Bruhat–Tits building

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Figures of Bruhat–Tits building

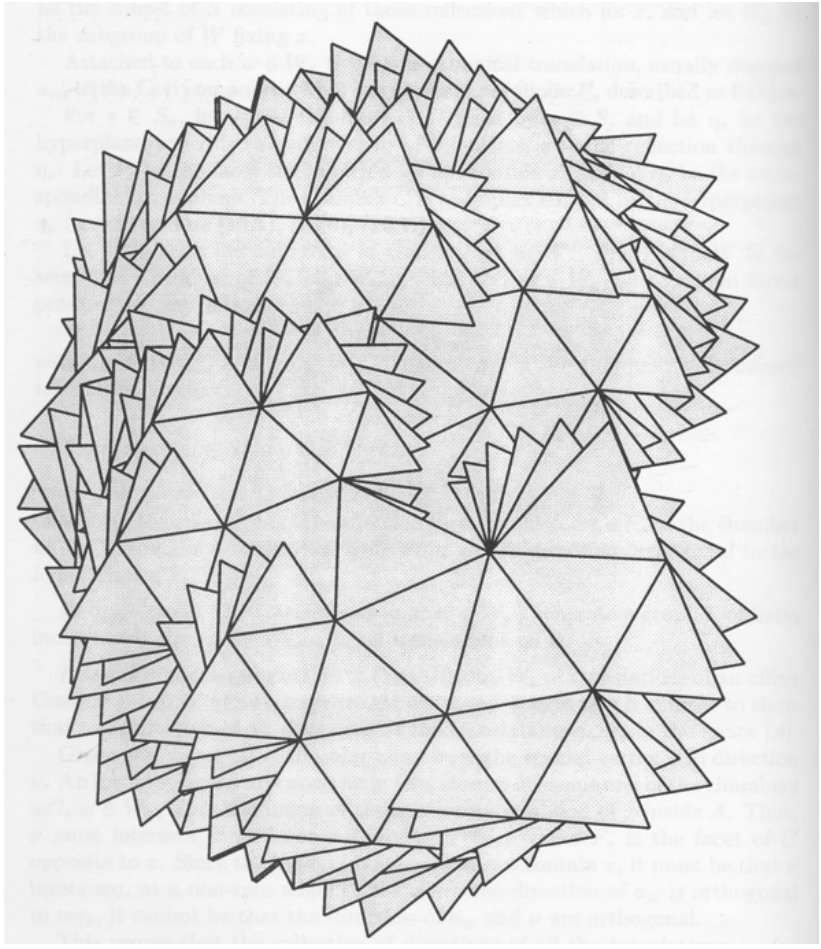

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from a talk by Annette Werner

Figures of Bruhat–Tits building

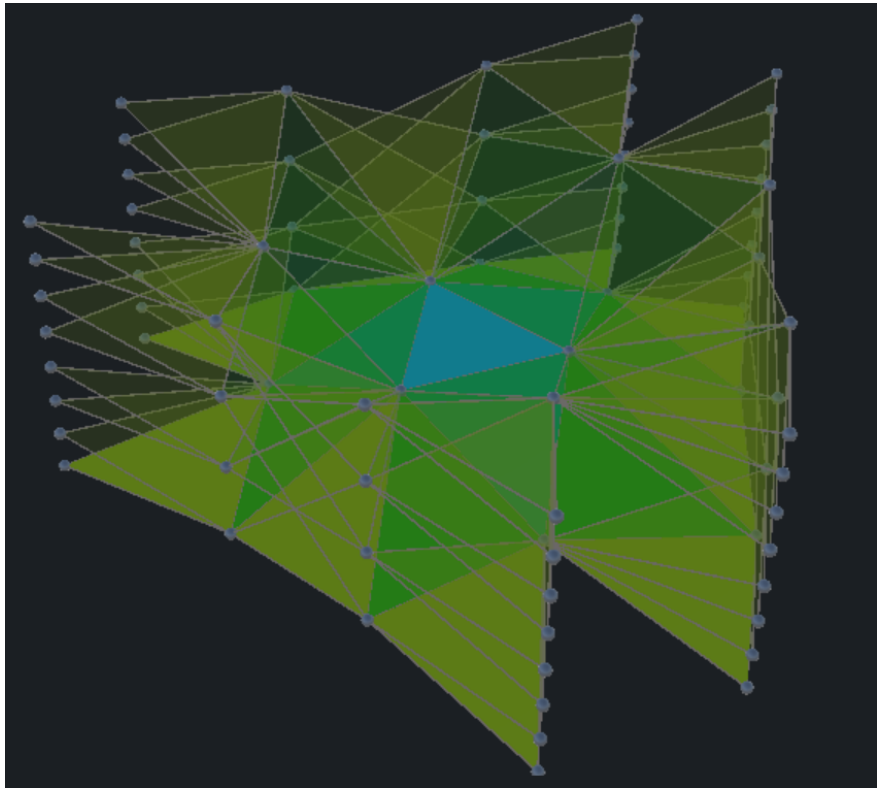

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

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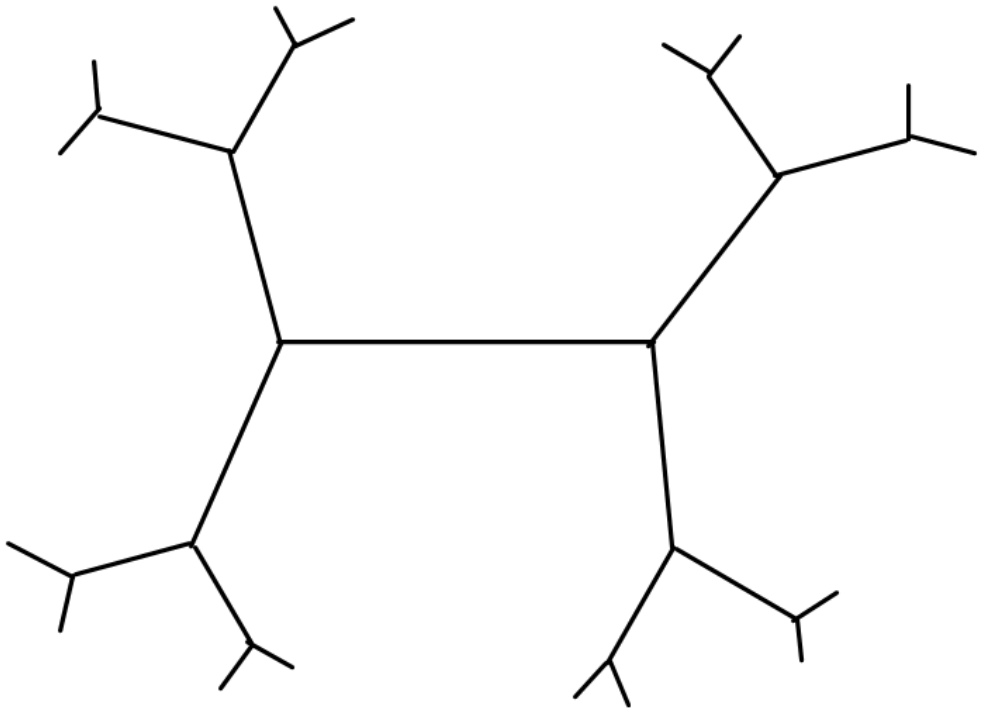

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

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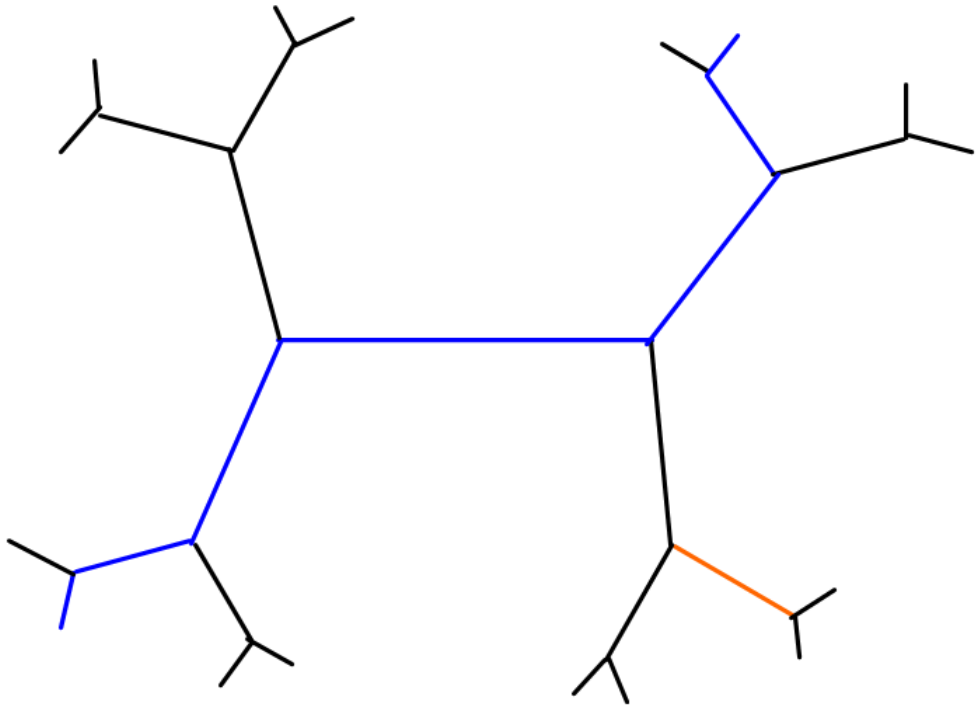

