

Thanks for coming to listen to my talk.
 If you have any question or can't see some symbols,
 just interrupt me.

My talk is about the Langlands correspondence (LC).
 There are many versions about LC,

from local to global,

from 1-dim to n-dim,

from GL_n to non-split gp,

← that's why an introduction to LC is usually hard and painful.

Not to say, today I have only 1 hour!

$$\begin{array}{ccc}
 \text{Irr}_{\mathbb{C}}(GL_n(F)) & \xleftarrow[1:1]{} & WDrep_{n\text{-dim}}(W_F) \\
 \text{Char}_{\mathbb{C}, \text{alg}}(F^\times \backslash A_F^\times) & \xleftarrow[1:1]{} & \text{Char}_{\overline{\mathbb{Q}_p}}(\Gamma_F) + \text{de Rham} \\
 \overline{\text{Irr}}_{\overline{\mathbb{A}_F^\times}, k, \eta}(GL_2(A_\mathbb{Q})) & \xrightarrow{ES} & \text{Irr}_{\overline{\mathbb{Q}_p}}(\Gamma_F) + \text{modular} \\
 \overline{\text{Irr}}_{\overline{\mathbb{A}_F^\times}, k, \eta}(G_D(A_\mathbb{Q})) & \longrightarrow & \dots
 \end{array}$$

This talk is divided into 4 part.

In each part, we discuss one type of the correspondence.

Before beginning discussing these correspondences,
let me fix notations.

Setting

F : NA local field \mathcal{O}_F x_F

Γ_F : $= \text{Gal}(F^{\text{sep}}/F)$

W_F : Weil gp of F

Rep	sm rep
Irr	irr sm rep
TI	adm irr sm rep
Char	1-dim sm rep
WDrep	Weil-Deligne rep
$\mathfrak{A}_{\text{cusp}}$	cuspidal automorphic forms

1. NA local field case ($F = \mathbb{Q}_p$)

Thm (LLC for $GL_n(F)$) \exists bij $(K = GL_n(\mathcal{O}_F), Z = F^\times)$

$$rec_{F, GL_n} : Irr_{\mathbb{C}}(GL_n(F)) \xleftrightarrow{1:1} WDrep_{n-\text{dim}}(W_F)$$

$n=1$

$n=2$

$$\chi : F^\times \rightarrow \mathbb{C}^\times \iff \chi : W_F \rightarrow W_F^{ab} \cong F^\times \rightarrow \mathbb{C}^\times$$

$$\chi \circ \det \iff \left(\begin{pmatrix} \chi|_{I_F^+}^{-\frac{1}{2}} & \\ & \chi|_{I_F^-}^{\frac{1}{2}} \end{pmatrix}, 0 \right)$$

$$n\text{-Ind}_B^{GL_n}(\chi_1, \chi_2) \iff \left(\begin{pmatrix} \chi_1 & \\ & \chi_2 \end{pmatrix}, 0 \right)$$

$$St \otimes (\chi \circ \det) \iff \left(\begin{pmatrix} \chi|_{I_F^+}^{-\frac{1}{2}} & \\ & \chi|_{I_F^-}^{\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right)$$

$\boxed{\begin{array}{l} \text{tempered} \\ \text{disc series} \\ \text{(super)cuspidal} \end{array}}$

$$c\text{-Ind}_{KZ}^{GL_n} \iff ?$$

Rmk.

$$\text{action by } \chi \quad - \otimes (\chi \circ \det) \iff - \otimes \chi$$

$$\text{unramified} \quad \forall k \neq \{0\} \iff \phi|_{I_F} = \mathbf{1}_{I_F}, N=0$$

~~disc~~ \iff disc series \iff indec rep

$$\text{cuspidal} \iff \text{irr rep}$$

$$\text{"n-Ind"} \iff \oplus$$

Rmk. For Archi field people have ~~also~~ clear statement classifications.

2. GLC, $n=1$ ($F = \mathbb{Q}$)

In order to state GLC, we need Adèle & idèle which collects all local information.

