CSC336 Assignment 2

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Problem 1

(a) Solution:

$$\begin{aligned}\kappa_\infty(A) &= \|A\|_\infty \|A^{-1}\|_\infty \\ &= \max\{6+13+17, 13+29+38, 17+38+50\} \cdot \max\{11, 22, 13\} \\ &= 105 \cdot 22 \\ &= 2310\end{aligned}$$

(b) Solution:

$Ax = b$, so $x = A^{-1}b$.
Let $r = b - A\hat{x}$, then $\|r\|_\infty \leq 0.01$.
We know that $\|\hat{x} - x\|_\infty = \|A^{-1}(A\hat{x} - b)\|_\infty = \|A^{-1}(-r)\|_\infty \leq \|A^{-1}\|_\infty \| - r\|_\infty$.
So $\|x - \hat{x}\|_\infty \leq \|A^{-1}\|_\infty \|r\|_\infty \leq 22 \cdot 0.01 = 0.22$.

(c) Solution:

$$
\begin{aligned}
\frac{\|x-\hat{x}\|_\infty}{\|x\|_\infty} &= \frac{\|A^{-1}b - A^{-1}A\hat{x}\|_\infty}{\|x\|_\infty} \\
&\leq \frac{\|A^{-1}\|_\infty \|b - A\hat{x}\|_\infty}{\|x\|_\infty} \\
&= \frac{\|A^{-1}\|_\infty \|A\|_\infty \|b - A\hat{x}\|_\infty}{\|A\|_\infty \|x\|_\infty} \\
&= \kappa_\infty(A) \frac{\|b - A\hat{x}\|_\infty}{\|A\|_\infty \|x\|_\infty} \\
&\leq \kappa_\infty(A) \frac{\|r\|_\infty}{\|b\|_\infty} \\
&= 2310 \cdot \frac{0.01}{\|b\|_\infty} \\
&= \frac{23.1}{\|b\|_\infty}
\end{aligned}
$$

Note that $\|b\|_\infty \leq \|A\|_\infty \|x\|_\infty$, then $\frac{1}{\|b\|_\infty} \geq \frac{1}{\|A\|_\infty \|x\|_\infty}$.

Problem 2

Solution:

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix},\quad b=\begin{pmatrix} 0.4313\times10^{-1} \\ 0.6757\times10^{1} \end{pmatrix}$$

(1). Without Pivoting

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \begin{pmatrix} 0.211\times10^{-2} & 0.8204\times10^{-1} \\ \boxed{0.1597\times10^{3}} & 0.1284\times10^{2}-0.8204\times0.1597\times10^{2} \end{pmatrix}$$

$$=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ \boxed{0.1597\times10^{3}} & -0.2600\times10^{0} \end{pmatrix},\ L=\begin{pmatrix} 1 & 0 \\ 0.1597\times10^{3} & 1 \end{pmatrix},\ U=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0 & -0.2600\times10^{0} \end{pmatrix}$$

take $fl(\cdot)$ each step

$$Ax=b \iff LUx=b \iff \begin{cases} Ly=b \\ Ux=y \end{cases}$$

$Ly=b$

$$\begin{pmatrix} 1 & 0 \\ 0.1597\times10^{3} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0.4313\times10^{-1} \\ 0.6757\times10^{1} \end{pmatrix} \Rightarrow \begin{cases} y_1=0.4313\times10^{-1} \\ y_2=0.6757\times10^{1}-0.1597\times0.4313\times10^{2} \\ \quad =0.6757\times10^{1}-0.6888\times10^{1} \\ \quad =-0.1310\times10^{0} \end{cases}$$

$Ux=y$

$$\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0 & -0.2600\times10^{0} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0.4313\times10^{-1} \\ -0.1310\times10^{0} \end{pmatrix} \Rightarrow \begin{cases} \hat{x}_1=\dfrac{0.4313\times10^{-1}-0.8204\times0.5038\times10^{-1}}{0.2110\times10^{-2}} \\ \quad =\dfrac{0.4313\times10^{-1}-0.4133\times10^{-1}}{0.2110\times10^{-2}} \\ \quad =\dfrac{0.1797\times10^{-2}}{0.2110\times10^{-2}} \\ \quad =0.8517\times10^{0} \\ \hat{x}_2=\dfrac{-0.1310}{-0.2600}=0.5038\times10^{0} \end{cases}$$

So $\hat{x}=(0.8517\times10^{0}, 0.5038\times10^{0})^T$.

