hw3

1

Problem 1: Problem 6.4 in R1 (i.e., Proakis 4th Edition)

(Note: To obtain the Fourier Transform of *x(n)=xa(nT)* in this problem, recall the "differentiation in the frequency domain" property of FT, i.e., $nx(n) \longleftrightarrow jdX(f)/(2\pi df)$)

6.4 Repeat Example 6.1.2 for the signal $x_a(t) = te^{-t}u_a(t)$.

EXAMPLE 6.1.2 **Sampling and Reconstruction of a Nonbandlimited Signal**

Consider the following continuous-time two-sided exponential signal:

$$x_a(t) = e^{-A|t|} \overset{\mathcal{F}}{\longleftrightarrow} X_a(F) = \frac{2A}{A^2 + (2\pi F)^2}, \quad A > 0$$

(a) Determine the spectrum of the sampled signal $x(n) = x_a(nT)$. **(b)** Plot the signals $x_a(t)$ and $x(n) = x_a(nT)$, for $T = 1/3$ sec and $T = 1$ sec, and their spectra. **(c)** Plot the continuous-time signal $\hat{x}_a(t)$ after reconstruction with ideal bandlimited interpolation.

Solution.

(a) If we sample $x_a(nT)$ with a sampling frequency $F_s = 1/T$, we have

$$x(n) = x_a(nT) = e^{-AT|n|} = (e^{-AT})^{|n|}, \quad -\infty < n < \infty$$

The spectrum of $x(n)$ can be found easily if we use a direct computation of the discrete-time Fourier transform. We find that

$$X(F) = \frac{1 - a^2}{1 - 2a\cos 2\pi(F/F_s) + a^2}, \quad a = e^{-AT}$$

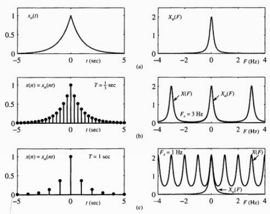


Figure 6.1.7 (a) Analog signal $x_a(t)$ and its spectrum $X_a(F)$; (b) $x(n) = x_a(nT)$ and its spectrum for $F_s = 3$ Hz; and (c) $x(n) = x_a(nT)$ and its spectrum for $F_s = 1$ Hz.

Clearly, since $\cos 2\pi(F/F_s)$ is periodic with period F_s, so is the spectrum $X(F)$.

(b) Since $X_a(F)$ is not bandlimited, aliasing cannot be avoided. Figure 6.1.7(a) shows the original signal $x_a(t)$ and its spectrum $X_a(F)$ for $A = 1$. The sampled signal $x(n)$ and its spectrum $X(F)$ are shown for $F_s = 3$ Hz and $F_s = 1$ Hz in Figures 6.1.7(b) and 6.1.7(c). The aliasing distortion is clearly noticeable in the frequency domain when $F_s = 1$ Hz and almost unnoticeable when $F_s = 3$ Hz.

(c) The spectrum $\hat{X}_a(F)$ of the reconstructed signal $\hat{x}_a(t)$ is given by

$$\hat{X}_a(F) = \begin{cases} TX(F), & |F| \le F_s/2 \\ 0, & \text{otherwise} \end{cases}$$

The values of $\hat{x}_a(t)$ can be evaluated for plotting purposes using the ideal bandlimited interpolation formula (6.1.20) for all significant values of $x(n)$ and $\sin(\pi t/T)/(\pi t/T)$. Figure 6.1.8 illustrates the reconstructed signal and its spectrum for $F_s = 3$ Hz and $F_s = 1$ Hz. It is interesting to note that in every case $\hat{x}_a(nT) = x_a(nt)$, but $\hat{x}_a(nt) \ne x_a(nT)$ for $t \ne nT$. The results of aliasing are clearly evident in the spectrum of $\hat{x}_a(t)$ for $F_s = 1$ Hz, where we note how the folding of the spectrum about $F = \pm 0.5$ Hz increases the high frequency content of $\hat{X}_a(F)$.

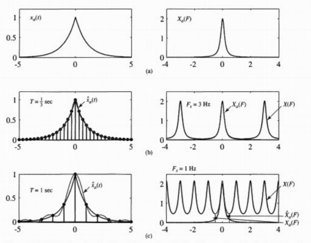


Figure 6.1.8 (a) Analog signal $x_a(t)$ and its spectrum $X_a(F)$; (b) reconstructed signal $\hat{x}_a(t)$ and its spectrum for $F_s = 3$ Hz; and (c) reconstructed signal $\hat{x}_a(t)$ and its spectrum for $F_s = 1$ Hz

a

$$X_a(t) = te^{-t} U_a(t)$$

$$X(n) = X_a(nT)$$

$$= nTe^{-nT} U_a(n)$$

$$X(n) \overset{\mathcal{F}}{\longleftrightarrow} X(F)$$

$$nTx_1(n) \overset{\mathcal{F}}{\longleftrightarrow} T\frac{j}{2\pi}\frac{d}{dF}X(F)$$

$$X(F) = \sum_{n=0}^{\infty} e^{nT} e^{-2\pi j F n}$$

$$= \frac{1}{1 - e^{-T}e^{-2\pi j F}}$$

$$= T\frac{j}{2\pi}\frac{\partial}{\partial F}\frac{1}{1 - e^{-T}e^{-2\pi j F}}$$

