DESSIN D'ENFANT: AN INTRODUCTION

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Abstract. In this talk, we will talk about the relation between Belyi map and dessin d'enfant, and than extract imformations from the dessin.

Contents: section 4.1-4.3 and some examples from section 4.6.

Last time, we talked about the Belyi's Theorem:

Theorem 0.1 (Thm 3.1). *Let S be a cpt RS, then S is defined over $\bar{\mathbb{Q}}$ iff S admits a Belyi fct.*

This time, we talked a specific Belyi fct (ramified at $0, 1, \infty$), and

- combine it with a kind of special graph (on S);
- extract information from this graph.

A black box is useful for us to familiar with Belyi fct (But not actually used in this talk):

$$\boxed{\textbf{Prop 3.34: } \text{Belyi fcts are defined over } \bar{\mathbb{Q}}.}$$

Remark 0.2. We can talk about the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts on $\{S \longrightarrow \mathbb{P}^1\}$.

Example 0.3. *When $S = \mathbb{P}^1$, then an Belyi fct $f(z)$ is an rational fct with coefficient in $\bar{\mathbb{Q}}$ such that f maps any zero or pole of $f'(z)$ to $0, 1, \infty$. In short,*

1. $f(z) \in \bar{\mathbb{Q}}(z)$;

2. For any z_0 such that $f'(z_0) = 0$ or ∞, $f(z_0) = 0, 1$ or ∞.

For example,

- $f(z) = z^n$;
- $f(z) = -\dfrac{256}{27} z^3(z-1)$;
- $f(z) = \dfrac{3+i}{5} z^3(z-1)^2 \left(z - \dfrac{4}{25}(4+3i) \right)$;
- $f(z) = \dfrac{4}{27} \dfrac{(1-z+z^2)^3}{z^2(z-1)^2}$;
- $f(z) = C \dfrac{z^4(z-1)^2}{z + \frac{9+2\sqrt{10}}{18}}$.

1. What is a dessin d'enfants? / Quel est un dessin d'enfants ?

We postponed the abstract definition of the dessin d'enfant. A better question: How to draw a dessin d'enfants from a Belyi fct?

Proposition 1.1.

the graph D is drawed on the RS S;
D is bicolored;
$X \setminus D$ is union of finitely many topo discs;
D is connected.

Abstractly, we have the following concept:

Definition 1.2 (Def 4.1)**.** *A dessin d'enfant, or simply a dessin, is a pair $(X, \mathcal{D})$ where X is an oriented compact topological surface, and $\mathcal{D} \subset X$ is a finite graph such that:*

(i) $\mathcal{D}$ is connected.
(ii) $\mathcal{D}$ is bicoloured, i.e. the vertices have been given either white or black colour and vertices connected by an edge have different colours.
(iii) $X \setminus \mathcal{D}$ is the union of finitely many topological discs, which we call faces of $\mathcal{D}$.

Example 1.3. *example pictures*

Proposition 1.4 (Prop 4.20)**.**

Example 1.5. *example pictures*

Remark 1.6. Belyi maps have some kind of rigid: they're decided by dessins (be viewed as skeleton of Belyi fcts)

Proof of Prop. 1. Given a pair (X, D), we need

- give a RS structure of X;
- give a fct $f : X \longrightarrow \mathbb{P}^1$.

f gives a RS structure of X (Riemann extension)

2. compability

3.

$$(S, f) \to (S, D_f) \to (S, f_{D_f}) \qquad (S, f) \sim (S, f_{D_f})?$$
$$(X, D) \to (X_D, f_D) \to (X_D, D_{f_D}) \qquad (X, D) \sim (X_D, D_{f_D})?$$

$\square$

2. Extract informations from the correspondence

{Belyi fct}/ $\sim$	{Dessin d'enfants}/ $\sim$
$\#f^{-1}(0) + \#f^{-1}(1)$	v
$\#f^{-1}(\infty)$	f
$\deg f$	e
$2 - 2g(S)$	$v + f - e$
Ram index of $x \in f^{-1}(0)$	# black dots adjecent to x
Ram index of $x \in f^{-1}(1)$	# white dots adjecent to x
Ram index of $x \in f^{-1}(\infty)$	$\frac{1}{2}$ # sides of face

Table 1: Correspondence

2.1. **basic information.**

$\{$Belyi fct$\}/\sim$	$\{$Dessin d'enfants$\}/\sim$	$\{$perm rep pair$\}/\sim$
$\#f^{-1}(0)$	$\#$ $\{$white dots$\}$	$\#$ $\{$cycles of $\sigma_0\}$
$\#f^{-1}(1)$	$\#$ $\{$black dots$\}$	$\#$ $\{$cycles of $\sigma_1\}$
$\#f^{-1}(\infty)$	f	$\#$ $\{$cycles of $\sigma_1\sigma_0\}$
$\deg f$	e	$N = \#$ $\{$ cycles of $Id\}$
$2-2g(S)$	$v+f-e$	$\#\{\ldots\sigma_0\}+\#\{\ldots\sigma_1\}+\#\{\ldots\sigma_1\sigma_0\}$-N
Ram index of $x \in f^{-1}(0)$	$\#$ $\{$black dots adjacent to $x\}$	length of a cycle on σ_0
Ram index of $x \in f^{-1}(1)$	$\#$ $\{$white dots adjacent to $x\}$	length of a cycle on σ_1
Ram index of $x \in f^{-1}(\infty)$	$\frac{1}{2}$ $\#$ $\{$sides of face$\}$	length of a cycle on $\sigma_0\sigma_1$

2.2. **monodromy.**

2.3. **Galois action.**

2.4. **construct new from old.** SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY OF SCIENCE AND TECHNOLOGY OF CHINA, HEFEI, 230026, P.R. CHINA,

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