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Faculty or School of Registration ______

Course MAT402
(e.g. PHL 202S, ANTA01Y)

Instructor 黎

Date of Examination Feb 27th

Place of Examination SF1101

BOOK NO. ______
TOTAL NUMBER OF BOOKS USED ______

TERM

INSTRUCTIONS

Write the information sought in the spaces above.

Write the answers on the RULED SIDE ONLY; all calculations or rough drafts of answers should be shown, preferably on the unruled side.

Clearly identify the question to which each answer applies; whenever the answer to a question is divided, note at the end of the first section "see also work on page ."

If a page is left blank write on it "see work on page ."

If more than one book is used, indicate the total number on the cover of each. At the conclusion of the examination, place all other books inside Book No. 1.

Do not tear any paper out of this book.

EXAMINER'S REPORT

1	19
2	20
3	19
4	20
5	20
6	
7	
8	
9	
10	
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Total	~~90~~ 98

THIS BOOK MUST NOT BE TAKEN FROM THE EXAMINATION ROOM

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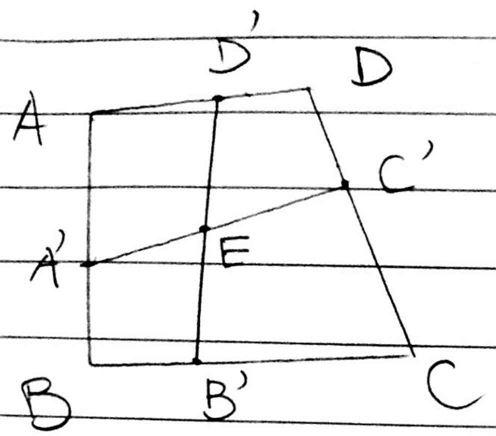
Problem 2 (~~mistakenly~~ sorry, problem 1 is on next page)



$$\frac{AA'}{A'B} = \frac{CB'}{B'B} = \frac{CC'}{C'D} = \frac{AD'}{D'D} = \frac{3}{2}$$

Solution: Suppose the intersection E of $A'C'$ & $B'D'$ is the center of mass of 4-gon ABCD.

Suppose A' is the center of mass of segment AB

since $\frac{AA'}{A'B} = \frac{3}{2}$ so $m_A \cdot AA' = m_B \cdot A'B$

then $\frac{m_A}{m_B} = \frac{2}{3}$

similarly we can find $\frac{m_B}{m_C} = \frac{3}{2}$, $\frac{m_C}{m_D} = \frac{2}{3}$, $\frac{m_D}{m_A} = \frac{3}{2}$

So there is $m_A + m_B$ amount of mass on A',

$m_C + m_D$ amount of mass on C'

and $\frac{m_A}{m_B} \cdot \frac{m_B}{m_C} = \frac{2}{3} \cdot \frac{3}{2} = 1 \Rightarrow m_A = m_C$

$\frac{m_B}{m_C} \cdot \frac{m_C}{m_D} = 1 \Rightarrow m_B = m_D$

So $(m_A + m_B) \cdot A'E = (m_C + m_D) \cdot EC'$

then $A'E : EC' = 1 : 1$

Similarly, we can have.

$(m_B + m_C) \cdot B'E = (m_A + m_D) \cdot ED'$

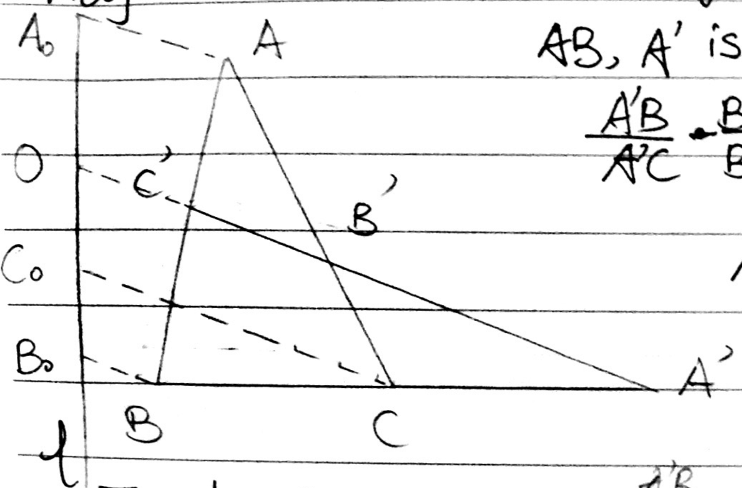
then $\frac{B'E}{ED'} = \frac{m_A + m_D}{m_B + m_C} = 1$

