Eine Woche, ein Beispiel

10.2 equivariant K-theory of Steinberg variety: notation

This document is written to reorganize the notations in Tomasz Przezdziecki's master thesis:
http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

We changed some notation for the convenience of writing.

Task.

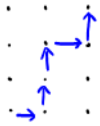
1. dimension vector
2. Weyl gp
3. alg group & Lie algebra
4. typical variety
5. (equivariant) stratifications
6. tangent space, Euler class
7. basis of Hecke alg

We may use two examples for the convenience of presentation.
Readers can easily distinguish them by the dim vectors.

1. dimension vector

$|d| = 5$

$d = (3,2)$

$\underline{d} = \begin{pmatrix} 3,2 \\ 2,2 \\ 2,1 \\ 1,1 \\ 0,0 \\ 0,0 \end{pmatrix} =$  $=$  $=$  $\in W_d \backslash \mathbb{W}_d$ or $Min(\mathbb{W}_{|d|}, W_d)$

Young Tableax $\quad r_\varpi = \pi_{\underline{d}}^{-1}(F_\varpi)$

2. Weyl group

Set	element	special element	others
$\mathbb{W}_{\vert d\vert} = S_5$	ϖ	$\varpi_{max} =$	$\Pi = \{s_1, s_2, s_3, s_4\}$
$W_d = S_3 \times S_2$	w	$w_{max} =$	$\Pi_d = \{s_1, s_2, \quad s_4\}$
$W_d \backslash \mathbb{W}_{\vert d\vert} = S_3 \times S_2 \backslash S_5$	$\varpi, \underline{d}$		(Comp$_d$)
$Min(\mathbb{W}_{\vert d\vert}, W_d) = \{$  $, \cdots \}$	u		(Shuffled)

$$\begin{tikzcd} & & & Min(\mathbb{W}_{|d|}, W_d) \arrow[dl] \arrow[d, "\cong"] & \\ 0 \arrow[r] & W_d \arrow[r] & \mathbb{W}_{|d|} \arrow[r] & \mathbb{W}_{|d|} \backslash W_d \arrow[r] & 0 \end{tikzcd}$$

$u \mapsto \underline{d}$

$\varpi = wu \longmapsto \underline{d}$

$\varpi =$
 $u =$
 $w =$ 

Another example: $\quad d = (1,2) \quad$ $a \longrightarrow b$, $\langle v_1 \rangle \longrightarrow \langle v_2, v_3 \rangle$

$\varpi = wu$					w	$\underline{d}$, u		order of basis	$l(\varpi)$	$l(w)$	$\mathbb{B}_\varpi$	B_ϖ	ϖB_ϖ^{-1}
Id	Id	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$				abb		$\{v_1, v_2, v_3\}$	0	0			
t	(23)	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$				abb		$\{v_1, v_3, v_2\}$	1	1			
s	(12)	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$				bab		$\{v_2, v_1, v_3\}$	1	0			
ts	(132)	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$				bab		$\{v_3, v_1, v_2\}$	2	1			
st	(123)	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$				bba		$\{v_2, v_3, v_1\}$	2	0			
sts	(13)	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$				bba		$\{v_3, v_2, v_1\}$	3	1			

3. alg group & Lie algebra

$G_{|d|}, B_{|d|}, T_{|d|}, N_{|d|}$ $\quad G_{|d|} = N_{|d|}/T_{|d|}$ $\quad GL_5(\mathbb{C}) = \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right)$

G_d, B_d, T_d, N_d $\quad G_d = N_d/T_d$ $\quad GL_3(\mathbb{C}) \times GL_2(\mathbb{C}) = \left(\begin{array}{c|c} * & \\ \hline & * \end{array}\right)$

$\mathbb{B}_\varpi = \varpi \mathbb{B}_{|d|} \varpi^{-1} = \mathrm{Stab}_{G_{|d|}}(F_\varpi)$

$B_\varpi \overset{\varpi = wu}{=\!=} w B_d w^{-1} = \mathrm{Stab}_{G_d}(F_\varpi)$ $\quad N_\varpi = R_u(B_\varpi)$

