

Eine Woche, ein Beispiel

4.13 lattices defining abelian variety

Ref:

[Deb99]: Complex tori and abelian varieties

[Mum74]: Mumford, David, Abelian varieties. Oxford university press Oxford, 1974.

[LR22]: Herbert Lange and Rubi E. Rodríguez. Decomposition of Jacobians by Prym Varieties. 2310.

[BL04]: Christina Birkenhake, and Herbert Lange. Complex Abelian Varieties. 2nd augmented ed. Grundlehren Math. Wiss. Berlin: Springer, 2004.

This document try to work out [Deb99, p28, Ex (3)]

Claim 1. [Mum74, p35, (1) \Leftrightarrow (4)]

Let $\Lambda \subseteq \mathbb{C}^g$ be a full lattice. Then

\mathbb{C}^g/Λ is an abelian variety

$\Leftrightarrow \exists$ an \mathbb{R} -bilinear alternating form $\omega: \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ s.t.

$$\begin{cases} \omega(\Lambda \times \Lambda) \subseteq \mathbb{Z} \\ \omega(x, ix) > 0 \quad \forall x \neq 0 \\ \omega(ix, iy) = \omega(x, y) \end{cases} \quad (*)$$

i.e. an integral Kähler form

$$\begin{aligned} \omega(u, v) &= \operatorname{Re} h(iu, v) = \operatorname{Im} h(u, v) & h(au, v) &= \bar{a} h(u, v) \\ &= -\operatorname{Re} h(u, iv) \end{aligned}$$

From now on. suppose $\Lambda = \langle \lambda_1, \dots, \lambda_{2g} \rangle_{\mathbb{Z}}$, we denote

$$\Pi := (a_{ij})_{i,j=1}^{2g} := (\lambda_1^*, \dots, \lambda_{2g}^*)^T \quad \Rightarrow \lambda_i^* = \sum a_{ij} e_j^*$$

The matrix Π encodes all information of the lattice (add a basis)

Q: For what kind of conditions of Π , can we find ω satisfying (*)?

Let $\omega = \sum_{i < j} c_{ij} \lambda_i^* \wedge \lambda_j^*$, then

$$\omega(\Delta \times \Delta) \subset \mathbb{Z} \Leftrightarrow c_{ij} \in \mathbb{Z} \quad \forall i, j.$$

Write $x = \sum x_i e_i$, $y = \sum y_i e_i$, we get

$$\begin{aligned} \omega(x, y) &= \sum_{k, l} x_k y_l \omega(e_k, e_l) \\ &= \sum_{k, l} x_k y_l \sum_{i < j} c_{ij} \lambda_i^* \wedge \lambda_j^*(e_k, e_l) \\ &= \sum_{k, l} x_k y_l \sum_{i < j} c_{ij} (a_{ik} a_{jl} - a_{jk} a_{il}) \\ \omega(x, iy) &= \sum_{k, l} x_k y_{l+g} \sum_{i < j} c_{ij} (a_{ik} a_{jl} - a_{jk} a_{il}) \\ &= \sum_{k, l} x_k y_l \sum_{i < j} c_{ij} (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \\ \omega(ix, iy) &= \sum_{k, l} x_{k+g} y_{l+g} \sum_{i < j} c_{ij} (a_{ik} a_{jl} - a_{jk} a_{il}) \\ &= \sum_{k, l} x_k y_l \sum_{i < j} c_{ij} (a_{i(k+g)} a_{j(l+g)} - a_{j(k+g)} a_{i(l+g)}) \end{aligned}$$

Therefore,

$\omega(x, ix) > 0$ is an open condition on Π .

$$\begin{aligned} \omega(ix, iy) = \omega(x, y) &\Leftrightarrow \sum_{i < j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0 \\ &\forall k, l \in \{1, \dots, 2g\} \end{aligned}$$