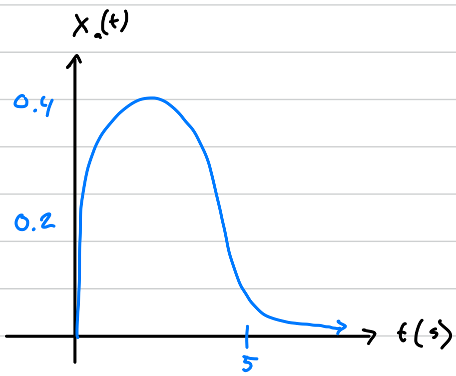
$$= T\frac{j}{2\pi}\frac{\partial}{\partial F}\frac{-2j\pi e^{2j\pi F}e^{T}}{(e^{2j\pi F}\cdot e^{T} - 1)^2}$$

$$= \frac{Te^{2j\pi F}\cdot e^{T}}{(e^{2j\pi F}\cdot e^{T} - 1)^2}$$

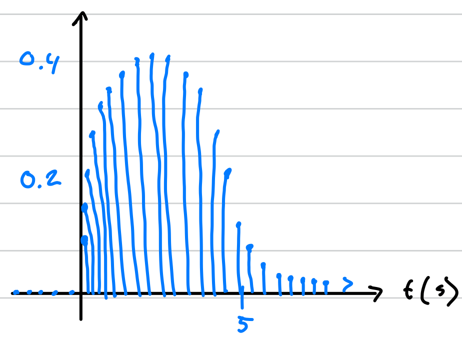
(b) Plot $x(n)$ & $x(n) = x_a(nT)$

$$t e^{-at} u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(\alpha + j2\pi f)^2} \quad // \quad X_a(F) = \frac{1}{(1+2\pi j F)^2}$$

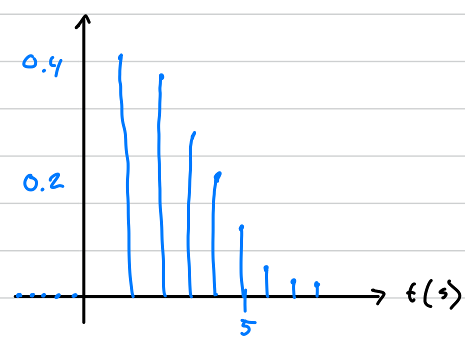
$$x(n) = \frac{n}{3} e^{-n/3} u_a\left(\frac{n}{3}\right) \qquad x(n) = n e^{-n} u_a(n)$$



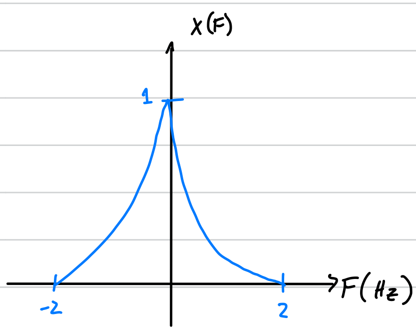
$x_a(nT) \rightarrow T = 1/3$

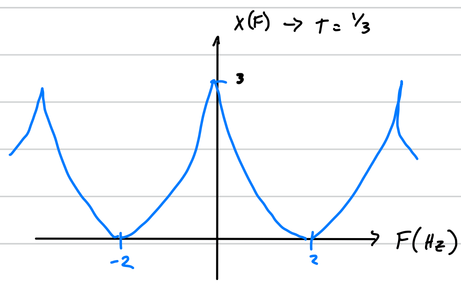


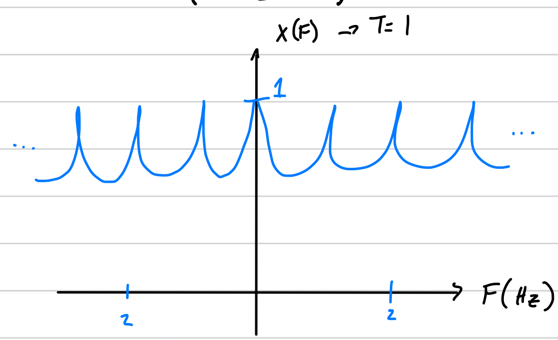
$x_a(nT) \rightarrow T = 1$



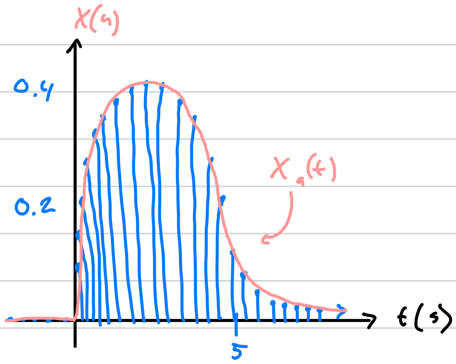
$$X(F) = \frac{1}{3}\left[\frac{e^{-1/3} e^{-2\pi j F}}{(1 - e^{-1/3} e^{-2\pi j F})^2}\right] \qquad X(F) = \frac{e^{-1} e^{-2\pi F}}{(1 - e^{-1} e^{-2\pi j F})^2}$$



