Dessin d'enfant: an Introduction

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Abstract

In this talk, we will talk about **the relation between Belyi map and dessin d'enfant**, and than extract imformations from the dessin.
Contents: section 4.1-4.3 and some examples from section 4.6.

Contents

Introduction

Last time, we talked about the Belyi's Theorem:

Theorem (Thm 3.1)

Let S be a cpt RS, then S is defined over $\bar{\mathbb{Q}}$ iff S admits a Belyi fct.

This time, we talked a specific Belyi fct (ramified at $0, 1, \infty$), and

- combine it with a kind of **special graph** (on S);
- extract information from this graph.

Contents

Black box

Prop 3.34: Belyi fcts are defined over $\bar{\mathbb{Q}}$.

Remark

We can talk about the Galois group

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \curvearrowright \left\{ \begin{array}{c} S \\ \downarrow f \\ \mathbb{P}^1 \end{array} \right\}$$

Example

When $S = \mathbb{P}^1$, then an Belyi fct $f(z)$ (not constant since we consider fct ramified on three points) is an rational fct with **coefficient in $\bar{\mathbf{Q}}$** such that f maps any zero or pole of $f'(z)$ to $0, 1, \infty$. In short,

- $f(z) \in \bar{\mathbb{Q}}(z)$;
- For any z_0 such that $f'(z_0) = 0$ or ∞, $f(z_0) = 0, 1$ or ∞.

For example,

- $f(z) = z^n$;
- $f(z) = -\dfrac{256}{27} z^3 (z-1)$;
- $f(z) = \dfrac{3+i}{5} z^3 (z-1)^2 \left(z - \dfrac{4}{25}(4+3i) \right)$;
- $f(z) = \dfrac{4}{27} \dfrac{(1-z+z^2)^3}{z^2(z-1)^2}$;
- $f(z) = C \dfrac{z^4(z-1)^2}{z - \frac{11+2\sqrt{10}}{18}}, \qquad C \approx -9.55063.$

Contents

Better question

We postponed the abstract definition of the dessin d'enfant.

A better question: How to draw a dessin d'enfants from a Belyi fct?

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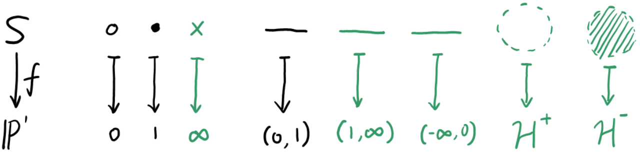
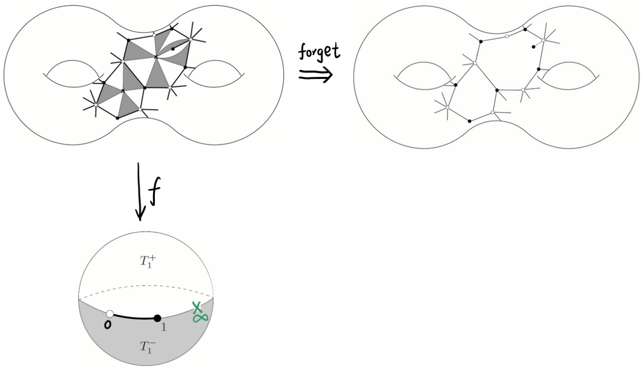


Better question

We postponed the abstract definition of the dessin d'enfant.

A better question: How to draw a dessin d'enfants from a Belyi fct?



Proposition

- *the graph D is drawed on the RS S;*
- *D is bicolored;* 
- *$X \setminus D$ is union of finitely many topo discs;*

- *D is connected.*

Abstract definition

Definition (Def 4.1)

A dessin d'enfant, *or simply* ***a dessin***, *is a pair* $(X, \mathcal{D})$ *where* X *is an* ***oriented*** *compact topological surface, and* $\mathcal{D} \subset X$ *is a finite graph such that:*

- $\mathcal{D}$ *is* ***connected***.
- $\mathcal{D}$ *is* ***bicoloured***, *i.e. the vertices have been given either white or black colour and vertices connected by an edge have different colours.*
- $X \setminus \mathcal{D}$ *is the union of finitely many* ***topological discs***, *which we call* ***faces*** *of* $\mathcal{D}$.

