Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases
4. conclusions on various forms

Setting. We work over conn red gp over F. ($G/_F$ conn red)

$\bar{F}$: the seperable closure of F mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_F := Gal(\bar{F}/F)$ $\sigma \in \Gamma_F$

$$Z^1(W,A) := \left\{ \begin{array}{l} {}^L\varphi : W \longrightarrow A \rtimes W \\ \quad \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{array} \middle| \begin{array}{l} {}^L\varphi : \textit{continuous group homo} \\ {}^L\varphi \textit{ is a section} \end{array} \right\}$$

$$= \{ \varphi : W \longrightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)\,{}^{\gamma}(\varphi(\gamma')) \}$$

$\varphi \in H^1(W,A)$

$$H^1(W,A) = Z^1(W,A)/A.$$

Def. (Split) G is split if
G has a maximal torus T over F which is split.

Def. (Quasi-split) G is quasi-split if
G has a Borel B over F.

Borel = maximal (Zar-closed) conn sol alg subgp
= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t G/H is projective
= closed subgp containing a Borel.

Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat{Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability
https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf

https://mathoverflow.net/questions/121959/classification-of-tori-of-gl2-up-to-conjugation

I make no claim to originality.

1. forms.

Def. $G_1, G_2/F$ are called forms, if
$\exists\ \alpha: G_{2,\bar F} \xrightarrow{\sim} G_{1,\bar F}$ as gps   not as Γ_F-gps!

α is considered as the information of forms.

Thm. $\{F\text{-forms of } G\}/\sim \longleftrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$

$[G_2,\ \alpha: G_{2,\bar F} \to G_{\bar F}] \longmapsto \varphi_\alpha := \alpha\sigma\alpha^{-1}\sigma^{-1}$

$G_2 \longleftarrow\!\shortmid\ \varphi$

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} \end{tikzcd}$$

$\Gamma_F \circlearrowleft G_{2,\bar F} \longrightarrow G_{\bar F} \circlearrowleft \Gamma_F, \Gamma_F$

φ_α measures the twist between two Γ_F-actions.

$G_2(F) := \{ g \in G(\bar F) \mid \underbrace{(\varphi(\sigma)\circ\sigma)}_{\text{induced action}} g = g \quad \forall \sigma \in \Gamma_F \}$

In general, $G_2(R) := \{ g \in G(\bar F \otimes_F R) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_F \}$

e.p. $G_2(\bar F) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_F} \in \prod_{\sigma\in\Gamma_F} G(\bar F) \mid g \in G(\bar F) \} \cong G(\bar F)$

$G(\bar F \otimes_F \bar F) \cong G(\bigoplus_{\sigma\in\Gamma_F} \bar F) \cong \prod_{\sigma\in\Gamma_F} G(\bar F)$

$(G_2, \alpha) \sim (G_2', \alpha')$, if
$\exists\ \beta: G_2 \to G_2'$ as an iso.

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

Functorial on F. (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])
Let E/F be finite Galois.

$$\begin{tikzcd} G_{2,E} & \{E\text{-forms of } G\} \arrow[r, leftrightarrow] & H^1(\Gamma_E, \mathrm{Aut}(G_{\bar F})) & \varphi|_{\Gamma_E} \\ G_2 \arrow[u, mapsto] & \{F\text{-forms of } G\} \arrow[u] \arrow[r, leftrightarrow] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[u] & \varphi \arrow[u, mapsto] \\ & \{G_2/F : G_{2,E} \cong G_E\} \arrow[u] \arrow[r, leftrightarrow] & H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(G_{\bar F})^{\Gamma_E}) \arrow[u] & \\ & 1 \arrow[u] & 1 \arrow[u] & \end{tikzcd}$$

$\mathrm{Aut}(G_{\bar F})^{\Gamma_E} = \mathrm{Aut}(G_E)$

Rmk. We have the classification of connected reductive gps:

$$\{\text{split red gp}/F\} \longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) \overset{\simeq}{\Longrightarrow} \Psi(G,B,T)$$

