Problem 2

Solution:

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix}, \quad b=\begin{pmatrix} 0.4313\times10^{-1} \\ 0.6757\times10^{1} \end{pmatrix}$$

(1). Without Pivoting

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix} \xrightarrow[k=1]{\text{elim}} \begin{pmatrix} 0.211\times10^{-2} & 0.8204\times10^{-1} \\ \boxed{0.1597\times10^{3}} & 0.1284\times10^{2}-0.8204\times0.1597\times10^{2} \end{pmatrix}$$

$$=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ \boxed{0.1597\times10^{3}} & -0.2600\times10^{0} \end{pmatrix}, \quad L=\begin{pmatrix} 1 & 0 \\ 0.1597\times10^{3} & 1 \end{pmatrix}, \quad U=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0 & -0.2600\times10^{0} \end{pmatrix}$$

take $fl(\cdot)$ each step

$$Ax=b \iff LUx=b \iff \begin{cases} Ly=b \\ Ux=y \end{cases}$$

$Ly=b$

$$\begin{pmatrix} 1 & 0 \\ 0.1597\times10^{3} & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}=\begin{pmatrix} 0.4313\times10^{-1} \\ 0.6757\times10^{1} \end{pmatrix} \Longrightarrow \begin{cases} y_1=0.4313\times10^{-1} \\ y_2=0.6757\times10^{1}-0.1597\times0.4313\times10^{2} \\ \quad =0.6757\times10^{1}-0.6888\times10^{1} \\ \quad =-0.1310\times10^{0} \end{cases}$$

$Ux=y$

$$\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0 & -0.2600\times10^{0} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=\begin{pmatrix} 0.4313\times10^{-1} \\ -0.1310\times10^{0} \end{pmatrix} \Longrightarrow \begin{cases} \hat{x}_1=\dfrac{0.4313\times10^{-1}-0.8204\times0.5038\times10^{-1}}{0.2110\times10^{-2}} \\ \quad =\dfrac{0.4313\times10^{-1}-0.4133\times10^{-1}}{0.2110\times10^{-2}} \\ \quad =\dfrac{0.1797\times10^{-2}}{0.2110\times10^{-2}} \\ \quad =0.8517\times10^{0} \\ \hat{x}_2=\dfrac{-0.1310}{-0.2600}=0.5038\times10^{0} \end{cases}$$

So $\hat{x}=(0.8517\times10^{0}, 0.5038\times10^{0})^T$.

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(2). With partial pivoting

$$A=\begin{pmatrix} 0.2110\times10^{-2} & 0.8204\times10^{-1} \\ 0.3370\times10^{0} & 0.1284\times10^{2} \end{pmatrix} \xrightarrow[k=1]{\text{piv}} \begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ 0.2110\times10^{-2} & 0.8204\times10^{-1} \end{pmatrix}, \quad P_1=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

ipiv=(2, ·),

$$\xrightarrow[k=1]{\text{elim}} \begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ \boxed{\dfrac{0.2110\times10^{-2}}{0.3370\times10^{0}}} & 0.8204\times10^{-1}-0.1284\times10^{2}\times0.6261\times10^{-2} \end{pmatrix}=\begin{pmatrix} 0.3370\times10^{0} & 0.1284\times10^{2} \\ \boxed{0.6261\times10^{-2}} & 0.1650\times10^{-2} \end{pmatrix}$$

$$L=\begin{pmatrix}1 & 0\\ 0.6261\times10^{-2} & 1\end{pmatrix},\quad U=\begin{pmatrix}0.3370\times10^{0} & 0.1284\times10^{2}\\ 0 & 0.1650\times10^{-2}\end{pmatrix}$$

$P_1Ax=LUx=P_1b$, note $P=P_1$

$$\begin{cases}Ly=P_1b\\ Ux=y\end{cases}$$

$$\begin{pmatrix}1 & 0\\ 0.6261\times10^{-2} & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0.4313\times10^{-1}\\ 0.6757\times10^{1}\end{pmatrix}\Rightarrow\begin{cases}y_1=0.6757\times10^{1}\\ y_2=0.4313\times10^{-1}-0.6261\times10^{-2}\times0.6757\times10^{1}\\ \quad=0.4313\times10^{-1}-0.4231\times10^{-1}\\ \quad=0.8200\times10^{-3}\end{cases}$$

$$\begin{pmatrix}0.3370\times10^{0} & 0.1284\times10^{2}\\ 0 & 0.1650\times10^{-2}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0.6757\times10^{1}\\ 0.82\times10^{-3}\end{pmatrix}$$

$$\Rightarrow\begin{cases}x_2=\dfrac{0.8200\times10^{-3}}{0.1650\times10^{-2}}=0.4970\times10^{0}\\[2ex] x_1=\dfrac{0.6757\times10^{1}-0.1284\times10^{2}\times0.4970\times10^{0}}{0.3370\times10^{0}}=\dfrac{0.6757\times10^{1}-0.6381\times10^{1}}{0.3370\times10^{0}}=\dfrac{0.3760}{0.3370}\\ \quad=0.1116\times10^{1}\end{cases}$$

