

## Informative hypotheses evaluation

Information criteria

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## ANOVA Example: Comparisons of 3 Means

Example Palmer and Gough (P&G)

Palmer and Gough (2007) examined the difference in the attribution of importance of defective education as an explanation for criminal behavior between three types of "offenders":

(1) non-offenders, (2) property offenders, and (3) person offenders.

#### **ANOVA**

$$y_j = \mu_1 d_{1j} + \mu_2 d_{2j} + \mu_3 d_{2j} + \epsilon_j,$$

where  $\mu_i$  (i=1,2,3) is the mean in group i,  $d_{ij}$  is a dummy variable denoting group membership,  $\epsilon_j$  is an error term and  $\epsilon_j \sim N(0,\sigma^2)$ .

$$(n_i = 20, 20, 31)$$

## Hypotheses of interest

Example Palmer and Gough (P&G)

#### Palmer and Gough (2007) expect that

- non-offenders (1) attribute more importance to defective education for explaining crime than the other two offenders (2 & 3): i.e.,  $\mu_1 > \mu_2$  and  $\mu_1 > \mu_3$ ,
- property offenders (2) attribute more importance to defective education than person offenders (3):
   i.e., μ<sub>2</sub> > μ<sub>3</sub>.

This leads to the theory-based hypothesis:

$$H_1: \mu_1 > \mu_2 > \mu_3.$$

Note that '<" denotes "smaller than" and ">" denotes "larger than".









## ANOVA Example: Comparisons of 3 Means

Simple, hypothetical example

Examine the difference in happiness between three types of "treatments":

(1) new treatment, (2) current treatment, and (3) no treatment.

Theory-based hypothesis:

$$H_1: \mu_1 > \mu_2 > \mu_3,$$

where ">" denotes "larger than".







## Testing the null hypothesis

Example 3 means

Test  $H_0$  with ANOVA F test:

$$H_0: \qquad \mu_1 = \mu_2 = \mu_3,$$

 $H_a$ : not  $H_0$ .

Then, reject or not-reject ('accept')  $H_0$ .

But, often not interested in  $H_0!$ 

Cannot say anything about  $H_1: \mu_1 > \mu_2 > \mu_3$ .











# Conclusions w.r.t. hypothesis of interest $H_1: \mu_1 > \mu_2 > \mu_3$ Example 3 means

With post-hoc test (e.g., Bonferroni) and on basis of the sample means, we could say a bit more.

But, what if

- $\mu_1 \neq \mu_2$ ,  $\mu_1 \neq \mu_3$ ,  $\mu_2 = \mu_3$ ,
- $\overline{y}_1 > \overline{y}_2$ ,

which implies  $\mu_1 > \mu_2 = \mu_3$ . But also not interested in this.

Furthermore, inconsistent results are possible (like in P&G example):  $\mu_1 \neq \mu_3$ ,  $\mu_1 = \mu_2$ , and  $\mu_2 = \mu_3$ , which is logically impossible, since latter two imply  $\mu_1 = \mu_3$ . Using planned contrasts would overcome this, but still = restrictions then.

## **NHST**

#### Example Palmer and Gough

```
PandG_data <- read.table("Data_PalmerAndGough.txt",</pre>
                          header=TRUE)
PandG_data$group <- factor(PandG_data$group)</pre>
pairwise.t.test(PandG_data$Importance, PandG_data$group,
                p.adj = 'bonferroni')
##
##
    Pairwise comparisons using t tests with pooled SD
##
   data: PandG_data$Importance and PandG_data$group
##
##
## 2 0.191 -
##
   3 0.011 1.000
##
## P value adjustment method: bonferroni
```

## **Exploration and Confirmation**

### Exploration (like in post hoc tests and possibly AIC)

Evaluate all possible pairs/subsets of means whether significant different (",") or not ("=").

For example, when k = 3:

$$H_{0E}: \mu_1 = \mu_2 = \mu_3$$
  
 $H_{1E}: \mu_1 = \mu_2, \mu_3$   
 $H_{2E}: \mu_1, \mu_2 = \mu_3$   
 $H_{3E}: \mu_1 = \mu_3, \mu_2$   
 $H_{a}: \mu_1, \mu_2, \mu_3$ 

When k = 5, there are even 52 hypotheses.

#### Confirmation

Limited set: Evaluate only prespecified hypotheses including order restrictions (<, >, but also =).



## Confirmatory methods

Most researchers are able to specify "order-restricted" / "informative" / "theory-based" hypotheses, like  $H_1: \mu_1 > \mu_2 > \mu_3$ . Use prior knowledge and/or expertise in hypothesis.

### Methods to evaluate theory-based hypotheses

- Hypothesis testing: Fbar  $(\bar{F})$  test (renders p-value and can test only one theory-based hypothesis)
- Confirmatory model selection using information criteria: GORIC and GORICA
- (Confirmatory) Bayesian model selection (BMS)

Note: 'model' refers to hypothesis.

### **GORIC**

#### Example Palmer and Gough, that is, 3 means

 $H_0: \mu_1 = \mu_2 = \mu_3,$ 

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

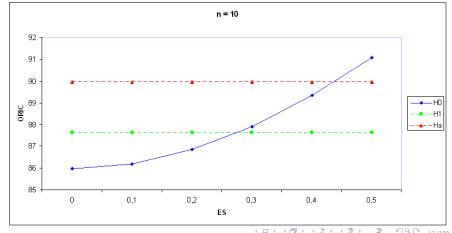
 $H_u: \mu_1, \ \mu_2, \ \mu_3.$ 

#### **GORIC**

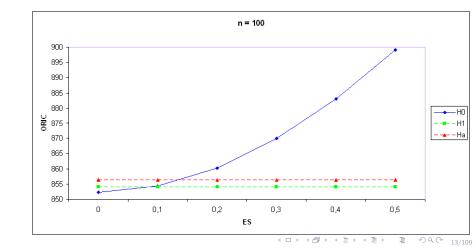
Model	Fit	Complexity	GORIC	GORIC weights
$H_0$	-196.36	2.00	396.71	0.02
$H_1$	-191.89	2.81	389.41	0.75
Hu	-191.89	4.00	391.79	0.23

Confirmatory methods (e.g., GORIC) have more "power" than their exploratory counterparts (e.g., AIC; cf. one- vs two-sided testing), (a, b) = (a, b) + (a,

# Confirmation more power: 1 data set. GORIC values for 3 groups, effect size ES, and n = 10 observations per group



# Confirmation more power: 1 data set. GORIC values for 3 groups, effect size ES, and $\mathbf{n}=\mathbf{100}$ observations per group



## Confirmatory methods - some of my references

- Kuiper, R. M., and Hoijtink, H. (2010). Comparisons of Means Using Exploratory and Confirmatory Approaches. *Psychological Methods*, 15(1), 69–86.
- Kuiper, R. M., Klugkist, I., and Hoijtink, H. (2010). A Fortran 90 Program for Confirmatory Analysis of Variance. *Journal of Statistical Software*, 34(8), 1–31.
- Kuiper, R.M., Hoijtink, H. and Silvapulle, M.J. (2011). An Akaike type information criterion for model selection under inequality constraints. *Biometrika*, 98, 495-501. (GORIC)
- Kuiper, R.M., Nederhof, T., and Klugkist, I. (2015). Properties of hypothesis testing techniques and (Bayesian) model selection for exploration-based and theory-based (order-restricted) hypotheses. *British Journal of Mathematical and Statistical Psychology*, 68(2), 220 245.
- Altınışık, Y., Van Lissa, C. J., Hoijtink, H., Oldehinkel, A. J., and Kuiper, R. M. (2021). Evaluation of inequality constrained hypotheses using a generalization of the AIC. *Psychological Methods*, 26(5), 599–621. https://doi.org/10.1037/met0000406 (GORICA)

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GORIC(A) in JASF

## Information criteria (ICs)

IC, like AIC and GORIC, balances fit and complexity.

