

Review + Tutorial 11 & Ex 10

t: not required

Space - $\langle \cdot, \cdot \rangle \Rightarrow \|\cdot\| \Rightarrow d(\cdot, \cdot) \Rightarrow U$

- open/closed, $A^\circ, \bar{A}, A', \partial A, \dots$

- connectedness:

connected, path connected,
simply connected $\rightsquigarrow \pi_1$

- compactness

+ separation axioms

+ countability

- convergence, completeness

Cauchy sequence are only
defined for metric space / TVS
topological v.s.

Differential - derivation, one variable

- directional differential

- total differential

- mean value thm

- Taylor expansion

- criterion of Max/Min

- Lagrange multiplier

} Task 1

} Task 3

$cpt \Leftrightarrow$ net cpt $\xrightleftharpoons{\text{metric space}}$ sequentially cpt
 $cts \Leftrightarrow$ net cts \Longrightarrow sequentially cont

<https://math.stackexchange.com/questions/152447/compactness-sequentially-compact>

<https://math.stackexchange.com/questions/44907/whats-going-on-with-compact-implies-sequentially-compact>

<https://math.stackexchange.com/questions/360419/f-brings-convergent-nets-to-convergent-nets-is-it-continuous>

<https://math.stackexchange.com/questions/324842/nets-and-compactness-in-topological-spaces>

Integration:

Calculation: indefinite integral

- multiple variable integral
 - change of variables
 - path integration
 - integrate v. f. on surfaces
 - + Stokes' formula
- } Task 2

Theory: - measure

- Riemann integral
- Lebesgue integral
- + de Rham cohomology

This time, the answer of the "PräsentzBlatt" is also available for you.

Task 1.

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2 (x_1^2 - x_2^2)}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0, 0) \\ 0 & (x_1, x_2) = (0, 0) \end{cases}$$

