

Eine Woche, ein Beispiel

2.6. six functors

Ref: <https://people.mpim-bonn.mpg.de/scholz/SixFunctors.pdf>

A preparation of exams.

$$\begin{array}{ccc} G & \xrightarrow{\mathcal{F}} & \mathcal{F}' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X \end{array} \quad \begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

Goal:

$$\begin{aligned} f^* &\dashv f_* \\ - \otimes \mathcal{F} &\dashv \underline{\mathrm{Hom}}(\mathcal{F}, -) \\ f_! &\dashv f^! \end{aligned}$$

$$\begin{aligned} f^*(- \otimes -) & \\ f^*(\mathcal{F} \otimes \mathcal{F}') &\cong f^* \mathcal{F} \otimes f^* \mathcal{F}' \\ f_* \underline{\mathrm{Hom}}(f^* \mathcal{F}, \mathcal{G}) &\cong \underline{\mathrm{Hom}}(\mathcal{F}, f_* \mathcal{G}) \end{aligned}$$

$$\begin{array}{ccc} & \otimes & \\ f^* & & f_! \\ \text{bc: } f^* g_! & \cong & g'_! f'^* \\ f_* g'_! & \cong & g'_! f_* \end{array}$$

proj formula

$$\begin{aligned} f_!(f^* \mathcal{F} \otimes \mathcal{G}) &\cong \mathcal{F} \otimes f_! \mathcal{G} \\ f_* \underline{\mathrm{Hom}}(\mathcal{G}, f^* \mathcal{F}) &\cong \underline{\mathrm{Hom}}(f_* \mathcal{G}, \mathcal{F}) \\ f'_! \underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{F}') &\cong \underline{\mathrm{Hom}}(f^* \mathcal{F}, f'^! \mathcal{F}') \end{aligned}$$

f^* : stalk

$f_* / f_!$: cohomology

$$\begin{aligned} I: f^* &= f^! \\ P: f_* &= f_! \end{aligned}$$

$$p: X \rightarrow pt$$

$$H^i(X; \mathbb{Z}) := p_* p^* \mathbb{1}$$

$$H_c^i(X; \mathbb{Z}) := p_! p^* \mathbb{1}$$

$$H^i(X; \mathbb{Z}) := p_! p^! \mathbb{1}$$

$$H^{BM}_i(X; \mathbb{Z}) := p_* p^! \mathbb{1}$$

$$H^i(X; \mathcal{F}) := p_* \mathcal{F}$$

$$H_c^i(X; \mathcal{F}) := p_! \mathcal{F}$$

$$H^i(X; \mathcal{F}) := p_!(p^! \mathbb{1} \otimes \mathcal{F})$$

$$H^{BM}_i(X; \mathcal{F}) := p_*(p^! \mathbb{1} \otimes \mathcal{F})$$

$$= R\Gamma(X; \mathcal{F})$$

$$= H_c^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

$$= H^i(X; p^! \mathbb{1} \otimes \mathcal{F})$$

App 1. (Künneth formula)

$$H_c^i(X; \mathcal{F}) \otimes H_c^j(Y; \mathcal{G}) \cong H_c^{i+j}(X \times Y; \mathcal{F} \boxtimes \mathcal{G})$$

$$\text{reduced to: } p_{X!} \mathcal{F} \otimes p_{Y!} \mathcal{G} \cong p_!(p_X^* \mathcal{F} \otimes p_Y^* \mathcal{G})$$

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_2} & Y \\ p_! \downarrow & \searrow p & \downarrow p_Y \\ X & \xrightarrow{p_X} & * \end{array}$$

App 2. (Poincaré duality)

X : a cpt oriented mfd of dim d , then

$$-^\vee = \underline{\mathrm{Hom}}_{\mathcal{D}(\mathbb{Z})}(-, \mathbb{Z})$$

$$\begin{aligned} \text{proper} & \quad p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ locally (Verdier duality)} \\ & \quad p^! \mathbb{Z} \cong \mathbb{Z}[d] \text{ globally} \end{aligned}$$

$$H^i(X; \mathbb{Z})[d] \cong H^i(X; \mathbb{Z})^\vee$$

$$\text{reduced to: } p_* \underline{\mathrm{Hom}}(A, p^* B \otimes p^! \mathbb{1}) \cong \underline{\mathrm{Hom}}(p_! A, B)$$

