Eine Woche, ein Beispiel

Ref: from [23.11.19]

## 5.12 sheaf version of ⊗ & Hom

1. def of sheaf Hom
2. def of sheaf ⊗

sheaf version of Tensor-Hom adjunction is left in the next document.

Compared with ⊗, Hom is more delicated, and it is harder than you expected.

### 1. def of sheaf Hom

$$\mathrm{Hom}_A(-,-) : (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod}$$

A: comm ring

$$\rightsquigarrow$$

$$\mathrm{Hom}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow \mathbb{Z}\text{-Mod}$$

$$\rightsquigarrow$$

$$\underline{\mathrm{Hom}}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$

$$\rightsquigarrow$$

$$R\underline{\mathrm{Hom}}_{\mathcal{D}^+(X)}(-,-) : \mathcal{D}^-(X)^{op} \times \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

Non-derived Hom

**Def** [FOAG, 2.3.1] For $\mathcal{F}, \mathcal{G} \in Sh(X)$, a morphism of sheave

$$\phi : \mathcal{F} \longrightarrow \mathcal{G}$$

is the data of maps

$$\phi(U) : \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \quad \text{for all } U \subseteq X \text{ open.}$$

which is compatible with restriction.

We write

$$\phi \in \mathrm{Hom}_{Sh(X)}(\mathcal{F}, \mathcal{G})$$

Similarly, one can define

$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$$

as the set of morphisms in $\mathcal{D}(X)$.

**Def** [FOAG, 2.3.C] (Sheaf Hom / Internal Hom)

For $\mathcal{F}, \mathcal{G} \in Sh(X)$, one gets a sheaf $\underline{Hom}(\mathcal{F}, \mathcal{G}) \in Sh(X)$ given by

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)(\mathcal{U}) = Hom_{Sh(\mathcal{U})}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}})$$

**Cor.**

$$Hom = \Gamma \circ \underline{Hom} : Sh(X)^{op} \times Sh(X) \xrightarrow{\underline{Hom}} Sh(X) \xrightarrow{\Gamma} Abel$$

⚠ Even though $(\mathcal{F} \otimes \mathcal{G})_p \cong \mathcal{F}_p \otimes \mathcal{G}_p$.

$\underline{Hom}$ does not commute with taking stalks:

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)_p \xrightarrow{\ncong} Hom(\mathcal{F}_p, \mathcal{G}_p)$$

It's neither inj nor surj.

[Left adj comm with limit, $\otimes \dashv \underline{Hom}$ ].

**Ex.** Try to compute  coefficient $\mathbb{Q}$.

$$\underline{Hom}_{Sh(X)}(\underline{\mathbb{Q}}_X, \mathcal{F}) \cong \mathcal{F}$$
$$\underline{Hom}_{Sh(X)}(j_! \underline{\mathbb{Q}}_{\mathcal{U}}, \mathcal{F}) \cong j_*(\mathcal{F}|_{\mathcal{U}})$$
$$\underline{Hom}_{Sh(\mathbb{C})}(sky_0(\mathbb{Q}), \underline{\mathbb{Q}}_{\mathbb{C}}) \cong 0$$