

# Springer Fibers for $SL_n(\mathbb{C})$

Xiaoxiang Zhou

Universität Bonn

January 28, 2022

# Recap: representation theory of finite groups

Restrict to **complex** representations, we have a nice theory:

- Any representation can be written as a direct sum of **irreducible representation**;
- We can extract information of irreducible representations from the **character table**:

$$\#\{\text{irreducible representations}\} = \#\{\text{conjugation classes}\}$$

$$\sum_{\chi:\text{irr}} (\dim \chi)^2 = \#G$$

However, in general,

- NO standard way finding an **explicit construction** of all irreducible representations;
- NO **one-to-one correspondence** between irreducible representations and conjugation classes.

In this talk, we use two methods to understand representations of  $S_n$ , and find connections/analogues between them.

methods	objects
combinatorial	Young diagram, Young tableau
geometrical	Springer fiber of $SL_n(\mathbb{C})$ , irreducible components

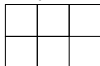
# Goal of the Part I

- Explicitly **construct irreducible representations** of  $S_n$  by Young diagram;
- Compute the character table;
  - $\dim \chi_i$  by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g.  $\otimes$ ,  $\text{Sym}^m$ ,  $\Lambda^m$ ;
  - e.g.  $M_\lambda$ .
  - restriction and induced representation
  - representation ring

# Notation

For boxes:

(Young) diagram



filling

11	78	11
6	8	

standard filling

3	5	4
1	2	

tableau

6	11	11
8	78	

standard tableau

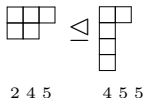
1	3	4
2	5	

Order of Young diagram:

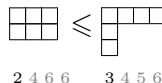
inclusion



dominance



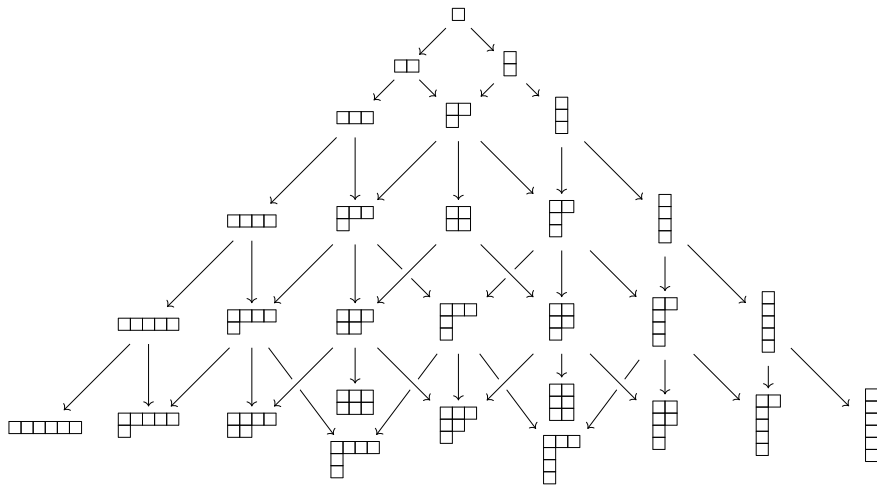
Lexicographic ordering



# tree of Young diagram

# tree of Young diagram

▶ skip

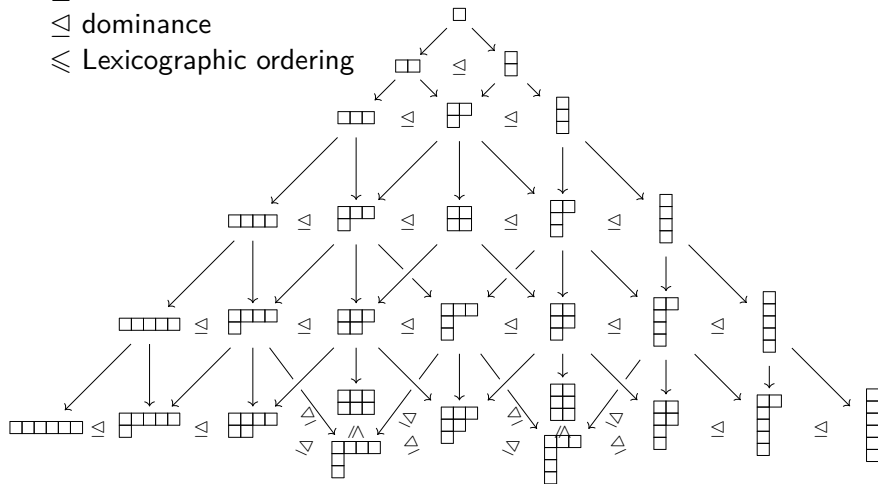


# Order

$\sqsubseteq$  inclusion

$\trianglelefteq$  dominance

$\preceq$  Lexicographic ordering





# $S_n$ & Young diagram

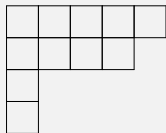
## Proposition

*We have the one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{Young diagram} \\ \text{of } n \text{ boxes} \end{array} \right\} \begin{array}{c} \xleftarrow{\text{partition of } n} \\ \xrightarrow{\lambda = \lambda_1^{v_1} \dots \lambda_k^{v_k}} \end{array} \left\{ \begin{array}{l} \text{Conjugation class} \\ \text{of } S_n \end{array} \right\}$$

## Example

$n = 11$ .



$$\begin{array}{c} \xleftarrow{11=5+4+1+1} \\ \xrightarrow{\lambda=5 \cdot 4 \cdot 1^2} \end{array} (12345)(6789)(10)(11)$$

# $S_n$ & Young diagram

## Proposition

*We have the one-to-one correspondence*

$$\left\{ \begin{array}{l} \text{Young diagram} \\ \text{of } n \text{ boxes} \end{array} \right\} \begin{array}{c} \xleftarrow{\text{partition of } n} \\ \xrightarrow{\lambda = \lambda_1^{v_1} \dots \lambda_k^{v_k}} \end{array} \left\{ \begin{array}{l} \text{Conjugation class} \\ \text{of } S_n \end{array} \right\}$$

## Claim

$$\left\{ \begin{array}{l} \text{Young diagram} \\ \text{of } n \text{ boxes} \end{array} \right\} \begin{array}{c} \xleftarrow{?} \\ \xrightarrow{?} \end{array} \left\{ \begin{array}{l} \text{Irreducible rep} \\ \text{of } S_n \end{array} \right\}$$

# $S_n$ & Young diagram

## Claim

$$\left\{ \begin{array}{l} \text{Young diagram} \\ \text{of } n \text{ boxes} \end{array} \right\} \xleftrightarrow{\quad ? \quad} \left\{ \begin{array}{l} \text{Irreducible rep} \\ \text{of } S_n \end{array} \right\}$$

## Remark

Reduced to: for each Young diagram  $\lambda$ ,  
construct an irreducible representation  $S^\lambda$ , and  
prove  $S^\lambda = S^{\lambda'} \Rightarrow \lambda = \lambda'$ .

