

# Bruhat–Tits building

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# Figures of Bruhat–Tits building

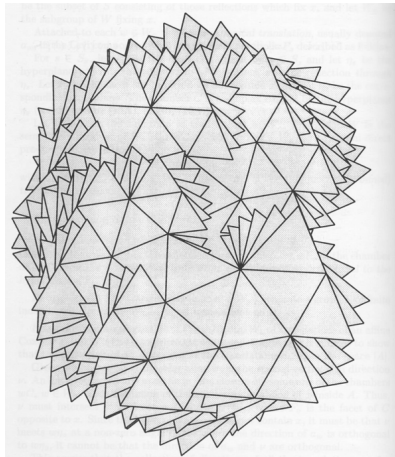


Figure:  $\mathcal{B}_{SL_3(\mathbb{Q}_p)}$ , from Annette Werner's talk

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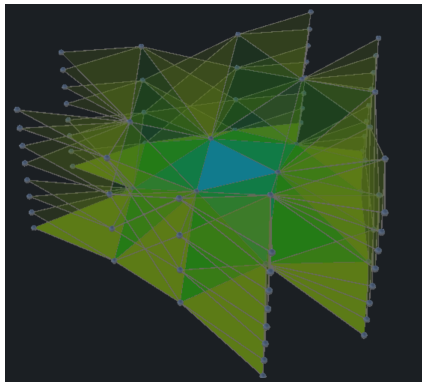


Figure:  $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$ , from buildings.gallery

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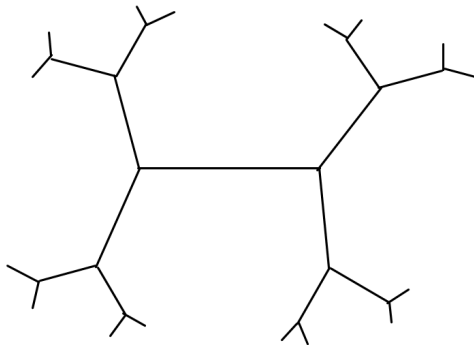


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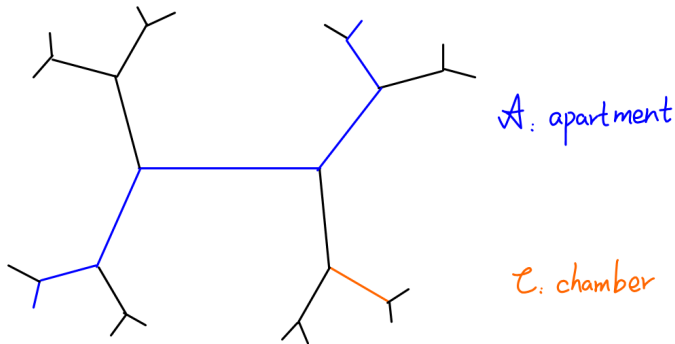


Figure:  $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

# Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \rightsquigarrow \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{flags of } \kappa^n \} \\ &= \{ V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i \} \end{aligned}$$

$$P = \begin{pmatrix} * & & * \\ \vdots & \ddots & \vdots \\ * & & * \end{pmatrix} \rightsquigarrow \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{ V \subset \kappa^n \mid \dim V = r \} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \rightsquigarrow \mathrm{GL}_n(\kappa)/T = \{ \kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1 \}$$

$T$  is comm, so every rep decomposes as direct sum of 1-dim reps.

$$X^*(T) =: \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n \quad \text{characters (1-dim reps)}$$

$$X_*(T) =: \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n \quad \text{cocharacters (1-parameter subgps)}$$

# Weyl group

## Definition (Weyl group)

$$W := N_G(T)/T.$$

## Example

When  $G = \mathrm{GL}_n(\kappa)$ ,

$$N_G(T) = \{ \text{monoidal matrixes} \}$$

$$N_G(T)/T \cong S_n \quad \text{Weyl group of type A}$$

## Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B.$$

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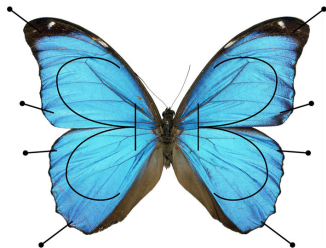
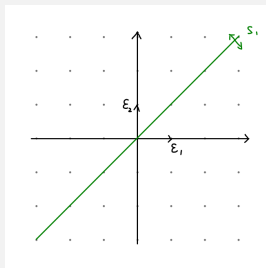


Figure: Pinned butterfly

# Weyl group action on cocharacter lattices

When  $G = \mathrm{GL}_2(\kappa)$ ,  $T = \begin{pmatrix} a & \\ & b \end{pmatrix}$ ,  $X_*(T) = \mathbb{Z}\varepsilon_1 \oplus \mathbb{Z}\varepsilon_2$ , where