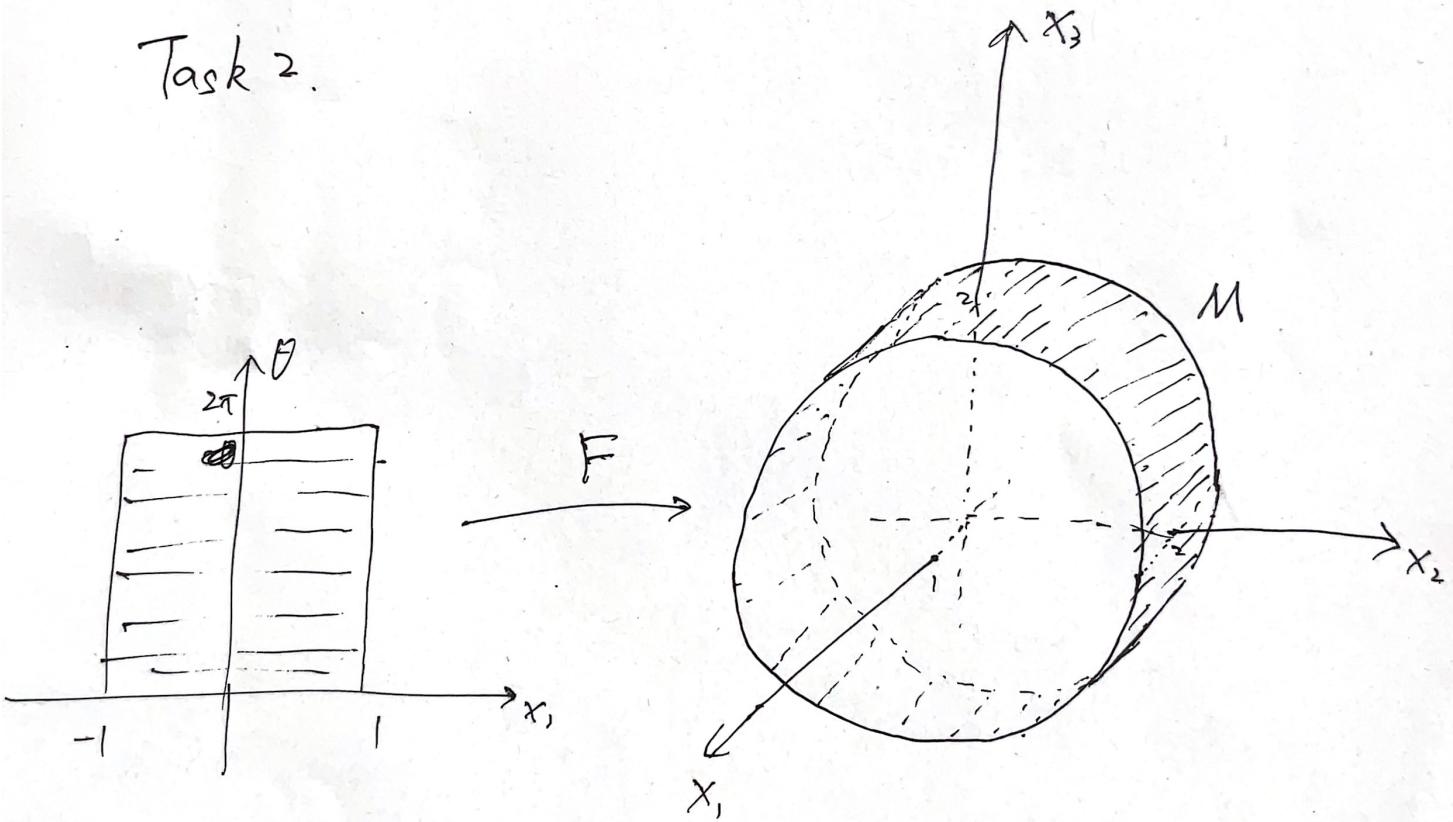
1) Compute $\partial_1 f$

2) Compute $\partial_2 \partial_1 f$

3) Show that $\partial_2 \partial_1 f \neq \partial_1 \partial_2 f$

4) Show that $\partial_1 \partial_1 f$ or $\partial_2 \partial_1 f$ is not cont.

Task 2.



$$F(x, \theta) = (x, 2 \cos \theta, 2 \sin \theta)$$

Compute

$$\int_M \vec{e}_3 \cdot d\vec{\sigma}, \quad \int_M \vec{e}_3 \cdot dA \quad \text{and} \quad \int_M \vec{e}_3 \cdot d\vec{\sigma}.$$

A. First, we compute dA & $d\vec{O}$.

$$\frac{\partial F}{\partial x} = (1, 0, 0)$$

$$\frac{\partial F}{\partial \theta} = (0, -2\sin\theta, 2\cos\theta)$$

$$\vec{O} = \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & -2\sin\theta & 2\cos\theta \\ i & j & k \end{vmatrix}$$

$$= \cancel{-2\cos\theta}j - 2\sin\theta \cdot k$$

$$= (0, -2\cos\theta, -2\sin\theta)$$

$$dA = \left| \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta} \right| dx d\theta = 2dx d\theta$$

$$d\vec{O} = \frac{\partial F}{\partial x} \times \frac{\partial F}{\partial \theta} dx d\theta = (0, -2\cos\theta, -2\sin\theta) dx d\theta$$

$$\begin{aligned} \int_M \vec{e}_3 dA &= \vec{e}_3 \int_M 1 dA \\ &= \vec{e}_3 \cdot \int_{-1}^1 \int_0^{2\pi} 2 d\theta dx \\ &= e_3 \cdot \int_{-1}^1 dx \cdot \int_0^{2\pi} 2 d\theta = 8\pi e_3 \end{aligned}$$

$$\begin{aligned} \int_M \vec{e}_3 \cdot d\vec{O} &= \int_0^{2\pi} \int_{-1}^1 \langle \vec{e}_3, (0, -2\cos\theta, -2\sin\theta) \rangle dx d\theta \\ &= \int_0^{2\pi} \int_{-1}^1 -2\sin\theta dx d\theta \\ &= \int_0^{2\pi} -2\sin\theta d\theta \cdot \int_{-1}^1 dx = 0 \end{aligned}$$

Task 3.

For $f(x, y) = x^2 + y^2$, the minimum is reached at $(0, 0)$.

