AN EQUAL-AREA MAP PROJECTION FOR POLYHEDRAL GLOBES

JOHN P SNYDER
United States Geological Survey/Reston, Virginia
United States

Abstract Numerous polyhedral shapes have been proposed as approximations for globes, and the projection most often used is the Gnomonic, with considerable scale and area distortion. Complicated conformal projections have been designed, but an equal-area projection has been used only once, for the icosahedron. The Lambert Azimuthal Equal-Area projection can be modified to provide an exactly fitting, perfectly equal-area projection for any polyhedral globe that has regular polygons, but is most satisfactory for the dodecahedron with 12 pentagons and for the truncated icosahedron with 20 hexagons and 12 pentagons. On the application to the truncated icosahedron, the angular deformation does not exceed 3.75°, and the scale variation is less than 3.3 percent. These advantages are at the expense of increased interruptions at the polygon edges when the polyhedral globe is unfolded.

Introduction

Polyhedral globes have been used as approximations for spherical globes for centuries. The artist Albrecht Dürer (1538) first called attention to them, although he did not discuss map projections. Many innovators of the 19th and 20th centuries applied the Gnomonic projection to most of the common polyhedra at one time or another (Snyder and Steward 1988). Folded polyhedral globes are easier to assemble without special techniques than spherical globes and serve as instructional tools, but they are bulky and small-scale, like globes. Unfolded and flattened polyhedral globes form world maps on projections which can have less distortion than other interrupted projections, but there are generally an increased number of interruptions and greater complications in plotting.

The Gnomonic is a perspective projection from the center of the globe and is rather easily adapted to polyhedral globes, but the distortion is extensive. On the cube (6 squares) or octahedron (8 triangles), linear scale has a 200 percent variation, and on the dodecahedron (12 pentagons) or icosahedron (20 triangles), the range is 58 percent (see Table 1). L.P. Lee (1976) projected the globe conformally onto the faces of all five Platonic solids (the tetrahedron of four triangles plus the four named above), but the equations for the transformations are very complicated.

An equal-area projection of the sphere onto a polyhedral globe was the subject of only one paper, published over 40 years ago and applied only to the icosahedron. Bradley (1946) devised a simple graticule consisting of segments of straight lines for meridians and parallels, although the faces do not quite correspond to 20 equilateral spherical triangles on the globe. In an editorial footnote to this paper, a previously unpublished equal-area transformation by Irving Fisher was described, with its triangle boundaries truly identical to those on the globe and with less distortion than Bradley's.

Fisher's approach was recently rederived to solve a practical problem, although equations are in a form different from Fisher's. A manuscript by White, Kimerling, and Overton (1992) led this author to apply Fisher's approach to the non-Platonic truncated icosahedron (with 20 hexagons and 12 pentagons), which the 1992 authors had selected for the EMAP sampling grid to be used for the U.S. Environmental Protection Agency. The approach was then generalized for the five Platonic solids. Inverse formulas and a distortion analysis were also included in the new derivations.

The development described is for the more complicated and less distorted truncated icosahedron; the application to Platonic polyhedral globes follows.

An equal-area truncated icosahedron

The truncated icosahedron is considered one of the 13 Archimedean polyhedra. Whereas the faces of each of the five Platonic solids are identical regular polygons, an Archimedean solid has faces of two or three types of identical regular polygons. The truncated icosahedron, which is formed by cutting off each corner of the icosahedron so that the triangular faces become hexagons and the twelve corners become pentagons, is the most spherical of these polyhedra, and is popularized as the basis of the soccerball design. The Lambert Azimuthal Equal-Area projection can be centered on each pentagon and hexagon, with appropriate centers, but the edges cannot be made to match correctly without changing the projection. There is a thin sliver of overlap or underlap, depending on the choice of scale.

JOHN P SNYDER, now retired, was a Research Scientist in the National Mapping Division, U.S. Geological Survey, Reston, Virginia. MS *submitted November 1991*

The Lambert projection can be modified by adjusting the radial distances from the center of each polygon so that the edges match. This removes the slivers, but the area scale is no longer correct. The author used this approach for terrestrial and celestial globes made from construction paper between 1945 and 1951. By adjusting the azimuths as well as the radial distances from the polygon centers, the slivers are eliminated and true area scale is maintained.

As applied to the icosahedron, this adjustment causes visible cusps or bends along the radii from the center of each polygon to its vertices, illustrated by Fisher. On the truncated icosahedron, the cusps are not noticeable.

There are two other minor problems with the equal-area truncated icosahedron described here. **1** Because the azimuth adjustment for the hexagons is numerically slightly different from that for the pentagons, there is an offset for features crossing any edge between a hexagon and a pentagon, but the maximum offset is less than one part in 100,000 of the length of a side. **2** The centers of the pentagons are 2.65 percent farther from the center of the polyhedron than the centers of the hexagons; the resulting area scale on the pentagons is 1.28 percent greater than that of the hexagons.

FORWARD FORMULAS FOR THE HEXAGON FACES

Because of symmetry, the derivation of equations for the equal-area projection onto a hexagon face (Figure 1) requires analysis only of one-twelfth of one hexagon, namely the part enclosed by one of the right triangles bounded by straight lines connecting the center of a face, a vertex, and the center of an adjacent edge (Figure 2), together with its corresponding right spherical triangle on the globe (Figure 3). The goals for the derivation consist of **A** making the total area of the right triangle ABC in Figure 2 equal to the total area of the right spherical triangle $A'B'C'$ in Figure 3, **B** making the area of scalene triangle ABD in Figure 2 equal to the area of scalene spherical triangle $A'B'D'$ in Figure 3 for a given azimuth Az from the vertex, and **C** positioning points at spherical distance z along all radius vectors AD so that the area scale will remain constant throughout triangle ABC.

Before obtaining the necessary equations, it is appropriate to determine the spherical distance g from the center of a hexagon to any of its vertices and the spherical angle G between radius vector $A'B'$ and edge $B'C'$. These calculations were made starting with the complete icosahedron and using the Laws of Sines and Cosines for spherical triangles in various rearrangements to determine the angular lengths of the radius vectors from the polygon centers to vertices. Truncating the icosahedron to produce hexagon and pentagon faces leads to the constants g and G in Table 1 for the hexagon faces. Angle θ at vertex B of Figure 2 is more obvious.

The area A_{GT} of triangle $A'B'C'$ on a globe of radius R is its 'spherical excess' over 180°, or

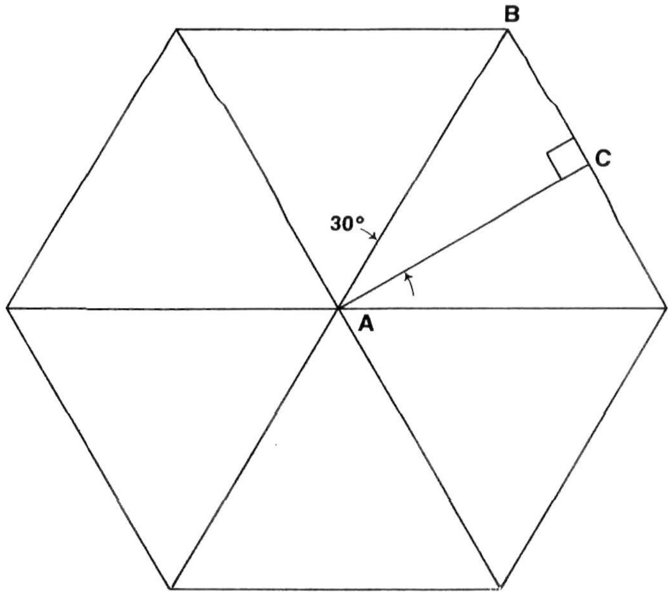

FIGURE 1. *Hexagon face on truncated icosahedron.*

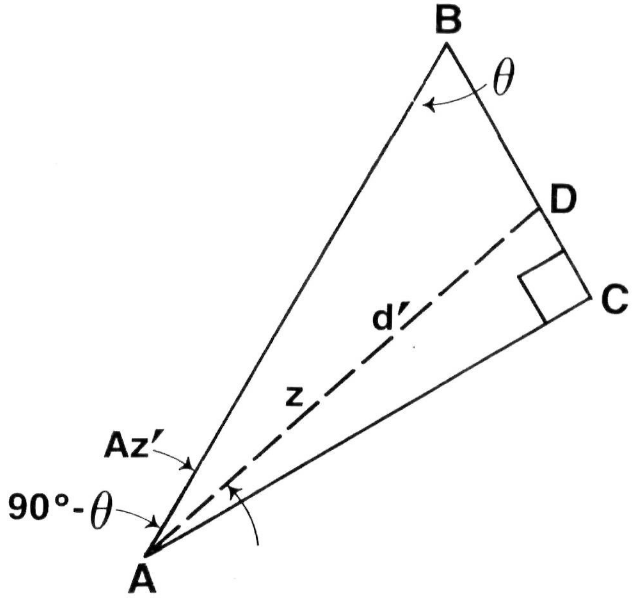

FIGURE 2. *Right triangle from Figure 1.*

$$A_{GT} = [(90^\circ - \theta) + 90^\circ + G - 180^\circ)]\pi R^2/180^\circ$$

$$= (G - \theta)\pi R^2/180^\circ \quad (1)$$

$$= 0.0376062647\ R^2 \quad (2)$$

The map area A_{MT} of triangle ABC, as part of a hexagon made tangent to a sphere of radius R', and therefore having a side $AB = R' \tan g$, is

$$A_{MT} = (1/2)(R'\tan g)^2 \sin\theta \cos\theta \quad (3)$$

$$= 0.0421171345\ R'^2 \quad (4)$$

Equating (2) and (4) to achieve the previously mentioned goal **A**, the size of the polyhedron is

$$R' = 0.9449322893\ R \quad (5)$$

TABLE 1. SPHERICAL CONSTANTS AND DISTORTION ON POLYHEDRAL GLOBES USING EQUAL-AREA AND GNOMONIC PROJECTIONS.

