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Schur - Horn Theorem

I. Intro

Given a Hermitian matrix $A\cancel{\ } = (a_{ij}) \in \mathbb{C}^n$ with eigenvalues

$\lambda = (\lambda_1, \lambda_2, \cdots \lambda_n)^T \in \mathbb{R}^n$

We want to see:

$\boxed{\text{Q: What do the diagonal elements } (a_{11}, a_{22}, \cdots a_{nn})^T \text{ look like?}}$

~~Obvious Facts: A^H~~

Facts (Obvious)

- $A^H = A \Rightarrow a_{11}, \cdots, a_{nn} \in \mathbb{R}$
- $A \overset{\text{unitary}}{\sim} \operatorname{diag}(\lambda_1, \cdots \lambda_n)$
  $\Rightarrow \sum_{i=1}^{n} a_{ii} = \operatorname{tr} A = \operatorname{tr}(\operatorname{diag}(\lambda_1, \cdots \lambda_n)) = \sum_{i=1}^{n} \lambda_i$
- $\forall \tau \in S_n$, $\operatorname{diag}(\lambda_1, \cdots \lambda_n) \overset{\text{uni}}{\sim} \operatorname{diag}(\lambda_{\tau(1)}, \cdots, \lambda_{\tau(n)})$
  $\Rightarrow$ WLOG, we can rearrange $(\lambda_1, \cdots \lambda_n)$ s.t
  $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \lambda_n$

NOTICE: After that we all assume $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$

Facts (Not Obvious)

- $\forall i \in \{1, \cdots, n\}$ ~~$\lambda_n \leq a_{ii}$~~ $\lambda_n \leq a_{ii} \leq \lambda_1$
- $\forall k \in \{1, \cdots, n\}$ $\sum_{i=1}^{k} a_{ii} \leq \sum_{i=1}^{k} \lambda_i$

Issai Schur (Russian) proved the above-mentioned inequalities in 1923.

- Issai Schur, 1875-1941,
  worked in Germany for most of his life
  student of Frobenius (SYLOW Thm. Proofed)
  ~~student~~:
  Schur lemma (representations & group)

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· ~~Thm (Schur-Horn)~~

denote $\lambda = (\lambda_1, \cdots \lambda_n)^T \in \mathbb{R}^n$

$\Lambda = \mathrm{diag}(\lambda_1, \cdots \lambda_n) \in \mathbb{R}^n$

$\mathcal{H}(n) = \{A \in \mathbb{C}^n \mid A^H = A\}$

$\mathcal{H}_\lambda = \{A \in \mathcal{H}(n) \mid A \overset{uni}{\sim} \Lambda\}$

$\pi : \mathcal{H}(n) \longrightarrow \mathbb{R}^n$

$A = (a_{ij})_{i,j=1}^{n} \mapsto (a_{11}, \cdots a_{nn})^T$

Thm. (Schur-Horn)

$\pi(\mathcal{H}_\lambda)$ is a convex polyhedron in $\mathbb{C}^n$
whose vertices are

$(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)})^T \in \mathbb{C}^n$

where $\tau \in S_n$.

Alfred Horn (American, UCLA) proved it in 1954.
- Doubly stochastic matrices & the diagonal of a rotation matrix
- lattice theory & universal algebra → logic programming

With these Facts in mind,
we will first discuss some examples.
- (trivial) when $\lambda = (\lambda_0, \cdots \lambda_0)$ we have

$\Lambda = \lambda_0 I$

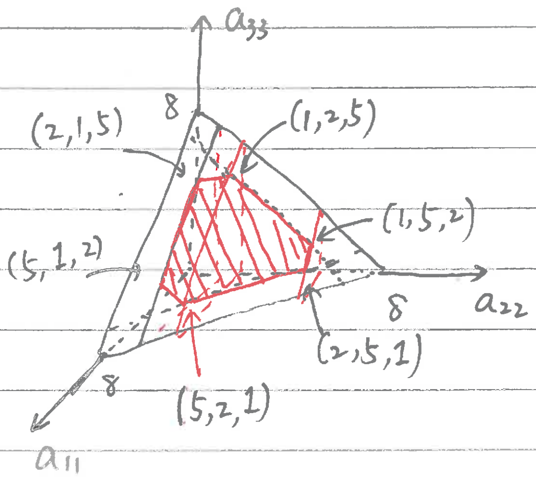
$\mathcal{H}_\lambda = \{A \in \mathbb{C}^{n \times n} \mid \exists U \in U(n),\ A = U(\lambda_0 I)U^H = \lambda_0 I\}$

$= \{\lambda_0 I\}$ only one element!

