as

Notation: - A: associate ring allowed to be non-commutative, contains 1 - There are two systems to write category of A-modules.

$$Mod_A = A - Mod$$
 $(Mod_A)^{\circ p} \neq Mod_{A^{\circ p}} = Mod - A = A^{\circ p} - Mod \Rightarrow M_A$ 
 $Mod_{A \otimes B^{\circ p}} = A - Mod - B \Rightarrow AM_B$ 

In this document, we want to emphasize left/right module, so we use the right version for the most of time.

$$\nabla$$
 Even though you can identify  $Ob(Ring) \cong Ob(Ring^{op})$ ,  $A^{op}$  is still a ring.

Be careful about the difference between "the opposite of category" and "the opposite of objects"

In this case, it is desirable to translate algebraic results into geometrical results. Q: How to see the geometry of noncommutative rings? It is still vague for me.

In section 4-6, we assume that A is a commutative ring for convenient.

1 definition recall for 
$$\otimes$$
 & Hom
2. adjunction
3. comparison between  $\otimes$ -1 Hom &  $f^*-1f_*$ 
4. definition recall for  $\otimes$  & Hom , derived version
5. adjunction , derived version
6. comparison between  $\otimes$ -1 Hom &  $f^*-1f_*$  , derived version

#### 1 definition recall for ⊗ & Hom

$$\otimes_A: \operatorname{Mod}_{A^{\circ P}} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$$
  
 $\operatorname{Hom}_A(-,-): (\operatorname{Mod}_A)^{\circ P} \times \operatorname{Mod}_A \longrightarrow \operatorname{Mod}_Z$ 

$$\otimes_{B}$$
:  $A - Mod - B \times B - Mod - C \longrightarrow A - Mod - C$   
 $Hom_{B}(-,-)$ :  $(A - Mod - B)^{\overline{0}} \times B - Mod - C \longrightarrow A - Mod - C$ 

$$Hom_{B}^{A}(-,-)$$
:  $(A-Mod-B)^{\overline{op}} \times B-Mod-A \longrightarrow \mathbb{Z}-Mod$ 

$$Hom_{B\otimes_{\mathbb{Z}}A^{op}}(-,-) (\mathbb{Z}-Mod-B\otimes_{\mathbb{Z}}A^{op})^{\overline{op}} \times (B\otimes_{\mathbb{Z}}A^{op}-Mod-\mathbb{Z})^{\overline{op}} \longrightarrow \mathbb{Z}-Mod-\mathbb{Z}$$

$$(X \otimes_{B} Y) \otimes_{C} Z \cong X \otimes_{B} (Y \otimes_{C} Z)$$

$$X \otimes_{B} Y \cong Y \otimes_{B^{op}} X$$

$$A \otimes_{A} X \cong X \cong X \otimes_{B} B$$

$$Hom_{A}(A, X) \cong X$$

in 
$$A-Mod-C = C^{op}-Mod-A^{op}$$

2 adjunction BXA, cYB, cZD, we get

 $Homc(Y \otimes_{B} X, Z) \cong Hom_{B}(X, Homc(Y, Z))$  in A-Mod-D.

Reason: both sides equal to the set  $f: Y \times X \longrightarrow Z \mid f(cyb,x) = cf(y,bx) \quad \forall b,c$ 

For A = D = Z, fix  $Y \in C$ -Mod-B, one gets adjunction fctors.

# slogan: adjunction & associativity

$$(A-Mod-B)^{\overline{op}} \times (B-Mod-C)^{\overline{op}} \times C-Mod-D \xrightarrow{(Id, Home)} (A-Mod-B)^{\overline{op}} \times B-Mod-D$$

$$(A-Mod-B) \times B-Mod-C)^{\overline{op}} \times C-Mod-D$$

$$(A-Mod-C)^{\overline{op}} \times C-Mod-D \xrightarrow{Hom_{C}} A-Mod-D$$

$$f^*-If_*.$$

$$Hom (f^*F, G) \cong Hom (F, f_*G)$$

$$Sh(X)^{\overline{op}} \times Mor(Y, X) \times Sh(Y) \xrightarrow{(Id, pushforward)} Sh(X)^{\overline{op}} \times Sh(X)$$

