The problematic nature of Gödel's theorem

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Abstract

In this paper we will show that some fundamental deductions Gödel's famous theorem is based upon, are highly problematic: Gödel begins the main part of his theorem by correlating to each formal expression (e.g. the variables) of a formal system a natural number. He calls the totality of these numbers x and he uses this x as a variable in recursive formulas. Then he correlates to each number x the formalized numeral $\mathbf{x}$. We will show that $\mathbf{x}$ can be nothing else than a variable in the formal system. In the well-known mapping ("Gödelization") he considers $\mathbf{x}$ as a predicate and correlates to it an expression, that is variable. This is contrary to the correlation of a fixed number to each variable, as the mapping is stated to be biunique. Gödel wanted the mapping itself not to be directly included in the proof, but in fact it is included and this is veiled by an incorrect predicate. In a further step (Corollary V) he effectively replaces $\mathbf{x}$ (that has a variable image) by the variable a (that has the image *17*). This problem shakes the foundation of Gödel's theorem and therefore its validity should be discussed again without any dogmatism.

1 Introduction

Gödel's theorem of incompleteness [3] is regarded as a fundamental part of metamathematics, and is not taken into question any more. One reason for this might be its age: One might believe that any mistake should have been discovered during more than seventy years. But one must consider the following: Gödel developed especially for his theorem complete new methods of proof, which could not easily be validated at the time. His sensational results themselves could be understood easily, the proof however was difficult to follow. Therefore many mathematicians accepted the proof without thorough examination. Though from time to time the theorem was criticized in manuscripts submitted to mathematical journals, but apparently none of these has been published.

Kleene in [4] performed a meticulous reworking of the proof. His intention was obviously to explain Gödel's theorem perfectly and to support its understanding, without however questioning the correctness of the proof. This might explain why such an outstanding mathematician took over uncritically a highly problematic *predicate* (see **p1** and **p2** in this paper) and made an incorrect deduction of it. Most later authors based their arguments on Kleene and did not correct this mistake either, using Gödel's methods without thorough examination to deduct other mathematical sentences. This of course does not demonstrate the correctness of the theorem, because one can deduct true as well as wrong sentences from a wrong sentence.

But the whole time it was problematic to understand, that the theorem can be correct, because the well known abridged version "a formula, that means its own improvability" is a circular statement. Such statements are highly problematic, for a statement on "something" must have a larger content than the mere naming of this "something", but both are identical here. But how can a statement be larger than itself? This becomes possible in Gödel's theorem, as we will show, because in the Gödelization he uses two essentially different mappings of the variables, one of them having a larger content than the other.

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On the other hand abridged versions were for many mathematicians a reason to accept the theorem, as the contents of the proof seems to be quite plausible there. In the first part of our paper (Sections **2.1** and **2.2**) we will show the reader that this is problematic, as the abridged versions are by nature not usable to take a final decision upon the truth of a theorem (see also discussion in [1]).

In Section **3** we will support our arguments by going back to Gödel's original paper. The crucial problematic of the theorem will be raised in Section **4**: In the proof the term **x**, that represents the "numerals" (formalized numbers), is a variable, the Gödelizing of which is highly problematic.

In the last section we will finally extend these realizations to a more general level and invite the readers to discuss its implications.

2 The problematic nature

2.1 The problematic nature of the first abridged version of Gödel's theorem

As is generally known the goal of Gödel's theorem is to produce an explicit number-theoretic formula, which is undecidable i.e. neither provable nor refutable. The existence of such a formula had not yet been proven and according to Hilbert's program was not possible at all.

The best known abridged version was formulated by Gödel himself: *"Wir haben also einen Satz vor uns, der seine eigene Unbeweisbarkeit behauptet."* ([3], p.175.) *Kleene* translates/formulates it as follows ([4], p. 205): "A means that A is unprovable." The word "mean" implies "is equivalent" i.e. the formula is then and only then unprovable, if it is true. If "true" would be equivalent to "provable" then the formula would be provable then an only then, if it is not provable. That would be a contradiction against formal logic. To avoid this contradiction one must assume that there exists a true and unprovable formula. This anticipates the result of the proof (because a true formula cannot be refused, if arithmetic is consistent).