(2). With partial pivoting

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix} \xrightarrow[k=1]{\text{piv}} \begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ 0.2110\times10^{-2} & 0.8204\times10^{-1} \end{pmatrix},\ P_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$ipiv=(2,\cdot)$,

$$\xrightarrow[k=1]{\text{elim}} \begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ \boxed{\dfrac{0.2110\times10^{-2}}{0.3370\times10^{0}}} & 0.8204\times10^{-1}-0.1284\times10^{2}\times0.6261\times10^{-2} \end{pmatrix}=\begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ \boxed{0.6261\times10^{-2}} & 0.1650\times10^{-2} \end{pmatrix}$$

$$L=\begin{pmatrix}1 & 0\\ 0.6261\times10^{-2} & 1\end{pmatrix},\quad U=\begin{pmatrix}0.3370\times10^{0} & 0.1284\times10^{2}\\ 0 & 0.1650\times10^{-2}\end{pmatrix}$$

$P_1Ax=LUx=P_1b$, note $P=P_1$

$$\begin{cases}Ly=P_1b\\ Ux=y\end{cases}$$

$$\begin{pmatrix}1 & 0\\ 0.6261\times10^{-2} & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0.4313\times10^{-1}\\ 0.6757\times10^{1}\end{pmatrix}\Rightarrow\begin{cases}y_1=0.6757\times10^{1}\\ y_2=0.4313\times10^{-1}-0.6261\times10^{-2}\times0.6757\times10^{1}\\ \quad=0.4313\times10^{-1}-0.4231\times10^{-1}\\ \quad=0.8200\times10^{-3}\end{cases}$$

$$\begin{pmatrix}0.3370\times10^{0} & 0.1284\times10^{2}\\ 0 & 0.1650\times10^{-2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0.6757\times10^{1}\\ 0.82\times10^{-3}\end{pmatrix}$$

$$\Rightarrow\begin{cases}x_2=\dfrac{0.8200\times10^{-3}}{0.1650\times10^{-2}}=0.4970\times10^{0}\\[2ex] x_1=\dfrac{0.6757\times10^{1}-0.1284\times10^{2}\times0.4970\times10^{0}}{0.3370\times10^{0}}=\dfrac{0.6757\times10^{1}-0.6381\times10^{1}}{0.3370\times10^{0}}=\dfrac{0.3760}{0.3370}\\ \qquad=0.1116\times10^{1}\end{cases}$$

So $\hat{x}=(0.1116\times10^{1},\ 0.4970\times10^{0})$

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(3). Complete pivoting

$$A=\begin{pmatrix}0.2110\times10^{-2} & 0.8204\times10^{-1}\\ 0.3370\times10^{0} & 0.1284\times10^{2}\end{pmatrix}\xrightarrow[P_1,\,Q_1]{\text{complete piv}}\begin{pmatrix}0.1284\times10^{2} & 0.3370\times10^{0}\\ 0.8204\times10^{-1} & 0.2110\times10^{-2}\end{pmatrix}$$

$$\xrightarrow[k=1]{\text{elim}}\left(\begin{array}{c|c}0.1284\times10^{2} & 0.3370\times10^{0}\\ \hline \dfrac{0.8204\times10^{-1}}{0.1284\times10^{2}} & 0.2110\times10^{-2}-0.3370\times10^{0}\times\dfrac{0.8204\times10^{-1}}{0.1284\times10^{2}}\end{array}\right)$$

$$=\left(\begin{array}{c|c}0.1284\times10^{2} & 0.3370\times10^{0}\\ \hline 0.6389\times10^{-2} & -0.4300\times10^{-4}\end{array}\right),\ L=\begin{pmatrix}1&0\\0.6389\times10^{-2}&1\end{pmatrix},\ U=\begin{pmatrix}0.1284\times10^{2} & 0.3370\times10^{0}\\ 0 & -0.4300\times10^{-4}\end{pmatrix}$$

$$P_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ P=P_1,\ Q_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ Q=Q_1,\ PAQ=LU$$

