Bayesian ML
Lecture # 4: Why would you want to be Bayesian?

Rémi Bardenet

CNRS & CRIStAL, Univ. Lille, France
`http://rbardenet.github.io`

The subjective expected utility principle

1. Choose $\mathcal{S}, \mathcal{Z}, \mathcal{A}$ and a loss function $L(a, s)$,
2. Choose a distribution p over $\mathcal{S}$,
3. Take the the corresponding Bayes action

$$a^\star \in \arg\min_{a \in \mathcal{A}} \mathbb{E}_{s \sim p} L(a, s). \tag{1}$$

Corollary: minimize the posterior expected loss

If we partition $s = (s_o, s_u)$, then

$$a^\star = a_{g^\star} \in \arg\min_{a \in \mathcal{A}} \mathbb{E}_{s_o} \mathbb{E}_{s_u | s_o} L(a, s).$$

Equivalently to (1), given s_o, we choose

$$g^\star(s_o) = \arg\min_{a \in \mathcal{A}} \mathbb{E}_{s_u | s_o} L(a, s).$$

The "formal" LP of Berger and Wolpert, 1988

Consider two statistical experiments

$$E_i = (X_i, \theta, \{p_i(\cdot|\vartheta), \vartheta \in \Theta\}), \quad i = 1, 2.$$

Assume that for some realizations x_1 and x_2,

$$p_1(x_1|\cdot) \propto p_2(x_2|\cdot).$$

If $\text{Ev}(E, x)$ denotes the "evidence on θ arising from E and x", then

$$\text{Ev}(E_1, x_1) = \text{Ev}(E_2, x_2).$$

Corollary

$\text{Ev}(E, x)$ can depend on x solely through $p(x|\cdot)$.

Standard Bayes satisfies the LP

▶ Take $p_i(s_i) = p_i(x_i, \theta) = p_i(x_i|\theta)p(\theta) = Z p_i(\theta|x_i)$, $i = 1, 2$.

▶ Then for $a : \mathcal{S} \to \mathcal{Z}$,

$$\int L(a, s_1) \frac{p_1(x_1|\theta)p(\theta)}{Z} \mathrm{d}\theta \propto \int L(a, s_2) \frac{p_2(x_2|\theta)p(\theta)}{Z} \mathrm{d}\theta,$$

so that Bayes actions coincide!

▶ However, full expected utilities are different in general:

$$\int L(a, s_1) p_1(x_1|\theta) p(\theta) \mathrm{d}x_1 \mathrm{d}\theta \neq \int L(a, s_2) p_2(x_2|\theta) p(\theta) \mathrm{d}x_2 \mathrm{d}\theta.$$

Pros and cons of the LP

- The LP is compelling to many (Berger and Wolpert, 1988), but it has its downsides.
- Being Bayesian is only one way to abide by the LP.
- I am personally uncomfortable with the stopping rule principle, probably because my frequentist intuition is still too strong.
- It is hard to make fully formal: is $\text{Ev}(E, x)$ even meaningful? See answer by LeCam to (Berger and Wolpert, 1988).
- It assumes we want to specify a likelihood, this prevents model-free Bayesianism.
- It separates the roles of the likelihood and the prior. For LP-abiding Bayesians, the prior is not allowed to depend on data.

The subjectivistic viewpoint

▶ Top requirement is internal coherence of decisions.

▶ Various attempts at proving that, internally, coherent decision makers minimize some expected utility; see (Parmigiani and Inoue, 2009).

Figure: Bruno de Finetti (1906–1985) and L. "Jimmie" Savage (1917–1971)

Savage's axioms

- Start with the triple $(\mathcal{S}, \mathcal{Z}, \mathcal{A} \subset \mathcal{F}(\mathcal{S}, \mathcal{Z}))$ as in Wald, 1950.
- Savage's idea is to list what we expect from a binary relation $\prec$ on $\mathcal{A} \times \mathcal{A}$ describing a decision maker's preferences.

Theorem; see (Schervish, 2012, Theorem 1.49)

Let $X_1, X_2, \dots$ be a sequence of exchangeable random variables on $\mathcal{X}$, i.e.

$$X_1, \dots, X_n \sim X_{\pi(1)}, \dots, X_{\pi(n)}, \forall n, \forall \pi \in \mathfrak{S}_n.$$

Then there exists a probability distribution μ on the set of probability measures $\mathcal{P}(\mathcal{X})$ on $\mathcal{X}$ such that

$$\mathbb{P}(X_1 \in A_1, \dots, X_n \in A_n) = \int Q(A_1) \dots Q(A_n) \mathrm{d}\mu(Q).$$

To a subjectivist, Savage's theorem says you should use SEU, and representation theorems like de Finetti's constrain your choice of p.