	Spherical constants			Equal-area proj.			Gnomonic proj.		
Polyhedron	g	G	θ	ω	a	b	ω	a	b
Truncated icosahedron									
Hexagon faces	23.80018260	62.15468023	60.	3.75	1.033	0.968	5.09	1.195	1.000
Pentagon faces	20.07675127	55.69063953	54.	2.65	1.030	0.983	3.59	1.141	1.027
Tetrahedron	70.52877937	60.	30.	52.07	1.601	0.624	60.00	9.000	1.000
Cube	54.73561032	60.	45.	25.17	1.248	0.801	31.08	3.000	1.000
Octahedron	54.73561032	45.	30.	34.45	1.357	0.737	31.08	3.000	1.000
Dodecahedron	37.37736814	60.	54.	10.24	1.094	0.914	13.14	1.584	1.000
Icosahedron	37.37736814	36.	30.	17.27	1.163	0.860	13.14	1.584	1.000

g = spherical distance, degrees, from center of polygon face to any of its vertices on the globe.
G = spherical angle, degrees, between radius vector to center and adjacent edge of spherical polygon on the globe.
θ = plane angle, degrees, between radius vector to center and adjacent edge of plane polygon.
ω = maximum value of maximum angular deformation, degrees, occurring along a radius to each vertex, but at the center on all equal-area polygons except the hexagons of the truncated icosahedron, where it occurs at each vertex. On the Gnomonic projection, this occurs at each vertex.
a = maximum value of the maximum scale factor, occurring where ω is maximum.
b = minimum value of the minimum scale factor, occurring where ω is maximum on equal-area polygons and at the centers of all Gnomonic polygons.
NOTE: The values of a and b on the pentagon faces of the equal-area truncated icosahedron reflect the higher area scale on those faces, taking the hexagon faces at true area scale. On the Gnomonic faces, all faces are considered tangent to the sphere except for the pentagons of the truncated icosahedron.

For goal **B**, the area A_G of triangle $A'B'D'$ is found after determining angle H at D' in Figure 3:

$$H = \arccos (\sin Az \sin G \cos g - \cos Az \cos G) \qquad (6)$$

$$A_G = (Az + G + H - 180°)\pi R^2/180° \qquad (7)$$

Area A_M of triangle ABD is

$$A_M = (1/2)(R' \tan g)^2 \sin \theta \sin Az'/\sin (Az' + \theta)$$
$$= (1/2)(R' \tan g)^2 \sin \theta \sin Az'/ \sin (Az' \cos \theta + \cos Az' \sin \theta)$$
$$= (1/2)(R' \tan g)^2 \tan Az'/(\tan Az' \cot \theta + 1)$$

Since A_M must equal A_G, transposing,

$$Az' = \text{arctan2}\ (2A_G, R'^2 \tan^2 g - 2A_G \cot \theta) \qquad (8)$$

Goal **C** is found as follows: Arc q of Figure 3 is found from Figure 2 with the Law of Sines for plane triangle ABD, but calculating AD for angle Az instead of Az' and using the Gnomonic projection:

$$AD/\sin \theta = AB/\sin (180° - Az - \theta)$$

Since $AD = R' \tan q$ and $AB = R' \tan g$, then

$$q = \arctan [\tan g/(\cos Az + \sin Az \cot \theta)] \qquad (9)$$

Point D' at the end of arc q on Figure 3 is placed at D on Figure 2 not at ($R' \tan q$) from A, but, due to azimuth shift to Az', at

$$d'/\sin \theta = AB/\sin (180° - Az' - \theta)$$

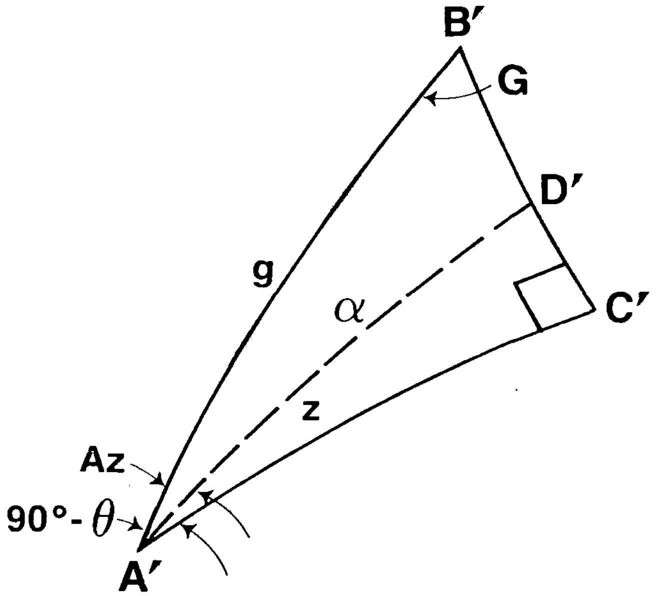

FIGURE 3. *Spherical right triangle corresponding to Figure 2.*

or

$$d' = R' \tan g/(\cos Az' + \sin Az' \cot \theta) \qquad (10)$$

If two of the arcs bounding each of a set of spherical triangles have a common vertex and are only differentially separated from each other, an equal-area map projection of the spherical triangles with these arcs shown as straight lines must also show these arcs with lengths ρ proportional to the sine of half their angular lengths z. For the Lambert Azimuthal Equal-Area projection, this proportion is constant for the entire 360° range of azimuths about the origin, or $\rho = 2R \sin (z/2)$. For the hexagon under discussion, the proportionality factor f varies with azimuth and is the ratio of d' to $[2R' \sin (q/2)]$, or

$$f = d'/2R' \sin (q/2) \tag{11}$$

For another point z along the same azimuth line Az (spherical) or Az' (plane),

$$\rho = 2R'f \sin (z/2) \tag{12}$$

The positioning of any given point on the spherical hexagon may then be plotted at the correct rectangular coordinates (x, y) on the polyhedral plane hexagon after several steps:

1 Calculate the azimuth Az and spherical distance z of the point (lat. ϕ, long. λ) from the geographic center (ϕ_0, λ_0) of the hexagon (see Table 2 and Figure 4), by using standard formulas for oblique transformations:

$$z = \arccos [\sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos (\lambda - \lambda_0)] \tag{13}$$

$$Az = \operatorname{arctan2} [\cos \phi \sin(\lambda - \lambda_0), \cos \phi_0 \sin \phi - \sin \phi_0 \cos \phi \cos (\lambda - \lambda_0)] \tag{14}$$

If z exceeds g, the point is too far from the center of the hexagon and is located on another polygon.

2 After initial adjustment to give some vertex an Az of 0, adjust Az for the given point to fall within the range 0° to $2(90° - \theta)$ or 60° for the hexagon (the above formulas for one-twelfth of the hexagon may by applied to one-sixth of it at a time, vertex to vertex) by subtracting or adding necessary multiples of 60° to Az and recording the amount of adjustment.

3 Calculate q from equation (9). If z exceeds q, it will not fit on this polygon and is located on another one.

4 Apply equations (5)–(8) and (10)–(12) in order. Add back the same 60°-multiple adjustment from step **2** to Az' and calculate rectangular coordinates from the equations

$$x = \rho \sin Az' \tag{15}$$

$$y = \rho \cos Az' \tag{16}$$

These coordinates must then be translated to the proper origin for the particular hexagon on the flattened polyhedral map plot (see Table 2 and Figure 4). It is convenient to let $R = 1$ until the final scaling of the map after the above calculations.

