Examples of (non-split) reductive gps

1. forms
2. torus case
3. other cases
4. conclusions on various forms

Setting. We work over conn red gp over F. (G/F conn red)

$\bar{F}$: the seperable closure of F mainly care about $\mathbb{R}$ & p-adic field case.

$\Gamma_F := \mathrm{Gal}(\bar{F}/F)$ $\sigma \in \Gamma_F$

$$Z^1(W,A) := \left\{ \begin{matrix} {}^L\varphi : W \longrightarrow A \rtimes W \\ \gamma \longmapsto (\gamma_0, \gamma) := {}^L\gamma \end{matrix} \;\middle|\; \begin{matrix} {}^L\varphi : \textit{continuous group homo} \\ {}^L\varphi \textit{ is a section} \end{matrix} \right\}$$

$$= \{ \varphi : W \longrightarrow A \mid \varphi(\gamma\gamma') = \varphi(\gamma)^{\gamma}(\varphi(\gamma')) \}$$

$\varphi \in H^1(W, A)$

$$H^1(W,A) = Z^1(W,A)/A.$$

Def. (Split) G is split if
G has a maximal torus T over F which is split.

Def. (Quasi-split) G is quasi-split if
G has a Borel B over F.

Borel = maximal (Zar-closed) conn sol alg subgp
= minimal parabolic subgp

Parabolic = $H \leq G$ closed subgp s.t G/H is projective
= closed subgp containing a Borel.

Ref:

[ECII] Silverman, The Arithmetic of Elliptic Curves

[Buzzard] Kevin Buzzard, Forms of reductive algebraic groups.
https://www.ma.imperial.ac.uk/~buzzard/maths/research/notes/forms_of_reductive_algebraic_groups.pdf

[KP] Tasho Kaletha and Gopal Prasad, Bruhat{Tits theory: a new approach. version from May 27, 2022.

[DR09] Stephen DeBacker and Mark Reeder, Depth-zero supercuspidal L-packets and their stability
https://annals.math.princeton.edu/wp-content/uploads/annals-v169-n3-p03.pdf

[SP98] T. A. Springer, Linear Algebraic Groups
https://link.springer.com/book/10.1007/978-0-8176-4840-4

https://mathoverflow.net/questions/121959/classification-of-tori-of-gl2-up-to-conjugation

I make no claim to originality.

1. forms.

Def. $G_1, G_2/F$ are called forms, if
$$\exists\ \alpha : G_{2,\bar F} \xrightarrow{\sim} G_{1,\bar F} \quad \text{as gps} \qquad \text{not as } \Gamma_F\text{-gps!}$$
α is considered as the information of forms.

Thm.
$$\{F\text{-forms of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$$
$$[G_2,\ \alpha : G_{2,\bar F} \longrightarrow G_{\bar F}] \longmapsto \varphi_\alpha : = \alpha\sigma\alpha^{-1}\sigma^{-1}$$
$$G_2 \longleftarrow\!\shortmid \varphi$$

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\sigma"'] & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[dr, "\alpha"'] & G_{\bar F} \\ & G_{\bar F} \end{tikzcd}$$

$$\overset{\Gamma_F}{\circlearrowleft} G_{2,\bar F} \longrightarrow \overset{\Gamma_F\ \Gamma_F}{\circlearrowleft\circlearrowleft} G_{\bar F}$$

φ_α measures the twist between two Γ_F-actions.

$$G_2(F) := \{ g \in G(\bar F) \mid \underbrace{(\varphi(\sigma)\circ\sigma)}_{\text{induced action}}\, g = g \quad \forall \sigma \in \Gamma_F \}$$

In general, $G_2(R) := \{ g \in G(\bar F \otimes_F R) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall\sigma\in\Gamma_F \}$

e.p. $G_2(\bar F) = \{ (\varphi(\sigma)^{-1} g)_{\sigma\in\Gamma_F} \in \prod_{\sigma\in\Gamma_F} G(\bar F) \mid g \in G(\bar F) \} \cong G(\bar F)$

$$G(\bar F \otimes_F \bar F) \cong G\Big(\bigoplus_{\sigma\in\Gamma_F} \bar F\Big) \cong \prod_{\sigma\in\Gamma_F} G(\bar F)$$

$(G_2, \alpha) \sim (G_2', \alpha')$, if
$\exists\ \beta : G_2 \longrightarrow G_2'$ as an iso.