Pros and cons of the subjectivist viewpoint

▶ Axiomatic derivations are powerful, and shed light on what coherence requires. In particular, coherence leads to SEU.

☹ ... Yet all axiomatic systems have a bit that is difficult to swallow.

▶ Priors should be elicited by expert knowledge, and should be *bona fide* probability distributions.

▶ Representation theorems can help design the joint distribution over states (**OrRo14**).

▶ Bayesian nonparametrics has revived the subjectivist viewpoint.

- A historical objection to Bayes is the need to choose a prior.
- By "objective", we mean that the prior is chosen by some external rule, and that this rule is relatively consensual.
- Take for instance, Jeffreys's "noninformative" priors.

A complete class theorem for estimation (Berger, 1985)

Under topological and Euclidean assumptions, if further

- $L(\theta, \cdot)$ is continuous,
- $\theta \mapsto \int L(\theta, \hat{\theta}) p(y_{1:n} | x_{1:n}, \theta) \mathrm{d}y_{1:n}$ is continuous for any $\hat{\theta}$,

then for any estimator $\tilde{\theta}$ there exists a prior and a corresponding Bayes estimator

$$\hat{\theta}_{\text{Bayes}} \in \arg\min_{\hat{\theta}} \mathbb{E}_{\theta | x_{1:n}, y_{1:n}} L(\theta, \hat{\theta})$$

such that

$$\forall \theta, \quad \mathbb{E}_{y_{1:n} | x_{1:n}, \theta} L(\theta, \hat{\theta}_{\text{Bayes}}) < E_{y_{1:n} | x_{1:n}, \theta} L(\theta, \tilde{\theta}).$$

Bayesian estimators thus have good frequentist properties

But finding the "right" prior can be difficult. Frequentists typically use Bayesian derivations with particular (often data-dependent) priors; see e.g. empirical Bayes procedures (Efron, 2012).

Sparse Bayesian regression

The Lasso

$$\hat{\theta}_{\mathsf{Lasso}} \in \arg\min_{\theta} \frac{1}{2}\|y - X\theta\|^2 + \lambda\|\theta\|_1.$$

▶ If $X|\theta \sim \mathcal{N}(X\theta, I)$ and $p(\theta) \propto e^{-\lambda\|\theta\|_1}$, the MAP is $\hat{\theta}_{\mathsf{Lasso}}$.
▶ But the posterior is not sparse.

Spike-and-slab priors

$$p(\theta|w) = \prod_{i=1}^{d} \left[(1 - w_i)\delta_0(\theta_i) + w_i q(\theta_i)\right].$$

The horseshoe prior (Carvalho, Polson, and Scott, 2010)

$$\lambda_1, \ldots, \lambda_d, \tau \sim C^+(0, 1),$$
$$\theta_j|\lambda_j, \tau \sim \mathcal{N}(0, \tau^2\lambda_j^2), \quad j = 1, \ldots, d.$$

The horseshoe prior (Carvalho, Polson, and Scott, 2010)

$$\lambda_1, \ldots, \lambda_d, \tau \sim C^+(0, 1),$$
$$\theta_j | \lambda_j, \tau \sim \mathcal{N}(0, \tau^2 \lambda_j^2), \quad j = 1, \ldots, d.$$
$$y_{1:n} | x_{1:n}, \theta \sim \mathcal{N}(X\theta, \sigma^2 I)$$

Theorem (van der Pas, Kleijn, and van der Vaart, 2014)

As $n, p_n \to \infty$, with $\tau = (p/n)^\alpha$, $\alpha \geqslant 1$,

$$\sup_{\|\theta\|_0^2 = p_n} \mathbb{E} \|g^*(x_{1:n}, y_{1:n}) - \theta\|^2 \asymp p_n \log \frac{n}{p_n}.$$

PAC bounds; see e.g. (Shalev-Shwartz and Ben-David, 2014)

