Inclusion $\subseteq$ $\longrightarrow$

dominance $\trianglelefteq$

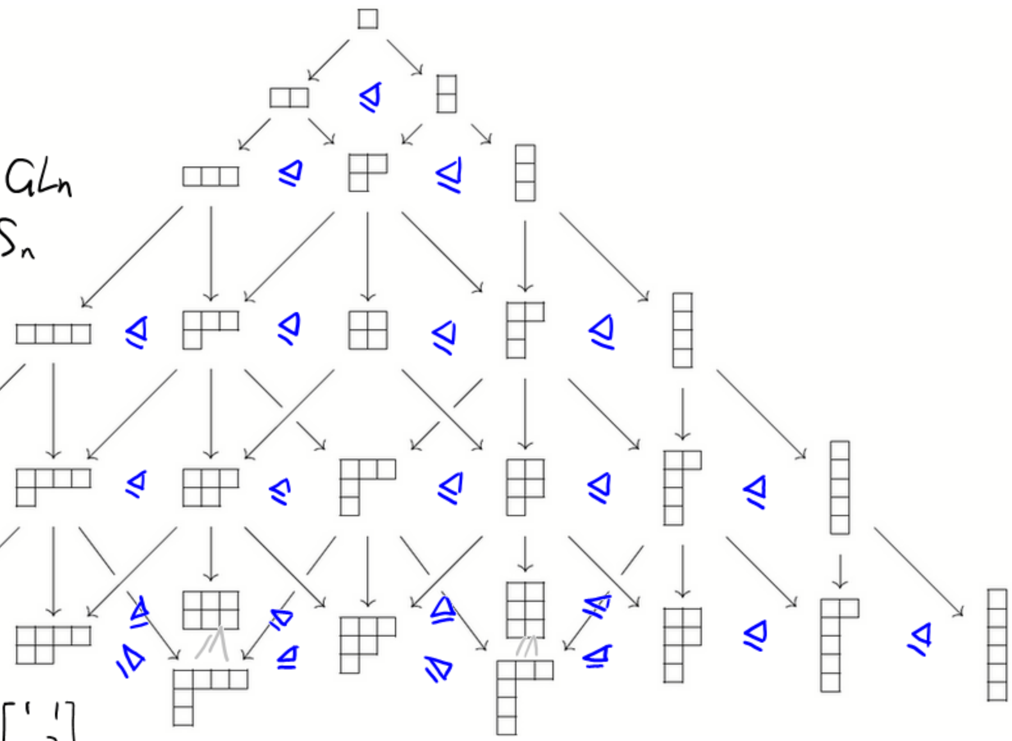
$$E_{\lambda_1,\ldots,\lambda_k} \cap \mathcal{F}\ell(n) \circlearrowleft GL_n$$
$$\circlearrowleft S_n$$

$N_G(T)/T$

"fix T"

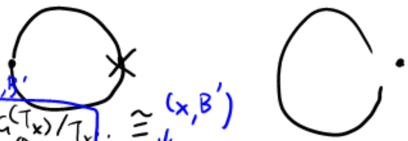
$S_n \subseteq GL_n$

$\uparrow$ fix basis



$\left(\begin{bmatrix}1 & \\ & 2\end{bmatrix}, \begin{bmatrix}* & * \\ & *\end{bmatrix}\right), \left(\begin{bmatrix}1 & \\ & 2\end{bmatrix}, \begin{bmatrix}* & \\ * & *\end{bmatrix}\right)$ $\circlearrowleft W$ $N_G(T)/T$ $\begin{bmatrix}1 & 1 \\ & 2\end{bmatrix}$

$\begin{bmatrix}1 & \\ & 2\end{bmatrix} \to \begin{bmatrix}* & \\ & *\end{bmatrix}$ $\begin{bmatrix}1 & 1 \\ & 2\end{bmatrix} \circlearrowleft$



$W_{x,B'}$ $\boxed{N_G(T_x)/T_x} \cong (x,B')$

$\tilde{\mathfrak{g}}$ $\boxed{(x,B) \in \mathfrak{g}\times\mathcal{B}}$ F $\circlearrowleft \boxed{N_G(T_x)/T_x} \cong W \cong N_G(T_{x'})/T_{x'} \circlearrowleft (x',B')$

