MA2001 Cheat Sheet

Chapter 1: Linear Systems & GE

Elementary Row Operations (ERO)

1) Multiply a row by a non-Zero constant [cR_i]
2) Interchange 2 rows [$R_i \leftrightarrow R_j$]
3) Add a multiple of 1 row to another [$R_i + aR_j$]

REF	RREF
• Pivot points → any no. • Entries above pivots can be any no. ** USE GC to CALCULATE	• Pivot points must = 1 • All other entries = 0 Note: Every matrix has a unique RREF but can have many different REF.

Linear Systems

System of linear equations can be represented by augmented matrices

$$Ax = b \Leftrightarrow (A \mid b)$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases} \Leftrightarrow \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

- A linear system has either NO solution, ONLY 1 solution or infinitely many solutions (Check their RREF/REF values to decipher)
- We can solve the augmented matrix to find the solutions to the linear system

Row Equivalence

- Two matrices are said to be row equivalent if one can be obtained from each other by EROs.
- If augmented matrices of two linear systems are row equivalent, then the two systems have the same set of solutions.

Geometric Interpretations:

- If the last non-zero row of **R** is of the form (0 0 0 *), where * is a non-zero number, then the system is inconsistent. i.e No solutions
- Suppose the system is consistent
  R has at most 3 non-zero rows (NZR):

REF of R	General Solution	Solution set in xyz-space
3 NZR	0 arbitrary para	A point
2 NZR	1 arbitrary para	A line
1 NZR	2 arbitrary para	A plane
0 NZR	3 arbitrary para	The whole space

*** No. of pivot columns = No. of non-zero rows

No Soln	1 Soln	Infinite Soln
Last column of REF = Pivot Column	Every column in REF (except last row) = pivot column	At least 1 non-pivot columns (except last column) = Arbitrary paras
$\left(\begin{array}{ccc\|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 0 & c \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 1 & c \end{array}\right)$	$\left(\begin{array}{ccc\|c} 1 & * & * & a \\ 0 & 1 & * & b \\ 0 & 0 & 0 & 0 \end{array}\right)$
Inconsistent	Consistent	

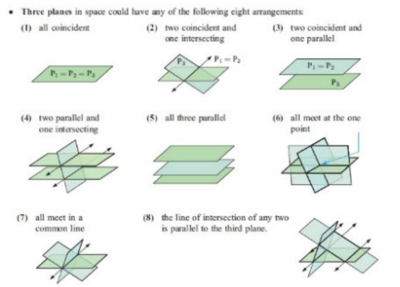
Lines

1) Parallel but non-intersecting
   ➢ No solution
2) Parallel but intersecting
   ➢ Infinitely many solutions
3) Non-parallel
   ➢ Intersect at a point (1 solution)

Planes

2 planes (3 Cases)
1) No solution
2) Infinite Solutions: Intersect, Common Line
3) Infinite Solutions: Intersect, Common Plane

3 Planes (8 Cases) (Diagram for visuals)



To extend the basis for $\mathbb{R}^n$, check |bases| ≤ n. Add standard bases to non-pivot columns s.t the matrix still preserves its linear

Solutions of Homogeneous Systems

- x = 0 is always a solution → Trivial solution
- A HLS has either only 1 trivial solution OR infinitely many solutions (in addition to the trivial solution)
- Any solution other than the trivial solution is called a non-trivial solution.
- A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.

Chapter 2: Matrices

** Convention, $A_{rows,columns}$

Matrix Addition/Subtraction

- We only can perform addition & subtraction to matrices of the same size/ dimension
- Add/subtract to the corresponding ai,j values

Matrix Multiplication

- We can only compute the scalar product **AB** when the #columns A = #rows B
- Size of Product matrix **AB** is (row A x column B)
  ➢ Criss-Cross the multiplication
  ➢ NOT COMMUTATIVE (**AB ≠ BA**)

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$$

- ***AB* = *0*** DOES NOT MEAN that ***A* = *0*** or ***B* = *0***

When matrix product = ***0*** it doesn't mean that the multiplicands = ***0*** (e.g ***A*** & ***B*** could be non-zero matrices and yet give a ***0*** product)
- In general, when we want to multiply, we put things to the left side (order matters)

Basic Properties of Matrices

Let ***A***, ***B***, ***C*** be matrices of the same size and *c*, *d* are scalars.
1) ***A*** + ***B*** = ***B*** + ***A***. (Commutative)
2) ***A*** + (***B*** + ***C***) = (***A*** + ***B***) + ***C***. (Associative)
3) ***A***(***BC***) = (***AB***)***C*** = ***ABC*** (Associative)

If ***A***, ***B₁*** and ***B₂*** are m x p, p x n & p x n matrices respectively:
4) $A(B_1 + B_2) = AB_1 + AB_2$ (Distributive)

If ***A***, ***C₁*** and ***C₂*** are p x n, m x p & m x p matrices respectively:
5) $(C_1 + C_2)A = C_1A + C_2A$ (Distributive)
6) *c*(***A*** + ***B***) = *c****A*** + *c****B***.
7) (*c* + *d*)***A*** = *c****A*** + *d****A***.
8) (*cd*)***A*** = *c*(*d****A***) = *d*(*c****A***).
9) ***A*** + **0** = **0** + ***A*** = ***A***.
10) ***A*** – ***A*** = **0**.
11) 0***A*** = **0**.
12) $AI_n = I_mA = A$
13) $A_{mxn}0_{nxq} = 0_{mxq}$ & $0_{pxm}A = 0_{pxn}$

Types of Matrices

Zero Matrix:
- All entries are 0

Square Matrices
- A square matrix is called a square matrix if it has the same number of rows and columns. (n x n matrix)

Diagonal Matrices
- A square matrix with non-diagonal entries = 0
- A square zero matrix is also considered diagonal

Scalar Matrix
- A diagonal matrix with all the same diagonal entries

$$a_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ c \text{ if } i = j \end{cases} \text{ for a constant c}$$

Identity Matrix
- A diagonal matrix with diagonal entries = 1

Symmetrix Matrix
- A square matrix with $a_{ij} = a_{ji}$
- A 1x1 matrix is considered symmetric $\quad$ *$A = A^T$

Triangular Matrix

Upper Triangular (UT)	A square matrix with all lower diagonal entries = 0 a_{ij} = 0 whenever i > j.
Lower Triangular (LT)	A square matrix with all upper diagonal entries = 0 a_{ij} = 0 whenever i < j.

