CS2040S
Data Structures and Algorithms

BFS, DFS, and Directed Graphs!

Roadmap

Last time: Graph Basics

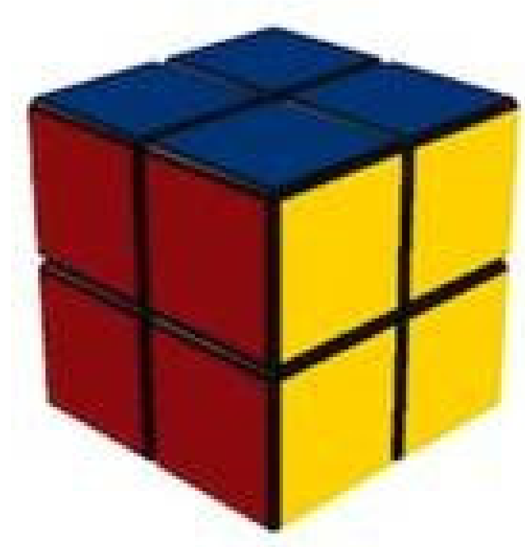
- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs: BFS

What is a graph?

Graph G = <V, E>

- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$
  - For all $e_1, e_2 \in E : e_1 \neq e_2$

2 x 2 x 2 Rubik's Cube



Configuration Graph

- Vertex for each possible state
- Edge for each basic move
  - 90 degree turn
  - 180 degree turn

Puzzle: given initial state, find a path to the solved state.

Trade-offs

Adjacency Matrix:

- Fast query: are v and w neighbors?
- Slow query: find me any neighbor of v.
- Slow query: enumerate all neighbors.

Adjacency List:

- Fast query: find me any neighbor.
- Fast query: enumerate all neighbors.
- Slower query: are v and w neighbors?

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

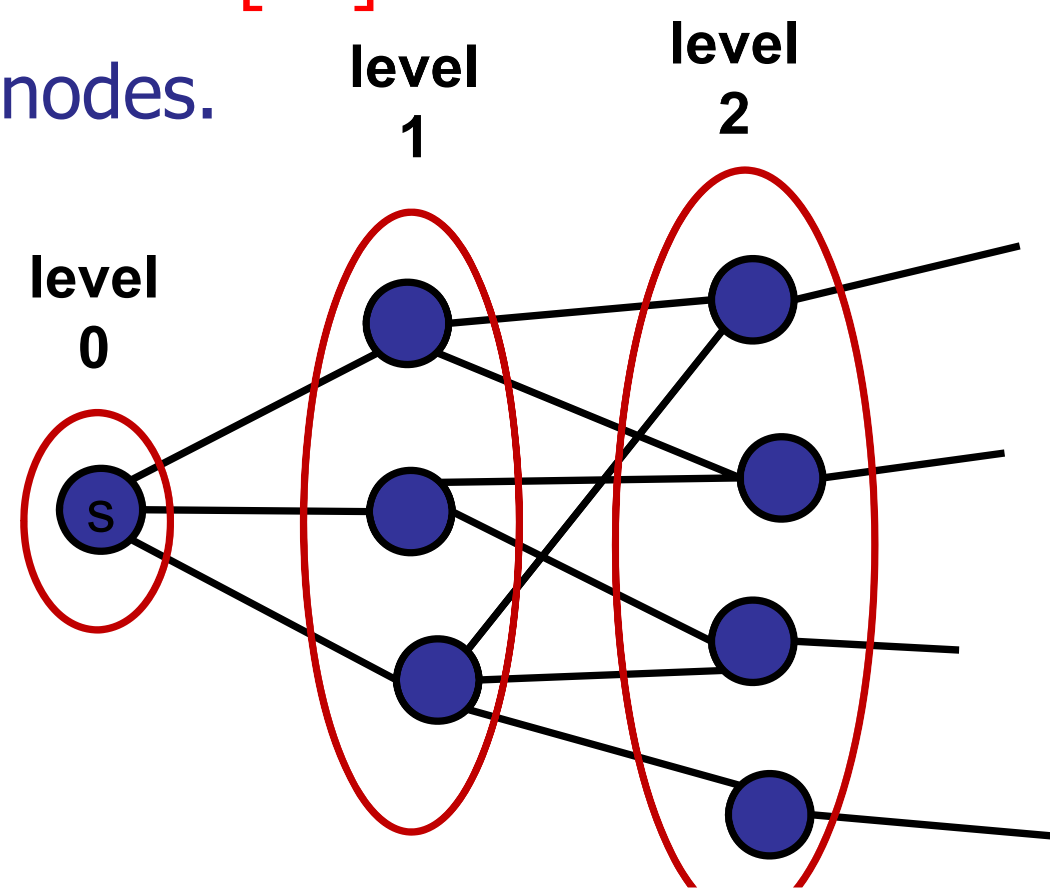
Graph representation:

- Adjacency list

Searching a graph

Breadth-First Search:

- Explore graph level by level.
- Calculate level[i] from level[i-1]
- Skip already visited nodes.

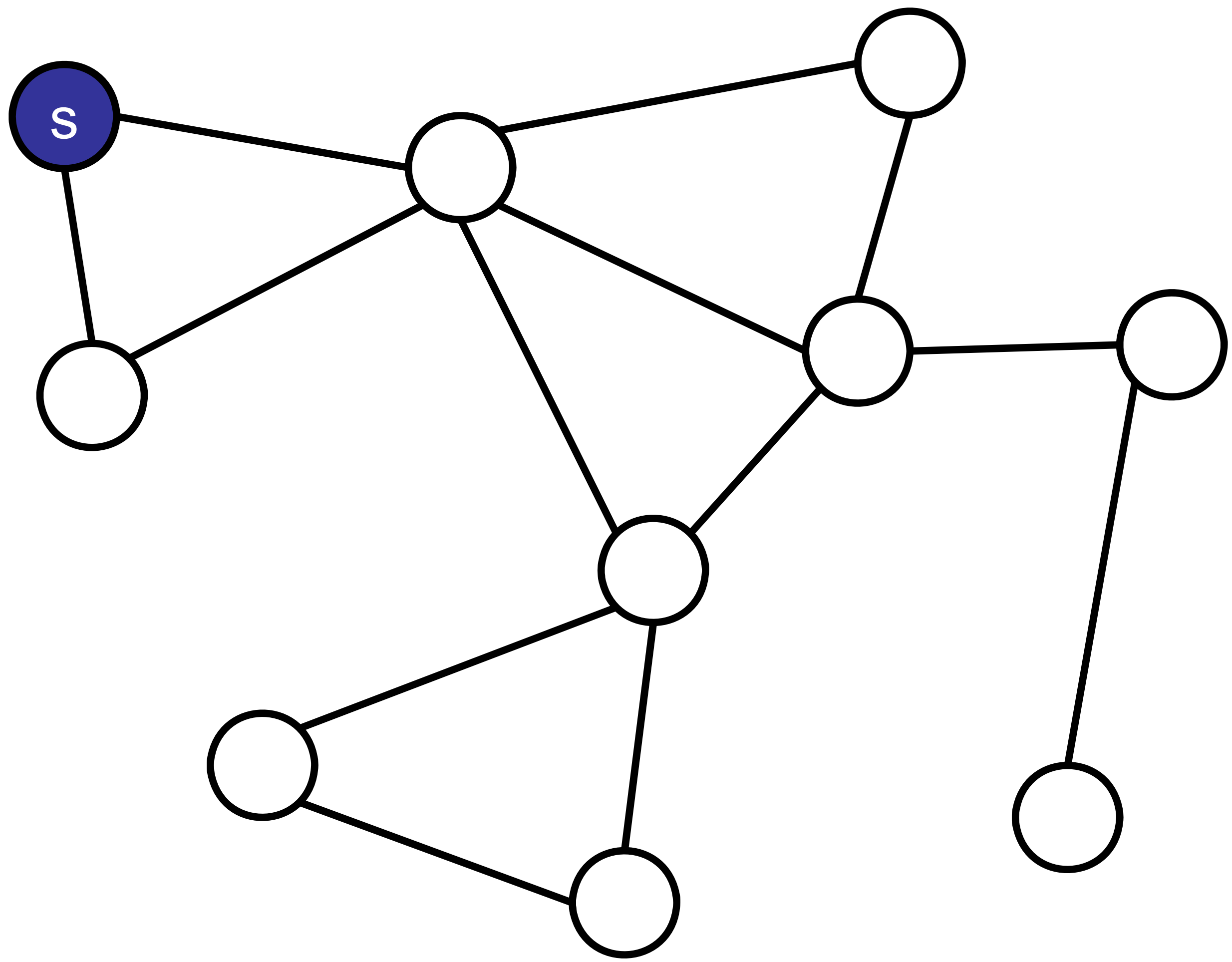


Searching a graph

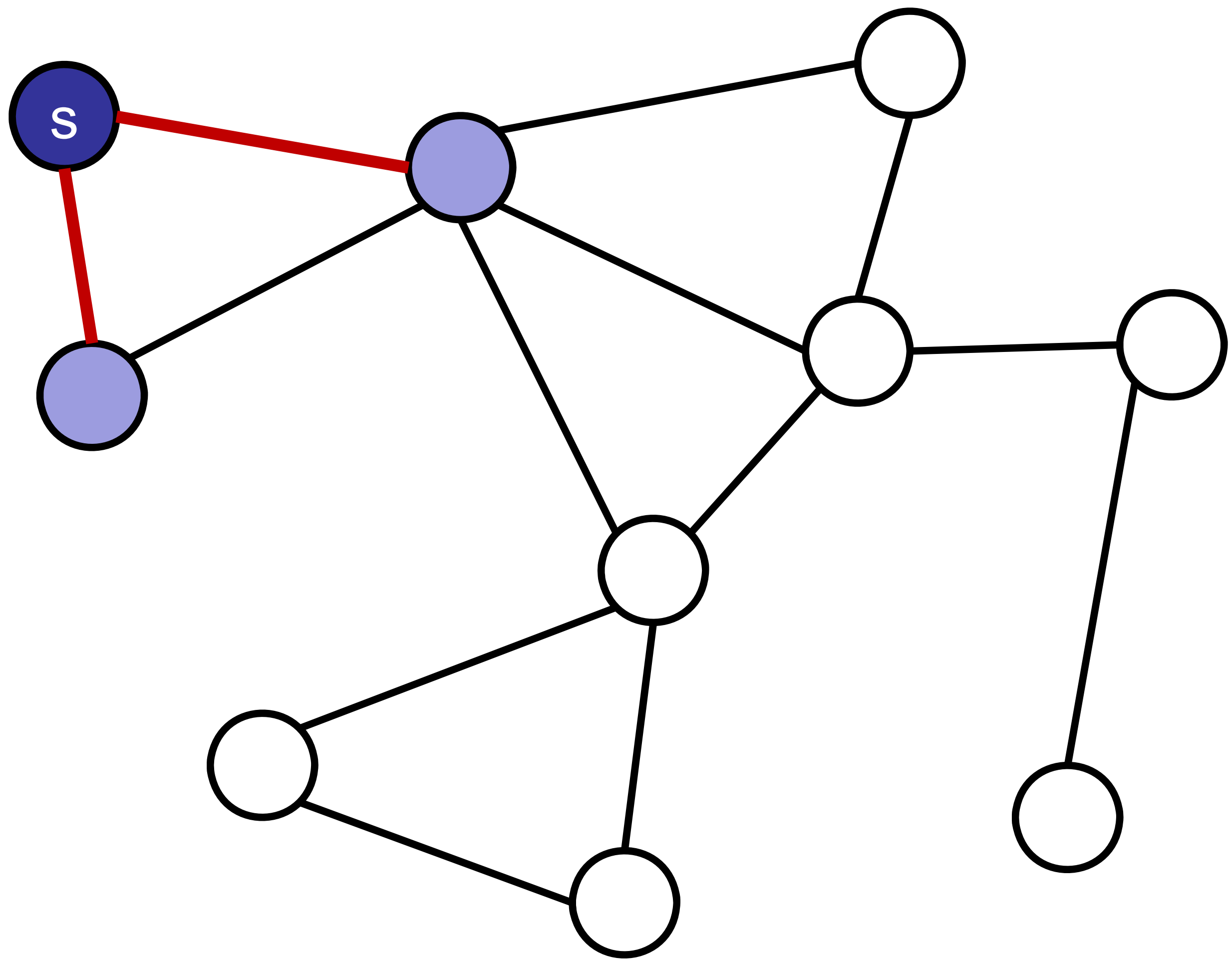
Breadth-First Search:

```
frontier = {s}
while frontier is not empty:
    next-frontier = {}
    for each node u in the frontier:
        for each edge (u,v) in the graph:
            if v has not yet been visited, add v to next-frontier
    frontier = next-frontier
```

