SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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Contents

This document is intended to collect the questions and doubts that arose during my research this year. For many of these problems, I have consulted my fellow students, my supervisor, and various other people I've met. However, most of them remain in the realm of folklore—problems that are likely known but for which I could not find a reference. On the other hand, some of the questions may not appear particularly interesting unless their underlying motivations are clearly explained. Therefore, I'll try to provide relevant background and outline some initial, perhaps naive, ideas while listing the problems along the way. Any responses, answers, or references are most welcome and will be added to keep this document updated.

1. Basic setting

For simplicity, we work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a integral separated scheme of finite type over $\mathbb{C}$. Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subseteq A$ be an irreducible closed subvariety of dimension r. We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.[1]

1.1. **Gauss map.** The goal of my research is to understand the geometry of Z, and the main tool for the subvariety geometry is the Gauss map. The Gauss map describe the tangent space information at each point:

$$\phi_Z : \mathrm{Z}^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longmapsto T_pZ \quad \subseteq T_pA \cong T_0A$$

Any map to the Grassmannian $\mathrm{Gr}(r, n)$ is induced by a rank r vector bundle together with n global sections. In this case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}$ and the sections

$$H^0(A, \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \twoheadrightarrow \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}.$$

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[1] I'm not sure whether we should consider the more general cases in the future—such as working over a field of characteristic p, letting A be a semiabelian variety or a complex torus, or allowing ι to be a covering onto its image. For now, I will omit these possibilities from this document.

1.2. **Conormal variety.** This concept may already be familiar to many readers, so we briefly recall the definition. On the smooth locus, the normal and conormal bundles behave well as vector bundles:[2]

$$\mathcal{N}_{Z^{\mathrm{sm}}/A} := \mathcal{T}_A|_{Z^{\mathrm{sm}}} \,/\, \mathcal{T}_{Z^{\mathrm{sm}}} \qquad \Lambda_{Z^{\mathrm{sm}}} := \mathcal{N}^*_{Z^{\mathrm{sm}}/A} = \ker\left(\Omega_A|_{Z^{\mathrm{sm}}} \to \Omega_{Z^{\mathrm{sm}}}\right).$$

The conormal variety Λ_Z is just the closure of $\Lambda_{Z^{\mathrm{sm}}}$ viewed as a subvariety in T^*A:

$$\Lambda_Z := \overline{\Lambda_{Z^{\mathrm{sm}}}} \subset T^*A \cong A \times T_0^*A$$

this is conical Lagrangian cycle in T^*A.

Moreover, the projectivized conormal variety

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A$$

is a Legendrian cycle in the contact variety $A \times \mathbb{P}T_0^*A$. $\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}$ is a $\mathbb{P}^{r-1}$-bundle over Z^{sm}, and the map

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A$$

is generically finite (i.e., clean) when Z is (an integral variety) of general type, see [2, Theorem 2.8 (1)].

A lot of geometry of Z is encoded in the map γ_Z. For instance, if Z is smooth and lies inside A, then

$$\deg \gamma_Z = (-1)^r \chi(Z)$$

tells us the Euler characteristic of Z.

Further insight can be gained by analyzing the fibers of γ_Z. These fibers, though finite, are not arbitrary—they obey hidden structural rules. For instance, if Z is preserved by a translation $t_v : A \longrightarrow A$, then each fiber $\gamma_Z^{-1}(\xi)$ is also invariant under t_v. Likewise, if $Z = -Z$, then the fiber satisfies $\gamma_Z^{-1}(\xi) = -\gamma_Z^{-1}(\xi)$. Outside these special configurations, it becomes more challenging to identify further constraints.[3]

An important invariant arising from the fiber $\gamma_Z^{-1}(\xi)$ is the monodromy group $\mathrm{Gal}(\gamma_Z)$; for completeness, we recall its definition below.

Definition 1.1. *Define*

$$U = \left\{\xi \in \mathbb{P}T_0^*A \,\middle|\, \#\gamma_Z^{-1}(\xi) = \deg \gamma_Z\right\}.$$

Moving along a loop in U induces a permutation of the points in the fiber $\gamma_Z^{-1}(\xi_0)$, which defines the map

$$\rho_{\gamma_Z} : \pi_1(U, \xi_0) \longrightarrow \mathrm{Aut}(\gamma_Z^{-1}(\xi_0)) = S_{\deg \gamma_Z}.$$

The monodromy group is then defined as the image of ρ_{γ_Z}, i.e.,

$$\mathrm{Gal}(\gamma_Z) := \mathrm{Im}\, \rho_{\gamma_Z}.$$

Question 1.2. *Suppose that the subvariety $Z \subset A$ is not stable under any translation on A. Are there known algorithms to compute the monodromy group $\mathrm{Gal}(\gamma_Z)$? Furthermore, what kinds of groups can appear as $\mathrm{Gal}(\gamma_Z)$ for suitable choices of $Z \subset A$?*

We will try to compute $\mathrm{Gal}(\gamma_Z)$ for a number of specific cases in Section 2. Three special cases are already treated in [3, Theorem 9], and we will generalize the strategies there.

