LaTeX TEMPLATE

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Contents

1. A small toolkit

$f : Y \longrightarrow \mathrm{pt}$ $f : p \hookrightarrow X$
f^* constant sheaf $\mathcal{F}_p$
Rf_* cohomology $\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$ cpt supp cohomology $\mathrm{sky}_p(\mathbb{Q})$
$f^!$ orientation sheaf $[n]$ $\mathcal{F}_p[-n]$
For $f^!$, assume Y, X are manifolds of dimension n.
$j_!j^*\mathcal{F}$ $\mathcal{F}$ $i_!i^*\mathcal{F}$

$$Z \xhookrightarrow{i} X \xhookleftarrow{j} U$$

$$\begin{tikzcd}
D(Z) \arrow[r, "i_*=i_!"] & D(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & D(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \xrightarrow{+1}$$

$$\begin{tikzcd}
& \otimes \arrow[dl, dash, "f^*(-\otimes-)"'] \arrow[dr, dash, "\text{proj formula}"] & \\
f^* \arrow[rr, dash, "\text{base change}"'] & & f_!
\end{tikzcd}$$

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f^*	constant sheaf	$\mathcal{F}_p$
Rf_*	cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$	cpt supp cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$f^!$	orientation sheaf	$[n]$ $\mathcal{F}_p[-n]$

$$\begin{tikzcd} & \otimes \arrow[dl, dash, "f^*(-\otimes-)"'] \arrow[dr, dash, "\text{proj formula}"] & \\ f^* \arrow[rr, dash, "\text{base change}"'] & & f_! \end{tikzcd}$$

2. A Short List of Applications

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:

$$\mathrm{H}^i_{\mathrm{c}}(X;\mathbb{Q}) \cong \mathrm{H}^i\left(\bar{X}, \{\infty\}; \mathbb{Q}\right)$$
$$\mathrm{H}^{\mathrm{BM}}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$
$$\mathrm{H}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_{\mathrm{c}}(X;\mathrm{Or}_X)$$

Also, define the cup and cap product structures.

2. Using the projection formula, show Poincaré duality:

$$\mathrm{H}^i_{\mathrm{c}}(X;\mathbb{Q})^* \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$
$$\mathrm{H}^i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_{\mathrm{c}}(X;\mathrm{Or}_X)^*$$

3. Derive the Gysin sequence for any oriented S^k-bundle $\pi : E \longrightarrow B$:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[eu_\pi]{+1}$$

Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.

4. Compute the upper shriek for singular spaces.

$$\begin{aligned}
\mathrm{H}^i(Y,\mathbb{Q}) &= \mathrm{H}^i(Y,\underline{\mathbb{Q}}_Y) &&= f_*\underline{\mathbb{Q}}_Y &&= f_*f^*\mathbb{Q} \\
\mathrm{H}^i_c(Y,\mathbb{Q}) &= \mathrm{H}^i_c(Y,\underline{\mathbb{Q}}_Y) &&= f_!\underline{\mathbb{Q}}_Y &&= f_!f^*\mathbb{Q} \\
\mathrm{H}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}_c(Y,\mathrm{Or}_Y) &&= f_!\,\mathrm{Or}_Y[n] &&= f_!f^!\mathbb{Q} \\
\mathrm{H}^{\mathrm{BM}}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}(Y,\mathrm{Or}_Y) &&= f_*\,\mathrm{Or}_Y[n] &&= f_*f^!\mathbb{Q}
\end{aligned}$$

six functor formalism $\approx$ cohomology theory

References

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