Modular form
5. moduli interpretation

1 level structure
2. moduli interpretation of $\Gamma\backslash\mathcal{H}$
3. cplx polarization
4. Siegel moduli space
5 Hilbert moduli space

Ex.

group	alg gp	act on	stabilizer at non-ell pt	gen & relation
$SL_2(\mathbb{Z})$	✓	$\mathcal{H}$	$\{\pm Id\}$	$\langle S,T \mid S^4=(ST)^6=Id\rangle$
$GL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	$\{\pm Id\}$	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$
$PSL_2(\mathbb{Z})$	×	$\mathcal{H}$	Id	$\langle S,T \mid S^2=(ST)^3=Id\rangle$
$PGL_2(\mathbb{Z})$	✓	$\mathcal{H}\sqcup\mathcal{H}^-$	Id	$\langle S,T,\begin{pmatrix}1&\\&-1\end{pmatrix}\rangle$

can't define $SL_2/\mathbb{G}_m$

https://arxiv.org/pdf/1605.07726.pdf
https://math.stackexchange.com/questions/1844504/why-is-this-isomophism-of-pgl2-mathbbz-with-a-coxeter-group-injective

See [https://mathoverflow.net/questions/181366/minimal-number-of-generators-for-gln-mathbbz] for a higher dimension generalization.

Ex. $A\leq B\leq C$ gp $\quad A\lhd C \Rightarrow A\lhd B$

no other restrictions. i.e. the following cases may happen:

$\underbrace{A\lhd B\lhd C}_{\lhd}$	$\underbrace{A\lhd B\leq C}_{\lhd}$	$A\lhd B\lhd C$	$A\lhd B\leq C$	$A\leq B\lhd C$	$A\leq B\leq C$
✓	✓	$C_2\lhd A_4\lhd S_4$	✓	✓	$S_2\leq S_3\leq S_4$

1 level structure

Def (congruence subgp) They're the preimage of some subgp of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

$$\begin{tikzcd}
\Gamma(N) \arrow[r, two heads] & \{Id\} \\
\Gamma_1(N) \arrow[r, two heads] \arrow[u, phantom, "\cap" rotate=180] & N(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \arrow[u, phantom, "\cap" rotate=180] \\
\Gamma_0(N) \arrow[r, two heads] \arrow[u, phantom, "\cap" rotate=180] & B(\mathbb{Z}/N\mathbb{Z}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \arrow[u, phantom, "\cap" rotate=180] \\
\Gamma(1) = SL_2(\mathbb{Z}) \arrow[r, two heads, "{[WWL, Prop 1.4.4]}"] \arrow[u, phantom, "\cap" rotate=180] & SL_2(\mathbb{Z}/N\mathbb{Z}) \arrow[u, phantom, "\cap" rotate=180] \\
\Gamma^0(N) \arrow[r, two heads] \arrow[u, phantom, "\cup" rotate=180] & \begin{pmatrix} * & 0 \\ * & * \end{pmatrix} \arrow[u, phantom, "\cup" rotate=180] \\
\Gamma^1(N) \arrow[r, two heads] \arrow[u, phantom, "\cup" rotate=180] & \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \arrow[u, phantom, "\cup" rotate=180]
\end{tikzcd}$$

⚠ $SL_2(\mathbb{Z}/N\mathbb{Z})$ is not $\mathbb{Z}/N\mathbb{Z}$-pt of $SL_2 = \operatorname{Spec} \mathbb{Z}[a_{11}, a_{12}, a_{21}, a_{22}]/(a_{11}a_{22} - a_{12}a_{21} - 1)$, but

$$SL_2(\mathbb{Z}/N\mathbb{Z}) := SL_{2,\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} a, b, c, d \in \mathbb{Z}/N\mathbb{Z} \\ ad - bc = 1 \end{array} \right\}$$

Ex. Verify the following tables (left comes from right)

$A \triangleleft B$ (row A, column B)	$\Gamma(N)$	$\Gamma_1(N)$	$\Gamma_0(N)$	$\Gamma(1)$
$\Gamma(N)$	–	✓	✓	✓
$\Gamma_1(N)$	–	–	✓	×
$\Gamma_0(N)$	–	–	–	×
$\Gamma(1)$	–	–	–	–

$A \triangleleft B$ (row A, column B)	N	B	G
N	–	✓	×
B	–	–	×
G	–	–	–

Ex. show [WWL, 练习 1.4.14]

练习 1.4.14 对所有正整数 N, 证明

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma(N)) = N^3 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N^2 \prod_{\substack{p:\text{素数}\\ p\mid N}} \left(1-\frac{1}{p^2}\right),$$

$$(\mathrm{SL}(2,\mathbb{Z}):\Gamma_0(N)) = |(\mathbb{Z}/N\mathbb{Z})^\times|^{-1}\cdot(\mathrm{SL}(2,\mathbb{Z}):\Gamma_1(N)) = N\prod_{p\mid N}\left(1+\frac{1}{p}\right).$$

A: Reduced to computation of $|SL_2(\mathbb{Z}/N\mathbb{Z})|$, $|B(\mathbb{Z}/N\mathbb{Z})|$, $|N(\mathbb{Z}/N\mathbb{Z})|$.

