

# Subvarieties in Complex Abelian Varieties

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## Tangent Gauss Map

Let  $A/\mathbb{C}$  be an abelian variety of dimension  $n$ , and let  $Z \subset A$  be a non-degenerate closed subvariety of dimension  $r$ .

To understand the geometry of  $Z$ , we encode the variation of its tangent spaces via the tangent Gauss map

$$\phi_Z : Z^{\text{sm}} \longrightarrow \text{Gr}(r, T_0 A) \quad p \longmapsto T_p Z \subset T_p A \cong T_0 A.$$

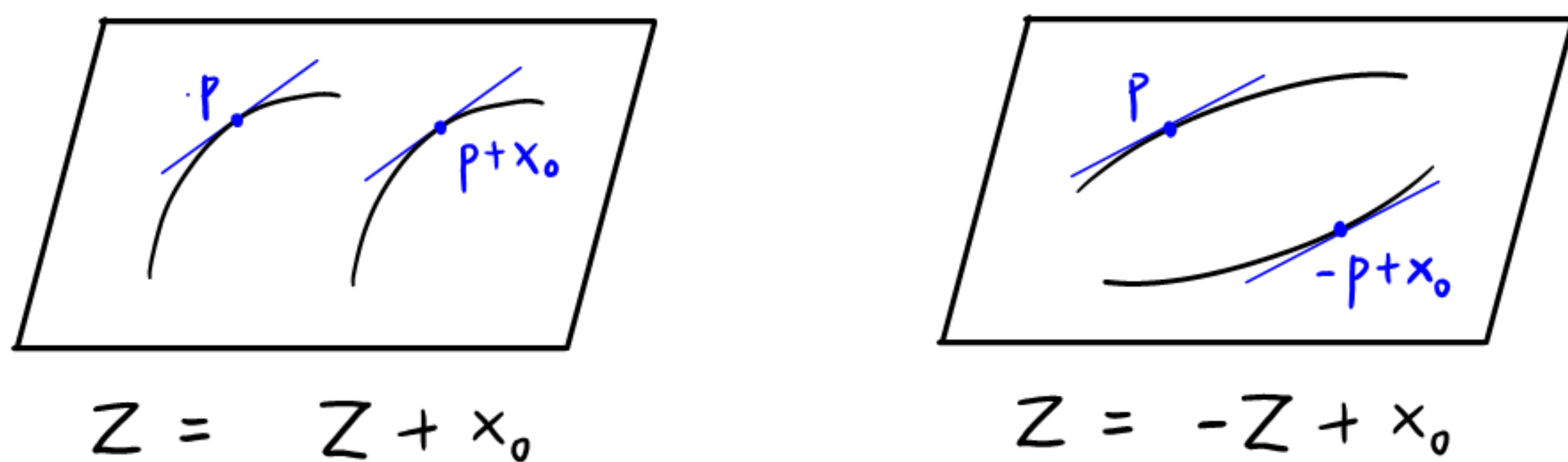
Its differential

$$d_p \phi_Z : T_p Z \longrightarrow \text{Hom}_{\mathbb{C}}(T_p Z, N_p Z)$$

is second fundamental form, which captures information about the curvature of  $Z$ .

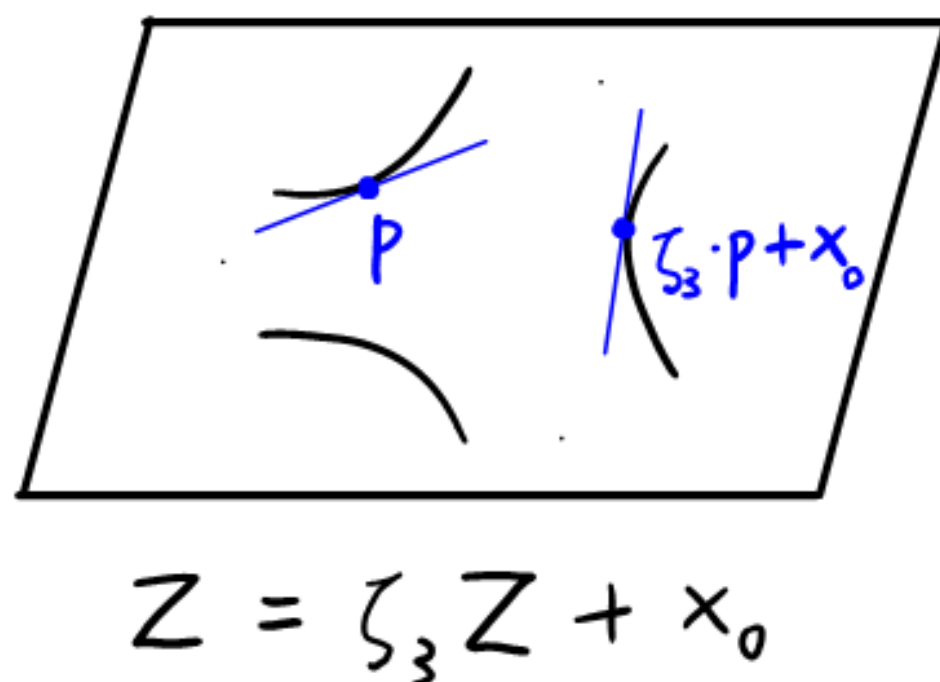
## Generic Injectivity of the Tangent Gauss Map