$PAQ=LU, Ax=b$

$PAQQ^Tx=Pb\Rightarrow LU\hat{x}=Pb,\ \hat{x}=Q^Tx$

$Ly=Pb$

$$\begin{pmatrix}1 & 0\\ 0.6389\times10^{-2} & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0.4313\times10^{-1}\\ 0.6757\times10^{1}\end{pmatrix}\Rightarrow\begin{cases}y_1=0.6757\times10^{1}\\ y_2=0.4313\times10^{-1}-0.6757\times10^{1}\times0.6389\times10^{-2}\\ \quad=0.4313\times10^{-1}-0.4317\times10^{-1}\\ \quad=-0.4\times10^{-4}\end{cases}$$

$Uz = y$

$$\begin{pmatrix} 0.1284\times10^{2} & 0.3370\times10^{0} \\ 0 & -0.4300\times10^{-4} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.6757\times10^{1} \\ -0.4\times10^{-4} \end{pmatrix}$$

$$\Rightarrow \begin{cases} z_2 = \dfrac{-0.4\times10^{-4}}{-0.4300\times10^{-4}} = 0.9302 \\ z_1 = \dfrac{0.6757\times10^{1} - 0.9302 \times 0.3370\times10^{0}}{0.1284\times10^{2}} \end{cases}$$

$$= \frac{0.6757\times10^{1} - 0.3135}{0.1284\times10^{2}} = \frac{0.6444\times10^{1}}{0.1284\times10^{2}} = 0.5019$$

$Q = Q^{-1} = Q^{T}$

$$x = Qz = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0.5019 \\ 0.9302 \end{pmatrix} = \begin{pmatrix} 0.9302 \\ 0.5019 \end{pmatrix}$$

so $\hat{x} = (0.9302 \,,\, 0.5019)^{T}$

Summarize

	$\frac{\lvert x_1-\hat{x}_1\rvert}{\lvert x_1\rvert}$	$\frac{\lvert x_2-\hat{x}_2\rvert}{\lvert x_2\rvert}$	$\frac{\lVert x-\hat{x}\rVert_\infty}{\lVert x\rVert_\infty}$
(1). without pivoting	0.1483	0.0076	0.1483
(2). partial pivoting	0.116	0.006	0.116
(3). Complete pivoting	0.0698	0.0038	0.0698

Comments:

① The complete pivoting has the least relative error in each component and in infinity norm

② Partial pivoting reduces round-off errors already.

③ Complete pivoting however, is not necessarily more accurate than partial pivoting.

Problem 3

Rewrite the three equations:

$$\begin{aligned} t_1 - m_1 t_n &= c_1 v - m_1 g \\ -t_{i-1} + t_i - m_i t_n &= c_i v - m_i g, \text{for } i \in [2, n-1] \\ -t_{n-1} - m_n t_n &= c_n v - m_n g \end{aligned}$$

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & -m_1 \\ -1 & 1 & 0 & \cdots & -m_2 \\ 0 & -1 & 1 & \cdots & -m_3 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & -m_n \end{pmatrix}, b = \begin{pmatrix} c_1 v - m_1 g \\ c_2 v - m_2 g \\ c_3 v - m_3 g \\ \vdots \\ c_n v - m_n g \end{pmatrix}$$

> Note that the direct results of running 3 scripts mentioned below can be found in the appendix at the end of this assignment.

(a) Solution
In case (i), we run the following script:

```
% Take g = 10, v = 6 as constants.
% (a) part i
kappa_vec = [];
t = zeros(32, 4);
ni = [4, 8, 16, 32];
for i = 1:4
    n = ni(i);
    m = linspace(50, 100, n);
    c = 25 - 10*linspace(0, 1, n);
    b = transpose(6.*c -10.*m);
    e = ones(n, 1);
    A = spdiags([-e, e], [-1, 0], n, n);
    A(:, n) = -m;
    tension = A\b;
    display(tension); % output of tension vector
    kappa = condest(A);
    display(kappa); % output of condition number of the matrix
    max_tension = max(tension);
    display(max_tension); % output of maximum tension computed
```

