THE DIMENSION OF Z_χ

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Contents

1. Background

In this section, we establish notation and provide background on the question. Experts may wish to skip the first two sections, which are likely to be revised.

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\mathrm{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in $\mathbb{Q}$. For any algebraic group G, we denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G.

Following the approach of [KW15], we work in the quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/N(A)$, where $N(A) \subset \mathrm{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex $\mathcal{F}$ is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [KW15]). In particular, for any perverse sheaf $\delta \in \overline{\mathrm{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G:

$$\langle \delta, * \rangle \cong \mathrm{Rep}(G).$$

Examples are abundant but intricate. For reference, we provide a brief list of known cases:

Proposition 1.1. *For any smooth projective variety X over $\mathbb{C}$, let $A := \mathrm{Alb}(X)$ be its Albanese variety. When the Albanese map*

$$\alpha : X \longrightarrow Alb(X)$$

is a closed embedding, this map defines a perverse sheaf

$$\delta := \alpha_*(\underline{\mathbb{Q}}[\dim X]) \in \overline{\mathrm{Perv}}(A).$$

In several cases, the Tannaka group is already well understood, as follows:

$$\langle \delta, * \rangle \cong \begin{cases} \mathrm{Rep}(\mathrm{SL}_{2g-2}(\mathbb{C})), & X = C \text{ non-hyperelliptic} \qquad A_{2g-3} \\ \mathrm{Rep}(\mathrm{Sp}_{2g-2}(\mathbb{C})), & X = C \text{ hyperelliptic} \qquad C_{g-1} \\ \mathrm{Rep}(\mathrm{E}_6(\mathbb{C})), & X = S \text{ Fano surface} \qquad E_6 \\ \mathrm{Rep}(\mathrm{SO}_{g!}(\mathbb{C})), & X = \Theta,\ g \text{ odd} \qquad D_{g!/2} \\ \mathrm{Rep}(\mathrm{Sp}_{g!}(\mathbb{C})), & X = \Theta,\ g \text{ even} \qquad C_{g!/2} \end{cases}$$

Here, $g := \dim_{\mathbb{C}}(A)$, and

- *C is a smooth projective curve over $\mathbb{C}$ with genus $g \geqslant 2$;*
- *S is the Fano surface of a smooth cubic threefold;*
- *Θ is the smooth theta divisor of a general principally polarized abelian variety.*

Date: October 30, 2024.

References

[1] Jens Niklas Eberhardt. K-motives and Koszul duality. *Bulletin of the London Mathematical Society*, 54(6):2232–2253, 2022.

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