Tuesday Tutorial at ES

MAT334 HW2 Rui Qiu #999292509

§ 1.6 #2 #12

§ 2.1 #10 #14 #16 #18 #20(b)

§ 2.2 #2 ~~#6~~10 #18

1. $\int_\gamma e^z dz$, where γ is the line ~~integ~~ segment from 0 to z_0

Solution: $\cancel{\int_\gamma e^z dz = e^z \Big|_0^{z_0} = e^{z_0} - e^0 = e^{z_0} - 1}$

1. Parametrize γ:

$\gamma(t) = t(z_0) + (1-t)\cdot 0 = tz_0$

where $t \in [0,1]$.

$\gamma'(t) = z_0$.

$$\int_\gamma e^z dt = \int_0^{z_0} e^{tz_0} z_0 dt$$

$$= e^{z_0 t} \Big|_0^{z_0}$$

$$= e^{z_0^2} - 1$$

$\cancel{= e^{z_0}}$

2. Use the result of (the extension of) Example 11 to compute the following integrals:

(a) $\int_\gamma (z^3 - 6z^2 + 4) dz$, γ is curve joining $-1+i$ to 1

(b). $\int_\gamma (z^4 + z^2) dz$, γ is curve joining $-i$ to $2+i$

Solution: (a). $\int_\gamma (z^3 - 6z^2 + 4) dz = \int_\gamma z^3 dz - \int_\gamma 6z^2 dz + \int_\gamma 4 dz$

$$= \frac{1}{3+1}\left(1^{3+1} - (-1+i)^{3+1}\right) - 6 \cdot \frac{1}{2+1}\left(1^{2+1} - (-1+i)^{2+1}\right) + 4(1 - (-1+i))$$

$$= \frac{1}{4}(1 - (-4)) - 2(1 - (2i+2)) + 4(2-i)$$

$$= \frac{5}{4} + 2 + 4i + 8 - 4i$$

$$= \frac{45}{4}$$

(b). $\int_\gamma (z^4 + z^2) dz = \int_\gamma z^4 dz + \int_\gamma z^2 dz$

$$= \frac{1}{5}\left[(2+i)^5 - (-i)^5\right] + \frac{1}{3}\left[(2+i)^3 - (-i)^3\right]$$

$$= \frac{1}{5}(\cancel{}^{-38+41i} + i) + \frac{1}{3}[\cancel{}_{2+11i} - i]$$

$\cancel{= 41i + 2 + \frac{1}{5}i + \frac{4}{3}i - \frac{10}{3}}$

$\cancel{= \frac{142}{15}i - \frac{4}{3}}$

$$= \frac{1}{5}[-38 + 42i] + \frac{1}{3}[2 + 10i]$$

$$= \frac{-104}{15} + \frac{176}{15} i$$

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3. Find analytic function F with $F'=f$.

$f(z)=\sin z\cos z$

Solution: $F(z)=\int f(z)\,dz=\int \sin z\cos z\,dz=\frac{1}{2}\int 2\sin z\cos z\,dz$

$=\frac{1}{2}\int \sin 2z\,dz$

$=\frac{1}{2}\left(-\frac{1}{2}\cos 2z\right)+C$

$=-\frac{1}{4}\cos 2z+C$

So $F(z)=-\frac{1}{4}\cos 2z+C$ where C is a constant. (i.e. $\underbrace{-\frac{1}{4}\cos 2z=\frac{1}{2}\sin^2 z}_{\text{can be written in other forms.}}$)

4. Let $P(z)=A(z-z_1)\cdots(z-z_n)$, where A and $z_1,\ldots,z_n$ are complex #s and $A\neq 0$.

Show that

$$\frac{P'(z)}{P(z)}=\sum_{j=1}^{n}\frac{1}{z-z_j}\qquad z\neq z_1,\ldots,z_n$$

Proof: Let $x_j=z-z_j$.

$$\frac{P'(z)}{P(z)}=\frac{A(x_1\cdots x_n)'}{A(x_1\cdots x_n)}=\frac{x_1'(x_2\cdots x_n)+x_1(x_2,\cdots x_n)'}{x_1\cdots x_n}$$

$$=\frac{x_1'}{x_1}+\frac{x_2'(x_3\cdots x_n)+x_2(x_3\cdots x_n)'}{x_2\cdots x_n}$$

$$=\frac{x_1'}{x_1}+\frac{x_2'}{x_2}+\frac{(x_3\cdots x_n)'}{x_3\cdots x_n}$$

$$=\cdots$$

$$=\frac{x_1'}{x_1}+\frac{x_2'}{x_2}+\cdots+\frac{x_n'}{x_n}$$

$$=\frac{1}{z-z_1}+\frac{1}{z-z_2}+\cdots+\frac{1}{z-z_n}$$

$$=\sum_{j=1}^{n}\frac{1}{z-z_j}$$

$\blacksquare$

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5. Find the derivative of the linear fractional transformation.

$T(z) = \frac{az+b}{cz+d}$, $ad \neq bc$.

In what way does the condition $ad-bc \neq 0$ enter?

