Springer Fibers for $SL_n(\mathbb{C})$

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June 15, 2021

Recap: representation theory of finite groups

Restrict to **complex** representations, we have a nice theory:

- Any representation can be written as a direct sum of **irreducible representation**;
- We can extract information of irreducible representations from the **character table**:

$$\#\{\text{irreducible representations}\} = \#\{\text{conjugation classes}\}$$

$$\sum_{\chi:\text{irr}} (\dim \chi)^2 = \#G$$

However, in general,

S_n

- NO standard way finding an **explicit construction** of all irreducible representations;
- NO **one-to-one correspondence** between irreducible representations and conjugation classes.

In this talk, we use two methods to understand representations of S_n, and find connections/analogs between them.

methods	objects
combinatorial	Young diagram, Young tableau
geometrical	Springer fiber of $SL_n(\mathbb{C})$, irreducible components

Goal of the Part I

- Explicitly construct irreducible representations of S_n by Young diagram;
- Compute the character table;
  - $\dim \chi_i$ by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g. $\otimes$, Sym^m, Λ^m;
  - e.g. M_λ.
  - restriction and induced representation

Notation

For boxes:

(Young) diagram

filling

$\begin{array}{|c|c|c|}\hline 11 & 78 & 11 \\ \hline 6 & 8 & \\ \cline{1-2} \end{array}$

standard filling

$\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 & \\ \cline{1-2} \end{array}$

tableau

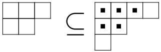
$\begin{array}{|c|c|c|}\hline 6 & 11 & 11 \\ \hline 8 & 78 & \\ \cline{1-2} \end{array}$

standard tableau

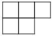
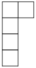
$\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \cline{1-2} \end{array}$

Order of Young diagram:

inclusion

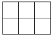
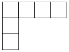


dominance

 $\trianglelefteq$ 

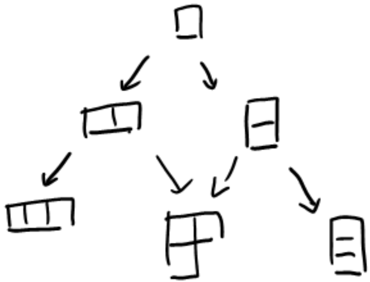
2 4 5   4 5 5

Lexicographic ordering

 $\leqslant$ 

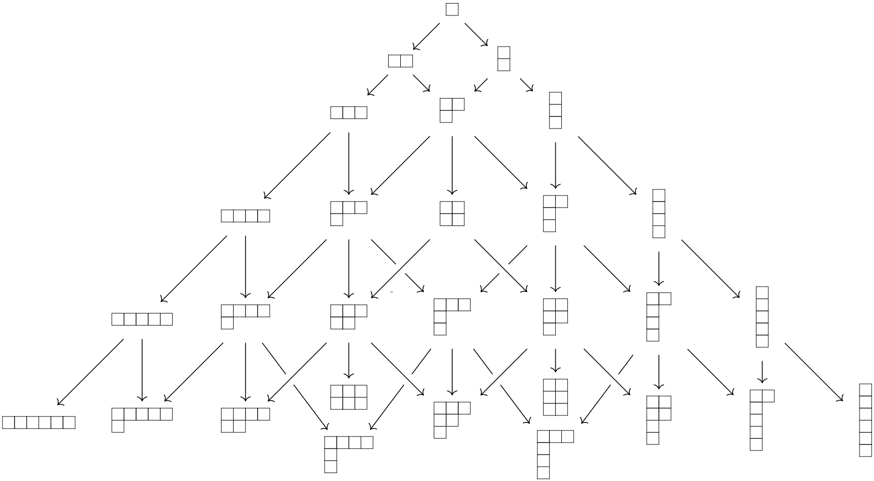
2 4 6 6   3 4 5 6

tree of Young diagram



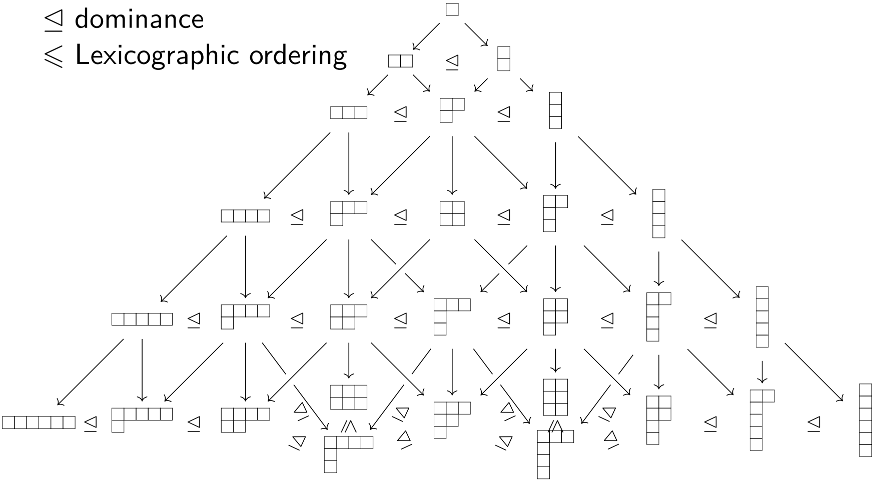
tree of Young diagram

skip



Order

- $\subseteq$ inclusion
- $\trianglelefteq$ dominance
- $\leqslant$ Lexicographic ordering



S_n & Young diagram

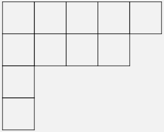
Proposition

We have the one-to-one correspondence

$$\left\{\begin{array}{c}\text{Young diagram}\\ \text{of } n \text{ boxes}\end{array}\right\} \xleftrightarrow[\lambda=\lambda_1^{v_1}\cdots\lambda_k^{v_k}]{\text{partition of } n} \left\{\begin{array}{c}\text{Conjugation class}\\ \text{of } S_n\end{array}\right\}$$

Example

$n = 11.$



$$\xleftrightarrow[\lambda=5\cdot4\cdot1^2]{11=5+4+1+1} (12345)(6789)(10)(11)$$

S_n & Young diagram

Proposition

We have the one-to-one correspondence

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow[\lambda=\lambda_1^{v_1}\cdots\lambda_k^{v_k}]{\textit{partition of } n} \left\{\begin{matrix}\textit{Conjugation class}\\ \textit{of } S_n\end{matrix}\right\}$$

Claim

$$\left\{\begin{matrix}\textit{Young diagram}\\ \textit{of } n \textit{ boxes}\end{matrix}\right\} \xleftrightarrow{?} \left\{\begin{matrix}\textit{Irreducible rep}\\ \textit{of } S_n\end{matrix}\right\}$$

S_n & Young diagram

Claim

$$\left\{\begin{matrix} \textit{Young diagram} \\ \textit{of } n \textit{ boxes} \end{matrix}\right\} \xleftrightarrow{?} \left\{\begin{matrix} \textit{Irreducible rep} \\ \textit{of } S_n \end{matrix}\right\}$$

λ S^λ

Remark

Reduced to: for each Young diagram λ,
construct an irreducible representation S^λ, and
prove $S^\lambda = S^{\lambda'} \Rightarrow \lambda = \lambda'$.