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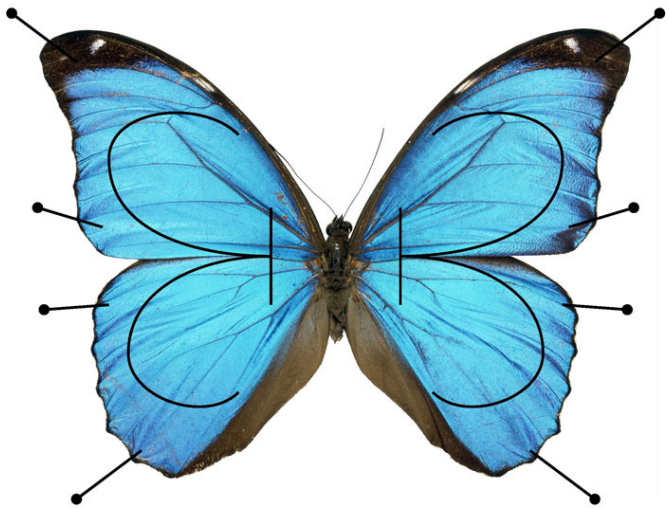

Figure: Pinned butterfly

Weyl group action on cocharacter lattices

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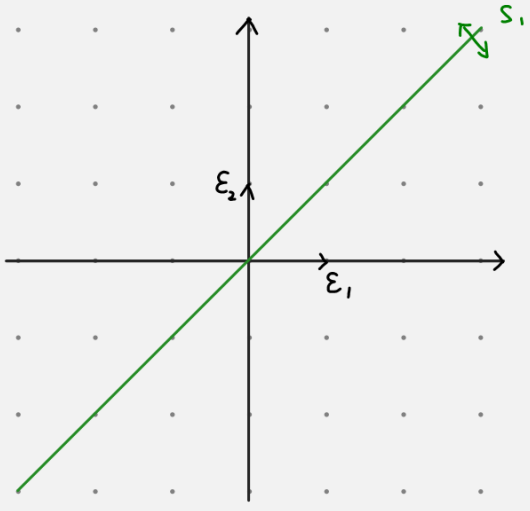

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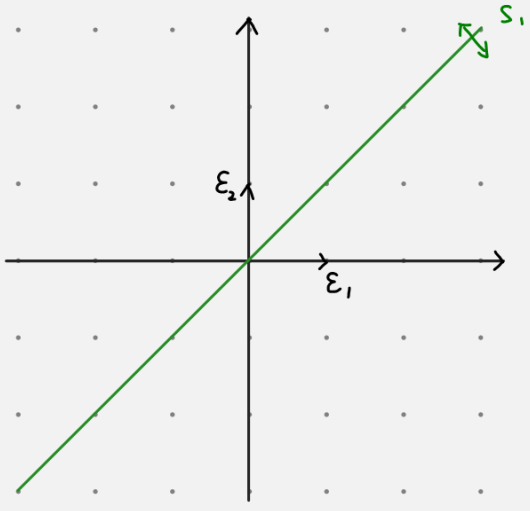

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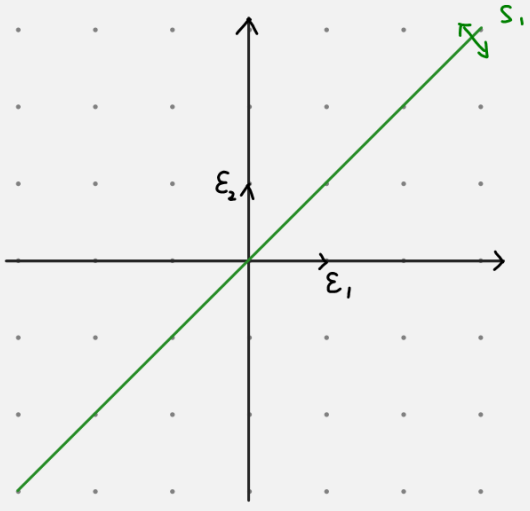

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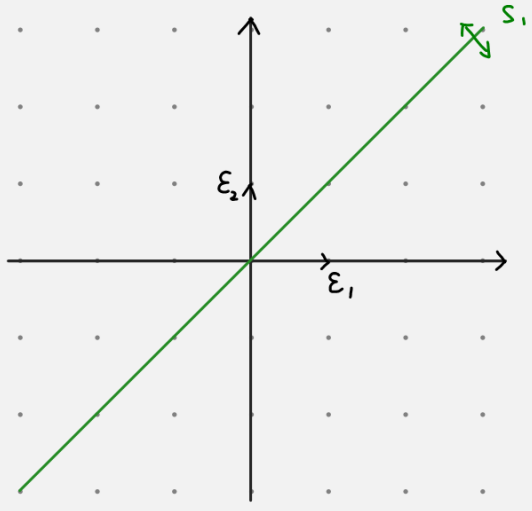

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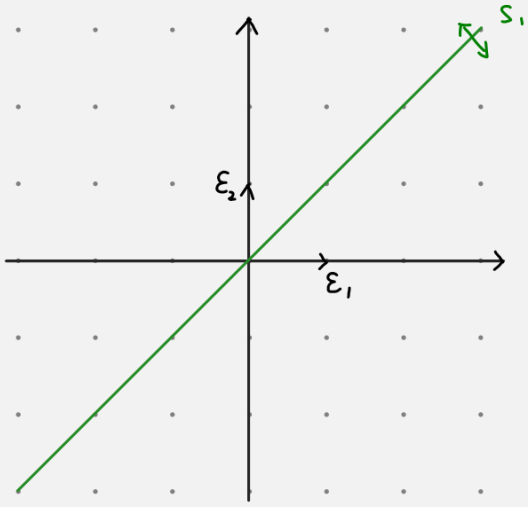

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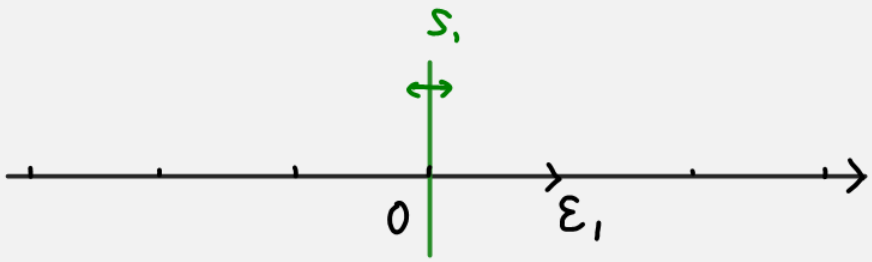

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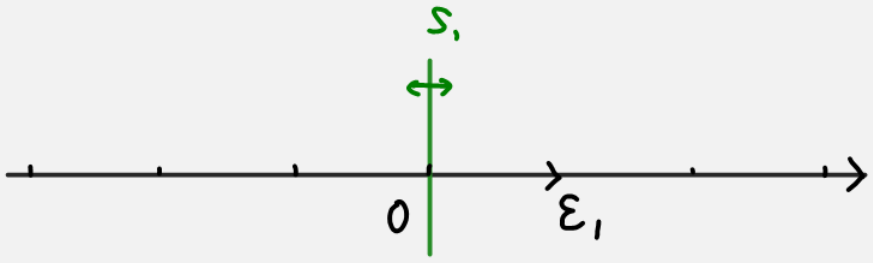

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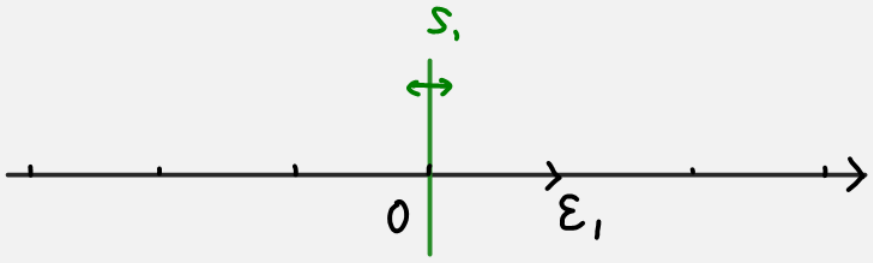

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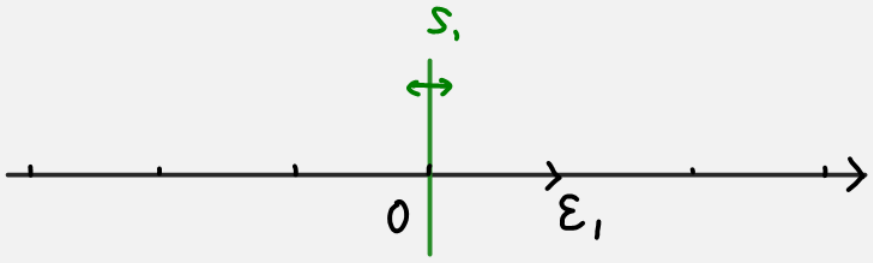

