

Eine Woche, ein Beispiel

1.23 Coxeter group

1. def & realizations

- def
- geometrical representation
- root system
- polytopes
- as subgp of S_n
- as Weyl gp of some Tits system

2. combinatorial results

3. Bruhat order

4. geometrical realization (faithfulness)

Roadmap

gen & relations  characteristic properties \rightarrow realizations.

Ref: [Building] Buildings, by Peter Abramenko and Kenneth S. Brown

[Bourbaki] (Elements of Mathematics) Nicolas Bourbaki - Lie Groups and Lie Algebras_ Chapters 4-9-Springer (2002)

[Comb] Combinatorics of Coxeter groups

[Flag] Flag Varieties An Interplay of Geometry, Combinatorics, and Representation Theory

In the first section, we omit technical details, which will be filled in later on. (Mainly: injectivity)

1. def & realizations

def

Def (Coxeter system) (W, S) gp + gen, $m(s, t) \in \mathbb{Z}_{>0} \cup \{+\infty\}$, $m(s, s) = 1$

$$W = \langle s \in S \rangle / (s^2 = (st)^{m(s,t)} = 1, \forall s, t \in S)$$

W is a Coxeter gp if $\exists S \subseteq W$, (W, S) is a Coxeter system.

E.g.

$$S_n \cong \langle s_i \rangle / (s_i^2 = (s_i s_j)^3 = (s_i s_{i+1})^3 = 1)$$

$|i-j| \geq 2$, and undefined relations (e.g. $(s_{n-1} s_n)^3$) should be removed.

Coxeter graph

$m(s, t)$	$m(s, t)$
2	
3	
4	
6	
$+\infty$	

Notation

$$S$$

$$l(w) = \min \{r \mid w = s_1 \dots s_r, s_i \in S\}$$

$$\mathcal{T} = \{wsw^{-1} \mid w \in W, s \in S\}$$

simple reflections/transpositions
length of $w \in W$
reflections/transpositions

geometrical representation $W \hookrightarrow GL(V_{\text{geo}})$

⚠ We suppose $|S| < \infty$, which is not necessary (but helpful for concentrating mind)

$$(W, S) \rightsquigarrow (\rho_{\text{geo}}, V_{\text{geo}}, \langle -, - \rangle) \in \text{Rep}_{\text{IR, ortho}}(W)$$

$$V_{\text{geo}} = \bigoplus_{s \in S} \text{IR} \alpha_s$$

$$\langle -, - \rangle: V_{\text{geo}} \otimes V_{\text{geo}} \longrightarrow \text{IR}$$

$$(\alpha_s, \alpha_t) \longmapsto \begin{cases} -\cos \frac{\pi}{m(s,t)} & m(s,t) \neq +\infty \\ -1 & m(s,t) = +\infty \end{cases}$$

$m(s,t)$	1	2	3	4	5	6	...	∞
(α_s, α_t)	1	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{5}+1}{4}$	$-\frac{\sqrt{3}}{2}$...	-1

$$\rho_{\text{geo}}: W \longrightarrow GL(V_{\text{geo}})$$

$$s \longmapsto r_{\alpha_s}$$

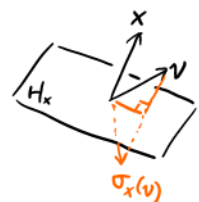
$$\text{For } x, v \in V_{\text{geo}}, \langle x, x \rangle = 1, \text{ define}$$

$$r_x(v) = v - 2\langle v, x \rangle x$$

$$\text{Check: } r_x(x) = -x$$

$$r_x(v) = v \iff v \in H_x, \text{ where}$$

$$H_x = \{v \in V_{\text{geo}} \mid \langle v, x \rangle = 0\}$$



Ex. Verify the well-definedness.