# The construction of $S^\lambda \subseteq M^\lambda$

Tabloid: equivalence class of standard filling

$$\begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 1 & 2 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 2 & 1 & \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline 3 & 4 & 5 \\ \hline 1 & 2 & \\ \hline \end{array} := \{345/12\}$$

$$\begin{array}{c} \left\{ \begin{array}{l} \text{standard filling} \\ \text{of shape } \lambda \end{array} \right\} \ni T = \begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 1 & 2 & \\ \hline \end{array} \\ \downarrow \\ T^\lambda := \left\{ \begin{array}{l} \text{Young tabloid} \\ \text{of shape } \lambda \end{array} \right\} \ni \{T\} = \left[ \begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 1 & 2 & \\ \hline \end{array} \right] = \{345/12\} \end{array}$$

# The construction of $S^\lambda \subseteq M^\lambda$

$$\mathcal{T}^\lambda := \{\text{Young tabloid of shape } \lambda\}$$

$$M^\lambda := \langle \{T\} \in \mathcal{T}^\lambda \rangle_{\mathbb{C}}$$

Choose a standard filling  $T$  of shape  $\lambda$ ,

$$C(T) := \{\sigma \in S_n \mid \sigma \text{ preserves numbers in each column}\}$$

$$v_T := \sum_{\sigma \in C(T)} \text{sgn}(\sigma) \{\sigma \cdot T\} \in M^\lambda$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda \quad \text{invariant subspace of } M^\lambda$$

# The construction of $S^\lambda \subseteq M^\lambda$

$$\mathcal{T}^\lambda := \{\text{Young tabloid of shape } \lambda\}$$

$$M^\lambda := \langle \{T\} \in \mathcal{T}^\lambda \rangle_{\mathbb{C}}$$

Example ( $\lambda = 3 \cdot 2$ )

$$\mathcal{T}^\lambda = \left\{ \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \right. \\ \left. \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \right\}$$

$$M^\lambda = \left\langle \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \right. \\ \left. \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \right\rangle_{\mathbb{C}}$$

# The construction of $S^\lambda \subseteq M^\lambda$

$$v_T := \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma) \{\sigma \cdot T\} \in M^\lambda \quad \sigma v_T = v_{\sigma T}$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda \quad \text{invariant subspace of } M^\lambda$$

Example ( $\lambda = 3 \cdot 2$ )

$$T = \begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 2 & 1 & \\ \hline \end{array}$$

$$C(T) = \{\operatorname{Id}, (23), (15), (23)(15)\}$$

$$v_T = \left\{ \begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 2 & 1 & \\ \hline \end{array} \right\} - \left\{ \begin{array}{|c|c|c|} \hline 2 & 5 & 4 \\ \hline 3 & 1 & \\ \hline \end{array} \right\} - \left\{ \begin{array}{|c|c|c|} \hline 3 & 1 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \right\} + \left\{ \begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \right\}$$

$$= \{345/12\} - \{245/13\} - \{134/25\} + \{124/35\} \in M^\lambda$$

$$S^\lambda = \langle v_T \rangle_{\mathbb{C}[S_n]} = \langle v_{T'} | T' : \text{standard filling} \rangle_{\mathbb{C}}$$

# Main theorem of $S^\lambda$

## Theorem

Fix the Young diagram  $\lambda$ , the corresponding representation  $S^\lambda$  has the following properties:

- 1 the linear space  $S^\lambda$  has a **basis**  $\{v_{T'} | T' : \text{standard tableau}\}$ , especially,  $\dim S^\lambda = \#\{\text{standard tableau}\}$ ;
- 2 the representation  $S^\lambda$  is **irreducible**;
- 3 for the Young diagram  $\lambda'$ ,  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ .



# Proof: basis

## Theorem

- 1 the linear space  $S^\lambda$  has a basis  $\{v_{T'} | T' : \text{standard tableau}\}$ , especially,  $\dim S^\lambda = \#\{\text{standard tableau}\}$ ;

## Idea of the proof

- $S^\lambda$  is generated by  $\{v_{T'} | T' : \text{standard filling}\}$ ,  
It's not an easy task to represent  $v_{T'}$  by linear combinations.

e.g. 
$$v_{\begin{array}{|c|c|c|} \hline 3 & 5 & 4 \\ \hline 2 & 1 & 1 \\ \hline \end{array}} \xrightarrow{\text{column}} v_{\begin{array}{|c|c|c|} \hline 2 & 1 & 4 \\ \hline 3 & 5 & 1 \\ \hline \end{array}} \xrightarrow{\text{row}} v_{\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 1 \\ \hline \end{array}} - v_{\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 1 \\ \hline \end{array}}$$

- $\{v_{T'} | T' : \text{standard tableau}\}$  are linear independent.

e.g. 
$$x_1 v_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 1 \\ \hline \end{array}} + x_2 v_{\begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 1 \\ \hline \end{array}} + x_3 v_{\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & 1 \\ \hline \end{array}} + x_4 v_{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 3 & 4 & 5 \\ \hline \end{array}} + x_5 v_{\begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & 1 \\ \hline \end{array}} = 0 \quad x_i \in \mathbb{C}$$

$$\{123/45\} \rightarrow x_1 = 0$$

$$\{134/25\} \rightarrow x_3 = 0$$

$$\{135/24\} \rightarrow x_5 = 0$$

$$\{124/35\} \rightarrow x_2 = 0$$

$$\{125/34\} \rightarrow x_4 = 0$$

# linear ordering

We use a linear ordering of standard fillings by

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \longrightarrow 54321$$

$\vee$

$$\begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \longrightarrow 52431$$

In the proof, we knock out the biggest one.

