Eine Woche, ein Beispiel

5.21 Hochchild (co)homology

1 Def. $C^{bar}_\cdot(A)$: $\quad \to A\otimes_k A\otimes_k A\otimes_k A \xrightarrow{\partial_2} A\otimes_k A\otimes_k A \xrightarrow{\partial_1} A\otimes_k A \to \cancel{A} \to 0$

a free $A^e := A\otimes_k A^{op}$-module resolution

$HH_\bullet(A,M) := \mathrm{Tor}^{A^e}_\bullet(A,M) = H_\bullet(M\otimes_{A^e} C^{bar}_\bullet(A))$ $\qquad HH_\cdot(A) = HH_\cdot(A,A)$

$HH^\bullet(A,M) := \mathrm{Ext}^\bullet_{A^e}(A,M) = H^\bullet(\mathrm{Hom}_{A^e}(C^{bar}_\bullet(A),M))$ $\qquad HH^\bullet(A) = HH^\bullet(A,A)$

$C_n(A,M) := \quad C^{bar}_\cdot(A)\otimes_{A^e} M$

$C^n(A,M) := \mathrm{Hom}_{A^e}(C^{bar}_\cdot(A),M)$

$\cdot$: covariant $\quad\rightsquigarrow\quad HH_\cdot(-)$ is a functor

$\cdot$: contravariant $\quad HH^\cdot(-)$ is not a functor, e.p. $Z(\text{ring})$ is not a functor.

$\partial_n: A^{\otimes n+2} \longrightarrow A^{\otimes n+1} \qquad a_0\otimes\cdots\otimes a_{n+1} \longmapsto \sum_{i=0}^{n}(-1)^i a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_{n+1}$

$\partial_n: C_n(A,M) \longrightarrow C_{n-1}(A,M)$

$A^{\otimes n+2}\otimes_{A^e}M \longrightarrow A^{\otimes n+1}\otimes_{A^e}M \qquad a_0\otimes\cdots\otimes a_{n+1}\otimes m \longmapsto \sum_{i=0}^{n}(-1)^i a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_{n+1}\otimes m$

$A^{\otimes n+1}\otimes_A M \longrightarrow A^{\otimes n}\otimes_A M \qquad a_0\otimes\cdots\otimes a_n\otimes m \longmapsto \sum_{i=0}^{n-1}(-1)^i a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_n\otimes m$
$\qquad + (-1)^n a_0\otimes\cdots\otimes a_{n-1}\otimes a_n m$

$A^{\otimes n}\otimes_k M \longrightarrow A^{\otimes n-1}\otimes_k M \qquad a_1\otimes\cdots\otimes a_n\otimes m \longmapsto \sum_{i=1}^{n-1}(-1)^i a_1\otimes\cdots a_ia_{i+1}\cdots\otimes a_n\otimes m$
$\qquad + (-1)^n a_1\otimes\cdots\otimes a_{n-1}\otimes a_n m$
$\qquad + a_2\otimes\cdots\otimes a_n\otimes m a_1$

$d^n: C^{n-1}(A,M) \longrightarrow C^n(A,M)$

$\mathrm{Hom}_{A^e}(A^{\otimes n+1},M) \longrightarrow \mathrm{Hom}_{A^e}(A^{\otimes n+2},M) \qquad d^nf(a_0\otimes\cdots\otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i f(a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_{n+1})$

$\mathrm{Hom}_A(A^{\otimes n},M) \longrightarrow \mathrm{Hom}_A(A^{\otimes n+1},M) \qquad d^nf(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i f(a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_n)$
$\qquad + (-1)^n f(a_0\otimes\cdots\otimes a_{n-1})a_n$

$\mathrm{Hom}_k(A^{\otimes n-1},M) \longrightarrow \mathrm{Hom}_k(A^{\otimes n},M) \qquad d^nf(a_1\otimes\cdots\otimes a_n) = \sum_{i=1}^{n-1}(-1)^i f(a_1\otimes\cdots a_ia_{i+1}\cdots\otimes a_n)$
$\qquad + (-1)^n f(a_1\otimes\cdots\otimes a_{n-1})a_n$
$\qquad + a_1 f(a_2\otimes\cdots\otimes a_n)$

$\|$ when $M = A^\vee := \mathrm{Hom}_k(A_{A^e},k)$

$\mathrm{Hom}_k(A^{\otimes n},k) \longrightarrow \mathrm{Hom}_k(A^{\otimes n+1},k) \qquad d^nf(a_0\otimes\cdots\otimes a_n) = \sum_{i=0}^{n-1}(-1)^i f(a_0\otimes\cdots a_ia_{i+1}\cdots\otimes a_n)$
$\qquad + (-1)^n f(a_na_0\otimes a_1\otimes\cdots\otimes a_{n-1})$

