Lecture 22

Arzelà–Ascoli theorem

Let K be a compact subset of $\mathbb{R}^n$. A subset $F \subset C(K, \mathbb{R}^m)$ is compact iff it closed, bounded & equicontinuous

* Example :

Prove that the series, $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly on $\mathbb{R}$.

Proof: $\left|\frac{x}{n(1+nx^2)}\right|$ is even

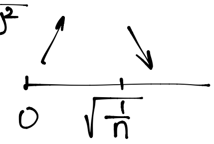
enough to consider on $[0,+\infty)$, $f(x) = \frac{x}{n(1+nx^2)}$

$$f'(x) = \frac{n(1-nx^2)}{n^2(1+nx^2)^2}$$

$1 = nx^2$

$x^2 = \frac{1}{n}$

$n = \sqrt{\frac{1}{n}}$



$$f\left(\frac{1}{\sqrt{n}}\right) = \frac{\sqrt{\frac{1}{n}}}{n(1+n\frac{1}{n})} = \frac{1}{2n^{3/2}}$$

$$\left|\frac{x}{n(1+nx^2)}\right| \leq \frac{1}{2n^{3/2}}$$

$\sum_{n=1}^{\infty} \frac{1}{2n^{3/2}}$ converges. $p = n^{\frac{3}{2}}$, p-series

converges uniformly by Weierstrass M test.

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Ex.: $G = \{g_n : n \geq 1\} \cup \{g\}$

G is compact

Let $a \in K$, let $\varepsilon > 0$ be given

I know that g_n is continuous, so I know that given $\varepsilon > 0, \exists \delta > 0$ s.t.

$\|g_n(x) - g_n(a)\| < \varepsilon$ whenever $|x - a| < \delta$

$\delta = \delta(\varepsilon, a\_h)$

g is continuous $\Rightarrow \exists\ r_0$ s.t. $\|g(x) - g(a)\| < \frac{\varepsilon}{3}$ whenever $\|x-a\| < r_0$. g_n converges to g uniformly $\Rightarrow \exists N$ $\|g_n - g\|_\infty < \frac{\varepsilon}{3}$ $\forall n \geq N$

Therefore:
$$\begin{aligned}\|g_n(x) - g_n(a)\| &= \|g_n(x) - g(x) + g(x) - g(a) + g(a) - g_n(a)\| \\ &\leq \|g_n(x) - g(x)\| + \|g(x) - g(a)\| + \|g(a) - g_n(a)\| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}\end{aligned}$$

for all $n \geq N$ and $\|x - a\| < r_0$

We have finitely many terms. $g_1, g_2, \cdots, g_{N-1}$ for which $r_1, \cdots, r_{N-1}$, r_0 might not work s.t. $\|g_i(x) - g_i(a)\| < \varepsilon$ whenever $\|x - a\| < r_i$.

Let $r = \min\{r_1, \dots, r_{N-1}, r_0\} \Rightarrow$ we have uniform r for all elements of a set G.

Def: A family of functions F from $S \subset \mathbb{R}^n \to \mathbb{R}^m$ is equicontinuous at a pt $a \in S$ if $\forall \varepsilon > 0\ \exists r > 0$ s.t. $\|f(x) - f(a)\| < \varepsilon$ whenever $\|x - a\| < r$ for all $f \in F$.

The family f is equicontinuous on S if it is equicontinuous at each pt of S.

Ex. The family F is uniformly equicontinuous on S if $\forall \varepsilon > 0\ \exists r > 0$ s.t.
$\|f(x) - f(y)\| < \varepsilon$ whenever $\|x - y\| < r$

<u>Ex</u> : $F = \{x^n : n \geq 1\}$ on $[0,1]$

F not equicontinuous on $[0,1]$

(F is closed & bdd, but not compact, recall last class)

F is closed, bounded, but not compact, therefore not equicontinuous by A-A Thm
Not difficult to see that F is not equicontinuous at 1.

Let $\varepsilon = \frac{1}{10}$, and let r be any positive number < 1. We'll show that $\exists x(r)$ s.t.
$|1 - x(r)^n| > \frac{1}{10}$ for large enough n.


Take $x = 1 - \frac{r}{2}$

$|1 - x^n| > \frac{1}{10}$ for suff. large n. $\blacksquare$

<u>Lemma</u> : Let K be a compact subset of $\mathbb{R}^n$. A compact subset F of $C(K, \mathbb{R}^m)$ is equicontinuous.

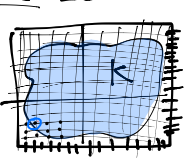
Proposition: If F is an equicontinuous family of functions on a compact set, then it is uniformly equicontinuous

A subset $S \subseteq K \subset \mathbb{R}^m$ is called an ε-net on K if $K \subset \bigcup_{a \in S} B_\varepsilon(a)$

A set K is totally bounded, if it has a finite ε-net for every $\varepsilon > 0$.

<u>Lemma</u>: Let K be a bounded set in $\mathbb{R}^n \Rightarrow$ it is totally bounded
<u>Pf</u>:

Take pt in intersection



$\mathbb{R}^n$

$$\underbrace{[-L, L] \times [-L, L] \times \cdots \times [-L, L]}_{n \text{ terms}}$$

"A Box"