2.2 The problematic nature of the second abridged version of the theorem

A second version that has been published in the journal *"Spektrum der Wissenschaft"* ([4], p.51) seems at first sight again plausible, but as we will show is wrong: The set of all proofs is enumerable and also the set of all formulas. It is possible to find a formula $Dem(a;b)$ that is provable then and only then, if b is the number of a proof and a is the number of the formula proved by this proof. Now one can build the formula:

$$f(a) \equiv \forall b\, (\neg\, Dem(a;b)), \qquad (1a)$$

which means, that the formula with number a is not provable. The number of formula (1a) now is named g, and then it is stated, that the formula

$$f(g) \equiv \forall b\, (\neg\, Dem(g;b)) \qquad (1b)$$

built by substituting g for a in formula (1a) means her own improvability. That is incorrect. In fact follows to the provability of formula (1b) that formula (1a) and not formula (1b) itself is unprovable.[3]

3 Main features of Gödel's theorem

Gödel begins his theorem with a formal mathematical system (NL) that comprises formal number theory and logic. (We call it upper level). The object language of this system has for its base seven[4] formal sym-

[3] (1a) is obviously refutable, because it contains a free variable and is refuted by any proof. Therefore (1b) is provable.

[4] It is useful (we think necessary) to add an eighth zero entity for "equal to" as Kleene did (We choose *15*). He also added zeroes for "plus" and "times" and "successor variable" and uses another sort of Gödelization.

bols and moreover an infinite number of symbols for variables, as presents the following table (Gödel calls them *"Grundzeichen"*, Kleene calls them "zero entities" or shortly "zeroes").

Zero entities in upper level NL	Name	Correlated numbers in lower level cn
0 (in bold)	zero	*1*
′	successor	*3*
¬	not	*5*
∨	or	*7*
∀	for all	*9*
(	opening bracket	*11*
)	closing bracket	*13*
=	equal to	*15*
a	first variable	*17*
b	second variable	*19*
x	third variable	*23*
y	fourth variable	*29*
etc.	etc.	etc.

All formal objects of the System NL are finite sequences of zero entities (*"Grundzeichenreihen"* by Gödel). E.g. the natural numbers greater than ***0*** are represented by ´***0***, ´´***0***, ´´´***0*** … .[5] i.e. by two zero entities. We write *N1*, *N2*, *N3* … *N*(*n*) for that. They are called "numerals" (*"Zahlzeichen"* by Gödel).

Formal expressions that are identical are connected by "≡". At cn (see below) this is correlated to "=".

Now distinct odd numbers are correlated to the zero entities (*"Zahlengrundzeichen"* by Gödel—see the third column of the table). To each formal object is in this way correlated a sequence of odd numbers[6]:

$$X \equiv n_1\, n_2\, n_3 \ldots\ldots n_k$$

By the well-known Gödelization now to each formal object is correlated by a bi-unique mapping a natural number $\tilde{x}$ (Gödel number)[7]:

$$X \rightarrow \tilde{x}\text{ , where}^8 \qquad \text{(2a)}$$

$$\tilde{x} = G[X] = G[n_1\, n_2\, n_3 \ldots n_k] = 2^{n_1} \bullet 3^{n_2} \bullet 5^{n_3} \bullet \ldots \bullet p_k^{n_k}\,, \qquad \text{(2b)}$$

where p_k is the k-th of the prime numbers in order of magnitude. For example is:

$$G[N2] = G[´´\mathbf{0}\,] = 2^3 \cdot 3^3 \cdot 5^1 = 1080$$

By this mapping and the theory of primitive recursive functions it becomes possible to develop for metamathematical predicates PR(X) in NL equivalent formulas pr($\tilde{x}$) in classical number-theory cn (we