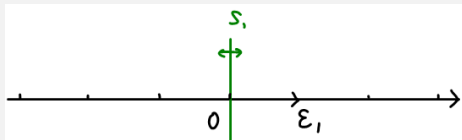


$$\begin{aligned} \varepsilon_1 : \mathbb{G}_m &\longrightarrow T \\ x &\longmapsto \begin{pmatrix} x & \\ & 1 \end{pmatrix} \end{aligned}$$

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$$W = S_2 = \{\mathrm{Id}, s_1\}$$

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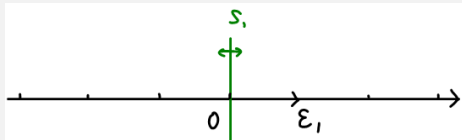


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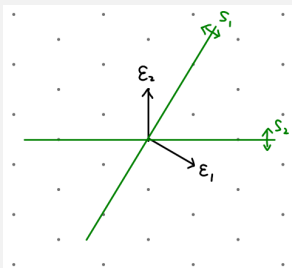
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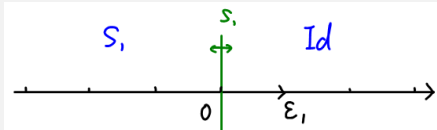
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$$W = S_3 = \langle s_1, s_2 \rangle$$

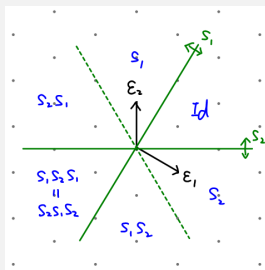
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# Non-standard subgroups

The subgroup  $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$  is not the only maximal torus.

## Fact

*All non-standard subgroups are conjugated to standard subgroups.  
Therefore,*

$$\{ \text{Borel subgroups} \} = \{ gBg^{-1} \} \cong G/B$$

$$\{ \text{parabolic subgroups} \} = \{ gPg^{-1} \} \cong G/P$$

$$\{ \text{maximal tori} \} = \{ gTg^{-1} \} \cong G/N_G(T)$$

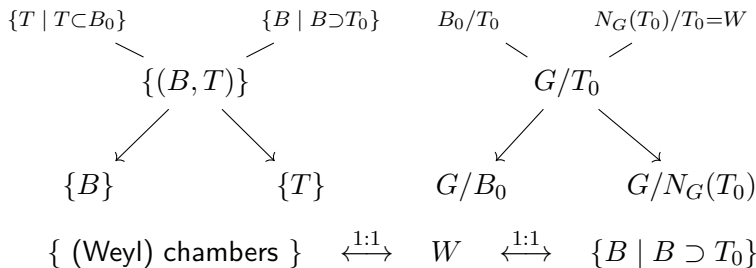
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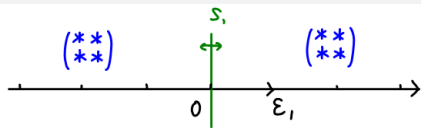
$$\{ \text{maximal tori} \} = \{ gTg^{-1} \} \cong G/N_G(T)$$

$$\{ (B, T) \mid B \supset T \} = \{ (gB_0g^{-1}, gT_0g^{-1}) \} \cong G/T_0$$



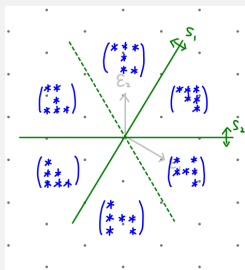
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## Definition (chamber, apartment and building)

Given a maximal torus  $T$ , the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{B \supset T} \mathcal{C}_B,$$

and the building is

$$\mathcal{B} := \left( \bigsqcup_T \mathcal{A}_T \right) / \sim = \bigcup_B \mathcal{C}_B.$$

## Example of spherical building

When  $G = \mathrm{SL}_2(\mathbb{F}_2)$ , the building  $\mathcal{B}$  has 3 apartments and 3 chambers.

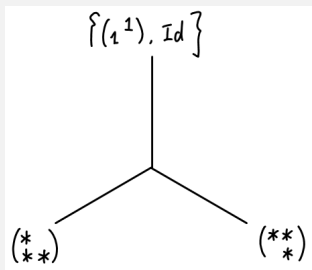


Figure:  $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When  $G = \mathrm{SL}_3(\mathbb{F}_2)$ , the building  $\mathcal{B}$  has 28 apartments and 21 chambers.

## Remark

$\mathcal{B}$  inherits the metric structure from  $\mathcal{A}_T = X_*(T)_{\mathbb{R}}$ .

$\mathcal{B}$  has also polysimplicial complex structure.

