SCHUR-HORN THEOREM

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Abstract. In this article, I will use the Atiyah-Guillemin-Sternberg Convexity theorem to prove the Schur-Horn theorem, which is a beautiful theorem in linear algebra, with deep symplectic geometry theory behind it. To introduce the AGM theorem, we first grasp the tools: the Lie bracket and the Exponential map; then we will focus on the vector field induced by the group action $\mathbb{T}^n \circlearrowleft \mathcal{H}_\lambda$, and use the symplectic structure on $\mathcal{H}_\lambda$ to convert the vector field to an exact 1-form, and then natually introduce the moment map on $\mathcal{H}_\lambda$. After that, we will state the AGM theorem and prove the Schur-Horn theorem.

1. Introduction

Given a Hermitian matrix $A = (a_{ij}) \in \mathbb{C}^n$ with eigenvalues

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}^n$$

We want to see:

Question:What do the diagonal elements

$$(a_{11}, a_{22}, \ldots, a_{nn})$$

look like?

Facts (Obvious)**.**

- $A^H = A \Rightarrow a_{11}, a_{22}, \ldots, a_{nn} \in \mathbb{R}$
- A *is* ***unitary similar*** *to* $\operatorname{diag}(\lambda_1, \ldots, \lambda_n)$
  $\Rightarrow \sum_{i=1}^n a_{ii} = \operatorname{tr} A = tr(\operatorname{diag}(\lambda_1, \ldots, \lambda_n)) = \sum_{i=1}^n \lambda_i$
- $\forall\, \tau \in S_n, \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *is unitary similar to* $\operatorname{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)})$
  $\Rightarrow$ *WLOG, we can rearrange* $(\lambda_1, \ldots, \lambda_n)$ *s.t.*
  $$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$$
  NOTICE:After that we will assume $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$.

Facts (Not Obvious)**.**

- $\forall\, i \in \{1, \ldots, n\}, \lambda_n \leqslant a_{ii} \leqslant \lambda_1$
- $\forall\, k \in \{1, \ldots, n\}, \sum_{i=1}^k a_{ii} \leqslant \sum_{i=1}^k \lambda_i$ [*]

Denote

$$\begin{aligned}
&\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}^n \\
&\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)^T \in \mathbb{R}^{n\times n} \\
&\mathcal{H}(n) = \{A \in \mathbb{C}^{n\times n} \mid A^H = A\} \\
&\mathcal{H}_\lambda = \{A \in \mathcal{H}(n) \mid A \text{ is unitary similar to } \Lambda\}
\end{aligned}$$

[*]Issai Schur (Russian,1875-1941) proved the above-mentioned inequalities in 1923.

$$\begin{array}{rccc} \pi: & \mathcal{H}(n) & \longrightarrow & \mathbb{R}^n \\ & A=(a_{ij})_{i,j=1}^n & \mapsto & (a_{11},a_{22},\ldots,a_{nn})^T \end{array}$$

Theorem 1.1 (Schur-Horn). *The image $\pi(\mathcal{H}_\lambda)$ is a* ***convex polyhedron*** *in $\mathbb{R}^n$ whose vertices are*

$$(\lambda_{\tau(1)},\ldots,\lambda_{\tau(n)})^T \in \mathbb{R}^n$$

where $\tau \in S_n$.†

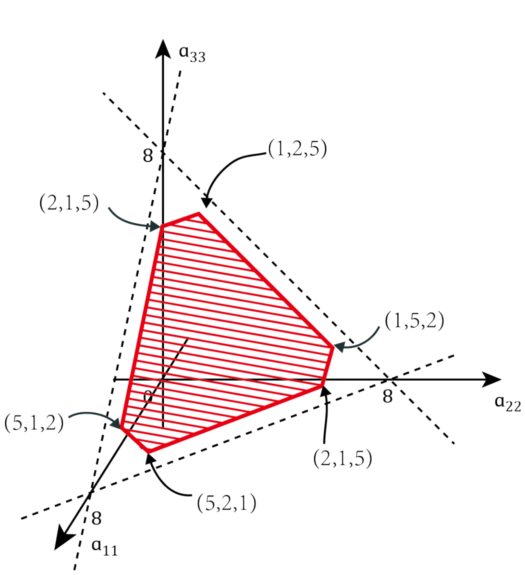
With these facts in mind, we will first discuss some examples.