INVERSE FORMULAS FOR THE HEXAGON FACES

Most of the above equations can be readily inverted to obtain Az and z from rectangular coordinates, but one iteration is involved. From equations (15) and (16),

$$Az' = \operatorname{arctan2} (x, y) \tag{17}$$

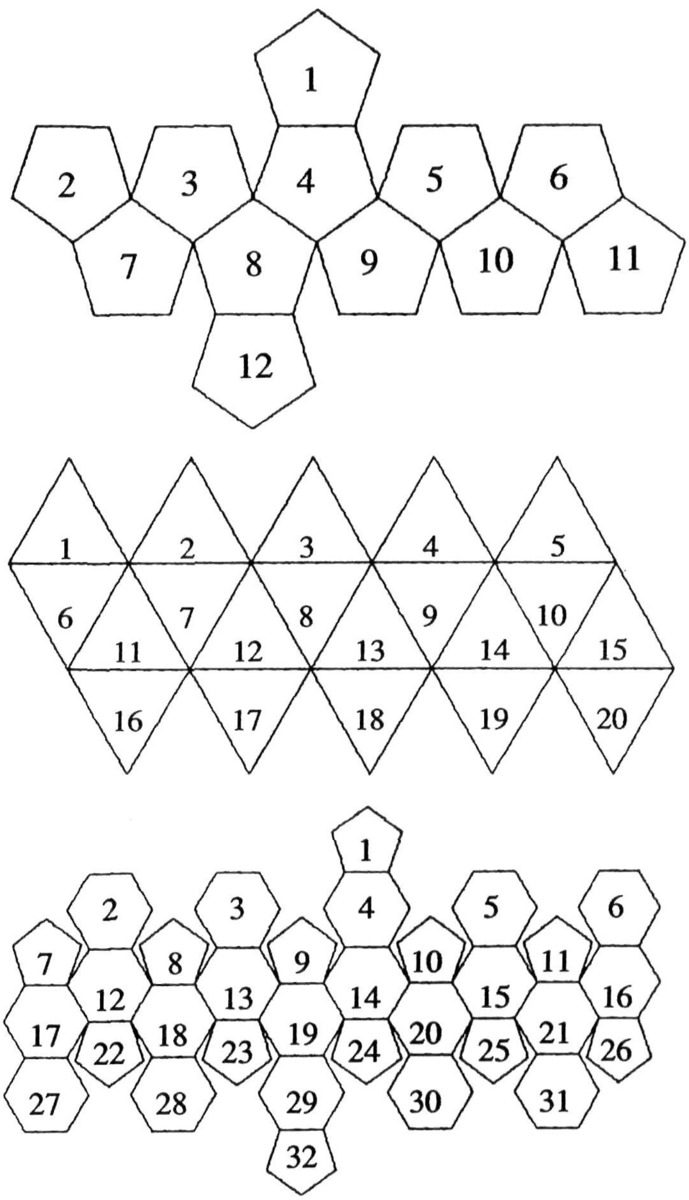

FIGURE 4. *Numbering of polygons for Table 2: (A) Dodecahedron, (B) Icosahedron, (C) Truncated icosahedron.*

$$\rho = (x^2 = y^2)^{1/2} \tag{18}$$

Transposing (8),

$$A_G = R'^2 \tan^2 g/2(\cot Az' + \cot \theta) \tag{19}$$

Transposing (7) for (20) differentiating (6) and (20) with respect to Az for (21), and applying the Newton-Raphson iteration formula to obtain (22),

$$F(Az) = (180° A_G/\pi R^2) - G - H - Az + 180° \tag{20}$$

$$F'(Az) = [(\cos Az \sin G \cos g + \sin Az \cos G)/\sin H] - 1 \tag{21}$$

$$\Delta Az° = -F(Az)/F'(Az) \tag{22}$$

Transposing (12),

$$z = 2 \arcsin (\rho/2R'f) \tag{23}$$

TABLE 2. GEOGRAPHICAL (LAT./LONG.) AND RECTANGULAR (x, y) COORDINATES OF CENTERS OF FACES FOR DODECAHEDRON, ICOSAHEDRON, AND TRUNCATED ICOSAHEDRON.
[Given for "normal" aspect as used in Figures 11, 12, and 6, respectively.]

Face no.	lat.	long.	x	y
Dodecahedron (see Figure 4A for polygon numbers; compare with Figure 11)				
1	+90°	0°	0	C
2	A	−144	−4B	D
3	A	−72	−2B	D
4	A	0	0	D
5	A	72	2B	D
6	A	144	4B	D
7	−A	−108	−3B	−D
8	−A	−36	−B	−D
9	−A	36	B	−D
10	−A	108	3B	−D
11	−A	180	5B	−D
12	−90	−36	−B	−C
Icosahedron (see Figure 4B for polygon numbers; compare with Figure 12)				
1	E	−144	−4G	5H
2	E	−72	−2G	5H
3	E	0	0	5H
4	E	72	2G	5H
5	E	144	4G	5H
6	F	−144	−4G	H
7	F	−72	−2G	H
8	F	0	0	H
9	F	72	2G	H
10	F	144	4G	H
11	−F	−108	−3G	−H
12	−F	−36	−G	−H
13	−F	36	G	−H
14	−F	108	3G	−H
15	−F	180	5G	−H
16	−E	−108	−3G	−5H
17	−E	−36	−G	−5H
18	−E	36	G	−5H
19	−E	108	3G	−5H
20	−E	180	5G	−5H
Truncated icosahedron (see Figure 4C for polygon numbers; compare with Figure 6)				
1	90	0	0	7K + L
2	E	−144	−6J	5K
3	E	−72	−3J	5K
4	E	0	0	5K
5	E	72	3J	5K
6	E	144	6J	5K
7	A	−180	−7.5J	K + L
8	A	−108	−4.5J	K + L
9	A	−36	−1.5J	K + L
10	A	36	1.5J	K + L
11	A	108	4.5J	K + L
12	F	−144	−6J	K
13	F	−72	−3J	K
14	F	0	0	K
15	F	72	3J	K
16	F	144	6J	K
17	−F	180	−7.5J	−K
18	−F	−108	−4.5J	−K
19	−F	−36	−1.5J	−K
20	−F	36	1.5J	−K
21	−F	108	4.5J	−K
22	−A	−144	−6J	−K − L
23	−A	−72	−3J	−K − L
24	−A	0	0	−K − L

TABLE 2. (*concluded*)

Face no.	lat.	long.	x	y
Truncated icosahedron (see Figure 4C for polygon numbers; compare with Figure 6)				
25	−A	72	3J	−K − L
26	−A	144	6J	−K − L
27	−E	180	−7.5J	−5K
28	−E	−108	−4.5J	−5K
29	−E	−36	−1.5J	−5K
30	−E	36	1.5J	−5K
31	−E	108	4.5J	−5K
32	−90	−36	−1.5J	−7K − L

A = 90° − 2 arctan (tan g cos 36°) = 26.56505118° (using g for dodecahedron)
B = R tan 36° = 0.726542528 R
C = R (0.5 + sec 36°) = 1.736067977 R
D = 0.5 R
E = 90° − g = 52.62263186° (using g for icosahedron)
F = 10.81231696°
G = R tan g sin 60° = 0.6615845383 R (using g for icosahedron)
H = 0.25 R tan g = 0.1909830056 R (using g for icosahedron)
J = 0.4167683948 R
K = 0.1804660087 R
L = 0.2868162417 R

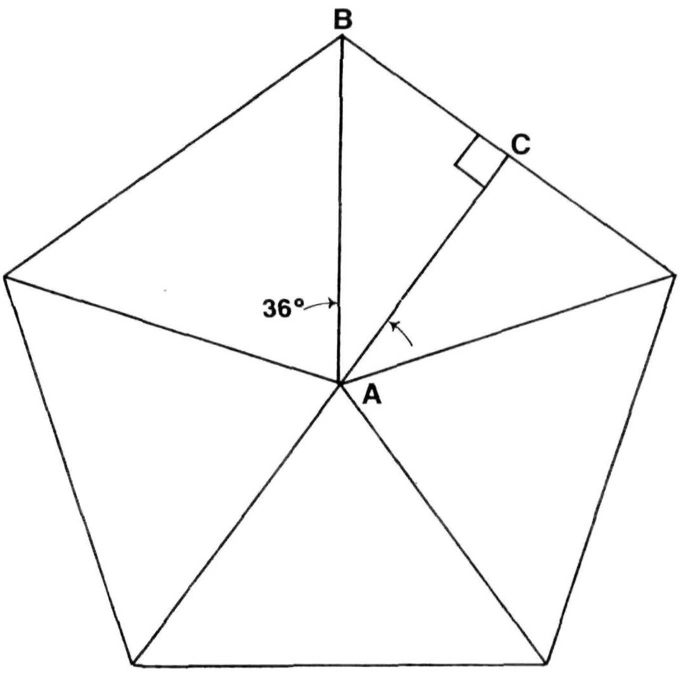

FIGURE 5. *Pentagon face corresponding to Figure 1.*

To use these equations, after translating the initial (x, y) to the origin of the hexagon, apply (17) and (18), adjust Az' to fall within 0° to 60° as in step **2** of forward calculations, then calculate A_G from (5) and (19). Now iterate (6) and (20)–(22) in order, with Az' as the first approximation for Az. This iteration converges even to 10^{-9} degrees in 3–4 cycles. After convergence, apply (9)–(11) and (23) to obtain z, add back the 60° adjustments to Az, and convert to the final latitude and longitude based on the center and orientation of the hexagon.

