$\mathcal{F}l(n)$

Springer Fibers for $SL_n(\mathbb{C})$

Xiaoxiang Zhou

Universität Bonn

June 13, 2021

Recap: representation theory of finite groups

Restrict to **complex** representations, we have a nice theory:

- Any representation can be written ~~of~~ as a direct sum of **irreducible representation**;
- We can extract information of irreducible representations from the **character table**:

$$\#\{\text{irreducible representations}\} = \#\{\text{conjugation classes}\}$$

$$\sum_{\chi:\text{irr}} (\dim \chi)^2 = \# G$$

However, in general,

- NO standard way finding an **explicit construction** of all irreducible representations;
- NO **one-to-one correspondence** between irreducible representations and conjugation classes.

In this talk, we use two methods to understand representations of S_n, and find connections/analogs between them.

methods	objects
combinatorial	Young diagram, Young tableau
geometrical	Springer fiber of $SL_n(\mathbb{C})$, irreducible components

Goal of the Part I

- Explicitly construct irreducible representations of S_n by Young diagram;
- Compute the character table;
  - $\dim \chi_i$ by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g. $\otimes$, Sym^m, Λ^m;
  - e.g. M_λ.

Notation

For boxes:

(Young) diagram

filling

11	78	11
6	8	

standard filling

3	5	4
1	2	

tableau

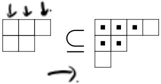
6	11	11
8	78	

standard tableau

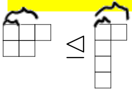
1	3	4
2	5	

Order of Young diagram:

inclusion

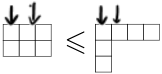


dominance



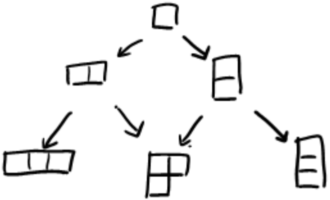
2 4 5   4 5 5

Lexicographic ordering

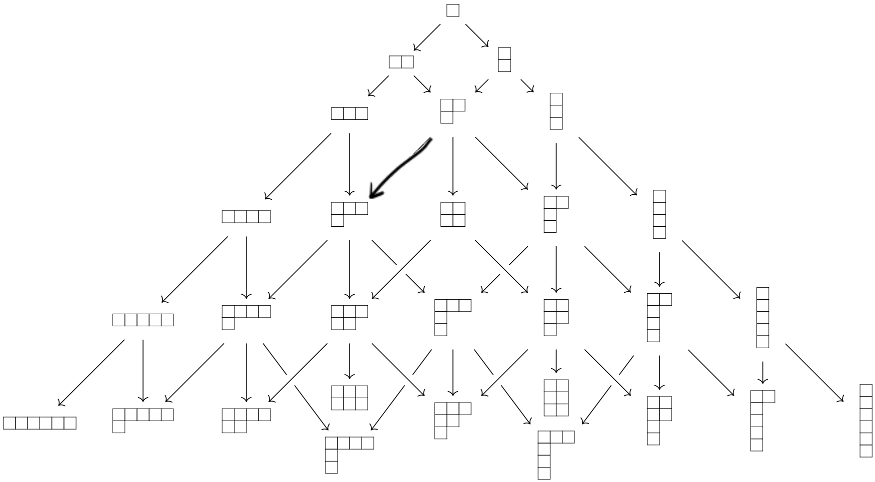


2 4 6 6   3 4 5 6

tree of Young diagram

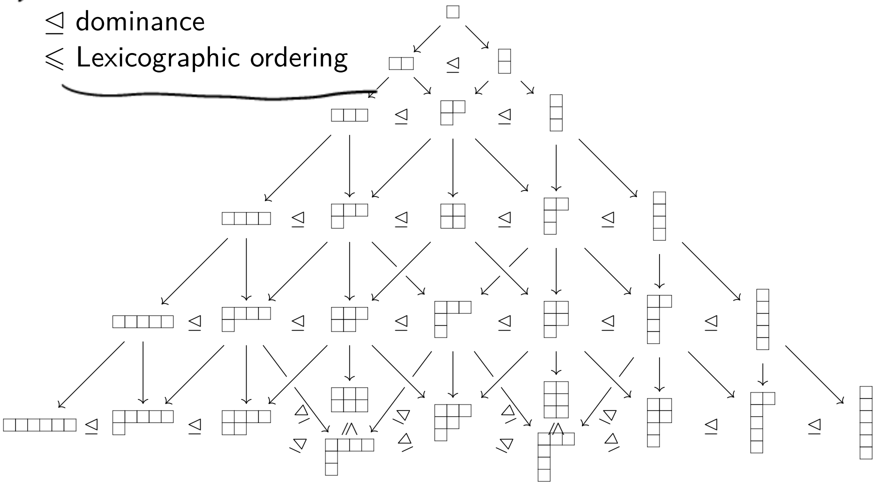


tree of Young diagram



Order

- $\subseteq$ inclusion
- $\trianglelefteq$ dominance
- $\leqslant$ Lexicographic ordering



S_n & Young diagram

Proposition

We have the one-to-one correspondence

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow[\lambda=\lambda_1^{v_1}\cdots\lambda_k^{v_k}]{\textit{partition of } n} \left\{\begin{matrix}\textit{Conjugation class}\\ \textit{of } S_n\end{matrix}\right\}$$

S_n & Young diagram

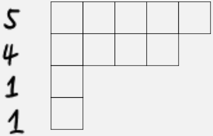
Proposition

We have the one-to-one correspondence

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow[\lambda=\lambda_1^{v_1}\cdots\lambda_k^{v_k}]{\text{partition of } n} \left\{\begin{matrix}\textit{Conjugation class}\\ \textit{of } S_n\end{matrix}\right\}$$

