Eine Woche, ein Beispiel

1.9. simplicial set

Ref:
[sSet]http://www.math.uni-bonn.de/~schwede/sset_vs_spaces.pdf
[6-Fctor]https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf

Some visual pictures can be seen here:
https://arxiv.org/pdf/0809.4221.pdf

Today: The category sSet and $\partial\Delta^n$, Λ^n_i, $sk^m X$, $\Delta^n/\partial\Delta^n$, $Hom(X,Y) \in Ob(sSet)$

Def $[n] = \{0, 1, \dots, n\}$ $\quad n \geq 0$

The simplex category Δ is defined by

$$Ob(\Delta) = \{[n] \mid n \geq 0\}$$

$$Mor_\Delta([m],[n]) = \{f : [m] \longrightarrow [n] \text{ weakly increasing}\}$$

The category of simplicial sets sSet is defined by

$$sSet = Fun(\Delta^{op}, Set)$$

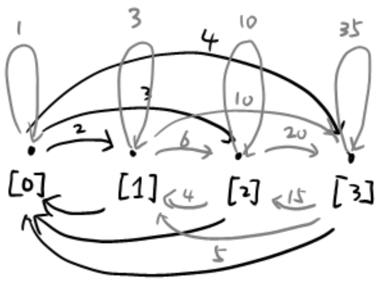
Notation in $Mor(\Delta)$: $d_i^n : [n-1] \longrightarrow [n]$ miss i $\quad 0 \leq i \leq n$

$s_i^n : [n] \longrightarrow [n-1]$ contracts i

$\Delta \hookrightarrow sSet \quad [n] \longmapsto \Delta^n := Mor_\Delta(-, [n])$

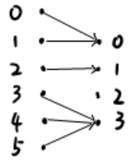
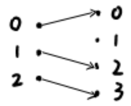
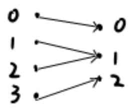
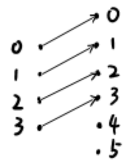
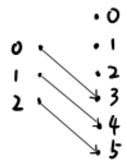
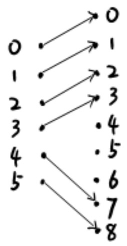
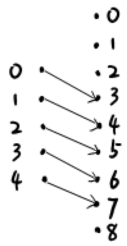
e.g. $\Delta^n_k = Mor_\Delta([k],[n])$

read from down to top



$\#\Delta^n_k$ (k \ n)	0	1	2	3
0	1	2	3	4
1	1	3	6	10
2	1	4	10	20
3	1	5	15	35

$$\#\Delta^n_k = \binom{n+k+1}{n}$$

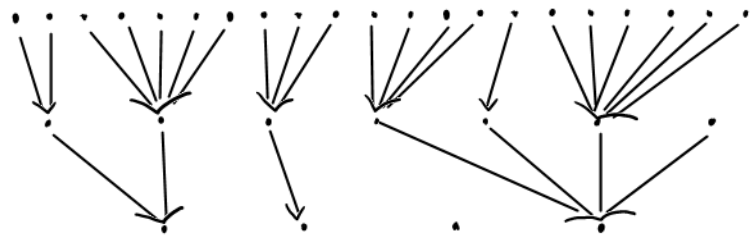
element	picture	list	count	other notations
$\alpha : [5] \to [3]$ $0 \mapsto 0$ $1 \mapsto 0$ $2 \mapsto 1$ $3 \mapsto 3$ $4 \mapsto 3$ $5 \mapsto 3$		$(0,0,1,3,3,3)$	$[2,1,0,3]$	
$d_1^3 : [2] \to [3]$ $0 \mapsto 0$ $1 \mapsto 2$ $2 \mapsto 3$		$(0,2,3)$	$[\underbrace{1}_{i},0,1,1]$	$d_i^n : [n-1] \to [n]$ δ^n
$s_1^3 : [3] \to [2]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 1$ $3 \mapsto 2$		$(0,1,1,2)$	$[\underbrace{1}_{i},2,1]$	$s_i^n : [n] \to [n-1]$
$d_{3,2} : [3] \to [5]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$		$(0,1,2,3)$	$[\underbrace{1,1,1}_{i},1,\underbrace{0,0}_{j}]$	$d_{i,j} : [i] \to [i+j]$ δ_i^f f=front
$d'_{3,2} : [2] \to [5]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$		$(3,4,5)$	$[\underbrace{0,0,0}_{i},1,\underbrace{1,1}_{j}]$	$d'_{i,j} : [j] \to [i+j]$ δ_i^b b=back
$\delta^{out}_{3,(5,4)} : [5] \to [8]$ $0 \mapsto 0$ $1 \mapsto 1$ $2 \mapsto 2$ $3 \mapsto 3$ $4 \mapsto 7$ $5 \mapsto 8$		$(0,1,2,3,7,8)$	$[\underbrace{1,1,1,1}_{i+1},0,0,0,\underbrace{1,1}_{p-i}]$	$\delta^{out}_{i,(p,q)} : [p] \to [p+q-1]$ δ_i^{out}
$\delta^{in}_{3,(5,4)} : [4] \to [8]$ $0 \mapsto 3$ $1 \mapsto 4$ $2 \mapsto 5$ $3 \mapsto 6$ $4 \mapsto 7$		$(3,4,5,6,7)$	$[\underbrace{0,0,0}_{i},\underbrace{1,1,1,1,1}_{q+1},0]$	$\delta^{in}_{i,(p,q)} : [q] \to [p+q-1]$ δ_i^{in}

