

# Eine Woche, ein Beispiel

## 1.9. simplicial set

Ref:

[sSet] [http://www.math.uni-bonn.de/~schwede/sset\\_vs\\_spaces.pdf](http://www.math.uni-bonn.de/~schwede/sset_vs_spaces.pdf)

[6-Fctor] <https://people.mpim-bonn.mpg.de/scholz/SixFunctors.pdf>

Some visual pictures can be seen here:

<https://arxiv.org/pdf/0809.4221.pdf>

Today: The category  $sSet$  and  $\partial \Delta^n, \Delta_i^n, sk^m X, \Delta^n / \partial \Delta^n, Hom(X, Y) \in Ob(sSet)$

Def  $[n] = \{0, 1, \dots, n\} \quad n \geq 0$

The simplex category  $\Delta$  is defined by

$$Ob(\Delta) = \{[n] \mid n \geq 0\}$$

$$Mor_{\Delta}([m], [n]) = \{f: [m] \rightarrow [n] \text{ weakly increasing}\}$$

The category of simplicial sets  $sSet$  is defined by

$$sSet = Fun(\Delta^{op}, Set)$$

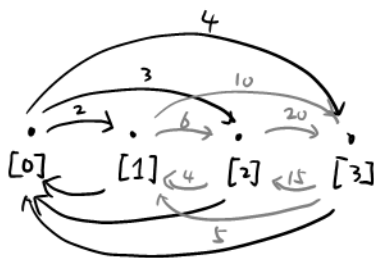
Notation in  $Mor(\Delta)$ :  $d_i^n: [n-1] \rightarrow [n]$  miss  $i$   $0 \leq i \leq n$

$s_i^n: [n] \rightarrow [n-1]$  contracts  $i$

$$\Delta \hookrightarrow sSet \quad [n] \mapsto \Delta^n = Mor_{\Delta}(-, [n])$$

$$\text{e.p. } \Delta_k^n = Mor_{\Delta}([k], [n])$$

$\uparrow$  read from down to top



$\# \Delta_k^n \begin{smallmatrix} n \\ k \end{smallmatrix}$	0	1	2	3
0	1	2	3	4
1	1	3	6	10
2	1	4	10	20
3	1	5	15	35

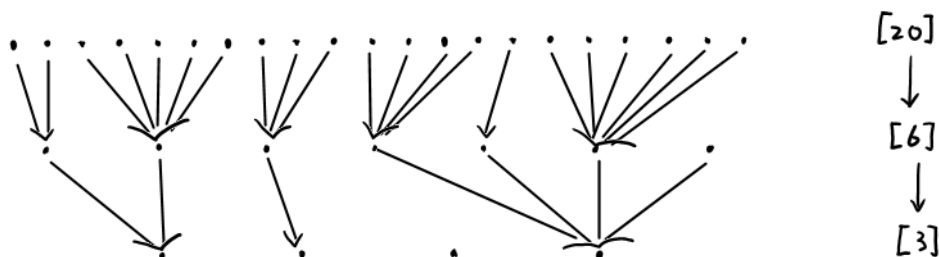
$$\# \Delta_k^n = \binom{n+k+1}{n}$$

element	picture	list	count	other notations
$d: [5] \rightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		$(0, 0, 1, 3, 3, 3)$	$[2, 1, 0, 3]$	
$d_1^3: [2] \rightarrow [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		$(0, 2, 3)$	$[1, 0, 1, 1]$	$d_i^n: [n-1] \rightarrow [n]$ $\delta^n$
$s_1^3: [3] \rightarrow [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		$(0, 1, 1, 2)$	$[1, 2, 1]$	$s_i^n: [n] \rightarrow [n-1]$
$d_{3,2}: [3] \rightarrow [5]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$		$(0, 1, 2, 3)$	$[1, 1, 1, 1, 0, 0]$	$d_{i,j}: [i] \rightarrow [i+j]$ $\delta_i^f$ $f = \text{front}$
$d'_{3,2}: [2] \rightarrow [5]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$		$(3, 4, 5)$	$[0, 0, 0, 1, 1, 1]$	$d'_{i,j}: [j] \rightarrow [i+j]$ $\delta_i^b$ $b = \text{back}$
$\delta_{3,(5,4)}^{\text{out}}: [5] \rightarrow [8]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$ $4 \mapsto 7$ $5 \mapsto 8$		$(0, 1, 2, 3, 7, 8)$	$[1, 1, 1, 1, 0, 0, 0, 1, 1]$	$\delta_{i,(p,q)}^{\text{out}}: [p] \rightarrow [p+q-1]$ $\delta_i^{\text{out}}$
$\delta_{3,(5,4)}^{\text{in}}: [4] \rightarrow [8]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$ $3 \mapsto 6$ $4 \mapsto 7$		$(3, 4, 5, 6, 7)$	$[0, 0, 0, 1, 1, 1, 1, 1, 0]$	$\delta_{i,(p,q)}^{\text{in}}: [q] \rightarrow [p+q-1]$ $\delta_i^{\text{in}}$

Table 1. Morphisms in  $\Delta$ .

