Bayesian machine learning

Bayesian nonparametrics: random probability measures

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http://github.com/rbardenet/bml-course

Outline

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Parametric versus nonparametric

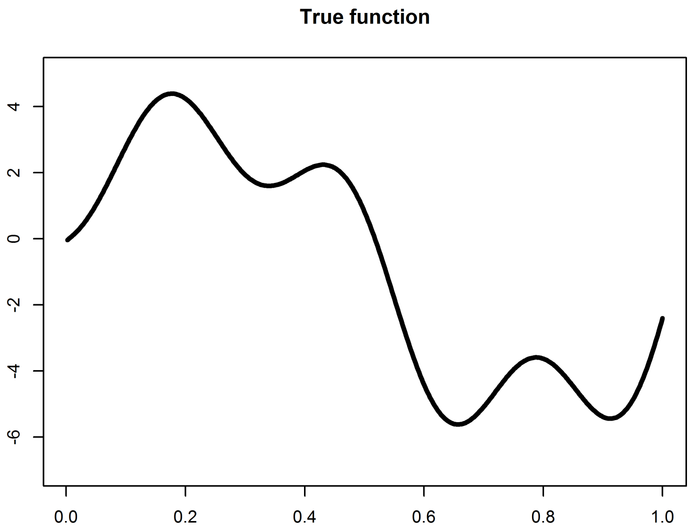
Parametric models

- Finite and fixed number of parameters
- Number of parameters is independent of the dataset

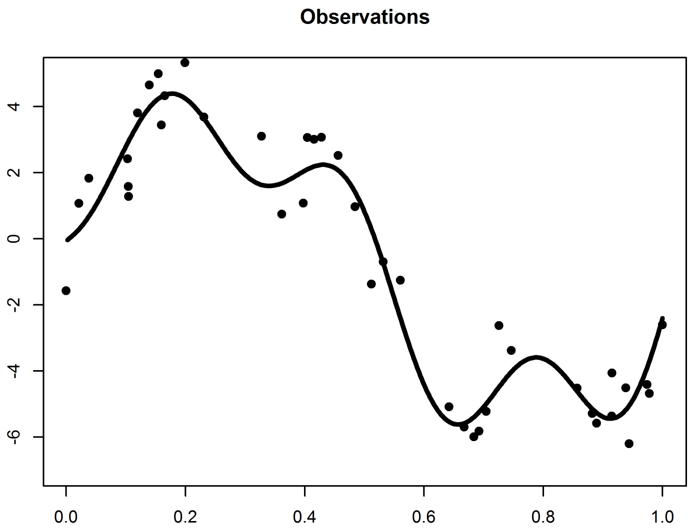
Nonparametric models

- Do have parameters
- Can be understood as having an infinite number of parameters
- Can be understood as having a random number of parameters
- Number of parameters can grow with the dataset

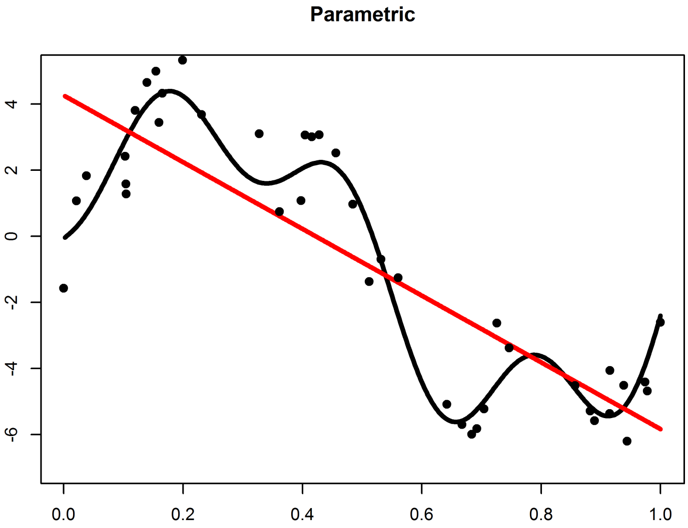
Underlying function



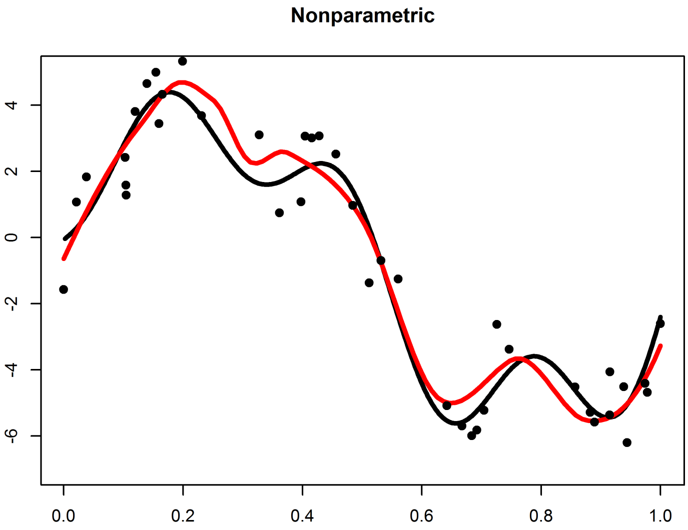
Data



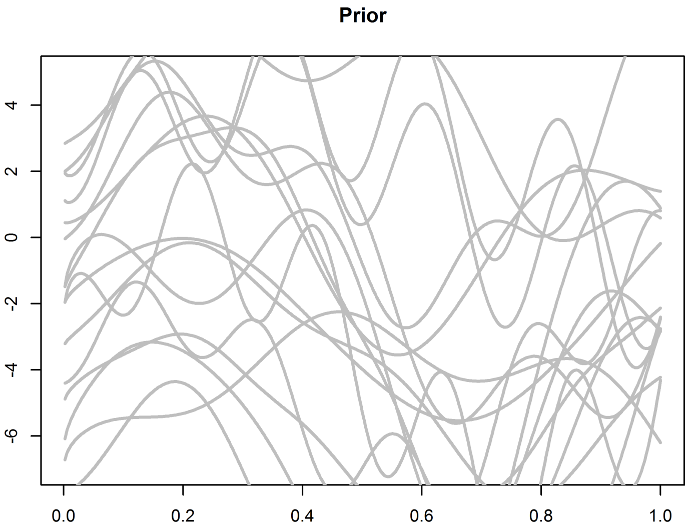
Parametric fitting



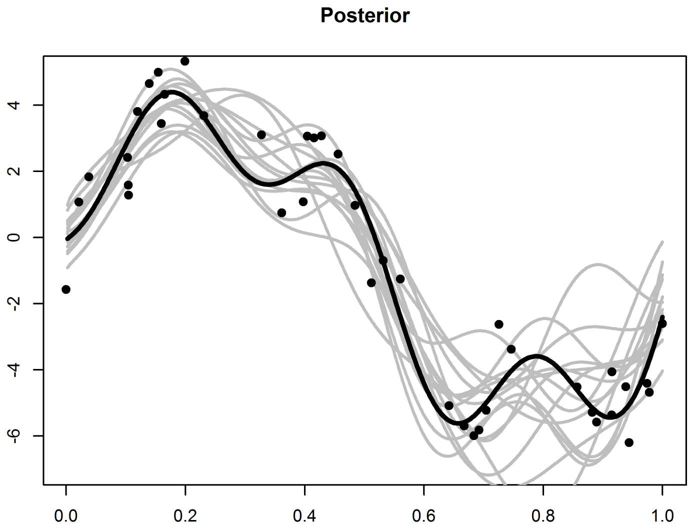
Nonparametric fitting



Prior



Posterior

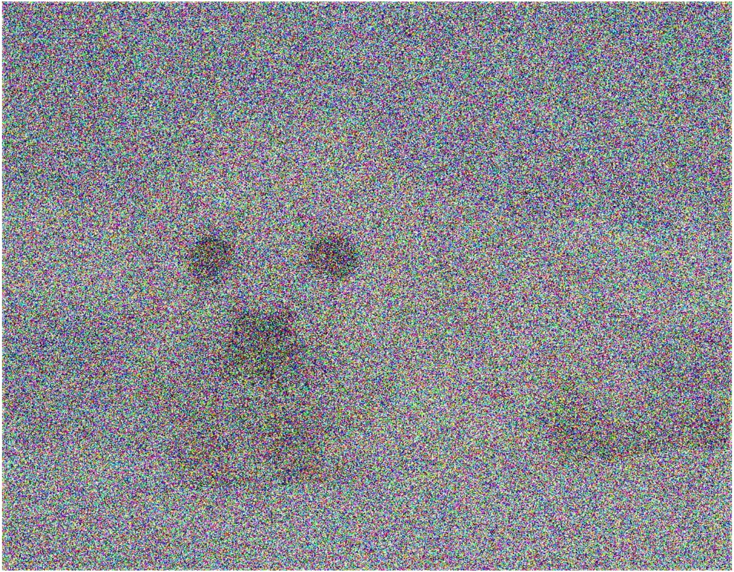


Parametric versus nonparametric

Complexity of the model $\{P_\theta : \theta \in \Theta\}$.

Models	**Parametric**	**Nonparametric**
Dimension	Finite dimensional Θ	Infinite dimensional Θ
Pros	Easier to handle and make interpretations of the results Computationally faster	Less chance for misspecifications More flexible
Cons	Without strong belief in the particular structure of the model not reliable	Computationally and analytically challenging
Examples	Poisson (number of car crashes, typos in a book) Normal distribution (grades of students, height, weight, foot-size of people)	Density, regression function estimation Clustering (unknown cluster size and number)

Noisy picture



Parametric

Nonparametric

Two main categories of priors depending on parameter spaces

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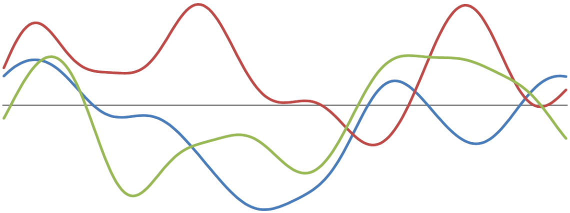
Spaces of functions
random functions

- Continuous stochastic processes
  e.g. Gaussian processes
- Random basis expansions
- Random densities (expon.)



[Wikipedia]

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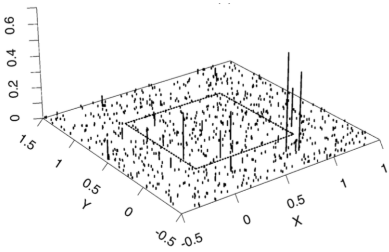
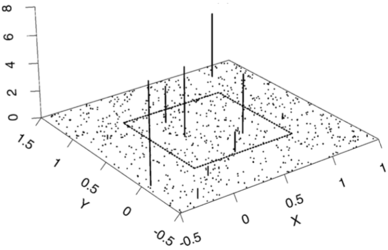
Spaces of probability measures
random probability measures (RPM)

- Often discrete proba. measures
  Cornerstone: Dirichlet process
  We'll see others: Pitman–Yor, Normalized generalized gamma process, Normalized stable process, Gibbs-type processes, Normalized random measures, etc





(**brix1999generalized**)

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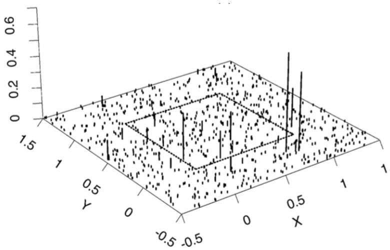

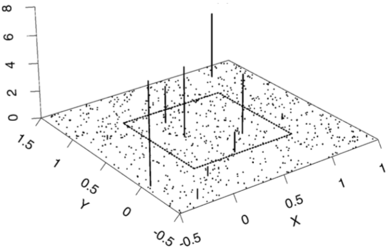

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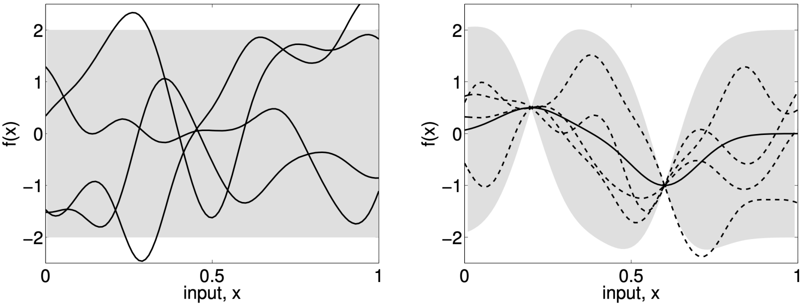
References

- One of the first textbooks: **Ghosh2003**
- One that reads very well: **hjort2010bayesian**
- Quite a comprehensive one on the theory side: **ghosal2017fundamentals**
- Chapter 31 on Nonparametric Bayesian models of **murphy2023probabilisticMLadvanced** (as of today, the full version of this chapter can be found in the supplementary of the book)

Outline

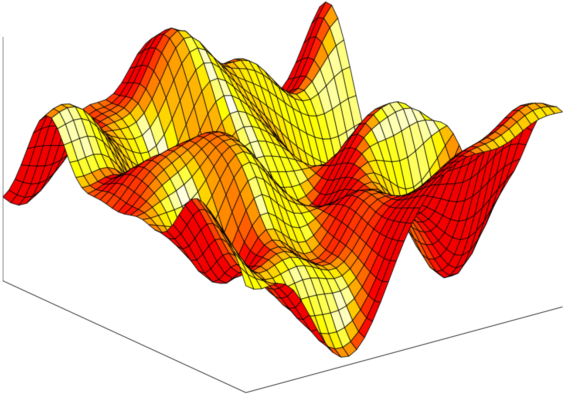
What comes to *your* mind when you hear "Gaussian processes"?

