CS2040S
Data Structures and Algorithms

Hashing II

Today: More Hashing!

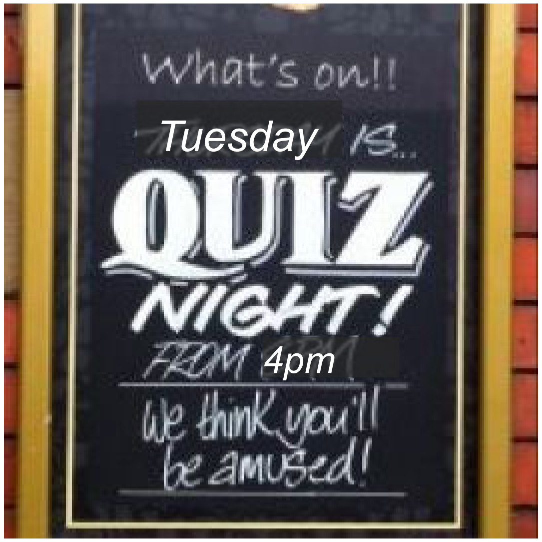
- Collision resolution: chaining (continued)

- Java hashing

- Collision resolution: open addressing

- Table (re)sizing

Announcements



Midterm : Tuesday March 9, 4pm

- In person: be there!
- Be on time.
- Locations TBA (mostly MPSH).

Bring to quiz:

- One sheet of paper with any notes you like.
- Pens/pencils.
- You may not use anything else.

Some topics:

Theory:

- Asymptotic analysis
- Simple recurrences
- Simple probability

Some topics:

Algorithms and data structures:

- Abstract Data Types
  - Stacks, Queues
- Divide-and-conquer
  - Binary search, Peak finding
- Sorting
  - BubbleSort, InsertionSort, SelectionSort, MergeSort, QuickSort
  - Pancacke Sorting, Reversal Sorting, etc.
  - Order Statistics (QuickSelect)

Some topics:

More algorithms and data structures:

- Trees
  - Binary Trees, Binary Search Trees, etc.
  - AVL Trees
  - (a,b)-trees
  - Order Statistics Trees, Interval Trees, etc.
- Hashing
  - Symbol tables
  - Hashing
  - Chaining

Announcements

Quiz Advice:

Announcements

Quiz Advice:

Get the maximum number of points you can.

- Do not leave easy questions blank.
- Bypass questions instead of getting stuck.

Announcements

Quiz Advice:

Be as clear as possible.

- Do not be ambiguous.
- Circle your final answer (if it is unclear).
- Cross out incorrect answers.
- Write neatly.

Announcements

Quiz Advice:

State your assumptions.

- If the question is ambiguous, state precisely what you are assuming.
- If your assumptions are reasonable, and your answer is correct subject to those assumptions, you will (most likely) get full credit!

Announcements

Quiz Advice:

Review the basics:

- Know the basic recurrences.
- Review how the algorithms we have studied work.
- Know the running time of the algorithms we have studied.

Announcements

Quiz Advice:

Review problem solving strategies:

- Review problems we have solved on problem sets, in tutorial, in recitation, in class.
- What is the basic strategy used in the question?
  - Binary search? Divide-and-conquer? Sorting?
- What strategy is good for which types of problems?

Announcements

Mid-Semester Survey:

Thanks for the feedback!

Announcements

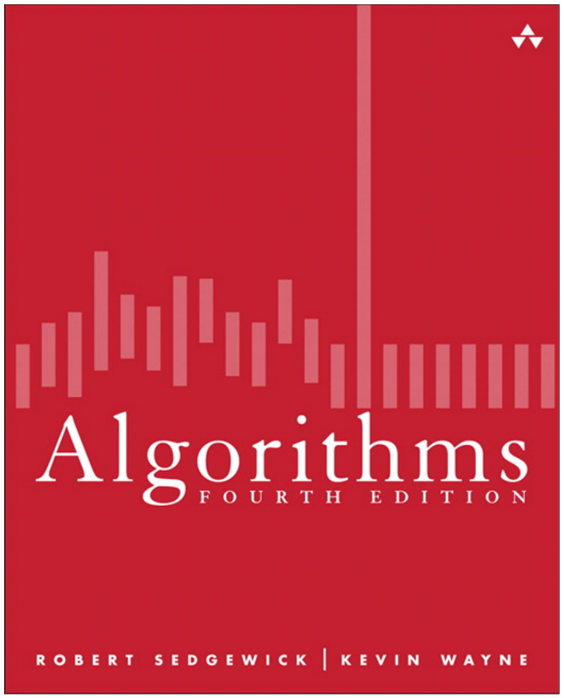
Mid-Semester Survey:

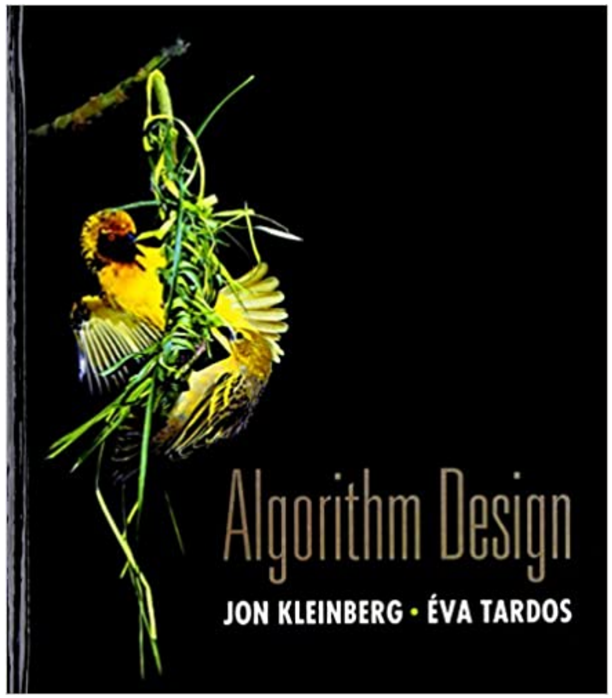
"Lecture notes are good for understanding how algorithms work, but..." (Thanks!)

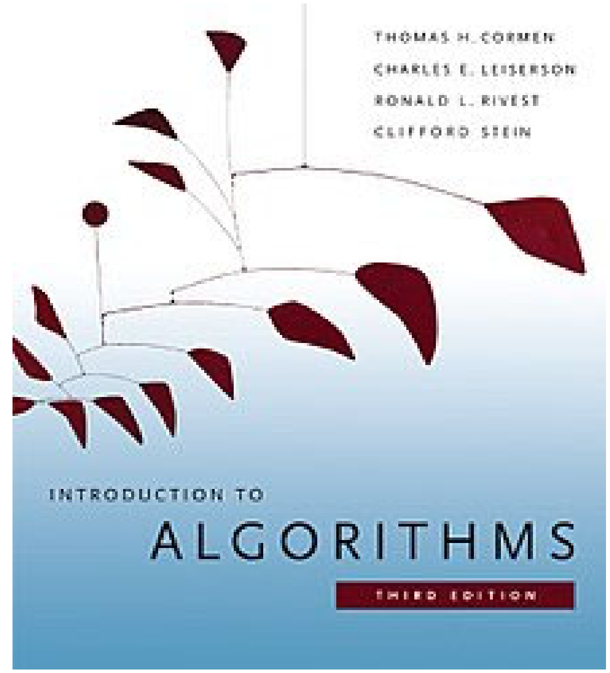
Lecture Slides	Textbook
optimized for oral explanation	optimized for reading
algorithm on 1-slide	full details
illustrated step-by-step	concise textual description
slides summarize important points	chapters contain detailed discussed of issues
divided into lectures	divided into chapters

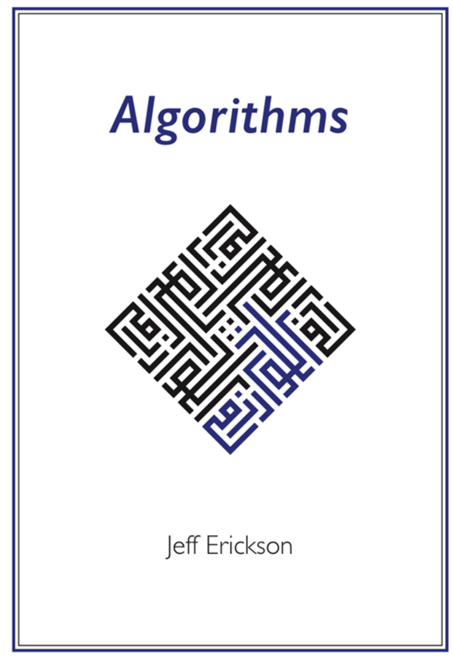
Announcements

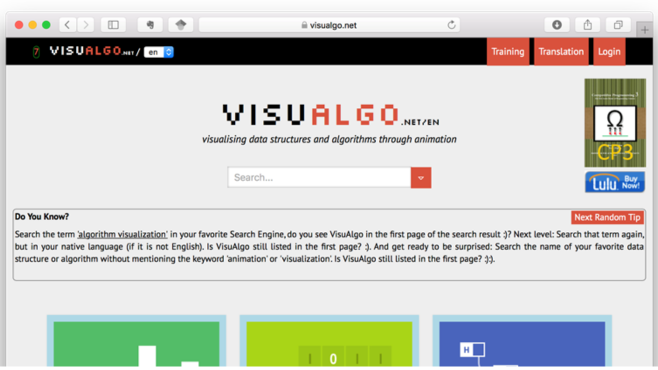
Quiz advice:











Today: More Hashing!

- Collision resolution: chaining (continued)
- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

Review: Symbol Table Abstract Data Type

Which of the following is *not* typically a symbol table operation?

1. insert(key, data)
2. delete(key)
3. successor(key)
4. search(key)
5. None of the above.

ARCHIPELAGO is open

Review: Symbol Table Abstract Data Type

Which of the following is *not* typically a symbol table operation?

1. insert(key, data)
2. delete(key)
3. successor(key)
4. search(key)
5. None of the above.

Abstract Data Types

Symbol Table

public interface	**SymbolTable**	
void	insert(Key k, Value v)	*insert (k,v) into table*
Value	search(Key k)	*get value paired with k*
void	delete(Key k)	*remove key k (and value)*
boolean	contains(Key k)	*is there a value for k?*
int	size()	*number of (k,v) pairs*

Note: no successor / predecessor queries.

Direct Access Tables

Attempt #1: Use a table, indexed by keys.

0	null
1	null
2	item1
3	null
4	null
5	item3
6	null
7	null
8	item2
9	null

Universe U={0..9} of size $m = 10$.

(key, value)

(2, item1)
(8, item2)
(5, item3)

Assume keys are distinct.

Direct Access Tables

Problems:

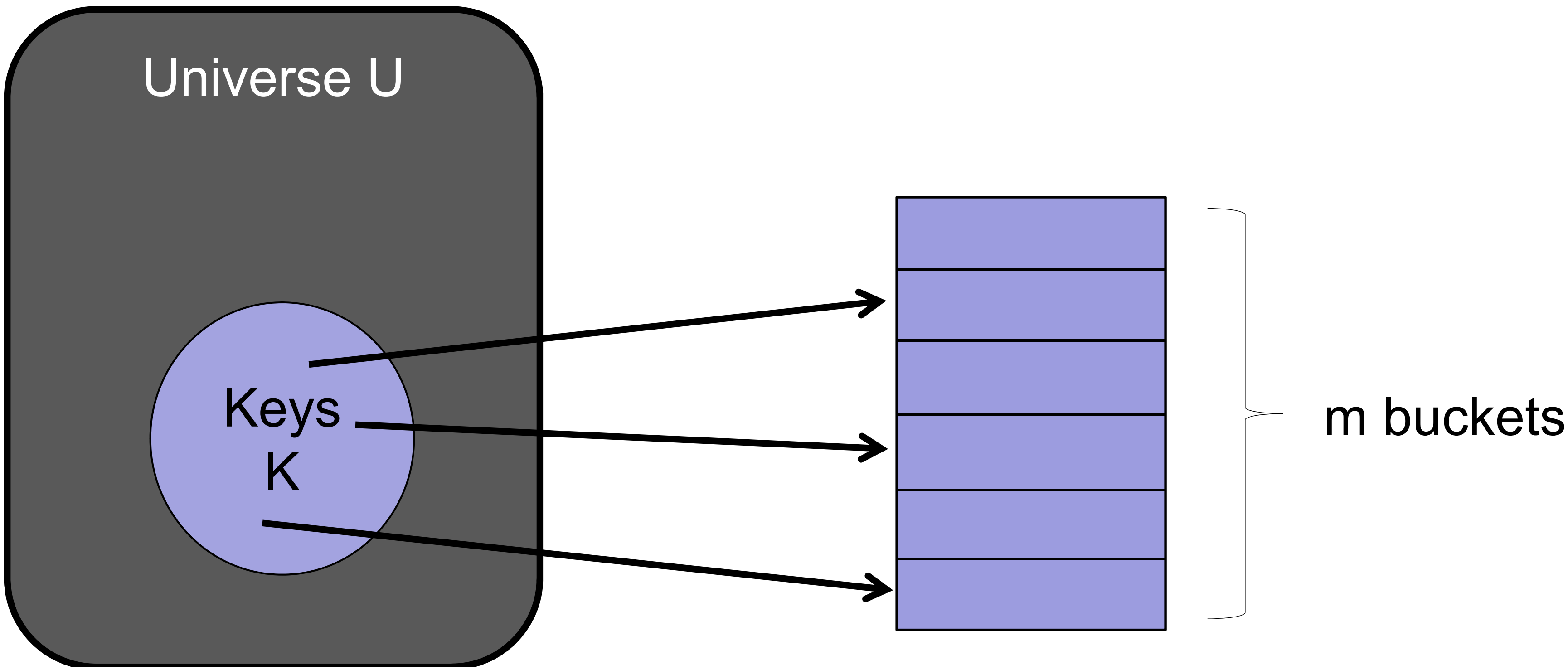
- Too much space
  - If keys are integers, then table-size > 4 billion

- What if keys are not integers?
  - Where do you put the key/value "**(hippopotamus, bob)**"?
  - Where do you put 3.14159…?

