Tutorial 10 & Ex 9.

Today we work on path integration.

Questions for Ex 9?

1. Topology: simply-connected

Def. For a ~~topo space $(X,\mathcal{T})$, assum~~ path connected topo space $(X,\mathcal{T})$,

we say that X is simply-connected (sc in this tutorial),

if the following equivalent conditions hold.

1) $\forall x_0 \in X$, $\forall \gamma : [a,b] \longrightarrow X$ with $\gamma(a) = \gamma(b) = x_0$,

$\exists F : [a,b] \times [0,1] \longrightarrow X$ s.t $F|_0 = \gamma$, $F|_1 = 1_{x_0}$

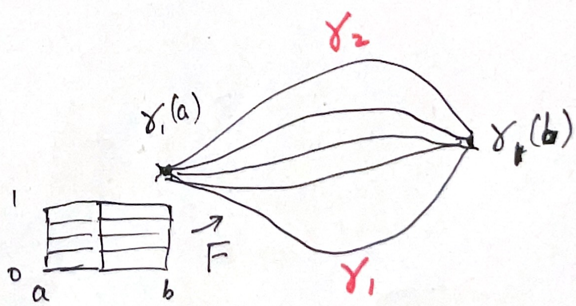
$(\forall x_0 \in X,\ \pi_1(X, x_0) = 0)$

2). $\forall \gamma : S^1 \longrightarrow X$,

$\exists F : S^1 \times [0,1] \longrightarrow X$ s.t. $F|_0 = \gamma$, $F|_1$ is const.

3). $\forall \gamma : S^1 \longrightarrow X$, $\exists \beta : D^2 \longrightarrow X$ s.t. $\gamma = \beta \circ \iota_{S^1}$

$$\begin{tikzcd} S^1 \arrow[r, "\gamma"] \arrow[d, "\iota_{S^1}"'] & X \\ D^2 \arrow[ur, dashed, "\exists\beta"'] \end{tikzcd}$$

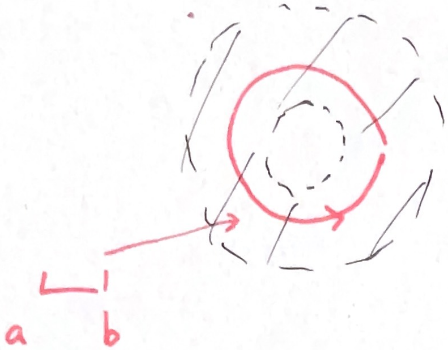


4). $\forall \gamma_1, \gamma_2 : [a,b] \longrightarrow X$ with $\gamma_1(a) = \gamma_2(a)$, $\gamma_1(b) = \gamma_2(b)$,

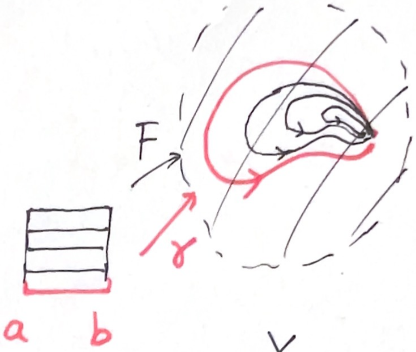
$\exists F : [a,b] \times [0,1] \longrightarrow X$ s.t. $F|_0 = \gamma_1$, $F|_1 = \gamma_2$,

$F|_{\{a\}\times[0,1]} \equiv \gamma_1(a)$, $F|_{\{b\}\times[0,1]} \equiv \gamma_2(a)$

E.g In $\mathbb{R}^2$,



X_1



X_2

$X_1 := B_2(0) - \overline{B_{\frac{1}{2}}(0)}$ is not simply-connected,

since you can not shrink $\gamma : [0, 2\pi] \longrightarrow X_1 \quad t \mapsto (\cos t, \sin t)$

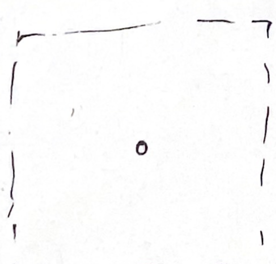
$X_2 := B_2(0)$ is simply-connected.

