April 2nd

Dec 2012, #5
$T: \mathbb{R}^4 \to \mathbb{R}^4$ given by $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & 0 & 2 & 1 \end{bmatrix}$ find ... <u>Q3 in PS7</u> in TUT

$T: P_1(\mathbb{R}) \to P_2(\mathbb{R})$, $T(p(x)) = (x+1)p(x)$
$\beta$ = standard basis of $P_2$. Find basis $\alpha$ of $P_1(\mathbb{R})$ s.t. $[T]_{\beta\alpha} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$

If $\alpha = \{u(x), v(x)\}$
$u = u_0 + u_1 x$, $v = v_0 + v_1 x$

If $[T]_{\beta\alpha} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$ then must have

$T(u) = 1 + 2x + x^2 = (x+1)(u_0 + u_1 x) = u_0 + (u_0 + u_1)x + u_1 x^2 \Rightarrow u_0 = u_1 = 1$
$T(v) = 2 + 3x + x^2 = (x+1)(v_0 + v_1 x) = v_0 + (v_0 + v_1)x + v_1 x^2 \Rightarrow v_0 = 2,\ v_1 = 1$

So $\alpha = \{1 + x, 2 + x\}$

April, 2012 #6
$V$ inner product space, $T: V \to V$ satisfies $T^2 = T$
Show that (a) $v - T(v) \in \ker(T)$
(b) Prove $V = \ker(T) \oplus \mathrm{Im}(T)$

(a). $T(v - T(v)) = T(v) - T(T(v))$
Since $T^2 = T$, $T(T(v)) = T(v)$ so
$T(v - T(v)) = 0$

(b). Need to show two things
(i) $\ker T \cap \mathrm{im}\, T = \{0\}$. Take $v \in \ker T \cap \mathrm{im}\, T$. let $w \in V$ be s.t. $v = T(w)$.
Since $v \in \ker T \Rightarrow 0 = T(v) = T^2(w) = T(w) = v$
(ii) Need to show every $v \in V$ can be written as $v = v_1 + v_2$ where $v_1 \in \ker T$
$v_2 \in \mathrm{im}\, T$
By (i) $v - T(v) \in \ker T$ so take $v_1 = v - T(v)$.
Let $v_2 = T(v) \in \mathrm{im}\, T$. Clearly $v_1 + v_2 = v$ ✓.

Dec 2011, #6
Let $V, W$ be inner product spaces over $\mathbb{R}$. Let $T: V \to W$. lin. tsfm & let
$x_1, \dots, x_k \in V$ s.t. $\{T(x_1), \dots, T(x_k)\}$ is a basis of $\mathrm{im}\, T$.

(a) Prove $\{T^*T(x_1), \dots, T^*T(x_k)\}$ is a lin. indpt subset of $\mathrm{im}\, T^*$ and show this implies $\mathrm{rank}(T) \leq \mathrm{rank}(T^*)$
(b). Explain why this $\Rightarrow \mathrm{rank}(T) = \mathrm{rank}(T^*)$

Hint: for part (a) show that

$T^*|_{\text{im}T} : \text{im}T \longrightarrow V$ injective

~~Solution~~: Sps $T^*|_{\text{im}T}$ injective then if $a_1T^*Tx_1 + \cdots + a_kT^*Tx_k = 0$ then

$T^*(a_1Tx_1 + \cdots + a_kTx_k) = 0$
Since $a_1Tx_1 + \cdots + a_kTx_k \in \text{im}T$ this implies $a_1Tx_1 + \cdots + a_kTx_k = 0$. Since $\{Tx_1, \cdots, Tx_k\}$ ind $\Rightarrow a_1 = \cdots = a_k = 0 \Rightarrow \{T^*Tx_1, \cdots, T^*Tx_k\}$ ind.

Proof Hint: $w \in \ker(T^*|_{\text{im}T})$
Want to show $w = 0$.
$w \in \text{im}T$ so $w = Tv$ for some $v \in V$.
and $T^*w = 0$, i.e. $T^*Tv = 0$
$\Rightarrow \langle v', T^*Tv \rangle = 0$ for any $v' \in V$
In particular, $\langle v, T^*Tv \rangle = 0 \Rightarrow \langle T(v), T(v) \rangle = 0 \Rightarrow T(v) = 0$
$\Rightarrow w = 0$ $\text{rank}(T) = k \leq \text{rank}(T^*)$.

(b). explain why this implies $\text{rank}(T) = \text{rank}(T^*)$.
Since $\{T^*Tx_1, \cdots, T^*Tx_k\}$ is a lin ind. subset of $\text{im}T^*$.

For (b), apply argument in (a) to $T^* : W \to V$. By (a) we have

$\text{rank}(T^*) \leq \text{rank}\,(T^*)^* = \text{rank}(T)$

So $\text{rank}(T^*) = \text{rank}(T)$

Dec 2011, #3

$A = \begin{bmatrix} 2 & 1-i \\ 1+i & 1 \end{bmatrix}$. Find spectral decomposition

General idea: $A : V \to V$ self-adj. then $A = \lambda_1 P_1 + \cdots + \lambda_k P_k$ where $\lambda_1, \cdots, \lambda_k$ eigenvalues. $P_i$ = orthog. proj. onto $E_{\lambda_i}$

Eigenvalues 0, 3.
In an orthonormal basis of eigenvectors of $A$. $\alpha$
$[A]_\alpha = 0 \cdot [P_1]_\alpha + 3[P_3]_\alpha$.

$$\begin{bmatrix} 0 & \\ & 3 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & \\ & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

Find $\alpha$ and then compute $[I]^\beta_\alpha \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [I]^\alpha_\beta$ to get $P_0$

where $\beta$ = std basis of $\mathbb{C}^2$. and to get $P_3$ we could either

① compute $[I]_\alpha^\beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} [I]_\beta^\alpha$ or

② since $A = 3P_3 \Rightarrow P_3 = \frac{1}{3}A$