Bruhat–Tits building

Xiaoxiang Zhou

Humboldt-Universität zu Berlin

January 26, 2026

Figures of Bruhat–Tits building

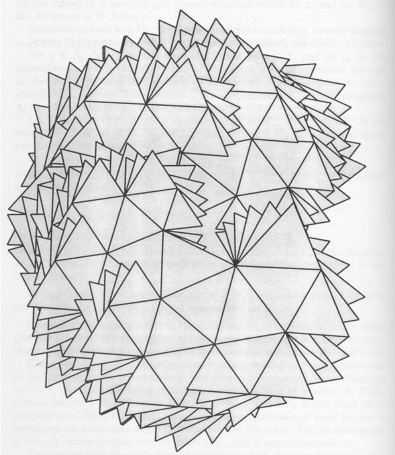


Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from a talk by Annette Werner

Figures of Bruhat–Tits building

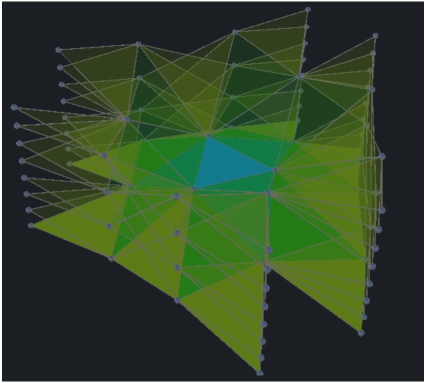


Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

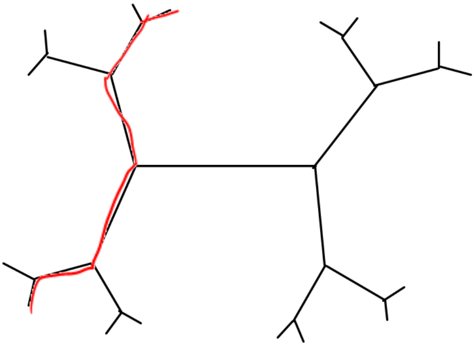


Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Figures of Bruhat–Tits building

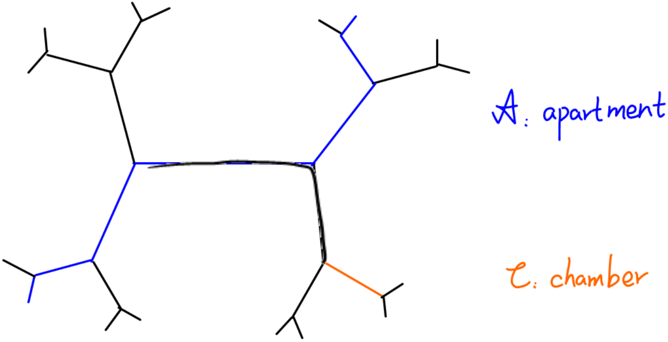


Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

Standard subgroups

G connected reductive

$\underline{GL_n(\mathbb{C}), SL_n}$
o

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/B = \{ \text{ flags of } \kappa^n \}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{ V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i \} \end{aligned}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$\begin{aligned} &\mathrm{Gr}(r, n) \\ = \;\; &\{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$\begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r,n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{ V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i \} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \ \mathrm{Gr}(r,n) \\ &= \ \{ V \subset \kappa^n \mid \dim V = r \} \end{aligned}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\ \text{flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \ \mathrm{Gr}(r, n) \\ &= \ \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

T is a torus, so every rep decomposes as direct sum of 1-dim reps.

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

T is a torus, so every rep decomposes as direct sum of 1-dim reps.

$$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r,n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

T is a torus, so every rep decomposes as direct sum of 1-dim reps.

$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ characters (1-dim reps)

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r,n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

T is a torus, so every rep decomposes as direct sum of 1-dim reps.

$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ characters (1-dim reps)

$X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n$

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{\text{ flags of } \kappa^n\} \\ &= \{V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i\} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r,n) \\ &= \{V \subset \kappa^n \mid \dim V = r\} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{\kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1\}$$

T is a torus, so every rep decomposes as direct sum of 1-dim reps.

$X^*(T) := \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ characters (1-dim reps)

$X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n$ cocharacters (1-parameter subgps)

Weyl group

Weyl group

Definition (Weyl group)

$$W := N_G(T)/T.$$

Weyl group

Definition (Weyl group)

$$W := N_G(T)/T.$$

Example

When $G = \mathrm{GL}_n(\kappa)$,

$$N_G(T) \quad = \{\ \textit{monoidal matrices}\ \} \qquad \begin{pmatrix} & * & & \\ & & * & \\ * & & & \\ & & & * \end{pmatrix}$$

$$N_G(T)/T \cong\ S_n \qquad \textit{Weyl group of type } A$$

Weyl group

$W G T$

$G \; G \; G$

$W = N_G(T)/T \curvearrowright G \quad T \rightsquigarrow W \circlearrowright X^*(T)$

$W \curvearrowright X_*(T)$

Definition (Weyl group)

$$W := N_G(T)/T.$$

Example

When $G = \mathrm{GL}_n(\kappa)$,

$$N_G(T) \quad = \{ \text{ monoidal matrices } \}$$

$$N_G(T)/T \cong S_n \qquad \textit{Weyl group of type } A$$

Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\underline{\omega \in W}} B\omega B. \qquad = \text{“}BWB\text{”}$$

So the Weyl group is the “heart” of the reductive group.

Weyl group

Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B.$$

So the Weyl group is the "heart" of the reductive group.

Weyl group

Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B.$$

So the Weyl group is the "heart" of the reductive group.