~~Hint~~ Proof: $\forall x_0 \in X,\ \forall \gamma : [a,b] \longrightarrow X$ with $\gamma(a) = \gamma(b) = x_0$,

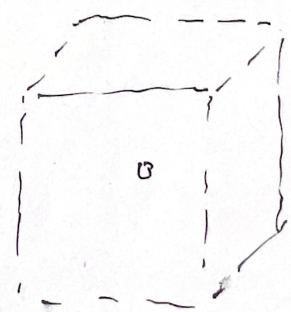
$\exists F : [a,b] \times [0,1] \longrightarrow X \qquad (s,t) \mapsto (1-t)\gamma(s) + t x_0$

s.t. $F|_0 = \gamma \quad F|_1 = \mathbb{1}_{x_0}$

Q: Is $\mathbb{R}^2 - \{0\}$ simply-connected? Is $\mathbb{R}^3 - \{0\}$ simply connected?



No



Yes

Def (Domain) $\emptyset \neq \mathcal{U} \subseteq \mathbb{R}^n$ is called a domain, if

$\mathcal{U}$ is open + (path) connected

E.g.

	open?	(path) connected?	domain?
$\mathbb{R}^2 - \{0\}$	✓	✓	✓
$B_2(0)$	✓	✓	✓
$\overline{B_2(0)}$	✗	✓	✗
$B_2(0) \cup B_2(4)$	✓	✗	✗

Draw them!

Thm. For a bounded domain $\mathcal{U} \subset \mathbb{R}^2$, we have

$\mathcal{U}$ is simply-connected $\Leftrightarrow$ $\partial \mathcal{U}$ is connected.

Rmk. These conditions are essential. Check the following cases.

$\mathcal{U}$	$\mathcal{U}$ simply-connected	$\partial\mathcal{U}$ is connected
$B_2(0) - \{0\}$ in $\mathbb{R}^2$	✗	✗
$B_2(0)$ in $\mathbb{R}^2$	✓	✓
$\mathbb{R}^2 - \{0\}$ in $\mathbb{R}^2$	✗	✓
$\partial B_2(0)$ $\cancel{\mathbb{R}^2 - ([-1,0) \cup (0,1]) \; S^1}$ in $\mathbb{R}^2$	✗	✓
$B_2(0) - \{0\}$ in $\mathbb{R}^3$	✓	✗

Task 3. Consider

$$M_1 = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 2\}$$

$$M_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2+(y-4)^2 < 9\}$$
$$- \{(x,y) \in \mathbb{R}^2 \mid x=0,\ \tfrac{7}{2} \leq y \leq \tfrac{9}{2}\}$$

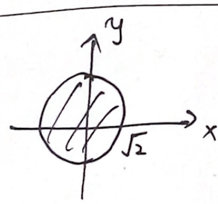
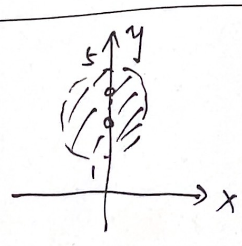
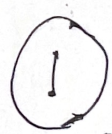
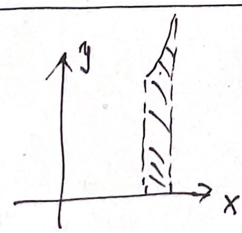
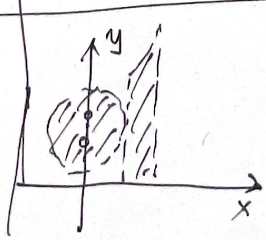
$$M_3 = \{(x,y) \in \mathbb{R}^2 \mid 3<x<4,\ 0<y<2^x\}$$

$$M_4 = M_2 \cup M_3.$$

a) Draw M_j

b) Which M_j is a domain?

c) Which M_j is a sc domain?

