STA447 Final Review

Markov Chains.

- A (discrete time, discrete space, time homogeneous) Markov chain is specified by 3 ingredients. ① A state space S, non empty finite or countable set. ② Transition probabilities $\{p_{ij}\}_{i,j\in S}$ ③ Initial probabilities $\{\nu_i\}_{i\in S}$
- $\mu_i^{(n)} = P(X_n = i)$, $\mu_i^{(0)} = \nu_i$, $\mu_j^{(1)} = P(X_1 = j) = \sum_{i\in S} P(X_0 = i, X_1 = j) = \sum_{i\in S} \nu_i p_{ij}$

"Law of Total Probability"

- "n-step transition": $p_{ij}^{(n)} = P(X_{m+n} = j \mid X_m = i)$
- "Chapman-Kolmogorov equation" $p_{ij}^{(m+n)} \geq p_{ik}^{(m)} p_{kj}^{(n)}$

Classification of States

- a state i is recurrent if $P_i(X_n = i \text{ for some } n \geq 1) = 1$ $\;(= P(\cdots \mid X_0 = i))$

otherwise, transient.

- RECURRENCE THEOREM: i recurrent iff $\sum_{n=1}^\infty p_{ii}^{(n)} = \infty$ iff $P_i(N(i) = \infty) = 1$

i transient iff $\sum_{n=1}^\infty p_{ii}^{(n)} < \infty$ iff $P_i(N(i) = \infty) = 0$

To prove this, let $f_{ij} = P_i(X_n = j \text{ for some } n \geq 1)$

Then iff recurrent $f_{ii} = 1$, if transient $f_{ii} < 1$

Communicating States

i communicates j: $f_{ij} > 0$.

A MC is irreducible if $\forall i,j \in S$, $i \to j$.

CASE THEOREM: For an irreducible MC either ① $\sum_{n=1}^\infty p_{ij}^{(n)} = \infty$ $\forall i,j \in S$, all states recurrent; ② $\sum_{n=1}^\infty p_{ij}^{(n)} < \infty$ $\forall i,j \in S$, all states transient.

SUM LEMMA: if $i \to k$, $l \to j$ & $\sum_{n=1}^\infty p_{kl}^{(n)} = \infty$, then $\sum_{n=1}^\infty p_{ij}^{(n)} = \infty$

FINITE SPACE THM: an irreducible MC on a finite state space always falls into case (a), i.e. $\sum_{n=1}^\infty p_{ij}^{(n)} = \infty$ $\forall i,j \in S$ and all states are recurrent.

- $T_i = \min\{n \geq 1 : X_n = i\}$ ($T_i = \infty$ if never hit i)

HIT LEMMA: If $j \to i$ with $j \neq i$. Then $P_j(T_i < T_j) > 0$.

F-LEMMA: If $j \to i$ & $f_{jj} = 1$, then $f_{ij} = 1$

STRONGER-RECURRENCE THM: If chain irreducible: TFRE

① $\exists k, l \in S$ with $\sum_{n=1}^\infty p_{kl}^{(n)} = \infty$

② $\forall i,j \in S$, $\sum_{n=1}^\infty p_{ij}^{(n)} = \infty$

③ $\exists k \in S$, $f_{kk} = 1$

④ $\forall j \in S$, $f_{jj} = 1$

⑤ $\forall i,j \in S$, $f_{ij} = 1$

Stationary Distribution

- Let π_i be a probability distribution on S, i.e. $\pi_i \geq 0\ \forall i \in S$, $\sum_{i\in S} \pi_i = 1$.
- π is a stationary for a Markov chain $P=(p_{ij})$ if $\sum_{i\in S} \pi_i p_{ij} = \pi_j\ \forall j \in S$.
- a Markov chain is reversible (or time reversible) w.r.t a prob dist'n $\{\pi_i\}$ if $\pi_i p_{ij} = \pi_j p_{ji}\ \forall i,j \in S$.
- PROPOSITION: if chain is ~~irreducible~~ reversible wrt $\{\pi_i\}$ then $\{\pi_i\}$ is a stationary distribution. (Converse is false).

Obstacles to Convergence

- The period of a state i is gcd of the set $\{n \geq 1;\ p_{ii}^{(n)} > 0\}$
- If ~~period~~ period of each state is 1, chain is aperiodic.
- if $p_{ii} > 0$, then ~~per~~ period of i is 1.
- Simple random walk / Ehrenfest's Urn period 2.
- ~~Equal per~~ EQUAL PERIODS LEMMA: if $i \leftrightarrow j$ then the periods of i & j are equal.

COR: if chain irreducible then all states the same period.

COR: if chain irreducible & $p_{ii} > 0$ for some state i, then the chain is aperiodic.

MARKOV CHAIN CONVERGENCE THEOREM

If a MC is irreducible and aperiodic and has a stationary distribution $\{\pi_j\}$ then $\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$ for $\forall i,j \in S$ and $\lim_{n\to\infty} P(X_n = j) = \pi_j$ for any initial prob $\{\nu_j\}$

- STATIONARY RECURRENCE LEMMA: If chain irreducible, and has stationary dist, then it is recurrent.

COR: If chain is irreducible and aperiodic, then at most one stationary dist.

- PERIODIC CONVERGENCE THM: Sps chain irreducible, period $b \geq 2$, stat dist $\{\pi_i\}$. $\forall i,j \in S$, $\lim_{n\to\infty} \frac{1}{b}[p_{ij}^{(n)} + \cdots + p_{ij}^{(n+b-1)}] = \pi_j$ also

$$\lim_{n\to\infty} \frac{1}{b} P[X_n = j \text{ or } X_{n+1} = j \text{ or} \cdots \text{or } X_{n+b-1} = j] = \pi_j$$

COR: If MC irreducible (not necessarily aperiodic), at most one stat dist.

