LaTeX TEMPLATE

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Contents

1. A small toolkit

$f : Y \longrightarrow \mathrm{pt}$ $f : p \hookrightarrow X$
f^* constant sheaf $\mathcal{F}_p$
Rf_* cohomology $\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$ cpt supp cohomology $\mathrm{sky}_p(\mathbb{Q})$
$f^!$ orientation sheaf $[n]$ $\mathcal{F}_p[-n]$
For $f^!$, assume Y, X are manifolds of dimension n.
$j_!j^*\mathcal{F}$ $\mathcal{F}$ $i_!i^*\mathcal{F}$

$$Z \xhookrightarrow{\ i\ } X \xhookleftarrow{\ j\ } U$$

$$\begin{tikzcd} D(Z) \arrow[r, "i_*=i_!"] & D(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & D(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \xrightarrow{+1}$$

$$\begin{tikzcd} & \otimes \arrow[dl, dash, "f^*(-\otimes-)"'] \arrow[dr, dash, "\text{proj formula}"] & \\ f^* \arrow[rr, dash, "\text{base change}"'] & & f_! \end{tikzcd}$$

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2. A Short List of Applications

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:

$$\mathrm{H}_c^i(X;\mathbb{Q}) \cong \mathrm{H}^i\left(\bar{X},\{\infty\};\mathbb{Q}\right)$$
$$\mathrm{H}_i^{\mathrm{BM}}(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$
$$\mathrm{H}_i(X;\mathbb{Q}) \cong \mathrm{H}_c^{n-i}(X;\mathrm{Or}_X)$$

Also, define the cup and cap product structures.

2. Using the projection formula, show Poincaré duality:

$$\mathrm{H}_c^i(X;\mathbb{Q})^* \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$
$$\mathrm{H}^i(X;\mathbb{Q}) \cong \mathrm{H}_c^{n-i}(X;\mathrm{Or}_X)^*$$

3. Derive the Gysin sequence for any oriented S^k-bundle $\pi : E \longrightarrow B$:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[eu_\pi]{+1}$$

Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.

4. Compute the upper shriek for singular spaces.

$$\begin{array}{llll} \mathrm{H}^i(Y,\mathbb{Q}) & = \mathrm{H}^i(Y,\underline{\mathbb{Q}}_Y) & = f_*\underline{\mathbb{Q}}_Y & = f_*f^*\mathbb{Q} \\ \mathrm{H}_c^i(Y,\mathbb{Q}) & = \mathrm{H}_c^i(Y,\underline{\mathbb{Q}}_Y) & = f_!\underline{\mathbb{Q}}_Y & = f_!f^*\mathbb{Q} \\ \mathrm{H}_{-i}(Y,\mathbb{Q}) & = \mathrm{H}_c^{n+i}(Y,\mathrm{Or}_Y) & = f_!\,\mathrm{Or}_Y[n] & = f_!f^!\mathbb{Q} \\ \mathrm{H}_{-i}^{\mathrm{BM}}(Y,\mathbb{Q}) & = \mathrm{H}^{n+i}(Y,\mathrm{Or}_Y) & = f_*\,\mathrm{Or}_Y[n] & = f_*f^!\mathbb{Q} \end{array}$$

six functor formalism $\approx$ cohomology theory

3. perverse sheaf

We will mix the usage of sheaves and complexes. For simplicity, let us fix a stratification $\mathcal{S}$:

$$\varnothing \overset{U_0}{\subset} Z_0 \overset{U_1}{\subset} \cdots \overset{U_n}{\subset} Z_n = X$$

Denote $D^b_{\mathrm{cons},\mathcal{S}}(X)$ as the category of constructible sheaves over X with respect to $\mathcal{S}$.

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure. We say that $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^i\left(\iota_{U_j}^*\mathcal{F}\right) = 0, & \text{for any } i > -j \\ \mathcal{H}^i\left(\iota_{U_j}^!\mathcal{F}\right) = 0, & \text{for any } i < -j \end{cases}$$

To determine whether a complex $\mathcal{F}$ is perverse, one simply needs to complete the following table:

$\iota_{U_j}^*(\mathcal{F})$

The local system supported on U_i (denoted by $\mathcal{L}$) converted to a perverse sheaf by truncations. This method is called Deligne's construction, and the constructed perverse sheaf is called the intersection cohomology complex(or the IC sheaf), denoted by $IC(\mathcal{L})$. IC sheaves are the simple objects in the category $Perv_{\mathcal{S}}(X)$.

References

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