Session 10 & 11 & Exercise 9

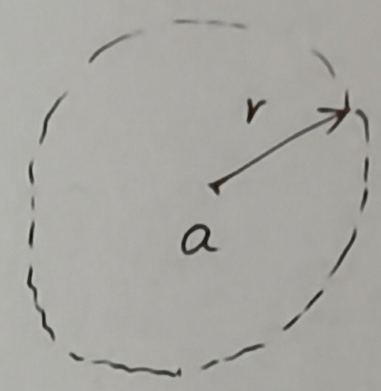
Today we have a lot of works to do.

In the first part, we will discuss local properties of fcts, and in the second part, we will discuss contour integrals.

1. Types of isolated singularities.

Def. $f \in \mathcal{O}(\mathcal{U})$, $a \in \mathbb{C} \setminus \mathcal{U}$ is called the <u>isolated singularity</u>, if

$\exists r > 0, \quad B_r(\check{a}) \subseteq \mathcal{U}$



$B_r(\check{a}) = \{ z \in \mathbb{C} \mid 0 < |z-a| < r \}$

(iso) singularities

- removable singularities:

  $\lim_{z \to a} f(z) \ \exists \iff |f(z)|$ is bounded near a

  $\iff f$ can be extended to holo fct on $B_r(a)$

- poles:

  $\lim_{z \to a} f(z) = \infty \iff \lim_{z \to a} |f(z)| = +\infty$

  $\iff \exists n \in \mathbb{N}_{>0}, \ (z-a)^n f(z)$ is removable at a

  (and $f(z)$ not removable) (minimal n: order of the pole)

- essential singularities:

  otherwise ($\lim_{z \to a} f(z) \nexists$ even for ∞)

  $\overset{\text{Great Picard's Thm}}{\iff}$ $\forall r > 0$, $f(B_r(\check{a}))$ takes on all possible cplx values, with at most a single exception (inf often)

Ex. 9.1. (1) $f(z) = \dfrac{3}{z-1-i} \in \mathcal{O}(\mathbb{C} - \{1+i\})$

$1+i$: pole of order 1

(2) $g(z) = \dfrac{z^3 + (-2+4i)z^2 + (5-8i)z - 10}{z^2 + (2-i)z - 2i} \in \mathcal{O}(\mathbb{C} - \{-2, i\})$

$$= \frac{(z-2)(z-i)(z-5i)}{(z+2)(z-i)}$$

$$= \frac{(z-2)(z-5i)}{z+2}$$

-2: pole of order 1

i: removable sing.

(3). $h(z) = e^{\frac{1}{z}} \in \mathcal{O}(\mathbb{C} - \{0\})$

0: essential sing.

2. Laurent expansion of iso singularities

removable sing. $\rightsquigarrow$ Taylor expansion

poles $\rightsquigarrow$ Taylor expansion of $(z-a)^n f(z)$, then divide $(z-a)^n$

essential sing. $\rightsquigarrow$ be careful about convergence!

e.g. $e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} z^{-n}$$

converge $\begin{cases} \text{for } z \neq 0 \\ \text{absolutely \& uniformly for } |z| \gg \varepsilon \end{cases}$

Ex 9.1. recurrence.

Laurant expansion

$$f(z) = \frac{3}{z-1-i}$$

$$= 3(z-1-i)^{-1} \qquad \text{at } 1+i$$

$$= -\frac{3}{1+i} \cdot \frac{1}{1-\frac{z}{1+i}}$$

$$= -\frac{3}{1+i} \sum_{k=0}^{+\infty} \left(\frac{z}{1+i}\right)^k \qquad \text{at } 0$$

$$h(z) = e^{\frac{1}{z}}$$

$$= \sum_{n=0}^{+\infty} \frac{1}{n!} z^{-n} \qquad \text{at } 0$$

Task 11.3 (ii) $f(z) = \dfrac{e^{\frac{1}{z}}}{(z-1)^2}$ at 0

$$f(z) = \left(\sum_{n=0}^{+\infty} \frac{1}{n!} z^{-n}\right)\left(\sum_{k=0}^{+\infty} (k+1) z^k\right)$$