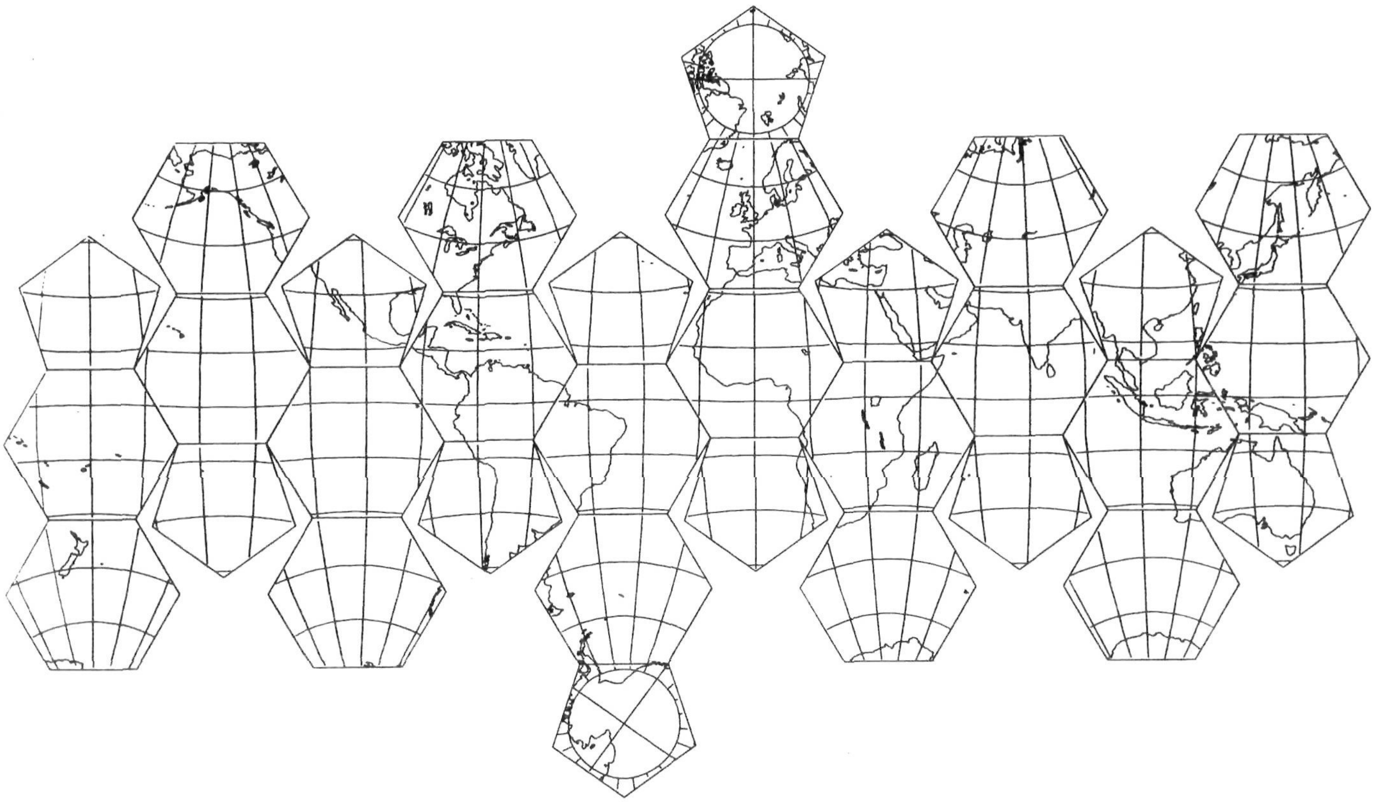

FIGURE 6. *Equal-area truncated icosahedron. Normal aspect, with poles centered on pentagons.*

EQUATIONS FOR PENTAGON FACES

The formulas for the pentagon faces (Figure 5) of the truncated icosahedron are very similar to those for the hexagons, altered by the fact that they have different shapes and a different area scale. Constants g, G, and θ are as listed in Table 1.

The area of triangle $A'B'C'$ for the pentagon, using equation (1) with the new constants, is

$$A_{GT} = 0.0295072264\ R^2 \tag{24}$$

The radius vector R'' from the center of the polyhedron to the center of the pentagon may be found by calculating the radius required to produce a pentagon with semiedge BC on Figure 5 equal in length to BC on Figure 2:

$$BC = R'\tan g \cos \theta \text{ for hexagon in Figure 2}$$
$$= R''\tan g \cos \theta \text{ for pentagon in Figure 5}$$

or

$$R'' = 1.026531522\ R' \tag{25}$$

$$= 0.970027811\ R \tag{26}$$

Then the area of one-tenth of the plane pentagon triangle ABC of Figure 5, using equation (3) with the new constants and R'' in place of R', is

$$A_{MT} = 0.031760883\ R''^2 \tag{27}$$

Combining equations (26) and (27),

$$A_{MT} = 0.0298839862\ R^2 \tag{28}$$

This plane area A_{MT} is not equal to A_{GT} in equation (24), but, by division, the ratio K of the area scale of the pentagons to that of the hexagons is

$$K = A_{MT}/A_{GT} = 1.01276833 \tag{29}$$

For other equations relating to the pentagon, equations (6)–(23) may be used exactly as they appear, except that the A_G of (8) is replaced with KA_G both places, R' in (8), (10)–(12), and (23) with R'' from (26), and R'^2 in (19) with R''^2/K. The steps given for the plotting concepts also apply to the pentagon formulas, except that $2(90° - \theta)$ or 72° steps of azimuth rotation are used instead of 60° steps.

An arbitrary but convenient arrangement of the faces of the truncated icosahedron is shown in Figure 6, with the poles in the centers of two pentagons for the 'normal' aspect. The North Pole can be shifted to any other location (for example see Figure 7) for an oblique aspect.

OFFSET ALONG PENTAGON-HEXAGON EDGES

Adapting the Law of Sines for plane triangles, the length of BD for a given Az can be determined for the Gnomonic projection on the hexagon of Figure 2. From this BD, the corresponding Az can be obtained for the Gnomonic projection on an adjacent pentagon. For the two values of Az, the two corresponding values of Az' can be calculated using the above formulas for the respective polygons, and two adjusted lengths of BD can be determined