Try $N=5,4,6$ if you don't understand the process.

Notation: $\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) := (\mathbb{Z}/N\mathbb{Z}^{\oplus 2})_{prim}/(\mathbb{Z}/N\mathbb{Z})^* \overset{[6.3\,M]}{=\!=} \mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}(\mathbb{Z}/N\mathbb{Z})$

See Def 5 here: https://arxiv.org/pdf/2010.15543v2.pdf

▽ $\mathbb{P}'_{\mathbb{Z}/N\mathbb{Z}}$ is covered by two $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$'s [4.5.N],

$\begin{bmatrix}3\\2\end{bmatrix} \in \mathbb{P}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) - \bigcup_{i=1,2} \mathbb{A}_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$, these do not contradict with each other.

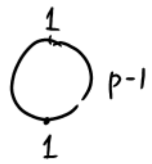
Reason: Spec $\mathbb{Z}/6\mathbb{Z}$ are two pts. They may lie in different piece of $\mathbb{A}_{\mathbb{Z}/N\mathbb{Z}}$.

① $|SL_2(\mathbb{F}_p)| = p^3-p$

$|B(\mathbb{F}_p)| = p^2-p$

$|N(\mathbb{F}_p)| = p$

$\# \mathbb{F}_p^\times = p-1$



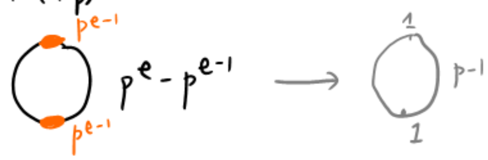
$\mathbb{P}'(\mathbb{F}_p)$

② $|SL_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e}-p^{3e-2}$

$|B(\mathbb{Z}/p^e\mathbb{Z})| = p^{2e}-p^{2e-1}$

$|N(\mathbb{Z}/p^e\mathbb{Z})| = p^e$

$\#(\mathbb{Z}/p^e\mathbb{Z})^\times = p^e-p^{e-1}$



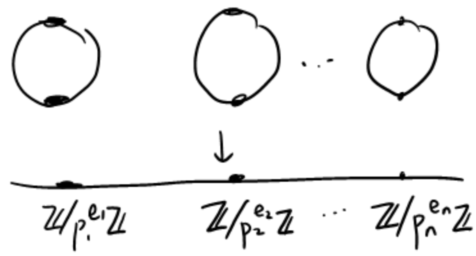
$0 \to p\mathbb{Z}/p^e\mathbb{Z} \to \mathbb{P}'(\mathbb{Z}/p^e\mathbb{Z}) \longrightarrow \mathbb{P}'(\mathbb{Z}/p\mathbb{Z}) \to 0$

③ $|SL_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{\substack{p \text{ prime}\\ p\mid N}} \left(1-\frac{1}{p^2}\right)$

$|B(\mathbb{Z}/N\mathbb{Z})| = N^2 \prod_{\substack{p \text{ prime}\\ p\mid N}} \left(1-\frac{1}{p}\right)$

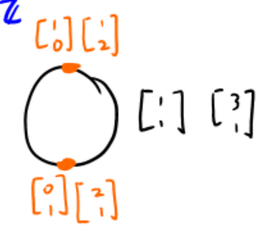
$|N(\mathbb{Z}/N\mathbb{Z})| = N$

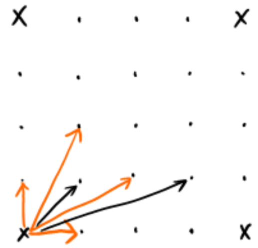
$\#(\mathbb{Z}/N\mathbb{Z})^\times = \varphi(N) = N \prod_{\substack{p \text{ prime}\\ p\mid N}} \left(1-\frac{1}{p}\right)$



$\mathbb{P}'(\mathbb{Z}/N\mathbb{Z}) \simeq \prod_{k=1}^n \mathbb{P}'(\mathbb{Z}/p_k^{e_k}\mathbb{Z})$

