A CRASH INTRODUCTION TO LANGLANDS CORRESPONDENCE

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Abstract. In these notes, we explore various versions of the Langlands correspondence, placing particular emphasis on modular forms, automorphic forms, and automorphic representations.

Contents

1. Introduction

These notes represent a faithful record of my talk at KleinAG. I have intentionally omitted sections that were not addressed during the actual presentation, making these notes somewhat incomplete. Readers may refer to my handwritten notes [6] for a more expressive and detailed account.

I want to acknowledge that there is nothing original in my presentation. I appreciate the organizers, the attentive audience, and fellow speakers for helping identify my mistakes. Please feel free to continue pointing out any more errors or issues.

Introducing the Langlands correspondence can often be a challenging and intricate endeavor. It encompasses numerous versions, spanning from local to global, from one dimension to n dimensions, and from GL_n to non-split groups. Today's talk is structured into four parts, each focusing on a specific version of Langlands correspondence, as outlined below:

$$\begin{aligned}
\mathrm{Irr}_{\mathbb{C}}\big(\mathrm{GL}_n(F)\big) &\xleftrightarrow{1:1} \mathrm{WDrep}_{\substack{n\text{-dim}\\ \text{Frob ss}}}(W_F) \\
\mathrm{Char}_{\mathbb{C},\mathrm{alg}}\big(F^\times\backslash\mathbb{A}_F^\times\big) &\xleftrightarrow{1:1} \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) \quad + \text{de Rham} \\
\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})\big) &\xrightarrow{ES} \mathrm{Irr}_{\overline{\mathbb{Q}}_p,2\text{-dim}}(\Gamma_F) + \text{modular} \\
\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(G_D(\mathbb{A}_{\mathbb{Q}})\big) &\longrightarrow \qquad \cdots
\end{aligned}$$

Before discussing these correspondings, let us fix some notations.

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Setting 1.1.

In Section 2, F is a non-Archimedean local field with integral ring O_F and residue field κ_F. Within this context, we also make use of the absolute Galois group Γ_F and the Weil group W_F associated with F.

Moving on to Section 3, we shift our focus to a number field, still denoted as F, with its integral ring denoted as O_F. For each place v of F, we equip with three complete local rings, namely, O_v, F_v and κ_v. The absolute Galois group of F remains denoted as Γ_F.

In Section 5, F will be a totally real field for simplicity.

We will use the following abbreviations for representations:

Rep	*smooth representation*
Irr	*irreducible smooth representation*
Π	*admissible irreducible smooth representation*
Char	*1-dim smooth representation*
WDrep	*Weil–Deligne representation*
$\mathcal{A}_{\text{cusp}}$	*cuspidal automorphic form*

For the definition of smooth/irreducible/admissible/Weil–Deligne representation, see [4] or (partially)[6, 22.04.17].

2. Non-Archimedean Local Field Case

Read [6, GL_n-case]. You may assume $F = \mathbb{Q}_p$ if you are not familiar with local fields.

In this instance, the Langlands correspondence is notably explicit, allowing for the classification of representations on both sides. Notably, it simplifies to a linear algebra task when considering the L-parameters of $\mathrm{GL}_{2,\mathbb{R}}$.

3. Global Langlands Correspondence, $n = 1$

To state the global Langlands correspondence, we rely on the concepts of adèles and idèles, which gather all the local information. A brief introduction to adèles and idèles can be found in [6, 21.08.28].

Observe that

$$\mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / \mathbb{R}_{>0} \;\cong\; \widehat{\mathbb{Z}}^\times \;\cong\; \mathrm{Gal}\left(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q}\right) := \Gamma_{\mathbb{Q}}^{\mathrm{ab}}.$$

In fact, we have Artin reciprocity:

$$\mathrm{Art} : F^\times \backslash \mathbb{A}_F^\times / \overline{\left(F_\infty^\times\right)^\circ} \;\cong\; \Gamma_F^{\mathrm{ab}},$$

which gives us global Langlands correspondence for $n = 1$:

$$\begin{tikzcd}
\mathrm{Char}_{\mathbb{C},\mathrm{alg},\mathrm{wt}\,0}\left(F^\times \backslash \mathbb{A}_F^\times\right) \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{Char}_{\mathbb{C}}(\Gamma_F) \arrow[d, hook, "\text{twist}"] \\
\mathrm{Char}_{\mathbb{C},\mathrm{alg}}\left(F^\times \backslash \mathbb{A}_F^\times\right) \arrow[r, leftrightarrow, "\text{twist}"] & \mathrm{Char}_{\overline{\mathbb{Q}}_p}(\Gamma_F) + \text{de Rham}
\end{tikzcd}$$

