Meaning-Imposers versus Meaning-Derivers

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The Geometric Algebra community has evolved into a large segment of meaning-imposers and a tiny segment of meaning-derivers. Meaning-imposers begin with an abstract mathematical formalism, viewed as a gift from heaven regardless of how it had actually been achieved historically; and then *impose* on it whatever geometric meaning seems convenient or appropriate. Such meaning, as many readers know, is called an *interpretation* informally, or a *model* formally. Meaning-derivers, as few readers may know, begin with geometric meaning, viewed as the primitive starting point, and then *derive* everything else from that, *including the mathematical formalism itself*.

Meaning imposition has been undeniably fruitful, but it generates subtle inconsistencies and confusions that have stalled us in the purely *free* Geometric Algebra. Hence we cannot articulate *bound* things, like points for example, except by imposing clever but clumsy artifices on the free language from *outside* it. Nearly two centuries ago Hermann Grassmann showed how to articulate bound things from *inside* the language,[1] but he suffered from the distinct inconsistencies and confusions of a creator who has not had time to polish his creation. By starting over and carefully re-deriving his *full* language from seminal geometric concepts, we can dispel the fog and gain a more expressive language.

Here are the seminal ideas: (1) the concept of geometric points, (2) what it means to summarize points, and (3) what it means to extend something from a point, to wit: (1*) Points have fixed distances among themselves. (2*) Summarizing points is like summarizing anything: order doesn't matter; grouping doesn't matter; a point summarized with Nothing is just the point itself. Finally, Grassmann's gem, somewhat polished: (3*) extending something from a point sweeps it from there directly back to its original position, filling in as it returns, which increments dimension. Hence, to begin at the beginning, extending a *point* from another point produces a directed line segment that has a dimension one higher than that of a point.

(So, clearly, Grassmann was the founding meaning-deriver; but he fell under the seductive spell of mathematical abstraction, and became a resolute meaning-*remover*. Since geometric meaning had already generated his symbolism, Grassmann never could have become a bona fide meaning-imposer. Such persons arrived later, after Grassmann's resolutely abstract symbolism had been cleaned up and unified by William Kingdon Clifford.[2])

At this juncture, meaning-imposers will ask what the three primitive concepts are, if not preliminary meaning imposition. Point well taken—we meaning-derivers are closet meaning-imposers; but we are *timid* ones who impose meaning only at the very beginning, before any symbolism has been established, and not just when it seems convenient or appropriate. If meaning ever comes to seem *overwhelmingly* convenient or appropriate, we go right back to the closet and start all over again, convinced that we, in our naivety, have neglected something important *that will change the symbolism*.

Well then, a meaning-imposer may say, I, the meaning-deriver, will shortly need to return to the closet if I expect to have free vectors in my language. There is just no way that *roving* directed line segments can ever be *derived* from *fixed* points!—the roving idea has to be *imposed*. Again, point well taken—it does seem implausible that securely *bound* things, unassisted, could ever produce something *free*. But let us just see if it might be true.

The primitive idea of point summary immediately generates some symbolism; and it looks exactly like the rules for elementary-school *addition*, applied unfamiliarly to points, with *zero* taking the role of Nothing. But it also looks like the rules for *logical or*, again applied unfamiliarly to points, with *false* taking the role of Nothing. So, which will it be?

One can't be sure immediately because, as mentioned, point summary is still unfamiliar, even these several centuries after Grassmann (and Mobius) introduced it. Indeed, it is commonly understood to be, for example, "non-geometric"; and it "makes no intrinsic sense".[3] Here is where an eager young meaning-deriver will probably have to return to the closet and start all over again because she, in her naivety, will decide that summarizing two points, $\boldsymbol{a}$ and $\boldsymbol{b}$ say, produces the ***m**idpoint*, $\boldsymbol{m}$, between them. What could be more natural?—this immediately establishes summary indifference to order, the *commutative law*. But it also implies that summarizing point $\boldsymbol{a}$, say, with itself simply reproduces point $\boldsymbol{a}$. This is clearly a kind of geometric *logic*, devoid of numbers; not arithmetic, wallowing in numbers.

With that satisfying thought, the young meaning-deriver begins to investigate how this relates to the primitive idea of fixed points. Whoa! *That* idea applied to the midpoint idea would invalidate summary indifference to grouping,[4] the *associative law*. Back to the closet: midpoint $\boldsymbol{m}$ summarizes *two* points, so it should have *twice* the significance of either *one* of them alone. So this is not *geometric logic*, it is *geometric addition*, and the symbolism now becomes $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{m}$ and $\boldsymbol{a} + \boldsymbol{a} = 2\boldsymbol{a}$, rather than naked $\boldsymbol{a}$, as before. This new symbolism now provides enough information to validate the associative law, and all the other laws of summary. (Terminology: $\boldsymbol{m}$ is a *location*; 2 is its *weight*, which is a kind of *magnitude* like length, area, and volume. The weight of a sum point is the sum of its summand weights. A naked location like $\boldsymbol{a}$ is called a *simple point*, or a *unit point* since $\boldsymbol{a} = 1\boldsymbol{a}$.)

To summarize, *points must be weighted* for the first two primitive concepts to be validated together—points must wallow in numbers. This is your very first *derived meaning*; and it may appear insignificant until you consider that it seems manifestly contrary to hoary Euclidean convention, which denies points "magnitude".[5] In retrospect, it is clear that this ancient convention must be misleading, at best: distances among points is *all about* numbers simply because distance is an ordered continuum; and summary of points should somehow cause points to inherit that continuum.

Your second derived meaning may not seem so insignificant: *a sum point always lies on the line thru its two summands*. This arises directly from previous equation $\boldsymbol{a} + \boldsymbol{b} = 2\boldsymbol{m}$, where $\boldsymbol{m}$ is still the ***m**idpoint* (to keep the commutative law valid); hence it necessarily *lies on the line thru* $\boldsymbol{a}$ *and* $\boldsymbol{b}$, tho it now has a weight of 2. You can use this equation to approximate any sum of two weighted points as closely as you wish by adjusting weights to express *midpoints* of *midpoints*, iterated; and such an approximation becomes exact in the limit. *Midpoints* of *midpoints* necessarily generate a result lying on the line thru the two summands.

What other derived concepts do the first two primitive concepts mandate? When you play with these concepts as Grassmann did,[6] you soon discover that they require a sum of two weighted points, $a\boldsymbol{a} + b\boldsymbol{b}$ say (having scalar weights a and b), to obey this simple rule:

weight–distance($\boldsymbol{a}$) = weight–distance($\boldsymbol{b}$)

This means that the weight of $\boldsymbol{a}$ times its distance to the sum point equals the weight of $\boldsymbol{b}$ times *its* distance to the sum point. This is your third derived meaning, and it is quite significant because it tells *exactly* how summary of points causes them to inherit the distance continuum. The rule is just as valid for negative weights as for positive ones provided you carefully distinguish signs as follows: give a summand-to-sum distance that crosses the other summand the opposite sign to a distance that doesn't.

The weight–distance rule induces the following intuitive concept of point summary: a sum point is physically a *balance point* so it always lies nearer the heavier summand (the one with greatest absolute value). When this idea is applied to a sum of points having opposite signs, the opposite-sign distinction kicks in, requiring the sum point to lie on the line *from* the lighter point (in absolute value) *thru and beyond* the heavier point. Which prepares you for...

Some magic: what is $\boldsymbol{a} - \boldsymbol{b}$?

Whatever the ***l***ocation of this sum, call it $\boldsymbol{l}$, its weight, $1 - 1$, is 0; so this sum has this form: $0\boldsymbol{l}$. Since anything multiplied by zero is just zero, the sum $\boldsymbol{a} - \boldsymbol{b}$ is clearly zero. Right?

There is a quick way to test this: give the entire sum a non-presupposing name, "v" say, meaning that v = $\boldsymbol{a} - \boldsymbol{b}$ ("$\boldsymbol{l}$" was presupposing since it was a ***location***). Now see if indifferent v acts like zero: Add v to the second nearest thing in sight, namely point $\boldsymbol{b}$. When you apply the primitive rules of point summary, you get point $\boldsymbol{a}$. Hmmm... Okay then, *subtract* v from the nearest thing in sight, point $\boldsymbol{a}$. You get point $\boldsymbol{b}$.

That is not how zero acts!—zero doesn't *change* things when it is added or subtracted with them. This v thing is *changing* points under addition and subtraction; in fact it is *moving them around*. Where did I go wrong reasoning that $\boldsymbol{a} - \boldsymbol{b}$ must be zero? I went wrong in assuming there was some location, "$\boldsymbol{l}$" I called it, for its zero weight to multiply. There's not. The point sum $\boldsymbol{a} - \boldsymbol{b}$ really *has no location*. It really *has no weight*. And yet it *is not zero*. It is truly bizarre to the modern mind, which has come to shun point summary, altho many minds born in the 1800s were comfortable with it. Let's reacquaint ourselves with their old friend:

You can sneak up on $\boldsymbol{a} - \boldsymbol{b}$ by approximating it with non-zero weights that approach zero. For example, start by giving $\boldsymbol{a}$ a weight of 1/2, and then successively halve $\boldsymbol{a}$'s actual-minus-approximate weight like this: 1/2, 3/4, 7/8, 15/16, etc. This will successively halve the approximate *sum* weight. At each weight *halving*, the weight–distance rule will scoot the approximate sum location *twice* as far away along the line thru $\boldsymbol{a}$ and $\boldsymbol{b}$.

This removes some of the mystery: as the sum point *weight* goes to zero, its *location* goes to infinity, in *lock-step*; so the diminishing weight and the receding location effectively cancel each other. Which is why $\boldsymbol{a} - \boldsymbol{b}$ is not zero: it is actually a peculiar kind of *zero times infinity*. The satisfying conclusion is that $\boldsymbol{a} - \boldsymbol{b}$ is a point at infinity. Right?

