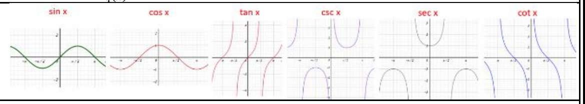
MA1521 Finals Cheat Sheet AY22/23 Sem2

Exponent Integral: $\int_b^a x^n e^{x^{n+1}}\,dx = \left[\frac{1}{n+1}e^{x^n}\right]_b^a$

Chapter 0: Numbers & Functions

ℕ: set of all natural numbers (no –'ve)	ℤ: set of all integers
ℚ: set of all rational numbers	ℝ: set of all real numbers
ℂ: set of all complex numbers	$0 \in \{\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{Q}\}$

Functions:

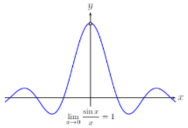
Poly-nomials	• $p(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$, where $a_0, a_1, \ldots, a_n$ are constants, is called a polynomial of degree n • Can be factored as a product of linear & quadratic factors • In general, a polynomial of degree n has at most n real roots
Rational	• Has a form $\frac{p(x)}{q(x)}$, where $p(x)$ & $q(x)$ are polynomials • Domain of $\frac{p(x)}{q(x)}$ consists of all ℝ except the roots of $q(x)$
Trigo	
Exp & Log	• Exponential: $f(x) = a^x$, where $a > 0$ • Logarithmic: $\log_a x$ ($a > 0$, $a \neq 1$) • They are inverses of each other

Chapter 1: Limits & Continuity

Limits of the form $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)} = \lim_{x\to\pm\infty}\frac{\overbrace{Ax^\alpha}^{\text{Leading term}} + \cdots}{\underbrace{Bx^\beta}_{\text{Leading term}} + \cdots} = \begin{cases} 0 & \text{if } \alpha < \beta \\ \frac{A}{B} & \text{if } \alpha = \beta \\ \infty \text{ or } -\infty & \text{if } \alpha > \beta \end{cases}$$

If $\lim_{x\to c} g(x) = 0$	$\lim_{x\to c}\frac{\sin(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\sin(g(x))} = 1$ $\lim_{x\to c}\frac{\tan(g(x))}{g(x)} = \lim_{x\to c}\frac{g(x)}{\tan(g(x))} = 1$
when $c = 0$ & $g(x) = x$	$\lim_{x\to 0}\frac{\sin(x)}{x} = \lim_{x\to 0}\frac{x}{\sin(x)} = 1$ $\lim_{x\to 0}\frac{\tan(x)}{x} = \lim_{x\to 0}\frac{x}{\tan(x)} = 1$



- Suppose $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a point c, except possibly at $x = c$, IF $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$, then $\lim_{x\to c} f(x) = L$
- If $\lim_{x\to c} g(x) = 0$, then for any function h, $\lim_{x\to c} g(x)\sin(h(x)) = 0$ and $\lim_{x\to c} g(x)\cos(h(x)) = 0$.

If f is continuous on [a, b] and k is a number between f(a) & f(b), then f(c) = k for some c ∈ [a, b]

** Compare the biggest n[th] term to find limit ** Try to convert to same n[th] term for numer and denom

L'Hospital when its $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Ind. form	Limit	Conditions	Conversion Rule
$\frac{0}{0}$	$\lim_{x\to a}\frac{f(x)}{g(x)}$	$\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = 0$	L Hospital
$\frac{\infty}{\infty}$	$\lim_{x\to a}\frac{f(x)}{g(x)}$	$\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} g(x) = \infty$	L Hospital
$0 * \infty$	$\lim_{x\to a} f(x)g(x)$	$\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = \infty$	Convert to $\lim_{x\to a}\frac{f(x)}{1/g(x)}$
$\infty - \infty$	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} g(x) = \infty$	Remove common factor, rationalise
0^0	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x) = 0$ & $\lim_{x\to a} g(x) = \infty$	$\lim_{x\to a} f(x)^{g(x)} = e^{g(x)\ln f(x)}$
∞^0	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x) = \infty$ & $\lim_{x\to a} g(x) = \infty$	OR
1^∞	$\lim_{x\to a} f(x)^{g(x)}$	$\lim_{x\to a} f(x) = 1$ & $\lim_{x\to a} g(x) = \infty$	let $y = f(x)^{g(x)}$, then $\ln y = g(x)\ln f(x)$