$W_{x,B}$ (x,B) x'

$\mathfrak{g} \ni x = (x,T_x)$ semi regular $\leftarrow (\mathrm{Ad}G)x$ simply connected. $(\mathrm{Ad}G)_x \ni x' = (x', T_{x'})$

$= G/Z_G(x) \leftarrow$ not a group

$Z_G(x) = T_x$

Lexicographic ordering

$\{\mathrm{Id}, \tau\}$

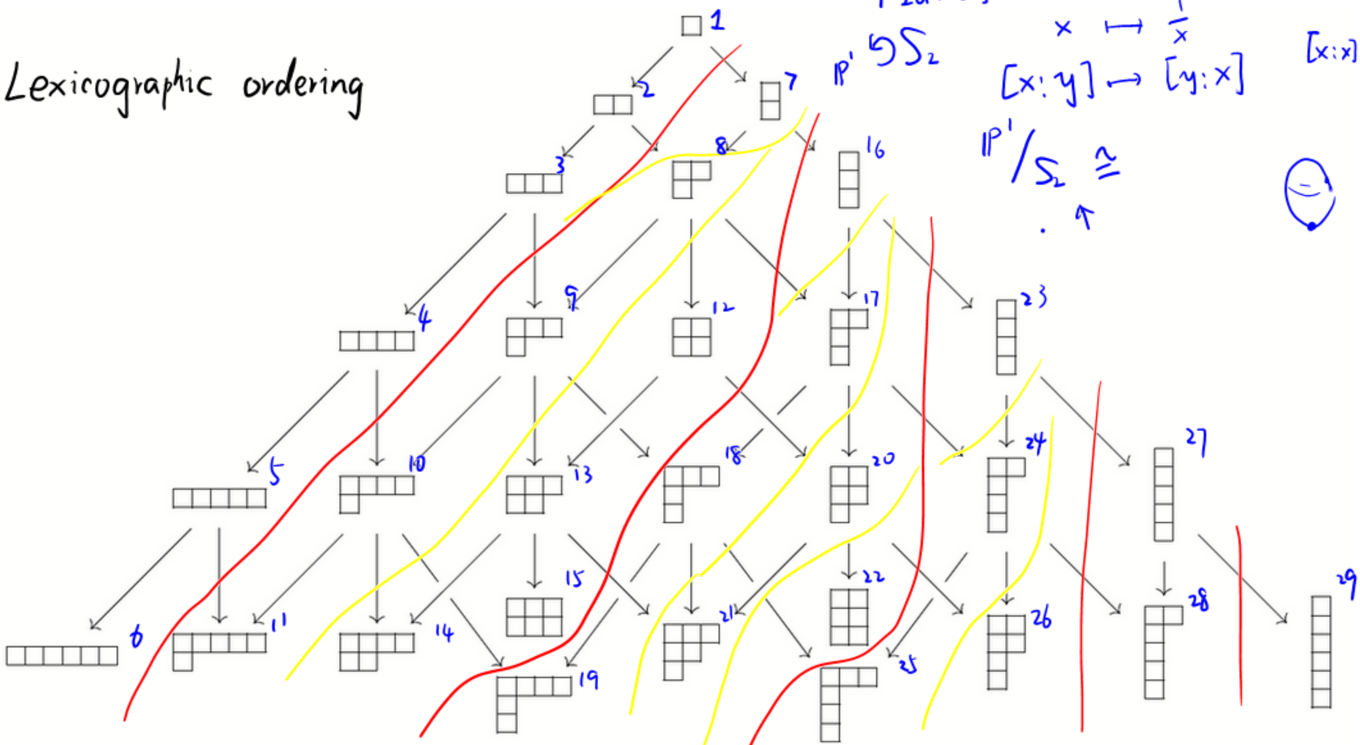
$\mathbb{P}^1 \circlearrowleft S_2$

$x \longmapsto \frac{1}{x}$

$[x:y] \longmapsto [y:x]$

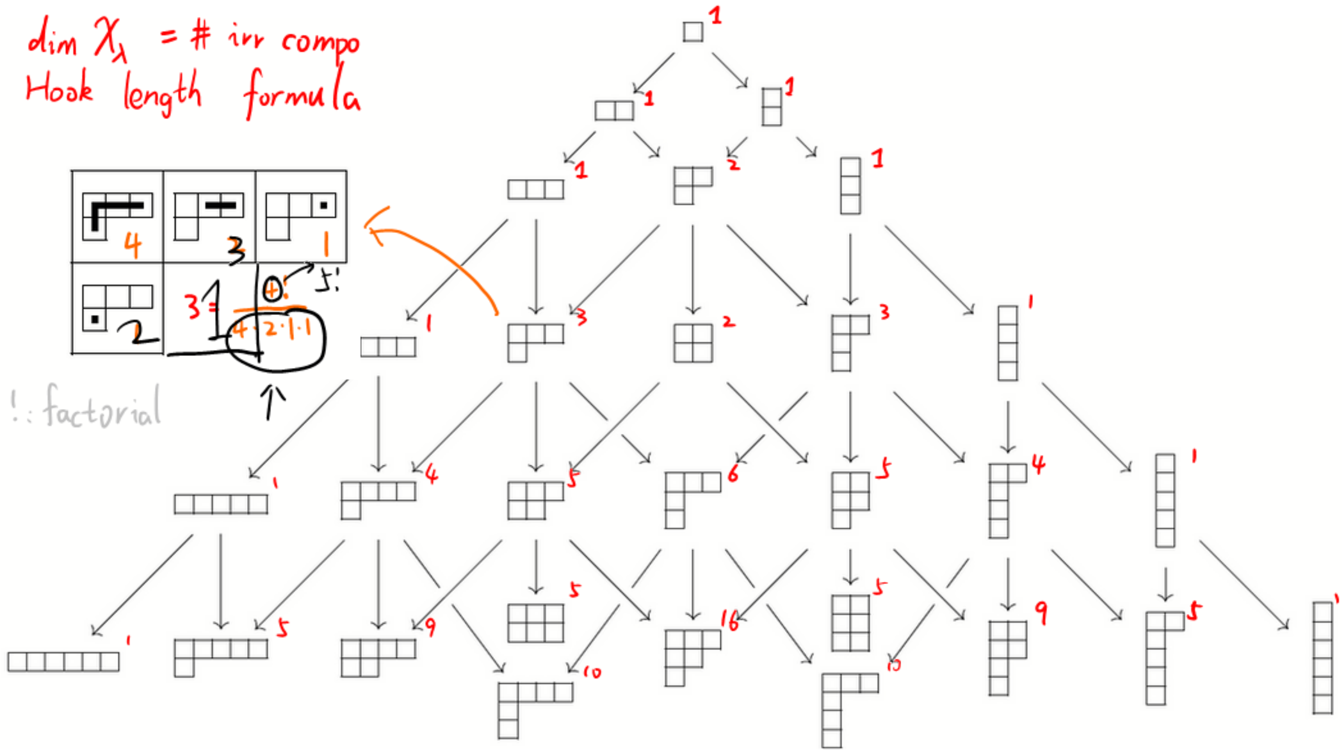
$[x:x]$

$\mathbb{P}^1/S_2 \cong$



