THE DIMENSION OF Z_χ

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Contents

1. Background

In this section, we establish notation and provide background on the question. Experts may wish to skip the first two sections, which are likely to be revised.

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\mathrm{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in $\mathbb{Q}$. For any algebraic group G, we denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G.

Following the approach of [KW15], we work in the quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/N(A)$, where $N(A) \subset \mathrm{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex $\mathcal{F}$ is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [KW15]). In particular, for any perverse sheaf $\delta \in \overline{\mathrm{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G:

$$\langle \delta, * \rangle \cong \mathrm{Rep}(G).$$

Examples are abundant but intricate. For reference, we provide a brief list of known cases:

Proposition 1.1. *For any smooth projective variety X over $\mathbb{C}$, let $A := \mathrm{Alb}(X)$ be its Albanese variety. When the Albanese map*

$$\alpha : X \longrightarrow Alb(X)$$

is a closed embedding, this map defines a perverse sheaf

$$\delta := \alpha_*(\underline{\mathbb{Q}}[\dim X]) \in \overline{\mathrm{Perv}}(A).$$

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In several cases, the Tannaka group is already well understood, as follows:

$$\langle\delta,*\rangle \cong \begin{cases} \mathrm{Rep}(\mathrm{SL}_{2g-2}(\mathbb{C})), & X = C \textit{ non-hyperelliptic} \\ \mathrm{Rep}(\mathrm{Sp}_{2g-2}(\mathbb{C})), & X = C \textit{ hyperelliptic} \\ \mathrm{Rep}(\mathrm{E}_6(\mathbb{C})), & X = S \textit{ Fano surface} \\ \mathrm{Rep}(\mathrm{SO}_{g!}(\mathbb{C})), & X = \Theta,\ g \textit{ odd} \\ \mathrm{Rep}(\mathrm{Sp}_{g!}(\mathbb{C})), & X = \Theta,\ g \textit{ even} \end{cases} \qquad \begin{matrix} A_{2g-3} \\ C_{g-1} \\ E_6 \\ D_{g!/2} \\ C_{g!/2} \end{matrix}$$

Here, $g := \dim_{\mathbb{C}}(A)$*, and*

- C *is a smooth projective curve over* $\mathbb{C}$ *with genus* $g \geqslant 2$*;*
- S *is the Fano surface of a smooth cubic threefold;*
- Θ *is the smooth theta divisor of a general principally polarized abelian variety.*

In [Kr20], any perverse sheaf $\mathcal{F}$ can be associated with its clean characteristic cycle

$$\mathrm{cc}(\mathcal{F}) = \sum_Z m_{\mathcal{F}}(Z)[\Lambda_Z].$$

This coinsides with the weight decomposition for $V \in \mathrm{Rep}(G)$:

$$V = \bigoplus_{\chi \in X^*(T)} V_\chi = \bigoplus_{[\chi] \in X^*(T)/W} \left(\bigoplus_{\chi \in [\chi]} V_\chi \right)$$

By comparing the following formulas (and applying induction on highest weight representations), we can associate each weight orbit $[\chi]$ with a subvariety Z of A:

$$\begin{cases} \chi(\mathcal{F}) = \sum_Z \deg \Lambda_Z \cdot m_{\mathcal{F}}(Z) \\ \dim_{\mathbb{C}} V = \sum_{[\chi]} \#[\chi] \cdot \dim_{\mathbb{C}} V_\chi \end{cases}$$

We may later denote this subvariety as Z_χ to indicate its correspondence with the weight orbit where χ lies.

In the case of curves, the conormal cone Λ_Z of $Z = Z_\chi$ has an explicit description as a Lagrangian cycle:

$$\Lambda_Z \subset T^*A \cong A \times \mathrm{H}^0(C, \omega_C).$$

In the next section, we will describe this Lagrangian cycle in detail, leading to an explicit description of Z_χ.