MCMC

$S = \mathbb{Z}$, $\{\pi_i\}$ be any prob dist on S. Assume $\pi_i > 0\ \forall i$.

Let
$$\left.\begin{aligned} p_{i,i+1} &= \tfrac{1}{2}\min[1, \tfrac{\pi_{i+1}}{\pi_i}] \\ p_{i,i-1} &= \tfrac{1}{2}\min[1, \tfrac{\pi_{i-1}}{\pi_i}] \\ p_{i,i} &= 1 - p_{i,i+1} - p_{i,i-1} \\ p_{ij} &= 0 \quad \text{o.w.} \end{aligned}\right\} \quad \text{s.t. } \lim_{n\to\infty} p_{ij}^{(n)} = \pi_j$$

• Equivalent algorithmic version: Given X_{n-1}, let Y_n equal $X_{n-1} \pm 1$ (prob $\frac{1}{2}$) each and $U_n \sim \text{Uniform}[0,1]$ (indep.) and

$$X_n = \begin{cases} Y_n, & U_n < \frac{\pi_{Y_n}}{\pi_{X_{n-1}}} \\ X_{n-1}, & \text{o.w.} \end{cases}$$

Then $\pi_i p_{i,i+1} = \pi_i \frac{1}{2} \min[1, \frac{\pi_{i+1}}{\pi_i}] = \frac{1}{2} \min[\pi_i, \pi_{i+1}]$

$\pi_{i+1} p_{i+1,i} = \pi_{i+1} \frac{1}{2} \min[1, \frac{\pi_i}{\pi_{i+1}}] = \frac{1}{2} \min[\pi_{i+1}, \pi_i]$

So $\pi_i p_{ij} = \pi_j p_{ji}$ if $j = i+1$ $\forall i,j \in S$, reversible. $\{\pi_i\} \to$ stationary --- (easy part)

Random walks on graphs

• $w(u,v)$ weight on edge (u,v), $d(u) = \sum_{v \in V} w(u,v)$ (degree of vertex u)

• $p_{uv} = \frac{w(u,v)}{d(u)}$

• $Z = \sum_{u \in V} d(u) = \sum_{u,v \in V} w(u,v)$ (unweighted case: $Z = 2 \times (\#$ of edges$)$)

$\pi_u = \frac{d(u)}{Z}$

• If graph is connected, then chain is irreducible.

• If graph is bipartite, then chain has period 2.

THM: for random walk on any connected graph with $Z < \infty$, $\lim_{n\to\infty} \frac{1}{2} [p_{uv}^{(n)} + p_{uv}^{(n+1)}] = \frac{d(v)}{Z}$

Gambler's Ruin

$T_i = \inf\{n \geq 0 : X_n = i\}$ be first time having i dollars.

write $P_a(T_c < T_0)$ as $s(a)$

Gambler's formula: $$\boxed{s(a) = \begin{cases} \frac{(\frac{1-p}{p})^a - 1}{(\frac{1-p}{p})^c - 1}, & p \neq \frac{1}{2} \\ \frac{a}{c}, & p = \frac{1}{2} \end{cases}}$$

Martingales

• A sequence $\{X_n\}_{n=0}^{\infty}$ of r.v. is a martingale if $E|X_n| < \infty$ for each n and also $E(X_{n+1} | X_0, \dots, X_n) = X_n$ (i.e. it stays same on average).

• SPECIAL CASE: if $\{X_n\}$ is a MC (with $E|X_n| < \infty$) then $E[X_{n+1} | X_0, \dots, X_n]$ $= \sum_j j P[X_{n+1} = j | X_0, \dots, X_n] = \sum_j j P_{X_n, j}$ so martingale if $\sum_j j P_{ij} = i$ for all i

• double-expectation formula

$E(X_{n+1}) = E[E(X_{n+1} | X_0, \dots, X_n)] = E(X_n)$. i.e. $E(X_n) = E(X_0)$ $\forall n$.

• A non-negative-integer-valued random variable T is a stopping time for $\{X_n\}$ if the event $\{T = n\}$ is determined by $X_0, X_1, \dots, X_n$.

"can't look into future before deciding to stop."

- OPTIONAL STOPPING LEMMA:

If $\{X_n\}$ martingale, with stopping time which is bdd ($\exists M<\infty$ with $P(T\leq M)=1$), then $E(X_T)=E(X_0)$.

- OPTIONAL STOPPING THM:

If $\{X_n\}$ martingale, with stopping time T and $P(T<\infty)=1$ and $E|X_T|<\infty$, and $\lim_{n\to\infty} E(X_n \mathbb{1}_{T>n})=0$, then $E(X_T)=E(X_0)$

- OPTIONAL STOPPING COR:

If $\{X_n\}$ is martingale with stopping time T, which is "bdd up to time T" (i.e. $\exists M<\infty$ with $P(|X_n|\mathbb{1}_{n\leq T}\leq M)=1$ for $\forall n$) and $P(T<\infty)=1$, then $E(X_T)=E(X_0)$

. WALD'S THM: Sps $X_n=a+Z_1+\cdots+Z_n$ with $\{Z_i\}$ iid, finite mean m.
Let T be a stopping time for $\{X_n\}$ which has finite mean, $E(T)<\infty$, then $E(X_T)=a+mE(T)$

- special case, if $m=0$, $\{X_n\}$ is a martingale.

LEMMA: Let $X_n=a+Z_1+\cdots+Z_n$, $\{Z_i\}$ iid with mean 0, variance $v<\infty$.
Let $Y_n=(X_n-a)^2-nv=(Z_1+\cdots+Z_n)^2-nv$, then $\{Y_n\}$ is a martingale.