dim χ_λ = # irr compo

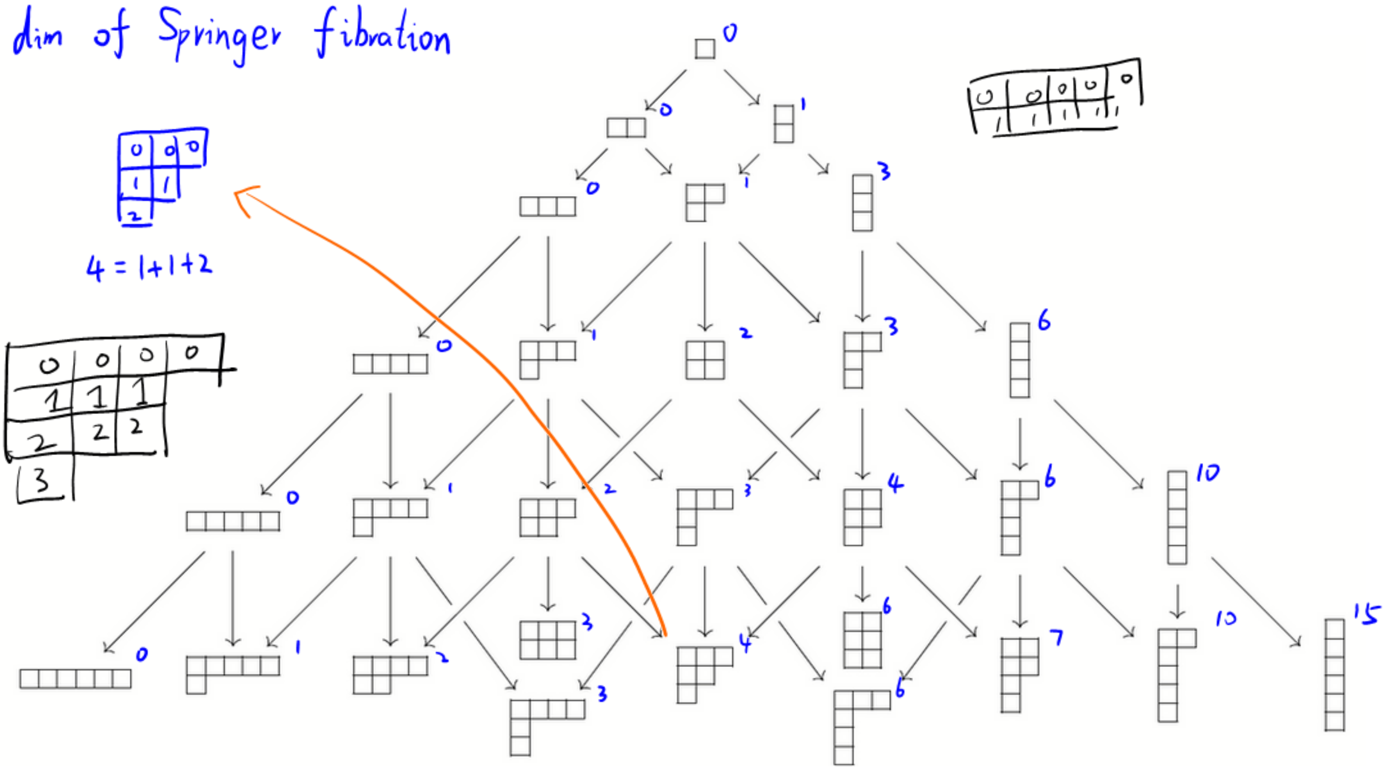
Hook length formula



$3 = \frac{4!}{4 \cdot 2 \cdot 1 \cdot 1}$

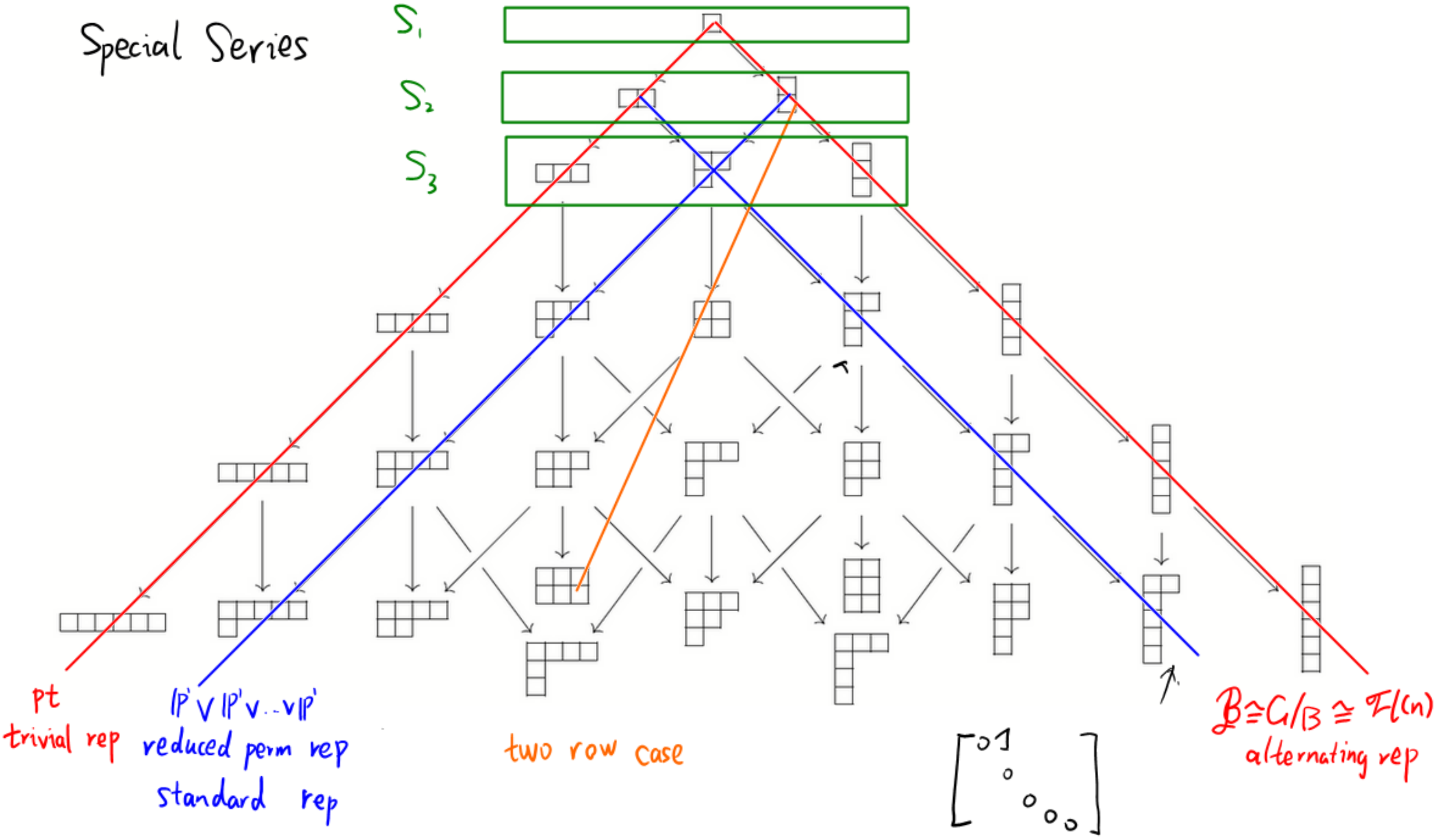
!: factorial

dim of Springer fibration



$4 = 1+1+2$

Special Series



Q: What's the dim of $\{\text{Cartan subalg of } \mathfrak{sl}_n(\mathbb{C})\}$?

$$
\begin{tikzcd}
G \curvearrowright \{\text{Borel subalg}\} \arrow[r, dashed, leftrightarrow] \arrow[d, "\text{finite surj?