$$(pullback, Id) \downarrow \qquad \qquad Hom_{Sh(Y)}(-, -)$$

$$Sh(Y)^{\overline{op}} \times Sh(Y) \xrightarrow{Hom_{Sh(Y)}(-, -)} Abel$$

$$(F, f, G) \longmapsto Hom_{Sh(Y)}(f^*F, G) \cong Hom_{Sh(X)}(F, f_*G)$$

$$f_*-If_* : similar.$$

3. comparison between ⊗-1 Hom & f\*-1 f\*

#### Forgetful fctor

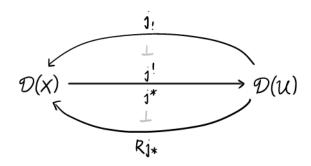
Prop. For ring homo 
$$\begin{picture}(1,0) \put(0,0){\line(1,0){150}} \put($$

one has adjunction fctors

djunction fctors
$$S_{R} \otimes_{R} - \frac{\sum_{S_{R} \otimes_{R} - 1}^{S_{R} \otimes_{R}} \otimes_{S_{R} - 1}}{\sum_{S_{R} \otimes_{S} - 1}^{S_{R} \otimes_{S}} \otimes_{S_{R} - 1}} R-Mod \qquad (3.1)$$

## Compare with j

Now, we compare (3.1) with part of the recollement diagram:



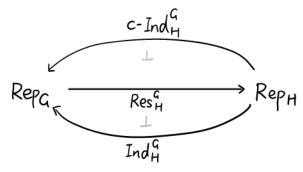
Vague slogan:  $u \approx$  "forget the information of Z".

In applications,  $U \longrightarrow X$  is a covering map. This change the feeling of the size between U & X.

E.g. For finite gps 
$$H \leq G$$
, one has Res-Ind adjunction.  
 $Res_{H}^{G} \dashv Ind_{H}^{G}$   
 $c-Ind_{H}^{G} \dashv Res_{H}^{G}$ 

It can be generalized for 
$$G: loc$$
 profinite  $gp$ ,  $H \leq G$  open If one only has  $H \leq G$  closed, then it's possible that  $j' \neq j^*$ . e.g.  $G = GL_1(\mathbb{Q}_p)$   $H = GL_2(\mathbb{Z}_p)$ 

In the diagram,



Ex Compare it with the recollement diagram & (3.1).

$$\mathcal{U}$$
 [\*/H]
$$\downarrow j$$
 "cover with fiber G/H"
$$X$$
 [\*/G]

translate the following geometrical results into algebraic statements.

1. One has natural fctor 
$$j_! \longrightarrow j_*$$
. When  $\# G/H < +\infty$ ,  $j_! = j_*$ 
 $c - Ind_H^G \longrightarrow Ind_H^G$ 

2. Even though Sho.v.([\*/G]) ≈ Repa = Q[G]-Mod. the "structure sheaf" of [\*/G] is Q. not Q[G].

Res<sub>f\*1</sub> 
$$Q = Q$$
, Res<sub>f\*1</sub>  $Q[G] = Q[G] \neq Q$ 

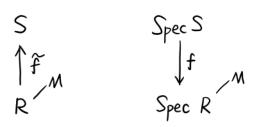
√ In this example, j\*Rj\* ≠ Id, j'j! ≠ Id.

Until now, we have met three types of six fctor formalism: top spaces, A-modules and stacks.

### Compare with i

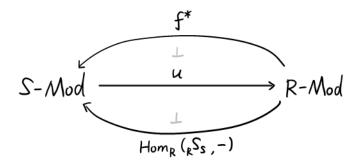
Now, assume S, R commutative in the scheme setting.

E.g. For ring homo

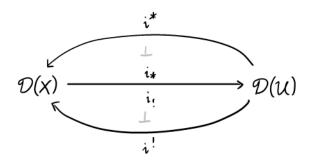


$$\exists$$
 "pullback fctor" 
$$f^*: R\text{-}Mod \longrightarrow S\text{-}Mod \qquad f^*M = sS_R \otimes_R M$$
 This is also called the base change.

Now, (31) can be rewritten as



compare it with another part of the recollement diagram.