Upgrade: ∞ -categories & sym monoidal structure

Idea: $\mathcal{D}_\bullet: \mathcal{C}^{op} \longrightarrow \text{Cat}_\infty$

$$\begin{array}{ccc} X & \longmapsto & \mathcal{D}(X) \\ f \downarrow & \Rightarrow & \uparrow f^* \\ Y & \longmapsto & \mathcal{D}(Y) \end{array}$$

e.g. $X :=$ nice top space,
 $\mathcal{D}(X) :=$ derived category of
 abelian sheaves over X .

extends to \hookrightarrow compatibility is encoded!

$$\mathcal{D}: \text{Corr}(C, E) \longrightarrow \text{Mon}(\text{Cat}_\infty)$$

$$[Y \xleftarrow{f} X = X] \longmapsto f^*$$

$$[X = X \xrightarrow{f \in E} X] \longmapsto f_!$$

$$[X \times X \xleftarrow{\epsilon} X = X] \longmapsto \otimes$$

Moreover, It factor through

$$\begin{array}{ccccc} \text{Corr}(C, E) & \longrightarrow & \text{LZ}_\mathcal{D} & \longrightarrow & \text{Mon}(\text{Cat}_\infty) \\ \text{Obj: } X & \longmapsto & X & \longmapsto & \mathcal{D}(X) \end{array}$$

Mor: $\left[\begin{array}{c} Y \\ X \xleftarrow{f} \quad \searrow g \\ Z \end{array} \right] \longmapsto \text{kernel} \longmapsto \text{FM-transformation}$

