Eine Woche, ein Beispiel

5.4. line bundles on abelian varieties

Ref: follows [2025.04.13].
Most contents in this document can be found in [BL04, Chap 2 and Appendix B].

Goal: For $A = V/\Lambda$, identify

$$\mathrm{Pic}(A) \underset{\text{hidden sheaf argument?}}{\cong} H^1(\Lambda, H^0(\mathcal{O}_V^*)) \overset{\text{def}}{=\!=} \mathcal{P}(\Lambda)$$

algebraic info — gp cohom info — analytic info

$$\mathcal{L} \qquad a_{\mathcal{L}} : \Lambda \times V \longrightarrow \mathbb{C} \qquad (H, \chi)$$

$$a_{\mathcal{L}}(\lambda, v) = \chi(\lambda)\exp\left(\pi H(\lambda, v) + \tfrac{\pi}{2} H(\lambda,\lambda)\right)$$

$$\chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)\exp(\pi i \,\mathrm{Im}\, H(\lambda,\mu))$$

https://mathoverflow.net/questions/30611/relation-between-sheaf-and-group-cohomology
This explains the iso between Pic(A) with gp cohom by using the spectral sequences on two functors:

$$(-)^{\pi_1(A)} \circ \pi_{A*}\pi_*\pi^* = \pi_{A,*}$$

$$V \xrightarrow{\pi} A \xrightarrow{\pi_A} \{*\}$$

I wonder if it is possible to express invariant functor as one in the six functor (like the pushforward from classifying space to one point), and argue this identity in abstract nonsense.

Thm (Appell-Humbert) [BL04, p32]

$$\begin{tikzcd}
 & \{\chi\} \arrow[d, phantom, "\shortparallel"] & \{(H,\chi)\} \arrow[d, phantom, "\shortparallel"] & \{H\} \arrow[d, phantom, "\shortparallel"] & \\
0 \arrow[r] & \mathrm{Hom}(\Lambda, S^1) \arrow[r] \arrow[d, "\cong"'] & \mathcal{P}(\Lambda) \arrow[r] \arrow[d, "\cong"'] & NS(A) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(A) \arrow[r] \arrow[d, equal, "\mathrm{def}"] & \mathrm{Pic}(A) \arrow[r] & NS(A) \arrow[r] & 0 \\
 & \hat{A} & & &
\end{tikzcd}$$

where a polarization lives

where

$$NS(A) = \left\{ H : V \times V \longrightarrow \mathbb{C} \;\middle|\; \begin{array}{l} H \text{ Hermitian} \\ \mathrm{Im}\, H(\Lambda \times \Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$\mathcal{P}(\Lambda) = \left\{ (H, \chi) \;\middle|\; \begin{array}{l} H \in NS(A) \\ \chi : \Lambda \longrightarrow S^1 \text{ semicharacter w.r.t. } H, \text{i.e.,} \\ \chi(\lambda+\mu) = \chi(\lambda)\chi(\mu)\exp(\pi i \,\mathrm{Im} H(\lambda,\mu)) \\ \qquad \forall \lambda, \mu \in \Lambda \end{array} \right\}$$

1. Cohomology of abelian varieties (Betti & Hodge)

Thm. [BL04, Thm 1.4.1 b)]

We have

$$
\begin{array}{rlllll}
\Omega & := \mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) \overset{V^*}{=} & = H^{1,0}(A) & \cong \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) & & f\,dz \\
\bar\Omega & := \mathrm{Hom}_{\bar{\mathbb{C}}}(V,\mathbb{C}) & = H^{0,1}(A) & \cong \mathrm{Hom}_{\mathbb{R}}(V,i\mathbb{R}) & & \bar f\,d\bar z \\
\Omega\oplus\bar\Omega & & = H^1(A;\mathbb{C}) & = \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) & &
\end{array}
$$

Proof.