Rmk. u is usually not f faithful, unless S = R/I. (In fact, only need S is R-idempotent, i.e.  $S \cong S \otimes_R S$ .) which croppeds to closed embedding. In that case,  $i^*i^* = Id : SS \otimes (SS \otimes_S M) \cong M$ 

$$i^*i_* = Id$$
:  ${}_{S}S_R \otimes_R ({}_{R}S_S \otimes_S M) \cong M$   
 $i^!i_* = Id$ :  $Hom_R ({}_{R}S_S, Hom({}_{S}S_R, M)) \cong M$ 

Slogan: in the comm alg., Spec  $R/I \longrightarrow Spec R$  is closed embedding. In general, if S is an R-idempotent algebra.  $S \cong S \otimes_R S$ then i. Spec  $S \longrightarrow Spec R$  can be viewed as "closed subset".

This poses a lot of bizarre phenomenons in six-fctors for coherent sheaves. Spec R/I is open instead?

E.g.  $R_p$ , R/I are idempotent R-algs.  $Z[\frac{1}{6}]$ ,  $F_p$ ,  $Z/p^2Z$ , Q,  $Z_p$ , are idem Z-algs. Usually R/1 is not an derived idem R-alg!

Rmk Following this slogan, original open/closed subsets are all closed. Also, i^! is not shifted (exists already in the non-derived category).

Q. What is the crspd "open subset"? A: (possibly) the Verdier quotient.
We will come back to this after we derive everything.

#### 4. LO -1 RHom

F	RF or LF	RiF or LiF	exact fctor
f*  f*  πx,*  f:  πx,:  πx,:	F & LF  Rf*  P(X,F)  Rf!  Pc(X,F)  f!	- Rif* H'(X;F) Rif! H'c(X;F) H'c(X;F)	f*-acyclic  \( \alpha - acyclic  \( \forall - acyclic  \( \forall c - acyclic  \)
- & - Hom <sub>R</sub> (-, -) MG MG MG MS M/[AM] MA A/[AA] Z(A)	- branched - branched - branched - control - c	Tork(-,-)  Extr(-,-)  Hi(G;M)  Hi(g;M)  Hi(g;M)  HHi(A,M)  HHi(A,M)  HHi(A)  HHi(A)	flat injective/projective

e.g. group coh

e.g. Lie alg coh

g/x: Lie alg

e.g. Hochschild coh

For calculations, see:

[23.04.09]: gp coh [wiki]: Lie algebra coh

[21.05.21]: Hochschild coh

[hidden]: quiver coh (there are also many books...)

Reminder: all the above fctors have adjoints.

For Hom(-,A), see https://math.stackexchange.com/questions/2010345/left-adjoint-to-hom-m.

Chenji Fu claimed that Hom(-,A) always has a left adjoint by SAFT, but we haven't found any explicit expression for that fctor.

https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod https://mathoverflow.net/questions/38080/what-are-examples-of-cogenerators-in-r-mod

https://math.stackexchange.com/questions/342534/when-to-use-projective-vs-injective-resolution

4. definition recall for ⊗ & Hom

, derived version

To define 6 & RHom, one needs to extend fctors

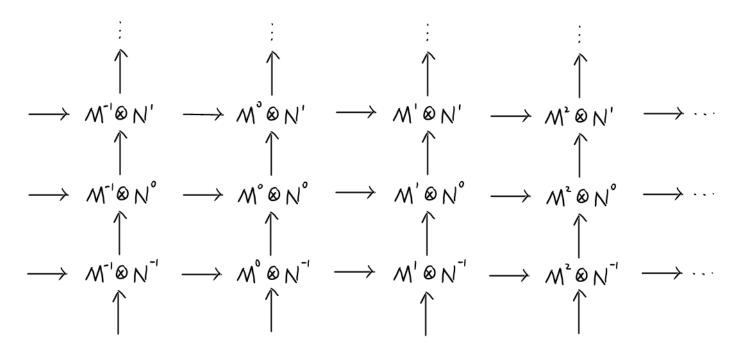
$$\otimes_{A}: A-Mod \times A-Mod \longrightarrow A-Mod$$
 $Hom_{A}(-,-): (A-Mod)^{op} \times A-Mod \longrightarrow A-Mod$ 