Example 1.2 (Trivial). *When $\boldsymbol{\lambda}=(\lambda_0,\lambda_0,\ldots,\lambda_0)^T$, we have*

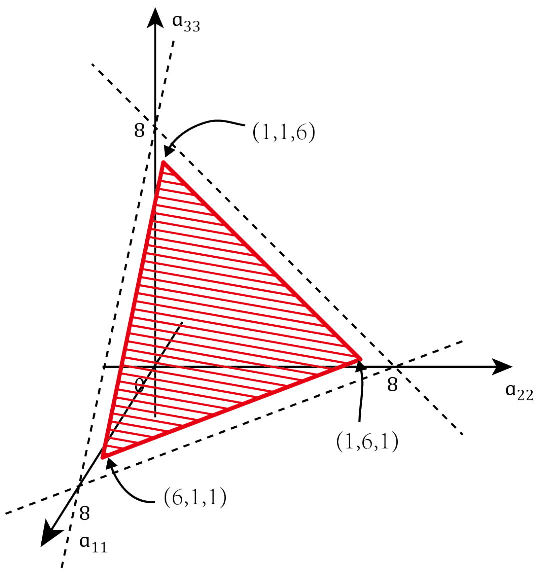
$$\begin{aligned} \Lambda &= \lambda_0 I \\ \mathcal{H}_\lambda &= \{A\in\mathbb{C}^{n\times n} \mid \exists\, U\in U(n), A=U(\lambda_0 I)U^H=\lambda_0 I\} \\ &= \{\lambda_0 I\} \qquad\qquad \textit{has only one element!} \end{aligned}$$

We leave 2-dimension example at last because it's computable.

Example 1.3 (3-dimension condition). *When $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\lambda_3)^T$, it's almost impossible to calculate, so we only draw out the final result:*



$(\lambda_1,\lambda_2,\lambda_3)^T=(5,2,1)^T$



$(\lambda_1,\lambda_2,\lambda_3)^T=(6,1,1)^T$
"degenerate"

Example 1.4 (2-dimension condition). *We have*

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} \in \mathcal{H}_\lambda \Leftrightarrow \exists\, U=\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22}\end{pmatrix} \in U(2),$$

†Alfred Horn (Amerian, UCLA) proved it in 1954.

$$
\begin{aligned}
A &= U\Lambda U^H \\
&= \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} \begin{pmatrix} \overline{u_{11}} & \overline{u_{21}} \\ \overline{u_{12}} & \overline{u_{22}} \end{pmatrix} \\
&= \begin{pmatrix} \lambda_1|u_{11}|^2 + \lambda_2|u_{12}|^2 & \lambda_1 u_{11}\overline{u_{21}} + \lambda_2 u_{12}\overline{u_{21}} \\ \lambda_1 u_{21}\overline{u_{11}} + \lambda_2 u_{22}\overline{u_{12}} & \lambda_1|u_{21}|^2 + \lambda_2|u_{22}|^2 \end{pmatrix} \\
&= \lambda_2 I + (\lambda_1 - \lambda_2) \begin{pmatrix} |u_{11}|^2 & u_{11}\overline{u_{21}} \\ \lambda_1 u_{21}\overline{u_{11}} & \lambda_1|u_{21}|^2 \end{pmatrix}
\end{aligned}
$$

Denote the line segment drawed in the figure 1 as Γ*, then*

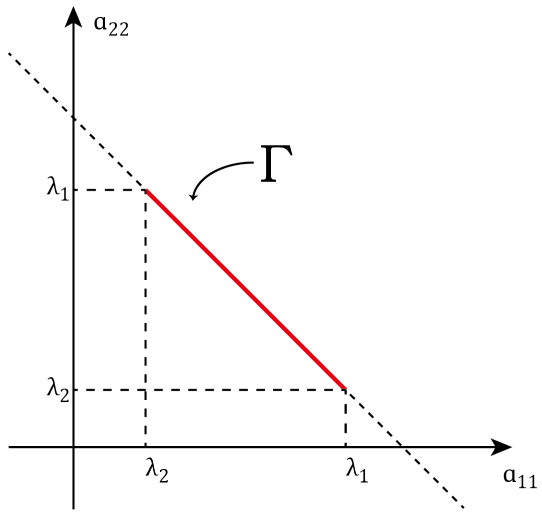


FIGURE 1

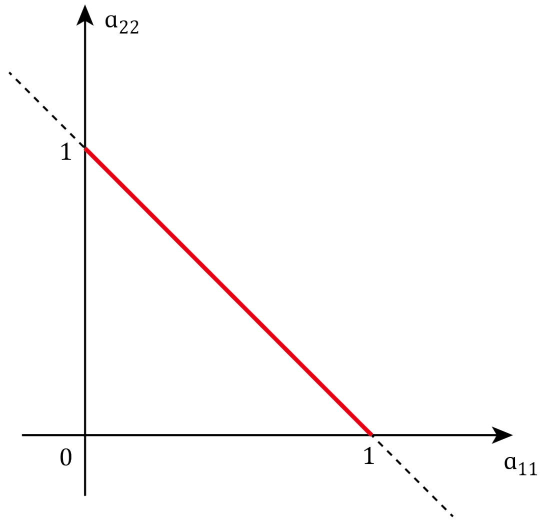


FIGURE 2

- $\pi(\mathcal{H}_\lambda) \subseteq \Gamma$ *because* $\lambda_1|u_{11}|^2 + \lambda_2|u_{12}|^2$ *is the convex combination of* λ_1, λ_2.
- $\Gamma \subseteq \pi(\mathcal{H}_\lambda)$ *because we can take*

$$\begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

.

