SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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Contents

1. Introduction

Let A be a complex abelian variety of dimension n. To any closed subvariety $Z \subset A$ of dimension r, one can associate a reductive group G_Z through the convolution of perverse sheaves. This correspondence allows to reformulate problems in algebraic geometry in representation-theoretic terms, thereby offering new geometric insights. A crucial role in this context is played by the characteristic cycle, a fundamental invariant attached to a perverse sheaf, [7, §5–§8]. Recent work demonstrates that one can approach these characteristic cycles directly as conic Lagrangian cycles, without invoking the full machinery of perverse sheaves.

In [9, Remark 2.9], Krämer establishes a correspondence between Weyl group orbits and conic Lagrangian cycles in the cotangent bundle which arise as the conormal varieties of certain subvarieties. Despite this correspondence, an explicit geometric description of these cycles themselves remains elusive. In this article, we propose a purely geometric construction of these cycles and analyze three key properties of the associated subvarieties: irreducibility, dimension, and homology class.

To present our results, we begin by recalling several fundamental notions, which will be discussed in detail in Section 2. On a complex abelian variety A, the *translations* and *reflections* are special automorphisms of A of the form $x \mapsto x+p$ and $x \mapsto -x+p$, respectively. For $\sigma \in \mathrm{Aut}(A)$ and a closed subvariety $Z \subset A$, we say that Z is *invariant* under σ if $\sigma(Z) = Z$. Given a subvariety $Z \subset A$, the *tangent Gauss map* ϕ_Z assigns to $p \in Z^{\mathrm{sm}}$ the embedded tangent space to Z at p (cf. Definition 2.1), while the *conormal Gauss map* $\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0A \longrightarrow \mathbb{P}T_0A$ (cf. Definition 2.3) arises from the projectivized conormal variety of Z and is typically generically finite. The *monodromy group* $\mathrm{Gal}(\gamma_Z)$ encodes the permutation of the sheets of the covering γ_Z around its branch locus, while W_Z denotes the *Weyl group* of the reductive group G_Z.

Concerning irreducibility, it follows from the constructions in [9, 2.c] that the irreducible components of the appearing cycles correspond precisely to the orbits of a certain monodromy group. The discrepancy between the monodromy group and a closely related Weyl group provides a measure of the failure of irreducibility of these subvarieties. A natural question is how big this discrepancy could be. For this, we formulate a conjecture that refines [8, Conjecture 8]:

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Conjecture 1.1. *Let A be an abelian variety of dimension n, and let $C \subset A$ be a non-degenerate curve that is not invariant under any non-trivial translation. Then, the monodromy group $\mathrm{Gal}(\gamma_C)$ is big (see Definition 3.1).*

For $n > 2$, this conjecture admits a purely geometric reformulation:

Conjecture 1.2. *Let A be an abelian variety of dimension $n > 2$, and let $C \subset A$ be a non-degenerate curve that is not invariant under any non-trivial translation in A. Then:*
- *If C is invariant under some reflection, the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ is a double cover onto its image;*
- *If C is not invariant under any reflection, the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ is generically injective.*

Outside some counterexamples like Example 3.16, Conjecture 1.1 is still open. We analyze the Prym case thoroughly, which will lead to the following result:

Theorem A *For any double cover $h : C \longrightarrow C'$ of smooth, projective, non-hyperelliptic curves, one may view C as a subvariety of the Prym variety $\mathrm{Prym}(C/C')$ via the Abel–Prym map. Assume that $n := \dim_{\mathbb{C}} \mathrm{Prym}(C/C') > 2$. Then the associated monodromy group is big, except in the case where C' is bielliptic and h arises as the pullback of an étale double cover of elliptic curves. In this exceptional situation, the curve $C \subset \mathrm{Prym}(C/C')$ is stable under a non-trivial translation.*

Theorem A follows from a combination of Proposition 3.9 and Corollary 3.11.

For the dimension and homology class of a subvariety Z, both invariants can be recovered from the homology class of $\mathbb{P}\Lambda_Z \subset \mathbb{P}T^*A$ — that is, from the total Chern–Mather class of Z. Standard computational techniques are available when the monodromy group $\mathrm{Gal}(\gamma_Z)$ is a full or signed symmetric group. The notions of the Chern–Mather class and the Pontryagin product will be recalled in Definition 5.1 and Definition 5.6, respectively.

Theorem B *Consider a non-degenerate subvariety $Z \subset A$ and a tuple $(m) = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. Let $Z^{(m)} \subset A$ be as in Definition 4.7. Let $d = \deg \gamma_Z$, $\widetilde{d} = d/2$, and let $c_i := c_{M,i}(\Lambda_Z)$ be the Chern–Mather class of Z. Let $*$ be the Pontryagin product. We write $\lambda \dashv l$ to indicate that $\lambda = [\lambda_1, \ldots, \lambda_{k'}]$ is a partition of l.*

(1) When $\mathrm{Gal}(\gamma_Z) = S_d$, the Chern–Mather classes of $Z^{(m)}$ can be written as
$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right)$$
for certain explicit coefficients $c_Z^{(m)} \in \mathbb{N}_{>0}$ and $\mu_d^{\lambda} \in \mathbb{Z}$.

(2) When $Z = -Z$ and $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, the Chern–Mather classes of $Z^{(m)}$ can be written as
$$c_{M,l}\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \widetilde{\mu}_{\widetilde{d}}^{\lambda} \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i} \right)$$
for certain explicit coefficients $c_Z^{(m)} \in \mathbb{N}_{>0}$ and $\widetilde{\mu}_{\widetilde{d}}^{\lambda} \in \mathbb{Z}$.

The proof of theorem B will occupy Section 5. In fact, we will show (1) in (5.1), and (2) in (5.2). The combinatorial coefficients μ_d^{λ} and $\widetilde{\mu}_{\widetilde{d}}^{\lambda}$ appearing in the formulas are defined in Definition 5.12, while the case of Jacobian varieties is discussed in detail in Example 5.14.

The structure of the article is as follows.

In Section 2, we fix notation and describe the Gauss map from three different perspectives. We also define the monodromy group and highlight its connection to the degree of the tangent Gauss map.

Section 3 is devoted to the explicit computation of the monodromy group in selected cases, with a particular focus on the Prym curve situation. Subsection **??** presents an example in which the monodromy group is small; however, this does not constitute a counterexample to Conjecture 1.1, as the curve remains invariant under a non-trivial translation, as discussed in Lemma **??**. In Subsection 3.1, we collect several sufficient conditions for the monodromy group to be large, which we then employ in Subsection 3.2 to show that, for all remaining Prym curve cases with $g(C') > 9$, the monodromy group is necessarily large.

In Section 4, we define the subvarieties $Z^{(m)}$, which serve as the central objects of this study. In Subsection 4.2, we briefly discuss how these varieties naturally arise as the characteristic cycles of certain perverse sheaves.

In Section 5, we focus on the computation of the dimension and the homology class of $Z^{(m)}$. After recalling the definition of the Chern-Mather class in Subsection 5.1, the task reduces to computing $[\mathbb{P}\Lambda_{Z^{(m)}}]$. Subsections 5.3 and 5.4 treat the type A case, where $\mathrm{Gal}(\gamma_Z) = S_d$, and the type C case, where $\mathrm{Gal}(\gamma_Z) = W(C_{d/2})$, respectively. The derivation of the formulas involves certain coefficients that are purely combinatorial in nature. These combinatorial contributions are treated separately in Subsection 5.5.

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2. Tangent Gauss map and conormal Gauss map

We work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a integral separated scheme of finite type over $\mathbb{C}$. Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subsetneq A$ be an irreducible closed subvariety of dimension r. We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.

2.1. Gauss map and monodromy group.

Definition 2.1. *For a subvariety $Z \subset A$, the tangent Gauss map of Z is the map*

$$\phi_Z : \mathrm{Z}^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longmapsto T_pZ \quad \subseteq T_pA \cong T_0A$$

sending a smooth point to the tangent space at that point (seen as a subspace of T_0A).

Remark 2.2. Any map $X \longrightarrow \mathrm{Gr}(r, V)$ is induced by a rank r vector bundle $\mathcal{E}$ on X together with an epimorphism $V^* \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{E}$. In our case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}$ and the sections

$$T_0^*A \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \to \mathrm{H}^0(\mathrm{Z}^{\mathrm{sm}}, \mathcal{T}^*_{\mathrm{Z}^{\mathrm{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \twoheadrightarrow \mathcal{T}^*_{\mathrm{Z}^{\mathrm{sm}}}.$$

The definition of the conormal Gauss map requires a brief recollection of the conormal variety. On the smooth locus, the normal and conormal bundles are the vector bundles $\mathcal{N}_{\mathrm{Z}^{\mathrm{sm}}/A}$ resp. $\mathcal{N}^*_{\mathrm{Z}^{\mathrm{sm}}/A}$ defined by the short exact sequences

$$0 \longrightarrow \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \mathcal{T}_A|_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \mathcal{N}_{\mathrm{Z}^{\mathrm{sm}}/A} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{N}^*_{\mathrm{Z}^{\mathrm{sm}}/A} \longrightarrow \Omega_A|_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow \Omega_{\mathrm{Z}^{\mathrm{sm}}} \longrightarrow 0$$

We write $\Lambda_{\mathrm{Z}^{\mathrm{sm}}}$ for the total space of $\mathcal{N}^*_{\mathrm{Z}^{\mathrm{sm}}/A}$. The conormal variety Λ_Z is the closure of $\Lambda_{\mathrm{Z}^{\mathrm{sm}}}$ viewed as a subvariety in T^*A:

$$\Lambda_Z := \overline{\Lambda_{\mathrm{Z}^{\mathrm{sm}}}} \subset T^*A \cong A \times T_0^*A$$

It is a conical Lagrangian subvariety of T^*A. The (affine) conormal Gauss map is defined as

$$\gamma_Z^{\mathrm{aff}} : \Lambda_Z \subset A \times T_0^*A \longrightarrow T_0^*A$$

For intersection-theoretic purposes it is more natural to work with projectivized conormal spaces. We therefore pass from the affine conormal Gauss map to its projectivized version:

Definition 2.3. *The projectivized conormal variety of a subvariety $Z \subset A$ is defined by*

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A,$$

and the (projectivized) conormal Gauss map is defined by

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A.$$

By [7, Theorem 2.8 (1)], γ_Z is generically finite when Z is (an integral variety) of general type.

A lot of geometry of Z is encoded in the map γ_Z. For instance, if $Z \subset A$ is smooth, then

$$\deg \gamma_Z = (-1)^r \chi(Z),$$

where $\chi(Z) = \sum_i (-1)^i b_i(Z)$ is the topological Euler characteristic of Z, see [10, Theorem 1.1.1].

Further insight can be gained by analyzing the fibers of γ_Z. For instance, if Z is preserved by a translation $t_v : A \longrightarrow A$, then each fiber $\gamma_Z^{-1}(\xi) \subset A$ is also invariant under t_v. Likewise, if $Z = -Z$, then the fiber satisfies $\gamma_Z^{-1}(\xi) = -\gamma_Z^{-1}(\xi)$. Finding further constraints is more challenging.

An important invariant arising from the fiber $\gamma_Z^{-1}(\xi)$ is the monodromy group $\mathrm{Gal}(\gamma_Z)$; for completeness, we recall its definition below.

Definition 2.4. *Let $f : Y \longrightarrow X$ be a generically finite morphism between algebraic varieties. Then there exists a non-empty open subset $U \subseteq X$ such that the restriction $f^{-1}(U) \longrightarrow U$ is a finite étale cover. Moving along a closed loop in U induces a permutation of the points in the fiber $f^{-1}(\xi_0)$, which defines the map*[1]

$$\rho_f : \pi_1(U, \xi_0) \longrightarrow \mathrm{Aut}(f^{-1}(\xi_0)) \cong S_{\deg f}.$$

The monodromy group is then defined as the image of ρ_f, i.e.,

$$\mathrm{Gal}(f) := \mathrm{Im}\, \rho_f.$$

When U is smooth, the isomorphism class of $\mathrm{Gal}(f)$ doesn't depend on the choice of U.

2.2. **Interpolation via hyperplanes.** In this subsection, we reinterpret γ_Z using a functorial and more transparent framework. This permits to define the conormal Gauss map and its monodromy group for any morphism $\phi_Z : Z \longrightarrow \mathrm{Gr}(r, n)$ with $\dim Z = r$, and clarifies the relation between the monodromy group and $\deg \phi_Z$.

To simplify notation, we abbreviate T_0^*A by W. Since each non-zero conormal vector $\xi \in T_0^*A$ determines a hyperplane $H_\xi \in \mathrm{Gr}(n-1, T_0A)$, we have the isomorphisms

$$\begin{aligned} \mathbb{P}T_0^*A &\cong \mathrm{Gr}(n-1, T_0A), \\ \mathbb{P}\Lambda_{Z^{\mathrm{sm}}} &= \left\{ (p, \xi) \in \mathrm{Z^{sm}} \times \mathbb{P}T_0^*A \;\middle|\; \xi|_{T_pZ} \equiv 0 \right\} \\ &\cong \left\{ (p, H) \in \mathrm{Z^{sm}} \times \mathrm{Gr}(n-1, W) \;\middle|\; \phi_Z(p) \subseteq H \right\} \\ &\cong (\phi_Z, \mathrm{Id})^{-1} I_{r,n-1}, \end{aligned}$$

where

$$I_{r,n-1} := \left\{ (V, H) \in \mathrm{Gr}(r, W) \times \mathrm{Gr}(n-1, W) \;\middle|\; V \subseteq H \right\}$$

is the incidence variety relating $\mathrm{Gr}(r, W)$ and $\mathrm{Gr}(n-1, W)$. In these terms,

$$\begin{aligned} \gamma_Z^{-1}(H) \cap \mathrm{Z^{sm}} &= \{ p \in \mathrm{Z^{sm}} \mid \phi_Z(p) \subseteq H \} \\ &\cong \phi_Z^{-1}\left(\mathrm{Gr}(r, H)\right) \end{aligned}$$

is the collection of points whose tangent spaces lie entirely within H.

[1] In the last isomorphism we implicitly give a labelling from $\{1, 2, \dots, \deg f\}$ to the fiber $f^{-1}(\xi)$.

Geometrically, the monodromy can be described as follows: Given a general hyperplane $H \in \mathbb{P}T_0^*A$, its preimage under γ_Z consists of d points $p_1, \dots, p_d$. Moving H continuously along a closed loop we obtain a permutation of these points, and the monodromy group $\mathrm{Gal}(\gamma_Z)$ consists of all permutations obtained this way. With this formulation, it suffices to consider the Gauss map ϕ_Z alone; the inclusion $\iota_Z : Z \hookrightarrow A$ is no longer required for computing the monodromy group.

