Eine Woche, ein Beispiel

3.16 Schubert calculus: subvariety with vb

This is a follow up of [2025.02.23].

Goal: relate subvarieties to some vector bundles, so that we can compute their homology class in terms of Chern class (when the dimension is correct).

The Chern class will be dealt with in the next document.

Concretely, we will write subvarieties as:

- the zero set of a section in a v.b.
- the degeneracy loci of a morphism $\mathcal{E} \longrightarrow \mathcal{F}$ among v.bs
- the preimage of known cycles in Grassmannian
- ...

1. Known subvarieties and known vector bundles
2. Subvariety as section
3. Subvariety as degeneracy loci

1. Known subvarieties and known vector bundles

Schubert variety

Recall that the Schubert variety has the expression $\quad \omega \overset{\text{cohom}}{\longleftrightarrow} (\lambda_1, \dots, \lambda_r)$

$$
\begin{aligned}
\Sigma_{\lambda_1,\dots,\lambda_r}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+i-\lambda_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{\omega_i} \geq i \quad \forall i\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{\omega_i} \leq n-\lambda_i \quad \forall i\}
\end{aligned}
$$

Especially,

$$
\begin{aligned}
\Sigma_{k^s}(\mathcal{V}) &= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+i-k} \leq n-k \quad \forall i \leq s\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k\} \\
&= \{\Lambda \in Gr(r,n) \mid \dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s\}
\end{aligned}
$$

For special k, s, one can further simplify the formulas:

	k	1	k	$n-r$
s	$Gr(r,n)$			
1		$\Lambda + \mathcal{V}_{n-r} \subseteq H$ or $\Lambda \cap \mathcal{V}_{n-r} \neq \{0\}$	$\Lambda \cap \mathcal{V}_{n-r+1-k} \neq \{0\}$	$\mathcal{V}_1 \subset \Lambda$
s		$\Lambda + \mathcal{V}_{n-r+s-1} \subseteq H$	$\dim \Lambda + \mathcal{V}_{n-r+s-k} \leq n-k$ or $\dim \Lambda \cap \mathcal{V}_{n-r+s-k} \geq s$	$\mathcal{V}_s \subset \Lambda$
r		$\Lambda \subset \mathcal{V}_{n-1}$	$\Lambda \subset \mathcal{V}_{n-k}$	$\{\mathcal{V}_r\}$

Vector bundles on Grassmannian

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}^{\oplus n} \xrightarrow{\pi_{\mathcal{S}^\vee}} \mathcal{S}^\vee \longrightarrow 0$$

$\mathcal{S}$: Subspace = tautological bundle
$\mathcal{Q}$: Quotient = quotient bundle

When $r=1$, $Gr(r,n) = \mathbb{P}^{n-1}$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{Q} \longrightarrow 0$$
$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow \mathcal{O}^{\oplus n} \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

With these basic v.bs, we can construct more bundles on $Gr(r,n)$:

$$T_{Gr} = Hom(\mathcal{S},\mathcal{Q}) = \mathcal{S}^\vee \otimes \mathcal{Q} \qquad \omega_{Gr}^\vee = \det \mathcal{S}^\vee \otimes \mathcal{Q}$$
$$\Omega_{Gr} = T_{Gr}^\vee = Hom(\mathcal{Q},\mathcal{S}) = \mathcal{Q}^\vee \otimes \mathcal{S} \qquad \omega_{Gr} = \det \mathcal{Q}^\vee \otimes \mathcal{S}$$

2. Subvariety as section

Hypersurface and its Fano variety of (r-1)-planes

Let $F \in k[z_1, \dots, z_n]$ be a homo poly of deg d. The hypersurface

$$Y_d := \{F = 0\} \subseteq \mathbb{P}^{n-1}$$

is given as a section of

$$\mathcal{O}(d) = \mathrm{Sym}^d \mathcal{O}(1)$$

In general, the Fano variety of (r-1)-planes ($\cong \mathbb{P}^{r-1}$)

$$F_{r-1}(Y_d) := \{W \in Gr(r,n) \mid F|_W = 0\} \subseteq Gr(r,n)$$

is given as a section of $\mathrm{Sym}^d \mathcal{S}^\vee$, through the map

$$\mathrm{Sym}^d \pi_{\mathcal{S}^\vee} : \underset{\substack{\parallel \\ (\mathrm{Sym}^d V^*) \otimes \mathcal{O}}}{\mathrm{Sym}^d(\mathcal{O}^{\oplus n})} \twoheadrightarrow \mathrm{Sym}^d(\mathcal{S}^\vee)$$

Map of section: $\quad F \otimes 1 \longmapsto s_F = \mathrm{Sym}^d \pi_{\mathcal{S}^\vee}(F \otimes 1)$

Fiberwise, $(\mathrm{Sym}^d \pi_{\mathcal{S}^\vee})_W : \mathrm{Sym}^d V^* \longrightarrow \mathrm{Sym}^d W^*$

We know that

$$F|_W \equiv 0$$
$$\Leftrightarrow (\mathrm{Sym}^d \pi_{\mathcal{S}^\vee})_W (F) = 0$$
$$\Leftrightarrow s_F = 0, \text{ i.e., } \quad [W] \text{ lies in the zero set of } s_F.$$

E.g. $F_0(Y_d) = Y_d$

$F_1(Y_d) \subseteq Gr(2,n)$ — Fano variety of lines

$F_m(Y_2) \subseteq Gr(m+1, 2m+2)$ or $Gr(m+1, 2m+3)$ — Last Grassmannian, orthogonal ...

Cor. $F_{r-1}(Y_d)$ has codimension $\leq \binom{d+r-1}{d}$ (when non-empty)