Clearly the tangent Gauss map has degree  $> 1$  whenever the subvariety  $Z \subset A$  is stable under a nontrivial translation or symmetric up to a translation, as shown below:



Are these the only cases where  $\gamma_Z$  fails to be generically injective, when  $Z = C$  is a curve? When  $n = 2$ ,  $\phi_C : C^{\text{sm}} \longrightarrow \mathbb{P}^1$  typically fails to be generically injective.

**Question 1.** Let  $C$  be a non-degenerate curve on an abelian variety  $A$  of dimension  $n > 2$ , and assume that  $C$  is not invariant under any non-trivial translation or reflection. Does it follow that  $\phi_C$  is generically injective?



**Example 1.** For any  $n$  the power  $A = E_\rho^{\oplus n}$  comes with a natural diagonal action of  $\mu_3 \cong \mathbb{Z}/3\mathbb{Z}$ . Computer experiments yield a non-degenerate  $\mu_3$ -invariant curve  $C \subset A$ , for which  $\phi_C$  is not generically injective.

We have found no counterexample to Question 1 when  $A$  is not isogenous to  $E_1^{\oplus n}$  or  $E_\rho^{\oplus n}$ . This suggests the following refinement:

**Conjecture 1.** Let  $C$  be a non-degenerate curve on an abelian variety  $A$  of dimension  $n > 2$ . Then  $\phi_C$  is generically injective unless there exists a non-trivial automorphism  $\tau \in \text{Aut}(A)$  which preserves  $C$  and acts via a scalar on  $T_0 A$ .

One may restate Conjecture 1 using Gauss curvature, yielding a slightly stronger statement:

