ECE113: DSP

Homework 6 Solutions

Problem 1: Problem 3.14 ((d), and (g) only) in R1

Solution:

(d)

$$
\begin{aligned}
X(z) &= \frac{1}{1+z^{-2}} + 2\frac{z^{-2}}{1+z^{-2}} \\
X(z) &= 2 - \frac{1}{1+z^{-2}} \\
x(n) &= cos\frac{\pi}{2}nu(n) + 2cos\frac{\pi}{2}(n-2)u(n-2) \\
x(n) &= 2\delta(n) - cos\frac{\pi}{2}nu(n)
\end{aligned}
$$

(g)

$$
\begin{aligned}
X(z) &= \frac{1+2z^{-1}+z^{-2}}{1+4z^{-1}+4z^{-2}} \\
&= 1 - \left(\frac{2z^{-1}+3z^{-2}}{(1+2z^{-1})(1+2z^{-1})}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 1 - \frac{2z^{-1}}{1+2z^{-1}} + \frac{z^{-2}}{(1+2z^{-1})^2} \\
x(n) &= \delta(n) - 2(-2)^{n-1}u(n-1) + (n-1)(-2)^{n-1}u(n-1) \\
&= \delta(n) + (n-3)(-2)^{n-1}u(n-1)
\end{aligned}
$$

Problem 2: Problem 3.16 ((d) only) in R1

Solution:

(d)

$$
\begin{aligned}
x_1(n) &= nu(n) \\
\Rightarrow X_1(z) &= \frac{z^{-1}}{(1-z^{-1})^2}, \\
x_2(n) &= 2^n u(n-1) \\
\Rightarrow X_2(z) &= \frac{2z^{-1}}{1-2z^{-1}} \\
Y(z) &= X_1(z)X_2(z) \\
&= \frac{2z^{-2}}{(1-z^{-1})^2(1-2z^{-1})} \\
&= \frac{-2}{1-z^{-1}} - \frac{-2z^{-1}}{(1-z^{-1})^2} + \frac{2}{1-2z^{-1}} \\
y(n) &= \left[-2(n+1)+2^{n+1}\right]u(n)
\end{aligned}
$$

Problem 3: Problem 3.18 ((d) only) in R1

Solution:

(d)

$$
\begin{aligned}
X_k(z) &= \sum_{n=-\infty, n/k integer}^{\infty} x(\frac{n}{k})z^{-n} \\
&= \sum_{m=-\infty}^{\infty} x(m)z^{-mk} \\
&= X(z^k)
\end{aligned}
$$

Problem 4: Problem 3.32 in R1

Solution:

(a)

$$
\begin{aligned}
Y(z)\left[1-0.2z^{-1}\right] &= X(z)\left[1-0.3z^{-1}-0.02z^{-2}\right] \\
\frac{Y(z)}{X(z)} &= \frac{(1-0.1z^{-1})(1-0.2z^{-1})}{1-0.2z^{-1}} \\
&= 1-0.1z^{-1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
Y(z) &= X(z)\left[1-0.1z^{-1}\right] \\
\frac{Y(z)}{X(z)} &= 1-0.1z^{-1}
\end{aligned}
$$

Therefore, (a) and (b) are equivalent systems.

Problem 5: Problem 3.35 ((c) and (g) only) in R1.

(Hint: In Part (c), a "," is obviously missing between x(n-1) and x(n). For Part (g), note that x(n) does not have a Z-transform and you should instead use the fundamental property of Transfer Functions relating to how an LTI system responds to a sinusoidal sequence)

Solution:

(c)

$$\begin{aligned} y(n) &= -0.1y(n-1) + 0.2y(n-2) + x(n) + x(n-1) \\ H(z) &= \frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}} \\ x(n) &= (\frac{1}{3})^n u(n) \\ X(z) &= \frac{1}{1-\frac{1}{3}z^{-1}} \\ Y(z) &= H(z)X(z) \\ &= \frac{1+z^{-1}}{(1-\frac{1}{3}z^{-1})(1+0.1z^{-1}-0.2z^{-2})} \\ &= \frac{-8}{1-\frac{1}{3}z^{-1}} + \frac{\frac{28}{3}}{1-0.4z^{-1}} + \frac{\frac{-1}{3}}{1+0.5z^{-1}} \end{aligned}$$