E.g. $\mathbb{Z}/4\mathbb{Z}$





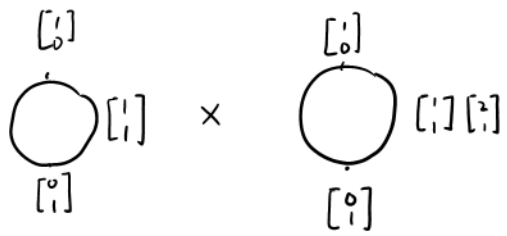
E.g. $\mathbb{Z}/6\mathbb{Z}$

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}/6\mathbb{Z}[x,y] = \bigcup_{\substack{f\in S_+ \\ f \text{ homogeneous}}} \operatorname{Spec}\left(\mathbb{Z}/6\mathbb{Z}[x,y]_f\right)_0$$

e.g. $(x-2,y-3) \lhd \mathbb{Z}/6\mathbb{Z}[x,y]$ is not prime.

$$\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{F}_3)$$
$$\cong \mathbb{P}^1_{\mathbb{Z}/2\mathbb{Z}}(\mathbb{F}_2) \times \mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}}(\mathbb{F}_3)$$

Ex. Use [Vakil, 6.3.M] to compute $\mathbb{P}^1_{\mathbb{Z}/6\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z})$. Enjoy it!



$\mathbb{P}^1(\mathbb{Z}/6\mathbb{Z})$ ╲ $\mathbb{P}^1(\mathbb{Z}/3\mathbb{Z})$ / $\mathbb{P}^1(\mathbb{Z}/2\mathbb{Z})$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}5\\4\end{bmatrix}$	$\begin{bmatrix}1\\4\end{bmatrix}$	$\begin{bmatrix}3\\4\end{bmatrix}$
$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}1\\3\end{bmatrix}$	$\begin{bmatrix}5\\1\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$	$\begin{bmatrix}3\\1\end{bmatrix}$
$\begin{bmatrix}0\\1\end{bmatrix}$	$\begin{bmatrix}2\\3\end{bmatrix}$	$\begin{bmatrix}2\\1\end{bmatrix}$	$\begin{bmatrix}4\\1\end{bmatrix}$	$\begin{bmatrix}0\\1\end{bmatrix}$

Rmk. The original proof is also good, but less geometrically obvious:
(Now you should understand the geometry in every step)

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & SL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[d] & SL_2(\mathbb{F}_p) \arrow[d] & \\
0 \arrow[r] & \overset{p^{4e-4}}{1+pM_2(\mathbb{Z}/p^e\mathbb{Z})} \arrow[r] & GL_2(\mathbb{Z}/p^e\mathbb{Z}) \arrow[r] \arrow[d] & \overset{(p^2-1)(p^2-p)}{GL_2(\mathbb{F}_p)} \arrow[r] \arrow[d] & 0 \\
& & (\mathbb{Z}/p^e\mathbb{Z})^\times \ {\scriptstyle p^e-p^{e-1}} \arrow[d] & \mathbb{F}_p^\times \ {\scriptstyle p-1} \arrow[d] & \\
& & 0 & 0 &
\end{tikzcd}$$

Finally, use Chinese remainder theorem to get

$$SL_2(\mathbb{Z}/N\mathbb{Z}) \cong \prod_{k=1}^n SL_2(\mathbb{Z}/p_k^{e_k}\mathbb{Z}) \qquad \square$$

Ex. do the exactly same thing with SL_2 replaced by GL_2 and PGL_2.

Ex. (hard) explore the Tits building & rep theory of $SL_2(\mathbb{Z}/N\mathbb{Z})$.

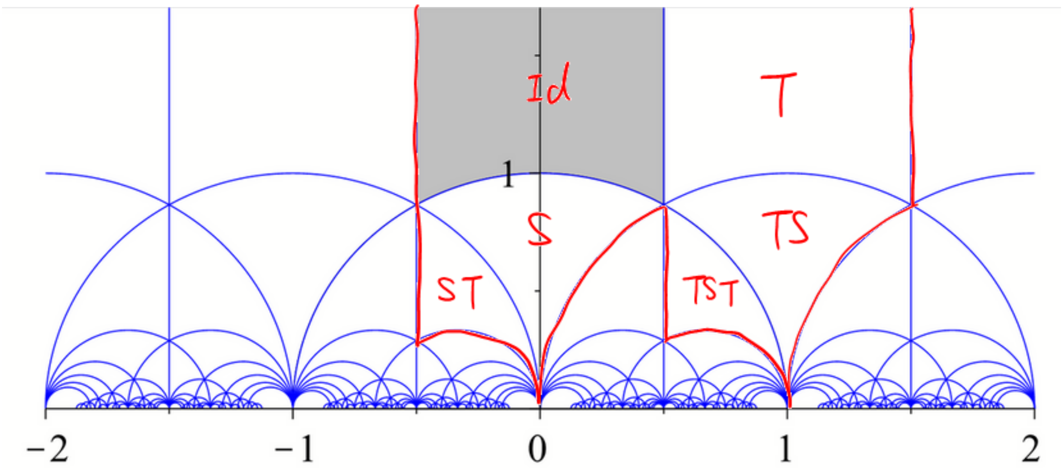
It will be used later on (I believe)

