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MAT334 Assignment #1

1. Let $z=-1+\sqrt{3}i$, $w=-2-2i$
(a). Find z, w in polar form.
(b). Find $\left(\frac{w}{z}\right)^{18}$. Express your final answer in $x+iy$ form.
(c). Find $z^{10}-2w^3$. Express your final answer in $x+iy$ form.

Solution:

(a). $z=-1+\sqrt{3}i$

$r=\sqrt{(-1)^2+(\sqrt{3})^2}=2$

$$\left.\begin{array}{l} 2\cos\theta=-1 \\ 2\sin\theta=\sqrt{3} \end{array}\right\} \Rightarrow \begin{array}{l} \cos\theta=-\frac{1}{2} \\ \sin\theta=\frac{\sqrt{3}}{2} \end{array} \qquad \theta=\frac{2}{3}\pi$$

so $z=-1+\sqrt{3}i=2e^{i\frac{2}{3}\pi}$

$w=-2-2i$

$r=\sqrt{(-2)^2+(-2)^2}=2\sqrt{2}$

$$\left.\begin{array}{l} 2\sqrt{2}\cos\theta=-2 \Rightarrow \cos\theta=-\frac{\sqrt{2}}{2} \\ 2\sqrt{2}\sin\theta=-2 \Rightarrow \sin\theta=-\frac{\sqrt{2}}{2} \end{array}\right\} \Rightarrow \theta=\frac{5\pi}{4}$$

So $w=-2-2i=2\sqrt{2}e^{i\frac{5}{4}\pi}$

(b). $\left(\frac{w}{z}\right)^{18}=\left(\frac{2\sqrt{2}e^{i\frac{5}{4}\pi}}{2e^{i\frac{2}{3}\pi}}\right)^{18}=\left(\sqrt{2}e^{i\frac{7}{12}\pi}\right)^{18}=(\sqrt{2})^{18}e^{i18\cdot\frac{7}{12}\pi}$

$=2^9e^{i\frac{21}{2}\pi}$

$=2^9e^{i\frac{1}{2}\pi}$

$=2^9\left(\cos(\tfrac{1}{2}\pi)+i\sin(\tfrac{1}{2}\pi)\right)$

$=2^9(0+i)$

$=512i$

(c). ~~$z^{10}-2w^3=2^{10}e^{i10\frac{2}{3}\pi}$~~

$z^{10}=(2e^{i\frac{2}{3}\pi})^{10}=2^{10}e^{i\frac{20}{3}\pi}=2^{10}e^{i\frac{2}{3}\pi}=2^{10}(\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi)$

$=2^{10}(-\frac{1}{2}+i\frac{\sqrt{3}}{2})$

$=-2^9+i\sqrt{3}\cdot2^9$

$-2w^3=-2\cdot(2\sqrt{2})^3e^{i\frac{15}{4}\pi}=-32\sqrt{2}\,e^{i\frac{7}{4}\pi}$

$=-32\sqrt{2}(\cos\frac{7}{4}\pi+i\sin\frac{7}{4}\pi)$

$=-32\sqrt{2}(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})$

$=-32+32i$

So $z^{10}-2w^3=-2^9+i\sqrt{3}\cdot2^9-32+32i=(-512-32)+(512\sqrt{3}+32)i=-544+i(512\sqrt{3}+32)$

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MAT334 Assignment#1

2. 1.1.18

Prove the identity $1+z+z^2+\cdots+z^n=\frac{1-z^{n+1}}{1-z}$ ~~with~~ valid for all z, $z\neq 1$.

Proof: Proof by induction.

Base case, when $n=1$, $1+z=\frac{1-z^2}{1-z}=\frac{(1+z)(1-z)}{1-z}=1+z$, done.

Inductive step, suppose, this is true for $n=k$.

$$1+z+\cdots+z^k=\frac{1-z^{k+1}}{1-z}$$

Want to show it's true for $n=k+1$.

$$1+z+\cdots+z^k+z^{k+1}=\frac{1-z^{k+1}}{1-z}+z^{k+1}=\frac{1-z^{k+1}+z^{k+1}-z^{k+2}}{1-z}$$

$$=\frac{1-z^{(k+1)+1}}{1-z}$$

So this is also true for $n=k+1$

Hence this is true for any $n\in\mathbb{Z}^+$, as $z\neq 1$.

3. 1.2.4

$\mathrm{Im}(2iz)=7$

Solution: suppose $z=x+iy$ where $x,y\in\mathbb{R}$.

$2iz=2i(x+iy)=2ix-2y$

$\mathrm{Im}(2iz)=\mathrm{Im}(-2y+2xi)=7$

So $2x=7$

$x=\frac{7}{2}$.

Therefore, the locus of z is the line $\mathrm{Re}(z)=\frac{7}{2}$, which is parallel to the imaginary line.

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4. 1.2.6

$|z-i| = \operatorname{Re} z$

Solution: Suppose $z = x + iy$.