[5] Gödel places the successor symbols before the zero symbol.
[6] *X is a variable Expression* of course.
[7] The tilde above *x* is placed to distinguish it from the variable *x* in NL. Gödel wrote *x* only. We later on will realize that $\tilde{x}$ is in fact a variable in cn. The exact formulation for "the number $\tilde{x}$ " therefore is : "a value of the variable $\tilde{x}$."
[8] *G*[*X*] is only a short expression für the Gödel number of *X*.

call it “lower level”). For example “X is a *numeral*”, “X is a *variable*”, “X is a *formula*”, “X is an *axiom*” etc. and finally (for two formal expressions X and Y) “Y is a proof for X” ([4], p. 253 f). For example the equivalent for the predicate $V(X) \equiv$ “X is a variable” is[9]:

$$v(\tilde{x}): \exists z\,(\tilde{x} = 2^z \wedge prim\, z \wedge z > 13). \qquad (3)$$

It means, that the number correlations z to the variables are the primes greater 13 and their Gödel numbers are 2^z.

The decisive step of deduction in Gödel’s proof (that was ignored in the abridged version **2.2)** is, that in each formula $X(a)$ with the single free variable a the “Gödel numeral” of $X(a)$ is substituted.

The Gödel numeral is defined as the formalized Gödel number i.e. the zero (***0***) with leading successor signs (´) their number being $\tilde{x}$.

We write according to Kleene **x** for the Gödel numeral[10] of the Gödel number $\tilde{x}$ (and **n** for the numeral of the number n).

The Gödel number of **x** is:

$$G[\mathbf{x}] = 2^3 \cdot 3^3 \cdot 5^3 \cdot \ldots \mathrm{p}^3_{\tilde{x}} \cdot \mathrm{p}^1_{\tilde{x}+1}. \qquad (4)$$

The result of the substitution one can write $R \equiv X_{\mathbf{x}}(\mathbf{x})$. We call it the “Richard formula” of the formula $X(a)$.

Let be e.g. $X(a) \equiv a = a$. Then is $G[X(a)] = \tilde{x} = 2^{17} \cdot 3^{15} \cdot 5^{17}$, and $\mathbf{x} = N(2^{17} \cdot 3^{15} \cdot 5^{17})$ (for the meaning of “N” see above) and the Richard formula of $X(a)$ is:

$$\mathbf{x} = \mathbf{x}, \text{ i.e.,}$$
$$N(2^{17} \cdot 3^{15} \cdot 5^{17}) = N(2^{17} \cdot 3^{15} \cdot 5^{17}).$$

We can shorten the substitution- predicate as follows (cf. ([4], p. 253 Dn5):

$$R \equiv X_{\mathbf{x}}(\mathbf{x}) \equiv SB(X,a,\mathbf{x}); \qquad (5a)$$

Its equivalent in cn is[11]:

$$\tilde{r} = sb\,(\tilde{x}, 17, \mathrm{G}\,[\mathbf{x}]), \qquad (5b)$$

that means in essence, to get $\tilde{r}$ one must replace on the right side of (2b) the potencies with exponent 17 by G [**x**] of (4) and then increase[12] the primes of G [**x**] and the following primes in this way that all primes are positioned in order of magnitude again (see [3], p. 184).

In order to carry out the substitution (5a) Gödel establishes a predicate that relates $\tilde{x}$ to **x,** the problematic connected with that will be shown in section **4**.

Ahead of this we will present the remaining part of the theorem. Gödel now creates in NL[13] the predicate “Y is a proof for $X_{\mathbf{x}}(\mathbf{x})$”[14]. The equivalent of this in cn can by recursions be formulated as

$$dem'(\tilde{x}\,; \tilde{y}), \qquad (6a)$$

where $\tilde{x}$ is the Gödel number of $X(a)$ and $\tilde{y}$ is the Gödel number of Y. Formula (6a) now is “formally expressed” i.e. an equivalent formula is created in NL[15] :

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[9] according to [3], p. 182, no. 11 for $n = 1$.

