Shapley-Shubik Power Index for Indian State Elections

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1 Abstract

This project aims to address the pivotal issue of equitable political representation in Indian National Elections through the utilization of Shapley-Shubik Power Index. The main notion is to assess how much each state influences the national elections in India. We mainly leverage two particular aspects, namely, how many seats each state has and how many seats the state requires to gain a majority. To accomplish this objective we vary the sequence in which each state is addressed to discern its significance and address its marginal contributions. We then incorporate the Shapley-Shubik index to average these observations. This, in return, will provide us with a power index that will help us identify the influence each particular state has in the electoral process.

2 Introduction

History:

- The Shapley-Shubik power index, was proposed by economist L. S. Shapley and mathematician Martin Shubik,It is particularly used to identify the winning coalitions that has the ability to form majority.
- Since 1954, it has been a crucial tool in the study of politics, particularly for analyzing coalition formation in the United States Supreme Court.
- Over the years, innovations and additions have made it an essential tool in Game Theory within Discrete Mathematics.

Formulas and Alterations:

- The Shapley-Shubik power index involves sequential coalitions and pivotal players.
- **Sequential Coalition:** Lists players in the order they joined the coalition.
- **Pivotal Player:** A player in a sequential coalition changing it from losing to winning. **Differences between Coalition and Sequence Coalition:**

  Coalition:
  - In the case of a coalition, we usually can take any number of elements into account.
  - We may regard more than one player as critical.
  - The ordering in set representation does not matter.

  Sequence Coalition:
  - We will have to take all the N players into account.
  - Exactly one pivotal player is to be chosen.
  - Furthermore, the order certainly matters in set representation.

- **Main Result:** More coalitions a player has, the more power they wield.

Determining the Shapley Shubik Power Index:

- For a weighting system with N players:
  - **Step 1:** Write out all Sequential Coalitions.
  - **Step 2:** Find out the Pivotal Player.
  - **Step 3:** Count the Number of times each player is pivotal; call it S.
  - **Step 4:** The Shapley Shubik Power Index would be $\sigma = S/N!$.

Representation using Mathematical/Logical Descriptors:

- The Shapley Shubik value, where the sum is taken over all coalitions of i, is given by:

$$\phi_i(v) = \sum \frac{(s-1)!(n-s)!}{n!} (v(s) - v(s-i))$$

Another means of quantifiers called the Banzhaf index, which serves a similar purpose to the Shapley Shubik Index, will be discussed here. The formula for the Banzhaf index is as follows:

$$Bz_i(v) = \frac{1}{2^{n-1}} \sum (v(s) - v(s-i))$$

Shapley and Banzhaf Indexes, although they provide a similar meaning to the routine problem we are trying to solve, Banzhaf Index is still growing, and its exploration is yet to be maximized. Based on several articles and their citations, we have come to the conclusion that there are certain attributes Shapley contributes to in terms of results, namely:

- Efficiency
- Sensitivity
- Linearity
- Symmetry

On the other hand, Banzhaf adheres to everything but has a little deviation from Shapley Shubik in terms of efficiency (efficiency sometimes translates to completeness).

Now we will discuss four different axioms that further define the working of the Power Index:

2.1 Efficiency

A value ψ is efficient if

$$\sum_{i \in N} (\psi_i(v) = v(N))$$

For the set of players N, define $\pi : N \to N$ to be a permutation of N.

2.2 Symmetry

The value ψ is symmetric if for all $i \in N$ and for all permutations π of N,

$$\Psi_\pi \cdot v(v) = \Psi_i(\pi \cdot v)$$

2.3 Dummy

The value ψ satisfies the dummy axiom if $\Psi_i(v) = 0$ whenever player i is a dummy in v.

2.4 Additive

The value ψ is additive if

$$\Psi(v + w) = \Psi(v) + \Psi(w)$$

2.5 Theorem:

Shapley value is the unique value satisfying all properties of Efficiency, Symmetry, Dummy, Additive as discussed above.
Assumptions and Terms used in the above notations:

- N refers simply to the set of players, where $N = \{1, \ldots, n\}$
- S refers to the sequential coalition hence acquired

- n, s refer to the cardinality of N, S respectively
- v refers to the real-valued set function defined on all coalitions
- π refers to the permutation being initialized
- ψ refers to the Ψ Greek letter

3 Scope of our Programming Project:

3.1 Abstract:

Our programming project simulates the permutations of states in India to analyze their impact on achieving a majority in a real-time election scenario. The "pivotal player" concept is quantified using the Shapley-Shubik power index. The results are visualized through pictorial representations such as pie and bar charts, offering insights into the relative influence of each state in shaping electoral outcomes.