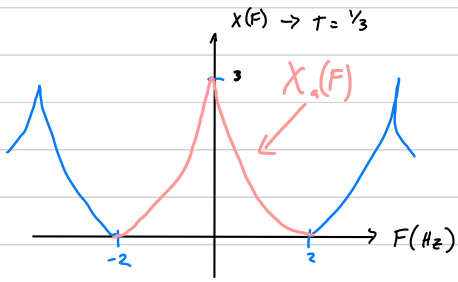




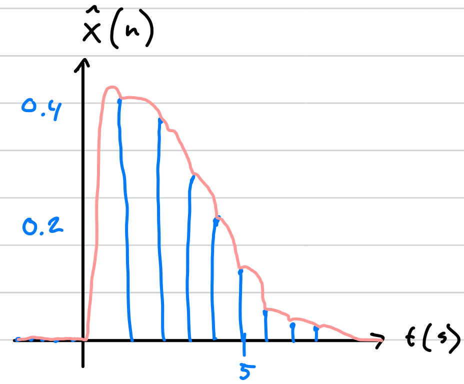
c) Reconstructed signal when T=1

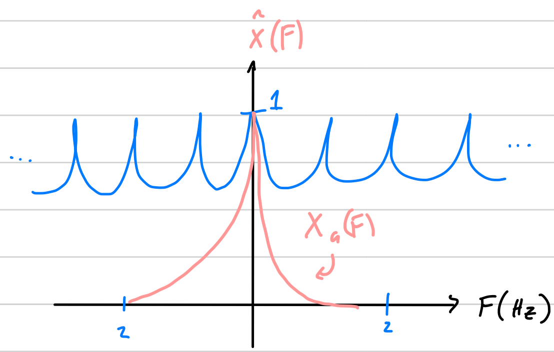


$$X(F) = \frac{1}{3}\left[\frac{e^{-1/3}e^{-2\pi jF}}{(1-e^{-1/3}e^{-2\pi jF})^2}\right]$$



Reconstructed signal when T=1





Problem 2:

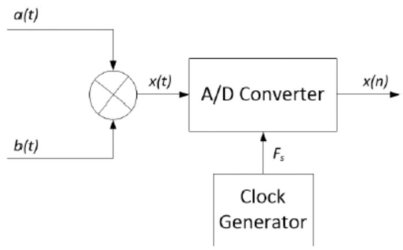
2

[redacted] Consider the two CW tones given by:

$$a(t) = \cos(4000\pi t)$$

$$b(t) = \cos(200\pi t)$$

These two tones are mixed (i.e., multiplied) and then sampled as shown in the following figure. What would be the minimum sampling rate, F_s, measured in Hz, that would result in a sequence $x(n)$ without any aliasing errors (i.e., no spectral replication overlap)?



$$a(t) = \cos(4000\pi t)$$
$$b(t) = \cos(200\pi t)$$

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\begin{aligned} x(t) &= a(t) \times b(t) \\ &= \tfrac{1}{2}\left(2\cos(4000\pi t)\cos(200\pi t)\right) \\ &= \tfrac{1}{2}\left(\cos(4000\pi t + 200\pi t) + \cos(4000\pi t - 200\pi t)\right) \\ &= \tfrac{1}{2}\left(\cos(4200\pi t) + \cos(3800\pi t)\right) \end{aligned}$$

$\omega_m = 4200\pi$ & $F_m = 2100$ Hz

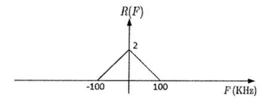
Nyquist $F_s \geq 2F_m$

$F_s \geq 4200$ Hz

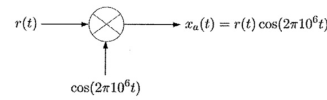
$\min(F_s) = 4200$ Hz

Problem 3:

Consider an information-bearing signal, $r(t)$, with the following frequency spectrum:

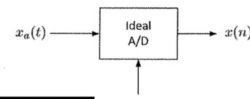


The signal $r(t)$ is then modulated onto a carrier with frequency 1.0 MHz:



(a) [redacted] Sketch the frequency spectrum of $x_a(t)$ (i.e., $X_a(F)$). Please show the values for all the important points on both axes.

(b) [redacted] Now, assume $x_a(t)$ is sampled as shown below. What would be the minimum *bandpass* sampling rate, F_s, in KHz, in order to avoid any aliasing error?

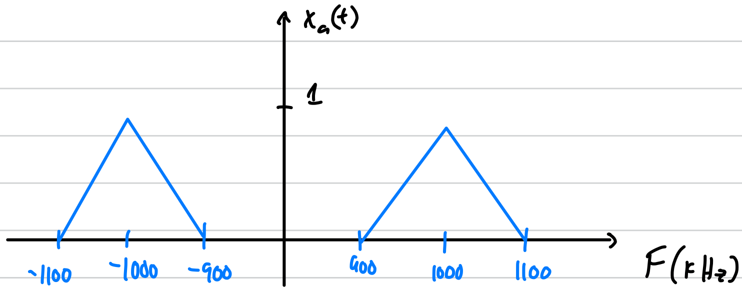


[redacted]

(c) [redacted] Now, assume $F_s = 800$ KHz. Sketch the frequency spectrum, $X(\omega)$, for $-2\pi \le \omega \le 2\pi$. Please show the values for all the important points on both axes. Also please show values on the frequency axis in both KHz and rad/sample scales.

3 (a) $\frac{1}{2}\left(R(F-F_0) + R(F+F_0)\right)$

$x_a(t) = r(t)\cdot\cos(2\pi 10^6 t) \longleftrightarrow X_a(F) = 0.5\left[(f - 10^6\,\text{Hz}) + (f+10^6)\right]$



(b) Sampling Formula (Bandpass): $\frac{2F_H}{k} \le F_s \le \frac{2F_L}{k-1}$

$F_H = 1100$ kHz, $F_L = 900$ Hz, $B = 200$ kHz

$k_{max} = \frac{F_H}{B} = \frac{1100}{200} = 5$

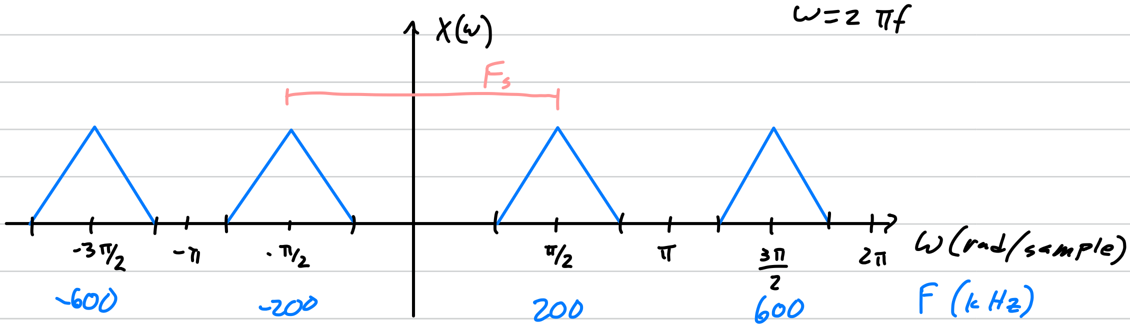
$F_{s,min} = \frac{2F_H}{k_{max}} = \frac{2200}{5} = 440$