The construction of $S^{\lambda} \subseteq M^{\lambda}$

Tabloid: equivalence class of standard filling

$$\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 2 & 1 \\ \cline{1-2} \end{array} = \begin{array}{|c|c|c|}\hline 3 & 4 & 5 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} := \{345/12\}$$

$$\begin{tikzcd} \left\{ \begin{array}{l} \text{standard filling} \\ \text{of shape } \lambda \end{array} \right\} \arrow[d] & \ni & T = \begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} \arrow[d, mapsto] \\ T^{\lambda} := \left\{ \begin{array}{l} \text{Young tabloid} \\ \text{of shape } \lambda \end{array} \right\} & \ni & \{T\} = \begin{array}{ccc}\hline 3 & 5 & 4 \\ \hline 1 & 2 \\ \cline{1-2} \end{array} = \{345/12\} \end{tikzcd}$$

The construction of $S^\lambda \subseteq M^\lambda$

$$\mathcal{T}^\lambda := \{\text{Young tabloid of shape } \lambda\}$$

$$M^\lambda := \left\langle \{T\} \in \mathcal{T}^\lambda \right\rangle_{\mathbb{C}}$$

Choose a standard filling T of shape λ,

$$C(T) := \{\sigma \in S_n | \, \sigma \text{ preserves numbers in each column}\}$$

$$v_T := \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\{\sigma \cdot T\} \in M^\lambda$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda$$

invariant subspace of M^λ

The construction of $S^\lambda \subseteq M^\lambda$

$$\mathcal{T}^\lambda := \{\text{Young tabloid of shape } \lambda\}$$
$$M^\lambda := \left\langle \{T\} \in \mathcal{T}^\lambda \right\rangle_{\mathbb{C}}$$

Example ($\lambda = 3 \cdot 2$)

$$\mathcal{T}^\lambda = \left\{ \begin{matrix} \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \\ \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \end{matrix} \right\}$$

$$\underline{M^\lambda} = \left\langle \begin{matrix} \{123/45\}, \{124/35\}, \{125/34\}, \{134/25\}, \{135/24\}, \\ \{145/23\}, \{234/15\}, \{235/14\}, \{245/13\}, \{345/12\} \end{matrix} \right\rangle_{\mathbb{C}}$$

The construction of $S^\lambda \subseteq M^\lambda$

$$v_T := \sum_{\sigma \in C(T)} \operatorname{sgn}(\sigma)\{\sigma \cdot T\} \in M^\lambda \qquad \sigma v_T = v_{\sigma T}$$

$$S^\lambda := \mathbb{C}[S_n] \cdot v_T \subseteq M^\lambda$$ invariant subspace of M^λ

Example ($\lambda = 3 \cdot 2$)

$$T = \begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}$$

$$C(T) = \{\mathrm{Id}, (23), (15), (23)(15)\}$$

$$v_T = \left\{\begin{array}{|c|c|c|}\hline 3 & 5 & 4 \\ \hline 2 & 1 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 2 & 5 & 4 \\ \hline 3 & 1 \\ \cline{1-2}\end{array}\right\} - \left\{\begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\right\} + \left\{\begin{array}{|c|c|c|}\hline 2 & 1 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\right\}$$

$$= \{345/12\} - \{245/13\} - \{134/25\} + \{124/35\} \in M^\lambda$$

$$S^\lambda = \langle v_T \rangle_{\mathbb{C}[S_n]} = \langle v_{T'} | T' : \textit{standard filling} \rangle_{\mathbb{C}}$$

Main theorem of S^λ

Theorem

Fix the Young diagram λ, the corresponding representation S^λ has the following properties:

1. *the linear space S^λ has a* ***basis*** $\{v_{T'} | T' : \text{standard tableau}\}$*, especially,* $\dim S^\lambda = \#\{\text{standard tableau}\}$*;*
2. *the representation S^λ is* ***irreducible****;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

Proof: basis

Theorem

1. *the linear space S^λ has a basis $\{v_{T'}|T' : \text{standard tableau}\}$, especially, $\dim S^\lambda = \#\{\text{standard tableau}\}$;*

Idea of the proof

- S^λ is generated by $\{v_{T'}|T' : \text{standard filling}\}$,
  It's not an easy task to represent $v_{T'}$ by linear combinations. $v_{T''}$

  e.g. $v_{\begin{smallmatrix}3&5&4\\2&1\end{smallmatrix}} \overset{\text{column}}{=} v_{\begin{smallmatrix}2&1&4\\3&5\end{smallmatrix}} \overset{\text{row}}{=} v_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} - v_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}}$

- $\{v_{T'}|T' : \text{standard tableau}\}$ are linear independent.

  e.g. $x_1 v_{\begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}} + x_2 v_{\begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}} + x_3 v_{\begin{smallmatrix}1&3&4\\2&5\end{smallmatrix}} + x_4 v_{\begin{smallmatrix}1&2&5\\3&4\end{smallmatrix}} + x_5 v_{\begin{smallmatrix}1&3&5\\2&4\end{smallmatrix}} = 0$ $\quad x_i \in \mathbb{C}$

  $\to \{123/45\} \to x_1 = 0$ $\quad \to \{134/25\} \to x_3 = 0$ $\quad \{135/24\} \to x_5 = 0$

  $\to \{124/135\} \to x_2 = 0$ $\quad \to \{125/135\} \to x_4 = 0$ $\quad -x_1 + x_5$

linear ordering

We use a linear ordering of standard fillings by

$$\begin{array}{ccc} \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline 4&5 \\ \cline{1-2}\end{array} & \longrightarrow & 5\,4\,3\,2\,1 \\ \vee & & \rotatebox{-90}{$\geq$} \\ \begin{array}{|c|c|c|}\hline 1&3&4\\ \hline 2&5 \\ \cline{1-2}\end{array} & \longrightarrow & 5\,2\,4\,3\,1 \end{array}$$

In the proof, we knock out the biggest one.

