MAT 335

CHAPTER 3 Orbits

§ 3.1 Iteration

§ 3.2 Orbits

Orbit of x_0 under F to be the sequence of points $x_0, x_1 = F(x_0)$.
$x_2 = F^2(x_0), \cdots x_n = F^n(x_0), \cdots$

x_0 : seed of the orbit.

§ 3.3 Types of orbits

I. fixed pt: $F(x_0) = x_0$

$x_0, x_0, x_0, \cdots$

II. periodic orbit or cycle.

The least such n that

$F^n(x_0) = x_0$ for some $n > 0$.

$x_0, F(x_0), \cdots, F^{n-1}(x_0), x_0, F(x_0), \cdots, F^{n-1}(x_0), \cdots$

we say $\underbrace{\text{period}}_{\text{prime}}$ n or a n-cycle for $F(x)$.

III. eventually fixed or eventually periodic.

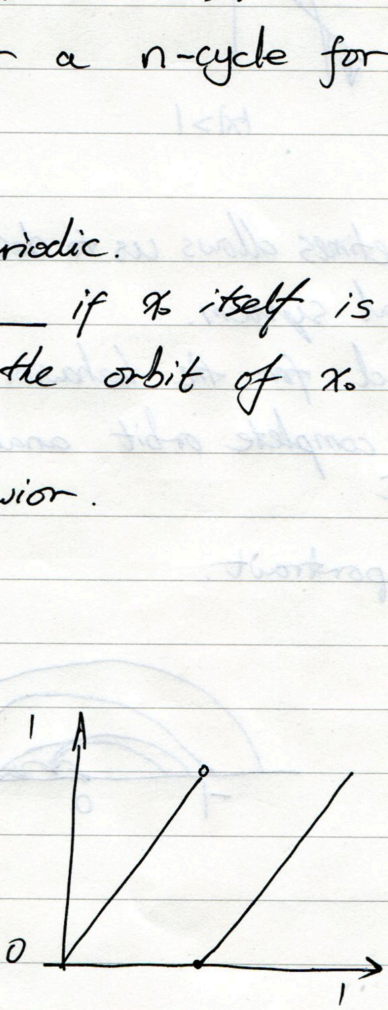
A point x_0 is called ________ if x_0 itself is not fixed or periodic, but some point on the orbit of x_0 is fixed or periodic.

IV. others -- ~~chat~~ chaotic behavior.

§ 3.4

§ 3.5. The doubling Function.

$$D(x) = \begin{cases} 2x & 0 \le x < 1/2 \\ 2x-1 & 1/2 \le x < 1 \end{cases}$$



$D : [0,1) \to [0,1)$.

or $D(x) = 2x \mod 1$.

CHAPTER 4 Graphical Analysis

§4.1 Graphical Analysis.

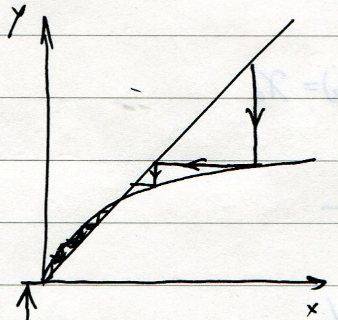
- begin at (x_0, x_0), draw a vertical line to the graph of F.
- get $(x_0, F(x_0))$, then draw a horizontal line from this to the diagonal
- get $(F(x_0), F(x_0))$

now we get the next point on the orbit of x_0.

then continue. ~~$(x_0, x_0) \longrightarrow$~~

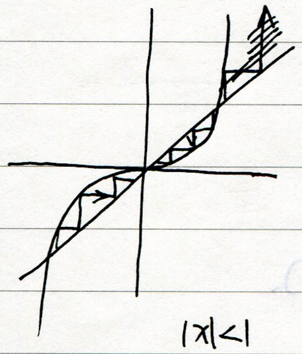
$$(F(x_0), F(x_0)) \longrightarrow (F(x_0), F^2(x_0)) \longrightarrow (F^2(x_0), F^2(x_0))$$

repeat ...

