Eine Woche, ein Beispiel

6.26. adic space

Ref: Berkeley notes by Peter Scholze.
Section 4, http://www.math.uni-bonn.de/people/ja/lubintate/lecture_notes_lubin_tate.pdf
I just feel the need to record these results, so that I won't check them again as time went by.
Also I just learned a little about the discrete version. It is still not obvious for me to find out all valuations up to equivalence.

1. Discrete Huber pairs.

Set level

(A, A^+): $A \in CRing$ $\quad A^+ \leq$ integrally closed subring (containing 1)

$$\mathrm{Spa}(A, A^+) = \left\{ v: A \longrightarrow \Gamma \cup \{+\infty\} \;\middle|\; \begin{array}{ll} v(f+g) \geq \min\{v(f), v(g)\} & v(0) = +\infty \\ v(fg) = v(f) + v(g) & v(1) = 0 \\ v(A^+) \geq 0 & \end{array} \right\} \Big/ \sim$$

$$= \left\{ |\cdot|: A \longrightarrow \Gamma' \cup \{0\} \;\middle|\; \begin{array}{ll} |f+g| \leq \max\{|f|, |g|\} & |0| = 0 \\ |fg| = |f||g| & |1| = 1 \\ |A^+| \leq 1 & \end{array} \right\} \Big/ \sim$$

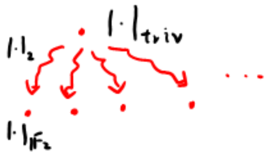
$v \sim v' \Leftrightarrow [v(a) \geq v(b) \quad \Leftrightarrow \quad v'(a) \geq v'(b)]$

$|\cdot| \sim |\cdot|' \Leftrightarrow [\,|a| \leq |b| \quad \Leftrightarrow \quad |a|' \leq |b|'\,]$

$(\Gamma, +, 0)$, $(\Gamma', \cdot, 1)$: totally ordered abelian group

We use the some notations as in the document [Berkovich space, 2021.11.7].

E.g. $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) = \{ |\cdot|_{triv}, |\cdot|_p, |\cdot|_{\mathbb{F}_p} \}$



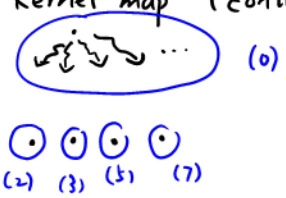
$\mathrm{Spa}(\mathbb{Q}, \mathbb{Z}) = \{ |\cdot|_{triv}, |\cdot|_p \}$

$\mathrm{Spa}(\mathbb{Q}, \mathbb{Q}) = \{ |\cdot|_{triv} \}$

• $|\cdot|_{triv}$

$\mathrm{supp}: \mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Spec}\,\mathbb{Z}$

$k = \mathrm{supp}$: kernel map (contraction)



Topology level.

Def (Rational open subsets of $\mathrm{Spa}(A, A^+)$) For $f, g_1, \cdots, g_n \in A$,

$$\begin{aligned} \mathcal{U}\left(\frac{g_1, \dots, g_n}{f}\right) &= \{ v \mid v(g_i) \geq v(f) \neq +\infty \quad \forall i \} \\ &= \{ |\cdot| \mid |g_i| \leq |f| \neq 0 \quad \forall i \} \\ &= \mathcal{U}\left(\frac{g_1}{f}\right) \cap \mathcal{U}\left(\frac{g_2}{f}\right) \cap \cdots \mathcal{U}\left(\frac{g_n}{f}\right) \end{aligned}$$

E.g. For $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$, Yellow: remove

$\mathcal{U}\left(\frac{0}{1}\right) = \mathcal{U}\left(\frac{9}{1}\right) = \mathcal{U}\left(\frac{3}{1}\right) = \mathcal{U}\left(\frac{1}{1}\right) = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$

$\mathcal{U}\left(\frac{0}{3}\right) = \mathcal{U}\left(\frac{9}{3}\right) = \mathcal{U}\left(\frac{3}{3}\right) = \{ \cdots \}$ $\qquad \mathcal{U}\left(\frac{1}{3}\right) = \{ \cdots \} = \mathcal{U}\left(\frac{1}{9}\right)$

$\mathcal{U}\left(\frac{45}{36}\right) = \{ \cdots \}$ $\qquad \mathcal{U}\left(\frac{a}{0}\right) = \emptyset$

For $\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$,

$\mathcal{U}\left(\frac{0}{1}\right) = \mathcal{U}\left(\frac{9}{1}\right) = \mathcal{U}\left(\frac{3}{1}\right) = \mathcal{U}\left(\frac{1}{1}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$

$\mathcal{U}\left(\frac{0}{3}\right) = \mathcal{U}\left(\frac{9}{3}\right) = \mathcal{U}\left(\frac{3}{3}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$ $\qquad \mathcal{U}\left(\frac{1}{3}\right) = \{ \cdots \} = \mathcal{U}\left(\frac{1}{9}\right)$

$\mathcal{U}\left(\frac{45}{36}\right) = \{ \cdots \}$ $\qquad \mathcal{U}\left(\frac{a}{0}\right) = \emptyset$

$\mathcal{U}\left(\frac{\frac{1}{9}}{1}\right) = \{ \cdots \}$ $\qquad \mathcal{U}\left(\frac{1}{\frac{1}{9}}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$ $\qquad \mathcal{U}\left(\frac{\frac{1}{9}}{\frac{1}{9}}\right) = \mathrm{Spa}(\mathbb{Q}, \mathbb{Z})$

It is actually easier in this case, we just take the reduction of a fraction, and remove valuations which correspond to primes on the denominators.

Rmk. Rational open subsets of $\mathrm{Spa}(A, A^+)$ is the basis of a topo, as

$$\mathcal{U}\left(\frac{g_1, \dots, g_n}{f}\right) \cap \mathcal{U}\left(\frac{g_1', \dots, g_m'}{f'}\right) = \mathcal{U}\left(\frac{g_1 f', \dots, g_n f', g_1' f, \dots g_m' f}{f f'}\right)$$

Now we can do the same formalism as two years ago.

- Topology
  - Closed subset of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$: $\bigcup_{p \in \Lambda_1} \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \} \cup \bigcup_{p \in \Lambda_2} \{ |\cdot|_{\mathbb{F}_p} \}$, or $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$

    $\Lambda_1, \Lambda_2 \subset \{\text{primes } p\}$ finite, $\Lambda_1 \cap \Lambda_2 = \emptyset$
  - Open subset of $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$: $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z}) - \bigcup_{p \in \Lambda_1} \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \} \cup \bigcup_{p \in \Lambda_2} \{ |\cdot|_{\mathbb{F}_p} \}$, or $\emptyset$.
  - closure of a pt:

    $\overline{\{ |\cdot|_{triv} \}} = \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ $\qquad \overline{\{ |\cdot|_p \}} = \{ |\cdot|_p, |\cdot|_{\mathbb{F}_p} \}$ $\qquad \overline{\{ |\cdot|_{\mathbb{F}_p} \}} = \{ |\cdot|_{\mathbb{F}_p} \}$
  - Topo properties: connected ✓ $\qquad$ Hausdorff ✗ $\qquad$ quasi-compact ✓

    irreducible ✓