- A diagonal matrix is both UT & LT
- A 1x1 matrix is also both UT & LT

Transposes

- Switch all rows with columns (r → c; c → r)
- The rows of A are the columns of A^T and vice versa.
- A square matrix A is symmetric iff A = A^T

Properties of Matrix Transpose
1) $(A^T)^T = A$
2) If ***B*** is an m x n matrix, then $(A + B)^T = A^T + B^T$
3) If c is a scalar, then $(cA)^T = cA^T$
4) If ***B*** be an n x p matrix, then $(AB)^T = B^TA^T$ (IMPT)

Powers of Square Matrices

$$A^n = \begin{cases} I & \text{if } n = 0 \\ \underbrace{AA \cdots A}_{n \text{ times}} & \text{if } n \geq 1 \end{cases}$$

- Let ***A*** be a square matrix and n, m non-negative integers. Then $A^mA^n = A^{m+n}$
- Since matrix multiplication is NOTE COMMUTATIVE, in general, for 2 square matrices ***A*** & ***B*** of the same size, $(AB)^2$ and A^2B^2 may be different.

Let ***A*** be an invertible matrix & n a positive integer:
$A^{-n} = (A^{-1})^n = A^{-1}A^{-1} \ldots A^{-1}$ (n times)

Invertible Matrix Theorem

Let ***A*** be an *n × n square* matrix. The following statements are equivalent:
1) ***A*** is invertible.
2) The linear system ***Ax*** = **0** has only the trivial solution.
3) The RREF of ***A*** is an identity matrix.
4) ***A*** can be expressed as a product of elementary matrices.
5) det(***A***) ≠ 0.
6) The rows of ***A*** form a basis for $\mathbb{R}^n$.
7) The columns of ***A*** form a basis for $\mathbb{R}^n$.
8) The column space of ***A*** = $\mathbb{R}^n$
9) rank(***A***) = n $\quad$ *Nullspace → ***A*** = ***0***; trivial solution
10) nullity(***A***) = 0
11) The nullspace of ***A*** is the zero vector. That is, {***0***}.
12) 0 is not an eigenvalue of ***A***

> Cancellation laws for matrix multiplication:
> Let ***A*** be an invertible m × m matrix.
> a) If ***B₁*** and ***B₂*** are m × n matrices s.t $AB_1 = AB_2$, then $B_1 = B_2$.
> b) If ***C₁*** and ***C₂*** are n × m matrices s.t $C_1A = C_2A$, then $C_1 = C_2$.
> ** Only works if matrix is invertible**

A square matrix is called SINGULAR if it has NO INVERSES.
Singular is a term used ONLY for square matrices.
Note that if ***AB*** is an identity, then it is guaranteed that ***BA*** is also an identity.
** If $A^{-1} = A^T$ → ***A*** is an orthogonal matrix of order n

Basic properties of Inverses

Let ***A***, ***B*** be two invertible matrices and c a non-zero scalar.
1. c***A*** is invertible and $(cA)^{-1} = \frac{1}{c}A^{-1}$.
2. A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.
3. A^{-1} is invertible and $(A^{-1})^{-1} = A$.
4. ***AB*** is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
5. $A^rA^s = A^{r+s}$ for any integers r & s
6. A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$

*** if $A_1, A_2, \ldots, A_k$ are invertible matrices, then $A_1A_2 \cdots A_k$ is invertible and $(A_1A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1}A_1^{-1}$.*

Method to find inverses

(***A*** | ***I***) → Gauss-Jordan Elimination → (***I*** | A^{-1})
If ***A*** doesn't "turn" into ***I***, → Not invertible & RREF not elementary

To check if matrix is invertible

- If the matrix has non-pivot columns OR have zero rows → NOT INVERTIBLE
  Generally, if Determinant = 0 → NOT invertible

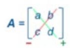
Elementary Matrices

- A square matrix is called an elementary matrix, IF it can be obtained from an identity matrix by performing A SINGLE Elementary Row Operations
- Multiplying elementary matrices (ordered to the left) to matrices yields row equivalent matrices.
  E.g $E_4E_3E_2E_1A = B \rightarrow A = (E_4E_3E_2E_1)^{-1}B = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}B$

Determinants

For 2x2 Matrices

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$



For 3x3 Matrices

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= aei + bfg + cdh - ceg - afh - bdi$$

For nxn Matrices (n > 3)
- Do co-factor expansion until you get a 3x3 matrix
- Compute the determinant using formula
- Tip: Expand by the column with zeroes
  → 0 x anything = 0, can save time ☺

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

$$= a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \cdot \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- Rmb to alternate the signs +/- ; $a_{1,1}$ always +'ve
- You can expand determinants by any row/columns

Identity Matrices
- det(***A***) = 1

Diagonal Matrices & Triangular Matrices
- det(***A***) = Product of Diagonal Entries

Note:
- If a matrix has identical rows/columns → det(***A***) = 0
- If ***A*** is a square matrix, then det(A^T) = det(***A***)