Gaussian processes



From **Rasmussen:2006aa**

Gaussian processes



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Links with other chapters:

- GPs are used are BNP priors on curves
- As such, the properties of the induced posterior are studied in the section on asymptotics
- Wide limit in Bayesian neural networks

References

- Main reference on GPs: **Rasmussen:2006aa**
- GPs in Bayesian inference: Chapter 11 of **ghosal2017fundamentals**
- Chapter 18 on Gaussian processes of **murphy2023probabilisticMLadvanced**

Supervized learning

Two common approaches to supervized learning:

- restrict the class of functions considered, for example only linear functions of the input
- give a prior probability to every possible function, where higher probabilities are given to functions that we consider to be more likely

Definition (Rasmussen:2006aa)

A Gaussian process is a collection of random variables, any finite number of which have a joint Gaussian distribution.

Definition (ghosal2017fundamentals)

A Gaussian process is a stochastic process $W = (W_t : t \in T)$ indexed by an arbitrary set T such that the vector $(W_{t_1}, \ldots, W_{t_k})$ possesses a multivariate normal distribution, for every $t_i \in T$ and $k \in \mathbb{N}$. A Gaussian process W indexed by $\mathbb{R}^d$ is called:

- self-similar of index α if $(W_{\sigma t} : t \in \mathbb{R}^d)$ is distributed like $(\sigma^\alpha W_t : t \in \mathbb{R}^d)$, for every $\sigma > 0$, and
- stationary if $(W_{t+h} : t \in \mathbb{R}^d)$ has the same distribution of $(W_t : t \in \mathbb{R}^d)$, for every $h \in \mathbb{R}^d$.

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Mean function and covariance kernel

Vectors $(W_{t_1}, \ldots, W_{t_k})$ are called marginals, and their distributions marginal distributions or finite-dimensional distributions

Mean function and covariance kernel

Finite-dimensional distributions are determined by the mean function and covariance kernel, defined by

$$\mu(t) = \mathbb{E}(W_t), \quad K(s,t) = \mathrm{Cov}(W_s, W_t), \quad s, t \in T.$$

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Scaling

If $W = (W_t : t \in \mathbb{R}^d)$ is a Gaussian process with covariance kernel K, then the process $(W_{\sigma t} : t \in \mathbb{R}^d)$ is another Gaussian process, with covariance kernel $K(\sigma s, \sigma t)$, for any $\sigma > 0$. A scaling factor $\sigma > 1$ shrinks the sample paths, whereas a factor $\sigma < 1$ stretches them.

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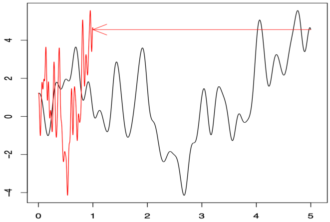

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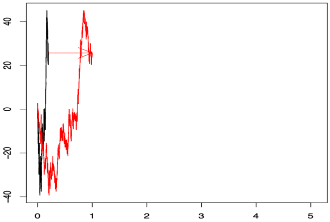

From **ghosal2017fundamentals**

Random series

If $Z_1, \ldots, Z_m \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$ and $a_1, \ldots, a_m$ are [deterministic] functions, then the *Random series* $W_t = \sum_{i=1}^{m} a_i(t) Z_i$ defines a Gaussian process with:

$\mu(t) =$

$K(s,t) =$

Examples

Brownian motion (or Wiener process)

The *Brownian motion* is the zero-mean Gaussian process, say on $[0, \infty)$, with continuous sample paths and covariance function $K(s, t) = \min(s, t)$.

Brownian motion properties

Let B_t be a Brownian motion, then $\forall s < t$:

- Stationarity: $B_t - B_s \sim \mathcal{N}(0, t - s)$
- Independent increments: $B_t - B_s \mathbf{1}(B_u, u \leq s)$

Thus it is a Lévy process.

- Self-similar of index $1/2$.

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Ornstein–Uhlenbeck

The standard *Ornstein–Uhlenbeck process* with parameter $\theta > 0$ is a mean-zero, stationary GP with time set $T = [0, \infty)$, continuous sample paths, and covariance function

$$K(s,t) = (2\theta)^{-1} \exp\left(-\theta|t-s|\right).$$

Properties of Ornstein–Uhlenbeck process

The standard Ornstein–Uhlenbeck process with parameter $\theta > 0$ can be constructed from a Brownian motion B through the relation

$$W_t = (2\theta)^{-1/2} \exp\left(-\theta t\right) B_{e^{2\theta t}}.$$

Relationship between [fixed learning rate] stochastic gradient descent (SGD) and Markov chain Monte Carlo (MCMC) through the Ornstein–Uhlenbeck process: see **mandt2017stochastic**.

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Examples

Square exponential

GP with covariance function (a.k.a. radial basis function kernel)

$$K(s,t) = \exp\left(-\frac{\|t-s\|^2}{2\ell^2}\right).$$

Parameter ℓ is called the *characteristic length-scale*.

Fractional Brownian motion

The *fractional Brownian motion* (fBm) with *Hurst parameter* $\alpha \in (0,1)$ is the mean zero Gaussian process $W = (W_t : t \in [0,1])$ with continuous sample paths and covariance function

$$K(s,t) = \frac{1}{2}\left(s^{2\alpha} + t^{2\alpha} - |t-s|^{2\alpha}\right).$$

- $\alpha = 2$ yields the standard Brownian motion.

Kriging

For a given Gaussian process $W = (W_t : t \in T)$ and fixed, distinct points $t_1, \ldots, t_m \in T$, the conditional expectations $W_t^\star = \mathbb{E}[W_t | W_{t_1}, \ldots, W_{t_m}]$ define another Gaussian process.

Exercise

Find the covariance function of $W_t^\star$, say $K^\star(t, s)$, as a function of $(t_1, \ldots, t_m)$.

Properties of Kriging

- If W has continuous sample paths, then so does $W^\star$.
- In that case the process $W^\star$ converges to W when $m \to \infty$ and the interpolating points $(t_1, \ldots, t_m)$ grow dense in T.

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To every Gaussian process corresponds a Hilbert space, determined by its covariance kernel. This space determines the support and shape of the process, and therefore is crucial for the properties of the Gaussian process as a prior.

Definition

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For a Gaussian process $W = (W_t : t \in T)$, let $\overline{\text{lin}}(W)$ be the closure of the set of all linear combinations $\sum_I \alpha_i W_{t_i}$ in the L_2-space of square-integrable variables. The space $\overline{\text{lin}}(W)$ is a Hilbert space.

Definition

The *reproducing kernel Hilbert space* (RKHS) of the mean-zero, Gaussian process $W = (W_t : t \in T)$ is the set $\mathbb{H}$ of all functions $z_H : T \to \mathbb{R}$ defined by $z_H(t) = \mathbb{E}(W_t H)$, for H ranging over $\overline{\text{lin}}(W)$. The corresponding inner product is

$$\langle z_{H_1}, z_{H_2} \rangle_{\mathbb{H}} = \mathbb{E}(H_1 H_2).$$

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Properties of RKHS

- Correspondance $z_H \leftrightarrow H$ is an isometry (by def of inner product), so the definition is well-posed (the correspondence is one-to-one), and H is indeed a Hilbert space.
- Function corresponding to $H = \sum_I \alpha_i W_{s_i}$ is $z_H =$
- For any $s \in T$, function $K(s, \cdot)$ is in RKHS $\mathbb{H}$ associated with $H = W_s$.

Reproducing kernel Hilbert space II

Reproducing formula

For a general function $z_H \in \mathbb{H}$ we have

$$\langle z_H, K(s, \cdot) \rangle_{\mathbb{H}} = \mathbb{E}(HW_s) = z_H(s).$$

That is to say, for any function $h \in \mathbb{H}$,

$$h(t) = \langle h, K(t, \cdot) \rangle_{\mathbb{H}}.$$

Example of RKHS: Euclidean space

Outline

References

- One of the first textbooks on the Dirichlet process: **Ghosh2003**
- One that reads very well: **hjort2010bayesian**
- Chapter 14 on Discrete Random Structures of **ghosal2017fundamentals**

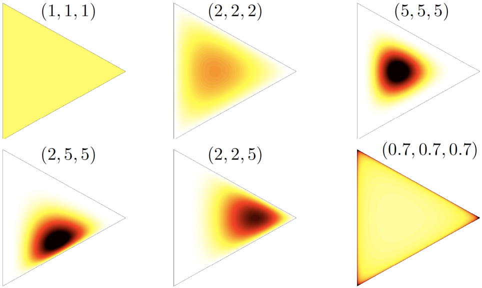
Outline

Dirichlet distribution

The *Dirichlet distribution* on the simplex Δ_K is a probability distribution with parameter $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_K)$ with $\alpha_j > 0$ and density function, for $\boldsymbol{x} = (x_1, \ldots, x_K) \in \Delta_K$,

$$f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{1}{B(\alpha)} \prod_{i=1}^{K} x_i^{\alpha_i - 1}.$$

The Dirichlet distribution is conjugate for the multinomial distribution.



[Image by Y.W. Teh]

Dirichlet process

A central Bayesian nonparametric prior (**ferguson1973bayesian**).

Definition (Dirichlet process)

A Dirichlet process on the space $\mathcal{Y}$ is a random process P such that there exist $\alpha > 0$ (precision parameter) and P_0 (base/centering distribution) such that for any finite partition $\{A_1, \ldots, A_k\}$ of $\mathcal{Y}$, the random vector $(P(A_1), \ldots, P(A_k))$ is Dirichlet distributed

$$(P(A_1), \ldots, P(A_k)) \sim \mathrm{Dir}(\alpha P_0(A_1), \ldots, \alpha P_0(A_k))$$

Notation: $P \sim \mathrm{DP}(\alpha, P_0)$

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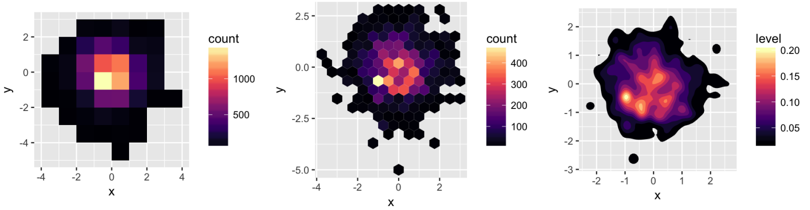


Moments of Dirichlet process I

Proposition

Let $P \sim DP(\alpha, P_0)$ then for every measurable sets A, B we have

$$\begin{aligned}
\mathbb{E}[P(A)] &= P_0(A), \\
\mathrm{Var}[P(A)] &= \frac{P_0(A)(1 - P_0(A))}{1+\alpha}, \\
\mathrm{Cov}(P(A), P(B)) &= \frac{P_0(A \cap B) - P_0(A)P_0(B)}{1+\alpha}.
\end{aligned}$$

Moments of Dirichlet process II

Proof. We will make use of $p(A) \sim \text{Beta}(\alpha P_0(A), \alpha(1 - P_0(A)))$. From this we obtain

$$\mathbb{E}(p(A)) = \frac{\alpha P_0(A)}{\alpha(P_0(A) + 1 - P_0(A))} = P_0(A)$$

and

$$\text{Var}(p(A)) = \frac{\alpha^2 P_0(A)(1 - P_0(A))}{\alpha^2(\alpha + 1)}.$$

We derive the covariance term in two cases, firstly taking into consideration the one with $A \cap B = \emptyset$. In that case any space Ω may be decomposed into three sets:

$$\Omega = \{A, B, (A \cup B)^c\}.$$

Using de Morgan's law the last can be written as $(A \cup B)^c = A^c \cap B^c =: C$. Therefore we may write a joint probability vector

$$\big(P(A), P(B), P(A^c \cap B^c)\big) \sim \text{Dir}\big(\alpha P_0(A), \alpha P_0(B), \alpha P_0(C)\big)$$

and hence $\text{Cov}(P(A), P(B)) = -P_0(A)P_0(B)/(1 + \alpha)$. In the more general case one may decompose

$$A = (A \cap B) \cup (A \cap B^c)$$
$$B = (B \cap A) \cup (B \cap A^c),$$

Moments of Dirichlet process III

so that

$$\mathrm{Cov}(P(A), P(B)) = \mathrm{Cov}(P(A \cap B) + P(A \cap B^c), P(B \cap A) + P(B \cap A^c))$$

and so forth using the linearity of covariance. $\square$

Marginalizing out the DP

Property $\mathbb{E}[P(A)] = P_0(A)$ can be written equivalently as

$$\mathbb{E}(P(A)) = P_0(A) = \int P(A)\mathrm{d}\mathrm{DP}(P).$$

A Dirichlet process model can be constructed as a two level sampling model:

$$\begin{cases} P \sim \mathrm{DP}(\alpha, P_0) \\ \quad X|P \sim P, \end{cases}$$

i.e. we sample a probability measure P from the Dirichlet process and then given P, we sample random variables X_i.

