CALDERÓN'S COMPLEX INTERPOLATION METHOD

EMPTY

Contents

1. Calderón's complex interpolation method (Due to Alberto Calderón)

In this section, we will try to show that

$$\mathcal{F}\left(L^p(\Omega)\right) \subseteq L^q(\Omega) \quad \text{for } 1 \leq p \leq 2,\ \frac{1}{p}+\frac{1}{q}=1 \tag{1.1}$$

under the help of complex interpolation method. Surprisingly, this method stems from a theorem in complex analysis, call the three-lines theorem.

Theorem 1.1 (Three lines theorem, due to Hadamard)**.** *Let*

$$\begin{aligned}&\Omega := \ \{z \in \mathbb{C} : 0 < \mathrm{Re} < 1\}\\ &E : \textit{Banach space}\\ &f : \overline{\Omega} \longrightarrow E \textit{ is bounded, continuous, and } f|_{\Omega} \textit{ is holomorphic.}\end{aligned}$$

For $0 \leq \theta \leq 1$*, define*

$$M_\theta(f) := \sup_{t\in\mathbb{R}} \|f(r+it)\| < +\infty,$$

then

$$M_\theta(f) \leq \left(M_0(f)\right)^{\theta}\left(M_1(f)\right)^{1-\theta}.$$

This is equivalent to say, the function

$$[0,1] \longrightarrow \mathbb{R} \qquad \theta \longmapsto \log M_\theta(f)$$

is convex.

Remark 1.2. We say $f : \Omega \longrightarrow E$ is **holomorphic** if f satisfies Riemann-Cauchy equation. Equivalently, $f : \Omega \longrightarrow E$ is holomorphic if for any $\phi \in E'$, the composition

$$\Omega \xrightarrow{f} E \xrightarrow{\phi} \mathbb{C}$$

is holomorphic.

The proof use the Phragmén–Lindelöf method. Before the proof, let me recall the maximum principle.

Theorem 1.3 (Maximum principle for holomorphic functions)**.** *Let* Ω *be a bounded open subset of* $\mathbb{C}$*,* $f : \overline{\Omega} \longrightarrow \mathbb{C}$ *be continuous and holomorphic (in* Ω*). Then for any* $z \in \overline{\Omega}$*,*

$$\|f(z)\|_E \leq \sup_{w\in\partial\Omega} |f(w)|.$$

Proof of Theorem 1.1, by Phragmén–Lindelöf method.

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Step 1. Prove the theorem for $E=\mathbb{C}$, $M_0(f)=M_1(f)=1$ case. In this case, we need to show

$$|f(z)|\le 1 \qquad \text{for any } z\in\Omega.$$

Idea: introduce a multiplicative factor $e^{\frac{z^2-1}{n}}$ to "subdue" the growth of f, so that we can use maximal principle to get the bound.

Let $f_n(z):=e^{\frac{z^2-1}{n}}f(z)$, then there exists $R>0$ (depend on n), such that

$$|f_n(z)|\le 1 \qquad \text{for } z\in\overline{\Omega},\quad |\operatorname{Im} z|\ge R.$$

By the maximal principle for $\left\{z\in\overline{\Omega}:|\operatorname{Im} z|\le R\right\}$,

$$|f_n(z)|\le 1 \qquad \text{for } z\in\overline{\Omega},\quad |\operatorname{Im} z|\le R.$$

Therefore, $\|f_n\|\le 1$. As a result,

$$|f(z)|=\lim_{n\to\infty}|f_n(z)|\le 1.$$

Step 2. Prove the theorem for $E=\mathbb{C}$ case.

Define

$$g:\overline{\Omega}\longrightarrow\mathbb{C} \qquad g(z):=M_0(f)^{z-1}M_1(f)^{-z}f(z),$$

and apply g to Step 1.

Step 3. General case.

For $\phi\in E'$, $\|\phi\|_{E'}\le 1$, define $h_\phi:=\phi\circ f$:

$$h_\phi:\overline{\Omega}\xrightarrow{f}E\xrightarrow{\phi}\mathbb{C},$$

then

$$|h_\phi(z)|=\left|\phi\left(f(z)\right)\right|\le\|\phi\|_{E'}\|f(z)\|_E\le\|f(z)\|_E.$$

Apply h_ϕ to Step 2, we get

$$|h_\phi(z)|\le M_0(h_\phi)^\theta M_1(h_\phi)^{1-\theta}\le M_0(f)^\theta M_1(f)^{1-\theta} \qquad \text{for any } z\in\overline{\Omega},\operatorname{Re} z=\theta,$$

so

$$\|f(z)\|_E=\sup_{\substack{\phi\in E'\\ \|\phi\|\le 1}}|\langle\phi,f(z)\rangle|=\sup_{\substack{\phi\in E'\\ \|\phi\|\le 1}}|h_\phi(z)|\le M_0(f)^\theta M_1(f)^{1-\theta}.$$

□

Somewhat surprising, Theorem 1.1 offers us a way to construct "a continuous deformation between two Banach spaces". Intuitively, these intermediate spaces must lie in the sum of these two Banach spaces. First, we try to give a norm to this ambiance space.

