

Eine Woche, ein Beispiel

8.1 irreducible representation of $GL_2(\mathbb{F}_q)/SL_2(\mathbb{F}_q)$, $PGL_2(\mathbb{F}_q)/PSL_2(\mathbb{F}_q)$

$$q = p^k, p \neq 2 \\ p \text{ prime} \\ k \in \mathbb{N}^+$$

Today: $q = 3, 5, 7$

This mainly follows Fulton's book "Representation theory -- a first course".

However, I would recommend "<http://math.mit.edu/~etingof/replect.pdf>" and "<https://zhuanlan.zhihu.com/p/24986434>" because of the comfortable notation and less typos.

What can we get from the character table? Answer(in Chinese):<https://zhuanlan.zhihu.com/p/27041214>

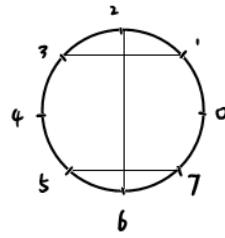
Key ingredient in computing the table:

Ind: We use the Mackey formula many times.

#C_i: We use the conjugation action which is transitive, and then use the orbit-stabilizer theorem to compute it.

$$\begin{array}{ccccc}
 & \circ & \circ & \circ & \\
 & \downarrow & \downarrow & \downarrow & \\
 \circ \rightarrow \{\pm 1\} & \longrightarrow & SL_2(\mathbb{F}_q) & \longrightarrow & PSL_2(\mathbb{F}_q) \longrightarrow \circ \\
 & \downarrow & \downarrow & \downarrow & \\
 \circ \rightarrow \mathbb{F}_q^\times & \longrightarrow & GL_2(\mathbb{F}_q) & \longrightarrow & PGL_2(\mathbb{F}_q) \longrightarrow \circ \\
 & \downarrow (-)^2 & \downarrow \det & \xrightarrow{(\frac{-}{q})} & \downarrow \\
 \circ \rightarrow \{\alpha \in \mathbb{F}_q^\times \mid (\frac{\alpha}{q}) = 1\} & \hookrightarrow & \mathbb{F}_q^\times & \longrightarrow & \{\pm 1\} \longrightarrow \circ \\
 & \downarrow & \downarrow & & \downarrow \\
 & \circ & \circ & & \circ
 \end{array}$$

$$1. \quad q = 3$$



$$\begin{array}{c} GL_2(\mathbb{F}_3) \\ \downarrow \\ \mathbb{F}_3^\times \hookrightarrow \mathbb{F}_9^\times \cong \mathbb{F}_3(\sqrt{2})^\times \\ \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/8\mathbb{Z} \end{array}$$

$$\begin{array}{ccccccccc} Id & \left(\begin{smallmatrix} 2 & 2 \\ 1 & 2 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 0 & 1 \\ 2 & 0 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix} \right) & \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right) \\ 1 & 2+\sqrt{2} & \sqrt{2} & 2+2\sqrt{2} & 2 & 1+2\sqrt{2} & 2\sqrt{2} & 1+\sqrt{2} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$