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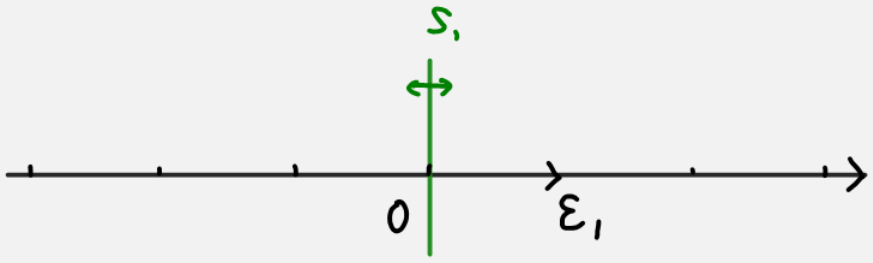

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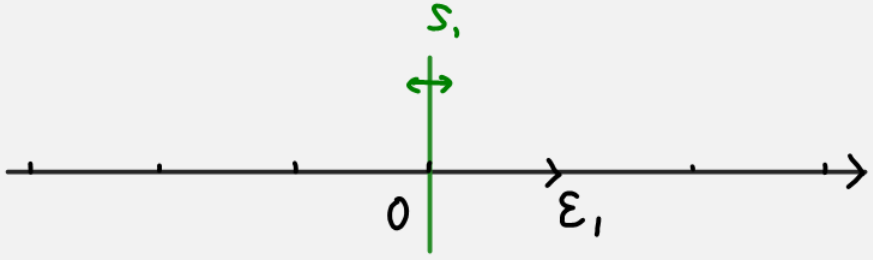

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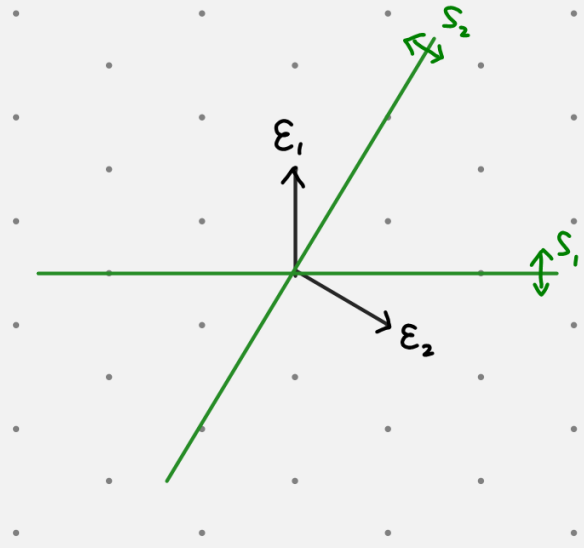

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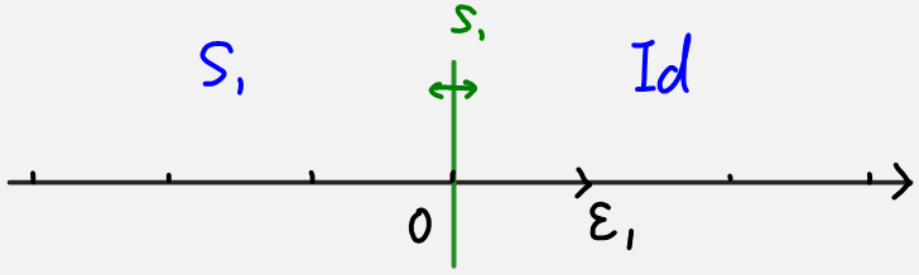

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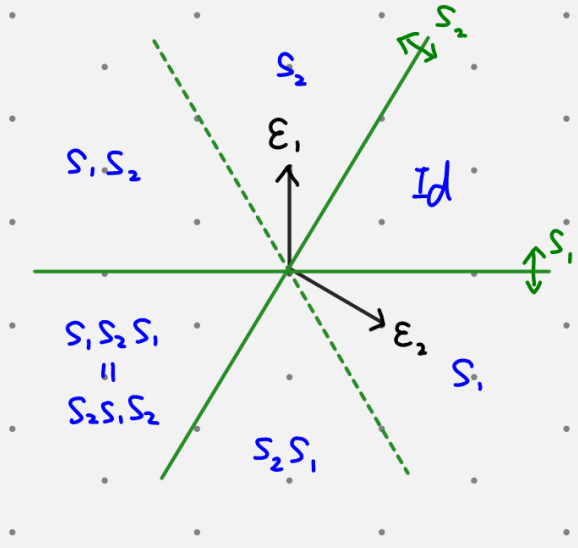

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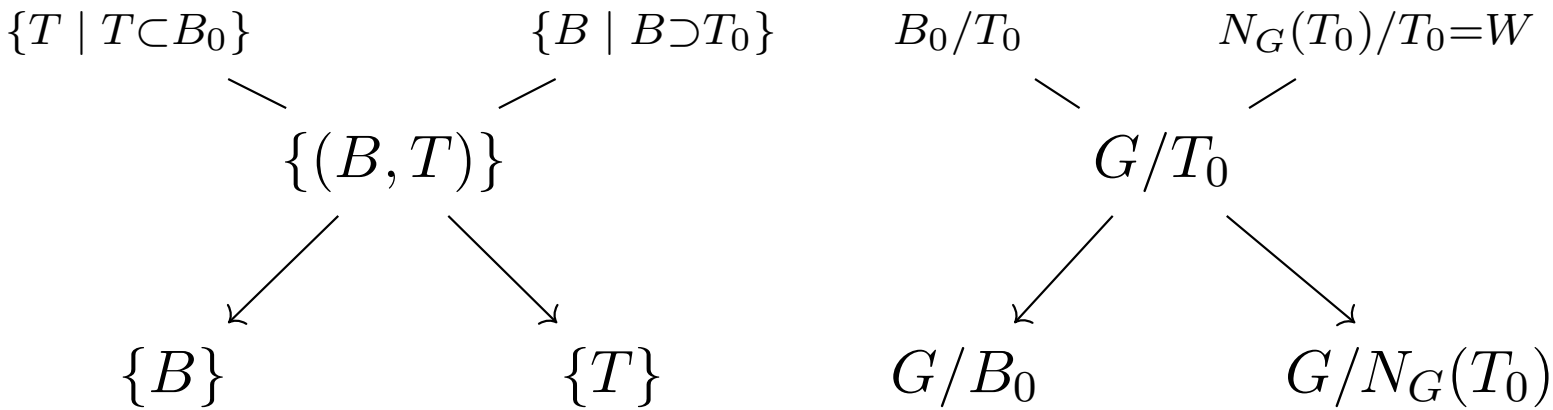

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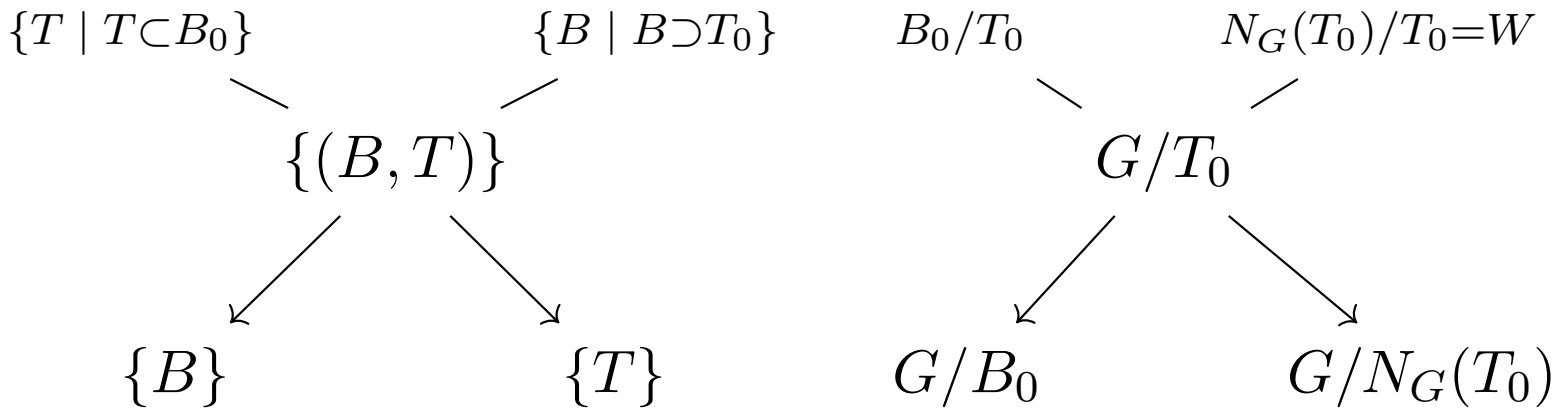

$$\{ \text{ (Weyl) chambers } \} \overset{1:1}{\longleftrightarrow} W \overset{1:1}{\longleftrightarrow} \{B \mid B \supset T_0\}$$

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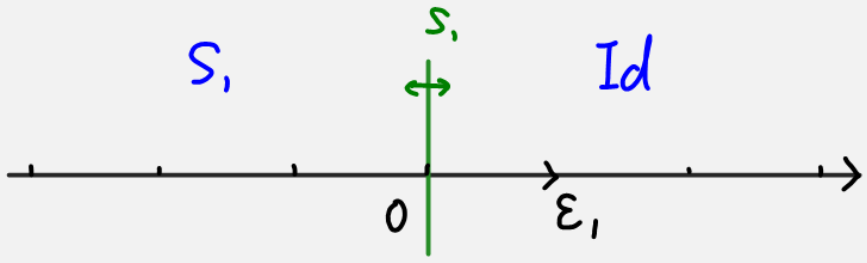