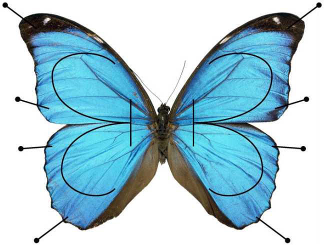


Figure: Pinned butterfly

Weyl group action on cocharacter lattices

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \end{aligned}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

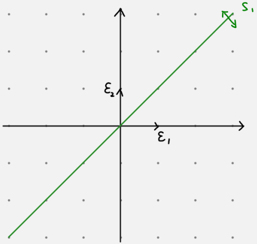
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right) \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



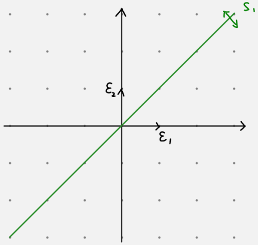
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

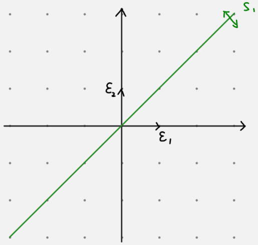
$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \left(\begin{smallmatrix} x & \\ & 1 \end{smallmatrix}\right) \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \left(\begin{smallmatrix} 1 & \\ & x \end{smallmatrix}\right) \end{aligned}$$

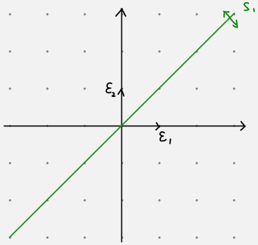
$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_2(\kappa)$, $T = \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right)$, $X_*(T) = \mathbb{Z}\varepsilon$, where

$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \left(\begin{smallmatrix} x & \\ & x^{-1} \end{smallmatrix}\right) \end{aligned}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

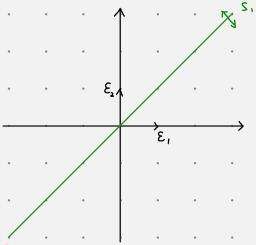
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where

$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

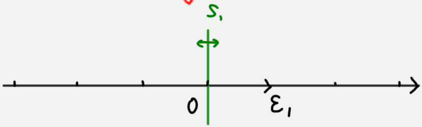


$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & \\ & x \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where

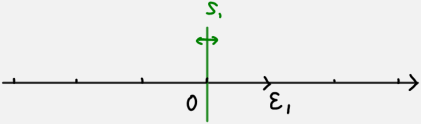


$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{SL}_2(\kappa)$, $T = \left(\begin{smallmatrix} a & \\ & a^{-1} \end{smallmatrix}\right)$, $X_*(T) = \mathbb{Z}\varepsilon$, where

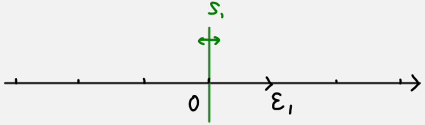


$$\begin{aligned} \varepsilon :\ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



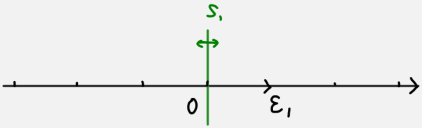
$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

Weyl group action on cocharacter lattices

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

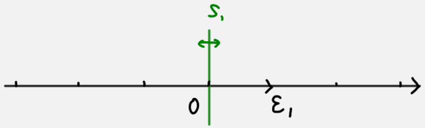
When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

Weyl group action on cocharacter lattices

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where

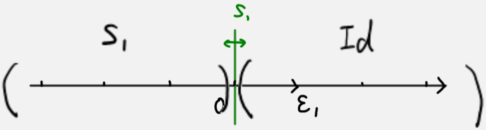
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle \underset{(12)}{s_1}, \underset{(23)}{s_2} \rangle$$

Weyl group action on cocharacter lattices

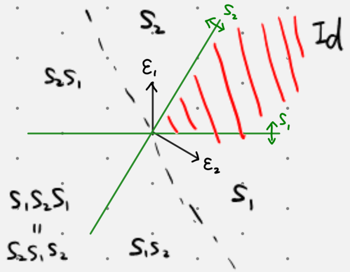
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



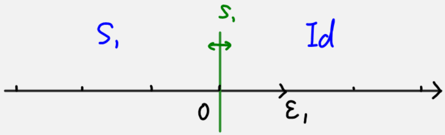
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle s_1, s_2 \rangle$$

Weyl group action on cocharacter lattices

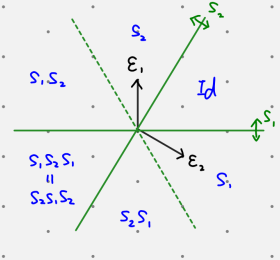
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle s_1, s_2 \rangle$$

Non-standard subgroups

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

gTg^{-1}

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\{ \textit{ Borel subgroups } \} = \left\{ gBg^{-1} \;\middle|\; g \in G \right\}$$

$$\{ \textit{ parabolic subgroups } \} = \left\{ gPg^{-1} \;\middle|\; g \in G \right\}$$

$$\{ \textit{ maximal tori } \} = \left\{ gTg^{-1} \;\middle|\; g \in G \right\}$$

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\{ \text{ \textit{Borel subgroups} } \} = \left\{ gBg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/B$$

$$\{ \text{ \textit{parabolic subgroups} } \} = \left\{ gPg^{-1} \;\middle|\; g \in G \right\}$$

$$\{ \text{ \textit{maximal tori} } \} = \left\{ gTg^{-1} \;\middle|\; g \in G \right\}$$