This is certainly a modern concept, quite familiar from Projective Geometry, which is redolent with classically imposed points at infinity. But I just said that the result of $\boldsymbol{a} - \boldsymbol{b}$ really *has no* *location*. How can it not have a location if it *resides at infinity*? Have I made another blunder? Or is this just an innocuous problem with our language?

There is an easy way to test this: start $\boldsymbol{a}$ with a weight of 1 1/2 (rather than 1/2) and then sneak up on $\boldsymbol{a} - \boldsymbol{b}$, as before. As before, halving the approximate sum weight scoots the approximate sum location twice as far away. *But it does so in the opposite direction*. This is again mandated by the weight–distance rule, which carefully notices, during the approximation, which summand is lighter, and which is heavier. In consequence, since both approximations approach $\boldsymbol{a} - \boldsymbol{b}$ in the limit, it appears that *this sum is infinitely distant from itself*!

However, if this sum *really* has no location, then the problem disappears because such a sum cannot be *any distance* from *anything*, let alone from itself. But if it "resides at infinity" then there is a problem with our language, and it definitely is *not* innocuous. It generates the subtle confusion that Geometric Algebra directly articulates "points at infinity". The full Geometric Algebra does not. It cannot. *It can articulate only* ***finite*** *representations*. Moreover, *such a finite representation cannot be a* **single** *non-decomposable thing*—it is *intrinsically* composite. (Foreshadowing query: Indifferent v therefore cannot be *a* sum, as naively assumed, so what is it?) To peek ahead, there is no "at" at infinity; rather there are "ats" at finity.

You have just seen that v, under addition and subtraction, can move the two points that compose it, $\boldsymbol{a}$ and $\boldsymbol{b}$. Look closely: from elementary-school rules of addition, $\mathrm{v} + \boldsymbol{b} = (\boldsymbol{a} - \boldsymbol{b}) + \boldsymbol{b} = \boldsymbol{a} + (\boldsymbol{b} - \boldsymbol{b}) = \boldsymbol{a} + 0 = \boldsymbol{a}$. So point $\boldsymbol{b}$ has effectively been carried from one end of v to the other end. And the reason is clear: under addition, $\boldsymbol{b}$ annihilates one of v's endpoints, *poof*, leaving the other endpoint as residue. It seems natural to call the *poofing* endpoint the *tail*, the residual endpoint the *head*; and say that $\mathrm{v} + \boldsymbol{b}$ carries point $\boldsymbol{b}$ from v's tail to v's head. Altho this nomenclature seems natural, One wonders how generally useful it might be since this obviously works only because v is being added with a copy of its *own* endpoint. Right?—v doesn't carry *other* points around under addition, does it?

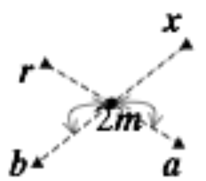
Well, let's just see: given an *arbitrary* point $\boldsymbol{r}$, what is $\mathrm{v} + \boldsymbol{r}$? To ask this question in the fresh young symbolism, solve this equation: $\mathrm{v} + \boldsymbol{r} = \mathrm{x}$, where x is unknown, *utterly* unknown as indicated by its generic font.

What is going to happen next is so important to your understanding of the full Geometric Algebra that I am going to present it in complete detail, with **pow**erful emphasis on the crucial part. If you hope to acquire a more expressive geometric language then you will have to wrestle with this until you understand it completely. How will you know whether you've understood it? If my experience is any indication, *you will become amazed*. If you don't, then you may be suffering from traditional meaning-imposing habits. To help overcome that, remember that we are articulating *fixed points*, and nothing else. We began *bound*; we are *bound* now; and it looks like we will *stay bound* because we are too timid to cavalierly *impose* any kind of geometric freedom. If freedom arises, it will be *entirely derived* from things that are *entirely bound*. Who would ever bet on that?

Okay, expand equation $\mathrm{v} + \boldsymbol{r} = \mathrm{x}$, giving $\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{r} = \mathrm{x}$. Now pull the purely-positive-equation trick by putting $\boldsymbol{b}$ on x's side of the equation: $\boldsymbol{a} + \boldsymbol{r} = \boldsymbol{b} + \mathrm{x}$. The left side has the sum of two simple points. The right side has the sum of a simple point and *something*, namely x. For the right side to equal the left side, this *something* must also be a simple point (do the weight calculation), so denote it in point font, $\boldsymbol{x}$. Hence, utterly unknown x has become somewhat known *simple point* $\boldsymbol{x}$. So, apparently v really does move arbitrary points around since that question will be answered by x's eventual location. To find this location, we need to visualize the transformed equation: $a + r = b + x$.

This equation involves *addition*, *equality* and *simple points*. These are the elements that have to be displayed geometrically. *Addition* of two points can be indicated by a dashed line connecting them. The *equals sign* is too imprecise about location to be useful on a geometric figure. Instead, a skinny curved line with tiny arrows on each end will be used. Call this the *geometric equals sign*. Its two tiny heads will just touch the things that are equal. *Simple points* are so useful that they should be distinguished from generic weighted points; let's use a little triangle for them and a little dot for generic points.

With these conventions, the transformed equation becomes geometrically obvious: two little triangles connected by a dashed line denote addition of two simple points, so their ***s***um, $2m$, lies at their ***m***idpoint. There are two of these additions connected by equality, so they share the *same* midpoint sum. Here is a picture:



This figure is an exact representation of the equation $a + r = 2m = b + x$. Given only the figure, you can directly write down the equation it represents. Try it.

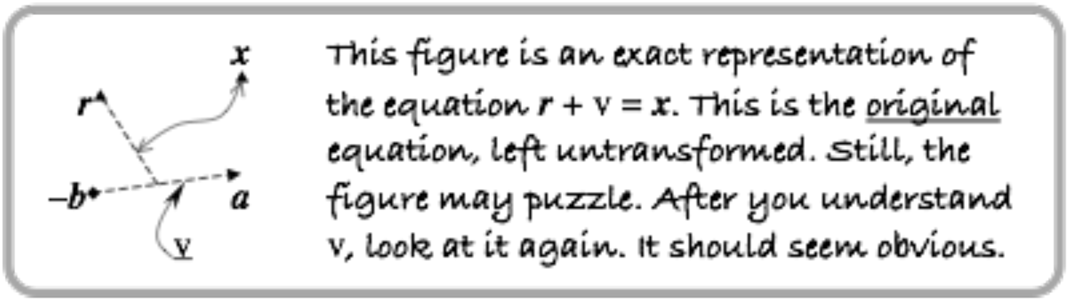
Visualizing simple point sums.

This kind of figure has seminal importance so let's dignify it as an *X-diagram*. It answers the question about v's ability to move points other than the two it comprises: x is the solution to the original equation v + $r = x$. In words, v added to r moves that over to x. Since the location of r is entirely a***r***bitrary, v moves *any* simple point in a similar way. Here is how to be sure you understand this completely:

Sketch weighted points a and $-b$. Connect them with a dashed line to indicate that they are being added together. This makes $a - b$ a kind of *cohesive bundle* (hint), deserving its own name, v; and deserving more intimate notation: $a–b$. Have a friend sketch a point r somewhere—*anywhere*. Now add your little bundle to it like this: **Wham!**—equate the summary result to x so you have a concrete result to work with. **Bang!**—unbundle v and swap $-b$ to x's side of the equation (by erasing the dashed-line addition and the minus sign). *This gives two simple point sums that equal each other*. **Pow!**—do the sum you immediately know, namely $a + r$. This gives midpoint $2m$, which is *also* the sum you didn't immediately know, namely $b + x$. So now you know it too, and you therefore know where x is. (For graphical precision, you should, of course, sketch the dashed-line additions as you do each sum, thereby making them neatly Xd together right in the middle.) Next, have your assistant sketch a different point somewhere, *anywhere*, **pow!** Another: **pow!**... If you can do ten distinct v+r sums in a row, correctly, without batting an eye, then you understand this. *Please* understand this—it is really quite simple; but the main reason we still suffer from geometric inconsistencies and confusion is nearly universal chronic ignorance of its various unexpected consequences.

Having understood this, you may think that it does not seem amazing. But it might seem surprising, or at least *peculiar*: Recall that v was able to move a copy of its own endpoint under addition by *poofing* it (to speak technically), leaving the other endpoint as residue. But here v is moving an arbitrary point under addition by a kind of *scissoring mechanism*, the X-diagram, in which nothing is being *poofed*. And yet...

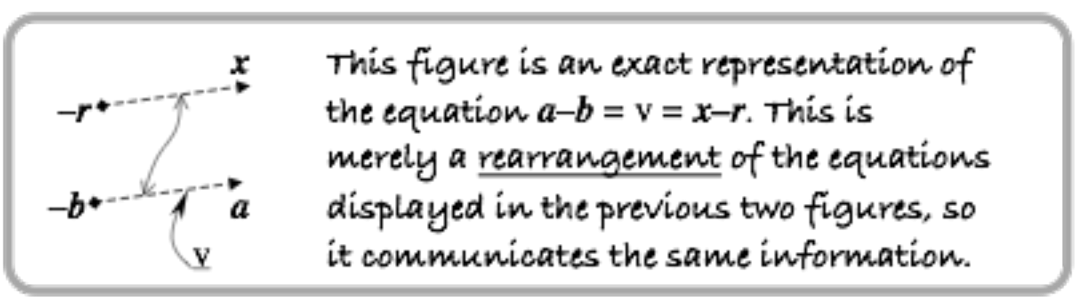
And yet v is moving point ***r*** *exactly as tho* v *could move parallel to itself* to place its tail over ***r***, and *then* do the addition; in which case v *would* be *poofing*, exactly as before. Dust off your high-school geometry and gaze at the X scissoring mechanism until you understand this in your bones. (You see the ***abm*** triangle, congruent to the ***rxm*** triangle, don't you?—take it from there.) To help understand this, here is the previous figure, *exactly as before*, geometrically, with the *exact same equation*, except that it has been left untransformed:



A direct visualization of $\boldsymbol{r} + v = \boldsymbol{x}$.

