MA1522 Reference Notes (midterms)

github.com/reidenong/cheatsheets, AY23/24 S1

Row Echelon Form

A matrix is in row echelon form if:

(1) All zero rows are at the bottom of the matrix.
(2) The leading entries are further to the right as we move down the rows.

It is in *Reduced Row Echelon form* if:

(1) The leading entries are 1.
(2) In each pivot column, all entries except the leading entry is zero.

Elementary Row Operations

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

Two Linear systems have the same solution set if their augmented matrices are row equivalent.

If matrix B is obtained from matrix A by

$$A \xrightarrow{r_1} \xrightarrow{r_2} \dots \xrightarrow{r_k} B$$

Then

$$B = E_k E_{k-1} \dots E_2 E_1 A$$

where E_i is the elementary matrix corresponding to r_i.

Systems of Linear Equations

The linear system of $A\boldsymbol{x} = \boldsymbol{b}$ is homogenous if $\boldsymbol{b} = 0$. If there is a nontrivial solution, it has infinitely many solutions.

A linear system is consistent if it has at least one solution. A homogenous equation $A\boldsymbol{x} = \mathbf{0}$ is always consistent, as it has at least the trivial solution.

Types of Matrices

Scalar Matrices

A scalar matrix is a diagonal matrix where all diagonal entries are equal.

Triangular Matrices

Upper Triangular A where $a_{\{ij\}} = 0$ for $i > j$.

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

Strictly Upper Triangular A where $a_{\{ij\}} = 0$ for $i \geq j$.

$$\begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lower Triangular A where $a_{\{ij\}} = 0$ for $i < j$.

$$\begin{pmatrix} * & 0 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & * \end{pmatrix}$$

Strictly Lower Triangular A where $a_{\{ij\}} = 0$ for $i \leq j$.

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & 0 \end{pmatrix}$$

Scalar Multiplication and Matrix Addition

Properties:

(1) Commutative: $A + B = B + A$
(2) Associative: $(A + B) + C = A + (B + C)$
(3) Additive identity: $A + 0 = A$
(4) Additive inverse: $A + (-A) = 0$
(5) Distributive: $c(A + B) = cA + cB$
(6) Scalar addition: $(c + d)A = cA + dA$
(7) Associative: $c(dA) = (cd)A$
(8) If $aA = \mathbf{0}$, then $a = 0$ or $A = \mathbf{0}$

Matrix Multiplication

For multiplication of a $m \times n$ matrix A and a $n \times p$ matrix B,

$$AB_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Properties:

(1) Associative: $A(BC) = (AB)C$
(2) Left distributive: $A(B + C) = AB + AC$
(3) Right distributive: $(A + B)C = AC + BC$
(4) Commutes with scalar multiplication:
$$c(AB) = (cA)B = A(cB)$$
(5) Not commutative: $AB \neq BA$ in general
(6) Multiplicative Identity: $I_n A = A I_m = A$
(7) Zero divisor: There exists nonzero matrices A and B such that $AB = \mathbf{0}$
(8) Zero matrix: $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$

Block Multiplication

$$AB = A(b_1 \ \ b_2 \ \ \dots \ \ b_n) = (Ab_1 \ \ Ab_2 \ \ \dots \ \ Ab_n)$$

$$AB = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ a_2 B \\ \dots \\ a_m B \end{pmatrix}$$

Transpose

The transpose of a $m \times n$ matrix A is a $n \times m$ matrix A^T where $A^T_{ij} = A_{ji}$.

Properties:

(1) $\left(A^T\right)^T = A$
(2) $(cA)^T = cA^T$
(3) $(A + B)^T = A^T + B^T$
(4) $(AB)^T = B^T A^T$

Inverse of a Matrix

A matrix A is invertible if there exists a unique matrix B such that $AB = BA = I$.