}"] & \{\text{Borel subgroup}\} \arrow[r, "\cong", "\text{fix } B"'] \arrow[d] & G/B \arrow[d, "\text{finite?}"] & \begin{bmatrix} \circ & \\ & \circ \end{bmatrix} \arrow[l, Rightarrow] \\
\{\text{Cartan subalg}\} \arrow[r, leftrightarrow] & \{\text{Maximal torus}\} \arrow[r, "\cong", "\text{fix } T"'] & G/N_G(T) &
\end{tikzcd}
$$

S_2

G transitive?

Examples of Springer fiber. $E_{\lambda_1,\dots,\lambda_n}$ = Springer fiber of $\begin{bmatrix} J_{\lambda_1} & & \\ & \ddots & \\ & & J_{\lambda_n}\end{bmatrix}$ $J_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$

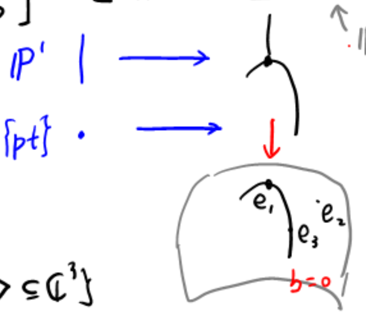
$\square$ $[0]$ $E_1 = \{0 \subseteq \mathbb{C}\} = \cdot$ $\mathcal{S}f$ $\mathbb{F}$ $\mathcal{F}$ $\mathcal{B}$ X $\mathbb{X}$ $\mathbb{V}$ $\mathbb{S}$ E $\lambda\times\lambda$

$\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$ $E_2 = \{0 \subseteq \langle v_1\rangle \subseteq \mathbb{C}^2\} = \cdot$ $\mathcal{B} = \mathcal{B}_{1,1,\dots,1}$

