

Subvarieties in Complex Abelian Varieties

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Tangent Gauss Map

Let A/\mathbb{C} be an abelian variety of dimension n , and let $Z \subset A$ be a non-degenerate closed subvariety of dimension r .

To understand the geometry of Z , we encode the variation of its tangent spaces via the tangent Gauss map

$$\phi_Z : Z^{\text{sm}} \longrightarrow \text{Gr}(r, T_0 A) \quad p \mapsto T_p Z \subset T_p A \cong T_0 A.$$

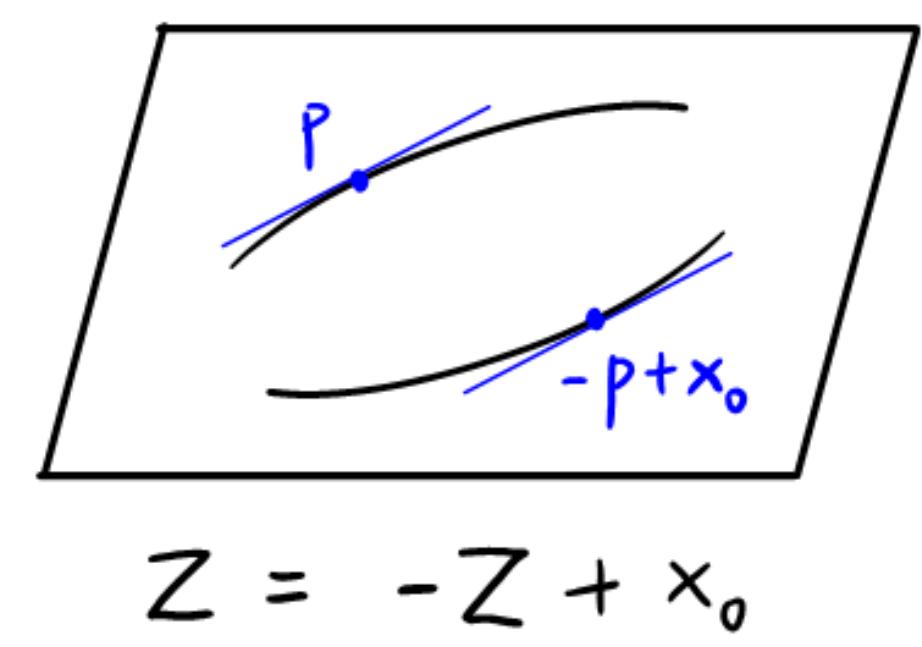
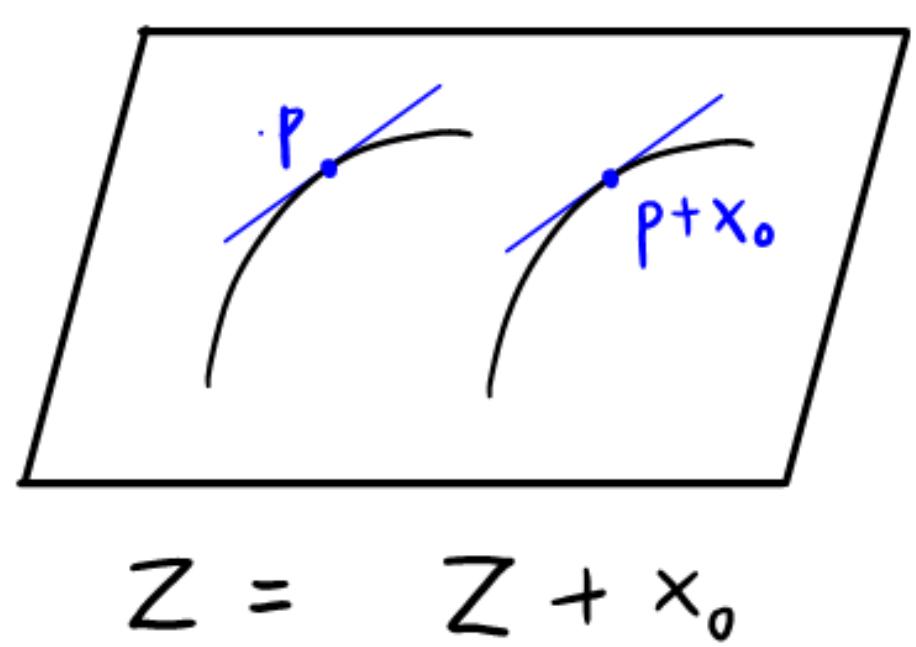
Its differential

$$d_p \phi_Z : T_p Z \longrightarrow \text{Hom}_{\mathbb{C}}(T_p Z, N_p Z)$$

is the second fundamental form, which captures information about the curvature of Z .

