Topological data analysis
Lecture 5

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Persistent homology: practical algorithm

- Previously we had $D_1, D_2, \ldots$ matrices of simplicial differentials $\partial_i \colon C_i(K) \to C_{i-1}(K)$
- D_i is the incidence matrix between i-dim simplices and $(i-1)$-dim simplices at least over $\mathbb{Z}_2$.
- Now we combine them all in a huge matrix D of size $N \times N$,
- where N is the total number of simplices in a filtration.
- The order of rows and columns as in the array Simplices.

Gauss elimination on columns

- Reduce D by elementary operations on columns.
- It is allowed to any column, to add another column multiplied by a scalar.
- We move from left to right checking all $1 \leqslant j \leqslant N$.
- Let $\mathrm{low}(j)$ denote the index of the lowest nonzero element in j-th column.
- For each j we loop through $r < j$, and look for r such that $\mathrm{low}(r) = \mathrm{low}(j)$.
- When we meet such r, we reduce j-th column by r-th column.
- We loop until $\mathrm{low}\colon [N] \to [N]$ becomes injective. This is called the reduced form of a matrix.
- Permutations are not allowed.

You have the Matrix...

	1	2	3	4	23	34	12	24	234	13	14	123	124	134	1234
1					0	0	1	0		1	1				
2					1	0	1	1		0	0				
3					1	1		0		1	0				
4						1		1	0		1				
23									1			1			
34									1			0	0	1	
12									0			1	1	0	
24									1			0	1	0	
234												0	0	0	1
13												1	0	1	0
14		0								0			1	1	0
123		0			0					0					1
124		0		0	0		0			0	0				1
134		0		0	0		0			0	0				1
1234		0		0	0		0			0	0				

After Gauss elimination in columns

	1	2	3	4	23	34	12	24	234	13	14	123	124	134	1234
1							1								
2					1		1								
3					1	1									
4						1									
23									1						
34									1						
12												1	1		
24									1				1		
234															1
13												1			
14													1		
123															1
124															1
134															1
1234															

After Gauss

Look at the reduced matrix M.

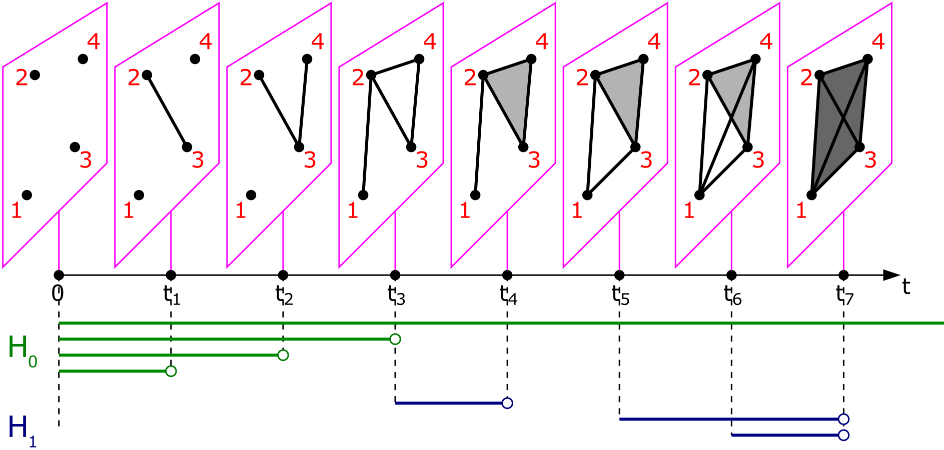
Theorem

- Let $j = \mathrm{low}(s) \neq 0$ in M. Then **Simplices[j]** gives birth to homology of rank dim Simplices[j] and **Simplices[s]** kills this homology. Each such pair (j, s) gives rise to the interval module $I_{[BirthTimes[j], BirthTimes[s])}$.
- If the whole r-th column vanish and moreover $r \notin \mathrm{low}([N])$, then Simplices[r] gives birth to homology of rank dim Simplices[r] which never dies. It gives the interval module $I_{[BirthTimes[r], +\infty)}$.

Persistent homology of the filtration is the direct sum of all the listed interval modules.

Example

Simplices = [1, 2, 3, 4, 23, 34, 12, 24, 234, 13, 14, 123, 124, 134, 1234]
BirthTimes = [0, 0, 0, 0, t1, t2, t3, t3, t4, t5, t6, t7, t7, t7, t7]



Our reduced matrix

	1	2	3	4	23	34	12	24	234	13	14	123	124	134	1234
1							1								
2					1		1								
3					1	1									
4						1									
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34									1						
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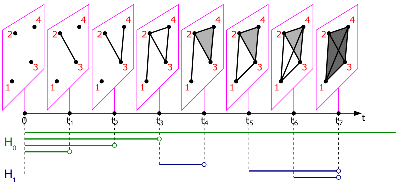
1234															

Circled elements have indices: $[3, 23]$, $[4, 34]$, $[2, 12]$, $[24, 234]$, $[13, 123]$, $[14, 124]$, $[134, 1234]$.

Reading births and deaths

Norns determine the fate of homology

Simplices = [1, 2, 3, 4, 23, 34, 12, 24, 234, 13, 14, 123, 124, 134, 1234]
BirthTimes = [0, 0, 0, 0, t1, t2, t3, t3, t4, t5, t6, t7, t7, t7, t7]



- We see pairs: $[3, 23]$, $[4, 34]$, $[2, 12]$, $[24, 234]$, $[13, 123]$, $[14, 124]$, $[134, 1234]$.
- For example, vertex 3 gives rise to homology which dies when 23 appears.
- 3 appears at time 0, and 23 — at time t_1.
- Hence we have interval module $I_{[0;t_1)}$ in 0-th homology.
- etc...

