Introduction to Lattice Based Cryptography

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Outline

- What is a lattice?
- Lattices in practice.
- Examples of hard problems on lattices.
- (Known) Algorithms for solving hard problems on lattices.
- (Maybe) NTRU cryptosystem.

Motivation - Post-Quantum Crypto

source: SafeCrypto Project

Overview of lattice-based constructions

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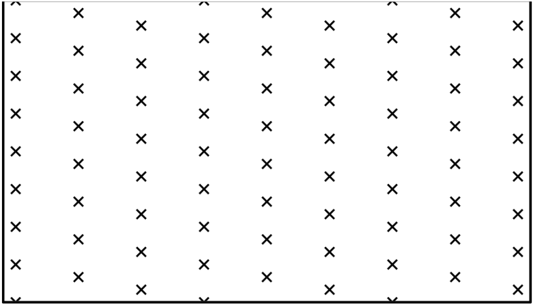
- Strong security proofs but not so fast (Learning with Errors).
- Searching for a solution from both worlds (Ring learning with Errors).

What is a lattice? v.1

Short Answer: A grid.



Lattice in R^2

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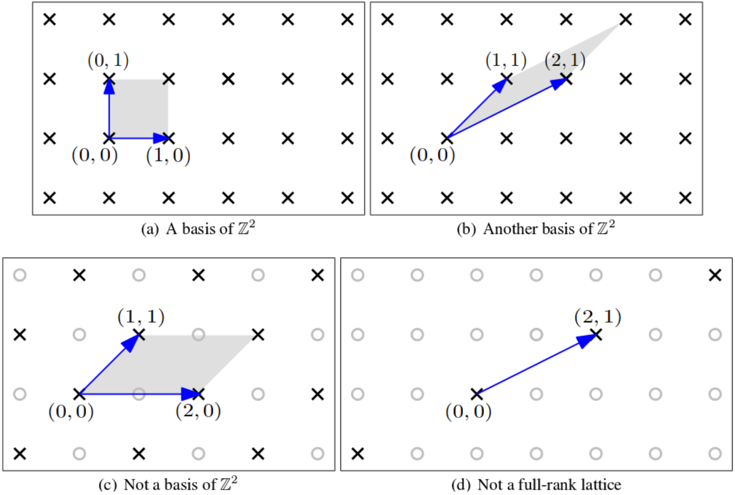
- We call $\mathbf{b}_1, \mathbf{b}_2, \ldots \mathbf{b}_n$ the basis of $\mathcal{L}$.
- Rewrite the definition as $\mathcal{L} = Bx$ where B has n columns: $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$.

Lattice Basis



(a) A basis of $\mathbb{Z}^2$

(b) Another basis of $\mathbb{Z}^2$

(c) Not a basis of $\mathbb{Z}^2$

(d) Not a full-rank lattice

Different bases - Source: Regev course

Lattice Basis

Fact

$\mathcal{L}(B) = \mathcal{L}(B')$ if and only if there exists an unimodular integer matrix $U \in \mathbb{Z}^{nxn}$ such that $B = B'U$.

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- Determinant of a lattice is inverse proportional to its density.

Succesive minima

Shortest Vector

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Upper bounds for $\lambda_1(\mathcal{L})$

For any lattice of dimension n, $\lambda_1(\mathcal{L}) \leq \sqrt{n}(det(\mathcal{L}))^{1/n}$.

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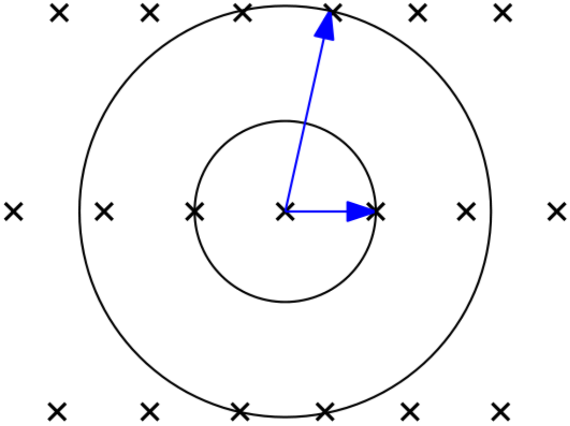
For any lattice of dimension n, $\lambda_1(\mathcal{L}) \leq \sqrt{n}(det(\mathcal{L}))^{1/n}$.

