Eine Woche, ein Beispiel

6.12 Condensed

Main Ref: https://people.mpim-bonn.mpg.de/scholze/Course%20Summer%2022.html

That's already so well written. I collect some notations here purely for self-study, and I believe this document is useless for other people.

Condensed set

Def (pro-étale site $*_{\text{proét}}$)

Category: Prof

Cover: for $S \in \text{Prof}$,

$$\text{Cov}(S) = \left\{ \{S_i \xrightarrow{f_i} S\}_{i\in I} \text{ in Prof} \;\middle|\; \begin{array}{l} I \text{ finite} \\ S = \bigcup_{i\in I} f_i(S_i) \end{array} \right\}$$

Naive def. ⚠ Caveat: Prof is large. Need minor modification.

$$\begin{aligned}
\text{CondSet} &= Sh(*_{\text{proét}}) \\
&= \left\{ X : \text{Prof}^{op} \longrightarrow \text{Set} \;\middle|\; \begin{array}{l} X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1) \times X(S_2) \\ 0 \to X(S) \to X(T) \rightrightarrows X(T\times_S T) \quad T \twoheadrightarrow S \end{array} \right\} \\
&= \{ X : \text{qcProj}^{op} \longrightarrow \text{Set} \mid X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1)\times X(S_2) \} \\
\text{CondAb} &= Sh(*_{\text{proét}}, \text{Ab}) \\
\text{Cond}(\mathcal{C}) &= Sh(*_{\text{proét}}, \mathcal{C}) \\
\text{Cond}(R) &= \text{Cond}(\text{Mod}(R)) = Sh(*_{\text{proét}}, \text{Mod}(R)) \qquad R \in \text{Ring}
\end{aligned}$$

when $R \in \text{Cond}(\text{Ring})$, require compatability.

Analytic ring and complete condensed A-module

Def. ⚠ Preliminary

An analytic ring is $\mathcal{A} = (A, \mathcal{M}_A(-), \delta)$, where

- $A \in \text{Cond}(\text{Ring})$
- $\mathcal{M}_A : \text{Prof} \longrightarrow \text{Cond}(A) \qquad S \longmapsto \mathcal{M}_A(S)$
- $\delta_S : S \longrightarrow \mathcal{M}_A(S) \qquad s \longmapsto \delta_s$
- $$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & A \end{tikzcd} \qquad \mu \mapsto \int f\mu$$
- $M \in \text{Cond}(A)$ is complete if
$$\begin{tikzcd} S \arrow[r, "\delta_S"] \arrow[dr, "f"'] & \mathcal{M}_A(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & M \end{tikzcd}$$

We require that the full subcategory

$\text{Cond}_{cpl}(A) := \{\text{complete condensed } A\text{-modules}\} \subseteq \text{Cond}(A)$

should be abelian category.

$0<p\le 1$ afterwards.

Liquid vector spaces. $S \in \mathrm{Prof}$.

$$\mathcal{M}(S) = \{ f : C(S;\mathbb{R}) \to \mathbb{R} \mid f \text{ cont} \} = \mathbb{R}[S]^{\blacksquare}$$

$$\mathcal{M}(S)_{\ell^p \le c} = \varprojlim_i \mathcal{M}(S_i)_{\ell^p\le c} \subseteq \varprojlim_i \mathbb{R}^{\oplus S_i}$$

$$\mathcal{M}_p(S) = \bigcup_{c>0} \mathcal{M}(S)_{\ell^p \le c}$$

$$\mathcal{M}_{<p}(S) = \bigcup_{0<q<p} \mathcal{M}_q(S)$$

Def. Let $V \in \mathrm{CondAb}$ and $0<p\le 1$.
V is p-liquid if

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_{<p}(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd}$$

equiv:

$$\begin{tikzcd} S \arrow[r, "\delta"] \arrow[dr, "f"'] & \mathcal{M}_q(S) \arrow[d, dashed, "\exists!\, \tilde{f}"] \\ & V \end{tikzcd} \quad \forall q<p$$

equiv:

$$\bigoplus_j \mathcal{M}_{<p}(T_j) \to \bigoplus_i \mathcal{M}_{<p}(S_i) \to V \to 0$$