Observe that

$$\mathbb{Q}^\times \backslash \widehat{\mathbb{A}_\mathbb{Q}}^x / \mathbb{R}_{>0} \cong \widehat{\mathbb{Z}}^x \cong \text{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \cong \widehat{\Gamma}_\mathbb{Q}^{ab}$$

which gives us GLC for $n=1$.

$$\text{Char}_{\mathbb{C}, \text{alg}, \text{wt } 0} (F^\times \backslash \widehat{\mathbb{A}_F}^x) \longleftrightarrow \text{Char}_{\mathbb{C}} (\Gamma_F)$$

$$\text{Char}_{\mathbb{C}, \text{alg}} (F^\times \backslash \widehat{\mathbb{A}_F}^x) \xleftarrow{\text{twist}} \text{Char}_{\overline{\mathbb{Q}_p}} (\Gamma_F) + \text{de Rham}$$

3. Adèlic MF

route:

Moduli space \leadsto MF \leadsto $\mathcal{A}_{\text{cusp}, k, \eta} \leadsto \prod_{\text{cusp}, k, \eta} \leadsto$ GLC

Ex. Shows that

$$\frac{\text{GL}_2(\mathbb{A}_\mathbb{Q})}{\widehat{\Gamma(N)} \cdot \mathbb{R}^\times \cdot \text{SO}_2} \cong \Gamma(N) \backslash \mathcal{H}^\pm$$

Def. (Cuspidal MF $S_{M_2(\mathbb{Q}), k, \eta}$)

For $k \geq 2$, $\eta \in \mathbb{Z}$, let
 \uparrow_{wt}

$$j_{k,\eta}(\gamma) := (\det \gamma)^{\eta-1} (c_i + d)^{\otimes k} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$$

$$S_{M_2(\mathbb{Q}), k, \eta} := \left\{ \gamma: GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_{\mathbb{Q}}) \rightarrow \mathbb{C} \text{ fcts} \right\} \\ \text{s.t. } \textcircled{1}-\textcircled{4} \text{ are true}$$

① (continuity) $\exists U_{\text{fin}} \subseteq GL_2(\mathbb{A}_{\mathbb{Q}, \text{fin}})$ open s.t.

$$\gamma(g\gamma) = \gamma(g) \quad \forall \gamma \in U_{\text{fin}}$$

② (automorphy)

$$\gamma(g\gamma) = \cancel{\gamma(g)} j_{k,\eta}(\gamma)^{-1} \gamma(g) \quad \forall \gamma \in \mathbb{R}^{\times} \times SO_2$$

$$\rightsquigarrow j_{k,\eta}(\gamma\gamma) \gamma(g\gamma) = j_{k,\eta}(\gamma') \gamma(g)$$

③ (holomorphy) $\forall g \in GL_2(\mathbb{A}_{\mathbb{Q}})$, the fact

$$f_{\gamma, g}: \mathcal{H}^{\pm} \rightarrow \mathbb{C} \quad \gamma_i \mapsto \gamma(g\gamma) j_{k,\eta}(\gamma)$$

is holomorphic

(holomorphic at ∞)

$f_{\gamma, g}(\tau) |Im \tau|^{\frac{k}{2}}$ is bounded

④ (cuspidal condition) $\forall g \in GL_2(\mathbb{A}_{\mathbb{Q}})$,

$$\int_{\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}} \gamma \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$

E.g. When $\mathcal{U}_{\text{fin}} = \widehat{\Gamma(N)}$, one has iso

$$S_{M_2(\mathbb{Q}), k, 1}^{\widehat{\Gamma(N)}} \cong S_k(\Gamma(N)) \quad \varphi \mapsto f_{\varphi, \text{Id}}$$

$GL_2(A_{\mathbb{Q}})$

$$\{f: GL_2(\mathbb{Q}) \backslash GL_2(A_{\mathbb{Q}}) \rightarrow \mathbb{C}\}$$

U

$A_{\text{cusp}, k, \eta}$ cusp auto forms of wt (k, η)

U

$S_{M_2(\mathbb{Q}), k, \eta}$ Adelic MF of wt (k, η)

Fact. $A_{\text{cusp}} \cong \bigoplus_{k \in \mathbb{Z}^2} A_{\text{cusp}, k, \eta}$

↑ Mass forms, wt 1 MF, ...

$$A_{\text{cusp}, k, \eta} = \bigoplus_{i \in I} \pi_i \quad \pi_i \in \Pi(GL_2(A_{\text{fin}}))$$

Def. We call

$$\Pi_{A_{\text{cusp}}, k, \eta} := \{\pi_i \mid i \in I\} \cong \Pi(GL_2(A_{\text{fin}}))$$

as cuspidal auto reps of wt (k, η) .

