Eine Woche, ein Beispiel

11.7. Berkovich space

Ref: Spectral theory and analytic geometry over non-Archimedean fields by Vladimir G. Berkovich(we mainly follow this article)
+courses from Junyi Xie
+An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves

Goal: understand the beautiful picture copied from "An introduction to Berkovich analytic spaces and non-archimedean potential theory on curves"

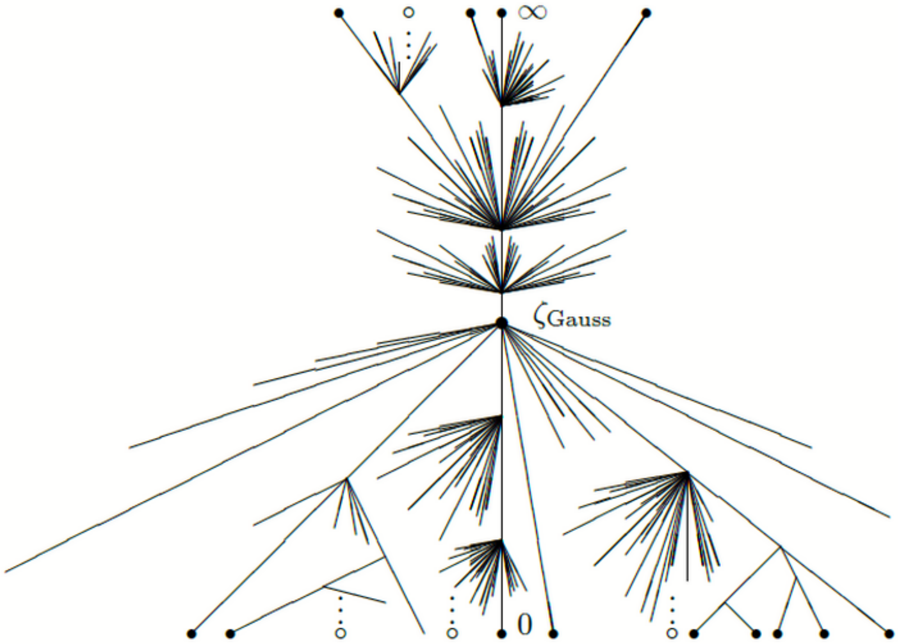


FIGURE 1. The Berkovich projective line (adapted from an illustration of Joe Silverman)

A: comm with 1 (for convenience)

extra condition	local		global	closed unit disc	open unit disc
—	Spec A	affine scheme	scheme		Spec $\mathbb{Z}[[T]]$
A: adic ring with f.g. ideal of def	Spf A	affine formal scheme	formal scheme		Spf $\mathbb{Z}_p[[T]]$
A: k-affinoid alg, i.e. $A = k\langle T_1, \dots, T_n\rangle / I$	MaxSpec A	k-affinoid space	rigid-analytic space over k	Max Spec $\mathbb{Q}_p\langle T\rangle$	$\mathcal{U} = \{ \lvert\cdot\rvert \in \mathrm{MaxSpec}\, \mathbb{Q}_p\langle T\rangle \mid \lvert T\rvert < 1\}$
(A, A^+): Huber pair	Spa(A, A^+)	affinoid adic space	adic space	Spa $(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$	
A: Banach ring	$\mathcal{M}(A)$	spectrum	Berkovich space		

Ref of table: Berkeley notes

Rmk. MaxSpec A has only a Grothendieck topology.
K (in K-affinoid space) is a NA field, but can also be generalized to k-Banach alg.
non-archimedean
k: NA field

1. Seminorm

1.1. Def (seminorm of abelian group) $\|\cdot\| : M \longrightarrow \mathbb{R}_{\geq 0}$ s.t

$\|0\| = 0$ norm: $\|m\| = 0 \Rightarrow m = 0$

$\|f-g\| \leq \|f\| + \|g\|$ non-Archimedean: $\|f-g\| \leq \max(\|f\|, \|g\|)$

- seminorm $\Rightarrow$ topology

  Prop. $(M, \|\cdot\|)$ is Hausdorff $\Leftrightarrow$ $\|\cdot\|$ is norm

  Def (equivalence of norm)

- sub, quotient, homomorphism

  Def (restricted seminorm)

  Def. (residue seminorm) $\pi : (M, \|\cdot\|_M) \longrightarrow M/N$ induce the seminorm on M/N:

  $$\|\bar{m}\|_{M/N} := \inf_{\pi(m') = \bar{m}} \|m'\|_M$$

  Def (bounded / admissible) $\varphi : (M, \|\cdot\|_M) \longrightarrow (N, \|\cdot\|_N)$

  - bounded: $\exists C > 0$, $\|\varphi(m)\|_N \leq C \|m\|_M$
  - admissible: $\bar{\varphi} : (M/\ker\varphi, \|\cdot\|_{quo}) \longrightarrow (\mathrm{Im}\,\varphi, \|\cdot\|_{res})$ induces equivalence of norm.