## Proof: part 2&3

### Theorem

- ② *the representation  $S^\lambda$  is irreducible;*
- ③ *for the Young diagram  $\lambda'$ ,  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ .*

We have to introduce element  $b_T$  in  $\mathbb{C}[S_n]$  by fix  $T$  of shape  $\lambda$

$$b_T := \sum_{q \in C(T)} \operatorname{sgn}(\sigma) \sigma$$

one can get

$$b_T S^\lambda = \mathbb{C} v_T \neq 0, \quad b_T S^{\lambda'} = 0 \quad \text{for } \lambda' > \lambda.$$

The results follow from these equations. ▶ skip

## Proof: part 2&3

### Theorem

- ② *the representation  $S^\lambda$  is irreducible;*
- ③ *for the Young diagram  $\lambda'$ ,  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ .*

We have to introduce element  $b_T$  in  $\mathbb{C}[S_n]$  by

$$b_T := \sum_{q \in C(T)} \text{sgn}(\sigma) \sigma$$

then

- $v_T = b_T \cdot \{T\};$
- $\tau(b_T) = \text{sgn}(\tau)b_T$  for any  $\tau \in C(T);$
- $b_T \cdot b_T = \#C(T) \cdot b_T;$
- $b_T M^\lambda = b_T S^\lambda = \mathbb{C}v_T \neq 0 ;$   
 $b_T M^{\lambda'} = b_T S^{\lambda'} = 0$  for  $\lambda' > \lambda.$

## Proof: part 2&3

### Theorem

- ② *the representation  $S^\lambda$  is irreducible;*
- ③ *for the Young diagram  $\lambda'$ ,  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ .*

$$\begin{aligned} b_T S^\lambda &= \mathbb{C}v_T \neq 0 ; \\ b_T S^{\lambda'} &= 0 \quad \text{for } \lambda' > \lambda \end{aligned}$$

✱To show  $S^\lambda$  is irreducible: only need to show indecomposability.  
If  $S^\lambda = V \oplus W$  as  $\mathbb{C}[S_n]$ -module, then

$$\begin{aligned} \mathbb{C}v_T &= b_T S^\lambda = b_T V \oplus b_T W \\ \Rightarrow b_T V &= \mathbb{C}v_T \quad (\text{or } b_T W = \mathbb{C}v_T) \\ \Rightarrow S^\lambda &= \mathbb{C}[S_n] \cdot v_T = \mathbb{C}[S_n] \cdot \mathbb{C}v_T = \mathbb{C}[S_n] \cdot b_T V \subseteq V \end{aligned}$$

## Proof: part 2&3

### Theorem

- ② *the representation  $S^\lambda$  is irreducible;*
- ③ *for the Young diagram  $\lambda'$ ,  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ .*

$$\begin{aligned} b_T S^\lambda &= \mathbb{C}v_T \neq 0 ; \\ b_T S^{\lambda'} &= 0 \quad \text{for } \lambda' > \lambda \end{aligned}$$

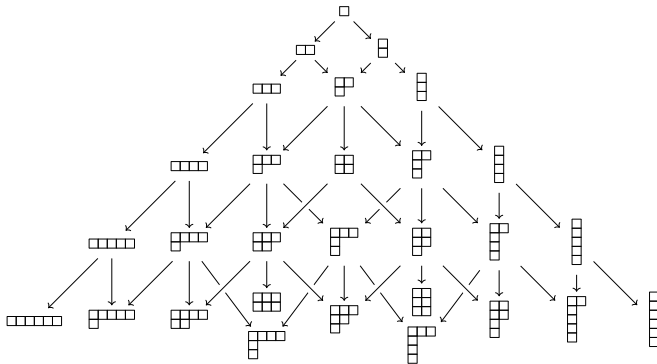
\*To show  $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$ :

If not w.l.o.g. suppose  $\lambda' > \lambda$ . Then

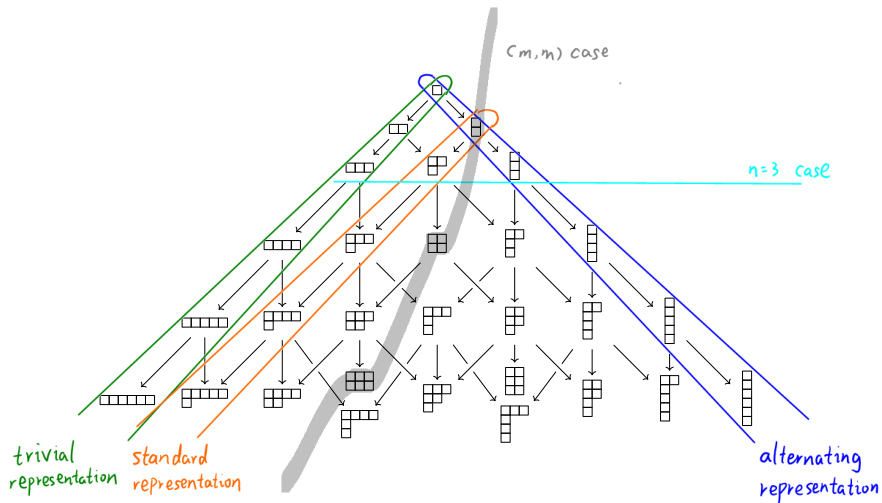
$$b_T S^{\lambda'} = b_T S^\lambda \implies \mathbb{C}v_T \cong 0,$$

contradiction!