Generic Injectivity of the Tangent Gauss Map

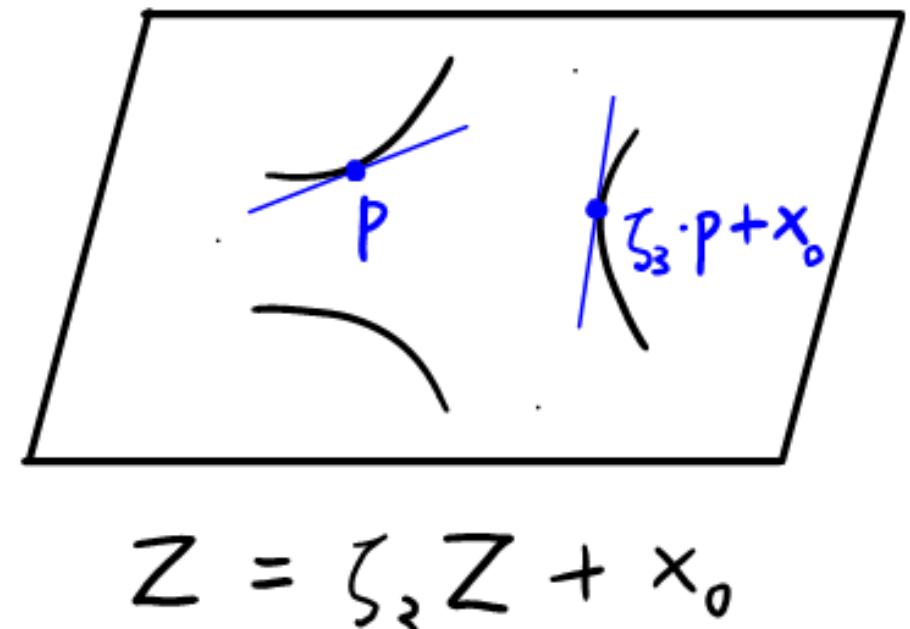
Clearly the Gauss map has degree > 1 whenever the subvariety $Z \subset A$ is stable under a nontrivial translation or symmetric up to a translation, as shown below:



Are these the only cases where the Gauss map fails to be generically injective?

We specialize to the case where $Z = C$ is a curve. When $n = 2$, $\phi_C : C^{\text{sm}} \longrightarrow \mathbb{P}^1$ typically fails to be generically injective.

Question 1. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$. If C is not invariant under any non-trivial translation or reflection, then ϕ_C is generically injective.



Example 1. For any n the power $A = E_{\rho}^{\oplus n}$ comes with a natural diagonal action of $\mu_3 \cong \mathbb{Z}/3\mathbb{Z}$. Computer experiments yield a non-degenerate μ_3 -invariant curve $C \subset A$, for which ϕ_C is not generically injective.

We have found no counterexample to Question 1 when A is not isogenous to $E_i^{\oplus n}$ or $E_{\rho}^{\oplus n}$. This suggests the following refinement:

Conjecture 2. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$. Then ϕ_C is generically injective unless there exists a non-trivial automorphism $\tau \in \text{Aut}(A)$ which preserves C and acts via a scalar on $T_0 A$.

One may restate the conjecture using Gauss curvature, yielding a slightly stronger statement:

Conjecture 3. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$. Then for general $p \in \text{Im } \phi_C$, the Gauss curvature of $C \subset A$ is the same at all points of the fiber $\phi_C^{-1}(p)$.

Known Cases

- If $A = \text{Jac}(C)$ and C is embedded via the Abel–Jacobi map, then $\phi_C = |\omega_C|$ is the canonical map, and one may check Conjectures 2 and 3 by direct inspection.