$$GL_2(\mathbb{F}_3)/SL_2(\mathbb{F}_3)$$

$$48/24$$

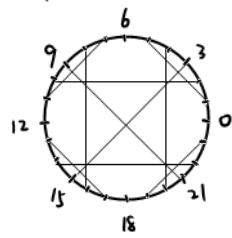
$$P = \{3\}$$

		scalar $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix} \right)$	parabolic $\left(\begin{smallmatrix} 1 & 1 \\ 8 & 1 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 1 \\ 2 & 2 \end{smallmatrix} \right)$	hyperbolic $\left(\begin{smallmatrix} 1 & 2 \\ 12 & 2 \end{smallmatrix} \right)$	elliptic $\left(\begin{smallmatrix} 1 & 2 \\ 6 & 6 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & 6 \end{smallmatrix} \right) \left(\begin{smallmatrix} 2 & 2 \\ 1 & 6 \end{smallmatrix} \right)$
1-dim		1 1	1 1	1	1 1 1
Principle series		3 3 3 3 3 3	0 0 0 0 0 0	1 -1 -1	-1 -1 -1 -1 1 1 -1
$V_{\lambda, \text{Id}}$		4 -4 2 -2 2 -2	1 -1 $-\rho^2$ $-\rho^2$ $-\rho^2$ $-\rho$	0 0 ρ	0 0 0 0 0 0 0 0 0
Complementary series		X_ρ X_{ρ^2} X_{ρ^3}	2 -2 2 -2 2 -2	-1 1 -1 1 -1 1	0 0 - $\sqrt{2}i$ - $\sqrt{2}i$ 0 $\sqrt{2}i$ $\sqrt{2}i$ 0
X_{ρ^2}		2 2 1 1 1 1	-1 -1 ρ^2 ρ^2 ρ ρ	0 0 0 2 0 0 1	2 0 0 1

$$PGL_2(\mathbb{F}_3)/PSL_2(\mathbb{F}_3) \cong S_4/A_4 \quad 24/12 \quad P = \{3\}$$

		scalar $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix} \right)$	parabolic $\left(\begin{smallmatrix} 1 & 1 \\ 8 & 1 \end{smallmatrix} \right)$	hyperbolic $\left(\begin{smallmatrix} 1 & 2 \\ 12 & 2 \end{smallmatrix} \right)$	elliptic $\left(\begin{smallmatrix} 1 & 2 \\ 6 & 6 \end{smallmatrix} \right) \left(\begin{smallmatrix} 1 & 2 \\ 1 & 6 \end{smallmatrix} \right)$
1-dim		1 1 1	1 1 1	1 -1 -1	1 1 1 1 -1 -1 1
Principle series		3 3 3	0 0 0 0 0 0	1 -1 -1	-1 -1 -1 -1 1 1 -1
Complementary series		X_ρ X_{ρ^2}	2 1 1	-1 -1 ρ^2 ρ^2 ρ ρ	0 0 0 2 0 0 1

2. $q=5$



$$\begin{array}{c} GL_2(\mathbb{F}_5) \\ \downarrow \\ \mathbb{F}_5^\times \hookrightarrow \mathbb{F}_{25}^\times \cong \mathbb{F}_5(\sqrt{2})^\times \\ \text{11S} \quad \text{11S} \\ \mathbb{Z}/5\mathbb{Z} \hookrightarrow \mathbb{Z}/24\mathbb{Z} \end{array}$$

$$\begin{array}{ccccccccc} \text{Id} & (1, 2)(2, 3)(2, 4)(4, 3)(3, 1) \\ & (1, 2)(2, 3)(2, 4)(4, 3)(3, 1) \\ 1 & \frac{2+4\sqrt{2}}{2-\sqrt{2}} & \sqrt{2} & 2\sqrt{2} & 4 & 4\sqrt{2} & 3 & 3\sqrt{2} \\ \downarrow & & & & & & & \\ 0 & 1 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 \\ & & & & & \uparrow & & & \\ & & & & & (4, 4) & & & \end{array}$$