Hash Functions

Problem:

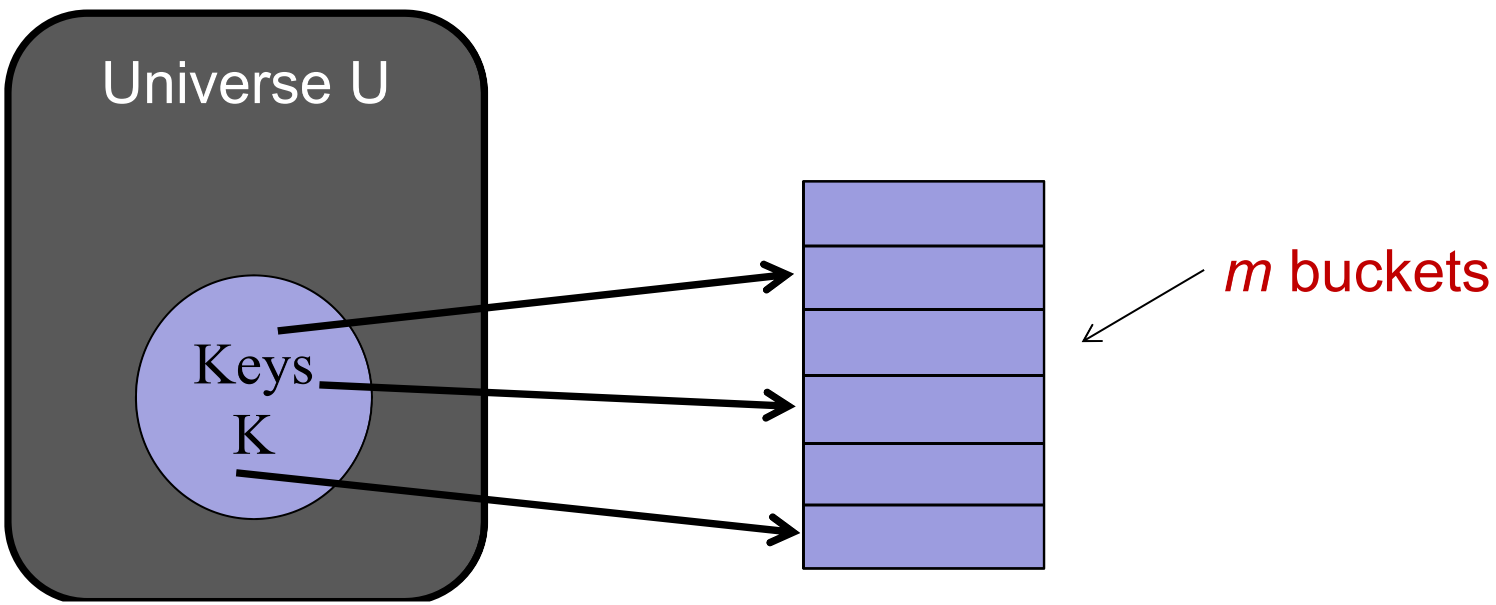
- Huge universe U of possible keys.
- Smaller number n of actual keys.
- How to map n keys to $m \approx n$ buckets?



Hash Functions

Define hash function $h : U \rightarrow \{1..m\}$

– Store key k in bucket $h(k)$.



Hash Functions

Collisions:

– We say that two distinct keys k_1 and k_2 collide if:

$$h(k_1) = h(k_2)$$

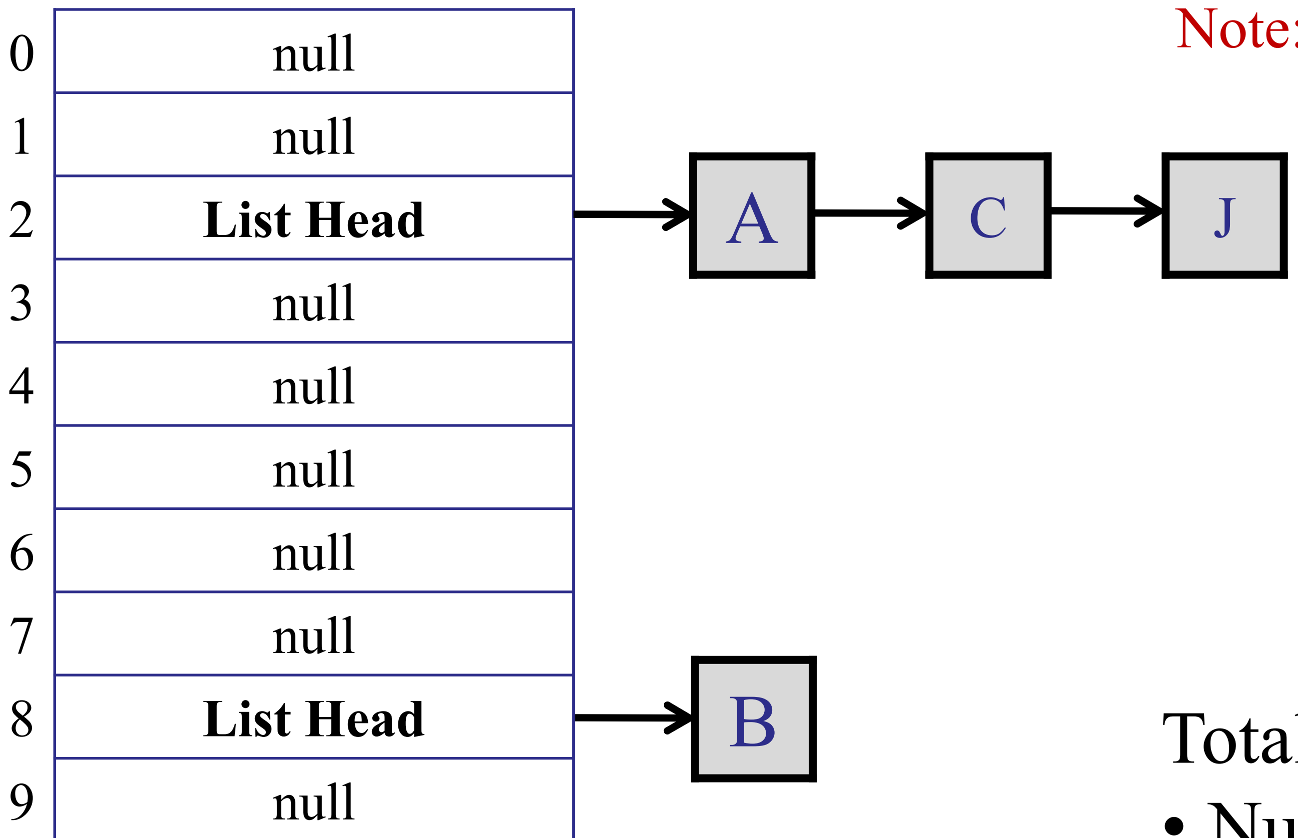
– Unavoidable!

- The table size is smaller than the universe size.
- The pigeonhole principle says:
  – There must exist two keys that map to the same bucket.
  – Some keys must collide!

Chaining

Each bucket contains a linked list of items.

Note: h(A) == h(C) == h(J)



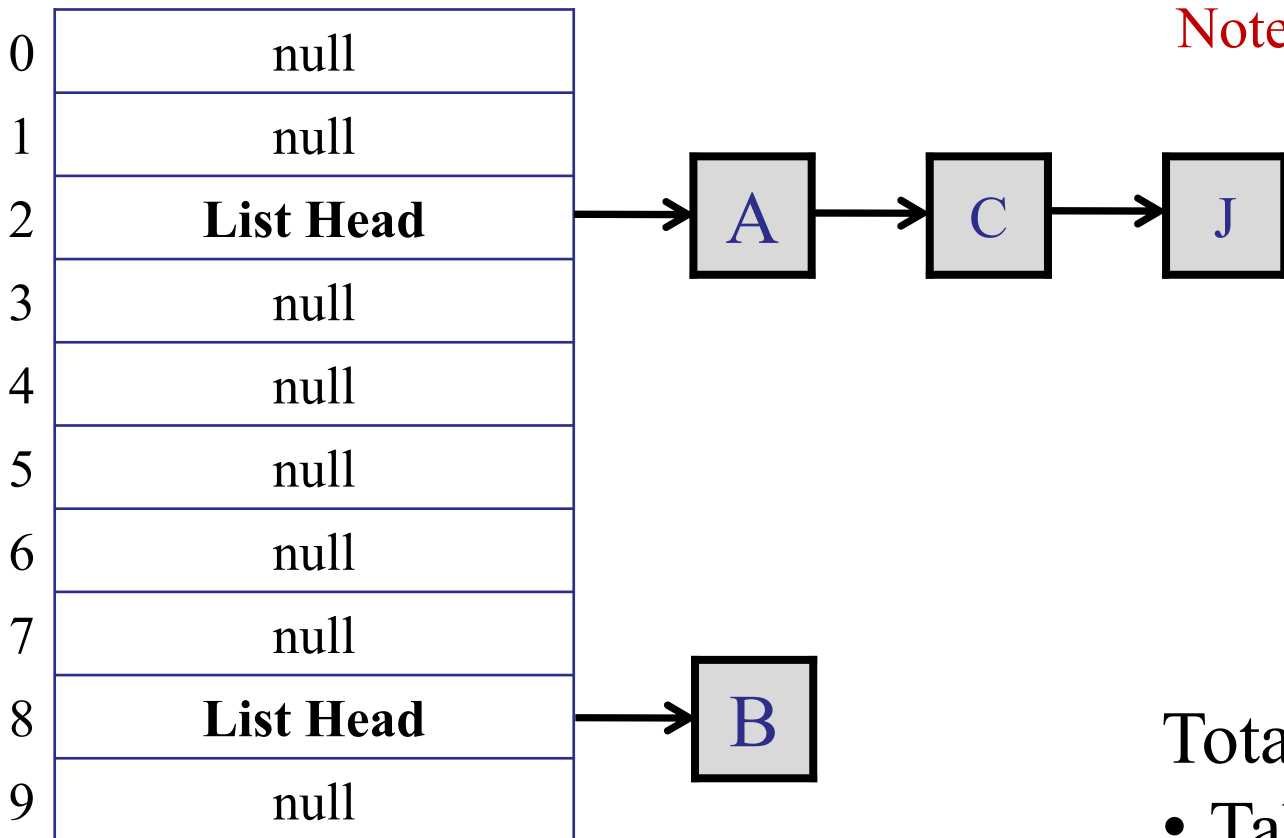
Total space:

- Number buckets: *m*
- Number entries: *n*

Chaining

Each bucket contains a linked list of items.