This figure may not seem as geometrically obvious as the X-diagram. Nevertheless, it directly represents the *same* equation, left untransformed. To understand that, superimpose the previous X-diagram on top of it—corresponding points will match exactly. The X-diagram should seem obvious if you understand how simple points add. When you understand how v acts, this figure should also seem obvious. Notice that the sum 2***m*** has disappeared from it because this point was not present in the original equation—it served as an illuminating *centered pivot point* in the transformed equation that you may now discard as a conceptual crutch. Notice also that, altho simple points are still depicted with tiny triangles, a *negative* simple point has been depicted with a tiny square. This makes it easy to determine the head and tail of v, as you see. Speaking of v, let me say again, for emphasis: in this figure it looks as tho v is able to move parallel to itself to engage the *poof* method of addition. If so, any puzzlement about the figure would evaporate.

It turns out that v actually is able to move parallel to itself; and a different transformation of the original equation will directly display this. In this new equation, don't unbundle v. Instead move ***r*** over to the ***x*** side of the equation: v = ***x*** – ***r***. **Kazaam!**—the right side of this equation has *exactly the same form as* v does, namely a simple point minus another simple point. Here is the figure representing this re-transformed equation:



geometric magic

To see that it communicates the same information, superimpose the X-diagram on top of it—corresponding points will again match exactly. In consequence, ***a–b*** *equals* ***x–r***. Hence, because the location of ***r*** is completely a***r***bitrary, v, in effect, *is able to move* **anywhere** *parallel to itself*. In short, *geometric freedom has been entirely derived from things that are entirely bound. That*, I submit, is amazing. Grassmann's protracted account of discovery seems to indicate that he found it amazing too.[7]

Apparently the strange *un-point-like* point v has a fixed *separation* and a fixed *direction* but no particular location. But exactly *what* has separation-and-direction? And exactly *what* has no location? Are these the *same thing*? No: the *summands* of v, considered as a separated–directed whole, can reside *anywhere* because the *sum itself* resides *nowhere*.

To be precise: v *is a peculiar addition of oppositely weighted points whose* ***sum loses****, in lock-step, both* ***magnitude and location****, which, in the limit, makes its* ***summands gain*** *both* ***separation and direction***. It's magic—without magic, to borrow John Wheeler's aphorism. This raises some...

Perplexities

- Roving v acts much like a conventional vector (so the name "v" was deviously presupposing), except that *it is not a line segment*. Right?...
- I mean, addition of weighted points always produces another *point*, at least *formally*, doesn't it?—addition never *changes dimension*, does it? (Everyone knows that a point is zero-dimensional, a line segment one-dimensional.)
- Speaking of changing things, the previous magic *changed focus* in the limit from a *sum* to its *summands*. Is this distinction important *formally*?
- For example, is this what makes v *intrinsically composite*?
- Is that why the full Geometric Algebra always articulates things *at finity*?
- If so, do the *conventional rules* of Geometric Algebra make the sum–summand distinction properly?
- If not, should they? Could they?

Some of these perplexities may be superfluous. After all, extending a point from another point produces a *directed line segment*, which would be *exactly a conventional vector* if it turns out to be as mobile, and as mobilizing, as v turned out to be; and that now seems likely, doesn't it? But that would raise another perplexity: what exactly, then, is the distinction between *subtracting* a point from another point, and *extending* that point from the other point? To find out requires specifying the...

Relationship between extension and addition

The relationship, like all healthy ones, is built on *mutual respect*: (4a) Addition respects extension enough not to change the properties that extension hath wrought. (4b) Extension respects addition enough to treat addition's *result* as a genuine summary of its *arguments*. These new concepts require a trip back to the closet to start all over again. Fortunately they merely augment the first three concepts, so, once we understand how they *change the symbolism*, we can just pick up where we left off.

Addition does not change the properties that extension hath wrought. This is just an unfamiliar specialization of the concept of summary. One naturally expects a summary *not to change any properties* of what it summarizes; to do otherwise would make a mockery of the concept. Refusal to change properties is really more central to the idea of summary than indifference to order, or indifference to grouping; but it remains unfamiliar because of our present narrow experience with summary of *homogeneous* things, ordinary numbers. Extension generates *inhomogeneous* things, things having *different dimensions*; and *that* is the property that addition does not change. Which removes one perplexity: (4a*) *Addition does not change dimension*. (This is a *semantic* formality that is surprisingly tricky to symbolize properly, as you shall see.)

And that, in turn, begins to remove the perplexity about v being intrinsically composite. When addition is presented with summands having different dimensions, it can't summarize them to anything simpler because it can't give them a common dimension. Hence, it merely bundles them into a summary list, with the plus sign serving as conjoining punctuation. This bundle is *intrinsically composite* because (1) its contents cannot reduce to just one thing, and (2) it can always be decomposed and re-bundled differently, using addition's associative law.

It is obvious to any student of Geometric Algebra that a sum of things having different dimensions is intrinsically composite: these things are obviously *too distinct* to merge in summary. But it is almost always a surprise that sometimes—*oftentimes*—even things of the *same dimension* are that distinct. This surprise can be blamed on a historical mishap: we have become stuck in the purely free Geometric Algebra. In that language, all *readily imaginable* things of the same dimension can always sum to a single thing simply because imaginable space just happens to be a perfect cage for free things.

If you go *just* beyond imaginable space, however, you bump into free things of the same dimension that are *too distinct* to sum to a single thing. Free bivectors in free 4-space, for example, are that distinct if the planes thru them intersect in just one point. This possibility arises naturally from the extra dimension (and has obvious expression in the *full* language), but it seems so bizarre to most students that they dismiss the idea of intrinsically composite same-dimensioned sums as too esoteric to worry about. Even Grassmann may have had that attitude during his early "geometry" phase,[8] as he dismissively called it.

If he did, he certainly revised his opinion after he encountered bound things late in his explorations. Grassmann began his language like all students today begin it, purely free; and perhaps humankind's roving spatial experience makes this approach natural. But his incredible curiosity and creativity eventually introduced him to *bound* points via *free* vectors![7] This is *exactly backward* logically; and it is truly, *astonishingly*, extraordinary as witnessed by the fact that in nearly two centuries of ignorance about Grassmann's bound language, no one else has made the trip backward and installed points within the formalities like Grassmann did. When he did that, he quickly discovered that there are *readily imaginable* same-dimensioned bound things that are too distinct to sum to a single thing.[9] They are not esoteric at all—in fact they are more common than same-dimensioned things that *can* sum to a single thing. (To peek ahead, they aren't simple points, are they?—they always sum to a *single* thing.) To really understand this, you need to know more about extension.

And that requires notation. Extension had initially been denoted in about as many different ways as there were authors writing about it—Grassmann himself used several distinct notations—but it has recently stabilized on Cartan's wedge, $\wedge$, meaning *extended to*. Unfortunately, that has two serious problems in the *full* Geometric Algebra, which must, above all, articulate points well since they are the generative elements:

(1) $\boldsymbol{a}\wedge\boldsymbol{b}$ would generate a directed line segment with tail at $\boldsymbol{a}$, head at $\boldsymbol{b}$; which is opposite to $\boldsymbol{a}-\boldsymbol{b}$, which generates separated–directed points with tail at $\boldsymbol{b}$, head at $\boldsymbol{a}$. This inconsistency would be confusing of course, but the worst of it is that these two expressions have an elegant relationship (coming up), *fundamental to the full language*, that would be obscured if they did not have their heads and tails in the same order. This really begs for extension to be *from* rather than *to*, which somewhat polishes Grassmann's gem. Consistency with point subtraction prompted Hamilton to adopt a similar convention.[10]

(2) From item 1, extension is clearly *directed*, so it really should have a *directed* symbol, rather than one with bilateral symmetry like $\wedge$. How about $\blacktriangleleft$? This clearly indicates *from*, and its *filled-in* form indicates *extension*. Hence $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ is "$\boldsymbol{a}$ *extended from* $\boldsymbol{b}$", like $\boldsymbol{a}-\boldsymbol{b}$ is "$\boldsymbol{a}$ subtracted from $\boldsymbol{b}$"; and these two expressions have their ducks aligned. As a bonus, this distinct notation should help clarify the transition from the conventional purely free language to Grassmann's full language for those readers crossing that bridge.

With notation established, we can pick up where we left off: *Extension respects addition enough to treat addition's* ***result*** *as a genuine summary of its* ***arguments***. Which is to say, extension with a point is indifferent to whether it operates on addition's *arguments*, or on addition's *result*. Here is how this augments the symbolism:

(4b.1*) $\quad (\mathsf{A}\blacktriangleleft\boldsymbol{c}) + (\mathsf{B}\blacktriangleleft\boldsymbol{c}) = (\mathsf{A}+\mathsf{B})\blacktriangleleft\boldsymbol{c} \quad$ and $\quad (\boldsymbol{c}\blacktriangleleft\mathsf{A}) + (\boldsymbol{c}\blacktriangleleft\mathsf{B}) = \boldsymbol{c}\blacktriangleleft(\mathsf{A}+\mathsf{B})$

Notice that there are two rules, commuted, because extension is *directed*, so extending *from* $\boldsymbol{c}$ on the left is generally different from extending *to* $\boldsymbol{c}$ (reading backward) on the right. Mathematicians call these rules *distributive laws*, which focuses on *syntax*. This may seem appropriate since these rules *are* part of the syntax, as explained shortly; but they, like all rules in this paper, were motivated by primitive semantics, so this paper will call them extension's *respect for summary* to emphasize their meaningful origin.