Definition 2.5. *Let Z be an r-dimensional variety and $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$ a morphism. Suppose that $\phi_*[Z] \cup [\mathrm{Gr}(r,H)] = d \neq 0$ in $\mathrm{H}^{2r(n-r)}(\mathrm{Gr}(r,n);\mathbb{Q})$. The monodromy group $\mathrm{Mon}(\phi)$ is defined as $\mathrm{Gal}(f_\phi)$, where*

$$f_\phi : (\phi, \mathrm{Id})^{-1} I_{r,n-1} \ \subset \ Z \times \mathrm{Gr}(n-1,n) \xrightarrow{\pi_2} \mathrm{Gr}(n-1,n)$$

is a generically finite morphism of degree d.

When ϕ is not generically injective, the monodromy group is subject to additional constraints, as captured by the next lemma.

Lemma 2.6. *Let $\phi : Z \longrightarrow \mathrm{Gr}(r,n)$ be generically k-to-1 onto its image. With a suitable ordering of the fibers, the associated monodromy group $\mathrm{Mon}(\phi)$ is contained in a wreath product*

$$S_k \wr S_{d/k} := \left(S_k^{\oplus d/k}\right) \rtimes S_{d/k}.$$

Proof. For a general hyperplane H, consider the Cartesian diagram below:

$$\begin{tikzcd} \phi : & Z \arrow[r, two heads, "k:1"] & \operatorname{Im} Z \arrow[r, hook] & \mathrm{Gr}(r,n) \\ & \cup & \cup & \cup \\ & \{p_1, \dots, p_d\} \arrow[r] & \{q_1, \dots, q_{d/k}\} \arrow[r] & \mathrm{Gr}(r,H) \end{tikzcd}$$

The fiber $\phi^{-1}(\mathrm{Gr}(r,H_0))$ splits into d/k groups of points, with the monodromy group acting by permutations within each group and among the groups. $\square$

2.3. **Interpolation via Albanese morphisms.** Examples of subvarieties of abelian varieties can be obtained in two ways: one may fix an abelian variety and consider cycles within it, or begin with a variety Z and construct a map to an abelian variety, such as the Albanese map. In the latter perspective, the setting changes slightly: X is a smooth projective variety of dimension r, $\iota_X : X \longrightarrow A$ is a morphism to an abelian variety A of dimension n, and $Z := \iota_X(X)$ denotes its image in A.

In this subsection, we begin with the Albanese morphism $X \longrightarrow \mathrm{Alb}(X)$, from which we derive some basic properties of the tangent Gauss map. Any other morphism to an abelian variety factors over the Albanese morphism, and then analogous methods apply, see Proposition 2.10.

Let X be a smooth complex projective variety of dimension r, and set $n := \dim_{\mathbb{C}} \mathrm{H}^0(X, \Omega_X) = h^{1,0}$. Recall that the Albanese variety of X is defined as

$$\mathrm{Alb}(X) := \mathrm{H}^0(X, \Omega_X)^* / \, \mathrm{H}_1(X, \mathbb{Z})_{\mathrm{free}},$$

and the Albanese map is given by (for some fixed base point $p_0 \in X$)

$$\alpha : X \longrightarrow \mathrm{Alb}(X) \qquad\qquad p \longmapsto \left[\ \omega \mapsto \int_{\gamma : p_0 \sim p} \omega \ \right].$$

One classical question is the dimension of $\alpha(X)$. Before proceeding, we fix the following notation: For any closed subscheme $S \subset X$, let $\mathcal{I}_S \subset \mathcal{O}_X$ be its ideal sheaf. Given a point $p \in X$ with inclusion $i_p : p \hookrightarrow X$, we denote by $i_{p,*}\mathcal{O}_p$ the skyscraper sheaf at p. We set $\Omega_X(-p) := \Omega_X \otimes \mathcal{I}_p$, and denote by $\Omega_X|_p := \Omega_X \otimes i_{p,*}\mathcal{O}_p = i_{p,*}i_p^*\Omega_X$ the fiber of Ω_X at p. With this notation, the cotangent map of α at $p \in X$ is

$$T_p^*\alpha : T_{\alpha(p)}^* \mathrm{Alb}(X) = H^0(X, \Omega_X) \longrightarrow T_p^*X = H^0(X, \Omega_X|_p).$$

Consider the short exact sequence of coherent sheaves on X

$$0 \longrightarrow \Omega_X(-p) \longrightarrow \Omega_X \longrightarrow \Omega_X|_p \longrightarrow 0$$

which induces a long exact sequence

$$
\begin{tikzcd}
 & \mathrm{H}^1(X,\Omega_X(-p)) \arrow[r] & \mathrm{H}^1(X,\Omega_X) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{H}^0(X,\Omega_X(-p)) \arrow[r] & \mathrm{H}^0(X,\Omega_X) \arrow[r, "T_p^*\alpha"] & \mathrm{H}^0(X,\Omega_X|_p) \arrow[llu, out=-10, in=180, looseness=1.5]
\end{tikzcd}
$$

The proposition below follows from standard arguments in homological algebra:

Proposition 2.7. *For a general point $p \in X$,*

$$
\begin{aligned}
\dim_{\mathbb{C}} \alpha(X) &= \operatorname{rank} T_p^*\alpha \\
&= n - h^0(X,\Omega_X(-p)) \\
&= h^1(X,\Omega_X(-p)) - h^1(X,\Omega_X).
\end{aligned}
$$

In particular,

$$
\begin{aligned}
&\alpha \textit{ is surjective} && \iff && h^0(X,\Omega_X(-p)) = 0 \\
&\alpha \textit{ is generically finite onto its image} && \iff && h^0(X,\Omega_X(-p)) = n - r \\
&\alpha \textit{ is constant} && \iff && h^0(X,\Omega_X(-p)) = n \\
& && \iff && h^1(X,\Omega_X(-p)) = h^1(X,\Omega_X) \\
& && \iff && n = 0.
\end{aligned}
$$

Proof. We know that

$$
\begin{aligned}
&\alpha \text{ is surjective} && \iff && \dim_{\mathbb{C}} \alpha(X) = n \\
&\alpha \text{ is generically finite onto its image} && \iff && \dim_{\mathbb{C}} \alpha(X) = r \\
&\alpha \text{ is constant} && \iff && \dim_{\mathbb{C}} \alpha(X) = 0 \\
& && \iff && \mathrm{Alb}(X) = 0.
\end{aligned}
$$

□

We will concentrate on the case where α is generically injective.[2] Under this assumption, we set $Z = \iota_X(X)$ and $A = \mathrm{Alb}(X)$. The corresponding Gauss map is then a rational map:

$$
\begin{aligned}
\phi_Z : Z &\dashrightarrow \mathrm{Gr}(r, T_0A) \cong \mathrm{Gr}\left(n-r, \mathrm{H}^0(X,\Omega_X)\right) \\
p &\longmapsto \mathrm{H}^0(X,\Omega_X(-p))
\end{aligned}
$$

and we have the isomorphisms

$$
\begin{aligned}
\mathbb{P}T_0^*A &\cong \mathbb{P}\,\mathrm{H}^0(X,\Omega_X), \\
\mathbb{P}\Lambda_Z &= \{\,(p,[\omega]) \in Z \times \mathbb{P}T_0^*A \mid \omega(p) = 0\,\} \\
&\cong \left\{\,(p,[\omega]) \in Z \times \mathbb{P}T_0^*A \;\middle|\; \omega \in \mathrm{H}^0(X,\Omega_X(-p))\,\right\} \\
&\cong (\phi_Z, \mathrm{Id})^{-1}\, I_{n-r,1},
\end{aligned}
$$

where

$$
I_{n-r,1} := \left\{\,(V,[\omega]) \in \mathrm{Gr}(n-r,n) \times \mathrm{Gr}(1,n) \;\middle|\; \omega \in V\,\right\}
$$

is the incidence variety relating $\mathrm{Gr}(n-r,n)$ and $\mathrm{Gr}(1,n)$. In that case,

$$
\gamma_Z^{-1}([\omega]) = \{\, p \in X \mid \omega(p) = 0\,\}
$$

[2] The general method remains valid in the broader setting, but the Gauss map then takes values in a different space. In fact, there is always a semicontinuous map

$$
\phi_X : X \longrightarrow \bigsqcup_{i=0}^{r} \mathrm{Gr}\left(n-i, \mathrm{H}^0(X,\Omega_X)\right) \qquad p \longmapsto \mathrm{H}^0(X,\Omega_X(-p)).
$$

is the zero set of the section $\omega \in \mathrm{H}^0(X, \Omega_X)$. The number $(-1)^r \deg \gamma_Z$ is the index (of the vector field) in the Poincaré–Hopf index formula (see [13, p35]), and the monodromy group $\mathrm{Gal}(\gamma_Z)$ serves as a more refined invariant.

The next proposition shows when the Gauss map ϕ_Z is not generically injective. We denote by $X^{[m]}$ the Hilbert scheme of m-points on X.

Proposition 2.8. *When α is generically finite onto its image,*

$$\begin{aligned} & \phi_Z \text{ is not generically injective} \\ \iff & \text{For general } p \in X, \text{ there exists } q \neq p \text{ such that } h^0(X, \Omega_X(-p-q)) = n-r \\ \overset{(1)}{\iff} & \text{For general } p \in X, \text{ there exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) = n-r \\ \overset{(2)}{\iff} & \text{For all } p \in X, \text{ there exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) \geqslant n-r. \end{aligned}$$

Proof. When $n = r$, the map φ_Z has a point as its target, so the equivalence is trivial. We shall thus restrict to the case $n > r$ in the subsequent discussion.

(1): The implication "$\Rightarrow$" is immediate. For the converse "$\Leftarrow$", it suffices to show that the image of

$$\left\{ S \in X^{[2]} \;\middle|\; h^0(X, \Omega_X \otimes \mathcal{I}_S) = n-r \right\}$$

under the map $\pi_X : X^{[2]} \longrightarrow X^{(2)}$ does not include the diagonal. Indeed, we can choose a section $s \in \mathrm{H}^0(X, \Omega_X)$ that cuts out finitely many reduced points $p_1, \ldots, p_d$ in X. (Well this maybe not so true, since ϕ_Z is not always regular, such a section s may vanish along some indeterminacy locus. However, for a general s, there will always be at least one isolated zero p_1, which is sufficient for our purposes.) Then for any $S \in \pi_X^{-1}(2p_1)$, we have

$$\mathrm{H}^0(X, \Omega_X \otimes \mathcal{I}_S) \subsetneq \mathrm{H}^0(X, \Omega_X(-p_1)).$$

(2): The implication "$\Rightarrow$" follows from the geometry of the projection $\mathrm{pr}_1 : X \times X^{[2]} \longrightarrow X$. By Lemma 2.9 and the closedness of the tautological correspondence in $X \times X^{[2]}$, the subset

$$I^{n-r} := \left\{ (p, S) \in X \times X^{[2]} \;\middle|\; p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) \geqslant n-r \right\}$$

is closed in $X \times X^{[2]}$. Consequently, its image

$$\mathrm{pr}_1(I^{n-r}) = \left\{ p \in X \;\middle|\; \text{there exists } S \in X^{[2]} \text{ such that } p \in S \text{ and } h^0(X, \Omega_X \otimes \mathcal{I}_S) = n-r \right\}$$

is closed in X. Since the hypothesis ensures that $\mathrm{pr}_1(I^{n-r})$ contains a nonempty open subset of X, we obtain $\mathrm{pr}_1(I^{n-r}) = X$.

For "$\Leftarrow$": as α is generically injective, one has $h^0(X, \Omega_X(-p)) = n-r$ for a general point $p \in X$. Therefore, for any subscheme S containing such a general point, $h^0(X, \Omega_X \otimes \mathcal{I}_S) \leqslant n-r$. □

Lemma 2.9. *Let $m \in \mathbb{Z}_{>0}$. The function*

$$h^\Omega : X^{[m]} \longrightarrow \mathbb{Z}_{\geqslant 0} \qquad S \longmapsto h^0(X, \Omega_X \otimes \mathcal{I}_S)$$

is Zariski upper semicontinuous.

Proof. Consider the coherent sheaf $\mathcal{F} \in \mathrm{Coh}(X^{[m]} \times X)$ characterized by the property

$$\mathcal{F}|_{\{S\} \times X} \cong \Omega_X \otimes \mathcal{I}_S,$$

The proposition then follows directly from the semicontinuity theorem [14, 28.1.1]. □

The stratification of $X^{[m]}$ by h^Ω offers a natural generalization of Brill–Noether theory beyond the setting of curves.

At the end of this subsection, let us turn to the setting of a general abelian variety A and a smooth subvariety $\iota_Z : Z \hookrightarrow A$, where $n = \dim_{\mathbb{C}} A$ and $r = \dim_{\mathbb{C}} Z$. Observe that ι_Z factors through the Albanese variety of Z:

$$\iota_Z : Z \xrightarrow{\alpha_Z} \mathrm{Alb}(Z) \xrightarrow{\pi} A$$

We shall also assume that Z generates A; it then follows that the map π is surjective. The cotangent map of ι_Z at a point $p \in Z$ factors through $\mathrm{H}^0(Z, \Omega_Z)$:

$$T_p^*\iota_Z : T_p^*A \hookrightarrow \mathrm{H}^0(Z, \Omega_Z) \longrightarrow T_p^*Z$$

For convenience, abbreviate $V := T_0^*A \cong T_p^*A$, and view V as a subspace of $\mathrm{H}^0(Z, \Omega_Z)$.

Proposition 2.10. *Assume that Z is embedded in A and generates A, and let $V := T_0^*A$. Then*

$$\dim_{\mathbb{C}} \mathrm{H}^0(Z, \Omega_Z(-p)) \cap V = n - r \qquad \text{for all } p \in Z$$

It follows that the Gauss map is a regular morphism

$$\begin{aligned} \phi_Z : Z &\longrightarrow \mathrm{Gr}(r, T_0A) \cong \mathrm{Gr}(n-r, V) \\ p &\longmapsto \mathrm{H}^0(Z, \Omega_Z(-p)) \cap V \end{aligned}$$

Furthermore,

$$\begin{aligned} & \phi_Z \text{ is not generically injective} \\ \iff & \text{For general } p \in Z, \text{ there exists } q \neq p \text{ such that } h^0(Z, \Omega_Z(-p-q)) \cap V = n - r \\ \iff & \text{For all } p \in Z, \text{ there exists } S \in Z^{[2]} \text{ such that } p \in S \text{ and } h^0(Z, \Omega_Z \otimes \mathcal{I}_S) \cap V = n - r. \end{aligned}$$

3. Monodromy group

3.1. Criteria for big monodromy group.

Definition 3.1 (big monodromy group). *We refer to the big monodromy group as any group of the following types:*

notation	*name*	*alias*
$W(A_{m+1}) = S_m$	*full symmetric group*	
$W(C_m) = S_2^{\oplus m} \rtimes S_m$	*signed symmetric group*	*hyperoctahedral group*
$W(D_m) = (S_2^{\oplus m})_0 \rtimes S_m$	*even-signed symmetric group*	*demihyperoctahedral group*

Table 1. big monodromy group

In practice, the term "big monodromy group" refers to the full symmetric group S_n when the subset $Z \subset A$ is not symmetric, and to the (even-)signed symmetric group when $Z \subset A$ is symmetric.