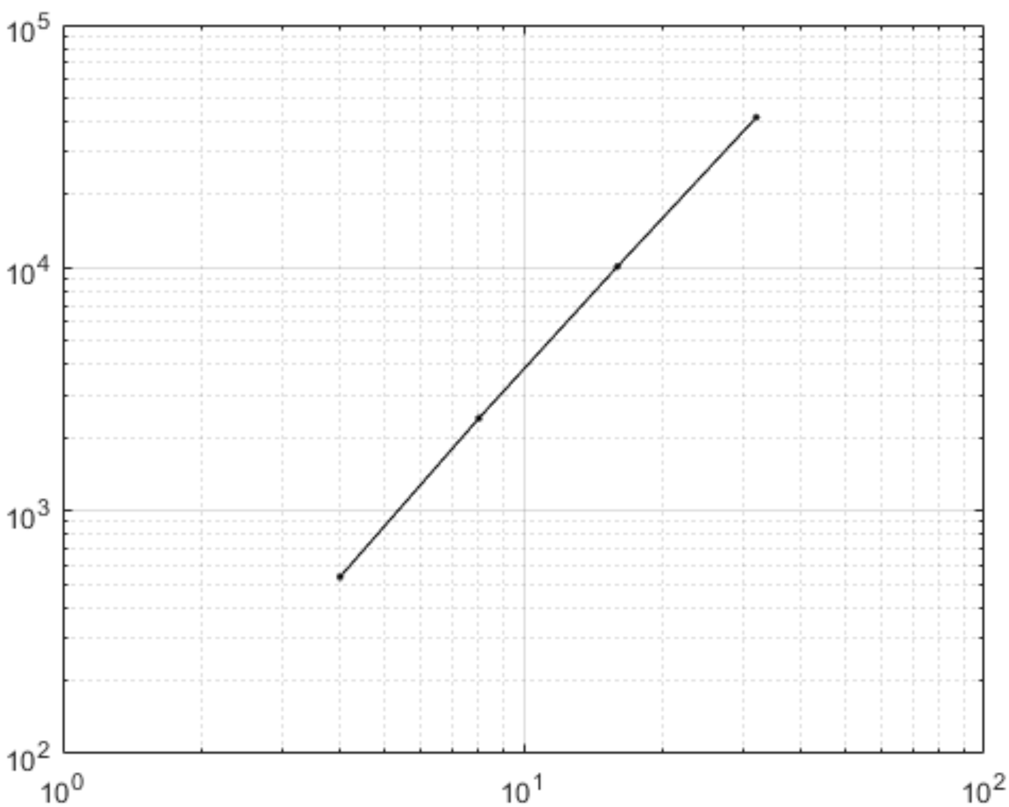
```
    kappa_vec = [kappa_vec, kappa]; % store condtion number
    t(1:n,i) = tension; % store tension vectors
end
figure
loglog([4 8 16 32], kappa_vec, 'k.-') % plot condition number vs. n
grid

figure
plot((1:ni(1)-1)/ni(1), t(1:ni(1)-1, 1), 'r-', ...
   (1:ni(2)-1)/ni(2), t(1:ni(2)-1, 2), 'g--', ...
   (1:ni(3)-1)/ni(3), t(1:ni(3)-1, 3), 'b-.', ...
   (1:ni(4)-1)/ni(4), t(1:ni(4)-1, 4), 'k.');
grid
```

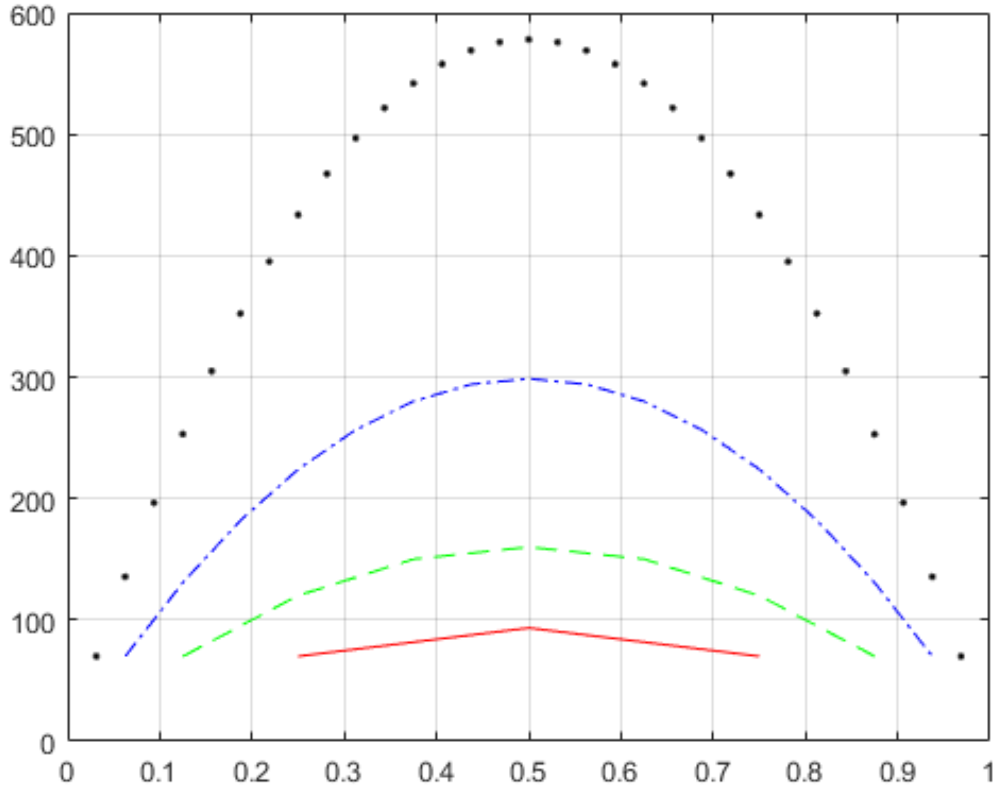
According to the output of this script, we can conclude the condition number of the matrix and maximum tension in one table:

	$n = 4$	$n = 8$	$n = 16$	$n = 32$
condition number	534	2401	10134	41601
maximum tension	93	160	299	578

The plot in log-log scale of condtiion numbers versus n is shown as below:



The plot of tension vectors components versus their normalized index is shown as:



In details,

- red curve (the lowest) indicates the case of $n = 4$,
- green curve (the 2nd lowest) indicates the case of $n = 8$,
- blue curve (the 3rd lowest) indicates the case of $n = 16$,
- black dot curve (the highest) indicates the case of $n = 32$.

In case (ii), we run the modified script instead, which randomly picks m_i and c_i:

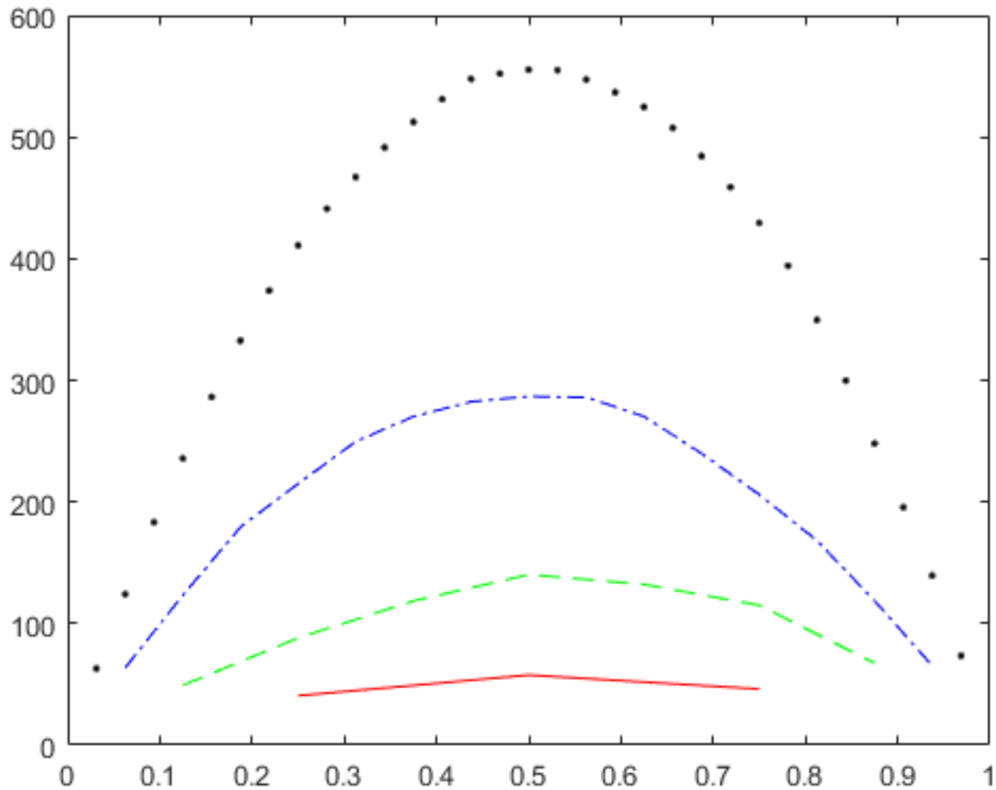
```
% (a) part ii


t = zeros(32, 4);
ni = [4, 8, 16, 32];
for i = 1:4
    n = ni(i);
    m = sort(50 + 50*rand(n, 1), 'ascend');
    c = sort(15 + 10*rand(n, 1), 'descend');
    b = 6.*c -10.*m;
    e = ones(n, 1);
    A = spdiags([-e, e], [-1, 0], n, n);
    A(:, n) = -m;
    tension = A\b;
    display(tension); % output of tension vector
    max_tension = max(tension);
    display(max_tension); % output of maximum tension computed
    t(1:n,i) = tension; % store tension vectors
end