Actually one can compute more:
WLOG(or take the coordinate trasformation), we only consider the condition when

- $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)^T = (1, 0)^T$
- $A = \begin{pmatrix} |u_{11}|^2 & u_{11}\overline{u_{21}} \\ \lambda_1 u_{21}\overline{u_{11}} & \lambda_1|u_{21}|^2 \end{pmatrix}$.

Now we can calculate out

$$\mathcal{H}_\lambda = \left\{ \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} \middle| a \in [0,1], 0 \leqslant \varphi < 2\pi \right\}$$

Now we know explicitly

$$\pi(\mathcal{H}_\lambda) = \{(a, 1-a) \mid 0 \leqslant a \leqslant 1\}$$

Moreover, $\pi(\mathcal{H}_\lambda)$ is a manifold diffeomorphic to S^2:

$$\begin{array}{cccc} \Phi: & \mathcal{H}_\lambda & \longrightarrow & S^2 \\ & \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} & \mapsto & (\varphi, a) \end{array}$$

Remark 1.5. What is a manifold? Manifold is a VERY GOOD geometric object which always look like R^n.

We will find out more information through this isomorphism.

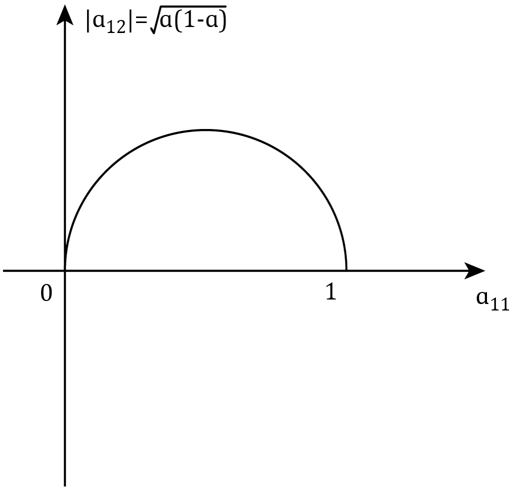


FIGURE 3

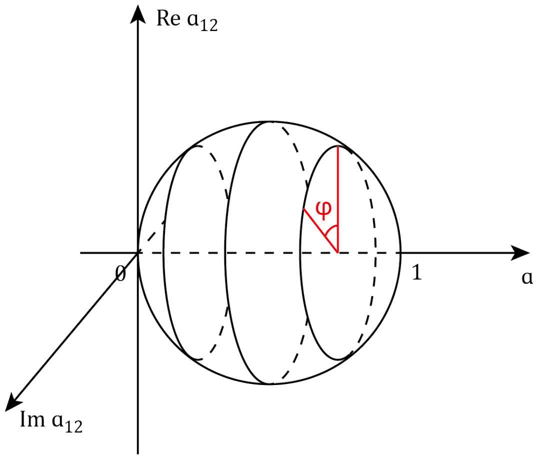


FIGURE 4

2. Simple Tools

Let us deriate from the phenomenon for a while to obtain the most basic tools:the **Lie bracket** and the **Exponential map**.

Lie bracket.

Definition 2.1. *the Lie bracket of $M_n(\mathbb{C})$ is*

$$\begin{array}{rccc} [\ \ ,\ \ ]: & M_n(\mathbb{C}) \times M_n(\mathbb{C}) & \longrightarrow & \mathbb{C} \\ & (A, B) & \mapsto & [A, B] := AB - BA \end{array}$$

Proposition 2.2. *For any $c_1, c_2 \in \mathbb{C}, A, A_1, A_2, B, C \in M_n(\mathbb{C})$, we have the following properties:*

- *(Skew-Symmetric)* $[A, B] = -[B, A]$;
- *(Linear)* $[c_1A_1 + c_2A_2, B] = c_1[A_1, B] + c_2[A_2, B]$;
- *(Jacobi-Identity)* $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$;
- $[A, B]^H = [B^H A^H]$;
- $\mathrm{tr}(A[B, C]) = \mathrm{tr}([A, B]C)$

Proof: Exercise.

Exponential map.

Definition 2.3 (The Exponentinal map for Matrix)**.** *Suppose $A \in M_n(\mathbb{C})$, then we define*

$$P_n(A) := \sum_{i=0}^{n} \frac{A^i}{i!}$$

$$\exp(A) := e^A := \lim_{n\to\infty} P_n(A)$$

Remark 2.4. about the definition

- By defining the norm on $M_n(\mathbb{C})$, one is easy to find out the existence and uniqueness of the definition.
- Generally $e^A e^B \neq e^{A+B}$. But we still have
$$AB = BA \Rightarrow e^A e^B = e^{A+B}$$
.
- Like polynomials, some properties are easily derived from the definition:
  - $\forall\, U \in U(n), Ue^X U^H = e^{UXU^H}$
  - $(e^X)^H = e^{X^H}$
  - $\frac{d}{dt}e^{tX} = Xe^{tX}$; especially $\frac{d}{dt}\big|_{t=0}e^{tX} = X$
- Sometimes we denote $\exp(X) = e^X$ to enlarge superscript.
- Someone may think the Exponential map as "walking along the vector field Xe^{tX} (in $GL_n(\mathbb{C})$) for t times". You can easily check (if you've learned about the Differential Manifold) that $\exp(tX)$ is just an integral curve $\gamma_X(t)$ in $GL_n(\mathbb{C})$.

