

Eine Woche, ein Beispiel

5.1 Extension of NA local field

F: NA local field

1 List of well-known results

- in general

- unramified / totally ramified

2. $\hat{\mathbb{Z}}$ = profinite completion (review)

3. Big picture

4. A detailed discussion concerning proofs.

5. Henselian ring

} not complete, I need time to check the proof

6. Cohomological dimension

7. Bonus: "plane geometry" for \mathbb{Q}_q .

Q: Is there any subfield of \mathbb{Q}_p with finite index?

Can we classify all subfield of $\mathbb{F}_p((t))$ with finite index?

<https://math.stackexchange.com/questions/211582/is-there-a-proper-subfield-k-subset-mathbb-r-such-that-mathbb-rk-is-fin>

Ref:

Initial motivation comes from

[AY]<https://alex-youcis.github.io/localglobalgalois.pdf>

which explains the relationships between local fields and global fields in a geometrical way.

main reference for cohomological dimension:

[NSW2e]<https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/>

[JPS96] Galois cohomology by Jean-Pierre Serre

<http://p-adic.com/Local%20Fields.pdf>

<https://people.clas.ufl.edu/rcrew/files/LCFT.pdf>

<http://www.mcm.ac.cn/faculty/tianyichao/201409/Wo20140919372982540194.pdf>

For existence and uniqueness of extension of valuation, see Theorem 3.2 here:

https://www.dpmms.cam.ac.uk/~ajs1005/ANT/notes_s3-4.pdf

1. List of well-known results

In general

F : NA local field E/F : finite extension

Rmk 1. E is also a NA local field with uniquely extended norm

$$\|x\|_v = \|N_{E/F}(x)\|_F^{\frac{1}{n}} \quad \text{resp. } v(x) := \frac{1}{n} v_F(N_{E/F}(x))$$

$$\text{E.g. } \|1 - \zeta_n\| = 1 \text{ in } \mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p \quad p \nmid n \quad v(1 - \zeta_n) = 0$$

$$\|1 - \zeta_p\| = \frac{1}{p} \quad \text{in } \mathbb{Q}_p(\zeta_p)/\mathbb{Q}_p \quad v(1 - \zeta_p) = \frac{1}{p}$$

$$\|1 - \zeta_5\| = \left\| (1 - \zeta_5)(1 - \zeta_5^2)(1 - \zeta_5^3)(1 - \zeta_5^4) \right\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \|5\|_{\mathbb{Q}_5}^{\frac{1}{5}} = \frac{1}{\sqrt[5]{5}} \quad \text{in } \mathbb{Q}_5(\zeta_5)$$

$$\|1 - \zeta_{p^n}\| = p^{-\frac{1}{p^{n+1}}} \quad \text{in } \mathbb{Q}_p(\zeta_{p^n})/\mathbb{Q}_p \quad v(1 - \zeta_{p^n}) = \frac{1}{p^{n+1}}$$

$\Rightarrow 1 - \zeta_{p^n}$ is a uniformizer of $\mathbb{Q}_p(\zeta_{p^n})$

Rmk 2. [AY, Thm 1.9]

\mathcal{O}_E is monogenic, i.e. $\mathcal{O}_E = \mathcal{O}_F[\alpha] \quad \exists \alpha \in \mathcal{O}_E$

A proof of this may be found here:

<https://math.stackexchange.com/questions/3406117/ring-of-integers-of-simple-field-extension-of-local-field-is-monogenic>

Cor. (primitive element thm for NA local field)

$$E = F[x]/(g(x)) \quad \exists x \in \mathcal{O}_E, g(x) \text{ min poly of } x.$$

Rmk: Every separable finite field extension has a primitive element, see wiki:

https://en.wikipedia.org/wiki/Primitive_element_theorem

Separable condition is necessary, see

<https://mathoverflow.net/questions/21/finite-extension-of-fields-with-no-primitive-element>

⚠ \mathcal{O}_E may be not a free \mathcal{O}_F -module.

See: <https://kconrad.math.uconn.edu/blubs/gradnumthy/notfree.pdf>

Rmk 3. Any finite extension of \mathbb{Q}_p is of form $\mathbb{Q}_p[x]/(g(x))$,

where $g(x) \in \mathbb{Q}[x]$ is an irr poly.

Any finite extension of $\mathbb{F}_q(t)$ is of form $\mathbb{F}_q((t))[x]/(g(x))$

where $g(x) \in \mathbb{F}_q((t))[x]$ is an irr poly..

Both are achieved by Krasner's lemma.

From [<https://math.mit.edu/classes/18.785/2017fa/LectureNotes11.pdf>]:

Remark 11.12. Krasner's lemma is another "Hensel's lemma" in the sense that it characterizes Henselian fields (fraction fields of Henselian rings);

<https://math.stackexchange.com/questions/1176495/the-maximal-unramified-extension-of-a-local-field-may-not-be-complete>

$$v = v_F = \frac{1}{e} v_E \quad \| \cdot \| = \| \cdot \|_F = \| \cdot \|_E^{\frac{1}{e}} \quad \wp_F \mathcal{O}_E = \wp_E^e$$

E	$v_E = ev$	$\ \cdot \ _E = \ \cdot \ ^{e^{-1}}$	$\pi_E = \pi_F^{\frac{1}{e}}$	$v(\pi_E) = \frac{1}{e}$
deg n				
F	v_F	$\ \cdot \ _F$	π_F	$v(\pi_F) = 1$

Unramified / totally ramified

Good ref: https://en.wikipedia.org/wiki/Finite_extensions_of_local_fields
It collects the equivalent conditions of unramified/totally ramified field extensions.

	tot ram	wild ram
		tame ram
	field ext	
		split in local case

When E/F is tot ramified.

$$e = n \quad v(\pi_E) = \frac{1}{n}$$

$\mathcal{O}_E = \mathcal{O}_F[\pi_E]$ $\min(\pi_E) \in \mathcal{O}_F[x]$ is Eisenstein poly.

Lemma. Let E/F : NA local field, $e = e(E/F)$, $r \in \mathbb{N}_{\geq 0}$. Easy to see

$$\begin{aligned} \wp_E^{1+r} \cap F &= \{x \in F \mid v_E(x) \geq \frac{1}{e}(1+r)\} \\ \wp_F^{1+\lceil \frac{r}{e} \rceil} &= \{x \in F \mid v_F(x) \geq 1 + \lceil \frac{r}{e} \rceil\} \end{aligned}$$

Then

$$Tr_{E/F}(\wp_E^{1+r}) \subseteq \wp_F^{1+\lceil \frac{r}{e} \rceil} \quad \text{when } E/F \text{ is tamely ramified}$$

Table for $e=3$: ("proof of lemma")

r	0	1	2	3	4	5	6	7
$\frac{1}{e}(1+r)$	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2	$\frac{7}{3}$	$\frac{8}{3}$
$1 + \lceil \frac{r}{e} \rceil$	1	1	1	2	2	2	3	3

E.g. $E/F = \mathbb{Q}_{49}/\mathbb{Q}_7 = \mathbb{Q}_7(\sqrt{3})/\mathbb{Q}_7$ is unramified.

