Eine Woche, ein Beispiel 19 Excision

Goal: state the homology/cohomology/homotopy excision than and give applications. Idon't want to see the proof, I'm lazy...

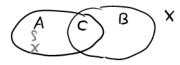
The main ref would be Hatcher, and we only consider the singular homology. http://www.math.uni-bonn.de/~schwede/Blakers-Massey.pdf

Def. An excisive triad  $(Y, Y_1, Y_2)$   $Y_1$  topo space,  $Y_1, Y_2 \subseteq Y$  s.t.  $Y = \mathring{Y}_1 \cup \mathring{Y}_2$  i.e. the open cores of  $Y_1$  and  $Y_2$  cover  $Y_1$ .

Thm (Homology excision thm, [Thm 2.20, Cor 2.24])

Let X space, A,BCX subspace, X = AUB, C = ADB

or, Let X CW cpk, A,BCX subsplx, X = AUB, C = ADB



then

 $L_*: H_i(A,C;\mathbb{Z}) \longrightarrow H_i(X,B;\mathbb{Z})$  is iso.

Cor [Prop 2.22] Let (X,A) be an NDR-pair then  $\pi_*: H_1(X,A;\mathbb{Z}) \longrightarrow \widetilde{H}_1(X_A;\mathbb{Z})$  is iso.

Proof.



 $H_{i}(X,A;Z) \stackrel{\text{LES}}{\cong} H_{i}(X \cup (A \times [0,1]) A;Z) \stackrel{\cong}{\longrightarrow} H_{i}(X \cup CA,CA;Z) \stackrel{\text{LES}}{\cong} \widehat{H_{i}}(X \cup CA;Z)$   $\sim \widehat{H_{i}}(X/A;Z)$ 

 $X \cup CA \longrightarrow (X \cup CA)_{CA} \cong X/A$  is a homotopy equivalence since [Prop 0.17]

· CA is contractable

· (X,A) satisfies HEP ⇒ (XUCA, CA) satisfies HEP

For the equivalent definitions of NPR-pair, see here: https://math.stackexchange.com/questions/3547820/neighborhood-deformation-retracts-vs-cofibrations or Prop A.6 in url:https://www.math.univ-parisi3.fr/~schwartz/FIMFA/Ando.pdf

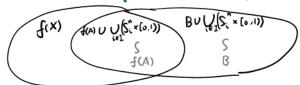
We also have the third version of excision thm. Thm. Suppose (X,A), (Y,B) are two relative CW-pairs,  $f:X \to Y$  send A to B, and the square  $A \longrightarrow X$ 

is a pushout of spaces, then

 $f_*: H_i(X, A; \mathbb{Z}) \longrightarrow H_i(Y, B; \mathbb{Z})$  is an iso.

Proof.  $Y = BU_{f,A}X$ , i.e. Y is gotten by attaching relative cells of (X,A) to B.

Step 1. If all relative cells of (X,A) are of dim n+1, then



 $H_{i}(Y, B; \mathbb{Z}) \stackrel{\text{LES}}{\cong} H_{i}(Y, B \cup \bigcup_{i \in I} (S_{i}^{n} \times [o, I])) \stackrel{\cong}{\longrightarrow} H_{i}(f(x), f(A) \cup \bigcup_{i \in I} (S_{i}^{n} \times [o, I])) \stackrel{\text{LES}}{\cong} H_{i}(f(x), f(A); \mathbb{Z})$   $H_{i}(f(x), f(A); \mathbb{Z}) \cong H_{i}(f(x)/f(A); \mathbb{Z}) \cong H_{i}(X/A; \mathbb{Z}) \cong H_{i}(X, A; \mathbb{Z})$ 

Step 2. Suppose that every square in the diagram

$$A \longrightarrow C \longrightarrow X$$

$$\downarrow fl_A \qquad \downarrow fl_e \qquad \downarrow f$$

$$B \longrightarrow D \longrightarrow Y$$

is a pushout of spaces, and

$$(f|c)_*$$
  $H_i(C, A, Z) \longrightarrow H_i(D, B, Z)$   
 $f_*$   $H_i(X, C, Z) \longrightarrow H_i(Y, D, Z)$ 

are iso, then by the naturality of LES and 5-lemma,

 $f_*: H_i(X,A;Z) \longrightarrow H_i(Y,B;Z)$ 

Step 3. Consider the diagram

$$A : X^{(-1)} \longrightarrow X^{(0)} \longrightarrow \cdots \longrightarrow X^{(n)} \longrightarrow \cdots$$

$$\beta : Y^{(-1)} \longrightarrow Y^{(0)} \longrightarrow \cdots \longrightarrow Y^{(n)} \longrightarrow \cdots$$

and take the colimit.

