A CRASH INTRODUCTION TO LANGLANDS CORRESPONDENCE

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Abstract. In these notes, we explore various versions of the Langlands correspondence, placing particular emphasis on modular forms, automorphic forms, and automorphic representations.

Contents

1. Introduction

These notes represent a faithful record of my talk at KleinAG. I have intentionally omitted sections that were not addressed during the actual presentation, making these notes somewhat incomplete. Readers may refer to my handwritten notes [2] for a more expressive and detailed account.

I want to acknowledge that there is nothing original in my presentation. I appreciate the organizers, the attentive audience, and fellow speakers for helping identify my mistakes. Please feel free to continue pointing out any more errors or issues.

Introducing the Langlands correspondence can often be a challenging and intricate endeavor. It encompasses numerous versions, spanning from local to global, from one dimension to n dimensions, and from GL_n to non-split groups. Today's talk is structured into four parts, each focusing on a specific version of Langlands correspondence, as outlined below:

$$\begin{aligned}
\mathrm{Irr}_{\mathbb{C}}\left(\mathrm{GL}_n(F)\right) &\xleftrightarrow{1:1} \mathrm{WDrep}_{\substack{n\text{-dim}\\ \text{Frob ss}}}(W_F) \\
\mathrm{Char}_{\mathbb{C},\mathrm{alg}}\left(F^\times\backslash\mathbb{A}_F^\times\right) &\xleftrightarrow{1:1} \mathrm{Char}_{\bar{\mathbb{Q}}_p}(\Gamma) \qquad + \text{de Rham} \\
\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\left(\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})\right) &\xrightarrow{ES} \mathrm{Irr}_{\bar{\mathbb{Q}}_p,2\text{-dim}}(\Gamma) + \text{modular} \\
\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\left(G_D(\mathbb{A}_{\mathbb{Q}})\right) &\longrightarrow \qquad \cdots
\end{aligned}$$

Before discussing these correspondings, let us fix some notations.

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Rep	*smooth representation*
Irr	*irreducible smooth representation*
Π	*admissible irreducible smooth representation*
Char	*1-dim smooth representation*
WDrep	*Weil–Deligne representation*
$\mathcal{A}_{\text{cusp}}$	*cuspidal automorphic form*

Setting 1.1.
In Section 2, F is a non-Archimedean local field with integral ring O_F and residue field κ_F. Within this context, we also make use of the absolute Galois group Γ_F and the Weil group W_F associated with F.

Moving on to Section 3, we shift our focus to a number field, still denoted as F, with its integral ring denoted as O_F. For each place v of F, we equip with three complete local rings, namely, O_v, F_v and κ_v. The absolute Galois group of F remains denoted as Γ_F.

We will use the following abbreviations for representations:

For the definition of smooth/irreducible/admissible/Weil–Deligne representation, see [1] or (partially)[2, 22.04.17].

2. Non-Archimedean Local Field Case

Read [2, GL_n-case]. You may assume $F = \mathbb{Q}_p$ if you are not familiar with local fields.

In this instance, the Langlands correspondence is notably explicit, allowing for the classification of representations on both sides. Notably, it simplifies to a linear algebra task when considering the L-parameters of $\mathrm{GL}_{2,\mathbb{R}}$.

3. Global Langlands Correspondence, $n = 1$

In order to state global Langlands correspondence, we need adèle and idèle which collects all local information. A short introduction of adèle and idèle can be found in A.

Observe that

$$\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{R}_{>0} \cong \widehat{\mathbb{Z}}^\times \cong \mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) := \Gamma_{\mathbb{Q}}^{\mathrm{ab}}.$$

In fact, we have Artin reciprocity:

$$\mathrm{Art} : F^\times \backslash \mathbb{A}_F^\times / \overline{\left(F_\infty^\times\right)^\circ} \cong \Gamma_F^{\mathrm{ab}},$$

which gives us global Langlands correspondence for $n = 1$:

For more information about the twist, see [2, Galois representation].(???Wait for updating)

4. Adèlic Modular Forms

4.1. Moduli space.

4.2. Automorphic forms and automorphic representations.

4.3. Global Langlands correspondence for $\mathrm{GL}_{2,F}$.

5. Adèlic Modular Forms on Quaternion Algebras

5.1. Quaternion algebras.

References

[1] Toby Gee. Modularity lifting theorems. *Essent. Number Theory*, 1(1):73–126, 2022.
[2] Xiaoxiang Zhou. Eine woche, ein beispiel. https://github.com/ramified/personal_handwritten_collection/tree/main/weeklyupdate, 2020-2023. [Online].

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