Lecture 4

Tools for Integration.

Working with complex numbers and the sophisticated machinery of contour integration will be needed in this course.

This week we are going to look at this beautiful area of Mathematics.

These tools will be very useful for calculations of the form

$$\int f(x)\, dF_y(x)$$

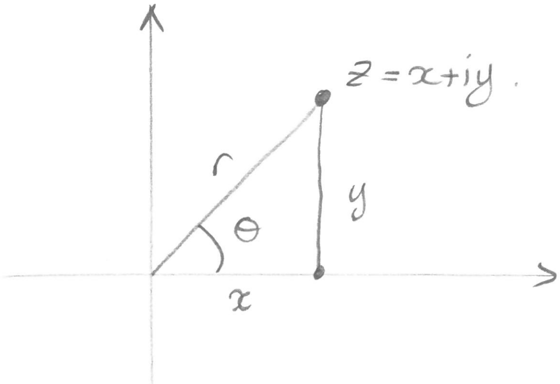
where F_y is the MP distribution, for example.

Complex numbers and elementary functions.

With $i^2 := -1$, a complex number is an expression of the form $z = x + iy$. We write $Re(z) = x$ and $Im(z) = y$.

We can also write complex numbers in Polar form (r, θ)

$$x = r\cos(\theta) \qquad y = r\sin(\theta) \qquad (r \geq 0)$$



The Complex plane $\mathbb{C}$.

A complex number z can be written

$$z = x + iy = r(\cos\theta + i\sin\theta) \qquad (*)$$

The radius $r = \sqrt{x^2+y^2} =: |z|$. (aka. modulus of z) and the angle θ is the argument of z, denoted $\arg z$.

When $z \neq 0$, we can find θ by trigonometry

$$\tan\theta = \frac{y}{x}.$$

$\theta = \arg z$ is multivalued as $\tan\theta$ is a periodic function of θ with period π.

Example: $z = -1 + i$ $\quad |z| = r = \sqrt{2}$ $\quad \theta = \frac{3\pi}{4} + 2n\pi, \; n = 0, \pm 1, \pm 2, \ldots$

We can define the (polar) exponential.

$$\cos\theta + i\sin(\theta) = e^{i\theta} \quad \overset{(\text{by } *)}{\Longrightarrow} \quad z = re^{i\theta}$$

Some beautiful formulas:

$$e^{2\pi i} = 1 \,, \quad e^{\pi i} = -1 \,, \quad \dots$$

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} \,, \quad \left(e^{i\theta}\right)^m = e^{im\theta} \,, \quad \left(e^{i\theta}\right)^{1/n} = e^{i\theta/n}$$

The <u>complex conjugate</u> of z is $\bar{z} = x - iy = re^{-i\theta}$

The usual rules apply: $z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$.

$$\begin{aligned} z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1). \end{aligned}$$

Notice that $z\bar{z} = \bar{z}z = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

so that

$$\begin{aligned} \frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} &= \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} \\ &= \frac{(x_1x_2 + y_1y_2) + i(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{(x_2y_1 - x_1y_2)}{x_2^2 + y_2^2} \end{aligned}$$

We can define some <u>elementary functions</u> of complex argument. The simplest is

$$f(z) = z^n \qquad n = 0, 1, 2, \dots \qquad \text{"Power function"}$$

A polynomial of order n:

$$P_n(z) = \sum_{j=0}^{n} a_j z^j = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

where a_j are complex numbers.

A rational function:

$$R(z) = \frac{P_n(z)}{Q_m(z)}, \quad P_n(z), Q_m(z) \text{ polynomials.}$$

In general the function $f(z)$ is complex-valued and can be written

$$f(z) = \underbrace{u(x,y)}_{\mathrm{Re}\, f} + i \underbrace{v(x,y)}_{\mathrm{Im}\, f}$$

Example: $z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{\substack{\mathrm{Re} \\ u(x,y)}} + i \underbrace{2xy}_{\substack{\mathrm{Im.} \\ v(x,y).}}$

We can define the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

and it is easy to show $e^z = e^x(\cos y + i \sin y)$.

$$e^{z_1+z_2} = e^{z_1} e^{z_2} \qquad (e^z)^n = e^{nz} \qquad n = 1, 2, \ldots$$

$$|e^z| = |e^x|\,|\cos y + i \sin y| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x$$

$$\overline{(e^z)} = e^{\bar{z}} = e^x(\cos y - i \sin y).$$

Trig functions:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2i}$$

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}, \quad \sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Hyperbolic functions: $\sinh z = \dfrac{e^z - e^{-z}}{2}$, $\cosh z = \dfrac{e^z + e^{-z}}{2}$

$$\tanh z = \frac{\sinh z}{\cosh z}, \quad \ldots$$

Note that

$$\sinh iz = i \sin z \qquad \sin iz = i \sinh z$$

$$\cosh iz = \cos z \qquad \cos iz = \cosh z.$$

Most of the definitions could have been introduced through the concept of power series.