Both series converge absolutely & uniformly in any cpt ~~se~~ subset of $B_1(0)$

$$= \sum_{n,k=0}^{+\infty} \frac{1}{n!} (k+1) z^{k-n}$$

$m = k-n$

$$= \sum_{m\in\mathbb{Z}} \left(\sum_{n=0}^{+\infty} \frac{n+m+1}{n!}\right) z^m = \sum_{m\in\mathbb{Z}} c_m z^m$$

$$\cancel{= \sum_{m\in\mathbb{Z}} c_m z^m}$$

$$c_{-1} = \sum_{n=0}^{+\infty} \frac{n}{n!} = e$$

Conclusion.

order		
$+\infty$	$\equiv 0$	removable sing.
$+$	zero	
0	nonzero value	
$-$	pole	
$-\infty$	$\equiv 0$	
X	essential sing.	

removable: $c_0 + c_1 z + \cdots$

pole: $c_{-n} z^{-n} + \cdots + \boxed{c_{-1}} z^{-1} + c_0 + c_1 z + \cdots$

essential: $\cdots + c_{-n} z^{-n} + \cdots + \boxed{c_{-1}} z^{-1} + c_0 + c_1 z + \cdots$

Residue

Def. $\operatorname{Res}_a f := c_{-1}$ for $f(z) = \sum_{k \in \mathbb{Z}} c_k (z-a)^k$ in $B_\delta(\check{a})$.

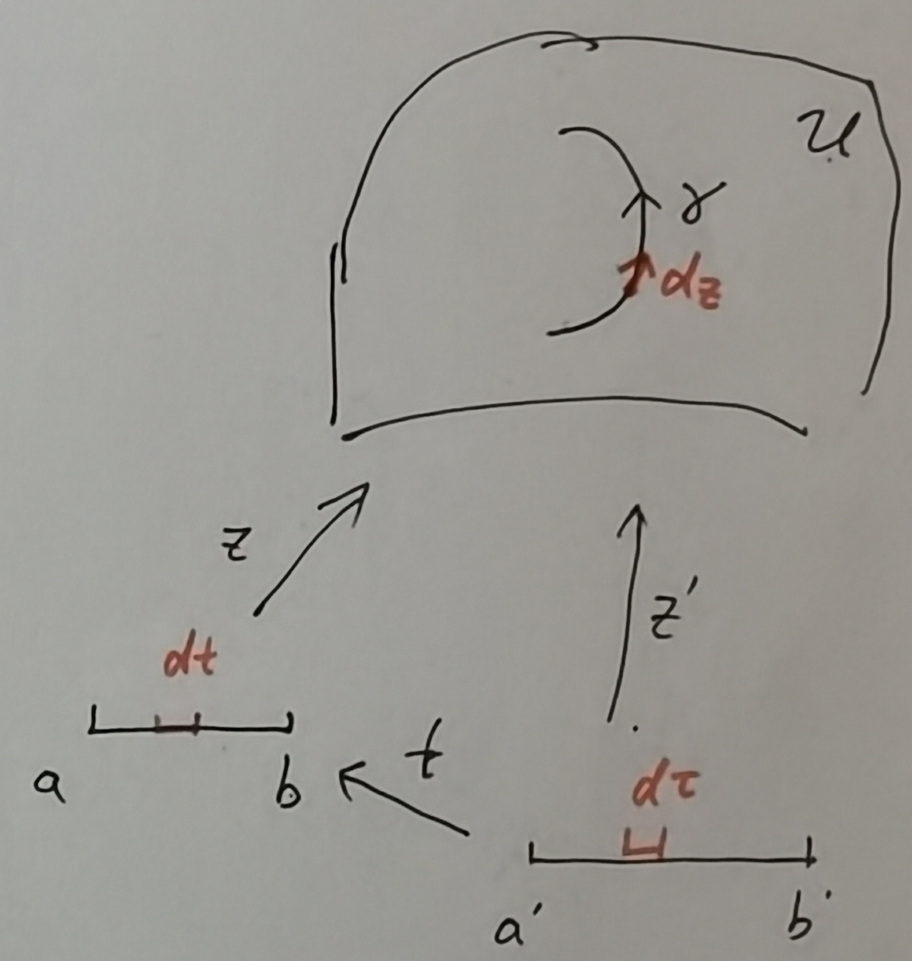
Have a Rest!

