Bruhat–Tits building

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Figures of Bruhat–Tits building

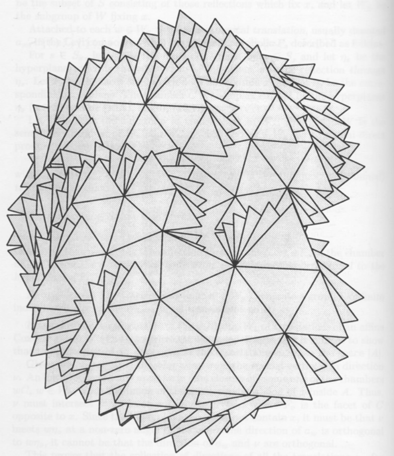

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from Annette Werner's talk

Figures of Bruhat–Tits building

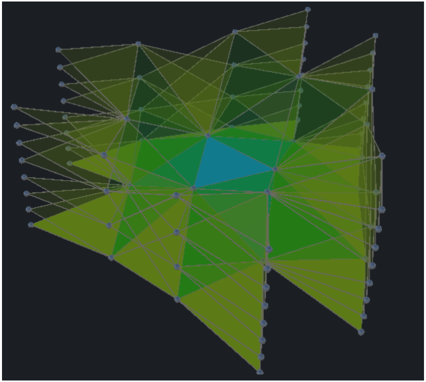

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

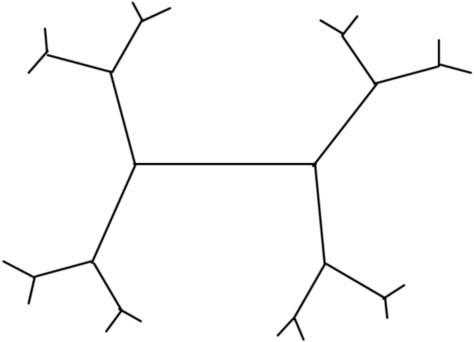

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

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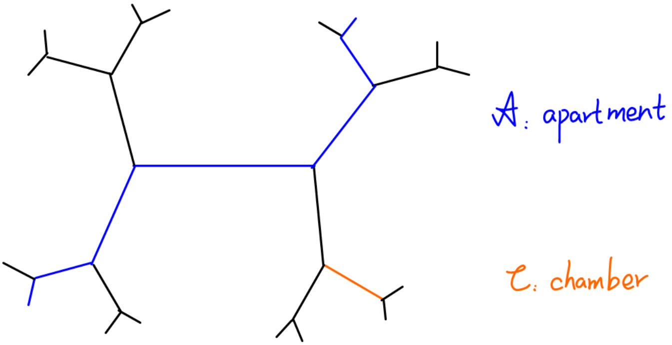

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2. p-adic building
3. Gromov-Schoen theorem

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We have Bruhat decomposition proved by Gauss elimination

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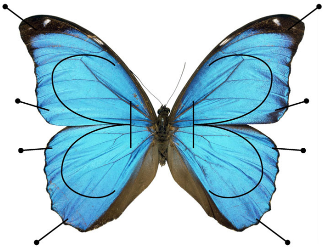

Figure: Pinned butterfly

Weyl group action on cocharacter lattices

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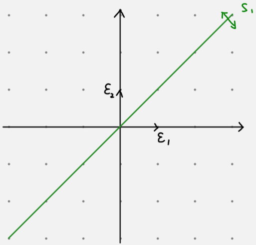

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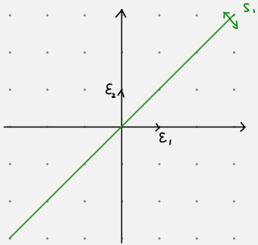

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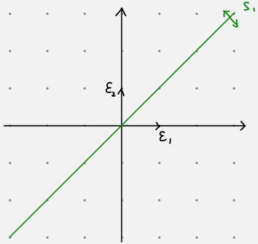

