Linearized MHD equations in 1D

Model:

- Homogenous plasma ($\rho_0, p_0 = const.$) in a constant magnetic field $\vec{B}_0$
- Consider small pertubations from equilibrium (e.g. $\rho = \rho_0 + \rho_1$)

$$\partial_t \underbrace{\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ \rho_1 \\ p_1 \end{pmatrix}}_{U} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & B_{oy}/\rho_0 & B_{oz}/\rho_0 & 0 & 1/2\rho_0 \\ 0 & 0 & 0 & 0 & -B_{0x}/\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_{0x}/\rho_0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{0y} & -B_{0x} & 0 & 0 & 0 & 0 & 0 & 0 \\ B_{0z} & 0 & -B_{0x} & 0 & 0 & 0 & 0 & 0 \\ \rho_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \gamma p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \partial_x \underbrace{\begin{pmatrix} u_{1x} \\ u_{1y} \\ u_{1z} \\ B_{1x} \\ B_{1y} \\ B_{1z} \\ \rho_1 \\ p_1 \end{pmatrix}}_{U} = 0$$

$$\partial_t U + A \partial_x U = 0 \quad \text{with} \quad A \in \mathbb{R}^{8x8}$$

Linearized MHD equations in 1D

Model:

- Homogenous plasma ($\rho_0, p_0 = const.$) in a constant magnetic field $\vec{B}_0$
- Consider small pertubations from equilibrium (e.g. $\rho = \rho_0 + \rho_1$)

$$\partial_t U + A\partial_x U = 0 \qquad \text{with} \qquad A \in \mathbb{R}^{8x8}$$

Analytic solution: $U = U_0 f(x - vt)$

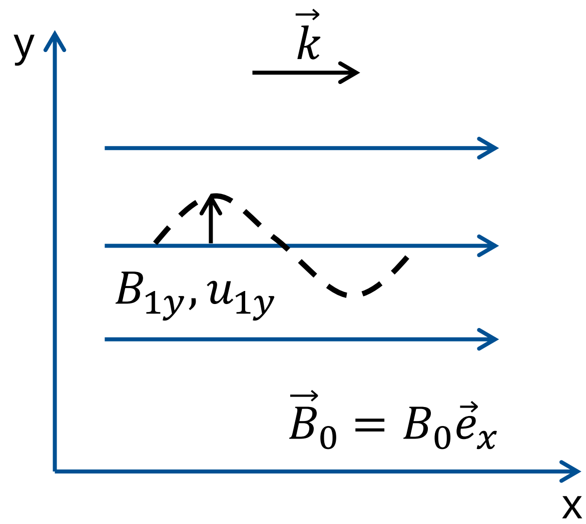
Eigenvalue problem: $AU_0 = vU_0$

- Eigenvalues v are the characteristic velocities of the system
- Eigenvectors U_0 define possible initial conditions

Transverse Alfvén wave

Model:

$$\partial_t \begin{pmatrix} u_{1y} \\ B_{1y} \end{pmatrix} + \begin{pmatrix} 0 & -B_0/\rho_0 \\ -B_0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1y} \\ B_{1y} \end{pmatrix} = 0$$

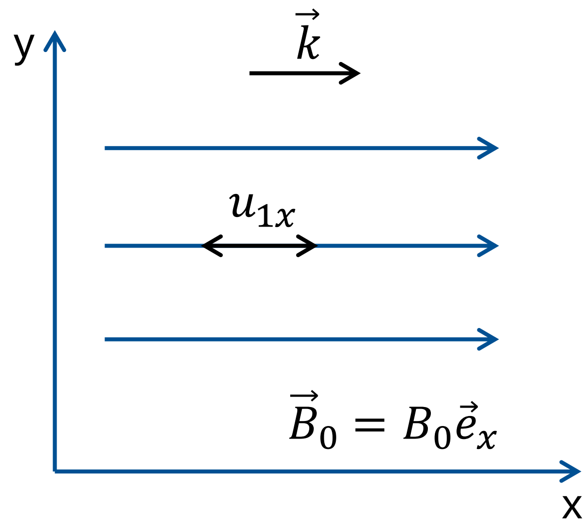


- Transverse wave: field line bending vs. ion inertia
- Characteristic velocity: Alfvén velocity $v_A = B_0/\sqrt{\rho_0}$

