Simplex Algorithm

No Author Given

No Institute Given

1 Diet problem

The diet problem is a classical application for describing a standard minimum problem in linear programming.

There are m types of food $F_1, \ldots, F_m$ and n types of nutrients $N_1, \ldots, N_n$. Let b_i be the price of food F_i per unit and c_j the minimal requirement for nutrient N_j. We are also given a matrix a_{ij} which maps the quantity of nutrient N_j in food type F_i.

Our goal is to find $y_1, \ldots, y_m$ such that $\sum_{i=1}^{m} y_i b_i$ subject to $y \geq 0$ and $\sum_{i=1}^{m} y_i a_{ji} \geq c_j$ for every $j \in 1, \ldots, n$.

2 Formalised versions of LP problems

2.1 Standard maximum problem

Given a matrix $A \in \mathbb{R}^{m \times n}$, a vector $b \in \mathbb{R}^m$ and a vector $c \in \mathbb{R}^n$. Find a vector $x \in \mathbb{R}^n$ such that $c^T x$ is maximum subject to: $Ax \geq b$, $x \geq 0$.

2.2 Standard minimum problem

Given a matrix $A \in \mathbb{R}^{m \times n}$, a vector $b \in \mathbb{R}^m$ and a vector $c \in \mathbb{R}^n$. Find a vector $y \in \mathbb{R}^m$ such that $y^T b$ is minimum subject to: $y^T A \geq c^T$, $y \geq 0$.

3 Duality theorem

The two problems from above are dual to each other and they are connected through the following result:

Theorem 1. *If a standard LP program has a boundfed feasible solution then it's optimal feasible solution is equal to the optimal feasible solution of it's dual LP.*

Pretty neat. We can use this theorem (among other things) to check our candidate solution for a standard problem is optimal by testing it through the dual. This is due to the following result:

Theorem 2. *Let x^* be a feasible solution for the standard maximum LP problem and y^* be a feasible solution for it's dual. x^* and y^* are optimal feasible iff:*

– $x_j^* = 0$ *for all* j *which* $\sum_{i=1}^{m} a_{ij} y_i^* > c_j$
– $y_i^* = 0$ *for all* i *which* $\sum_{j=1}^{n} a_{ij} x_j^* < b_i$

Proof. Let's prove the only if part:

Following from the first and the the second equation we rewrite $\sum_{i=1}^{m} y_i^* b_i$ as $\sum_{i=1}^{m} y_i^* \sum_{j=1}^{n} a_{ij} x_j^* = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} y_i^* x_j^* = \sum_{j=1}^{n} c_j x_j^*$.

4 Solving the maximum standard problem with the simplex tableau

First we introduce a slack variable $u = b - Ax$.

The original problem now becomes:

$$\begin{aligned} &\max(c^T x) \text{ subject to} \\ &u \geq 0 \text{ and } x \geq 0 \end{aligned}$$

Now we write the corresponding simplex tableau:

$$\begin{array}{c} \\ -u_1 \\ -u_2 \\ -u_3 \\ c \end{array} \left[\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & b \\ \hline 0 & 0 & 1.6 & 2.2 & 0 & b_1 \\ 1 & 1 & 0.4 & -0.2 & 0 & b_2 \\ 0 & 1 & -0.2 & 0.6 & 0 & b_m \\ \hline -c_1 & -c_2 & -c_3 & -c_4 & -c_n & 0 \end{array} \right] \quad (1)$$

This method is very similar to the gaussian elimination procedure.

5 Solve general problems using Simplex Method

6 Different approaches to solving LP problems