MA1522 Reference Notes

~github/reidenong/cheatsheets~, AY23/24 S1

Row Echelon Form

A matrix is in row echelon form if:

(1) All zero rows are at the bottom of the matrix.
(2) The leading entries are further to the right as we move down the rows.

It is in *Reduced Row Echelon form* if:

(1) The leading entries are 1.
(2) In each pivot column, all entries except the leading entry is zero.

Elementary Row Operations

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

Two Linear systems have the same solution set if their augmented matrices are row equivalent.

If matrix B is obtained from matrix A by

$$A \xrightarrow{r_1} \xrightarrow{r_2} \dots \xrightarrow{r_k} B$$

Then

$$B = E_k E_{k-1} \dots E_2 E_1 A$$

where E_i is the elementary matrix corresponding to r_i.

Systems of Linear Equations

The linear system of $A\boldsymbol{x} = \boldsymbol{b}$ is homogenous if $\boldsymbol{b} = 0$. If there is a nontrivial solution, it has infinitely many solutions.

A linear system is consistent if it has at least one solution. A homogenous equation $A\boldsymbol{x} = \mathbf{0}$ is always consistent, as it has at least the trivial solution.

Types of Matrices

Scalar Matrices

A scalar matrix is a diagonal matrix where all diagonal entries are equal.

Triangular Matrices

Upper Triangular A where $a_{\{ij\}} = 0$ for $i > j$.

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

Strictly Upper Triangular A where $a_{\{ij\}} = 0$ for $i \geq j$.

$$\begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lower Triangular A where $a_{\{ij\}} = 0$ for $i < j$.

$$\begin{pmatrix} * & 0 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & * \end{pmatrix}$$

Strictly Lower Triangular A where $a_{\{ij\}} = 0$ for $i \leq j$.

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & 0 \end{pmatrix}$$

Scalar Multiplication and Matrix Addition

Properties:

(1) Commutative: $A + B = B + A$
(2) Associative: $(A + B) + C = A + (B + C)$
(3) Additive identity: $A + 0 = A$
(4) Additive inverse: $A + (-A) = 0$
(5) Distributive: $c(A + B) = cA + cB$
(6) Scalar addition: $(c + d)A = cA + dA$
(7) Associative: $c(dA) = (cd)A$
(8) If $aA = \mathbf{0}$, then $a = 0$ or $A = \mathbf{0}$

Matrix Multiplication

For multiplication of a $m \times n$ matrix A and a $n \times p$ matrix B,

$$AB_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Properties:

(1) Associative: $A(BC) = (AB)C$
(2) Left distributive: $A(B + C) = AB + AC$
(3) Right distributive: $(A + B)C = AC + BC$
(4) Commutes with scalar multiplication:
$$c(AB) = (cA)B = A(cB)$$
(5) Not commutative: $AB \neq BA$ in general
(6) Multiplicative Identity: $I_n A = A I_m = A$
(7) Zero divisor: There exists nonzero matrices A and B such that $AB = \mathbf{0}$
(8) Zero matrix: $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$

Block Multiplication

$$AB = A(b_1 \ \ b_2 \ \ \dots \ \ b_n) = (Ab_1 \ \ Ab_2 \ \ \dots \ \ Ab_n)$$

$$AB = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ a_2 B \\ \dots \\ a_m B \end{pmatrix}$$

Transpose

The transpose of a $m \times n$ matrix A is a $n \times m$ matrix A^T where $A^T_{ij} = A_{ji}$.

Properties:

(1) $(A^T)^T = A$
(2) $(cA)^T = cA^T$
(3) $(A + B)^T = A^T + B^T$
(4) $(AB)^T = B^T A^T$

Inverse of a Matrix

A matrix A is invertible if there exists a unique matrix B such that $AB = BA = I$.

Properties:

(1) $(A^{-1})^{-1} = A$
(2) $(cA)^{-1} = c^{-1}A^{-1}, \forall c \in \mathbb{R}$
(3) $(A^T)^{-1} = (A^{-1})^T$
(4) $(AB)^{-1} = B^{-1}A^{-1}$ if A, B are both invertible
(5) Left Cancellation Law: $AB = AC \to B = C$
(6) Right Cancellation Law: $BA = CA \to B = C$

To find an inverse, consider

$$(A \mid I) \overset{RREF}{\longrightarrow} (I \mid A^{-1})$$

Invertible Matrix Theorem

Let A be a $n \times n$ matrix. The following statements are equivalent:

(1) A is invertible.
(2) A has a left inverse
(3) A has a right inverse
(4) RREF of A is I_n
(5) A can be expressed as a product of elementary matrices
(6) Homogenous system $Ax = \mathbf{0}$ has only the trivial solution
(7) for any b, the system $Ax = b$ has a unique solution
(8) The determinant of A is nonzero
(9) The columns/rows of A are linearly independent
(10) The columns/rows of A span $\mathbb{R}^n$

LU Decomposition

Suppose $A \overset{r_1}{\to}\overset{r_2}{\to} \dots \overset{r_k}{\to} U$, where each row operation is of the form $R_i + cR_j$ and U is a row echelon form of A. Then A can be decomposed into a *unit* lower triangular matrix and an upper triangular matrix.

$$A = LU = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \dots & 1 \end{pmatrix} \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1}\dots E_k^{-1}$$

To solve $LUx = Ax = b$, solve $Ly = b$, then $Ux = y$.

Determinant

Properties:

(1) $\det(A^T) = \det(A)$
(2) $\det(AB) = \det(A)\det(B)$ for A, B of same size
(3) $\det(A^{-1}) = \frac{1}{\det(A)}$
(4) $\det(cA) = c^n \det(A)$ for $n \times n$ matrix A
(5) $\det(diag(a_1, a_2, \dots, a_n)) = a_1 \cdot a_2 \cdot \dots \cdot a_n$
(6) Determinant and Row Elementary Operations:

$A \overset{R_i + cR_j}{\longrightarrow} B$	$\det(A) = \det(B)$	$\det(B) = \det(A)$
$A \overset{cR_i}{\longrightarrow} B$	$\det(A) = \frac{1}{c}\det(B)$	$\det(B) = c\det(A)$
$A \overset{R_i \leftrightarrow R_j}{\longrightarrow} B$	$\det(A) = -\det(B)$	$\det(B) = -\det(A)$

Finding Determinants:

1. for n = 2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$
2. for n = 3, $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,
   $\det(A) = aei + bfg + cdh - ceg - bdi - afh$
3. for $n \geq 3$, use Cofactor Expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{jk}A_{jk}$$

where A_{ij} is the (i,j) cofactor of A, given by

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

where M_{ij} is the (i, j) matrix minor of A, the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

Adjoint

With a order n square matrix A, the adjoint of A is

$$adj(A) = (A_{ij})^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & -M_{21} & \dots & \pm M_{n1} \\ -M_{12} & M_{22} & \dots & \mp M_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \pm M_{1n} & \mp M_{2n} & \dots & \pm M_{nn} \end{pmatrix}$$

Adjoint Formula:

$$A \cdot adj(A) = \det(A)I$$

Cramer's Rule

Let A be a invertible $n \times n$ matrix.
For any $b \in \mathbb{R}^n$, the unique solution of $Ax = b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

where $A_i(b)$ is the matrix obtained by replacing the ith column of A with b.

Linear Span

$$span(u_1 \ u_2 \ \dots \ u_n) = \{c_1u_1 + c_2u_2 + \dots + c_nu_n \mid c_i \in \mathbb{R}, \forall i\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}, \forall i \right\}$$

$\mathbb{R}^n$ is the set of all vectors with n-coordinates.

Theorem:

(1) $\boldsymbol{v} \in span(u_1 \ u_2 \ \dots \ u_n) \leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n)x = \boldsymbol{v}$ is consistent $\leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n \mid \boldsymbol{v})$ is consistent.

(2) $span(u_1 \ u_2 \ \dots \ u_n) = \mathbb{R}^n \leftrightarrow$
The reduced row echelon form of A has no zero rows.

Properties:

Let $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$.

(1) Contains Origin: $\mathbf{0} \in span(S)$

(2) Closed under addition:
$\forall u, v \in span(S), u + v \in span(S)$

(3) Closed under scalar multiplication:
$\forall u \in span(S), \forall c \in \mathbb{R}, c \cdot u \in span(S)$

(4) Contains all linear combinations:
$\forall u_1, u_2, ..., u_n \in span(S),$
$\forall c_1, c_2, ..., c_n \in \mathbb{R},$
$c_1 u_1 + c_2 u_2 + ... + c_n u_n \in span(S)$

Span equality:

Let $S = \{u_1, u_2, ..., u_k\}$ and $T = \{v_1, v_2, ..., v_n\}$. Then,

$$span(T) \subseteq span(S) \leftrightarrow$$
$$\forall v \in T, v \in span(S) \leftrightarrow$$
$$(S \mid T) \text{ is consistent.}$$

For equality, we need to show that $span(S) \subseteq span(T)$ and $span(T) \subseteq span(S)$.