Therefore,

$$y(n) = \left[-8(\frac{1}{3})^n + \frac{28}{3}(\frac{2}{5})^n - \frac{1}{3}(\frac{1}{2})^n\right] u(n)$$

(g)

$$\begin{aligned} h(n) &= (\frac{1}{2})^n u(n) \\ H(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} \\ x(n) &= (-1)^n, \qquad -\infty < n < \infty \\ &= cos\pi n, \qquad -\infty < n < \infty \end{aligned}$$

$x(n)$ is periodic sequence and its z-transform does not exist.

$$\begin{aligned} y(n) &= |H(w_0)|cos[\pi n + \Theta(w_0)], w_0 = \pi \\ H(z) &= \frac{1}{1-\frac{1}{2}e^{-jw}} \\ H(\pi) &= \frac{1}{1+\frac{1}{2}} \\ &= \frac{2}{3}, \qquad \Theta = 0. \\ \text{Hence, } y(n) &= \frac{2}{3}cos\pi n, \qquad -\infty < n < \infty \end{aligned}$$

Problem 6: Problem 3.38 ((b) only) in R1

Solution:

(b)

$$\begin{aligned} y(n) &= y(n-1) - \frac{1}{2}y(n-2) + x(n) + x(n-1) \\ Y(z) &= \frac{1+z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}}X(z) \end{aligned}$$

$H(z)$ has zeros at $z = 0, 1$, and poles at $z = \frac{1\pm j}{2}$. Hence, the system is stable.

$$\begin{aligned} \text{Impulse Response: } X(z) &= 1 \\ Y(z) &= \frac{1-(\sqrt{2})^{-1}cos\frac{\pi}{4}z^{-1}}{1-2(\sqrt{2})^{-1}cos\frac{\pi}{4}z^{-1}+(\frac{1}{\sqrt{2}})^2z^{-2}} + \frac{\frac{3}{2}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} \\ \Rightarrow y(n) = h(n) &= (\frac{1}{\sqrt{2}})^n\left[cos\frac{\pi}{4}n + sin\frac{\pi}{4}n\right]u(n) \\ \text{Step Response: } X(z) &= \frac{1}{1-z^{-1}} \\ Y(z) &= \frac{1+z^{-1}}{(1-z^{-1})(1-z^{-1}+\frac{1}{2}z^{-2}} \\ &= \frac{-(1-\frac{1}{2}z^{-1})}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{\frac{1}{2}z^{-1}}{1-z^{-1}+\frac{1}{2}z^{-2}} + \frac{2}{1-z^{-1}} \\ y(n) &= (\frac{1}{\sqrt{2}})^n\left[sin\frac{\pi}{4}n - cos\frac{\pi}{4}n\right]u(n) + 2u(n) \end{aligned}$$

Problem 7: Problem 3.40 in R1

Solution:

$$
\begin{aligned}
x(n) &= (\frac{1}{2})^n u(n) - \frac{1}{4}(\frac{1}{2})^{n-1}u(n-1) \\
X(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{4}\frac{z^{-1}}{1-\frac{1}{2}z^{-1}} \\
&= \frac{1-\frac{1}{4}z^{-1}}{1-\frac{1}{2}z^{-1}} \\
y(n) &= (\frac{1}{3})^n u(n) \\
Y(z) &= \frac{1}{1-\frac{1}{3}z^{-1}}
\end{aligned}
$$

(a)

$$
\begin{aligned}
H(z) &= Y(z)X(z) \\
&= \frac{1-\frac{1}{2}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})} \\
&= \frac{3}{1-\frac{1}{4}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} \\
h(n) &= \left[3(\frac{1}{4})^n - 2(\frac{1}{3})^n\right] u(n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
H(z) &= \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} \\
y(n) &= \frac{7}{12}y(n-1) - \frac{1}{12}y(n-2) + x(n) - \frac{1}{2}x(n-1)
\end{aligned}
$$

(c) Refer to fig 3.40-1.

(d) The poles of the system are inside the unit circle. Hence, the system is stable.