Problem solved.

Problem 1: Menelaus's thm:

For triangle ABC, B' is on side AC, C' is on side AB, A' is on the extension of BC, then $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$ if and only if A', B', C' are on the same line.

Proof:



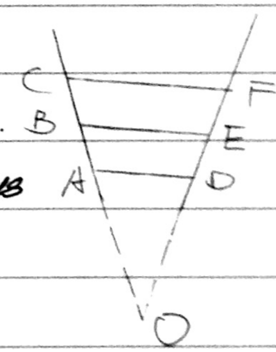
Two directions: $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B}$

① Suppose A', B', C' are on the same line, then want to prove $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$.

We construct a non-parallel line l out of $\triangle ABC$, say it's near AB (WLOG), then we extend $A'C'$, ~~interte~~ intersects with l at point O. Then we project A, B, C onto line l s.t. the projection lines are parallel with $A'C'$, then we get points A_0, B_0, C_0 respectively.

Now we have a ~~state~~ Claim:

suppose l_1, l_2 are 2 non-parallel lines, then AD, BE, CF are parallel to each other, we always have $\frac{BC}{AB} = \frac{EF}{DE}$ ⊛



This is proved by similar triangles.

$\triangle OAD \sim \triangle OBE \sim \triangle OCF$.

$\frac{OB}{OC} = \frac{OE}{OF}$ $\quad \frac{OA}{OC} = \frac{OD}{OF}$

$\frac{OB - OA}{OC} = \frac{OE - OD}{OF} \Rightarrow \frac{AB}{OC} = \frac{DE}{OF}$

Similarly $\frac{BC}{OC} = \frac{EF}{OF}$ Hence $\frac{BC}{AB} = \frac{EF}{DE}$ (claim proved)

(see next page)

So, we can use claim $(*)$

Since $B_0B \parallel CC_0 \parallel A'O \parallel AA_0$

Then $\frac{A'B}{A'C} = \frac{OB_0}{OC_0}$, $\frac{B'C}{B'A} = \frac{OC_0}{OA_0}$, $\frac{C'A}{C'B} = \frac{OA_0}{OB_0}$

Therefore $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = \cancel{\frac{OB_0}{OC_0}} \cdot \cancel{\frac{OC_0}{OA_0}} \cdot \cancel{\frac{OA_0}{OB_0}} = 1.$

(one direction done)

(2) Suppose we have the equation just proved 2 lines above, then we want to know C', B', A' are on the same line.

Assume C', B' ~~are on the desired line~~ are given

we can connect $C'B'$ and extend it, say it intersects the extension of BC at point A''.

by the equation proved above.

$$\frac{A''B}{A''C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$$

and we have condition $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$

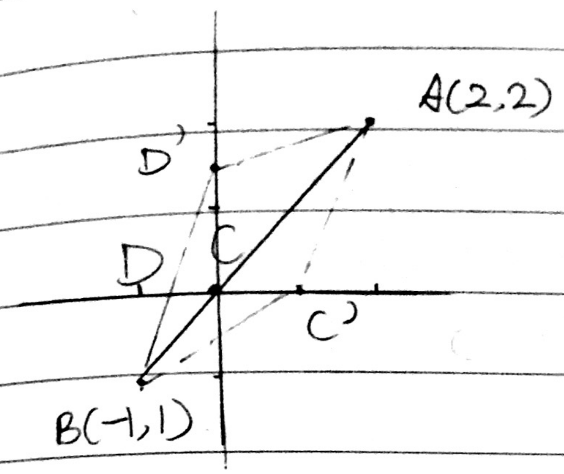
Hence $\frac{A''B}{A''C} = \frac{A'B}{A'C}$

Therefore A' is identically A''.

i.e. A', B', C' are on the same line.

