ECE113: DSP

Homework 4

Due 04/30/2021 11:59pm

Problem 1: Problem 4.3 in R1

Solution:

(a) Refer to fig 4.3-1.

$$x(t) = \begin{cases} 1 - \frac{|t|}{\tau}, & |t| \leq \tau \\ 0, & \text{otherwise} \end{cases}$$

$$X_a(F) = \int_{-\tau}^{0} (1 + \frac{t}{\tau}) e^{-j2\pi F t} dt + \int_{0}^{\tau} (1 - \frac{t}{\tau}) e^{-j2\pi F t} dt$$

Alternatively, we may find the fourier transform of

$$y(t) = x'(t) = \begin{cases} \frac{1}{\tau}, & -\tau < t \leq 0 \\ \frac{1}{\tau}, & 0 < t \leq \tau \end{cases}$$

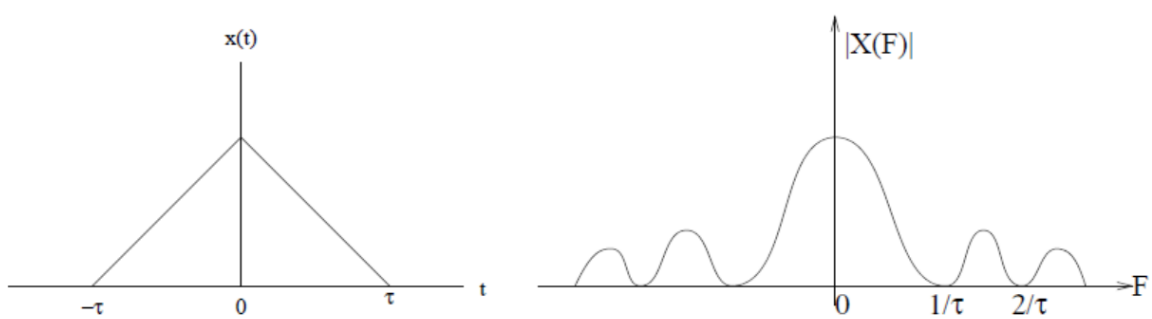


Figure 4.3-1:

Then,

$$\begin{aligned}
Y(F) &= \int_{-\tau}^{\tau} y(t)e^{-j2\pi Ft}dt \\
&= \int_{-\tau}^{0} \frac{1}{\tau}e^{-j2\pi Ft}dt + \int_{0}^{\tau}(\frac{-1}{\tau})e^{-j2\pi Ft}dt \\
&= -\frac{2sin^2\pi F\tau}{j\pi F\tau} \\
\text{and } X(F) &= \frac{1}{j2\pi F}Y(F) \\
&= \tau\left(\frac{sin\pi F\tau}{\pi F\tau}\right)^2 \\
|X(F)| &= \tau\left(\frac{sin\pi F\tau}{\pi F\tau}\right)^2 \\
\angle X_a(F) &= 0
\end{aligned}$$

(b)

$$\begin{aligned}
c_k &= \frac{1}{T_p}\int_{-T_p/2}^{T_p/2} x(t)e^{-j2\pi kt/T_p}dt \\
&= \frac{1}{T_p}\left[\int_{-\tau}^{0}(1+\frac{t}{\tau})e^{-j2\pi kt/T_p}dt + \int_{0}^{\tau}(1-\frac{t}{\tau})e^{-j2\pi kt/T_p}dt\right] \\
&= \frac{\tau}{T_p}\left[\frac{sin\pi k\tau/T_p}{\pi k\tau/T_p}\right]^2
\end{aligned}$$

(c) From (a) and (b), we have $c_k = \frac{1}{T_p}X_a(\frac{k}{T_p})$

Problem 2: Problem 4.6 (Part (b) only) in R1

<u>Solution:</u>

(b)

$$\begin{aligned} x(n) &= cos\frac{2\pi n}{3} + sin\frac{2\pi n}{5} \Rightarrow N = 15 \\ c_k &= c_{1k} + c_{2k} \end{aligned}$$

where c_{1k} is the DTFS coefficients of $cos\frac{2\pi n}{3}$ and c_{2k} is the DTFS coefficients of $sin\frac{2\pi n}{5}$. But

$$cos\frac{2\pi n}{3} = \frac{1}{2}(e^{\frac{j2\pi n}{3}} + e^{\frac{-j2\pi n}{3}})$$