- Let $h : C \longrightarrow C'$ be a cyclic k -fold cover defined by $\eta \in \text{Pic}(C')$ with $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. If $A = \text{Prym}(C/C')$ and $C \rightarrow A$ is the Abel–Prym map, then

$$\phi_C : C \longrightarrow \mathbb{P} T_0 A \cong \mathbb{P} \left(\bigoplus_{i=1}^{k-1} H^0(\omega_{C'} \otimes \eta^i) \right)$$

- $k = 2$: C is invariant under the Prym involution, and $\phi_C = |\omega_{C'} \otimes \eta| \circ h$. If C' is non-hyperelliptic with $g(C') \geq 4$, then $\deg \phi_C = 2$ or 4, and

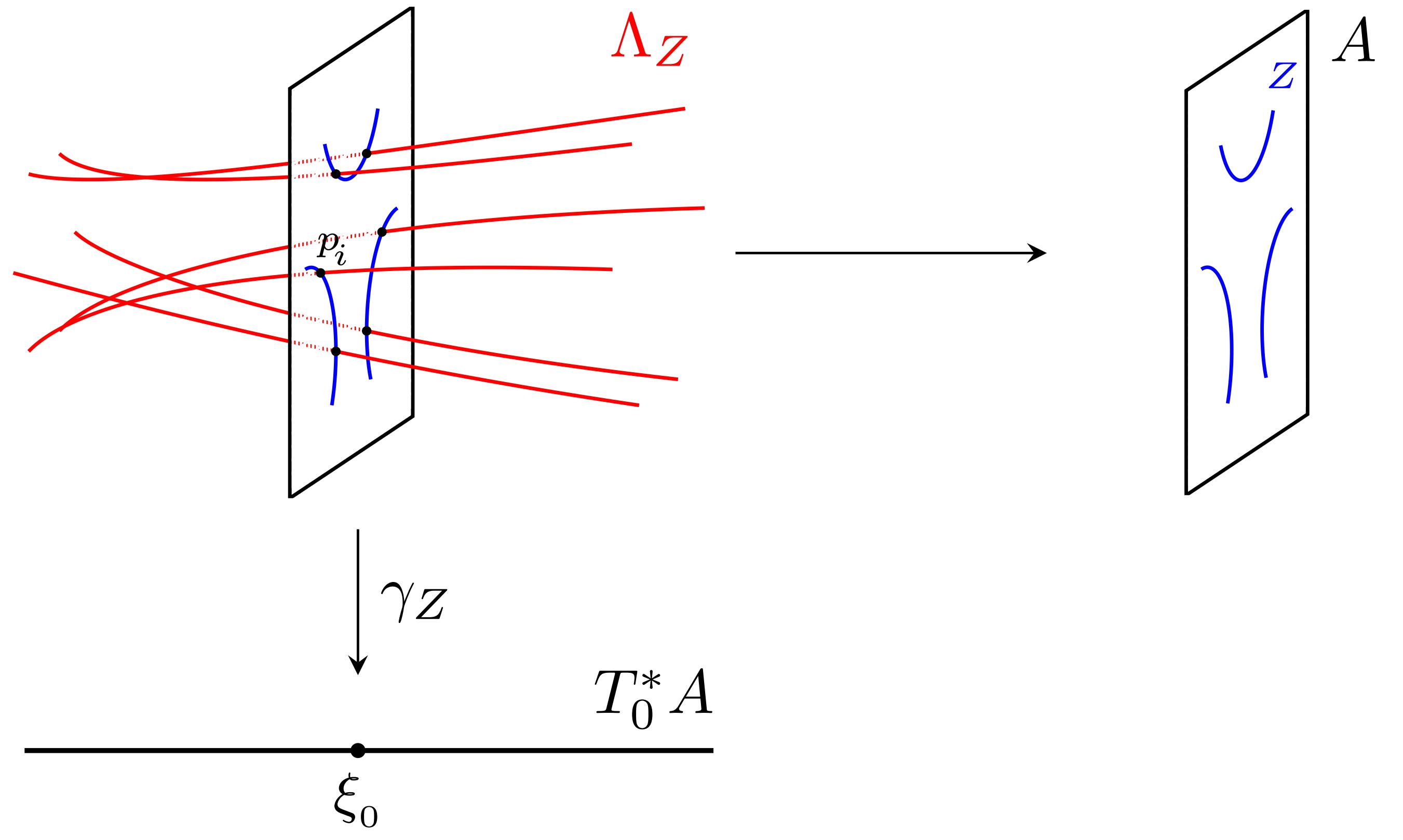
$$\deg \phi_C = 4 \iff B = \emptyset, C' \text{ is bielliptic and } \eta \text{ pulled back from EC.}$$

- $k > 2$: if $g(C') \geq 1$ and $|\omega_{C'} \otimes \eta|$ is generically injective, then ϕ_C is generically injective.

- If $C \subset A$ is smooth and either $\deg \phi_C = 2$ or ϕ_C is unramified, Conjecture 3 also holds.

