SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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Contents

1. Introduction

Let A be a complex abelian variety of dimension n. To any closed subvariety $Z \subset A$ of dimension r, one can associate a reductive group G_Z through the convolution of perverse sheaves. This correspondence allows to reformulate problems in algebraic geometry in representation-theoretic terms, thereby offering new geometric insights. A crucial role in this context is played by the characteristic cycle, a fundamental invariant attached to a perverse sheaf, [8, §5–§8]. Recent work demonstrates that one can approach these characteristic cycles directly as conic Lagrangian cycles, without invoking the full machinery of perverse sheaves.

In [11], Krämer establishes a correspondence between Weyl group orbits and conic Lagrangian cycles in the cotangent bundle which arise as the conormal varieties of certain subvarieties. Despite this correspondence, an explicit geometric description of these cycles themselves remains elusive. In this article, we propose a purely geometric construction of these cycles and analyze three key properties of the associated subvarieties: irreducibility, dimension, and homology class.

To present our results, we begin by recalling several fundamental notions, which will be discussed in detail in Section 2. On a complex abelian variety A, the *translations* and *reflections* are special automorphisms of A of the form $x \mapsto x + p$ and $x \mapsto -x + p$, respectively. For $\sigma \in \mathrm{Aut}(A)$ and a closed subvariety $Z \subset A$, we say that Z is *invariant* under σ if $\sigma(Z) = Z$. Given a subvariety $Z \subset A$, the *tangent Gauss map* ϕ_Z assigns to $p \in Z^{\mathrm{sm}}$ the embedded tangent space to Z at p (cf. Definition 2.1), while the *conormal Gauss map* $\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0 A \to \mathbb{P}T_0 A$ (cf. Definition 2.3) arises from the projectivized conormal variety of Z and is typically generically finite. The *monodromy group* $\mathrm{Gal}(\gamma_Z)$ encodes the permutation of the sheets of the covering γ_Z around its branch locus, while W_Z denotes the *Weyl group* of the reductive group G_Z.

Concerning irreducibility, it follows from the constructions in [11, 2.c] that the irreducible components of the appearing cycles correspond precisely to the orbits of a certain monodromy group. The discrepancy between the monodromy group and a closely related Weyl group provides a measure of the failure of irreducibility of these subvarieties. A natural question is how big this discrepancy could be. For this, we formulate a conjecture that refines [10, Conjecture 8]:

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Conjecture 1.1. *Let A be an abelian variety of dimension n, and let $C \subset A$ be a non-degenerate curve that is not invariant under any non-trivial translation. Then, the monodromy group* $\mathrm{Gal}(\gamma_C)$ *is big (see Definition 3.1).*

For $n > 2$, this conjecture admits a purely geometric reformulation:

Conjecture 1.2. *Let A be an abelian variety of dimension $n > 2$, and let $C \subset A$ be a non-degenerate curve that is not invariant under any non-trivial translation in A. Then:*

- *If C is invariant under some reflection, the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ is a double cover onto its image;*
- *If C is not invariant under any reflection, the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ is generically injective.*

In general Conjecture 1.1 is still open. We analyze the Prym case thoroughly, which will lead to the following result:

Theorem A *For any étale double cover $h : C \longrightarrow C'$ of smooth, projective, non-hyperelliptic curves, one may view C as a subvariety of the Prym variety* $\mathrm{Prym}(C/C')$ *via the Abel–Prym map. Then the associated monodromy group is big, except in the case where C' is bielliptic and h arises as the pullback of an étale double cover of elliptic curves. In this exceptional situation, the curve $C \subset \mathrm{Prym}(C/C')$ is stable under a non-trivial translation.*

Theorem A follows from a combination of Proposition 3.16 and Lemma 3.17. The bielliptic case, which constitutes an exception, is analyzed thoroughly in Subsection 3.3.

For the dimension and homology class of a subvariety Z, both invariants can be recovered from the homology class of $\mathbb{P}\Lambda_Z \subset \mathbb{P}T^*A$ — that is, from the total Chern–Mather class of Z. Standard computational techniques are available when the monodromy group $\mathrm{Gal}(\gamma_Z)$ is a full or signed symmetric group. The notions of the Chern–Mather class and the Pontryagin product will be recalled in Definition 5.1 and Definition 5.6, respectively.

Theorem B *Consider a non-degenerate subvariety $Z \subset A$ and a tuple $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$. Let $Z^{(m)} \subset A$ be as in Definition 4.7. Let $d = \deg \gamma_Z$, $\widetilde{d} = d/2$, and let $c_i := c_{M,i}(\Lambda_Z)$ be the Chern–Mather class of Z. Let $*$ be the Pontryagin product. We write $\lambda \dashv l$ to indicate that $\lambda = [\lambda_1, \dots, \lambda_{k'}]$ is a partition of l.*

(1) When $\mathrm{Gal}(\gamma_Z) = S_d$, the Chern–Mather classes of $Z^{(m)}$ can be written as

$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right)$$

for certain explicit coefficients $c_Z^{(m)} \in \mathbb{N}_{>0}$ and $\mu_d^{\lambda} \in \mathbb{Z}$.

(2) When $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, the Chern–Mather classes of $Z^{(m)}$ can be written as

$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \widetilde{\mu}_{\widetilde{d}}^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right)$$

for certain explicit coefficients $c_Z^{(m)} \in \mathbb{N}_{>0}$ and $\widetilde{\mu}_{\widetilde{d}}^{\lambda} \in \mathbb{Z}$.

The proof of theorem B will occupy Section 5. In fact, we will show (1) in (5.1), and (2) in (5.2). The combinatorial coefficients μ_d^{λ} and $\widetilde{\mu}_{\widetilde{d}}^{\lambda}$ appearing in the formulas are defined in Definition 5.12, while the case of Jacobian varieties is discussed in detail in Example 5.14.

The structure of the article is as follows.

In Section 2, we fix notation and describe the Gauss map from three different perspectives. We also define the monodromy group and highlight its connection to the degree of the tangent Gauss map.

Section 3 is devoted to the explicit computation of the monodromy group in selected cases, with a particular focus on the Prym curve situation. Subsection 3.3 presents an example in which the monodromy group is small; however, this does not constitute a counterexample to Conjecture 1.1, as the curve remains invariant under a non-trivial translation, as discussed in Lemma 3.12. In Subsection 3.1, we collect several sufficient conditions for the monodromy group to be large, which we then employ in Subsection 3.2 to show that, for all remaining Prym curve cases with $g(C') > 9$, the monodromy group is necessarily large.

In Section 4, we define the subvarieties $Z^{(m)}$, which serve as the central objects of this study. In Subsection 4.2, we briefly discuss how these varieties naturally arise as the characteristic cycles of certain perverse sheaves.

In Section 5, we focus on the computation of the dimension and the homology class of $Z^{(m)}$. After recalling the definition of the Chern-Mather class in Subsection 5.1, the task reduces to computing $[\mathbb{P}\Lambda_{Z^{(m)}}]$. Subsections 5.3 and 5.4 treat the type A case, where $\mathrm{Gal}(\gamma_Z) = S_d$, and the type C case, where $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, respectively. The derivation of the formulas involves certain coefficients that are purely combinatorial in nature. These combinatorial contributions are treated separately in Subsection 5.5.

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2. Tangent Gauss map and conormal Gauss map

We work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a integral separated scheme of finite type over $\mathbb{C}$. Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subseteq A$ be an irreducible closed subvariety of dimension r. We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.

2.1. **Gauss map and monodromy group.**

Definition 2.1. *For a subvariety $Z \subset A$, the tangent Gauss map of Z is the map*

$$\phi_Z : \mathrm{Z}^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longmapsto T_pZ \quad \subseteq T_pA \cong T_0A$$

sending a smooth point to the tangent space at that point (seen as a subspace of T_0A).

Remark 2.2. Any map $X \longrightarrow \mathrm{Gr}(r, V)$ is induced by a rank r vector bundle $\mathcal{E}$ on X together with an epimorphism $V^* \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{E}$. In our case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}$ and the sections

$$T_0^*A \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \to \mathrm{H}^0(\mathrm{Z}^{\mathrm{sm}}, \mathcal{T}^*_{\mathrm{Z}^{\mathrm{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \twoheadrightarrow \mathcal{T}^*_{\mathrm{Z}^{\mathrm{sm}}}.$$

The definition of the conormal Gauss map requires a brief recollection of the conormal variety. On the smooth locus, the normal and conormal bundles are the vector bundles $\mathcal{N}_{\mathrm{Z}^{\mathrm{sm}}/A}$ resp. $\mathcal{N}^*_{\mathrm{Z}^{\mathrm{sm}}/A}$ defined by the short exact sequences

$$0 \longrightarrow \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \mathcal{T}_A|_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \mathcal{N}_{\mathrm{Z}^{\mathrm{sm}}/A} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}^*_{\mathrm{Z}^{\mathrm{sm}}/A} \longrightarrow \Omega_A|_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \Omega_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow 0$$

We write $\Lambda_{Z^{\mathrm{sm}}}$ for the total space of $\mathcal{N}^*_{Z^{\mathrm{sm}}/A}$. The conormal variety Λ_Z is the closure of $\Lambda_{Z^{\mathrm{sm}}}$ viewed as a subvariety in T^*A:

$$\Lambda_Z := \overline{\Lambda_{Z^{\mathrm{sm}}}} \subset T^*A \cong A \times T_0^*A$$

It is a conical Lagrangian subvariety of T^*A. The (affine) conormal Gauss map is defined as

$$\gamma_Z^{\mathrm{aff}} : \Lambda_Z \subset A \times T_0^*A \longrightarrow T_0^*A$$

For intersection-theoretic purposes it is more natural to work with projectivized conormal spaces. We therefore pass from the affine conormal Gauss map to its projectivized version:

Definition 2.3. *The projectivized conormal variety of a subvariety $Z \subset A$ is defined by*

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A,$$

and the (projectivized) conormal Gauss map is defined by

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A.$$

By [8, Theorem 2.8 (1)], γ_Z is generically finite when Z is (an integral variety) of general type.

A lot of geometry of Z is encoded in the map γ_Z. For instance, if $Z \subset A$ is smooth, then

$$\deg \gamma_Z = (-1)^r \chi(Z),$$

where $\chi(Z) = \sum_i (-1)^i b_i(Z)$ is the topological Euler characteristic of Z, see [12, Theorem 1.1.1].

Further insight can be gained by analyzing the fibers of γ_Z. For instance, if Z is preserved by a translation $t_v : A \longrightarrow A$, then each fiber $\gamma_Z^{-1}(\xi) \subset A$ is also invariant under t_v. Likewise, if $Z = -Z$, then the fiber satisfies $\gamma_Z^{-1}(\xi) = -\gamma_Z^{-1}(\xi)$. Finding further constraints is more challenging.

An important invariant arising from the fiber $\gamma_Z^{-1}(\xi)$ is the monodromy group $\mathrm{Gal}(\gamma_Z)$; for completeness, we recall its definition below.

Definition 2.4. *Let $f : Y \to X$ be a generically finite morphism between algebraic varieties. Then there exists a non-empty open subset $U \subseteq X$ such that the restriction $f^{-1}(U) \to U$ is a finite étale cover. Moving along a closed loop in U induces a permutation of the points in the fiber $f^{-1}(\xi_0)$, which defines the map*[1]

$$\rho_f : \pi_1(U, \xi_0) \longrightarrow \mathrm{Aut}(f^{-1}(\xi_0)) \cong S_{\deg f}.$$

The monodromy group is then defined as the image of ρ_f, i.e.,

$$\mathrm{Gal}(f) := \mathrm{Im}\, \rho_f.$$

When U is smooth, the isomorphism class of $\mathrm{Gal}(f)$ *doesn't depend on the choice of U.*

2.2. **Interpolation via hyperplanes.** In this subsection, we reinterpret γ_Z using a functorial and more transparent framework. This permits to define the conormal Gauss map and its monodromy group for any morphism $\phi_Z : Z \to \mathrm{Gr}(r,n)$ with $\dim Z = r$, and clarifies the relation between the monodromy group and $\deg \phi_Z$.