Example

$n = \cancel{10}.$ 11



$$\xleftrightarrow[\lambda=5\cdot4\cdot1^2]{11=5+4+1+1} (12345)(6789)(10)(11)$$

S_n & Young diagram

Proposition

We have the one-to-one correspondence

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow[\lambda=\lambda_1^{v_1}\cdots\lambda_k^{v_k}]{\textit{partition of } n} \left\{\begin{matrix}\textit{Conjugation class}\\ \textit{of } S_n\end{matrix}\right\}$$

Claim

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow{?} \left\{\begin{matrix}\textit{Irreducible rep}\\ \textit{of } S_n\end{matrix}\right\}$$

S_n & Young diagram

Claim

$$\left\{ \begin{matrix} \textit{Young diagram} \\ \textit{of } n \textit{ boxes} \end{matrix} \right\} \xleftrightarrow{?} \left\{ \begin{matrix} \textit{Irreducible rep} \\ \textit{of } S_n \end{matrix} \right\}$$

Remark

Reduced to: for each Young diagram λ,
construct an irreducible representation S^λ, and
prove $S^\lambda = S^{\lambda'} \Rightarrow \lambda = \lambda'$.

The construction of $S^\lambda \subseteq M^\lambda$

Tabloid: equivalence class of standard filling

$$\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} \overset{\longleftrightarrow}{=} \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} \overset{\longleftrightarrow}{=} \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} := \underline{\{345/12\}}$$

The construction of $S^\lambda \subseteq M^\lambda$

Tabloid: equivalence class of standard filling

$$\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 1 \\ \cline{1-2}\end{array} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2}\end{array} := \{345/12\}$$

$C(T) = \{Id, (13), (25), (13)(25)\}$

$\mathcal{T}^\lambda :=$ {Young tabloid} = {standard filling $\{T\}$}/$\sim$

$M^\lambda := \left\langle \{T\} \in \mathcal{T}^\lambda \right\rangle_{\mathbb{C}}$ $\quad \cancel{C(T) := \{\sigma \in S_n | \{\sigma \cdot T\} \simeq \{T\}\}}$

$$v_T := \sum_{q \in C(T)} \mathrm{sgn}(q)\{q \cdot T\} \in M^\lambda$$

$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda$ $\quad$ invariant subspace of M^λ

The construction of $S^\lambda \subseteq M^\lambda$

$$\mathcal{T}^\lambda := \{\text{Young tabloid}\} = \{\text{standard filling } \{T\}\}$$

$$M^\lambda := \left\langle \{T\} \in \mathcal{T}^\lambda \right\rangle_{\mathbb{C}} \qquad C(T) := \{\sigma \in S_n | \{\sigma \cdot T\} \sim \{T\}\}$$

$$v_T := \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \in M^\lambda$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda \qquad \text{invariant subspace of } M^\lambda$$

Example ($\lambda = 3 \cdot 2$) 

$$\mathcal{T}^\lambda = \left\{ \begin{matrix} \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \\ \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \end{matrix} \right\}$$

$$M^\lambda = \left\langle \begin{matrix} \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \\ \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \end{matrix} \right\rangle_{\mathbb{C}}$$

The construction of $S^\lambda \subseteq M^\lambda$

$$v_T := \sum_{q \in C(T)} \operatorname{sgn}(q)\{q \cdot T\} \in M^\lambda \qquad \boxed{\sigma v_T = v_{\sigma T}} \ \forall \sigma \in S_n$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda$$

invariant subspace of M^λ

Example ($\lambda = 3 \cdot 2$)

$$T = \begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$$

$$C(T) = \{\mathrm{Id}, (23), (15), (23)(15)\}$$

$$v_T = \left\{\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 2 & 5 & 4 \\ \hline 3 & 1 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\right\}$$

$$= \{345/12\} - \{245/13\} - \{134/25\} + \{124/35\} \in M^\lambda$$

$$S^\lambda = \langle v_T \rangle_{\mathbb{C}[S_n]} = \langle v_{T'} | T' : \textit{standard tableau} \rangle_{\mathbb{C}}$$

Main theorem of S^λ

Proposition

Fix the Young diagram λ, the corresponding representation S^λ has the following properties:

1. *the linear space S^λ has a **basis** $\{v_{T'} | T'$: standard tableau$\}$, e.p. $\dim S^\lambda = \#\{$standard tableau$\}$;*
2. *the representation S^λ is **irreducible**;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

Proof: basis

Proposition

1. *the linear space S^λ has a basis $\{v_{T'} | T' :$ standard tableau$\}$, e.p. $\dim S^\lambda = \#\{$standard tableau$\}$;*

→ especially.

Proof

- *S^λ is generated by $\{v_{T'} | T' :$ standard filling$\}$, eliminate the relations, we get $v_{T'} = \pm v_{T'_0}$.*

  T_0' : a standard tableau

  e.g. $v_{\begin{smallmatrix}3&5&4\\2&1\end{smallmatrix}} \overset{\text{column}}{=} v_{\begin{smallmatrix}2&1&4\\3&5\end{smallmatrix}} \overset{\text{row}}{=} -v_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}}$

- *$\{v_{T'} | T' :$ standard tableau$\}$ are linear independent.*

  e.g. $x_1 v_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} + x_2 v_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} + x_3 v_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} + x_4 v_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} + x_5 v_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} = 0$ $\quad x_i \in \mathbb{C}$

  $-\begin{smallmatrix}1&5&3\\4&2\end{smallmatrix}$!

  $\{123/45\} \to x_1 = 0$ $\quad \{134/25\} \to x_3 = 0$ $\quad \{135/24\} \to x_5 = 0$

  $\{124/35\} \to x_2 = 0$ $\quad \{125/35\} \to x_4 = 0$

linear ordering

We use a linear ordering of standard filling by

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} \end{array} \longrightarrow 5\,4\,3\,2\,1$$

$$\vee$$

$$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} \end{array} \longrightarrow 5\,2\,4\,3\,1$$

In the proof, we knock out the biggest one.