Chapter 2: Derivatives

$$f'(x_0) = \lim_{h\to 0}\frac{f(x_0+h) - f(x_0)}{h}$$

Function	Derivative
Elementary Functions	
x^n	nx^{n-1}
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
Trigonometric Functions	
$\cos(x)$	$-\sin(x)$
$\sin(x)$	$\cos(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\tan(x)$
$\cot(x)$	$-\csc^2(x)$
Inverse Trigonometric Functions	
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$
$\cot^{-1}(x)$	$-\frac{1}{1+x^2}$
$\sec^{-1}(x)$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 1$
$\csc^{-1}(x)$	$-\frac{1}{\lvert x\rvert\sqrt{x^2-1}}, \lvert x\rvert > 1$

Function	Derivative
Elementary Functions	
$(f(n))^n$	$nf'(x)\,(f(n))^{n-1}$
$e^{f(x)}$	$f'(x)e^{f(x)}$
$\ln(f(x))$	$\frac{f'(x)}{f(x)}$
Trigonometric Functions	
$\cos(f(x))$	$-f'(x)\sin(f(x))$
$\sin(f(x))$	$f'(x)\cos(f(x))$
$\tan(f(x))$	$f'(x)\sec^2(f(x))$
$\sec(f(x))$	$f'(x)\sec(f(x))\tan(f(x))$
$\csc(f(x))$	$-f'(x)\csc(f(x))\tan(f(x))$
$\cot(f(x))$	$-f'(x)\csc^2(f(x))$
Inverse Trigonometric Functions	
$\sin^{-1}(f(x))$	$\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\cos^{-1}(f(x))$	$-\frac{f'(x)}{\sqrt{1-f(x)^2}}$
$\tan^{-1}(f(x))$	$\frac{f'(x)}{1+f(x)^2}$
$\cot^{-1}(f(x))$	$-\frac{f'(x)}{1+f(x)^2}$
$\sec^{-1}(f(x))$	$\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}}, \lvert f(x)\rvert > 1$
$\csc^{-1}(f(x))$	$-\frac{f'(x)}{\lvert f(x)\rvert\sqrt{f(x)^2-1}}, \lvert f(x)\rvert > 1$

Global/Local Min/Max (self-explanatory)

Rules of Differentiation: Let u and v be differentiable functions of x and c be a constant

Constant Rule	$\frac{d}{dx}(c) = 0$	**Sum Rule**	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Constant Multiple	$\frac{d}{dx}(cu) = c\frac{du}{dx}$	**Product Rule**	$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$	**Chain Rule**	$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Implicit Differentiation:

- Differentiate both sides wrt x and solving the resultant equation for $\frac{dy}{dx}$
  - Obtain $\frac{dy}{dx}$ by differentiating every term by x
- When differentiating a function in y wrt x: $\frac{d}{dx}g(y) = g'(y) \times \frac{dy}{dx}$

Change Base Formula: $\log_a x = \frac{\ln x}{\ln a}$, $a > 0$ & $a \neq 1$

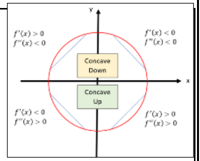
Inverse Functions:	Higher Order Derivatives
$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))} = \frac{1}{f'(b)}$	$f^{(n)}(x) = \frac{d^ny}{dx^n} = D^ny = D^nf(x)$
Parametric Form	$f(x)^{g(x)}$
$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$ $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$	$\frac{d}{dx}f(x)^{g(x)} =$ $f(x)^{g(x)}\left(g'(x)\ln(f(x)) + \frac{f'(x)}{f(x)}g(x)\right)$

Chapter 3: Application of Derivatives (Stationary Point → $f'(x) = 0$)

Turning Pt

1[st] Derivative Test:

x	α^-	α	α^+	α^-	α	α^+	α^-	α	α^+	α^-	α	α^+
$\frac{dy}{dx}$	>0	$=0$	<0	<0	$=0$	>0	>0	$=0$	>0	<0	$=0$	<0
Graph	/	—	\	\	—	/	/	—	/	\	—	\
Nature of stationary point	Maximum turning point			Minimum turning point			Stationary point of inflexion					



2[nd] Derivative Test: Check signs of $\frac{d^2y}{dx^2}$ at $x = k$

If $\frac{d^2y}{dx^2}\big\rvert_{x=k} < 0$	Then $(k, f(k))$ is a MAX turning point
If $\frac{d^2y}{dx^2}\big\rvert_{x=k} > 0$	Then $(k, f(k))$ is a MIN turning point
If $\frac{d^2y}{dx^2}\big\rvert_{x=k} = 0$	Use 1[st] derivative test. f''(x) may be inflection point

