ECE113: DSP

Homework 1 Solutions

Problem 1:

1.1 Classify the following signals according to whether they are (1) one- or multi-dimensional; (2) single or multichannel, (3) continuous time or discrete time, and (4) analog or digital (in amplitude). Give a brief explanation.

(a) Closing prices of utility stocks on the New York Stock Exchange.
(b) A color movie.
(c) Position of the steering wheel of a car in motion relative to car's reference frame.
(d) Position of the steering wheel of a car in motion relative to ground reference frame.
(e) Weight and height measurements of a child taken every month.

Solution:

(a) One dimensional, multichannel, discrete time, and digital.
(b) Multi dimensional, single channel, continuous-time, analog.
(c) One dimensional, single channel, continuous-time, analog.
(d) One dimensional, single channel, continuous-time, analog.
(e) One dimensional, multichannel, discrete-time, digital.

Problem 2:

1.3 Determine whether or not each of the following signals is periodic. In case a signal is periodic, specify its fundamental period.

(a) $x_a(t) = 3\cos(5t + \pi/6)$
(b) $x(n) = 3\cos(5n + \pi/6)$
(c) $x(n) = 2\exp[j(n/6 - \pi)]$
(d) $x(n) = \cos(n/8)\cos(\pi n/8)$
(e) $x(n) = \cos(\pi n/2) - \sin(\pi n/8) + 3\cos(\pi n/4 + \pi/3)$

Solution:

(a) Periodic with period $T_p = \frac{2\pi}{5}$.
(b) $f = \frac{5}{2\pi} \Rightarrow$ non-periodic.
(c) $f = \frac{1}{12\pi} \Rightarrow$ non-periodic.
(d) $cos(\frac{n}{8})$ is non-periodic; $cos(\frac{\pi n}{8})$ is periodic; Their product is non-periodic.
(e) $cos(\frac{\pi n}{2})$ is periodic with period N_p=4
$sin(\frac{\pi n}{8})$ is periodic with period N_p=16
$cos(\frac{\pi n}{4} + \frac{\pi}{3})$ is periodic with period N_p=8
Therefore, x(n) is periodic with period N_p=16. (16 is the least common multiple of 4,8,16).

Problem 3:

1.5 Consider the following analog sinusoidal signal:

$$x_a(t) = 3\sin(100\pi t)$$

(a) Sketch the signal $x_a(t)$ for $0 \le t \le 30$ ms.
(b) The signal $x_a(t)$ is sampled with a sampling rate $F_s = 300$ samples/s. Determine the frequency of the discrete-time signal $x(n) = x_a(nT)$, $T = 1/F_s$, and show that it is periodic.
(c) Compute the sample values in one period of $x(n)$. Sketch $x(n)$ on the same diagram with $x_a(t)$. What is the period of the discrete-time signal in milliseconds?
(d) Can you find a sampling rate F_s such that the signal $x(n)$ reaches its peak value of 3? What is the minimum F_s suitable for this task?

Solution:

(a) Refer to fig 1.5-1
(b)

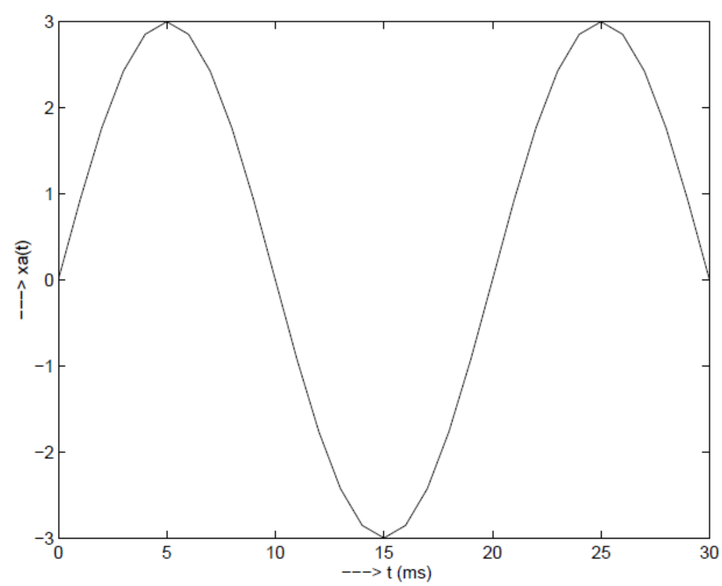

Figure 1.5-1:

$$\begin{aligned} x(n) &= x_a(nT) \\ &= x_a(n/F_s) \\ &= 3sin(\pi n/3) \Rightarrow \\ f &= \frac{1}{2\pi}(\frac{\pi}{3}) \\ &= \frac{1}{6}, N_p = 6 \end{aligned}$$

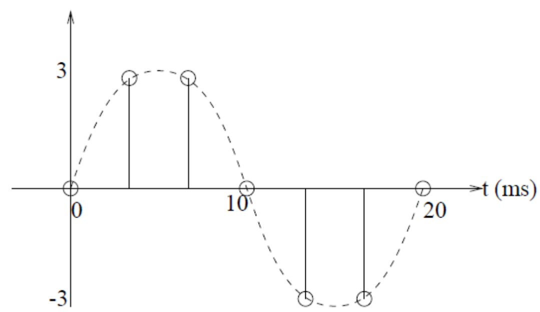

Figure 1.5-2:

(c)Refer to fig 1.5-2
$x(n) = \left\{0, \frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 0, -\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right\}, N_p = 6.$
(d) Yes.

$$x(1) = 3 = 3sin(\frac{100\pi}{F_s}) \Rightarrow F_s = 200 \text{ samples/sec.}$$

Problem 4:

2.1 A discrete-time signal $x(n)$ is defined as

$$x(n) = \begin{cases} 1 + \frac{n}{3}, & -3 \leq n \leq -1 \\ 1, & 0 \leq n \leq 3 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Determine its values and sketch the signal $x(n)$.
(b) Sketch the signals that result if we:
(1) First fold $x(n)$ and then delay the resulting signal by four samples.
(2) First delay $x(n)$ by four samples and then fold the resulting signal.
(c) Sketch the signal $x(-n + 4)$.
(d) Compare the results in parts (b) and (c) and derive a rule for obtaining the signal $x(-n + k)$ from $x(n)$.
(e) Can you express the signal $x(n)$ in terms of signals $\delta(n)$ and $u(n)$?

<u>Solution:</u>

(a)

$$x(n) = \left\{\ldots 0, \frac{1}{3}, \frac{2}{3}, \underset{\uparrow}{1}, 1, 1, 1, 0, \ldots\right\}$$

. Refer to fig 2.1-1.

(b) After folding s(n) we have

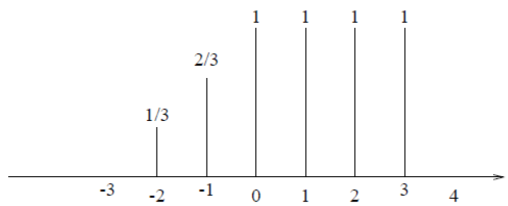

Figure 2.1-1:

$$x(-n) = \left\{\ldots 0, 1, 1, 1, \underset{\uparrow}{1}, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}.$$

After delaying the folded signal by 4 samples, we have

$$x(-n+4) = \left\{\ldots 0, \underset{\uparrow}{0}, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}.$$

On the other hand, if we delay x(n) by 4 samples we have

$$x(n-4) = \left\{\ldots \underset{\uparrow}{0}, 0, \frac{1}{3}, \frac{2}{3}, 1, 1, 1, 1, 0, \ldots\right\}.$$

Now, if we fold $x(n-4)$ we have

$$x(-n-4) = \left\{\ldots 0, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \underset{\uparrow}{0}, \ldots\right\}$$

(c)

$$x(-n+4) = \left\{\ldots \underset{\uparrow}{0}, 1, 1, 1, 1, \frac{2}{3}, \frac{1}{3}, 0, \ldots\right\}$$

(d) To obtain $x(-n+k)$, first we fold $x(n)$. This yields $x(-n)$. Then, we shift $x(-n)$ by k samples to the right if $k > 0$, or k samples to the left if $k < 0$.

