Session 5 & Exercise 3

Today we try to catch up with the course.

Namely, we need to introduce

- Topological space
- Properties
  - Continuity and Convergence
  - Connectness
  - Compactness
  - (Axioms of countability, Seperation Axioms, ...)
- Integral: computation ~~a~~ and definition.

For a ~~detailed~~ deep understanding of topo space and its properties, see Prof. Zuoqin Wang's notes.

http://staff.ustc.edu.cn/~wangzuoq/Courses/19S-Topology/index.html

1. Topological space

Recall that

$$\langle -,- \rangle \Rightarrow \|\cdot\| \Rightarrow d(-,-) \Rightarrow \text{topology } \mathcal{T}$$

Roughly speaking, a topology tells us which subset is an open subset.

Def. X: a set. A <u>topology</u> on X is a set

$$\mathcal{T} \subset \{\text{subsets of } X\}$$

s.t.

1) $\emptyset, X \in \mathcal{T}$
2) If $A_\alpha \in \mathcal{T}$ $(\forall \alpha)$, then $\bigcup_\alpha A_\alpha \in \mathcal{T}$
3) If $A_1, A_2 \in \mathcal{T}$, then $A_1 \cap A_2 \in \mathcal{T}$

$(X, \mathcal{T})$ is called a topological space (we will omit $\mathcal{T}$).

$A \in \mathcal{T}$ is called an open set.

E.g. (X, d) a metric space. Then

$$\mathcal{T}_d := \{A \subseteq X \mid \forall a \in A,\ \exists r > 0 \text{ s.t. } B_r(a) \subseteq A\}$$

is a topology on X, called the topology induced by the metric.

Ex. For $X \in \text{Set}$, check that

$$\mathcal{T}_{disc} := \{A \subseteq X\} \quad \text{(e.g. } (\mathbb{Z}, d) \text{ induce disc topo)}$$

$$\mathcal{T}_{trivial} := \{\emptyset, X\}$$

are two topologies on X.

Ex 0.1. For (X,d) a metric space, where

$$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

find all the open balls.

Verify that (X,d) has discrete topology.

E.g. Let $(X,\mathcal{T})$ be a topological space, $Y \subseteq X$ be any subset.

Then the collection

$$\mathcal{T}_Y := \{\mathcal{U} \cap Y \subset Y \mid \mathcal{U} \in \mathcal{T}\}$$

forms a topology on Y.

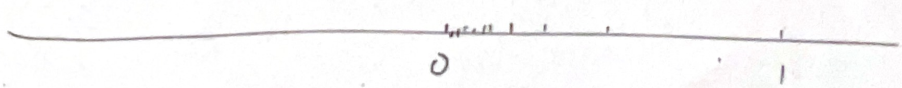
In ptc,

$\mathcal{U}' \subset Y$ is open $\Leftrightarrow \exists \mathcal{U} \subset X$ open s.t. $\mathcal{U} \cap Y = \mathcal{U}'$.

E.x. For $Y_1 := \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}_{>0}\} \subseteq \mathbb{R}$

$Y_2 := \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}_{>0}\} \cup \{0\} \subseteq \mathbb{R}$

Verify that Y_1 is discrete, while Y_2 is not.



2. Continuity and convergence

Def. Let X, Y are two topo spaces.

We say $f: X \to Y$ is <u>continuous</u>, if

$$\forall \mathcal{U} \subseteq Y \text{ open}, \quad f^{-1}(\mathcal{U}) \subseteq X \text{ is open}.$$

Rmk. For (X, d_X), (Y, d_Y) two metric spaces,

we've defined the continuous map.

$$f: X \to Y \quad \text{is cont, if}$$

$$\forall x_0 \in X \quad \forall \varepsilon > 0, \ \exists \delta > 0 \text{ s.t. } f(B_\delta(x_0)) \subseteq B_\varepsilon(f(x_0))$$

i.e.,

$\forall x_0 \in X, \ \forall \varepsilon > 0, \ \exists \delta > 0$ s.t.

$$\forall x \in B_\delta(x_0), \qquad d_Y(f(x), f(x_0)) < \varepsilon$$

In case $X = \mathbb{R}$, $Y = \mathbb{R}$, this is equiv to

$\forall x_0 \in \mathbb{R} \quad \forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$$\forall x \in (x_0 - \delta, x_0 + \delta), \quad |f(x) - f(x_0)| < \varepsilon$$

All the def of continuous are compatible.