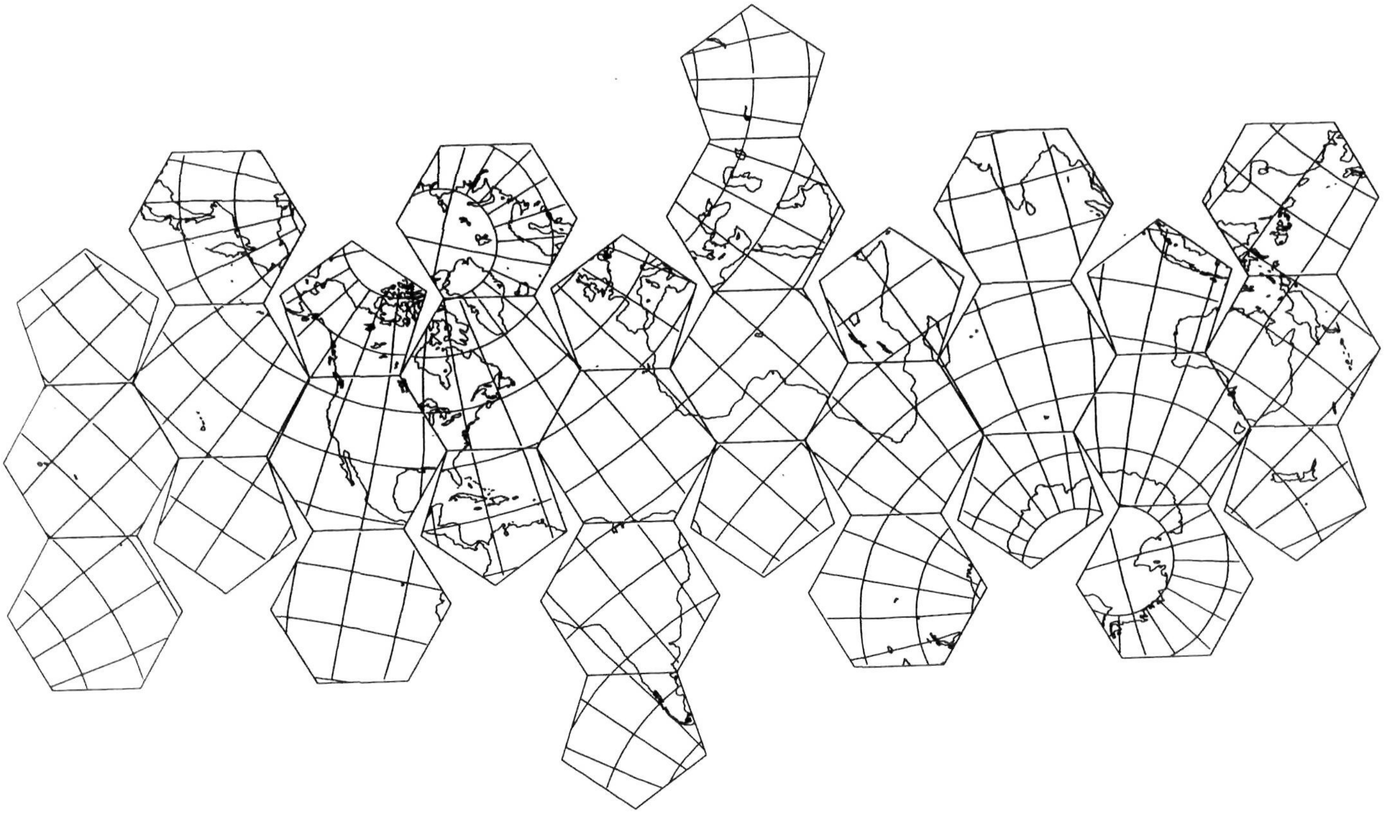

FIGURE 7. *Equal-area truncated icosahedron. oblique aspect, North Pole at +45°, −90° on normal aspect.*

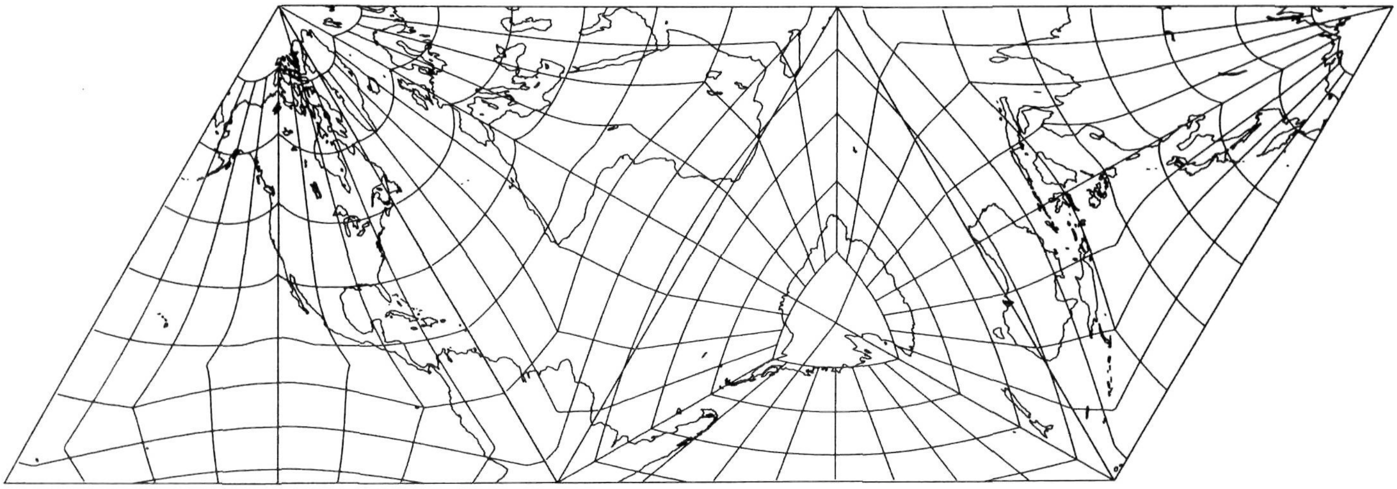

FIGURE 8. *Equal-area tetrahedron, North Pole at vertex, South Pole centered on triangle.*

for the two polygons using the respective Az'. If this is done for each 5° of Az on the hexagon from 0° to 30°, the difference between the two values of BD, matching at $Az = 0°$ and 30° for the hexagon, is less than 0.000018 of length BC (half a side) near $Az = 10°$ on the hexagon. Because $BD = (R' \tan g)/2 = 0.208\ R$ using g for the hexagon, this difference is less than 24 meters on the ground as a lateral offset only, not an overlap or underlap.

Application to the Platonic Polyhedral Globes

The above derivation may be applied to globes made from the five regular Platonic solids, using the equations for the hexagon, changing g, G, and θ, but without the