COR: If $\{X_n\}$ is Gambler's Ruin with $p=\frac{1}{2}$, and $T=\inf\{n\geq 0: X_n=0 \text{ or } c\}$ then $E(T)=Var(X_T)=a(c-a)$

~~MA~~

$\boxed{\text{Martingale Convergence Thm}}$

Any non-negative martingale converges w.p. 1 to some random variable X.

- non-negative, can't spread out forever, since it's a martingale, can't "drift" anywhere.

s.s.r.w. – martingale but not non-negative, does not converge

s.s.r.w. stopped at zero – converges.

s.r.w. with $p=\frac{2}{3}$ stopped at zero – non-negative but not converge b/c not a martingale.

$\boxed{\text{Branching Process}}$

Brownian Motion

$\{B_t\}_{t\geq 0}$ is the limit as $M\to\infty$ of $\{Y_t^{(M)}\}$

$Y_t^{(M)} = \frac{1}{\sqrt{M}}(Z_1+Z_2+\cdots+Z_{tM})$

Thus $E(Y_t^{(M)})=0$, $Var(Y_t^{(M)}) = \frac{1}{M}(tM)=t$, as $M\to\infty$ by CLT, $Y_t^{(M)}\to N(0,t)$

so $B_t \sim N(0,t)$

Also $B_s - B_t \sim Normal(0, s-t)$ independent of B_t $(s\geq t>0)$

- $Cov(B_t, B_s) = E(B_tB_s) = E(B_t[B_s - B_t + B_t]) = E(B_t[B_s - B_t]) + E((B_t)^2)$

$= E(B_t)E(B_s - B_t) + E((B_t)^2)$

$= 0\cdot 0 + t$

$= t$

- In general, $Cov(B_t, B_s) = \min(t,s)$ (covariance structure)
- Definition: Brownian motion is a process $\{B_t\}_{t\geq 0}$ satisfying the above property and with continuous sample paths (i.e. $t\to B_t$ is continuous)

e.g. $E[(B_2+B_3+1)^2] = E[(B_2)^2] + E[(B_3)^2] + 1^2 + 2E[B_2B_3] + 2E[B_2\cdot(1)] + 2E[B_3(1)] = 2+3+1+2(2)$ $=10$

$Var(B_3+B_5+7) = E[(B_3+B_5)^2] = E(B_3^2) + E(B_5^2) + 2E(B_3B_5) = 3+5+2(3) = 14$

week 8 p31.

$0<t<s$

- $E[Y_s \mid \{Y_r\}_{r\leq t}] = E[E[Y_s \mid \{B_r\}_{r\leq t}] \mid \{Y_r\}_{r\leq t}] = E[Y_t \mid \{Y_r\}_{r\leq t}] = Y_t$ "law of iterated expectations"

Financial Modeling

- common model for stock price: $X_t = x_0 \exp(\mu t + \sigma B_t)$

"fair pair" Black-Scholes formula"

$x_0\Phi\left(\frac{(r+\sigma^2/2)S - \log(K/x_0)}{\sigma\sqrt{S}}\right) - e^{-rS}K\Phi\left(\frac{(r-\sigma^2/2)S - \log(K/x_0)}{\sigma\sqrt{S}}\right)$

RETURN TIME THM: If an irreducible MC on a discrete S has stat. dist π, $\forall i\in S$, mean return time $m_i = E_i(T_i) = \frac{1}{\pi_i}$

Sequence Waiting Times

τ be the first time the sequence "HTH" is completed, $E(\tau)$?

Let X_n be the amount of the desired sequence (HTH) that the chain has "achieved" so far. If begins with HHTTHT, then $X_0=0$, $X_1=1$, $X_2=1$, $X_3=2$, $X_4=0$, $X_5=1$, $X_6=2$

So. $S=\{0,1,2,3\}$ $X_0=0$, $X_\tau=3$

- $p_{01}=p_{12}=p_{23}=\frac{1}{2}$ (prob of continuing the process)
- $p_{00}=p_{20}=\frac{1}{2}$. But $p_{11}=\frac{1}{2}$
- $p_{31}=p_{30}=\frac{1}{2}$, o.w. 0.

$$P=\begin{pmatrix} \frac12 & \frac12 & 0 & 0 \\ 0 & \frac12 & \frac12 & 0 \\ \frac12 & 0 & 0 & \frac12 \\ \frac12 & \frac12 & 0 & 0 \end{pmatrix}$$

Then stat dist'n:
$$\begin{cases} \pi_0 = \pi_1\cdot p_{10} + \pi_2\cdot p_{20} + \pi_3\cdot p_{30} + \pi_0\cdot p_{00} \\ \pi_3 = \pi_1 p_{13} + \pi_0\cdot p_{03} + \pi_2 p_{23} + \pi_3 p_{33} \\ \cdot\ - \qquad - \\ - \ - \end{cases}$$

$\tau = \frac{1}{\pi_i} = \frac{1}{0.1} = 10$

$\pi_i \to \pi_3$

$\pi = (0.3, 0.4, 0.2, 0.1)$

But the prob from 3→3 is the same as 0→3. ✓

~~• Another approach: (HTH) using martingale.~~
~~(Assume at ~~time~~ each n, a better, bets 1 on H, then if win bet 2 on T, if win bet 4 on H. Stops as soon as win 3 in a row or lose once)~~
~~S_n be total amount won by all betters by time n.~~
~~$\{S_n\}$ is a martingale with stopping time τ.~~
~~$S_\tau = -(\tau-3)+(-1)+1+7$~~

Poisson Process

$\lambda =$ average fires

$P(\#\text{ fires} = k) = \boxed{e^{-\lambda}\frac{(\lambda)^k}{k!}}$

• $T_0=0$, $T_n = Y_1+Y_2+\cdots+Y_n \ \forall n\geq 1$ (nth arrival time)

• $N(t) = \max\{n\geq 0 : T_n \leq t\} = \#\{n\geq 1 : T_n\leq t\} = \#$ arrivals up to time t.