Example (2, 2, 2)

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} > \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} > \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} > \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array} > \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline 5 & 6 \\ \hline \end{array}$$

654321 654231 645231 635241 635241

Proof: part 2&3

Proposition

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

We have to introduce element b_T in $\mathbb{C}[S_n]$ by

$$b_T := \sum_{q \in C(T)} \operatorname{sgn}(\sigma)\sigma$$

then

- $v_T = b_T \cdot \{T\}$;
- $\tau(b_T) = \operatorname{sgn}(\tau) b_T$ for any $\tau \in C(T)$;
- $b_T \cdot b_T = \#C(T) \cdot b_T$;
- $b_T M^\lambda = b_T S^\lambda = \mathbb{C} v_T \neq 0$;
  $b_T M^{\lambda'} = b_T S^{\lambda'} = 0$ for $\lambda' > \lambda$.

Proof: part 2&3

Proposition

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

$$\begin{cases} b_T M^\lambda = b_T S^\lambda = \mathbb{C} v_T \neq 0 \ ; & \\ b_T M^{\lambda'} = b_T S^{\lambda'} = 0 & \text{for } \lambda' > \lambda \end{cases}$$

⋇To show S^λ is irreducible: only need to show indecomposablility.
If $S^\lambda = V \oplus W$ as $\mathbb{C}[S_n]$-module, then

$$\mathbb{C} v_T = b_T S^\lambda = b_T V \oplus b_T W$$

$$\Rightarrow b_T V = \mathbb{C} v_T \qquad \text{(or } b_T W = \mathbb{C} v_T\text{)}$$

$$\Rightarrow S^\lambda = \mathbb{C}[S_n] \cdot v_T = \mathbb{C}[S_n] \cdot \mathbb{C} v_T = \mathbb{C}[S_n] \cdot b_T V \subseteq V$$

Proof: part 2&3

Proposition

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

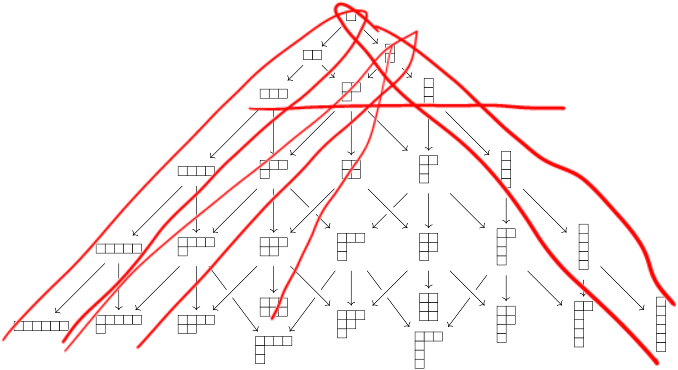
$$b_T M^\lambda = b_T S^\lambda = \mathbb{C} v_T \neq 0 \;;$$
$$b_T M^{\lambda'} = b_T S^{\lambda'} = 0 \qquad \text{for } \lambda' > \lambda$$

⋇To show $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$:
If not w.l.o.g. suppose $\lambda' > \lambda$. Then

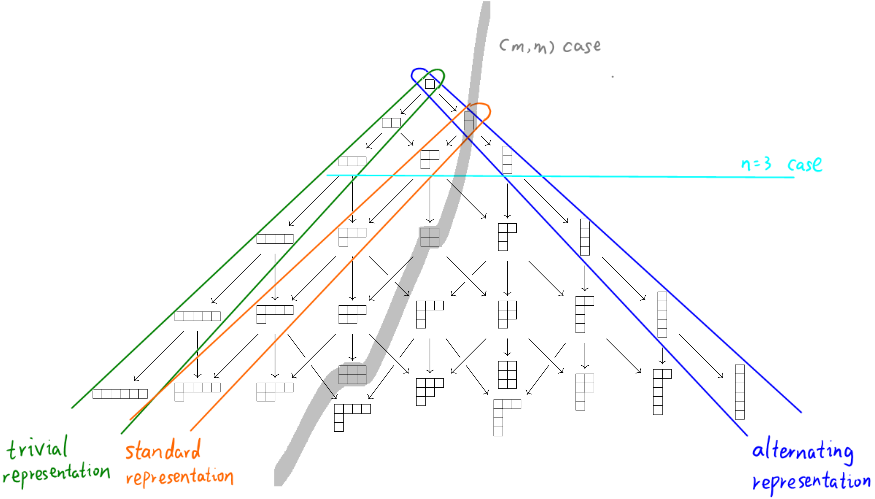
$$\underline{b_T} S^{\lambda'} = b_T S^\lambda \Longrightarrow \underline{\mathbb{C} v_T} \cong 0,$$

contradiction!

Example



Example



(m,m) case
n=3 case
trivial representation
standard representation
alternating representation

Example: trivial representation

$$\lambda = \square\square\square = 3^1$$

$$\underline{M^\lambda} = \langle \{123\} \rangle = \underline{\mathbb{C}}$$

$$\underline{T} = \boxed{1}\boxed{2}\boxed{3}$$

$$\underline{C(T)} = \mathrm{Id}$$

$$\underline{v_T} = \left\{ \boxed{1}\boxed{2}\boxed{3} \right\}$$

$$\underline{S^\lambda} = \mathbb{C}[S_3] \cdot v_T = \underline{\mathbb{C}v_T}$$

$$\sigma \qquad = v_T$$

Example: alternating representation

$$\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} = 1^3$$

$$M^\lambda = \langle \{1/2/3\}, \{1/3/2\}, \{2/1/3\}, \{2/3/1\}, \{3/1/2\}, \{3/2/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

$$\begin{aligned} C(T) &= S_3 \\ v_T &= \{1/2/3\} - \{1/3/2\} - \{2/1/3\} \\ &\qquad + \{2/3/1\} + \{3/1/2\} - \{3/2/1\} \\ S^\lambda &= \mathbb{C}[S_3] \cdot v_T = \mathbb{C} v_T \\ (23) v_T &= \{1/3/2\} - \{1/2/3\} - \{3/1/2\} \\ &\qquad + \{3/2/1\} + \{2/1/3\} - \{2/3/1\} = -v_T \end{aligned}$$