Pictures

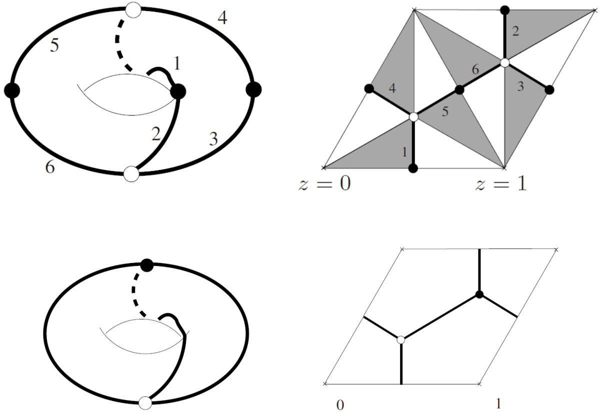


Fig. 4.8. Another dessin inducing the Riemann surface of Example 4.21.

Pictures

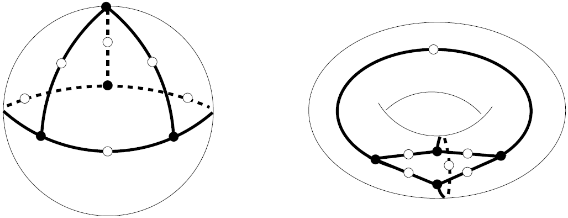


Fig. 4.1. Two dessins with the same underlying abstract graph.

Pictures

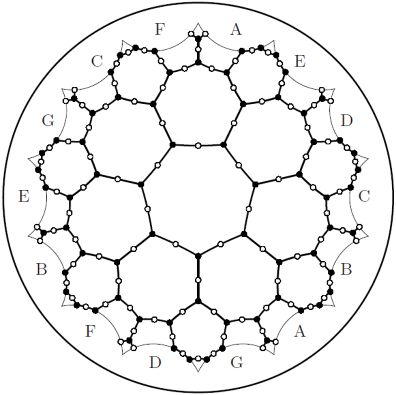
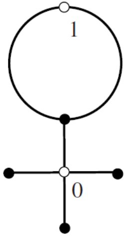
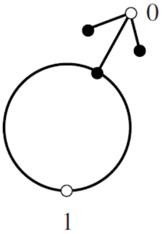


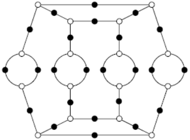
Fig. 4.16. A $(2, 3, 7)$-regular dessin having Klein's surface of genus 3 as the underlying Riemann surface. The side-pairing identifications defining the group K are represented here by the letters at the sides of the 14-gon.

Pictures

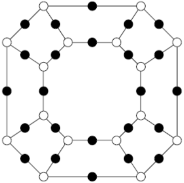




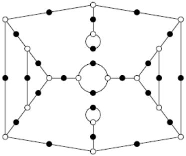
Pictures



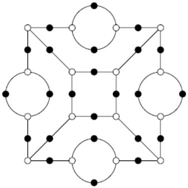
$\Gamma_1(8) \cap \Gamma(2)$
8, 8, 8, 8, 4, 4, 2, 2, 2, 2



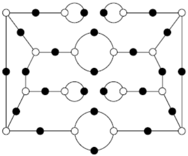
$\Gamma(8; 2, 1, 2)$
8, 8, 4, 4, 4, 4, 4, 4, 4, 4



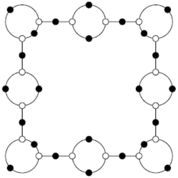
$\Gamma_1(12)$
12, 12, 6, 4, 4, 3, 3, 2, 1, 1