Marginalizing out P, we obtain the marginal distribution of X:

$$X \sim P_0.$$

Posterior distribution I

Let $X_{1:n} := (X_1, \dots, X_n)$ be sampled from the hierarchical model

$$\begin{cases} P \sim \mathrm{DP}(\alpha, P_0) \\ X_{1:n} | P \overset{\text{iid}}{\sim} P. \end{cases}$$

This model is usually used as a building block in a larger hierarchical model, e.g. mixture models, graphs, etc.

Theorem (DP posterior distribution)

The DP is conjugate, with posterior equal to

$$P | X_{1:n} \sim DP(\alpha P_0 + \sum_{i=1}^{n} \delta_{X_i}).$$

The predictive distribution, called Pólya urn or Blackwell–MacQueen scheme, is given by

$$\mathbb{P}(X_{n+1} | X_{1:n}) = \frac{\alpha}{\alpha + n} P_0 + \frac{1}{\alpha + n} \sum_{i=1}^{n} \delta_{X_i}.$$

Posterior distribution II

Proof. The posterior distribution of $\boldsymbol{a} = (a_1, \dots, a_k) = (P(A_1), \dots, P(A_k))$ depends on the observations only via their cell counts $\boldsymbol{N} = (N_1, \dots, N_k)$, $N_j = \#\{i : X_i \in A_j\}$ (it comes from *tail–free* property), so

$$\boldsymbol{a}|X_{1:n} \sim \boldsymbol{a}|N_{1:k}.$$

The prior and model are

$$\begin{cases} \boldsymbol{a} \sim \mathrm{Dir}_k(\alpha P_0(A_1), \dots, \alpha P_0(A_k)) \\ \boldsymbol{N}|P \sim \mathrm{Multinom}_k(\boldsymbol{a}). \end{cases}$$

This results in the posterior of form

$$\begin{aligned} p(\boldsymbol{a}|\boldsymbol{N}) &\propto a_1^{\alpha P_0(A_1)+N_1-1} \cdots a_k^{\alpha P_0(A_k)+N_k-1} \\ &= \mathrm{Dir}_k\big(\alpha P_0(A_1) + N_1, \dots, \alpha P_0(A_k) + N_k\big). \end{aligned}$$

$\square$

Combinatorial properties: Number of distinct values I

Assume that the base measure P_0 is non-atomic. Then with probability 1:

$$X_i \notin \{X_1, \dots, X_{i-1}\} \Leftrightarrow X_i \sim P_0.$$

Let $D_i = \mathbb{I}(X_i \text{ is a new value})$ and lets denote $K_n = \sum_{i=1}^n D_i$, a number of distinct values $X_1, \dots, X_n$ with distribution $\mathcal{L}(K_n)$.

Proposition (Asymptotics for K_n)

Random variables D_i are distributed i.i.d. with respect to Bernoulli$(\alpha/(\alpha+i-1))$. Therefore for fixed α and for $n \to \infty$ we have:

- **i)** $\mathbb{E}K_n \sim \alpha \log n \sim \mathrm{Var}(K_n)$
- **ii)** $K_n / \log(n) \xrightarrow{a.s.} \alpha$
- **iii)** $(K_n - \mathbb{E}K_n)/sd(K_n) \to N(0,1)$
- **iv)** $d_{TV}\big(\mathcal{L}(K_n), Poisson(\mathbb{E}K_n)\big) = o\big(1/\log(n)\big)$ *where*

$$d_{TV}(P,Q) = sup|P(A) - Q(A)|$$

over measurable partition A

Proof.

- **i)** $\mathbb{E}K_n = \sum_{i=1}^{n} \frac{\alpha}{\alpha+i-1}$ and $\mathrm{Var}(K_n) = \sum_{i=1}^{n} \frac{\alpha(i-1)}{(\alpha+i-1)^2}$.
- **ii)** Since D_i's are $\mathbb{I}$ one may use Kolmogorov law of strong numbers and

$$\sum_{i=1}^{\infty} \frac{\mathrm{Var}(D_i)}{(\log i)^2} = \sum_{i=1}^{\infty} \frac{\alpha(i-1)}{(\alpha+i-1)^2(\log i)^2} < \infty$$

  by e.g. the fact that $\sum_i (1/i(\log i)^2)$ converges.
- **iii)** By Lindeberg central limit theorem.
- **iv)** This is implied from Chein–Stein approximation.

$\square$

A central limit theorem for independent random variables (possibly not identically distributed).

Theorem (Lindeberg central limit theorem)

Suppose X_i *are i.i.d. such that* $\mathbb{E}X_i = \mu_i$ *and* $\mathrm{Var}X_i = \sigma_i^2 < \infty$. *Define* $Y_i = X_i - \mu_i$, $T_n = \sum_{i=1}^n Y_i$, $s_n^2 = \mathrm{Var}(T_n) = \sum_{i=1}^n \sigma_i^2$. *Then provided that*

$$\forall \epsilon > 0 \quad \frac{1}{s_n^2} \sum_{i=1}^n \mathbb{E}\big(Y_i^2 \mathbb{I}(|Y_i| > \epsilon s_n)\big) \xrightarrow{n\to\infty} 0 \textit{[Lindeberg condition]},$$

we have the central limit theorem: $T_n/s_n \xrightarrow{d} N(0,1)$.

Combinatorial properties: Distribution of distinct values I

We have now the limits of K_n and we know its approximate distribution $\mathcal{L}(K_n)$. The exact distribution of K_n is:

Proposition (Distribution of K_n)

If P_0 is non-atomic then

$$\mathbb{P}(K_n = k) = \mathfrak{C}_n(k) n! \alpha^k \frac{\Gamma(\alpha)}{\Gamma(\alpha + n)}, \qquad (1)$$

where

$$\mathfrak{C}_n(k) = \frac{1}{n!} \sum_{S \in \mathfrak{I}_n(k)} \prod_{j \in S} j \qquad (2)$$

and $\mathfrak{I}_n(k) = \{S \subset \{1, \ldots, n-1\},\ |S| = n - k\}$.

Recall the definition of the Gamma function $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$.

Combinatorial properties: Distribution of distinct values II

Let us consider when we may deal with events $K_n = k$: we have two cases

$$\begin{cases} K_{n-1} = k-1 \text{ and } X_n \text{ is a new value} \\ K_{n-1} = k \text{ and } X_n \text{ is not a new value.} \end{cases}$$

This results in

$$p_n(k,\alpha) := \mathbb{P}(k_n = k|\alpha) = \frac{\alpha}{\alpha+n-1} p_{n-1}(k-1,\alpha) + \frac{n-1}{\alpha+n-1} p_{n-1}(k,\alpha). \tag{3}$$

Now let us remark that $\mathfrak{C}_n(k) = p_n(k, \alpha = 1)$. Therefore

$$\mathfrak{C}_n(k) = \frac{1}{n}\mathfrak{C}_{n-1}(k-1) + \frac{n-1}{n}\mathfrak{C}_{n-1}(k). \tag{4}$$

By induction over n: first we check case $n = 1$:

$$p_1(1,\alpha) = \mathfrak{C}_1(1)\frac{\alpha}{\alpha} = \mathfrak{C}_1(1).$$

Combinatorial properties: Distribution of distinct values III

To check case $n > 1$ we use (1) and then (3):

$$\begin{aligned}
p_n(k,\alpha) &= \frac{\alpha}{\alpha+n-1} p_{n-1}(k-1,\alpha) + \frac{n-1}{\alpha+n-1} p_{n-1}(k,\alpha) \\
&= \frac{\alpha}{\alpha+n-1} \mathfrak{C}_{n-1}(k-1)(n-1)!\alpha^{k-1} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)} + \\
&+ \frac{n-1}{\alpha+n-1} \mathfrak{C}_{n-1}(k)(n-1)!\alpha^{k} \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)} \\
&= \frac{\alpha^k}{\alpha+n-1}(n-1)! \frac{\Gamma(\alpha)}{\Gamma(\alpha+n-1)} n \left(\frac{1}{n} \mathfrak{C}_{n-1}(k-1) + \frac{n-1}{n} \mathfrak{C}_{n-1}(k) \right) \\
&= \mathfrak{C}_n(k) n! \alpha^k \frac{\Gamma(\alpha)}{\Gamma(\alpha+n)},
\end{aligned}$$

which proves property (1).

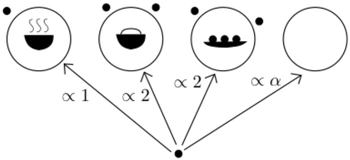
To prove (2) let use define a polynomial $A_n(s)$ as $A_n(s) = \sum_{i=1}^{\infty} \mathfrak{C}_n(k)s^k$. Then using (4) polynomial $A_n(s)$ can be written as

$$
\begin{aligned}
A_n(s) &= \sum_{k=1}^{\infty} \left(\frac{1}{n} \mathfrak{C}_{n-1}(k-1) + \frac{n-1}{n} \mathfrak{C}_{n-1}(k) \right) s_k \\
&= \frac{1}{n}(sA_{n-1}(s) + (n-1)A_{n-1}(s)) = \frac{s+n-1}{n} A_{n-1}(s) \\
&= \ldots = A_1(s) \prod_{j=2}^{n} \frac{s+j-1}{j} = \frac{s(s+1) \cdot \ldots \cdot (s+n-1)}{n!}.
\end{aligned}
$$

Last equality implies from the fact that $\mathfrak{C}_1(k) = \mathbf{1}\{k=1\}$ and hence $A_1(s) = s$. Checking terms after the expansion finishes the proof of (2).

Combinatorial properties: Chinese Restaurant process I

A culinary metaphor of the random partition induced by the DP. Customers join tables with probability proportional to n_j, the number of clients already sitting, or sit at new table with probability proportional to α.



Proposition (Chinese Restaurant process)

A random sample $X_{1:n}$ from a DP with precision parameter α induces a partition of $\{1, \ldots, n\}$ into k sets of sizes $n_1, \ldots, n_k$ with probability

$$p(n_1, \ldots, n_k) = p(\{n_1, \ldots, n_k\}) = \alpha^k \frac{\Gamma(\alpha)}{\Gamma(\alpha + n)} \prod_{j=1}^{k} \Gamma(n_j).$$

Combinatorial properties: Chinese Restaurant process II

Proof. We use the Pólya urn scheme slightly changed by using $n_1, \ldots, n_k$

$$\mathbb{P}(X_{n+1}|X_{1:n}) = \frac{\alpha}{\alpha+n} P_0 + \frac{1}{\alpha+n} \sum_{j=1}^{k} n_j \delta_{X_j^*}.$$

By exchangeability, the distribution of $\{n_1, \ldots, n_k\}$ does not depend on the order of the observations. Let's compute $p(n_1, \ldots, p_k)$ as the probability of one draw where the first table consists of first n_1 observations etc.

To proceed, let us use Pólya urn scheme: we denote $\bar{n}_j = \sum_{i=1}^{j} n_i$ and hence $\bar{n}_k = n$, the total number of observations. We can observe the following pattern: first ball open new table, following $n_j - 1$ ones fill in that table and so forth. That quantity can be rewritten as

$$\frac{\alpha^k}{\alpha(\alpha+1)\ldots(\alpha+n-1)} \prod_{j=1}^{k} (n_j - 1)!,$$

where one can rewrite both terms using Gamma function $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u} du$: the first term can be written as

$$\frac{\alpha^k}{\alpha(\alpha+1)\ldots(\alpha+n-1)} = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

Combinatorial properties: Chinese Restaurant process III

while the second one as $(n_j - 1)! = \Gamma(n_j)$.

One should remark that for ordered partitions we have

$$\bar{p}(n_1, \ldots, n_k) = \frac{p(n_1, \ldots, n_k)}{k!}.$$

$\square$

Combinatorial properties: Ewens sampling formula I

Ewens sampling formula (ESF), presented originally by **ewens1972sampling**, is the distribution of multiplicities $m = (m_1, \ldots, m_n)$, m_ℓ is the number of groups of size ℓ. Also known as allelic partitions in population genetics, when there is no selective difference between types: null hypothesis in non Darwinian theory. See also **antoniak1974mixtures**.

Proposition (Ewens sampling formula)

The distribution of the multiplicities $(m_1, \ldots, m_n)$ *induced by a DP is*

$$p(m_1, \ldots, m_n) = \frac{\alpha^k}{\alpha_{(n)}} \frac{n!}{\prod_{\ell=1}^n \ell^{m_\ell} m_\ell!}.$$

Notation $n_{(k)} := n(n-1)\cdots(n-k+1)$.

Combinatorial properties: Ewens sampling formula II

Proof. Two steps: 1) Compute probability of particular sequence of $X_1, \ldots, X_n$ in given class $(m_1, \ldots, m_n)$, note that all such sequences are equally likely and 2) multiply obtained quantity by the number of such sequences.

