ECE113: DSP

Homework 2 Solutions

Problem 1: Problem 2.24 in R1 (i.e., Proakis 4th Edition)

Solution:

If

$$\begin{aligned} y_1(n) &= ny_1(n-1) + x_1(n) \text{ and} \\ y_2(n) &= ny_2(n-1) + x_2(n) \text{ then} \\ x(n) &= ax_1(n) + bx_2(n) \end{aligned}$$

produces the output

$$y(n) = ny(n-1) + x(n), \text{ where}$$

$$y(n) = ay_1(n) + by_2(n).$$

Hence, the system is linear. If the input is $x(n-1)$, we have

$$\begin{aligned} y(n-1) &= (n-1)y(n-2) + x(n-1). \text{ But} \\ y(n-1) &= ny(n-2) + x(n-1). \end{aligned}$$

Hence, the system is time variant. If $x(n) = u(n)$, then $|x(n)| \leq 1$. But for this bounded input, the output is

$$y(0) = 1, \qquad y(1) = 1 + 1 = 2, \qquad y(2) = 2\text{x}2 + 1 = 5, \ldots$$

which is unbounded. Hence, the system is unstable.

Problem 2: Problem 2.32 in R1

<u>Solution:</u>

(a) $L_1 = N_1 + M_1$ and $L_2 = N_2 + M_2$
(b) Partial overlap from left:

$$\text{low } N_1 + M_1 \qquad \text{high } N_1 + M_2 - 1$$

$$\text{Full overlap: } \text{low } N_1 + M_2 \qquad \text{high } N_2 + M_1$$

Partial overlap from right:

$$\text{low } N_2 + M_1 + 1 \qquad \text{high } N_2 + M_2$$

(c)

$$\begin{aligned} x(n) &= \left\{1, 1, \underset{\uparrow}{1}, 1, 1, 1, 1\right\} \\ h(n) &= \left\{2, \underset{\uparrow}{2}, 2, 2\right\} \\ N_1 &= -2, \\ N_2 &= 4, \\ M_1 &= -1, \\ M_2 &= 2, \end{aligned}$$

Partial overlap from left: $n = -3 \qquad n = -1 \qquad L_1 = -3$

Full overlap: $n = 0 \qquad n = 3$

Partial overlap from right: $n = 4 \qquad n = 6 \qquad L_2 = 6$

Problem 3: Problem 2.35 in R1

Solution:

(a) $h(n) = h_1(n) * [h_2(n) - h_3(n) * h_4(n)]$
(b)

$$
\begin{aligned}
h_3(n) * h_4(n) &= (n-1)u(n-2) \\
h_2(n) - h_3(n) * h_4(n) &= 2u(n) - \delta(n) \\
h_1(n) &= \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n-2) \\
\text{Hence } h(n) &= \left[\frac{1}{2}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{2}\delta(n-2)\right] * [2u(n) - \delta(n)] \\
&= \frac{1}{2}\delta(n) + \frac{5}{4}\delta(n-1) + 2\delta(n-2) + \frac{5}{2}u(n-3)
\end{aligned}
$$

(c)

$$
\begin{aligned}
x(n) &= \left\{1, 0, \underset{\uparrow}{0}, 3, 0, -4\right\} \\
y(n) &= \left\{\frac{1}{2}, \frac{5}{4}, \underset{\uparrow}{2}, \frac{25}{4}, \frac{13}{2}, 5, 2, 0, 0, \ldots\right\}
\end{aligned}
$$

(Looks like y(1)=4 is missing from the above answer)

Problem 4: Problem 2.57 in R1

Solution:

$$
\begin{aligned}
y(n) - 4y(n-1) + 4y(n-2) &= x(n) - x(n-1) \\
\text{The characteristic equation is} & \\
\lambda^2 - 4\lambda + 4 &= 0 \\
\lambda &= 2, 2. \text{ Hence,} \\
y_h(n) &= c_1 2^n + c_2 n 2^n
\end{aligned}
$$