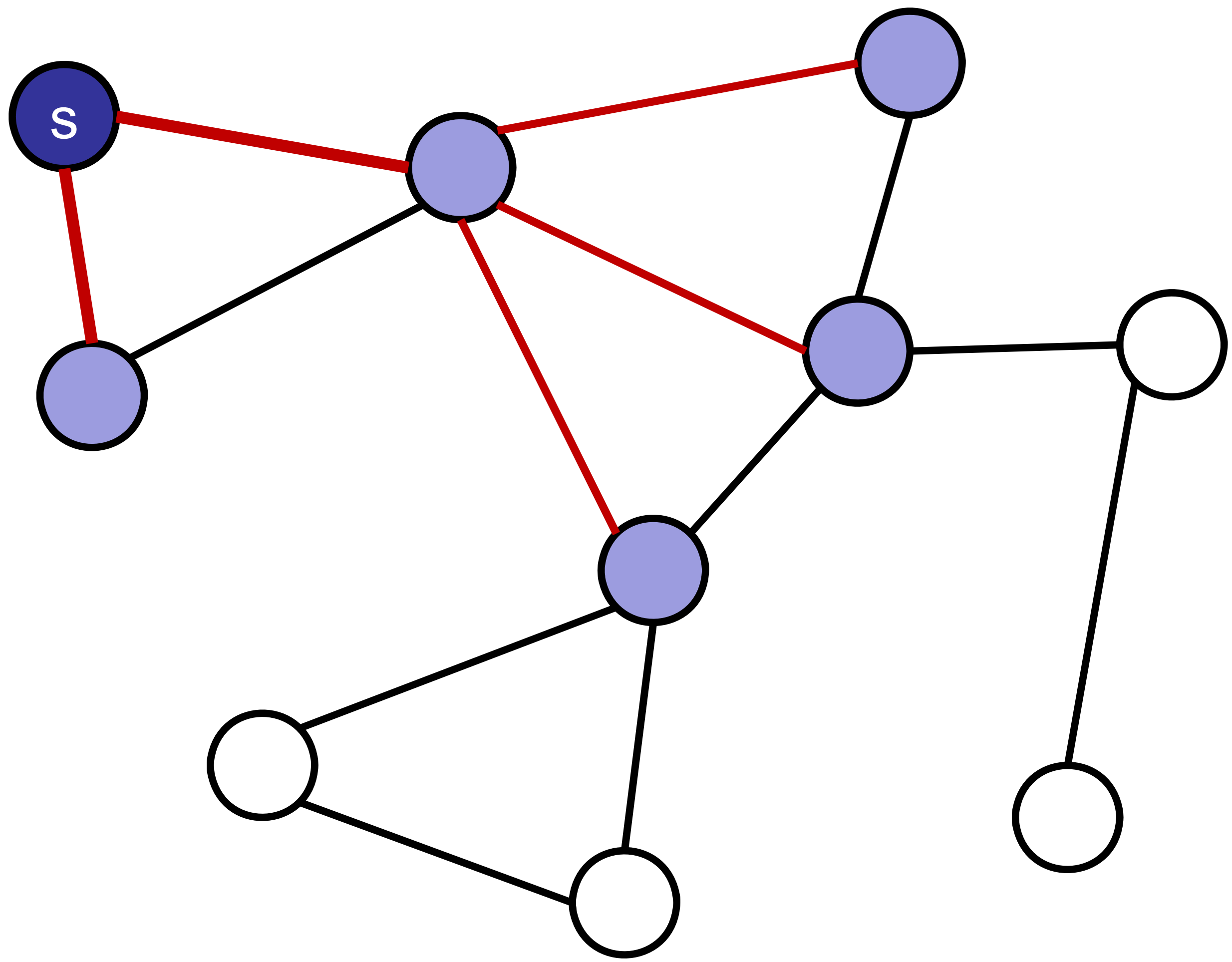
Breadth First Search



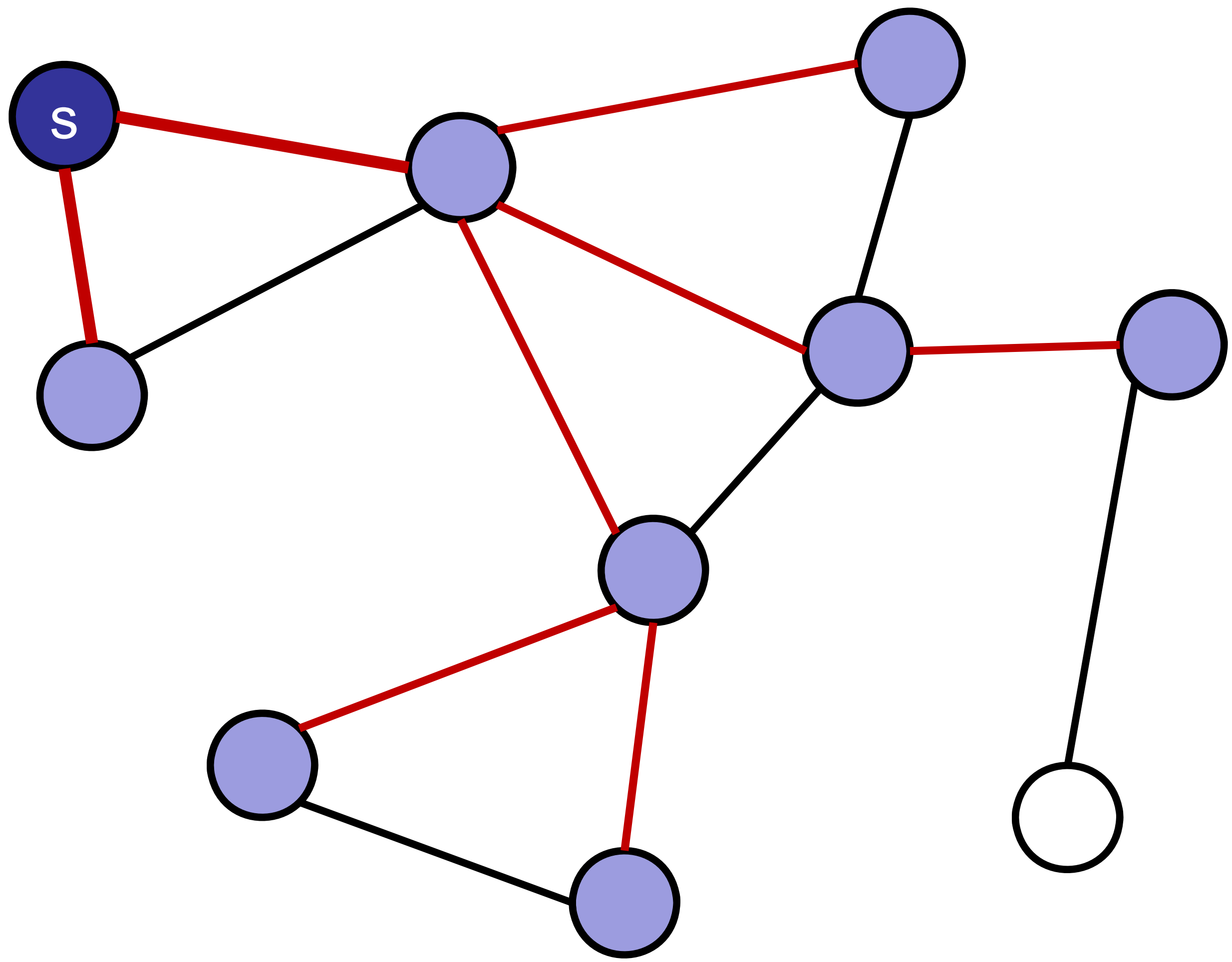
Breadth First Search



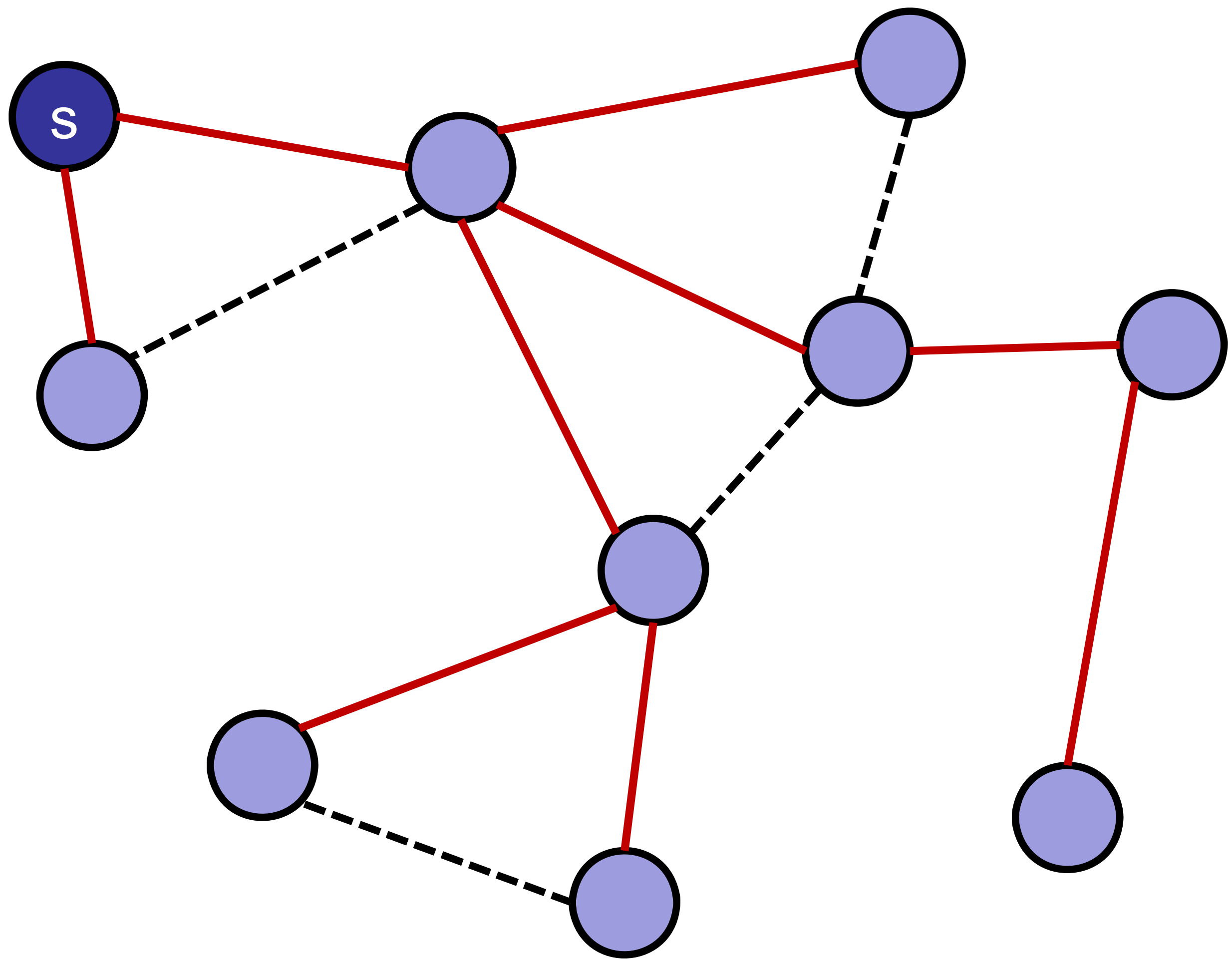
Breadth First Search



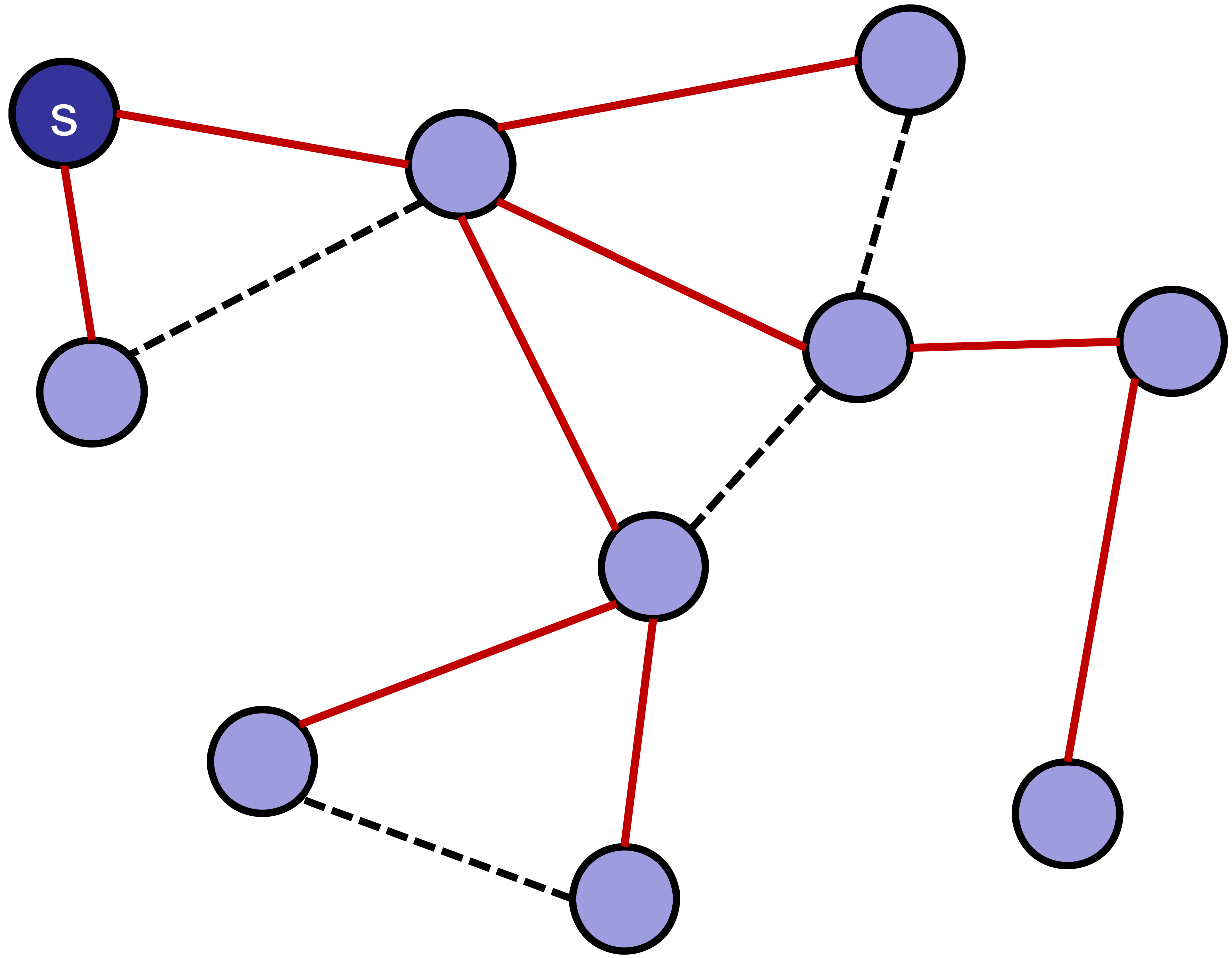
Breadth First Search



Breadth First Search



Breadth First Search



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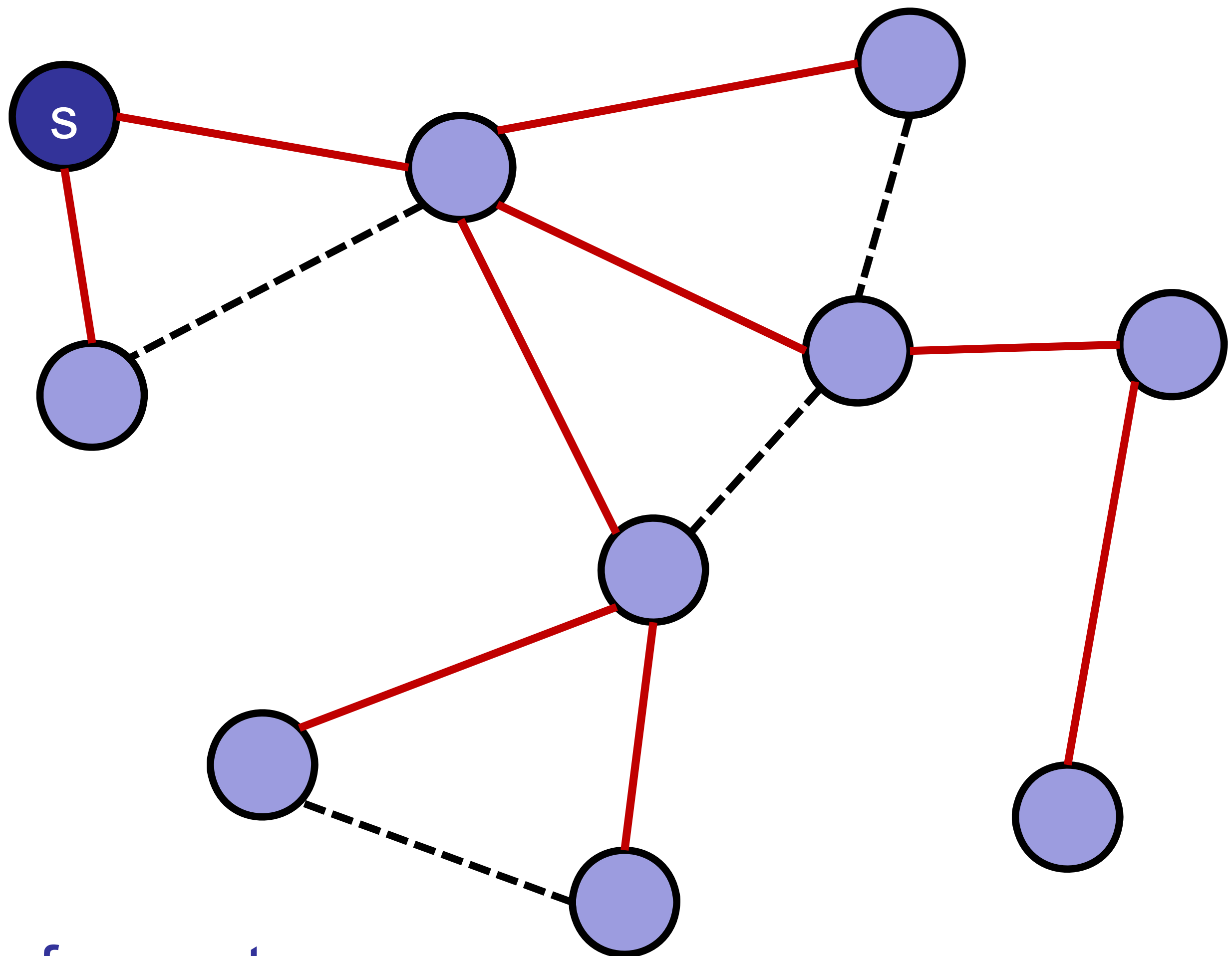
is open

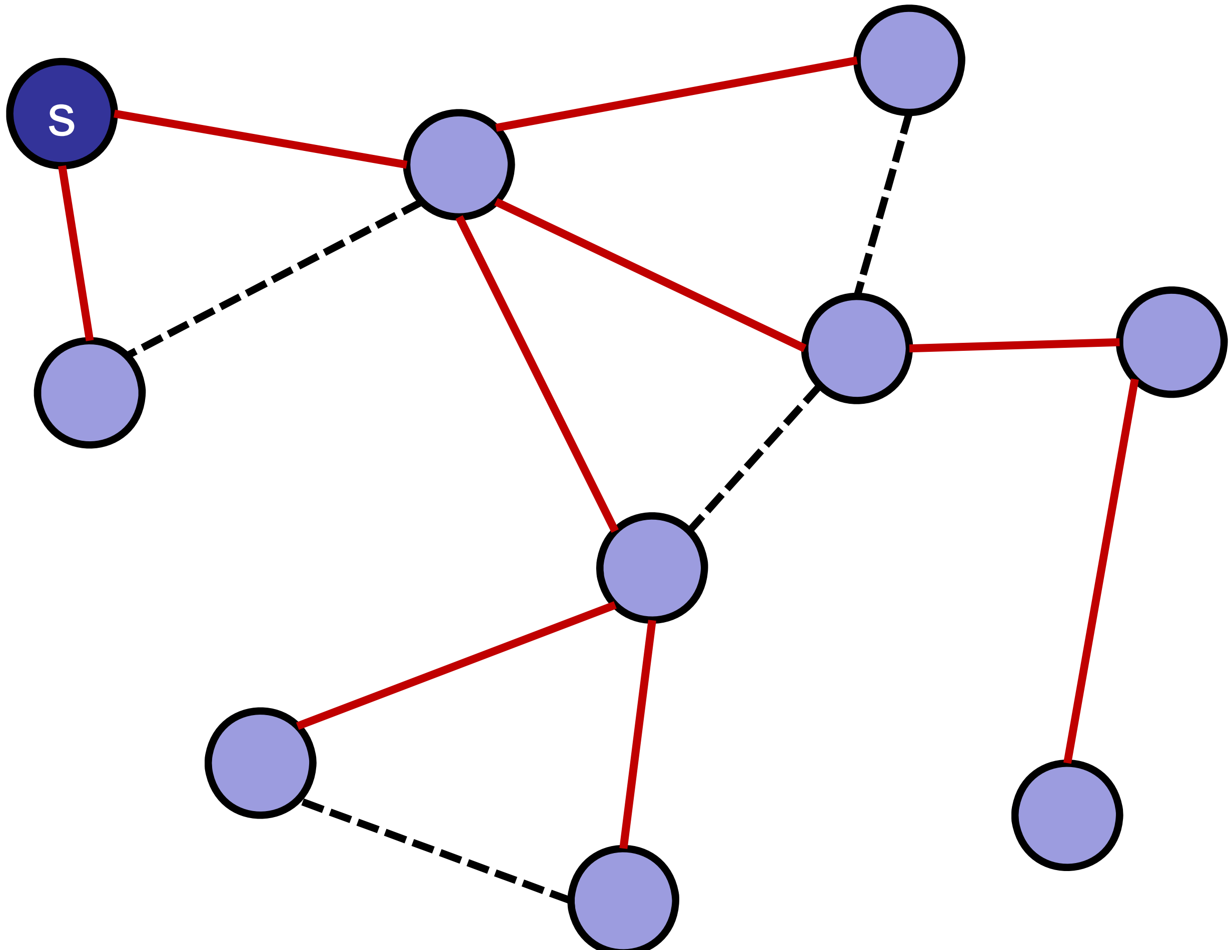
What are the properties of the parent edges?

Breadth First Search



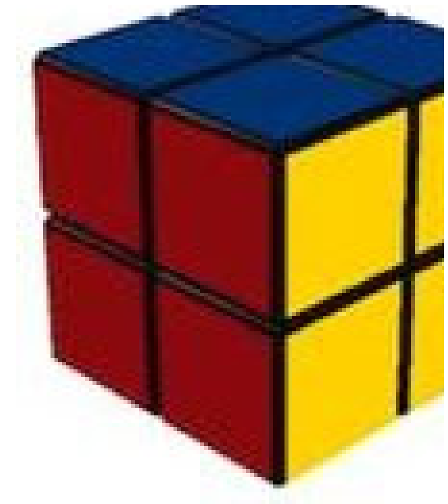
1. Parent edges form a tree
   (No cycles.)