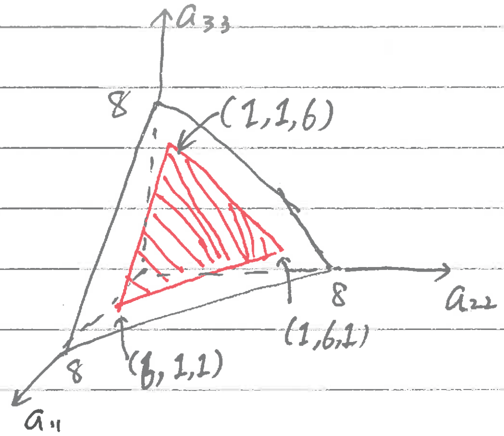
We leave 2-dim example at last because it's computable.

– (3-dim) when $\lambda = (\lambda_1, \lambda_2, \lambda_3)$

It's almost impossible to calculate, So we only draw out the final result:



$(\lambda_1, \lambda_2, \lambda_3) = (5,2,1)$



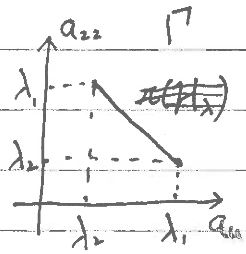
$(\lambda_1, \lambda_2, \lambda_3) = (6,1,1)$
"degenerate"

– (2-dim)

$A = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \in \mathbb{C}^{2\times 2}$ has eigenvalues $\lambda = (\lambda_1, \lambda_2)^T \in \mathbb{R}^2$

$\Leftrightarrow \exists U = (u_{ij})_{i,j=1}^{2} \in U(2),$

$$A = U \Lambda U^H = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \lambda_1 & \\ & \lambda_2 \end{bmatrix} \begin{bmatrix} \bar{u}_{11} & \bar{u}_{21} \\ \bar{u}_{12} & \bar{u}_{22} \end{bmatrix}$$



$$= \begin{bmatrix} \lambda_1 |u_{11}|^2 + \lambda_2 |u_{12}|^2 & \lambda_1 u_{11}\bar{u}_{21} + \lambda_2 u_{12}\bar{u}_{22} \\ \lambda_1 u_{21}\bar{u}_{11} + \lambda_2 u_{22}\bar{u}_{12} & \lambda_1 |u_{21}|^2 + \lambda_2 |u_{22}|^2 \end{bmatrix}$$

$$= \lambda_2 I + (\lambda_1 - \lambda_2) \begin{bmatrix} |u_{11}|^2 & u_{11}\bar{u}_{21} \\ u_{21}\bar{u}_{11} & |u_{21}|^2 \end{bmatrix}$$

· $\pi(\mathcal{H}_\lambda) \subseteq \Gamma$ ← because $\lambda_1 |u_{11}|^2 + \lambda_2 |u_{12}|^2$ is the convex combination of λ_1, λ_2

· $\Gamma \subseteq \pi(\mathcal{H}_\lambda)$ because we can take $\begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

· ~~More~~ Actually we can compute more.

→ WLOG, we only consider the condition when

(take the coordinate transformation)

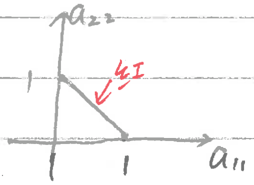
- $\lambda = (\lambda_1, \lambda_2)^T = (1,0)^T$
- $A = \begin{pmatrix} |u_{11}|^2 & u_{11}\bar{u}_{21} \\ u_{21}\bar{u}_{11} & |u_{21}|^2 \end{pmatrix}$

- $a \hat{=} a_{11}$
- $a_{22} = 1-a$
- $a_{21} = \bar{a}_{12}$
- $|a_{12}| = |u_{11}||u_{21}| = \sqrt{a_{11}a_{22}}$