Describe data as good as possible (fit) with fewest number of parameters (simplicity / non-complexity).

### Generalized Order-Restricted Information Criterion

#### **GORIC**

'IC' = -2 fit + 2 complexity

#### Fit = Maximized order-restricted log likelihood

Maximized log likelihood based on parameters in agreement with  $H_m$ .

#### Complexity = Penalty

Represents: Expected number of distinct/free parameters.

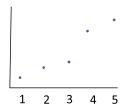
Here, expected number of distinct mean values plus 1 (because of the unknown variance term).

Details: Function of level probabilities.

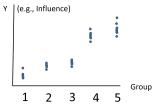
# Intermezzo: Balance Fit and Complexity (1/6)

Example 5 means

Data for 5 groups – Oversimplified representation



Data for 5 groups - Better representation (scatter of data points)



# Intermezzo: Balance Fit and Complexity (2/6) Example 5 means

# parameters: 1 mean parameter (i.e.,  $\mu$ ) Fit: Bad fit

### Intermezzo: Balance Fit and Complexity (3/6)Example 5 means

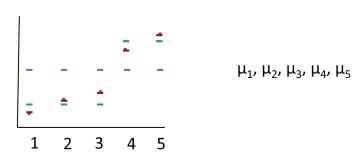
$$\mu_{1} = \mu_{2} = \mu_{3} = \mu_{low}$$

$$\mu_{4} = \mu_{5} = \mu_{high}$$
1 2 3 4 5

 $\mu_4 = \mu_5 = \mu_{high}$ 

2 mean parameters (i.e.,  $\mu_{low}$  and  $\mu_{high}$ ) # parameters: Better than with one parameter, even good fit. Fit:

# Intermezzo: Balance Fit and Complexity (4/6) Example 5 means



# parameters: 5 mean parameter (complex)

Fit: Better than with two parameters, even best possible fit.

Thus: Best fit (= highest likelihood), but also most complex (= highest penalty).

# Intermezzo: Balance Fit and Complexity (5/6) Example 5 means

$$\mu_{1} = \mu_{2} = \mu_{3} = \mu_{4} = \mu_{5} = \mu$$

$$\mu_{1} = \mu_{2} = \mu_{3} = \mu_{4} = \mu_{5} = \mu$$

$$\mu_{1} = \mu_{2} = \mu_{3} = \mu_{low} \&$$

$$\mu_{4} = \mu_{5} = \mu_{high}$$

$$\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}$$

$$\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}$$

In this example: Two means may be best trade-off between fit & complexity.

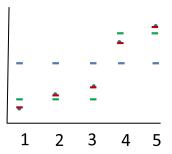
# Intermezzo: Balance Fit and Complexity (6a/6) Order-restrictions

Previous slides demonstrated balancing fit and complexity for equality ('=') restrictions.

For inequality/order ('<', '>', and possible some '=') restrictions, the calculation of the fit and complexity is more complex :-), but the idea of balancing is the same.

# Intermezzo: Balance Fit and Complexity (6b/6)

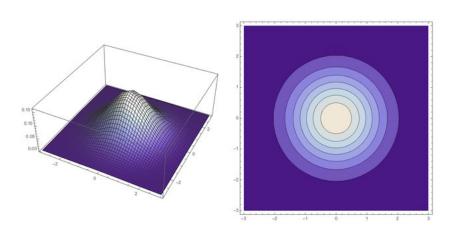
Order-restrictions: From "Example 5 means' to 'Example 2 means'



In theory-based hypotheses, we also incorporate **order-restrictions** (e.g.,  $\mu_1 < \mu_2$ ). Then, helpful to look at likelihood using **contour plots** (not scatter plot, as done here).

## Intermezzo: Contour plot (1/2)

Example 2 means

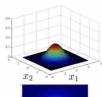


## Intermezzo: Contour plot (2/2)

Example 2 means

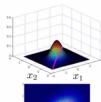
## Multivariate Gaussian (Normal) examples

$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



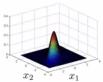


$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Sigma = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$$





$$\mu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Sigma = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 1 \end{bmatrix}$$

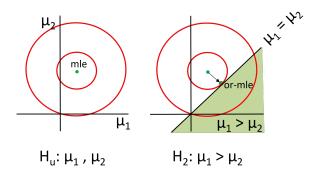




Andrew

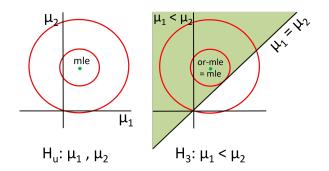


# Idea fit: order-restricted maximum likelihood (or-ml) Example 2 means



mle not in  $H_2$ : Find highest likelihood in allowable (= green) space. The resulting estimated mean  $\hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2)$  is referred to as or-mle.

# Note: if in agreement, or-mle = mle Example 2 means



mle in  $H_3$ : highest likelihood is in allowable (= green) space. Hence, or-mle = mle.

### Idea fit

#### Example Palmer and Gough, that is, 3 means

 $H_0: \qquad \mu_1 = \mu_2 = \mu_3,$ 

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_2: \qquad \mu_1 > \mu_2 < \mu_3,$ 

 $H_3: \mu_1 < \mu_2 < \mu_3,$ 

 $H_u: \mu_1, \mu_2, \mu_3.$ 

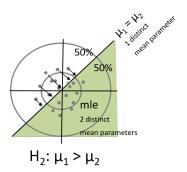
## group1 group2 group3 ## 11.95 9.75 8.77

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_0$	-196.36	2.00	396.71	0.01
$H_1$	-191.89	2.81	389.41	0.56
$H_2$	-192.34	3.19	391.05	0.25
$H_3$	-196.36	2.81	398.34	0.01
$H_u$	-191.89	4.00	391.79	0.17

## Idea complexity: penalty (PT)

Example 2 means

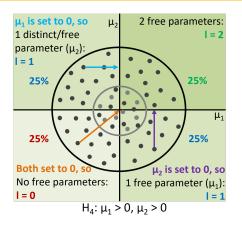


Note: sampling from distribution with  $\mu_1=\mu_2=0$ . complexity  $H_2=PT_2=1+0.5\times 1+0.5\times 2=2.5$ . PT = number of expected distinct/free parameters.



