Eine Woche, ein Beispiel
3.26. double coset decomposition

Double coset decompositions are quite impressive!
This document follows and repeats 2022.09.04_Hecke_algebra_for_matrix_groups. Some new ideas come, so I have to write a new.

Ref:
Wiki: Symmetric space, Homogeneous space and Lorentz group

[JL18]: John M. Lee, Introduction to Riemannian Manifolds

[Gorodski]: Claudio Gorodski, An Introduction to Riemannian Symmetric Spaces
https://www.ime.usp.br/~gorodski/ps/symmetric-spaces.pdf

[KWL10]: Kai-Wen Lan: An example-based introduction to Shimura varieties
https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf

https://www.mathi.uni-heidelberg.de/~pozzetti/References/Iozzi.pdf
https://www.mathi.uni-heidelberg.de/~lee/seminarSS16.html

1. G-space
2. double coset decomposition: schedule
3. examples (draw Table)
4. special case: v.b on $\mathbb{P}^1$.

In this document, stratification = disjoint union of sets

1. G-space

Recall: Group action $G \circlearrowright X$

discrete $\Rightarrow$ foundamental domain $\qquad \Lambda \circlearrowright \mathbb{C} \qquad SL_2(\mathbb{Z}) \circlearrowright \mathcal{H}$

non discrete $\Rightarrow$ stratification by $G/G_x \qquad S^1 \circlearrowright S^2 \qquad \mathbb{C}^\times \circlearrowright \mathbb{CP}^1$

Rmk. Many familiar spaces are homogeneous spaces.

E.g. $\mathrm{Flag}(V) \cong GL(V)/P$ $\qquad$ e.g. Grassmannian, $\mathbb{P}^n$

$S^n \cong O(n+1)/O(n) \cong SO(n+1)/SO(n)$

$O(n) := O(n,\mathbb{R}) \qquad \rightsquigarrow$ Stiefel mfld [21.11.14]

$SO(n) := SO(n,\mathbb{R})$

$\mathbb{A}^n = \mathbb{A}^n$

$\mathcal{H}^n \cong O^+(1,n)/O(n)$

$\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO_2(\mathbb{R})$

$\rightsquigarrow$ Hermitian symmetric space

where $\quad \mathcal{H}^n := \{v = (v_i)_{i=1}^{n+1} \in \mathbb{R}^{n+1} \mid \langle v, v\rangle = -1,\ v_{n+1} > 0\}$

$\langle\ ,\ \rangle : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R} \qquad \langle v, w\rangle = v^T \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} w$

$O(n,1) := \mathrm{Aut}(\mathbb{R}^{n+1}, \langle\ ,\ \rangle) \subseteq GL_{n+1}(\mathbb{R})$

$O^+(n,1) := \{g \in O(n,1) \mid g\mathcal{H}^n \subset \mathcal{H}^n\}$

For more informations about $\mathcal{H}^n$, see [JL18, P62-67].

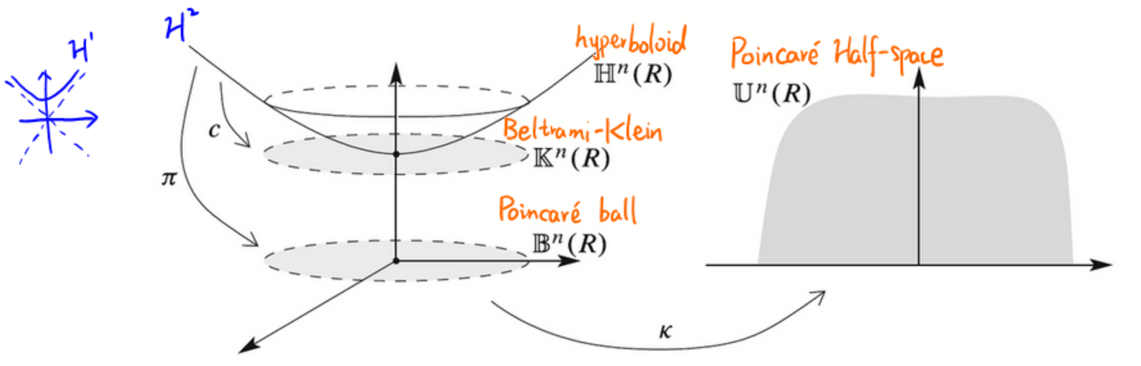


Fig. 3.3: Isometries among the hyperbolic models [JL18, P63]

https://math.stackexchange.com/questions/3340992/sl2-mathbbr-as-a-lorentz-group-01-2

Terminology.