Breadth First Search

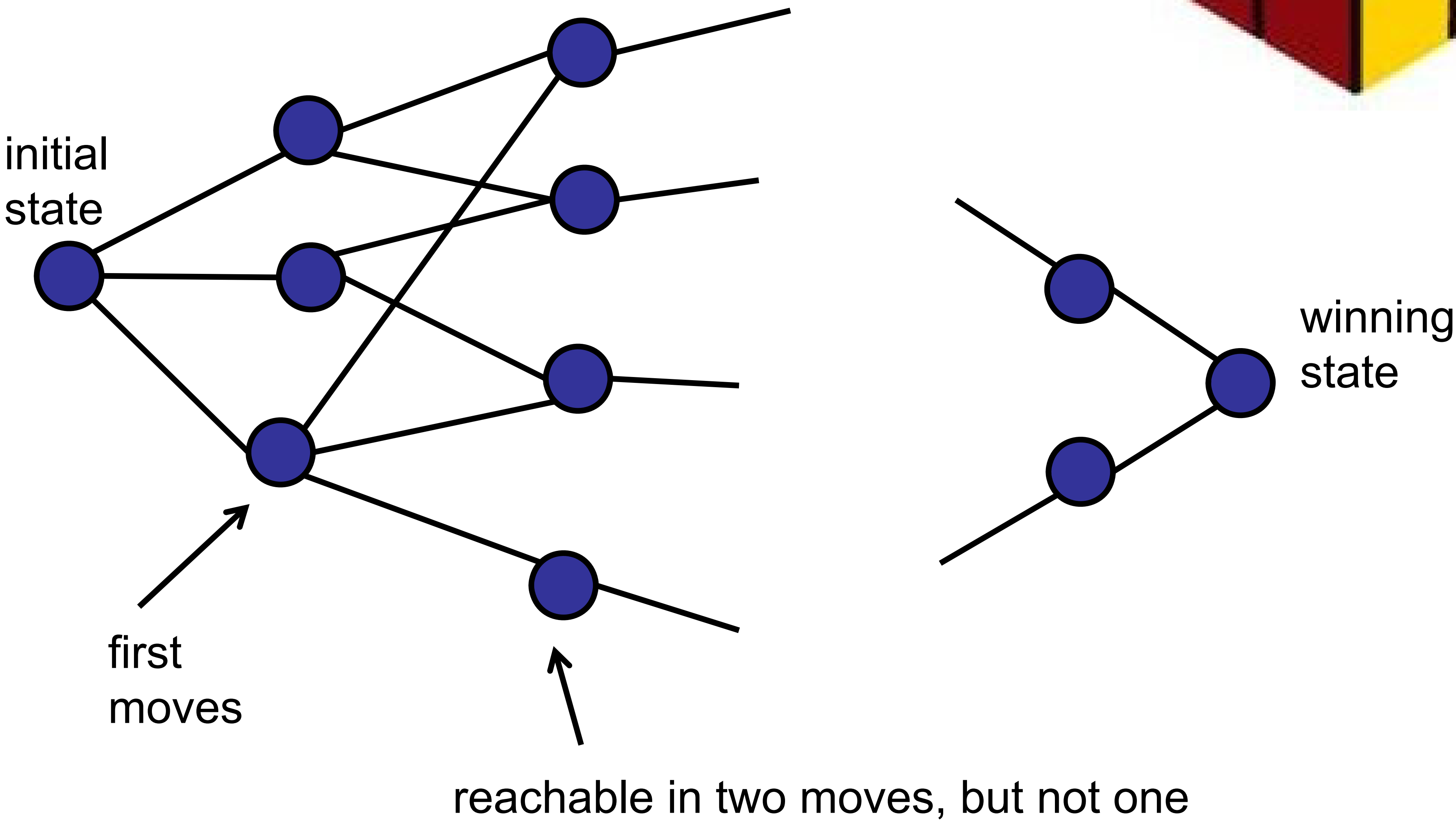


2. Parent edges are shortest paths from s.

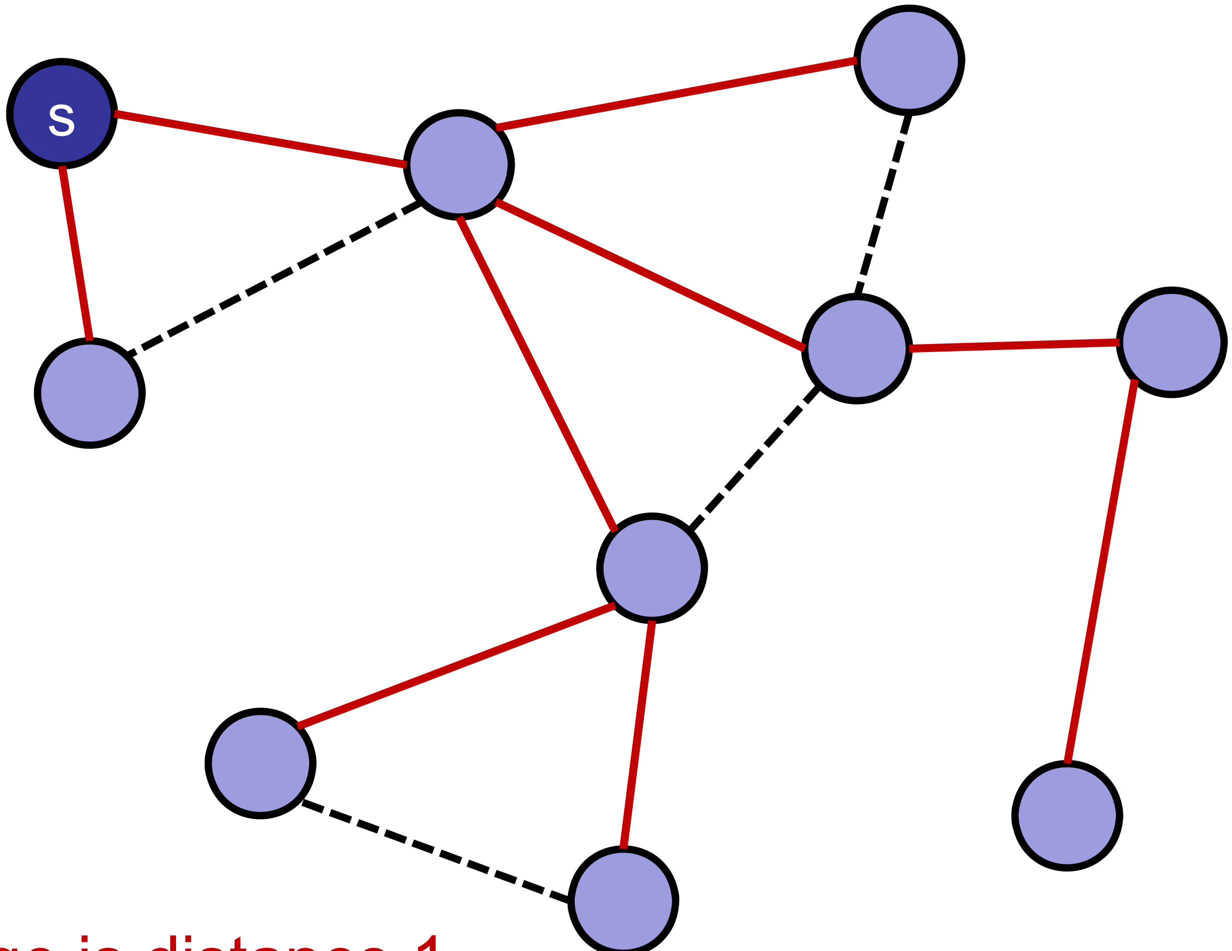
2 x 2 x 2 Rubik's Cube



Layers define shortest paths:



Breadth First Search



Beware: each edge is distance 1.

Next week: graphs with distances on the edges.

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

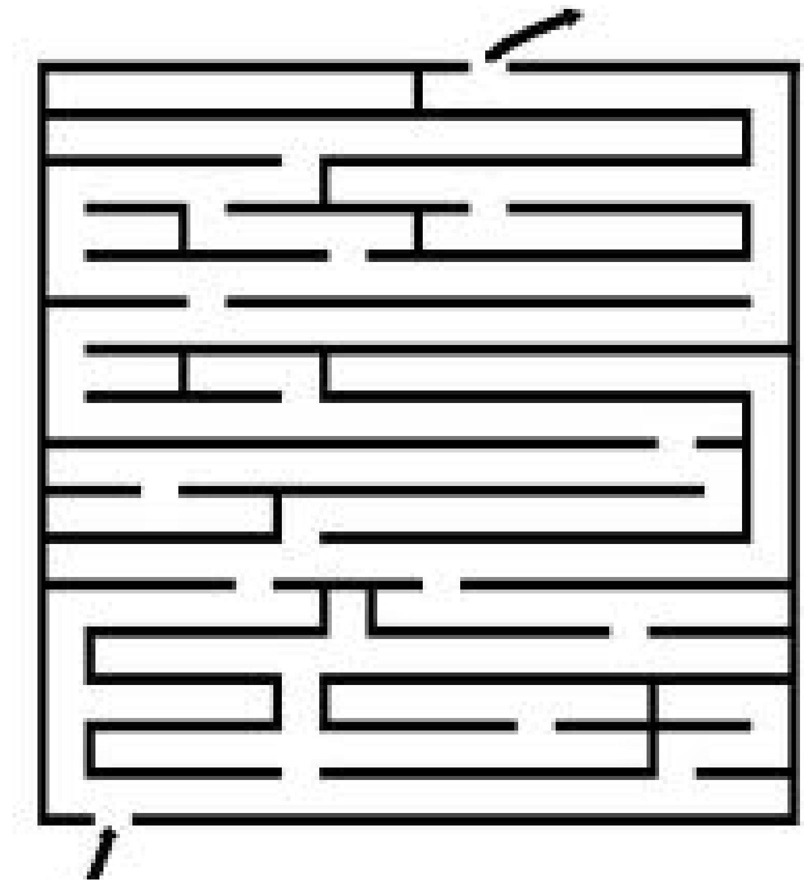
Graph representation:

- Adjacency list

Depth-First Search

Exploring a maze:

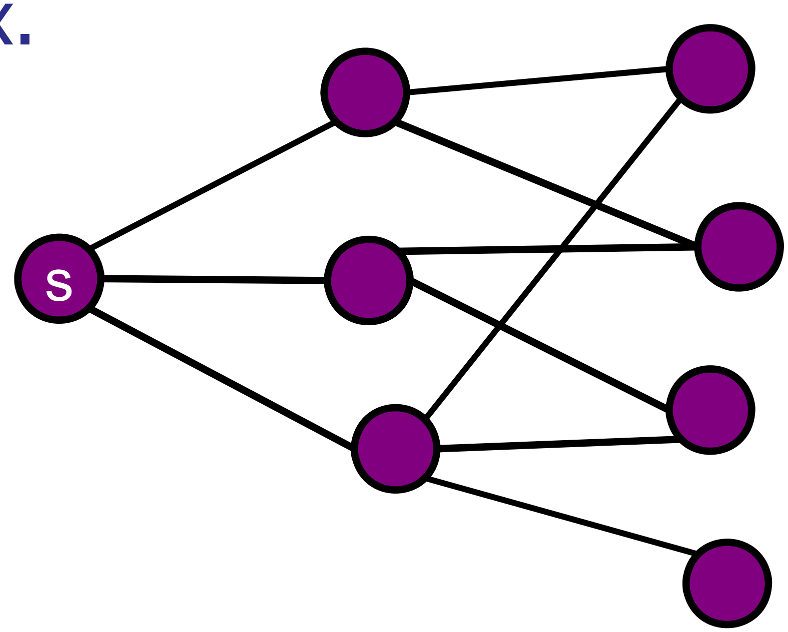
- Follow path until stuck.
- Backtrack along breadcrumbs until reach unexplored neighbor.
- Recursively explore.



Searching a graph

Depth-First Search:

- Follow path until you get stuck
- Backtrack until you find a new edge
- Recursively explore it
- Don't repeat a vertex.



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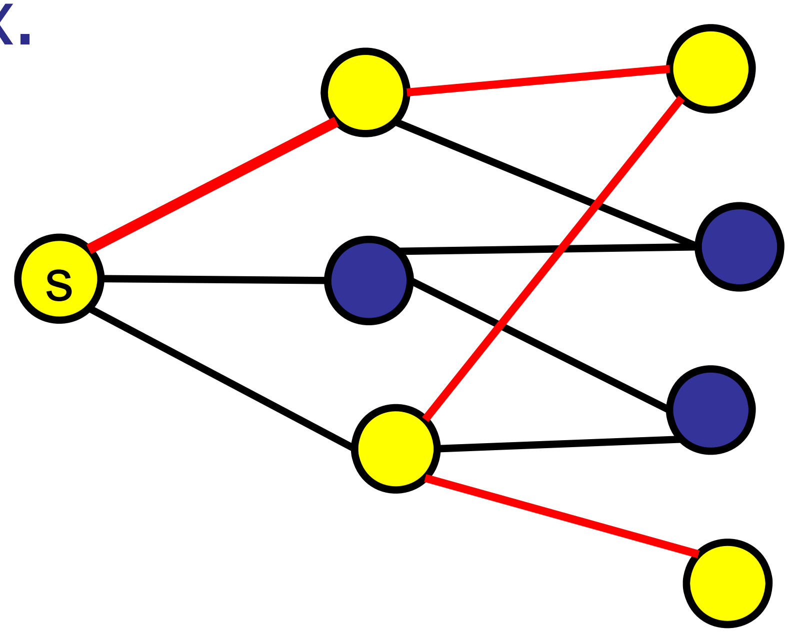

Searching a graph

Depth-First Search:

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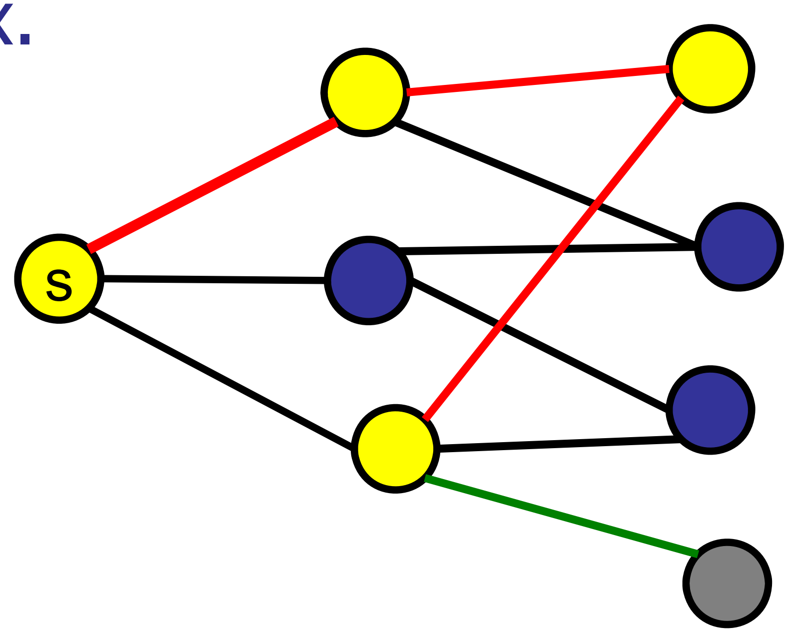

Searching a graph

Depth-First Search:

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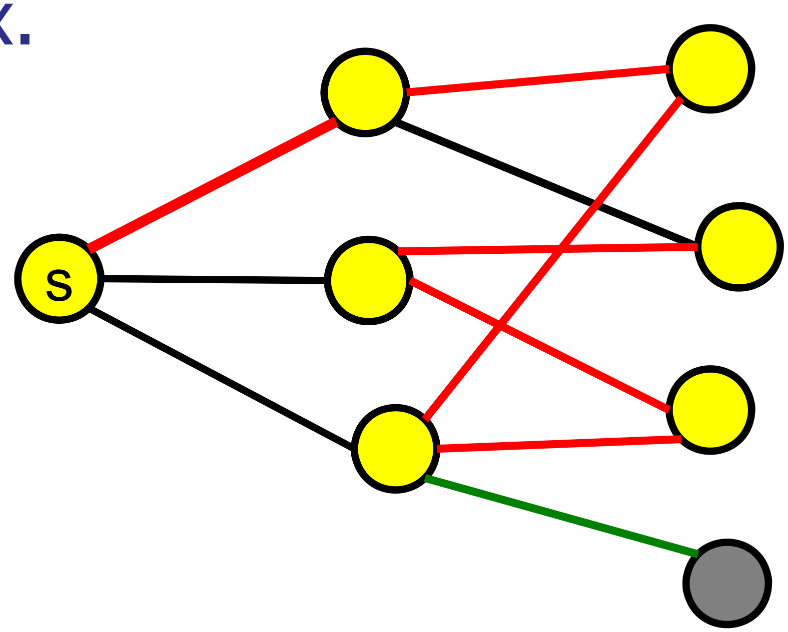

Searching a graph

Depth-First Search:

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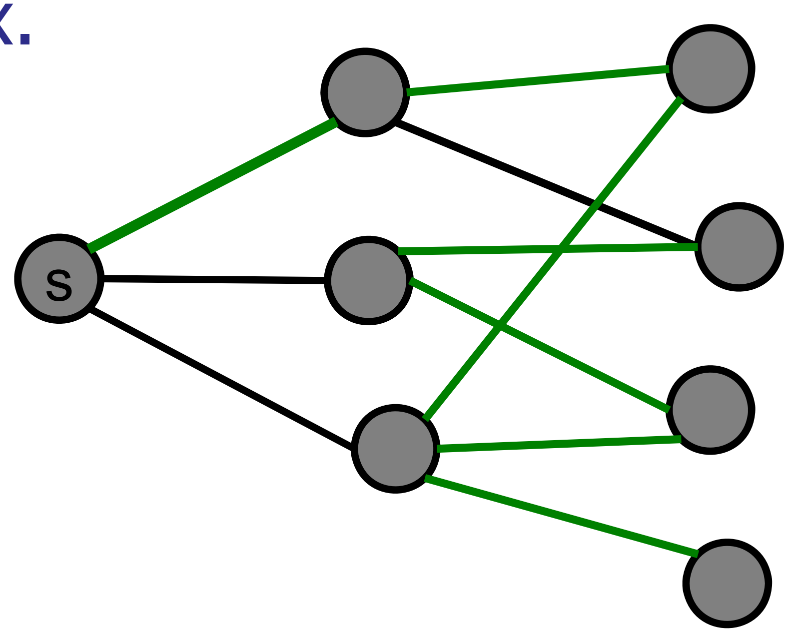