Hence,

$$c_{1k} = \begin{cases} \frac{1}{2}, & k = 5, 10 \\ 0, & \text{otherwise} \end{cases}$$

Similarly,

$$sin\frac{2\pi n}{5} = \frac{1}{2j}(e^{\frac{j2\pi n}{5}} - e^{\frac{-j2\pi n}{5}}).$$

Hence,

$$c_{2k} = \begin{cases} \frac{1}{2j}, & k = 3 \\ \frac{-1}{2j}, & k = 12 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$c_k = c_{1k} + c_{2k} \begin{cases} \frac{1}{2j}, & k = 3 \\ \frac{1}{2}, & k = 5 \\ \frac{1}{2}, & k = 10 \\ \frac{-1}{2j}, & k = 12 \\ 0, & \text{otherwise} \end{cases}$$

Problem 3: Problem 4.7 (Part (a) only) in R1

Solution:

(a)

$$\begin{aligned} x(n) &= \sum_{k=0}^{7} c_k e^{\frac{j2\pi nk}{8}} \\ \text{Note that if } c_k &= e^{\frac{j2\pi pk}{8}}, \text{ then} \\ \sum_{k=0}^{7} e^{\frac{j2\pi pk}{8}} e^{\frac{j2\pi nk}{8}} &= \sum_{n=0}^{7} e^{\frac{j2\pi(p+n)k}{8}} \\ &= 8, \quad p = -n \\ &= 0, \quad p \neq -n \\ \text{Since } c_k &= \frac{1}{2}\left[e^{\frac{j2\pi k}{8}} + e^{\frac{-j2\pi k}{8}}\right] + \frac{1}{2j}\left[e^{\frac{j6\pi k}{8}} - e^{\frac{-j6\pi k}{8}}\right] \\ \text{We have } x(n) &= 4\delta(n+1) + 4\delta(n-1) - 4j\delta(n+3) + 4j\delta(n-3), -3 \leq n \leq 5 \end{aligned}$$

Problem 4: Problem 4.9 (Part (d) only) in R1

Solution:

(d)

$$\begin{aligned} x(n) &= \alpha^n sinw_0 nu(n), |\alpha| < 1 \\ X(w) &= \sum_{n=0}^{\infty} \alpha^n \left[\frac{e^{jw_0 n} - e^{-jw_0 n}}{2j}\right] e^{-jwn} \\ &= \frac{1}{2j}\sum_{n=0}^{\infty}\left[\alpha e^{-j(w-w_0)}\right]^n - \frac{1}{2j}\sum_{n=0}^{\infty}\left[\alpha e^{-j(w+w_0)}\right]^n \\ &= \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(w-w_0)}} - \frac{1}{1-\alpha e^{-j(w+w_0)}}\right] \\ &= \frac{\alpha sinw_0 e^{-jw}}{1 - 2\alpha cosw_0 e^{-jw} + \alpha^2 e^{-j2w}} \end{aligned}$$

Problem 5: Problem 4.10 (Parts (c) and (d) only) in R1

Solution:

(c)

$$\begin{aligned} x(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w)e^{jwn}dw \\ &= \frac{1}{2\pi}\int_{w_0-\frac{\delta w}{2}}^{w_0+\frac{\delta w}{2}} e^{jwn}dw \\ &= \frac{2}{\pi}\delta w\left(\frac{sin(n\delta w/2)}{n\delta w/2}\right)e^{jnw_0} \end{aligned}$$