ERO on Determinants

A → ERO → B	Determinant
cR_i	det(***B***) = c det(***A***)
$R_i \leftrightarrow R_j$	det(***B***) = - det(***A***)
$R_i + cR_j$	det(***B***) = det(***A***)

If ***E*** is an elementary matrix of the same size as ***A***,
then det(***EA***) = det(***E***)det(***A***)

Properties of Determinants

Let ***A***, ***B*** be invertible matrices and *c* a non-zero scalar.
1) *det*(***A***) = *det*(A^T)
2) *det*(c***A***) = c^n*det*(***A***) ** n → size of square matrix
3) *det*(***AB***) = *det*(***A***)*det*(***B***)
4) If ***A*** is an invertible matrix, then $det(A^{-1}) = \frac{1}{\det(A)}$
5) If ***A*** is invertible, then $A^{-1} = \frac{1}{\det(A)} adj(A)$

Adjoints/Classical

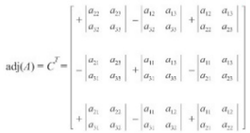
- The adjoint of a matrix → transpose of the cofactor matrix of that particular matrix
- Rmb to alternate + & -

Let ***A*** be a square matrix of order n

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Where $A_{i,j}$ which is the (i, j) – cofactor of ***A***

Example:



Cramers Rule

Example:

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3x + 9y \quad = 3 \end{cases} \Leftrightarrow \begin{bmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}$$

By Cramer's Rule:

$$x = \frac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}, y = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}, z = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}$$

x = 2.2, y = -0.4, z = -0.6

Chapter 3: Vector Spaces

Euclidean n-space (n-space)

$$\mathbb{R}^n = \{(u_1, u_2, \ldots, u_n) \mid u_1, u_2, \ldots, u_n \in \mathbb{R}\}$$

A vector $\mathbb{R}^n$ can be identified as a matrix

Row Vector	Column Vector
Vector $\mathbb{R}^n$ is written as $(u_1, u_2, \ldots, u_n)$ 1 x n matrix	Vector $\mathbb{R}^n$ is written as $(u_1, u_2, \ldots, u_n)^T$ n x 1 matrix

Properties of Vectors

Let **u**, **v**, **w** be n-vectors and c, d real numbers.
1. **u** + **v** = **v** + **u**.
2. **u** + (**v** + **w**) = (**u** + **v**) + **w**.
3. **u** + **0** = **u** = **0** + **u**.
4. **u** + (–**u**) = **0**.
5. c(d**u**) = (cd)**u**. (scalar product)
6. c(**u** + **v**) = c**u** + c**v**. (scalar multiplication)
7. (c + d)**u** = c**u** + d**u**.
8. 1**u** = **u**

Implicit/Explicit forms of expressions

Lines in $\mathbb{R}^2$ $\quad$ **explicit → general formula

Implicit	$\{(x, y) \mid ax + by = c\}$
Explicit	$\left\{\left(\frac{c - bt}{a}, t\right) \mid t \in \mathbb{R}\right\}$ or $\left\{\left(\frac{c - at}{b}, t\right) \mid t \in \mathbb{R}\right\}$

Planes in $\mathbb{R}^3$

Implicit	$\{(x, y, z) \mid ax + by + cz = d\}$
Explicit	$\left\{\left(\frac{d - bs - ct}{a}, s, t\right) \mid s, t \in \mathbb{R}\right\}$ if $a \neq 0$; $\left\{\left(s, \frac{d - bs - ct}{b}, t\right) \mid s, t \in \mathbb{R}\right\}$ if $b \neq 0$; $\left\{\left(s, t, \frac{d - bs - ct}{c}\right) \mid s, t \in \mathbb{R}\right\}$ if $c \neq 0$;

Linear combinations

A vector V, $a_1\mathbf{u_1} + a_2\mathbf{u_2} + \ldots + a_k\mathbf{u_k}$ is the linear combination of $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$

Linear Spans

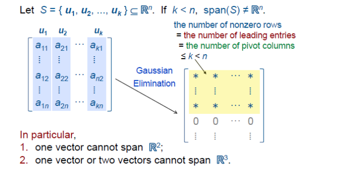
- Linear Span → The set of all possible linear combination of $u_1, u_2, \ldots, u_k$ $\{a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_k\mathbf{u_k} \mid a_1, a_2, \ldots, a_k \in \mathbb{R}\}$
- Denoted by: span(S) or span $\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\}$
- Span is a collection of vectors that assumes the form of the matrix. Usually, if the equation is linear, there should be a span; Span → no redundant vectors.

V = span(S) = span{$u_1, u_2, \ldots, u_k$} →S spans V;{$u_1, u_2, \ldots, u_k$} spans V

To check if the vector is a linear combination(LC)/span?

- Using row space, perform GE, if the augmented matrix is consistent → vector is LC/span
- If matrix inconsistent → not LC/span

Span(S) = $\mathbb{R}^n$



Let S = {$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$} ⊆ $\mathbb{R}^n$.
1) **0** ∈ span(S)
2) For any $v_1, v_2, \ldots, v_r$ ∈ span(S) and $c_1, c_2, \ldots, c_r \in \mathbb{R}$, $c_1v_1 + c_2v_2 + \cdots + c_rv_r$ ∈ span(S)
   (span(s) is closed under linear combinations)
3) The span of the zero space is the zero space

When span(S_1) ⊆ span(S_2)

Let S_1 = {$u_1, u_2, \ldots, u_k$} & S_2 = {$v_1, v_2, \ldots, v_m$} are subsets of $\mathbb{R}^n$
Then span(S_1) ⊆ span(S_2) iff each $\mathbf{u_i}$ is a linear combination of $v_1, v_2, \ldots v_m$
Then span(S_2) ⊆ span(S_1) iff each v_i is a linear combination of $u_1, u_2, \ldots u_k$

Subspaces

Let V be a subset of $\mathbb{R}^n$.
V is called a subspace of $\mathbb{R}^n$ if V = span(S),
where S = {$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$} for some vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k} \in \mathbb{R}^n$.
→ V is a subspace spanned by S OR $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$.
→ S spans V.
Alternative:
A subset V is a subspace of $\mathbb{R}^n$ if and only if
1) V is non-empty
2) for all ***u***, ***v*** ∈ V and c, d ∈ ℝ, c***u*** + d***v*** ∈ V.