For more information about the twist, see [6, Galois representation].(???Wait for updating)

4. Adèlic Modular Forms

In this section, we want to discuss global Langlands correspondence for GL_2. The route is as follows:

$$\text{moduli space} \rightsquigarrow \text{MF} \rightsquigarrow \mathcal{A}_{\text{cusp},k,\eta} \rightsquigarrow \Pi_{\mathcal{A}_{\text{cusp}},k,\eta} \rightsquigarrow \text{GLC}$$

4.1. **Moduli space.** Recall:

$$\begin{array}{ccccc}
\Gamma(N) & \subset & \Gamma_1(N) & \subset & \mathrm{GL}_2(\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{not surj}} \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \subset & \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} & \subset & \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})
\end{array}$$

One can define subgroups of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ in a similar way:

$$\begin{array}{ccccc}
\widehat{\Gamma(N)} & \subset & \widehat{\Gamma_1(N)} & \subset & \mathrm{GL}_2(\widehat{\mathbb{Z}}) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \text{not surj}} \\
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \subset & \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} & \subset & \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})
\end{array}$$

Proposition 4.1. *As a topological space,*

$$\mathrm{GL}_2(\mathbb{Q})\backslash^{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})}/\widehat{\Gamma_1(N)}\cdot\mathbb{R}^\times\cdot\mathrm{SO}_2 \;\cong\; \Gamma_1(N)\backslash^{\mathcal{H}^\pm}.$$

As a result, the moduli space can be realized adèlically.

Proof. We use the strong approximation theorem[1] for SL_2:

$$\mathrm{SL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}}) = \mathrm{SL}_2(\mathbb{Q})\cdot\widehat{\Gamma_1(N)}_{\det=1}.$$

With this in hand, one can show that

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}}) = \mathrm{GL}_2(\mathbb{Q})\cdot\widehat{\Gamma_1(N)}.$$

Therefore,

$$\begin{aligned}
&\mathrm{GL}_2(\mathbb{Q})\backslash^{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})}/\widehat{\Gamma_1(N)}\cdot\mathbb{R}^\times\cdot\mathrm{SO}_2 \\
\cong\;& \mathrm{GL}_2(\mathbb{Q})\backslash\left(^{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})}/\widehat{\Gamma_1(N)} \;\times\; ^{\mathrm{GL}_2(\mathbb{R})}/\mathbb{R}^\times\cdot\mathrm{SO}_2\right) \\
\cong\;& \mathrm{GL}_2(\mathbb{Q})\backslash\left(^{\mathrm{GL}_2(\mathbb{Q})\cdot\widehat{\Gamma_1(N)}}/\widehat{\Gamma_1(N)} \;\times\; ^{\mathrm{GL}_2(\mathbb{R})}/\mathbb{R}^\times\cdot\mathrm{SO}_2\right) \\
\cong\;& \mathrm{GL}_2(\mathbb{Q})\backslash\left(^{\mathrm{GL}_2(\mathbb{Q})}/\Gamma_1(N) \;\times\; \mathcal{H}^\pm\right) \\
\cong\;& \left(\Gamma_1(N)\backslash^{\mathrm{GL}_2(\mathbb{Q})}\right) \times_{\mathrm{GL}_2(\mathbb{Q})} \mathcal{H}^\pm \\
\cong\;& \Gamma_1(N)\backslash^{\mathcal{H}^\pm}.
\end{aligned}$$

□

Remark 4.2. One don't have strong approximation theorem for GL_2. In fact, for $N \geqslant 2$,

$$\mathbb{A}^\times_{\mathbb{Q},\mathrm{fin}} = \bigsqcup_{t\in I_N}\mathbb{Q}^\times\cdot t\cdot\ker\chi_N$$

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}}) = \bigsqcup_{t\in I_N}\mathrm{GL}_2(\mathbb{Q})\cdot\begin{pmatrix} 1 & \\ & t\end{pmatrix}\cdot\widehat{\Gamma(N)}$$

where

$$\chi_N : \widehat{\mathbb{Z}}^\times \longrightarrow (\mathbb{Z}/N\mathbb{Z})^\times \qquad (a_n)_n \longmapsto a_N$$

$$I_N := \{\pm1\}\backslash^{\widehat{\mathbb{Z}}^\times}/\ker\chi_N \;\cong\; \{\pm1\}\backslash^{(\mathbb{Z}/N\mathbb{Z})^\times} \qquad \#I_N = \begin{cases} 1, & N = 2, \\ \phi(N)/2, & N > 2.\end{cases}$$

[1]See [2, 1] for the sketch of proof.