3. Group Actions

3.1. **Group action on $\mathcal{H}(n)$.** We have **VERY NICE** group actron on $\mathcal{H}(n)$:

$$U(n) \circlearrowright \mathcal{H}(n)$$

$$U \cdot H = UHU^H$$

Remark 3.1. One can easily check that this is really the group action:

- $UHU^H \in \mathcal{H}(n)$
- $I \cdot H = H$
- $(U_1U_2) \cdot H = U_1 \cdot (U_2 \cdot H)$

Question: What is the orbit of this action?
Answer: From the linear algebra theory,

$$A \in \mathcal{H}_\lambda \Leftrightarrow \exists\, U \in U(n), A = U\Lambda U^H$$

As a result,

Proposition 3.2. *The orbit of the group action is*

$$\mathcal{H}_\lambda = \{A \in \mathcal{H}(n) \mid A \text{ has eigenvalues } \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)^T\}$$

Moreover

- $\mathcal{H}(n)$ is a $\mathbb{R}$-linear space, thus naturally a manifold

- $U(n)$ is a Lie group

So from the Lie group's theory we can obtain

Proposition 3.3. *$\mathcal{H}_\lambda$ is a manifold.*

This is not so surprising because we have calculated the $\mathcal{H}_{(1,0)^T}$ and "verified" that this is a manifold diffeomorphic to S^2. Later we will see more structures on $\mathcal{H}_\lambda$, and these structures in all will help us to find out more informations about $\pi(\mathcal{H}_\lambda)$.

3.2. **Subgroup actions.** We have found

$$\begin{aligned}
S^1 &= \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} : \theta \in \mathbb{R} \right\} \subseteq U(n) \\
\mathbb{T}^n &= S^1 \times S^1 \times \cdots \times S^1 \\
&= \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \theta_1, \ldots, \theta_n \in \mathbb{R} \right\} \subseteq U(n)
\end{aligned}$$

Then $S^1 \subseteq \mathbb{T}^n \subseteq U(n)$.

We have the induced subgroup actions:

$$S^1 \circlearrowleft \mathcal{H}_\lambda$$
$$A \cdot H = AHA^H$$
$$\theta \cdot H = \begin{pmatrix} e^{i\theta} & \\ & I_{n-1} \end{pmatrix} H \begin{pmatrix} e^{-i\theta} & \\ & I_{n-1} \end{pmatrix}$$

$$\mathbb{T}^n \circlearrowleft \mathcal{H}_\lambda$$
$$A \cdot H = AHA^H$$
$$\theta \cdot H = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} H \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix}$$

We may split the matrix H into 4 different parts:

$$H = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array} \right)$$

Then

$$\begin{aligned}
\theta \cdot H &= \begin{pmatrix} e^{i\theta} & \\ & I_{n-1} \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & I_{n-1} \end{pmatrix} = \begin{pmatrix} H_{11} & e^{i\theta} H_{12} \\ e^{-i\theta} H_{21} & H_{22} \end{pmatrix} \\
\frac{d}{d\theta}(\theta \cdot H) &= \begin{pmatrix} 0 & ie^{i\theta} H_{12} \\ -ie^{-i\theta} H_{21} & 0 \end{pmatrix}
\end{aligned}$$

Remark 3.4. Notice that the group action $S^1 \circlearrowleft \mathcal{H}_\lambda$ doesn't change the diagonal components.Similarly, one can easily verify that the group action $\mathbb{T}^n \circlearrowleft \mathcal{H}_\lambda$ also keeps the diagonal components. Thus we may think "the group actions decrease the other unrelated degree of freedom", and thus "gives the invariance" of $\mathcal{H}_\lambda$.