Example: standard representation

$$\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} = 2 \cdot 1$$

$$M^\lambda = \langle \{12/3\}, \{13/2\}, \{23/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

$$C(T) = \{\mathrm{Id}, (13)\}$$

$$v_T = \{12/3\} - \{23/1\} \qquad 3-1$$

$$S^\lambda = \mathbb{C}[S_3] \cdot v_T \cong \mathbb{C}^3$$

$$(12)v_T = \{12/3\} - \{13/2\} \qquad 3-2$$

$$(13)v_T = \{23/1\} - \{12/3\} = -v_T$$

$1 - 3$

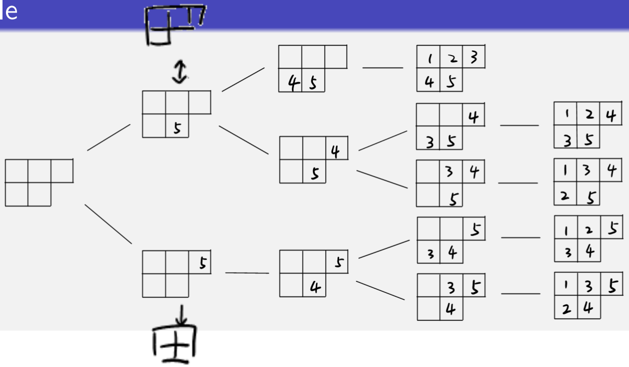
Goal of the Part 1

- Explicitly construct irreducible representations of S_n by Young diagram;
- Compute the character table;
  - $\dim \chi_i$ by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g. $\otimes$, Sym^m, Λ^m;
  - e.g. M_λ.

Example: dimension of irreducible representation

$$\dim S^\lambda = \#\{\text{standard tableau of } \lambda\} = ?$$

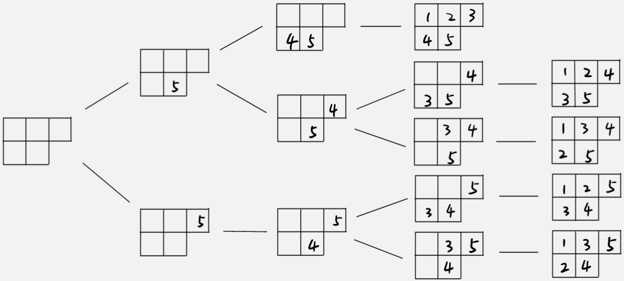
Example



Example: dimension of irreducible representation

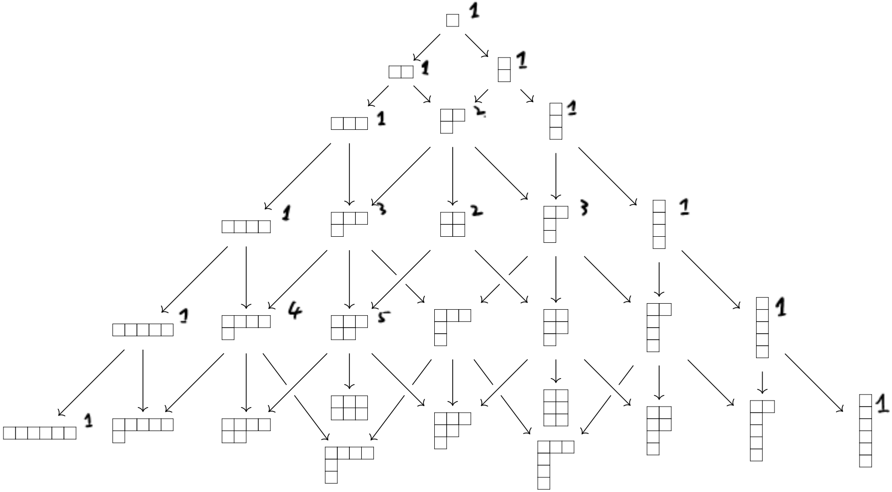
$$\dim S^{\lambda} = \#\{\text{standard tableau of } \lambda\} = \,?$$

Example

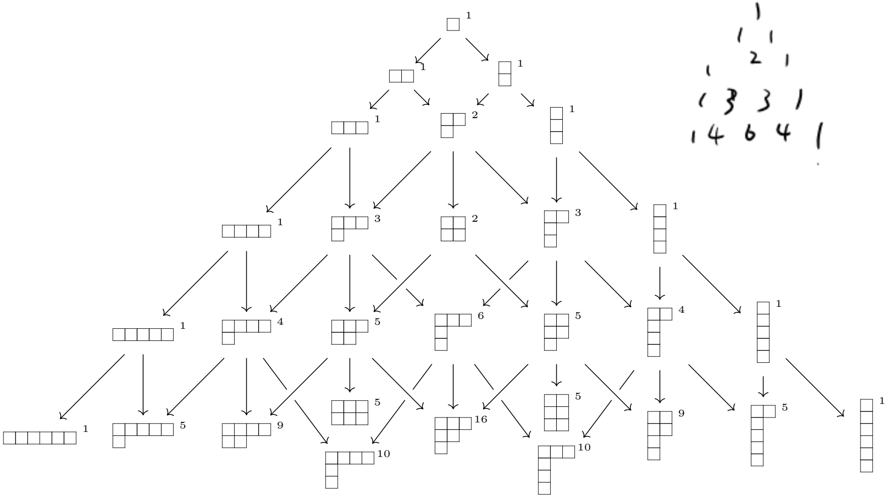


$$\rightarrow \dim S^{\lambda} = \sum_{\substack{\lambda' \subseteq \lambda \\ |\lambda'| = n-1}} \dim S^{\lambda'}$$

Example: dimension of irreducible representation

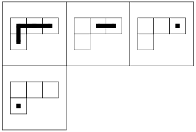


Example: dimension of irreducible representation



Hook length formula

It helps us compute the dimension of S^λ without induction.
Step 1: count the length of hook.