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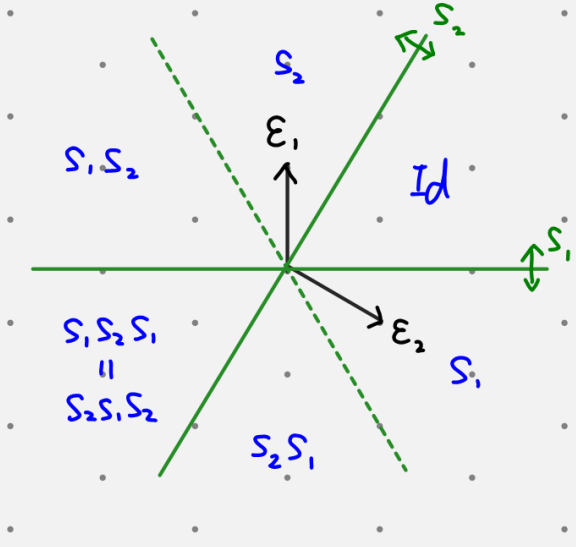

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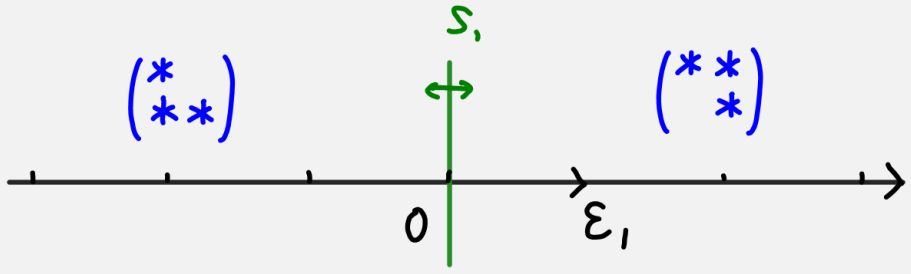

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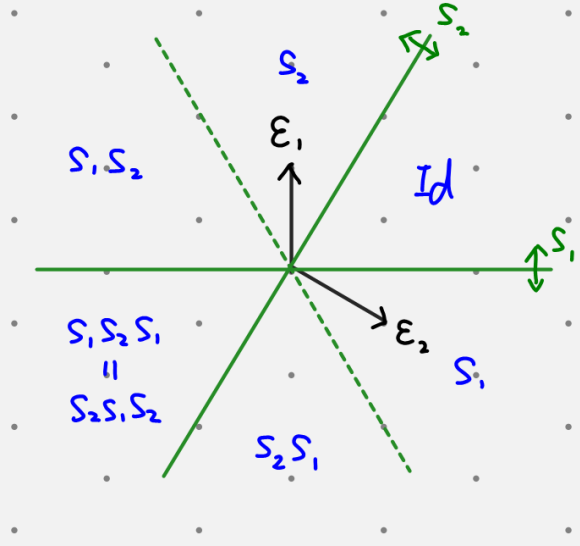

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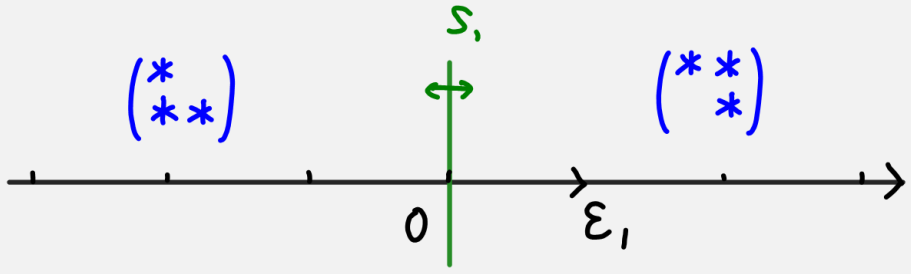

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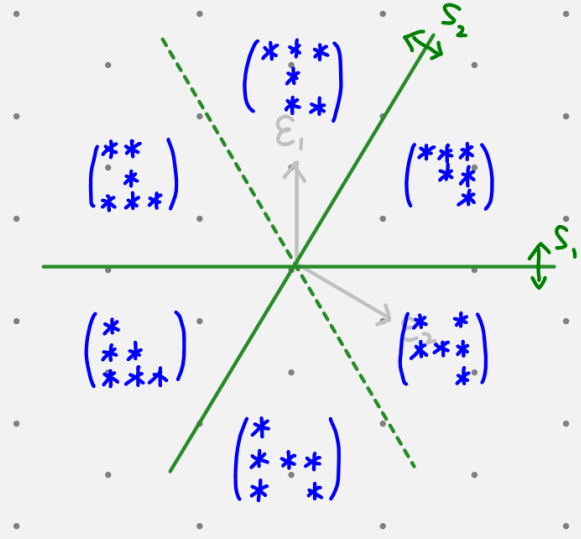

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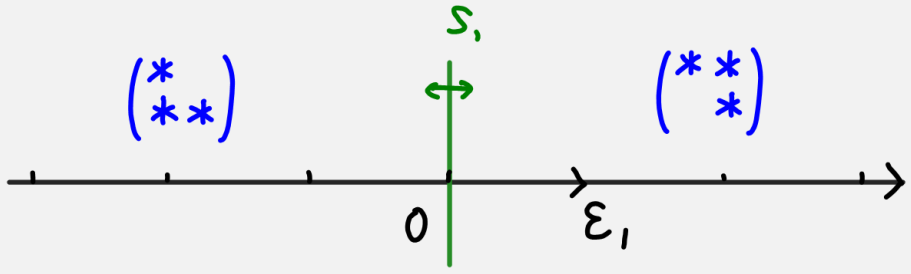

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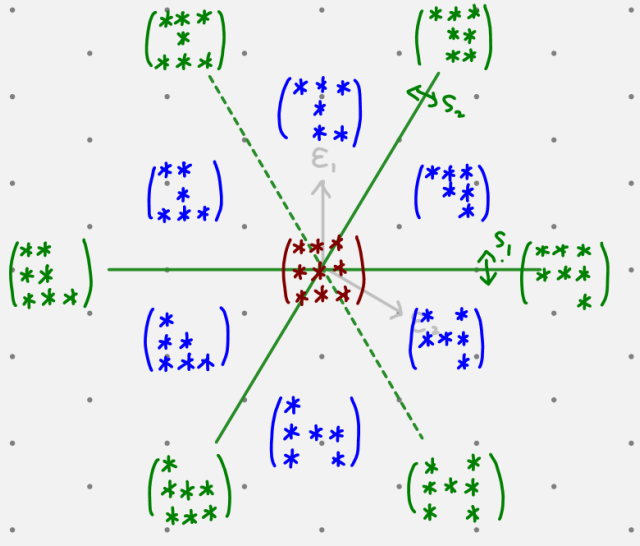

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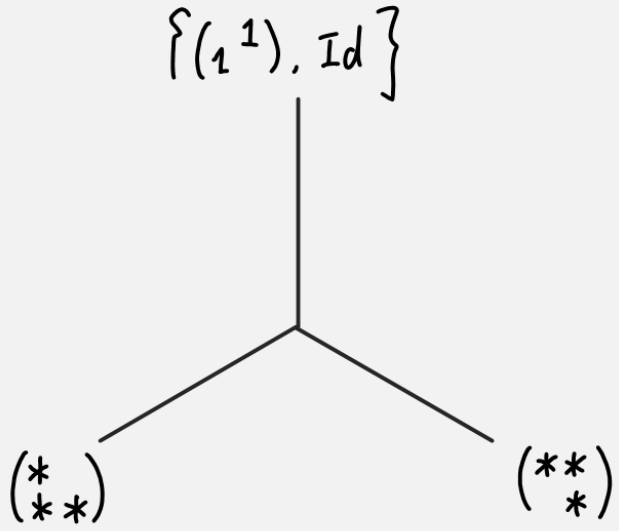

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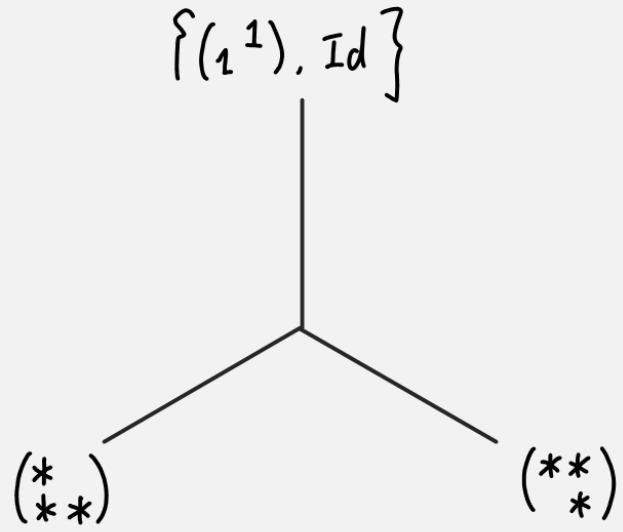

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

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Proposition

- *Any two chambers lie in one apartment.*
- *There is a unique geodesic through any two points* $p_1, p_2 \in \mathcal{B}$.

