ECE113: DSP

Homework 5 Solutions

Problem 1: Problem 7.4 in R1

Solution:

(a)

$$
\begin{aligned}
x_1(n) &= \frac{1}{2}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right) \\
X_1(k) &= \frac{N}{2}\left[\delta(k-1) + \delta(k+1)\right] \\
\text{also } X_2(k) &= \frac{N}{2j}\left[\delta(k-1) - \delta(k+1)\right] \\
\text{So } X_3(k) &= X_1(k)X_2(k) \\
&= \frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right] \\
\text{and } x_3(n) &= \frac{N}{2} sin(\frac{2\pi}{N}n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
\tilde{R}_{xy}(k) &= X_1(k)X_2^*(k) \\
&= \frac{N^2}{4j}\left[\delta(k-1) - \delta(k+1)\right] \\
\Rightarrow \quad \tilde{r}_{xy}(n) &= -\frac{N}{2} sin(\frac{2\pi}{N}n)
\end{aligned}
$$

(c)

$$\begin{aligned} \tilde{R}_{xx}(k) &= X_1(k)X_1^*(k) \\ &= \frac{N^2}{4}\left[\delta(k-1)+\delta(k+1)\right] \\ \Rightarrow \quad \tilde{r}_{xx}(n) &= \frac{N}{2}cos(\frac{2\pi}{N}n) \end{aligned}$$

(d)

$$\begin{aligned} \tilde{R}_{yy}(k) &= X_2(k)X_2^*(k) \\ &= \frac{N^2}{4}\left[\delta(k-1)+\delta(k+1)\right] \\ \Rightarrow \quad \tilde{r}_{yy}(n) &= \frac{N}{2}cos(\frac{2\pi}{N}n) \end{aligned}$$

Problem 2: Problem 7.8 in R1

Solution:

$$\begin{aligned} y(n) &= x_1(n) \, 4 \, x_2(n) \\ &= \sum_{m=0}^{3} x_1(m)_{\text{mod}4} x_2(n-m)_{\text{mod}4} \\ &= \{17, 19, 22, 19\} \end{aligned}$$

Problem 3: Problem 7.9 in R1

Solution:

$$\begin{aligned} X_1(k) &= \{7, -2-j, 1, -2+j\} \\ X_2(k) &= \{11, 2-j, 1, 2+j\} \\ \Rightarrow X_3(k) &= X_1(k)X_2(k) \\ &= \{17, 19, 22, 19\} \end{aligned}$$

Problem 4: Problem 8.13 in R1

Solution:

(a) "gain" $= W_8^0 W_8^0(-1)W_8^2 = -W_8^2 = j$
(b) Given a certain output sample, there is one path from every input leading to it. This is true for every output.
(c) $X(3) = x(0) + W_8^3 x(1) - W_8^2 x(2) + W_8^2 W_8^3 x(3) - W_8^0 x(4) - W_8^0 W_8^3 x(5) + W_8^0 W_8^2 x(6) + W_8^0 W_8^2 W_8^3 x(7)$

Problem 5: Problem 8.17 in R1

Solution:

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$\text{Hence, } X(z_k) = \sum_{n=0}^{N-1} x(n) r^{-n} e^{-j\frac{2\pi}{N}kn}$$

where $z_k = re^{-j\frac{2\pi}{N}k}, k = 0, 1, \ldots, N-1$ are the N sample points. It is clear that $X(z_k), k = 0, 1, \ldots, N-1$ is equivalent to the DFT (N-pt) of the sequence $x(n)r^{-n}, n \in [0, N-1]$.