We also have the concept of convergence.

Def. Let (X, d_X), (Y, d_Y) be two metric spaces, $x_0 \in X$ not isolated, $a \in Y$.

$f: X \setminus \{x_0\} \longrightarrow Y$.

We write $\lim_{x \to x_0} f(x) = a$, if

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $f(B_\delta(\check{x}_0)) \subseteq B_\varepsilon(a)$

Ex. 3.4. For $\alpha \in (0, +\infty)$ define

$$f_\alpha : \mathbb{R}^2 \longrightarrow \mathbb{R} \qquad f_\alpha(x,y) = \begin{cases} \dfrac{|xy|^\alpha}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

(a) Compute $\lim_{t \to 0} f_\alpha(t,t)$ $\qquad \lim_{t \to 0} f_\alpha(t, 2t)$

$$= \lim_{t \to 0} \frac{t^{2\alpha}}{2t^2} \qquad = \lim_{t \to 0} \frac{2^\alpha t^{2\alpha}}{5t^2}$$

$$= \lim_{t \to 0} \frac{1}{2} t^{2\alpha - 2} \qquad = \lim_{t \to 0} \frac{2^\alpha}{5} t^{2\alpha - 2}$$

$$= \begin{cases} 0 & \alpha > 1 \\ \frac{1}{2} & \alpha = 1 \\ +\infty & \alpha < 1 \end{cases} \qquad = \begin{cases} 0 & \alpha > 1 \\ \frac{2}{5} & \alpha = 1 \\ +\infty & \alpha < 1 \end{cases}$$

(c) Find $\{\alpha \in (0, +\infty) \mid f_\alpha(x,y) \text{ is cont at } (0,0)\}$

A: $(1, +\infty)$

Need: $\lim_{(x,y) \to (0,0)} \dfrac{|xy|^\alpha}{x^2+y^2} = 0$ when $\alpha > 1$

Hint: $|xy| \leq \frac{1}{2}(x^2+y^2)$ + squeeze Theorem (sandwich Thm)

Try: use any software to draw the picture of f_α.

Rmk. A number sequence $\{x_n\}_{n\in\mathbb{N}_{>0}}$ can be viewed as a fct

$$\phi : \mathbb{N}_{>0} \longrightarrow \mathbb{R} \qquad n \longmapsto x_n$$

When compute $\lim\limits_{n\to\infty} x_n$, we view $\mathbb{N}_{>0} \subseteq \underbrace{\mathbb{N}_{>0} \cup \{\infty\}}_{\text{same topo as } Y_2}$

When we write $\lim\limits_{n\to\infty} x_n = +\infty$ or $-\infty$, we view $\mathbb{R} \subset \mathbb{R} \sqcup \{\pm\infty\}$

When we write $\lim\limits_{n\to\infty} x_n = \infty$, we view $\mathbb{R} \subset \mathbb{R} \sqcup \{\infty\}$

$\mathbb{R} \sqcup \{\pm\infty\}$ $\qquad$ $\mathbb{R} \sqcup \{\infty\}$

Ex. 3.3 For a number sequence $\{x_n\}_{n\in\mathbb{N}_{>0}}$, define

$$M = \{x_n \in \mathbb{R} \mid n \in \mathbb{N}\}$$

(a) Find a bounded seq $\{x_n\}_n$ s.t.

$$\overline{\lim_{n\to\infty}}\, x_n \notin M', \quad \underline{\lim_{n\to\infty}}\, x_n \in M'$$

