Lecture 8

* Gamma Distribution

Gamma Function

$$\boxed{\Gamma(x) = \int_0^\infty y^{x-1} e^{-y} dy} \qquad x > 0$$

$\boxed{\text{Properties} \quad ① \ \Gamma(n) = (n-1)! \quad n \in \mathbb{Z} \quad ② \ \Gamma(x+1) = x\Gamma(x)}$

Def: A r.v. X is called a Gamma (α, β) r.v. if it has density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}, \quad x > 0$$

Here $\alpha > 0$ and $\beta > 0$.

HW: ① Verify that $\int_0^\infty f(x) dx = 1$

② Calculate EX & VX

Proposition: If $X_1, \cdots X_n$ are iid. with exponential(λ) distribution, then $X_1 + X_2 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$

Proof: Let $Y_n = X_1 + \cdots + X_n$

Then $M_{Y_n}(x) = \prod_{i=1} M_{X_i}(x)$

Note: $M_X(x) = \frac{1}{1-\lambda x} = \frac{1}{(1-\lambda x)^n}$

Now for a $X \sim \text{Gamma}(\alpha, \beta)$

$$M_X(x) = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{\frac{-y}{\beta}} e^{xy} dy$$

$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \boxed{\int_0^\infty y^{\alpha-1} e^{-(\frac{1}{\beta}-x)y} dy} \rightarrow ⊛$$

do a change of variables. let $z = (\frac{1}{\beta} - x)y$

Then $y = \frac{z}{\frac{1}{\beta} - x} = rz$ where $r = \frac{1}{\frac{1}{\beta} - x}$

$$⊛ = \int_0^\infty (rz)^{\alpha-1} e^{-z} d(rz)$$

$$= r^\alpha \boxed{\int_0^\infty z^{\alpha-1} e^{-z} dz}$$

$$= r^\alpha \boxed{\Gamma(\alpha)} \leftarrow \text{by def'n}$$

Then $\frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot r^\alpha \Gamma(\alpha) = (\frac{r}{\beta})^\alpha = (\frac{1}{1-\beta x})^\alpha$

Recall that the mgf of $X_1 + \cdots + X_n$

$$\left(\frac{1}{1-\lambda X}\right)^n \Rightarrow X_1 + \cdots + X_n \sim \text{Gamma}(n, \lambda)$$

Normal distribution/Gaussian distribution

Def: X is called a $N(\mu, \sigma^2)$ r.v. if

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right\} \qquad -\infty < x < \infty$$

Properties

① $E[X] = \mu$

② $V[X] = \sigma^2$

③ If $X \sim N(\mu, \sigma^2)$ Then $\frac{X-\mu}{\sigma} \sim N(0,1)$

standard normal

standard distribution

④ MGF of $N(0,1)$

$$M_{N(0,1)}(x) = e^{\frac{1}{2}x^2}$$

Properties of MGF

① $M_{x+a}(t) = E[e^{t(x+a)}] = e^{ta} M_x(t)$

② $M_{bx}(t) = E[e^{t(bx)}] = E[e^{(tb)x}] = M_x(tb)$

From ① & ②, If $X \sim N(\mu, \sigma^2)$

Then $M_x(t) = M_{\mu + \sigma Z}(t)$, where $Z \sim N(0,1)$

$$M_x(t) = e^{t\mu} M_{\sigma Z}(t) = e^{t\mu} M_Z(\sigma t) = e^{t\mu} e^{\frac{1}{2}(\sigma t)^2} = e^{\frac{1}{2}\sigma^2 t^2 + \mu t}$$

CHAPTER 5 Conditioning

Sps X is a r.v. & A is an event

Define $$E[X|A] = \frac{E[X I(A)]}{E(I(A))} = \frac{E[X I(A)]}{P(A)}$$

Suppose $X \sim \text{Unif}[0,1]$

Calculate $E[X | X > \frac{1}{4}]$

$$= \frac{E[X I(X > \frac{1}{4})]}{P(X > \frac{1}{4})}$$

$$= \frac{\int_{\frac{1}{4}}^{1} 1 \cdot x \, dx}{\frac{3}{4}} = \frac{\frac{1}{2}x^2 \Big|_{\frac{1}{4}}^{1}}{\frac{3}{4}} = \frac{5}{8}$$

Thm: ① If $P(A)>0$. Then $E(\cdot \mid A)$ follows the 5 axioms of probabilities

①a. If $X \geq 0$. $E[X|A] \geq 0$

①b. if c is a constant, then $E[cX|A] = cE[X|A]$

①c.

①d.

①e.