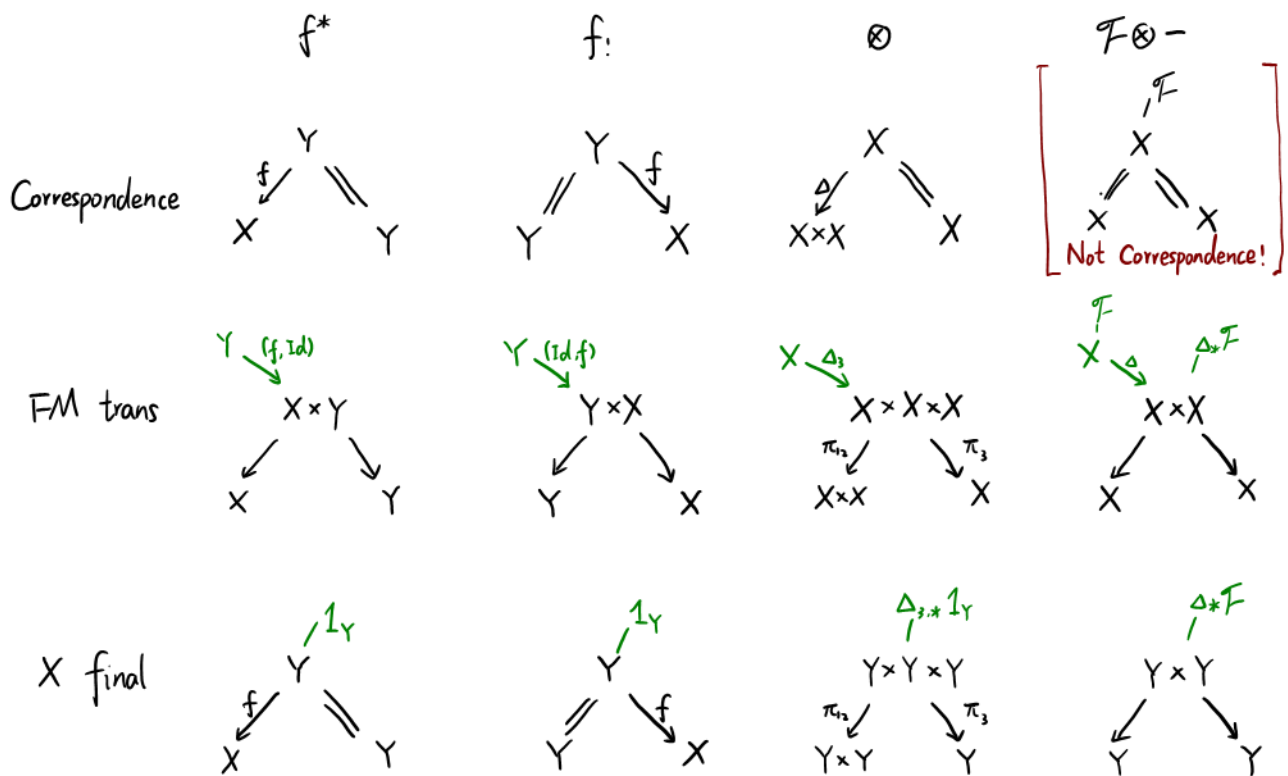
composition = convolution

2-Mor: $\mathcal{E} \rightarrow \mathcal{E}' \longmapsto \Phi_{\mathcal{E}} \longrightarrow \Phi_{\mathcal{E}'}$

$$\left[\begin{array}{c} \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \\ \cong \downarrow F \\ \begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X_1 & \xleftarrow{Y_1} & & \xrightarrow{Y_2} & X_3 \\ & \searrow & X_2 & \swarrow & \\ & & Z & & \end{array} \end{array} \right] \mapsto \left[\begin{array}{c} \mathcal{E}_{12} * \mathcal{E}_{23} \\ \downarrow \\ \mathcal{E}_{13} \end{array} \right]$$

Goal: framework of ∞ -category & \otimes

$$\leadsto \text{Corr}(C, E) \text{ \& } \text{Corr}(C, E)^{\otimes}$$



Ex. Explain base change, projection formula and Poincaré duality.

A:

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ g' \downarrow & & \downarrow g \\ Y & \xrightarrow{f} & X \end{array} \quad f^* g_! \cong g'_! f'^*: \mathcal{D}(X') \rightarrow \mathcal{D}(Y)$$

$$\begin{array}{ccc} & Y' & \\ f' \swarrow & & \searrow g' \\ X' & & Y \\ \swarrow & \searrow f & \swarrow \\ X & & Y \end{array} \quad \cong \quad \begin{array}{ccc} & Y' & \\ f' \swarrow & & \searrow g' \\ X' & & Y \\ \swarrow & \searrow f' & \swarrow \\ X & & Y \end{array}$$

$$(*, X') \xrightarrow{g'_!} (*, X) \xrightarrow{f^*} (*, Y) \quad \cong \quad (*, X') \xrightarrow{f'^*} (*, Y') \xrightarrow{g'_!} (*, Y)$$

$$\begin{array}{ccc} & \mathcal{F} & \mathcal{G} \\ & \swarrow & \searrow \\ Y & \xrightarrow{f} & X \end{array} \quad f^*(\mathcal{F} \otimes \mathcal{G}) \cong f^* \mathcal{F} \otimes f^* \mathcal{G}: \mathcal{D}(X) \otimes \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$$

$$\begin{array}{ccc} & Y & \\ f \swarrow & & \searrow \\ X & & Y \\ \swarrow \Delta & \searrow f & \swarrow \\ X \times X & & X \end{array} \quad \begin{array}{ccc} & Y & \\ \Delta \swarrow & & \searrow \\ Y \times Y & & Y \\ \swarrow (f, f) & \searrow \Delta & \swarrow \\ X \times X & & Y \times Y \end{array}$$

$$(\{1, 2\}, (X, X)) \xrightarrow{\Delta^*} (*, X) \xrightarrow{f^*} (*, Y) \quad \begin{array}{ccc} (\{1, 2\}, (X, X)) \xrightarrow{(f, f)^*} (\{1, 2\}, (Y, Y)) \xrightarrow{\Delta^*} (*, Y) \end{array}$$

$$\begin{array}{ccc} \mathcal{F} & & \mathcal{G} \\ | & & | \\ Y & \xrightarrow{f} & X \end{array}$$