# Example



# Example





## Example: trivial representation

$$\lambda = \square\square\square = 3^1$$

$$M^\lambda = \langle \{123\} \rangle = \mathbb{C}$$

$$T = \boxed{1 \mid 2 \mid 3}$$

$$C(T) = \text{Id}$$

$$v_T = \{123\}$$

$$S^\lambda = \mathbb{C}[S_3] \cdot v_T = \mathbb{C}v_T$$

## Example: alternating representation

$$\lambda = \begin{array}{|c|} \hline \square \\ \hline \square \\ \hline \square \\ \hline \end{array} = 1^3$$

$$M^\lambda = \langle \{1/2/3\}, \{1/3/2\}, \{2/1/3\}, \{2/3/1\}, \{3/1/2\}, \{3/2/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|} \hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$$

$$C(T) = S_3$$

$$\begin{aligned} v_T = & \{1/2/3\} - \{1/3/2\} - \{2/1/3\} \\ & + \{2/3/1\} + \{3/1/2\} - \{3/2/1\} \end{aligned}$$

$$S^\lambda = \mathbb{C}[S_3] \cdot v_T = \mathbb{C}v_T$$

$$\begin{aligned} (23)v_T = & \{1/3/2\} - \{1/2/3\} - \{3/1/2\} \\ & + \{3/2/1\} + \{2/1/3\} - \{2/3/1\} = -v_T \end{aligned}$$

## Example: standard representation

$$\lambda = \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} = 2 \cdot 1$$

$$M^\lambda = \langle \{12/3\}, \{13/2\}, \{23/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array}$$

$$C(T) = \{\text{Id}, (13)\}$$

$$v_T = \{12/3\} - \{23/1\}$$

$$S^\lambda = \mathbb{C}[S_3] \cdot v_T \cong \mathbb{C}^2$$

$$(12)v_T = \{12/3\} - \{13/2\}$$

$$(13)v_T = \{23/1\} - \{12/3\} = -v_T$$

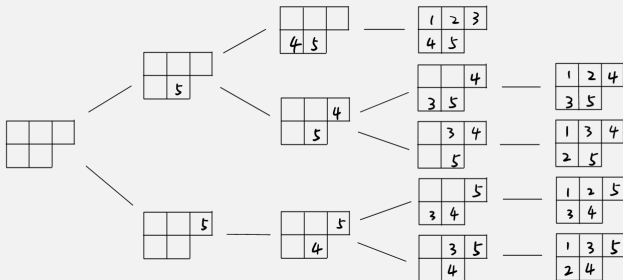
# Goal of the Part 1

- Explicitly construct irreducible representations of  $S_n$  by Young diagram;
- Compute the character table;
  - $\dim \chi_i$  by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g.  $\otimes, \text{Sym}^m, \Lambda^m$ ;
  - e.g.  $M_\lambda$ .
  - restriction and induced representation
  - representation ring

# Example: dimension of irreducible representation

$$\dim S^\lambda = \#\{\text{standard tableau of } \lambda\} = ?$$

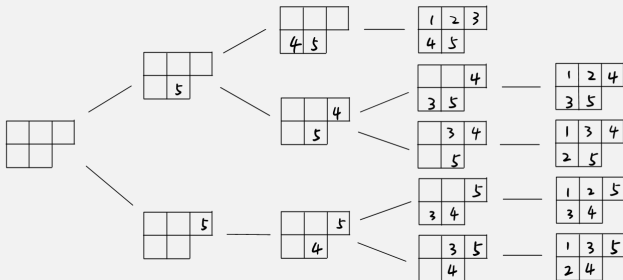
## Example



# Example: dimension of irreducible representation

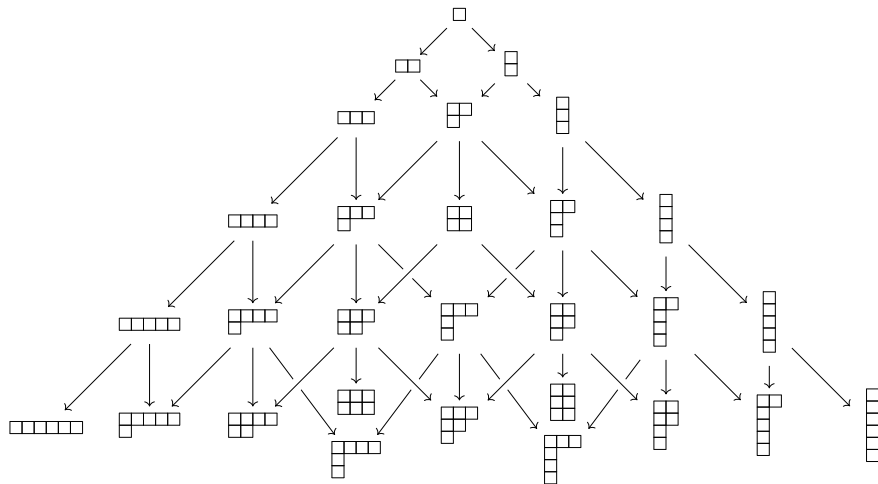
$$\dim S^\lambda = \#\{\text{standard tableau of } \lambda\} = ?$$

## Example

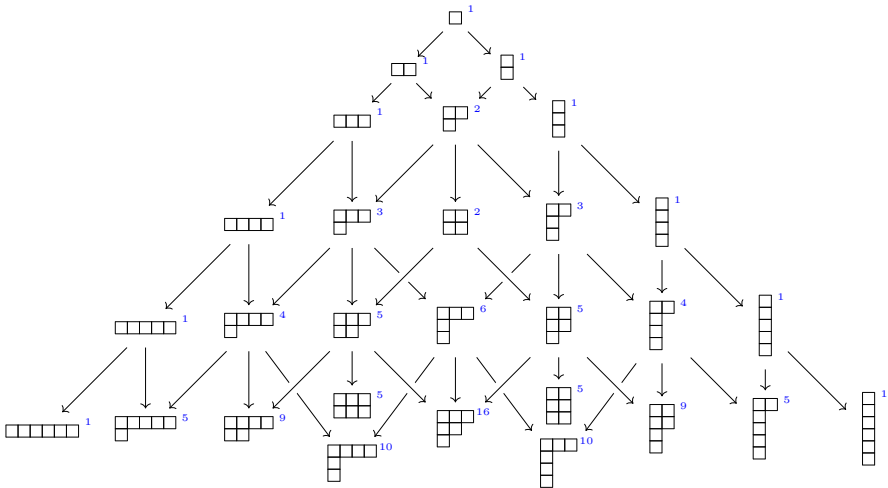


$$\dim S^\lambda = \sum_{\substack{\lambda' \subseteq \lambda \\ |\lambda'| = n-1}} \dim S^{\lambda'}$$

# Example: dimension of irreducible representation



## Example: dimension of irreducible representation

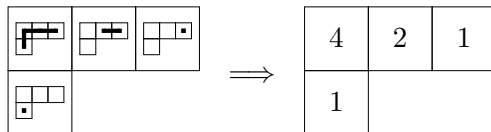




# Hook length formula

It helps us compute the dimension of  $S^\lambda$  without induction.

