Thanks for coming to listen to my talk.

If you have any question or can't see some symbols, just interrupt me.

My talk is about the Langlands correspondence (LC). There are many versions about LC,

from local to global,

from 1-dim to n-dim,

from GL_n to non-split gp,

$\leftarrow$ that's why an introduction to LC is usually hard and painful. Not to say, today I have only 1 hour!

$$\mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \xleftrightarrow{1:1} \underset{\text{Frob ss}}{\mathrm{WDrep}_{n\text{-dim}}}(W_F)$$

$$\mathrm{Char}_{\mathbb{C},alg}(F^\times \backslash \mathbb{A}_F^\times) \xleftrightarrow{1:1} \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) \ + \text{de Rham}$$

$$\Pi_{\mathcal{A}cusp,k,\eta}(GL_2(\mathbb{A}_{\mathbb{Q}})) \xrightarrow{ES} \mathrm{Irr}_{\overline{\mathbb{Q}}_p}(\Gamma_F) \ + \text{modular}$$

$$\Pi_{\mathcal{A}cusp,k,\eta}(G_D(\mathbb{A}_{\mathbb{Q}})) \longrightarrow \cdots$$

This talk is divided into 4 part.

In each part, we discuss one type of the correspondence.

Before beginning discussing these correspondences, let me fix notations.

Setting

F : NA local field $\mathcal{O}_F$ κ_F

Γ_F : $= Gal(F^{sep}/F)$

W_F : $=$ Weil gp of F

Rep	sm rep
Irr	irr sm rep
Π	adm irr sm rep
Char	1-dim sm rep
WDrep	Weil-Deligne rep
$\mathcal{A}_{cusp}$	cuspidal automorphic forms

1. NA local field case $(F = \mathbb{Q}_p)$

Thm (LLC for $GL_n(F)$) $\exists$ bij $\quad (K = GL_n(\mathcal{O}_F),\ Z = F^\times)$

$$\mathrm{rec}_{F,GL_n} : \mathrm{Irr}_{\mathbb{C}}(GL_n(F)) \xleftrightarrow{1:1} \mathrm{WDrep}_{\substack{n\text{-dim} \\ \text{Frob ss}}}(W_F)$$

$n=1$
$$\chi : F^\times \to \mathbb{C}^\times \longleftrightarrow \chi : W_F \twoheadrightarrow W_F^{ab} \cong F^\times \to \mathbb{C}^\times$$

$n=2$

$$\chi \circ \det \longleftrightarrow \left(\begin{pmatrix} \chi|\cdot|_F^{-\frac{1}{2}} & \\ & \chi|\cdot|_F^{\frac{1}{2}} \end{pmatrix}, 0 \right)$$

tempered

$$n\text{-}\mathrm{Ind}_B^{GL_n}(\chi_1, \chi_2) \longleftrightarrow \left(\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}, 0 \right)$$

disc series

$$\mathrm{St} \otimes (\chi \circ \det) \longleftrightarrow \left(\begin{pmatrix} \chi|\cdot|_F^{-\frac{1}{2}} & \\ & \chi|\cdot|_F^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

(super) cuspidal

$$c\text{-}\mathrm{Ind}_{KZ}^{GL_n} \rho \longleftrightarrow ?$$

Rmk.

action by χ	$-\otimes(\chi\circ\det)$	$\longleftrightarrow$	$-\otimes\chi$
unramified	$V^K \neq \{0\}$	$\longleftrightarrow$	$\phi\|_{I_F} = \mathbb{1}_{I_F}$, $N = 0$
~~disc se~~	disc series	$\longleftrightarrow$	indec rep
	cuspidal	$\longleftrightarrow$	irr rep
	"n-Ind"	$\longleftrightarrow$	$\oplus$

Rmk. For Archi field people have ~~de~~ also clear ~~statemen~~ classifications.

2. GLC, $n=1$ $(F=\mathbb{Q})$

In order to state GLC, we need ~~A~~ adèle & idèle which collects all local information.

