EC ENGR113-1 (2018 Winter) Final Exam Solutions
(March 19, 2018)

1. Consider the design of a causal discrete-time LTI system with the property that if the input is:

$$x[n] = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

then the output is:

$$y[n] = \left(\frac{1}{3}\right)^n u(n)$$

a) Determine the impulse response $h[n]$ and the system function $H(z)$ of a system that satisfies this input output condition
b) Find the difference equation that characterizes this system
c) Determine a realization of the system that requires the minimum possible amount of memory
d) Determine if the system is stable

This is problem 3.40 from Proakis, assigned as homework.

a) Taking z-transforms of $x[n]$ and $[y[n]$, we get:

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4}\cdot\frac{z^{-1}}{1-\frac{1}{2}z^{-1}}$$

$$= \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} \quad \text{ROC: } |z| > \frac{1}{2}$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \quad \text{ROC: } |z| > \frac{1}{3}$$

Therefore the transfer function is $H(z) = \frac{Y(z)}{X(z)}$

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

$$= \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$h(n) = \left[3\left(\frac{1}{4}\right)^n - 2\left(\frac{1}{3}\right)^n\right]u(n)$$

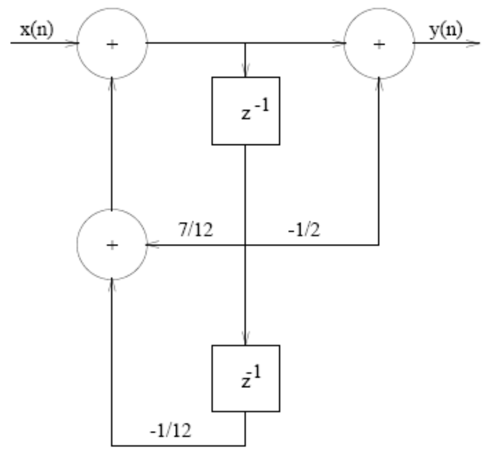
b) Note that $H(z) = \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}}$. Therefore:

$$Y(z)\cdot\left(1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}\right) = X(z)\cdot(1-\frac{1}{2}z^{-1})$$

$$Y(z) - \frac{7}{12}z^{-1}Y(z) + \frac{1}{12}z^{-2}Y(z) = X(z) - \frac{1}{2}z^{-1}X(z)$$

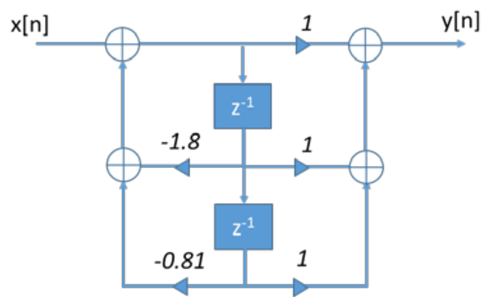
Thus the difference equation is: $y[n] = \frac{7}{12}y[n-1] - \frac{1}{12}y[n-2] + x[n] - \frac{1}{2}x[n-1]$

c) The Direct Form II realization requires the minimum possible amount of memory, and the structure is shown below:

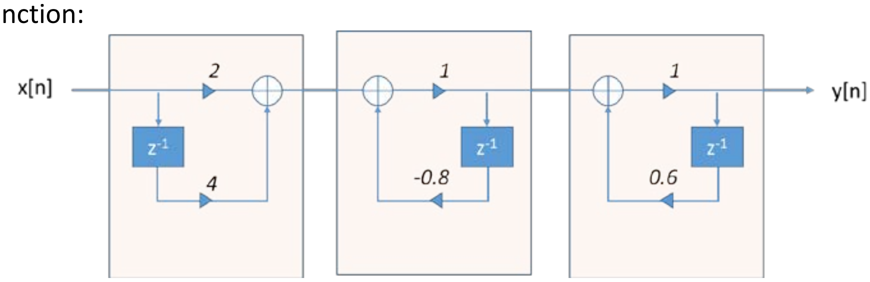


d) Notice that both poles of the system are inside the unit circle, hence the system is stable.