3.2 Project Details:

1. **Increment Pivot Key Function:**
   - **Function:** Adjusts a state's key when the cumulative weight surpasses a specified value (majority).
   - **Purpose:**It Identifies the pivotal key players by marking states contributing to.
2. **Shuffle Function:**
   - **Purpose:**It makes sure all different and diverse state orderings are utilized.

3.2.1 Randomization Algorithm:

In our code we had the problem of handling 36 factorial permutations of all states and union territories in India to create an effective solution we used randomized Sampling,Yates Algorithm,and Shuffle Function.

The Fisher Yates algorithm in our code starts from the last element of the set the algorithm then goes through each element and swaps it with the randomly chosen element.Hence,the final result obtained is a uniform random permutation of set.

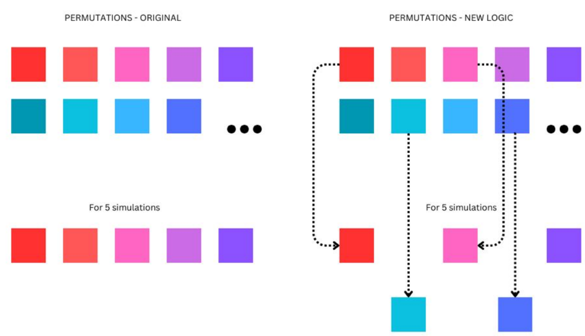


Figure 1: For instance here we have used our permutation logic to account for 36 factorial states by using randomization and Yates shuffling

3.3 Utilization of Shapley-Shubik Index in the Code

The Shapley-Shubik index is used in our code to quantify the (pivotal player) concept. Mainly used in the calculation of key values for each state, considering their cumulative weight(alias seats in Lok Sabha) in the permutation. The Shapley-Shubik index contributes to identifying the state/s that has the possibility to gather maximum majority, guiding the simulation towards possible outcomes over varied permutations.

3 Existing Literature

3.1 Survey of Existing Literature on Shapley-Shubik Index in Sustainable Food-Energy-Water Systems

- This study utilizes the Shapley-Shubik Power Index in sustainable food-energy-water systems, focusing on the palm oil industry.
- The research integrates multi-objective optimizations and Sapley-Shubik Power aiming for economic and environmental enhancements in palm oil mill processes.

3.2 Computational Aspects of Shapley Value in Networks

- This research investigates the computational aspects of the Shapley value.
- Exact analytical formulae in both weighted and unweighted networks are introduced using Shapley value.
- Real-time applications include the Western States Power Grid and astrophysics collaboration networks.

3.3 Exploring Set-Weighted Games in Network Cover Problems

- This paper delves into the computational complexity of network covering problems in transport, social networks.
- It introduces a special class of simple games called set-weighted games, where minimal winning coalitions align with the least covers, utilizing sets for weights.
- It then applies these concepts to real-world scenarios, including public transport networks.

4 Analysis and Findings

Here are the key findings and analysis of our project.