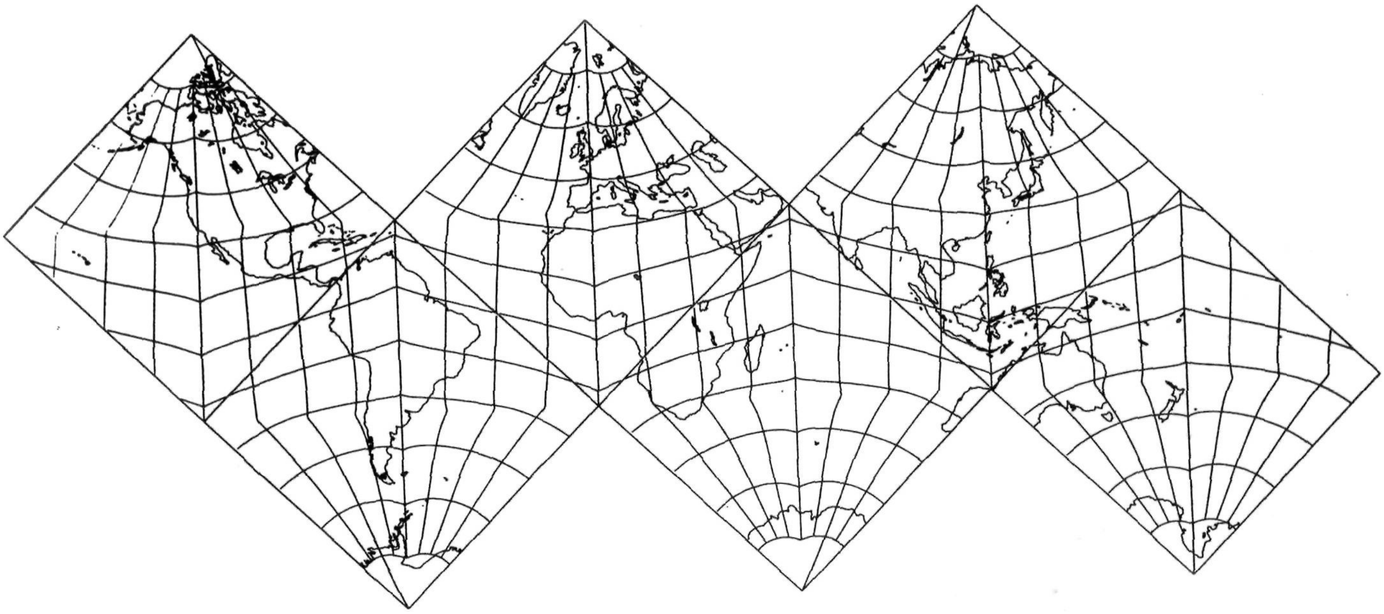

FIGURE 9. *Equal-area cube, poles at vertices.*

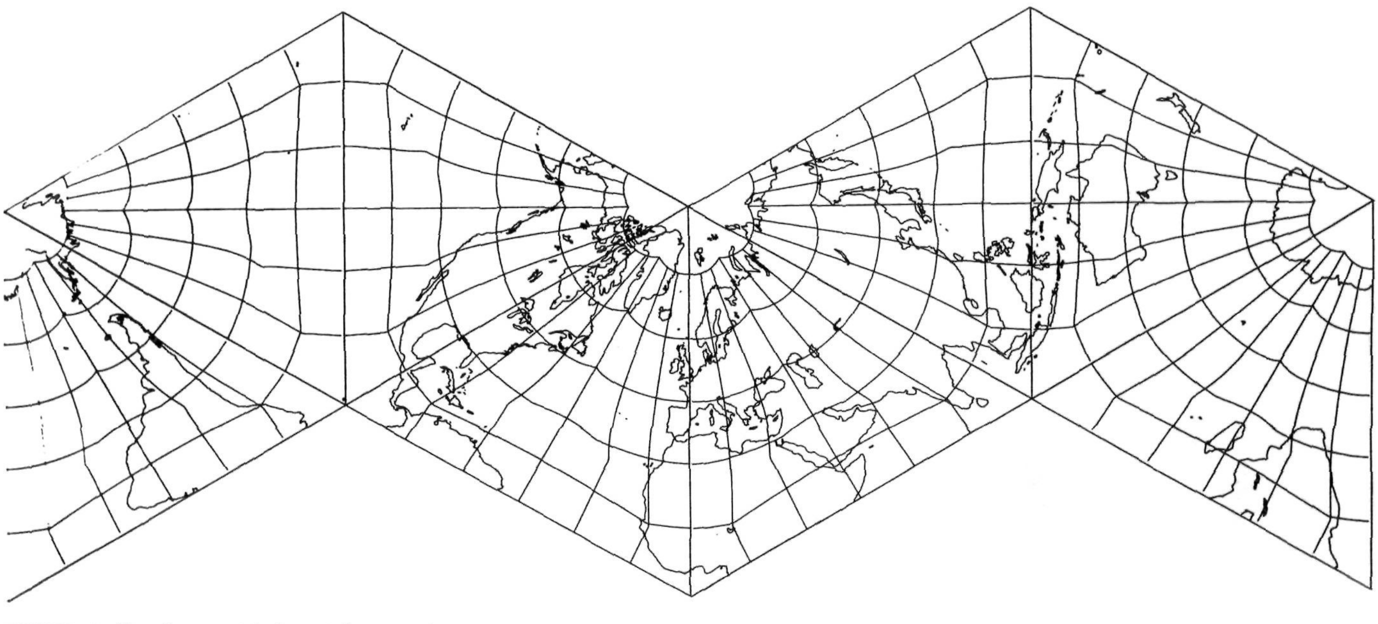

FIGURE 10. *Equal-area octahedron, poles at vertices.*

complications of the change in area scale involved in the pentagon faces of the truncated icosahedron. The calculation of the initial constants for each polyhedron requires rather straightforward manipulation of spherical geometry, leading to the values given in Table 1 for substitution into equations (1)–(23).

The values of tan g for the various polyhedra may be expressed as quadratic surds, namely $(3 - \sqrt{5})/\sqrt{3}$, $(4\sqrt{5} - 2)/19$, $2\sqrt{2}$, $\sqrt{2}$, and $\sqrt{(14 - 6\sqrt{5})}$, respectively, for the five successive tabular values. The cusps or breaks that Fisher pointed out for the icosahedron (Bradley 1946) are sufficiently more pronounced for the tetrahedron, cube, and octahedron to make equal-area projections onto any of them no more than novelties, although the same can be said for corresponding conformal or Gnomonic projections. Although cusps occur along all radii from polygon center to vertex, there is no visible change in direction of meridians and parallels as they cross edges of a flattened equal-area polyhedron; the breaks are pronounced when the Gnomonic projection is used.

Arbitrary orientations of the equal-area projections are shown in Figures 8–12, with a Gnomonic projection onto a tetrahedron shown in Figure 13 for comparison with Figure 8.

