Problem 1: Problem 7.4 in R1

Problem 2: Problem 7.8 in R1

Problem 3: Problem 7.9 in R1

Problem 4: Problem 8.13 in R1

Problem 5: Problem 8.17 in R1

Problem 6: Problem 3.2 ((b) and (h) only) in R1

Problem 7: Problem 3.4 ((b) only) in R1

Problem 8: Problem 3.8 in R1

Problem 9: Problem 3.9 in R1

Hw 5

7.4 For the sequences

$$x_1(n) = \cos\frac{2\pi}{N}n, \qquad x_2(n) = \sin\frac{2\pi}{N}n, \qquad 0 \le n \le N-1$$

determine the N-point:

(a) Circular convolution $x_1(n) \circledN x_2(n)$

(b) Circular correlation of $x_1(n)$ and $x_2(n)$

(c) Circular autocorrelation of $x_1(n)$

(d) Circular autocorrelation of $x_2(n)$

ⓐ $X_1(n) = \cos\left(\frac{2\pi}{N}n\right) = \frac{1}{2}\left(e^{j\frac{2\pi}{N}n} + e^{-j\frac{2\pi}{N}n}\right)$

$X_1(k) = \frac{N}{2}\left(\delta(k-1) + \delta(k+1)\right)$

$X_2(k) = \frac{N}{2j}\left(\delta(k-1) - \delta(k+1)\right)$

$X_3(k) = X_1(k)\,X_2(k) = \frac{N}{2}\left(\delta(k-1)+\delta(k+1)\right) \cdot \frac{N}{2j}\left(\delta(k-1) - \delta(k+1)\right)$

$= N^2/4j\left(\delta(k-1)^2 - \delta(k+1)^2\right) = N^2/4j\left(\delta(k-1) - \delta(k+1)\right)$

$X_3(k) = \frac{N}{2}\sin\left(\frac{2\pi}{N}n\right)$

ⓑ $\tilde{R}_{x_1x_2}(k) = X_1(k)\,X_2^*(k)$

$= -N^2/4j\left(\delta(k-1) - \delta(k+1)\right)$

$\tilde{r}_{x_1x_2}(k) = -N/2\,\sin\left(\frac{2\pi}{N}n\right)$

ⓒ $\tilde{R}_{x_1x_1}(k) = X_1(k)\,X_1^*(k)$

$= \frac{N^2}{4}\left(\delta(k-1) + \delta(k+1)\right)$

$\tilde{r}_{x_1x_1}(n) = \frac{N}{2}\cos\left(\frac{2\pi}{N}n\right)$

ⓓ $\tilde{R}_{x_2x_2}(k) = X_2(k)\,X_2^*(k)$

$= \frac{N^2}{4}\left(\delta(k-1) + \delta(k+1)\right)$

$\tilde{r}_{x_2x_2}(n) = N/2\,\cos\left(\frac{2\pi}{N}n\right)$

7.8 Determine the circular convolution of the sequences

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$$

using the time-domain formula in (7.2.39).

$$y(n) = x_1(n) \; ④ \; x_2(n)$$

$$= \sum_{m=0}^{3} x_1(m)\, x_2((n-m))_4$$

$$= \sum_{m=0}^{3} x_1(m)\, x_2(4-m-n)$$

$$y(0) = \sum_m x_1(m)\, x_2(4-m+0) = \{1,2,3,1\} * \{4,2,2,3\} = 17$$

$$y(1) = \sum_m x_1(m)\, x_2(4-m+1) = \{1,2,3,1\} * \{3,4,2,2\} = 19$$

$$y(2) = \sum_m x_1(m)\, x_2(4-m+2) = \{1,2,3,1\} * \{2,3,4,2\} = 22$$

$$y(3) = \sum_m x_1(m)\, x_2(4-m+3) = \{1,2,3,1\} * \{2,2,3,4\} = 19$$

$$y(n) = \{\underset{\uparrow}{17}, 19, 22, 19\}$$

7.9 Use the four-point DFT and IDFT to determine the sequence

$$x_1(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

$$x_3(n) = x_1(n) \circledN x_2(n)$$

$$x_2(n) = \{\underset{\uparrow}{4}, 3, 2, 2\}$$

where $x_1(n)$ and $x_2(n)$ are the sequence given in Problem 7.8.