2. Description of Z_χ via correspondence

From now on, we focus on the curve case, where $G = \mathrm{SL}_{2g-2}(\mathbb{C})$ or $\mathrm{Sp}_{2g-2}(\mathbb{C})$. In both cases, δ corresponds to the minuscule representation $L(\omega)$ for some highest weight $\omega \in X^(T)$. The Weyl group $W = S_{2g-2}$ or $S_{g-1} \times (\mathbb{Z}/2\mathbb{Z})^{g-1}$ acts on the character lattice $X^*(T)$. Letting

$$[\omega] = \{\lambda_1, \ldots, \lambda_{2g-2}\} \subset X^*(T)$$

denote the orbit of ω, we have

$$X^*(T) = \langle \lambda_1, \ldots, \lambda_{2g-2} \rangle_{\mathbb{Z}\text{-mod}}.$$

In other words, any character $\chi \in X^*(T)$ can be written as $\chi = \sum_{i=1}^{2g-2} m_i \lambda_i$ for some tuple $(m) = (m_1, \ldots, m_{2g-2}) \in \mathbb{Z}^{2g-2}$.

For any $(m) \in \mathbb{Z}^k$, we can construct a map

$$\begin{aligned} a^{(m)}: \quad C^k &\longrightarrow \mathrm{Pic}^{\Sigma m_i}(C) \cong A \\ (p_1, \ldots, p_k) &\longmapsto \textstyle\sum_{i=1}^k m_i p_i \mapsto \sum_{i=1}^k m_i (p_i - p_0) \end{aligned}$$

For simplicity, we write $a := a^{(1,\dots,1)}$ and let

$$K \in \mathrm{Pic}^{2g-2}(C) \cong A$$

denote the class corresponding to the line bundle ω_C of degree $2g-2$.

Proposition 2.1. *Assume the curve is non-hyperelliptic. For $\chi \in X^*(T)$, express χ as $\chi = \sum_{i=1}^{2g-2} m_i\lambda_i$ for some tuple $(m) \in \mathbb{Z}^{2g-2}$.*

1) The conormal cone Λ_{Z_χ} is given by

$$\Lambda_{Z_\chi} = \left\{ \left(a^{(m)}(p), \eta\right) \in A \times \mathrm{H}^0(C, \omega_C) \;\middle|\; p \in C^{2g-2}, \sum p_i = \operatorname{div}\eta \right\}.$$

2) The subvariety Z_χ is described by $Z_\chi = a^{(m)}\left(a^{-1}(K)\right)$.

Proof.

1) This can first be checked on the fundamental weights and then extended linearly.
2) Take the projection $\pi_A : A \times \mathrm{H}^0(C, \omega_C) \longrightarrow A$, then

$$\begin{aligned} Z_\chi &= \pi_A(\Lambda_{Z_\chi}) \\ &= \left\{ a^{(m)}(p) \in A \;\middle|\; \operatorname{div}\eta = \sum p_i \text{ for some } \eta \in \mathrm{H}^0(C, \omega_C) \right\} \\ &= \left\{ a^{(m)}(p) \in A \;\middle|\; a(p) = K \right\} \\ &= a^{(m)}\left(a^{-1}(K)\right) \end{aligned}$$

□

In the hyperelliptic case, the statement differs slightly. Assume $X^*(T) = \bigoplus_{i=1}^{g-1} \mathbb{Z}\lambda_i$ with $\lambda_{i+g-1} = -\lambda_i$. For $\chi = \sum_{i=1}^{g-1} m_i\lambda_i + \sum_{i=g}^{2g-2} 0 \cdot \lambda_i$, let $(n) = (m_1, \dots, m_{g-1}) \in \mathbb{Z}^{g-1}$. Then the normal cone Λ_{Z_χ} is given by

$$\begin{aligned} \Lambda_{Z_\chi} &= \left\{ \left(a^{(m)}(p), \eta\right) \in A \times \mathrm{H}^0(C, \omega_C) \;\middle|\; p \in C^{2g-2}, p_{i+g} = p_i, \sum p_i = \operatorname{div}\eta \right\} \\ &= \left\{ \left(a^{(n)}(p), \eta\right) \in A \times \mathrm{H}^0(C, \omega_C) \;\middle|\; p \in C^{g-1}, \sum 2p_i = \operatorname{div}\eta \right\} \end{aligned}$$

and the subvariety Z_χ is described by

$$Z_\chi = \mathrm{Im}\left(a^{(n)} : C^{g-1} \longrightarrow A\right).$$

The primary difference here arises from the distinct symmetry type. The divisor $\operatorname{div}(\eta)$ exhibits certain internal constraints; the closer the Weyl group is to the full symmetric group, the fewer such constraints we observe. Fortunately, most results for hyperelliptic curves have already been discussed in detail in [Kr20]. For this reason, we will mainly focus on the non-hyperelliptic case from now on.