How to check if subset is subspace?

1) Show that every vector ∈ V can be written in the form $a_1\mathbf{u_1} + a_2\mathbf{u_2} + \cdots + a_k\mathbf{u_k}$ where $a_1, a_2, \ldots, a_k$ are arbitrary parameters and $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ are constant vectors(their entries do not consist of any arbitrary parameters).
2) Show that V is closed under linear combinations
   a) **0** ∈ V
   b) For any $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_r}$ ∈ V and $c_1, c_2, \ldots, c_r$ ∈ R, $c_1\mathbf{v_1} + c_2v_2 + \ldots + c_rv_r$ ∈ V (closure)
   (i.e closed under vector addition & scalar multiplication)

No show NOT subspace:
- To show that V is not a subspace, we usually show that it violates some property of vector spaces, e.g. if V does not contain **0** or not closed under vector addition & scalar x

Linear Independence & Dependence

1. *S* is linearly dependent iff at least one vector $u_i \in S$ can be written as a linear combination of other vectors in *S*. (i.e. $u_i = a_1u_1 + \cdots + a_{i-1}u_{i-1} + u_i \text{ (redundant)} + a_{i+1}u_{i+1} + \cdots + a_ku_k$ for some real numbers $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k$) (set has redundant vector) – has non-trivial solutions
2. *S* is linearly independent if no vector in *S* can be written as a linear combination of other vectors in *S* – has only the trivial solution (no redundant vectors)

* In span{$\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$}, there are NO redundant vectors among $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ iff $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ are linearly independent.

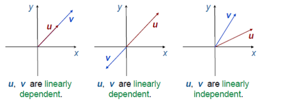
Linearly independent → $c_1\mathbf{u_1} + c_2\mathbf{u_2} + \ldots + c_k\mathbf{u_k} = \mathbf{0}$ (trivial solution)

Adding an Independent Vector:

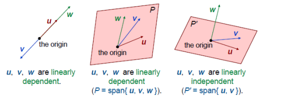
Let $u_1, u_2, \ldots, u_k$ be linearly independent vectors in R^n.
If u_{k+1} is not a linear combination of $u_1, u_2, \ldots, u_k$, then $u_1, u_2, \ldots, u_k, u_{k+1}$ are linearly independent.

Geometric Interpretation of Independence/Dependence

$\mathbb{R}^2$: **u** & **v** are linearly dependent iff they lie on the same line (when they are placed with their initial points at the origin)



$\mathbb{R}^3$: **u, v** & **w** are linearly dependent iff they lie on the same line or the same plane (when they are placed with their initial points at the origin).



Let $S = \{u_1, u_2, \ldots, u_k\} \subseteq R^n$
If $k > n$, then S is linearly dependent. (*if n > k, it does not guarantee independence). In particular,

- In R^2, a set of 3< vectors must be linearly dependent;
- In R^3, a set of 4 <vectors must be linearly dependent.

Solution Spaces

The solution set of a homogeneous system of linear equation in n variables is a subspace of $\mathbb{R}^n$

Bases

Let V be a vector space and S= { $u_1, u_2, \ldots, u_k$} a subset of V. Then S is called a basis(plural bases) for V if

1) S is linearly independent and
2) S spans V. (no redundancies)
3) Empty set $\emptyset$ → basis of zero space {**0**}; dim($\emptyset$) = 0

Size of Bases

dim(V) = k

Let V be a vector space which has a basis with k vectors.

1) Subsets of V with > k vectors → linearly dependent.
2) Subsets of V with < k vectors → cannot spans V.
3) Every basis for V MUST have the same size k.

**To check if S is a basis of V, just have to check 1 of the 3

Dimension

1) dim($\mathbb{R}^n$) = n
2) If a vector V has a basis S, then the dim(V) = |S|
3) Dimension of zero space = 0
4) No. of non-pivot columns = Dim of solution space

Given $A = t_1u_1 + t_2u_2 + \cdots + t_ku_k$, where $u_1, u_2, \ldots, u_k$ are linearly independent. {$u_1, u_2, \ldots, u_k$} is basis of sol space (SS)

➢ Dim(SS) = k = no. of arbitrary parameters = no of non-pivot columns in **A**

Equivalent Relations Bases & Dimensions

Let V be a vector space of dimension k and S a subset of V. The following are equivalent:

1. S is a basis for V(S is linearly independent& S spans V)
2. S is linearly independent and |S| = k.
3. S spans V and |S| = k.

Dimension of subspaces

Let U be a subspace of a vector space V.
Then dim(U) ≤ dim(V). If $U \neq V$, then dim(U) < dim(V).