- "counting process" e.g. $N(t) = \#$ fires between times 0 & t.
  "poisson process with intensity λ"

$P(N(t)=k) = \boxed{e^{-\lambda t}\frac{(\lambda t)^k}{k!}}$, $k=0,1,2,\dots$

$E(N(t)) = \lambda t$, $Var(N(t)) = \lambda t$

• "memoryless" property of the $Exp(\lambda)$ dist'n: $\forall a,b>0$. $P(Y_n > a+b \mid Y_n > a) = P(Y_n > b) = e^{-\lambda b}$

$N(t+s) - N(s) \sim N(t) \sim Poisson(\lambda t)$

if $0\leq a < b\leq c < d$ then $N(d)-N(c)$ independent of $N(b)-N(a)$ "independent increments"

• more generally: $0\leq t_1 \leq s_1 \leq t_2 \leq s_2 \leq \cdots \leq t_k \leq s_k$

$N(s_i) - N(t_i) \sim Poisson(\lambda(s_i - t_i))$ and $\{N(s_i)-N(t_i)\}_{i=1}^{k}$ are all independent. ("independent Poisson increments")

• A poisson process with intensity $\lambda > 0$ is a collection $\{N(t)\}_{t\geq 0}$ of random non-decreasing integer counts $N(t)$, satisfying

(a). $N(0)=0$.

(b). $N(t)\sim Poisson(\lambda t)\ \forall t\geq 0$

(c). independent Poisson increments.

• $0<t<s$. $P(N(t)=1 \mid N(s)=1) = \frac{t}{s}$

$P(N(4)=1 \mid N(5)=3) = \boxed{\frac{P(N(4)=1, N(5)-N(4)=2)}{P(N(5)=3)}}$

C-time, MC, large $\underline{t}$. $P^{(t)} = ?$

Date

· Sum of poisson is Poisson.

$\boxed{\text{Continuous-Time, discrete-space MC}}$

· Brownian motion & Poisson processes defined in continuous time.

· A ~~a~~ continuous-time (time-homogeneous, non-explosive) Markov process, on a countable (discrete) state space S, is a collection $\{X(t)\}_{t\geq 0}$ of r.v. s.t.

$P(X_0 = i_0, X_{t_1} = i_1, \cdots, X_{t_n} = i_n) = \nu_{i_0} p_{i_0 i_1}^{(t_1)} p_{i_1 i_2}^{(t_2 - t_1)} \cdots p_{i_{n-1} i_n}^{(t_n - t_{n-1})}$

· initial prob. & transition prob.

· $P^{(t)} = (p_{ij}^{(t)})_{i,j\in S}$ = matrix version

$p_{ij}^{(s+t)} = \sum_{k\in S} p_{ik}^{(s)} p_{kj}^{(t)}$. $P^{(s+t)} = P^{(s)} P^{(t)}$ Chapman-Kolmogorov Equation)

· process's generator $g_{ij} = \lim_{t\searrow 0} \frac{p_{ij}^{(t)} - \delta_{ij}}{t} = p_{ij}'(0)$

$G = P'^{(0)} = \lim_{t\searrow 0} \frac{P^{(t)} - I}{t}$

· if $t>0$ is small, $G \approx \frac{P^{(t)} - I}{t}$ so $P^{(t)} \approx I + tG$ i.e. $p_{ij}^{(t)} \approx \delta_{ij} + t g_{ij}$

δ_{ij}: $=1$ if $i=j$, 0 o.w.

· reversibility still implies stationary ($\pi G = 0$)

CONTINUOUS-TIME MARKOV CONVERGENCE THM.

if a continuous-time MC is irreducible, has a stationary distribution π, then $\lim_{t\to\infty} p_{ij}^{(t)} = \pi_j$ for all $i,j \in S$.

(automatically aperiodic)

· Generator of Markov process $\{X_t\}$

$\boxed{\text{Queueing Theory}}$

· T_n = time of arrival of n^{th} customer ($T_0 = 0$)

· $Y_n = T_n - T_{n-1}$ = inter-arrival time b/w $(n-1)$st and n^{th} customers.

· S_n = time it takes to serve the n^{th} customer.

· $Q(t)$ = number of customers in the system at time $t \geq 0$.

M/M/1 QUEUE:

$T_n - T_{n-1} \sim Exp(\lambda)$, $S_n \sim Exp(\mu)$, all indep., $\lambda, \mu > 0$.

(So $\{T_n\}$ are arrival times of a Poisson process with intensity λ)

Then $\{Q(t)\}$ is a Markov process.

· General (G/G/1) Queue:

D_n = time of departure of the nth customer.

$W_n = \max(0, D_{n-1} - T_n)$ = the amount of time that the n^{th} customer has to wait. (With $W_0 = 0$)

· LINDLEY'S EQ: For $n \geq 1$, $W_n = \max(0, W_{n-1} + S_{n-1} - Y_n)$

· LINDLEY'S COR: $W_n = \max_{0 \leq k \leq n} \sum_{i=k+1}^{n} (S_{i-1} - Y_i)$

THM: For a general (G/G/1)

(a). if $E(Y_n) < E(S_n)$ then $\lim_{n\to\infty} W_n = \infty$ w.p. 1.