For $s \in \Pi$ s.t. $\varpi s \varpi^{-1} \in W_d$ (i.e. $W_d \varpi = W_d \varpi s$), define

$$P_{\varpi,\varpi s} :\overset{\varpi = wu}{=\!=} w(B_d usu^{-1} B_d \cup B_d) w^{-1} = B_\varpi \varpi s \varpi^{-1} B_\varpi \cup B_\varpi$$

$N_{\varpi,\varpi s} = R_u(B_{\varpi,\varpi s}) = N_\varpi \cap N_{\varpi s}$

$M_{\varpi,\varpi s} = N_\varpi / N_{\varpi,\varpi s}$

Ex. Show that

$$u s_i u^{-1} \in W_d \Rightarrow u s_i u^{-1} = s_{u(i)} \in \Pi_d$$

We can generalize the unipotent part.

$N_{\varpi,\varpi'} := N_\varpi \cap N_{\varpi'}$

$M_{\varpi,\varpi'} := N_\varpi / N_{\varpi,\varpi'}$

Their Lie algebras are collected here.

$\mathfrak{g}_{|d|}, \mathbb{b}_{|d|}, \mathfrak{t}_{|d|}, \mathfrak{n}_{|d|}$

$\mathfrak{g}_d \quad \mathfrak{b}_d \quad \mathfrak{t}_d \quad \mathfrak{n}_d$

$\mathbb{b}_\varpi$		$\mathbb{b}^-_\varpi$		$\mathbb{b}^-_\varpi = \mathbb{b}_{\varpi_{max}\varpi}$
$\mathfrak{b}_\varpi$	$\mathfrak{n}_\varpi$	$\mathfrak{b}^-_\varpi$	$\mathfrak{n}^-_\varpi$	$\mathfrak{b}^-_\varpi = \mathfrak{b}_{w_{max}\varpi}$ ...
$\mathfrak{p}_{\varpi,\varpi s}$	$\mathfrak{n}_{\varpi,\varpi'}$	$\mathfrak{p}^-_{\varpi,\varpi s}$	$\mathfrak{n}^-_{\varpi,\varpi'}$	$\mathfrak{p}^-_{\varpi,\varpi s} = \mathfrak{p}_{w_{max}\varpi, w_{max}\varpi s}$ ...
	$\mathfrak{m}_{\varpi,\varpi'}$		$\mathfrak{m}^-_{\varpi,\varpi'}$	...

$$\mathrm{Rep}_d(Q) := \prod_{e \in Q_1} \mathrm{Hom}(V_{s(e)}, V_{t(e)}) = \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix} \subseteq \mathfrak{g}_{|d|}$$

in general case, lies in $\mathfrak{g}_{|d|}^{\oplus k}$

$$\mathfrak{r}_\varpi = \{ f \in \mathrm{Rep}_d(Q) \mid f.F_{\varpi,i} \subseteq F_{\varpi,i} \} = \mu_{\underline{d}} \pi_{\underline{d}}^{-1}(F_\varpi)$$

$$= \begin{array}{c} \\ v_5 \\ v_4 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} v_3 & v_1 & v_2 \end{array} \\ \begin{pmatrix} * & * & * \\ & & * \end{pmatrix} \end{array} = \begin{array}{c} \\ v_4 \\ v_5 \end{array} \!\!\begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} & * & \\ * & * & * \end{pmatrix} \end{array}$$

$v_{w(i)}$

$\mathfrak{r}_{\varpi,\varpi'} = \mathfrak{r}_\varpi \cap \mathfrak{r}_{\varpi'}$

$\mathfrak{d}_{\varpi,\varpi'} = \mathfrak{r}_\varpi / \mathfrak{r}_{\varpi,\varpi'}$

