BML lecture #5: Foundations

http://github.com/rbardenet/bml-course

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Misc

- You can still apply to the PhD position I advertise on my website with me and Subhro Ghosh (NUS Singapore).
- Stay tuned: we will announce projects on Monday, papers go on a first come first served basis.
- I'm late with writing up the solutions to the exercises, but they are coming!

Outline

What comes to *your* mind when you hear "Foundations "?

Recap: posterior expected utility

The subjective expected utility principle

1. Choose $\mathcal{S}, \mathcal{Z}, \mathcal{A}$ and a loss function $L(a, s)$,
2. Choose a distribution p over $\mathcal{S}$,
3. Take the the corresponding Bayes action

$$a^\star \in \arg\min_{a\in\mathcal{A}} \mathbb{E}_{s\sim p} L(a, s). \tag{1}$$

Corollary: minimize the posterior expected loss

If we partition $s = (s_o, s_{\mathrm{u}})$, then

$$a^\star \in \arg\min_{a\in\mathcal{A}} \mathbb{E}_{s_o} \mathbb{E}_{s_{\mathrm{u}}|s_o} L(a, s).$$

Equivalently to (1), given s_o, we choose

$$a^\star = \delta(s_o) = \arg\min_{a\in\mathcal{A}} \mathbb{E}_{s_{\mathrm{u}}|s_o} L(a, s).$$

The likelihood principle (J. O. Berger and Wolpert, 1988)

The "formal" LP

Consider two statistical experiments

$$E_i = (X_i, \theta, \{p_i(\cdot|\vartheta), \vartheta \in \Theta\}), \quad i = 1, 2.$$

Assume that for some realizations x_1 and x_2,

$$p_1(x_1|\cdot) \propto p_2(x_2|\cdot).$$

If $\mathrm{Ev}(E, x)$ denotes the "evidence on θ arising from E and x", then

$$\mathrm{Ev}(E_1, x_1) = \mathrm{Ev}(E_2, x_2).$$

Corollary

$\mathrm{Ev}(E, x)$ can depend on x solely through $p(x|\cdot)$.

Standard Bayes satisfies the LP

- Take $p_i(s_i) = p_i(x_i, \theta) = p_i(x_i|\theta)p(\theta) = Zp_i(\theta|x_i)$.
- Then for $a : \mathcal{S} \to \mathcal{Z}$,

$$\int L(a, s_1)\frac{p_1(x_1|\theta)p(\theta)}{Z}\mathrm{d}\theta \propto \int L(a, s_2)\frac{p_2(x_2|\theta)p(\theta)}{Z}\mathrm{d}\theta,$$

  so that Bayes actions coincide: $a^\star = \delta_1(x_1) = \delta_2(x_2)$.
- However, full expected utilities are different in general:

$$\int L(a, s_1)p_1(x_1|\theta)p(\theta)\mathrm{d}x_1\mathrm{d}\theta \neq \int L(a, s_2)p_2(x_2|\theta)p(\theta)\mathrm{d}x_2\mathrm{d}\theta.$$

Some downsides of the LP

- The LP is compelling to many (J. O. Berger and Wolpert, 1988), but it has its downsides.
- It doesn't lead all the way to Bayes.
- I am (personally) uncomfortable with the stopping rule principle: it seems to good to be the right answer.
- It is hard to make fully formal: is $\mathrm{Ev}(E, x)$ even meaningful? See answer by LeCam to (J. O. Berger and Wolpert, 1988).
- It assumes well-specification: $x \sim p(\cdot|\theta^\star)$ for some $\theta^\star$. This is often false in ML.
- It separates the roles of the likelihood and the prior. For LP-abiding Bayesians, the prior is not allowed to depend on data.

Outline

- Top requirement is internal coherence of decisions.
- Various attempts at proving that internally coherent decision-makers minimize some expected utility; see (Parmigiani and Inoue, 2009).

Figure: Bruno de Finetti (1906–1985) and L. Jimmie Savage (1917–1971)

Savage's axioms 1/2

- Start with the triple $(\mathcal{S}, \mathcal{Z}, \mathcal{A} \subset \mathcal{F}(\mathcal{S}, \mathcal{Z}))$ as in Wald, 1950.
- Savage's idea is to list what we expect from a binary relation $\prec$ on $\mathcal{A} \times \mathcal{A}$ describing a decision maker's preferences.

Savage's axioms 2/2

De Finetti's theorem (Hewitt-Savage form)

Theorem: exchangeable $\leftrightarrow$ conditionally i.i.d.; see(Sch95)

Let $X_1, X_2, \dots$ be a sequence of exchangeable random variables on $\mathcal{X}$, i.e.

$$X_1, \dots, X_n \sim X_{\pi(1)}, \dots, X_{\pi(n)}, \forall n, \forall \pi \in \mathfrak{S}_n.$$

Then there exists a probability distribution μ on the set of probability measures $\mathcal{P}(\mathcal{X})$ on $\mathcal{X}$ such that

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \int Q(A_1) \dots Q(A_n) \mathrm{d}\mu(Q).$$

Furthermore, if $Q \sim \mu$,

$$Q(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} 1_A(X_i).$$

To a subjectivist, Savage's theorem says you should use SEU, and representation theorems like de Finetti's constrain your choice of p.