[JL18, Cor 3.18]: Nice hom space can have compatible Rie mfld structures.

$$
\begin{tikzcd}
 & X + G \curvearrowright X \arrow[d] & \\
 & G\text{-space} \arrow[d, "\text{transitive}"] & \\
 & \text{homogeneous space} \arrow[dl, "\substack{\text{Riemannian mfld} \\ G \text{ preserve } g}"'] \arrow[dr, "\substack{\text{Hermitian mfld} \\ G \text{ preserve } g \leftarrow \text{metric}}"] & \\
\text{Riemannian homogeneous space} \arrow[rr, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\substack{\text{geodesic symmetry} \\ \text{globally defined, preserve } g}"'] & & \text{Hermitian homogeneous space} \arrow[d, "\substack{\text{geodesic symmetry} \\ \text{globally defined, preserve } g}"] \\
\text{Symmetric space} \arrow[rr, "\mathbb{R} \rightsquigarrow \mathbb{C}"] \arrow[d, "\text{transitive on frames}"'] & & \text{Hermitian symmetric space} \\
\text{frame homogeneous space} & &
\end{tikzcd}
$$

$\mathbb{C}\mathbb{P}^n$ $n \geq 1$

Rmk. Sym spaces & Hermitian sym spaces are fully classified.
See [Gorodski, Thm 2.38] and [KWL10, §3] for the result.

Q: Can we define and classify sym spaces in p-adic world?

2. double coset decomposition: schedule

$$G = \bigsqcup_{\alpha \in I} H \alpha K$$

usually, H, K are easier than G.

- comes from (usually) Gauss elimination
- I is the "foundamental domain"
- produces stratifications on G/K and ${}_H\backslash G$ indexed by I.

To be exact,

$$G/K = \bigsqcup_{\alpha \in I} H\alpha K/K \cong \bigsqcup_{\alpha \in I} H/H_{[\alpha K]} = \bigsqcup_{\alpha \in I} H/(H \cap \alpha K \alpha^{-1})$$

$$_H\backslash G = \bigsqcup_{\alpha \in I} {}_H\backslash H\alpha K \cong \bigsqcup_{\alpha \in I} K_{[H\alpha]}\backslash K = \bigsqcup_{\alpha \in I} (K \cap \alpha^{-1} H \alpha)\backslash K$$

$H_{[\alpha K]}$: stabilizer of H on $[\alpha K] \in G/K$

$K_{[H\alpha]}$: stabilizer of K on $[H\alpha] \in {}_H\backslash G$

$$\# \, H/H \cap \alpha K \alpha^{-1} = \# \left\{ \begin{array}{l} \text{single cosets } [gK] \\ \text{in one double coset } H\alpha K \end{array} \right\} < +\infty$$

Therefore, the dec helps us to understand the geometry of

$$G/K \quad \& \quad {}_H\backslash G \qquad \text{individually}$$

- can be viewed as stack quotient.