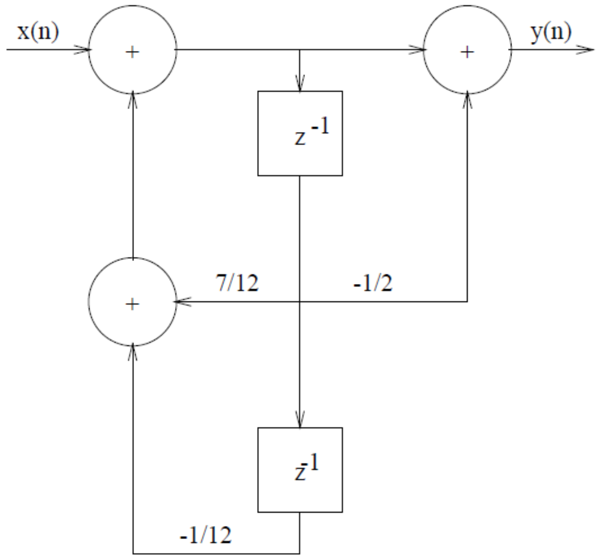

Figure 3.40-1:

Problem 8: Problem 3.42 in R1

Solution:

(a)

$$\begin{aligned} H(z) &= z^{-1}\left[\frac{-\frac{7}{2}}{1-\frac{1}{5}z^{-1}} + \frac{\frac{9}{2}}{1-\frac{2}{5}z^{-1}}\right] \\ h(n) &= \left[-\frac{7}{2}(\frac{1}{5})^{n-1} + \frac{9}{2}(\frac{2}{5})^{n-1}\right] u(n-1) \end{aligned}$$

(b)

$$\begin{aligned} Y(z) &= H(z)X(z) \\ X(z) &= \frac{1}{1-z^{-1}} \\ Y(z) &= \frac{\frac{25}{8}}{1-z^{-1}} + \frac{\frac{7}{8}}{1-\frac{1}{5}z^{-1}} + \frac{-3}{1-\frac{2}{5}z^{-1}} \\ y(n) &= \left[\frac{25}{8} + \frac{7}{8}(\frac{1}{5})^{n} - 3(\frac{2}{5})^{n}\right] u(n) \end{aligned}$$

(c) Determine the response caused by the initial conditions and add it to the response in (b).

$$\begin{aligned}
y(n)-\frac{3}{5}y(n-1)+\frac{2}{25}y(n-2) &= 0 \\
Y^+(z)-\frac{3}{5}\left[Y^+(z)z^{-1}+1\right]+\frac{2}{25}\left[Y^+(z)z^{-2}+z^{-1}+2\right] &= 0 \\
Y^+(z) &= \frac{\frac{2}{25}z^{-1}-\frac{11}{25}}{(1-\frac{1}{5}z^{-1})(1-\frac{2}{5}z^{-1})} \\
&= \frac{\frac{1}{25}}{1-\frac{1}{5}z^{-1}}+\frac{\frac{-12}{25}}{1-\frac{2}{5}z^{-1}} \\
y^+(n) &= \left[\frac{1}{25}(\frac{1}{5})^n-\frac{12}{25}(\frac{2}{5})^n\right]u(n)
\end{aligned}$$

Therefore, the total step response is

$$y(n) = \left[\frac{25}{8}+\frac{33}{200}(\frac{1}{5})^n-\frac{87}{25}(\frac{2}{5})^n\right]u(n)$$

Problem 9: Problem 3.51 in R1

Solution:

(a)

$$H(z)=\frac{z-1}{(z+\frac{1}{2})(z+3)(z-2)}, \qquad \text{ROC: } \frac{1}{2}<|z|<2$$

(b) The system can be causal if the ROC is $|z|>3$, but it cannot be stable.

(c)

$$H(z)=\frac{A}{1+\frac{1}{2}z^{-1}}+\frac{B}{1+3z^{-1}}+\frac{C}{1-2z^{-1}}$$

(1) The system can be causal; (2) The system can be anti-causal; (3) There are two other noncausal responses.The corresponding ROC for each of these possibilities are :

$\text{ROC}_1 : |z|>3;\quad \text{ROC}_2 : |z|<3;\quad \text{ROC}_3 : \frac{1}{2}<|z|<2;\quad \text{ROC}_4 : 2<|z|<3;$

Problem 10: Problem 5.20 in R1

Solution:

$$\begin{aligned}
y(n) &= (cos\pi n)x(n) \Rightarrow \text{ This is a time-varying system} \\
Y(w) &= \frac{1}{2\pi}\left[\pi\delta(w-\pi)+\pi\delta(w+\pi)\right]*X(w) \\
&= \frac{1}{2}\left[X(w-\pi)+X(w+\pi)\right] \\
&= 0, \quad |w|\leq\frac{3\pi}{4} \\
&= \frac{1}{2}, \quad \frac{3\pi}{4}\leq|w|\leq\pi
\end{aligned}$$

<u>MATLAB:</u>

P4.11 Determine the following inverse z-transforms using the partial fraction expansion method.

