CS1231 Reference Notes (midterms)

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Chapter 1. Propositional Logic

Rules of Inference:

Modus Ponens	$p \to q$ p $\therefore q$
Modus Tollens	$p \to q$ $\sim q$ $\therefore \sim p$
Generalization	p $\therefore p \vee q$
Specialization	$p \wedge q$ $\therefore p$
Conjunction	p q $\therefore p \wedge q$
Elimination	$p \vee q$ $\sim p$ $\therefore q$
Transitivity	$p \to q$ $q \to r$ $\therefore p \to r$
Proof by Division into Cases	$p \vee q$ $p \to r$ $q \to r$ $\therefore r$
Contradiction	$\sim p \to F$ $\therefore p$

Chapter 3. Quantified Statements

Expressions of if-then statements:

$r(x)$ sufficient condition for $s(x)$	$r(x) \to s(x)$
$r(x)$ necessary condition for $s(x)$	$\neg r(x) \to \neg s(x)$ $s(x) \to r(x)$
$r(x)$ only if $s(x)$	$r(x) \to s(x)$

Truth Set:
If $P(x)$ is a predicate and x has domain D, the truth set of $P(x)$ is the set of all elements in D that make $P(x)$ true.

Universal Statements:
$\forall x \in D, Q(x)$ where D is the domain of x and $Q(x)$ is a predicate.

Existential Statements:
$\exists x \in D, Q(x)$ where D is the domain of x and $Q(x)$ is a predicate.

Validity of Arguments:
An argument is valid if and only if the truth of its premises implies the truth of its conclusion.

Soundness of Arguments:
An argument is sound if and only if it is valid and all its premises are true.

Showing Invalidity of Arguments:

1. Show there exist a counterexample of predicates such that all premises are true but the conclusion is false.
   - May be easier to work backwards, ie. find how the $\therefore$ can be false before constructing premises
2. Boolean algebra

Rule of Inference for Quantified Statements:

Universal Instantiation	$\forall x \in D, P(x)$ $\therefore P(c)$ if $c \in D$
Universal Generalization	$P(c)$ for every $c \in D$ $\therefore \forall x \in D, P(x)$
Existential Instantiation	$\exists x \in D, P(x)$ $\therefore P(c)$ for some $c \in D$
Existential Generalization	$P(c)$ for some $c \in D$ $\therefore \exists x \in D, P(x)$

Proving Existential Statements
$\exists x \in D, P(x)$:
1. Constructive proof to find a $x \in D$
2. Constructive proof by giving directions to find such an $x \in D$

Disproving Universal Conditional Statement
$\forall x \in D, P(x) \to Q(x)$:
1. Show negation is true by counterexample, ie. prove $\exists x \in D, P(x) \wedge \sim Q(x)$

Chapter 5. Sets

Set Builder Notation:
The set of all x such that $P(x)$ is true is denoted by

$$A = \{x \in D \mid P(x)\}$$

where D is the domain of x and $P(x)$ is a predicate.

Set Replacement Notation:
The set of all x such that $P(x)$ is true is denoted by

$$A = \{f(x) \mid x \in D\}$$

where D is the domain of x and $f(x)$ is a function.

Roster Notation:
Listing all elements, we have

$$A = \{a, b, c, ...\}$$

Disjoint Sets:
Two sets A and B are disjoint if and only if they have no elements in common, ie. $A \cap B = \emptyset$.

Power Sets
The power set of a set $P(A)$ is the set of all subsets of A. For a set A with n elements, $P(A)$ has 2^n (Theorem 5.2.4)

For all sets A and B,

$$P(A \cap B) = P(A) \cap P(B)$$

However, there exists A and B such that

$$P(A \cup B) \neq P(A) \cup P(B)$$

Proving with Sets
1. Work with the universal set U, convert to boolean algebra and then back to sets
   eg. let $z \in U$, then $z \in$ {Given Set} ...
2. Work with set notation

Chapter 6. Relations
Definition of a relation:
A relation R from a set A to a set B is a subset of the Cartesian product AxB.

Given statement P(x,y), we have

$$\forall (x, y) \in A \times B, ((x, y) \in R \leftrightarrow P(x, y))$$
$$\forall x \in A, \forall y \in B, (xRy \leftrightarrow P(x, y))$$

Inverse Relations:
Given a relation R from a set A to a set B, the inverse relation $R^{\{-1\}}$ from B to A is defined as

$$R^{-1} = \{(y, x) \in B \times A : (x, y) \in R\}$$
$$\forall x \in A, \forall y \in B, (xRy \leftrightarrow yR^{-1}x)$$

Domain, Co-Domain and Range:
Domain of R is the set of all first elements of ordered pairs in R, ie.

$$\{x \in A : \exists y \in B, (x, y) \in R\}$$

Co-domain of R is the set of all second elements of ordered pairs in R, ie. B

Range of R is the set of all second elements of ordered pairs in R, ie.

$$\{y \in B : \exists x \in A, (x, y) \in R\}$$

Compositions of Relations:
Relation starting in R and ending in S
$=$ Composition of R with S
$= S \circ R$

$$\forall x \in A, \forall z \in C, (x(\boldsymbol{S} \circ \boldsymbol{R})z \leftrightarrow \exists y \in B, (xRy \wedge ySz))$$

Composition is associative, ie.

$$(S \circ R) \circ T = S \circ (R \circ T) = S \circ R \circ T$$

Inverse of Composition is given as

$$(S \circ R)^{-1} = R^{-1} \circ S^{-1}$$

Properties of Relations

Reflexive	$\forall x \in A, xRx$
Irreflexive	$\forall x \in A, (x, x) \notin R$
Symmetric	$\forall x \in A, \forall y \in A, (xRy \to yRx)$
Anti-Sym	$\forall x \in A, \forall y \in A, ((xRy \wedge yRx) \to x = y)$
Asymmetric	$\forall x \in A, \forall y \in A, (xRy \to y\not{R}x)$ ie. Anti-Sym and Irreflexive
Transitive	$\forall x, y, z \in A, ((xRy \wedge yRz) \to xRz)$
Equivalence	Reflexive, Symmetric, Transitive
Partial Order	Reflexive, Anti-Sym, Transitive

Transitive Closure:
The transitive closure of a relation R on a set A is the smallest transitive relation on A that contains R.