[10] I will show later on **x** to be a variable in NL. Today $\ulcorner x \urcorner$ is written for **x**.

[11] See in [3], p. 188 formula (8.1) the expression on the right side in square brackets, where stands y instead of x (and 19 instead of 17) . Gödel writes $Z(x)$ instead of $G[\mathbf{x}]$ and sets the number 17 above instead of ahead of it. Let it be noted that $X_{\mathbf{x}}(\mathbf{x})$ is not a formal Expression in NL, but its equivalent in cn is a term there.

[12] Only if 17 is the first exponent $G[\mathbf{x}]$ remains unchanged.

[13] For *Gödel´*s mode of formulation see section **4.1**.

[14] The exact predicate is “X is a proof of $Y_{\mathbf{y}}$“ – X and Y are exchanged.

$$DEM'(\mathbf{x}; \mathbf{y}), \qquad (6b)$$

where **x** it the numeral of $\tilde{x}$ and **y** that of $\tilde{y}$.

This is combined with the second problematic step, which effectively comes to replace **x** by the variable *a* and **y** by the variable *b*. The result of this step is (cf. [4], p. 206, LEMMA 21):

$$DEM'(a; b). \qquad (6c)$$

Formula (6c) is (if we accept the deduction) provable then and only then, if there is substituted for *a* the Gödel numeral **x** of a formula $X(a)$ and for *b* is substituted the Gödel numeral **y** of a proof of the Richard formula $X_{\mathbf{x}}(\mathbf{x})$ of $X(a)$.

The next formula created is:

$$f(a) \equiv \forall b\, (\neg\, DEM'(a;b)). \qquad (1a')$$

In this formula finally its Gödel numeral ***g*** is substituted for the variable *a*:

$$f(\boldsymbol{g}) \equiv \forall b\, (\neg\, DEM'(\mathbf{g};b)). \qquad (1b')$$

(1b') means, that the Richard formula of the formula with the Gödel numeral **g** is unprovable. The formula with the Gödel numeral **g** is formula (1a'); its Richard formula is formula (1b') itself. It means, that it is unprovable. Now it is easy to demonstrate, that formula (1b') is formally undecidable in a consistent (ω-consistent respectively) arithmetic system.

4 The fundamental problematic nature of the theorem

The key problematic of the theorem has to do with the meaning of **x**. The question is, whether in the proof **x** can be anything else than a variable in NL. We deny this and will justify our conclusion soon. At first we would like to explain the consequences for Gödel's proof: The Gödelization correlates to **x** the variable expression $G[\mathbf{x}]$ of formula (4). If **x** is a variable it must be correlated a definite natural number to it. There results the contradiction $G[\mathbf{x}] = \text{const}$ (see **4.3.3**).

4.1 The problematic relation and its correction

We will now show how Gödel establishes the predicate concerning the relation between $\tilde{x}$ and **x,** earlier referred to be problematic. Since Gödel's mode of formulation is very unusual, we will equally refer to Kleene´s reworking afterwards.

As is generally known, statements on itself can produce antinomies by their *"Zirkelhaftigkeit"* (basing on a circular argument). The abridged version above (Section **2.1**) is such a statement. Gödel comments on this problem with the words:

> "Ein solcher Satz hat entgegen dem Anschein nichts Zirkelhaftes, denn er behauptet zunächst die Unbeweisbarkeit einer ganz bestimmten Formel und erst nachträglich (gewissermaßen zufällig) stellt sich heraus, daß diese Formel gerade die ist, in der er selbst ausgedrückt wurde." ([3], p. 175, footnote 15).
>
> Such a theorem against appearance has nothing to do with a circular argument, because it first of all states the improvability of a definite formula and later only (so to speak coincidentally) it emerges, that this formula is just the one, that expresses the theorem itself.