Proposition 3.2 (See [2, p111] for a detailed proof). *Suppose that $\iota_C : C \subseteq \mathbb{P}^{n-1}$ is an irreducible nondegenerate*[3] *curve of degree d, then $\mathrm{Mon}(\iota_C) \cong S_d$.*

Sketch of proof. Because S_d is generated by its transpositions, we are reduced to verifying that:
- $\mathrm{Mon}(\iota_C)$ acts doubly transitively on the fiber;
- $\mathrm{Mon}(\iota_C)$ contains a transposition.

□

In fact, a degree $2:1$ map does not give rise to any exceptional monodromy groups beyond those listed in Table 1.

[3] A curve $C \subseteq \mathbb{P}^{n-1}$ is said to be nondegenerate if it is not contained in any hyperplane $H \subseteq \mathbb{P}^{n-1}$.

Proposition 3.3. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \longrightarrow C'$ be a degree 2 ramified covering. Then*

$$\mathrm{Mon}(\iota_{C'} \circ h) \cong W(C_{d/2}) \text{ or } W(D_{d/2}).$$

Sketch of proof. By Lemma 2.6 we know that $\mathrm{Mon}(\iota_{C'} \circ h) \subseteq W(C_{d/2})$. By Lemma 3.4, we are reduced to verifying that:

- The quotient map $\mathrm{Mon}(\iota_{C'} \circ h) \longrightarrow \mathrm{Mon}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- (signed doubly transitive) $\mathrm{Mon}(\iota_{C'} \circ h)$ acts transitively on pairs (x, y) with $x \neq \pm y$.

□

Lemma 3.4. *Let G be a subgroup of $W(C_m)$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \longrightarrow S_m$ is surjective then*

$$G \cong W(C_m) \text{ or } W(D_m) \text{ or } S_2 \times S_m \text{ or } S_m.$$

Sketch of proof. Let H denote the kernel of the natural quotient map $G \longrightarrow S_m$. Then $H \subseteq (S_2)^{\oplus m}$ is stable under the action of S_n. There are only four possible forms that H can take:[4]

- $H = 0$. Then $G \cong S_m$.
- $H = \langle(-1, \dots, -1)\rangle \cong S_2$. Then $G \cong S_2 \times S_m$.
- $H = (S_2^{\oplus m})_0$. Then G is a index 2 subgroup of $W(C_m)$, so $G \cong W(D_m)$.[5]
- $H = S_2^{\oplus m}$. Then $G = W(C_m)$.

□

By incorporating further information about the covering, we are able to determine the monodromy group.

Proposition 3.5. *Let $\iota_{C'} : C' \hookrightarrow \mathbb{P}^{n-1}$ be an irreducible nondegenerate curve of degree $d/2$, and let $h : C \longrightarrow C'$ be a degree 2 ramified covering, with ramification occurring at at least one smooth point of C'. Then $\mathrm{Mon}(\iota_{C'} \circ h) \cong W(C_{d/2})$ is the hyperoctahedral group/signed symmetric group.*

Sketch of proof. By Lemma 2.6 we know that $\mathrm{Mon}(\iota_{C'} \circ h) \subseteq W(C_{d/2})$. By Lemma 3.6, we are reduced to verifying that:

- The quotient map $\mathrm{Mon}(\iota_{C'} \circ h) \longrightarrow \mathrm{Mon}(\iota_{C'}) \cong S_{d/2}$ is surjective;
- $\mathrm{Mon}(\iota_{C'} \circ h)$ contains a transposition of a pair of points in the fiber of h.

□

Lemma 3.6. *Let G be a subgroup of $W(C_m)$, acting naturally on the set $\pm 1, \dots, \pm m$. If the projection $G \longrightarrow S_m$ is surjective and the transposition σ_0 of ± 1 lies in G, then $G = W(C_m)$.*

Sketch of proof. Let ε_i denote the transposition of $\pm i$. For any $\sigma \in S_m$, choose a lift $\tilde{\sigma} \in G$, then

$$\varepsilon_{\sigma(1)} = \tilde{\sigma} \circ \sigma_0 \circ \tilde{\sigma}^{-1} \in G.$$

Thus, $S_2^{\oplus m} \subset G$, and since G maps onto S_m, we obtain $G = W(C_m)$. □

Proof that Conjecture 1.1 is equivalent to Conjecture 1.2 when $n > 2$. If ϕ_C is generically injective, Proposition 3.2 yields $\mathrm{Gal}(\gamma_C) \cong S_d$. When $C = -C$ and ϕ_C is a double cover, Proposition 3.3 implies $\mathrm{Gal}(\gamma_C) \cong W(C_{d/2})$ or $W(D_{d/2})$, both of which are big monodromy groups. In all other cases, Lemma 2.6 shows that $\mathrm{Gal}(\gamma_C) \subseteq S_k \wr S_{d/k}$, which is not big whenever $k > 2$ and $k \neq d$. Note that $k = d$ occurs only when $n = 2$. □

[4]Here is a brief argument showing that H must be one of 0, S_2, $(S_2^{\oplus m})_0$, or $S_2^{\oplus m}$. If H contains an element $h = (a_1, \dots, a_m)$ with $a_i \neq a_j$ for some $i \neq j$, then

$$(ij)h + h = (1, \dots, \underset{\substack{\uparrow \\ i\text{-th}}}{-1}, \dots, \underset{\substack{\uparrow \\ j\text{-th}}}{-1}, \dots, 1) \in H,$$

implying $(S_2^{\oplus m})_0 \subseteq H$. Hence, H must be either $(S_2^{\oplus m})_0$ or $S_2^{\oplus m}$. Otherwise, if all $h \in H$ have identical coordinates, then H is either 0 or S_2.

[5]Check stackexchange discussions

3.2. **Curves with big monodromy group.** The availability of these criteria permits the systematic construction of numerous cases where the associated monodromy group is big.

Example 3.7. *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$. In this case, the tangent Gauss map is the canonical map $|\omega_C|$.*

When C is non-hyperelliptic, the corresponding Gauss map

$$|\omega_C| : C \longrightarrow \mathbb{P}^{g-1}$$

makes C as an irreducible nondegenerate curve of degree $2g-2$, by Proposition 3.2 we get

$$\mathrm{Gal}(\gamma_C) \cong S_{2g-2}.$$

When C is hyperelliptic, the corresponding Gauss map is $2:1$ onto a rational normal curve $R \subset \mathbb{P}^{g-1}$:

$$|\omega_C| : C \xrightarrow{2:1} R \hookrightarrow \mathbb{P}^{g-1}$$

By Proposition 3.5 we get

$$\mathrm{Gal}(\gamma_C) \cong S_2^{\oplus g-1} \rtimes S_{g-1}.$$

The Prym case is more intricate, since the associated monodromy group may fail to be large.

Setting 3.8. *Let $C'/\mathbb{C}$ be a smooth projective curve, $\eta \in \mathrm{Pic}(C')$ a line bundle, and B a reduced effective (possibly zero) divisor on C'. Suppose $k > 1$ and that there is an isomorphism $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. According to [3, §17], these data determine a cyclic k-fold covering $h : C \longrightarrow C'$ of smooth projective curves, branched exactly along B. Denote by σ the generator of the Galois group $\mathrm{Gal}(C/C') \cong \mathbb{Z}/k\mathbb{Z}$.*

Recall that the Prym variety $A := \mathrm{Prym}(C/C')$ is defined as the connected component of the identity in

$$\ker \left[\mathrm{Nm}: \ \mathrm{Jac}(C) \longrightarrow \mathrm{Jac}(C') \right]$$

Its dimension is denoted by n.

The Abel–Prym map is defined by

$$\mathrm{AP}_{C/C'} : C \longrightarrow A \qquad p \longmapsto \mathcal{O}_C(p - \sigma(p)).$$

The following example, which essentially restates [5, Corollary 2.2], illustrates the case where the cover is an étale double cover.

Proposition 3.9. *In Setting 3.8, assume further that $k = 2$, $B = \varnothing$, and that C' is non-hyperelliptic of genus $g(C') \geqslant 4$.*

- *If C' is bielliptic and $\eta = \mathrm{pr}^* \eta_0$, where $\mathrm{pr} : C' \longrightarrow E'$ is a bielliptic map and $\eta_0 \in \mathrm{Pic}^0(E')[2]$, then the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ has degree 4, and the curve $C \subset A$ is invariant under a 2-torsion translation in A.*
- *Otherwise, the tangent Gauss map $\phi_C : C \longrightarrow \mathbb{P}^{n-1}$ is a double cover onto its image, and $\mathrm{Gal}(\gamma_C)$ is big.*

Proof. When $k = 2$, the corresponding Gauss map γ_C factors through h:

$$\begin{aligned} \gamma_C : C \xrightarrow{\ h\ } C' \xrightarrow{|\omega_{C'} \otimes \eta|} \mathbb{P}^{n-1} &= \mathbb{P}\big(\mathrm{H}^0(\omega_{C'} \otimes \eta)^*\big) \\ &\cong \mathrm{Gr}\left(n-1, \mathrm{H}^0(\omega_{C'} \otimes \eta)\right) \\ \tilde{p} \longmapsto p \longmapsto \ & \mathrm{H}^0\left(\omega_{C'} \otimes \eta(-p)\right) \end{aligned}$$

where $|\omega_{C'} \otimes \eta|$ denotes the Prym–canonical map. According to [5, Corollary 2.2], one has $\deg |\omega_{C'} \otimes \eta| \in \{1, 2\}$, and the case $\deg |\omega_{C'} \otimes \eta| = 2$ occurs precisely when C' is bielliptic and $\eta = \mathrm{pr}^* \eta_0$. In this situation, let $h_E\colon E \longrightarrow E'$ denote the unramified double covering corresponding to η_0, and let $\iota \in \mathrm{Gal}(E/E')$ be the associated involution. The following diagram is Cartesian:

$$\begin{tikzcd}[column sep=small] & \eta \arrow[dl, dash] & & & & C \arrow[dl, "h"'] \arrow[dr, "p"] & \\ C' \arrow[dr, "\mathrm{pr}"'] & & \eta_0 \arrow[dl, dash] & & C' \arrow[dr, "\mathrm{pr}"'] & & E \arrow[dl, "h_E"] \\ & E' & & & & E' & \end{tikzcd}$$

For any $p_0 \in E$, set

$$\mathcal{L}_0 := \mathcal{O}_E\left(p_0 - \iota(p_0)\right) \in \mathrm{Pic}^0(E)[2],$$

which does not depend on the choice of p_0. The curve $C \subset A$ is then invariant under translation by $p^*\mathcal{L}_0 \in A$. □

We now examine all possible cases of double covers. Motivated by [12, Lemma 2.1], we formulate the following result:

Lemma 3.10. *Let C' be a smooth projective curve of genus $g(C') \geqslant 4$, and let $\eta \in \mathrm{Pic}(C')$ be a line bundle of degree $s \geqslant 0$. Then the linear system $|\omega_{C'} \otimes \eta|$ is not generically injective if and only if one of the following holds:*

(a) $s = 2$ and for all $x \in C'$, there exists $y \in C'$ such that $\eta \cong \mathcal{O}_{C'}(x + y)$;
(b) $s = 1$ and for all $x \in C'$, there exist $y, u \in C'$ such that $\eta \cong \mathcal{O}_{C'}(x + y - u)$;
(c) $s = 0$ and for all $x \in C'$, there exist $y, u, v \in C'$ such that $\eta \cong \mathcal{O}_{C'}(x + y - u - v)$.

Furthermore, in cases (a) and (b), the curve C' must be hyperelliptic. In case (c), either $g(C') = 4$ and the linear system $|\omega_{C'} \otimes \eta|$ has degree 3, or C' is bielliptic and $\eta = \mathrm{pr}^*\,\eta_0$, where $\mathrm{pr} : C' \longrightarrow E'$ denotes the bielliptic map and $\eta_0 \in \mathrm{Pic}^0(E')$.

Proof. This is a consequence of Riemann–Roch. In case (a), since $\eta \in g^1_2$, the curve C' is hyperelliptic. In case (b), one obtains a family of g^1_3's of the form $\mathcal{O}_{C'}(x + y + u)$, which implies that $\mathcal{O}_{C'}(x + y) \in g^1_2$. Case (c) is a reformulation of [5, Corollary 2.2]. The key argument is to bound the degree of the morphism defined by $|\omega_{C'} \otimes \eta| : C' \longrightarrow \mathbb{P}^{g-2}$. When this degree equals 2, the image is a curve of almost minimal degree, which in this situation is a smooth elliptic curve. □

Corollary 3.11. *In Setting 3.8, assume further that $k = 2$, $B \neq \varnothing$, and that C' is non-hyperelliptic of genus $g(C') \geqslant 4$. Then the Prym-canonical map $|\omega_{C'} \otimes \eta|$ is generically injective, and therefore $\mathrm{Gal}(\gamma_C)$ is big.*

3.3. **Extend automorphism from curve to abelian variety.** In the process of persuing proving Conjecture 1.2, we propose the following lemma which will roll out a lot of trivial cases.