2. a) Find the poles of the following filter and determine if the system is bounded-input bounded-output (BIBO) stable or not:



b) Write the linear difference equation representing the following discrete time system, and determine the transfer function:



a) To determine the stability of the system, we calculate the location of the poles of the system from the transfer function. The structure is in standard Direct Form II realization, so we can directly write down the polynomial numerator and denominator of the transfer function:

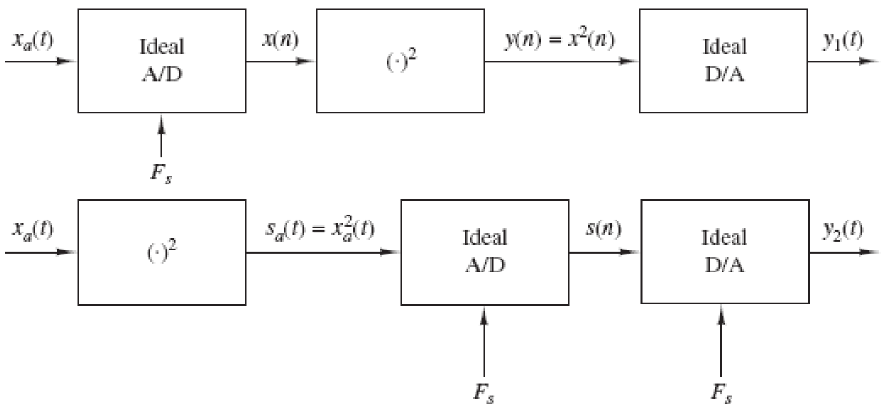
$$H(z) = \frac{1 + z^{-1} + z^{-2}}{1 + 1.8z^{-1} + 0.81z^{-2}}$$

Solving the quadratic for the denominator, the poles of the system are at: $-\frac{1.8}{2} \pm \sqrt{1.8^2 - 4 \times 0.81} = -0.9$ i.e. repeated poles at -0.9. Thus the region of convergence for $H(z)$ is for $|z| > 0.9$ which includes the unit circle. Thus both poles are inside the unit circle and the system is BIBO stable and causal.

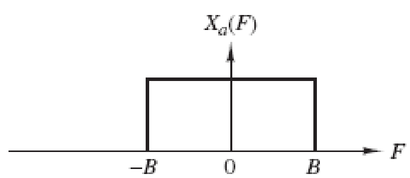
b) We first determine the transfer function of each shaded block and then multiply to obtain the response of the cascaded LTI system:

$$
\begin{aligned}
H(z) &= H_{1st\ block}(z) \cdot H_{2nd\ block}(z) \cdot H_{3rd\ block}(z) \\
&= (2 + 4z^{-1}) \cdot \left(\frac{1}{1 + 0.8z^{-1}}\right) \cdot \left(\frac{1}{1 - 0.6z^{-1}}\right) \\
&= \frac{2 + 4z^{-1}}{1 + 0.2z^{-1} - 0.48z^{-2}} \\
\therefore y[n] &= -0.2y[n-1] + 0.48y[n-2] + 2x[n] + 4x[n-1]
\end{aligned}
$$

3. Consider the two systems shown below.
   a) Sketch the spectra of the various signals if $x_a(t)$ has the Fourier transform as shown in figure (b), and $F_s = 2B$. How are $y_1(t)$ and $y_2(t)$ related to $x_a(t)$?
   b) Determine $y_1(t)$ and $y_2(t)$ if $x_a(t) = \cos(2\pi F_0 t)$, where $F_0(t) = 20$ kHz, and $F_s = 30$ Hz.