(e) Yes.

$$x(n) = \frac{1}{3}\delta(n-2) + \frac{2}{3}\delta(n+1) + u(n) - u(n-4)$$

(The first term should be (1/3)δ(n+2))

Problem 5:

2.5 Show that the energy (power) of a real-valued energy (power) signal is equal to the sum of the energies (powers) of its even and odd components.

Solution:

First, we prove that

$$\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) = 0$$

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) &= \sum_{m=-\infty}^{\infty} x_e(-m)x_o(-m) \\
&= -\sum_{m=-\infty}^{\infty} x_e(m)x_o(m) \\
&= -\sum_{n=-\infty}^{\infty} x_e(n)x_o(n) \\
&= \sum_{n=-\infty}^{\infty} x_e(n)x_o(n) \\
&= 0
\end{aligned}$$

Then,

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^2(n) &= \sum_{n=-\infty}^{\infty} [x_e(n) + x_o(n)]^2 \\
&= \sum_{n=-\infty}^{\infty} x_e^2(n) + \sum_{n=-\infty}^{\infty} x_o^2(n) + \sum_{n=-\infty}^{\infty} 2x_e(n)x_o(n) \\
&= E_e + E_o
\end{aligned}$$

Problem 6:

2.10 The following input–output pairs have been observed during the operation of a time-invariant system:

$$x_1(n) = \{\underset{\uparrow}{1}, 0, 2\} \overset{\mathcal{T}}{\longleftrightarrow} y_1(n) = \{\underset{\uparrow}{0}, 1, 2\}$$

$$x_2(n) = \{\underset{\uparrow}{0}, 0, 3\} \overset{\mathcal{T}}{\longleftrightarrow} y_2(n) = \{\underset{\uparrow}{0}, 1, 0, 2\}$$

$$x_3(n) = \{\underset{\uparrow}{0}, 0, 0, 1\} \overset{\mathcal{T}}{\longleftrightarrow} y_3(n) = \{1, \underset{\uparrow}{2}, 1\}$$

Can you draw any conclusions regarding the linearity of the system. What is the impulse response of the system?

Solution:

The system is nonlinear. This is evident from observation of the pairs

$$x_3(n) \leftrightarrow y_3(n) \text{ and } x_2(n) \leftrightarrow y_2(n).$$

If the system were linear, $y_2(n)$ would be of the form

$$y_2(n) = \{3, 6, 3\}$$

because the system is time-invariant. However, this is not the case.

x3(n) is a delayed impulse. And since the system is time-invariant, the impulse response is simply obtained as h(n)={1,2,1,0,0} where the value of 2 now occurs at n=-3. Note that, given that this system is nonlinear, the impulse response is not very useful, and specifically it would NOT characterize the general input-output relationship for this system.

Problem 7:

2.21 Compute the convolution $y(n) = x(n) * h(n)$ of the following pairs of signals.

(a) $x(n) = a^n u(n)$, $h(n) = b^n u(n)$ when $a \neq b$ and when $a = b$

(b) $x(n) = \begin{cases} 1, & n = -2, 0, 1 \\ 2, & n = -1 \\ 0, & \text{elsewhere} \end{cases}$

$h(n) = \delta(n) - \delta(n-1) + \delta(n-4) + \delta(n-5)$

Solution:

(a)

$$y(n) = \sum_{k=0}^{n} a^k u(k) b^{n-k} u(n-k) = b^n \sum_{k=0}^{n} (ab^{-1})^k$$

$$y(n) = \begin{cases} \frac{b^{n+1}-a^{n+1}}{b-a} u(n), & a \neq b \\ b^n (n+1) u(n), & a = b \end{cases}$$

(b)

$$\begin{aligned} x(n) &= \left\{1, 2, \underset{\uparrow}{1}, 1\right\} \\ h(n) &= \left\{\underset{\uparrow}{1}, -1, 0, 0, 1, 1\right\} \\ y(n) &= \left\{1, 1, -\underset{\uparrow}{1}, 0, 0, 3, 3, 2, 1\right\} \end{aligned}$$

Problem 8:

2.23 Express the output *y(n)* of a linear time-invariant (LTI) system with impulse response *h(n)* in terms of its step response *s(n)=h(n)*u(n)* and the input *x(n)*.

