ECEn 671: Mathematics of Signals and Systems
Moon: Chapter 4.

Randal W. Beard

Brigham Young University

August 29, 2023

Table of Contents

Section 1

Linear Operators

Linear Operators

Recall from Chapter 3 the definition of a Linear operator:

Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be vector spaces, then $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is a linear operator if

$$\mathcal{A}[\alpha_1 x_1 + \alpha_2 x_2] = \alpha_1 \mathcal{A}[x_1] + \alpha_2 \mathcal{A}[x_2]$$

$\forall x_1, x_2 \in \mathbb{X}$ and $\forall \alpha_1, \alpha_2 \in \mathbb{F}$

See chapter 2 notes (slides 79–83) for examples of linear operators.

Norm of a Linear Operator

An important concept is the norm of an operator. There are several ways to define norms for operators. The most important is the "induced" or "subordinate" norm.

Definition

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ then

$$\begin{aligned} \|\mathcal{A}\| &= \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{\mathbb{Y}}}{\|x\|_{\mathbb{X}}} \\ &= \sup_{\|x\|_{\mathbb{X}}=1} \|\mathcal{A}[x]\|_{\mathbb{Y}} \end{aligned}$$

Different norms on $\mathcal{A}$ are defined by taking different norms in $\mathbb{X}$ and $\mathbb{Y}$.

Norm of a Linear Operator, Examples

Example

Let $\mathcal{A} : L_2 \to L_2$ then

$$\begin{aligned}\|\mathcal{A}\|_2 &= \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_2}}{\|x\|_{L_2}} \\ &= sup_{\|x\|_{L_2}=1} \|\mathcal{A}[x]\|_{L_2}\end{aligned}$$

Example

Let $\mathcal{A} : L_\infty \to L_\infty$ then

$$\begin{aligned}\|\mathcal{A}\|_\infty &= \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_\infty}}{\|x\|_{L_\infty}} \\ &= \sup_{\|x\|_{L_\infty}=1} \|\mathcal{A}[x]\|_{L_\infty}\end{aligned}$$

Norm of a Linear Operator, Examples

Example

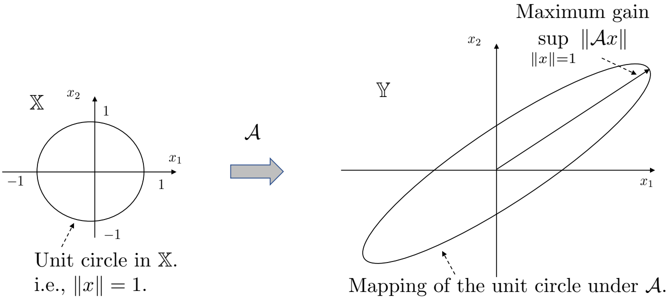
Let $\mathcal{A} : L_p \to L_p$ then

$$\begin{aligned} \|\mathcal{A}\|_p &= \sup_{x \neq 0} \frac{\|\mathcal{A}[x]\|_{L_p}}{\|x\|_{L_p}} \\ &= \sup_{\|x\|_{L_p}=1} \|\mathcal{A}[x]\|_{L_p} \end{aligned}$$

Why is it called the induced or subordinate norm? The norm on the operator is induced by the vector norm.

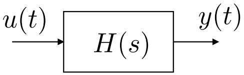
Norm of a Linear Operator, Geometric Interpretation

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$



Norm of a Linear Operator, System Interpretation

Given a linear system



The norm of the system $H(s)$ is the maximum gain of the system.

Norm of BIBO System

Let $\mathcal{A} : L_\infty \to L_\infty$ be an LTI system that is BIBO stable with impulse response $h(t)$, then

$$
\begin{aligned}
y(t) &= \int_0^t h(t-\tau)u(\tau)d\tau \\
&\triangleq \mathcal{A}[u]
\end{aligned}
$$

Find $\|\mathcal{A}\|_\infty$.

Norm of BIBO System, cont

Lemma

$$\|\mathcal{A}\|_\infty = \|h\|_{L_1[0,\infty]} \triangleq \int_0^\infty |h(t)|dt$$

Proof.

We need to prove two things

1. $\|\mathcal{A}\|_\infty \leq \int_0^\infty |h(t)|dt$
2. $\int_0^\infty |h(t)|dt \leq \|\mathcal{A}\|_\infty$

$\square$

Norm of BIBO System, Proof

Proof of 1.

$$
\begin{aligned}
\sup_{\|x\|_\infty=1} \|\mathcal{A}[u]\|_\infty &= \sup_{\|u\|_\infty=1} \left\| \int_0^t h(t-\tau)u(\tau)d\tau \right\|_\infty \\
&= \sup_{\|u\|_\infty=1} \left[\sup_{t>0} \left| \int_0^t h(t-\tau)u(\tau)d\tau \right| \right] \\
&\leq \sup_{\|u\|_\infty=1} \left[\sup_{t>0} \int_0^t |h(t-\tau)u(\tau)| \, d\tau \right] \\
&\leq \sup_{\|u\|_\infty=1} \left[\|u\|_\infty \sup_{t>0} \int_0^t |h(t-\tau)| \, d\tau \right] \\
&\leq \int_0^\infty |h(\tau)| \, d\tau = \|h\|_{L_1[0,\infty]}
\end{aligned}
$$

Norm of BIBO System, Proof

Proof of 2.

Let $\hat{u}_t(\tau) = \begin{cases} 1 & \text{if } h(t-\tau) \geq 0 \\ -1 & \text{otherwise} \end{cases}$.

Note that $\|\hat{u}_t\|_\infty = 1 \ \forall t > 0$, we have that

$$\int_0^t h(t-\tau)\hat{u}_t(\tau)d\tau = \int_0^t |h(t-\tau)|\, d\tau.$$

Therefore for this particular choice of $\hat{u}_t$ we have that

$$\sup_{t>0}\left[\int_0^t |h(t-\tau)|\, d\tau\right] = \|A\hat{u}_\infty\|_\infty = \int_0^\infty |h(\tau)|\, d\tau.$$

By definition of sup

$$\int_0^\infty |h(\tau)|\, d\tau = \|A\hat{u}_\infty\|_\infty \leq \sup_{\|u\|=1} \|Au\|_\infty .$$

Operator Norm: Proof Technique

The proof technique shown here is the general approach to show that the norm of an operator is some value.

Suppose that you would like to prove that

$$\|\mathcal{A}\| = M.$$

You need to show two things

1. $\|\mathcal{A}\| \leq M$
2. $M \leq \|\mathcal{A}\|$.

Operator Norm: Proof Technique

To show (1) use triangle and other inequalities to show that

$$\|\mathcal{A}x\| \leq M\,\|x\|$$

which implies that

$$\sup_{\|x\|=1} \|\mathcal{A}x\| \leq \sup_{\|x\|=1} M\,\|x\| = M$$

To show (2), construct a specific $\hat{x}$ such that

$$\|\hat{x}\| = 1 \text{ and } \|\mathcal{A}\hat{x}\| = M.$$

This implies that

$$M \leq \sup_{\|x\|=1} \|\mathcal{A}x\| = \|\mathcal{A}\|\,.$$

Properties of Linear Operators

Lemma

For any induced operator norm,

$$\|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| \, .$$

Proof.

$$\|\mathcal{A}\| = \sup_{x \neq 0} \frac{\|\mathcal{A}x\|}{\|x\|}.$$

Therefore for any $x \neq 0$ we must have that

$$\|\mathcal{A}\| \geq \frac{\|\mathcal{A}x\|}{\|x\|}$$
$$\Rightarrow \|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| \, .$$

$\square$

Properties of Linear Operators, cont

Lemma

All induced operator norms satisfy the "submultiplicative property," i.e.,

$$\|\mathcal{AB}\| \leq \|\mathcal{A}\| \, \|\mathcal{B}\|$$

Proof.

$$\begin{aligned}
\|\mathcal{AB}\| &= \sup_{\|x\|=1} \|\mathcal{AB}x\| \\
&\leq \sup_{\|x\|=1} \|\mathcal{A}\| \, \|\mathcal{B}x\| \\
&\leq \sup_{\|x\|=1} \|\mathcal{A}\| \, \|\mathcal{B}\| \, \|x\| \\
&= \|\mathcal{A}\| \, \|\mathcal{B}\|
\end{aligned}$$

$\square$

Properties of Linear Operators, cont

Definition

An operator $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is <u>bounded</u> if $\|\mathcal{A}\| < \infty$

Definition

The following three statements are equivalent

1. $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ is <u>continuous</u>
2. $x_n \to x^* \Rightarrow \mathcal{A}[x_n] \to \mathcal{A}[x^*]$ for all convergent sequences in $\mathbb{X}$
3. $\forall \epsilon > 0, \quad \exists \delta > 0$ such that

$$\|x - y\| \leq \delta \quad \Rightarrow \quad \|\mathcal{A}[x] - \mathcal{A}[y]\| < \epsilon \qquad \forall x, y \in \mathbb{X}$$

Properties of Linear Operators, cont

Theorem (Moon Theorem 4.1)

A linear operator is bounded iff it is continuous.