Thm (MV sequence, LES)

Let X space, U, V = X open subset, UUV=X. Then we get a LES

Hatl(X; Z) =

 $\widehat{Fh}(unV; Z) \longrightarrow \widehat{H}_n(u; Z) \oplus \widehat{H}_n(V; Z) \longrightarrow \widehat{H}_n(x; Z)$ 

MV->excision: https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision excision->MV: https://www.math.ru.nl/~gutierrez/files/homology/Lectureo6.pdf
Please be aware of the conditions in the theorems. We have many versions of theorems when we slightly change the conditions, but I don't want to go to the most generality(Actually I don't know the most general condition).

 $\begin{array}{lll} E.g. & H_{n}\left(\Delta^{n},\partial\Delta^{n};\mathbb{Z}\right) \overset{SES}{\cong} H_{n-1}\left(\partial\Delta^{n},\Lambda;\mathbb{Z}\right) \cong H_{n-1}(\Delta^{n-1},\partial\Delta^{n-1};\mathbb{Z}) \overset{induction}{\cong} \mathbb{Z}. \\ E.g. & The local homology groups & H_{n}(x)_{:} = H_{n}(x,X-x;\mathbb{Z}) & H$ 

Given a sm map  $f: M \to N$  between mflds of dim n, a pt  $y \in N$  s.t.  $f^{-1}(y) \stackrel{?}{=} \{x_1, \dots, x_n\}$  is finite, we can define the local degree  $\deg_x f \in \mathbb{Z}$  at  $x \in f^{-1}(y)$ .

 $(U \wedge f^{-1}(y) = \{x\}) \qquad f_*: H_n (U, U - \{x\}; Z) \longrightarrow H_n(N, N - \{y\}; Z) \qquad [U] \mapsto \deg_x f^{-1}[N]$  When M, N are cpt. we can also define the global degree  $\deg_x f \in Z$ .

 $f_*: H_n(M; \mathbb{Z}) \xrightarrow{\mathbb{Z}^{\mathbb{Z}}} H_n(N; \mathbb{Z})$   $[M] \mapsto \deg f[N]$  we have the equality  $\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$ .

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Thm (Homotopy excision thm [Thm 4:3])
                 X: CW cplx, A,B subsplx, X=AUB, C=ANB nonempty and connected.
                 (A,C) is m-connected,
                  (B.C) is n-connected,
                 \pi_i(A,C,\kappa) \longrightarrow \pi_i(X,B,\kappa_0) is \begin{cases} iso & i < m+n \\ surj & i = m+n \end{cases}
                                                                                                       ×oe C
Cor [Prop 428] Suppose X.A are CW cplxs, ACX is sub-cplx
         If (X,A) is r-connected, A is s-connected,
      then the map Tx. Ti (X, A, xo) -> Ti (X/A, xo) is siso ix ++s+1
           \pi_i(X, A, x_o) \longrightarrow \pi_i(X UCA, CA, x_o) \stackrel{LES}{\hookrightarrow} \pi_i(X UCA, x_o) \stackrel{homotopy}{\sim} \pi_i(X/A, x_o)
Thm (Freudenthal suspension thm [Cor 4.24])
                  n≥1, X be an (n-1)-connected CW cplx, then the suspension map
                  \Sigma_{i} \pi_{i}(X_{i}, x_{o}) \longrightarrow \pi_{i+1}(\Sigma X_{i}, x_{o})
                    [f: S^i \to X] \longrightarrow [\Sigma f: S^{i+1} \cong \Sigma S^i \longrightarrow \Sigma X]
                \begin{cases} iso & i < 2n-1 \\ Surj & i = 2n-1 \end{cases}
Rmk. Freudenthal suspension thm was concept of the stable homotopy gp.
         \pi_{i}(X, X_{o}) \cong \pi_{i+1}(CX, X, X_{o}) \longrightarrow \pi_{i+1}(\Sigma X, CX, X_{o}) \cong \pi_{i+1}(\Sigma X)
                         itl=nth
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