1.2. Def (seminorm of ring non-comm, with 1): seminorm group +

$\|1\| = 1$ norm $\xrightarrow{\text{+completed}}$ Banach ring

$\|fg\| \leq \|f\| \|g\|$ power-multi: $\|f^n\| = \|f\|^n$

multiplicative: $\|fg\| = \|f\|\|g\|$ $\Big\}$ $\Rightarrow$ absolute value

- quotient, $\prod_{\text{infinite}}$, $\mathcal{A}\langle r^{-1}T\rangle$, ...
- comparison among norms: bounded.
- Def related to valuation field

https://math.stackexchange.com/questions/2151779/normed-vector-spaces-over-finite-fields

1.3. Def (seminorm of $\mathcal{A}$-module, where $\mathcal{A}$: normed ring)

seminorm group + $\exists C > 0$, $\|fm\| \leq C \|f\| \|m\|$

- $\hat{\otimes}_{\mathcal{A}}$

seminormed ring

$\uparrow$ $(\mathbb{Z}, |\ |_p)$, $(\mathbb{Q}, |\ |_\infty)$, $(\mathbb{Q}, |\ |_p)$

Banach ring

$\uparrow$ (R, triv) R: not field, $(\mathbb{Z}, |\ |_\infty)$, $(\mathbb{C}\langle r^{-1}T\rangle, \|\ \|_{ind})$

valuation field

Arch field	non-Arch field
$(\mathbb{R}, |\ |_\infty)$ $(\mathbb{R}/\mathbb{C}, |\ |_\infty^{\varepsilon})$ $0 < \varepsilon \leq 1$	$(\mathbb{F}_q, \text{triv})$ (K, triv)
$(\mathbb{C}, |\ |_\infty)$	$(\mathbb{Q}_p, |\cdot|_p)$

⚠ In analysis, the word "seminorm" is defined in a "totally" different way:

Definition 1.1.3. A **seminorm** on a $\mathbb{K}$-vector space E is a function $p : E \to \mathbb{R}$ such that

(1) $p(x+y) \leq p(x) + p(y)$ for all $x, y \in E$.

(2) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{K}$, $x \in E$.

In analysis, the ring usually has no unit (e.g. $\mathcal{L}^1(\mathbb{R})$), and (semi)norms are absolute homogeneous.

Moreover, we don't require semimultiplicative.

e.g. in $\mathcal{L}^p(\mathbb{R})$, one don't have $\|fg\|_p \leq \|f\|_p \|g\|_p$

Apart from analysis, the terminology is concluded as follows.

seminorm	
(multiplicative) norm = absolute value = places	
valuation (Bourbaki) exponential valuation NA absolute value ultrametric absolute value	Archi absolute value

I prefer Bourbaki's terminology, because valuations are always written additive, and the natural triangular inequality is the ultrametric inequality, i.e.,

$$v(a+b) \geq \min(v(a), v(b)), \quad \text{with equality if } v(a) \neq v(b)$$

In the main ref (as well as this document, e.g. no example found yet) the norm can be not multiplicative, but I assume norm to be multiplicative in other documents.

2. Affine case

suppose $\mathcal{A}$: Banach ring (comm + 1)

$\mathcal{M}(\mathcal{A}) := \{\text{bounded mult seminorms on } \mathcal{A}\}$

with top basis generated by $U_{m,(a,b)} := \{\|\cdot\| \in \mathcal{M}(\mathcal{A}) \mid \|m\| \in (a,b)\}$

$m \in \mathcal{A}$, $(a,b) \subseteq \mathbb{R}$

$\mathcal{M}(\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)) := \{\text{mult seminorms on } \mathcal{A}\}$

E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

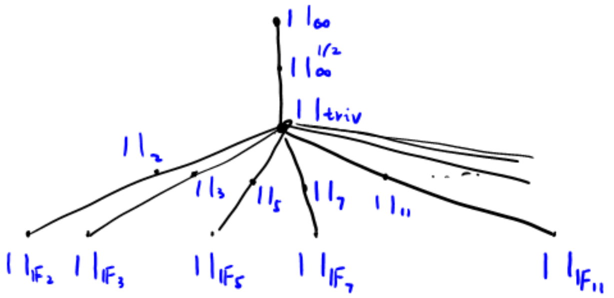
Don't confuse with $|\cdot|_p = |\cdot|_{L^p}$ in functional analysis!