Properties:

(1) $(A^{-1})^{-1} = A$
(2) $(cA)^{-1} = c^{-1}A^{-1}, \forall c \in \mathbb{R}$
(3) $(A^T)^{-1} = (A^{-1})^T$
(4) $(AB)^{-1} = B^{-1}A^{-1}$ if A, B are both invertible
(5) Left Cancellation Law: $AB = AC \to B = C$
(6) Right Cancellation Law: $BA = CA \to B = C$

To find an inverse, consider

$$(A \mid I) \overset{RREF}{\longrightarrow} (I \mid A^{-1})$$

Invertible Matrix Theorem

Let A be a $n \times n$ matrix. The following statements are equivalent:

(1) A is invertible.
(2) A has a left inverse
(3) A has a right inverse
(4) RREF of A is I_n
(5) A can be expressed as a product of elementary matrices
(6) Homogenous system $Ax = \mathbf{0}$ has only the trivial solution
(7) for any b, the system $Ax = b$ has a unique solution
(8) The determinant of A is nonzero
(9) The columns/rows of A are linearly independent
(10) The columns/rows of A span $\mathbb{R}^n$

LU Decomposition

Suppose $A \overset{r_1}{\to}\overset{r_2}{\to} \dots \overset{r_k}{\to} U$, where each row operation is of the form $R_i + cR_j$ and U is a row echelon form of A. Then A can be decomposed into a *unit* lower triangular matrix and an upper triangular matrix.

$$A = LU = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \dots & 1 \end{pmatrix} \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1}\dots E_k^{-1}$$

To solve $LUx = Ax = b$, solve $Ly = b$, then $Ux = y$.

Determinant

Properties:

(1) $\det(A^T) = \det(A)$
(2) $\det(AB) = \det(A)\det(B)$ for A, B of same size
(3) $\det(A^{-1}) = \frac{1}{\det(A)}$
(4) $\det(cA) = c^n \det(A)$ for $n \times n$ matrix A
(5) $\det(diag(a_1, a_2, \dots, a_n)) = a_1 \cdot a_2 \cdot \dots \cdot a_n$
(6) Determinant and Row Elementary Operations:

$A \overset{R_i + cR_j}{\longrightarrow} B$	$\det(A) = \det(B)$	$\det(B) = \det(A)$
$A \overset{cR_i}{\longrightarrow} B$	$\det(A) = \frac{1}{c}\det(B)$	$\det(B) = c\det(A)$
$A \overset{R_i \leftrightarrow R_j}{\longrightarrow} B$	$\det(A) = -\det(B)$	$\det(B) = -\det(A)$

Finding Determinants:

1. for n = 2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$
2. for n = 3, $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,
   $\det(A) = aei + bfg + cdh - ceg - bdi - afh$
3. for $n \geq 3$, use Cofactor Expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{jk}A_{jk}$$

where A_{ij} is the (i,j) cofactor of A, given by

$$A_{ij} = (-1)^{i+j}\det(M_{ij})$$

where M_{ij} is the (i, j) matrix minor of A, the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

Adjoint

With a order n square matrix A, the adjoint of A is

$$adj(A) = (A_{ij})^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & -M_{21} & \dots & \pm M_{n1} \\ -M_{12} & M_{22} & \dots & \mp M_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \pm M_{1n} & \mp M_{2n} & \dots & \pm M_{nn} \end{pmatrix}$$

Adjoint Formula:

$$A \cdot adj(A) = \det(A)I$$

Cramer's Rule

Let A be a invertible $n \times n$ matrix.
For any $b \in \mathbb{R}^n$, the unique solution of $Ax = b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

where $A_i(b)$ is the matrix obtained by replacing the ith column of A with b.

Linear Span

$$span(u_1 \ u_2 \ \dots \ u_n) = \{c_1u_1 + c_2u_2 + \dots + c_nu_n \mid c_i \in \mathbb{R}, \forall i\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}, \forall i \right\}$$

$\mathbb{R}^n$ is the set of all vectors with n-coordinates.

Theorem:

(1) $\boldsymbol{v} \in span(u_1 \ u_2 \ \dots \ u_n) \leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n)x = \boldsymbol{v}$ is consistent $\leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n \mid \boldsymbol{v})$ is consistent.

(2) $span(u_1 \ u_2 \ \dots \ u_n) = \mathbb{R}^n \leftrightarrow$
The reduced row echelon form of A has no zero rows.