3. Contour integral

Def. For $f \in C(\mathcal{U})$, $\gamma \subset \mathcal{U}$,

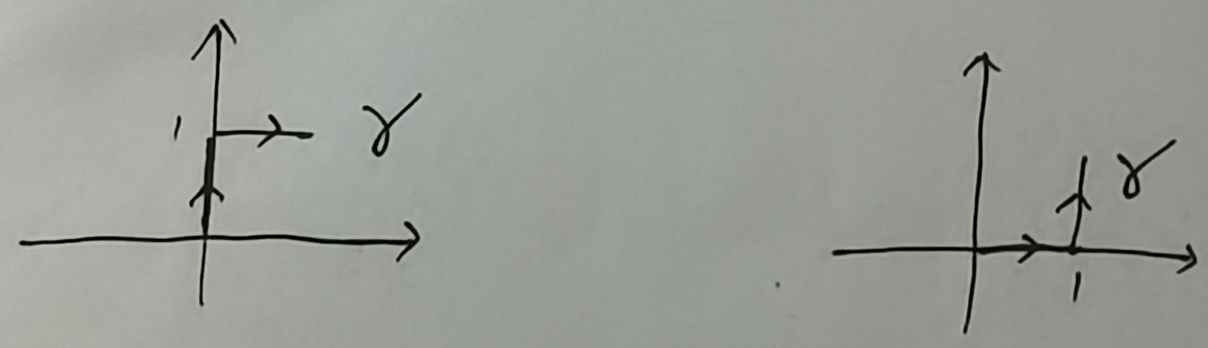
$$\int_\gamma f(z)\,dz := \int_a^b f(z(t)) \frac{dz}{dt}(t)\,dt$$

(does not depend on the choice of ~~parameter~~ ~~are length parametrization~~ parameterization)



Task 10.1

Task 10.2.

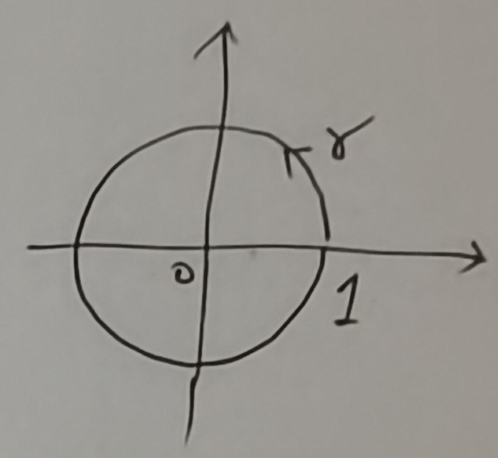


$f(z) = \bar{z}$

$$\int_\gamma f(z)\,dz = \int_0^1 f(it)\,d(it) + \int_0^1 f(i+t)\,d(i+t)$$

Task 10.3. $\left|\int_\gamma f(z)\,dz\right| \nleq \int_\gamma |f(z)|\,dz$ $\in \mathbb{C}$

Eg. For $f(z) = z^n$ $(n \in \mathbb{Z})$, compute



$$\int_\gamma f(z)\,dz = \int_0^{2\pi} f(e^{it})\,d(e^{it})$$

$$= \int_0^{2\pi} e^{nit} \cdot e^{it} \cdot i\,dt$$

$$= \int_0^{2\pi} e^{(n+1)it} \cdot i\,dt$$

$$= \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1 \end{cases}$$

Idea: for $f(z) = c_{-n} z^{-n} + \cdots + c_n z^n$ $\quad \in \mathcal{O}(B_1(\check{0}))$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \frac{1}{2\pi i}\int_\gamma \Big(\sum_{k \text{ fin}} c_k z^k\Big)\,dz$$

$$= \frac{1}{2\pi i}\sum_{k \text{ fin}} \Big(c_k \int_\gamma z^k\,dz\Big)$$

$$= c_{-1}$$ $\rightsquigarrow$ residue number

Similarly,

$$\frac{1}{2\pi i}\int_\gamma f'(z)\,dz = 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z^k}\,dz = c_{k-1}$$ $\rightsquigarrow$ "recover coefficient from f"