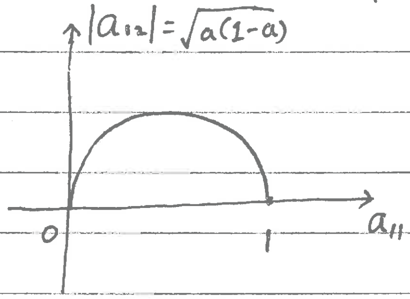
$$\Rightarrow \mathcal{H}_\lambda \underset{\supseteq}{\subseteq} \left\{ \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} \middle| a \in [0,1],\ 0 \le \varphi < 2\pi \right\}$$

calculate the eigenvalue

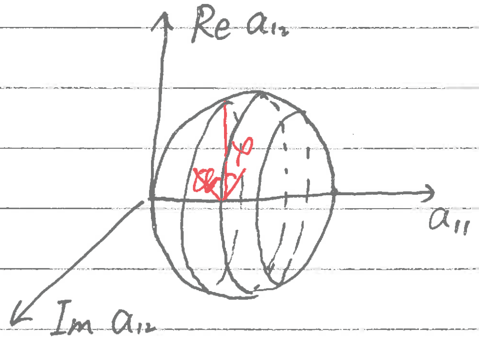
$$U = \begin{pmatrix} \cos\theta & -e^{i\varphi}\sin\theta \\ e^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix}$$



Now $\pi(\mathcal{H}_\lambda) = \{(a, 1-a) \mid 0 \le a \le 1\}$



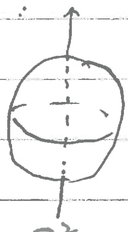
$\rightsquigarrow$



Moreover, $\mathcal{H}_\lambda$ is a mfld diffeomorphic to S^2:

$$\Phi : \mathcal{H}_\lambda \longrightarrow S^2$$

$$\begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} \mapsto (\cos\varphi, \sin\varphi, a)$$



S^2

[What is a mfld? Mfld is a VERY GOOD geometric structure which locally always look like $\mathbb{R}^n$]

We will find out more informations through this isomorphism.

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II. Simple Tools

~~Before exploring the phenomenon~~, let us deviate from this phenomenon for a while to obtain the most basic tools: the Lie bracket & Exponential map.

1. Lie bracket

Def: the Lie bracket of $M_n(\mathbb{C})$ is

$$[\ ,\ ]: M_n(\mathbb{C}) \times M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$$
$$(A, B) \mapsto [A,B] := AB - BA$$

Prop: (basic)
i) (Skew-Symmetric) $[A,B] = -[B,A]$
ii) (linear) $[c_1A_1 + c_2A_2, B] = c_1[A_1, B] + c_2[A_2, B]$
iii) (Jacobi-identity) $[A,[B,C]] + [B[C,A]] + [C,[A,B]] = 0$

Prop.' (used today)
iv) $([AB])^H = [B^H A^H]$
v) $tr(A[BC]) = tr([AB]C)$

Proof: Exercise.

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2. The Exponential ~~map~~ Map for Matrix.

Def. Suppose $A \in M_n(\mathbb{C})$, then we define

$$P_n(A) := \sum_{i=0}^{n} \frac{A^i}{i!}$$

$$\exp(A) := e^A := \lim_{n\to+\infty} P_n(A)$$

Rmk. i) By defining the norm on $M_n(\mathbb{C})$; one is easy to find out the existence & uniqueness of the limit.

ii) Generally $e^A \cdot e^B \neq e^{A+B}$. We still have

$$AB = BA \Rightarrow e^A \cdot e^B = e^{A+B}$$

iii) like polynomials, some properties are easily derived from the definition:

- $\forall U \in U(n),\ Ue^X U^{-1} = e^{UXU^{-1}}$
- $(e^X)^H = e^{X^H}$
- $\frac{d}{dt} e^{tX} = Xe^{tX}$

  especially $\frac{d}{dt}\Big|_{t=0} e^{tX} = X$

iv) sometimes we denote $\exp(X) = e^X$ to enlarge superscript.

v). Someone may think the Exp map as "walking along the v.f. Xe^{tX} (in ~~$M_n(\mathbb{C})$~~ $GL_n(\mathbb{C})$) for t times". You can easily check (if you've learned about mfld) that $\exp(tX)$ is just an integral curve $\gamma_X(t)$ in $GL_n(\mathbb{C})$.