$\Gamma(12; 6, 1, 2)$
12, 6, 6, 6, 6, 4, 2, 2, 2, 2



$\Gamma_0(16) \cap \Gamma_1(8)$
16, 16, 4, 4, 2, 2, 1, 1, 1, 1



$\Gamma(16; 8, 2, 2)$
16, 16, 2, 2, 2, 2, 2, 2, 2, 2

Equivalence

Proposition (Prop 4.20)

*We have an **equivalence***

$$\left\{ \begin{matrix}\text{Belyi fcts}\\ f: S \longrightarrow \mathbb{P}^1\end{matrix}\right\} \Big/ \sim \quad \Longleftrightarrow \quad \left\{\begin{matrix}\text{Dessins d'enfants}\\ D \subset X\end{matrix}\right\} \Big/ \sim$$

$\sim$:

$$\begin{tikzcd} S \arrow[rr, "\sim"] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & S' \arrow[dl] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \qquad \begin{tikzcd} X \arrow[r, "\sim"] & X' \\ D \arrow[u, phantom, sloped, "\subset"] \arrow[r, "\sim"] \arrow[ur, phantom, "\circlearrowleft"] & D' \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

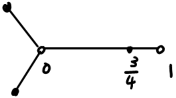
Example: $S = X = \mathbb{P}^1$

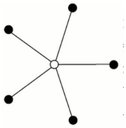
$$f(z) = z^n$$

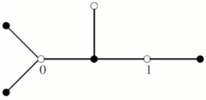
$$f(z) = -\frac{256}{27} z^3 (z-1)$$

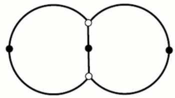
$$f(z) = \frac{3+i}{5} z^3 (z-1)^2 \left(z - \frac{4}{25}(4+3i) \right)$$

$$f(z) = \frac{4}{27} \frac{(1-z+z^2)^3}{z^2(z-1)^2}$$



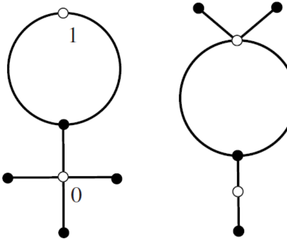






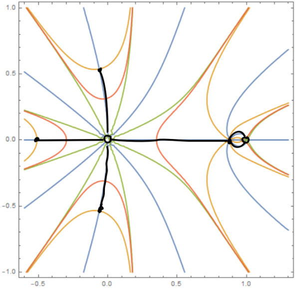
Example: $S = X = \mathbb{P}^1$

Which one is which?

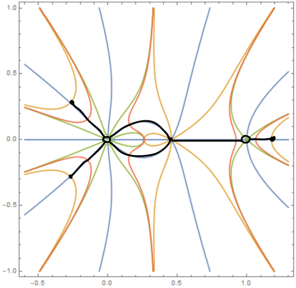


$$f(z) = C\frac{z^4(z-1)^2}{z - \frac{11+2\sqrt{10}}{18}} \qquad f(z) = C'\frac{z^4(z-1)^2}{z - \frac{11-2\sqrt{10}}{18}}$$

Example: $S = X = \mathbb{P}^1$



$$f(z) = C\frac{z^4(z-1)^2}{z - \frac{11+2\sqrt{10}}{18}}$$

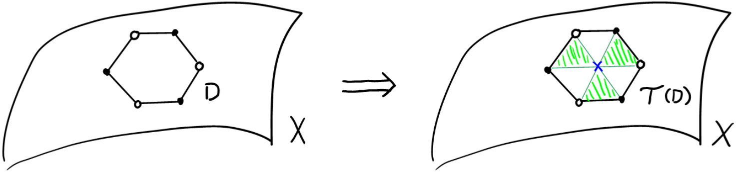


$$f(z) = C'\frac{z^4(z-1)^2}{z - \frac{11-2\sqrt{10}}{18}}$$

Proof of Prop 4.20: Step 1

Given a pair (X, D), we need

- give a RS structure of X;
- give a fct $f\colon X \longrightarrow \mathbb{P}^1$.