$$
\begin{array}{ccc}
\mathrm{Hom}_{\mathbb{C}}(V,\mathbb{C}) & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{R}) \\
l & \longmapsto & \mathrm{Re}\, l \\
k(-)-ik(i-) & \longleftarrow & k
\end{array}
\qquad
\begin{array}{l}
dz = dx+idy \longmapsto dx \\
idz = -dy+idx \longmapsto -dy
\end{array}
$$

$$
\begin{array}{ccc}
\mathrm{Hom}_{\bar{\mathbb{C}}}(V,\mathbb{C}) & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,i\mathbb{R}) \\
l & \longmapsto & i\,\mathrm{Im}\, l \\
-k(i-)+ik(-) & \longleftarrow & ik
\end{array}
\qquad
\begin{array}{l}
d\bar z = dx-idy \longmapsto -i\,dy \\
id\bar z = dy+idx \longmapsto i\,dx
\end{array}
$$

$$
\begin{array}{ccc}
\Omega\oplus\bar\Omega & \longleftrightarrow & \mathrm{Hom}_{\mathbb{R}}(V,\mathbb{C}) \\
(f\,dz,\bar g\,d\bar z) & \longmapsto & ?
\end{array}
$$

$$
\begin{array}{ccl}
(dz, -id\bar z) & \longleftarrow & dz = dx+idy \\
(idz, id\bar z) & \longleftarrow & idz = idx-dy \\
(dz, d\bar z) & \longleftarrow & d\bar z = dx-idy \\
(idz, -id\bar z) & \longleftarrow & id\bar z = idx+dy
\end{array}
$$

$$
\begin{array}{ccl}
((f_1+if_2)dz, 0) & \longmapsto & f_1dx - f_2dy \\
(0, (g_1-ig_2)d\bar z) & \longmapsto & -i(g_1dy+g_2dx)
\end{array}
\qquad f_1,f_2,g_1,g_2\in C^\infty(A;\mathbb{R})
$$

Cor. $H^q(A;\Omega_A^p) \cong \Lambda^p\Omega \otimes \Lambda^q\overline{\Omega}$ $\qquad \Omega_A^p := \mathrm{Alt}^p\,\Omega_A$

Proof Sketch

$$\begin{aligned} H^q(A;\Omega_A^p) &\cong H_{\bar\partial}^{p,q}(A) \\ &= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V/\Lambda\}/\sim \\ &= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } \Lambda\}/\sim \\ &= \{\bar\partial\text{-closed } (p,q)\text{-forms on } V \text{ invariant under } V\} \\ &= \Lambda^p\Omega \otimes \Lambda^q\overline{\Omega} \end{aligned}$$

Another proof, though essentially the same:

Step 1 $\quad \Omega_A$ is a free $\mathcal{O}_A$-module with rank n, so

$$\Omega_A \cong \mathcal{O}_A \otimes_{\mathbb{C}} V^*$$

$$\Rightarrow \Omega_A^p = \Lambda^p\Omega_A = \mathcal{O}_A \otimes_{\mathbb{C}} \Lambda^p\Omega$$

Step 2 $\quad$ By Dolbeault resolution,

$$H^q(A;\mathcal{O}_A) \cong H^q(\mathcal{A}^{0,\bullet}_{A\times\mathbb{C}}(A)) \cong H_{\bar\partial}^{0,q}(A) \cong \Lambda^q\overline{\Omega}$$

trivial l.b. over A

Dual abelian variety

Rmk. We have canonical bilinear pairing

$$\langle -,- \rangle : \overline{\Omega} \times \underset{V}{\Omega^*} \longrightarrow \mathbb{R} \qquad \langle l, v\rangle = \mathrm{Im}\,(l(v))$$

which restricts to

$$\langle -,- \rangle : \hat{\Lambda} \times \Lambda \longrightarrow \mathbb{Z}$$

where

$$\begin{aligned} \hat{\Lambda} &:= \{l \in \overline{\Omega} \mid \langle l, \Lambda\rangle \subseteq \mathbb{Z}\} \\ &\cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \\ &\cong H^1(A;\mathbb{Z}) \end{aligned}$$

Prop. The abelian variety A and its dual $\hat{A}$ have following expressions:

$$\begin{aligned} A &= \Omega^*/\Lambda = H^0(A;\Omega_A)^*/H_1(A;\mathbb{Z}) \\ \hat{A} &= \overline{\Omega}/\hat{\Lambda} = H^1(A;\mathcal{O}_A)/H^1(A;\mathbb{Z}) \\ &= \mathrm{Pic}^0(A) \cong \mathrm{Hom}(\Lambda, S^1) = \{\chi\} \end{aligned}$$