$\Longrightarrow$

4	2	1
1		

Step 2: $\dim S^\lambda = \dfrac{n!}{\prod(\text{hook lengths})}$

Special case: (m, l)

Ballot problem

In an election where candidate A receives m votes and candidate B receives l votes with $m \geqslant l$, what is the probability that A will be (non-strictly) ahead of B throughout the count?

Proposition

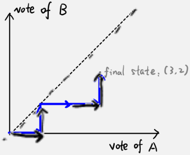
Each process of the count corresponds to each standard tableau of form (m, l).

Example

$(m.l) = 3,2$

A	1	3	4
B	2	5	

$\longleftrightarrow$



Special case: (m, m)

Corollary

$$\underline{\dim S^{(m,m)}} = C_m = \frac{1}{m+1}\binom{2m}{m}.$$

where C_m is the n-th Catalan number.

Special case: (m, m)

Corollary

$$\dim S^{(m,m)} = C_m = \frac{1}{m+1}\binom{2m}{m}.$$

where C_m is the n-th Catalan number.

Catalan number has many interpretations. For example, it counts the number of crossingless matchings of $2n$ points.

Ex. $m=3$

crossingless matchings of 6 points

$\{(1,2),(3,4),(5,6)\}$

$\{(1,4), \cdots\}$

Special case: (m, m)

Corollary

$$\dim S^{(m,m)} = C_m = \frac{1}{m+1}\binom{2m}{m}.$$

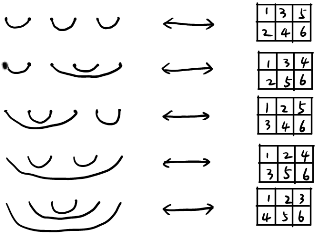
where C_m is the n-th Catalan number.

Catalan number has many interpretations. For example, it counts the number of crossingless matchings of $2n$ points.

Ex. $m=3$

crossingless matchings of 6 points

Young tableau of (3,3) type



5 2

Goal of the Part II

- Definition of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension? $X = \sqcup \mathbb{C}^k$
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top cohomology.

Goal of the Part II

- Definition of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top cohomology.

Definition

$$\tilde{\mathfrak{g}} \subseteq \mathfrak{g} \times \mathcal{B} \longrightarrow \mathcal{B}_{:=\mathcal{F}\ell(n)} \qquad\qquad \tilde{\mathcal{N}} \qquad \mu^{-1}(X)$$

$$\downarrow \mu \qquad\qquad \rightsquigarrow \qquad\qquad \downarrow \mu|_{\mathcal{N}\times \mathcal{B}} \qquad \downarrow$$

$$\mathfrak{g}_{:=\mathfrak{sl}_n(\mathbb{C})} \qquad\qquad\qquad\qquad \mathcal{N} \qquad\qquad X$$

resolution of nilpotent cone

Let $X \in \mathfrak{g}$ be a nilpotent element. The Springer fiber B_X over X is defined as

$$\begin{aligned}\mathfrak{B}_X :=& \mu^{-1}(X)\\ \overset{\text{def}}{=}& \{B \in \mathfrak{B} \mid X \in B\} \qquad\qquad \curvearrowleft X\\ \overset{\mathfrak{g}=\mathfrak{sl}_n(\mathbb{C})}{=}& \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n | XV_i \subseteq V_{i-1}\} \quad \dim V_i = i\end{aligned}$$

$\uparrow$

By the Jordan normal form, we have

$$\left\{\begin{matrix}\text{Nilpotent element}\\ \text{in } \mathfrak{gl}_n(\mathbb{C})\end{matrix}\right\}\Big/_{\text{conj}} \longleftrightarrow \left\{\begin{matrix}\text{Young diagram}\\ \text{of } n \text{ boxes}\end{matrix}\right\}$$

$$X_\lambda = \mathrm{diag}(\underbrace{J_{\lambda_1}, \ldots, J_{\lambda_1}}_{v_1}, J_{\lambda_2}, \ldots, J_{\lambda_k}) \longleftrightarrow \lambda = \lambda_1^{v_1} \cdots \lambda_k^{v_k}$$

$$J_{\lambda_i} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}_{\lambda_i \times \lambda_i}$$

Denote $B_\lambda := B_{X_\lambda}$. $B_X \cong B_{gXg^{-1}}$ for any $g \in G$

Theorem (we will not give the proof.)

As S_n-representation, $S^\lambda \cong H^{\cancel{top}}_{top.}(B_\lambda)$.

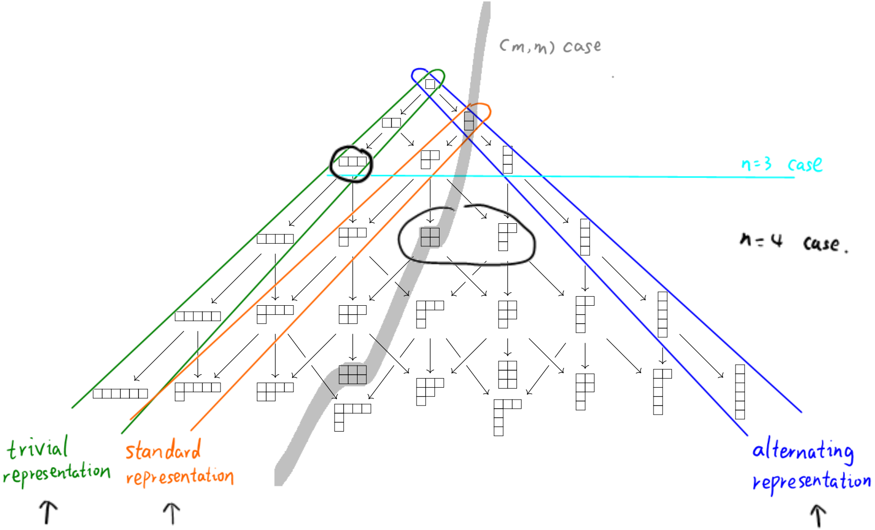
Corollary

$\#\{$*irreducible component of* $B_\lambda\} = \dim S^\lambda$

Goal of the Part II

- Definition of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top cohomology.