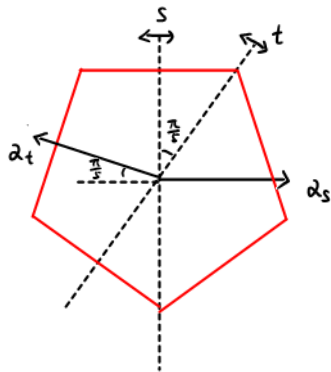
- $\rho_{\text{geo}}(s)$ is linear & orthogonal;
- $\rho_{\text{geo}}(\text{relations}) = \text{Id}$

Also, $\langle wv, wv' \rangle = \langle v, v' \rangle$.

Thm. ρ_{geo} is faithful (sketch of proof: later on)

E.g. $W = W(I_5)$

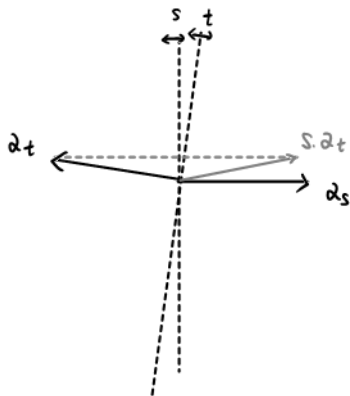
$$\begin{array}{c} s \\ \circ \text{---} \circ \\ t \\ I_5 \end{array}$$



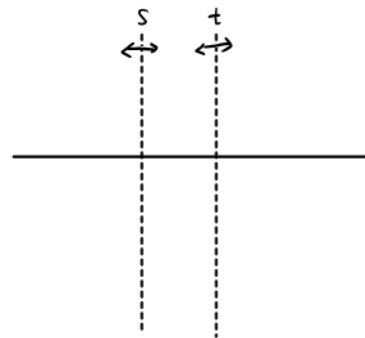
$$\rho_{\text{geo}}(W) \cong D(5) \quad \text{Dihedral gp}$$

$W = W(I_\infty)$

$$\begin{array}{c} s \\ \circ \text{---} \circ \\ t \\ I_\infty \end{array}$$



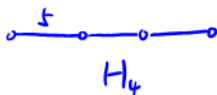
$$s \cdot \alpha_t = \alpha_t + 2\alpha_s$$



$$\begin{aligned} s(\alpha_s, \alpha_t) &= (\alpha_s, \alpha_t) \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \\ t(\alpha_s, \alpha_t) &= (\alpha_s, \alpha_t) \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \end{aligned}$$

$$\rho_{\text{geo}}(W) \cong \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

E.g.



$$\begin{pmatrix} 1 & -\frac{\sqrt{5}+1}{4} \\ -\frac{\sqrt{5}+1}{4} & 1 \\ & -\frac{1}{2} & -\frac{1}{2} \\ & -\frac{1}{2} & 1 \end{pmatrix} \text{ is pos-def}$$

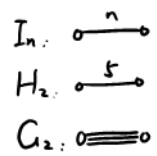
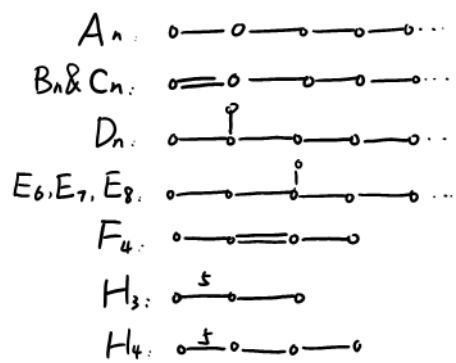
E.x.

$\# W < +\infty$

\Leftrightarrow The bilinear form $\langle -, - \rangle$ is pos-def

\Leftrightarrow Cartan matrix is pos-def

\Leftrightarrow Coxeter graph is finite disjoint union of following shape.



Root system $W \sim \text{Aut}_R(V_{\text{geo}})$

⚠ Not the same as in Lie alg! E.g. here, every root has length 1.
That's why we don't use Φ here.

$$R = \{v \in V_{\text{geo}} \mid v = w \cdot \alpha_s \text{ for some } w \in W, s \in S\}$$

⚠ $\sigma \xrightarrow{\rho_{\text{geo}}} \{ \sigma \in GL(V_{\text{geo}}) \mid \sigma = r_x \text{ for some } x \in V_{\text{geo}}, \langle x, x \rangle = 1, \sigma(R) = R \}$

can be not surj when the irr root system is not simply laced.