(b) ______________________________

$$\overline{\lim_{n\to\infty}}\, x_n = \underline{\lim_{n\to\infty}}\, x_n \notin M'$$

A. (a) $x_n = \frac{1}{n}$ $\qquad$ (b) $x_n \equiv 0$

Hint: $\overline{\lim\limits_{n\to\infty}}\, x_n := \lim\limits_{n\to\infty} \left(\sup\limits_{m\geq n} \{x_m\}\right)$

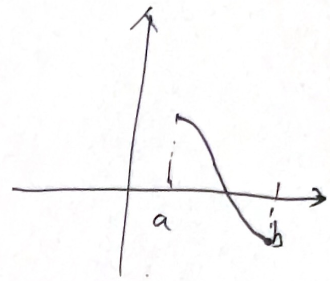
$\underline{\lim\limits_{n\to\infty}}\, x_n := \lim\limits_{n\to\infty} \left(\inf\limits_{m\geq n} \{x_m\}\right)$

This hint will be used for Ex 4.3.

To motivate the path connectness, we state the mean value theorem.

Thm. Let $f: [a,b] \longrightarrow \mathbb{R}$ be a cont fct with $f(a)f(b) \leq 0$, then

$\exists\, c \in [a,b]$ s.t. $f(c) = 0$



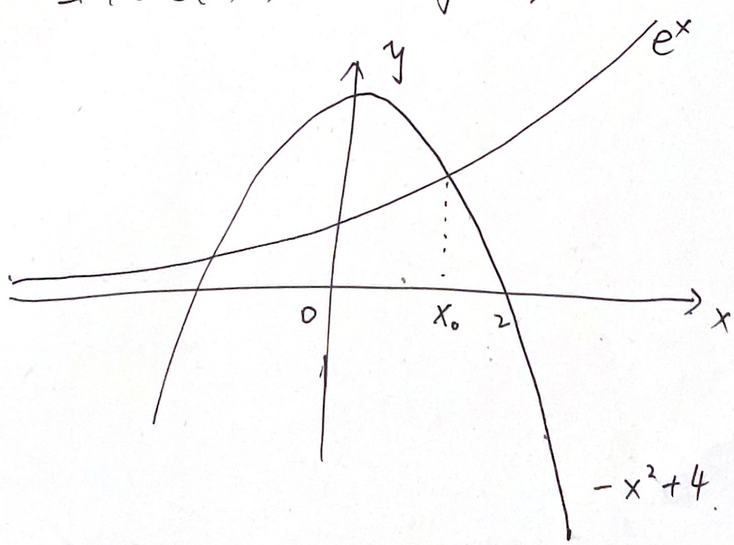
This will be used in Ex 5.2

E.g. the equation $e^x = -x^2 + 4$ has one solution in $(0,2)$.

A: $f(x) := e^x - (-x^2+4)$ satisfies (f cont)

$f(0) = 1 - 4 = -3 < 0 \qquad f(2) = e^2 - 0 > 0$

$\Rightarrow \exists\, x_0 \in (0,2)$ s.t. $f(x_0) = 0$



3. Connectness

(Connectness)

Def. $(X, \mathcal{T})$ topo space. X is <u>connected</u>, if

$$\{U \in \mathcal{T} \mid X \setminus U \in \mathcal{T}\} = \{\emptyset, X\}$$

(except for $\emptyset$ and X, there are no closed and open subsets)

This is equiv to

$$\boxed{X = U_1 \sqcup U_2 \text{ open} \Rightarrow U_1 = \emptyset \text{ or } X.}$$

Def. (Path connectness)

$(X, \mathcal{T})$ topo space. X is <u>path connected</u>, if

$\forall x, y \in X$, $\exists$ cont. map $f : [0,1] \to X$ s.t.

$$f(0) = x \quad f(1) = y.$$

Rmk. path connected $\underset{\text{loc path connected}}{\overset{\nleftarrow}{\Longrightarrow}}$ connected

Ex 5.3 for a counterexample.