$\begin{bmatrix} 0 & \\ & 0 \end{bmatrix}$ $E_{1,1} = \{0 \subseteq \langle ?\rangle \subseteq \mathbb{C}^2\} = \mathbb{P}^1$

$\rightarrow$ $\begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$ $E_3 = \{0 \subseteq \langle v_1\rangle \subseteq \langle v_1, v_2\rangle \subseteq \mathbb{C}^3\} = \cdot$

$\begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$ $E_{2,1} = \{0 \subseteq \underset{\uparrow \mathbb{P}^2}{\langle ?\rangle} \subseteq \langle ?, ?\rangle \subseteq \mathbb{C}^3\} = \mathbb{P}^1 \vee \mathbb{P}^1$ $ae_1 + 0e_2 + ce_3$



$E_{1,1} \rightarrow E_{2,1}$
$E_2 \rightarrow$
$\downarrow$
$\mathbb{P}^1 \subseteq \mathbb{P}^2$

$\{0 \subseteq \langle e_1\rangle \subseteq \langle e_1, e_3\rangle \subseteq \mathbb{C}^3\}$

$\{0 \subseteq \langle e_1\rangle \subseteq \langle e_1, ?\rangle \subseteq \mathbb{C}^3\}$

$\{0 \subseteq \langle ae_1 + e_3\rangle \subseteq \langle e_1, e_3\rangle \subseteq \mathbb{C}^3\}$

$F_{e_1} = \{0 \subseteq \langle e_1\rangle \subseteq \langle e_1, ?\rangle \subseteq \mathbb{C}^3\} \circlearrowleft \begin{bmatrix} \cancel{0} & \cancel{1} & \\ & 0 & \\ & & 0 \end{bmatrix}$ $\leftarrow$ compatible.

$\Updownarrow$

$\mathbb{P}^1 = E_{1,1} = \{0 \subseteq \langle ?\rangle \subseteq \mathbb{C}^2\} \circlearrowleft \begin{bmatrix} 0 & \\ & 0 \end{bmatrix}$

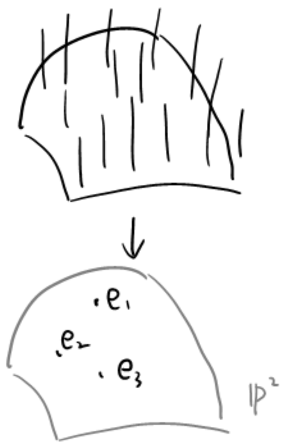
$F_{e_3} = \{0 \subseteq \langle e_3\rangle \subseteq \langle e_3, ?\rangle \subseteq \mathbb{C}^3\} \circlearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$

$\Updownarrow$

$\{pt\} = E_2 = \{0 \subseteq \langle ?\rangle \subseteq \mathbb{C}^2\} \circlearrowleft \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$ $E_{1,1,1} = \{0 \subseteq \underset{\uparrow \mathbb{P}^2}{\langle ?\rangle} \subseteq \langle ?, ?\rangle \subseteq \mathbb{C}^3\} = \mathcal{F}l(3)$

$\mathcal{F}l(2)$ $|$ $\longrightarrow$



* $\mathcal{F}l(1) - \mathcal{F}l(2) - \mathcal{F}l(3) - \mathcal{F}l(4) - \mathcal{F}l(5)$

$|\ \mathbb{P}^1$ $|\ \mathbb{P}^2$ $|\ \mathbb{P}^3$ $|\ \mathbb{P}^4$

$E_1 - E_{1,1} - E_{1,1,1} - E_{1,1,1,1} - E_{1,1,1,1,1}$

$\vdots$ $\vdots$ $\vdots$ $\vdots$

$E_{2,1}$ $\quad F_{ae_1+be_3} = \{0 \subseteq \langle ae_1+be_3\rangle \subseteq \langle ae_1+e_3, \underset{xe_1+ye_2}{?}\rangle \subseteq \mathbb{C}^3\} \circlearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$

$\perp$

$\updownarrow$ basis $\langle \underset{f_1}{e_1}, \underset{f_2}{e_2}, \underset{f_3}{ae_1+e_3}\rangle$