Solution:

We can express the unit sample in terms of the unit step function as $\delta(n) = u(n) - u(n-1)$. Then,

$$\begin{aligned} h(n) &= h(n) * \delta(n) \\ &= h(n) * (u(n) - u(n-1)) \\ &= h(n) * u(n) - h(n) * u(n-1) \\ &= s(n) - s(n-1) \end{aligned}$$

Using this definition of $h(n)$

$$\begin{aligned} y(n) &= h(n) * x(n) \\ &= (s(n) - s(n-1)) * x(n) \\ &= s(n) * x(n) - s(n-1) * x(n) \end{aligned}$$

Problem 9:

(a) Let $x[n]$ and $y[n]$ be real-valued sequences both of which are even-symmetric: $x[n] = x[-n]$ and $y[n] = y[-n]$. Under these conditions, prove that $r_{xy}[\ell] = r_{yx}[\ell]$ for all ℓ.

(b) Express the autocorrelation sequence $r_{zz}[\ell]$ for the complex-valued signal $z[n] = x[n] + jy[n]$ where $x[n]$ and $y[n]$ are real-valued sequences, in terms of $r_{xx}[\ell]$, $r_{xy}[\ell]$, $r_{yx}[\ell]$, and $r_{yy}[\ell]$.

Solution:

(a) For real-valued sequences:

$r_{xy}[\ell] = x[\ell] * y[-\ell]$

$r_{yx}[\ell] = y[\ell] * x[-\ell]$

If both $x[n] = x[-n]$ and $y[n] = y[-n]$ are even-symmetric

$r_{xy}[\ell] = x[\ell] * y[\ell]$

$r_{yx}[\ell] = y[\ell] * x[\ell]$

$= x[\ell] * y[\ell]$ since convolution is commutative

Thus, $r_{xy}[\ell] = r_{yx}[\ell]$

(b) $r_{zz}[\ell] = z[\ell] * z^*[-\ell]$

$$= \left(x[\ell] + j\,y[\ell]\right) * \left(x[-\ell] - j\,y[-\ell]\right)$$

$$= x[\ell] * x[-\ell] + y[\ell] * y[-\ell]$$

$$+ j\left(y[\ell] * x[-\ell] - x[\ell] * y[-\ell]\right)$$

$$= r_{xx}[\ell] + r_{yy}[\ell] + j\left(r_{yx}[\ell] - r_{xy}[\ell]\right)$$

Note, if $x[-n] = x[n]$ and $y[n] = y[-n]$, then it follows from part (a):

$$r_{zz}[\ell] = r_{xx}[\ell] + r_{yy}[\ell]$$

MATLAB Exercises:

Please submit your MATLAB script source code along with any necessary plots and discussion.

P2.8 The operation of *signal dilation* (or *decimation* or *down-sampling*) is defined by

$$y(n) = x(nM)$$

in which the sequence $x(n)$ is down-sampled by an integer factor M. For example, if

$$x(n) = \{\ldots, -2, 4, \underset{\uparrow}{3}, -6, 5, -1, 8, \ldots\}$$

then the down-sampled sequences by a factor 2 are given by

$$y(n) = \{\ldots, -2, \underset{\uparrow}{3}, 5, 8, \ldots\}$$

1. Develop a MATLAB function **dnsample** that has the form

```
function [y,m] = dnsample(x,n,M)
% Downsample sequence x(n) by a factor M to obtain y(m)
```

   to implement the above operation. Use the indexing mechanism of MATLAB with careful attention to the origin of the time axis $n = 0$.
2. Generate $x(n) = \sin(0.125\pi n), \quad -50 \le n \le 50$. Decimate $x(n)$ by a factor of 4 to generate $y(n)$. Plot both $x(n)$ and $y(n)$ using **subplot** and comment on the results.
3. Repeat the above using $x(n) = \sin(0.5\pi n), \quad -50 \le n \le 50$. Qualitatively discuss the effect of down-sampling on signals.