1) Consider a sequence $X_1, \ldots, X_n$ such that $X_1, \ldots, X_{m_1}$ occur each only once, then the next m_2 occur only twice and so on. This sequence has probability which may be obtained by the Pólya Urn scheme in the same fashion as CRP:

$$\frac{\alpha^{m_1}(\alpha \cdot 1)^{m_2} \ldots \left(\alpha \cdot 1 \cdot \ldots \cdot (n-1)\right)^{m_n}}{\alpha_{(n)}} = \frac{\alpha^k}{\alpha_{(n)}} \prod_{\ell=1}^{n} ((\ell-1)!)^{m_\ell}.$$

2) Number of sequences $X_1, \ldots, X_n$ with frequencies $(m_1, \ldots, m_n)$ is a number of ways of putting n distinct objects into bins, so called multinomial coefficient. Since ordering of the m_ℓ bins of frequency ℓ is irrelevant, divide by $m_\ell!$:

$$\frac{1}{\prod_{l=1}^{n}(m_\ell)!} \binom{n}{1 \times \# m_1, 2 \times \# m_2, \ldots, n \times \# m_n} = \frac{n!}{\prod_{\ell=1}^{n} m_\ell! (\ell!)^{m_\ell}}$$

To finish one needs to multiply results obtained in 1) and 2). $\square$

Stick-breaking representation

The DP has almost surely discrete realizations (Sethuraman, 1994)

$$P = \sum_{j=1}^{\infty} \pi_j \delta_{\theta_j}$$

- locations $\theta_j \overset{\text{iid}}{\sim} G_0$
- weights $\pi_j = \tilde{\pi}_j \prod_{l<j}(1 - \tilde{\pi}_l)$ with $\tilde{\pi}_j \overset{\text{iid}}{\sim} Beta(1, \alpha)$,

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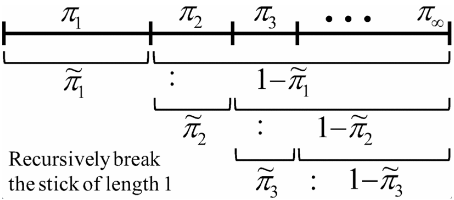

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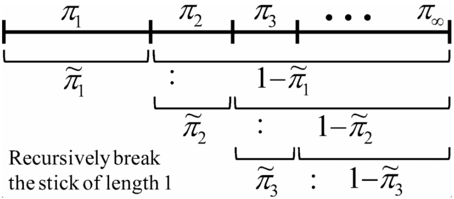

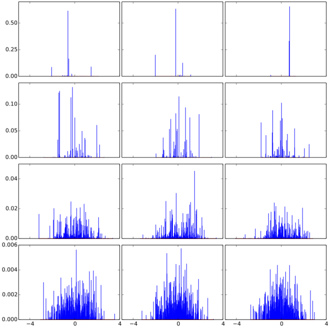

Stick-breaking representation I

A constructive representation of the DP due to **sethuraman1994constructive**.

Theorem (Stick-breaking)

If $V_1, V_2, \ldots \overset{iid}{\sim} Be(1, \alpha)$ *and* $\phi_1, \phi_2, \ldots \overset{iid}{\sim} P_0$ *are i.i.d. variables, then define* $p_1 = V_1$ *and*

$$p_j = V_j \prod_{1 \leq l \leq j} (1 - V_l)$$

then

$$P = \sum_{i=1}^{\infty} p_i \delta_{\phi_i} \sim DP(\alpha, P_0).$$

Lemma

For independent $\phi \sim P_0$ *and* $V \sim Be(1, \alpha)$ *the DP is the only solution of the distributional equation*

$$P \sim V\delta_\phi + (1 - V)P,$$

where $P \sim DP(\alpha, P_0)$*.*

Stick-breaking representation II

Proof. 1) The weights $(p_1, p_2, \dots)$ need to form a probability vector. The leftover mass at stage j is

$$1 - \left(\sum_{i=1}^{j} p_i\right) = \prod_{i=1}^{j}(1 - V_i) =: R_j.$$

One may notice that R_j is decreasing and for every j we have $R_j \in [0, 1]$, hence we obtain almost sure convergence which is equivalent with convergence in mean. Therefore

$$\mathbb{E}R_j = \mathbb{E}\prod_{j}(1 - V_j) = \prod_{j}\mathbb{E}(1 - V_j) = \left(\frac{\alpha}{\alpha + 1}\right)^j \to 0.$$

So $(p_1, \dots)$ is a probability vector almost surely and P is a probability measure almost surely.

Stick-breaking representation III

2) Now one may write

$$P = p_1\delta_{\phi_1} + \sum_{j=2}^{\infty} p_j\delta_{\phi_j} = V_1\delta_{\phi_1} + (1 - V_1)\sum_{j=1}^{\infty} \tilde{p}_j\delta_{\tilde{\phi}_j},$$

where $\tilde{p}_j = \frac{p_{j+1}}{1-V_1} = V_{j+1}\prod_{l=2}^{j}(1 - V_l)$ and $\tilde{\phi}_j = \phi_{j+1}$, then $(\tilde{p}_j)$ and $(\tilde{\phi}_j)$ satisfy the same distributional definitions as (p_j) and (ϕ_j), hence $\tilde{P} \sim P$ and so P is solution of the Lemma equation (4) whose only solution is the DP. $\square$

DP as a normalized Gamma process I

The DP can be obtained by normalizing a Gamma process. It is a generic way to obtain random probability measures from almost surely finite random measures. Let us restrict to $\mathcal{Y} = \mathbb{R}$.

Definition

Gamma process on $\mathbb{R}_+$ is a process $(S(u) : u \geq 0)$ with independent increments satisfying

$$\forall u_1 : 0 \leq u_1 \leq u_2 : \quad S(u_2) - S(u_1) \overset{\text{ind}}{\sim} Ga(u_2 - u_1, 1).$$

This ensures that the process has non-decreasing right continuous sample path $u \mapsto S(u)$.

Theorem

For every $\alpha > 0$ and for every cumulative distribution function G, a random cumulative distribution function such that

$$F(t) = \frac{S(\alpha G(t))}{S(\alpha)}$$

is the distribution of a $\mathrm{DP}(\alpha, G)$.

DP as a normalized Gamma process II

Proof. For any set of t_i satisfying $-\infty = t_0 < t_1 < \ldots < t_k = \infty$ we have

$$S(\alpha G(t_i)) - S(\alpha G(t_{i-1})) \sim Ga\big(\alpha G(t_i) - \alpha G(t_{i-1}), 1\big).$$

Use property that if $Y_i \stackrel{\text{ind}}{\sim} Ga(\alpha_i, 1)$ then $(Y_1, \ldots, Y_n)/\sum_i Y_i \sim \mathrm{Dir}_n(\alpha_1, \ldots, \alpha_n)$ to obtain

$$\big(F(t_1)-F(t_0), \ldots, F(t_k)-F(t_{k-1})\big) \sim \mathrm{Dir}_k\big(\alpha G(t_1)-\alpha G(t_0), \ldots, \alpha G(t_k)-\alpha G(t_{k-1})\big).$$

Hence the definition of DP holds for every partition in intervals. These form a measure determining class, so that the definition holds for every partition in general. $\square$

Definition via the Pólya urn scheme

A Pólya sequence with parameter αP_0 is a sequence of random variables $X_1, \ldots, X_n$ whose joint distribution satisfies

$$X_1 \sim P_0,\ \ X_{n+1} | X_1, \ldots, X_n \sim \frac{\alpha}{\alpha + n} P_0 + \frac{1}{\alpha + n} \sum_{i=1}^{n} \delta_{X_i}.$$

Theorem

If $X_1, X_2, \ldots$ is a Pólya sequence then exists random probability measure P such that $X_i | P \overset{iid}{\sim} P$ and $P \sim DP(\alpha, P_0)$.

Proof. We can consider Pólya sequence as an outcome of Pólya urn, we see that it is exchangeable. By de Finetti theorem exists such probability measure P such that $X_i | P \overset{\text{iid}}{\sim} P$. So far we have proved existence of the DP and know that DP generates a Pólya sequence. Since the RPM given by de Finetti's theorem is unique this proves that $P \sim \mathrm{DP}(\alpha, P_0)$. $\square$

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Outline

A parametric approach

Mixture model with K components

$$G = \sum_{k=1}^{K} \pi_k \delta_{\phi_k}$$

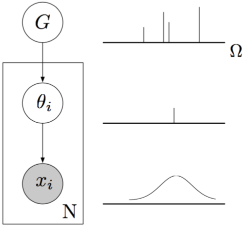
δ_{ϕ_k} is a point mass at ϕ_k.

G is to be understood as a K-faceted dice. The mixture density is:

$$p(X|\pi, \phi) = \sum_{k=1}^{K} \pi_k p(x|\phi_k)$$

Then

$$\begin{aligned} \theta_i &\sim G \\ x_i &\sim p(x|\theta_i) \end{aligned}$$



A Bayesian parametric approach

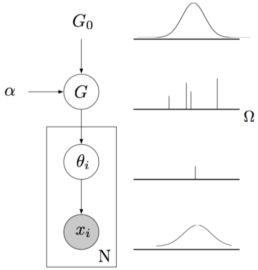
Bayesian mixture Models with K components

We need a distribution over the probability measure (aka dice) G, that is a distribution over weights or classes $\pi = (\pi_1, \ldots, \pi_K)$ and over mean and covariance (for 2-dimensional data) $\phi_k = (\mu_k, \Sigma_k)$

- $\pi \sim \text{Dirichlet}(\alpha/K, \ldots, \alpha/K)$
- $(\mu_k, \Sigma_k) \sim \text{Normal} \times \text{Inverse-Wishart}$

This makes $G = \sum_{k=1}^{K} \pi_k \delta_{\phi_k}$ a random dice

$$
\begin{aligned}
\phi_k &\sim G_0 \\
\pi &\sim \text{Dirichlet}(\alpha/K, \ldots, \alpha/K) \\
G &= \sum_{i=1}^{K} \pi_k \delta_{\phi_k} \\
\theta_i &\sim G \\
x_i &\sim p(x|\theta_i)
\end{aligned}
$$



Choosing K

There are several options for choosing K

- Model selection with information criteria: AIC, BIC, or cross-validation, etc
- Hierarchical model, with a prior on K
- Be nonparametric, and let K get large... possibly infinite.

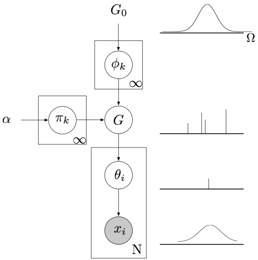
A Bayesian nonparametric approach

Bayesian nonparametric mixture Models

We now move to G being an infinite sum $G = \sum_{k=1}^{\infty} \pi_k \delta_{\phi_k}$

We need a distribution over this infinite dice G, that is exactly what the Dirichlet process does. It is parameterized by the precision parameter α and the base measure G_0.

- $\pi = (\pi_1, \pi_2, \ldots) \sim \mathrm{GEM}(\alpha)$
- $\phi_k \sim G_0$



Markov chain Monte Carlo (MCMC) methods:

- Marginal methods: marginalizing over the posterior DP P, and sampling using the posterior Pólya urn scheme (**neal2000markov**)
- Conditional methods: sampling a finite but sufficient number of parameters (**ishwaran2001gibbs**). BNPdensity R package (**arbel2021BNPdensity**).

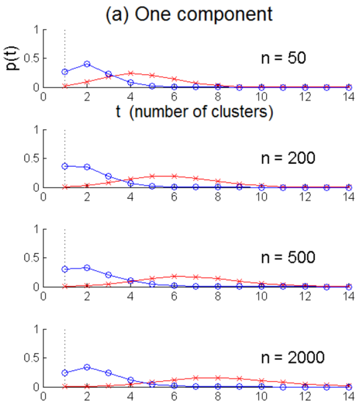
Variational approximations (**blei2006variational**)

Warning on interpretation of K_n I

Consider a simple DP mixture model with

- Gaussian base measure,
- Gaussian kernel,
- where data are sampled iid from some distribution.

Then the posterior on K_n is inconsistent (**miller2013simple**).



Warning on interpretation of K_n II

From **miller2013simple** (here K_n is denoted T_n):

Theorem 4.1. *If $X_1, X_2, \ldots \in \mathbb{R}$ are i.i.d. from any distribution with $\mathbb{E}|X_i| < \infty$, then with probability 1, under the standard normal DPM with $\alpha = 1$ as defined above, $p(T_n = 1 \mid X_{1:n})$ does not converge to 1 as $n \to \infty$.*

Theorem 5.1. *If $X_1, X_2, \ldots \sim \mathcal{N}(0, 1)$ i.i.d. then*

$$p(T_n = 1 \mid X_{1:n}) \xrightarrow{\text{Pr}} 0 \quad \text{as } n \to \infty$$

under the standard normal DPM with concentration parameter $\alpha = 1$.

But there is some hope...

Bayesian decision theory

From decision theory: a Bayes estimator minimizes a posterior expected loss.

$$\hat{a}_L = \arg\inf_{a \in A} \mathbb{E}_{\pi(\theta)}[L_a(\theta)].$$

Examples with Euclidean parameter spaces:

- L^2, squared loss $\longrightarrow$ posterior mean
- L^1, absolute loss $\longrightarrow$ posterior median
- $0-1$ loss $\longrightarrow$ mode a posteriori (MAP)

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Deriving an optimal clustering

The posterior expected loss of clustering c', denoted by $L(c')$, is obtained by averaging the loss with respect to posterior weight

$$L(c') = \sum_{c \in \mathcal{A}_n} L(c, c') p(c|\boldsymbol{x}),$$

and the decision is taken by choosing the best

$$\hat{c} = \arg\min_{c' \in \mathcal{A}_n} \sum_{c \in \mathcal{A}_n} L(c, c') p(c|\boldsymbol{x})$$

Several losses have been considered:

- 0-1 loss (**rajkowski2019analysis**),
- Binder loss (**dahl2006model**),
- Variation of information (**wade2018bayesian**).

