Lecture 1

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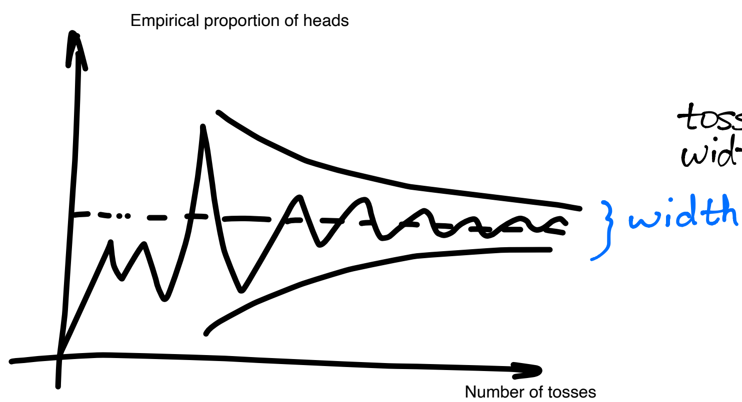
Office hour: M 3–5pm SS6026B

Probability tries to explain structured randomness.

What is "structured"? It's a property of some event happening converges as we repeat the experiment many many times.



toss n times

width ① $\propto \frac{1}{n}$

② $\propto \frac{1}{\sqrt{n}}$

③ $\propto \frac{1}{\sqrt[3]{n}}$

Ω sample space

ω elements of sample space
realization

A event a subset of Ω

Random Variable X is a r.v. if it is a function from $\mathbb{R}$ to $\mathbb{R}$.

$X(\omega) = \begin{cases} 1 & \text{if } \omega = \text{head} \\ 0 & \omega = \text{tail} \end{cases}$

If Ω is the population of all Canadians

$X(\omega) =$ the height of ω.

Expectation of a r.v. is the long-run average of repeated realiz-tions of X.

Def: Indicator Functions. If A is an event

$$I(A,\omega)=\begin{cases}1 & \text{if } \omega\in A\\ 0 & \omega\notin A\end{cases}$$

Def: For any event $A\subseteq\Omega$

$P(A)=E\,I(A,\omega)$

Axioms of Expectations

1. If $X\geq 0$ then $E[X]\geq 0$
2. $E[X_1+X_2]=E[X_1]+E[X_2]$
3. If c is a constant then $E[cX]=cE[X]$
4. $E[1]=1$

E is called expectational operator.

5. If a sequence of r.v's $X_i(\omega)$ changes monotonically to a limit $X(\omega)$ then $E[X]=\lim_{n\to\infty}E[X_n]$

Corollaries

a). $E[C_1X_1+C_2X_2+\cdots+C_nX_n]=C_1E[X_1]+\cdots+C_nE[X_n]$

Here C_i's are constant.

Proof: take home exercise.

b). If $X\leq Y$ then $E[X]\leq E[Y]$

Proof: Define $Z=Y-X\geq 0$

According to Axiom 1

$E[Z]\geq 0$, $E[Z]=E[Y]-E[X]$

So $E[Y]-E[X]\geq 0$

c). $|E[X]|\leq E|X|$

1 $E[X]\leq E|X|$

2 $-E[X]\leq E|X| \iff E[-X]\leq E|X|$

$X\leq|X| \xRightarrow{b)} E[X]\leq E|X|$

$-X\leq|X| \xRightarrow{b)} E[-X]\leq E|X|$

d). If X_i's are nonnegative random variables such that Fatou's Lemma

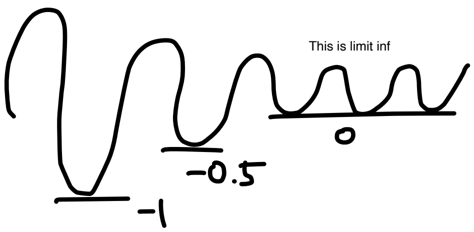
$X_i(\omega)\to X(\omega)$ for some r.v X

this convergence means convergence for every ω in Ω

Then $\liminf_{n\to\infty} E[X_n] \geq E[X]$

Def: $\liminf a_n = \lim_{n\to\infty} [\inf_{k\geq n} a_k]$

a_n is any sequence of real numbers.



We need to prove that $\lim_{n\to\infty} \left[\inf_{k\geq n} E[X_k]\right] \geq E[X]$ (*)

Define new random variables
$Y_n = \inf_{k\geq n} X_k$ Then Y_n is a well-defined r.v. and cannot take $\pm\infty$ values. (b/c inf is bdd)

Observation: $Y_n(\omega)$ is a non-decreasing sequence

$Y_n(\omega) \to X(\omega)$ (b/c $X_n(\omega) \to X(\omega)$)
Hence using Axiom 5, we have $E[Y_n] \xrightarrow{n\to\infty} E[X]$ ①
On the other hand note that $Y_n \leq X_k$ for any $k \geq n$
By b). $E[Y_n] \leq E[X_k]$ for any $k\geq n$ ②

Hence $E[Y_n] \leq \inf_{k\geq n} E[X_k]$

$$\boxed{\begin{array}{c}\lim_{n\to\infty} E[Y_n] \\ \wedge \\ \lim_{n\to\infty}\inf_{k\geq n} E[X_k]\end{array}} = E[X] \quad (①)$$

b/c of ②

$\Longrightarrow \liminf_{n\to\infty} E[X_n] \geq E[X]$

Corollary of d) (Dominated Convergence Thm)
If a sequence $X_n(\omega) \xrightarrow{n\to\infty} X(\omega)$ and $|X_n(\omega)| \leq Y(\omega)$ with $E[Y] < \infty$

then $E[X_n] \to E[X]$

Proof: Define $Z_n(\omega) = Y(\omega) + X_n(\omega) \geq 0$
$Z_n(\omega) \xrightarrow{n\to\infty} Y(\omega) + X(\omega) = Z(\omega)$
According to Fatou's lemma
$$\underline{\lim_{n\to\infty} E[Z_n(\omega)]} \geq E[Z(\omega)] = E[Y] + E[X] \quad \alpha$$
$$\|$$
$$\liminf_{n\to\infty} [E[X_n] + E[Y]]$$

$\liminf(E[X_n]+E[Y])$

$= \liminf E[X_n] + EY$ By ②

$\liminf E[X_n] \geq E[X]$ (**)

On the other hand, define

$Q_n(\omega) = Y(\omega) - X_n(\omega)$

Note $Q_n(\omega) \geq 0$

$Q_n(\omega) \to Q = Y(\omega) - X(\omega)$

According to Fatou's lemma

$$(\liminf E[Q_n]) \geq E[Q] = E[Y] - E[X]$$

$$\begin{array}{c} \| \\ \liminf(E[Y]-E[X_n]) \\ \| \\ E[Y] - \limsup E[X_n] \end{array}$$

Proof:

Hence $E[Y] - \limsup E[X_n] \geq E[Y] - E[X]$

$\Rightarrow \limsup E[X_n] \leq E[X]$ (***)

Combining (**) & (***)

$\Rightarrow \limsup E[X_n] = \liminf E[X_n] = E[X]$

$\Rightarrow \lim E[X_n] = E[X]$

∎