$$f_!(\mathcal{F} \otimes f^* \mathcal{G}) \cong f_! \mathcal{F} \otimes \mathcal{G} : \mathcal{D}(Y) \times \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

$$\begin{array}{c} Y \\ \parallel \quad \parallel \\ Y \quad Y \\ \Delta \swarrow \quad \searrow \quad \parallel \quad \parallel \\ Y \times Y \quad Y \quad Y \\ \downarrow (Id, f) \quad \parallel \quad \Delta \quad \parallel \quad \searrow f \\ Y \times X \quad Y \times Y \quad Y \quad X \\ (f_!, z], (Y, X) \xrightarrow{(Id, f)^*} (f_!, z], (Y, Y) \xrightarrow{\Delta^*} (*, Y) \xrightarrow{f_!} (*, X) \end{array}$$

$$\begin{array}{c} Y \\ \swarrow (Id, f) \quad \searrow f \\ Y \times X \quad X \\ \parallel \quad \searrow (f, Id) \quad \Delta \quad \parallel \\ Y \times X \quad X \times X \quad X \\ (f_!, z], (Y, X) \xrightarrow{(f, Id)_!} (f_!, z], (X, X) \xrightarrow{\Delta^*} (*, X) \end{array}$$

∞ -category

$$\Delta \hookrightarrow \text{Fun}(\Delta^{\text{op}}, \text{Set}) \stackrel{\Delta}{=} \text{sSet} \supseteq \text{Cat}_{\infty} \supseteq \text{An}$$

$$\begin{array}{ccc} \Delta^n & \xrightarrow{h} & X \\ \downarrow & \nearrow \exists K_{n,i}(h) & \\ \Delta^n & & \end{array} \quad \begin{array}{ll} \forall 0 \leq i < n & \text{in } \text{Cat}_{\infty} \\ \forall 0 \leq i \leq n & \text{in } \text{An} \end{array}$$

Notation

Set (0,0)-category set
 Cat (1,1)-category category
 An (∞ ,0)-category anima / Kan cplx / ∞ -groupoid
 Cat $_{\infty}$ (∞ ,1)-category

Ex. Realize $\text{Corr}(C, E)$ as an ∞ -category.

Monoidal structure

In (1,1)-category:

Monoidal structure on \mathcal{C} :

$$\begin{array}{ll} m_{\mathcal{C}}: \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C} & u_{\mathcal{C}}: 1 \longrightarrow \mathcal{C} \\ (\mathcal{F}, \mathcal{G}) \longmapsto \mathcal{F} \otimes \mathcal{G} & * \longmapsto 1_{\mathcal{C}} \end{array}$$

Monoidal object in (\mathcal{C}, \otimes) : $X \in \text{Ob}(\mathcal{C})$ with

$$m_X: X \times X \longrightarrow X \quad u_X: 1_{\mathcal{C}} \longrightarrow X$$

In (∞ ,1)-category:

$$(C, \otimes) \stackrel{\text{def}}{\longleftrightarrow} \left[\begin{array}{ccc} X: \text{Fin}^{\text{part}} & \longrightarrow & \text{Cat}_{\infty} \\ I & \longmapsto & X(I) \\ \text{comm monoid} & & \end{array} \right] \stackrel{\text{"straightening"}}{\longleftrightarrow} \left[\begin{array}{ccc} \pi^{\otimes}: Y^{\otimes} & \longrightarrow & \text{Fin}^{\text{part}} \\ \text{coCartesian fibration} & & \\ Y_I^{\otimes} \xrightarrow{\sim} \prod_i Y_i^{\otimes} & & \end{array} \right]$$