Setting 3.12. *Let A be a complex abelian variety of dimension $n \geqslant 2$, and let $\tilde{C}$ be a smooth projective curve of genus g_ν. Consider a morphism $\nu : \tilde{C} \longrightarrow A$ whose image $C = \nu(\tilde{C})$ is non-degenerate, and let $\phi_\nu = \phi_C \circ \nu$ denote the tangent Gauss map of $\tilde{C}$. Fix an automorphism $\tau \in \mathrm{Aut}(\tilde{C})$ of order k, inducing automorphisms*

$$\tilde{\tau} : \mathrm{H}^0(\omega_{\tilde{C}}) \longrightarrow \mathrm{H}^0(\omega_{\tilde{C}}) \qquad \tilde{\tau} : \mathrm{Jac}(\tilde{C}) \longrightarrow \mathrm{Jac}(\tilde{C}).$$

Denote $J = \mathrm{Jac}(\tilde{C})$, $\tilde{W} = \mathrm{H}^0(\omega_{\tilde{C}}) = T_0^*J$, and $W = T_0^*A$. We regard W as a subspace of $\tilde{W}$ via the fixed linear embedding $\iota_W : W \hookrightarrow \tilde{W}$, which is induced by the cotangent map associated to the morphism $J \longrightarrow A$. We would assume $\phi_\nu \circ \tau = \phi_\nu$ as the rational map.

Lemma 3.13. *In Setting 3.12, suppose that for there exists a uniform number $c \in \mathbb{C}^*$ such that the following diagram commutes for every $p \in \tilde{C}$:*

$$\begin{tikzcd} T_p\tilde{C} \arrow[r, "T_p\nu"] \arrow[d, "T_p\tau"'] & T_{\nu(p)}A \arrow[r, phantom, "\cong"] & W^* \arrow[d, "c"] \\ T_{\tau(p)}\tilde{C} \arrow[r, "T_{\tau(p)}\nu"] & T_{\nu(\tau(p))}A \arrow[r, phantom, "\cong"] & W^* \end{tikzcd} \tag{3.1}$$

In this situation, W is contained in the eigenspace of $\tilde{\tau} \in \mathrm{GL}(\tilde{W})$ corresponding to the eigenvalue c. Furthermore, ν admits a lift $\tilde{\nu} : \tilde{C} \longrightarrow \tilde{A}$ to an isogeny of A, where τ extends to an automorphism of $\tilde{A}$ as a variety (not necessarily as an algebraic group), and this extension satisfies the compatibility relation:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\tilde{\nu}"] \arrow[d, "\tau"'] & \tilde{A} \arrow[d] \\ \tilde{C} \arrow[r, "\tilde{\nu}"] & \tilde{A} \end{tikzcd}$$

Proof. Consider the following diagram:

$$\begin{tikzcd} \tilde{C} \arrow[r, "|\omega_{\tilde{C}}|"] \arrow[d, "\tau"'] & \mathbb{P}\tilde{W}^* \arrow[d, "\mathbb{P}\tilde{\tau}^*"'] \arrow[dr, dashed, "\mathbb{P}\iota_W^*"] & \\ & & \mathbb{P}W^* \\ \tilde{C} \arrow[r, "|\omega_{\tilde{C}}|"] & \mathbb{P}\tilde{W}^* \arrow[ur, dashed, "\mathbb{P}\iota_W^*"'] & \end{tikzcd}$$

where $\iota_W^* : \tilde{W}^* \longrightarrow W^*$ is the dual linear map of ι_W. By functoriality of $|\omega_{\tilde{C}}|$, the left square of the diagram commutes. From the definition of ϕ_ν, we have $\phi_\nu = \mathbb{P}\iota_W^* \circ |\omega_{\tilde{C}}|$, which ensures the outer boundary of the diagram commutes. Under condition (3.1), the right triangular part also commutes, and furthermore

$$\iota_W^* \circ \tilde{\tau}^* = c \cdot \iota_W^*.$$

Taking duals gives

$$\tilde{\tau} \circ \iota_W = c \cdot \iota_W,$$

so W lies in the eigenspace of $\tilde{\tau}$ with eigenvalue c.

$$\begin{tikzcd} \tilde{W}^* \arrow[r, two heads, "\iota_W^*"] \arrow[d, "\tilde{\tau}^*"'] & W^* \arrow[d, "c"] \\ \tilde{W}^* \arrow[r, two heads, "\iota_W^*"] & W^* \end{tikzcd} \qquad \begin{tikzcd} \tilde{W} & W \arrow[l, hook', "\iota_W"'] \\ \tilde{W} \arrow[u, "\tilde{\tau}"] & W \arrow[l, hook', "\iota_W"'] \arrow[u, "c"'] \end{tikzcd}$$

Fix $p_0 \in \tilde{C}$ and assume, for convenience, that $\nu(p_0) = 0$. By the universal property of J, there exists a morphism $\iota_W^* : J \longrightarrow A$ induced by $\iota_W^* : \tilde{W}^* \longrightarrow W^*$ such that $\nu = \iota_W^* \circ \mathrm{AJ}_{\tilde{C},p_0}$. Furthermore, again by the universal property of J, there exists an automorphism $\tau^* \in \mathrm{Aut}(J)$ induced by $\tilde{\tau}^* \in \mathrm{GL}(\tilde{W}^*)$ for which the corresponding diagram commutes:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\mathrm{AJ}_{\tilde{C},p_0}"] \arrow[dd, "\tau"'] & J \arrow[r, phantom, "\cong"] \arrow[d, "\tau^*"] & \tilde{W}^*/\mathrm{H}_1(\tilde{C};\mathbb{Z}) \arrow[d, "\tilde{\tau}^*"] \\ & J \arrow[r, phantom, "\cong"] \arrow[d, "t_{\tau(p_0)}"] & \tilde{W}^*/\mathrm{H}_1(\tilde{C};\mathbb{Z}) \\ \tilde{C} \arrow[r, "\mathrm{AJ}_{\tilde{C},p_0}"] & J & \end{tikzcd}$$

Since the subgroup $\iota_W^* \mathrm{H}_1(\tilde{C};\mathbb{Z}) \subseteq \mathrm{H}_1(A;\mathbb{Z})$ has finite index, the quotient $\tilde{A} := W^*/\iota_W^* \mathrm{H}_1(\tilde{C};\mathbb{Z})$ defines an abelian variety isogenous to $A = W^*/\mathrm{H}_1(A;\mathbb{Z})$, and the commutative diagram extends:

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\mathrm{AJ}_{\tilde{C},p_0}"] \arrow[dd, "\tau"'] & J \arrow[d, "\tau^*"] & \cong & \tilde{W}^*/\mathrm{H}_1(\tilde{C};\mathbb{Z}) \arrow[d, "\tilde{\tau}^*"] \arrow[r, two heads, "\iota_W^*"] & W^*/\iota_W^* \mathrm{H}_1(\tilde{C};\mathbb{Z}) \arrow[d, "c"] & \cong & \tilde{A} \arrow[d] \\
 & J \arrow[d, "t_{\tau(p_0)}"] & \cong & \tilde{W}^*/\mathrm{H}_1(\tilde{C};\mathbb{Z}) \arrow[r, two heads, "\iota_W^*"] & W^*/\iota_W^* \mathrm{H}_1(\tilde{C};\mathbb{Z}) & \cong & \tilde{A} \arrow[d, "t_{\tau(p_0)-p_0}"] \\
\tilde{C} \arrow[r, "\mathrm{AJ}_{\tilde{C},p_0}"] & J \arrow[rrrrr] & & & & & \tilde{A}
\end{tikzcd}$$

□

Remark 3.14. If $\tau \in \mathrm{Aut}(\tilde{C})$ extends to an automorphism $\tau_A : A \to A$ with linear part τ_A^L, then condition (3.1) holds automatically. In fact, the following diagram commutes:

$$\begin{tikzcd}
T_p\tilde{C} \arrow[r, "T_p\nu"] \arrow[d, "T_p\tau"'] & T_{\nu(p)}A \arrow[d, "T_{\nu(p)}\tau_A"] & \cong & W^* \arrow[d, "T_0\tau_A^L"] \\
T_{\tau(p)}\tilde{C} \arrow[r, "T_{\tau(p)}\nu"] & T_{\nu(\tau(p))}A & \cong & W^*
\end{tikzcd}$$

Since τ is of finite order, the endomorphism $T_0\tau_A^L$ is diagonalizable. If $T_0\tau_A^L$ possesses at least two eigenvalues, then by the non-degeneracy of C there exists a point $p \in \tilde{C}$ such that $\phi_\nu(\tau(p)) \neq \phi_\nu(p)$.

The resulting full automorphism of $\tilde{A}$ is always a rotation by c composed with a translation. Such automorphisms occur only rarely, as established in the following lemma.

Lemma 3.15. *Let A be an abelian variety of dimension n. Each automorphism $\tau \in \mathrm{Aut}(A)$ induces a linear automorphism $\tilde{\tau}^* \in \mathrm{GL}(T_0A)$. If $\tilde{\tau}^* = \zeta_k \cdot \mathrm{Id}_{T_0A}$ for some k-th root of unity ζ_k, then $k \in \{1,2,3,4,6\}$. Moreover, if $k \in \{3,4,6\}$, there exists an isogeny*

$$E_1 \times \ldots \times E_n \longrightarrow A,$$

where each E_i is an elliptic curve satisfying $\mathrm{Aut}(E_i) \supseteq \mathbb{Z}/k\mathbb{Z}$.

Proof. If $\tilde{\tau}^*$ acts by the scalar ζ_k on T_0A, then the induced action on $\mathrm{H}_1(A;\mathbb{Q})$ has complex eigenvalues ζ_k and ζ_k^{-1}, each occurring with multiplicity n. Hence, the characteristic polynomial of $\tilde{\tau}$ on $\mathrm{H}_1(A;\mathbb{Q})$ is

$$(T-\zeta_k)^g(T-\zeta_k^{-1})^g \in \mathbb{Q}[T].$$

The rationality of this polynomial forces $k \in \{1,2,3,4,6\}$.

When $k \in \{3,4,6\}$, one may choose an $\mathbb{R}$-basis of T_0A of the form $\{v_1, \zeta_k v_1, \ldots, v_n, \zeta_k v_n\}$ by successively selecting $v_i \in \mathrm{H}_1(A;\mathbb{Z})$ outside the complex span $\langle v_1, \ldots, v_{i-1}\rangle_{\mathbb{C}}$. Writing $E_i := \mathbb{C}/\mathbb{Z}[\zeta_k]$, we have an isogeny

$$E_1 \times \ldots \times E_n \longrightarrow A \qquad (z_1, \ldots, z_n) \longrightarrow \sum_{i=1}^n z_i v_i.$$

□

We can now provide a counterexample for Conjecture 1.2.

Example 3.16. *Consider the elliptic curve $E = \mathbb{C}/\mathbb{Z}[\zeta_3]$ and the abelian threefold $A = E^3$. Let $\sigma \in \mathrm{Aut}(E)$ be an automorphism of order 3, and define $\tau = (\sigma,\sigma,\sigma) \in \mathrm{Aut}(A)$. If a curve $C \subset A$ is invariant under τ, then its Gauss map $\phi_C : C \longrightarrow \mathbb{P}^2$ factors through the quotient morphism $C \longrightarrow C/\tau$. Applying a Bertini argument yields the existence of a smooth curve $C \subset A$ invariant under τ and under no nontrivial translation or reflection. For instance, with $\mathcal{L} = \mathcal{O}_E(6\cdot[0])^{\boxtimes 3} \in \mathrm{Pic}(A)$, two general sections $s_1, s_2 \in H^0(A,\mathcal{L})^\tau$ cut out such a smooth τ-invariant curve C.*

Example 3.17. *If we replace A with $\mathbb{C}^3$ in Conjecture 1.2, we can construct curves $C \subset \mathbb{C}^3$ that are not preserved under any linear transformation but whose Gauss maps fail to be generically injective. Here is one such example:*

Consider the curve $C \subset \mathbb{C}^3$ defined by the parametrization

$$r(t) = \left(t, (t-1)e^t, \frac{1}{4}(2t^2 - 2t + 1)e^{2t}\right).$$

Its derivative is

$$r'(t) = \left(1, te^t, t^2 e^{2t}\right),$$

so the Gauss map fails to be generically injective.

3.4. **Comparing tangent spaces via automorphism.** Assuming $\phi_\nu \circ \tau = \phi_\nu$ in Setting 3.12 allows us to compare $T_p\tilde{C}$ and $T_{\tau(p)}\tilde{C}$ either through the action of τ or via parallel transport along the flat connection on A. The difference between these two identifications defines a rational map $c : \tilde{C} \dashrightarrow \mathbb{G}_m$, as made precise below.

Definition 3.18. *Let $p \in \tilde{C}$ such that $T_p\nu$ and $T_{\tau(p)}\nu$ are injective. The twisting coefficient $c(p) \in \mathbb{C}^*$ associated with τ is the unique scalar for which the diagram*

$$\begin{tikzcd} T_p\tilde{C} \arrow[r, "T_p\nu"] \arrow[d, "T_p\tau"'] & T_{\nu(p)}A \cong W^* \arrow[d, "c(p)"] \\ T_{\tau(p)}\tilde{C} \arrow[r, "T_{\tau(p)}\nu"] & T_{\nu(\tau(p))}A \cong W^* \end{tikzcd} \tag{3.2}$$

commutes.

Proposition 3.19. *The twisting map*

$$c : \tilde{C} \dashrightarrow \mathbb{G}_m \qquad p \longmapsto c(p)$$

is algebraic. Consequently, it admits a unique extension to a morphism $c : \tilde{C} \longrightarrow \mathbb{P}^1$. If $T_p\nu$ is injective for all $p \in \tilde{C}$, then c is constant; moreover, this constant must be a root of unity.

Proof. Since

$$t^{-1} : \tilde{C} \times A \longrightarrow A \qquad (p, x) \longmapsto t_p^{-1}(x) := -\nu(p) + x$$

is algebraic, its differential

$$dt^{-1} : \tilde{C} \times TA \longrightarrow TA \qquad (p, v_x) \longmapsto (dt_p^{-1})(v_x)$$

is also algebraic. In fact, every algebraic morphism $f : X \longrightarrow Y$ induces an algebraic morphism $df : TX \longrightarrow TY$. It follows that all maps in diagram (3.3) are algebraic,

$$\begin{tikzcd} T\tilde{C} \times \mathbb{G}_m \arrow[r, "(\pi_{\tilde{C}}, d\nu)\times \mathrm{Id}_{\mathbb{G}_m}"] \arrow[d, "\pi_1"'] & \tilde{C} \times TA \times \mathbb{G}_m \arrow[r, "dt^{-1}\times \mathrm{Id}_{\mathbb{G}_m}"] & TA \times \mathbb{G}_m \arrow[dd, "m"] \\ T\tilde{C} \arrow[d, "d\tau"'] & & \\ T\tilde{C} \arrow[r, "(\pi_{\tilde{C}}, d\nu)"] & \tilde{C} \times TA \arrow[r, "dt^{-1}"] & TA \end{tikzcd} \tag{3.3}$$

and the subset $\mathcal{I}_{\mathrm{tw}} \subset T\tilde{C} \times \mathbb{G}_m$ where the diagram (3.3) commutes is algebraic. The graph of c can be expressed algebraically as $(\pi_{\tilde{C}} \times \mathrm{Id}_{\mathbb{G}_m})\left(\mathcal{I}_{\mathrm{tw}} \smallsetminus \tilde{C} \times \mathbb{G}_m\right)$ outside finitely many points of $\tilde{C}$; therefore, c itself is algebraic. □

Conjecture 3.20. *In Setting 3.12, the twisting map c is a constant map when $n \geqslant 3$.*

Proposition 3.21. *Conjecture 3.20 implies a refined form of Conjecture 1.2. To exclude the counterexamples of the type illustrated in Example 3.16, we require that either $\deg \phi_C = 2$ or that the abelian variety A is not isogenous to $E_{\mathrm{i}}^{\oplus n}$ or $E_\rho^{\oplus n}$, where $E_{\mathrm{i}} = \mathbb{C}/\mathbb{Z}[i]$ and $E_\rho = \mathbb{C}/\mathbb{Z}[\zeta_3]$ are the only elliptic curves with extra automorphism groups.*

Proof. In Conjecture 1.2, set $C' = \phi_C(C)$ and let $\nu : \tilde{C} \longrightarrow C$ be the natural map from the Galois closure $\tilde{C}$ of the covering C/C'. Then we are in Setting 3.12, and every $\tau \in \mathrm{Gal}(\tilde{C}/C')$ satisfies $\phi_\nu \circ \tau = \phi_\nu$.