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Simplest loss: L_{0-1}

$$L_{0-1}(c') = \sum_{c \in \mathcal{A}_n} L_{0-1}(c, c') p(c|\boldsymbol{x}) = \sum_{c \in \mathcal{A}_n,\, c \neq c'} p(c|\boldsymbol{x}),$$
$$= 1 - p(c'|\boldsymbol{x})$$

which is to say that the expected loss of c' is all the posterior mass except that of c'. So that it is easily minimized at the value c' which has maximum posterior weight:

$$\hat{c} = \arg\min_{c' \in \mathcal{A}_n} L_{0-1}(c') = \arg\max_{c' \in \mathcal{A}_n} p(c'|x) := MAP.$$

Negative results by **rajkowski2019analysis** show that the mode a posteriori (MAP) is inconsistent.

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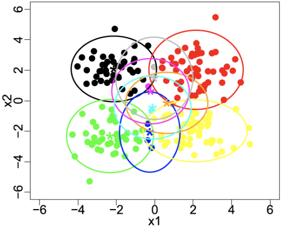
Variation of information (VI) by **meila2007comparing** for cluster comparison. From information theory, compares information in two clusterings with information shared between the two clusterings:

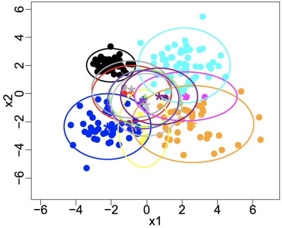
$$\mathrm{VI}(c, \widehat{c}) = \mathrm{H}(c) + \mathrm{H}(\widehat{c}) - 2\mathcal{I}(c, \widehat{c})$$

Variation of information

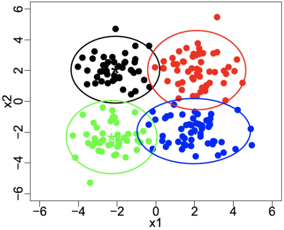
wade2018bayesian compare Binder and VI:



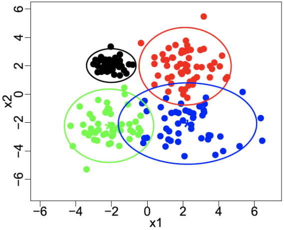
(a) Ex 1 Binder's: 9 clusters



(b) Ex 2 Binder's: 12 clusters



(c) Ex 1 VI: 4 clusters



(d) Ex 2 VI: 4 clusters

wade2018bayesian also provide credible balls around the estimated clustering, based on Hasse diagram:

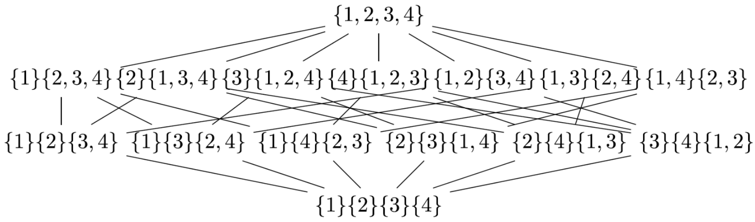


Figure 1: Hasse diagram for the lattice of partitions with a sample of size $N = 4$. A line is drawn from $\mathbf{c}$ up to $\widehat{\mathbf{c}}$ when $\mathbf{c}$ is covered by $\widehat{\mathbf{c}}$.

Outline

Need for a power-law for K_n

newman2005power; **clauset2009power** show that "*Power-law distributions occur in many situations of scientific interest and have significant consequences for our understanding of natural and man-made phenomena*".

Hence the need to depart from $K_n \sim \alpha \log n$ induced by a Dirichlet process.

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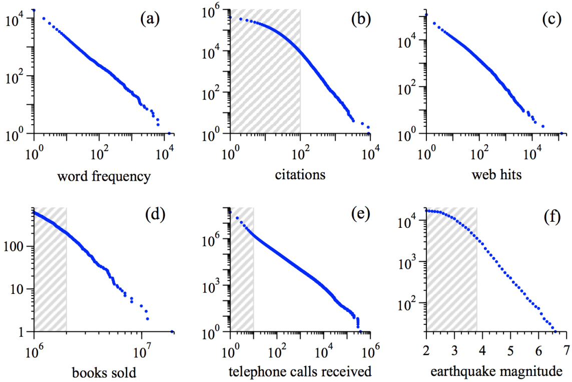

[Image from **newman2005power**]

Hence the need to depart from $K_n \sim \alpha \log n$ induced by a Dirichlet process.

Chinese restaurant process

Consider discrete data $X_1, \ldots X_n | P \overset{\text{iid}}{\sim} P$, and $P \sim \mathcal{Q}$

Features $k_n \leq n$ unique values $X_1^*, \ldots, X_{k_n}^*$ with resp. frequencies $n_1, \ldots, n_{k_n}$

Discrete random probability measures are characterized by predictive distr.

Dirichlet process by **ferguson1973bayesian**: $P \sim DP(\alpha, G_0)$

$$\mathbb{P}[X_{n+1} \in \cdot \,|\, X_1, \ldots X_n] = \frac{\alpha}{\alpha + n} G_0(.) + \frac{1}{\alpha + n} \sum_{j=1}^{k_n} n_j \delta_{X_j^*}(.)$$

Log rate for number of clusters $k_n \asymp \alpha \log n$

Product form exchangeable partition probability function

$$p(n_1, \ldots, n_{k_n}) = \alpha^{k_n} \frac{\Gamma(\alpha)}{\Gamma(\alpha + k_n)} \prod_{j=1}^{k_n} (n_j - 1)!$$

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Pitman–Yor process by **pitman1997two**: $P \sim PY(\sigma, \alpha, G_0)$, $\sigma \in (0, 1)$

$$\mathbb{P}[X_{n+1} \in \cdot \,|\, X_1, \dots X_n] = \frac{\alpha + \sigma k_n}{\alpha + n} G_0(.) + \frac{1}{\alpha + n} \sum_{j=1}^{k_n} (n_j - \sigma) \delta_{X_j^*}(.)$$

Power law rate for number of clusters $k_n \asymp S n^{\sigma}$

Product form exchangeable partition probability function

$$p(n_1, \dots, n_{k_n}) = \frac{\prod_{i=1}^{k_n - 1} (\alpha + i\sigma)}{(\alpha + 1)_{(n-1)}} \prod_{j=1}^{k_n} (1 - \sigma)_{(n_j - 1)}$$

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Discrete random probability measures are characterized by predictive distr.

Gibbs-type processes by **pitman2003poisson**:
$P \sim Gibbs(\sigma, (V_{n,k})_{n,k}, G_0)$, $\sigma < 1$

$$\mathbb{P}[X_{n+1} \in \cdot \,|\, X_1, \dots X_n] = \frac{V_{n+1,k_n+1}}{V_{n,k_n}} G_0(.) + \frac{V_{n+1,k_n}}{V_{n,k_n}} \sum_{j=1}^{k_n} (n_j - \sigma) \delta_{X_j^*}(.)$$

$$\text{Rate for number of clusters } k_n \asymp \begin{cases} K \text{ random variable a.s. finite if } \sigma < 0 \\ \alpha \log n \text{ if } \sigma = 0 \\ S n^{\sigma} \text{ if } \sigma \in (0,1), (S \text{ random variable}). \end{cases}$$

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Beyond the DP from predictive function viewpoint

A discrete random probability measure P can be classified in 3 main categories according to $\mathbb{P}[X_{n+1} \text{ is "new"} \mid \boldsymbol{X}_n]$

1) $\mathbb{P}[X_{n+1} \text{ is "new"} \mid \boldsymbol{X}_n] = f(n, \text{model parameters})$
$\iff$ depends on n but not on k_n and $(n_1, \dots, n_{k_n})$
$\iff$ Dirichlet process (**ferguson1973bayesian**);

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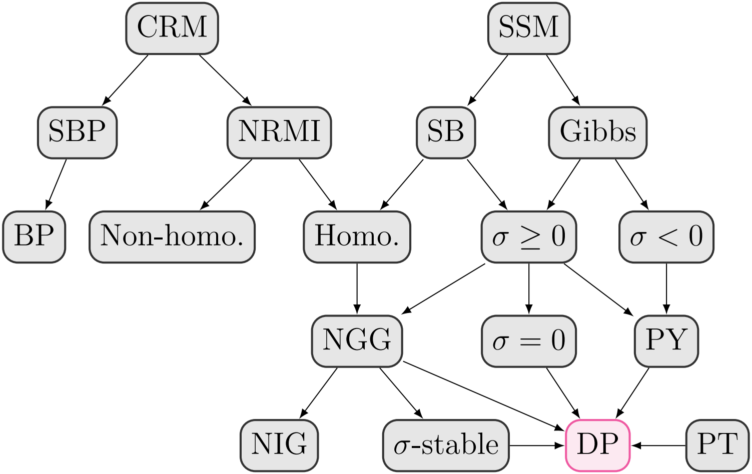
Tree of discrete random probability measures



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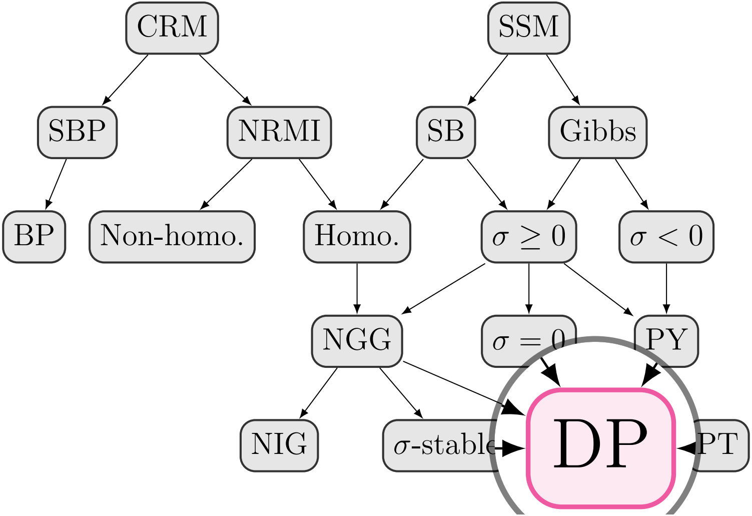


Proposition (Pitman Sampling formula)

The multiplicities $(m_1, \ldots, m_n)$ *in* $X_1, \ldots, X_n | P \overset{iid}{\sim} P, \; P \sim PY(\sigma, \alpha P_0)$ *have distribution*

$$p(m_1, \ldots, m_n) = \frac{n!}{(1+\alpha)_{(n-1)}} (\alpha+\sigma) \cdots (\alpha+(k-1)\sigma) \prod_{\ell=1}^{n} \frac{1}{m_\ell!} \left(\frac{(1-\sigma)_{(\ell-1)}}{\ell!} \right)^{m_\ell}$$

Proof. Same technique as for the DP ESF.

Pitman–Yor process II

Proposition (Power law and σ-diversity)

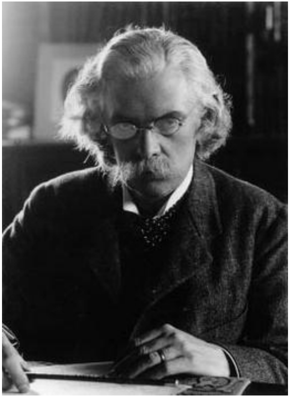
For $\sigma > 0$ we have the almost sure convergence

$$n^{-\sigma} K_n \to S_{\sigma,\alpha},$$

where $S_{\sigma,\alpha}$ is called σ-diversity of the PY, whose density is a polynomially tilted Mittag–Leffler density (ML):

$$g_{\sigma,\alpha}(x) \propto x^{\alpha/\sigma} g_{\alpha}(x),$$

and g_{α} is ML density.



[Image: Wikipedia]

Theorem (Stick breaking representation for PY)

If $V_j \overset{ind}{\sim} Be(1-\sigma, \alpha + j\sigma)$ *and* $p_1 = V_1,\ p_j = V_j \prod_{l<j}(1-V_l)$ *and further we have* $\phi_j \overset{iid}{\sim} P_0$ *then*

$$P = \sum_{j=1}^{\infty} p_j \delta_{\phi_j} \sim PY(\sigma, \alpha P_0).$$

Pitman–Yor process IV

Proposition (Moments of PY)

If $P \sim PY(\sigma, \alpha P_0)$, then for every measurable sets A, B we have

1) $\mathbb{E}[P(A)] = P_0(A)$,
2) $\mathbb{E}[P(A)P(B)] = (1-\sigma)/(1+\alpha)P_0(A \cap B) + (\alpha+\sigma)/(1+\alpha)P_0(A)P_0(B)$,
3) $\mathrm{Cov}[P(A), P(B)] = (1-\sigma)/(1+\alpha)\big(P_0(A \cap B) - P_0(A)P_0(B)\big)$.

Proof.