To simplify notation, we abbreviate T_0^*A by W. Since each non-zero conormal vector $\xi \in T_0^*A$ determines a hyperplane $H_\xi \in \mathrm{Gr}(n-1, T_0A)$, we have the isomorphisms

$$\begin{aligned}
\mathbb{P}T_0^*A &\cong \mathrm{Gr}(n-1, T_0A), \\
\mathbb{P}\Lambda_{Z^{\mathrm{sm}}} &= \left\{ (p,\xi) \in Z^{\mathrm{sm}} \times \mathbb{P}T_0^*A \;\middle|\; \xi|_{T_pZ} \equiv 0 \right\} \\
&\cong \left\{ (p,H) \in Z^{\mathrm{sm}} \times \mathrm{Gr}(n-1, W) \;\middle|\; \phi_Z(p) \subseteq H \right\} \\
&\cong (\phi_Z, \mathrm{Id})^{-1}\, I_{r,n-1},
\end{aligned}$$

where

$$I_{r,n-1} := \left\{ (V,H) \in \mathrm{Gr}(r,W) \times \mathrm{Gr}(n-1,W) \;\middle|\; V \subseteq H \right\}$$

[1] In the last isomorphism we implicitly give a labelling from $\{1, 2, \ldots, \deg f\}$ to the fiber $f^{-1}(\xi)$.

is the incidence variety relating $\mathrm{Gr}(r,W)$ and $\mathrm{Gr}(n-1,W)$. In these terms,

$$\begin{aligned}\gamma_Z^{-1}(H)\cap \mathrm{Z}^{\mathrm{sm}} &= \{\, p\in \mathrm{Z}^{\mathrm{sm}} \mid \phi_Z(p)\subseteq H \,\}\\ &\cong \phi_Z^{-1}\left(\mathrm{Gr}(r,H)\right)\end{aligned}$$

is the collection of points whose tangent spaces lie entirely within H.

Geometrically, the monodromy can be described as follows: Given a general hyperplane $H\in \mathbb{P}T_0^*A$, its preimage under γ_Z consists of d points $p_1,\dots,p_d$. Moving H continuously along a closed loop we obtain a permutation of these points, and the monodromy group $\mathrm{Gal}(\gamma_Z)$ consists of all permutations obtained this way. With this formulation, it suffices to consider the Gauss map ϕ_Z alone; the inclusion $\iota_Z : Z\to A$ is no longer required for computing the monodromy group.

Definition 2.5. *Let Z be an r-dimensional variety and $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$ a morphism. Suppose that $\phi_*[Z]\cup[\mathrm{Gr}(r,H)] = d\neq 0$ in $\mathrm{H}^{2r(n-r)}(\mathrm{Gr}(r,n);\mathbb{Q})$. The monodromy group $\mathrm{Mon}(\phi)$ is defined as $\mathrm{Gal}(f_\phi)$, where*

$$f_\phi : (\phi,\mathrm{Id})^{-1} I_{r,n-1} \ \subset\ Z\times \mathrm{Gr}(n-1,n) \xrightarrow{\pi_2} \mathrm{Gr}(n-1,n)$$

is a generically finite morphism of degree d.

When ϕ is not generically injective, the monodromy group is subject to additional constraints, as captured by the next lemma.

Lemma 2.6. *Let $\phi : Z\to \mathrm{Gr}(r,n)$ be generically k-to-1 onto its image. With a suitable ordering of the fibers, the associated monodromy group $\mathrm{Mon}(\phi)$ is contained in a wreath product*

$$S_k \wr S_{d/k} := \left(S_k^{\oplus d/k}\right) \rtimes S_{d/k}.$$

Proof. For a general hyperplane H, consider the Cartesian diagram below:

$$\begin{tikzcd} \phi: & Z \arrow[r, two heads, "k:1"] & \mathrm{Im}\, Z \arrow[r, hook] & \mathrm{Gr}(r,n) \\ & \cup & \cup & \cup \\ & \{p_1,\dots,p_d\} \arrow[r] & \{q_1,\dots,q_{d/k}\} \arrow[r] & \mathrm{Gr}(r,H) \end{tikzcd}$$

The fiber $\phi^{-1}\left(\mathrm{Gr}(r,H_0)\right)$ splits into d/k groups of points, with the monodromy group acting by permutations within each group and among the groups. $\square$

2.3. **Interpolation via Albanese morphisms.** Examples of subvarieties of abelian varieties can be obtained in two ways: one may fix an abelian variety and consider cycles within it, or begin with a variety Z and construct a map to an abelian variety, such as the Albanese map. In the latter perspective, the setting changes slightly: X is a smooth projective variety of dimension r, $\iota_X : X\to A$ is a morphism to an abelian variety A of dimension n, and $Z := \iota_X(X)$ denotes its image in A.

In this subsection, we begin with the Albanese morphism $X\to \mathrm{Alb}(X)$, from which we derive some basic properties of the tangent Gauss map. Any other morphism to an abelian variety factors over the Albanese morphism, and then analogous methods apply, see Proposition 2.10.

Let X be a smooth complex projective variety of dimension r, and set $n := \dim_{\mathbb{C}} \mathrm{H}^0(X,\Omega_X) = h^{1,0}$. Recall that the Albanese variety of X is defined as

$$\mathrm{Alb}(X) := \mathrm{H}^0(X,\Omega_X)^* / \mathrm{H}_1(X,\mathbb{Z})_{\mathrm{free}},$$

and the Albanese map is given by (for some fixed base point $p_0\in X$)

$$\alpha : X \longrightarrow \mathrm{Alb}(X) \qquad\qquad p \longmapsto \left[\ \omega \mapsto \int_{\gamma: p_0\sim p} \omega\ \right].$$

One classical question is the dimension of $\alpha(X)$. Before proceeding, we fix the following notation: For any closed subscheme $S\subset X$, let $\mathcal{I}_S\subset \mathcal{O}_X$ be its ideal sheaf. Given a point $p\in X$ with inclusion $i_p : p\hookrightarrow X$, we denote by $i_{p,*}\mathcal{O}_p$ the skyscraper sheaf at p. We set $\Omega_X(-p) := \Omega_X\otimes \mathcal{I}_p$, and denote by $\Omega_X|_p := \Omega_X \otimes i_{p,*}\mathcal{O}_p = i_{p,*}i_p^*\Omega_X$ the fiber of Ω_X at p. With this notation, the cotangent map of α at $p \in X$ is

$$T_p^*\alpha : T_{\alpha(p)}^* \operatorname{Alb}(X) = H^0(X, \Omega_X) \longrightarrow T_p^* X = H^0(X, \Omega_X|_p).$$

Consider the short exact sequence of coherent sheaves on X

$$0 \longrightarrow \Omega_X(-p) \longrightarrow \Omega_X \longrightarrow \Omega_X|_p \longrightarrow 0$$

which induces a long exact sequence

$$\begin{tikzcd} & \mathrm{H}^1(X, \Omega_X(-p)) \arrow[r] & \mathrm{H}^1(X, \Omega_X) \arrow[r] & 0 \\ 0 \arrow[r] & \mathrm{H}^0(X, \Omega_X(-p)) \arrow[r] & \mathrm{H}^0(X, \Omega_X) \arrow[r, "T_p^*\alpha"] & \mathrm{H}^0(X, \Omega_X|_p) \arrow[ull, out=0, in=180] \end{tikzcd}$$

The proposition below follows from standard arguments in homological algebra:

Proposition 2.7. *For a general point $p \in X$,*

$$\begin{aligned} \dim_{\mathbb{C}} \alpha(X) &= \operatorname{rank} T_p^*\alpha \\ &= n - h^0(X, \Omega_X(-p)) \\ &= h^1(X, \Omega_X(-p)) - h^1(X, \Omega_X). \end{aligned}$$

In particular,

$$\begin{aligned} &\alpha \textit{ is surjective} &&\iff h^0(X, \Omega_X(-p)) = 0 \\ &\alpha \textit{ is generically finite onto its image} &&\iff h^0(X, \Omega_X(-p)) = n - r \\ &\alpha \textit{ is constant} &&\iff h^0(X, \Omega_X(-p)) = n \\ & &&\iff h^1(X, \Omega_X(-p)) = h^1(X, \Omega_X) \\ & &&\iff n = 0. \end{aligned}$$

Proof. We know that

$$\begin{aligned} &\alpha \text{ is surjective} &&\iff \dim_{\mathbb{C}} \alpha(X) = n \\ &\alpha \text{ is generically finite onto its image} &&\iff \dim_{\mathbb{C}} \alpha(X) = r \\ &\alpha \text{ is constant} &&\iff \dim_{\mathbb{C}} \alpha(X) = 0 \\ & &&\iff \operatorname{Alb}(X) = 0. \end{aligned}$$

□

We will concentrate on the case where α is generically injective.[2] Under this assumption, we set $Z = \iota_X(X)$ and $A = \operatorname{Alb}(X)$. The corresponding Gauss map is then a rational map:

$$\begin{aligned} \phi_Z : Z &\dashrightarrow \operatorname{Gr}(r, T_0 A) \cong \operatorname{Gr}\left(n - r, \mathrm{H}^0(X, \Omega_X)\right) \\ p &\longmapsto \mathrm{H}^0(X, \Omega_X(-p)) \end{aligned}$$

[2]The general method remains valid in the broader setting, but the Gauss map then takes values in a different space. In fact, there is always a semicontinuous map

$$\phi_X : X \longrightarrow \bigsqcup_{i=0}^{r} \operatorname{Gr}\left(n - i, \mathrm{H}^0(X, \Omega_X)\right) \qquad p \longmapsto \mathrm{H}^0(X, \Omega_X(-p)).$$

and we have the isomorphisms

$$\begin{aligned} \mathbb{P}T_0^*A &\cong \mathbb{P}\,\mathrm{H}^0(X,\Omega_X), \\ \mathbb{P}\Lambda_Z &= \{\,(p,[\omega]) \in Z \times \mathbb{P}T_0^*A \mid \omega(p) = 0\,\} \\ &\cong \left\{\,(p,[\omega]) \in Z \times \mathbb{P}T_0^*A \;\middle|\; \omega \in \mathrm{H}^0(X,\Omega_X(-p))\,\right\} \\ &\cong (\phi_Z,\mathrm{Id})^{-1}\, I_{n-r,1}, \end{aligned}$$

where

$$I_{n-r,1} := \left\{\,(V,[\omega]) \in \mathrm{Gr}(n-r,n) \times \mathrm{Gr}(1,n) \;\middle|\; \omega \in V\,\right\}$$

is the incidence variety relating $\mathrm{Gr}(n-r,n)$ and $\mathrm{Gr}(1,n)$. In that case,

$$\gamma_Z^{-1}([\omega]) = \{\,p \in X \mid \omega(p) = 0\,\}$$

is the zero set of the section $\omega \in \mathrm{H}^0(X,\Omega_X)$. The number $(-1)^r \deg \gamma_Z$ is the index (of the vector field) in the Poincaré–Hopf index formula (see [14, p35]), and the monodromy group $\mathrm{Gal}(\gamma_Z)$ serves as a more refined invariant.

The next proposition shows when the Gauss map ϕ_Z is not generically injective. We denote by $X^{[m]}$ the Hilbert scheme of m-points on X.

Proposition 2.8. *When α is generically finite onto its image,*

ϕ_Z is not generically injective

$\iff$ *For general $p \in X$, there exists $q \neq p$ such that $h^0(X,\Omega_X(-p-q)) = n-r$*

$\overset{(1)}{\iff}$ *For general $p \in X$, there exists $S \in X^{[2]}$ such that $p \in S$ and $h^0(X,\Omega_X \otimes \mathcal{I}_S) = n-r$*

$\overset{(2)}{\iff}$ *For all $p \in X$, there exists $S \in X^{[2]}$ such that $p \in S$ and $h^0(X,\Omega_X \otimes \mathcal{I}_S) \geqslant n-r$.*

Proof. When $n = r$, the map φ_Z has a point as its target, so the equivalence is trivial. We shall thus restrict to the case $n > r$ in the subsequent discussion.