Finding Bases

Let u_1= (1, 2, 2, 1), u_2= (3, 6, 6, 3), u_3= (4, 9, 9, 5), u_4= (−2, −1, −1, 1), u_5= (5, 8, 9, 4), u_6= (4, 2, 7, 3).
Find a basis for W = span{$u_1, u_2, u_3, u_4, u_5, u_6$}.
ANS: {(1,2,2,1), (4,9,9,5), (5,8,9,4,)} is a basis for W

*Which ever method used, basis obtained will be the same

Row Method	Column Method
➢ Arrange the matrix in rows (row space) ➢ Perform GE ➢ Basis are the NZR	➢ Arrange the matrix in columns (col space) ➢ Perform GE ➢ Basis → Corresponding pivot columns
$\begin{bmatrix} 1&2&2&1\\3&6&6&3\\4&9&9&5\\-2&-1&-1&1\\5&8&9&4\\4&2&7&3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1&2&2&1\\0&1&1&1\\0&0&1&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}$	$\begin{bmatrix} 1&3&4&-2&5&4\\2&6&9&-1&8&2\\2&6&9&-1&9&7\\1&3&5&1&4&3 \end{bmatrix} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{bmatrix} 1&3&4&-2&5&4\\0&0&1&3&-2&-6\\0&0&0&0&1&5\\0&0&0&0&0&0 \end{bmatrix}$ pivot columns

Transition Matrices

Let S and T be two bases for a vector space and let **P** be the transition matrix from S to T.

1) For any vector **w** ∈ V, [**w**]$_T$ = **P**[**w**]$_S$
2) **P** is invertible
3) P^{-1} (& P^T is orthogonal) is the TM from T to S.
4) If S & T are orthonormal bases → **P** orthonormal matrix

Composition of Transition Matricess

Let $\boldsymbol{A}$→ TM from $\boldsymbol{U}$ to $\boldsymbol{V}$, $\boldsymbol{B}$ from $\boldsymbol{V}$ to $\boldsymbol{W}$, & $\boldsymbol{C}$ from $\boldsymbol{U}$ to $\boldsymbol{W}$

1) BA = C
2) For any vector x, $[x]_W = BA[w]_U = C[x]_U$

The transition matrix from S to T is defined to be the matrix

$$P = \begin{bmatrix} [u_1]_T & [u_2]_T & \cdots & [u_k]_T \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}$$

e.g $[v_1]_S = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$, then v_1 = $-2\boldsymbol{u_1} + \boldsymbol{u_2} + 3\boldsymbol{u_3}$

How to find Transition Matrices?

1) Use Augmented matrices & Perform GJE
2) (T | S) → GJE → (**I** | **P**) — **P** → TM from S to T
3) [**w**]$_T$ = P[**w**]$_S$ — **TP = S**
4) **P** = [u_j]$_T$ = $u_j \cdot v_i = v_i \cdot u_j$ = [v_i]$_S$ — P^{-1} → TM from T to S

Chapter 4: Vector Spaces Associated with Matrices

Row Space

- Space spanned by the rows of the matrix

Suppose $A \xrightarrow[\text{Elimination}]{\text{Gaussian}} R$ where R is a row-echelon form of A.

Then the row space of A = the row space of R.

The nonzero rows of R form a basis for the row space of R
⇒ the nonzero rows of R form a basis for the row space of A.

Column Space

- Space Spanned by the columns of the matrix

Suppose $A \xrightarrow[\text{Elimination}]{\text{Gaussian}} R$ where R is a row-echelon form of A.

In general, the column space of A ≠ the column space of R.

The pivot columns of R form a basis for the column space of R
⇒ the corresponding columns of A form a basis for the column space of A.

Remarks:

→ERO don't change row spaces, BUT changes column space
→ERO preserves linear independence of column spaces
→The column space of **A** = the row space of A^T, and the row space of **A** = the column space of A^T

Nullspaces

Fancy name for solution set of homogeneous linear system

Rank & Nullities

Suppose **A** – GE → **R**, where **R** is REF of A

rank(**A**)	=	dim(the row space of **A**)
	=	dim(the column space of **A**)
	=	The no. of zero-rows of **R**
	=	The no. of pivot columns of **R**

Let **A** and **B** be $m \times n$ and $m \times p$ matrices respectively
rank(**AB**) ≤ min{ rank(**A**), rank(**B**) }

nullity(**A**)	=	dim(the nullspace of **A**)
	=	The no. of non-pivot columns of **R**

Since nullspace is a subspace of $\mathbb{R}^n$
*nullity(**A**) = dim(the nullspace of **A**) ≤ dim(n) = n*

Dimension Theorem for Matrices:
rank(**A**) + nullity(**A**) = number of columns in **A**

System of Linear Equations

Ax = **b** is consistent iff **b** lies in the col space of **A**.
Suppose **x** = **v** is a solution to **Ax** = **b**
Then the solution set of the system **Ax** = **b** is given by
{**u** + **v**| **u** ∈ the nullspace of **A**}
General Solution of Ax = b:
x = (general solution **Ax = 0**) + (1 particular solution **Ax = b**)

Chapter 5: Orthogonality

Dot Product

dot product	$u \cdot v = uv^T = u_1v_1 + u_2v_2 + \cdots + u_nv_n$.
norm/length	$\lVert u \rVert = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2}$ $\lVert u \rVert^2 = u \cdot u \geq 0$; $u \cdot u = 0$ iff $u = 0$ Vectors of norm 1 ($\lVert u \rVert = 1$) → unit vectors.
Distance between **u** and **v**	$d(u, v) = \lVert u - v \rVert = \sqrt{(u - v) \cdot (u - v)}$ $= \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2}$
Angle between **u** and **v**	$\cos^{-1}\left(\frac{u \cdot v}{\lVert u \rVert \lVert v \rVert}\right)$ The angle is well-defined as $-1 \leq \frac{u \cdot v}{\lVert u \rVert \lVert v \rVert} \leq 1$

Recall Cosine Rule:
$\lVert u - v \rVert^2 = \lVert u \rVert^2 + \lVert v \rVert^2 - 2\lVert u \rVert \lVert v \rVert \cos(\theta)$

$$\theta = \cos^{-1}\left(\frac{\lVert u \rVert^2 + \lVert v \rVert^2 - \lVert u - v \rVert^2}{2\lVert u \rVert \lVert v \rVert}\right)$$

Principal values:
Cos → $0 \leq \theta \leq \pi$

Basic Properties of dot products

Let **u**, **v**, **w** be vectors in *n* and *c* a scalar.