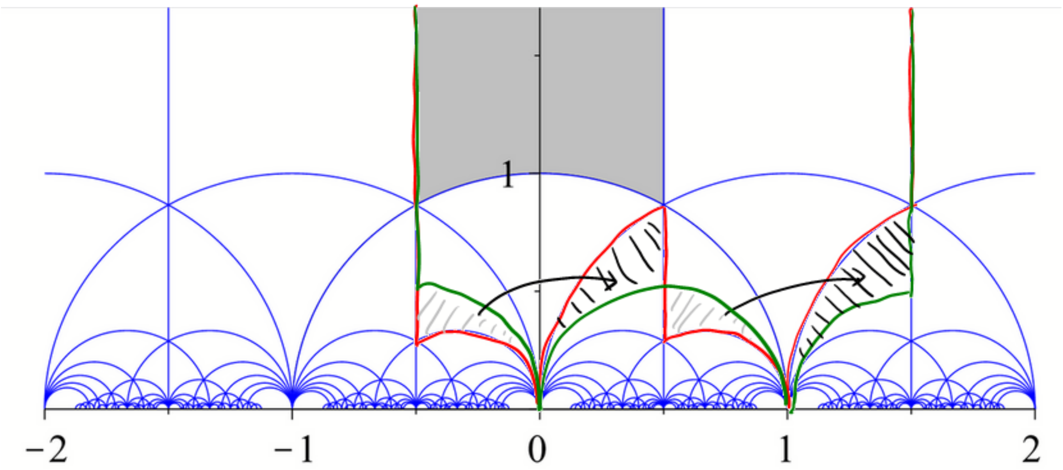
Is the Tits building of $SL_2(\mathbb{Z}) \longrightarrow SL_2(\mathbb{Z}/N\mathbb{Z})$ functorial?

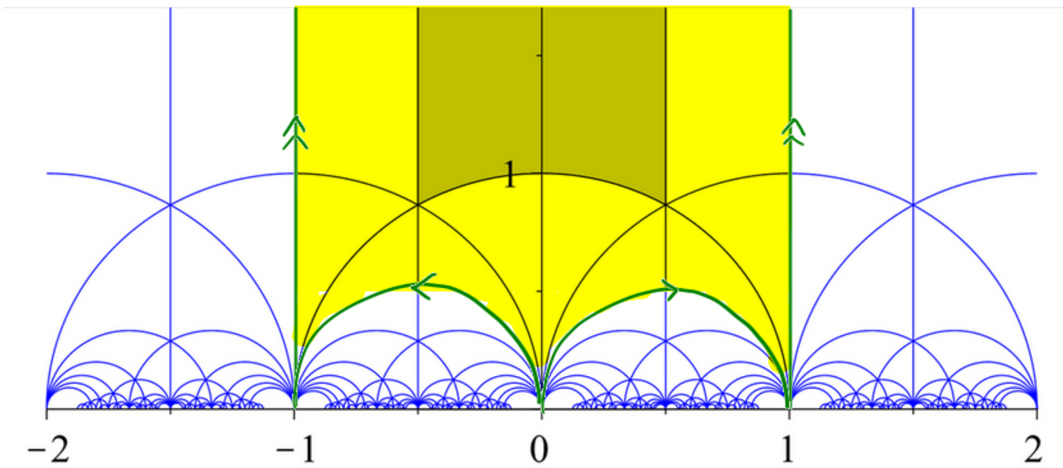
we write left quotient from now on, since it's a left action

Ex. Draw the fundamental domain of $\Gamma(2)\backslash\mathcal{H}$.

