

SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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This document is intended to collect the questions and doubts that arose during my research this year. For many of these problems, I have consulted my fellow students, my supervisor, and various other people I've met. However, most of them remain in the realm of folklore—problems that are likely known but for which I could not find a reference. On the other hand, some of the questions may not appear particularly interesting unless their underlying motivations are clearly explained. Therefore, I'll try to provide relevant background and outline some initial, perhaps naive, ideas while listing the problems along the way. Any responses, answers, or references are most welcome and will be added to keep this document updated.

1. BASIC SETTING

For simplicity, we work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a integral separated scheme of finite type over \mathbb{C} . Let A/\mathbb{C} be an abelian variety of dimension n , and let $Z \subseteq A$ be an irreducible closed subvariety of dimension r . We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.¹

1.1. Gauss map. The goal of my research is to understand the geometry of Z , and the main tool for the subvariety geometry is the Gauss map. The Gauss map describe the tangent space information at each point:

$$\phi_Z : Z^{\text{sm}} \longrightarrow \text{Gr}(r, T_0 A) \quad p \longmapsto T_p Z \subseteq T_p A \cong T_0 A$$

Any map to the Grassmannian $\text{Gr}(r, n)$ is induced by a rank r vector bundle together with n global sections. In this case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{Z^{\text{sm}}}$ and the sections

$$H^0(A, \mathcal{T}_{Z^{\text{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{Z^{\text{sm}}} \twoheadrightarrow \mathcal{T}_{Z^{\text{sm}}}.$$

Date: June 7, 2025.

¹I'm not sure whether we should consider the more general cases in the future—such as working over a field of characteristic p , letting A be a semiabelian variety or a complex torus, or allowing ι to be a covering onto its image. For now, I will omit these possibilities from this document.

1.2. Conormal variety. This concept may already be familiar to many readers, so we briefly recall the definition. On the smooth locus, the normal and conormal bundles behave well as vector bundles:²

$$\mathcal{N}_{Z^{\text{sm}}/A} := \mathcal{T}_A|_{Z^{\text{sm}}} / \mathcal{T}_{Z^{\text{sm}}} \quad \Lambda_{Z^{\text{sm}}} := \mathcal{N}_{Z^{\text{sm}}/A}^* = \ker \left(\Omega_A|_{Z^{\text{sm}}} \rightarrow \Omega_{Z^{\text{sm}}} \right).$$

The conormal variety Λ_Z is just the closure of $\Lambda_{Z^{\text{sm}}}$ viewed as a subvariety in T^*A :

$$\Lambda_Z := \overline{\Lambda_{Z^{\text{sm}}}} \subset T^*A \cong A \times T_0^*A$$

this is a conical Lagrangian cycle in T^*A .

Moreover, the projectivized conormal variety

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\text{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A$$

is a Legendrian cycle in the contact variety $A \times \mathbb{P}T_0^*A$. $\mathbb{P}\Lambda_{Z^{\text{sm}}}$ is a \mathbb{P}^{r-1} -bundle over Z^{sm} , and the map

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A$$

is generically finite (i.e., clean) when Z is (an integral variety) of general type, see [5, Theorem 2.8 (1)].

A lot of geometry of Z is encoded in the map γ_Z . For instance, if Z is smooth and lies inside A , then

$$\deg \gamma_Z = (-1)^r \chi(Z)$$

tells us the Euler characteristic of Z .

Further insight can be gained by analyzing the fibers of γ_Z . These fibers, though finite, are not arbitrary—they obey hidden structural rules. For instance, if Z is preserved by a translation $t_v : A \rightarrow A$, then each fiber $\gamma_Z^{-1}(\xi)$ is also invariant under t_v . Likewise, if $Z = -Z$, then the fiber satisfies $\gamma_Z^{-1}(\xi) = -\gamma_Z^{-1}(\xi)$. Outside these special configurations, it becomes more challenging to identify further constraints.³

An important invariant arising from the fiber $\gamma_Z^{-1}(\xi)$ is the monodromy group $\text{Gal}(\gamma_Z)$; for completeness, we recall its definition below.