4. typical variety

	Id corres to
$\mathcal{F}_{\lvert d\rvert} \cong G_{\lvert d\rvert}/B_{\lvert d\rvert}$	F_{Id}
$\mathcal{F}_{\underline{d}} \cong G_d/B_{\underline{d}}$	F_u
$\mathcal{F}_{\varpi} \cong G_d/B_{\varpi}$	F_{ϖ}
$\mathcal{F}_d := \coprod_{\underline{d}} \mathcal{F}_{\underline{d}}$	—

$$F_{\varpi} := \varpi(F_{Id}) = F_{\{v_{\varpi(1)}, v_{\varpi(2)}, \cdots, v_{\varpi(\lvert d\rvert)}\}}$$
$$= F_{\{v_5, v_3, v_1, v_4, v_2\}}$$

⚠ The action on $\mathcal{F}lag$ is not the same as in http://www.math.uni-bonn.de/ag/stroppel/Master%27s%20Thesis_Tomasz%20Przezdziecki.pdf

$\mathcal{F}_{\lvert d\rvert} \neq \coprod_{d} \mathcal{F}_{d}$

$\mathcal{F}_{\varpi} \cong \mathcal{F}_{\underline{d}}$ with different base pt. Base pt makes difference!

$\mathcal{F}_{\lvert d\rvert} \times \mathcal{F}_{\lvert d\rvert}$	$F_{Id,Id}$
$\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$	$F_{u,u'}$
$\mathcal{F}_{\varpi} \times \mathcal{F}_{\varpi'}$	$F_{\varpi,\varpi'}$
$\mathcal{F}_d \times \mathcal{F}_d := \coprod_{\underline{d},\underline{d}'} (\mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'})$	—

$$F_{\varpi,\varpi'} := (F_{\varpi}, F_{\varpi'})$$

$$\begin{tikzcd} \widetilde{Rep}_{\underline{d}}(Q) \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \arrow[d, "\mu_{\underline{d}}"'] \arrow[dr, "\pi_{\underline{d}}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \end{tikzcd} \qquad \begin{tikzcd} \widetilde{Rep}_d(Q) \subset Rep_d(Q) \times \mathcal{F}_d \arrow[d, "\mu_d"'] \arrow[dr, "\pi_d"] & \\ Rep_d(Q) & \mathcal{F}_d \end{tikzcd}$$

$\mu_{\underline{d}}^{-1}(M) \cong \mathcal{F}lag_{\underline{d}}(M) \subseteq \mathcal{F}_{\underline{d}}$ is the Springer fiber.

$$\begin{tikzcd} Z_{\underline{d},\underline{d}'} \subset Rep_d(Q) \times \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \arrow[d, "\mu_{\underline{d},\underline{d}'}"'] \arrow[dr, "\pi_{\underline{d},\underline{d}'}"] & \\ Rep_d(Q) & \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'} \end{tikzcd} \qquad \begin{tikzcd} Z_d \subset Rep_d(Q) \times \mathcal{F}_d \times \mathcal{F}_d \arrow[d, "\mu_{d,d}"'] \arrow[dr, "\pi_{d,d}"] & \\ Rep_d(Q) & \mathcal{F}_d \times \mathcal{F}_d \end{tikzcd}$$

$$\widetilde{Rep}_{\underline{d}}(Q) \subseteq Rep_d(Q) \times \mathcal{F}_{\underline{d}} \qquad \widetilde{Rep}_{\varpi}(Q) \cong G_d \times^{B_{\varpi}} r_{\varpi}$$
$$\widetilde{Rep}_d(Q) := \bigsqcup_{\underline{d}} \widetilde{Rep}_{\underline{d}}(Q)$$

$$Z_{\underline{d},\underline{d}'} := \widetilde{Rep}_{\underline{d}}(Q) \times_{Rep_d(Q)} \widetilde{Rep}_{\underline{d}'}(Q) \qquad Z_{\varpi,\varpi'} := Z_{u,u'}$$
$$Z_d = \bigsqcup_{\underline{d},\underline{d}'} Z_{\underline{d},\underline{d}'}$$
$$= \widetilde{Rep}_d(Q) \times_{Rep_d(Q)} \widetilde{Rep}_d(Q)$$

5. (equivariant) stratifications.

