Auslander–Reiten theory

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Jan Schröer's lecture notes should be a perfect reference.

In this talk, we dive into the huge forest of Auslander–Reiten theory.

	Last time	This time
Central concepts	quiver rep	ind rep & AR quiver
Proofs	relative easy	most skipped
Goal	comprehend	enjoy

Review

Exercise

Q:

$$\begin{tikzcd} & 1 \arrow[dl, "a"'] \arrow[dr, "c"] & \\ 2 \arrow[dr, "b"'] & & 3 \arrow[dl, "d"] \\ & 4 & \end{tikzcd}$$

$I = (ab - cd)$ Compute $\mathrm{Ext}^i_{KQ/I}(S(1), S(4))$

Definition

$$\underline{\dim}\, M := (\dim_K M_i)_{i \in v(Q)} \qquad \textit{for } M \in \mathrm{mod}(KQ/I)$$

Process

- Find more representations.
  - knitting process
  - introduction to root system
  - relations among indecomposable representations (Compute Hom, $\ker$, coker in a fancy way)
  - starting function

- From Dynkin quiver to affine quiver.
  - knitting process
  - new root system
  - tube
  - other cases

E.g. A_5 $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xleftarrow{c} 4 \xrightarrow{d} 5$

$$\begin{tikzcd}
1 \arrow[dr] & & & & \\
& 2 \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & \\
& & 3 & & 5
\end{tikzcd}$$

$$\begin{tikzcd}
& & P(3) \arrow[dl] \arrow[dr] & & P(5) \arrow[dl] \\
& P(2) \arrow[dl] & & P(4) & \\
P(1) & & & &
\end{tikzcd}$$

Exercise

$a\colon i \to j \implies f^a\colon P(j) \to P(i)$ is unique up to (nonzero) scalar.

E.g. A_5 $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xleftarrow{c} 4 \xrightarrow{d} 5$

$$\begin{tikzcd}[column sep=small]
1 \arrow[dr] & & & & & & \\
& 2 \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & & & \\
& & 3 & & 5 & &
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
& & 00100 \arrow[dl, "f^b"'] \arrow[dr, "f^c"] & & 00001 \arrow[dl, "f^d"'] \\
& 01100 \arrow[dl, "f^a"'] & & 00111 & \\
11100 & & & &
\end{tikzcd}$$

E.g. A_5 $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xleftarrow{c} 4 \xrightarrow{d} 5$

$$\begin{tikzcd}[column sep=small]
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& 01100 \arrow[dl, "f^a"'] & & 00111 & \\
11100 & & & &
\end{tikzcd}$$

(Initial case)

$$0 \longrightarrow 00001 \xrightarrow{f^d} 00111 \longrightarrow \operatorname{coker} f^d \longrightarrow 0$$

$$0 \longrightarrow 00100 \xrightarrow{\begin{pmatrix} f^b \\ f^c \end{pmatrix}} 01100 \oplus 00111 \longrightarrow \operatorname{coker} \begin{pmatrix} f^b \\ f^c \end{pmatrix} \longrightarrow 0$$

E.g. A_5 $\quad 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xleftarrow{c} 4 \xrightarrow{d} 5$

$$
\begin{tikzcd}
1 \arrow[dr] & & & & \\
& 2 \arrow[dr] & & 4 \arrow[dl] \arrow[dr] & \\
& & 3 & & 5
\end{tikzcd}
$$

$$
\begin{tikzcd}
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11100 & & 01111 & & 00110
\end{tikzcd}
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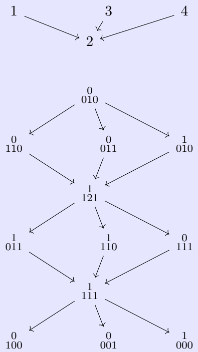
$$

$$
\begin{tikzcd}
& & 00100 \arrow[dl] \arrow[dr] & & 00001 \arrow[dl] \\
& 01100 \arrow[dl] \arrow[dr] & & 00111 \arrow[dl] \arrow[dr] & \\
11100 \arrow[dr] & & 01111 \arrow[dl] \arrow[dr] & & 00110 \arrow[dl] \\
& 11111 \arrow[dl] \arrow[dr] & & 01110 \arrow[dl] \arrow[dr] & \\
00011 \arrow[dr] & & 11110 \arrow[dl] \arrow[dr] & & 01000 \arrow[dl] \\
& 00010 & & 11000 \arrow[dr] & \\
& & & & 10000
\end{tikzcd}
$$

The constructed quiver is called the **Auslander–Reiten quiver**, and the process is called the **knitting algorithm**.

Another example: D_4

$$\begin{tikzcd} & 4 \arrow[d] & \\ 1 \arrow[r] & 2 & 3 \arrow[l] \end{tikzcd}$$



For other examples, see here.

Questions

- How many indecomposable representations are there?
- Do those dimension vectors follow any patterns?
- Where are those irreducible/projective/injective representations?