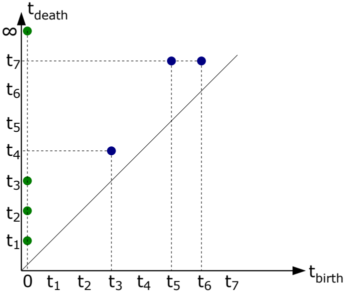
We have decomposition: $I_{[0;t_1)} \oplus I_{[0;t_2)} \oplus I_{[0;t_3)} \oplus I_{[t_3;t_4)} \oplus I_{[t_5;t_7)} \oplus I_{[t_6;t_7)}$.

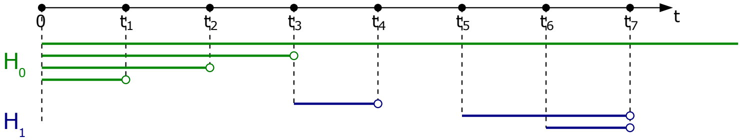
Real times

- Let K_t, $t \in \mathbb{R}$ be a collection of simplicial complexes on vertex set $[m]$,
- such that $t_1 < t_2$ implies $K_{t_1} \subseteq K_{t_2}$.
- This is called a filtration with real time.
- Changes may occur only at discrete time moments.
- Therefore, the only difference is that BirthTimes stores real values.
- The algorithm above outputs the interval decomposition.

How to encode persistent homology?

Persistence diagram instead of barcode.





Philosophy

- If some homology lives long, then it is **a meaningful homology**.
- The lifetime = death time - birth time.
- Lifetime is long, if the point of persistence diagram is far from the diagonal $y = x$.
- Points close to the diagonal are considered noise.
- Long-live-homology are important topological features of a filtration.

Filtrations: Čech

Demonstration: press to play in browser

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- Let $X = \{x_1, \ldots, x_m\} \subset \mathbb{R}^d$ be a point-cloud.
- Let $X_t = \bigcup B_{t/2}(x_i)$.
- X_t is a filtration, but not of simplicial complexes.

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- Let $X_t = \bigcup B_{t/2}(x_i)$.
- X_t is a filtration, but not of simplicial complexes.
- Replace X_t by its **nerve**!

Nerve complex

Let $K_t^{\check{C}}$ be a simplicial complex on $[m]$, such that $\{i_1, \dots, i_s\} \in K_t^{\check{C}}$ iff

$$B_{t/2}(x_{i_1}) \cap \dots \cap B_{t/2}(x_{i_s}) \neq \varnothing$$

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- Therefore X_t and $K_t^{\check{C}}$ have the same homology for each t.
- Hence they have the same persistent homology.
- We can work with simplicial filtration $\{K_t^{\check{C}}\}$ called **Čech filtration**.

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- Simplex I is born at the time = minimal t for which balls of radii $t/2$ around x_i, $i \in I$, intersect.
- It may be difficult to find this number in practice.

Vietoris–Rips filtration

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Vietoris–Rips filtration

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- K_t^{VR} be a simplicial complex on $[m]$ such that
- $I \in K_t^{VR}$ iff all pairwise distances between x_i and x_j, for $i \neq j$, $i, j \in I$, are less than t.
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- Very easy to compute birth times!
- Can be adapted to any finite metric space, e.g. metric graph.

How to compare filtrations and diagrams?

- We have two filtrations $\{K_t\}$ and $\{L_t\}$ on the same set $[m]$.
- Set $\mathrm{dist}(\{K_t\},\{L_t\}) = \max_{I\subset[m]} |\mathrm{BirthTime}_K[I] - \mathrm{BirthTime}_L[I]|$.

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- If $X = \{x_1,\dots,x_m\}$ and $Y = \{y_1,\dots,y_m\}$ are two data clouds and
- $\mathrm{dist}(X,Y) = \max_i \mathrm{dist}(x_i,y_i)$, and
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- **Exercise:** $\mathrm{dist}(\{K_t^{VR}\}, \{L_t^{VR}\}) \leqslant 2\,\mathrm{dist}(X, Y)$.

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- **Exercise:** $\mathrm{dist}(\{K_t^{VR}\},\{L_t^{VR}\}) \leqslant 2\,\mathrm{dist}(X, Y)$.
- What about their persistent diagrams?

Metric on diagrams

Bottleneck distance (or Wasserstein, or Kantorovich metric)

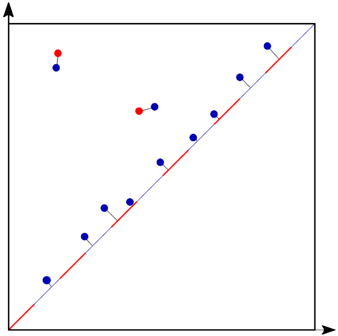

Fig.from GUDHI Library

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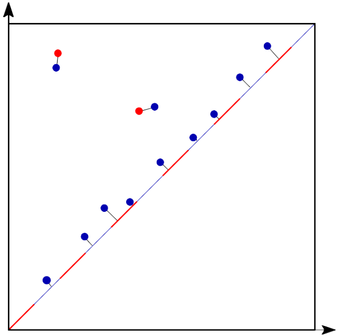

Fig.from GUDHI Library

Stability theorem

$$\mathrm{dist}(\mathrm{PD}(\{K_t\}), \mathrm{PD}(\{L_t\})) \leqslant \mathrm{dist}(\{K_t\}, \{L_t\}).$$

Generalizations of persistent modules

We can hardly see this slide. However if we do, then it is time to switch to whiteboard

Sources

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