See 1084790 for more details.

⚠ Here, $W \neq \text{Aut}(R)$! See example on $W(I_5)$.

Ex. Verify the following properties.

(R1) R spans V_{geo} , does not contain 0

(R2) $-R = R$

(R3) $r_v R = R \quad \forall v \in R$

Define $R^+ = \left(\sum_{s \in S} \mathbb{R}_{\geq 0} \alpha_s \right) \cap R$ $R^- = \left(\sum_{s \in S} \mathbb{R}_{\leq 0} \alpha_s \right) \cap R$

one can check $R = R^+ \sqcup R^-$ by hand.

Lemma. $r_{w \cdot \alpha_s} = \rho_{\text{geo}}(w s w^{-1}) \quad w \in W, s \in S$

Proof. $r_{w \cdot \alpha_s}(x) = x - 2 \langle w \cdot \alpha_s, x \rangle w \cdot \alpha_s$
 $= w \cdot (w^{-1} x - 2 \langle \alpha_s, w^{-1} x \rangle \alpha_s)$
 $= w \cdot \sigma_{\alpha_s}(w^{-1} x)$
 $= \rho_{\text{geo}}(w s w^{-1}) x.$

Prop. We have bijection

$$\begin{array}{ccc} R & \xrightarrow{\quad} & \mathcal{T} \times \{\pm 1\} \\ w \cdot \alpha_s & \longmapsto & (w s w^{-1}, \eta(w, s)) \end{array} \quad \begin{array}{l} R^+ \leftrightarrow \mathcal{T} \times \{+1\} \\ R^- \leftrightarrow \mathcal{T} \times \{-1\} \end{array}$$

where $\eta(s, t) = \begin{cases} -1 & s=t \\ 1 & s \neq t \end{cases} \quad \eta(w'w, t) = \eta(w'; w t w^{-1}) \eta(w, t)$

For the well-definedness of η , we postpone to next section.

See <https://math.stackexchange.com/questions/1084790/weyl-groups-correspondence-of-reflections-and-roots> and [Building, Prop 1.113].

Polytopes

$$W \cong \text{Aut}(\text{Polytopes})$$

(fundamental domain, chambers)

▽ For Dynkin - Coxeter graph.

Others can be viewed as mosaic in spaces with constant curv ≤ 0 .

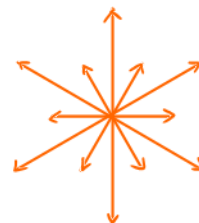
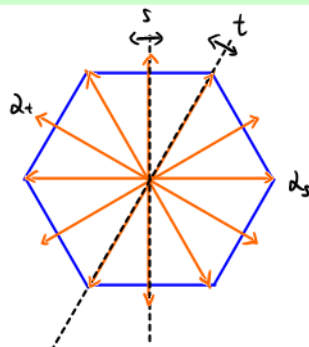
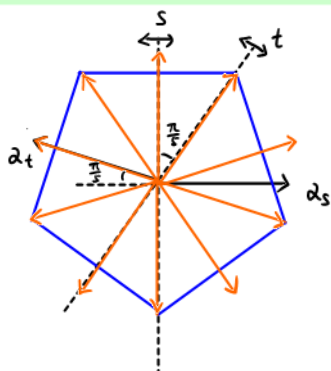
It comes from the geo rep.

Ref:

<https://syntopia.github.io/Polytopia/polytopes.html>

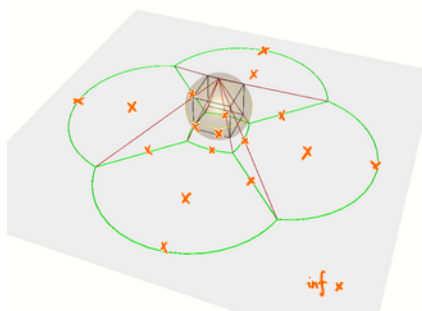
<https://www3.mpi-fr-bonn.mpg.de/staff/pfreire/polyhedra/index.html>

