

Eine Woche, ein Beispiel

1.9 Excision

Goal: state the homology/cohomology/homotopy excision thm and give applications.
^ don't want to see the proof, I'm lazy...

The main ref would be Hatcher, and we only consider the singular homology.

<http://www.math.uni-bonn.de/~schwede/Blakers-Massey.pdf>

Def. An excisive triad (Y, Y_1, Y_2) Y : topo space, $Y_1, Y_2 \subset Y$ s.t.
 $Y = \dot{Y}_1 \cup \dot{Y}_2$
 i.e. the open cores of Y_1 and Y_2 cover Y .

Thm (Homology excision thm, [Thm 2.20, Cor 2.24])

Let X space, $A, B \subset X$ subspace, $X = \dot{A} \cup \dot{B}$, $C = A \cap B$
 or: Let X CW cplx, $A, B \subset X$ subcplx, $X = A \cup B$, $C = A \cap B$



then $L_*: H_i(A, C; \mathbb{Z}) \longrightarrow H_i(X, B; \mathbb{Z})$ is iso.

Cor [Prop 2.22] Let (X, A) be an NDR-pair. then

$\pi_*: H_i(X, A; \mathbb{Z}) \longrightarrow \tilde{H}_i(X/A; \mathbb{Z})$ is iso.

Proof.



$$H_i(X, A; \mathbb{Z}) \xrightarrow{LES} H_i(X \cup (A \times [0, 1)), A; \mathbb{Z}) \xrightarrow{\cong} H_i(X \cup CA, CA; \mathbb{Z}) \xrightarrow{LES} \tilde{H}_i(X \cup CA; \mathbb{Z}) \sim \tilde{H}_i(X/A; \mathbb{Z})$$

$X \cup CA \longrightarrow (X \cup CA)/CA \cong X/A$ is a homotopy equivalence since [Prop 0.17]

• CA is contractible

• (X, A) satisfies HEP $\Rightarrow (X \cup CA, CA)$ satisfies HEP

For the equivalent definitions of NDR-pair, see here:

<https://math.stackexchange.com/questions/3547820/neighborhood-deformation-retracts-vs-cofibrations>
 or Prop A.6 in url: <https://www.math.univ-paris13.fr/~schwartz/FIMFA/Ando.pdf>

We also have the third version of excision thm.

Thm. Suppose (X, A) , (Y, B) are two relative CW-pairs, $f: X \rightarrow Y$ send A to B , and the square

$$\begin{array}{ccc} A & \longrightarrow & X \\ f|_A \downarrow & & \downarrow f \\ B & \longrightarrow & Y \end{array}$$

is a pushout of spaces, then

$$f_*: H_i(X, A; \mathbb{Z}) \longrightarrow H_i(Y, B; \mathbb{Z}) \text{ is an iso.}$$

Proof. $Y = B \cup_{f|_A} X$, i.e. Y is gotten by attaching relative cells of (X, A) to B .

Step 1. If all relative cells of (X, A) are of dim $n+1$, then

$$\begin{array}{ccc} f(X) & \supset & B \cup \bigcup_{i \in I} (S_i^n \times [0, 1]) \\ & \supset & \bigcup_{i \in I} (S_i^n \times [0, 1]) \\ & \supset & f(A) \end{array}$$

$$\begin{aligned} H_i(Y, B; \mathbb{Z}) &\stackrel{LES}{\cong} H_i(Y, B \cup \bigcup_{i \in I} (S_i^n \times [0, 1])) \xrightarrow{\cong} H_i(f(X), f(A) \cup \bigcup_{i \in I} (S_i^n \times [0, 1])) \stackrel{LES}{\cong} H_i(f(X), f(A); \mathbb{Z}) \\ &= H_i(f(X), f(A); \mathbb{Z}) \cong \tilde{H}_i(f(X)/f(A); \mathbb{Z}) \cong \tilde{H}_i(X/A; \mathbb{Z}) \cong H_i(X, A; \mathbb{Z}) \end{aligned}$$

Step 2. Suppose that every square in the diagram

$$\begin{array}{ccccc} A & \longrightarrow & C & \longrightarrow & X \\ \downarrow f|_A & & \downarrow f|_C & & \downarrow f \\ B & \longrightarrow & D & \longrightarrow & Y \end{array}$$

is a pushout of spaces, and

$$(f|_C)_*: H_i(C, A; \mathbb{Z}) \longrightarrow H_i(D, B; \mathbb{Z})$$

$$f_*: H_i(X, C; \mathbb{Z}) \longrightarrow H_i(Y, D; \mathbb{Z})$$

are iso, then by the naturality of LES and 5-lemma,

$$f_*: H_i(X, A; \mathbb{Z}) \longrightarrow H_i(Y, B; \mathbb{Z}) \text{ is iso.}$$

Step 3. Consider the diagram

$$\begin{array}{ccccccc} A = X^{(-1)} & \longrightarrow & X^{(0)} & \longrightarrow & \dots & \longrightarrow & X^{(n)} \longrightarrow \dots \\ & & \downarrow & & & & \downarrow \\ B = Y^{(-1)} & \longrightarrow & Y^{(0)} & \longrightarrow & \dots & \longrightarrow & Y^{(n)} \longrightarrow \dots \end{array}$$

and take the colimit.

