Un exemple par jour

4.1. the complex torus of form $\mathbb{C}^\times/\mathbb{Z}\gamma$

$C := \mathbb{C}^\times/\mathbb{Z}\gamma \overset{topo}{=\!=\!=} \mathbb{T}^2$ is a cpt Riemannian surface of genus 1.

$\gamma \in \text{Aut}(\mathbb{C}^\times)$ $\gamma(z) = az$ $a \in \mathbb{C}^\times$ $|a| > 1$

Today: $a = 2$

1. fundamental set:
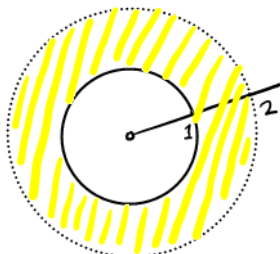
$\Rightarrow$ only need 2 local chart

2. $0 \to \mathbb{Z} \hookrightarrow \mathbb{C} \overset{f:\ z \mapsto e^{2\pi i z}}{\longrightarrow} \mathbb{C}^\times \to 1$

$\downarrow +\frac{1}{2\pi i}\ln 2$ $\downarrow +\frac{1}{2\pi i}\ln 2$ $\downarrow \times 2$

$0 \to \mathbb{Z} + \frac{1}{2\pi i}\ln 2 \to \mathbb{C} \longrightarrow \mathbb{C}^\times \to 1$

another local chart

$\mathbb{C} \overset{f}{\longrightarrow} \mathbb{C}^\times$
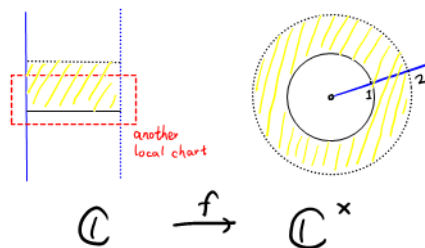
$\mathbb{C}^\times = \mathbb{C}/\mathbb{Z} \Rightarrow \mathbb{C}^\times/\mathbb{Z}\gamma = \mathbb{C}/(\mathbb{Z} \oplus \frac{1}{2\pi i}\ln 2\,\mathbb{Z})$

better: $a = e^{2\pi} \approx 535.49$

$a = e^{-2\pi i w} \approx -230.765$

3. line bundle on C

$b \in \mathbb{C}^\times$ $\mathcal{L}_b := \mathbb{C}^\times \times \mathbb{C} / (z, \xi) \sim (2z, b\xi)$

$\downarrow$

$C = \mathbb{C}^\times / z \sim 2z$

$\Rightarrow$ ① $\mathcal{L}_b \in \text{Pic}_0(C)$; $(\mathcal{L}_b \sim \mathcal{L}_1 \cong \mathcal{O}_C)$ cont. deformation

② $\text{Pic}_0(C) \overset{\sim}{\cong} C = \mathbb{C}^\times/\mathbb{Z}\gamma$ (naive, base pt $1 \in \mathbb{C}^\times/\mathbb{Z}\gamma$)

$\mathcal{L}_b \longmapsto b$

Reduced to: find a section s on $\mathcal{L}_b$ st div $s = [b] - [1]$

Reduced to: find a meromorphic functions g on $\mathbb{C}^\times$ s.t

① $g(2z) = b\,g(z)$ $b \in \mathbb{C}^\times,\ b \ne 2^k$; e.g. $b = 3$

② g has simple poles on 2^n, and simple zeros on $2^n b$ $n \in \mathbb{Z}$

$b = e^{2\pi i c},\ c \in \mathbb{C}$

$\tau := \frac{1}{2\pi i}\ln 2$

$w(z) = \frac{1}{2\pi i}\ln z$

$g(z) = \dfrac{\theta\left[\begin{smallmatrix}1\\1-2c\end{smallmatrix}\right](w(z),\tau)}{\theta\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right](w(z),\tau)}$ is the required one.

Blue — example
Orange — more than this example
Red — important results
Purple — I don't know the answer/proof
Green — sketsch of proof . in a minimal way
Grey — some supplementary explanation. Unimportant assumptions.
Hell grey — explanation on well-known notations.
Brown — small title in subsections.