(d)

$$\begin{aligned} x(n) &= \frac{1}{2\pi}Re\left\{\int_0^{\pi/8} 2e^{jwn}dw + \int_{\pi/8}^{3\pi/8} e^{jwn}dw + \int_{6\pi/8}^{7\pi/8} e^{jwn}dw + \int_{7\pi/8}^{\pi} e^{jwn}dw\right\} \\ &= \frac{1}{\pi}\left[\int_0^{\pi/8} 2coswndw + \int_{\pi/8}^{3\pi/8} coswndw + \int_{6\pi/8}^{7\pi/8} coswndw + \int_{7\pi/8}^{\pi} 2coswndw\right] \\ &= \frac{1}{n\pi}\left[sin\frac{7\pi n}{8} + sin\frac{6\pi n}{8} - sin\frac{3\pi n}{8} - sin\frac{\pi n}{8}\right] \end{aligned}$$

Problem 6: Problem 7.18 in R1

Solution:

$$\begin{aligned}
x(n) &= \sum_{i=-\infty}^{\infty} \delta(n-iN) \\
y(n) &= \sum_{m} h(m)x(n-m) \\
&= \sum_{m} h(m)\left[\sum_{i} \delta(n-m-iN)\right] \\
&= \sum_{i} h(n-iN)
\end{aligned}$$

Therefore, $y(.)$ is a periodic sequence with period N. So

$$\begin{aligned}
Y(k) &= \sum_{n=0}^{N-1} y(n)W_N^{kn} \\
&= H(w)|_{w=\frac{2\pi}{N}k} \\
Y(k) &= H(\frac{2\pi k}{N}) \qquad k=0,1,\ldots,N-1
\end{aligned}$$

Problem 7: Problem 7.23 (Part (h) only, assume *N* odd) in R1

Solution:

(h)

$$\begin{aligned}
X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk}(\text{ assume N odd }) \\
&= 1+e^{-j\frac{2\pi}{N}2k}+e^{-j\frac{2\pi}{N}4k}+\ldots+e^{-j\frac{2\pi}{N}(n-1)k} \\
&= \frac{1-(e^{-j\frac{2\pi}{N}2k})^{\frac{N+1}{2}}}{1-e^{-j\frac{2\pi}{N}2k}} \\
&= \frac{1-e^{-j\frac{2\pi}{N}k}}{1-e^{-j\frac{4\pi}{N}k}} \\
&= \frac{1}{1-e^{-j\frac{2\pi}{N}k}}
\end{aligned}$$

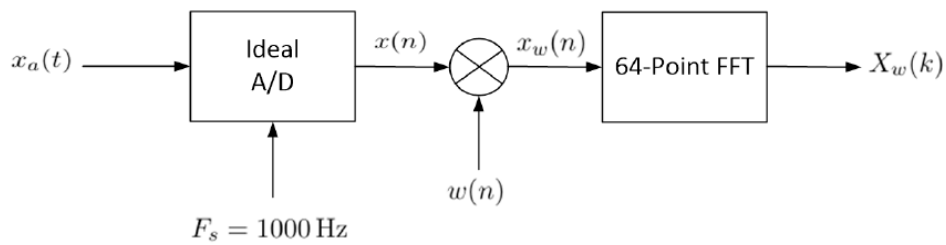
Problem 8: Assume that an *N*-point DFT, performed on an *N*-sample *x(n)* sequence, results in a DFT frequency sample spacing of 100Hz. What would be the DFT frequency domain sample spacing in Hz if the *N*-sample sequence *x(n)* is zero padded with *4N* zero-valued samples and we perform DFT on that extended time sequence?

Solution:

The frequency spacing between any two consecutive bins in an N-point DFT is $2\pi/N$ rad/sample or F_s/N Hz. So when the signal is zero-padded with 4N zeros, the new length would be 5N, and as such, the frequency spacing will reduce to $F_s/5N$, or, in this case, 100/5=20Hz.