Definition 1.1. *Define*

$$U = \{ \xi \in \mathbb{P}T_0^*A \mid \# \gamma_Z^{-1}(\xi) = \deg \gamma_Z \}.$$

Moving along a loop in U induces a permutation of the points in the fiber $\gamma_Z^{-1}(\xi_0)$, which defines the map

$$\rho_{\gamma_Z} : \pi_1(U, \xi_0) \longrightarrow \text{Aut}(\gamma_Z^{-1}(\xi_0)) = S_{\deg \gamma_Z}.$$

The monodromy group is then defined as the image of ρ_{γ_Z} , i.e.,

$$\text{Gal}(\gamma_Z) := \text{Im } \rho_{\gamma_Z}.$$

Question 1.2. *Suppose that the subvariety $Z \subset A$ is not stable under any translation on A . Are there known algorithms to compute the monodromy group $\text{Gal}(\gamma_Z)$? Furthermore, what kinds of groups can appear as $\text{Gal}(\gamma_Z)$ for suitable choices of $Z \subset A$?*

We will try to compute $\text{Gal}(\gamma_Z)$ for a number of specific cases in Section 2. Three special cases are already treated in [6, Theorem 9], and we will generalize the strategies there.

²This is more symmetric when writing them as short exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{T}_{Z^{\text{sm}}} \longrightarrow \mathcal{T}_A|_{Z^{\text{sm}}} \longrightarrow \mathcal{N}_{Z^{\text{sm}}/A} \longrightarrow 0 \\ 0 &\longrightarrow \Lambda_{Z^{\text{sm}}} \longrightarrow \Omega_A|_{Z^{\text{sm}}} \longrightarrow \Omega_{Z^{\text{sm}}} \longrightarrow 0 \end{aligned}$$

³You can imagine the fiber $\gamma_Z^{-1}(\xi)$ as a cluster of stars projected onto a celestial dome. As ξ varies, these points shift, tracing out paths much like stars drifting across the night sky. The constraints that govern them are subtle, like the imagined lines that shape constellations. And in the long arc of variation, monodromy emerges—like the slow turning that replaces Kochab with Polaris among the stars.

1.3. Interpolation via hyperplanes. Before delving into examples, we reinterpret γ_Z using a functorial and more transparent framework, enabling a decomposition of Question 1.2 into two primary subquestions.

Recognizing that each non-zero conormal vector $\xi \in T_0^*A$ determines a hyperplane $H_\xi \in \text{Gr}(n-1, T_0A)$, we establish the isomorphisms

$$\begin{aligned} \mathbb{P}T_0^*A &\cong \text{Gr}(n-1, T_0A) \doteq (\mathbb{P}^{n-1})^\vee, \\ \mathbb{P}\Lambda_{Z^{\text{sm}}} &= \{ (p, \xi) \in Z^{\text{sm}} \times \mathbb{P}T_0^*A \mid \xi|_{T_pZ} \equiv 0 \} \\ &\cong \{ (p, H) \in Z^{\text{sm}} \times \text{Gr}(n-1, n) \mid \phi_Z(p) \subseteq H \} \\ &\cong (\phi_Z, \text{Id})^{-1} I_{r, n-1}, \end{aligned}$$

where

$$I_{r, n-1} := \{ (V, H) \in \text{Gr}(r, n) \times \text{Gr}(n-1, n) \mid V \subseteq H \}$$

is the incidence variety relating $\text{Gr}(r, n)$ and $\text{Gr}(n-1, n)$. In that case,

$$\begin{aligned} \gamma_Z^{-1}(H) \cap Z^{\text{sm}} &= \{ p \in Z^{\text{sm}} \mid \phi_Z(p) \subseteq H \} \\ &\cong \phi_Z^{-1}(\text{Gr}(r, H)) \end{aligned}$$

is the collection of points whose tangent spaces lie entirely within H .

Geometrically, the monodromy can be described as follows: given a general hyperplane H , its preimage consists of d points p_1, \dots, p_d . Moving H continuously along a loop causes these points to permute, and the monodromy group $\text{Gal}(\gamma_Z)$ consists of all permutations obtained this way. With this new formulation, it suffices to consider the Gauss map ϕ_Z alone; the inclusion $\iota_Z : Z \rightarrow A$ is no longer required for computing the monodromy group.

Definition 1.3. Let Z be an r -dimensional variety and $\phi : Z \rightarrow \text{Gr}(r, n)$ a morphism. Suppose that for some $d \in \mathbb{N}_{>0}$, the set

$$U := \{ H \in \text{Gr}(n-1, n) \mid \#\phi^{-1}(\text{Gr}(r, H)) = d \}$$

is non-empty open in $\text{Gr}(n-1, n)$ ⁴. The monodromy group $\text{Gal}(\phi)$ is defined as the image of

$$\rho_{\gamma_Z} : \pi_1(U, H_0) \rightarrow \text{Aut}(\phi^{-1}(\text{Gr}(r, H_0))) \cong S_d.$$

When ϕ is not generically finite onto its image, the monodromy group is subject to additional constraints, as captured by the next lemma.