When you apply these rules multiple times to *scalar*-weighted points via a valid limiting process you get, for *scalar* c:

(4b.2*) $\quad c(\boldsymbol{a}\blacktriangleleft\boldsymbol{b}) = (c\boldsymbol{a})\blacktriangleleft\boldsymbol{b} = \boldsymbol{a}\blacktriangleleft(c\boldsymbol{b})$

This will be called extension's *respect for multiple summary*, again focusing on geometric meaning (and it can also be generalized to generic A and B). To indicate that extension has both kinds of respect, let's say that it has *strong respect for summary*. Strong respect for summary makes the language versatile and expressive by decoupling extension from addition, and from addition's infinitesimal multiple limit, scalar multiplication. This prepares you to start...

Extending things

Here is where timid meaning derivation begins to really pay off. A meaning-deriver has to start with the primitive concept for extension, the third concept, which requires extension from a point to *increment dimension*. That, in turn, requires establishing the *primitive dimension*, the dimension of a *point*.

Meaning-imposers long ago agreed that a point is zero-dimensional, but that poses a serious conundrum for meaning-derivers: Since extension from a point *increments dimension*, shouldn't the dimension of a point therefore establish the *dimensional increment* that gives everything else a dimension? This is simply a natural requirement for the dimension of an extension result to be the sum of its argument dimensions. If so, then *points must be one-dimensional*. This would imply that line segments are really *two*-dimensional; patches of plane are *three*-dimensional; and so on. This seems silly—we have known for millennia that lines are one-dimensional, planes are two-dimensional, and on up. Nevertheless, in a last-gasp nag, the meaning-deriver asks, *what about on down*?

If points were one-dimensional, then scalars would be zero-dimensional. Suddenly it is the meaning-imposers who have a serious conundrum: They have recently reached universal agreement that scalars are indeed zero-dimensional. If *points* were zero-dimensional, as *also* agreed, then, by dimensional decrement scalars would have a dimension of minus one. (Unless points *are* scalars. Well, are they? If meaning-imposing habits incline you to think so, please ponder the elegant relationship (coming up) between points and scalars before deciding.) Here you have an example of the inconsistency that meaning imposition generates. Exposed like this, *zero is not minus one*, it doesn't seem subtle, does it?

Why haven't we noticed this problem for the last several thousand years? First, only recently have scalars acquired a dimension, when they were belatedly recognized to be full-fledged *geometric objects* like lines, planes and so on. When scalars interacted with geometric things, it was seen that they must have a dimension of zero because they do not change the dimension of what they multiply. Strangely, second, points have yet to be emancipated like that—*points have not yet become full-fledged geometric objects*, like scalars! Meaning-imposers have so far *refused to allow points into the formalities* alongside scalars, vectors, etc; except as outcasts, undesirables who are denied full geometric rights. It is tempting to blame this on Euclid, who refused to grant points "magnitude", which effectively exiled them to the interpretation where, third, they have been neglected, orphaned from their geometric family, and underfed to the extent that they literally have no weight at all. *Exile to the interpretation*—let David Hilbert describe that:

"One should always be able to say, instead of 'points, lines, and planes', 'tables, chairs, and beer mugs'."[11] Well, lines long ago managed to escape from Hilbert's beer hall by dressing up as vectors, able to participate in black-tie formalities. Planes have recently pulled off the same formal getaway by dressing up as bivectors; but points are still stuck in the pub in their underwear. Since they are, who really cares what their dimension is? Apparently it is very much like the dimension of a table, or perhaps a beer mug?—who cares? Meaning-derivers care, and they want to get the orphan point out of the unruly interpretation and into the ruly formalities alongside its geometric kin: scalars, vectors, bivectors and so on. Transition into the formalities has been a paradigm for mathematical progress for thousands of years; but unexpectedly, for points it will require, *gazook*, ***meaning inside the language***, *formal semantics*.

Which, for distinction, requires *formal syntax*. Some of this syntax has already been presented: it is just the collection of *conventional rules* for Geometric Algebra—the equations, like the commutative, associative and distributive laws, that serve as axioms. These equations establish the valid sentences in the language.

All the rest is semantics, which traditionally—*dogmatically*—has resided almost entirely within the mind of the person composing the sentences. That turns out to be woefully inadequate for the *full* language, where *bound* points generate *free* things. The important *formal* distinction between bound and free requires *formal* semantics because the syntax intentionally ignores the distinction, for good reason. Moreover, such semantics rest, in an unanticipated way, on *formal* dimension, which, because it cannot be defined by equations, is also part of the semantics.

Formal dimension presents a rare opportunity to please everyone. To distinguish it from the previous decidedly *informal* dimension, give it a distinguished name: *extent*, which means *number of points required in an extension*. This will please the meaning-imposer since a line segment obviously requires *two* points, so it has extent two; a patch of plane requires *three* points, so it has extent three; and on up. Certainly, a meaning-deriver is pleased because this gets the *foundational* dimensions right: a point requires *one* point in the trivial do-nothing extension, so it has *extent one*; a scalar requires (dare I say) *zero* points, so it has *extent zero*. The meaning-imposer might be doubly pleased to discover that formal extent, in its intrinsically *separated* form, automatically articulates conventional dimension. Hence, conventional geometric dimension is not wrong, it is just a *special kind* of dimension.

To be specific, addition in the full language makes a distinction it could not have made with points absent, namely the distinction between the *separated* extent of free things, and the *filled-in* extent of bound things. This distinction is definitely part of the semantics because the syntax—the *conventional rules* of Geometric Algebra—simply cannot make it. To begin understanding that, investigate filled-in extent from the beginning:

Extending a point from another point produces a directed line segment that has a dimension one higher than that of a point. Start formalizing this by expressing it in the young symbolism: $\boldsymbol{a} \blacktriangleleft \boldsymbol{b}$.

Now proceed to formalize dimension by making extent an operator that accepts an argument, so that, for example, extent($\boldsymbol{a}$) produces $\{\mathbf{1}\}$ since $\boldsymbol{a}$ is a simple point. Curly braces indicate a *list* of extents, necessary because extent()'s argument might be intrinsically composite. For example, extent($b + \boldsymbol{a} + \boldsymbol{a} \blacktriangleleft \boldsymbol{b}$) produces $\{0, \mathbf{1}, \mathbf{2}\}$ if simple point $\boldsymbol{a}$ has a different location than $\boldsymbol{b}$. (If these points had the same location, the extension would produce Nothing with extent $\{\mathbf{2}\}$ (a line segment with no length); in which case extent($b + \boldsymbol{a} + \boldsymbol{a} \blacktriangleleft \boldsymbol{b}$) would have been $\{0, \mathbf{1}\}$. Such potential disappearance is just one of the reasons extent() is semantic—discovering disappearance requires computation.)

So, extension from a point increments extent, as required; and what does *addition* of two points do?—what, for example, is extent($\boldsymbol{a} + \boldsymbol{b}$)? You already know: since $\boldsymbol{a} + \boldsymbol{b}$ generates a single thing, and since addition does not change dimension, this extent must be $\{\mathbf{1}\}$. Which brings up a subtle but very important point: because $\boldsymbol{a} + \boldsymbol{b}$ generates a point with a weight of 2, the extent() operator clearly ignores weight; so, in general, for *any* generic weighted point $x\boldsymbol{x}$, extent($x\boldsymbol{x}$) produces $\{\mathbf{1}\}$.

In consequence, *weight is not Euclidian "magnitude"*. When Euclid asserted that a point "has no magnitude" he meant that it has no *spatial extent* like a line does, like a plane does, like a volume does... Euclid was asserting, in the technical language of the full Geometric Algebra, that a point has no extent greater than one. This is true: it has precisely extent $\{\mathbf{1}\}$, and *this has nothing whatsoever to do with the point's weight*, which specifies its potential scaling relations with its geometric kinfolk. These distinct concepts have been confused for millennia because there was no terminology that clearly distinguished them. The full Geometric Algebra remedies that by quantifying Euclidian "magnitude" with *extent*, and scaling relations with *weight, length, area, volume...* (Each of which is a formal kind of *magnitude*, un-scare-quoted—see how deviously confusing the vernacular is?)

Now for the conventional-vector question: how does directed line segment $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ move around? Meaning-imposers long ago asserted that directed line segments are free to move anywhere parallel to themselves; and that seems to have been wildly successful. Nevertheless, a meaning-deriver is not that bold; in fact he is *so* timid that he won't let $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ move *at all*, unless the primitive semantics allow it. Fortunately, the primitive semantics have already generated things that can move points around; so the meaning-deriver can try moving the endpoints of $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ to see what happens, like this:

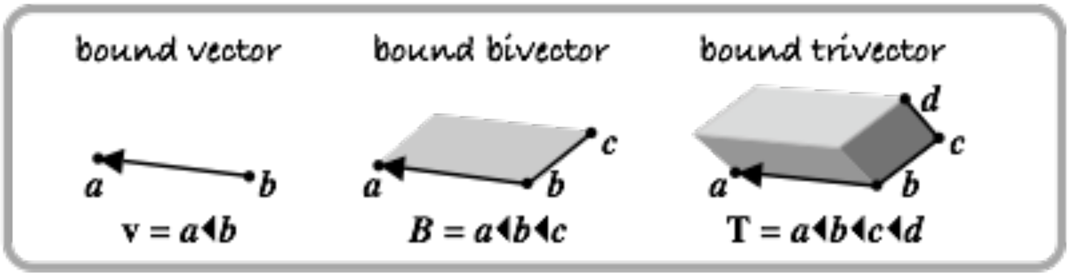
Generate a roving separated–directed pair of points, add it to both $\boldsymbol{a}$ and $\boldsymbol{b}$, and then extend them, thereby translating $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ parallel to itself. *This parallel-translated version of* $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ *will almost never equal* $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ for a simple reason: To be equal to $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$, it would, for starters, obviously have to be expressible entirely in terms of points $\boldsymbol{a}$ and $\boldsymbol{b}$. This is generally not possible because the translator itself is generally not so expressible.