```

```
plot((1:ni(1)-1)/ni(1), t(1:ni(1)-1, 1), 'r-', ...
   (1:ni(2)-1)/ni(2), t(1:ni(2)-1, 2), 'g--', ...
   (1:ni(3)-1)/ni(3), t(1:ni(3)-1, 3), 'b-.', ...
   (1:ni(4)-1)/ni(4), t(1:ni(4)-1, 4), 'k.');
```

This time we only need to show the plot of tension vectors components versus their normalized index:



And we notice that the plot is not as perfectly symmetical as above, but it stays in a very similar shape.

- **Comments on:**
  - **how the acceleration and the maximum tension behave with n.**
    - The acceleration a stays the same as n increases. Note that in this problem we substitute a with t_n, which is the last part of tension, i.e. the last tension vector component.
    - The maximum tension increases as n increases.
  - **how the components of the tension vectors vary with their index.**
    - By observing two "bell-shape" plots we can easily notice that, no matter what n value we pick, the general trend as bell-curve remains. And for larger n, the curve is more concave (with higher top).
  - **where (for which i) the max tension occurs.**
    - The max tension (component) tends to sit in the middle of its tension vector. Specifically, our n is even, index $i = \frac{n}{2}$ indicates max tension.
  - **how the condition numbers behave with n.**
    - The condition numbers increase as n increases. And on log-log scale, such increase is almost proportional.

(b) Solution:

To help us get an understanding of L, U, P, we suppose $n = 4$, and we apply the case (i) script.

```
% (b)
ni = 4;
m = linspace(50, 100, ni);
c = 25 - 10*linspace(0, 1, ni);
b = 6.*c -10.*m;
e = ones(ni, 1);
A = spdiags([-e, e], [-1, 0], ni, ni);
A(:, ni) = -m;
[L, U, P] = lu(A);
display(L);
display(U);
display(P);
```

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}, U = \begin{pmatrix} 1 & 0 & 0 & -50 \\ 0 & 1 & 0 & -116.6667 \\ 0 & 0 & 1 & -200 \\ 0 & 0 & 0 & -300 \end{pmatrix}, P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we change n to 8, the corresponding L, U, P are:

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -50 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -107.1429 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -171.4286 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -242.8571 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -321.4286 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -407.1429 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -500.0000 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -600.0000 \end{pmatrix},$$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The form of P would be identity matrix $I_{n\times n}$. This can be explained by the form of matrix A where each row is perfect (no need to reorder for partial pivoting), so the permutation matrix should be identity matrix in fact.
- The form of matrices L and U can be calculated through tedious process. Here we only present the final result:
  - The form of L would be an $n \times n$ matrix with ones on diagonal line and entries $l_{ij} = -1$ for all $i - j = 1$.
  - The form of U would be an $n \times n$ matrix with ones on diagonal line and the last column substituted by a column vector of sums of negative m_i's.
- To represent the three matrices above explicitly and directly, we can write:

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix},$$

$$
U = \begin{pmatrix} 1 & 0 & \cdots & 0 & -m_1 \\ 0 & 1 & \cdots & 0 & -m_1 - m_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\sum_{i=1}^{n-1} m_i \\ 0 & 0 & \cdots & 0 & -\sum_{i=1}^{n} m_i \end{pmatrix},
$$

$$
P = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},
$$

Can also check that $PA = LU$.

