Unidad 5. Cuadraturas de Gauss

Polinomios ortogonales [1]

$$\int_a^b w(x)\varphi_m(x)\varphi_n(x)dx = 0, \quad m \neq n$$

Lista de polinomios ortogonales más utilizados

Name	Symbol	a	b	$w(x)$	$\int_a^b w(x)\left[\varphi_n(x)\right]^2 dx$
Legendre	$p_n(x)$	-1	1	1	$2/(2n+1)$
Chebyshev	$T_n(x)$	-1	1	$(1-x^2)^{-1/2}$	$\pi/2 \quad (n>0)$
Laguerre	$L_n(x)$	0	∞	e^{-x}	1
Hermite	$H_n(x)$	$-\infty$	∞	e^{-x^2}	$\sqrt{\pi}2^n n!$

Table 6.1. Classical orthogonal polynomials

Contrucción del conjunto de polinomios utilizando dos de ellos y la tabla de más abajo

Orthogonal polynomials obey recurrence relations of the form

$$a_n\varphi_{n+1}(x) = (b_n + c_n x)\varphi_n(x) - d_n\varphi_{n-1}(x)$$

Name	$\varphi_0(x)$	$\varphi_1(x)$	a_n	b_n	c_n	d_n
Legendre	1	x	$n+1$	0	$2n+1$	n
Chebyshev	1	x	1	0	2	1
Laguerre	1	$1-x$	$n+1$	$2n+1$	-1	n
Hermite	1	$2x$	1	0	2	2

Relaciones entre los polinomios y sus derivadas

$$p_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}\left[\left(1-x^2\right)^n\right]$$

$$T_n(x) = \cos(n\cos^{-1}x), \quad n>0$$

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}(e^{-x^2})$$

$$(1-x^2)\,p_n'(x) = n\left[-xp_n(x) + p_{n-1}(x)\right]$$

$$(1-x^2)\,T_n'(x) = n\left[-xT_n(x) + nT_{n-1}(x)\right]$$

$$x\,L_n'(x) = n\left[L_n(x) - L_{n-1}(x)\right]$$

$$H_n'(x) = 2n\,H_{n-1}(x)$$

Integración por cuadratura de Gauss

$$\int_a^b w(x)\,P_m(x)dx = \sum_{i=0}^{n} A_i\,P_m(x_i), \quad m \le 2n+1$$

The truncation error in Gaussian quadrature

$$E = \int_a^b w(x)f(x)dx - \sum_{i=0}^{n} A_i f(x_i)$$

has the form $E = K(n)f^{(2n+2)}(c)$, where $a < c < b$ (the value of c is unknown; only its bounds are given). The expression for $K(n)$ depends on the particular quadrature being used. If the derivatives of $f(x)$ can be evaluated, the error formulas are useful in estimating the error bounds.

Los puntos nodales xi (hasta n = 5) y abscisas Ai se dan en tabla.
Se recomienda revisar la siguiente bibliografía para más detalle:

1) Siam. Germund Dahlquist, Åke Björck . Numerical Methods in Scientific Computing: Volume 1 (2007).
2) Abramowitz, M. and Stegun, I.A, Handbook of Mathematical Functions, Dover Publications, (1965)
3) Stroud, A.H.and Secrest, D., Gaussian Quadrature Formulas, Prentice-Hall, 1966.
4) W.H. Press et al, Numerical Recipes in Fortran 90, Cambridge University Press, 1996.

Cuadratura de Gauss-Legendre

$$\int_{-1}^{1} f(\xi)d\xi \approx \sum_{i=0}^{n} A_i f(\xi_i)$$

$$E = \frac{2^{2n+3}\left[(n+1)!\right]^4}{(2n+3)\left[(2n+2)!\right]^3} f^{(2n+2)}(c), \quad -1 < c < 1$$

$\pm\xi_i$	A_i	$\pm\xi_i$	A_i
$n=1$		$n=4$	
0.577 350	1.000 000	0.000 000	0.568 889
$n=2$		0.538 469	0.478 629
0.000 000	0.888 889	0.906 180	0.236 927
0.774 597	0.555 556	$n=5$	
$n=3$		0.238 619	0.467 914
0.339 981	0.652 145	0.661 209	0.360 762
0.861 136	0.347 855	0.932 470	0.171 324

Table 6.3. Nodes and weights for Gauss–Legendre quadrature.

