Subvarieties in Complex Abelian Varieties

Xiaoxiang Zhou
Advisor: Prof. Dr. Thomas Krämer
Humboldt-Universität zu Berlin

Berlin Mathematical School BMS

Tangent Gauss Map

Let $A/\mathbb{C}$ be an abelian variety of dimension n,
and let $Z \subset A$ be a non-degenerate closed subvariety of dimension r.

To understand the geometry of Z, we encode the variation of its tangent spaces via the tangent Gauss map

$$\phi_Z : Z^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longmapsto T_pZ \subset T_pA \cong T_0A.$$

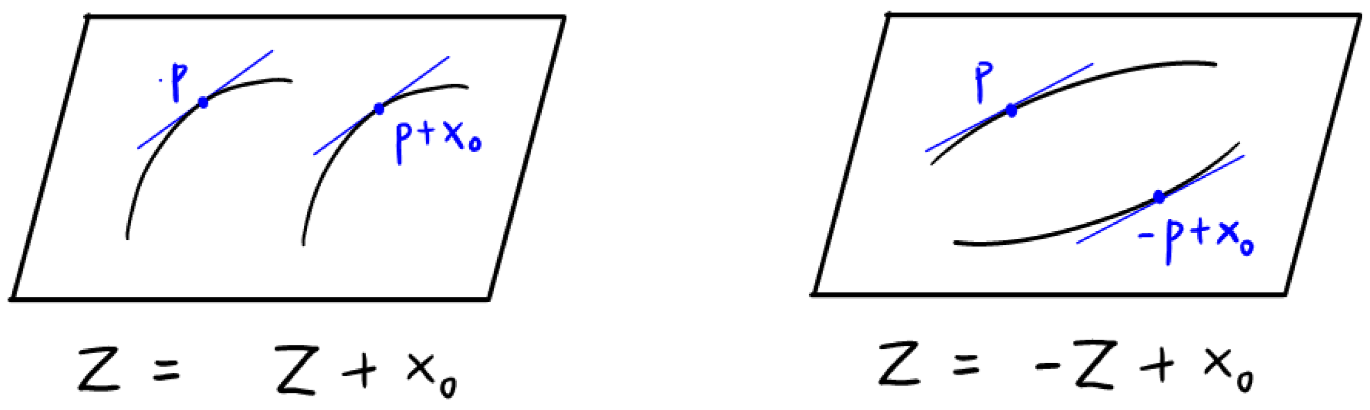
Its differential

$$d_p\phi_Z : T_pZ \longrightarrow \mathrm{Hom}_{\mathbb{C}}(T_pZ, N_pZ)$$

is second fundamental form, which captures information about the curvature of Z.

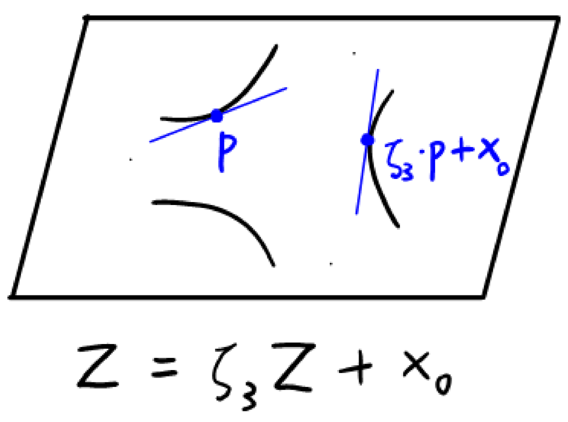
Generic Injectivity of the Tangent Gauss Map

Clearly the tangent Gauss map has degree > 1 whenever the subvariety $Z \subset A$ is stable under a nontrivial translation or symmetric up to a translation, as shown below:



Are these the only cases where γ_Z fails to be generically injective, when $Z = C$ is a curve? When $n = 2$, ϕ_C: $C^{\mathrm{sm}} \longrightarrow \mathbb{P}^1$ typically fails to be generically injective.

Question 1. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$, and assume that C is not invariant under any non-trivial translation or reflection. Does it follow that ϕ_C is generically injective?



Example 1. For any n the power $A = E_\rho^{\oplus n}$ comes with a natural diagonal action of $\mu_3 \cong \mathbb{Z}/3\mathbb{Z}$. Computer experiments yield a non-degenerate μ_3-invariant curve $C \subset A$, for which ϕ_C is not generically injective.

We have found no counterexample to Question 1 when A is not isogenous to $E_i^{\oplus n}$ or $E_\rho^{\oplus n}$. This suggests the following refinement:

Conjecture 1. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$. Then ϕ_C is generically injective unless there exists a non-trivial automorphism $\tau \in \mathrm{Aut}(A)$ which preserves C and acts via a scalar on T_0A.