In the remainder of this document, we will address one central question:

$$\boxed{\text{What is the dimension of } Z_\chi\text{?}}$$

We will analyze this question from two perspectives: one that focuses on the local geometry of Z_χ and another that examines its global characteristics.

3. Description of the tangent map

The first approach attempts to determine $\dim_{\mathbb{C}} Z_\chi$ by analyzing its tangent space.

For a general point $p = (p_1, \ldots, p_{2g-2})$ in $a^{-1}(K)$, $a^{-1}(K)$ is smooth at p, and Z_χ is also smooth at $q := a^{(m)}(p)$. This allows us to derive the diagram

$$\begin{tikzcd} & a^{-1}(K) \arrow[dl] \arrow[d] \arrow[dr] & \\ \{K\} \arrow[dd] & & Z_\chi \arrow[dd] \\ & C^{2g-2} \arrow[dl, "a"'] \arrow[dr, "a^{(m)}"] & \\ A & & A \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & T_p a^{-1}(K) \arrow[dl] \arrow[d] \arrow[dr] & \\ 0 \arrow[dd] & & T_q Z_\chi \arrow[dd] \\ & T_p C^{2g-2} \arrow[dl, "d_p a"'] \arrow[dr, "d_p a^{(m)}"] & \\ T_K A & & T_q A \end{tikzcd}$$

and get

$$T_q Z_\chi = d_p a^{(m)}(T_p a^{-1}(K)).$$

In the remainder of this section, we will analyze $T_q Z_\chi$ for general points $p \in a^{-1}(K)$, breaking down the process into three steps:

Step 1. Analyze the form of $d_p a^{(m)}$.
Step 2. Verify that $T_p a^{-1}(K) = \ker d_p a$.
Step 3. Compute $\dim_{\mathbb{C}} T_q Z_\chi$, and transform it to a linear algebra question.

Step 1. The following lemma provides a foundational result for analyzing the tangent map up to scalar.

Lemma 3.1 (???). *The projectivized differential of the Abel-Jacobi map $\iota_C : C \longrightarrow A$ is the canonical embedding $\varphi_C : C \longrightarrow \mathbb{P}^{g-1}$, i.e.,*

$$\varphi_C(p) = \mathrm{Im}(d_p \iota_C) \in \mathbb{P}(T_p A) \cong \mathbb{P}\left(\mathrm{H}^0(C, \omega_C)^*\right).$$

For convenience, at each point $p = (p_1, \ldots, p_{2g-2}) \in C^{2g-2}$, we select nonzero elements $\alpha_i \in T_{p_i} C \subset \oplus_i T_{p_i} C$. Then, α_i forms a basis for $\oplus_i T_{p_i} C$, and $\mathrm{Im}\, d_p a$ is generated by

$$\beta_i := d_p \iota_C(\alpha_i) \in \mathrm{H}^0(C, \omega_C)^*.$$

By Lemma 3.1, $[\beta_i] = \varphi_C(p_i)$.

Lemma 3.2. *For any tuple $(m) \in \mathbb{Z}^{2g-2}$, the differential of the map $a^{(m)} : C^{2g-2} \longrightarrow A$ at p is given by*

$$\begin{aligned} d_p a^{(m)} &= (m_i d_{p_i} \iota_C)_{i=1}^{2g-2} : \bigoplus_i T_{p_i} C \longrightarrow \mathrm{H}^0(C, \omega_C)^* \\ &= (m_i \beta_i)_{i=1}^{2g-2} : \bigoplus_i \mathbb{C} \longrightarrow \mathrm{H}^0(C, \omega_C)^* \end{aligned}$$

Proof. This is done by first checking that $(m) = (0, \ldots, 1, \ldots, 0)$, and then extending linearly. □

The next lemma integrates Lemma 3.1, Lemma 3.2, and the general position theorem:

Lemma 3.3.

1) For any point $p \in a^{-1}(K)$, the associated differential $\eta \in H^0(C, \omega_C)$ determines a hyperplane H in $H^0(C, \omega_C)^*$, which contains the image of $d_p a$.
2) For a general point $p \in a^{-1}(K)$, any selection of $g - 1$ elements from $\{\beta_1, \ldots, \beta_{2g-2}\}$ is linearly independent and spans the hyperplane H.