- Unfortunately, no constructive proof.
- Also, a loose bound. Think about the lattice generated in $\mathbb{R}^2$ by
$$\begin{bmatrix} 0 & \epsilon \\ 1/\epsilon & 0 \end{bmatrix}$$

Succesive minima



$\lambda_1(\mathcal{L}), \lambda_2(\mathcal{L})$ - Source: Regev course

Dream world: Gram-Schmidt for Lattices

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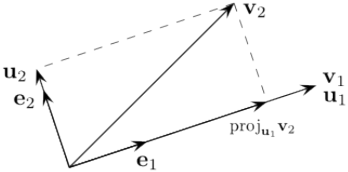
- What?

Gram-Schmidt Ortogonalization



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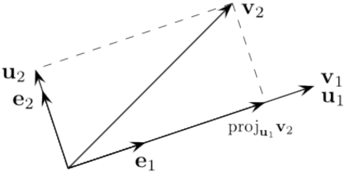

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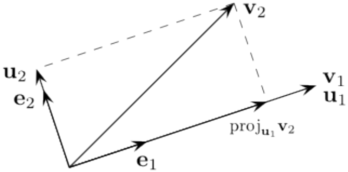

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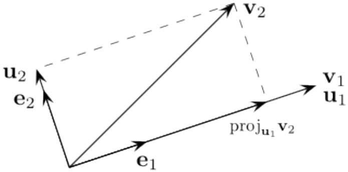

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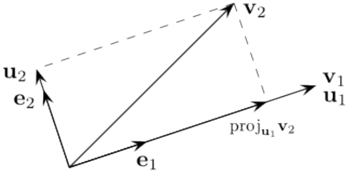

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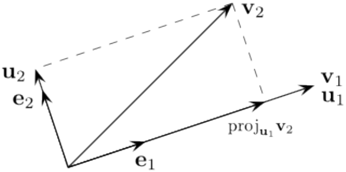

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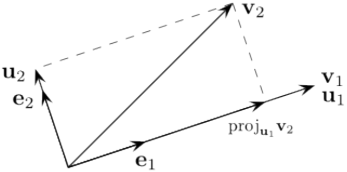

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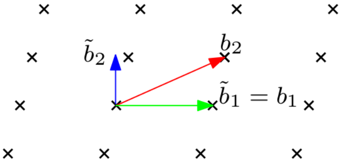
- Cool! Now plug-in a lattice and find an orthogonal basis! What is wrong with this approach?

Gram-Schmidt for Lattices - LLL Reduction

By changing the basis, we change the spanned lattice.



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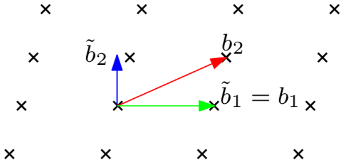

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Solution: Round the projection to the nearest integer!

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$\frac{3}{4}-$LLL Reduced basis

A basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$ is called LLL-reduced if and only if:

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Why these conditions?

1. Used to prove that the algorithm runs in polynomial time.
2. The vector b_{i+1} is not too shorter that b_i.

Gram-Schmidt for Lattices - LLL Reduction

LLL Reduction

Input: Basis $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\}$
Output: $\frac{3}{4}$-LLL reduced basis.

Algorithm 1 LLL Algorithm

```
Reduction Step:
for $i = 1$ to $N$ do
    for $j = i - 1$ to 1 do
        $b_i = b_i - \lfloor c_{i,j} \rceil b_j,\ c_{i,j} = \frac{\langle \mathbf{b}_i, \tilde{\mathbf{b}}_j \rangle}{\langle \tilde{\mathbf{b}}_j, \tilde{\mathbf{b}}_j \rangle}$
    end for
    Swap Step:
    if $\exists i$ s.t. $\frac{3}{4} \left\| \tilde{b}_i \right\|^2 > \left\| \mu_{i+1,i} \tilde{b}_i + \tilde{b}_{i+1} \right\|^2$ then
        Swap $b_i, b_{i+1}$; goto Reduction Step
    end if
end for
```

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Hard problems in crypto

Cryptography requires that underlying problems are hard to solve on average, i.e. from a specific distribution

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Interesting fact about LLL - in practice

It can approximate the shortest vector by a smaller factor 1.022^n - still exponential.
In 2011 Chen and Nguyen showed that in practice you can approximate the shortest vector by 1.005^n with a variant of LLL.