(1): The implication "$\Rightarrow$" is immediate. For the converse "$\Leftarrow$", it suffices to show that the image of

$$\left\{S \in X^{[2]} \;\middle|\; h^0(X,\Omega_X \otimes \mathcal{I}_S) = n-r\right\}$$

under the map $\pi_X : X^{[2]} \longrightarrow X^{(2)}$ does not include the diagonal. Indeed, we can choose a section $s \in \mathrm{H}^0(X,\Omega_X)$ that cuts out finitely many reduced points $p_1,\ldots,p_d$ in X. (Well this maybe not so true, since ϕ_Z is not always regular, such a section s may vanish along some indeterminacy locus. However, for a general s, there will always be at least one isolated zero p_1, which is sufficient for our purposes.) Then for any $S \in \pi_X^{-1}(2p_1)$, we have

$$\mathrm{H}^0(X,\Omega_X \otimes \mathcal{I}_S) \subsetneq \mathrm{H}^0(X,\Omega_X(-p_1)).$$

(2): The implication "$\Rightarrow$" follows from the geometry of the projection $\mathrm{pr}_1 : X \times X^{[2]} \to X$. By Lemma 2.9 and the closedness of the tautological correspondence in $X \times X^{[2]}$, the subset

$$I^{n-r} := \left\{(p,S) \in X \times X^{[2]} \;\middle|\; p \in S \text{ and } h^0(X,\Omega_X \otimes \mathcal{I}_S) \geqslant n-r\right\}$$

is closed in $X \times X^{[2]}$. Consequently, its image

$$\mathrm{pr}_1(I^{n-r}) = \left\{p \in X \;\middle|\; \text{there exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X,\Omega_X \otimes \mathcal{I}_S) = n-r\right\}$$

is closed in X. Since the hypothesis ensures that $\mathrm{pr}_1(I^{n-r})$ contains a nonempty open subset of X, we obtain $\mathrm{pr}_1(I^{n-r}) = X$.

For "$\Leftarrow$": as α is generically injective, one has $h^0(X,\Omega_X(-p)) = n-r$ for a general point $p \in X$. Therefore, for any subscheme S containing such a general point, $h^0(X,\Omega_X \otimes \mathcal{I}_S) \leqslant n-r$. $\square$

This last step relies on the following lemma, combined with the closedness of the tautological correspondence in $X \times X^{[2]}$.

Lemma 2.9. *Let $m \in \mathbb{Z}_{>0}$. The function*
$$h^\Omega : X^{[m]} \longrightarrow \mathbb{Z}_{\geqslant 0} \qquad S \longmapsto h^0(X, \Omega_X \otimes \mathcal{I}_S)$$
is Zariski upper semicontinuous.

Proof. Consider the coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X^{[m]} \times X)$ characterized by the property
$$\mathcal{F}|_{\{S\}\times X} \cong \Omega_X \otimes \mathcal{I}_S,$$
The proposition then follows directly from the semicontinuity theorem [15, 28.1.1]. ☐

The stratification of $X^{[m]}$ by h^Ω offers a natural generalization of Brill–Noether theory beyond the setting of curves.

At the end of this subsection, let us turn to the setting of a general abelian variety A and a smooth subvariety $\iota_Z : Z \hookrightarrow A$, where $n = \dim_{\mathbb{C}} A$ and $r = \dim_{\mathbb{C}} Z$. Observe that ι_Z factors through the Albanese variety of Z:
$$\iota_Z : Z \xrightarrow{\ \alpha_Z\ } \mathrm{Alb}(Z) \xrightarrow{\ \pi\ } A$$
We shall also assume that Z generates A; it then follows that the map π is surjective. The cotangent map of ι_Z at a point $p \in Z$ factors through $\mathrm{H}^0(Z, \Omega_Z)$:
$$T_p^*\iota_Z : T_p^* A \hookrightarrow \mathrm{H}^0(Z, \Omega_Z) \longrightarrow T_p^* Z$$
For convenience, abbreviate $V := T_0^* A \cong T_p^* A$, and view V as a subspace of $\mathrm{H}^0(Z, \Omega_Z)$.

Proposition 2.10. *Assume that Z is embedded in A and generates A, and let $V := T_0^* A$. Then*
$$\dim_{\mathbb{C}} \mathrm{H}^0(Z, \Omega_Z(-p)) \cap V = n - r \qquad \text{for all } p \in Z$$
It follows that the Gauss map is a regular morphism
$$\begin{aligned} \phi_Z : Z &\longrightarrow \mathrm{Gr}(r, T_0 A) \quad \cong \quad \mathrm{Gr}(n-r, V) \\ p &\longmapsto \mathrm{H}^0(Z, \Omega_Z(-p)) \cap V \end{aligned}$$
Furthermore,
$$\begin{aligned} & \phi_Z \text{ is not generically injective} \\ \iff & \text{For general } p \in Z, \text{ there exists } q \neq p \text{ such that } h^0(Z, \Omega_Z(-p-q)) \cap V = n - r \\ \iff & \text{For all } p \in Z, \text{ there exists } S \in Z^{[2]} \text{ such that } p \in S \text{ and } h^0(Z, \Omega_Z \otimes \mathcal{I}_S) \cap V = n - r. \end{aligned}$$

3. Monodromy group

3.1. Criteria for big monodromy group.

Definition 3.1 (big monodromy group)**.** *We refer to the big monodromy group as any group of the following types:*

notation	*name*	*alias*
$W(A_{m+1}) = S_m$	*full symmetric group*	
$W(C_m) = S_2^{\oplus m} \rtimes S_m$	*signed symmetric group*	*hyperoctahedral group*
$W(D_m) = (S_2^{\oplus m})_0 \rtimes S_m$	*even-signed symmetric group*	*demihyperoctahedral group*

Table 1. big monodromy group

In practice, the term "big monodromy group" refers to the full symmetric group S_n when the subset $Z \subset A$ is not symmetric, and to the (even-)signed symmetric group when $Z \subset A$ is symmetric.

Proposition 3.2 (See [2, p111] for a detailed proof)**.** *Suppose that $\iota_C : C \subseteq \mathbb{P}^{n-1}$ is an irreducible nondegenerate*[3] *curve of degree d, then $\mathrm{Mon}(\iota_C) \cong S_d$.*

Sketch of proof. Because S_d is generated by its transpositions, we are reduced to verifying that:

- $\mathrm{Mon}(\iota_C)$ acts doubly transitively on the fiber;
- $\mathrm{Mon}(\iota_C)$ contains a transposition.

□

In fact, a degree $2:1$ map does not give rise to any exceptional monodromy groups beyond those listed in Table 1.

Proposition 3.3. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \to C'$ be a degree 2 ramified covering. Then*

$$\mathrm{Mon}(\iota_{C'} \circ h) \cong W(C_{d/2}) \text{ or } W(D_{d/2}).$$

Sketch of proof. By Lemma 2.6 we know that $\mathrm{Mon}(\iota_{C'} \circ h) \subseteq W(C_{d/2})$. By Lemma 3.4, we are reduced to verifying that:

- The quotient map $\mathrm{Mon}(\iota_{C'} \circ h) \longrightarrow \mathrm{Mon}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- (signed doubly transitive) $\mathrm{Mon}(\iota_{C'} \circ h)$ acts transitively on pairs (x, y) with $x \neq \pm y$.

□

Lemma 3.4. *Let G be a subgroup of $W(C_m)$, acting naturally on the set $\pm 1, \ldots, \pm m$. If the projection $G \to S_m$ is surjective then*

$$G \cong W(C_m) \text{ or } W(D_m) \text{ or } S_2 \times S_m \text{ or } S_m.$$

Sketch of proof. Let H denote the kernel of the natural quotient map $G \to S_m$. Then $H \subsetneq (S_2)^{\oplus m}$ is stable under the action of S_n. There are only four possible forms that H can take:[4]

- $H = 0$. Then $G \cong S_m$.
- $H = \langle(-1, \ldots, -1)\rangle \cong S_2$. Then $G \cong S_2 \times S_m$.
- $H = (S_2^{\oplus m})_0$. Then G is a index 2 subgroup of $W(C_m)$, so $G \cong W(D_m)$.[5]
- $H = S_2^{\oplus m}$. Then $G = W(C_m)$.

□

By incorporating further information about the covering, we are able to determine the monodromy group.

Proposition 3.5. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \to C'$ be a degree 2 ramified covering, with ramification occurring at at least one smooth point of C'. Then $\mathrm{Mon}(\iota_{C'} \circ h) \cong W(C_{d/2})$ is the hyperoctahedral group/signed symmetric group.*

Sketch of proof. By Lemma 2.6 we know that $\mathrm{Mon}(\iota_{C'} \circ h) \subseteq W(C_{d/2})$. By Lemma 3.6, we are reduced to verifying that:

- The quotient map $\mathrm{Mon}(\iota_{C'} \circ h) \longrightarrow \mathrm{Mon}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- $\mathrm{Mon}(\iota_{C'} \circ h)$ contains a transposition of a pair of points in the fiber of h.

□

[3] A curve $C \subseteq \mathbb{P}^{n-1}$ is said to be nondegenerate if it is not contained in any hyperplane $H \subseteq \mathbb{P}^{n-1}$.

[4] Here is a brief argument showing that H must be one of 0, S_2, $(S_2^{\oplus m})_0$, or $S_2^{\oplus m}$. If H contains an element $h = (a_1, \ldots, a_m)$ with $a_i \neq a_j$ for some $i \neq j$, then

$$(ij)h + h = (1, \ldots, \underset{\substack{\uparrow \\ i\text{-th}}}{-1}, \ldots, \underset{\substack{\uparrow \\ j\text{-th}}}{-1}, \ldots, 1) \in H,$$

implying $(S_2^{\oplus m})_0 \subseteq H$. Hence, H must be either $(S_2^{\oplus m})_0$ or $S_2^{\oplus m}$. Otherwise, if all $h \in H$ have identical coordinates, then H is either 0 or S_2.

[5] Check stackexchange discussions

Lemma 3.6. *Let G be a subgroup of $W(C_m)$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \to S_m$ is surjective and the transposition σ_0 of ± 1 lies in G, then $G = W(C_m)$.*

Sketch of proof. Let ε_i denote the transposition of $\pm i$. For any $\sigma \in S_m$, choose a lift $\tilde{\sigma} \in G$, then

$$\varepsilon_{\sigma(1)} = \tilde{\sigma} \circ \sigma_0 \circ \tilde{\sigma}^{-1} \in G.$$

Thus, $S_2^{\oplus m} \subset G$, and since G maps onto S_m, we obtain $G = W(C_m)$. □

Proof that Conjecture 1.1 is equivalent to Conjecture 1.2 when $n > 2$. If ϕ_C is generically injective, Proposition 3.2 yields $\mathrm{Gal}(\gamma_C) \cong S_d$. When $C = -C$ and ϕ_C is a double cover, Proposition 3.3 implies $\mathrm{Gal}(\gamma_C) \cong W(C_{d/2})$ or $W(D_{d/2})$, both of which are big monodromy groups. In all other cases, Lemma 2.6 shows that $\mathrm{Gal}(\gamma_C) \subseteq S_k \wr S_{d/k}$, which is not big whenever $k > 2$ and $k \neq d$. Note that $k = d$ occurs only when $n = 2$. □

3.2. **Curves with big monodromy group.** The availability of these criteria permits the systematic construction of numerous cases where the associated monodromy group is large.

Example 3.7. *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$. In this case, the tangent Gauss map is the canonical map $|\omega_C|$.*

When C is non-hyperelliptic, the corresponding Gauss map

$$|\omega_C| : C \longrightarrow \mathbb{P}^{g-1}$$

makes C as an irreducible nondegenerate curve of degree $2g-2$, by Proposition 3.2 we get

$$\mathrm{Gal}(\gamma_C) \cong S_{2g-2}.$$

When C is hyperelliptic, the corresponding Gauss map is $2:1$ onto a rational normal curve $R \subset \mathbb{P}^{g-1}$:

$$|\omega_C| : C \xrightarrow{2:1} R \hookrightarrow \mathbb{P}^{g-1}$$

By Proposition 3.5 we get

$$\mathrm{Gal}(\gamma_C) \cong S_2^{\oplus g-1} \rtimes S_{g-1}.$$

The Prym case is more intricate, since the associated monodromy group may fail to be large.

Setting 3.8. *Let $C'/\mathbb{C}$ be a smooth projective curve, $\eta \in \mathrm{Pic}(C')$ a line bundle, and B a reduced effective (possibly zero) divisor on C'. Suppose $k > 1$ and that there is an isomorphism $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. According to [3, §17], these data determine a cyclic k-fold covering $h : C \to C'$ of smooth projective curves, branched exactly along B. Denote by σ the generator of the Galois group $\mathrm{Gal}(C/C') \cong \mathbb{Z}/k\mathbb{Z}$.*

Recall that the Prym variety $A := \mathrm{Prym}(C/C')$ is defined as the connected component of the identity in

$$\ker \left[\mathrm{Nm} : \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C') \right]$$

Its dimension is denoted by n.