$$X_1(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}k\cdot N} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \{7, -2-j, 1, -2+j\}$$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 3 \\ 2 \\ 2 \end{bmatrix} = \{11, 2-j, 1, 2+j\}$$

$$X_3(k) = X_1(k)\, X_2(k) = \{77, -5, 1, -5\}$$

$$\text{IDFT}\{X_3(k)\} = \frac{1}{4} \sum_{k=0}^{3} X_3(k)\, e^{j\frac{2\pi}{4}nk}$$

$$x_3(0) = \frac{1}{4} \sum_{k=0}^{3} X_3(k)\, e^0 = 17$$

$$x_3(1) = \frac{1}{4} \sum X_3(k)\, e^{j\frac{2\pi}{4}k} = 19$$

$$x_3(2) = \frac{1}{4} \sum X_3(k)\, e^{j\frac{2\pi}{4}k} = 22$$

$$x_3(3) = \frac{1}{4} \sum X_3(k)\, e^{j\frac{2\pi}{4}k} = 19$$

$$x_3(n) = \{17, 19, 22, 19\}$$

8.13 Consider the eight-point decimation-in-time (DIT) flow graph in Fig. 8.1.6.

(a) What is the gain of the "signal path" that goes from $x(7)$ to $X(2)$?

(b) How many paths lead from the input to a given output sample? Is this true for every output sample?

(c) Compute $X(3)$ using the operations dictated by this flow graph.

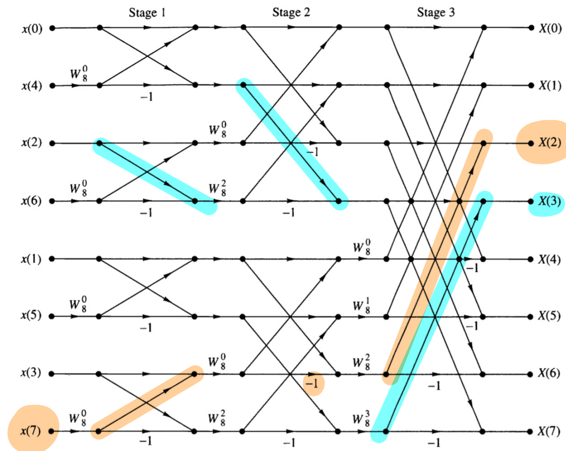


Figure 8.1.6 Eight-point decimation-in-time FFT algorithm.

(a) gain $= W_8^0 \, W_8^0 \, (-1) \, W_8^2 = j$

$$W_N^k = e^{-j\frac{2\pi}{N}k}$$

(b) For every output sample there is only one path to it from each input. It is True.

(c) $X(3) = x(0)$
$+ x(1)\, W_8^3$
$+ x(2)\, W_8^2$
$+ x(3)\, W_8^2\, W_8^3$
$- x(4)\, W_8^0$
$- x(5)\, W_8^0\, W_8^3$
$+ x(6)\, W_8^0\, W_8^2$
$+ x(7)\, W_8^0\, W_8^2\, W_8^3$

8.17 Explain how the DFT can be used to compute N equispaced samples of the z-transform of an N-point sequence, on a circle of radius r.

$$X(z) = \sum_{n=-\infty}^{n=\infty} x(n) z^{-n}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

Let circle have radius r $\Rightarrow$ $z = r e^{-j\frac{2\pi}{N}k}$ for $k = 0, 1, \ldots, N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n) r^{-n} e^{-j\frac{2\pi}{N}kn} \quad \text{for } k = 0, 1, \ldots, N-1$$