3.3. **The induced vector field of group action.**

Definition 3.5. *Suppose $j \in \{1,\dots,n\}$ the group $\mathbb{T}^n$ acts on $\mathcal{H}_\lambda$, then the induced vector field X_j at $H \in \mathcal{H}_\lambda$ is the matrix*

$$X_j(H) = \frac{d}{dt}\bigg|_{t=0} ((0,\dots,t,\dots,0)\cdot H)$$

Example 3.6. *We have computed*

$$X_1(H) = \frac{d}{dt}\bigg|_{t=0} ((t,0,\dots,0)\cdot H) = \begin{pmatrix} & iH_{12} \\ -iH_{21} & \end{pmatrix}$$

Similarly, if $H = (h_{ij})_{i,j=1}^n$, then

$$X_j(H) = \begin{pmatrix} & & ih_{1j} & & \\ & & \vdots & & \\ -ih_{j1} & \cdots & 0 & \cdots & -h_{jn} \\ & & \vdots & & \\ & & ih_{nj} & & \end{pmatrix}$$

Example 3.7. *When $n=2$, $H = \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix}$,*

$$\begin{aligned} X_1(H) &= \begin{pmatrix} 0 & ih_{12} \\ -ih_{21} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ie^{i\varphi}\sqrt{a(1-a)} \\ -ie^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix} \end{aligned}$$

Notice that

$$\frac{\partial H}{\partial \varphi} = \begin{pmatrix} 0 & ie^{i\varphi}\sqrt{a(1-a)} \\ -ie^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix}$$

4. New Tools

4.1. **Symplectic manifold.** Roughly speaking, the symplectic manifold is the manifold with a 2-form which locally looks like $\sum_{i=1}^n dx^i \wedge dy^i$.

Now suppose M is a manifold of dimension 2n.

Definition 4.1. *A symplectic form on M is a 2-form $w \in \Lambda^2 T^*M$ on M such that*

- *w is closed: $dw = 0$.*
- *w is non-degenerate: $w \wedge w \wedge \cdots \wedge w \neq 0$ is a volume form on M.*

The pair (M, w) is called a ***symplectic manifold****.*

Remark 4.2. Compared with Riemann metric g:

- g can be defined on any manifold, while w can't (dimension =2n, orientable, and so on).
- g is symmetric while w is skew-symmetric.
- By Darboux theorem, w looks like $\sum_{i=1}^n dx^i \wedge dy^i$ near any $p \in M$, while g has plenty of local geometric structrues (such as curvature and connection)

- g_p gives an isomorphism
$$g_p^{\#}: T_pM \longrightarrow T_p^*M$$
$$X_p \mapsto g_p(X_p, -)$$
While w also gives an isomorphism
$$w_p^{\#}: T_pM \longrightarrow T_p^*M$$
$$X_p \mapsto w_p(X_p, -)$$
We will use this isomorphism to convert a vector field (which I have mentioned, induced by group action) to an exact 1-form.

Example 4.3. *$(\mathbb{R}^{2n}, w)$ is a symplectic manifold with chart coordinate $(x_1, \dots, x_n, y_1, \dots, y_n)$*
$$w = \sum_{i=1}^{n} dx^i \wedge dy^i$$
Verify:

- $w \in \Lambda^2 T^*M$
- $dw = \sum_{i=1}^{n} d1 \wedge dx^i \wedge dy^i = 0$
- $w \wedge w \wedge \cdots \wedge w = n!\, dx^1 \wedge dy^1 \wedge \cdots dx^n \wedge dy^n \neq 0$

Example 4.4. *(S^2, w) is a symplectic manifold where w is the canonical volume form of S^2. in $S^2 \setminus \{North, South\}$, $d\theta \wedge dh$ is the local representation of w.*

Verify:

- $w \in \Lambda^2 T^*M$
- $dw = 0$ *because w is a top form.*
- w *is no-degenerate since it is already a volume form.*

Example 4.5. *From the diffeomorphism*
$$\begin{array}{cccc} \Phi: & \mathcal{H}_{(1,0)^T} & \longrightarrow & S^2 \\ H(a,\varphi) \hat{=} & \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} & \mapsto & (\varphi, a) \end{array}$$
one can obtain a natural symplectic form on $\mathcal{H}_{(1,0)^T}$:
$$\begin{array}{cccc} \Phi^*: & \Omega^2(S^2) & \longrightarrow & \Omega^2(\mathcal{H}_{(1,0)^T}) \\ & w = d\theta \wedge dh & \mapsto & w_{can} \end{array}$$
We can calculate $(a \neq 0, 1)$
$$(d\Phi)^{-1}(\frac{\partial}{\partial\theta}) = \begin{pmatrix} 0 & ie^{i\varphi}\sqrt{a(1-a)} \\ -ie^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix} \hat{=} \frac{\partial}{\partial\phi} = X_1(H(a,\phi))$$
$$(d\Phi)^{-1}(\frac{\partial}{\partial h}) = \begin{pmatrix} 1 & e^{i\varphi}\dfrac{1-2a}{2\sqrt{a(1-a)}} \\ e^{-i\varphi}\dfrac{1-2a}{2\sqrt{a(1-a)}} & -1 \end{pmatrix} \hat{=} \frac{\partial}{\partial a}$$
$$T_{(e^{i\varphi},a)}S^2 = \left\langle \frac{\partial}{\partial\theta}\bigg|_{(e^{i\varphi},a)}, \frac{\partial}{\partial h}\bigg|_{(e^{i\varphi},a)} \right\rangle_{span} \Rightarrow T_{H(a,\varphi)}\mathcal{H}_{(1,0)^T} = \left\langle \frac{\partial}{\partial\phi}\bigg|_{H(a,\varphi)}, \frac{\partial}{\partial a}\bigg|_{H(a,\varphi)} \right\rangle_{span}$$

$$1 = w(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial h}) = w_{can}(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial a}) = w_{can}^{\#}(\frac{\partial}{\partial\phi})(\frac{\partial}{\partial a}) = w_{can}^{\#}(X_1)(\frac{\partial}{\partial a}) \Rightarrow w_{can}^{\#}(X_1) = da$$

