Eine Woche, ein Beispiel 4.28 naive &- Hom adjunction

Ref: from [23.11.19]

Notation: - A: associate ring allowed to be non-commutative, contains 1 - There are two systems to write category of A-modules.

$$Mod_A = A - Mod$$
 $(Mod_A)^{\circ p} \neq Mod_{A^{\circ p}} = Mod - A = A^{\circ p} - Mod \Rightarrow M_A$ 
 $Mod_{A \otimes B^{\circ p}} = A - Mod - B \Rightarrow A^{M_B}$ 

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

$$\nabla$$
 Even though you can identify  $Ob(Ring) \cong Ob(Ring^{op})$ ,  $A^{op}$  is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

In this case, it is desirable to translate algebraic results into geometrical results. Q: How to see the geometry of noncommutative rings? It is still vague for me.

- 1 definition recall for ⊗ & Hom
- 2 adjunction
- 3. comparison between ⊗-1 Hom & f\*-1 f\*

6. comparison between ⊗-1 Hom & f\*-1 f\*, derived version

## 1 definition recall for ⊗ & Hom

$$\otimes_A: \operatorname{Mod}_{A^{\circ P}} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$$
  
 $\operatorname{Hom}_A(-,-): (\operatorname{Mod}_A)^{\circ P} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$ 

$$\otimes_{B}$$
:  $A - Mod - B \times B - Mod - C \longrightarrow A - Mod - C$   
 $Hom_{B}(-,-)$ :  $(A - Mod - B)^{\overline{P}} \times B - Mod - C \longrightarrow A - Mod - C$ 

$$Hom_{B}^{A}(-,-)$$
:  $(A-Mod-B)^{\overline{op}} \times B-Mod-A \longrightarrow \mathbb{Z}-Mod$ 

$$Hom_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) (\mathbb{Z}-Mod-B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}-Mod-\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}-Mod-\mathbb{Z}$$

$$(X \otimes_{B} Y) \otimes_{C} Z \cong X \otimes_{B} (Y \otimes_{C} Z)$$

$$X \otimes_{B} Y \cong Y \otimes_{B^{op}} X$$

$$A \otimes_{A} X \cong X \cong X \otimes_{B} B$$

$$Hom_{A}(A, X) \cong X$$

in 
$$A-Mod-C = C^{op}-Mod-A^{op}$$

2 adjunction BXA, cYB, cZD, we get

 $Homc(Y \otimes_{B} X, Z) \cong Hom_{B}(X, Homc(Y, Z))$  in A-Mod-D.

Reason: both sides equal to the set  $f: Y \times X \longrightarrow Z \mid f(cyb,x) = cf(y,bx) \quad \forall b,c$ 

For A = D = Z, fix  $Y \in C$ -Mod-B, one gets adjunction fctors.

# slogan: adjunction & associativity

3. comparison between ⊗-1 Hom & f\*-1 f\*

#### Forgetful fctor

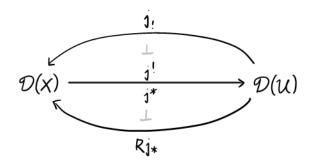
Prop. For ring homo 
$$\begin{array}{ll} S \\ R \\ R \end{array} = \begin{array}{ll} S \\ -Mod \\ U(M) = \begin{array}{ll} R \\ -Mod \\ R \end{array} = \begin{array}{ll} M \\ -Mod \\ R \end{array} = \begin{arra$$

one has adjunction fctors

djunction fctors
$$S_{R} \otimes_{R} - \frac{\sum_{S_{R} \otimes_{R} - 1}^{S_{R} \otimes_{R}} \otimes_{S_{R} - 1}}{\sum_{S_{R} \otimes_{S} - 1}^{S_{R} \otimes_{S}} \otimes_{S_{R} - 1}} R-Mod \qquad (3.1)$$

## Compare with j

Now, we compare (3.1) with part of the recollement diagram:



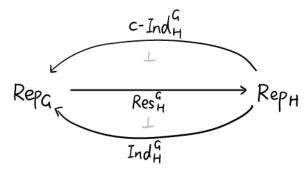
Vague slogan:  $u \approx$  "forget the information of Z".

In applications.  $U \longrightarrow X$  is a covering map. This change the feeling of the size between U & X.

E.g. For finite gps 
$$H \leq G$$
, one has Res-Ind adjunction.  
 $Res_{H}^{G} \dashv Ind_{H}^{G}$   
 $c-Ind_{H}^{G} \dashv Res_{H}^{G}$ 

It can be generalized for 
$$G: loc$$
 profinite  $gp$ ,  $H \leq G$  open If one only has  $H \leq G$  closed, then it's possible that  $j' \neq j^*$ . e.g.  $G = GL_1(\mathbb{Q}_p)$   $H = GL_2(\mathbb{Z}_p)$ 

In the diagram,



Ex Compare it with the recollement diagram & (3.1).

$$U$$
 [\*/H]   
 $\downarrow$  "cover with fiber G/H"   
 $X$  [\*/G]

translate the following geometrical results into algebraic statements.

1. One has natural fctor 
$$j_! \longrightarrow j_*$$
. When  $\# G/H < +\infty$ ,  $j_! = j_*$ 
 $c - Ind_H^G \longrightarrow Ind_H^G$ 

2. Even though

Sho.v.([\*/G]) ≈ Repa = Q[G]-Mod. the "structure sheaf" of [\*/G] is Q. not Q[G].

$$\operatorname{Res}_{f*}^{G} Q = Q$$
,  $\operatorname{Res}_{f*}^{G} Q[G] = Q[G] \neq Q$ 

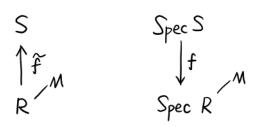
√ In this example, j\*Rj\* ≠ Id, j'j! ≠ Id.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

### Compare with i

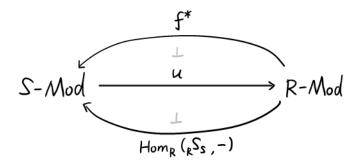
Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

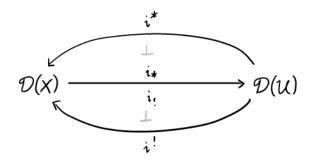


 $\exists$  "pullback fctor"  $f^*: R\text{-}Mod \longrightarrow S\text{-}Mod \qquad f^*M = sS_R \otimes_R M$  This is also called the base change.

Now, (31) can be rewritten as



compare it with another part of the recollement diagram.



Rmk. u is usually not f faithful, unless S = R/I. (In fact, only need S is R-idempotent, i.e.  $S \cong S \otimes_R S$ .) which croppds to closed embedding. In that case,

$$i^*i_* = Id$$
:  ${}_{S}S_R \otimes_R ({}_{R}S_S \otimes_S M) \cong M$   
 $i^*i_* = Id$ :  $Hom_R ({}_{R}S_S, Hom({}_{S}S_R, M)) \cong M$ 

Slogan: in the comm alg., Spec  $R/I \longrightarrow Spec R$  is closed embedding. In general, if S is an R-idempotent algebra.  $S \cong S \otimes_R S$ then i. Spec  $S \longrightarrow Spec R$  can be viewed as "closed subset".

E.g.  $R_p$ , R/I are idempotent R-algs.  $Z[\frac{1}{6}]$ ,  $F_p$ ,  $Z/p^2Z$ , Q,  $Z_p$ , are idem Z-algs. Usually R/1 is not an derived idem R-alg!

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

Rmk Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"? A: (possibly) the Verdier quotient.
We will come back to this after we derive everything.