Proof. We only need to show that $\bar{\Omega}/\hat{\Lambda} \cong \mathrm{Hom}_{\mathbb{Z}}(\Lambda, S')$, which follows from applying $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, -)$ to $0 \to \mathbb{Z} \to \mathbb{R} \to S' \to 0$:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \arrow[r] \arrow[ddd, equal] & \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{R}) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \mathrm{Hom}(\Lambda, S') \arrow[r] & 0 \\
 & & \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \arrow[d, phantom, "\cong" rotate=-90] & & \\
 & & \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) \arrow[d, equal] & & \\
0 \arrow[r] & \hat{\Lambda} \arrow[r] & \bar{\Omega} \arrow[r] & \bar{\Omega}/\hat{\Lambda} \arrow[r] & 0
\end{tikzcd}
$$

NS(A): more descriptions

Lemma [BL04, Prop 2.1.6]

Let

$$NS(A) := Pic(A)/Pic^0_{red}(A) = H^2(A;\mathbb{Z}) \cap H^{1,1}(A)$$

$$NS'(A) := \left\{ \omega : V\times V \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} \omega \ \mathbb{R}\text{-bilinear alternating form} \\ \omega(ix, iy) = \omega(x,y) \\ \omega(\Lambda\times\Lambda) \subset \mathbb{Z} \end{array} \right\}$$

$$NS''(A) = \left\{ H : V\times V \longrightarrow \mathbb{R} \;\middle|\; \begin{array}{l} H \ \text{Hermitian} \\ \operatorname{Im} H(\Lambda\times\Lambda) \subset \mathbb{Z} \end{array} \right\}$$

↑imaginary part

Then

$$NS(A) \cong NS'(A) \cong NS''(A).$$

As a reminder, H Hermitian:

$$H(au, bv) = \bar{a}b\, H(u,v) \quad + \mathbb{R}\text{-linear}$$

$$H(u,v) = \overline{H(v,u)}$$

crspds to the matrix M s.t. $M^H = M$

$H(u,v) = \omega(u,iv) + i\omega(u,v)$

$\omega(u,v) = \operatorname{Im} H(u,v)$

Rmk. A Hermitian form is equiv to a $\mathbb{C}$-linear map

$$\begin{array}{rcccl} \phi : \Omega^* = V & \longrightarrow & Hom_{\mathbb{R}}(V,\mathbb{R}) & \cong & Hom_{\overline{\mathbb{C}}}(V,\mathbb{C}) = \overline{\Omega} \\ v_2 & \longmapsto & \omega(-,v_2) & \mapsto & H(-,v_2) \end{array}$$

$$\phi(iv) = H(-,iv) = iH(-,v) = i\phi(v)$$

or a cplx bilinear map $\overline{V}\times V \longrightarrow \mathbb{C}$

or an element in $\Omega\otimes_{\mathbb{C}}\overline{\Omega} \cong H^{1,1}(A)$

Moreover,

H is non-deg $\iff$ ϕ is an iso

$\omega(\Lambda\times\Lambda)\subseteq\mathbb{Z}$ $\iff \langle\phi(\Lambda),\Lambda\rangle \subseteq \mathbb{Z} \iff \phi(\Lambda)\subseteq\widehat{\Lambda}$

Therefore, $\mathrm{NS}'(A) = \{\phi \in \mathrm{Hom}(\Omega^*, \bar{\Omega}) \mid \phi(\Lambda) \subset \hat{\Lambda}\}$
$= \mathrm{Hom}(\Omega^*/\Lambda, \bar{\Omega}/\hat{\Lambda})$
$= \mathrm{Hom}(A, \hat{A})$.

This explains why some references would like to call a morphism

$$\phi : A \longrightarrow \hat{A}$$

(induced by an ample l.b.) as the polarization of A.