Rmk. It works similarly for group cohomology.
In gp coh case, one even has identification

$$
\begin{array}{ccc}
G^{\otimes n} & \longrightarrow & G^{\otimes n,\, homo} \leftarrow \text{homogeneous} \\
a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \longmapsto & a_0 \otimes a_0 a_1 \otimes \cdots \otimes a_0 a_1 \cdots a_{n-1} \\
b_0 \otimes b_0^{-1} b_1 \otimes \cdots \otimes b_{n-2}^{-1} b_{n-1} & & b_0 \otimes b_1 \otimes \cdots \otimes b_{n-1}
\end{array}
$$

and therefore more expressions for ∂_n & d^n.

Rmk.

$$
\begin{aligned}
H^n(G,M) &= \mathrm{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z},M) & \qquad & R\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z},M) \\
H_n(G,M) &= \mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},M) & & \mathbb{Z} \overset{L}{\otimes}_{\mathbb{Z}[G]} M \\[1em]
HH^n(A,M) &= \mathrm{Ext}^n_{A^e}(A,M) & & R\mathrm{Hom}_{A^e}(A,M) \\
HH_n(A,M) &= \mathrm{Tor}_n^{A^e}(A,M) & & A \overset{L}{\otimes}_{A^e} M
\end{aligned}
$$

2. Cyclic cohomology

$$C_\lambda^n(A, A^\vee) := \{ f \in C^n(A, A^\vee) \mid f(a_0 \otimes \cdots \otimes a_n) = (-1)^n f(a_1 \otimes \cdots \otimes a_n) \}$$

$$HC^i(A) := H^i(C_\lambda^\bullet(A, A^\vee))$$

We have a LES:

$$\begin{tikzcd}
HC^{n+1}(A) \arrow[r] & HH^{n+1}(A, A^\vee) \arrow[r] & H^{n+1}(C^\bullet/C_\lambda^\bullet) \arrow[r] & {} \\
HC^n(A) \arrow[r, "I"] & HH^n(A, A^\vee) \arrow[r, "B"] & H^n(C^\bullet/C_\lambda^\bullet) \arrow[ulll, "S"']
\end{tikzcd}$$

$H^{n+1}(C^\bullet/C_\lambda^\bullet) = HC^n(A)$, $\quad H^n(C^\bullet/C_\lambda^\bullet) = HC^{n-1}(A)$

2. In low-dimension

n	$HH_n(A,M)$	$HH_n(A)$	$HH^n(A,M)$	$HH^n(A)$
0	$A\otimes_{A^e}M$ $=M/[AM]$	$A\otimes_{A^e}A$ $=A/[AA]$	$\mathrm{Hom}_{A^e}(A,M)$ $=M^A=\{m\in M \mid am=ma\}$	$\mathrm{Hom}_{A^e}(A,A)$ $=Z(A)$
1	—	no alg struc $\Omega_{A/k}$ when A is comm	Lie alg $\mathrm{OutDer}(A,M)$	$\mathrm{OutDer}(A)$
2	—	—	$\mathrm{AlgExt}(A,M)$	Square-zero deformation $=$ flat $k[t]/(t^2)$-alg E s.t. $E\otimes_{k[t]/(t^2)}k\cong A$
3	—	—	crossed bimodules	obstruction space