The power series of $f(z)$ around the point $z=z_0$ is

$$f(z) = \lim_{n\to\infty} \sum_{j=0}^{n} a_j (z-z_0)^j = \sum_{j=0}^{\infty} a_j (z-z_0)^j$$

where a_j, a_0 are constants.

Remember convergence only occurs within some radius, ie, within some circle $|z-z_0| = R$

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| .$$

We have the power series representations:

$$e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!} , \quad \sin z = \sum^{\infty} \frac{(-1)^j z^{2j+1}}{(2j+1)!} ,$$

$$\cos z = \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{(2j)!} , \quad \sinh z = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!} , \quad \cosh z = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$$

Let $f(z)$ be defined in some region R containing the neighborhood of a point z_0. The <u>derivative</u> of $f(z)$ at $z = z_0$, denoted by $f'(z_0)$ or $\frac{d}{dz} f(z_0)$ is defined by.

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right)$$

provided the limit exists. Alternatively, writing $\Delta z = z - z_0$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

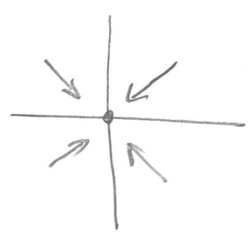
<u>Caution</u>: A continuous function is not necessarily differentiable as complex functions have a two-dimensional character.

For example, $f(z) = \overline{z}$.

$$\lim_{\Delta z \to 0} \frac{\overline{(z_0 + \Delta z)} - \overline{z_0}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = c$$

and a unique value for c cannot be found!

<u>NOT DIFFERENTIABLE!</u>



Differentiable complex functions are called <u>analytic</u>.

If f and g have derivatives:

$$(f+g)' = f' + g', \quad (fg)' = f'g + g'f$$

$$\left(\frac{f}{g}\right)' = (f'g - fg')/g^2 \quad (g \neq 0)$$

If $f'(g(z))$ and $g'(z)$ exist, then

$$[f(g(z))]' = f'(g(z))g'(z).$$

Since $\dfrac{(z+\Delta z)^n - z^n}{\Delta z} = nz^{n-1} + a_1 z^{n-2}\Delta z + a_2 z^{n-3}(\Delta z)^2 + \cdots + (\Delta z)^n \to nz^{n-1}$

as $\Delta z \to 0$, where a_j are binomial coefs. of $(a+b)^n$.

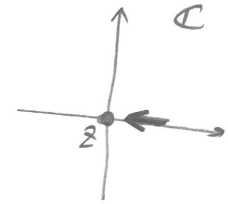
we have $\dfrac{d}{dz}(z^n) = nz^{n-1}$.

It follows (formally) that

$$\frac{d}{dz}\left(\sum_{n=0}^{\infty} a_n z^n\right) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

inside radius of convergence.

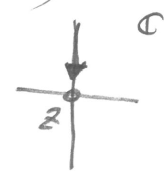
Writing $f(z) = u(x,y) + iv(x,y)$,



$$f'(z) = \lim_{\Delta x \to 0} \left(\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x,y)}{\Delta x} \right)$$

$$= u_x(x,y) + i v_x(x,y) \qquad u_x := \frac{\partial u}{\partial x}, \quad v_x := \frac{\partial v}{\partial x}$$

$$f'(z) = \lim_{\Delta y \to 0} \left(\frac{u(x, y+\Delta y) - u(x,y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x,y)}{i\Delta y} \right)$$



$$= -i u_y(x,y) + v_y(x,y)$$

Hence, equating these two expressions we have the <u>Cauchy-Riemann equations:</u>

$$u_x = v_y, \quad v_x = -u_y.$$

that are necessarily satisfied if $f(z)$ is differentiable.

<u>Theorem:</u> The function $f(z) = u(x,y) + iv(x,y)$ is differentiable at a point $z = x+iy$ of a region in the complex plane if and only if u_x, u_y, v_x, v_y are continuous and satisfy the Cauchy-Riemann equations.