1. $X_1(z) = (1 - z^{-1} - 4z^{-2} + 4z^{-3})/(1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3})$. The sequence is rightsided.

4. $X_4(z) = z/(z^3 + 2z^2 + 1.25z + 0.25)$, $|z| > 1$

Note: For PFE, you can use the "residuez" function in MATLAB.

Solution:

1. $X_1(z) = (1 - z^{-1} - 4z^{-2} + 4z^{-3})/(1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3})$. The sequence is right-sided.
MATLAB script:

```
% P0611a: Inverse z-Transform of X1(z)
clc; close all;
b1 = [1,-1,-4,4]; a1 = [1,-11/4,13/8,-1/4];
[R,p,k] = residuez(b1,a1)
 R =
     0.0000
   -10.0000
    27.0000
 p =
     2.0000
     0.5000
     0.2500
 k =
    -16
```

or

$$X_1(z) = \frac{1 - z^{-1} - 4z^{-2} + 4z^{-3}}{1 - \frac{11}{4}z^{-1} + \frac{13}{8}z^{-2} - \frac{1}{4}z^{-3}} = -16 + \frac{0}{1 - 2z^{-1}} - \frac{10}{1 - 0.5z^{-1}} + \frac{27}{1 - 0.25z^{-1}}, \ |z| > 0.5$$

Note that from the second term on the right, there is a pole-zero cancellation. Hence

$$x_1(n) = -16\delta(n) - 10(0.5)^n u(n) + 27(0.25)^n u(n)$$

4. $X_4(z) = z/(z^3 + 2z^2 + 1.25z + 0.25)$, $|z| > 1$. Consider

$$X_4(z) = \frac{z}{z^3 + 2z^2 + 1.25z + 0.25} = \frac{z^{-2}}{1 + 2z^{-1} + 1.25z^{-2} + 0.25z^{-3}}$$

MATLAB script for the PFE:

```
% P0611d: Inverse z-Transform of X4(z)
clc; close all;
b4 = [0,0,1]; a4 = [1,2,1.25,0.25];
[R,p,k] = residuez(b4,a4)
 R =
    4.0000
   -0.0000 + 0.0000i
   -4.0000 - 0.0000i
 p =
   -1.0000
   -0.5000 + 0.0000i
   -0.5000 - 0.0000i
 k =
      []
```

or

$$X_4(z) = \frac{4}{1 + z^{-1}} - \frac{4}{(1 + 0.5z^{-1})^2} = \frac{4}{1 + z^{-1}} - 8z\frac{0.5z^{-1}}{(1 + 0.5z^{-1})^2}, \ |z| > 1$$

Hence

$$x_4(n) = 4(-1)^n u(n) - 8(n + 1)(0.5)^{n+1} u(n + 1)$$

P4.21 A digital filter is described by the frequency response function

$$H(e^{j\omega}) = [1 + 2\cos(\omega) + 3\cos(2\omega)]\cos\left(\frac{\omega}{2}\right)e^{-j5\omega/2}$$

1. Determine the difference equation representation.
2. Using the **freqz** function, plot the magnitude and phase of the frequency response of the filter. Note the magnitude and phase at $\omega = \pi/2$ and at $\omega = \pi$.
3. Generate 200 samples of the signal $x(n) = \sin(\pi n/2) + 5\cos(\pi n)$, and process through the filter to obtain $y(n)$. Compare the steady-state portion of $y(n)$ to $x(n)$. How are the amplitudes and phases of two sinusoids affected by the filter?

<u>Note:</u>

For Part (1), you can use Euler equation along with the relationship between DTFT and Z-transform, i.e., the relationship between z and ω, to find the associated Transfer Function H(z) and from that, obtain the LCCDE.

For Part (3), you can use the "filter" function in MATLAB.

