Lecture 13 (Continue § 5.3 Properties of Continuous Functions)

Theorem :
Let $f: X \to Y$ be a continuous map between metric spaces. If $C \subseteq X$ is compact, then $f(C)$ is compact.

Suppose (X, δ) is a metric space, $C \subseteq X$, then $\mathcal{U} = \{U_\alpha, \alpha \in \Lambda\}$ is called an open cover of C if U_α is open for $\alpha \in \Lambda$ & $\bigcup_{\alpha \in \Lambda} U_\alpha \supset C$
C is compact if any open cover has a finite subcover.

Proof:
Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be an open cover of $f(C)$
Let $V = \{V_\alpha = f^{-1}(U_\alpha), \alpha \in \Lambda\}$
Claim V is an open cover of C.
(Collection of open sets, since f is continuous & U_α is open $\forall \alpha \in \Lambda$)
$C \subset \bigcup_\alpha V_\alpha$

Theorem: Since C is compact, $\exists$ a finite subcover $V_{\alpha_1}, \cdots, V_{\alpha_k}$
$\Rightarrow U_{\alpha_1}, \cdots, U_{\alpha_k}$ is a finite subcover of $f(C)$

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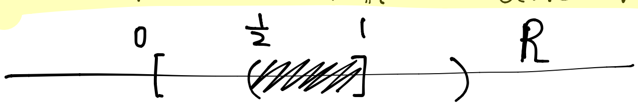
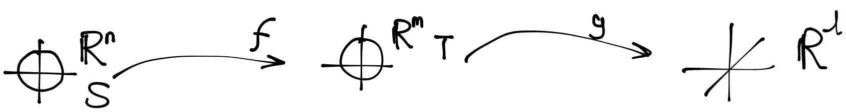
Def: A subset $V \subset S \subset \mathbb{R}^n$ is open in S (rel open wrt S)
if $\exists$ an open set $\mathcal{U}$ in $\mathbb{R}^n$ s.t. $U \cap S = V$



Thm: Let f, g be continuous functions $f, g: S \subset \mathbb{R}^n \to \mathbb{R}^m$ at $a \in S$, let $\alpha \in \mathbb{R}$
Then (1) $f+g$ is continuous at a,
(2) αf is continuous at a
(3) fg is continuous at a, when range of f & g is contained in $\mathbb{R}$
(4) f/g is continuous at a when $g(a) \neq 0$.

Rational Functions
$\frac{P(x)}{Q(x)}$

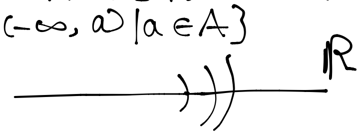


Thm: Sps that f maps a domain $S \subset \mathbb{R}^n$ into a subset T of $\mathbb{R}^m$, $g: T \to \mathbb{R}^1$. If f is continuous at $a \in S$ and g is continuous at $f(a) \Rightarrow g \circ f$ is continuous at a.

§ 5.4 Compactness and Extreme Values EVT

Let $f: X \to \mathbb{R}$ be continuous, where X is a compact metric space. then there exist points c and d in X s.t. $f(c) \leq f(x) \leq f(d)$, $\forall x \in X$.

Proof:
We will show that there is the largest element $M \in A$, $M = f(x)$
Then since $M \in A$, we must "have" $M = f(d)$ for some point $d \in X$.
Suppose A does not have the largest element
$\{(-\infty, a) \mid a \in A\}$



Let $\alpha \in A$. since A has no largest element $\exists a \in A$ s.t. $a > \alpha \Rightarrow \alpha \in (-\infty, a)$
$\Rightarrow \exists$ a finite subcover
$(-\infty, a_1), \cdots, (-\infty, a_k)$
Let $t = \max\{a_1, \cdots, a_k\}$
$\Rightarrow t \notin (-\infty, a_i)$, $t \in A$
contradiction.

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§5.5 Uniform continuity

Def: A function $f: S \subset \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|f(x) - f(a)\| < \varepsilon$ whenever $\|x - a\| < \delta$, $x, a \in S$. [δ here does not depend on a]

$f(X, \varphi) \to (Y, \sigma)$ is uniformly continuous if given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall$ pair of points, $x_0, x_1 \in X$
$\varphi(x_0, x_1) < \delta \Rightarrow \sigma(f(x_0), f(x_1)) < \varepsilon$

Theorem: Let $f: X \to Y$ be a continuous map of the compact metric space (X, d_X) to the metric space (Y, d_Y), Then f is uniformly continuous.

Proof (use Lebesgue # lemma)
Given $\varepsilon > 0$, take an open cover of Y by metric balls of radius $\varepsilon/2$
$B_{\frac{\varepsilon}{2}}(y)$, we can cover X by the inverse image of the metric balls under f.
Since f is continuous, this is an open cover of X. By the Lebesgue # lemma
$\exists \delta > 0$ s.t. for any set of diameter $< \delta$ $\exists$ an element in the cover that contains it.
If $d_X(x, y) < \delta$, $\exists y \in Y$ s.t. $x, y \in f^{-1}(B_{\frac{\varepsilon}{2}}(y)) \Rightarrow f(x), f(y) \in B_{\frac{\varepsilon}{2}}(y) \Rightarrow d_Y(f(x), f(y)) < \varepsilon$