Example of Hard Problems based on Lattices

Shortest Vector Problem (*SVP*)

Given an arbitrary lattice basis **B** of a n dimensional lattice $\mathcal{L}$ output a shortest non-zero lattice vector, $v \in \mathcal{L} - \{0\}$ for which $\|v\| = \lambda_1(\mathcal{L})$.

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Approximate Shortest Vector Problem (SVP_γ)

Given an arbitrary lattice basis **B** of a n dimensional lattice $\mathcal{L}$ output a shortest non-zero lattice vector bounded by a polyonimal function in n, i.e. $\|v\| \leq \gamma(n)\lambda_1(\mathcal{L})$.

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Approximate Decisional SVP ($GapSVP_\gamma$)

Given a lattice basis $\mathcal{B}$ of n dimensional lattice $\mathcal{L}$ for which $\lambda_1(\mathcal{L}) \leq 1$ or $\lambda_1(\mathcal{L}) > \gamma(n)$ decide which is the case.

Example of Hard Problems based on Lattices

Approximate Bounded Distance Decoding (BDD_γ)

Given a lattice basis $\mathcal{B}$ and a vector $t \in \mathbf{R}^n$ find the unique vector $v \in \mathcal{L}$ s.t. $dist(v, t) \ \leq \lambda_1(\mathcal{L})/2\gamma(n)$.

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Closest Vector Problem *CVP*

Currently there isn't any cryptosystem based on *CVP* - maybe because it's just too hard.

Not More Problems...

Membership Problem

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Equivalence Problem

Given 2 lattice bases $\mathbf{B}_1 \in \mathbb{R}^{nxn}$, $\mathbf{B}_2 \in \mathbb{R}^{nxn}$ decide if $\mathcal{L}(\mathbf{B}_1) = \mathcal{L}(\mathbf{B}_2)$.

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Something Wrong?

Those are easy problems!

Ideals in Rings look alike lattices

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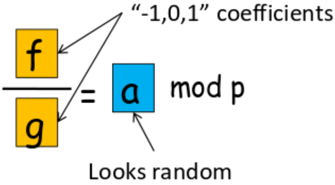
- Polynomial Ring in $Z_p[X]/(x^n + 1)$.
- Elements are polynomials of degree $n - 1$ with coefficients in range $[-(p-1)/2, (p-1)/2]$. Just think about n dimensional vectors with values in Z_p.

NTRU cryptosystem

Next slides with the NTRU cryptosystem belong to Vadim Lyubashevki.