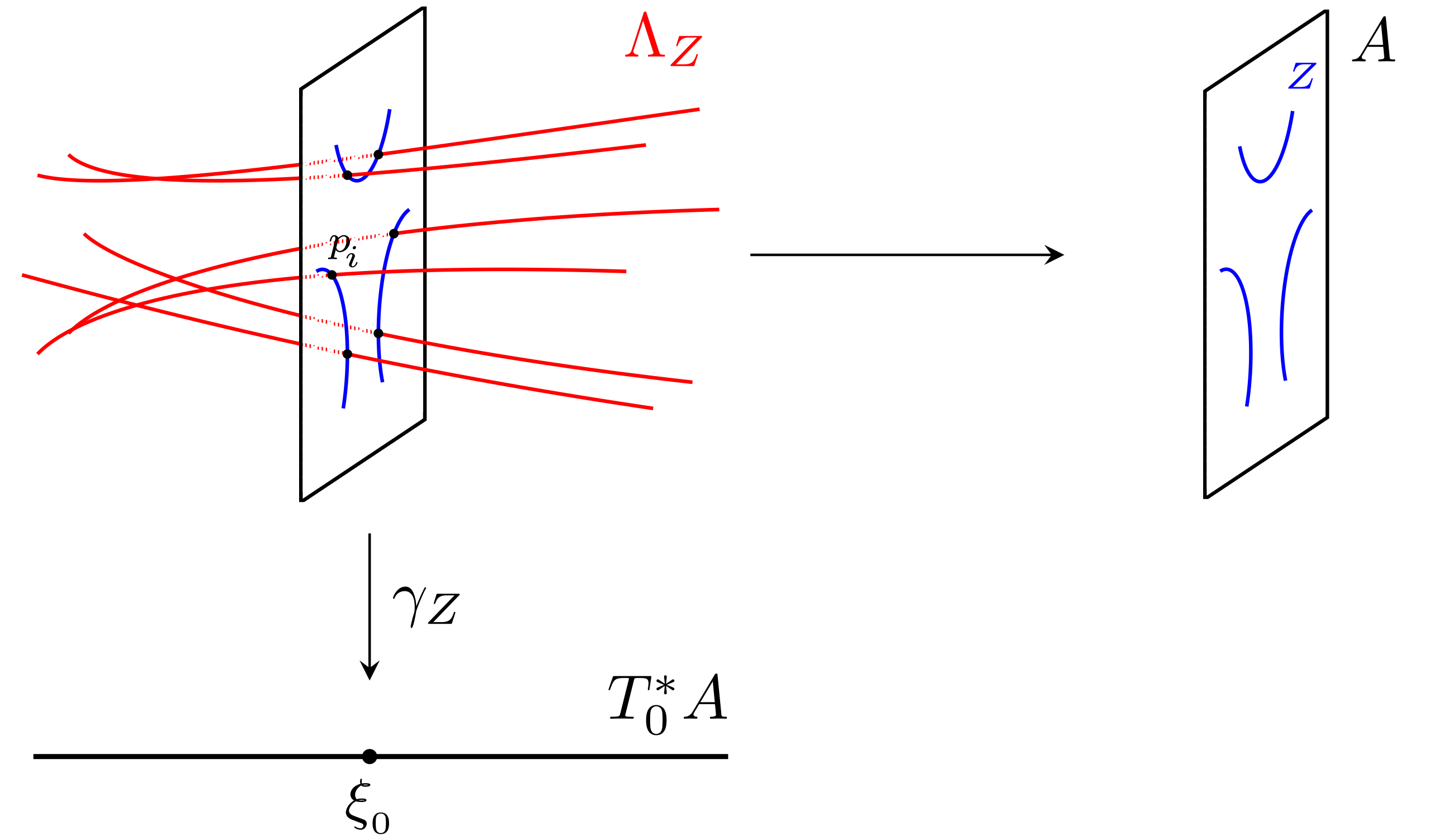
**Conjecture 2.** Let  $C$  be a non-degenerate curve on an abelian variety  $A$  of dimension  $n > 2$ . Then for general  $p \in \text{Im } \phi_C$ , the Gauss curvature of  $C \subset A$  is the same at all points of the fiber  $\phi_C^{-1}(p)$ .

## Known Cases

- If  $A = \text{Jac}(C)$  and  $C$  is embedded via the Abel–Jacobi map, then  $\phi_C = [\omega_C]$  is the canonical map, and one may check Conjectures 1 and 2 by direct inspection.
- Let  $h : C \longrightarrow C'$  be a cyclic  $k$ -fold cover defined by  $\eta \in \text{Pic}(C')$  with  $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$ . If  $A = \text{Prym}(C/C')$  and  $C \rightarrow A$  is the Abel–Prym map, then
 
$$\phi_C : C \longrightarrow \mathbb{P}T_0 A \cong \mathbb{P} \left( \bigoplus_{i=1}^{k-1} H^0(\omega_{C'} \otimes \eta^i) \right)$$
  - $k = 2$ :  $C$  is invariant under the Prym involution, and  $\phi_C = [\omega_{C'} \otimes \eta] \circ h$ . If  $C'$  is non-hyperelliptic with  $g(C') \geq 4$ , then  $\deg \phi_C = 2$  or  $4$ , and  $\deg \phi_C = 4 \iff B = \emptyset, C'$  is bielliptic and  $\eta$  pulled back from EC.
  - $k > 2$ : if  $g(C') \geq 1$  and  $[\omega_{C'} \otimes \eta]$  is generically injective, then  $\phi_C$  is generically injective.
- If  $C \subset A$  is smooth and either  $\deg \phi_C = 2$  or  $\phi_C$  is unramified, Conjecture 2 also holds.

## Conormal Gauss Map

Passing to the cotangent perspective, one is naturally led to the conormal variety and the associated conormal Gauss map. Consider the conormal variety  $\Lambda_Z \subset T^*A \cong A \times T_0^*A$ . The natural projection is the conormal Gauss map  $\gamma_Z^{(\text{aff})} : \Lambda_Z \longrightarrow T_0^*A$ , which is generically finite whenever  $Z$  is of general type.