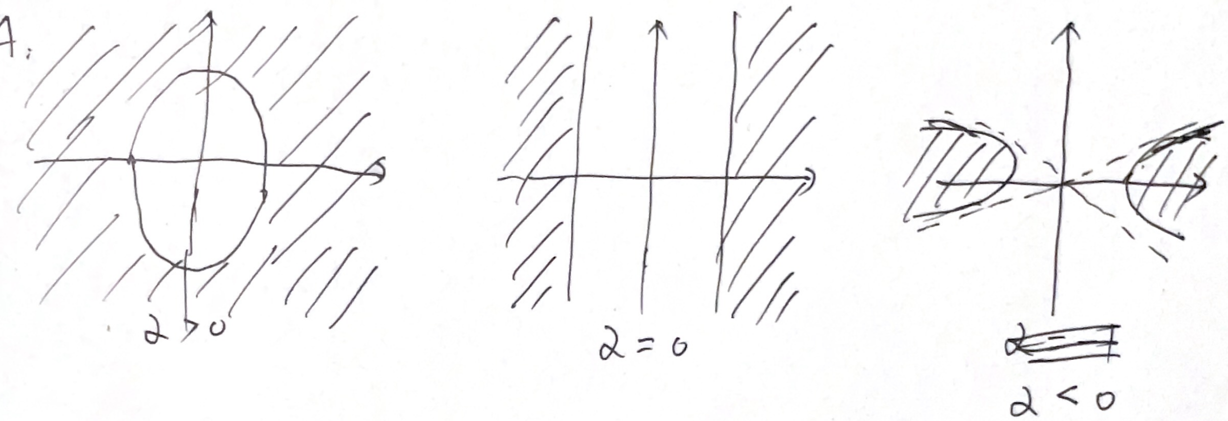
See 1878410 & 4104271 for more information.

Task 1. For $\alpha \in \mathbb{R}$, define

$$E_\alpha = \{(x,y) \in \mathbb{R}^2 \mid x^2 + \alpha y^2 = 1\}$$
$$M_\alpha = \{(x,y) \in \mathbb{R}^2 \mid x^2 + \alpha y^2 > 1\}$$

draw E_α, M_α and decide when they are (path) connected.

A:



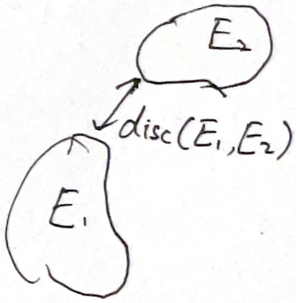
$$\{\alpha \in \mathbb{R} \mid E_\alpha \text{ is connected}\} = (0, +\infty)$$
M_α (path)

Ex 5.1.

Task 2. (game of distance)

For $E_1, E_2 \subseteq \mathbb{R}^2$, define

$$\operatorname{disc}(E_1, E_2) := \inf_{x \in E_1, y \in E_2} |x - y|$$



Find $E_1, E_2 \subseteq \mathbb{R}^2$ s.t.

$$\operatorname{disc}(E_1, E_2) = 0 \quad \text{and} \quad E_1 \cap E_2 = \emptyset$$

Moreover, we can require one of the following properties.

- $E_1, E_2 \subseteq \mathbb{R}^2$ open
- $E_1, E_2 \subseteq \mathbb{R}^2$ closed
- E_1 cpt, E_2 bounded

(If E_1, E_2 are cpt, $E_1 \cap E_2 = \emptyset$, then $\operatorname{disc}(E_1, E_2) > 0$.)

4. Compactness

Let $(X, \mathcal{T})$ be a topo space.

<u>Def.</u> A subset $\{\mathcal{U}_\alpha\}_{\alpha\in\Lambda} \subseteq \mathcal{T}$ is called an open cover of X. if

$$\bigcup_{\alpha\in\Lambda} \mathcal{U}_\alpha = X$$

For $I \subseteq \Lambda$ finite, $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ is called a finite subcover, if

$$\bigcup_{\alpha\in I} \mathcal{U}_\alpha = X$$

X is called <u>compact</u>, if any open cover of X has a finite ~~subo~~ subcover.