Problem 9:

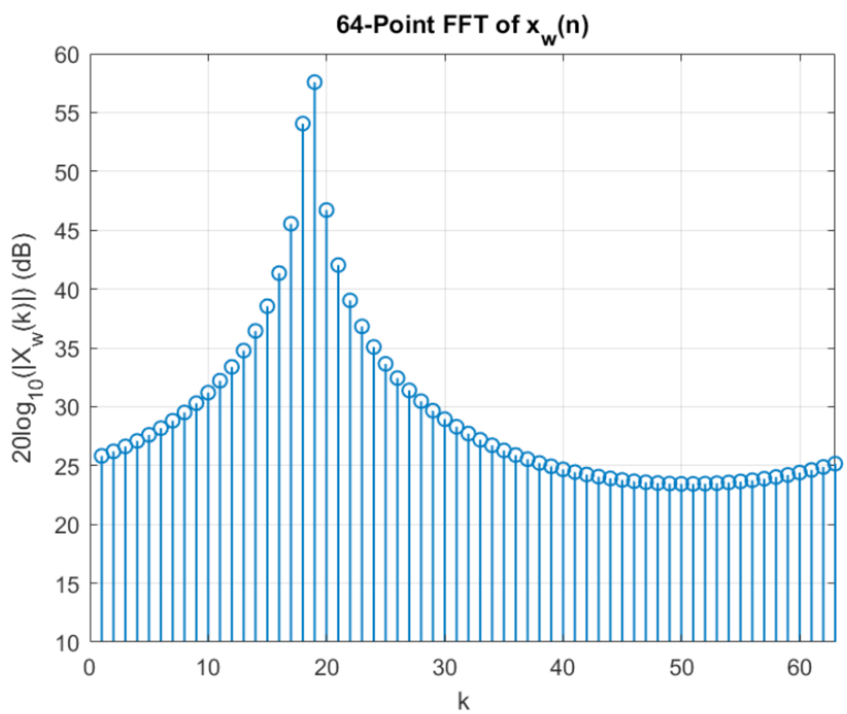
A system for discrete-time spectral analysis of a continuous-time signal is shown below:



where $w(n)$ is a rectangular window:

$$w(n) = \begin{cases} \frac{1}{64}, & \text{for } 0 \le n \le 63 \\ 0, & \text{otherwise} \end{cases}$$

We have obtained the 64-point FFT, $X_w(k)$, the magnitude of which is shown below with the vertical axis in dB scale:



The associated continuous-time input signal, $x_a(t)$, could be one or more of the following signals. Identify which one(s) of the signals below could have produced this FFT. And clearly explain your reasoning for your choice(s).

Hint: Do not try to analyze the signals one at a time. Instead, first look at all the signals and, from what you know about DFT's, try to divide and conquer!

Solution:

$\boxtimes$ $x_{a1}(t) = 10\cos(550\pi t)$

$\boxtimes$ $x_{a2}(t) = 1000\cos(550\pi t)$

$\boxtimes$ $x_{a3}(t) = 10e^{j550\pi t}$

$\checkmark$ $x_{a4}(t) = 1000e^{j550\pi t}$

$\boxtimes$ $x_{a5}(t) = 10\cos(531.25\pi t)$

$\boxtimes$ $x_{a6}(t) = 1000\cos(531.25\pi t)$

$\boxtimes$ $x_{a7}(t) = 1000e^{j531.25\pi t}$

$\boxtimes$ $x_{a8}(t) = 1000e^{j562.5\pi t}$

$\boxtimes$ $x_{a9}(t) = 1000\cos(562.5\pi t)$

$\boxtimes$ $x_{a10}(t) = 1000e^{j2562.5\pi t}$

$\boxtimes$ $x_{a11}(t) = 1000\cos(2550\pi t)$

$\checkmark$ $x_{a12}(t) = 1000e^{j2550\pi t}$

* Only one peak. So input cannot include cos function. That excludes $x_{a_1}, x_{a_2}, x_{a_5}, x_{a_6}, x_{a_9}, x_{a_{11}}$

* The amplitude of the rectangular window is set to $\frac{1}{N} = \frac{1}{64}$. So it normalized the DFT magnitudes.

