Formal Analysis

Lecture 2: Strategic Settings (Following in part slides by Ethan Bueno de Mesquita, and in part slides by Dimitri Landa, and in part my own.)

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January 25, 2022

Today's roadmap

Strategic form games and how to solve them: strategic form games, iterated elimination of dominated strategies, Nash equilibrium

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Models, beautiful (mathematical) models, but why?

What is Game Theory?

A mathematical language developed to study situations of *strategic interdependence*.

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Strategic interdependence means that the consequences of your actions do not only depend on what you do but also on what others do.

Components of a Game

Players: Who is involved in the strategic interaction ?

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Strategies: What can each player do?

Payoffs: What do the players want?

A few Examples

- Prisoner's dilemma

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- Game of Chicken

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- Prisoner's dilemma
- Game of Chicken
- Coordination games: (i) pure, (ii) distributional conflict, (iii) Pareto dominated
- Matching pennies

Choosing a Number

N players

Each player "bids" a real number in $[0, 10]$

If the bids sum to 10 or less, each player's payoff is her bid

Otherwise players' payoffs are 0

How to Solve a Game?

Nash Equilibrium
A strategy profile where no individual has a unilateral incentive to change her behavior

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Before we talk about why this is our central solution concept, let's formalize it

Notation

Player i's strategy

- s_i

Set of all possible strategies for Player i

- S_i

Strategy profile (one strategy for each player)

- $\mathbf{s} = (s_1, s_2, \ldots, s_N)$

Strategy profile for all players except i

- $\mathbf{s_{-i}} = (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$

Different notation for strategy profile

- $\mathbf{s} = (\mathbf{s_{-i}}, s_i)$

Choosing a number with 3 players

$S_i = [0, 10]$

- Player i can choose any real number between 0 and 10

$\mathbf{s} = (s_1 = 1, s_2 = 4, s_3 = 7) = (1, 4, 7)$

- An example of a strategy profile

$\mathbf{s_{-2}} = (1, 7)$

- Same strategy profile, with player 2's strategy omitted

$\mathbf{s} = (\mathbf{s_{-2}}, s_2) = ((1, 7), 4)$

- Reconstructing the strategy profile

Notating Payoffs

Players' payoffs are defined over strategy profiles

- A strategy profile implies an outcome of the game

Player i's payoff from the strategy profile $\mathbf{s}$ is

$$u_i(\mathbf{s})$$

Player i's payoff if she chooses s_i and others play as in $\mathbf{s_{-i}}$

$$u_i((s_i, \mathbf{s_{-i}}))$$

Nash Equilibrium

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if, for every player i

$$u_i(s_i^*, \mathbf{s_{-i}}^*) \geq u_i(s_i', \mathbf{s_{-i}}^*)$$

for all $s_i' \in S_i$

Best Responses

A strategy, s_i, is a **best response** by Player i to a profile of strategies for all other players, $\mathbf{s_{-i}}$, if

$$u_i(s_i, \mathbf{s_{-i}}) \geq u_i(s_i', \mathbf{s_{-i}})$$

for all $s_i' \in S_i$

Best Response Correspondence

Player i's **best response correspondence**, BR_i, is a mapping from strategies for all players other than i into subsets of S_i satisfying the following condition:

- For each $\mathbf{s_{-i}}$, the mapping yields a set of strategies for Player i, $BR_i(\mathbf{s_{-i}})$, such that s_i is in $BR_i(\mathbf{s_{-i}})$ if and only if s_i is a best response to $\mathbf{s_{-i}}$

An Equivalent Definition of NE

Consider a game with N players. A strategy profile $\mathbf{s}^* = (s_1^*, s_2^*, \ldots, s_N^*)$ is a **Nash equilibrium** of the game if s_i^* is a best response to $\mathbf{s_{-i}}^*$ for each $i = 1, 2, \ldots, N$

Solving for pure-strategy NE in N-player games

Basic idea:

1. For each player, identify the best responses to each (pure) strategy by other player

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2. Identify strategy profiles in which each players' strategy is a best response
   - solving a system of conditions for best responses – one for each player

Optimization

In any maximum of a continuous function $F(x, \cdot)$ with respect to the argument x:

1. the first derivative of F with respect to x must be zero (first-order condition (FOC)):

$$\frac{\partial F(x, \cdot)}{\partial x} = 0$$

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   AND

2. the second derivative of F with respect to x must be negative (second-order condition (SOC)):

$$\frac{\partial^2 F(x, \cdot)}{\partial x^2} < 0$$

An Example with Continuous Action Space

- Context Success Function (CSF)
  - (m_1, m_2) – investments by players 1 and 2
  - $p(m_1, m_2)$ – probability of state 1 winning as a function of (m_1, m_2):
    $$p(m_1, m_2) = \frac{m_1}{m_1 + m_2},$$
    unless $m_1 = m_2 = 0$, in which case $p(m_1, m_2) = 1/2$

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  $$u_i = \frac{m_1}{m_1 + m_2} - \gamma_i m_i,$$
  where γ_i is the marginal costs of investment
  - "almost" continuous – except at $m_1 = m_2 = 0$, but $m_i = 0$ is never a best response

Solving the Contest Game I

To get the FOC, take a derivative of player 1's utility with respect to m_1:

$$\frac{\partial u_1(m_1, m_2)}{\partial m_1} = \frac{m_2}{(m_1 + m_2)^2} - \gamma_1$$

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Setting equal to 0 and solving for m_1, we get the optimal value of m_1 as a function of m_2 (i.e., player 1's best response to m_2):

$$m_1(m_2) = \sqrt{\frac{m_2}{\gamma_1}} - m_2$$

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Check the SOC to make sure you have the max u_i and not the min u_i.