\rightsquigarrow See next page for details

where $\text{Ob}(\text{Fin}^{\text{part}}) = \text{Ob}(\text{Fin})$

$$\text{Mor}_{\text{Fin}^{\text{part}}}(I, J) = \{\alpha: I \dashrightarrow J\}$$

$$\text{commutative monoid: } X(I) \xrightarrow{\sim} \prod_i X(i)$$

$$\mathcal{F} \boxtimes \mathcal{G} \longleftarrow (\mathcal{F}, \mathcal{G}) \quad |I|=2$$

coCartesian fibration: see [Def 3.5]

$$\left[\begin{array}{c} (C, \otimes) \\ \text{monoidal} \\ \text{co-cat} \end{array} \right] \xleftrightarrow{\text{def}} \left[\begin{array}{c} X: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty} \\ I \longmapsto X(I) \\ \text{comm monoid} \end{array} \right] \xleftrightarrow{\text{"straightening"}} \left[\begin{array}{c} \pi^{\otimes}: Y^{\otimes} \longrightarrow \text{Fin}^{\text{part}} \\ \text{coCartesian fibration} \\ Y_I^{\otimes} \xrightarrow{\sim} \prod_i Y_i^{\otimes} \end{array} \right]$$

$$(C, \otimes) \longmapsto \mathcal{C}^{(-)}: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty}$$

$$I \longmapsto \mathcal{C}^I = \prod_{i \in I} \mathcal{C}$$

$$\downarrow \alpha \Rightarrow \downarrow$$

$$J \qquad \mathcal{C}^J$$

$$(X_i)_{i \in I} \downarrow (\bigotimes_{i \in \alpha^{-1}(j)} X_i)_{j \in J}$$

$$\xrightarrow{\qquad \qquad \qquad} \pi^{\otimes}: \mathcal{C}^{\otimes} \longrightarrow \text{Fin}^{\text{part}}$$

$$\mathcal{C}^{\otimes} = \{(I, (X_i)_i) \mid I \in \text{Fin}^{\text{part}}, X_i \in \mathcal{C}\} \qquad (I, (X_i)_i) \longmapsto I$$

$$\text{Mor}^{\otimes}((I, X_i), (J, Y_j)) = \left\{ \alpha: I \dashrightarrow J, \{ \bigotimes_{i \in \alpha^{-1}(j)} X_i \rightarrow Y_j \}_j \right\}$$

$$(X(*), X(\{1,2\} \rightarrow \{*\})) \longleftarrow X \longrightarrow X^{\otimes} = \{(I, A) \mid I \in \text{Fin}^{\text{part}}, A \in X(I)\}$$

$$\text{Mor}((I, A), (J, B)) = \left\{ \alpha: I \dashrightarrow J, f: (X(\alpha)A \rightarrow B \text{ in } X(J)) \right\}$$

$$Y: \text{Fin}^{\text{part}} \longrightarrow \text{Cat}_{\infty} \longleftarrow Y^{\otimes}$$

$$I \longmapsto Y^I = Y_I^{\otimes}$$

$$\downarrow \alpha \Rightarrow \downarrow \text{loc. coCartesian lift}$$

$$J \qquad Y^J$$

$$(\pi^{\otimes^{-1}}(*), \text{loc. coCartesian lift of } \{1,2\} \rightarrow \{*\}) \longleftarrow Y^{\otimes}$$

X encodes the monoidal structure

$$\begin{array}{ccc} & \{1,2,3\} & \\ & \nearrow \searrow & \\ \{1,2,3\} & & * \\ & \nwarrow \nearrow & \\ & \{1,23\} & \end{array}$$

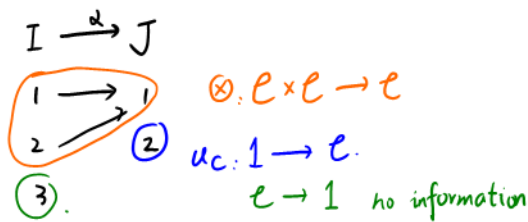
$$\begin{array}{ccccc} & & \mathcal{C} \times \mathcal{C} & & \\ & \nearrow & & \searrow & \\ \mathcal{C} \times \mathcal{C} \times \mathcal{C} & & & & \mathcal{C} \\ & \searrow & & \nearrow & \\ & & \mathcal{C} \times \mathcal{C} & & \end{array}$$

$$\begin{array}{ccc} & (X \otimes Y, Z) & \xrightarrow{\sim} (X \otimes Y) \otimes Z \\ & \nearrow \searrow & \\ (X, Y, Z) & & X \otimes (Y \otimes Z) \\ & \nwarrow \nearrow & \\ & (X, Y \otimes Z) & \end{array}$$

$$\begin{array}{ccc} \{1,2\} & & * \\ \downarrow & \nearrow & \\ \{1,2\} & & \\ \downarrow & \nearrow & \\ \{1\} & & * \\ \downarrow & \nearrow & \\ \{1,2\} & & \end{array}$$