We shall now look at how to evaluate integrals of complex-valued functions along curves in the complex plane.

First, consider the case of the complex-valued function f of the real variable t, on the interval $a \leq t \leq b$.

$$f(t) = u(t) + i\,v(t).$$

We say that f is <u>integrable</u> if u & v are integrable, then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

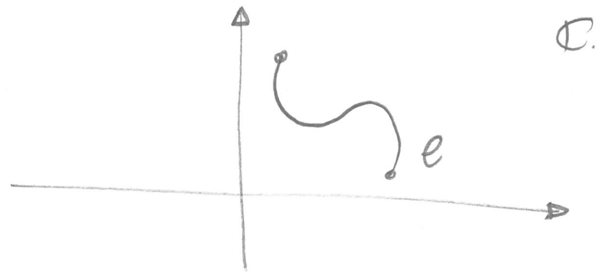
Usual rules apply:

$$\frac{d}{dt}\int_a^t f(\tau)\,d\tau = f(t)$$

"Fundamental theorem of calculus".

and if $f'(t)$ is continuous, $\int_a^b f'(t)\,dt = f(b) - f(a).$

Integration along a curve.



A curve in $\mathbb{C}$ can be decribed by the <u>parametrisation</u>

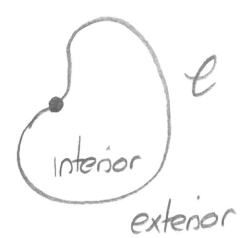
$$z(t) = x(t) + iy(t), \quad a \le t \le b.$$

A curve $\mathcal{C}$ is called <u>simple</u> if it does not intersect itself.
It is called <u>differentiable curve</u> if $z'(t) = x'(t) + iy'(t)$ is non-null.

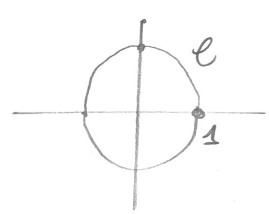
A <u>piecewise differentiable curve</u> (or <u>path</u>) is obtained by joining a finite number of differentiable curves.

Let $\mathcal{C}$ be a path, we call it <u>closed</u> if $z(a) = z(b)$.

A closed path is also called a <u>contour</u>.



<u>Example:</u> The unit circle in $\mathbb{C}$ is a contour and is parametrised by $z(t) = e^{it}$, $0 \le t \le 2\pi$



The <u>contour integral</u> of a piecewise continuous function on a <u>smooth contour</u> (ie. differentiable) is defined to be

$$\int_{\mathcal{C}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

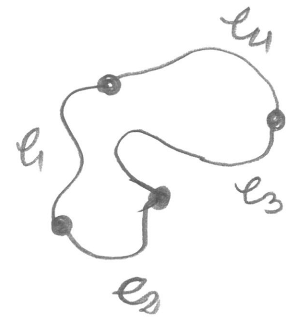
$dz \approx z'(t)\,dt.$

Usual properties apply:

$$\int_{\mathcal{C}} [\alpha f(z) + \beta g(z)]\,dz = \alpha \int_{\mathcal{C}} f(z)\,dz + \beta \int_{\mathcal{C}} g(z)\,dz$$

If we traverse the contour in the opposite direction (ie. from $t=b$ to $t=a$) then this is denoted $-\mathcal{C}$. and

$$\int_{-\mathcal{C}} f(z)\,dz = -\int_{\mathcal{C}} f(z)\,dz.$$



And if $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \dots \cup \mathcal{C}_n$.

then $$\int_{\mathcal{C}} f = \sum_{j=1}^{n} \int_{\mathcal{C}_j} f$$

From the fundamental theorem of calculus we get:

<u>Theorem:</u> Suppose $F(z)$ is an analytic function such that $f(z) = F'(z)$ is continuous in a domain $\mathcal{D}$. Then for the contour $\mathcal{C}$ lying in $\mathcal{D}$ with endpoints z_1 and z_2

$$\int_{\mathcal{C}} f(z)\,dz = F(z_2) - F(z_1).$$

$$z(a) = z_1$$
$$z(b) = z_2.$$

<u>Proof:</u>
$$\int_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} F'(z)\,dz = \int_a^b F'(z(t))\,z'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}\left[F(z(t))\right] dt = F(z(b)) - F(z(a)) = F(z_2) - F(z_1).$$
$\blacksquare$

Q: What happens if $\mathcal{C}$ is a <u>closed contour</u>?

$$\oint_{\mathcal{C}} f(z)\,dz = \int_{\mathcal{C}} F'(z)\,dz = 0.$$

" $\oint$ " denotes integration along a closed contour $\mathcal{C}$.