Thm (Eichler - Shimura)

Let $\rho \in \mathcal{H}_{\text{Acusp}, k, \eta} (GL_2(A_F))$

\exists a CM field L (big enough),

$\forall \lambda$: fin place of L ,

$\exists r_\lambda(\pi) \in \text{Irr}_{\overline{\mathbb{Q}_\lambda}}(\Gamma_F)$ 2-dim s.t.

1) π_v unramified $\left. \begin{array}{l} \\ \text{char } k_v \neq \text{char } k_\lambda \end{array} \right\} \Rightarrow \left. \begin{array}{l} r_\lambda(\pi)|_{G_{F_v}} \text{ is unramified} \\ \text{char poly(Frob}_v) = X^2 - t_v X + (\# k_v) s_v \end{array} \right\}$

2) compatible with LLC in $l \neq p, l=p$ case

3). ρ is geometric

4). $\forall v | \infty$, s.t. $\overset{F_v \cong \mathbb{R}}{\underset{\text{denote}}{\Gamma_{F_v}}} = \{1, \sigma_v\} \subseteq \Gamma_F$,

$$\det(r_\lambda(\pi)(\sigma_v)) = -1$$

5) $r_\lambda(\pi)$ forms a strictly compatible system.

Def $r \in \text{Irr}_{\overline{\mathbb{Q}_\lambda}}(G_F)$ is modular, if $r = r_\lambda(\pi)$ for some π .

4. Adèlic MF on quaternion algebras

F : totally real field	\mathbb{Q}	$\mathbb{Q}(\sqrt{3})$
D/F : quaternion alg (with center F)	$M_2(\mathbb{Q})$	$\left(\frac{-1, -1}{\mathbb{Q}(\sqrt{3})}\right)$
$S(D) = \{v : \text{places of } F \mid D \otimes_F F_v \not\cong M_2(F_v)\}$	\emptyset	$\{\infty_1, \infty_2\}$
$G_D := (D \otimes_{\mathbb{Q}} -)^{\times} = \text{Res}_{D/\mathbb{Q}} \mathbb{G}_m$	GL_2	$\left(\frac{-1, -1}{-\otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{3})}\right)^{\times}$
$G_D(\mathbb{Q}) = D^{\times}$	$GL_2(\mathbb{Q})$	$\left(\frac{-1, -1}{\mathbb{Q}(\sqrt{3})}\right)^{\times}$
$G_D(\mathbb{R})$	$GL_2(\mathbb{R})$	$H^{\times} \times H^{\times}$

\mathbb{C} ? \mathbb{Q}_p ? $A_{\mathbb{Q}}$?

For $v \neq \infty$, define \mathcal{U}_v & $(\tau_v, W_v) \in \text{rep}(\mathcal{U}_v)$ as follows.

	$v \notin S(D)$	$v \in S(D)$
\mathcal{U}_v	$\mathbb{R}^{\times} \otimes_{\mathbb{Z}} SO_2$	H^{\times}
W_v	\mathbb{C}	$(\text{Sym}^{k-2} \mathbb{C}^2) \otimes (\Lambda^2 \mathbb{C}^2)^{\dagger}$ $(\Lambda^2 \mathbb{C}^2)^{\dagger} \otimes (\text{Sym}^{k-2} \mathbb{C}^2)$
τ_v	$\mathbb{R}^{\times} SO_2 \xrightarrow{j_k, \eta} \mathbb{C}^{\times} \subset \mathbb{C}$	$H \hookrightarrow GL_2(\mathbb{C}) \subset \mathbb{C}^2$ (construct from)

Let $\mathcal{U}_{\infty} = \prod_{v \mid \infty} \mathcal{U}_v$, $(\tau_{\infty}, W_{\infty}) = \bigotimes_{v \mid \infty} (W_v, \tau_v)$.