Proof.

1) This is simply a tautology.
2) This is a consequence of the general position theorem; see [Ar85] for further details.

□

Step 2. It is easy to check that $T_p a^{-1}(K) \subseteq \ker d_p a$, and the equality is established through dimension counting. Observe that Lemma 3 implies $\operatorname{Im} d_p a = H$ for a generic point $p \in a^{-1}(K)$.

Step 3. Notice that for a generic point $p \in a^{-1}(K)$,

$$\begin{aligned}\dim_{\mathbb{C}} T_q Z_\chi &= \dim_{\mathbb{C}} T_p a^{-1}(K) - \dim_{\mathbb{C}} \ker \left(d_p a^{(m)}|_{T_p a^{-1}(K)} \right) \\ &= g - 1 - \dim_{\mathbb{C}} \left(\ker d_p a^{(m)} \cap \ker d_p a \right) \\ &= g - 1 - \dim_{\mathbb{C}} \ker \begin{pmatrix} d_p a \\ d_p a^{(m)} \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} d_p a \\ d_p a^{(m)} \end{pmatrix} - (g-1)\end{aligned}$$

where the map

$$\begin{pmatrix} d_p a \\ d_p a^{(m)} \end{pmatrix} : \mathbb{C}^{2g-2} \longrightarrow H \oplus H \cong \mathbb{C}^{2g-2}$$

has the matrix coefficient expression

$$\begin{pmatrix} \beta_1 & \cdots & \beta_{2g-2} \\ m_1\beta_1 & \cdots & m_{2g-2}\beta_{2g-2} \end{pmatrix} \in M^{(2g-2)\times(2g-2)}(\mathbb{C}).$$

Now it is time to work on some special cases.

Example 3.4.

1) When $m_1 \cdots m_i \neq 0$ and $m_{i+1} = \cdots = m_{2g-2} = 0$ for some $i \leqslant g-1$,

$$\begin{aligned}\operatorname{rank} \begin{pmatrix} \beta_1 & \cdots & \beta_{2g-2} \\ m_1\beta_1 & \cdots & m_{2g-2}\beta_{2g-2} \end{pmatrix} &= \operatorname{rank} \begin{pmatrix} \beta_1 & \cdots & \beta_i & \beta_{i+1} & \cdots & \beta_{2g-2} \\ m_1\beta_1 & \cdots & m_i\beta_i & 0 & \cdots & 0 \end{pmatrix} \\ &= \operatorname{rank} \begin{pmatrix} \beta_{i+1} & \cdots & \beta_{2g-2} \end{pmatrix} + \operatorname{rank} \begin{pmatrix} m_1\beta_1 & \cdots & m_i\beta_i \end{pmatrix} \\ &= g - 1 + i\end{aligned}$$

   As a result, we have $\dim_{\mathbb{C}} T_q Z_\chi = i$.

2) In cases where at least $g-1$ of the m_i's are equal, let i denote the cardinality of the remaining elements, one also gets $\dim_{\mathbb{C}} T_q Z_\chi = i$.

These examples illustrate only a fraction of the possible cases. In the next section, we will address all other cases and prove that they are all divisors.

4. First proof by representation

In this section, we assume that at most $g-2$ of the m_i values are equal. We also recall that any arbitrary selection of $g-1$ elements from $\{\beta_1, \ldots, \beta_{2g-2}\} \subset H \cong \mathbb{C}^{g-1}$ are linearly independent.

The goal is to establish the following linear algebra result:

Theorem 4.1. *Let $m_i \in \mathbb{Z}$ and $\beta_i \in \mathbb{C}^{g-1}$ as defined above, and fix these values. Then there exists a permutation $\sigma \in S_{2g-2}$ such that*

$$\det \begin{pmatrix} \beta_{\sigma(1)} & \cdots & \beta_{\sigma(2g-2)} \\ m_1\beta_{\sigma(1)} & \cdots & m_{2g-2}\beta_{\sigma(2g-2)} \end{pmatrix} \neq 0.$$

Corollary 4.2. *For a non-hyperelliptic curve, where $\chi = \sum_{i=1}^{2g-2} m_i\lambda_i$ with at most $g-2$ identical m_i values, we have $\dim_{\mathbb{C}} Z_\chi = g-1$.*

To prove this theorem, several preliminary steps are required.