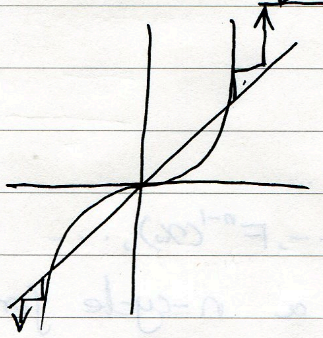


graphical analysis for $F(x)=\sqrt{x}$

§4.2 orbit analysis



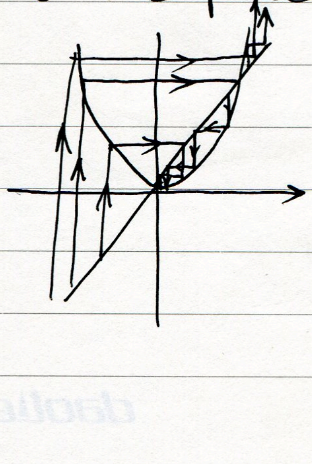
$|x|<1$

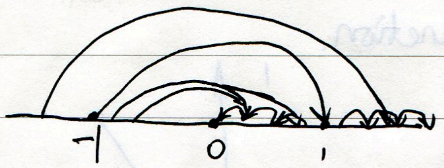


$|x|>1$

Graphical analysis sometimes allows us to describe the behavior of all orbits of a dynamical system.
When we have accounted for the behavior of the orbits of all pts.
↳ say we have $\hat{}$ complete orbit analysis.
a

§4.3 The phase portrait.





CHAPTER 5 Fixed and periodic points

§5.1 A Fixed Point Thm

The IVT: Sps $F:[a,b]\to\mathbb{R}$ is continuous. Suppose y_0 lies between $F(a)$ and $F(b)$. Then there is an x_0 in the interval $[a,b]$ with $F(x_0)=y_0$

• Fixed Point Thm: Suppose $F:[a,b]\to[a,b]$ is continuous. Then there is a fixed point for F in $[a,b]$.

Remarks: ① "<u>at least</u> one"
② continuity and F takes the interval to itself.
③ closed interval

§5.2 Attraction and Repulsion.

§5.3 Calculus of fixed points

Def: x_0 is fixed pt for F. Then x_0 is an attracting fixed point if $|F'(x_0)|<1$. The point x_0 is a repelling fixed pt if $|F'(x_0)|>1$. if $|F'(x_0)|=1$, x_0 is neutral or indifferent.

§5.4 Why is this True?

Two Corollaries of FPT.

Attracting Fixed Point Thm:

Sps x_0 is an attracting fixed point for F. Then there is an interval I that contains x_0 in its interior and in which the following condition is satisfied: if $x\in I$, then $F^n(x)\in I$ for all n and, moreover, $F^n(x)\to x_0$ as $n\to\infty$.

Repelling fixed Point Thm:

---- if $x\in I$ and $x\neq x_0$, then there is an integer $n>0$ s.t. $F^n(x)\notin I$.

• neutral fixed points sometimes may attract or repel all nearby orbits "weakly attracting/repelling". (as its convergence/divergence is quite slow)

§5.5 Period Points

Chain Rule Along a Cycle: Suppose $x_0, x_1, \dots, x_{n-1}$ lie on a cycle of period n for F with $x_i = F^i(x_0)$. Then.

$$(F^n)'(x_0) = F'(x_{n-1}) \cdots F'(x_1) F'(x_0)$$

Corollary: Sps $x_0, x_1, \dots, x_{n-1}$ lie on an n-cycle for F. Then

$$(F^n)'(x_0) = (F^n)'(x_1) = \cdots = (F^n)'(x_{n-1})$$

CHAPTER 6 Bifurcations

$Q_c(x) = x^2 + c$, c is a constant parameter.

§6.1 Dynamics of the Quadratic Map

$x^2 + c = x$, $x^2 - x + c = 0$

$p_+ = \frac{1+\sqrt{1-4c}}{2}$. $p_- = \frac{1-\sqrt{1-4c}}{2}$, $c \leq \frac{1}{4}$

when $c > \frac{1}{4}$, simple to understand, no fixed point, all orbits tend to infinity

$c = \frac{1}{4}$, one fixed pt. at $x = \frac{1}{2} = p_+ = p_-$

when $c < \frac{1}{4}$, we have first bifurcation.

