Subvarieties in Complex Abelian Varieties

Xiaoxiang Zhou
Advisor: Prof. Dr. Thomas Krämer
Humboldt-Universität zu Berlin

Tangent Gauss Map

Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subset A$ be a non-degenerate closed subvariety of dimension r.

To understand the geometry of Z, we encode the variation of its tangent spaces via the tangent Gauss map

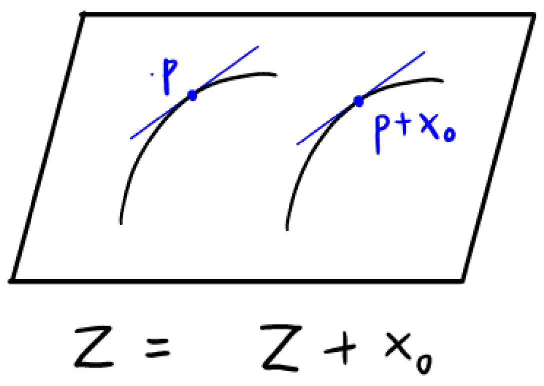
$$\phi_Z : Z^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0A) \qquad p \longrightarrow T_pZ \subset T_pA \cong T_0A.$$

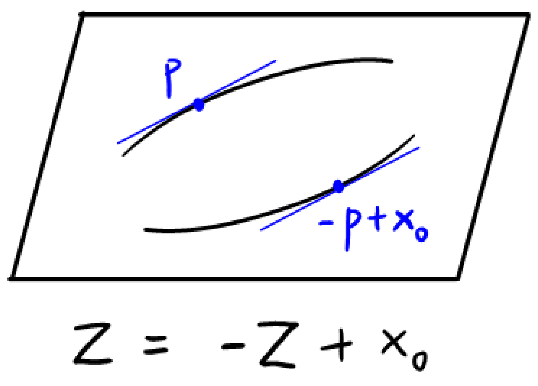
Its differential

$$d_p\phi_Z : T_pZ \longrightarrow \mathrm{Hom}_{\mathbb{C}}(T_pZ, N_pZ)$$

is the second fundamental form, from which curvature invariants can be extracted.

Besides the obvious examples (illustrated below), when can the Gauss map ϕ_Z fail to be generically injective?

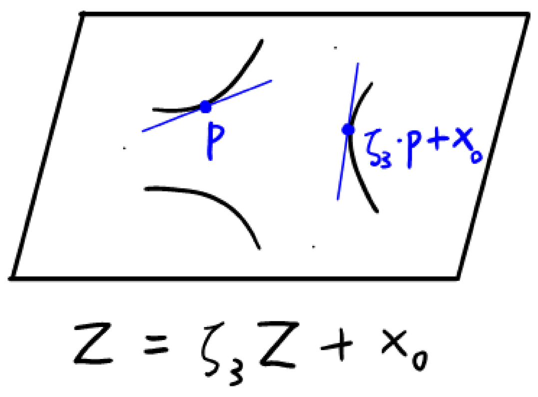




We specialize to the case where $Z = C$ is a curve. When $n = 2$, ϕ_C: $C^{\mathrm{sm}} \longrightarrow \mathbb{P}^1$ typically fails to be generically injective.

Conjecture 1. Let $C \subset A$ be a non-degenerate curve, $n > 2$. If C is not invariant under any non-trivial translation or reflection, then ϕ_C is generically injective.

A Counterexample for Conjecture 1



Example 1. For $A = E_\rho^{\oplus n}$, ζ_3 acts on A (and hence on T_0A) by scalar multiplication. Computer experiments yield a non-degenerate ζ_3-invariant curve $C \subset A$, for which ϕ_C is not generically injective.

We have found no counterexample to Conjecture 1 when A is not isogenous to $E_i^{\oplus n}$ or $E_\rho^{\oplus n}$. This suggests the following refinement:

Conjecture 2. Let $C \subset A$ be a non-degenerate curve, $n > 2$. If no non-trivial $\tau \in \mathrm{Aut}(A)$ preserves C and acts by scalar multiplication on T_0A, then ϕ_C is generically injective.

One may restate the conjecture using Gauss curvature, yielding a slightly stronger statement:

Conjecture 3. Let $C \subset A$ be a non-degenerate curve, $n > 2$. For a general point $p \in \mathrm{Im}\,\phi_C$, all points in $\phi_C^{-1}(p)$ exhibit the same Gauss curvature.