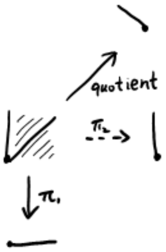
In the following tables, $u w' = \tilde{\omega}' \tilde{u}$.

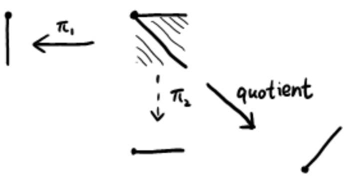
$F_\varpi \in \widetilde{Rep}_d(Q)$ means (ρ_0, F_ϖ); $(F_\varpi, F_{\varpi'}) \in \mathcal{Z}_d$ means $(\rho_0, F_\varpi, F_{\varpi'})$.

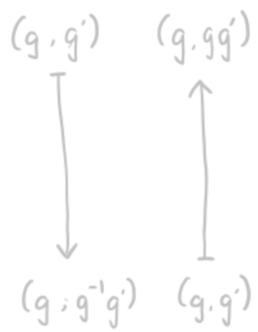
⚠ $G \times G$ acts on $\mathcal{F} \times \mathcal{F}$ in a twisted way

e.g. $(g_1, g_2) F_{\varpi, \varpi'} = F_{g_1 \varpi,\, g_1 \varpi g_2 \varpi^{-1} \varpi'}$

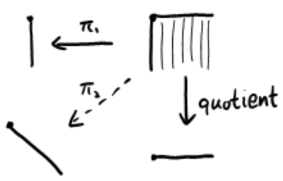
variety / base point (stratification, stabilizer / type)		B-orbit	B×B-orbit	B×G-orbit	G×B-orbit	Remark
$\mathcal{B}$ F_g	$\mathcal{B} \times \mathcal{B}$ $(F_g, F_{gg'})$	Ω_g $B \cap gBg^{-1}$	$\Omega_{g,g'}$ $(B \cap gBg^{-1}) \times (B \cap g'Bg'^{-1})$	$pr_1^{-1}(\Omega_g)$ $(B \cap gBg^{-1}) \times g'Bg'^{-1}$	$\Omega_{g'}$ $gBg^{-1} \times (B \cap g'Bg'^{-1})$	
$\mathcal{F}_{\lvert d \rvert}$ F_ϖ	$\mathcal{F}_{\lvert d \rvert} \times \mathcal{F}_{\lvert d \rvert}$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{V}_\varpi$ $\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi$	$\mathcal{V}_{\varpi,\varpi'}$ $(\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi) \times (\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_{\varpi'})$	$pr_1^{-1}(\mathcal{V}_\varpi)$ $(\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_\varpi) \times \mathbb{B}_{\varpi'}$	$\mathcal{V}_{\varpi'}$ $\mathbb{B}_\varpi \times (\mathbb{B}_{\lvert d \rvert} \cap \mathbb{B}_{\varpi'})$	
$\mathcal{F}_u$ F_{wu}	$\mathcal{F}_u \times \mathcal{F}_{u'}$ $(F_{wu}, F_{ww'u'})$	Ω^u_w $B_d \cap B_w$	$\Omega^{u,u'}_{w,w'}$ $(B_d \cap B_w) \times (B_d \cap B_{w'})$	$pr_{1,u'}^{-1}(\Omega^u_w)$ $(B_d \cap B_w) \times B_{w'}$	$\Omega^{u,u'}_{w'}$ $B_w \times (B_d \cap B_{w'})$	
$\mathcal{F}_d$ F_ϖ $\parallel$ F_{wu}	$\mathcal{F}_d \times \mathcal{F}_d$ $(F_\varpi, F_{\varpi'})$ $\parallel$ $(F_{wu}, F_{\omega\tilde{\omega}'\tilde{u}u'})$	Ω^u_w $B_d \cap B_w$	$\Omega^{u,\tilde{u}u'}_{w,\tilde{w}'}$ $(B_d \cap B_w) \times (B_d \cap B_{\tilde{\omega}'})$	$pr_{1,\tilde{u}u'}^{-1}(\Omega^u_w)$ $(B_d \cap B_w) \times B_{\tilde{\omega}'}$	$\mathcal{O}^u_{\varpi'} := \Omega^{u,\tilde{u}u'}_{\tilde{w}'}$ $B_w \times (B_d \cap B_{\tilde{\omega}'})$	compatability
The following may not be single orbit, but derived from the above definition.						
$\mathcal{F}_d$ F_ϖ	$\mathcal{F}_d \times \mathcal{F}_d$ $(F_\varpi, F_{\varpi\varpi'})$	$\mathcal{O}_\varpi$	$\mathcal{O}_{\varpi,\varpi'}$	$pr_1^{-1}(\mathcal{O}_\varpi)$	$\mathcal{O}_{\varpi'}$ $\parallel$ $\bigsqcup_u \mathcal{O}^u_{\varpi'}$	preimage of $\mathcal{F}_d \times \mathcal{F}_d \hookrightarrow \mathcal{F}_{\lvert d \rvert} \times \mathcal{F}_{\lvert d \rvert}$
$\widetilde{Rep}_{\underline{d}}(Q)$ F_{wu}	$\mathcal{Z}_{\underline{d},\underline{d}'}$ $(F_{wu}, F_{w'u'})$	$\widetilde{\Omega}^u_w$	$\widetilde{\Omega}^{u,u'}_{w,w'}$	$pr_{1,u'}^{-1}(\widetilde{\Omega}^u_w)$	$\widetilde{\Omega}^{u,u'}_{w'}$	preimage of $\mathcal{Z}_{\underline{d},\underline{d}'} \longrightarrow \mathcal{F}_{\underline{d}} \times \mathcal{F}_{\underline{d}'}$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi'})$				$\widetilde{\mathcal{O}}^u_{\varpi'}$ $\parallel$ $\widetilde{\Omega}^{u,\tilde{u}u'}_{\tilde{w}'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d \times \mathcal{F}_d$
$\widetilde{Rep}_d(Q)$ F_ϖ	$\mathcal{Z}_d$ $(F_\varpi, F_{\varpi'})$	$\widetilde{\mathcal{O}}_\varpi$	$\widetilde{\mathcal{O}}_{\varpi,\varpi'}$	$pr_1^{-1}(\widetilde{\mathcal{O}}_\varpi)$	$\widetilde{\mathcal{O}}_{\varpi'}$ $\parallel$ $\bigsqcup_u \widetilde{\mathcal{O}}^u_{\varpi'}$	preimage of $\mathcal{Z}_d \longrightarrow \mathcal{F}_d \times \mathcal{F}_d$