<https://www.mdpi.com/2073-8994/11/3/391/pdf?version=1552904082> (Some vague pictures of 5D polytopes)



G_2

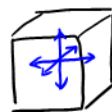
Ex. draw roots in Δ , \square . (Bad picture for Δ !)



as subgp of S_n

strand description

▽ For type A ~ D, since they have "nice" shapes of polytopes.



$$W(A_3) \xrightarrow{\cong} S_4$$

$$W(B_3) \hookrightarrow S_6$$

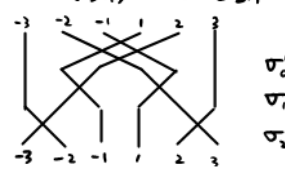
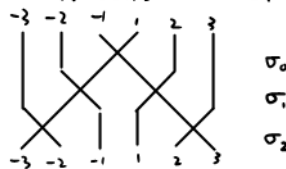
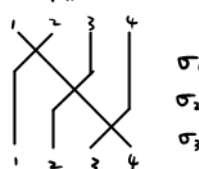
$$W(D_3) \hookrightarrow S_6$$

$$W(A_n) \cong S_{n+1}$$

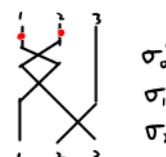
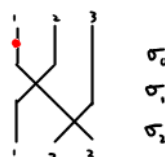
$$W(B_n) \hookrightarrow S_{2n}$$

$$W(D_n) \hookrightarrow S_{2n}$$

Standard



with dot



as Weyl gp of some Tits system (later)

Ex. for the section

1. Verify the gen & rel in each case.
2. Describe element, reflection, simple reflection, length, roots, ... in each realization.

e.g. how to see $|\Gamma| = \ell(w_0)$?

3. (Finite) group study:

- $\#G$
- conj class
- $Z(G)$, $[G, G]$
- char table (Rep theory)
- simple?
- subgp, quotient, central series, ...

4. Generalize everything to affine diagram.
e.g. find a strand description of \widehat{A}_n .

2. combinatorial results

Lemma. For $(W, S) \in \text{Cosgp}$, $\exists!$ gp homo

$$\begin{aligned} \text{sgn}: W &\longrightarrow \{\pm 1\} \\ s &\longmapsto -1 \end{aligned}$$

$$\text{s.t. } \text{sgn}(w) = (-1)^{l(w)} \quad \forall w \in W$$

Cor. $\forall w \in W, s \in S, l(ws) \equiv l(sw) \equiv l(w) + 1 \pmod{2}$

In ptc, $l(ws) \neq l(w)$

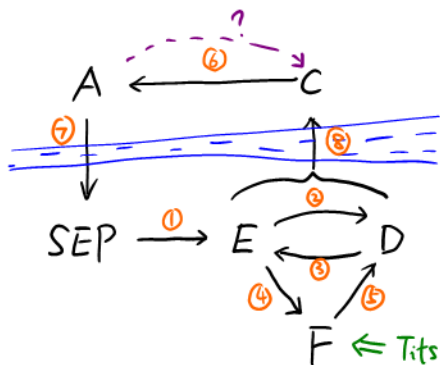
Setting In this section, W is a gp, S is a set of gen of order 2.

Still,

$$\begin{aligned} l(w) &:= \min \{r \mid w = s_1 \dots s_r, s_i \in S\} && \text{length of } w \in W \\ \mathcal{T} &:= \{ws w^{-1} \mid w \in W, s \in S\} && \text{reflections / transpositions} \end{aligned}$$

We have $l(w^{-1}) = l(w)$, but it is possible that $l(ws) = l(w)$ now.