The particular solution is

$$y_p(n) = k(-1)^n u(n).$$

Substituting this solution into the difference equation, we obtain

$$k(-1)^n u(n) - 4k(-1)^{n-1}u(n-1) + 4k(-1)^{n-2}u(n-2) = (-1)^n u(n) - (-1)^{n-1}u(n-1)$$

For $n = 2, k(1+4+4) = 2 \Rightarrow k = \frac{2}{9}$. The total solution is

$$y(n) = \left[c_1 2^n + c_2 n 2^n + \frac{2}{9}(-1)^n\right] u(n)$$

From the initial condtions, we obtain $y(0) = 1, y(1) = 2$. Then,

$$
\begin{aligned}
c_1 + \frac{2}{9} &= 1 \\
\Rightarrow c_1 &= \frac{7}{9}, \\
2c_1 + 2c_2 - \frac{2}{9} &= 2 \\
\Rightarrow c_2 &= \frac{1}{3},
\end{aligned}
$$

Problem 5: Problem 5.5 in R1

Solution:

(a)

$$
\begin{aligned}
y(n) &= x(n) + x(n-10) \\
Y(w) &= (1 + e^{-j10w})X(w) \\
H(w) &= (2cos5w)e^{-j5w}
\end{aligned}
$$

Refer to fig 5.5-1.

(b)

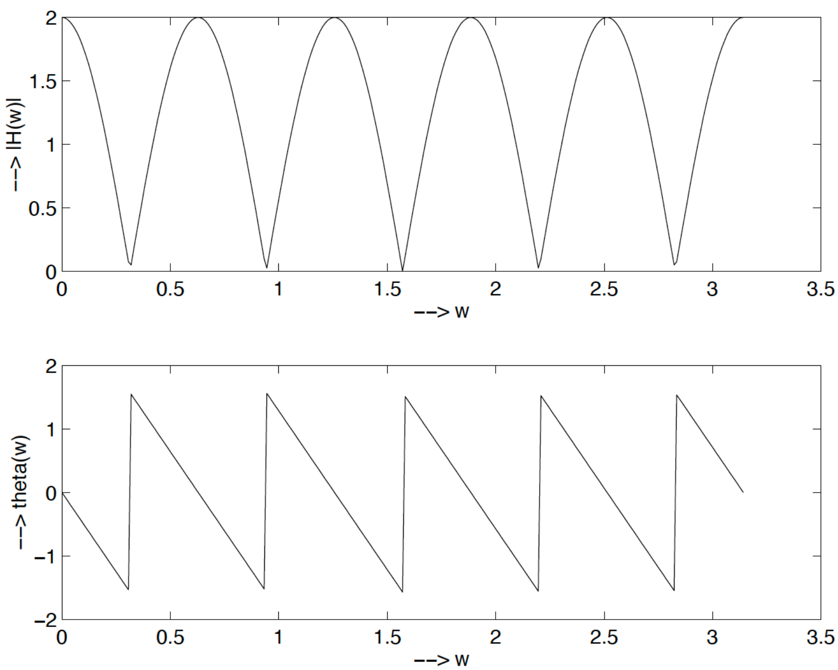

Figure 5.5-1:

$$
\begin{aligned}
H(\frac{\pi}{10}) &= 0 \\
H(\frac{\pi}{3}) &= (2cos\frac{5\pi}{3})e^{-j\frac{5\pi}{3}} \\
y(n) &= (6cos\frac{5\pi}{3})sin(\frac{\pi}{3}+\frac{\pi}{10}-\frac{5\pi}{3}) \\
&= (6cos\frac{5\pi}{3})sin(\frac{\pi}{3}-\frac{47\pi}{30})
\end{aligned}
$$

(c)

$$
\begin{aligned}
H(0) &= 2 \\
H(\frac{4\pi}{10}) &= 2 \\
y(n) &= 20+10cos\frac{2\pi n}{5}+\frac{\pi}{2}
\end{aligned}
$$