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \end{array}$$

$$\begin{array}{ccc} (X, Y) & \longmapsto & X \otimes Y \\ \downarrow & & \downarrow \\ (Y, X) & \longmapsto & Y \otimes X \\ \downarrow & & \downarrow \\ X & \longmapsto & X \\ \downarrow & & \downarrow \\ (X, 1_{\mathcal{C}}) & \longmapsto & X \otimes 1_{\mathcal{C}} \end{array}$$



Fctor. (lax) sym monoidal fctors

Special case: $[F: (\mathcal{C}, \otimes) \longrightarrow (\mathcal{D}, \times)] \iff [F: \mathcal{C}^{\otimes} \longrightarrow \mathcal{D} \text{ with conditions}]$

Ex. Realize $\text{Corr}(\mathcal{C}, \mathcal{E})^{\otimes}$, and show $f^*(- \otimes -)$, bc & proj formula.

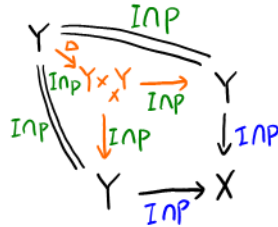
Why is $f: \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ $\mathcal{D}(Y)$ -linear?

| Category | Object | Morphism |
|-------------------------------|------------------------------|--|
| \mathcal{C} | $X \quad Y$ | $X \rightarrow Y$ |
| \mathcal{C}^{op} | $X \quad Y$ | $X \leftarrow Y$ in \mathcal{C} |
| $(\mathcal{C}^{op})^L$ | $(I, (X_i)_i), (J, (Y_j)_j)$ | $\alpha: I \rightarrow J, \{ \bigotimes_{i \in I}^{op} X_i \rightarrow Y_j \}_j$ in \mathcal{C}^{op} |
| | | \parallel |
| | | $\{ \prod_{i \in I} X_i \leftarrow Y_j \}_j$ in \mathcal{C} |
| $((\mathcal{C}^{op})^L)^{op}$ | $(I, (X_i)_i), (J, (Y_j)_j)$ | $\alpha: I \leftarrow J, \{ \prod_{i \in I} X_i \rightarrow Y_j \}_j$ in \mathcal{C} |

Construction "Uniqueness of $f_!$ "

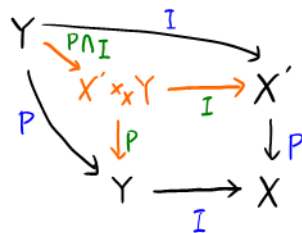
Const 1. $f: Y \rightarrow X \quad f \in \text{INP} \quad \Rightarrow f_! \cong f_*$

By induction.

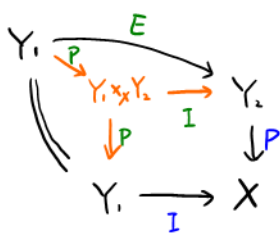
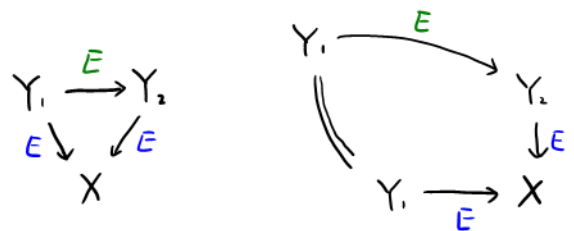