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

p-adic notation

symbol	name	example
F	NA local field	
$\mathcal{O} = \mathcal{O}_F$	ring of integers	
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Remark

They also have moduli intepretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{\mathfrak{p}L = L_0 \subset L_1 \subset \cdots \subset L_n = L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

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$$W_{\mathrm{ext}} = \{ \textit{ monoidal matrixes } \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

Extended Weyl group action

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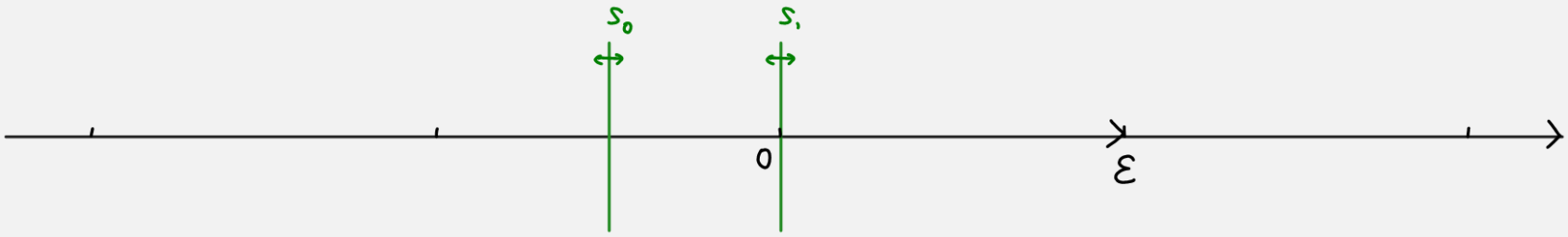

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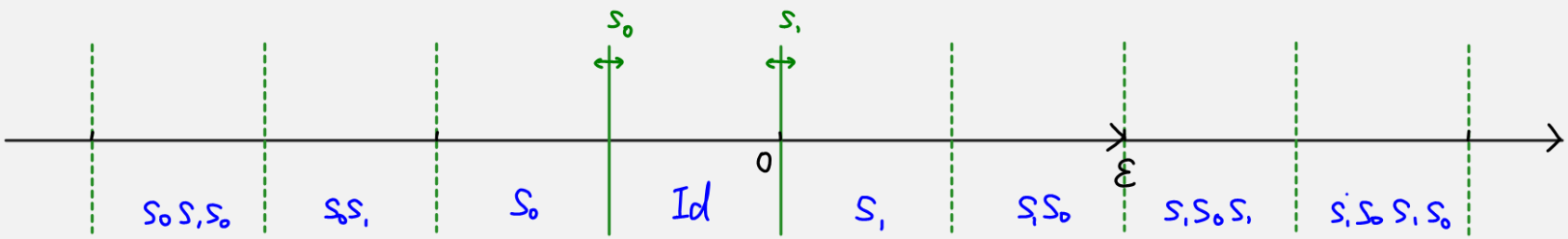

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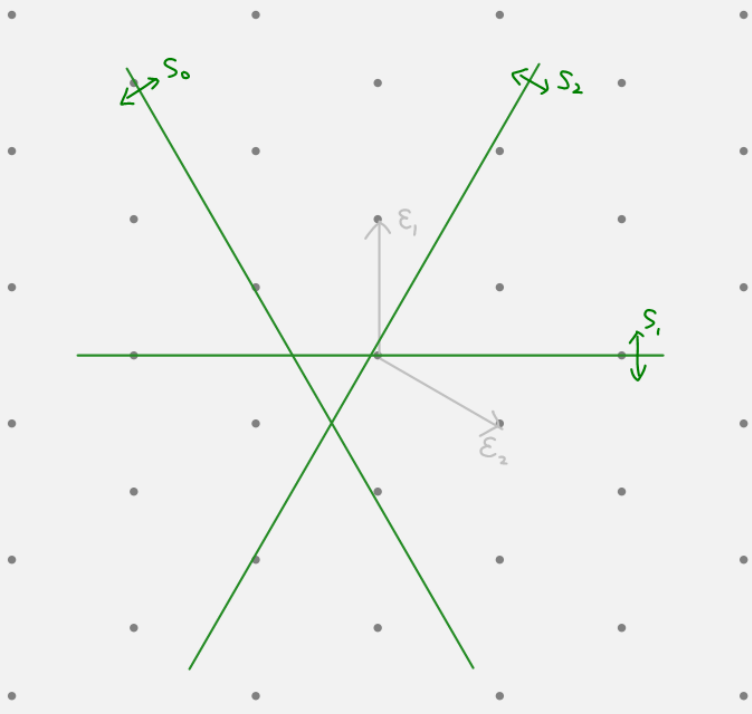

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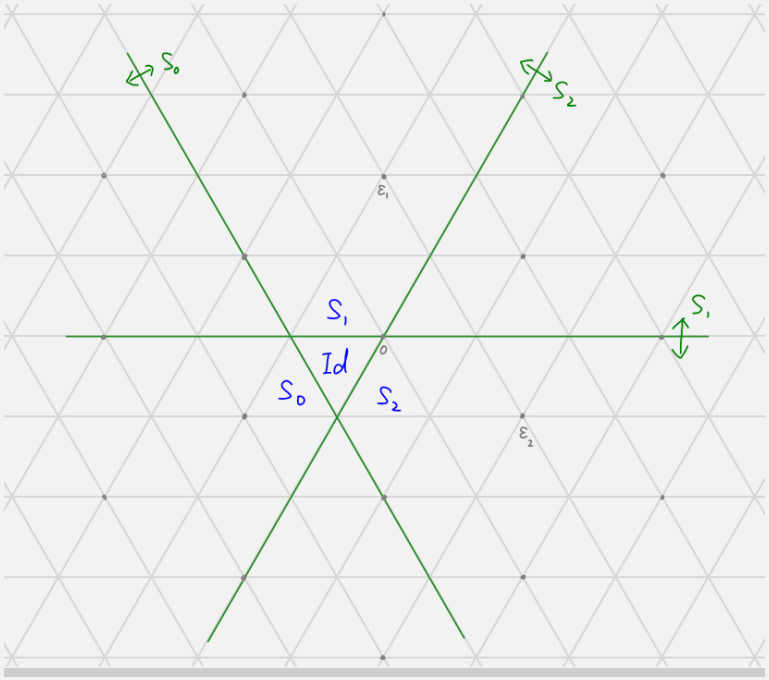

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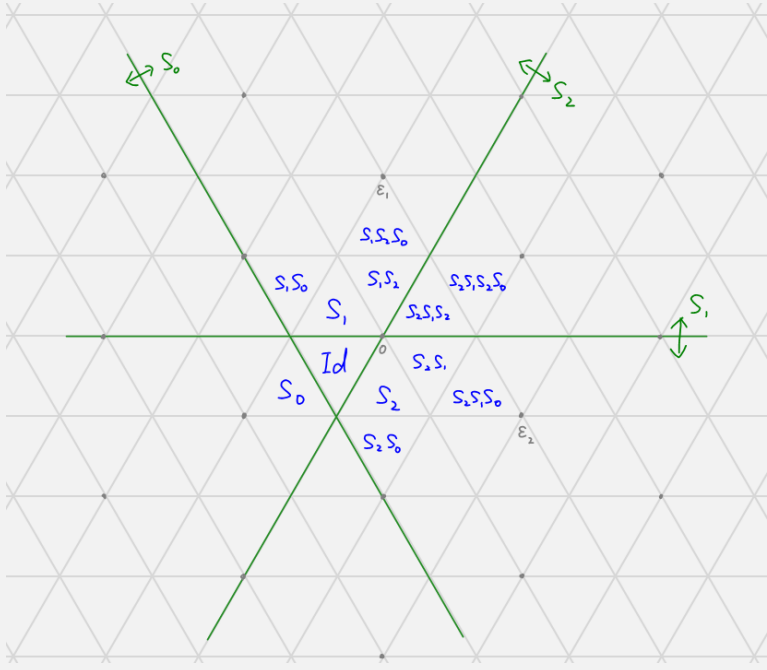