Note: h(A) == h(C) == h(J)



Total space: $O(m + n)$

- Table size: m
- Linked list size: n

Hashing with Chaining

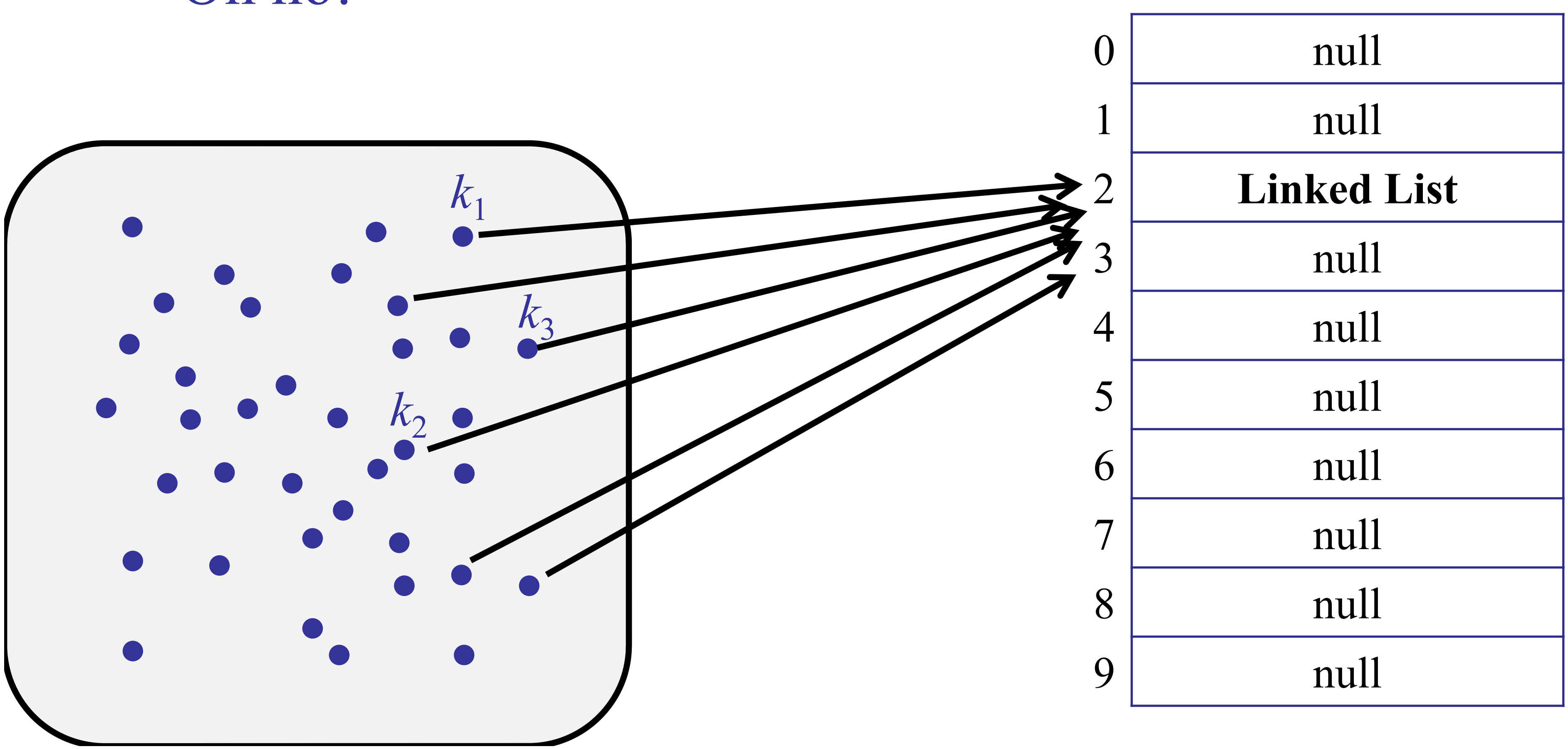
Operations:

- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.
- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

Hashing with Chaining

What if all keys hash to the same bucket!

- Worst-case search costs O(n)
- Oh no!



Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.
- Keys are mapped independently.

Intuition:

- Each key is put in a random bucket.
- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

Why don’t we just insert each key into a random bucket (instead of using a hash function h)?

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is open

Why don't we just insert each key into a random bucket (instead of using h)?

Slow to insert? No…

Where to get random numbers from? Not a problem in practice…

Might cause more collisions? No…

✓ Searching would be very slow. Yes! How do you find the item?

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / bucket.

- Expected search time = 1 + *expected # items per bucket*

hash function + array access

linked list traversal

Probability Theory

Set of outcomes for $X = (e_1, e_2, e_3, \ldots, e_k)$:

- $\Pr(e_1) = p_1$
- $\Pr(e_2) = p_2$
- …
- $\Pr(e_k) = p_k$

Expected outcome:

$$E[X] = e_1p_1 + e_2p_2 + \ldots + e_kp_k$$

Probability Theory

Linearity of Expectation:

- $E[A + B] = E[A] + E[B]$

Example:

- A = # heads in 2 coin flips
- B = # heads in 2 coin flips
- $A + B$ = # heads in 4 coin flips

$$E[A+B] = E[A] + E[B] = 1 + 1 = 2$$

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / bucket.

- Expected search time = 1 + *expected # items per bucket*

linked list traversal

hash function + array access

A little more probability

Indicator random variables

$$X(i, j) = 1 \text{ if item } i \text{ is put in bucket } j$$
$$= 0 \text{ otherwise}$$

Pr(X(i, j) == 1) = ?

1. 1/m
2. 1/n
3. 1/(m+n)
4. m/n
5. n/m
6. log(n)

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.
- Keys are mapped independently.

Intuition:

- Each key is put in a random bucket.
- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$= 0$ otherwise

$\mathbf{Pr}(X(i, j)==1) = 1/m$

A little probability

Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\mathbf{Pr}(X(i, j) == 1) = 1/m$$

$$\mathbf{E}(X(i, j)) = \mathbf{??}$$

A little probability

Indicator random variables

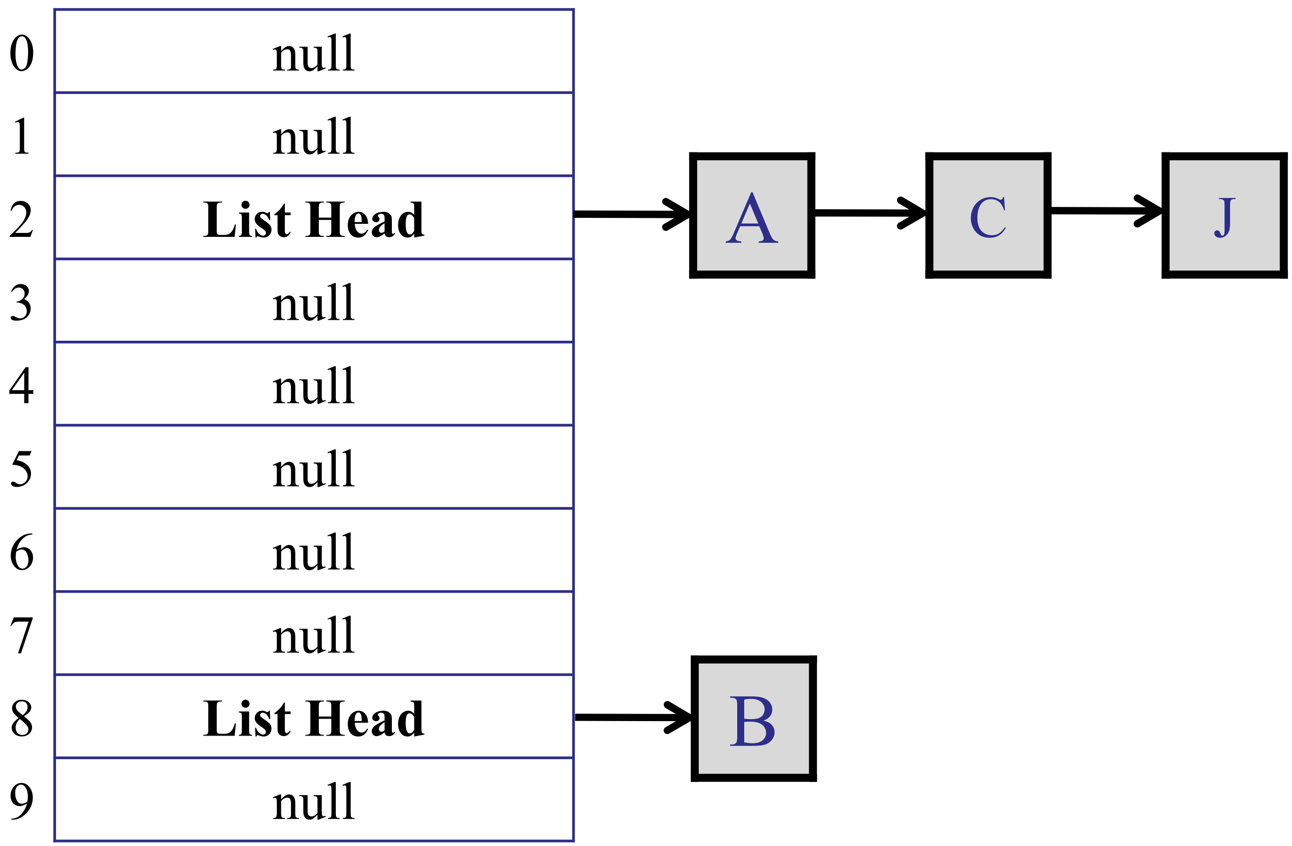
$$X(i, j) = 1 \text{ if item } i \text{ is put in bucket } j$$
$$= 0 \text{ otherwise}$$

$$\mathbf{Pr}(X(i, j)==1) = 1/m$$

$$\mathbf{E}(X(i, j)) = \mathbf{Pr}(X(i, j)==1)*1 + \mathbf{Pr}(X(i, j)==0)*0$$
$$= \mathbf{Pr}(X(i, j)==1)$$
$$= 1/m$$

A little probability

What is the expected number of items in a bucket?



A little probability

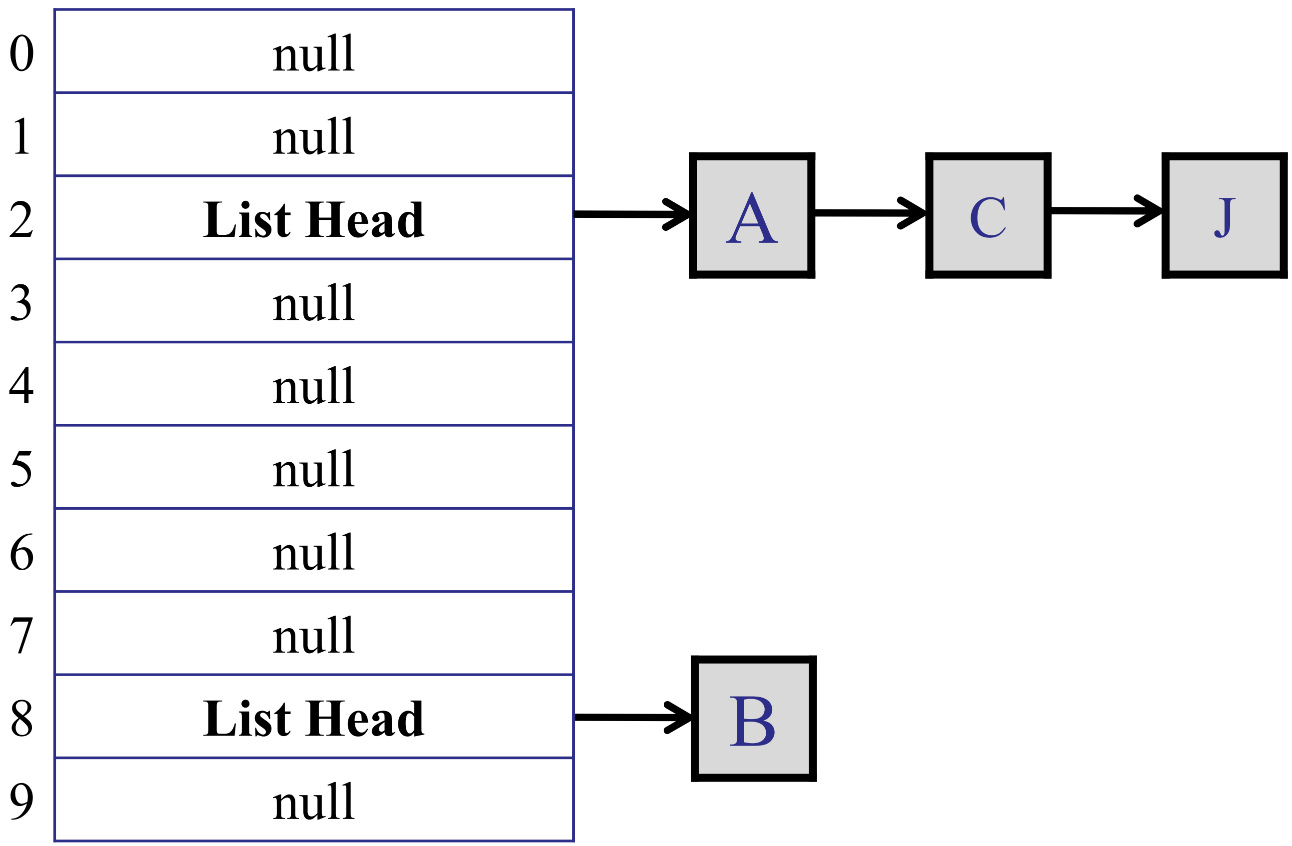
Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\Sigma_i\, X(i, b) = \text{number of items in bucket } b$$

A little probability

Each item contributes `1' to the bucket it is in..