Se puede mapear el problema al intervalo [a,b]

Cuadratura de Gauss-Chebyshev

$$\int_{-1}^{1} \left(1-x^2\right)^{-1/2} f(x)\,dx \approx \frac{\pi}{n+1}\sum_{i=0}^{n} f(x_i)$$

Note that all the weights are equal: $A_i = \pi/(n+1)$. The abscissas of the nodes, which are symmetric about $x=0$, are given by

$$x_i = \cos\frac{(2i+1)\pi}{2n+2} \tag{6.32}$$

$$E = \frac{2\pi}{2^{2n+2}(2n+2)!} f^{(2n+2)}(c), \quad -1 < c < 1$$

Cuadratura de Gauss-Laguerre

$$\int_{0}^{\infty} e^{-x} f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)$$

$$E = \frac{\left[(n+1)!\right]^2}{(2n+2)!} f^{(2n+2)}(c), \quad 0 < c < \infty$$

x_i	A_i	x_i	A_i
$n=1$		$n=4$	
0.585 786	0.853 554	0.263 560	0.521 756
3.414 214	0.146 447	1.413 403	0.398 667
$n=2$		3.596 426	(−1)0.759 424
0.415 775	0.711 093	7.085 810	(−2)0.361 175
2.294 280	0.278 517	12.640 801	(−4)0.233 670
6.289 945	(−1)0.103 892	$n=5$	
$n=3$		0.222 847	0.458 964
0.322 548	0.603 154	1.188 932	0.417 000
1.745 761	0.357 418	2.992 736	0.113 373
4.536 620	(−1)0.388 791	5.775 144	(−1)0.103 992
9.395 071	(−3)0.539 295	9.837 467	(−3)0.261 017
		15.982 874	(−6)0.898 548

Table 6.4. Nodes and weights for Gauss–Laguerre quadrature (Multiply numbers by 10^k, where k is given in parentheses.)

Cuadratura de Gauss-Hermite

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)dx \approx \sum_{i=0}^{n} A_i f(x_i) \qquad ($$

$\pm x_i$	A_i	$\pm x_i$	A_i
$n=1$		$n=4$	
0.707 107	0.886 227	0.000 000	0.945 308
$n=2$		0.958 572	0.393 619
0.000 000	1.181 636	2.020 183	(−1) 0.199 532
1.224745	0.295 409	$n=5$	
$n=3$		0.436 077	0.724 629
0.524 648	0.804 914	1.335 849	0.157 067
1.650 680	(−1)0.813 128	2.350 605	(−2)0.453 001

Table 6.5. Nodes and weights for Gauss–Hermite quadrature (Multiply numbers by 10^k, where k is given in parentheses.)

$$E = \frac{\sqrt{\pi}\,(n+1)!}{2^2(2n+2)!} f^{(2n+2)}(c), \quad 0 < c < \infty$$

Cuadratura de Gauss con singularidad logarítmica

$$\int_0^1 f(x)\ln(x)\,dx \approx -\sum_{i=0}^{n} A_i f(x_i)$$

$$E = \frac{k(n)}{(2n+1)!} f^{(2n+1)}(c), \quad 0 < c < 1$$

where $k(1) = 0.00\,285$, $k(2) = 0.000\,17$, and $k(3) = 0.000\,01$.

x_i	A_i	x_i	A_i
$n = 1$		$n = 4$	
0.112 009	0.718 539	(−1)0.291 345	0.297 893
0.602 277	0.281 461	0.173 977	0.349 776
$n = 2$		0.411 703	0.234 488
(−1)0.638 907	0.513 405	0.677314	(−1)0.989 305
0.368 997	0.391 980	0.894 771	(−1)0.189 116
0.766 880	(−1)0.946 154	$n = 5$	
$n = 3$		(−1)0.216 344	0.238 764
(−1)0.414 485	0.383 464	0.129 583	0.308 287
0.245 275	0.386 875	0.314 020	0.245 317
0.556 165	0.190 435	0.538 657	0.142 009
0.848 982	(−1)0.392 255	0.756 916	(−1)0.554 546
		0.922 669	(−1)0.101 690

Table 6.6. Nodes and weights for quadrature with logarithmic singularily (Multiply numbers by 10^k, where k is given in parentheses.)

Referencias

[1] Jaan Kiusalaas. Numerical Methods in Engineering with Python 3 3rd Edition (2013). Cambridge University Press. ISBN-10: 1107033853. ISBN-13: 978-1107033856