not confuse with $[n]$

Table 1. Morphisms in Δ.

How to compute the composition?

e.g. $[2,1,0,4] \circ [\underbrace{2,5},\underbrace{3},\underbrace{4},\underbrace{1,6,0}] = [7,3,0,11]$



$[20] \to [6] \to [3]$

$[1,2,1] \circ [\underbrace{2},\underbrace{1,0},\underbrace{3}] = [2,1,3]$

$[4] \circ [\underbrace{2,1,0,3}] = [6]$

$s_1^3 \circ d_1^3 = [1,2,1] \circ [\underbrace{1},\underbrace{0,1},\underbrace{1}] = [1,1,1]$

$d_1^3 \circ s_1^3 = [1,0,1,1] \circ [\underbrace{1},\underbrace{},\underbrace{2},\underbrace{1}] = [1,0,2,1]$

Rmk. In Δ we don't have finite colimit, while in $sSet = Fun(\Delta^{op}, Set)$ we have finite colimit because Set is (complete +) cocomplete.

For a construction, see
https://math.stackexchange.com/questions/3837844/limits-and-colimits-are-computed-pointwise-in-functor-categories

Notice that

$$\partial\Delta^n,\ \Lambda_i^n,\ sk^m\Delta^n,\ \Delta^n/\partial\Delta^n \in sSet - \Delta$$

Conclusion: sSet is a Grothendieck topos.
It is Cartesian closed, complete and cocomplete.
In sSet, we can glue objects ($\approx$pushforward), which is impossible in Δ.

Slogan: sSet $\sim$ simplicial complex
X_n $\sim$ the index set of n-dim cells

Rmk. ([sSet]) If you have strong enough geometrical background, you will find out the adjoint pair

$$\begin{tikzcd} \mathrm{sSet} \arrow[r, bend left, "|-|"] \arrow[r, phantom, "\perp"] & \mathrm{Top} \arrow[l, bend left, "S"] \end{tikzcd}$$

quite useful, where

$$|X| := \Big(\coprod_{n\geq 0} X_n \times \nabla^n\Big)/\sim$$

$$S(A)_n := \mathrm{Mor}_{\mathrm{Top}}(\nabla^n, A)$$

$$\alpha^* : S(A)_n \longrightarrow S(A)_m \qquad x \longmapsto x \circ S(\alpha)$$

$\alpha : [m] \to [n]$

$S(\alpha) : \nabla^m \longrightarrow \nabla^n$

Moreover, we have eqvlc of categories

$$\begin{tikzcd} & \mathrm{Ho}(\mathrm{Top}_{cw}) \arrow[d] \\ \mathrm{sSet}[\mathrm{weq}^{-1}] \arrow[ur, "|-|"] & \mathrm{Top}[\mathrm{weq}^{-1}] \arrow[l, "S"'] \end{tikzcd}$$

where

Top_{cw} is the full subcategory of Top with objects the top spaces admitting a CW-cplx structure, and

$\mathrm{Ho}(\mathrm{Top}_{cw})$ is the homotopy category of CW cplxes.