Definition 4.3. *For $\sigma \in S_{2g-2}$, define*

$$f_\sigma : (\mathbb{C}^{g-1})^{2g-2} \longrightarrow \mathbb{C} \qquad (v_1, \ldots, v_{2g-2}) \longmapsto \det(v_{\sigma(1)} \cdots v_{\sigma(g-1)}) \det(v_{\sigma(g)} \cdots v_{\sigma(2g-2)})$$

as a polynomial in $(g-1) \times (2g-2)$ variables, and let

$$V := \langle f_\sigma \rangle_{\sigma \in S_{2g-2}}$$

denote the vector subspace of the polynomial ring generated by f_σ. The symmetric group S_{2g-2} acts naturally on V, defined by

$$\tau f(v_1, \ldots, v_{2g-2}) = f(v_{\tau(1)}, \ldots, v_{\tau(2g-2)}).$$

Lemma 4.4. *The S_{2g-2}-representation V is irreducible. In fact, it is exactly the Specht module associated with the Young diagram of shape $(2, 2, \ldots, 2)$, see [???].*

Lemma 4.5. *The polynomial*

$$f^{(m)} := \det \begin{pmatrix} v_1 & \cdots & v_{2g-2} \\ m_1 v_1 & \cdots & m_{2g-2} v_{2g-2} \end{pmatrix} \in V$$

is nonzero.

Proof. The multilinearity property of the determinant allows us to show that $f^{(m)} \in V$. For proving that $f^{(m)} \neq 0$, we evaluate $f^{(m)}$ at specially chosen values. Since at most $g-2$ of the m_i terms are identical, we can always select a permutation $\sigma \in S_{2g-2}$ such that

$$\prod_{k=1}^{g-1} \left(m_{\sigma(2l-1)} - m_{\sigma(2l)} \right) \neq 0.$$

By setting $v_{\sigma(2l-1)} = v_{\sigma(2l)} = e_k$, it follows that

$$\det \begin{pmatrix} v_1 & \cdots & v_{2g-2} \\ m_1 v_1 & \cdots & m_{2g-2} v_{2g-2} \end{pmatrix} \Bigg|_{\substack{v_{\sigma(2l-1)} = e_k \\ v_{\sigma(2l)} = e_k}} = \pm \prod_{k=1}^{g-1} \left(m_{\sigma(2l-1)} - m_{\sigma(2l)} \right) \neq 0.$$

$\square$

Proof of Theorem 4.1. We prove it by contradiction. If

$$\det \begin{pmatrix} \beta_{\sigma(1)} & \cdots & \beta_{\sigma(2g-2)} \\ m_1 \beta_{\sigma(1)} & \cdots & m_{2g-2} \beta_{\sigma(2g-2)} \end{pmatrix} = 0 \quad \text{for any } \sigma \in S_{2g-2},$$

then

$$\sigma(f^{(m)})(\beta_1, \ldots, \beta_{2g-2}) = 0 \quad \text{for any } \sigma \in S_{2g-2}.$$

Since $f^{(m)} \neq 0$ and V is irreducible, $V = \left\langle \sigma(f^{(m)}) \right\rangle_{\sigma \in S_{2g-2}}$, one gets

$$f(\beta_1, \ldots, \beta_{2g-2}) = 0 \quad \text{for any } f \in S_{2g-2}.$$

Taking $f = f_{\mathrm{Id}}$ implies that

$$\det(\beta_1, \ldots, \beta_{g-1}) \det(\beta_g, \ldots, \beta_{2g-2}) = 0.$$

However, this contradicts the fact that any arbitrary selection of $g-1$ elements from $\{\beta_1, \ldots, \beta_{2g-2}\} \subset H \cong \mathbb{C}^{g-1}$ must be linearly independent. $\square$

5. Description of homology class

In the second argument, we seek to identify the homology class $[Z_\chi]$. When Z_χ is a divisor,

$$a_*^{(m)}[a^{-1}(K)] = \deg\left(a^{(m)}|_{a^{-1}(K)}\right)\cdot[Z_\chi]\in \mathrm{H}^2(A;\mathbb{Z}).$$