Lemma 1.4. When $\phi : Z \rightarrow \text{Gr}(r, n)$ is generically k -to-1 onto its image, the monodromy group $\text{Gal}(\phi)$ is contained in the wreath product

$$S_k \wr S_{d/k} := \left(S_k^{\oplus d/k} \right) \rtimes S_{d/k}.$$

Proof. Consider the diagram below:

$$\begin{array}{ccccc} \phi : & Z & \xrightarrow{k:1} & \text{Im } Z & \hookrightarrow & \text{Gr}(r, n) \\ & \cup & & \cup & & \cup \\ & \{p_1, \dots, p_d\} & \longrightarrow & \{q_1, \dots, q_{d/k}\} & \longrightarrow & \text{Gr}(r, H) \end{array}$$

The fiber $\phi^{-1}(\text{Gr}(r, H_0))$ splits into d/k groups of points, with the monodromy group acting by permutations within each group and among the groups. \square

Based on the discussion above, Question 1.2 reduces to Question 1.5 and Question 1.7, with Question 1.6 appearing as a special case of Question 1.5.

Question 1.5. In the situation of Definition 1.3, how can we compute $\text{Gal}(\phi)$?

Question 1.6. Let $Z \subseteq \text{Gr}(r, n)$ be a subvariety of dimension r such that $[Z] \neq 0$ in $H_r(\text{Gr}(r, n); \mathbb{Z})$. What can be said about its monodromy group?

⁴Here, the fiber is understood set-theoretically; multiplicities are not taken into account.

Question 1.7. For a map $\phi : Z \rightarrow \mathrm{Gr}(r, n)$, when is it induced from some inclusion $Z \subset A$?

2. SEARCHING FOR EXAMPLES

In this section, we discuss examples drawn from my ongoing work, focusing on the construction of subvarieties and the computation of their monodromy groups.

Broadly speaking, there are two approaches to constructing examples. One may start with a given variety Z and attempt to embed it into an abelian variety—this always factors through its Albanese $\mathrm{Alb}(Z)$. Alternatively, one may begin with an abelian variety A and construct subvarieties by intersecting suitable divisors. Unfortunately, even in the case where Z is a curve, we have not fully resolved Question 1.2.

2.1. Curves, basic results. A complete answer to Question 1.6 is known when $Z = C$ is a curve.

Proposition 2.1 (See [1, p111] for a detailed proof). *Suppose that $\iota_C : C \subseteq \mathbb{P}^{n-1}$ is an irreducible nondegenerate curve of degree d , then $\mathrm{Gal}(\iota_C) \cong S_d$.*

Sketch of proof. Because S_d is generated by its transpositions, we are reduced to verifying that:

- $\mathrm{Gal}(\iota_C)$ acts doubly transitively on the fiber;
- $\mathrm{Gal}(\iota_C)$ contains a transposition.

□

Proposition 2.2. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \rightarrow C'$ be a degree 2 ramified covering, with ramification occurring at at least one smooth point of C' . Then $\mathrm{Gal}(\iota_{C'} \circ h) \cong S_2^{\oplus d/2} \rtimes S_{d/2}$ is the hyperoctahedral group/signed symmetric group.*

Sketch of proof. By Lemma 1.4 we know that $\mathrm{Gal}(\iota_{C'} \circ h) \subseteq S_2^{\oplus d/2} \rtimes S_{d/2}$. By Lemma 2.3, we are reduced to verifying that:

- The quotient map $\mathrm{Gal}(\iota_{C'} \circ h) \rightarrow \mathrm{Gal}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- $\mathrm{Gal}(\iota_{C'} \circ h)$ contains a transposition of a pair of points in the fiber of h .

□

Lemma 2.3. *Let G be a subgroup of $S_2^{\oplus m} \rtimes S_m$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \rightarrow S_m$ is surjective and the transposition σ_0 of ± 1 lies in G , then $G = S_2^{\oplus m} \rtimes S_m$.*

Sketch of proof. (with help from Chenji Fu) Let ε_i denote the transposition of $\pm i$. For any $\sigma \in S_m$, choose a lift $\tilde{\sigma} \in G$, then

$$\varepsilon_{\sigma(1)} = \tilde{\sigma} \circ \sigma_0 \circ \tilde{\sigma}^{-1} \in G.$$

Thus, $S_2^{\oplus m} \subset G$, and since G maps onto S_m , we obtain $G = S_2^{\oplus m} \rtimes S_m$. □