tree of Young diagram



Example: $\lambda = 3$

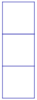
$\{e_1, \dots e_n\} \subseteq \mathbb{C}^3$

$$X_\lambda = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle \overset{e_1}{?} \rangle \subseteq \langle \overset{e_1}{?}, \overset{e_2}{?} \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \quad = \{*\}$$

In general, $B_\lambda = \{*\}$ when λ has only one row.

Example: $\lambda = (1,1,1)$



$$X_\lambda = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

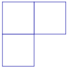
$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \quad = \mathcal{F}\ell(3)$$

In general, $B_\lambda = \mathcal{F}\ell(n)$ when $\lambda = 1^n$.

Properties of $B_\lambda = \mathcal{F}\ell(n)$

- irreducible: ✓
- $dim B_\lambda =$ $\frac{n(n-1)}{2}$
- CW complex: Schubert Cell.
- cohomology group: ring. [Xiong]
- smooth: ✓
- explicit description: $\begin{cases} ① \text{ local chart.} \\ ② \text{ fiber bundle.} \end{cases}$ $\overset{\{*\}}{\mathcal{F}\ell(1)} - \underset{\substack{| \\ \mathbb{P}^1}}{\mathcal{F}\ell(2)} - \underset{\substack{| \\ \mathbb{P}^2}}{\mathcal{F}\ell(3)} - \underset{\substack{| \\ \mathbb{P}^3}}{\mathcal{F}\ell(4)}$
- Weyl group action on $H^{\cancel{top}}_{top}(B_\lambda) \cong \mathbb{C}$:

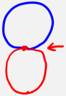
Example: $\lambda = (2,1)$



$$\underline{X_\lambda} = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\underline{B_\lambda} = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \quad = \underline{\mathbb{P}^1 \vee \mathbb{P}^1}$$

$$B_\lambda = \begin{array}{c} \{0 \subseteq \langle ae_1+e_3 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3\} \xrightarrow{a \in \mathbb{C}} \\ \cup \\ \{0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, be_2+ce_3 \rangle \subseteq \mathbb{C}^3\} \longrightarrow \end{array}$$



$\leftarrow \{0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3\}$

In general, $B_\lambda = \underbrace{\mathbb{P}^1 \vee \cdots \vee \mathbb{P}^1}_{n-1}$ when $\lambda = (n-1, 1)$.

Properties of $B_\lambda = \underbrace{\mathbb{P}^1 \vee \cdots \vee \mathbb{P}^1}_{n-1}$

- irreducible component: $n-1$
- $dim B_\lambda = 1$
- affine paving: Yes
- cohomology group:
- smooth: ✓
- explicit description: ✓
- Weyl group action on $H^{\cancel{\text{top}}}_{\text{top}}(B_\lambda) \cong \mathbb{C}^{n-1}$:

$$H^i(B_\lambda) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i=1 \\ \mathbb{C}^{n-1} & i=2 \end{cases}$$

$$H_i(B_\lambda) = \begin{cases} \mathbb{C} & i=0 \\ 0 & i=1 \\ \mathbb{C}^{n-1} & i=2 \end{cases}$$

Tool: stratification/cellular fibration/affine paving

$$\begin{array}{l} \phi \longrightarrow \\ B_{\lambda'} \longrightarrow \\ \vdots \\ B_{\lambda''} \longrightarrow \end{array} \quad \begin{array}{c} B_\lambda \\ \downarrow \pi \\ \mathbb{P}^{n-1} \end{array} \qquad \begin{array}{c} \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n\} \circlearrowleft X_\lambda \\ \downarrow \\ [V_1] \end{array}$$

Remark

In general, We don't have a natural CW complex structure.
We don't understand the ring structure.

Return!

For $\lambda = 1^3$, $B_\lambda \cong \mathcal{F}\ell(3)$ can be viewed as $\mathcal{F}\ell(2)$-bundle over $\mathbb{P}^2$.

$$\begin{tikzcd} \mathcal{F}\ell(2) \arrow[r] & \mathcal{F}\ell(3) \arrow[d, two heads, "\pi"] \\ & \mathbb{P}^2 \end{tikzcd}$$

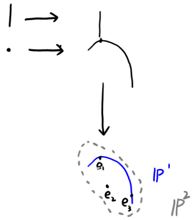
$[v]$

$$\pi^{-1}([v]) = \left\{0 \subseteq \langle v \rangle \subseteq \langle v,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \cong \mathcal{F}\ell(2)$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:

$$\begin{tikzcd} \mathbb{P}^1 = B_{1,1} \arrow[r] & B_{2,1} \arrow[d, "\pi"] \\ \{*\} = B_2 \arrow[ur] & \mathbb{P}^1 \end{tikzcd}$$



$$\pi^{-1}([e_1]) = \left\{0 \subseteq \langle e_1 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\cong \left\{0 \subseteq \langle\ ?\ \rangle \subseteq \mathbb{C}^2\right\} \curvearrowleft \begin{bmatrix} 0 & \cancel{1}\ 0 \\ & 0 \end{bmatrix} \qquad = B_{\cancel{2}\ 1,1}$$