Subspaces

A subset $V \subseteq \mathbb{R}^n$ is a subspace if:

(1) Contains Origin: $\mathbf{0} \in V$

(2) Closed under linear combination:
$\forall u, v \in V, \forall c, d \in \mathbb{R}, cu + dv \in V$

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span,$V = span(S)$ for some finite set $S = \{u_1, u_2, ..., u_n\}$.

Solution Sets of linear systems

Solution sets of linear systems can be expressed implicitly or explicitly.

Implicit form:

$$\left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid Ax = \boldsymbol{b} \right\}$$

Explicit form:

$$\{\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k} \mid s_i \in \mathbb{R}, \forall i\}$$

where $\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k}$ are the general solutions of $Ax = \boldsymbol{b}$.

The solution set $V = \{u \mid Au = \boldsymbol{b}\}$ is a subspace if and only if **b** = 0, ie. the system is homogenous.

The solution set $W = \{w \mid Aw = \boldsymbol{b}\}$ of a linear system $Ax = \boldsymbol{b}$ is given by **u** + V, where

(1) $V = \{v \mid Av = \mathbf{0}\}$ is the solution set of the homogenous
system $Ax = \mathbf{0}$ and

(2) **u** is a particular solution of $Au = \boldsymbol{b}$.

Linear Independence

A set of vectors $S = \{u_1, u_2, ..., u_n\}$ is linearly independent if the only solution to $c_1 u_1 + c_2 u_2 + ... + c_n u_n = \mathbf{0}$ is $c_1 = c_2 = ... = c_n = 0$. A set is linearly independent iff the RREF of S has no non-pivot columns.

Special Cases:

1. $\{\mathbf{0}\}$ is always linearly dependent.
2. {v_1, v_2} is linearly dependent iff v_1 is a scalar multiple of v_2.
3. {} = $\emptyset$ is linearly independent.
4. Any subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.
6. Any superset of a linearly dependent set is linearly dependent.
7. Any subset of a linearly independent set is linearly independent.
8. A set S containing n vectors in $\mathbb{R}^n$ is linearly independent iff it spans $\mathbb{R}^n$

Basis

Let $V \subseteq \mathbb{R}^n$ be a subspace. A set $B = \{u_1, u_2, ..., u_k\}$ is a basis of V if:

(1) B is linearly independent

(2) B spans V, ie. span(B) = V

A subset $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$ iff $|S| = n$ and $A = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$ is an invertible matrix.

Coordinates Relative to a basis

With basis $S = \{u_1, u_2, ..., u_n\}$, every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be expressed uniquely as a linear combination of the basis vectors.

$$\boldsymbol{v} = c_1 u_1 + c_2 u_2 + ... + c_n u_n \leftrightarrow [\boldsymbol{v}]_s = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Change of Basis / Transition Matrix

Suppose there exist bases $S = \{u_1, u_2, u_3\}$ and $T = \{v_1, v_2, v_3\}$. Then, the transition matrix from T to S is

$$\text{RREF}(S \mid T) = (I_k \mid P_{T \to S})$$

Then $[w]_S = P_{T \to S} [w]_T$, and

$$T = S \times P_{T \to S}$$

Dimension

The dimension of a subspace $V \subseteq \mathbb{R}^n$ is the number of vectors in any basis of V. The dimension of a solution space $V = \{u \mid Au = 0\}$ is the number of non-pivot columns in the RREF of A.

Column and Row Space

The column space of a matrix A is the subspace of $\mathbb{R}^m$ spanned by its column vectors.

$$v \in \text{colspace}(A) \leftrightarrow A\boldsymbol{x} = v \text{ is consistent.}$$

The row space of a matrix A is the subspace of $\mathbb{R}^n$ spanned by its row vectors. Row operations preserve the row space of a matrix. The nonzero rows of the RREF of A form a basis for the row space of A.

Rank

rank(A) = dimension of colspace(A)
= dimension of rowspace(A)
= number of pivot columns in RREF
= number of nonzero rows in RREF

1. rank(A^T) = rank(A)
2. rank($A_{m,n}$) $\leq$ min(m, n), with equality when *full rank*.
3. rank(AB) $\leq$ min(rank(A), rank(B))

Nullspace

Nullspace of A is the solution space to $Ax = \mathbf{0}$.

$$\begin{aligned}\text{null}(A) &= \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} \\ &= \text{null}(A^T A)\end{aligned}$$

$$\begin{aligned}\text{nullity}(A) &= \dim(\text{null}(A)) \\ &= \text{number of non-pivot columns in RREF}\end{aligned}$$

Then by the *Rank-Nullity Theorem*,

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns in A}$$

Dot Product, norm

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T v$$
$$\|u\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$$
$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|u\|\|v\|}$$

Where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Orthogonality

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. A set of vectors are orthogonal if every pair of vectors in the set are orthogonal. It is linearly independent if it does not have the zero vector.