Example 2.4. *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$.*

When C is non-hyperelliptic, the corresponding Gauss map

$$|\omega_C| : C \rightarrow \mathbb{P}^{g-1}$$

makes C as an irreducible nondegenerate curve of degree $2g - 2$, by Proposition 2.1 we get

$$\mathrm{Gal}(\gamma_C) \cong S_{2g-2}.$$

When C is hyperelliptic, the corresponding Gauss map is $2 : 1$ onto a rational normal curve $R \subset \mathbb{P}^{g-1}$:

$$|\omega_C| : C \xrightarrow{2:1} R \hookrightarrow \mathbb{P}^{g-1}$$

By Proposition 2.2 we get

$$\mathrm{Gal}(\gamma_C) \cong S_2^{\oplus g-1} \rtimes S_{g-1}.$$

In fact, a degree $2 : 1$ map does not give rise to any exceptional monodromy groups beyond those listed in Table 1.

Definition 2.5 (big monodromy group). *We refer to the big monodromy group as any group of the following types:*

notation	name	alias
$W(A_{m+1}) = S_m$	full symmetric group	
$W(C_m) = S_2^{\oplus m} \rtimes S_m$	signed symmetric group	hyperoctahedral group
$W(D_m) = (S_2^{\oplus m})_0 \rtimes S_m$	even-signed symmetric group	demihyperoctahedral group

TABLE 1. big monodromy group

Proposition 2.6. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \rightarrow C'$ be a degree 2 ramified covering. Then*

$$\mathrm{Gal}(\iota_{C'} \circ h) \cong W(C_{d/2}) \text{ or } W(D_{d/2}).$$

Sketch of proof. By Lemma 1.4 we know that $\mathrm{Gal}(\iota_{C'} \circ h) \subseteq W(D_{d/2})$. By Lemma 2.7, we are reduced to verifying that:

- The quotient map $\mathrm{Gal}(\iota_{C'} \circ h) \rightarrow \mathrm{Gal}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- (signed doubly transitive) $\mathrm{Gal}(\iota_{C'} \circ h)$ acts transitively on pairs (x, y) with $x \neq \pm y$.

□

Lemma 2.7. *Let G be a subgroup of $W(D_m)$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \rightarrow S_m$ is surjective then*

$$G \cong W(C_m) \text{ or } W(D_m) \text{ or } S_m.$$

Sketch of proof. (with help from Chenji Fu) Let H denote the kernel of the natural quotient map $G \rightarrow S_m$. Then H is stable under the action of S_n . There are only three possible forms that H can take:

- $H = 0$. Then $G \cong S_m$.
- $H = (S_2^{\oplus m})_0$. Then G is a index 2 subgroup of $W(C_m)$, so $G \cong W(D_m)$.⁵
- $H = S_2^{\oplus m}$. Then $G = W(C_m)$.

□

We are particularly interested in identifying cases where the monodromy group is small(i.e., not big), which can occur only when $k > 2$, where k denotes the degree of the map $\phi : C \rightarrow \mathbb{P}^{n-1}$. On the other hand, when both $k > 2$ and $n > 2$, Lemma 1.4 ensures that the resulting monodromy group is indeed small. Thus, in the case of curves, Question 1.2 is resolved except in the following three cases:

- (1) $n > 2$ and $k = 2$. By Proposition 2.6, the monodromy group is known to be large, although its precise structure remains undetermined.
- (2) $n = 2$. Here, A is an abelian surface and C corresponds to a divisor.
- (3) $n > 2$ and $k > 2$. Finding an example of this would guarantee a case of small monodromy.

Question 2.8. *Can we find any curve $C \subset A$ not stable under any translation on A , and for which the associated monodromy group is small?*

Assume that $n > 2$. Can we find any curve $C \subset A$ not stable under any translation on A , whose Gauss map has degree $d > 2$?

⁵Check [stackexchange discussions](#)

2.2. Curves in Prym variety. To address Question 2.8, we seek explicit examples, and the case of a Prym curve mapping to its associated Prym variety offers a natural starting point for investigation.

Notations 2.9. Suppose C' is a smooth projective curve of genus $g(C')$, and let B be an effective (possibly zero) divisor on C' . For any line bundle $\eta \in \text{Pic}(C')$ such that $\eta^{\otimes 2} \cong \mathcal{O}_{C'}(B)$, one obtains a double cover $h : C \rightarrow C'$ of smooth projective curves, ramified precisely over B . The associated involution on C is denoted by ι , and the triple (C, C', h) is referred to as the Prym pair.