Proof: part 2&3

Theorem

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

We have to introduce element b_T in $\mathbb{C}[S_n]$ by   fix T of shape λ

$$b_T := \sum_{q \in C(T)} \mathrm{sgn}(\sigma)\sigma$$

one can get

$$b_T S^\lambda = \mathbb{C} v_T \neq 0, \qquad b_T S^{\lambda'} = 0 \quad \text{for } \lambda' > \lambda.$$

The results follow from these equations. skip

Proof: part 2&3

Theorem

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We have to introduce element b_T in $\mathbb{C}[S_n]$ by

$$b_T := \sum_{q \in C(T)} \mathrm{sgn}(\sigma)\sigma$$

then

- $v_T = b_T \cdot \{T\}$;
- $\tau(b_T) = \mathrm{sgn}(\tau) b_T$ for any $\tau \in C(T)$;
- $b_T \cdot b_T = \#C(T) \cdot b_T$;
- $b_T M^\lambda = b_T S^\lambda = \mathbb{C} v_T \neq 0$;
  $b_T M^{\lambda'} = b_T S^{\lambda'} = 0$ for $\lambda' > \lambda$.

Proof: part 2&3

Theorem

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

$$b_T S^\lambda = \mathbb{C} v_T \neq 0 \;;$$
$$b_T S^{\lambda'} = 0 \qquad \text{for } \lambda' > \lambda$$

⋇To show S^λ is irreducible: only need to show indecomposablility.
If $S^\lambda = V \oplus W$ as $\mathbb{C}[S_n]$-module, then

$$\mathbb{C} v_T = b_T S^\lambda = b_T V \oplus b_T W$$
$$\Rightarrow b_T V = \mathbb{C} v_T \qquad (\text{or } b_T W = \mathbb{C} v_T)$$
$$\Rightarrow S^\lambda = \mathbb{C}[S_n] \cdot v_T = \mathbb{C}[S_n] \cdot \mathbb{C} v_T = \mathbb{C}[S_n] \cdot b_T V \subseteq V$$

Proof: part 2&3

Theorem

2. *the representation S^λ is irreducible;*
3. *for the Young diagram λ', $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$.*

$b_T S^\lambda = \mathbb{C} v_T \neq 0$;

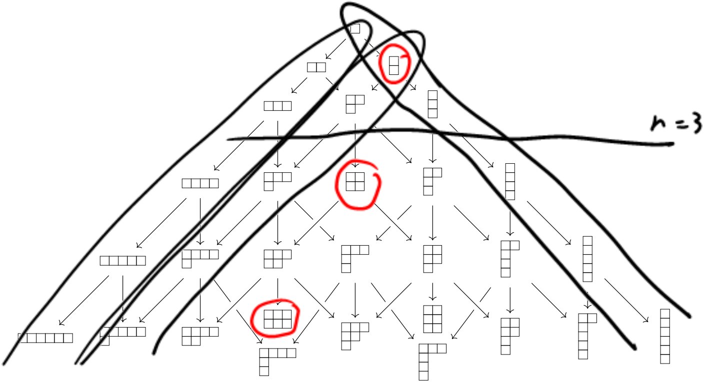
$b_T S^{\lambda'} = 0$ for $\lambda' > \lambda$

⋇To show $S^{\lambda'} \cong S^\lambda \Rightarrow \lambda' = \lambda$:
If not w.l.o.g. suppose $\lambda' > \lambda$. Then

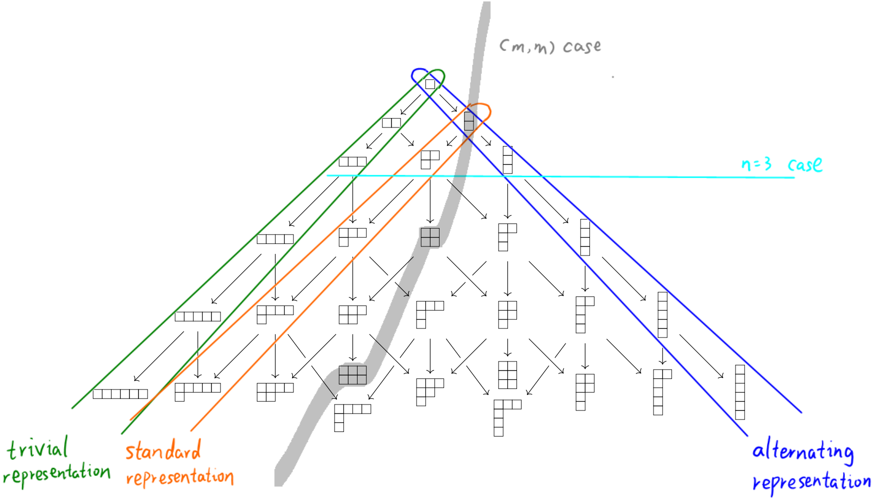
$$b_T S^{\lambda'} = b_T S^\lambda \Longrightarrow \mathbb{C} v_T \cong 0,$$

contradiction!

Example



Example



(m,m) case
n=3 case
trivial representation
standard representation
alternating representation

Example: trivial representation

$$\lambda = \boxed{}\boxed{}\boxed{} = 3^1$$

$$M^\lambda = \langle\{123\}\rangle = \mathbb{C}$$

$$T = \boxed{1}\boxed{2}\boxed{3}$$

$$C(T) = \mathrm{Id}$$

$$v_T = \{123\}$$

$$S^\lambda = \mathbb{C}[S_3] \cdot v_T = \mathbb{C}v_T$$

Example: alternating representation

$$\lambda = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array} = 1^3$$

$$M^\lambda = \langle \{1/2/3\}, \{1/3/2\}, \{2/1/3\}, \{2/3/1\}, \{3/1/2\}, \{3/2/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$$

$$\begin{aligned} C(T) =& S_3 \\ v_T =& \{1/2/3\} - \{1/3/2\} - \{2/1/3\} \\ &+ \{2/3/1\} + \{3/1/2\} - \{3/2/1\} \\ S^\lambda =& \mathbb{C}[S_3] \cdot v_T = \mathbb{C}v_T \\ (23)v_T =& \{1/3/2\} - \{1/2/3\} - \{3/1/2\} \\ &+ \{3/2/1\} + \{2/1/3\} - \{2/3/1\} = -v_T \end{aligned}$$