Solution:

1. MATLAB function:

```
function [y,m] = dnsample(x,n,M)
% [y,m] = dnsample(x,n,M)
% Downsample sequence x(n) by a factor M to obtain y(m)
mb = ceil(n(1)/M)*M; me = floor(n(end)/M)*M;
nb = find(n==mb); ne = find(n==me);
y = x(nb:M:ne); m = fix((mb:M:me)/M);
```

2. $x_1(n) = \sin(0.125\pi n), \quad -50 \le n \le 50$. Decimation by a factor of 4.

```
% P0208b: x1(n) = sin(0.125*pi*n),-50 <= n <= 50
%         Decimate x(n) by a factor of 4 to obtain y(n)
clc; close all;
n1 = [-50:50]; x1 = sin(0.125*pi*n1); [y1,m1] = dnsample(x1,n1,4);
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0208b');
subplot(2,1,1); Hs = stem(n1,x1); set(Hs,'markersize',2);
xlabel('n','FontSize',LFS); ylabel('x(n)','FontSize',LFS);
title('Original sequence x_1(n)','FontSize',TFS);
axis([min(n1)-5,max(n1)+5,min(x1)-0.5,max(x1)+0.5]);
ytick = [-1.5:0.5:1.5]; ntick = [n1(1):10:n1(end)];
set(gca,'XTick',ntick); set(gca,'YTick',ytick);
subplot(2,1,2); Hs = stem(m1,y1); set(Hs,'markersize',2);
xlabel('n','FontSize',LFS); ylabel('y(n) = x(4n)','FontSize',LFS);
title('y_1(n)  =  Original sequence x_1(n) decimated by a factor of 4',...
      'FontSize',TFS);
axis([min(m1)-2,max(m1)+2,min(y1)-0.5,max(y1)+0.5]);
ytick = [-1.5:0.5:1.5]; ntick = [m1(1):2:m1(end)];
set(gca,'XTick',ntick); set(gca,'YTick',ytick);
```

3. $x(n) = \sin(0.5\pi n), \quad -50 \le n \le 50$. Decimation by a factor of 4.

```
% P0208c: x2(n) = sin(0.5*pi*n),-50 <= n <= 50
%         Decimate x2(n) by a factor of 4 to obtain y2(n)
clc; close all;
n2 = [-50:50]; x2 = sin(0.5*pi*n2); [y2,m2] = dnsample(x2,n2,4);
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0208c');
subplot(2,1,1); Hs = stem(n2,x2); set(Hs,'markersize',2);
xlabel('n','FontSize',LFS); ylabel('x(n)','FontSize',LFS);
axis([min(n2)-5,max(n2)+5,min(x2)-0.5,max(x2)+0.5]);
title('Original sequence x_2(n)','FontSize',TFS);
ytick = [-1.5:0.5:1.5]; ntick = [n2(1):10:n2(end)];
set(gca,'XTick',ntick); set(gca,'YTick',ytick);
subplot(2,1,2); Hs = stem(m2,y2); set(Hs,'markersize',2);
xlabel('n','FontSize',LFS); ylabel('y(n) = x(4n)','FontSize',LFS);
axis([min(m2)-1,max(m2)+1,min(y2)-1,max(y2)+1]);
title('y_2(n) = Original sequence x_2(n) decimated by a factor of 4',...
      'FontSize',TFS);
ntick = [m2(1):2:m2(end)]; set(gca,'XTick',ntick);
```