If Conjecture 1.2 fails, there exists $\tau \in \mathrm{Gal}(\tilde{C}/C')$ that does not fix C/ι (when C is invariant under a reflection ι) or C (when C is not invariant under any reflection). Equivalently, one can fix $\tau \in \mathrm{Gal}(\tilde{C}/C')$ that does not extend to any reflection or translation of A. Note that, by assumption, the only translation preserving C is Id_A, and the reflection preserving C is unique when it exists.

By Conjecture 3.20, the condition (3.1) in Lemma 3.13 is satisfied. Hence, τ extends to a full automorphism $\tilde{\tau}$ of an abelian variety $\tilde{A}$ that is isogenous to A, and the induced automorphism on $T_0\tilde{A}$ takes the form $\tilde{\tau}^* = \zeta_k \cdot \mathrm{Id}_{T_0\tilde{A}}$ for some k-th root of unity ζ_k. By Lemma 3.15, one has $k \in \{1, 2, 3, 4, 6\}$, and the additional degree/isogeny condition restricts k to $\{1, 2\}$. For either value of k, the automorphism $\tilde{\tau} : \tilde{A} \longrightarrow \tilde{A}$ descends to an automorphism $\tau_A : A \longrightarrow A$, making the diagram

$$\begin{tikzcd}
\tilde{C} \arrow[r, "\tilde{\nu}"] \arrow[d, "\tau"'] & \tilde{A} \arrow[r] \arrow[d, "\tilde{\tau}"] & A \arrow[d, "\tau_A"] \\
\tilde{C} \arrow[r, "\tilde{\nu}"] & \tilde{A} \arrow[r] & A
\end{tikzcd}$$

commute. However, τ_A is necessarily a translation (if $k = 1$) or a reflection (if $k = 2$), contradicting our choice of τ. $\square$

Corollary 3.22. *Conjecture 3.20 (and thus the refined Conjecture 1.2) holds in the following cases:*

- *$C \subset A$ is smooth and $\deg \phi_C = 2$;*
- *$C \subset A$ is smooth and ϕ_C is unramified.*

Here, we set $C' = \phi_C(C)$ and let $\nu : \tilde{C} \longrightarrow C$ be the natural map from the Galois closure $\tilde{C}$ of the covering C/C', with $\tau \in \mathrm{Gal}(\tilde{C}/C')$ fixed arbitrarily.

Proof. In both cases, ν is unramified. Together with the smoothness of $C \subset A$, this ensures that

$$T_p\nu : T_p\tilde{C} \cong T_{\nu(p)}C \longrightarrow T_{\nu(p)}A$$

is injective for every $p \in \tilde{C}$. It follows from Proposition 3.19 that c is constant. $\square$

Proposition 3.23. *In Setting 3.8, suppose in addition that $k > 2$, $g(C') \geqslant 1$, and that the Prym–canonical map $|\omega_{C'} \otimes \eta|$ is generically injective. Under these assumptions, the refined Conjecture 1.2 is satisfied.*

Proof. By [12, Proposition 3.1], the Abel–Prym map $\mathrm{AP}_{C/C'} : C \to A$ is generically injective. Set $C'' = \phi_C(C)$; then the following diagram commutes:

$$\begin{tikzcd}
C \arrow[r, "\phi_C"] \arrow[d, "h"'] & C'' \subset \mathbb{P}T_0^*A \arrow[dl, dotted, "\exists"] \arrow[d, dashed] \\
C' \arrow[r, "|\omega_{C'} \otimes \eta|"'] & \mathbb{P}\,\mathrm{H}^0(\omega_{C'} \otimes \eta)^*
\end{tikzcd}$$

Since the linear system $|\omega_{C'} \otimes \eta|$ is generically injective, the map h factors through ϕ_C. As C/C' is Galois, it follows that C/C'' is Galois, $\tilde{C} = C$, and every $\tau \in \mathrm{Gal}(\tilde{C}/C'')$ fixes C'. Moreover, each $\tau \in \mathrm{Gal}(C/C')$ extends to an automorphism of A, so by Remark 3.14, the twisting map c is constant whenever $\phi_\nu \circ \tau = \phi_\nu$. The remaining argument proceeds as in the proof of Proposition 3.21. $\square$

3.5. **Conjecture in terms of curvature.** The twisting map may also be interpreted as comparing the second fundamental forms (and hence the curvature) at p and $\sigma(p)$. Indeed, the differential of ϕ_C at p induces

$$T_p\phi_C : T_pC \longrightarrow T_{\phi_C(p)}\,\mathrm{Gr}(1,n) = \mathrm{Hom}(T_pC, N_pC),$$

where $N_pC := T_pA/T_pC$ denotes the normal space at p. This map is equivalent to specifying the second fundamental form

$$\mathrm{I\!I}_p : T_pC \times T_pC \longrightarrow N_pC.$$

The construction is compatible with the action of σ in the sense that the following diagram commutes:

$$\begin{array}{ccc} T_pC \times T_pC & \xrightarrow{\mathrm{I\!I}_p} & N_pC \\ {\scriptstyle T_p\sigma}\downarrow \qquad \downarrow\cong & & \downarrow\cong \\ T_{\sigma(p)}C \times T_{\sigma(p)}C & \xrightarrow{\mathrm{I\!I}_{\sigma(p)}} & N_{\sigma(p)}C \end{array}$$

Upon identifying T_pC with $T_{\sigma(p)}C$ via parallel transport, one obtains the relation

$$\mathrm{I\!I}_p = c(p)\cdot \mathrm{I\!I}_{\sigma(p)}.$$

From now on, we fix a Hermitian metric on T_0A, which induces a flat Hermitian metric on A. Its pullback along $C \subset A$ endows C^{sm} with a Hermitian metric as well. For any point $p \in \mathrm{C}^{\mathrm{sm}}$, we denote by $\kappa(p)$ the Gauss curvature of C^{sm} at p.

Lemma 3.24. *The curve C^{sm} is a minimal surface, and its Gauss curvature satisfies*

$$\kappa(p) = -2\,(\mathrm{Nm}\,\mathrm{I\!I}_p)^2.$$

Proof. We begin by comparing the complex second fundamental form $\mathrm{I\!I}_p$ with its real counterpart

$$\mathrm{I\!I}_{p,\mathbb{R}} : T_pC_{\mathbb{R}} \times T_pC_{\mathbb{R}} \longrightarrow N_pC_{\mathbb{R}}.$$

Both admit a natural common extension to a J-bilinear form

$$\mathrm{I\!I} : T_pC \otimes_{\mathbb{R}} \mathbb{C} \times T_pC \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow N_pC \otimes_{\mathbb{R}} \mathbb{C}$$

via the canonical identification

$$T_pC \otimes_{\mathbb{R}} \mathbb{C} \cong T_pC_{\mathbb{R}} \oplus T^{0,1}C.$$

Writing $\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \in T_pC$, with orthonormal real tangent vectors $\partial_x, \partial_y \in T_pC_{\mathbb{R}}$, we obtain the following expressions for the mean curvature vector:

$$\begin{aligned} \kappa_{\mathrm{mean}}(p) &= \frac{1}{2}\Big(\mathrm{I\!I}(\partial_x,\partial_x) + \mathrm{I\!I}(\partial_y,\partial_y)\Big) \\ &= \frac{1}{2}\Big(\mathrm{I\!I}(\partial_x,\partial_x) + \mathrm{I\!I}(J\partial_x,J\partial_x)\Big) \\ &= \frac{1}{2}\Big(\mathrm{I\!I}(\partial_x,\partial_x) + \mathrm{I\!I}(\partial_x,\partial_x)\Big) \\ &= 0 \end{aligned}$$

and for the Gauss curvature:

$$\begin{aligned} \kappa(p) &= \Big\langle \mathrm{I\!I}(\partial_x,\partial_x), \mathrm{I\!I}(\partial_y,\partial_y)\Big\rangle - \Big\langle \mathrm{I\!I}(\partial_x,\partial_y), \mathrm{I\!I}(\partial_y,\partial_x)\Big\rangle \\ &= \Big\langle \mathrm{I\!I}(\partial_x,\partial_x), -\mathrm{I\!I}(\partial_x,\partial_x)\Big\rangle - \Big\langle J\mathrm{I\!I}(\partial_x,\partial_x), J\mathrm{I\!I}(\partial_x,\partial_x)\Big\rangle \\ &= -\Big\langle \mathrm{I\!I}(\partial_x,\partial_x), \mathrm{I\!I}(\partial_x,\partial_x)\Big\rangle - J\cdot(-J)\Big\langle \mathrm{I\!I}(\partial_x,\partial_x), \mathrm{I\!I}(\partial_x,\partial_x)\Big\rangle \\ &= -2\,|\mathrm{I\!I}(\partial_x,\partial_x)|^2 \\ &= -2\,(\mathrm{Nm}\,\mathrm{I\!I})^2 \\ &= -2\,(\mathrm{Nm}\,\mathrm{I\!I}_p)^2 \end{aligned}$$

$\square$

Conjecture 3.25. *Let A be an abelian variety of dimension $n > 2$. For a curve $C \subset A$ and a general point $p \in \operatorname{Im}\phi_C$, all points in $\phi_C^{-1}(p)$ exhibit the same Gauss curvature.*

Proposition 3.26. *Conjecture 3.25 is equivalent to Conjecture 3.20.*

Proof. By Lemma 3.24, we obtain the relation $\kappa(p) = |c(p)|^2 \cdot \kappa(\sigma(p))$.[6] By Proposition 3.19,

$$\begin{aligned} &c \text{ is a constant map} \\ \iff &|c(p)| \equiv 1 \text{ for all } p \\ \iff &\kappa(p) = \kappa(\sigma(p)) \text{ for all } p \end{aligned}$$

□

4. Families of subvarieties

In this section, we move from the study of monodromy groups to a more direct analysis of the subvarieties themselves. Given an initial subvariety, one can naturally generate a family of subvarieties. Our goal here is to define these families and investigate their properties.

4.1. Clean Lagrangian cycles.

Proposition 4.1. *All irreducible conic Lagrangian cycles in T^*A are of the form Λ_Z for some irreducible subvariety $Z \subset A$. This yields a one-to-one correspondence between irreducible conic Lagrangian cycles in T^*A and irreducible subvarieties of A:*

$$\{ \textit{ irreducible conic Lagrangian cycles in } T^*A \} \longleftrightarrow \{ \textit{ irreducible subvarieties in } A \}$$

Sketch of proof. For any irreducible conic Lagrangian cycle $\mathbf{\Lambda} \subset T^*A$, let Z denote the image of $\mathbf{\Lambda}$ under the natural projection $T^*A \longrightarrow A$. Our goal is to show that $\mathbf{\Lambda} = \Lambda_Z$.

- By definition, $\mathbf{\Lambda} \subset T^*A|_Z$.
- Since $\mathbf{\Lambda}$ is conic, we have $s(Z) \subset \mathbf{\Lambda}$, where $s : A \longrightarrow T^*A$ denotes the zero section.
- Since $\mathbf{\Lambda}$ is Lagrangian and $s(Z) \subset \mathbf{\Lambda}$, we have $\mathbf{\Lambda} \subset \Lambda_Z$.
- Since Λ_Z is irreducible with $\dim_{\mathbb{C}} \mathbf{\Lambda} = \dim_{\mathbb{C}} \Lambda_Z = n$, we have $\mathbf{\Lambda} = \Lambda_Z$.

□

Why do we shift attention from Z to Λ_Z as the main object of study? One reason is the uniformity of Λ_Z: it always has dimension n, and in most cases, the natural map $\Lambda_Z \longrightarrow T_0^*A$ is generically finite, with fibers lying inside A.

Definition 4.2 (Clean cycle)**.** *An irreducible Lagrangian cycle $\mathbf{\Lambda} \subset T^*A$ is called clean if the composed projection*

$$\mathbf{\Lambda} \longrightarrow T^*A \twoheadrightarrow T_0^*A$$

is generically finite.

---

[6]This identity may also be verified directly from the curvature formula expressed in terms of the Hermitian metric h:

$$\kappa = -\frac{1}{h}\partial_z\partial_{\bar z}\ln h.$$

The pullback metric σ^*h satisfies

$$\sigma^*h(p) = |c(p)|^2 h(p),$$

so

$$\begin{aligned} \kappa(\sigma(p)) &= -\frac{1}{\sigma^*h(p)}\left(\partial_z\partial_{\bar z}\ln\sigma^*h(z)\right)\Big|_{z=p} \\ &= -\frac{1}{|c(p)|^2 h(p)}\left(\partial_z\partial_{\bar z}\left(\ln h(z) + \ln|c(z)|^2\right)\right)\Big|_{z=p} \\ &= \frac{1}{|c(p)|^2}\cdot -\frac{1}{h(p)}\left(\partial_z\partial_{\bar z}\ln h(z)\right)\Big|_{z=p} \\ &= \frac{1}{|c(p)|^2}\cdot\kappa(p) \end{aligned}$$

Another important reason is that the space of weighted clean conic Lagrangian cycles naturally acquires a convolution structure, arising from the group law on A, which plays a central role in the analysis.