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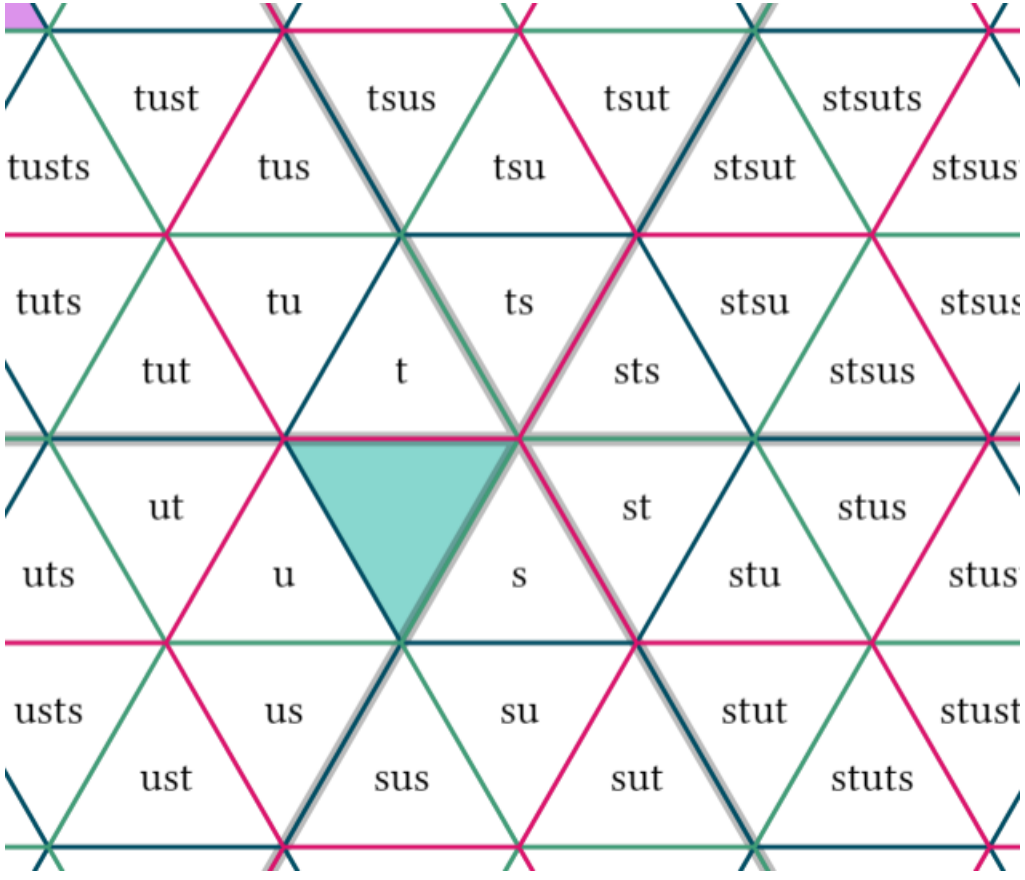

Figure: Reduced expressions labels, from Lievis

Non-standard subgroups in the p-adic world

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Similarly,

$$\{ \text{ Iwahori subgroups } \} = \left\{ g I_0 g^{-1} \right\} \qquad \cong G/I_0$$

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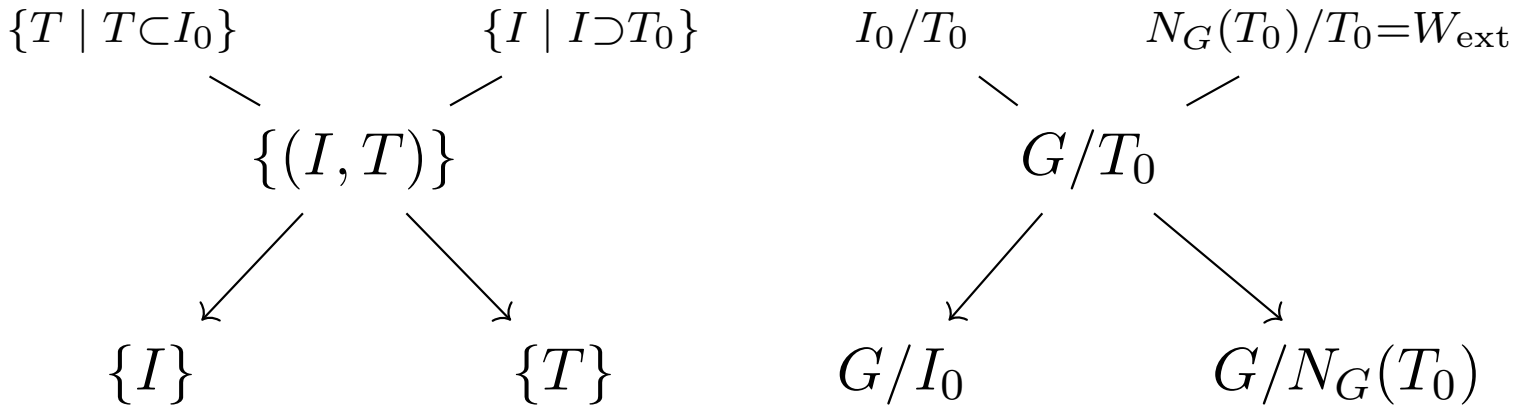

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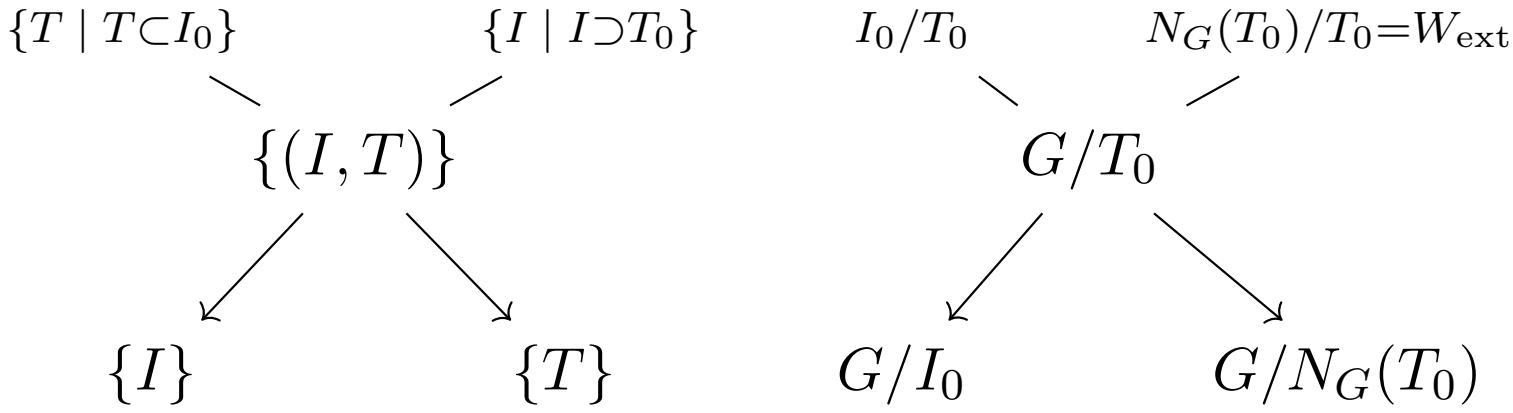
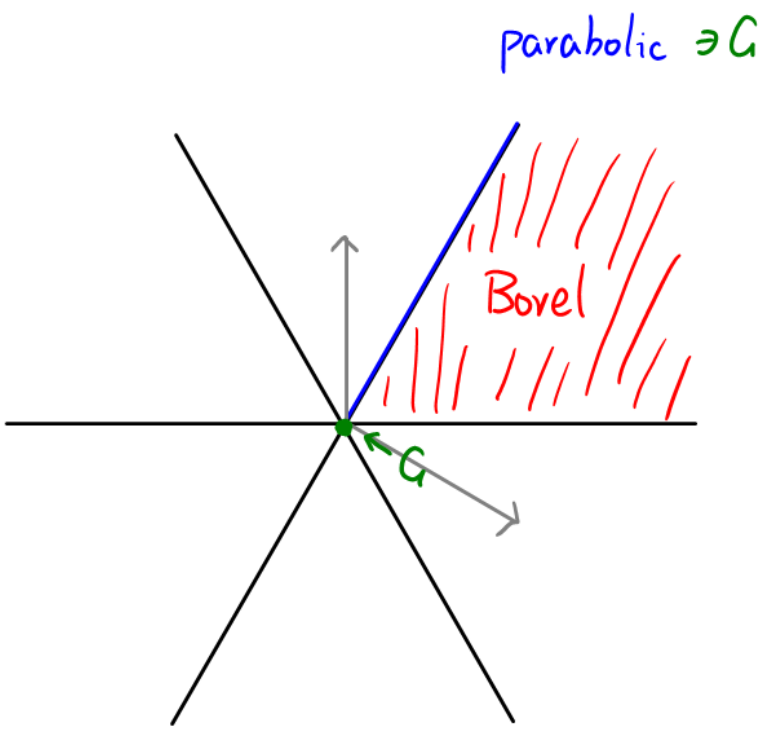
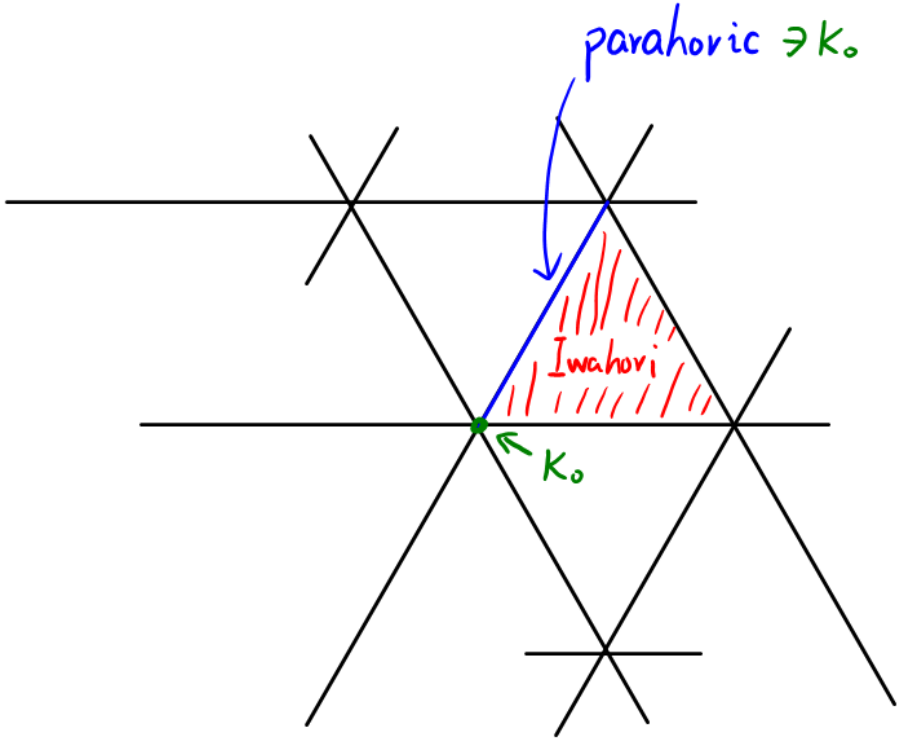


$$\{ \text{ chambers } \} \overset{1:1}{\longleftrightarrow} W_{\text{ext}} \overset{1:1}{\longleftrightarrow} \{I \mid I \supset T_0\}$$