$$\begin{tikzcd} T(D) \arrow[r] \arrow[d, phantom, "\cap" rotate=0] & \mathbb{R} \cup \{\infty\} \arrow[d] \\ X \arrow[r, dashed, "\exists f"'] & \mathbb{P}^1 \end{tikzcd}$$

f gives a RS structure of X (Riemann compactification)

Proof of Prop 4.20: Step 2

Proposition (Prop 4.20)

We have an ***equivalence***

$$\left\{\begin{array}{c}\text{Belyi fcts}\\ f: S\longrightarrow \mathbb{P}^1\end{array}\right\}\Big/\sim \quad\Longleftrightarrow\quad \left\{\begin{array}{c}\text{Dessins d'enfants}\\ D\subset X\end{array}\right\}\Big/\sim$$

$$\begin{tikzcd} S \arrow[rr, "\varphi", "\sim"'] \arrow[dr, "f"'] & & S' \arrow[dl, "g"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} D' \arrow[r] \arrow[d, phantom, "\cap"] & \mathbb{R}\cup\{\infty\} \arrow[d, hook] \\ X \arrow[r, dashed, shift left, "f"] \arrow[r, dashed, shift right, "g"'] & \mathbb{P}^1 \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\varphi", "\sim"'] \arrow[d, phantom, "\cup"] & X' \arrow[d, phantom, "\cup"] \\ D \arrow[r, "\sim"] & D' \end{tikzcd}$$

$$\varphi: f^{-1}(X) \xrightarrow{\sim} g^{-1}(X)$$

$$\Downarrow$$

$$\varphi|_D: D \xrightarrow{\sim} D'$$

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & X \arrow[dl, "g"] \\ & \mathbb{P}^1 & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & X \arrow[dl, "f\circ\varphi^{-1}"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

Proof of Prop 4.20: Step 3

$$(X, D) \to (X_D, f_D) \to (X_D, D_{f_D}) \qquad (X, D) \sim (X_D, D_{f_D})?$$

$$(S, f) \to (S, D_f) \to (S, f_{D_f}) \qquad (S, f) \sim (S, f_{D_f})?$$

Remark

Belyi maps have some kind of rigid: they're decided by dessins (be viewed as **skeleton** of Belyi fcts)

Questions

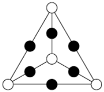
Even though the equivalence seems natural and trivial, I still have some questions unsolved about this.

- Given a Belyi map, can we carry the dessin d'enfant out **in algorithm**, rather than see with eyes? (For example, write down the polynomial representation)
- Given a complex dessin d'enfants defined on $\mathbb{P}^1$, how do we calculate out the **corresponding rational fuctions**? Is there an algorithm for this?

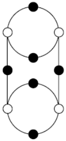
What's corresponding Belyi map?

$$J^{\lambda_2}_{\Gamma(3)}(z) = \frac{(z^4 - 2\sqrt{3}\, i z^2 + 1)^3}{(z^4 + 2\sqrt{3}\, i z^2 + 1)^3}$$

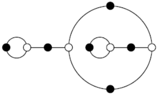
$$J_{\Gamma_0(9)}(z) = \frac{[\, z^4 - 40(1+\sqrt{3}\, i) z^3 - 120(1-\sqrt{3}\, i) z^2 + 248 z - 80(1+\sqrt{3}\, i)]^3}{\left[z + \left(\frac{1}{2}(1+\sqrt{3}\, i)\right)\right]^9 (z-1) \left[z + \frac{1}{2}(1-\sqrt{3}\, i)\right]}$$