Remark 4.6. In general $\mathcal{H}_\lambda$ is also a symplectic manifold whose symplectic form can be written as (if $H = U\Lambda U^H, X = A\Lambda U^H + U\Lambda A^H, Y = B\Lambda U^H + U\Lambda B^H$)

$$w_\lambda\big|_H(X,Y) = i\,\mathrm{tr}(\Lambda[U^H A, U^H B])$$

Moreover, $w_\lambda^{\#}(X_i)$ is exact, i,e

$$\exists\, f \in C^\infty(\mathcal{H}_\lambda) \text{ such that } w_\lambda^{\#}(X_i) = df$$

This function f will bw denoted "**the moment map**".

We will verify that when $\boldsymbol{\lambda} = (1,0)^T$, this symplectic structure defined coincide with w_{can} we've encountered. This is shown as follows:

$$\begin{aligned}
H(a,\varphi) &= \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} \xlongequal[0<\theta<\pi/2]{a=\cos^2\theta} \begin{pmatrix} \cos^2\theta & e^{i\varphi}\sin\theta\cos\theta \\ e^{-i\varphi}\sin\theta\cos\theta & \sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & -e^{i\varphi}\sin\theta \\ e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & \\ & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \\
&\Rightarrow U = \begin{pmatrix} \cos\theta & -e^{i\varphi}\sin\theta \\ e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \qquad U^H = \begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial H(a,\varphi)}{\partial\varphi} &= \frac{\partial(U\Lambda U^H)}{\partial\varphi} = \frac{\partial U}{\partial\varphi}\Lambda U^H + U\Lambda\left(\frac{\partial U}{\partial\varphi}\right)^H \\
\Rightarrow A = \frac{\partial U}{\partial\varphi} &= \begin{pmatrix} 0 & -ie^{i\varphi}\sin\theta \\ -ie^{-i\varphi}\sin\theta & 0 \end{pmatrix} \\
\Rightarrow U^H A &= \begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 & -ie^{i\varphi}\sin\theta \\ -ie^{-i\varphi}\sin\theta & 0 \end{pmatrix} \\
&= -i\sin\theta \begin{pmatrix} \sin\theta & -e^{i\varphi}\cos\theta \\ e^{-i\varphi}\cos\theta & -\sin\theta \end{pmatrix} \\
\frac{\partial H(a,\varphi)}{\partial a} &= \frac{\partial(U\Lambda U^H)}{\partial a} = \frac{\partial U}{\partial a}\Lambda U^H + U\Lambda\left(\frac{\partial U}{\partial a}\right)^H \\
\Rightarrow B = \frac{\partial U}{\partial a} &= \frac{1}{2\cos\theta\sin\theta}\frac{\partial U}{\partial\theta} = \frac{1}{2\cos\theta\sin\theta}\begin{pmatrix} -\sin\theta & -e^{i\varphi}\cos\theta \\ e^{-i\varphi}\cos\theta & -\sin\theta \end{pmatrix} \\
\Rightarrow U^H B &= \frac{1}{2\cos\theta\sin\theta}\begin{pmatrix} \cos\theta & e^{i\varphi}\sin\theta \\ -e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} -\sin\theta & -e^{i\varphi}\cos\theta \\ e^{-i\varphi}\cos\theta & -\sin\theta \end{pmatrix} \\
&= \frac{1}{2\cos\theta\sin\theta}\begin{pmatrix} & -e^{i\varphi} \\ -e^{-i\varphi} & \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
[U^H A, U^H B] =& -\frac{i}{2\cos\theta}\left[\begin{pmatrix}\sin\theta & -e^{i\varphi}\cos\theta\\ e^{-i\varphi}\cos\theta & -\sin\theta\end{pmatrix},\begin{pmatrix} & -e^{i\varphi}\\ -e^{-i\varphi} & \end{pmatrix}\right]\\
=& -\frac{i}{2\cos\theta}\left\{\begin{pmatrix}\cos\theta & -e^{i\varphi}\sin\theta\\ e^{-i\varphi}\sin\theta & \cos\theta\end{pmatrix}-\begin{pmatrix}-\cos\theta & e^{i\varphi}\sin\theta\\ -e^{-i\varphi}\sin\theta & -\cos\theta\end{pmatrix}\right\}\\
=& -\frac{i}{\cos\theta}\begin{pmatrix}\cos\theta & -e^{i\varphi}\sin\theta\\ e^{-i\varphi}\sin\theta & \cos\theta\end{pmatrix}\\
w_\lambda\big|_H(X,Y) =& i\,\mathrm{tr}(\Lambda[U^H A, U^H B])\\
=& \frac{1}{\cos\theta}\,\mathrm{tr}\left(\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\cos\theta & -e^{i\varphi}\sin\theta\\ e^{-i\varphi}\sin\theta & \cos\theta\end{pmatrix}\right)\\
=& 1
\end{aligned}
$$