Step 1: count the length of hook.



Step 2:  $\dim S^\lambda = \frac{n!}{\prod(\text{hook lengths})}$

# Special case: $(m, l)$

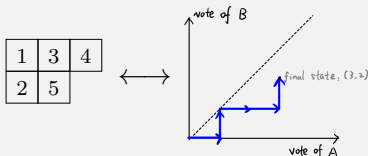
## Ballot problem

In an election where candidate A receives  $m$  votes and candidate B receives  $l$  votes with  $m \geq l$ , what is the probability that A will be (non-strictly) ahead of B throughout the count?

## Proposition

*Each process of the count corresponds to each standard tableau of form  $(m, l)$ .*

## Example



## Special case: $(m, m)$

### Corollary

$$\dim S^{(m,m)} = C_m = \frac{1}{m+1} \binom{2m}{m}.$$

where  $C_m$  is the  $m$ -th Catalan number.

## Special case: $(m, m)$

### Corollary

$$\dim S^{(m,m)} = C_m = \frac{1}{m+1} \binom{2m}{m}.$$

where  $C_m$  is the  $m$ -th Catalan number.

Catalan number has many interpretations. For example, it counts the number of crossingless matchings of  $2m$  points.

Ex.  $m=3$

crossingless matchings  
of 6 points



## Special case: $(m, m)$

### Corollary

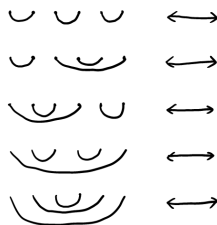
$$\dim S^{(m,m)} = C_m = \frac{1}{m+1} \binom{2m}{m}.$$

where  $C_m$  is the  $m$ -th Catalan number.

Catalan number has many interpretations. For example, it counts the number of crossingless matchings of  $2m$  points.

Ex.  $m=3$

crossingless matchings  
of 6 points



Young tableau  
of  $(3,3)$  type

1	3	5
2	4	6

1	3	4
2	5	6

1	2	5
3	4	6

1	2	4
3	5	6

1	2	3
4	5	6

# Goal of the Part II

- Definition of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top homology.

# Goal of the Part II

- **Definition** of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top homology.

## Definition

$$\begin{array}{ccc}
 \widehat{\mathfrak{g}} \subseteq \mathfrak{g} \times \mathcal{B} & \longrightarrow & \mathcal{B}_\bullet = \mathcal{F}(n) \\
 \downarrow \mu & \rightsquigarrow & \downarrow \mu|_{\mathcal{N} \times \mathcal{B}} \\
 \mathfrak{g}_\bullet = \mathfrak{sl}_n(\mathbb{C}) & & \mathcal{N} \\
 & & \text{resolution of nilpotent cone}
 \end{array}$$

Let  $X \in \mathfrak{g}$  be a nilpotent element. The Springer fiber  $B_X$  over  $X$  is defined as

$$\begin{aligned}
 B_X &:= \mu^{-1}(X) \\
 &= \{B \in \mathfrak{B} \mid X \in B\} \\
 &= \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \mid XV_i \subseteq V_{i-1}\} \quad \dim V_i = i
 \end{aligned}$$



By the Jordan normal form, we have

$$\left\{ \begin{array}{c} \text{Nilpotent element} \\ \text{in } \mathfrak{gl}_n(\mathbb{C}) \end{array} \right\} / \text{conj} \longleftrightarrow \left\{ \begin{array}{c} \text{Young diagram} \\ \text{of } n \text{ boxes} \end{array} \right\}$$

$$X_\lambda = \text{diag}(\underbrace{J_{\lambda_1}, \dots, J_{\lambda_1}}_{v_1}, J_{\lambda_2}, \dots, J_{\lambda_k}) \longleftrightarrow \lambda = \lambda_1^{v_1} \cdots \lambda_k^{v_k}$$

Denote  $B_\lambda := B_{X_\lambda}$ .  $B_X \cong B_{gXg^{-1}}$  for any  $g \in G$

Theorem (we will not give the proof.)

As  $S_n$ -representation,  $S^\lambda \cong H_{\text{top}}(B_\lambda)$ .

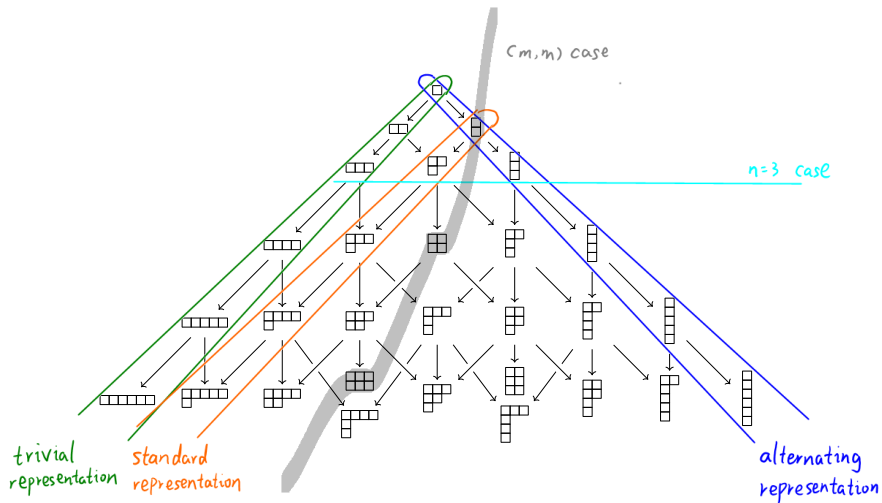
Corollary

$\#\{\text{irreducible component of } B_\lambda\} = \dim S^\lambda$

# Goal of the Part II

- Definition of Springer fiber;
- Some **examples** of Springer fiber;
- **Properties:** (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top homology.