Recall that the Prym variety $A := \text{Prym}(C/C')$ is defined as the connected component of the identity in

$$\ker [\text{Nm} : \text{Jac}(C) \rightarrow \text{Jac}(C')].$$

The Abel–Prym map is defined by

$$\text{AP}_{C/C'} : C \rightarrow A \quad p \mapsto \mathcal{O}_C(p - \iota(p)).$$

The classical theory treats the case where the Prym variety A is principally polarized—this occurs exactly when $B = \emptyset$ or $B = \{p_0, q_0\}$, that is, when h is either unramified or ramified at two points.⁶ For clarity and focus, we restrict our attention to these cases. Table 2 lists the relevant numerical data.⁷

	$g(C)$	$g(C')$	$\deg \eta$
$B = \emptyset$	$2n + 1$	$n + 1$	0
$B = \{p_0, q_0\}$	$2n$	n	1

TABLE 2. numerical data of Prym pair

The Abel–Prym map $\text{AP}_{C/C'}$ does not always behave as nicely as the Abel–Jacobi map; it may fail to be an embedding. When C is hyperelliptic, the map $\text{AP}_{C/C'}$ fails to be generically injective, and therefore falls outside the scope of our discussion. When C is non-hyperelliptic, we may regard it as a subvariety of A for our purposes, although it may be not strictly embedded.⁸ However, it remains unclear whether C is stable under any translation in A .

In the Prym setting, the corresponding Gauss map γ_C factors through h :

$$\begin{aligned} \gamma_C : C &\xrightarrow{h} C' \xrightarrow{|\omega_{C'} \otimes \eta|} \mathbb{P}^{n-1} = \mathbb{P}(\text{H}^0(\omega_{C'} \otimes \eta)^*) \\ &\cong \text{Gr}(n-1, \text{H}^0(\omega_{C'} \otimes \eta)) \\ \tilde{p} &\longmapsto p \longmapsto \text{H}^0(\omega_{C'} \otimes \eta(-p)) \end{aligned}$$

Unfortunately, in the Prym setting, no example has been found for Question 2.8, and such an example is likely to exist only when the gonality $\text{gon}(C)$ is sufficiently small.

Lemma 2.10. When $\text{gon}(C) > 4$, $|\omega_{C'} \otimes \eta|$ is injective. As a result, the monodromy group is big.

Proof. Suppose that $|\omega_{C'} \otimes \eta|$ is not injective, we need to find a line bundle of degree 4 and rank ≥ 1 . In fact, for $p \neq q$,

$$\begin{aligned} |\omega_{C'} \otimes \eta|(p) &= |\omega_{C'} \otimes \eta|(q) \\ \iff h^0(\omega_{C'} \otimes \eta) - h^0(\omega_{C'} \otimes \eta(-p - q)) &= 1 && \text{By [9, 19.2.8]} \\ \iff h^0(\omega_{C'} \otimes \eta(-p - q)) &= n - 1 && \text{Since } \dim_{\mathbb{C}} A = n \\ \iff h^0(\eta^{\vee}(p + q)) &= 1 && \text{Via Riemann–Roch} \end{aligned}$$

⁶See [3, Theorem 3.2.6] for the proof.

⁷As usual, $n = \dim_{\mathbb{C}} A$ is the dimension of the abelian variety.

⁸For a detailed description of the map $\text{AP}_{C/C'}$, see [2, Proposition 12.5.2, Corollary 12.5.6]. Although $\text{AP}_{C/C'}$ may collapse p_0 and q_0 , this does not affect the relevant computation of $\text{Gal}(\gamma_Z)$.

When $B = \emptyset$, write $\eta^\vee(p + q) = \mathcal{O}_{C'}(p' + q')$, then

$$\mathcal{O}_{C'}(2p + 2q) = (\eta^\vee)^{\otimes 2}(2p + 2q) = \mathcal{O}_{C'}(2p' + 2q') \in g_4^1;$$

When $B = \{p_0, q_0\}$, write $\eta^\vee(p + q) = \mathcal{O}_{C'}(p')$, then

$$\mathcal{O}_{C'}(2p + 2q) = (\eta^\vee)^{\otimes 2}(2p + 2q + p_0 + q_0) = \mathcal{O}_{C'}(2p' + p_0 + q_0) \in g_4^1.$$