$GL_2(\mathbb{F}_5) / SL_2(\mathbb{F}_5)$

480 / 120

 $\S = \S_{24}$

3 8 4 11 1

		scalar	parabolic	hyperbolic	elliptic
		$(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix})$... $(\begin{smallmatrix} 4 & 4 \\ & 4 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & \\ & 24 \end{smallmatrix})$... $(\begin{smallmatrix} 4 & 1 \\ & 24 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & \\ & 30 \end{smallmatrix})$... $(\begin{smallmatrix} 3 & 4 \\ & 30 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & \\ & 20 \end{smallmatrix})$... $(\begin{smallmatrix} 4 & 4 \\ & 20 \end{smallmatrix})$ $(\begin{smallmatrix} 2 & 3 \\ & 20 \end{smallmatrix})$
		$(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix})$ $(\begin{smallmatrix} 4 & 4 \\ & 4 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & 1 \\ & 12 \end{smallmatrix})(\begin{smallmatrix} 1 & 1 \\ & 12 \end{smallmatrix})$ $(\begin{smallmatrix} 4 & 1 \\ & 12 \end{smallmatrix})(\begin{smallmatrix} 4 & 2 \\ & 12 \end{smallmatrix})$	$(\begin{smallmatrix} 2 & 3 \\ & 30 \end{smallmatrix})$	$(\begin{smallmatrix} 2 & 4 \\ & 20 \end{smallmatrix})$ $(\begin{smallmatrix} 3 & 4 \\ & 20 \end{smallmatrix})$
1-dim representations		1 1 1 1	1 1 1 1	1 ... 1	1 ... 1 1 1
		1 -1 -1 1	1 -1 -1 1	i ... i	-i ... -i i
		1 1 1 1	1 1 1 1	-1 ... -1	-1 ... -1 -1
		1 -1 -1 1	1 -1 -1 1	-i ... -i	i ... i -i
		1 1 1 1	1 1 1 1	i ... i	i ... i -i
Principle series representations		5 5 5 5	0 0 0 0	1 ... 1	-1 ... -1 -1
		5 -5 -5 5	0 0 0 0	i ... i	i ... i -i
		5 5 5 5	0 0 0 0	-1 ... -1	1 ... 1 1
		5 -5 -5 5	0 0 0 0	-i ... -i	-i ... -i i
		5 5 5 5	0 0 0 0	i ... i	-1 ... -1
	$V_{\lambda^1, 1d}$	6 bi -6i 6	1 i -i -1	1+i ... -1-i	0 ... 0 0
	$V_{\lambda^1, \lambda}$	6 -bi bi 6	1 -i i -1	-1+ti ... 1-i	0 ... 0 0
	V_{λ^3, λ^2}	6 bi -6i 6	1 i -i -1	-1-i ... 1+ti	0 ... 0 0
	V_{1d, λ^3}	6 -bi bi 6	1 -i i -1	1-i ... -1+ti	0 ... 0 0
	$V_{\lambda^2, 1d}$	6 -6 -6 6	1 -1 -1 1	0 ... 0	0 ... 0 0
Complementary series representations	$V_{\lambda^2, \lambda}$	6 6 6 6	1 1 1 1	0 ... 0	0 ... 0 0
	X_{ρ}	4 4i -4i -4	-1 -i i 1	0 ... 0	0 ... -($\xi^6 + \xi^7$) - ($\xi + \xi^5$)
	X_{ρ^2}	4 -4i 4i -4	-1 i -i 1	0 ... 0	0 ... -($\xi + \xi^3$) - ($\xi^7 + \xi^1$)
	X_{ρ^3}	4 4i -4i -4	-1 -i i 1	0 ... 0	0 ... -($\xi^9 + \xi^{21}$) - ($\xi^3 + \xi^7$)
	X_{ρ^9}	4 -4i 4i -4	-1 i -i 1	0 ... 0	0 ... -($\xi^3 + \xi^7$) - ($\xi^9 + \xi^{21}$)
	X_{ρ^2}	4 -4 -4 4	1 -1 -1 1	0 ... 0	-2i ... i -i
	X_{ρ^8}	4 4 4 4	1 1 1 1	0 ... 0	2 ... 1 1
	$X_{\rho^{14}}$	4 -4 -4 4	1 -1 -1 1	0 ... 0	2i ... -i i
	X_{ρ^4}	4 4 4 4	1 1 1 1	0 ... 0	-2 ... -1 -1
	X_{ρ^3}	4 -4i -4i 4	-1 i -i 1	0 ... 0	0 ... 0 0
	X_{ρ^9}	4 4i 4i 4	-1 -i i 1	0 ... 0	0 ... 0 0
	2	-2	$\frac{-1+\sqrt{5}}{2} \frac{-1-\sqrt{5}}{2} \frac{1-\sqrt{5}}{2} \frac{1+\sqrt{5}}{2}$	0	-1 1
	2	-2	$\frac{-1-\sqrt{5}}{2} \frac{-1+\sqrt{5}}{2} \frac{1+\sqrt{5}}{2} \frac{1-\sqrt{5}}{2}$	0	-1 1