Proof.

□

($\Rightarrow$) Suppose $\|\mathcal{A}\| = M < \infty$,let $\{x_n\}$ be any convergent sequence with limit $x^* \in \mathbb{X}$, then

$$\begin{aligned}\|\mathcal{A}x_n - \mathcal{A}x^*\| &= \|\mathcal{A}(x_n - x^*)\| \leq \|\mathcal{A}\| \, \|x_n - x^*\| \\ &= M \, \|x_n - x^*\| \to 0 \Rightarrow \|\mathcal{A}x_n - \mathcal{A}x^*\| \to 0.\end{aligned}$$

Therefore $\mathcal{A}$ is continuous.

Proof, cont

($\Leftarrow$) Assume $\mathcal{A}$ is continuous and let $\epsilon = 1$ and $y = 0$ then $\exists \delta$ such that $\|x\| \leq \delta \Rightarrow \|\mathcal{A}x\| < 1$

Now let $0 \neq x \in \mathbb{X}$ be arbitrary, then

$$\left\| \frac{\delta x}{\|x\|} \right\| = \frac{\delta}{\|x\|} \|x\| = \delta \leq \delta$$

implies that

$$\left\| \mathcal{A} \left(\frac{\delta x}{\|x\|} \right) \right\| = \frac{\delta}{\|x\|} \|\mathcal{A}x\| < 1$$

which implies that

$$\|\mathcal{A}x\| \leq \frac{1}{\delta} \|x\|$$

Therefore $\mathcal{A}$ is bounded.

Properties of Linear Operators, cont

Theorem (Moon Theorem 4.2)

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a linear operator. If $\mathbb{X}$ is a finite dimensional Hilbert space, then $\mathcal{A}$ is bounded.

Proof.

$\square$

Let $\dim(\mathbb{X}) = n$ and let $\{p_1, \cdots p_n\}$ be an orthonormal basis for $\mathbb{X}$, then

$$x = \sum_{k=1}^{n} \langle x, p_k \rangle \, p_k$$

Proof, cont.

Define $D = \max\{\|\mathcal{A}p_1\|, \|\mathcal{A}p_2\|, \ldots, \|\mathcal{A}p_n\|\}$ then

$$\begin{aligned}
\|\mathcal{A}x\| &= \left\|\mathcal{A}\left(\sum_{k=1}^{n} \langle x, p_k \rangle p_k\right)\right\| \\
&\leq \sum_{k=1}^{n} |\langle x, p_k \rangle| \, \|\mathcal{A}p_k\| \\
&\leq D \sum_{k=1}^{n} |\langle x, p_k \rangle| \\
&\leq D \sum_{k=1}^{n} \|x\| \, \|p_k\| \qquad (Caucy - Schwartz) \\
&= Dn \|x\|
\end{aligned}$$

Therefore $\mathcal{A}$ is bounded.

Section 2

Neumann Expansion

Geometric Series

One of the most important series in analysis is the geometric series

$$S = 1 + x + x^2 + \ldots = \sum_{i=0}^{\infty} x^i$$

Noting that

$$\begin{aligned} & 1 + xS = 1 + x + x^2 + \ldots = S \\ \Rightarrow \qquad & S(1-x) = 1 \end{aligned}$$

Therefore

$$S = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = (1-x)^{-1}$$

The series converges if $|x| < 1$.

Geometric Series for Operators (Neumann Expansion)

For operators we have a similar expression:

Theorem (Moon Theorem 4.3)

Suppose $\|\cdot\|$ *is a norm satisfying the submultiplicative property and* $\|\mathcal{A}\| < 1$. *Then* $(I - \mathcal{A})^{-1}$ *exists and*

$$(I - \mathcal{A})^{-1} = \sum_{i=0}^{\infty} \mathcal{A}^i = I + \mathcal{A} + \mathcal{A}^2 + \mathcal{A}^3 + \dots$$

where

$$\mathcal{A}^2 = \mathcal{A}\mathcal{A}$$
$$\mathcal{A}^3 = \mathcal{A}\mathcal{A}^2$$
$$\mathcal{A}^k = \mathcal{A}\mathcal{A}^{k-1}.$$

Neumann Expansion, Proof

Suppose that $(I - \mathcal{A})^{-1}$ does not exist. Then $\mathcal{N}(I - A)$ is non-trivial.
Therefore, $\exists x \neq 0$ such that

$$
\begin{aligned}
(I - \mathcal{A})x = 0 \quad &\Longleftrightarrow \quad x = \mathcal{A}x \\
&\Longleftrightarrow \quad \|x\| = \|\mathcal{A}x\| \leq \|\mathcal{A}\| \, \|x\| < \|x\| \,,
\end{aligned}
$$

which is a contradiction.

Therefore $(I - \mathcal{A})^{-1}$ exists.

Neumann Expansion, cont.

Note that $\left\|\mathcal{A}^k\right\| \leq \|\mathcal{A}\|^k$ since $\|\cdot\|$ satisfies the submultiplication property.

Since $\|\mathcal{A}\| < 1$

$$\lim_{k\to\infty} \left\|\mathcal{A}^k\right\| = 0 \quad \Longleftrightarrow \quad \lim_{k\to\infty} \mathcal{A}^k = 0$$

Note that

$$(I - \mathcal{A})(I + \mathcal{A} + \mathcal{A}^2 + \cdots + \mathcal{A}^{k-1}) = I - \mathcal{A}^k$$

$k \to \infty$ gives

$$(I - \mathcal{A}) \left(\sum_{i=0}^{\infty} \mathcal{A}^i \right) = I$$

Therefore

$$\sum_{i=0}^{\infty} \mathcal{A}^i = (I - \mathcal{A})^{-1}.$$

Section 3

Matrix Norms

Matrix Norms

For matrices $A : \mathbb{C}^m \to \mathbb{C}^n$ we have the following induced norm:

$$\|A\|_\infty = \max_{\|x\|_\infty = 1} \|Ax\|_\infty$$

(Why max not sup?)

Lemma

$$\|A\|_\infty = \max_{i=1:m} \sum_{j=1:n} |a_{ij}|$$

i.e., the largest row sum.

Proof

First show that $\|A\|_\infty \le \max_{i=1:m} \sum_{i=1:n} |a_{ij}|$:

$$
\begin{aligned}
\|A\|_\infty &= \max_{\|x\|_\infty=1} \left\| \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\|_\infty \\
&= \max_{\|x\|_\infty=1} \left[\max \begin{pmatrix} \left|\sum_{j=1}^n a_{1j} x_j\right| \\ \vdots \\ \left|\sum_{j=1}^n a_{mj} x_j\right| \end{pmatrix} \right] \\
&\le \max_{x \text{ s.t. } \max|x_i|=1} \left[\max\left(\sum_{j=1}^n |a_{1j}|\,|x_j|, \quad \cdots, \quad \sum_{j=1}^n |a_{mj}|\,|x_j| \right) \right] \\
&\le \max_{\|x\|_\infty=1} \left[\max\left(\|x\|_\infty \sum_{j=1}^n |a_{1j}|, \quad \cdots, \quad \|x\|_\infty \sum_{j=1}^m |a_{mj}| \right) \right] \\
&= \max_{i=1:m} \sum_{j=1}^m |a_{ij}|
\end{aligned}
$$

Proof, cont.