Observe that

$$\mathbb{Q}^\times \backslash \mathbb{A}_\mathbb{Q}^\times / \mathbb{R}_{>0} \cong \hat{\mathbb{Z}}^\times \cong \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \hat{=} \Gamma_\mathbb{Q}^{ab}$$

which gives us GLC for $n=1$:

$$\begin{array}{ccc}
\mathrm{Char}_{\mathbb{C},alg,\,wt\,0}(F^\times \backslash \mathbb{A}_F^\times) & \longleftrightarrow & \mathrm{Char}_\mathbb{C}(\Gamma_F) \\
\cap & & \downarrow \text{twist} \\
\mathrm{Char}_{\mathbb{C},alg}(F^\times \backslash \mathbb{A}_F^\times) & \xleftrightarrow{\text{twist}} & \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) + \text{de Rham}
\end{array}$$

3. Adèlic MF

route:

$$\text{Moduli space} \rightsquigarrow MF \rightsquigarrow \mathcal{A}_{cusp,k,\eta} \rightsquigarrow \Pi_{\mathcal{A}_{cusp,k,\eta}} \rightsquigarrow GLC$$

Ex. Shows that

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) / \widehat{\Gamma_1(N)} \cdot \mathbb{R}^\times \cdot SO_2 \cong \Gamma_1(N) \backslash \mathcal{H}^\pm$$

Def. (Cuspidal MF $S_{M_2(\mathbb{Q}),k,\eta}$)

For $k \geq 2$, $\eta \in \mathbb{Z}$, let
wt

$$j_{k,\eta}(\gamma) := (\det \gamma)^{\eta-1}(ci+d)^{k} \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$$

$$S_{M_2(\mathbb{Q}),k,\eta} := \left\{ \begin{array}{l} \varphi : GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C} \text{ fcts} \\ \text{s.t. ①–④ are true} \end{array} \right\}$$

① (continuity) $\exists\, \mathcal{U}_{fin} \leq GL_2(\mathbb{A}_{\mathbb{Q},fin})$ open s.t.
level

$$\varphi(g\gamma) = \varphi(g) \qquad \forall \gamma \in \mathcal{U}_{fin}$$

② (automorphy)

$$\varphi(g\gamma) = \cancel{\varphi(g)}\, j_{k,\eta}(\gamma)^{-1}\varphi(g) \qquad \forall \gamma \in \mathbb{R}^\times \times SO_2$$

$$\rightsquigarrow j_{k,\eta}(\gamma'\gamma)\varphi(g\gamma) = j_{k,\eta}(\gamma')\varphi(g)$$

③ (holomorphy) $\forall g \in GL_2(\mathbb{A}_\mathbb{Q})$, the fct

$$f_{\varphi,g} : \mathcal{H}^\pm \longrightarrow \mathbb{C} \qquad \gamma i \longmapsto \varphi(g\gamma)\, j_{k,\eta}(\gamma)$$

is holomorphic

(holomorphic at ∞)

$$f_{\varphi,g}(\tau)\,|\mathrm{Im}\,\tau|^{\frac{k}{2}} \text{ is bounded}$$

④ (cuspidal condition) $\qquad \forall g \in GL_2(\mathbb{A}_\mathbb{Q})$,

$$\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

E.g. When $\mathcal{U}_{fin} = \widehat{\Gamma_1(N)}$, one has iso

$$S^{\widehat{\Gamma_1(N)}}_{M_2(\mathbb{Q}),k,1} \cong S_k(\Gamma_1(N)) \qquad \varphi \longmapsto f_{\varphi,\mathrm{Id}}$$

$$\begin{array}{c} GL_2(\mathbb{A}_\mathbb{Q}) \\ \circlearrowright \end{array} \{f : GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}\}$$

$$\cup$$

$$\mathcal{A}_{cusp,k,\eta} \qquad \underline{\text{cusp auto forms}} \text{ of wt } (k,\eta)$$

$$\cup$$

$$S_{M_2(\mathbb{Q}),k,\eta} \qquad \underline{\text{Adèlic MF}} \text{ of wt } (k,\eta)$$

Fact. $\mathcal{A}_{cusp} \supsetneqq \bigoplus_{k_v \geq 2, \dots} \mathcal{A}_{cusp,k,\eta}$

$\uparrow$ Maass forms, wt 1 MF, ...

$$\mathcal{A}_{cusp,k,\eta} = \bigoplus_{i\in I} \cancel{\pi_i}\ \rho_i \qquad \cancel{\pi_i}\ \rho_i \in \Pi(GL_2(\mathbb{A}_{fin}))$$

Def. We call

$$\Pi_{\mathcal{A}_{cusp,k,\eta}} := \{\cancel{\pi_i}\ \rho_i \mid i \in I\} \subseteq \Pi(GL_2(\mathbb{A}_{fin}))$$

as $\underline{\text{cuspidal auto reps}}$ of wt (k,η).