A set of vectors is orthonormal if it is orthogonal and every vector in the set has norm 1. It is then guaranteed to be linearly independent.

Orthogonal Basis

A basis is orthogonal if it is a orthogonal set.
Let $S = \{u_1, u_2, ..., u_n\}$ be an orthogonal basis. Then for any $v \in V$,

$$v = \left(\frac{v \cdot u_1}{\|u_1\|^2}\right) u_1 + \left(\frac{v \cdot u_2}{\|u_2\|^2}\right) u_2 + ... + \left(\frac{v \cdot u_n}{\|u_n\|^2}\right) u_n$$

Orthogonal Projection

vector $n \in \mathbb{R}^n$ is orthogonal to subspace V if $\forall v \in V, n \cdot v = 0$.

Consider subspace $V \subseteq \mathbb{R}^n$. Every vector $w \in \mathbb{R}^n$ can be decomposed uniquely as $w = v + n$, where n is orthogonal to V and $v \in V$ is the orthogonal projection of w onto V.

Let subspace V have basis $S = \{v_1, ..., u_n\}$. Let $A = (v_1 ... v_n)$. Then the orthogonal projection of w onto V is

$$\text{proj} = A(A^T A)^{-1} A^T w = A\hat{w}$$

Where $\hat{w}$ is a least Square solution of $Ax = w$. Alternatively, one can use the method above of expressing v as a linear combination of the (orthogonal) basis vectors to show the projection of w onto v

Gram-Schmidt Process

Let $S = \{u_1, u_2, ..., u_n\}$ be a linearly independent set.

$$v_1 = u_1$$
$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right) v_1$$
$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right) v_2$$
$$\vdots$$
$$v_n = u_n - \left(\frac{u_n \cdot v_1}{\|v_1\|^2}\right) v_1 - \left(\frac{u_n \cdot v_2}{\|v_2\|^2}\right) v_2 - ... - \left(\frac{u_n \cdot v_{n-1}}{\|v_{n-1}\|^2}\right) v_{n-1}$$

Then $S' = \{v_1, v_2, ..., v_n\}$ is an orthogonal basis for $span(S)$.

Orthogonal Matrices

A square matrix A is orthogonal if $A^T = A^{-1} \leftrightarrow A^T A = I$.
Then the columns and rows of A form an orthonormal basis for $\mathbb{R}^n$.

QR Factorization

If matrix A has linearly independent columns, A can be uniquely written as $A = QR$, where Q is an orthogonal matrix and R is an invertible upper triangular matrix with +ve diagonals.

Algorithm for QR Factorization:
(1) Gram-Schmidt on A to obtain orthonormal set Q.
(2) $R = Q^T A$, ensuring R has +ve diagonals by multiplying the column of Q by −1 as needed.

Least Square Approximation

A vector $u \in \mathbb{R}^n$ is a least square solution to $Ax = b$ if for every vector $v \in \mathbb{R}^n$, $\|Au - b\| \leq \|Av - b\|$.
The least square solution is given by the solution set of $A^T Ax = A^T b$.

Eigenvalues and Eigenvectors

With square matrix A, λ /v are eigenvalue/eigenvector if $v \neq 0$ and $Av = \lambda v$.

The nontrivial solutions to $(\lambda I - A)x = 0$ are the eigenvectors of A with eigenvalue λ. If λ is a eigenvalue of A, it is a eigenvalue of A^T as they share the same characteristic polynomial.

The algebraic multiplicity of λ is the number of times λ appears as a root of the characteristic polynomial of A.

The geometric multiplicity of λ is the dimension of the eigenspace of λ, null($A - \lambda I$).

$$\text{geometric multiplicity} \leq \text{algebraic multiplicity}, \forall \lambda_i$$

Diagonalization

A is diagonalizable if there exists invertible P such that $P^{-1}AP$ is a diagonal matrix.