Conclude that $T'(z)$ is never zero, $z \neq \frac{-d}{c}$

Solution: $T(z) = \frac{az+b}{cz+d} = \frac{az + a\frac{d}{c} + (b - \frac{ad}{c})}{cz+d}$

$= \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$

$= \frac{a}{c} + \frac{bc-ad}{c^2 z + cd}$

$$T'(z) = \left(\frac{az+b}{cz+d}\right)' = \frac{a(cz+d) - (az+b)\cdot c}{(cz+d)^2} = \frac{acz + ad - acz - bc}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

~~since $ad-bc \neq 0$, then $T'(z) \neq 0$, and $z \neq -\frac{d}{c}$ (o.w. the denominator is zero).~~ If $ad = bc$, then $T'(z) = 0$ for sure, $T(z)$ is constant.

So $ad \neq bc$, i.e, $T'(z)$ is not zero.

In the meanwhile, $z \neq \frac{-d}{c}$ since if so, the demoninator is zero and $T(z)$ DNE.

6. Show $h(z) = \bar{z}$ is not analytic on any domain.

Proof: $h(z) = x - iy$ where $z = x + iy$, $x, y \in \mathbb{R}$.

so we note that $u = x$, $v = -y$

By Cauchy-Riemann equations.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Check: $\frac{\partial u}{\partial x} = \frac{\partial x}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$ for any domain.

Hence the function is not analytic on any domain.

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7. Let $f=u+iv$ be analytic. find v given u.

(b). $u=\frac{x}{x^2+y^2}$

$$\frac{\partial u}{\partial x}=\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y}=\frac{0-x\cdot 2y}{(x^2+y^2)^2}=\frac{-2xy}{(x^2+y^2)^2}=-\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}\Rightarrow v=\int\frac{y^2-x^2}{(x^2+y^2)^2}\,dy=\frac{y}{-(x^2+y^2)}+C$$

$$\frac{\partial v}{\partial x}=\frac{2xy}{(x^2+y^2)^2}\Rightarrow v=\int\frac{2xy}{(x^2+y^2)^2}\,dx=\frac{y}{-(x^2+y^2)}+C$$

Hence $v=\frac{y}{-(x^2+y^2)}+C$, where C is a constant.

8. Find radius of convergence of (a). $\sum_{k=0}^{\infty}\frac{(k!)^2}{(2k)!}(z-2)^k$

Solution: Check $\lim_{k\to\infty}\left|\frac{((k+1)!)^2/(2(k+1))!}{(k!)^2/(2k)!}\right|$ $\lim_{k\to\infty}\left|\frac{\frac{1\cdot 2\cdots(k+1)\cdot 1\cdot 2\cdots(k+1)}{1\cdot 2\cdots(k+1)(k+2)\cdots(2k+2)}}{\frac{k!\,k!}{k!(k+1)\cdots 2k}}\right|$

$$=\lim_{k\to\infty}\left|\frac{(k+1)!}{(k+2)\cdots(2k+2)}\cdot\frac{(k+1)\cdots 2k}{k!}\right|$$

$$=\lim_{k\to\infty}\left|\frac{(k+1)^2}{(2k+1)(2k+2)}\right|$$

$\frac{k^2+2k+1}{4k^2+6k+2}$

since $(k+1)<(2k+1)$
$(k+1)<(2k+2)$

So $\lim_{k\to\infty}\left|\frac{(k+1)^2}{(2k+1)(2k+2)}\right|=\frac{1}{4}$ by L'Hopital Rule

$\frac{1}{R}=\frac{1}{4}$, so the radius of convergence is 4. i.e. $R=4$

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9. Find the power series about the origin ~~that~~ for $\frac{1+z}{1-z}$, $|z|<1$.

Solution: Since $\frac{1}{1-z} = 1+z+z^2+z^3+\cdots$. $|z|<1$. $\sum_{n=0}^{\infty} z^n = \frac{1(1-z^n)}{1-z}$

$$\frac{1+z}{1-z} = (1+z)(1+z+z^2+\cdots)$$

$$= 1+z+z^2+z^3+\cdots + z+z^2+z^3+\cdots$$

$$= 1+2(z+z^2+z^3+\cdots)$$

$$= \sum_{n=0}^{\infty} a_n z^n \quad \text{where } a_n = \begin{cases} 1 & \text{when } n=0 \\ 2 & \text{otherwise} \end{cases}$$

10. Find a "closed form" for (18). $\sum_{n=2}^{\infty} n(n-1) z^n$ (Divide by z^2)

Solution: Since $n(n-1)z^{n-2}$ is the 2nd derivative of z^n

and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

So $\sum_{n=2}^{\infty} n(n-1) z^n = z^2 \sum_{n=2}^{\infty} n(n-1) z^{n-2}$

Suppose $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

$f'(z) = \sum_{n=0}^{\infty} n z^{n-1}$

$f''(z) = \sum_{n=2}^{\infty} n(n-1) z^{n-2} = (-2)(1-z)^{-3}(-1) = \frac{2}{(1-z)^3}$

So $\sum_{n=2}^{\infty} n(n-1) z^n = z^2 \cdot \frac{2}{(1-z)^3} = \frac{2z^2}{(1-z)^3}$