Conormal Gauss Map

Consider the conormal variety $\Lambda_Z \subset T^* A \cong A \times T_0^* A$. The natural projection is the conormal Gauss map $\gamma_Z^{\text{aff}} : \Lambda_Z \longrightarrow T_0^* A$, which is generically finite whenever Z is of general type.



Conjecture 4. Suppose A is not isogenous to $E_i^{\oplus n}$ or $E_{\rho}^{\oplus n}$, and let $C \subset A$ be a non-degenerate curve which is not stable under any non-trivial translation or reflection. Then the monodromy group $\text{Gal}(\gamma_C)$ is big – namely, a Weyl group of type A , C , or D .

When $n > 2$, Conjecture 4 follows from Conjecture 3.

The monodromy group $\text{Gal}(\gamma_C)$ helps us to determine controls the Tannaka group of the tensor category generated by the IC sheaf on C ; see [Krä22, Theorem 2.1].

The Subvariety $Z^{(m)}$

The convolution structure on perverse sheaves gives rise to numerous cycles in A . They admit a simple geometric description: by fiberwise summing points in $\gamma_Z^{-1}(\xi_0)$ and projecting, one obtains new subvarieties of varying dimensions.

Fix a general point $\xi_0 \in T_0^* A$ and choose an ordering $\gamma_Z^{-1}(\xi_0) = \{p_1, \dots, p_d\} \subset Z$. For $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$, let $Z^{(m)}$ denote the irreducible component of the subvariety obtained by this construction that contains the point $\sum_i m_i p_i$.

Theorem 1. Let $c_i := c_{M,i}(\Lambda_Z)$ be the Segre class of the cone Λ_Z , and let $*$ denote the Pontryagin product.

When $\text{Gal}(\gamma_Z) = S_d$, the Segre classes of $\Lambda_{Z^{(m)}}$ can be written as

$$c_{M,l}(\Lambda_{Z^{(m)}}) = \frac{1}{d_Z^{(m)}} \sum_{\lambda \vdash l} \mu_d^{\lambda} \left(\underset{i=1}{\overset{k'}{*}} c_{\lambda_i} \right)$$

where $\lambda = [\lambda_1, \dots, \lambda_{k'}]$ ranges over all partitions of l and where

- $d_Z^{(m)} \in \mathbb{N}_{>0}$ is the degree of a certain addition map;
- $\mu_d^{\lambda} = \sum_{\alpha \in \mathcal{P}(d)} \sum_{\substack{l: \text{length } k \\ \text{length } k}} \mu(\hat{0}, \alpha) \alpha(m)^{2l} d^{k-k'} \in \mathbb{Z}[m_1, \dots, m_d]^{S_d}$
- $\alpha = \{A_1, \dots, A_k\} \in \mathcal{P}(d)$, $\mathbf{l} = (l_1, \dots, l_k) \in \mathbb{Z}^k$;
- $\mu(\hat{0}, \alpha) = (-1)^{d-k} \prod_{i=1}^k (|A_i| - 1)$;
- $\alpha(m)^{2l} = (\sum_{i \in A_1} m_i)^{2l_1} \cdots (\sum_{i \in A_k} m_i)^{2l_k}$.

Remarks.

- One can recover both $\dim Z$ and $[Z] \in H^{2(n-\dim Z)}(A)$ from the Segre classes of Λ_Z :

$$\dim Z = \max \{i \in \mathbb{Z} \mid c_i \neq 0\}, \quad [Z] = c_{\dim Z}.$$

- If $Z = -Z$, then a similar formula also holds for $\text{Gal}(\gamma_Z) = W(C_{d/2})$, but the method does not extend to the case where $\text{Gal}(\gamma_Z) = W(D_{d/2})$.

- The formula simplifies a lot for curves inside their Jacobian.

For example, if C is non-hyperelliptic, we obtain:

$$c_l(\Lambda_{Z^{(m)}}) = \frac{1}{c_Z^{(m)}} \frac{1}{2^l (g-l)!} \sum_{\sigma \in S_{2g-2}} \prod_{i=1}^l (m_{\sigma(2i-1)} - m_{\sigma(2i)})^2 \cdot \Theta^{g-l}$$

$$\dim Z^{(m)} = \min_{k \in \mathbb{Z}} \{g-1, \# \{i \in [2g-2] \mid m_i \neq k\}\}$$

References

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 [Krä22] Thomas Krämer. Characteristic cycles and the microlocal geometry of the Gauss map. I. *Ann. Sci. Éc. Norm. Supér.* (4), 55(6):1475–1527, 2022.
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