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$$m_1(m_2) + m_2 = m_2(m_1) + m_1 = \sqrt{\frac{m_1}{\gamma_2}} = \sqrt{\frac{m_2}{\gamma_1}}$$

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Substituting now into player 1's best response, $m_1(m_2)$, we can solve for equilibrium value of m_2:

$$m_2^* = \frac{\gamma_1}{(\gamma_1 + \gamma_2)^2}$$

and similarly,

$$m_1^* = \frac{\gamma_2}{(\gamma_1 + \gamma_2)^2}$$

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  - when a player randomizes this way, she makes the rival indifferent between playing heads or tails, and so the rival is also willing to randomize between heads and tails
  - indifference among strategies played with positive probability is a general feature of mixed strategy equilibria

Randomized/Mixed strategies

Definition (Mixed Strategy)

A mixed strategy for player i, $\sigma_i : S_i \to [0,1]$ assigns to each pure strategy $s_i \in S_i$, a probability $\sigma_i(s_i) \geq 0$ that it will be played, where

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The revised definition of a normal form game:

Definition (Normal Form Representation Allowing for Mixed Strategies)

$$\{N, (\Delta(S_i)), (u_i(\cdot))\}.$$

Mixed Strategy Equilibrium

Definition (Mixed Strategy Nash Equilibrium)

A mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_N)$ constitutes a Nash equilibrium of game $\{N, (\Delta(S_i)), (u_i(\cdot))\}$, if for every $i = 1, ..., N$,

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \text{ for all } \sigma_i' \in \Delta(S_i),$$

Nash's Theorem

Proposition

Every finite strategic game has a mixed strategy Nash equilibrium.

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This is what John Nash got his Nobel Prize for.

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Proposition

Let $S_i^+ \subset S_i$ denote the set of pure strategies that player i plays with positive probability in a mixed strategy profile $\sigma = (\sigma_1, ..., \sigma_N)$.
Strategy profile $\boldsymbol{\sigma}$ is a Nash equilibrium in game $\{N, (\Delta(S_i)), (u_i(\cdot))\}$ if and only if for all $i = 1, ..., N$,

$$u_i(s_i, \sigma_{-i}) = u_i(s_i', \sigma_{-i}), \text{ for all } s_i, s_i' \in S_i^+, \tag{1}$$

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- To test if σ is a NE need to consider only pure strategy deviations (i.e., changes in a player's strategy σ_i to some pure strategy s_i').
- To identify the pure strategy equilibria of game $\{N, (\Delta(S_i)), (u_i(\cdot))\}$, it suffices to restrict attention to the game $\{N, (S_i), (u_i(\cdot))\}$.

A (Symmetric) Coordination Game: Meeting in New York

- *Players*: Mr. Thomas and Mr. Schelling.
- *Strategies*: They are supposed to meet in New York City at noon for lunch but have forgotten to specify where and cannot reach each other. Each must choose one place where to go.
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Pure strategies?

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- If Mr. Thomas is to randomize between ES and GC, he must be indifferent between them.
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- For Mr. Schelling to set $\sigma_s = 1/11$, he must also be indifferent between his two pure strategies.
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=> Each player going to GC with a probability of $1/11$ and to ES with probability of $10/11$ is a Nash Equilibrium.

Interpreting MSE

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2. if there was a bit of uncertainty about each other's preferences, equilibrium mixed strategy would capture individual players' beliefs about what preference "types" they are playing against, determining when they switch their own strategies in response to those beliefs (Harsanyi's Purification Theorem).

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Social learning

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Self-enforcing agreements

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Analyst humility

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2. Did the analysis of the model teach me something about aspects of the world that I didn't know before?
3. Is there something about the world that is missing from my model that I believe would materially change the conclusions of the model were it included?

Why mathematical models?

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We need the math to make sure that we think straight–to ensure that our conclusions follow from our premises and that we haven't left loose ends hanging in our argument.

In other words, we use math not because we are smart, but because we are not smart enough. We are just smart enough to recognize that we are not smart enough. (Dani Rodrik)

Why game theory?

Solve for the equilibrium!

Take Aways

A Nash Equilibrium is a strategy profile where each player is best responding to what all other players are doing

You find a NE by calculating each player's best response correspondence and seeing where they intersect

NE is our main *solution concept* for strategic situations