4.2. **Moment Map.**

Definition 4.7. *Suppose $S^1 \circlearrowleft \mathcal{H}_\lambda$,then the moment map is a map*

$$\mu : \mathcal{H}_\lambda \longrightarrow \mathbb{R}$$

such that $w_{can}^{\#}(X_1) = d\mu$.

From Example 4.5 we can see, the moment map of $S^1 \circlearrowleft \mathcal{H}_{(1,0)^T}$ is

$$
\begin{aligned}
\mu: \quad & \mathcal{H}_\lambda \longrightarrow \mathbb{R}\\
& \begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix} \mapsto a_{11}
\end{aligned}
$$

Definition 4.8. *Suppose $\mathbb{T}^n \circlearrowleft \mathcal{H}_\lambda$,then the moment map is a map*

$$
\begin{aligned}
\mu\colon \mathcal{H}_\lambda \quad & \longrightarrow \quad \mathbb{R}^n\\
A \quad & \mapsto (\mu_1(A), \ldots, \mu_n(A))^T
\end{aligned}
$$

such that for any $i \in \{1, \ldots, n\}, w_{can}^{\#}(X_i) = d\mu_i$.

Remark 4.9. Like the examples we have seen, in general, if $\mathbb{T}^n \circlearrowleft \mathcal{H}_\lambda$ in a canonical way, then

$$
\begin{aligned}
\mu = \pi\colon \quad \mathcal{H}_\lambda \quad & \longrightarrow \quad \mathbb{R}^n\\
A = (a_{ij})_{i,j=1}^n & \mapsto (a_{11}, \ldots, a_{nn})^T
\end{aligned}
$$

is just the projection to its diagonal components! Its proof require the knowledge of coadjoint orbit, so I regret that I'll skip it.

Definition 4.10. *We will call $(\mathcal{H}_\lambda, w_\lambda, \mathbb{T}^r, \mu)$ as the **Hamiltonian** $\mathbb{T}^r$**-manifold**.*

5. Proof of the Schur-Horn Theorem

After we've introduced all conceptions, we state the last theorem which is ingenious formally but its proof need deep symplectic geometry knowledge.

Theorem 5.1 (Atiyah-Guillemin-Sternberg convexity theorem)**.** *Suppose $(\mathcal{H}_\lambda, w_\lambda, \mathbb{T}^r, \mu)$ be a Hamiltonian $\mathbb{T}^r$-manifold.If M is compact and connected, then*

$\boxed{\mathcal{H}_\lambda \textit{ is a convex polyhedron in } \mathbb{R}^n \textit{ whose vertices are the images of the } \mathbb{T}^r\textit{-fixed points.}}$

Proof of Schur-Horn theorem:

- $(\mathcal{H}_\lambda, w_\lambda, \mathbb{T}^n, \mu)$ be a Hamiltonian $\mathbb{T}^n$-manifold.
- $\mathcal{H}_\lambda$ is compact:
  - $\mathcal{H}_\lambda$ is bounded by λ_1;
  - $\mathcal{H}_\lambda$ is closed. You can see $\mathcal{H}_\lambda$ as the zero set of some algebraic functions on $\mathcal{H}(n)$, or you can realize it as the orbit of the compact Lie groups $U(n)$,thus by the theory of Lie group's theory a closed set in $\mathcal{H}(n)$.
- $\mathcal{H}_\lambda$ is connected:for any $A \in \mathcal{H}_\lambda$, there exists $U \in U(n)$ such that $A = U\Lambda U^H$.

$$\begin{aligned} & U(n) \text{ is connected} \\ \Rightarrow & \text{ there exists } U_t : [0,1] \to U(n) \text{ such that } U_0 = I, U_1 = U \\ \Rightarrow & \text{ there exists } A_t := U_t\Lambda U_t^H : [0,1] \to \mathcal{H}_\lambda \text{ such that } A_0 = \Lambda, A_1 = A \\ \Rightarrow & \mathcal{H}_\lambda \text{ is connected} \end{aligned}$$

$\rightsquigarrow \pi(\mathcal{H}_\lambda)$ is a convex polyhedron in $\mathbb{R}^n$.