to fctors on double cplxes. C(A): = complex of A-modules, temporate notation

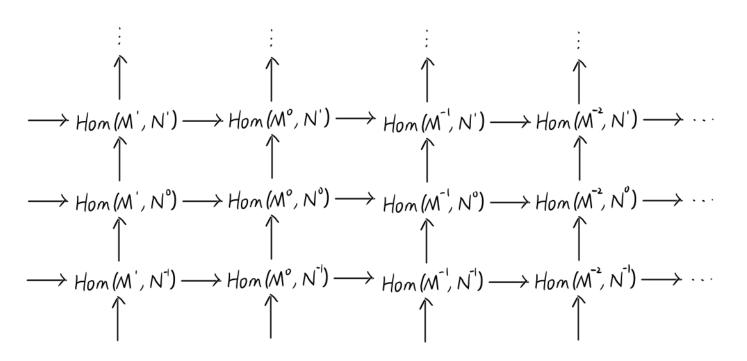
$$\begin{array}{cccc}
\otimes & \mathcal{L}(A) & \times & \mathcal{L}(A) & \longrightarrow & \mathcal{L}(A) \\
\text{Hom } (-,-): & (& \mathcal{L}(A) &)^{op} & \times & \mathcal{L}(A) & \longrightarrow & \mathcal{L}(A)
\end{array}$$

But how?

Wishes:



Tot  $(M \otimes N)$ , the double complex of  $M \otimes N$ .



Tot (Hom(M', N')), the double complex of Hom(M', N').

Def For M, N' 
$$\in$$
  $\mathcal{L}(A)$ , define  $M' \otimes N'$ ,  $Hom_A(M', N') \in \mathcal{L}(A)$  by  $(M' \otimes_{\mathcal{L}(A)} N')^n = \bigoplus_{i \neq j = n} M^i \otimes_A N^j$   $(Hom_{\mathcal{L}(A)}(M', N'))^n = \bigoplus_{i \neq j = n} Hom_A(M^{-i}, N^j)$  and morphisms given by  $d + (-1)^j S$ .

$$E_{x}$$
. Let  $M' = \begin{bmatrix} \mathbb{Z} \xrightarrow{\times^{3}} \mathbb{Z} \end{bmatrix}$ ,  $N' = \begin{bmatrix} \mathbb{Z} \xrightarrow{\times^{2}} \mathbb{Z} \end{bmatrix}$   
compute  $M' \otimes_{e(\mathbb{Z})} N' \otimes_{e(\mathbb{Z})} M' \otimes_{e(\mathbb{Z$ 

Tot (Hom (M', N'))

Now, we can define L⊗ & RHom.

Def. For M, N ∈ A-Mod, one can define

 $M^{L}\otimes_{A}N:=M\otimes_{e(A)}P'$  when  $N\stackrel{\cong}{\leftarrow}P'$  flat resolution in general,  $M',N'\in\mathcal{D}^{-}(A-M\circ d)$ 

 $\begin{array}{lll} \text{RHom}_{A}\left(M,\,N\right):=\, \text{Hom}_{e(A)}(M,I') & \text{when} & N\stackrel{\cong}{\to} I' & \text{inj} & \text{resolution} \\ &:=\, \text{Hom}_{e(A)}\left(P',N\right) & \text{when} & M\stackrel{\cong}{\leftarrow} P' & \text{proj} & \text{resolution} \\ &\text{in general}, & M' \in \mathcal{D}^{-}(A\text{-Mod}), & N' \in \mathcal{D}^{+}(A\text{-Mod}) \end{array}$ 

Side Rmk. Proj module is flat. Since free module is flat https://math.stackexchange.com/questions/4322028/three-ways-to-to-prove-that-projective-modules-are-flat

Ex Compute  $F_2 \otimes_{\mathbb{Z}} F_2 & R + Com_{\mathbb{Z}} (F_2, F_2),$ and get  $Tov_{\mathbb{Z}}^2 (F_2, F_2) & Ext_{\mathbb{Z}}^2 (F_2, F_2)$ 

Ex. Shows that  $Hom_{\mathcal{D}(A)}(M', N') = R^{\circ} Hom_{\mathcal{D}(A)}(M', N') + Hom_{\mathcal{D}(A)}(M', N') = R^{\circ} Hom_{\mathcal{D}(A)}(M', N') = R^{\circ} Hom_{\mathcal{D}(A)}(M, N).$ 

# A wrong proof for "flat -> proj"

"Proof" when P is flat,

$$P \otimes_A - Hom_A(P, -)$$
 $P \stackrel{\square}{\otimes}_A - Hom_A(P, -)$ 

by the uniqueness of the adjunction,  $Hom_A(P, -) = RHom_A(P, -)$ , so P is flat.

This is wrong.  $Q \in \mathbb{Z}$ -Mod is flat but not proj. In the proof, we only have

Ex. Compute  $RHom_Z(Q,-)$ , and shows that