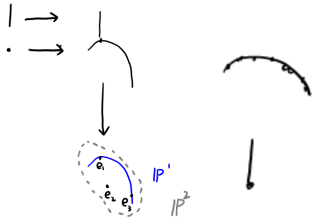
$$\pi^{-1}([e_3]) = \left\{0 \subseteq \langle e_3 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\cong \underline{\left\{0 \subseteq \langle\ ?\ \rangle \subseteq \mathbb{C}^2\right\}} \curvearrowleft \underline{\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix}} \qquad = B_{\cancel{1,1}\ 2}$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:

$$\begin{array}{rcl} \mathbb{P}^1 = B_{1,1} & \longrightarrow & \\ \{*\} = B_2 & \longrightarrow & B_{2,1} \\ & & \downarrow \\ & & \mathbb{P}^1 \end{array}$$



$$\pi^{-1}([e_1]) \cong B_{1,1} \qquad \pi^{-1}([e_3]) \cong B_2$$

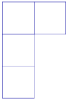
$$\pi^{-1}([ae_1 + e_3]) = \left\{ 0 \subseteq \langle ae_1 + e_3 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$(f_1, f_2, f_3) = (e_1, e_2, ae_1 + e_3)$

$$\cong \underline{\left\{ 0 \subseteq \langle f_3 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\}} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \quad \cong \pi^{-1}([e_3])$$

By this way, $\pi^{-1}(\mathbb{P}^1 \setminus \{[e_1]\}) \cong B_2 \times \mathbb{C}$ induces an affine paving.

Example: $\lambda = (2,1,1)$



$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowleft X_\lambda$$

$\mathcal{F}\ell(3) = B_{1,1,1} \longrightarrow$

$\mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \longrightarrow B_{2,1,1}$

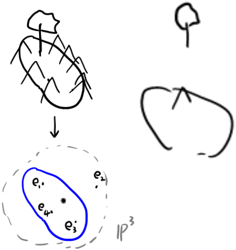
$\downarrow$

$\mathbb{P}^2$

$\mathbb{C}$ $\quad$ $\mathbb{C}^2$

$0 \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^1 \subseteq \mathbb{P}^2 \subseteq \mathbb{P}^3 \subseteq$

3-dim



3-dim

1

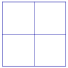
+ 3-dim

2

=

3 compone

Example: $\lambda = (2,2)$



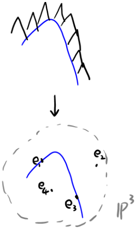
$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

ker X_λ : $\langle e_1, e_3 \rangle$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \circlearrowleft X_\lambda$$

$$\mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \longrightarrow B_{2,2}$$
$$\downarrow$$
$$\mathbb{P}^1$$

$\Lambda \longrightarrow$



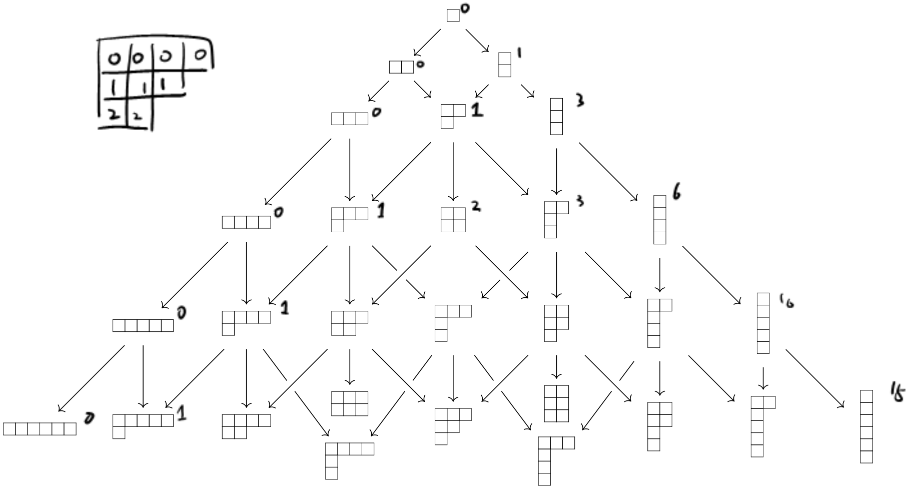
$\# \mathrm{Irr} = 2$

$\dim = 2$

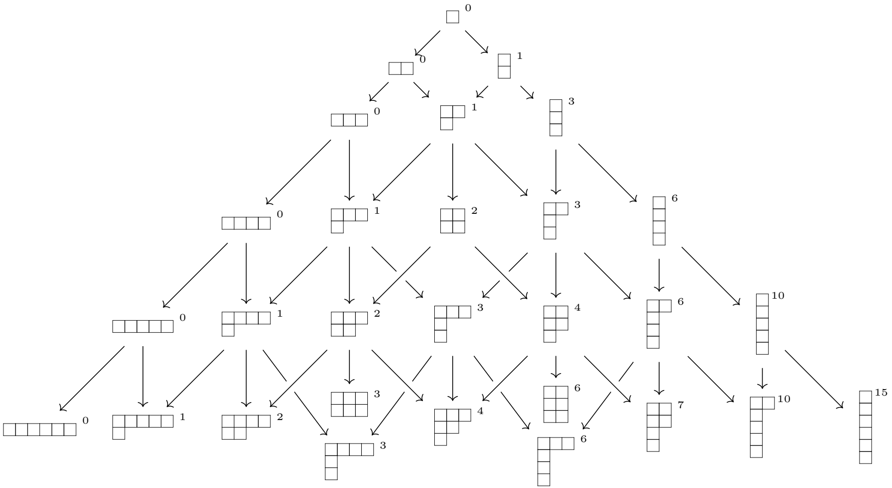
Using the same technique, we can get

- B_λ has an affine paving $\rightsquigarrow$ cohomology;
- Each irreducible component in B_λ has same dimension;
- It's easy to compute the dimension and the number of irreducible component.

