Feb 12th

Def: $T: V \to V$. The char. poly of T is $\det(\lambda I - [T]_\alpha)$ where α is a basis of V and $[T]_\alpha$ is the matrix of T in α.

Why does $p(\lambda)$ not depend on α?

Suppose β another basis of V

Want: $\det(\lambda I - [T]_\beta) = \det(\lambda I - [T]_\alpha)$

know $[T]_\beta = P[T]_\alpha P^{-1}$ when $P = [I]_\alpha^\beta$

$\det(\lambda I - P[T]_\alpha P^{-1}) \longrightarrow$

$$\det(\lambda P P^{-1} - P[T]_\alpha P^{-1}) = \det(P(\lambda I - [T]_\alpha)P^{-1})$$
$$= \det(P)\det(\lambda I - [T]_\alpha)\det(P^{-1})$$
$$= \det(\lambda I - [T]_\alpha)$$

$\det(AB) = \det(A)\det(B)$

Ex: $T: M_2(\mathbb{C}) \to M_2(\mathbb{C})$.

$T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$

$$\alpha = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$[T]_\alpha = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \quad p(\lambda) = \begin{vmatrix} (\lambda-1) & 0 & -1 & 0 \\ 0 & (\lambda-1) & 0 & -1 \\ 0 & 0 & (\lambda-1) & 0 \\ 0 & 0 & 0 & (\lambda-1) \end{vmatrix} = (\lambda-1)^4$$

$E_1 =$

$$A = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{null}(A) = \text{span}\{e_1, e_2\}$$

$$E_1 = \text{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

Def: $T: V \to V$ is diagonalizable if there is a basis of V which consists of eigenvectors of T.

Ex: $T: M_2(\mathbb{C}) \to M_2(\mathbb{C})$ $T(A) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} A$ not diagonalizable.

Prop: $T: V \to V$ is diagonalizable $\iff$ for any basis α of V $[T]_\alpha$ is diagonalizable

Proof: If T is diagonalizable then there is a basis β of eigenvectors
$\beta=\{v_1,\cdots,v_n\}$ $Tv_i=\lambda_i v_i$

$$[T]_\beta=\begin{bmatrix}\lambda_1 & & & 0\\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n\end{bmatrix}$$

$T(v_1)=\lambda_1 v_1$
$T(v_2)=\lambda_2 v_2$

Let α be any basis of V.
$P[T]_\alpha P^{-1}=[T]_\beta$
$P=[I]_\alpha^\beta$ $P^{-1}=[I]_\beta^\alpha$

This proves one direction ... Skip the other

Thm 1: $T:V\to V$
$\{v_1,\cdots,v_k\}$ eigenvectors
$T(v_i)=\lambda_i v_i$
Suppose $\lambda_i\neq\lambda_j$ for all $i\neq j$
Then $\{v_1,\cdots,v_k\}$ are lin. indpt.

We will use induction to prove thm

Proof: Induct on n.
Base case is easy, clearly $\{v_1\}$ is lin. ind.

Ind hyp: sps that if $\{v_1,\cdots,v_{n-1}\}$ are eigenvectors with distinct evalues, then they're ind.

Ind step: We want to show $\{v_1,\cdots,v_n\}$ ind.

sps $a_1v_1+\cdots+a_nv_n=0$ $(*)$

Apply T to both sides: $a_1\lambda_1v_1+a_2\lambda_2v_2+\cdots+a_n\lambda_n v_n=0$

Multiply both sides of $(*)$ by λ_n: $a_1\lambda_n v_1+\cdots+a_n\lambda_n v_n=0$

Subtracting second eqtn from first:

$0=a_1(\lambda_1-\lambda_n)v_1+a_2(\lambda_2-\lambda_n)v_2+\cdots+a_{n-1}(\lambda_{n-1}-\lambda_n)v_n$

By ind. hyp get $0=a(\lambda_1-\lambda_n)=\cdots=a_{n-1}(\lambda_{n-1}-\lambda_n)$

Because $\lambda_i \neq \lambda_j$ for $i \neq j$

Then $a_1 = a_2 = \cdots = a_{n-1} = 0 \Rightarrow a_n = 0$ as well. $\blacksquare$

def: Sps λ is an eigenvalue of $T: V \to V$.
The multiplicity of λ is its multiplicity as a root of the characteristic poly. of T.