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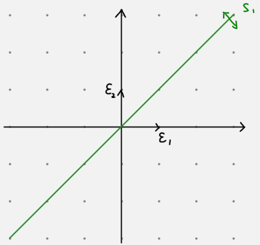

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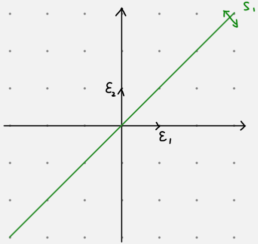

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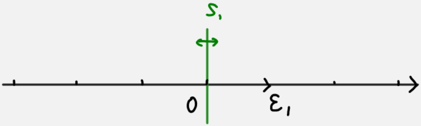

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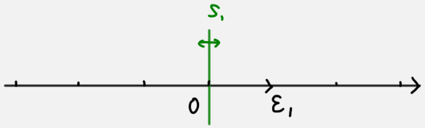

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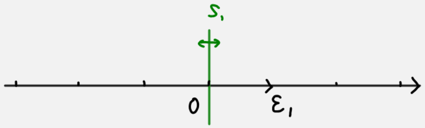

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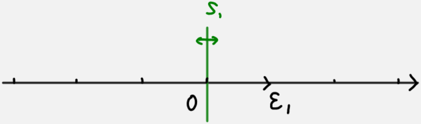

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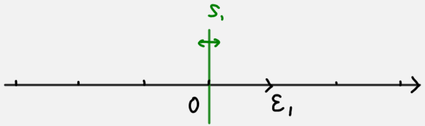

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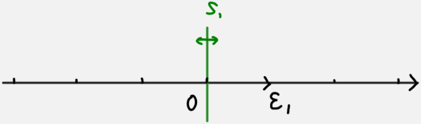

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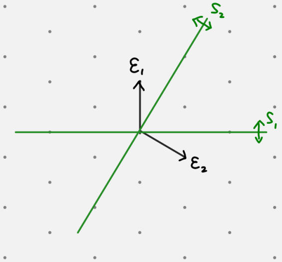

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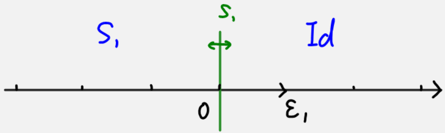

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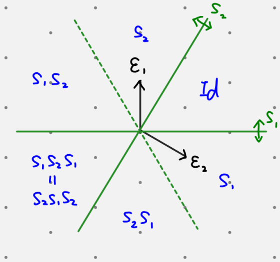

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\{B\} & & \{T\}
\end{tikzcd}
\qquad
\begin{tikzcd}
& G/T_0 \arrow[dl, "B_0/T_0"'] \arrow[dr, "N_G(T_0)/T_0=W"] & \\
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$\{$ (Weyl) chambers $\}$ $\overset{1:1}{\longleftrightarrow}$ W $\overset{1:1}{\longleftrightarrow}$ $\{B \mid B \supset T_0\}$

Weyl group action on cocharacter lattices(revisit)

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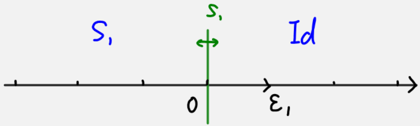

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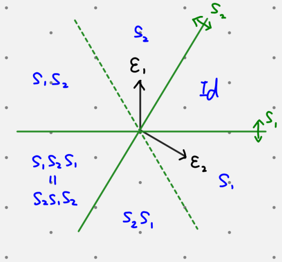

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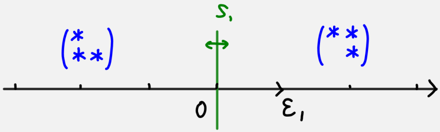

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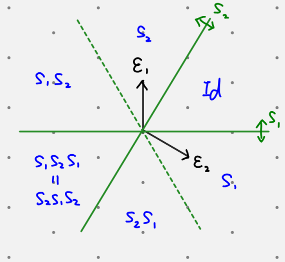