Critical pt	a) It is NOT an end-point b) Either $f'(c) = 0$ OR $f'(c)$ DOES NOT exist
Differentiation with L'Hospital	If $\lim_{x\to c} f(x) = 0 = \lim_{x\to c} g(x)$ OR $\lim_{x\to c} f(x) = \infty = \lim_{x\to c} g(x)$, then $\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$
Rolle's Theorem	If $f(a) = f(b)$ → at least 1 no. c in (a, b) such that $f'(c) = 0$
Mean Value Theorem	$gradient = f'(c) = \frac{f(b) - f(a)}{b - a}$

Chapter 4: Integrals

Integration of Elementary Functions		
1.	$\int (ax+b)^n\,dx$	$\frac{(ax+b)^{n+1}}{(n+1)a} + C \quad (n \neq -1)$
2.	$\int \frac{1}{ax+b}\,dx$	$\frac{1}{a}\ln\lvert ax+b\rvert + C$
3.	$\int e^{ax+b}\,dx$	$\frac{1}{a}e^{ax+b} + C$
Integration of Trigonometric Functions		
4.	$\int \sin(ax+b)\,dx$	$-\frac{1}{a}\cos(ax+b) + C$
5.	$\int \cos(ax+b)\,dx$	$\frac{1}{a}\sin(ax+b) + C$
6.	$\int \tan(ax+b)\,dx$	$\frac{1}{a}\ln\lvert\sec(ax+b)\rvert + C$
7.	$\int \sec(ax+b)\,dx$	$\frac{1}{a}\ln\lvert\sec(ax+b) + \tan(ax+b)\rvert + C$
8.	$\int \csc(ax+b)\,dx$	$-\frac{1}{a}\ln\lvert\csc(ax+b) + \cot(ax+b)\rvert + C$
9.	$\int \cot(ax+b)\,dx$	$-\frac{1}{a}\ln\lvert\csc(ax+b)\rvert + C$
10.	$\int \sec^2(ax+b)\,dx$	$\frac{1}{a}\tan(ax+b) + C$
11.	$\int \csc^2(ax+b)\,dx$	$-\frac{1}{a}\cot(ax+b) + C$
12.	$\int \sec(ax+b)\tan(ax+b)\,dx$	$\frac{1}{a}\sec(ax+b) + C$
13.	$\int \csc(ax+b)\cot(ax+b)\,dx$	$-\frac{1}{a}\csc(ax+b) + C$
Integration of Polynomials (Partial Fractions)		
14.	$\int \frac{1}{a^2+(x+b)^2}\,dx$	$\frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right) + C$
15.	$\int \frac{1}{\sqrt{a^2-(x+b)^2}}\,dx$	$\sin^{-1}\left(\frac{x+b}{a}\right) + C$
16.	$\int \frac{-1}{\sqrt{a^2-(x+b)^2}}\,dx$	$\cos^{-1}\left(\frac{x+b}{a}\right) + C$
17.	$\int \frac{1}{a^2-(x+b)^2}\,dx$	$\frac{1}{2a}\ln\left\lvert\frac{x+b+a}{x+b-a}\right\rvert + C$
18.	$\int \frac{1}{(x+b)^2-a^2}\,dx$	$\frac{1}{2a}\ln\left\lvert\frac{x+b-a}{x+b+a}\right\rvert + C$
19.	$\int \frac{1}{\sqrt{(x+b)^2+a^2}}\,dx$	$\ln\left\lvert(x+b) + \sqrt{(x+b)^2+a^2}\right\rvert + C$
20.	$\int \frac{1}{\sqrt{(x+b)^2-a^2}}\,dx$	$\ln\left\lvert(x+b) + \sqrt{(x+b)^2-a^2}\right\rvert + C$
21.	$\int \sqrt{a^2-x^2}\,dx$	$\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + C$
22.	$\int \sqrt{x^2-a^2}\,dx$	$\frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left\lvert x + \sqrt{x^2-a^2}\right\rvert + C$
Modulus		
23.	$\int_{-a}^{a}\lvert bx^n\rvert\,dx$	$\left[\frac{x^{n+1}}{b(n+1)}\right]_0^a - \left[\frac{x^{n+1}}{b(n+1)}\right]_{-a}^0$
Result from tutorials (prob only works with trigonometry)		
24.	$\int_a^b f(x)\,dx$	$bf(b) - af(a) - \int_{f(a)}^{f(b)} f^{-1}(x)\,dx$
25.	$\int_0^a f(x)\,dx$	$\int_0^a f(a-x)\,dx$