From $\phi : \Omega^* \longrightarrow \bar{\Omega}$ to $H(-,-)$:

In fact, once we fixed the $\mathbb{C}$-linear map $\phi : \Omega^* \longrightarrow \bar{\Omega}$, the Hermitian from comes from the canonical bilinear pairing:

$$\begin{array}{ll} \langle -,- \rangle : \bar{\Omega} \times \Omega^* \longrightarrow \mathbb{R} & \langle l, v \rangle = \mathrm{Im}\, l(v) \\ \quad\quad \uparrow (\phi, \mathrm{Id}) & \\ \omega(-,-) : \Omega^* \times \Omega^* \longrightarrow \mathbb{R} & \omega(v_1, v_2) = \mathrm{Im}\, H(v_1, v_2) \end{array}$$

Hint for the main lemma. Consider the ambient spaces:

$$\begin{array}{ccccccc} & & \mathrm{NS}(A) & & \mathrm{NS}'(A) & & \mathrm{NS}''(A) \\ & & \cap & & \cap & & \cap \\ H^1(\mathcal{O}_A^*) & \xrightarrow{c_1} & H^2(A;\mathbb{Z}) & \hookrightarrow & H^2(A;\mathbb{R}) & \hookrightarrow & H^2(A;\mathbb{C}) \end{array}$$

Prop (Identifying symmetric l.bs) [BL04, Cor 2.3.7, Lemma 4.6.2]

Suppose $\mathcal{L} = \mathcal{L}(H, \chi) \in \mathrm{Pic}(A)$. Then

$$\mathcal{L} \text{ is symmetric } \overset{\text{def}}{\iff} [-1]^* \mathcal{L} \cong \mathcal{L}$$
$$\iff \chi(\Lambda) \subseteq \{\pm 1\}$$

Furthermore,

$$\{\mathcal{L} \in \mathrm{Pic}^0(A) \mid \mathcal{L} \text{ sym}\} = \hat{A}[2]$$
$$\{\mathcal{L} \in \mathrm{Pic}^H(A) \mid \mathcal{L} \text{ sym}\} \text{ is a torsor of } \hat{A}[2].$$

2. Miscellaneous

I will collect some basic results as well as its proofs here for my personal reference. They show the strategy to work with these line bundles, dual varieties stuff, but not the most important thing to put in the main goal.
Maybe someday I will move them to better places.

Prop. (dual variety with SES) [BL04, Prop 2.4.2]
If

$$0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow 0$$

is a SES of cplx tori, then

$$0 \longleftarrow \widehat{X}_1 \longleftarrow \widehat{X}_2 \longleftarrow \widehat{X}_3 \longleftarrow 0$$

is exact.

Proof. Let $X_i = V_i/\Lambda_i$, then

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Lambda_1 \arrow[r] \arrow[d] & \Lambda_2 \arrow[r] \arrow[d] & \Lambda_3 \arrow[d] \arrow[dddll, bend left] & & \text{exact} \\
0 \arrow[r] & V_1 \arrow[r] \arrow[d] & V_2 \arrow[r] \arrow[d] & V_3 \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & 0 & &
\end{tikzcd}$$

Taking $\mathrm{Hom}_{\mathbb{Z}}(-, S^1)$,

$$0 \leftarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda_1, S^1) \leftarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda_2, S^1) \leftarrow \mathrm{Hom}_{\mathbb{Z}}(\Lambda_3, S^1) \leftarrow 0$$

is exact, i.e.,

$$0 \longleftarrow \widehat{X}_1 \longleftarrow \widehat{X}_2 \longleftarrow \widehat{X}_3 \longleftarrow 0$$

is exact.

Prop. (dual variety with isogeny) [BL04, Prop 2.4.3]
If $f: Y \longrightarrow X$ is an isogeny of cplx tori, then
$\hat{f}: \hat{X} \longrightarrow \hat{Y}$ is an isogeny of cplx tori, with

$$\operatorname{Ker} \hat{f} = \operatorname{Hom}(\operatorname{Ker} f, S^1).$$

Proof. Let $Y = V/\Lambda_2$, $X = V/\Lambda_3$, then

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \Lambda_2 \arrow[r] \arrow[d] & \Lambda_3 \arrow[d] \arrow[dddll, bend left=80] & & \text{exact} \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & V \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] & \ker f \arrow[r] \arrow[d] & Y \arrow[r, "f"] \arrow[d] & X \arrow[r] \arrow[d] & 0 & \\
 & \ker f \arrow[r] & 0 \arrow[r] & 0 & &
\end{tikzcd}$$