Q: Do we have the following comm diag: (as eqvlc of categories)

$$\begin{tikzcd} \mathrm{sSet}[\mathrm{weq}^{-1}] \arrow[r, "|-|"] & \mathrm{Ho}(\mathrm{Top}_{cw}) \arrow[d] \\ \mathrm{An} \arrow[u, hook] & \mathrm{Top}[\mathrm{weq}^{-1}] \arrow[l, "S"'] \end{tikzcd}$$

Q: For $\mathcal{C} \in \mathrm{Cat}_\infty \subseteq \mathrm{sSet}$, how to view $\mathcal{C}$ as a topo space?

e.p. compute $\pi_n(\mathcal{C})$?

Roughly, we have three ways to define/determine a simplicial set:

1. By writing down their def directly; brutal force
2. By universal property (pullback, pushforward, ...) abstract construction
3. By its geometrical realization name

Let us see how they're compatible with each other.

E.g. 1. For $A \in Top$ discrete, define $X = \mathcal{S}(A)$, i.e.,

$X_n = A \qquad \partial^* = Id_A \qquad \forall \partial : [m] \longrightarrow [n]$

$$
\begin{aligned}
|\mathcal{S}(A)| &= \Big(\coprod_k X_k \times \nabla^k\Big)/\sim \\
&\sim A \times \nabla^0 \\
&\sim A
\end{aligned}
$$

Eg. 2. $\Delta^n_k = Mor_\Delta([k],[n]) = \{x : [k] \longrightarrow [n] \text{ weakly increasing}\}$

$$
\begin{aligned}
|\Delta^n| &= \Big(\coprod_k \Delta^n_k \times \nabla^k\Big)/\sim \\
&\sim (\Delta^n_n \times \nabla^n)/\sim \\
&\sim \nabla^n
\end{aligned}
$$

Eg. 3. $\Delta^{n-1}_{(i)} := Im\,(d^n_i : \Delta^{n-1} \longrightarrow \Delta^n)$ in sSet

$\Rightarrow (\Delta^{n-1}_{(i)})_k = \{x \in \Delta^n_k \mid \exists\, y \in \Delta^{n-1}_k \text{ s.t. } x = d^n_i \circ y\}$

$$
\begin{aligned}
|\Delta^{n-1}_{(i)}| &= \Big(\coprod_k (\Delta^{n-1}_{(i)})_k \times \nabla^k\Big)/\sim \\
&\sim \Big((\Delta^{n-1}_{(i)})^{nondeg}_{n-1} \times \nabla^{n-1}\Big)/\sim \\
&\sim (Sd^n_i)(\nabla^{n-1})
\end{aligned}
$$

Denote $|\Delta^{n-1}_{(i)}|$ by $\nabla^{n-1}_{(i)}$, i.e. $\nabla^{n-1}_{(i)} := Im\,(Sd^n_i : \nabla^{n-1} \longrightarrow \nabla^n)$

Eg. 4. $(\partial\Delta^n)_k = \{x \in \Delta^n_k \mid x \text{ is not surj}\}$

$\partial\Delta^n = \bigcup_{i=0}^n \Delta^{n-1}_{(i)}$ = colimit of ⋯

e.g. $\partial\Delta^2 = \Bigg[$ colimit of
$$
\begin{tikzcd}
\Delta^0 \arrow[d] \arrow[dr] & \Delta^0 \arrow[dl] \arrow[dr] & \Delta^0 \arrow[dl] \arrow[d] \\
\Delta^1 & \Delta^1 & \Delta^1
\end{tikzcd}
$$
$\Bigg]$

$$
\begin{aligned}
|\partial\Delta^n| &= \Big(\coprod_k (\partial\Delta^n)_k \times \nabla^k\Big)/\sim \\
&\sim \big(Mor_\Delta^{nondeg}([n-1],[n]) \times \nabla^{n-1}\big)/\sim \\
&\sim \Big(\coprod_{i=0}^n Sd^n_i\,(\nabla^{n-1})\Big)/\sim \\
&= \bigcup_{i=0}^n \nabla^{n-1}_{(i)} \\
&= \partial\nabla^n
\end{aligned}
$$