[2] This is more symmetric when writing them as short exact sequences:

$$0 \longrightarrow \mathcal{T}_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{T}_A|_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{N}_{Z^{\mathrm{sm}}/A} \longrightarrow 0$$
$$0 \longrightarrow \Lambda_{Z^{\mathrm{sm}}} \longrightarrow \Omega_A|_{Z^{\mathrm{sm}}} \longrightarrow \Omega_{Z^{\mathrm{sm}}} \longrightarrow 0$$

[3] You can imagine the fiber $\gamma_Z^{-1}(\xi)$ as a cluster of stars projected onto a celestial dome. As ξ varies, these points shift, tracing out paths much like stars drifting across the night sky. The constraints that govern them are subtle, like the imagined lines that shape constellations. And in the long arc of variation, monodromy emerges—like the slow turning that replaces Kochab with Polaris among the stars.

1.3. **Interpolation via hyperplanes.** Before delving into examples, we reinterpret γ_Z using a functorial and more transparent framework, enabling a decomposition of Question 1.2 into two primary subquestions.

Recognizing that each non-zero conormal vector $\xi \in T_0^*A$ determines a hyperplane $H_\xi \in \mathrm{Gr}(n-1, T_0A)$, we establish the isomorphisms

$$\begin{aligned} \mathbb{P}T_0^*A &\cong \mathrm{Gr}(n-1, T_0A) \mathrel{\hat{=}} \left(\mathbb{P}^{n-1}\right)^\vee, \\ \mathbb{P}\Lambda_{\mathrm{Z}^{\mathrm{sm}}} &= \left\{ (p,\xi) \in \mathrm{Z}^{\mathrm{sm}} \times \mathbb{P}T_0^*A \;\middle|\; \xi|_{T_pZ} \equiv 0 \right\} \\ &\cong \left\{ (p,H) \in \mathrm{Z}^{\mathrm{sm}} \times \mathrm{Gr}(n-1,n) \;\middle|\; \phi_Z(p) \subseteq H \right\} \\ &\cong (\phi_Z, \mathrm{Id})^{-1} I_{r,n-1}, \end{aligned}$$

where

$$I_{r,n-1} := \left\{ (V,H) \in \mathrm{Gr}(r,n) \times \mathrm{Gr}(n-1,n) \;\middle|\; V \subseteq H \right\}$$

is the incidence variety relating $\mathrm{Gr}(r,n)$ and $\mathrm{Gr}(n-1,n)$. In that case,

$$\begin{aligned} \gamma_Z^{-1}(H) \cap \mathrm{Z}^{\mathrm{sm}} &= \{ p \in \mathrm{Z}^{\mathrm{sm}} \mid \phi_Z(p) \subseteq H \} \\ &\cong \phi_Z^{-1}\left(\mathrm{Gr}(r,H)\right) \end{aligned}$$

is the collection of points whose tangent spaces lie entirely within H.

Geometrically, the monodromy can be described as follows: given a general hyperplane H, its preimage consists of d points $p_1, \ldots, p_d$. Moving H continuously along a loop causes these points to permute, and the monodromy group $\mathrm{Gal}(\gamma_Z)$ consists of all permutations obtained this way. With this new formulation, it suffices to consider the Gauss map ϕ_Z alone; the inclusion $\iota_Z : Z \to A$ is no longer required for computing the monodromy group.

Definition 1.3. *Let Z be an r-dimensional variety and $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$ a morphism. Suppose that for some $d \in \mathbb{N}_{>0}$, the set*

$$U := \left\{ H \in \mathrm{Gr}(n-1,n) \;\middle|\; \#\phi^{-1}\left(\mathrm{Gr}(r,H_0)\right) = d \right\}$$

is non-empty open in $\mathrm{Gr}(n-1,n)$[4]*. The monodromy group $\mathrm{Gal}(\phi)$ is defined as the image of*

$$\rho_{\gamma_Z} : \pi_1(U, H_0) \longrightarrow \mathrm{Aut}\left(\phi^{-1}\left(\mathrm{Gr}(r,H_0)\right)\right) \cong S_d.$$

When ϕ is not generically finite onto its image, the monodromy group is subject to additional constraints, as captured by the next lemma.

