Linearized MHD equations in 1D

Model:

- Homogenous plasma ($\rho_0, p_0 = const.$) in a constant magnetic field $\vec{B}_0$
- Consider small pertubations from equilibrium (e.g. $\rho = \rho_0 + \rho_1$)

$$\partial_t \underbrace{\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ \rho_1 \\ p_1 \end{pmatrix}}_{U} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & B_{oy}/\rho_0 & B_{oz}/\rho_0 & 0 & 1/2\rho_0 \\ 0 & 0 & 0 & 0 & -B_{0x}/\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_{0x}/\rho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{0y} & -B_{0x} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{0z} & 0 & -B_{0x} & 0 & 0 & 0 & 0 & 0 \\ \rho_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \partial_x \underbrace{\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ \rho_1 \\ p_1 \end{pmatrix}}_{U} = 0$$

$$\partial_t U + A \partial_x U = 0 \qquad \text{with} \qquad A \in \mathbb{R}^{8x8}$$

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$$\partial_t U + A\partial_x U = 0 \qquad \text{with} \qquad A \in \mathbb{R}^{8x8}$$

Analytic solution: $U = U_0 f(x - vt)$

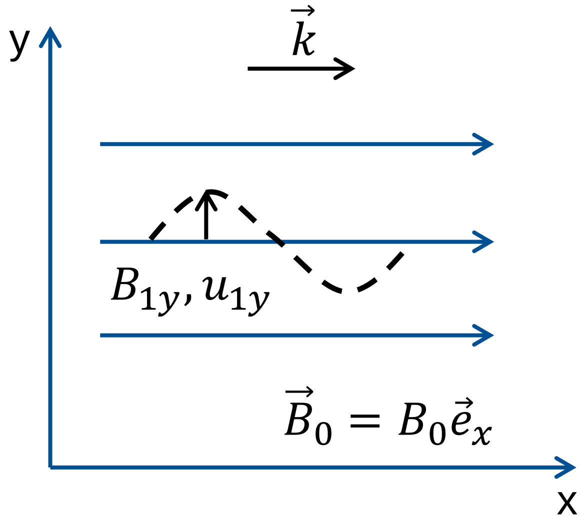
Eigenvalue problem: $AU_0 = vU_0$

- Eigenvalues v are the characteristic velocities of the system
- Eigenvectors U_0 define possible initial conditions

Transverse Alfvén wave

Model:

$$\partial_t \begin{pmatrix} u_{1y} \\ B_{1y} \end{pmatrix} + \begin{pmatrix} 0 & -B_0/\rho_0 \\ -B_0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1y} \\ B_{1y} \end{pmatrix} = 0$$

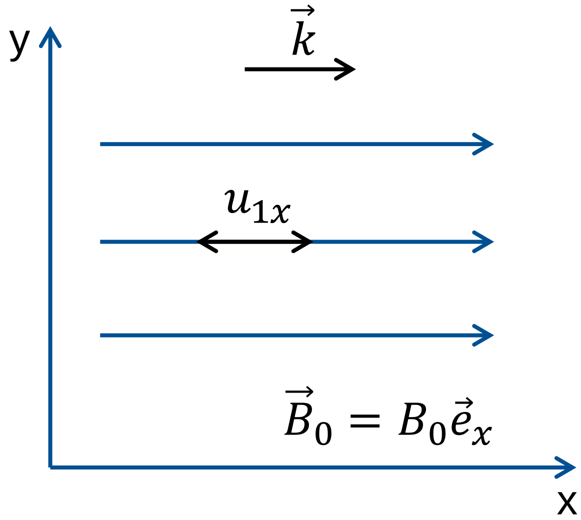


- Transverse wave: field line bending vs. ion inertia
- Characteristic velocity: Alfvén velocity $v_A = B_0/\sqrt{\rho_0}$

Sound wave

Model:

$$\partial_t \begin{pmatrix} u_{1x} \\ \rho_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1/2\rho_0 \\ \rho_0 & 0 & 0 \\ \gamma p_0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1x} \\ \rho_1 \\ p_1 \end{pmatrix} = 0$$