The Abel–Prym map is defined by

$$\mathrm{AP}_{C/C'} : C \longrightarrow A \qquad p \longmapsto \mathcal{O}_C(p - \sigma(p)).$$

The following example, which essentially restates [5, Corollary 2.2], illustrates the case where the cover is an étale double cover.

Proposition 3.9. *In Setting 3.8, assume further that $k = 2$, $B = \varnothing$, and that C' is non-hyperelliptic of genus $g(C') \geqslant 4$.*

- *If C' is bielliptic and $\eta = \mathrm{pr}^* \eta_0$, where $\mathrm{pr} : C' \to E'$ is a bielliptic map and $\eta_0 \in \mathrm{Pic}^0(E')[2]$, then the tangent Gauss map $\phi_C : C \to \mathbb{P}^{n-1}$ has degree 4, and the curve $C \subset A$ is invariant under a 2-torsion translation in A.*
- *Otherwise, the tangent Gauss map $\phi_C : C \to \mathbb{P}^{n-1}$ is a double cover onto its image, and $\mathrm{Gal}(\gamma_C)$ is big.*

Proof. When $k = 2$, the corresponding Gauss map γ_C factors through h:

$$\begin{aligned}\gamma_C : C \xrightarrow{\ h\ } C' \xrightarrow{|\omega_{C'}\otimes\eta|} \mathbb{P}^{n-1} &= \mathbb{P}\big(\mathrm{H}^0(\omega_{C'}\otimes\eta)^*\big)\\ &\cong \mathrm{Gr}\left(n-1, \mathrm{H}^0(\omega_{C'}\otimes\eta)\right)\\ \tilde{p} \longmapsto p \longmapsto\ & \mathrm{H}^0\left(\omega_{C'}\otimes\eta(-p)\right)\end{aligned}$$

where $|\omega_{C'}\otimes\eta|$ denotes the Prym–canonical map. According to [5, Corollary 2.2], one has $\deg|\omega_{C'}\otimes\eta| \in \{1,2\}$, and the case $\deg|\omega_{C'}\otimes\eta| = 2$ occurs precisely when C' is bielliptic and $\eta = \mathrm{pr}^*\eta_0$. $\square$

3.3. **Curves in Prym variety with small monodromy group(Delete later).** This subsection is devoted to presenting an example that has small monodromy group.

The classical theory treats the case where the Prym variety A is principally polarized—this occurs exactly when $B = \varnothing$ or $B = \{p_0, q_0\}$, that is, when h is either unramified or ramified at two points.[6] For clarity, we restrict our attention to these cases. Table 2 lists the relevant numerical data.[7]

	$g(C)$	$g(C')$	$\deg\eta$
$B = \varnothing$	$2n+1$	$n+1$	0
$B = \{p_0, q_0\}$	$2n$	n	1

TABLE 2. numerical data of Prym pair

The Abel–Prym map $\mathrm{AP}_{C/C'}$ does not always behave as nicely as the Abel–Jacobi map; it may fail to be an embedding. When C is hyperelliptic, the map $\mathrm{AP}_{C/C'}$ fails to be generically injective, and therefore falls outside the scope of our discussion. When C is non-hyperelliptic, we may regard it as a subvariety of A for our purposes, although it may be not strictly embedded.[8] However, it remains unclear whether C is stable under any translation in A.

Example 3.10. *Let C' be a non-hyperelliptic bielliptic curve, and let $\mathrm{pr} : C' \to E$ denote a $2:1$ covering onto an elliptic curve. For any nontrivial 2-torsion line bundle $\eta_0 \in \mathrm{Pic}^0(E)[2]$, the pullback $\eta := \mathrm{pr}^*\eta_0$ satisfies $\eta^{\otimes 2} \cong \mathcal{O}_{C'}$ and the map $|\omega_{C'}\otimes\eta|$ factors through pr.*

Proof of Example 3.10. For any $x \in C'$, write $x_0 := \mathrm{pr}(x)$, then

$$\deg \eta_0^\vee(x_0) = 1 \quad\Longrightarrow\quad \text{there exists } y_0 \in E \text{ such that } \eta_0^\vee(x_0) = \mathcal{O}_E(y_0).$$

Write $\mathrm{pr}^{-1}(x_0) = \{x, x'\}$, $\mathrm{pr}^{-1}(y_0) = \{y, y'\}$, we get

$$\begin{aligned} &\eta_0^\vee(x_0) = \mathcal{O}_E(y_0) \\ \Longrightarrow\ &\eta^\vee(x+x') = \mathcal{O}_{C'}(y+y') && \text{Via pulling back along pr}\\ \Longrightarrow\ &h^0\left(\eta^\vee(x+x')\right) = 1 \\ \Longleftrightarrow\ &h^0\left(\omega_{C'}\otimes\eta(-x-x')\right) = n-1 && \text{Via Riemann–Roch}\\ \Longleftrightarrow\ &\mathrm{H}^0\left(\omega_{C'}\otimes\eta(-x)\right) \cong \mathrm{H}^0\left(\omega_{C'}\otimes\eta(-x')\right)\\ \Longleftrightarrow\ &|\omega_{C'}\otimes\eta|(x) = |\omega_{C'}\otimes\eta|(x') \end{aligned}$$

$\square$

The next proposition tells us, if we put some Galois condition for the covering, then the resulting curve $C \subset A$ is not invariant under any non-trivial translation of A.

[6]See [6, Theorem 3.2.6] for the proof.

[7]As usual, $n = \dim_{\mathbb{C}} A$ is the dimension of the abelian variety.

[8]For a detailed description of the map $\mathrm{AP}_{C/C'}$, see [4, Proposition 12.5.2, Corollary 12.5.6]. Although $\mathrm{AP}_{C/C'}$ may collapse p_0 and q_0, this does not affect the relevant computation of $\mathrm{Gal}(\gamma_Z)$.

Proposition 3.11. *In Example 3.10, let C be the curve corresponding to $\eta \in \mathrm{Pic}^0(C')[2]$. If $\mathrm{Gal}(C/E) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, then C admits three intermediate quotients over E. Denote them by C', C_1, and C_2, where C_1 and C_2 are the two intermediate curves different from C'.*

$$\begin{tikzcd} C \arrow[d, "h"] \\ C' \arrow[d] \\ E \arrow[d] \\ \mathbb{P}^1 \end{tikzcd} \qquad\qquad \begin{tikzcd} & C \arrow[dl, "h_1"'] \arrow[d, "h"] \arrow[dr, "h_2"] & \\ C_1 \arrow[dr] & C' \arrow[d] & C_2 \arrow[dl] \\ & E \arrow[d] & \\ & \mathbb{P}^1 & \end{tikzcd}$$

$$\mathrm{Gal}(C/E) \cong \mathbb{Z}/4\mathbb{Z} \qquad\qquad \mathrm{Gal}(C/E) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

If either $\mathrm{Gal}(C/E) \cong \mathbb{Z}/4\mathbb{Z}$, or $\mathrm{Gal}(C/E) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ with the coverings $h_i : C \to C_i$ ramified, then the curve $C \subset A$ is not fixed under any non-trivial translation on A.

Proof. We prove by contradiction. Assume there exists a non-trivial translation $\sigma : A \to A$ preserving C. Then the restriction $\sigma|_C : C \to C$ is an automorphism, and the induced map

$$h_\sigma : C \longrightarrow C/\sigma$$

is an unramified double covering. Since σ is a translation, the Gauss map γ_C necessarily factors through h_σ:

$$\begin{tikzcd} \gamma_C : C \arrow[r, "2:1"] \arrow[dr, "h_\sigma"'] & C' \arrow[r, "2:1"] & E \subset \mathbb{P}^{n-1} \\ & C/\sigma \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Without loss of generality, we may assume that $\deg(h_\sigma) = 2$. By assumption, C/σ coincides with C', and $\sigma|_C$ is the involution ι associated with the cover $h : C \to C'$, which extends to an involution $\tilde{\iota}$ of A. According to [4, Proposition 12.4.2], $\tilde{\iota}$ is a reflection. Hence $\sigma \circ \tilde{\iota}^{-1}$ is another reflection fixing C, which is impossible.

□

Unfortunately, the condition in Proposition 3.11 is never met, and consequently the curve $C \subset A$ is preserved by a 2-torsion translation of A, as explained below.

Lemma 3.12. *In Example 3.10, let $(\mathcal{L}_3, \mathcal{L}_3^{\otimes 2} \cong \mathcal{O}_E(R))$ denote the line bundle associated with the double cover $C' \to E$, where R is its branch divisor. Set $\mathcal{L}_1 := \eta_0$, $\mathcal{L}_2 := \eta_0 \otimes \mathcal{L}_3$, and let $u_i \colon C_i \to E$ be the ramified covering determined by $\mathcal{L}_i$.*

(1) We have $C_3 = C'$, $g(C_1) = 1$ and $g(C_2) = g(C_3) = n + 1$.

(2) The curve C arises as the fiber product $C = C_3 \times_E C_1$. The covering $C \to E$ is Galois with
$$\mathrm{Gal}(C/E) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \mathrel{\hat{=}} \{\mathrm{Id}, \sigma_1, \sigma_2, \sigma_3\},$$
where each σ_i is the involution associated with the projection $h_i : C \to C_i$ (so that $\iota = \sigma_3$). By the Riemann–Hurwitz formula, the maps u_1, h_2, and h_3 are unramified.

(3) With $\sigma = \{\sigma_2, \sigma_3\} \in \mathrm{Gal}(C_1/E)$ the involution of u_1, one can define
$$\mathcal{L}_0 := \mathcal{O}_{C_1}\left(p_0 - \sigma(p_0)\right) \in \mathrm{Pic}^0(C_1)[2],$$
independent of the choice of $p_0 \in C_1$. The curve $C \subset A$ is invariant under the translation corresponding to $h_1^*\mathcal{L}_0$. Concretely, for every $p \in C$, one has
$$\mathcal{O}_C\left(p - \sigma_3(p)\right) \cong \mathcal{O}_C\left(\sigma_2(p) - \sigma_3\sigma_2(p)\right) \otimes h_1^*\mathcal{L}_0.$$

Proof.

(1) Since
$$\mathcal{L}_1^{\otimes 2} \cong \mathcal{O}_E, \qquad \mathcal{L}_2^{\otimes 2} \cong \mathcal{O}_E(R), \qquad \mathcal{L}_3^{\otimes 2} \cong \mathcal{O}_E(R),$$
he Riemann–Hurwitz formula yields the genus of C_i.
(2) Notice that the curve C corresponds to the line bundle $\eta := \mathrm{pr}^* \eta_0$.
(3) Since u_1 is unramified, the involution $\sigma \in \mathrm{Aut}(C_1/E)$ acts by translation and thus preserves the Jacobian:
$$\mathcal{O}_{C_1}(p_0 - q_0) \cong \mathcal{O}_{C_1}(\sigma(p_0 - q_0)), \qquad \text{for any } p_0, q_0 \in C_1.$$
For $p \in C$, write $p_0 = h_1(p)$, one can computes that
$$\begin{aligned} h_1^* \mathcal{L}_0 &\cong h_1^* \mathcal{O}_{C_1}\big(p_0 - \sigma(p_0)\big) \\ &\cong \mathcal{O}_C\big(p + \sigma_1(p) - \sigma_2(p) - \sigma_3(p)\big) \\ &\cong \mathcal{O}_C\big(p - \sigma_3(p)\big) \otimes \mathcal{O}_C\big(-\sigma_2(p) + \sigma_3\sigma_2(p)\big) \end{aligned}$$

□

Lemma 3.13. *When* $\mathrm{gon}(C') > 4$, $|\omega_{C'} \otimes \eta|$ *is injective. As a result, the monodromy group is big.*

Proof. Suppose that $|\omega_{C'} \otimes \eta|$ is not injective, we need to find a line bundle of degree 4 and rank $\geqslant 1$. In fact, for $p \neq q$,
$$\begin{aligned} & |\omega_{C'} \otimes \eta|(p) = |\omega_{C'} \otimes \eta|(q) && \\ \iff & h^0(\omega_{C'} \otimes \eta) - h^0(\omega_{C'} \otimes \eta(-p-q)) = 1 && \text{By [15, 19.2.8]} \\ \iff & h^0(\omega_{C'} \otimes \eta(-p-q)) = n-1 && \text{Since } \dim_{\mathbb{C}} A = n \\ \iff & h^0(\eta^\vee(p+q)) = 1 && \text{Via Riemann–Roch} \end{aligned}$$
When $B = \varnothing$, write $\eta^\vee(p+q) = \mathcal{O}_{C'}(p' + q')$, then
$$\mathcal{O}_{C'}(2p+2q) = (\eta^\vee)^{\otimes 2}(2p+2q) = \mathcal{O}_{C'}(2p'+2q') \qquad \in g_4^1;$$
When $B = \{p_0, q_0\}$, write $\eta^\vee(p+q) = \mathcal{O}_{C'}(p')$, then
$$\mathcal{O}_{C'}(2p+2q) = (\eta^\vee)^{\otimes 2}(2p+2q+p_0+q_0) = \mathcal{O}_{C'}(2p'+p_0+q_0) \quad \in g_4^1.$$