1. $u \cdot v = v \cdot u$ (Commutative)
2. $(u + v) \cdot w = u \cdot w + v \cdot w$ and $w \cdot (u + v) = w \cdot u + w \cdot v$ (Distributive)
3. $(cu) \cdot v = u \cdot (cv) = c(u \cdot v)$ (Associative)
4. $\lVert cu \rVert = |c| \lVert u \rVert$
5. $u \cdot u \geq 0$;
6. $u \cdot u = 0$ iff $u = 0$

Orthogonality

1. 2 vector **u** and **v** in R^n are orthogonal if **u** · **v** = 0. (**u** & **v** are perpendicular to each other → $\theta = \frac{\pi}{2}$
2. A set S of vectors in R^n is called an orthogonal set if every pair of distinct vectors in S are orthogonal.
3. A set S of vectors in R^n is called an orthonormal set if S is an orthogonal set and every vector in S is a unit
4. If S is an orthogonal set of non-zero vector, S is linearly independent
5. Orthogonality implies (→) linear independence
6. If you want to prove that a vector **w** is orthogonal to V, you just need to prove it for the basis V
7. A square matrix A is orthogonal is $A^{-1} = A^T$

Orthogonal & Orthonormal basis

- A basis S for a vector space is called an orthogonal basis if S is orthogonal.
- A basis S for a vector space is called an orthonormal basis if S is orthonormal.
- Normalising: $\frac{1}{\lVert u \rVert}u$ (Preserves orthogonality)

How to check if S is an orthogonal/orthonormal basis for V

1. Check if S is orthogonal (respectively, orthonormal)
2. Check if |S| = dim(V) (if we know the dimension) OR Check if span(S) = V (if we don't know the dim)

Gram Schmidt Process

Let {$u_1, u_2, \ldots, u_k$} be a basis for a vector space V.

Let $v_1 = u_1$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1}v_1$$

Now, {v_1, v_2} is an orthogonal basis for spans {u_1, u_2} ⊆ V

$$v_3 = u_3 - \frac{u_3 \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_3 \cdot v_2}{v_2 \cdot v_2}v_2$$

Normalise to get an orthonormal basis

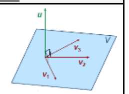
$$\vdots \qquad \vdots$$

$$v_k = u_k - \frac{u_k \cdot v_1}{v_1 \cdot v_1}v_1 - \frac{u_k \cdot v_2}{v_2 \cdot v_2}v_2 - \cdots - \frac{u_k \cdot v_{k-1}}{v_{k-1} \cdot v_{k-1}}v_{k-1}$$

Then {$v_1, v_2, \ldots, v_k$} is an orthogonal basis for V
$\left\{\frac{1}{\lVert v_1 \rVert}v_1, \frac{1}{\lVert v_2 \rVert}v_2, \ldots, \frac{1}{\lVert v_k \rVert}v_k\right\}$ is an orthonormal basis for V
To simplify, let $w_1 = \frac{1}{\lVert v_1 \rVert}v_1, w_2 = \frac{1}{\lVert v_2 \rVert}v_2, \ldots, w_k = \frac{1}{\lVert v_k \rVert}v_k$,
Then {$w_1, w_2, \ldots, w_k$} is an orthonormal basis for V

Vectors Orthogonal to subspace

→Vectors perpendicular to subspace



Let V be a subspace of $\mathbb{R}^n$.
A vector **u** ∈ $\mathbb{R}^n$ is said to be orthogonal to V if **u** is orthogonal to all vectors in V
Let V = span{$v_1, v_2, \ldots, v_k$}.
A vector **u** is orthogonal to V iff $v_i \cdot u = 0$ for i = 1, 2, …, k.
Note that $V^{\perp}$ = { **u** | **u** is orthogonal to V}
= { **u**| $v_i \cdot u$ = 0 for i = 1, 2, …, k} is a subspace of $\mathbb{R}^n$.

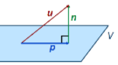
Standard Basis

Let e_1=(1, 0, 0, …, 0), e_2=(0, 1, 0, …, 0),…, e_n=(0, 0, …, 0, 1)be vectors in $\mathbb{R}^n$
E = {$e_1, e_2, \ldots, e_n$} is a basis for $\mathbb{R}^n$ →standard basis of $\mathbb{R}^n$
→ For any v ∈ $\mathbb{R}^n$, $(v)_E$ = **v**
→ Let **A** be an m×n matrix. Then $A_{e_i}^T$ = the i^{th} column of **A**.

Projections

Let V be a subspace of $\mathbb{R}^n$.
Every **u** ∈ $\mathbb{R}^n$ can be written uniquely as

$$u = n + p$$



where **p** is a vector in V
and **n** is a vector orthogonal to V.
p → (orthogonal) projection of **u** onto V.

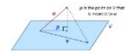
Let V be a subspace of R^n and **w** a vector in R^n (any **w** ∈ R^n)

1) If {$u_1, u_2, \ldots, u_k$} is an orthogonal basis for V, then
$$p = \frac{w \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k}u_k$$
➔ Is the projection of **w** onto V
2) If {$v_1, v_2, \ldots, v_k$} is an orthonormal basis for V, then
$$p = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$$
➔ Is the projection of **w** onto V

Note: The formula for **p** only works when you have an orthogonal basis for V; Length of projection = | |**p**| |

Best Approximations

Let V be a subspace of $\mathbb{R}^n$
Take anu **u** ∈ $\mathbb{R}^n$ and let **p** be the projection of **u** onto *V*.
d(**u**, **p**) ≤ *d*(**u**, **v**) for all **v** ∈ *V*



➔ Dist btwn **u** & **p** ≤ Dist btwn **u** & **v**
➔ **p** is the best approximation of **u** in *V*.