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$(\mathcal{H}_\lambda, \Omega_{-i\lambda}, \mathbb{T}^n, \pi)$

III. Group actions

1. Look at $\mathcal{H}(n)$.

We have VERY NICE group action on $\mathcal{H}(n)$.

$$U(n) \circlearrowright \mathcal{H}(n)$$
$$U \cdot H = UHU^H$$

Rmk. One can check this is really the group action:
- $UHU^H \in \mathcal{H}(n)$
- $I \cdot H = H$
- $(U_1 U_2) \cdot H = U_1 \cdot (U_2 \cdot H)$

Q: What is the orbit of this action?

A: From the linear algebra theory,
$A \in \mathcal{H}(n)$ has eigenvalues $\lambda = (\lambda_1, \cdots \lambda_n) \in \mathbb{R}^n$
$\Leftrightarrow \exists U \in U(n) \quad A = U \Lambda U^H$. As a result,

Prop. The orbit of the group action is
$\mathcal{H}_\lambda = \{A \in \mathcal{H}(n) \mid A \text{ has eigenvalues } \lambda = (\lambda_1, \cdots \lambda_n)\}$.

Moreover, • $\mathcal{H}(n)$ is a $\mathbb{R}$-linear space $\Rightarrow$ mfld
• $U(n)$ is a Lie group
~~• $U(n)$ acts on $\mathcal{H}(n)$ properly & freely (Good!)~~

So from the Lie group's theory we can obtain

Prop. $\mathcal{H}_\lambda$ is a manifold.

This is not so surprising because we have calculated the $\mathcal{H}_{(1,0)}$ and "verified" that this is a mfld diffeo to S^2. Later we will see more structures on $\mathcal{H}_\lambda$, and these ~~actions~~ structures in all will help us to find out more informations about $\pi(\mathcal{H}_\lambda)$.

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2. Subgroup actions.

We have found

$$S^1 = \left\{ \begin{pmatrix} e^{i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \middle| \theta \in \mathbb{R} \right\} \subseteq U(n)$$

$$\mathbb{T}^n = S^1 \times S^1 \times \cdots S^1$$

$$= \left\{ \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} \middle| \theta_1, \cdots \theta_n \in \mathbb{R} \right\} \subseteq U(n)$$

Then $S^1 \subseteq \mathbb{T}^n \subseteq U(n)$

We have the induced subgroup actions.

$S^1 \curvearrowright \mathcal{H}_\lambda$	$\mathbb{T}^n \curvearrowright \mathcal{H}_\lambda$
$A \cdot H = AHA^H$	$A \cdot H = AHA^H$
$\theta \cdot H = \begin{pmatrix} e^{i\theta} & \\ & I_{n-1} \end{pmatrix} H \begin{pmatrix} e^{-i\theta} & \\ & I_{n-1} \end{pmatrix}$	$(\theta_1, \cdots \theta_n) \cdot H = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} H \begin{pmatrix} e^{-i\theta_1} & & \\ & \ddots & \\ & & e^{-i\theta_n} \end{pmatrix}$

分块：$H = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline H_{21} & H_{22} \end{array} \right)$
(partition matrices)

$$\theta \cdot H = \begin{pmatrix} e^{i\theta} & \\ & I_{n-1} \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} e^{-i\theta} & \\ & I_{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} H_{11} & e^{i\theta} H_{12} \\ e^{-i\theta} H_{21} & H_{22} \end{pmatrix}$$

$$\frac{d}{d\theta}(\theta \cdot H) = \begin{pmatrix} 0 & ie^{i\theta} H_{12} \\ -ie^{-i\theta} H_{21} & 0 \end{pmatrix}$$

3. The induced v.f. of group action

Def Suppose $j \in \{1,\dots,n\}$, $\mathbb{T}^n \curvearrowright \mathcal{H}_\lambda$, then the induced v.f. X_j at point H in $\mathcal{H}_\lambda$ is the matrix

$$X_j(H) = \frac{d}{dt}\Big|_{t=0}\big((0,\dots,\underset{j\text{-th}}{t},\dots)\cdot H\big)$$