Road map



- A. Action [Building, p65]
- C. Coxeter
- D. DP = Deletion property
- E. EP = Exchange property
- F. Folding condition [Building, p79]

(Coxeter) (W, S) is a Coxeter system

(SEP) $w = s_1 \dots s_r, s_i \in S, t \in \mathcal{T}, l(tw) < l(w)$

$$\Rightarrow tw = s_1 \dots \hat{s}_i \dots s_r \quad \exists i$$

(EP) $w = s_1 \dots s_r, s_i \in S, t \in S, l(tw) < l(w)$

$$\Rightarrow tw = s_1 \dots \hat{s}_i \dots s_r \quad \exists i$$

(DP) $w = s_1 \dots s_r, s_i \in S, l(w) < r$

$$\Rightarrow w = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_r \quad \exists i, j$$

(Folding) For $w \in W, s, t \in S$ s.t. $l(tw) = l(w) + 1, l(ws) = l(w) + 1,$

$$\Rightarrow l(tws) = l(w) + 2 \text{ or } tws = w$$

(Action) $\exists \rho: W \rightarrow \mathcal{T} \times \{\pm 1\}$ s.t. $\forall s \in S,$

$$\rho_s(t, \varepsilon) = \begin{cases} (s, -\varepsilon) & s = t \\ (sts, \varepsilon) & s \neq t \end{cases}$$

In ptc, $\rho_w(t, \varepsilon) = (wtw^{-1}, \eta(w; t)\varepsilon)$ where

$$\eta(s; t) = \begin{cases} -1 & s = t \\ 1 & s \neq t \end{cases}$$

$$\eta(w'w; t) = \eta(w'; wt w^{-1}) \eta(w; t)$$

Def. (Reduced expression)

$w = s_1 \dots s_r$ is reduced, if $l(w) = r$.

① Obvious

② Choose i maximal s.t. $s_i \dots s_r$ is not reduced.

$$\Rightarrow l(s_i s_{i+1} \dots s_r) < l(s_{i+1} \dots s_r)$$

$$\stackrel{(EP)}{\Rightarrow} s_i \dots s_r = s_{i+1} \dots \hat{s}_j \dots s_r$$

$$\Rightarrow s_i \dots s_r = s_i \dots \hat{s}_i \dots \hat{s}_j \dots s_r$$

③ $l(tw) < l(w) \leq r$

$$\stackrel{(DP)}{\Rightarrow} tw = t s_i \dots \hat{s}_i \dots \hat{s}_j \dots s_r \quad \text{or} \quad s_i \dots \hat{s}_i \dots s_r$$

\downarrow
 $l(w) \leq r-2$

Then use induction on r .

④ Take $w = s_1 \dots s_r$. If $l(tws) \neq l(w) + 2$, then $l(tws) < l(ws)$

$$\stackrel{(EP)}{\Rightarrow} tws = s_1 \dots \hat{s}_j \dots s_r s \quad \text{or} \quad s_1 \dots s_r$$

$$\downarrow$$

$$tw = s_1 \dots \hat{s}_j \dots s_r \quad \swarrow$$

$$\downarrow$$

$$tws = w$$

⑤ By using induction on r , we can assume

$$l(s_1 \dots s_{r-1}) = l(s_2 \dots s_r) = r-1. \quad \text{Obviously } l(s_1 \dots s_{r-1}) = r-2.$$

$$\text{Since } l(s_1 s_2 \dots s_{r-1} s_r) \neq l(s_2 \dots s_r) + 2, \quad s_1 \dots s_r = s_2 \dots s_{r-1}$$

⑥ By direct calculation.

⑦ Lemma. $\eta(w; t) = -1 \Leftrightarrow l(wt) < l(w)$

$$\eta(w; t) = 1 \Leftrightarrow l(wt) > l(w)$$

Proof. Suppose $l(w) = r$, $w = s_1 \dots s_r$.	
$\eta(w; t) = -1$	$\stackrel{\text{def}}{\Rightarrow} t = s_r \dots s_j \dots s_r \quad \exists j$
	$\Rightarrow wt = s_1 \dots \hat{s}_j \dots s_r$
	$\Rightarrow l(wt) < l(w)$
$\eta(w; t) = 1$	$\Rightarrow \eta(wt; t) = \eta(w; t) \eta(t; t) = -1$
	$\Rightarrow l(wt \cdot t) < l(wt)$

$$\text{So } l(wt) < l(w) \Rightarrow \eta(w; t) = -1 \stackrel{\text{def}}{\Rightarrow} wt = s_1 \dots \hat{s}_j \dots s_r \quad \exists j.$$

⑧ For (W, S) , define Coxeter gp (W_{cox}, S) s.t.

$$m(s, t) = \min \left(\left\{ r \in \mathbb{Z}_{\geq 0} \mid (st)^r = \text{Id in } W \right\} \cup \{+\infty\} \right)$$

then

$$W_{\text{cox}} = \langle s \in S \rangle / \left((st)^{m(s,t)} \quad s, t \in S \right)$$

and we have surj map

$$\Phi: W_{\text{cox}} \longrightarrow W$$

Need: injectivity.