Ex: $T: V \to V$
$p(\lambda) = (\lambda - i)^3 (\lambda - 1)(\lambda - 3)^5$ is an eigenvalue of multiplicity

Prop: $T: V \to V$
λ evalue
$\dim E_\lambda \leq$ multiplicity of λ

Proof: $d = \dim E_\lambda$ $\quad \{v_1, \cdots, v_d\}$ basis of E_λ
Extend basis to $\alpha = \{v_1, \cdots, v_n\}$ of V

$$[T]_\alpha = \begin{array}{c c} \left[\begin{array}{cccc|c} \lambda & & & 0 & \\ & \lambda & & & B \\ & & \ddots & & \\ & & & \lambda & \\ \hline 0 & 0 & & \vdots & A \end{array}\right] & \begin{array}{c} d \\ \\ \\ \\ n-d \end{array} \\ \underbrace{\qquad\qquad\qquad}_{d} & \end{array}$$

A: $(n-d) \times (n-d)$
B: $d \times (n-d)$

$$p(x) = \det(xI - [T]_\alpha)$$

$$= \left[\begin{array}{ccc|c} x-\lambda & & 0 & \\ & \ddots & & -B \\ 0 & & x-\lambda & \\ \hline & 0 & & \lambda I - A \end{array}\right] = (x-\lambda)^d \det(\lambda I - A)$$

multiplicity of λ is biggest n s.t.
$(x-\lambda)^n$ divides $p(x)$.

Ex: $p(x) = (x-i)^3 (x-1)(x-3)^5$

$(x-\lambda)^d$ divides $p(x)$, so the ...

$\blacksquare$

Thm: (Diagonalizability thm)

$T: V \to V$
$\lambda_1, \cdots, \lambda_k$ are its eigenvalues
$m_1, \cdots, m_k$ multiplicities

diagonalizable $\Longleftrightarrow$ ① $m_1 + \cdots + m_k = \dim V$
② $\dim E_{\lambda_i} = m_i$

This thm based on two ideas

① eigenvectors with distinct eigenvalues are indep.

② $\lambda_1, \cdots, \lambda_k$ distinct evalues
for each E_{λ_i} have a basis $\{v_{1,i}, \cdots, v_{n,i}\}$

Then $\{v_{1,1}, \cdots v_{n,1}, v_{1,2}, \cdots v_{n,2}, \cdots\}$
this is lin. indpt.

Ex: When is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ diagonalizable?

$p(\lambda) = (\lambda - a)^2$ char. poly

evalue a with mult 2.

so A diag'ble $\Leftrightarrow$ $\dim E_a = 2$

$E_a = \text{null}\left(\begin{bmatrix} 0 & -b \\ 0 & 0 \end{bmatrix}\right)$ has dim 2 $\Leftrightarrow b = 0$

Ex: Let $V = \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}$

$E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $T: V \to V$ given by $T(A) = EA - AE$

Is T diag'ble?

$$\alpha = \Big\{ \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_1}, \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A_2}, \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{A_3} \Big\}$$

$T(A_1) = -A_2$, $T(A_2) = 0$, $T(A_3) = A_2$

$$[T]_\alpha = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$p(\lambda) = \det \begin{bmatrix} \lambda & 0 & 0 \\ 1 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix} = \lambda^3$$

T has eigenvalue 0 with multiplicity 3

Ex: $\dim E_0 = 2$

$$\text{null}\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\right) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

We need $\dim E_0 =$ mult of 0

In order for T to be diagl'ble

<u>Conclusion</u>: T is not diagl'ble.