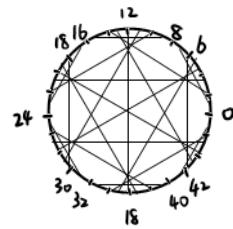
$$PGL_2(\mathbb{F}_5)/PSL_2(\mathbb{F}_5) \cong S_5/A_5$$

$$120/60$$

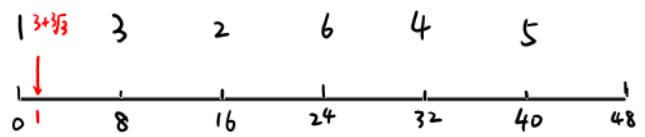
$$\S = \S_{24}$$

		scalar $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	parabolic $\begin{pmatrix} 1 & \\ & 24 \end{pmatrix}$	hyperbolic $\begin{pmatrix} 1 & \\ 30 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & \\ 30 & 4 \end{pmatrix}$	elliptic $\begin{pmatrix} 1 & 2 \\ 20 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 2 \\ 20 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 2 \\ 20 & 1 \end{pmatrix}$ $\begin{pmatrix} 2 & 4 \\ 20 & 1 \end{pmatrix}$
1-dim		1	1	1 1	1 1 1
		1	1	-1 1	-1 1 -1
		1	1	1 1	-1 -1 -1
Principle series		5	0	1 1	-1 -1 -1
		5	0	-1 1	1 -1 1
		5	0 0	1	1
	$V_{\lambda, \lambda}$	6	1	0 -2	0 0 0
		3	$\frac{1+\sqrt{5}}{2}$ $\frac{1-\sqrt{5}}{2}$	-1	0
		3	$\frac{1-\sqrt{5}}{2}$ $\frac{1+\sqrt{5}}{2}$	-1	0
Complementary series	X_p	4	1	0 0	2 1 1
	X_p^*	4	1	0 0	-2 1 -1
		4	1 1	0	1

2. $q=7$



$$\begin{array}{c} GL_2(\mathbb{F}_7) \\ \downarrow \\ \mathbb{F}_7^\times \hookrightarrow \mathbb{F}_{49}^\times \cong \mathbb{F}_7(\mathbb{F}_2)^\times \\ \text{IIS} \qquad \text{IIS} \\ \mathbb{Z}/6\mathbb{Z} \hookrightarrow \mathbb{Z}/48\mathbb{Z} \end{array}$$