$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$

Functorial on F. (Inflation - Restriction seq. [ECII, Appendix B, Prop 1.3])
Let E/F be finite Galois.

$$\begin{tikzcd} G_{2,E} & \{E\text{-forms of } G\} \arrow[r, leftrightarrow] & H^1(\Gamma_E, \mathrm{Aut}(G_{\bar F})) & \varphi|_{\Gamma_E} \\ G_2 \arrow[u, mapsto] & \{F\text{-forms of } G\} \arrow[u] \arrow[r, leftrightarrow] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[u] & \varphi \arrow[u, mapsto] \\ & \{G_2/F : G_{2,E} \cong G_E\} \arrow[u] \arrow[r, leftrightarrow] & H^1(\mathrm{Gal}(E/F), \mathrm{Aut}(G_{\bar F})^{\Gamma_E}) \arrow[u] & \\ & 1 \arrow[u] & 1 \arrow[u] & \end{tikzcd}$$

$\mathrm{Aut}(G_{\bar F})^{\Gamma_E} = \mathrm{Aut}(G_E)$

Rmk. We have the classification of connected reductive gps:

$$\begin{aligned}
\{\text{split red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) \quad \cong \Psi(G,B,T) \\
G \text{ split} \qquad \{\text{qs red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + \Gamma_F\text{-action} \\
&= (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, \mathrm{Out}(G_{\bar F})) \\
\{\text{red gp}/F\} &\longleftrightarrow (X^*, \Delta, X_*, \Delta^\vee) + H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))
\end{aligned}$$

To understand the result, the following isos are needed:

$$\mathrm{Aut}(G_{\bar F}) \cong \mathrm{Inn}(G_{\bar F}) \rtimes \mathrm{Aut}(G, B, T, \{u_\alpha\})$$

$$\begin{aligned}
\mathrm{Out}(G_{\bar F}) &\cong \mathrm{Aut}(G, B, T, \{u_\alpha\}) && \text{for embedding} \\
&\cong \mathrm{Aut}(\Psi(G,B,T)) && \text{for combinatorics}
\end{aligned}$$

Also, by the Hilbert 90, one has

$$H^1(\Gamma_F, \mathrm{Aut}(G,B,T,\{u_\alpha\})) \cong H^1(\Gamma_F, \mathrm{Aut}(G,B,T))$$

2. torus case

Let us try to find all the forms of the split torus $\mathbb{G}_m^n$.

They're called (non-split) torus.

We know

$$\begin{array}{ccc} \mathrm{Aut}(\mathbb{G}_m^n) & \subseteq & \mathrm{End}(\mathbb{G}_m^n) \\ \| & & \| \\ GL_n(\mathbb{Z}) & \subseteq & M^{n\times n}(\mathbb{Z}) \end{array}$$

$\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) \cong \mathbb{Z}$

$(-)^n \leftarrow n$

Therefore,

$$\begin{aligned} H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}_{m,\overline{F}}^n)) &= H^1(\Gamma_F, GL_n(\mathbb{Z})) \\ &= \mathrm{Hom}_{Grp}(\Gamma_F, GL_n(\mathbb{Z}))/_{GL_n(\mathbb{Z})\text{-conj}} \\ &\underset{\text{when } F=\mathbb{R}}{=\!=} \{g \in GL_n(\mathbb{Z}) \mid g^2 = Id\}/_{GL_n(\mathbb{Z})\text{-conj}} \end{aligned}$$

[Γ_F acts on $\mathrm{Aut}(\mathbb{G}_{m,\overline{F}}^n) \subseteq \mathrm{End}(\mathbb{G}_{m,\overline{F}}^n)$ trivially:

see $\overline{F}$-pts, $n=1$:

$$\begin{array}{ccc} \overline{F}^\times & \xrightarrow{\alpha} & \overline{F}^\times \\ \sigma\downarrow & & \downarrow\sigma \\ \overline{F}^\times & \xrightarrow{{}^\sigma\alpha} & \overline{F}^\times \end{array} \qquad \begin{array}{ccc} x & \longmapsto & x^n \\ \downarrow & & \downarrow \\ \sigma(x) & & \sigma(x^n) = \sigma(x)^n \end{array}$$

$\Rightarrow {}^\sigma\alpha = \alpha$]