Least Square Solutions

Let **Ax** = **b** be a linear system where **A** is an $m \times n$ matrix.
A vector **u** ∈ $\mathbb{R}^n$ is called a least square solution to the linear system **Ax** = **b** if

| |**b** – **Au**| | ≤ | |**b** – **Av**| |for all **v** ∈ $\mathbb{R}^n$ ------- (#)

i.e **u** is a least square solution to **Ax** = **b**
⇔ **p** = **Au** is the projection of **b** onto the column space of **A**
⇔ **u** is a solution to $A^TAx = A^Tb$ (use this, no need to find **p**)

Let V = {**Av** | **v** ∈ $\mathbb{R}^n$} and **p** = **Au**.
Then (#) is rewritten as
d(**b**, **p**) ≤ *d*(**b**, **w**) for all **w** ∈ V,
i.e. **p** = **Au** is the best approximation of **b** onto *V*

Coordinate Systems

Let S = {$u_1, u_2, \ldots, u_k$} be a basis for a vector space *V*.
v ∈ V → **v** = $c_1u_1 + c_2u_2 + \ldots + c_ku_k$(unique)
$(v)_S$ = ($c_1, c_2, \ldots, c_k$) (row); $[v]_S$ = $(c_1, c_2, \ldots, c_k)^T$ (column) is called the coordinate vector **v** relative to S

S = {$u_1, u_2, \ldots, u_k$} is an orthogonal basis for V,then any **w** ∈ V

$$w = \frac{w \cdot u_1}{u_1 \cdot u_1}u_1 + \frac{w \cdot u_2}{u_2 \cdot u_2}u_2 + \cdots + \frac{w \cdot u_k}{u_k \cdot u_k}u_k$$

i.e $(w)_S = \left(\frac{w \cdot u_1}{u_1 \cdot u_1}, \frac{w \cdot u_2}{u_2 \cdot u_2}, \ldots, \frac{w \cdot u_k}{u_k \cdot u_k}\right)$
T = {$v_1, v_2, \ldots, v_k$} is an orthonormal basis for V, then any **w**∈V
$$w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdot v_k)v_k$$
i.e $(w)_S = (w \cdot v_1, w \cdot v_2, \ldots, w \cdot v_k)$

Chapter 6: Diagonalisation

Eigenvalues & Eigenvectors

$$Au = \lambda u \leftrightarrow Au = (\lambda I)u = \lambda\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}u$$

λ → eigenvalue of **A**; **u** → eigenvector of **A** *associated with* λ

- λ is an eigenvalue of **A** iff det(λ**I** – **A**) = 0
- If expanded, det(λ**I** – **A**) → polynomial in λ of degree n

Characteristic Equation	Characteristic Polynomial
det(λ**I** – **A**) = 0	det(λ**I** – **A**)

If **A** = $(a_{ij})_{n \times n}$ → triangular matrix, then
det(λ**I** – **A**) = (λ – a_{11})(λ – a_{22}) ⋯ (λ – a_{nn}).
The eigenvalues of **A** are $a_{11}, a_{22}, \ldots, a_{nn}$.
Given a square matrix **A, you cannot find eigenvalues by reducing **A** to a triangular matrix by using ERO. But you can use ERO to reduce (λ**I** – **A**) to a triangular matrix in order to find det(λ**I** – **A**).

Eigenspaces

Eigenspace is the solution space of the linear system
Let **A** be a square matrix of order n & λ an eigenvalue of **A**.
Then the solution space of the linear system (λ**I** – **A**)**x** = **0** is called the eigenspace of **A** associated with the eigenvalue λ, denoted by E_λ or E_λ(**A**)

- If **u** is a nonzero vector in E_λ, then **u** is an eigenvector of **A** associated with λ. (λ**I** – **A**)**x** = **0** ⇔ λ**u** – **Au** = **0** ⇔ λ**u** = **Au**
- Eigenspace is always a subspace of R^n
- If dim(eigenspace) ≠ power→ matrix cannot be diagonalised

Diagonalisation

- A square matrix **A** is called diagonalizable if there exists an invertible matrix **P** such that $P^{-1}AP$ is a diagonal matrix. (recall: $P^{-1}AP = D$)
- **A** is diagonalisable iff **A** has n linearly independent eigenvectors
- A square matrix **A** is called orthogonally diagonalisable if there exist an orthogonal matrix **P** s.t P^TAP is a diagonal matrix
- **A** is orthogonally diagonalisable iff **A** is symmetric
- The dimension of the eigenspaces MUST = power for the matrix to be diagonalisable

Power of Matrices

Let A be a square matrix of order n and P an invertible matrix such that

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

For any integer m (if A is singular, then m has to be non-negative),

$$A^m = P\begin{bmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^m \end{bmatrix}P^{-1}.$$

The transition matrix from S to T:

$$P = \begin{bmatrix} u_1 \cdot v_1 & u_2 \cdot v_1 & \cdots & u_k \cdot v_1 \\ u_1 \cdot v_2 & u_2 \cdot v_2 & \cdots & u_k \cdot v_2 \\ \vdots & \vdots & & \vdots \\ u_1 \cdot v_k & u_2 \cdot v_k & \cdots & u_k \cdot v_k \end{bmatrix}$$

The transition matrix from T to S:

$$Q = \begin{bmatrix} v_1 \cdot u_1 & v_2 \cdot u_1 & \cdots & v_k \cdot u_1 \\ v_1 \cdot u_2 & v_2 \cdot u_2 & \cdots & v_k \cdot u_2 \\ \vdots & \vdots & & \vdots \\ v_1 \cdot u_k & v_2 \cdot u_k & \cdots & v_k \cdot u_k \end{bmatrix}$$

For limits (lim), recall: fractions converge to 0

How to Diagonalise a Matrix?