Claim 2. \mathbb{C}^g / Δ is an abelian variety

$$\begin{aligned} \Delta &= \langle \lambda_1, \dots, \lambda_{2g} \rangle_{\mathbb{Z}} \\ \Pi &= (a_{ij})_{i,j=1}^{2g} = (\lambda_1^*, \dots, \lambda_{2g}^*)^T \end{aligned}$$

$\Leftrightarrow \exists c_{ij} \in \mathbb{Z}$ for all $i, j \in \{1, \dots, 2g\}$ s.t.

$$\textcircled{1} \sum_{i < j} c_{ij} \left(\begin{vmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{vmatrix} - \begin{vmatrix} a_{i(k+g)} & a_{i(l+g)} \\ a_{j(k+g)} & a_{j(l+g)} \end{vmatrix} \right) = 0 \quad \forall k, l \in \{1, \dots, 2g\}$$

$$\textcircled{2} \left(\sum_{i < j} c_{ij} (a_{ik} a_{j(l+g)} - a_{jk} a_{i(l+g)}) \right)_{k, l=1}^{2g} \text{ is def positive.}$$

Rmk. The equations in ① are not linear independent.
 There are at most $\binom{g}{2} \cdot \frac{4 \cdot 2}{4} = g(g-1)$ equations.

Reason: When $l=k$ or $l=k+g$, we get 0.
 e.p. when $g=1$, we get no condition.

When $l \neq k$ & $l \neq k+g$, denote $\{k, l, k+g, l+g\} = \{k_1, k_2, k_3, k_4\}$,

$$a_{ik_1} a_{jk_2} - a_{ik_2} a_{jk_1} - a_{ik_3} a_{jk_4} + a_{ik_4} a_{jk_3}$$

$k_1 \leftrightarrow k_2$	$k_3 \leftrightarrow k_4$	neg
$k_1 \leftrightarrow k_3$	$k_2 \leftrightarrow k_4$	neg
$k_1 \leftrightarrow k_4$	$k_2 \leftrightarrow k_3$	same

Cor. Since

$$A_g \subseteq \{C^n / \Delta \text{ a.v.}\} \cong \{\pi \in \mathbb{R}^{2g \times 2g} \mid \pi \text{ satisfies } ① \oplus \} / GL_g(\mathbb{C})$$

for some (a_{ij})

We know

$$\dim A_g = \frac{1}{2} ((2g)^2 - g(g-1)) - g^2 = \frac{1}{2} g(g+1).$$

Rmk. [BL04, 4.2]

We are actually reconstructing the Riemann relations.

Let $X = \mathbb{C}^g / \Lambda$, $\Lambda = \langle \lambda_1, \dots, \lambda_{2g} \rangle$, $\lambda_i \in \mathbb{C}^g$
 $\pi = (\lambda_1, \dots, \lambda_{2g})_{g \times 2g} \in M^{g \times 2g}(\mathbb{C})$
 For $\omega: \mathbb{C}^g \times \mathbb{C}^g \rightarrow \mathbb{R}$ an nondeg alternating form, denote
 $A := (\omega(\lambda_i, \lambda_j))_{2g \times 2g} \in M^{2g \times 2g}(\mathbb{Z})$.
 Then, $\omega(\Lambda \times \Lambda) \subseteq \mathbb{Z}$

- ① $\omega(ix, iy) = \omega(x, y) \Leftrightarrow \pi A^{-1} \pi^T = 0$
- ② $\omega(x, ix) > 0 \quad \forall x \neq 0 \Leftrightarrow i \pi A^{-1} \pi^H > 0$
 $\uparrow \pi^H = \overline{\pi}^T$

With the symplectic basis, $\mathcal{L} = \mathcal{L}(H, \chi)$ is of type D
 the Riemann relations become:

- ① $\pi_2 D^{-1} \pi_1^T - \pi_1 D^{-1} \pi_2^T = 0$
- ② $i \pi_2 D^{-1} \pi_1^H - i \pi_1 D^{-1} \pi_2^H > 0$