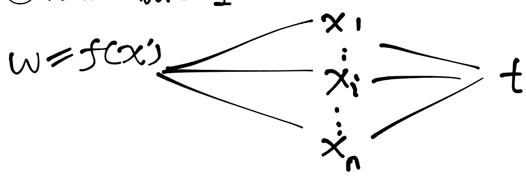
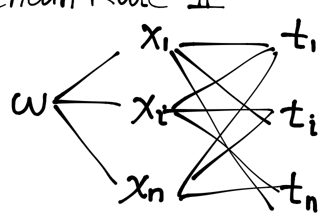
June 6th

Chain Rule I



$$\frac{dw}{dt}=\frac{\partial w}{\partial x_1}\frac{dx_1}{dt}+\cdots+\frac{\partial w}{\partial x_n}\cdot\frac{dx_n}{dt}$$

Chain Rule II



$f(\vec{x})=f(x_1,\dots,x_i,\dots,x_n)$

$x_i=g_i(t_1,\dots,t_j,\dots,t_m)$

$\vec{x}=\vec{g}(t_1,\dots,t_i,t_m)$

$\varphi(\vec{t})=f(\vec{g}(\vec{t}))$

$g_i: \mathbb{R}^m\to\mathbb{R}$

$\vec{g}: \mathbb{R}^m\to\mathbb{R}^n$

$f: \mathbb{R}^n\to\mathbb{R}$

$\varphi: \mathbb{R}^m\to\mathbb{R}^n\to\mathbb{R}$

at $\vec{b}$

$$\frac{\partial\varphi}{\partial t_j}=\frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_j}+\cdots+\frac{\partial f}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j}=\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t_j}$$

at $\vec{a}$

If f is diff. at $\vec{b}=\vec{g}(\vec{a})$, $\vec{g}$ is diff at $\vec{a}$, then φ is diff. at $\vec{a}$.
And the above statement is true

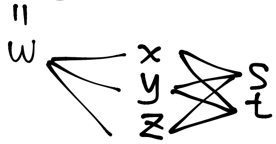
– can replace "differentiable" with "C^1 about ____". Also True.

Ex: $f(x,y,z)=x^2y+z$, $x=x(s,t)=s+t$

$y(s,t)=t^2$

$z(s,t)=2s$

$f(x,y,z)$
$\parallel$



$w: \mathbb{R}^2\to\mathbb{R}^3\to\mathbb{R}^1$

$$\frac{\partial w}{\partial s}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial s}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial s}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial s}=2xy\Big|_{\substack{x=s+t\\y=t^2}}\cdot 1+x^2\Big|_{x=s+t}\cdot 0$$

$$+1\cdot 2$$

$$=2(s+t)^2t+2$$

$$\frac{\partial w}{\partial t} = 2xy\Big|_{\substack{x=s+t \\ y=t^2}} \cdot 1 + x^2\Big|_{x=s+t} \cdot 2t + 1\cdot 0 = 2(s+t)t^2 + (s+t)^2\cdot 2t$$

$w = f(\vec{x})$, $\vec{x} = \vec{g}(\vec{t})$, $\vec{t} = (t_1, \cdots, t_m)$

$w = w(\vec{x})$ also think as $w = w(\vec{t})$

$$A: dw = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i} dx_i$$

$$B: dw = \sum_{j=1}^{m} \frac{\partial w}{\partial t_j} dt_j$$

But can write, $dx_i = \sum_{j=1}^{m} \frac{\partial x_i}{\partial t_j} dt_j \leftarrow C$

subbing C into A, $dw = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\left(\sum_{j=1}^{m} \frac{\partial x_j}{\partial t_j} dt_j\right)$

$$= \sum_{j=1}^{m}\left(\sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\frac{\partial x_i}{\partial t_j}\right) dt_j \qquad \leftarrow D$$

saying $B \Leftrightarrow D$ is saying $\frac{\partial w}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial w}{\partial x_i}\frac{\partial x_i}{\partial t_j}$ $\Leftarrow$ Chain Rule

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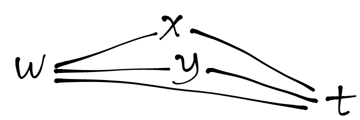
$f(x(t), y(t), t) = f(x(t), g(t), u(t))$

