Bruhat–Tits building

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Figures of Bruhat–Tits building

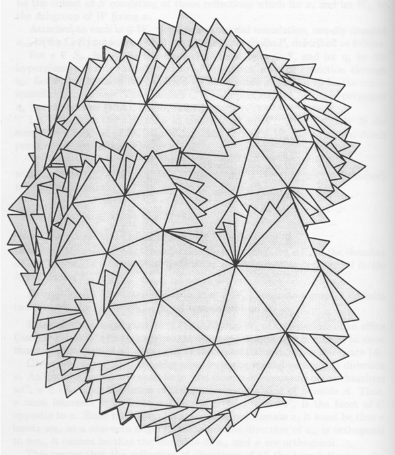

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from a talk by Annette Werner

Figures of Bruhat–Tits building

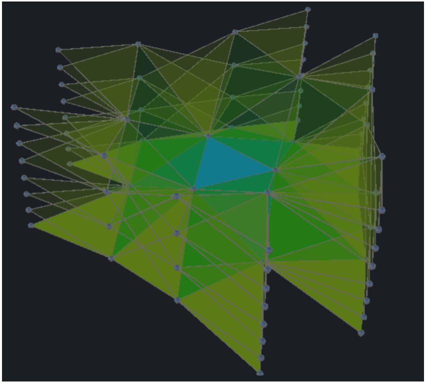

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

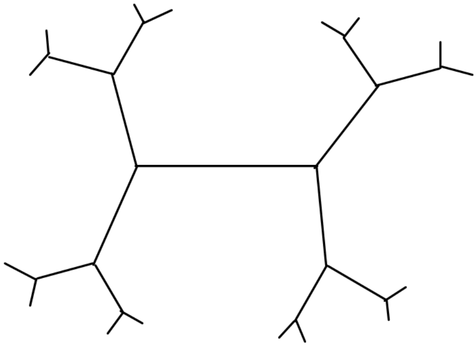

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

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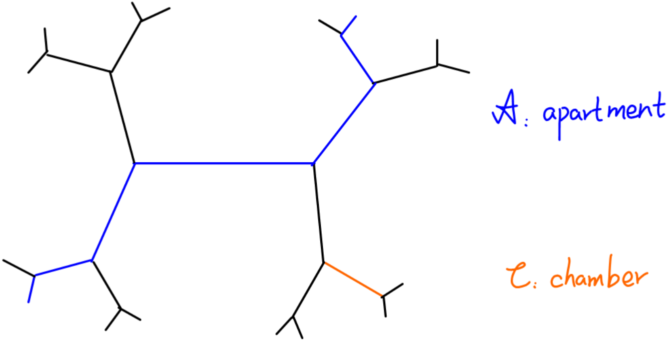

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We have Bruhat decomposition proved by Gauss elimination

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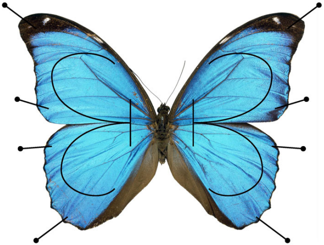

Figure: Pinned butterfly

Weyl group action on cocharacter lattices

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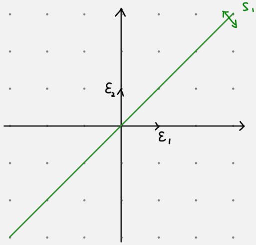

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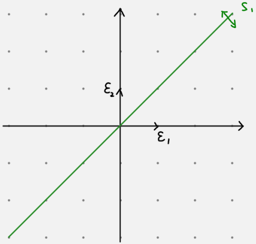

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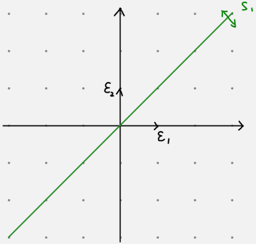

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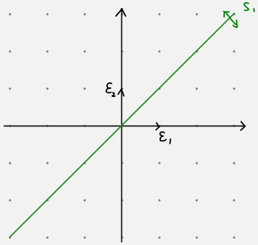