Yet the peak is near 60dB = 1000. So the input signal cannot have an amplitude of 10. That excludes x_{a_3} (in addition to x_{a_1} and x_{a_5} which had already been excluded).

* The F.F.T. does not show a single nonzero sample. So we clearly have leakage. As such the input freq. cannot coincide with any of the freq bins which are located at multiples of $\frac{F_s}{N} = \frac{1000}{64} = 15.625$ Hz

That excludes x_{a_7} $\left(f_0 = \frac{531.25}{2} = 17 \times 15.625\right)$ and

x_{a_8} $\left(f_0 = \frac{562.5}{2} = 18 \times 15.625\right)$ Also $x_{a_{10}}$ $\left(\frac{2562.5}{2} = 82 \times 15.625\right)$

* would alias back onto one of the bins and could not have been our input signal.

* That leaves us with $\boxed{x_{a_4}}$ and $\boxed{x_{a_{12}}}$ both of which could have been our input. Notice that the freq. of $x_{a_{12}}$ is exactly $F_s = 1000$ Hz apart from the freq of x_{a_4} and as such would alias back to exactly the same samples in the freq domain.

MATLAB: (Please submit your source code along with any necessary analysis and plots)

P5.10 Plot the DTFT magnitude and angle of each of the following sequences using the DFT as a computation tool. Make an educated guess about the length N so that your plots are meaningful.

3. $x(n) = [\cos(0.5\pi n) + j\sin(0.5\pi n)][u(n) - u(n-51)]$.
4. $x(n) = \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\}$.

Note: For DFT calculation in MATLAB for this problem, you can either use the built-in "fft" function in MATLAB, or the following DFT function:

```
function [Xk] = dft(xn,N)
% Computes Discrete Fourier Transform
% -----------------------------------
% [Xk] = dft(xn,N)
% Xk = DFT coeff. array over 0 <= k <= N-1
% xn = N-point finite-duration sequence
%  N = Length of DFT
%
n = [0:1:N-1];                      % row vector for n
k = [0:1:N-1];                      % row vecor for k
WN = exp(-j*2*pi/N);                % Wn factor
nk = n'*k;                          % creates a N by N matrix of nk values
WNnk = WN .^ nk;                    % DFT matrix
Xk = xn * WNnk;                     % row vector for DFT coefficients
```

Solution:

3. $x_3(n) = \cos(0.5\pi n) + j\sin(0.5\pi n)$, $0 \le n \le 50$. MATLAB script:

```
n3 = [0:50]; x3 = cos(0.5*pi*n3)+j*sin(0.5*pi*n3); N = 500;% Length of DFT
N3 = length(n3); x3 = [x3,zeros(1,N-N3)]; % Assemble x3
[X3] = dft(x3,N); w = (0:N/2)*2*pi/N;
mag_X3 = abs(X3(1:N/2+1)); pha_X3 = angle(X3(1:N/2+1))*180/pi;
Hf_3 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_3,'NumberTitle','off','Name','P5.10.3');
subplot(2,1,1); plot(w/pi,mag_X3,'g','linewidth',1); %axis([0,1,0,7000]);
title('Magnitude of DTFT X_3(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X3,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_3(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510c
```

The plot of the DTFT $X_3(e^{j\omega})$ is shown in 5.10.