By Parts	By Substitution
$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}\,dx$	$u = f(x)$

$\frac{dv}{dx}$ ← L I A T E → u

Point of inflection Test:

- $f' \neq 0$ & $f'' = 0$
- If sign changes, it is a point of inflection

X	x⁻	x	x⁺
f''	-'ve	0	+'ve
f''	+'ve	0	-'ve

Partial Fractions:

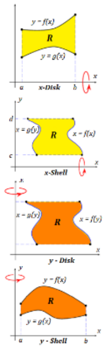
Non-Repeated Linear Factors	$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$
Repeated Linear Factors	$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$
Non-Repeated Quadratic Factors	$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{B}{(x^2+c^2)}$

Riemann Sums & Definite Integrals

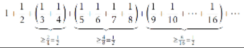
Riemann Sum of f	Approximation (For large n, ≈)
$\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a + k\left(\frac{b-a}{n}\right)\right)$	$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left\{\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k\left(\frac{b-a}{n}\right)\right)\right\} \approx \sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f\left(a+k\left(\frac{b-a}{n}\right)\right)$

Chapter 5: Applications of Integration

Area	**x-axis**		$A = \int_a^b \lvert f(x) - g(x)\rvert\,dx$
	y-axis		$A = \int_c^d \lvert f(y) - g(y)\rvert\,dy$
Volume	**Resolve about x-axis**	**Disk**	$V = \pi\int_a^b f(x)^2\,dx - \pi\int_a^b g(x)^2\,dx$
		Shell	$V = 2\pi\int_c^d y\lvert f(y) - g(y)\rvert\,dy$
	Resolve about y-axis	**Disk**	$V = \pi\int_c^d f(y)^2\,dy - \pi\int_c^d g(y)^2\,dy$
		Shell	$V = 2\pi\int_a^b x\lvert f(x) - g(x)\rvert\,dx$
Arc Length			$\int_a^b \sqrt{1 + f'(x)^2}\,dx$

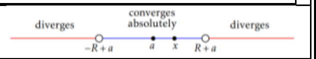


Chapter 6: Sequences & Series

AP	$\{a+(n-1)d\}_{n=1}^{\infty}$ $S_n=\frac{n}{2}(2a+(n-1)d)=\frac{n}{2}(u_1+u_n)$	Where $a \rightarrow 1^{st}$ term of the sequence $d \rightarrow$ common difference
GP	$\{ar^{n-1}\}_{n=1}^{\infty}$ $S_n=\frac{a(r^n-1)}{r-1}=\frac{a(1-r^n)}{1-r}, r\neq 1$ $S_\infty=\frac{a}{1-r}, \lvert r\rvert<1$	Where $a \rightarrow 1^{st}$ term of the sequence $r \rightarrow$ common ratio (of the GP); for GP to be convergent, $\lvert r\rvert<1$
Harmonic Series	$\sum_{n=1}^{\infty}\frac{1}{n}$ *is divergent*	

$$\sum_{n=1}^{\infty} an$$

$\sum_{n=1}^{\infty}\frac{1}{n^p}\begin{cases}\text{diverges}: & p\le 1\\ \text{converges}: & p<1\end{cases}$	$\sum_{n=1}^{\infty}\frac{1}{a^p}\begin{cases}\text{diverges}: & 0<p\le 1\\ \text{converges}: & a>1\end{cases}$
Comparison Test	**Ratio Test**
$\sum_{n=1}^{\infty} an<+\infty,\quad an>0,\quad \lvert b_n\rvert<an$ $\Rightarrow\sum b_n$ *Absolute Comparison*	$b_n=p^n$ $\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert=\begin{cases}>1 & \text{diverges}\\ =1 & \text{undetermined}\\ <1 & \text{converges}\end{cases}$
Root Test	**Radius of Convergence**
$b_n=p^n\quad p=\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}$ $=\begin{cases}>1 & \text{diverges}\\ =1 & \text{undetermined}\\ <1 & \text{converges}\end{cases}$ $\sum_{n=1}^{\infty} an<+\infty\ (\text{converges})\Rightarrow\lim_{n\to\infty} an=0$	$R=\frac{1}{\lim_{n\to\infty}\sqrt[n]{\lvert a_n\rvert}}\qquad \sum_{n=1}^{\infty}a_n(x-a)^n$ $\forall\lvert x-a\rvert<R\ \Rightarrow$ *Absolute Convergence* $\forall\lvert x-a\rvert=R\ \Rightarrow$ *Undetermined* $\forall\lvert x-a\rvert>R\ \Rightarrow$ *Diverge*