In order to avoid antinomy on any rate, Gödel intends, to carry out all calculations on the lower level cn, but to name there the terms and formulas after their meaning in the upper level NL *written in italics* ([3], p. 179, line 20f).

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[15] This step is left out in the referee of *Nagel* and *Newman* [6], p. 82 – 89, therefore it is not correct.

For examples: The sentence in the metalanguage of NL "*a* is a variable" Gödel formulates[16]: *"17* is a *variable* (italic)". The sentence "*´0* is a numeral" Gödel would formulate: "$2^3 \cdot 3^1$ is a *numeral* (italic)" that means: "$2^3 \cdot 3^1$ is correlated (by Gödelization) to a numeral."

If that procedure is carried out consequently, any antinomy can be avoided in fact. But Gödel is inconsequent, for he defines the following predicate ([3] p.183, nr. 16 and 17): "G[**n**] is the *numeral* (italics) for the natural number (not italics!) *n*". One can write for this:

$$nu(\mathrm{G}[\mathbf{n}]; n). \tag{7a}$$

The inconsequence is, that "natural number" is not written in italics. It is an expression of the lower level cn and is named at this level. Gödel here contradicts his declared intention, to name only after the meaning in the upper level NL. The consequent formulation of the meaning of (7a) becomes obvious, if one pursue, how Gödel uses it further: He replaces *n* by $\tilde{x}$ and G[**n**] by G[**x**] (i.e. *Z*(*n*) by *Z*(*x*) in his formulation) without mentioning that particularly[17]. We formulate the result analogous to the predicate and (7a):

"G[**x**] is the *numeral* for the natural number $\tilde{x}$", and **(p1)**

$$nu(\mathrm{G}[\mathbf{x}]; \tilde{x}). \tag{7b}$$

(7b) is a relation at the lower level and corresponds with formula (4) in this paper. A consequent formulation of its meaning (according to Gödel´s declared intention) one finds by considering, that $\tilde{x}$ is the Gödel number, correlated to the formal expressions *X*(*a*) of NL[18]. The correct formulation in Gödel´s diction therefore is: "G[**x**] is the *numeral* (italics) correlated to the *formal Expression* (italics!) *X*(*a*)." It means at the upper level "**x** is the numeral (not italics) correlated to the formal expression (not italics) *X*(*a*)." The consequences of this we will clarify after having referred its reworking by Kleene.

4.2 Reworking by Kleene

Gödel simultaneously works in both levels, as we saw. Kleene does not adopt this. He formulates the metatheory at the upper level NL and wants to separate strictly the number-theoretic equivalents at the lower level cn. But he is equally inconsequent in connection with the problematic predicate (**p1**). He formulates at the upper level the predicate:

"**x** is the numeral for the natural number $\tilde{x}$"[19] (abbreviation *Nu*(**x**; $\tilde{x}$). **(p2)**

That is inadmissible, for $\tilde{x}$ does not exist at this level. (**p2**) is a **mixed predicate.** The equivalent in cn Kleene formulates effectively[20]: $nu(\mathrm{G}[\mathbf{x}]; \tilde{x})$, i.e., he correlates $\tilde{x}$ to $\tilde{x}$, which is wrong.

The correction is only possible at the upper level. As we already have seen, $\tilde{x}$ must be replaced there by the formal object *X*(*a*):

$$NU(\mathbf{x}; X(a)). \tag{8}$$

Therefore it must be possible to relate the formal objects directly at the upper level to their Gödel numeral **x** without the detour via $\tilde{x}$. We can write:

$$X(a) \to \mathbf{x}. \tag{9}$$

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[16] He writes *17* instead of 2^{17}.

[17] See [3], p. 188 formula (8.1). It is notable, that *Gödel* names *n* a natural number, where *x* is an argument of the recursive relation *Q* (*x*;*y*), i.e. it is a variable (see section **4. 3. 3**).