E.g. $n=1$, $F=\mathbb{R}$

$$H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}_{m,\overline{F}})) \cong \{1, -1\}$$

$\rightsquigarrow \varphi(\sigma) = (-)^{-1}$

$$\begin{array}{cc} \Downarrow & \Downarrow \\ \mathbb{G}_m & G = \ ?\ SO_{2,\mathbb{R}} \end{array}$$

$$\begin{aligned} G(\mathbb{R}) &= \{g \in \mathbb{G}_m(\mathbb{C}) \mid (\varphi(\sigma)\circ\sigma)\, g = g \quad \forall \sigma \in \Gamma_{\mathbb{R}}\} \\ &= \{g \in \mathbb{C}^\times \mid (\bar{g})^{-1} = g\} \\ &= \{g \in \mathbb{C}^\times \mid |g| = 1\} \\ &= S^1 \end{aligned}$$

$$G(\mathbb{C}) = \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$$

$$\Rightarrow G = \mathrm{Spec}\, \mathbb{R}[x,y]/(x^2+y^2-1) = SO_{2,\mathbb{R}}$$

Check:
$$\begin{aligned} G(\mathbb{C}) &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid x^2+y^2-1 = 0\} \\ &= \{(x,y) \in \mathbb{C}\times\mathbb{C} \mid (x+iy)(x-iy) = 1\} \\ &= \left\{ (x,y,t) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^\times \;\middle|\; \begin{array}{l} x+iy = t \\ x-iy = \frac{1}{t} \end{array} \right\} \\ &\cong \mathbb{C}^\times \end{aligned}$$

$$SO_2(K) = \left\{ \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \;\middle|\; x,y \in K,\ x^2+y^2 = 1 \right\}$$

E.g. $n=2,\ F=\mathbb{R}$

$$H^1(\Gamma_F, \mathrm{Aut}(\mathbb{G}_{m,\bar{F}}^2)) \cong \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \right\}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow \qquad \Downarrow$$

$$\mathbb{G}_m^2 \qquad \mathbb{G}_m \times SO_{2,\mathbb{R}} \qquad (SO_{2,\mathbb{R}})^2 \qquad G = ? \quad \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

$$G(\mathbb{R}) = \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; \underbrace{(\varphi(\sigma) \circ \sigma) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}}_{= \varphi(\sigma) \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 \\ \bar{z}_1 \end{pmatrix}} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^\times \times \mathbb{C}^\times \;\middle|\; z_1 = \bar{z}_2 \right\}$$

$$= \mathbb{C}^\times$$

$$G(\mathbb{C}) = \mathbb{G}_m^2(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$$

$$\Rightarrow \quad G = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$$

Rmk. As a scheme,

$$\begin{aligned} \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m &= \mathrm{Spec}\ \mathbb{C}[\mathbb{Z} \times \mathbb{Z}]^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\left(\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[s, s^{-1}] \right)^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\ \mathbb{C}[t, s, t^{-1}, s^{-1}]^{\Gamma_{\mathbb{R}}} \\ &= \mathrm{Spec}\ \mathbb{R}\left[\tfrac{t+s}{2}, \tfrac{t-s}{2i}, t^{-1}s^{-1}\right] \\ &\cong \mathrm{Spec}\ \mathbb{R}[x, y, u] / ((x^2+y^2)u - 1) \end{aligned}$$

Fact. Any (conn) $\mathbb{R}$-torus is product of

$$\begin{array}{ccc} \mathbb{G}_m, & SO_{2,\mathbb{R}}, & \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \\ \updownarrow & \updownarrow & \updownarrow \\ 1 & -1 & \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ \updownarrow & \updownarrow & \updownarrow \end{array}$$

Fact$^{\text{dual}}$: $\mathrm{Ind}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}) = \{\mathbb{Z}_{\text{triv}}, \mathbb{Z}_{\text{sign}}, \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]\}$

i.e., $\mathbb{Z}/2\mathbb{Z}$ has 3 indecomposable integral reps.