Lemma 1.4. *When $\phi : Z \to \mathrm{Gr}(r,n)$ is generically k-to-1 onto its image , the monodromy group $\mathrm{Gal}(\phi)$ is contained in the wreath product*

$$S_k \wr S_{d/k} := \left(S_k^{\oplus d/k}\right) \rtimes S_{d/k}.$$

Proof. Consider the diagram below:

$$\begin{array}{cccccc} \phi: & Z & \xrightarrow{k:1} & \mathrm{Im}\, Z & \hookrightarrow & \mathrm{Gr}(r,n) \\ & \cup & & \cup & & \cup \\ & \{p_1,\ldots,p_d\} & \longrightarrow & \{q_1,\ldots,q_{d/k}\} & \longrightarrow & \mathrm{Gr}(r,H) \end{array}$$

The fiber $\phi^{-1}\left(\mathrm{Gr}(r,H_0)\right)$ splits into d/k groups of points, with the monodromy group acting by permutations within each group and among the groups. ☐

Based on the discussion above, Question 1.2 reduces to Question 1.5 and Question 1.7, with Question 1.6 appearing as a special case of Question 1.5.

Question 1.5. *For a map $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$ satisfying the conditions in Definition 1.3, how can we compute $\mathrm{Gal}(\phi)$?*

[4] Here, the fiber is understood set-theoretically; multiplicities are not taken into account.

Question 1.6. *Let $Z \subseteq \mathrm{Gr}(r,n)$ be a subvariety of dimension r such that $[Z] \neq 0$ in $H_r(\mathrm{Gr}(r,n);\mathbb{Z})$. What can be said about its monodromy group?*

Question 1.7. *For a map $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$, when is it induced from some inclusion $Z \subset A$?*

2. Searching for examples

In this section, we discuss examples drawn from my ongoing work, focusing on the construction of subvarieties and the computation of their monodromy groups.

Broadly speaking, there are two approaches to constructing examples. One may start with a given variety Z and attempt to embed it into an abelian variety—this always factors through its Albanese $\mathrm{Alb}(Z)$. Alternatively, one may begin with an abelian variety A and construct subvarieties by intersecting suitable divisors. Unfortunately, even in the case where Z is a curve, we have not fully resolved Question 1.2.

2.1. Curves, basic results.

A complete answer to Question 1.6 is known when $Z = C$ is a curve.

Proposition 2.1 (See [1, p111] for a detailed proof)**.** *Suppose that $\iota_C : C \subseteq \mathbb{P}^{n-1}$ is an irreducible nondegenerate curve of degree d, then $\mathrm{Gal}(\iota_C) \cong S_d$.*

Sketch of proof. Because S_d is generated by its transpositions, we are reduced to verifying that:

- $\mathrm{Gal}(\iota_C)$ acts doubly transitively on the fiber;
- $\mathrm{Gal}(\iota_C)$ contains a transposition.

$\square$

Proposition 2.2. *Let $\iota'_C : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \to C'$ be a degree 2 ramified covering, with ramification occurring at at least one smooth point of C'. Then $\mathrm{Gal}(\iota_{C'} \circ h) \cong S_2^{\oplus d/2} \rtimes S_{d/2}$ is the hyperoctahedral group/signed symmetric group.*

Sketch of proof. By Lemma 1.4 we know that $\mathrm{Gal}(\iota_{C'} \circ h) \subseteq S_2^{\oplus d/2} \rtimes S_{d/2}$. By Lemma 2.3, we are reduced to verifying that:

- The quotient map $\mathrm{Gal}(\iota_{C'} \circ h) \longrightarrow \mathrm{Gal}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- $\mathrm{Gal}(\iota_{C'} \circ h)$ contains a transposition of a pair of points in the fiber of h.

