SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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Contents

This document is intended to collect the questions and doubts that arose during my research this year. For many of these problems, I have consulted my fellow students, my supervisor, and various other people I've met. However, most of them remain in the realm of folklore—problems that are likely known but for which I could not find a reference. On the other hand, some of the questions may not appear particularly interesting unless their underlying motivations are clearly explained. Therefore, I'll try to provide relevant background and outline some initial, perhaps naive, ideas while listing the problems along the way. Any responses, answers, or references are most welcome and will be added to keep this document updated.

1. Basic setting

For simplicity, we work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a reduced, separated scheme of finite type over $\mathbb{C}$. Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subseteq A$ be an irreducible closed subvariety of dimension r. We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.[1]

1.1. **Gauss map.** The goal of my research is to understand the geometry of Z, and the main tool for the subvariety geometry is the Gauss map. The Gauss map describe the tangent space information at each point:

$$\phi_Z : \mathrm{Z}^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longmapsto T_pZ \quad \subseteq T_pA \cong T_0A$$

Any map to the Grassmannian $\mathrm{Gr}(r,n)$ is induced by a rank r vector bundle together with n global sections. In this case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}$ and the sections

$$H^0(A, \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \twoheadrightarrow \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}.$$

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[1] I'm not sure whether we should consider the more general cases in the future—such as working over a field of characteristic p, letting A be a semiabelian variety or a complex torus, or allowing ι to be a covering onto its image. For now, I will omit these possibilities from this document.

1.2. **Conormal variety.** This concept may already be familiar to many readers, so we briefly recall the definition. On the smooth locus, the normal and conormal bundles behave well as vector bundles:[2]

$$\mathcal{N}_{Z^{\mathrm{sm}}/A} := \mathcal{T}_A|_{Z^{\mathrm{sm}}} \;/\; \mathcal{T}_{Z^{\mathrm{sm}}} \qquad \Lambda_{Z^{\mathrm{sm}}} := \mathcal{N}^*_{Z^{\mathrm{sm}}/A} = \ker\left(\Omega_A|_{Z^{\mathrm{sm}}} \to \Omega_{Z^{\mathrm{sm}}}\right).$$

The conormal variety Ω_Z is just the closure of $\Lambda_{Z^{\mathrm{sm}}}$ viewed as a subvariety in T^*A:

$$\Lambda_Z := \overline{\Lambda_{Z^{\mathrm{sm}}}} \subset T^*A \cong A \times T_0^*A$$

this is conical Lagrangian cycle in T^*A.

Moreover, the projectivized conormal variety

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A$$

is a Legendrian cycle in the contact variety $A \times \mathbb{P}T_0^*A$. $\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}$ is a $\mathbb{P}^{r-1}$-bundle, and the map

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A$$

is generically finite (i.e., clean) when Z is (an integral variety) of general type, see [1, Theorem 2.8 (1)].

A lot of geometry of Z is encoded in the map γ_Z. For instance, if Z is smooth and lies inside A, then

$$\deg \gamma_Z = (-1)^r \chi(Z)$$

tells us the Euler characteristic of Z.

Further insight can be gained by analyzing the fibers of γ_Z. These fibers, though finite, are not arbitrary—they obey hidden structural rules. For instance, if Z is preserved by a translation $t_v : A \longrightarrow A$, then each fiber $\gamma_Z^{-1}(\xi)$ is also invariant under t_v. Likewise, if $Z = -Z$, then the fibers satisfy $\gamma_Z^{-1}(\xi) = -\gamma_Z^{-1}(\xi)$. Outside these special configurations, it becomes more challenging to identify further constraints.[3]

An important invariant arising from the fiber $\gamma_Z^{-1}(\xi)$ is the monodromy group $\mathrm{Gal}(\gamma_Z)$; for completeness, we recall its definition below.

Definition 1.1. *Define*

$$U = \left\{\xi \in \mathbb{P}T_0^*A \,\middle|\, \#\gamma_Z^{-1}(\xi) = \deg \gamma_Z\right\}.$$

Moving along a loop in U induces a permutation of the points in the fiber $\gamma_Z^{-1}(\xi)$, which defines the map

$$\rho_{\gamma_Z} : \pi_1(U, \xi_0) \longrightarrow \mathrm{Aut}(\gamma_Z^{-1}(\xi)) = S_{\deg \gamma_Z}.$$

The monodromy group is then defined as the image of ρ, i.e.,

$$\mathrm{Gal}(\gamma_Z) := \mathrm{Im}\, \rho_{\gamma_Z}.$$

Question 1.2. *Suppose that the subvariety $Z \subset A$ is not stable under any translation on A. Are there known algorithms to compute the monodromy group $\mathrm{Gal}(\gamma_Z)$? Furthermore, what kinds of groups can appear as $\mathrm{Gal}(\gamma_Z)$ for suitable choices of $Z \subset A$?*

We will try to compute $\mathrm{Gal}(\gamma_Z)$ for a number of specific cases in Section 2. Three special cases are already treated in [2, Theorem 9], and we will generalize the strategies there.