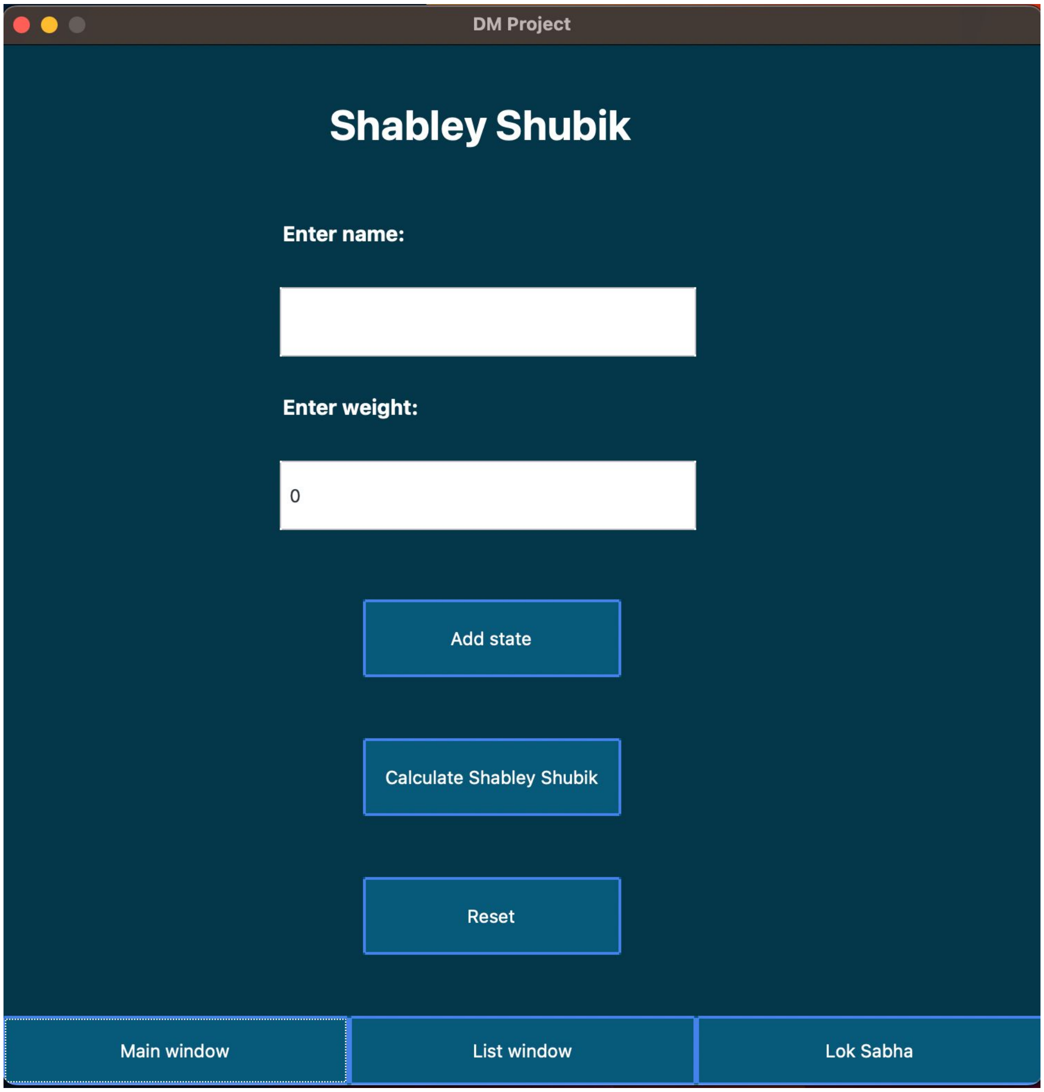


Figure 2: Page 1 with features to add the The State's Name and its Weight alias seats in Lok Sabha