Relations

$$\begin{tikzcd} & \mathrm{Solid} \arrow[d, shift left] & \\ \mathrm{Liq}_p \arrow[r, shift right] & \mathrm{CondAb} \arrow[l, shift right, "(-)^{liq}"'] \arrow[u, shift left, "(\ )^{\blacksquare}"] \arrow[r, shift right] \arrow[l, phantom, "\perp"] \arrow[u, phantom, "\perp"] & \mathrm{CondSet} \arrow[l, shift right, "\mathbb{Z}[-]"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

Liq_p ⇒

- Abelian
- $-\otimes_{\mathbb{R}_{<p}} -$, $-\otimes^{\mathbb{L}}_{\mathbb{R}_{<p}} -$, $\underline{\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$, $\underline{R\mathrm{Hom}}_{\mathbb{R}_{<p}}(-,-)$
- flatness
- projective objects

CondAb ⇒

- Abelian
- $-\otimes -$, $-\otimes^{\mathbb{L}} -$, $\underline{\mathrm{Hom}}(-,-)$, $\underline{R\mathrm{Hom}}(-,-)$
- flatness
- projective objects

Prop 3.4. $M \in \mathrm{CondAb}$ is flat $\Leftrightarrow$ $M(S)$ is torsion free $\forall S \in \mathrm{qcProj}$
$\Leftrightarrow$ $M(S)$ is torsion free $\forall S \in \mathrm{Prof}$

Prop 3.5. $M \in \mathrm{CondAb}$ is cpt proj $\Leftrightarrow$ M is a retract of $\mathbb{Z}[S]$ for some $S \in \mathrm{Extr}$
$\Leftrightarrow$ M is a retract of $\mathbb{Z}[\beta I]$ for some infinite set I with discrete topology

$\Rightarrow M \oplus \mathbb{Z}[\beta I] \simeq \mathbb{Z}[\beta I]$

Recall that for $A, B \in \mathrm{Ob}(\mathcal{C})$, A is a retract of B if $\exists\,(r,i)$ s.t

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[d, "\mathrm{Id}_A"] \\ B \arrow[r, "r"] & A \end{tikzcd} \quad \text{commutes}$$

Thm 3.14. $\mathcal{M}_{<p}(S) \in \mathrm{Liq}_p$ is flat $\forall S \in \mathrm{Prof}$.

Moreover, $\forall V \in \mathrm{Liq}_p$ qs. we have an iso

$$\mathcal{M}_{<p}(S) \otimes_{\mathbb{R}_{<p}} V \cong \bigcup_{q<p} \bigcup_{\substack{K \subset V \\ q\text{-convex}}} \mathcal{M}_q(S, K)$$

Here, for S finite,

$$\mathcal{M}_q(S, K) = \langle s \otimes K \mid_{s \in S} \rangle_{q\text{-convex hull}} \subseteq \mathbb{R}[S] \otimes_{\mathrm{Cond}(\mathbb{R})} V ;$$

for $S = \varprojlim_i S_i$ profinite,

$$\mathcal{M}_q(S, K) = \varprojlim_i \mathcal{M}_q(S_i, K) \subseteq \mathbb{R}[S] \otimes_{\mathrm{Cond}(\mathbb{R})} V .$$

More examples of p-liquid spaces. $\mathcal{M}_{<p}(X)$, $\ell^{<p}(I)$ and $\mathcal{O}(\mathbb{D})$

Def. ($\mathcal{M}_{<p}(X)$ for $X \in$ loc. Prof)

$$\begin{aligned} \mathcal{M}(X)_{\ell^p \le c} &= \{ p\text{-measures on } X \text{ with upper bound } C \} \\ &= \left\{ \mu : \{K \subset X \text{ cpt open}\} \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \mu(K_1 \sqcup K_2) = \mu(K_1) + \mu(K_2) \\ \sum_{i \text{ finite}} |\mu(K_i)|^p \le C \end{array} \right\} \\ &\subset \prod_{\substack{K \subset X \\ \text{cpt open}}} [-C^{\frac{1}{p}}, C^{\frac{1}{p}}] \quad \text{with product topology} \end{aligned}$$

$\Rightarrow \mathcal{M}(X)_{\ell^p \le c} \in \mathrm{CHaus} \le \mathrm{CondSet}$

$$\begin{aligned} \mathcal{M}_p(X) &= \{ p\text{-measures on } X \} \\ &= \bigcup_{c>0} \mathcal{M}(X)_{\ell^p \le c} \\ \mathcal{M}_{<p}(X) &= \bigcup_{0<q<p} \mathcal{M}_q(X) \end{aligned}$$