Hint. $\Gamma(1)/\Gamma(2) = \{Id, T, S, TS, ST, TST\}$





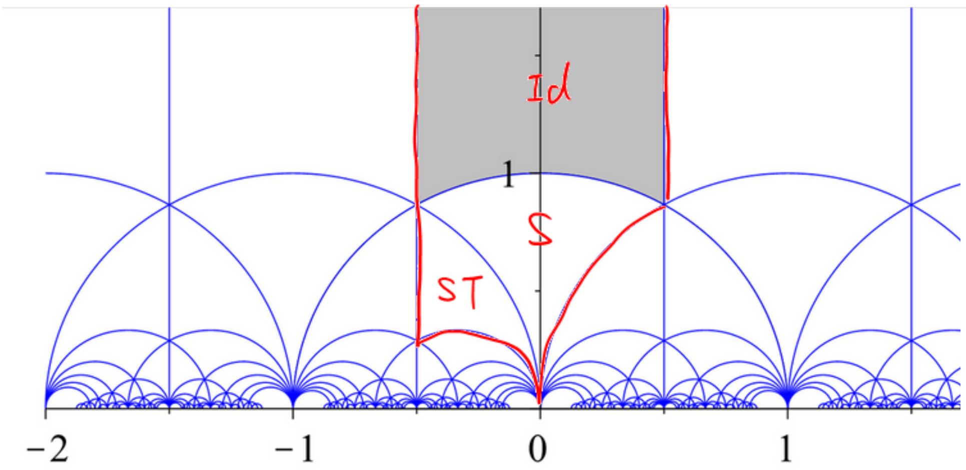


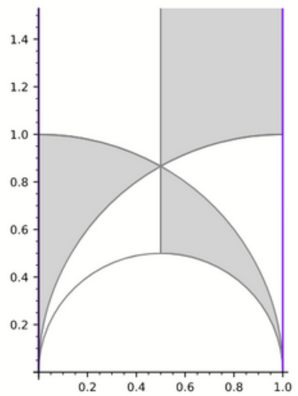
Cor. $\Gamma(2)/\{\pm Id\} = \mathbb{Z} * \mathbb{Z} = \langle \begin{pmatrix}1&2\\0&1\end{pmatrix}, \begin{pmatrix}1&0\\2&1\end{pmatrix} \rangle$

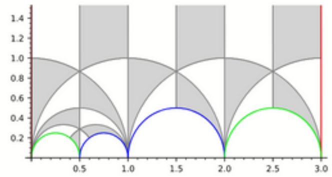
Ex. Draw the fundamental domain of $\Gamma_0(2)\backslash\mathcal{H}$. $\Gamma_0(2)=\Gamma_1(2)$

∇ $SL_2(\mathbb{Z}) \curvearrowright \Gamma_0(2)\backslash\mathcal{H}$ is not well-defined. e.g. $S\,i \neq S(i+1)$ in $\Gamma_0(2)\backslash\mathcal{H}$.

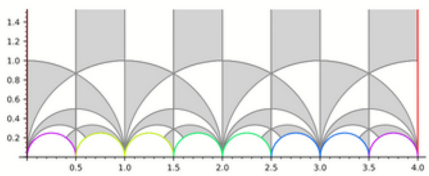
Hint. $\Gamma_0(2)\backslash\Gamma(1) = \{Id, S, ST\}$