Thm (Eichler – Shimura)

Let $\rho \in \Pi_{\mathrm{Acusp},k,\eta}(GL_2(\mathbb{A}_F))$

$\exists$ a CM field L (big enough),

$\forall \lambda$: fin place of L,

$\exists\ r_\lambda(\pi) \in \mathrm{Irr}_{\overline{L}_\lambda}(\Gamma_F)$ 2-dim s.t.

1) $\left.\begin{array}{l} \pi_v \text{ unramified} \\ \mathrm{char}\ k_v \neq \mathrm{char}\ k_\lambda \end{array}\right\} \Rightarrow \left\{\begin{array}{l} r_\lambda(\pi)|_{G_{F_v}} \text{ is unramified} \\ \text{char poly}(\mathrm{Frob}_v) = X^2 - t_v X + (\# k_v) s_v \end{array}\right.$

2) compatible with LLC in $l \neq p$, $l = p$ case

3). ρ is geometric

4). $\forall\, v | \infty$ s.t. $F_v \cong \mathbb{R}$, ~~denote~~ $\Gamma_{F_v} := \{1, \sigma_v\} \subseteq \Gamma_F$,

$$\det(r_\lambda(\pi)(\sigma_v)) = -1$$

5) $r_\lambda(\pi)$ forms a strictly compatible system.

<u>Def.</u> $r \in \mathrm{Irr}_{\overline{\mathbb{Q}}_l}(G_F)$ is <u>modular</u>, if $r = r_\lambda(\pi)$ for some π.

4. Adèlic MF on quaternion algebras

F: totally real field	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{3})$
D/F: quaternion alg (with center F)	$M_2(\mathbb{Q})$	$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)$
$S(D) = \{v : \text{places of } F \mid D \otimes_F F_v \not\cong M_2(F_v)\}$	$\emptyset$	$\{\infty_1, \infty_2\}$
$G_D := (D \otimes_{\mathbb{Q}} -)^\times$ " $= \mathrm{Res}_{D/\mathbb{Q}} \mathbb{G}_m$ "	GL_2	$\left(\frac{-1,-1}{- \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})}\right)^\times$
$G_D(\mathbb{Q}) = D^\times$	$GL_2(\mathbb{Q})$	$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)^\times$
$G_D(\mathbb{R})$	$GL_2(\mathbb{R})$	$\cancel{\emptyset}\ \mathbb{H}^\times \times \mathbb{H}^\times$
$\mathbb{C}$? $\mathbb{Q}_p$? $\mathbb{A}_{\mathbb{Q}}$?		

For $v \mid \infty$, define $\mathcal{U}_v$ & $(\tau_v, W_v) \in \mathrm{rep}(\mathcal{U}_v)$ as follows.

	$v \notin S(D)$	$v \in S(D)$
$\mathcal{U}_v$	$\mathbb{R}^\times \cancel{SO_2}\, SO_2$	$\mathbb{H}^\times$
W_v	$\mathbb{C}$	$\cancel{(\mathrm{Sym}^{k-2} \mathbb{C}^2) \otimes (\Lambda^2 \mathbb{C}^2)^\eta}$ $(\Lambda^2 \mathbb{C}^2)^{\eta_v} \otimes (\mathrm{Sym}^{k_v-2} \mathbb{C}^2)$
τ_v	$\mathbb{R}^\times SO_2 \xrightarrow{j_{k_v,\eta_v}} \mathbb{C}^\times \circlearrowright \mathbb{C}$	$\mathbb{H} \hookrightarrow GL_2(\mathbb{C}) \circlearrowright \mathbb{C}^2$ (construct from)

Let $\mathcal{U}_\infty = \prod_{v \mid \infty} \mathcal{U}_v$, $\quad (\tau_\infty, W_\infty) = \bigotimes_{v \mid \infty} (\tau_v, W_v)$.