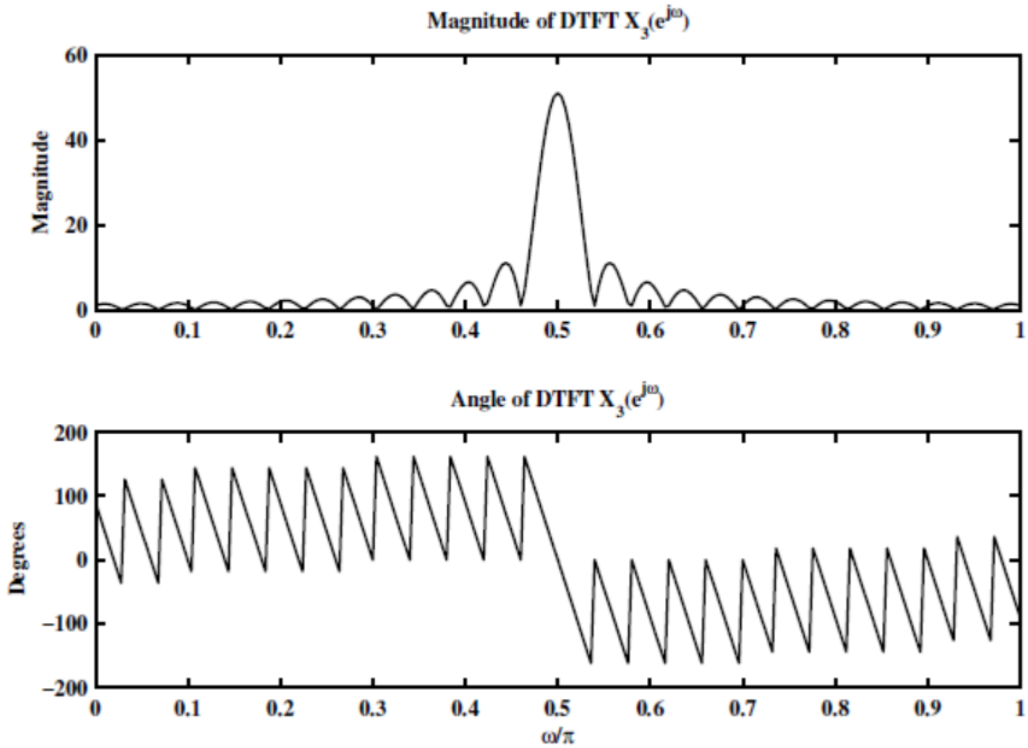


Figure 5.10: Plots of DTFT magnitude and phase in Problem 5.10.3

Below is the original solution I had:

4. $x(n) = \{1, 2, 3, \underset{\uparrow}{4}, 3, 2, 1\}$. MATLAB script:

```
n4 = [-3:3]; x4 = [1,2,3,4,3,2,1]; N4 = length(n4); N = 100;   % Length of DFT
[X4] = dft([x4, zeros(1,N-N4)],N); w = (0:N/2)*2*pi/N;
mag_X4 = abs(X4(1:N/2+1)); pha_X4 = angle(X4(1:N/2+1))*180/pi;
Hf_4 = figure('Units','inches','position',[1,1,6,4],...
    'color',[0,0,0],'paperunits','inches','paperposition',[0,0,6,4]);
set(Hf_4,'NumberTitle','off','Name','P5.10.4');
subplot(2,1,1); plot(w/pi,mag_X4,'g','linewidth',1); axis([0,1,0,20]);
title('Magnitude of DTFT X_4(e^{j\omega})'); ylabel('Magnitude');
subplot(2,1,2); plot(w/pi,pha_X4,'g','linewidth',1); axis([0,1,-200,200]);
title('Angle of DTFT X_4(e^{j\omega})'); ylabel('Degrees'); xlabel('\omega/\pi');
print -deps2 ../EPSFILES/P0510d
```

The plot of the DTFT $X_4(e^{j\omega})$ is shown in 5.11.