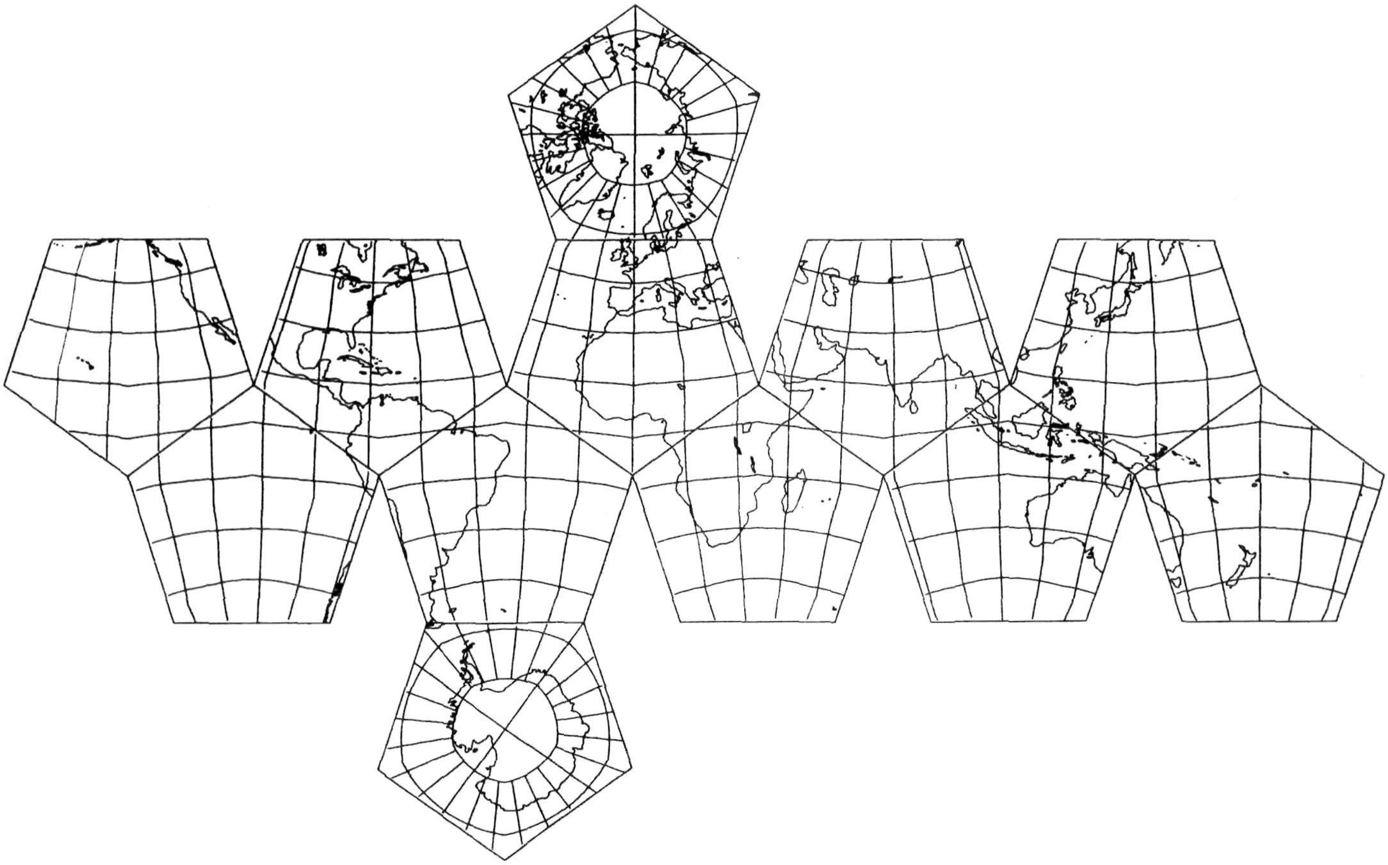

FIGURE 11. *Equal-area dodecahedron, poles centered on pentagons.*

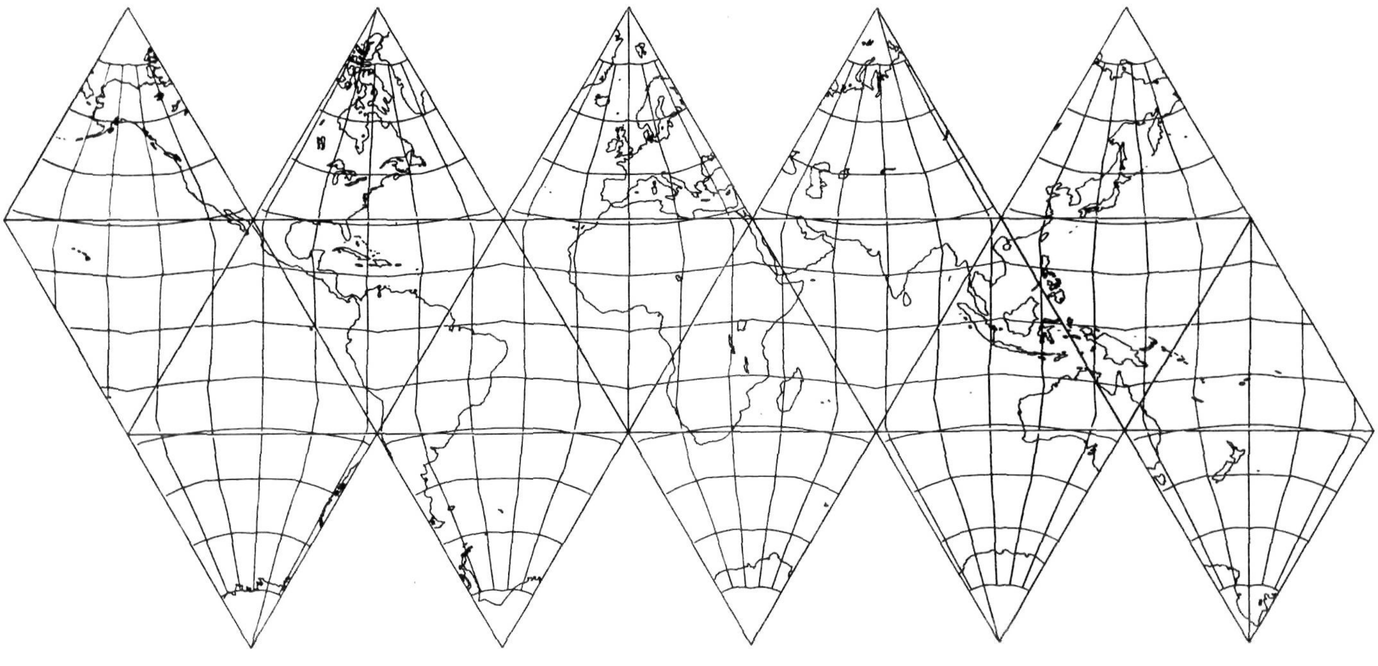

FIGURE 12. *Equal-area icosahedron, poles at vertices.*

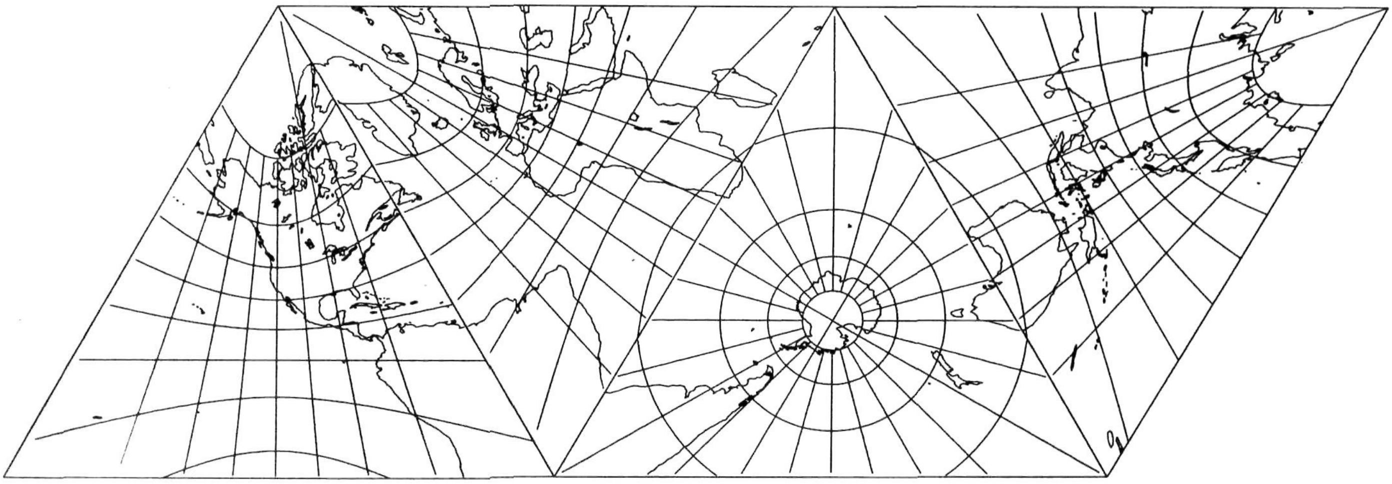

FIGURE 13. *Gnomonic tetrahedron, North Pole at vertex, South Pole centered on triangle.*

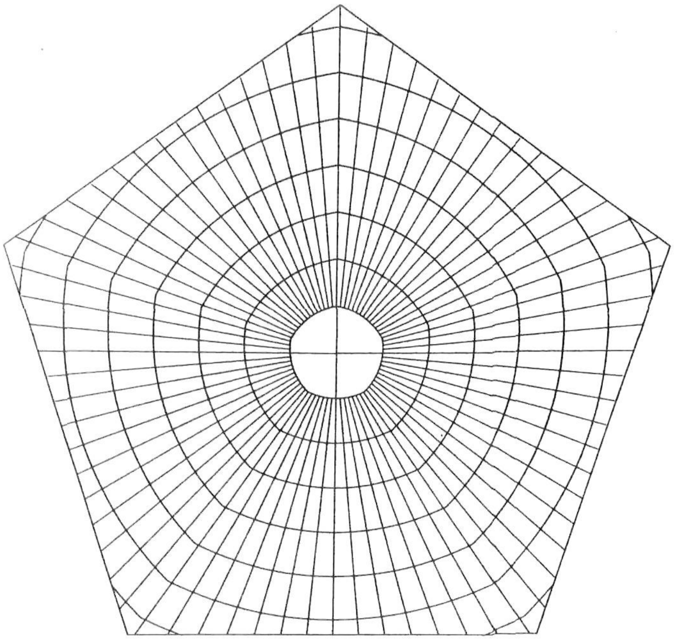

FIGURE 14. *Single face of equal-area dodecahedron, pole at center, 5° graticule.*

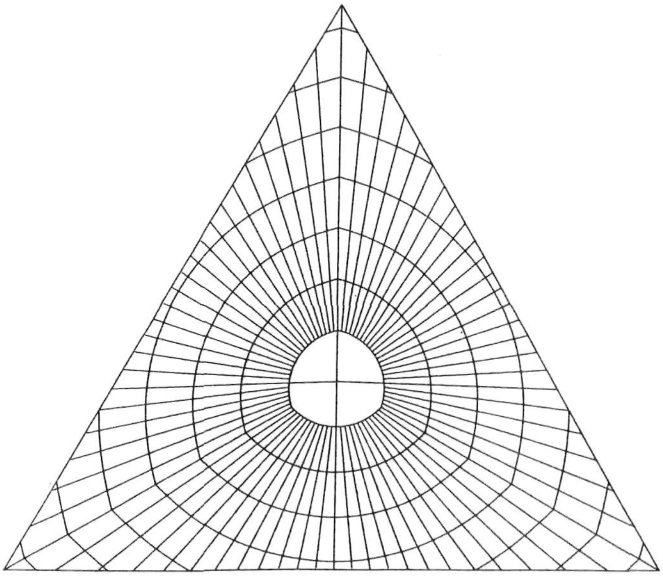

FIGURE 15. *Single face of equal-area icosahedron, pole at center, 5° graticule.*

Distortion on Equal-area Polyhedral Globes

An analysis of the distortion obtained when using the equal-area projection on polyhedral globes leads to values also shown in Table 1. The derivations are given in the appendix to this paper. Of the globes from the five Platonic solids, the dodecahedron, not the icosahedron, is the most distortion-free in the equal-area form, with the maximum angular deformation not exceeding 10.24° and linear scale not varying more than 9.4 percent from the nominal map scale. The maximum angular deformations for the equal-area tetrahedron and icosahedron exceed those of the Gnomonic versions, but the linear scale varies far less. The truncated icosahedron allows this distortion to be reduced so that it does not exceed 3.75° of angular deformation and 3.3 percent of linear scale deviation in the equal-area version (versus 5.09° and 19.5 percent in the Gnomonic form), at the expense of numerous additional interruptions at the edges.

The relative cusp problems and angular deformations are shown in Figures 14–17 with enlarged faces on which a pole has been centered and surrounded with a 5° graticule. Polygons are shown for the three least-distorted equal-area polyhedra of the group discussed. For the two polygons of the truncated icosahedron (Figures 16 and 17), the patterns of lines of constant angular distortion are also shown for one section between vertices. This pattern is repeated for each such section on the same polygon, regardless of geographic center.

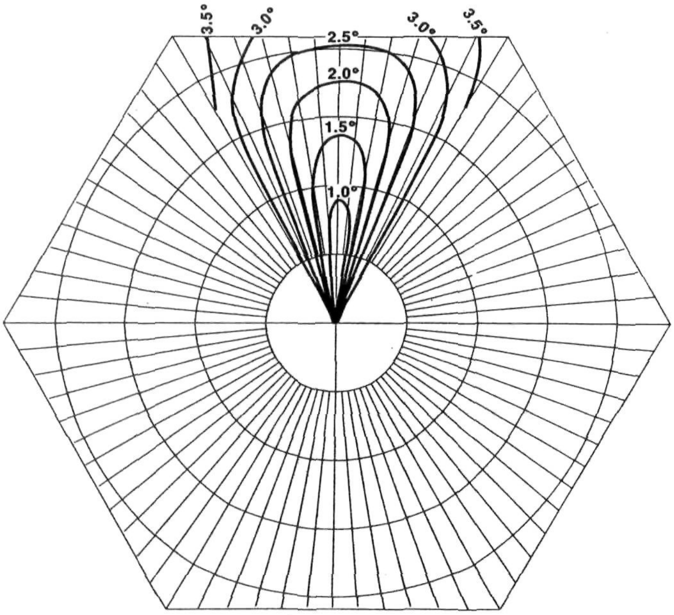

FIGURE 16. *Single hexagon face of equal-area truncated icosahedron, pole at center, 5° graticule, lines of constant maximum angular deformation between two vertices (pattern repeated).*