How to compute the composition?

e.g.  $[2, 1, 0, 4] \circ [\underline{2}, \underline{5}, \underline{3}, \underline{4}, \underline{1}, \underline{6}, \underline{0}] = [7, 3, 0, 11]$



$$[1, 2, 1] \circ [\underline{2}, \underline{1}, \underline{0}, \underline{3}] = [2, 1, 3]$$

$$[4] \circ [\underline{2}, \underline{1}, \underline{0}, \underline{3}] = [6]$$

$$s_i^3 \circ d_i^3 = [1, 2, 1] \circ [\underline{1}, \underline{0}, \underline{1}, \underline{1}] = [1, 1, 1]$$

$$d_i^3 \circ s_i^3 = [1, 0, 1, 1] \circ [\underline{1}, \underline{2}, \underline{1}] = [1, 0, 2, 1]$$

Rmk. In  $\Delta$  we don't have finite colimit, while in  $sSet = Fun(\Delta^{op}, Set)$  we have finite colimit because  $Set$  is (complete +) cocomplete.

For a construction, see

<https://math.stackexchange.com/questions/3837844/limits-and-colimits-are-computed-pointwise-in-functor-categories>

Notice that

$$\partial \Delta^n, \Delta_i^n, sk^m \Delta^n, \Delta^n / \partial \Delta^n \in sSet - \Delta$$

Conclusion:  $sSet$  is a Grothendieck topos.

It is Cartesian closed, complete and cocomplete.

In  $sSet$ , we can glue objects ( $\approx$  pushforward), which is impossible in  $\Delta$ .

Slogan:  $sSet \sim$  simplicial complex

$X_n \sim$  the index set of  $n$ -dim cells

Rmk. ([sSet]) If you have strong enough geometrical background, you will find out the adjoint pair

$$sSet \begin{matrix} \xrightarrow{|-|} \\ \xleftarrow{S} \end{matrix} Top$$

quite useful, where

$$|X| := \left( \coprod_{n \geq 0} X_n \times \nabla^n \right) / \sim$$

$$S(A)_n := \text{Mor}_{Top}(\nabla^n, A)$$

$$\alpha^* : S(A)_n \longrightarrow S(A)_m \quad x \longmapsto x \circ S(\alpha)$$

$$\alpha : [m] \rightarrow [n] \\ S(\alpha) : \nabla^m \rightarrow \nabla^n$$

Moreover, we have eqv/c of categories

$$sSet[weq^{-1}] \begin{matrix} \xrightarrow{|-|} \\ \xleftarrow{S} \end{matrix} \begin{matrix} Ho(Top_{CW}) \\ \downarrow \\ Top[weq^{-1}] \end{matrix}$$

where

$Top_{CW}$  is the full subcategory of  $Top$  with objects the top spaces admitting a CW-cplx structure, and

$Ho(Top_{CW})$  is the homotopy category of CW cplxes.

Q: Do we have the following comm diag. (as eqv/c of categories)

$$\begin{array}{ccc} sSet[weq^{-1}] & \xrightarrow{|-|} & Ho(Top_{CW}) \\ \uparrow & & \downarrow \\ An & \xleftarrow{S} & Top[weq^{-1}] \end{array}$$

Q: For  $\mathcal{C} \in Cat_{\infty} \subseteq sSet$ , how to view  $\mathcal{C}$  as a topo space?  
e.p. compute  $\pi_n(\mathcal{C})$ ?

Roughly, we have three ways to define/determine a simplicial set:

1. By writing down their def directly; brutal force
2. By universal property (pullback, pushforward, ...) abstract construction
3. By its geometrical realization name

Let us see how they're compatible with each other.

E.g.1. For  $A \in \text{Top}$  discrete, define  $X = \mathcal{S}(A)$ , i.e.,  
 $X_n = A$   $\quad \quad \quad \partial^* = \text{Id}_A$   $\quad \quad \quad \forall \alpha: [m] \rightarrow [n]$   
 $|\mathcal{S}(A)| = (\coprod_k X_k \times \nabla^k) / \sim$   
 $\sim A \times \nabla^0$   
 $\sim A$

E.g.2.  $\Delta_k^n = \text{Mor}_\Delta([k], [n]) = \{x: [k] \rightarrow [n] \text{ weakly increasing}\}$   
 $|\Delta^n| = (\coprod_k \Delta_k^n \times \nabla^k) / \sim$   
 $\sim (\Delta^n \times \nabla^n) / \sim$   
 $\sim \nabla^n$