□

Remark 2.11. Based on the strategy used in the proof of Lemma 2.10, we can in fact obtain stronger results. For $p \in C$,

$$\begin{aligned} |\omega_{C'} \otimes \eta| &\text{ is ramified at } p && \text{(i.e., the tangent map is 0 at } p) \\ \iff h^0(\omega_{C'} \otimes \eta) - h^0(\omega_{C'} \otimes \eta(-2p)) &= 1 && \text{By [9, 19.2.9]} \\ \iff h^0(\omega_{C'} \otimes \eta(-2p)) &= n - 1 && \text{Since } \dim_{\mathbb{C}} A = n \\ \iff h^0(\eta^\vee(2p)) &= 1 && \text{Via Riemann–Roch} \end{aligned}$$

For the remainder of this discussion, we focus exclusively on the case $B = \emptyset$. Combining both,

$$\begin{aligned} &|\omega_{C'} \otimes \eta| \text{ is not generically injective} \\ \iff &\text{For any } p \in C', \text{ there exist } q \in C' \text{ such that } h^0(\eta^\vee(p + q)) = 1 \\ \iff &\text{For any } p \in C', \text{ there exist } q \in C' \text{ such that } \eta^\vee + p \in C' + C' - q \\ \iff &\eta^\vee + C' \subseteq C' + C' - C'. \end{aligned}$$

It is expected that

$$\eta^\vee + C' \subseteq C' + C' - C' \iff \eta^\vee \in C' - C'. \quad (2.1)$$

If so, writing $\eta^\vee = \mathcal{O}_{C'}(p - q)$, we obtain

$$\mathcal{O}_{C'}(2p) = \mathcal{O}_{C'}(2q) \in g_2^1,$$

which implies that C' is hyperelliptic. By Fact 2.12, this forces C to be hyperelliptic as well, and hence it lies outside the scope of our discussion.

Unfortunately, (2.1) does not hold in general. A counterexample, communicated to me by Gavril Farkas, will be presented in Proposition 2.14.

Fact 2.12 (From [8, p346]). *We have an explicit classification of $2 : 1$ unramified covers of a hyperelliptic curve C' . Let $\pi : C' \rightarrow \mathbb{P}^1$ be the double cover ramified at $\{p_1, \dots, p_{2g+2}\}$, then every such unramified double cover of C' arises from some line bundle in C of the form*

$$\eta := \mathcal{O}_{C'}(p_{i_1} + \dots + p_{i_l} - p_{i_{l+1}} - \dots - p_{i_{2l}}) = \mathcal{O}_{C'}\left(\sum_{j=1}^{2l} p_{i_j} - 2l \cdot p_1\right).$$

When $l = 1$, the associated covering curve C is also hyperelliptic.

Question 2.13. *Can one find a non-hyperelliptic curve C' and a nontrivial line bundle $\eta \in \text{Jac}(C')$ satisfying $\eta^{\otimes 2} \cong \mathcal{O}_{C'}$, such that*

$$\eta^\vee + C' \subseteq C' + C' - C'?$$

If such a pair exists, can we find a case where the resulting curve $C \subset A$ is not preserved under any translation on A ?

As a key example relevant to Question 2.13, I include here a result of Gavril Farkas.

Proposition 2.14 (See [4, Proposition 2.8]). *Let C be a general smooth curve of genus $g = 2i$, and suppose $D \in W_{i+1}^1(C)$ is a pencil of minimal degree—i.e., a line bundle $\mathcal{O}_C(D) \in \text{Pic}^{i+1}(C)$ with $h^0(C, \mathcal{O}_C(D)) = 2$. Define*

$$\mathcal{L} := K_C - 2D \in \text{Pic}^{2i-4}(C),$$

then \mathcal{L} satisfies

$$\mathcal{L} + C \subset C_{2i-2} - C, \quad \text{but} \quad \mathcal{L} \notin C_{2i-3} - C.$$

Remark 2.15. Take $i = 2$, the line bundle \mathcal{L} provides a counterexample to (2.1). However, this example does not directly yield a counterexample to Question 2.13. This is due to the uncertainty of whether there exist C and D satisfying $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_C$, and whether the curve C is sufficiently general for Proposition 2.14 to hold.

I intend to revisit the proof in detail, especially the use of the base point free pencil trick and the Petri map. Should this yield any useful observations, they will be included here.

