Bruhat–Tits building

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January 22, 2026

Figures of Bruhat–Tits building

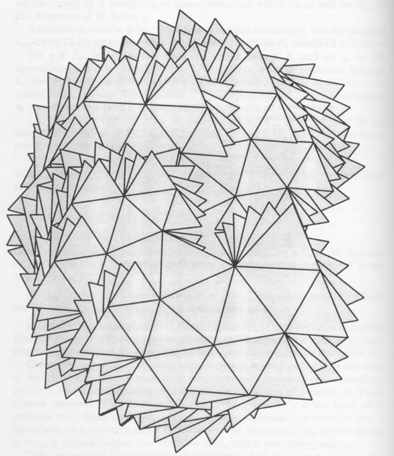

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from Annette Werner's talk

Figures of Bruhat–Tits building

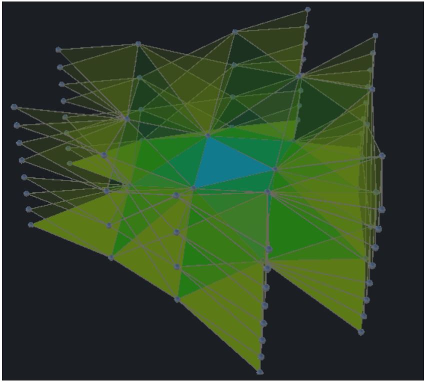

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

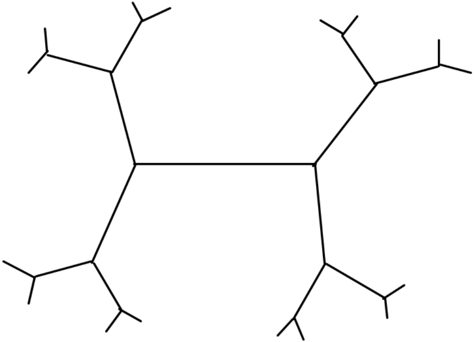

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

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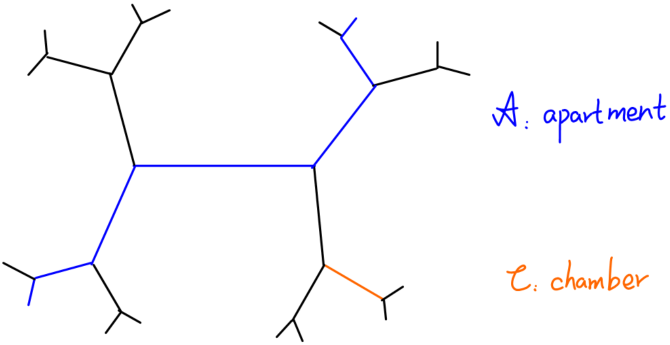

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Recap: standard Lie theory

Restrict to **complex** representations, we have a nice theory:

- Any representation can be written as a direct sum of **irreducible representation**;
- We can extract information of irreducible representations from the **character table**:

$$\#\{\text{irreducible representations}\} = \#\{\text{conjugation classes}\}$$
$$\sum_{\chi:\text{irr}} (\dim \chi)^2 = \#G$$

However, in general,

- NO standard way finding an **explicit construction** of all irreducible representations;
- NO **one-to-one correspondence** between irreducible representations and conjugation classes.

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{ V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i \} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{ V \subset \kappa^n \mid \dim V = r \} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{ \kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1 \}$$

T is comm, so every rep decomposes as direct sum of 1-dim reps.

$X^*(T) =: \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ characters (1-dim reps)

$X_*(T) =: \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n$ cocharacters (1-parameter subgps)

Weyl group

Definition (Weyl group)

$$W := N_G(T)/T.$$

Example

When $G = \mathrm{GL}_n(\kappa)$,

$$\begin{aligned} N_G(T) \quad &= \{ \textit{ monoidal matrixes } \} \\ N_G(T)/T &\cong \ S_n \qquad\qquad \textit{Weyl group of type } A \end{aligned}$$

Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B.$$

So the Weyl group is the "heart" of the reductive group.

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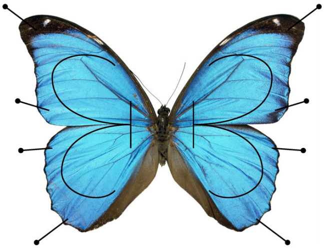
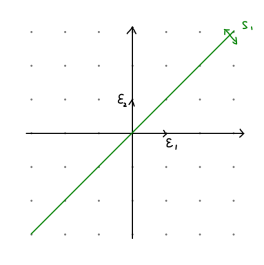


Figure: Pinned butterfly

Weyl group action on cocharacter lattices

When $G = \mathrm{GL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$, where



$$\begin{aligned} \varepsilon_1 : \ \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

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$$W = S_2 = \{\mathrm{Id}, s_1\}$$

