LaTeX TEMPLATE

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Contents

1. A small toolkit

$f : Y \longrightarrow \mathrm{pt}$ $f : p \hookrightarrow X$
f^* constant sheaf $\mathcal{F}_p$
Rf_* cohomology $\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$ cpt supp cohomology $\mathrm{sky}_p(\mathbb{Q})$
$f^!$ orientation sheaf $[n]$ $\mathcal{F}_p[-n]$
For $f^!$, assume Y, X are manifolds of dimension n.
$j_!j^*\mathcal{F}$ $\mathcal{F}$ $i_!i^*\mathcal{F}$

$$Z \xhookrightarrow{i} X \xhookleftarrow{j} U$$

$$\begin{tikzcd} D(Z) \arrow[r, "i_*=i_!"] & D(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & D(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \xrightarrow{+1}$$

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$Rf_!$	cpt supp cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$f^!$	orientation sheaf	$[n]$ $\mathcal{F}_p[-n]$

$$\begin{tikzcd}
& \otimes \arrow[dl, dash, "f^*(-\otimes-)"'] \arrow[dr, dash, "\text{proj formula}"] & \\
f^* \arrow[rr, dash, "\text{base change}"'] & & f_!
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\end{tikzcd}$$

2. A Short List of Applications

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:

$$\mathrm{H}^i_c(X;\mathbb{Q}) \cong \mathrm{H}^i\left(\bar{X},\{\infty\};\mathbb{Q}\right)$$

$$\mathrm{H}^{\mathrm{BM}}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$

$$\mathrm{H}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_c(X;\mathrm{Or}_X)$$

Also, define the cup and cap product structures.

2. Using the projection formula, show Poincaré duality:

$$\mathrm{H}^i_c(X;\mathbb{Q})^* \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$

$$\mathrm{H}^i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_c(X;\mathrm{Or}_X)^*$$

3. Derive the Gysin sequence for any oriented S^k-bundle $\pi : E \longrightarrow B$:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[eu_\pi]{+1}$$

Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.

4. Compute the upper shriek for singular spaces.

$$\begin{array}{llll}
\mathrm{H}^i(Y,\mathbb{Q}) & = \mathrm{H}^i(Y,\underline{\mathbb{Q}}_Y) & = f_*\underline{\mathbb{Q}}_Y & = f_*f^*\mathbb{Q} \\
\mathrm{H}^i_c(Y,\mathbb{Q}) & = \mathrm{H}^i_c(Y,\underline{\mathbb{Q}}_Y) & = f_!\underline{\mathbb{Q}}_Y & = f_!f^*\mathbb{Q} \\
\mathrm{H}_{-i}(Y,\mathbb{Q}) & = \mathrm{H}^{n+i}_c(Y,\mathrm{Or}_Y) & = f_!\,\mathrm{Or}_Y[n] & = f_!f^!\mathbb{Q} \\
\mathrm{H}^{\mathrm{BM}}_{-i}(Y,\mathbb{Q}) & = \mathrm{H}^{n+i}(Y,\mathrm{Or}_Y) & = f_*\,\mathrm{Or}_Y[n] & = f_*f^!\mathbb{Q}
\end{array}$$

six functor formalism $\approx$ cohomology theory

3. perverse sheaf

We will mix the usage of sheaves and complexes. For simplicity, let us fix a stratification $\mathcal{S}$:

$$\varnothing \overset{U_0}{\subset} Z_0 \overset{U_1}{\subset} \cdots \overset{U_n}{\subset} Z_n = X$$

Denote $D^b_{\mathrm{cons},\mathcal{S}}(X)$ as the category of constructible sheaves over X with respect to $\mathcal{S}$.

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure. We say that $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^i\left(\iota_{U_j}^*\mathcal{F}\right) = 0, & \text{for any } i > -j \\ \mathcal{H}^i\left(\iota_{U_j}^!\mathcal{F}\right) = 0, & \text{for any } i < -j \end{cases}$$

To determine whether a complex $\mathcal{F}$ is perverse, one simply needs to complete the following table:

$\iota_{U_j}^*(\mathcal{F})$

The local system supported on U_i (denoted by $\mathcal{L}$) converted to a perverse sheaf by truncations. This method is called Deligne's construction, and the constructed perverse sheaf is called the intersection cohomology complex(or the IC sheaf), denoted by $IC(\mathcal{L})$. IC sheaves are the simple objects in the category $Perv_{\mathcal{S}}(X)$.

4. Nearby Cycle

A perverse sheaf may not be so "perverse", but a nearby cycle is definitely "nearby".

Given $\mathcal{F} \in D^b(\mathbb{C})$, one can construct the nearby cycle

$$\psi\mathcal{F} := i^*Rj_*p_*p^*j^*\mathcal{F} \in D^b(\{0\}),$$

which can be roughly viewed as the fiber $\mathcal{F}_x$ for x sufficiently close to 0. By quotienting out the non-vanishing cycle $i^*\mathcal{F}$, one obtains the vanishing cycle

$$\varphi\mathcal{F} := \text{cone}\left[i^*\mathcal{F} \xrightarrow{sp} \psi\mathcal{F}\right] \in D^b(\{0\}).$$

In general, $\mathcal{C}$ can be replaced by any disk $\mathcal{D}$, as the problem is local, and $\mathcal{F}$ can be a sheaf over any space X over $\mathcal{D}$.

The same construction yields a distinguished triangle in $D^b(X_0)$:

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \varphi_f\mathcal{F} \xrightarrow{+1}$$

5. Characteristic Cycle

With normal Morse data, one can define the characteristic cycle

$$\mathrm{CC}(\mathcal{F}) := \sum_S m_S[T_S^*\mathbb{C}^n] \in H_{2n}^{\mathrm{BM}}(\cup_S T_S^*\mathbb{C}^n),$$

where

$$m_S := \chi(\mathrm{NMD}(\mathcal{F}, S)[-\dim S]) \in \mathbb{Z}.$$

Notably, $\mathrm{CC}(\mathcal{F})$ does not depend on the stratification $\mathcal{S}$.

The characteristic cycle can be computed when the geometry is well-understood, such as when X is a cone over a smooth hyperplane.

6. nmd

We work with a fixed complex variety embedding $X \subseteq \mathbb{C}^n$, equipped with a Whitney stratification $\mathcal{S}$. Let $S \subseteq X$ be a connected component of some U_i. Fix $x_0 \in S$, and let N be a normal slice of S at x_0.

For any sheaf $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$, the normal Morse data(NMD) is defined as

$$\mathrm{NMD}(\mathcal{F}, S) := \left(\varphi_{g|_{N\cap X}}\left(\mathcal{F}|_{N\cap X}\right)\right)_{x_0}[-1]$$

where $g : \mathbb{C}^n \longrightarrow \mathbb{C}$ is a holomorphic function, and $f := Re(g)$ such that

- $g(x_0) = 0$;
- $df_{x_0} \in T_S^*\mathbb{C}^n$, and $df_{x_0} \notin T_{S'}^*\mathbb{C}^n$ for any $S' \neq S$;
- x_0 is a non-degenerate critical point of $f|_S$.

References

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