Proposition 4.3. *The group of weighted clean conic Lagrangian cycles*

$$\begin{aligned}\mathcal{L}(A) &:= \{\textit{weighted clean conic Lagrangian cycles in } T^*A\}\\ &= \left\{ \sum_{\substack{Z_i\subset A\\ \textit{irr clean}}} n_i\Lambda_{Z_i} \;\middle|\; n_i\in\mathbb{Z}\right\}\end{aligned}$$

has a natural convolution structure as follows:

$$\begin{aligned}\Lambda_{Z_1}\circ\Lambda_{Z_2} &= \textit{ the clean part of } (a,\mathrm{Id}_{T_0^*A})_*\left(\Lambda_{Z_1}\times_{T_0^*A}\Lambda_{Z_2}\right)\\ &= \overline{(a,\mathrm{Id}_U)_*\left(\Lambda_{Z_1}|_U\times_U\Lambda_{Z_2}|_U\right)}\end{aligned}$$

where

$$U := \left\{\xi\in T_0^*A \;\middle|\; \deg\phi_{Z_i} = \#\phi_{Z_i}^{-1}(\xi) \textit{ for } i=1,2\right\}$$

and $a : A\times A \longrightarrow A$ is the addition map in A. The general convolution is defined by $\mathbb{Z}$-linear extension.

Sketch of proof. To establish the claim, it suffices to show that $\Lambda_{Z_1}\circ\Lambda_{Z_2}$ defines a weighted conic Lagrangian cycle. The conic property follows directly from the definition, while the Lagrangian condition can be verified at a general point $(p_1+p_2,\xi)\in\Lambda_{Z_1}\circ\Lambda_{Z_2}$. □

We now consider the projective versions of all objects involved, so that we may make use of properness. To simplify notation, we abbreviate $\mathbb{P}T_0^*A$ by $\mathbb{P}^\vee$.

Lemma 4.4.

(1) Suppose that $\mathbb{P}\Lambda_{Z_1},\mathbb{P}\Lambda_{Z_1}\subset\mathbb{P}T^*A$ admit monodromy representations

$$\rho_{\gamma_{Z_i}} : \pi_1(U,\xi_0)\longrightarrow \mathrm{Aut}\left(\gamma_{Z_i}^{-1}(\xi_0)\right),$$

then $\mathbb{P}\Lambda_{Z_1}\times_{\mathbb{P}^\vee}\mathbb{P}\Lambda_{Z_2}\subset A\times A\times\mathbb{P}^\vee$ admits monodromy representation given by

$$(\rho_{\gamma_{Z_1}},\rho_{\gamma_{Z_2}}) : \pi_1(U,\xi_0)\longrightarrow \mathrm{Aut}\left(\gamma_{Z_1}^{-1}(\xi_0)\times\gamma_{Z_2}^{-1}(\xi_0)\right),$$

(2) When $Z_1=Z_2=Z$, we obtain an one-to-one correspondence:

$$\left\{\begin{matrix}\text{irr components of } \mathbb{P}\Lambda_Z\times_{\mathbb{P}^\vee}\mathbb{P}\Lambda_Z\\ \text{with a surjection to } \mathbb{P}^\vee\end{matrix}\right\}\longleftrightarrow\left\{\begin{matrix}\mathrm{Gal}(\gamma_Z)\text{-orbits of}\\ \gamma_Z^{-1}(\xi_0)\times\gamma_Z^{-1}(\xi_0)\end{matrix}\right\}$$

Sketch of proof. Statement (1) holds by definition. The proof of (2) reduces to the following purely topological statement:

Claim 4.5. *Let $\pi : E\longrightarrow B$ be a (unramified) covering space over a manifold B with deck transformation group G, then*

$$\{\ \textit{connected components of } E\times_B E\ \}\longleftrightarrow\{\ G\textit{-orbits of } \pi^{-1}(b_0)\times\pi^{-1}(b_0)\ \}.$$

The claim follows directly from the correspondence between covering spaces over B and $\pi_1(B)$-sets; see [6, Theorem 1.38]. □

Generalizing the argument of Lemma 4.4, we arrive at the following lemma.

Lemma 4.6. *For $d=\deg\gamma_Z$, $\xi_0\in T_0^*A$ a general point, write*

$$\begin{aligned}\mathbb{P}\Lambda_Z^{\times d} &:= \mathbb{P}\Lambda_Z\times_{\mathbb{P}^\vee}\mathbb{P}\Lambda_Z\times_{\mathbb{P}^\vee}\cdots\times_{\mathbb{P}^\vee}\mathbb{P}\Lambda_Z &&\subset A^d\times\mathbb{P}^\vee\\ \gamma_Z^{-1}(\xi_0)^d &:= \gamma_Z^{-1}(\xi_0)\times\gamma_Z^{-1}(\xi_0)\times\cdots\times\gamma_Z^{-1}(\xi_0) &&\subset A^d\end{aligned}$$

(1) we obtain an one-to-one correspondence:

$$\left\{\begin{array}{l}\text{irr components of } \mathbb{P}\Lambda_Z^{\times d}\\ \text{with a surjection to } \mathbb{P}^\vee\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\mathrm{Gal}(\gamma_Z)\text{-orbits of}\\ \gamma_Z^{-1}(\xi_0)^d\end{array}\right\}$$

(2) Write

$$\begin{aligned}\Delta_d &:= \left\{(p_1,\dots,p_d)\in A^d \;\middle|\; p_i=p_j \text{ for some } i\neq j\right\} && \subset A^d\\ \mathbb{P}\Lambda_Z^{[d]} &:= \overline{\left(\mathbb{P}\Lambda_Z^{\times d}\smallsetminus(\Delta_d\times\mathbb{P}^\vee)\right)|_U} && \subset A^d\times\mathbb{P}^\vee\end{aligned}$$

Fix a general point $\xi_0\in\mathbb{P}^\vee$ and a well-order for $\gamma_Z^{-1}(\xi_0)$, one can identify $S_d\cong\gamma_Z^{-1}(\xi_0)^d\smallsetminus\Delta_d$, and

$$\left\{\text{ irr components of } \mathbb{P}\Lambda_Z^{[d]}\right\}\longleftrightarrow\left\{\mathrm{Gal}(\gamma_Z)\text{-orbits of } S_d\right\}$$

In reference, Δ_d is usually called the big diagonal.

From this point on, we fix an irreducible component of $\mathbb{P}\Lambda_Z^{[d]}$, denoted by $\mathbb{P}\Lambda_Z^{\mathrm{univ}}$. As we will see in Definition 4.7, this variety generates all subvarieties within the families under consideration.

Definition 4.7 (The subvariety $Z^{(m)}$)**.** *For any tuple $(m)=(m_1,\dots,m_d)\in\mathbb{Z}^d$, we define the weighted sum map*

$$a^{(m)}:A^d\longrightarrow A\qquad (p_1,\dots,p_d)\longmapsto\sum_{i=1}^d m_ip_i.$$

We also define

$$\mathbb{P}\Lambda_Z^{(m)}:=\left(a^{(m)},\mathrm{Id}_{\mathbb{P}^\vee}\right)_*\mathbb{P}\Lambda_Z^{\mathrm{univ}}.$$

as the (projectivized) weighted Lagrangian cycle in $\mathbb{P}T^*A$. The projective cycle $\mathbb{P}\Lambda_Z^{(m)}$ is irreducible but may appear with multiplicities. We can therefore write

$$\mathbb{P}\Lambda_Z^{(m)}=c_Z^{(m)}\mathbb{P}\Lambda_{Z^{(m)}}$$

where $c_Z^{(m)}\in\mathbb{Z}_{>0}$ and $Z^{(m)}\subset A$ are uniquely determined. This gives rise to a family of subvarieties parametrized by $\mathbb{Z}^d$.

The next lemma gathers some basic properties of $Z^{(m)}$. Observe that $S_d=\mathrm{Aut}\left(\gamma_Z^{-1}(\xi_0)\right)$ acts naturally on $\mathbb{Z}^d$ via

$$g(m)=\left(m_{g(1)},\dots,m_{g(d)}\right)\in\mathbb{Z}^d.$$

Lemma 4.8.

(1) For all $g\in\mathrm{Gal}(\gamma_Z)$, we have $Z^{g(m)}=Z^{(m)}$, $c_Z^{g(m)}=c_Z^{(m)}$;
(2) For $(m)=(1,0,\dots,0)\in\mathbb{Z}^d$, $Z^{(m)}=Z$;
(3) For all $(m),(m')\in\mathbb{Z}^d$, we have $\mathbb{P}\Lambda_Z^{(m)}\circ\mathbb{P}\Lambda_Z^{(m')}\supseteq\mathbb{P}\Lambda_Z^{(m+m')}$;
(4) The group $\langle\mathbb{P}\Lambda_{Z^{(m)}}\rangle_{\mathrm{Abel}}$ is closed under the convolution product.

4.2. **Realized as characteristic cycles.** In fact, the Lagrangian cycles $\mathbb{P}\Lambda_{Z^{(m)}}$ coincide with the irreducible components of the clean cycles described in [9, 2.c] and [11, p5, Theorem 1.7], leading to the following relations:

$$\begin{array}{ccccc}\mathrm{Perv}(A)/N(A) & \supset & \langle\delta_Z\rangle & \cong & \mathrm{Rep}\left(G_u\right)\\ & & \big\downarrow{\scriptstyle \mathrm{cc}} & & \big\downarrow\\ \mathcal{L}(A) & \supset & \langle\mathrm{cc}(\delta_Z)\rangle & \overset{7}{\cong} & K_0\,\mathrm{Rep}\left(T_u\rtimes\mathrm{Gal}(\gamma_Z)\right)\end{array}$$

Here, δ_Z denotes the perverse intersection complex associated with the subvariety Z (in particular, $\delta_Z = \iota_{Z,*}\underline{\mathbb{Q}}_Z[-\dim Z]$ when Z is smooth), and $\mathcal{L}(A)$ stands for the λ-ring of clean conic Lagrangian cycles on T^*A [10, p5].

Remark 4.9. Suppose that $Z \subset A$ is smooth of general type. Then the characteristic cycle $\mathrm{cc}(\delta_Z)$ is irreducible and equals $\mathbb{P}\Lambda_Z$. In this situation, let $\lambda_1, \ldots, \lambda_d \subset X^*(T_u)$ denote the weights corresponding to the points $p_1, \ldots, p_d \in \gamma_Z^{-1}(\xi_0)$. For each tuple $(m) \in \mathbb{Z}^d$, define $\lambda^{(m)} = \sum m_i \lambda_i \in X^*(T_u)$, and consider the "highest weight representation"

$$V_{\lambda^{(m)}} = \bigoplus_{\lambda \in \mathrm{Gal}(\gamma_Z)\cdot\lambda^{(m)}} \mathbb{C}_\lambda \in \mathrm{Rep}\left(T_u \rtimes \mathrm{Gal}(\gamma_Z)\right)$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of T_u with weight λ. Under this correspondence, one has an explicit identification

$$\langle \mathbb{P}\Lambda_Z \rangle \cong \mathrm{Rep}\left(T_u \rtimes \mathrm{Gal}(\gamma_Z)\right) \qquad \mathbb{P}\Lambda_{Z^{(m)}} \longleftrightarrow V_{\lambda^{(m)}}.$$

Moreover, the Weyl group W_Z acts naturally on $X^(T)$. For the orbit $W_Z \cdot \chi_0 \subset X^(T)$, the associated Lagrangian cycle is given by

$$\sum_{\sigma \in W_Z / \mathrm{Gal}(\gamma_Z)} \mathbb{P}\Lambda_{Z^{\sigma(\chi_0)}}.$$

5. Dimension and homology class

In this section, all cohomology groups are taken with $\mathbb{Q}$-coefficients for convenience in applying the Künneth formula.

5.1. **Reminder on the (homological) Chern–Mather class.** We begin by recalling the definition of the Chern–Mather class. Suppose $\dim A = n$, and denote by

$$p : \mathbb{P}T^*A \cong A \otimes \mathbb{P}^\vee \longrightarrow \mathbb{P}^\vee$$

the natural projection.

Definition 5.1. *For a conic Lagrangian cycle Λ on T^*A and $i \geqslant 0$, the Chern–Mather class is defined as*

$$c_{M,i}(\Lambda) = p_*\left([\mathbb{P}\Lambda] \cdot [A \times H_i]\right) \in \mathrm{H}_{2i}(A) \cong \mathrm{H}^{2(n-i)}(A),$$

where $H_i \subseteq \mathbb{P}^\vee$ denotes a general linear subspace of dimension i. For brevity, we may later write $c_{M,i}(\Lambda)$ as $c_i(\Lambda)$, and $c_{M,i}(\Lambda_Z)$ simply as c_i. [8]

By the Künneth formula, we may write

$$[\mathbb{P}\Lambda] = \sum_{i=0}^{n-1} a_i \otimes H^i,$$

where $H \in H^2(\mathbb{P}^\vee)$ denotes the hyperplane class and $a_i \in H^{2(n-i)}(A)$. A direct computation gives

$$\begin{aligned} c_i(\Lambda) &= p_*\left([\mathbb{P}\Lambda] \cdot [A \times H_i]\right) \\ &= p_*\left(\sum_{j=0}^{n-1} \left(a_j \otimes H^j\right) \cup \left(1 \otimes H^{n-1-i}\right)\right) \\ &= p_*\left(\sum_{j=0}^{n-1} a_j \otimes H^{n-1-i+j}\right) \\ &= a_i. \end{aligned}$$

[7] I believe that this isomorphism should be already known, so I should probably cite it somewhere(rather than making it up all by myself).

[8] Note that although c_i arises as the pushforward of the classical Chern class in the smooth case, the indexing order is reversed. In particular, $c_0 = d$, rather than 1.

Consequently,

$$[\mathbb{P}\Lambda] = \sum_{i=0}^{n-1} c_i(\Lambda) \otimes H^i \in \mathrm{H}^{2n}(A \times \mathbb{P}^\vee),$$

showing that the Chern–Mather classes $c_i(\Lambda)$ are precisely the coefficients of the class $[\mathbb{P}\Lambda]$ in the Künneth decomposition. [9]

Remark 5.2. For a subvariety $Z \subset A$, both its dimension $\dim Z$ and its cohomology class $[Z] \in \mathrm{H}^{2(n-\dim Z)}(A)$ can be determined from $[\mathbb{P}\Lambda] \in \mathrm{H}^{2n}(A \times \mathbb{P}^\vee)$, as shown in [10, Lemma 3.1.2(2)]. Indeed,

$$\begin{aligned} \dim Z &= \max\{i \in \mathbb{Z} \mid c_i \neq 0\}, \\ [Z] &= c_{\dim Z}. \end{aligned}$$

5.2. **The homology class of $\mathbb{P}\Lambda_Z^{\times d}$.** By Remark 5.2, it suffices to consider the homology class of $\mathbb{P}\Lambda_{Z^{(m)}}$. If we are not concerned with the scalar factor $c_Z^{(m)}$, we may equivalently compute $[\mathbb{P}\Lambda_Z^{(m)}]$, which by definition is the pushforward of $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}]$. Consequently, the problem reduces to determining $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}] \in \mathrm{H}^{2dn}(A^d \times \mathbb{P}^\vee)$.

