LATEX TEMPLATE

XIAOXIANG ZHOU

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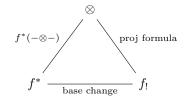
1. A SMALL TOOLKIT

$$\begin{split} f: Y &\longrightarrow \operatorname{pt} f: p \hookrightarrow X \\ f^* & \operatorname{constant sheaf} \mathcal{F}_p \\ Rf_* & \operatorname{cohomology} \operatorname{sky}_p(\mathbb{Q}) \\ Rf_! & \operatorname{cpt} \operatorname{supp} \operatorname{cohomology} \operatorname{sky}_p(\mathbb{Q}) \\ f^! & \operatorname{orientation} \operatorname{sheaf} [n] \ \mathcal{F}_p[-n] \\ & \operatorname{For} \ f^!, \operatorname{assume} \ Y, X \ \operatorname{are} \ \operatorname{manifolds} \ \operatorname{of} \ \operatorname{dimension} \ n. \\ j_! j^* \mathcal{F} \ \mathcal{F} \ i_! i^* \mathcal{F} \end{split}$$

$$Z \stackrel{i}{\longleftarrow} X \stackrel{j}{\longleftarrow} U$$

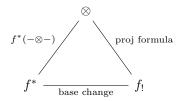
$$D(Z) \xrightarrow{i_* = i_!} D(X) \xrightarrow{j^* = j^!} D(U)$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \stackrel{+1}{\longrightarrow}$$



Date: September 5, 2024.

	$f: Y \longrightarrow \mathrm{pt}$	$f:p\hookrightarrow X$
f^*	constant sheaf	\mathcal{F}_p
Rf_*	cohomology	$\operatorname{sky}_p(\mathbb{Q})$
$Rf_!$	cpt supp cohomology	$\operatorname{sky}_p(\mathbb{Q})$
f!	orientation sheaf	$[n] \mathcal{F}_p[-n]$



2. A SHORT LIST OF APPLICATIONS

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:

$$\begin{aligned} & \mathrm{H}_{\mathrm{c}}^{i}(X;\mathbb{Q}) \cong \mathrm{H}^{i}\left(\bar{X},\{\infty\};\mathbb{Q}\right) \\ & \mathrm{H}_{i}^{\mathrm{BM}}(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_{X}) \\ & \mathrm{H}_{i}(X;\mathbb{Q}) \cong \mathrm{H}_{\mathrm{c}}^{n-i}(X;\mathrm{Or}_{X}) \end{aligned}$$

Also, define the cup and cap product structures.

2. Using the projection formula, show Poincaré duality:

$$\mathrm{H}^{i}_{\mathrm{c}}(X;\mathbb{Q})^{*} \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_{X})$$

 $\mathrm{H}^{i}(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_{\mathrm{c}}(X;\mathrm{Or}_{X})^{*}$

3. Derive the Gysin sequence for any oriented S^k -bundle $\pi: E \longrightarrow B$:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow{e u_\pi}^{+1}$$

Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.

4. Compute the upper shriek for singular spaces.

$$\begin{array}{lll} \mathrm{H}^i(Y,\mathbb{Q}) &= \mathrm{H}^i(Y,\underline{\mathbb{Q}}_Y) &= f_*\underline{\mathbb{Q}}_Y &= f_*f^*\mathbb{Q} \\ \mathrm{H}^i_c(Y,\mathbb{Q}) &= \mathrm{H}^i_c(Y,\underline{\mathbb{Q}}_Y) &= f_!\underline{\mathbb{Q}}_Y &= f_!f^*\mathbb{Q} \\ \mathrm{H}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}_c(Y,\mathrm{Or}_Y) &= f_!\operatorname{Or}_Y[n] &= f_!f^!\mathbb{Q} \\ \mathrm{H}^{\mathrm{BM}}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}(Y,\mathrm{Or}_Y) &= f_*\operatorname{Or}_Y[n] &= f_*f^!\mathbb{Q} \end{array}$$

six functor formalism

3. PERVERSE SHEAF
We will mix the usage of sheaves and complexes. For simplicity, let us fix a stratification \mathcal{S} :

$$\varnothing \stackrel{U_0}{\subset} Z_0 \stackrel{U_1}{\subset} \cdots \stackrel{U_n}{\subset} Z_n = X$$

Denote $D^b_{cons,\mathcal{S}}(X)$ as the category of constructible sheaves over X with respect to \mathcal{S} .

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure. We say that $\mathcal{F} \in D^b_{\text{cons.}\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^{i}\left(\iota_{U_{j}}^{*}\mathcal{F}\right)=0, & \text{for any } i>-j\\ \mathcal{H}^{i}\left(\iota_{U_{j}}^{!}\mathcal{F}\right)=0, & \text{for any } i<-j \end{cases}$$

To determine whether a complex \mathcal{F} is perverse, one simply needs to complete the following table:

$$\iota_{U_i}^*(\mathcal{F})$$

The local system supported on U_i (denoted by \mathcal{L}) converted to a perverse sheaf by truncations. This method is called Deligne's construction, and the constructed perverse sheaf is called the intersection cohomology complex(or the IC sheaf), denoted by $IC(\mathcal{L})$. IC sheaves are the simple objects in the category $Perv_{\mathcal{S}}(X)$.

References

School of Mathematical Sciences, University of Bonn, Bonn, 53115, Germany, $Email\ address:$ email:xx352229@mail.ustc.edu.cn