Example: standard representation

$$\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} = 2 \cdot 1$$

$$M^{\lambda} = \langle \{12/3\}, \{13/2\}, \{23/1\} \rangle_{\mathbb{C}}$$

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

$$C(T) = \{\mathrm{Id}, (13)\}$$

$$v_T = \{12/3\} - \{23/1\}$$

$$S^{\lambda} = \mathbb{C}[S_3] \cdot v_T \cong \mathbb{C}^2$$

$$(12)v_T = \{12/3\} - \{13/2\}$$

$$(13)v_T = \{23/1\} - \{12/3\} = -v_T$$

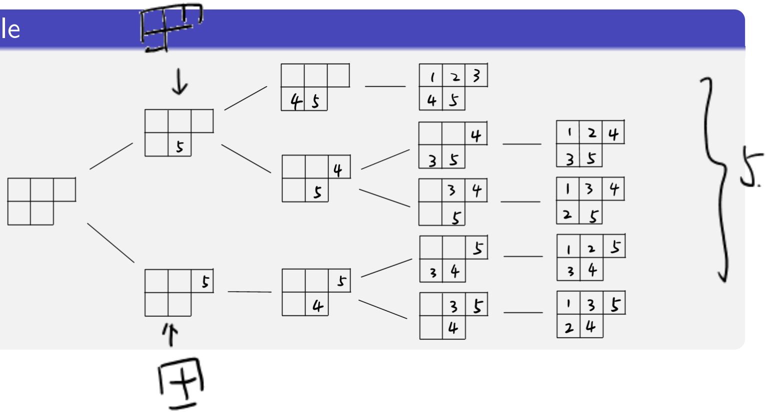
Goal of the Part 1

- Explicitly construct irreducible representations of S_n by Young diagram;
- Compute the character table;
  - $\dim \chi_\lambda$ S^λ by recursion / Hook length formula
  - character by Frobenius formula
- Compute other representations.
  - e.g. $\otimes$, Sym^m, Λ^m;
  - e.g. M_λ.
  - restriction and induced representation

Example: dimension of irreducible representation

$$\dim S^\lambda = \#\{\text{standard tableau of } \lambda\} = ?$$

Example



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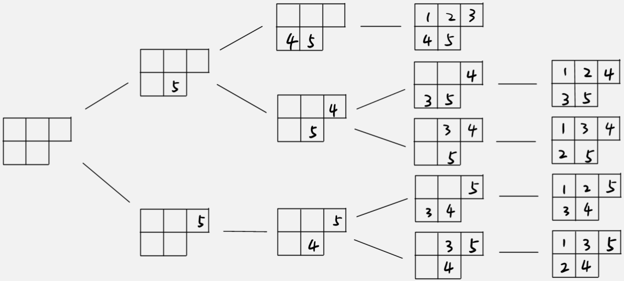
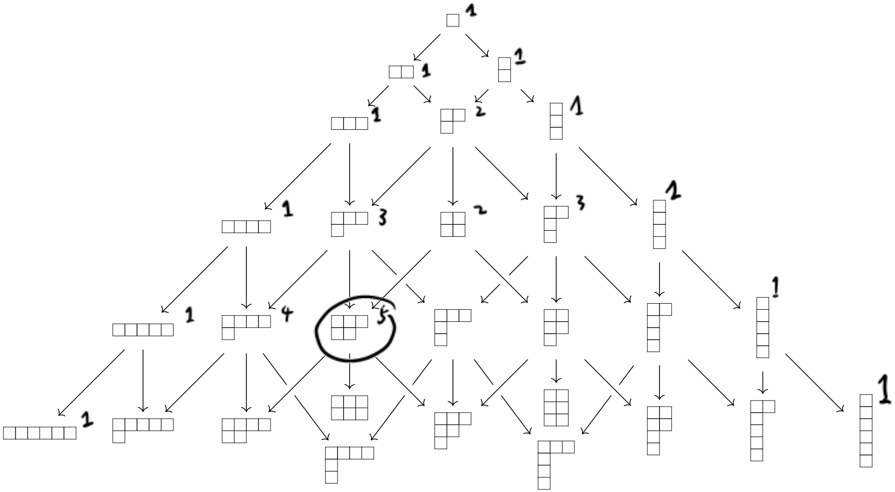


$$\dim S^\lambda = \sum_{\substack{\lambda' \subseteq \lambda \\ |\lambda'| = n-1}} \dim S^{\lambda'}$$

Example: dimension of irreducible representation



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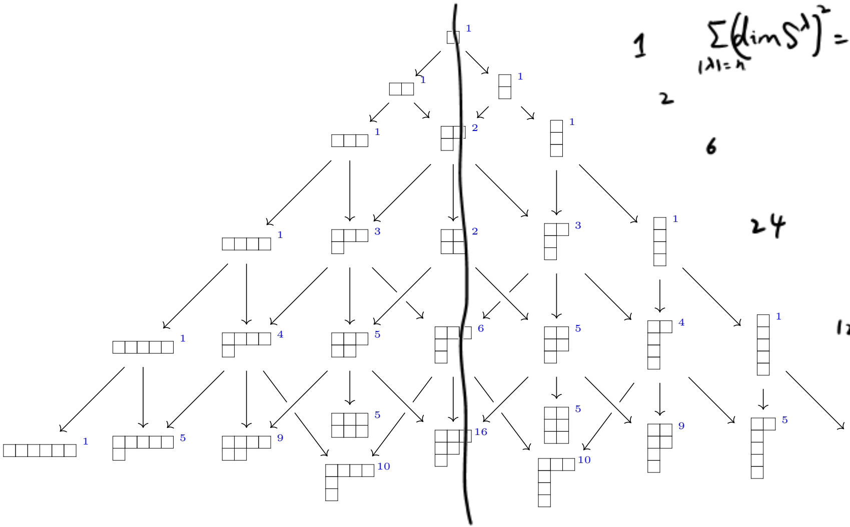


$$\sum_{|\lambda|=n} (\dim S^\lambda)^2 = n!$$

1

2

6

24

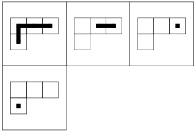
120

600

Hook length formula

It helps us compute the dimension of S^λ without induction.