G split

$$\{\text{qs red gp}/F\} \longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + \Gamma_F\text{-action}$$
$$= (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, \mathrm{Out}(G_{\bar{F}}))$$
$$\{\text{red gp}/F\} \longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, \mathrm{Aut}(G_{\bar{F}}))$$

To understand the result, the following isos are needed:

$$\mathrm{Aut}(G_{\bar{F}}) \cong \mathrm{Inn}(G_{\bar{F}}) \rtimes \mathrm{Aut}(G, B, T, \{u_\alpha\})$$

$$\mathrm{Out}(G_{\bar{F}}) \cong \mathrm{Aut}(G, B, T, \{u_\alpha\}) \quad \text{for embedding}$$
$$\cong \mathrm{Aut}(\Psi(G,B,T)) \quad \text{for combinatorics}$$

Also, by the Hilbert 90, one has
$$H^1(\Gamma_F, \mathrm{Aut}(G,B,T,\{u_\alpha\})) \cong H^1(\Gamma_F, \mathrm{Aut}(G,B,T))$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} Aut(\mathbb{G}_m^n) & \subseteq & End(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$Hom(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow\!\shortmid n$

Therefore,

$$\begin{aligned} H^1(\Gamma_F, Aut(\mathbb{G}_{m,\bar{F}}^n)) &= H^1(\Gamma_F, GL_n(\mathbb{Z})) \\ &= Hom_{Grp}(\Gamma_F, GL_n(\mathbb{Z}))/GL_n(\mathbb{Z})\text{-conj} \\ &\underset{\text{when } F=\mathbb{R}}{=\!=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = Id\}/GL_n(\mathbb{Z})\text{-conj} \end{aligned}$$

[Γ_F acts on $Aut(\mathbb{G}_{m,\bar{F}}^n) \subseteq End(\mathbb{G}_{m,\bar{F}}^n)$ trivially:

see $\bar{F}$-pts, $n=1$:

$$\begin{tikzcd} \bar{F}^\times \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & \bar{F}^\times \arrow[d, "\sigma"] \\ \bar{F}^\times \arrow[r, "{}^\sigma\alpha"] & \bar{F}^\times \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, mapsto] \arrow[d, mapsto] & x^n \arrow[d, mapsto] \\ \sigma(x) & \sigma(x^n) = \sigma(x)^n \end{tikzcd}$$

$\Rightarrow {}^\sigma\alpha = \alpha$]

E.g. $n=1$, $F=\mathbb{R}$

$$H^1(\Gamma_F, Aut(\mathbb{G}_{m,\bar{F}})) \cong \{1, -1\}$$

$\rightsquigarrow \varphi(\sigma) = (-)^{-1}$

$1 \Downarrow \mathbb{G}_m$, $\quad -1 \Downarrow G = ?\ SO_{2,\mathbb{R}}$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\} \\ &= \{g \in \mathbb{C}^\times \mid |g| = 1\} \\ &= S^1 \\ G(\mathbb{C}) &= \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times \\ \Rightarrow G &= \operatorname{Spec} \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}} \end{aligned}$$

Check:
$$\begin{aligned} G(\mathbb{C}) &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1 = 0\} \\ &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1\} \\ &= \left\{(x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \,\middle|\, \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array}\right\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \,\middle|\, x,y \in K,\ x^2+y^2 = 1\right\}$$

E.g. $n=2$, $F=\mathbb{R}$

$$H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}_{m,\bar F}^2)) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

$$\mathbb{G}_m^2 \quad \mathbb{G}_m \times SO_{2,\mathbb{R}} \quad (SO_{2,\mathbb{R}})^2 \quad G = ? \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; \underbrace{(\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{= \varphi(\sigma) \begin{pmatrix} \bar z_1 \\ \bar z_2 \end{pmatrix} = \begin{pmatrix} \bar z_2 \\ \bar z_1 \end{pmatrix}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar z_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\mathbb{G}_m, \quad SO_{2,\mathbb{R}}, \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

$$1 \quad -1 \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

$$\updownarrow \quad \updownarrow \quad \updownarrow$$

Fact^{dual}: $\mathrm{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{ \mathbb{Z}_{triv}, \mathbb{Z}_{sign}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}] \}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