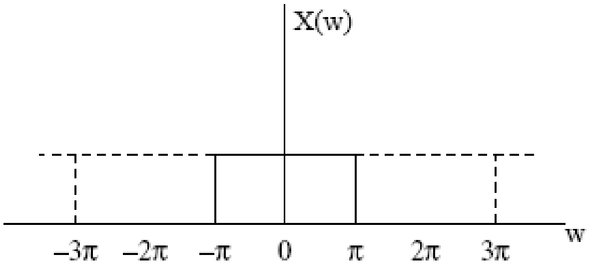
(a)

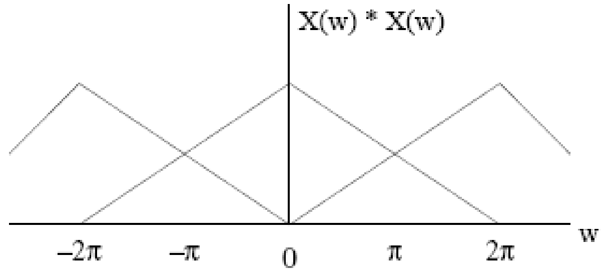


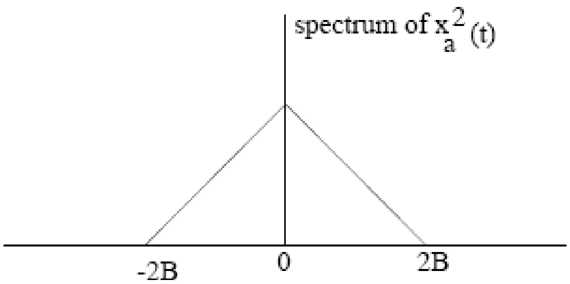
(b)

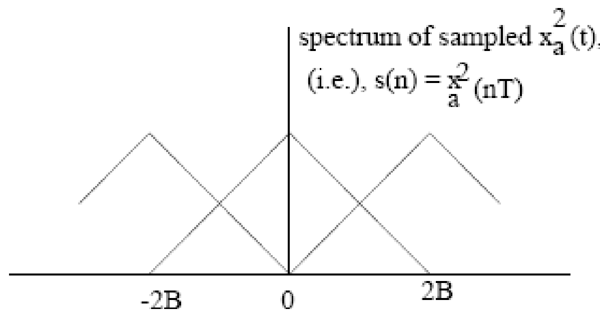
This is problem 6.12 from Proakis.

a) If $X(\omega)$ is taken as the Fourier transform of $x[n]$, then the Fourier transform of $x^2[n]$ will the convolution $X(\omega) \star X(\omega)$. So the output $y_1(t)$ is basically the square of the input signal $y_a(t)$. For the 2nd system, note that $x_a^2(t) \leftrightarrow X(\omega) \star X(\omega)$, thus the bandwidth basically doubles to 2B. The spectrum plots are shown below:









b) $x_a(t) = \cos(40\pi t) \to x[n] = \cos\left(\frac{40\pi n}{30}\right) = \cos\left(\frac{2\pi n}{3}\right)$

$$\therefore\ y[n] = x^2[n] = \cos^2\left(\frac{2\pi n}{3}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{3}\right)$$

$$y_1(t) = \frac{1}{2} + \frac{1}{2}\cos(20\pi t)$$

$$s_a(t) = x_a^2(t) = \cos^2(40\pi t) = \frac{1}{2} + \frac{1}{2}\cos(80\pi t)$$

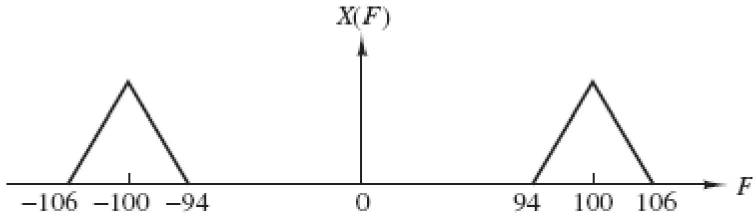
$$s(n) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{80\pi n}{30}\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\pi n}{3}\right)$$

$$\therefore\ y_2(t) = \frac{1}{2} + \frac{1}{2}\cos(20\pi t)$$

4. a) Complete the following table:

Signals	Fourier	Transform Characteristics
Continuous in t and periodic	Fourier Series	Discrete in ω
Continuous in t	Continuous-time Fourier Transform	Continuous in ω
Discrete in t	Disctete-time Fourier transform	Continuous in ω and periodic
Discrete in t and periodic	Discrete Fourier Transform	Discrete in ω and periodic

b) Consider the sampling of the bandpass signal whose spectrum is illustrated below. Determine the minimum sampling rate F_s to avoid aliasing.



$$F_c = 100, B = 12$$
$$r = \left\lfloor \frac{\left(F_c + \frac{B}{2}\right)}{2} \right\rfloor = \left\lfloor \frac{106}{2} \right\rfloor = \lfloor 8.83 \rfloor = 8$$
$$B' = \frac{F_c + \frac{B}{2}}{2} = \frac{106}{8} = \frac{53}{4}$$
$$F_s = 2B' = \frac{53}{2} \text{ Hz}$$

c) Show that the following filter has linear phase:

$$H(z) = a_0(z^0 + z^{-2n}) + a_1(z^{-1} + z^{-2n+1}) + \cdots + a_{n-1}(z^{-n+1} + z^{-n-1}) + a_n z^{-n}$$

We substitute $z = e^{j\omega T}$ to evaluate the discrete time Fourier transform (DTFT) from the z-transform:

$$H(\omega) = a_0\left(e^0 + e^{-j2n\omega T}\right) + a_1\left(e^{-j\omega T} + e^{-j2(n-1)\omega T}\right) + \cdots + a_{n-1}\left(e^{-j(n-1)\omega T} + e^{-j(n+1)\omega T}\right) + a_n e^{-jn\omega T}$$
$$H(\omega) = e^{-jn\omega T}\left[a_0\left(e^{jn\omega T} + e^{-jn\omega T}\right) + a_1\left(e^{j(n-1)\omega T} + e^{-j(n-1)\omega T}\right) + \cdots + a_{n-1}\left(e^{j\omega T} + e^{-j\omega T}\right) + a_n\right]$$

Recall that the conjugate of $e^{j\theta}$ is $e^{-j\theta}$. Therefore the expression inside the square bracket is always real. Hence the overall phase of the system is simply the phase of the $e^{-jn\omega T}$ term, which is $-n\omega T$, which is linear in ω. Thus the filter has linear phase.

5. a) Consider the filter $y[n] = 0.9y(n-1) + bx[n]$. Determine b so that $|H(0)|$ =1 and determine the frequency at which $|H(\omega)| = \frac{1}{\sqrt{2}}$. What sort of a filter is this?

   b) Consider the FIR filter $y[n] = x[n] - x[n-4]$. Compute the magnitude and phase response and compute the response to the input $x[n] = \cos\left(\frac{\pi}{2}n\right) + \cos(\frac{\pi}{4}n)$ where $-\infty < n < \infty$. Explain the result in terms of the magnitude/phase response computed earlier.

a) We take the Fourier transform and calculate the transfer function magnitude:

$$Y(\omega) = 0.9e^{-j\omega}Y(\omega) + bX(\omega)$$
$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{b}{1 - 0.9e^{-j\omega}}$$
$$|H(0)| = 1 \ \Rightarrow \ b = \pm 0.1$$
$$\Theta(\omega) = \begin{cases} -\dfrac{\omega M}{2}, & \cos\left(\dfrac{\omega M}{2}\right) > 0 \\ \pi - \dfrac{\omega M}{2}, & \cos\left(\dfrac{\omega M}{2}\right) < 0 \end{cases}$$

