Eine Woche, ein Beispiel

12.1 weights of type E

It feels incomplete to discuss only the type E case without addressing the other classical cases.

Hence, this document serves as a complement to [2024.12.01].

⚠ There are some new phenomenons outside type E (which are not essential).

1. The formula becomes

$$2\frac{\langle \varpi_i , \alpha_j \rangle}{\langle \alpha_j , \alpha_j \rangle} = \delta_{ij} \qquad \text{i.e.,} \quad \langle \varpi_i \frac{2}{\langle \alpha_i, \alpha_i \rangle} , \alpha_j \rangle = \delta_{ij}$$

when $i \neq j$, $\frac{2}{\langle \alpha_i, \alpha_i \rangle}$ won't impact, as $\delta_{ij} = 0$

$$s_k v = v - 2\frac{\langle \alpha_k , v \rangle}{\langle \alpha_k , \alpha_k \rangle} \alpha_k$$

2. $A = (\langle \alpha_i , \alpha_j \rangle)_{i,j}$ is not Cartan matrix. It is $\left(2\frac{\langle \alpha_i , \alpha_j \rangle}{\langle \alpha_i , \alpha_i \rangle}\right)_{i,j}$.

3. The minuscule weight orbit may not generate the whole lattice in type A, B, D

4. The minuscule weight orbit may not be the wts nearest to the origin in type A, B, C, D

Since the coordinate itself already offers good symmetry(compared with type E case), we will omit many details.

1. A_n E.g. $n=5$

– Weights nearest to the origin

There are n many minuscule representations of A_n:

#	typical coordinates	symbol	
$\binom{6}{1}$	$\frac{1}{6}(5,-1,-1,-1,-1,-1)^T$	v_i	●—○—○—○—○
$\binom{6}{2}$	$\frac{1}{6}(4,4,-2,-2,-2,-2)^T$	v_{ij}	○—●—○—○—○
$\binom{6}{3}$	$\frac{1}{6}(3,3,3,-3,-3,-3)^T$	v_{ijk}	○—○—●—○—○
$\binom{6}{4}$	$\frac{1}{6}(2,2,2,2,-4,-4)^T$	v_{ijkl}	○—○—○—●—○
$\binom{6}{5}$	$\frac{1}{6}(1,1,1,1,1,-5)^T$	v_{ijklm}	○—○—○—○—●

$$|v_i|^2 = \langle v_i, v_i \rangle \in \left\{\frac{5}{6}, \frac{4}{3}, \frac{3}{2}\right\}$$

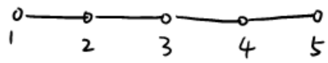
in general, in $\binom{n+1}{k}$, $\langle v_i, v_i \rangle = \frac{k(n+1-k)}{n+1}$ in $\left\{\sum_{i=1}^{n+1} z_i = 0\right\} \cong \mathbb{R}^n$

Restrict to the standard rep case,

$$\langle v_i, v_j \rangle \in \left\{\frac{n}{n+1}, -1\right\}.$$

The graph has no edges.

- Simple roots



$$\{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5\}$$
$$= \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6\}$$
$$= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \right\}$$

Ex. Verify that all the $2\binom{n+1}{2} = 30$ roots are given by

#	typical coordinates	symbol
30	$(1, -1, 0, 0, 0, 0)^T$	α_{1-2}

- Fundamental weights

$$\{\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi_4,\ \varpi_5\}$$
$$= \{v_1,\ v_{12},\ v_{123},\ v_{1234},\ v_{12345}\}$$
$$= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{6}\begin{pmatrix}4\\4\\-2\\-2\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}3\\3\\3\\-3\\-3\\-3\end{pmatrix}, \frac{1}{6}\begin{pmatrix}2\\2\\2\\2\\-4\\-4\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\}$$
$$= \left\{ \frac{1}{6}\begin{pmatrix}5\\-1\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}2\\2\\-1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\-1\\-1\\-1\end{pmatrix}, \frac{1}{3}\begin{pmatrix}1\\1\\1\\1\\-2\\-2\end{pmatrix}, \frac{1}{6}\begin{pmatrix}1\\1\\1\\1\\1\\-5\end{pmatrix}, \right\}$$

- Weyl group action

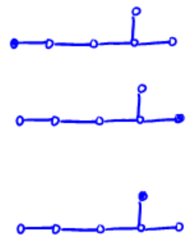
S_{n+1} acts by permutations. Nothing special.

