Bruhat–Tits building

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Figures of Bruhat–Tits building

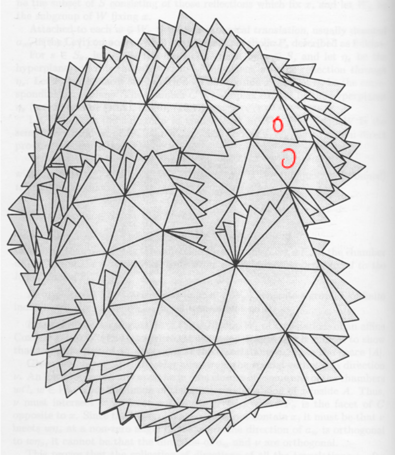


Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from a talk by Annette Werner

Figures of Bruhat–Tits building

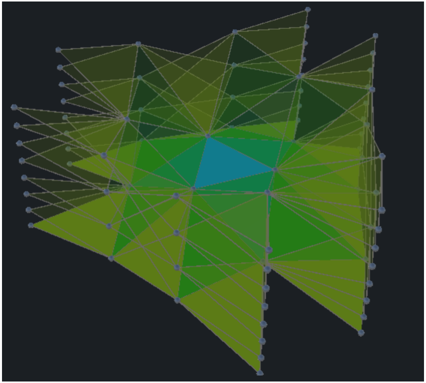


Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

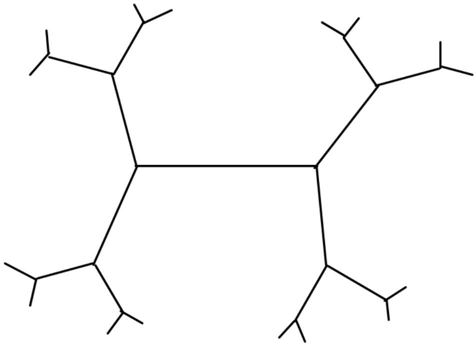
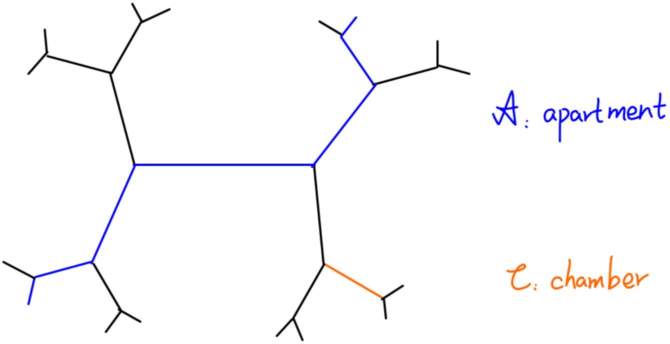


Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Figures of Bruhat–Tits building



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1. Spherical buildings
2. p-adic buildings
3. The Gromov-Schoen theorem

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Standard subgroups

$G/_k$ red gp $\qquad G = \underline{GL_n, SL_n,}$ $Sp_n, O(n), \ldots$

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$$W := N_G(T)/T.$$

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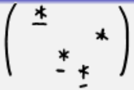
Example

When $G = \mathrm{GL}_n(\kappa)$,

$N_G(T) = \{$ *monoidal matrixes* $\}$



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We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B. \qquad = \text{“}BWB\text{”}$$

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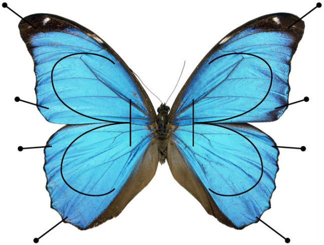

Figure: Pinned butterfly

Weyl group action on cocharacter lattices

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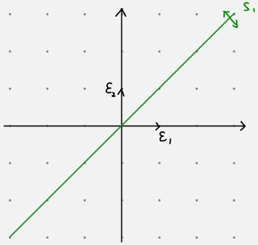

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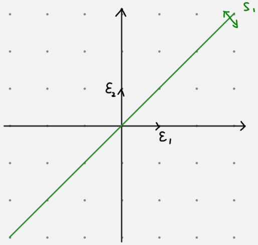

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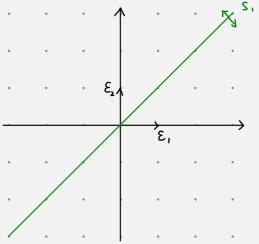