Combine (1) and (2), we are done. $\blacksquare$

Problem 3.



Solution: rough idea. we want to find $\min F(C,D) = \min(|AC+CB| - |AD-BD|)$

so the absolute value $|AC+CB|$ should be as small as possible,

and $|AD-BD|$ should be as large as possible.

i.e. we are looking for $C=(x,0)$ s.t. we have a ~~max~~ min $|AC+CB|$

$D=(0,y)$ s.t. we have a ~~min~~ max $|AD-BD|$

① Since C is on x-axis, A, B on different sides of it.

Then just simply draw a segment from A to B, which intersects x-axis at point $C=(0,0)$.

such that $|AC+CB|$ is minimized (length is $\sqrt{3^2+3^2} = 3\sqrt{2}$

Because if C is shifted to any other point, say C' on x-axis. then for $\triangle ABC'$, by triangular inequality,

$|AC'+BC'| > |AB| = 3\sqrt{2}$. So we are done.

② Similarly, D is on y-axis, and A, B on different sides of it.

D is still the same point. i.e. $D=(0,0)$.

and $|AD-DB|$ is maximized here, which is $2\sqrt{2}$.

Because if we choose another D' except $D=(0,0)$.

by the proposition of triangular inequalition:

(see next page)

$\cancel{|BD'-D'C|+|D'C-D'A| \geq |BD'-D'C+D'C-D'A|}$

$\cancel{=|BD'-D'A|}$

$\cancel{|BD'-D'C| < |BD|}$

$\cancel{|D'C-D'A| < |DA|}$

$\cancel{|D'C-D'A|+|BD'-D'C| < |BD| > |BD'-D'A|}$

$$
\begin{aligned}
|BD-AD| &\geq \big||BD'-DD'|-AD\big| \\
&\geq |BD'-(DD'+AD)| \\
&\geq |BD'-AD'|
\end{aligned}
$$

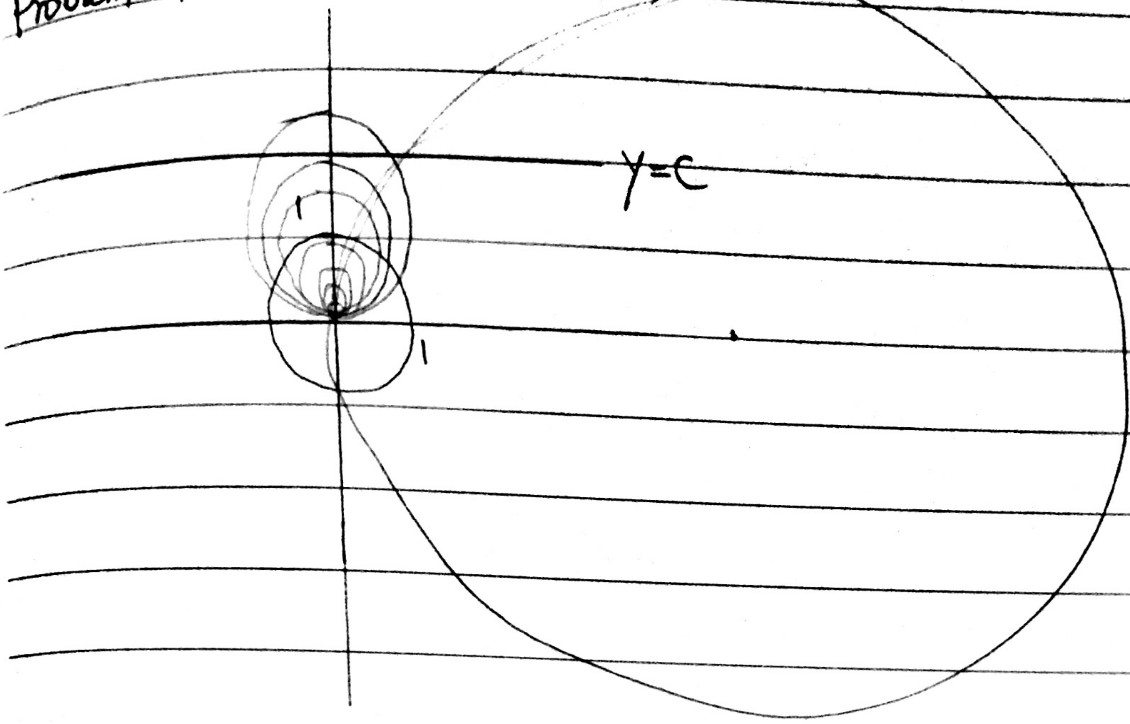
(equality holds iff D ~~is~~ and D' are the same point).