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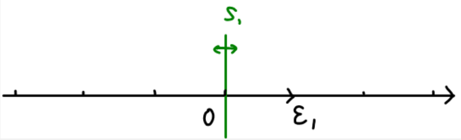

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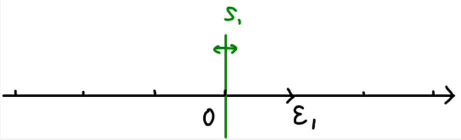

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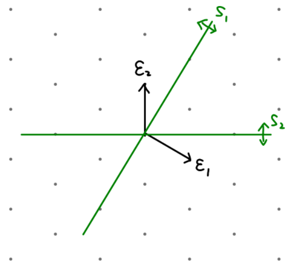

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$$W = S_3 = \langle s_1, s_2 \rangle$$

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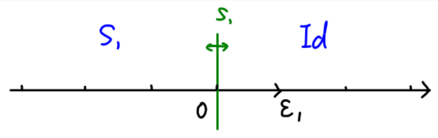

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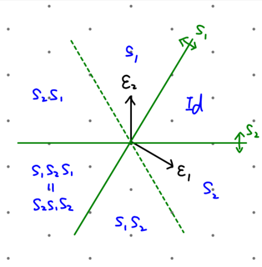

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Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\{ \textit{ Borel subgroups } \} = \left\{ gBg^{-1} \right\} \cong G/B$$
$$\{ \textit{ parabolic subgroups } \} = \left\{ gPg^{-1} \right\} \cong G/P$$
$$\{ \textit{ maximal tori } \} = \left\{ gTg^{-1} \right\} \cong G/N_G(T)$$

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\{ \text{ maximal tori } \} &= \left\{ gTg^{-1} \right\} && \cong G/N_G(T) \\
\{(B, T) \mid B \supset T\} &= \left\{ (gB_0g^{-1}, gT_0g^{-1}) \right\} && \cong G/T_0
\end{aligned}
$$

Definition (chamber, apartment and building)

Given a maximal torus T, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{B \supset T} \mathcal{C}_B,$$

and the building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_{\sim} = \bigcup_B \mathcal{C}_B.$$

Example of spherical building

When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

Remark

$\mathcal{B}$ inherits the metric structure from $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$.
$\mathcal{B}$ has also polysimplicial complex structure.
When $\kappa = \mathbb{F}_p$, $\mathcal{B}$ is finite.

Proposition

- *Two chambers lie in one apartment.*
- *There is a unique geodesic passing any two points $p_1, p_2 \in \mathcal{B}$.*

p-adic notation

symbol	name	example
F	local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	integral ring	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

standard subgroups in p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} \qquad \text{Iwahori subgroup}$$

$$\widetilde{P} = \pi^{-1}(P) = \left(\begin{array}{c:c} \mathcal{O} & \mathcal{O} \\ \hdashline \mathfrak{p} & \mathcal{O} \end{array}\right) \qquad \text{Parahoric subgroup}$$

Remark

They also have moduli intepretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{L = L_0 \subset L_1 \subset \cdots \subset L_n = \mathfrak{p}L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

Extended Weyl group

To get the Iwahori decompositionn

$$G(F) = \bigsqcup_{\varpi \in W_{\text{ext}}} I \varpi I,$$

we define the extended Weyl group as

$$W_{\text{ext}} := N_G(T(\mathcal{O}))/T(\mathcal{O}) \cong X_*(T) \rtimes W_f.$$

Example

When $G = \mathrm{GL}_n(F)$,

$$W_{\text{ext}} = \{ \text{ monoidal matrixes } \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

Extended Weyl group action

W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

Non-standard subgroups in p-adic world

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p-adic building

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Given a maximal torus T over $\mathcal{O}$, the apartment is

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and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_\sim \; = \bigcup_I \mathcal{C}_I.$$

Remark

Similarly, two chambers lie in one apartment,
and there is a unique geodesic passing $p_1, p_2 \in \mathcal{B}$.

Gromov-Schoen theorem

Theorem

Let F be a local field, (M, g) be a cpt conn Riemannian manifold with the universal covering space $\widetilde{M}$.
For any reductive map

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}_n(F),$$

there exists a $\pi_1(M)$-equivariant Lipschitz continuous regular harmonic map

$$h_\rho : \widetilde{M} \longrightarrow \mathcal{B}_{\mathrm{GL}_n(F)}$$

regularity

Definition

h_ρ is regular at $x \in \widetilde{M}$ if
a neighbourhood of x is contained in an apartment of $\mathcal{A}$.

h_ρ is regular if

$$\operatorname{codim}_{\widetilde{M}} \left\{ x \in \widetilde{M} \;\middle|\; h_\rho \text{ is not regular at } x \right\} \geqslant 2.$$

Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.

test

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