Eg. 5. $(\Lambda^n_i)_k = \left\{ x \in \Delta^n_k \;\middle|\; x = \alpha^*(y) \text{ for some } \begin{array}{l} y \in \Delta^n_{n-1} \\ y \neq d^n_i \end{array} \text{ and } \alpha : [k] \to [n-1] \right\}$

$\Lambda^n_i = \bigcup_{j \neq i} \Delta^{n-1}_{(j)}$ = colimit of $\cdots$

e.g. $\Lambda^2_1 = \left[\text{colimit of} \quad \begin{tikzcd} & & \Delta^0 \arrow[d, dashed, "d^1_0"'] \\ & \Delta^0 \arrow[r, "d^1_1"] \arrow[d, "d^1_0"'] & \Delta^1 \\ \Delta^0 \arrow[r, dashed, "d^1_1"] & \Delta^1 & \end{tikzcd} \right]$

$= \Delta^1 \sqcup_{\Delta^0} \Delta^1$

ex. write down $(X \sqcup_Y Z)_k$ for $X, Y, Z \in sSet$

$$
\begin{aligned}
|\Lambda^n_i| &= \Big(\coprod_k (\Lambda^n_i)_k \times \nabla^k \Big) / \sim \\
&\sim \big((\Lambda^n_i)^{nondeg}_{n-1} \times \nabla^{n-1} \big) / \sim \\
&\sim \Big(\coprod_{j \neq i} (Sd^n_j)(\nabla^{n-1}) \Big) / \sim \\
&\sim \bigcup_{j \neq i} \nabla^{n-1}_{(j)}
\end{aligned}
$$

E.g. 6. $(sk^m \Delta^n)_k = \left\{ x \in \Delta^n_k \;\middle|\; x = \alpha^*(y) \text{ for some } y \in \Delta^n_m \text{ and } \alpha : [k] \to [m] \right\}$

$sk^m \Delta^n = \bigcup_{\beta : [m] \to [n]} \beta(\Delta^m)$ = colimit of $\cdots$

$$
\begin{aligned}
|sk^m \Delta^n| &= \Big(\coprod_k (sk^m \Delta^n)_k \times \nabla^k \Big) / \sim \\
&\sim \big((sk^m \Delta^n)^{nondeg}_m \times \nabla^m \big) / \sim \\
&\sim \big(Mor^{nondeg}([m],[n]) \times \nabla^m \big) / \sim \\
&\sim \bigcup_{\beta : [m] \to [n]} (S\beta)(\nabla^m)
\end{aligned}
$$

E.g. 7. $(\Delta^n / \partial \Delta^n)_k = \Delta^n_k / (\partial \Delta^n)_k = \Delta^n_k / \sim$

$x \sim y \iff x, y \in (\partial \Delta^n)_k$ or $x = y$

Universal property:

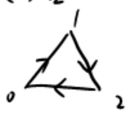
$$
\begin{tikzcd}
\partial \Delta^n \arrow[r] \arrow[rr, dashed, bend right, "\text{contract to one pt}"'] & \Delta^n \arrow[r] \arrow[dr] & \Delta^n / \partial \Delta^n \arrow[r] \arrow[d, dashed, "\exists !"] & 0 \\
& & X &
\end{tikzcd}
$$

$$
\begin{aligned}
|\Delta^n/\partial\Delta^n| &= \left(\coprod_k (\Delta^n/\partial\Delta^n)_k \times \nabla^k\right)/\sim \\
&\sim \left((\Delta^n/\partial\Delta^n)_n^{\text{nondeg}} \times \nabla^n\right)/\sim \\
&\sim \nabla^n/\sim \\
&\sim \nabla^n/\partial\nabla^n
\end{aligned}
$$