More details can be found here: https://personal.math.ubc.ca/~cass/research/pdf/realtori.pdf

The classification is much more general than we may think of. When I really understand this I would make a new document for this.
See: Dieudonné–Manin classification theorem in wiki
http://faculty.bicmr.pku.edu.cn/~dingyiwen/dm.pdf

Rmk. Using the same argument, one can show that

$$\{T/\mathbb{F}_p \text{ s.t } T_{\mathbb{F}_{p^2}} \cong \mathbb{G}^n_{m,\mathbb{F}_{p^2}}\} = \text{products of } \mathbb{G}_m, \begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix}, \mathrm{Res}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\mathbb{G}_m$$

The torus G crspd to -1: Assume $\xi \in \mathbb{F}_{p^2} \setminus \mathbb{F}_p$, $\xi^2 = \varepsilon \in \mathbb{F}_p$, $\left(\frac{\varepsilon}{p}\right) = -1$

$$\begin{aligned} G(\mathbb{F}_p) &= \{g \in \mathbb{G}_m(\mathbb{F}_{p^2}) \mid (\varphi(\sigma)\circ\sigma) g = g \quad \forall \sigma \in \Gamma_k\} \\ &= \{a+b\xi \in \mathbb{F}_{p^2}^\times \mid \underset{\substack{\| \\ (a-b\xi)^{-1}}}{\varphi(\sigma)}(a-b\xi) = a+b\xi\} \\ &= \{a+b\xi \in \mathbb{F}_{p^2}^\times \mid a^2 - b^2\varepsilon = 1\} \\ &\cong \left\{\begin{pmatrix} a & b \\ \varepsilon b & a \end{pmatrix} \subseteq GL_2(\mathbb{F}_p)\right\} \end{aligned}$$

3. other cases.

By using the same methods introduced in last section, we can compute the (inner) forms of reductive gps.

	inner forms	outer forms
$\mathbb{G}_m$ $(\mathbb{G}_m)^2$ $(\mathbb{G}_m)^n$	$\emptyset$	SO_2 $SO_2\times\mathbb{G}_m$, $(SO_2)^2$, $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$ product of lower rank torus
$GL_{2,\mathbb{R}}$ $SL_{2,\mathbb{R}}$ $PGL_{2,\mathbb{R}}$	$\mathbb{H}^\times = GL_1(\mathbb{H}\otimes_{\mathbb{R}}-)$ $\mathbb{H}^{Nm=1} = SU_{2,\mathbb{C}/\mathbb{R}}$?	$(\ U_{2,\mathbb{C}/\mathbb{R},\omega} = U(1,1)\quad U(2,0)\ )$ $\emptyset$ $\emptyset$
$GL_{n,\mathbb{R}}$ $SL_{n,\mathbb{R}}$ $PGL_{n,\mathbb{R}}$	? $GL_{n/2}(\mathbb{H}\otimes_{\mathbb{R}}-)^{Nm=1}$ when n even ?	$\left(U_{n,\mathbb{C}/\mathbb{R},\omega} = \begin{cases} U(\frac{n}{2},\frac{n}{2}) & n \text{ even} \\ U(\frac{n+1}{2},\frac{n-1}{2}) & n \text{ odd} \end{cases}\quad U(p,q)\right)$ ... $SU(a,n-a)$ e.p. $SU(2,1)$ ← need clarification ?
$(SL_2)^2/\mathbb{R}$ $(SL_2)^3/\mathbb{R}$	$SL_2\times SU_2$, $(SU_2)^2$, $\emptyset$? $(8-1)$ possibilities	$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$?

? : I have no time to compute / don't know any symbol to represent

: quasi-split gp

See [SP98, Chapter 17] for a full classification of reductive groups of all types.

Compute $\mathrm{Aut}(G_{\bar{F}})$

Lemma. We understand $\mathrm{Aut}(G_{\bar{F}})$ quite well.

$$1 \longrightarrow \underset{\substack{\| \\ G(\bar F)/Z(G(\bar F)) = G^{ad}(\bar F)}}{\mathrm{Inn}(G_{\bar F})} \longrightarrow \mathrm{Aut}(G_{\bar F}) \longrightarrow \underset{\substack{\| \\ \mathrm{Aut}(\Psi_0)}}{\mathrm{Out}(G_{\bar F})} \longrightarrow 1$$