Well then, suppose the translator *were* so expressible. Then it would be a scaled version of $\boldsymbol{a}–\boldsymbol{b}$, *which translates* $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ *somewhere along the line thru itself*. In this case, however, extension utterly ignores the translation, thereby making the translated version of $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ equal to it. This is mathematical poetry arising from extension's strong respect for summary, which, in particular, requires that a point extended from itself vanishes. (For pleasure and education, you might compose this simple poetry yourself.)

In consequence, $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$ is *not* a conventional vector, even tho is looks like one (since it is a directed line segment). Rather it is a *bound vector*, bound to the line thru itself, which will be called its *confining space*. Contrariwise, $\boldsymbol{a}–\boldsymbol{b}$ actually *is* a conventional *free vector*, even tho it does *not* look like one (it is *not* a directed line segment). It does not look like a conventional vector because it has been *unconventionally disciplined to treat points as bona fide geometric objects*.

This long-overdue discipline prepares you for a hint about the elegant relationship between bound and free: What is a free vector extended from a simple point? For example, what is $\boldsymbol{a}–\boldsymbol{b}$ extended from $\boldsymbol{b}$? Extended from $\boldsymbol{a}$? Extended from $\boldsymbol{c}$, not on $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$'s confining line? You can easily do the math for the first two questions by appealing to respect for summary (and you'll get the same answer); but to really understand the elegant relationship, you need to make acquaintance with all readily imaginable bound things, and then watch how they generate their free counterparts.

First, extend bound vector $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$, call it $\mathbf{v}$, from point $\boldsymbol{c}$ not on $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}$'s confining line. This sweeps $\mathbf{v}$ directly back from $\boldsymbol{c}$, filling in as it returns, thereby generating *bound bivector* $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}\blacktriangleleft\boldsymbol{c}$, call it $\boldsymbol{B}$. This bivector is bound to the *plane* thru itself for the same reason a bound vector is bound to the *line* thru itself: a parallel-translated version of $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}\blacktriangleleft\boldsymbol{c}$ cannot equal $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}\blacktriangleleft\boldsymbol{c}$ unless the translating free vector is expressible entirely in terms of generating points $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$. Next, extend bound bivector $\boldsymbol{B}$ from point $\boldsymbol{d}$ not on $\boldsymbol{B}$'s confining plane. This generates *bound trivector* $\boldsymbol{a}\blacktriangleleft\boldsymbol{b}\blacktriangleleft\boldsymbol{c}\blacktriangleleft\boldsymbol{d}$, call it $\mathbf{T}$, which is bound to the linear space thru *itself*. This space happens to model extent-$\{\mathbf{4}\}$ physical space (count the points required in the extension), so a bound trivector is a *ceiling* for that space. Here is a picture:



Readily imaginable bound things

Now for the free counterparts to these bound things. You have already seen the free counterpart to bound vector **v** = ***a***◂***b***, namely free vector v = ***a***–***b***. Note these two important properties:

(1) Bound vector **v** is free vector v *extended from a simple point on the confining space* thru the bound vector, *exactly*. You just discovered this if you took the previous hint about the relationship between bound and free. This establishes the elegant relationship between ***a***–***b*** and ***a***◂***b***, which removes the perplexity about the exact distinction between them. (Is it now clear why these corresponding vectors should be articulated in the same order?) To dignify the relationship, call free vector v = ***a***–***b*** the *free part* of bound vector **v** = ***a***◂***b***.

(2) The free vector is composed of *separate, but otherwise exactly opposite bound things added together*.

The emphasized phrases are *universal attributes* of the free-bound relationship, so it will be useful to ponder them briefly before examining that relationship in detail.

First, to extend free vector v from a simple point, the simplest strategy is to place v's tail right over the point before extending. *Poof*, the tail-on-point part of the extension will vanish because a point extended from itself vanishes. This leaves the head of v extended from the point, which is just bound vector **v**. This is the *poof* method of point extension, even more wonderful than the *poof* method of point addition because it will apply to things of even higher extent.

Second, ponder what it will mean for *separate but otherwise exactly opposite bound things* of higher extent to be *added together*. As with primitive things, it will mean that sum magnitude diminishes to zero as sum location recedes to infinity, which will, in the limit, shift focus from sum to summands. There is a transparent way to demonstrate this: successively extend by the independent free vectors hidden within these higher-extent things. This will automatically produce roving things having separate but otherwise exactly opposite bound ends because free vectors have those properties. As a bonus, it will show that even tho bound generates free, free does not generate bound, which is one reason we are still stuck in the free language. (Being stuck there impels us to persistently try to represent bound with free, typically points with vectors, which is *inherently contradictory* because free *cannot* generate bound.)

Here are the free vectors hidden in the previous readily imaginable bound things: v = ***a***–***b***, w = ***b***–***c*** and x = ***c***–***d*** (gaze at the previous figure).

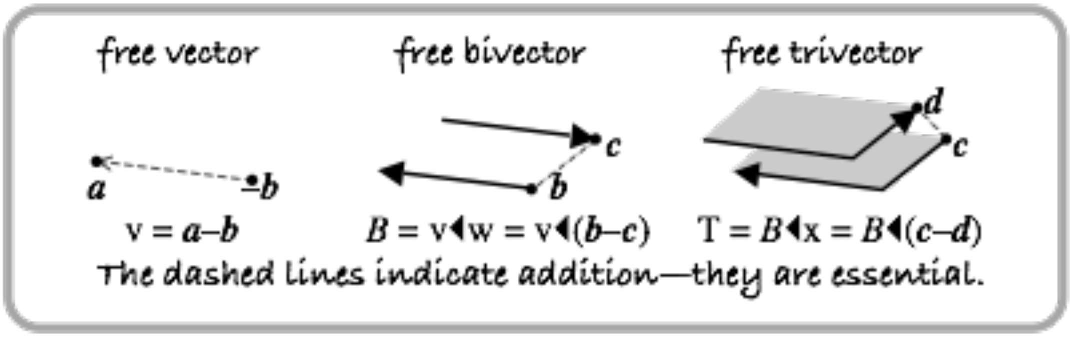
To begin, extend v from w: v◂w = v◂(***b***–***c***) = v◂***b*** – v◂***c***. It would be enlightening to descend further toward points, but there is no real need to do so because you know, from the *poof* method of free-vector extension, that this is a pair of separate but otherwise exactly opposite bound vectors added together. You also know, from freedom of the v and w arguments, that there are countless other exactly opposite bound vector pairs equivalent to this one, differing only in location, all having the same separation and direction. (This is *area* separation, as dimensionally distinguished from the *length* separation of a free vector; just as *area* magnitude is dimensionally distinguished from *length* magnitude, and so on.)

Here is an intuitively appealing way to understand why these variously located $\mathrm{v}\blacktriangleleft(\boldsymbol{b}-\boldsymbol{c})$ extensions are all equal: incrementally approximate them in unison by sneaking up on free vector $\boldsymbol{b}-\boldsymbol{c}$ as before, but extend at each step. The various bound vector results will dwindle away in lock step as they recede to infinity until they all seem to merge together as a tiny directed dash on the horizon. In the limit this dash loses (length) magnitude and location, which causes its various summand pairs to gain (area) separation and direction. At that limit, the focus necessarily shifts from non-existent sum to existent summands—these summand pairs have suddenly become a *free* bivector, call it B.

Does free bivector B have an elegant relationship with bound bivector $\boldsymbol{B}$? Here is where the *poof* method of point extension really shines: To extend free B from a simple point on bound $\boldsymbol{B}$'s confining plane, take advantage of B's freedom to place one of its bound vector ends over the point, then extend from there. *Poof*, the summand-on-point part of the extension vanishes because it produces no area (technically: it produces Nothing, *zero*, having extent {**3**}). This leaves only the other summand extended from the point—a **bound** bivector that is just a filled-in version of the *free* one with an incremented extent. This is indeed bound bivector $\boldsymbol{B}$. To dignify this elegant relationship, call free B the *free part* of bound $\boldsymbol{B}$. Query: do you still get bound $\boldsymbol{B}$ if you put free B's *other* end on the point before extending? What if you don't place *either* end on the point?

Now here is a curiosity: to get bound $\boldsymbol{B}$ from free B, you had to use something **bound**, namely a simple point. However, the game we are now playing is to extend by *free* vectors hidden within bound things; and so far that has generated something *free*, namely B.

To continue this game, extend free B from the last hidden free vector, x: $B\blacktriangleleft\mathrm{x} = B\blacktriangleleft(\boldsymbol{c}-\boldsymbol{d}) = B\blacktriangleleft\boldsymbol{c} - B\blacktriangleleft\boldsymbol{d}$. You could descend further toward points, but there is no need to do so because you know, from a paragraph ago, that this is a pair of separate but otherwise exactly opposite bound bivectors added together. You also know, from freedom of the B and x arguments, that there are countless other exactly opposite bound bivector pairs equivalent to this one, differing only in location, all having the same volume separation and direction. Here is an intuitively appealing way to understand why these variously located $B\blacktriangleleft(\boldsymbol{c}-\boldsymbol{d})$ extensions are all equal: incrementally approximate them in unison by sneaking up on free vector $\boldsymbol{c}-\boldsymbol{d}$ as before, but extend at each step. The various bound bivector results will dwindle away in lock step as they recede to infinity until they all seem to merge together as a tiny directed patch on the horizon. In the limit this patch loses (area) magnitude and location, which causes its various summand pairs to gain (volume) separation and direction. At that limit the focus necessarily shifts from non-existent sum to existent summands—these summand pairs have suddenly become a *free* trivector, call it T. Here is a picture of it with the free consorts that led to it:



Extension generates free things from free vectors.