4. Residue ~~Theorem~~ Formula

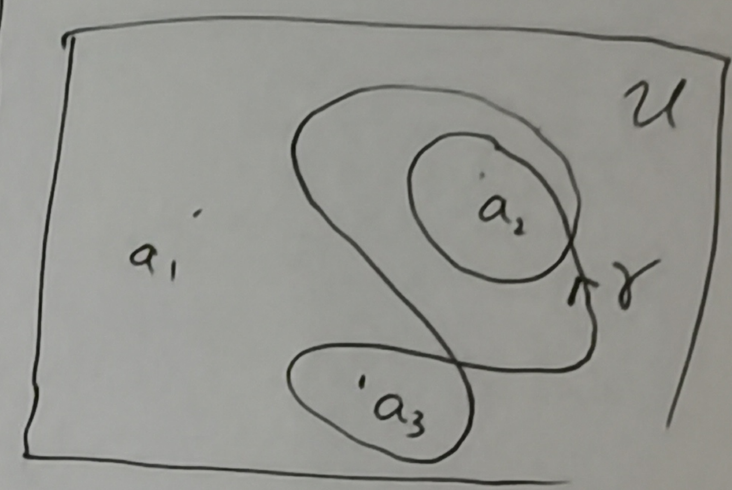
Thm. $\mathcal{U}$ simply connected. $\gamma \subset \mathcal{U} - \{a_1, \dots a_n\}$

$f \in \mathcal{O}(\mathcal{U} - \{a_1, \dots, a_n\})$.

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_{j=1}^{n} n(\gamma, a_j)\,\mathrm{Res}_{a_j} f$$

Don't forget!

usually 1



$n(\gamma, a_1) = 0$, $n(\gamma, a_2) = 2$, $n(\gamma, a_3) = -1$

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = 2\,\mathrm{Res}_{a_2} f - \mathrm{Res}_{a_3} f$$

Cor (Cauchy integral formula [p113, p119]. Average value formula)

$\mathcal{U}$ s.c ~~$\gamma \subset \mathcal{U} - \{a_1, \dots a_n\}$~~ $\gamma \subset \mathcal{U} - \{a\}$

$f \in \mathcal{O}(\mathcal{U})$

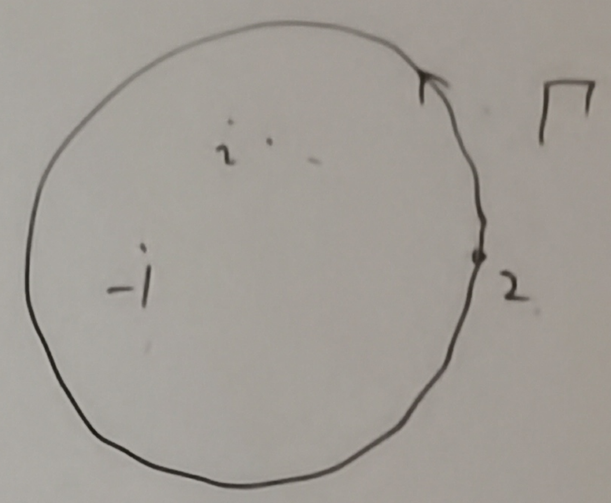
$$\int_\gamma f(z)\,dz = 0$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz = n(\gamma, z)\,f(a)$$

e.p. $\frac{1}{2\pi}\int_0^{2\pi} f(a + re^{it})\,dt = f(a)$.