3.2 Determine the z-transforms of the following signals and sketch the corresponding pole–zero patterns.

(b) $x(n) = (a^n + a^{-n})u(n)$, a real

(h) $x(n) = (\frac{1}{2})^n[u(n) - u(n-10)]$

Problem 6: Problem 3.2 ((b) and (h) only) in R1

(b)

$$X(z) = \sum_{n=-\infty}^{\infty} (a^n + a^{-n})\, u(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} (a^n + a^{-n})\, z^{-n}$$

$$= \underbrace{\sum_{n=0}^{\infty} a^n z^{-n}}_{X_1(z)} + \underbrace{\sum_{n=0}^{\infty} a^{-n} z^{-n}}_{X_2(z)}$$

$$X_1(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a z^{-1})^n = \frac{1}{1 - a z^{-1}} \; ; \; ROC\ |z| > a$$

$$X_2(z) = \sum_{n=0}^{\infty} a^{-n} z^{-n} = \sum_{n=0}^{\infty} (a^{-1} z^{-1})^n = \frac{1}{1 - a^{-1} z^{-1}} \; ; \; ROC\ |z| > \frac{1}{a}$$

$$X(z) = \frac{1}{1 - a z^{-1}} + \frac{1}{1 - a^{-1} z^{-1}} = \frac{1 - a^{-1} z^{-1} + 1 - a z^{-1}}{(1 - a z^{-1})(1 - a^{-1} z^{-1})} = \frac{\frac{az - 1 + az - a^2}{az}}{\frac{(z-a)(z - 1/a)}{z^2}}$$

$$= \frac{2z^2 - z\left(\frac{1}{a} + a\right)}{(z-a)(z - 1/a)} \; ; \; ROC\ |z| > a$$

zeroes

$z = 0$

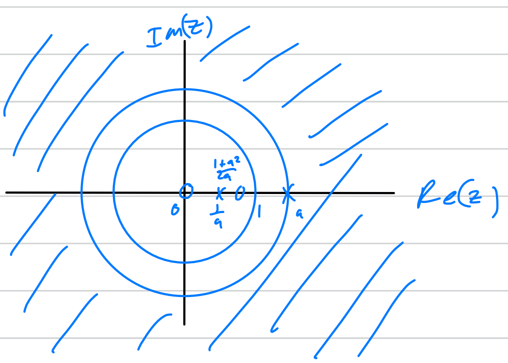
$z = \frac{1 + a^2}{2a}$

pole

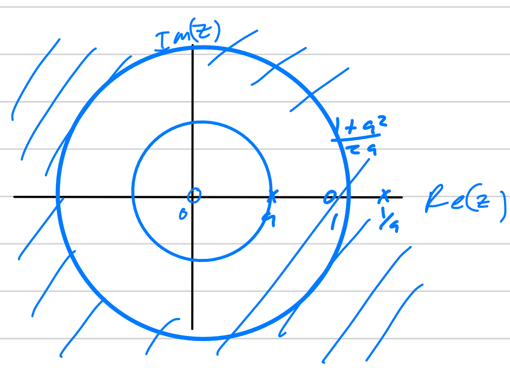
$z = a$

$z = 1/a$

$a > 1$



$a < 1$



3.2 Determine the z-transforms of the following signals and sketch the corresponding pole–zero patterns.

(h) $x(n) = (\frac{1}{2})^n[u(n) - u(n-10)]$

$$X(z) = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n \left(u(n) - u(n-10)\right) z^{-n}$$

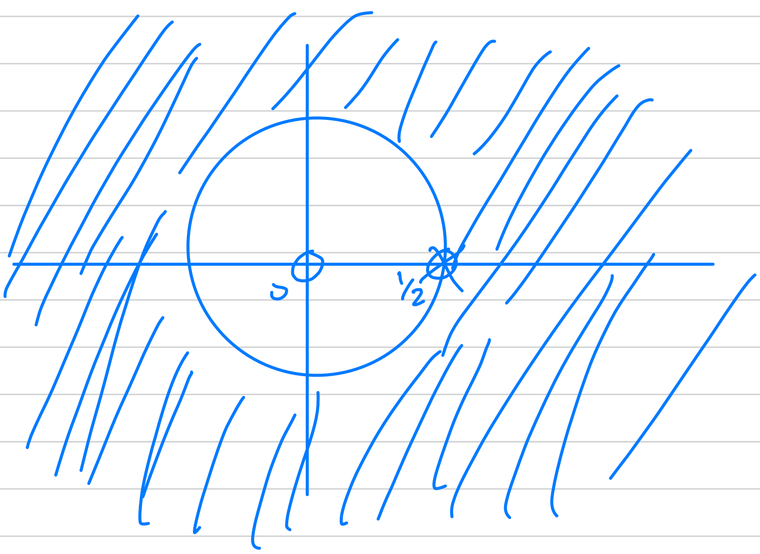
$$= \sum_{n=0}^{\infty} \left(\frac{z^{-1}}{2}\right)^n - \sum_{n=10}^{\infty} \left(\frac{z^{-1}}{2}\right)^n$$