Using the same method, one would get

$$\mathrm{GL}_2(\mathbb{Q})\backslash^{\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})}/\widehat{\Gamma(N)}\cdot\mathbb{R}^\times\cdot\mathrm{SO}_2 \;\cong\; \bigsqcup_{t\in I_N}\Gamma(N)\backslash^{\mathcal{H}^\pm}.$$

You may need the following fact during the proof:

$$\begin{aligned} &\mathrm{GL}_2(\mathbb{Q})\cap\begin{pmatrix}1&\\&t\end{pmatrix}\widehat{\Gamma(N)}\begin{pmatrix}1&\\&t\end{pmatrix}^{-1}\\ =\ &\mathrm{GL}_2(\mathbb{Q})\cap\widehat{\Gamma(N)}\\ =\ &\Gamma(N). \end{aligned}$$

4.2. **Adèlic cuspidal modular forms.** In this subsection, we define modular form in an adèlic way.

Definition 4.3 (Cuspidal modular form $S_{M_2(\mathbb{Q}),k,\eta}$)**.** *For $k\geqslant 2$, $\eta\in\mathbb{Z}$, let*

$$j_{k,\eta}(\gamma):=(\det\gamma)^{\eta-1}(ci+d)^k\qquad\gamma=\begin{pmatrix}a\,b\\c\,d\end{pmatrix}\in\mathrm{GL}_2(\mathbb{R}).$$

We define the space of cuspidal modular form

$$S_{M_2(\mathbb{Q}),k,\eta}:=\left\{\begin{matrix}\phi\colon\ \mathrm{GL}_2(\mathbb{Q})\backslash^{\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})}\longrightarrow\mathbb{C}\quad \textit{as functions}\\ \textit{such that } (1) \text{ to } (4) \textit{ are ture}\end{matrix}\right\}$$

(1) (continuity) There exists an open subset $U_{\mathrm{fin}}\leqslant\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ such that

$$\phi(g\gamma)=\phi(g)\qquad\qquad\text{for any }\gamma\in U_{\mathrm{fin}}.$$

(2) (automorphy)

$$\phi(g\gamma)=j_{k,\eta}(\gamma)^{-1}\phi(g)\qquad\text{for any }\gamma\in\mathbb{R}^\times\cdot\mathrm{SO}_2\,.$$

This formula can also be formulated as

$$j_{k,\eta}(\gamma'\gamma)\phi(g\gamma)=j_{k,\eta}(\gamma')\phi(g).$$

(3) (holomorphy) For any $g\in\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, the function

$$f_{\phi,g}:\mathcal{H}^\pm\longrightarrow\mathbb{C}\qquad\gamma i\longmapsto j_{k,\eta}(\gamma)\phi(g\gamma)$$

is holomorphic.

(holomorphic at ∞) $f_{\phi,g}(\tau)|\mathrm{Im}\,\tau|^{\frac{k}{2}}$ is bounded.

(4) (cuspidal condition) For any $g\in\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$,

$$\int_{\mathbb{Q}\backslash\mathbb{A}_\mathbb{Q}}\phi\left(\begin{pmatrix}1\,x\\0\,1\end{pmatrix}g\right)dx=0.$$

Example 4.4. *When $U_{\mathrm{fin}}=\widehat{\Gamma_1(N)}$, one has isomorphism*

$$S^{\widehat{\Gamma_1(N)}}_{M_2(\mathbb{Q}),k,\eta}\cong S_k\big(\Gamma_1(N)\big)\qquad\phi\longmapsto f_{\phi,\mathrm{Id}},$$

where

$$S_k\big(\Gamma_1(N)\big)=\left\{f:\mathcal{H}^\pm\longrightarrow\mathbb{C}\;\middle|\;\begin{matrix}f(\gamma z)=(c\tau+d)^kf(z)\quad \textit{for any }\gamma\in\Gamma_1(N)\\ f\textit{ has zeros in the cusps }+\cdots\end{matrix}\right\}.$$

Remark 4.5. The integer k works as the weight while the subgroup U_{fin} works as the level. The integer η is not too important: one has isomorphism

$$S_{M_2(\mathbb{Q}),k,\eta}\longrightarrow S_{M_2(\mathbb{Q}),k,\eta-1}\qquad\phi(-)\longmapsto\phi(-)\cdot|\det(-)|_{\mathbb{A}_\mathbb{Q}^\times}$$

which shifts the weight η.

