Subvarieties in Complex Abelian Varieties

Xiaoxiang Zhou
Advisor: Prof. Dr. Thomas Krämer
Humboldt-Universität zu Berlin

Tangent Gauss Map

Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subset A$ be a non-degenerate closed subvariety of dimension r.

To understand the geometry of Z, we encode the variation of its tangent spaces via the tangent Gauss map

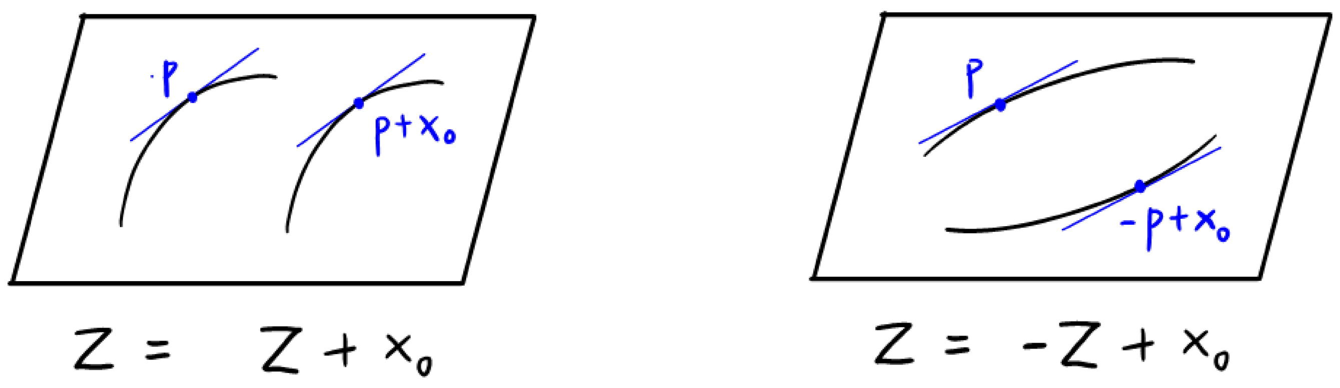
$$\phi_Z : Z^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longrightarrow T_pZ \subset T_pA \cong T_0A.$$

Its differential

$$d_p\phi_Z : T_pZ \longrightarrow \mathrm{Hom}_{\mathbb{C}}(T_pZ, N_pZ)$$

is the second fundamental form, from which curvature invariants can be extracted.

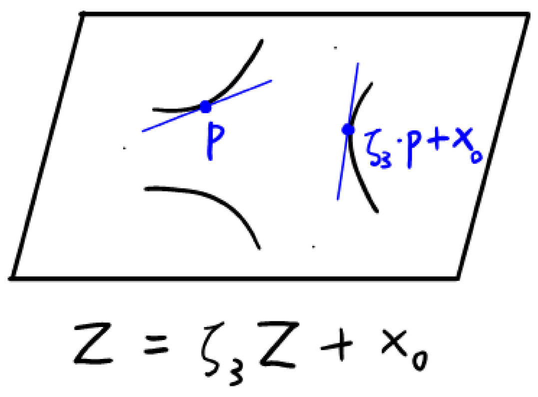
Besides the obvious examples (illustrated below), when can the Gauss map ϕ_Z fail to be generically injective?



We specialize to the case where $Z = C$ is a curve. When $n = 2$, ϕ_C: $C^{\mathrm{sm}} \longrightarrow \mathbb{P}^1$ typically fails to be generically injective.

Conjecture 1. Let $C \subset A$ be a non-degenerate curve, $n > 2$. If C is not invariant under any non-trivial translation or reflection, then ϕ_C is generically injective.

A Counterexample for Conjecture 1



Example 1. For $A = E_\rho^{\oplus n}$, ζ_3 acts on A (and hence on T_0A) by scalar multiplication. Computer experiments yield a non-degenerate ζ_3-invariant curve $C \subset A$, for which ϕ_C is not generically injective.

We have found no counterexample to Conjecture 1 when A is not isogenous to $E_i^{\oplus n}$ or $E_\rho^{\oplus n}$. This suggests the following refinement:

Conjecture 2. Let $C \subset A$ be a non-degenerate curve, $n > 2$. If no non-trivial $\tau \in \mathrm{Aut}(A)$ preserves C and acts by scalar multiplication on T_0A, then ϕ_C is generically injective.

One may restate the conjecture using Gauss curvature, yielding a slightly stronger statement:

Conjecture 3. Let $C \subset A$ be a non-degenerate curve, $n > 2$. For a general point $p \in \mathrm{Im}\,\phi_C$, all points in $\phi_C^{-1}(p)$ exhibit the same Gauss curvature.