## Extra: Levels and level probabilities

Example 2 means



Level (I): number of distinct/free parameters. Level probability ( $LP_I$ ): probability that there are I levels.

$$PT_4 = 1 + 0.25 \times 0 + (0.25 + 0.25) \times 1 + 0.25 \times 2 = 2.2 \times 2 = 2$$













## Idea complexity: loose interpretation

Example Palmer and Gough, that is, 3 means

$$H_1: \mu_1 > \mu_2 > \mu_3$$

contains 1 ordering of three means, 1-2-3. Thus, not complex (i.e., parsimonious).

$$H_2: \mu_1 > (\mu_2, \mu_3)$$

contains 2 orderings of three means: 1-2-3 and 1-3-2. Thus, more complex (less parsimonious).

$$H_u: \mu_1, \mu_2, \mu_3$$

contains all six possible orderings of three means. Thus, is most complex one (least parsimonious).

## Idea complexity

Example Palmer and Gough, that is, 3 means

 $H_0:$   $\mu_1 = \mu_2 = \mu_3,$   $H_1:$   $\mu_1 > \mu_2 > \mu_3,$   $H_2:$   $\mu_1 > \mu_2 < \mu_3,$   $H_3:$   $\mu_1 < \mu_2 < \mu_3,$  $H_{u}:$   $\mu_1, \mu_2, \mu_3.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_0$	-196.36	2.00	396.71	0.01
$H_1$	-191.89	2.81	389.41	0.56
$H_2$	-192.34	3.19	391.05	0.25
$H_3$	-196.36	2.81	398.34	0.01
$H_u$	-191.89	4.00	391.79	0.17

#### **GORIC**

$$IC_m = -2 \ fit_m + 2 \ complexity_m$$

#### Broad type of restrictions

More or less: any linear restriction.

e.g., the interaction  $H_1: \mu_1 - \mu_2 < \mu_3 - \mu_4$ .

#### Note

If no inequalities (< and/or>), then (G)ORIC = AIC.

#### Reference:

Kuiper, R.M., Hoijtink, H. and Silvapulle, M.J. (2011). An Akaike type information criterion for model selection under inequality constraints. *Biometrika*, *98*, 495-501.

## GORIC: Lowest value is best

GORIC is like AIC expected distance from the truth (KL-distance).

Or: One wants the smallest 'misfit + complexity'.

Note: GORIC = -2 fit + 2 complexity = 2 misfit + 2 complexity

Hence, smallest value is best.

### **GORIC**

#### Example Palmer and Gough

 $H_0$ :  $\mu_1 = \mu_2 = \mu_3$  (include only when of interest)

 $H_1: \mu_1 > \mu_2 > \mu_3$ 

 $H_2$ :  $\mu_2 > \mu_1 > \mu_3$  (if competing theory/hypothesis)

 $H_u: \mu_1, \mu_2, \mu_3.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC
$H_0$	-196.36	2.00	396.71
$H_1$	-191.89	2.81	389.41
$H_2$	-193.70	2.81	393.03
Hu	-191.89	4.00	391.79

# Hands-on/Demo (1a): GORIC

#### Let's practice.

- Go to https://github.com/rebeccakuiper/Tutorials:
  - 1. Click on green button called Code.
  - 2. Download zip (last option in list).
  - 3. Unzip it on your machine (that folder is now your working dir.).
- Start Rstudio. Optional: make project.
- Open 'Tutorial\_GORIC\_restriktor\_ANOVA.html',
  - 'PalmerAndGough\_and\_Lucas.R', and/or

'Hands-on\_1\_GORIC\_Unc\_ANOVA\_restriktor.R' ('Hands-on files').

- Install packages and load them.
- Read and inspect data.

Use **Data\_PalmerAndGough.txt** and/or Data\_Lucas.txt.

- Run model (lm()).
- Specify hypotheses (make up your own). Note: Use names used in the model.
- Run goric().
- Inspect and interpret output.













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## Interpretation: GORIC weights

#### **GORIC** values

GORIC values cannot be interpreted, only compared: Smallest is best.

## GORIC weights $(w_m)$

 $w_m$  quantifies relative support for  $H_m$  versus others in the set. Values between 0 and 1, and they sum to 1. The bigger, the better.

## GORIC weight ratios $(w_m/w_{m'})$

 $w_m/w_{m'}$  quantifies relative support of  $H_m$  vs  $H_{m'}$ . Values between 0 and infinity:

- \* If < 1, lack of support for  $H_m$  (nl.  $w_{m'}/w_m$ , support for  $H_{m'}$ ).
- \* If > 1, support for  $H_m$ .



# Illustration of the GORIC weights $(w_m)$

Example Palmer and Gough

 $H_0$ :  $\mu_1 = \mu_2 = \mu_3$  (include only when of interest)

 $H_1$ :  $\mu_1 > \mu_2 > \mu_3$ 

 $\mu_2 > \mu_1 > \mu_3$  (if competing theory/hypothesis)  $H_2$ :

 $H_{ij}$ :  $\mu_1, \ \mu_2, \ \mu_3.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_1$	-191.89	2.81	389.41	0.68
$H_2$	-193.70	2.81	393.03	0.11
$H_u$	-191.89	4.00	391.79	0.21

# Hands-on/Demo (1b): GORIC weights

#### Let's practice.

- If needed: Start Rstudio again (optional: make project) and then also load packages again.
- Read and inspect data.
- Run model (Im()).
- Specify hypotheses (make up your own).
   Note: Use names used in the model.
- Run goric().
- Inspect and interpret output: Focus on GORIC weights.

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GORIC(A) in JASF

## R code: Data and fit object

Palmer & Gough

## R code: Hypotheses

Palmer & Gough

 $H_1: \qquad \mu_1 > \mu_2 > \mu_3$  $H_2: \qquad \mu_2 > \mu_1 > \mu_3$  $H_u: \qquad \mu_1, \ \mu_2, \ \mu_3$ 

```
# Hypotheses
H1 <- 'group1 > group2 > group3'
H2 <- 'group2 > group1 > group3'
# and the unconstrained
# (default in case of multiple hypotheses)
```

## R code: GORIC

Palmer & Gough

 $H_1: \qquad \mu_1 > \mu_2 > \mu_3$   $H_2: \qquad \mu_2 > \mu_1 > \mu_3$  $H_u: \qquad \mu_1, \ \mu_2, \ \mu_3$ 

## GORIC output and interpretation

Palmer & Gough

 $H_1$ :  $\mu_1 > \mu_2 > \mu_3$  $H_2: \mu_2 > \mu_1 > \mu_3$  $H_{ii}$ :  $\mu_1, \ \mu_2, \ \mu_3$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_1$	-191.89	2.81	389.41	0.68
$H_2$	-193.70	2.81	393.03	0.11
Hu	-191.89	4.00	391.79	0.21

 $H_1$  and  $H_2$  are not weak (nl, better than Hu - will be explained in a bit).  $H_1$  is  $0.68 / 0.11 \approx 6.11$  times more supported than  $H_2$ .