$$= \frac{1}{1-\frac{1}{2}z^{-1}} - \frac{\left(\frac{1}{2}\right)^{10} z^{-10}}{1-\frac{1}{2}z^{-1}}$$

$$X(z) = \frac{1-\left(\frac{1}{2}z^{-1}\right)^{10}}{1-\frac{1}{2}z^{-1}} \quad ; \quad \text{ROC } |z| > \frac{1}{2}$$

$$1-\left(\frac{1}{2}z^{-1}\right)^{10} = 0 \implies \left(\frac{1}{2}\right)^{10} - z^{10} = 0$$

$$z_n = \frac{1}{2} e^{j\frac{2\pi n}{10}} \qquad n = 1, 2, \ldots, k$$



3.4 Determine the z-transform of the following signals.

(b) $x(n) = n^2 u(n)$

Problem 7: Problem 3.4 ((b) only) in R1

(a) Let $y_1(n) = u(n) \Rightarrow Y_1(z) = \sum_{n=-\infty}^{\infty} \overbrace{y_1(n)}^{u(n)} z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$

$|z^{-1}| < 1$

$|z| < 1$

$y_2(n) = n\,u(n) \Rightarrow Y_2(z) = -z \frac{d}{dz} Y(z) = -z \frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)$

$= n\, y_1(n)$

$= \frac{z^{-1}}{(1-z^{-1})^2}$

$x(n) = n^2 u(n) = n\left(y_2(n)\right) \Rightarrow X(z) = -z \frac{d}{dz}\left(\frac{z^{-1}}{(1-z^{-1})^2}\right) = \frac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$

$X(n) = \frac{z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$; ROC $|z| > 1$

3.8 Use the convolution property to:

(a) Express the z-transform of

$$y(n) = \sum_{k=-\infty}^{n} x(k)$$

in terms of $X(z)$.

(b) Determine the z-transform of $x(n) = (n+1)u(n)$. [*Hint:* Show first that $x(n) = u(n) * u(n)$.]

(a) $y(n) = \sum_{k=-\infty}^{n} x(k) = x(-\infty) + \dots + x(n)$

$$y(n-1) = x(-\infty) + \dots + x(n-2) + x(n-1)$$

$$y(n) - y(n-1) = x(n)$$

$$Y(z) - z^{-1} Y(z) = X(z)$$

$$Y(z)(1 - z^{-1}) = X(z)$$

$$Y(z) = \frac{X(z)}{1 - z^{-1}}$$

(b) $x(n) = u(n) * u(n)$

$$= \sum_{k=-\infty}^{\infty} u(k)\, u(n-k)$$

$$= \sum_{k=0}^{n} 1$$

$$= (n+1)\, u(n)$$

$$u(n) \leftrightarrow \frac{1}{1 - z^{-1}} \qquad |z| > 1$$

$$u(n) * u(n) \leftrightarrow \left(\frac{1}{1 - z^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right)$$

$$X(z) = \left(\frac{1}{1 - z^{-1}}\right)^2 \quad ; \quad |z| > 1$$

ROC

$$X(z) = \frac{1}{(1 - z^{-1})^2} \quad ; \quad |z| > 1$$

3.9 The z-transform $X(z)$ of a real signal $x(n)$ includes a pair of complex-conjugate zeros and a pair of complex-conjugate poles. What happens to these pairs if we multiply $x(n)$ by $e^{j\omega_0 n}$? (*Hint*: Use the scaling theorem in the z-domain.)

$$\mathcal{Z}\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum x(n) (a^{-1} z)^n$$

$$y(n) = x(n) e^{j\omega_0 n}$$

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(n) e^{j\omega_0 n} \xleftrightarrow{z} ??$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n) e^{j\omega_0} z^{-n}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n) \left(e^{-j\omega_0} z\right)^{-n}$$

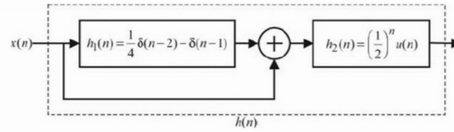
$$Y(z) = X\left(e^{-j\omega_0} z\right) \longrightarrow$$ poles + zeroes are phase rotated by ω_0

Problem 10:

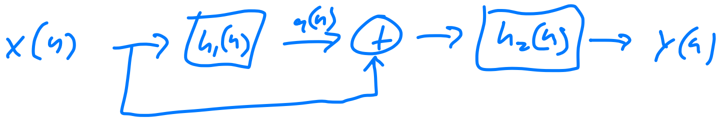
Consider the system shown below.