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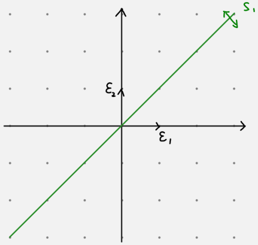

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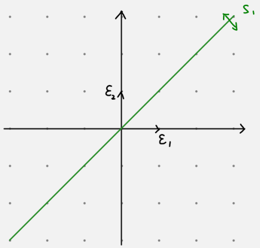

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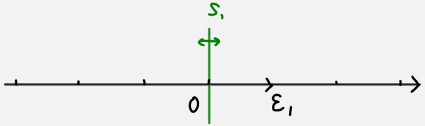

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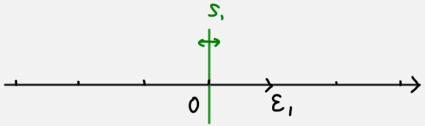

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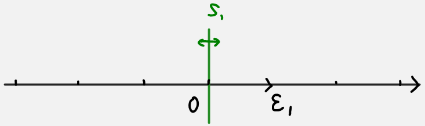

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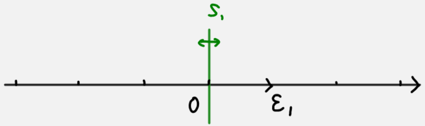

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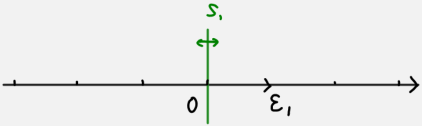

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$$W = S_3 = \langle s_1, s_2 \rangle$$

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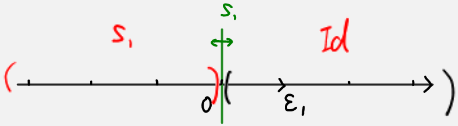

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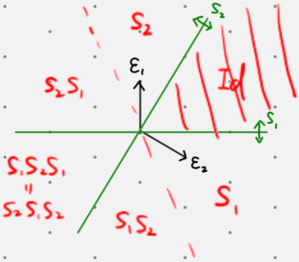

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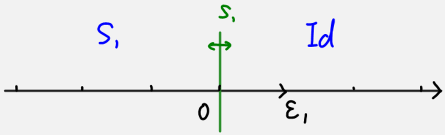

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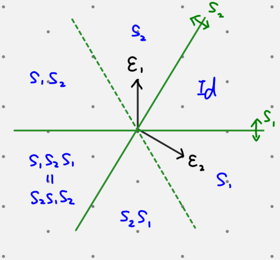

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\end{aligned}
$$

$B_0' = gB_0g^{-1}$

$$
\begin{tikzcd}
\{T \mid T \subset B_0'\} \arrow[dr, dash] & & \{B \mid B \supset T_0\} \arrow[dl, dash] \\
& \{(B,T)\} \arrow[dl] \arrow[dr] & \\
B_0' \in \{B\} & & \{T\}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{A}^{\frac{n(n-1)}{2}} \cong gB_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 \cong W \arrow[dl, dash] \\
& G/T_0 \arrow[dl] \arrow[dr] & \\
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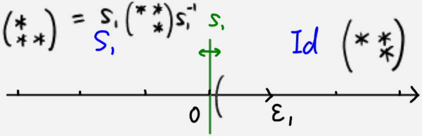
\end{tikzcd}$$

$$\{\text{ (Weyl) chambers }\} \overset{1:1}{\longleftrightarrow} W \overset{1:1}{\longleftrightarrow} \{B \mid B \supset T_0\}$$

Weyl group action on cocharacter lattices(revisited)

When $G = \mathrm{SL}_2(\kappa)$, $T = \begin{pmatrix} a & \\ & a^{-1} \end{pmatrix}$, $X_*(T) = \mathbb{Z}\varepsilon$, where



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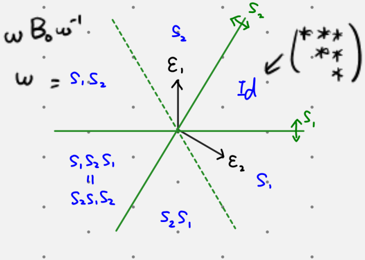