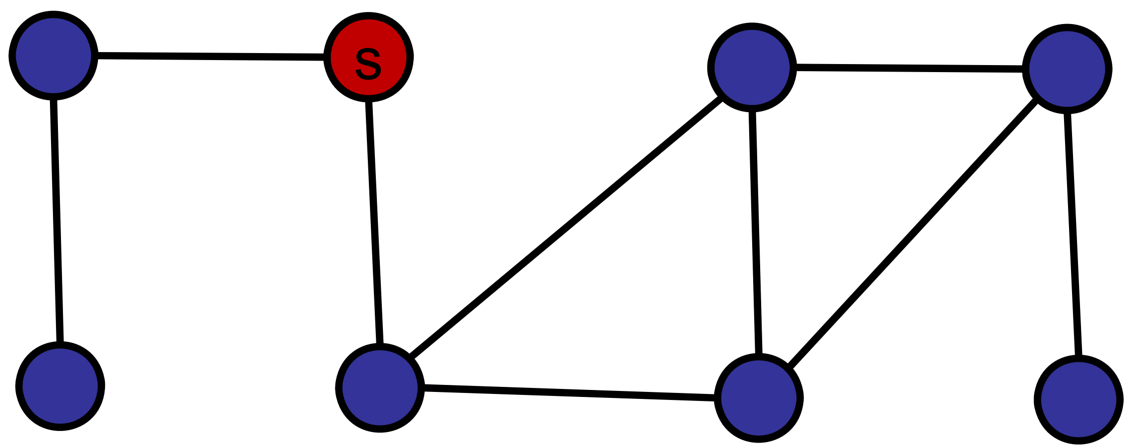
Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
           visited[v] = true;
           DFS-visit(nodeList, visited, v);
     }
  }
}
```

Depth-First Search

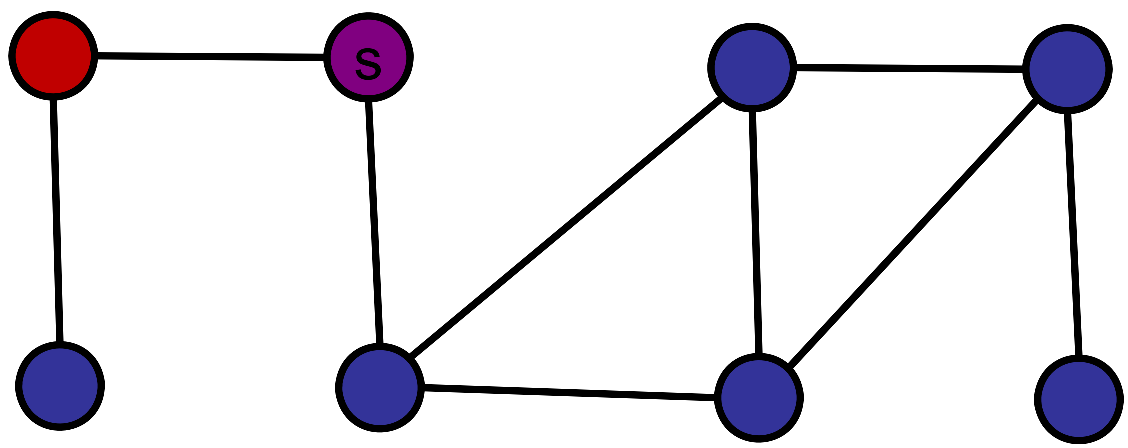
```
DFS(Node[] nodeList){
  boolean[] visited = new boolean[nodeList.length];
  Arrays.fill(visited, false);