Problem 6: Problem 5.24 in R1

Solution:

$$\begin{aligned} y(n) &= \frac{1}{2}y(n-1) + x(n) + \frac{1}{2}x(n-1) \\ Y(z) &= \frac{1}{2}z^{-1}Y(z) + X(z) + \frac{1}{2}z^{-1}X(z) \\ H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{1+\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

(a)

$$\begin{aligned} H(z) &= \frac{2}{1-\frac{1}{2}z^{-1}} - 1 \\ h(n) &= 2(\frac{1}{2})^n u(n) - \delta(n) \end{aligned}$$

(b)

$$\begin{aligned} H(w) &= \sum_{n=0}^{\infty} h(n)e^{-jwn} \\ &= \frac{2}{1-\frac{1}{2}e^{-jw}} - 1 \\ &= \frac{1+\frac{1}{2}e^{-jw}}{1-\frac{1}{2}e^{-jw}} \\ &= H(z)|_{z=e^{jw}} \end{aligned}$$

(c)

$$\begin{aligned} H(\frac{\pi}{2}) &= \frac{1+\frac{1}{2}e^{-j\frac{\pi}{2}}}{1-\frac{1}{2}e^{-j\frac{\pi}{2}}} \\ &= \frac{1-j\frac{1}{2}}{1+j\frac{1}{2}} \\ &= 1e^{-j2tan^{-1}\frac{1}{2}} \\ \text{Hence, } y(n) &= cos(\frac{\pi}{2}n + \frac{\pi}{4} - 2tan^{-1}\frac{1}{2}) \end{aligned}$$

Problem 7: Problem 9.5 in R1

<u>Solution:</u>

$$
\begin{aligned}
H(z) &= \frac{6 + \frac{9}{2}z^1 - \frac{5}{3}z^{-2}}{(1+\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})} \\
&= \frac{6 + \frac{9}{2}z^1 - \frac{5}{3}z^{-2}}{1 - \frac{1}{6}z^1 - \frac{1}{6}z^{-2}}
\end{aligned}
$$

Refer to fig 9.5-1

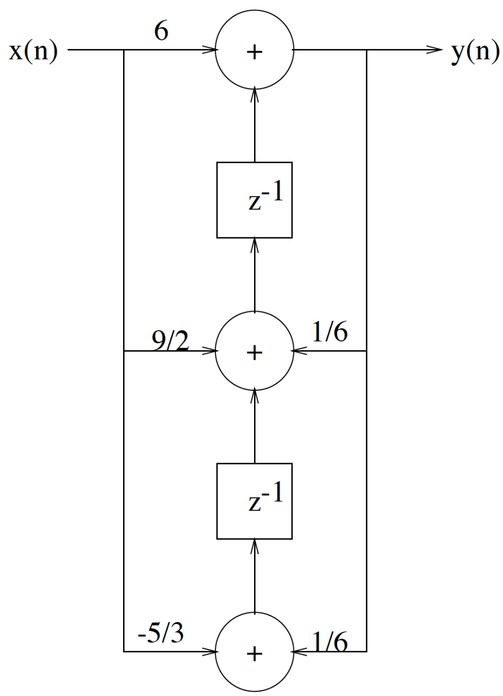

Figure 9.5-1:

Problem 8: Problem 9.9 in R1 (Part b)

Solution:

$$
\begin{aligned}
H(z) &= \frac{0.7(1-0.36z^{-2})}{1+0.1z^{-1}-0.72z^{-2}} \\
&= \frac{0.7(1-0.6z^{-1})(1+0.6z^{-1})}{(1+0.9z^{-1})(1-0.8z^{-1})} \\
&= 0.35 - \frac{0.1647}{1+0.9z^{-1}} - \frac{0.1853}{1-0.8z^{-1}}
\end{aligned}
$$