□

Remark 3.14. Based on the strategy used in the proof of Lemma 2.10, we can in fact obtain stronger results. For $p \in C'$,
$$\begin{aligned} & |\omega_{C'} \otimes \eta| \text{ is ramified at } p && \text{(i.e., the tangent map is 0 at } p) \\ \iff & h^0(\omega_{C'} \otimes \eta) - h^0(\omega_{C'} \otimes \eta(-2p)) = 1 && \text{By [15, 19.2.9]} \\ \iff & h^0(\omega_{C'} \otimes \eta(-2p)) = n-1 && \text{Since } \dim_{\mathbb{C}} A = n \\ \iff & h^0(\eta^\vee(2p)) = 1 && \text{Via Riemann–Roch} \end{aligned}$$
For the remainder of this discussion, we focus exclusively on the case $B = \varnothing$. Combining both,
$$\begin{aligned} & |\omega_{C'} \otimes \eta| \text{ is not generically injective} \\ \iff & \text{For any } p \in C', \text{ there exist } q \in C' \text{ such that } h^0(\eta^\vee(p+q)) = 1 \\ \iff & \text{For any } p \in C', \text{ there exist } q \in C' \text{ such that } \eta^\vee + p \in C' + C' - q \\ \iff & \eta^\vee + C' \subseteq C' + C' - C'. \end{aligned}$$

Gonality is not the only invariant forcing the monodromy group to be large. Indeed, the Castelnuovo–Severi inequality implies that Example 3.10 is the sole instance of a non-trivial small monodromy group when $g(C') > 9$.

Fact 3.15 (Castelnuovo–Severi inequality, [9, p26, Corollary])**.** *Let C be a smooth projective curve equipped with two ramified coverings $f_i : C \to C_i$ of degrees d_i $(i = 1, 2)$. Suppose that there is no morphism $h : C \to \tilde{C}$ with $\deg(h) > 1$ such that both f_1 and f_2 factor through h. Then*

$$g(C) \leqslant d_1 \cdot g(C_1) + d_2 \cdot g(C_2) + (d_1 - 1)(d_2 - 1).$$

Proposition 3.16. *If, in Setting 3.8, we additionally require that C' be non-hyperelliptic and non-bielliptic, and $h : C \longrightarrow C'$ is étale, then any such curve with $g(C') > 9$ must have big monodromy group.*

Proof. Assume that C' and η satisfies

$$\eta^\vee + C' \subseteq C' + C' - C'.$$

Step 1. we can find two distinct g^1_4 of C'.

For any $p \in C'$, there exist $q, p', q' \in C'$ such that

$$\eta^\vee(p + q) = \mathcal{O}_{C'}(p' + q').$$

This implies

$$\mathcal{O}_{C'}(2p + 2q) = \mathcal{O}_{C'}(2p' + 2q'),$$

which defines a degree-4 covering $f_1 \colon C' \to \mathbb{P}^1$ ramified at p, q, p', q'. Choosing $\tilde{p} \in C'$ outside the ramification locus of f_1 and repeating the construction yields another g^1_4, denoted $f_2 \colon C' \to \mathbb{P}^1$.

Step 2. If f_1 and f_2 factor through a common map $h \colon C' \to \tilde{C}$ with $\deg(h) > 1$, then C' is hyperelliptic or bielliptic.

Indeed, in this case $\deg(h) = 2$, and the Castelnuovo–Severi inequality applied to $(\tilde{C}, \tilde{f}_1, \tilde{f}_2)$ gives $g(\tilde{C}) \leqslant 1$, so $\tilde{C}$ is either $\mathbb{P}^1$ or an elliptic curve.

$$\begin{tikzcd} & C \arrow[d, "h"'] \arrow[ddl, bend right, "f_1"'] \arrow[ddr, bend left, "f_2"] & \\ & \tilde{C} \arrow[dl, "\tilde{f}_1"] \arrow[dr, "\tilde{f}_2"'] & \\ \mathbb{P}^1 & & \mathbb{P}^1 \end{tikzcd}$$

Step 3. If C' is neither hyperelliptic nor bielliptic, then the Castelnuovo–Severi inequality yields $g(C') \leqslant 9$. □

The following lemma tells us that all bielliptic curve case with small monodromy group are contained in Example 3.10, when $g(C') > 9$.

Lemma 3.17. *Let C' be a non-hyperelliptic bielliptic curve with $g(C') > 9$, and let $\mathrm{pr} : C' \to E$ be the double covering onto an elliptic curve. Assume that there exists a nontrivial 2-torsion line bundle $\eta \in \mathrm{Pic}^0(C')[2]$ such that the linear system $|\omega_{C'} \otimes \eta|$ fails to be generically injective. Then η arises as the pullback of some $\eta_0 \in \mathrm{Pic}^0(E)[2]$, i.e. $\eta = \mathrm{pr}^* \eta_0$.*

Proof. When $|\omega_{C'} \otimes \eta|$ is not generically injective, one has

$$\eta^\vee + C' \subseteq C' + C' - C'.$$

Let R' denote the ramification locus of pr, which is a finite subset of C'. Since C' is non-hyperelliptic, the set

$$U := \{p \in C' \mid p \notin R', \eta^\vee + p \notin R' + R' - C'\}$$

is a non-empty open subset. Consequently, there exist points $p, p' \in C' \setminus R'$ and $q, q' \in C'$ with

$$\eta^\vee(p + q) = \mathcal{O}_{C'}(p' + q').$$

Following the same strategy as in Proposition 3.16, the map $f : C' \to \mathbb{P}^1$ of degree 4, corresponding to the linear system $g^1_4 = \mathcal{O}_{C'}(2p+2q) = \mathcal{O}_{C'}(2p'+2q')$, must factor through pr.[9] Let $\iota \in \mathrm{Gal}(C'/E)$ be the involution associated with pr. Since p and $\iota(p)$ are contained in the same fiber of f, we obtain $q = \iota(p)$. Similarly, one has $q' = \iota(p')$. It follows that

$$\begin{aligned}\eta^\vee &\cong \mathcal{O}_{C'}(p' + q' - p - q)\\ &\cong \mathcal{O}_{C'}\big(p' + \iota(p') - p - \iota(p)\big)\\ &\cong \mathrm{pr}^* \mathcal{O}_E(p' - p).\end{aligned}$$

□

4. Families of subvarieties

In this section, we move from the study of monodromy groups to a more direct analysis of the subvarieties themselves. Given an initial subvariety, one can naturally generate a family of subvarieties. Our goal here is to define these families and investigate their properties.

4.1. Clean Lagrangian cycles.

Proposition 4.1. *All irreducible conic Lagrangian cycles in T^*A are of the form Λ_Z for some irreducible subvariety $Z \subset A$. This yields a one-to-one correspondence between irreducible conic Lagrangian cycles in T^*A and irreducible subvarieties of A:*

$$\{\textit{ irreducible conic Lagrangian cycles in } T^*A \} \longleftrightarrow \{\textit{ irreducible subvarieties in } A \}$$

Sketch of proof. For any irreducible conic Lagrangian cycle $\mathbf{\Lambda} \subset T^*A$, let Z denote the image of $\mathbf{\Lambda}$ under the natural projection $T^*A \to A$. Our goal is to show that $\mathbf{\Lambda} = \Lambda_Z$.

- By definition, $\mathbf{\Lambda} \subset T^*A|_Z$.
- Since $\mathbf{\Lambda}$ is conic, we have $s(Z) \subset \mathbf{\Lambda}$, where $s : A \to T^*A$ denotes the zero section.
- Since $\mathbf{\Lambda}$ is Lagrangian and $s(Z) \subset \mathbf{\Lambda}$, we have $\mathbf{\Lambda} \subset \Lambda_Z$.
- Since Λ_Z is irreducible with $\dim_{\mathbb{C}} \mathbf{\Lambda} = \dim_{\mathbb{C}} \Lambda_Z = n$, we have $\mathbf{\Lambda} = \Lambda_Z$.

□

Why do we shift attention from Z to Λ_Z as the main object of study? One reason is the uniformity of Λ_Z: it always has dimension n, and in most cases, the natural map $\Lambda_Z \to T^*_0 A$ is generically finite, with fibers lying inside A.

Definition 4.2 (Clean cycle)**.** *An irreducible Lagrangian cycle $\mathbf{\Lambda} \subset T^*A$ is called clean if the composed projection*

$$\mathbf{\Lambda} \longrightarrow T^*A \twoheadrightarrow T^*_0 A$$

is generically finite.

Another important reason is that the space of weighted clean conic Lagrangian cycles naturally acquires a convolution structure, arising from the group law on A, which plays a central role in the analysis.

Proposition 4.3. *The group of weighted clean conic Lagrangian cycles*

$$\begin{aligned}\mathcal{L}(A) &:= \{\textit{weighted clean conic Lagrangian cycles in } T^*A\}\\ &= \left\{ \sum_{\substack{Z_i \subset A\\ \textit{irr clean}}} n_i \Lambda_{Z_i} \;\middle|\; n_i \in \mathbb{Z} \right\}\end{aligned}$$

has a natural convolution structure as follows:

$$\begin{aligned}\Lambda_{Z_1} \circ \Lambda_{Z_2} &= \textit{ the clean part of } (a, \mathrm{Id}_{T^*_0 A})_* \left(\Lambda_{Z_1} \times_{T^*_0 A} \Lambda_{Z_2}\right)\\ &= \overline{(a, \mathrm{Id}_U)_* \left(\Lambda_{Z_1}|_U \times_U \Lambda_{Z_2}|_U\right)}\end{aligned}$$

[9] For $g(C') > 5$, Castelnuovo–Severi implies that C' admits exactly one ramified double cover onto an elliptic curve.

where

$$U := \left\{ \xi \in T_0^* A \;\middle|\; \deg \phi_{Z_i} = \# \phi_{Z_i}^{-1}(\xi) \text{ for } i = 1, 2 \right\}$$

and $a : A \times A \longrightarrow A$ is the addition map in A. The general convolution is defined by $\mathbb{Z}$-linear extension.

Sketch of proof. To establish the claim, it suffices to show that $\Lambda_{Z_1} \circ \Lambda_{Z_2}$ defines a weighted conic Lagrangian cycle. The conic property follows directly from the definition, while the Lagrangian condition can be verified at a general point $(p_1 + p_2, \xi) \in \Lambda_{Z_1} \circ \Lambda_{Z_2}$. □

We now consider the projective versions of all objects involved, so that we may make use of properness. To simplify notation, we abbreviate $\mathbb{P}T_0^* A$ by $\mathbb{P}^\vee$.

Lemma 4.4.

(1) Suppose that $\mathbb{P}\Lambda_{Z_1}, \mathbb{P}\Lambda_{Z_1} \subset \mathbb{P}T^* A$ admit monodromy representations
$$\rho_{\gamma_{Z_i}} : \pi_1(U, \xi_0) \longrightarrow \operatorname{Aut}\left(\gamma_{Z_i}^{-1}(\xi_0)\right),$$
then $\mathbb{P}\Lambda_{Z_1} \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_{Z_2} \subset A \times A \times \mathbb{P}^\vee$ admits monodromy representation given by
$$(\rho_{\gamma_{Z_1}}, \rho_{\gamma_{Z_2}}) : \pi_1(U, \xi_0) \longrightarrow \operatorname{Aut}\left(\gamma_{Z_1}^{-1}(\xi_0) \times \gamma_{Z_2}^{-1}(\xi_0)\right),$$

(2) When $Z_1 = Z_2 = Z$, we obtain an one-to-one correspondence:
$$\left\{ \begin{matrix} \text{irr components of } \mathbb{P}\Lambda_Z \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_Z \\ \text{with a surjection to } \mathbb{P}^\vee \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \operatorname{Gal}(\gamma_Z)\text{-orbits of} \\ \gamma_Z^{-1}(\xi_0) \times \gamma_Z^{-1}(\xi_0) \end{matrix} \right\}$$

Sketch of proof. Statement (1) holds by definition. The proof of (2) reduces to the following purely topological statement:

Claim 4.5. *Let $\pi : E \longrightarrow B$ be a (unramified) covering space over a manifold B with deck transformation group G, then*

$$\{ \text{ connected components of } E \times_B E \ \} \longleftrightarrow \{ \ G\text{-orbits of } \pi^{-1}(b_0) \times \pi^{-1}(b_0) \ \}.$$

The claim follows directly from the correspondence between covering spaces over B and $\pi_1(B)$-sets; see [7, Theorem 1.38]. □

Generalizing the argument of Lemma 4.4, we arrive at the following lemma.