↳ a division in two, a splitting apart, and that is exactly what has happened to the fixed points of Q_c. namely p_- and p_+

$Q_c'(x) = 2x$

Then $Q_c'(p_+) = 2p_+ = 1 + \sqrt{1-4c}$

$Q_c'(p_-) = 2p_- = 1 - \sqrt{1-4c}$

$Q_c'(p_\pm) = 1$ if $c = \frac{1}{4}$ (netural)

$Q_c'(p_+) > 1$ if $c < \frac{1}{4}$ (repelling)

$Q_c'(p_-) = 1$ if $c = \frac{1}{4}$

--- { other two si

Note: $c < -\frac{5}{4}$, repelling 2-cycle.

§6.2 The saddle-node bifurcation.

One-parameter family of functions. F_λ.

Def: A one-para family of functions F_λ undergoes a saddle node (or tangent) ~~bifur~~ bifurcation at the parameter value λ_0 if there is an open interval I and an $\varepsilon > 0$ such that.

① For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, no fixed points in the interval I.

② For $\lambda = \lambda_0$, F_λ has one neutral fixed point in I.

③ For $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, F_λ has two fixed points in I. one attracting one repelling.

E.g. First bifurcation of ~~quadic~~ quadratic ~~the~~ family

§6.3 The period-doubling Bifurcation

E.g. Second bifurcation of ~~quard~~ quadratic family.

Def: A one-para - - - - F_λ - - - period-doubling ~~function~~ bifurcation at $\lambda = \lambda_0$ if - - - I and $\varepsilon > 0$ s.t.

① For each λ in the interval $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, there is a unique fixed point p_λ for F_λ in I.

② For $\lambda_0 - \varepsilon < \lambda < \lambda_0$, no cycles of period 2 in I and p_λ is attracting (resp. repelling).

③ For $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, there is a unique 2-cycle q_λ^1, q_λ^2 in I with $F_\lambda(q_\lambda^1) = q_\lambda^2$, it is attracting (resp. repelling). Meanwhile the fixed point p_λ is repelling (resp. attracting).

④ As $\lambda \to \lambda_0$, we have ~~$q_\lambda^1 \to q_\lambda^2$~~. $q_\lambda^i \to p_\lambda$.

• ~~One best method to tell a bifurcation is a tangent~~ or period-doubling is that to check it's derivative.

If ~~$F_\lambda'(p_{\lambda_0}) = 1$ - - - saddle-node bifurcation~~

$F_\lambda'(p_{\lambda_0}) = -1$ - - - period-doubling bifurcation.

Proposition: The First Bifurcation:
$Q_c(x) = x^2 + c$.

1. All orbits tend to infinity if $c > \frac{1}{4}$.

2. When $c = \frac{1}{4}$, Q_c has a single fixed point at $p_+ = p_- = \frac{1}{2}$ that is neutral.

3. For $c < \frac{1}{4}$. Q_c has two fixed points at p_+ and p_-. The fixed point p_+ is always repelling.

a. If $\frac{-3}{4} < c < \frac{1}{4}$, p_- is attracting.

b. If $c = \frac{-3}{4}$, p_- is neutral. *

c. If $c < -\frac{3}{4}$, p_- is repelling. *

• Note: $Q_c(-p_+) = Q_c(p_+)$
so $-p_+$ is eventually fixed.

And a cycle of period 2 appears when $c < -\frac{3}{4}$.

$$x^4 + 2cx^2 - x + c^2 + c = 0$$

$$(x - p_+)(x - p_-) = x^2 + c - x$$

$$\frac{x^4 + 2cx^2 - x + c^2 + c}{x^2 + c - x} = x^2 + x + c + 1$$

$$x = q_\pm = \frac{1}{2}(-1 \pm \sqrt{-4c-3})$$

$q_\pm$ exist iff $-4c - 3 \geq 0 \iff c \leq \frac{-3}{4}$

A new bifurcation: period doubling bifurcation.

Proposition: The Second Bifurcation: $Q_c(x) = x^2 + c$.

① For $-3/4 < c < 1/4$. Q_c has an attracting fixed point at p_- and no 2-cycles.

② For $c = -\frac{3}{4}$, Q_c has a neutral fixed point at $p_- = q_\pm$ and no 2-cycles.

③ For $\frac{-5}{4} < c < -\frac{3}{4}$, Q_c has repelling fixed points at $p_\pm$ and attracting 2-cycle at $q_\pm$.

Note: $c < -\frac{5}{4}$, repelling 2-cycle.

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