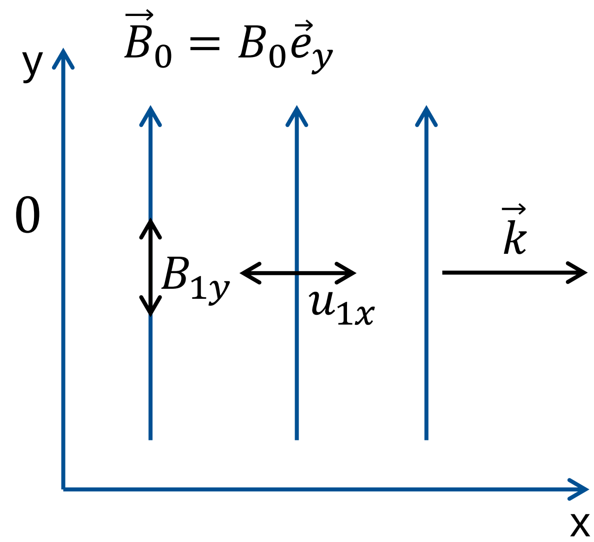


- Longitudinal wave: Periodic compression parallel to magnetic field
- Characteristic velocity: speed of sound $c_S = \sqrt{\gamma p_0 / 2\rho_0}$

Longitudinal Alfvén wave (fast magnetosonic wave)

Model:

$$\partial_t \begin{pmatrix} u_{1x} \\ B_{1y} \\ \rho_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1/2\rho_0 \\ B_0 & 0 & 0 & 0 \\ \rho_0 & 0 & 0 & 0 \\ \gamma p_0 & 0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1x} \\ B_{1y} \\ \rho_1 \\ p_1 \end{pmatrix} = 0$$



- Mixture of Alfvén and sound wave
- Characteristic velocity: $v = \sqrt{v_A^2 + c_S^2}$

Discretization

Model:

$$\partial_t U + A\partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

1. Explicit upwind scheme

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + A\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0$$

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + A\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $U^{n+1} = TU^n$, $\qquad T \in \mathbb{R}^{8Nx8N}$

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Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $U^{n+1} = TU^n,$ $\qquad T \in \mathbb{R}^{8Nx8N}$

Observations:

- Strong diffusion if $\frac{v\delta t}{\delta x} < 1$, no diffusion if $\frac{v\delta t}{\delta x} = 1$
- Unstable if $\frac{v\delta t}{\delta x} > 1$
- Becomes eventually unstable even if $\frac{v\delta t}{\delta x} < 1$

Discretization

Model:

$$\partial_t U + A\partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

2. Implicit upwind scheme

$$\frac{U_j^n - U_j^{n-1}}{\delta t} + A\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0$$

$$\frac{U_j^n - U_j^{n-1}}{\delta t} + A\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $TU^{n+1} = U^n$, $\qquad T \in \mathbb{R}^{8Nx8N}$

Discretization

2. Implicit upwind scheme

$$\left.\begin{aligned}
&\frac{U_j^n - U_j^{n-1}}{\delta t} + A\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0 \\
&\frac{U_j^n - U_j^{n-1}}{\delta t} + A\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0
\end{aligned}\right\}$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $TU^{n+1} = U^n$, $\qquad T \in \mathbb{R}^{8Nx8N}$

Observations:

- Strong diffusion if $\frac{v\delta t}{\delta x} \geq 1$
- Unstable if $\frac{v\delta t}{\delta x} < 1$

Discretization

Model:

$$\partial_t U + A\partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

3. Lax-Wendroff scheme

$$U_j^{n+1} = \left[\mathbb{1}_8 - \left(\frac{\delta t}{\delta x}\right)^2 A^2\right] U_j^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 - \frac{\delta t}{2\delta x} A\right] U_{j+1}^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 + \frac{\delta t}{2\delta x} A\right] U_{j-1}^n$$