**Conjecture 3.** Suppose  $A$  is not isogenous to  $E_1^{\oplus n}$  or  $E_\rho^{\oplus n}$ , and let  $C \subset A$  be a non-degenerate curve which is not stable under any non-trivial translation or reflection. Then the monodromy group  $\text{Gal}(\gamma_C)$  is big — namely, a Weyl group of type  $A$ ,  $C$ , or  $D$ .

When  $n > 2$ , Conjecture 3 follows from Conjecture 2.

The monodromy group  $\text{Gal}(\gamma_C)$  helps us to control the Tannaka group of the tensor category generated by the IC sheaf on  $C$ ; see [Krä22, Theorem 2.1].

## The Subvariety $Z^{(m)}$

The convolution structure on perverse sheaves gives rise to numerous cycles in  $A$ . They admit a simple geometric description: by fiberwise summing points in  $\gamma_Z^{-1}(\xi_0)$  and projecting, one obtains new subvarieties of varying dimensions.

Fix a general point  $\xi_0 \in T_0^*A$  and choose an ordering  $\gamma_Z^{-1}(\xi_0) = \{p_1, \dots, p_d\} \subset Z$ . For  $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$ , let  $Z^{(m)}$  denote the irreducible component of the subvariety obtained by this construction that contains the point  $\sum_i m_i p_i$ .

**Theorem 1.** Let  $c_i := c_{M,i}(\Lambda_Z)$  be the Segre class of the cone  $\Lambda_Z$ , and let  $*$  denote the Pontryagin product.

When  $\text{Gal}(\gamma_Z) = S_d$ , the Segre classes of  $\Lambda_{Z^{(m)}}$  can be written as

$$c_{M,l}(\Lambda_{Z^{(m)}}) = \frac{1}{d_Z^{(m)}} \sum_{\lambda \vdash l} \mu_d^\lambda \left( \bigstar_{i=1}^{k'} c_{\lambda_i} \right)$$

where  $\lambda = [\lambda_1, \dots, \lambda_{k'}]$  ranges over all partitions of  $l$  and where

- $d_Z^{(m)} \in \mathbb{N}_{>0}$  is the degree of a certain addition map;
- $\mu_d^\lambda = \sum_{\alpha \in \mathcal{P}(d)} \sum_{\mathbf{l}: \text{length } k} \mu(\hat{0}, \alpha) \alpha(m)^{2\mathbf{l}} d^{k-k'} \in \mathbb{Z}[m_1, \dots, m_d]^{S_d}$ ;
- $\alpha = \{A_1, \dots, A_k\} \in \mathcal{P}(d)$ ,  $\mathbf{l} = (l_1, \dots, l_k) \in \mathbb{Z}^k$ ;
- $\mu(\hat{0}, \alpha) = (-1)^{d-k} \prod_{i=1}^k (|A_i| - 1)$ ;
- $\alpha(m)^{2\mathbf{l}} = (\sum_{i \in A_1} m_i)^{2l_1} \cdots (\sum_{i \in A_k} m_i)^{2l_k}$ .

## Remarks.

- One can recover both  $\dim Z$  and  $[Z] \in H^{2(n-\dim Z)}(A)$  from the Segre classes of  $\Lambda_Z$ :
 
$$\dim Z = \max \{i \in \mathbb{Z} \mid c_i \neq 0\}, \quad [Z] = c_{\dim Z}.$$
- If  $Z = -Z$ , then a similar formula also holds for  $\text{Gal}(\gamma_Z) = W(C_{d/2})$ , but the method does not extend to the case where  $\text{Gal}(\gamma_Z) = W(D_{d/2})$ .
- The formula simplifies a lot for curves inside their Jacobian. For example, if  $C$  is non-hyperelliptic, we obtain:

$$c_l(\Lambda_{Z^{(m)}}) = \frac{1}{c_Z^{(m)}} \frac{1}{2^l (g-l)!} \sum_{\sigma \in S_{2g-2}} \prod_{i=1}^l (m_{\sigma(2i-1)} - m_{\sigma(2i)})^2 \cdot \Theta^{g-l}$$

$$\dim Z^{(m)} = \min_{k \in \mathbb{Z}} \{g-1, \# \{i \in [2g-2] \mid m_i \neq k\}\}$$

## References

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- [Krä22] Thomas Krämer. Characteristic cycles and the microlocal geometry of the Gauss map. I. *Ann. Sci. Éc. Norm. Supér. (4)*, 55(6):1475–1527, 2022.
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