Known Cases

1. If $A = \mathrm{Jac}(C)$ and C is embedded via the Abel-–Jacobi map, then $\phi_C = |\omega_C|$ is the canonical map:
   - When C is hyperelliptic, C is invariant under the hyperelliptic involution, and $\deg \phi_C = 2$;
   - When C is non-hyperelliptic, ϕ_C is an embedding.
2. Let $h : C \longrightarrow C'$ be a cyclic k-fold cover defined by $\eta \in \mathrm{Pic}(C')$ with $\eta^{\otimes k} \cong \mathcal{O}_{C'}(B)$. If $A = \mathrm{Prym}(C/C')$ and $C \to A$ is the Abel–Prym map, then

$$T_0A \cong \mathrm{H}^0(\omega_C)/\,\mathrm{H}^0(\omega_{C'}) \cong \bigoplus_{i=1}^{k-1} \mathrm{H}^0(\omega_{C'} \otimes \eta^i)$$

$$\phi_C : C \longrightarrow \mathbb{P}T_0A \cong \mathbb{P}\left(\bigoplus_{i=1}^{k-1} \mathrm{H}^0(\omega_{C'} \otimes \eta^i)\right)$$

   - $k = 2$: C is invariant under the Prym involution, and $\phi_C = |\omega_{C'} \otimes \eta| \circ h$. If C' is non-hyperelliptic with $g(C') \geq 4$, then $\deg \phi_C = 2$ or 4, and

     $$\deg \phi_C = 4 \iff B = \varnothing, C' \text{ is bielliptic and } \eta \text{ pulled back from EC.}$$

   - $k > 2$: if $g(C') \geq 1$ and $|\omega_{C'} \otimes \eta|$ is generically injective, then ϕ_C is generically injective.
3. If $C \subset A$ is smooth and either $\deg \phi_C = 2$ or ϕ_C is unramified, Conjecture 3 also holds.

Perverse Sheaf

We will mix the usage of sheaves and complexes. For simplicity, let us fix a Whitney stratification $\mathcal{S}$:

$$\varnothing \overset{U_0}{\subset} Z_0 \overset{U_1}{\subset} \cdots \overset{U_n}{\subset} Z_n = X$$

Denote $D^b_{\mathrm{cons},\mathcal{S}}(X)$ as the category of constructible sheaves over X with respect to $\mathcal{S}$.

Definition

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure.

We say that $\mathcal{F} \in D^b_{\mathrm{cons},\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^i\left(\iota_{U_j}^*\mathcal{F}\right) = 0, & \text{for any } i > -j \\ \mathcal{H}^i\left(\iota_{U_j}^!\mathcal{F}\right) = 0, & \text{for any } i < -j \end{cases}$$

Deligne's construction

Any local system $\mathcal{L}$ supported on U_i can be converted into a perverse sheaf through truncations. This process is known as **Deligne's construction**, and the resulting perverse sheaf is called the intersection cohomology complex, or the IC sheaf, denoted by $\mathrm{IC}(\mathcal{L})$. IC sheaves are the simple objects in the category $\mathrm{Perv}_{\mathcal{S}}(X)$.

To determine whether a complex $\mathcal{F}$ is perverse or an IC sheaf, one simply needs to complete Table 2.

Nearby Cycle

A perverse sheaf may not be so "perverse", but a nearby cycle is definitely "nearby".

$$\begin{tikzcd} i^*\mathcal{F}\ \psi\mathcal{F}\ \varphi\mathcal{F} \arrow[d, dash] & \mathcal{F} \arrow[d, dash] & & \\ \{0\} \arrow[r, hook, "i"] & \mathbb{C} & \mathbb{C}^\times \arrow[l, hook', "j"'] & \widetilde{\mathbb{C}}^\times \cong \mathbb{C} \arrow[l, "p"'] \end{tikzcd}$$

Given $\mathcal{F} \in D^b(\mathbb{C})$, one can construct the **nearby cycle**

$$\psi\mathcal{F} := i^*Rj_*p_*p^*j^*\mathcal{F} \in D^b(\{0\}),$$

which can be roughly viewed as the fiber $\mathcal{F}_x$ for x sufficiently close to 0. By quotienting out the **non-vanishing cycle** $i^*\mathcal{F}$, one obtains the **vanishing cycle**

$$\varphi\mathcal{F} := \mathrm{cone}\left[i^*\mathcal{F} \xrightarrow{sp} \psi\mathcal{F}\right] \in D^b(\{0\}).$$

$$\begin{tikzcd} & \mathcal{F} \arrow[d, dash] & & \\ X_0 \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] & X^* \arrow[l, hook', "j"'] \arrow[d] & \widetilde{X}^* \arrow[l, "p"'] \arrow[d] \\ \{0\} \arrow[r, hook] & D & D^* \arrow[l, hook'] & \widetilde{D}^* \arrow[l] \end{tikzcd}$$

In general, $\mathcal{C}$ can be replaced by any disk $\mathcal{D}$, as the problem is local, and $\mathcal{F}$ can be a sheaf over any space X over $\mathcal{D}$.

The same construction yields a distinguished triangle in $D^b(X_0)$:

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \varphi_f\mathcal{F} \xrightarrow{+1}$$