E.g. We have computed

$$X_1(H) = \frac{d}{dt}\Big|_{t=0}\big((t,0,\dots 0)\cdot H\big) = \begin{pmatrix} & iH_{12} \\ -iH_{21} & \end{pmatrix}$$

Similarly, if $H = (h_{ij})_{i,j=1}^{n}$, then

$$X_j(H) = \begin{pmatrix} & & ih_{1j} & & \\ & & \vdots & & \\ -ih_{j1} & \cdots & 0 & \cdots & -ih_{jn} \\ & & \vdots & & \\ & & ih_{nj} & & \end{pmatrix}$$

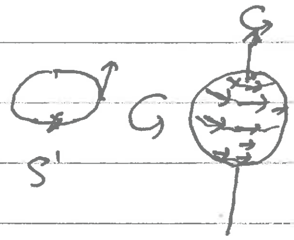
E.g. When $n=2$, $H = \begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix}$

$$X_1(H) = \begin{pmatrix} 0 & ih_{12} \\ -ih_{21} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & ie^{i\varphi}\sqrt{a(1-a)} \\ -ie^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix}$$

Notice that

$$\frac{\partial H}{\partial \varphi} = \begin{pmatrix} 0 & ie^{i\varphi}\sqrt{a(1-a)} \\ -ie^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix}$$



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IV. New Tools.

Symplectic Manifold

~~Def.~~ Suppose M is a mfld of dim $2n$.

Def. A symplectic form on M is a 2-form $\omega \in \Lambda^2 T^*M$ on M s.t

1) ω is closed: $d\omega = 0$

2) ω is non-degenerate:

$\underbrace{\omega \wedge \omega \wedge \cdots \wedge \omega}_{n} \neq 0$ is a volume form on M.

The pair (M, ω) is called a symplectic mfld.

Rmk. Compared with Riemann metric g:

i) g can be defined on ANY mfld, while ω CAN'T (dim $= 2n$, orientable, so on)

ii) g is symmetric while ω is skew-symmetric

iii) By Darboux thm, ω looks like $\sum_{i=1}^{n} dx^i \wedge dy^i$ near ANY $p \in M$. while g has plenty of local geometric structure (e.g curvature & connection)

iv) g_p gives an isomorphism

$$g_p^{\#} : T_pM \longrightarrow T_p^*M$$
$$X_p \longmapsto g_p(X_p, -)$$

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while ω_p also gives an isomorphism.

$$\omega_p^{\#}: T_pM \longrightarrow T_p^*M$$
$$X_p \longmapsto \omega_p(X_p, -)$$

We will use this isomorphism to convert a v.f. (which I have mentioned, induced by group action) to a exact 1-form.

E.g. 1. $(\mathbb{R}^{2n}, \omega)$ is a sym mfld with

local chart coord $(x^1, \dots x^n, y^1, \dots y^n)$

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

Verify: · $\omega \in \Lambda^2 T^*M$

· $d\omega = \sum_{i=1}^{n} d1 \wedge dx^i \wedge dy^i = 0$

· $\omega \wedge \omega \wedge \cdots \wedge \omega = n!\, dx^1 \wedge dy^1 \wedge \cdots dx^n \wedge dy^n \neq 0$

E.g. 2. $(S^2, \omega = d\theta \wedge dh)$ is a sym mfld.

the volume form locally

Verify: · $\omega \in \Lambda^2 T^*M$

· $d\omega = 0$ because ω is a top form

· ω is non-degenerate since it is already a volume.

