Lecture 5 §3.3 absolute & conditional convergence

Rearrangement Theorem: If $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series, then $\forall L \in \mathbb{R}$, $\exists$ a rearrangement that converges to L.

Proof: $b_1, b_2, b_3, \cdots$ positive terms sequence, $c_1, c_2, c_3 \cdots$ negative terms sequence

$\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$, $\lim_{n\to\infty} c_n = 0$

Claim: $\sum_{k=1}^{\infty} b_k = +\infty$ & $\sum_{k=1}^{\infty} |c_k| = +\infty$

Impossible that both $\sum_{k=1}^{\infty} b_k$, $\sum |c_k|$ converge

Sps $\sum b_k$ diverges but $\sum |c_k|$ converges to L. $\Rightarrow \forall n > 0, \exists N$ s.t.

$\sum_{n=1}^{N} b_k > R + L \Rightarrow$ if M large enough s.t. $b_1, b_2, \cdots, b_n$ are contained among terms: $a_1, \cdots, a_m$.

$\sum_{i=1}^{m} a_i \geq \sum_{k=1}^{N} b_k - \sum_{k=1}^{\infty} |c_k| > R + L - L = R. \Rightarrow \sum_{i=1}^{\infty} a_i$ diverges

contradiction.

Let $L \in \mathbb{R}$, choose smallest M_1 s.t. $u_1 = b_1 + b_2 + \cdots + b_{M_1} > L$

choose smallest N_1 s.t. $v_1 = u_1 + c_1 + c_2 + \cdots + c_{N_1} < L$

Continue in such a way

$u_1, u_2, u_3, \cdots$

$v_1, v_2, v_3, \cdots$

Claim: a rearranged series converges to L.

$u_i - b_{m_i} \leq L < u_i$

$v_j < L \leq v_j - c_{n_j}$

$L + c_{n_j} \leq v_j < L < u_j \leq L + b_{m_i}$ by sequeezing Thm, $u_i, v_j \to L$

If S_k is any partial sum, sps k between m_1 & $m_1 + n_1$

$u_1 \geq S_k \geq v_1$

if S_k is between $m_1 + n_1$ & $m_2 + n_1$

$\Rightarrow v_1 \leq S_k \leq u_2$

In general $v_{i-1} \leq S_k \leq u_i$ for $m_{i-1} + n_{i-1} \leq k \leq m_i + n_{i-1}$

$v_i \leq S_k \leq u_i$, when $m_i + n_{i-1} \leq k \leq m_i + n_i$

Since $v_i, v_j \to L$ by S.T. then $S_k \to L$

Ex: $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$ try to show $L = 1$

$1, \frac{1}{3}, \frac{1}{5}, \cdots$

$-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, \cdots$

$1, \frac{1}{3}, -\frac{1}{2}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, -\frac{1}{4}, -\frac{1}{6}, \cdots$

Determine whether or not the following series converge or diverge

i). $\sum \frac{(n!)^2}{(2n)!}$ Ratio Test

$a_{n+1} = \frac{[(n+1)!]^2}{(2n+2)!}$ $\quad a_n = \frac{(n!)^2}{(2n)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)} \to \frac{1}{4} < 1$$

Ex: $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \cdot 4^n$: Root Test

$(a_n)^{\frac{1}{n}} = \left(\frac{n}{n+1}\right)^n \cdot 4 = \left(1-\frac{1}{n+1}\right)^n \cdot 4 = \left(1-\frac{1}{n+1}\right)^{(n+1)-1} \cdot 4 = e^{-1} \cdot 4$ diverges

$\sum_{n=1}^{\infty} \frac{2+\sin^3(n+1)}{2^n+n^2}$ Comparison Test

$$0 \leq \frac{2+\sin^3(n+1)}{2^n+n^2} \leq \frac{3}{n^2} \left(\frac{3}{2^n}\right)$$

$\sum_{n=1}^{\infty} \frac{3}{2^n}$, $r = \frac{1}{2}$ geometric series $\Rightarrow$ converges

How about $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$?

Limit Comparison Test.