Compact: $U^{n+1} = TU^n,$ $\qquad T \in \mathbb{R}^{8Nx8N}$

Discretization

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$$U_j^{n+1} = \left[\mathbb{1}_8 - \left(\frac{\delta t}{\delta x}\right)^2 A^2\right] U_j^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 - \frac{\delta t}{2\delta x} A\right] U_{j+1}^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 + \frac{\delta t}{2\delta x} A\right] U_{j-1}^n$$

Compact: $U^{n+1} = TU^n,\qquad T \in \mathbb{R}^{8Nx8N}$

Observations:

- Strong diffusion if $\frac{v\delta t}{\delta x} < 1$, no diffusion if $\frac{v\delta t}{\delta x} = 1$
- Unstable if $\frac{v\delta t}{\delta x} > 1$
- Becomes eventually unstable even if $\frac{v\delta t}{\delta x} < 1$

Nonlinear MHD in 1D

Conservative form:

$$\partial_t U + \partial_x F(U) = 0$$

Start with nonlinear sound wave (i.e. $\vec{B} = 0$):

$$\partial_t \underbrace{\begin{pmatrix} \rho \\ \rho u_x \\ \frac{1}{2}\rho u_x^2 + \frac{p}{2(\gamma-1)} \end{pmatrix}}_{U} + \partial_x \underbrace{\begin{pmatrix} \rho u_x \\ \rho u_x^2 + \frac{1}{2} p \\ \frac{1}{2}\rho u_x^3 + \frac{\gamma}{2(\gamma-1)} p u_x \end{pmatrix}}_{F(U)} = 0$$

$$\Rightarrow U = (\rho, \rho u_x, \epsilon)^T = (U_0, U_1, U_2)^T \qquad \epsilon = \frac{1}{2}\rho u_x^2 + \frac{p}{2(\gamma-1)}$$

Nonlinear MHD in 1D

Transformation formulas:

$$\begin{pmatrix} \rho \\ u_x \\ p \end{pmatrix} = \begin{pmatrix} U_0 \\ U_1/U_0 \\ 2(\gamma-1)U_2 - (\gamma-1)U_1^2/U_0 \end{pmatrix}$$

Flux $F(U)$:

$$F(U) = \begin{pmatrix} U_1 \\ (\gamma-1)U_2 + U_1^2/U_0(3/2-\gamma/2) \\ \dfrac{(1-\gamma)\,U_1^3}{2U_0^2} + \dfrac{\gamma U_2 U_1}{U_0} \end{pmatrix}$$

Nonlinear MHD in 1D

Jacobian of Flux:

$$J = \left(\frac{\partial F}{\partial U}\right)_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ U_1^2/U_0^2(\gamma/2 - 3/2) & U_1/U_0(3-\gamma) & \gamma - 1 \\ (\gamma-1)U_1^3/U_0^3 - \gamma U_2 U_1/U_0^2 & \dfrac{3(1-\gamma)U_1^2}{2U_0^2} + \gamma U_2/U_0 & \gamma U_1/U_0 \end{pmatrix}$$

Nonlinear MHD in 1D

Discretization with Lax-Wendroff scheme:

$$U_j^{n+1} = U_j^n - \frac{\delta t}{2\delta x}\left[F\left(U_{j+1}^n\right) - F\left(U_{j-1}^n\right)\right]$$

$$+ \frac{\delta t^2}{2\delta x^2}\left[J_{j+1/2}\left(F\left(U_{j+1}^n\right) - F\left(U_j^n\right)\right) - J_{j-1/2}\left(F\left(U_j^n\right) - F\left(U_{j-1}^n\right)\right)\right]$$

Evaluation of Jacobian:

$$J_{j\pm 1/2} = J\left(1/2\left(U_j^n + U_{j\pm 1}\right)\right)$$

Next steps

Extension of linear code to 2D:

$$\partial_t U + A\partial_x U + B\partial_y U = 0$$