Problem 6: Problem 3.2 ((b) and (h) only) in R1

Solution:

(b)

$$\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(a^n + a^{-n})z^{-n} \\
&= \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=0}^{\infty} a^{-n} z^{-n} \\
\text{But } \sum_{n=0}^{\infty} a^n z^{-n} &= \frac{1}{1 - az^{-1}} \text{ ROC: } |z| > |a| \\
\text{and } \sum_{n=0}^{\infty} a^{-n} z^{-n} &= \frac{1}{(1 - \frac{1}{a}z^{-1})^2} \text{ ROC: } |z| > \frac{1}{|a|} \\
\text{Hence, } X(z) &= \frac{1}{1 - az^{-1}} + \frac{1}{1 - \frac{1}{a}z^{-1}} \\
&= \frac{2 - (a + \frac{1}{a})z^{-1}}{(1 - az^{-1})(1 - \frac{1}{a}z^{-1})} \text{ ROC: } |z| > \max(|a|, \frac{1}{|a|})
\end{aligned}$$

(h)

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty}(\frac{1}{2})^n z^{-n} - \sum_{n=10}^{\infty}(\frac{1}{2})^n z^{-n} \\
&= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{(\frac{1}{2})^{10}z^{-10}}{1-\frac{1}{2}z^{-1}} \\
&= \frac{1-(\frac{1}{2}z^{-1})^{10}}{1-\frac{1}{2}z^{-1}}, \qquad |z| > \frac{1}{2}
\end{aligned}
$$

The pole-zero patterns are as follows:

(b) Poles at $z = a$ and $z = \frac{1}{a}$. Zeros at $z = 0$ and $z = \frac{1}{2}(a + \frac{1}{a})$.

(h) $X(z)$ has a pole of order 9 at $z = 0$. For nine zeros which we find from the roots of

$$
\begin{aligned}
1-(\frac{1}{2}z^{-1})^{10} &= 0 \\
\text{or, equivalently, } (\frac{1}{2})^{10} - z^{10} &= 0 \\
\text{Hence, } z_n &= \frac{1}{2}e^{\frac{j2\pi n}{10}}, n = 1, 2, \dots, k.
\end{aligned}
$$

Note the pole-zero cancellation at $z = \frac{1}{2}$.

Problem 7: Problem 3.4 ((b) only) in R1

Solution:

(b)

$$
\begin{aligned}
X(z) &= \sum_{n=0}^{\infty} n^2 z^{-n} \\
&= z^2 \frac{d^2}{dz^2} \sum_{n=0}^{\infty} z^{-n} \\
&= z^2 \frac{d^2}{dz^2}\left[\frac{1}{1-z^{-1}}\right] \\
&= -\frac{z^{-1}}{(1-z^{-1})^2} + \frac{2z^{-1}}{(1-z^{-1})^3} \\
&= \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}, |z| > 1
\end{aligned}
$$

Problem 8: Problem 3.8 in R1

Solution:

(a)

$$
\begin{aligned}
y(n) &= \sum_{k=-\infty}^{n} x(k) \\
&= \sum_{k=-\infty}^{\infty} x(k)u(n-k) \\
&= x(n) * u(n) \\
Y(z) &= X(z)U(z) \\
&= \frac{X(z)}{1-z^{-1}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
u(n) * u(n) &= \sum_{k=-\infty}^{\infty} u(k)u(n-k) \\
&= \sum_{k=-\infty}^{n} u(k) = (n+1)u(n) \\
\text{Hence,} x(n) &= u(n) * u(n) \\
\text{and} X(z) &= \frac{1}{(1-z^{-1})^2}, |z| > 1
\end{aligned}
$$

Problem 9: Problem 3.9 in R1

Solution:

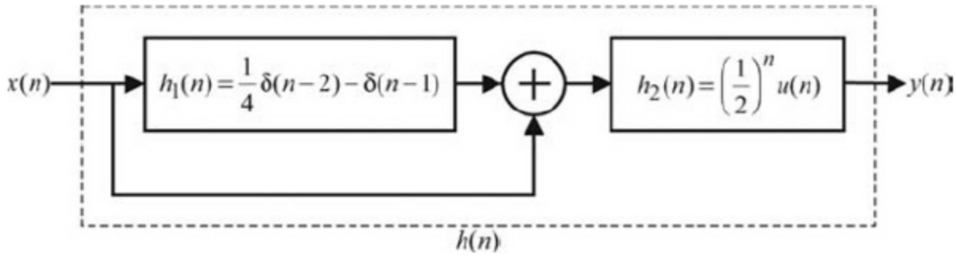
$y(n) = x(n)e^{jw_0 n}$. From the scaling theorem, we have $Y(z) = X(e^{-jw_0}z)$. Thus, the poles and zeros are phase rotated by an angle w_0.