Refer to fig 9.9-2

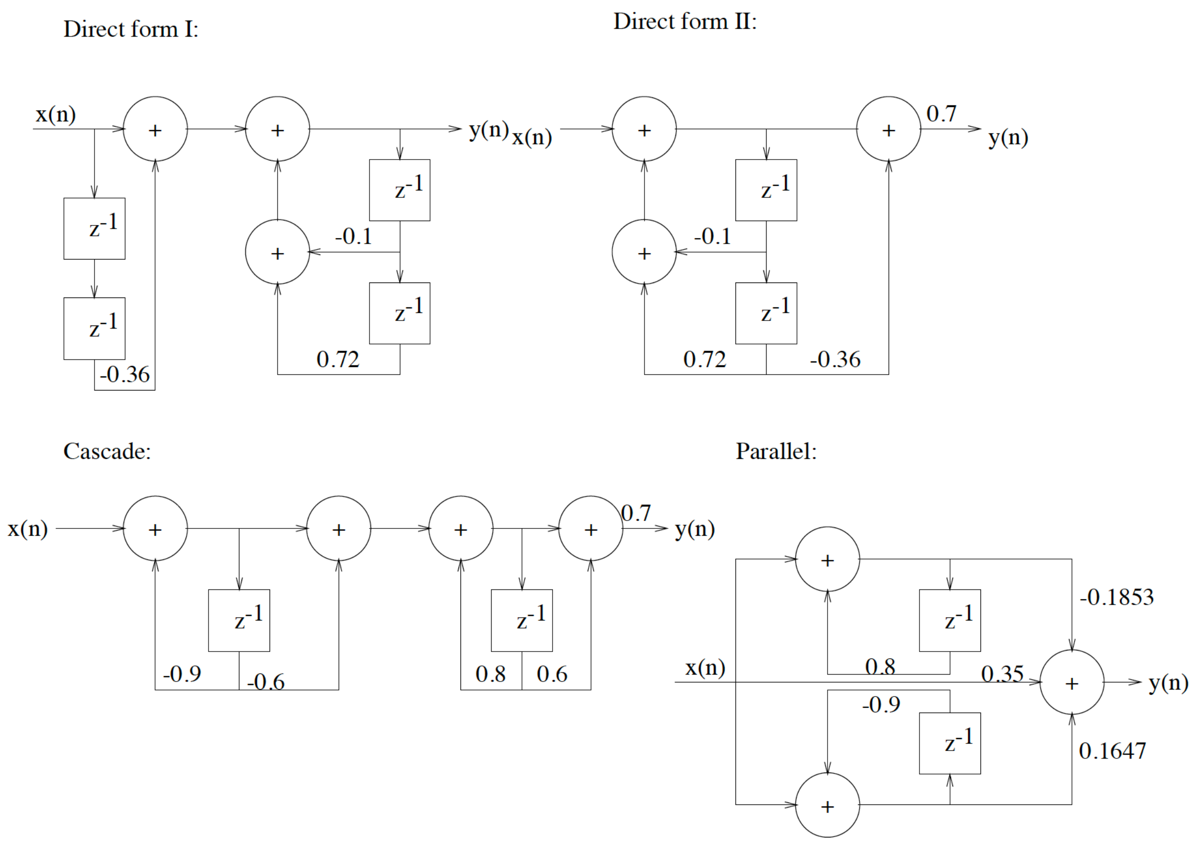

Figure 9.9-2:

Problem 9:

Problem 9. Consider a discrete-time sinewave sequence defined by $x(n) = \sin(\pi n/4)$ which was obtained by sampling a CW tone $x(t) = \sin(2\pi F_0 t)$ with the frequency F_0 Hz. If the sampling rate was $F_s = 160$ Hz, what are the possible positive frequency values for F_0, measured in Hz, that would result in the sequence $x(n)$?