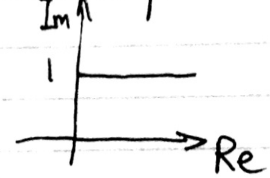
$$|z-i| = |x + (y-1)i| = \sqrt{x^2 + (y-1)^2} = x = \operatorname{Re} z$$

$$x^2 + (y-1)^2 = x^2$$

so $(y-1)^2 = 0$ and since $\operatorname{Re} z \geq 0$

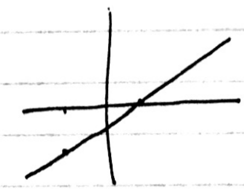
$y = 1$

Hence the locus of z is line $\operatorname{Im}(z) = 1$, which is parallel to the real line, as the graph shows:



5. 1.2.12

The straight line through 1 and $-1-i$.



Solution: Suppose the line is

$$\operatorname{Re}((m+i)z + b) = 0$$

plug in $z=1$ and $z = -1-i$

$$\operatorname{Re}((m+i) + b) = 0$$

$$\operatorname{Re}((m+i)(-1-i) + b) = 0$$

So $m + b = 0$

$$\operatorname{Re}(-m - i - im + 1 + b) = \operatorname{Re}(1 + b - m + i(-1-m)) = 0$$

$$1 + b - m = 0$$

so $b = -\frac{1}{2}$, $m = \frac{1}{2}$

The line is $\operatorname{Re}((\frac{1}{2} + i)z - \frac{1}{2}) = 0$

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6. 1.2.24

$(z+1)^4=(1-i)$

Solution: This is equivalent to solve

$$z+1=(1-i)^{\frac{1}{4}}$$

$$1-i=\sqrt{2}\left(\cos\left(\tfrac{7}{4}\pi\right)+i\sin\left(\tfrac{7}{4}\pi\right)\right)$$

$$(1-i)^{\frac{1}{4}}=(\sqrt{2})^{\frac{1}{4}}\left(\cos\tfrac{7}{16}\pi+i\sin\tfrac{7}{16}\pi\right)$$

So one of the roots is located with polar angle $\frac{7}{16}\pi$, and the others have ~~pl~~ polar angles

$\frac{7}{16}\pi+\frac{\pi}{2}$, $\frac{7}{16}\pi+\pi$, $\frac{7}{16}\pi+\frac{3}{2}\pi$, respectively.

so $z+1=(1-i)^{\frac{1}{4}}=2^{\frac{1}{8}}\left(\cos\frac{7}{16}\pi+i\sin\frac{7}{16}\pi\right)$

or $2^{\frac{1}{8}}\left(\cos\frac{15}{16}\pi+i\sin\frac{15}{16}\pi\right)$

or $2^{\frac{1}{8}}\left(\cos\frac{23}{16}\pi+i\sin\frac{23}{16}\pi\right)$

or $2^{\frac{1}{8}}\left(\cos\frac{31}{16}\pi+i\sin\frac{31}{16}\pi\right)$

then $z=2^{\frac{1}{8}}\left(\cos\frac{7}{16}\pi+i\sin\frac{7}{16}\pi\right)-1$

or $2^{\frac{1}{8}}\left(\cos\frac{15}{16}\pi+i\sin\frac{15}{16}\pi\right)-1$

or $2^{\frac{1}{8}}\left(\cos\frac{23}{16}\pi+i\sin\frac{23}{16}\pi\right)-1$

or $2^{\frac{1}{8}}\left(\cos\frac{31}{16}\pi+i\sin\frac{31}{16}\pi\right)-1$

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7. 1.3.2

$B = \{z : |z| < 1 \text{ or } |z-3| \le 1\}$

Solution:

(a). $\text{int } B = \{z : |z| < 1 \text{ or } |z-3| < 1\}$

$\partial B = \{z : |z-3| = 1\} \cup \{z : |z| = 1\}$

(b). B is neither open nor closed.

(c). int B is disjoint, i.e. not connected.

8. 1.3.10

Describe the set of points z^2 as z varies over the second quadrant: $\{z = x+iy : x<0 \text{ and } y>0\}$. Show that this is an open, connected set. (Hint: use the polar representation of z).

Solution: Write z in polar form

$z = |z| e^{i\theta}$

then $z^2 = |z|^2 e^{2i\theta}$

Since z is in the second quadrant, so $\theta \in (\frac{\pi}{2}, \pi)$

then $2\theta \in (\pi, 2\pi)$

i.e. $\{z^2\}$ is the lower ~~two~~ half plane without x-axis, the 3rd & 4th quadrant.

- Now, we can prove "open".

  $\forall q = x' + iy' \in \{z^2\}$ where $y', x' \in \mathbb{R}$ and $y' < 0$

  Let $\varepsilon = -\frac{y'}{2} > 0$.

  $B_\varepsilon(q) \subseteq \{z^2\}$, so it's open $\in$

  (or we can show each point in $\{z^2\}$ is an interior point, just like what we do in ~~ex~~ example 2 in the textbook).

- Now, show "connectedness"

  $\forall q_1, q_2 \in \{z^2\}$, the line segment joining q_1, q_2 lies entirely in $\{z^2\}$, since $\{z^2\}$ is a half-plane.

  i.e. $(1-t)q_1 + tq_2 \in \{z^2\}$ where $t \in [0,1]$.