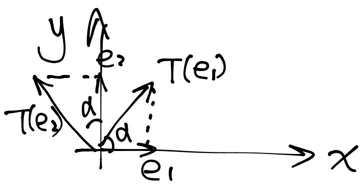
Lecture 11

Recall some sine & cosine in usual geo.

Spherical Geometry



$T(\alpha)$: rotate by angle α

$$T(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Consider

$$T(\beta) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$$

Then $T(\alpha+\beta) = \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$ (*)

$T(\alpha+\beta) = T(\beta)T(\alpha)$: First by α, then by β

How to get (*)

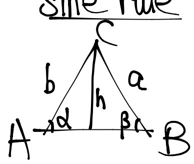
do matrix multiplication:

and we then have

$$\left.\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \end{aligned}\right\} \text{need to use them later}$$

Law of sines/cosines

sine rule

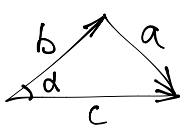


$h = b\sin\alpha$

$h = a\sin\beta$

so $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b}$ similarly $= \dfrac{\sin\gamma}{c}$

cosine rule



$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

if $\alpha = 90°$, we have Pythagorean Thm $a^2 = b^2 + c^2$

Consider a, b, c vectors

$\vec{c} = \vec{a} + \vec{b}$

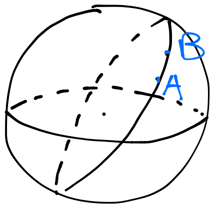
$\vec{a} = \vec{c} - \vec{b}$

$a^2 = \langle a, a\rangle$

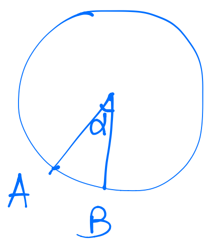
$\langle c-b, c-b\rangle = c^2 + b^2 - 2bc\cos\alpha$

recall inner product

Consider sphere of radius 1.



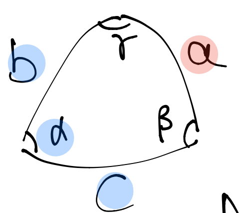
define "line": A, B, the "great circle pass through them & centered at the spherical center.



2π

$$\overset{\frown}{AB} = \frac{\alpha}{2\pi} \cdot 2\pi = \alpha \quad \ldots \text{ arc length}$$

assume 3 pts on sphere



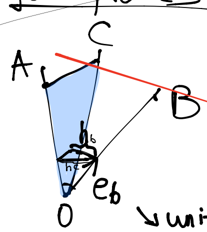
α, β, γ are angles between planes

Now if we have angle α and arc lengths b & c.

What is a?

Prove sine rule

for planes :



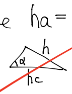
project unit vector $\vec{e_b}$ onto plane OAC

↘ unit vector

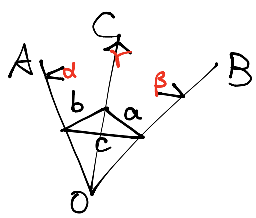
$\angle AOB = c$

since $h_a = 1 \cdot \sin c$

&

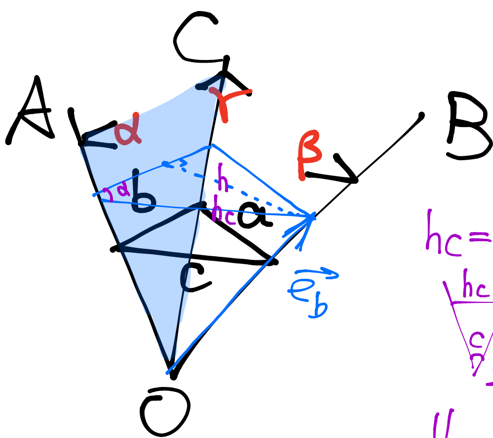


$h = h_a \sin\alpha = \sin c \sin\alpha$

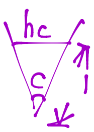


$OA = ABO \cap ACO$
$OB = ABO \cap BCO$
$OC = ACO \cap BCO$

have unit vector $\vec{e_b}$



$hc = \sin c$



(unit vector)