## Another illustration: one-way ANOVA

Lucas: 5 groups

Lucas (2003) investigated difference between female and male leadership w.r.t. influence of the leader.

#### Five experimental groups:

- 1. a randomly selected male leader
- 2. a randomly selected female leader
- 3. male leader selected via task
- 4. female leader selected via task
- female leader selected via task + institutionalized female leadership via movie



## (Two informative) hypotheses of interest

 $H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$  (include only when of interest)

 $H_1: \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$  $H_2: \mu_3 > \mu_1 > \mu_4 = \mu_5 > \mu_2$ 

 $H_u: \mu_1, \ \mu_2, \ \mu_3, \ \mu_4, \ \mu_5$  (included as failsafe)

## Descriptive statistics of Lucas' Data

Group		Mean Influence	SD	n
1	(male, random)	2.33	1.86	30
2	(female, random)	1.33	1.15	30
3	(male, selected)	3.20	1.79	30
4	(female, selected)	2.23	1.45	30
5	(female, selected+)	3.23	1.50	30

# R code: Data and fit object

```
# Data
lucas_data <- read.table("Data_Lucas.txt", header=TRUE)
lucas_data$group <- factor(lucas_data$group)

# fit object (needed as input for goric function)
fit.lucas <- lm(Influence ~ group - 1, data = lucas_data)</pre>
```

# R code: Hypotheses

 $\begin{array}{lll} H_0: & \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5 \\ H_1: & \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2 \\ H_2: & \mu_3 > \mu_1 > \mu_4 = \mu_5 > \mu_2 \\ H_u: & \mu_1, \ \mu_2, \ \mu_3, \ \mu_4, \ \mu_5 \end{array}$ 

```
# Hypotheses

#HO <- 'group1 = group2 = group3 = group4 = group5'

H1 <- 'group5 = group3 > (group1, group4) > group2'

# Stated otherwise:

#H1 <- 'group5 = group3 > group1 > group2;

# group3 > group4 > group2'

H2 <- 'group3 > group1 > group4 = group5 > group2'

# and the unconstrained (default here)
```

## R code: GORIC

Lucas

 $H_1: \qquad \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$   $H_2: \qquad \mu_3 > \mu_1 > \mu_4 = \mu_5 > \mu_2$  $H_u: \qquad \mu_1, \ \mu_2, \ \mu_3, \ \mu_4, \ \mu_5$ 











## **GORIC** output

Lucas

 $H_1: \qquad \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$ 

 $H_2: \qquad \mu_3 > \mu_1 > \mu_4 = \mu_5 > \mu_2,$ 

 $H_u: \mu_1, \mu_2, \mu_3, \mu_4, \mu_5.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_1$	-278.05	3.46	563.02	0.90
$H_2$	-281.76	3.14	569.79	0.03
$H_u$	-278.05	6.00	568.10	0.07



# GORIC weights $(w_m)$

Lucas

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_1$	-278.05	3.46	563.02	0.90
$H_2$	-281.76	3.14	569.79	0.03
H <sub>u</sub>	-278.05	6.00	568.10	0.07

 $H_1$  is 0.9 / 0.03  $\approx$  29.59 times more supported than  $H_2$ . Thus, there is quite some evidence that  $H_1$  is the best of this set.

#### Note:

 $H_1$  is 0.9 / 0.07 > 1 times more supported than  $H_u$ . Hence,  $H_1$  not weak and thus comparable to the other (weak and non-weak) informative hypotheses (as explained in a bit).



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GORIC(A) in JASF

# Include "unconstrained" hypothesis

If set of hypotheses does not contain a reasonable/good one: Select the best of set of weak hypotheses.

E.g.: 
$$w_1 = .8$$
 and  $w_2 = .2$ .

#### Prevent choosing a weak hypothesis

Include unconstrained hypothesis  $H_u$  (or  $H_a$ ):

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_2: \mu_2 > \mu_1 > \mu_3$ 

 $H_u: \mu_1, \mu_2, \mu_3,$ 

(i.e., no restrictions).

 $H_u$  highest fit but also most complex, thus failsafe/safeguard.

E.g.:  $w_1 = .08$ ,  $w_2 = .02$ , and  $w_u = .90$ .



## What if another hypothesis is true?

#### What if another informative hypothesis is true?

The failsafe hypothesis will be preferred over the informative hypotheses in the set, if the sample size is large enough.

E.g.: 
$$w_1 = .08$$
,  $w_2 = .02$ , and  $w_u = .90$ .

### Use of $H_u$ Palmer & Gough

 $H_1: \qquad \mu_1 > \mu_2 > \mu_3,$  $H_2: \qquad \mu_2 > \mu_1 > \mu_3$ 

 $H_u: \mu_1, \mu_2, \mu_3.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
$H_1$	-191.89	2.81	389.41	0.68
$H_2$	-193.70	2.81	393.03	0.11
$H_u$	-191.89	4.00	391.79	0.21

If at least one informative hypothesis not weak (i.e.,  $w_m > w_u$  or  $w_m/w_u > 1$ ), then compare informative hypotheses. Hence:  $H_u$  is only a failsafe, not another hypothesis of interest.



# $H_1$ vs $H_u$

What if only one informative hypothesis:

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_u: \mu_1, \mu_2, \mu_3.$ 

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
H <sub>1</sub>	-191.89	2.81	389.41	0.77
$H_u$	-191.89	4.00	391.79	0.23

 $H_1$  is 0.77 / 0.23 pprox 3.27 times more supported than  $H_u$ .

BUT:  $H_u$  includes  $H_1$ .

So, support for  $H_u$  contains support for  $H_1$ .



# $H_1$ vs $H_u$ ctd.

Palmer & Gough

#### **GORIC**

	Model	Fit	Complexity	GORIC	GORIC weights
F	$I_1$	-191.89	2.81	389.41	0.77
ŀ	$H_{u}$	-191.89	4.00	391.79	0.23

Unconstrained is always true, so will always receive support. Support for  $H_1$  (if any).

Now: Fit  $H_1$  and  $H_u$  the same; thus, GORIC weights are based on only complexity.

Consequently, 0.77 / 0.23  $\approx$  3.27 is an upper bound.



# $H_1$ vs $H_u$ : Upper bound

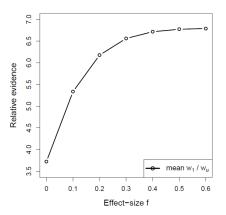


Figure: IC weights can have an upper bound, when informative hypothesis has maximum fit (i.e., is fully in agreement with the data).

# Alternative failsafe: Complement of $H_m$

### Alternatively (in the case of one hypothesis of interest)

Evaluate hypothesis of interest against its complement; that is, all other possible hypotheses.

More powerful than against the unconstrained if  $H_m$  has maximum fit.