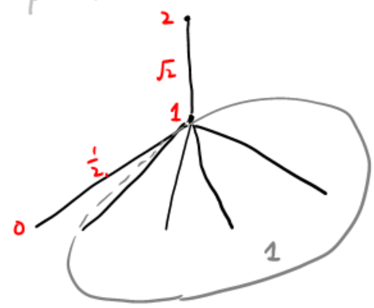
We have

$$\mathcal{M}(\mathbb{Z}, |\ |_\infty) = \left\{ \begin{array}{ll} |\ |_{triv} := \text{trivial norm} & \\ |\ |_p^t \ : \ t \in (0, +\infty] & |m|_{\mathbb{F}_p} := |m|_p^\infty = \begin{cases} 0 & p \mid m \\ 1 & p \nmid m \end{cases} \\ |\ |_\infty^\varepsilon \ : \ \varepsilon \in (0,1] & \end{array} \right\}$$

$|\ |_{triv} = |\ |_p^0 = |\ |_\infty^0$

Picture:

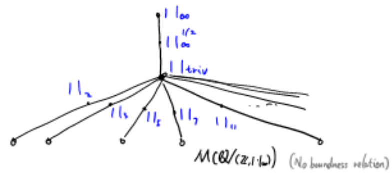




value of 2.

From this picture, we want to get:
Bound relations among seminorms
Topology properties: Hausdorff? compact?
Residue field, injection and contraction
... See next page

Rmk. When we do not identify the norm, we mean $\mathcal{A}/(\mathbb{Z}, |\cdot|_\infty)$.



E.g. $\mathcal{A} = (\mathbb{Q}, \|\cdot\|_{any})$, $\mathcal{M}(\mathcal{A}) = \{*\}$

E.g. $\mathcal{A} = (\mathbb{F}_q, \|\cdot\|_{triv})$ $\quad \mathcal{M}(\mathbb{F}_q) = \{*\}$

E.g. $\mathcal{A} = \mathbb{R}/\mathbb{C}$ $\quad$ continuous seminorms are $\|\cdot\|_\infty^\varepsilon$, $\varepsilon \in [0,1]$.

Do we have any other cont seminorms? No.

E.g. $\mathcal{A} = \mathbb{Z}_p$ $\quad$ continuous seminorms are $\|\cdot\|_p^t$, $t \in [0, +\infty]$. ($\mathcal{A} = \mathbb{Q}_p$ is also interesting)

Do we have any other cont seminorms?

E.g. $\mathcal{A} = \mathbb{C}_p$

E.g. $\mathcal{A} = \mathbb{C}[x]$ $\quad$ If we only consider the norm which restricted to $\mathbb{C}$ is $|\ |_\infty$, we would get $\mathbb{C}$. Need to verify...

What would happen in the other cases?

If we only consider the norm which restricted to $\mathbb{C}$ is $|\ |_{triv}$, we would get $\mathbb{C}\mathbb{P}^1$.

E.g. $\mathcal{A} = \mathbb{C}_p\langle r^{-1}T\rangle$ or $\mathbb{P}^1_{\mathbb{C}_p}$

E.g. $\mathcal{A} = (\mathbb{Z}[i], \|\ \|_\infty)$

I'm very happy to do the homework one years ago.

E.g. $\mathcal{A} = (\mathbb{Z}, |\ |_\infty)$

Try to answer the following questions.

- Set
  - $\mathcal{M}(\mathbb{Z})$ = ✓
  - partial order ⇝ bound order
  - Picture ✓
  - maximal / minimal seminorm   max: $\|\cdot\|_{\mathbb{F}_p}$ min $\|\cdot\|_\infty$
  - Berkovich structure of $\|\cdot\| \in \mathcal{M}(\mathbb{Z})$?

- Archi or non Archi?
  !