E.g 3. From the diffeomorphism

$$\Phi : \mathcal{H}_{(1,0)^T} \longrightarrow S^2$$

$$\begin{pmatrix} a & e^{i\varphi}\sqrt{a(1-a)} \\ e^{-i\varphi}\sqrt{a(1-a)} & 1-a \end{pmatrix} \mapsto (\cos\varphi, \sin\varphi, a)$$

$$\hat{=}\; H(a,\varphi) \qquad\qquad \hat{=}\; (e^{i\varphi}, a)$$

One can obtain a natural Symp form on $\mathcal{H}_{(1,0)^T}$.

$$\Phi^* : \Omega^2(S^2) \longrightarrow \Omega^2(\mathcal{H}_{(1,0)^T})$$

$$\omega = d\theta \wedge dh \longmapsto \omega_{can}$$

We can calculate $(a \neq 0, 1)$

$$(d\Phi)^{-1}\left(\frac{\partial}{\partial\theta}\right) = \begin{pmatrix} 0 & i e^{i\varphi}\sqrt{a(1-a)} \\ -i e^{-i\varphi}\sqrt{a(1-a)} & 0 \end{pmatrix} \hat{=} \frac{\partial}{\partial\varphi} = X_1(H(a,\varphi))$$

$$(d\Phi)^{-1}\left(\frac{\partial}{\partial h}\right) = \begin{pmatrix} 1 & e^{i\varphi}\frac{1-2a}{2\sqrt{a(1-a)}} \\ e^{-i\varphi}\frac{1-2a}{2\sqrt{a(1-a)}} & -1 \end{pmatrix} \hat{=} \frac{\partial}{\partial a}$$

$$\because\; T_{(e^{i\varphi},a)}S^2 = \left\langle \frac{\partial}{\partial\theta}\Big|_{(e^{i\varphi},a)}, \frac{\partial}{\partial h}\Big|_{(e^{i\varphi},a)} \right\rangle$$

$$\therefore\; T_{H(a,\varphi)}\mathcal{H}_{(1,0)^T} = \left\langle \frac{\partial}{\partial\varphi}\Big|_{H(a,\varphi)}, \frac{\partial}{\partial a}\Big|_{H(a,\varphi)} \right\rangle$$

$$\omega\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial h}\right) = 1$$

$$\Rightarrow 1 = \omega_{can}\left(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial a}\right)$$

$$= \omega_{can}^{\#}\left(\frac{\partial}{\partial\varphi}\right)\left(\frac{\partial}{\partial a}\right)$$

$$\Rightarrow \omega_{can}^{\#}\left(\frac{\partial}{\partial\varphi}\right) = da \;\Rightarrow\; \omega_{can}^{\#}(X_1) = da.$$

Rmk. In general $\mathcal{H}_\lambda$ is also a symp mfld whose symp form $(H = U\Lambda U^H)$

$$\omega_\lambda|_H(X,Y) = i\,tr(\Lambda[U^H X, U^H Y])$$

Moreover, $\omega_\lambda^{\#}(X_i)$ is exact, i.e $\exists f \in C^\infty(\mathcal{H}_\lambda)$ s.t $\omega_\lambda^{\#}(X_i) = df$.
This will be denoted "The moment map".

Moment Map.

Def. || Suppose $S^1 \curvearrowright \mathcal{H}_\lambda$, then the moment map is a map
$$\mu : \mathcal{H}_\lambda \longrightarrow \mathbb{R} \quad \text{s.t}$$
$$\omega_\lambda^{\#}(X_1) = d\mu$$

From E.g.3 we can see, the moment map of $S^1 \curvearrowright \mathcal{H}_{(1,0)^T}$ is
$$\mu : \mathcal{H}_{(1,0)^T} \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto a_{11}$$

Def. || Suppose $\mathbb{T}^n \curvearrowright \mathcal{H}_\lambda$, then the moment map of the group action is a map
$$\mu : \mathcal{H}_\lambda \longrightarrow \mathbb{R}^n \quad \text{s.t}$$
$$A \longmapsto (\mu_1(A), \cdots \mu_n(A))$$
$$\omega_\lambda^{\#}(X_i) = d\mu_i \quad \forall i \in \{1, \cdots n\}$$

Rem. Like the examples we have seen, in general, if $\mathbb{T}^n \curvearrowright \mathcal{H}_\lambda$ in a canonical way, then
$$\mu := \pi : \mathcal{H}_\lambda \longrightarrow \mathbb{R}^n$$
$$A = (a_{ij})_{i,j=1}^n \longmapsto \cancel{\mathrm{diag}}(a_{11}, \cdots a_{nn})^T$$

is just the projections to its diagonal items! Its proof require the knowledge of coadjoint orbit, so I regret that I'll skip it.