NTRU Cryptosystem

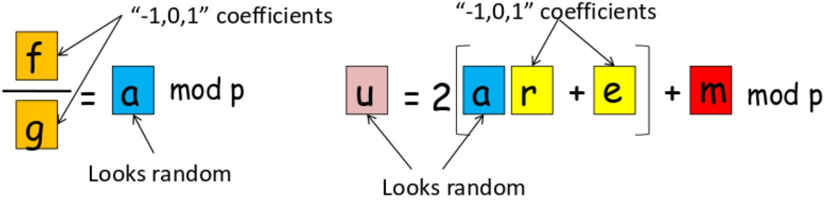


$$\frac{f}{g} = a \mod p$$

“-1,0,1” coefficients

Looks random

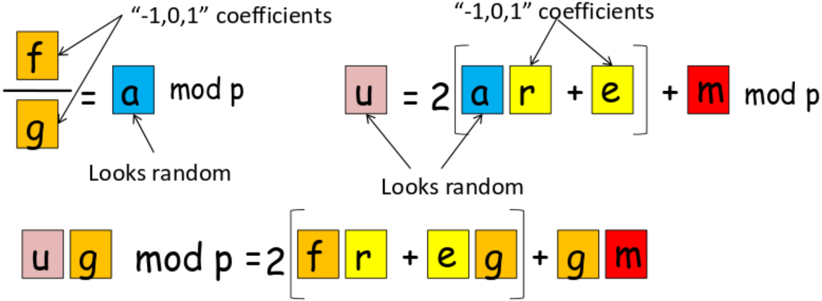
NTRU Cryptosystem



$$\frac{f}{g} = a \mod p$$

$$u = 2\left[a\,r + e\right] + m \mod p$$

NTRU Cryptosystem



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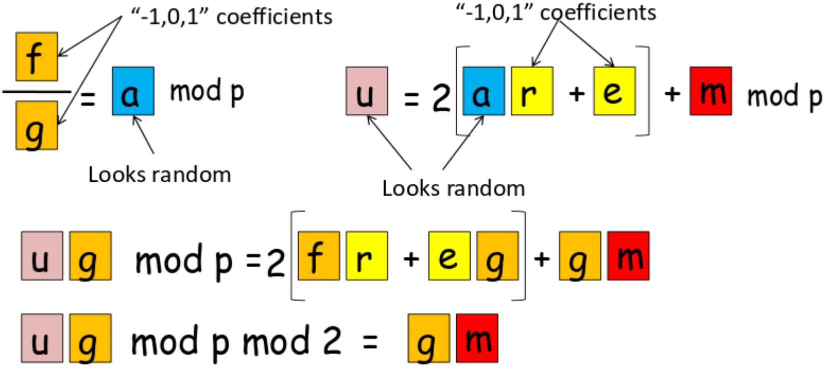

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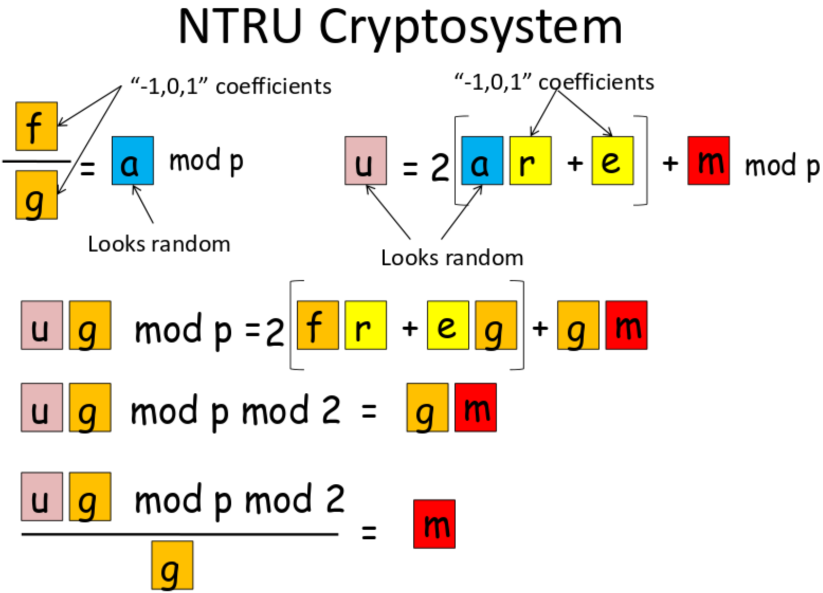
$$u\, g \mod p = 2\left[f\, r + e\, g\right] + g\, m$$

$$u\, g \mod p \mod 2 = g\, m$$



NTRU Cryptosystem

“-1,0,1” coefficients

$$\frac{f}{g} = a \bmod p$$

Looks random

“-1,0,1” coefficients

$$u = 2\left[a\,r + e\right] + m \bmod p$$

Looks random

$$u\,g \bmod p = 2\left[f\,r + e\,g\right] + g\,m$$

$$u\,g \bmod p \bmod 2 = g\,m$$

$$\frac{u\,g \bmod p \bmod 2}{g} = m$$

Facts about NTRU

Security proofs

Until 2011 there was no proof of NTRU security. The proof is based on the hardness of Ring-LWE distribution.

Thank you!