1. Using the z-transform approach, show that the impulse response, $h(n)$, of the overall system is given by

$$h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$



2. Determine the difference equation representation of the overall system that relates the output $y(n)$ to the input $x(n)$.
3. Is this system causal? BIBO stable? Explain clearly to receive full credit.
4. Determine the frequency response $H(e^{j\omega})$ of the overall system.
5. Using MATLAB, provide a plot of this frequency response over $0 \le \omega \le \pi$.

① 

$$x(n) * h_1(n) = q(n)$$

$$\big(q(n) + x(n)\big) * h_2(n) = y(n)$$

$$\big(A(z) + X(z)\big) H_2(z) = Y(z)$$

$$A(z) = X(z)\,H_1(z) \quad // \quad H_1(z) = \frac{1-4z}{4z^2} \quad // \quad H_2(z) = \frac{2z}{2z-1}$$

$$H(z) = \left(\frac{1-4z}{4z^2} + 1\right)\frac{2z}{2z-1} = 1 - \frac{1}{2z} \Rightarrow h(n) = \delta(n) - \tfrac{1}{2}\,\delta(n-1)$$

$$\delta(n-1) \leftrightarrow z^{-1}$$

$$\delta(n) \leftrightarrow 1$$

②

$$x(n) * \delta(n-\tau) = x(n-\tau)$$

$$h(n) = \delta(n) - \tfrac{1}{2}\,\delta(n-1)$$

$$\therefore \; y(n) = x(n) - \tfrac{1}{2}\,x(n-1)$$

③ Causality

$$y(n) = x(n) - \tfrac{1}{2}\,x(n-1)$$

$\uparrow$ current val

$\uparrow$ past val

$\therefore$ Causal

BIBO stability

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

+ poles @ $z = 0$ + $z = \tfrac{1}{2}$

$\therefore$ BIBO Stable

④

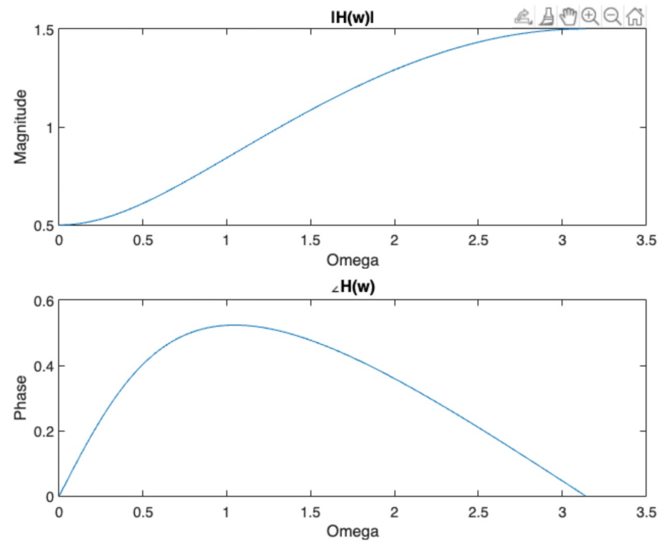
$$H(z) = 1 - \tfrac{1}{2}\,z^{-1}$$

$$H(e^{j\omega}) = 1 - \tfrac{1}{2}e^{-j\omega}$$

hw5_10.m

```
clear variables;
close all;

w = linspace(0, pi);

h_w = 1- 0.5 .* exp(-1i .* w);

figure();

subplot(2,1,1);
plot(w, abs(h_w));
title("|H(w)|");
xlabel("Omega");
ylabel("Magnitude");

subplot(2,1,2);
plot(w, angle(h_w));
title("\angleH(w)");
xlabel("Omega");
ylabel("Phase");
```

Figure 1



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