Rmk. Using the same argument, one can show that
$\{ T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}_{m,\mathbb{F}_{p^2}}^n \}$ = products of $\mathbb{G}_m$, $\begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}$, $\mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \mathbb{G}_m$

The torus G crspd to -1: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$G(\mathbb{F}_p) = \{ g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma) \circ \sigma) g = g \quad \forall \sigma \in \Gamma_k \}$$

$$= \{ a + b\xi \in \mathbb{F}_{p^2}^\times \mid \underbrace{\varphi(\sigma)(a - b\xi)}_{= (a - b\xi)^{-1}} = a + b\xi \}$$

$$= \{ a + b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2 \varepsilon = 1 \}$$

$$\cong \left\{ \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p) \right\}$$

3. other cases.

By using the same methods introduced in last section, we can compute the (inner) forms of reductive gps.

	inner forms	outer forms
$\mathbb{G}_m$ $(\mathbb{G}_m)^2$ $(\mathbb{G}_m)^n$	ϕ	SO_2 $SO_2 \times \mathbb{G}_m$, $(SO_2)^2$, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ product of lower rank torus
$GL_{2,\mathbb{R}}$ $SL_{2,\mathbb{R}}$ $PGL_{2,\mathbb{R}}$	$\mathbb{H}^\times = GL_1(\mathbb{H} \otimes_{\mathbb{R}} -)$ $\mathbb{H}^{Nm=1} = SU_{2,\mathbb{C}/\mathbb{R}}$?	$(U_{2,\mathbb{C}/\mathbb{R},\omega} = U(1,1)\ \ U(2,0))$ ϕ ϕ
$GL_{n,\mathbb{R}}$ $SL_{n,\mathbb{R}}$ $PGL_{n,\mathbb{R}}$	? $GL_{n/2}^{Nm=1}(\mathbb{H} \otimes_{\mathbb{R}} -)$ when n even ?	$\left(U_{n,\mathbb{C}/\mathbb{R},\omega} = \begin{cases} U(\frac{n}{2},\frac{n}{2}) & n \text{ even} \\ U(\frac{n+1}{2},\frac{n-1}{2}) & n \text{ odd} \end{cases} \quad U(p,q)\right)$ $SU(a,n-a)$ e.p. $SU(2,1)$?
$(SL_2)^2/\mathbb{R}$ $(SL_2)^3/\mathbb{R}$	$SL_2 \times SU_2$, $(SU_2)^2$, ϕ? $(8-1)$ possibilities	$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$?

← need clarification

? : I have no time to compute / don't know any symbol to represent

(shaded box) : quasi-split gp

Compute $\mathrm{Aut}(G_{\bar F})$

Lemma. We understand $\mathrm{Aut}(G_{\bar F})$ quite well.

$$\begin{array}{ccccccccc}
 & & G(\bar F)/_{Z(G(\bar F))} = G^{ad}(\bar F) & & & & \mathrm{Aut}(\Psi_0) & & \\
 & & \| & & & & \| & & \\
G & 1 \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow & 1
\end{array}$$

G	$\mathrm{Inn}(G_{\bar F})$	$\mathrm{Aut}(G_{\bar F})$	$\mathrm{Out}(G_{\bar F})$	
T rk n	1	$GL_n(\mathbb{Z})$	$GL_n(\mathbb{Z})$	
$GL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$SL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$PGL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$n \geq 3$				
$GL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$	$\{\pm 1\}^{\oplus 2}$	
$SL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$PGL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$(SL_2)^2/\mathbb{R}$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	with different $\Gamma_{\mathbb{R}}$-action
$(SL_2)^n/\mathbb{R}$	$PGL_n(\mathbb{C})^n$	$PGL_n(\mathbb{C})^n \rtimes S^n$	S^n	

Compute $H^1(\Gamma_F, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP] **Theorem 12.7.7** *There is a functorial isomorphism* $\mathrm{H}^1(k, G) \to \pi_1(G)_{\Theta,\mathrm{tor}}$.