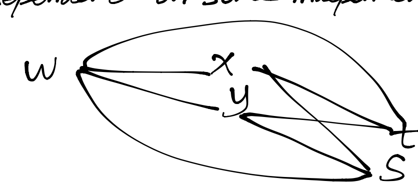
mixes dep. & indep. in one formula. $u(t) = t$

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \underbrace{\frac{\partial f}{\partial u}}_{\to \frac{\partial w}{\partial t}} \cdot \overbrace{\frac{du}{dt}}^{1}$$



$w = f(x, y, t, s)$ — x, y: dep; t, s: indep

$x = \varphi(s,t)$
$y = \psi(s,t)$ $\quad$ x, y are dependent on some independent s, t.



$$\frac{\partial w}{\partial s} = \partial_1 f\,\partial x_1 + \partial_2 f\,\partial_1 y + \partial_4 f$$

Using numbers removes ambiguities.

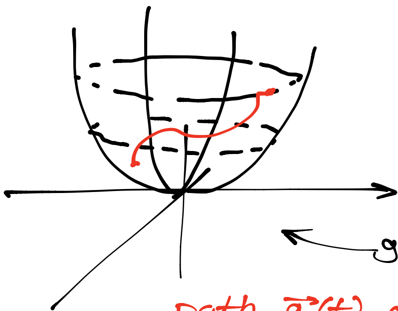
Ex: $f(x,y,t,s) = x^2 + ys + t$, $x(s,t) = s+t$
$y(s,t) = 2t$

$$\frac{\partial w}{\partial s} = 2x\Big|_{x=s+t} \cdot 1 + s\cdot 0 + y\Big|_{y=2t}$$

$F(x,y,z)=0$ is different from $z=f(x,y)$

$z-x^2-y^2=0$

$F(x,y,z)=z-x^2-y^2$



← generically a "smooth surface" when F diff.

path $\vec{g}(t)$ on surface

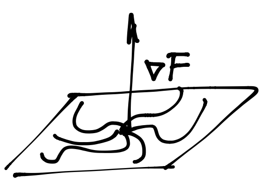
$F(\vec{g}(t))=0$ ← why? b/c $\vec{g}(t)$ is on the surface.

$\frac{d}{dt}F(\vec{g}(t))=0$

$\frac{d}{dt}F(\vec{g}(t))=\nabla F\cdot\vec{g}'(t)$ → via chain rule

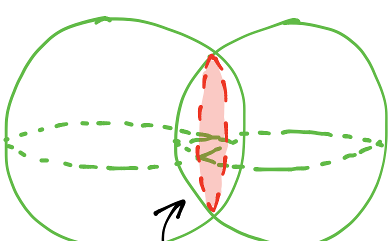
So $\nabla F\cdot\vec{g}'(t)=0 \Rightarrow \nabla F$ is orthogonal to $\vec{g}'(t)$

- $\vec{g}'(t)$ is known tangent to the curve given by $\vec{g}(t)$
- as ∇f is $\perp$ for all such $\vec{g}'(t)$, then ∇F as being "normal" to our surface.



- Recall that ∇f is pointing in direction of maximum slope

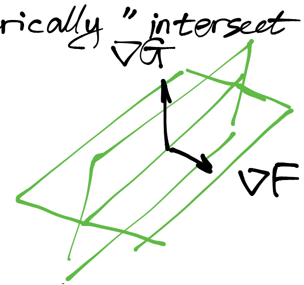
The equation of the tangent plane : $\underbrace{\nabla F}_{\text{normal}}\cdot(\vec{x}-\vec{a})=0$



intersection circle

For $F(x,y,z)=0$
$G(x,y,z)=0$

"generically" intersect in a curve



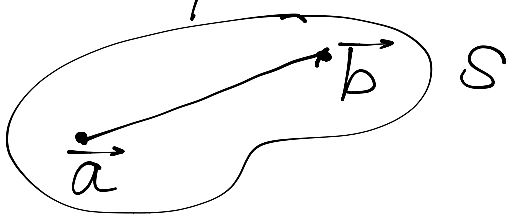
• If ∇F & ∇G are linearly independent, they span the plane $\perp$ to intersection curve.