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\{ \textit{ Borel subgroups } \} = \left\{ gBg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/B$$

$$\{ \textit{ parabolic subgroups } \} = \left\{ gPg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/P$$

$$\{ \textit{ maximal tori } \} = \left\{ gTg^{-1} \;\middle|\; g \in G \right\}$$

Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\{\ \textit{Borel subgroups}\ \} = \left\{ gBg^{-1} \,\middle|\, g \in G \right\} \cong G/B$$

$$\{\ \textit{parabolic subgroups}\ \} = \left\{ gPg^{-1} \,\middle|\, g \in G \right\} \cong G/P$$

$$\{\ \textit{maximal tori}\ \} = \left\{ gTg^{-1} \,\middle|\, g \in G \right\} \cong G/N_G(T)$$

G/T

Non-standard subgroups

$$\{\text{ Borel subgroups }\} = \left\{ gBg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/B$$

$$\{\text{ parabolic subgroups }\} = \left\{ gPg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/P$$

$$\{\text{ maximal tori }\} = \left\{ gTg^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/N_G(T)$$

Non-standard subgroups

$$\{\text{ Borel subgroups }\} = \left\{ gB_0g^{-1} \;\middle|\; g \in G \right\} \cong G/B_0$$

$$\{\text{ parabolic subgroups }\} = \left\{ gP_0g^{-1} \;\middle|\; g \in G \right\} \cong G/P_0$$

$$\{\text{ maximal tori }\} = \left\{ gT_0g^{-1} \;\middle|\; g \in G \right\} \cong G/N_G(T_0)$$

$$\{(B,T) \mid B \supset T\} = \left\{ (gB_0g^{-1}, gT_0g^{-1}) \right\} \cong G/T_0$$

Non-standard subgroups

$$
\begin{aligned}
\{\text{ Borel subgroups }\} &= \left\{ gB_0g^{-1} \,\middle|\, g \in G \right\} && \cong G/B_0 \\
\{\text{ parabolic subgroups }\} &= \left\{ gP_0g^{-1} \,\middle|\, g \in G \right\} && \cong G/P_0 \\
\{\text{ maximal tori }\} &= \left\{ gT_0g^{-1} \,\middle|\, g \in G \right\} && \cong G/N_G(T_0) \\
\{(B,T) \mid B \supset T\} &= \left\{ (gB_0g^{-1}, gT_0g^{-1}) \right\} && \cong G/T_0
\end{aligned}
$$

$$
\begin{tikzcd}
\{T \mid T \subset B_0\} \arrow[dr, dash] & & \{B \mid B \supset T_0\} \arrow[dl, dash] \\
& \{(B,T)\} \arrow[dl] \arrow[dr] & \\
\{B\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{A}^{\frac{n(n-1)}{2}} \cong B_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 = W \arrow[dl, dash] \\
& G/T_0 \arrow[dl] \arrow[dr] & \\
G/B_0 & & G/N_G(T_0)
\end{tikzcd}
$$

Non-standard subgroups

$$\{\text{ Borel subgroups }\} = \left\{ gB_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/B_0$$

$$\{\text{ parabolic subgroups }\} = \left\{ gP_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/P_0$$

$$\{\text{ maximal tori }\} = \left\{ gT_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/N_G(T_0)$$

$$\{(B,T) \mid B \supset T\} = \left\{ (gB_0g^{-1}, gT_0g^{-1}) \right\} \qquad \cong G/T_0$$

$$\begin{tikzcd}
\{T \mid T \subset B_0\} \arrow[dr, dash] & & \{B \mid B \supset T_0\} \arrow[dl, dash] \\
& \{(B,T)\} \arrow[dl] \arrow[dr] & \\
\{B\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 = W \arrow[dl, dash] \\
& G/T_0 \arrow[dl] \arrow[dr] & \\
G/B_0 & & G/N_G(T_0)
\end{tikzcd}$$

Non-standard subgroups

$$\{\text{ Borel subgroups }\} = \left\{ gB_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/B_0$$

$$\{\text{ parabolic subgroups }\} = \left\{ gP_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/P_0$$

$$\{\text{ maximal tori }\} = \left\{ gT_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/N_G(T_0)$$

$$\{(B,T) \mid B \supset T\} = \left\{ (gB_0g^{-1}, gT_0g^{-1}) \right\} \qquad \cong G/T_0$$

$$\begin{tikzcd}[column sep=small] & \{(B,T)\} \arrow[dl] \arrow[dr] & \\ \{B\} & & \{T\} \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & G/T_0 \arrow[dl] \arrow[dr] & \\ G/B_0 & & G/N_G(T_0) \end{tikzcd}$$

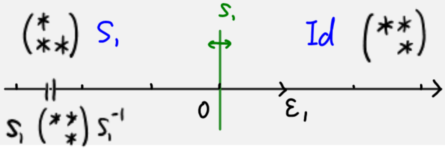
Fibres: $\{T \mid T \subset B_0\}$, $\{B \mid B \supset T_0\}$; B_0/T_0, $N_G(T_0)/T_0 = W$

$$\{\text{ (Weyl) chambers }\} \overset{1:1}{\longleftrightarrow} \underline{W} \overset{1:1}{\longleftrightarrow} \{B \mid B \supset T_0\}$$

$\mathcal{C}_B$ $\qquad\qquad$ B

Weyl group action on cocharacter lattices(revisited)

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where

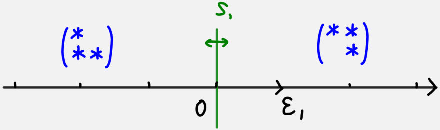


$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices(revisited)