  for (start = i; start<nodeList.length; start++) {
     if (!visited[start]){
           visited[start] = true;
           DFS-visit(nodeList, visited, start);
     }
  }
}
```

Depth-First Search Example



Red = active frontier
Purple = next
Gray = visited
Blue = unvisited

Depth-First Search Example



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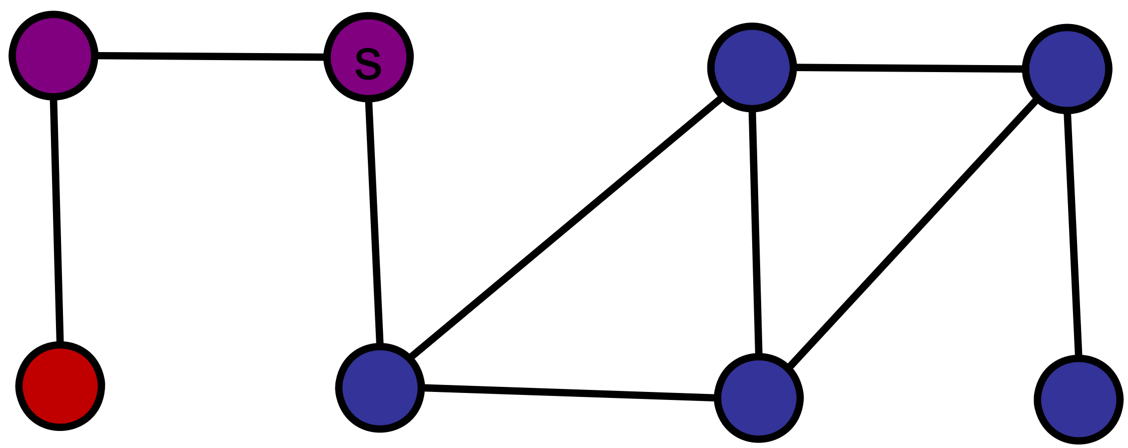

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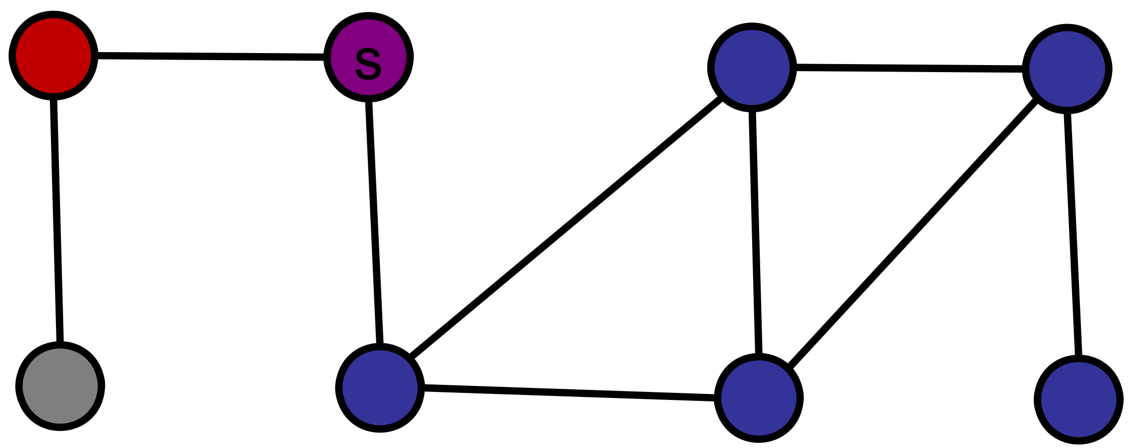

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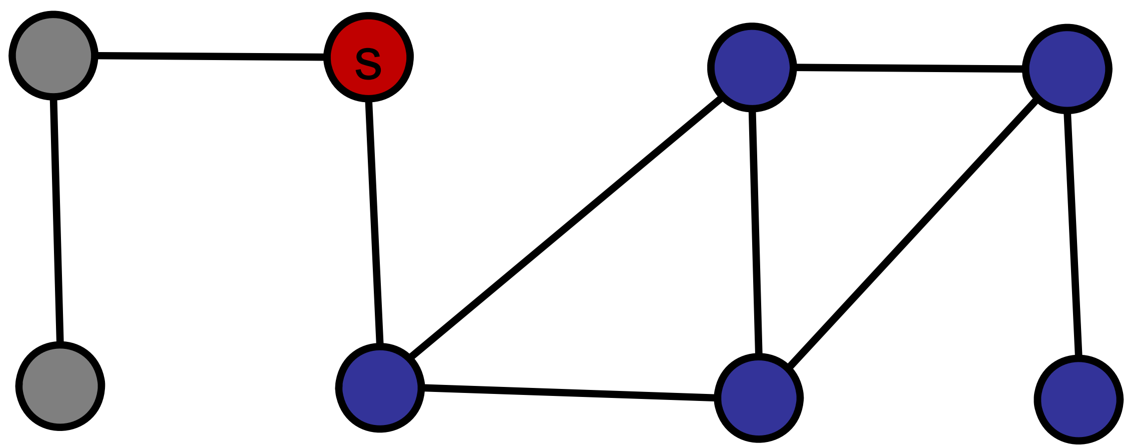

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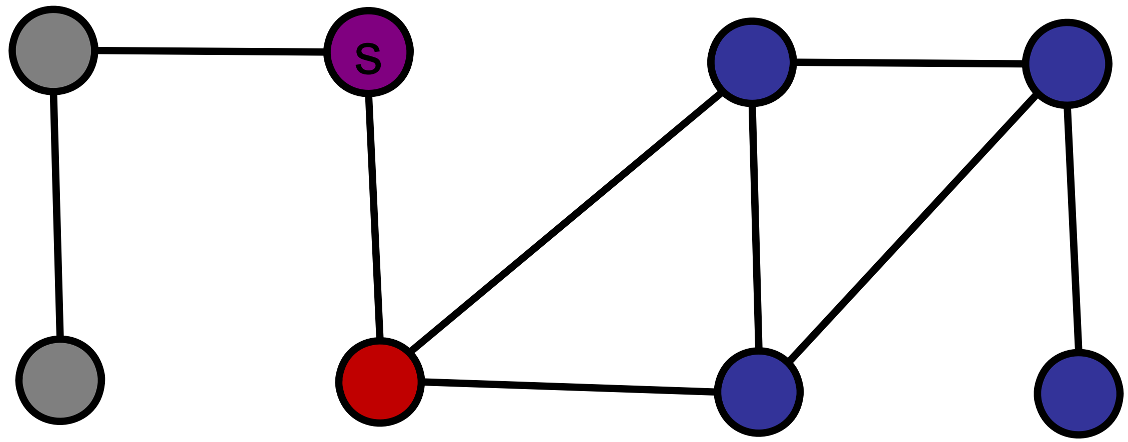

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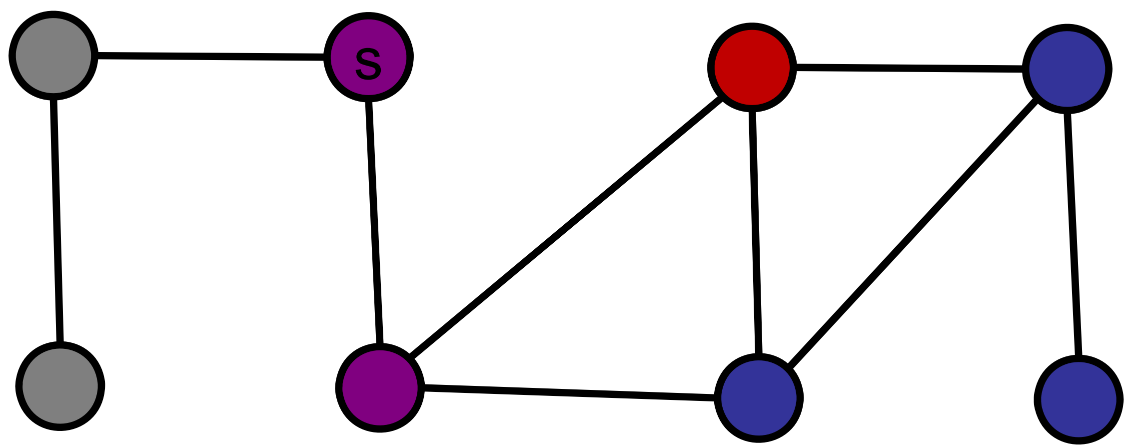

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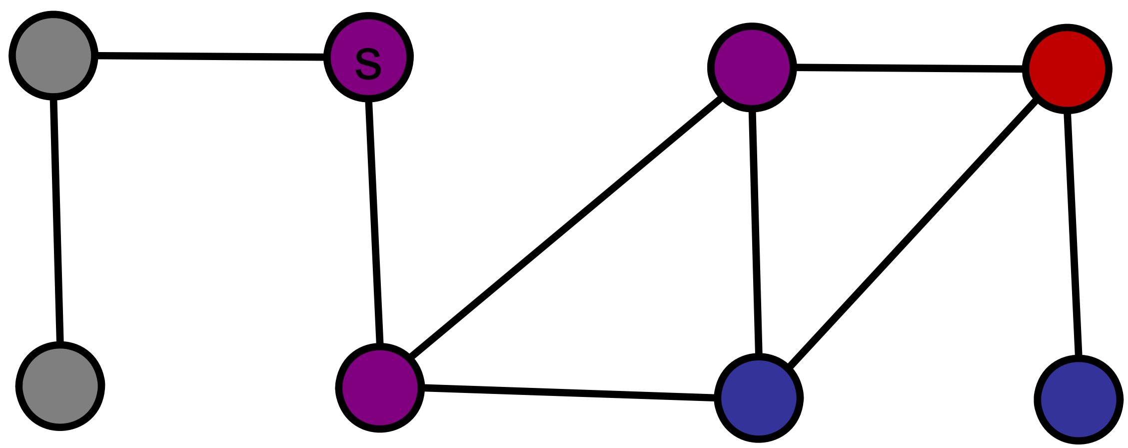

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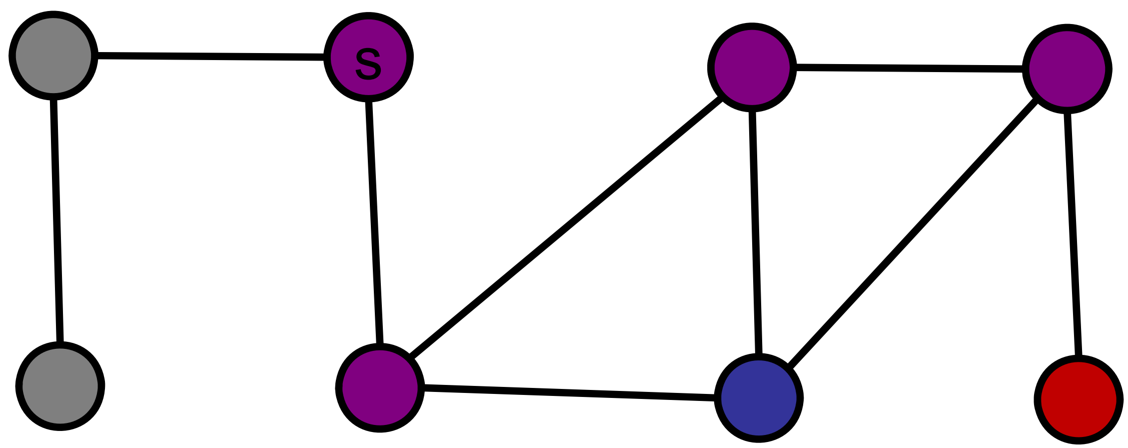

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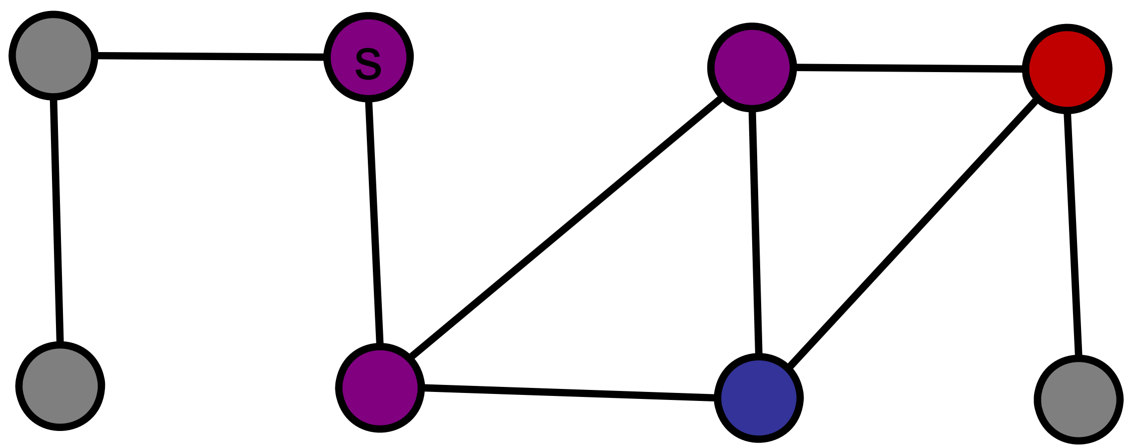

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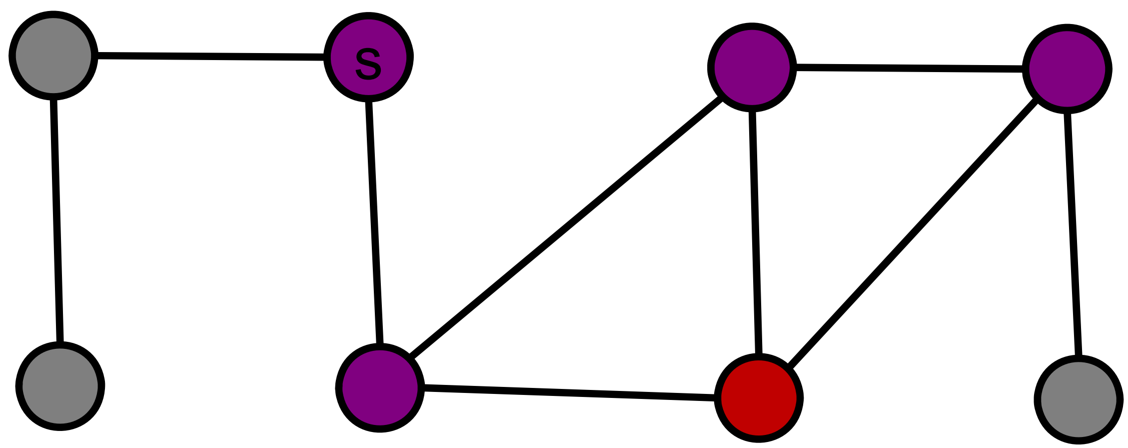

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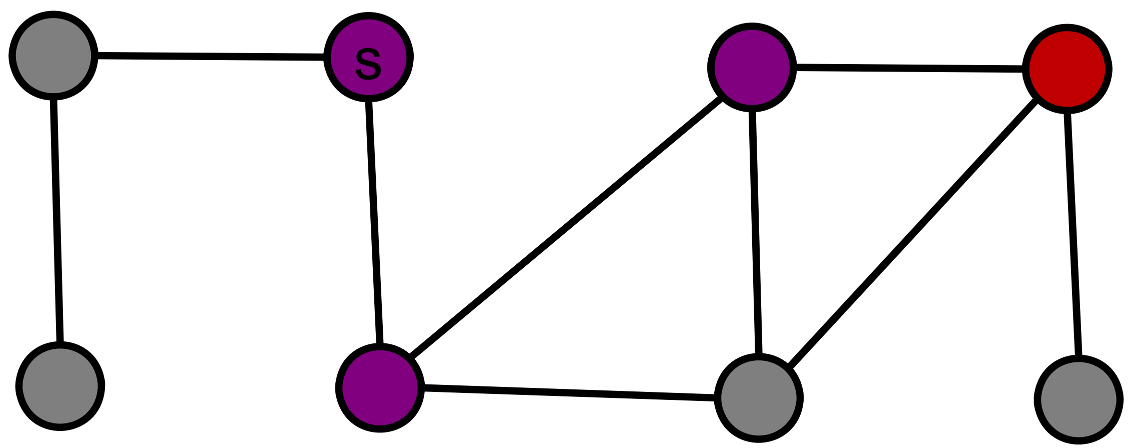

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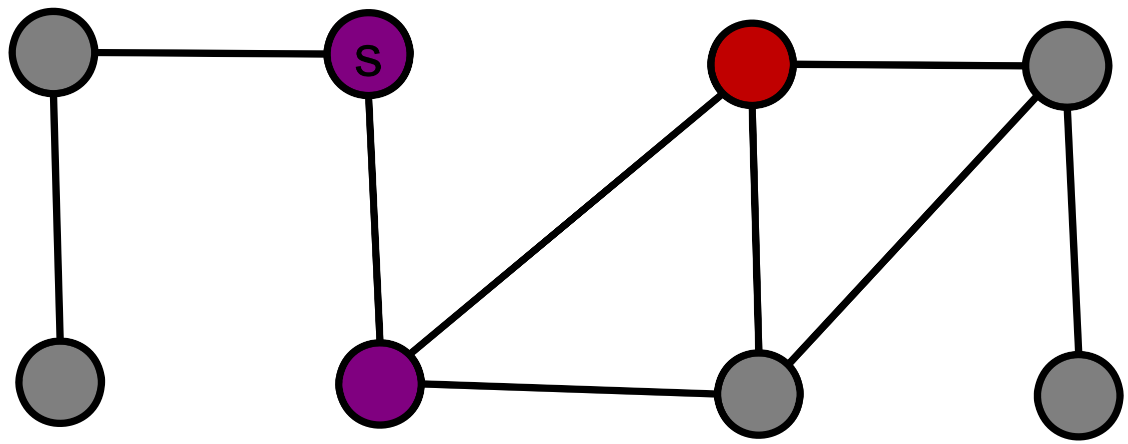

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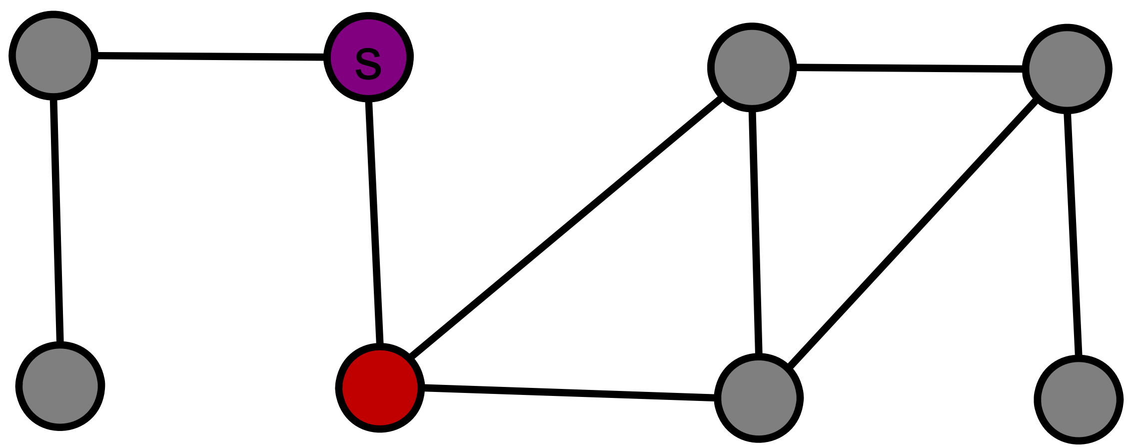

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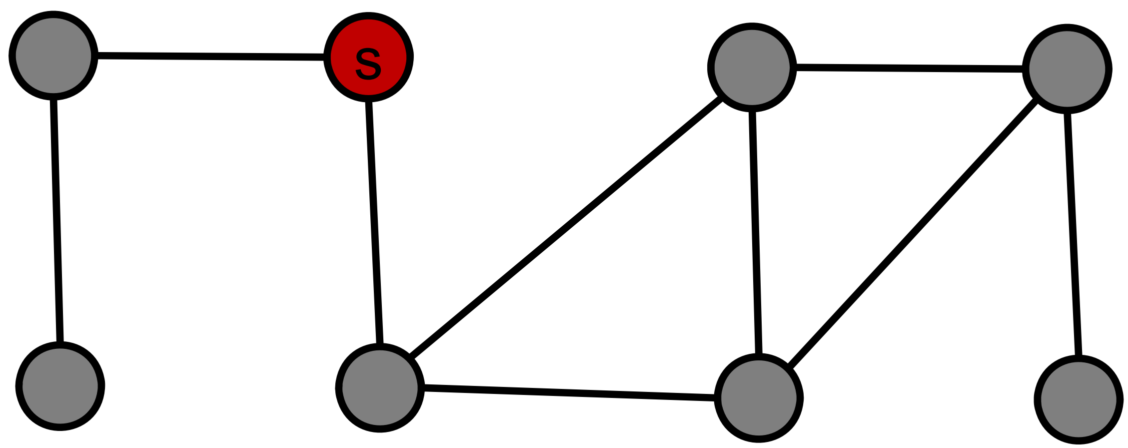

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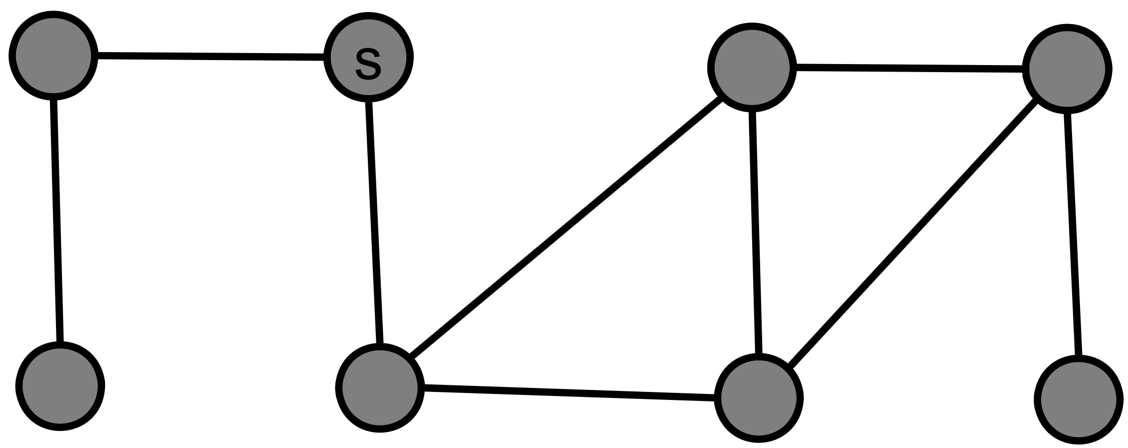

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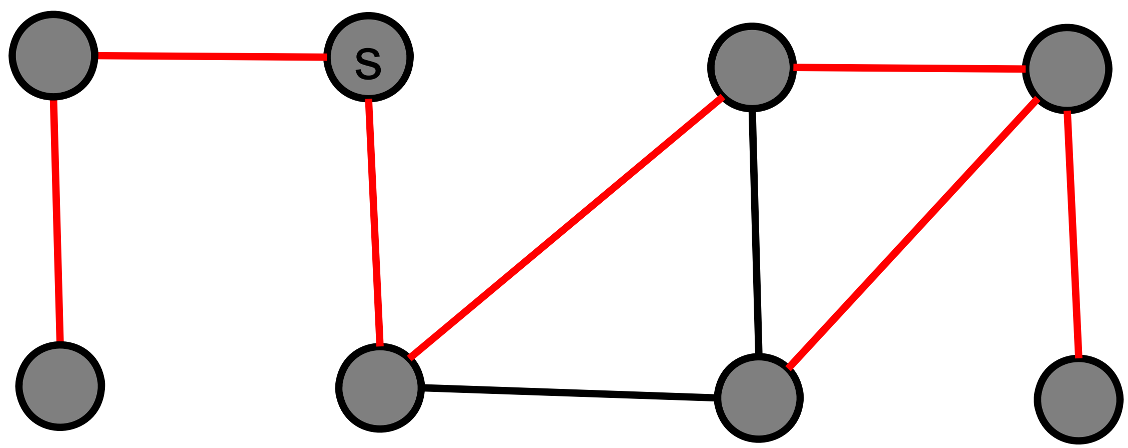

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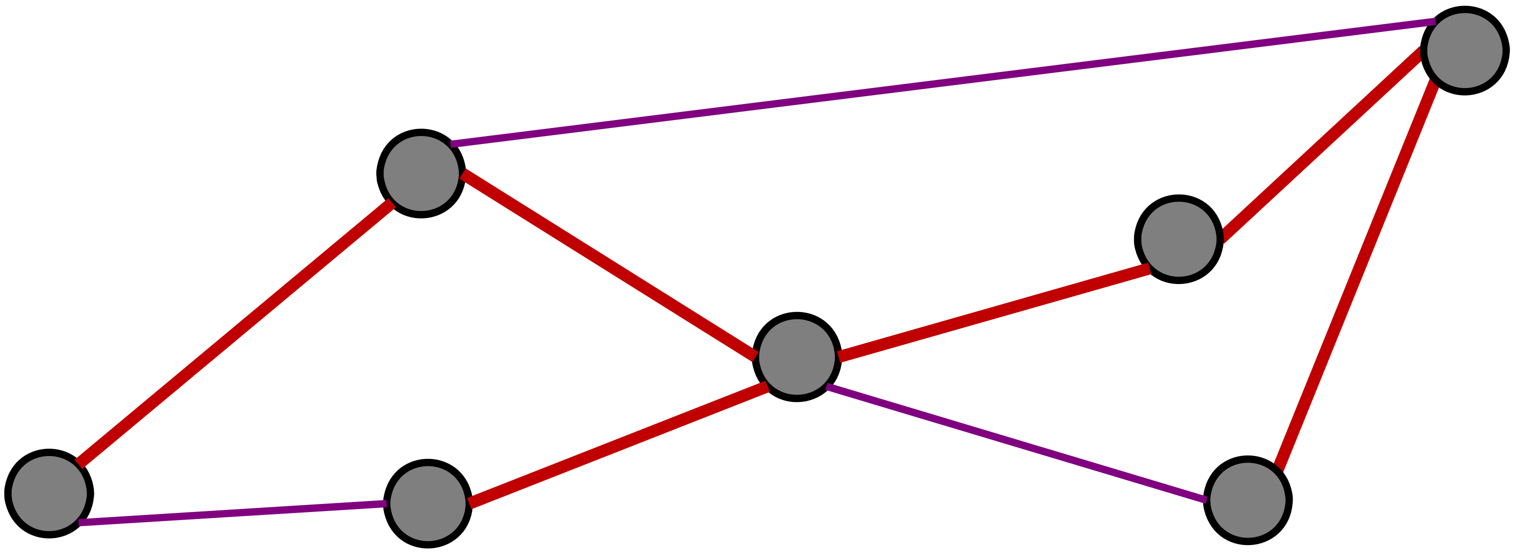
DFS parent edges



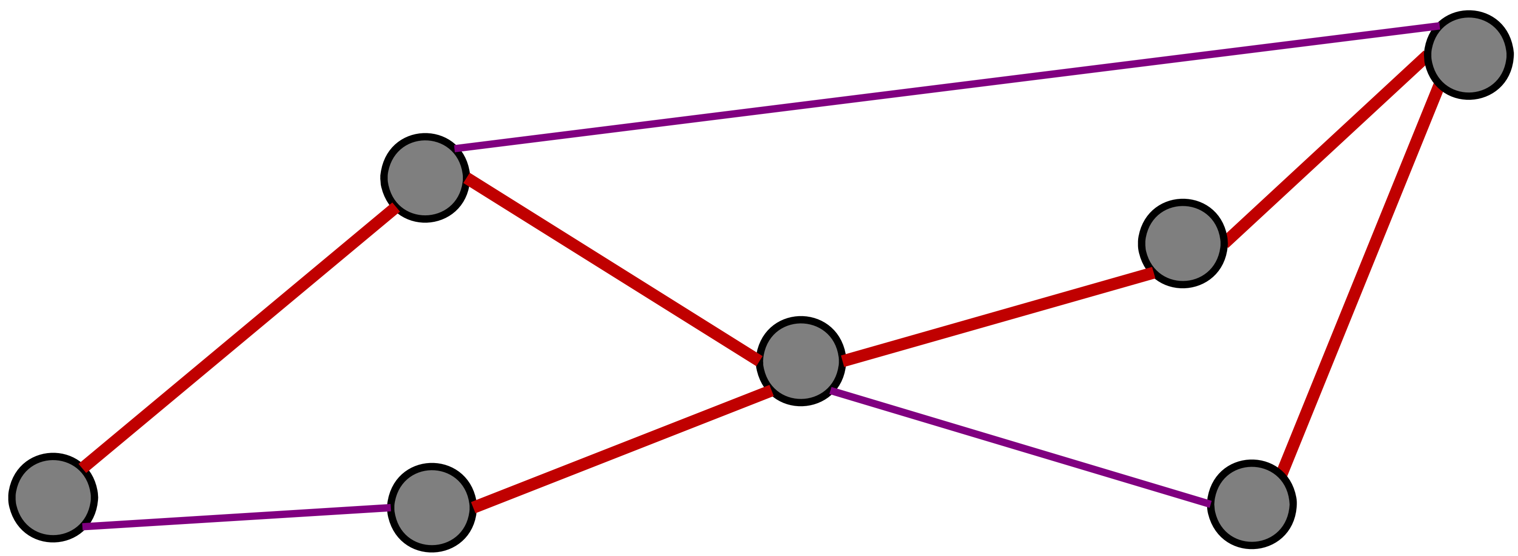
Red = Parent Edges
Purple = Non-parent edges

Which is true? (More than one may apply.)

1. DFS parent graph is a cycle.
2. DFS parent graph is a tree.
3. DFS parent graph has low-degree.
4. DFS parent graph has low diameter.
5. None of the above.

ARCHIPELAGO is open

DFS parent edges = tree



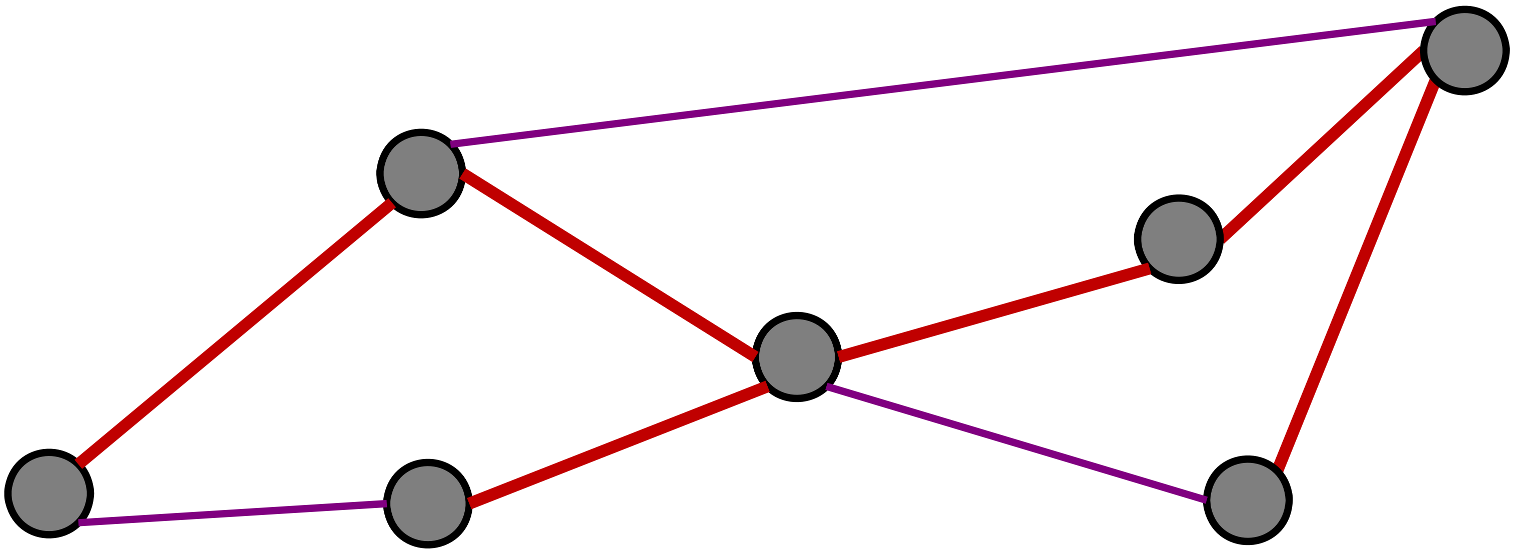
Red = Parent Edges
Purple = Non-parent edges

True or false:
DFS parent graph contains shortest paths.

1. True
2. False

ARCHIPELAGO
is open

DFS parent edges = tree



Red = Parent Edges
Purple = Non-parent edges

Note: not shortest paths!

The running time of DFS is:

1. O(V)
2. O(E)
3. O(V+E)
4. O(VE)
5. O(V^2)
6. I have no idea.

ARCHIPELAGO

is open

Depth-First Search

Analysis:

- DFS-visit called only once per node. O(V)
  - After visited, never call DFS-visit again.
- In DFS-visit, each neighbor is enumerated. O(E)

If the graph is stored as an adjacency matrix, what is the running time of DFS?

1. $O(V)$
2. $O(E)$
3. $(V+E)$
4. $O(VE)$
5. $O(V^2)$
6. $O(E^2)$

Depth-First Search

Analysis:

- DFS-visit called only once per node.
  - After visited, never call DFS-visit again.

O(V)

- In DFS-visit, each neighbor is enumerated.

O(V)
per
node

To implement an iterative version of DFS:

1. Use a queue.
2. Use a stack.
3. Use a set.
4. Use a tree.

Graph Search

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.
- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Graph Search

Breadth-first search:

Same algorithm, implemented with a queue:

Add start-node to queue.

Repeat until queue is empty:

- Remove node v from the front of the queue.
- Visit v.
- Explore all outgoing edges of v.
- Add all unvisited neighbors of v to the queue.

Graph Search

Depth-first search:

Same algorithm, implemented with a stack:

Add start-node to stack.

Repeat until stack is empty:

- Pop node v from the front of the stack.
- Visit v.
- Explore all outgoing edges of v.
- Push all unvisited neighbors of v on the front of the stack.

Graph Search

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.
- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Common Mistake

What do BFS and DFS solve?

- They visit every node in the graph?
- They visit every edge in the graph?
- They visit every path in the graph?

Common Mistake

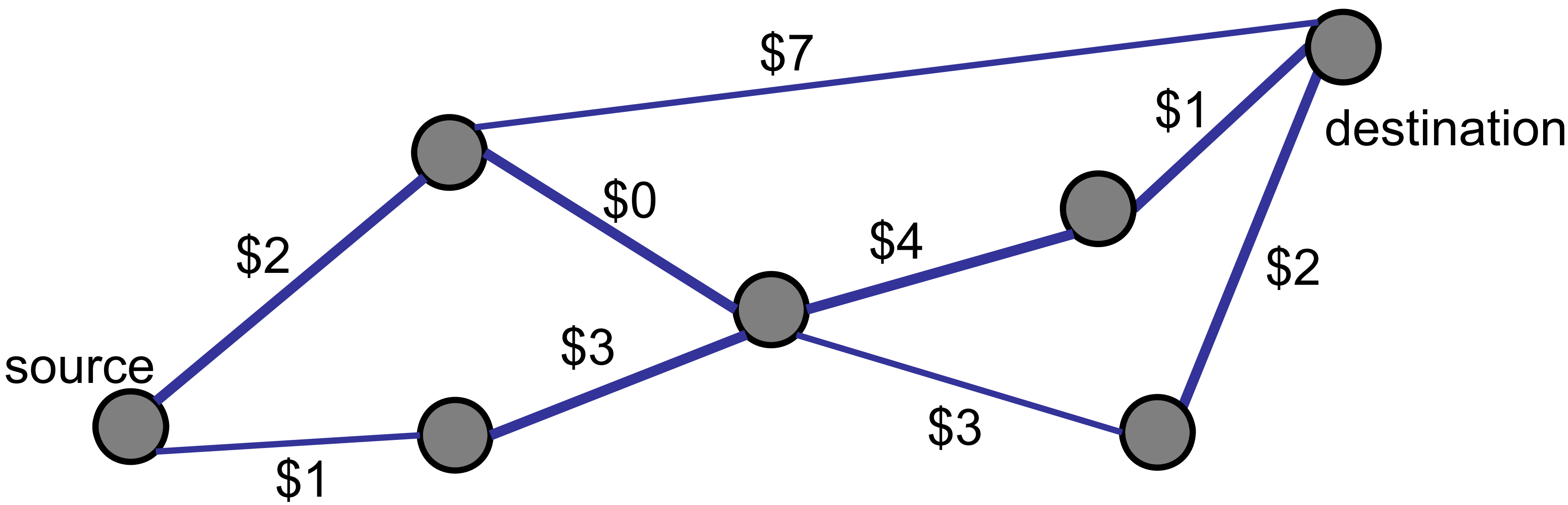
What do BFS and DFS solve?

- They visit every node in the graph? Yes.
- They visit every edge in the graph? Yes.
- ~~They visit every path in the graph?~~

Example

Problem: Make Money

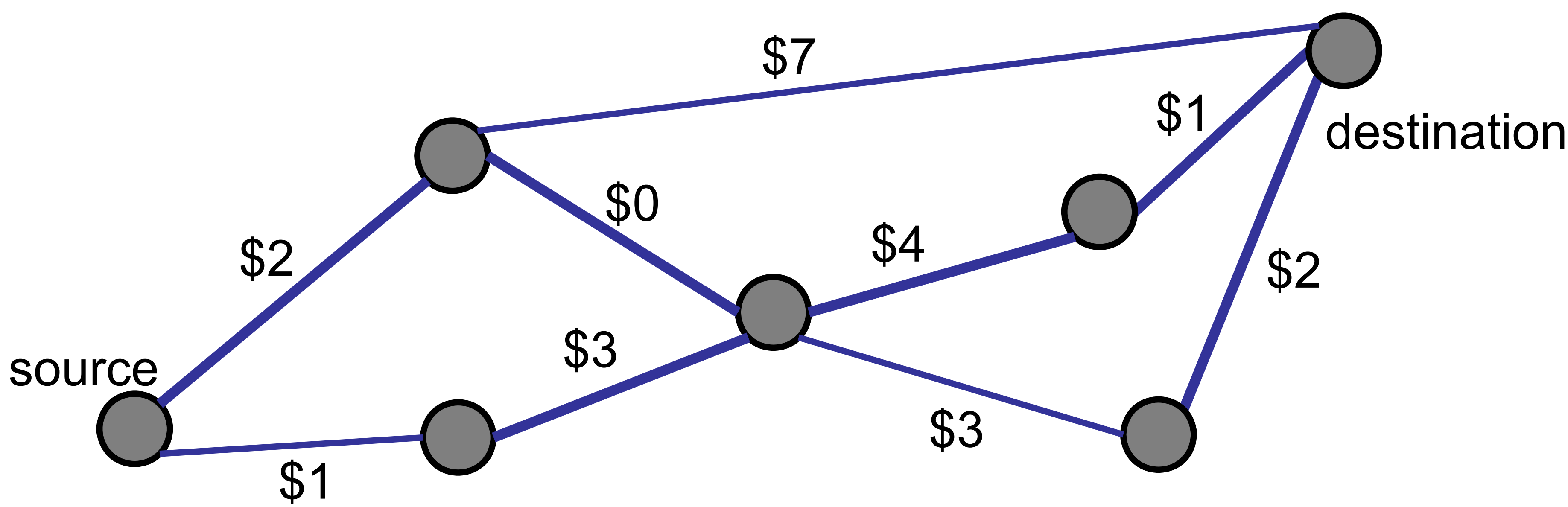
- Start at source s.
- Go to destination d.
- Each edge e earns money m(e).
- Find the path that makes the most money.



Example

NOT a solution:

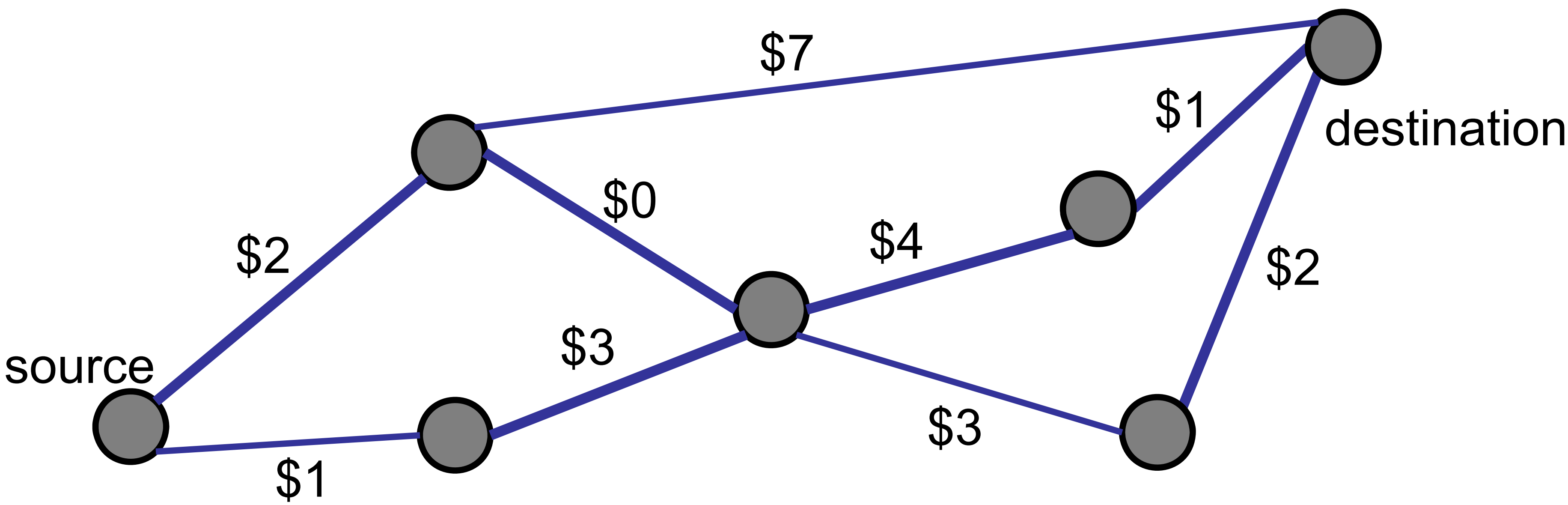
- Start at source s.
- Run BFS (or DFS) to explore every path.
- Keep track of the best path.



Example

Problem 1: Does not work.

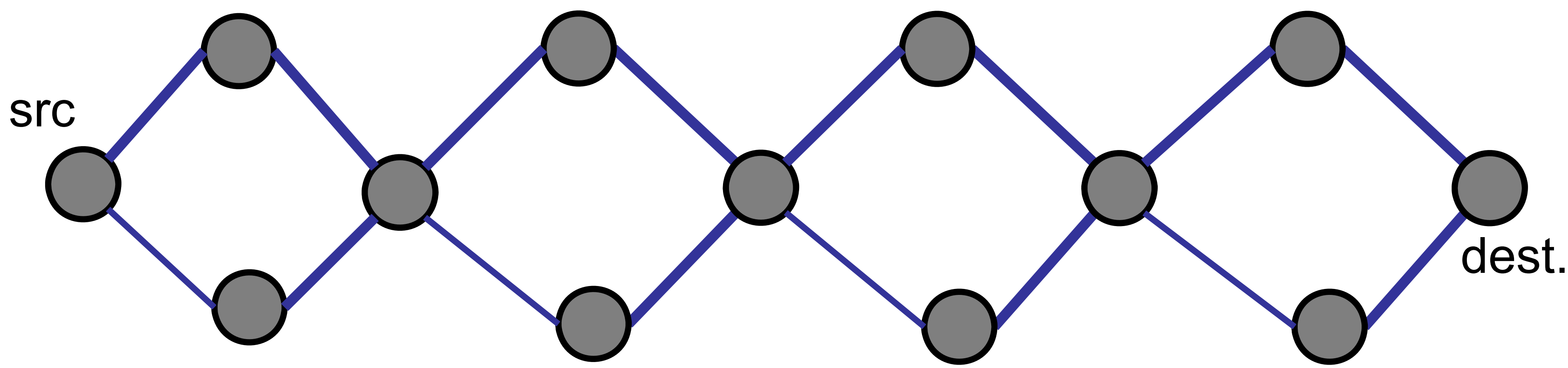
- DFS or BFS do NOT explore every path.
- Once a node is visited, it is never explored again.



Example

Problem 2: Too expensive.

- Some graphs have an exponential number of paths.
- It takes exponential time to explore all paths.



Example: $2^4 > 2^{n/4}$ different s->d paths.

Common Mistake

What do BFS and DFS solve?

- They visit every node in the graph? Yes.
- They visit every edge in the graph? Yes.
- ~~They visit every path in the graph?~~

Roadmap

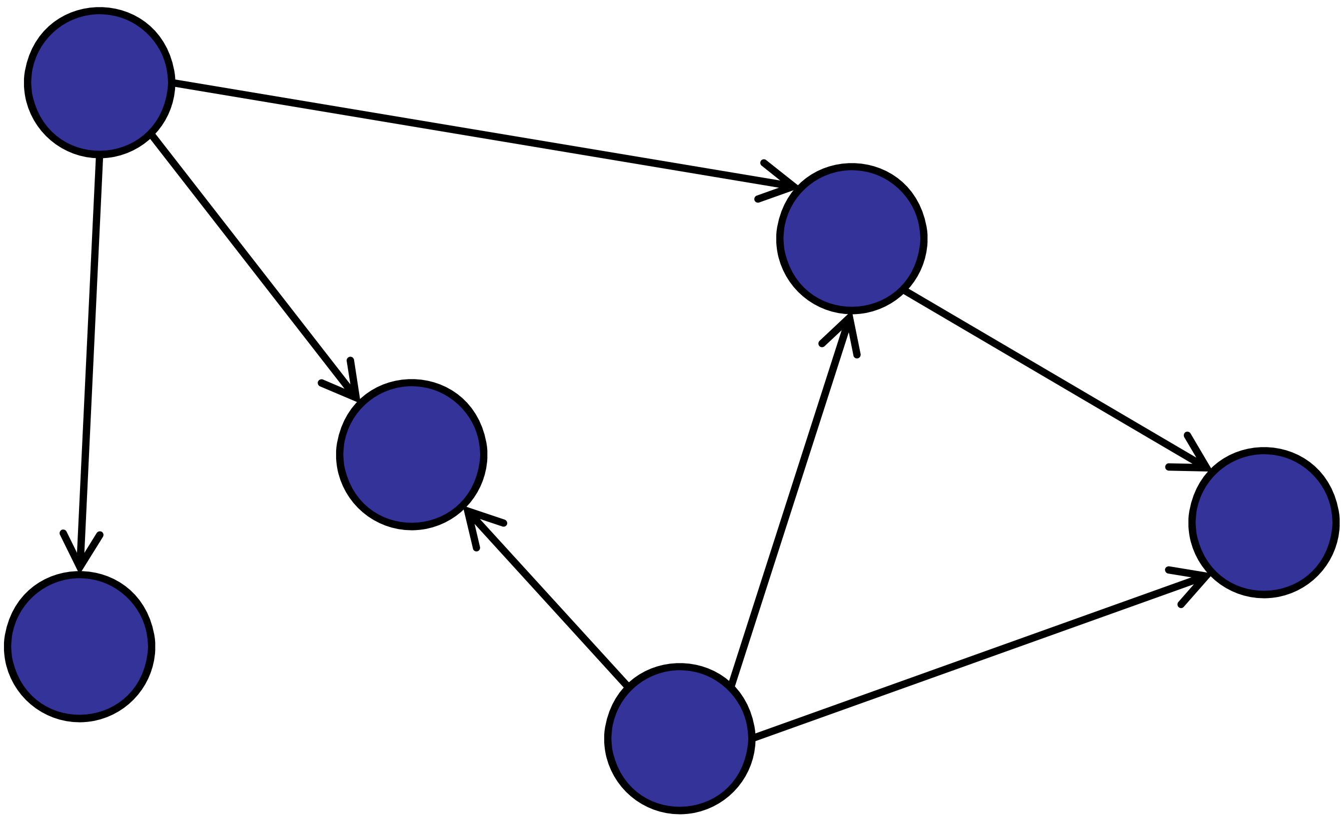
Directed Graphs

- What is a directed graph?
- Searching directed graphs (DFS / BFS)
- Topological Sort
- Connected Components

What is a **directed** graph? (Digraph)

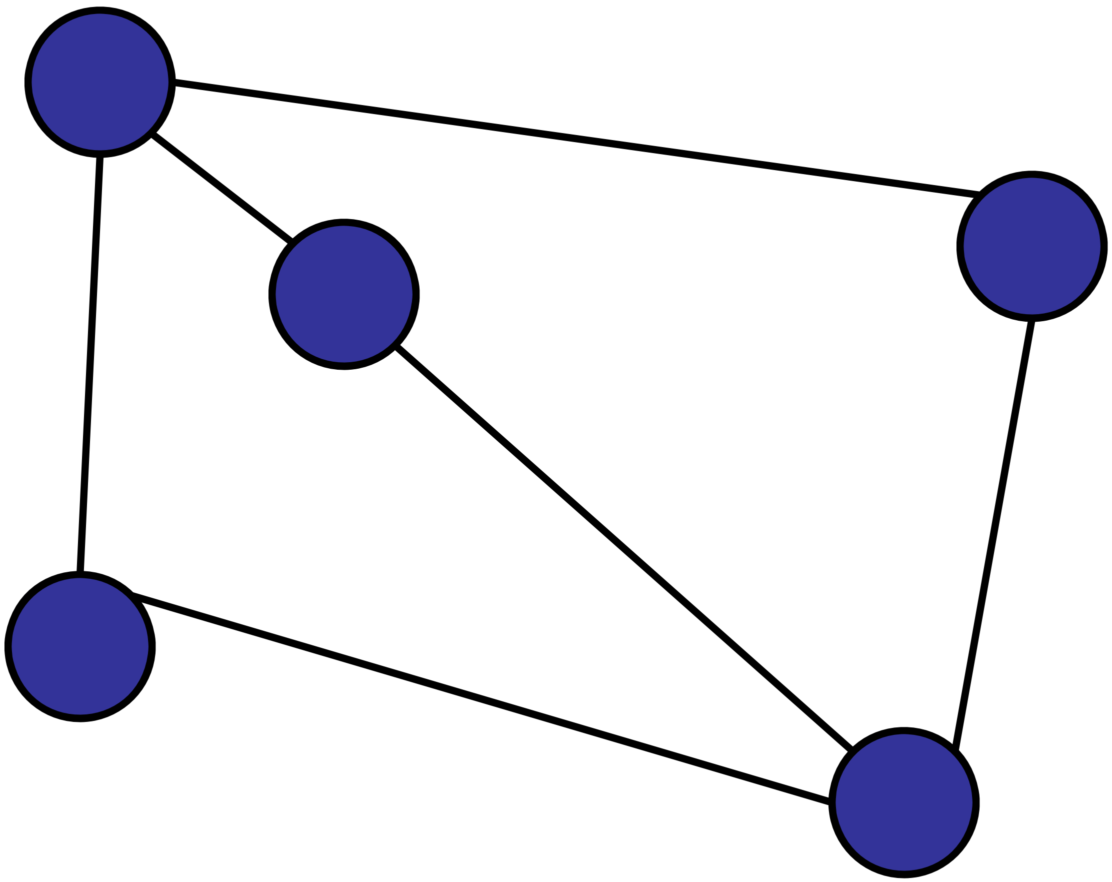
Is it a directed graph?

1. Yes
2. No.



Is it a directed graph?

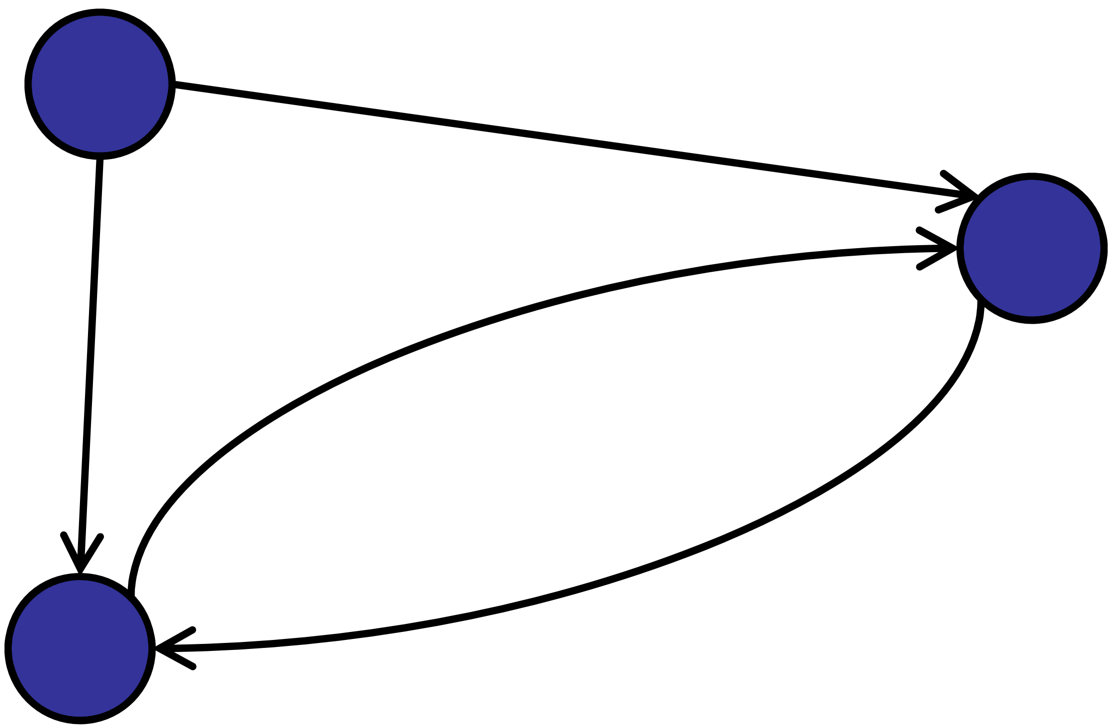
1. Yes
2. No.



Is it a directed graph?

1. Yes
2. No.

Yes or No on Zoom.



What is a directed graph?

Graph consists of two types of elements:

Nodes (or vertices)

- At least one.

Edges (or arcs)

- Each edge connects two nodes in the graph
- Each edge is unique.
- Each edge is **directed**.

What is a directed graph?

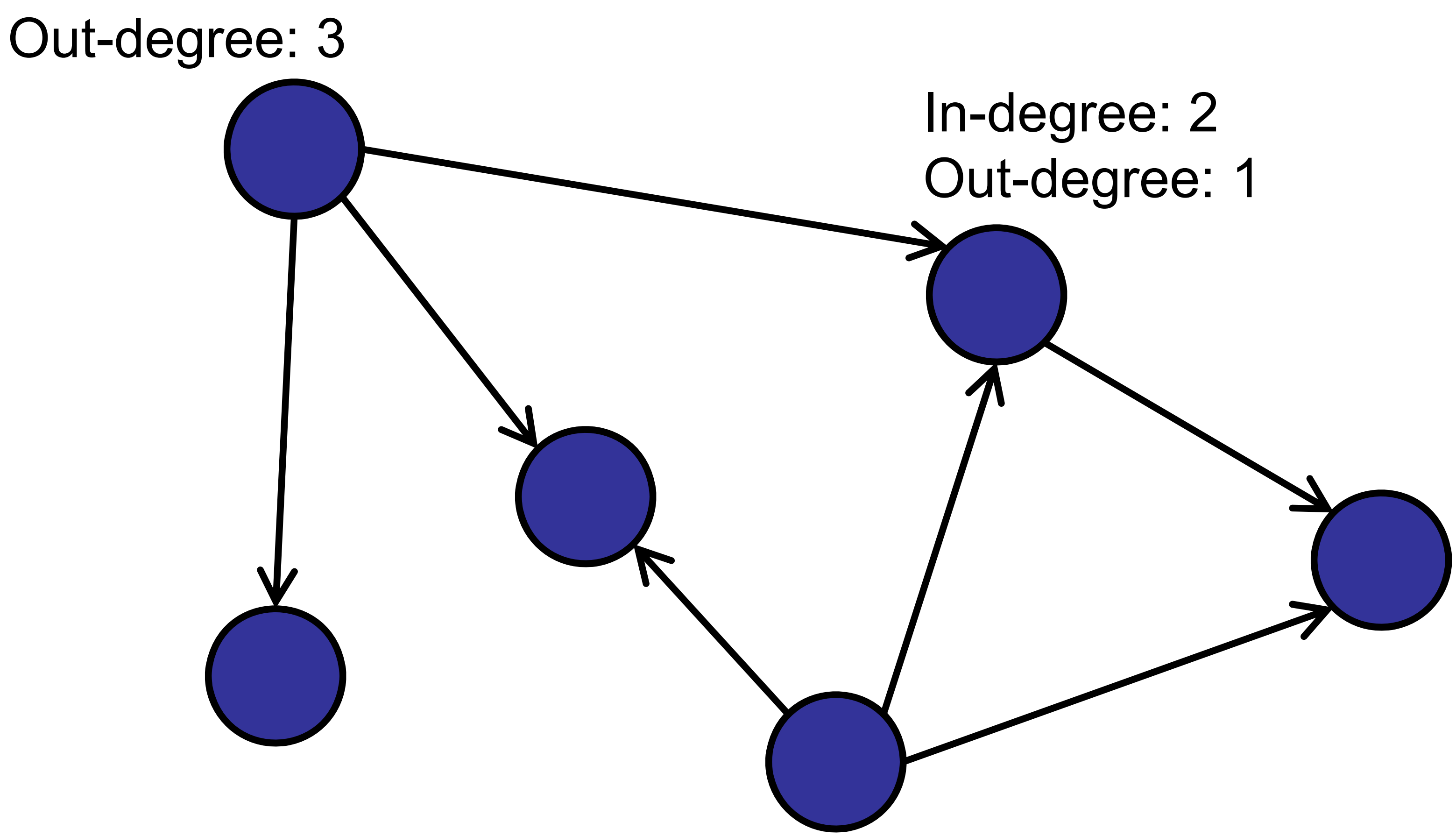
Graph G = <V, E>

- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$ Order matters!
  - For all $e_1, e_2 \in E : e_1 \neq e_2$

What is a directed graph?

In-degree: number of incoming edges

Out-degree: number of outgoing edges



Representing a (Directed) Graph

Adjacency List:

- Array of nodes
- Each node maintains a list of neighbors
- Space: $O(V + E)$

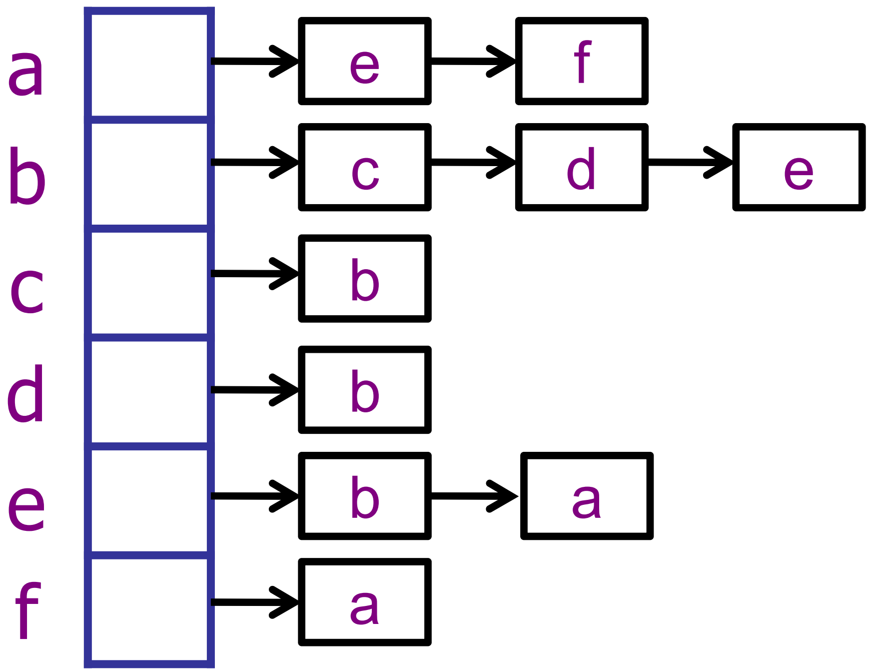
Adjacency Matrix:

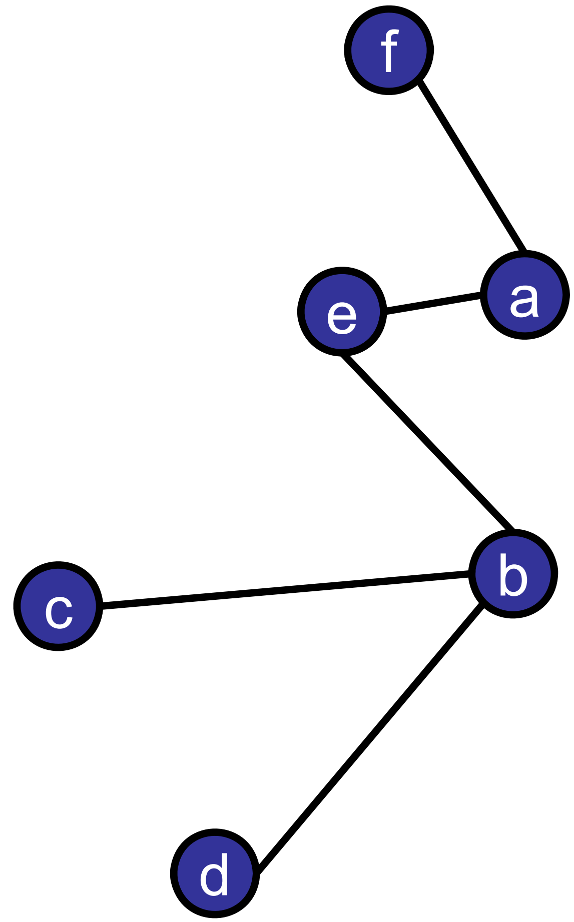
- Matrix $A[v,w]$ represents edge (v,w)
- Space: $O(V^2)$

Adjacency List

Graph consists of:

- Nodes: stored in an array
- Edges: linked list per node

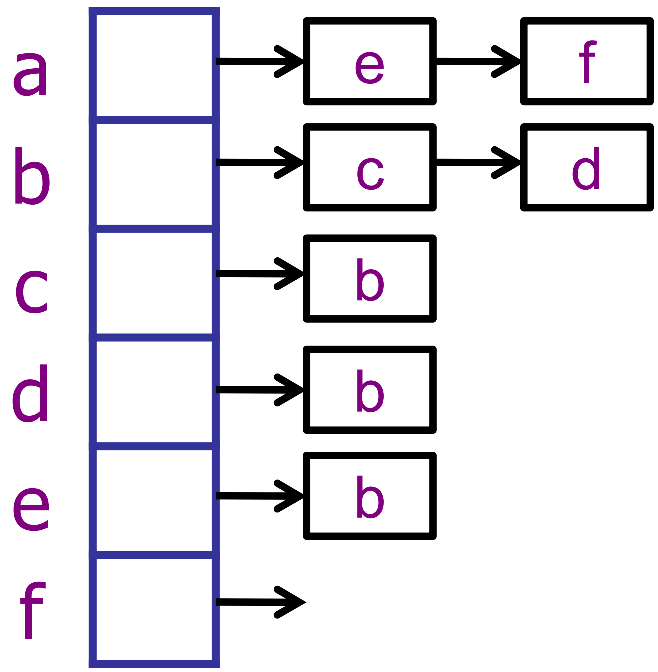


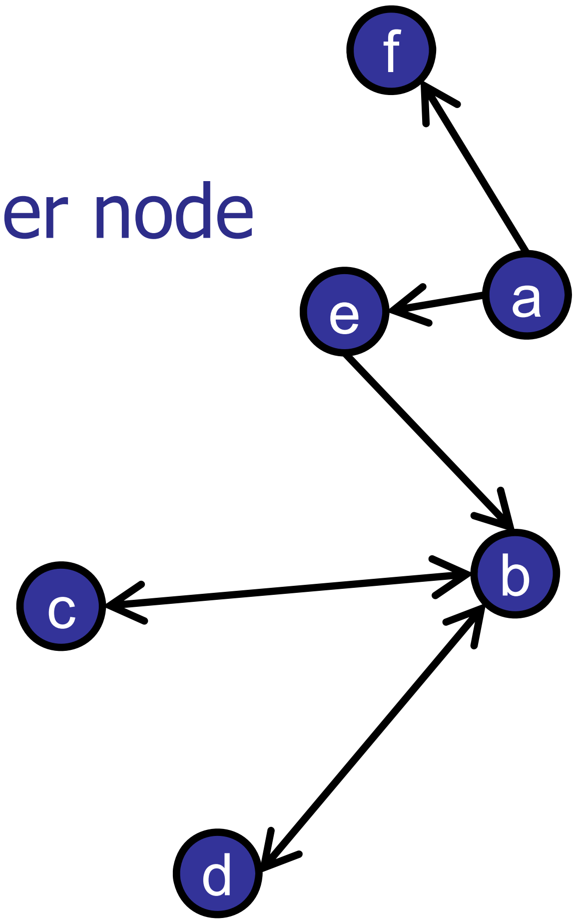


Adjacency List

Directed Graph consists of:

- Nodes: stored in an array
- **Outgoing** Edges: linked list per node





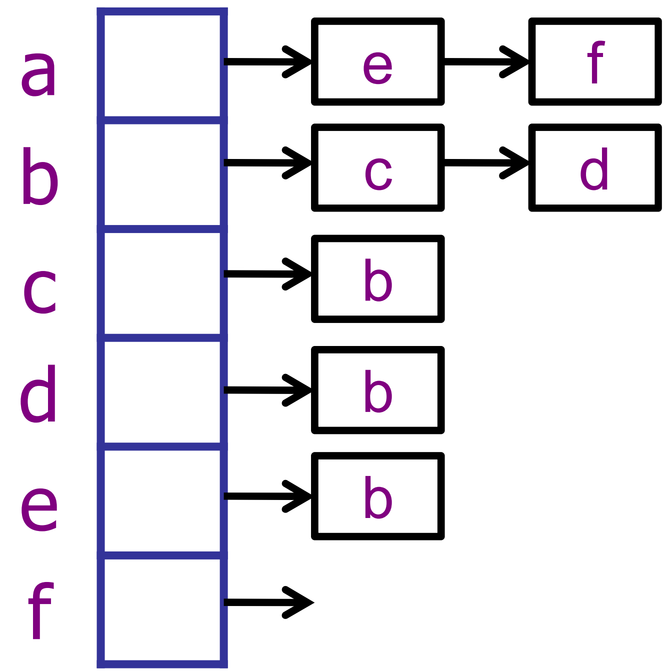
Adjacency List in Java

