MAT334 Final Review

• $\boxed{\text{BASICS of COMPLEX NUMBERS}}$

Polar : $e^{i\theta} = \cos\theta + i\sin\theta$

$(e^{i\theta})^n = e^{in\theta}$

$e^{i\pi} + 1 = 0$

argument : the angle θ such that $z = e^{i\theta}$.

Argument : Argument $\rightarrow$ angle θ such that $z = e^{i\theta}$
and $\theta \in [-\pi, \pi)$

Take n-th roots of complex number.

• $\boxed{\text{GEOMETRY AND TOPOLOGY OF THE PLANE}}$

open, closed, connected, convex, boundary

disks, annuli, half planes.

D is a domain if it's open & connected.

Line: for the line $y = mx + b$ (in real $\mathbb{R}^2$ plane)
the complex version is defined as:

$$\mathrm{Re}((m+i)z + b) = 0$$

• $\boxed{\text{Complex Functions and Limits}}$

Ratio / n-th root test $\Rightarrow$ determine series convergent or not.

exponential $e^z = e^x e^{iy} = e^x(\cos y + i\sin y)$

Logarithm $\log z = \ln|z| + i\arg z$

$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z$

Trig: $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

· $\boxed{\text{Line Integrals and Green's Theorem}}$

simple (no self-intersections)
closed, smooth, ... piecewise smooth.
positively oriented (inside on LHS)
parametrizations
line : $z_0 \to z_1$

$$\gamma(z) = (1-t)z_0 + t(z_1)$$

$$\gamma : [0,1] \to \mathbb{C}$$

circle/arc of circle:

$$\gamma(t) = z_0 + re^{it}$$

Let γ be p.w smooth curve and $f: D \to \mathbb{C}$ (D-domain) contains so that $\gamma \subseteq D$. Define the line integral:

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

· useful facts:

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

$$\text{Length}(\gamma) = \int_a^b |\gamma'(t)|dt$$

$$\left|\int_\gamma f(z)dz\right| \leq \text{length}(\gamma) \cdot \max_{z \text{ on } \gamma} |f(z)|$$

$$\int_\gamma z^m dz = \frac{1}{m+1} z^{m+1} \Big|_{\text{initial pt}}^{\text{end pt}}$$

Green's thm: (complex version)

Ω: domain with boundary $\partial\Omega = \gamma_1 \cup \cdots \cup \gamma_n$
γ's p.w. smooth, simple closed curves oriented positively.
Let f be a cts function

$$\int_{\partial\Omega} f(z)\, dz = i \iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) dx\,dy$$

where $\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

note: $f = u + iv$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial u}{\partial y}$$

- $\boxed{\text{Analytic and Harmonic Functions, Cauchy-Riemman Equations}}$

· analytic, entire
· Cauchy-Riemann Equations

analytic $\Leftrightarrow$ cauchy-Riemann.

if $f = u + iv$, $u(x,y)$, $v(x,y)$ are 2 functions of variables x, y.
then C-R equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

Harmonic : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, $u \Rightarrow$ harmonic

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ - Laplacian, write ~~$\Delta = 0$~~ $\Delta u = 0$ as shorthand

THM: If $f = u + iv$ is analytic $\Rightarrow$ u & v are both harmonic.

If u & v are harmonic. ~~iso~~ and $f = u + iv$ is analytic, we say u & v are conjugate functions.

Thm: if $f = u + iv$ analytic on D & $u^2 + v^2$, u, v are constant on D, then f is constant.

$\boxed{\text{Sequences, Series and Power Series}}$

powerseries: $\sum_{n=0}^{\infty} a_n (z - z_0)^n$

Radius of convergence (R) only converges for $|z - z_0| < R$.
and $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \sqrt[n]{a_n} = \frac{1}{R}$.

• a convergent power series is an analytic function on its domain of convergence.

p.w. expansion of :

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

~~cos z~~

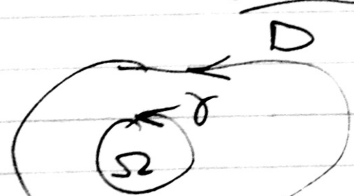
$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots$$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

• $\boxed{\text{Cauchy Theorem and Cauchy Formula}}$

Thm: (Cauchy Theorem) Let f be analytic in D, γ a simple curve in D, Ω = inside of γ, $\Omega \subseteq D$

Then $\int_\gamma f(z)dz = 0$



"simply connected": if $\forall$ simply closed γ in S, inside of γ is also contained in S (I.e. S is not punctured!)