4.3. **Automorphic forms and automorphic representations.** In this subsection, we introduce the space of cuspidal automorphic forms $\mathcal{A}_{\mathrm{cusp},k,\eta}$ and the space of cuspidal automorphic representations $\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}$.

Definition 4.6. *For $k \geqslant 2$, $\eta \in \mathbb{Z}$, the space of cuspidal automorphic forms of weight (k,η) is defined as the minimal $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ representation containing $S_{M_2(\mathbb{Q}),k,\eta}$, i.e.,*

$$\mathcal{A}_{\mathrm{cusp},k,\eta} = \left\langle S_{M_2(\mathbb{Q}),k,\eta} \right\rangle_{\mathrm{Rep}_\mathbb{C}(\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}))}$$

$$\begin{array}{lll}
\mathrm{GL}_2(\mathbb{Q}) \circlearrowright & & \\
\left\{\phi\colon \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathbb{C}\right\} & & \\
\cup & & \\
\mathcal{A}_{\mathrm{cusp},k,\eta} & & \textbf{cuspidal automorphic forms} \text{ of weight } (k,\eta) \\
\cup & & \\
S_{M_2(\mathbb{Q}),k,\eta} & & \textbf{adèlic modular forms} \text{ of weight } (k,\eta)
\end{array}$$

Remarks.

1. (see [4, Remark 4.14]) People have defined the space of cuspidal automorphic forms, denoted as $\mathcal{A}_{\mathrm{cusp}}$, which encompasses a broader range of elements compared to the definitions provided earlier. One get

$$\mathcal{A}_{\mathrm{cusp}} \supsetneqq \bigoplus_{\substack{k \geqslant 2 \\ \eta \in \mathbb{Z}}} \mathcal{A}_{\mathrm{cusp},k,\eta},$$

where Maass forms (a special case of regular algebraic cuspidal automorphic forms) and weight-1 modular forms (a special case of regular algebraic cuspidal automorphic forms) are missing on the right hand side.

2. (see [4, Fact 4.12]) $\mathcal{A}_{\mathrm{cusp},k,\eta}$ can be written as direct sums of irreducible admissible representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$,[2] i.e.,[3]

$$\mathcal{A}_{\mathrm{cusp},k,\eta} = \bigoplus_{i \in I} \pi_i \qquad \pi_i \in \Pi\big(\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})\big).$$

Definition 4.7. *We call*

$$\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta} := \{\pi_i | i \in I\} \subseteq \Pi\big(\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})\big)$$

*as the set of **cuspidal automorphic representations** of weight (k,η).*

Remark 4.8. We did not delve into the Hecke operator theory [4, 4.6-4.7], strong multiplicity one [4, 4.15], and the theory of newforms [4, 4.16] in this discussion. Interested readers are encouraged to explore these topics independently.

Remark 4.9. To generalize the above results to $\mathrm{GL}_{2,F}$, substitute $\mathbb{Q}$ with $\mathbb{F}$. If any issues arise, apply $D = M_2(F)$ in the following section to observe the generalization.

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[2] I'm not sure if replacing $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ with $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is possible, but I don't think using $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ here sounds natural.

[3] In this document, I consistently use symbols $\pi \in \Pi$ and $\varphi \in \Phi$ to ensure clarity and prevent any confusion between their respective memberships.

4.4. **Global Langlands correspondence for** $\mathrm{GL}_{2,F}$**.** In this subsection, we present the Eichler–Shimura theorem without providing a proof. For more information about global Langlands correspondence, see the discussion in Mathoverflow:127157.

Theorem 4.10 (Eichler–Shimura, [4, 4.20])**.**
Fix $\pi \in \Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}(\mathrm{GL}_2(\mathbb{A}_F))$*, and take* L *as some CM-field containing all eigenvalues of Hecke operators. For any finite place* λ *of* L*, there exists* $\varphi_\lambda(\pi) \in \mathrm{Irr}_{\overline{\mathbb{Q}}_p,2\text{-dim}}(\Gamma_F)$ *such that*

