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MAT246 HW9 Rui Qiu #999292509

P112 #5, 6, 11, 12, 18, 23, 24, 27

Q5. Prove that the set of all finite subsets of $\mathbb{Q}$ is countable.

?
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Proof Let S_k be the set of ~~the set~~ $\mathbb{Q}$ consisting of k elements.
Then $S = \bigcup_{k=1}^{\infty} S_k$.

Let $f_k = S_k \longrightarrow \mathbb{Q}^k$, given a set of k rational numbers $A = \{x_1 < x_2 < \cdots < x_k\}$ define $f_k(A) = (x_1, x_2, \cdots, x_k)$.

$|\{1\}|, |\{2\}| \le |\{3\}|$ f_k is one-to-one.

but

Thus $|S_k| \le |\mathbb{Q}^k| = |\mathbb{Q}|$ Which fact?

By the theorem from class, $|S| \le |\mathbb{Q}|$.

$|\{1,2\}| \not\le |\{3\}|$ And it's obvious that $|S| \ge |S_1| = |\mathbb{Q}|$ why

By Shroeder - Berenstein Theorem, $|S| = |\mathbb{Q}|$.

Now we only need to show $|\mathbb{Q}| = |\mathbb{N}|$. Fact from class?

Define ~~$\Phi: \mathbb{Q} \to \mathbb{Z}\times\mathbb{N}$~~ $\Phi: \mathbb{Q} \to \mathbb{N}$

~~$\forall \frac{p}{q} \in \mathbb{Q}$ $\Phi(\frac{p}{q}) = (p, q)$ where~~ $\frac{p}{q}$ is in canonical form. $p, q \in \mathbb{Z}$

~~The Φ is injective.~~

And obviously we order the $\frac{p}{q}$ list as this:

$\frac{0}{1}, \frac{1}{1}, \frac{-1}{1}, \frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}, \frac{1}{3}, \frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{-3}{1}, \frac{-3}{2}, \cdots$

It's clear that every rational number will appear in this list.

So it is injective.

Also, obviously Φ is surjective.

Therefore $|\mathbb{Q}| = |\mathbb{N}|$

Hence $|S| = |\mathbb{Q}| = |\mathbb{N}|$.

By the definition, S is countable.

Q6. What is the cardinality of the set of all functions from N to $\{1, 2\}$?

Solution: ~~Consider a subset S of N (an element of $P(N)$)~~
~~Now consider the function~~

$$\cancel{f_s(a) = \begin{cases} 2 & \text{if } a \in S \\ 1 & \text{if } a \notin S \end{cases}} \quad \text{for } a \in S$$

~~Then by Theorem 10.3.32 for any the set of N, the set of all characteristic function, namely $\{f_s(a) \mid a \in S\}$ has the cardinality as $P(N)$.~~

~~Hence $|\{f_s(a)=1\}| = |P(N)|$~~

~~Now we only need to show $|\{f_s(a)=1\}| = |\{f_s(a)\}|$~~

First we need to redefine a "characteristic function" by if S is a set and $S_0 \subset S$, then the characteristic function of S_0 as a subset of S would be the function with domain S defined by

$f(s) = 2$ if $s \in S_0$,
$f(s) = 1$ if $s \notin S_0$.

Then by the theorem 10.3.32,
for set N, the set of all "characteristic functions" with domain N has the same cardinality ~~of~~ as, $P(N)$.
\# why? ① each subset does have one "characteristic function".
② if two "characteristic functions" are equal, ~~their subsets~~ they must be the ~~same~~ "characteristic functions" of the same subset.
Thus the correspondence between the set of subsets

of N and characteristic functions with domain N is 1-1 & onto.

Therefore, the cardinality of the set of all functions from N to $\{1, 2\}$ is the same as the cardinality of $P(N)$. ✓
(Can also say R) ✓

Q11. Find the cardinality of the set $\{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{Q}\}$.

Solution: Say $S = \{ \cancel{\text{set}}\ (x,y) \mid x \in R, y \in Q \}$

$$
\begin{aligned}
\text{Then } |S| &= \cancel{|R| \times}\ |R \times Q| \\
&= |R| \times |Q| \\
&= |R| \times |N| \\
&\cancel{= c\aleph_0 \text{ where } c} \\
&= |P(N)| \times |N| \\
&= c\,\aleph_0
\end{aligned}
$$

Q12. What is the cardinality of the set of all numbers in the interval $[0, 1]$ which have decimal expansions with a finite number of non-zero digits.