Thm (MV sequence, LES)

Let X space, $U, V \subseteq X$ open subset, $U \cup V = X$. Then we get a LES

$$\tilde{H}_{n+1}(X; \mathbb{Z}) \longrightarrow$$

$$\tilde{H}_n(U \cup V; \mathbb{Z}) \longrightarrow \tilde{H}_n(U; \mathbb{Z}) \oplus \tilde{H}_n(V; \mathbb{Z}) \longrightarrow \tilde{H}_n(X; \mathbb{Z}) \longrightarrow$$

MV \rightarrow excision: <https://mathoverflow.net/questions/97621/mayer-vietoris-implies-excision>

excision \rightarrow MV: <https://www.math.ru.nl/~gutierrez/files/homology/Lecture06.pdf>

Please be aware of the conditions in the theorems. We have many versions of theorems when we slightly change the conditions, but I don't want to go to the most generality (Actually I don't know the most general condition).

E.g. $H_n(\Delta^n, \partial\Delta^n; \mathbb{Z}) \xrightarrow{SES} H_{n-1}(\partial\Delta^n, \Lambda; \mathbb{Z}) \cong H_{n-1}(\Delta^{n-1}, \partial\Delta^{n-1}; \mathbb{Z}) \xrightarrow{\text{induction}} \mathbb{Z}.$

E.g. The local homology groups $H_n(x) := H_n(X, X - \{x\}; \mathbb{Z})$ can be defined locally, i.e. $H_n(X, X - \{x\}; \mathbb{Z}) \cong H_n(U, U - \{x\}; \mathbb{Z})$ for $U \subseteq X$ open and $x \in U$.

Given a sm map $f: M \rightarrow N$ between mflds of dim n ,
a pt $y \in N$ s.t. $f^{-1}(y) \stackrel{\Delta}{=} \{x_1, \dots, x_n\}$ is finite,
we can define the local degree $\deg_x f \in \mathbb{Z}$ at $x \in f^{-1}(y)$.

$(U \cap f^{-1}(y) = \{x\}) \quad f_*: H_n(U, U - \{x\}; \mathbb{Z}) \xrightarrow{\cong \mathbb{Z}} H_n(N, N - \{y\}; \mathbb{Z}) \xrightarrow{\cong \mathbb{Z}} [N] \quad [U] \mapsto \deg_x f \cdot [N]$

When M, N are cpt, we can also define the global degree $\deg f \in \mathbb{Z}$:

$f_*: H_n(M; \mathbb{Z}) \xrightarrow{\cong \mathbb{Z}} H_n(N; \mathbb{Z}) \xrightarrow{\cong \mathbb{Z}} [N] \quad [M] \mapsto \deg f \cdot [N]$

we have the equality $\deg f = \sum_{x \in f^{-1}(y)} \deg_x f$.

Thm (Homotopy excision thm [Thm 4.23])

Let X : CW cplx, A, B subcplx, $X = A \cup B$, $C = A \cap B$ nonempty and connected.

If (A, C) is m -connected,

(B, C) is n -connected,

then $\pi_i(A, C, x_0) \longrightarrow \pi_i(X, B, x_0)$

is $\begin{cases} \text{iso} & i < m+n \\ \text{surj} & i = m+n \end{cases}$

$x_0 \in C$



Cor [Prop 4.28] Suppose X, A are CW cplx, $A \subset X$ is subcplx.

If (X, A) is r -connected, A is s -connected,

then the map $\pi_i(X, A, x_0) \longrightarrow \pi_i(X/A, x_0)$ is $\begin{cases} \text{iso} & i < r+s+1 \\ \text{surj} & i = r+s+1 \end{cases}$

[Proof.



$$\pi_i(X, A, x_0) \longrightarrow \pi_i(X \cup CA, CA, x_0) \xrightarrow{\text{LES}} \pi_i(X \cup CA, x_0) \xrightarrow{\text{homotopy}} \pi_i(X/A, x_0)$$

Thm (Freudenthal suspension thm [Cor 4.24])

Let $n \geq 1$, X be an $(n-1)$ -connected CW cplx, then the suspension map

$$\Sigma: \pi_i(X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, x_0)$$

$$[f: S^i \rightarrow X] \longmapsto [\Sigma f: S^{i+1} \rightarrow \Sigma X]$$

is $\begin{cases} \text{iso} & i < 2n-1 \\ \text{surj} & i = 2n-1 \end{cases}$

Rmk. Freudenthal suspension thm \leadsto concept of the stable homotopy gp.

[Proof.

$$\pi_i(X, x_0) \xrightarrow{\text{LES}} \pi_{i+1}(CX, X, x_0) \longrightarrow \pi_{i+1}(\Sigma X, CX, x_0) \cong \pi_{i+1}(\Sigma X)$$

is $\begin{cases} \text{iso} & i+1 < n+n \\ \text{surj} & i+1 = n+n \end{cases}$