```
class NeighborList extends ArrayList<Integer> {
}

class Node {
  int key;
  NeighborList nbrs;
}

class Graph {
  Node[] nodeList;


}
```



Representing a (Directed) Graph

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Adjacency Matrix:

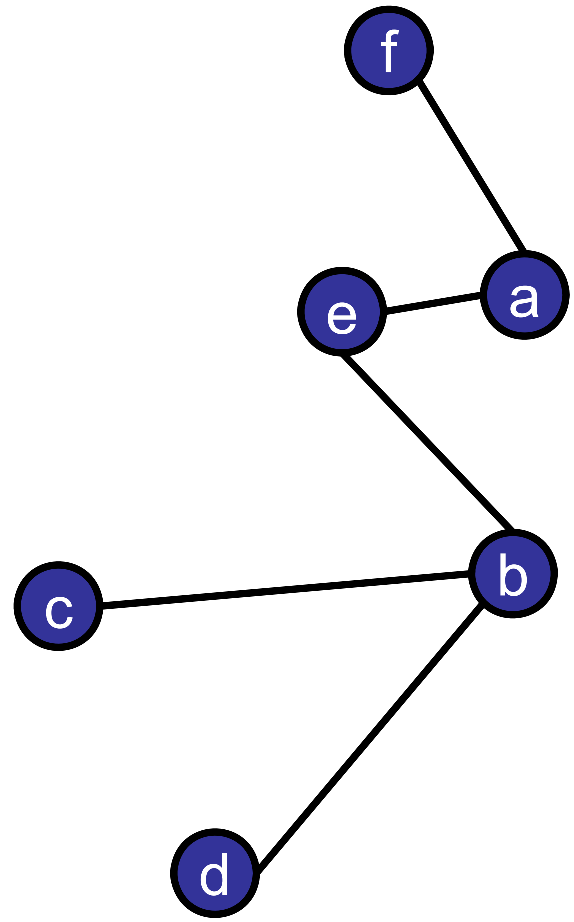
- Matrix $A[v,w]$ represents edge (v,w)
- Space: $O(V^2)$

Adjacency Matrix

Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**1**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**1**	**1**	0	0	0	0
f	**1**	0	0	0	0	0

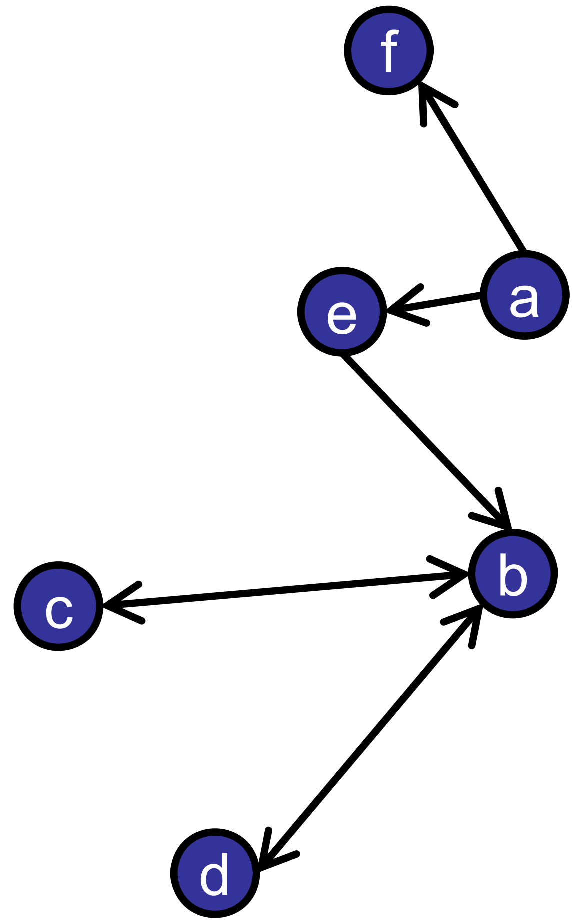


Adjacency Matrix

Directed Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**0**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**0**	**1**	0	0	0	0
f	**0**	0	0	0	0	0



Adjacency Matrix

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**0**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**0**	**1**	0	0	0	0
f	**0**	0	0	0	0	0

Searching a (Directed) Graph

Breadth-First Search:

- Search level-by-level
- Follow outgoing edges
- Ignore incoming edges

Depth-First Search:

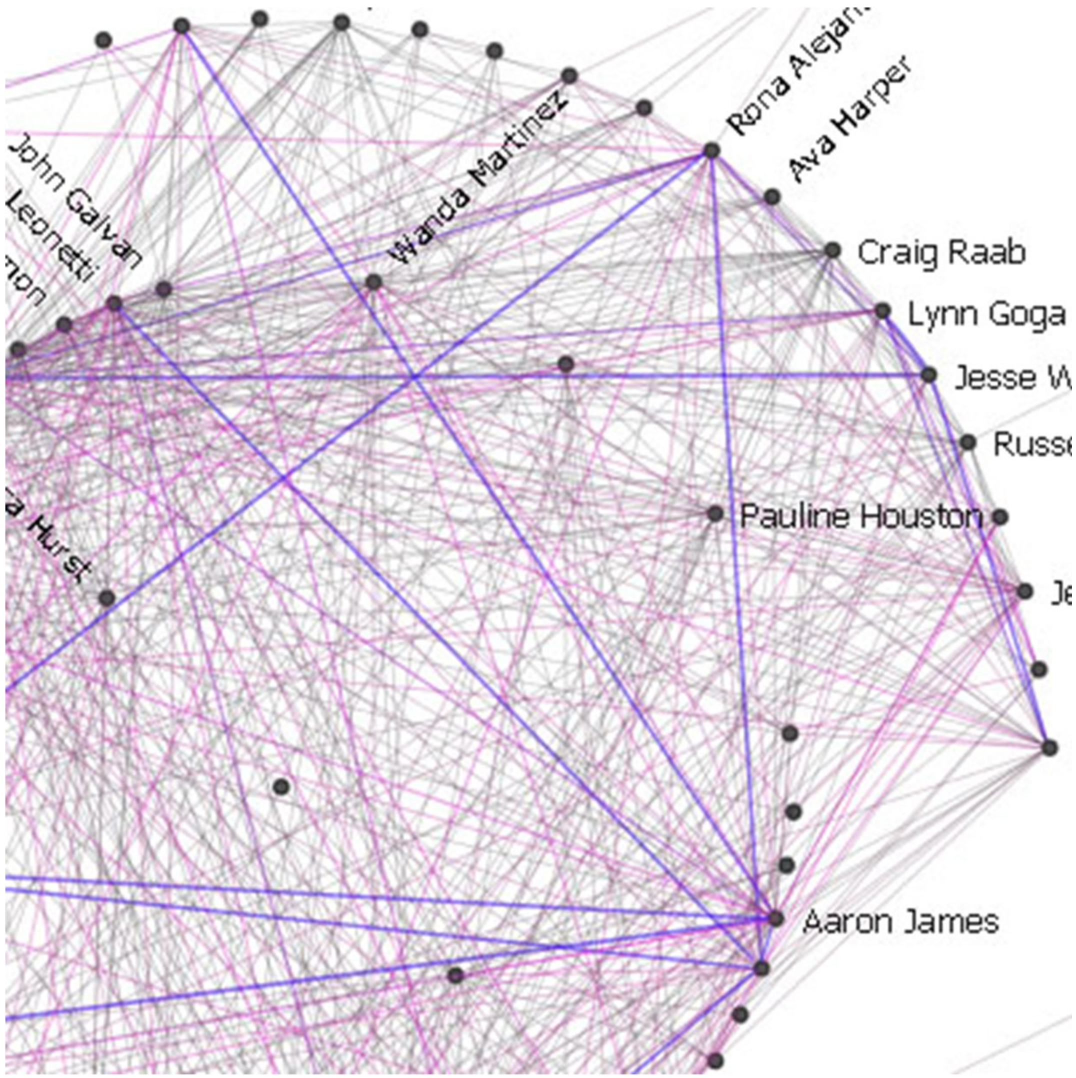
- Search recursively
- Follow outgoing edges
- Backtrack (through incoming edges)

Example of directed graphs

Directed Graphs

Is friendship always bidirectional?:

- Nodes are people
- Edge = friendship



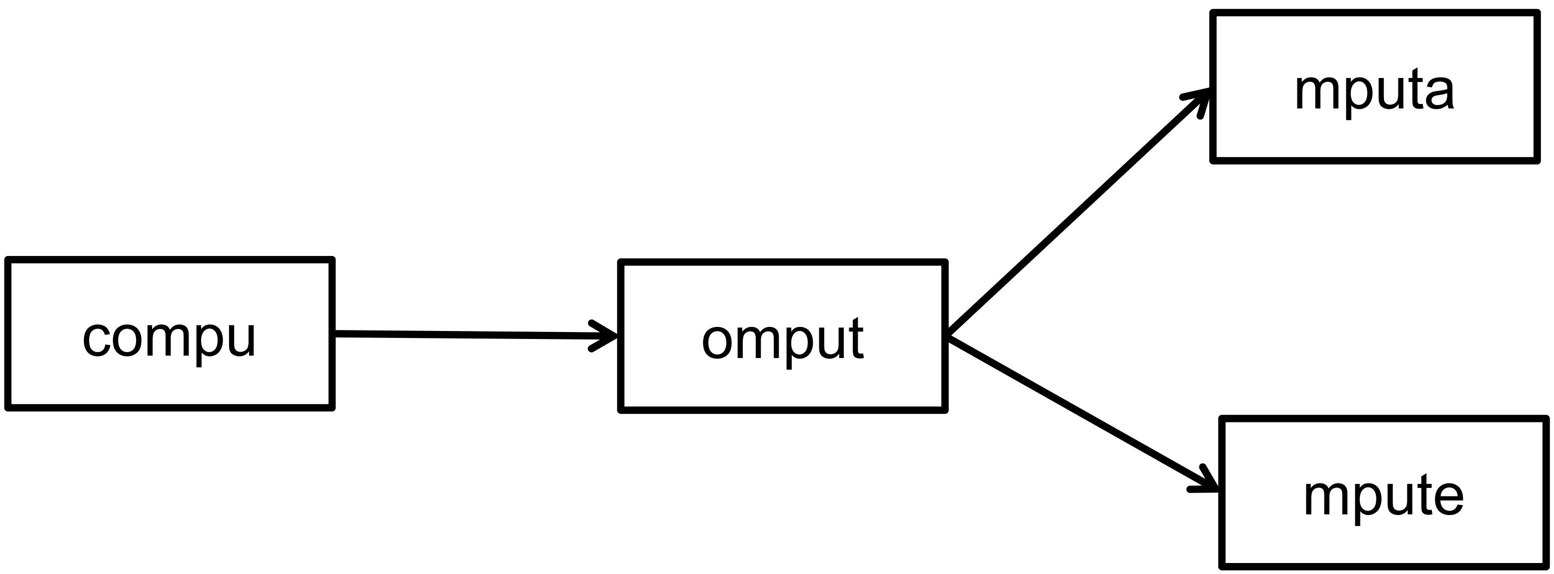
Facebook: yes

Twitter: no

Directed Graphs

Markov text generation:

- Nodes are kgrams
- Edge = one kgram follows another



Scheduling

Set of tasks for baking cookies:

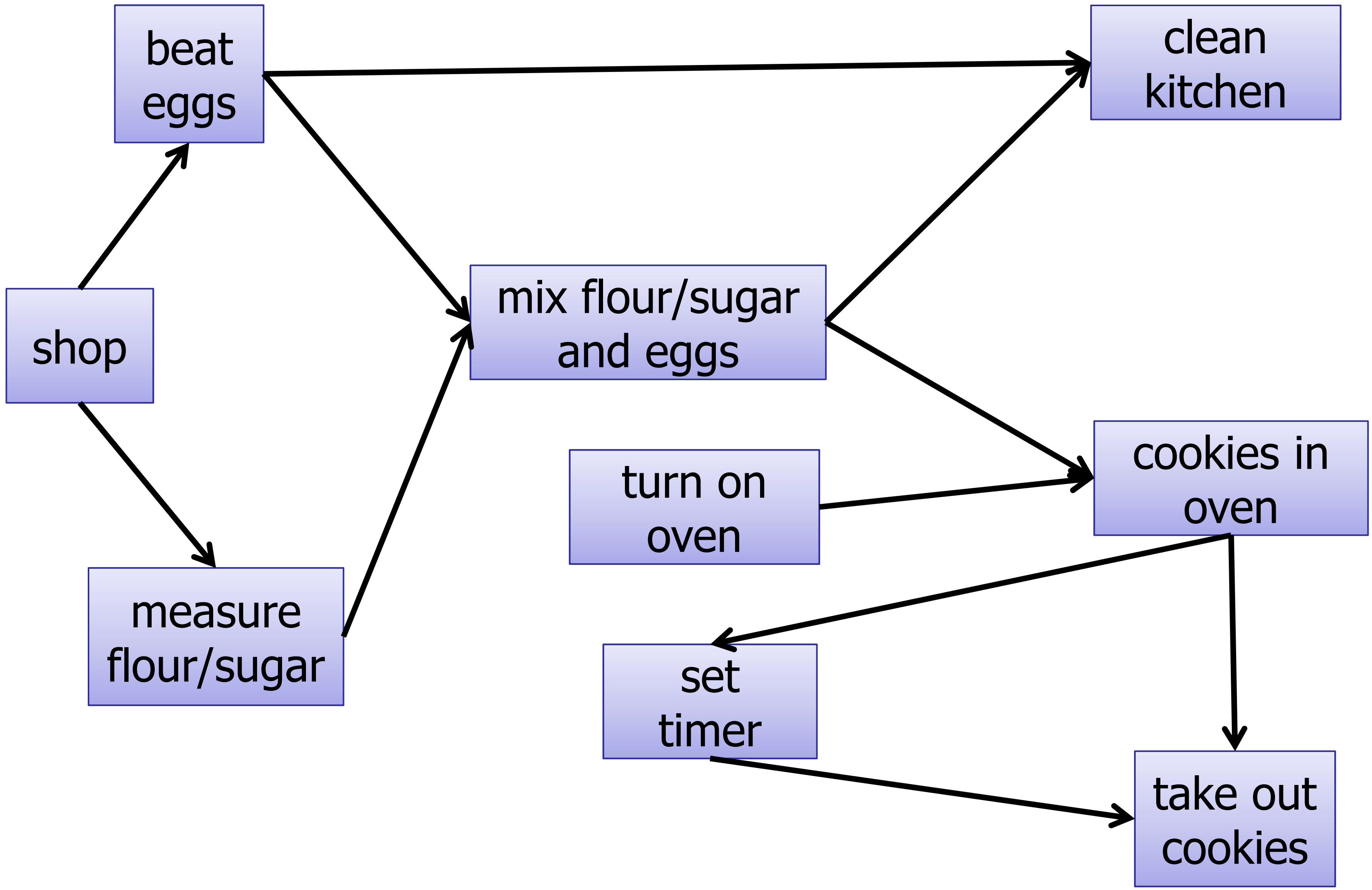
- Shop for groceries
- Put the cookies in the oven
- Clean the kitchen
- Beat the eggs in a bowl
- Measure the flour and sugar in a bowl
- Mix the eggs with the flour and sugar
- Turn on the oven
- Set the timer
- Take out the cookies

Scheduling

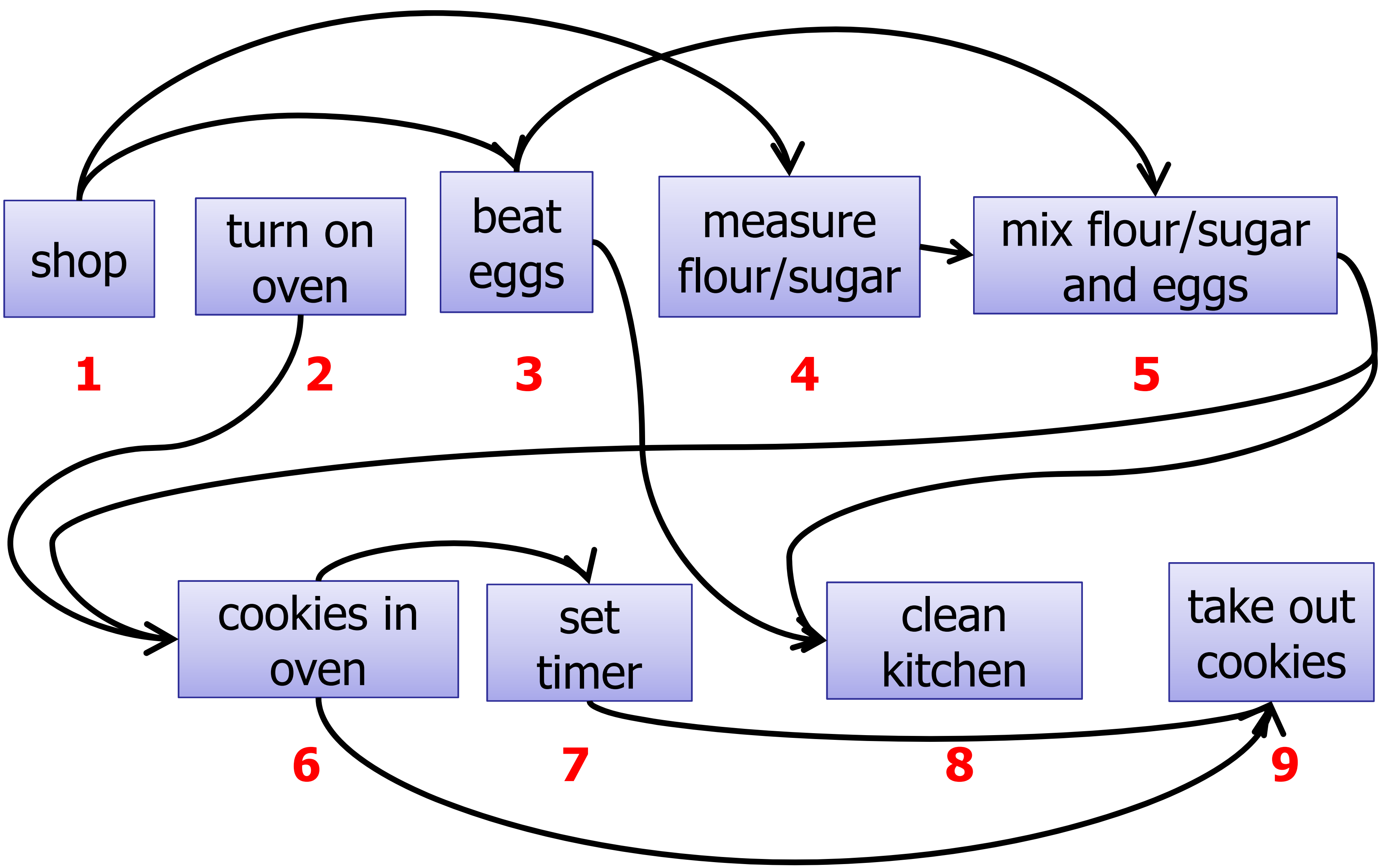
Ordering:

- Shop for groceries before beat the eggs
- Shop for groceries before measure the flour
- Turn on the oven before put the cookies in the oven
- Beat the eggs before mix the eggs with the flour
- Measure the flour before mix the eggs with the flour
- Put the cookies in the oven before set the timer
- Measure the flour before clean the kitchen
- Beat the eggs before clean the kitchen
- Mix the flour and the eggs before clean the kitchen

Scheduling


beat eggs
clean kitchen
shop
mix flour/sugar and eggs
turn on oven
cookies in oven
measure flour/sugar
set timer
take out cookies

Topological Ordering



Topological Order

Properties:

1. Sequential total ordering of all nodes

1. shop

2. turn on oven

3. measure flour/sugar

4. eggs

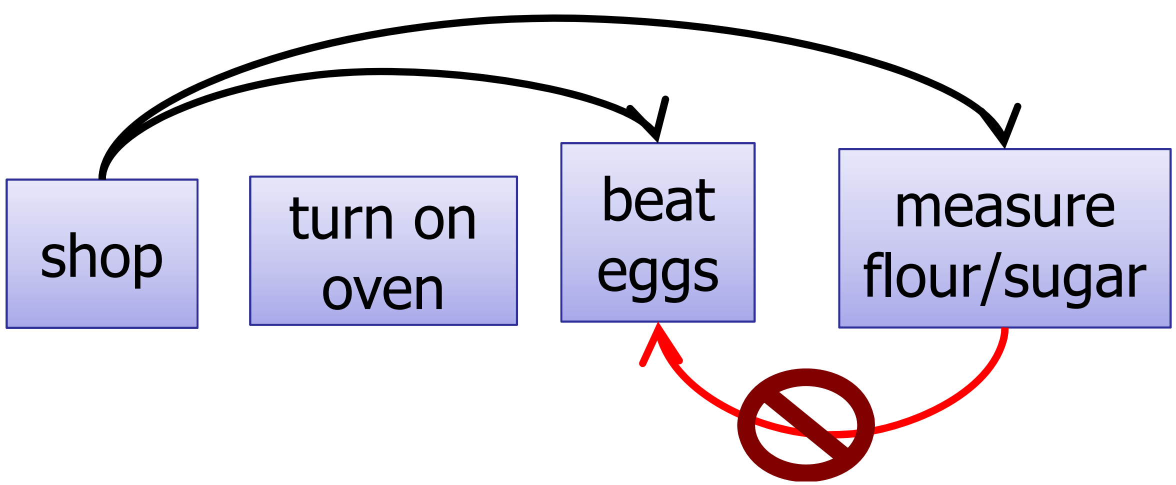
Topological Order

Properties:

1. Sequential total ordering of all nodes

1. shop | 2. turn on oven | 3. measure flour/sugar | 4. eggs

2. Edges only point forward



Does every directed graph have a topological ordering?

1. Yes
2. No
3. Only if the adjacency matrix has small second eigenvalue.

Today

Searching a graph

- BFS (finding shortest paths)
- DFS

Directed Graphs

- What is a directed graph?
- Searching directed graphs (DFS / BFS)
- Topological Sort
- Connected Components