A little probability

Indicator random variables

$X(i, j) = 1$ if item i is put in bucket j

$= 0$ otherwise

$\Sigma_i X(i, b) =$ number of items in bucket b

A little probability

Calculate expected number of items per bucket:

Expected (Σ_i X(i, b)) =

A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i X(i, b)) = \Sigma_i \mathbf{E}(X(i, b))$$

Linearity of expectation: $E(A + B) = E(A) + E(B)$

A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i \; X(i, b)) = \Sigma_i \; \mathbf{E} \; (X(i, b))$$

$$= \Sigma_i \; 1/m$$

$$= n/m$$

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - m buckets
- Define: load(hash table) = n/m

  = average # items / buckets.

- Expected search time = $1 + n/m$

hash function + array access

linked list traversal

Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - n items
  - $m = \Omega(n)$ buckets, e.g., $m = 2n$

- Expected search time $= 1 + n/m$

$$= O(1)$$

Hashing with Chaining

Searching:

- Expected search time = $1 + n/m = O(1)$
- Worst-case search time = O(n)

Inserting:

- Worst-case insertion time = O(1)

** In this case, inserting allows duplicates…
Preventing duplicates requires searching.

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: O(log n)

Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw n balls in m = n bins.
  - What is the maximum number of balls in a bin?

Cost: $\Theta(\log n / \log\log n)$

Hashing: Recap

Problem: coping with large universe of keys

- Number of possible keys is very, very large.
- Direct Access Table takes too much space

Hash functions

- Use hash function to map keys to buckets.
- Sometimes, keys collide (inevitably!)
- Use linked list to store multiple keys in one bucket.

Analyze performance with simple uniform hashing.

- Expected number of keys / bucket is $O(n/m) = O(1)$.

Today

- Collision resolution: chaining
- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();

hmap.put(foo, 8);
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Java Object

Every class implicitly extends Object

`public class Object`

`Object`	`clone()`	*creates a copy*
`boolean`	**`equals(Object obj)`**	***is obj equal to this?***
`void`	`finalize()`	*used by garbage collector*
`Class`	`getClass()`	*returns class*
`int`	**`hashCode()`**	***calculates hash code***
`void`	`notify()`	*wakes up a waiting thread*
`void`	`notifyAll()`	*wakes up all waiting threads*
`String`	**`toString()`**	***returns string representation***
`void`	`wait(…)`	*wait until notified*

Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();

int hash = foo.hashCode();

hmap.put(foo, 8);
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

Is it legal for every object to return 32?

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

Is it legal for every object to return 32? (YES)

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Default Java implementation:

- hashCode returns the memory location of the object
- Every object hashes to a different location

Must implement/override `hashCode()` for your class.

Java Library Classes

`Integer`

`Long`

`String`

Integer

```
public int hashCode() {
   return value;
}
```

Note: hashcode is always a 32-bit integer.

Note: every 32-bit integer gets a unique hashcode.

What do you do for smaller hash tables?
Can there be collisions?

Long

```
public int hashCode() {
   return (int)(value ^ (value >>> 32));
}
```

$$\text{hash}(\overbrace{0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0}^{\text{32 bits}}\,\overbrace{0\,0\,1\,0\,1\,0\,0\,0\,0\,1\,1\,0\,0\,1\,0\,0}^{\text{32 bits}})$$

```
     0 1 1 0 0 1 0 1 1 0 0 1 1 1 0 0
XOR  0 0 1 0 1 0 0 0 0 1 1 0 0 1 0 0
-------------------------------------
     0 1 0 0 1 1 0 1 1 1 1 1 1 0 0 0
```

String

```
public int hashCode() {
   int h = hash; // only calculate hash once
   if (h == 0 && count > 0) {  // empty = 0
        int off = offset;
        char val[] = value;
        int len = count;
        for (int i = 0; i < len; i++) {
             h = 31*h + val[off++];
        }
        hash = h;
   }
   return h;
}
```

String

HashCode calculation:

```
hash = s[0]*31^(n-1) +
       s[1]*31^(n-2) +
       s[2]*31^(n-3) +
       ... +
       s[n-2]*31 +
       s[n-1]
```

Why did they choose 31?

String

HashCode calculation:

```
hash = s[0]*31^(n-1) +
       s[1]*31^(n-2) +
       s[2]*31^(n-3) +
       ... +
       s[n-2]*31 +
       s[n-1]
```

Why did they choose 31? Prime, 2^5-1

Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }
}
```

Creating a new class

```
public void testPair() {

  HashMap<Pair, Integer> htable =
          new HashMap<Pair, Integer>();

  Pair one = new Pair(20, 40);
  htable.put(one, 7);

  Pair two = new Pair(20, 40);
  int question = htable.get(two);
}
```

htable.get(new Pair(20, 20)) == ?

1. 1
2. 7
3. 11
4. null

Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);

one.hashCode() != two.hashCode()
```

Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, "first item");

htable.get(one) → "first item"

htable.get(two) → null
```

Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }

  int hashCode(){
     return (first ^ second);
  }
}
```

Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, "first item");

htable.get(one) → "first item"
htable.get(two) → null

one.equals(two) → false
```

Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.
- **Must redefine .equals to be consistent with hashCode.**

Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, “first item”);

htable.get(one) => “first item”

htable.get(two) => null
```

Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

Rules:

- **Reflexive**: x.equals(x) ➔ true
- **Symmetric**: x.equals(y) == y.equals(x)
- **Transitive**: x.equals(y), y.equals(z) ➔ x.equals(z)
- **Consistent**: always returns the same answer
- **Null is null**: x.equals(null) ➔ false

Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

```
boolean equals(Object p){
  if (p == null) return false;
  if (p == this) return true;

  if (!(p instanceOf Pair)) return false;
  Pair pair = (Pair)p;

  if (pair.first != first) return false;
  if (pair.second != second) return false;
  return true;
}
```

Java HashMap

```
public V get(Object key) {
   if (key == null) return getForNullKey();
   int hash = hash(key.hashCode());
   for (Entry<K,V> e = table[indexFor(hash,table.length)];
        e != null;
        e = e.next)
   {
      Object k;
      if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
         return e.value;
   }
   return null;
}
```

Java HashMap

```
// This function ensures that hashCodes that differ only
// by constant multiples at each bit position have a
// bounded number of collisions (approximately 8 at
// default load factor).

static int hash(int h) {
   h ^= (h >>> 20) ^ (h >>> 12);
   return h ^ (h >>> 7) ^ (h >>> 4);
}
```

Java HashMap

```
public V get(Object key) {
   if (key == null) return getForNullKey();
   int hash = hash(key.hashCode());
   for (Entry<K,V> e = table[indexFor(hash,table.length)];
        e != null;
        e = e.next)
   {
      Object k;
      if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
         return e.value;
   }
   return null;
}
```