$$\begin{aligned} v(a+b\sqrt{3}) &= \frac{1}{2} v(N_{E/F}(a+b\sqrt{3})) \\ &= \frac{1}{2} v(a^2 - 3b^2) \\ &= \frac{1}{2} \min(v(a^2), v(b^2)) \\ &= \min(v(a), v(b)) \end{aligned} \quad a, b \in \mathbb{Q}_7$$

$$\begin{aligned} \mathcal{O}_E &= \mathbb{Z}_7(\sqrt{3}) & \mathfrak{p}_E &= (7, 7\sqrt{3}) = (7) & k_E &= \mathbb{Z}_7(\sqrt{3})/(7) \\ &&&&&\cong \mathbb{Z}_7[\alpha]/(\alpha^2 - 3, 7) \cong \mathbb{F}_7(\sqrt{3}) \cong \mathbb{F}_{49} \end{aligned}$$

$$\beta_E^{1+r} = (7)^{1+r} = (7^{1+r}) \quad \text{Tr}_{E/F}(\beta_E^{1+r}) = \beta_F^{1+r} = \beta_E^{1+r} \cap F \quad r \geq 0$$

E.g. $E/F = \mathbb{Q}_7(\sqrt{7})/\mathbb{Q}_7$ is tamely ramified.

$$\begin{aligned} v(a+b\sqrt{7}) &= \frac{1}{2} v(N_{E/F}(a+b\sqrt{7})) \\ &= \frac{1}{2} v(a^2 - 7b^2) \\ &= \frac{1}{2} \min(v(a^2), 1+v(b^2)) \\ &= \min(v(a), \frac{1}{2} + v(b)) \end{aligned} \quad a, b \in \mathbb{Q}_7$$

$$\begin{aligned} \mathcal{O}_E &= \mathbb{Z}_7(\sqrt{7}) & \mathfrak{p}_E &= (7, \sqrt{7}) = (\sqrt{7}) & k_E &= \mathbb{Z}_7(\sqrt{7})/(\sqrt{7}) \\ &&&&&\cong \mathbb{Z}_7[\alpha]/(\alpha^2 - 7, \alpha) \cong \mathbb{Z}_7/(7) \cong \mathbb{F}_7 \end{aligned}$$

$$\beta_E^{1+r} = (\sqrt{7})^{1+r} = \begin{cases} (7^{\frac{1+r}{2}}) & r \text{ odd} \\ \sqrt{7} \cdot (7^{\frac{r}{2}}) & r \text{ even} \end{cases} \quad \text{Tr}(\sqrt{7}^{\frac{1+r}{2}}) = 2 \cdot 7^{\frac{1+r}{2}} \quad r \geq 0$$

$$\text{So } \text{Tr}_{E/F}(\beta_E^{1+r}) = \beta_E^{1+r} \cap F = \beta_F^{1+\lceil \frac{r}{2} \rceil}.$$

2. $\widehat{\mathbb{Z}} = \text{profinite completion of } \mathbb{Z}$ (Recall 2022.2.13 outer auto...)

$$\widehat{\mathbb{Z}} := \prod_l \mathbb{Z}_l$$

$$\widehat{\mathbb{Z}}^{\times} := \prod_l \mathbb{Z}_l^{\times}$$

$$\widehat{\mathbb{Z}}^{(p)} := \prod_{l \neq p} \mathbb{Z}_l$$

$$(\widehat{\mathbb{Z}}^{\times})^{(p)} := \prod_{l \neq p} \mathbb{Z}_l^{\times} = (\widehat{\mathbb{Z}}^{(p)})^{\times}$$

Prop. ① $\text{Hom}_{\text{pro-gp}}(\mathbb{Z}_l, \mathbb{Z}_m) = \begin{cases} \mathbb{Z}_l & l=m \\ 0 & l \neq m \end{cases} \quad l, m \text{ prime.}$

② $\text{Aut}(\mathbb{Z}_p) = \mathbb{Z}_p^{\times}$

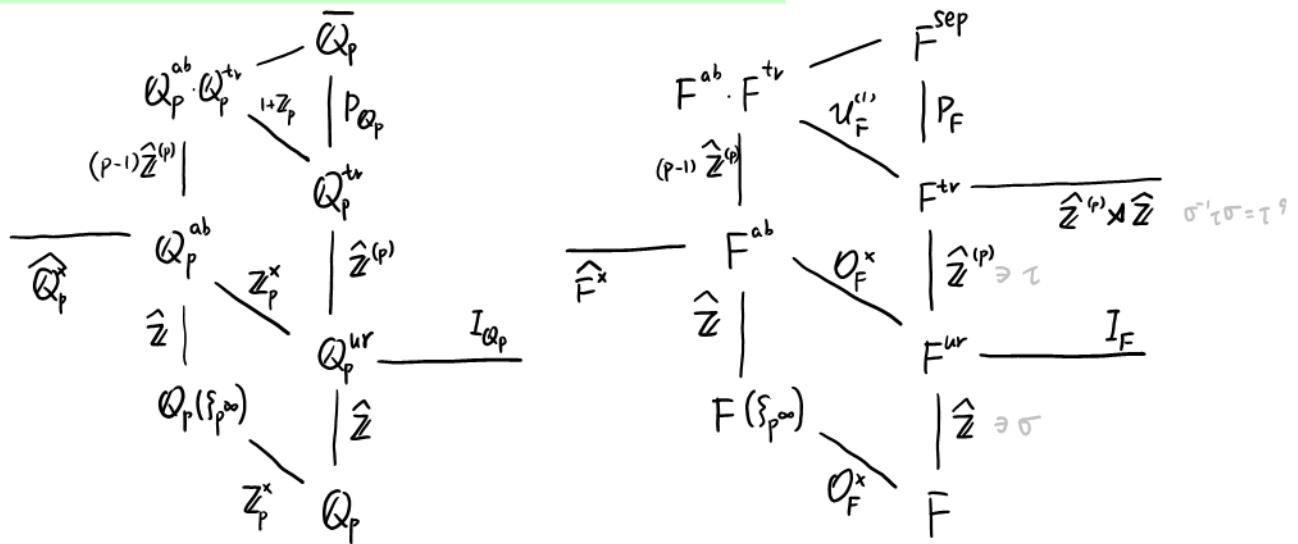
$\text{Aut}(\widehat{\mathbb{Z}}) = \widehat{\mathbb{Z}}^{\times}$ in the category of profinite gps.

$\text{Aut}(\widehat{\mathbb{Z}}^{(p)}) = \widehat{\mathbb{Z}}^{(p)\times}$

③ $\mathcal{O}_F, \mathcal{O}_F^{\times}$ are profinite groups, so $\widehat{\mathcal{O}}_F = \mathcal{O}_F \quad \widehat{\mathcal{O}}_F^{\times} = \mathcal{O}_F^{\times}$.