One may restate Conjecture 1 using Gauss curvature, yielding a slightly stronger statement:

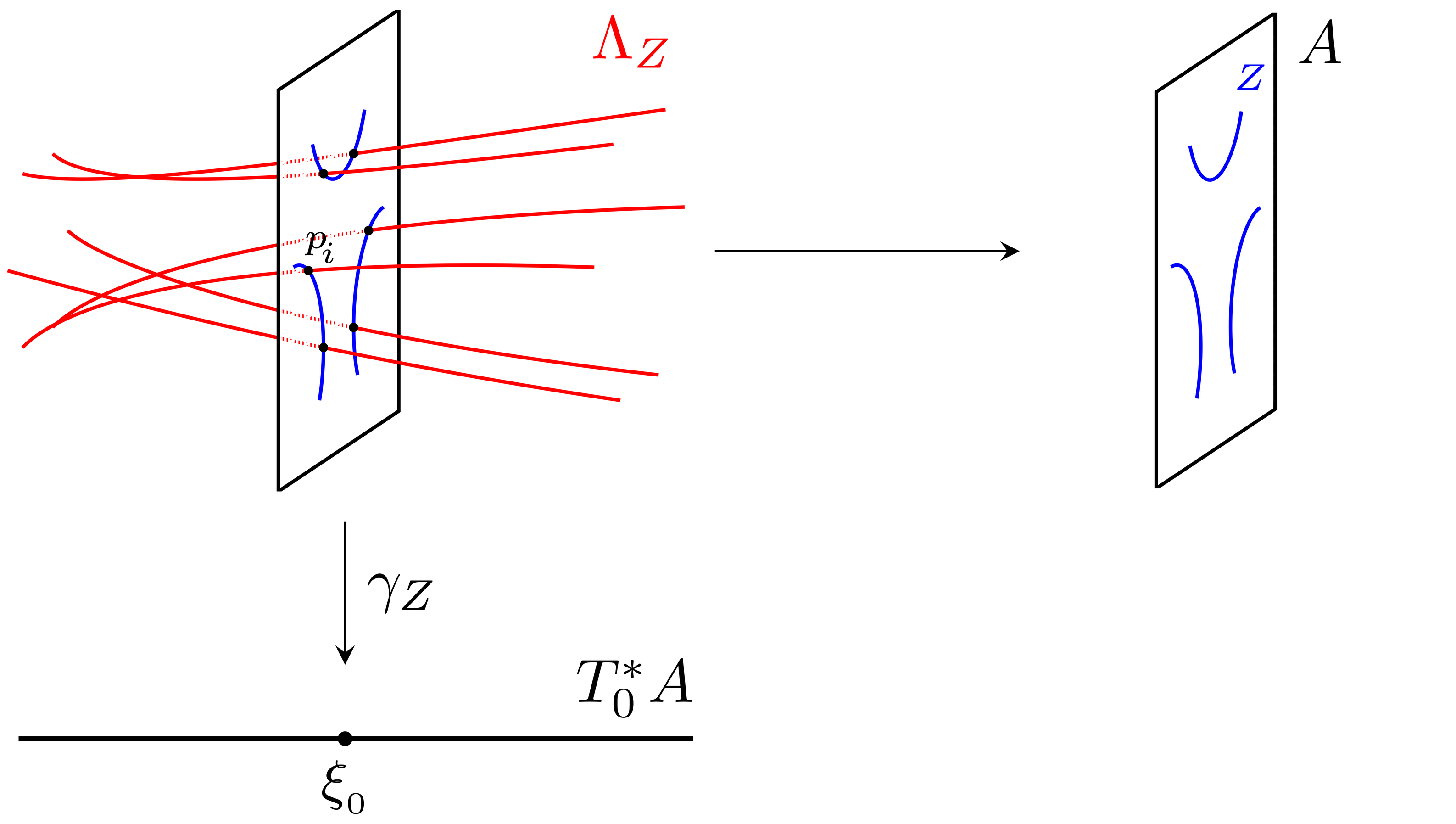
Conjecture 2. Let C be a non-degenerate curve on an abelian variety A of dimension $n > 2$. Then for general $p \in \mathrm{Im}\,\phi_C$, the Gauss curvature of $C \subset A$ is the same at all points of the fiber $\phi_C^{-1}(p)$.

Known Cases

1. If $A = \mathrm{Jac}(C)$ and C is embedded via the Abel–Jacobi map, then $\phi_C = |\omega_C|$ is the canonical map, and one may check Conjectures 1 and 2 by direct inspection.
2. Let $h : C \longrightarrow C'$ be a cyclic k-fold cover defined by $\eta \in \mathrm{Pic}(C')$ with $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. If $A = \mathrm{Prym}(C/C')$ and $C \to A$ is the Abel–Prym map, then
$$\phi_C : C \longrightarrow \mathbb{P}T_0A \cong \mathbb{P}\left(\bigoplus_{i=1}^{k-1} \mathrm{H}^0(\omega_{C'} \otimes \eta^i)\right)$$
   - $k = 2$: C is invariant under the Prym involution, and $\phi_C = |\omega_{C'} \otimes \eta| \circ h$.
   If C' is non-hyperelliptic with $g(C') \geq 4$, then $\deg \phi_C = 2$ or 4, and
   $$\deg \phi_C = 4 \iff B = \varnothing, C' \text{ is bielliptic and } \eta \text{ pulled back from EC.}$$
   - $k > 2$: if $g(C') \geq 1$ and $|\omega_{C'} \otimes \eta|$ is generically injective, then ϕ_C is generically injective.
3. If $C \subset A$ is smooth and either $\deg \phi_C = 2$ or ϕ_C is unramified, Conjecture 2 also holds.

