Formal Analysis: Maths Refresher

Centre for Experimental Social Sciences

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Preliminaries: set theory

A set is a collection of elements. $x \in A$, $y \in [0, 1]$
For discrete elements, a set $s_1 = \{1, 3, 8\}$, for continuous elements, $s_2 = [0, 100]$ or $s_3 = (0, 100)$.
Set properties:

- Finite or infinite
- Countable or uncountable
- Bounded or unbounded
- Open or closed

Sets can contain or be contained in other sets as (proper) subsets, i.e. $A \subset B$, $A \subseteq B$

Preliminaries: set theory II

$A \setminus B$ is the difference between two sets, i.e. $x \in A \setminus B$ if $x \in A, x \notin B$

A^c is the complement of set A (i.e. difference between universal set and A

Intersection $A \cap B$, i.e. $x \in A \cap B$ if $x \in A$ and $x \in B$

Union $A \cup B$, i.e. $x \in A \cup B$ if $x \in A$ or $x \in B$

Preliminaries: set theory III

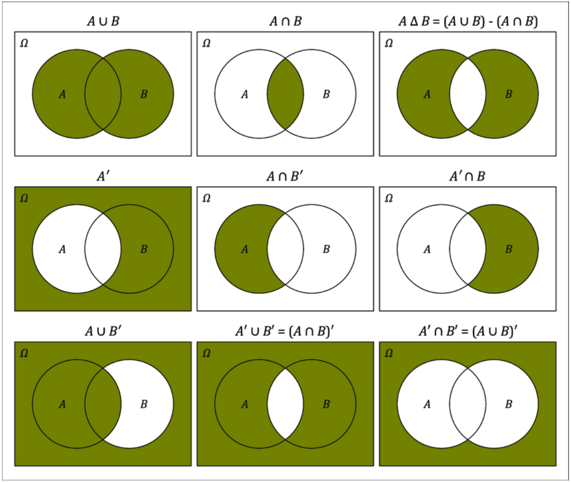


Figure: `https://www.researchgate.net/figure/A-Venn-diagram-of-unions-and-intersections-for-two-sets-A-and-B fig1_332453167`

Preliminaries: Functions I

A function is a mapping (transformation) between sets A and B. It maps an element of A (domain) onto an element of B (range), i.e. define f as $f(x) : A \to B$

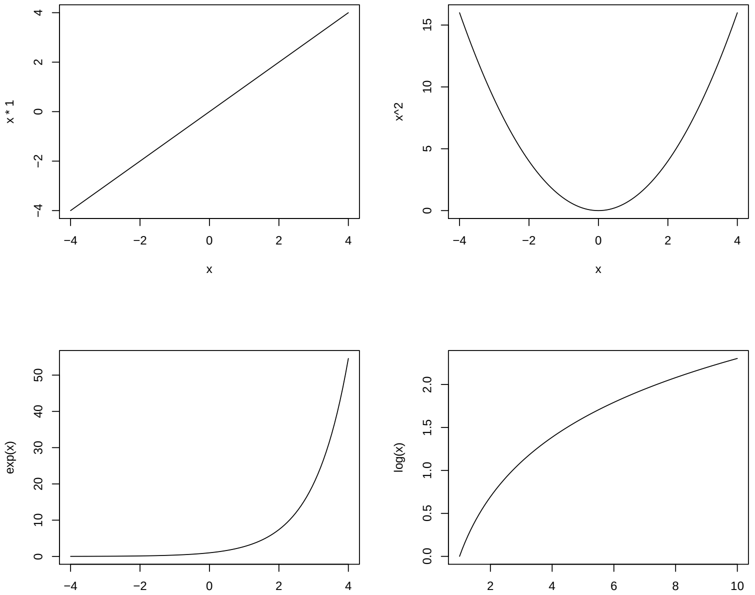
Assigns one element of the domain to each element of the range, such that $y = f(x)$.

A function is bijective if it is

- surjective (onto), i.e. every value in the range is produced by some value in the domain
- injective (1-to-1), i.e. each value in the range is produced by only one value in the domain

Bijective function are invertible, i.e. have an inverse: $f^{-1}(x) : B \to A$

Preliminaries: Functions II



Limits, sequences and series

A sequence is an ordered list of numbers, $\{x_i\}_{i=1}^{N}$
A series is the sum of a sequence, $\sum_{i=1}^{N} x_i$ (payoff streams, discounting future payoffs,...)
Limits of

- sequences: $\lim_{i \to \infty} x_i = L$
- series: $\lim_{N \to \infty} \sum_{i=1}^{N} x_i = S$
- functions: $\lim_{x \to c} f(x) = L$

Differential calculus

Consider the slope of the secant

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \tag{1}$$

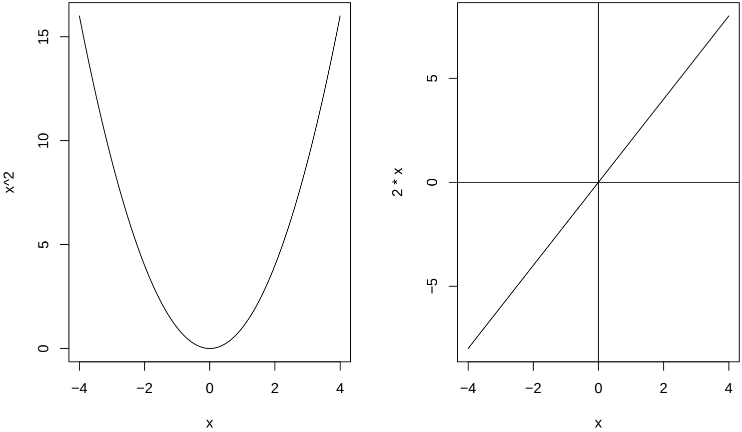
Substituting $h = x_2 - x_1$, we can write the slope as

$$m = \frac{f(x_1 + h) - f(x_1)}{h} \tag{2}$$

What happens as $h \to 0$?