1) We use the stick-breaking representation:

$$\mathbb{E}P(A) = \sum_j \mathbb{E}p_j \mathbb{E}\delta_{\phi_j} = \sum_j \mathbb{E}(p_j) P_0(A) = P_0(A)\mathbb{E}(\sum_j p_j) = P_0(A).$$

2) Let $X_1, X_2 | P \overset{\text{iid}}{\sim} P$, then

$$\mathbb{E}(P(A)P(B)) = \mathbb{P}(X_1 \in A, X_2 \in B) = \mathbb{P}(X_1 \in A)\mathbb{P}(X_2 \in B | X_1 \in A).$$

Lets investigate two terms above: from 1) we know that $\mathbb{P}(X_1 \in A) = P_0(A)$. We know the predictive of PY:

$$X_2 | X_1 \sim \frac{\alpha + \sigma}{\alpha + 1} P_0 + \frac{1 - \sigma}{\alpha + 1} \delta_{X_1},$$

and hence

$$\mathbb{P}(X_2 \in B | X_1 \in A) = \frac{\alpha + \sigma}{\alpha + 1} P_0(B) + \frac{1 - \sigma}{\alpha + 1} P_{0A}(B),$$

when we used notation $P_{0A}(B) = P_0(B|A) = P_0(A \cap B)/P_0(A)$ for a conditional measure.

3) It is straightforward combination of 1) and 2).

Pitman–Yor process VI

Unlike the DP, PY is not conjugate under incoming independent samples. However, the posterior can be explicited.

Theorem (Posterior distribution of PY)

If $P \sim PY(\sigma, \alpha P_0)$ then the posterior of P based on observations $X_{1:n} | P \overset{iid}{\sim} P$ has the distribution of the random probability measure

$$(1 - q_n) P_n + q_n \sum_{j=1}^{K_n} p_j^* \delta_{X_j^*},$$

where $X_{1:n}^*$ are the K_n distinct values in $X_{1:n}$, frequencies are referred to as $n_1, \ldots, n_{K_n}$ and

- $q_n \sim Beta(n - K_n \sigma, \alpha + K_n \sigma)$,
- $(p_1^*, \ldots, p_{K_n}^*) \sim Dir_{K_n}(n_1 - \sigma, \ldots, n_{K_n} - \sigma)$,
- $P_n \sim PY(\sigma, (\alpha + \sigma K_n) P_0)$.

Impact of the stability parameter σ

Prior distribution of the number of clusters k_n

- α controls the location (as for the DP)
- σ controls the flatness (or variability)

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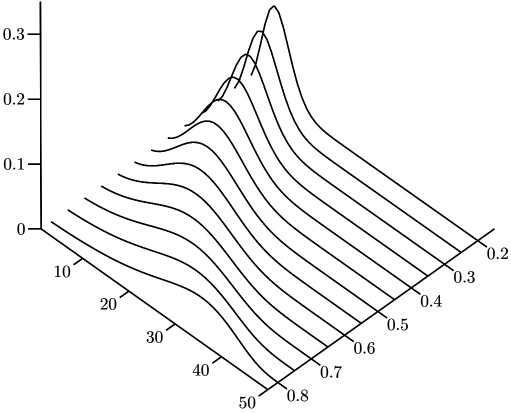
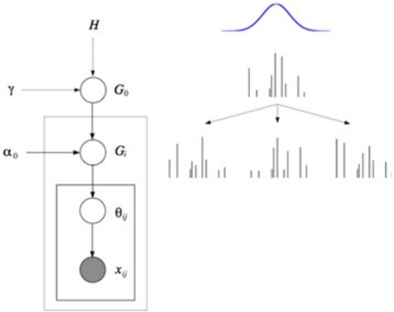
Example with $n = 50, \alpha = 1$ and $\sigma = 0.2, 0.3, \dots, 0.8$



[Image by **deblasi2015gibbs**]

Hierarchical Dirichlet process

A nonparametric version of **Latent dirichlet allocation** (**blei2003latent**) due to **teh2006hierarchical**



$$
\begin{aligned}
G_0|\gamma, H &\sim DP(\gamma H) \\
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[Image by M. Jordan]

Associated partition distr. called Chinese Restaurant Franchise

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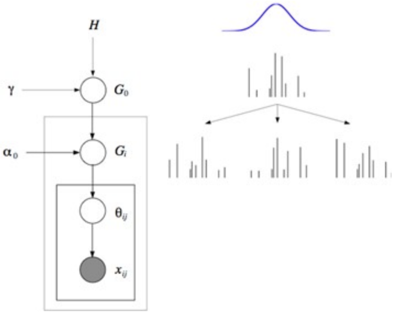

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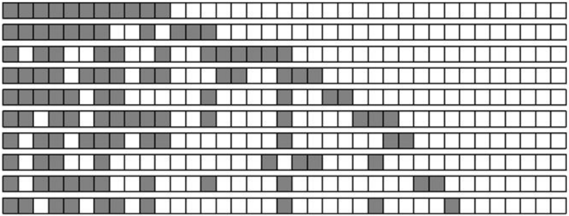
Indian Buffet process

Feature allocation model by **ghahramani2006infinite**, where observations may share several features. Generative model is as follows

- first customer samples Poisson(γ) dishes
- second customer chooses every dish of first customer *wp* 1/2, plus Poisson($\gamma/2$) new dishes
- ...
- ith step: K dishes have been sampled, each by $m_1, \ldots, m_K$ customers; ith customer chooses jth dish *wp* m_j/i, plus Poisson(γ/i) new dishes

Log growth: $K_n \sim \text{Poisson}(\gamma \log n)$.



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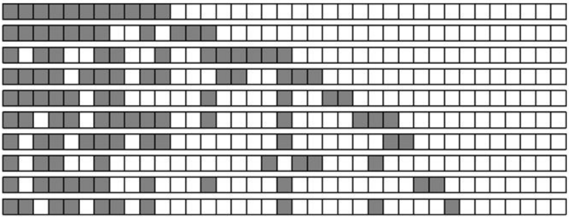

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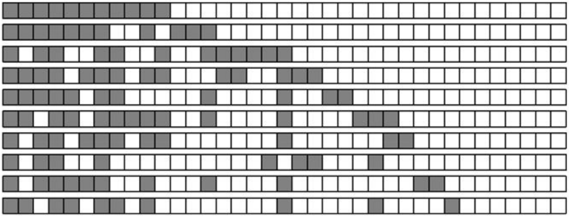

[Image by M. Jordan]

Indian Buffet process

Feature allocation model by **ghahramani2006infinite**, where observations may share several features. Generative model is as follows

- first customer samples Poisson(γ) dishes
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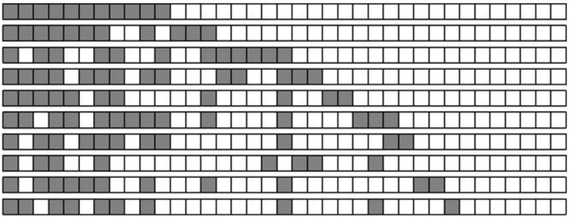

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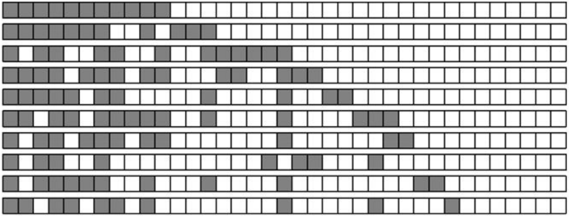

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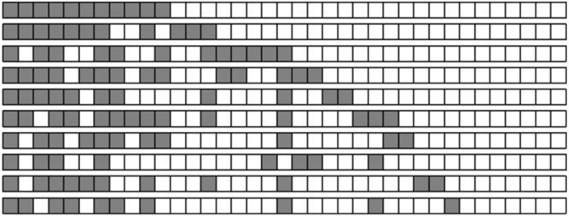

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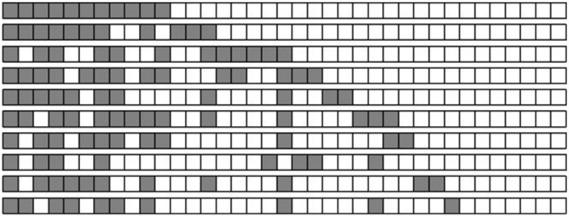

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Outline

What comes to *your* mind when you hear "Asymptotics"?

Why Asymptotics

- Construction of a prior on a nonparametric space is difficult
- We cannot hope to cover all the space of density (for example) with our prior (the prior does not have full support)
- We need to check that our inference is not completely off!

Parametric setting

We have the celebrated Bernstein-von Mises theorem that implies that the effect of the prior on the posterior inference vanishes when the amount of information grows.

This is not true anymore in a nonparametric setting!

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A first order approximation is to consider the asymptotic setting:

- Adopt a Frequentist point of view: "There exists a *true* parameter θ_0, and we study the posterior distribution with data generated w.r.t. θ_0."
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References

- **Ghosh2003**
- **hjort2010bayesian**
- **ghosal2017fundamentals**

Outline

Consistency

Setting:

- $\forall n \in \mathbb{N}$, let X^n be some observations in a sample space $\{\mathcal{X}^n, \mathcal{A}^n\}$ with distribution P_θ
- $\theta \in \Theta$ with (Θ, d) a (semi-)metric space

Let Π be a prior distribution on Θ and $\Pi(\cdot|X^n)$ a version of its posterior distribution.

Definition (Consistency)

The posterior distribution $\Pi(\cdot|X^n)$ is said to be weakly consistent at θ_0 if for all $\epsilon > 0$

$$\Pi\left(d(\theta, \theta_0) > \epsilon | X^n\right) \xrightarrow[n \to \infty]{P_{\theta_0}} 0.$$

If the convergence is almost sure, then the posterior is said to be strongly consistent.

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Naturally one will hope that posterior consistency implies that some summary of the posterior location would be a consistent estimator.

Theorem

Let $\Pi(\cdot|X^n)$ be a posterior distribution on Θ and suppose that it is consistent at θ_0 relative to a metric d on Θ. For $\alpha \in (0,1)$, define $\hat{\theta}_n$ as the centre of the smallest ball containing at least α of the posterior mass. Then

$$d(\hat{\theta}_n, \theta_0) \xrightarrow[n\to\infty]{P_{\theta_0},\ or\ P_{\theta_0}\ a.s.} 0.$$

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Extra notes I

Take $\alpha = 1/2$ for simplicity and consistency in probability. Define $B(\theta, r)$ the closed ball of radius r centred around θ, and let

$$\hat{r}(\theta) = \inf\{r, \Pi(B(\theta, r)|X^n) \geq 1/2\}$$

(and inf over the empty set is ∞). Now let $\hat{\theta}_n$ be such that

$$\hat{r}(\hat{\theta}_n) \leq \inf_{\theta\in\Theta} r(\theta) + 1/n$$

Consistency implies that $\Pi(B(\theta_0, \epsilon)|X^n) \to 1$ so $\hat{r}(\theta_0) \leq \epsilon$ with probability tending to 1. Furthermore, $\hat{r}(\hat{\theta}_n) \leq \hat{r}(\theta_0) + 1/n$ thus $\hat{r}(\hat{\theta}_n) \leq \epsilon + 1/n$ with probability tending to 1.

In addition, $B(\theta_0, \epsilon) \cap B(\hat{\theta}_n, \hat{r}(\hat{\theta}_n)) \neq \emptyset$ otherwise $\Pi(B(\theta_0, \epsilon) \cup B(\hat{\theta}_n, \hat{r}(\hat{\theta}_n))|X^n) = \Pi(B(\theta_0, \epsilon)|X^n) + \Pi(B(\hat{\theta}_n, \hat{r}(\hat{\theta}_n))|X^n) \to 1 + 1/2$. So we have

$$d(\theta_0, \hat{\theta}_n) \leq \hat{r}(\hat{\theta}_n) + \epsilon \leq 2\epsilon + 1/n$$

with probability that goes to 1.

Point estimator

- If Θ is a vector space, then one might want to use the posterior mean.
- But... weak convergence to a Dirac does not imply convergence of moments.
- Consistency of the posterior mean requires additional assumptions such as boundedness of posterior moments in probability or a.s. for some $p > 1$ would be sufficient.

Point estimator

Theorem (Posterior mean)

Assume that the balls of the metric space (Θ, d) are convex. Suppose that for any sequence $\theta_{1,n}, \theta_{2,n}$ in Θ and $\lambda_n \to 0$

$$d(\theta_{1,n}, (1 - \lambda_n)\theta_{1,n} + \lambda_n \theta_{2,n}) \to 0$$

Then consistency of the posterior distribution implies consistency of the posterior mean.

Extra notes I

Let $\epsilon > 0$ and write $\hat{\theta}_n = \int \theta \Pi(d\theta|X^n)$. We decompose

$$\hat{\theta}_n = \int_{B(\theta_0,\epsilon)} \theta \Pi(d\theta|X^n) + \int_{B(\theta_0,\epsilon)^c} \theta \Pi(d\theta|X^n) = \theta_{1,n}(1-\lambda_n) + \lambda_n \theta_{2,n}$$

where $\theta_{1,n} = \int_{B(\theta_0,\epsilon)} \theta \frac{\Pi(d\theta|X^n))}{\Pi(B(\theta_0,\epsilon)|X^n)}$, $\lambda_n = \Pi(B(\theta_0,\epsilon)|X^n)$ and similarly for $\theta_{2,n}$ on the complement of $B(\theta_0,\epsilon)$. Using Jensen inequality we have

$$d(\theta_{n,1},\theta_0) \leq \int_{B(\theta_0,\epsilon)} d(\theta,\theta_0) \frac{\Pi(d\theta|X^n))}{\Pi(B(\theta_0,\epsilon)|X^n)} \leq \epsilon$$

In addition we have

$$d(\hat{\theta}_n,\theta_0) \leq d(\theta_{n,1},\theta_0) + d(\theta_{n,1},\theta_{1,n}(1-\lambda_n) + \lambda_n \theta_{2,n}).$$

Using the fact that $\lambda_n \to 0$ since the posterior is consistent, we have the desired result.