real case: 

in table: 



We want gp action to be compatible with π_1 and the quotient map.
Therefore, we would do a twist.



The following tables may help you to understand the notations.

dim; $\mathbb{B}_{\|d\|}\cdot F_{\varpi'}$ ϖ'; $\mathbb{B}_{\|d\|}\times\mathbb{B}_{\|d\|}\cdot(F_{\varpi},F_{\varpi'})$; $\mathbb{B}_{\|d\|}\cdot F_{\omega}$ ϖ	0 ϑ_{Id}	1 ϑ_{t}	1 ϑ_{s}	2 ϑ_{ts}	2 ϑ_{st}	3 ϑ_{sts}
0 ϑ_{Id}	0 $\vartheta_{Id,Id}$	1 $\vartheta_{Id,t}$	1 $\vartheta_{Id,s}$	2 $\vartheta_{Id,s}$	2 $\vartheta_{Id,st}$	3 $\vartheta_{Id,sts}$
1 ϑ_{t}	2 $\vartheta_{t,t}$	1 $\vartheta_{t,Id}$	3 $\vartheta_{t,ts}$	2 $\vartheta_{t,s}$	4 $\vartheta_{t,sts}$	3 $\vartheta_{t,st}$
1 ϑ_{s}	2 $\vartheta_{s,s}$	3 $\vartheta_{s,st}$	1 $\vartheta_{s,Id}$	4 $\vartheta_{s,sts}$	2 $\vartheta_{s,t}$	3 $\vartheta_{s,ts}$
2 ϑ_{ts}	4 $\vartheta_{ts,st}$	3 $\vartheta_{ts,s}$	5 $\vartheta_{ts,sts}$	2 $\vartheta_{ts,Id}$	4 $\vartheta_{ts,ts}$	3 $\vartheta_{ts,t}$
2 ϑ_{st}	4 $\vartheta_{st,ts}$	5 $\vartheta_{st,sts}$	3 $\vartheta_{st,t}$	4 $\vartheta_{st,st}$	2 $\vartheta_{st,Id}$	3 $\vartheta_{st,s}$
3 ϑ_{sts}	6 $\vartheta_{sts,sts}$	5 $\vartheta_{sts,ts}$	5 $\vartheta_{sts,st}$	4 $\vartheta_{sts,t}$	4 $\vartheta_{sts,s}$	3 $\vartheta_{sts,Id}$