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where

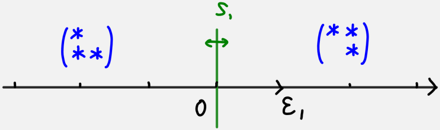


$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

Weyl group action on cocharacter lattices(revisited)

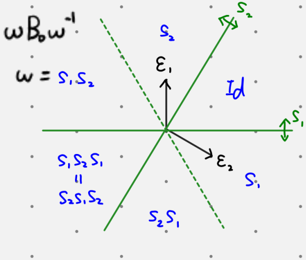
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



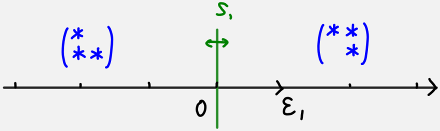
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle s_1, s_2 \rangle$$

Weyl group action on cocharacter lattices(revisited)

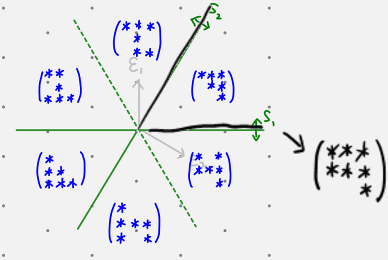
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



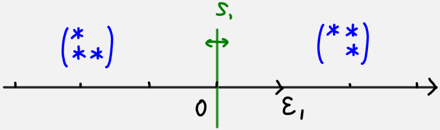
$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle s_1, s_2 \rangle$$

Weyl group action on cocharacter lattices(revisited)

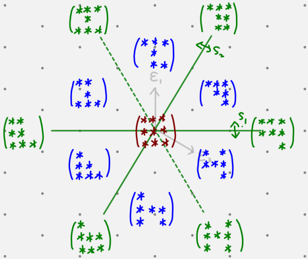
When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



$$\begin{aligned} \varepsilon : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_2 = \{\mathrm{Id}, s_1\}$$

When $G = \mathrm{SL}_3(\kappa)$, $T = \begin{pmatrix} a & & \\ & b & \\ & & a^{-1}b^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & & \\ & x^{-1} & \\ & & 1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \varepsilon_2 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} 1 & & \\ & x & \\ & & x^{-1} \end{pmatrix} \end{aligned}$$

$$W = S_3 = \langle s_1, s_2 \rangle$$

Definition (chamber, apartment and building)

Given a maximal torus T, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{B \supset T} \mathcal{C}_B,$$

Definition (chamber, apartment and building)

Given a maximal torus T, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{B \supset T} \mathcal{C}_B,$$

and the building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) / \sim \; = \bigcup_B \mathcal{C}_B.$$

e.g. $k = \mathbb{F}_p$

$k = \mathbb{C}$

Example of spherical building

Example of spherical building

When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.

$\mathcal{B}_{SL_2(\mathbb{F}_3)}$

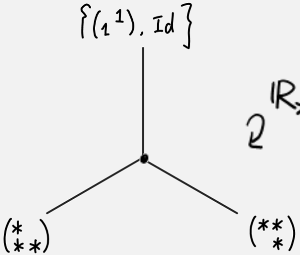


Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

$\circlearrowleft \mathbb{R}_{>0}$

$\left(\mathcal{B} - \{0\}\right) / \mathbb{R}_{>0}$

Example of spherical building

When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.

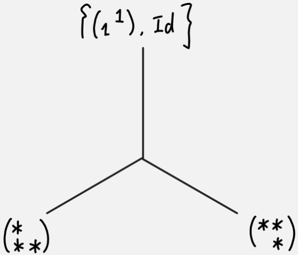


Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

Remark

$\mathcal{B}$ inherits the metric structure from $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$.

Remark

$\mathcal{B}$ inherits the metric structure from $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$.
$\mathcal{B}$ has also polysimplicial complex structure.
When $\kappa = \mathbb{F}_p$, $\mathcal{B}$ is finite. i.e., having finite many chambers

Remark

$\mathcal{B}$ inherits the metric structure from $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$.
$\mathcal{B}$ has also polysimplicial complex structure.
When $\kappa = \mathbb{F}_p$, $\mathcal{B}$ is finite. i.e., having finite many chambers

Proposition

- *Any two chambers lie in one apartment.*

B_1 , B_2 , $\exists T \subset B_1 \cap B_2$

$B_0 \cap g B_0 g^{-1} = B_0 \cap b_1 w B_0 w^{-1} b_1^{-1} \supset b_1 T b_1^{-1}$

$g = b_1 w b_2$

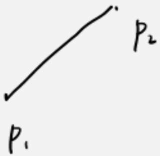
Remark

$\mathcal{B}$ inherits the metric structure from $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$.
$\mathcal{B}$ has also polysimplicial complex structure.
When $\kappa = \mathbb{F}_p$, $\mathcal{B}$ is finite. i.e., having finite many chambers

Proposition

- *Any two chambers lie in one apartment.*
- *There is a unique geodesic through any two points $p_1, p_2 \in \mathcal{B}$.*

$$p_1 \in \mathcal{C}_{B_1} \subset \mathcal{A}_T$$
$$p_2 \in \mathcal{C}_{B_2} \subset$$



Plan of the talk

p-adic notation

symbol	name	example
F	NA local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	ring of integers	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v(\frac{a}{b}p^k) = k$

p-adic notation

symbol	name	example
F	NA local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	ring of integers	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \smallsetminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

standard subgroups in the p-adic world

$$\underbrace{GL_n(\mathcal{O})}_{\text{maximal compact subgp}} \subset GL_n(F)$$