M_j	Picture	domain?	∂M_j	sc?
M_1		✗ not open		✓
M_2		✓		✗ ∂M_2 not conn
M_3		✓		✓
M_4		✗ not connected		✗

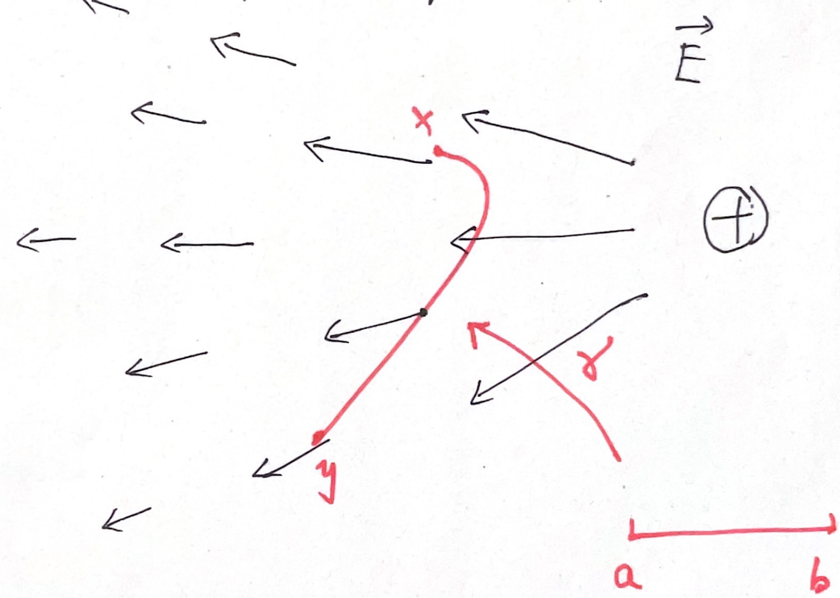
2. Computation: path integration

Intuition: A particle moves in an electric field $\vec{E}$, and get the electrostatic force proportional to $\vec{E}$.

The energy obtained from $\vec{E}$ is

$$\int_\gamma \vec{E} \cdot d\vec{s} = \int_a^b \underset{\substack{\uparrow \\ \text{force}}}{\vec{E}(\gamma(t))} \cdot \underset{\substack{\uparrow \\ \text{distance}}}{\frac{d\gamma}{dt}(t)}\, dt$$

~~This motivate the definition of path~~ in



Def. For $\mathcal{U} \subset \mathbb{R}^n$ open. $\gamma : [a,b] \longrightarrow \mathcal{U}$,

$\vec{F} := (F_1, \dots, F_n) : \mathcal{U} \longrightarrow \mathbb{R}^n$ as a vector field,

we define the path integration.

$$\int_\gamma \vec{F} \cdot d\vec{s} := \int_a^b \vec{F}(\gamma(t)) \frac{d\gamma}{dt}(t)\, dt$$

$$= \int_\gamma \sum_{i=1}^n F_i \, dx_i = \int_a^b \sum_{i=1}^n F_i(\gamma(t)) \frac{dx_i}{dt}(t)\, dt$$

For $f : \mathcal{U} \longrightarrow \mathbb{R}$ a fct, we can define

$$\int_\gamma f\, ds = \int_a^b f(\gamma(t)) \left| \frac{d\gamma}{dt}(t) \right| dt$$

Rmk. $\int_\gamma \vec{F} \cdot d\vec{s}$, $\int_\gamma f\, ds$ do not depend on the parameterization of γ

(As long as the reparameterization preserves orientation)

Ex. For $\mathcal{U} := \mathbb{R}^2 - \{0\}$,

$$\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$$

$$\gamma : [0, 2\pi] \longrightarrow \mathcal{U} \qquad t \mapsto (\cos t, \sin t)$$

Compute $\int_\gamma \vec{F} \cdot d\vec{s}$, $\int_\gamma ds$.