Step 1: count the length of hook.



$\Longrightarrow$

4	2	1
1		

Step 2: $\dim S^\lambda = \dfrac{n!}{\prod(\text{hook lengths})}$

Special case: (m, l)

Ballot problem

In an election where candidate A receives m votes and candidate B receives l votes with $m \geqslant l$, what is the probability that A will be (non-strictly) ahead of B throughout the count?

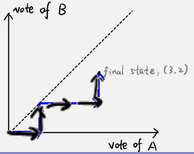
Proposition

Each process of the count corresponds to each standard tableau of form (m, l).

Example

A	1	3	4
B :	2	5	

$\longleftrightarrow$



Special case: (m,m)

Corollary

$$\dim S^{(m,m)} = C_m = \frac{1}{m+1}\binom{2m}{m}.$$

where C_m is the m-th Catalan number.

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Catalan number has many interpretations. For example, it counts the number of crossingless matchings of $2m$ points.

Ex. $m=3$

crossingless matchings of 6 points

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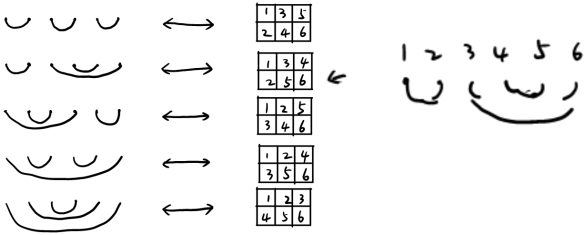
crossingless matchings of 6 points ⟷ Young tableau of (3,3) type



Goal of the Part II

- Definition of Springer fiber;
- Some examples of Springer fiber;
- Properties: (closely connected with combinatorics)
  - irreducible component?
  - dimension?
  - affine paving? – CW complex?
  - cohomology? – ring structure?
  - smooth?
  - explicit description?
- Weyl group action on top homology.

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Definition

$$\begin{array}{ccc} \tilde{\mathfrak{g}} \subseteq \mathfrak{g} \times \mathcal{B} \longrightarrow \mathcal{B}_{:=\mathcal{F}\ell(n)} & & \tilde{\mathcal{N}} \\ \big\downarrow \mu & \rightsquigarrow & \big\downarrow \mu|_{\mathcal{N}\times\mathcal{B}} \\ \mathfrak{g}_{:=\mathfrak{sl}_n(\mathbb{C})} & & \mathcal{N} \end{array}$$

resolution of nilpotent cone

Let $X \in \mathfrak{g}$ be a nilpotent element. The Springer fiber B_X over X is defined as

$$\begin{aligned} B_X :=& \mu^{-1}(X) \\ =& \{B \in \mathfrak{B} \mid X \in B\} \\ =& \{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{C}^n | XV_i \subseteq V_{i-1}\} \ \dim V_i = i \end{aligned}$$

By the Jordan normal form, we have

$$\left\{\begin{matrix}\text{Nilpotent element} \\ \text{in } \mathfrak{gl}_n(\mathbb{C})\end{matrix}\right\} \Big/ \text{conj} \longleftrightarrow \left\{\begin{matrix}\text{Young diagram} \\ \text{of } n \text{ boxes}\end{matrix}\right\}$$

$$X_\lambda = \mathrm{diag}(\underbrace{J_{\lambda_1}, \ldots, J_{\lambda_1}}_{v_1}, J_{\lambda_2}, \ldots, J_{\lambda_k}) \longleftrightarrow \lambda = \lambda_1^{v_1} \cdots \lambda_k^{v_k}$$

$$J_{\lambda_1} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}_{\lambda_1 \times \lambda_1}$$

Denote $B_\lambda := B_{X_\lambda}$. $B_X \cong B_{gXg^{-1}}$ for any $g \in G$

Theorem (we will not give the proof.)

As S_n-representation, $S^\lambda \cong H_{top}(B_\lambda)$.

Corollary

$$\#\{\textit{irreducible component of } B_\lambda\} = \dim S^\lambda$$

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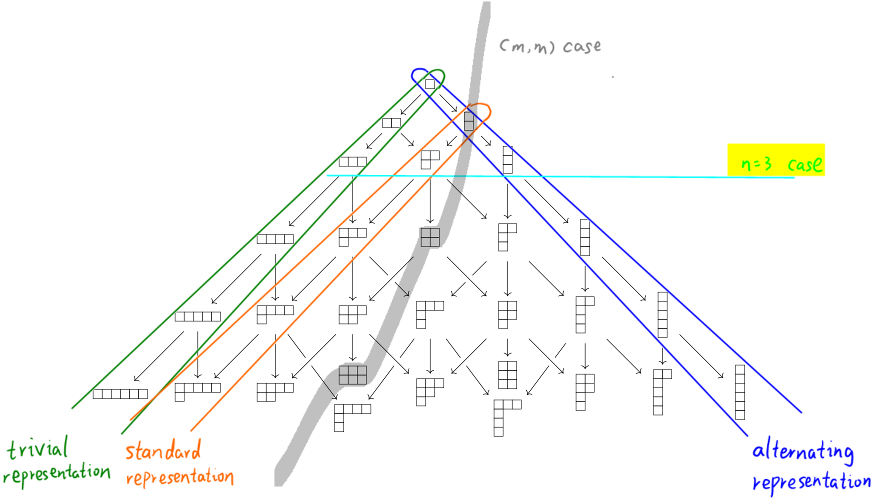
- Weyl group action on top homology.

tree of Young diagram



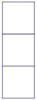
Example: $\lambda = 3$ 

$$X_\lambda = \begin{bmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle \overset{e_1}{?} \rangle \subseteq \langle \overset{e_1}{?}, \overset{e_2}{?} \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \quad = \underline{\{*\}}$$

In general, $B_\lambda = \{*\}$ when λ has only one row.

Example: $\lambda = (1,1,1)$



$$X_\lambda = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

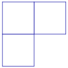
$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \quad = \mathcal{F}\ell(3)$$

In general, $B_\lambda = \mathcal{F}\ell(n)$ when $\lambda = 1^n$.

