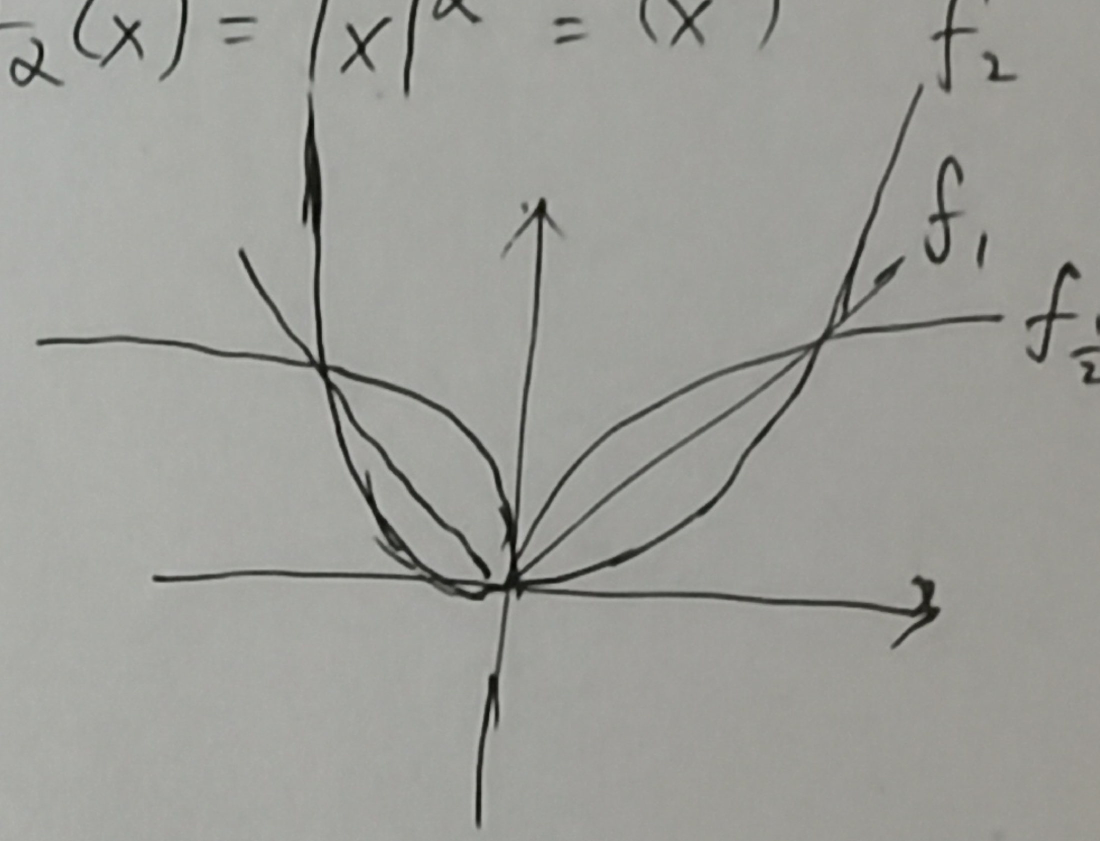
Session 5 & ~~Tutorial 3~~

Exercise 3.

Ex 1. Let $\alpha > 0$ $\quad f_\alpha : \mathbb{R} \to \mathbb{R} \quad f_\alpha(x) = |x|^\alpha = (x^2)^{\frac{\alpha}{2}}$



(i) When is f_α differentiable?

When $x_0 \neq 0$, $\frac{d f_\alpha}{d x}(x_0) = \frac{\alpha}{2}(x_0^2)^{\frac{\alpha}{2}-1} \cdot 2x_0$

$= \alpha |x_0|^{\alpha-2} x_0$

$= \alpha (\operatorname{sgn} x_0) |x_0|^{\alpha-1}$

When $x_0 = 0$, $\frac{d f_\alpha}{dx}(x_0) = \lim_{t \to 0} \frac{f_\alpha(x_0+t) - f_\alpha(x_0)}{t}$

$= \lim_{t \to 0} \frac{|t|^\alpha}{t}$

$= \begin{cases} \nexists & \alpha < 1 \\ \nexists & \alpha = 1 \\ 0 & \alpha > 1 \end{cases}$

(ii), $f_\alpha(x) = f_\alpha(x_0) + \left(\alpha(\operatorname{sgn} x_0)|x_0|^{\alpha-1}\right)(x - x_0) + o(|x - x_0|)$ for $x_0 \neq 0$.