		scalar	parabolic	hyperbolic	elliptic
		$(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}) \dots (\begin{smallmatrix} 6 & \\ & 6 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & \\ & 48 \end{smallmatrix}) \dots (\begin{smallmatrix} 6 & \\ & 48 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & \\ & 56 \end{smallmatrix}) \dots (\begin{smallmatrix} 5 & \\ & 56 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & 3 \\ & 42 \end{smallmatrix}) \dots (\begin{smallmatrix} 6 & 2 \\ & 42 \end{smallmatrix})(\begin{smallmatrix} 3 & 2 \\ & 42 \end{smallmatrix})$
		$(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}) (\begin{smallmatrix} 6 & \\ & 6 \end{smallmatrix})$	$(\begin{smallmatrix} 1 & 1 \\ & 24 \end{smallmatrix})(\begin{smallmatrix} 1 & 3 \\ & 24 \end{smallmatrix}) (\begin{smallmatrix} 6 & 1 \\ & 24 \end{smallmatrix})(\begin{smallmatrix} 6 & 3 \\ & 24 \end{smallmatrix})$	$(\begin{smallmatrix} 2 & 4 \\ & 56 \end{smallmatrix})(\begin{smallmatrix} 3 & 5 \\ & 56 \end{smallmatrix})$	$(\begin{smallmatrix} 2 & 3 \\ & 42 \end{smallmatrix})(\begin{smallmatrix} 5 & 3 \\ & 42 \end{smallmatrix})(\begin{smallmatrix} 0 & 2 \\ & 42 \end{smallmatrix})$
1-dim representations		1 1 ... 1	1 1 ... 1	1 ... 1	1 ... 1 1 1
		1 p^4 ... 1	1 p^4 ... 1	p^2 ... p^2	p^4 ... $p^2 p$
		1 p^2 ... 1	1 p^2 ... 1	p^4 ... p^4	p^2 ... $p^4 p^2$
		1 1 ... 1	1 1 ... 1	1 ... 1	1 ... 1 -1
		1 p^4 ... 1	1 p^4 ... 1	p^2 ... p^2	p^4 ... $p^2 p^4$
		1 p^2 ... 1	1 p^2 ... 1	p^4 ... p^4	p^2 ... $p^4 p^5$
		1	1	p^4	p^2
Principle series representations	$V_{\lambda, \text{Id}}$	7 7 ... 7	0 0 ... 0	1 ... 1	-1 ... -1 -1 -1
		7	0 0 ... 0 0 0	1	-1 -1 -1
	$V_{\lambda^2, \text{Id}}$	8 8 p^2 ... -8	1 p^2 ... -1	p ... - p	0 ... 0 0 0
		8	-8	-1 -1 -1 -1	-1 1 0 0 0
	$V_{\lambda^3, \text{Id}}$	8 8 p^4 ... 8	1 p^4 ... 1	p^5 ... p^5	0 ... 0 0 0
		8	8	-1 -1 -1 -1	-1 -1 0 0 0
	$V_{\lambda^4, \text{Id}}$	8 8 ... -8	1 1 ... -1	2 ... -2	0 ... 0 0 0
Complementary series representations	X_p	6 6 p^2 ... -6	-1 $-p^2$... 1	0 ... 0	0 ... $-(\frac{1}{2} + \frac{\sqrt{-3}}{2})$ $-(\frac{1}{2} + \frac{\sqrt{-3}}{2})$
		6	-6	-1 -1 1 1	0 0 $\sqrt{2}$ $-\sqrt{2}$ 0
	X_{p^2}	6 6 p^4 ... 6	-1 $-p^4$... -1	0 ... 0	-2 p ... 0 $-(\frac{1}{2} + \frac{\sqrt{-3}}{2})$
		6	6	-1 -1 1 1	0 0 0 2
	X_{p^3}	6 6 ... -6	-1 -1 ... 1	0 ... 0	0 ... $\sqrt{2}$ $-(\frac{1}{2} + \frac{\sqrt{-3}}{2})$
		6	-6	-1 -1 1 1	0 0 $-\sqrt{2}$ $\sqrt{2}$ 0
	X_{p^4}	6 6 p^2 ... 6	-1 $-p^2$... -1	0 ... 0	-2 p^2 ... -2 p 0
		3	3	$\frac{-1+\sqrt{-3}}{2} \frac{-1-\sqrt{-3}}{2} \frac{-1-\sqrt{-3}}{2} \frac{-1+\sqrt{-3}}{2}$	1 1 -1
		3	3	$\frac{-1-\sqrt{-3}}{2} \frac{-1+\sqrt{-3}}{2} \frac{-1+\sqrt{-3}}{2} \frac{-1-\sqrt{-3}}{2}$	1 1 -1