— Initial case
— Deduced case

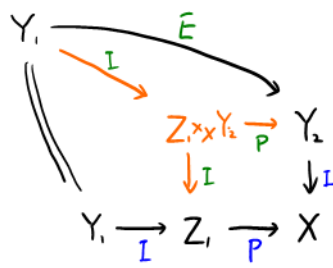
Const 2



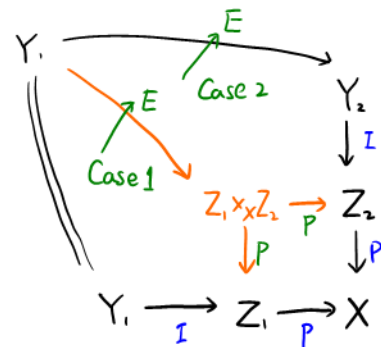
Const 3. E satisfies 2-out-of-3, i.e.



Case 1



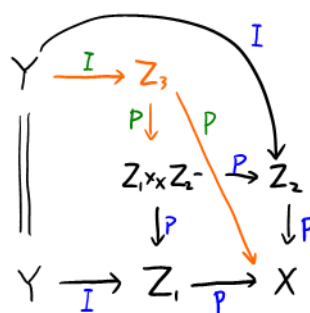
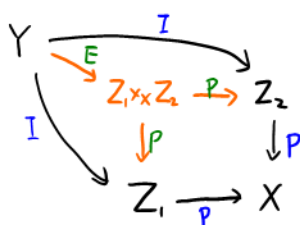
Case 2



Case 3

Const 4. $Y \xrightarrow{i_2} Z_2$
 $i_1 \downarrow I \quad P \downarrow f_!$
 $Z_1 \xrightarrow{f_!} X$

want: $f_{1*} j_{1!} \cong f_{2*} j_{2!}$



Construction

$$\begin{aligned}
 \text{Corr}(C, E) &= \delta_+^* \text{Corr}(C, E)_{ct, co} \\
 &\underset{\text{Fun}(-, E)}{\sim} \delta^* \text{Corr}(C, E)_{ct, co} \\
 &\sim \delta^* \text{Corr}(C, I, P)_{ct, co, co} \\
 &\longleftarrow \delta^* \text{Corr}(C, I, P)_{ct, ct, ct}
 \end{aligned}$$

$$\begin{array}{ccc}
 \Delta^n \times \Delta^n \times \Delta^n & \longrightarrow & \text{Cat}_\infty \\
 \Delta^n \times \Delta^n & \longrightarrow & \text{Fun}(\Delta^n, \text{Cat}_\infty) \\
 & \searrow & \downarrow \\
 & & \text{Fun}^L(\Delta^n, \text{Cat}_\infty) \\
 & & \downarrow \\
 & & \text{Fun}(\mathbb{A}^n{}^{op}, \text{Cat}_\infty)
 \end{array}$$

f -smooth (= f -admissible)

$$f: Y \rightarrow X$$

$$\begin{array}{ccc} \begin{array}{c} A \\ \swarrow \\ Y \\ \swarrow \quad \searrow \\ Y \quad X \end{array} & & \begin{array}{c} B \\ \swarrow \\ Y \\ \swarrow \quad \searrow \\ X \quad Y \end{array} \\ F: f_!(A \otimes -) \dashv G: B \otimes f^* - & & \\ \dashv \text{Hom}(A, f^! -) & & \end{array}$$

$$\textcircled{1} \quad B \otimes f^* - \cong \text{Hom}(A, f^! -)$$

$$\text{App 1. } \Delta! 1_Y \text{ cpt} \Rightarrow A \text{ cpt}$$

[Proof. $\text{Hom}(\Delta! 1_Y, B \otimes f^* -) \cong \text{Hom}(A, -)$ preserves filtered colimit.]

$$\textcircled{2} \quad B \cong \text{Hom}(A, f^! 1_X)$$

$$p_2^* B \otimes p_1^* - \cong \text{Hom}(p_2^* A, p_1^! -)$$

[Verdier's diagonal trick]

$$\text{Prop. } A \text{ is } f\text{-smooth} \Leftrightarrow p_2^* B \otimes p_1^* A \cong \text{Hom}(p_2^* A, p_1^! A) \quad \textcircled{2b}$$

$$\text{where } B \cong \text{Hom}(A, f^! 1_X) \quad \textcircled{2a}$$

$\Rightarrow: \checkmark$

\Leftarrow : Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = \text{Id}$,

$$A \text{ is } f\text{-smooth} \Leftrightarrow \text{Hom}(A, 1_X) \otimes A \cong \text{Hom}(A, A) \\ \Leftrightarrow A \text{ is dualizable}$$

App 3. When $A = 1_Y$, $B = f^! 1_X$.