$f_\alpha(x) = f_\alpha(x_0) + 0 \cdot (x - x_0) + o(|x - x_0|)$ for ~~$x_0 \neq 1$~~ $x_0 = 0, \alpha > 1$

$\therefore (\nabla f_\alpha)(x_0) = \begin{cases} \alpha(\operatorname{sgn} x_0)|x_0|^{\alpha-1} & x_0 \neq 0 \\ 0 & x_0 = 0, \alpha > 1 \\ \nexists & x_0 = 0, \alpha \leq 1 \end{cases}$

(iii) $(\nabla f_\alpha)(x_0) = 0 \iff x_0 = 0 \ \& \ \alpha > 1$

(iv) When ∇f_α is defined ($x_0 \neq 0$ or $\alpha > 1$)

x_0 is local extremum $\Rightarrow (\nabla f_\alpha)(x_0) = 0 \iff x_0 = 0 \ \& \ \alpha > 1$.

When $x_0 = 0$, f_α is local minimum at 0.

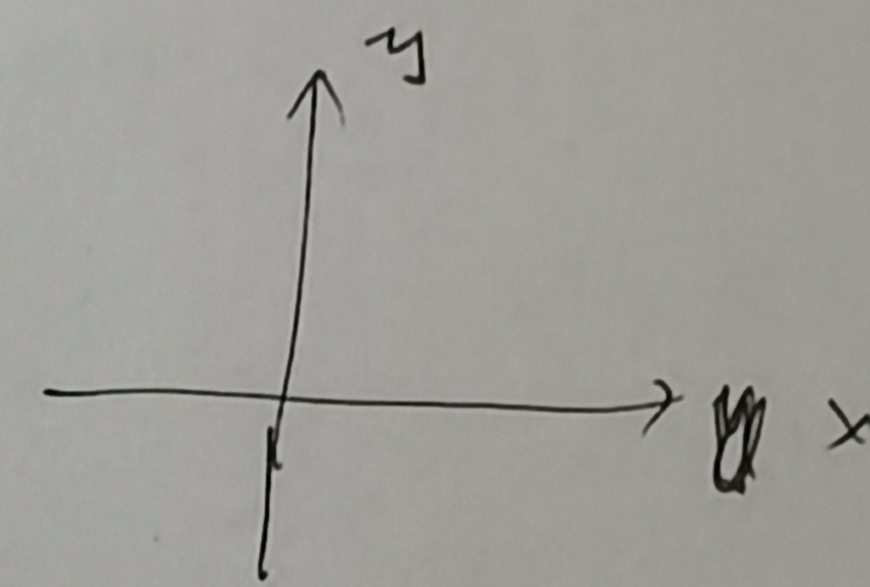
In conclusion, $x_0 = 0$ is the only local extremum.

Ex. do the same ~~analysis~~ calculation for

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

e.p. compute $f'(0)$, $f''(0)$.

Ex 2. $f(x,y) = (y-x^2)(y-2x^2)$



(i) $f(0,0) = 0$

$$f(x, kx^2) = (kx^2 - x^2)(kx^2 - 2x^2)$$
$$= (k-1)(k-2)x^4$$

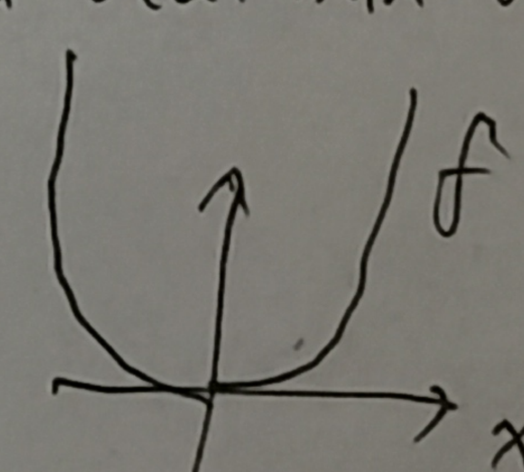
$$\begin{cases} < 0 & \text{when } 1 < k < 2 \\ > 0 & \text{when } k < 1 \text{ or } k > 2 \end{cases}$$

$\therefore$ f is not a local minimum.