Def. || We call $(\mathcal{H}_\lambda, \omega, \mathbb{T}^n, \mu)$ as the Hamiltonian $\mathbb{T}^n$- mfld.

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After we've introduced ~~the~~ all conceptions, we state the last theorem which is ingenious formally but its proof need deep symp geometry knowledges.

Thm. (~~AGS~~ Atiyah-Guillemin-Sternberg Convexity thm)
Suppose $(M, \omega, \mathbb{T}^r, \mu)$ be a Hamiltonian $\mathbb{T}^r$-mfld.
If M is compact & connected, then
$\mu(M)$ is a convex polyhedron in $\mathbb{R}^n$
whose vertices are the images of the $\mathbb{T}^r$-fixed points.

Proof of Schur-Horn thm

- $(\mathcal{H}_\lambda, \omega_\lambda, \mathbb{T}^n, \mu)$ is a Hamiltonian $\mathbb{T}^n$-mfld
- $\mathcal{H}_\lambda$ is compact: $\mathcal{H}_\lambda$ is closed $(U(n) \curvearrowright \mathcal{H}_\lambda)$
  (Algebraic functions)
  $\mathcal{H}_\lambda$ is bounded by λ.
- $\mathcal{H}_\lambda$ is connected:

$\forall A \in \mathcal{H}_\lambda \ \exists U \in U(n) \quad A = U \Lambda U^H$

$U(n)$ is connected

$\Rightarrow \exists U(t): [0,1] \to U(n)$ s.t. $U(0) = I \quad U(1) = U$

$\Rightarrow A(t) := U(t) \Lambda U^H(t): [0,1] \to \mathcal{H}(n)$ s.t.

$A(0) = \Lambda \quad A(1) = A$

$\Rightarrow \mathcal{H}_\lambda$ is connected

$\rightsquigarrow \pi(\mathcal{H}_\lambda)$ is a convex polyhedron in $\mathbb{R}^n$.

· $\mathbb{T}^n$-fixed points.

Suppose ~~$A \in \mathcal{H}_\lambda$~~ $A = (a_{ij})_{i,j=1}^n \in \mathcal{H}_\lambda$

- If $(\theta_1, \cdots \theta_n) \cdot A = A$ for $\forall (\theta_1, \cdots \theta_n) \in \mathbb{R}^n$

$$\Rightarrow \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} (a_{ij}) = (a_{ij}) \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} e^{i\theta_1} a_{11} & \cdots & e^{i\theta_1} a_{1n} \\ \vdots & \ddots & \vdots \\ e^{i\theta_n} a_{n1} & \cdots & e^{i\theta_n} a_{nn} \end{pmatrix} = \begin{pmatrix} e^{i\theta_1} a_{11} & \cdots & e^{i\theta_n} a_{1n} \\ \vdots & \ddots & \vdots \\ e^{i\theta_1} a_{n1} & \cdots & e^{i\theta_n} a_{nn} \end{pmatrix}$$

$\Rightarrow a_{ij} = 0 \quad \forall i \neq j \qquad A = \mathrm{diag}(a_{11}, \cdots, a_{nn})$

$\xRightarrow{A \in \mathcal{H}_\lambda} A = \mathrm{diag}(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)})$ where $\tau \in S_n$.

- On the other hand, If $A = \mathrm{diag}(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)})$, then $(\theta_1, \cdots \theta_n) \cdot A = A$ for $\forall (\theta_1, \cdots \theta_n) \in \mathbb{R}^n$

$\therefore$ $\mathbb{T}^n$-fixed points are

$$\mathbb{T}^n_{fix} = \{\mathrm{diag}(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)}) \in \mathcal{H}_\lambda \mid \tau \in S_n\}$$

$$\Rightarrow \pi(\mathbb{T}^n_{fix}) = \{(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)})^T \mid \tau \in S_n\}$$

$\therefore$ By the AGM-convexity thm.

$\pi(\mathcal{H}_\lambda)$ is a convex polyhedron in $\mathbb{R}^n$ whose vertices are $(\lambda_{\tau(1)}, \cdots \lambda_{\tau(n)})^T \in \mathbb{R}^n$ where $\tau \in S_n$. $\square$