Now we need to show that $\max_{i=1:m} \sum_{j=1:n} |a_{ij}| \leq \|A\|_\infty$:

Let $k = \arg\max\limits_{i=1:m} \sum_{j=1:n} |a_{ij}|$
and let $\hat{x}$ be such that

$$\hat{x}_j = \begin{cases} 1 & \text{if} \quad a_{kj} \geq 0 \\ -1 & \quad \text{otherwise} \end{cases}$$

then $\|\hat{x}\|_\infty = 1$ and then

$$\|A\hat{x}\|_\infty = \max_{i=1:m} \sum_{j=1:n} |a_{ij}| \leq \max_{\|x\|_\infty = 1} \|Ax\|_\infty = \|A\|_\infty .$$

Other Matrix Norms

Lemma

$$\begin{aligned}\|A\|_1 &= \max_{\|x\|_1=1} \|Ax\|_1 \\ &= \max_{j=1:n} \sum_{i=1}^{m} |a_{ij}| \ \ \textit{(largest column sum)}\end{aligned}$$

Lemma

$$\|A\|_2 = \max_i \sqrt{\lambda_i(A^H A)} = \ \textit{largest singular value of } A$$

More discussion of this in Chapter 7.

Norm of A^{-1}

Theorem

For induced matrix norms, where A^{-1} exists we have

$$\|A^{-1}\| = \frac{1}{\min\limits_{x\neq 0} \frac{\|Ax\|}{\|x\|}} = \frac{1}{\min\limits_{\|x\|=1} \|Ax\|}$$

Proof.

Let $Ax = b \Rightarrow x = A^{-1}b$ then

$$\|A^{-1}\| = \max_{b\neq 0} \frac{\|A^{-1}b\|}{\|b\|} = \max_{x\neq 0} \frac{\|x\|}{\|Ax\|} = \max_{x\neq 0} \frac{1}{\frac{\|Ax\|}{\|x\|}}$$

$$= \frac{1}{\min\limits_{x\neq 0} \frac{\|Ax\|}{\|x\|}} = \frac{1}{\min\limits_{\|x\|=1} \|Ax\|}$$

Frobenius Norm

Definition

The Frobenius norm of a matrix is given by

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|^2 \right)^{\frac{1}{2}}$$
$$= \sqrt{tr(A^H A)}$$

Fact: The Frobenius norm is NOT an induced norm.

Matrix Convergence

For matrices: convergence in any norm implies convergence in any other norm. In particular

$$
\begin{aligned}
\|A\|_2 \leq \|A\|_F &\leq \sqrt{n}\,\|A\|_2 \\
\max|a_{ij}| \leq \|A\|_2 &\leq \sqrt{mn}\max|a_{ij}| \\
\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 &\leq \sqrt{m}\,\|A\|_\infty \\
\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 &\leq \sqrt{n}\,\|A\|_1
\end{aligned}
$$

Section 4

Adjoint Operators

Adjoint Operator

Definition

Let $\mathcal{A} : \mathbb{X} \to \mathbb{Y}$ be a bounded linear operator from Hilbert space $\mathbb{X}$ to Hilbert space $\mathbb{Y}$, then the adjoint of $\mathcal{A}$ ($\mathcal{A}^*$) is the linear operator $\mathcal{A}^* : \mathbb{Y} \to \mathbb{X}$ such that

$$\langle \mathcal{A}x, y \rangle_{\mathbb{Y}} = \langle x, \mathcal{A}^* y \rangle_{\mathbb{X}}$$

$\forall x \in \mathbb{X}$ and $\forall y \in \mathbb{Y}$.

$\mathcal{A}$ is self-adjoint if $\mathcal{A}^* = \mathcal{A}$

Adjoint Operator, Example

Example (Complex matrices)

$A : \mathbb{C}^n \to \mathbb{C}^m$

What is A^*?

By definition:

$$\begin{aligned} & \langle Ax, y \rangle_{\mathbb{C}^m} = \langle x, A^* y \rangle_{\mathbb{C}^n} \\ \iff & y^H A x = y^H (A^*)^H x \\ \iff & A^* = A^H \end{aligned}$$

Note $A^H : \mathbb{C}^m \to \mathbb{C}^n$

Adjoint Operator, Example

Example (Real matrices)

$A : \mathbb{R}^n \to \mathbb{R}^m$

What is A^*?

By definiton,

$$
\begin{aligned}
& \langle Ax, y \rangle_{\mathbb{R}^m} = \langle x, A^* y \rangle_{\mathbb{R}^n} \\
\iff & x^\top A^\top y = x^\top A^* y \\
\iff & A^* = A^\top
\end{aligned}
$$

Adjoint Operator, Example

Example (Convolution)

$$\mathcal{A} : L_2 \to L_2$$

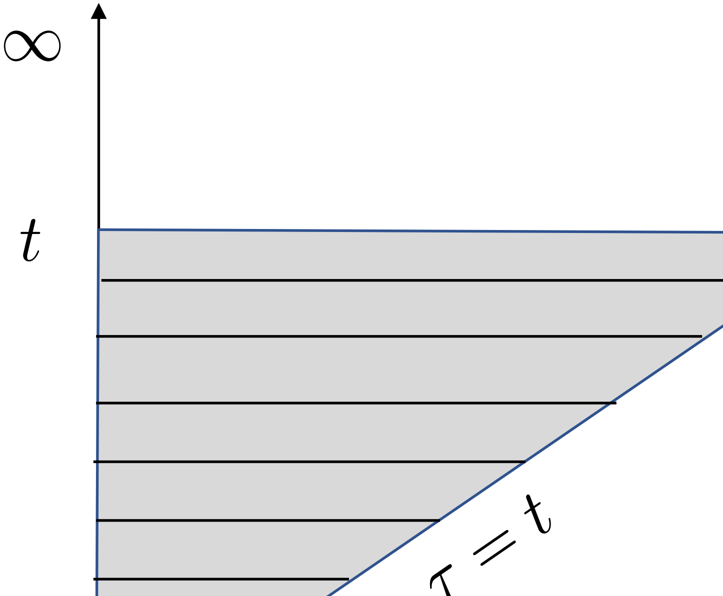
$$\mathcal{A}[x](t) = \int_0^{\top} h(t-\tau)x(\tau)d\tau$$

Let $x \in L_2[0,\infty]$ and $y \in L_2[0,\infty]$ then $\mathcal{A}^*$ is defined by

$$\langle \mathcal{A}x, y \rangle_{L_2} = \langle x, \mathcal{A}^* y \rangle_{L_2}$$

$$\iff \int_{t=0}^{\infty} \left[\int_{\tau=0}^{t} h(t-\tau)x(\tau)d\tau \right] y(t)dt = \int_0^{\infty} x(t)\mathcal{A}^*[y](t)dt$$

Adjoint Operator, Example, Convolution, cont.



Adjoint Operator, Example

Example (linear ode's)

$$\dot{x} = Fx \quad ; \quad x(0) = x_0$$

The solution is $x(t) = e^{Ft}x_0$
Let $\mathcal{A}[x_0](t) = e^{Ft}x_0$, then

$$\mathcal{A} : \mathbb{R}^n \to L_{2[0,T]}$$

What is $\mathcal{A}^*$?

Adjoint Operator, Example, linear ODE, cont.

Let $x \in \mathbb{R}^n$ and let $y \in L_2[0, T]$ then by definition,

$$\langle \mathcal{A}[x_0], y \rangle_{L_2[0,T]} = \langle x_0, \mathcal{A}^* y \rangle_{\mathbb{R}^n}$$

$$\iff \int_0^T x_0^\top (e^{Ft})^\top y(t)dt = x_0^\top \mathcal{A}^* y$$

$$\iff x_0^\top \int_0^T e^{F^\top t} y(t)dt = x_0^\top \mathcal{A}^* y$$

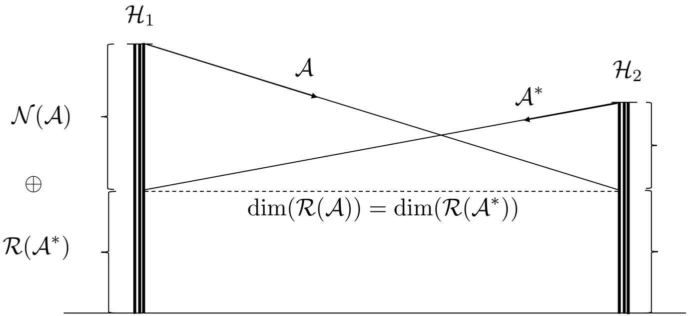
$$\Rightarrow \boxed{\mathcal{A}^*[y] = \int_0^T e^{F^\top t} y(t)dt}$$

Section 5

Fundamental Subspaces

Fundamental Subspaces

Let $\mathcal{A} : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces.
Then $\mathcal{A}^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$ and we have the following picture:



Fundamental Subspaces, cont.