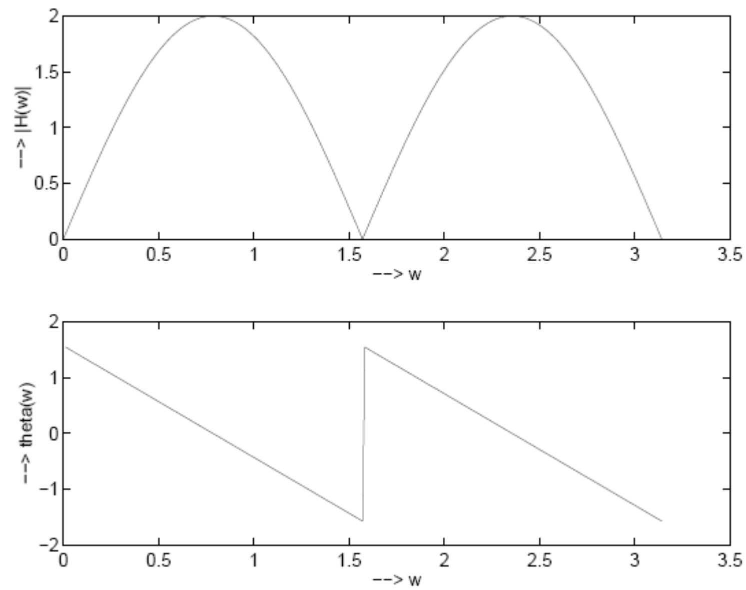
Since $|H(\omega_0)| = \frac{1}{\sqrt{2}}$, we solve for ω_0:

$$|H(\omega_0)|^2 = \frac{1}{2} \ \Rightarrow \ \frac{b^2}{1.81 - 1.8\cos\omega_0} = \frac{1}{2} \ \Rightarrow \ \omega_0 = 0.105$$

Note that the magnitude response is decreasing from $\omega = 0$ onwards, so the filter is a low pass filter.

b) We take the Fourier transform again and plot the transfer function magnitude and phase response to see the effect on the input frequencies:

$$y[n] = x[n] - x[n-4]$$
$$H(\omega) = 1 - e^{-j4\omega} = 2e^{-j2\omega}e^{\frac{j\pi}{2}}\sin(2\omega)$$



$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \cos\left(\frac{\pi}{4}n\right), \quad H\left(\frac{\pi}{2}n\right) = 0$$
$$y[n] = 2\cos\left(\frac{\pi}{4}n\right), \quad H\left(\frac{\pi}{4}\right) = 2, \quad \angle H\left(\frac{\pi}{4}\right) = 0$$

Thus the filter blocks the frequency at $\omega = \frac{\pi}{2}$

6. Consider the two 3-point sequences given below:

$$x[n] = \begin{cases} 1 & \text{for} \quad n=0 \\ 2 & \text{for} \quad n=1 \\ 1 & \text{for} \quad n=2 \end{cases} \quad y[n] = \begin{cases} -1 & \text{for} \quad n=0 \\ 2 & \text{for} \quad n=1 \\ 1 & \text{for} \quad n=2 \end{cases}$$

   a) Compute the 3-point circular convolution between $z[n]$ between $x[n]$ and $y[n]$
   b) Compute the 5-point DFT $X[k]$ for x[n]. You do not need to simplify your answers
   c) In addition to $X[k]$, suppose you also computed the 5-point DFT $Y[k]$ for $y[n]$. Without carrying out any computations, comment on whether applying a 5-pointDFT on $z[n]$ in part (a) will result in the product of $X[k]$ and $Y[k]$.

a) The circular convolution can be done graphically, or by explicitly writing the output sum:

$$z[0] = 1\cdot(-1) + 2\cdot 1 + 1\cdot 2 = 3$$
$$z[1] = 1\cdot 2 + 2\cdot(-1) + 1\cdot 1 = 1$$
$$z[2] = 1\cdot 1 + 2\cdot 2 + 1\cdot(-1) = 4$$