$[*/G]$: groupoid

$${}_H\backslash G/K \overset{\text{def}}{=} [*/H] \times_{[*/G]} [*/K] \quad \text{with groupoid structure}$$

$$H_H^*(G/K) \cong H^*({}_H\backslash G/K) \cong H_K^*({}_H\backslash G)$$

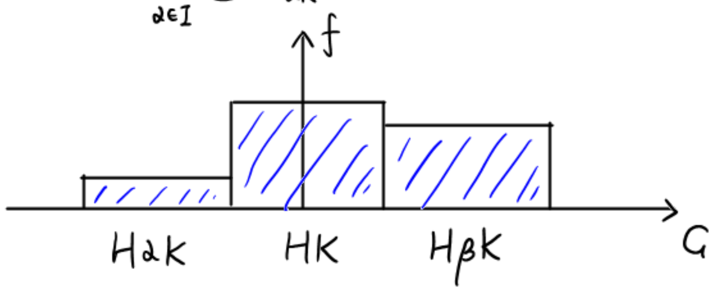
slogan: the (equiv) cohomology of G/K and ${}_H\backslash G$ are connected.

- Hecke algebra $\mathcal{H}(H\backslash G/K)$

for $H=K$. You can also do $\mathcal{H}(H_1\backslash G/H_2) \hookrightarrow \bigoplus_{i,j=1}^{2} \mathcal{H}(H_i\backslash G/H_j)$

$\mathcal{H}(H\backslash G/K)$: reasonable subspaces of

$$\mathbb{C}[H\backslash G/K] = \{ f: G \to \mathbb{C} \mid f(hgk) = f(g) \quad \forall h\in H, g\in G, k\in K \}$$
$$\underset{\text{"o-dim"}}{\supseteq} \bigoplus_{\alpha\in I} \mathbb{C}\, \mathbb{1}_{H\alpha K}$$



with reasonable convolution structure

$$*: \mathcal{H}(H_1\backslash G/H_2) \times \mathcal{H}(H_2\backslash G/H_3) \longrightarrow \mathcal{H}(H_1\backslash G/H_3)$$

which are often computable (but hard)

It encodes important informations of double coset decomposition.

Vague: $\mathcal{H}(H\backslash G/K) \sim H^*(H\backslash G/K)$ should be a type of cohomology

$\mathcal{H}(G) \overset{G \text{ fin}}{=\!=} \mathbb{C}[G]$

$\mathcal{H}(K\backslash G/K) \cong (\mathrm{End}(\text{c-}\mathrm{Ind}_K^G \mathbb{1}_K))^{op}$ should be a type of base ring

Generalize: $\mathrm{Ind}_H^G \chi \approx \mathcal{H}_\chi(H\backslash G/K) \subseteq \{ f: G\to\mathbb{C} \mid f(hgk) = \chi(h) f(g) \}$

depth of χ

3. examples (after [22.09.04])

Works over:

- list of possibilities
- moduli intepretation
- typical examples

finite field, $GL_n(\mathbb{F}_q)$ (Applies to any field κ, actually)

- subgps can be

Borel	max split torus	unipotent
B	T	N
parabolic	Levi	unipotent
P	L	M
	nonsplit torus	
	T'	

$$\square \backslash GL_n(\kappa) / \square$$

- moduli intepretation $\quad V = \kappa^{\oplus n}$

$$
\begin{aligned}
G/B &= \{ \text{cpl flags in } V \} \\
G/T &= \{ (V_i)_{i=1}^n \mid V = \textstyle\bigoplus_{i=1}^n V_i,\ \dim V_i = 1 \} \\
G/N &= \left\{ (\mathcal{F}, m_i) \;\middle|\; \begin{array}{l} \mathcal{F}:\ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = V \ \text{cpl} \\ 0 \neq m_i \in M_i/M_{i-1} \end{array} \right\} \\
G/P &= \{ \text{flags in } V \} \\
G/L &= \{ (V_i)_{i \in I} \mid V = \textstyle\bigoplus_{i \in I} V_i \} \\
G/M &= \left\{ (\mathcal{F}, \mathcal{B}_i) \;\middle|\; \begin{array}{l} \mathcal{F}:\ 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_d = V \\ \mathcal{B}_i:\ \text{a basis of } M_i/M_{i-1} \end{array} \right\}
\end{aligned}
$$

Rmk. We have a fiber bundle

$$
\begin{array}{ccc}
\mathbb{A}^{\oplus\binom{n}{2}} \cong B/N & \longrightarrow & G/N \\
& & \downarrow \\
& & G/B
\end{array}
$$

which makes G/N a $\mathbb{A}^{\oplus\binom{n}{2}}$-torsor over G/B

⚠ G/N is not a $\underbrace{\text{rk } \binom{n}{2} \text{ v.b}}_{\text{i.e. } GL_{\binom{n}{2}}(\kappa)\text{-torsor}}$ over G/B, so G/N can be affine space.