Taking $\operatorname{Hom}(-, S^1)$,

$$\begin{tikzcd}
0 & \operatorname{Hom}(\Lambda_2, S^1) \arrow[l] \arrow[d, equal] & \operatorname{Hom}(\Lambda_3, S^1) \arrow[l, "\hat{f}"'] \arrow[d, equal] & \operatorname{Hom}(\ker f, S^1) \arrow[l] & 0 \arrow[l] \\
 & \hat{Y} & \hat{X} & &
\end{tikzcd}$$

so
$$\operatorname{Ker} \hat{f} = \operatorname{Hom}(\operatorname{Ker} f, S^1).$$

Prop (dual variety with ρ_a & ρ_r) [BL04, Ex. 2.6.(13)]
For $\rho: Y \longrightarrow X$ morphism between abelian varieties,

$$\rho_a(\hat{f}) = \rho_a(f)^H$$
$$\rho_r(\hat{f}) = \rho_r(f)^T$$

Hint.

$$\bar{\Omega} = \operatorname{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C})$$
$$\hat{\Lambda} = \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})$$

Prop (Image of f^* via isogeny) [BL04, Cor 2.4.4]

Let $f: X_1 \longrightarrow X_2$ be an isogeny of cplx tori, $X_i = V/\Lambda_i$.
For $\mathcal{L} = \mathcal{L}(H, \chi) \in \mathrm{Pic}(X_1)$,

imaginary part

$$\begin{array}{c} \mathcal{L} = f^*\mathcal{M} \\ \text{for some } \mathcal{M} \in \mathrm{Pic}(X_2) \end{array} \iff \mathrm{Im}\, H(\Lambda_2 \times \Lambda_2) \subseteq \mathbb{Z}$$

Proof. Diagram chasing. Find H' and χ'.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}^0(X_1) \arrow[r] & \mathrm{Pic}(X_1) \arrow[r] & NS(X_1) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Pic}^0(X_2) \arrow[r] \arrow[u] & \mathrm{Pic}(X_2) \arrow[r] \arrow[u] & NS(X_2) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
\chi_1 \arrow[r, mapsto] & \left\{ \begin{array}{c} \mathcal{L} \\ f^*\mathcal{L}_2 \end{array} \right. \arrow[r, mapsto] & H \\
\chi' \arrow[r, mapsto] \arrow[u, mapsto] & \mathcal{L}_2 \arrow[r, mapsto] \arrow[u, mapsto] & H \arrow[u, mapsto]
\end{tikzcd}$$

Prop. (line bundles under pullbacks) [BL04, Lemma 2.3.2 & Lemma 2.3.4]

For $\mathcal{L} = \mathcal{L}(H, \chi) \in \mathrm{Pic}(A)$ and $[v] \in A$
$f: \widetilde{A} \longrightarrow A$ homo of cplx tori,

① $t_{[v]}^* \mathcal{L}(H, \chi) = \mathcal{L}\left(H, \chi \exp(2\pi i\, \mathrm{Im}\, H(-, v))\right)$

② $f^* \mathcal{L}(H, \chi) = \mathcal{L}(f_a^* H, f_r^* \chi)$

Prop. ($c_1(\mathcal{L})$ in terms of basis) [BL04, Ex. 2.6.(2)]

Suppose $A = V/\Lambda$ of dim n,
$\mathcal{L} = \mathcal{L}(H, \chi) \in \mathrm{Pic}(A)$ of type $(d_1, \dots, d_n)$.

1) For

$$V = \langle e_1, \dots, e_n \rangle_{\mathbb{C}}, \qquad z = \sum z_i e_i,$$

$$c_1(\mathcal{L}) = \frac{i}{2} \sum_{\nu,\mu=1}^{n} H(e_\nu, e_\mu)\, dz_\nu \wedge d\bar{z}_\mu$$

2) For

Sympletic basis for $\mathcal{L}$

$$\Lambda = \langle \lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n \rangle_{\mathbb{Z}}, \qquad z = \sum x_i \lambda_i + \sum y_i \mu_i$$

$$c_1(\mathcal{L}) = - \sum_{\nu=1}^{n} d_\nu \cdot dx_\nu \wedge dy_\nu$$

Cor. (equiv def of polarization)
the diagram commutes:

$$\begin{tikzcd}
V \arrow[r, "\phi_H"] \arrow[d] & \overline{\Omega} \arrow[d, "\pi_{\overline{\Omega}}"] & v \longmapsto H(-,v) & \\
V/\Lambda \arrow[r, "\phi_H"] \arrow[d, equal] & \overline{\Omega}/\hat{\Lambda} \arrow[d, equal] & {[v]} \longmapsto H(-,v) & l \arrow[d, mapsto] \\
A \arrow[r, "\phi_{\mathcal{L}}"] & \hat{A} = \mathrm{Pic}^0(A) & {[v]} \longmapsto t_{[v]}^* \mathcal{L} \otimes \mathcal{L}^{-1} & \exp(2\pi i \, \mathrm{Im}\, l(-))
\end{tikzcd}$$