By now there should be no need to explain the relationship between free T and bound **T** by describing how the former is the free part of the latter. In general, it should now be clear that generic bound **a** is just its free part a extended from simple point ***a*** lying within its confining space. This fills in free a and binds it thru point ***a***. In algebra, **a** = a◀***a***. This is truly elegant because it *universally* relates free with bound, a relationship that applies *even to the latecomers*, scalars and points. It is an algebraic fact that weighted point *a**a*** equals *a*◀***a***.[12] Which is to say, *a weighted point is a **bound** scalar*, and the typeface emphasizes this, as you may have noticed. So, even tho scalars aren't points, since they have lower dimension, nevertheless, by extension they give magnitude to simple points; and that is the deep reason points wallow in numbers in the full Geometric Algebra. Hence scalars are really, *really*, full-fledged geometric objects.

The previous derivations showed that when extension's arguments are free, its result is also free. This holds for addition as well, and here is a peek at the reason: When two elementary free things are so distinct that their sum cannot coalesce to a single thing, then that composite sum of *free* things is naturally declared *free* by fiat. On the other hand, when such a sum *can* coalesce to a single thing, it does so by algebraically pre-shaping and positioning the two free summands so that, when added, an end of one *cancels* an end of the other, *poof* (imagine adding, for example, two free bivectors from the previous figure). The canceling pre-shaping ensures that the two surviving ends are exactly opposite (and separate, else they didn't survive)—again something free. So, if you begin with free vectors as your primitive elements, then you will be stuck in the free sub-language of the full Geometric Algebra. Therefore, don't imagine that you can represent points with the conventional free language alone—that will set you up for inconsistency and confusion.

The free sub-language has the lovely property that its elements can always be juxtaposed, which allows you to not only *extend* to higher dimensions, but to also *retract* to lower dimensions. Extension and retraction have complementary symmetries that, together, provide full information about geometric relationships. It was Clifford's genius to conjoin them into a very informative, widely celebrated *Clifford product*.[2] In Geometric Algebra this is called the *geometric product*, and it makes the free sub-language extremely versatile and expressive—it's a wonderful place to get stuck in.

How not to distinguish free from bound.

When you examine any contemporary book on Geometric Algebra, you discover that the vectors, bivectors and trivectors within it *are all depicted filled-in*, as tho they were bound. And yet they are allowed to roam around, *as tho they were not filled in*. How can these books get away with such blatant inconsistency? *By refusing to allow points into the formalities except as outcasts*, that's how. The precise formal distinction between bound and free (coming up) disappears with points left in the interpretation. This renders the inconsistency so subtle it doesn't get noticed.

It's not as tho there were a malicious conspiracy to exclude points; it's more subtle than that. Geometers, despite mathematics' renowned proud rejection of meaning, approach their intrinsically meaningful subject with deeply held preconceptions that are fertile and mostly correct. Points within the symbolism would crumble these preconceptions around the edges like this: *Points would need the same dimension as free vectors* to add properly with them. This would *require free vectors not to be filled-in* for Grassmann's gem to be able to assign dimension consistently. Points? One-dimensional? Don't kid me. Free vectors? Not filled-in? Ha!—how could such things have *fixed* length and direction?

When reason and logical consistency nudge comfortable misconception, misconception typically remains complacent and unmoved; so points remain in their underwear breathing beer fumes. Except in one glorious yet sobering case: the curious case of Hermann Gunther Grassmann.

When he happened on points late in his investigations, he quickly realized that they must have the same *order* as free vectors, namely one. This of course had generative consequences for everything on up, so he gave them *orders* too, corresponding to the formal dimensions developed in this paper. His supple accommodation of points within the formalities has yet to be matched. That's the glorious part.

Here's the sobering part: his *order* was for him not *dimension*, but rather a way to get things to interact properly, with no other meaning. Listen to this:

> There are seven types of spatial magnitude, divided into four orders:
>
> 1st order 1. Simple or multiple points
>
> 2. Straight lines of definite length and direction
>
> 2nd order 3. Definite parts of definite infinite straight lines
>
> 4. Plane areas of definite magnitude and direction
>
> 3rd order 5. Definite parts of definite infinite planes
>
> 6. Definite volumes
>
> 4th order 7. Definite volumes
>
> Volumes appear twice here, once as magnitudes of third order, once as magnitudes of fourth order, according as they are regarded as products of three straight lines of definite direction and length or as products of four points.[13]

Do you recognize these things? Here's a hint: the first item in each pair is bound, the second item is free with the same extent (neglecting separation, which *must not* be neglected, as explained shortly). So, in the first order there are points and free vectors, in the second order there are bound vectors and free bivectors, and so on. Since each pair has the same numerical extent, you see that Grassmann's *order* indeed corresponds to my *formal dimension*; but *his* dimension is, again, decidedly *informal*. This is most evident from his perplexing comment on "volumes", in which he explicates the generative distinction between a free trivector and a bound one (in that order), *without making a dimensional distinction between them*.[14]

Why did he fail to do that? Remember, he arrived at points via *roving arrows*, the very same imagery we still have of free vectors. Even tho he explicitly discarded this imagery in favor of abstract algebra; nevertheless, free-vectors-as-roving-arrows must have become for him an inviolable concept, given how incredibly fruitful it had been. When he happened on points he was already comfortable with meaningless abstraction. Indeed, by then he embraced it; so he left *order* as an abstract formality that merely oiled the gears in his algebraic machinery. To have interpreted *order* geometrically would have required him to remove the shafts of his roving arrows—his "straight lines of definite direction and length"—leaving only arrow-heads and arrow-tails possessing mysteriously fixed separation and direction. But he clearly had no inclination to interpret *order* geometrically; and almost certainly no inclination to dismantle his fertile preconceptions. So, here again, comfortable misconception remained complacent and unmoved—Grassmann was human after all. That's my guess.

Finite intrinsically composite semantic formalities

Intrinsically composite, as mentioned, is hard to imagine for same-extent free things, but easy to do so for *bound* things. Vectors bound to skewed non-intersecting lines, for example, do not have a common-enough extension factor to sum to a *single* thing; so their sum has extent {**2**, **2**}. These vectors can, however, sum to *two* things in many different ways, and the most perspicuous is a free bivector perpendicular to a bound vector. In physical terms, the free bivector can articulate an angular velocity while the bound vector articulates a velocity along some line. Or these things can articulate a torque combined with a linear force. In short, addition of skewed lines generates an expressive *screw algebra*, reinvented by just about everyone who has really understood Grassmann.[15]

Are there any other imaginable same-extent bound sums that are intrinsically composite? How about the simple point sum $\boldsymbol{a}+\boldsymbol{b}$? It certainly is not intrinsically composite because it reduces to single midpoint $2\boldsymbol{m}$. In fact, the humdrum sum of most weighted points reduces to a single thing. Well then, how about the *magic* sum, free vector $\boldsymbol{a}-\boldsymbol{b}$? It *is* intrinsically composite because (1) it cannot reduce to just one thing, and (2) it can always be decomposed by being unbundled and re-bundled differently (that's how the *poof* interactions work). To put this intuitively and generalize it, *separate but otherwise exactly opposite bound things are* ***too distinct*** *to sum to a single thing*. This should not come as a surprise—exactly opposite *is* quite distinct. Therefore, a free vector has extent {**1**, **1**}; a free bivector has extent {**2**, **2**}; and so on.

This notation transparently displays the *separated* extent of free things, but it fails to make a further crucial distinction. For example, it would give a free bivector the same extent as the sum of skewed vectors bound to non-intersecting lines, extent {**2**, **2**}. A free bivector is a sum of *parallel* vectors bound to non-intersecting lines, which generally produces a single extent {**2**} result (as previously demonstrated by approximation); except when the bound vectors are *exactly opposite*, in which case you get *exactly opposite* extent {**2**, **2**}. *Exactly opposite* addition is what distinguishes free from bound; and such addition refuses to reduce to a single result by attempting to assign *contrary* properties to that result: zero magnitude and infinite "location". The infinite "location" cannot be computed because it does not exist, but the zero magnitude is straightforward to compute. It is the *semantic formality* that shifts focus from sum to summands, from infinite to finite, from bound to free. Which motivates a peculiar yet precise definition of a *free thing*: *a non-zero thing with zero magnitude*. *That* is the crucial distinction.

Let's put it to use: To be non-zero, a free thing, a bivector for example, must have formal *separation* (also readily computable), transparently annotated by *composite* extent $\{\mathbf{2}, \mathbf{2}\}$. To further indicate that this is *yin-yang* composite in a cohesive *exactly opposite* way, One could call it extent $\{\mathbf{2}, \mathbf{2}\}$-with-zero-magnitude. This is an accurate but clumsy way of specifying that it is a free bivector. Since it is *free*, why not instead distinguish it with *free non-bold* notation?—extent $\{2\}$. Hence, extent $\{2\}$ means cohesive extent $\{\mathbf{2}, \mathbf{2}\}$-with-zero-magnitude. Similarly extent $\{1\}$ means cohesive extent $\{\mathbf{1}, \mathbf{1}\}$-with-zero-magnitude, and so on.

You can think of cohesive free extent as addition's respectful way of "extending". When addition is presented with separate but otherwise exactly opposite bound summands, it leaves them "extended" not by *incrementing* extent, but rather by *separating* it in a formal way. So, a free vector has formal separated extent $\{1\}$, a free bivector has separated extent $\{2\}$, and so on, just the numerical dimensions meaning-imposers have been declaring all along. Which is to say, free extent is conventional geometric dimension (called *grade* in the conventional free language), now well distinguished.