Def. (Cuspidal MF $S_{D,k,\eta}$)

For $k=(k_v)_{v|\infty}$, $\eta=(\eta_v)_{v|\infty}$. define

$$S_{D,k,\eta} := \left\{ \begin{array}{c} \varphi: G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C} \text{ fcts} \\ \text{s.t. ①–④ are true} \end{array} \right\}$$

① (continuity) $\exists\, \mathcal{U}_{fin} \leq G_D(\mathbb{A}_{\mathbb{Q},fin})$ open s.t

$$\varphi(g\gamma) = \varphi(g) \qquad \forall \gamma \in \mathcal{U}_{fin}$$

② (automorphy)

$$\varphi(g\gamma) = \tau_\infty(\gamma)^{-1}\varphi(g) \qquad \forall \gamma \in \mathcal{U}_\infty$$

$\rightsquigarrow \tau_\infty(\gamma\gamma')\varphi(g\gamma) = \tau_\infty(\gamma')\varphi(g)$

③ (holomorphy) $\forall g \in \cancel{GL_2(\mathbb{A}_\mathbb{Q})}\ G_D(\mathbb{A}_\mathbb{Q})$, the fct

$$f_{\varphi,g}: (\mathcal{H}^\pm)^{S_\infty - S(D)} \longrightarrow W_\infty \quad \gamma(i,\dots,i) \longmapsto \tau_\infty(\gamma)\varphi(g\gamma)$$

is holomorphic

(holomorphic at ∞) When $D = M_2(\mathbb{Q})$,

$$f_{\varphi,g}(\tau)\ |\mathrm{Im}\,\tau|^{\frac{k}{2}} \text{ is bounded.}$$

④ (cuspidal condition) When $\cancel{G_D = GL_2(-\otimes_\mathbb{Q} F)}$ $G_D = GL_2(-\otimes_\mathbb{Q} F)$ $\cancel{D^\times = GL_2(F)}$, $\forall g \in G_D(\mathbb{A}_\mathbb{Q}) = GL_2(\mathbb{A}_F)$,

$$\int_{F\backslash\mathbb{A}_F} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0$$

Rmk. For assuring $S_{D,k,\eta} \neq \{0\}$, one should require that

$$k_v \geq 2 \qquad k_v + 2\eta_v = k_w + 2\eta_w \quad \forall v,w|\infty$$

E.g. For $D = \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)$, conditions ③ & ④ are auto satisfied.

Furthermore,

$$S_{D,2,0}^{\mathcal{U}_{fin}} = \mathbb{C}\left[G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q}})/\mathcal{U}_{fin}\,\mathcal{U}_\infty\right]$$

$$= \mathbb{C}\left[G_D(\mathbb{Q})\backslash G_D(\mathbb{A}_{\mathbb{Q},fin})/\mathcal{U}_{fin}\right]$$

is a f.d. v.s.

Def. D is definite, if $\{v|\infty\} \subset S(D)$, i.e. $G_D(\mathbb{R}) \cong \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times$.

E.g. $\left(\frac{-1,-1}{F}\right)$ is definite.

Rmk. One can define $\mathcal{A}_{cusp,k,\eta}(G_D)$, $\Pi_{\mathcal{A}_{cusp,k,\eta}}(G_D)$ in a similar way, and show that they satisfy similar properties.

Fact (The JL ~~corr~~ crspd) $(v \in S(D), v \nmid \infty)$

$$\begin{tikzcd} \left\{ \begin{array}{l} (\rho, V) \in \Pi_{\mathcal{A}_{cusp},k,\eta}(G_D(\mathbb{A}_{\mathbb{Q},fin})) \\ \dim_{\mathbb{C}} V = +\infty \end{array} \right\} \arrow[r, leftrightarrow, "JL"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \left\{ \begin{array}{l} (\rho, V) \in \Pi_{\mathcal{A}_{cusp},k,\eta}(GL_2(\mathbb{A}_{\mathbb{Q},fin})) \\ \rho|_{GL_2(F_v)} \text{ disc series } \forall v \in S(D), v \nmid \infty \end{array} \right\} \arrow[d] \\ \mathrm{Irr}(G_D(F_v)) \arrow[r, leftrightarrow, "JL_{loc}"] & \mathrm{Irr}(GL_2(F_v)) + \text{disc series} \end{tikzcd}$$