Lemma 4.6. *For $d = \deg \gamma_Z$, $\xi_0 \in T_0^* A$ a general point, write*

$$\begin{aligned} \mathbb{P}\Lambda_Z^{\times d} &:= \mathbb{P}\Lambda_Z \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_Z \times_{\mathbb{P}^\vee} \cdots \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_Z && \subset A^d \times \mathbb{P}^\vee \\ \gamma_Z^{-1}(\xi_0)^d &:= \gamma_Z^{-1}(\xi_0) \times \gamma_Z^{-1}(\xi_0) \times \cdots \times \gamma_Z^{-1}(\xi_0) && \subset A^d \end{aligned}$$

(1) we obtain an one-to-one correspondence:
$$\left\{ \begin{matrix} \text{irr components of } \mathbb{P}\Lambda_Z^{\times d} \\ \text{with a surjection to } \mathbb{P}^\vee \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \operatorname{Gal}(\gamma_Z)\text{-orbits of} \\ \gamma_Z^{-1}(\xi_0)^d \end{matrix} \right\}$$

(2) Write
$$\begin{aligned} \Delta_d &:= \left\{ (p_1, \ldots, p_d) \in A^d \;\middle|\; p_i = p_j \text{ for some } i \neq j \right\} && \subset A^d \\ \mathbb{P}\Lambda_Z^{[d]} &:= \overline{\left(\mathbb{P}\Lambda_Z^{\times d} \smallsetminus (\Delta_d \times \mathbb{P}^\vee)\right)|_U} && \subset A^d \times \mathbb{P}^\vee \end{aligned}$$
Fix a general point $\xi_0 \in \mathbb{P}^\vee$ and a well-order for $\gamma_Z^{-1}(\xi_0)$, one can identify $S_d \cong \gamma_Z^{-1}(\xi_0)^d \smallsetminus \Delta_d$, and
$$\left\{ \text{ irr components of } \mathbb{P}\Lambda_Z^{[d]} \ \right\} \longleftrightarrow \left\{ \ \operatorname{Gal}(\gamma_Z)\text{-orbits of } S_d \ \right\}$$
In reference, Δ_d is usually called the big diagonal.

From this point on, we fix an irreducible component of $\mathbb{P}\Lambda_Z^{[d]}$, denoted by $\mathbb{P}\Lambda_Z^{\mathrm{univ}}$. As we will see in Definition 4.7, this variety generates all subvarieties within the families under consideration.

Definition 4.7 (The subvariety $Z^{(m)}$)**.** *For any tuple $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$, we define the weighted sum map*

$$a^{(m)} : A^d \longrightarrow A \qquad (p_1, \dots, p_d) \longmapsto \sum_{i=1}^{d} m_i p_i.$$

We also define

$$\mathbb{P}\Lambda_Z^{(m)} := \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* \mathbb{P}\Lambda_Z^{\mathrm{univ}}.$$

as the (projectivized) weighted Lagrangian cycle in $\mathbb{P}T^*A$. The projective cycle $\mathbb{P}\Lambda_Z^{(m)}$ is irreducible but may appear with multiplicities. We can therefore write

$$\mathbb{P}\Lambda_Z^{(m)} = c_Z^{(m)} \mathbb{P}\Lambda_{Z^{(m)}}$$

where $c_Z^{(m)} \in \mathbb{Z}_{>0}$ and $Z^{(m)} \subset A$ are uniquely determined. This gives rise to a family of subvarieties parametrized by $\mathbb{Z}^d$.

The next lemma gathers some basic properties of $Z^{(m)}$. Observe that $S_d = \mathrm{Aut}\left(\gamma_Z^{-1}(\xi_0)\right)$ acts naturally on $\mathbb{Z}^d$ via

$$g(m) = \left(m_{g(1)}, \dots, m_{g(d)}\right) \in \mathbb{Z}^d.$$

Lemma 4.8.

(1) For all $g \in \mathrm{Gal}(\gamma_Z)$, we have $Z^{g(m)} = Z^{(m)}$, $c_Z^{g(m)} = c_Z^{(m)}$;
(2) For $(m) = (1, 0, \dots, 0) \in \mathbb{Z}^d$, $Z^{(m)} = Z$;
(3) For all $(m), (m') \in \mathbb{Z}^d$, we have $\mathbb{P}\Lambda_Z^{(m)} \circ \mathbb{P}\Lambda_Z^{(m')} \supseteq \mathbb{P}\Lambda_Z^{(m+m')}$;
(4) The group $\langle \mathbb{P}\Lambda_{Z^{(m)}} \rangle_{\mathrm{Abel}}$ is closed under the convolution product.

4.2. **Realized as characteristic cycles.** In fact, the Lagrangian cycles $\mathbb{P}\Lambda_{Z^{(m)}}$ coincide with the irreducible components of the clean cycles described in [11, 2.c] and [13, p5, Theorem 1.7], leading to the following relations:

$$\begin{tikzcd}
\mathrm{Perv}(A)/N(A) & \supset & \langle \delta_Z \rangle \arrow[d, "\mathrm{cc}"] & \cong & \mathrm{Rep}\left(G_u\right) \arrow[d] \\
\mathcal{L}(A) & \supset & \langle \mathrm{cc}(\delta_Z) \rangle & \overset{10}{\cong} & \mathrm{Rep}\left(T_u \rtimes \mathrm{Gal}(\gamma_Z)\right)
\end{tikzcd}$$

Here, δ_Z denotes the perverse intersection complex associated with the subvariety Z (in particular, $\delta_Z = \iota_{Z,*}\underline{\mathbb{Q}}_Z[-\dim Z]$ when Z is smooth), and $\mathcal{L}(A)$ stands for the λ-ring of clean conic Lagrangian cycles on T^*A [12, p5].

Remark 4.9. Suppose that $Z \subset A$ is smooth of general type. Then the characteristic cycle $\mathrm{cc}(\delta_Z)$ is irreducible and equals $\mathbb{P}\Lambda_Z$. In this situation, let $\lambda_1, \dots, \lambda_d \subset X^*(T_u)$ denote the weights corresponding to the points $p_1, \dots, p_d \in \gamma_Z^{-1}(\xi_0)$. For each tuple $(m) \in \mathbb{Z}^d$, define $\lambda^{(m)} = \sum m_i \lambda_i \in X^*(T_u)$, and consider the "highest weight representation"

$$V_{\lambda^{(m)}} = \bigoplus_{\lambda \in \mathrm{Gal}(\gamma_Z)\cdot\lambda^{(m)}} \mathbb{C}_\lambda \in \mathrm{Rep}\left(T_u \rtimes \mathrm{Gal}(\gamma_Z)\right)$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of T_u with weight λ. Under this correspondence, one has an explicit identification

$$\langle \mathbb{P}\Lambda_Z \rangle \cong \mathrm{Rep}\left(T_u \rtimes \mathrm{Gal}(\gamma_Z)\right) \qquad \mathbb{P}\Lambda_{Z^{(m)}} \longleftrightarrow V_{\lambda^{(m)}}.$$

[10] I believe that this isomorphism should be already known, so I should probably cite it somewhere(rather than making it up all by myself).

Moreover, the Weyl group W_Z acts naturally on $X^(T)$. For the orbit $W_Z \cdot \chi_0 \subset X^(T)$, the associated Lagrangian cycle is given by

$$\sum_{\sigma \in W_Z / \operatorname{Gal}(\gamma_Z)} \mathbb{P}\Lambda_{Z^{\sigma(\chi_0)}}.$$

5. Dimension and homology class

In this section, all cohomology groups are taken with $\mathbb{Q}$-coefficients for convenience in applying the Künneth formula.

5.1. **Reminder on the (homological) Chern–Mather class.** We begin by recalling the definition of the Chern–Mather class. Suppose $\dim A = n$, and denote by

$$p : \mathbb{P}T^*A \cong A \otimes \mathbb{P}^\vee \longrightarrow \mathbb{P}^\vee$$

the natural projection.

Definition 5.1. *For a conic Lagrangian cycle Λ on T^*A and $i \geqslant 0$, the Chern–Mather class is defined as*

$$c_{M,i}(\Lambda) = p_*\left([\mathbb{P}\Lambda] \cdot [A \times H_i]\right) \in \mathrm{H}_{2i}(A) \cong \mathrm{H}^{2(n-i)}(A),$$

where $H_i \subseteq \mathbb{P}^\vee$ denotes a general linear subspace of dimension i. For brevity, we may later write $c_{M,i}(\Lambda)$ as $c_i(\Lambda)$, and $c_{M,i}(\Lambda_Z)$ simply as c_i. [11]

By the Künneth formula, we may write

$$[\mathbb{P}\Lambda] = \sum_{i=0}^{n-1} a_i \otimes H^i,$$

where $H \in H^2(\mathbb{P}^\vee)$ denotes the hyperplane class and $a_i \in H^{2(n-i)}(A)$. A direct computation gives

$$\begin{aligned} c_i(\Lambda) &= p_*\left([\mathbb{P}\Lambda] \cdot [A \times H_i]\right) \\ &= p_*\left(\sum_{j=0}^{n-1} \left(a_j \otimes H^j\right) \cup \left(1 \otimes H^{n-1-i}\right)\right) \\ &= p_*\left(\sum_{j=0}^{n-1} a_j \otimes H^{n-1-i+j}\right) \\ &= a_i. \end{aligned}$$

Consequently,

$$[\mathbb{P}\Lambda] = \sum_{i=0}^{n-1} c_i(\Lambda) \otimes H^i \in \mathrm{H}^{2n}(A \times \mathbb{P}^\vee),$$

showing that the Chern–Mather classes $c_i(\Lambda)$ are precisely the coefficients of the class $[\mathbb{P}\Lambda]$ in the Künneth decomposition. [12]

Remark 5.2. For a subvariety $Z \subset A$, both its dimension $\dim Z$ and its cohomology class $[Z] \in \mathrm{H}^{2(n-\dim Z)}(A)$ can be determined from $[\mathbb{P}\Lambda] \in \mathrm{H}^{2n}(A \times \mathbb{P}^\vee)$, as shown in [12, Lemma 3.1.2(2)]. Indeed,

$$\begin{aligned} \dim Z &= \max\left\{i \in \mathbb{Z} \mid c_i \neq 0\right\}, \\ [Z] &= c_{\dim Z}. \end{aligned}$$

[11] Note that although c_i arises as the pushforward of the classical Chern class in the smooth case, the indexing order is reversed. In particular, $c_0 = d$, rather than 1.

[12] If the Chern–Mather classes are considered in the Chow ring, this argument does not apply, since the Künneth decomposition is unavailable at the level of Chow groups.

5.2. **The homology class of $\mathbb{P}\Lambda_Z^{\times d}$.** By Remark 5.2, it suffices to consider the homology class of $\mathbb{P}\Lambda_{Z^{(m)}}$. If we are not concerned with the scalar factor $c_Z^{(m)}$, we may equivalently compute $[\mathbb{P}\Lambda_Z^{(m)}]$, which by definition is the pushforward of $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}]$. Consequently, the problem reduces to determining $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}] \in \mathrm{H}^{2dn}(A^d \times \mathbb{P}^\vee)$.

Typically, for an initial subvariety $Z \subset A$, the Chern–Mather classes are known. Our ultimate goal is to express $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}]$ in terms of these classes; as a preparatory step, we first examine $[\mathbb{P}\Lambda_Z^{\times d}]$.