G	$\mathrm{Inn}(G_{\bar F})$	$\mathrm{Aut}(G_{\bar F})$	$\mathrm{Out}(G_{\bar F})$	
T rk n	1	$GL_n(\mathbb{Z})$	$GL_n(\mathbb{Z})$	
$GL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$SL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$PGL_{2,\mathbb{R}}$	$PGL_2(\mathbb{C})$	$PGL_2(\mathbb{C})$	1	
$n \geq 3$				
$GL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}^{\oplus 2}$	$\{\pm 1\}^{\oplus 2}$	
$SL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$PGL_{n,\mathbb{R}}$	$PGL_n(\mathbb{C})$	$PGL_n(\mathbb{C}) \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$(SL_2)^2/\mathbb{R}$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	
$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} SL_2$	$PGL_n(\mathbb{C})^2$	$PGL_n(\mathbb{C})^2 \rtimes \{\pm 1\}$	$\{\pm 1\}$	with different $\Gamma_{\mathbb{R}}$-action
$(SL_2)^n/\mathbb{R}$	$PGL_n(\mathbb{C})^n$	$PGL_n(\mathbb{C})^n \rtimes S^n$	S^n	

Compute $H^1(\Gamma_F, -)$

Method 1: Hilbert 90 + LES

Method 2: Use black box

p-adic field: Kottwitz's isomorphism

[KP] **Theorem 12.7.7** *There is a functorial isomorphism* $\mathrm{H}^1(k, G) \to \pi_1(G)_{\Theta,\mathrm{tor}}$.

[DR09] COROLLARY 2.4.3. *The composition*

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} \xrightarrow{\sim} H^1(\mathrm{F}, \Omega_C) \xrightarrow{a_*^{-1}} H^1(\mathrm{F}, N_C) \xrightarrow{b_*} H^1(\mathrm{F}, G)$$

is a bijection.

$$[\bar{X}/(1-\vartheta)\bar{X}]_{\mathrm{tor}} = \mathrm{Irr}[\pi_0(\hat{Z}^{\hat{\vartheta}})].$$

real field:

Mikhail Borovoi, Galois cohomology of reductive algebraic groups over the field of real numbers
https://arxiv.org/abs/1401.5913

3.1. Theorem. *Let G, T_0, T, and W_0 be as above. The map*

$$H^1(\mathbb{R}, T)/W_0(\mathbb{R}) \to H^1(\mathbb{R}, G)$$

induced by the map $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G)$ is a bijection.

global field:

Mikhail Borovoi, Tasho Kaletha, Galois cohomology of reductive groups over global fields
https://arxiv.org/pdf/2303.04120.pdf

E.g. $G = SL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{llllllll} G & 1 \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\ SL_{2,\mathbb{R}} & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) & & 1 & \end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(SL_{2,\mathbb{C}})) = H^1(\Gamma_{\mathbb{R}}, PGL_2(\mathbb{C})) = \{ 1, \ \omega(\ )\omega^{-1} \}$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$1 \Downarrow SL_{2,\mathbb{C}}$, $\quad \omega(\ )\omega^{-1} \Downarrow G = ?$ $\quad \mathbb{H}^{N_m = 1}$

$$\begin{aligned} G(\mathbb{R}) &= \{ g \in SL_2(\mathbb{C}) \mid (\varphi(\sigma) \circ \sigma)(g) = g \quad \forall g \in \Gamma_{\mathbb{R}} \} \\ &= \{ g \in SL_2(\mathbb{C}) \mid \omega \bar g \omega^{-1} = g \} \\ &= \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) \mid \underbrace{\begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} \bar a & \bar b \\ \bar c & \bar d \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}}_{\begin{pmatrix} \bar d & -\bar c \\ -\bar b & \bar a \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \} \\ &= \{ \begin{pmatrix} a & b \\ -\bar b & \bar a \end{pmatrix} \in SL_2(\mathbb{C}) \} \\ &= \mathbb{H}^{N_m = 1} \end{aligned}$$

E.g. $G = GL_{2,\mathbb{R}}$, $F = \mathbb{R}$

$$\begin{array}{llllllll} G & 1 \longrightarrow & \mathrm{Inn}(G_{\bar F}) & \longrightarrow & \mathrm{Aut}(G_{\bar F}) & \longrightarrow & \mathrm{Out}(G_{\bar F}) & \longrightarrow 1 \\ GL_{2,\mathbb{R}} & & PGL_2(\mathbb{C}) & & PGL_2(\mathbb{C}) \rtimes \{\pm 1\} & & \{\pm 1\} & \end{array}$$

$$H^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}(GL_{2,\mathbb{C}})) = \{ 1, \ \omega(\ )\omega^{-1}, \ (\ )^{H,-1}, \ \omega(\ )^{H,-1}\omega^{-1} \}$$