Reason: $f(x, y) = x^2 + y^2 \geq 0 = f(0, 0)$

$(\Rightarrow \partial_x f(0, 0) = \partial_y f(0, 0) = 0)$

Q. What would happen when we restrict to the set

$$B = \{ (x, y) \in \mathbb{R}^2 \mid x - y = 1 \} ?$$

Bad news: We can no longer find possible max/mins by solving

$$\partial_x f = \partial_y f = 0$$

since in general, the solutions $z \notin B$

Good news: We can perturb f by

$$\tilde{f}(x, y) := f(x, y) + a(x - y - 1) \text{ for some } a \in \mathbb{R}$$

so that the solutions of the equations

$$\partial_x \tilde{f} = \partial_y \tilde{f} = 0$$

lie in B . These solutions are possible max/mins. of \tilde{f} .

Moreover, $\tilde{f}|_B = f|_B$, so

$$\min_{z \in B} \tilde{f}(z) = \min_{z \in B} f(z), \quad \max_{z \in B} \tilde{f}(z) = \max_{z \in B} f(z).$$

<https://math.stackexchange.com/questions/1682481/lagrange-multiplier-problem-why-doesnt-the-method-work>

<https://math.stackexchange.com/questions/4377669/why-does-the-method-of-lagrange-multipliers-fail-when-nabla-g-o>

Ex. Check that the following three steps work.

$$\begin{array}{l} \textcircled{1} \quad \left\{ \begin{array}{l} (\partial_x \tilde{f})(x,y) = 0 \\ (\partial_y \tilde{f})(x,y) = 0 \\ x - y = 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} 2x + a = 0 \\ 2y - a = 0 \\ x - y - 1 = 0 \end{array} \right. \\ \qquad \qquad \qquad \Leftrightarrow \left\{ \begin{array}{l} a = 1 \\ x = -\frac{1}{2} \\ y = \frac{1}{2} \end{array} \right. \end{array}$$

\textcircled{2} Let $a=1$, then

$$\textcircled{2} \quad \tilde{f}\left(-\frac{1}{2}, \frac{1}{2}\right) = \min_{z \in \mathbb{R}^2} \tilde{f}(z)$$

$$= \min_{z \in B} \tilde{f}(z)$$

$$= \min_{z \in B} \cancel{\tilde{f}(z)} f(z)$$

$$\textcircled{3} \quad \lim_{\substack{(x,y) \rightarrow \infty \\ (x,y) \in B}} f(x,y) = +\infty \quad \Rightarrow \text{no maximum.}$$

In conclusion, $f|_B$ has minimum $\frac{1}{2}$ at $(-\frac{1}{2}, \frac{1}{2})$,
 $f|_B$ has no maximum.

Ex. Use the GeoGebra (online, free) to draw the function

$$z = x^2 + y^2 - a(x - y - 1).$$

Ex. For $g(x, y) = 2x + y$, compute

$$\min_{(x,y) \in S'} g(x, y) \quad \max_{(x,y) \in S'} g(x, y)$$

where

$$S' = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

Hint: Let $\tilde{g}(x, y) = g(x, y) + \alpha(x^2 + y^2 - 1)$, then

$$\begin{cases} (\partial_x \tilde{g})(x, y) = 0 \\ (\partial_y \tilde{g})(x, y) = 0 \\ x^2 + y^2 = 1 \end{cases} \Leftrightarrow \begin{cases} 2 + 2\alpha x = 0 \\ 1 + 2\alpha y = 0 \\ x^2 + y^2 = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha = \frac{\sqrt{5}}{2} \\ x = -\frac{2\sqrt{5}}{5} \\ y = -\frac{\sqrt{5}}{5} \end{cases} \quad \text{or} \quad \begin{cases} \alpha = -\frac{\sqrt{5}}{2} \\ x = \frac{2\sqrt{5}}{5} \\ y = \frac{\sqrt{5}}{5} \end{cases}$$

Can check that

$$g\left(-\frac{2\sqrt{5}}{5}, -\frac{\sqrt{5}}{5}\right) = -\sqrt{5} \quad \text{minimum}$$

$$g\left(\frac{2\sqrt{5}}{5}, \frac{\sqrt{5}}{5}\right) = \sqrt{5} \quad \text{maximum.}$$

$$\therefore f = (x^2 + y^2)(4+1) \geq (2x+y)^2$$

Ex. Read wiki: Lagrange multiplier,
try to understand examples there.