Distinguished by *separation*—that's what a tag of zero magnitude means: *this thing is not filled in*. Extension from a simple point fills it in, suddenly giving it *non-zero magnitude* equal to its just-departed separation. This magnitude becomes annotated with incremented **bold** singleton extent. Hence, separation describes a *pair of opposite summands*, something free; magnitude describes *one result*, something bound. The lowest extent magnitude is weight.

Weight: a scalar extended from a simple point generates weight—*non-zero magnitude* annotated with incremented bold singleton extent $\{\mathbf{1}\}$. So, algebraically, a scalar is free, meaning that it is a non-zero thing[16] tagged with a formal magnitude of zero, like all free things in the full Geometric Algebra. Such lowest-extent "separation" is the *value* of the scalar, which gets "filled in"—acquires locus—by extension with a simple point.

All this expressive formal distinction arises from finally letting points enter the symbolism as full-fledged geometric objects. To see that this emancipation is well worthwhile, examine the machinations necessary to keep points out while trying nonetheless to gain their expressive power. You need to impose...

Models

The three most popular models are the *vector space model*, the *homogeneous model,* and the *conformal model*. They are easiest to understand by the way they represent the plane. (Technically, the collection of primitive semantics is a model itself, *Grassmann's point model*; but it, unlike these, is not a clever artifice *imposed* on the symbolism, rather it is the DNA from which the symbolism is *derived*: Grassmann's point model—seed for a growing symbolism; conventional models—straightjackets for an inert symbolism.)

The vector space model of the plane begins with a formal algebra of two free vectors, whose inherent limitations are traditionally overcome informally. First, since these vectors are *free*, where are you going to put them? Answer: *implicitly* anchor them to a point, the *origin*. With their tails firmly fixed in one place, their heads can represent points—you get free vectors *and* points! Free vectors?—but you just bound them! No, let them roam around when you need them to. But then you can't use them to represent points! No, just attach them to the origin when you need points. And so on. This has worked surprisingly well because, even tho the various fleeting distinctions all reside *outside* the symbolism, they nonetheless reside *inside* a human mind, which is superb with fleeting distinctions.

Fleeting distinctions won't do for a *model*, however, so the modeling community has decided that free vectors shall be *explicitly* anchored to the origin. This allows the fertile vector space idea to be unambiguously implemented on a computer. It has the ironic consequence that all the *free* elements in the language are effectively *bound* thru the origin, which has become *semi-formal* since it now has explicit representation in the software, even tho it has none in the algebra proper. Modeling enthusiasts don't mind this self-imposed handicap because they have more spiffy models that overcome it.

There is a different way to overcome the handicap that should be clear by now: having moved the origin from the informalities into the semi-formalities, why not continue this advance by moving it right into the formalities? As previously explained, this effectively moves some of the semantics into the symbolism. Here are the advantages of a meaningful symbolism: nebulous distinctions *outside* the language become precise distinctions *inside* the language, which now lets free things move parallel to themselves but requires bound things to stay in their confining spaces. Moreover, with the origin formal, everyone will have to implement it in the same way, as specified by the syntax of Geometric Algebra. (With the origin semi-formal, this is left to the digression of the implementer—need I say more?) The natural and expressive origin-in-the-formalities solution is obvious only in retrospect because comfortable misconception has rendered it almost inconceivable.

Consequently, modelers overcome the handicaps inherent in the vector space model in a different way, by moving to the *homogeneous model*. They always describe this by saying that you must move up an *extra* dimension above the plane. **Not!**—*a healthy plane* ***already*** *requires three dimensions*. You just saw that a plane with only two dimensions is crippled. By formally introducing the origin to heal it, *you increased its dimension by one*; but this is not an *extra* dimension, it is a *missing* dimension!

The origin increases dimension by one because it is just as variable as a free vector—that's what enables it to make a formal getaway from Hilbert's beer hall. Speaking abstractly, *dimension* just counts the number of variables available. With two free vectors, you have two variables available corresponding to the separation ("length") of each vector. But *you don't have points yet*—you don't *really* have a plane. To get points, you need a point to refer your free vectors to; and if that is done *outside* the symbolism, as it always has been in the last century, then you mangle your free vectors in the alleyway, as just described. By formally introducing a point for your free vectors to collaborate with, your symbolism suddenly acquires *the missing variable*, weight, which generates weighted points thruout the plane—this is now a *genuine* plane; it is not pointless anymore. It has abstract dimension 3 corresponding to formal extent $\{\mathbf{3}\}$.

Consequently, "*move up an extra dimension*" really means "*use another free vector to stand in for the missing variable that a point would supply, had not preconception abandoned it in the gutter*." Stating it baldly (badly?) like this makes achieving it obvious: let the separation of the "extra" free vector basis element correspond to the weight of the missing origin. Hence, distance above the plane corresponds to point weight; so unit points—*locations*—can be represented by anchored free vectors whose heads lie one unit above the original plane. The new unit-separated plane becomes a model-with-"points". (Modelers call them *points*, un-quoted, but their "points" are always free vectors masquerading as points. Such things aren't real points—they are undesirables, tacitly denied full geometric rights. This is easy to demonstrate: real points with *full* rights would, for consistency, induce free vectors with *separation*, absent in every model except the generative one, Grassmann's point model.)

The homogeneous model is fun to play with because it shows, in an unexpected way, how perpendicular distance can precisely represent point weight. By applying some mind-boggling dimension hopping, you can use this model to articulate both free and bound things, thereby overcoming the shortcomings of the vector space model. Of course such contortions make sense only if you are *absolutely determined* to keep real points out of your formalities.

Semi-formally anchored free vectors give the homogeneous model its own peculiar handicaps. To bypass my point sympathies, let a model enthusiast expose them:

> ... the geometric algebra approach exposes some weaknesses in the homogeneous model. It turns out that we cannot really define a useful inner product in the representation space R^{n+1} that represents the metric aspects of the original space R^n well; we can only revert to the inner product of R^n. As a consequence, we also have no compelling geometric product and our geometric algebra of R^{n+1} is impoverished ...[17]

Not to worry—there is another model that overcomes this fresh impoverishment, the *conformal model*, "which requires *two* extra dimensions",[17] meaning *one* dimension above the defective homogeneous model, which constitutes one *genuine* dimension above Grassmann's formal point space. The genuine extra dimension is given negative distances, thereby causing the augmented space to curve in such a way that extension in it can be projected down to *rounds* in the original space.[18] So, in a conformal representation of physical space, the extension of three points generates the unique circle thru them; the extension of four points generates the unique sphere thru *them*. "Points" themselves are rounds with zero radius (null vectors). Clever, huh? It gets even better: by including a special "point at infinity", ∞, you can generate *flats*, rounds with infinite radius. Moreover...

> Our model also solves another problem that perplexed Grassmann throughout his life. He was finally forced to conclude that it is impossible to define a geometrically meaningful inner product between points. The solution eluded him because it requires the concept of indefinite metric that accompanies the concept of null vector. Our model supplies an inner product $a \cdot b$ that directly represents the Euclidean distance between the points. This is a boon to distance geometry, because it greatly facilitates computation of distances among many points.[19]

Altho it is true that the bondage of Grassmann's points $\boldsymbol{a}$ and $\boldsymbol{b}$ naturally precludes an inner product for them (since they cannot be juxtaposed), it is not true that this precludes finding the distance between them. Conjure up free vector $\boldsymbol{a}-\boldsymbol{b}$, and then use Grassmann's inner product—there is no need to hop on the conformal pony, lovely tho it may be, to access its high-dimensional inner product.

There is no question that models are lovely, with beautiful, fruitful mathematics generated by incredibly curious, creative mathematicians whom I deeply admire. But models typically solve problems they have inflicted on themselves by leaving the origin in the semi-formalities, where it cannot interact properly with its geometric kin. Even worse, they solve problems in an indirect, obscure and inefficient way that Grassmann's full language can solve in a direct, transparent and efficient way. The *formal* point-generated distinction between bound and free (obviously lacking in the previous purely free models) enables this. This distinction, coupled with Clifford's unification of the free sublanguage, gives you an exceptionally expressive way to articulate geometric concepts: hop in the free sublanguage when you need its services; hop in the bound part when you need things in certain places; stay in the free sublanguage as much as possible because it is most versatile and expressive.

To illustrate, simple subtraction of points $\boldsymbol{a}$ and $\boldsymbol{b}$ moved these *bound* things into the *free* language where distance calculations are available. Simple subtraction can also generate rounds by moving points into the free language. For example, to generate the circle thru three points, subtract the points pairwise to form three free vectors, then apply symmetry to find the center point. Finally, do a direct, transparent and efficient fixed-radius computation. Or just apply symmetry *directly* to generate peripheral points iteratively—this is even more direct, transparent and efficient. (As for *flats*, why not generate them with ordinary low-dimensional extension? This avoids the superfluous imposed "point at infinity" and is (need I say?) direct, transparent and efficient.)

Simple subtraction of separate but otherwise identical bound things can always be used to move them into the versatile free sublanguage. This is seldom convenient for anything but points, however, and seldom necessary either because the elegant relationship between bound and free offers an easier way to hop into the free sublanguage: extract free parts.

Extracting free parts is such a crucial bridge from the full Geometric Algebra into its free sublanguage that I like to consider it a primitive operation, on par with extension, retraction and their unification, the geometric product. This requires a pithy notation for extracting free parts; and it also requires the elegant relationship, $\mathbf{a} = \mathsf{a}\blacktriangleleft\boldsymbol{a}$, to be added to the symbolism as an axiom. (For the purpose of generating free parts, I'm guessing it really is an axiom: If you don't want it as an axiom, then you have to isolate free a on the right to directly generate free parts; and good luck with that. Remember, point $\boldsymbol{a}$ cannot participate in a retraction (an inner product), nor a geometric product, so how are you going to *un-extend* it to the scalar unit to isolate free a? If this intrigues you, study how Whitehead did it by crippling his language with tacit context.[20])

As a practical matter, free parts discard locus information so they are easy to compute. *Especially* easy if you keep your basis as free as possible by allowing just one point in it, the origin. With this discipline, the origin is the sole source of bondage; so extracting a free part amounts to extricating a (generally translated) origin.