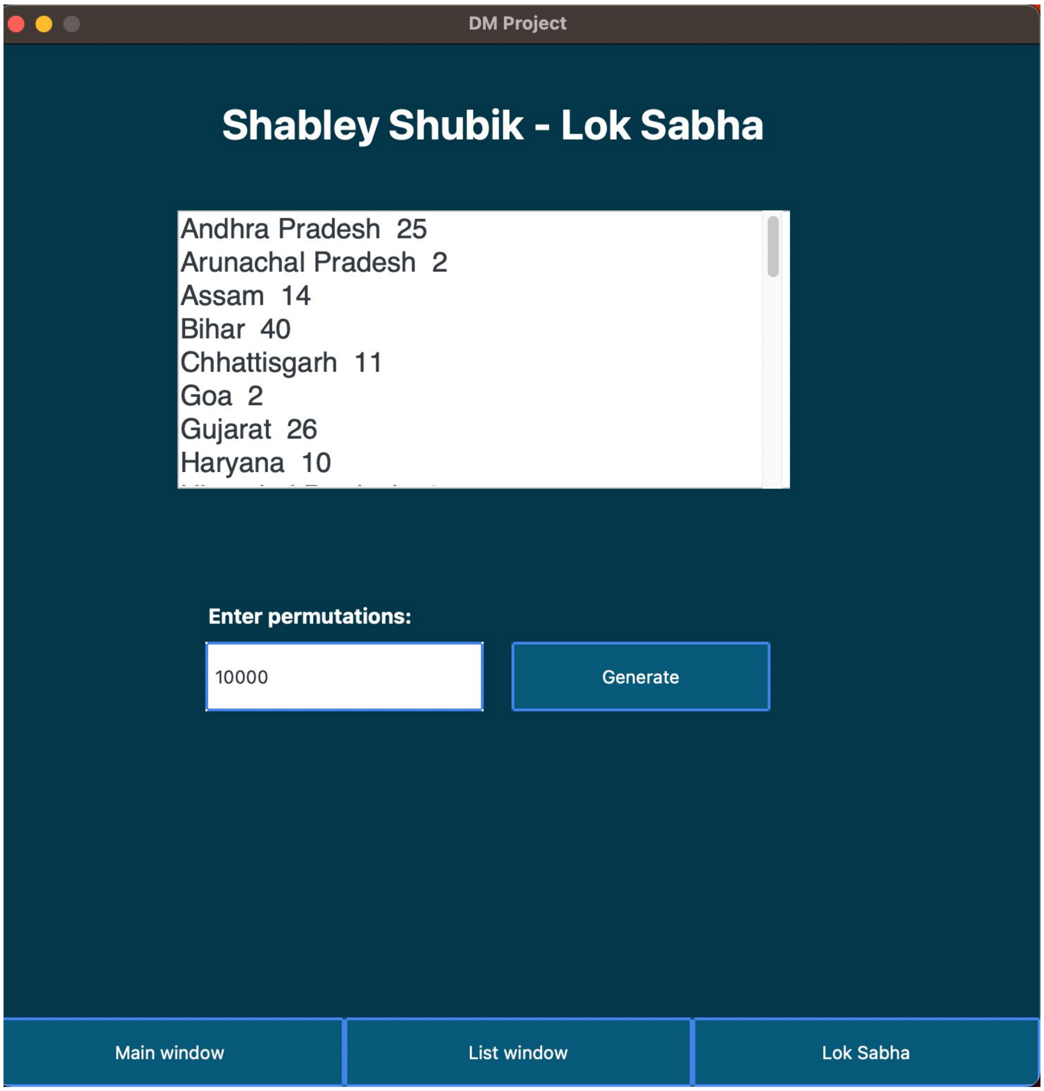


Figure 3: Page 2 with feature to enter the required number of permutation required

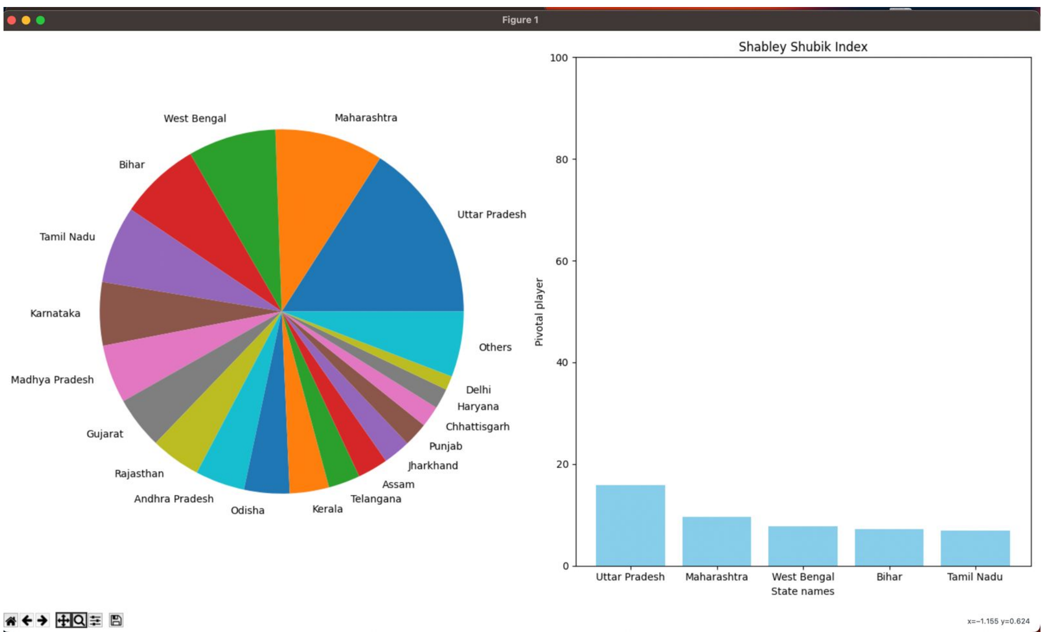
4.1 Simulation Process

- The simulation process involves adding states with their corresponding weights (seats in Lok Sabha).
- Users have the flexibility to input the state names (S) and weights (W).

- Following the addition of states, users input the desired number of permutations (P) for analysis.

4.2 Graphical Representation

The graphical representation includes a pie chart and a bar chart:



1. **Pie Chart Insights:**

    - Here are the results from the pie chart after running 10,000 permutations.