@ $k = 5$ // $440\text{ kHz} \le F_s \le 450\text{ kHz}$

$\min(F_s) = 440$ kHz

(c)

$\omega = 2\pi f$



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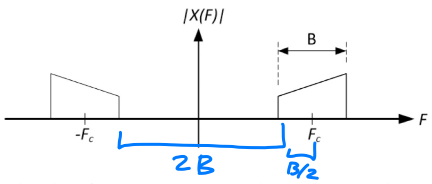
Problem 4:

a) Consider a band-pass signal with the following frequency spectrum. What would be the minimum center frequency F_c, in terms of the signal bandwidth B that would enable us to do bandpass sampling while avoiding any aliasing?



b) If a person wants to be classified as a soprano in classical opera, she must be able to sing notes in the frequency range of 247 Hz to 1175 Hz. Is *bandpass* sampling of full audio spectrum of a singing soprano possible? If yes, what would be the minimum F_s sampling rate allowable for bandpass sampling? If no, why not?

a From BP sampling inequalities

$$K_{max} = \left\lfloor \frac{F_H}{B} \right\rfloor = 2 \qquad \text{because } K = 1 \text{ is nyquist rate}$$

$$\frac{F_c + B/2}{B} = \frac{F_c}{B} + \frac{1}{2} = 2 \Rightarrow F_c = 3/2\, B$$

b Given $F_H = 1175$ Hz, $F_L = 247$ Hz

$$B = F_H - F_L = 1175 - 247 = 928 \text{ Hz}$$

$$K_{max} = \frac{\lfloor F_H \rfloor}{B} = \left\lfloor \frac{175}{928} \right\rfloor = 1$$

Because $K_{max} = 1$ Band pass Sampling is Not Possible

we must sample at the Nyquist Rate $2F_H = 2350$ Hz

There is not enough room from 0 to $F_L = 247$ Hz to put an entire bandwidth $B = 947$ Hz $\therefore$ it would cause aliasing.

Problem 5: Problem 11.1 in R1

11.1 An analog signal $x_a(t)$ is bandlimited to the range $900 \le F \le 1100$ Hz. It is used as an input to the system shown in Fig. P11.1. In this system, $H(\omega)$ is an ideal lowpass filter with cutoff frequency $F_c = 125$ Hz.

5

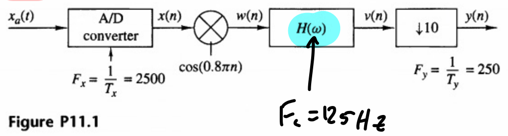
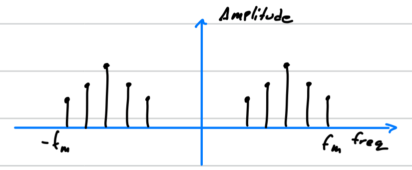
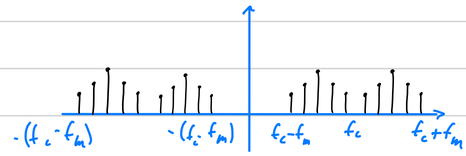


Figure P11.1





(a) Determine and sketch the spectra for the signals $x(n)$, $w(n)$, $v(n)$, and $y(n)$.

(b) Show that it is possible to obtain $y(n)$ by sampling $x_a(t)$ with period $T = 4$ milliseconds.

(a) $X_a(t) \rightarrow$ bandlimited to $900 \le F \le 1100$ Hz

$$f(t) = m(t) \times \cos(\omega c t)$$

$$F(f) = \frac{1}{2}\left[M(f - fc) + M(f + fc)\right]$$

$$X_a(\Omega) = 1/2\left[X_b(\Omega - 2000\pi) + X_b(\Omega + 2000\pi)\right]$$

normalize frequencies to F_x: $\omega' = \frac{\Omega}{F_x}$

DTFT $x(n)$

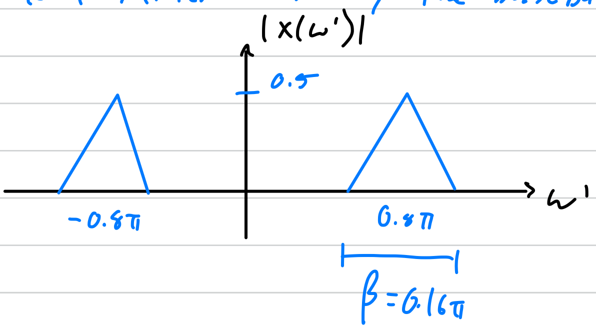
$$X(\omega') = \sum_{q=-\infty}^{\infty} X(\omega' - 2\pi q)$$