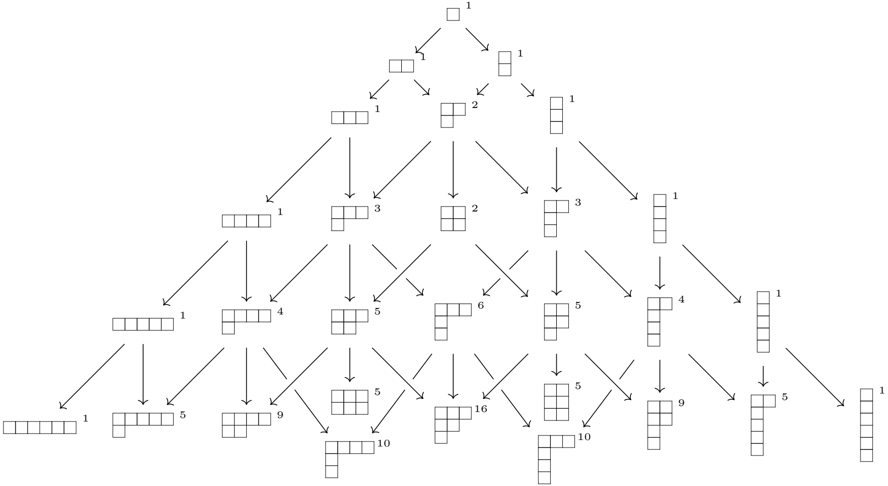
Game: compute!



Answer: dimension



Answer: the number of irreducible component



Smooth problem

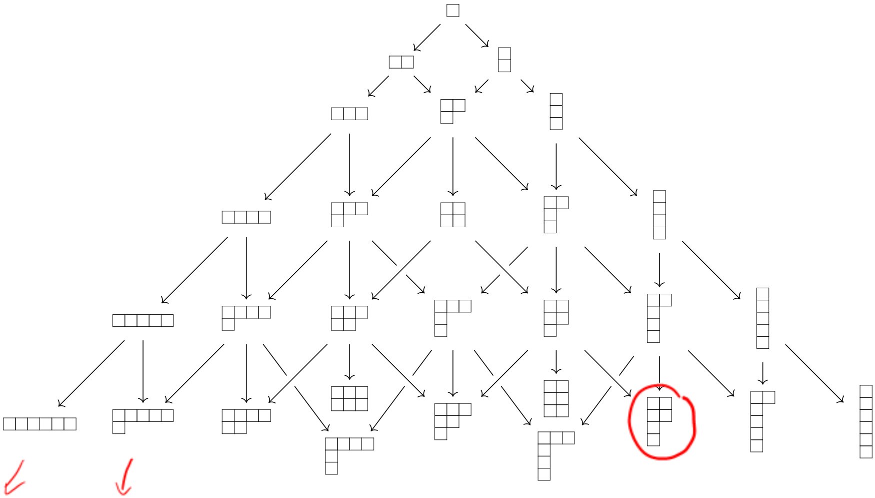
B_λ

Results

- Not all the the irreducible components of B_λ are smooth; For example, one component of $\underline{B_{2,2,1,1}}$ is not smooth. 
- All the components of B_λ are nonsingular iff

$$\lambda \in \{(\lambda_1, 1, 1, \ldots), (\lambda_1, \lambda_2), (\lambda_1, \lambda_2, 1), (2, 2, 2)\}$$

tree of Young diagram



(n, n) case

We have an explicit description in the 2-row case when we forget the variety structure. Use this description, we can get the cohomology group structure.

Definition and Theorem

Let α be a crossingless matching, define 

$$\tilde{B}_{\alpha;\,m,m} := \left\{ (x_1, \ldots, x_{2m}) \in (\mathbb{P}^1)^{2m} \,\middle|\, x_i = x_j \text{ if } (i,j) \in \alpha \right\} \subseteq (\mathbb{P}^1)^{2m}$$

$$\tilde{B}_{m,m} := \bigcup_{\alpha} \tilde{B}_{\alpha;\,m,m} \subseteq (\mathbb{P}^1)^{2m}$$

then we have a homeomorphism

$$B_{m,m} \cong \tilde{B}_{m,m}$$

(n, n) case

Definition and Theorem

Let α be a crossingless matching, define

$$\tilde{B}_{\alpha;\,m,m} := \left\{ (x_1, \ldots, x_{2m}) \in (\mathbb{P}^1)^{2m} \middle| x_i = x_j \text{ if } (i,j) \in \alpha \right\} \subseteq (\mathbb{P}^1)^{2m}$$

$$\tilde{B}_{m,m} := \bigcup_{\alpha} \tilde{B}_{\alpha;\,m,m} \subseteq (\mathbb{P}^1)^{2m}$$

then we have a homeomorphism

$$B_{m,m} \cong \tilde{B}_{m,m}$$

Example (m=2)

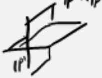
$$\alpha = \{(1,2),(3,4)\} \qquad \tilde{B}_{\alpha;\,2,2} = \left\{ (x_1, x_1, x_2, x_2) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

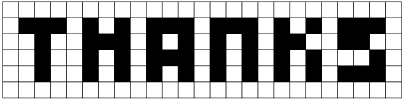
$$\beta = \{(1,4),(2,3)\} \qquad \tilde{B}_{\beta;\,2,2} = \left\{ (x_1, x_2, x_2, x_1) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

$$B_{2,2} \cong \tilde{B}_{2,2} \cong (\mathbb{P}^1)^2 \bigvee_{\mathbb{P}^1} (\mathbb{P}^1)^2$$

$\mathbb{P}^1$-bundle over $\mathbb{P}^1$ $\quad$ $\mathbb{P}^1$ $\quad$ $\mathbb{P}^2$

$\mathbb{P}^1 \times \mathbb{P}^1$





Thank you for listening!

Thank Rui Xiong for providing the package of Young diagram,
Thank my roommate David Cueto for pointing out typos,
Thank Prof. Eberhart for offering valuable materials and advice!