(Hence, $\lim_{n\to\infty} W_n = \infty$ in prob, so $\forall M < \infty$, $\lim_{n\to\infty} P(W_n > M) = 1$).

(b). if $E(Y_n) > E(S_n)$, then $\{W_n\}$ is bdd in probability, i.e. for any $\varepsilon > 0$ $\exists M < \infty$ s.t. $P(W_n > M) < \varepsilon$ for all $n \in N$.

(c). if $E(Y_n) = E(S_n)$, and S_{n-1} & T_n are not both constant (i.e. $Var(S_{n-1} - Y_n) > 0$) then $W_n \to \infty$ in prob. (but not w.p. 1)

Quantum Mechanics

Renewal Theory:

arrival times $\{T_n\}$, $T_0 = 0$, $T_n = Y_1 + Y_2 + \cdots + Y_n$ where $\{Y_n\}_{n=1}^{\infty}$ are independent interarrival times with $\{Y_n\}_{n=2}^{\infty}$ i.i.d.

· $N(t) = \max\{n \geq 0 ; T_n \leq t\} = \#\{n \geq 1 ; T_n \leq t\}$ is a "renewal process"

· If $\{Y_n\}_{n=1}^{\infty}$ all i.i.d then the process is zero-deplayed

· $\{Y_n\}$ i.i.d $\sim Exp(\lambda)$, then $\{N(t)\}$ is Markovian ("memoryless property") in fact it is Poisson.

· Poisson process is the only Markovian renewal process

e.g. T_n to be the nth time we replace a lightbulb. ($T_0 = 0$).

$Y_n = T_n - T_{n-1}$ is the lifetime of the n-th bulb.

$\{Y_n\}_{n=2}^{\infty}$ are iid, $N(t)$ is # of bulbs replaced by time t.

$\{N(t)\}$ is a renewal process.

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ELEMENTARY RENEWAL THM:

For a renewal ~~thm~~ process. if $P(Y_1 < \infty) = 1$, if "mean interarrival time" $\mu := E(Y_2) < \infty$, then (a). $\boxed{\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu}}$ w.p. 1

(b). $\boxed{\lim_{t\to\infty} \frac{E(N(t))}{t} = \frac{1}{\mu}}$

· Practice: Let $Y_1, Y_2, \ldots$ iid $\sim Unif[0,10]$.

Let $T_0 = 0$. $T_n = Y_1 + Y_2 + \cdots + Y_n$ $n \geq 1$. Let $N(t) = \max\{n \geq 0 : T_n < t\}$

(a). Compute $\lim_{t\to\infty} N(t)/t$.

(b). Approximate $E(\#\{n \geq 1 : 1234 < T_n < 1236\})$

BlackWELL RENEWAL THM: For a renewal process, sps $P(Y_1 < \infty)$ & $\mu = E(Y_2) < \infty$, Y_2 not arithmetic i.e. no $\lambda > 0$ s.t. $P(Y_2 = k\lambda$ for some $k \in \mathbb{Z}) = 1$

Then $\forall h > 0$, $\lim_{t\to\infty} E[N(t+h) - N(t)] = \frac{h}{\mu}$.

~~Rew~~Renewal Reward Process

renewal process $P(Y_1 < \infty) = 1$. mean interarrival time $\mu := E(Y_2) < \infty$

Sps kth renewal time T_k, receive an reward (or cost) R_k. $\{R_k\}$ iid

Renewal Reward THM:

$\boxed{\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E[R_1]}{\mu}}$ w.p. 1

Car purchases

~~Ca~~ $L \sim Unif[0,10]$. buy new car when broke down or after S years

$0 \leq S \leq 10$. 30 - cost, if broke down b4 sell it, cost 5.

· How many cars will you buy each year?

$\mu = E(Y_2) = E[\min(L, S)]$

$\mu = SP(L > S) + E[L \mid L < S] P(L < S) = S[(10-S)/10] + \frac{S}{2}\frac{S}{10} = S - S^2\frac{1}{20}$

By Elementary Renewal Thm part(a) $\lim_{t\to\infty} \frac{N(t)}{t} = \frac{1}{\mu} = \frac{1}{S - S^2\frac{1}{20}}$

· How many cars will you buy between 562 & 566 years.

$t=562,\ t+h=566,\ h=4.$

By Blackwell Renewal THM:

$$\frac{h}{\mu} = \frac{4}{S - S^2\frac{1}{20}}$$

· Long-run average car ~~#~~ cost per year?

T_k be the time of purchase of kth ~~#~~ car.

$Y_k = T_k - T_{k-1}$ be k^{th} interarrival time

$R(t)$ total car cost by t.

$$E(R_1) = 30 + 5P(L \le S) = 30 + 5\frac{S}{10} = 30 + \frac{S}{2}$$

$$\mu = E(Y_2) = S - S^2\frac{1}{20}$$

Renewal Reward Thm: $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{E(R_1)}{\mu} = \frac{30+S/2}{S-S^2/20}$

Discrete-time Markov Chains on Continuous State Spaces

· ~~state~~ state space $\mathcal{X}$: any non-empty (probably uncountable) set. e.g. $\mathbb{R}$.