#### Reference:

Vanbrabant, L., Van Loey, N., and Kuiper, R. M. (2020). Evaluating a Theory-Based Hypothesis Against Its Complement Using an AIC-Type Information Criterion With an Application to Facial Burn Injury. *Psychological Methods*, *25*(2), 129-142. https://doi.org/10.1037/met0000238

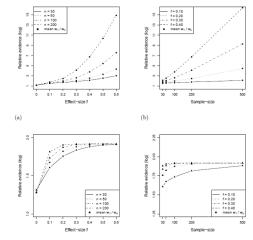
# Alternative failsafe: Complement of $H_m$

Based on simulation





(c)



## Note: unconstrained vs complement

#### $H_1$ contains 1 ordering of means:

1. 
$$\mu_1 > \mu_2 > \mu_3$$

#### $H_c$ contains 5 orderings of means:

- 2.  $\mu_1 > \mu_3 > \mu_2$
- 3.  $\mu_2 > \mu_1 > \mu_3$
- 4.  $\mu_2 > \mu_3 > \mu_1$
- 5.  $\mu_3 > \mu_1 > \mu_2$
- 6.  $\mu_3 > \mu_2 > \mu_1$

 $H_u$  combines  $H_1$  and  $H_c$ , and thus consists of all 6 possble orderings.

## R code: complement

Palmer & Gough

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_c$ : not  $H_1$ .

#### **GORIC**

### $H_1$ VS $H_C$ Palmer & Gough

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_c$ : not  $H_1$ ,

complement of  $H_1$ .

#### **GORIC**

Model	Fit	Complexity	GORIC	GORIC weights
H1	-191.89	2.81	389.41	0.79
complement	-192.34	3.69	392.05	0.21

 $H_1$  is 0.79 / 0.21  $\approx$  3.73 times more supported than its complement, that is, any other hypothesis (i.e., any other ordering).



# Note on use of safeguard hypotheses (1/2)

- Unconstrained  $(H_u)$  is failsafe. If at least one hypothesis  $(H_m)$  not weak (i.e., better than  $H_u$ ), then compare that with the other, competing hypotheses  $(H_{m'})$ . Note: Do not compare  $H_m$  with  $H_u$  (since support is bounded).
- Complement  $(H_c)$  acts as competing hypothesis. Note 1: Relative support  $H_m$  vs  $H_{m'}$  same as in case above. Note 2: Can compare  $H_m$  with  $H_c$ .

# Note on use of safeguard hypotheses (2/2)

My advise and default in goric function:

- If multiple informative, theory-based hypotheses:
   Goal: Comparison of informative, theory-based hypotheses.
   Safeguard: Unconstrained.
- If one informative, theory-based hypothesis  $(H_m)$ : Goal: Compare  $H_m$  with all other possibilities / theories. Safeguard: Its complement.

# Extra: complement & 'Heq' (1/4)

Palmer & Gough

 $H_1$  and  $H_c$  do not really overlap, but they do share a border. The border is defined by boundary hypotheses, that is, all possible hypotheses where  $\geq 1$  inequalities ('<', '>') are replaced by '='.

If one wishes to also evaluate the boundary hypothesis were all inequalities are replaced by equalities  $(H_{eq})$ :

$$H_{eq}: \quad \mu_1 = \mu_2, \ \mu_3, \ H_1: \quad \mu_1 > \mu_2, \ \mu_3, \ H_c: \quad \text{not } H_1.$$

then one can use 'Heq = TRUE', as demonstrated on the next slide.

## Extra: complement & 'Heq' (2/4)

Palmer & Gough

#### R code

# Extra: complement & 'Heq' (3/4)

Palmer & Gough

### R output (partly)

```
## restriktor (0.6-15): generalized order-restricted information criterion:
##
## Results:
##
          model
                  loglik
                           penalty goric
                                            loglik.weights penalty.weights
## 1
            Heg -193.703 3.000 393.406
                                                     0.123
                                                                      0.452
## 2
             H1 -191.893 3.500 390.786
                                                     0.753
                                                                      0.274
     complement -193.703 3.500 394.406
## 3
                                                     0.123
                                                                      0.274
##
     goric.weights goric.weights_without_heq
## 1
             0.188
## 2
             0.698
                                       0.859
             0.114
                                        0.141
## 3
##
## Conclusion:
## - The order-restricted hypothesis H1 is the best in the set, as it has the h
    * 'H1' is 6.109 times more supported than 'complement'.
##
```

# Extra: complement & 'Heq' (4/4)

 $H_{eq}$  is only one boundary hypothesis, and the most strict one.

I would only include  $H_{eq}$  when you a priori want to evaluate this (possibly to rule it out).

When the truth lies on the border of  $H_1$  and  $H_c$ , then the loglik.weights are approximate equal (i.e., approx. .5). If you find this, then I would go into an exploration phase (and evaluate all (or some) boundary hypotheses).

# Hands-on/Demo (2): GORIC - $H_m$ vs complement

Let's practice.

- If needed: Start Rstudio again (optional: make project) and then also load packages again.
- Optional: Open 'Tutorial\_GORIC\_restriktor\_ANOVA.html', 'PalmerAndGough\_and\_Lucas.R', and/or 'Hands-on\_2\_GORIC\_Compl\_ANOVA\_restriktor.R'.
- Read and inspect data.
   Use Data\_PalmerAndGough.txt and/or Data\_Lucas.txt.
- Run model (lm()).
- Specify hypotheses (make up your own). E.g., for Lucas:

$$H_1: \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$$

Note: Use names used in the model.

- Run goric()
- Inspect and interpret output.



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### **GORICA**

GORIC(A) in JASF

### **GORICA**

#### GORIC: Normal linear models

GORIC can easily be applied to normal linear models (e.g., ANOVA models or regression models).

#### GORIC: Other statistical models

In case of other statistical models (e.g., a SEM model), more cumbersome to calculate maximized order-restricted log likelihood and thus GORIC.

### GORICA: All statistical models

https://doi.org/10.1037/met0000406

Therefore, GORICA: asymptotic expression for GORIC. Can be used for all types of statistical models.

#### Reference:

Altınışık, Y., Van Lissa, C. J., Hoijtink, H., Oldehinkel, A. J., and Kuiper, R. M. (2021). Evaluation of inequality constrained hypotheses using a generalization of the AIC. *Psychological Methods*, 26(5), 599–621.





#### Similarities with GORIC

- Form:  $GORICA_m = -2$  fit + 2 complexity.
- Broad type of restrictions.

### Differences compared to GORIC

- Uses asymptotic expression of the likelihood (is a normal): can therefore be easily applied to all types of statistical models.
   Disadvantage: might work less well in case of small samples.
- Does not need data set; mle's and their covariance matrix suffice.
- Can leave out nuisance parameters (i.e., not part of hypotheses).

#### Note

In case of normal linear models and/or not too small samples: GORICA weights = GORIC weights.

## R code: GORICA

Palmer & Gough

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_c$ : not  $H_1$ .

## GORICA: type = "gorica" (not default for Im objects)

### **GORICA**

Palmer & Gough

 $H_1: \mu_1 > \mu_2 > \mu_3,$ 

 $H_c$ : not  $H_1$ .