$\omega = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$

$1 \Downarrow (GL_{2,\mathbb{C}}$, $\quad \omega(\ )\omega^{-1} \Downarrow \mathbb{H}^\times)$, $\quad (\ )^{H,-1} \Downarrow (\mathcal{U}(2,0))$, $\quad \omega(\ )^{H,-1}\omega^{-1} \Downarrow \mathcal{U}_{2,\mathbb{C}/\mathbb{R},\omega})$

4. conclusions on various forms

$H^1(\Gamma_F, -)$ as parameter space

$$\begin{tikzcd}[column sep=small] & & & 1 \arrow[dr] & & 1 & & \\ 1 \arrow[r] & Z(G(\bar F)) \arrow[r] & G(\bar F) \arrow[r] & \mathrm{Inn}(G_{\bar F}) \arrow[ur] \arrow[dr] & & & & \\ & & & & \mathrm{Aut}(G_{\bar F}) \arrow[r] & \mathrm{Out}(G_{\bar F}) \arrow[r] & 1 & \end{tikzcd}$$

$$\rightsquigarrow H^1(\Gamma_F, Z(G(\bar F))) \to \underset{\text{pure inner twist}}{H^1(\Gamma_F, G(\bar F))} \to \underset{\text{inner twist}}{H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))} \to \underset{\text{form}}{H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))} \to H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))$$

$$H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \to H^2(\Gamma_F, Z(G(\bar F)))$$

$\{F\text{-forms of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_K, \mathrm{Aut}(G_{\bar K}))$

$\{F\text{- inner forms of } G\}/\sim \;\longleftrightarrow\; \mathrm{Im}\left(H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))\right)$

$\{F\text{-inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$

$\{F\text{-pure inner twists of } G\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, G(\bar F))$

G split: $\left\{\begin{matrix}F\text{-forms of } G \\ \text{which are quasi-split}\end{matrix}\right\}/\sim \;\longleftrightarrow\; H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) \cong H^1(\Gamma_F, \mathrm{Out}(G_{\bar F}))$

$\quad\quad H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}, B, T)) = \Gamma_F\text{-actions on } (X^*, \Delta, X_*, \Delta^\vee)$

Q: Do we have

$$\begin{tikzcd} & H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \arrow[r, hook] & H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \arrow[r] & H^1(\Gamma_F, \mathrm{Out}(G_{\bar F})) \\ 1 \arrow[r] & \underset{=\mathrm{Inn}(G_F)}{\mathrm{Inn}(G_{\bar F})^{\Gamma_F}} \arrow[r] & \underset{=\mathrm{Aut}(G_F)}{\mathrm{Aut}(G_{\bar F})^{\Gamma_F}} \arrow[r] & \underset{=\mathrm{Out}(G_F)}{\mathrm{Out}(G_{\bar F})^{\Gamma_F}} \arrow[ull, bend right, "0"'] \end{tikzcd}$$

Give one example s.t. $H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \longrightarrow H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$ is not inj ?

Categorification of $H^1(\Gamma_F, -)$

These categories are all groupoids.

These $H^1(\Gamma_F, -)$ are all achieved as isomorphism classes.

<table>
<tr><th></th><th>Obj</th><th>Mor $((G_2,\alpha),(G_2',\alpha'))$</th></tr>
<tr>
<td>form

$H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F}))$</td>
<td>$(G_2,\ \alpha : G_{2,\bar F} \to G_{\bar F})$

$\Rightarrow$
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"] \\ G_{2,\bar F} \arrow[r, "\alpha"] \arrow[urr, "\sigma(\alpha)"] & G_{\bar F} & \end{tikzcd}$$
commutes $\forall \sigma \in \Gamma_F$</td>
<td>$\beta : G_2 \longrightarrow G_2'$ iso

$\Rightarrow$
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
commutes</td>
</tr>
<tr>
<td>inner form

$\mathrm{Im}\left(\begin{array}{c} H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F})) \\ \downarrow \\ H^1(\Gamma_F, \mathrm{Aut}(G_{\bar F})) \end{array}\right)$

full subcategory of "form"</td>
<td>$(G_2,\ \alpha : G_{2,\bar F} \to G_{\bar F})$

s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$
$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.</td>
<td>$\beta : G_2 \longrightarrow G_2'$ iso