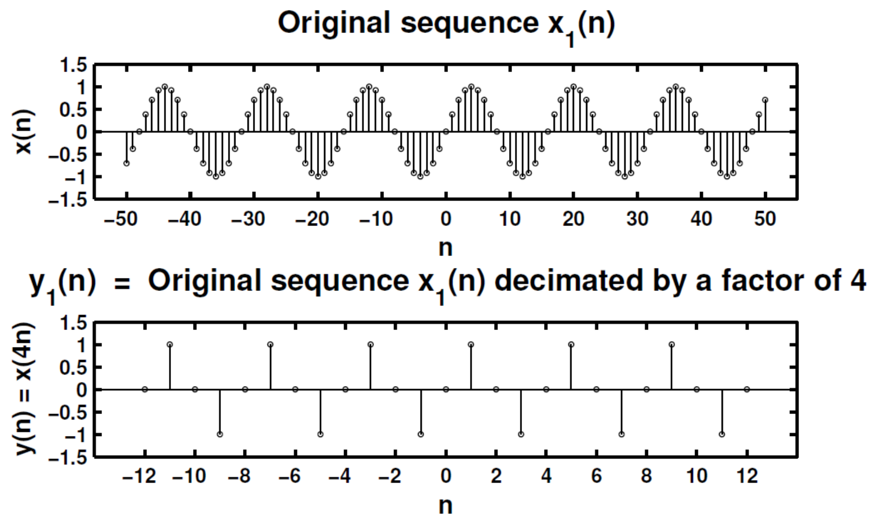

Figure 2.28: Problem P2.8.2 sequence plot

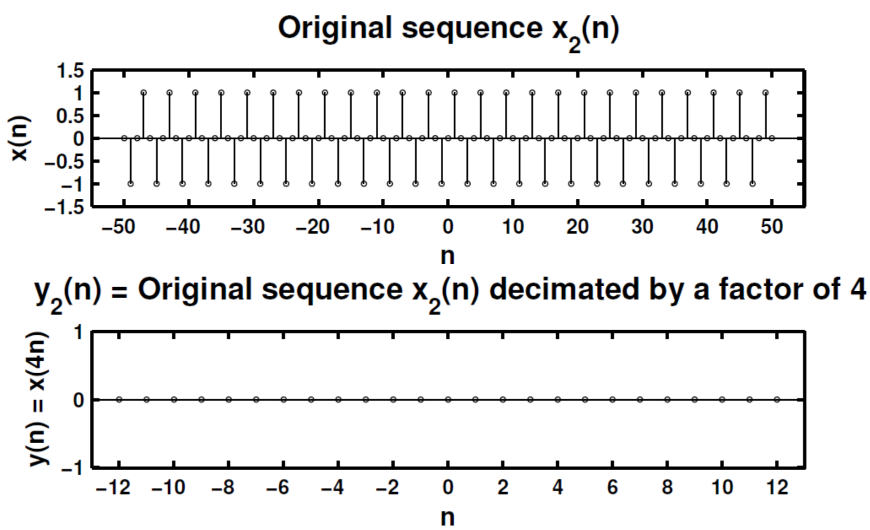

Figure 2.29: Problem P2.8.3 sequence plot

The plots of $x_2(n)$ and $y_2(n)$ are shown in Figure 2.29. Observe that the downsampled signal is a signal with zero frequency. Thus the original signal $x_2(n)$ is lost.

P2.16 Let $x(n) = (0.8)^n u(n)$, $h(n) = (-0.9)^n u(n)$, and $y(n) = h(n) * x(n)$. Use 3 columns and 1 row of subplots for the following parts.

1. Determine $y(n)$ analytically. Plot first 51 samples of $y(n)$ using the `stem` function.
2. Truncate $x(n)$ and $h(n)$ to 26 samples. Use `conv` function to compute $y(n)$. Plot $y(n)$ using the `stem` function. Compare your results with those of part 1.
3. Using the `filter` function, determine the first 51 samples of $x(n) * h(n)$. Plot $y(n)$ using the `stem` function. Compare your results with those of parts 1 and 2.