($\mathcal{M}(\mathbb{Z})$, graph)
↓
($\mathcal{M}(\mathbb{Z})$, weak)
is continuous

weak top is a little weaker then graph top

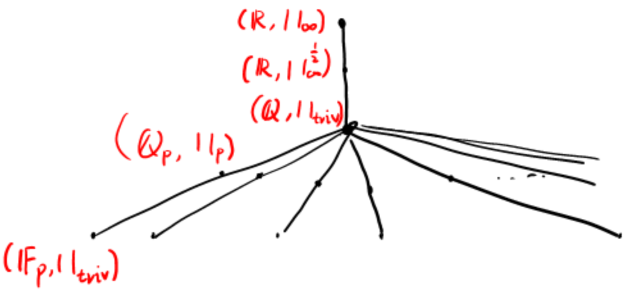
- Topo
  - Close set   not contain $|\ |_{triv}$: normal way + contain only finite $\|\cdot\|_p^t$
    contain $|\ |_{triv}$: normal way
  - Open set   not contain $|\ |_{triv}$: normal way
    contain $|\ |_{triv}$: normal way + contain all $\|\cdot\|_p^t$ except finite p

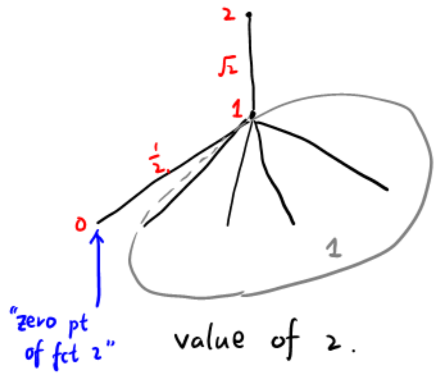
  - Topo properties: connected? ✓ Hausdorff? ✓ (quasi) compact? ✓
    $X = Y \sqcup Z$
    irreducible? ✗
    $X = Y \cup Z$

Def. $p \in X$ is a closed pt iff $\{p\}$ is closed

Then every pt is closed pt

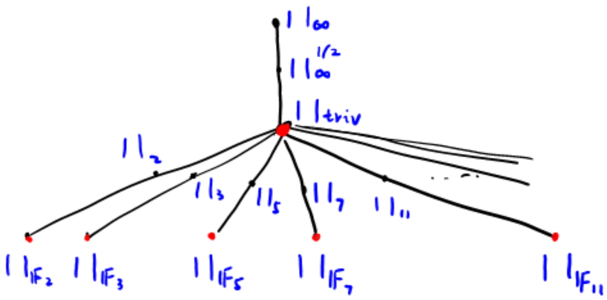
The definitions of Residue field, injection and contraction follows from [3.1.1, https://arxiv.org/abs/2105.13587v3]

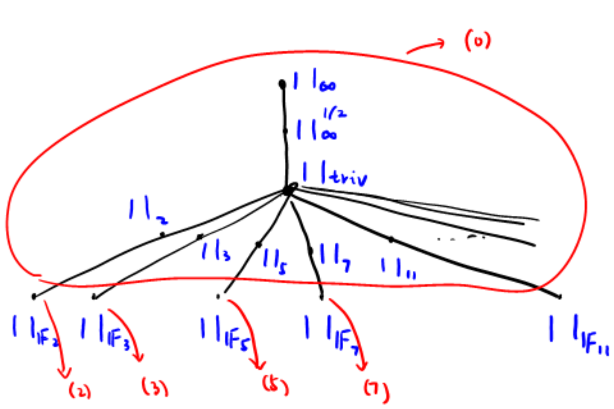




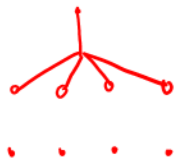
value of 2.

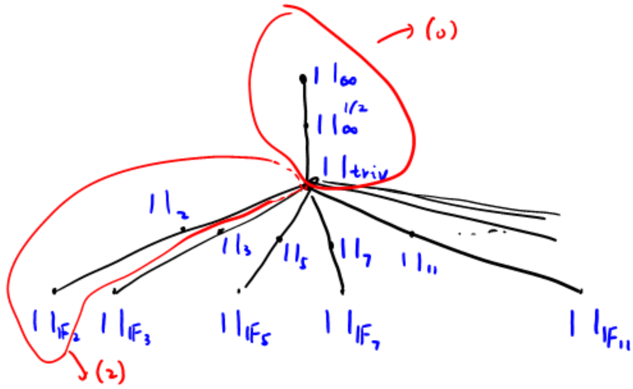
Residue field





k: kernel map (contraction)





r: reduction map (specialization)