LATEX TEMPLATE

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1. A small toolkit

$$\begin{split} f: Y &\longrightarrow \operatorname{pt} \, f: p \hookrightarrow X \\ f^* & \operatorname{constant \, sheaf} \, \mathcal{F}_p \\ Rf_* & \operatorname{cohomology \, sky}_p(\mathbb{Q}) \\ Rf_! & \operatorname{cpt \, supp \, cohomology \, sky}_p(\mathbb{Q}) \\ f^! & \operatorname{orientation \, sheaf \, } [n] \, \mathcal{F}_p[-n] \\ & \operatorname{For} \, f^!, & \operatorname{assume} \, Y, X & \operatorname{are \, manifolds \, of \, dimension \, } n. \\ j_! j^* \mathcal{F} \, \mathcal{F} \, i_! i^* \mathcal{F} \end{split}$$

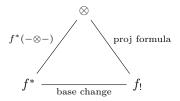
$$Z \stackrel{i}{\longleftarrow} X \stackrel{j}{\longleftarrow} U$$

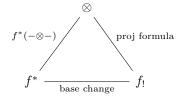
$$D(Z) \xrightarrow{i_* = i_!} D(X) \xrightarrow{j^* = j^!} D(U)$$

$$j_!j^*\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_!i^*\mathcal{F} \stackrel{+1}{\longrightarrow}$$

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	$f: Y \longrightarrow \mathrm{pt}$	$f:p\hookrightarrow X$
f^*	constant sheaf	\mathcal{F}_p
Rf_*	cohomology	$\mathrm{sky}_p(\mathbb{Q})$
$Rf_!$	cpt supp cohomology	$\mathrm{sky}_p(\mathbb{Q})$
f!	orientation sheaf	$[n] \mathcal{F}_p[-n]$





2. A Short List of Applications

Assuming the six-functor formalism (and everything derived), let X be a smooth manifold of dimension n.

1. Define four types of cohomology and the relative cohomology. Verify that:

$$\mathrm{H}^{i}_{\mathrm{c}}(X;\mathbb{Q}) \cong \mathrm{H}^{i}\left(\bar{X}, \{\infty\}; \mathbb{Q}\right)$$

$$\mathrm{H}_{i}^{\mathrm{BM}}(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_{X})$$

$$H_i(X; \mathbb{Q}) \cong H_c^{n-i}(X; Or_X)$$

Also, define the cup and cap product structures.

2. Using the projection formula, show Poincaré duality:

$$\mathrm{H}^i_\mathrm{c}(X;\mathbb{Q})^* \cong \mathrm{H}^{n-i}(X;\mathrm{Or}_X)$$

$$\mathrm{H}^{i}(X;\mathbb{Q}) \cong \mathrm{H}^{n-i}_{c}(X;\mathrm{Or}_{X})^{*}$$

3. Derive the Gysin sequence for any oriented S^k -bundle $\pi: E \longrightarrow B$:

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \xrightarrow{\pi_*} H^{n-k}(B) \xrightarrow{eu_{\pi^*}}$$

Derive the Mayer-Vietoris sequence and the relative cohomology sequence, and verify the equivalence of different cohomology groups.

4. Compute the upper shriek for singular spaces.

$$\begin{array}{lll} \mathrm{H}^i(Y,\mathbb{Q}) &= \mathrm{H}^i(Y,\underline{\mathbb{Q}}_Y) &= f_*\underline{\mathbb{Q}}_Y &= f_*f^*\mathbb{Q} \\ \mathrm{H}^i_c(Y,\mathbb{Q}) &= \mathrm{H}^i_c(Y,\underline{\mathbb{Q}}_Y) &= f_!\underline{\mathbb{Q}}_Y &= f_!f^*\mathbb{Q} \\ \mathrm{H}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}_c(Y,\mathrm{Or}_Y) &= f_!\,\mathrm{Or}_Y[n] &= f_!f^!\mathbb{Q} \\ \mathrm{H}^{\mathrm{BM}}_{-i}(Y,\mathbb{Q}) &= \mathrm{H}^{n+i}(Y,\mathrm{Or}_Y) &= f_*\,\mathrm{Or}_Y[n] &= f_*f^!\mathbb{Q} \end{array}$$

six functor formalism \approx cohomology theory

3. PERVERSE SHEAF

We will mix the usage of sheaves and complexes. For simplicity, let us fix a stratification \mathcal{S} :

$$\varnothing \stackrel{U_0}{\subset} Z_0 \stackrel{U_1}{\subset} \cdots \stackrel{U_n}{\subset} Z_n = X$$

Denote $D^b_{cons,S}(X)$ as the category of constructible sheaves over X with respect to S.

