Subvarieties in Abelian Variety

Xiaoxiang Zhou

Supervisor: Thomas Krämer

Humboldt-Universität zu Berlin

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Setting

- $A/\mathbb{C}$: an abelian variety of dim n
- $Z \subsetneq A$: a (nondegenerate) subvariety of dim r
  Z is a curve C in our talk.

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Goal

- *Construct a family indexed by $\mathbb{Z}^d$ of subvarieties in A.*
- *Find their dimension and homology class.*

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Example (Jacobian case)

When C is a smooth projective curve over $\mathbb{C}$ of genus $g \geqslant 2$,

$$\begin{aligned} A &:= \mathrm{Jac}(C) && \textit{the Jacobian of } C \\ \mathrm{AJ}_C : C &\hookrightarrow A && \textit{Abel–Jacobi map} \end{aligned}$$

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Example (Prym case)

When $h : C \longrightarrow C'$ is an unramified double cover of smooth projective curves, we can define

$$A := \mathrm{Prym}(C/C') \qquad \textit{the Prym variety of } h$$

$$\mathrm{AP}_{C/C'} : C \longrightarrow A \qquad \textit{Abel–Prym map}$$

We need to assume C is non-hyperelliptic so that $\mathrm{AP}_{C/C'}$ is injective.

Construct new subvarieties

Since A has addition structure, one defines

$$\begin{aligned} C+C &:= \{p+q \,|\, p,q \in C\} \quad &\subseteq A \\ 2C &:= \{2p \,|\, p \in C\} \quad &\subseteq A \end{aligned}$$

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Remark

Since C is nondegenerate,

$$\underbrace{C + C + \cdots + C}_{\geqslant n \text{ many}} = A.$$

Construct new subvarieties

Question

Can we define a family of subvarieties analogous to

$$\{m_1C + \cdots + m_dC \subseteq A \mid m_1, \ldots, m_d \in \mathbb{Z}\}$$

but constructed in a way that reflects the additive structure of A more faithfully?

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In fact, we can construct a family of subvarieties

$$\left\{Z_\chi \subseteq A \,\middle|\, \chi \in \mathbb{Z}^d\right\}$$

via the conormal variety.

Conic Lagrangian cycle

For a (smooth) subvariety $Z \subset A$, one can define the conormal variety $\Lambda_Z \subset T^*A \cong A \times T_0^*A$ by

$$\Lambda_Z := \left\{ (p, \xi) \in T^*A \mid p \in Z, \xi|_{T_pZ} = 0 \right\}.$$

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Facts

- Λ_Z is a conic Lagrangian cycle in T^*A;
- We have one-to-one correspondence

$$\begin{array}{ccc} \{\text{irr conic Lagrangian cycles in } T^*A\} & \cong & \{\text{irr subvarieties in } A\} \\ \Lambda_Z & \longleftrightarrow & Z \end{array}$$

- The map $\gamma_Z : \Lambda_Z \subset A \times T_0^*A \longrightarrow T_0^*A$ is a generically finite map, when Z is nondegenerate.

Family of subvarieties

Definition

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$$\underbrace{\Lambda_Z \times_{T_0^* A} \cdots \times_{T_0^* A} \Lambda_Z}_{d \text{ many}} \subset A \times \cdots \times A \times T_0^* A$$

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For $\chi = (m_1, \dots, m_d) \in \mathbb{Z}^d$, define $\Lambda_{Z_\chi} := f(\Lambda_Z^{\mathrm{univ}})$, where

$$\begin{aligned} f : A \times \cdots \times A \times T_0^*A &\longrightarrow A \times T_0^*A \\ (q_1, \dots, q_d, \xi) &\longmapsto (\textstyle\sum_i m_i q_i, \xi) \end{aligned}$$

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Z_χ is then the corresponding subvariety of Λ_{Z_χ}.

Our work

We determine $\dim Z_\chi$ and $[Z_\chi] \in H_*(A;\mathbb{Z})$ in special cases.

Example

In the Jacobian case, $d = \deg \gamma_C = 2g-2$.
Assume that C is non-hyperelliptic. For $\chi = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, when no g of m_i equal to each other, we get

$$\dim Z_\chi = \ g-1$$

$$[Z_\chi] = \ \frac{1}{\deg f|_{\Lambda_Z^{\mathrm{univ}}}} \left(\frac{1}{2^{g-1}} \sum_{\sigma \in S_{2g-2}} \prod_{l=1}^{g-1} \left(m_{\sigma(2l-1)} - m_{\sigma(2l)} \right)^2 \right) \cdot [\Theta]$$

Q & A

Thank you for your listening!

Any questions?

This slide is online available: https://shorturl.at/qqWDe
For more infos (and references) about this topic, please check my note1 and note2.

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