A. $\int_\gamma \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \cdot \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} ds = \int_0^{2\pi} 1\, ds = 2\pi$

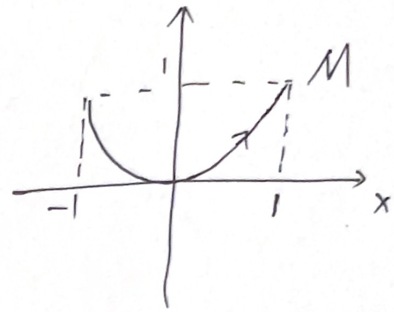
$\int_\gamma ds = \int_0^{2\pi} \left\| \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} \right\| ds = \int_0^{2\pi} 1\, ds = 2\pi$.

Task 1. Let $M = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} x \in [-1,1], \\ y = x^2 \end{array} . \right\}$

(a.) Find two non-equivalent parametrization of M.

A: $\gamma : [-1,1] \to M \quad t \mapsto (t, t^2)$

$\xi : [-1,1] \to M \quad t \mapsto (-t, t^2)$



We have comm diag

$$\begin{tikzcd} t \arrow[d, mapsto] & {[-1,1]} \arrow[d, "\psi"] \arrow[dr, "\gamma"] & \\ -t & {[-1,1]} \arrow[r, "\xi"'] & M \end{tikzcd}$$

But ψ does not preserve the orientation.

(b) For $\vec{F}(x,y) = (x+y, x-y)$, compute

$$\int_\gamma \vec{F} \cdot d\vec{s} \qquad \int_\xi \vec{F} \cdot d\vec{s}$$

A: $\int_\gamma \vec{F} \cdot d\vec{s} = \int_\gamma (x+y)dx + (x-y)dy$

$$\overset{\substack{x=t \\ y=t^2}}{=\!=} \int_{-1}^{1} (t+t^2)dt + (t-t^2)dt^2$$

$$= \int_{-1}^{1} \left((t+t^2) + (t-t^2) \cdot 2t \right) dt$$

$$= \int_{-1}^{1} 3t^2 dt = t^3 \Big|_{-1}^{1} = 2$$

"or": $\int_\gamma \vec{F} \cdot d\vec{s} = \int_{-1}^{1} \vec{F}(\gamma(t)) \frac{d\gamma}{dt}(t)\, dt$

$$= \int_{-1}^{1} \begin{pmatrix} x(t)+y(t) \\ x(t)-y(t) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_{-1}^{1} \left((t+t^2) + (t-t^2) \cdot 2t \right) dt = 2$$

Similarly, $\int_\xi \vec{F} \cdot d\vec{s} = -2$. $\square$

3. Independence of path

In many cases, the path integral only depends on the starting point and the ending point.

We know that

$$\int_a^b f'(t)\,dt = f(b) - f(a) \quad \text{for } f \in C'[a,b]$$

Similarly, for $\mathcal{U} \subseteq \mathbb{R}^n$, $\Phi \in C'(\mathcal{U})$, $\forall \gamma : [a,b] \to \mathcal{U}$, one gets

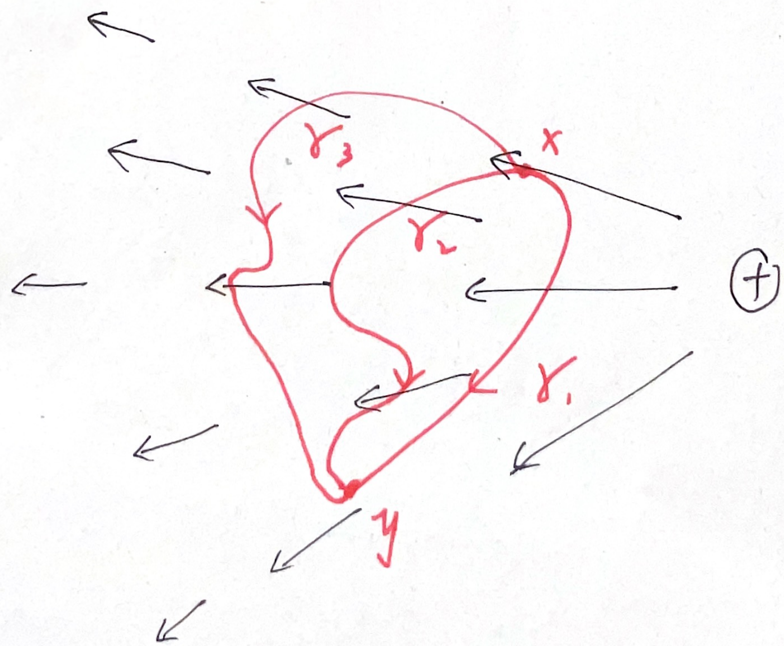
$$\int_\gamma \nabla \Phi \cdot d\vec{\gamma} = \Phi(\gamma(b)) - \Phi(\gamma(a))$$

