Session 12 & Ex 10

Recall

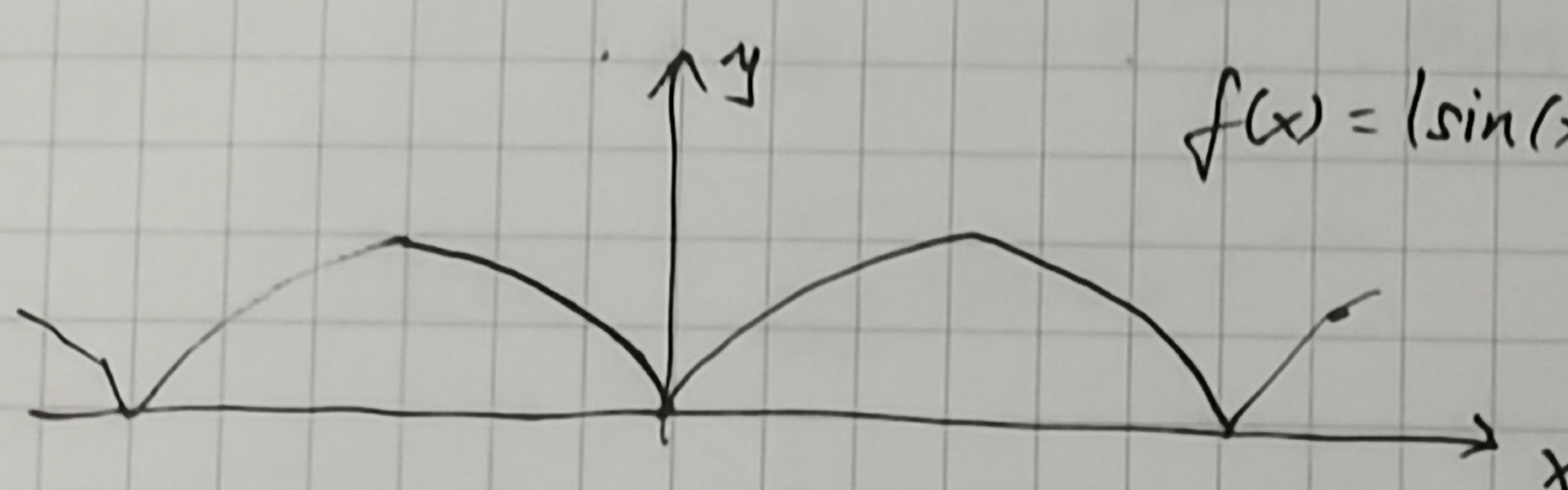
Tayler: $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k\geq 0} a_k z^k$

Laurant: $f(z) = \cdots + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{k} a_k z^k$

Fourier: $f(x) = \sum_{k} c_k e^{ikx}$

$= c_0 + \sum_{k\geq 1} a_k \cos kx + b_k \sin kx$

Task 1.



$f \in C(\mathbb{R})$, $f(x+2\pi) = f(x)$, compute $f(x) = \sum_{k\in\mathbb{Z}} c_n e^{ikx}$

Hint. $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{int}\,dt = \begin{cases} 1 & n=0 \\ 0 & n\neq 0 \end{cases}$

$$
\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-int}\,dt \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} f(t) e^{-int}\,dt + \int_{-\pi}^{0} f(t) e^{-int}\,dt\right) \\
&= \frac{1}{2\pi}\left(\int_0^{\pi} f(t) e^{-int}\,dt + \int_0^{\pi} f(t-\pi) e^{-in(t-\pi)}\,d(t-\pi)\right) \\
&= \frac{1}{2\pi}\int_0^{\pi} f(t)\left(e^{-int} + e^{-in(t-\pi)}\right) dt \\
&= \begin{cases} \frac{2}{\pi}\frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}
\end{aligned}
$$

Therefore,

$$f(x) = \sum_{n \text{ even}} \frac{2}{\pi} \frac{1}{1-n^2} e^{inx}$$

$$\overset{n=2k}{=\!=} \sum_{k\in\mathbb{Z}} \frac{2}{\pi} \frac{1}{1-(2k)^2} e^{2ikx}$$

$$= \frac{2}{\pi} \left(1 + \sum_{k=1}^{\infty} \frac{2\cos(2kx)}{1-(2k)^2}\right)$$

(ii). Solve the equation

$$u''(x) + u(x) = |\sin x|$$

Suppose $u(x) = \sum_{n\in\mathbb{Z}} d_n e^{inx}$, then

$$u'(x) = \sum_{n\in\mathbb{Z}} (in) d_n e^{inx}$$

$$u''(x) = \sum_{n\in\mathbb{Z}} -n^2 d_n e^{inx}$$

Substitute,

$$-n^2 d_n + d_n = \begin{cases} \frac{2}{\pi} \frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\Rightarrow d_n = \begin{cases} \frac{2}{\pi} \frac{1}{(1-n^2)^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\therefore u(x) = \sum_{n \text{ even}} \frac{2}{\pi} \frac{1}{(1-n^2)^2} e^{inx}$$

$$\overset{n=2k}{=\!=} \sum_{k\in\mathbb{Z}} \frac{2}{\pi} \frac{1}{(1-4k^2)^2} e^{2ikx}$$

$$= \frac{2}{\pi} \left(1 + \sum_{k=1}^{\infty} \frac{2\cos(2kx)}{(1-4k^2)^2}\right).$$

Task 2. ~~Show the~~

Assuming the axiom of choice, show the existence of non-measurable set.

Idea: "$\infty \cdot C = 0$ or ∞" for C constant number.

Define relation on $[0,1]$: $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$

get

$$\begin{tikzcd} {[0,1]} \arrow[d, "\pi"'] \\ {[0,1]/\sim} \arrow[u, dashed, bend right, "f"'] \end{tikzcd}$$

Choose a section f of π, define $A := \mathrm{Im} f$, we show that A is ~~no~~ non-measurable. If not,

Claim: $$[0,1] \subseteq \bigsqcup_{q \in [-1,1] \cap \mathbb{Q}} (A+q) \subseteq [-1,2]$$

$$\overset{m(\ )}{\Rightarrow} \quad 1 \leq \sum_{q} m^*(A+q) \leq 3$$