(c) Solution:

The acceleration $a = t_n$ is the last component in vector t which satisfies $At = b$.
The tensions are $t_i, i = 1, \ldots, n-1$.
So we want closed form formula for all components in t in terms of n, m_i, c_i, v, g. Also find i for max t_i.
So far, we have:

$$
\begin{aligned}
t_1 - m_1 t_n &= c_1 v - m_1 g \\
-t_1 + t_2 - m_2 t_n &= c_2 v - m_2 g \\
-t_2 + t_3 - m_3 t_n &= c_3 v - m_3 g \\
&\cdots \\
-t_{n-2} + t_{n-1} - m_{n-1} t_n &= c_{n-1} v - m_{n-1} g \\
-t_{n-1} - m_n t_n &= c_n v - m_n g
\end{aligned}
$$

If we sum up all the equations, we would have this:

$$
\begin{aligned}
LHS = -(m_1 + \cdots + m_n)t_n &= (c_1 + \cdots + c_n)v - (m_1 + \cdots + m_n)g = RHS \\
(m_1 + \cdots + m_n)t_n &= (m_1 + \cdots + m_n)g - (c_1 + \cdots + c_n)v \\
(c_1 + \cdots + c_n)v &= (m_1 + \cdots + m_n)(g - t_n) \\
g - t_n &= \frac{c_1 + \cdots + c_n}{m_1 + \cdots + m_n} v \\
\therefore t_n &= g - \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i} v
\end{aligned}
$$

Then we can plug t_n back into the first equation:

$$
\begin{aligned}
t_1 &= c_1 v - m_1 g + m_1 t_n \\
&= c_1 v + m_1(t_n - g) \\
&= c_1 v + m_1\left(-\frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i} v\right) \\
&= \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v.
\end{aligned}
$$

Similarly for t_2 in the second equation, we can plug in t_1 and t_n:

$$
\begin{aligned}
t_2 &= c_2 v - m_2 g + m_2 t_n + t_1 \\
&= c_2 v + m_2(t_n - g) + \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_2 - m_2 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v + \left[c_1 - m_1 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_1 + c_2 - (m_1 + m_2) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v.
\end{aligned}
$$

Then for t_3:

$$
\begin{aligned}
t_3 &= c_3 v - m_3 g + m_3 t_n + t_2 \\
&= \left[c_3 - m_3 \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v + \left[c_1 + c_2 - (m_1 + m_2) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v \\
&= \left[c_1 + c_2 + c_3 - (m_1 + m_2 + m_3) \frac{\sum_{i=1}^n c_i}{\sum_{i=1}^n m_i}\right] v
\end{aligned}
$$

For all $2 \leq i \leq n - 1$,

$$t_i = \left[\sum_{k=1}^{i} c_k - \sum_{k=1}^{i} m_k \frac{\sum_{j=1}^{n} c_j}{\sum_{j=1}^{n} m_j} \right] v.$$

One of the other task of the problem is to find the index for max tension.
Observe:

$$\begin{aligned} t_1 &= c_1 v - m_1 g + m_1 t_n \\ -t_1 + t_2 &= c_2 v - m_2 g + m_2 t_n \\ -t_2 + t_3 &= c_3 v - m_3 g + m_3 t_n \\ &\dots \\ -t_{n-1} &= c_n v - m_n g + m_n t_n \end{aligned}$$

We know that c_i is monotone decreasing, while m_i is monote increasing. So on the RHS of all these equations, $c_i v - m_i g$ is monotone decreasing, and $m_i t_n$ is monotone increasing. By basic calculus knowledge, the sum of such two parts is concave down, i.e., there exists a local maximum. We also know that t_i are values on this concave curve. And the LHS are differences between each consecutive t_i values.
Therefore we want to find the smallest i such that $t_i - t_{i-1} = c_i v + m_i(t_n - g) < 0$.
Now we take the condition of case (i):

$$\begin{aligned} m_i &= 50 + 50 \frac{i-1}{n-1}, \\ c_i &= 25 - 10 \frac{i-1}{n-1}. \end{aligned}$$