Note that the right hand side depends only on the endpoints.

Reason: $\displaystyle \int_\gamma \nabla\Phi \cdot d\vec{\gamma} = \int_a^b \begin{pmatrix} (\partial_x \Phi)(\gamma(t)) \\ (\partial_y \Phi)(\gamma(t)) \end{pmatrix} \frac{d\gamma}{dt}(t)\,dt$

$$= \int_a^b \cancel{d\Phi(\gamma(t))} \; \frac{d(\Phi\circ\gamma)}{dt}(t)\,dt$$

$$= \Phi(\gamma(b)) - \Phi(\gamma(a))$$



Q. For $\mathcal{U} \subseteq \mathbb{R}^n$, $\vec{F}$: a vector field on $\mathcal{U}$, when

$$\boxed{\exists\, \Phi \in C^2(\mathcal{U}) \text{ s.t. } \vec{F} = \nabla \Phi \qquad (*)}$$

conservative

In ~~case~~ case $n=2$, when $(*)$ is true,

$$\left.\begin{array}{l} \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{\partial \Phi}{\partial x} \\ \frac{\partial \Phi}{\partial y} \end{pmatrix} \\ \frac{\partial}{\partial y}\frac{\partial \Phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \Phi}{\partial y} \end{array}\right\} \Rightarrow \boxed{\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \qquad (**)}$$

curl-free

Def. For $\mathcal{U} \subseteq \mathbb{R}^n$, $\vec{F}$: a vector field on $\mathcal{U}$.

$\vec{F}$ is called a conservative vector field, when

$$\vec{F} = \nabla \Phi \quad \text{for some } \Phi \in C^2(\mathcal{U}).$$

$\vec{F}$ is called a curl-free vector field, when

$$\operatorname{rot} \vec{F} = 0. \quad \left(\text{e.p. in } \mathbb{R}^2, \text{ equivalent to } \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}\right)$$

Rmk. It is easily seen that

$$\text{conservative} \Rightarrow \text{curl-free}$$

Task 2.

(a) For $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} (x+y+1)e^x - e^y \\ e^x - (x+y+1)e^y \end{pmatrix}$,

show that $\operatorname{rot} \vec{F} = 0$, i.e. $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

(b) For $\alpha \geq 1$, $\gamma_\alpha : [0,1] \longrightarrow \mathbb{R}^2 \quad t \longmapsto (t, t^\alpha)$,

compute $\int_{\gamma_\alpha} \vec{F} \cdot d\vec{s}$.

(c) Let $\beta \in \mathbb{R}$,

$$\xi_\beta : [0,1] \longrightarrow \mathbb{R}^2 \qquad t \longmapsto (\beta - t, \beta - t)$$

determines when we can define a path $\gamma_4 \oplus \xi_\beta$.

(A: $\beta = 1$)

(d) Compute ($\beta \triangleq 1$)

$$\int_{\gamma_4 \oplus \xi_\beta} \vec{F} \cdot d\vec{s}$$

Rmk. (b) & (d) would be much easier if we find $\Phi \in C^2(\mathbb{R}^2)$ s.t. $\nabla \Phi = \vec{F}$.

How could we find it?