Solution:

1. Convolution $y(n) = h(n) * x(n)$:

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=0}^{\infty} (-0.9)^k (0.8)^{n-k} u(n-k) \\
&= \left[\sum_{k=0}^{n} (-0.9)^k (0.8)^n (0.8)^{-k} \right] u(n) = (0.8)^n \left[\sum_{k=0}^{n} \left(-\frac{9}{8} \right)^k \right] u(n) \\
&= \frac{0.8^{n+1} - (-0.9)^{n+1}}{1.7}
\end{aligned}
$$

MATLAB script:

```
clc; close all;run defaultsettings;
n = [0:50]; x = 0.8.^n; h = (-0.9).^n;
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0216');

% (a) Plot of the analytical convolution
y1 = ((0.8).^(n+1) - (-0.9).^(n+1))/(0.8+0.9);
subplot(1,3,1); Hs1 = stem(n,y1,'filled'); set(Hs1,'markersize',2);
title('Analytical'); xlabel('n'); ylabel('y(n)');
```

2. Computation using convolution of truncated sequences: MATLAB script

```
% (b) Plot using the conv function and truncated sequences
x2 = x(1:26); h2 = h(1:26); y2 = conv(h2,x2);
subplot(1,3,2); Hs2 = stem(n,y2,'filled'); set(Hs2,'markersize',2);
title('Using conv function'); xlabel('n'); %ylabel('y(n)');
```

3. To use the MATLAB's `filter` function we have to represent the $h(n)$ sequence by coefficients an equivalent difference equation. MATLAB script:

```
% (c) Plot of the convolution using the filter function
y3 = filter([1],[1,0.9],x);
subplot(1,3,3); Hs3 = stem(n,y3,'filled'); set(Hs3,'markersize',2);
title('Using filter function'); xlabel('n'); %ylabel('y(n)');
```

The plots of this solution are shown in Figure 2.33. The analytical solution to the convolution in 1 is the exact answer. In the `filter` function approach of 2, the infinite-duration sequence $x(n)$ is exactly represented by coefficients of an equivalent filter. Therefore, the filter solution should be exact except that it is evaluated up to the length of the input sequence. The truncated-sequence computation in 3 is correct up to the first 26 samples and then it degrades rapidly.

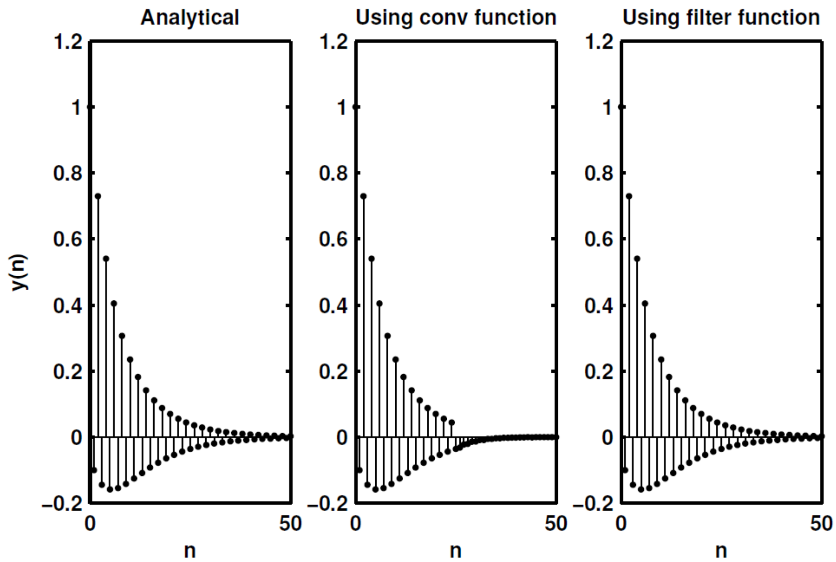

Figure 2.33: Problem P2.16 convolution plots