Power Series	$\sum_{n=0}^{\infty}a_nx^n$	$\sum_{n=0}^{\infty}c_n(x-a)^n$	

Series

Power Series		$\sum_{n=0}^{\infty}x^n=1+x+x^2+\cdots+x^n=\frac{1}{1-x}, \lvert r\rvert<1$
	Sequence Differential	$f'(x)=\sum_{n=0}^{\infty}nc_n(x-a)^{n-1}$, *for all* $\lvert x-a\rvert<R$
	Sequence Integral	$\int f(x)\,dx=\sum_{n=0}^{\infty}c_n\frac{(x-a)^{n+1}}{n+1}+C$, *for* $\lvert x-a\rvert<R$
Taylor Series	Power Series	$f(x)=\sum_{n=0}^{\infty}c_n(x-a)^n, \lvert x-a\rvert<R$, *for some* $R>0$
	Coefficient	$c_n=\frac{f^{(n)}(a)}{n!}$
	Function	$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-a)^n$
Maclaurin Series		$f(x)=\sum_{n=0}^{\infty}f^{(n)}(0)\frac{(x-0)^n}{n!}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$ If you see a $\frac{1}{1+x}$, try to convert it to $\frac{1}{1-(x\pm a)}$, then $\sum_{n=0}^{\infty}(x\pm a)^n$

Common Maclaurin Series:

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

$$\sin(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots+\frac{(-1)^nx^{2n+1}}{(2n+1)!}+\cdots$$

$$\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots+\frac{(-1)^nx^{2n}}{(2n)!}+\cdots$$

for all x

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+\cdots+x^n+\cdots,\qquad for\ \lvert x\rvert<1$$

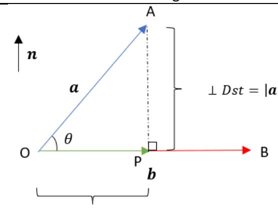
$$ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots+\frac{(-1)^{n+1}x^n}{n}+\cdots,\qquad for\ -1<x\le 1$$

$$1+x^2+x^4+x^6+\cdots=\frac{1}{1-x^2}$$

Pythagorean Identity		
$\sec^2x=1+\tan^2x$	$\csc^2x=1+\cot^2x$	$\sin^2x+\cos^2x=1$
Addition/Subtraction Identities		
$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$	$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$	$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$
Double Angle Formula/ Half Angle Formula		
$\sin 2A=2\sin A\cos A$ $\sin^2A=\frac{1}{2}(1-\cos 2A)$	$\cos 2A=\cos^2A-\sin^2A$ $=2\cos^2A-1$ $=1-2\sin^2A$ $\cos^2A=\frac{1}{2}(1+\cos 2A)$	$\tan 2A=\frac{2\tan A}{1-\tan^2A}$ $\tan\left(\frac{x}{2}\right)=\frac{\sin(x)}{1+\cos(x)}$

Chapter 7: Vectors

Distance Formula	**Eqn Sphere**
$\lvert P_1P_2\rvert=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$	$(x-a)^2+(y-b)^2+(z-c)^2=r^2$
Distance point to π	**Length $\boldsymbol{u}$**
$D=\frac{\lvert ax_0+by_0+cz_0-d\rvert}{\sqrt{a^2+b^2+c^2}}$	$\lVert\boldsymbol{u}\rVert=\sqrt{\boldsymbol{u_1}^2+\boldsymbol{u_2}^2+\boldsymbol{u_3}^2}$ $\lVert\boldsymbol{a}\times\boldsymbol{b}\rVert=\lVert\boldsymbol{a}\rVert\lVert\boldsymbol{b}\rVert\sin\theta$