[18] The restriction to *X*(*a*) instead of *X* is insignificant of course.

[19][4] p.254, DN11 and [7] p. 38 V_{11}

[20] according to [4], p. 258, example 2 and explicit [7] p. 42, line 14

This way the mapping of Gödelization is directly included in the proof, although Gödel wanted to avoid this.

4.3 Consequence for the meaning of x

Predicate (8) requires **x** to be defined immediately in NL. Therefore we have to search the meaning of **x** in NL.

4.3.1 Metamathematical symbol

Can one interpret **x** as a metamathematical symbol for a numeral, which can be replaced by a numeral? (cf. [4], p. 82f.)

Such a metamathematical symbol however is not a formal expression in NL and therefore has not a number-theoretic image in cn, while **x** has the image G[**x**] there.

4.3.2 Infinite set

Another possible interpretation would be to interpret **x** (and **y** also) as a symbol for an infinite set of numerals in NL. According to that *DEM´* (**x**; **y**) would be an infinite set of formulas in NL. But Gödel´s proof requires drawing a conclusion from this infinite set to a formal expression and this is not expressible in a formal deduction that must be finite.

The problem will be clarified by giving the gist of the central corollary V at Gödel (([3] p.186) applied to the step of deduction from formula (6a) to formula (6b) in this paper respectively (6c) (we refer to it in a diction according to the one used in this paper hitherto):

For the recursive relation *dem´*($\tilde{x}$; $\tilde{y}$) there exists a relation (*"Relatiosszeichen"* by Gödel) *DEM´*(*a*; *b*) with the free variables *a* and *b* and for all pairs of numbers $\tilde{x}$; $\tilde{y}$ the following is valid: From *dem´*($\tilde{x}$; $\tilde{y}$) follows that *DEM´* (**x**; **y**) is provable.

The proof of this corollary (not given by Gödel, but by Kleene in ([4] p. 238 to p.245) represents an infinite set of proofs, each having the result "*DEM´* (**x**; **y**) is provable", where **x** and **y** are elements of infinite sets of numerals corresponding to the numbers $\tilde{x}$ and $\tilde{y}$. But these sets are not expressible in the formal system NL and therefore a conclusion to a relation *DEM´*(*a*; *b*) is not possible there.

This objection is also valid for the mathematical sketch of the proof, that Gödel sets ahead of it ([3], p. 174f, where the "class *K*" is an infinite set of natural numbers), but for which he however did not demand exactness.

4.3.3 Variable

In Gödel's proof $\tilde{x}$ and $\tilde{y}$ are variables, for he uses them as arguments of recursive relations basing on recursive functions, whose arguments are variables of course. Using (7a) in the formulation of (7b), he defines a recursive relation *Q*($\tilde{x}$; $\tilde{y}$), where $\tilde{x}$ and $\tilde{y}$ are variables.[21]

The correct interpretation of (2b) therefore is: By the Gödelization the variable $\tilde{x}$ is correlated to the variable formal expression *X* in this sense, that to each formal expression is correlated a value of the variable. The equation of this correlation can be expressed by (2b).

According to that **x** is correlated to *X* in NL (8 and 9) and therefore **x** can be nothing else than a variable in NL e.g. *x*.[22] This result moreover can be realized by the following deduction:

The mixed predicate *Nu*(**x**; $\tilde{x}$) (**p2**) according to Kleene ([4], p. 254) is defined as follows[23]:

$$Nu(\boldsymbol{0};\, 0) \wedge ((Nu(\mathbf{x};\, \tilde{x}\,) \Rightarrow Nu(\acute{}\mathbf{x};\, \tilde{x} + 1)).$$

By induction proof results:

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[21] cf. [4] p. 254 – 258

[22] We can write X_x, where *x* is a parameter.

