Bruhat–Tits building

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Figures of Bruhat–Tits building

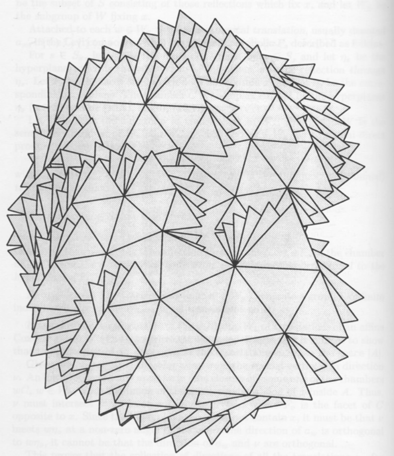

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from Annette Werner's talk

Figures of Bruhat–Tits building

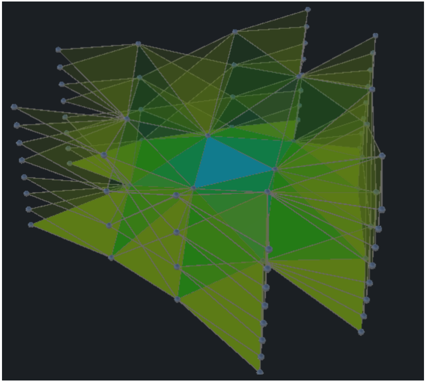

Figure: $\mathcal{B}_{\mathrm{SL}_3(\mathbb{Q}_p)}$, from buildings.gallery

Figures of Bruhat–Tits building

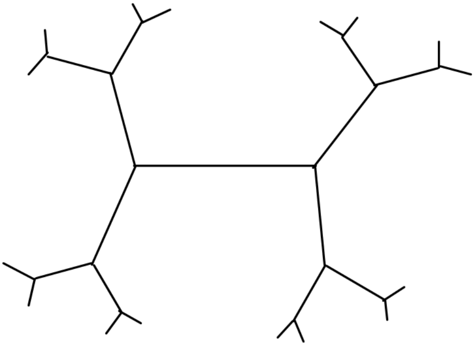

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Figures of Bruhat–Tits building

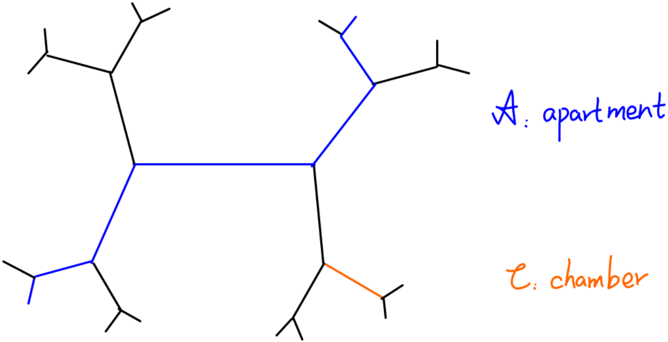

Figure: $\mathcal{B}_{\mathrm{SL}_2(\mathbb{Q}_2)}$

Recap: standard Lie theory

Restrict to **complex** representations, we have a nice theory:

- Any representation can be written as a direct sum of **irreducible representation**;
- We can extract information of irreducible representations from the **character table**:

$$\#\{\text{irreducible representations}\} = \#\{\text{conjugation classes}\}$$
$$\sum_{\chi:\text{irr}} (\dim \chi)^2 = \#G$$

However, in general,

- NO standard way finding an **explicit construction** of all irreducible representations;
- NO **one-to-one correspondence** between irreducible representations and conjugation classes.

Standard subgroups

$$B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/B &= \{ \text{ flags of } \kappa^n \} \\ &= \{ V_1 \subset \cdots \subset V_n = \kappa^n \mid \dim V_i = i \} \end{aligned}$$

$$P = \left(\begin{array}{c:c} * & * \\ \hdashline & * \end{array}\right) \quad \rightsquigarrow \quad \begin{aligned} \mathrm{GL}_n(\kappa)/P &= \mathrm{Gr}(r, n) \\ &= \{ V \subset \kappa^n \mid \dim V = r \} \end{aligned}$$

$$T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \quad \rightsquigarrow \quad \mathrm{GL}_n(\kappa)/T = \{ \kappa^n = W_1 \oplus \cdots \oplus W_n \mid \dim W_i = 1 \}$$

T is comm, so every rep decomposes as direct sum of 1-dim reps.

$X^*(T) =: \mathrm{Hom}(T, \mathbb{G}_m) \cong \mathbb{Z}^n$ characters (1-dim reps)

$X_*(T) =: \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n$ cocharacters (1-parameter subgps)

Weyl group

Definition (Weyl group)

$$W := N_G(T)/T.$$

Example

When $G = \mathrm{GL}_n$,

$$\begin{aligned} N_G(T) &= \{ \textit{ monoidal matrixes } \} \\ N_G(T)/T &\cong S_n \qquad \textit{Weyl group of type } A \end{aligned}$$

Remark

We have Bruhat decomposition proved by Gauss elimination

$$G = \bigsqcup_{\omega \in W} B\omega B.$$

So the Weyl group is the "heart" of the reductive group.

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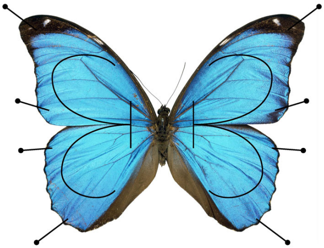

Figure: Pinned butterfly

Weyl group action on cocharacter lattices

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Non-standard subgroups

The subgroup $T = \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix}$ is not the only maximal torus.

Fact

All non-standard subgroups are conjugated to standard subgroups. Therefore,

$$\begin{aligned} \{ \textit{ Borel subgroups } \} &= \left\{ gBg^{-1} \right\} \cong G/B \\ \{ \textit{ parabolic subgroups } \} &= \left\{ gPg^{-1} \right\} \cong G/P \\ \{ \textit{ maximal tori } \} &= \left\{ gTg^{-1} \right\} \cong G/N_G(T) \end{aligned}$$

p-adic notation

symbol	name	example
F	local field	$\mathbb{Q}_p$
$\mathcal{O} = \mathcal{O}_F$	integral ring	$\mathbb{Z}_p$
$\mathfrak{p} = \mathfrak{p}_F$	maximal ideal	$p\mathbb{Z}_p$
$\kappa = \mathcal{O}/\mathfrak{p}$	residue field	$\mathbb{F}_p$
$\pi \in \mathfrak{p} \setminus \mathfrak{p}^2$	uniformizer	p
$v : F^* \longrightarrow \mathbb{Z}$	valuation	$v\left(\frac{a}{b}p^k\right) = k$

standard subgroups in p-adic world

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Extended Weyl group

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Non-standard subgroups in p-adic world

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p-adic building

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Gromov-Schoen theorem

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test

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