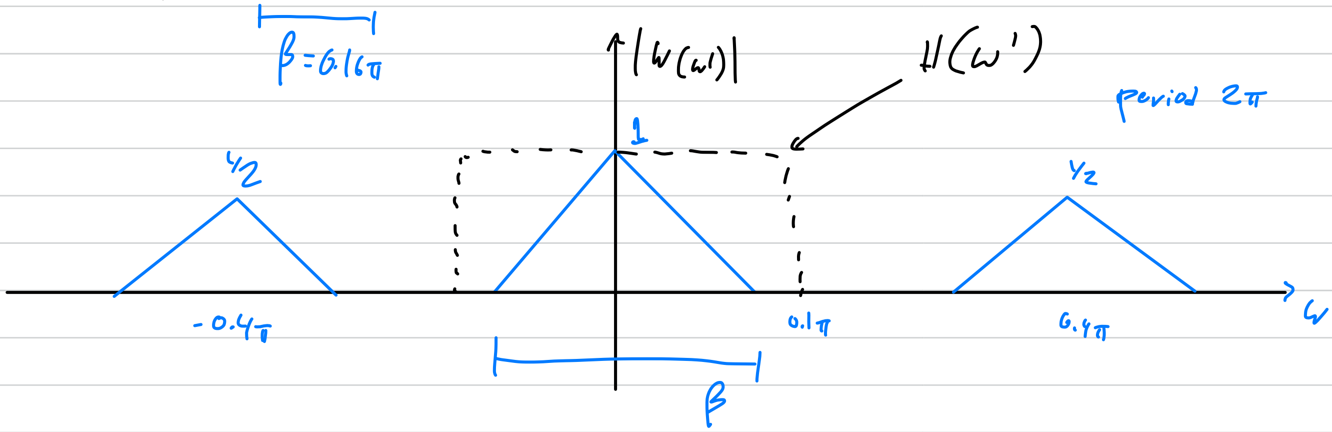
$$= \sum_{q=-\infty}^{\infty}\left[X_a(\omega' - 0.8\pi - 2\pi q) + X_b(\omega' + 0.8\pi - 2\pi q)\right]$$

modulate by $\cos(0.8\pi)$ scale by $1/2$

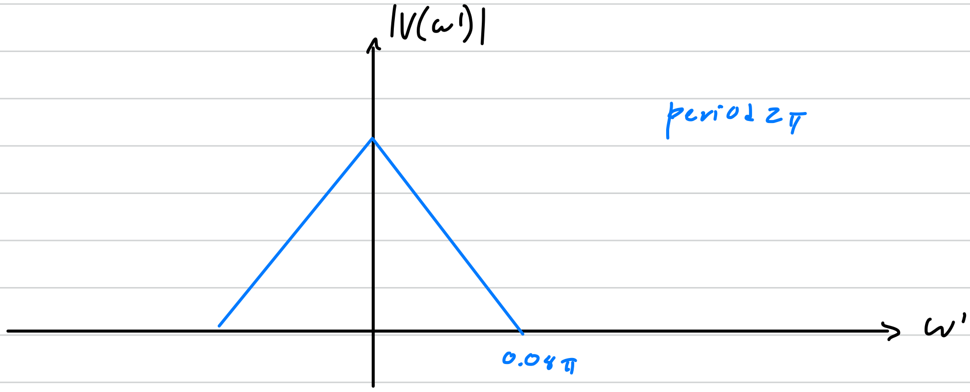
low pass filter saves only the baseband spectrum of ea. period

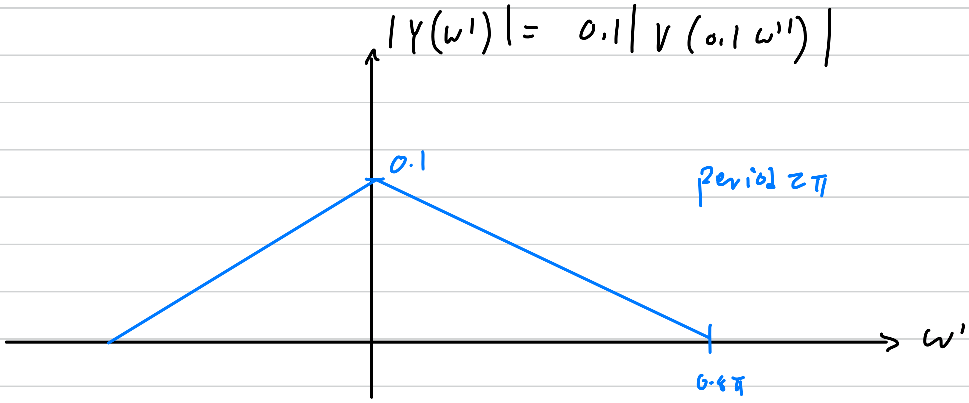
shift to 0.8





We assume band signal $X_a(\Omega)$ is limited to $-0.4\pi \le f \le 0.4\pi$
if we send $w(t)$ through the LPF we can get the baseband signal $X_a(\Omega)$





by downsampling we expand each $|V(\omega')|$ by a factor of M with the ω' axis. we also reduce gain by factor of M.

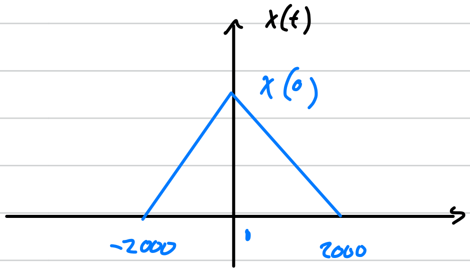
$$\omega' = \frac{\Omega}{F_y} \quad // \text{ then downsample}$$

$$\omega'' = \frac{\Omega D}{F_y} = \frac{\Omega}{F_y} \cdot 10 = 10\omega'$$

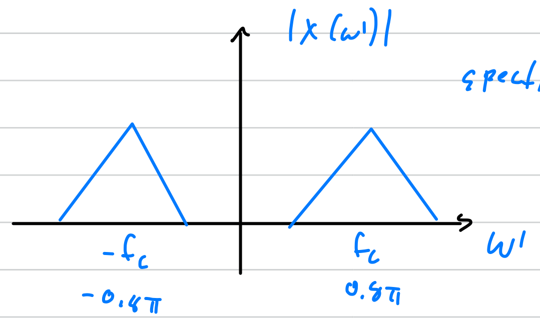
(b) $X_a(t)$ limited to $-2000 \leq f \leq 2000$