#### **GORIC**

Model	Fit	Complexity	GORICA	GORICA weights
H1	-1.96	1.81	7.55	0.79
complement	-2.39	2.69	10.15	0.21

 $\textit{H}_{1}$  is 0.79 / 0.21  $\approx$  3.67 times more supported than its complement, that is, any other hypothesis.

Note: GORIC weights are the same (ratio may differ a bit).



## More information GORIC(A) output

#### More information GORIC(A) output:

- 'Guidelines\_output\_GORIC.html'
- Possibly: 'Guidelines\_GORIC-Benchmarks'

from https://github.com/rebeccakuiper/Tutorials.
On github site, go to Code (green button) and download zip.

## Hands-on/Demo (3): GORICA

#### Let's practice.

- If needed: Start Rstudio again (optional: make project) and then also load packages again.
- Optional: Open 'Tutorial\_GORIC\_restriktor\_ANOVA.html', 'PalmerAndGough\_and\_Lucas.R', and/or 'Hands-on\_3\_GORICA\_UncAndCompl\_ANOVA\_restriktor.R'.
- Read and inspect data.
   Use Data\_PalmerAndGough.txt and/or Data\_Lucas.txt.
- Run model (Im()).
- Specify hypotheses (make up your own). E.g., for Lucas:

$$H_1: \qquad \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$$

Note: Use names used in the model (or overwrite those).

- Run goric(); now, add (since Im object):
  - type = "gorica"
- Inspect and interpret output.



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### GORIC(A) in JASP





## GORIC in JASP: instructions

Palmer & Gough

- Extract data file 'Data\_PalmerAndGough\_JASP.txt'
   from 'data\_JASP' subfolder (github.com/rebeccakuiper/Tutorials).
- Open JASP and 'Data\_PalmerAndGough\_JASP.txt'.
   Check measurement levels.
- Go to ANOVA and denote variables of interest.
- Go to Order Restricted Hypotheses tab.
- Specify hypotheses. For example,

$$H_1: \mu_1 > \mu_2 > \mu_3$$

JASP: group1 > group2 > group3

Press Ctrl+Enter. Inspect and interpret output.











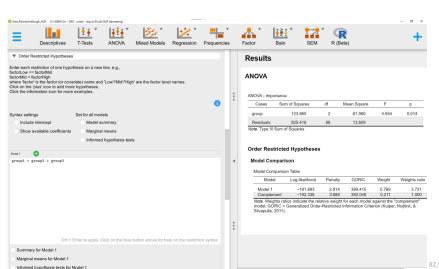






## GORIC in JASP: screenshot

Palmer & Gough



## GORIC in JASP: instructions

Lucas

- Open JASP.
   Use 'Data\_Lucas\_JASP.txt' ('data\_JASP' subfolder).
   Check measurement levels.
- Go to ANOVA and denote variables of interest.
- Go to Order Restricted Hypotheses tab.
- Specify hypotheses. For example,

$$H_1: \qquad \mu_5 = \mu_3 > \{\mu_1, \mu_4\} > \mu_2$$
   
  $JASP: \qquad group5 = group3 > group1 > group2$    
  $group3 > group4 > group2$ 

Press Ctrl+Enter. Inspect and interpret output.







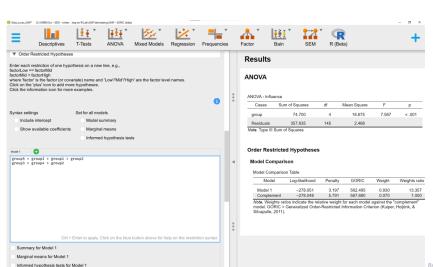






## GORIC in JASP: screenshot

Lucas



## GORIC and GORICA in JASP

	GORIC R	GORIC JASP	GORICA R	GORICA JASP
ANOVA	<b>~</b>	<b>~</b>	<b>~</b>	
RM-ANOVA			<b>~</b>	<b>~</b>
Regression	<b>~</b>	Not yet	<b>~</b>	
SEM			<b>~</b>	Future?
Other model			<b>\</b>	Future?



## GORIC and GORICA in JASP

#### GORIC or GORICA in the following statistical models (as part of that model) ▼ ANCOVA GORIC ▶ Order Restricted Hypotheses ANOVA Dependent Variable ▶ Post Hoc Tests pi site A sex Fixed Factors ▶ Descriptives Plots **\** age if viewcat ▶ Raincloud Plots A treat ► Marginal Means nebody > prelet ♦ preform ▶ Order Restricted Hypothese nerumb > ANCOVA nerelat GORIC ▶ Post Hoc Tests preclast postbody ▶ Descriptives Plots oostet . Mean Square ♦ postform ▶ Raincloud Plots oruntecq 🧳 postrelat ► Marginal Means Note: Type III Sum of Square WLS Weights ocatcles! nontrote ➤ Assumption Checks Repeated Measures ANOVA Descriptive statistics ▶ Contrasts Estimates of effect size In Order Restricted Hypotheses Within Subjects Effects Vovk-Selike maximum p-ratio Mean Squan **GORICA** ▶ Post Hoc Tests ► Model ▶ Descriptives Plots ➤ Assumption Checks ► Raincloud Plots Between Subjects Effects Sum of Squares Mean Square ▶ Marginal Means ▶ Order Restricted Hypotheses ▶ Simple Main Effects Note: Type III Sum of Squares

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# The End GORIC(A)

## Thanks for listening!

## Are there any questions?

#### Websites

https://github.com/rebeccakuiper/Tutorials www.uu.nl/staff/RMKuiper/Extra1 www.uu.nl/staff/RMKuiper/Extra2 informative-hypotheses.sites.uu.nl/software/goric/

#### E-mail

r.m.kuiper@uu.nl

## Notes

## Note on comparable estimates

Until now: comparing means.

### Continuous predictors

If compare relative strength/importance of parameters (e.g.,  $\beta_1 > \beta_2$ ), then make sure comparable: e.g., standardize continuous predictors.

### Multiple outcomes

If compare parameters across outcomes, then (also) standardize outcomes.

## Note on using complement

Complement not always higher weight

In case  $H_m$  is almost true, but not true:

- 1.  $H_m$  does not have maximum fit (but much lower penalty).
- 2. Support for  $H_m$  is less when evaluating it against its complement (than  $H_u$ ).

This is of course a good thing, since  $H_m$  is not true.

#### Explanation:

Penalty of  $H_c$  is smaller than that of  $H_u$ .

Against  $H_u$ , you choose  $H_m$  'sooner' because of low penalty for  $H_m$ .

## Note on type of hypotheses

## Absolute strength

Compare strength/importance of parameters:  $\beta_1 > \beta_2$ . Compare absolute strength/importance of parameters:  $abs(\beta_1) > abs(\beta_2)$ .

### About-equality

Equality:  $\beta_1 = \beta_2$ .

About-equality:  $-0.01 < \beta_1 - \beta_2 < 0.01$ .

#### Minimum difference

Difference:  $abs(\beta_1) > abs(\beta_2)$ .

Minimum difference:  $abs(\beta_1) - abs(\beta_2) > 0.01$ .