- For the $\mathbb{T}^n$-fixed points, we will find that they're just

$$\mathrm{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)}) \in \mathbb{R}^{n\times n} \qquad \text{where } \tau \in S_n$$

  Now suppose $A = (a_{ij})_{i,j=1}^n \in \mathcal{H}_\lambda$.
  - If $(\theta_1, \ldots, \theta_n) \cdot A = A$ for any $(\theta_1, \ldots, \theta_n) \in \mathbb{R}$, then

$$\Rightarrow \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\theta_n} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} e^{i\theta_1}a_{11} & \cdots & e^{i\theta_1}a_{1n} \\ \vdots & \ddots & \vdots \\ e^{i\theta_n}a_{n1} & \cdots & e^{i\theta_n}a_{nn} \end{pmatrix} \begin{pmatrix} e^{i\theta_1}a_{11} & \cdots & e^{i\theta_n}a_{1n} \\ \vdots & \ddots & \vdots \\ e^{i\theta_1}a_{n1} & \cdots & e^{i\theta_n}a_{nn} \end{pmatrix}$$

$$\begin{aligned} &\Rightarrow a_{ij} = 0 \text{ for any } i \neq j, \qquad A = \mathrm{diag}(a_{11}, \ldots, a_{nn}) \\ &\Rightarrow \mathrm{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)}) \qquad \text{where } \tau \in S_n \end{aligned}$$

  - On the other hand, if $A = \mathrm{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)})$ where $\tau \in S_n$, then

$$(\theta_1, \ldots, \theta_n) \cdot A = A \qquad \text{for any } (\theta_1, \ldots, \theta_n) \in \mathbb{R}$$

  - In a word, all the $\mathbb{T}^n$-fixed points are

$$\begin{aligned} \mathbb{T}^n_{fix} &= \{\mathrm{diag}(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)}) \in \mathcal{H}_\lambda \mid \tau \in S_n\} \\ \Rightarrow \pi(\mathbb{T}^n_{fix}) &= \{(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)})^T \in \mathbb{R}^n \mid \tau \in S_n\} \end{aligned}$$

Thus by the AGM-convexity theorem,
$\pi(\mathcal{H}_\lambda)$ is a **convex polyhedron** in $\mathbb{R}^n$ whose **vertices** are

$$(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)})^T \in \mathbb{R}^n$$

where $\tau \in S_n$.

6. Miscellaneous

Using deeper results in symplectic geometry, one is able to prove more results in linear algebra. Take one for example:

Denote the principal $k \times k$ minor of a matrix $A \in \mathcal{H}(n+1)$, denote the eigenvalues of A_k by $\mu_{1k}, \mu_{2k}, \ldots, \mu_{kk}$, and assume that they are arranged in decreasing order: $\mu_{1k} \geqslant \mu_{2k} \geqslant \cdots \geqslant \mu_{kk}$.

Theorem 6.1 (Gelfand-Cetlin,[3]). *The μ_{ik}'s satisfy the interlacing conditions. Moreover, for every sequence of μ_{ik}'s satisfying these interlacing conditions*

$$\begin{array}{ccccccccc}
\lambda_1 & & & & \lambda_2 & & & & \lambda_3 \ \cdots\cdots\cdots\cdots\cdots \ \lambda_{n+1} \\
& \geqslant & & \leqslant & & \geqslant & & \leqslant & \cdots\cdots\cdots\cdots\cdots \leqslant \\
& & \mu_{1n} & & & & \mu_{2n} & & \cdots\cdots\cdots\cdots\cdots \ \mu_{nn} \\
& & & \geqslant & & \leqslant & & & \cdots\cdots\cdots\cdots\cdots \leqslant \\
& & & & \mu_{1(n-1)} & & & & \cdots\cdots\cdots \ \mu_{(n-1)(n-1)} \\
& & & & & & \cdots\cdots\cdots & & \\
& & & & & & \cdots\cdots & & \\
& & & & & & \mu_{1k} \cdots\cdots \mu_{kk} & &
\end{array}$$

there exists a matrix $A \in \mathcal{H}_\lambda$, for which the eigenvalues of its k-th principal minor are $\mu_{1k}, \mu_{2k}, \ldots, \mu_{kk}$.

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References

[1] Zuoqin Wang. "Introduction to Symplectic Geometry". 2015. url:http://staff.ustc.edu.cn/∼wangzuoq/Courses/15S-Symp/SympGeom.html

[2] Zuoqin Wang: Geometry of Moment Maps. 2016.

[3] V. Guillemin, R. Sjamaar: Convexity properties of Hamiltonian group actions. 2005. url:http://pi.math.cornell.edu/∼sjamaar/papers/polytope.ps

[4] V. Guillemin and S. Sternberg: Convexity Properties of the Moment Mapping. Inventiones Mathematicae 67, pp. 491-513 (1982)

[5] Ana Rita Pires: Convexity in Symplectic Geometry: The Atiyah-Guillemin-Sternberg Theorem. 2007. url:http://faculty.fordham.edu/apissarrapires/convexitytalk.pdf

[6] A. Cannas da Silva, Lecture Notes on Symplectic Geometry. Lecture Notes in Mathematics 1764, Springer Verlag, Berlin (2001)

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