$\Rightarrow$
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
commutes</td>
</tr>
<tr>
<td>inner twist

$H^1(\Gamma_F, \mathrm{Inn}(G_{\bar F}))$

less isomorphisms compared with inner form</td>
<td>$(G_2,\ \alpha : G_{2,\bar F} \to G_{\bar F})$

s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\sigma(\alpha)\circ\alpha^{-1}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$
$\sigma(\alpha)\circ\alpha^{-1}$ is inner auto.</td>
<td>$\beta : G_2 \longrightarrow G_2'$ iso

s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
$\alpha'\circ\beta_{\bar F}\circ\alpha^{-1}$ is inner auto.</td>
</tr>
<tr>
<td>pure inner twist

$H^1(\Gamma_F, G(\bar F))$</td>
<td>$(G_2,\ \alpha : G_{2,\bar F} \to G_{\bar F},\ \phi)$
$\phi \in Z^1(\Gamma_F, G(\bar F))$

s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[dd, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ & & G_{\bar F} \arrow[dl, "\phi(\sigma)\text{-conj}"'] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & \end{tikzcd}$$
commutes</td>
<td>(β, δ)
$\beta : G_2 \longrightarrow G_2'$ iso $\quad \delta \in G(\bar F)$

s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
commutes, and
$\phi_1(\sigma) = \delta^{-1}\phi_2(\sigma)\,\sigma(\delta)$</td>
</tr>
</table>

<table>
<tr><th></th><th>Obj</th><th>Mor $((G_2,\alpha),(G_2',\alpha'))$</th></tr>
<tr>
<td>rigid inner twist

$H^1\begin{pmatrix} u(\bar F)\to\mathcal{E}^{rig}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$</td>
<td>$(G_2,\ \alpha: G_{2,\bar F}\to G_{\bar F},\ Z,\ z)$
$Z\subset Z(G)$ finite gp subscheme
$z\in Z^1\begin{pmatrix} u(\bar F)\to\mathcal{E}^{rig}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$
s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & G_{\bar F} \arrow[l, "z(\sigma)\text{-conj}"'] \end{tikzcd}$$
commutes</td>
<td>(β,δ)
$\beta: G_2\longrightarrow G_2'$ iso $\quad \delta\in G(\bar F)$
s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
commutes, and
$z_1(\sigma)=\delta^{-1}z_2(\sigma)\,\sigma(\delta)$</td>
</tr>
<tr>
<td>when $Z=Z(G)$ $\left\{\begin{array}{l}\text{basic Kottwitz set} \\ B(G)_{basic} \\ H^1_{basic}(\mathcal{E}^{iso},G)\end{array}\right.$
$H^1\begin{pmatrix} \mathbb{D}(\bar F)\to\mathcal{E}^{iso}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$</td>
<td>$(G_2,\ \alpha: G_{2,\bar F}\to G_{\bar F},\ Z,\ z)$
$Z\subset Z(G)$ finite gp subscheme
$z\in Z^1\begin{pmatrix} \mathbb{D}(\bar F)\to\mathcal{E}^{iso}, \\ Z(\bar F)\to G(\bar F)\end{pmatrix}$
s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[rr, "\alpha"] \arrow[d, "\sigma"'] & & G_{\bar F} \arrow[d, "\sigma"] \\ G_{2,\bar F} \arrow[r, "\alpha"] & G_{\bar F} & G_{\bar F} \arrow[l, "z(\sigma)\text{-conj}"'] \end{tikzcd}$$
commutes</td>
<td>(β,δ)
$\beta: G_2\longrightarrow G_2'$ iso $\quad \delta\in G(\bar F)$
s.t.
$$\begin{tikzcd} G_{2,\bar F} \arrow[r, "\alpha"] \arrow[d, "\beta_{\bar F}"'] & G_{\bar F} \arrow[d, "\delta\text{-conj}"] \\ G'_{2,\bar F} \arrow[r, "\alpha'"] & G_{\bar F} \end{tikzcd}$$
commutes, and
$z_1(\sigma)=\delta^{-1}z_2(\sigma)\,\sigma(\delta)$</td>
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https://mathoverflow.net/questions/117033/center-of-the-algebraic-group-g-mathbbr-for-a-centerless-g
https://math.stackexchange.com/questions/953526/relative-center-of-relative-group-scheme