Solution: Let S be the set of all numbers in the interval $[0, 1]$ which have decimal expansions with a finite number of non-zero digits.

We construct $f: S \to \mathbb{N}$ as follow:

$f(0.1) = 1$

$f(0) = 0$

$f(0.11) = 11$

$f(0.12) = 21$

$f(0.13) = 31$

$f(0.02) = 20$

$f(0.0875) = 5780$

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... (we reverse all the digits after decimal mark)

Now we consider $a, b \in [0, 1]$

say $f(a) = f(b)$

Then $a = b$ automatically since if ~~the~~ two reverse form" of numbers are equal, then the two numbers are equal.

Hence f is injective.

And obviously every number in $\mathbb{N}$ can have a "decimal reverse form" in S.

Hence f is surjective.

Therefore $|S| = |\mathbb{N}|$.

Q02. Find the cardinality of each of the following sets.
(a). The set of roots of polynomials with constructible coefficients.
(c). The set of all points (x, y) in the plane such that x is constructible ~~any~~ and y is irrational.

Solution:

(a). Let S be such a set.

maybe I should explain it in details in the last page

① It's obvious to see that $|S| \geq |N|$

② We have proved in class that a root of a polynomial with constructible ~~numbers~~ coefficients is also a root of a polynomial with rational coefficients.
Hence all the elements in S are algebraic.
then $|S| \leq |N|$.
By S-B theorem, $|S| = |N|$.

(b). Say the set of all such points is
$S = \{(x, y) \mid x \in C, y \text{ is irrational}\}$

since $|C| = |N|$, $|R \setminus Q| = |R \times R| = |R|$

$|S| = |C \times (R \setminus Q)| = |C| \times |R \setminus Q| = |N| \times |R| = |N| \cdot |R|$

For Q20:

Since we proved that $|C| = |N| < |R|$

So in this problem, we are drawing a line in a plane which has real coordinates.

Hence, no matter how, the line, which is continuous and infinitely long at both ends will pass a point or points with Coordinates that are not constructible.

For example, for a line $y = kx + b$, $k, b \in R$

~~If k≠0~~, it will pass $(\pi, k\pi + b)$

~~If k=0, it will pass (π, b)~~

We also proved that π is not constructible (if it is then we can square a circle).

Then every line in the plane will pass points which are not constructible somehow. Such lines don't exist.

For Q22 (a):

~~Why a root of a polynomial with constructible coefficients is also a root of a polynomial with rational coefficients?~~

~~We'll prove it by induction in k:~~

~~∃ a tower of fields:~~

~~$Q = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$, $F_i = F_{i-1}(\sqrt{r_i})$~~

~~① base of induction~~

By theorem, let x be a root of a polynomial with constructible ~~coef~~ coefficients then x is algebraic.

So $|S| = |A| = |N|$ (since the cardinality of algebraic numbers is equal to that of natural numbers, or we say the set of algebraic numbers is countable, we proved this in ~~Ch~~ Chapter 10).

~~Q18. Let t be a transcendental number. Prove that t cannot be a root of any equation of the form $x^2+ax+b=0$ where a and b are constructible numbers.~~

~~Proof.~~

Q19. What is the cardinality of the set of all constructible points in the plane?

Solution: First we set the set of all constructible numbers C.

And we set the set of all constructible points in the plane $S=\{(x,y) \mid x\in C, y\in C\}$

Then $|S|=|C\times C|=|C|\cdot|C|$

Now we need $|C|$.

By theorem 12.2.11, C is a number field.

Then $Q=F_0\subset F_1\subset F_2\subset\cdots\subset F_n=C$

$r_1\notin Q \leftarrow F_1=Q(\sqrt{r_1}) \Rightarrow |F_1|=|Q\times Q|=|Q|\times|Q|$
$=|N|\times|N|$
$=|N\cdot N|$
$=|N|$

$r_2\notin F_1 \quad F_2=F_1(\sqrt{r_2}) \Rightarrow |F_2|=|F_1\times F_1|=|N|\times|N|=|N|$

...

$r_n\notin F_{n-1} \quad F_n=F_{n-1}(\sqrt{r_n}) \Rightarrow |F_n|=|F_{n-1}\times F_{n-1}|=|N|\times|N|=|N|$

$|C|=|F_n|=|N|$

Therefore $|S|=|C|\cdot|C|=|N|\cdot|N|=|N|$.