2. D_n E.g. $n=6$ $n>4$ for avoiding special cases.

- Weights nearest to the origin

There are 3 minuscule representations of D_n:

	#	typical coordinates		symbol
vector	$2\cdot 6$	$(1, 0, 0, 0, 0, 0)^T$		v_i & $-v_i$
half spin	2^5	$\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)^T$	odd (neg) sign	$v_{\pm\pm\pm\pm\pm\pm}$
		$\frac{1}{2}(1, 1, 1, 1, 1, -1)^T$		v_-
	2^5	$\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1)^T$	even (neg) sign	$v_{\pm\pm\pm\pm\pm\pm}$
		$\frac{1}{2}(1, 1, 1, 1, 1, 1)^T$		v_+



in general, $\langle v_i, v_i \rangle \in \{1, \frac{n}{4}\}$ in $\mathbb{R}^n$.

Restrict to the standard rep case,

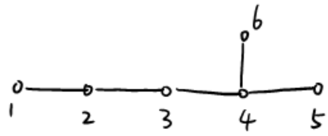
$\langle v_i, v_j \rangle \in \{1, 0, -1\}$.

edge

The graph is ≣.

Here, the weights corresponding to standard reps does not generate all other weights.

– Simple roots



$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$
$$= \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5 - v_6,\ v_5 + v_6\}$$
$$= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\\1\end{pmatrix}, \right\}$$

Ex. Verify that all the $4\binom{n}{2} = 60$ roots are given by

#	typical coordinates	symbol
60	$(\pm1, \pm1, 0, 0, 0, 0)^T$	$\alpha_{\pm i \pm j}$
	$(-1, 1, 0, 0, 0, 0)^T$	α_{-1+2}

– Fundamental weights

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5, \varpi_6\}$$
$$= \{v_1,\ v_1 + v_2,\ \sum_{i=1}^{3} v_i,\ \sum_{i=1}^{4} v_i,\ v_-,\ v_+\}$$
$$= \left\{ \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\\-1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix}, \right\}$$

– Weyl group action

$$s_k = s_{(k,k+1)} \quad \text{for } k = 1, \dots, n-1.$$

$$s_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & & -1 \\ & & & -1 & \end{pmatrix}$$

$$W(D_n) \cong (\mathbb{Z}/2\mathbb{Z})^{n-1} \rtimes S_n \subseteq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

3. D_4

– Weights nearest to the origin

D_4 is more symmetric.

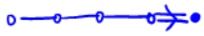
#	typical coordinates	symbol	
$8 = 2\cdot 4$	$(1, 0, 0, 0)^T$	v_i & $-v_i$	
$8 = 2^3$	$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$ odd sign	$v_{\pm\pm\pm\pm}$	
	$\frac{1}{2}(1, 1, 1, 1)^T$	v_-	
$8 = 2^3$	$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$ even sign	$v_{\pm\pm\pm\pm}$	
	$\frac{1}{2}(1, 1, 1, 1)^T$	v_+	

If not restricted to the standard representation case,

$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1\}.$$

4. B_n E.g. $n=5$

– minuscule weights

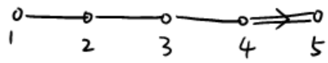


The minuscule weight of B_n is usually not the nearest weight orbit:

	#	typical coordinates	symbol
Spin	$32 = 2^5$	$\frac{1}{2}(\pm1, \pm1, \pm1, \pm1, \pm1)^T$	$v_{\pm\pm\pm\pm\pm}$
		$\frac{1}{2}(1, -1, 1, 1, -1)^T$	v_{+-++-}

$$\langle v_i, v_j \rangle \in \left\{ \frac{n}{4}, \frac{n-2}{4}, \cdots, \frac{-n}{4} \right\}. \quad \text{in } \mathbb{R}^n$$