3. Examples

in $k[t]/(t^n)$, k

A	$HH_n(A, M)$	$HH^n(A, M)$	proj resolution
k	$\begin{cases} M & n=0 \\ 0 & n\geq 1 \end{cases}$	$\begin{cases} M & n=0 \\ 0 & n\geq 1 \end{cases}$	$0 \longrightarrow K \longrightarrow \cancel{K} \longrightarrow 0$ $1 \longmapsto 1$
$k[x]$	$\begin{cases} k[x] & n=0,1 \\ 0 & n\geq 2 \end{cases}$	$\begin{cases} k[x] & n=0,1 \\ 0 & n=2 \end{cases}$	$0 \longrightarrow A^e \longrightarrow A^e \longrightarrow \cancel{A} \longrightarrow 0$ $1 \longmapsto 1$ $1 \longmapsto x_2 - x_1$
$k[x,y]$	$\begin{cases} k[x,y] & n=0 \\ k[x,y]^{\oplus 2} & n=1 \\ k[x,y] & n=2 \\ 0 & n\geq 3 \end{cases}$	$\begin{cases} k[x,y] & n=0 \\ k[x,y]^{\oplus 2} & n=1 \\ k[x,y] & n=2 \\ 0 & n\geq 3 \end{cases}$	$0 \longrightarrow A^e \longrightarrow A^{e^{\oplus 2}} \longrightarrow A^e \longrightarrow \cancel{A} \longrightarrow 0$ $1 \mapsto (-(y_1-y_2), x_1-x_2)$ $\quad 1 \longmapsto 1$ $(1,0) \longmapsto x_1 - x_2$ $(0,1) \longmapsto y_1 - y_2$
$k[t]/(t^k)$ char $k \nmid n$	$\begin{cases} M & n=0 \\ tM & 2\nmid n \\ M/t^{k-1}M & 2\nmid n \ n>0 \end{cases}$	$\begin{cases} M & n=0 \\ tM & 2\nmid n \\ M/t^{k-1}M & 2\nmid n \ n>0 \end{cases}$	$\cdots \xrightarrow{u} A^e \xrightarrow{v} A^e \xrightarrow{u} A^e \longrightarrow \cancel{A} \longrightarrow 0$ $1 \mapsto \sum_{i+j=k-1} r^i s^j$ $\quad 1 \longmapsto 1$ $1 \longmapsto r-s$
$k[t]/(t^k)$ char $k \mid n$	M	M	
$k\langle x,y\rangle/(yx-xy-1)$ char $k=0$	$\begin{cases} 0 & n=0,1 \\ K & n=2 \\ 0 & n\geq 3 \end{cases}$	$\begin{cases} K & n=0 \\ 0 & n\geq 1 \end{cases}$	$0 \longrightarrow A^e \longrightarrow A^{e^{\oplus 2}} \longrightarrow A^e \longrightarrow \cancel{A} \longrightarrow 0$ $1 \mapsto (-(y_2-y_1), x_2-x_1)$ $\quad 1 \longrightarrow 1$ $(1,0) \longrightarrow x_2 - x_1$ $(0,1) \longmapsto y_2 - y_1$
KQ Q: connected acyclic	$\begin{cases} K^{\oplus r} & n=0 \\ 0 & n\geq 1 \end{cases}$	—	$0 \to \bigoplus_{\alpha\in Q_1} A^e e_{t(\alpha)} \otimes e_{s(\alpha)} \to \bigoplus_{v\in Q_0} A^e e_v \otimes e_v \to \cancel{A} \to 0$ $e_v \otimes e_v \longmapsto e_v$ $e_{t(\alpha)} \otimes e_{s(\alpha)} \longmapsto \alpha \otimes 1 (e_{s(\alpha)} \otimes e_{s(\alpha)})$ $- 1 \otimes \alpha (e_{t(\alpha)} \otimes e_{t(\alpha)})$
$K[G]$ G: finite gp	$\begin{cases} K^{\oplus \# \text{conj class}} & n=0 \\ 0 & n\geq 1 \end{cases}$	—	

Morita equivalence

$$A \overset{\text{morita}}{\sim} B \quad \overset{\text{def}}{\Longleftrightarrow} \quad \mathrm{Mod}_A \overset{\text{equiv}}{\sim} \mathrm{Mod}_B$$

Thm. Each f.d. alg over $k=\bar{k}$ is Morita equiv to a basic alg.

[Rep notes 1, Cor 24.5]

Theorem 24.4. *Let A be a finite-dimensional K-algebra. Let P be a projective generator of* $\mathrm{mod}(A)$*, and set* $B := \mathrm{End}_A(P)^{\mathrm{op}}$*. Then*

$$F := P \otimes_B -\colon\ \mathrm{mod}(B) \to \mathrm{mod}(A)$$

and

$$G := \mathrm{Hom}_A(P, -)\colon\ \mathrm{mod}(A) \to \mathrm{mod}(B)$$

are equivalences of categories which are quasi-inverses of each other.

e.g. $M^{n\times n}(\mathbb{C}) \sim \mathbb{C}$ $\qquad \mathbb{C}[S_3] \sim \mathbb{C}^{\oplus 3}$

$$X^{\oplus n} := X \otimes_{\mathbb{C}} \mathbb{C}^{\oplus n} \longleftarrow\!\!\shortmid X$$

$$Y \longmapsto \mathrm{Hom}_{M^{n\times n}(\mathbb{C})\text{-mod}}(\mathbb{C}^{\oplus n}, Y)$$

or

$$\mathrm{Hom}_{\mathbb{C}\text{-mod}}(\mathbb{C}^{\oplus n}, X) \longleftarrow\!\!\shortmid X$$

$$Y \longmapsto Y \otimes_{M^n(\mathbb{C})} \mathbb{C}^{\oplus n}$$