(For Q20, I have better way to solve this in the last page)

Q20 Is there a line in the plane such that every point on it is contructible?

Solution: First we can devide the lines in a plane into two types, one is ~~parr~~ parallel to y-axis, the other is not parallel to y-axis.

① If line l is ~~parrel~~ parallel to y-axis then the points on l can be described as a set $S_1 = \{(a, y) \mid a \in R$ and a is constant, $y \in R\}$
Say ~~...~~, $y = \sqrt[3]{2}$. (Since $\sqrt[3]{2}$ is in R, l is continuous)

Since we cannot double a unit cube with length 1.
thus $\sqrt[3]{2}$ is not constructible.
So the point $(a, \sqrt[3]{2})$ is not constructible.
Therefore we cannot draw a $\overbrace{\text{line}}^{\text{type ①}}$ which consists of only constructible numbers.

② If line l is not parallel to y-axis
then the set
$S_2 = \{(x, b) \mid x \in R, b \in R$ and b is constant$\}$
Say $x = \sqrt[3]{2}$ (since $\sqrt[3]{2}$ is in R. l is continuous)

Similarly, $x = \sqrt[3]{2}$ is not constructible
So the point $(\sqrt[3]{2}, b)$ is not constructible
Therefore we cannot draw a $\overbrace{\text{line}}^{\text{type ②}}$ which concists of only constructible numbers.

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Q3 Let F be the field consisting real numbers of the form $p+q\sqrt{2+\sqrt{2}}$ where p, q are of the form $a+b\sqrt{2}$ with a, b rational. Represent $\dfrac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}}$ in this form.

Solution.
$$\frac{1+\sqrt{2+\sqrt{2}}}{2-3\sqrt{2+\sqrt{2}}} = \frac{\left(1+\sqrt{2+\sqrt{2}}\right)\left(2+3\sqrt{2+\sqrt{2}}\right)}{\left(2-3\sqrt{2+\sqrt{2}}\right)\left(2+3\sqrt{2+\sqrt{2}}\right)}$$

$$= \frac{2+3(2+\sqrt{2})+5\sqrt{2+\sqrt{2}}}{4-9(2+\sqrt{2})}$$

$$= \frac{(8+3\sqrt{2})+5\sqrt{2+\sqrt{2}}}{-14-9\sqrt{2}} \cdot \frac{-14+9\sqrt{2}}{-14+9\sqrt{2}}$$

$$= \frac{(8+3\sqrt{2})(-14+9\sqrt{2})+(-70+45\sqrt{2})\sqrt{2+\sqrt{2}}}{34}$$

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$$= \frac{-58+30\sqrt{2}+(-70+45\sqrt{2})\sqrt{2+\sqrt{2}}}{34}$$

$$= \left(-\frac{29}{17}+\frac{15}{17}\sqrt{2}\right)+\left(-\frac{35}{17}+\frac{45}{34}\sqrt{2}\right)\sqrt{2+\sqrt{2}}$$

with $p = -\frac{29}{17}+\frac{15}{17}\sqrt{2}$, $q = -\frac{35}{17}+\frac{45}{34}\sqrt{2}$ ✓

Problems on textbook.
P165.
#17, 19, 20, 22a, c.

Q17. Let t be a transcendental number. Prove that $\{a+bt \mid a, b \in \mathbb{Q}\}$ is not a number field.

Proof. Assume $\{a+bt \mid a, b \in \mathbb{Q}\}$ is a number field. Then for $a, b, c, d \in \mathbb{Q}$, t is a transcendental number.

$$(a+bt)(c+dt)$$
$$= ac + (bc+ad)t + bdt^2$$

① If $bdt^2 \in \mathbb{Q}$, then $ac + bdt^2 = x$, $x \in \mathbb{Q}$

$$t = \pm\sqrt{\frac{x-ac}{bd}}, \quad x, ac, b, d \in \mathbb{Q}$$

contradicts the definition of transcendental number

② If $bdt^2 \notin \mathbb{Q}$, then $(bc+ad)t + bdt^2$
$= (bc+ad+bdt)t$

then $bc+ad+bdt = y \in \mathbb{Q}$

$$t = \frac{y-bc-ad}{bd}, \quad y, a, b, c, d \in \mathbb{Q}$$

also contradicts the definition of transcendental number.