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx} \tag{3}$$

Extrema in one dimension I



Extrema in one dimension II

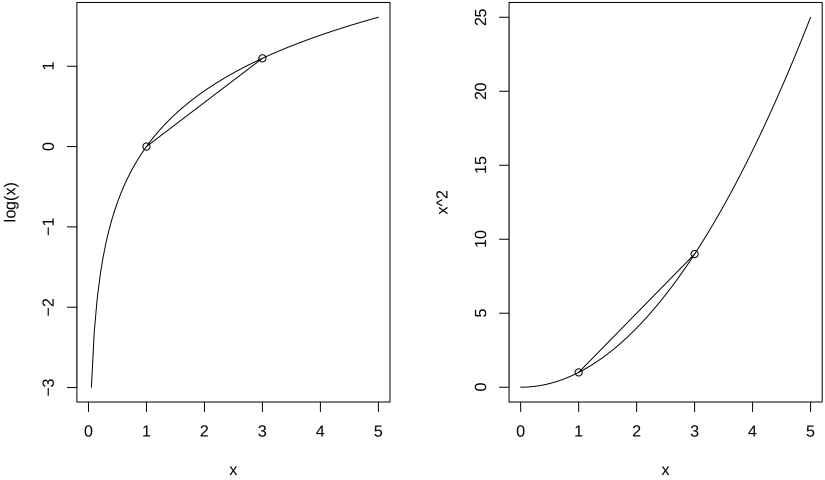
First-order derivative = instantaneous rate of change
Second-order derivative = rate of change of the instantaneous rate of change
How exactly?
A function is concave if for any points x_1, x_2 in its domain and any weight $\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2) \tag{4}$$

A function is convex if for any points x_1, x_2 in its domain and any weight $\lambda \in [0, 1]$

$$f(\lambda x_1 + (1-\lambda)x_2 \leq \lambda f(x_1) + (1-\lambda)f(x_2) \tag{5}$$

Extrema in one dimension III



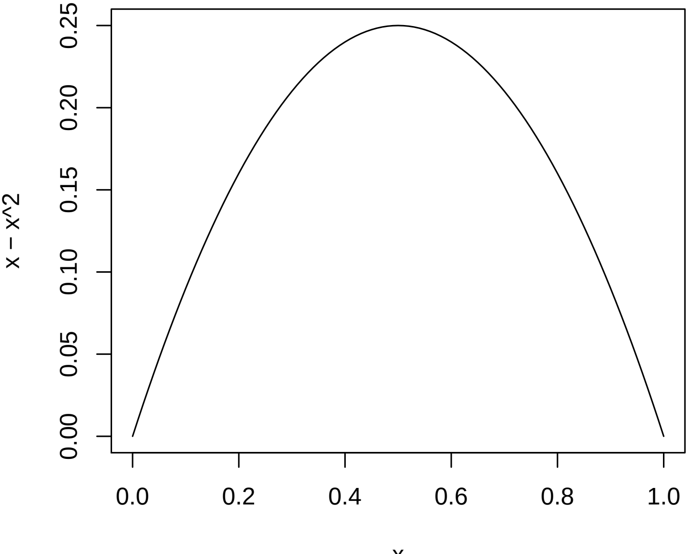
Extrema in one dimension IV

First-order condition (FOC): $f'(x*) = 0$
Second-order condition (SOC):
$f''(x*) < 0 \rightarrow$ local maximum at $x*$
$f''(x*) > 0 \rightarrow$ local minimum at $x*$
$f''(x*) = 0 \rightarrow$ inflection point at $x*$?
Global extremum? (1) substitute local extremum into $f(x)$; (2) substitute lower and upper bounds of domain into $f(x)$; (3) find smallest and largest value

Extrema in one dimension V

Campaign donation: choose x amount to donate
Benefit: same amount x
Opportunity cost: x^2
The general maximization problem is $\max_x u(x)$ where
$u(x) = x - x^2$
Find $u'(x) = 1 - 2x$; solve for $x* : 1 - 2x* = 0 \Rightarrow x* = \frac{1}{2}$
Find $u''(x) = -2 < 0 \rightarrow$ local maximum
Check $u(\frac{1}{2} = \frac{1}{4}, u(0) = u(1) = 0 \rightarrow$ global maximum

Extrema in one dimension V



Extrema in one dimension VI

Restaurant pricing decision (from Gelman, Hill and Vehtari 2020)
Cost per dinner: \$11
Net profit per customer: $x - 11$
Expected number of customers (per night): $5000/x^2$
Expected net profit is then $u(x) = (5000/x^2) * (x - 11)$