3. Big picture

Main ref: [AY] <https://alex-youscis.github.io/localglobalgalois.pdf>



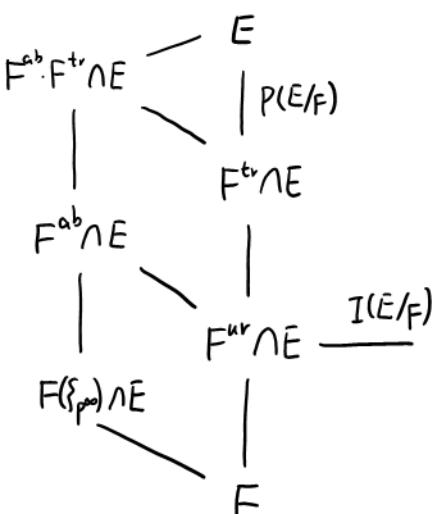
unramified $F^{ur} = \bigcup_{n \geq 1} F(\{s_{p^n}\})$ Fermat's little thm $\bigcup_{\substack{n \geq 1 \\ p \nmid n}} F(\{s_n\})$ **local field with char k = p**

tame ramified $F^{tr} = F^{ur}(\pi_F^{\frac{1}{n}} |_{(n,p)=1})$ $= F(\pi_F^{\frac{1}{n}}, s_n |_{(n,p)=1})$ Notice that $s_p \in F^{tr}$!

abelian $F^{ab} = F(\{s_\infty\}) := \bigcup_{n \geq 1} F(\{s_n\})$

$$F^{ab} \cdot F^{tr} = F(\pi_F^{\frac{1}{n}}, s_\infty |_{(n,p)=1})$$

<https://math.stackexchange.com/questions/507671/the-galois-group-of-a-composite-of-galois-extensions>



$$\begin{array}{ccc}
\begin{array}{c}
\mathbb{Q}_p(\{p^n\}) \\
| \\
\mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^2\mathbb{Z})^\times \cap \mu_{p-1}} \{p^n\}\right) \\
| \\
(\mathbb{Z}/p^2\mathbb{Z})^\times \\
| \\
(\mathbb{Z}/p\mathbb{Z})^\times \\
| \\
\mathbb{Q}_p(\{p\}) \\
| \\
\mathbb{Q}_p \\
\end{array} & \dots &
\begin{array}{c}
\mathbb{Q}_p(\{p^\infty\}) \\
| \\
\mathbb{Q}_p\left(\sum_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^2\mathbb{Z})^\times \cap \mu_{p-1}} \{p^n\}\right)\right) \\
| \\
(\mathbb{Z}/p^2\mathbb{Z})^\times \\
| \\
(\mathbb{Z}/p\mathbb{Z})^\times \\
| \\
\mathbb{Q}_p(\{p\}) \\
| \\
\mathbb{Q}_p \\
\end{array} & \dots
\end{array}$$

$\xrightarrow{(\mathbb{Z}/p^2\mathbb{Z})^\times}$ $\xrightarrow{\mathbb{Z}/(p-1)\mathbb{Z}}$ $\xrightarrow{\mathbb{Z}/(p-1)\mathbb{Z}}$ $\xrightarrow{\mathbb{Z}/(p-1)\mathbb{Z}}$
 $\xrightarrow{1+p\mathbb{Z}}$ $\xrightarrow{1+p^n\mathbb{Z}}$ $\xrightarrow{1+p\mathbb{Z}}$ $\xrightarrow{1+p\mathbb{Z}}$
 $\xrightarrow{(\mathbb{Z}/p\mathbb{Z})^\times}$ $\xrightarrow{(\mathbb{Z}/p\mathbb{Z})^\times}$ $\xrightarrow{(\mathbb{Z}/p\mathbb{Z})^\times}$ $\xrightarrow{(\mathbb{Z}/p\mathbb{Z})^\times}$

$$E/F = \mathbb{Q}_p(\{p^n\})/\mathbb{Q}_p$$

$$E/F = \mathbb{Q}_p(\{p^\infty\})/\mathbb{Q}_p$$

$$\begin{array}{ccc}
\mathbb{Q}_p(\{p\}) & \mathbb{Q}_p(\{p\}) & \mathbb{Q}_p(\{p\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & & \\
| & | & | \\
\mathbb{Q}_p(\{p + p^{-1}\}) & \mathbb{Q}_p(\{p\}) & \mathbb{Q}_p(\{p\}) \\
| & | & | \\
| & | & | \\
\mathbb{Q}_p & \mathbb{Q}_p & \mathbb{Q}_p
\end{array}$$

$$\begin{array}{ccc}
\mathbb{Q}_p(\{p^\infty\}) & \mathbb{Q}_p(\{p^\infty\}) & \mathbb{Q}_p(\{p^\infty\}) \\
| & | & | \\
\mathbb{Z}/2\mathbb{Z} & & \\
| & | & | \\
\mathbb{Q}_p(\{p^n + p^{-n}\}) & \mathbb{Q}_p^{abwr}(\{p\}) & \mathbb{Q}_p^{abwr}(\{p\}) \\
| & | & | \\
| & | & | \\
\mathbb{Q}_p & \mathbb{Q}_p^{abwr} & \mathbb{Q}_p^{abwr}
\end{array}$$

$\xrightarrow{\mathbb{Z}/2\mathbb{Z}}$ $\xrightarrow{\mathbb{Z}/2\mathbb{Z}}$ $\xrightarrow{\mathbb{Z}/2\mathbb{Z}}$
 $\xrightarrow{\cup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p(\{p^n + p^{-n}\})}$ $\xrightarrow{\mathbb{Z}/2\mathbb{Z}}$ $\xrightarrow{\mathbb{Z}/2\mathbb{Z}}$
 $\xrightarrow{\mathbb{Q}_p^{abwr} := \cup_{n \in \mathbb{Z}_{\geq 0}} \mathbb{Q}_p\left(\sum_{i \in (\mathbb{Z}/p^2\mathbb{Z})^\times \cap \mu_{p-1}} \{p^n\}\right)}$ $\xrightarrow{\mathbb{Q}_p^{abwr}}$ $\xrightarrow{\mathbb{Q}_p^{abwr}}$

$$p \text{ odd}$$

$$p \equiv 1 \pmod{4}$$

$$p \equiv 3 \pmod{4}$$

There are only finite isomorphism classes of degree n extensions of \mathbb{Q}_p , see here for a discussion:
<https://math.stackexchange.com/questions/1118068/finitely-many-extensions-of-fixed-degree-of-a-local-field>

Except for the filtrations as well as cohomology dimensions, the Artin-Schreier theory also gives us a better understanding of the wild inertia group. For example, there are exactly p^2 ramified degree p field extensions of \mathbb{Q}_p (for p odd prime). A detailed discussion (and Table 2.1) can be seen here:

<https://www.sciencedirect.com/science/article/pii/S0747717105001276?via%3Dihub>

For a ref of the Artin-Schreier theory, you can see

https://en.wikipedia.org/wiki/Artin-Schreier_theory

<https://math.stackexchange.com/questions/50041/reference-book-for-artin-schreier-theory> (gives the proof of $x^p - x - a$)

Q: How many degree p field extensions of $\mathbb{F}_p((t))$ are there?