Properties of $B_\lambda = \mathcal{F}\ell(n)$

- irreducible: ✓
- $\dim B_\lambda = \frac{n(n-1)}{2}$
- CW complex: Schubert Cell.
- cohomology group: ✓
- smooth: ✓
- explicit description: $\begin{cases} \text{local chart} \\ \text{fiber bundle} \end{cases}$
- Weyl group action on $H_{\text{top}}(B_\lambda) \cong \mathbb{C}$:

$$\begin{array}{ccc} \mathcal{F}\ell(n-1) & - & \mathcal{F}\ell(n) \\ & & | \\ & & \mathbb{P}^{n-1} \end{array}$$

Example: $\lambda = (2,1)$ 

truncate.

$$X_\lambda = \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft X_\lambda \qquad = \mathbb{P}^1 \vee \mathbb{P}^1$$

$e_1 \qquad e_1\ ,\ ?$

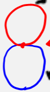
$\Leftrightarrow \{ 0 \subseteq \langle ? \rangle \subseteq \mathbb{C}^2 \} \curvearrowleft \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbb{P}^1$

$\mathbb{P}^1$

$B_\lambda = \{ 0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, be_2 + ce_3 \rangle \subseteq \mathbb{C}^3 \} \longrightarrow$

$\uplus$

$\{ 0 \subseteq \langle ae_1 + e_3 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3 \} \longrightarrow$



$\leftarrow \{ 0 \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_3 \rangle \subseteq \mathbb{C}^3 \}$

$\mathbb{C}$

In general, $B_\lambda = \underbrace{\mathbb{P}^1 \vee \cdots \vee \mathbb{P}^1}_{n-1}$ when $\lambda = (n-1, 1)$.

Properties of $B_\lambda = \underbrace{\mathbb{P}^1 \vee \cdots \vee \mathbb{P}^1}_{n-1}$

- irreducible component: $n-1$
- $\dim B_\lambda = 1$
- affine paving: ✓
- cohomology group: ✓

- smooth: ✗
- explicit description: ✓
- Weyl group action on $H_{\text{top}}(B_\lambda) \cong \mathbb{C}^{n-1}$:

Tool: stratification / cellular fibration / affine paving

$$\begin{array}{l} B_{\lambda'} \longrightarrow \\ \quad \vdots \\ B_{\lambda''} \longrightarrow \\ \phi \longrightarrow \end{array} \quad B_\lambda \xrightarrow{\ \pi\ } \mathbb{P}^{n-1}$$

$$\{0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq \mathbb{C}^n\} \circlearrowleft X_\lambda \longmapsto [V_1]$$

Remark

In general, we don't understand the ring structure of the cohomology group.

Return!

For $\lambda = 1^3$, $B_\lambda \cong \mathcal{F}\ell(3)$ can be viewed as $\mathcal{F}\ell(2)$-bundle over $\mathbb{P}^2$.

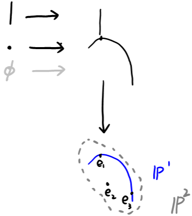
$$\begin{tikzcd}
\mathcal{F}\ell(1) \arrow[r] \arrow[d, phantom, "= \{*\}"] & \mathcal{F}\ell(2) \arrow[r, "\cong \mathbb{P}^1"'] \arrow[d] & \mathcal{F}\ell(3) \arrow[r, "B_{,,}"] \arrow[d, "\pi"] & \mathcal{F}\ell(4) \arrow[d] \\
 & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3
\end{tikzcd}$$

$$\pi^{-1}([v]) = \left\{0 \subseteq \langle v \rangle \subseteq \langle v,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \cong \underline{\mathcal{F}\ell(2)}$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:

$$\begin{tikzcd} \mathbb{P}^1 = B_{1,1} \arrow[r] & B_{2,1} \arrow[d, "\pi"] \\ \{*\} = B_2 \arrow[ur] & \mathbb{P}^2 \\ \varnothing \arrow[uur] & \end{tikzcd}$$



$$\pi^{-1}([e_1]) = \left\{ 0 \subseteq \langle e_1 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\cong \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \mathbb{C}^2 \right\} \curvearrowleft \begin{bmatrix} 0 & \\ & 0 \end{bmatrix} \qquad = \underline{B_{1,1}} \quad \mathbb{P}^1$$

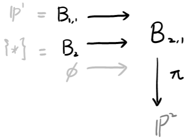
$$\pi^{-1}([e_3]) = \left\{ 0 \subseteq \langle e_3 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

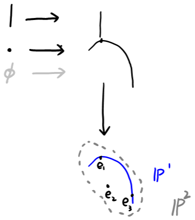
$$\cong \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \mathbb{C}^2 \right\} \curvearrowleft \begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \qquad = B_2 \quad = \{*\}$$

$$\pi^{-1}([e_2]) = \left\{ 0 \subseteq \langle e_2 \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3 \right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix} \qquad = \varnothing$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:

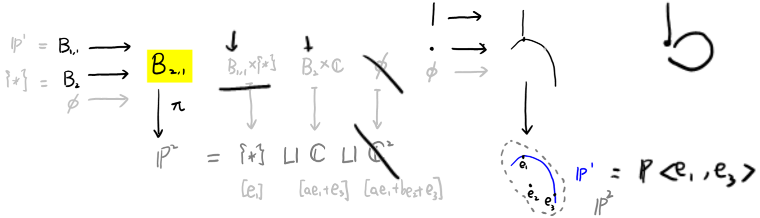




$$\pi^{-1}([e_1]) \cong B_{1,1} \qquad \pi^{-1}([e_3]) \cong B_2 \qquad \pi^{-1}([e_2]) \cong \varnothing$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:



$$\pi^{-1}([e_1]) \cong B_{1,1} \qquad \pi^{-1}([e_3]) \cong B_2 \qquad \pi^{-1}([e_2]) \cong \varnothing$$

$$\pi^{-1}([ae_1+e_3]) = \left\{0 \subseteq \langle ae_1+e_3\rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$(f_1, f_2, f_3) = (e_1, e_2, ae_1+e_3)$

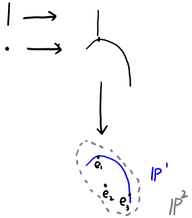
$$\cong \left\{0 \subseteq \langle f_3\rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\cong \pi^{-1}([e_3]) = \{*\}$$

Return!