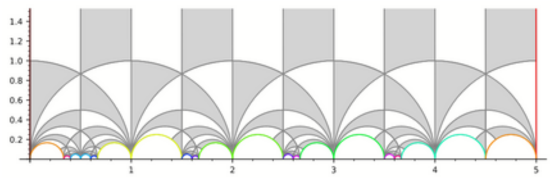




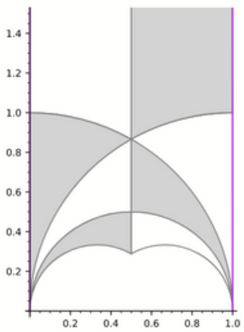
$\Gamma(3)$



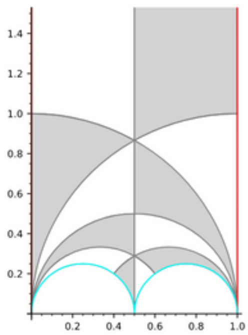
$\Gamma(4)$



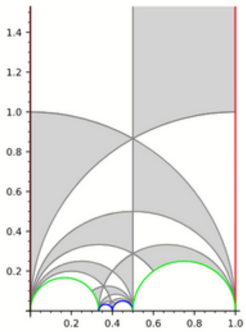
$\Gamma(5)$



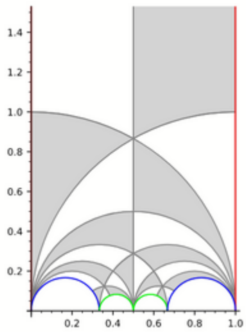
$\Gamma_1(3)=\Gamma_0(3)$



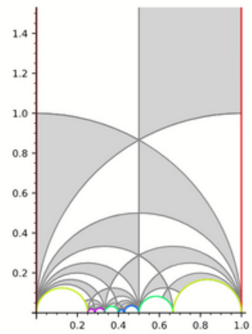
$\Gamma_1(4)=\Gamma_0(4)$



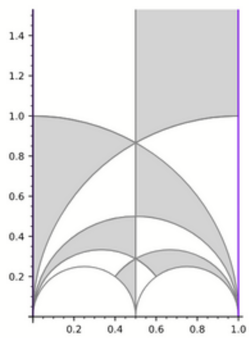
$\Gamma_1(5)$



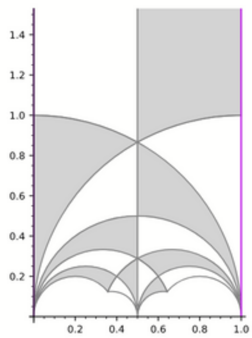
$\Gamma_1(6)=\Gamma_0(6)$



$\Gamma_1(7)$



$\Gamma_0(5)$



$\Gamma_0(7)$

2. moduli interpretation of $\Gamma\backslash\mathcal{H}$

Def. A basis (v_1,v_2) of a lattice $\Lambda\subseteq\mathbb{C}$ is called oriented if $\operatorname{Im}\frac{v_1}{v_2}>0$.

Def (Weil pairing) [WWL, 注记8.5.9, 定义3.8.9, 练习3.8.10]

For $N\in\mathbb{Z}_{\geq1}$, $E=\mathbb{C}/\Lambda$, $\Lambda=\mathbb{Z}u\oplus\mathbb{Z}v$, $\operatorname{Im}\frac{u}{v}>0$, we define the Weil pairing e_N:

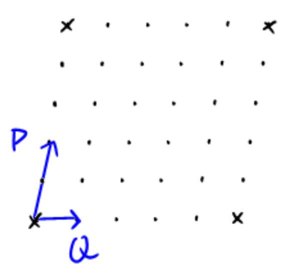
$$\begin{tikzcd}
& E[N]\times E[N] \arrow[d, phantom, "\wr\|"] \arrow[ddr, "e_N"] & \\
a\frac{u}{N}+c\frac{v}{N} \arrow[dd] & \frac{1}{N}\Lambda/\Lambda\times\frac{1}{N}\Lambda/\Lambda \arrow[d, phantom, "\wr\|"] & \\
& (\mathbb{Z}/N\mathbb{Z})^{\oplus2}\times(\mathbb{Z}/N\mathbb{Z})^{\oplus2} \arrow[r] & \mu_N \cong (\mathbb{Z}/N\mathbb{Z},+) \\
\binom{a}{c} & \left(\binom{a}{c},\binom{b}{d}\right) \arrow[r, mapsto] & \zeta_N^{\left|\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right|} \mapsto \left|\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right|
\end{tikzcd}$$

Ex. Let $e_1,e_2\in E[N]$.

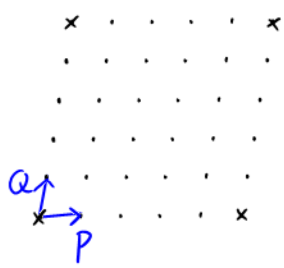
1. e_N is antisymmetric and bilinear.
   $e_N(\gamma(e_1,e_2)) = \zeta_N^{\det\gamma}\, e_N(e_1,e_2)$ $\quad\forall\gamma\in GL_2(\mathbb{Z}/N\mathbb{Z})$
   e.p. e_N only depends on E and N (does not depend on Λ and u,v)
2. $e_N(e_1,e_2)\in\mu_N^\times \cong (\mathbb{Z}/N\mathbb{Z})^\times \iff E[N]=\langle e_1,e_2\rangle_{\mathbb{Z}}$
   $e_N(e_1,e_2)=\zeta_N \mapsto 1 \iff \exists P,Q\in\frac{1}{N}\Lambda$, $\bar P=e_1$, $\bar Q=e_2$, (NP,NQ) is an oriented basis of Λ.

Def. (e_1,e_2) is called a pretty oriented basis of $E[N]$, if $e_N(e_1,e_2)=\zeta_N$.

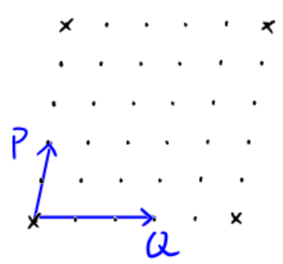
Ex. $N=5$



$e_N(\bar P,\bar Q)=\zeta_5^2$
$(5P,5Q)$ is not a basis of Λ.

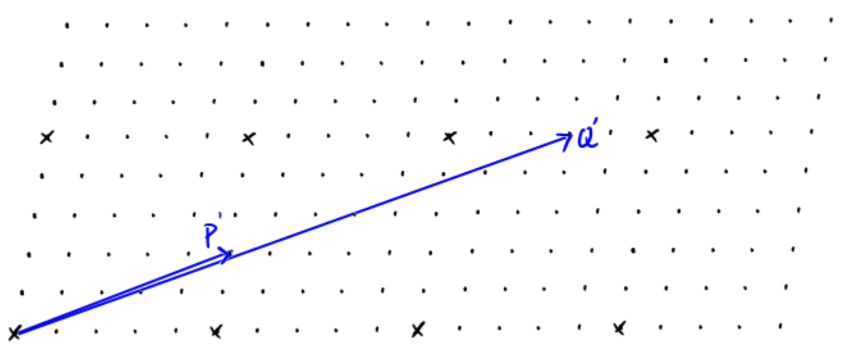


$e_N(\bar P,\bar Q)=\zeta_5^4=\zeta_5^{-1}$
$(5P,5Q)$ is a basis of Λ, but not an oriented basis.