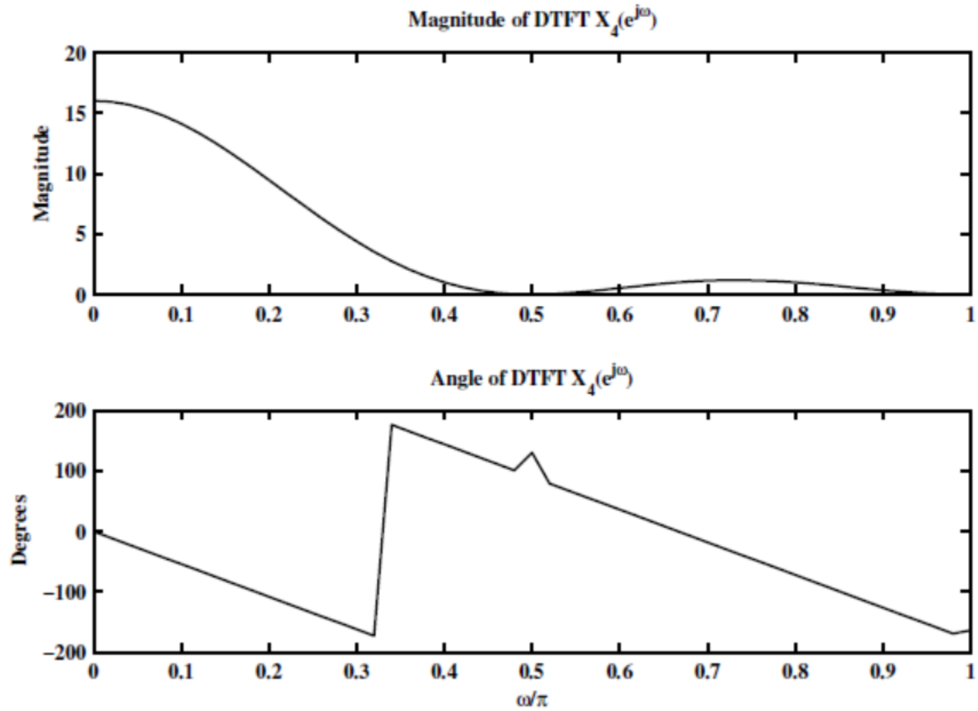


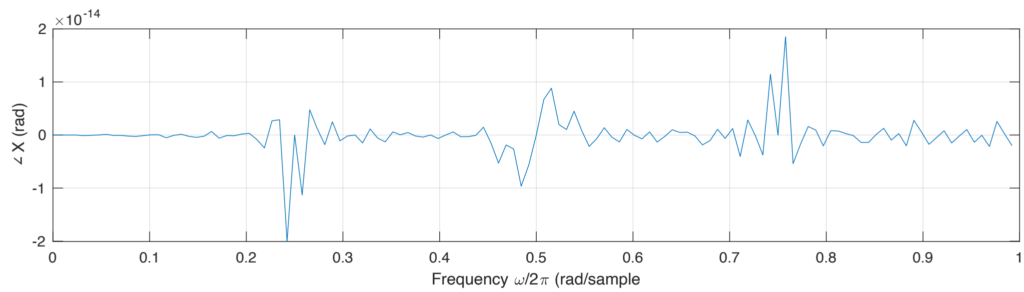
Figure 5.11: Plots of DTFT magnitude and phase in Problem 5.10.4

As you notice, it has not accounted for the time delay, given that x(0)=4. As a result, even though our original sequence was real-valued and even-symmetric, we have ended up with non-zero phase! So below is the correct version of the code and the plot. Notice the very small phase shown on the plot is due to the numerical precision of my computer.

```
N=128;
w=0:2*pi/N:2*pi*(N-1)/N;
x=[1 2 3 4 3 2 1];

Xd=fft(x,N);
X=Xd.*exp(1i*w*3);   % To account for the time delay, since x(0)=4

subplot(211)
plot(w/2/pi,abs(X));
grid on;
xlabel('Frequency \omega/2\pi (rad/sample');
ylabel('|X|')
subplot(212)
plot(w/2/pi,angle(X));
grid on;
xlabel('Frequency \omega/2\pi (rad/sample');
ylabel('\angleX (rad)')
```



P5.38 An analog signal $x_a(t) = 2\sin(4\pi t) + 5\cos(8\pi t)$ is sampled at $t = 0.01n$ for $n = 0, 1, \ldots, N-1$ to obtain an N-point sequence $x(n)$. An N-point DFT is used to obtain an estimate of the magnitude spectrum of $x_a(t)$.

1. From the following values of N, choose the one that will provide the accurate estimate of the spectrum of $x_a(t)$. Plot the real and imaginary parts of the DFT spectrum $X(k)$. (a) $N = 40$, (b) $N = 50$, (c) $N = 60$.