Lemma

1. $\mathcal{H}_1 = \mathcal{N}(\mathcal{A}) \oplus \mathcal{R}(\mathcal{A}^*)$
2. $\mathcal{H}_2 = \mathcal{N}(\mathcal{A}^*) \oplus \mathcal{R}(\mathcal{A})$
3. $\dim(\mathcal{H}_1) = \dim(\mathcal{N}(\mathcal{A})) + \dim(\mathcal{R}(\mathcal{A}^*))$
4. $\dim(\mathcal{H}_2) = \dim(\mathcal{N}(\mathcal{A}^*)) + \dim(\mathcal{R}(\mathcal{A}))$
5. $\dim(\mathcal{R}(\mathcal{A})) = \dim(\mathcal{R}(\mathcal{A}^*))$

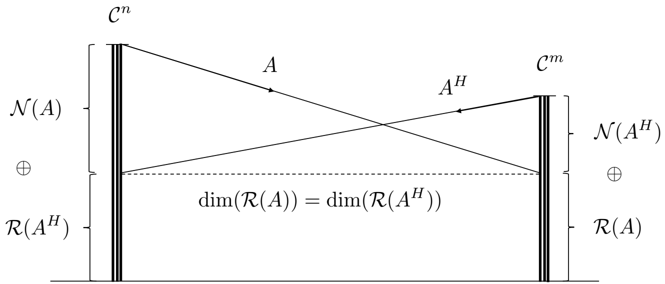
Proofs to follow.

Fundamental Subspaces for Matrices

For matrices, the picture looks as follows:

$$A : \mathbb{C}^n \to \mathbb{C}^m$$

$$A^* = A^H : \mathbb{C}^m \to \mathbb{C}^n$$



$$\dim(\mathcal{R}(A^H)) = \dim(\mathcal{R}(A))$$

Fundamental Subspaces, cont

Theorem (Moon Theorem 4.5)

Let $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ *be bounded and let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces and let* $\mathcal{R}(\mathcal{A})$ *and* $\mathcal{R}(\mathcal{A}^*)$ *be closed, then*

1. $[\mathcal{R}(\mathcal{A})]^\perp = \mathcal{N}(\mathcal{A}^*)$
2. $[\mathcal{R}(\mathcal{A}^*)]^\perp = \mathcal{N}(\mathcal{A})$

Theorem 4.5, Proof

(1): To show that $[\mathcal{R}(\mathcal{A})]^{\perp} = \mathcal{N}(\mathcal{A}^*)$ we need to show that $\mathcal{N}(\mathcal{A}^{\perp}) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}$ and $[\mathcal{R}(\mathcal{A})]^{\perp} \subseteq \mathcal{N}(\mathcal{A}^*)$.

We first show that $\mathcal{N}(\mathcal{A}^*) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}$:

Select any $y \in \mathcal{N}(\mathcal{A}^*)$ and any $\hat{y} \in \mathcal{R}(\mathcal{A})$. Then $\exists \hat{x} \in \mathcal{H}_1$ such that $\hat{y} = \mathcal{A}\hat{x}$. Therefore

$$
\begin{aligned}
\langle \hat{y}, y \rangle &= \langle \mathcal{A}\hat{x}, y \rangle \\
&= \langle \hat{x}, \mathcal{A}^* y \rangle \\
&= \langle \hat{x}, 0 \rangle = 0 \\
\Rightarrow \quad & y \in [\mathcal{R}(\mathcal{A})]^{\perp} \\
\Rightarrow \quad & \mathcal{N}(\mathcal{A}^*) \subseteq [\mathcal{R}(\mathcal{A})]^{\perp}
\end{aligned}
$$

Theorem 4.5, Proof, cont.

We first show that $[\mathcal{R}(\mathcal{A})]^\perp \subseteq \mathcal{N}(\mathcal{A}^*)$:

Select any $y \in [\mathcal{R}(\mathcal{A})]^\perp$. For every $\hat{x} \in \mathcal{H}_1$ we have $\hat{y} = \mathcal{A}\hat{x} \in \mathcal{R}(\mathcal{A})$, and therefore

$$\langle \hat{y}, y \rangle = \langle \mathcal{A}\hat{x}, y \rangle = 0$$

By definition of the adjoint, we therefore have that

$$\langle \hat{x}, \mathcal{A}^* y \rangle = 0$$

Since this is true for every $\hat{x} \in \mathcal{H}_1$ it must be that $\mathcal{A}^* y = 0$. Therefore

$$y \in \mathcal{N}(\mathcal{A}^*),$$

which implies that

$$[\mathcal{R}(\mathcal{A})]^\perp \subseteq \mathcal{N}(\mathcal{A}^*).$$

Item (2) is shown similarly.

Fundamental Subspaces, cont

Theorem 2.10 states that if $\mathcal{H}$ is a Hilbert space and if $\mathbb{V}$ a closed subspace in $\mathcal{H}$ then

$$\mathcal{H} = \mathbb{V} \oplus \mathbb{V}^{\perp}$$

Therefore Theorem 4.5 implies that

$$\mathcal{H}_1 = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A})$$
$$\mathcal{H}_2 = \mathcal{R}(\mathcal{A}) \oplus \mathcal{N}(\mathcal{A}^*)$$

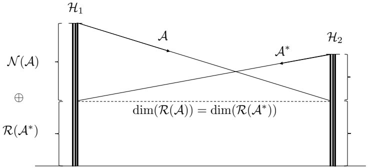
Which also implies that

$$\dim(\mathcal{H}_1) = \dim(\mathcal{R}(\mathcal{A}^*)) + \dim(\mathcal{N}(\mathcal{A}))$$
$$\dim(\mathcal{H}_2) = \dim(\mathcal{R}(\mathcal{A})) + \dim(\mathcal{N}(\mathcal{A}^*))$$

Fundamental Subspaces, cont

Lemma

- $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}\mathcal{A}^*)$
- $\mathcal{R}(\mathcal{A}^*) = \mathcal{R}(\mathcal{A}^*\mathcal{A})$



Proof.
We will prove (1) by showing that:

(a) $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A}\mathcal{A}^*)$

(b) $\mathcal{R}(\mathcal{A}\mathcal{A}^*) \subseteq \mathcal{R}(\mathcal{A})$

Fundamental Subspaces, cont

Proof (cont.)

(a) Let $y \in \mathcal{R}(\mathcal{A}) \Rightarrow \exists x \in \mathcal{H}_1$ such that $y = \mathcal{A}x$
Since $\mathcal{H}_1 = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A})$, $\quad x = x_n + x_r$ where

$$x_n \in \mathcal{N}(\mathcal{A}) \text{ and } x_r \in \mathcal{R}(\mathcal{A}^*)$$

$$\Rightarrow \exists \hat{y} \in \mathcal{H}_2 \text{ such that } x_r = \mathcal{A}^* \hat{y}$$

so

$$y = \mathcal{A}x = \mathcal{A}(x_n + x_r) = \mathcal{A}\mathcal{A}^* \hat{y}$$

$$\Rightarrow y \in \mathcal{R}(\mathcal{A}\mathcal{A}^*)$$

(b) let $y \in \mathcal{R}(\mathcal{A}\mathcal{A}^*) \Rightarrow \exists \hat{y} \in \mathcal{H}_2$ such that

$$y = \mathcal{A}\mathcal{A}^* \hat{y} \Rightarrow y = \mathcal{A}\hat{x} \text{ where } \hat{x} \in \mathcal{H}_1$$

$$\Rightarrow y \in \mathcal{R}(\mathcal{A}).$$

$\square$

Fundamental Subspaces, cont

Theorem

$$\dim(\mathcal{R}(\mathcal{A})) = \dim(\mathcal{R}(\mathcal{A}^*))$$

Proof.

We need to show that

(a) $\dim(\mathcal{R}(\mathcal{A})) \leq \dim(\mathcal{R}(\mathcal{A}^*))$

(b) $\dim(\mathcal{R}(\mathcal{A}^*)) \leq \dim(\mathcal{R}(\mathcal{A}))$

Fundamental Subspaces, cont

Proof (cont.)