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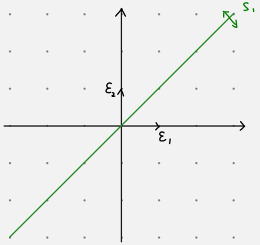

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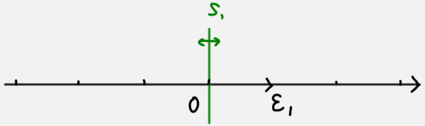

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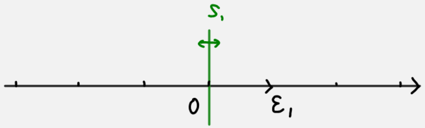

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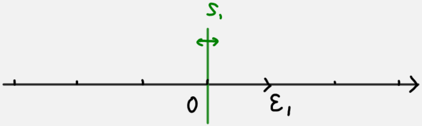

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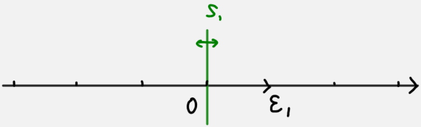

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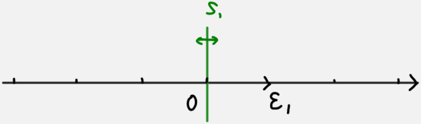

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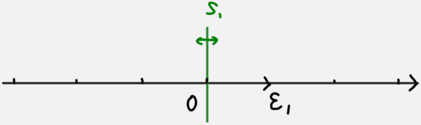

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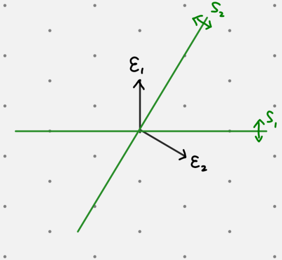

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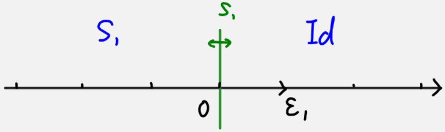

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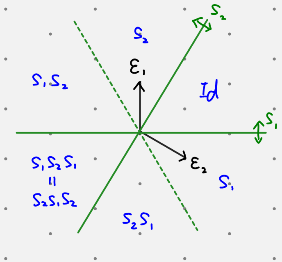

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\end{tikzcd}
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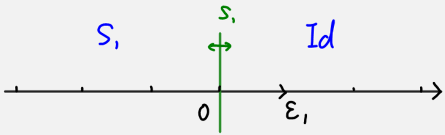

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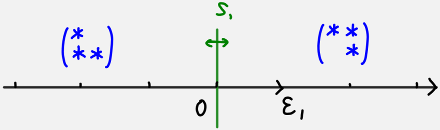

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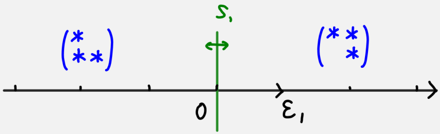

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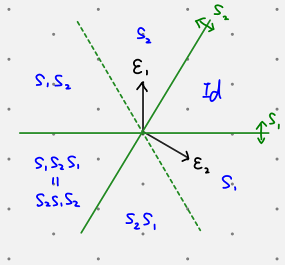

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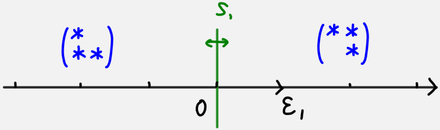

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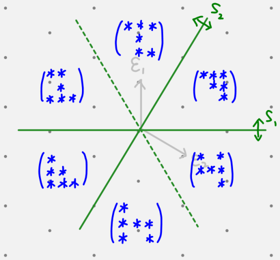

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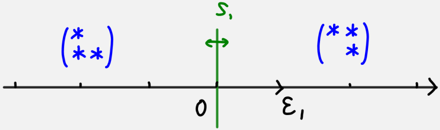

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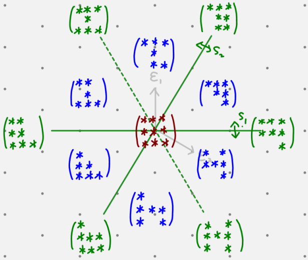

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When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.