$\triangledown$ $(\nabla f)_{(z_0)} = 0$, $(\text{Hess} f)_{(z_0)}$ not positive definite $\not\Rightarrow$ f is not a local minimum at z_0

$(\nabla f)(z_0) = 0$, $(\text{Hess} f)(z_0)$ pos def $\Rightarrow$ f is a $\overset{\text{isolated}}{\text{local min}}$ at z_0

E.g. $f(x) = x^4$. $(\nabla f)(0) = 0$, $(\text{Hess} f)(0)$. f is a local min at 0.



(ii) $g(t) \stackrel{\circ}{=} f(tx_0, ty_0) = (ty_0 - t^2x_0^2)(ty_0 - 2t^2x_0^2)$

$$= t^2(y_0 - tx_0^2)(y_0 - 2tx_0^2)$$

$$\cancel{g'(t) = 2t(y_0 - tx_0^2)(y_0} = 2x_0^4t^4 - 3x_0^2y_0t^3 + y_0^2t^2$$

$g'(0) = 0$ $\qquad g'(t) = 8x_0^4t^3 - 9x_0^2y_0t^2 + 2y_0^2t.$

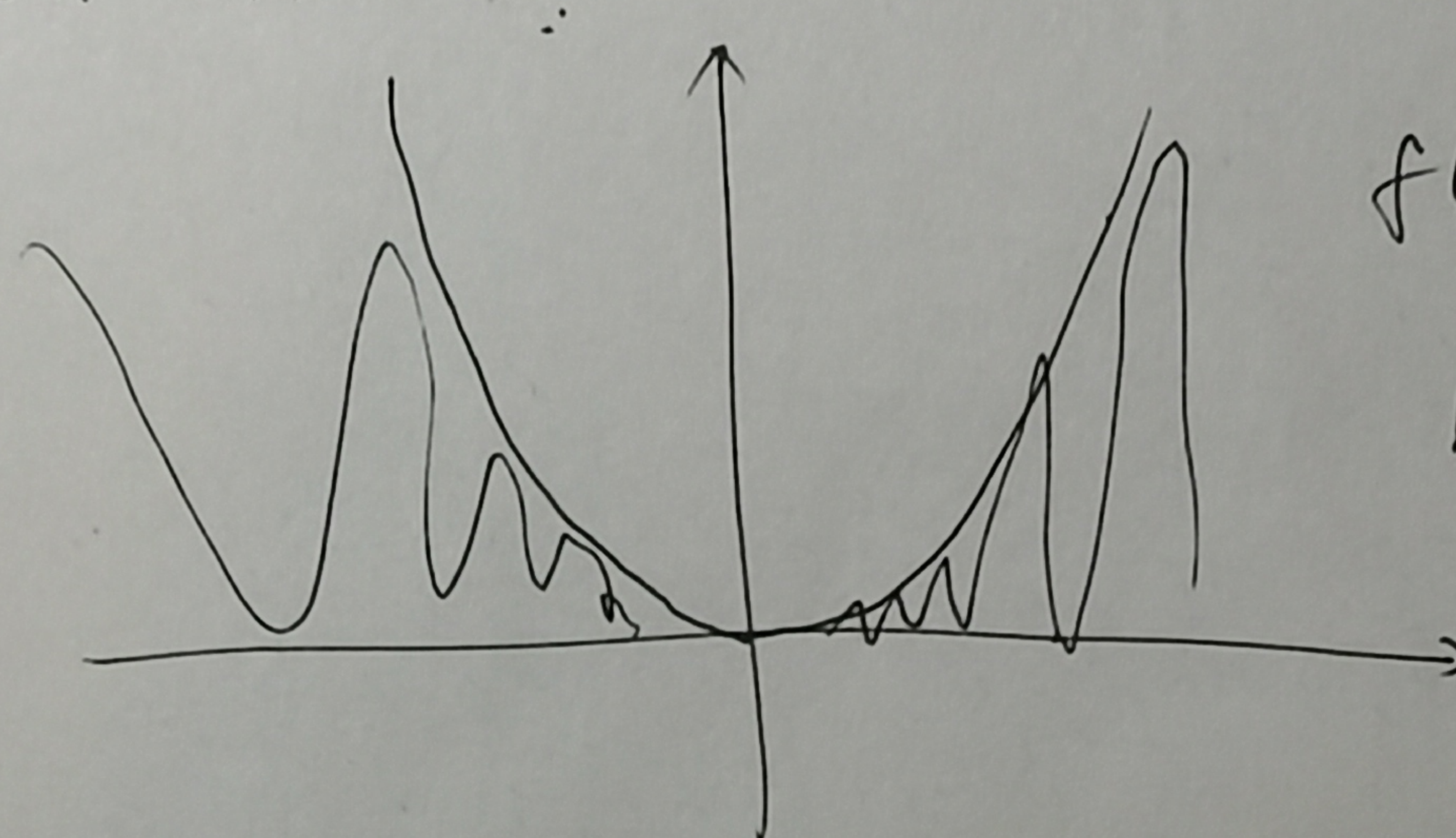
$g''(0) = y_0^2$

When $y_0 \neq 0$, $g(t)$ has (isolated) local minimum at 0;

When $y_0 = 0$, $\underbrace{\cancel{f(tx_0, ty_0)}}_{g(t)} = 2x_0^4t^4$ also has local min at 0

it is isolated local minimum because there's no other local minimum.