Let $(x_{1:n}, y_{1:n}) \sim \mathbb{P}^{\otimes n}$, and independently $(x, y) \sim \mathbb{P}$, we want an algorithm $g(\cdot; x_{1:n}, y_{1:n}) \in \mathcal{G}$ such that if $n \geqslant n(\delta, \varepsilon)$,

$$\mathbb{P}^{\otimes n}\left[\mathbb{E}_{(x,y)\sim\mathbb{P}}L(a_g, s) \leqslant \varepsilon\right] \geqslant 1 - \delta.$$

McAllester's bound for 0-1 loss (Chapter 31 of the above book)

For any two distributions P, Q on $\mathcal{G}$, with $\mathbb{P}^{\otimes n}$-probability $1 - \delta$,

$$\mathbb{E}_{g\sim Q}\mathbb{P}(g(x) \neq y) \leqslant \mathbb{E}_{g\sim Q}\frac{1}{n}\sum_{i=1}^{n}1_{g(x_i)\neq y_i} + \sqrt{\frac{\mathrm{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

This suggests taking the "posterior" Q to be in

$$\arg\min \mathbb{E}_{g\sim Q}\frac{1}{n}\sum_{i=1}^{n}1_{g(x_i)\neq y_i} + \sqrt{\frac{\mathrm{KL}(Q, P) + \log(n/\delta)}{2(n-1)}}.$$

One possible hybrid view, e.g. (Robert, 2007)

▶ The starting point is Wald's decision setting, adding integration with respect to a prior.

▶ It is simple, widely applicable, has good frequentist properties.

▶ It satisfies the likelihood principle when priors do not depend on data.

▶ It is tempting to interpret it as follows: beliefs are
  ▶ represented by probabilities,
  ▶ updated using Bayes' rule,
  ▶ integrated when making decisions.

▶ It is easy to communicate your uncertainty
  ▶ Simply give your posterior.
  ▶ When making a decision, make sure that the priors of everyone involved would yield the same decision.
  ▶ Alternately, perform a prior sensitivity analysis.

What kind of Bayesian are you?

▶ I've only scratched the surface. See e.g. (Mayo, 2018).

▶ Posterior expected utility is conceptually simple and unifying. Beyond that, many intepretations get partial philosophical support.

▶ The role of the likelihood, the prior, your update mechanism, etc. depend on the intepretation that you choose.

☹ Many people do not care.

▶ Hybrid views have become common (Robert, 2007; Gelman et al., 2013). This arguably makes the role of priors fuzzy.

▶ In ML, the development of Bayesian nonparametrics is reviving the subjectivist view, while objective approaches like PAC-Bayes are also increasingly popular.

▶ A great door to subjective Bayes is (Parmigiani and Inoue, 2009).

[1] J. O. Berger. *Statistical decision theory and Bayesian analysis*. Springer, 1985.

[2] J. O. Berger and R. L. Wolpert. *The likelihood principle: A review, generalizations, and statistical implications*. Vol. 6. Institute of Mathematical Statistics, 1988.

[3] C. M. Carvalho, N. G. Polson, and J. G. Scott. "The horseshoe estimator for sparse signals". In: *Biometrika* 97.2 (2010), pp. 465–480.

[4] B. Efron. *Large-scale inference: empirical Bayes methods for estimation, testing, and prediction*. Vol. 1. Cambridge University Press, 2012.

[5] A. Gelman et al. *Bayesian data analysis*. 3rd. CRC press, 2013.

[6] D. G. Mayo. *Statistical inference as severe testing: How to get beyond the statistics wars*. Cambridge University Press, 2018.

[7] G. Parmigiani and L. Inoue. *Decision theory: principles and approaches*. Vol. 812. John Wiley & Sons, 2009.

[8]   C. P. Robert. *The Bayesian choice: from decision-theoretic foundations to computational implementation*. Springer Science & Business Media, 2007.

[9]   M. J. Schervish. *Theory of statistics*. Springer Science & Business Media, 2012.

[10] S. Shalev-Shwartz and S. Ben-David. *Understanding machine learning: From theory to algorithms*. Cambridge university press, 2014.

[11] S. L. van der Pas, B. J. K. Kleijn, and A. W. van der Vaart. "The horseshoe estimator: Posterior concentration around nearly black vectors". In: *Electronic Journal of Statistics* 8.2 (2014), pp. 2585–2618.

[12] A. Wald. *Statistical decision functions*. Wiley, 1950.