Comparison





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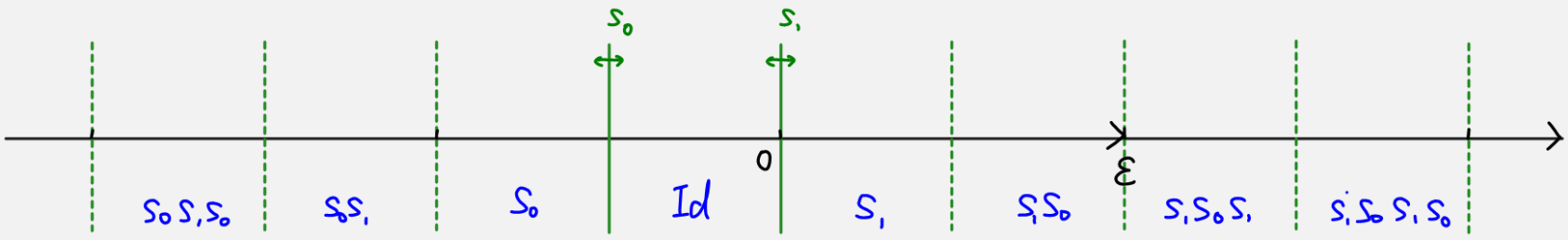

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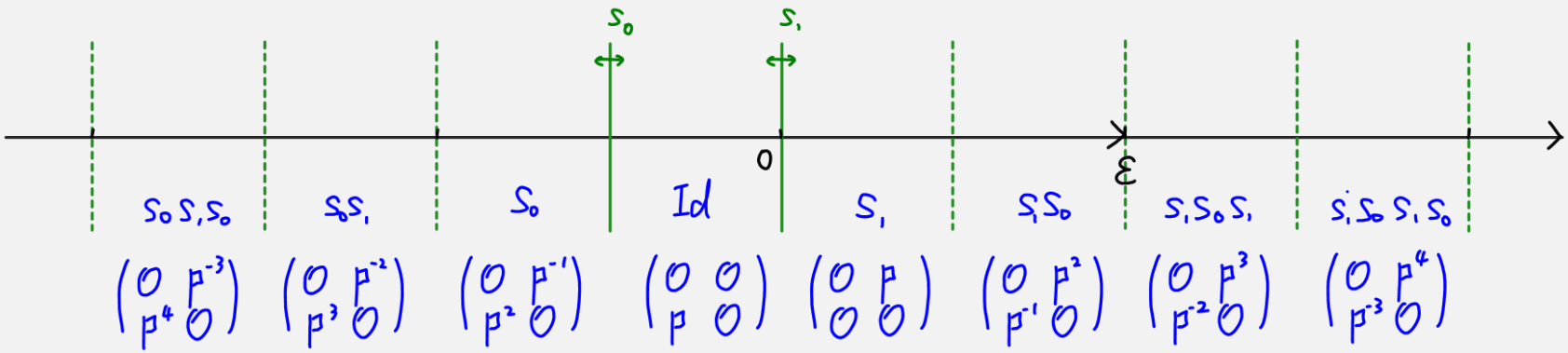

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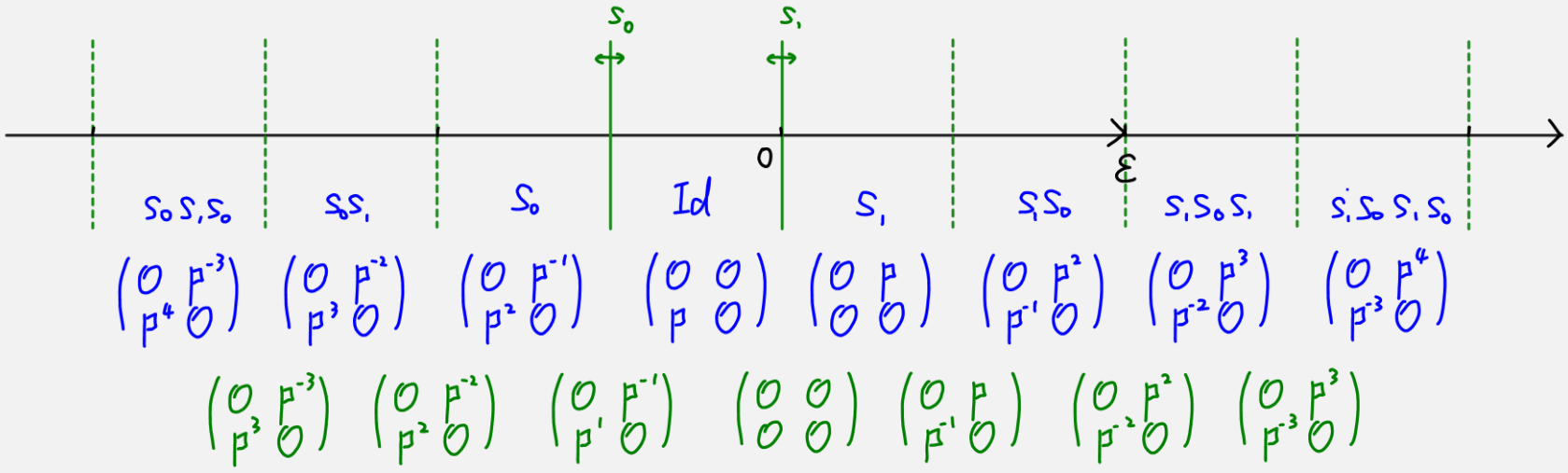

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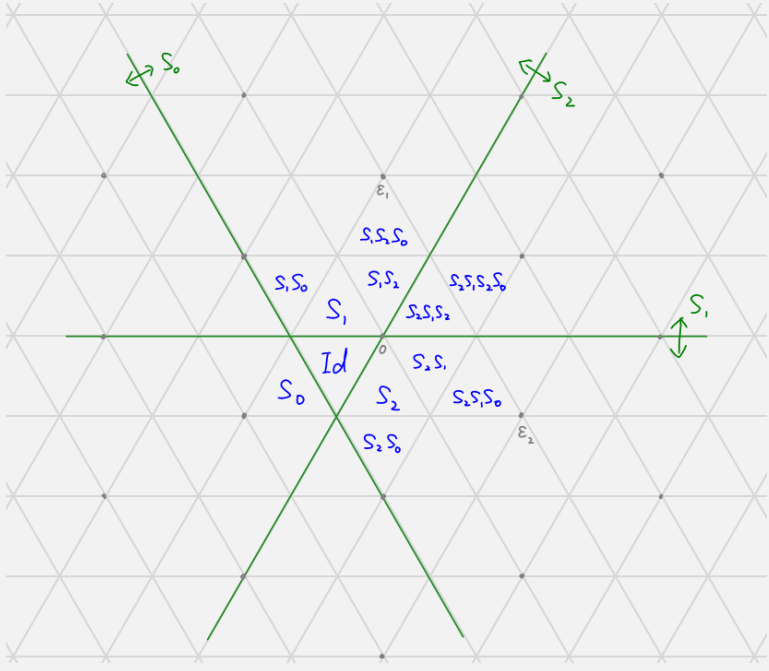