Hence when $C=(0,0)$, $D=(0,0)$,

$$
\begin{aligned}
M=F(C,D) &= |2\sqrt{2}+\sqrt{2}|-|2\sqrt{2}-\sqrt{2}| \\
&= 3\sqrt{2}-\sqrt{2} \\
&= 2\sqrt{2}
\end{aligned}
$$

which is a minimum.

Problem 4



Solution: We are looking for something that becomes horizontal lines after inversion about unit circle.

Know that distance from $y=c$ to $(0,0)$ is c, so before inversion point $(0,c)$ used to be $(0,\frac{1}{c})$, and since line $y=c$ extends to infinity at both sides, so they are point $(0,0)$ before inversion.

To conclude, for line $y=c$, it used to be a circle passing point $(0,0)$ and $(0,\frac{1}{c})$ with radius $\frac{1}{2c}$ i.e. centered at $(0,\frac{1}{2c})$

* But there is one exception, which is, when $c=0$, then before inversion, the line used to be $y=0$ itself.

To find circle S orthogonal to all circles passing through $A=(10,0)$, we are going to use 2 properties of inversion:

1. Inversion preserves angles.
2. If we do inversion twice (about the same unit circle), then we get back to what we have at the beginning.

So. circle S ~~orthogot~~ orthogonal to all circles.

$\Rightarrow$ circle S orthogonal to all horizontal lines $y=c$.

$\Rightarrow$ do one inversion to S then we get a line orthogonal to horizontal lines, so S now becomes a vertical line.

(*) (assume S is a new circle, passing origin).

which passing through the point of $(10,0)$ after inversion, i.e. $(\frac{1}{10}, 0)$.

So S is now $x = \frac{1}{10}$.

Therefore S is a circle passing $(10,0)$ and $(0,0)$ with radius 5. So S is $(x-5)^2+y^2=25$.

Why must passing origin? b/c if not, the circle may not intersect with those circles, as the circles close to origin are really small.

Problem 6: Solution:

this is because each vertex has 3 edges but each edge is calculated twice.

(1). $f_1 = 33$ edges, since for a simple convex polyhedron. $\frac{3}{2} f_0 = f_1$

So $f_0 = 22$. Then since $f_0 - f_1 + f_2 - f_3 = 1$

$$22 - 33 + f_2 - 1 = 1$$

$$f_2 = 2 - 22 + 33 = 13$$

So there are 22 vertices and 13 faces.

(2). $f(t) = 22 + 33t + 13t^2 + t^3$

$$h(t) = f(t-1) = 22 + 33(t-1) + 13(t-1)^2 + (t-1)^3$$

$$= 22 + 33t - 33 + 13t^2 - 26t + 13 + t^3 - 3t^2 + 3t - 1$$

$$= 1 + 10t + 10t^2 + t^3$$

So $h_0 = h_3 = 1$, $h_1 = h_2 = 10$.

Therefore, 10 vertices have index 1 w.r.t. L.