to get $|X(w')|$ for $x(f)$

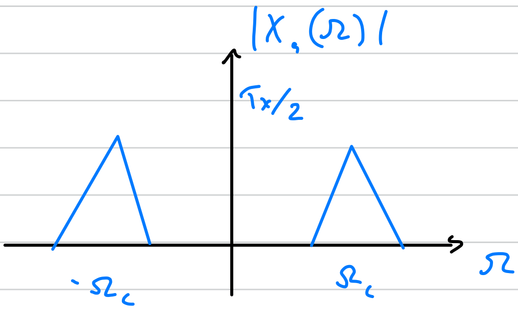
we changed the scale in the modulation which translates the base band signal by $\pm f_c$



base band signal



spectrum of $|X(w')|$



The sample rate is the same as F_y

$$\overset{s}{Y}(\Omega) = \frac{1}{T_y} \sum_{q} X_a(\Omega - q\Omega_y)$$

$$\Omega_y = 2\pi F_y$$

$$W'' = \Omega T_y$$

$$W'' = \Omega \frac{1}{F_y} = \Omega / F_y$$

$$\therefore \tilde{Y}(\omega'') = \frac{1}{T_y} \sum_{q} X_a(\omega'' - q2\pi) \quad \text{s.t. } \tilde{y}(n) = y(n)$$

6

Problem 6:

Given the input sequence $x_r(n)$, as shown in (a) in the following figure, whose magnitude spectrum is shown in (b) below, draw a rough sketch of the magnitude spectrum $|X_c(F)|$ of the system's complex output sequence: $x_c(m)=x_I(m)+jx_Q(m)$. The frequency magnitude responses of the *complex* bandpass filter $h_{BP}(k)$, and the real-valued highpass filter $h_{HP}(k)$ are shown in (c) and (d) below.

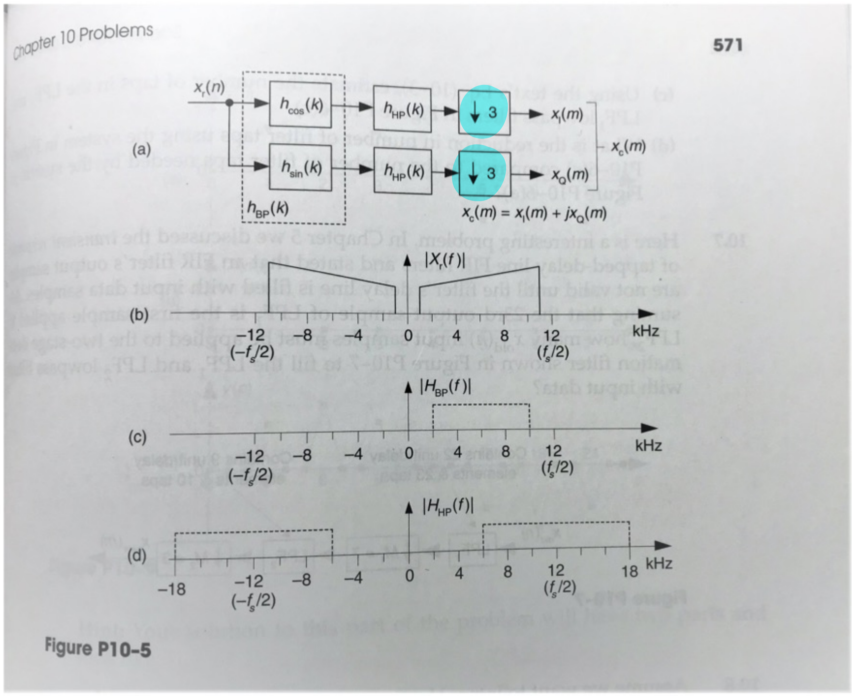
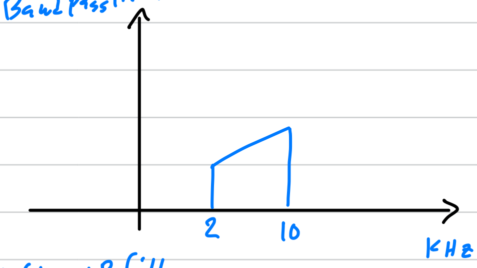


