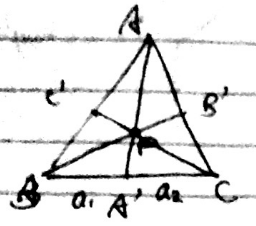
Some questions & problems related to lectures 1-3.

1. Ceva's theorem

concurrent $\Rightarrow \frac{a_1}{a_2} \cdot \frac{b_1}{b_2} \cdot \frac{c_1}{c_2} = 1.$



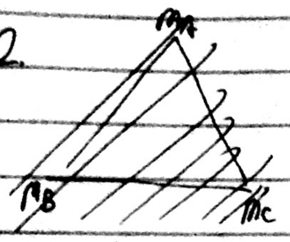
$\frac{a_1}{a_2} = \frac{S_{\Delta ABA'}}{S_{\Delta ACA'}} = \frac{S_\Delta ABP}{S_\Delta ACP}.$

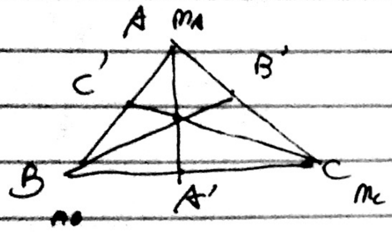
$\frac{b_1}{b_2} = \dots$

$\frac{c_1}{c_2} = \dots$

then $\dots = 1.$

2.





$m_A \cdot AC' = m_B C'B$

$m_B \cdot BA' = m_C A'C$

let $m_A = 1$, $m_B = \frac{C'B}{AC'}$

$m_C = \frac{C'B}{AC'} \cdot \frac{BA'}{A'C}$

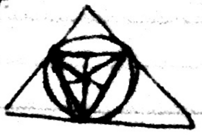
$m_A B'A = m_C CB'$

$\Leftrightarrow \frac{m_A}{m_C} = \frac{CB'}{B'A}$

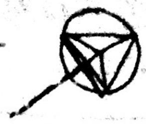
$\Leftrightarrow \frac{1}{\frac{C'B}{AC'} \cdot \frac{BA'}{A'C}} = \frac{CB'}{B'A}$

$\frac{AC' \cdot A'C}{C'B \cdot BA'} \cdot \frac{B'A}{CB'} = 1$ ✓

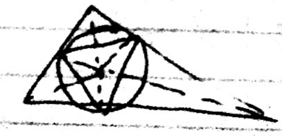
4



want to prove ceva for

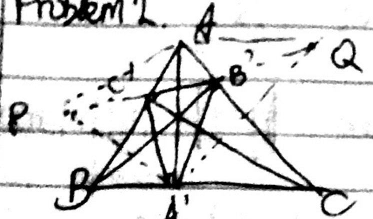


3 height pass 1 point



Problems of Lecture 4.

Problem 1.



$C'B' + B'A' + A'C' < 2AA'$
$< 2BB'$
$< 2CC'$

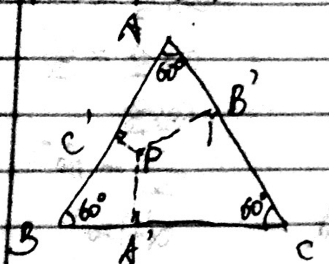
we wanna know this true.

WLOG say AA' is shortest among AA', BB' & CC'

in $\triangle PAQ$, $PA = QA = AA'$

$PA + QA = 2AA' > C'B' + B'A' + A'C' = PQ$ - Perimeter.

Problem 2.



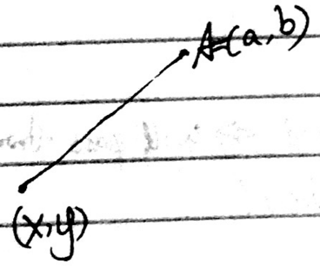
Prove: $\forall P$ in $\triangle ABC$. $PA' + PB' + PC'$ fixed.

note $\triangle APB = \frac{1}{2} AB \cdot PC'$
$\triangle BPC = \frac{1}{2} BC \cdot PA'$
$\triangle APC = \frac{1}{2} AC \cdot PB'$
add up

$\triangle ABC = \frac{1}{2}$ side $\times \underbrace{(PA' + PB' + PC')}$

$\downarrow$ fixed $\quad \downarrow$ fixed $\quad \downarrow$ fixed.

Problem 3.



$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[(x-a)^2 + (y-b)^2\right]^{\frac{1}{2}}, \frac{\partial}{\partial y}\left[(x-a)^2 + (y-b)^2\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{2}\left[(x-a)^2 + (y-b)^2\right]^{-\frac{1}{2}} \cdot (2x-2a), \frac{1}{2}\left[(x-a)^2 + (y-b)^2\right]^{-\frac{1}{2}}(2y-2b)\right]$$

$$= \left[(x-a)^2 + (y-b)^2\right]^{-\frac{1}{2}} (x-a, y-b)$$

$\downarrow$ looking away from A

$$\|\nabla f\| = \frac{1}{(x-a)^2 + (y-b)^2} \cdot \left[(x-a)^2 + (y-b)^2\right] = 1.$$

$\downarrow$ length 1.

Problem 4.

x, y, z on plane.

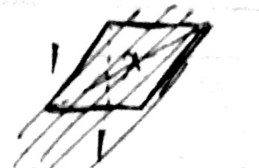
$x+y+z=0$

$\|x\|+\|y\|+\|z\|=1$.