When  $\kappa = \mathbb{F}_p$ ,  $\mathcal{B}$  is finite.

## Proposition

- *Two chambers lie in one apartment.*
- *There is a unique geodesic passing any two points  $p_1, p_2 \in \mathcal{B}$ .*

# p-adic notation

symbol	name	example
$F$	local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	integral ring	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	$p$
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

## standard subgroups in p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix} \quad \text{Iwahori subgroup}$$

$$\tilde{P} = \pi^{-1}(P) = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \hline \mathfrak{p} & \mathcal{O} \end{pmatrix} \quad \text{Parahoric subgroup}$$

### Remark

They also have moduli interpretations. For example,

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{L = L_0 \subset L_1 \subset \cdots \subset L_n = \mathfrak{p}L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

# Extended Weyl group

To get the Iwahori decomposition

$$G(F) = \bigsqcup_{\varpi \in W_{\text{ext}}} I\varpi I,$$

we define the extended Weyl group as

$$W_{\text{ext}} := N_G(T(\mathcal{O}))/T(\mathcal{O}) \cong X_*(T) \rtimes W_f.$$

## Example

When  $G = \text{GL}_n(F)$ ,

$$W_{\text{ext}} = \{ \text{monoidal matrixes} \} / \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

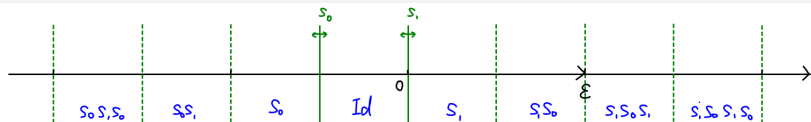
# Extended Weyl group action

$W_{\text{ext}}$  acts on  $X_*(T)$  by “twisted conjugation”:

$$W_{\text{ext}} \times X_*(T) \longrightarrow X_*(T) \quad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

When  $G = \text{SL}_2(F)$ ,  $W_{\text{ext}} = \langle s_0, s_1 \rangle$ , where

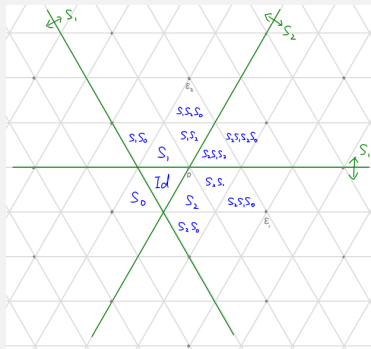
$$s_1 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \quad s_0 = \begin{pmatrix} & \pi^{-1} \\ -\pi & \end{pmatrix}$$



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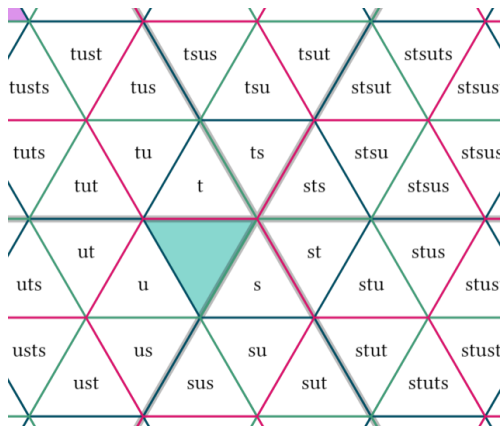


Figure: Reduced expressions labels, from Lievis

# Non-standard subgroups in p-adic world

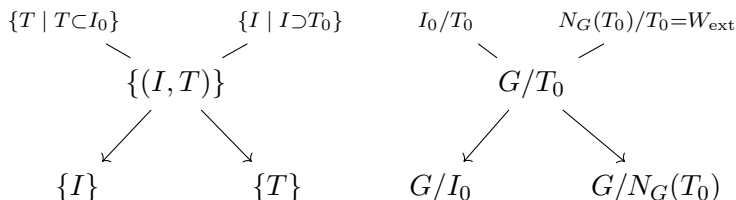
Similarly,

$$\{ \text{Iwahori subgroups} \} = \{ gI_0g^{-1} \} \cong G/I_0$$

$$\{ \text{paraholic subgroups} \} = \{ g\tilde{P}_0g^{-1} \} \cong G/\tilde{P}_0$$