Warning:

Even though every degree p field ext of \mathbb{Q}_p can be written of the form
$$\mathbb{Q}_p[x]/(x^p - x - \alpha), \quad \alpha \in \mathbb{Q}_p, \quad \alpha + \beta^p - \beta \text{ for } \beta \in K$$

it's not feasible to do so when we do for examples (no good parameters)

e.g. It's not easy to find a canonical $\alpha \in \mathbb{Q}_p$ s.t.

$$\mathbb{Q}_p[x]/(x^p - p) \cong \mathbb{Q}_p[y]/(y^p - y - \alpha).$$

4. A detailed discussion concerning proofs.

Let us elaborate our understanding on $\text{Gal}(E/F)$, G_F
 (Rather than viewing them as black box)

Setting: E/F be a finite Galois ext of NA local fields.

<https://math.stackexchange.com/questions/4385377/ramified-extension-of-local-field-which-is-not-galois>

Finite extension

Lemma. $\forall \sigma \in \text{Gal}(E/F), \quad v_E(\sigma(-)) = v_E(-).$

$$\text{e.g. } \sigma(\mathcal{O}_E) = \mathcal{O}_E \quad \sigma(\mathfrak{p}_E) = \mathfrak{p}_E,$$

$$\text{Gal}(E/F) \xrightarrow{\sim} \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E)$$

[Proof. The valuation of F extends uniquely to E . (use completeness)
 In addition, $v_E(\sigma(-)), v_E(-)$ are two valuations.]

Rmk. Using this, one can show that

$$\mathcal{O}_E = \text{integral closure of } \mathcal{O}_F \text{ in } E.$$

Completeness is necessary, see <https://math.stackexchange.com/questions/4065594/integer-ring-and-valuation-ring-of-local-fields>

Using the lemma, we construct a map

$$\text{Gal}(E/F) \cong \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E) \cong \text{Gal}(k_E/k_F)$$

and extends it to a SES

$$1 \longrightarrow I(E/F) \longrightarrow \text{Gal}(E/F) \longrightarrow \text{Gal}(k_E/k_F) \longrightarrow 1$$

where

$$I(E/F) := \{\sigma \in \text{Gal}(E/F) \mid \sigma(x) \equiv x \pmod{\mathfrak{p}_E} \quad \forall x \in \mathcal{O}_E\}$$

Q. How to show that

$$\text{Gal}(E/F) \longrightarrow \text{Gal}(k_E/k_F)$$

$$\text{Gal}(E/F) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

$$\text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E)$$

are surjective? (Now: take it as a black box)

In general, define $r \in \mathbb{N}_{\geq 0}$

$$\text{Gal}(E/F) \cong \text{Aut}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_E) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1})$$

we get SES

$$1 \longrightarrow I_r(E/F) \longrightarrow \text{Gal}(E/F) \longrightarrow \text{Aut}_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \longrightarrow 1$$

where

$$I_r(E/F) := \{\sigma \in \text{Gal}(E/F) \mid \sigma(x) \equiv x \pmod{\mathfrak{p}_E^{r+1}} \quad \forall x \in \mathcal{O}_E\}$$

Comparing those constructions, we get a filtration

$$\cdots \subseteq I_r(E/F) \subseteq \cdots \subseteq I_1(E/F) \subseteq I_0(E/F) \subseteq I_{-1}(E/F)$$

|| || || def

$$P(E/F) \subseteq I(E/F) \subseteq Gal(E/F)$$

with $\bigcap_{r \in \mathbb{N}} I_r(E/F) = 1$

and surjections

$$\rightarrow Aut_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \rightarrow \cdots \rightarrow Aut_{\mathcal{O}_F/\mathfrak{p}_F^1\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^1) \rightarrow Aut_{k_F\text{-alg}}(k_E) \rightarrow 1$$

with $\varprojlim_{r \in \mathbb{N}} Aut_{\mathcal{O}_F/\mathfrak{p}_F^{r+1}\text{-alg}}(\mathcal{O}_E/\mathfrak{p}_E^{r+1}) \cong Gal(E/F)$

Prop. We understand filtrations well:

$$I_{-1}(E/F) / I_0(E/F) \xrightarrow{\sim} Gal(k_E/k_F)$$

cyclic

$$I_0(E/F) / I_1(E/F) \hookrightarrow (\mathcal{O}_E/\mathfrak{p}_E)^* \cong k_E^*$$

$\sigma \mapsto \frac{\sigma(\pi)}{\pi}$

i.e. $[\sigma : \pi_E \mapsto a_1\pi_E + a_2\pi_E^2 + \dots] \xrightarrow{\quad} a_1$

$$I_r(E/F) / I_{r+1}(E/F) \hookrightarrow \mathcal{O}_E/\mathfrak{p}_E^r \cong k_E$$

$\sigma \mapsto \frac{\sigma(\pi)}{\pi} - 1 \quad r \in \mathbb{N}_{\geq 1}$

i.e. $[\sigma : \pi_E \mapsto \pi_E + a_r\pi_E^r + \dots] \xrightarrow{\quad} a_r$ Not cyclic!

Cor. $P(E/F)$ is the Sylow-p-subgp of $I(E/F)$.

Cor. When E/F is tot ramified with deg r , we have an iso

$$I_0(E/F)/I_1(E/F) \cong \mu_r(k_E)$$

In tame case, $\#\mu_r(k_E) = r$.

Absolute Galois gp

⚠ Some naive ideas are surprisingly complicated, so we list possible mistakes here, and don't touch those ideas...

- Don't want to mention $\overline{\mathcal{O}_E}$, since \overline{E} is not a NA local field.

Also, I don't know if

$$\text{the integral closure } \overline{\mathcal{O}_E} = \{x \in \overline{E} \mid v(x) \geq 0\}$$

- Can't define $I_{F,v} = \varprojlim_{E/F \text{ fin Gal}} I_v(E/F)$, so we switch to upper indexing.
In that case,

$$I_F = I_F^\circ \quad P_F = I_F^{\geq 0} = \bigcup_{v \geq 0} I_F^v$$

$$\bigcap_{v \in [0, \infty)} I_F^v = 1 \quad \bigcap_{v' < v} I_F^{v'} = I_F^v \quad v, v' \in [0, \infty)$$

We define I_F and P_F directly. Denote $\phi_E: G_F \rightarrow \text{Gal}(E/F)$

$$I_F := \left\{ \sigma \in G_F \mid \phi_E(\sigma) \in I(E/F) \quad \forall E/F \text{ fin Gal} \right\}$$

$$P_F := \left\{ \sigma \in G_F \mid \phi_E(\sigma) \in P(E/F) \quad \forall E/F \text{ fin Gal} \right\}$$

We have SES

$$1 \longrightarrow I_F \longrightarrow G_F \longrightarrow \begin{matrix} G_{K_F} \\ \cong \\ \widehat{\mathbb{Z}} \end{matrix} \longrightarrow 1$$

$$1 \longrightarrow P_F \longrightarrow I_F \longrightarrow \varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(K_E) \longrightarrow 1$$

where $\varprojlim_{\substack{E/F \\ \text{fin Gal} \\ \text{rami index } e}} \mu_e(K_E) \cong \varprojlim_{\substack{E=F(\pi_F^{\frac{1}{e}}) \\ (m, p)=1}} \mathbb{Z}/m\mathbb{Z} \cong \widehat{\mathbb{Z}}^{(p)}$

Transforming to the field side, we have clear descriptions on F^{ur} & F^{tr} :

$$F^{ur} := F(\{s_n\}_{(n,p)=1})$$

$$F^{tr} := F(\pi_F^{\frac{1}{e}}, s_n|_{(n,p)=1})$$

Claim: For $F' = F^{ur} \cap E$, $\# E/F' = e$, if $(e,p) = 1$, then $E \subseteq F^{tr}$

[Lemma 1.3.2, 1.3.3]: https://kskedlaya.org/cft/sec_loalkronweb.html

Try: Only need to show $\pi_E \in F^{tr}$.

