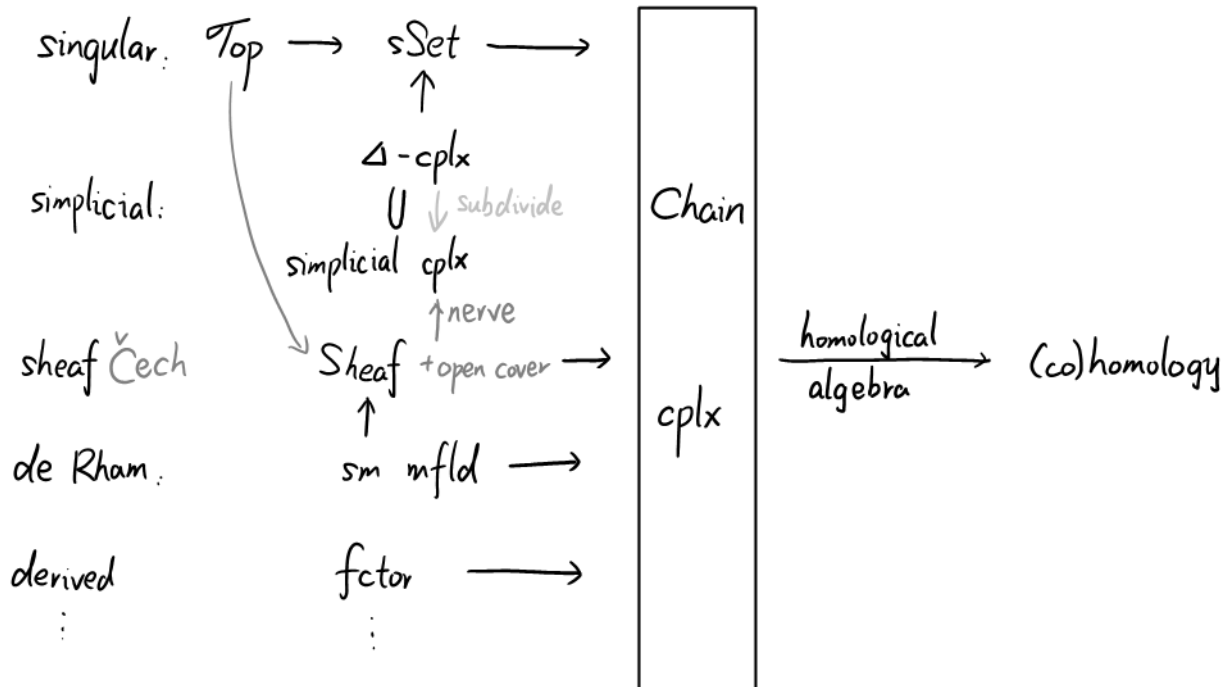


Eine Woche, ein Beispiel

6.25 (co)homology of simplicial set

<https://ncatlab.org/nlab/show/simplicial+complex>
<https://mathoverflow.net/questions/18544/sheaves-over-simplicial-sets>



Today: $sSet \longrightarrow \text{chain cplx} \dashrightarrow (co)homology$

1. definition and basic examples
2. connection with simplicial complexes
3. more structures
4. connection with sheaf cohomology + derived category

1. definition and basic examples

We use \mathbb{Z} here because
we are considering $X = \Delta^n$ case.
May change to x in the future.

Def. For $X \in \mathbf{sSet}$, $G \in \mathbf{Mod}(\mathbb{Z})$, define

$$C_n(X; G) = \bigoplus_{\alpha \in X_n} G \quad 0 \leftarrow \bigoplus_{\alpha \in X_0} G \xleftarrow{(d_0' - d_1')^*} \bigoplus_{\alpha \in X_1} G \xleftarrow{(d_0'' - d_1'' + d_2'')^*} \bigoplus_{\alpha \in X_2} G \dots$$

$$C^n(X; G) = \prod_{\alpha \in X_n} G \quad 0 \longrightarrow \prod_{\alpha \in X_0} G \xrightarrow{\text{dual}} \prod_{\alpha \in X_1} G \longrightarrow \prod_{\alpha \in X_2} G \dots$$

$$C_n^{\text{BM}}(X; G) =$$

$$C_c^n(X; G) =$$

$$\text{Hom}_{\mathbb{Z}\text{-mod}}(\bigoplus_{\alpha \in X_n} \mathbb{Z}, G) \cong \prod_{\alpha \in X_n} \text{Hom}_{\mathbb{Z}\text{-mod}}(\mathbb{Z}, G) \cong \prod_{\alpha \in X_n} G$$