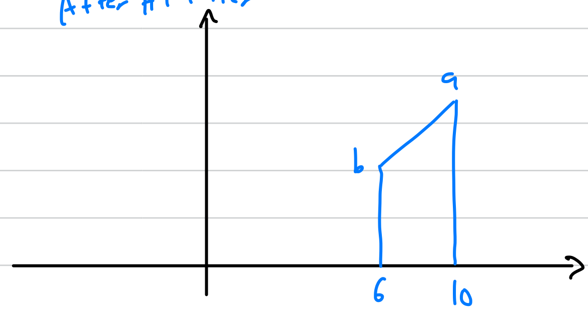
Figure P10-5

$X_c(m) = X_I(m) + j X_Q(m)$

Band Pass filter

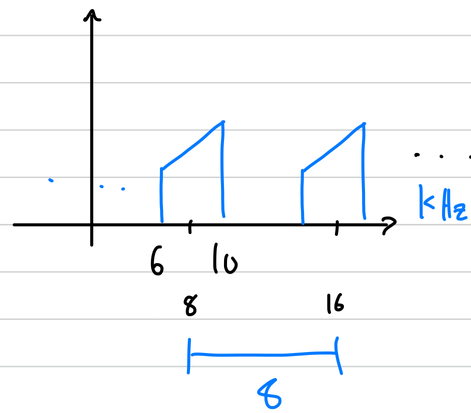


After HP Filter



Down Sample by 3

$X_c(m)$



homework 3 Matlab

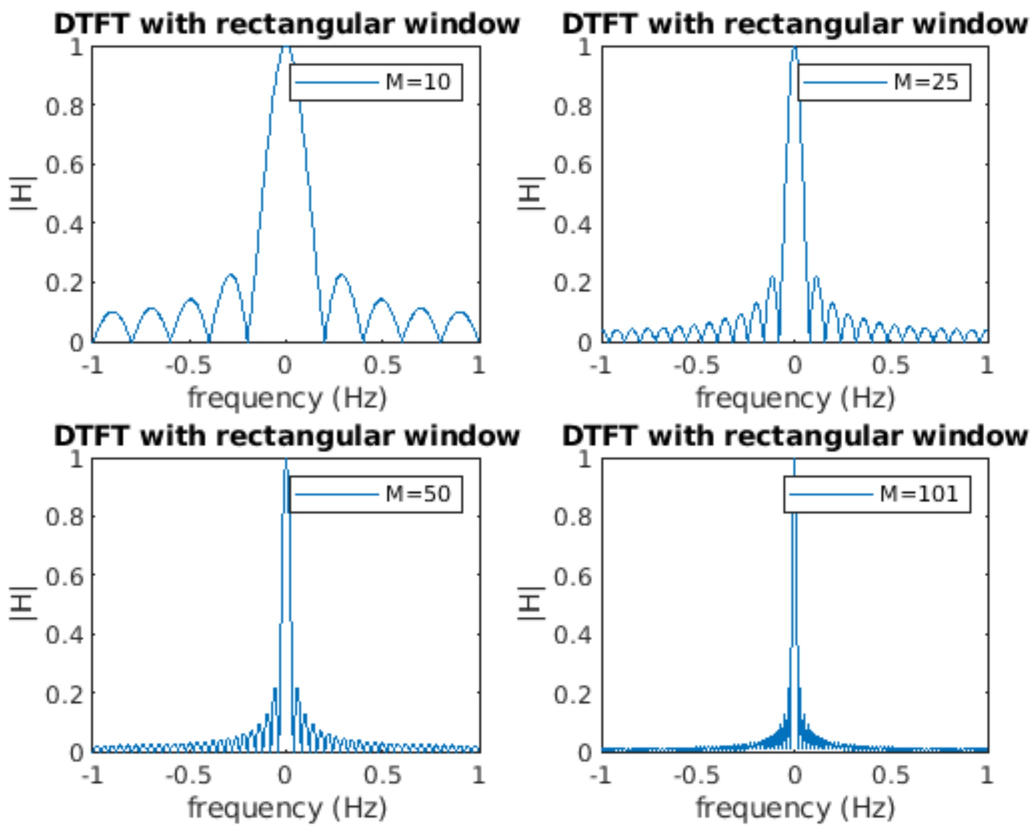
DTFT

```
function [X] = dtft(x,n,w)
% computes the discrete-time fourier transform
% X = dtft values computed at w frequencies
% n = sample position vector
% w = frequency location vector
    X = x * exp(-1j * n' * w);
end
```

Rectangular

```
i = 0;
for M = [10 25 50 101]
    n = 0:1:M-1;
    Rm = (n>=0) - (n>=M);
    w = linspace(-pi,pi,501);
    H = dtft(Rm,n,w);
    h_magnitude = abs(H)/max(abs(H));

%     plots
    i = i + 1;
    subplot(2,2,i);
    plot(w/pi,h_magnitude);
    xlabel('frequency (Hz)');
    ylabel('|H|');
    title('DTFT with rectangular window');
    legend(sprintf('M=%d',M));
end
```