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$\cup$$

$$B, P$$

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

Iwahori subgroup

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

Iwahori subgroup

$$\widetilde{P} = \pi^{-1}(P) = \left(\begin{array}{c:c} \mathcal{O} & \mathcal{O} \\ \hdashline \mathfrak{p} & \mathcal{O} \end{array}\right)$$

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

Iwahori subgroup

$$\widetilde{P} = \pi^{-1}(P) = \left(\begin{array}{c:c} \mathcal{O} & \mathcal{O} \\ \hdashline \mathfrak{p} & \mathcal{O} \end{array}\right)$$

Parahoric subgroup

=Parabolic Iwahori subgroup

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$ Iwahori subgroup

$$\widetilde{P} = \pi^{-1}(P) = \left(\begin{array}{c:c} \mathcal{O} & \mathcal{O} \\ \hdashline \mathfrak{p} & \mathcal{O} \end{array}\right)$$ Parahoric subgroup

$=$Parabolic Iwahori subgroup

Remark

They also have moduli interpretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{\mathfrak{p}L = L_0 \subset L_1 \subset \cdots \subset L_n = L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

Extended Weyl group

Extended Weyl group

To get the Iwahori decomposition

$$G(F) = \bigsqcup_{\varpi \in W_{\mathrm{ext}}} I \varpi I, \qquad = \text{“} I W_{ext} I \text{”}$$

we define the extended Weyl group as

$$W_{\mathrm{ext}} := N_G(T(\mathcal{O}))/T(\mathcal{O}) \cong X_*(T) \rtimes W_f. \leftarrow \text{finite}$$

Extended Weyl group

To get the Iwahori decomposition

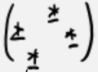
$$G(F) = \bigsqcup_{\varpi \in W_{\text{ext}}} I \varpi I,$$

we define the extended Weyl group as

$$W_{\text{ext}} := N_G(T(\mathcal{O}))/T(\mathcal{O}) \cong X_*(T) \rtimes W_f.$$

Example

When $G = \mathrm{GL}_n(F)$,



$$W_{\text{ext}} = \{\ \textit{monoidal matrices}\ \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

Extended Weyl group action

Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\mathrm{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\mathrm{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1 \rangle$, where

Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\mathrm{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$

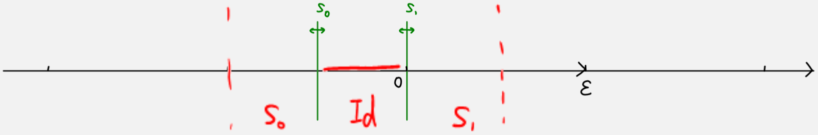
Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\text{ext}} = \langle s_0, s_1 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



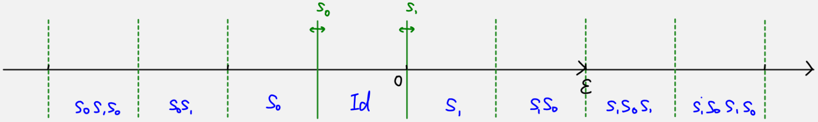
Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\text{ext}} = \langle s_0, s_1 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



Extended Weyl group action

When $G = \mathrm{SL}_3(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1, s_2 \rangle$, where

Extended Weyl group action

When $G = \mathrm{SL}_3(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1, s_2 \rangle$, where

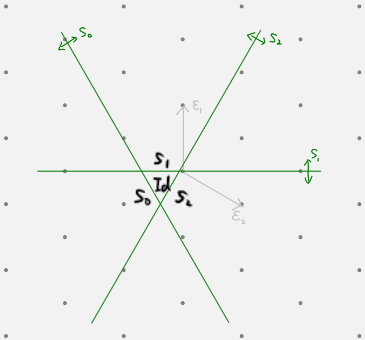
$$s_1 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ -\pi & & \end{pmatrix}$$

(12) (23) (31)

Extended Weyl group action

When $G = \mathrm{SL}_3(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1, s_2 \rangle$, where

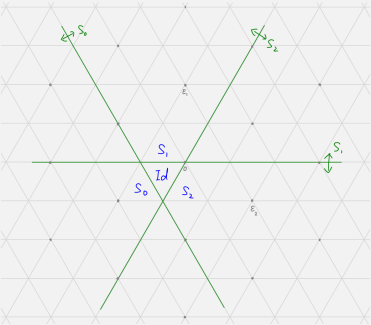
$$s_1 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ -\pi & & \end{pmatrix}$$



Extended Weyl group action

When $G = \mathrm{SL}_3(F)$, $W_{\text{ext}} = \langle s_0, s_1, s_2 \rangle$, where

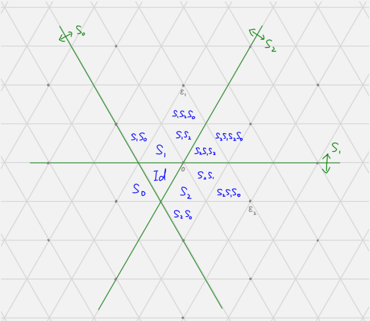
$$s_1 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ -\pi & & \end{pmatrix}$$



Extended Weyl group action

When $G = \mathrm{SL}_3(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1, s_2 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ -\pi & & \end{pmatrix}$$