Problem 10:

Consider the system shown below.

1. Using the z-transform approach, show that the impulse response, $h(n)$, of the overall system is given by

$$
h(n) = \delta(n) - \frac{1}{2}\delta(n-1)
$$



2. Determine the difference equation representation of the overall system that relates the output $y(n)$ to the input $x(n)$.
3. Is this system causal? BIBO stable? Explain clearly to receive full credit.
4. Determine the frequency response $H(e^{j\omega})$ of the overall system.
5. Using MATLAB, provide a plot of this frequency response over $0 \leq \omega \leq \pi$.

Solution:

1. The overall system impulse response, $h(n)$, using the z-transform approach: The above system is given by

$$H(z) = H_2(z)\left[1 + H_1(z)\right] = \frac{1}{1-0.5z^{-1}}\left[1 + 0.25z^{-2} - z^{-1}\right]$$
$$= \frac{\left(1-0.5z^{-1}\right)^2}{1-0.5z^{-1}} = 1 - 0.5z^{-1},\ |z| \neq 0$$

Hence after taking inverse z-transform , we obtain

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

2. Difference equation representation of the overall system: From the overall system function $H(z)$,

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} \Rightarrow y(n) = x(n) - 0.5x(n-1)$$

3. Causality and stability: Since $h(n) = 0$ for $n < 0$, the system is causal. Since $h(n)$ is of finite duration (only two samples), $h(n)$ is absolutely summable. Hence BIBO stable.
4. Frequency response $H\left(e^{j\omega}\right)$ of the overall system.

$$H\left(e^{j\omega}\right) = \mathcal{F}\left[h(n)\right] = \mathcal{F}\left[\delta(n) - \frac{1}{2}\delta(n-1)\right] = 1 - \frac{1}{2}e^{-j\omega}$$

5. Frequency response over $0 \leq \omega \leq \pi$ is shown in Figure 4.11.

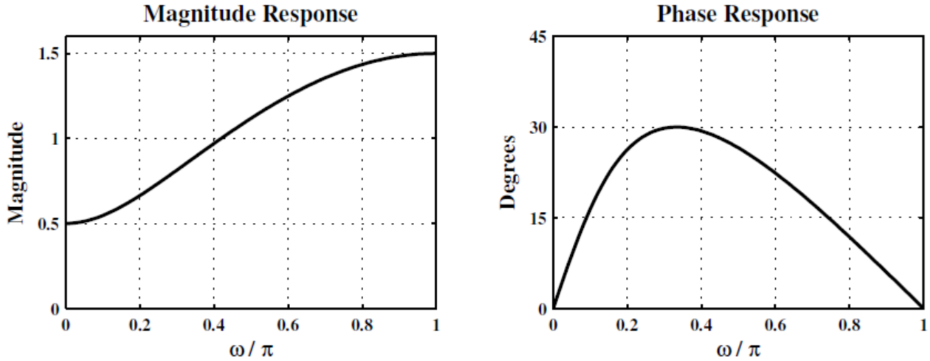


Figure 4.11: Problem P4.16 frequency-response plot

<u>MATLAB:</u>

For this HW, you do not need to submit anything. But this would be a very good exercise for you to better understand the importance of frequency-domain signal analysis in practice (with an audio signal used as an example), and appreciate the significance of the information contained in the magnitude and the phase of the frequency spectrum.

In MATLAB, please run the following command:

```
openExample('signal/FrequencyAnalysisExample')
```

And carefully read all of the text (you can also read it here: https://www.mathworks.com/help/signal/examples/practical-introduction-to-frequency-domain-analysis.html).

Then please run the code, and generate all the plots for yourself. You may want to place breakpoints and step through the code and try to understand what is happening at every step.