$$PGL_2(\mathbb{F}_7) / PSL_2(\mathbb{F}_7)$$

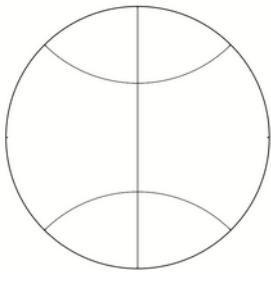
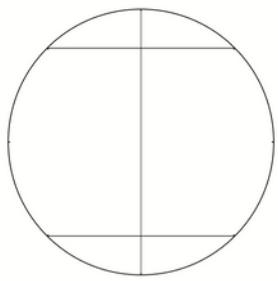
$$336/168$$

$$P = \{6\} \quad \{ = \{48\}$$

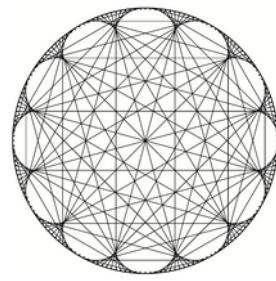
$$41 \quad 18 \quad 45$$

		scalar $(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix})$	parabolic $(\begin{smallmatrix} 1 & \\ & 48 \end{smallmatrix})$	hyperbolic $(\begin{smallmatrix} 1 & \\ 56 & 2 \end{smallmatrix}) (\begin{smallmatrix} 1 & \\ 56 & 3 \end{smallmatrix}) (\begin{smallmatrix} 1 & \\ 28 & 6 \end{smallmatrix})$	elliptic $(\begin{smallmatrix} 1 & 3 \\ 21 & 1 \end{smallmatrix}) (\begin{smallmatrix} 1 & 3 \\ 42 & 1 \end{smallmatrix}) (\begin{smallmatrix} 2 & 3 \\ 42 & 2 \end{smallmatrix}) (\begin{smallmatrix} 3 & 3 \\ 42 & 3 \end{smallmatrix})$
1-dim		1	1	1 1 1	1 1 1 1
		1	1	1 -1 -1	1 -1 1 -1
		1	1	1	1
Principle series	7	0	1	1 1 1	-1 -1 -1 -1
	7	0	1	1 -1 -1	-1 1 -1 1
	7	0 0	1	1	-1
Complementary series	V_{λ^2, λ^4}	8	1	-1 -1 2	0 0 0 0
	V_{λ^1, λ^5}	8	1	-1 1 -2	0 0 0 0
	V_{λ^3}	8	1 1	-1	0 0 0 0
X_{φ^8}	6	-1	0 0 0	2 $\sqrt{2}$ 0 $-\sqrt{2}$	
	X_{φ^6}	6	-1 -1	0 0 0	2 $-\sqrt{2}$ 0 $\sqrt{2}$
	X_{φ^6}	6	-1 -1	0	2 $-\sqrt{2}$ 0 0
$X_{\varphi^{12}}$	6	-1	0 0 0	-2 0 2 0	
	3	$\frac{-1+\sqrt{7}i}{2}$ $\frac{-1-\sqrt{7}i}{2}$	0	1	
	3	$\frac{-1-\sqrt{7}i}{2}$ $\frac{-1+\sqrt{7}i}{2}$	0	1	

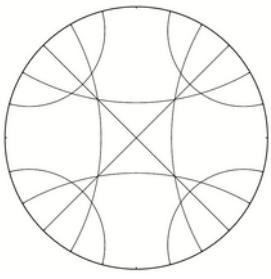
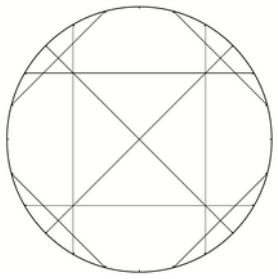
Bonus: Frobenius map in $\mathbb{F}_{q^2}/\mathbb{F}_q$