Orthogonal	2 vectors $\boldsymbol{a}$ & $\boldsymbol{b}$ are orthogonal IFF $\boldsymbol{a}\cdot\boldsymbol{b}=0$ The vector $\boldsymbol{a}\times\boldsymbol{b}$ is orthogonal to BOTH $\boldsymbol{a}$ & $\boldsymbol{b}$
Perp Distance **Foot of Perp** **Length of Projection**	 $\perp Dst=\lvert\boldsymbol{a}\rvert\sin\theta=\lvert\overrightarrow{AP}\rvert=\frac{\lvert\boldsymbol{a}\times\boldsymbol{b}\rvert}{\lvert\boldsymbol{b}\rvert}=\frac{\lvert\overrightarrow{AP}\cdot\boldsymbol{n}\rvert}{\lvert\boldsymbol{n}\rvert}$ $L.O.P=\frac{\lvert\overrightarrow{AP}\cdot\boldsymbol{b}\rvert}{\lvert\boldsymbol{b}\rvert}=\frac{\lvert\overrightarrow{AP}\times\boldsymbol{n}\rvert}{\lvert\boldsymbol{n}\rvert}$ • Adjacent → use dot product (scalar): $\lvert\overrightarrow{AP}\cdot\hat{\boldsymbol{b}}\rvert$ • Opposite → use cross product (vectors): $\lvert\overrightarrow{AP}\times\hat{\boldsymbol{b}}\rvert$
Cross Product	$\boldsymbol{a}\times\boldsymbol{b}=\begin{vmatrix}\boldsymbol{i}&\boldsymbol{j}&\boldsymbol{k}\\a_1&a_2&a_3\\b_1&b_2&b_3\end{vmatrix}=(a_2b_3-a_3b_2)\boldsymbol{i}+(a_3b_1-a_1b_3)\boldsymbol{j}+(a_1b_2-a_2b_1)\boldsymbol{k}$
Line Eqn	$x=x_0+at;\quad y=y_0+bt;\quad z=z_0+ct$
Plane Eqn	$\langle a,b,c\rangle\cdot\langle x,y,z\rangle=\langle a,b,c\rangle\cdot\langle x_0,y_0,z_0\rangle\Leftrightarrow ax+by+cz=ax_0+by_0+cz_0$ $ax+by+cz+d=0\Leftrightarrow d=-(ax_0+by_0+cz_0)$
Skew Lines	• Non-Parallel (Check if directional vector has common factor) • Non-Intersecting (Equate P_1 into L_2, if no solution → non-intersecting)

Chapter 8: Functions of Several Variables

Vector Val Fn $\boldsymbol{r(t)}$	$\boldsymbol{r}(t)=\boldsymbol{r}_0+t\boldsymbol{v}=\langle x_0,y_0,z_0\rangle+t\langle a,b,c\rangle=f(t)\boldsymbol{i}+g(t)\boldsymbol{j}+h(t)\boldsymbol{k}$ $=\langle f(t),g(t),h(t)\rangle$
$\boldsymbol{r'(t)}$	$\boldsymbol{r}'(t)=\lim_{\Delta t\to 0}\frac{\boldsymbol{r}(t+\Delta t)-\boldsymbol{r}(t)}{\Delta t}$ If all components *f,g,h* are differentiable at *t = a* → r is differentiable at *t = a* $\boldsymbol{r}'(a)=\langle f'(a),g'(a),h'(a)\rangle$
Arc Length	$s=\int_a^b\sqrt{f'^{(t)^2}+g'^{(t)^2}+h'^{(t)^2}}\,dt=\int_a^b\lVert\boldsymbol{r}'(t)\rVert\,dt$
Quadric Surfaces	**Elliptic Paraboloid**: Symmetric About z-axis, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$. Traces: $x-plane$: A parabola; $y-plane$: A parabola; $z-plane$: An ellipse **Ellipsoid**: If a = b = c → Sphere, $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Traces: $x-plane$, $y-plane$, $z-plane$: An ellipse
Partial Derivatives (Order matters $f_{xy}\neq f_{yx}$)	$f_x(x,y)=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\qquad f_y(x,y)=\lim_{h\to 0}\frac{f(x,h+y)-f(x,y)}{h}$ Partial Derivatives: f_x; f_y Derivatives of Partial Derivatives: $(f_x)_x$ and $(f_x)_y$; $(f_y)_x$ and $(f_y)_y$
Clairaut's Theorem	Order matters, however, if f_{xy} and f_{yx} are both continuous on D, then: General: $f_{xy}(a,b)=f_{yx}(a,b)$ Applicable to higher deriv: E.g $f_{xyy}=f_{yxy}=f_{yyx}$
Equation of Tangent Plane	**Normal Vector to Tangent Plane:** $\langle f_x(a,b),f_y(a,b),-1\rangle$ **Eqn of Tangent Plane:** $f_x(a,b)(x-a)+f_y(a,b)(y-a)-(z-f(a,b))=0$
Implicit Diff 2Independent	$\frac{\partial z}{\partial x}=-\frac{F_x(x,y,z)}{F_z(x,y,z)}\qquad\frac{\partial z}{\partial y}=-\frac{F_y(x,y,z)}{F_z(x,y,z)}$
Incr, Diff in z	$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)\qquad dz=f_x(x,y)dx+f_y(x,y)dy$
Δz approx	$\Delta z\approx dz=f_x(x,y)dx+f_y(x,y)dy=f_x(x,y)\Delta x+f_y(x,y)\Delta y$
2D Directional Derivatives	$D_{\boldsymbol{u}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ha,y_0+hb)-f(x_0,y_0)}{h}$ $\nabla f(x,y)=\langle f_x,f_y\rangle=\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}$ → Gradient $D_{\boldsymbol{u}}f(x,y)=f_x(x,y)a+f_y(x,y)b=\langle f_x,f_y\rangle\cdot\langle a,b\rangle=\langle f_x,f_y\rangle\cdot\boldsymbol{u}=\nabla f(x,y)\cdot\boldsymbol{u}$
3D Directional Derivatives	$D_{\boldsymbol{u}}f(x_0,y_0,z_0)=\lim_{h\to 0}\frac{f(x_0+ha,y_0+hb,z_0+hc)-f(x_0,y_0,z_0)}{h}=\nabla f(x,y,z)\cdot\boldsymbol{u}$ Gradient: $\nabla f\langle f_x,f_y,f_z\rangle=\frac{\partial f}{\partial x}\boldsymbol{i}+\frac{\partial f}{\partial y}\boldsymbol{j}+\frac{\partial f}{\partial z}\boldsymbol{k}$ Tangent plane to level surface $F(x,y,z)=k$ at (x_0,y_0,z_0) is: $\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0,y-y_0,z-z_0\rangle=0$
Rates ↑/↓	$D_uf(P)=\lVert\nabla f(P)\rVert\cos\theta$ Max: $\lVert\nabla f(P)\rVert$ *Min:* $-\lVert\nabla f(P)\rVert$