For $\lambda = (2,1)$, $B_\lambda \cong \mathbb{P}^1 \vee \mathbb{P}^1$:

$$\begin{tikzcd}
\mathbb{P}^1 = B_{1,1} \arrow[r] & B_{2,1} \arrow[d, "\pi"] & B_{1,1}\times\{*\} \arrow[d] & B_2\times\mathbb{C} \arrow[d] \\
\{*\} = B_2 \arrow[ru] & \mathbb{P}^1 = & \{*\} & \sqcup\ \mathbb{C} \\
 & & {[e_1]} & {[ae_1+e_3]}
\end{tikzcd}$$

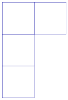


$$\pi^{-1}([e_1]) \cong B_{1,1} \qquad \pi^{-1}([e_3]) \cong B_2 \qquad \pi^{-1}([e_2]) \cong \varnothing$$

$$\pi^{-1}([ae_1+e_3]) = \left\{0 \subseteq \langle ae_1+e_3\rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

$$\cong \left\{0 \subseteq \langle f_3\rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^3\right\} \curvearrowleft \begin{bmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Example: $\lambda = (2,1,1)$

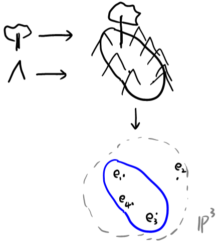


$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowleft X_\lambda$$

$$\begin{array}{lll} \mathcal{F}\ell(3) = B_{1,1,1} \longrightarrow & & \\ & & B_{2,1,1} \\ \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \longrightarrow & & \end{array}$$

$$B_{2,1,1} \xrightarrow{\pi} \mathbb{P}^2$$

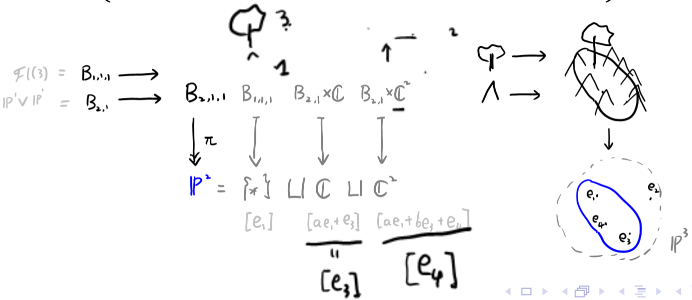


Example: $\lambda = (2,1,1)$

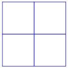
e_2	e_1
e_4	
e_3	

$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowleft X_\lambda$$



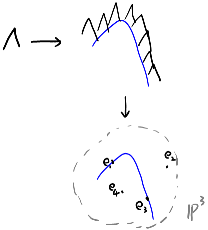
Example: $\lambda = (2,2)$



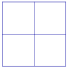
$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowleft X_\lambda$$

$$\begin{tikzcd} \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \arrow[r] & B_{2,2} \arrow[d, "\pi"] \\ & \mathbb{P}^3 \circlearrowleft S_4 \end{tikzcd}$$



Example: $\lambda = (2,2)$

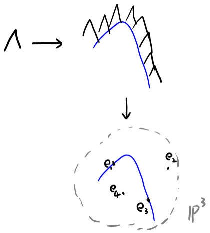


$$X_\lambda = \begin{bmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{bmatrix}$$

$$B_\lambda = \left\{ 0 \subseteq \langle\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ \rangle \subseteq \langle\ ?\ ,\ ?\ ,\ ?\ \rangle \subseteq \mathbb{C}^4 \right\} \curvearrowleft X_\lambda$$

$1 + 1 \overset{?}{=} 2$

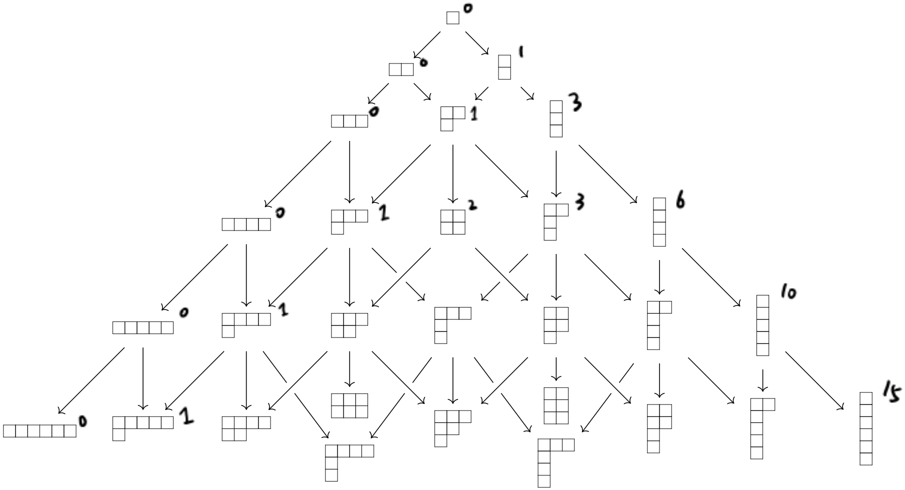
$$\begin{array}{ccccccc} \mathbb{P}^1 \vee \mathbb{P}^1 = B_{2,1} \longrightarrow & B_{2,2} & & B_{2,1}\times\{*\} & & B_{2,1}\times\mathbb{C} \\ & \downarrow \pi & & \downarrow & & \downarrow \\ & \mathbb{P}^1 & = & \{*\} & \sqcup & \mathbb{C} \\ & & & [e_1] & & [ae_1+e_3] \end{array}$$



Using the same technique, we can get

- B_λ has an affine paving $\rightsquigarrow$ cohomology;
- Each irreducible component in B_λ has same dimension;
- It's easy to compute the dimension and the number of irreducible component.

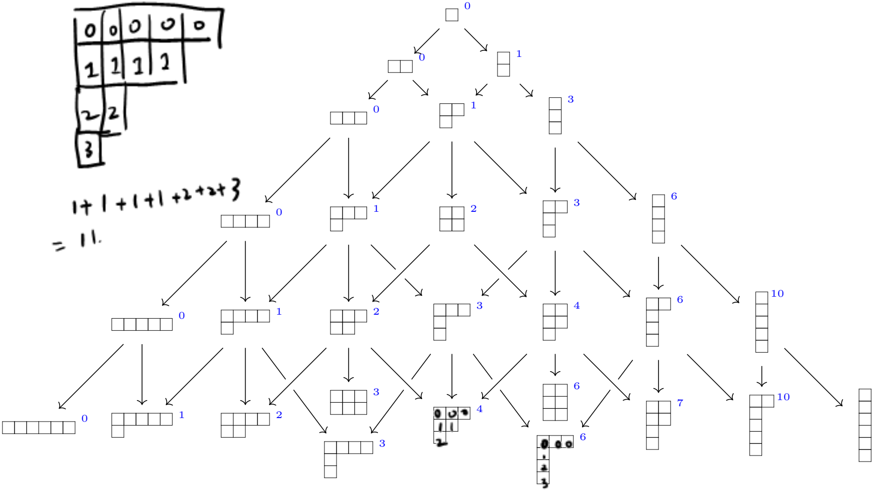
Game: compute!