By definition,

$$\mathbb{P}\Lambda_Z^{\times d} = \underbrace{\mathbb{P}\Lambda_Z \times_{\mathbb{P}^\vee} \cdots \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_Z}_{d \text{ factors}} = \bigcap_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{-1}(\mathbb{P}\Lambda_Z),$$

where $\pi_{i,\mathbb{P}^\vee} : A^d \times \mathbb{P}^\vee \to A \times \mathbb{P}^\vee$ denotes the projection onto the i-th factor. The transversality of these intersections is immediate, so

$$\begin{aligned}
[\mathbb{P}\Lambda_Z^{\times d}] &= \cup_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{*}[\mathbb{P}\Lambda_Z] \\
&= \cup_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{*}\left(\sum_{j=0}^{n-1} c_j \otimes H^j\right) \\
&= \cup_{i=1}^{d} \sum_{j=0}^{n-1}\left(1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{-th}}}{c_j} \otimes \cdots \otimes H^j\right) \\
&= \sum_{j=0}^{n-1}\left(\sum_{\sum_{k=1}^{d} j_k = j} c_{j_1} \otimes \cdots \otimes c_{j_d}\right) \otimes H^j.
\end{aligned}$$

5.3. **The homology class of $\mathbb{P}\Lambda_Z^{[d]}$.** This subsection explains how to eliminate the contribution of the big diagonal and how to compute $\mathbb{P}\Lambda_Z^{[d]}$ from $\mathbb{P}\Lambda_Z^{\times d}$. For this purpose, we introduce some combinatorial preliminaries.

Definition 5.3. *For $n \in \mathbb{N}_{>0}$, let $[n] := \{1, \ldots, n\}$, and denote by $\mathcal{P}(n)$ the lattice of partitions of $[n]$, ordered by refinement: $\alpha' \leqslant \alpha$ if and only if any two elements i, j belonging to the same block of α' also belong to the same block of α. For a partition $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}(d)$, we associate a surjective map*

$$f_\alpha : [d] \longrightarrow [k] \qquad a \longmapsto j \quad \text{if } a \in A_j$$

which is well-defined up to the natural S_k-action; this indeterminacy will not affect our discussion. Each map f_α naturally gives rise to a partial diagonal embedding

$$\Delta_\alpha : A^k \times \mathbb{P}^\vee \longrightarrow A^d \times \mathbb{P}^\vee \qquad \big((p_i), \xi\big) \longmapsto \big((p_{f_\alpha(i)}), \xi\big).$$

This construction determines a subvariety of $\mathbb{P}\Lambda_Z^{\times d}$, defined by

$$\begin{aligned}
\mathbb{P}\Lambda_Z^{\geqslant \alpha} &:= \Delta_\alpha\left(\mathbb{P}\Lambda_Z^{\times k}\right) \\
&= \left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{\times d} \;\middle|\; p_i = p_j \text{ if } i \sim_\alpha j\right\}.
\end{aligned}$$

We can in addition define the locus corresponding precisely to α:

$$\mathbb{P}\Lambda_Z^{\alpha} := \overline{\left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{\times d} \;\middle|\; p_i = p_j \text{ iff } i \sim_\alpha j\right\}}.$$

Remark 5.4. Denote by

$$\hat{0} := \big\{\{1\}, \ldots, \{d\}\big\} \in \mathcal{P}(d)$$

the finest partition. Then $\mathbb{P}\Lambda_Z^{\times d} = \mathbb{P}\Lambda_Z^{\geqslant \hat{0}}$, and $\mathbb{P}\Lambda_Z^{[d]} = \mathbb{P}\Lambda_Z^{\hat{0}}$.

By definition,

$$[\mathbb{P}\Lambda_Z^{\geqslant\alpha}] = \sum_{\alpha'\geqslant\alpha} [\mathbb{P}\Lambda_Z^{\alpha'}].$$

Applying Möbius inversion on the partition lattice yields

$$[\mathbb{P}\Lambda_Z^{\alpha}] = \sum_{\alpha'\geqslant\alpha} \mu(\alpha,\alpha')\,[\mathbb{P}\Lambda_Z^{\geqslant\alpha'}],$$

where $\mu(\alpha,\alpha')$ denotes the Möbius function as defined in [1, p141].

Fact 5.5 (See [1, IV.3])**.** *Let $\alpha' \geqslant \alpha$ be two partitions of $[d]$, where $\alpha' = \{A_1,\dots,A_k\}$. For each i, denote by r_i the number of blocks of α that are contained in A_i. Then*

$$\mu(\alpha,\alpha') = (-1)^{|\alpha|-k}\prod_{i=1}^{k}(r_i-1)!$$

In particular,

$$\mu(\hat{0},\alpha') = (-1)^{d-k}\prod_{i=1}^{k}\big(|A_i|-1\big)!$$

With Fact 5.5, one can compute $[\mathbb{P}\Lambda_Z^{[d]}] = [\mathbb{P}\Lambda_Z^{\hat{0}}]$ by the Möbius inverse formula:

$$\begin{aligned}[\mathbb{P}\Lambda_Z^{[d]}] &= \sum_{\alpha'} \mu(\hat{0},\alpha')\,[\mathbb{P}\Lambda_Z^{\geqslant\alpha'}]\\ &= \sum_{k=1}^{d}\sum_{\alpha'=\{A_1,\dots,A_k\}}(-1)^{d-k}\prod_{i=1}^{k}\big(|A_i|-1\big)!\ \cdot\Delta_{\alpha,*}[\mathbb{P}\Lambda_Z^{\times k}]\end{aligned}$$

The computation of pushforwards along diagonal embeddings can be rather cumbersome. However, upon composing with $a^{(m)}$, it suffices to compute those of certain weighted sum maps. These pushforwards admit a more transparent description via the Pontryagin product, whose definition we now recall.

Definition 5.6 (Pontryagin product)**.** *Let A be an abelian variety, and denote by $a : A\times A\to A$ the addition map. The Pontryagin product on A is defined by*

$$\mathrm{H}^{2n-i}(A)\times\mathrm{H}^{2n-j}(A)\subseteq\mathrm{H}^{4n-i-j}(A\times A)\xrightarrow{a_*}\mathrm{H}^{2n-i-j}(A)\qquad a\otimes b\longmapsto a*b.$$

Remark 5.7. The Pontryagin product is unital and associative, but in general only anti-commutative [4, 1.5.(7) b)]. In the context of our work, we are concerned with Chern–Mather classes, which have even degrees; consequently, the anti-commutativity of the Pontryagin product does not pose any complications.

In particular, for any sum map $a : A^k \longrightarrow A$, one can unambiguously write

$$a_*(a_1\otimes\cdots\otimes a_k) = a_1*\cdots*a_k \mathrel{\hat{=}} \mathop{\ast}_{i=1}^{k} a_i$$

where no additional parentheses are required.

Lemma 5.8. *For any tuple $(m) = (m_1,\dots,m_k)\in\mathbb{Z}^k$, $a_i\in\mathrm{H}^{2(n-l_i)}(A)$,*

$$a_*^{(m)}(a_1\otimes\cdots\otimes a_k) = \left(\prod_{i=1}^{k} m_i^{2l_i}\right)\cdot\mathop{\ast}_{i=1}^{k} a_i.$$

Proof. Notice that $a^{(m)}$ can be written as compositions of basic functions:

$$a^{(m)} : A^k \xrightarrow{(m_1,\dots,m_k)} A^k \xrightarrow{\quad a\quad} A$$

□

Definition 5.9. *For any tuple* $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$ *and any partition* $\alpha = \{A_1, \dots, A_k\} \in \mathcal{P}(d)$*, we define*

$$\alpha(m) := \left(\sum_{i \in A_1} m_i, \dots, \sum_{i \in A_k} m_i \right) \in \mathbb{Z}^k.$$

Moreover, for $\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{N}_{\geqslant 0}^k$ *with* $\sum l_i = l$*, we set*

$$\alpha(m)^{2\boldsymbol{l}} := \left(\sum_{i \in A_1} m_i \right)^{2l_1} \cdots \left(\sum_{i \in A_k} m_i \right)^{2l_k}$$

which defines a homogeneous polynomial of degree $2l$ *in* $\mathbb{Z}[m_1, \dots, m_k]$*.*

We can now determine $[\mathbb{P}\Lambda_Z^{(m)}]$ in the case where the monodromy group is S_d:

$$\begin{aligned}
[\mathbb{P}\Lambda_Z^{(m)}] &= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{\mathrm{univ}}] \\
&= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{[d]}] \\
&= \sum_\alpha \mu(\hat{0}, \alpha) \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* \Delta_{\alpha,*} [\mathbb{P}\Lambda_Z^{\times k}] \\
&= \sum_\alpha \mu(\hat{0}, \alpha) \left(a^{\alpha(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{\times k}] \\
&= \sum_\alpha \mu(\hat{0}, \alpha) \left(a^{\alpha(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* \left(\sum_{l=0}^{n-1} \left(\sum_{\sum l_i = l} c_{l_1} \otimes \cdots \otimes c_{l_k} \right) \otimes H^l \right) \\
&= \sum_\alpha \mu(\hat{0}, \alpha) \sum_{l=0}^{n-1} \sum_{\sum l_i = l} \alpha(m)^{2\boldsymbol{l}} \left(\mathop{\ast}_{i=1}^{k} c_{l_i} \otimes H^l \right) \\
&= \sum_{l=0}^{n-1} \left(\sum_\alpha \sum_{\sum l_i = l} \left(\mu(\hat{0}, \alpha) \alpha(m)^{2\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \right) \otimes H^l
\end{aligned}$$

Therefore,

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_\alpha \sum_{\sum l_i = l} \left(\mu(\hat{0}, \alpha) \alpha(m)^{2\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \tag{5.1}$$

The following corollary collects some quantitative implications of (5.1).

Corollary 5.10. *Assume that* $\mathrm{Gal}(\gamma_Z) = S_d$*.*

(1) The Chern–Mather classes

$$c_l\left(\Lambda_{Z^{(m)}}\right) \in \mathrm{H}^{2(n-l)}(A)[m_1, \dots, m_d]^{S_d}$$

are polynomials of degree $2l$ with exponent at most $2r$, when expressed in the variables $m_1, \dots, m_d$.

(2) If $Z \subset A$ is a smooth curve, there exists a homogeneous symmetric polynomial $f_{Z,l} \in \mathbb{Q}[m_1, \dots, m_d]^{S_d}$ of degree $2l$ and exponent at most 2, such that

$$c_l\left(\Lambda_{Z^{(m)}}\right) = f_{Z,l}(m_1, \dots, m_d) \left(\mathop{\ast}_{i=1}^{k} c_1 \right).$$

In this case,

$$Z^{(m)} \subset A \text{ is a divisor} \iff f_{Z,n-1}(m_1, \dots, m_d) \neq 0.$$

5.4. **The homology class in type C case.** In many instances, the subvariety $Z \subset A$ is invariant under a reflection, and we may, without loss of generality, assume that the reflection is taken with respect to the origin, so that $Z = -Z$. In this situation, the degree $d = \deg \gamma_Z$ is necessarily even, and the locus $[\mathbb{P}\Lambda_Z^{[d]}]$ admits a decomposition into a finite union of subvarieties. To describe these components, let

$$\mathcal{P}_2(d) := \left\{\alpha \in \mathcal{P}(d) \;\middle|\; |A| = 2 \text{ for all } A \in \alpha\right\}$$

denote the set of all perfect matchings of $[d]$. Each $\alpha \in \mathcal{P}_2(d)$ induces an involution τ_α of $[d]$, and we define

$$\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}} := \left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{[d]} \;\middle|\; p_{\tau_\alpha(i)} = -p_i \text{ for all } i\right\}$$

$$\Delta_{\widetilde{\alpha}} :\; A^{d/2} \times \mathbb{P}^\vee \longrightarrow A^d \times \mathbb{P}^\vee \qquad \big((p_i), \xi\big) \longmapsto \big((\pm p_{f_\alpha(i)}), \xi\big)$$

where the sign is chosen so that $p_{\tau_\alpha(i)} = -p_i$. Furthermore, there exists a distinguished subvariety in $\mathbb{P}\Lambda_Z^{\times d/2}$:

$$\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}} := \overline{\left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{\times d/2} \;\middle|\; p_i \neq \pm p_j \text{ iff } i \neq j\right\}}.$$

With these conventions, we can now express the following decomposition:

$$\begin{aligned}\mathbb{P}\Lambda_Z^{[d]} &= \bigcup_{\alpha \in \mathcal{P}_2(d)} \mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}} \\ &= \bigcup_{\alpha \in \mathcal{P}_2(d)} \Delta_{\widetilde{\alpha}}\left(\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}}\right)\end{aligned}$$

Evidently, determining $[\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}]$ amounts to determining $[\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}}]$, and the latter computation reduces again to combinatorics.