De Finetti's theorem and LDA

The Blackwell-McQueen urn scheme (aka the CRP)

Start with an urn containing a single black ball with weight α. Repeat: draw a ball from the urn with probability $\propto$ its weight. Then,

- If the ball is black, return it to the urn along with another ball of weight 1, with a new color sampled from some base measure H.
- If the ball is colored, return it to the urn along with another ball of weight 1 of the same color.

Denote by $X_1, \ldots$ the color of the ball added.

- Exercise: show that $X_1, X_2, \ldots$ are exchangeable.
- The corresponding prior μ on $\mathcal{P}(\mathcal{X})$ is the Dirichlet process with concentration α and base measure H.

Flaws in the subjectivist viewpoint

Objective (or consensual) Bayes

- A historical objection to Bayes is the need to choose a prior.
- By "objective", we mean that the prior is chosen by some external rule, and that this rule is relatively consensual.
- Take for instance, Jeffreys's "noninformative" priors.

Outline

Complete class theorems state that there is a good prior (but which one?)

A complete class theorem for estimation (James O Berger, 1985)

Under topological and Euclidean assumptions, if further

- $L(\theta, \cdot)$ is continuous,
- $\theta \mapsto \int L(\theta, \hat{\theta}) p(y_{1:n}|x_{1:n}, \theta) \mathrm{d}y_{1:n}$ is continuous for any $\hat{\theta}$,

then for any estimator $\tilde{\theta}$ there exists a prior and a corresponding Bayes estimator

$$\hat{\theta}_{\mathsf{Bayes}} \in \arg\min_{\hat{\theta}} \mathbb{E}_{\theta|x_{1:n}, y_{1:n}} L(\theta, \hat{\theta})$$

such that

$$\forall \theta, \quad \mathbb{E}_{y_{1:n}|x_{1:n},\theta} L(\theta, \hat{\theta}_{\mathsf{Bayes}}) < E_{y_{1:n}|x_{1:n},\theta} L(\theta, \tilde{\theta}).$$

Bayesian estimators thus have good frequentist properties

But finding the "right" prior can be difficult. Frequentists typically use Bayesian derivations with particular (often data-dependent) priors; see e.g. empirical Bayes procedures (Efron, 2012).

PAC bounds; see e.g. (Shalev-Shwartz and Ben-David, 2014)

Let $(x_{1:n}, y_{1:n}) \sim \mathbb{P}^{\otimes n}$, and independently $(x, y) \sim \mathbb{P}$, we want an algorithm $g(\cdot; x_{1:n}, y_{1:n}) \in \mathcal{G}$ such that if $n \geqslant n(\delta, \varepsilon)$,

$$\mathbb{P}^{\otimes n}\left[\mathbb{E}_{(x,y)\sim\mathbb{P}} L(a_g, s) \leqslant \varepsilon\right] \geqslant 1 - \delta.$$

McAllester's bound for 0-1 loss (Chapter 31 of the above book)

For any two distributions P, Q on $\mathcal{G}$, with $\mathbb{P}^{\otimes n}$-probability $1 - \delta$,

$$\mathbb{E}_{g\sim Q}\mathbb{P}(g(x) \neq y) \leqslant \mathbb{E}_{g\sim Q}\frac{1}{n}\sum_{i=1}^{n} 1_{g(x_i)\neq y_i} + \sqrt{\frac{\mathrm{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

This suggests taking the "posterior" Q to be in

$$\arg\min \mathbb{E}_{g\sim Q}\frac{1}{n}\sum_{i=1}^{n} 1_{g(x_i)\neq y_i} + \sqrt{\frac{\mathrm{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

Outline

One possible hybrid view, e.g. (Robert, 2007)

- The starting point is Wald's decision setting, adding integration with respect to a prior.
- It is simple, widely applicable, has good frequentist properties.
- It satisfies the likelihood principle.
- It is tempting to interpret it as follows: beliefs are
  - represented by probabilities,
  - updated using Bayes' rule,
  - integrated when making decisions.
- It is easy to communicate your uncertainty
  - Simply give your posterior.
  - When making a decision, make sure that the priors of everyone involved would yield the same decision.
  - Alternately, perform a prior sensitivity analysis.

Outline

What kind of Bayesian are you?

- I've only scratched the surface. See e.g. (Mayo, 2018).
- Posterior expected utility is conceptually simple and unifying. Beyond that, many intepretations get (partial) philosophical support.
- The role of the likelihood, the prior, your update mechanism, etc. depend on the intepretation that you choose.
- ☹ Many people do not care.
- Hybrid views have become common among statisticians (Robert, 2007; Gelman et al., 2013), but this arguably makes the role of priors fuzzy.
- In ML, the development of Bayesian nonparametrics is reviving the subjectivist view, while objective approaches like PAC-Bayes are also increasingly popular.
- A great entry on subjective Bayes is (Parmigiani and Inoue, 2009).

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