Solution:

A digital filter is described by the frequency response function

$$H(e^{j\omega}) = [1 + 2\cos(\omega) + 3\cos(2\omega)]\cos(\omega/2)e^{-j5\omega/2}$$

which can be written as

$$\begin{aligned} H(e^{j\omega}) &= \left[1 + 2\left(\frac{e^{j\omega} + e^{-j\omega}}{2}\right) + 3\left(\frac{e^{j2\omega} + e^{-j2\omega}}{2}\right)\right]\frac{e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega}}{2}e^{-j\frac{5}{2}\omega} \\ &= \frac{3}{4} + \frac{5}{4}e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega} + \frac{5}{4}e^{-j4\omega} + \frac{3}{4}e^{-j5\omega} \end{aligned}$$

or after substituting $e^{-j\omega} = z^{-1}$, we obtain

$$H(z) = \frac{3}{4} + \frac{5}{4}z^{-1} + z^{-2} + z^{-3} + \frac{5}{4}z^{-4} + \frac{3}{4}z^{-5}$$

1. The difference equation representation: From $H(z)$ above

$$y(n) = \frac{3}{4}x(n) + \frac{5}{4}x(n-1) + x(n-2) + x(n-3) + \frac{5}{4}x(n-4) + \frac{3}{4}x(n-5)$$

2. The magnitude and phase response plots are shown in Figure 4.22.

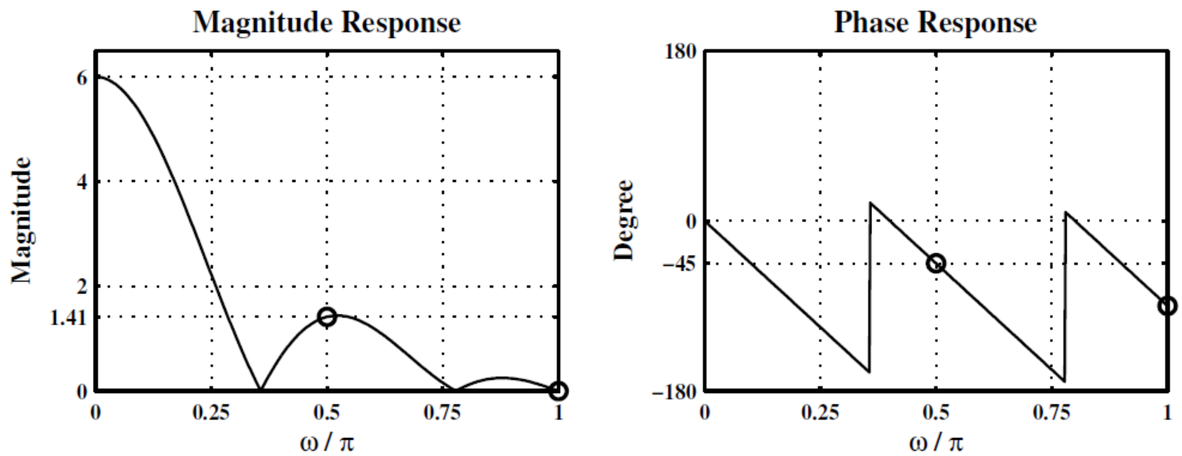

Figure 4.22: Problem P4.21.2 frequency-response plots

The magnitude and phase at $\omega = \pi/2$ are $\sqrt{2}$ and $-45°$, respectively. The magnitude at $\omega = \pi$ is zero.

3. The output sequence $y(n)$ for the input $x(n) = \sin(\pi n/2) + 5\cos(\pi n)$: MATLAB script:

```
clc; close all;  set(0,'defaultfigurepaperposition',[0,0,7,5]);
b = [3/4 5/4 1 1 5/4 3/4]; a = [1 0];
n = 0:200; x = sin(pi*n/2)+5*cos(pi*n); y = filter(b,a,x);
Hf_1 = figure; set(Hf_1,'NumberTitle','off','Name','P0421c');
subplot(2,1,1); Hs = stem(n,x); set(Hs,'markersize',2); axis([-2 202 -7 6]);
xlabel('n','FontSize',LFS); ylabel('x(n)','FontSize',LFS);
title('x(n) = sin(\pi \times n / 2)+5 \times cos(\pi \times n)',...
      'FontSize',TFS);
subplot(2,1,2); Hs = stem(n,y); set(Hs,'markersize',2); axis([-2 202 -2 4]);
xlabel('n','FontSize',LFS); ylabel('y(n)','FontSize',LFS);
title('Output sequence after filtering','FontSize',TFS);
print -deps2 ../epsfiles/P0421c;
```