Def. Let $I := \mathbb{N}_{\ge 0}$ be a countable set with discrete topo, so $I \in$ loc. Prof.

$$\ell(I)_{\ell^p \le c} = \{ (x_i)_{i \in I} \in \mathbb{R}^I \mid \sum_i |x_i|^p \le C \} \quad \text{with } \ell^p\text{-topology}$$

$\Rightarrow \ell(I)_{\ell^p \le c} \in \mathrm{Top} \longrightarrow \mathrm{CondSet}$

$$\begin{aligned} \ell^p(I) &= \{ (x_i)_{i \in I} \in \mathbb{R}^I \mid \sum_i |x_i|^p < +\infty \} \\ &= \bigcup_{c>0} \ell(I)_{\ell^p \le c} \\ \ell^{<p}(I) &= \bigcup_{0<q<p} \ell^q(I) \end{aligned}$$

We have maps among topological spaces:

$$\begin{array}{ccccccccccccc} \cdots \subset & \mathcal{M}_{\frac{1}{3}}(I) & \subset & \mathcal{M}_1(I) & \subset & \mathcal{M}_2(I) & \subset & \cdots & \subset & \mathcal{M}_\infty(I) & \subset & \prod_I \mathbb{R} \\ & \cup & & \cup & & \cup & & & & \cup & & \\ & & & & & & & & & \ell^\infty(I) & & \\ & & & & & & & & & \cup & & \\ \cdots \subset & \ell^{\frac{1}{3}}(I) & \subset & \ell^1(I) & \subset & \ell^2(I) & \subset & \cdots & \subset & c_0(I) & & \end{array}$$

Def.

$$\begin{aligned}
\mathcal{O}(\mathbb{D})_{\mathbb{R}} &= \Big\{ \sum_{i\geq 0} c_i T^i \in \mathbb{R}[[T]] \;\Big|\; c_i r^i \to 0 \quad \forall r<1 \Big\} \\
&= \Big\{ \sum_{i\geq 0} c_i T^i \in \mathbb{R}[[T]] \;\Big|\; c_i r^i \text{ is bounded} \quad \forall r<1 \Big\} \\
&\cong \Big\{ (c_i)_{i\in I} \in \mathbb{R}^I \;\Big|\; \sum_i |c_i r^i|^p < +\infty \quad \forall r<1 \Big\} \\
&= \Big\{ (c_i)_{i\in I} \in \mathbb{R}^I \;\Big|\; \sum_i |c_i r_n^i|^p < +\infty \quad \forall n\in\mathbb{N}_{\geq 0} \Big\}
\end{aligned}$$

(Here we choose $r_n = 1-\frac{1}{2n}$ $\quad r_0 = \frac{1}{3}$ $\quad , \mathbb{N} = \mathbb{N}_{\geq 0}$)

$$\cong \varprojlim_n \mathcal{M}_{<p}(\mathbb{N})$$

where

$$\longrightarrow \mathcal{M}_{<p}(\mathbb{N}) \xrightarrow{\lambda^{(2)}} \mathcal{M}_{<p}(\mathbb{N}) \xrightarrow{\lambda^{(1)}} \mathcal{M}_{<p}(\mathbb{N})$$

$$\{b_n^{(1)}\}_{n\in\mathbb{N}} \longmapsto \Big\{ b_n^{(1)} \cdot \Big(\frac{r_0}{r_1}\Big)^n \Big\}_{n\in\mathbb{N}}$$

$$\{b_n^{(2)}\}_{n\in\mathbb{N}} \longmapsto \Big\{ b_n^{(2)} \Big(\frac{r_1}{r_2}\Big)^n \Big\}_{n\in\mathbb{N}}$$

[Think in this way:

$$\{c_n r_1^n\}_{n\in\mathbb{N}} \longmapsto \Big\{ c_n r_1^n \Big(\frac{r_0}{r_1}\Big)^n \Big\}_{n\in\mathbb{N}}$$

$$\{c_n r_2^n\}_{n\in\mathbb{N}} \longmapsto \Big\{ c_n r_2^n \Big(\frac{r_1}{r_2}\Big)^n \Big\}_{n\in\mathbb{N}}$$
]

We can discuss these examples with $\otimes_{\mathbb{R}_{<p}}$ & flatness.