For $(x_0, y_0) \in \mathbb{R}^2$, define

$$\Phi(x_0, y_0) := \int_{\gamma:\, 0 \sim (x_0, y_0)} \vec{F} \cdot d\vec{s}$$

$$= \int_{(0,0)}^{(x,0)} F_1\, dx + \int_{(x_0,0)}^{(x_0,y_0)} F_2\, dy$$

$$= \int_0^{x_0} \left((x+1)e^x - 1\right) dx + \int_0^{y_0} e^{x_0} - (x_0 + y + 1)e^y\, dy$$

$$= (xe^x - x)\Big|_{x=0}^{x_0} + \left(e^{x_0} y - x_0 e^y - e^y\right)\Big|_{y=0}^{y_0}$$

$$= (x_0 e^{x_0} - x_0) - 0 + \left(e^{x_0} y_0 - x_0 e^{y_0} - e^{y_0}\right) - (-x_0 - 1)$$

$$= (x_0 + y_0)e^{x_0} - (x_0 + 1)e^{y_0} + 1$$

$$\Rightarrow \Phi(x, y) = (x + y)e^x - (x+1)e^y + 1$$

Notice that we use

$$\boxed{\begin{array}{ll} \int e^x dx = e^x + C & \int x e^x dx = xe^x - e^x + C \\ & \int (x+1) e^x dx = xe^x + C \end{array}}$$

in the computation.

Ex. 1) Verify that $\nabla \Phi = \vec{F}$, i.e. $\begin{cases} \partial_x \Phi = F_1 \\ \partial_y \Phi = F_2 \end{cases}$

2) Show that

$$\int_{\gamma_2} \vec{F} \cdot d\vec{s} = \Phi(1,1) - \Phi(0,0) = 1$$

$\square$

Rmk. For a curl-free vector field $\vec{F}$ on $\mathcal{U} \subseteq \mathbb{R}^n$,

when we try to find $\Phi \in C^1(\mathcal{U})$ s.t. $\nabla \Phi = \vec{F}$,

we always try to define (fix $z_0 \in \mathcal{U}$)

$$\Phi(z) := \int_{\gamma: z_0 \sim z} \vec{F} \cdot d\vec{s}$$

However, this may be not well-defined.

The problem is, we need to choose γ, and different (non-homotopy equivalent) paths may give you different values.

E.g. $\mathcal{U} = \mathbb{R}^2 - \{0\}$, $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{-y}{x^2+y^2} \\ \frac{x}{x^2+y^2} \end{pmatrix}$

$$\gamma_1 : [0, 2\pi] \longrightarrow \mathcal{U} \qquad t \longmapsto (\cos t, \sin t)$$

$$\gamma_0 : [0, 2\pi] \longrightarrow \mathcal{U} \qquad t \longmapsto (1, 0)$$

$$\int_{\gamma_1} \vec{F} \cdot d\vec{s} = 2\pi \neq 0 = \int_{\gamma_0} \vec{F} \cdot d\vec{s}$$

This inequality prohibits $\vec{F}$ from being conservative.

Rmk. In future, you may see that

$$0 \longrightarrow \Omega^0(\mathcal{U}) \xrightarrow{\nabla} \Omega^1(\mathcal{U}) \xrightarrow{\text{rot}} \Omega^2(\mathcal{U}) \rightarrow \cdots \rightarrow \Omega^{n-1}(\mathcal{U}) \xrightarrow{\text{div}} \Omega^n(\mathcal{U}) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im} \nabla \longrightarrow \ker \text{rot} \longrightarrow H^1(\mathcal{U}; \mathbb{R}) \longrightarrow 0$$

$\uparrow$ conservative $\quad$ $\uparrow$ curl-free $\quad$ $\uparrow$ homology gp

[characterizing how curl-free v.f. is far away from conservative v.f.]

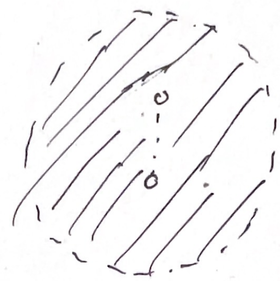
e.g. When $\mathcal{U}$ is simply connected,

conservative $\Leftrightarrow$ curl-free

Task 3.(d)

For M_2, construct

a curl-free but not conservative vector field.



M_2