$e_N(\bar P,\bar Q)=\zeta_5^6=\zeta_5$
$(5P,5Q)$ is not a basis of Λ but $(5P',5Q')$ is an oriented basis.

$$\underset{\cap\ SL_2(\mathbb{Z})}{\begin{pmatrix}2&5\\5&13\end{pmatrix}} \overset{\bmod 5}{\equiv}\begin{pmatrix}2&0\\0&3\end{pmatrix}$$



Recall: For $E = \mathbb{C}/\Lambda$, $E[N] \cong \frac{1}{N}\Lambda/\Lambda \cong \Lambda/N\Lambda$

Main Thm. We have the following moduli interpolations (E: any cplx EC curve)

$$
\begin{tikzcd}
\left\{(E,\alpha) \,\middle|\, \begin{array}{l} \alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \xrightarrow{\sim} E[N] \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N \end{array}\right\}/\sim \atop \text{E with a pretty oriented basis } (e_1,e_2) \arrow[r, "\sim"] \arrow[d, "\beta = \alpha(0,-)"] & \Gamma(N)\backslash\mathcal{H} \arrow[d, "N"] \\
\{(E,\beta) \mid \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow E[N]\}/\sim \atop \text{E with an N-torsion pt of E} \arrow[r] \arrow[d, "C = \operatorname{Im}\beta"] & \Gamma_1(N)\backslash\mathcal{H} \arrow[d, "\begin{cases} \frac{N}{2}\prod_{p\mid N}(1-\frac{1}{p}) & N\neq 2 \\ 1 & N=2 \end{cases}"] \\
\{(E,\mathcal{F}) \mid \mathcal{F}: 0 \subseteq C \subseteq E[N]\}/\sim \atop \text{E with } C\subseteq E[N],\ C\cong \mathbb{Z}/N\mathbb{Z} \arrow[r] \arrow[d] & \Gamma_0(N)\backslash\mathcal{H} \arrow[d, "N\prod_{p\mid N}(1+\frac{1}{p})"] \\
\{\text{cplx EC } E\}/\sim \arrow[r] & \Gamma(1)\backslash\mathcal{H}
\end{tikzcd}
$$

Idea. A pretty oriented basis on $E[N]$ gives us a oriented basis on E up to $\Gamma(N)$-action;

coefficient has to be 1, so that (v_1+bv_2, v_2) is a pretty oriented basis.

$\begin{pmatrix}1&b\\0&1\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}v_1+bv_2\\v_2\end{pmatrix}$ $\Rightarrow$ an n-torsion pt v_2

$\begin{pmatrix}a&b\\0&d\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix} = \begin{pmatrix}av_1+bv_2\\dv_2\end{pmatrix}$ $\Rightarrow$ a flag $0 \subseteq \underset{\langle v_2\rangle}{\underset{\parallel}{C}} \subseteq \underset{\langle v_1,v_2\rangle}{\underset{\parallel}{E[N]}}$

Proof. For $\Gamma(N)\backslash\mathcal{H}$,

$$
\begin{aligned}
\text{LHS} &\cong \left\{(\Lambda,\alpha) \,\middle|\, \begin{array}{l}\alpha: (\mathbb{Z}/N\mathbb{Z})^{\oplus 2} \longrightarrow \frac{1}{N}\Lambda/\Lambda \\ e_N(\alpha(1,0),\alpha(0,1)) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,e_1,e_2) \,\middle|\, \begin{array}{l} e_1,e_2 \in \frac{1}{N}\Lambda/\Lambda \\ e_N(e_1,e_2) = \zeta_N\end{array}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,z_1,z_2,e_1,e_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ e_1 = \frac{z_1}{N} \quad e_2 = \frac{z_2}{N} \\ e_N(e_1,e_2) = \zeta_N\end{array}\right\} {\circlearrowleft \mathbb{C}^\times \atop \circlearrowright \Gamma(N)} \\
&\cong \left\{(\Lambda,z_1,z_2) \,\middle|\, \begin{array}{l} \Lambda = \mathbb{Z}z_1 \oplus \mathbb{Z}z_2 \\ (z_1,z_2) \text{ is an oriented basis of } \Lambda\end{array}\right\} {\circlearrowleft \mathbb{C}^\times \atop \circlearrowright \Gamma(N)} \\
&\cong \left\{(z_1,z_2) \in (\mathbb{C}-\{0\})^2 \,\middle|\, \operatorname{Im}\frac{z_1}{z_2} > 0\right\} {\circlearrowleft \mathbb{C}^\times \atop \circlearrowright \Gamma(N)} \\
&\cong \Gamma(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}
$$