## Note on hypotheses

- Only include hypotheses with sound theoretical and/or empirical basis.
  - Often a null hypothesis is not of interest.
- 2. Keep the number of hypotheses included as small as possible.
- 3. This is a subjective endeavor, aim for inter-peer / inter-subjective agreement.

## Note on possibilities multiple studies

- Update GORIC(A) values & weights.
   More data collected: (re-)calculate.
- Update hypotheses.

First data set (or a part of it) generates one or more hypotheses. Other data set (or part) used to determine evidence / support.

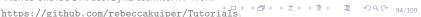
Download 'Tutorial\_GORIC\_restriktor\_UpdateHypo.html' and/or 'Hands-on\_4\_GORIC\_UpdateHypo\_restriktor.R' from https://github.com/rebeccakuiper/Tutorials.

Aggregate evidence for hypotheses.

Aggregate the support for theories (diverse designs allowed). Bear in mind: Meta-analysis aggregates parameter estimates or effect sizes which need to be comparable (often same designs required).

 $Download\ `Tutorial\_GORIC\_restriktor\_evSyn.html'\ and/or$ 

'Hands-on\_4\_GORIC\_evSyn\_restriktor.R' from



## Extra material (1/2)

#### Logistic Regression Modeling

- Article: doi.org/10.1037/met0000406

#### GORICA on SEM

- Article: www.tandfonline.com/doi/full/10.1080/10705511.2020.1836967.
- R scripts: github.com/rebeccakuiper/GORICA\_in\_SEM.

#### GORICA on cross-lagged panel model (CLPM)

- Article: doi.org/10.1111/bjep.12455.
- R scripts: github.com/rebeccakuiper/GORICA\_in\_CLPM.

#### GORICA on Random-Intercept CLPM (RI-CLPM)

- Article: Sukpan, C. and Kuiper, R.M. (2023). How to evaluate causal dominance hypotheses in lagged effects models.
- R scripts: github.com/rebeccakuiper/GORICA%20in%20RI-CLPM.

#### GORICA for EffectLiteR

- R scripts: https://github.com/rebeccakuiper/Tutorials/tree/main/ GORICA%20for%20EffectLiteR.

Note: On github site, go to Code (green button) and download zip. 🗸 🤈 🤈 💍 95/109



## Extra material (2/2)

#### GORICA on CTmeta

- Article: doi.org/10.1080/10705511.2020.1823228.
- R scripts: github.com/rebeccakuiper/GORICA\_on\_CTmeta.

#### GORICA on Meta-analysis

- Article: doi.org/10.3390/e24111525.
- $\ {\sf R} \ {\sf scripts:} \ {\sf github.com/rebeccakuiper/GORICA\_on\_MetaAn}.$

Note: On github site, go to Code (green button) and download zip.

## Note on BF as an IC

The fit in GORICA refers to the maximum log likelihood. The fit in BF is related to the maximum likelihood.

Notably, BF can be written as an IC (and vice versa):

$$-2 \log BF_{1u} = -2 \log \frac{f_1}{c_1}$$

$$= -2 \log f_1 + 2 \log c_1$$

$$= -2 \log \text{ fit } H_1 + 2 \log \text{ complexity } H_1.$$

Note: complexity value in BF depends on prior.

## Note on GORIC weights vs BF and PMPs

ratio GORIC weights  $(w_m/w_{m'}) \sim$  Bayes factor  $(BF_{mm'})$ .

GORIC weight  $(w_m) \sim$  posterior model probability (PMP).

1 -  $w_m =$  conditional error probability. Like PMP,  $w_m$  depends on set of hypotheses.

## Note on conditional error probability (1/3)

using PMPs and GORIC(A) weights

$H_m$	weights
H <sub>1</sub> : Sex Match	.04
$H_2$ : Gender Role Match	.81
$H_3$ : Sex Mismatch	.01
$H_4$ : Gender Role Mismatch	.00
$H_u$ :	.14

The conditional error probability for preferred hypotheses  $H_2$  is 1-.81=.19.

## Note on conditional error probability (2/3)

using PMPs and GORIC(A) weights

What if we compare many hypotheses?

H <sub>m</sub>	weights
H <sub>1</sub> : Sex Match	.013
$H_2$ : Gender Role Match	.270
$H_3$ : Sex Mismatch	.003
$H_4$ : Gender Role Mismatch	.000
$H_5$ : Lets try this one too	.180
$H_{12}$ : Don't miss something	.040
$H_u$ :	.047

The conditional error probability for  $H_2$  becomes 1 - .27 = .77 (was .19).

Namely, included additional hypotheses may obtain some support. Hence, the conditional error probabilities becomes larger.



## Note on conditional error probability (3/3)

using PMPS and GORIC(A) weights

What does conditional error probability reflect when hypotheses overlap? Notably, when hypotheses overlap they can also share/divide support (like  $H_u$  does with any hypothesis).

#### My advise:

Only use error probabilities if one hypothesis versus its complement. (Or when you are sure that there is no overlap in hypotheses;

and they preferably do cover all possibilities)

H <sub>m</sub>	weights
H <sub>2</sub> : Gender Role Match	.82
$H_c$ : complement $H_2$	.18

## Simulation study

## Properties of Methods: Simulation study

- 1. All methods (e.g., (G)ORIC and BMS) programmed in Fortran.
- 2. Generate data sets with know structure (conditions).
- 3. Apply methods to all data sets (within one condition).
- 4. Summarize results (per condition): proportion of (correct) chosen hypotheses.

#### Reference:

Kuiper, R.M., Nederhof, T., and Klugkist, I. (2015). Properties of hypothesis testing techniques and (Bayesian) model selection for exploration-based and theory-based (order-restricted) hypotheses. *British Journal of Mathematical and Statistical Psychology, 68*(2), 220 – 245.

## Population Values in Simulated Data Sets when k = 3

Values of Population Means  $(\mu_i)$  and Standard Deviations  $(\sigma_i)$  Used to Simulate 1000 Data Sets

	C 11 (1501)				== (c + )
k = 3	Case Number ('ES')	$\mu_1$	$\mu_2$	$\mu_{3}$	ES (Cohen)
	0	0.00	0.00	0.00	
	1	-0.10	0.00	0.10	.1
	2	-0.20	0.00	0.20	.2
	3	-0.30	0.00	0.30	.3
	4	-0.40	0.00	0.40	.4
	5	-0.50	0.00	0.50	.5
		$\sigma_1$	$\sigma_2$	$\sigma_3$	
	0 - 5	1.00	1.00	1.00	

NB in Case 0  $H_{30C}$ :  $\mu_1 = \mu_2 = \mu_3$  is true in Cases 1 to 5  $H_{31C}$ :  $\mu_1 < \mu_2 < \mu_3$  is true.