Extended Weyl group action

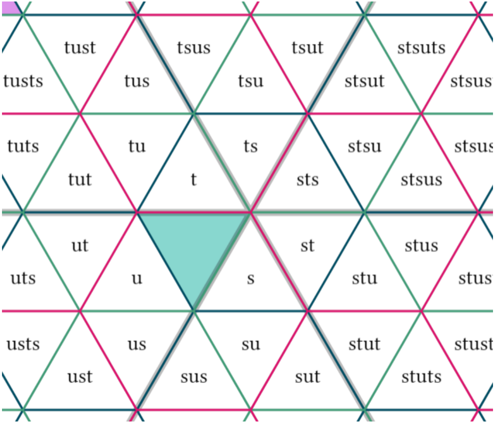


Figure: Reduced expressions labels, from Lievis

Non-standard subgroups in the p-adic world

Non-standard subgroups in the p-adic world

Similarly,

$$\{\text{ Iwahori subgroups }\} = \left\{ gI_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/I_0$$

$$\{\text{ paraholic subgroups }\} = \left\{ g\widetilde{P}_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/\widetilde{P}_0$$

$$\{\text{ maximal tori over } \mathcal{O} \text{ }\} = \left\{ gT_0g^{-1} \;\middle|\; g \in G \right\} \qquad \cong G/N_G(T_0(\mathcal{O}))$$

Non-standard subgroups in the p-adic world

Similarly,

$$\{\text{ Iwahori subgroups }\} = \left\{gI_0g^{-1} \;\middle|\; g \in G\right\} \cong G/I_0$$

$$\{\text{ paraholic subgroups }\} = \left\{g\widetilde{P}_0g^{-1} \;\middle|\; g \in G\right\} \cong G/\widetilde{P}_0$$

$$\{\text{ maximal tori over } \mathcal{O} \;\} = \left\{gT_0g^{-1} \;\middle|\; g \in G\right\} \cong G/N_G(T_0(\mathcal{O}))$$

$$\{(I, T) \mid I \supset T\} = \left\{(gI_0g^{-1}, gT_0g^{-1})\right\} \cong G/T_0(\mathcal{O})$$

Non-standard subgroups in the p-adic world

Similarly,

$$\{\text{ Iwahori subgroups }\} = \left\{ gI_0g^{-1} \;\middle|\; g \in G \right\} \cong G/I_0$$

$$\{\text{ paraholic subgroups }\} = \left\{ g\widetilde{P}_0g^{-1} \;\middle|\; g \in G \right\} \cong G/\widetilde{P}_0$$

$$\{\text{ maximal tori over } \mathcal{O} \} = \left\{ gT_0g^{-1} \;\middle|\; g \in G \right\} \cong G/N_G(T_0(\mathcal{O}))$$

$$\{(I,T) \mid I \supset T\} = \left\{ (gI_0g^{-1}, gT_0g^{-1}) \right\} \cong G/T_0(\mathcal{O})$$

$$\begin{tikzcd}[column sep=small]
\{T \mid T \subset I_0\} \arrow[dr, dash] & & \{I \mid I \supset T_0\} \arrow[dl, dash] \\
& \{(I,T)\} \arrow[dl] \arrow[dr] & \\
\{I\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
I_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 = W_{\mathrm{ext}} \arrow[dl, dash] \\
& G/T_0 \arrow[dl] \arrow[dr] & \\
G/I_0 & & G/N_G(T_0)
\end{tikzcd}$$

Non-standard subgroups in the p-adic world

Similarly,

$$\{\text{ Iwahori subgroups }\} = \left\{ gI_0g^{-1} \,\middle|\, g \in G \right\} \qquad \cong G/I_0$$

$$\{\text{ paraholic subgroups }\} = \left\{ g\widetilde{P}_0g^{-1} \,\middle|\, g \in G \right\} \qquad \cong G/\widetilde{P}_0$$

$$\{\text{ maximal tori over } \mathcal{O} \} = \left\{ gT_0g^{-1} \,\middle|\, g \in G \right\} \qquad \cong G/N_G(T_0(\mathcal{O}))$$

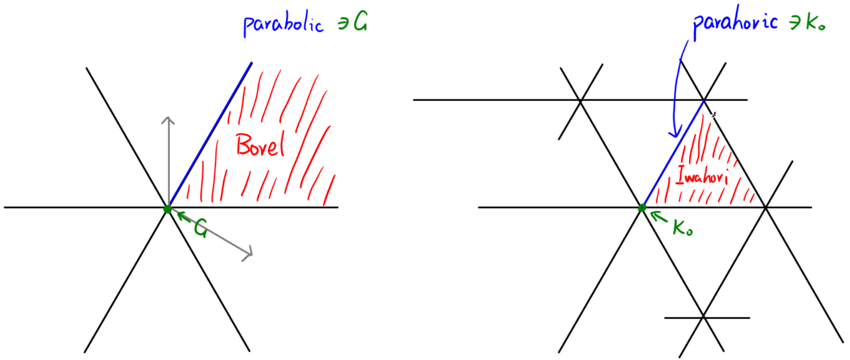
$$\{(I,T) \mid I \supset T\} = \left\{ (gI_0g^{-1}, gT_0g^{-1}) \right\} \qquad \cong G/T_0(\mathcal{O})$$

$$\begin{tikzcd}[column sep=small]
\{T \mid T\subset I_0\} \arrow[dr, dash] & & \{I \mid I\supset T_0\} \arrow[dl, dash] \\
& \{(I,T)\} \arrow[dl] \arrow[dr] & \\
\{I\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
I_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 = W_{\text{ext}} \arrow[dl, dash] \\
& G/T_0 \arrow[dl] \arrow[dr] & \\
G/I_0 & & G/N_G(T_0)
\end{tikzcd}$$

$$\{\text{ chambers }\} \xleftrightarrow{1:1} W_{\text{ext}} \xleftrightarrow{1:1} \{I \mid I \supset T_0\}$$

$$\mathcal{C}_I \longleftarrow\!\!\shortmid \; I$$

Comparison



Extended Weyl group action(revisited)