Take min poly of π_E over F'

$$f(T) = T^e + a_{e-1}T^{e-1} + \dots + a_1T + a_0 \in O_F[T]$$

$\Rightarrow f(T)$ is an Eisenstein poly

I think that we can use Newton's method to show that, the equation

$$f(T) = 0$$

has a root in F^{tr} .

Wrong try: $[E : E \cap F^{tr}]$ is p -power, and $[E : E \cap F^{tr}] \mid e \Rightarrow E = E \cap F^{tr}$
 $\Rightarrow E \subseteq F^{tr}$

Reason: we don't know yet if F^{tr} is max tame ram ext.

Actually, we can write down the iso explicitly:

$$t_{Fr}: G_F/I_F \longrightarrow \widehat{\mathbb{Z}} \quad \sigma \mapsto t_{Fr}(\sigma) \text{ satisfying}$$

$\forall \text{ field ext } E/F, \quad \forall x \in \mathcal{O}_E, \quad \# K_F = q$

$$\sigma(x) \equiv x^{q^{t_{Fr}(\sigma)}} \pmod{p_E}$$

Fix compatible generators

$$\{s_m\}_{(m,p)=1} \in \varprojlim_{(m,p)=1} \mu_m(F^{\text{tr}}),$$

one can define

$$t_s: I_F/p_F \longrightarrow \widehat{\mathbb{Z}}^{(p)} \quad \sigma \mapsto t_s(\sigma) \text{ satisfying}$$

$$\frac{\sigma(\pi_F^{\frac{1}{m}})}{\pi_F^{\frac{1}{m}}} = s_m^{t_s(\sigma) \pmod{m}} \quad \text{in } F^{\text{tr}} \quad \forall (m,p)=1$$

Task. Take $F_r := t_{Fr}^{-1}(1)$ (lift to G_F) $\tau_r := t_s^{-1}(1)$, $m \in \mathbb{N}_{\geq 1}$, $(m,p)=1$.

Show that

$$F_r(s_m) = s_m^q \quad \tau_r(s_m) = s_m$$

$$F_r(\pi_F^{\frac{1}{m}}) = s_m^l \pi_F^{\frac{1}{m}} \quad \tau_r(\pi_F^{\frac{1}{m}}) = s_m^l \pi_F^{\frac{1}{m}} \quad \exists l \in (\mathbb{Z}/m\mathbb{Z})^\times$$

Therefore,

$$F_r \tau_r F_r^{-1} = \tau_r^q \quad \text{in } F_r(s_m, \pi_F^{\frac{1}{m}})$$

$$\Rightarrow F_r \tau_r F_r^{-1} = \tau_r^q \quad \text{in } F^{\text{tr}}.$$

For completeness, one also have the iso

$$t_{ab}: \text{Gal}(F^{\text{ab}}/F^{\text{ur}}) \xrightarrow{\sim} \widehat{\mathbb{Z}}^{(p)} \quad \sigma \mapsto t_{ab}(\sigma) \text{ satisfying}$$

$$\sigma(s_{p^r}) = s_{p^r}^{t_{ab}(\sigma) \pmod{p^r}} \quad \forall r \in \mathbb{N}_{\geq 0}$$

5. Henselian ring

Main ref: https://en.wikipedia.org/wiki/Henselian_ring

R comm with 1 (local in this section)

Def. A local ring (R, \mathfrak{m}) is Henselian if Hensel's lemma holds, i.e.

$$\begin{array}{ccc} \text{for } P \in R[x] & \exists f_i \in P[x] & \bullet P = f_1 \dots f_n \\ \downarrow & \downarrow \circ & \\ \bar{P} = g_1 \dots g_n \in R/\mathfrak{m}[x] & g_i \in R/\mathfrak{m}[x] & \end{array}$$

(R, \mathfrak{m}) is strictly Henselian if additionally $(R/\mathfrak{m})^{\text{sep}} = R/\mathfrak{m}$.

- E.g.
- Fields/Complete Hausdorff local rings are Henselian.
e.g. \mathcal{O}_F are Henselian
 - R is Henselian $\Leftrightarrow R/\mathfrak{m}(R)$ is Henselian
 $\Leftrightarrow R/I$ is Henselian for $\forall I \triangleleft R$
e.g. when $\text{Spec } R = \{*\}$, R is Henselian.

Denote $\text{StrHense} \subset \text{Hense} \subset \text{LocRing} \subset \text{CommRing}$ full subcategories

$$\begin{array}{ccccc} & & \text{zero} & & (-)^{\text{sh}} \\ & \swarrow & & \searrow & \\ \text{Str Hense} & \xleftarrow{\quad \text{forget} \quad} & \text{Hense} & \xrightarrow{\perp} & \text{LocRing} \\ & \xleftarrow{\text{Stack OSL}} & & \xrightarrow{(-)^h} & \end{array}$$

E.g. $F^h = F$ $F^{\text{sh}} = F^{\text{un}}$

Geometrically, Henselian means $\text{Spec } R/\mathfrak{m} \rightarrow \text{Spec } R$ has a section.

6. Cohomological dimension

main reference for cohomological dimension:
 [NSW2e] <https://www.mathi.uni-heidelberg.de/~schmidt/NSW2e/>

<https://mathoverflow.net/questions/349484/what-is-known-about-the-cohomological-dimension-of-algebraic-number-fields>

This section is initially devoted to the following result.

Prop. [(7.5.1)] The wild inertia gp P_F is free pro-p-group of countably infinite rank.

See [Galois Theory of p-Extensions, Chap 4] for the definition and construction of free pro-p-groups.

Q: Do we have the adjoint

$$\begin{array}{ccc} \text{Pro-p-gp} & \xrightleftharpoons[\text{forget}]{\perp} & \text{Set} \\ & \xleftarrow{(\)^{\text{free}}} & \end{array}$$

?

Now let

$$\begin{aligned} G: & \text{ profinite gp} \\ \text{Mod}(G): & \text{ category of discrete } G\text{-modules} \\ \text{full subcategory} \\ \text{of Mod}(G) & \left. \begin{array}{ll} \text{Mod}_t(G): & \text{torsion} \\ \text{Mod}_p(G) & \text{p-torsion} \\ \text{Mod}_f(G) & \text{finite} \end{array} \right\} \text{viewed as abelian gp} \end{aligned}$$

Lemma For abelian torsion gp X , denote

$$X(p) := \{x \in X \mid x^{p^k} = 1 \quad \exists k \in \mathbb{N}_{>0}\}$$

we have $X = \bigoplus_p X(p)$.

