

## Session 12 & Ex 10

Recall

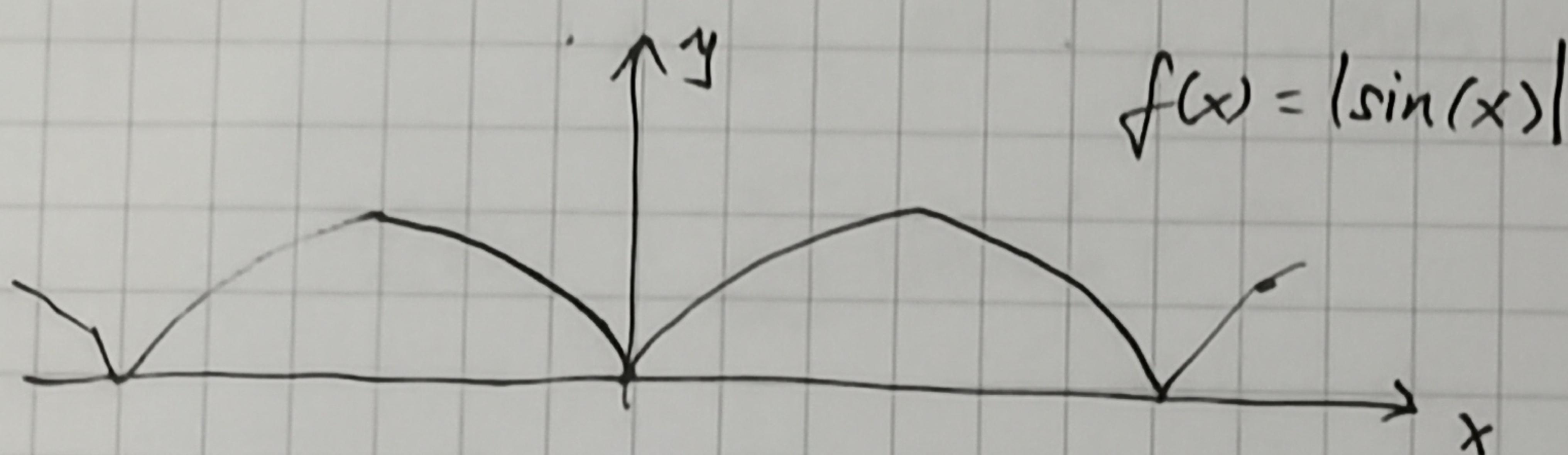
$$\text{Taylor: } f(z) = a_0 + a_1 z + a_2 z^2 + \dots = \sum_{k \geq 0} a_k z^k$$

$$\text{Laurant: } f(z) = \dots + a_{-1} z^{-1} + a_0 + a_1 z + a_2 z^2 + \dots = \sum_k a_k z^k$$

$$\text{Fourier: } f(x) \approx \sum_k c_k e^{ikx}$$

$$= \cancel{c_0} + \sum_{k \geq 1} a_k \cos kx + b_k \sin kx$$

Task 1.



$f \in C(\mathbb{R})$ ,  $f(x+2\pi) = f(x)$ , compute  $f(x) \approx \sum_{k \in \mathbb{Z}} c_k e^{ikx}$

$$\text{Hint. } \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{int} dt = \begin{cases} 1 & n=0 \\ 0 & n \neq 0 \end{cases}$$

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int} dt \\ &= \frac{1}{2\pi} \left( \int_0^\pi f(t) e^{-int} dt + \int_{-\pi}^0 f(t) e^{-int} dt \right) \\ &= \frac{1}{2\pi} \left( \int_0^\pi f(t) e^{-int} dt + \int_0^\pi f(t-\pi) e^{-in(t-\pi)} dt \right) \\ &= \frac{1}{2\pi} \left( \int_0^\pi f(t) (e^{-int} + e^{-in(t-\pi)}) dt \right) \\ &= \begin{cases} \frac{2}{\pi} \frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \sum_{n \text{ even}} \frac{2}{\pi} \frac{1}{1-n^2} e^{inx} \\ &\stackrel{n=2k}{=} \sum_{k \in \mathbb{Z}} \frac{2}{\pi} \frac{1}{1-(2k)^2} e^{2ikx} \\ &= \frac{2}{\pi} \left( 1 + \sum_{k=1}^{\infty} \frac{2 \cos(2kx)}{1-(2k)^2} \right) \end{aligned}$$

(ii). Solve the equation

$$u''(x) + u(x) = |\sin x|$$

Suppose  $u(x) = \sum_{n \in \mathbb{Z}} d_n e^{inx}$ , then

$$u'(x) = \sum_{n \in \mathbb{Z}} (in)d_n e^{inx}$$

$$u''(x) = \sum_{n \in \mathbb{Z}} -n^2 d_n e^{inx}$$

Substitute,

$$-n^2 d_n + d_n = \begin{cases} \frac{2}{\pi} \frac{1}{1-n^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\Rightarrow d_n = \begin{cases} \frac{2}{\pi} \frac{1}{(1-n^2)^2} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\therefore u(x) = \sum_{n \text{ even}} \frac{2}{\pi} \frac{1}{(1-n^2)^2} e^{inx}$$

$$\stackrel{n=2k}{=} \sum_{k \in \mathbb{Z}} \frac{2}{\pi} \frac{1}{(1-4k^2)^2} e^{2ikx}$$

$$= \frac{2}{\pi} \left( 1 + \sum_{k=1}^{\infty} \frac{2 \cos(2kx)}{(1-4k^2)^2} \right).$$

Task 2. ~~Show the~~

Assuming the axiom of choice, show  
the existence of ~~a~~ non-measurable set.

Idea: " $\infty \cdot \mathbb{Q} = 0$  or  $\infty$ " for ~~a~~ constant number  $C$ .

Define relation on  $[0,1]$ :  $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ .

get  $[0,1]$

$$\begin{array}{c} \downarrow \pi \\ [0,1] \end{array} \stackrel{f}{\sim}$$

~~Choose a section~~ Choose a section  $f$  of  $\pi$ , define  $A := \text{Im } f$ ,  
we show that  $A$  is ~~not~~ non-measurable. If not,

Claim:  $[0,1] \subseteq \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A+q) \subseteq [-1,2]$

$$\begin{aligned} m(\ ) \\ \Rightarrow 1 &\leq \sum_q m^*(A+q) \leq 3 \end{aligned}$$