⚠ Later, all elements s_i are in W_{cox} ,

$$l(s_1 \dots s_r) = l(\Phi(s_1 \dots s_r)) \stackrel{!}{=} l_{\text{cox}}(s_1 \dots s_r)$$

Don't know yet

and also

$$\Phi((st)^r) = \text{Id} \Leftrightarrow r(s, t) \mid r \Leftrightarrow (st)^r = \text{Id} \quad (\star)$$

Claim 1. $\forall r \in \mathbb{Z}_{\geq 0}$, if $\Phi(s_1 \dots s_{2r}) = \text{Id}$, then $s_1 \dots s_{2r} = \text{Id}$
 i.e. if $\Phi(s_1 \dots s_r) = \Phi(s'_1 \dots s'_r)$, then $s_1 \dots s_r = s'_1 \dots s'_r$

Proof of Claim 1.

Use induction on r . $r=0$ ✓

By induction, assume $s_i \neq s'_i$

By (DP) & induction, assume $s_1 \dots s_r, s'_1 \dots s'_r$ are reduced

$$l(s_1 s_i \dots s_r) = l(s'_1 \dots s'_r) = r-1 < r = l(s_1 \dots s_r)$$

$$\stackrel{(\text{EP})}{\Rightarrow} \Phi(s'_1 s_i \dots s_r) = \Phi(s_1 \dots \hat{s}_j \dots s_r) \quad \exists j$$

$$\Rightarrow \Phi(s'_1 s_i \dots s_{j-1}) = \Phi(s_1 \dots s_j)$$

If we show $s'_1 s_i \dots s_{j-1} = s_1 \dots s_j$, (For $j < r$, get by induction)

$$\Phi(s'_1 s_i \dots \hat{s}_j \dots s_r) = \Phi(s_1 \dots s_r) = \Phi(s'_1 \dots s'_r)$$

$$\Rightarrow \Phi(s_1 \dots \hat{s}_j \dots s_r) = \Phi(s'_1 \dots s'_r)$$

$$\stackrel{\text{induction}}{\Rightarrow} s_1 \dots \hat{s}_j \dots s_r = s'_1 \dots s'_r$$