Rmk. For metric space, X is cpt $\Leftrightarrow$ X is sequence cpt.

E.g. $[0,1]$ is cpt, $(0,1)$ is not cpt, $[0,+\infty)$ is not cpt.

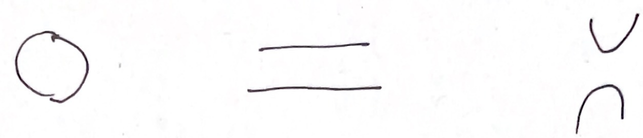
Ex. show that $(0,1)$ is not cpt.

Rmk. For $X \subseteq \mathbb{R}^n$,

$$X \text{ is cpt} \Leftrightarrow X \text{ is bounded \& closed.}$$

This is not true for other spaces. (4.4.9.7)

Ex 3.2.



	$\alpha > 0$	$\alpha = 0$	$\alpha < 0$
bounded	✓	×	×
closed	✓	✓	✓
cpt	✓	×	×

Verify one case — show: not seq compact.

Ex. 3.1. $M := \{x \in l^2 \mid \|x\| \leq 1\} \subseteq l^2$

Show that M is bounded & closed, but not cpt.

bounded: $\forall x \in M$, $\|x\| \leq 1$

closed: $\forall x_0 \notin M$, $\exists r := \frac{\|x_0\| - 1}{2} > 0$ s.t. $B_r(x_0) \subseteq M^c$

(check $B_r(x_0) \subseteq M^c$!)

not cpt: equiv to show M is not seq cpt.

Take $\{e_n\}_{n=1}^{\infty} \subseteq l^2$, where

$$e_i = (0, 0, \cdots, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \cdots)$$

For any subsequence $\{e_{n_k}\}_{k=1}^{\infty} \subseteq l^2$, need to show

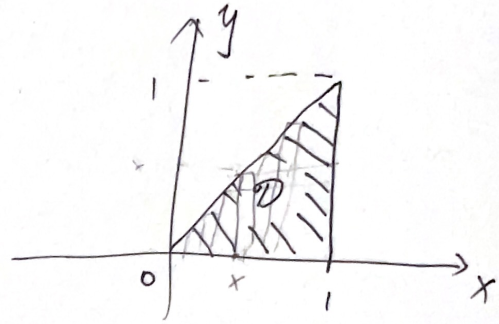
$\{e_{k_n}\}_{\cancel{k}n=1}^{\infty} \subseteq l^2$ is not a Cauchy seq.

Integral.

Compute multiple integral is not as hard as you think!

Ex. Compute

$$\int_{\mathcal{D}} xy\,dx\,dy \quad (\star)$$



Idea: cut it into some pieces

$$\star = \int_0^1 \left(\int_0^x xy\,dy\right) dx$$

$$= \int_0^1 \left(x \int_0^x y\,dy\right) dx$$

$$= \int_0^1 x \left(\tfrac{1}{2} y^2\right)\Big|_{y=0}^{x} dx$$

$$= \int_0^1 x \cdot \tfrac{1}{2} x^2\,dx$$

$$= \frac{1}{2} \int_0^1 x^3\,dx$$

$$= \frac{1}{8}$$

Ex. compute

$$V(\mathcal{D}) := \int_{\mathcal{D}} dx\,dy$$

Ex. 5.4.

Task 3.

Define

$$\chi_{\mathbb{Q}} : \mathbb{R} \longrightarrow \mathbb{R} \qquad \chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

a). Show that $\chi_{\mathbb{Q}}$ is not Riemann-integrable over $[0,1]$.

b). Find all $[a,b]$ s.t. $\chi_{\mathbb{Q}}(x)$ is Riemann-integrable over $[a,b]$.