(a) Let $P = \{p_1, p_2, \ldots\}$ be a Hamel basis for $\mathcal{R}(\mathcal{A})$ so $\dim(\mathcal{R}(\mathcal{A})) =$ *cardinality* of P.

$$p_i \in \mathcal{R}(\mathcal{A}) \Rightarrow \exists \hat{q}_i \in \mathcal{H}_1 \text{ such that } p_i = \mathcal{A}\hat{q}_i$$

$$\mathcal{H}_1 = \mathcal{R}(\mathcal{A}^*) \oplus \mathcal{N}(\mathcal{A}) \Rightarrow \hat{q}_i = q_{i,n} + q_i$$

$$\text{where } q_{i,n} \in \mathcal{N}(\mathcal{A}) \text{ and } q_i \in \mathcal{R}(\mathcal{A}^*)$$

$$\Rightarrow p_i = \mathcal{A}q_i,$$

let

$$Q = \{q_1, q_2, \ldots\}$$

we will show that Q is linearly independent $\Rightarrow$ any Hamel basis of $\mathcal{R}(A^*)$ contains $Q \Rightarrow \dim(\mathcal{R}(A^*)) \geq \dim(\mathcal{R}(A))$,

Fundamental Subspaces, cont

Proof (cont.)

P is a Hamel basis $\Rightarrow$ all finite subsets of P are linearly independent, i.e.

$$\sum_{i \in I} c_i p_i = 0 \iff c_i = 0, i \in I$$

where I is a finite index set. But,

$$\sum_I c_i p_i = 0 \iff \sum_I c_i \mathcal{A} q_i = 0 \iff \mathcal{A}(\sum_I c_i q_i) = 0$$

but $\sum_I c_i q_i \in \mathcal{R}(\mathcal{A}^*) \perp \mathcal{N}(\mathcal{A})$
so

$$\iff \sum_I c_i q_i = 0 \iff c_i = 0, i \in I$$

$$\Rightarrow Q \text{ is linearly independent}$$

(b) Substitute $\mathcal{A}$ for $\mathcal{A}^*$ and $\mathcal{A}^*$ for $\mathcal{A}$ is above argument.

Solution of Operator Equations

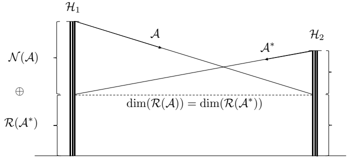
We turn to solutions to the linear operator equation

$$\mathcal{A}x = y$$

where $\mathcal{A} : \mathcal{H}_1 \to \mathcal{H}_2$ is bounded, $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert and $\mathcal{R}(\mathcal{A})$ is closed.

Fact 1. $\mathcal{A}x = y$ has a solution $\iff y \in \mathcal{R}(\mathcal{A})$

Fact 2. $\mathcal{A}x = y$ has a solution $\iff y \perp \mathcal{N}(\mathcal{A}^*)$

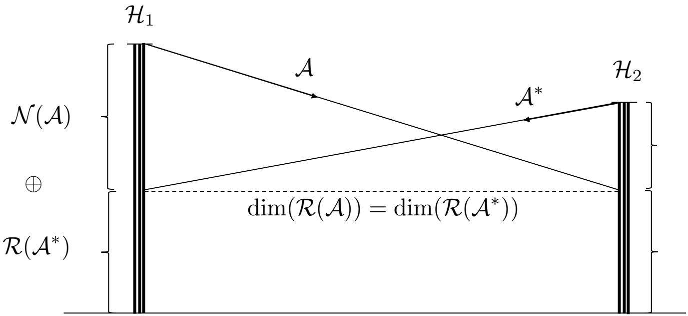


Solution of Operator Equations

Fact 3. If $\mathcal{A}x = y$ has a solution then it is unique $\iff \mathcal{N}(\mathcal{A}) = \{0\}$

Fact 4. If $\mathcal{N}(\mathcal{A}) \neq \{0\}$ and $y \in \mathcal{R}(\mathcal{A})$ then $\mathcal{A}x = y$ has an infinite number of solutions.

Fact 5 . $\mathcal{A}^{-1}$ exists $\Rightarrow \mathcal{N}(\mathcal{A}) = \{0\}$ (otherwise can't get back to all of $\mathcal{H}$.



Matrix Rank

Definition (Row Rank)

The <u>row rank</u> of $A \in \mathbb{C}^{m \times n}$ is the number of linearly independent rows.

Definition (Column Rank)

The <u>column rank</u> of $A \in \mathbb{C}^{m \times n}$ is the number of linearly independent columns.

- Since $\mathcal{R}(A) = span\{\text{columns of } A\}$ we have that $\dim(\mathcal{R}(A)) = \text{ column rank}$
- Since $\mathcal{R}(A^H) = span\{\text{rows of } A\}$ we have that $\dim(\mathcal{R}(A^*)) = \text{ row rank}$
- Therefore $\dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A^H))$ implies that $\text{column rank} = \text{row rank}$

Matrix Rank

Definition

The rank of A is the number of linearly independent rows or columns.

Lemma

$$rank(A) = rank(A^H)$$

Definition

$A : \mathbb{C}^n \to \mathbb{C}^m$ is full rank if $rank(A) = \min(n, m)$

Sylvester's Inequality

Lemma (Sylvester's Inequality)

Let $A \in \mathbb{C}^{q \times n}$ *and* $B \in \mathbb{C}^{n \times p}$ *then*

$$rank(A) + rank(B) - n \leq rank(AB) \leq \min(rank(A), rank(B)).$$

Example

Let $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ then

$$rank(xy^\top) = 1$$

Section 6

Matrix Inverses

Matrix Inverses

Definition

$A \in \mathbb{C}^{m\times n}$ has a <u>left</u> inverse if $\exists B \in \mathbb{C}^{n\times m}$ such that

$$\underset{n\times m}{B}\ \underset{m\times n}{A} = \underset{n\times n}{I}$$

Definition

$A \in \mathbb{C}^{m\times n}$ has a <u>right</u> inverse if $\exists D \in \mathbb{C}^{n\times m}$ such that

$$\underset{m\times n}{A}\ \underset{n\times m}{C} = \underset{m\times m}{I}$$

Matrix Inverses, cont

Example

The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 7 & 0 \end{pmatrix}.$$

has an infinite number of right inverses, namely

$$C = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{7} \\ c_1 & c_2 \end{pmatrix} \qquad \forall c_1, c_2 \in \mathbb{R}$$

since

$$AC = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Matrix Inverses, cont

- Suppose A has a left inverse, then

$$Ax = b \iff BAx = Bb \iff x = Bb$$

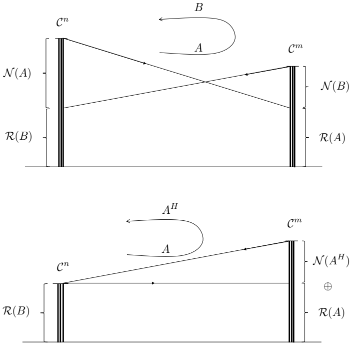
- Suppose A has a right inverse, then let

$$x = Cb \Rightarrow Ax = ACb = b$$

so $x = Cb$ is a solution.

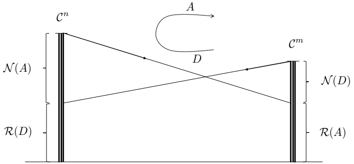
Left Inverse

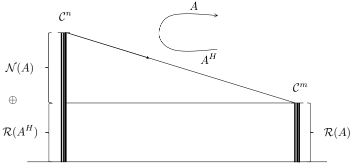
- Let B be a left inverse of A.
- Then $BA = I : \mathbb{C}^n \to \mathbb{C}^n$.
- Of necessity we must have that $\mathcal{N}(A) = \{0\}$, otherwise there are vectors $x \in \mathcal{N}(\mathcal{A}) \subseteq \mathbb{C}^n$ such that $BAx = B0 = 0 \neq x$, i.e., $BA \neq I$.
- Therefore $Ax = b$ has <u>at most</u> one solution (since b may not be in $\mathcal{R}(A)$).



Right Inverse

- Let D be a right inverse of A.
- Then $AD = I : \mathbb{C}^m \rightarrow \mathbb{C}^m$.
- Of necessity we must have that $\mathcal{N}(A^H) = \{0\}$, otherwise $D^H A^H = I$ is impossible.
- $\mathcal{N}(A)$ may be nontrivial therefore if $\hat{x}$ is a solutions so is $\hat{x} + x_n$ where $x_n \in \mathcal{N}(A)$ since $A(\hat{x} + x_n) = A\hat{x} = b$. Therefore, there is at least one solution.