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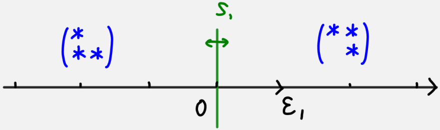

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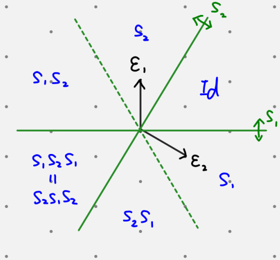

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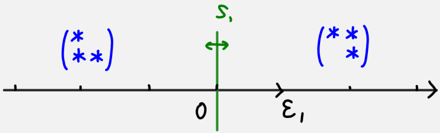

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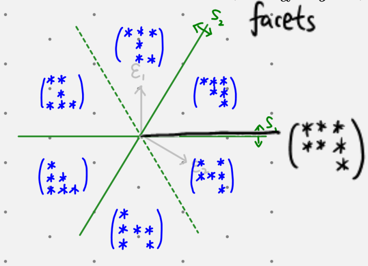

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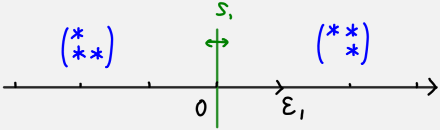

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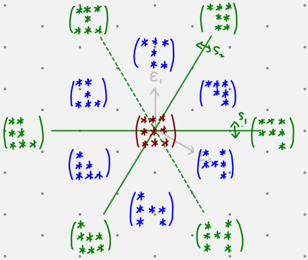

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Given a maximal torus T, the apartment is

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$k = \mathbb{F}_p$

$= \mathbb{C}$

Example of spherical building

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When $G = \mathrm{SL}_2(\mathbb{F}_2)$, the building $\mathcal{B}$ has 3 apartments and 3 chambers.

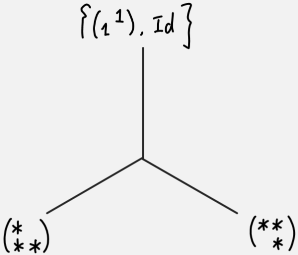

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

$SL_2(\mathbb{F}_3)$

$\# SL_2(\mathbb{F}_3)/B = \# \mathbb{P}^1(\mathbb{F}_3)$
$= 4$

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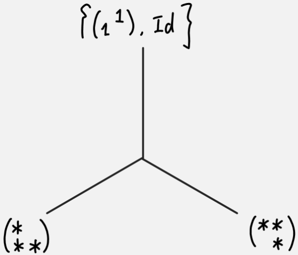

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{F}_2)}$

When $G = \mathrm{SL}_3(\mathbb{F}_2)$, the building $\mathcal{B}$ has 28 apartments and 21 chambers.

Remark

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Proposition

- *Any two chambers lie in one apartment.*

$B_1, B_2, \quad \exists T \subset B_1 \cap B_2$

$B_0 \cap g B_0 g^{-1} = B_0 \cap b_1 \omega B_0 \omega^{-1} b_1^{-1} \supset b_1 T_0 b_1^{-1}$

$g = b_1 \omega b_2^{-1}$

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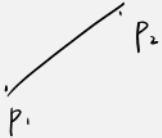
Proposition

- *Any two chambers lie in one apartment.*
- *There is a unique geodesic through any two points $p_1, p_2 \in \mathcal{B}$.*

$$p_1 \in \mathcal{C}_{B_1} \subset \mathcal{A}_T$$
$$p_2 \in \mathcal{C}_{B_2} \subset \mathcal{A}_T$$



Plan of the talk

p-adic notation

symbol	name	example
F	NA local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	ring of integers	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b} p^k\right) = k$

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standard subgroups in the p-adic world

$\underline{\underline{GL_n(\mathcal{O})}} \subset GL_n(F)$

standard subgroups in the p-adic world

$$\begin{array}{ccc}
\pi : \mathrm{GL}_n(\mathcal{O}) & \longrightarrow & \mathrm{GL}_n(\kappa) \\
\cup & & \cup \\
I & \longrightarrow & B
\end{array}$$

standard subgroups in the p-adic world

$$\pi : \mathrm{GL}_n(\mathcal{O}) \longrightarrow \mathrm{GL}_n(\kappa)$$

$$I = \pi^{-1}(B) = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathcal{O} & \mathcal{O} \\ \mathfrak{p} & \mathfrak{p} & \mathcal{O} \end{pmatrix}$$