This is trivial when X is finite, but I don't know how to prove this in the general case. It should be not too hard.

Def [(3.3.1)] (cohomological dimension) p prime

$$cd G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A) \neq 0\}$$

$$tcd G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A) \neq 0\}$$

$$cd_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}_t(G), H^i(G, A)(p) \neq 0\}$$

$$tcd_p G = \sup \{i \in \mathbb{N}_{\geq 0} \mid \exists A \in \text{Mod}(G), H^i(G, A)(p) \neq 0\}$$

Prop. (local to global) $cd G = \sup_p cd_p G$ $scd G = \sup_p scd_p G$

Prop. [(3.3.2)] $cd_p G \leq n \iff H^{n+1}(G, A) = 0 \quad \forall \text{ simple } G\text{-mod } A \text{ with } pA = 0$

e.g. for G : pro-p-gp,

$$cd_p G \leq n \iff H^{n+1}(G, \mathbb{Z}/p\mathbb{Z}) = 0$$

E.g. $cd_p \mathbb{Z} = 1$ $scd_p \mathbb{Z} = 2$

Prop. [(3.3.5)] For $H \leq G$ closed,

$$cd_p H \leq cd_p G \quad scd_p H \leq scd_p G$$

When $p \nmid [G:H]$ or $[H \text{ open} + cd_p G < +\infty]$, the equality holds.

Weaker condition, see [(3.3.5, Serre)]

Cor. G : profinite gp, then

$$cd_p G = 0 \iff p \nmid \#G$$

Prop. [(3.5.17)] A pro-p-gp G is free iff $cd G \leq 1$.

Prop [7.1.8] (i) F NA local field with $\text{char } k = p$.

$$cd_l(F) = \begin{cases} 2 & \text{if } l \neq \text{char } F, \\ 1 & \text{if } l = \text{char } F. \end{cases}$$

For any E/F field extension s.t. $l^{\infty} \mid \deg E/F$, $cd_l(E) \leq 1$.

(ii) Fix $n \in \mathbb{N}_{>0}$ s.t. $\text{char } F \nmid n$.

$$H^i(F, \mu_n) = \begin{cases} F^\times / (F^\times)^n & i=1 \\ \frac{1}{n} \mathbb{Z}/\mathbb{Z} & i=2 \\ 0 & i \geq 3 \end{cases}$$

[Proof for Prop (7.5.1)]

$$\text{Now } l^{\infty} \mid \deg F^{\text{tr}}/F \stackrel{(7.1.8)}{\Rightarrow} cd_l(F^{\text{tr}}) \leq 1 \quad \forall \text{ prime } l$$

$$\Leftrightarrow cd_l(F) \leq 1$$

$\Leftrightarrow P_F$ is free pro- p -group.

□

7. Bonus: "plane geometry" for \mathbb{Q}_9

In this section, the picture comes from [<https://www.nt.th-koeln.de/fachgebiete/mathe/knospe/p-adic/>] by Heiko Knospe.

I want to define:

Compare \mathbb{Q}_9 and $\mathbb{Q}_3(\sqrt{3})$

triangle (Actually we just consider 3 points, and they may be "collinear")

disk

sphere

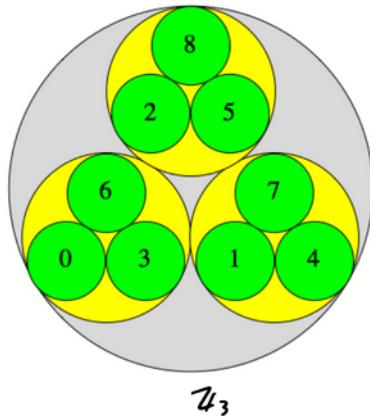
line (in higher dimension, like \mathbb{Q}_9 or $\mathbb{Q}_3(\sqrt{3})$)

no angle, no perpendicular, but parallel lines

$P^1(\mathbb{Q}_3)$ (should characterize all lines in \mathbb{Q}_9 passing through 0, parameterized by a line not passing the origin)

intersections of disks, spheres and lines

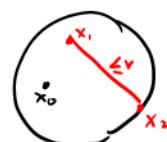
sphere packing? Symmetric group of the objects considered? connection with the tree-structures/Bruhat--Tits building?



\mathbb{Z}_3



Triangles: $a = a \geq b$



disks: every pt can be center
(even pts on the edge)

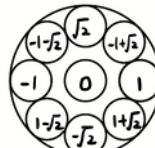
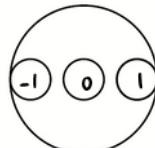
$B_{x_0}(r)$

I personally would like to draw it more "compatible with arithmetic":

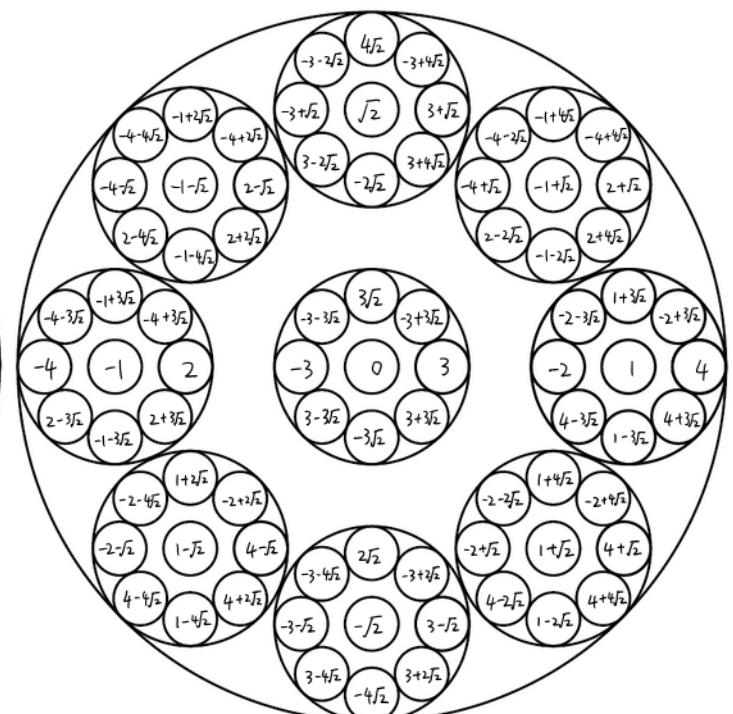
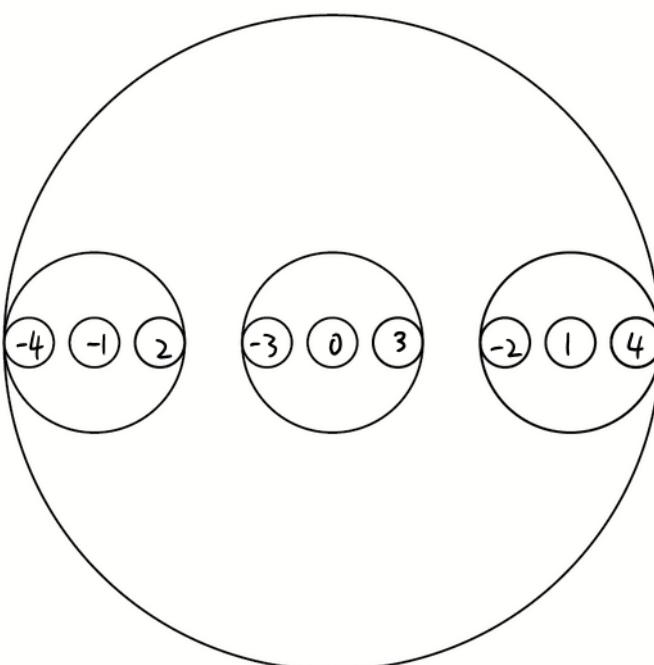
E.g. \mathbb{Z}_3 vs. $\mathbb{Z}_9 = \mathbb{Z}_3(\sqrt{2})$