Notice that this holds for <u>any</u> closed contour $\mathcal{C}$. So this integral is independent of the path.

<u>Theorem:</u> Let $f(z)$ be analytic interior to and on a simple closed contour $\mathscr{C}$. Then at any interior point z

$$f(z) = \frac{1}{2\pi i} \oint_{\mathscr{C}} \frac{f(\xi)}{\xi - z} \, d\xi .$$

This is the <u>Cauchy integral formula</u>.

" The function f is completely determined by the points $z \in \mathscr{C}$ "

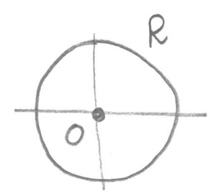
Further, we can also say something about all the derivatives of f.

<u>Theorem:</u> If $f(z)$ is analytic interior to and on a simple closed contour $\mathscr{C}$, then all the derivatives $f^{(k)}(z)$, $k = 1, 2, \dots$ exist in the domain $\mathscr{D}$ interior to $\mathscr{C}$ and

$$f^{(k)}(z) = \frac{k!}{2\pi i} \oint_{\mathscr{C}} \frac{f(\xi)}{(\xi - z)^{k+1}} \, d\xi .$$

If $f(z)$ is an analytic function, we can establish its <u>Taylor series</u> on its domain $\mathscr{D} = \{z : |z| \leq R\}$:

$$f(z) = \sum_{j=0}^{\infty} b_j z^j , \quad b_j := \frac{f^{(j)}(0)}{j!} .$$

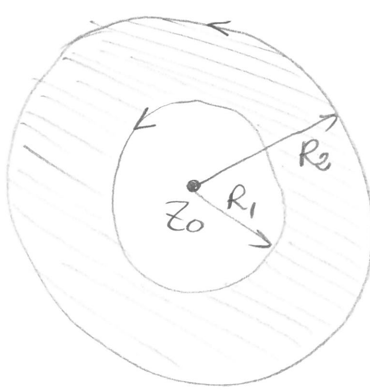


<u>Example:</u> $e^z = \sum_{j=0}^{\infty} \frac{z^j}{j!}$, $|z| < \infty$

In many situations we encounter functions that are not analytic everywhere in $\mathbb{C}$. Typically, they are not analytic at a point, or in some region.

This means that Taylor series <u>cannot</u> be applied.

Luckily, another series representation can sometimes be found in terms of positive and negative powers of $(z - z_0)$.



Circular annulus
$R_1 \le |z - z_0| \le R_2$.

<u>Theorem:</u> (Laurent Series) A function $f(z)$ analytic in an annulus $R_1 \le |z - z_0| \le R_2$ may be represented by the expansion

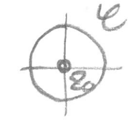
$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

in the region $R_1 < R_a \le |z - z_0| \le R_b < R_2$ where

$c_n := \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{f(z)}{(z - z_0)^{n+1}} dz$ and $\mathcal{C}$ is any simple closed contour in region of analyticity enclosing $|z - z_0| < R_1$

So $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$ and c_{-1} is called the <u>residue</u> of f. The negative powers are called the <u>principal part</u> of f.

$$c_{-1} := \frac{1}{2\pi i} \oint_{\mathcal{C}} f(z)\, dz$$



We call a point z_0 an <u>isolated singularity</u> of a function f if the function is analytic in the punctured disk

$$\mathcal{D} = \{ z : 0 < |z - z_0| \leq R \}.$$

Three types of singular points exist:

- A <u>removable singularity point</u> is when the Laurent series at the point has no terms with negative power $n<0$.
- A <u>pole of order m</u> is an isolated singularity point such that
$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n \qquad a_{-m} \neq 0.$$
- An <u>essential singularity point</u> is an isolated singularity point where the Laurent series has infinitely many terms with negative power $n<0$.

<u>Theorem</u>: Let $f(z)$ be analytic inside and on a simple closed contour $\mathscr{C}$, except for a finite number of isolated singular points $z_1, z_2, \dots, z_N$ located inside $\mathscr{C}$. Then

$$\oint f(z)\,dz = 2\pi i \sum_{j=1}^{N} a_j$$

where a_j is the <u>residue</u> of $f(z)$ at $z = z_j$, denoted by

$$a_j := \mathrm{Res}(f(z), z_j).$$