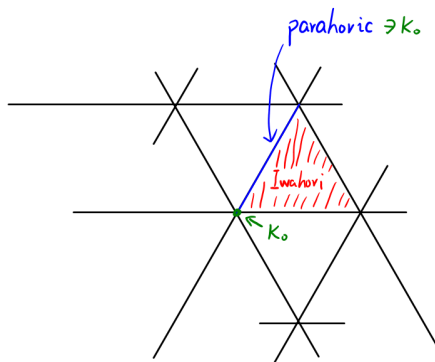
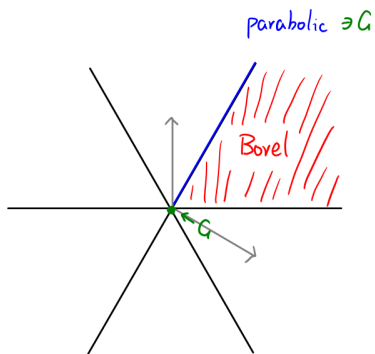
$$\{ \text{maximal tori over } \mathcal{O} \} = \{ gT_0g^{-1} \} \cong G/N_G(T_0(\mathcal{O}))$$

$$\{(I, T) \mid I \supset T\} = \{(gI_0g^{-1}, gT_0g^{-1})\} \cong G/T_0(\mathcal{O})$$



$$\{ \text{chambers} \} \xleftrightarrow{1:1} W_{\text{ext}} \xleftrightarrow{1:1} \{ I \mid I \supset T_0 \}$$

# Comparison



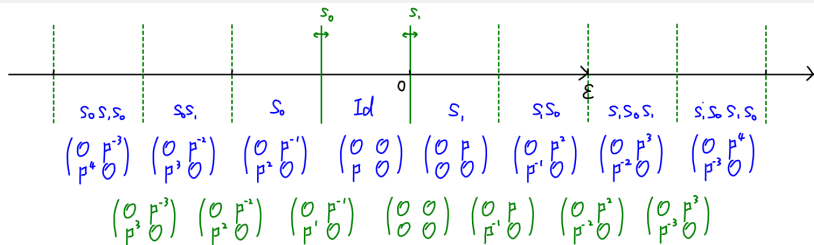
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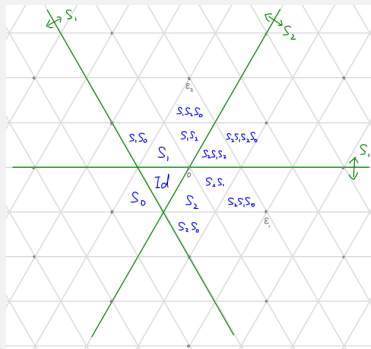
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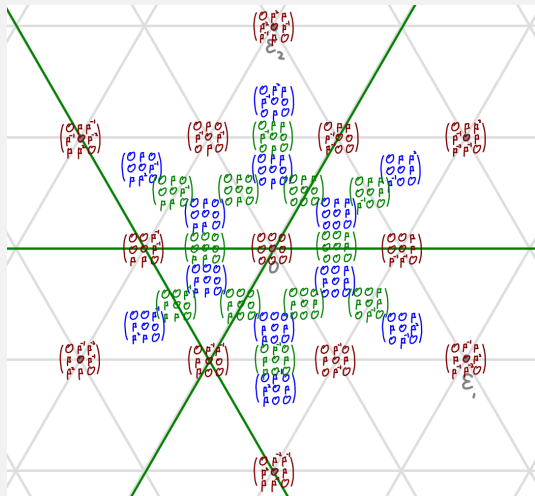
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# Extended Weyl group action



# p-adic building

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and the p-adic building is

$$\mathcal{B} := \left( \bigsqcup_T \mathcal{A}_T \right) / \sim = \bigcup_I \mathcal{C}_I.$$

## Remark

Similarly, two chambers lie in one apartment,  
and there is a unique geodesic passing  $p_1, p_2 \in \mathcal{B}$ .

# Gromov-Schoen theorem

## Theorem

*Let  $F$  be a local field,  $(M, g)$  be a cpt conn Riemannian manifold with the universal covering space  $\widetilde{M}$ .*

*For any reductive map*

$$\rho : \pi_1(M) \longrightarrow \mathrm{GL}_n(F),$$

*there exists a  $\pi_1(M)$ -equivariant Lipschitz continuous regular harmonic map*

$$h_\rho : \widetilde{M} \longrightarrow \mathcal{B}_{\mathrm{GL}_n(F)}$$

We call  $\rho$  reductive when  $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$  is reductive.

# regularity

## Definition

$h_\rho$  is regular at  $x \in \widetilde{M}$  if  
a neighbourhood of  $x$  is contained in an apartment of  $\mathcal{A}$ .

$h_\rho$  is regular if

$$\operatorname{codim}_{\widetilde{M}} \left\{ x \in \widetilde{M} \mid h_\rho \text{ is not regular at } x \right\} \geq 2.$$

## Example

*The map*

$$f : \mathbb{R}^2 \longrightarrow \{y^2 = x^2\} \quad (a, b) \longmapsto (a|b|, b|a|)$$

*is regular.*

test

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