· The one-step transition probabilities are ~~then~~ $P(x, A)$, for each $x \in \mathcal{X}$ and $A \subseteq \mathcal{X}$

· discrete-time, general state space, time-homogeneous MC $X_0, X_1, X_2, \ldots$

where $P(X_0 \in A_0, X_1 \in A_1, \ldots, X_n \in A_n)$

$$= \int_{x_0 \in A_0} \nu(dx_0) \int_{x_1 \in A_1} P(x_0, dx_1) \cdots \int_{x_{n-1} \in A_{n-1}} P(x_{n-2}, dx_{n-1}) \int_{x_n \in A_n} P(x_{n-1}, dx_n)$$

simplify. $\int_{x_n \in A_n} P(x_{n-1}, dx_n) = P(x_{n-1}, A_n)$

· A stationary distribution for MC is a probability measure $\pi(\cdot)$ on $\mathcal{X}$

s.t. $\pi(A) = \int_{\mathcal{X}} \pi(dx) P(x, A)\ \forall A \subseteq \mathcal{X}$

A MC on a general state space $\mathcal{X}$ is ϕ-irreducible if $\exists$ a non-zero (σ-finite) measure ψ on $\mathcal{X}$ s.t. if $\psi(A) > 0 \Rightarrow P_x(\tau_A < \infty) > 0\ \forall x \in \mathcal{X}$

$\tau_A = \inf\{n \ge 0;\ X_n \in A\}$ 1st hitting time of subset A.

Date

Mid-term review.

EQUAL PERIOD LEMMA: if a chain is irreducible ~~& there are~~ all states must have the same period.

STAT RECURRENCE LEMMA: stationary dist'n + irreducible $\Rightarrow$ recurrent

- Let $S=\{1,2,3\}$ with $\pi_1=\frac{1}{2}, \pi_2=\frac{1}{3}, \pi_3=\frac{1}{6}$. Find irreducible transition prob. $\{p_{ij}\}_{i,j\in S}$ s.t. π is stationary dist.

MCMC $\Rightarrow$ for $i\neq j$, want $p_{ij} = \frac{1}{2}\min(1, \frac{\pi_j}{\pi_i})$

$p_{21}=p_{32}=\frac{1}{2}$, $p_{12}=\frac{1}{3}$, $p_{23}=\frac{1}{4}$,

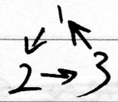
Then we satisfy $\sum_j p_{ij}=1 \ \forall i\in S$

so $p_{11}=\frac{2}{3}$, $p_{22}=\frac{1}{4}$, $p_{33}=\frac{1}{2}$

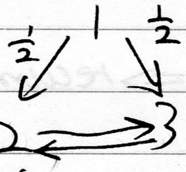
Check $\pi_i p_{ij}=\pi_j p_{ji}$. ✓

01. Final.

1.



2.



3. $f_{12} = p_{12} + p_{11} f_{12}$

$\frac{1}{3} = p_{12} + \frac{1}{3} p_{11}$

$1 = 3p_{12} + p_{11}$

but $p_{12} + p_{11} = 1$

So impossible

4.

Name: ______________________ Student Number: ______________

STA 447 / 2006 (circle one), Winter 2012, Mid-Term Test

(February 16, 2012. Duration: 60 minutes. Questions: 2. Pages: 3. Total points: 36.)

Notes: You should fully explain your answers! Continue on the back if necessary. NO AIDS ALLOWED. You may use theorems from class. Point values in [square-brackets].

1. Consider a Markov chain with state space $S = \{1, 2\}$, and transition probabilities $p_{11} = 2/3$, $p_{12} = 1/3$, $p_{21} = 1/4$, and $p_{22} = 3/4$.

$$(P_{ij}) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

(a) [3] Compute $p_{12}^{(2)}$.

$$p_{12}^{(2)} = p_{11} p_{12} + p_{12} p_{22}$$
$$= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{3}{4}$$
$$= \frac{2}{9} + \frac{1}{4} = \frac{8+9}{36} = \frac{17}{36}$$

(b) [2] Determine whether or not this chain is irreducible.

$\forall i, j \in S,\ f_{ij} > 0.$

in details: $f_{11} = 1 = f_{22} > 0$

$f_{21} = f_{12} = 1. > 0$

So chain is irreducible.

(c) [4] Let $\pi_1 = 3/7$ and $\pi_2 = 4/7$. Prove that $\{\pi_i\}$ is a stationary probability distribution for this chain.

$\cancel{\pi_1 p_{12} = p}$

$$\cancel{\sum_{j=1,2} \pi_i p_{ij} = \pi_1 p_{12} + \pi_1 p_{11} = \frac{3}{7} \times \frac{1}{3} + \frac{3}{7} \times \frac{2}{3}}$$

$\cancel{= \frac{1+2}{7}}$

$\cancel{= \frac{3}{7}}$

reversability

$\pi_i p_{ij} = \pi_j p_{ji}$

$\frac{3}{7} \times \frac{1}{3} = \frac{4}{7} \times \frac{1}{4}$

$\pi_1 p_{12} + \pi_2 p_{22} = \pi_2$

$\frac{3}{7} \times \frac{1}{3} + \frac{4}{7} \times \frac{3}{4} = \frac{4}{7}.$

OR,

defn: $\cancel{\pi_1 p_{12}\ \pi_1 (p_{11} + p_{12}) = \frac{3}{7}}$ ×

1. (cont'd) Recall that $S = \{1, 2\}$, $p_{11} = 2/3$, $p_{12} = 1/3$, $p_{21} = 1/4$, and $p_{22} = 3/4$.

(d) [3] Determine whether or not $f_{11} = 1$.

$$\left.\begin{aligned} f_{11} &= P_{11} + P_{12} f_{21} = \tfrac{2}{3} + \tfrac{1}{3} = 1 \\ f_{21} &= P_{21} + P_{22} f_{21} \\ (1-P_{22}) f_{21} &= P_{21} \Rightarrow f_{21} = \frac{P_{21}}{1-P_{22}} = \frac{\frac{1}{4}}{\frac{1}{4}} = 1 \end{aligned}\right\} \checkmark$$

(e) [3] Determine whether or not $f_{21} = 1$.

同理.