1) If π_v is unramified and $\mathrm{char}\,\kappa_v \neq \mathrm{char}\,\kappa_\lambda$, then $\varphi_\lambda(\pi)\big|_{G_{F_v}}$ is unramified, and
$$\mathrm{char\,poly}(\mathrm{Frob}) = X^2 - t_v X + (\#\kappa_v)s_v,$$
where t_v and s_v are the eigenvalues of T_v and S_v.
2) It is compatible with the local Langlands correspondence in both the cases when $l \neq p$ and when $l = p$.
3) $\varphi_\lambda(\pi)$ is geometric.[4]
4) For any $\lambda|\infty$ satisfying $F_v \cong \mathbb{R}$, if we denote $\Gamma_{F_v} := \{1, \sigma_v\} \subseteq \Gamma_F$, then
$$\det\left(\varphi_\lambda(\pi)(\sigma_v)\right) = -1.$$
5) $\{\varphi_\lambda(\pi)\}_\lambda$ forms a strictly compatible system.[5]

Definition 4.11. $\varphi \in \mathrm{Irr}_{\overline{\mathbb{Q}}_p,2\text{-dim}}(\Gamma_F)$ *is modular, if* $\varphi = \varphi_\lambda(\pi)$ *for some* π, λ.

Question 4.12. *Are all geometric representations modular?*

5. Adèlic Modular Forms on Quaternion Algebras

In this section, we try to generalize all the results in Section 4 to quaternion algebras. In another word, we are trying to do global Langlands correspondence for inner forms of GL_2.

For simplicity, F is a totally real field in the whole section.

5.1. **Quaternion algebras.** There are many stories about Quaternion algebras, see Topics on Quaternions(Chinese).

Definition 5.1. *A* ***quaternion algebra*** *over* F *is a* 4-dim *central simple algebra(CSA) over* F.

Exercise 5.2. *For* $a, b \in F^\times$, $\mathrm{char}\,F \neq 2$*, denote*
$$\left(\frac{a,b}{F}\right) = F \oplus Fi \oplus Fj \oplus Fk$$
with relations
$$i^2 = a, j^2 = b, ij = k = -ji.$$
Then $\left(\frac{a,b}{F}\right)$ *is a quaternion algebra over* F*.*[6]

Remark 5.3. All the quaternion algebras can be written as the form $\left(\frac{a,b}{F}\right)$. See [3, Theorem 5.5] for a proof of this fact.

The quaternion algebra is closely related to the Brauer group and Galois cohomology. Consequently, I may not be able to provide a comprehensive presentation of all aspects. Here, I have compiled some key facts that may aid in understanding and computations.

[4] See [4, 2.28] for the definition of geometric representations.
[5] See [4, 2.32] for the definition of a strictly compatible system.
[6] One can also define $\left(\frac{a,b}{F}\right)$ for a ring R.

Black box.

$$
\begin{aligned}
&1)\quad \left(\frac{a,b}{F}\right) \cong M_2(F) && \textit{when } F \textit{ is finite aor algebraic closed.}\\
&2)\quad \left(\frac{a,b}{\mathbb{R}}\right) \cong M_2(\mathbb{R}) &\iff\ & a>0 \textit{ or } b>0\\
&3)\quad \left(\frac{a,b}{F_v}\right) \cong M_2(F_v) &\iff\ & (a,b)_v = 1\\
& &\iff\ & z^2 = ax^2+by^2 \textit{ has a non-zero solution } (x,y,z)\in F_v^{\oplus 3}\\
& &\iff\ & b \in \operatorname{Im}\operatorname{Nm}_{F_v(\sqrt{a})/F_v}
\end{aligned}
$$

Here,

$$
(a,b)_v = \begin{cases} (-1)^{\alpha\beta\varepsilon(p)}\left(\dfrac{u}{p}\right)^{\beta}\left(\dfrac{v}{p}\right)^{\alpha}, & F_v \cong \mathbb{Z}_p, p\geqslant 3, a = p^\alpha u, b = p^\beta v\\ (-1)^{\varepsilon(u)\varepsilon(v)+\alpha\omega(v)+\beta\omega(u)}, & F_v\cong \mathbb{Z}_2, \qquad a = 2^\alpha u, b = 2^\beta v\\ ? & \textit{other cases}\end{cases}
$$

is the Hilbert symbol, where

$$
\varepsilon(n) = \frac{n-1}{2}, \qquad \omega(n) = \frac{n^2-1}{8}.
$$

Quaternion algebras over number fields are global objects, for which we can study the “ramification information” locally.

Definition 5.4 (Ramification for quaternion algebras)**.** *Let F be a number field, D be a quaternion algebra over F, and v be a place of F. We say that D is ramified at v, if*

$$
D\otimes_F F_v \not\cong M_2(F_v)
$$

as quaternion algebras over F_v. Denote

$$
S(D) := \big\{ v : \textit{places of } F \,\big|\, D\otimes_F F_v \not\cong M_2(F_v)\big\}
$$

as the set of places of F at which D is ramified.