shape; $B_d\cdot F_{\varpi'}$ ϖ'; $B_d\times B_d\cdot(F_{\varpi},F_{\varpi'})$; $B_d\cdot F_{\varpi}$ ϖ		$\mathcal{F}_{Id}$		$\mathcal{F}_{s}$		$\mathcal{F}_{st}$	
		• $\mathcal{O}_{Id}$	— $\mathcal{O}_{t}$	• $\mathcal{O}_{s}$	— $\mathcal{O}_{ts}$	• $\mathcal{O}_{st}$	— $\mathcal{O}_{sts}$
$\mathcal{F}_{Id}$	• $\mathcal{O}_{Id}$	• $\Omega^{Id,Id}_{Id,Id}$	— $\Omega^{Id,Id}_{Id,t}$	• $\Omega^{Id,s}_{Id,Id}$	— $\Omega^{Id,s}_{Id,t}$	• $\Omega^{Id,st}_{Id,Id}$	— $\Omega^{Id,st}_{Id,t}$
	$\vert$ $\mathcal{O}_{t}$	▱ $\Omega^{Id,Id}_{t,t}$	\ $\Omega^{Id,Id}_{t,Id}$	▱ $\Omega^{Id,s}_{t,t}$	\ $\Omega^{Id,s}_{t,Id}$	▱ $\Omega^{Id,st}_{t,t}$	\ $\Omega^{Id,st}_{t,Id}$
$\mathcal{F}_{s}$	• $\mathcal{O}_{s}$	• $\Omega^{s,Id}_{Id,Id}$	— $\Omega^{s,Id}_{Id,t}$	• $\Omega^{s,s}_{Id,Id}$	— $\Omega^{s,s}_{Id,t}$	• $\Omega^{s,st}_{Id,Id}$	— $\Omega^{s,st}_{Id,t}$
	$\vert$ $\mathcal{O}_{ts}$	▱ $\Omega^{s,Id}_{t,t}$	\ $\Omega^{s,Id}_{t,Id}$	▱ $\Omega^{s,s}_{t,t}$	\ $\Omega^{s,s}_{t,Id}$	▱ $\Omega^{s,st}_{t,t}$	\ $\Omega^{s,st}_{t,Id}$
$\mathcal{F}_{st}$	• $\mathcal{O}_{ts}$	• $\Omega^{st,Id}_{Id,Id}$	— $\Omega^{st,Id}_{Id,t}$	• $\Omega^{st,s}_{Id,Id}$	— $\Omega^{st,s}_{Id,t}$	• $\Omega^{st,st}_{Id,Id}$	— $\Omega^{st,st}_{Id,t}$
	$\vert$ $\mathcal{O}_{sts}$	▱ $\Omega^{st,Id}_{t,t}$	\ $\Omega^{st,Id}_{t,Id}$	▱ $\Omega^{st,s}_{t,t}$	\ $\Omega^{st,s}_{t,Id}$	▱ $\Omega^{st,st}_{t,t}$	\ $\Omega^{st,st}_{t,Id}$

6. tangent space, Euler class.