[23] I use the formal symbols, though the predicates are mixed ones.

$$\forall n(\tilde{x} = n \Rightarrow \mathbf{x} = \mathbf{n}).$$

The correct interpretation of this predicate is: "**x** is a corresponding variable in NL to the variable $\tilde{x}$ in cn". Therefore **x** is identical with a variable in NL (e.g. $\mathbf{x} \equiv x$).

But now in cn is correlated to this the contradiction G[**x**] = const (e.g. G[**x**] = 2^{17}) for all **x**. Therefore the nature of **x** is highly problematic.

If we could accept **x** and **y** to be variables, the last statement of corollary V could be formulated very simply:

… for all pairs of number values of (the pair of variables) $\tilde{x};\tilde{y}$ the following is valid: From $dem'(\tilde{x};\tilde{y})$ follows that *DEM´* (**x**; **y**) for the corresponding pair of numeral values (of the pair of variables **x**; **y**) is provable. The transition to *DEM´*(*a*; *b*) then only changes the names of variables.

In Gödel's theorem the problematic nature of **x** and **y** is transmitted to all other variables e.g. to *a* and *b*. Only if one ignores this, one can correlate to formula (1a') a fixed Gödel numeral **g**. Only then one can substitute this numeral in (1a') for the variable *a* and create the formula (1b'), that states something about itself.

5 The depth of the problem

The key problematic represented in this article concerns the problem of mapping the formal expressions of NL into the natural numbers of cn. For such a mapping the basic elements need to be independent of each other. This is not the case here. The signs correlated to "zero", "successor" and "variable" are not independent. We can define the recursive predicate "$S(x)$ is the x-th successor of $\boldsymbol{0}$" as follows. $S(\boldsymbol{0}) = \boldsymbol{0}$ and $S(´x) = ´S(x)$. Then is valid: $S(x) = x \Rightarrow S(´x) = ´S(x) = ´x$. Therefore it results (by induction proof):

$$\forall x(S(x) = x), \text{ i.e. } S(x) \equiv x$$

Therefore two different images (G[**x**] and a constant) of a variable are possible, although the mapping is stated to be unique and therefore circular deductions and finally contradictions result.

The problem is connected to the notion of variables in general. Are variables entities for themselves or vacant places only for concrete numbers (respective numerals)? Both opinions are possible, but both are one-sided. Their combination only makes the notion of variables comprehensible. At the Gödelization however it is impossible to combine both opinions. An entity for itself should have a definite Gödel number, whilst to a vacant place the Gödelization should correlate a vacant place.[24] Therefore a unique mapping of the full notion of variables is impossible. A discussion of this problem is necessary and welcome.

The existence of undecidable formulas has tried to be proven with the aid of Turing machines. If such proofs use a predicate like (**p2**) they are equally problematic.[25]

However there are proofs of this subject, which are based on the theory of machine-numbers. These are not discussed here.[26]

6 Summary

As we have shown, Gödel's theorem has key problematic that asks for further discussions. As we pointed out these problems are not obvious, as the formulas of the lower level as well as the predicates of the higher level are by themselves correct. If Gödelization is accepted as a fact one can then indeed deduct

[24] The problem has relations even to the discussion about formalistic and realistic conception of ideas (cf. [1], p. 16).
[25] *Kleen* formulates such a proof in [4], p. 376-386.
[26] cf [2], Kap. 10.

without any further contradiction formulas, that state something about themselves, e.g. that a formula states itself to be provable, decidable, refutable, not existent"(!) etc. But the mapping itself and only this is contradictory: The expression **x** is a variable and must have a fixed natural number as an image and not the variable expression G[**x**]. As a consequence the deduction of Gödels theorem becomes impossible, as the variable a has then a variable image and the image of $X(a)$ for a fixed X is variable too. Only by using two essentially different images of the variables it becomes possible to make statements about themselves. The incorrect deductions (**p1** in this paper and corollary V in [3]) veil the notion of **x** as a variable.

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