Solution:

$$x(n) = \sin(\frac{n\pi}{4}) = \sin(2\pi F_0 n T_s) = \sin(\frac{2\pi F_0 n}{F_s})$$
$$\implies \frac{n\pi}{4} = \frac{2\pi F_0 n}{F_s}$$
$$F_0 = \frac{F_s}{n}\pi/42\pi n = \frac{F_s}{8} = \frac{160}{8} = 20\,\text{Hz}$$

But we have:

$$x(n) = \sin(\frac{n\pi}{4}) = \sin(2\pi F_0 n T_s) = \sin(2\pi(F_0 + kF_s)nT_s)$$

And therefore:

$$F_0 = 20 + k \times 160\,\text{Hz, for any integer } k$$

MATLAB:

P2.19 A linear and time-invariant system is described by the difference equation

$$y(n) - 0.5y(n-1) + 0.25y(n-2) = x(n) + 2x(n-1) + x(n-3)$$

1. Using the `filter` function, compute and plot the impulse response of the system over $0 \le n \le 100$.
2. Determine the stability of the system from this impulse response.
3. If the input to this system is $x(n) = [5 + 3\cos(0.2\pi n) + 4\sin(0.6\pi n)]\, u(n)$, determine the response $y(n)$ over $0 \le n \le 200$ using the `filter` function.

Solution:

(a) Impulse response using the Using the `filter` function.

```
% P0219a: System response using the filter function
clc; close all;

b = [1 2 0 1]; a = [1 -0.5 0.25]; [delta,n] = impseq(0,0,100);
h = filter(b,a,delta);

Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0219a');
Hs = stem(n,h,'filled'); set(Hs,'markersize',2);
axis([min(n)-5,max(n)+5,min(h)-0.5,max(h)+0.5]);
xlabel('n','FontSize',LFS); ylabel('h(n)','FontSize',LFS);
title('Impulse response','FontSize',TFS);
```

There are various ways to represent a discrete impulse in MATLAB. Below is a general function that is called in the above script, and you may find it useful:

```
function [x,n] = impseq(n0,n1,n2)
% Generates x(n) = delta(n-n0); n1 <= n,n0 <= n2
% ----------------------------------------------
% [x,n] = impseq(n0,n1,n2)
%
if ((n0 < n1) | (n0 > n2) | (n1 > n2))
    error('arguments must satisfy n1 <= n0 <= n2')
end
n = [n1:n2];
%x = [zeros(1,(n0-n1)), 1, zeros(1,(n2-n0))];
x = [(n-n0) == 0];
```

The plots of the impulse response $h(n)$ is shown in Figure 2.34.

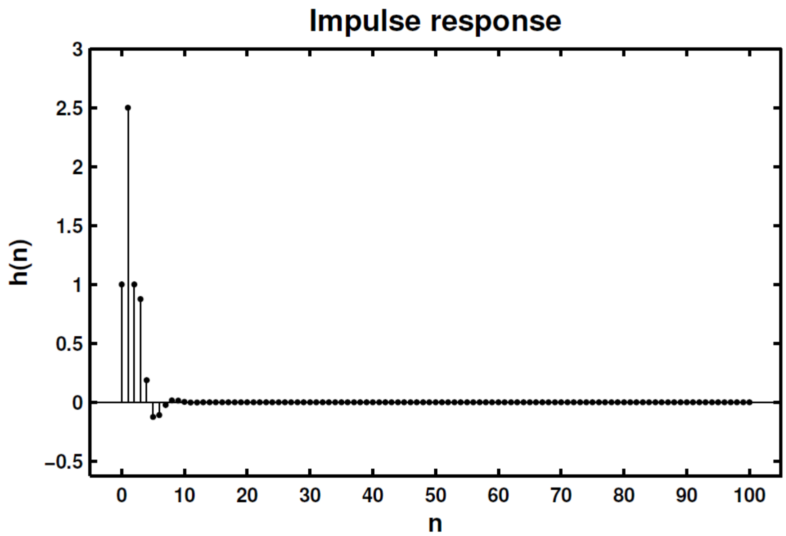

Figure 2.34: Problem P2.19.1 impulse response plot

(b) Clearly from Figure 2.34 the system is stable.