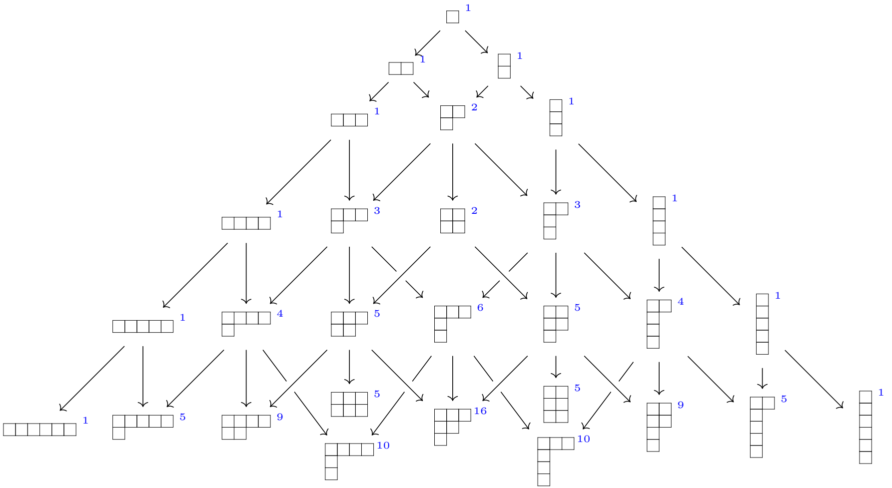
Answer: dimension

0	0	0	0	0
1	1	1	1	
2	2			
3				

$1+1+1+1+2+2+3$
$= 11.$



Answer: the number of irreducible component



Smooth problem

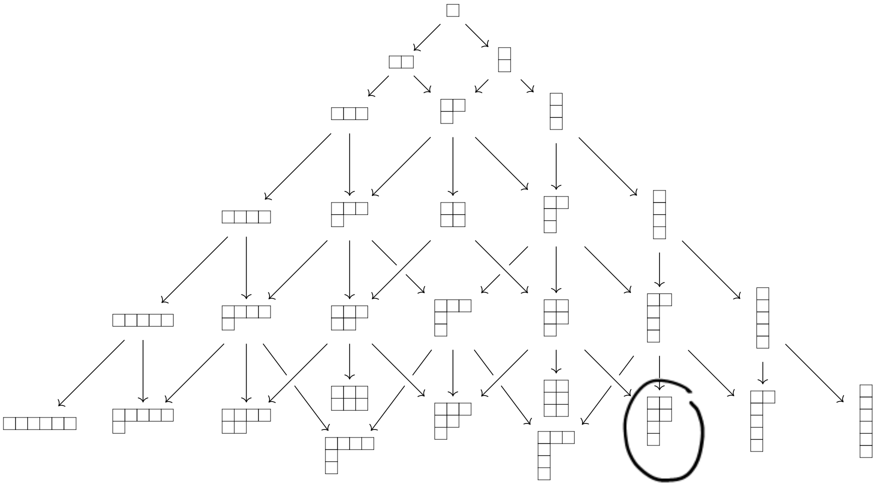
Results

- Not all the the irreducible components of B_λ are smooth; For example, one component of $B_{2,2,1,1}$ is not smooth.

- All the components of B_λ are nonsingular iff

$$\lambda \in \{(\lambda_1, 1, 1, \ldots), (\lambda_1, \lambda_2), (\lambda_1, \lambda_2, 1), (2, 2, 2)\}$$

tree of Young diagram



(m, m) case

We have an explicit description in the 2-row case when we forget the variety structure. Use this description, we can get the cohomology group structure.

Definition and Theorem

Let α be a crossingless matching, define

$$\tilde{B}_{\alpha;\, m,m} := \left\{ (x_1, \ldots, x_{2m}) \in (\mathbb{P}^1)^{2m} \middle| x_i = x_j \text{ if } (i,j) \in \alpha \right\} \subseteq (\mathbb{P}^1)^{2m}$$

$$\tilde{B}_{m,m} := \bigcup_{\alpha} \tilde{B}_{\alpha;\, m,m} \subseteq (\mathbb{P}^1)^{2m}$$

then we have a homeomorphism

$$B_{m,m} \cong \tilde{B}_{m,m}$$

(m, m) case

Definition and Theorem

Let α be a crossingless matching, define

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then we have a homeomorphism

$$B_{m,m} \cong \tilde{B}_{m,m}$$

$\mathbb{F}_2 = \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1})$

$n = 2m$

$\mathbb{F}_{2m}(\mathbb{R})$ = torus 

$n = 2m+1$

$\mathbb{F}_{2m+1}(\mathbb{R})$ = klein bottle 

Example (m=2)

$m=2$

1 2 3 4

$$\alpha = \{(1,2), (3,4)\} \qquad \tilde{B}_{\alpha;\, 2,2} = \left\{ (x_1, x_1, x_2, x_2) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

1 2 3 4

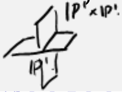
$$\beta = \{(1,4), (2,3)\} \qquad \tilde{B}_{\beta;\, 2,2} = \left\{ (x_1, x_2, x_2, x_1) \in (\mathbb{P}^1)^4 \right\} \cong (\mathbb{P}^1)^2$$

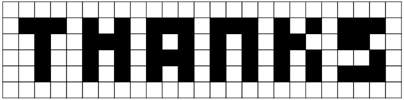
$\mathbb{P}^1 \times \mathbb{P}^1 \bigvee_{\mathbb{P}^1}$

$\mathbb{P}^1 - \mathbb{F}_2$

$\mathbb{P}^1$

$$B_{2,2} \cong \tilde{B}_{2,2} \cong (\mathbb{P}^1)^2 \bigvee_{\mathbb{P}^1} (\mathbb{P}^1)^2$$





Thank you for listening!
Thank Rui Xiong for providing the package of Young diagram,
Thank my roommate David Cueto for pointing out typos,
Thank Prof. Eberhart for offering valuable materials and advice!