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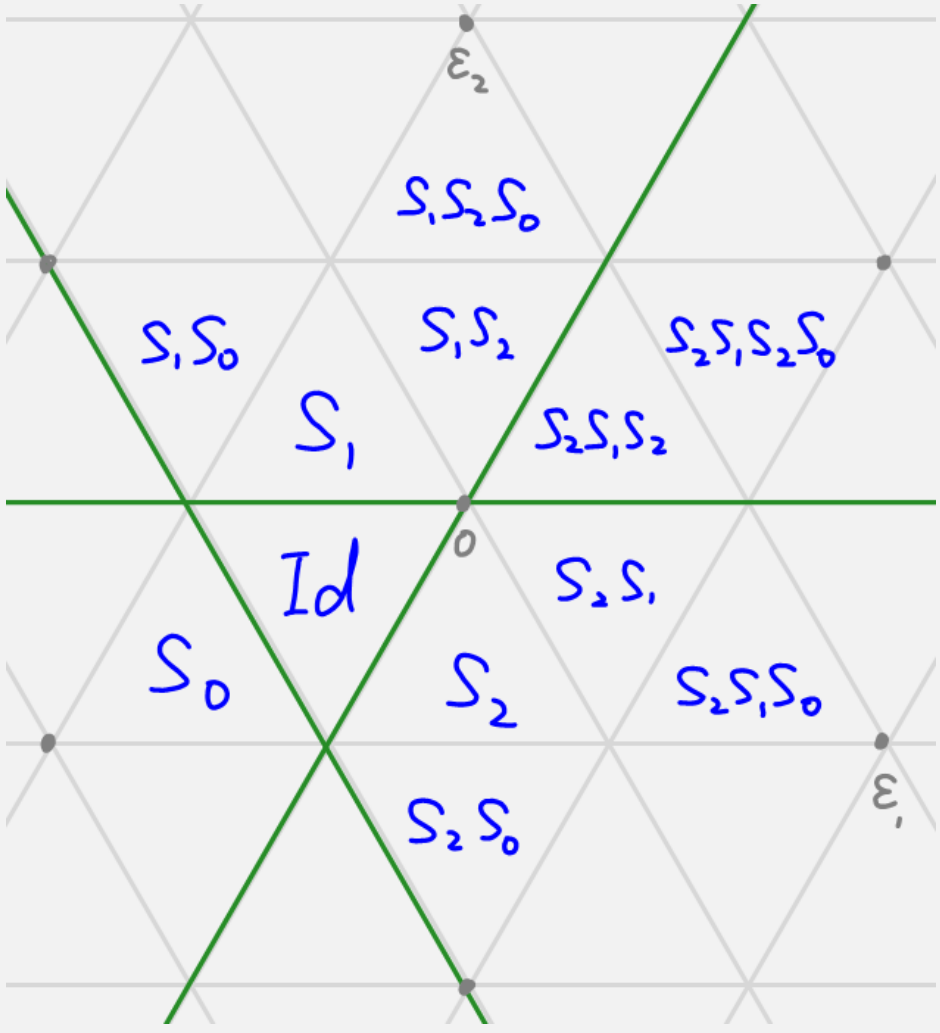

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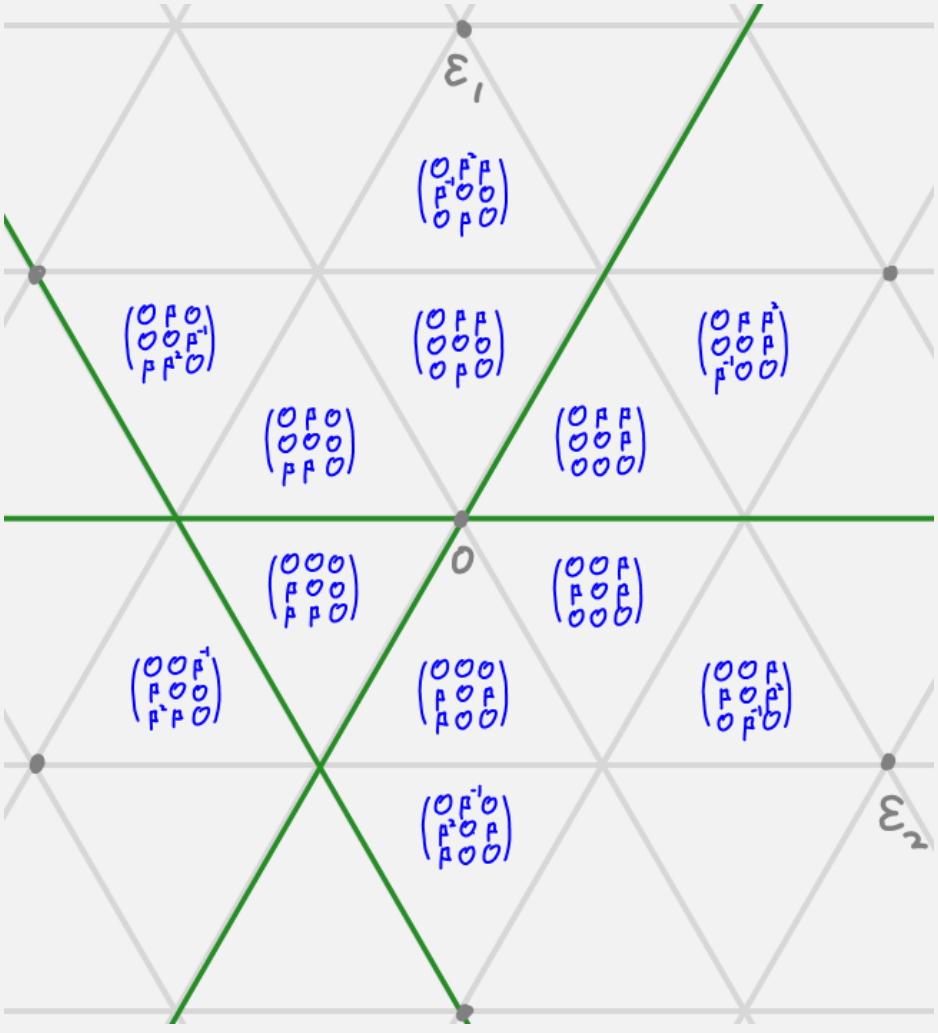

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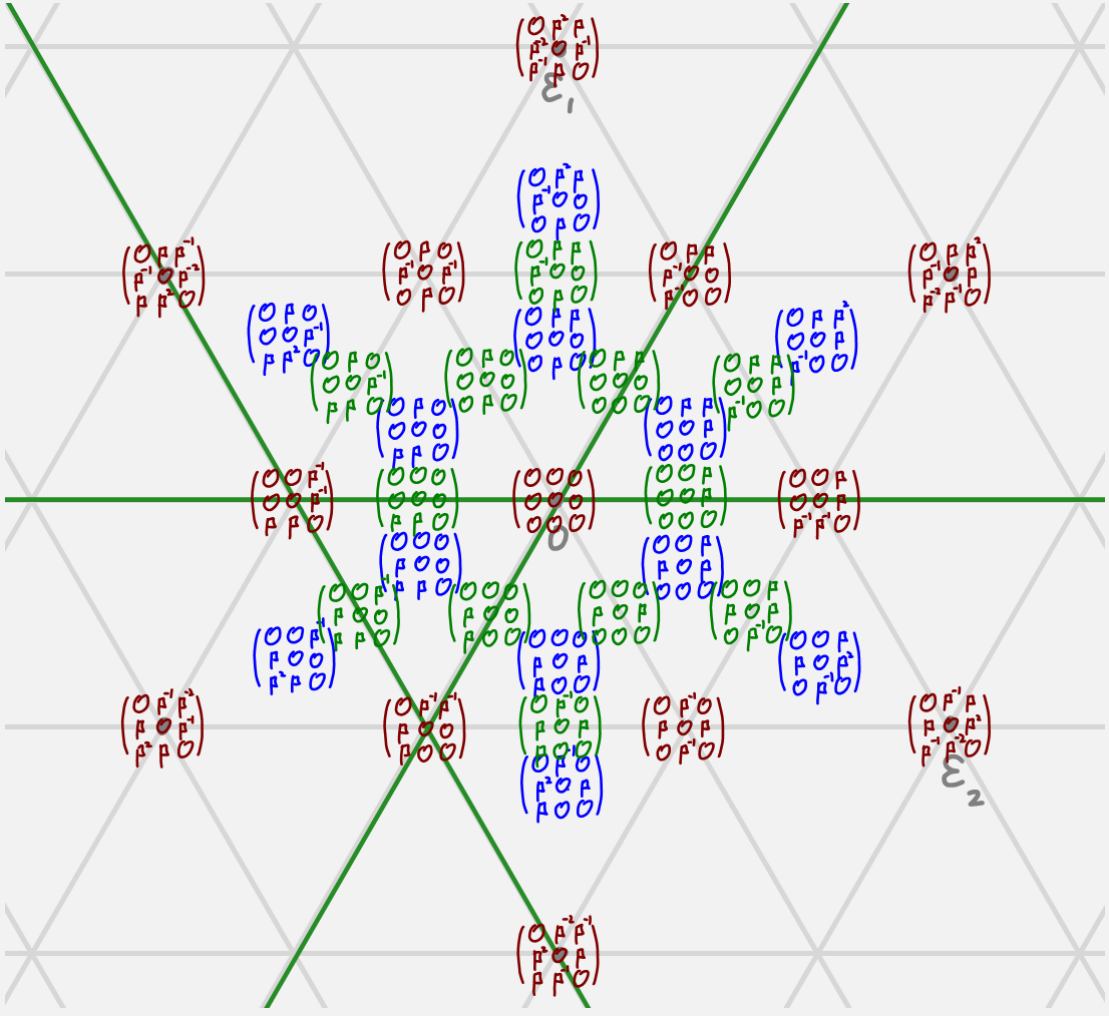

p-adic building

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Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

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Remark

Similarly, any two chambers lie in one apartment,
and there is a unique geodesic through $p_1, p_2 \in \mathcal{B}$.

Plan of the talk

1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

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Theorem

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We call ρ reductive when $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$ is reductive.

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Example

The map

$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.

Thanks for listening!

You can get this slide at:
https://github.com/ramified/personal_tex_collection/raw/main/Bruhat–Tits_building/Bruhat–Tits_building.pdf