$$\Rightarrow s_1 \dots s_j \dots s_r = s'_1 s'_i \dots s'_r$$

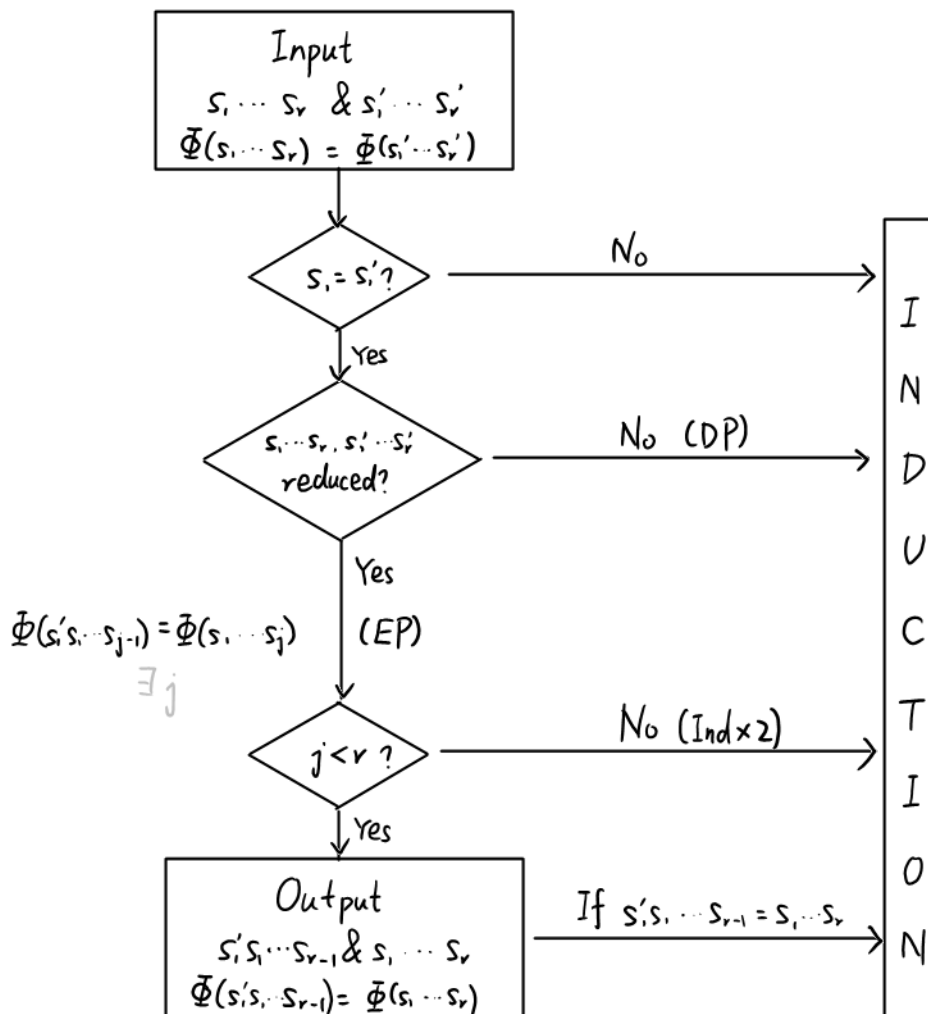
If $j=r$, we reduce the Claim 1 to Claim 2.

Claim 2. If $\Phi(s'_1 s_i \dots s_{r-1}) = \Phi(s_1 \dots s_r)$, then $s'_1 s_i \dots s_{r-1} = s_1 \dots s_r$.

Take this reduction repeatedly, we get reduced to

Claim 2. If $\Phi(s'_1 s'_1 s'_1 \dots) = \Phi(s_1 s_1 s_1 \dots)$, then $s'_1 s'_1 s'_1 \dots = s_1 s_1 s_1 \dots$

This is (\star) .



Rmk. The claim is stronger than we stated here.

By using the same method, we can show that

Prop. (Matsumoto) $\forall r \in \mathbb{Z}_{\geq 0}$, if

$$\Phi(s_1 \dots s_r) = \Phi(s'_1 \dots s'_r), \quad l(s_1 \dots s_r) = r, \quad \text{i.e. } \Phi(s_1) \dots \Phi(s_r) \text{ is reduced}$$

then $s_1 \dots s_r \sim s'_1 \dots s'_r$ by braid relations.

Q. Try to show ④ directly.

Possible idea: mimic the proof of faithfulness of geo rep.

Rmk. By (DP),

$$w = s_1 \dots s_r \text{ is reduced} \Leftrightarrow w = s_1 \dots s_r \text{ can't be shorter}$$

$$\text{i.e. } w = s_1 \dots \hat{s}_{i_1} \dots \hat{s}_{i_{r'}} \dots s_r \Rightarrow r' = 0$$

Ex. Translate all results in terms of models introduced in the last section.