Example 5.5. *Using the black box, one can show that*

$$
S\big(M_2(F)\big) = \varnothing, \qquad S\left(\left(\frac{-1,-1}{\mathbb{Q}}\right)\right) = \{2,\infty\}, \qquad S\left(\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)\right) = \{\infty_1,\infty_2\}.
$$

Remark 5.6. It is claimed that the map

$$
\begin{aligned}
\{\text{quaternion algebras over } F\}/_{\cong} &\xrightarrow{\ \cong\ } \big\{A \subsetneqq \{\text{places of } F\} \,\big|\, \#A \text{ is even}\big\}\\
D &\longmapsto S(D)
\end{aligned}
$$

is a bijection by the theory on Brauer group, but I don’t know how to construct the inverse map explicitly. The set $S(D)$ is not easy to compute neither.

Question 5.7. *How to understand the ramification theory of quaternion algebras geometrically? My understanding of ramifications in field extensions* [6, ramified covering] *may be helpful. As a beginning point, I want to consider quaternion algebras over $\mathbb{C}(t)$ as some twists of the trivial* $\mathrm{GL}_2(\mathbb{C})$*-bundle over $\mathbb{P}^1_{\mathbb{C}}$.*

Roughly, the quaternion algebra gives us inner forms of $\mathrm{GL}_{2,F}$.

Definition 5.8 (Algebraic group associated with D)**.** *For a quaternion algebra D over F, the functor*

$$G_D := (- \otimes_{\mathbb{Q}} D)^\times : \mathbb{Q}\text{-Alg}^{\mathrm{op}} \longrightarrow \mathrm{Set} \qquad A \longmapsto (A \otimes_{\mathbb{Q}} D)^\times$$

is represented by an algebraic group over $\mathbb{Q}$, still denoted by G_D.

Remark 5.9. It can be shown that G_D is an inner form of $\prod_{[F:\mathbb{Q}]} \mathrm{GL}_{2,\mathbb{Q}}$. Moreover, every inner form of $\mathrm{GL}_{2,\mathbb{Q}}$ has the form G_D, for some quaternion algebra D over $\mathbb{Q}$. These facts can be found in [5, Chapter 17].

Remark 5.10. I personally would like to write

$$G_D = \mathrm{Res}_{D/\mathbb{Q}} \mathbb{G}_m = \mathrm{Res}_{F/\mathbb{Q}} \mathrm{Res}_{D/F} \mathbb{G}_m,$$

where

$$\mathrm{Res}_{D/F} : \mathrm{Fun}(D\text{-Alg}^{\mathrm{op}}, \mathrm{Set}) \longrightarrow \mathrm{Fun}(F\text{-Alg}^{\mathrm{op}}, \mathrm{Set}) \qquad G \longmapsto G(- \otimes_{\mathbb{Q}} D)$$

is defined in a similar way as the Weil restriction.

The comparison between quaternion algebra and field extension can be perplexing. On one hand, it is suggested that GL_2 may not be as intimidating as it seems, as it can be seen as "the restriction of a torus". On the other hand, $\mathbb{G}_{m,D}$ faces challenges in being represented by a scheme due to the non-commutativity of D. Can one really understand the geometry of non-commutative rings?

We conclude this subsection by a table of examples.

F	$\mathbb{Q}$	$\mathbb{Q}(\sqrt{3})$
D/F	$M_2(\mathbb{Q})$	$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)$
$S(D)$	$\varnothing$	$\{\infty_1, \infty_2\}$
$G_D := (D \otimes_{\mathbb{Q}} -)^\times$	GL_2	$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3}) \otimes_{\mathbb{Q}} -}\right)^\times$
$G_D(\mathbb{Q}) = D^\times$	$\mathrm{GL}_2(\mathbb{Q})$	$\left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)^\times$
$G_D(\mathbb{R})$	$\mathrm{GL}_2(\mathbb{R})$	$\mathbb{H}^\times \times \mathbb{H}^\times$

Exercise 5.11. *Work out $G_D(A)$ for $A = \mathbb{C}, \mathbb{Q}_p$ and $\mathbb{A}_{\mathbb{Q}}$.*

5.2. **Modular form of quaternion algebras.** For mimicking Definition 4.3, one has to give some new definitions.