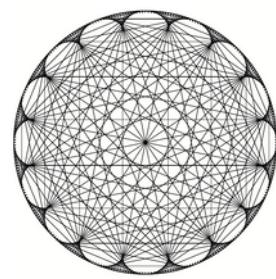
$$\mathbb{F}_9/\mathbb{F}_3$$



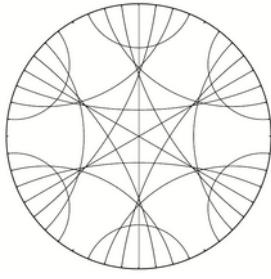
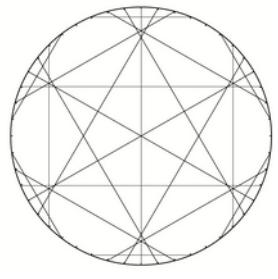
$$\mathbb{F}_{169}/\mathbb{F}_{13}$$



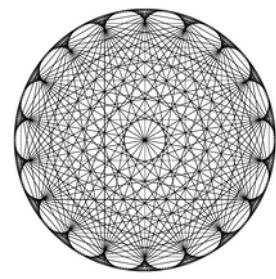
$$\mathbb{F}_{25}/\mathbb{F}_5$$



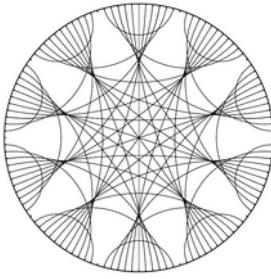
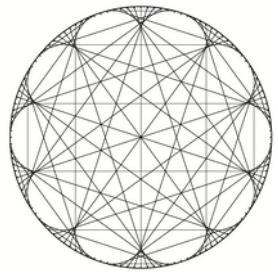
$$\mathbb{F}_{289}/\mathbb{F}_{17}$$



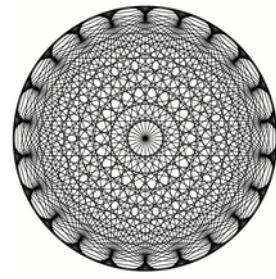
$$\mathbb{F}_{49}/\mathbb{F}_7$$



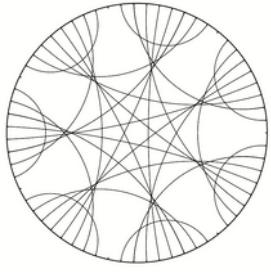
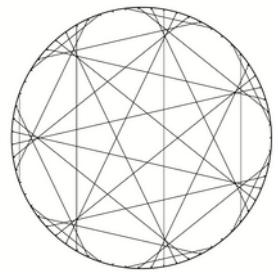
$$\mathbb{F}_{361}/\mathbb{F}_{19}$$



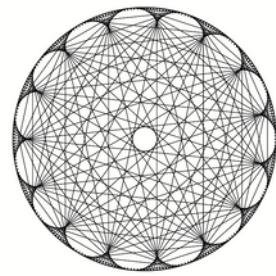
$$\mathbb{F}_{121}/\mathbb{F}_{11}$$



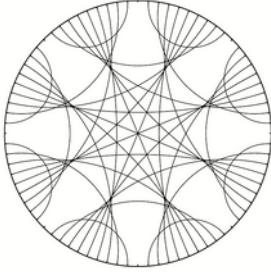
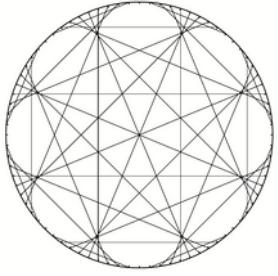
$$\mathbb{F}_{529}/\mathbb{F}_{23}$$



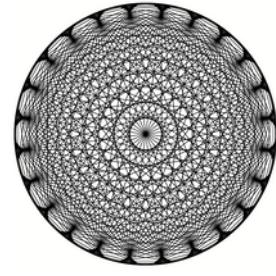
$$\mathbb{F}_{64}/\mathbb{F}_8$$



$$\mathbb{F}_{286}/\mathbb{F}_{16}$$



$$\mathbb{F}_{81}/\mathbb{F}_9$$



$$\mathbb{F}_{645}/\mathbb{F}_{25}$$