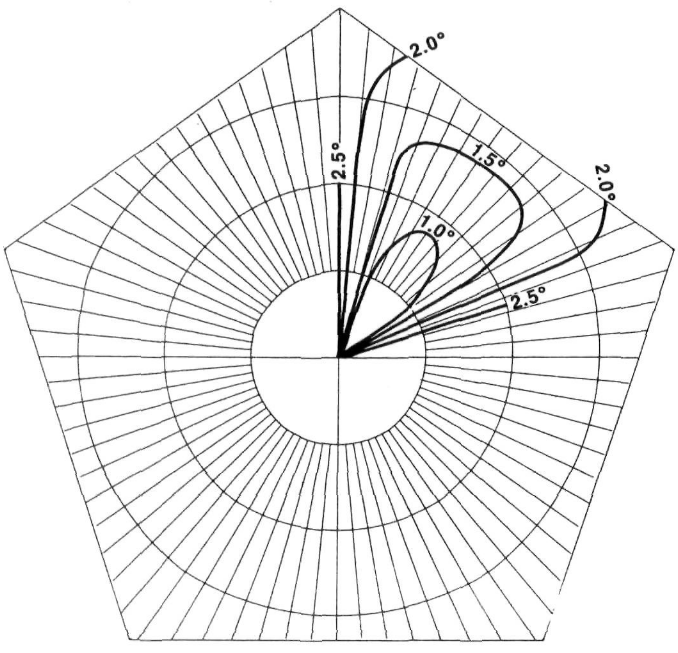

FIGURE 17. *Single pentagon face of equal-area truncated icosahedron, otherwise as Figure 15.*

Summary

Equal-area map projections on polyhedral globes can be prepared most usefully for the dodecahedron and truncated icosahedron with relatively low scale and angular distortion. The equations involved are relatively straightforward, and for certain instructional tools and data bases, the projections are useful for world maps. The interruptions remain a disadvantage, as with any low-error projection system applied to the entire globe.

ACKNOWLEDGEMENTS

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Appendix

Distortion Analysis for the Equal-Area Projection on Polyhedral Globes

The general formulas for the scale factors h along meridians of longitude λ and k along parallels of latitude ϕ are

$$h = (1/R)[(\partial x/\partial\phi)^2 + (\partial y/\partial\phi)^2]^{1/2} \tag{A1}$$

$$k = (1/R\cos\phi)[\partial x/\partial\lambda)^2 + (\partial y/\partial\lambda)^2]^{1/2} \tag{A2}$$

To adapt them to the polar coordinates (Az, z) of a hexagonal face initially azimuthal, calling the scale factor h' along a straight line radiating from the center and k' in a direction perpendicular to it, requires:

$$h' = (1/R)[(\partial x/\partial z)^2 + (\partial y/\partial z)^2]^{1/2} \tag{A3}$$

$$k' = (1/R\sin z)[(\partial x/\partial Az)^2 + (\partial y/\partial Az)^2]^{1/2} \tag{A4}$$

From (15) and (16),

$$\partial x/\partial z = \sin Az'\ (\partial\rho/\partial z) \tag{A5}$$

$$\partial x/\partial Az = \rho\cos Az'(\partial Az'/\partial Az) + \sin Az'\ (\partial\rho/\partial Az) \tag{A6}$$

$$\partial y/\partial z = \cos Az'(\partial\rho/\partial z) \tag{A7}$$

$$\partial y/\partial Az = -\rho\sin Az'(\partial Az'/\partial Az) + \cos Az'\ (\partial\rho/\partial Az) \tag{A8}$$

Differentiating (7) and (8) with respect to Az and rearranging,

$$dAz'/dAz = 2\,R^2\,(1 + dH/dAz)\cos^2 Az'/R'^2\tan^2 g \times (1 - 2A_G\cot\theta/R'^2\tan^2 g)^2 \tag{A9}$$

From (11) and (10), however,

$$f = \tan g/[2\cos Az'(1 + \cot\theta\tan Az')\sin(q/2)]$$

Substituting from (8) and rearranging,

$$f = \tan g\,(1 - 2\,A_G\cot\theta/R'^2\tan^2 g)/2\cos Az'\sin(q/2) \tag{A10}$$

Squaring (A10) and substituting it into (A9), and because $\sin^2 (q/2) = (1 - \cos q)/2$,

$$dAz'/dAz = R^2 (1 + dH/dAz)/R'^2 f^2 (1 - \cos q) \quad \text{(A11)}$$

From the spherical triangle $A'B'C'$ in Figure 3,

$$\cos G = \cos q \sin Az \sin H - \cos Az \cos H$$

Solving for $\cos q$, substituting into (A11), and differentiating (6) for dH/dAz, then substituting from (6) for the $\cos H$ term, equation (A11) simplifies to

$$dAz'/dAz = (R/R'f)^2 \quad \text{(A12)}$$

From (12),

$$\partial\rho/\partial z = R' f \cos (z/2) \quad \text{(A13)}$$

$$\partial\rho/\partial Az = 2 R' \sin (z/2)(df/dAz) \quad \text{(A14)}$$

The derivation of the last term of (A14), using (9)–(11) and a few simplifying steps, becomes

$$df/dAz = (2 R^2/R'^2 \tan g) \sin (q/2)(\sin Az' - \cot \theta \cos Az') - [f/2 \tan (q/2)](\sin^2 q/\tan g) \times (\sin Az - \cot \theta \cos Az) \quad \text{(A15)}$$

The equation for the equal-area condition on the sphere can be written

$$(\partial x/\partial Az)(\partial y/\partial z) - (\partial x/\partial z)(\partial y/\partial Az) = R^2 \sin z \quad \text{(A16)}$$

By substituting into (A16) from (A5)–(A8), (12), (A12), and (A13), this equation is found to be satisfied.

Equations (A3) and (A4) can also be simplified. From (A3), (A5), and (A7),

$$h' = \partial\rho/R\, \partial z \quad \text{(A17)}$$

From (A4), (A6), and (A8),

$$k' = [(\rho dAz'/dAz)^2 + (\partial\rho/\partial Az)^2]^{1/2}/R \sin z \quad \text{(A18)}$$

These values can be converted to maximum and minimum scale factors a and b, respectively, and maximum angular deformation ω, using intermediate variables a' and b' as follows for an equal-area map projection only:

$$a' = (h'^2 + k'^2 + 2)^{1/2} \quad \text{(A19)}$$

$$b' = (h'^2 + k'^2 - 2)^{1/2} \quad \text{(A20)}$$

$$a = (a' + b')/2 \quad \text{(A21)}$$

$$b = 1/a \quad \text{(A22)}$$

$$\sin (\omega/2) = b'/a' \quad \text{(A23)}$$

The equations in order of use for finding a, b, and ω, after steps (1)–(3) of the forward calculations, are (5)–(12), (A12), (A13), (A15), (A14), and (A17)–(A23).

For the pentagon faces, an analogous derivation produces the above equations, but with R' changed to R''; "2" replaced with "$2K$" both places in (A9) and in only the numerator of (A10); and the right-hand sides of (A11), (A12), and (A16) multiplied by K; as well as the use of different values for constants g, G, and θ. For the Platonic polyhedra, substitution of the values in Table 1 into equations (A3)–(A23), as well as in the earlier equations, suffices.

Résumé On a proposé de nombreuses formes polyédriques comme approximation de globes et la projection gnomonique est la plus souvent utilisée, avec de considérables distortions d'échelles et de superficies. On a conçu des projections conformes compliquées, mais on n'a utilisé qu'une seule fois une projection équivalente pour l'icosaèdre. La projection équivalente azimutale de Lambert peut être modifiée pour permettre un ajustement exact, parfaitement équivalent pour tout globe polyédrique doté de polygones réguliers, mais elle est plus que satisfaisante pour le dodécaèdre à douze pentagones et pour l'icosaèdre tronqué à vingt hexagones et douze pentagones. Dans l'application de l'icosaèdre tronqué, la déformation angulaire ne dépasse pas 3.75° et la variation d'échelle est inférieure à 3.3 pour cent. Ces avantages s'obtiennent aux dépens d'une augmentation des interruptions aux limites des polygones, lorsque l'on déplie le globe polyédrique.

Zusammenfassung Man hat zahlreiche Polyederformen als Näherung für die Erdkugel vorgeschlagen. Die am meisten verwendete Abbildung ist die gnomonische Projektion, die beträchtliche Maßstabs- und Flächenverzerrungen aufweist. Man hat komplizierte konforme Projektionen entwickelt, aber eine flächentreue Abbildung wurde nur einmal, für den Ikosaeder, verwendet. Lamberts flächentreue Azimutalabbildung kann man modifizieren, um eine genau passende, perfekt flächentreue Projektion für jeden Polyederglobus zu erhalten, der reguläre Polygone besitzt. sie ist jedoch am besten geeignet für ein Dodekaeder mit zwölf Fünfecken und für ein abgestumpftes Ikosaeder mit zwanzig Sechsecken und zwölf Fünfecken. Bei der Anwendung auf einem abgestumpften Ikosaeder bleibt die Winkelverzerrung unterhalb 3.75°, und die Maßstabsänderung ist kleiner als 3.3%. Diese Vorteile gehen auf Kosten von vermehrten Unterbrechungen an den Polygonkanten, weenn der Polygonglobus entwickelt wird.