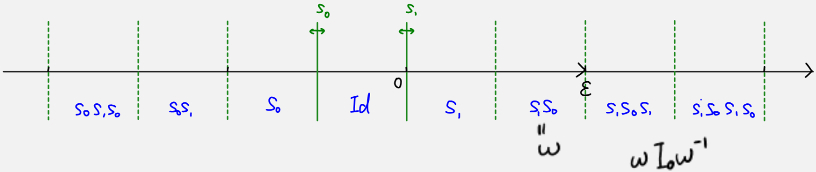
$T_{\mathcal{O}}$ $\qquad T\otimes_{\mathcal{O}} F \simeq T'\otimes_{\mathcal{O}} F$

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\text{ext}} = \langle s_0, s_1 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



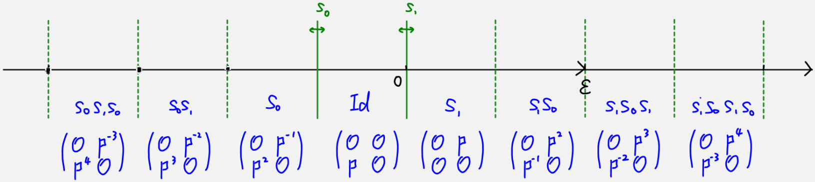
Extended Weyl group action(revisited)

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\mathrm{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1 \rangle$, where

$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



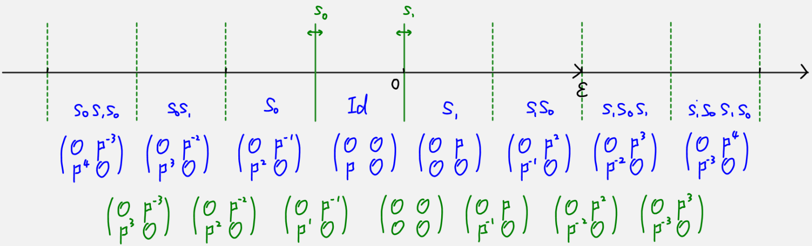
Extended Weyl group action(revisited)

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When $G = \mathrm{SL}_2(F)$, $W_{\text{ext}} = \langle s_0, s_1 \rangle$, where

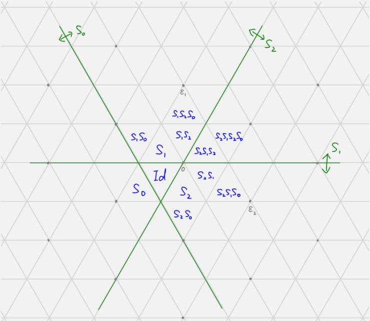
$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \qquad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



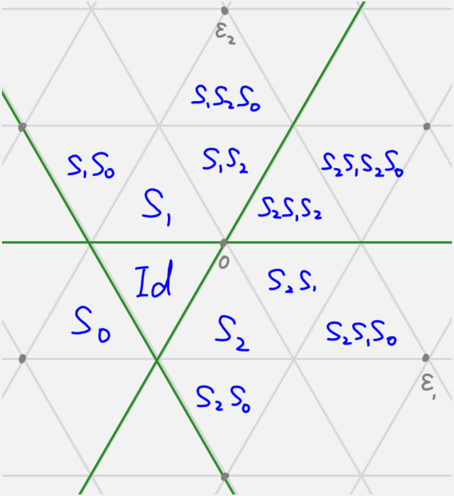
Extended Weyl group action(revisited)

When $G = \mathrm{SL}_3(F)$, $W_{\mathrm{ext}} = \langle s_0, s_1, s_2 \rangle$, where

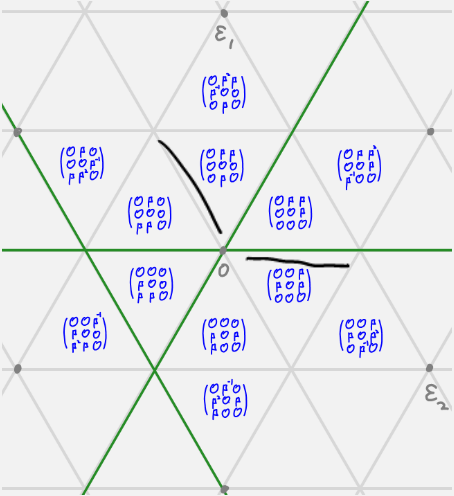
$$s_1 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix} \quad s_2 = \begin{pmatrix} 1 & & \\ & & 1 \\ & -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & & \pi^{-1} \\ & 1 & \\ -\pi & & \end{pmatrix}$$



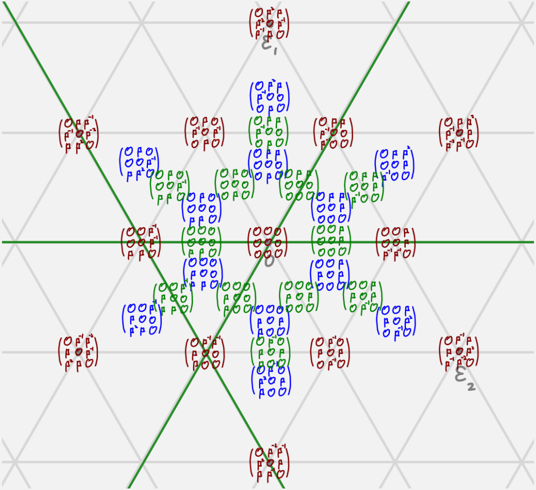
Extended Weyl group action(revisited)



Extended Weyl group action(revisited)



Extended Weyl group action(revisited)



p-adic building

p-adic building

Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_{\sim} = \bigcup_I \mathcal{C}_I.$$

p-adic building

Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_{\sim} = \bigcup_I \mathcal{C}_I.$$

Remark

Similarly, any two chambers lie in one apartment,
and there is a unique geodesic through $p_1, p_2 \in \mathcal{B}$.