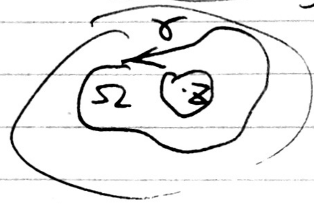
Thm: D simply connected, $f: D \to \mathbb{C}$ analytic. γ is any closed curve consisting only horizontal & vertical line segments.
then $\int_\gamma f(z)dz = 0$.

Thm: $f: D \to \mathbb{C}$ analytic & s. connected
$\Rightarrow \exists F: D \to \mathbb{C}$ s.t. $F' = f$, F is analytic.

Morera's thm: f cts in D (s. connected) & $\int_\gamma f(z)dz = 0$ $\forall$ all triangles in $D \Rightarrow f$ is analytic.

Cauchy's Formula: If f is analytic in D, γ any closed curve whose inside is contained in D, then

, p is any point in γ.



$$f(p) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-p}\, dz$$

The line integral of an analytic function depends only on the endpoints of the curve, not actual path

- Application of Cauchy Theorem & Formula

Sps f is analytic in D, $z_0 \in D$, can find $D_R(z_0)$ & a power series of f, $f(z) \sum a_k (z-z_0)^k$ which cvgs abs in $D_R(z_0)$

Moreover, $a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\, dz$, γ is simple closed curve around z_0

Cor: if f analytic, so is f'. $f \Rightarrow$ infinitely diff.

Cor: f is analytic in D & $\exists$ $z_0 \in D$ we have

$$f^{(k)}(z_0) = 0 \ \forall k, \text{ then } f(z) = 0 \ \forall z \in D.$$

$$\boxed{a_k = \frac{f^{(k)}(z_0)}{k!}}$$

· $\boxed{\text{ZEROS}}$

zero of order m. for $n<m$, $f^{(n)}(z)=0$, but ~~f~~ $f^{(m)}(z)\neq 0$.

Liouville's Thm:
If f ~~is~~ is bdd & entire, then it's a constant.

· when finding zeros of $\frac{P}{Q}$ => need to fully factor

· f: ~~order~~ m at z_0, g: order n at z_0
  $f\cdot g$: order $m+n$ at ~~z~~ z_0

· $\boxed{\text{Isolated Singularities}}$

· $f(z)=\frac{z^2-1}{z-1}$ removable singularity b/c
  $\lim|f(z)|$ is bdd ~~at~~ as $z\to z_0$

· $f(z)=\frac{1}{z+1}$ ~~is~~ pole b/c
  $\lim|f(z)|$ is ∞ as $z\to z_0$

· $f(z)=e^{\frac{1}{z-1}}$ essential singularity as $z\to z_0$
  not (i) nor (ii)

· $\boxed{\text{RESIDUES}}$

Sps f is analytic in a punctured disk $0<|z-z_0|<r$.
then $\frac{1}{2\pi i}\int_{\gamma_s} f(z)dz$ does not depend on γ_s.

γ_s is a circle of radius s centered at z_0 $(s<r)$
we call
$\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \text{Res}(f; z_0)$ the residue of f at z_0.

Method 1

Residue $(f: z_0) = C_{m-1}$ where $g(z) = \sum_{k=0}^{\infty} C_k (z-z_0)^k$ & z_0 is a pole of order m.

and $f(z) = \frac{1}{(z-z_0)^m} g(z)$

Method 2

ONLY IN THIS CASE:

If $f(z) = \frac{P(z)}{Q(z)}$ and z_0 is a pole of order 1,

then Res $(f: z_0) = \frac{P(z_0)}{Q'(z_0)}$

~~Laurent~~ Laurent Series

If f has a pole of order m at z_0, we can write

$f(z) = \frac{1}{(z-z_0)^m} g(z)$ (and g is analytic)

$= \frac{1}{(z-z_0)^m} \sum_{k=0}^{\infty} C_k (z-z_0)^k$

$= \frac{1}{(z-z_0)^m} (C_0 + C_1(z-z_0) + C_2(z-z_0)^2 + \cdots + C_m(z-z_0)^m + \cdots)$

"double ~~series~~ infinite" sum

$\sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$

- A Laurent Series converges in an annulus / punctured disk.