Isolated local minimum:

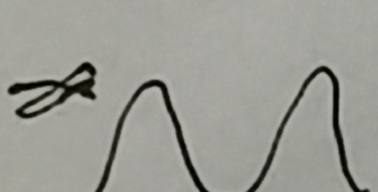


$$f(x) = \begin{cases} x^2\sin^2\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

has local minimum at $\{0, \frac{1}{\pi n} \mid n \in \mathbb{Z} - \{0\}\}$

$0 \in \{0, \frac{1}{\pi n} \mid n \in \mathbb{Z} - \{0\}\}$ is not isolated.

$\sin^2 at = ,1+$



Task 3. Let $V_1, \dots, V_m \in \mathrm{Vect}_K$, $K = \mathbb{R}$ or $\mathbb{C}$,

$$\varphi : V_1 \times \cdots \times V_m \longrightarrow K \quad m\text{-linear map}$$

$$\vec{x} = (x_1, \cdots, x_m) \in V_1 \times \cdots \times V_m$$

$$\vec{h} = (h_1, \cdots, h_m) \in V_1 \times \cdots \times V_m$$

Compute $\dfrac{\partial \varphi}{\partial \vec{h}}(\vec{x})$.

Hint.

$$\frac{\partial \varphi}{\partial \vec{h}}(\vec{x}) = \lim_{t \to 0} \frac{\varphi(\vec{x} + t\vec{h}) - \varphi(\vec{x})}{t}$$

$$= \lim_{t \to 0} \frac{\varphi(x_1 + th_1, \cdots, x_m + th_m) - \varphi(x_1, \cdots x_m)}{t}$$