Definition 5.12. *For $v|\infty$, k_v, $\eta_v \in \mathbb{Z}$, define U_v and $(\tau_v, W_v) \in \mathrm{rep}(U_v)$ as follows:*

	$v \notin S(D)$	$v \in S(D)$
U_v	$\mathbb{R}^\times \mathrm{SO}_2$	$\mathbb{H}^\times$
W_v	$\mathbb{C}$	$(\Lambda^2\mathbb{C}^2)^{\eta_v} \otimes_{\mathbb{C}} (\mathrm{Sym}^{k_v-2}\mathbb{C}^2)$
τ_v	$\mathbb{R}^\times \mathrm{SO}_2 \xrightarrow{j_{k_v,\eta_v}} \mathbb{C}^\times \curvearrowright \mathbb{C}$	*constructed from* $\mathbb{H}^\times \hookrightarrow \mathrm{GL}_{2,\mathbb{C}} \curvearrowright \mathbb{C}^2$

We denote

$$U_\infty = \prod_{v|\infty} U_v, \qquad (\tau_\infty, W_\infty) = \bigotimes_{v|\infty} (\tau_v, W_v).$$

Now we can modify Definition 4.3.

Definition 5.13 (Cuspidal modular form $S_{D,k,\eta}$, [4, 4.8])**.** *For $k = (k_v)_{v|\infty}$, $\eta = (\eta_v)_{v|\infty}$, define the space of cuspidal modular form for a quaternion algebra D:*

$$S_{D,k,\eta} := \left\{ \begin{matrix} \phi\colon {}^{G_D(\mathbb{Q})}\backslash^{G_D(\mathbb{A}_\mathbb{Q})} \longrightarrow \mathbb{C} \quad \textit{as functions} \\ \textit{such that } (1) \text{ to } (4) \textit{ are ture} \end{matrix} \right\}$$

(1) (continuity) There exists an open subset $U_{\mathrm{fin}} \leqslant G_D(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})$ such that

$$\phi(g\gamma) = \phi(g) \qquad\qquad \text{for any } \gamma \in U_{\mathrm{fin}}.$$

(2) (automorphy)

$$\phi(g\gamma) = \tau_\infty(\gamma)^{-1}\phi(g) \qquad \text{for any } \gamma \in U_\infty.$$

This formula can also be formulated as

$$\tau_\infty(\gamma'\gamma)\phi(g\gamma) = \tau_\infty(\gamma')\phi(g).$$

(3) (holomorphy) For any $g \in G_D(\mathbb{A}_\mathbb{Q})$, the function

$$f_{\phi,g} : (\mathcal{H}^\pm)^{\{v|\infty\}\smallsetminus S(D)} \longrightarrow \mathbb{C} \qquad \gamma(i,\dots,i) \longmapsto \tau_\infty(\gamma)\phi(g\gamma)$$

is holomorphic.
(holomorphic at ∞) When $D = M_2(\mathbb{Q})$, $f_{\phi,g}(\tau)|\mathrm{Im}\,\tau|^{\frac{k}{2}}$ is bounded.

(4) (cuspidal condition) When $G_D = \mathrm{GL}_2(F \otimes_\mathbb{Q} -)$, for any $g \in G_D(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{A}_F)$,

$$\int_{F\backslash\mathbb{A}_F} \phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

Remark 5.14. For assuring $S_{D,k,\eta} \neq \{0\}$, one should require that

$$k_v \geqslant 2, \qquad k_v + 2\eta_v = k_w + 2\eta_w, \qquad \text{for any } v, w|\infty.$$

Example 5.15. *For $D = \left(\frac{-1,-1}{\mathbb{Q}(\sqrt{3})}\right)$, conditions (3) and (4) are automatically satisfied. Furthermore,*

$$\begin{aligned} S^{U_{\mathrm{fin}}}_{D,2,0} &= \mathbb{C}\left[{}^{G_D(\mathbb{Q})}\backslash^{G_D(\mathbb{A}_\mathbb{Q})}/_{U_{\mathrm{fin}}U_\infty}\right] \\ &= \mathbb{C}\left[{}^{G_D(\mathbb{Q})}\backslash^{G_D(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})}/_{U_{\mathrm{fin}}}\right] \end{aligned}$$

is a finite dimensional $\mathbb{C}$-vector space.

Definition 5.16. *D is definite, if $\{v|\infty\} \subseteq S(D)$, i.e., $G_D(\mathbb{R}) \cong \mathbb{H}^\times \times \cdots \times \mathbb{H}^\times$.*

Example 5.17. *The quaternion algebra $\left(\frac{-1,-1}{F}\right)$ is definite.*

Modular forms are easier in the definite quaternion algebra case. In the proof of modularity lifting, we will reduce the problem to this case.

