Eine Woche, ein Beispiel

5.12 sheaf version of $\otimes$ & Hom

1. def of sheaf Hom
2. def of sheaf $\otimes$

sheaf version of Tensor-Hom adjunction is left in the next document.

Compared with $\otimes$, Hom is more delicated, and it is harder than you expected.

1. def of sheaf Hom

$$\mathrm{Hom}_A(-,-) : (A\text{-Mod})^{op} \times A\text{-Mod} \longrightarrow A\text{-Mod} \qquad A: \text{comm ring}$$
$$\rightsquigarrow$$
$$\mathrm{Hom}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow \mathbb{Z}\text{-Mod}$$
$$\rightsquigarrow$$
$$\underline{\mathrm{Hom}}_{Sh(X)}(-,-) : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$
$$\rightsquigarrow$$
$$R\underline{\mathrm{Hom}}_{\mathcal{D}^+(X)}(-,-) : \mathcal{D}^-(X)^{op} \times \mathcal{D}(X) \longrightarrow \mathcal{D}(X)$$

non-derived sheaf Hom

Def [Vakil, 2.3.1] For $\mathcal{F}, \mathcal{G} \in Sh(X)$, a morphism of sheave
$$\phi : \mathcal{F} \longrightarrow \mathcal{G}$$
is the data of maps
$$\phi(\mathcal{U}) : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}) \quad \text{for all } \mathcal{U} \subseteq X \text{ open.}$$
which is compatible with restriction.

We write
$$\phi \in \mathrm{Hom}_{Sh(X)}(\mathcal{F}, \mathcal{G})$$

Similarly, one can define
$$\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F}^{\cdot}, \mathcal{G}^{\cdot})$$
as the set of morphisms in $\mathcal{D}(X)$.

Def [Vakil, 2.3.C] (Sheaf Hom / Internal Hom)

For $\mathcal{F}, \mathcal{G} \in Sh(X)$, one gets a sheaf $\underline{Hom}(\mathcal{F}, \mathcal{G}) \in Sh(X)$ given by

$$\left(\underline{Hom}(\mathcal{F}, \mathcal{G})\right)(\mathcal{U}) = Hom_{Sh(\mathcal{U})}(\mathcal{F}|_{\mathcal{U}}, \mathcal{G}|_{\mathcal{U}})$$

Cor

$$Hom = \Gamma \circ \underline{Hom} : Sh(X)^{op} \times Sh(X) \xrightarrow{\underline{Hom}} Sh(X) \xrightarrow{\Gamma} Abel$$

⚠ Even though $(\mathcal{F} \otimes \mathcal{G})_p \cong \mathcal{F}_p \otimes \mathcal{G}_p$,

$\underline{Hom}$ does not commute with taking stalks.

$$(\underline{Hom}(\mathcal{F}, \mathcal{G}))_p \xrightarrow{\ncong} Hom(\mathcal{F}_p, \mathcal{G}_p)$$

It's neither inj nor surj.

[Left adj comm with limit, $\otimes \dashv \underline{Hom}$].

Ex. Try to compute   coefficient $\mathbb{Q}$.

$$\underline{Hom}_{Sh(X)}(\underline{\mathbb{Q}}_X, \mathcal{F}) \cong \mathcal{F}$$
$$\underline{Hom}_{Sh(X)}(j_! \underline{\mathbb{Q}}_{\mathcal{U}}, \mathcal{F}) \cong j_*(\mathcal{F}|_{\mathcal{U}})$$
$$\underline{Hom}_{Sh(\mathbb{C})}(sky_0(\mathbb{Q}), \underline{\mathbb{Q}}_{\mathbb{C}}) \cong 0$$

derived sheaf Hom

Def. For $\mathcal{F}, \mathcal{G} \in Sh(X)$, the derived internal Hom

in general, $\mathcal{F} \in \mathcal{D}(X)^-$, $\mathcal{G} \in \mathcal{D}^+(X)$

$$R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) \in \mathcal{D}^+(X)$$

is given by

$$\begin{array}{llll} \underline{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \mathcal{I}^\cdot) & \text{when } \mathcal{G} \xrightarrow{\simeq} \mathcal{I}^\cdot & \text{inj resolution} \\ \underline{Hom}_{\mathcal{C}(X)}(\mathcal{P}^\cdot, \mathcal{G}) & \text{when } \mathcal{F} \xleftarrow{\simeq} \mathcal{P}^\cdot & \text{proj resolution} \end{array}$$

Here,

$$\underline{Hom} : Sh(X)^{op} \times Sh(X) \longrightarrow Sh(X)$$

is extended to the double complex

$\mathcal{C}(X) :=$ complex of sheaves on X, temperate notation

$$\underline{Hom}_{\mathcal{C}(X)} : \mathcal{C}(X)^{op} \times \mathcal{C}(X) \longrightarrow \mathcal{C}(X)$$