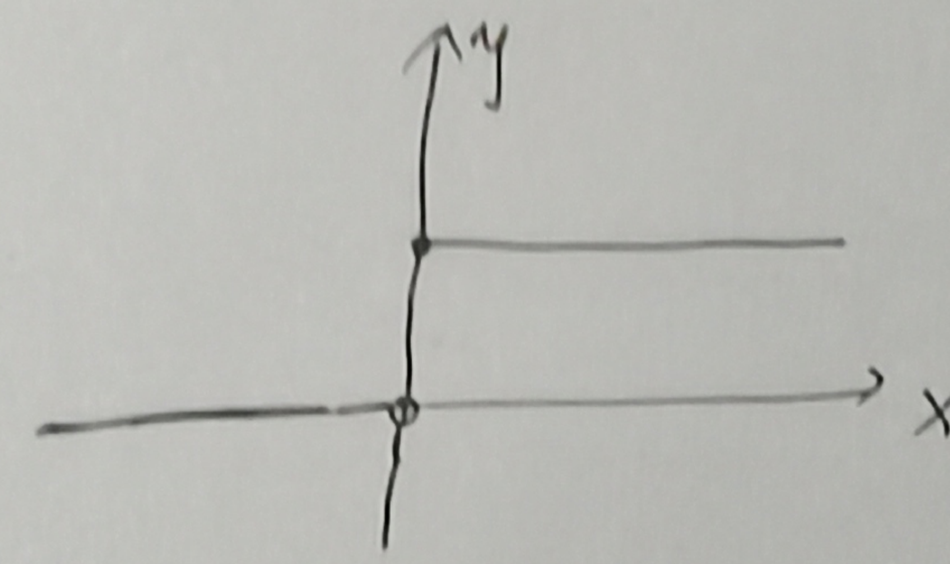
$$= \lim_{t \to 0} \frac{t\left(\varphi(h_1, x_2, \cdots x_m) + \cdots + \varphi(x_1, x_2, \cdots h_m)\right) + t^2(\cdots) + t^3(\cdots) + \cdots}{t}$$

$$= \sum_{j=1}^{m} \varphi(x_1, \cdots, x_{j-1}, h_j, x_{j+1}, \cdots x_m).$$

Task 1.2.

Motivation for the distribution

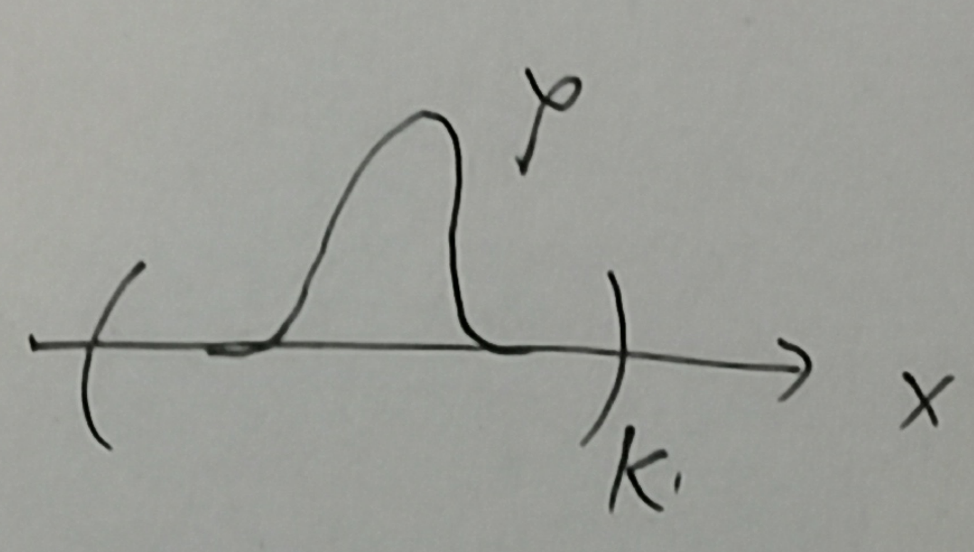
We want to differentiate not differentiable fcts, like this.



idea: embed fct space into a bigger space $\mathcal{D}'(\mathbb{R}^n)$.