Typically, for an initial subvariety $Z \subset A$, the Chern–Mather classes are known. Our ultimate goal is to express $[\mathbb{P}\Lambda_Z^{\mathrm{univ}}]$ in terms of these classes; as a preparatory step, we first examine $[\mathbb{P}\Lambda_Z^{\times d}]$.

By definition,

$$\mathbb{P}\Lambda_Z^{\times d} = \underbrace{\mathbb{P}\Lambda_Z \times_{\mathbb{P}^\vee} \cdots \times_{\mathbb{P}^\vee} \mathbb{P}\Lambda_Z}_{d \text{ factors}} = \bigcap_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{-1}(\mathbb{P}\Lambda_Z),$$

where $\pi_{i,\mathbb{P}^\vee} : A^d \times \mathbb{P}^\vee \longrightarrow A \times \mathbb{P}^\vee$ denotes the projection onto the i-th factor. The transversality of these intersections is immediate, so

$$\begin{aligned} [\mathbb{P}\Lambda_Z^{\times d}] &= \cup_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{*}[\mathbb{P}\Lambda_Z] \\ &= \cup_{i=1}^{d} \pi_{i,\mathbb{P}^\vee}^{*}\left(\sum_{j=0}^{n-1} c_j \otimes H^j\right) \\ &= \cup_{i=1}^{d} \sum_{j=0}^{n-1}\left(1 \otimes \cdots \otimes \underset{\substack{\uparrow \\ i\text{-th}}}{c_j} \otimes \cdots \otimes H^j\right) \\ &= \sum_{j=0}^{n-1}\left(\sum_{\sum_{k=1}^{d} j_k = j} c_{j_1} \otimes \cdots \otimes c_{j_d}\right) \otimes H^j. \end{aligned}$$

5.3. **The homology class of $\mathbb{P}\Lambda_Z^{[d]}$.** This subsection explains how to eliminate the contribution of the big diagonal and how to compute $\mathbb{P}\Lambda_Z^{[d]}$ from $\mathbb{P}\Lambda_Z^{\times d}$. For this purpose, we introduce some combinatorial preliminaries.

Definition 5.3. *For $n \in \mathbb{N}_{>0}$, let $[n] := \{1, \ldots, n\}$, and denote by $\mathcal{P}(n)$ the lattice of partitions of $[n]$, ordered by refinement: $\alpha' \leqslant \alpha$ if and only if any two elements i, j belonging to the same block of α' also belong to the same block of α. For a partition $\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}(d)$, we associate a surjective map*

$$f_\alpha : [d] \longrightarrow [k] \qquad a \longmapsto j \quad \text{if } a \in A_j$$

which is well-defined up to the natural S_k-action; this indeterminacy will not affect our discussion. Each map f_α naturally gives rise to a partial diagonal embedding

$$\Delta_\alpha : A^k \times \mathbb{P}^\vee \longrightarrow A^d \times \mathbb{P}^\vee \qquad \big((p_i), \xi\big) \longmapsto \big((p_{f_\alpha(i)}), \xi\big).$$

[9] If the Chern–Mather classes are considered in the Chow ring, this argument does not apply, since the Künneth decomposition is unavailable at the level of Chow groups.

This construction determines a subvariety of $\mathbb{P}\Lambda_Z^{\times d}$*, defined by*

$$\begin{aligned}\mathbb{P}\Lambda_Z^{\geqslant\alpha} &:= \Delta_\alpha\left(\mathbb{P}\Lambda_Z^{\times k}\right)\\ &= \left\{\left((p_i),\xi\right)\in\mathbb{P}\Lambda_Z^{\times d}\;\middle|\;p_i=p_j \text{ if } i\sim_\alpha j\right\}.\end{aligned}$$

We can in addition define the locus corresponding precisely to α*:*

$$\mathbb{P}\Lambda_Z^{\alpha} := \overline{\left\{\left((p_i),\xi\right)\in\mathbb{P}\Lambda_Z^{\times d}\;\middle|\;p_i=p_j \text{ iff } i\sim_\alpha j\right\}}.$$

Remark 5.4. Denote by

$$\hat{0} := \left\{\{1\},\ldots,\{d\}\right\}\in\mathcal{P}(d)$$

the finest partition. Then $\mathbb{P}\Lambda_Z^{\times d} = \mathbb{P}\Lambda_Z^{\geqslant\hat{0}}$, and $\mathbb{P}\Lambda_Z^{[d]} = \mathbb{P}\Lambda_Z^{\hat{0}}$.

By definition,

$$[\mathbb{P}\Lambda_Z^{\geqslant\alpha}] = \sum_{\alpha'\geqslant\alpha}[\mathbb{P}\Lambda_Z^{\alpha'}].$$

Applying Möbius inversion on the partition lattice yields

$$[\mathbb{P}\Lambda_Z^{\alpha}] = \sum_{\alpha'\geqslant\alpha}\mu(\alpha,\alpha')\,[\mathbb{P}\Lambda_Z^{\geqslant\alpha'}],$$

where $\mu(\alpha,\alpha')$ denotes the Möbius function as defined in [1, p141].

Fact 5.5 (See [1, IV.3])**.** *Let* $\alpha'\geqslant\alpha$ *be two partitions of* $[d]$*, where* $\alpha'=\{A_1,\ldots,A_k\}$*. For each* i*, denote by* r_i *the number of blocks of* α *that are contained in* A_i*. Then*

$$\mu(\alpha,\alpha') = (-1)^{|\alpha|-k}\prod_{i=1}^{k}(r_i-1)!$$

In particular,

$$\mu(\hat{0},\alpha') = (-1)^{d-k}\prod_{i=1}^{k}\left(|A_i|-1\right)!$$

With Fact 5.5, one can compute $[\mathbb{P}\Lambda_Z^{[d]}] = [\mathbb{P}\Lambda_Z^{\hat{0}}]$ by the Möbius inverse formula:

$$\begin{aligned}[\mathbb{P}\Lambda_Z^{[d]}] &= \sum_{\alpha'}\mu(\hat{0},\alpha')\,[\mathbb{P}\Lambda_Z^{\geqslant\alpha'}]\\ &= \sum_{k=1}^{d}\sum_{\alpha'=\{A_1,\ldots,A_k\}}(-1)^{d-k}\prod_{i=1}^{k}\left(|A_i|-1\right)!\;\cdot\Delta_{\alpha,*}[\mathbb{P}\Lambda_Z^{\times k}]\end{aligned}$$

The computation of pushforwards along diagonal embeddings can be rather cumbersome. However, upon composing with $a^{(m)}$, it suffices to compute those of certain weighted sum maps. These pushforwards admit a more transparent description via the Pontryagin product, whose definition we now recall.

Definition 5.6 (Pontryagin product)**.** *Let* A *be an abelian variety, and denote by* $a: A\times A\longrightarrow A$ *the addition map. The Pontryagin product on* A *is defined by*

$$\mathrm{H}^{2n-i}(A)\times\mathrm{H}^{2n-j}(A)\subseteq\mathrm{H}^{4n-i-j}(A\times A)\xrightarrow{a_*}\mathrm{H}^{2n-i-j}(A)\qquad a\otimes b\longmapsto a*b.$$

Remark 5.7. The Pontryagin product is unital and associative, but in general only anti-commutative [4, 1.5.(7) b)]. In the context of our work, we are concerned with Chern–Mather classes, which have even degrees; consequently, the anti-commutativity of the Pontryagin product does not pose any complications.

In particular, for any sum map $a: A^k\longrightarrow A$, one can unambiguously write

$$a_*(a_1\otimes\cdots\otimes a_k) = a_1*\cdots*a_k \mathrel{\hat{=}} \mathop{\ast}_{i=1}^{k} a_i$$

where no additional parentheses are required.

Lemma 5.8. *For any tuple $(m) = (m_1, \dots, m_k) \in \mathbb{Z}^k$, $a_i \in \mathrm{H}^{2(n-l_i)}(A)$,*

$$a_*^{(m)}(a_1 \otimes \cdots \otimes a_k) = \left(\prod_{i=1}^{k} m_i^{2l_i} \right) \cdot \mathop{\ast}_{i=1}^{k} a_i.$$

Proof. Notice that $a^{(m)}$ can be written as compositions of basic functions:

$$a^{(m)} : A^k \xrightarrow{(m_1,\dots,m_k)} A^k \xrightarrow{\quad a \quad} A$$

$\square$

Definition 5.9. *For any tuple $(m) = (m_1, \dots, m_d) \in \mathbb{Z}^d$ and any partition $\alpha = \{A_1, \dots, A_k\} \in \mathcal{P}(d)$, we define*

$$\alpha(m) := \left(\sum_{i \in A_1} m_i, \dots, \sum_{i \in A_k} m_i \right) \in \mathbb{Z}^k.$$

Moreover, for $\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{N}_{\geqslant 0}^k$ with $\sum l_i = l$, we set

$$\alpha(m)^{2\boldsymbol{l}} := \left(\sum_{i \in A_1} m_i \right)^{2l_1} \cdots \left(\sum_{i \in A_k} m_i \right)^{2l_k}$$

which defines a homogeneous polynomial of degree $2l$ in $\mathbb{Z}[m_1, \dots, m_k]$.

We can now determine $[\mathbb{P}\Lambda_Z^{(m)}]$ in the case where the monodromy group is S_d:

$$\begin{aligned}
[\mathbb{P}\Lambda_Z^{(m)}] &= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{\mathrm{univ}}] \\
&= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{[d]}] \\
&= \sum_{\alpha} \mu(\hat{0}, \alpha) \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* \Delta_{\alpha,*} [\mathbb{P}\Lambda_Z^{\times k}] \\
&= \sum_{\alpha} \mu(\hat{0}, \alpha) \left(a^{\alpha(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* [\mathbb{P}\Lambda_Z^{\times k}] \\
&= \sum_{\alpha} \mu(\hat{0}, \alpha) \left(a^{\alpha(m)}, \mathrm{Id}_{\mathbb{P}^\vee} \right)_* \left(\sum_{l=0}^{n-1} \left(\sum_{\sum l_i = l} c_{l_1} \otimes \cdots \otimes c_{l_k} \right) \otimes H^l \right) \\
&= \sum_{\alpha} \mu(\hat{0}, \alpha) \sum_{l=0}^{n-1} \sum_{\sum l_i = l} \alpha(m)^{2\boldsymbol{l}} \left(\mathop{\ast}_{i=1}^{k} c_{l_i} \otimes H^l \right) \\
&= \sum_{l=0}^{n-1} \left(\sum_{\alpha} \sum_{\sum l_i = l} \left(\mu(\hat{0}, \alpha) \alpha(m)^{2\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \right) \otimes H^l
\end{aligned}$$

Therefore,

$$c_l \left(\Lambda_{Z^{(m)}} \right) = \frac{1}{c_Z^{(m)}} \sum_{\alpha} \sum_{\sum l_i = l} \left(\mu(\hat{0}, \alpha) \alpha(m)^{2\boldsymbol{l}} \mathop{\ast}_{i=1}^{k} c_{l_i} \right) \tag{5.1}$$

The following corollary collects some quantitative implications of (5.1).

Corollary 5.10. *Assume that* $\mathrm{Gal}(\gamma_Z) = S_d$.

(1) The Chern–Mather classes

$$c_l \left(\Lambda_{Z^{(m)}} \right) \in \mathrm{H}^{2(n-l)}(A)[m_1, \dots, m_d]^{S_d}$$

are polynomials of degree $2l$ with exponent at most $2r$, when expressed in the variables $m_1, \dots, m_d$.

(2) If $Z \subset A$ is a smooth curve, there exists a homogeneous symmetric polynomial $f_{Z,l} \in \mathbb{Q}[m_1, \ldots, m_d]^{S_d}$ of degree $2l$ and exponent at most 2, such that

$$c_l\left(\Lambda_{Z^{(m)}}\right) = f_{Z,l}(m_1, \ldots, m_d)\left(\mathop{\ast}_{i=1}^{k} c_1\right).$$

In this case,

$$Z^{(m)} \subset A \text{ is a divisor} \quad \Longleftrightarrow \quad f_{Z,n-1}(m_1, \ldots, m_d) \neq 0.$$

5.4. **The homology class in type C case.** In many instances, the subvariety $Z \subset A$ is invariant under a reflection, and we may, without loss of generality, assume that the reflection is taken with respect to the origin, so that $Z = -Z$. In this situation, the degree $d = \deg \gamma_Z$ is necessarily even, and the locus $[\mathbb{P}\Lambda_Z^{[d]}]$ admits a decomposition into a finite union of subvarieties. To describe these components, let

$$\mathcal{P}_2(d) := \left\{\alpha \in \mathcal{P}(d) \,\middle|\, |A| = 2 \text{ for all } A \in \alpha\right\}$$

denote the set of all perfect matchings of $[d]$. Each $\alpha \in \mathcal{P}_2(d)$ induces an involution τ_α of $[d]$, and we define

$$\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}} := \left\{\left((p_i), \xi\right) \in \mathbb{P}\Lambda_Z^{[d]} \,\middle|\, p_{\tau_\alpha(i)} = -p_i \text{ for all } i\right\}$$

$$\Delta_{\widetilde{\alpha}} : \; A^{d/2} \times \mathbb{P}^\vee \longrightarrow A^d \times \mathbb{P}^\vee \qquad \left((p_i), \xi\right) \longmapsto \left((\pm p_{f_\alpha(i)}), \xi\right)$$

where the sign is chosen so that $p_{\tau_\alpha(i)} = -p_i$. Furthermore, there exists a distinguished subvariety in $\mathbb{P}\Lambda_Z^{\times d/2}$:

$$\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}} := \overline{\left\{\left((p_i), \xi\right) \in \mathbb{P}\Lambda_Z^{\times d/2} \,\middle|\, p_i \neq \pm p_j \text{ iff } i \neq j\right\}}.$$

With these conventions, we can now express the following decomposition:

$$\begin{aligned} \mathbb{P}\Lambda_Z^{[d]} &= \bigcup_{\alpha \in \mathcal{P}_2(d)} \mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}} \\ &= \bigcup_{\alpha \in \mathcal{P}_2(d)} \Delta_{\widetilde{\alpha}}\left(\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}}\right) \end{aligned}$$

Evidently, determining $[\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}]$ amounts to determining $[\mathbb{P}\Lambda_Z^{\widetilde{[d/2]}}]$, and the latter computation reduces again to combinatorics.