(f) [3] Determine whether or not $\sum_{n=1}^{\infty} p_{21}^{(n)} = \infty$.

irreducible + finite state $\Rightarrow$ all recurrent

$\Downarrow$

$\sum_{n=1}^{\infty} P_{21}^{(n)} = \infty$

(g) [3] Determine whether or not this chain is aperiodic.

~~gcd~~ b/c $P_{11} \neq 0$, $P_{22} \neq 0$. $\checkmark$ aperiodic $\checkmark$

(h) [2] Determine whether or not $\lim_{n\to\infty} p_{12}^{(n)} = \pi_2$.

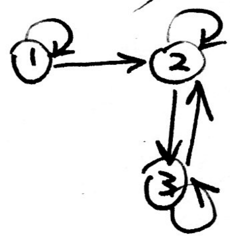
b/c aperodic.

by MC convergence Thm:

$\lim_{n\to\infty} P_{12}^{(n)} = \pi_2$

2. Consider a Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities $p_{11} = p_{12} = p_{22} = p_{23} = p_{32} = p_{33} = 1/2$, with $p_{ij} = 0$ otherwise.

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



(a) [3] Determine whether or not this chain is irreducible.

Not irreducible

b/c $3 \not\to 1$.

$f_{31} = 0$

(b) [4] Compute f_{ii} for each $i \in S$.

$f_{11} = p_{11} + p_{12} f_{21} = \frac{1}{2}$

$f_{12} = p_{12} + p_{11} f_{12} \Rightarrow f_{12} = \frac{p_{12}}{1-p_{11}} = 1$

~~$f_{13} = p_{12} f_{23} + \ldots p_{11}$~~

$f_{21} = 0$

$f_{22} = p_{22} + p_{23} f_{32} = 1$

$f_{23} = p_{23} + p_{22} f_{23} \Rightarrow f_{23} = 1$

Similarly: $f_{31} = 0$

$f_{32} = f_{33} = 1$

$f_{32} = p_{32} + p_{33} f_{32}$

(c) [3] Specify which states are recurrent, and which states are transient.

recurrent: 2, 3

transient: 1

(d) [3] Compute the value of f_{13}.

$f_{13} = p_{13} + \cancel{p_{23} f_{23}}\ p_{12} f_{23} + p_{11} f_{13}$

$= 0 + \frac{1}{2} \cdot 1 + \frac{1}{2} f_{13}$

$\Rightarrow f_{13} = 1$

[END]

CIRCLE ONE: **STA447** **STA2006** Student #: ____________

Surname: ____________ Given name(s): ____________

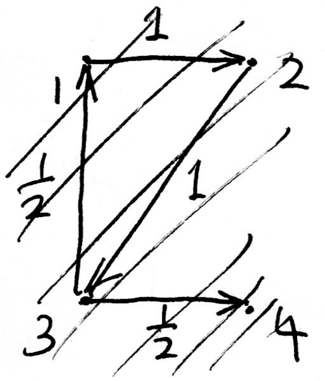
STA 447/2006S, Winter 2008: In-Class Test

(February 28, 2008, 6:10 p.m. Time: 130 minutes.)

(Questions: 6; Pages: 6; Total points: 65.)

NO AIDS ALLOWED – NOT EVEN CALCULATORS.

1. [8 points] Let (p_{ij}) be the transition probabilities for random walk on the graph whose vertices are $V = \{1,2,3,4\}$, with a single edge between each of the four pairs (1,2), (2,3), (3,1), and (3,4), and no other edges. Compute (with full explanation) $\lim_{n\to\infty} p_{13}^{(n)}$.



$\pi_3 = \frac{3}{4\times 2} = \frac{3}{8} = \lim_{n\to\infty} p_{13}^{(n)}$ by MCCT

$\cancel{\lim_{n\to\infty} \frac{1}{3}\left(p_{13}^{(n)} + p_{13}^{(n+1)} + p_{13}^{(n+2)}\right) = \frac{3}{8}}$

irreducible ✓

aperiodic ✓

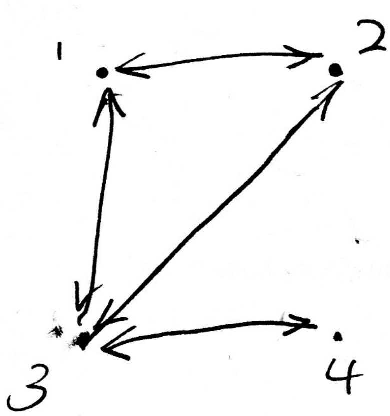
stationary ? ✓

b/c

$\pi_1 =$

~~fixed~~ b/c reversibility

so by MCCT. ✓



2. Consider the Markov chain with state space $S = \{1, 2, 3\}$, and transition probabilities given by $p_{11} = 1/6$, $p_{12} = 1/3$, $p_{13} = 1/2$, $p_{22} = p_{33} = 1$, and $p_{ij} = 0$ otherwise.

(a) [4 points] Compute (with explanation) f_{12} (i.e., the probability, starting from 1, that the chain will eventually visit 2).

(b) [3 points] Prove that $p_{12}^{(n)} \geq 1/3$, for any positive integer n.

(c) [2 points] Compute $\sum_{n=1}^{\infty} p_{12}^{(n)}$.

(d) [3 points] Relate the answers in parts (a) and (c) to theorems from class about when $f_{ij} = 1$ and when $\sum_{n=1}^{\infty} p_{ij}^{(n)} = \infty$.

3. Let $S = \mathbf{Z}$ (the set of all integers), and let $h : S \to [0,1]$ with $\sum_{i \in S} h(i) = 1$. Consider the transition probabilities on S given by $p_{ij} = (1/4)\min(1,\ h(j)/h(i))$ if $j = i-2, i-1, i+1$, or $i+2$, and $p_{ii} = 1 - p_{i,i-2} - p_{i,i-1} - p_{i,i+1} - p_{i,i+2}$, and $p_{ij} = 0$ whenever $|j - i| \geq 3$.