Sound wave

Model:

$$\partial_t \begin{pmatrix} u_{1x} \\ \rho_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1/2\rho_0 \\ \rho_0 & 0 & 0 \\ \gamma p_0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1x} \\ \rho_1 \\ p_1 \end{pmatrix} = 0$$

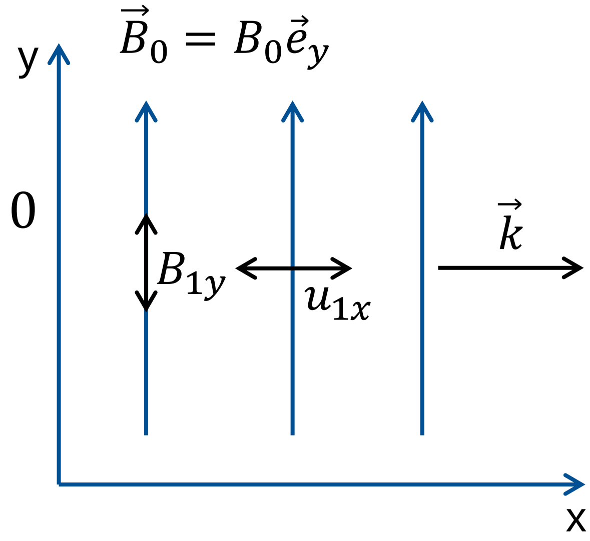


- Longitudinal wave: Periodic compression parallel to magnetic field
- Characteristic velocity: speed of sound $c_S = \sqrt{\gamma p_0 / 2\rho_0}$

Longitudinal Alfvén wave (fast magnetosonic wave)

Model:

$$\partial_t \begin{pmatrix} u_{1x} \\ B_{1y} \\ \rho_1 \\ p_1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 1/2\rho_0 \\ B_0 & 0 & 0 & 0 \\ \rho_0 & 0 & 0 & 0 \\ \gamma p_0 & 0 & 0 & 0 \end{pmatrix} \partial_x \begin{pmatrix} u_{1x} \\ B_{1y} \\ \rho_1 \\ p_1 \end{pmatrix} = 0$$



- Mixture of Alfvén and sound wave
- Characteristic velocity: $v = \sqrt{v_A^2 + c_S^2}$

Discretisation

Model:

$$\partial_t U + A\partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

1. Explicit upwind scheme

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + A\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0$$

$$\frac{U_j^{n+1} - U_j^n}{\delta t} + A\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $U^{n+1} = TU^n,$ $\qquad T \in \mathbb{R}^{8Nx8N}$

Discretisation

1. Explicit upwind scheme

$$\left.\begin{aligned} &\frac{U_j^{n+1} - U_j^n}{\delta t} + A\,\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0 \\ &\frac{U_j^{n+1} - U_j^n}{\delta t} + A\,\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0 \end{aligned}\right\}$$

Periodic boundary conditions:
$$U_N^n = U_0^n$$

Compact: $U^{n+1} = TU^n,$ $\qquad T \in \mathbb{R}^{8Nx8N}$

Observations/Problems:

- Strong diffusion if $\frac{v\delta t}{\delta x} < 1$, no diffusion if $\frac{v\delta t}{\delta x} = 1$
- Unstable if $\frac{v\delta t}{\delta x} > 1$
- Unstable for some configurations (e.g. long. Alfvén wave) even if $\frac{v\delta t}{\delta x} < 1$

Discretisation

Model:

$$\partial_t U + A \partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

2. Implicit upwind scheme

$$\frac{U_j^n - U_j^{n-1}}{\delta t} + A \frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0$$

$$\frac{U_j^n - U_j^{n-1}}{\delta t} + A \frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $TU^{n+1} = U^n$, $\qquad T \in \mathbb{R}^{8Nx8N}$