Define the space of cpt supported fcts

$$C_c^\infty(\mathbb{R}^n) = \{\varphi : \mathbb{R}^n \longrightarrow \mathbb{R} \mid \varphi \text{ sm \& cpt supported}\}$$



For $\varphi_j, \varphi \in C_c^\infty(\mathbb{R}^n)$, denote $\lim\limits_{j\to+\infty} \varphi_j = \varphi$, if

(a) $\exists K \subseteq \mathbb{R}^n$ cpt s.t. φ_j is supported ~~at~~ in K (i.e. $\varphi_j|_{\mathbb{R}^n \setminus K} \equiv 0$)

(b) $\forall \alpha \in \mathbb{N}^n$, $x \in \mathbb{R}^n$, $\lim\limits_{j\to+\infty} (\partial^\alpha \varphi_j)(x) = (\partial^\alpha \varphi_j)(x)$

$l \in \mathcal{L}(C_c^\infty(\mathbb{R}^n), \mathbb{R})$ is called sequence continuous, if

$$\lim_{j\to+\infty} \varphi_j = \varphi \quad \Rightarrow \quad \lim_{j\to+\infty} l(\varphi_j) = l(\varphi)$$

Def. (Distribution)

$$\mathcal{D}'(\mathbb{R}^n) := \{ l \in \mathcal{L}(C_c^\infty(\mathbb{R}^n), \mathbb{R}) \mid l \text{ is seq continuous} \}$$

This space satisfies our requirements.

(i) We have an embedding

$$\begin{aligned} \Phi : L^1_{loc}(\mathbb{R}^n) &\hookrightarrow \mathcal{D}'(\mathbb{R}^n) \\ f &\longmapsto \left[l_f : \begin{array}{l} C_c^\infty(\mathbb{R}^n) \to \mathbb{R} \\ \varphi \mapsto \int_{\mathbb{R}^n} f\varphi \, dx \end{array} \right] \end{aligned}$$

(ii) We can take derivative in $\mathcal{D}'(\mathbb{R}^n)$ freely.

$$\begin{aligned} \partial_i : \mathcal{D}'(\mathbb{R}^n) &\longrightarrow \mathcal{D}'(\mathbb{R}^n) \\ l &\longmapsto \left[\partial_i l : \begin{array}{l} C_c^\infty(\mathbb{R}) \longrightarrow \mathbb{R} \\ \varphi \longmapsto -l(\partial_i \varphi) \end{array} \right] \end{aligned}$$

(iii) We have more elements.

$$\begin{aligned} \delta_0 : C_c^\infty(\mathbb{R}^n) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \varphi(0) \end{aligned}$$

is a typical element in $\mathcal{D}'(\mathbb{R}^n) - L^1_{loc}(\mathbb{R}^n)$

Q. ~~What do we a is ne~~

How many things are needed to be checked?

① $\delta_0, l_f, \partial_i l \in \mathcal{D}'(\mathbb{R}^n)$

② Φ is injective

③ No clash of term for ∂_i (if $\partial_i f$ exists, then $l_{\partial_i f} = \partial_i l_f$)

④ $\delta_0 \notin L^1_{loc}(\mathbb{R}^n)$

Hint l_f is seq cont. Suppose $\lim_{j\to+\infty} \varphi_j = \varphi$. Fix $K \subseteq \mathbb{R}^n$ cpt s.t. $\varphi_j|_{\mathbb{R}-K} = 0$.

Need: $\lim_{j\to+\infty} l(\varphi_j) = l(\varphi)$, i.e.

$$\lim_{j\to+\infty} \int_{\mathbb{R}^n} f\varphi_j \, dx = \int_{\mathbb{R}^n} f\varphi \, dx$$

$$\left| \int_{\mathbb{R}^n} f\varphi_j \, dx - \int_{\mathbb{R}^n} f\varphi \, dx \right| \le \int_K |f(\varphi_j - \varphi)| \, dx$$

$$\cancel{\le \int_K \left|\max_{y\in K} f(y)\right| |\varphi_j - \varphi| \, dx}$$

$$\cancel{\le \left|\max_{y\in K} f\right|}$$

$$\le \int_K |f| \left|\sup_{y\in K} (\varphi_j - \varphi)(y)\right| dx$$

$$\le \left(\int_K |f| \, dx\right) \left|\sup_{y\in K} (\varphi_j - \varphi)(y)\right| \xrightarrow{j\to+\infty} 0$$

Ex. Let $f(x) = \begin{cases} 0 & x<0 \\ 1 & x \ge 1 \end{cases}$. Compute ∂l_f.