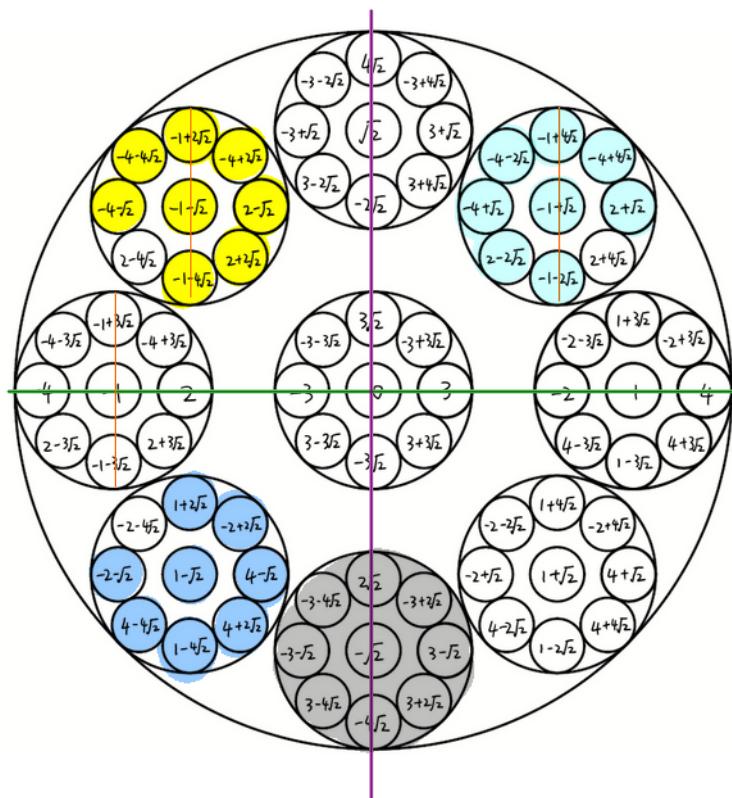
$\left. \begin{array}{l} \text{mod } 3 \\ +, \text{ inverse} \\ \text{multi by } 3 \\ \text{Frobenius} \end{array} \right\}$

basic buildings:



It's more canonical to use Teichmüller lift rather than 1~p, but I don't do so because of my limited computation ability.





$$x_0 = 2 + 4\sqrt{2} \quad \text{Gal}(\mathbb{Q}_9/\mathbb{Q}_3) = \{1, \sigma\}$$

$$A := \{x \in \mathbb{Q}_9 \mid d(x, x_0) = \frac{1}{3}\}$$

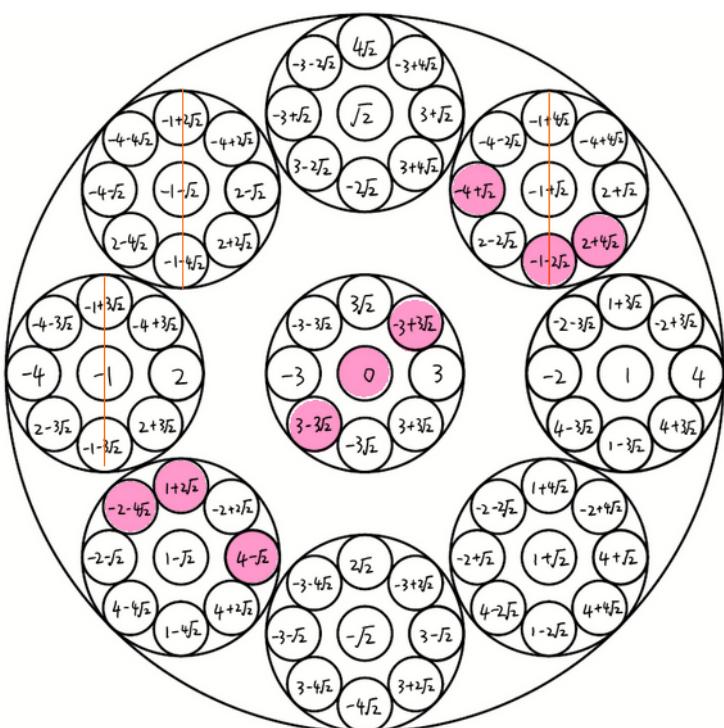
● σA
● $-A$

$$x_1 = 2\sqrt{2}$$

$$B := B(x_1, \frac{1}{3})$$

$$= B(x, \frac{1}{3}) \text{ for } \forall x \in B$$

— \mathbb{Q}_3 — \mathbb{Q}_3 -v.s. generated by x_1
— $\{ax_1 - 1 \mid a \in \mathbb{Q}_3\}$



(smallest)
circles containing elements in $\mathbb{Q}_3 \cdot x_0$

Observation: for \forall disk $D = \bigcup_{i=1}^9 D_i \subset \mathbb{Q}_9$
if $D \cap (\mathbb{Q}_3 \cdot x_0) \neq \emptyset$, then
 $\#\{i \in \{1, \dots, 9\} \mid D_i \cap (\mathbb{Q}_3 \cdot x_0) \neq \emptyset\} = 3$.