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Parahoric subgroup

"parabolic Iwahori"

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→ Parahoric subgroup

→ (Parabolic Iwahori)

Remark

They also have moduli intepretations. For example,

L : $\mathcal{O}$-lattice in F^n

$$\begin{aligned} \mathrm{GL}_n(F)/I &\cong \{\mathfrak{p}L = L_0 \subset L_1 \subset \cdots \subset L_n = L \mid L_{i+1}/L_i \cong \kappa\} \\ &= \{\mathcal{O}\text{-lattice chains in } F^n\} \end{aligned}$$

Extended Weyl group

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To get the Iwahori decompositionn

$$G(F) = \bigsqcup_{\varpi \in W_{\text{ext}}} I \varpi I,$$

we define the extended Weyl group as

$$\underline{W_{\text{ext}}} := N_G(T(\mathcal{O}))/T(\mathcal{O}) \cong X_*(T) \rtimes \underline{W_f}. \quad \text{finite.}$$

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Example

When $G = \mathrm{GL}_n(F)$,

$$\begin{pmatrix} & \underline{*} & & \\ \underline{*} & & & \\ & & & \underline{*} \\ & & \underline{*} & \end{pmatrix}$$

$$W_{\mathrm{ext}} = \{ \textit{ monoidal matrixes } \} \Big/ \begin{pmatrix} \mathcal{O}^* & & \\ & \ddots & \\ & & \mathcal{O}^* \end{pmatrix} \cong \mathbb{Z}^n \rtimes S_n.$$

Extended Weyl group action

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W_{ext} acts on $X_*(T)$ by "twisted conjugation":

$$W_{\mathrm{ext}} \times X_*(T) \longrightarrow X_*(T) \qquad (\mu \rtimes u, \lambda) \longmapsto \mu + u\lambda u^{-1}$$

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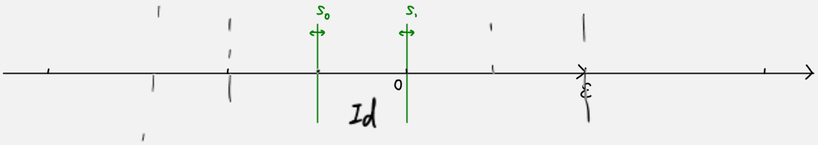

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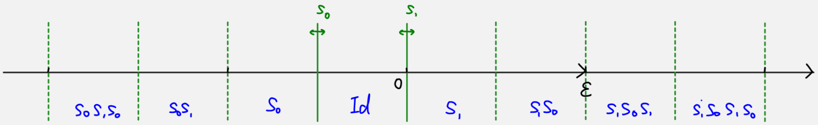

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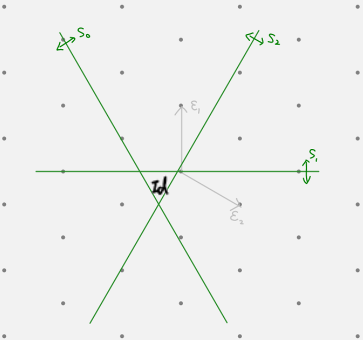

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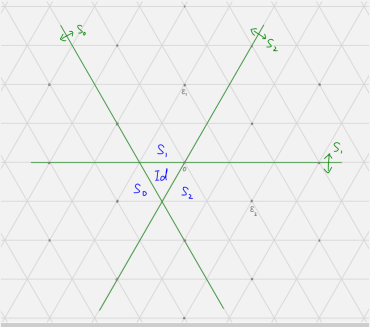

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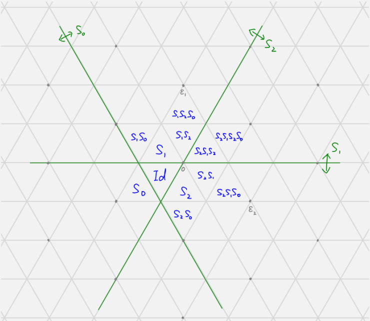