<https://math.stackexchange.com/questions/102725/calculating-the-cohomology-with-compact-support-of-the-open-mc3%b6bius-strip?rq=1>
<https://math.stackexchange.com/questions/3215960/cohomology-with-compact-supports-of-infinite-trivalent-tree>

Eg. 1 For $A \in \mathbf{Top}$ discrete, $X := \mathcal{S}(A) \in \mathbf{sSet}$, one can compute

$$\begin{aligned} C_*(X; G): \quad 0 &\leftarrow \bigoplus_{\alpha \in A} G \xleftarrow{0} \bigoplus_{\alpha \in A} G \xleftarrow{\text{Id}} \bigoplus_{\alpha \in A} G \xleftarrow{0} \bigoplus_{\alpha \in A} G \xleftarrow{\text{Id}} \dots \\ C^*(X; G): \quad 0 &\longrightarrow \prod_{\alpha \in A} G \xrightarrow{0} \prod_{\alpha \in A} G \xrightarrow{\text{Id}} \prod_{\alpha \in A} G \xrightarrow{0} \prod_{\alpha \in A} G \xrightarrow{\text{Id}} \dots \\ C_*^{\text{BM}}(X; G): \quad 0 &\leftarrow \prod_{\alpha \in A} G \xleftarrow{0} \prod_{\alpha \in A} G \xleftarrow{\text{Id}} \prod_{\alpha \in A} G \xleftarrow{0} \prod_{\alpha \in A} G \xleftarrow{\text{Id}} \dots \\ C_c^*(X; G): \quad 0 &\longrightarrow \bigoplus_{\alpha \in A} G \xrightarrow{0} \bigoplus_{\alpha \in A} G \xrightarrow{\text{Id}} \bigoplus_{\alpha \in A} G \xrightarrow{0} \bigoplus_{\alpha \in A} G \xrightarrow{\text{Id}} \dots \end{aligned}$$

Therefore,

$$\begin{aligned} H_n(X; G) &= \begin{cases} \bigoplus_{\alpha \in A} G & n=0 \\ 0 & n>0 \end{cases} & H_n^{\text{BM}}(X; G) &= \begin{cases} \prod_{\alpha \in A} G & n=0 \\ 0 & n>0 \end{cases} \\ H^n(X; G) &= \begin{cases} \prod_{\alpha \in A} G & n=0 \\ 0 & n>0 \end{cases} & H_c^n(X; G) &= \begin{cases} \bigoplus_{\alpha \in A} G & n=0 \\ 0 & n>0 \end{cases} \end{aligned}$$

Eq 2. We want to compute $H_n(\Delta'; G)$ & $H^n(\Delta'; G)$.

Notice that $\#\Delta'_k = k+2$, so

$C(\Delta'; G): 0 \leftarrow G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}} G^{\oplus 5} \dots$

basis: $d'_0 \triangleq x_0, \dots, x_4$
remember indexes: $d'_1 \triangleq x_1, \dots, x_4$

$0 = x_0 - x_0 \longleftarrow x_0$
 $x_0 - x_1 = x_0 - x_1 \longleftarrow x_1$
 $0 = x_1 - x_1 \longleftarrow x_2$

$0 = x_0 - x_0 + x_0 - x_0 \longleftarrow x_0$
 $x_0 - x_1 = x_0 - x_1 + x_1 - x_1 \longleftarrow x_1$
 $0 = x_1 - x_1 + x_2 - x_2 \longleftarrow x_2$
 $x_2 - x_3 = x_2 - x_2 + x_2 - x_3 \longleftarrow x_3$
 $0 = x_3 - x_3 + x_3 - x_3 \longleftarrow x_4$