- Simple roots



$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$$
$$= \{v_1 - v_2,\ v_2 - v_3,\ v_3 - v_4,\ v_4 - v_5,\ v_5\}$$
$$= \left\{ \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix} \right\}$$

Ex. Verify that all the $2 \cdot n^2 = 50$ roots are given by

#	typical coordinates	symbol
$20 = 2 \cdot \binom{5}{2}$	$(1, -1, 0, 0, 0)^T$	v_{1-2}
$10 = 2 \cdot 5$	$(1, 0, 0, 0, 0)^T$	v_1
$20 = 2 \cdot \binom{5}{2}$	$(1, 1, 0, 0, 0)^T$	v_{1+2} & v_{-1-2}

- Fundamental weights

$$\{\varpi_1, \varpi_2, \varpi_3, \varpi_4, \varpi_5\}$$
$$= \{v_1,\ v_1 + v_2,\ \sum_{i=1}^{3} v_i,\ \sum_{i=1}^{4} v_i,\ v_{+++++}\}$$
$$= \left\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} \right\}$$

- Weyl group action

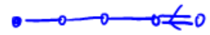
$$s_k = s_{(k,k+1)} \quad \text{for } k = 1, \dots, n-1.$$

$$s_n = \begin{pmatrix}1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1\end{pmatrix}$$

$$W(B_n) \cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$$

5. C_n E.g. $n=5$

- minuscule weights

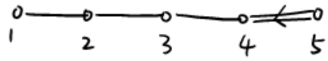


The minuscule representation of C_n is the standard representation:

	#	typical coordinates	symbol
vector	$10 = 2 \cdot 5$	$(1, 0, 0, 0, 0)^T$	v_i & $-v_i$

$\langle v_i, v_j \rangle \in \{1, 0\}$. in $\mathbb{R}^n$

– Simple roots



$$
\begin{aligned}
&\{\ \alpha_1\ ,\ \alpha_2\ ,\ \alpha_3\ ,\ \alpha_4\ ,\ \alpha_5\ \} \\
=&\{\ \nu_1-\nu_2,\ \nu_2-\nu_3,\ \nu_3-\nu_4,\ \nu_4-\nu_5,\ \nu_5\ \} \\
=&\left\{ \begin{pmatrix}1\\-1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\\2\end{pmatrix}, \right\}
\end{aligned}
$$

Ex. Verify that all the $2 \cdot n^2 = 50$ roots are given by

#	typical coordinates	symbol
$20 = 2 \cdot \binom{5}{2}$	$(1, -1, 0, 0, 0)^T$	ν_{1-2}
$10 = 2 \cdot 5$	$(2, 0, 0, 0, 0)^T$	$2\nu_1$
$20 = 2 \cdot \binom{5}{2}$	$(1, 1, 0, 0, 0)^T$	ν_{1+2} & ν_{-1-2}

– Fundamental weights

$$
\begin{aligned}
&\{\ \varpi_1\ ,\ \varpi_2\ ,\ \varpi_3\ ,\ \varpi_4\ ,\ \varpi_5\ \} \\
=&\{\ \nu_1\ ,\ \nu_1+\nu_2,\ \sum_{i=1}^{3}\nu_i,\ \sum_{i=1}^{4}\nu_i\ ,\ 2\nu_{+++++}\} \\
=&\left\{ \begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}, \right\}
\end{aligned}
$$

– Weyl group action

$s_k = s_{(k,k+1)}$ for $k = 1, \dots, n-1$.