Remark 5.18. One can define $\mathcal{A}_{\mathrm{cusp},k,\eta}(G_D)$ and $\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}(G_D)$ in a manner analogous to Section 4, demonstrating that they exhibit similar properties.

5.3. **Global Langlands correspondence for inner forms of** $\mathrm{GL}_{2,F}$**.** Finally, we state the Jacquet–Langlands correspondence without proof, which may be viewed as the global Langlands correspondence for inner forms of GL_2.

Theorem 5.19 (The Jacquet–Langlands correspondence, [4, 4.17])**.** *One has the following* 1*-to-*1 *correspondence:*

$$\mathrm{JK}:\left\{\begin{matrix}(\pi,V)\in\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(G_D(\mathbb{A}_{\mathbb{Q},\mathrm{fin}})\big)\\ \dim_{\mathbb{C}}V=+\infty\end{matrix}\right\}\xrightarrow{\cong}\left\{\begin{matrix}(\pi,V)\in\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(\mathrm{GL}_2\left(\mathbb{A}_{\mathbb{Q},\mathrm{fin}}\right)\big)\\ \pi|_{\mathrm{GL}_2(F_v)}:\textit{discrete series, for any } v\in S(D),v\nmid\infty\end{matrix}\right\}$$

Moreover,

- *when* $v\notin S(D)$*, both sides gives the same representation;*
- *when* $v\in S(D)$*,* $v\nmid\infty$*, the Jacquet–Langlands map is compatible with the local Jacquet–Langlands map:*

$$\mathrm{JK}_{\mathrm{loc}}:\mathrm{Irr}\big(G_D(F_v)\big)\xrightarrow{\cong}\mathrm{Irr}\big(\mathrm{GL}_2\left(F_v\right)\big)+\ \textit{discrete series.}$$

We also state the base change property here.

Theorem 5.20 (Base change, [4, 4.23])**.** *Let* E/F *be a totally real field extension with* $\mathrm{Gal}(E/F)\cong\mathbb{Z}/\tilde{p}\mathbb{Z}$*, where* $\tilde{p}$ *is a prime number. We get a map*

$$\mathrm{BC}_{E/F}:\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(\mathrm{GL}_2\left(\mathbb{A}_{F,\mathrm{fin}}\right)\big)/\sim\ \xrightarrow{\cong}\!\!\!\!\rightarrow\Pi_{\mathcal{A}_{\mathrm{cusp}},k,\eta}\big(\mathrm{GL}_2\left(\mathbb{A}_{E,\mathrm{fin}}\right)\big)^{\mathrm{Gal}(E/F)}$$

which is compatible with local Langlands correspondence, where

$$\pi\sim\pi'\quad\Longleftrightarrow\quad\pi\ \cong\ \pi'\otimes(\chi\circ\det)$$
$$\chi\circ\det:\mathrm{GL}_2\left(\mathbb{A}_{F,\mathrm{fin}}\right)\xrightarrow{\det}\mathbb{A}_{F,\mathrm{fin}}^{\times}\xrightarrow{\mathrm{Art}_F}\Gamma_F^{\mathrm{ab}}\twoheadrightarrow\mathrm{Gal}(E/F)\xrightarrow{\chi}\mathbb{C}^{\times}.$$

After extensive effort, we have comprehended the Langlands correspondence in these four versions. All the versions presented in this document have been rigorously proven. To gain a deep understanding and avoid errors, readers are encouraged to explore the extensive literature for the detailed proofs. However, this is far away from the goal of this talk.

References

[1] Strong approximation and class number in the adelic setting. Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3057117 (version: 2018-12-30).

[2] Brian Conrad. Strong approximation in algebraic groups. http://virtualmath1.stanford.edu/~conrad/248BPage/handouts/strongapprox.pdf. [Online].

[3] Pete L Clark. Non-commutative algebra. *University of Georgia: Athens, GA, USA*, 2012.

[4] Toby Gee. Modularity lifting theorems. *Essent. Number Theory*, 1(1):73–126, 2022.

[5] T. A. Springer. *Linear algebraic groups.*, volume 9 of *Prog. Math.* Boston, MA: Birkhäuser, 2nd ed. edition, 1998.

[6] Xiaoxiang Zhou. Eine woche, ein beispiel. https://github.com/ramified/personal_handwritten_collection/tree/main/weeklyupdate, 2020-2023. [Online].

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