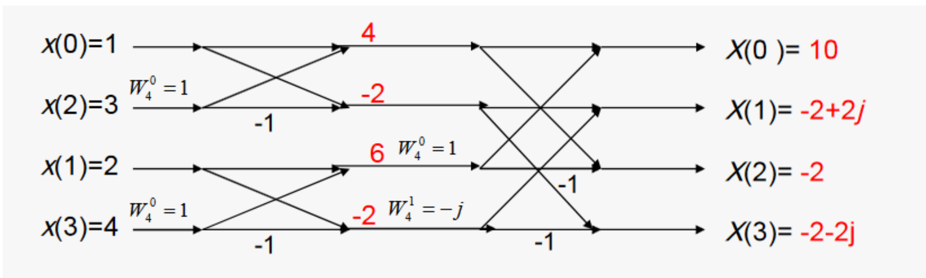
b) Using the definition of the DFT (and zero padding x[n] to get 5 point sequence: {1, 2, 1, 0, 0}):

$$X(0) = 1 + 2 + 1 = 4$$
$$X(1) = 1 + 2e^{-\frac{j2\pi}{5}} + e^{-\frac{j4\pi}{5}}$$
$$X(2) = 1 + 2e^{-\frac{j4\pi}{5}} + e^{-\frac{j8\pi}{5}}$$
$$X(3) = 1 + 2e^{-\frac{j6\pi}{5}} + e^{-\frac{j12\pi}{5}}$$
$$X(4) = 1 + 2e^{-\frac{j8\pi}{5}} + e^{-\frac{j16\pi}{5}}$$

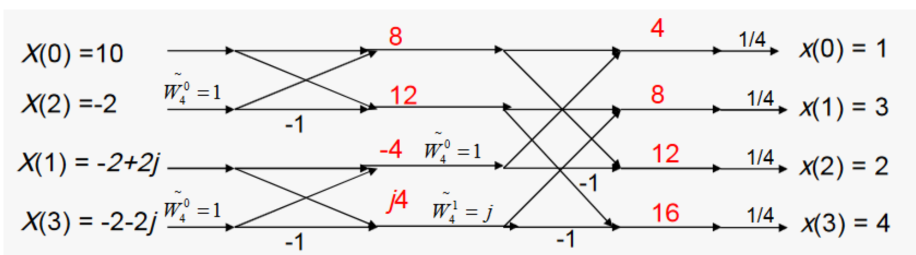
c) No. When multiplying X(k) with Y(k), we obtain the 5-point DFT for the linear convolution between x[n] and y[n], not the 3 point circular convolution z[n].

7. a) Given the following sequence: $x[n] = \{1, 2, 3, 4\}$ where $x[0] = 1$. Use the decimation in time FFT algorithm to compute the 4-point DFT of the sequence, $X[k]$. Draw the signal flow & butterfly structure and clearly label the branches with the intermediate values and variables$W_N = e^{-\frac{j2\pi}{N}}$
b) The inverse discrete Fourier transform can be calculated using the same structure and method but after appropriately changing the variable W_N and multiplying the result by 1/N. Redraw the signal flow diagram in (a) to show the IFFT structure and show that you get back the original sequence $x[n]$ from input $X[k]$.

a) We draw the butterfly structure for the FFT as follows:



b) The IFFT is the same structure with the appropriate modifications to the input output values and complex exponential factors as per the IFFT formula:



8. Design an FIR linear-phase, digital high pass filter approximating the ideal frequency response:

$$H_d(\omega) = \begin{cases} 0 \text{ for } & |\omega| < \frac{\pi}{6} \\ 1 \text{ for } & \frac{\pi}{6} < |\omega| < \pi \end{cases}$$

a) Determine the coefficients of a 25-tap filter based on the window method with a rectangular window, and explicitly write the values for the first 4 terms

b) A Hamming window is defined by the window function: $w(n) = 0.54 - 0.46 \cos \frac{2\pi n}{M-1}$ for filter of length M. For the same tap filter as part (a), write the expression for the filter using this window, and calculate the first 4 terms.