The input and output sequence plots are shown in Figure 4.23. It shows that the sinusoidal sequence with the input frequency $\omega = \pi$ is completely suppressed in the steady-state output. The steady-state response of $x(n) = sin(\pi n/2)$ should be (using the magnitude and phase at $\omega = \pi/2$ computed in part 2. above)

$$\begin{aligned} y_{ss}(n) &= \sqrt{2}\sin(\pi n/2 - 45°) = \sqrt{2}\cos(45°)\sin(\pi n/2) - \sqrt{2}\sin(45°)\cos(\pi n/2) \\ &= sin(\pi n/2) - cos(\pi n/2) = \{\ldots, \underset{\uparrow}{-1}, 1, 1, -1, -1, \ldots\} \end{aligned}$$

as verified in the bottom plot of Figure 4.23.

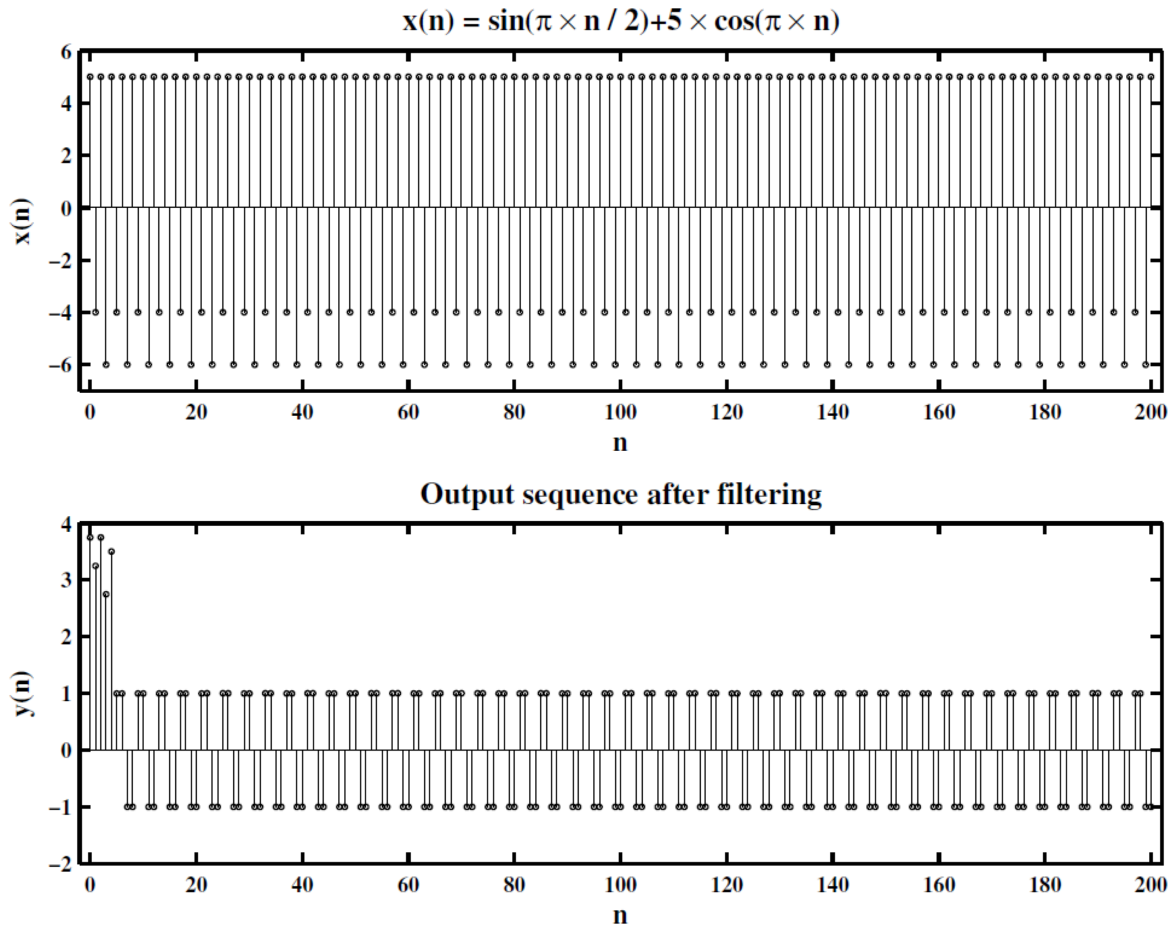

Figure 4.23: Problem P4.21.3 input and output sequence plots