E.g.3.  $\Delta_{(i)}^{n-1} := \text{Im}(d_i^n: \Delta^{n-1} \rightarrow \Delta^n)$  in  $\text{sSet}$

$\Rightarrow (\Delta_{(i)}^{n-1})_k = \{x \in \Delta_k^n \mid \exists y \in \Delta_k^{n-1} \text{ s.t. } x = d_i^n \circ y\}$

$|\Delta_{(i)}^{n-1}| = (\coprod_k (\Delta_{(i)}^{n-1})_k \times \nabla^k) / \sim$

$\sim ((\Delta_{(i)}^{n-1})_{n-1}^{\text{nondeg}} \times \nabla^{n-1}) / \sim$

$\sim (\text{Sd}_i^n(\nabla^{n-1}))$

Denote  $|\Delta_{(i)}^{n-1}|$  by  $\nabla_{(i)}^{n-1}$ , i.e.  $\nabla_{(i)}^{n-1} := \text{Im}(\text{Sd}_i^n: \nabla^{n-1} \rightarrow \nabla^n)$

E.g.4.  $(\partial \Delta^n)_k = \{x \in \Delta_k^n \mid x \text{ is not surj}\}$   
 $\partial \Delta^n = \bigcup_{i=0}^n \Delta_{(i)}^{n-1} = \text{colimit of } \dots$

e.g.  $\partial \Delta^2 = \left[ \text{colimit of } \begin{array}{ccc} \Delta^0 & \Delta^0 & \Delta^0 \\ \downarrow & \searrow & \swarrow \\ \Delta^1 & \Delta^1 & \Delta^1 \end{array} \right]$

$|\partial \Delta^n| = (\coprod_k (\partial \Delta^n)_k \times \nabla^k) / \sim$

$\sim (\text{Mor}_\Delta^{\text{nondeg}}([n-1], [n]) \times \nabla^{n-1}) / \sim$

$\sim (\coprod_{i=0}^n \text{Sd}_i^n(\nabla^{n-1})) / \sim$

$= \bigcup_{i=0}^n \nabla_{(i)}^{n-1}$

$= \partial \nabla^n$

Eg. 5.  $(\Delta_i^n)_k = \{ x \in \Delta_k^n \mid x = \alpha^*(y) \text{ for some } y \in \Delta_{n-1}^n \text{ and } \alpha: [k] \rightarrow [n-1] \}$   
 $y \neq d_i^n$

$$\Delta_i^n = \bigcup_{j \neq i} \Delta_{(j)}^{n-1} = \text{colimit of } \dots$$

e.g.  $\Delta_i^2 = \left[ \text{colimit of } \begin{array}{ccc} & \Delta^0 & \\ & \downarrow d_i' & \\ \Delta^0 & \xrightarrow{d_i'} & \Delta^1 \end{array} \right]$   
 $= \Delta^1 \sqcup_{\Delta^0} \Delta^1$

ex. write down  $(X \sqcup_Y Z)_k$  for  $X, Y, Z \in \mathbf{sSet}$

$$\begin{aligned} |\Delta_i^n| &= \left( \bigsqcup_k (\Delta_i^n)_k \times \nabla^k \right) / \sim \\ &\sim ((\Delta_i^n)_{n-1}^{\text{nondeg}} \times \nabla^{n-1}) / \sim \\ &\sim \left( \bigsqcup_{j \neq i} (Sd_j^n)(\nabla^{n-1}) \right) / \sim \\ &\sim \bigcup_{j \neq i} \nabla_{(j)}^{n-1} \end{aligned}$$

Eg. 6.  $(sk^m \Delta^n)_k = \{ x \in \Delta_k^n \mid x = \alpha^*(y) \text{ for some } y \in \Delta_m^n \text{ and } \alpha: [k] \rightarrow [m] \}$

$$sk^m \Delta^n = \bigcup_{\beta: [m] \rightarrow [n]} \beta(\Delta^m) = \text{colimit of } \dots$$

$$\begin{aligned} |sk^m \Delta^n| &= \left( \bigsqcup_k (sk^m \Delta^n)_k \times \nabla^k \right) / \sim \\ &\sim ((sk^m \Delta^n)_m^{\text{nondeg}} \times \nabla^m) / \sim \\ &\sim (Mor^{\text{nondeg}}([m], [n]) \times \nabla^m) / \sim \\ &\sim \bigcup_{\beta: [m] \rightarrow [n]} (S\beta)(\nabla^m) \end{aligned}$$