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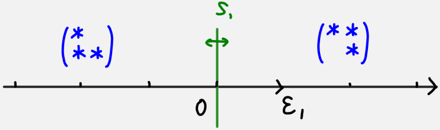

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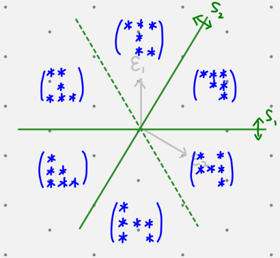

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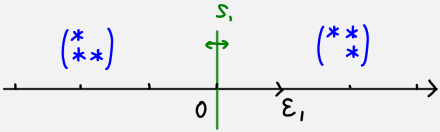

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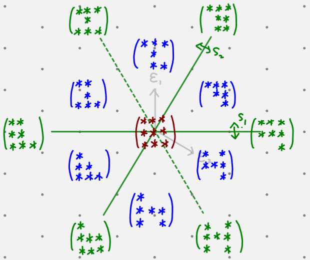

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Given a maximal torus T, the apartment is

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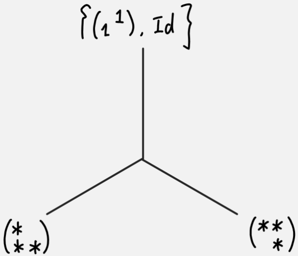
When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.



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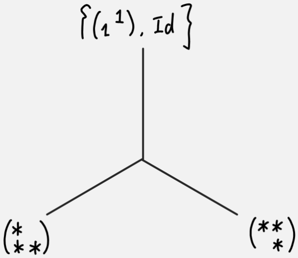

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

Remark

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Proposition

- *Two chambers lie in one apartment.*
- *There is a unique geodesic passing any two points $p_1, p_2 \in \mathcal{B}$.*

Process

1 Spherical building

2 p-adic building

3 Gromov-Schoen theorem

p-adic notation

symbol	name	example
F	NA local field	
$\mathcal{O} = \mathcal{O}_F$	integral ring	
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	
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$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

standard subgroups in p-adic world

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They also have moduli intepretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{L = L_0 \subset L_1 \subset \cdots \subset L_n = \mathfrak{p}L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

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we define the extended Weyl group as

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Example

When $G = \mathrm{GL}_n(F)$,

$$W_{\mathrm{ext}} = \{\ \textit{monoidal matrixes}\ \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

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W_{ext} acts on $X_*(T)$ by "twisted conjugation":

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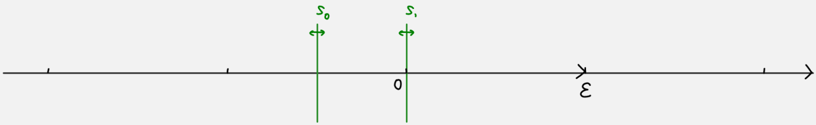

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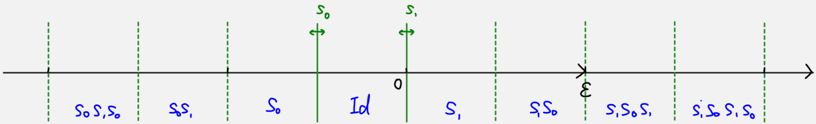

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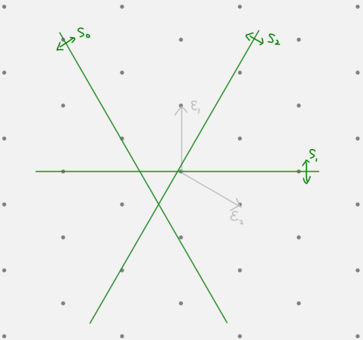

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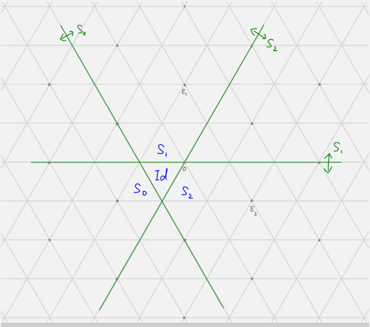