A is diagonalizable iff

(1) the characteristic polynomial of A splits into linear factors

(2) the algebraic multiplicity of each eigenvalue equals its geometric multiplicity

If $A_{n\times n}$ diagonalizable, then $\bigcup^k S_{\lambda_k} = \mathbb{R}^n$, where S_{λ_k} is the basis for each eigenspace E_{λ_k}. The eigenvectors for A form a basis for $\mathbb{R}^n$, ie $\forall u \in \mathbb{R}^n, v = c_1 v_1 + \ldots + c_n v_n$ where v is the eigenvectors that form the basis for each eigenspace.

Algorithm for diagonalization:

(1) Find Eigenvalues of A.

(2) For each Eigenvalue, find a basis for its Eigenspace.

(3) The bases of the Eigenvalues form the columns of P.

(4) D is a diagonal matrix of Eigenvalues, where each align to their corresponding bases in P.

Powers of Diagonalizable matrices

$$A^k = P \cdot D^k \cdot P^{-1} = P \begin{pmatrix} \lambda_1^k & 0 & \ldots & 0 \\ 0 & \lambda_2^k & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n^k \end{pmatrix} P^{-1}$$

Orthogonally Diagonalizable / Symmetric

A is orthogonally diagonalizable if $A = PDP^T$ for some orthogonal matrix P. Algorithm is the same as diagonalization, except the basis of each eigenspace is change to an orthonormal basis. Then $\boldsymbol{A} = \boldsymbol{A^T}$.

Stochastic Matrices

(1) A is square.

(2) Sum of the columns of A is 1.

(3) All entries of A are nonnegative.

(4) 1 is an eigenvalue of A, as $A^T(1; \ldots; 1) = (1; \ldots; 1)$

A Markov chain is a sequence of probability vectors $x_0, x_1, x_2, \ldots$ such that $x_{k+1} = Ax_k$ for some stochastic matrix A.

Singular Value Decomposition

Every $m \times n$ matrix can be written as $A = \boldsymbol{U\Sigma V}^T$, where

(1) U is a order m orthogonal matrix

(2) V is a order n orthogonal matrix

(3) Σ is of the form

$$\Sigma_{m,n} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$$

where D has the roots of the eigenvalues of $A^T A$.

SVD Algorithm (autoSVD) :

(1) find the eigenvalues of $A^T A$, arranging the nonzero ones in descending order with duplicates. Find Σ by using the roots of these eigenvalues.

(2) Find an orthogonal basis for each eigenspace, then set $V = (v_1 v_2 \ldots v_n)$ where v_i is the unit vector associated to the ith eigenvalue.

(3) Let $u_i = \frac{1}{\sigma_i} A v_i$ for i = 1,2,...,r. Extend $\{u_1, \ldots, u_r\}$ to an orthonormal basis $\{u_1, \ldots, u_r, \ldots, u_m\}$ of $\mathbb{R}^m$ to get U.

Linear Transformations

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if $\forall u, v \in \mathbb{R}^n$

(1) $T(u+v) = T(u) + T(v)$

(2) $T(cu) = cT(u), \forall c \in \mathbb{R}$

(3) $T(\mathbf{0}) = \mathbf{0}$

Range of T:

$$R(T) = \{T(u) \mid u \in \mathbb{R}^n\}$$

$$\text{rank}(T) = \dim(R(T)) = \dim(\text{colspace}(A)) = \text{rank}(A)$$

Kernel of T:

$$\ker(T) = \{u \in \mathbb{R}^n \mid T(u) = \mathbf{0}\}$$

$$\text{nullity}(T) = \dim(\ker(T)) = \text{nullity}(A)$$

Rank-Nullity Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

$$\text{rank}(T) + \text{nullity}(T) = n$$

One-to-One (Injective)

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *one-to-one* if $T(u) = T(v) \rightarrow u = v$.

$$\text{T is One-to-One} \leftrightarrow \ker(T) = \{\mathbf{0}\} \leftrightarrow \text{nullity}(T) = 0$$

Onto (Surjective)

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *onto* if $R(T) = \mathbb{R}^m$.

$$\text{T is Onto} \leftrightarrow R(T) = \mathbb{R}^m \leftrightarrow \text{rank}(T) = m$$

Invertible Matrix Theorem (extended)

if *invertible* A describes a linear transformation T,

(xi) rank(A) = n, ie. A has full rank

(xii) nullity(A) = 0

(xiii) 0 is not an eigenvalue of A

(xiv) T is one-to-one

(xv) T is onto

Finding Standard Matrix A

we need $\{T(u_1), \ldots, T(u_n)\}$ for a *basis* $\{u_1, \ldots, u_n\}$ of $\mathbb{R}^n$.

Then

$$A = (T(u_1) \ \ldots \ T(u_n))(u_1 \ldots u_n)^{-1}$$