Eg. 7.  $(\Delta^n / \partial \Delta^n)_k = \Delta_k^n / (\partial \Delta^n)_k = \Delta_k^n / \sim$

$$x \sim y \Leftrightarrow x, y \in (\partial \Delta^n)_k \text{ or } x = y$$

Universal property:

$$\begin{array}{ccccccc} \partial \Delta^n & \longrightarrow & \Delta^n & \longrightarrow & \Delta^n / \partial \Delta^n & \longrightarrow & 0 \\ & \searrow \text{contract to} & \searrow & & \downarrow \exists! & & \\ & \text{one pt} & & & X & & \end{array}$$

$$\begin{aligned}
|\Delta^n / \partial \Delta^n| &= (\coprod_k (\Delta^n / \partial \Delta^n)_k \times \nabla^k) / \sim \\
&\sim ((\Delta^n / \partial \Delta^n)^{\text{nondeg}}_n \times \nabla^n) / \sim \\
&\sim \nabla^n / \sim \\
&\sim \nabla^n / \partial \nabla^n
\end{aligned}$$

Eq. 8. Define  $X = \left[ \text{colimit of } \Delta^1 \begin{matrix} \xrightarrow{d_1^1} \\ \xleftarrow{-d_1^1} \\ \xrightarrow{d_0^1} \end{matrix} \Delta^2 \right]$

$$\begin{aligned}
X_k &= \Delta_k^2 / \sim \quad \text{here we identify } d_2^2 x = -d_1^2 x = d_0^2 x \\
|X| &= (\coprod_k X_k \times \nabla^k) / \sim \\
&\sim (X_2^{\text{nondeg}} \times \nabla^2) / \sim \\
&\sim \triangle_2
\end{aligned}$$

Similarly, one can consider  $\Delta^2 \cup_{\partial \Delta^2} \Delta^2 \cong S^2$



Ex. Shows that

$\partial \Delta^3$ ,  $\Delta^2 / \partial \Delta^2$ ,  $\Delta^2 \cup_{\partial \Delta^2} \Delta^2$   
are homotopy equivalent as simplicial sets.



Eg. 9  $(\text{Hom}(X, Y))_n = \text{Hom}_{\text{sSet}}(\Delta^n \times X, Y)$

$\alpha^*: \text{Hom}_{\text{sSet}}(\Delta^n \times X, Y) \longrightarrow \text{Hom}_{\text{sSet}}(\Delta^m \times X, Y)$  for  $\alpha: [m] \rightarrow [n]$   
 $\alpha: \Delta^m \rightarrow \Delta^n$

$$\text{sSet} \begin{array}{c} \xrightarrow{- \times X} \\ \text{\textcolor{red}{\perp}} \\ \xleftarrow{\text{Hom}(X, -)} \end{array} \text{sSet}$$

"Proof"  $\text{Hom}_{\text{sSet}}(Z, \text{Hom}(X, Y)) \cong \left[ \begin{array}{l} \{ g_m: Z_m \longrightarrow \text{Hom}_{\text{sSet}}(\Delta^m \times X, Y) \} \\ + \dots \\ \{ h_{m,k}: Z_m \times \Delta_k^m \times X_k \longrightarrow Y_k \} \\ + \dots \\ \{ h_k: Z_k \times X_k \longrightarrow Y_k \} \\ + \dots \\ \cong \text{Hom}_{\text{sSet}}(Z \times X, Y) \end{array} \right]$

Cor.  $\text{Hom}(Z \times X, Y) \cong \text{Hom}(Z, \text{Hom}(X, Y))$

e.g.  $\text{Hom}(\Delta^0, Y) \cong Y$   $(\text{Hom}(\Delta^n, Y))_m \cong (\text{Hom}(\Delta^m, Y))_n$   
 $\text{Hom}(X, \Delta^0) \cong \Delta^0$  e.p.  $(\text{Hom}(\Delta^n, Y))_0 \cong Y_n$

$$|\text{Hom}(X, Y)| = \left( \coprod_k \text{Hom}_{\text{sSet}}(\Delta^k \times X, Y) \times \nabla^k \right) / \sim$$

$$= ?$$

Remaining: Compute  $\# (\text{Hom}(\Delta^n, \Delta^m))_k$   
 Compute  $(\text{Hom}(\Delta^n / \Delta^n, Y))_k$ . How is it related to  $Y_{k+n}$  or  $\pi_n(|Y|)$ ?  
 How to see the geometrical realization of  $\text{Hom}(X, Y)$ ,  
 e.p. in these examples?

Eg. 10. Let  $X$  be a subset of  $\Delta$  whose realization is as follows.  
Write down  $X_k$  for  $k \leq 3$ .

$$\text{eg. } X_1 = \left\{ \begin{array}{l} [2, 0, 0, 0], [0, 2, 0, 0], [0, 0, 2, 0], [0, 0, 0, 2], \\ [1, 1, 0, 0], [1, 0, 1, 0], [1, 0, 0, 1], [0, 1, 0, 1], [0, 0, 1, 1] \end{array} \right\}$$



Eg. 11. BG

Eg. 12.

Realize Hochschild homology as simplicial homology:  
<https://arxiv.org/pdf/1802.03076.pdf>