Proposition 1.4. *Let E_0, E_1 be two Banach spaces contained in some topological vector space V.??? Then $E_0\oplus E_1$ is a Banach space with norm*

$$\|(x,y)\|=\|x\|_{E_0}+\|y\|_{E_1},$$

$E_0\oplus E_1$ is a Banach space with norm

$$\|x+y\|=\inf_{\xi\in E_0\cap E_1}\left\{\|x+\xi\|_{E_0}+\|y-\xi\|_{E_1}\right\}.$$

Proof. Conditions on norm are relatively easy to check, but I don't know how to show completeness. □

Lemma 1.5. *The injection $j:E_0\hookrightarrow E_0+E_1$ is continuous of norm ≤ 1.*

Proof. We have the estimation

$$\|x+0\|_{E_0+E_1}=\inf_{\xi\in E_0\cap E_1}\left\{\|x+\xi\|_{E_0}+\|-\xi\|_{E_1}\right\}\le\|x\|_{E_0}.$$

□

Warning 1.6. *The injection j may be not topological embedding, i.e., $E_0 \hookrightarrow \operatorname{Im} j$ may be not homeomorphism.*

Definition 1.7 (Interpolation spaces)**.** *For two Banach spaces E_0, E_1 contained in some vector space V, we define the space*

$$\mathcal{H} := \mathcal{H}(E_0, E_1) := \left\{ f : \overline{\Omega} \longrightarrow E_0 + E_1 \;\middle|\; \begin{array}{l} f \text{ is continuous and bounded} \\ f|_\Omega \text{ is holomorphic} \\ f(it) \in E_0,\ f(1+it) \in E_1, \text{ for any } t \in \mathbb{R} \end{array} \right\}$$

to be the Banach space with norm

$$\|f\|_{\mathcal{H}} := \max\left(M_0(f), M_1(f)\right) = |||f|||_\infty.$$

For $0 < \theta < 1$, define the ***interpolation space***

$$E_\theta := [E_0, E_1]_\theta := \mathcal{H}(E_0, E_1) / \{f \in \mathcal{H} : f(\theta) = 0\},$$

i.e., the image of the map

$$\mathrm{ev}_\theta : \mathcal{H}(E_0, E_1) \longrightarrow E_0 + E_1 \qquad f \longmapsto f(\theta).$$

Notice that we only take the norm of E_θ as the residue norm of $\mathcal{H}$, instead of the subspace norm of $E_0 + E_1$.

Question 1.8. *Are these two norms the same norm?*

Remark 1.9. It is natural to set $\theta = 0$, and guess $[E_0, E_1]_0 = E_0$, but this is false in general. Consider $E_0 = \mathbb{C}, E_1 = 0$, then

$$\mathcal{H}(E_0, E_1) = 0 \quad \Longrightarrow \quad [E_0, E_1]_\theta = 0 \quad \text{for any } \theta.$$

Here we list some immediate properties of the interpolation spaces.

Lemma 1.10. $[E_0, E_1]_\theta = [E_1, E_0]_{1-\theta}$.

Lemma 1.11. *For $\xi \in [E_0, E_1]_\theta$,*

$$\|\xi\|_\theta = \inf_{\substack{f \in \mathcal{H} \\ \bar{f} = \xi}} \left\{ M_0(f)^{1-\theta} M_1(f)^\theta \right\}.$$

Proof. For the easy direction,

$$\text{LHS} = \inf_{\substack{f \in \mathcal{H} \\ \bar{f} = \xi}} \|f\|_{\mathcal{H}} = \inf_{\substack{f \in \mathcal{H} \\ \bar{f} = \xi}} \left\{ \max\left(M_0(f), M_1(f)\right) \right\} \geq \text{RHS}.$$

To show RHS $\leq$ LHS, one needs to show that

$$\text{For } f \in \mathcal{H}, \text{ there exists } g \in \mathcal{H}, g(\theta) = f(\theta), \text{ such that } \|g\|_{\mathcal{H}} \leq M_0(f)^{1-\theta} M_1(f)^\theta.$$

Then $g(z) := M_0(f)^{z-1} M_1(f)^{-z} f(z)$ satisfy the condition. ??? □

The next theorem gives us a perfect example.

Theorem 1.12 (Riesz-Thorin interpolation theorem)**.** *Let $(X, \mathcal{A}, \mu)$ be a σ-finite space, $1 \leq p < q \leq \infty$, $0 \leq \theta \leq 1$, p', q' be the conjugate indices of p, q. Let $r \in \mathbb{R}$ such that $\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}$, then*

$$[L^p(X), L^q(X)]_\theta \cong L^r(X)$$

as Banach spaces.

Example 1.13.

When $q = \infty$, $r = \frac{1}{1-\theta} \cdot p$.