Therefore the assumption is wrong, hence $\{a+bt \mid a, b \in \mathbb{Q}\}$ is not a number field. ∎

Then we want to prove $F_3 = F_2(\sqrt[4]{8})$ is a number field.
Similarly, we now suppose $a, b, c, d \in F_2 = F_1(\sqrt{2})$
Obviously, $0, 1 \in F_3$ and we can prove

for $a + b\sqrt[4]{8} \in F_3$

$c + d\sqrt[4]{8} \in F_3$

$(a+c) + (b+d)\sqrt[4]{8} \in F_3$

$(a-c) + (b-d)\sqrt[4]{8} \in F_3$

and

$$(a + b\sqrt[4]{8})(c + d\sqrt[4]{8})$$
$$= ac + bd \cdot 2\sqrt{2} + (ad+bc)\sqrt[4]{8}$$
$$= (ac + 2\sqrt{2}\,bd) + (ad+bc)\sqrt[4]{8}$$

since $a, b, c, d \in F_2 = F_1(\sqrt{2})$, F_2 is a number field, $2\sqrt{2} \in F_2$.
So $ac + 2\sqrt{2}\,bd \in F_2$ and $ad + bc \in F_2$.
Hence $(a+b\sqrt[4]{8})(c+d\sqrt[4]{8}) \in F_3$

and $$\frac{a+b\sqrt[4]{8}}{c+d\sqrt[4]{8}} = \frac{(a+b\sqrt[4]{8})(c-d\sqrt[4]{8})}{c^2 - d^2 \cdot 2\sqrt{2}}$$

(Note: $c + d\sqrt[4]{8} \neq 0$)

$$= \frac{ac + (bc-ad)\sqrt[4]{8} - bd \cdot 2\sqrt{2}}{c^2 - d^2 \cdot 2\sqrt{2}}$$

$$= \frac{ac - bd \cdot 2\sqrt{2}}{c^2 - d^2 \cdot 2\sqrt{2}} + \frac{bc - ad}{c^2 - d^2 2\sqrt{2}}\sqrt[4]{8}$$

with $\frac{ac - bd 2\sqrt{2}}{c^2 - d^2 2\sqrt{2}}$, $\frac{bc-ad}{c^2 - d^2 2\sqrt{2}} \in F_2$ since $a, b, c, d \in F_2$ which is a number field.

Hence $\frac{a+b\sqrt[4]{8}}{c+d\sqrt[4]{8}} \in F_3$

Therefore F_3 is also a number field.

Hence the set of the form $a + b\sqrt{2} + c\sqrt[4]{2} + d\sqrt[4]{8}$, for $a, b, c, d \in \mathbb{Q}$ is a number field.

(4). The set of irrational numbers. NO.

Answer.

Take irrational numbers π and π.
Both belong to the set of irrational numbers,
but $\pi-\pi=0$ is not an irrational number.
Hence the set is not a number field.

Q2. Let x_0 be a root of the polynomial
$a_nx^n+\cdots+a_1x+a_0$ where each a_i has the form
$a_i=b_i+c_i\sqrt{2}$ where $b_i, c_i \in Q$.
Prove that x_0 is a root of a polynomial with rational coefficients.

Proof: Suppose $f(x_0)=a_nx_0^n+\cdots+a_1x_0+a_0$

$$=(b_n+c_n\sqrt{2})x_0^n+\cdots+(b_1+c_1\sqrt{2})x_0+b_0+c_0\sqrt{2}$$

$$=0$$

$$b_nx_0^n+\cdots+b_1x_0+b_0=-\sqrt{2}(c_nx_0^n+\cdots+c_1x_0+c_0)$$

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$$(b_nx_0^n+\cdots+b_1x_0+b_0)^2=2(c_nx_0^n+\cdots+c_1x_0+c_0)^2$$

$$(b_nx_0^n+\cdots+b_1x_0+b_0)^2-2(c_nx_0^n+\cdots+c_1x_0+c_0)^2=0$$

For the polynomial

$$p=(b_nx^n+\cdots b_1x+b_0)-2(c_nx^n+\cdots+c_1x+c_0)$$

$b_i, c_i \in Q$, $2\in Q$, Q is a number field.
So all the coefficients in p are also in Q.
Hence x_0 is a root of a polynomial