Task 11.1. Compute

$$\int_\Gamma \frac{z^3+5}{z-i}\,dz \quad , \quad \int_\Gamma \frac{z^3+5}{(z+1)^3}\,dz$$



$$= 2\pi i \operatorname{Res}_i \frac{z^3+5}{z-i} \qquad = 2\pi i \boxed{\operatorname{Res}_{-1} \frac{z^3+5}{(z+1)^3}}.$$

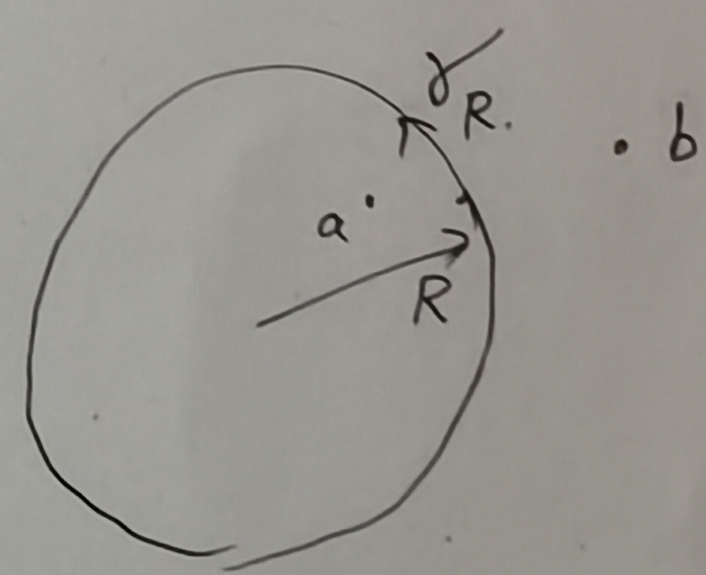
$$= 2\pi i(-5-i) \qquad = -6\pi i$$

$$\frac{z^3+5}{(z+1)^3} \overset{z+1=w}{=\!=} \frac{(w-1)^3+5}{w^3} = 4w^{-3}+3w^{-2}-3w^{-1}+1$$

$$= 4(z+1)^{-3}+3(z+1)^{-2}-3(z+1)^{-1}+1$$

Task. 10.2. (iii) Compute

$$\int_{\Gamma_R} \frac{1}{(z-a)(z-b)}\,dz$$



$$= 2\pi i \operatorname{Res}_a \frac{1}{(z-a)(z-b)}$$

$$= 2\pi i \frac{1}{a-b}$$

Task 11.2. Shows that for f harmonic in $B_{r'}(a)$

$$\cancel{\int_{\gamma_R} \frac{f(z)}{z}\, dz = f(0)}$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt = f(a) \qquad \forall r<r'$$

Proof. Denote $F(r) := \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{it})\,dt$. $\quad 0<r<r'$, ~~we know~~ need to show

$$F(r) \equiv f(a)$$

① $\lim_{r\to 0} F(r) = \frac{1}{2\pi}\int_0^{2\pi} \lim_{r\to 0} f(a+re^{it})\,dt = f(a)$

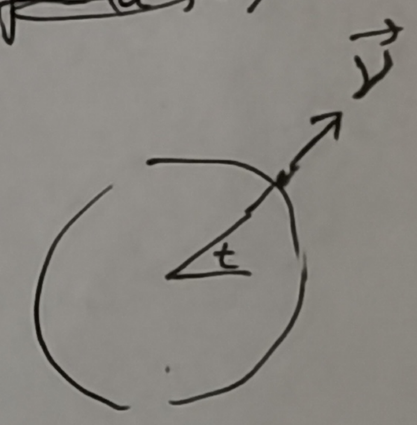
$a = x+iy$

② $\frac{dF}{dr} = \cancel{\frac{1}{2\pi}\int_0^{2\pi} e^{it} f'(a+re^{it})\,dt}$ $\qquad x+r\cos t,\ y+r\sin t.$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\partial f}{\partial x}\cdot\cos t + \frac{\partial f}{\partial y}\sin t\right)dt$$