Partitions
A partition of a set A is a collection of non-empty, mutually disjoint subsets of A such that every element of A is in exactly one of these subsets.

λ is a partiton of set A if
1. λ is a set of non-empty subsets of A
2. Every element of A is in exactly one element of λ, ie.

$$\forall x \in A, \exists S \in \lambda(x \in S)$$
$$\forall x \in A, \forall S, T \in \lambda((x \in S \wedge x \in T) \to S = T)$$

Equivalence Relations
Relations induced by set partitions are equivalence relations.

The set of all elements $x \in A$ such that A is ~-related to x is known as the *equivalence class* of x and is denoted by $[x]$.

$$[a] = \{x \in A : x \sim a\}$$
$$\forall x \in A, \forall y \in A, ([x] = [y] \leftrightarrow x \sim y)$$

Order Relations

Maximal Element	c is a maximal element iff $\forall x \in A, c \preccurlyeq x \to c = x$ ie. no larger element exists
Largest / Greatest/ Maximum Element	c is a largest element iff $\forall x \in A, x \preccurlyeq c$ ie. all other elements are smaller

Minimal Element	c is a minimal element iff $$\forall x \in A, x \preccurlyeq c \to c = x$$ ie. no smaller element exists
Smallest / Least / Minimum Element	c is a smallest element iff $$\forall x \in A, c \preccurlyeq x$$ ie. all other elements are larger

Compatible and Comparable

Consider a partial order $\preccurlyeq$ on a set A, with $a, b \in A$.

$$\text{a, b comparable} \leftrightarrow a \preccurlyeq b \vee b \preccurlyeq a$$
$$\text{a, b compatible} \leftrightarrow \exists c \in A, (a \preccurlyeq c \wedge b \preccurlyeq c)$$

Definition of a Total Order:

$$\forall x, y \in A, (xRy \vee yRx)$$

Appendix

Laws of Boolean Algebra:

Commutative Law	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
Associative Law	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
Distributive Law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity Law	$p \wedge T \equiv p$	$p \vee F \equiv p$
Negation Law	$p \wedge {\sim}p \equiv F$	$p \vee {\sim}p \equiv T$
Double Negation Law	${\sim}({\sim}p) \equiv p$	
Idempotent Law	$p \wedge p \equiv p$	$p \vee p \equiv p$
Universal Bound Law	$p \vee T \equiv T$	$p \wedge F \equiv F$
De Morgan's Law	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
Absorption Law	$p \wedge (p \vee q) \equiv p$	$p \vee (p \wedge q) \equiv p$
Negation of T and F	${\sim}T \equiv F$	${\sim}F \equiv T$
Implication Law	$p \to q \equiv {\sim}p \vee q$	
Contrapositive Law	$p \to q \equiv {\sim}q \to {\sim}p$	
Converse Law	$\text{converse}(p \to q) \equiv q \to p$	
Inverse Law	$\text{inverse}(p \to q) \equiv {\sim}p \to {\sim}q$	

Consensus Theorem	$(p \wedge q) \vee (\neg p \wedge r) \vee (q \wedge r) \equiv (p \wedge q) \vee (\neg p \wedge r)$
Proof	$(p \wedge q) \vee \underline{(q \wedge r)} \vee (\neg p \wedge r)$ $\equiv (p \wedge q) \vee \underline{\{(\neg p \vee p) \wedge (q \wedge r)\}} \vee (\neg p \wedge r)$ $\equiv (p \wedge q) \vee (p \wedge q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge r)$ $\equiv (p \wedge q) \vee (\neg p \wedge r)$

Laws of Set Algebra

Commutative Law	$A \cup B = B \cup A$	$A \cap B = .B \cap A$
Associative Law	$A \cup (B \cup C) = (A \cup B) \cup C$	$A \cap (B \cap C) = (A \cap B) \cap C$
Distributive Law	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Law	$A \cup \emptyset = A$	$A \cap U = A$
Complement Law	$A \cup \overline{A} = U$	$A \cap \overline{A} = \emptyset$
Idempotent Law	$A \cup A = A$	$A \cap A = A$
Universal Bound Law	$A \cup U = U$	$A \cap \emptyset = \emptyset$
De Morgan's Law	$\overline{A \cup B} = \overline{A} \cap \overline{B}$	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
Absorption Law	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Double Complement Law	$\overline{\overline{A}} = A$	
Complement of Universal Set Law	$\overline{U} = \emptyset$	
Set Difference Law	$A \setminus B = A \cap \overline{B}$	

Quick Power Set References

$P(\emptyset)$	$\{\emptyset\} = \{\{\}\}$
$P(\{a\})$	$\{\emptyset, \{a\}\}$
$P(\{a, b\})$	$\{\emptyset, \{a\}, \{b\},$ $\{a, b\}\}$
$P(\{a, b, c\})$	$\{\emptyset, \{a\}, \{b\}, \{c\},$ $\{a, b\}, \{a, c\}, \{b, c\},$ $\{a, b, c\}\}$
$P(\{a, b, c, d\})$	$\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\},$ $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\},$ $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\},$ $\{a, b, c, d\}\}$