$\square$

Lemma 2.3. *Let G be a subgroup of $S_2^{\oplus m} \rtimes S_m$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \to S_m$ is surjective and the transposition σ_0 of ± 1 lies in G, then $G = S_2^{\oplus m} \rtimes S_m$.*

Sketch of proof. (with help from Chenji Fu) Let ε_i denote the transposition of $\pm i$. For any $\sigma \in S_m$, choose a lift $\tilde{\sigma} \in G$, then

$$\varepsilon_{\sigma(1)} = \tilde{\sigma} \circ \sigma_0 \circ \tilde{\sigma}^{-1} \in G.$$

Thus, $S_2^{\oplus m} \subset G$, and since G maps onto S_m, we obtain $G = S_2^{\oplus m} \rtimes S_m$. $\square$

Example 2.4. *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$.*

When C is non-hyperelliptic, the corresponding Gauss map

$$|\omega_C| : C \longrightarrow \mathbb{P}^{g-1}$$

makes C as an irreducible nondegenerate curve of degree $2g-2$, by Proposition 2.1 we get

$$\mathrm{Gal}(\gamma_C) \cong S_{2g-2}.$$

When C is hyperelliptic, the corresponding Gauss map is $2:1$ onto a rational normal curve $R \subset \mathbb{P}^{g-1}$:

$$|\omega_C| : C \xrightarrow{2:1} R \hookrightarrow \mathbb{P}^{g-1}$$

By Proposition 2.2 we get

$$\mathrm{Gal}(\gamma_C) \cong S_2^{\oplus g-1} \rtimes S_{g-1}.$$

In fact, a degree $2:1$ map does not give rise to any exceptional monodromy groups beyond those listed in Table 1.

Definition 2.5 (big monodromy group). *We refer to the big monodromy group as any group of the following types:*

notation	*name*	*alias*
$W(A_{m+1}) = S_m$	*full symmetric group*	
$W(C_m) = S_2^{\oplus m} \rtimes S_m$	*signed symmetric group*	*hyperoctahedral group*
$W(D_m) = (S_2^{\oplus m})_0 \rtimes S_m$	*even-signed symmetric group*	*demihyperoctahedral group*

TABLE 1. big monodromy group

Proposition 2.6. *Let $\iota'_C : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \to C'$ be a degree 2 ramified covering. Then*

$$\mathrm{Gal}(\iota_{C'} \circ h) \cong W(C_{d/2}) \text{ or } W(D_{d/2}).$$

Sketch of proof. By Lemma 1.4 we know that $\mathrm{Gal}(\iota_{C'} \circ h) \subseteq W(D_{d/2})$. By Lemma 2.7, we are reduced to verifying that:

- The quotient map $\mathrm{Gal}(\iota_{C'} \circ h) \longrightarrow \mathrm{Gal}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- (signed doubly transitive) $\mathrm{Gal}(\iota_{C'} \circ h)$ acts transitively on pairs (x, y) with $x \neq \pm y$.

□

Lemma 2.7. *Let G be a subgroup of $W(D_m)$, acting naturally on the set $\pm 1, \ldots, \pm m$. If the projection $G \to S_m$ is surjective then*

$$G \cong W(C_m) \text{ or } W(D_m) \text{ or } S_m.$$

Sketch of proof. (with help from Chenji Fu) Let H denote the kernel of the natural quotient map $G \to S_m$. Then H is stable under the action of S_n. There are only three possible forms that H can take:

- $H = 0$. Then $G \cong S_m$.
- $H = (S_2^{\oplus m})_0$. Then G is a index 2 subgroup of $W(C_m)$, so $G \cong W(D_m)$.[5]
- $H = S_2^{\oplus m}$. Then $G = W(C_m)$.

□

We are particularly interested in identifying cases where the monodromy group is small(i.e., not big), which can occur only when $k > 2$, where k denotes the degree of the map $\phi : C \to \mathbb{P}^{n-1}$. On the other hand, when both $k > 2$ and $n > 2$, Lemma 1.4 ensures that the resulting monodromy group is indeed small. Thus, in the case of curves, Question 1.2 is resolved except in the following three cases:

(1) $n > 2$ and $k = 2$. By Proposition 2.6, the monodromy group is known to be large, although its precise structure remains undetermined.
(2) $n = 2$. Here, A is an abelian surface and C corresponds to a divisor.
(3) $n > 2$ and $k > 2$. Finding an example of this would guarantee a case of small monodromy.

Question 2.8. *Can we find any curve $C \subset A$ not stable under any translation on A, and for which the associated monodromy group is small?*

Assume that $n > 2$. Can we find any curve $C \subset A$ not stable under any translation on A, whose Gauss map has degree $d > 2$?

[5] Check stackexchange discussions

3. Families of subvarieties

4. Tannakian formalism

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\mathrm{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in $\mathbb{Q}$. For any algebraic group G, we denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G.

Following the approach of [4], we work in the quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/N(A)$, where $N(A) \subset \mathrm{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex $\mathcal{F}$ is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [4, Thm 7.1 & Cor 9.2]). In particular, for any perverse sheaf $\delta \in \overline{\mathrm{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G:

$$\langle \delta, * \rangle \cong \mathrm{Rep}(G).$$

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