  $r < |z-z_0| < R$ (annulus)

  $0 < |z-z_0| < R$ (punctured disk)

Thm: ① Sps f analytic in $r < |z-z_0| < R$. write $f(z) = f_1(z) + f_2(z)$

where f_1 is analytic in $r < |z-z_0|$

f_2 — " — " — $|z-z_0| < R$

② f_1 has p.s. in $\frac{1}{z-z_0}$ which converges inside $r < |z-z_0|$

f_2 — p.s. in $z-z_0$ which cvgs inside $|z-z_0| < R$

③ can write Laurent series using ① & ②

④ If $f(z) = \sum_{-\infty}^{\infty} a_k (z-z_0)^k$ then $a_k = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-z_0)^{k+1}} dz$

γ simple closed curve inside $r < | \quad | < R$

Method 3

THM: $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$ then $\text{Res}(f; z_0) = a_{-1}$

coefficient of $\frac{1}{z-z_0}$

• $\boxed{\text{REAL INTEGRALS}}$

~~Residual~~

Residue Theorem:

Sps f is analytic in a simply-connected region except for poles $z_1, \dots, z_N$. If γ is a simple closed curve which does not pass through the poles then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \text{Res}(f; z_k)$$

INSIDE

• ~~Let γ be the circle of~~ r

If a removable singularity inside γ, you don't need to add it up b/c the residue of it (think about Laurent series with only removable singularities? Just another power series.

Improper integral:

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx \text{ with } \deg Q \geq \deg P + 2$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \sum_{z_k \in \text{Upper halfplane}} \text{Res}(f; z_k)$$

* For $\int \frac{P(x)}{Q(x)} \cos x\,dx$ / $\int \frac{P(x)}{Q(x)} \sin x\,dx$

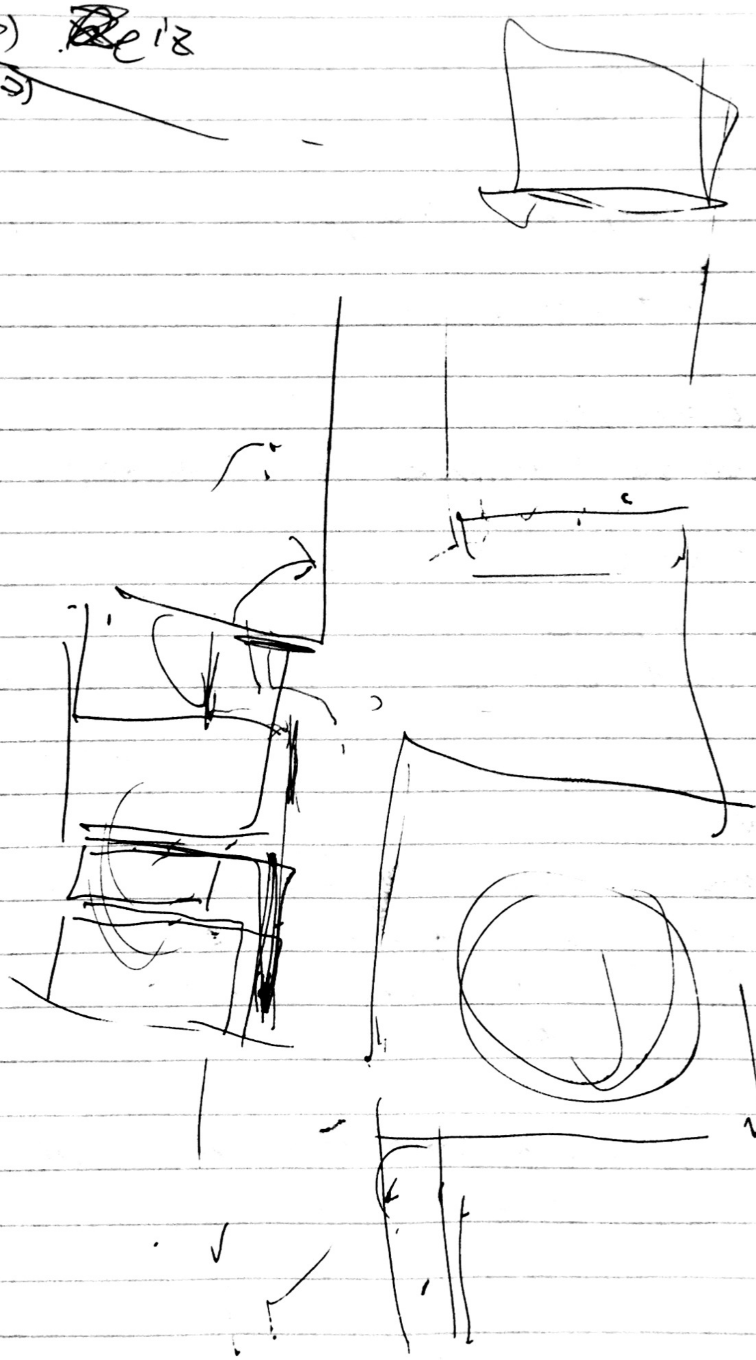
Check textbook for example. (This part I'm not sure)