Proof. Write $\mathcal{L} = \mathcal{L}(H, \chi)$, then

$$\begin{aligned}
t_{[v]}^* \mathcal{L} \otimes \mathcal{L}^{-1} &= \mathcal{L}\left(H, \chi \exp(2\pi i \, \mathrm{Im}\, H(-,v))\right) \otimes \mathcal{L}(H,\chi)^{-1} \\
&= \mathcal{L}\left(0, \exp(2\pi i \, \mathrm{Im}\, H(-,v))\right) \\
&= \pi_{\overline{\Omega}} \circ \phi_H(v).
\end{aligned}$$

Prof & Def (Kernel of polarization) [BL04, Prop 2.4.9]

Let $\mathcal{L} \in \mathrm{Pic}(A)$ ample, then

$$\begin{tikzcd}
0 \arrow[r] & \Lambda(L)/\Lambda \arrow[r] \arrow[d, equal] & \Omega^*/\Lambda \arrow[r, "\overline{\phi}_H"] \arrow[d, equal] & \overline{\Omega}/\hat{\Lambda} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}\, \phi_{\mathcal{L}} \arrow[r] \arrow[d, equal, "\mathrm{def}"] & A \arrow[r, "\phi_{\mathcal{L}}"] & \hat{A} \arrow[r] & 0 \\
& K(L) & & &
\end{tikzcd}$$

Where

$$\Lambda(L) = \phi_H^{-1}(\hat{\Lambda}) = \{v \in V \mid \mathrm{Im}\, H(\Lambda, v) \subseteq \mathbb{Z}\}.$$

We know that (for any l.b.)

$$\phi_{\mathcal{L}} = 0 \iff H \equiv 0 \iff \mathcal{L} \in \mathrm{Pic}^0(X)$$

$\# \mathrm{Ker}\, \phi_{\mathcal{L}} < +\infty \iff \phi_{\mathcal{L}}$ is isogeny $\iff H$ is nondeg $\Leftarrow$ $\mathcal{L}$ is ample

When $\# \mathrm{Ker}\, \phi_{\mathcal{L}} < +\infty$,

$$\deg \phi_{\mathcal{L}} = \# \mathrm{Ker}\, \phi_{\mathcal{L}} = [\Lambda(L) : \Lambda] = \det(\mathrm{Im}\, H).$$

Prop. (analytic construction of Poincaré bundle)

We have a l.b. $\mathcal{P} \in \mathrm{Pic}(A \times \widehat{A})$ s.t.

i) $\mathcal{P}|_{A \times \{L\}} \cong L$

ii) $\mathcal{P}|_{\{0\} \times \widehat{A}} \cong \mathcal{O}_{\widehat{A}}$

Hint. We define $\mathcal{P} = \mathcal{P}(H, \chi)$, where

$$H : (V \times \overline{\Omega}) \times (V \times \overline{\Omega}) \longrightarrow \mathbb{C} \quad H((v_1, l_1), (v_2, l_2)) = \overline{l_2(v_1)} + l_1(v_2)$$

$$\chi : \Lambda \times \widehat{\Lambda} \longrightarrow S^1 \quad \chi(\lambda, l_0) = \exp(\pi i \operatorname{Im} l_0(\lambda))$$

For $L = L(0, \exp(2\pi i \operatorname{Im} l(-)))$ $l \in \overline{\Omega}$, need to check i) ii).