Since we need a 25-tap filter, a delay of $\frac{25-1}{2} = 12$ is incorporated into $H_d(\omega)$. Thus:

$$H_d(\omega) = e^{-j12\omega} \text{ for } \frac{\pi}{6} < |\omega| < \pi, \text{ and 0 elsewhere}$$

We notice that the high pass filter transfer function can be neatly connected to the response of the corresponding low pass filter exercise in Proakis (problem 10.1) by the following relationship:

$$H_d(\omega) = e^{-j12\omega} - H_{lowpass}(\omega)$$

where, for the lowpass filter for the same tap filter, we know $H_{lowpass}(\omega) = e^{-j12\omega}$ for $|\omega| < \frac{\pi}{6}$ (this step reduces the number of integration computations needed). Thus

$$h_{lowpass}[n] = \frac{1}{2\pi}\int_{-\pi/6}^{\pi/6} H_{lowpass}(\omega)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/6}^{\frac{\pi}{6}} e^{j\omega(n-12)}\,d\omega$$

$$= \frac{\sin\frac{\pi}{6}(n-12)}{\pi(n-12)}, n \neq 12$$

So the high pass filter response $H_d(\omega)$ is the inverse Fourier transform of $\left[e^{-j12\omega} - H_{lowpass}(\omega)\right]$ and thus:

$$h_d[n] = \delta[n-12] - \frac{\sin\frac{\pi}{6}(n-12)}{\pi(n-12)}$$

Using a rectangular window just truncates this filter response to 24 terms ($n = 0$ to 23), so the first 4 terms are, using a calculator: $\{0, 0.0145, 0.0276, 0.0354\}$

b) Using a Hamming window (or any other window) simply implies multiplication of the time domain filter response with the window function: $h_o[n] = h_d[n] \cdot w[n]$. Since the same tap filter is used, we set M to be same length = 25. So:

$$h_d[n] = \left[\delta[n-12] - \frac{\sin\frac{\pi}{6}(n-12)}{\pi(n-12)}\right] \cdot \left[0.54 - 0.46\cos\frac{\pi n}{12}\right]$$

To calculate the first 4 terms of this filter response, we simply calculate the Hamming window coefficients for the first 4 terms and multiply with the results already calculated in part (a). We obtain: $\{0, 0.0014, 0.0039, 0.0076\}$

9. The signal $x[n]$ is distorted by a causal LTI system $H(z)$ described by:

$$y[n] + \frac{1}{4}y[n-2] = x[n] + 3x[n-1] - 10x[n-2]$$

a) Design a causal, stable filter $H_c(z)$ to compensate for the magnitude distortion caused by this system so that $|H(\omega)||H_c(\omega)| = 1$ for all ω.
b) Check your answer by showing that the DC gain (i.e. at $\omega = 0$) of the compensation filter is the reciprocal of the DC gain of the distortion filter H.

a) We take the z-transform and compute the transfer function:

$$Y(z) + \frac{1}{4}z^{-2}Y(z) = X(z) + 3z^{-1}X(z) - 10z^{-2}X(z) = (1+5z^{-1})(1-2z^{-1})X(z)$$

$$\therefore H(z) = \frac{(1+5z^{-1})(1-2z^{-1})}{1+\frac{1}{4}z^{-2}}$$

So to cancel the effect of the distortion, we need a filter $H_c(z)$ such that $H_c(z) = \frac{1}{H(z)}$. Thus:

$$H_c(z) = \frac{1+\frac{1}{4}z^{-2}}{(1+5z^{-1})(1-2z^{-1})}$$

b) To compute the DC gain, we set $z = e^{j\omega}\big|_{\omega=0} = 1$, and show that it is the reciprocal of the distortion.

$$|H_c(1)| = \frac{5}{24}$$

$$|H(1)| = \frac{24}{5}$$

thus: $|H_c(1)||H(1)| = 1$