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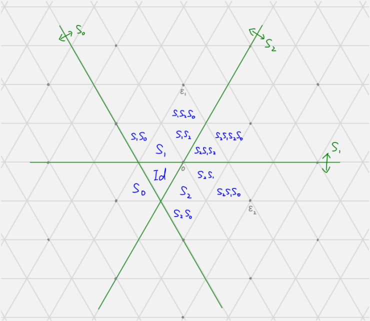

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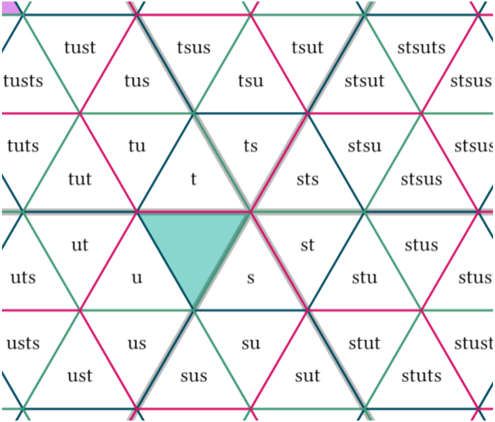

Figure: Reduced expressions labels, from Lievis

Non-standard subgroups in p-adic world

Non-standard subgroups in p-adic world

Similarly,

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\end{tikzcd}
\qquad
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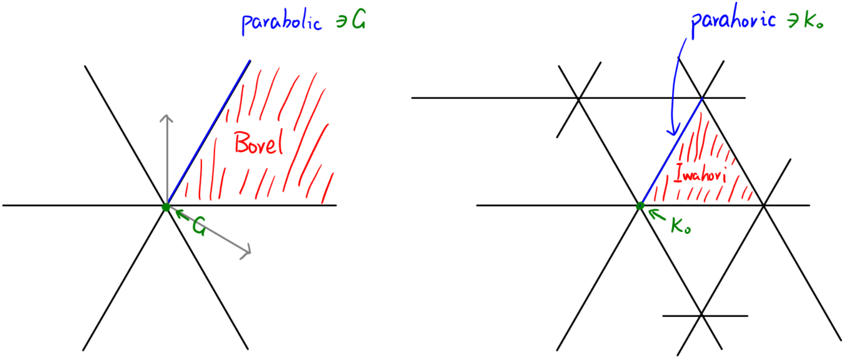
$$\{ \text{ chambers } \} \overset{1:1}{\longleftrightarrow} W_{\text{ext}} \overset{1:1}{\longleftrightarrow} \{I \mid I \supset T_0\}$$

Comparison



Extended Weyl group action(revisit)

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

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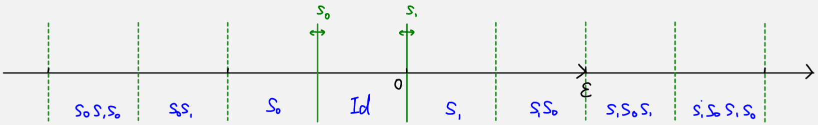

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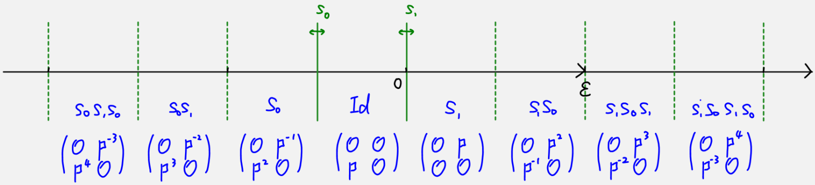