Plan of the talk

The Gromov-Schoen theorem

The Gromov-Schoen theorem

Theorem

Let F be a NA local field, (M, g) be a cpt conn Riemannian manifold with the universal covering space $\widetilde{M}$.

The Gromov-Schoen theorem

Theorem

Let F be a NA local field, (M, g) be a cpt conn Riemannian manifold with the universal covering space $\widetilde{M}$.

For any reductive homomorphism

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}_n(F),$$

The Gromov-Schoen theorem

Theorem

Let F be a NA local field, (M, g) be a cpt conn Riemannian manifold with the universal covering space $\widetilde{M}$.

For any reductive homomorphism

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}_n(F),$$

We call ρ reductive when $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$ is reductive.

The Gromov-Schoen theorem

Theorem

Let F be a NA local field, (M, g) be a cpt conn Riemannian manifold with the universal covering space $\widetilde{M}$.

For any reductive homomorphism

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}_n(F),$$

there exists a ① *$\pi_1(M)$-equivariant* ② *Lipschitz continuous* ④ *regular* ③ *harmonic map*

$$h_\rho : \widetilde{M} \longrightarrow \mathcal{B}_{\mathrm{GL}_n(F)}$$

$h_\rho : \mathcal{U} \longrightarrow \mathbb{R}^n$

$\exists C$

$d(h_\rho(x), h_\rho(y)) \leq C d(x, y)$

We call ρ reductive when $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$ is reductive.

$\Delta h_\rho = 0$

regularity

regularity

Definition

h_ρ is regular at $x \in \widetilde{M}$ if

a neighbourhood of x has image inside an apartment $\mathcal{A}_T$ of $\mathcal{B}$.

regularity

Definition

h_ρ is regular at $x \in \widetilde{M}$ if

a neighbourhood of x has image inside an apartment $\mathcal{A}_T$ of $\mathcal{B}$.

h_ρ is regular if

$$\operatorname{codim}_{\widetilde{M}} \left\{ x \in \widetilde{M} \;\middle|\; h_\rho \text{ is not regular at } x \right\} \geqslant 2.$$

regularity

Definition

h_ρ is regular at $x \in \widetilde{M}$ if

a neighbourhood of x has image inside an apartment $\mathcal{A}_T$ of $\mathcal{B}$.

h_ρ is regular if

$$\operatorname{codim}_{\widetilde{M},\mathbb{R}} \left\{ x \in \widetilde{M} \;\middle|\; h_\rho \text{ is not regular at } x \right\} \geqslant 2.$$

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.

regularity

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{y^2 = x^2\right\} \qquad (a,b) \longmapsto (a|b|, b|a|)$$

is regular.

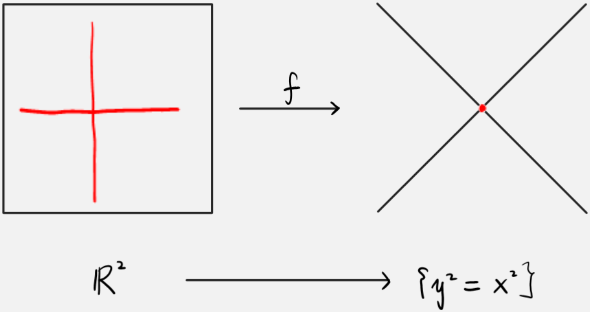
regularity

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{y^2 = x^2\right\} \qquad (a,b) \longmapsto (a|b|, b|a|)$$

is regular.



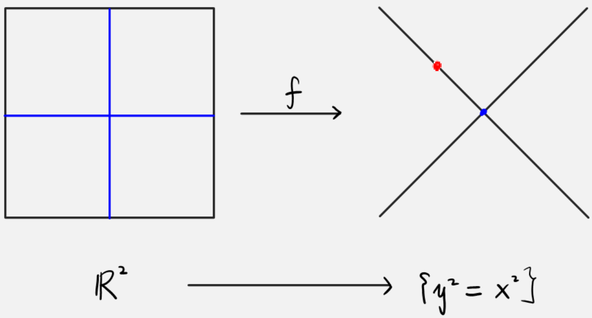
regularity

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{y^2 = x^2\right\} \qquad (a,b) \longmapsto (a|b|, b|a|)$$

is regular.



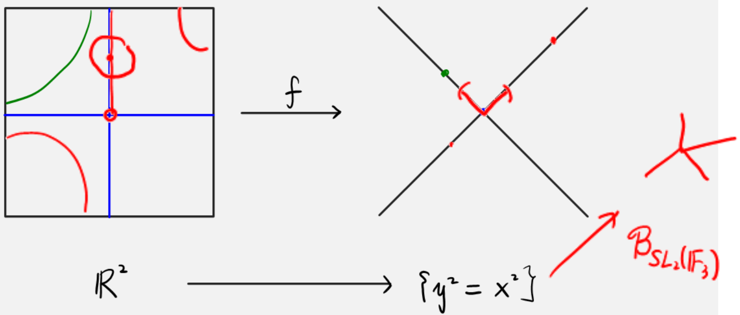
regularity

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{y^2 = x^2\right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.



Thanks for listening!

You can get this slide at:
https://github.com/ramified/personal_tex_collection/raw/main/Bruhat–Tits_building/Bruhat–Tits_building.pdf