Known Cases

1. If $A = \mathrm{Jac}(C)$ and C is embedded via the Abel-‐Jacobi map, then $\phi_C = |\omega_C|$ is the canonical map:
   - When C is hyperelliptic, C is invariant under the hyperelliptic involution, and $\deg \phi_C = 2$;
   - When C is non-hyperelliptic, ϕ_C is an embedding.
2. Let $h : C \longrightarrow C'$ be a cyclic k-fold cover defined by $\eta \in \mathrm{Pic}(C')$ with $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. If $A = \mathrm{Prym}(C/C')$ and $C \to A$ is the Abel–Prym map, then

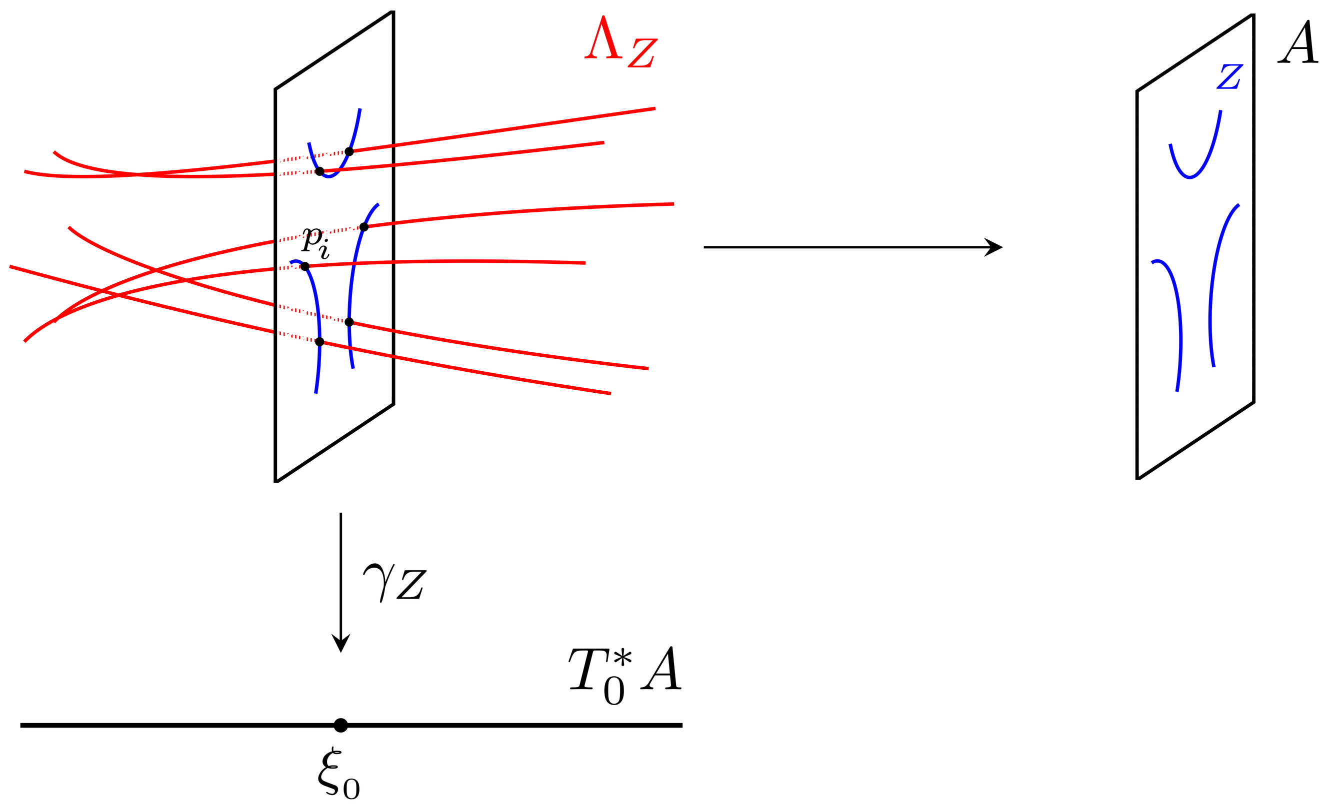
$$T_0A \cong \mathrm{H}^0(\omega_C)/\,\mathrm{H}^0(\omega_{C'}) \cong \bigoplus_{i=1}^{k-1} \mathrm{H}^0(\omega_{C'} \otimes \eta^i)$$

$$\phi_C : C \longrightarrow \mathbb{P}T_0A \cong \mathbb{P}\left(\bigoplus_{i=1}^{k-1} \mathrm{H}^0(\omega_{C'} \otimes \eta^i)\right)$$