Properties:
Let $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$.
(1) Contains Origin: $\mathbf{0} \in span(S)$
(2) Closed under addition:
$\forall u, v \in span(S), u + v \in span(S)$
(3) Closed under scalar multiplication:
$\forall u \in span(S), \forall c \in \mathbb{R}, c \cdot u \in span(S)$
(4) Contains all linear combinations:
$\forall u_1, u_2, ..., u_n \in span(S)$,
$\forall c_1, c_2, ..., c_n \in \mathbb{R}$,
$c_1 u_1 + c_2 u_2 + ... + c_n u_n \in span(S)$

Span equality:
Let $S = \{u_1, u_2, ..., u_k\}$ and $T = \{v_1, v_2, ..., v_n\}$. Then,

$$span(T) \subseteq span(S) \leftrightarrow$$
$$\forall v \in T, v \in span(S) \leftrightarrow$$
$$(S \mid T) \text{ is consistent.}$$

For equality, we need to show that $span(S) \subseteq span(T)$ and $span(T) \subseteq span(S)$.

Subspaces

A subset $V \subseteq \mathbb{R}^n$ is a subspace if:
(1) Contains Origin: $\mathbf{0} \in V$
(2) Closed under linear combination:
$\forall u, v \in V, \forall c, d \in \mathbb{R}, cu + dv \in V$

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span,$V = span(S)$ for some finite set $S = \{u_1, u_2, ..., u_n\}$.

Solution Sets of linear systems

Solution sets of linear systems can be expressed implicitly or explicitly.

Implicit form:

$$\left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid Ax = \boldsymbol{b} \right\}$$

Explicit form:

$$\{\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k} \mid s_i \in \mathbb{R}, \forall i\}$$

where $\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k}$ are the general solutions of $Ax = \boldsymbol{b}$.

The solution set $V = \{u \mid Au = \boldsymbol{b}\}$ is a subspace if and only if **b** = 0, ie. the system is homogenous.

The solution set $W = \{w \mid Aw = \boldsymbol{b}\}$ of a linear system $Ax = \boldsymbol{b}$ is given by **u** + V, where
(1) $V = \{v \mid Av = \mathbf{0}\}$ is the solution set of the homogenous
system $Ax = \mathbf{0}$ and
(2) **u** is a particular solution of $Au = \boldsymbol{b}$.

Linear Independence

A set of vectors $S = \{u_1, u_2, ..., u_n\}$ is linearly independent if the only solution to $c_1 u_1 + c_2 u_2 + ... + c_n u_n = \mathbf{0}$ is $c_1 = c_2 = ... = c_n = 0$.
A set is linearly independent iff the RREF of S has no non-pivot columns.

Special Cases:
1. $\{\mathbf{0}\}$ is always linearly dependent.
2. {v_1, v_2} is linearly dependent iff v_1 is a scalar multiple of v_2.
3. {} = $\emptyset$ is linearly independent.
4. Any subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.
6. Any superset of a linearly dependent set is linearly dependent.
7. Any subset of a linearly independent set is linearly independent.
8. A set S containing n vectors in $\mathbb{R}^n$ is linearly independent iff it spans $\mathbb{R}^n$

Basis

Let $V \subseteq \mathbb{R}^n$ be a subspace. A set $B = \{u_1, u_2, ..., u_k\}$ is a basis of V if:
(1) B is linearly independent
(2) B spans V, ie. span(B) = V

A subset $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$ iff $|S| = n$ and $A = \begin{pmatrix} u_1 & u_2 & \cdots & u_n \end{pmatrix}$ is an invertible matrix.

Coordinates Relative to a basis

With basis $S = \{u_1, u_2, ..., u_n\}$, every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be expressed uniquely as a linear combination of the basis vectors.

$$\boldsymbol{v} = c_1 u_1 + c_2 u_2 + ... + c_n u_n \leftrightarrow [\boldsymbol{v}]_s = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Change of Basis / Transition Matrix

Suppose there exist bases $S = \{u_1, u_2, u_3\}$ and $T = \{v_1, v_2, v_3\}$. Then, the transition matrix from T to S is

$$\text{RREF}(S \mid T) = (I_k \mid P_{T \to S})$$

Then $[w]_S = P_{T \to S}[w]_T$, and

$$T = S \times P_{T \to S}$$

Dimension

The dimension of a subspace $V \subseteq \mathbb{R}^n$ is the number of vectors in any basis of V. The dimension of a solution space $V = \{u \mid Au = 0\}$ is the number of non-pivot columns in the RREF of A.