(a) [10 points] Assuming that $h(i) > 0$ for all i, prove that $\lim_{n\to\infty} p_{ij}^{(n)} = h(j)$ for all $i, j \in S$. (Carefully justify each step.)

$\boxed{\text{treat } h(i) \text{ as } \pi_i}$

$$p_{ij} = \begin{cases} \frac{1}{4}\min\left(1, \frac{h(j)}{h(i)}\right) & \text{if } 0 < |i-j| \leq 2 \\ 1 - p \cdots p \cdots & \text{if } i = j \\ 0 & \text{if } |j-i| \geq 3 \end{cases}$$

irreducable: state $i \to j$ if $0 < |i-j| \leq 2$

aperiodic: ~~gcd~~ $= 1$. $\left.\begin{array}{l} \gcd(2,3) \\ \text{or } \gcd(1, \ldots) \end{array}\right\} = 1.$

stationary:

$h(i)p_{ij} = h(j)p_{ji}$

if $|j-i| \geq 3$. trivial

if $0 < |j-i| \leq 2$, $h(i) \cdot \frac{1}{4}\min\left(1, \frac{h(j)}{h(i)}\right) = \frac{1}{4}h(j)$ Sps. WLOG. $h(i) \geq h(j)$

$h(j) \cdot \frac{1}{4}\min\left(1, \frac{h(i)}{h(j)}\right)$ ✓

trivial for $p_{ii}\,h(i) = \ldots$

(b) [5 points] Show by example that part (a) might be false if we do not assume that $h(i) > 0$ for all i. [For definiteness, we take $\min(1,\ h(j)/h(i)) \equiv 1$ whenever $h(i) = 0$.]

3 4 5 6 7

$h(5) = h(6) = 0$

$p_{35} = p_{45} = p_{46} = 0$

$f_{ij} = 0$ if $i \leq 4$ and $j \geq 5$

$\Downarrow$

$p_{47}^{(n)} = 0 \quad \forall n.$

but $h(7) > 0$.

4. Consider a Markov chain $\{X_n\}$ with state space $S = \{1, 2, 3, 4, 5\}$, $X_0 = 4$, and transition probabilities specified by $p_{11} = p_{55} = 1$, $p_{21} = 5/7$, $p_{24} = p_{25} = 1/7$, $p_{31} = p_{32} = p_{33} = p_{34} = p_{35} = 1/5$, and $p_{43} = p_{45} = 1/2$. Let $T = \min\{n \geq 1 : X_n = 1 \text{ or } 5\}$.

(a) [8 points] Determine (with full explanation) whether or not $\{X_n\}$ is a martingale.

$$\sum_j j\, p_{ij} = i \quad \forall i$$

$i=1, \quad 1 \times p_{11} + 2 \times p_{12} + 3 \times p_{13} + 4 p_{14} + 5 p_{15} = 1$

$i=2, \quad 1 \times p_{21} + 2 p_{22} + 3 p_{23} + 4 p_{24} + 5 p_{25} = \frac{1}{7} \times 4 + \frac{1}{7} \times 5 + \frac{5}{7} = 2$

$i=3, \quad \frac{1}{5}(15) = 3$

$i=4, \quad 3 \times \frac{1}{2} + 5 \times \frac{1}{2} = 4$

$i=5, \quad 5$ ✓

Martingale ✓.

(b) [4 points] Compute $\mathbf{P}(X_T = 5)$. [Hint: part (a) might help.]

~~$E(X_T=5) = E(X_0) = 4$~~.

$E(X_T) = 5 \cdot P(X_T = 5) + 1 P(X_T = 1) = E(X_0) = 4$

$$P(X_T=5) = \frac{4 - P(X_T=1)}{5}$$

$$= \frac{4 - (1 - P(X_T=5))}{5}$$

$$= \frac{3 + P}{5}$$

$5P = 3 + P$

$4p = 3$

$P(X_T=5) = \frac{3}{4}$

5. Consider a Markov chain $\{X_n\}$ on the state space $S = \{0, 1, 2, 3, \ldots\}$, with $X_0 = 100$, and $p_{ij} = 1/(2i+1)$ if $0 \le j \le 2i$, otherwise $p_{ij} = 0$.

(a) [5 points] Prove that $\{X_n\}$ is a martingale. (You may assume without proof that $\mathbf{E}|X_n| < \infty$ for all n.)

$i=0$ 0 ✓

~~$i=1$~~

~~$1p_{11}+2p_{12}+0$~~

$\cancel{= \frac{1}{2\times1+1} + \frac{1}{2\times2+1}\times 2 = \frac{1}{3}+\frac{2}{3} \cdot \frac{11}{15}}$

$\sum p_{ij} \cdot j = \sum_{j=0}^{2i} \frac{j}{2i+1} = \frac{(2i+1)2i/2}{2i+1} = i$ ✓

(b) [5 points] Prove that $\mathbf{P}(\exists n \ge 1 : X_n = 1000) < 1/6$. [Hint: the martingale maximal inequality might help.]

6. Let $\{N(t)\}_{t\geq 0}$ be a Poisson process with rate $\lambda > 0$.

(a) [6 points] Compute the conditional probability $q_\lambda \equiv \mathbf{P}(N(4) = 1 \,|\, N(5) = 3)$.

(b) [2 points] Compute $q_{2\lambda} \,/\, q_\lambda$. (That is, determine the fraction by which the probability in part (a) changes if we replace λ by 2λ.)

[END]