A User-Friendly Introduction to Six-Functor Formalism

Xiaoxiang Zhou

Advisor: Prof. Dr. Thomas Krämer

Humboldt-Universität zu Berlin

A Small Toolkit

- **Basic examples.**
  For $f^!$, assume Y, X are manifolds of dimension n.

	$f: Y \longrightarrow \mathrm{pt}$	$f: p \hookrightarrow X$
f^*	constant sheaf	$\mathcal{F}_p$
Rf_*	cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$	cpt supp cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$f^!$	orientation sheaf$[n]$	$\mathcal{F}_p[n]$

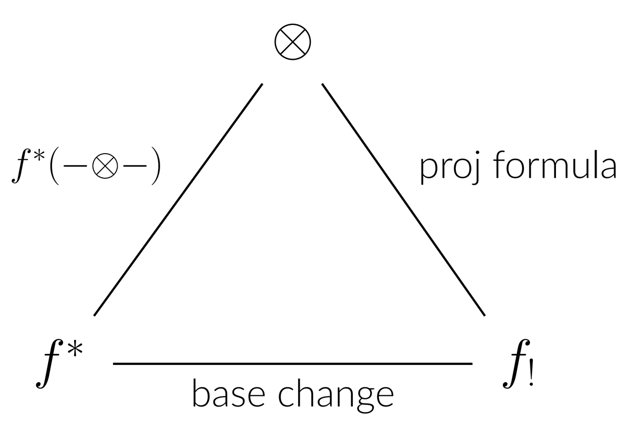
- **Recollement diagram.**

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$$

$$\begin{tikzcd} D(Z) \arrow[r, "i_*=i_!"] & D(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*=j^!"] & D(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \end{tikzcd}$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \xrightarrow{+1}$$

- **Compatability among functors.**



A Short List of Applications

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:
$$\mathrm{H}^i_c(X;\mathbb{Q}) \cong \mathrm{H}^i\left(\bar{X}, \{\infty\}; \mathbb{Q}\right)$$
$$\mathrm{H}^{\mathrm{BM}}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X; \mathrm{Or}_X)$$
$$\mathrm{H}_i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_c(X; \mathrm{Or}_X)$$
Also, define the cup and cap product structures.
2. Using the projection formula, show Poincaré duality:
$$\mathrm{H}^i_c(X;\mathbb{Q})^* \cong \mathrm{H}^{n-i}(X; \mathrm{Or}_X)$$
$$\mathrm{H}^i(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_c(X; \mathrm{Or}_X)^*$$
3. Derive the Gysin sequence for any oriented S^k-bundle $\pi: E \longrightarrow B$:
$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow[eu_\pi]{+1}$$
Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.
4. Compute the upper shriek for singular spaces.

Perverse Sheaf

We will mix the usage of sheaves and complexes. For simplicity, let us fix a Whitney stratification $\mathcal{S}$:

$$\varnothing \overset{U_0}{\subset} Z_0 \overset{U_1}{\subset} \cdots \overset{U_n}{\subset} Z_n = X$$

Denote $D^b_{\mathrm{cons},\mathcal{S}}(X)$ as the category of constructible sheaves over X with respect to $\mathcal{S}$.

Definition

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure.

We say that $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^i\left(\iota^*_{U_j}\mathcal{F}\right) = 0, & \text{for any } i > -j \\ \mathcal{H}^i\left(\iota^!_{U_j}\mathcal{F}\right) = 0, & \text{for any } i < -j \end{cases}$$

Deligne's construction

Any local system $\mathcal{L}$ supported on U_i can be converted into a perverse sheaf through truncations. This process is known as **Deligne's construction**, and the resulting perverse sheaf is called the intersection cohomology complex, or the IC sheaf, denoted by $\mathrm{IC}(\mathcal{L})$. IC sheaves are the simple objects in the category $\mathrm{Perv}_{\mathcal{S}}(X)$.

To determine whether a complex $\mathcal{F}$ is perverse or an IC sheaf, one simply needs to complete Table 1.