Extrema	**1st order derivative (min/max)** $f_x(a,b)=f_y(a,b)=0$	**Critical/Stationary Point** • $\boldsymbol{f_x(a,b)=0}$ and $\boldsymbol{f_y(a,b)=0}$ OR • One of the partial deriv doesn't exist
	Saddle Point • It is a critical point of f AND • Every open disk centred at (a,b) contains points $(x,y)\in D$ where $f(x,y)<f(a,b)$ & points $(x,y)\in D$ where $f(x,y)>f(a,b)$	**2nd Derivative Test** *Discriminant* $D=D(a,b)$ Or $f_{yy}(a,b)$ $=f_{xx}(a,b)f_{yy}(a,b)-[f_{xy}(a,b)]^2$ 1. D > 0 & $f_{xx}(a,b)>0$ → $f(a,b)$ MIN 2. D > 0 & $f_{xx}(a,b)<0$ → $f(a,b)$ MAX 3. D < 0 → (a,b) is a **saddle point** of f 4. D = 0 → no conclusions

Chapter 9: Double Integrals

Double Integrals	$\iint_R f(x,y)\,dA=\lim_{m,n\to\infty}\left(\sum_{i=1}^{m}\sum_{j=1}^{n}f(x_{ij}^*,y_{ij}^*)\,\triangle A\right)$ $\int_a^b A(x)=\int_a^b\left[\int_c^d f(x,y)\,dy\right]dx=\int_a^b\int_c^d f(x,y)\,dy\,dx$
Volume	$V=\iint_R f(x,y)\,dA$
Fubini's Theorem	$\iint_R f(x,y)\,dA=\int_a^b\int_c^d f(x,y)\,dy\,dx=\int_c^d\int_a^b f(x,y)\,dx\,dy$ **Special Case:** $f(x,y)=g(x)h(y)$ [factored out as product of only x or y] $\iint_R g(x)h(y)\,dA=\left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$
Type I Region	Lies between the graphs of 2 continuous functions of x, that is: $D=\{(x,y):a\le x\le b,g_1(x)\le y\le g_2(x)\}$ $\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)}f(x,y)\,dy\,dx$
Type II Region	Lies between the graphs of 2 continuous functions of y, that is: $D=\{(x,y):c\le y\le d,h_1(y)\le x\le h_2(y)\}$ $\iint_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)}f(x,y)\,dx\,dy$
Area of π	$A(D)=\iint_D 1\,dA$
Polar Form	$r^2=x^2+y^2$, $x=r\cos\theta,\quad y=r\sin\theta$ If f is continuous on a polar rectangle R given by: $R=\{(r,\theta):0\le a\le r\le b,\alpha\le\theta\le\beta\}$ where $0\le\beta-\alpha\le 2\pi$ then: $\iint_R f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta,r\sin\theta)r\,dr\,d\theta$
Surface Area	$S.A=\iint_D dS=\iint_D\sqrt{f_x^2+f_y^2+1}\,dA$

