二阶椭圆 PDE 作业

第一周

1 Exercise 1

取 $H = H_0^2(\Omega), u \in H_0^2(\Omega)$, 则 $\|u\|_H = \|\Delta u\|_{L^2(\Omega)}$. 再取

$$a(u,v) := \int_\Omega \Delta u \Delta v dx$$

$$F : H_0^1(\Omega) \longrightarrow \mathbb{R} \qquad v \longmapsto \int_\Omega f v dx$$

则有

- $F \in H^{-1}(\Omega)$
- $$\begin{aligned} |a(u,v)| &\leqslant \int_\Omega |\Delta u||\Delta v| dx \\ &\leqslant C\|\Delta u\|_{L^2}\|\Delta v\|_{L^2} \\ &\leqslant \tilde{C}\|u\|_{H_0^2(\Omega)}\|v\|_{H_0^2(\Omega)} \end{aligned}$$
- $$\begin{aligned} a(u,u) &= \int_\Omega \Delta u \Delta u dx \\ &= \|\Delta u\|_{L^2(\Omega)} \\ &\geqslant C'\|u\|_{H_0^2(\Omega)} \end{aligned}$$

则由 lax-milgram 引理, 我们有唯一的 $u \in H$, 使得 $a(u,v) = \langle F, v\rangle$,i.e.

$$\int_\Omega \Delta u \Delta v dx = \int_\Omega f v dx \text{ for any } v \in H_0^2(\Omega)$$

故此双调和方程存在唯一的弱解.

2 Exercise 2

回忆

$$W^{1,p}(B_1) = \{u \in L^p(B_1) \mid u_{x_i} \in L^p(B_1)\}$$

我们有

$$\begin{aligned}
& u(x) = |x|^{-\alpha} \in L^p(B_1) \\
\Leftrightarrow & \int_{B_1} |x|^{-\alpha} dV < +\infty \\
\Leftrightarrow & \int_0^1 r^{n-1} r^{-\alpha p} dr < +\infty \\
\Leftrightarrow & (n-1) - \alpha p > -1 \\
\Leftrightarrow & \alpha < \frac{n}{p}
\end{aligned}$$

$$\begin{aligned}
u_{x_i}(x) &= -\frac{\alpha}{2}|x|^{-\alpha-2}\frac{\partial |x|^2}{\partial x_i} \\
&= -\alpha |x|^{-\alpha-2} x_i
\end{aligned}$$

$$\begin{aligned}
& u_{x_i}(x) = -\alpha |x|^{-\alpha-2} x_i \in L^p(B_1) \\
\Leftrightarrow & \int_{B_1} \alpha^p (|x|^{-2\alpha-4}|x_i|^2)^{p/2} dV < +\infty \\
\Leftrightarrow & \sum_{i=1}^{n} \int_{B_1} (|x|^{-2\alpha-4}|x_i|^2)^{p/2} dV < +\infty \\
\Leftrightarrow & \int_{B_1} (|x|^{-\alpha-1})^p dV < +\infty \\
\Leftrightarrow & u(x) = |x|^{-\alpha-1} \in L^p(B_1) \\
\Leftrightarrow & \alpha < \frac{n-p}{p}
\end{aligned}$$

故

$$u(x) = |x|^{-\alpha} \in W^{1,p}(B_1) \Longleftrightarrow \alpha < \frac{n-p}{p}$$

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