(c) Response $y(n)$ when the input is $x(n) = [5 + 3\cos(0.2\pi n) + 4\sin(0.6\pi n)]\,u(n)$:

```
% P0219c: Output response of a system using the filter function.
clc; close all;

b = [1 2 0 1]; a = [1 -0.5 0.25]; n = 0:200;
x = 5*ones(size(n))+3*cos(0.2*pi*n)+4*sin(0.6*pi*n); y = filter(b,a,x);

Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0219c');
Hs = stem(n,y,'filled'); set(Hs,'markersize',2); axis([-10,210,0,50]);
xlabel('n','FontSize',LFS); ylabel('y(n)','FontSize',LFS);
title('Output response','FontSize',TFS);
```

The plots of the response $y(n)$ is shown in Figure 2.35.

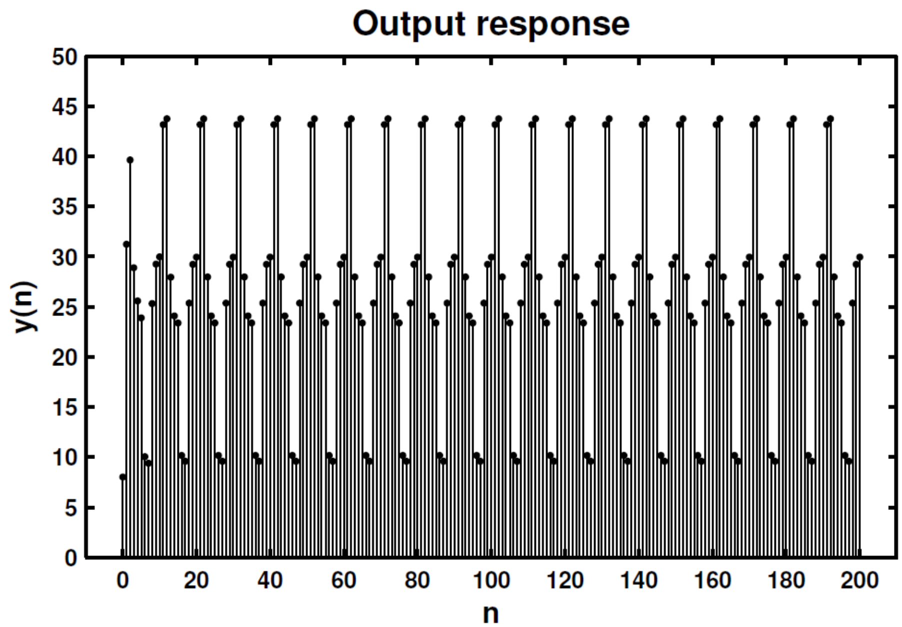

Figure 2.35: Problem P2.19.3 response plot

P3.16 For a linear, shift-invariant system described by the difference equation

$$y(n) = \sum_{m=0}^{M} b_m x(n-m) - \sum_{\ell=1}^{N} a_\ell y(n-\ell)$$

the frequency-response function is given by

$$H(e^{j\omega}) = \frac{\sum_{m=0}^{M} b_m e^{-j\omega m}}{1 + \sum_{\ell=1}^{N} a_\ell e^{-j\omega \ell}}$$

Write a MATLAB function **freqresp** to implement this relation. The format of this function should be

```
function [H] = freqresp(b,a,w)
% Frequency response function from difference equation
%   [H] = freqresp(b,a,w)
% H = frequency response array evaluated at w frequencies
% b = numerator coefficient array
% a = denominator coefficient array (a(1)=1)
% w = frequency location array
```

Solution:

MATLAB function freqresp.

```
function [H] = freqresp(b,a,w)
% Frequency response function from difference equation
% [H] = freqresp(b,a,w)
%  H  = frequency response array evaluated at w frequencies
%  b  = numerator coefficient array
%  a  = denominator coeeficient array (a(1) = 1)
%  w  = frequency location array
%
b = reshape(b,1,length(b));
a = reshape(a,1,length(a));
w = reshape(w,1,length(w));
m = 0:length(b)-1; num = b*exp(-j*m'*w);
l = 0:length(a)-1; den = a*exp(-j*l'*w);
H = num./den;
```