So $\hat{x}=(0.1116\times10^{1},\ 0.4970\times10^{0})$

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(3). Complete pivoting

$$A=\begin{pmatrix}0.2110\times10^{-2} & 0.8204\times10^{-1}\\ 0.3370\times10^{0} & 0.1284\times10^{2}\end{pmatrix}\xrightarrow[P_1,\,Q_1]{\text{complete piv}}\begin{pmatrix}0.1284\times10^{2} & 0.3370\times10^{0}\\ 0.8204\times10^{-1} & 0.2110\times10^{-2}\end{pmatrix}$$

$$\xrightarrow[k=1]{\text{elim}}\left(\begin{array}{c|c}0.1284\times10^{2} & 0.3370\times10^{0}\\ \hline \dfrac{0.8204\times10^{-1}}{0.1284\times10^{2}} & 0.2110\times10^{-2}-0.3370\times10^{0}\times\dfrac{0.8204\times10^{-1}}{0.1284\times10^{2}}\end{array}\right)$$

$$=\left(\begin{array}{c|c}0.1284\times10^{2} & 0.3370\times10^{0}\\ \hline 0.6389\times10^{-2} & -0.4300\times10^{-4}\end{array}\right),\quad L=\begin{pmatrix}1 & 0\\ 0.6389\times10^{-2} & 1\end{pmatrix},\quad U=\begin{pmatrix}0.1284\times10^{2} & 0.3370\times10^{0}\\ 0 & -0.4300\times10^{-4}\end{pmatrix}$$

$$P_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ P=P_1,\ Q_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\ Q=Q_1,\quad PAQ=LU$$

$PAQ=LU, Ax=b$

$PAQQ^Tx=Pb\Rightarrow LU\hat{x}=Pb,\ \hat{x}=Q^Tx$

$Ly=Pb$

$$\begin{pmatrix}1 & 0\\ 0.6389\times10^{-2} & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0.4313\times10^{-1}\\ 0.6757\times10^{1}\end{pmatrix}\Rightarrow\begin{cases}y_1=0.6757\times10^{1}\\ y_2=0.4313\times10^{-1}-0.6757\times10^{1}\times0.6389\times10^{-2}\\ \quad=0.4313\times10^{-1}-0.4317\times10^{-1}\\ \quad=-0.4\times10^{-4}\end{cases}$$

$Uz = y$

$$\begin{pmatrix} 0.1284\times10^{2} & 0.3370\times10^{0} \\ 0 & -0.4300\times10^{-4} \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.6757\times10^{1} \\ -0.4\times10^{-4} \end{pmatrix}$$

$$\Rightarrow \begin{cases} z_2 = \dfrac{-0.4\times10^{-4}}{-0.4300\times10^{-4}} = 0.9302 \\ z_1 = \dfrac{0.6757\times10^{1} - 0.9302 \times 0.3370\times10^{0}}{0.1284\times10^{2}} \end{cases}$$

$$= \frac{0.6757\times10^{1} - 0.3135}{0.1284\times10^{2}} = \frac{0.6444\times10^{1}}{0.1284\times10^{2}} = 0.5019$$

$Q = Q^{-1} = Q^T$

$$x = Qz = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0.5019 \\ 0.9302 \end{pmatrix} = \begin{pmatrix} 0.9302 \\ 0.5019 \end{pmatrix}$$

so $\hat{x} = (0.9302,\ 0.5019)^T$

Summarize

	$\frac{\lvert x_1-\hat{x}_1\rvert}{\lvert x_1\rvert}$	$\frac{\lvert x_2-\hat{x}_2\rvert}{\lvert x_2\rvert}$	$\frac{\lVert x-\hat{x}\rVert_\infty}{\lVert x\rVert_\infty}$
(1). without pivoting	0.1483	0.0076	0.1483
(2). partial pivoting	0.116	0.006	0.116
(3). Complete pivoting	0.0698	0.0038	0.0698

Comments:

① The complete pivoting has the least relative error in each component and in infinity norm

② Partial pivoting reduces round-off errors already.

③ Complete pivoting however, is not necessarily more accurate than partial pivoting.