Right and Left Inverses

Lemma

1. *If A has a left inverse then $Ax = b$ has at most one solution.*
2. *If A has a right inverse then $Ax = b$ has at least one solution.*

Regular Inverse

If $A \in \mathbb{C}^{n \times n}$ when the following statements are equivalent:

1. A^{-1} exists
2. $\mathcal{N}(A) = \{0\}$ and $\mathcal{N}(A^H) = \{0\}$.
3. $rank(A) = n$
4. $det(A) \neq 0$
5. (right inverse of A) $=$ (left inverse of A) $= A^{-1}$
6. there are no zero eigenvalues of A
7. $A^H A$ is positive definite
8. A is nonsingular

Regular Inverse, cont.

If A^{-1} exists then

$$\boxed{A^{-1} = \frac{adj(A)}{det(A)}}$$

where $adj(A)$ is the adjugate of A where $adj(A) = [B_{ij}]^\top$ and $B_{ij} = (-1)^{i+j} det(M_{ij})$ and M_{ij} is the $(i,j)^{th}$ minor of A.

Example

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$adj(A) = \begin{pmatrix} (-1)^2|d| & (-1)^3|c| \\ (-1)^3|b| & (-1)^4|a| \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

so $A^{-1} = \dfrac{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}{det(A)} = \dfrac{\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}}{ad-cb}$

Matrix Rank

Lemma

Let $A : \mathbb{C}^n \to \mathbb{C}^m$ then

$$rank(\underset{m\times n}{A}) = rank(\underset{n\times m}{A^H}) = rank(\underset{n\times n}{A^H A}) = rank(\underset{m\times m}{AA^H})$$

Proof.

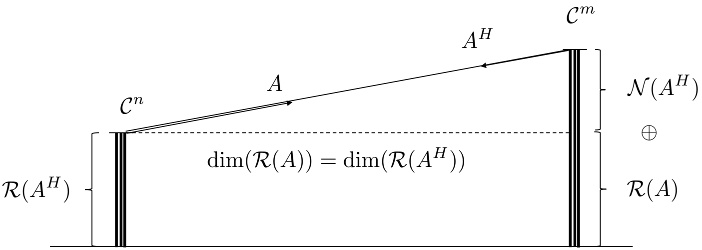
$$\begin{aligned} rank(B) &= \ \# \text{ of linearly independent columns} = \dim(\mathcal{R}(B)) \\ &= \ \# \text{ of linearly independent rows } = \dim(\mathcal{R}(B^H)). \end{aligned}$$

Therefore

$$\begin{aligned} rank(A) &= \dim(\mathcal{R}(A)) = \dim(\mathcal{R}(A^H)) = rank(A^H) \\ &= \dim(\mathcal{R}(AA^H)) = rank(AA^H) \text{ Since } \mathcal{R}(A^*) = \mathcal{R}(AA^*) \\ &= \dim(\mathcal{R}(A^H A)) = rank(A^H A) \text{ Since } \mathcal{R}(A) = \mathcal{R}(A^* A) \end{aligned}$$

Left Inverse: Least Squares

- Consider the solution of $Ax = b$ where $m > n$, i.e., A is tall.
- Assume A is full rank, i.e., $rank(A) = n$.
- Assume $b \in \mathcal{R}(A)$



- Map b to $\mathcal{R}(A^*)$: $A^H b = A^H A x$
- Since $rank(A) = n \iff rank(A^H A) = n$ so $(A^H A)^{-1}$ exists

$$\Rightarrow \boxed{x = (A^H A)^{-1} A^H b}$$

Left Inverse: Least Squares, cont.

What if $b \notin \mathcal{R}(A)$? This is the least squares problem, e.g.

$$\underbrace{\underbrace{\begin{pmatrix} a_1 & 1 \\ a_2 & 1 \\ \vdots & \vdots \\ a_n & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}}_{b}}_{\text{linear regression}}$$

Since there is no solution, it is reasonable to find x that minimizes $\|e\|_2$ where

$$e = Ax - b$$

.

Left Inverse: Least Squares, cont.

- Note that $b = b_r + b_n$ where $b_r \in \mathcal{R}(A)$ and $b_n \in \mathcal{N}(A^H)$ so $e = Ax - b_r - b_n$.
- Since $Ax - b_r \in \mathcal{R}(A) \perp \mathcal{N}(A^H)$ the best we can do is make $Ax = b_r \Rightarrow e = b_n$.
- Since $b_n \in \mathcal{N}(A^H)$ we have

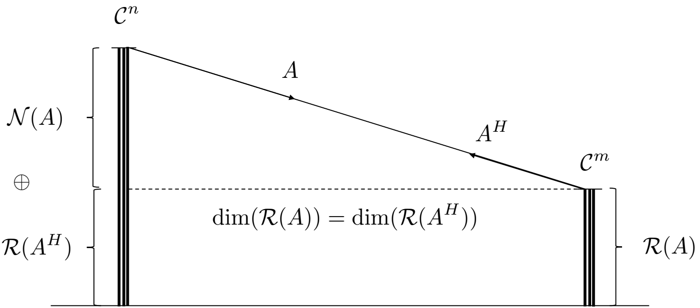
$$
\begin{aligned}
& 0 = A^H Ax - A^H b_r \\
\Rightarrow & \underbrace{A^H Ax}_{\text{projection of } x \text{ onto } \mathcal{R}(A^H)} = A^H b_r = \underbrace{A^H b}_{\text{projection of } b \text{ onto } \mathcal{R}(A^H)}
\end{aligned}
$$

- Since $rank(A^H A) = rank(A) = n$ we have

$$
\underbrace{x = (A^H A)^{-1} A^H b}_{\text{least square solution}}
$$

Right Inverse: Min-Norm Solution

- Consider the solution of $Ax = b$ where $m < n$, i.e., A is fat.
- Assume A is full rank, i.e., $rank(A) = m$.



We would like to solve $Ax = b$ note that since $x = x_r + x_n$ where $x_r \in \mathcal{R}(A^H)$ and $x_n \in \mathcal{N}(A)$ and $\mathcal{N}(A) \neq \{0\}$ there are an infinite number of solutions (i.e. add any thing in $\mathcal{N}(A)$ to a solution). The minimum norm solution will be the element of $\mathcal{R}(A^H)$ that satisfies $Ax_r = b$.

Right Inverse: Min-norm Solution, cont.

$$x_r \in \mathcal{R}(A^H) \Rightarrow x_r = A^H y \text{ where } y \in \mathbb{C}^m$$

so we need to solve

$$(\underset{m\times n}{A}\underset{n\times m}{A^H})\underset{m\times 1}{y} = \underset{m\times 1}{b}$$

Since $rank(A) = rank(AA^H) = m$, $(AA^H)^{-1}$ exists.

$$\Rightarrow y = (AA^H)^{-1} b$$

$$\Rightarrow \boxed{x_r = A^H(AA^H)^{-1} b}$$

Note that this is the same solution as

$$\begin{aligned} \min \quad & \|x\|_2 \\ \text{s.t.} \quad & Ax = b \end{aligned}$$

Right and Left Inverses

Lemma
If $A \in \mathbb{C}^{m\times n}$ where $m > n$ and A is full rank, then $(A^H A)^{-1} A^H$ is a left inverse of A.

Proof.

$$(A^H A)^{-1} A^H A = I_n \qquad \square$$

Lemma
If $A \in \mathbb{C}^{m\times n}$ where $m < n$ and A is full rank, then $A^H (AA^H)^{-1} b$ is a right inverse of A.

Proof.

$$AA^H (AA^H)^{-1} = I_m \qquad \square$$

- Both are examples of pseudo-inverses.
- $A^H (AA^H)^{-1}$ is called the Moore-Penrose pseudo-inverse.
- In Matlab type `pinv(A)`.

Section 7

Matrix Condition Number

Matrix Condition Number

- Suppose that $A \in \mathbb{C}^{n\times n}$ is full rank and A^{-1} is to be computed numerically. How reliable is the computation?
- $Ax = b$ can be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

- Therefore, the solution x is the intersection of $n-$ hyperplanes:

$$\begin{aligned} a_{11}x_1 + \ldots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{n1}x_1 + \ldots + a_{nn}x_n &= b_n \end{aligned}$$

Matrix Condition Number, cont.