Hint. $(\partial l_f)(\varphi) = -l_f(\partial\varphi) = -\int_{\mathbb{R}} f \partial\varphi \, dx$

$$= -\int_{[0,+\infty)} \partial\varphi \, dx \qquad \partial\varphi = \frac{d\varphi}{dx}$$

$$= -\int_{[0,+\infty)} d\varphi$$

$$= -\varphi\Big|_0^{+\infty}$$

$$= \varphi(0)$$

$\therefore \partial l_f = \delta_0$

Ex. Let $f(x) = |x|$. Compute Δl_f.

Task 2. Let

$$f : \mathbb{R}^3 - \{0\} \longrightarrow \mathbb{R} \qquad f(x) = \frac{1}{4\pi |x|}$$

View $f \in L^1_{loc}(\mathbb{R}^3)$, compute Δl_f. (Recall $\Delta f = 0$ in $\mathbb{R}^3 - \{0\}$)

Hint. $(\Delta l_f)(\varphi)$

$$= \left(\sum_i \partial_i^2 l_f\right)(\varphi)$$

$$= \sum_i (\partial_i l_f)(-\partial_i \varphi)$$

$$= \sum_i l_f(\partial_i^2 \varphi)$$

$$= l_f(\Delta\varphi)$$

$$= \int_{\mathbb{R}^3} f \Delta\varphi \, dV$$

$$= \int_{K - B_\varepsilon(0)} f \Delta\varphi \, dV + \underbrace{\int_{B_\varepsilon(0)} f \Delta\varphi \, dV}_{I_\varepsilon}$$

$$|I_\varepsilon| \leq \int_{B_\varepsilon(0)} |f| \, |\Delta\varphi| \, dV$$

$$\leq \int_{B_\varepsilon(0)} |f| \sup_{y \in B_\varepsilon(0)} |\Delta\varphi| \, dV$$

$$= \sup_{y \in B_\varepsilon(0)} |\Delta\varphi| \cdot \int_{B_\varepsilon(0)} |f| \, dV \xrightarrow{\varepsilon \to 0} 0$$

$$\int_{K-B_\varepsilon(0)} f \Delta \varphi \, dV = -\int_{\partial B_\varepsilon(0)} f \nabla \varphi \vec{\gamma} \, dA - \int_{K-B_\varepsilon(0)} \nabla f \cdot \nabla \varphi \, dV$$

$$= -\int_{\partial B_\varepsilon(0)} f \nabla \varphi \vec{\gamma} \, dA + \int_{\partial B_\varepsilon(0)} \nabla f \cdot \varphi \vec{\gamma} \, dA + \int_{K-B_\varepsilon(0)} \underbrace{\Delta f}_{= 0} \cdot \varphi \, dV$$

$$= -\frac{1}{4\pi\varepsilon} \int_{\partial B_\varepsilon(0)} \nabla \varphi \vec{\gamma} \, dA + \left(-\frac{1}{4\pi\varepsilon^2}\right) \int_{\partial B_\varepsilon(0)} \varphi \, dA$$

$$\sim -\frac{1}{4\pi\varepsilon} (\nabla \varphi)(0) \cdot 4\pi\varepsilon^2 + \left(-\frac{1}{4\pi\varepsilon^2}\right) \cancel{4\pi\varepsilon^2} \varphi(0) \cdot 4\pi\varepsilon^2$$

$$\xrightarrow{\varepsilon \to 0} -\varphi(0)$$

$$\therefore (\Delta l_f)(\varphi) = \lim_{\varepsilon \to 0} \left(\int_{K-B_\varepsilon(0)} f \Delta \varphi \, dV + I_\varepsilon \right) = -\varphi(0)$$

$$\cancel{= \frac{1}{4\pi\varepsilon} \varphi(0)}$$

i.e. $\Delta l_f = -\delta_0$ $\square$