2.3. Albanese variety. Let X be a smooth complex projective variety of dimension r , and set $n := \dim_{\mathbb{C}} H^0(X, \Omega_X) = h^{1,0}$. Recall that the Albanese variety of X is defined as

$$\text{Alb}(X) := H^0(X, \Omega_X)^* / H_1(X, \mathbb{Z})_{\text{free}},$$

and the Albanese map is given by (pick a base point $p_0 \in X$)

$$\alpha : X \longrightarrow \text{Alb}(X) \quad p \longmapsto \left[\omega \mapsto \int_{\gamma: p_0 \sim p} \omega \right].$$

One classical question is the dimension of $\alpha(X)$. For this, consider the cotangent map of α at $p \in X$:

$$T_p^* \alpha : H^0(X, \Omega_X) \longrightarrow T_p^* X = H^0(X, \Omega_X|_p).$$

Consider the short exact sequence of coherent sheaves on X

$$0 \longrightarrow \Omega_X(-p) \longrightarrow \Omega_X \longrightarrow \Omega_X|_p \longrightarrow 0$$

which induces a long exact sequence

$$\begin{array}{ccccccc} & & H^1(X, \Omega_X(-p)) & \longrightarrow & H^1(X, \Omega_X) & \longrightarrow & 0 \\ & \nearrow & & & & & \\ 0 & \longrightarrow & H^0(X, \Omega_X(-p)) & \longrightarrow & H^0(X, \Omega_X) & \xrightarrow{T_p^* \alpha} & H^0(X, \Omega_X|_p) \end{array}$$

The proposition below follows from standard arguments in homological algebra:

Proposition 2.16. *For a general point $p \in X$,*

$$\begin{aligned} \dim_{\mathbb{C}} \alpha(X) &= \text{rank } T_p^* \alpha \\ &= n - h^0(X, \Omega_X(-p)) \\ &= h^1(X, \Omega_X(-p)) - h^1(X, \Omega_X). \end{aligned}$$

In particular,

$$\begin{array}{ll} \alpha \text{ is surjective} & \iff h^0(X, \Omega_X(-p)) = 0 \\ \alpha \text{ is finite onto image} & \iff h^0(X, \Omega_X(-p)) = n - r \\ \alpha \text{ is constant} & \iff h^0(X, \Omega_X(-p)) = n \\ & \iff h^1(X, \Omega_X(-p)) = h^1(X, \Omega_X) \\ & \iff n = 0. \end{array}$$

We will concentrate on the case where α is finite onto its image.⁹ Under this assumption, we set $Z = X$ and $A = \text{Alb}(Z)$. The corresponding Gauss map is then a rational map:

$$\begin{array}{ccc} \phi_Z : Z & \dashrightarrow & \text{Gr}(r, T^0 A) \cong \text{Gr}(n - r, H^0(X, \Omega_X)) \\ p & \longmapsto & H^0(X, \Omega_X(-p)) \end{array}$$

⁹The general method remains valid in the broader setting, but the Gauss map then takes values in a different space.

and we have the isomorphisms

$$\begin{aligned}\mathbb{P}T_0^*A &\cong \mathbb{P}H^0(X, \Omega_X), \\ \mathbb{P}\Lambda_Z &= \{ (p, [\omega]) \in Z \times \mathbb{P}T_0^*A \mid \omega(p) = 0 \} \\ &\cong \{ (p, [\omega]) \in Z \times \mathbb{P}T_0^*A \mid \omega \in H^0(X, \Omega_X(-p)) \} \\ &\cong (\phi_Z, \text{Id})^{-1} I_{n-r,1},\end{aligned}$$

where

$$I_{n-r,1} := \{ (V, [\omega]) \in \text{Gr}(n-r, n) \times \text{Gr}(1, n) \mid \omega \in V \}$$

is the incidence variety relating $\text{Gr}(n-r, n)$ and $\text{Gr}(1, n)$. In that case,

$$\gamma_Z^{-1}([\omega]) = \{ p \in X \mid \omega(p) = 0 \}$$

is the zero set of section $\omega \in H^0(X, \Omega_X)$. The number $(-1)^r \deg \gamma_Z$ is the index in the Poincaré–Hopf index formula, and the monodromy group $\text{Gal}(\gamma_Z)$ serves as a more refined invariant, encoding subtler aspects of the geometry.

Remark 2.17. One can define the monodromy group in the real (or topological) setting in an analogous manner. Consider the set

$$U := \{ \omega \in \Gamma(\Omega_M) \mid \text{the zeros of } \omega \text{ are isolated with local index } 1 \}.$$

A loop in U induces a permutation of the zeros of ω , and the monodromy group is defined as the group of all such permutations.

However, when $\dim_{\mathbb{C}} M > 1$, the corresponding monodromy group is always the full symmetric group (as all transpositions occur). As a result, it does not yield any further insight into the topology of the manifold.

The next proposition shows when the Gauss map ϕ_Z is not generic injective.