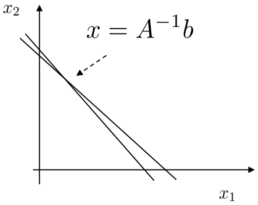
- The problem comes when these hyperplanes are almost parallel.
- In two dimensions we have two lines

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}$$

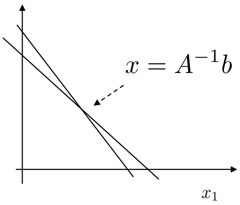
which can be rewritten as

$$\begin{aligned} x_2 &= -\frac{a_{11}}{a_{12}}x_1 + \frac{b_1}{a_{12}} \\ x_2 &= \underbrace{-\frac{a_{21}}{a_{22}}}_{\text{slope}}x_1 + \underbrace{\frac{b_2}{a_{22}}}_{\text{x-intercept}} \end{aligned}$$

Matrix Condition Number, cont.



Small change in y-intercept of second line has large impact on solution.



If the two lines are almost parallel then small changes in the slope or x_2-intercept of either line will result in large changes in $x = A^{-1}b$.

Matrix Condition Number, cont.

- Since computers must represent numbers to finite precision, representation errors could significantly change the numerical solution to the equation $Ax = b$.
- The condition number quantifies this effect.

Definition

The condition number of a square matrix is defined to be

$$\mathcal{K}(A) = \|A\| \left\|A^{-1}\right\|$$

where $\|\cdot\|$ is an induced matrix norm usually taken to be the induced 2-norm.

Matrix Condition Number: Derivation

- Given the two equations $Ax = b$ and $(A + \epsilon E)x = b$ where ϵE is a "small" perturbation of A (introduced by finite machine precision of A)
- Let $x_0 = A^{-1}b$ and

$$
\begin{aligned}
x_E &= (A + \epsilon E)^{-1} b \\
&= [A(I + \epsilon A^{-1}E)]^{-1} b \\
&= (I + \epsilon A^{-1}E)^{-1} A^{-1} b \\
&= \underbrace{(I + \epsilon A^{-1}E)^{-1}}_{\text{perturbation}} x_0
\end{aligned}
$$

Matrix Condition Number: Derivation, cont.

Using the Neumann expansion gives

$(I + \epsilon A^{-1}E)^{-1} = \sum_{i=0}^{\infty} (-\epsilon A^{-1}E)^i$. Therefore

$$\begin{aligned} x_E &= (I + \epsilon A^{-1}E)^{-1} A^{-1} b \\ &= (I - \epsilon A^{-1}E) A^{-1} b + O(\|\epsilon E\|^2 x_0) \\ &= A^{-1}b - \epsilon A^{-1}EA^{-1}b + O(\|\epsilon E\|^2 x_0) \\ &= x_0 - \epsilon A^{-1}Ex_0 + O(\|\epsilon E\|^2 x_0) \end{aligned}$$

Therefore

$$\underbrace{\frac{\|x_E - x_0\|}{\|x_0\|}}_{\text{relative change in the solution}} \leq \underbrace{\epsilon \left\|A^{-1}\right\| \|E\|}_{\text{want to relate to relative change in } A} + O(\|\epsilon E\|^2)$$

Matrix Condition Number: Derivation, cont.

What is the relative change in A?

$$\frac{\|A - (A + \epsilon E)\|}{\|A\|} = \frac{\epsilon \|E\|}{\|A\|} \triangleq \rho$$

Therefore

$$\frac{\|x_E - x_0\|}{\|x_0\|} \leq \rho \underbrace{\|A^{-1}\| \, \|A\|}_{\mathcal{K}(A)} + O(\|\epsilon E\|^2)$$

The condition number $\mathcal{K}(A)$ relates (approximately) the relative change in A to the relative change in the solution x_0.

Matrix Condition Number: Implication

Rule of Thumb:

If the solution is computed to n digits then only

$$n - \log_{10} \mathcal{K}(A)$$

can be considered to be accurate.

Section 8

Schur Complement and the Matrix Inversion Lemma

Schur Complement

Definition

Consider the partitioned matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}.$$

1. When A_{11} is non-singular,

   $$S_{ch}(A_{11}) \triangleq A_{22} - A_{21}A_{11}^{-1}A_{12}$$

   is called the Schur Complement of A_{11} in A.

2. When A_{22} is non-singular,

   $$S_{ch}(A_{22}) \triangleq A_{11} - A_{12}A_{22}^{-1}A_{21}$$

   is called the Schur Complement of A_{22} in A.

Schur Complement, cont.

Lemma

When A_{11} is nonsingular, A is nonsingular if and only if $S_c h(A_{11})$ is nonsingular, in which case

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} S_{ch}^{-1}(A_{11}) \\ -S_{ch}^{-1}(A_{11}) A_{21} A_{11}^{-1} & S_{ch}^{-1}(A_{11}) \end{bmatrix}$$

Lemma

When A_{22} is nonsingular, A is nonsingular if and only if $S_c h(A_{22})$ is nonsingular, in which case

$$A^{-1} = \begin{bmatrix} S_{ch}^{-1}(A_{22}) & -S_{ch}^{-1}(A_{22}) A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{12} S_{ch}^{-1}(A_{22}) & A_{22}^{-1} + A_{22}^{-1} A_{21} S_{ch}^{-1}(A_{22}) A_{12} A_{22}^{-1} \end{bmatrix}$$

Proof.

By direct manipulation. $\square$

Matrix Inversion Lemma

Lemma (Matrix Inversion Lemma)

If $A \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{m\times m}$ *are invertible, and* $X \in \mathbb{R}^{n\times m}$ *and* $Y \in \mathbb{R}^{m\times n}$ *then*

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1}$$

Proof.

Equate the $(2,2)$ elements of A^{-1} in the previous slide, and re-label matrices. $\square$

Matrix Inversion Lemma, cont.

- A special case of this matrix inversion lemma is the formula

$$(A + xy^H)^{-1} = A^{-1} - \frac{A^{-1}xy^H A^{-1}}{1 + y^H A^{-1}x}$$

where x and y are vectors.

- Sylvester's inequality gives

$$rank(x) + rank(y) - 1 \leq rank(xy^H) \leq min(rank(x), rank(y)).$$

But

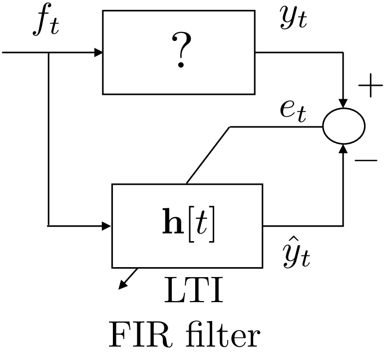
$$rank(x) + rank(y) - 1 = 1$$
$$min(rank(x), rank(y)) = 1$$

- Therefore $rank(xy^H) = 1$

Section 9

Recursive Least Squares Filtering

Least Squares Filtering Problem



Problem Statement: Given the input data f_t and y_t, find the FIR filter coefficients $\mathbf{h}[t]$ that minimize the running least squared error e_t.

Least Squares Filtering Problem

Definition (Least Squares Filtering Problem)

Given the filter

$$\hat{y}_t = \sum_{i=1}^{m} h_i f_{t-i}$$

where the inputs f_t are known and we measure the actual outputs y_t, find the coefficients h_i such that the mean squared error

$$E = \sum_{i=1}^{m} (y_i - \hat{y}_i)^2$$

is minimized.

Batch Least Squares Filtering

If we assume $f_t = 0, t \leq 0$ we get

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} = \begin{pmatrix} f_1 & 0 & \cdots & \cdots & 0 \\ f_2 & f_1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & \\ f_m & f_{m-1} & \cdots & \cdots & f_1 \\ f_{m+1} & f_m & f_{m-1} & \cdots & f_2 \\ \vdots & & & \ddots & \\ f_N & f_{N-1} & \cdots & \cdots & f_{N-m+1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_m \end{pmatrix}$$

Batch Least Squares Filtering, cont.