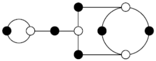
$\Gamma(3)$
$3,3,3,3$



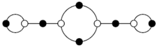
$\Gamma_0(4) \cap \Gamma(2)$
$4,4,2,2$



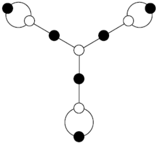
$\Gamma_1(5)$
$5,5,1,1$



$\Gamma_0(6)$
$6,3,2,1$



$\Gamma_0(8)$
$8,2,1,1$



$\Gamma_0(9)$
$9,1,1,1$

Contents

$\{\text{Belyi fct}\}/\sim$	$\{\text{Dessin d'enfants}\}/\sim$
$\#f^{-1}(0) + \#f^{-1}(1)$	v
$\#f^{-1}(\infty)$	f
$\deg f$	e
$2 - 2g(S)$	$v + f - e$
Ram index of $x \in f^{-1}(0)$	$\#$ {black dots adjacent to x}
Ram index of $x \in f^{-1}(1)$	$\#$ {white dots adjacent to x}
Ram index of $x \in f^{-1}(\infty)$	$\frac{1}{2}$ $\#$ {sides of face}

$\{$Belyi fct$\}/\sim$	$\{$Dessin d'enfants$\}/\sim$
$\#f^{-1}(0)+\#f^{-1}(1)$	v
$\#f^{-1}(\infty)$	f
$\deg f$	e
$2-2g(S)$	$v+f-e$
Ram index of $x\in f^{-1}(0)$	$\#$ $\{$black dots adjacent to $x\}$
Ram index of $x\in f^{-1}(1)$	$\#$ $\{$white dots adjacent to $x\}$
Ram index of $x\in f^{-1}(\infty)$	$\frac{1}{2}$ $\#$ $\{$sides of face$\}$
monodromy	perm representation pair
"Evenly ramified fct"	uniform dessin
Galois/normal/regular covering	regular dessin
Deck transformation $\mathrm{Aut}(S,f)$	$\mathrm{Homeo}^{+}(X,D)$
Galois action	
construct new from old	

Contents

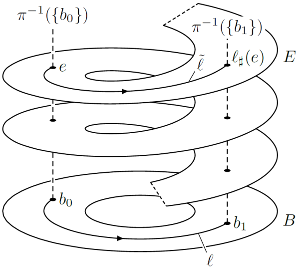
Recall: the monodromy of covering

Proposition

For a covering map $\pi \colon E \longrightarrow B$ *and* $b_0 \in B$*, we have an action*

$$\rho := \pi_1(B, b_0)^{op} \longrightarrow \mathrm{Aut}(\pi^{-1}(b_0))$$

which is ***transitive****. We call* $Mon(\pi) := \mathrm{Im}\,\rho$ ***the monodromy group****.*



Recall: the monodromy of covering

Proposition

For a covering map $\pi \colon E \longrightarrow B$ and $b_0 \in B$, we have an action

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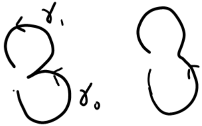
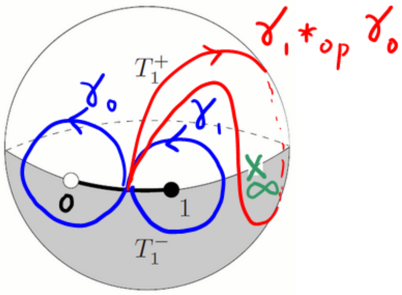


$$\gamma_1 *_{op} \gamma_0 := \gamma_0 * \gamma_1$$

For Belyi map

When $B = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, then $\pi_1(B, b_0) = \langle \gamma_0, \gamma_1 \rangle_{free}$.



Denote $\sigma_0 := \rho(\gamma_0), \sigma_1 := \rho(\gamma_1)$, then $Mon(\pi) = \langle \sigma_0, \sigma_1 \rangle$.