$f = u+iv$

$$= \int_{\partial B_r(a)} \nabla f \cdot \vec{\gamma}\; d\left(\frac{t}{2\pi}\right)$$



$$= \int_{B_r(a)} \nabla\cdot\nabla f \; dA$$

$$= \int_{B_r(a)} \Delta f \; dA$$

$$= 0.$$

$\square$

5. Laurent series

Prop. Suppose $f \in \mathcal{O}(K_r^{r,r'}(a))$, then

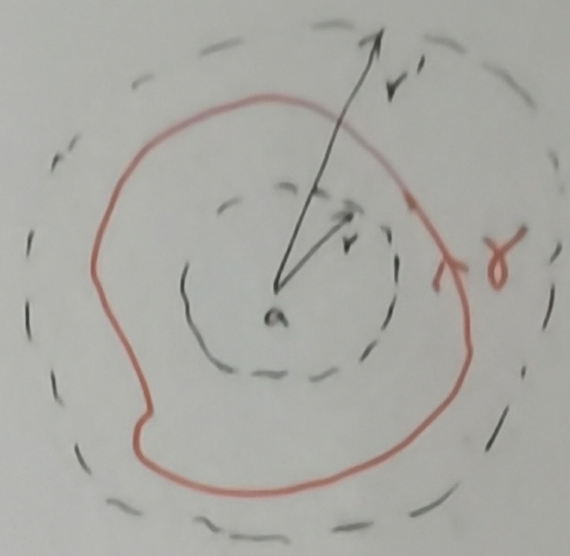
$$f(z) = \sum_{k\in\mathbb{Z}} b_k (z-a)^k \quad \text{in } K_r^{r,r'}(a)$$

where

$$b_k = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{k+1}}\,dz$$

with $\gamma \subset K_r^{r,r'}(a)$, $n(\gamma, a) = 1$

(does not depend on the choice of γ by Cauchy formula)



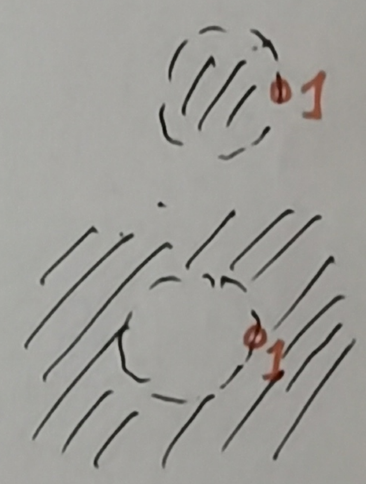
$K_r^{r,r'}(a) := \{z \in \mathbb{C} \mid r < |z-a| < r'\}$

$K_r^{0,r'}(a) = B_{r'}(a)$

Rmk. Be careful about the convergence!

E.g. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$

$= -z^{-1}\frac{1}{1-z^{-1}} = -z^{-1} - z^{-2} - z^{-3} - z^{-4} - \cdots$



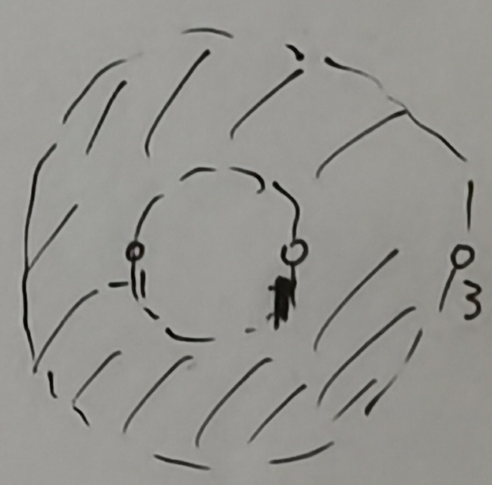
"When you go through the singularity, you get different Laurent series"

Task 10.3. (i)

$$\frac{1}{1-z^2} + \frac{1}{3-z}$$

$$= -z^{-2}\frac{1}{1-z^{-2}} + \frac{1}{3}\frac{1}{1-\frac{z}{3}}$$

$$= -z^{-2}\sum_{k=0}^{+\infty}(z^{-2})^k + \frac{1}{3}\sum_{k=0}^{+\infty}\left(\frac{z}{3}\right)^k.$$



$|z^{-2}| < 1$, $\left|\frac{z}{3}\right| < 1$