For $\Gamma_1(N)\backslash\mathcal{H}$,

$$\begin{aligned}
\text{LHS} &\cong \left\{(\Lambda,\beta) \,\middle|\, \beta: \mathbb{Z}/N\mathbb{Z} \hookrightarrow \tfrac{1}{N}\Lambda/\Lambda\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,e_2) \,\middle|\, e_2 \in \tfrac{1}{N}\Lambda/\Lambda,\ \mathrm{order}(e_2)=N\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,z_1,z_2,e_2) \,\middle|\, \begin{array}{l} \Lambda=\mathbb{Z}z_1\oplus\mathbb{Z}z_2 \\ e_2=\frac{z_2}{N} \\ Im\,\frac{z_1}{z_2}>0 \end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \\ \circlearrowleft \Gamma_1(N)\end{array} \\
&\cong \left\{(\Lambda,z_1,z_2) \,\middle|\, \begin{array}{l} \Lambda=\mathbb{Z}z_1\oplus\mathbb{Z}z_2 \\ Im\,\frac{z_1}{z_2}>0 \end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowleft \Gamma_1(N)\end{array} \\
&\cong \Gamma_1(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}$$

If not, set $(z_1,z_2)\mapsto(-z_1,z_2)$

For $\Gamma_0(N)\backslash\mathcal{H}$,

$$\begin{aligned}
\text{LHS} &\cong \left\{(\Lambda,C) \,\middle|\, C\subseteq \tfrac{1}{N}\Lambda/\Lambda,\ C\cong\mathbb{Z}/N\mathbb{Z}\right\} \circlearrowleft \mathbb{C}^\times \\
&\cong \left\{(\Lambda,z_1,z_2,C) \,\middle|\, \begin{array}{l} \Lambda=\mathbb{Z}z_1\oplus\mathbb{Z}z_2 \\ C=\left(\frac{z_2}{N}\right)\subseteq\frac{1}{N}\Lambda/\Lambda \\ Im\,\frac{z_1}{z_2}>0 \end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \\ \circlearrowleft \Gamma_0(N)\end{array} \\
&\cong \left\{(\Lambda,z_1,z_2) \,\middle|\, \begin{array}{l} \Lambda=\mathbb{Z}z_1\oplus\mathbb{Z}z_2 \\ Im\,\frac{z_1}{z_2}>0 \end{array}\right\} \begin{array}{l}\circlearrowleft \mathbb{C}^\times \\ \circlearrowleft \Gamma_0(N)\end{array} \\
&\cong \Gamma_0(N)\backslash\mathcal{H} = \text{RHS}
\end{aligned}$$

$\square$

Rmk. If you observe carefully, you will find out that what we prove actually is

$\mathbb{C}^\times = SO_2(\mathbb{R})\times\mathbb{R}_{>0}$

$$\begin{array}{ccccccc}
\{(E,\alpha)\} & \xrightarrow{\sim} & & & \Gamma(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{(E,\beta)\} & \xrightarrow{\sim} & \Gamma_1^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_1(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_1(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{(E,\mathcal{F})\} & \xrightarrow{\sim} & \Gamma_0^{\pm}(N)\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_0(N)\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & \Gamma_0(N)\backslash\mathcal{H} \\
\downarrow & & & & & & \downarrow \\
\{E\} & \xrightarrow{\sim} & GL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & SL_2(\mathbb{Z})\backslash GL_2(\mathbb{R})^+/SO_2(\mathbb{R})\cdot\mathbb{R}_{>0} & \cong & SL_2(\mathbb{Z})\backslash\mathcal{H}
\end{array}$$

where

$$\begin{array}{ccc}
\Gamma_1^{\pm}(N) & \longrightarrow & \begin{pmatrix}1 & *\\ 0 & \pm1\end{pmatrix} \\
\cap & & \cap \\
\Gamma_0^{\pm}(N) & \longrightarrow & \begin{pmatrix}* & *\\ 0 & *\end{pmatrix} \\
\cap & & \cap \\
GL_2(\mathbb{Z}) & \twoheadrightarrow & GL_2(\mathbb{Z}/N\mathbb{Z})_{\det=\pm1} \subseteq GL_2(\mathbb{Z}/N\mathbb{Z})
\end{array}$$