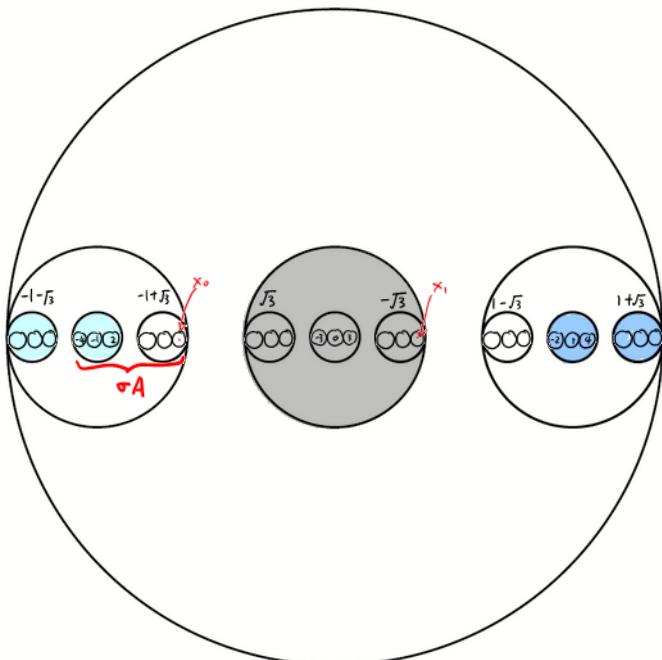
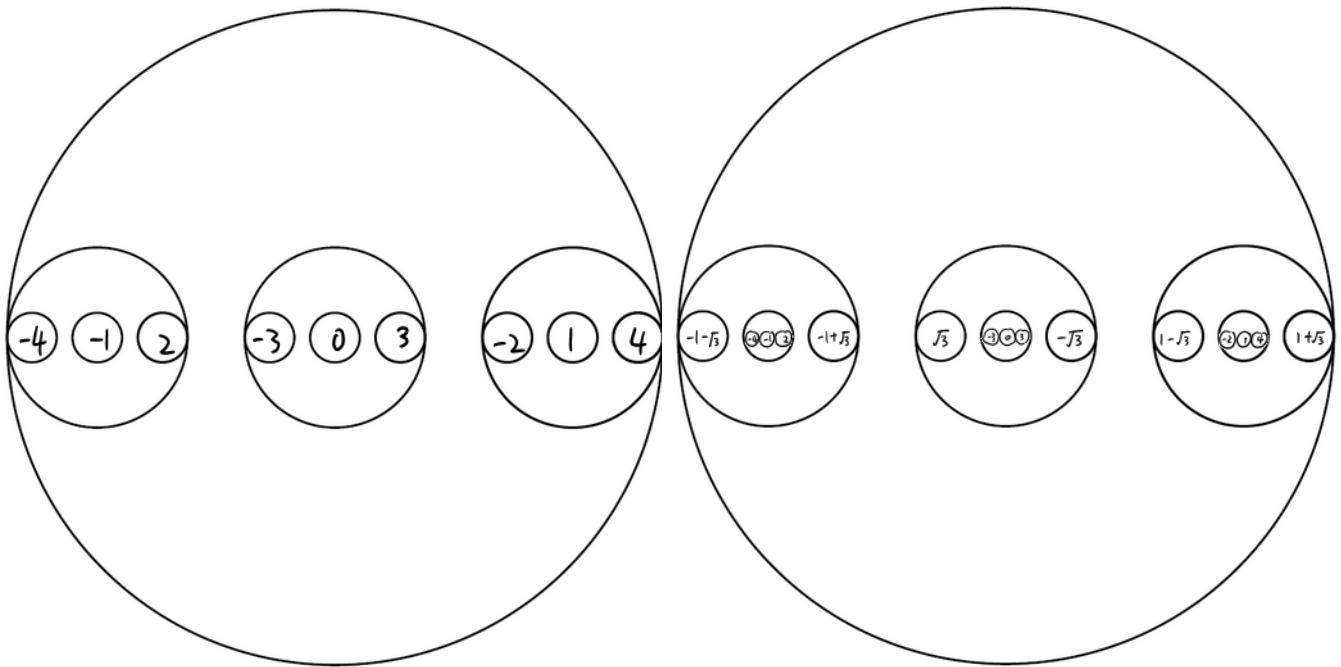
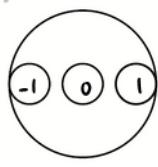
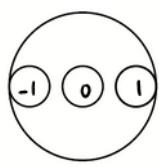
Q: Can we recover \mathbb{Q}_3 -v.s V from the set

$$\left\{ D \subset \mathbb{Q}_9 \mid \begin{array}{l} D = B_x(r) \text{ for some } x \in \mathbb{Q}_9, r \geq 1 \\ D \cap V \neq \emptyset \end{array} \right\}?$$

E.g. \mathbb{Z}_3 vs. $\mathbb{Z}_3(\sqrt{3})$

$$\|\cdot\| = \|\cdot\|_{\mathbb{Z}_3} = \|\cdot\|_{\mathbb{Z}_3(\sqrt{3})}^{\frac{1}{2}}$$

basic buildings:



$x_0 = 2 + \sqrt{3}$ $\text{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q}_3) = \{1, \sigma\}$

$A := \{x \in \mathbb{Q}_9 \mid d(x, x_0) = \frac{1}{3}\}$

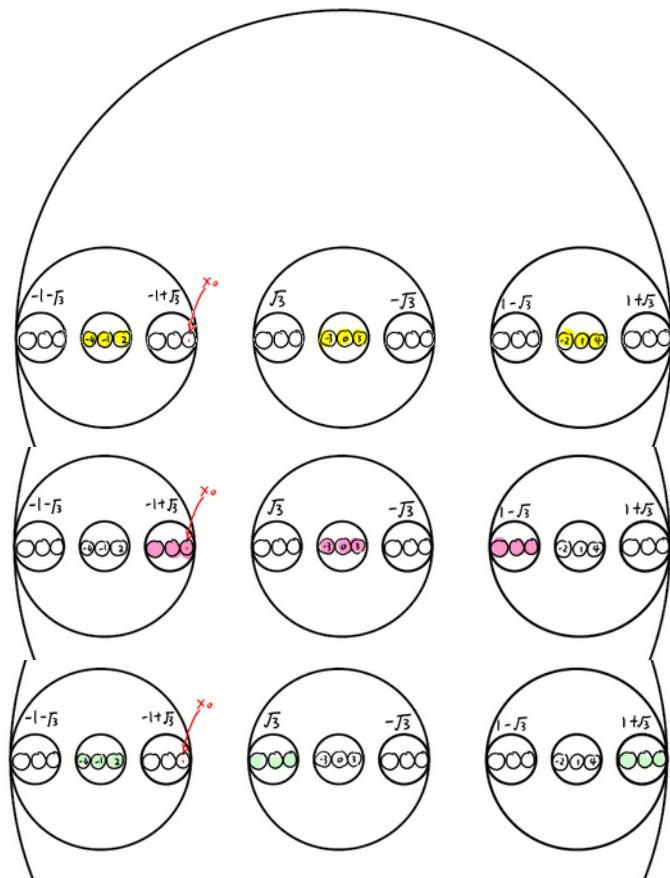
σA

$-A$

$x_1 = 2\sqrt{3}$

$B := B(x_1, \frac{1}{\sqrt{3}})$

$= B(x, \frac{1}{\sqrt{3}}) \text{ for } \forall x \in B$



smallest

- circles containing elements in $Q_3 \cdot 1$
- circles containing elements in $Q_3 \cdot x_0$
- circles containing elements in $\{2x_0 - 1\}$

Observation: for \forall disk $D = \bigcup_{i=1}^3 D_i \subset Q_3(\sqrt{3})$

if $D \cap Q_3 \cdot x_0 \neq \emptyset$, then
 $\#\{i \in \{1, \dots, 3\} \mid D_i \cap Q_3 \cdot x_0 \neq \emptyset\} = 3$.

Tasks for interesting readers: figure out all the cases of quadratic extension of Q_2 .