Def. (Cuspidal MF $S_{D,k,\eta}$)

For $k = (k_v)_{v \neq \infty}$, $\eta = (\eta_v)_{v \neq \infty}$. define

$$S_{D,k,\eta} := \left\{ \varphi : G_D(\mathbb{Q}) \backslash G_D(\mathbb{A}_{\mathbb{Q}}) \longrightarrow \mathbb{C} \text{ fcts} \right\} \\ \text{s.t. } \textcircled{1} - \textcircled{4} \text{ are true}$$

① (continuity) $\exists U_{\text{fin}} \leq G_D(\mathbb{A}_{\mathbb{Q}, \text{fin}})$ open s.t.

$$\varphi(gr) = \varphi(g) \quad \forall r \in U_{\text{fin}}$$

② (automorphy)

$$\varphi(gr) = \tau_{\infty}(r)^{-1} \varphi(g) \quad \forall r \in U_{\infty}$$

$$\rightsquigarrow \tau_{\infty}(r') \varphi(gr) = \tau_{\infty}(r') \varphi(g)$$

③ (holomorphy) $\forall g \in \mathbb{G}_m \backslash G_D(\mathbb{A}_{\mathbb{Q}})$, the fct

$$f_{\varphi,g} : (H^{\pm})^{S_{\infty} - S(D)} \longrightarrow W_{\infty} \quad r(i, \dots, i) \mapsto \tau_{\infty}(r) \varphi(gr)$$

is holomorphic

(holomorphic at ∞) When $D = M_2(\mathbb{Q})$,

$$f_{\varphi,g}(\tau) | \operatorname{Im} \tau |^{\frac{k}{2}} \text{ is bounded.}$$

$$G_D = GL_2(-\mathfrak{O}_{\mathbb{Q}} F)$$

④ (cuspidal condition) When $G_D = GL_2(-\mathfrak{O}_{\mathbb{Q}} F)$, $\forall g \in G_D(\mathbb{A}_{\mathbb{Q}}) = GL(\mathbb{A}_{\mathbb{Q}})$,
 $\mathbb{D}^{\times} = GL_2(F)$

$$\int_{F \backslash A_F} \varphi \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx = 0$$

Rmk. For assuring $S_{D,k,\eta} \neq \{\emptyset\}$, one should require that

$$k_v \geq 2 \quad k_v + 2\eta_v = k_w + 2\eta_w \quad \forall v, w \in \infty$$

E.g. For $D = \begin{pmatrix} -1 & -1 \\ Q(F) \end{pmatrix}$, conditions ③ & ④ are auto satisfied.

Furthermore,

$$\begin{aligned} S_{D,2,0}^{\mathcal{U}_{fin}} &= \mathbb{C}[G_D(\mathbb{Q}) \backslash G_D(\mathbb{A}_\infty) / \mathcal{U}_{fin} \mathcal{U}_\infty] \\ &= \mathbb{C}[G_D(\mathbb{Q}) \backslash G_D(\mathbb{A}_{\mathbb{Q}, fin}) / \mathcal{U}_{fin}] \end{aligned}$$

is a f.d. v.s.

Def. D is definite, if $\{\nu \mid \infty\} \subset S(D)$, i.e. $G_D(\mathbb{R}) \cong \mathbb{H}^X \times \dots \times \mathbb{H}^X$.

E.g. $\begin{pmatrix} -1 & -1 \\ F \end{pmatrix}$ is definite.

Rmk. One can define $A_{cusp,k,\eta}(G_D)$, $\mathcal{T} A_{cusp,k,\eta}(G_D)$ in a similar way, and show that they satisfy similar properties.

Fact (The JL correspd) $(v \in S(D), v \neq \infty)$

$$\left\{ \begin{array}{l} (\rho, V) \in \prod_{A_{cusp}, k, \eta} (G_D(A_{Q, fin})) \\ \dim_{\mathbb{C}} V = +\infty \end{array} \right\} \xleftrightarrow{\text{JL}} \left\{ \begin{array}{l} (\rho, V) \in \prod_{A_{cusp}, k, \eta} (GL_2(A_{Q, fin})) \\ \rho|_{GL_2(F_v)} \text{ disc series } \forall v \in S(D), \\ v \neq \infty \end{array} \right\}$$

$$\downarrow \quad \quad \quad \hookrightarrow \quad \quad \quad \downarrow$$

$$\text{Irr}(G_D(F_v)) \xleftarrow{\text{JL}_{loc}} \text{Irr}(GL_2(F_v)) + \text{disc series}$$