Remark

For the condition on d to hold, one can assume it to be convex and uniformly bounded.

A first consistent posterior

Example (Dirichlet process)

Assume the following model

$$\begin{aligned} X_1, \ldots, X_n &\overset{iid}{\sim} P, \\ P &\sim \mathrm{DP}(M\alpha) \end{aligned}$$

Consider the semi-metric $d_A(P, Q) = |P(A) - Q(A)|$ for some measurable event A on Θ, then $\Pi(\cdot|X^n)$ is strongly consistent at any P_0 for d_A.

From this result, we can easily obtain consistency under the weak topology. We could also obtain stronger consistency using Glivenko–Cantelli theorem.

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Extra notes I

Consider $\Pi(|P(A) - P_0(A)| \geq \epsilon | X^n)$ which calls for applying Markov inequality. Properties of the Dirichlet process imply that

$$P|X^n \sim \mathrm{DP}(M\alpha + n\mathbb{P}_n),$$

thus

$$P(A)|X^n \sim \mathrm{Beta}(M\alpha(A) + n\mathbb{P}_n(A), M\alpha(A^c) + n\mathbb{P}_n(A^c)).$$

We thus have

$$\mathbb{E}(P(A)|X^n) = \frac{M}{M+n}\alpha(A) + \frac{n}{M+n}\mathbb{P}_n(A) := \bar{P}(A)$$
$$\mathrm{var}(P(A)|X^n) = \frac{\bar{P}(A)\bar{P}(A^c)}{1+n+M} \leq \frac{1}{4(1+n+M)}.$$

Markov inequality gives

$$\Pi(|P(A) - P_0(A)| \geq \epsilon | X^n) \leq \frac{1}{\epsilon^2}\left(|\bar{P}(A) - P_0(A)|^2 + \mathrm{var}(P(A)|X^n)\right)$$
$$\to 0 \; [P_0, a.s.]$$

using the law of large numbers on $\mathbb{P}(A)$.

Bayesian modelling perspective

From a Bayesian point of view, a Dirac measure at θ_0 corresponds to perfect knowledge of the parameter.

- Prior and posterior distributions model our knowledge about the parameter.
- Consistency thus implies that when the amount of information grows, we tend towards perfect knowledge of the parameter.

A validation of Bayesian methods

The frequentist setting where there exists a *true* parameter θ_0 that generates the data can be seen as an idealized set-up.

- An experimenter feeds a Bayesian with some data using the same data-generating mechanism.
- When the number of observation grows, a Bayesian should be able to pin-point the data-generating mechanism, whatever their prior.
- A prior that does not lead to a consistent posterior should not be used.

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Robustness

Two Bayesians walk into a bar... with *almost* the same prior, then their posterior inference should not differ that much.

- Let Π_1 be the prior of Bayesian number 1
- Bayesian number 2 uses an "ϵ-corrupted" prior $\Pi_2 = (1-\epsilon)\Pi_1 + \epsilon\delta_{p_0}$ for some $p_0 \in \Theta$

The posterior of Bayesian number 2 is consistent at p_0 (to be seen later), now what if Π_1 is not consistent at p_0? Let d_W be the metric for the weak topology, then $d_W(\Pi_1(\cdot|X^n), \Pi_2(\cdot|X^n))$ would not go to 0.

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Extra notes I

There exists some $\varepsilon_0 > 0$ such that

$$\Pi_{n,1}(B(\theta_0, \varepsilon_0)|X^n) \not\to 0$$

Thus

$$|\Pi_{n,1}(B(\theta_0, \varepsilon_0)|X^n) - \Pi_{n,2}(B(\theta_0, \varepsilon_0)|X^n)| \not\to 0$$

since $\Pi_{n,2}(B(\theta_0, \varepsilon_0)|X^n) \to 0$.

Doob's Theorem

Can one get general conditions on the prior to ensure that it is consistent?

- A first answer: Doob's Theorem
- The posterior is consistent at every θ Π-a.s.

Consider the case of *i.i.d.* observations

Theorem (Doob's Theorem)

Let $\{\mathcal{X}^n, P_\theta, \Theta\}$ be a statistical model where $\{\mathcal{X}^n, \mathcal{A}^n\}$ is a Polish space with Borel σ-field and Θ a Borel subset of a Polish space. Suppose that the map $\theta \mapsto P_\theta(A)$ is Borel measurable for evey $A \in \mathcal{A}$ and $\theta \mapsto P_\theta$ is one-to-one. Then for any prior distribution Π on Θ, if $X_1, \ldots, X_n \overset{iid}{\sim} P_\theta$, $\theta \sim \Pi$, the posterior is strongly consistent at any θ Π-a.s.

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Some remarks on Doob's Theorem

- The conditions of the theorem are extremely weak
- And no conditions on the prior
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- Note: the Π-null set can be quite big! we can be happy with this result only if we are confident that the parameters are on the support of the prior. In general no one can be sure that the parameter generating the data inside the support of the prior, this is a real problem in fact in general the support of the prior can be quite thin.
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Setting

Doob's approach is not enough to show consistency of the posterior. For simplicity we focus on the density estimation setting.

- Θ is the set of probability density functions on $\mathcal{X}$ w.r.t. a common dominating measure ν. We denote the parameter p (instead of θ) and P the associated probability measure.
- Observations follow $X_1, \ldots, X_n \overset{iid}{\sim} p$, and $p \sim \Pi$.

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Considering density estimation makes things easier without being too simplistic. The same results can be extended to nonparametric regression.

To achieve consistency, we do not want to require that the true parameter p_0 is inside the support of Π. However we still require some prior mass *near* p_0.

Definition (Kullback–Leibler)

Let p and p_0 be two p.d.f. with respect to a common measure such that $p_0 \ll p$. Then the Kullback–Leibler divergence between p and p_0 is

$$\mathrm{KL}(p, p_0) = \int p_0 \log(p_0/p) d\nu.$$

KL property

Definition (KL property)

We say that a prior distribution Π satisfies the Kullback–Leibler property at p_0 if for every $\epsilon > 0$,

$$\Pi(p : \mathrm{KL}(p, p_0) \geq \epsilon) > 0$$

We note $p_0 \in \mathrm{KL}(\Pi)$ and alternatively will say that p_0 is in the KL-support of Π.

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This extends quite a lot the parameters at which the posterior can be consistent.

The other requirement is that the parameter set is not too complex.

Definition (Exponentially consistent tests)

We say that a sequence of tests ϕ_n for $H_0 : p = p_0$ versus $H_1 : p \in U^c$ is exponentially consistent if

$$P_0^n(\phi_n) \lesssim e^{-Cn}, \quad \sup_{p \in U^c} P^n(1 - \phi_n) \lesssim e^{-Cn}$$

A test is understood as a measurable map $\mathcal{X}^n \to [0, 1]$ and the corresponding statistic $\phi_n(X_1, \ldots, X_n)$. ϕ_n is interpreted as the probability that the null is rejected.

Extra notes I

The existence of tests means that we can differentiate between p_0 and parameter in U^c.

It is enough to have uniformly consistent sequence of test

$$P_0(\phi_n) \to 0, \ \sup_{p \in U^c} P(1-\phi_n) \to 0.$$

Since the test is uniformly consistent then there exists $k \in \mathbb{N}$ such that $P_0^k(\phi_k) \leq 1/4$, $P^k(1-\phi_k) \leq 1/4$. Now for n large, write $n = mk + r$. Slice $X^n = (X_1, \ldots, X_n)$ into m sub-sample of size k $X_l^n = (X_{(l-1)k+1}, \ldots, X_{lk})$ and define $Y_{l,n} = \phi_k(X_l^n)$. Now create a new test $\psi_n = \mathcal{I}\{\bar{Y}_m > 1/2\}$. We have for every $p \in U^c$, $P(1-Y_j) \leq 1/4$

$$P(\psi_n) = P(\bar{Y} \leq 1/2) = P(1-\bar{Y} \geq 1/2) =$$
$$P(1-\bar(Y) \geq 1/2) \leq e^{-2m/16} \lesssim e^{-Cn}$$

Using Hoeffding inequality: $\mathbb{P}(\bar{X} - \mathbb{E}(X) \geq \epsilon) \leq \exp\left\{-2\epsilon^2 m\right\}$.

Schwartz Theorem

Theorem

Let Π be a prior distribution on Θ such that $p_0 \in KL(\Pi)$. Let U be a neighbourhood of p_0 such that there exists an exponentially consistent sequence of tests for p_0 against U^c. Then

$$\Pi(U^c | X^n) \to 0 \; [P_0 a.s].$$

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Extra notes I

$$\Pi(U^c|X^n) = \frac{\int_{U^c} \prod_{i=1}^n \frac{p}{p_0}(X_i) d\Pi(p)}{\int_{\mathcal{P}} \prod_{i=1}^n \frac{p}{p_0}(X_i) d\Pi(p)} := \frac{N_n}{D_n}.$$

We first show $\liminf D_n e^{n\epsilon}/\Pi(KL(p,p_0) > \epsilon) \geq 1$, $P_0[a.s.]$. Let $\Pi_0(\cdot) = \Pi(\cdot \cap \mathrm{KL}(p,p_0) > \epsilon)/\Pi(KL(p,p_0) > \epsilon)$. Then

$$\begin{aligned}
\log(D_n) &\geq \log\left(\int_{KL(p,p_0)>\epsilon} \frac{p}{p_0}(X_i) d\Pi_0(p)\right) + \log(\Pi(KL(p,p_0) < \epsilon)) \\
&\geq \int_{KL(p,p_0)>\epsilon} \log\left(\prod_{i=1}^n \frac{p}{p_0}(X_i)\right) d\Pi_0(p) + \log(\Pi(KL(p,p_0) < \epsilon)) \\
&= \sum_{i=1}^n \int \log \frac{p}{p_0}(X_i) d\Pi_0(p) + \log(\Pi(KL(p,p_0) < \epsilon))
\end{aligned}$$

The law of large numbers implies

$$\frac{1}{n}\sum_{i=1}^n \int \log \frac{p}{p_0}(X_i) d\Pi_0(p) \to P_0 \int \frac{p}{p_0}(X_i) d\Pi_0(p), \ \ P_0[a.s.]$$

Extra notes II

which is $-\int \mathrm{KL}(p, p_0) d\Pi_0(p) > -\epsilon$. Thus

$$\liminf D_n e^{n\epsilon} / \Pi(KL(p, p_0) > \epsilon) \geq 1, \;\; P_0[a.s.]$$

For n large enough we have the following $P_0[a.s.]$

$$\begin{aligned} \Pi(U^c | X^n) &\leq \phi_n + (1 - \phi_n) \frac{N_n}{D_n} \\ &\leq \phi_n + (1 - \phi_n) N_n e^{\epsilon n} \Pi(KL(p, p_0) > \epsilon) \end{aligned}$$

Furthermore we have that

$$\begin{aligned} P_0^n N_n (1 - \phi_n) &= P_0^n \int_{U^c} (1 - \phi_n) \prod_{i=1}^{n} \frac{p}{p_0}(X_i) \Pi(dp) \\ &= \int_{U^c} P^n (1 - \phi_n) \Pi(dp) \leq e^{-Cn} \end{aligned}$$

We thus get $P_0 \Pi(U^c | X^n) \leq e^{-C'n}$ for $\epsilon < C$ and for $C' = C - \epsilon$. Using Borel–Cantelli we get that $\Pi(U^c | X^n) \to 0 P_0[a.s.]$.

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- Need to test away all densities in U^c
- Might not be possible for strong neighbourhood of p_0 (L_1 metrics)

Extension of Schwartz theorem

The idea is that not *all* functions in U^c matters and we can discard function with very low prior probabilities.

Theorem

The results of the previous theorem are still valid if we replace the assumption on the existence of tests by:

$$\Theta_n \subset \Theta$$
$$\Pi(\Theta_n) \leq e^{-Cn}, \; P_0^n \phi_n \leq e^{-Cn}, \quad \sup_{p \in U^c \cap \Theta_n} P(1 - \phi_n) \leq e^{-Cn}$$

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Existence of tests

Schwartz' theorem requires the existence of exponentially consistent tests.

- We can differentiate between θ_0 and U^c
- The model is not too complex

Question

When do such tests exist?

Let's see the example of iid observations.

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Sketch of the proof

- Cannot directly construct test against $U^c = \{p, d(p, p_0) > \epsilon\}$...
- Construct an exponentially consistent test against a generic ball that is at least at distance ϵ
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Consistency under Entropy bound

We combine the preceding results to get general conditions ¡2-¿on the prior and ¡2-¿on the model, that ensure consistency.