Eg. 8. Define

$$
X = \left[\ \text{colimit of} \quad \begin{tikzcd} \Delta^1 \arrow[r, bend left, "d_2^2"] \arrow[r, "-d_1^2"] \arrow[r, bend right, "d_0^2"'] & \Delta^2 \end{tikzcd}\ \right]
$$

$$
\begin{aligned}
X_k &= \Delta_k^2/\sim \qquad \text{here we identify } d_2^2 x = -d_1^2 x = d_0^2 x \\
|X| &= \left(\coprod_k X_k \times \nabla^k\right)/\sim \\
&\sim \left(X_2^{\text{nondeg}} \times \nabla^2\right)/\sim \\
&\sim
\end{aligned}
$$



Similarly, one can consider $\Delta^2 \cup_{\partial\Delta^2} \Delta^2 \cong S^2$

Ex. Shows that

$$\partial\Delta^3, \quad \Delta^2/\partial\Delta^2, \quad \Delta^2 \cup_{\partial\Delta^2} \Delta^2$$

are homotopy equivalent as simplicial sets.

Eg. 9 $\quad (\mathrm{Hom}(X,Y))_n = \mathrm{Hom}_{sSet}(\Delta^n \times X, Y)$

$$\alpha^*: \mathrm{Hom}_{sSet}(\Delta^n \times X, Y) \longrightarrow \mathrm{Hom}_{sSet}(\Delta^m \times X, Y)$$

for $\alpha: [m] \to [n]$
$\alpha: \Delta^m \to \Delta^n$

$$\begin{tikzcd} sSet \arrow[r, bend left, "- \times X"] \arrow[r, phantom, "\perp"] & sSet \arrow[l, bend left, "\mathrm{Hom}(X,-)"] \end{tikzcd}$$

"Proof" $\quad \mathrm{Hom}_{sSet}(Z, \mathrm{Hom}(X,Y)) \cong \left\{ \begin{matrix} g_m : Z_m \longrightarrow \mathrm{Hom}_{sSet}(\Delta^m \times X, Y) \\ + \cdots \end{matrix} \right\}$

$$\cong \left\{ \begin{matrix} h_{m,k}: Z_m \times \Delta^m_k \times X_k \longrightarrow Y_k \\ + \cdots \end{matrix} \right\}$$

$$\cong \left\{ \begin{matrix} h_k : Z_k \times X_k \longrightarrow Y_k \\ + \cdots \end{matrix} \right\}$$

$$\cong \mathrm{Hom}_{sSet}(Z \times X, Y)$$

Cor. $\quad \mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \mathrm{Hom}(X,Y))$

e.g. $\quad \mathrm{Hom}(\Delta^0, Y) \cong Y \qquad (\mathrm{Hom}(\Delta^n, Y))_m \cong (\mathrm{Hom}(\Delta^m, Y))_n$

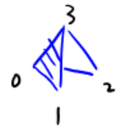
$\mathrm{Hom}(X, \Delta^0) \cong \Delta^0 \qquad$ e.p. $(\mathrm{Hom}(\Delta^n, Y))_0 \cong Y_n$

$$|\mathrm{Hom}(X,Y)| = \left(\coprod_k \mathrm{Hom}_{sSet}(\Delta^k \times X, Y) \times \nabla^k \right) / \sim$$

$$= ?$$

Remaining: Compute $\# \, (\mathrm{Hom}(\Delta^n, \Delta^m))_k$
Compute $(\mathrm{Hom}(\Delta^n/\partial\Delta^n, Y))_k$. How is it related to Y_{k+n} or $\pi_n(|Y|)$?
How to see the geometrical realization of $\mathrm{Hom}(X,Y)$,
e.p. in these examples?

Eg. 10. Let X be a subset of Δ whose realization is as follows. Write down X_k for $k \leq 3$.



e.g. $X_1 = \left\{ \begin{array}{l} [2,0,0,0], [0,2,0,0], [0,0,2,0], [0,0,0,2], \\ [1,1,0,0], [1,0,1,0], [1,0,0,1], [0,1,0,1], [0,0,1,1] \end{array} \right\}$

Eg. 11. BG

Eg. 12.

Realize Hochschild homology as simplicial homology:
https://arxiv.org/pdf/1802.03076.pdf