ccccc} \mathrm{Sym}^3 \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^5}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \boxtimes \mathrm{Sym}^2 \mathcal{O}_{\mathbb{P}^5}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \\ \| & & \| & & \| \\ \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^5}^{\oplus 10} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^5}^{\oplus 18} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1) \end{array}$$

induces the map

$$\phi : \mathbb{P}^2 \times \mathbb{P}^5 \longrightarrow \mathbb{P}^{17} \dashrightarrow \mathbb{P}^9$$

$$(\underset{\sum x_i z_i}{\overset{}{F_1}}, \underset{\sum y_{ij} z_i z_j}{F_2}) \longmapsto \underset{\sum_{i,j,k} x_i y_{jk} z_i z_j z_k}{F_1 F_2}$$

$\mathrm{Im}\,\phi$ describes when the cubic curve contains a line.
Since $\phi : \mathbb{P}^2 \times \mathbb{P}^5 \longrightarrow \mathrm{Im}\,\phi$ is generically injective,

$$\phi^*([H]^7) = \binom{7}{2} = 21 \quad \Rightarrow \quad \mathrm{Im}\,\phi \subset \mathbb{P}^9 \text{ is of degree } 21.$$

E.g. [3264, 2.2.2]

$$\begin{array}{ccccc}
\mathrm{Sym}^3\mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\boxtimes\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\boxtimes\mathcal{O}_{\mathbb{P}^2}^{\oplus 3} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{O}_{\mathbb{P}^2}(1) \\
\| & & \| & & \| \\
\mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 10} & \longrightarrow & \mathcal{O}_{(\mathbb{P}^2)^3}^{\oplus 27} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{O}_{\mathbb{P}^2}(1)\boxtimes\mathcal{O}_{\mathbb{P}^2}(1)
\end{array}$$

induces the map

$$\phi: \mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2 \longrightarrow \mathbb{P}^{26} \dashrightarrow \mathbb{P}^9 \quad (F_1, F_2, F_3) \longmapsto F_1F_2F_3$$

$\mathrm{Im}\,\phi$ describes when the cubic curve degenerates as union of three lines.
Since $\phi: \mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2 \longrightarrow \mathrm{Im}\,\phi$ is a $6:1$ ramified cover,

$$\phi^*([H]^6) = \binom{6}{4}\binom{4}{2} = \frac{6!}{2!2!2!} = 90 \Rightarrow \mathrm{Im}\,\phi \subset \mathbb{P}^9 \text{ is of degree } 15.$$