- E.g. Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

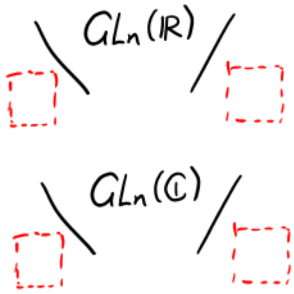
  - Gauss elimination gives "$\subseteq$",
    while the observation of process gives "$\sqcup$" (Something is invariant)
  - the "fundamental domain" W has a gp structure, and crsp to B-orbits of G/B.
    gp structure comes from Tits system
  - produces an affine paving of G/B, and the Zariski topo gives Bruhat order
    works also for Euclidean topo, $k = \mathbb{R}$ or $\mathbb{C}$.
  - $B \backslash G / B = [*/B] \times_{[*/G]} [*/B]$, with
    $H_B^*(G/B) \cong H_T^*(G/B) \cong \bigoplus_w H_T^*(pt)$ [my master thesis]
  - $\mathcal{H}(G, B)$: see [22.09.04]

- possible exercise:
  - Work out $T \backslash G / B$
    $P_1 \backslash G / P_2$, $GL_m \times GL_n \backslash GL_{m+n} / GL_m \times GL_n$, $S_m \times S_n \backslash S_{m+n} / S_m \times S_n$ [22.11.13]
    $\mathbb{F}_{q^n}^\times \backslash GL_n(\mathbb{F}_q) / B$, ...
    $k = \mathbb{F}_1$; $GL_n \rightsquigarrow$ other gps
  - Computation of cardinals.

Archi field, $\mathbb{R}$ or $\mathbb{C}$

- subgps can be

nearly affine			cpt
Borel B	max split torus T	unipotent N	$O(n)$
parabolic P	Levi L	unipotent M	or $SO(n)$
	nonsplit torus T'		$\mathcal{U}(n) := \mathcal{U}_{\mathbb{C}/\mathbb{R}}(n)$ or $S\mathcal{U}(n)$



⚠ $M^{sym}_{n\times n}(\mathbb{R})$, $M^{sym,>0}_{n\times n}(\mathbb{R})$ are not gps!

- moduli intepretation

$V := \mathbb{R}^{\oplus n}$ In $\mathbb{C}^{\oplus n}$ case, replace inner product by Hermittian prod.

$$G/O(n) \cong \{\text{inner products on } V\} \cong M^{sym,>0}_{n\times n}(\mathbb{R})$$

$$g = (v_1, \dots, v_n) \longmapsto \langle\, , \rangle \text{ s.t. } \{v_1, \dots, v_n\} \text{ is an ortho basis} \longmapsto (\langle e_i, e_j\rangle)_{i,j=1}^n$$

$v_i = ge_i$ i.e. $\langle x, y\rangle := x^T (g^{-1})^T g^{-1} y$

$$g \longmapsto (g^{-1})^T g^{-1}$$

as G-sets, where

$g \cdot x := gx$ $\qquad g \cdot \langle\, , \rangle := \langle g^{-1} -, g^{-1} - \rangle$ $\qquad g \cdot A := (g^{-1})^T A g^{-1}$

i.e. $\langle gx, gy\rangle_g = \langle x, y\rangle$

action on inner product

Ex. $\mathcal{H} \cong GL_2(\mathbb{R})/O(2)\cdot\mathbb{R}_{>0} \cong SL_2(\mathbb{R})/SO(2)$

$\cong$

4. special case: v.b on $\mathbb{P}^1$.

https://en.wikipedia.org/wiki/Birkhoff_factorization