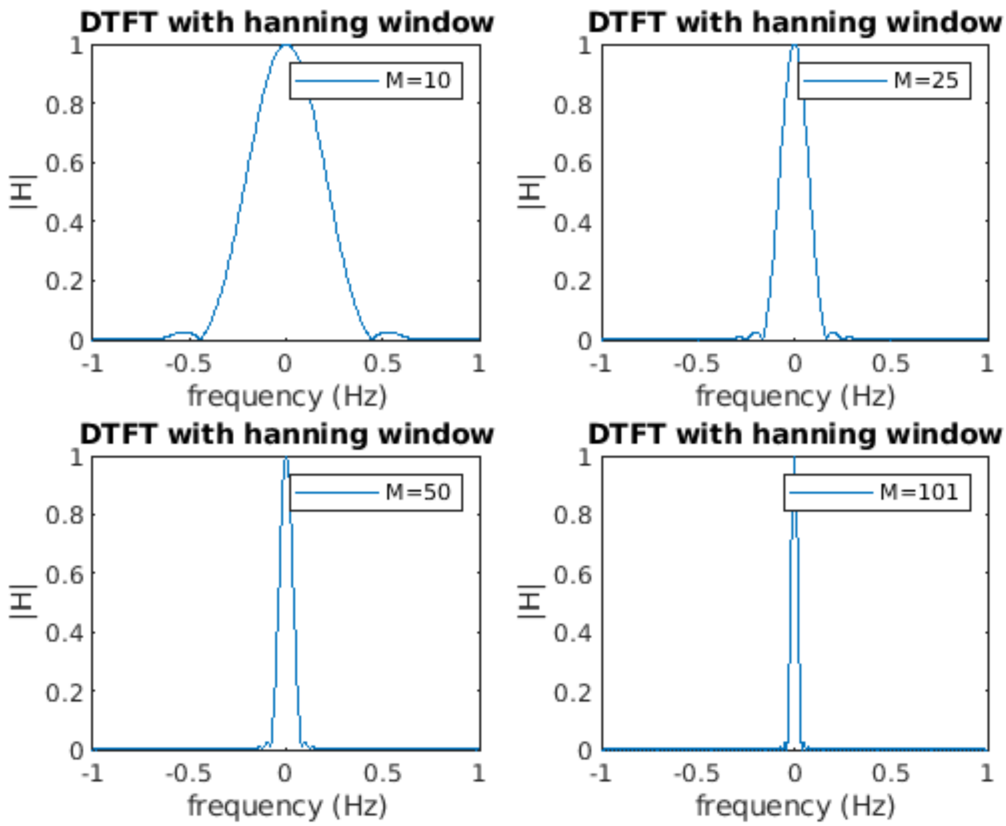


Hanning

```
i = 0;
for M = [10 25 50 101]
    n = 0:1:M-1;
    Rm = (n>=0)-(n>=M);
    Cm = 0.5* (1-cos((2*pi*n)/(M-1))) .* Rm;
    w = linspace(-pi,pi,501);
    H = dtft(Cm,n,w);
    h_magnitude = abs(H)/max(abs(H));

%     plots
    i = i + 1;
    subplot(2,2,i);
    plot(w/pi,h_magnitude);
    xlabel('frequency (Hz)');
    ylabel('|H|');
    title('DTFT with hanning window');
```

```
    legend(sprintf('M=%d',M));
end
```

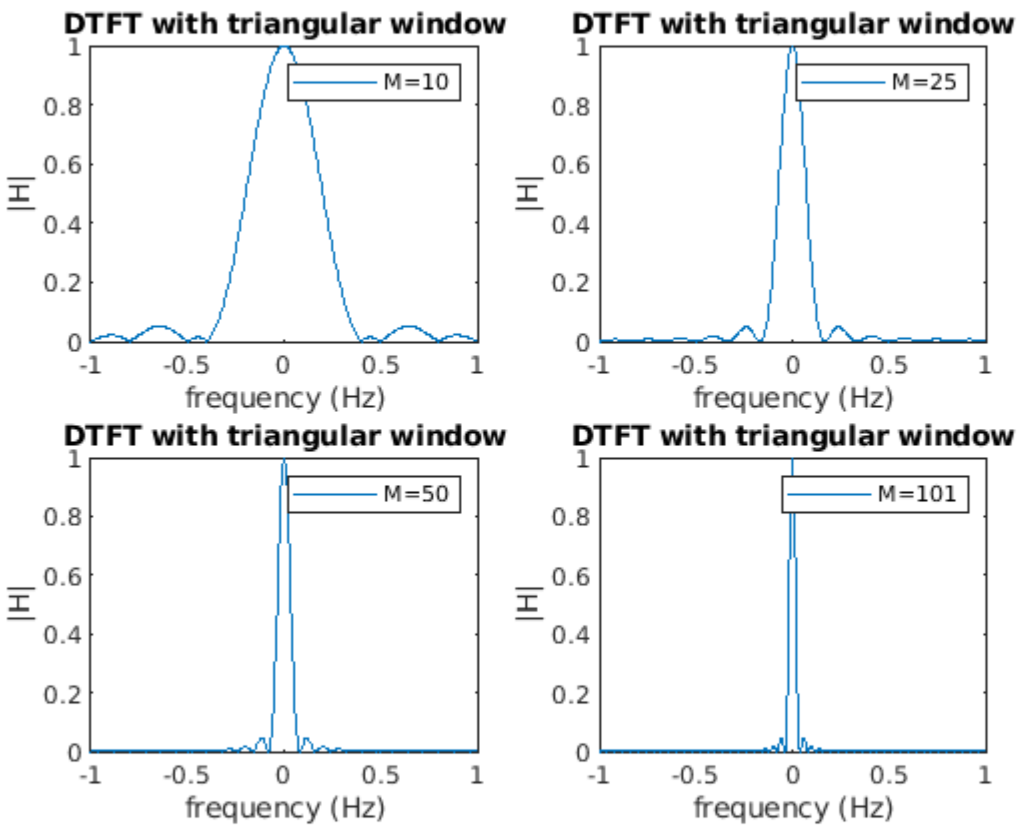


Triangle

```
i = 0;
for M = [10 25 50 101]
    n = 0:1:M-1;
    Rm = (n>=0) - (n>=M);
    Tm = (1-(abs(M-1-2*n)/(M-1))) .* Rm;
    w = linspace(-pi,pi,501);
    H = dtft(Tm,n,w);
    h_magnitude = abs(H)/max(abs(H));

%     plots
    i = i + 1;
    subplot(2,2,i);
    plot(w/pi,h_magnitude);
    xlabel('frequency (Hz)');
    ylabel('|H|');
```

```
    title('DTFT with triangular window');
    legend(sprintf('M=%d',M));
end
```



Hamming

```
i = 0;
for M = [10 25 50 101]
    n = 0:1:M-1;
    Rm = (n>=0) - (n>=M);
    Hm = (0.54 - 0.46 * cos((2*pi*n)/(M-1))) .* Rm;
    w = linspace(-pi,pi,501);
    H = dtft(Hm,n,w);
    h_magnitude = abs(H)/max(abs(H));

%     plots
    i = i + 1;
    subplot(2,2,i);
    plot(w/pi,h_magnitude);
    xlabel('frequency (Hz)');
```

```
    ylabel('|H|');
    title('DTFT with Hamming window');
    legend(sprintf('M=%d',M));
end
```