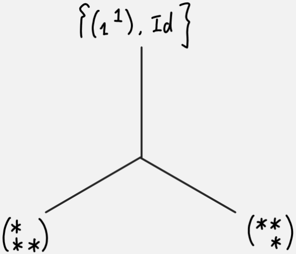

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

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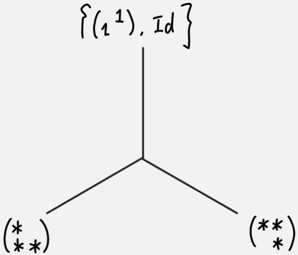

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

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Proposition

- *Any two chambers lie in one apartment.*
- *There is a unique geodesic through any two points $p_1, p_2 \in \mathcal{B}$.*

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

p-adic notation

symbol	name	example
F	NA local field	
$\mathcal{O} = \mathcal{O}_F$	ring of integers	
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	
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$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

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Remark

They also have moduli interpretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{\mathfrak{p}L = L_0 \subset L_1 \subset \cdots \subset L_n = L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

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To get the Iwahori decomposition

$$G(F) = \bigsqcup_{\varpi \in W_{\mathrm{ext}}} I \varpi I,$$

we define the extended Weyl group as

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Example

When $G = \mathrm{GL}_n(F)$,

$$W_{\mathrm{ext}} = \{ \textit{ monoidal matrices } \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

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W_{ext} acts on $X_*(T)$ by "twisted conjugation":

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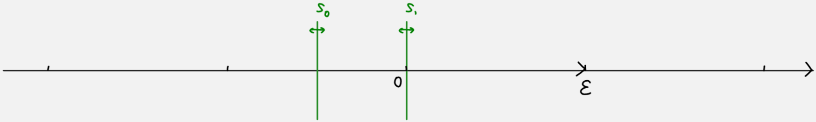

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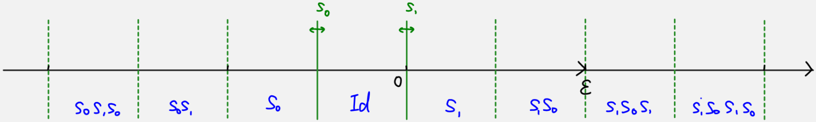

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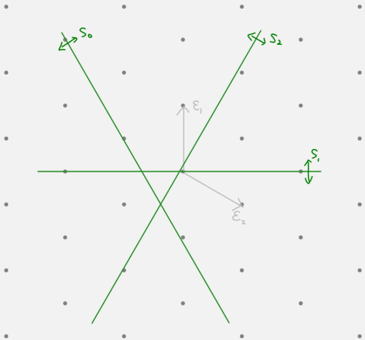

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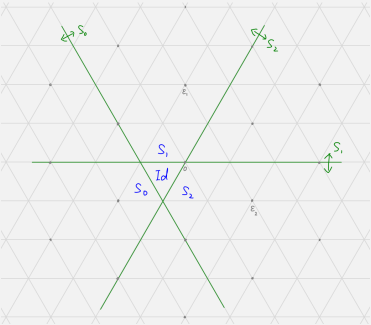

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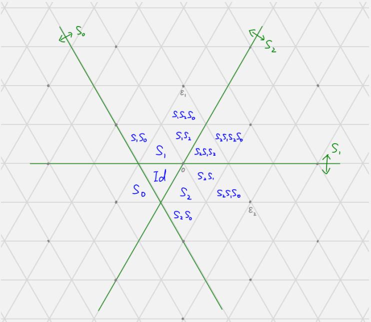

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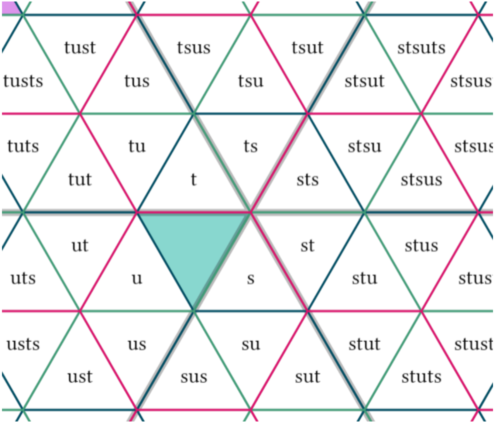

