Preview of semisimple

Semisimple modules: best modules

Theorem 16.1. *For a module V the following are equivalent:*

Verify & structure

(i) *V is semisimple;* Zorn

(ii) *V is a sum of simple modules;* (not direct sum)

zorn

(iii) *Every submodule of V is a direct summand.*

(iv) $\mathrm{soc}(V) = V$ (iv)* $\mathrm{rad}(V) = 0$, i.e. $\mathrm{top}(V) = V$

for SES: $2 \to 1, 3$

Semisimple algebras: "best" algebras

Theorem 16.6. *Let A be a K-algebra. Then the following are equivalent:*

Verify def Structure

(i) *The regular representation ${}_AA$ is semisimple;* easier to veryfy, {A}_A is direct sum of simple modules

(ii) *Every A-module is semisimple;* def.

(iii) *There exist K-skew fields D_i and natural numbers n_i where $1 \le i \le s$ such that*

$$A \cong \prod_{i=1}^{s} M_{n_i}(D_i).$$

(v). Any A-module is proj ((v)* injective) $\Leftrightarrow$ $\mathrm{gl\,dim}(A) = 0$

Remark. (iii) $\Rightarrow$ $\begin{cases} (-)^{op}, \text{ finite prod keeps semisimple ring (not infinite prod)} \\ \text{classification of simple } A\text{-modules.} \end{cases}$

(iii) $\overset{k=\bar{K}}{\Rightarrow}$ $A \cong \prod_{i=1}^{s} M_{n_i}(K)$

E.g $K[G]$ is semisimple when $k \nmid |G|$. **Theorem 16.12** (Maschke).

Semisimple Lie alg

In mathematics, a Lie algebra is **semisimple** if it is a direct sum (i) of simple Lie algebras (non-abelian Lie algebras without any non-zero proper ideals).

Throughout the article, unless otherwise stated, a Lie algebra is a finite-dimensional Lie algebra over a field of characteristic 0. For such a Lie algebra $\mathfrak{g}$, if nonzero, the following conditions are equivalent:

- $\mathfrak{g}$ is semisimple;
- the Killing form, $\kappa(x,y) = \mathrm{tr}(\mathrm{ad}(x)\mathrm{ad}(y))$, is non-degenerate;
- $\mathfrak{g}$ has no non-zero abelian ideals;
- $\mathfrak{g}$ has no non-zero solvable ideals;
- the radical (maximal solvable ideal) of $\mathfrak{g}$ is zero. (iv)*

socle & radical: approximation of semisimple modules

$\mathrm{soc}(V) = \sum_{i} S_i = \sum_{i\text{ semi}} S_i = \bigcap_{U \subseteq V \text{ large}} U$ maximal semisimple submodule

$\mathrm{rad}(V) = \bigcap_{U: V/U \text{ simple}} U = \bigcap_{U: V/U \text{ semi}} U = \sum_{U \subseteq V \text{ small}} U$ $\mathrm{top}(V)$: maximal "semisimple" factor module. ↑ f.l ✓

$V/\mathrm{rad}(V)$

Rmk. soc & rad are both neither left adjoints nor right adjoints but still we have:

- $f: V \to W \rightsquigarrow f(\mathrm{soc}(V)) \subset \mathrm{soc}(W) \qquad f(\mathrm{rad}(V)) \subseteq \mathrm{rad}(W)$
- $V = \bigoplus_{i \in I} V_i \rightsquigarrow \mathrm{Soc}(V) = \bigoplus_{i \in I} \mathrm{soc}(V_i) \qquad \mathrm{rad}(V) = \bigoplus_{i \in I} \mathrm{rad}(V_i)$

A submodule U of a module V is **large** in V if
$$U \cap U' \neq 0$$
for all non-zero submodules U' of V. $\Rightarrow U \supset \mathrm{soc}(V) \supset$ semisimple $\supset$ simple

$V_1 \cap V_2$ still large

A submodule U of a module V is **small** in V if
$$U + U' \neq V$$
for all proper submodules U' of V. $\Rightarrow U \subset \mathrm{rad}(V) \subset$ maximal

$U_1 + U_2$ still small

$\forall U \subseteq V, \exists U' \subseteq V$ s.t $U \oplus U \subseteq V$ is large.