   - $k = 2$: C is invariant under the Prym involution, and $\phi_C = |\omega_{C'} \otimes \eta| \circ h$. If C' is non-hyperelliptic with $g(C') \geq 4$, then $\deg \phi_C = 2$ or 4, and
   $$\deg \phi_C = 4 \iff B = \varnothing, C' \text{ is bielliptic and } \eta \text{ pulled back from EC.}$$
   - $k > 2$: if $g(C') \geq 1$ and $|\omega_{C'} \otimes \eta|$ is generically injective, then ϕ_C is generically injective.
3. If $C \subset A$ is smooth and either $\deg \phi_C = 2$ or ϕ_C is unramified, Conjecture 3 also holds.

Conormal Gauss Map

Consider the conormal variety $\Lambda_Z \subset T^*A \cong A \times T_0^*A$. The natural projection is the conormal Gauss map $\gamma_Z^{(\mathrm{aff})} : \Lambda_Z \longrightarrow T_0^*A$, which is generically finite whenever Z is of general type.



Conjecture 4. Suppose A is not isogenous to $E_i^{\oplus n}$ or $E_\rho^{\oplus n}$. For any non-degenerate curve $C \subset A$ that is invariant under no non-trivial translation or reflection, the monodromy group $\mathrm{Gal}(\gamma_C)$ is big — namely, a Weyl group of type A, C, or D.

When $n > 2$, Conjecture 4 follows from Conjecture 3.

The monodromy group $\mathrm{Gal}(\gamma_C)$ helps us to determine controls the Tannaka group of the perverse sheaf category generated by the IC sheaf on C; see [Krä22, Thm 2.1].

The Subvariety $Z^{(m)}$

The convolution structure on perverse sheaves gives rise to numerous cycles in A. They admit a simple geometric description: by fiberwise summing points in $\gamma_Z^{-1}(\xi_0)$ and projecting, one obtains new subvarieties of varying dimensions.

Fix a general point $\xi_0 \in T_0^*A$ and choose an ordering $\gamma_Z^{-1}(\xi_0) = \{p_1, \ldots, p_d\} \subset Z$. For $(m) = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, let $Z^{(m)}$ denote the irreducible component of the resulting subvariety containing $\sum_i m_i p_i$.

Theorem 1. Let $c_i := c_{M,i}(\Lambda_Z)$ be the Chern–Mather class of Z, $*$ be the Pontryagin product, and we write $\lambda \dashv l$ to indicate that $\lambda = [\lambda_1, \ldots, \lambda_{k'}]$ is a partition of l. When $\mathrm{Gal}(\gamma_Z) = S_d$, the Chern–Mather classes of $Z^{(m)}$ can be written as

$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^\lambda \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right)$$

where

- $c_Z^{(m)} \in \mathbb{N}_{>0}$ is the degree of a certain addition map;
- $\mu_d^\lambda = \sum_{\alpha \in \mathcal{P}(d)}; \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}\text{: length } k}} \mu(\hat{0}, \alpha)\alpha(m)^{2\boldsymbol{l}} d^{k-k'} \in \mathbb{Z}[m_1, \ldots, m_d]^{S_d}$
- $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}(d)$, $\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$;
- $\mu(\hat{0}, \alpha) = (-1)^{d-k} \prod_{i=1}^k (|A_i| - 1)$;
- $\alpha(m)^{2\boldsymbol{l}} = \left(\sum_{i \in A_1} m_i\right)^{2l_1} \cdots \left(\sum_{i \in A_k} m_i\right)^{2l_k}$.

Remarks.

1. One can recover both $\dim Z$ and $[Z] \in \mathrm{H}^{2(n-\dim Z)}(A)$ from the Chern–Mather class of Λ_Z:
$$\dim Z = \max\{i \in \mathbb{Z} \mid c_i \neq 0\}, \qquad [Z] = c_{\dim Z}.$$
2. A similar formula holds when $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, but the method does not extend to the case $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) = W(D_{d/2})$.
3. The formula simplifies significantly in the Jacobian case. For example, if C is non-hyperelliptic, we obtain:
$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \frac{1}{2^l (g-l)!} \sum_{\sigma \in S_{2g-2}} \prod_{i=1}^{l} \left(m_{\sigma(2i-1)} - m_{\sigma(2i)}\right)^2 \cdot \Theta^{g-l}$$
$$\dim Z^{(m)} = \min_{k \in \mathbb{Z}} \{g-1, \#\{i \in [2g-2] \mid m_i \neq k\}\}$$

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