Java HashMap

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public V get(Object key) {
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   for (Entry<K,V> e = table[indexFor(hash,table.length)];
        e != null;
        e = e.next)
   {
      Object k;
      if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
         return e.value;
   }
   return null;
}
```

Java checks if the key is equal to the item in the hash table before returning it!

Today

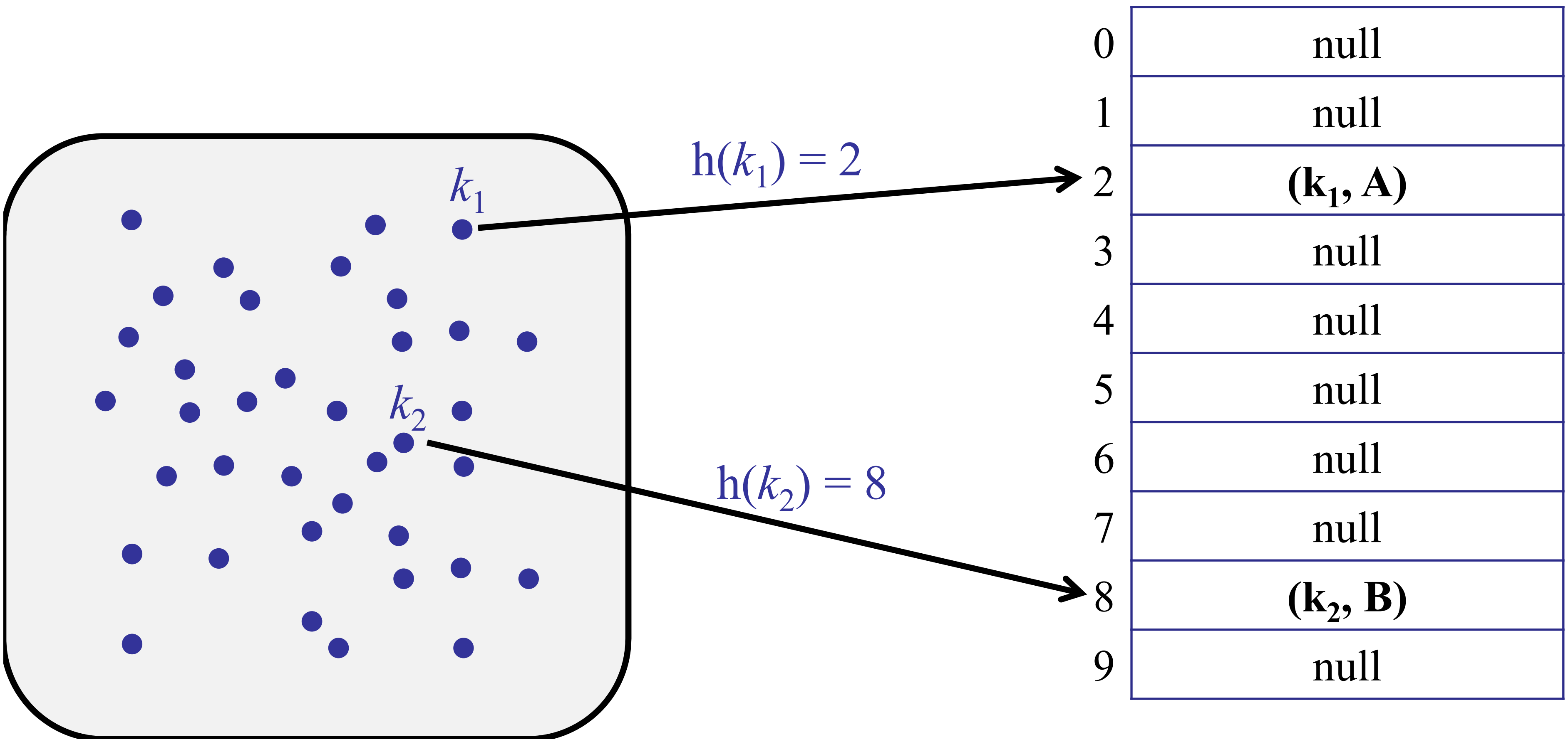
- Collision resolution: chaining
- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

Review

Hash Tables

– Store each item from the symbol table in a table.

– Use hash function to map each key to a bucket.



Resolving Collisions

- Basic problem:
  - What to do when two items hash to the same bucket?
- Solution 1: Chaining
  - Insert item into a linked list.
- Solution 2: Open Addressing
  - Find another free bucket.

Open Addressing

Advantages:

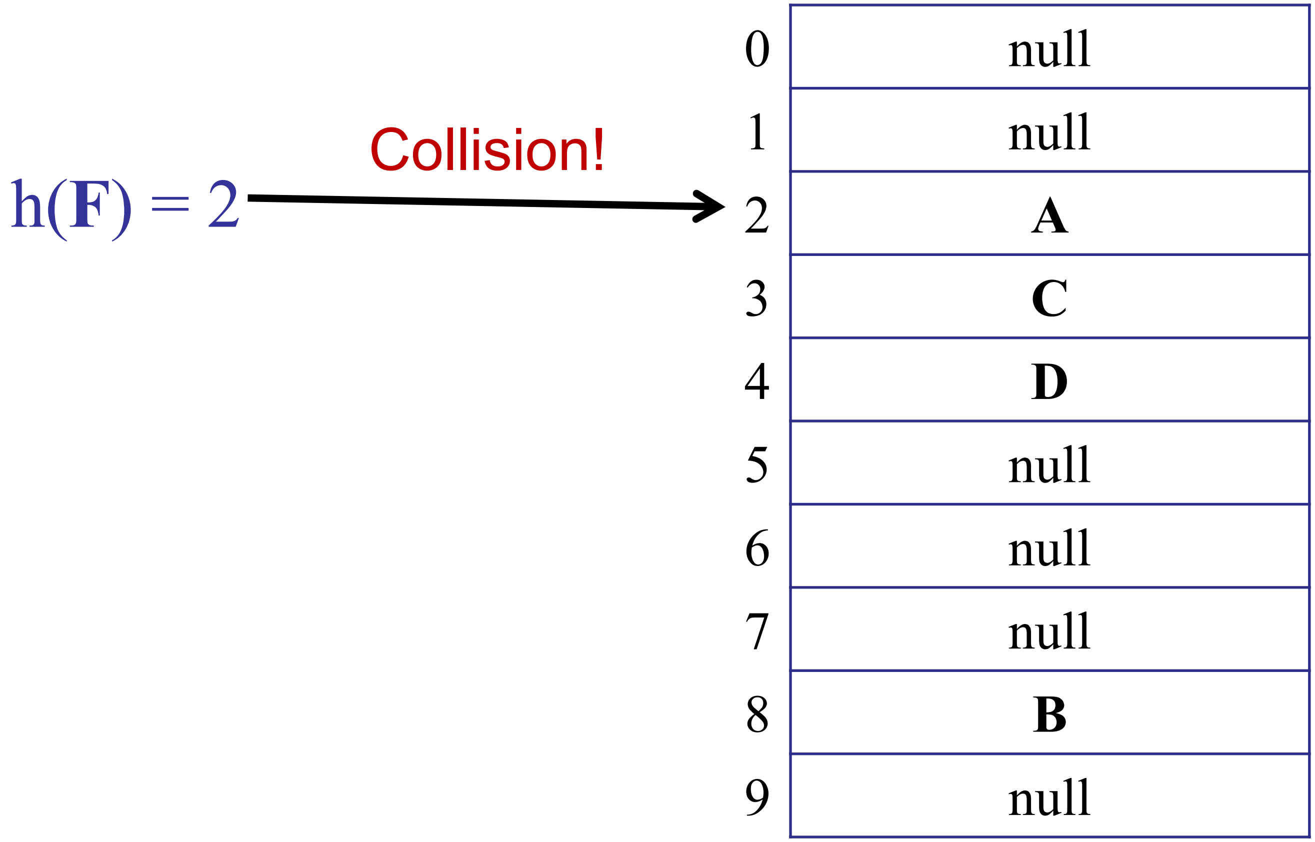
- No linked lists!
- All data directly stored in the table.
- One item per slot.

0	null
1	null
2	**A**
3	null
4	null
5	null
6	null
7	null
8	**B**
9	null

Open Addressing

On collision:

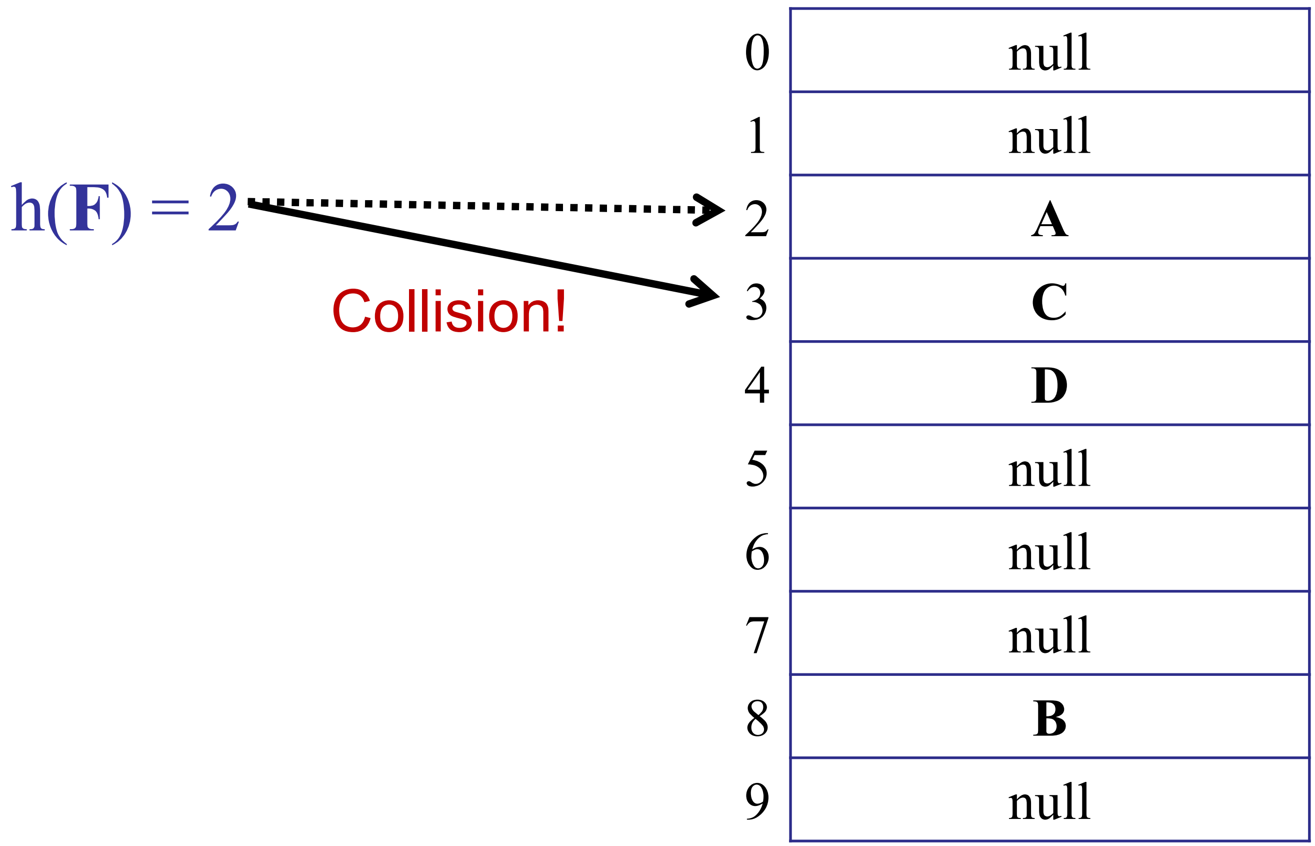
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

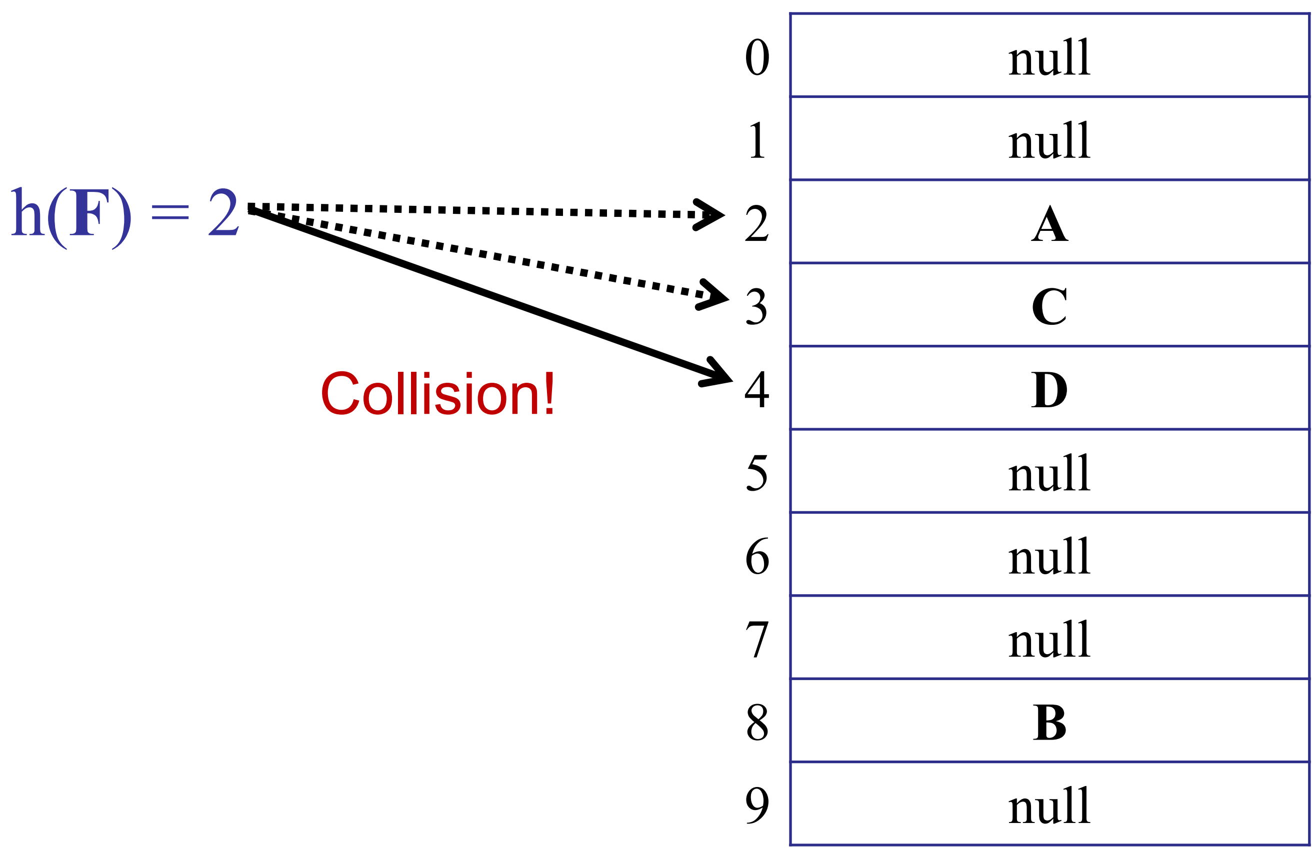
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

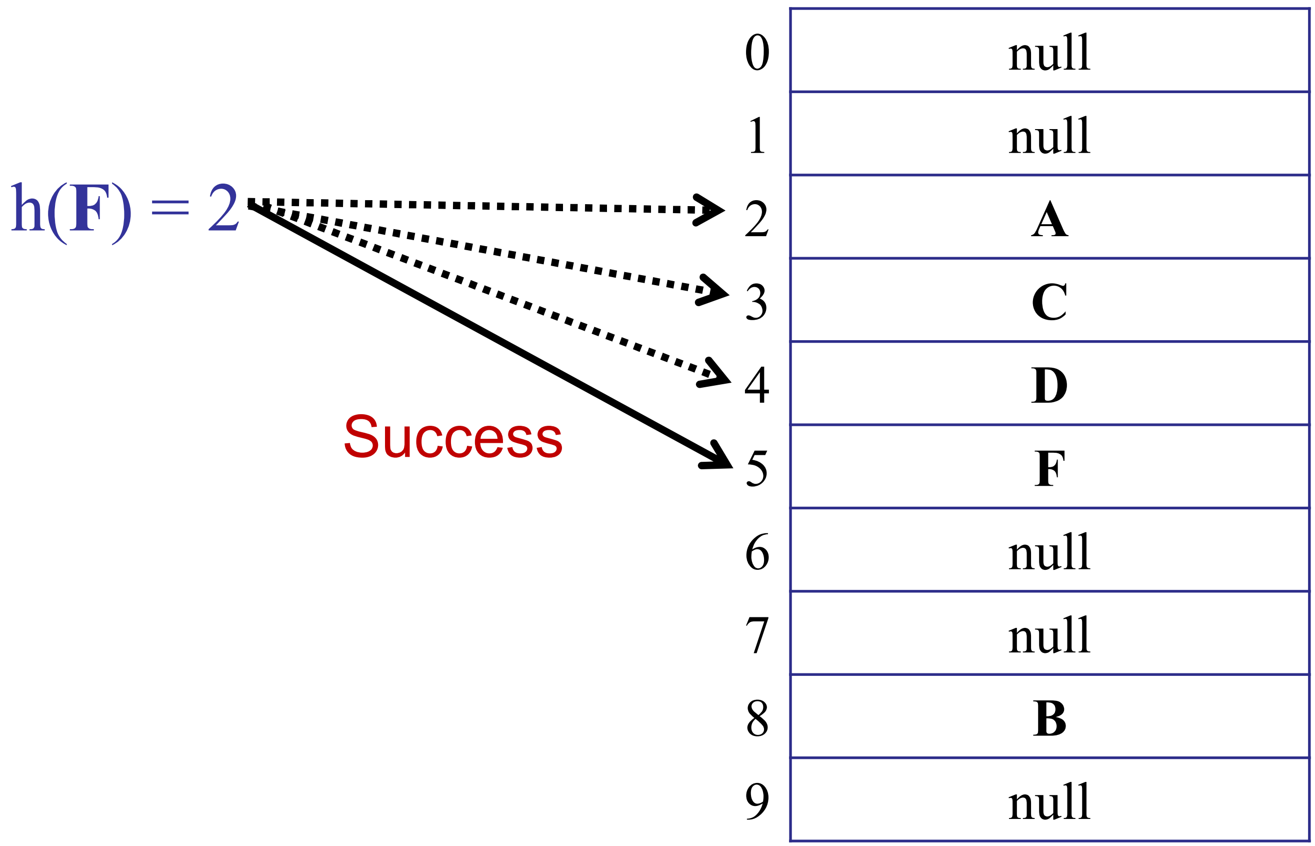
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

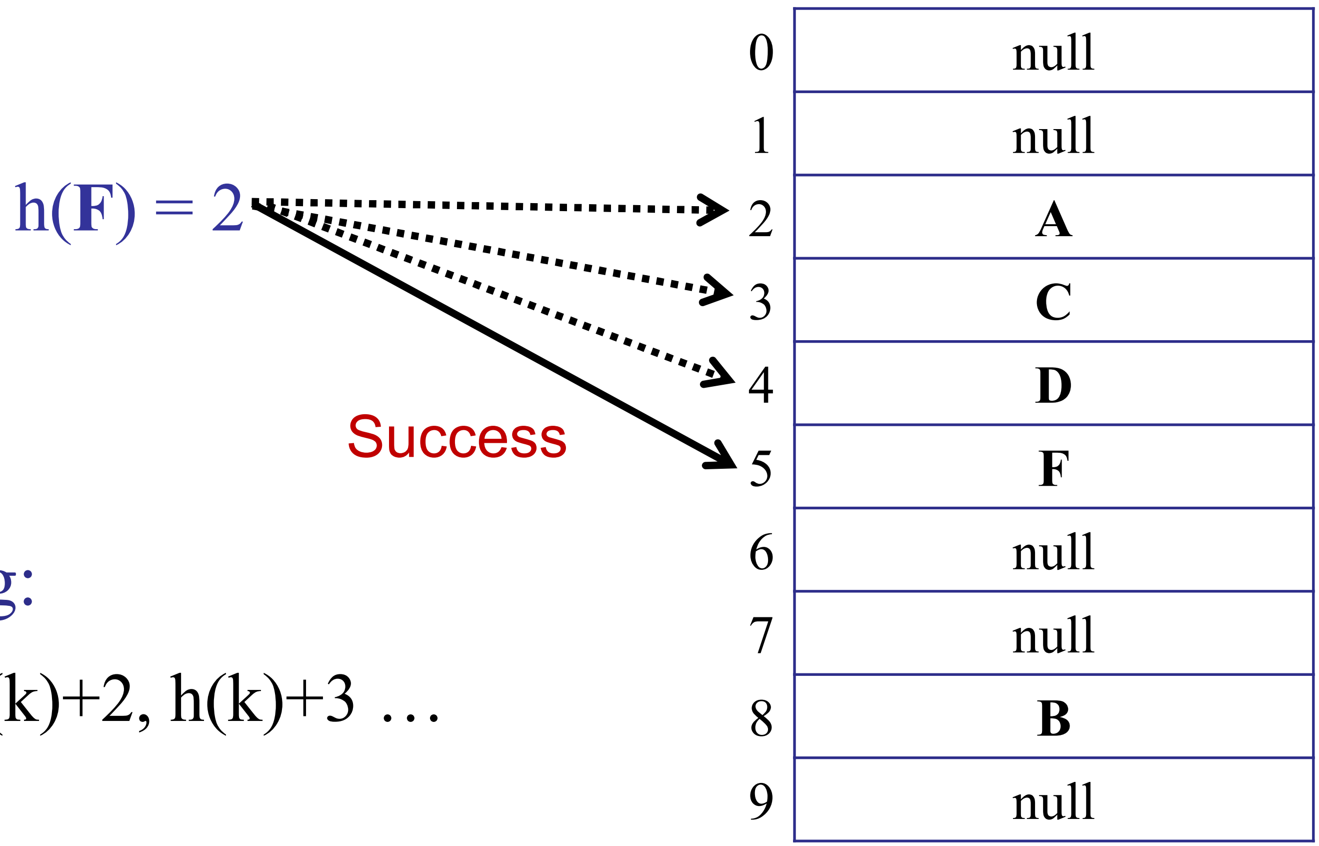
Probe a sequence of buckets until you find an empty one.



Open Addressing

On collision:

Probe a sequence of buckets until you find an empty one.



Linear Probing:

- h(k)+1, h(k)+2, h(k)+3 …

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Two parameters:

- key : the thing to map
- i : number of collisions

Open Addressing

Hash Function re-defined:

$h(\text{key}, i) : U \rightarrow \{1..m\}$

Example: Linear Probing

- $h(k, 1)$ = hash of key k
- $h(k, 2) = h(k, 1) + 1$
- $h(k, 3) = h(k, 1) + 2$
- $h(k, 4) = h(k, 1) + 3$
- …
- $h(k, i) = h(k, 1) + i \bmod m$

0	null
1	null
2	**A**
3	**C**
4	**D**
5	**F**
6	null
7	null
8	**B**
9	null

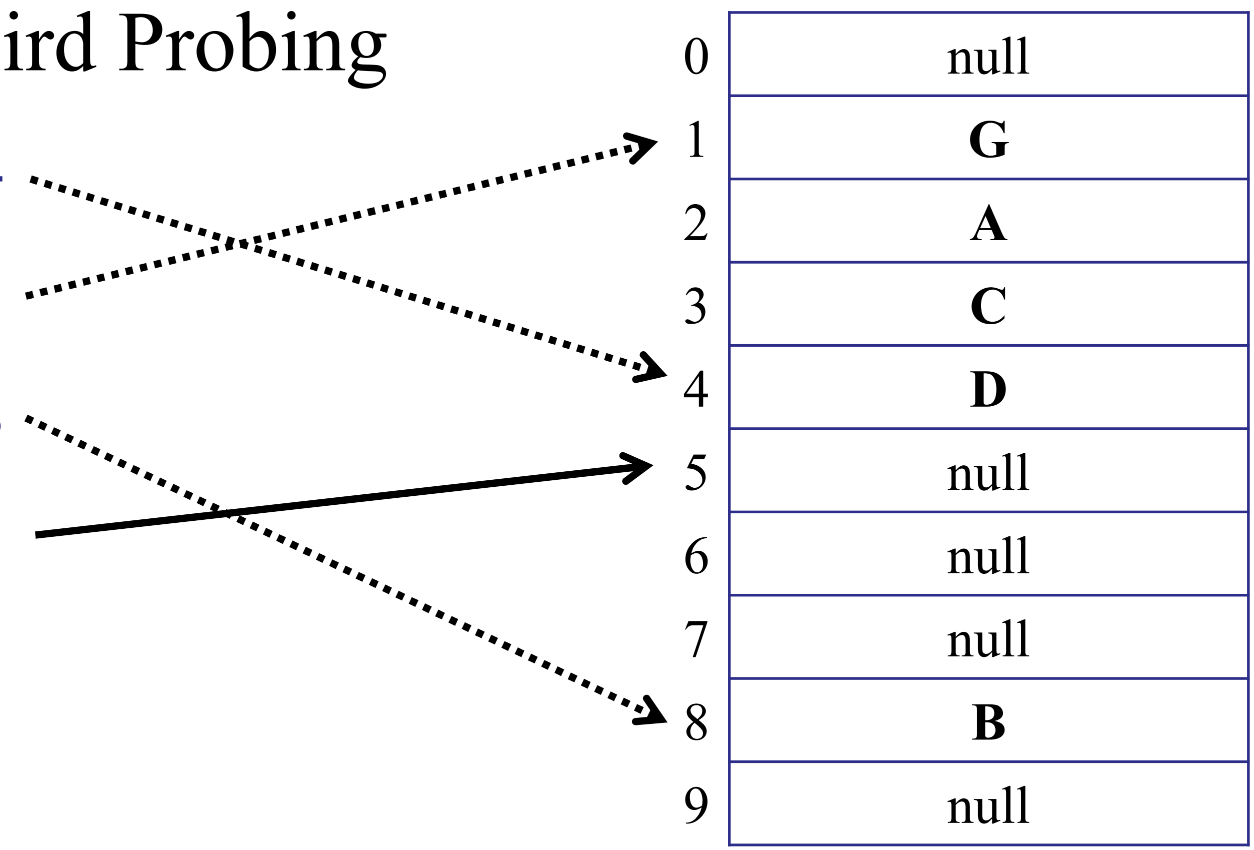
Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Example: Weird Probing

- $h(k, 1) = 4$
- $h(k, 2) = 1$
- $h(k, 3) = 8$
- $h(k, 4) = 5$



Open Addressing