$F(x,y,z)=0$, $z=f(x,y)$ can't always pull the eqn apart, namely the two eqns are not necessarily equal.

– In 1-D, called "curve"
2-D, "surface"
3-D "hypersurface"

Mean Value Theorem §2.4

Set S – points $\vec{a}, \vec{b}$



Sps f defined on S, continuous on L, diff. on L, diff. on $L \setminus \{\vec{a}, \vec{b}\}$

$$L=\{\vec{x}\in S \mid \vec{a}+t(\vec{b}-\vec{a}), t\in[0,1]\}, \exists \vec{c}\in L$$

Then $f(\vec{b})-f(\vec{a})=\nabla f(\vec{c})\cdot(\vec{b}-\vec{a})$ ← directional derivative in $(\vec{b}-\vec{a})$ direction (but not normalized)

$\varphi(t)=f(\vec{a}+t\vec{h})$, $\vec{h}=\vec{b}-\vec{a}$, $t\in[0,1]$.
– so composition of diff. functions on appropriate places

$$\varphi'(t)=\nabla f(\vec{a}+t\vec{h})\cdot\vec{h} \quad \leftarrow \text{by chain Rule}$$
$$=\nabla f(\vec{a}+t\vec{h})\cdot(\vec{b}-\vec{a})$$

$\varphi(1)=f(\vec{b})$
$\varphi(0)=f(\vec{a})$ $\quad 1-0=1$

$$\exists u - f(\vec{b})-f(\vec{a})=\varphi(1)-\varphi(0)=\varphi'(u)(1-0)=\varphi'(u)=\nabla f(\vec{a}+u\vec{h})(\vec{b}-\vec{a})$$

letting $\vec{c}=\vec{a}+u\vec{h}$

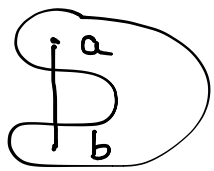
from MVT for $R\to R$

Although MVT doesn't work all the time, it works for two vectors which have a straight line in between.

$\Downarrow$

CONVEX

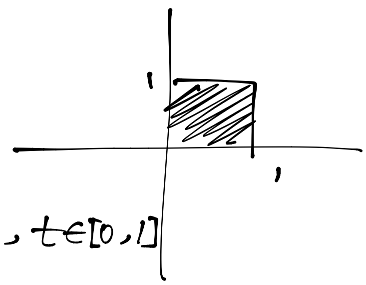
Def: A set is convex if $\forall \vec{a}, \vec{b}$ then $L=\{\vec{x}\in R^n \mid \vec{x}=\vec{a}+t(\vec{b}-\vec{a}), t\in[0,1]\} \subset S$



not convex

- If convex, surely it's path-connected.
- Proved $B(r,x)$ is convex as we proved it is pathconnected using straight lines.
- likewise for $\mathbb{R}^n$.

Ex:
$S=\{(x,y)\mid 0\le x\le 1,\ 0\le y\le 1\}$



Claim: S is convex (hence path connected)

Proof: Take $\vec{a},\vec{b}\in S$, consider $\vec{a}+t(\vec{b}-\vec{a})$, $t\in[0,1]$

want $\vec{x}\in S\ \forall t$. $0\le(1-t)a_1+tb_1\le(1-t)\cdot 1+t\cdot 1=1$
$0\le(1-t)a_2+tb_2\le(1-t)+t=1$
$\Rightarrow \vec{x}\in S\Rightarrow S$ is convex

Thm: Let S be open & convex, f diff on S. Suppose $|\nabla f|\le M$ then $|f(\vec{b})-f(\vec{a})|\le M|\vec{b}-\vec{a}|$

Proof: Mean Value Theorem $\Rightarrow\exists\,\vec{c}$ s.t.
$f(\vec{b})-f(\vec{a})=\nabla f(\vec{c})\cdot(\vec{b}-\vec{a})$
Then $|f(\vec{b})-f(\vec{a})|\le|\nabla f(\vec{c})||\vec{b}-\vec{a}|\le M|\vec{b}-\vec{a}|$

Cauchy-Schwartz

∎

Corollary: If f diff. on open convex sets and $\nabla f=0$ everywhere in S then S is constant on S.

