

BML lecture #5: Foundations

<http://github.com/rbardenet/bml-course>

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- ▶ You can still apply to the PhD position I advertise on my website with me and Subhro Ghosh (NUS Singapore).
- ▶ Stay tuned: we will announce projects on Monday, papers go on a first come first served basis.
- ▶ I'm late with writing up the solutions to the exercises, but they are coming!

- 1 Introduction
- 2 The likelihood principle
- 3 Subjective (ot "personalist") Bayesians
- 4 Objective Bayes
- 5 Frequentist Bayes
- 6 Most people are hybrid Bayesians
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What comes to *your* mind when you hear “Foundations ”?

The subjective expected utility principle

- 1 Choose $\mathcal{S}, \mathcal{Z}, \mathcal{A}$ and a loss function $L(a, s)$,
- 2 Choose a distribution p over \mathcal{S} ,
- 3 Take the the corresponding Bayes action

$$a^* \in \arg \min_{a \in \mathcal{A}} \mathbb{E}_{s \sim p} L(a, s). \quad (1)$$

Corollary: minimize the posterior expected loss

If we partition $s = (s_o, s_u)$, then

$$a^* \in \arg \min_{a \in \mathcal{A}} \mathbb{E}_{s_o} \mathbb{E}_{s_u | s_o} L(a, s).$$

Equivalently to (1), given s_o , we choose

$$a^* = \delta(s_o) = \arg \min_{a \in \mathcal{A}} \mathbb{E}_{s_u | s_o} L(a, s).$$

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The “formal” LP

Consider two statistical experiments

$$E_i = (X_i, \theta, \{p_i(\cdot|\vartheta), \vartheta \in \Theta\}), \quad i = 1, 2.$$

Assume that for some realizations x_1 and x_2 ,

$$p_1(x_1|\cdot) \propto p_2(x_2|\cdot).$$

If $\text{Ev}(E, x)$ denotes the “evidence on θ arising from E and x ”, then

$$\text{Ev}(E_1, x_1) = \text{Ev}(E_2, x_2).$$

Corollary

$\text{Ev}(E, x)$ can depend on x solely through $p(x|\cdot)$.

An example: model-based classification

- ▶ Take $p_i(s_i) = p_i(x_i, \theta) = p_i(x_i|\theta)p(\theta) = Z p_i(\theta|x_i)$.
- ▶ Then for $a : \mathcal{S} \rightarrow \mathcal{Z}$,

$$\int L(a, s_1) \frac{p_1(x_1|\theta)p(\theta)}{Z} d\theta \propto \int L(a, s_2) \frac{p_2(x_2|\theta)p(\theta)}{Z} d\theta,$$

so that Bayes actions coincide: $a^* = \delta_1(x_1) = \delta_2(x_2)$.

- ▶ However, full expected utilities are different:

$$\begin{aligned} \int L(a, s_1) p_1(x_1|\theta) p(\theta) dx_1 d\theta &= \int L(a, s_2) C(x_2) p_2(x_2|\theta) p(\theta) dx_2 d\theta \\ &\neq \int L(a, s_2) p_2(x_2|\theta) p(\theta) dx_2 d\theta. \end{aligned}$$

- ▶ The LP is compelling to many (J. O. Berger and Wolpert, 1988), but it has its downsides.
- ▶ It doesn't lead all the way to Bayes.
- ▶ I am (personally) uncomfortable with the stopping rule principle: it seems too good to be the right answer.
- ▶ It is hard to make fully formal: is $\text{Ev}(E, x)$ even meaningful? See answer by LeCam to (J. O. Berger and Wolpert, 1988).
- ▶ It assumes **well-specification**: $x \sim p(\cdot | \theta^*)$ for some θ^* . This is often false in ML.
- ▶ It separates the roles of the likelihood and the prior. For LP-abiding Bayesians, **the prior is not allowed to depend on data**.

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The subjectivistic viewpoint

- ▶ Top requirement is **internal coherence** of decisions.
- ▶ Various attempts at proving that internally coherent decision-makers minimize some expected utility; see (Parmigiani and Inoue, 2009).



Figure: Bruno de Finetti (1906–1985) and L. Jimmie Savage (1917–1971)

- ▶ Start with the triple $(\mathcal{S}, \mathcal{Z}, \mathcal{A} \subset \mathcal{F}(\mathcal{S}, \mathcal{Z}))$ as in Wald, 1950.
- ▶ Savage's idea is to list what we expect from a binary relation \prec on $\mathcal{A} \times \mathcal{A}$ describing a decision maker's preferences.

Theorem: exchangeable \leftrightarrow conditionally i.i.d.; see(Sch95)

Let X_1, X_2, \dots be a sequence of exchangeable random variables on \mathcal{X} , i.e.

$$X_1, \dots, X_n \sim X_{\pi(1)}, \dots, X_{\pi(n)}, \forall n, \forall \pi \in \mathfrak{S}_n.$$

Then there exists a probability distribution μ on the set of probability measures $\mathcal{P}(\mathcal{X})$ on \mathcal{X} such that

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \int Q(A_1) \dots Q(A_n) d\mu(Q).$$

Furthermore, if $Q \sim \mu$,

$$Q(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n 1_A(X_i).$$

To a subjectivist, Savage's theorem says you should use SEU, and representation theorems like de Finetti's constrain your choice of p .

The Blackwell-McQueen urn scheme (aka the CRP)

Start with an urn containing a single black ball with weight α . Repeat: draw a ball from the urn with probability \propto its weight. Then,

- ▶ If the ball is black, return it to the urn along with another ball of weight 1, with a new color sampled from some base measure H .
- ▶ If the ball is colored, return it to the urn along with another ball of weight 1 of the same color.