Chapter 10: Ordinary Differential Equations (ODE)

Separable ODE	**Reduction to Separable Form**	
$\frac{dy}{dx}=f(x)g(y)$ Separating the variables: $\frac{1}{g(y)}dy=f(x)dx$ Integrating both sides: $\int\frac{1}{g(y)}dy=\int f(x)\,dx+C$	$y'=g\left(\frac{y}{x}\right)$, where g is any function of $\frac{y}{x}$. • Let $v=\frac{y}{x}$, then $y=vx$, $y'=v+xv'$ • Then the equation $y'=g\left(\frac{y}{x}\right)$ can be written as: $v+xv'=g(v)$, which is separable! • That is $\frac{dv}{g(v)-v}=\frac{dx}{x}$ • Solving for v, we obtain y	$y'=f(ax+by+c)$ Where f is continuous and $b\neq 0$ (If $b=0$, eqn is separable) can be solved by setting: $u=ax+by+c$
Linear 1st Order ODE	$\frac{dy}{dx}+P(x)y=Q(x)$ $y\cdot I(x)=\int Q(x)\cdot I(x)\,dx$	Integrating Factor: $I(x)=e^{\int P(x)dx}$ (× both sides by $I(x)$, compute and integrate)
Bernoulli Equation	$y'+p(x)y=q(x)y^n$ where $n\neq 0,1$ is called the Bernoulli equation. *Let* $u=y^{1-n}$ Subst into Bernoulli equation: $\underline{u'+(1-n)p(x)u=(1-n)q(x)}$ This is a first order linear ODE.	
Malthus Model	$\frac{1}{N}=\frac{1}{N_\infty}+\left(\frac{1}{N}-\frac{1}{N_\infty}\right)e^{-Bt}\Rightarrow N=\frac{N_\infty}{1+\left(\frac{N_\infty}{N}-1\right)e^{-Bt}}$	

Basic Identities	
$\frac{1}{\sin x}=\csc x$	$\frac{1}{\cos x}=\sec x$
$\frac{1}{\tan x}=\cot x$	$\frac{\sin x}{\cos x}=\tan x$
Product to Sum Formulas/Sum to Product Formulas	
$\sin A\cos B=\frac{1}{2}(\sin(A+B)+\sin(A-B))$	$\cos A\sin B=\frac{1}{2}(\sin(A+B)-\sin(A-B))$
$\sin A\sin B=-\frac{1}{2}(\cos(A+B)+\cos(A-B))$	$\cos A\cos B=\frac{1}{2}(\cos(A+B)+\cos(A-B))$
R-Formula (α is always +'ve)	
$A\sin\theta\pm B\cos\theta=R\sin(\theta\pm\alpha)$ $R=\sqrt{A^2+B^2}$ $\alpha=\tan^{-1}\left(\frac{B}{A}\right)$	$A\cos\theta\pm B\sin\theta=R\cos(\theta\mp\alpha)$ $R=\sqrt{A^2+B^2}$ $\alpha=\tan^{-1}\left(\frac{B}{A}\right)$

Half-Life	
$y=Ae^{-kt}$ $k=\frac{\ln 2}{T}$	$T\rightarrow t_{1/2}\ of\ x$ $t\rightarrow time\ elapsed$ $A\rightarrow constant\ \in\mathbb{R}$

$$\sqrt{1+(x-y)^2}=\sqrt{(x+y)^2}$$

If integral forms a cycle:

$$\int f(x)g(x)\,dx=A\pm B\pm\int f(x)g(x)\,dx$$

Ans: $\frac{f(x)}{2}(A\pm B)$

$\nabla f(x_0,y_0)$ is the normal to the level curve $f(x,y)=k$ at the point (x_0,y_0) where $f(x_0,y_0)=k$

$\nabla F(x_0,y_0,z_0)$ is normal to tangent vector $r'(t_0)$ to **any** curve C on the surface S that passes through (x_0,y_0,z_0)