For $\beta \in \mathbb{P}(d/2)$, define

$$\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}} := \left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{\times d/2} \;\middle|\; p_i = \pm p_j \text{ if } i \sim_\beta j\right\}$$

$$\mathbb{P}\Lambda_Z^{\widetilde{\beta}} := \overline{\left\{\big((p_i), \xi\big) \in \mathbb{P}\Lambda_Z^{\times d/2} \;\middle|\; p_i = \pm p_j \text{ iff } i \sim_\beta j\right\}}$$

Then one has the relations

$$\begin{aligned}[][\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}] &= \sum_{\beta' \geqslant \beta} [\mathbb{P}\Lambda_Z^{\widetilde{\beta'}}], \\ [\mathbb{P}\Lambda_Z^{\widetilde{\beta}}] &= \sum_{\beta' \geqslant \beta} \mu(\beta, \beta')\, [\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta'}}],\end{aligned}$$

where $\mu(\beta, \beta')$ denotes the Möbius function of the poset $\mathcal{P}(d/2)$.

Furthermore, the classes $[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}]$ admit a decomposition as sums of pushforwards of $[\mathbb{P}\Lambda_Z^{\times k}]$ under appropriately defined signed diagonal maps. Explicitly, for every pair of maps $f_\beta : [d/2] \longrightarrow [k]$ and $\eta : [d/2] \longrightarrow \{\pm 1\}$, one introduces the signed diagonal embedding

$$\Delta_\beta^\eta : A^k \times \mathbb{P}^\vee \longrightarrow A^{d/2} \times \mathbb{P}^\vee \qquad \big((p_i), \xi\big) \longmapsto \big((\eta(i) p_{f_\beta(i)}), \xi\big)\,,$$

which yields the expression

$$[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}] = \frac{1}{2^k} \sum_{\eta : [d/2] \longrightarrow \{\pm 1\}} \Delta_{\beta,*}^\eta [\mathbb{P}\Lambda_Z^{\times k}].$$

Assuming that the monodromy group is the signed symmetric group, we may identify $\mathbb{P}\Lambda_Z^{\mathrm{univ}}$ with one of the components $\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}$. Without loss of generality, we choose

$$\alpha = \left\{\{i, i + d/2\} \;\middle|\; i = 1, \ldots, d/2\right\}$$

and fix the sign convention for $\Delta_{\widetilde{\alpha}}$:

$$\Delta_{\widetilde{\alpha}} : A^{d/2} \times \mathbb{P}^\vee \xhookrightarrow{(\Delta, \mathrm{Id})} A^{d/2} \times A^{d/2} \times \mathbb{P}^\vee \xrightarrow{(\mathrm{Id}, -\mathrm{Id}, \mathrm{Id})} A^d \times \mathbb{P}^\vee$$

Under this convention, the tuples $(m_1, \ldots, m_d)$ and $(m_1 - m_{d/2+1}, \ldots, m_{d/2} - m_d, 0, \ldots, 0)$ define the same subvariety. Hence, without loss of generality, we may assume that

$$(m) := (m_1, \ldots, m_{d/2}, 0, \ldots, 0) \in \mathbb{Z}^d.$$

We now introduce some notations for later computations:

$$\begin{aligned}
(\widetilde{m}) &:= (m_1, \ldots, m_{d/2}) \in \mathbb{Z}^{d/2} \\
\beta^\eta(\widetilde{m}) &:= \left(\sum_{i \in A_1} \eta(i) m_i, \ldots, \sum_{i \in A_k} \eta(i) m_i \right) \in \mathbb{Z}^k \\
\beta^\eta(\widetilde{m})^{2\boldsymbol{l}} &:= \left(\sum_{i \in A_1} \eta(i) m_i \right)^{2l_1} \cdots \left(\sum_{i \in A_k} \eta(i) m_i \right)^{2l_k} \\
\beta(\widetilde{m}^2)^{\boldsymbol{l}} &:= \left(\sum_{i \in A_1} m_i^2 \right)^{l_1} \cdots \left(\sum_{i \in A_k} m_i^2 \right)^{l_k}
\end{aligned}$$

where $\beta = \{A_1, \ldots, A_k\} \in \mathcal{P}(d/2)$ is a partition of length k, $\eta : [d/2] \longrightarrow \{\pm 1\}$ is a sign function, and $\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{N}^k_{\geqslant 0}$ with $\sum l_i = l$. Notice that

$$\sum_\eta \beta^\eta(\widetilde{m})^{2\boldsymbol{l}} = 2^{d/2} \, \beta(\widetilde{m}^2)^{\boldsymbol{l}}.$$

We can now determine $[\mathbb{P}\Lambda_Z^{(m)}]$ in the case where $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) \cong W(C_{d/2})$:

$$\begin{aligned}
[\mathbb{P}\Lambda_Z^{(m)}] &= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}] \\
&= \left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{[d/2]}] \\
&= \left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{\widetilde{\hat{0}}}] \\
&= \sum_\beta \mu(\hat{0}, \beta) \left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* \left[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant \beta}} \right] \\
&= \sum_{\beta, \eta} \frac{1}{2^k} \mu(\hat{0}, \beta) \left(a^{\beta^\eta(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* \left[\mathbb{P}\Lambda_Z^{\times k} \right] \\
&= \sum_{\beta, \eta} \frac{1}{2^k} \mu(\hat{0}, \beta) \left(a^{\beta^\eta(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* \left(\sum_{l=0}^{n-1} \left(\sum_{\sum l_i = l} c_{l_1} \otimes \cdots \otimes c_{l_k} \right) \otimes H^l \right) \\
&= \sum_{\beta, \eta} \frac{1}{2^k} \mu(\hat{0}, \beta) \sum_{l=0}^{n-1} \sum_{\sum l_i = l} \beta^\eta(\widetilde{m})^{2\boldsymbol{l}} \left(\mathop{\ast}_{i=1}^{k} c_{l_i} \otimes H^l \right) \\
&= \sum_{l=0}^{n-1} \left(\sum_{\beta, \eta} \sum_{\sum l_i = l} \left(\frac{1}{2^k} \mu(\hat{0}, \beta) \beta^\eta(\widetilde{m})^{2\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \right) \otimes H^l \\
&= \sum_{l=0}^{n-1} \left(\sum_{\beta} \sum_{\sum l_i = l} \left(2^{d/2-k} \mu(\hat{0}, \beta) \beta(\widetilde{m}^2)^{\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \right) \otimes H^l
\end{aligned}$$

Therefore,

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\beta} \sum_{\sum l_i = l} \left(2^{d/2-k}\, \mu(\hat{0},\beta)\beta(\widetilde{m}^2)^{\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \tag{5.2}$$

The following corollary collects some quantitative implications of (5.2).

Corollary 5.11. *Suppose that $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) \cong W(C_{d/2})$, and assume that $(m) := (m_1, \dots, m_{d/2}, 0, \dots, 0) \in \mathbb{Z}^d$.*

(1) The Chern–Mather classes

$$c_l\left(\Lambda_{Z^{(m)}}\right) \in \mathrm{H}^{2(n-l)}(A)[m_1^2, \dots, m_{d/2}^2]^{S_{d/2}}$$

are polynomials of degree l with exponent at most $2r$, when expressed in the variables $m_1^2, \dots, m_{d/2}^2$.

(2) If $Z \subset A$ is a smooth curve, there exists a homogeneous symmetric polynomial $\widetilde{f}_{Z,l} \in \mathbb{Q}[m_1^2, \dots, m_{d/2}^2]^{S_{d/2}}$ of degree l and exponent at most 2, such that

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \widetilde{f}_{Z,l}(m_1^2, \dots, m_{d/2}^2) \left(\mathop{\ast}_{i=1}^{k} c_1 \right).$$

In this case,

$$Z^{(m)} \subset A \text{ is a divisor} \quad \Longleftrightarrow \quad \widetilde{f}_{Z,n-1}(m_1^2, \dots, m_{d/2}^2) \neq 0.$$

5.5. **Möbius-transforms block-sum polynomial.** The formulas (5.1) and (5.2) admit further simplifications in certain special situations, such as the case of curves. In what follows, we extract the corresponding polynomial coefficient, which may be of independent combinatorial interest.

Definition 5.12. *Fix d and $\tilde{d} = d/2$, and let $l \in \mathbb{N}_{\geqslant 0}$. For a partition $\lambda = [\lambda_1, \dots, \lambda_{k'}]$ of l with $\lambda_1 \geqslant \cdots \geqslant \lambda_{k'} > 0$, the Möbius-transforms block-sum polynomial of type A (resp. type C) is defined by*

$$\mu_d^{\lambda} := \sum_{\alpha \in \mathcal{P}(d)} \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}:\ \text{length } k}} \mu(\hat{0},\alpha)\alpha(m)^{2\boldsymbol{l}} d^{k-k'} \qquad \in \mathbb{Z}[m_1, \dots, m_d]^{S_d}$$

$$\widetilde{\mu}_{\tilde{d}}^{\lambda} := \sum_{\beta \in \mathcal{P}(\tilde{d})} \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}:\ \text{length } k}} 2^{\tilde{d}-k}\mu(\hat{0},\beta)\beta(\widetilde{m}^2)^{\boldsymbol{l}}(2\tilde{d})^{k-k'} \qquad \in \mathbb{Z}[m_1^2, \dots, m_{\tilde{d}}^2]^{S_{\tilde{d}}}$$

In this definition:

- *$\alpha = \{A_1, \dots, A_k\} \in \mathcal{P}(d)$ is a set partition of $[d]$ with $k = \#\alpha$ blocks (analogously for β);*
- *$\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{Z}^k$, and $\boldsymbol{l} \dashv \lambda$ indicates that $\boldsymbol{l}$ is of type λ, meaning that its nonzero components, counted with multiplicities, form the partition λ;*
- *$\mu(\hat{0},\alpha) = (-1)^{d-k}\prod_{i=1}^{k}\left(|A_i|-1\right)$ is the Möbius function;*
- *$\alpha(m)^{2\boldsymbol{l}} = \left(\sum_{i\in A_1} m_i\right)^{2l_1} \cdots \left(\sum_{i\in A_k} m_i\right)^{2l_k}$ and $\beta(\widetilde{m}^2)^{\boldsymbol{l}} = \left(\sum_{i\in A_1} m_i^2\right)^{l_1} \cdots \left(\sum_{i\in A_k} m_i^2\right)^{l_k}$.*

Now, the formulas (5.1) and (5.2) can be written as

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right) \qquad \text{type } A \text{ case}$$

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \widetilde{\mu}_{\tilde{d}}^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right) \qquad \text{type } C \text{ case}$$

The following identities have been verified in low-degree cases using SageMath, though a purely combinatorial explanation, as well as a general formulation for the remaining types, is still lacking.

Fact 5.13. *Let $1^{k'} := [1, \dots, 1] \dashv k'$. When $k' < \widetilde{d}$, we have*

$$\begin{aligned} \mu_d^{1^{k'}} &= \frac{1}{2^{k'}k'!}\sum_{\sigma\in S_d}\prod_{i=1}^{k'}\left(m_{\sigma(2i-1)} - m_{\sigma(2i)}\right)^2 \\ \widetilde{\mu}_{\widetilde{d}}^{1^{k'}} &= 2^{\widetilde{d}}\frac{1}{2^{k'}k'!}\sum_{\sigma\in S_d}\prod_{i=1}^{k'} m_{\sigma(i)}^2 \\ &= 2^{\widetilde{d}-k'}(\tilde{d}-k')!\sum_{1\leqslant i_1<\cdots l_{k'}\leqslant d} m_{i_1}^2\cdots m_{i_{k'}}^2 \end{aligned}$$

Example 5.14 (Continuation of Example 3.7)**.** *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$.*
When C is non-hyperelliptic, we get

$$\dim Z^{(m)} = \min_{k\in\mathbb{Z}}\left\{g-1, \#\left\{i\in[2g-2] \mid m_i \neq k\right\}\right\}.$$

When C is hyperelliptic, with $(m) = (m_1, \dots, m_{g-1}, 0, \dots, 0)$, we get

$$\dim Z^{(m)} = \#\left\{i\in[g-1] \mid m_i \neq 0\right\}.$$

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