# tree of Young diagram



Example:  $\lambda = 3$  

$$X_\lambda = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle ? \rangle \subseteq \langle ? , ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright X_\lambda = \{*\}$$

In general,  $B_\lambda = \{*\}$  when  $\lambda$  has only one row.

Example:  $\lambda = (1, 1, 1)$



$$X_\lambda = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle ? \rangle \subseteq \langle ? , ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright X_\lambda = \mathcal{F}\ell(3)$$

In general,  $B_\lambda = \mathcal{F}\ell(n)$  when  $\lambda = 1^n$ .

# Properties of $B_\lambda = \mathcal{F}\ell(n)$

- irreducible:
- $\dim B_\lambda =$
- CW complex:
- cohomology group:
- smooth:
- explicit description:
- Weyl group action on  $H_{\text{top}}(B_\lambda) \cong \mathbb{C}$ :

$$X_\lambda = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle ? \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright X_\lambda = \mathbb{P}^1 \vee \mathbb{P}^1$$

$$B_\lambda = \left\{ \begin{array}{l} \{0 \subseteq \langle e_i \rangle \subseteq \langle e_1, be_2 + ce_3 \rangle \subseteq \mathbb{C}^3\} \longrightarrow \text{red circle} \\ \{0 \subseteq \langle ae_1 + e_3 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3\} \longrightarrow \text{blue circle} \end{array} \right. \xrightarrow{(\cup)} \text{red circle} \leftarrow \{0 \subseteq \langle e_i \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3\}$$

In general,  $B_\lambda = \underbrace{\mathbb{P}^1 \vee \cdots \vee \mathbb{P}^1}_{n-1}$  when  $\lambda = (n-1, 1)$ .

# Properties of $B_\lambda = \underbrace{\mathbb{P}^1 \vee \dots \vee \mathbb{P}^1}_{n-1}$

- irreducible component:
- $\dim B_\lambda =$
- affine paving:
- cohomology group:
- smooth:
- explicit description:
- Weyl group action on  $H_{\text{top}}(B_\lambda) \cong \mathbb{C}^{n-1}$ :



# Tool: stratification/cellular fibration/affine paving

$$\begin{array}{ccc}
 \begin{array}{l} B_\lambda \longrightarrow \\ \vdots \\ B_{\lambda'} \longrightarrow \\ \phi \longrightarrow \end{array} & B_\lambda & \\
 & \downarrow \pi & \\
 & \mathbb{P}^{n-1} & 
 \end{array}
 \qquad
 \begin{array}{ccc}
 \{ \emptyset \subseteq V_1 \subseteq V_2 \subseteq \dots \subseteq \mathbb{C}^n \}^{\hookrightarrow X_\lambda} & & \\
 \downarrow & & \\
 [V_\bullet] & & 
 \end{array}$$

## Remark

In general, we don't understand the ring structure of the cohomology group.

# Return!

For  $\lambda = 1^3$ ,  $B_\lambda \cong \mathcal{F}\ell(3)$  can be viewed as  $\mathcal{F}\ell(2)$ -bundle over  $\mathbb{P}^2$ .

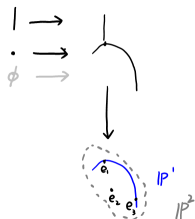
$$\begin{array}{ccc} \mathcal{F}\ell(2) & \longrightarrow & \mathcal{F}\ell(3) \\ & & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$

$$\pi^{-1}([v]) = \{0 \subseteq \langle v \rangle \subseteq \langle v, ? \rangle \subseteq \mathbb{C}^3\} \cong \mathcal{F}\ell(2)$$

# Return!

For  $\lambda = (2, 1)$ ,  $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$ :

$$\begin{array}{ccc} \mathbb{P}^1 & = & B_{\bullet, \bullet} \longrightarrow \\ \{e_i\} & = & B_{\bullet} \longrightarrow B_{\bullet, \bullet} \\ \emptyset & \longrightarrow & \downarrow \pi \\ & & \mathbb{P}^2 \end{array}$$



$$\begin{aligned} \pi^{-1}([e_1]) &= \{0 \subseteq \langle e_1 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3\} \curvearrowright \begin{bmatrix} 0 & 1 \\ & 0 \\ & & 0 \end{bmatrix} \\ &\cong \{0 \subseteq \langle ? \rangle \subseteq \mathbb{C}^2\} \curvearrowright \begin{bmatrix} 0 & \\ & 0 \end{bmatrix} &= B_{1,1} \end{aligned}$$

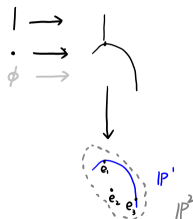
$$\begin{aligned} \pi^{-1}([e_3]) &= \{0 \subseteq \langle e_3 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3\} \curvearrowright \begin{bmatrix} 0 & 1 \\ & 0 \\ & & 0 \end{bmatrix} \\ &\cong \{0 \subseteq \langle ? \rangle \subseteq \mathbb{C}^2\} \curvearrowright \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} &= B_2 \end{aligned}$$

$$\pi^{-1}([e_2]) = \{0 \subseteq \langle e_2 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3\} \curvearrowright \begin{bmatrix} 0 & 1 \\ & 0 \\ & & 0 \end{bmatrix} = \emptyset$$

# Return!

For  $\lambda = (2, 1)$ ,  $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$ :

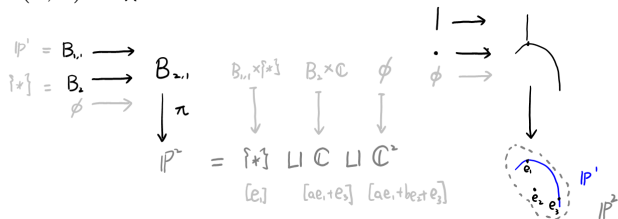
$$\begin{array}{ccc} \mathbb{P}^1 = B_{\bullet, \bullet} & \longrightarrow & B_{\bullet, \bullet} \\ \{e_i\} = B_{\bullet} & \longrightarrow & \downarrow \pi \\ \emptyset & \longrightarrow & \mathbb{P}^2 \end{array}$$



$$\pi^{-1}([e_1]) \cong B_{1,1} \quad \pi^{-1}([e_3]) \cong B_2 \quad \pi^{-1}([e_2]) \cong \emptyset$$