Other versions of sheaf Hom

$$\begin{array}{ccc} Hom_A(-,-) & \rightsquigarrow & RHom_A(-,-) \\ \wr\downarrow & & \wr\downarrow \\ Hom_{Sh(X)}(-,-) & \rightsquigarrow & RHom_{\mathcal{D}^+(X)}(-,-) \\ \wr\downarrow & & \wr\downarrow \\ \underline{Hom}_{Sh(X)}(-,-) & \rightsquigarrow & R\underline{Hom}_{\mathcal{D}^+(X)}(-,-) \end{array}$$

$$Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}^\cdot, \mathcal{G}^\cdot)$$

$$Hom_{Sh(X)}(\mathcal{F}, \mathcal{G}) = Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R^0 Hom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}).$$

$$RHom_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G}) = R\Gamma \circ R\underline{Hom}_{\mathcal{D}^+(X)}(\mathcal{F}, \mathcal{G})$$

2. def of sheaf $\otimes$

$$M\otimes - \dashv \mathrm{Hom}_A(M,-) : A\text{-Mod} \longrightarrow A\text{-Mod} \qquad A:\text{comm ring}$$

$$\Downarrow$$

$$\mathcal{F}\otimes \pi_X^*(-) \dashv \mathrm{Hom}_{Sh(X)}(\mathcal{F},-) : \mathbb{Z}\text{-Mod} \longrightarrow Sh(X)$$

$$\Downarrow$$

$$\mathcal{F}\otimes - \dashv \underline{\mathrm{Hom}}_{Sh(X)}(\mathcal{F},-) : Sh(X) \longrightarrow Sh(X)$$

$$\Downarrow$$

$$\mathcal{F}^{\cdot}\otimes^{L} - \dashv R\underline{\mathrm{Hom}}_{D^+(X)}(\mathcal{F}^{\cdot},-) : D(X) \longrightarrow D(X)$$

non-derived sheaf $\otimes$

Def [Vakil 2.6.J] For $\mathcal{F},\mathcal{G}\in Sh(X)$,
$\mathcal{F}\otimes\mathcal{G}\in Sh(X)$ is given by sheafification of

$$\mathcal{F}\ \ \mathcal{G}$$
$$\searrow\ \swarrow$$
$$X$$

$$(\mathcal{F}\otimes^{pre}\mathcal{G})(U) = \mathcal{F}(U)\otimes_{\mathbb{Z}}\mathcal{G}(U),$$

this defines a bifctor

$$-\otimes- : Sh(X)\times Sh(X)\longrightarrow Sh(X)$$

Cor. $$(\mathcal{F}\otimes\mathcal{G})_p = \varinjlim_{p\in U}\left(\mathcal{F}(U)\otimes_{\mathbb{Z}}\mathcal{G}(U)\right) = \mathcal{F}_p\otimes_{\mathbb{Z}}\mathcal{G}_p$$

i.e., $\otimes$ commutes with taking stalks.

Ex. Verify that

$$(\mathcal{F}\otimes\mathcal{G})(U) = (\mathcal{F}|_U\otimes\mathcal{G}|_U)(U) = \Gamma(\mathcal{F}|_U\otimes\mathcal{G}|_U)$$

and compare it with the formula

$$\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})(U) = \mathrm{Hom}(\mathcal{F}|_U,\mathcal{G}|_U) = \Gamma(\underline{\mathrm{Hom}}(\mathcal{F}|_U,\mathcal{G}|_U))$$

Can't define $\otimes$ in this way though.

In general, one has formula

$$f^*(\mathcal{F}\otimes\mathcal{F}') \cong f^*\mathcal{F}\otimes f^*\mathcal{F}'.$$

$$\begin{array}{ccc} \mathcal{G} & & \mathcal{F}\ \ \mathcal{F}' \\ | & & \backslash\,/ \\ f: Y & \longrightarrow & X \end{array}$$

Combined with formulas

$$\begin{aligned} f^*\underline{\mathbb{Q}}_X &= \underline{\mathbb{Q}}_Y \\ \underline{\mathbb{Q}}_X \otimes \mathcal{F} &= \mathcal{F}, \end{aligned}$$

it means that

$$f^*: (Sh(X), \otimes_{Sh(X)}) \longrightarrow (Sh(Y), \otimes_{Sh(Y)})$$

is a (strong) monoidal fctor.

Ex. Try to compute

$$\begin{aligned} j_!\underline{\mathbb{Q}}_U \otimes_{Sh(X;\mathbb{Q})} \mathcal{F} &\cong j_!(\mathcal{F}|_U) \\ sky_p(\mathbb{Q}) \otimes_{Sh(\mathbb{C};\mathbb{Q})} \mathcal{F} &\cong i_*\mathcal{F}_p \end{aligned}$$

In general, one has projection formula

$$f_!(f^*\mathcal{F}\otimes\mathcal{G}) \xleftarrow{\cong} \mathcal{F}\otimes f_!\mathcal{G} \quad \text{when } \mathcal{F} \text{ is flat.}$$

So this formula doesn't cover the Ex. (ˇ∩ˇ)
Even so, their proofs are similar: checking stalkwise.