Finally, hopping into the bound part of Geometric Algebra from its free sublanguage is trivial in two ways: (1) Extend from a simple point. Since this point can be smoothly moved, any bound thing can be smoothly moved. (2) Decompose the free thing using addition's associative law. Suddenly you have two *relatively bound* things, one of which gets associated with *something else*, thereby transferring the relative bondage to it. The screw algebra illustrates this well, as the following section explains.

The free–bound distinction is intrinsically semantic.

Can the conventional rules of Geometric Algebra, the axioms, make the free–bound distinction? If so, they would have to distinguish between a sum and its summands. But they can't—as far as the axioms are concerned, *a sum and its added summands are literally equal*. That is a great boon because it allows free and bound to be articulated together, and intermingled. For example, altho the semantics make a clear distinction between magnitude and separation, the axioms can't because they cannot distinguish a sum from its summands. Instead, the axioms simply articulate magnitude and separation *simultaneously*, indifferently; and automatically switch from one to the other as the situation dictates. To begin understanding this, scale a free vector as you sneak up on it. During the approximation you will be scaling a diminishing *weight*. At the limit, however, you will suddenly be scaling a *separation*—a startling revelation for me. (At that limit you will be *simultaneously* scaling a zero weight, which will of course remain zero.) The axioms' indifferent automatic switching allows free things to be decomposed into bound things when need be, or vice versa. In short, the axioms do their syntactic duty, which is: let you express any valid sentence in Geometric Algebra, and let you transform that into more informative sentences.

The free–bound distinction *requires* distinguishing between a sum and its summands; so if syntax can't do it, semantics will have to. You have just seen that this is done by a formal zero–nonzero distinction. To illustrate just how adamantly semantic such distinction is, let's resolve the final dangling perplexity:

"Vectors bound to skewed non-intersecting lines, for example, cannot sum to a *single* thing; consequently their sum has extent $\{\mathbf{2}, \mathbf{2}\}$. These vectors can, however, sum to *two* things in many different ways, and the most perspicuous is a free bivector perpendicular to a bound vector." ???

How can that be? A free bivector plus a bound vector seems to have extent $\{2, \mathbf{2}\}$, which would expand into extent $\{\{\mathbf{2}, \mathbf{2}\}$-with-zero-magnitude, $\mathbf{2}\}$. Can these *three* things (when fully decomposed) possibly reduce to extent $\{\mathbf{2}, \mathbf{2}\}$? Yes—the perspicuous sum is a convenient and illuminating *canonical* form, not a *minimal* form. A minimal form has extent $\{\mathbf{2}, \mathbf{2}\}$ and this is easy to see: move the free bivector so that the tail of one of its ends is right on the tail of the bound vector. Conjoining these two vectors like this gives them a common extension factor that engages extension's respect for summary. This collapses the two vectors to one, leaving two skewed bound vectors. (Here you see *relative* bondage, transparently exposed. Reverse the procedure orthogonally to get the canonical form.)

So you see, for dimension to be well defined, addition must present the `extent()` operator with a minimal form. This is inherently computational—intrinsically semantic. Magnitude is an important part of this computation since it distinguishes free from bound, an essential distinction for a minimal form.

(If you want addition to give *you* a canonical form, you will have to ask for it—that is how semantics works; and it is just one more reason an expressive geometric language requires semantics. Whether your request should be formal or semi-formal is a question we haven't pondered adequately because we have shunned semantics.)

The computer people know that, in pathological cases, computation cannot distinguish between zero and darn-close-to-zero. With respect to magnitude this means that, in pathological cases, the full Geometric Algebra cannot distinguish between free and bound—it will not produce a genuine minimal form. This does not, however, invalidate the distinction that magnitude makes; it just requires extra care to do well, as with all formal semantics. Keeping free things bundled goes a long way toward minimizing this problem—to repeat, stay as free as possible and take care to represent free bundles by individual names, which keeps their bundles intact during computation. This forces their zero magnitude to *stay* zero, unambiguously. The extreme way to do that is to remain in the comfortable conventional free language; but then you are back where you started—bigoted against points.

Finally, it is not as if mathematics has been immaculately devoid of formal semantics, tho many mathematicians are reluctant to admit it. What, for example, is a *metric* but a precise assignment of *meaning* to distance? That is just as semantic, and just as formal, as the distinction between free and bound; in fact it helps establish that distinction in the full Geometric Algebra. Mathematicians should come out of the closet about semantics. Computer scientists outed long ago and they feel liberated now.

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Gras.18??

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Gras.1862

Hermann Grassmann, *Extension Theory*, American Mathematical Society. Translated by Lloyd Kannenberg, 2000. Grassmann's attempt to appeal to mathematicians after his 1844 opus had failed to do so.

Harp.201?

Gary Harper, *Playing with Geometric Algebra—Stalking a coherent language*. Forthcoming. My own modest contribution, the hardest thing I've ever tried. Foundational chapters may be downloaded from gary-harper.com/ Some of the prose makes me wince now.

Hest.198?-20??

David Hestenes, modelingNTS.la.ASU.edu/ Collects most of Hestenes's papers on mathematics and physics. Hestenes is the prolific lion of the free Geometric Algebra and has become an enthusiast of geometric models.

[1]Gras.1844, Part Two, "Elementary Magnitudes", devoted to the bound language. Part One, "Extensive Magnitudes" is devoted to the free language.

[2]Clif. "Applications of Grassmann's Extensive Algebra", 1878, and "On the classification of Geometric Algebras", 1876.

[3]Trust me on this—to embarrass these *particular* authors would unfairly single them out from among the *many* others who hold similar opinions, saying, for example, that you can't add London to Paris.

[4]Harp. "Speaking of Space" p5.

[5]John Playfair, (Euclid's) *Elements of Geometry*, J. B. Lippincott, 1857. p8: "A point is that which has position, but not magnitude." This is Playfair's elucidation of Euclid's "that which has no parts, or which has no magnitude."

[6]Grass.1844 p162, Grass.1847 p326. He developed point sums obliquely in terms of center of gravity, and displacements from an arbitrary origin. For a direct midpoints-of-midpoints development, see Harp, "Adding Points".

[7]Grass.1844, p154–161. This discovery prompted Grassmann to abandon the notation he had used for displacements in the first half of his book. For commentary on the mathematics, see Harp, "Speaking of Space", p43.

[8]Grass.1844 p11: "I found that the analysis I had discovered did not touch only on the subject of geometry, as it seemed before. Rather, I soon realized that I had come upon the domain of a new science, of which geometry itself is only a special application."

[9]Grass.1844 p184–185, 192–198.

[10]William Rowan Hamilton, *On Symbolical Geometry*, p2. Available from maths.tcd.ie/pub/HistMath/People/Hamilton/

[11]Otto Blumenthal, *Lebensgeschichte,* Berlin, 1935 in David Hilbert, "Gesammelte Abhandlungen", p403.

[12]Set $\grave{\boldsymbol{a}} = a\boldsymbol{a}$, then $\grave{\boldsymbol{a}} = \grave{a}◂\boldsymbol{a}$ in parallel with $\mathbf{a} = \mathrm{a}◂\boldsymbol{a}$. In this *uniform notation*, $\grave{\boldsymbol{a}}$ denotes *both* magnitude and location, as bold generic $\mathbf{a}$ does. Similarly, non-bold $\grave{a}$ is the *free part* of bold $\grave{\boldsymbol{a}}$, namely its weight a.

[13]Grass.1845 p289. Written by request for clarification from his editor.

[14]Altho a bound trivector has extent {**4**}, a free trivector has extent {3}, meaning extent {**3**, **3**}-with-zero-magnitude.

[15]Clifford was working on a screw algebra he called *biquaternions* when he encountered Grassmann's ideas; and he realized Grassmann could unify his screw algebra. This is clear form the main subsection of Clifford's previously cited "Applications of Grassmann's Extensive Algebra"[2], which is titled "*On the Relation of Grassmann's Method to Quaternions and Biquaternions; and on the Generalization of these Systems*". See also Clif.1873 p181, "Preliminary Sketch of Biquaternions".

[16]In the full Geometric Algebra zero cannot be a scalar or any other kind of number. See Harp. "Speaking of Space" p45.

[17]Dors.2007 p246.

[18]Hence, you can import the conformal model into Grassmann's language of imaginable space by augmenting that language with an extra dimension having Minkowski metric. This would require *explicit* projection back into imaginable space, which is certainly more expressive than the *implicit* projection of the conformal model, but more cumbersome when imaginable space is the only space of interest. But in that narrow case, why even bother with the conformal ploy?—Grassmann's origin-in-the-formalities already articulates imaginable space directly, transparently, efficiently.

[19]Hest.2001 *Unified Algebraic Framework for Classical Geometry* (UAFCG.html). I am guessing that Grassmann would have been as appalled by the conformal model as he was by Hamilton's vector algebra: it's not *his baby*, and it demonstrates for the n^{th} time that we still haven't understood the bound part of *his baby*.

[20]Alfred North Whitehead, *Universal Algebra*, Cambridge, 1898, p516. Whitehead called extracting a free part the "operation of taking the vector". He achieved it by setting the free ceiling, the unit trivector, equal to scalar one, which introduces a tacit context that precludes moving to other dimensions. Even worse, it obscures important dimensional distinctions. "Vector" had its Latin meaning, "*carrier*", for Whitehead; and because free things are able to *carry* bound things of the same extent, *vector* became synonymous with *free* for him, which makes for perplexing reading.