## Results Exploration vs Confirmatory (k = 3 and n = 50)

$$\begin{array}{lll} \textit{H}_{30E}: \mu_1 = \mu_2 = \mu_3 & \textit{H}_{30C}: \mu_1 = \mu_2 = \mu_3 \\ \textit{H}_{31E}: \mu_1 = \mu_2, \ \mu_3 & \textit{H}_{31C}: \mu_1 < \mu_2 < \mu_3 \\ \textit{H}_{32E}: \mu_1, \ \mu_2 = \mu_3 & \textit{H}_{32C}: \mu_1 = \mu_2 < \mu_3 \\ \textit{H}_{33E}: \mu_1 = \mu_3, \ \mu_2 & \textit{H}_{33C}: \mu_1 < \mu_2 > \mu_3 \\ \textit{H}_{34E}: \mu_1, \ \mu_2, \ \mu_3 & \textit{H}_{34C}: \mu_1, \ \mu_2, \ \mu_3 \end{array}$$

'ES'	Method	$H_{30E}$	$H_{31E}$	$H_{32E}$	$H_{33E}$	$H_{34E}$
0	PCIC: AIC	0.658	0.113	0.115	0.112	0.002
2	PCIC: AIC	0.201	0.356	0.358	0.028	0.057
5	PCIC: AIC	0.000	0.135	0.136	0.000	0.729

'ES'	Method	$H_{30C}$	$H_{31C}$	$H_{32C}$	$H_{33C}$	$H_{34C}$
0	ORIC	0.662	0.065	0.119	0.120	0.034
2	ORIC	0.148	0.506	0.311	0.029	0.006
5	ORIC	0.000	0.955	0.044	0.001	0.000

## Results Confirmation (k = 3 and n = 50)

 $H_{30C}: \mu_1 = \mu_2 = \mu_3$   $H_{31C}: \mu_1 < \mu_2 < \mu_3$   $H_{32C}: \mu_1 = \mu_2 < \mu_3$   $H_{33C}: \mu_1 < \mu_2 > \mu_3$  $H_{34C}: \mu_1, \mu_2, \mu_3$ 

'ES'	Method	'Prior'	$H_{30C}$	$H_{31C}$	$H_{32C}$	$H_{33C}$	$H_{34C}$
0	ORIC		0.662	0.065	0.119	0.120	0.034
0	BMS	3	0.865	0.012	0.075	0.036	0.012
0	BMS	2	0.774	0.022	0.124	0.058	0.022
0	BMS	1	0.656	0.047	0.147	0.120	0.033
2	ORIC		0.148	0.506	0.311	0.029	0.006
2	BMS	3	0.299	0.299	0.371	0.031	0.001
2	BMS	2	0.252	0.328	0.381	0.040	0.000
2	BMS	1	0.125	0.494	0.321	0.063	0.000
5	ORIC		0.000	0.955	0.044	0.001	0.000
5	BMS	3	0.000	0.887	0.112	0.001	0.000
5	BMS	2	0.001	0.890	0.107	0.002	0.000
5	BMS	1	0.000	0.928	0.070	0.002	0.000

## Results Confirmation (k = 3 and n = 50)

```
H_{30C}: \mu_1 = \mu_2 = \mu_3
```

 $H_{31C}: \mu_1 < \mu_2 < \mu_3$ 

 $H_{32C}: \mu_1 = \mu_2 < \mu_3$ 

 $H_{33C}: \mu_1 < \mu_2 > \mu_3$ 

 $H_{34C}: \mu_1, \ \mu_2, \ \mu_3$ 

1-Type	error	(here):	
preferring t	the true	e null hy <sub>l</sub>	pothesis
given the s	et of h	ypothese	s (!).

'ES'	Method	'Prior'	$H_{30C}$	$H_{31C}$	$H_{32C}$	$H_{33C}$	$H_{34C}$
0	ORIC		0.662	0.065	0.119	0.120	0.034
0	BMS	3	0.865	0.012	0.075	0.036	0.012
0	BMS	2	0.774	0.022	0.124	0.058	0.022
0	BMS	1	0.656	0.047	0.147	0.120	0.033
2	ORIC		0.148	0.506	0.311	0.029	0.006
2	BMS	3	0.299	0.299	0.371	0.031	0.001
2	BMS	2	0.252	0.328	0.381	0.040	0.000
2	BMS	1	0.125	0.494	0.321	0.063	0.000
5	ORIC		0.000	0.955	0.044	0.001	0.000
5	BMS	3	0.000	0.887	0.112	0.001	0.000
5	BMS	2	0.001	0.890	0.107	0.002	0.000
5	BMS	1	0.000	0.928	0.070	0.002	0.000

## Results Confirmation (k = 3 and n = 50)

 $H_{30C}: \mu_1 = \mu_2 = \mu_3$ 

 $H_{31C}: \mu_1 < \mu_2 < \mu_3$ 

 $H_{32C}: \mu_1 = \mu_2 < \mu_3$  $H_{33C}: \mu_1 < \mu_2 > \mu_3$ 

 $H_{34C}: \mu_1, \ \mu_2, \ \mu_3$ 

1 - Type II error (here):

preferring the true hypothesis, given the set of hypotheses (!).

'ES'	Method	'Prior'	$H_{30C}$	$H_{31C}$	$H_{32C}$	$H_{33C}$	$H_{34C}$
0	ORIC		0.662	0.065	0.119	0.120	0.034
0	BMS	3	0.865	0.012	0.075	0.036	0.012
0	BMS	2	0.774	0.022	0.124	0.058	0.022
0	BMS	1	0.656	0.047	0.147	0.120	0.033
2	ORIC		0.148	0.506	0.311	0.029	0.006
2	BMS	3	0.299	0.299	0.371	0.031	0.001
2	BMS	2	0.252	0.328	0.381	0.040	0.000
2	BMS	1	0.125	0.494	0.321	0.063	0.000
5	ORIC		0.000	0.955	0.044	0.001	0.000
5	BMS	3	0.000	0.887	0.112	0.001	0.000
5	BMS	2	0.001	0.890	0.107	0.002	0.000
5	BMS	1	0.000	0.928	0.070	0.002	0.000

## Extra: Prior choice for BMS

#### Prior

$$P(\mu, \sigma^2) = P(\mu_1) \times ... \times P(\mu_k) \times P(\sigma^2),$$
  
where  $P(\mu_i) = P(\mu) \ \forall \ i = 1, ..., k.$ 

 $P(\mu)$  is a data-based normal distribution, with parameters  $\beta_0$  and  $\tau_0^2$ , and  $P(\sigma^2)$  an inverse chi-squared distribution (see Klugkist et al., 2005)

## $\beta_0$ and $\tau_0^2$ data-based hyperparameters

For each  $\mu_i$  (i = 1, ..., k) a credibility interval is computed: estimated group mean  $\pm PV \times$  the group standard error. The credibility interval for  $\mu_i$  is represented by " $[LB_i, UB_i]$ ".

Lowest lower bound: LB<sub>min</sub>.

Highest upper bound:  $UB_{max}$ . Then,  $\beta_0 = \frac{LB_{min} + UB_{max}}{2}$  and  $\tau_0^2 = [\frac{UB_{max} - LB_{min}}{2}]^2$ .

Three types of prior vagueness: BMS PV 1, BMS PV 2 and BMS PV 3