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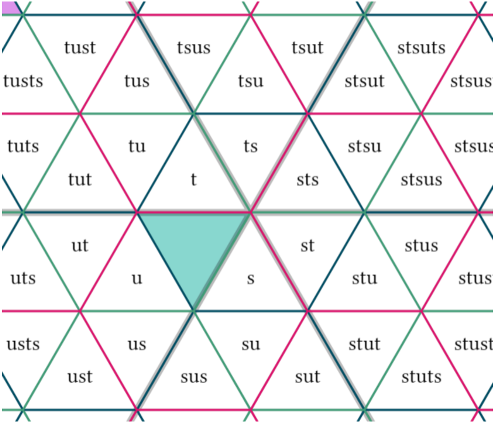

Figure: Reduced expressions labels, from Lievis

Non-standard subgroups in the p-adic world

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Similarly,

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$$

$$
\begin{tikzcd}
\{T \mid T \subset I_0\} \arrow[dr, dash] & & \{I \mid I \supset T_0\} \arrow[dl, dash] \\
& \{(I,T)\} \arrow[dl] \arrow[dr] & \\
\{I\} & & \{T\}
\end{tikzcd}
\qquad
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I_0/T_0 \arrow[dr, dash] & & N_G(T_0)/T_0 = W_{\mathrm{ext}} \arrow[dl, dash] \\
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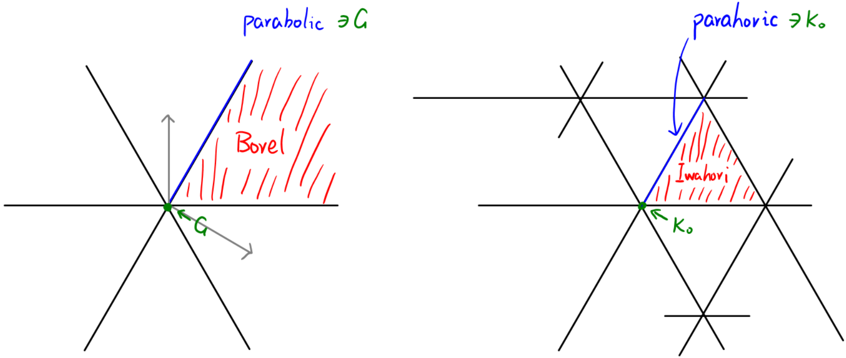
$$\{\text{ chambers }\} \overset{1:1}{\longleftrightarrow} W_{\text{ext}} \overset{1:1}{\longleftrightarrow} \{I \mid I \supset T_0\}$$

Comparison



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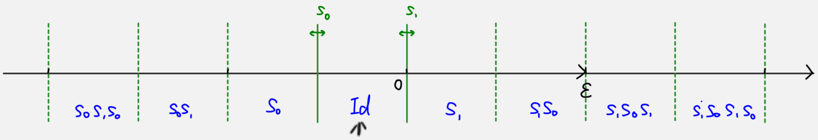

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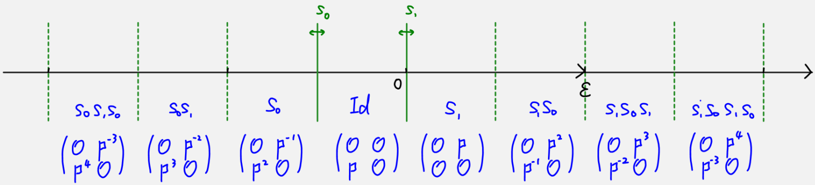

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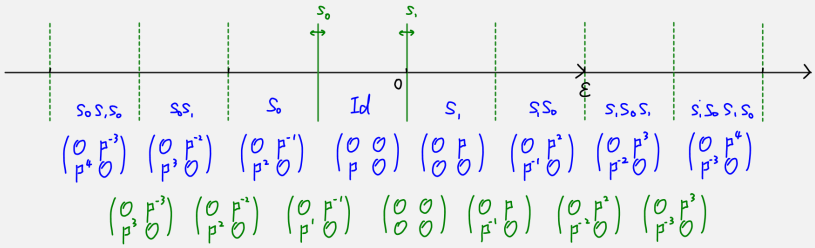

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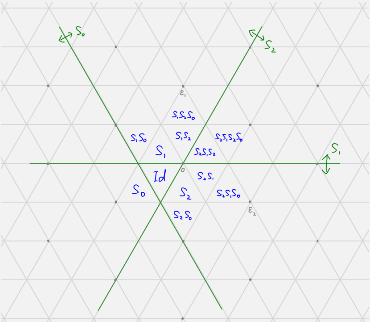