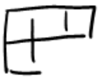
$\mathbb{P}^1 \cong F_{f_3} = \{ \quad f_3, \quad \subseteq \langle f_3 \quad xf_1+yf_2 \quad \} \quad \begin{bmatrix} 0 & \cdot & \\ & 0 & \\ & & 0 \end{bmatrix}$

$(g_1, g_2, g_3) = \langle \quad e_2,$

F

$E_{2,2}.$

$-a$



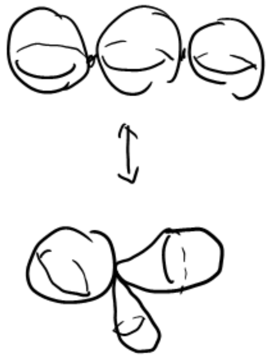
$$\begin{array}{c} \downarrow \quad\quad \downarrow \\ \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \\ & & & 0 \end{bmatrix} \end{array} \begin{array}{l} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \Longrightarrow \begin{array}{|c|c|c|} \hline v_3 & v_2 & v_1 \\ \hline v_4 & & \\ \cline{1-1} \end{array}$$

$E_{3,1} = \{0 \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \mathbb{C}^4\} = \mathbb{P}^1 \vee \mathbb{P}^1 \vee \mathbb{P}^1$ ↻ X.

$\mathbb{P}^1 \vee \mathbb{P}^1$ ↖ $\mathbb{P}^3$

$E_{2,1}$ Λ

E_3 •



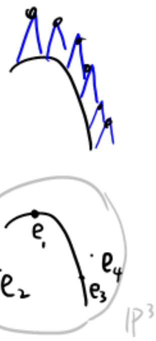
 e_3

$$\begin{array}{c} e_1 \quad e_3 \\ \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix} \end{array}$$

$E_{2,2} = \{0 \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \mathbb{C}^4\} = \mathbb{P}^1\text{-bundle} \vee_{\mathbb{P}^1} \mathbb{P}^1\text{-bundle}$

$E_{2,1}$ Λ

$\mathbb{P}^1 \vee \mathbb{P}^1$.



$$\begin{array}{ccc} \mathbb{P}^1 \vee \mathbb{P}^1 & \longrightarrow & E_{2,2} \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$

$$\begin{array}{ccl} \mathbb{P}^1 \to & \mathcal{F}\ell(4) & = \{0 \subseteq \langle\ \rangle \subseteq \langle ?? \rangle \subseteq \mathbb{C}^4\} \\ & \downarrow & \\ & & \{0 \subseteq \langle ? \rangle \subseteq \langle ??? \rangle \subseteq \mathbb{C}^4\} \end{array}$$

→  $\begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$

$E_{2,1,1} = \{0 \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \langle ? \rangle \subseteq \mathbb{C}^1\}$

$\mathcal{F}(3) = E_{1,1,1}$ 

$\mathbb{P}^1 \vee \mathbb{P}^1 = E_{2,1}$ Λ

 ↻ S_4

→ Q: Can we construct an affine paving from this kind of fibration?

Q: How to understand the Weyl group action on Springer fibration?

1	3	4	6
2	5	7	8

" Bertrand's ballot theorem "

1 2 3 4 5 6 7 8

( ) (( ) ( ))

"bracket pairing"

$(\mathbb{P}^1)^8$

$x_1 = x_2 \quad x_3 = x_8$

$x, x, y, z, z, w, w,$ '

$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$\cup \quad \mathbb{P}$

$\cap$

Define M^λ to be the complex vector space with basis the tabloids $\{T\}$ of shape λ, with λ a partition of n.

$$(3) \qquad a_T = \sum_{p \in R(T)} p, \quad b_T = \sum_{q \in C(T)} \operatorname{sgn}(q) q. \qquad v_T = b_T \cdot \{T\} = \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\}.$$

These elements, and the product

$$c_T = b_T \cdot a_T,$$

Define the ***Specht module*** S^λ to be the subspace of M^λ spanned by the elements v_T, as T varies over all numberings of λ.

Young tabloid

Proposition 1 *For each partition λ of n, S^λ is an irreducible representation of S_n. Every irreducible representation of S_n is isomorphic to exactly one S^λ.*

整系数矩阵.

Proposition 2 *The elements v_T, as T varies over the standard tableaux on λ, form a basis for S^λ.*