Denote by X_1, \dots the color of the ball added.

- ▶ Exercise: show that X_1, X_2, \dots are exchangeable.
- ▶ The corresponding prior μ on $\mathcal{P}(\mathcal{X})$ is the Dirichlet process with concentration α and base measure H .

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Objective (or consensual) Bayes

- ▶ A historical objection to Bayes is the need to choose a prior.
- ▶ By “objective”, we mean that the prior is chosen by some external rule, and that this rule is relatively consensual.
- ▶ Take for instance, Jeffreys’s “noninformative” priors.

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A complete class theorem for estimation (James O Berger, 1985)

Under topological and Euclidean assumptions, if further

- ▶ $L(\theta, \cdot)$ is continuous,
- ▶ $\theta \mapsto \int L(\theta, \hat{\theta}) p(y_{1:n} | x_{1:n}, \theta) dy_{1:n}$ is continuous for any $\hat{\theta}$,

then **for any estimator $\tilde{\theta}$** there exists a prior and a corresponding Bayes estimator

$$\hat{\theta}_{\text{Bayes}} \in \arg \min_{\hat{\theta}} \mathbb{E}_{\theta | x_{1:n}, y_{1:n}} L(\theta, \hat{\theta})$$

such that

$$\forall \theta, \quad \mathbb{E}_{y_{1:n} | x_{1:n}, \theta} L(\theta, \hat{\theta}_{\text{Bayes}}) < \mathbb{E}_{y_{1:n} | x_{1:n}, \theta} L(\theta, \tilde{\theta}).$$

Bayesian estimators thus have good frequentist properties

But finding the “right” prior can be difficult. Frequentists typically use Bayesian derivations with particular (often data-dependent) priors; see e.g. empirical Bayes procedures (Efron, 2012).

PAC bounds; see e.g. (Shalev-Shwartz and Ben-David, 2014)

Let $(x_{1:n}, y_{1:n}) \sim \mathbb{P}^{\otimes n}$, and independently $(x, y) \sim \mathbb{P}$, we want an algorithm $g(\cdot; x_{1:n}, y_{1:n}) \in \mathcal{G}$ such that if $n \geq n(\delta, \varepsilon)$,

$$\mathbb{P}^{\otimes n} [\mathbb{E}_{(x,y) \sim \mathbb{P}} L(a_g, s) \leq \varepsilon] \geq 1 - \delta.$$

McAllester's bound for 0-1 loss (Chapter 31 of the above book)

For any two distributions P, Q on \mathcal{G} , with $\mathbb{P}^{\otimes n}$ -probability $1 - \delta$,

$$\mathbb{E}_{g \sim Q} \mathbb{P}(g(x) \neq y) \leq \mathbb{E}_{g \sim Q} \frac{1}{n} \sum_{i=1}^n 1_{g(x_i) \neq y_i} + \sqrt{\frac{\text{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

This suggests taking the “posterior” Q to be in

$$\arg \min \mathbb{E}_{g \sim Q} \frac{1}{n} \sum_{i=1}^n 1_{g(x_i) \neq y_i} + \sqrt{\frac{\text{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

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One possible hybrid view, e.g. (Robert, 2007)

- ▶ The starting point is Wald's decision setting, adding integration with respect to a prior.
- ▶ It is simple, widely applicable, has good frequentist properties.
- ▶ It satisfies the **likelihood principle**.
- ▶ It is tempting to interpret it as follows: beliefs are
 - ▶ represented by probabilities,
 - ▶ updated using Bayes' rule,
 - ▶ integrated when making decisions.
- ▶ It is easy to communicate your uncertainty
 - ▶ Simply give your posterior.
 - ▶ When making a decision, make sure that the priors of everyone involved would yield the same decision.
 - ▶ Alternately, perform a **prior sensitivity analysis**.

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What kind of Bayesian are you?

- ▶ I've only scratched the surface. See e.g. (Mayo, 2018).
- ▶ Posterior expected utility is conceptually simple and unifying. Beyond that, **many interpretations get (partial) philosophical support.**
- ▶ The role of the likelihood, the prior, your update mechanism, etc. depend on the interpretation that you choose.
- ☹ **Many people do not care.**
- ▶ Hybrid views have become common among statisticians (Robert, 2007; Gelman et al., 2013), but this arguably makes the role of priors fuzzy.
- ▶ In ML, the development of **Bayesian nonparametrics is reviving the subjectivist view**, while objective approaches like **PAC-Bayes are also increasingly popular.**
- ▶ A great entry on subjective Bayes is (Parmigiani and Inoue, 2009).

- [1] J. O. Berger and R. L. Wolpert. *The likelihood principle: A review, generalizations, and statistical implications*. Vol. 6. Institute of Mathematical Statistics, 1988.
- [2] James O Berger. *Statistical decision theory and Bayesian analysis*. Springer, 1985.
- [3] B. Efron. *Large-scale inference: empirical Bayes methods for estimation, testing, and prediction*. Vol. 1. Cambridge University Press, 2012.
- [4] A. Gelman et al. *Bayesian data analysis*. 3rd. CRC press, 2013.
- [5] D. G. Mayo. *Statistical inference as severe testing: How to get beyond the statistics wars*. Cambridge University Press, 2018.
- [6] G. Parmigiani and L. Inoue. *Decision theory: principles and approaches*. Vol. 812. John Wiley & Sons, 2009.
- [7] C. P. Robert. *The Bayesian choice: from decision-theoretic foundations to computational implementation*. Springer Science & Business Media, 2007.

- [8] S. Shalev-Shwartz and S. Ben-David. *Understanding machine learning: From theory to algorithms*. Cambridge university press, 2014.
- [9] A. Wald. *Statistical decision functions*. Wiley, 1950.