SUBVARIETIES IN COMPLEX ABELIAN VARIETIES

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This document is intended to collect the questions and doubts that arose during my research this year. For many of these problems, I have consulted my fellow students, my supervisor, and various other people I've met. However, most of them remain in the realm of folklore—problems that are likely known but for which I could not find a reference. On the other hand, some of the questions may not appear particularly interesting unless their underlying motivations are clearly explained. Therefore, I'll try to provide relevant background and outline some initial, perhaps naive, ideas while listing the problems along the way. Any responses, answers, or references are most welcome and will be added to keep this document updated.

1. Basic setting

For simplicity, we work over the base field $\kappa = \mathbb{C}$, and by a variety we mean a reduced, separated scheme of finite type over $\mathbb{C}$. Let $A/\mathbb{C}$ be an abelian variety of dimension n, and let $Z \subseteq A$ be an irreducible closed subvariety of dimension r. We denote by $\iota_Z : Z \hookrightarrow A$ the inclusion morphism.[1]

1.1. Gauss map.

The goal of my research is to understand the geometry of Z, and the main tool for the subvariety geometry is the Gauss map. The Gauss map describe the tangent space information at each point:

$$\phi_Z : \mathrm{Z}^{\mathrm{sm}} \longrightarrow \mathrm{Gr}(r, T_0 A) \qquad p \longmapsto T_p Z \quad \subseteq T_p A \cong T_0 A$$

Any map to the Grassmannian $\mathrm{Gr}(r, n)$ is induced by a rank r vector bundle together with n global sections. In this case, the map ϕ_Z is induced by the tangent bundle $\mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}$ and the sections

$$H^0(A, \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}) \otimes_{\mathbb{C}} \mathcal{O}_{\mathrm{Z}^{\mathrm{sm}}} \twoheadrightarrow \mathcal{T}_{\mathrm{Z}^{\mathrm{sm}}}.$$

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[1] I'm not sure whether we should consider the more general cases in the future—such as working over a field of characteristic p, letting A be a semiabelian variety or a complex torus, or allowing ι to be a covering onto its image. For now, I will omit these possibilities from this document.

1.2. **Conormal variety.** This concept may already be familiar to many readers, so we briefly recall the definition. On the smooth locus, the normal and conormal bundles behave well as vector bundles:[2]

$$\mathcal{N}_{Z^{\mathrm{sm}}/A} := \mathcal{T}_A|_{Z^{\mathrm{sm}}} \,/\, \mathcal{T}_{Z^{\mathrm{sm}}} \qquad \Lambda_{Z^{\mathrm{sm}}} := \mathcal{N}^*_{Z^{\mathrm{sm}}/A} = \ker\left(\Omega_A|_{Z^{\mathrm{sm}}} \to \Omega_{Z^{\mathrm{sm}}}\right).$$

The conormal variety Ω_Z is just the closure of $\Lambda_{Z^{\mathrm{sm}}}$ viewed as a subvariety in T^*A:

$$\Lambda_Z := \overline{\Lambda_{Z^{\mathrm{sm}}}} \subset T^*A \cong A \times T_0^*A$$

this is conical Lagrangian cycle in T^*A.

Moreover, the projectivized conormal variety

$$\mathbb{P}\Lambda_Z := \overline{\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}} \subset \mathbb{P}T^*A \cong A \times \mathbb{P}T_0^*A$$

is a Legendrian cycle in the contact variety $A \times \mathbb{P}T_0^*A$. $\mathbb{P}\Lambda_{Z^{\mathrm{sm}}}$ is a $\mathbb{P}^{r-1}$-bundle, and the map

$$\gamma_Z : \mathbb{P}\Lambda_Z \subset A \times \mathbb{P}T_0^*A \longrightarrow \mathbb{P}T_0^*A$$

is generically finite (i.e., clean) when Z is (an integral variety) of general type, see [1, Theorem 2.8 (1)].

2. Searching for examples

3. Families of subvarieties

4. Tannakian formalism

For simplicity, we work over the base field $\kappa = \mathbb{C}$. Let A denote a fixed complex abelian variety, and let $\mathrm{Perv}(A)$ denote the category of perverse sheaves on A with coefficients in $\mathbb{Q}$. For any algebraic group G, we denote by $\mathrm{Rep}(G)$ the category of algebraic representations of G.

Following the approach of [2], we work in the quotient category $\overline{\mathrm{Perv}}(A) = \mathrm{Perv}(A)/N(A)$, where $N(A) \subset \mathrm{Perv}(A)$ is the Serre subcategory of negligible complexes. A complex $\mathcal{F}$ is defined to be negligible if $\chi(A, \mathcal{F}) = 0$. This quotient category admits a natural convolution structure, and every finitely generated tensor subcategory of it is Tannakian, with a reductive Tannaka group G (see [2, Thm 7.1 & Cor 9.2]). In particular, for any perverse sheaf $\delta \in \overline{\mathrm{Perv}}(A)$, the full subcategory generated by δ is categorically equivalent to the representation category of an algebraic group G:

$$\langle \delta, * \rangle \cong \mathrm{Rep}(G).$$

References

[1] Ariyan Javanpeykar, Thomas Krämer, Christian Lehn, and Marco Maculan. The monodromy of families of subvarieties on abelian varieties. Preprint, arXiv:2210.05166 [math.AG] (2022), 2022.

[2] Thomas Krämer and Rainer Weissauer. Vanishing theorems for constructible sheaves on abelian varieties. *J. Algebr. Geom.*, 24(3):531–568, 2015.

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[2] This is more symmetric when writing them as short exact sequences:

$$0 \longrightarrow \mathcal{T}_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{T}_A|_{Z^{\mathrm{sm}}} \longrightarrow \mathcal{N}_{Z^{\mathrm{sm}}/A} \longrightarrow 0$$

$$0 \longrightarrow \Lambda_{Z^{\mathrm{sm}}} \longrightarrow \Omega_A|_{Z^{\mathrm{sm}}} \longrightarrow \Omega_{Z^{\mathrm{sm}}} \longrightarrow 0$$