$\forall U \subseteq V$, $\quad U$ is small $\overset{\Longrightarrow}{\underset{V \text{ f.g} / U \text{ cyclic} / A \text{ f.d } k\text{-alg}}{\Longleftarrow}} U \subseteq \mathrm{rad}(V)$

Corollary 17.16. *For a finitely generated module V the following hold:*

✓ (i) $\mathrm{rad}(V)$ *is small in* V; $\quad \Rightarrow$ small modules are $\{U \mid U \subseteq \mathrm{rad}(V)\}$

(ii) *If* $V \neq 0$, *then* $\mathrm{rad}(V) \subset V$;

✓ (iii) *If* $V \neq 0$, *then* V *has a maximal submodule.* $\quad$ for non f.g modules, may has no maximal submodule...

e.g 

Lemma 17.22. *For a module V of finite length, the socle series and the radical series of V are both finite, and the factors* $\mathrm{soc}_i(V)/\mathrm{soc}_{i-1}(V)$ *and* $\mathrm{rad}^i(V)/\mathrm{rad}^{i+1}(V)$ *are semisimple for all* $i \geq 0$.

Jacobson radical: $J(A) := \mathrm{rad}({}_A A) = \bigcap_{S \text{ simple}} \mathrm{Ann}_A(S)$ $\quad A/J(A)$ is semi when A is f.l.

is an two-sided ideal.

Lemma 18.10. *Let $x \in A$. The following statements are equivalent:* $\quad \Rightarrow \mathrm{Nil}(A) \subseteq J(A)$

(i) $x \in J(A)$;

(ii) *For all* $a_1, a_2 \in A$, *the element* $1 + a_1 x a_2$ *has an inverse;*

(iii) *For all* $a \in A$, *the element* $1 + ax$ *has a left inverse;*

Corollary 18.8 (Nakayama Lemma). *If V is a finitely generated A-module such that $J(A)V = V$, then $V = 0$.*

$J(eAe) = eJ(A)e = J(A) \cap eAe$. used to simplify calculation

E.g. For a module V, we want to find out:

- Is V semisimple (best). If not,
  - $\mathrm{soc}(V)$, $\mathrm{rad}(V)$, $\mathrm{top}(V)$ $\rightsquigarrow \mathrm{rad}^i(V)$ & $\mathrm{soc}^i(V)$
  - small & large module
  - $\mathrm{End}_A(V)$, Is das decomposable, and so on...

For an alg A, we want $J(A)$.

E.g. 1. $k[G]$ generally semisimple

2. quiver

17.6.1. Let Q be a quiver without oriented cycles, and let $V = (V_i, V_a)$ be a representation of Q. Show that

$$\operatorname{soc}(V) = \bigoplus_{i \in Q_0} \left(\bigcap_{\substack{a \in Q_1 \\ s(a)=i}} \operatorname{Ker}(V_a) \right) \quad \text{and} \quad \operatorname{rad}(V) = \bigoplus_{i \in Q_0} \left(\sum_{\substack{a \in Q_1 \\ t(a)=i}} \operatorname{Im}(V_a) \right).$$

Convention: If $\{a \in Q_1 \mid s(a) = i\} = \varnothing$, then

$$\bigcap_{\substack{a \in Q_1 \\ s(a)=i}} \operatorname{Ker}(V_a) = V_i$$

18.6.1. Let Q be a quiver, and let $A = KQ$. Show that $J(A)$ has as a K-basis the set of all paths from i to j such that there is no path from j to i, where i and j run through the set of vertices of Q.

module

alg

3. $N(\infty)$ & $k[T]$

4. upper triangle matrix.