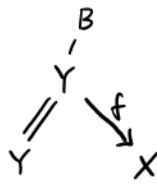
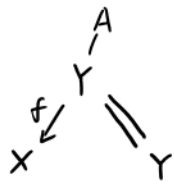
$$1_Y \text{ is } f\text{-smooth} \Leftrightarrow p_2^* f^! 1_X \cong p_1^! 1_Y \\ \xLeftrightarrow{+f^! 1_X \text{ inv}} f \text{ is coh smooth}$$

Using this, one can prove results on coh étale.

Write $B = \mathbb{D}_f(A)$. we get $\mathbb{D}_f(\mathbb{D}_f(A)) \cong A$. (adjunction is symmetric in A & B).

$$B = \text{SP}_f(A) \quad w_f := \text{SD}_f(1_Y) \\ \uparrow \text{smooth dual}$$

f -proper (f-coadmissible) $f: Y \rightarrow X$



$$F: A \otimes f^* - \dashv G: f_*(B \otimes -) \\ \dashv f_* \text{Hom}(A, -)$$

① $f_!(B \otimes -) \cong f_* \text{Hom}(A, -)$

App 1. $1_X \text{ cpt} \Rightarrow A \text{ cpt}$

[Proof. $\text{Hom}(1_X, f_!(B \otimes -)) \cong \text{Hom}(A, -)$ preserves filtered colimit.]

② $p_{1*}(p_2^* B \otimes -) \cong p_{1*} \text{Hom}(p_2^* A, -)$ [Verdier's diagonal trick]
 $B \cong p_{1*} \text{Hom}(p_2^* A, \Delta_! 1_Y)$

Prop. A is f -proper $\Leftrightarrow f_!(B \otimes A) \cong f_* \text{Hom}(A, A)$ (2b)
 where $B \cong p_{1*} \text{Hom}(p_2^* A, \Delta_! 1_Y)$ (2a)

$\Rightarrow: \checkmark$

\Leftarrow : Writing down adjunctions in 2-category.

App 2. When $Y = X$, $f = \text{Id}$,

A is f -proper $\Leftrightarrow \text{Hom}(A, 1_X) \otimes A \cong \text{Hom}(A, A)$
 $\Leftrightarrow A$ is dualizable

App 3. When $A = 1_Y$, $B = p_{1*} \Delta_! 1_Y$

1_Y is f -proper $\Leftrightarrow f_! p_{1*} \Delta_! 1_Y \cong f_* 1_Y$

Using this, one can prove results on coh proper.

Write $B = \mathbb{D}_f^{\text{pro}}(A)$. we get $\mathbb{D}_f^{\text{pro}}(\mathbb{D}_f^{\text{pro}}(A)) \cong A$. (adjunction is symmetric in A & B)

$B = \text{PD}_f(A)$ $\omega_f := \text{PD}_f(1_Y)$
 \uparrow proper dual

When $\Delta_! = \Delta_*$, $\mathbb{D}_f^{\text{pro}} = \text{Hom}(-, 1_Y)$ is the naive dual.

Relations

open immersion \rightarrow coh smooth $\xLeftrightarrow[\text{if } f^! 1_X \text{ inv}]{\text{if } \Delta \text{ coh étale } f \text{ is n-truncated}} 1_Y \text{ is } f\text{-sm} \xLeftrightarrow[\text{if } \Delta \text{ coh étale } f \text{ is n-truncated}] \text{ coh étale}$

proper $\xrightarrow{\text{if } \Delta_! = \Delta_*} 1_Y \text{ is } f\text{-proper} \xLeftrightarrow[\text{if } \Delta \text{ coh proper } f \text{ is n-truncated}] \text{ coh proper}$