MVT $\Rightarrow\exists\,\vec{c}$, $f(\vec{b})-f(\vec{a})=\nabla f(\vec{c})(\vec{b}-\vec{a})=0$
$\Rightarrow f(\vec{b})=f(\vec{a})\quad\forall a,b\in S$

∎

Thm: If f is diff on open, connected set S, then f constant.

Proof: Take S_1 as a disconnection of S.
$S_1=\{x\mid f(x)=f(a)\}$ after choosing some $a\in S$.
$S_2=\{x\mid f(x)\ne f(a)\}$
(S_1,S_2) must fail to be a disconnection for some property of a disconnection.

- $a\in S_1\Rightarrow S_1\ne\emptyset$ ✓ trivially
- $S_1\cup S_2=S$ ✓ as $f(x)=$ or $\ne f(a)$ exhausts possibilities
- Claim: $\overline{S_1}\cap S_2=S_1\cap\overline{S_2}=\emptyset$
- $\Rightarrow S_2=\emptyset$
- $S_2=\emptyset\Rightarrow S_1=S\Rightarrow f(x)=f(a)\ \forall x\in S\Rightarrow f$ is constant.

Lemma: Sps A,B open, $A\cap B=\emptyset$, then $\overline{A}\cap B=\emptyset$, $\overline{B}\cap A=\emptyset$.

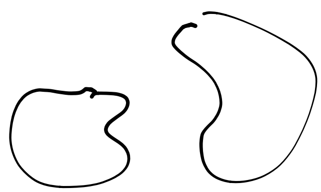
Proof: WLOG, say $x \in \bar{A} \cap B$. As B open, $\exists r . B(r,x) \subset B$ as $x \in \bar{A}$ $\Rightarrow \forall r'$, in particular r, $B(r,x) \cap A^c \Rightarrow B(r,x) \not\subset B$ ∎

If I can proof S_2 & S_1 open as $S_1 \cap S_2 = \emptyset$, satisfies Lemma $\Rightarrow \bar{S_1} \cap S_2 = S_1 \cap \bar{S_2} = \emptyset$

- S_2 is open, as: $\mathbb{R}^n \setminus \{f(a)\}$ is open as singleton sets are closed. as f is diff, it is <u>cont.</u> $\Rightarrow f^{-1}(\text{open})$ is open $\Rightarrow S_2$ is open (as intersecting with S open stays open)

Can't do the same trick for S_1 as closed & open is ambiguous.

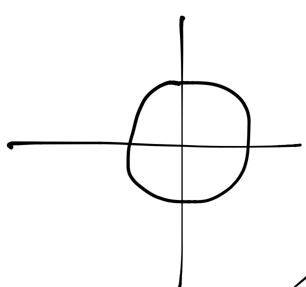
Instead for $x \in S$, $\exists B(r,x)$ s.t. $B(r,x) \subset S$. However, as $B(r,x)$ is convex, open, f is diff. & $\nabla f = 0$. $\Rightarrow f$ is constant on $B(r,x)$ by previous corollary.
$\Rightarrow B(r,x) \subset S_1 \Rightarrow S_1$ is open. ∎



$S = S_1 \cup S_2$

$$f(x) = \begin{cases} 1, & x \in S_1 \\ 0, & x \in S_2 \end{cases}$$

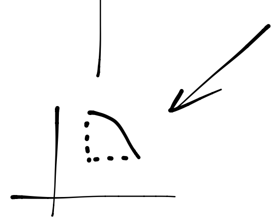
Counterexample to show why connected is needed.



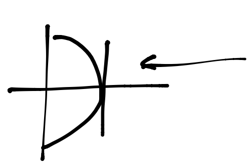
$x^2 + y^2 = 1$

- not a function it fails vertical line test

But take a portion, it passes the test
$y = \sqrt{1-x^2}$



There is not a single function describes the whole part of $x^2+y^2=1$.



always fails vertical test here. (and, the opposite point)

So for all points <u>except</u> $(1,0)$, $(-1,0)$ can represent a portion of curve about a point, as the graph of a function

- Can generalize, except for problems in the derivative

$F(\overbrace{x_1, \cdots, x_n}^{\text{dep}}, \overset{\text{indpt}}{y})=0$, F is diff.

when can we write $y = g(x_1, \cdots, x_n)$?

Answer : Given by Implicit Function Theorem