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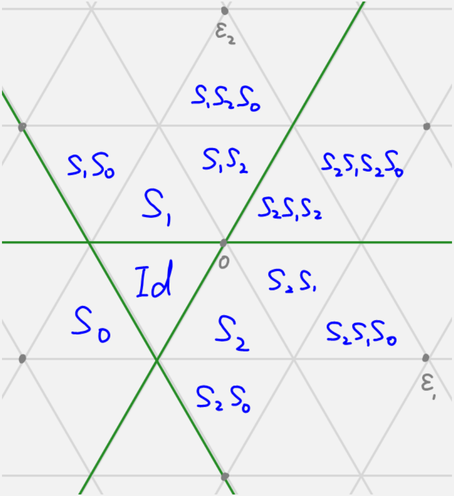

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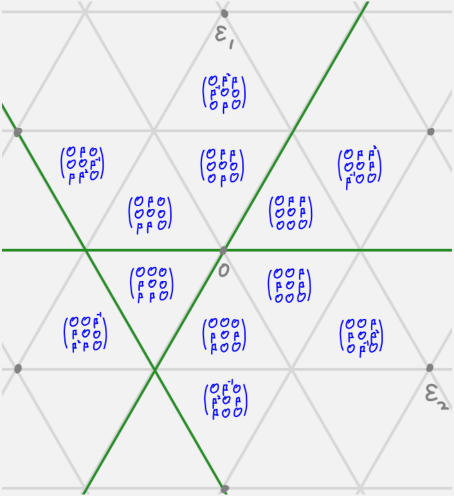

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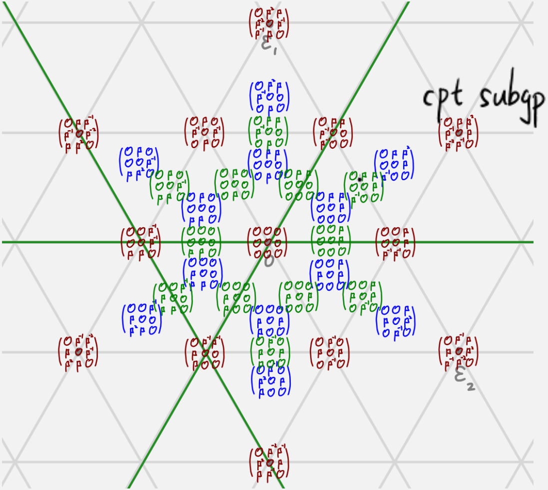

p-adic building

p-adic building

Definition (chamber, apartment and building)

Given a maximal torus T over $\mathcal{O}$, the apartment is

$$\mathcal{A}_T := X_*(T)_{\mathbb{R}} = \bigcup_{I \supset T} \mathcal{C}_I,$$

and the p-adic building is

$$\mathcal{B} := \left(\bigsqcup_{T} \mathcal{A}_T \right) /\!\sim \; = \bigcup_{I} \mathcal{C}_I.$$

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Remark

Similarly, any two chambers lie in one apartment,
and there is a unique geodesic through $p_1, p_2 \in \mathcal{B}$.

Plan of the talk

The Gromov-Schoen theorem

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X
$\downarrow$
M

$\pi_1(M) = \{*\}$

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$H^i_{ét}(X_s; \mathbb{Q}_p)$

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$\Delta f = 0$

$$\boxed{h_\rho} : \widetilde{M} \longrightarrow \mathcal{B}_{\mathrm{GL}_n(F)}$$

We call ρ reductive when $\overline{\rho(\pi_1(M))}^{\mathrm{Zar}} \subseteq \mathrm{GL}_n(F)$ is reductive.

$M \dashrightarrow \mathcal{B}_{\mathrm{GL}_n(F)} / \mathrm{GL}_n(F)$

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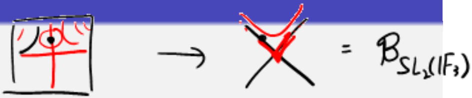
Example

The map



$$f : \mathbb{R}^2 \longrightarrow \left\{ y^2 = x^2 \right\} \qquad (a, b) \longmapsto (a|b|, b|a|)$$

is regular.

Thanks for listening!

You can get this slide at:
https://github.com/ramified/personal_tex_collection/raw/main/Bruhat–Tits_building/Bruhat–Tits_building.pdf