Compute

$$\int_{\mathbb{Z}} \chi_{\mathbb{Q}}(x)\, dx.$$

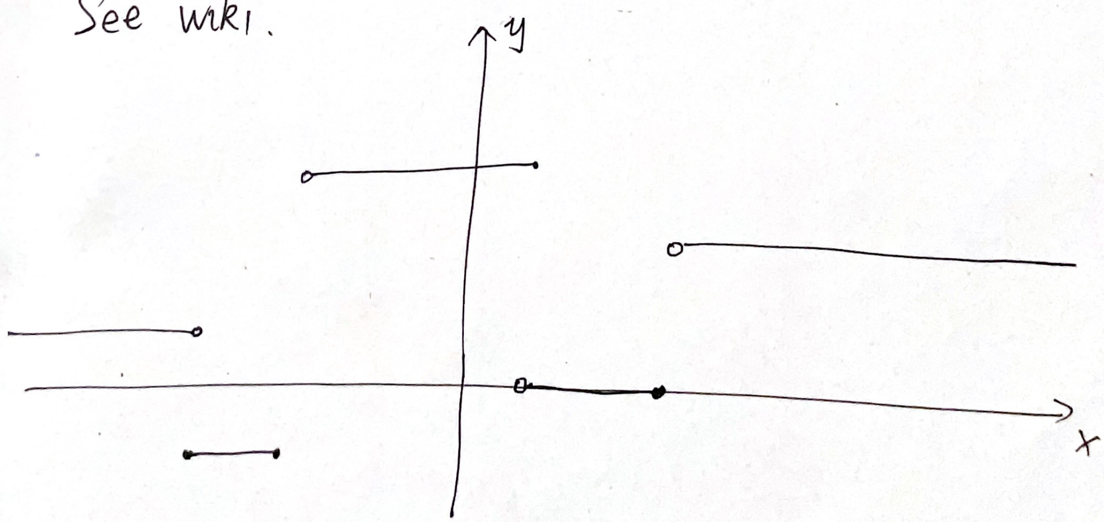
c) Is $\chi_{\mathbb{Q}}(\cdot)$ a step function?

Def. $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a step fct if

$$f(x) = \sum_{i=0}^{n} \alpha_i \chi_{A_i}(x) \text{ for some } n \in \mathbb{N}_{\geq 0},\ \alpha_i \in \mathbb{R},$$

A_i intervals.

See wiki.



Def. (Riemann integral)

$f : [0,1] \longrightarrow \mathbb{R}$ is called Riemann integrable, if

$\exists\, a \in \mathbb{R} :\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0$ s.t.

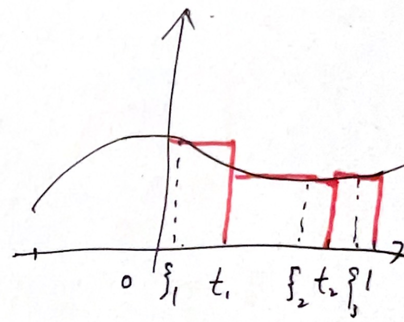
$\forall\, n \in \mathbb{N},\ \forall\, 0 = t_0 < t_1 < \cdots < t_n = 1$ with $|t_{i+1} - t_i| < \delta$, $\forall\, \xi_i \in [t_i, t_{i+1}]$,

we have

$$\left| \sum_{i=0}^{n-1} f(\xi_i)(t_{i+1} - t_i) - a \right| < \varepsilon .$$

If so, we denote

$$\int_0^1 f(x)\, dx := a$$



a is called the integral of f over $[0,1]$.

Ex. Give a def for $f : [0,1] \longrightarrow \mathbb{R}$ to be not Riemann integrable.

Rmk. $\mu(\mathbb{Q}) = 0$, i.e. $\mathbb{Q}$ has measure 0.

So it is expected that

$$\int_0^1 \chi_{\mathbb{Q}}(x)\, dx = 0$$

That's why we finally replace Riemann integral by Lebesgue integral.