For $\beta \in \mathbb{P}(d/2)$, define

$$\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}} := \left\{\left((p_i), \xi\right) \in \mathbb{P}\Lambda_Z^{\times d/2} \,\middle|\, p_i = \pm p_j \text{ if } i \sim_\beta j\right\}$$

$$\mathbb{P}\Lambda_Z^{\widetilde{\beta}} := \overline{\left\{\left((p_i), \xi\right) \in \mathbb{P}\Lambda_Z^{\times d/2} \,\middle|\, p_i = \pm p_j \text{ iff } i \sim_\beta j\right\}}$$

Then one has the relations

$$[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}] = \sum_{\beta' \geqslant \beta} [\mathbb{P}\Lambda_Z^{\widetilde{\beta'}}],$$

$$[\mathbb{P}\Lambda_Z^{\widetilde{\beta}}] = \sum_{\beta' \geqslant \beta} \mu(\beta, \beta')\, [\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta'}}],$$

where $\mu(\beta, \beta')$ denotes the Möbius function of the poset $\mathcal{P}(d/2)$.

Furthermore, the classes $[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}]$ admit a decomposition as sums of pushforwards of $[\mathbb{P}\Lambda_Z^{\times k}]$ under appropriately defined signed diagonal maps. Explicitly, for every pair of maps $f_\beta : [d/2] \longrightarrow [k]$ and $\eta : [d/2] \longrightarrow \{\pm 1\}$, one introduces the signed diagonal embedding

$$\Delta_\beta^\eta : A^k \times \mathbb{P}^\vee \longrightarrow A^{d/2} \times \mathbb{P}^\vee \qquad \left((p_i), \xi\right) \longmapsto \left((\eta(i) p_{f_\beta(i)}), \xi\right),$$

which yields the expression

$$[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}] = \frac{1}{2^k} \sum_{\eta:[d/2]\longrightarrow\{\pm 1\}} \Delta^{\eta}_{\beta,*}[\mathbb{P}\Lambda_Z^{\times k}].$$

Assuming that the monodromy group is the signed symmetric group, we may identify $\mathbb{P}\Lambda_Z^{\mathrm{univ}}$ with one of the components $\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}$. Without loss of generality, we choose

$$\alpha = \big\{\{i, i+d/2\} \bigm| i = 1, \dots, d/2\big\}$$

and fix the sign convention for $\Delta_{\widetilde{\alpha}}$:

$$\Delta_{\widetilde{\alpha}} : A^{d/2} \times \mathbb{P}^\vee \xhookrightarrow{(\Delta,\mathrm{Id})} A^{d/2} \times A^{d/2} \times \mathbb{P}^\vee \xrightarrow{(\mathrm{Id},-\mathrm{Id},\mathrm{Id})} A^d \times \mathbb{P}^\vee$$

Under this convention, the tuples $(m_1, \dots, m_d)$ and $(m_1 - m_{d/2+1}, \dots, m_{d/2} - m_d, 0, \dots, 0)$ define the same subvariety. Hence, without loss of generality, we may assume that

$$(m) := (m_1, \dots, m_{d/2}, 0, \dots, 0) \in \mathbb{Z}^d.$$

We now introduce some notations for later computations:

$$\begin{aligned}
(\widetilde{m}) &:= (m_1, \dots, m_{d/2}) \in \mathbb{Z}^{d/2} \\
\beta^\eta(\widetilde{m}) &:= \left(\sum_{i\in A_1} \eta(i)m_i, \dots, \sum_{i\in A_k} \eta(i)m_i\right) \in \mathbb{Z}^k \\
\beta^\eta(\widetilde{m})^{2\boldsymbol{l}} &:= \left(\sum_{i\in A_1} \eta(i)m_i\right)^{2l_1} \cdots \left(\sum_{i\in A_k} \eta(i)m_i\right)^{2l_k} \\
\beta(\widetilde{m}^2)^{\boldsymbol{l}} &:= \left(\sum_{i\in A_1} m_i^2\right)^{l_1} \cdots \left(\sum_{i\in A_k} m_i^2\right)^{l_k}
\end{aligned}$$

where $\beta = \{A_1, \dots, A_k\} \in \mathcal{P}(d/2)$ is a partition of length k, $\eta : [d/2] \longrightarrow \{\pm 1\}$ is a sign function, and $\boldsymbol{l} = (l_1, \dots, l_k) \in \mathbb{N}^k_{\geqslant 0}$ with $\sum l_i = l$. Notice that

$$\sum_\eta \beta^\eta(\widetilde{m})^{2\boldsymbol{l}} = 2^{d/2}\, \beta(\widetilde{m}^2)^{\boldsymbol{l}}.$$

We can now determine $[\mathbb{P}\Lambda_Z^{(m)}]$ in the case where $Z=-Z$ and $\mathrm{Gal}(\gamma_Z)\cong W(C_{d/2})$:

$$\begin{aligned}
[\mathbb{P}\Lambda_Z^{(m)}] &= \left(a^{(m)}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{\widetilde{[\alpha]}}] \\
&= \left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{[d/2]}] \\
&= \left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{\widetilde{\hat{0}}}] \\
&= \sum_{\beta} \mu(\hat{0},\beta)\left(a^{(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* \left[\mathbb{P}\Lambda_Z^{\widetilde{\geqslant\beta}}\right] \\
&= \sum_{\beta,\eta} \frac{1}{2^k}\,\mu(\hat{0},\beta)\left(a^{\beta^\eta(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* [\mathbb{P}\Lambda_Z^{\times k}] \\
&= \sum_{\beta,\eta} \frac{1}{2^k}\,\mu(\hat{0},\beta)\left(a^{\beta^\eta(\widetilde{m})}, \mathrm{Id}_{\mathbb{P}^\vee}\right)_* \left(\sum_{l=0}^{n-1}\left(\sum_{\sum l_i=l} c_{l_1}\otimes\cdots\otimes c_{l_k}\right)\otimes H^l\right) \\
&= \sum_{\beta,\eta} \frac{1}{2^k}\,\mu(\hat{0},\beta)\sum_{l=0}^{n-1}\sum_{\sum l_i=l} \beta^\eta(\widetilde{m})^{2\boldsymbol{l}}\left(\mathop{\ast}_{i=1}^{k} c_{l_i}\otimes H^l\right) \\
&= \sum_{l=0}^{n-1}\left(\sum_{\beta,\eta}\sum_{\sum l_i=l}\left(\frac{1}{2^k}\,\mu(\hat{0},\beta)\beta^\eta(\widetilde{m})^{2\boldsymbol{l}}\mathop{\ast}_{i=1}^{k} c_{l_i}\right)\right)\otimes H^l \\
&= \sum_{l=0}^{n-1}\left(\sum_{\beta}\sum_{\sum l_i=l}\left(2^{d/2-k}\,\mu(\hat{0},\beta)\beta(\widetilde{m}^2)^{\boldsymbol{l}}\mathop{\ast}_{i=1}^{k} c_{l_i}\right)\right)\otimes H^l
\end{aligned}$$

Therefore,

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}}\sum_{\beta}\sum_{\sum l_i=l}\left(2^{d/2-k}\,\mu(\hat{0},\beta)\beta(\widetilde{m}^2)^{\boldsymbol{l}}\mathop{\ast}_{i=1}^{k} c_{l_i}\right) \tag{5.2}$$

The following corollary collects some quantitative implications of (5.2).

Corollary 5.11. *Suppose that $Z=-Z$ and $\mathrm{Gal}(\gamma_Z)\cong W(C_{d/2})$, and assume that $(m) := (m_1,\ldots,m_{d/2},0,\ldots,0)\in\mathbb{Z}^d$.*

(1) The Chern–Mather classes

$$c_l\left(\Lambda_{Z^{(m)}}\right)\in \mathrm{H}^{2(n-l)}(A)[m_1^2,\ldots,m_{d/2}^2]^{S_{d/2}}$$

are polynomials of degree l with exponent at most $2r$, when expressed in the variables $m_1^2,\ldots,m_{d/2}^2$.

(2) If $Z\subset A$ is a smooth curve, there exists a homogeneous symmetric polynomial $\widetilde{f}_{Z,l}\in\mathbb{Q}[m_1^2,\ldots,m_{d/2}^2]^{S_{d/2}}$ of degree l and exponent at most 2, such that

$$c_l\left(\Lambda_{Z^{(m)}}\right)=\widetilde{f}_{Z,l}(m_1^2,\ldots,m_{d/2}^2)\left(\mathop{\ast}_{i=1}^{k} c_1\right).$$

In this case,

$$Z^{(m)}\subset A \text{ is a divisor} \quad\Longleftrightarrow\quad \widetilde{f}_{Z,n-1}(m_1^2,\ldots,m_{d/2}^2)\neq 0.$$

5.5. **Möbius-transforms block-sum polynomial.** The formulas (5.1) and (5.2) admit further simplifications in certain special situations, such as the case of curves. In what follows, we extract the corresponding polynomial coefficient, which may be of independent combinatorial interest.

Definition 5.12. *Fix d and $\tilde{d} = d/2$, and let $l \in \mathbb{N}_{\geqslant 0}$. For a partition $\lambda = [\lambda_1, \ldots, \lambda_{k'}]$ of l with $\lambda_1 \geqslant \cdots \geqslant \lambda_{k'} > 0$, the Möbius-transforms block-sum polynomial of type A (resp. type C) is defined by*

$$\mu_d^\lambda := \sum_{\alpha \in \mathcal{P}(d)} \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}\colon \text{ length } k}} \mu(\hat{0}, \alpha)\alpha(m)^{2\boldsymbol{l}} d^{k-k'} \qquad \in \mathbb{Z}[m_1, \ldots, m_d]^{S_d}$$

$$\widetilde{\mu}_{\tilde{d}}^\lambda := \sum_{\beta \in \mathcal{P}(\tilde{d})} \sum_{\substack{\boldsymbol{l} \dashv \lambda \\ \boldsymbol{l}\colon \text{ length } k}} 2^{\tilde{d}-k}\mu(\hat{0}, \beta)\beta(\widetilde{m}^2)^{\boldsymbol{l}} (2\tilde{d})^{k-k'} \qquad \in \mathbb{Z}[m_1^2, \ldots, m_{\tilde{d}}^2]^{S_{\tilde{d}}}$$

In this definition:

- *$\alpha = \{A_1, \ldots, A_k\} \in \mathcal{P}(d)$ is a set partition of $[d]$ with $k = \#\alpha$ blocks (analogously for β);*
- *$\boldsymbol{l} = (l_1, \ldots, l_k) \in \mathbb{Z}^k$, and $\boldsymbol{l} \dashv \lambda$ indicates that $\boldsymbol{l}$ is of type λ, meaning that its nonzero components, counted with multiplicities, form the partition λ;*
- *$\mu(\hat{0}, \alpha) = (-1)^{d-k} \prod_{i=1}^k \left(|A_i| - 1\right)$ is the Möbius function;*
- *$\alpha(m)^{2\boldsymbol{l}} = \left(\sum_{i \in A_1} m_i\right)^{2l_1} \cdots \left(\sum_{i \in A_k} m_i\right)^{2l_k}$ and $\beta(\widetilde{m}^2)^{\boldsymbol{l}} = \left(\sum_{i \in A_1} m_i^2\right)^{l_1} \cdots \left(\sum_{i \in A_k} m_i^2\right)^{l_k}$.*

Now, the formulas (5.1) and (5.2) can be written as

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \mu_d^\lambda \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i}\right) \qquad \text{type } A \text{ case}$$

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \sum_{\lambda \dashv l} \widetilde{\mu}_{\tilde{d}}^\lambda \left(\mathop{\ast}_{i=1}^{k'} c_{\lambda_i}\right) \qquad \text{type } C \text{ case}$$

The following identities have been verified in low-degree cases using SageMath, though a purely combinatorial explanation, as well as a general formulation for the remaining types, is still lacking.

Fact 5.13. *Let $1^l := [1, \ldots, 1] \dashv l$. When $l < \tilde{d}$, we have*

$$\mu_d^{1^l} = \frac{1}{2^l l!} \sum_{\sigma \in S_d} \prod_{i=1}^{l} \left(m_{\sigma(2i-1)} - m_{\sigma(2i)}\right)^2$$

$$\begin{aligned}\widetilde{\mu}_{\tilde{d}}^{1^l} &= 2^{\tilde{d}} \frac{1}{2^l l!} \sum_{\sigma \in S_{\tilde{d}}} \prod_{i=1}^{l} m_{\sigma(i)}^2 \\ &= 2^{\tilde{d}-l}(\tilde{d}-l)! \sum_{1 \leqslant i_1 < \cdots i_l \leqslant d} m_{i_1}^2 \cdots m_{i_l}^2\end{aligned}$$

Example 5.14 (Continuation of Example 3.7)**.** *Let C be a smooth curve of genus g embedded in its Jacobian $A := \mathrm{Jac}(C)$ via the Abel–Jacobi map $\mathrm{AJ}_C : C \hookrightarrow A$. In this case, write Θ as the theta divisor of A, we have formula* [4, p323]

$$\mathop{\ast}_{i=1}^{l} c_1 = \frac{l!}{(g-l)!} \cdot \Theta^{g-l}.$$

When C is non-hyperelliptic, we get

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \frac{1}{2^l (g-l)!} \sum_{\sigma \in S_{2g-2}} \prod_{i=1}^{l} \left(m_{\sigma(2i-1)} - m_{\sigma(2i)}\right)^2 \cdot \Theta^{g-l}$$

$$\dim Z^{(m)} = \min_{k \in \mathbb{Z}} \left\{g-1, \#\left\{i \in [2g-2] \mid m_i \neq k\right\}\right\}$$

When C is hyperelliptic, with $(m) = (m_1, \ldots, m_{g-1}, 0, \ldots, 0)$, we get

$$c_l\left(\Lambda_{Z^{(m)}}\right) = \frac{1}{c_Z^{(m)}} \frac{2^{g-1-l} l!}{g-l} \sum_{1 \leqslant i_1 < \cdots i_l \leqslant g-1} m_{i_1}^2 \cdots m_{i_l}^2 \cdot \Theta^{g-l}$$

$$\dim Z^{(m)} = \#\left\{i \in [g-1] \mid m_i \neq 0\right\}$$

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