Figure: Reduced expressions labels, from Lievis

Non-standard subgroups in the p-adic world

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Similarly,

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\{I\} & & \{T\}
\end{tikzcd}
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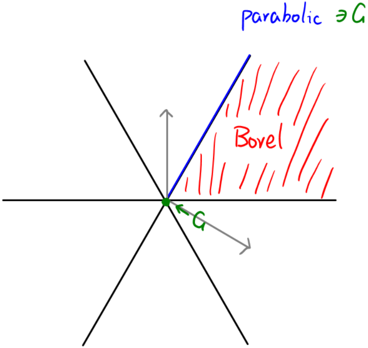
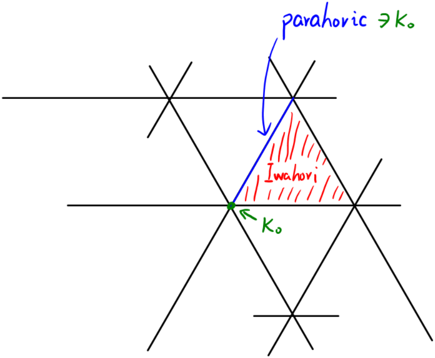
$$\{ \text{ chambers } \} \quad \overset{1:1}{\longleftrightarrow} \quad W_{\text{ext}} \quad \overset{1:1}{\longleftrightarrow} \quad \{I \mid I \supset T_0\}$$

Comparison





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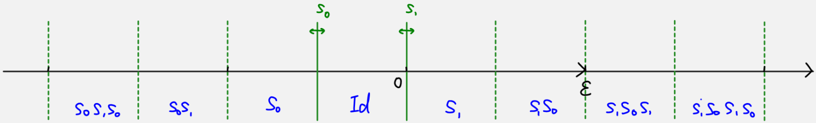

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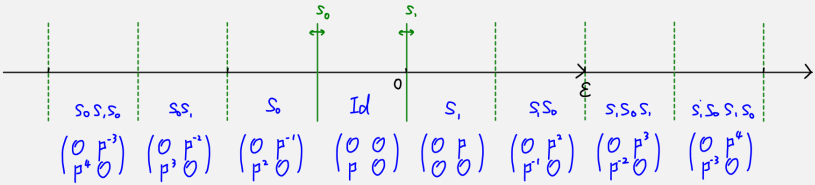

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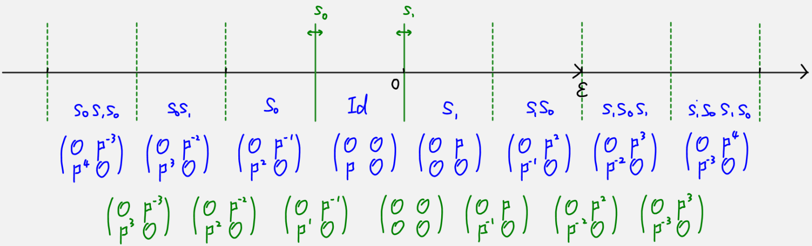

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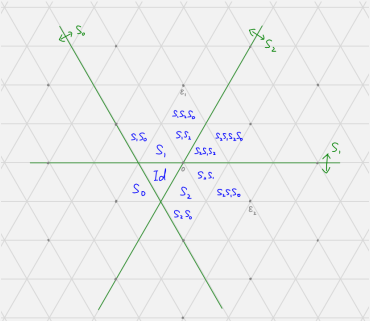

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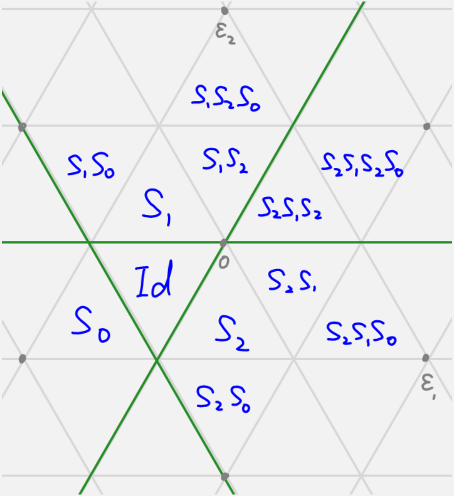