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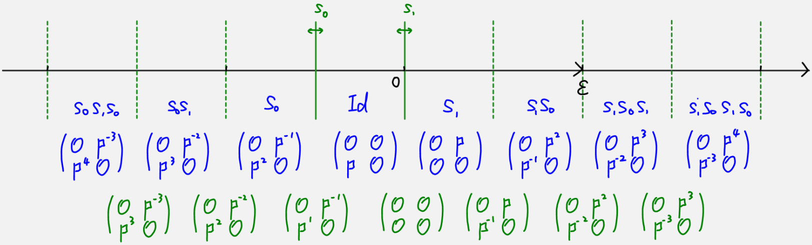

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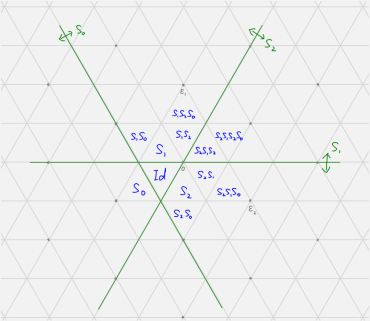

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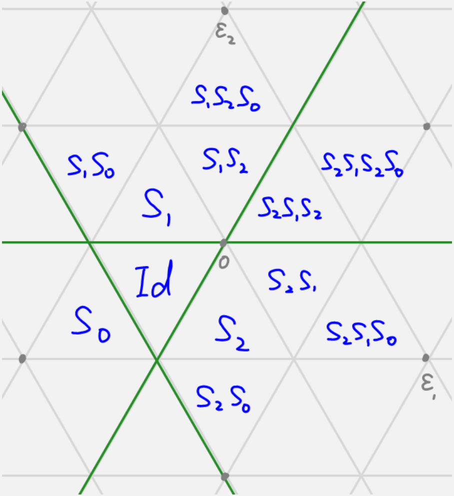

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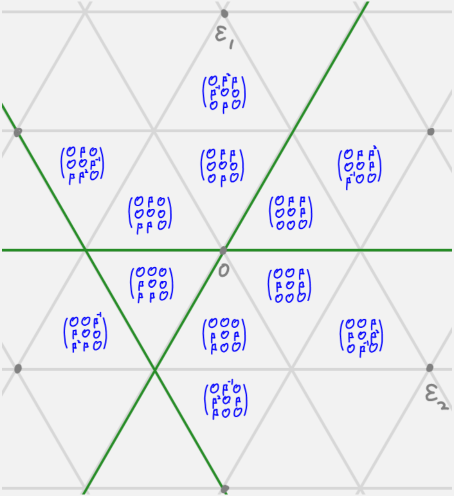

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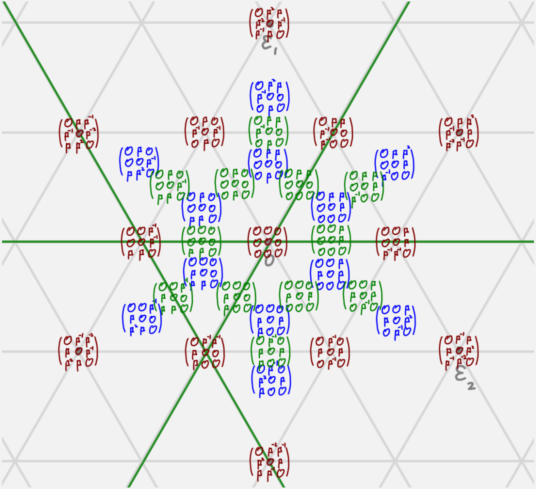

p-adic building

p-adic building

Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_{\sim} = \bigcup_I \mathcal{C}_I.$$

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Remark

Similarly, two chambers lie in one apartment,
and there is a unique geodesic passing $p_1, p_2 \in \mathcal{B}$.

Process

1. Spherical building
2. p-adic building
3. Gromov-Schoen theorem

Gromov-Schoen theorem

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Theorem

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We call ρ reductive when $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$ is reductive.

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Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.

Thanks for listening!

You can get this slide at:
https://github.com/ramified/personal_tex_collection/raw/main/Bruhat–Tits_building/Bruhat–Tits_building.pdf