$\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, $b_n \geq 0$

$\limsup_{n\to\infty} \frac{a_n}{b_n} < \infty$ and $\sum b_n < \infty$

Ex. $\sum_{n=1}^{\infty} \frac{1}{n}(\sqrt{n+1} - \sqrt{n}) = \frac{1}{n}(\sqrt{n+1} - \sqrt{n}) = \frac{1}{n} \cdot \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n} \cdot n} = \frac{1}{2n^{\frac{3}{2}}}$

$\sum_{n=1}^{\infty} \frac{1}{2n^{\frac{3}{2}}}$ integral Test converges

Ex: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$

converges: Alternating series Test conditionally

$\left|\frac{(-1)^{n-1}}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}}$ integral test

Chapter 4

Topology of $\mathbb{R}^n$ §4.1 §4.1 n-dimensional space

$\mathbb{R}^n$ = the set of n-vector

$\{\vec{x} = (x_1, \cdots, x_n) : x_i \in \mathbb{R}, \forall 1 \leq i \leq n\}$

0-vector $(0, 0, \cdots, 0)$

$\vec{x} + \vec{y}$

$a \cdot \vec{x}$

$\|\vec{x}\|$ **Euclidean norm**

$= \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ length of $\vec{x}$

Let $\vec{x}, \vec{y} \in \mathbb{R}^n$

$\|\vec{x} - \vec{y}\| = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$

$\langle \vec{x}, \vec{y} \rangle = \langle (x_1, \cdots, x_n), (y_1, \cdots, y_n) \rangle$

inner product $= \sum_{i=1}^{n} x_i y_i$

$\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

inner product is bilinear: linear in both of the entries

Schwarz Inequality: $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$, $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$

Proof: $\vec{x} = (x_1, \cdots, x_n)$, $\vec{y} = (y_1, \cdots, y_n)$

$$2\|x\|^2 \cdot \|y\|^2 - 2|\langle x, y \rangle|^2$$

$$= 2\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) - 2\left(\sum_{i=1}^{n} x_i y_i\right)^2$$

$$= \sum_{i,j=1}^{n} (x_i^2 y_j^2 + x_j^2 y_i^2) - 2\sum_{i,j=1}^{n} 2x_i x_j y_j y_i$$

(Remark: equality holds iff $\vec{x}, \vec{y}$ are colinear)

$$= \sum_{i,j=1}^{n} (x_i^2 y_j^2 - 2x_i x_j y_i y_j + x_j^2 y_i^2)$$

$$= \sum_{i,j=1}^{n} (x_i y_j - x_j y_i)^2 \geq 0$$

equality holds $\iff x_i y_j - x_j y_i = 0$

case 1: $x = y = 0$

case 2: at least one of the coefficient is different from 0.

W.L.O.G assume $x_i \neq 0$, $y_j = \frac{y_i}{x_i} x_j$ for all $i \leq j \leq n \Rightarrow x, y$ are colinear

Triangle Inequality

$\|x+y\| \leq \|x\| + \|y\|$

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle = \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &\leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle \\ &\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| + \|y\|)^2\end{aligned}$$

If equality holds, $\Rightarrow \langle x,y\rangle = \|x\| \cdot \|y\|$

either $x = 0$, or $y = cx$

$\langle x, cx\rangle = \|x\| \cdot \|cx\|$

$c\|x\|^2 = \|x\| \cdot \|cx\| \quad \Rightarrow c = \frac{\|y\|}{\|x\|} > 0$

Lemma

A $\{v_1, \cdots, v_m\} \in \mathbb{R}^n$ is orthonormal

if $\langle v_i, v_j\rangle = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

Let $\{x_1, \cdots, x_n\}$ be orthonormal

$\|\sum_{i=1}^{m} a_i x_i\| = (\sum_{i=1}^{m} |a_i|^2)^{\frac{1}{2}}$

orthonormal set is always L.I.

$\sum a_i v_i = 0$

$\Rightarrow \sum a_i^2 = 0 \Rightarrow a_i = 0$

§4.2

Convergence and completeness in $\mathbb{R}^n$

Def'n: A sequence of pts $(\vec{x_n}) \in \mathbb{R}^n$ converges to $\vec{a}$ if $\forall \varepsilon > 0\ \exists N \in \mathbb{Z}$ s.t

$\|\vec{x_k} - \vec{a}\| < \varepsilon$, $\forall k \geq N$, $\lim_{k\to\infty} \vec{x_k} = \vec{a}$