While I have not developed a way to calculate this degree, we do have a clear description of the homology class $a_*^{(m)}[a^{-1}(K)]$:

Theorem 5.1. *There exists a homology class $[Z]\in \mathrm{H}^2(A;\mathbb{Q})$, such that for any tuple $(m)\in\mathbb{Z}^{2g-2}$,*

$$a_*^{(m)}[a^{-1}(K)] = \left(\frac{1}{2^{g-1}}\sum_{\sigma\in S_{2g-2}}\prod_{l=1}^{g-1}\left(m_{\sigma(2l-1)}-m_{\sigma(2l)}\right)^2\right)\cdot[Z].$$

Remark 5.2. Take $m=(\underbrace{1,\dots,1}_{g-1},\underbrace{0,\dots,0}_{g-1})$, then Theorem 5.1 tells us that

$$a_*^{(m)}[a^{-1}(K)] = \left((g-1)!\right)^2\cdot[Z]. \tag{5.1}$$

In the same time, $Z_\chi=\Theta$ and $\deg\left(a^{(m)}|_{a^{-1}(K)}\right)=\left((g-1)!\right)^2$, so

$$a_*^{(m)}[a^{-1}(K)] = \left((g-1)!\right)^2\cdot[\Theta]. \tag{5.2}$$

Combining (5.1) and (5.2), one gets $[Z]=[\Theta]$ in the main theorem.

Notice that the homology calculation aligns with the dimension calculation, as shown by the equivalence:

$$\dim_{\mathbb{C}} Z_\chi = g-1 \quad\iff\quad a_*^{(m)}[a^{-1}(K)]\neq 0.$$

To prove Theorem 5.1, we start with a few preparatory steps. We start by fixing a basis for the cohomology and then express both the pullback and pushforward with respect to this basis. The homology class $[a^{-1}(K)]\in \mathrm{H}^{2g-2}(C^{2g-2};\mathbb{Z})$ is not yet fully understood, adding complexity to the problem. We address this issue using certain symmetry methods in Section 6.

5.1. Basis.

The cohomology ring of a curve $C/\mathbb{C}$ is well-known:

$$\mathrm{H}^*(C;\mathbb{Z}) = \mathbb{Z}\ \oplus\ \bigoplus_{i=1}^{g}\left(\mathbb{Z}a_i\oplus\mathbb{Z}b_i\right)\ \oplus\ \mathbb{Z}e.$$

In this structure, $e=a_i\cap b_i=-b_i\cap a_i$ are the only non-trivial cup products. To simplify, we relabel $(a_1,b_1,\dots,a_g,b_g)$ as $(c_1,c_2,\dots,c_{2g})$, while also setting $c_0=1$ and $c_{2g+1}=e$.

Observe that the Abel–Jacobi map $\iota_C: C\longrightarrow A$ yields an isomorphism

$$\iota_C^*:\mathrm{H}^1(A;\mathbb{Z})\longrightarrow \mathrm{H}^1(C;\mathbb{Z})\cong\bigoplus_{i=1}^{2g}\mathbb{Z}c_i.$$

We retain the notation $\{c_i\}_{i=1}^{2g}$ as a basis of $\mathrm{H}^1(A;\mathbb{Z})$ as well, and define

$$c_I := c_{i_1}\cap\cdots\cap c_{i_d}\in \mathrm{H}^d(A;\mathbb{Z})$$

for $I=\{i_1,\dots,i_d\}\subseteq\{1,\dots,2g\}$, $i_1<\cdots<i_d$. This sets up the basis for our calculations:

$$\begin{cases}\mathrm{H}^*(C;\mathbb{Z})\cong \displaystyle\bigoplus_{i=0}^{2g+1}\mathbb{Z}c_i\\ \mathrm{H}^*(A;\mathbb{Z})\cong\displaystyle\bigoplus_{I\subseteq\{1,\dots,2g\}}\mathbb{Z}c_I\end{cases}$$

By Künneth formula, one can directly write down a basis for the cohomology of product spaces:

$$\begin{cases} \mathrm{H}^*(C^k;\mathbb{Z}) \cong \bigoplus_{(i_1,\dots,i_k)} \mathbb{Z}\left(c_{i_1}\otimes\cdots\otimes c_{i_k}\right) \\ \mathrm{H}^*(A^k;\mathbb{Z}) \cong \bigoplus_{(I_1,\dots,I_k)} \mathbb{Z}\left(c_{I_1}\otimes\cdots\otimes c_{I_k}\right) \end{cases}$$