Roughly speaking, a perverse sheaf is a type of sheaf that lies between $\pi^*\mathbb{Q}$ and $\pi^!\mathbb{Q}$. More rigorously, a perverse sheaf is a complex that belongs to the heart of the perverse t-structure. We say that $\mathcal{F} \in D^b_{\text{cons},\mathcal{S}}(X)$ is perverse if

$$\begin{cases} \mathcal{H}^{i}\left(\iota_{U_{j}}^{*}\mathcal{F}\right)=0, & \text{for any } i>-j\\ \mathcal{H}^{i}\left(\iota_{U_{j}}^{!}\mathcal{F}\right)=0, & \text{for any } i<-j \end{cases}$$

To determine whether a complex \mathcal{F} is perverse, one simply needs to complete the following table:

$$\iota_{U_i}^*(\mathcal{F})$$

The local system supported on U_i (denoted by \mathcal{L}) converted to a perverse sheaf by truncations. This method is called Deligne's construction, and the constructed perverse sheaf is called the intersection cohomology complex(or the IC sheaf), denoted by $IC(\mathcal{L})$. IC sheaves are the simple objects in the category $Perv_{\mathcal{S}}(X)$.

4. Nearby Cycle

A perverse sheaf may not be so "perverse", but a nearby cycle is definitely "nearby". Given $\mathcal{F} \in D^b(\mathbb{C})$, one can construct the nearby cycle

$$\psi \mathcal{F} := i^* R j_* p_* p^* j^* \mathcal{F} \in D^b(\{0\}),$$

which can be roughly viewed as the fiber \mathcal{F}_x for x sufficiently close to 0. By quotienting out the non-vanishing cycle $i^*\mathcal{F}$, one obtains the vanishing cycle

$$\varphi \mathcal{F} := \operatorname{cone} \left[i^* \mathcal{F} \xrightarrow{sp} \psi \mathcal{F} \right] \in D^b(\{0\}).$$

In general, C can be replaced by any disk D, as the problem is local, and F can be a sheaf over any space X over D.

The same construction yields a distinguished triangle in $D^b(X_0)$:

$$i^*\mathcal{F} \longrightarrow \psi_f\mathcal{F} \longrightarrow \varphi_f\mathcal{F} \stackrel{+1}{\longrightarrow}$$

5. Characteristic Cycle

With normal Morse data, one can define the characteristic cycle

$$\mathrm{CC}(\mathcal{F}) := \sum_{S} m_{S}[T_{S}^{*}\mathbb{C}^{n}] \in H_{2n}^{\mathrm{BM}}(\cup_{S} T_{S}^{*}\mathbb{C}^{n}),$$

where

$$m_S := \chi(\text{NMD}(\mathcal{F}, S)[-\dim S]) \in \mathbb{Z}.$$

Notably, $CC(\mathcal{F})$ does not depend on the stratification \mathcal{S} .

The characteristic cycle can be computed when the geometry is well-understood, such as when X is a cone over a smooth hyperplane.

We work with a fixed complex variety embedding $X \subseteq \mathbb{C}^n$, equipped with a Whitney stratification S. Let $S \subseteq X$ be a connected component of some U_i . Fix $x_0 \in S$, and let N be a normal slice of S at x_0 .

For any sheaf $\mathcal{F} \in D^b_{\text{cons},\mathcal{S}}(X)$, the normal Morse data(NMD) is defined as

$$NMD(\mathcal{F}, S) := \left(\varphi_{g|_{N \cap X}} \left(\mathcal{F}|_{N \cap X}\right)\right)_{x_0} [-1]$$

where $g:\mathbb{C}^n\longrightarrow\mathbb{C}$ is a holomorphic function, and f:=Re(g) such that

- $g(x_0) = 0;$
- $df_{x_0} \in T_S^* \mathbb{C}^n$, and $df_{x_0} \notin T_{S'}^* \mathbb{C}^n$ for any $S' \neq S$;
- x_0 is a non-degenerate critical point of $f|_S$.

References

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