When $\theta = \frac{1}{2}$, $\frac{2}{r} = \frac{1}{p} + \frac{1}{q}$, $(0, p, r, q)$ is a harmonic range.

Proof. We do the case $q < +\infty$. Let

$$L^0(X) := \{ \text{ measurable functions } \} / \text{ null functions}$$

be an ambiance space, and $f \in L^r(X)$ be a representative (i.e., a function). We need three steps:

Step 1. Let $f \in L^r(X)$, construct $\phi \in \mathcal{H}\left(L^p(X), L^q(X)\right)$ such that $\phi(\theta) = f$.

For this, define

$$\phi : \overline{\Omega} \longrightarrow L^p(X) + L^q(X) \qquad \phi(z) = \frac{f(-)}{|f(-)|}|f(-)|^{r\left(\frac{1-z}{p} + \frac{z}{q}\right)} \mathbb{1}_{\{|f|>0\}}.$$

We need to verify:

- For a fixed z, $\phi(z) \in L^p(X) + L^q(X)$;
- $\phi \in \mathcal{H}\left(L^p(X), L^q(X)\right)$;
- $\phi(\theta) = f$.

Step 2. For $\phi \in \mathcal{H}\left(L^p(X), L^q(X)\right)$, show that $\phi(\theta) \in L^r(X)$.

For proving this, we need a fact from the duality theory:

Fact 1.14. *Given $h \in L^0(X)$ and r, r' as conjugate indices. If for all simple functions*[1] *g we have*

$$h \cdot g \in L^1, \qquad \int |h \cdot g| d\mu \leq C \cdot \|g\|_{r'},$$

then $h \in L^r$ and $\|h\|_{L^r} \leq C$.

From this fact, one needs to estimate

$$\int |\phi(\theta) \cdot g| d\mu \leq C \cdot \|g\|_{r'}$$

for any simple function g. Now fix g, define

$$\psi : \overline{\Omega} \longrightarrow L^{p'}(X) + L^{q'}(X) \qquad \psi(z) = \frac{g(-)}{|g(-)|}|g(-)|^{r'\left(\frac{1-z}{p'} + \frac{z}{q'}\right)} \mathbb{1}_{\{|g|>0\}}$$

$$H : \overline{\Omega} \longrightarrow \mathbb{C} \qquad H(z) := \int_{L^1} \phi(z)\psi(z) d\mu.$$

???

Step 3. For $\xi \in [L^p(X), L^q(X)]_\theta$, show that $\|\xi\|_\theta = \|\xi\|_{L^r}$.

??? □

Finally, we state the main theorem of this section. The inclusion 1.1 is a natural corollary of Theorem 1.15.

Theorem 1.15 (Abstract Riesz-Thorin)**.** *Given E_0, E_1; F_0, F_1 two pairs of Banach spaces as before???, $0 < \theta < 1$. Suppose $T : E_0 + E_1 \longrightarrow F_0 + F_1$ is linear with*

$$T(E_0) \subseteq F_0, \quad T(E_1) \subseteq F_1,$$

then

$$T([E_0, E_1]_\theta) \subseteq [F_0, F_1]_\theta.$$

Moreover, if $T|_{E_0}$, $T|_{E_1}$ are bounded, then $T|_{E_\theta}$ is bounded, and

$$\|T\|_\theta \leq \|T\|_0^{1-\theta} \|T\|_1^\theta.$$

Proof. Let $xi \in [E_0, E_1]_\theta$, we need to show $T(\xi) \in [F_0, F_1]_\theta$, and give an estimation of $T(\xi)$. For any $\varepsilon > 0$, we choose $f \in \mathcal{H}(E_0, E_1)$, $\bar{f} = \xi$ such that

$$\|\xi\|_\theta \leq \|f\|_{\mathcal{H}} \leq \|\xi\|_\theta + \varepsilon.$$

[1] For simple functions, we mean the function of form

$$\sum_{\text{fin}} a_i \, \mathbb{1}_{A_i}$$

where all A_i are measurable sets with finite measure.

Then $T(f) \in \mathcal{H}(F_0, F_1)$ $(\Rightarrow T(\xi) \in [F_0, F_1]_\theta)$, and

$$\begin{aligned} M_0(T(f)) &\leq \|T\|_0 M_0(f) \qquad M_1(T(f)) \leq \|T\|_1 M_1(f) \\ \Longrightarrow M_\theta(T(f)) &\leq \|T\|_0^{1-\theta} \|T\|_1^\theta M_0(f)^{1-\theta} M_1(f)^\theta \\ &\leq \|T\|_0^{1-\theta} \|T\|_1^\theta \left(\|\xi\|_\theta + \varepsilon\right) \end{aligned}$$

Let $\varepsilon \to 0$, we get the bound. □

School of Mathematical Sciences, University of Bonn, Bonn, 53115, Germany,
Email address: `email:xx352229@mail.ustc.edu.cn`