[DR09] COROLLARY 2.4.3. *The composition*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

is a bijection.

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field:

Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
https://arxiv.org/abs/1401.5913

3.1. Theorem. *Let G, T_0, T, and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

global field:

Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
https://arxiv.org/pdf/2303.04120.pdf

E.g. $G = SL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{lccccccc} G & 1 \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\ SL_{2,\mathbb{R}} & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) & & 1 & \end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(SL_{2,\mathbb{C}})) = H^1(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C}))$$
$$= \{1, \quad \omega(\ )\omega^{-1}\}$$
$$\Downarrow \qquad \Downarrow$$
$$SL_{2,\mathbb{C}} \qquad G = ?$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\quad \mathbb{H}^{N_m = 1}$

$$\begin{aligned} G(\mathbb{R}) &= \{ g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}} \} \\ &= \{ g \in SL_2(\mathbb{C}) \mid \omega \bar{g} \omega^{-1} = g \} \\ &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \mid \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \} \\ &= \{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in SL_2(\mathbb{C}) \} \\ &= \mathbb{H}^{N_m = 1} \end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{lccccccc} G & 1 \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\ GL_{2,\mathbb{R}} & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) \rtimes \{\pm 1\} & & \{\pm 1\} & \end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(GL_{2,\mathbb{C}})) = \{1, \quad \omega(\ )\omega^{-1}, \quad (\ )^{H,-1}, \quad \omega(\ )^{H,-1}\omega^{-1}\}$$
$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$
$$(GL_{2,\mathbb{C}} \qquad \mathbb{H}^\times) \qquad (\mathcal{U}(2,0) \qquad \mathcal{U}_{2,\mathbb{C}/\mathbb{R},\omega})$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

4. conclusions on various forms

$H^1(\Gamma_F,-)$ as parameter space

$$
\begin{tikzcd}
 & & & 1 \arrow[dr] & & 1 & & \\
1 \arrow[r] & Z(G(\bar F)) \arrow[r] & G(\bar F) \arrow[r] & \mathrm{Inn}(G_{\bar F}) \arrow[ur] \arrow[r] & \mathrm{Aut}(G_{\bar F}) \arrow[r] & \mathrm{Out}(G_{\bar F}) \arrow[r] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & & & & H^2(\Gamma_F, Z(G(\bar F))) & \\
 & & & H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \arrow[ur] \arrow[dr] & & \\
\rightsquigarrow H^1(\Gamma_F, Z(G(\bar F))) \arrow[r] & H^1(\Gamma_F, G(\bar F)) \arrow[urr] & & \text{inner twist} & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[r] & H^1(\Gamma_F, \mathrm{Out}(G_{\bar F})) \\
 & \text{pure inner twist} & & & \text{form} &
\end{tikzcd}
$$

$\{F\text{-forms of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$

$\{F\text{- inner forms of } G\}/\sim \;\longleftrightarrow\; \mathrm{Im}\big(H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))\big)$

$\{F\text{- inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$

$\{F\text{-pure inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, G(\bar F))$

G split: $\left\{\begin{matrix}F\text{-forms of } G \\ \text{which are quasi-split}\end{matrix}\right\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) \cong H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))$

$$H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) = \Gamma_F\text{-actions on } (X^*, \Delta, X_*, \Delta^\vee)$$

Q: Do we have

$$
\begin{tikzcd}
 & H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \arrow[r, hook] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[r] & H^1(\Gamma_F, \mathrm{Out}(G_{\bar F})) \\
1 \arrow[r] & \mathrm{Inn}(G_{\bar F})^{\Gamma_F} \arrow[r] \arrow[d, equal] & \mathrm{Aut}(G_{\bar F})^{\Gamma_F} \arrow[r] \arrow[d, equal] & \mathrm{Out}(G_{\bar F})^{\Gamma_F} \arrow[d, equal] \arrow[ull, bend right] \\
 & \mathrm{Inn}(G_F) & \mathrm{Aut}(G_F) & \mathrm{Out}(G_F)
\end{tikzcd}
$$

Give one example s.t. $H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$ is not inj ?