- Use $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$.

  estimates for $|z| = R$ with R very large.

  $\frac{1}{2}|a_n|R^n \leq |p(z)| \leq 2|a_n|R^n$

$f(z) = \frac{P(z)}{Q(z)} e^{iz}$



*(What about improper integral with Log functions?)

- ARUMENT PRINCIPLES

zeros of analytic functions.

THM. Sps f is analytic in D (a domain) and $z_1, z_2, z_3, \cdots$ is a sequence of zeros of f in D. If $\lim_{n\to\infty} z_n = z_0 \in D$. then $f(z) = 0 \ \forall z$.

- argument principle.

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \#\text{ zeros inside } \gamma - \#\text{ poles inside } \gamma.$$

h analytic except at poles p_k, γ simple closed curve.

where we count with multiplicity.

THM (argument principle)

$\frac{1}{2\pi}$ (net change in $\arg h(z)$ as z traverses γ)

$= \#$ zeros inside γ $- \#$ poles inside γ

If h has no poles

$\frac{1}{2\pi}$ (net change in $\arg h(z)$) $= \#$ zeros inside γ.

Rouché's Theorem:

f, g holomorphic in D, γ simple closed in D.

Sps that $|f(z)+g(z)| < |f(z)|$ on γ.

then f & g have the same number of zeros inside γ (with multiplicity).

- $\boxed{\text{Rouché's} \Rightarrow p(z) \text{ \& } z^n \text{ have the same \# of zeros}}$ inside S_r. $\rightarrow z^n$ has n zeros.

• $\boxed{\text{Maximum Modulus Principle}}$

~~Thm (open mapping) Let f be holomorphic on D. $U \subseteq D$ open set then $f(U)$ is open.~~

Let D be an open domain, f is holomorphic on D.
then ① $|f|$ has no max on D
② Re[illegible] has no max on D
③ Im[illegible]s no max on D
④ ②[illegible] but for "min"

Cor (mmp). If D is a bdd region, & f extends to a cts. function on ∂D, then
① $|f|$ attains its max ∂D.
② $\operatorname{Re} f$, $\operatorname{Im} f$ attain their max & min on ∂D.

• $\boxed{\text{FLT}}$

$T(z) = \dfrac{az+b}{cz+d}$, $ad-bc \neq 0$

"North pole"

• $\boxed{\text{CONFORMAL MAPPINGS}}$

$f: \mathbb{C} \to \mathbb{C}$ ~~for~~ conformal at $z_0 \in \mathbb{C}$ if it's analytic at z_0 & preserves angles.

FLT is conformal.

Thm: f analytic on D, $f'(z_0) \neq 0$, $\Rightarrow$ f conformal at z_0.
Thm: f analytic in D & 1-1 (injective) $\Rightarrow$ conformal
Thm: composition of conformal $\Rightarrow$ conformal
Thm (Riemann Mapping Theorem): S_1 & S_2 are 2 simply-connected domains, then $\exists$ a conformal bijection $f: S_1 \to S_2$.