# Return!

For  $\lambda = (2, 1)$ ,  $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$ :



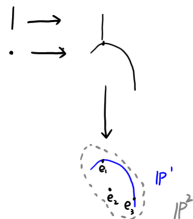
$$\pi^{-1}([e_1]) \cong B_{1,1} \quad \pi^{-1}([e_3]) \cong B_2 \quad \pi^{-1}([e_2]) \cong \emptyset$$

$$\begin{aligned} \pi^{-1}([ae_1 + e_3]) &= \left\{ 0 \subseteq \langle ae_1 + e_3 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \\ &\cong \left\{ 0 \subseteq \langle f_3 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \end{aligned}$$

# Return!

For  $\lambda = (2, 1)$ ,  $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$ :

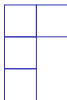
$$\begin{array}{ccc}
 \mathbb{P}^1 = B_{\bullet, \bullet} & \longrightarrow & B_{\bullet, \bullet} \\
 \{*\} = B_{\bullet} & \longrightarrow & B_{\bullet, \bullet} \\
 & & \downarrow \pi \\
 & & \mathbb{P}^1
 \end{array}
 \quad
 \begin{array}{ccc}
 B_{\bullet, \bullet} \times \{*\} & & B_{\bullet} \times \mathbb{C} \\
 \downarrow & & \downarrow \\
 \{*\} & \sqcup & \mathbb{C} \\
 [e_1] & & [ae_1 + e_3]
 \end{array}$$



$$\pi^{-1}([e_1]) \cong B_{1,1} \quad \pi^{-1}([e_3]) \cong B_2 \quad \pi^{-1}([e_2]) \cong \emptyset$$

$$\begin{aligned}
 \pi^{-1}([ae_1 + e_3]) &= \left\{ 0 \subseteq \langle ae_1 + e_3 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 &\cong \left\{ 0 \subseteq \langle f_3 \rangle \subseteq \langle ?, ? \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowright \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

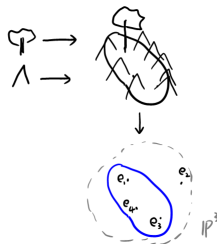
Example:  $\lambda = (2, 1, 1)$



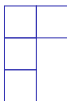
$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \{0 \subseteq \langle ? \rangle \subseteq \langle ?, ? \rangle \subseteq \langle ?, ?, ? \rangle \subseteq \mathbb{C}^4\} \curvearrowright X_\lambda$$

$$\begin{array}{ccc} \mathcal{F}(3) = B_{1,1,1} & \longrightarrow & B_{1,1,1} \\ \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} & \longrightarrow & B_{1,1,1} \\ & & \downarrow \pi \\ & & \mathbb{P}^3 \end{array}$$



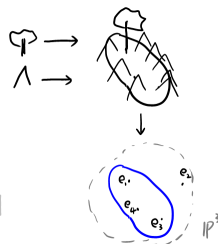
Example:  $\lambda = (2, 1, 1)$



$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \{0 \subseteq \langle ? \rangle \subseteq \langle ?, ? \rangle \subseteq \langle ?, ?, ? \rangle \subseteq \mathbb{C}^4\} \curvearrowright X_\lambda$$

$$\begin{array}{ccccccc} \mathcal{F}(3) = B_{1,1,1} & \longrightarrow & B_{2,1,1} & B_{1,1,1} & B_{2,1} \times \mathbb{C} & B_{2,1} \times \mathbb{C}^* \\ \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} & \longrightarrow & & & & \\ & & \downarrow \pi & \downarrow & \downarrow & \downarrow \\ \mathbb{P}^3 = & \{ \cdot \} & \sqcup \mathbb{C} & \sqcup \mathbb{C}^2 & & \\ & [e_1] & [ae_1 + e_3] & [ae_1 + be_3 + e_4] & & \end{array}$$





Example:  $\lambda = (2, 2)$



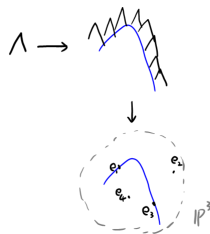
$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle ? \rangle \subseteq \langle ?, ? \rangle \subseteq \langle ?, ?, ? \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowright X_\lambda$$

$$\mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \longrightarrow B_{2,2}$$

$$\downarrow \pi$$

$$\mathbb{P}^1$$



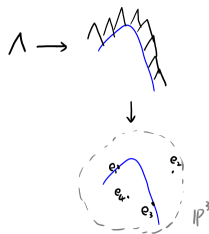
Example:  $\lambda = (2, 2)$



$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \{0 \subseteq \langle ? \rangle \subseteq \langle ?, ? \rangle \subseteq \langle ?, ?, ? \rangle \subseteq \mathbb{C}^4\} \curvearrowright X_\lambda$$

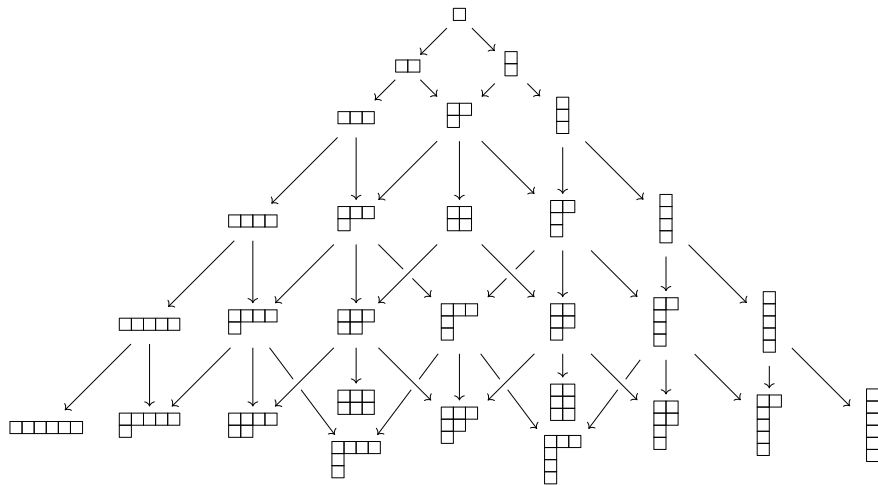
$$\begin{array}{ccccc} \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} & \longrightarrow & B_{2,2} & B_{2,1} \times \mathbb{P}^1 & B_{2,1} \times \mathbb{C} \\ \downarrow \pi & & \downarrow & \downarrow & \downarrow \\ \mathbb{P}^1 = \{*\} & & \{*\} & \mathbb{C} & \\ & & [e_1] & [ae_1 + e_2] & \end{array}$$