Categorification of $H^1(\Gamma_F, -)$

These categories are all groupoids.
These $H^1(\Gamma_F, -)$ are all achieved as isomorphism classes.

	Obj	Mor $((G_2,\alpha),(G_2',\alpha'))$
form $H^1(\Gamma_F, Aut(G_{\bar F}))$	$(G_2, \alpha: G_{2,\bar F} \to G_{\bar F})$ $\Rightarrow$ $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha"] \arrow[urr, "\sigma(\alpha)"] & G_{\bar F} & \end{tikzcd}$ commutes $\forall \sigma \in \Gamma_F$	$\beta: G_2 \to G_2'$ iso $\Rightarrow$ $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ commutes
inner form $Im\left(\begin{array}{c} H^1(\Gamma_F, Inn(G_{\bar F})) \\ \downarrow \\ H^1(\Gamma_F, Aut(G_{\bar F})) \end{array}\right)$ full subcategory of "form"	$(G_2, \alpha: G_{2,\bar F} \to G_{\bar F})$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$ $\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.	$\beta: G_2 \to G_2'$ iso $\Rightarrow$ $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ commutes
inner twist $H^1(\Gamma_F, Inn(G_{\bar F}))$ less isomorphisms compared with inner form	$(G_2, \alpha: G_{2,\bar F} \to G_{\bar F})$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$ $\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.	$\beta: G_2 \to G_2'$ iso s.t $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ $\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}$ is inner auto.
pure inner twist $H^1(\Gamma_F, G(\bar F))$	$(G_2, \alpha: G_{2,\bar F} \to G_{\bar F}, \phi)$ $\phi \in Z^1(\Gamma_F, G(\bar F))$ s.t. $\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\phi(\sigma)\text{-conj}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$ commutes	(β, δ) $\beta: G_2 \to G_2'$ iso $\delta \in G(\bar F)$ s.t $\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$ commutes, and $\phi_1(\sigma) = \delta^{-1}\phi_2(\sigma)\sigma(\delta)$

<table>
<tr><th></th><th>Obj</th><th>Mor $((G_2, \alpha), (G_2', \alpha'))$</th></tr>
<tr>
<td>rigid inner twist

$H^1\begin{pmatrix} u(\bar{F}) \to \mathcal{E}^{rig}, \\ Z(\bar{F}) \to G(\bar{F}) \end{pmatrix}$</td>
<td>$(G_2, \alpha : G_{2,\bar{F}} \to G_{\bar{F}}, Z, z)$
$Z \subset Z(G)$ finite gp subscheme
$z \in Z^1\begin{pmatrix} u(\bar{F}) \to \mathcal{E}^{rig}, \\ Z(\bar{F}) \to G(\bar{F}) \end{pmatrix}$
s.t. $$\begin{tikzcd} G_{2,\bar{F}} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar{F}} \arrow[d, "\sigma"] \\ G_{2,\bar{F}} \arrow[r, "\alpha"] & G_{\bar{F}} & G_{\bar{F}} \arrow[l, "\bar{z}(\sigma)\text{-conj}"'] \end{tikzcd}$$
commutes</td>
<td>(β, δ)
$\beta : G_2 \longrightarrow G_2'$ iso $\quad \delta \in G(\bar{F})$
s.t $$\begin{tikzcd} G_{2,\bar{F}} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar{F}}"'] & G_{\bar{F}} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar{F}} \arrow[r, "\alpha'"] & G_{\bar{F}} \end{tikzcd}$$
commutes, and
$z_1(\sigma) = \delta^{-1} z_2(\sigma)\, \sigma(\delta)$</td>
</tr>
</table>

https://mathoverflow.net/questions/117033/center-of-the-algebraic-group-g-mathbbr-for-a-centerless-g
https://math.stackexchange.com/questions/953526/relative-center-of-relative-group-scheme