Define

$$\mathbf{q}_i = \begin{pmatrix} f_i & f_{i-1} & \dots f_{i-m+1} \end{pmatrix}^H$$

$$\mathbf{y}_N = \begin{pmatrix} \bar{y}_1 & \bar{y}_2 & \dots \bar{y}_N \end{pmatrix}^H$$

$$\mathbf{h}[N] = \begin{pmatrix} \bar{h}_1[N] & \bar{h}_2[N] & \dots \bar{h}_m[N] \end{pmatrix}^H$$

$$A_N = \begin{pmatrix} \mathbf{q}_1^H \\ \vdots \\ \mathbf{q}_m^H \end{pmatrix},$$

then the least squares problem reduces to

$$\mathbf{e}_N = \mathbf{y}_N - \underbrace{A_N \mathbf{h}[N]}_{\hat{\mathbf{y}}_N}$$

where $\mathbf{e}_N$ is the error to be minimized. From the projection theorem, $\|\mathbf{e}\|_2$ is minimized when

$$\underset{m\times 1}{\mathbf{h}[N]} = (\underset{m\times N}{A_N^H}\,\underset{N\times m}{A_N})^{-1}\,\underset{m\times N}{A_N^H}\,\underset{N\times 1}{\mathbf{y}_N}.$$

Batch Least Squares Filtering

- Note that the size of y_N and A_N grow linearly with time N.
- Therefore, each time step requires more computation than the last step. This is obviously problematic as $N \to \infty$.
- For some N, batch least squares is no longer a real-time algorithm.
- Note that at time $N + 1$ the data include new samples, but includes all of the data available at time N.

??? Is is possible to design an algorithm with fixed computational cost at each time step, that produces the same least squares solution?

Recursive Least Squares Filtering

Define

$$
\begin{aligned}
\mathbf{q}_t &= \begin{pmatrix} f_i & f_{i-1} & \ldots f_{i-m+1} \end{pmatrix}^H \\
\mathbf{y}_t &= \begin{pmatrix} \bar{y}_1 & \bar{y}_2 & \ldots \bar{y}_t \end{pmatrix}^H \\
\mathbf{h}[t] &= \begin{pmatrix} \bar{h}_1[t] & \bar{h}_2[t] & \ldots \bar{h}_m[t] \end{pmatrix}^H \\
A_t &= \begin{pmatrix} \mathbf{q}_1^H \\ \vdots \\ \mathbf{q}_t^H \end{pmatrix}.
\end{aligned}
$$

Then at time t we have $\mathbf{e}_t = \mathbf{y}_t - A_t \mathbf{h}[t]$. From the projection theorem, the error is minimized when

$$\mathbf{h}[t] = (A_t^H A_t)^{-1} A_t^H \mathbf{y}_t.$$

Recursive Least Squares Filtering, cont.

Let

$$R_{t-1} \triangleq A_{t-1}^H A_{t-1} = \begin{pmatrix} \mathbf{q}_1 & \cdots & \mathbf{q}_{t-1} \end{pmatrix} \begin{pmatrix} \mathbf{q}_1^H \\ \vdots \\ \mathbf{q}_{t-1}^H \end{pmatrix}$$
$$= \sum_{i=1}^{t-1} \mathbf{q}_i \mathbf{q}_i^H$$

be the associated Grammian when there are $t-1$ samples.

Suppose that we receive new data q_t and y_t at time t.

Then

$$\begin{aligned} R_t &= \sum_{i=1}^{t} \mathbf{q}_i \mathbf{q}_i^H \\ &= \sum_{i=1}^{t-1} \mathbf{q}_i \mathbf{q}_i^H + \mathbf{q}_t \mathbf{q}_t^H \\ &= R_{t-1} + \mathbf{q}_t \mathbf{q}_t^H. \end{aligned}$$

Recursive Least Squares Filtering, cont.

In the solution $\mathbf{h}_t = (A_t^H A_t)^{-1} A_t^H \mathbf{y}_t$, we need $R_t^{-1} \triangleq (A_t^H A_t)^{-1}$. Note that

$$R_t^{-1} = (\underbrace{R_{t-1}}_{A} + \underbrace{q_t}_{X} \underbrace{}_{R=1} \underbrace{q_t^H}_{Y})^{-1}$$

and recall the matrix inversion lemma:

$$(A + XRY)^{-1} = A^{-1} - A^{-1} X (R^{-1} + Y A^{-1} X)^{-1} Y A^{-1}$$

Therefore

$$R_t^{-1} = R_{t-1}^{-1} - R_{t-1}^{-1} \mathbf{q}_t (1 + \mathbf{q}_t^H R_{t-1}^{-1} \mathbf{q}_t)^{-1} \mathbf{q}_t^H R_{t-1}^{-1}.$$

Recursive Least Squares Filtering, cont.

Defining $P_t = R_t^{-1}$ gives

$$P_t = P_{t-1} - \frac{P_{t-1}\mathbf{q}_t\mathbf{q}_t^H P_{t-1}}{1 + \mathbf{q}_t^H P_{t-1}\mathbf{q}_t}.$$

Define the (Kalman) gain as

$$\mathbf{k}_t = \frac{P_{t-1}\mathbf{q}_t}{1 + \mathbf{q}_t^H P_{t-1}\mathbf{q}_t}$$

Then

$$P_t = P_{t-1} - \mathbf{k}_t\mathbf{q}_t^H P_{t-1}.$$

Note that we have found a fixed computational scheme to update

$$P_t = (A_t^H A_t)^{-1}$$

using old data P_{t-1} and new data $\mathbf{q}_t$.

Recursive Least Squares Filtering, cont.

In the solution $\mathbf{h}[t] = (A_t^H A_t)^{-1} A_t^H \mathbf{y}_t$, we have found a clever way to update $P_t = (A_t^H A_t)^{-1}$ recursively. Define

$$\mathbf{z}_t \triangleq A_t^H \mathbf{y}_t.$$

We need a recursive update for $\mathbf{z}_t$.
Toward that end note that

$$\begin{aligned}
\mathbf{z}_t &= A_t^H \mathbf{y}_t \\
&= \sum_{i=1}^{t} \mathbf{q}_i y_i \\
&= \sum_{i=1}^{t-1} \mathbf{q}_i y_i + \mathbf{q}_t y_t \\
&= \mathbf{z}_{t-1} + \mathbf{q}_t y_t
\end{aligned}$$

Recursive Least Squares Filtering, cont.

Therefore

$$\begin{aligned}
\mathbf{h}_t &= (A_t^H A_t)^{-1} A_t^H \mathbf{y}_t \\
&= P_t \mathbf{z}_t \\
&= (P_{t-1} - \mathbf{k}_t \mathbf{q}_t^H P_{t-1})(\mathbf{z}_{t-1} + \mathbf{q}_t y_t) \\
&= P_{t-1}\mathbf{z}_{t-1} - \mathbf{k}_t \mathbf{q}_t^H P_{t-1}\mathbf{z}_{t-1} + P_{t-1}\mathbf{q}_t y_t - \mathbf{k}_t \mathbf{q}_t^H P_{t-1}\mathbf{q}_t y_t \\
&= \mathbf{h}_{t-1} - \mathbf{k}_t \mathbf{q}_t^H \mathbf{h}_{t-1} + \underbrace{\left(P_{t-1} - \mathbf{k}_t \mathbf{q}_t^H P_{t-1}\right)}_{P_t} \mathbf{q}_t y_t \\
&= \mathbf{h}_{t-1} + \mathbf{k}_t (y_t - \mathbf{q}_t^H \mathbf{h}_{t-1}) \\
\implies \mathbf{h}_t &= \mathbf{h}_{t-1} + \mathbf{k}_t (y_t - \hat{y}),
\end{aligned}$$

where we have used the fact that $P_t q_t = \mathbf{k}_t$.
Note that $\hat{y}_t = \mathbf{q}_t^H \mathbf{h}_{t-1}$ is the predicted output, and $e_t = y_t - \hat{y}_t$ is the quantity that is being minimized.

Summary: Recursive Least Squares Filtering

At time $t = 0$ initialize algorithm with

$$\begin{aligned} P_0 &= \alpha I, \text{ where } \alpha > 0 \text{ is a large number} \\ \mathbf{h}_0 &= 0. \end{aligned}$$

At time t, get y_t, f_t, and compute $\mathbf{q}_t$ from f_t. Update the least squares estimate using

$$\begin{aligned} \mathbf{k}_t &= \frac{P_{t-1}\mathbf{q}_t}{1 + \mathbf{q}_t^H P_{t-1}\mathbf{q}_t} \\ P_t &= P_{t-1} - \mathbf{k}_t \mathbf{q}_t^H P_{t-1} \\ \mathbf{h}_t &= \mathbf{h}_{t-1} + \mathbf{k}_t (y_t - \mathbf{q}_t^H \mathbf{h}_{t-1}). \end{aligned}$$

This is equivalent to a discrete time Kalman filter with stationary dynamics.