    From the pie chart, it is evident that the states with the maximum power index/influence to gather seats in the majority are:

    1. Uttar Pradesh
    2. Maharashtra
    3. West Bengal

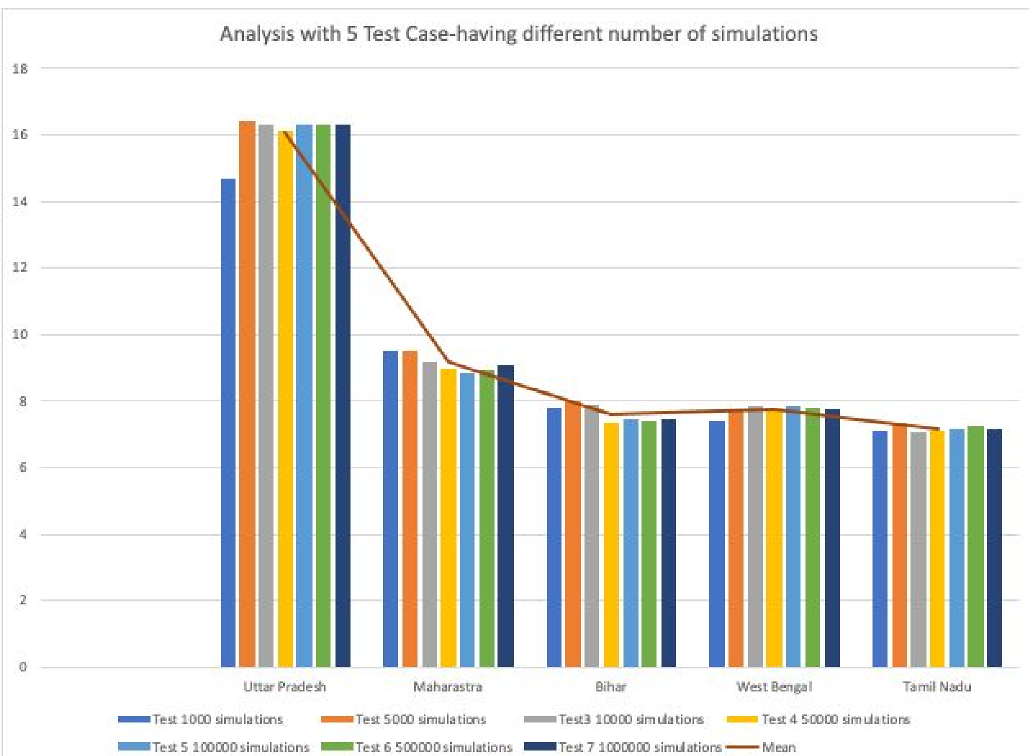
4. Bihar
5. Tamil Nadu

2. **Bar Chart Insights:**

   - From the bar chart results, we have a further exploration into the key pivotal players.

   According to the bar chart, pivotal players are ranked as:

   1. Uttar Pradesh 18 times as pivotal player
   2. Maharashtra 10 times as pivotal player
   3. West Bengal 8 times as pivotal player
   4. Bihar 8 times as pivotal player
   5. Tamil Nadu 8 times as pivotal player



- Here is the outcome after running the The Algorithm For 5 Testcases with varied Number of Simulations
- As we can clearly see in this graph it shows how to Algorithm calculates the power index with the leading 5 States in the Above Chart

4.3 Significance and Application

- The project's outcome helps in providing valuable insights into the plethora of outcomes for the elections depending on the number of state permutations.
- This application can be modified with context and can be used in several domains and especially be redesigned to use for various other countries as well for political analysis.

Overall, the simulation tool empowers users with a comprehensive understanding of the role each state plays in shaping electoral outcomes, helping in proper decision making and strategic planning.

5 Conclusion

- In conclusion,we have used the following DM topics from our course syllabus,Randomization,Weighted Nodes,Heap based permutations and Game Theory.
- We have been able to use the Shapley-Shubik index and Fisher's algorithm effectively in pertaining to this task. Our easy-to-use GUI and pictorial representation further makes it easy for the users to have a vivid representation of the entire project and help in concluding the various sets of outcomes that can occur.
- Future expansion of development of this project can pertain that it can be modified and developed in such a way that it can be used for comprehensive political analysis and can also be developed to be used in various other domains and elections not only accustomed to India but to several other countries as well.
- We would also like to incorporate further algorithms like the Banzhaf index and others, which would help in creating a complex political analysis which has the potential to produce groundbreaking results.
- We would also like it to incorporate in future various Ml models that can increase the precision of the algorithm and can just buy itself using the current data and produce an enhanced version with no margin of error. These further explorations would hopefully delve into an interest pertaining to Game theory and its vivid applications.

6 Glossary

[1] https://www.sciencedirect.com/science/article/pii/S1877050922011590

[2] https://arxiv.org/pdf/2108.04126.pdf

[3] https://www.jstor.org/stable/29761488?seq=12

[4] https://www.math.colostate.edu/ spriggs/m130/power2.pdf

[5] https://jlmartin.ku.edu/courses/math105-F11/Lectures/chapter2-part3.pdf

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