Let **A** be a square matrix of order n.
Step 1: Find all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (by solving the characteristic eqn, det(λ**I** – **A**) = **0**).
Step 2: For each eigenvalue λ_i, find a basis $S_{\lambda i}$ for the eigenspace $E_{\lambda i}$
Step 3: Let S = $S_{\lambda 1} \cup S_{\lambda 2} \cup \ldots \cup S_{\lambda k}$
a) If |S| < n, then **A** is not diagonalizable
b) If |S| = n, say, S = {$u_1, u_2, \ldots, u_n$},
✓ then A is diagonalizable

and P = [u_1 u_2 … u_n] is an invertible matrix that diagonalizes **A**

How to Orthogonally Diagonalise a Matrix?

Let **A** be a symmetric matrix of order n.
Step 1: Find all distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ (by solving the characteristic eqn, det(λ**I** – **A**) = **0**).
Step 2: For each eigenvalue λ_i,
a) Find a basis $S_{\lambda i}$ for the eigenspace $E_{\lambda i}$ and then
b) Use the Gram-Schmidt Process to transform $S_{\lambda i}$ to an orthonormal basis $T_{\lambda i}$
Step 3: Let T = $T_{\lambda 1} \cup T_{\lambda 2} \cup \ldots \cup T_{\lambda k}$, say T = {$v_1, v_2, \ldots v_n$} **If you start w a symmetric matrix, |T| = n
- Then **P** = [v_1 v_2… v_n] is an orthogonal matrix that orthogonally diagonalises **A**

Matrix must be symmetric b4 it can be orthogonally diagonalised

Chapter 7: Linear Transformation

Linear Transformations (LT)

Definition 1:
A linear transformation is a mapping T: $\mathbb{R}^n$ → $\mathbb{R}^m$ of the form

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & a_{1n}x_n \\ a_{22}x_1 & + & a_{22}x_2 & + & \cdots & a_{2n}x_n \\ \vdots & & \vdots & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & + & a_{12} & + & \cdots & a_{1n} \\ a_{22} & + & a_{22} & + & \cdots & a_{2n} \\ \vdots & & \vdots & & & \vdots \\ a_{m1} & + & a_{m2} & + & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, for \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in R^n$$

- If n = m, then T is the linear operator on R^n
- The matrix $(a_{ij})_{m \times n}$ is called the standard matrix for T

Definition 2:
Let V & W be vector spaces
A mapping T: V → W is called a linear transformation iff:
T(c**u** + d**v**) = cT(**u**) + dT(**v**) for all **u**, **v** ∈ V and c, d ∈ R

Definition 1 & 2 are the same if V = R^n & W = R^m
Definition 2 is a useful way to check is something is a LT

- If T(c**u** + d**v**) = cT(**u**) + dT(**v**) holds → LT
- If T(c**u** + d**v**) = cT(**u**) + dT(**v**) does not hold → not LT

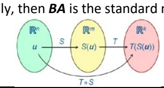
How to find Standard Matrix, A?

1) Solve the vector equation, then sub in the values i.e Find **A** by computing the formula of T directly
2) Gauss Jordan Elimination with standard basis to find A^{-1} i.e Find **A** using images of basis vectors of the standard basis

Composition of Mappings

Let S: $\mathbb{R}^n$ → $\mathbb{R}^m$ and T: $\mathbb{R}^m$ → $\mathbb{R}^k$ be mappings.
The composition of T with S, denoted by T ∘ S, is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^k$ defined by: (T ∘ S)(**u**) = T(S(**u**)) for **u** ∈ $\mathbb{R}^n$
Suppose S: $\mathbb{R}^n$ → $\mathbb{R}^m$ and T: $\mathbb{R}^m$ → $\mathbb{R}^k$are LT. → T ∘ S is also a LT.

- Furthermore, if **A** and **B** are standard matrices for S & T respectively, then **BA** is the standard matrix for T ∘ S.



Ranges & Kernels

Let T: R^n → R^m be a linear transformation.

- The range of T, which is denoted by R(T) is the set of images of T
R(T) = { T(**u**) | **u**∈ R^n } ⊆ R^m
- The kernel of T, which is denoted by Ker(T), is the set of vectors in R^n whose image is the zero vector in R^m
Ker(T) = { **u**| T(**u**) = **0**} ⊆ R^n

IF A is the standard matrix of T, then
R(T) = column space of **A**
Ket(T) = nullspace of **A**

- The rank of T, which is denoted by rank(T), is the dimension of R(T).
- The nullity of T, which is denoted by nullity(T), is the dimension of Ker(T).

If **A** is the standard matrix for T, then
rank(T) = rank(**A**).
nullity(T) = nullity(**A**)

Dimension Theorem for Linear Transformations:
rank(T) + nullity(T) = n

**Row space basis → RREF non-zero rows
**Column space basis → row matrix (i.e transpose corresponding pivot cols)

Basis for nullspace/zerospace → GJE→ find $x_1, x_2, \ldots, x_n$
Column space A = Row space A^T; vice-versa

Basis for range space → corresponding column space basis (NO NEED TRANSPOSE)

* $P^{-1}AP = D$
$A = PDP^{-1}$