Let x be the positive direction

$x=-(y+z)$. let α

Then ~~$|y+z|$~~ $|y||z|$



~~$\cos\alpha = |x| \cdot 1$~~

~~$\sin\alpha \cdot 1 = x \cdot \frac{1}{2}\sin\frac{\alpha}{2}$~~

~~$2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} = x \cdot \frac{1}{2}\sin\frac{\alpha}{2}$~~

~~$4\cos\frac{\alpha}{2} = x$~~

~~$\cos\frac{\alpha}{2} = \frac{1}{4}x$~~

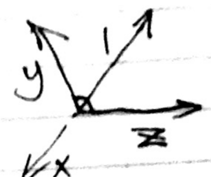
~~and $x = 1$~~

~~$\cos\frac{\alpha}{2} = \frac{1}{4}$~~

~~$\cos\alpha =$~~

~~$\cos\alpha = 1$~~

$\alpha =$



$\cos\frac{\alpha}{2} = \frac{1}{2}$

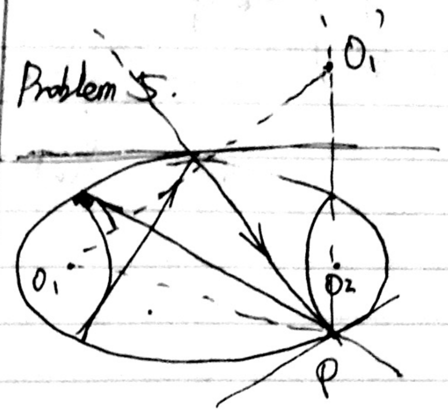
$2\cos\frac{\alpha}{2} = 1$

$\cos\frac{\alpha}{2} = \frac{1}{2}$

$\frac{\alpha}{2} = 60^\circ$

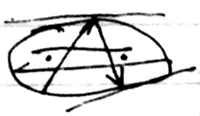
$\alpha = 120^\circ$

Problem 5.



$O_1'O_2 = O_1O_2 = PO_1 - PO_2$ fixed

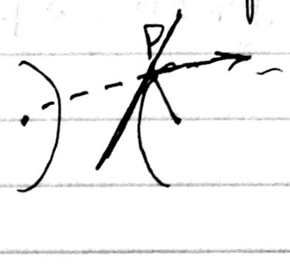
optical property: ray intersects a segment joining O_1, O_2



Know: In ellipse, if ray pass through 1 foci ⇒ will pass through one of the foci after every reflection!

Know if initial does not pass through foci, then after infinitely many reflections, still doesn't pass $O_1 O_2$.
(small ellipse area inside) untouchable

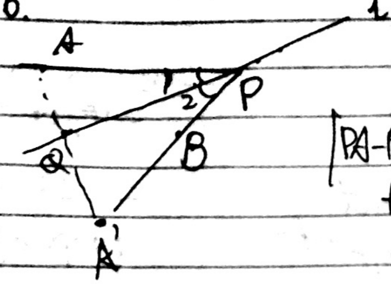
Hyperbola



"looks like from another foci"

Problem 6.

(1).



$|PA-PB|$ max

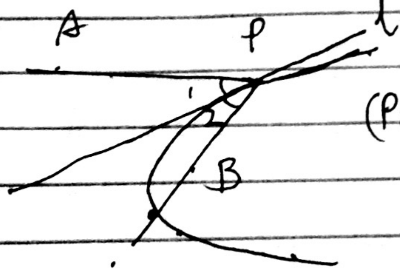
then the reflection point A' about l is on the line PB.

Therefore.

$$\left.\begin{array}{l} PA=PA' \\ QA=QA' \\ QP=QP \end{array}\right\} \Rightarrow \triangle APQ \cong \triangle A'PQ$$

$\Rightarrow \angle 1 = \angle 2$.

(2).

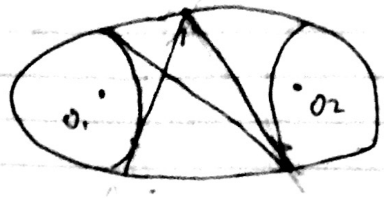


$(PA-PB)$ constant

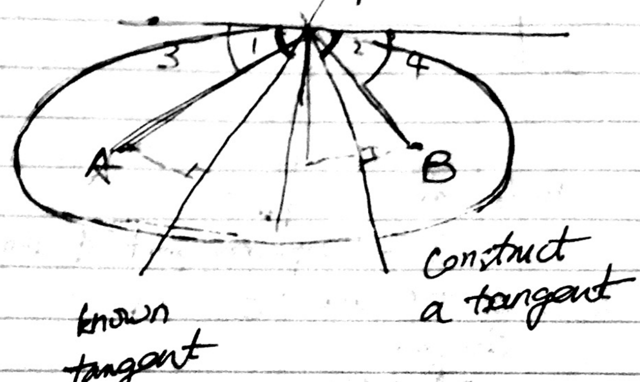
hyperbola $\Rightarrow A, B$ focus

Property. P tangent $\Rightarrow$ l bisector of $\angle APB$.

Problem 5



prove it's a reflection



~~$\angle 1 = \angle 2$ since tangent to ellipse.~~

$\angle 3 = \angle 4$ due to property of ellipse

want $\angle 1 = \angle 2$