Discretisation

2. Implicit upwind scheme

$$\left.\begin{aligned} &\frac{U_j^n - U_j^{n-1}}{\delta t} + A\,\frac{U_j^n - U_{j-1}^n}{\delta x} = 0, \qquad v > 0 \\ &\frac{U_j^n - U_j^{n-1}}{\delta t} + A\,\frac{U_{j+1}^n - U_j^n}{\delta x} = 0, \qquad v < 0 \end{aligned}\right\}$$

Periodic boundary conditions:

$$U_N^n = U_0^n$$

Compact: $TU^{n+1} = U^n,$ $\qquad T \in \mathbb{R}^{8Nx8N}$

Problem:

- Update matrix T not invertible since det(T)=Tr(T)=0

Discretisation

Model:

$$\partial_t U + A\partial_x U = 0, \qquad A \in \mathbb{R}^{8x8} = const.$$

3. Lax-Wendroff scheme

$$U_j^{n+1} = \left[\mathbb{1}_8 - \left(\frac{\delta t}{\delta x}\right)^2 A^2\right] U_j^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 - \frac{\delta t}{2\delta x} A\right] U_{j+1}^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 + \frac{\delta t}{2\delta x} A\right] U_{j-1}^n$$

Compact: $U^{n+1} = TU^n, \qquad T \in \mathbb{R}^{8Nx8N}$

Discretisation

3. Lax-Wendroff scheme

$$U_j^{n+1} = \left[\mathbb{1}_8 - \left(\frac{\delta t}{\delta x}\right)^2 A^2\right] U_j^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 - \frac{\delta t}{2\delta x} A\right] U_{j+1}^n + \left[\frac{1}{2}\left(\frac{\delta t}{\delta x}\right)^2 A^2 + \frac{\delta t}{2\delta x} A\right] U_{j-1}^n$$

Compact: $U^{n+1} = TU^n,$ $T \in \mathbb{R}^{8Nx8N}$

Observations/Problems:

- Strong diffusion if $\frac{v\delta t}{\delta x} < 1$, no diffusion if $\frac{v\delta t}{\delta x} = 1$
- Unstable if $\frac{v\delta t}{\delta x} > 1$
- Works only for $v > 0$

Next steps

Nonlinear equations with finite volumes:

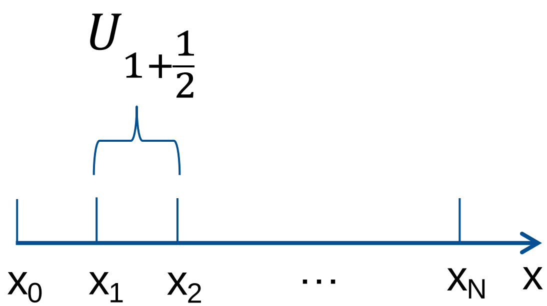
$$\partial_t U + \partial_x F(U) = G(U)$$

e.g. nonlinear sound wave:

$$\partial_t \underbrace{\begin{pmatrix} u_x \\ \rho u_x \\ p \end{pmatrix}}_{U} + \partial_x \underbrace{\begin{pmatrix} \rho u_x \\ \rho u_x^2 + \frac{1}{2} p \\ p u_x \end{pmatrix}}_{F(U)} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ -(\gamma - 1) p \partial_x u_x \end{pmatrix}}_{?}$$

Next steps

Nonlinear equations with finite volumes:



$$\partial_t U + \partial_x F(U) = 0$$

Integration from x_i to x_{i+1} yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\frac{1}{\delta x}\int_{x_i}^{x_{i+1}} U\,\mathrm{d}x}_{U_{i+\frac{1}{2}}} + \frac{1}{\delta x}\left[F\big(U(t,x_{i+1})\big) - F\big(U(t,x_i)\big)\right] = 0$$

Forward difference in time and upwind flux:

$$U^{n+1}_{i+\frac{1}{2}} = U^{n}_{i+\frac{1}{2}} - \frac{\delta t}{\delta x}\left[F^n\left(U_{i+\frac{1}{2}}\right) - F^n\left(U_{i-\frac{1}{2}}\right)\right]$$

Next steps

Extension of linear code to 2D:

$$\partial_t U + A\partial_x U + B\partial_y U = 0$$