Nearby Cycle

A perverse sheaf may not be so "perverse", but a nearby cycle is definitely "nearby".

$$\begin{array}{ccccccc} i^*\mathcal{F}\ \ \psi\mathcal{F}\ \ \varphi\mathcal{F} & & \mathcal{F} & & & & \\ \{0\} & \overset{i}{\hookrightarrow} & \mathbb{C} & \overset{j}{\hookleftarrow} & \mathbb{C}^\times & \overset{p}{\longleftarrow} & \widetilde{\mathbb{C}}^\times \cong \mathbb{C} \end{array}$$

Given $\mathcal{F} \in D^b(\mathbb{C})$, one can construct the **nearby cycle**

$$\psi\mathcal{F} := i^*Rj_*p_*p^*j^*\mathcal{F} \in D^b(\{0\}),$$

which can be roughly viewed as the fiber $\mathcal{F}_x$ for x sufficiently close to 0. By quotienting out the **non-vanishing cycle** $i^*\mathcal{F}$, one obtains the **vanishing cycle**

$$\varphi\mathcal{F} := \mathrm{cone}\left[i^*\mathcal{F} \xrightarrow{sp} \psi\mathcal{F}\right] \in D^b(\{0\}).$$

$$\begin{tikzcd} & \mathcal{F} & & \\ X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X^* \arrow[l, hook', "j"'] \arrow[d] & \widetilde{X}^* \arrow[l, "p"'] \arrow[d] \\ \{0\} \arrow[r, hook] & D & D^* \arrow[l, hook'] & \widetilde{D}^* \arrow[l] \end{tikzcd}$$

In general, $\mathbb{C}$ can be replaced by any disk $\mathcal{D}$, as the problem is local, and $\mathcal{F}$ can be a sheaf over any space X over $\mathcal{D}$.

The same construction yields a distinguished triangle in $D^b(X_0)$:

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \varphi_f\mathcal{F} \xrightarrow{+1}$$

NMD and CC

Normal Morse Data

We work with a fixed complex variety embedding $X \subseteq \mathbb{C}^n$, equipped with a Whitney stratification $\mathcal{S}$. Let $S \subseteq X$ be a connected component of some U_i. Fix $x_0 \in S$, and let N be a normal slice of S at x_0.

For any sheaf $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$, the normal Morse data is defined as

$$\mathrm{NMD}(\mathcal{F}, S) := \left(\varphi_{g|_{N\cap X}}(\mathcal{F}|_{N\cap X})\right)_{x_0}[-1]$$

where $g: \mathbb{C}^n \longrightarrow \mathbb{C}$ is a holomorphic function, and $f := Re(g)$ such that

- $g(x_0) = 0$;
- $df_{x_0} \in T^*_S\mathbb{C}^n$, $df_{x_0} \notin T^*_{S'}\mathbb{C}^n$ for any $S' \neq S$;
- x_0 is a non-degenerate critical point of $f|_S$.

Characteristic Cycle

With normal Morse data, one can define the characteristic cycle

$$\mathrm{CC}(\mathcal{F}) := \sum_S m_S[T^*_S\mathbb{C}^n] \in H^{\mathrm{BM}}_{2n}\Big(\bigcup_S T^*_S\mathbb{C}^n\Big),$$

where

$$m_S := \chi\left(\mathrm{NMD}(\mathcal{F}, S)[-\dim S]\right) \in \mathbb{Z}.$$

The characteristic cycle can be computed when the geometry is well-understood. For example, we can compute $\mathrm{CC}(\mathrm{IC}(\underline{\mathbb{Q}}_{X\setminus\{0\}}))$ when $X \subseteq \mathbb{C}^m$ is a cone over a smooth hyperplane.

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$\mathcal{H}^i(-)$	-3	-2	-1	0	1	2	3
$\iota^*_{U_2}\mathcal{F} = \iota^!_{U_2}\mathcal{F}$	×		×	×	×	×	×
$\iota^*_{U_1}\mathcal{F}$			×	×	×	×	×
$\iota^!_{U_1}\mathcal{F}$	×	×	×				
$\iota^*_{Z_0}\mathcal{F}$				×	×	×	×
$\iota^!_{Z_0}\mathcal{F}$	×	×	×	×			

Table 1. Sheaf verification for $\dim_{\mathbb{C}} X = 2$.