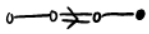
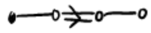
$$
s_n = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{pmatrix}
$$

$$
W(C_n) \cong (\mathbb{Z}/_{2\mathbb{Z}})^n \rtimes S_n
$$

6. F_4

– Weights nearest to the origin

We make a list of the root lattices:

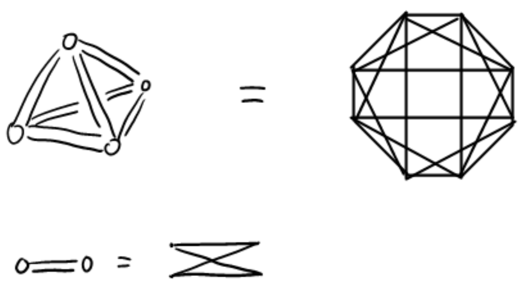
#	typical coordinates		symbol		
$8 = 4\cdot 2$	$(\pm 1, 0, 0, 0)^T$		e_1	short	
$16 = 2^4$	$\frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1)^T$	even sign / odd sign	$\alpha_{\pm\pm\pm\pm}$	short	
$24 = 4\cdot\binom{4}{2}$	$(\pm 1, \pm 1, 0, 0)^T$		$\alpha_{\pm 1 \pm 2}$	long	

Restrict to the short roots:

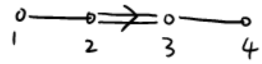
$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -1\} \quad \text{in } \mathbb{R}^4$$

edge

Restrict to thhe short roots, the graph constructed has 24 vertices and 72 edges. It is not connected, and has 3 components. The connected component has HoG Id 176.



- Simple roots



$$\{\ \alpha_1\ ,\quad \alpha_2\ ,\quad \alpha_3\ ,\quad \alpha_4\ \}$$
$$=\{\ \alpha_{2-3}\ ,\quad \alpha_{3-4}\ ,\quad e_4\ ,\quad \alpha_{+---}\ \}$$
$$=\left\{ \begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix},\ \begin{pmatrix}0\\0\\0\\0\end{pmatrix}, \right\}$$

- Fundamental weights

$$\{\ \varpi_1\ ,\quad \varpi_2\ ,\quad \varpi_3\ ,\quad \varpi_4\ \}$$
$$\{\ \alpha_{1+2}\ ,\ \alpha_{1+2}+\alpha_{1+3}\ ,\ e_1+\alpha_{++++}\ ,\quad e_1\ \}$$
$$=\left\{ \begin{pmatrix}1\\1\\0\\0\end{pmatrix},\ \begin{pmatrix}2\\1\\1\\0\end{pmatrix},\ \frac{1}{2}\begin{pmatrix}3\\1\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \right\}$$

- Weyl group action

$$s_1 = s_{(2,3)} \qquad s_2 = s_{(3,4)} \qquad s_3 \rightsquigarrow \begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&-1\end{pmatrix} \qquad s_4 \rightsquigarrow \frac{1}{4}\begin{pmatrix}3&1&1&1\\1&3&-1&-1\\1&-1&3&-1\\1&-1&-1&3\end{pmatrix}$$

7. G_2

– Weights nearest to the origin

We make a list of the root lattices:

#	typical coordinates	symbol	
6	$\frac{1}{\sqrt{2}}(1,\ -1,\ 0)^T$	α_{1-2}	short
6	$\frac{1}{\sqrt{2}}(2,\ -1,\ -1)^T$	β_1 & $-\beta_1$	long

Restrict to the short roots:

$$\langle v_i, v_j \rangle \in \{1, \tfrac{1}{2}, -\tfrac{1}{2}, -1\}$$

$$\text{in} \quad \Big\{\sum_{i=1}^{3} z_i = 0\Big\} \cong \mathbb{R}^2$$

– Simple roots

$$\begin{aligned} &\{\alpha_1,\ \alpha_2\} \\ = &\{\alpha_{1-2},\ \beta_2\} \\ = &\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}\right\} \end{aligned}$$

– Fundamental weights

$$\begin{aligned} &\{\varpi_1,\ \varpi_2\} \\ = &\{\alpha_{3-2},\ -\beta_3\} \\ = &\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix},\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\-2\end{pmatrix}\right\} \end{aligned}$$

– Weyl group action

$$s_1 = S_{(1,2)} \qquad s_2 \rightsquigarrow \frac{1}{3}\begin{pmatrix}-1 & 2 & 2\\ 2 & 2 & -1\\ 2 & -1 & 2\end{pmatrix}$$