Therefore,

$$t_i - t_{i-1} = (25 - 10\,\frac{i-1}{n-1})v + (50 + 50\,\frac{i-1}{n-1})\,\frac{\sum c}{\sum m}\,v < 0$$

$$5(n-1) - 2(i-1) < \left[10(n-1) + 10(i-1)\right]\frac{\sum c}{\sum m}$$

$$i - 1 > (n-1)\cdot\frac{5 - 10\cdot\frac{\sum c}{\sum m}}{2 + 10\cdot\frac{\sum c}{\sum m}}$$

$$\because \sum c = 25n - 10\cdot\frac{(0 + n - 1)n}{(n-1)2} = 25n - 5n = 20n$$

$$\because \sum m = 50n + 50\cdot\frac{(0 + n - 1)n}{(n-1)2} = 50n + 25n = 75n$$

$$\therefore \frac{5 - 10\cdot\frac{\sum c}{\sum m}}{2 + 10\cdot\frac{\sum c}{\sum m}} = \frac{5 - 10\cdot 4/15}{2 + 10\cdot 4/15} = \frac{7/3}{14/3} = \frac{1}{2}$$

$$\therefore i - 1 > \frac{1}{2}\,(n-1)$$

$$\therefore i > \frac{1}{2}\,(n-1) + 1$$

So the largest index we can pick to keep the difference between values t_i, t_{i-1} positive is

$$i = \lceil\frac{1}{2}\,(n-1)\rceil = \begin{cases} n/2, & \text{if } n \text{ is even} \\ \frac{1}{2}\,(n-1), & \text{if } n \text{ is odd} \end{cases}$$

.

Problem 4

Solution:

The core of efficient computation of the problem is to avoid calculating B^{-1} directly. Instead, we try to treat some parts including B^{-1} as a whole, and solve that part using LU factorization which genuinely saves steps.

First, we need to expand z:

$$\begin{aligned} z &= B^{-1}(2A+I)(B^{-1}+A)b \\ &= (2B^{-1}A+B^{-1})(B^{-1}+A)b \\ &= 2B^{-1}B^{-1}Ab + B^{-1}B^{-1}b + 2B^{-1}A^2b + B^{-1}Ab \end{aligned}$$

Now we assume $B^{-1}B^{-1}A = x, B^{-1}A = y, B^{-1}B^{-1}b = m$, then

$$\begin{aligned} A &= B^2x \\ A &= By \\ b &= B^2m \end{aligned}$$

Note that x, y are two $n \times n$ matrices, m is a $n \times 1$ vector.
Thereafter we can solve for x, y, m by LU factorization.

- We need to calculate B^2, which takes $2n-1$ flops for each entry. And the result of B^2 is an $n \times n$ matrix which has n^2 entries. So in total, the calculation of B^2 takes $(2n-1) \cdot n^2 = 2n^3 - n^2$ flops.
  - Also note that all $n \times n$ matrix multiplication takes $2n^3 - n^2$ flops.
- To calculate x, apply LU factorization on B^2 ($n^3/3$ flops), then do forward and backward substitutions (n^3 flops). So in total, $\frac{4}{3}n^3$ flops.
- Similarly for calculating y, also $\frac{4}{3}n^3$ flops.
- But for m, apply LU factorization on B^2 ($n^3/3$ flops), then do forward and backward substitutions ($n \cdot 1 \cdot n \cdot 1 = n^2$ flops). So this time in total $n^3/3 + n^2$ flops.
  - **Not sure whether we can save this calculation of B^2, i.e. saving $n^3/3$ flops since we already have that previously.**

So the original equation can be rewritten as:

$$z = 2xb + m + 2yAb + yb$$

Now we are able to know the flop counts of z:

- xb is a $n \times n$ matrix times $n \times 1$ vector, $(2n-1)n = 2n^2 - n$ flops. Scalar takes extra n flops. So in total $2n^2 + \frac{4}{3}n^3$ flops for $2xb$ (including calculating x).
- m takes $\frac{n^3}{3} + n^2$ flops.
- Scalar 2 takes n flops, y times A takes $2n^3 - n^2$ flops, times b takes another $2n^2 - n$ flops. So in total $n + 2n^3 - n^2 + 2n^2 - n + \frac{4}{3}n^3 = \frac{10}{3}n^3 + n^2$ flops (including calculating y).
- yb takes another $2n^2 - n$ flops.
- Each addition takes n flops. Three additions take $3n$ flops.

Hence, we have:

$$2n^2 + \frac{4}{3}n^3 + \frac{1}{3}n^3 + n^2 + \frac{10}{3}n^3 + n^2 + 2n^2 - n + 3n = 5n^3 + 6n^2 + 2n$$

flops for calculating z.