Prop. ($\mathcal{P}_A$ with duality & polarization) [BL04, Ex. 2.6.(16)(17) & Lemma 14.1.1]

Denote

$$s: \hat{A}\times A \longrightarrow A \times \hat{A} \qquad (l, v) \longmapsto (v, l)$$
$$\kappa: A \longrightarrow \hat{\hat{A}}$$

$$\begin{array}{ccl} & & A\times\hat{A} - \mathcal{P}_A \\ & \overset{s}{\nearrow} & \\ \hat{A}\times A & \xrightarrow{Id_{\hat{A}}\times\kappa} & \hat{A}\times\hat{\hat{A}} - \mathcal{P}_{\hat{A}} \end{array}$$

$$A\times\hat{A} \xrightarrow{\phi_{\mathcal{P}_A}} (A \times \hat{A})^\wedge \cong \hat{A}\times\hat{\hat{A}} - \mathcal{P}_{\hat{A}}$$

$$\begin{array}{ccl} & \overset{p_1}{\nearrow} & A - \mathcal{L} \\ A\times A & \xrightarrow{(Id_A,\, \phi_{\mathcal{L}})} & A\times\hat{A} - \mathcal{P}_A \\ & \overset{\mu}{\searrow} & A - \mathcal{L} \\ & \overset{p_2}{\searrow} & A - \mathcal{L} \end{array}$$

we have

$$s^*\mathcal{P}_A \cong (Id_A \times \kappa)^*\mathcal{P}_{\hat{A}}$$
$$\phi^*_{\mathcal{P}_A}\mathcal{P}_{\hat{A}} \cong \mathcal{P}_A$$
$$(Id_A, \phi_{\mathcal{L}})^*\mathcal{P}_A \cong \mu^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}$$

As a result,

$$\begin{aligned} \mathcal{L}\in \mathrm{Pic}^0(A) &\iff \phi_{\mathcal{L}} = \{*\} \\ &\iff H \equiv 0 \\ &\iff (Id_A, \phi_{\mathcal{L}})^*\mathcal{P}_A = \mathcal{O}_{A\times A} \\ &\iff \mu^*\mathcal{L} \cong p_1^*\mathcal{L} \otimes p_2^*\mathcal{L}. \end{aligned}$$

Hodge structures [BL04, 17.1 & 17.2]

Suppose $V \in \mathrm{Vect}_{\mathbb{R}}$, $\dim_{\mathbb{R}} V = 2n$, $\Lambda \subseteq V$ lattice.
$E : \Lambda^2 V \longrightarrow \mathbb{R}$ non-deg s.t. $E(\Lambda, \Lambda) \subseteq \mathbb{Z}$.

str = structure

$$\left\{ \begin{array}{l} h : S^1 \longrightarrow GL_{\mathbb{R}}(V) \\ V_{\mathbb{C}} \cong V_+ \oplus V_- \\ h(z)v = z^{\pm 1} v \quad \forall v \in V_\pm \end{array} \right\}$$

$$\{ J \in GL_{\mathbb{R}}(V) \mid J^2 = -Id_V \}$$

$$\{\text{cplx str on } V\} \cong \left\{ \begin{array}{l} h : S^1 \longrightarrow GL_{\mathbb{R}}(V) \\ \text{with wt} \quad 1 \quad -1 \\ \quad \text{mult} \quad g \quad g \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Hodge str on } V \\ \text{with wt } -1 \end{array} \right\}$$

$$\cup \qquad \cup \qquad \cup$$

$$\{ J \in Sp(V,E) \mid J^2 = -Id_V \}$$

$$\{\text{sympletic cplx str on } V\} \cong \left\{ \begin{array}{l} h : S^1 \longrightarrow Sp(V,E) \\ \text{with wt} \quad 1 \quad -1 \\ \quad \text{mult} \quad g \quad g \end{array} \right\} \cong \left\{ \begin{array}{l} \text{Hodge str on } V \\ \text{with wt } -1 \\ E(V^{-1,0}, V^{0,-1}) \equiv 0 \end{array} \right\}$$

$$\cup$$

$$C_0(Sp(V,E)) \qquad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \qquad \begin{pmatrix} Y^{-1}X & Y^{-1}D \\ -D^{-1}(XY^{-1}X+Y) & -D^{-1}XY^{-1}D \end{pmatrix}$$

$$\cong \qquad \downarrow \qquad \uparrow$$

$$\mathcal{H}_g \qquad (D\beta^{-1}\alpha + iD\beta^{-1}, D) \qquad (X+iY, D)$$

Siegel upper half plane