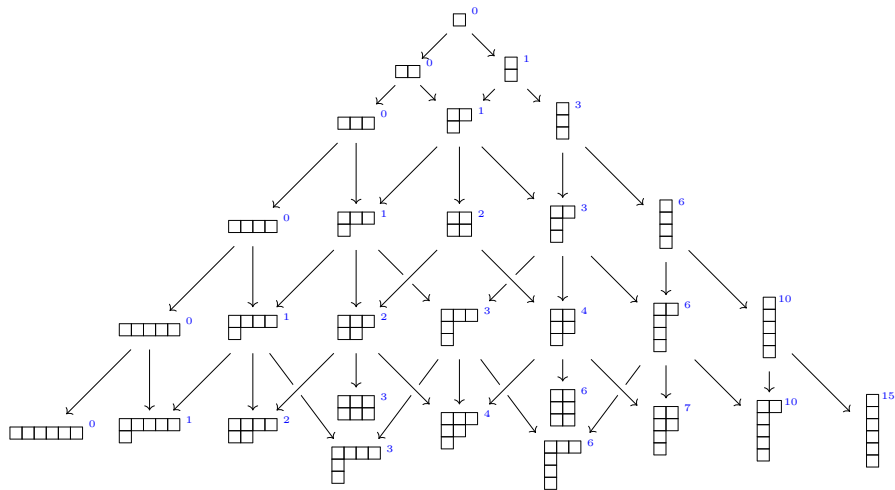
Using the same technique, we can get

- $B_\lambda$  has an affine paving  $\rightsquigarrow$  cohomology;
- Each irreducible component in  $B_\lambda$  has same dimension;
- It's easy to compute the dimension and the number of irreducible component.

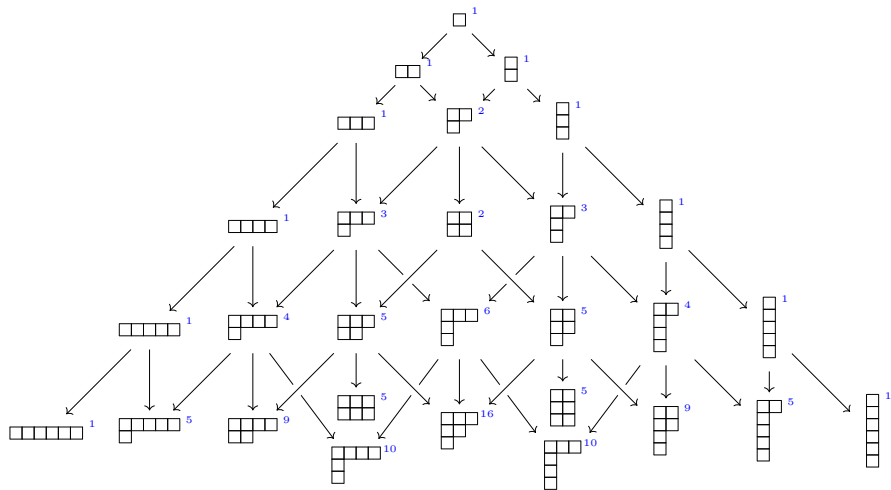
# Game: compute!



Answer: dimension



Answer: the number of irreducible component



# Smooth problem

## Results

- Not all the the irreducible components of  $B_\lambda$  are smooth;  
For example, one component of  $B_{2,2,1,1}$  is not smooth.
- All the components of  $B_\lambda$  are nonsingular iff
$$\lambda \in \{(\lambda_1, 1, 1, \dots), (\lambda_1, \lambda_2), (\lambda_1, \lambda_2, 1), (2, 2, 2)\}$$





## $(m, m)$ case

We have an explicit description in the 2-row case when we forget the variety structure. Use this description, we can get the cohomology group structure.

### Definition and Theorem

Let  $\alpha$  be a crossingless matching, define

$$\tilde{B}_{\alpha; m, m} := \left\{ (x_1, \dots, x_{2m}) \in (\mathbb{P}^1)^{2m} \mid x_i = x_j \text{ if } (i, j) \in \alpha \right\} \subseteq (\mathbb{P}^1)^{2m}$$

$$\tilde{B}_{m, m} := \bigcup_{\alpha} \tilde{B}_{\alpha; m, m} \subseteq (\mathbb{P}^1)^{2m}$$

then we have a homeomorphism

$$B_{m, m} \cong \tilde{B}_{m, m}$$

## $(m, m)$ case

### Definition and Theorem

Let  $\alpha$  be a crossingless matching, define

$$\tilde{B}_{\alpha; m, m} := \left\{ (x_1, \dots, x_{2m}) \in (\mathbb{P}^1)^{2m} \mid x_i = x_j \text{ if } (i, j) \in \alpha \right\} \subseteq (\mathbb{P}^1)^{2m}$$

$$\tilde{B}_{m, m} := \bigcup_{\alpha} \tilde{B}_{\alpha; m, m} \subseteq (\mathbb{P}^1)^{2m}$$

then we have a homeomorphism

$$B_{m, m} \cong \tilde{B}_{m, m}$$

### Example (m=2)

$$\alpha = \{(1, 2), (3, 4)\} \quad \tilde{B}_{\alpha; 2, 2} = \left\{ (x_1, x_1, x_2, x_2) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

$$\beta = \{(1, 4), (2, 3)\} \quad \tilde{B}_{\beta; 2, 2} = \left\{ (x_1, x_2, x_2, x_1) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

$$B_{2, 2} \cong \tilde{B}_{2, 2} \cong (\mathbb{P}^1)^2 \bigvee_{\mathbb{P}^1} (\mathbb{P}^1)^2$$

THANKS

Thank you for listening!

Thank Rui Xiong for providing the package of Young diagram,

Thank my roommate David Cueto for pointing out typos,

Thank Prof. Eberhart for offering valuable materials and advice!