Conormal Gauss Map

Passing to the cotangent perspective, one is naturally led to the conormal variety and the associated conormal Gauss map. Consider the conormal variety $\Lambda_Z \subset T^*A \cong A \times T_0^*A$. The natural projection is the conormal Gauss map $\gamma_Z^{(\mathrm{aff})} : \Lambda_Z \longrightarrow T_0^*A$, which is generically finite whenever Z is of general type.



Conjecture 3. Suppose A is not isogenous to $E_i^{\oplus n}$ or $E_\rho^{\oplus n}$, and let $C \subset A$ be a non-degenerate curve which is not stable under any non-trivial translation or reflection. Then the monodromy group $\mathrm{Gal}(\gamma_C)$ is big — namely, a Weyl group of type A, C, or D.

When $n > 2$, Conjecture 3 follows from Conjecture 2.

The monodromy group $\mathrm{Gal}(\gamma_C)$ helps us to control the Tannaka group of the tensor category generated by the IC sheaf on C; see [Krä22, Theorem 2.1].

The Subvariety $Z^{(m)}$

The convolution structure on perverse sheaves gives rise to numerous cycles in A. They admit a simple geometric description: by fiberwise summing points in $\gamma_Z^{-1}(\xi_0)$ and projecting, one obtains new subvarieties of varying dimensions.

Fix a general point $\xi_0 \in T_0^*A$ and choose an ordering $\gamma_Z^{-1}(\xi_0) = \{p_1, \ldots, p_d\} \subset Z$. For $(m) = (m_1, \ldots, m_d) \in \mathbb{Z}^d$, let $Z^{(m)}$ denote the irreducible component of the subvariety obtained by this construction that contains the point $\sum_i m_i p_i$.

Theorem 1. Let $c_i := c_{M,i}(\Lambda_Z)$ be the Segre class of the cone Λ_Z, and let $*$ denote the Pontryagin product.
When $\mathrm{Gal}(\gamma_Z) = S_d$, the Segre classes of $\Lambda_{Z^{(m)}}$ can be written as

$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{d_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^\lambda \left(\mathop{*}_{i=1}^{k'} c_{\lambda_i} \right)$$

where $\lambda = [\lambda_1, \ldots, \lambda_{k'}]$ ranges over all partitions of l and where

- $d_Z^{(m)} \in \mathbb{N}_{>0}$ is the degree of a certain addition map;
- $\mu_d^\lambda = \sum_{\alpha \in \mathcal{P}(d)} \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}:\text{ length } k}} \mu(\hat{0}, \alpha)\alpha(m)^{2\boldsymbol{l}} d^{k-k'} \in \mathbb{Z}[m_1, \ldots, m_d]^{S_d}$;
- $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}(d)$, $\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$;
- $\mu(\hat{0}, \alpha) = (-1)^{d-k} \prod_{i=1}^k (|A_i| - 1)$;
- $\alpha(m)^{2\boldsymbol{l}} = \left(\sum_{i \in A_1} m_i\right)^{2l_1} \cdots \left(\sum_{i \in A_k} m_i\right)^{2l_k}$.

Remarks.

1. One can recover both $\dim Z$ and $[Z] \in \mathrm{H}^{2(n-\dim Z)}(A)$ from the Segre classes of Λ_Z:
$$\dim Z = \max\{i \in \mathbb{Z} \mid c_i \neq 0\}, \qquad [Z] = c_{\dim Z}.$$
2. If $Z = -Z$, then a similar formula also holds for $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, but the method does not extend to the case where $\mathrm{Gal}(\gamma_Z) = W(D_{d/2})$.
3. The formula simplifies a lot for curves inside their Jacobian. For example, if C is non-hyperelliptic, we obtain:
$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \frac{1}{2^l (g-l)!} \sum_{\sigma \in S_{2g-2}} \prod_{i=1}^{l} \left(m_{\sigma(2i-1)} - m_{\sigma(2i)}\right)^2 \cdot \Theta^{g-l}$$
$$\dim Z^{(m)} = \min_{k \in \mathbb{Z}} \{g-1, \#\{i \in [2g-2] \mid m_i \neq k\}\}$$

References

[Krä20] Thomas Krämer. Summands of theta divisors on Jacobians. *Compos. Math.*, 156(7):1457–1475, 2020.

[Krä22] Thomas Krämer. Characteristic cycles and the microlocal geometry of the Gauss map. I. *Ann. Sci. Éc. Norm. Supér. (4)*, 55(6):1475–1527, 2022.

[Zho26] Xiaoxiang Zhou. Subvarieties in complex abelian varieties. `https://tinyurl.com/AVramified`, 2026.