```
hash-insert(key, data)
int i = 1;
while (i <= m) {                               // Try every bucket
      int bucket = h(key, i);
      if (T[bucket] == null){           // Found an empty bucket
            T[bucket] = {key, data};    // Insert key/data
            return success;             // Return
      }
      i++;
}
throw new TableFullException();         // Table full!
```

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

search(key)

- $h(key, 1) = 4$
- $h(key, 2) = 1$
- $h(key, 3) = 8$
- $h(key, 4) = 5$

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

```
hash-search(key)
int i = 1;
while (i <= m) {
    int bucket = h(key, i);
    if (T[bucket] == null)  // Empty bucket!
        return key-not-found;
    if (T[bucket].key == key)  // Full bucket.
            return T[bucket].data;
    i++;
}
return key-not-found;  // Exhausted entire table.
```

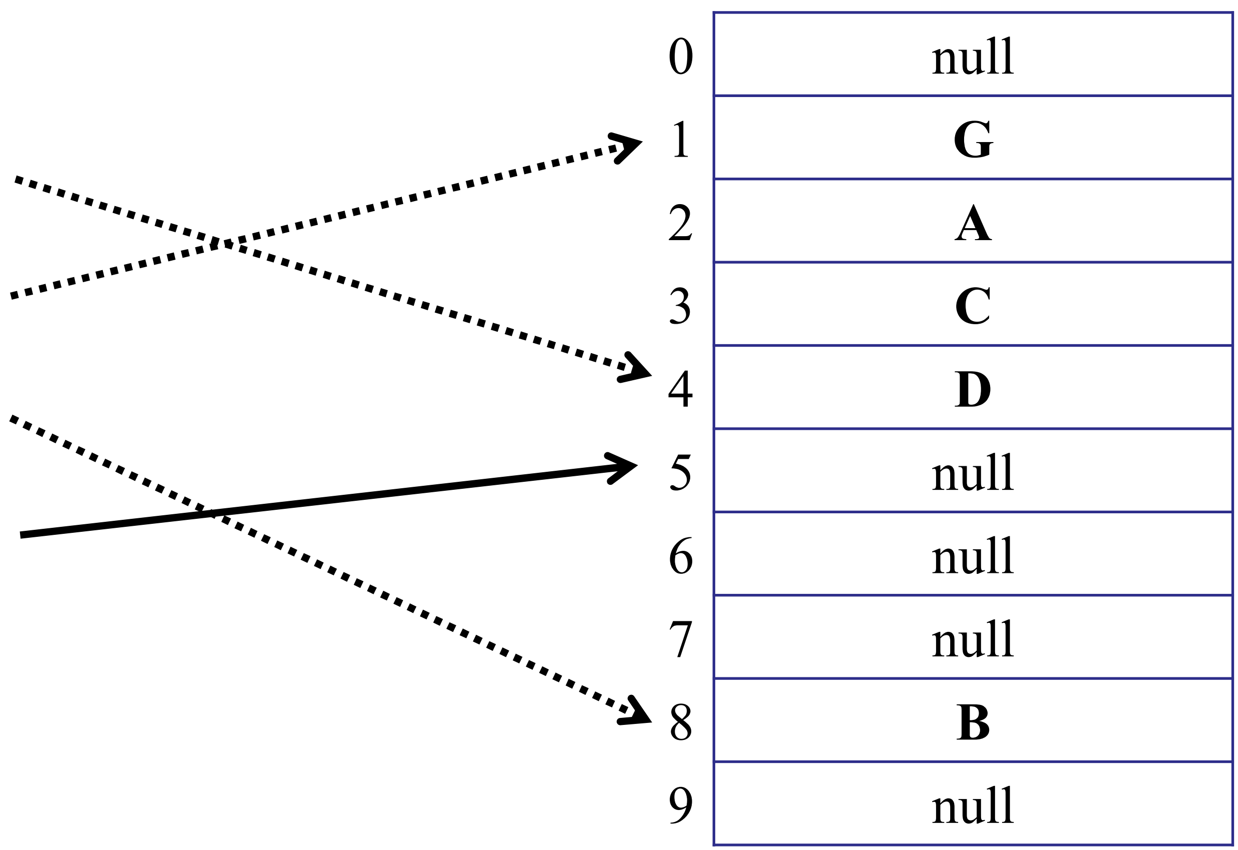
Open Addressing

Hash Function re-defined:

h(key, i) : U → {1..m}

search(key)

– h(key, 1) = 4
– h(key, 2) = 1
– h(key, 3) = 8
– h(key, 4) = 5



Open Addressing

Hash Function re-defined:

h(key, i) : U → {1..m}

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to null.

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	**NULL**
6	null
7	null
8	**B**
9	null

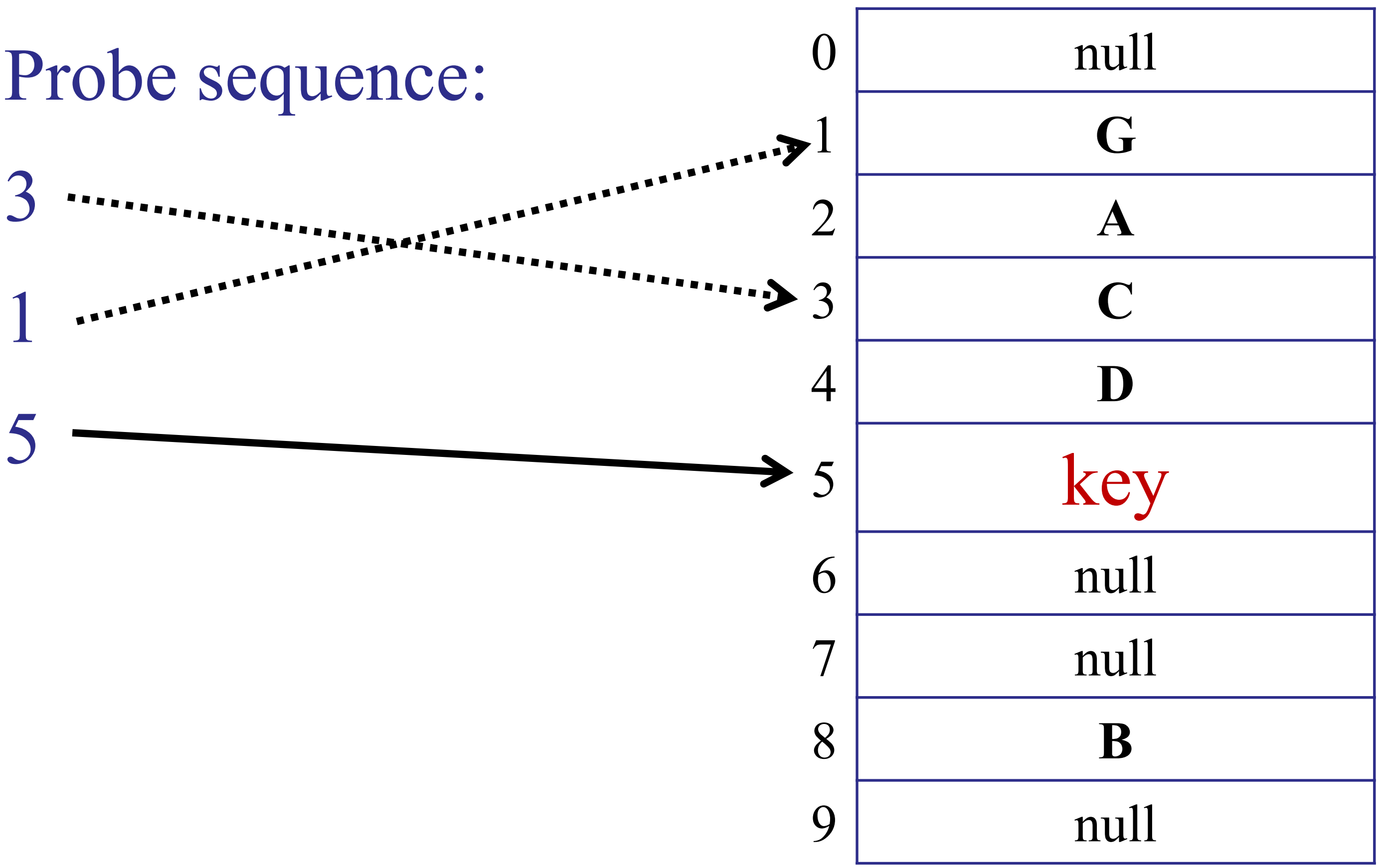
What is wrong with delete?

✓ 1. Search may fail to find an element.
2. The table will have gaps in it.
3. Space is used inefficiently.
4. If the key is inserted again, it may end up in a different bucket.

Open Addressing

insert(key)

Probe sequence:



Open Addressing

insert(key)

delete(G)

0	null
1	**G → NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

5

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

Not found!

0	null
1	**NULL**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to DELETED.

(Tombstone value.)

0	null
1	**G**
2	**A**
3	**C**
4	**D**
5	**DELETED**
6	null
7	null
8	**B**
9	null

Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

5

0	null
1	**DELETED**
2	**A**
3	**C**
4	**D**
5	key
6	null
7	null
8	**B**
9	null

What happens when an insert finds a DELETED cell?

1. Overwrite the deleted cell.
2. Continue probing.
3. Fail.

Hash Functions

Two properties of a good hash function:

1. h(*key*, *i*) enumerates all possible buckets.
   - For every bucket *j*, there is some *i* such that:

     $$h(key, i) = j$$

   - The hash function is permutation of $\{1..m\}$.
   - For linear probing: true!

What goes wrong if the sequence is not a permutation?

1. Search incorrectly returns key-not-found.
2. Delete fails.
3. Insert puts a key in the wrong place
4. Returns table-full even when there is still space left.

Hash Functions

Two properties of a good hash function:

2. Simple Uniform Hashing Assumption

   Every key is equally likely to be mapped to every bucket, independently of every other key.

For h(*key*, 1)?

For every h(*key*, *i*)?

Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

n! permutations for probe sequence: e.g.,

- 1 2 3 4
- 1 2 4 3
- 1 4 2 3
- 1 4 3 2
- …

Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

n! permutations for probe sequence: e.g.,

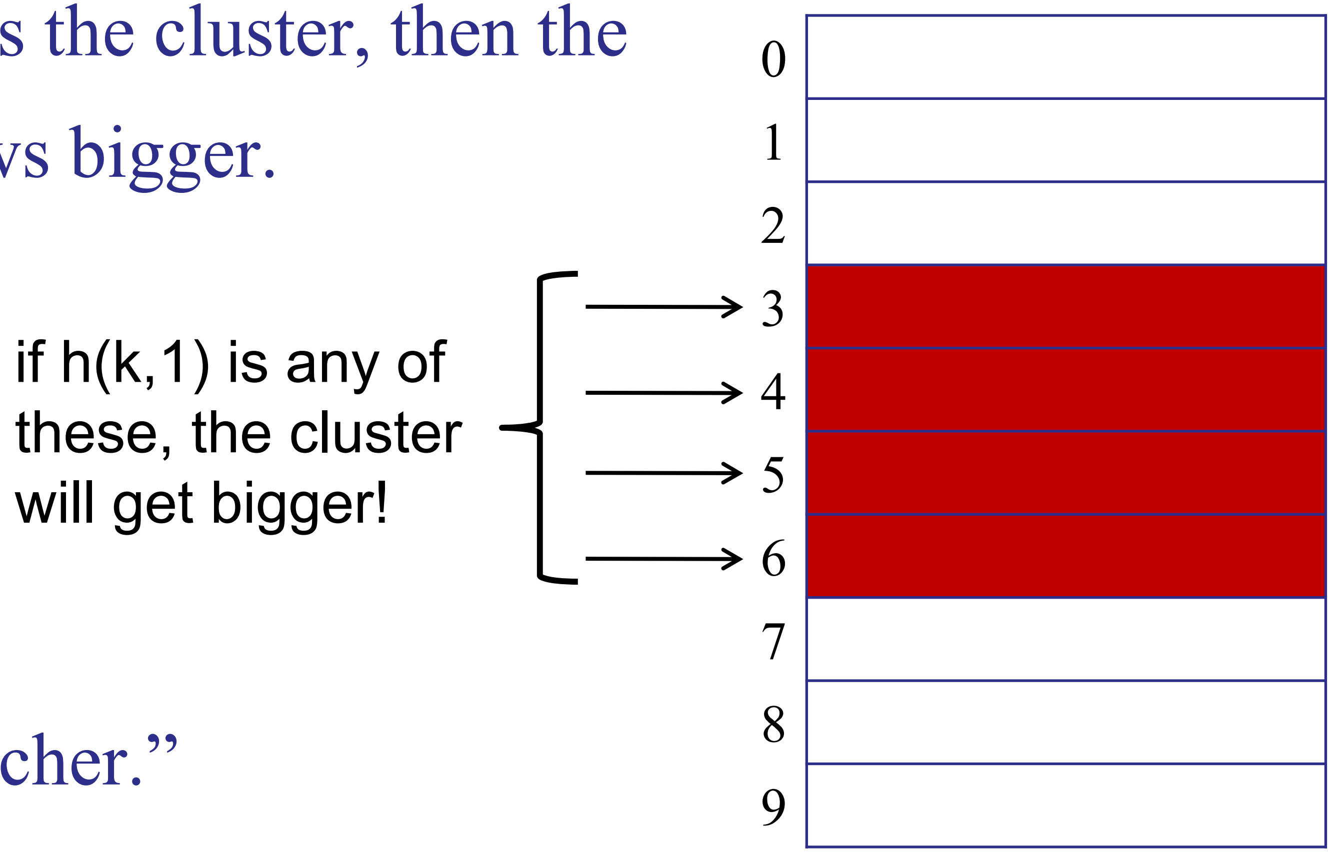
- 1 2 3 4 Pr(1/m)
- 1 2 4 3 Pr(0)
- 1 4 2 3 Pr(0)
- 1 4 3 2 Pr(0)
- …

NOT Linear Probing

Linear Probing

Problem with linear probing: ***clusters***

- If there is a cluster, then there is a higher probability that the next h(k) will hit the cluster.
- If h(k,1) hits the cluster, then the cluster grows bigger.



- "Rich get richer."

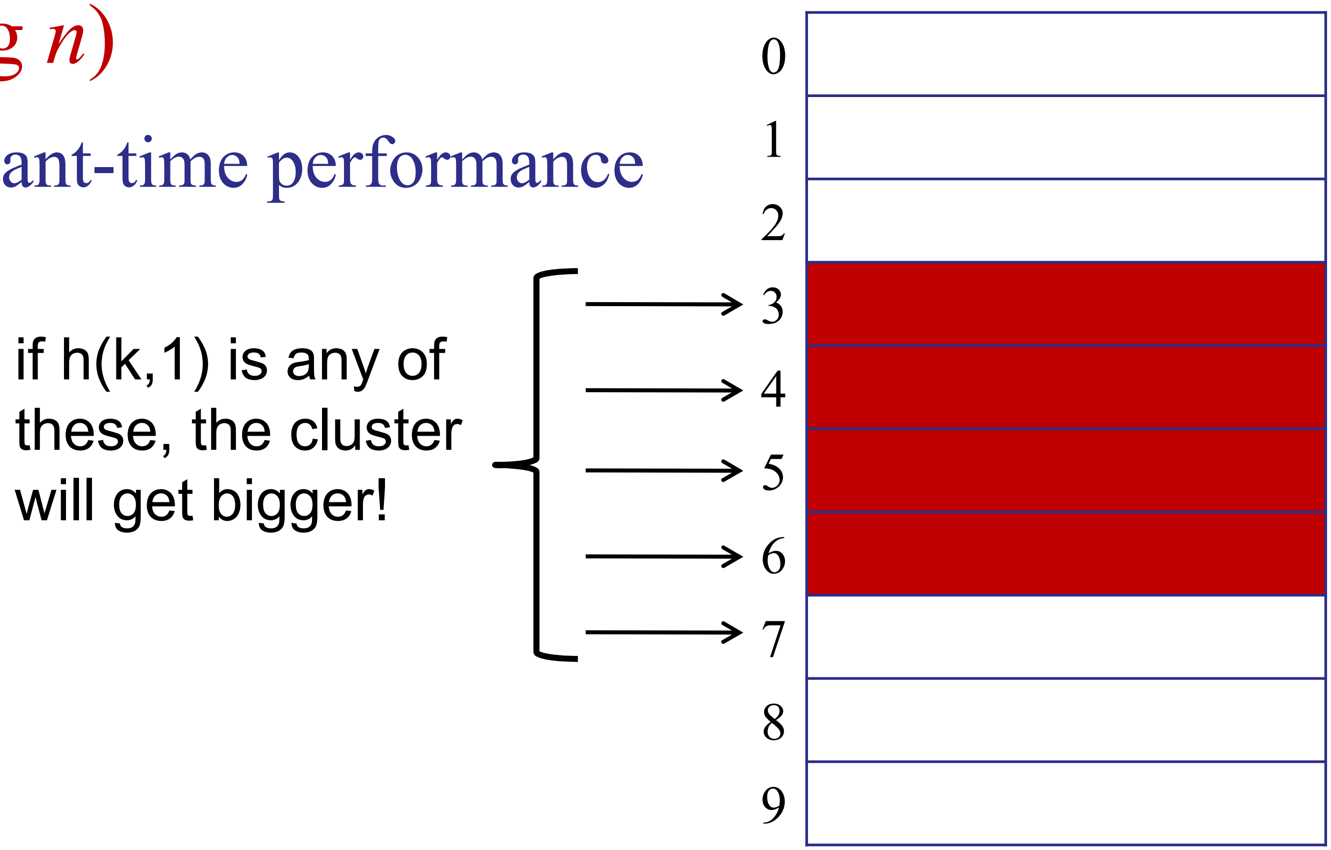
Linear Probing

Problem with linear probing: ***clusters***

- If the table is 1/4 full, then there will be clusters of size:

  $\theta(\log n)$

- Ruins constant-time performance



Linear probing

In practice, linear probing is very fast!

- Why? Caching!
- It is *cheap* to access nearby array cells.
  - Example: access T[17]
  - Cache loads: T[10..50]
  - Almost 0 cost to access T[18], T[19], T[20], …
- If the table is 1/4 full, then there will be clusters of size: $\theta(\log n)$
  - Cache may hold entire cluster!
  - No worse than wacky probe sequence.

Open Addressing

Properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

n! permutations for probe sequence: e.g.,

- 1 2 3 4
- 1 2 4 3
- 1 4 2 3
- 1 4 3 2
- …

Double Hashing

- Start with two ordinary hash functions:

$$f(k),\ g(k)$$

- Define new hash function:

$$h(k, i) = f(k) + i \cdot g(k) \mod m$$

- Note:
  - Since f(k) is good, f(k, 1) is “almost” random.
  - Since g(k) is good, the probe sequence is “almost” random.

Double Hashing

Hash function

$$h(k, i) = f(k) + i \cdot g(k) \mod m$$

Claim: if $g(k)$ is relatively prime to m, then $h(k, i)$ hits all buckets.

- Assume not: then for some distinct $i, j < m$:

$$f(k) + i \cdot g(k) = f(k) + j \cdot g(k) \mod m$$

$$\Rightarrow i \cdot g(k) = j \cdot g(k) \mod m$$

$$\Rightarrow (i - j) \cdot g(k) = 0 \mod m$$

$\Rightarrow$ g(k) not relatively prime to m, since $(i, j < m)$

Double Hashing

Hash function

$$h(k, i) = f(k) + i \cdot g(k) \mod m$$

Claim: if g(k) is relatively prime to m, then $h(k, i)$ hits all buckets.

Example: if $(m = 2^r)$, then choose $g(k)$ odd.

Performance of Open Addressing

If (m==n), what is the expected insert time, under uniform hashing assumption?

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n^2)$
5. None of the above.

Performance of Open Addressing

- Chaining:
  - When (m==n), we can still add new items to the hash table.
  - We can still search efficiently.

- Open addressing:
  - When (m==n), the table is full.
  - We cannot insert any more items.
  - We cannot search efficiently.

Performance of Open Addressing

Define:

- Load $\alpha = n / m$

  Average # items / bucket

- Assume $\alpha < 1$.

Performance of Open Addressing

Define:

- Load $\alpha = n / m$ ← Average # items / bucket
- Assume $\alpha < 1$.

Claim:

For n items, in a table of size m, assuming *uniform hashing*, the expected cost of an operation is:

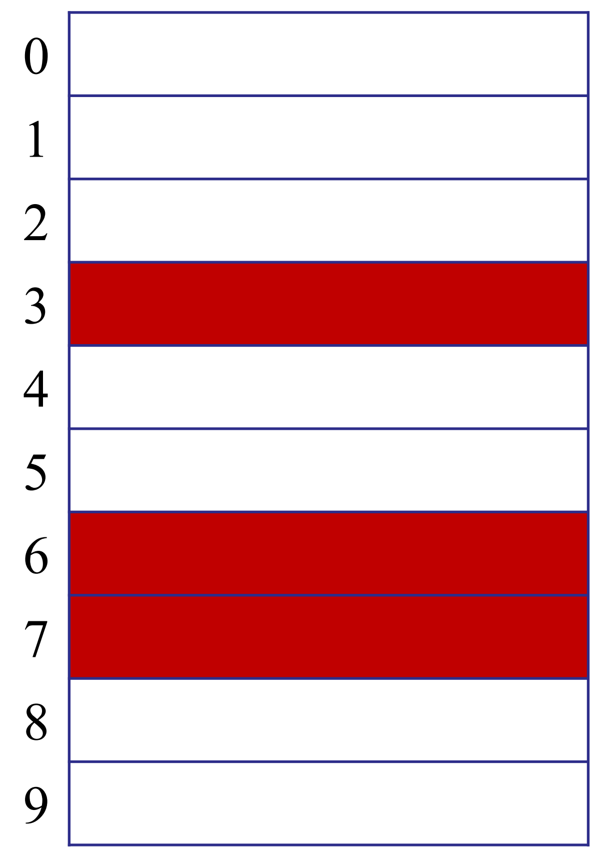
$$\leq \frac{1}{1-\alpha}$$

Example: if (α=90%), then E[# probes] = 10

Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: *n/m*

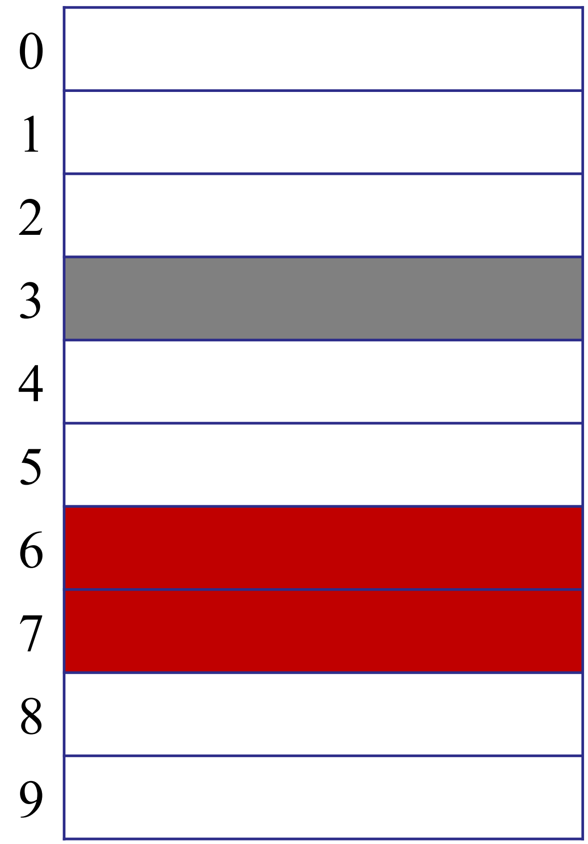


Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: n/m

- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n-1)/(m-1)$



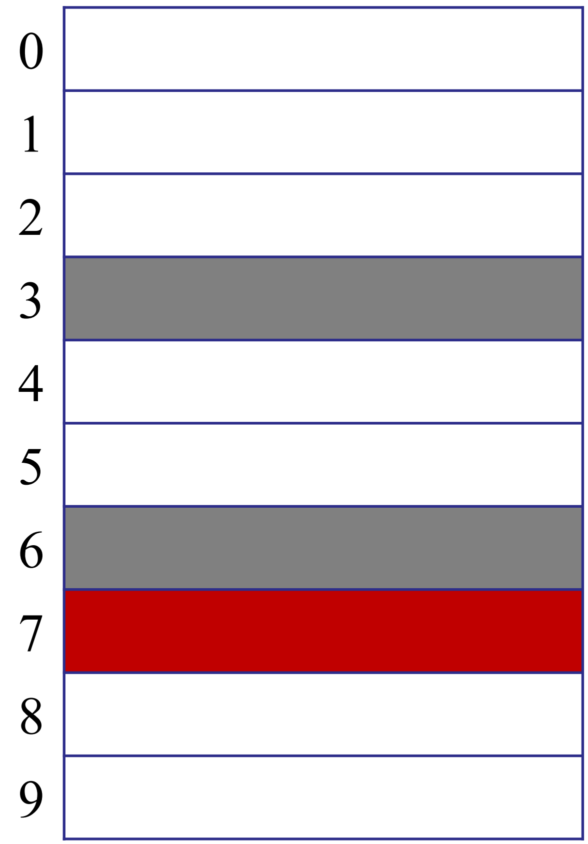
Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: n/m

- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n - 1) / (m - 1)$

- Third probe: probability is full: $(n - 2) / (m - 2)$



Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$1+\frac{n}{m}\left(\boxed{\text{Expected cost of remaining probes}}\right)$$

First probe

Probability of collision on first probe

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(\boxed{\text{Expected cost of remaining probes}}\right)\right)$$

First probe

Probability of collision on first probe

Probability of collision on second probe

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\right)\right)\right)$$

First probe Second probe Third probe

Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\right)\right)\right)$$

- Note:

$$\frac{n-i}{m-i}\leq\frac{n}{m}\leq\alpha$$

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\ \square\ \square\ \square\ \right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\quad \right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

$$\leq 1+\alpha+\alpha^2+\alpha^3+\cdots$$

Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\cdots\right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

$$\leq 1+\alpha+\alpha^2+\alpha^3+\cdots$$

$$\leq \frac{1}{1-\alpha}$$

Performance of Open Addressing

Define:

- Load $\alpha = n / m$ — Average # items / bucket
- Assume $\alpha < 1$.

Claim:

For n items, in a table of size m, assuming *uniform hashing*, the expected cost of an operation is:

$$\leq \frac{1}{1-\alpha}$$

Example: if (α=90%), then E[# probes] = 10

Advantages…

Open addressing:

- Saves space
  - Empty slots vs. linked lists.
- Rarely allocate memory
  - No new list-node allocations.
- Better cache performance
  - Table all in one place in memory
  - Fewer accesses to bring table into cache.
  - Linked lists can wander all over the memory.

Disadvantages…

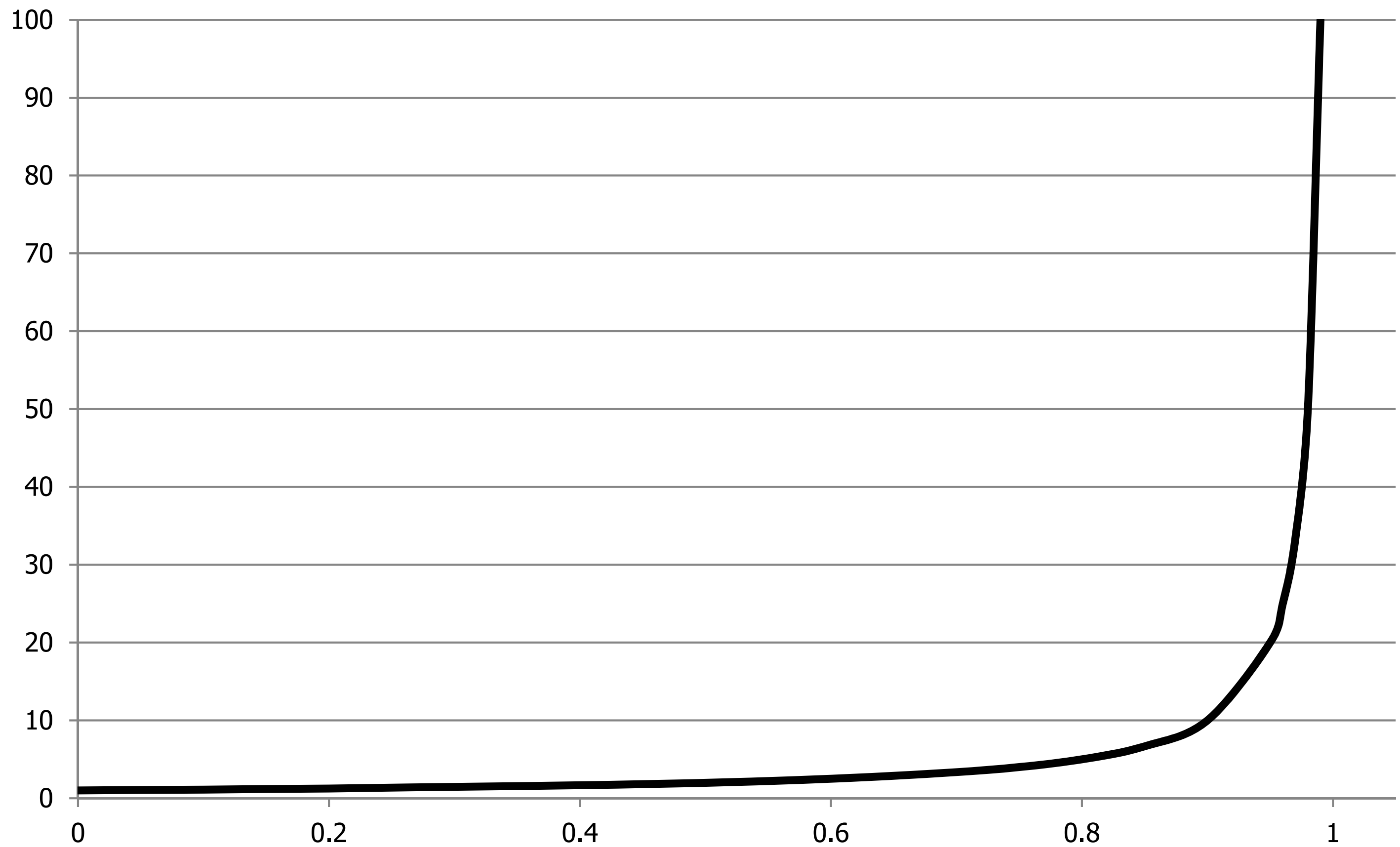
Open addressing:

- More sensitive to choice of hash functions.
  - Clustering is a common problem.
  - See issues with linear probing.
- More sensitive to load.
  - Performance degrades badly as $\alpha \rightarrow 1$.

Disadvantages…

Open addressing:

– Performance degrades badly as $\alpha \rightarrow 1$.



Today

- Collision resolution: chaining
- Java hashing
- Collision resolution: open addressing
- Table (re)sizing