HW: PROOF

② If $\{A_k\}_{k=1}^{K}$ is a decomposition of Ω then

$$E(X) = \sum_{k=1}^{K} P(A_k) E[X|A_k] \quad (*)$$

Proof: RHS of $(*) = \sum_{k=1}^{K} P(A_k) \frac{E[X I(A_k)]}{P(A_k)} = \sum_{k=1}^{K} E[X I(A_k)] = E[X \sum I(A_k)]$

$= E[X]$

Def: Contional Prob. of events

If A, B are events & $P(A)>0$. Then $P(B|A) \overset{\text{Def}}{=} E(I(B)|A)$

Corollary:

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Proof: $P(B|A) = E(I(B)|A) = \frac{E(I(B) \times I(A))}{P(A)} = \frac{E\, I(AB)}{P(A)} = \frac{P(AB)}{P(A)}$

We see that if A, B are independent. $P(B|A) = \frac{P(A)P(B)}{P(A)} = P(B)$

Proposition: If $\{A_k\}$ is a decomposition of Ω. Then $P(B) = \sum P(A_k) P(B|A_k)$

LAW OF TOTAL PROBABILITY

statistical model

Baye's Formula

$$\boxed{P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{k=1}^{K} P(B|A_k)P(A_k)}}$$

prior

parameter data

Ex: In genetics, hair color is decided by a pair of genes A & a. A person will have blond hair iff he/she has aa genes.

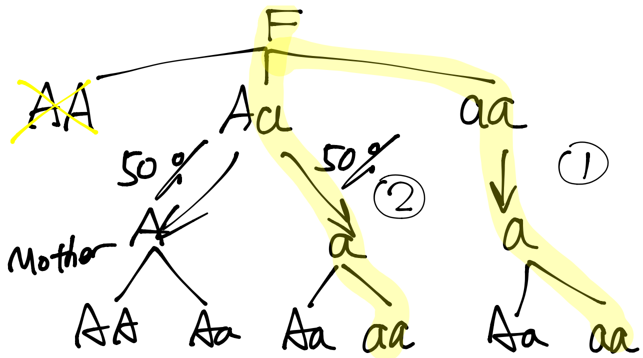
Sps in the population A appears 80%

a ··· 20%

and we know that a child has blond hair. What's the prob that his/her father has blond hair.

Prob. of intereset $= P(X|Y=aa)$

X = father's geno type, Y = child's gene type.



$P[\text{path}①] = 20\% \times 20\% \times 100\% \times 20\%$

$P[\text{path}②] = 80\% \times 20\% \times 2 \times 50\% \times 20\%$

Rememeber father blond <=> Genes follow by Bayes rule desired prob

$$\frac{p(\text{path}①)}{p(\text{path}①)+p(\text{path}②)} = 20\%$$

Conditional distribution as a r.v.

Ex: Sps one's blood pressure (X) is related to age Y. Then we know that for each given y we can calculate $E[X|Y=y]$

Note: When y changes. $E[X|Y=y]$ also changes

Therefore, guesses of one's blood pressure is changing with this person's age, in other words, it's a function of age.

Now Sps: $E[X|Y=y] = f(y)$

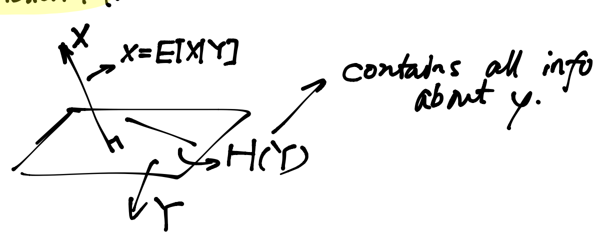
Then we define $E[X|Y] = f(Y)$ which says that the conditional Exp. of X given Y is a function of Y. in particular it is a random variable.

Once Y is discrete. $E[X|Y=y] = \dfrac{E[X\,I(Y=y)]}{P[Y=y]} \overset{\text{Def}}{=} f(y)$ (*)

$$E[X|Y] \overset{\text{def}}{=\!=} f(Y)$$

DEF: X, Y are two r.v. $E[X|Y]$ is defined as a function of Y s.t. $E[(X-E[X|Y])H(Y)]=0$ for any function H.

projection of X onto Y plane is $E[X|Y]$



$*$ Consistency in the discrete case. Let $f(Y)$ be defined in $*$ and Sps Y is discrete. Then I only need to show that $f(Y)$ satisfies $\checkmark$.

Need to show $E[(X-f(Y))H(Y)]=0$. for any H $(**)$

Now let us consider each case $Y=y$. for instance y can take $y_1, \cdots, y_K$

$$(**) = \sum_{i=1}^{K} \{E[X \mid Y=y_i] - f(y_i)\} H(y_i) \cdot P(Y=y_i)$$

$$= \sum_{i=1}^{K} \underbrace{\left[\frac{E[X I\{Y=y_i\}]}{P[Y=y_i]} - f(y_i)\right]}_{=0 \text{ by def}} \times H(y_i) \times P(Y=y_i)$$

So $(**)=0$