Theorem

The posterior is strongly consistent relative to the L_1 distance at every p_0 in the KL-support of the prior if for every $\epsilon > 0$ there exist Θ_n such that for $C > 0$ and $0 < c < 1/2$

$$\Pi(\Theta_n^c) \leq e^{-Cn}, \ \log N(\epsilon, \Theta_n, \|\cdot\|_1) \leq cn\epsilon_n^2,$$

for n large enough.

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Outline

Definition

Contraction rates are a refinement of posterior consistency.

- How fast posterior concentrates its mass around the true parameter
- Helps to see how much the prior influences the posterior

Definition

Let ϵ_n be a positive sequence. The posterior contracts at the rate ϵ_n at θ_0 if for any $M_n \to \infty$

$$\Pi(d(\theta, \theta_0) > M_n \epsilon_n | X^n) \xrightarrow[n \to \infty]{P_{\theta_0}} 0$$

If all the experiments share the same probability space and the convergence is $P_{\theta_0}[a.s]$ we say that the posterior contracts in the strong sense.

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Remarks

- Any slower rate than ϵ_n also fits the definition so we will say *a* posterior contraction rate
- We will naturally try to find the fastest possible rate!

Regarding M_n

- The sequence M_n plays virtually no role in the posterior rate. In many cases it can be fixed to a constant M.
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Consequences of posterior contraction

Point Estimator

- Let $\hat{\theta}_n$ = centre of the smallest ball that contains at least $1/2$ of the posterior mass.
- Assume that the posterior contracts at θ_0 with rate ϵ_n for the metric d

Then $d(\hat{\theta}_n, \theta) = O_P(\epsilon_n)$ in P_0 probability (or a.s. if strong contraction).

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If the metric d is bounded and $\theta \mapsto d^s(\theta, \theta_0)$ is convex for some $s \geq 1$ then the posterior mean $\tilde{\theta}_n$ satisfies

$$d(\tilde{\theta}_n, \theta_0) \leq M_n \epsilon_n + \|d\|_\infty^{1/s} \Pi_n(d(\theta, \theta_0) \geq M_n \epsilon_n | X^n)^{1/s}.$$

- First term is the dominating term
- The second term is exponentially small in general

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Some first Examples - Parametric models

- Let $X_1, \ldots, X_n|\theta \overset{iid}{\sim} \mathcal{B}(\theta)$, and $\theta \sim \text{Beta}(\alpha, \beta)$. The posterior contracts at a rate $n^{-1/2}$
- Let $X_1, \ldots, X_n|\theta \overset{iid}{\sim} \mathcal{U}([0, \theta])$ and $\pi(\theta) \propto \theta^{-a}$. The posterior contracts at a rate n^{-1}.

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In fact for all regular finite dimensional models the Bernstein von-Mises theorem implies a posterior rate of $n^{-1/2}$.

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Nonparametric example: Dirichlet Process

- $X_1, \dots, X_n | P \overset{iid}{\sim} P$
- $P \sim \mathrm{DP}(M\alpha)$ for α a probability measure on $\mathcal{X}$.

The posterior distribution is $P|X^n \sim \mathrm{DP}(M\alpha + n\mathbb{P}_n)$.

Local semi-metric[1]

For a measurable set A, let $d(P, Q) = |P(A) - Q(A)|$. The posterior distribution is consistent at P_0 at a rate $n^{-1/2}$.

Global metric

For ν a σ-finite measure and F and G two c.d.f. let $d(F, G) = \|F - G\|_\nu^2 = \int (F(t) - G(t))^2 d\nu(t)$. The posterior contracts at rate $n^{-1/2}$ at P_0 for this metric.

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Nonparametric example: White Noise

Consider the following model for W_t a white noise

$$X_t = f(t) + n^{-1/2} W_t.$$

Projecting this model onto the Fourier basis if $f \in L_2$, we have the equivalent formulation

$$X_{i,n} = \theta_i + n^{-1/2} \epsilon_i, \; i \in \mathbb{N}^*$$

$\theta \in \ell_2(\mathbb{L})$. Assume the following prior

$$\theta_i \overset{ind.}{\sim} \mathcal{N}(0, i^{-2\alpha-1}).$$

If $\theta_0 \in \mathcal{S}_\beta^{2,2}$ then the posterior contracts at θ_0 at the rate $n^{-\min(\alpha,\beta)/(2\alpha+1)}$.

General theorem

- Result similar to Schwartz theorem?
- We focus on the case of i.i.d observations $X_1, \ldots, X_n \overset{iid}{\sim} P$
- The parameter set Θ is the set of probability densities with respect to a common dominating measure μ.

Let Π_n be a sequence of priors. We study the sequence of posterior distributions $\Pi_n(\cdot|X^n)$ under the assumption that the data are generated from P.

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General Theorem

We follow the same steps as for Schwartz' Theorem:

- Existence of tests to separate p_0 from the complement of balls
- KL condition: the prior puts enough mass on neighbourhood of p_0

Define $V_{2,0}$, the 2nd KL variation

$$V_2 = P_0 \left(\log^2 \left(\frac{p_0}{p}(X) \right) \right),$$

and define two KL neighbourhoods as

$$B_0(p_0, \epsilon) = \{p, \mathrm{KL}(p_0, p) \leq \epsilon^2\},$$
$$B_2(p_0, \epsilon) = \{p, \mathrm{KL}(p_0, p) \leq \epsilon^2, V_2(p_0, p) \leq \epsilon^2\}.$$

Theorem (Ghosal, Ghosh and van der Vaart)

Let $d \leq h$ be a metric on Θ for which balls are convex, and let $\Theta_n \subset \Theta$. The posterior contracts at a rate ϵ_n for all ϵ_n such that $n\epsilon_n^2 \to \infty$ and such that for positive constants c_1, c_2 and any $\underline{\epsilon}_n \leq \epsilon_n$

$$\log N(\epsilon_n, \Theta_n, d) \leq c_1 n\epsilon_n^2,$$

$$\Pi_n(B_{2,0}(p_0, \underline{\epsilon}_n^2)) \geq e^{-c_2 n\underline{\epsilon}_n^2}$$

$$\Pi(\Theta_n^c) \leq e^{-(c_2+3)n\underline{\epsilon}_n^2}$$

General Theorem

- The KL condition can be refined, but the idea is basically the same
- Entropy condition is useful for the existence of tests
- Entropy condition can be replaced by a local entropy, which is more like a *dimension of* Θ_n

Interpretation

Assume that d and KL are equivalent

- We need $e^{n\epsilon_n^2}$ balls to cover Θ_n.
- If the prior spread evenly the mass on these balls, we have $e^{-Cn\epsilon_n^2}$ mass on each of these balls thus KL condition is satisfied
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- If the prior spread evenly the mass on these balls, we have $e^{-Cn\epsilon_n^2}$ mass on each of these balls thus KL condition is satisfied
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General Theorem

- The KL condition can be refined, but the idea is basically the same
- Entropy condition is useful for the existence of tests
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General observations

- The previous theorem can be generalized to other models (like regression for instance)
- But we have to be careful with the metric we use, and the existence of test is not guaranteed!
- To be general we will have to assume that we can test away parameters

Existence of tests

Let d_n and e_n be two semi-metrics on Θ. For $\epsilon > 0$, and for all $\theta_1 \in \Theta$ such that ¡6-¿ $d_n(\theta_0, \theta_1) > \epsilon$ there exists ϕ_n

$$P^n_{\Theta_0}\phi_n \leq e^{-Kn\epsilon^2}, \qquad \sup_{\theta, <6-> e_n(\theta,\theta_1)\leq \xi\epsilon} P^n_\theta(1-\phi_n) \leq e^{-Kn\epsilon^2}$$

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General theorem

Define the following KL-neighbourhood

$$V_{k,0}(f,g) = \int f|\log(f/g) - \mathrm{KL}(f,g)|^k d\mu$$

$$B_n(\theta_0,\epsilon,k) = \left\{\theta \in \Theta \mathrm{KL}(p^n_{\theta_0}, p^n_\theta) \leq n\epsilon^2, V_{k,0}(p^n_{\theta_0}, p^n_\theta) \leq n^{k/2}\epsilon^k\right\}$$

General theorem

Theorem

Let d_n and e_n be two semi-metrics on Θ, such that tests exists, $\epsilon_n \to 0$, $n\epsilon_n^2 \to \infty$, $k > 1$, $\Theta_n \subset \Theta$ such that for sufficiently large $j \in \mathbb{N}$

$$\sup_{\epsilon \geq \epsilon_n} \log N\left(\frac{1}{2}\xi\epsilon, \{\theta \in \Theta_n d_n(\theta_0, \theta) \leq \epsilon\}, e_n\right) \leq n\epsilon_n^2$$

$$\frac{\Pi_n(\theta \in \Theta_n, j\epsilon_n \leq d_n(\theta, \theta_0) \leq 2j\epsilon_n)}{\Pi_n(B_n(\theta_0, \epsilon_n, k))} \leq e^{Kn\epsilon_n^2 j^2/2}$$

$$\frac{\Pi_n(\Theta_n^c)}{\Pi_n(B_n(\theta_0, \epsilon_n, k))} \leq e^{-2n\epsilon_n}$$

then $P_{\theta_0}^n \Pi_n(d_n(\theta_0, \theta) \geq M_n\epsilon_n) = o(1)$

Independent observations

- Assume that the measure $P_\theta^n = \bigotimes_{i=1}^n P_{i,\theta}$ on some product space $\bigotimes_{i=1}^n \{\mathcal{X}_i, \mathcal{A}_i\}$.
- Assume that each measures $P_{i,\theta}$ are absolutely continuous w.r.t μ_i
- Define the Root average Hellinger distance

$$d_{n,H}(\theta,\theta') = \left(\frac{1}{n}\sum_{i=1}^n \int (\sqrt{dP_{i,\theta}} - \sqrt{dP_{i,\theta'}})^2\right)^{1/2}$$

Lemma

For all here exists tests ϕ_n such that

$$P_{\theta_0}^n \phi_n \le e^{-nd_{n,H}(\theta_0,\theta_1)}, \;\; P_\theta^n \le e^{-nd_{n,H}(\theta_0,\theta_1)}$$

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We can also simplify the KL condition in this case. Note that

$$KL(p^n_{\theta_0}), p^n_\theta) = \sum_{i=1}^n \mathrm{KL}(p_{i,\theta_0}, p_{i,\theta})$$

Furthermore for the KL-variation term we have that

$$V_{k,0}(p^n_{\theta_0}, p^n_\theta) \le n^{k/2} C_k \frac{1}{n} \sum_{i=1}^n V_{k,0}(p_{i,\theta_0}, p_{i,\theta})$$

Thus the KL condition can be re-written

$$B^*_n(\theta_0, \epsilon, k) = \left\{\theta, \frac{1}{n}\sum_{i=1}^n \mathrm{KL}(p_{i,\theta_0}, p_{i,\theta}) \le \epsilon^2, \frac{1}{n}\sum_{i=1}^n V_{k,0}(p_{i,\theta_0}, p_{i,\theta}) \le C_k \epsilon^2\right\}$$

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NP Regression with splines

Consider the model

$$X_i = f(z_i) + \epsilon_i, \ \ i = 1, \ldots, n$$

where $\epsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2)$ and the $z_i \in \mathbb{L}$ are known fixed covariates. For simplicity σ^2 is also assume to be known. Let $\mathbb{P}_n^z = \frac{1}{n}\sum_{i=1}^n \delta_{z_i}$ and $\|\cdot\|_n$ the $L_2(\mathbb{P}_n^z)$ norm

Lemma

We have the following results

$$KL(P_{f,i}, P_{g,i}) = \frac{1}{2\sigma^2}(f(z_i) - g(z_i))^2$$

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Assume that $f_0 \in \mathcal{H}(\alpha, L)$ such that $\|f_0\|_\infty \leq H$, then the $d^2_{n,H}$ and $\|\cdot\|^2_n$ are equivalent.

Spline prior

Consider $(B_j)_{j=1}^J$ the B-splines basis with J equally spaced nodes, and consider

$$f_\beta(\cdot) = \sum_{j=1}^J \beta_j B_j(\cdot)$$

and induce a prior on f by choosing a prior on β, $\beta_j \overset{iid}{\sim} g$.

Approximation techniques with splines gives us that for $\beta^* \in \mathbb{L}^J$ the coefficient of the projection of f_0 in $\mathrm{Span}(B_j)$,

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We also need to impose conditions on the design. Let Σ_n be such that $\Sigma_{n,i,j} = \int B_i B_j d\mathbb{P}_n^z$. We assume that

$$J^{-1}\|\beta\|^2 \asymp \beta' \Sigma_n \beta$$

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Theorem

Assume that g is a standard Gaussian distribution, and assume that $J = J_n \asymp n^{1/(2\alpha+1)}$, then the posterior contracts at a rate $\epsilon_n = n^{-\alpha/(2\alpha+1)}$.

- This is the minimax rate, in addition this rate is uniform over all bounded $\mathcal{H}(\alpha, L)$ functions.
- Some condition can be relaxed, in particular, g could be any distribution such that for every β^* such that $\|\beta^*\|_\infty \leq C$ $\Pi(\|\beta - \beta^*\| \leq \epsilon) \geq e^{-cJ\log(1/\epsilon)}$. Some log factor may appear in the rate.
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Acknowledgements

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References I