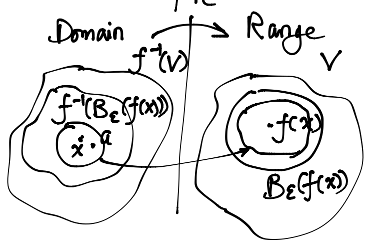
Lecture 7

<u>def'n</u>: Let $(X,\mathcal{T})$ & $(Y,\mathcal{U})$ be top spaces.
We say that a function $f: X \to Y$ is <u>continuous</u> if $\forall V \in \mathcal{U}$, $f^{-1}(V) \in \mathcal{T}$
In English "a function is continuous (cts) if the <u>preimage</u> of every open set is open."

Example: If $f: \mathbb{R} \to \mathbb{R}$ is a cts function (in 1st year calc sense) then $f$ cts in the top. sense (where both have the usual topology).

Proof: Sps $f$ is 1st year cts. Let $V \subseteq \mathbb{R}$ (in the range) be open.
WTS $f^{-1}(V)$ is open. Fix $x \in f^{-1}(V)$ (WT find an open disc around $x$ contained in $f^{-1}(V)$)

Pic




Since $V$ is open, find an $\varepsilon > 0$, s.t. $B_\varepsilon(f(x)) \subseteq V$

It's enough to show.

Claim: $f^{-1}(B_\varepsilon(f(x)))$ is open.
Find a $\delta > 0$ s.t. if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

This is the same as saying if $a \in B_\delta(x)$, then $f(a) \in B_\varepsilon(f(x))$

[Small details not withstanding]

<u>Examples</u>:
- $f: \mathbb{R}_{usual} \to \mathbb{R}_{usual}$ given by $f(x) = x^3 + 7$ is cts.
- The identity function $id: \mathbb{R}_{sorgenfrey} \to \mathbb{R}_{usual}$ given by $f(x) = x$, is cts.

Nonexample:
- The $id: \mathbb{R}_{usual} \to \mathbb{R}_{sorgenfrey}$ is not cts.
  As $[1,7)$ is open in Sorgenfrey Line, but $id^{-1}[[1,7)] = [1,7)$ which is not open in $\mathbb{R}_{usual}$.

def'n: Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be top spaces, let $f: X \to Y$ be a function, and let $a \in X$.
We say that $f$ is <u>cts at a</u>, provided that for open set $V$ containing $f(a)$, there is an open set $\mathcal{U}$ containing $a$ such that $f(\mathcal{U}) \subseteq V$.

<u>Thm</u>: Fix $(X,\mathcal{T})$, $(Y,\mathcal{U})$ and $f: X \to Y$. TFAE.
① $f$ is continuous
② $f$ is continuous at $a$, for all $a \in X$.
③ $f^{-1}(C)$ is closed in $X$, for all $C$ closed in $Y$.

Proof is straightforward, try it on your own.
Try ② → ① → ③ → ②

∎

[In 1st year, we said $f$ is continuous if $\lim_{n \to \infty} f(a_n) = f(a)$, where $a_n \to a$]

Thm: $f: X \to Y$ is continuous iff for every $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$
<u>Proof</u>: Straightforward.

∎

Thm: Let $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ be top spaces. Let $\mathcal{B}$ be a basis for $(Y,\mathcal{U})$ and $\mathcal{S}$ be a subbasis for $(Y,\mathcal{U})$. TFAE. and $f: X \to Y$.

① $f$ is cts.
② $f^{-1}(B) \in \mathcal{T}$, for all $B \in \mathcal{B}$
③ $f^{-1}(S) \in \mathcal{T}$, for all $S \in \mathcal{S}$

Proof: ① ⇒ ②, ① ⇒ ③ are obvious.
Let's see ② ⇒ ①
Assume ②, fix an open set $U \in \mathcal{U}$. Find $\mathcal{C} \subseteq \mathcal{B}$ s.t. $U = \bigcup_{C \in \mathcal{C}} C$
Now $f^{-1}(U) = f^{-1}\left(\bigcup_{C \in \mathcal{C}} C\right) = \bigcup_{C \in \mathcal{C}} \underbrace{f^{-1}(C)}_{\text{open}}$ ∎

Cool example: Define plus: $\mathbb{R}^2_{\text{usual}} \to \mathbb{R}$ by $\text{plus}(x,y) = x+y$
Claim: plus is a cts function.

We'll check that the preimage of every subbasic open set is open.
Recall: $\mathcal{S} = \{(a,+\infty) : a \in \mathbb{R}\} \cup \{(-\infty, b) : b \in \mathbb{R}\}$
Case 1: Fix $S = (a,+\infty)$, check $\text{plus}^{-1}(S)$ is open
$\text{plus}^{-1}(a,+\infty) = \{(x,y) \in \mathbb{R}^2 : x+y > a\}$
$= \{(x,y) \in \mathbb{R}^2 : y > a - x\}$ is open.

defn: Let $f: X \to Y$ (top spaces)
We say $f$ is an open map if $f(U)$ is open in $Y$, for all open sets $U$ in $X$.
In English, "open functions map open sets to open sets."

Fact: Open functions are not the same as cts functions.
(In fact, you have all proved this).