$\chi_0 = x_0 - x_0 + x_0 \longleftarrow x_0$
 $\chi_0 = x_0 - x_1 + x_1 \longleftarrow x_1$
 $\chi_2 = x_1 - x_1 + x_2 \longleftarrow x_2$
 $\chi_2 = x_2 - x_2 + x_2 \longleftarrow x_3$

By taking the transpose, one get

$C^*(\Delta'; G): 0 \rightarrow G^{\oplus 2} \xrightarrow{\begin{pmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \end{pmatrix}} G^{\oplus 3} \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} G^{\oplus 4} \xrightarrow{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}} G^{\oplus 5} \dots$

Therefore,

$$H_n(\Delta'; G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

$$H^n(\Delta'; G) = \begin{cases} G & n=0 \\ 0 & n>0 \end{cases}$$

Rmk. Actually, we have chain homotopy equivalence between $C.(\Delta'; G)$ and $C.(\Delta^0; G)$.

$$\begin{array}{ccccccc}
 \Delta' & C.(\Delta'; G) : & 0 \leftarrow & G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}} & G^{\oplus 5} \dots \\
 \downarrow s' & \downarrow s'_{0,*} & & \downarrow (11) & \downarrow (111) & \downarrow (1111) & \downarrow (11111) \\
 \Delta^0 & C.(\Delta^0; G) : & 0 \leftarrow & G \xleftarrow{0} & G \xleftarrow{Id} & G \xleftarrow{0} & G \dots \\
 \Delta^0 & C.(\Delta^0; G) : & 0 \leftarrow & G \xleftarrow{0} & G \xleftarrow{Id} & G \xleftarrow{0} & G \dots \\
 \downarrow d'_0 & \downarrow d'_{0,*} & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \\
 \Delta' & C.(\Delta'; G) : & 0 \leftarrow & G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}} & G^{\oplus 5} \dots
 \end{array}$$

s.t. $s'_{0,*} \circ d'_{0,*} = Id_{C.(\Delta'; G)}$, $d'_{0,*} \circ s'_{0,*} \sim Id_{C.(\Delta^0; G)}$.

In fact, we have

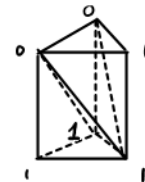
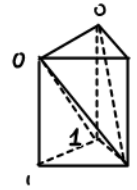
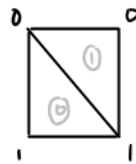
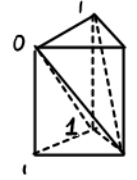
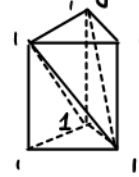
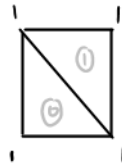
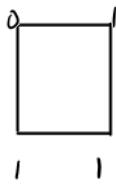
$$\begin{array}{ccccccc}
 C.(\Delta'; G) : & 0 \leftarrow & G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}} & G^{\oplus 5} \dots \\
 \downarrow Id & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{*} & \downarrow Id & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}_{*} & \downarrow Id & \downarrow \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}_{*} & \downarrow Id \\
 C.(\Delta'; G) : & 0 \leftarrow & G^{\oplus 2} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}} & G^{\oplus 5} \dots
 \end{array}$$

$$\begin{array}{l}
 x_0 \mapsto x_0 \\
 x_1 \mapsto x_1
 \end{array}$$

$$\begin{array}{l}
 x_0 \mapsto x_0 - x_0 + x_0 = x_0 \\
 x_1 \mapsto x_0 - x_1 + x_1 = x_1 \\
 x_2 \mapsto x_1 - x_2 + x_2 = x_1 \\
 x_3 \mapsto x_1 - x_2 + x_3
 \end{array}$$

$$\begin{array}{l}
 x_0 \mapsto x_0 - x_0 = 0 \\
 x_1 \mapsto x_1 - x_1 = 0 \\
 x_2 \mapsto x_1 - x_2
 \end{array}$$

Ex. Observe the picture, try to translate the calculation in geometrical language.



2. connection with simplicial complexes.

Continuation of Eq. 2.