Proposition 2.18. *When α is finite onto its image,*

ϕ_Z is not generic injective

$$\iff \text{For general } p \in X, \text{ exists } q \neq p \text{ such that } h^0(X, \Omega_X(-p-q)) = n-r$$

$$\iff \text{For general } p \in X, \text{ exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) = n-r$$

$$\iff \text{For all } p \in X, \text{ exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) = n-r.$$

This last step relies on the following lemma, combined with the closedness of the tautological correspondence in $X \times X^{[2]}$.

Lemma 2.19. *Let $m \in \mathbb{Z}_{>0}$. The function*

$$h^\Omega : X^{[m]} \longrightarrow \mathbb{Z}_{\geq 0} \quad S \longmapsto h^0(X, \Omega_X \otimes \mathcal{I}_S)$$

is Zariski upper semicontinuous.

Sketch of proof. Consider the coherent sheaf $\mathcal{F} \in \text{Coh}(X^{[m]} \times X)$ characterized by the property

$$\mathcal{F}|_{\{S\} \times X} \cong \Omega_X \otimes \mathcal{I}_S,$$

The proposition then follows directly from the semicontinuity theorem [9, 28.1.1]. \square

The stratification of $X^{[m]}$ by h^Ω offers a natural generalization of Brill–Noether theory beyond the setting of curves.

At the end of this subsection, let us turn to the setting of a general abelian variety A and a smooth subvariety $\iota_Z : Z \hookrightarrow A$, where $n = \dim_{\mathbb{C}} A$ and $r = \dim_{\mathbb{C}} Z$. Observe that ι_Z factors through the Albanese variety of Z :

$$\iota_Z : Z \xrightarrow{\alpha_Z} \text{Alb}(Z) \xrightarrow{\pi} A$$

We shall also assume that Z generates A ; it then follows that the map π is surjective. The cotangent map of ι_Z at a point $p \in Z$ factors through $H^0(Z, \Omega_Z)$:

$$T_p^* \iota_Z : T_p^* A \hookrightarrow H^0(Z, \Omega_Z) \longrightarrow T_p^* Z$$

For convenience, abbreviate $V := T_0^* A \cong T_p^* A$, and view V as a subspace of $H^0(Z, \Omega_Z)$.

Proposition 2.20. *Assume that Z is embedded in A and generates A , and let $V := T_0^* A$. Then*

$$\dim_{\mathbb{C}} H^0(Z, \Omega_Z(-p)) \cap V = n - r \quad \text{for all } p \in Z$$

It follows that the Gauss map is a regular morphism

$$\begin{array}{ccc} \phi_Z : Z & \dashrightarrow & \text{Gr}(r, T^0 A) \cong \text{Gr}(n - r, V) \\ p & \longmapsto & H^0(Z, \Omega_Z(-p)) \cap V \end{array}$$

Furthermore,

ϕ_Z is not generic injective

\iff For general $p \in X$, exists $q \neq p$ such that $h^0(X, \Omega_X(-p - q)) \cap V = n - r$

\iff For all $p \in X$, exists $S \in X^{[2]}$ such that $p \in S$ and $h^0(X, \Omega_X \otimes \mathcal{I}_S) \cap V = n - r$.

Question 2.21. *For an r -dimensional smooth variety Z whose Albanese map α_Z is an embedding, which subspaces $V \subset H^0(Z, \Omega_Z)$ can arise as the cotangent space of some quotient variety of $\text{Alb}(Z)$?*

The precise formulation of this question is somewhat ambiguous, and depending on the interpretation, it may turn out to be either relatively straightforward or extremely difficult. On the one hand, the classification of abelian subvarieties of A via symmetric idempotents in $\text{End}_{\mathbb{Q}}(A)$ is well established [2, Theorem 5.3.2]. On the other hand, explicitly computing $\text{End}_{\mathbb{Q}}(A)$ can be highly nontrivial when A is a non-simple Albanese variety—even in the case where A arises from a curve.

3. FAMILIES OF SUBVARIETIES

4. TANNAKIAN FORMALISM

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\text{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in \mathbb{Q} . For any algebraic group G , we denote by $\text{Rep}(G)$ the category of algebraic representations of G .

Following the approach of [7], we work in the quotient category $\overline{\text{Perv}}(A) = \text{Perv}(A)/N(A)$, where $N(A) \subset \text{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex \mathcal{F} is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [7, Thm 7.1 & Cor 9.2]). In particular, for any perverse sheaf $\delta \in \overline{\text{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G :

$$\langle \delta, * \rangle \cong \text{Rep}(G).$$

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