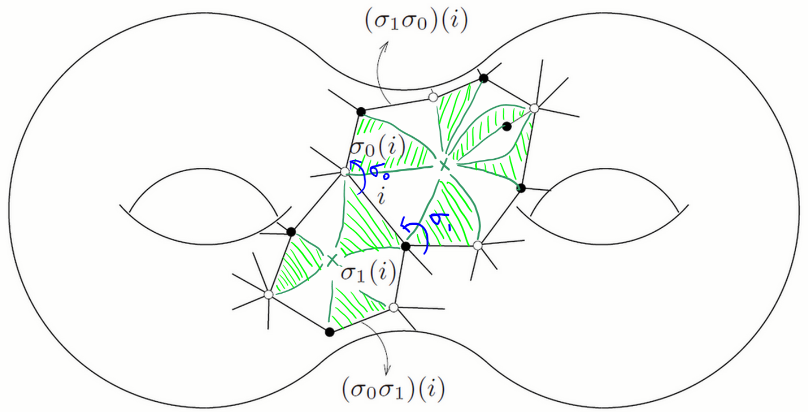
For Belyi map

When π is the covering of Belyi map, let $b_0 = \frac{1}{2}$, then

- $\pi^{-1}(b_0) \longleftrightarrow \{\text{edges of dessin}\}$
- σ_0, σ_1 are permutations of edges, as followed:

(σ_0, σ_1) are called **the permutation representation pair** of the dessin.

σ_0, σ_1



Example: Fig 4.3

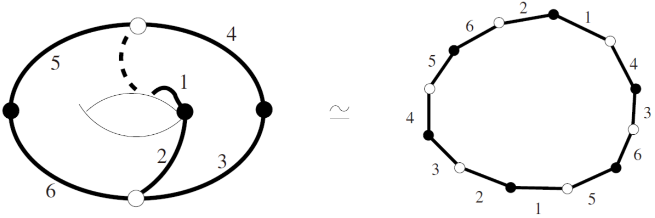
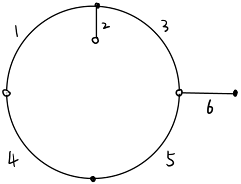


Fig. 4.3. A dessin in a topological torus.

$$\begin{aligned}
\sigma_0 &= (263)(154) \\
\sigma_1 &= (12)(34)(56) \\
\sigma_1\sigma_0 &= (253164) \\
\sigma_0\sigma_1 &= (253164)
\end{aligned}$$

Example: Fig 4.3



$$\begin{aligned}
\sigma_0 &= (14)(2)(356) \\
\sigma_1 &= (123)(45)(6) \\
\sigma_1\sigma_0 &= (156)(234) \\
\sigma_0\sigma_1 &= (125)(346)
\end{aligned}$$

Correspondence

Remark

$$\{(X,D)\}/_{\sim} \longleftrightarrow \{(\sigma_0,\sigma_1)\in\Sigma_N\}/_{\text{conj}}$$

$\{$Belyi fct$\}/\sim$	$\{$Dessin d'enfants$\}/\sim$	$\{$perm rep pair$\}/\sim$
$\#f^{-1}(0)$	$\#$ $\{$white dots$\}$	$\#$ $\{$cycles of $\sigma_0\}$
$\#f^{-1}(1)$	$\#$ $\{$black dots$\}$	$\#$ $\{$cycles of $\sigma_1\}$
$\#f^{-1}(\infty)$	f	$\#$ $\{$cycles of $\sigma_1\sigma_0\}$
$\deg f$	e	$N = \#$ $\{$ cycles of $Id\}$
$2-2g(S)$	$v+f-e$	$\#\{\ldots\sigma_0\}+\#\{\ldots\sigma_1\}+\#\{\ldots\sigma_1\sigma_0\}$-N
Ram index of $x\in f^{-1}(0)$	$\#$ $\{$black dots adjacent to $x\}$	length of a cycle on σ_0
Ram index of $x\in f^{-1}(1)$	$\#$ $\{$white dots adjacent to $x\}$	length of a cycle on σ_1
Ram index of $x\in f^{-1}(\infty)$	$\frac{1}{2}$ $\#$ $\{$sides of face$\}$	length of a cycle on $\sigma_0\sigma_1$