Even more, we have chain homotopy between $C_*(\Delta'; G)$ and $C_*(\Delta'; G)^\diamond$:

non-degenerate
↓

$$\begin{array}{ccccccc}
 C_*(\Delta'; G) : & 0 & \leftarrow & G^{\oplus 2} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} & \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 5} & \dots \\
 \downarrow \text{projection} & & & \downarrow \text{Id} & & \downarrow (111) & & \downarrow 0 & & \downarrow 0 & & \\
 C_*(\Delta'; G)^\diamond : & 0 & \leftarrow & G^{\oplus 2} & \xleftarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} & G & \xleftarrow{0} & 0 & \xleftarrow{0} & 0 & \dots \\
 \downarrow \text{inclusion} & & & \downarrow \text{Id} & & \downarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix} & & \downarrow 0 & & \downarrow 0 & & \\
 C_*(\Delta'; G) : & 0 & \leftarrow & G^{\oplus 2} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} & \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 5} & \dots
 \end{array}$$

In fact, we have

$$\begin{array}{ccccccc}
 C_*(\Delta'; G) : & 0 & \leftarrow & G^{\oplus 2} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} & \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 5} & \dots \\
 \text{Id} \parallel & & & \text{Id} \parallel \text{Id} & & \text{Id} \parallel \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} & & \text{Id} \parallel 0 & & \text{Id} \parallel 0 & & \\
 C_*(\Delta'; G) : & 0 & \leftarrow & G^{\oplus 2} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 3} & \xleftarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}} & G^{\oplus 4} & \xleftarrow{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}} & G^{\oplus 5} & \dots
 \end{array}$$

(Note: Red dashed arrows labeled 0 connect the two rows of the chain complex.)

Q: How could one find the homotopy in the general case?

Def (Stratification by skeletons)
For $X \in sSet$, define

\diamond : non-degenerate
 ζ : degenerate

$$\begin{aligned} X_k^\diamond &:= \{x \in X_k \mid x \text{ non-degenerate}\} &= X_k - (sk^{k-1}X)_k \\ X_k^\zeta &:= \{x \in X_k \mid x \text{ degenerate}\} &= (sk^{k-1}X)_k \\ X_k^{\zeta i} &:= \left\{ x \in X_k \mid x = \alpha^*(y) \text{ for some } y \in X_{k-i}^\diamond, \alpha: [k-i] \rightarrow [k] \right\} &= (sk^{k-i}X)_k - (sk^{k-i-1}X)_k \end{aligned}$$

$$0 = (sk^{-1}X)_k \subset \underbrace{(sk^0X)_k \subset (sk^1X)_k \subset \dots \subset (sk^{k-1}X)_k}_{X_k^\zeta} \subset \underbrace{(sk^kX)_k}_{X_k^\diamond} = X_k$$

Def. For $X \in sSet$, $G \in \text{Abel}$, define the chain cplx

$$\begin{aligned} C_n(X; G)^\diamond &= \bigoplus_{\alpha \in X_n^\diamond} G & 0 \leftarrow \bigoplus_{\alpha \in X_0^\diamond} G \xleftarrow{(d_0' - d_1')^*} \bigoplus_{\alpha \in X_1^\diamond} G \xleftarrow{(d_0' - d_0' + d_2')^*} \bigoplus_{\alpha \in X_2^\diamond} G \dots \\ C_n(X; G)^\zeta &= \bigoplus_{\alpha \in X_n^\zeta} G & 0 \leftarrow \bigoplus_{\alpha \in X_0^\zeta} G \xleftarrow{(d_0' - d_1')^*} \bigoplus_{\alpha \in X_1^\zeta} G \xleftarrow{(d_0' - d_0' + d_2')^*} \bigoplus_{\alpha \in X_2^\zeta} G \dots \end{aligned}$$

and $H_*(X; G)^\diamond$, $H_*(X; G)^\zeta$ as crspd homology.

By definition, $C_*(X; G) \cong C_*(X; G)^\diamond \oplus C_*(X; G)^\zeta$

Claim 1. $H_*(X; G)^\zeta = 0$, so

$$H_*(X; G) \cong H_*(X; G)^\diamond. \quad (*)$$

Rmk. Roughly, $(*)$ says that

singular homology \approx simplicial homology.

Finally, one can compute the (co)homology of $sSets$ without too much pain.

To prove Claim 1, one has to expend $C_*(X; G)$ by double complex.

Def (Double complex of $C.(X; G)$)