3. Bruhat order.

Def (Bruhat order)

$(W, S) \in \text{Coxgp}$, $v, w \in W$, TFAE:

① \forall red exp $w = s_1 \dots s_r$, $\exists 1 \leq i_1 < \dots < i_r \leq r$ st
 $v = s_{i_1} \dots s_{i_r}$

② \exists red exp $w = s_1 \dots s_r$, $\exists 1 \leq i_1 < \dots < i_r \leq r$ st
 $v = s_{i_1} \dots s_{i_r}$

③ $v = y_0 \xrightarrow{t_1} y_1 \xrightarrow{t_2} \dots \xrightarrow{t_k} y_k = w$
 where $y \xrightarrow{t} y'$ iff $y' = yt$ $(\ell(y') > \ell(y))$ $y, y' \in W, t \in T$

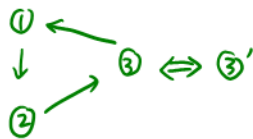
③' $v = y_0 \xrightarrow{t_1} y_1 \xrightarrow{t_2} \dots \xrightarrow{t_k} y_k = w$
 where $y \xrightarrow{t} y'$ iff $y' = ty$ $(\ell(y') > \ell(y))$ $y, y' \in W, t \in T$

④ $\overline{BvB} \subseteq \overline{BwB}$

(W, S) should be from the Tits system (G, B, N, S) .

Also, appropriate topology of G is needed.

Proof of



① \rightarrow ②: \checkmark

② \rightarrow ③: Idea: Write $v = s_1 \dots \hat{s}_{j_1} \dots \hat{s}_{j_k} \dots s_r$, then denote

$$t_k = s_r \dots s_{j_k} \dots s_r$$

$$t_{k-1} = s_r \dots s_{j_{k-1}} \dots s_r$$

\vdots

Problem: It is possible that $t_k v = s_1 \dots \hat{s}_{j_1} \dots \hat{s}_{j_{k-1}} \dots s_r$ is no longer reduced.

Solution: Write $v = s_1 \dots \hat{s}_{j_1} \dots \hat{s}_{j_k} \dots s_r$ where j_1 is minimal in all expressions.

③ \rightarrow ①: $\ell(s_1 \dots s_r t_k) < \ell(s_1 \dots s_r) = r$

$$\Rightarrow s_1 \dots s_r t_k = s_1 \dots \hat{s}_{j_k} \dots s_r$$

$$\Rightarrow t_k = s_r \dots s_{j_k} \dots s_r$$

Do that by induction.

③ \Leftrightarrow ③': Take inverse and notice that $\ell(w^{-1}) = \ell(w)$.

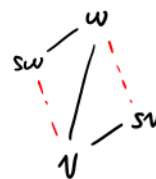
Cor. Bruhat order is really a partial order.

Prop. (Lifting property)

$(W, S) \in \text{Coxgp}$, $w, v \in W$, $s \in S$.

$v \leq w$, $l(sw) < l(w)$ $l(v) < l(sv)$

$\Rightarrow sv \leq w$, $v \leq sw$



Proof. For $sw = s_1 \dots s_r$ reduced, $w = ss_1 \dots s_r$.

$\Rightarrow v = s_{i_1} \dots s_{i_r}$

$\Rightarrow sv \leq w$, $v \leq sw$.

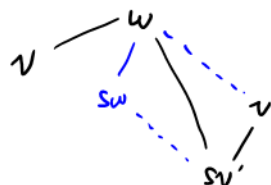
Prop. Bruhat order is directed, i.e.,

$\forall v, v' \in W$, $\exists w \in W$ s.t. $v, v' \leq w$.

Proof. Induction on $l(v) + l(v')$. For $v' \neq \text{Id}$, take $s \in S$ s.t. $l(sv') < l(v')$.



Case 1



Case 2

4. geometrical realization

Thm p_{geo} is faithful.

Proof.

Recall the root system.

$$\begin{array}{c} \nearrow GL(V_{\text{geo}}) \\ W \longrightarrow GL_R(V_{\text{geo}}) \longrightarrow \text{Aut}(R) \end{array}$$

$$GL_R(V_{\text{geo}}) = \{ \sigma \in GL(V_{\text{geo}}) \mid \sigma(R) = R \}$$

Lemma $(W, S) \in \text{Coxgp}$, $w \in W, s \in S$.

$$w \cdot \alpha_s \in R^- \iff l(ws) < l(w)$$

$$w \cdot \alpha_s \in R^+ \iff l(ws) > l(w)$$

Cor. $W \rightarrow \text{Aut}(R)$ is injective. As a result, p_{geo} is faithful.