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[2]This is more symmetric when writing them as short exact sequences:

$$0 \longrightarrow \mathcal{T}_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{T}_A|_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{N}_{Z^{\mathrm{sm}}/A} \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{Z^{\mathrm{sm}}} \longrightarrow \Omega_A|_{Z^{\mathrm{sm}}} \longrightarrow \Omega_{Z^{\mathrm{sm}}} \longrightarrow 0$$

[3]You can imagine the fiber $\gamma_Z^{-1}(\xi)$ as a cluster of stars projected onto a celestial dome. As ξ varies, these points shift, tracing out paths much like stars drifting across the night sky. The constraints that govern them are subtle, like the imagined lines that shape constellations. And in the long arc of variation, monodromy emerges—like the slow turning that replaces Kochab with Polaris among the stars.

1.3. **Interpolation via hyperplanes.** Before delving into examples, we reinterpret γ_Z using a functorial and more transparent framework, enabling a decomposition of Question 1.2 into two primary subquestions.

Recognizing that each non-zero conormal vector $\xi \in T_0^*A$ determines a hyperplane $H_\xi \in \mathrm{Gr}(n-1, T_0A)$, we establish the isomorphisms

$$\begin{aligned}\mathbb{P}T_0^*A &\cong \mathrm{Gr}(n-1, T_0A) \mathrel{\hat{=}} \left(\mathbb{P}^{n-1}\right)^\vee, \\ \mathbb{P}\Lambda_{\mathrm{Z}^{\mathrm{sm}}} &= \left\{ (p,\xi) \in \mathrm{Z}^{\mathrm{sm}} \times \mathbb{P}T_0^*A \;\middle|\; \xi|_{T_pZ} \equiv 0 \right\} \\ &\cong \left\{ (p,H) \in \mathrm{Z}^{\mathrm{sm}} \times \mathrm{Gr}(n-1,n) \;\middle|\; \phi_Z(p) \subseteq H \right\} \\ &\cong (\phi_Z, \mathrm{Id})^{-1} I_{r,n-1},\end{aligned}$$

where

$$I_{r,n-1} := \left\{ (V,H) \in \mathrm{Gr}(r,n) \times \mathrm{Gr}(n-1,n) \;\middle|\; V \subseteq H \right\}$$

is the incidence variety relating $\mathrm{Gr}(r,n)$ and $\mathrm{Gr}(n-1,n)$. In that case,

$$\begin{aligned}\gamma_Z^{-1}(H) \cap \mathrm{Z}^{\mathrm{sm}} &= \{ p \in \mathrm{Z}^{\mathrm{sm}} \mid \phi_Z(p) \subseteq H \} \\ &\cong \phi_Z^{-1}\left(\mathrm{Gr}(r,H)\right)\end{aligned}$$

is the collection of points whose tangent spaces lie entirely within H.

Geometrically, the monodromy can be described as follows: given a hyperplane H, its preimage consists of d points $p_1, \dots, p_d$. Moving H continuously along a loop causes these points to permute, and the monodromy group $\mathrm{Gal}(\gamma_Z)$ consists of all permutations obtained this way.

2. Searching for examples

In this section, we discuss examples drawn from my ongoing work, focusing on the construction of subvarieties and the computation of their monodromy groups.

3. Families of subvarieties

4. Tannakian formalism

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\mathrm{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in $\mathbb{Q}$. For any algebraic group G, we denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G.

Following the approach of [3], we work in the quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/N(A)$, where $N(A) \subset \mathrm{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex $\mathcal{F}$ is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [3, Thm 7.1 & Cor 9.2]). In particular, for any perverse sheaf $\delta \in \overline{\mathrm{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G:

$$\langle \delta, * \rangle \cong \mathrm{Rep}(G).$$

References

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[3] Thomas Krämer and Rainer Weissauer. Vanishing theorems for constructible sheaves on abelian varieties. *J. Algebr. Geom.*, 24(3):531–568, 2015.

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