There's a small issue here. Due to the non-commutativity of the cohomology ring, expressions often involve -1 coefficients, which, while algebraically necessary, can obscure the core insights. To simplify matters, we adopt a modified basis:

$$\begin{aligned} \left(c_{i_1}\otimes\cdots\otimes c_{i_k}\right)^c &:= (-1)^{\mathrm{sign}}\left(c_{i_1}\otimes\cdots\otimes c_{i_k}\right) \\ \left(c_{I_1}\otimes\cdots\otimes c_{I_k}\right)^c &:= (-1)^{\mathrm{sign}}\left(c_{I_1}\otimes\cdots\otimes c_{I_k}\right) \end{aligned}$$

where the sign accounts for all coefficients arising when rearranging terms to the standard grading order $(c_0,\dots,c_{2g+1})$. With this adjustment, the permutation action no longer introduces any extraneous coefficients:

$$\sigma\cdot\left(c_{i_1}\otimes\cdots\otimes c_{i_k}\right)^c = \left(c_{\sigma(i_1)}\otimes\cdots\otimes c_{\sigma(i_k)}\right)^c.$$

5.2. **Pullback.** The pullback is more straightforward to compute than the pushforward, as we only need to determine it on the basis elements. We collect the building blocks here: [1]

$$\begin{array}{lllrl}
1) & \iota_C: & C \longrightarrow A & c_i & \longleftarrow c_i \\
2) & +: & A\times A \longrightarrow A & c_i\otimes 1 + 1\otimes c_i & \longleftarrow c_i \\
3) & \Delta: & A \longrightarrow A\times A & c_i & \longleftarrow c_i\otimes 1,\ 1\otimes c_i \\
4) & [m]: & A\times A \longrightarrow A & mc_i & \longleftarrow c_i \\
5) & +: & A^k \longrightarrow A & \sum_{j=1}^k 1\otimes\cdots \underset{\underset{j\text{-th}}{\uparrow}}{c_i} \cdots\otimes 1 & \longleftarrow c_i
\end{array}$$

Now, for a tuple $(m)\in\mathbb{Z}^k$, the map

$$a^{(m)}: C^k \longrightarrow A \qquad (p_1,\dots,p_k)\longrightarrow \sum_{i=1}^k m_i(p_i-p_0)$$

can be written as compositions of basic functions:

$$a^{(m)}: C^k \xrightarrow{(\iota_C,\dots,\iota_C)} A^k \xrightarrow{(m_1,\dots,m_k)} A^k \xrightarrow{\quad +\quad} A$$

We get

$$a^{(m),*}c_i = \sum_{j=1}^k 1\otimes\cdots \underset{\underset{j\text{-th}}{\uparrow}}{m_i c_i}\cdots\otimes 1$$

As a result,

$$\begin{aligned} a^{(m),*}c_I &= \sum_{I=\sqcup I_i}\left(m_1^{|I_1|}\iota_C^* c_{I_1}\otimes\cdots\otimes m_k^{|I_k|}\iota_C^* c_{I_k}\right)^c \\ &= \sum_{\substack{I=\sqcup I_i\\ |I_i|\leqslant 2}}\left(\prod_{i=1}^k m_i^{|I_i|}\right)\left(c_{I_1}\otimes\cdots\otimes c_{I_k}\right)^c \end{aligned}$$

where $c_{\{i,j\}} = c_i\cap c_j \in \mathrm{H}^0(C;\mathbb{Z})$ in the last expression.

5.3. **Pushforward.**

[1] For 2), it reduces to the addition map $S^1\times S^1\longrightarrow S^1$.

6. Second proof by symmetric functions

Ideally, if we could express $[a^{-1}(K)] \in \mathrm{H}^{2g-2}(C^{2g-2}; \mathbb{Z})$ as a linear combination of the basis elements $c_{i_1} \otimes \cdots \otimes c_{i_{2g-2}}$, we could then compute it directly to obtain the answer. As of now, however, this formulation is unknown.

Fortunately, there are several symmetries that simplify calculations. For instance, the subset $a^{-1}(K) \subset C^{2g-2}$ is preserved under the action of S_{2g-2}.

References

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