2. From the following values of N, choose the one that will provide the least amount of leakage in the spectrum of $x_a(t)$. Plot the real and imaginary parts of the DFT spectrum $X(k)$. (a) $N = 90$, (b) $N = 95$, (c) $N = 99$.

Solution:

1. Out of the given three values, $N = 50$ provides complete cycles of both the sine and the cosine components. Thus $N = 50$ provides the most accurate estimate as shown in Figure 5.36.

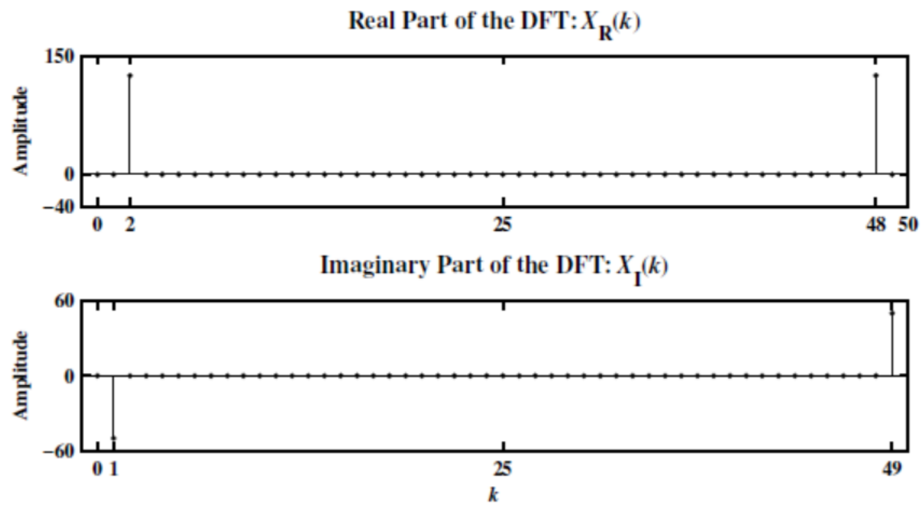


Figure 5.36: The accurate spectrum of the signal in Problem P5.38.1

2. Out of the given three values, $N = 99$ provides almost complete cycles of both the sine and the cosine components. Thus $N = 99$ provides the least amount of leakage as shown in Figure 5.37.

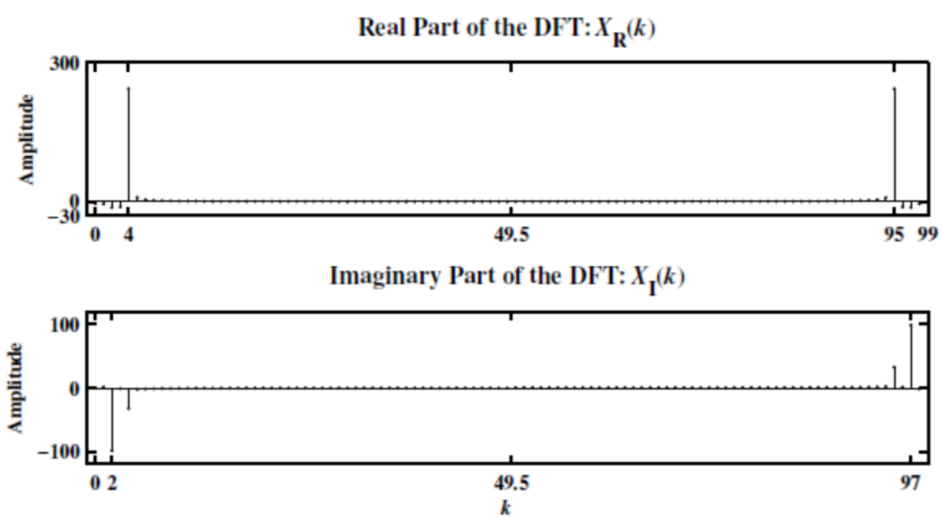


Figure 5.37: The least amount of leakage in the spectrum of the signal in Problem P5.38.2