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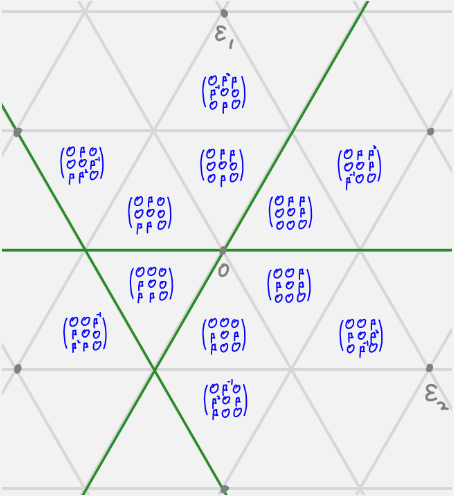

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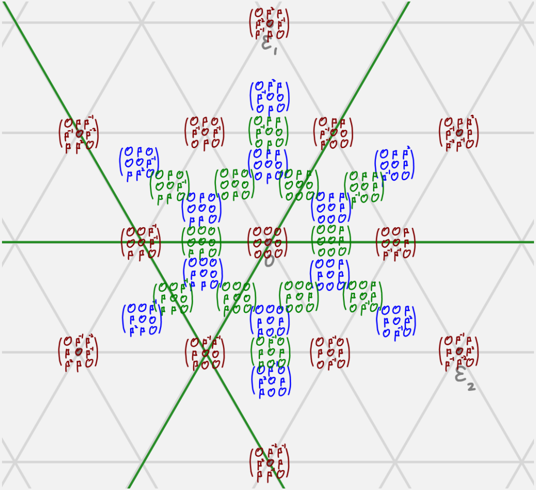

p-adic building

p-adic building

Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_T \mathcal{A}_T \right) /_{\sim} = \bigcup_I \mathcal{C}_I.$$

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Remark

Similarly, any two chambers lie in one apartment,
and there is a unique geodesic through $p_1, p_2 \in \mathcal{B}$.

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

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The map

$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

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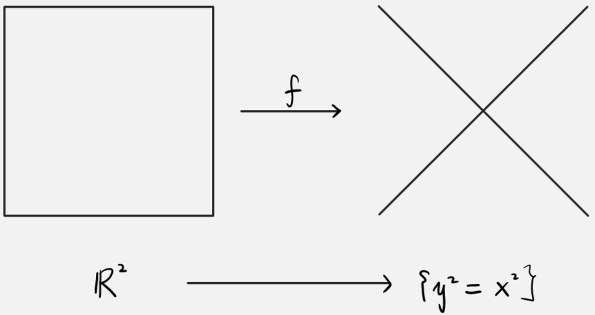

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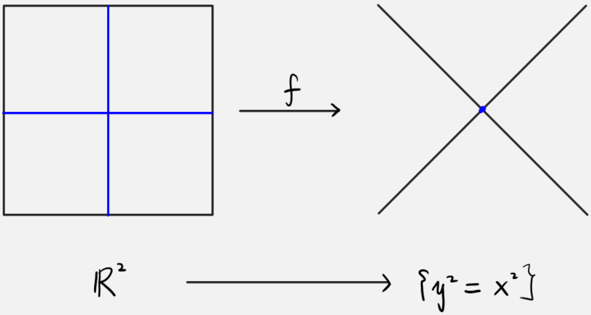

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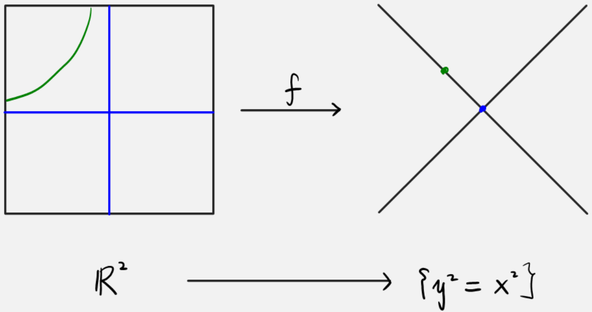

Thanks for listening!

You can get this slide at:
https://github.com/ramified/personal_tex_collection/raw/main/Bruhat–Tits_building/Bruhat–Tits_building.pdf