Let $\sin\angle AOB = \sin C$
$\sin\angle BOC = \sin A$
$\sin\angle COA = \sin B$

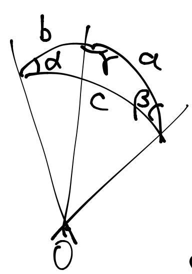
then $h = \sin\alpha \sin C$

similarly $h = \sin\gamma \sin A$

Hence: $\dfrac{\sin\beta}{\sin B} = \dfrac{\sin\gamma}{\sin C} = \dfrac{\sin\alpha}{\sin A}$

Why important?

Think about:



claim:
same property holds

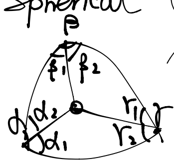
$\dfrac{\sin\alpha}{\sin a} = \dfrac{\sin\beta}{\sin b} = \dfrac{\sin\gamma}{\sin c}$

if we take very small spherical triangle, we can consider it "flat",
Then we have our usual sine rule/property.

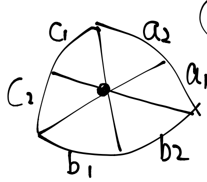
$\boxed{\text{Definition}}$ line is a part of "great circle":

V

Spherical Ceva Thm:



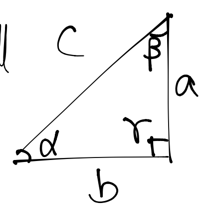
① $\dfrac{\sin\alpha_1 \sin\beta_1 \sin\gamma_1}{\sin\alpha_2 \sin\beta_2 \sin\gamma_2} = 1$ iff pass 1 pt.



② $\dfrac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = 1$ iff pass 1 pt.

} 2 statements

recall

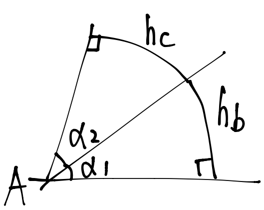


$a = c\sin\alpha$

In sph. geo, $\dfrac{\sin c}{\sin 90^\circ} = \dfrac{\sin a}{\sin\alpha}$

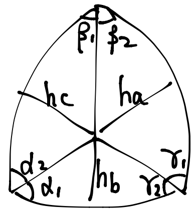
$\Rightarrow \sin a = \sin c \sin\alpha$

Proof of ① statement of Ceva in sph. geo.



$\dfrac{\sin h_b}{\sin h_c} = \dfrac{b\sin\alpha_1 \sin b}{b\sin\alpha_2 \sin b} = \dfrac{\sin\alpha_1}{\sin\alpha_2}$ } Tool:

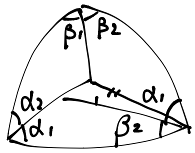
Assume 3 lines do pass 1 pt.



$$\left.\begin{aligned} \frac{\sin\alpha_1}{\sin\alpha_2} &= \frac{\sin h_b}{\sin h_c} \\ \frac{\sin\beta_1}{\sin\beta_2} &= \frac{\sin h_c}{\sin h_a} \\ \frac{\sin\gamma_1}{\sin\gamma_2} &= \frac{\sin h_a}{\sin h_b} \end{aligned}\right\} \Longrightarrow \frac{\sin\alpha_1}{\sin\alpha_2}\frac{\sin\beta_1}{\sin\beta_2}\frac{\sin\gamma_1}{\sin\gamma_2} = 1$$

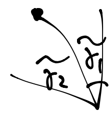
(1 direction done)

The other direction:



prove + & ++ coincide

say 

& 

$$\gamma_1 + \gamma_2 = \tilde{\gamma}_1 + \tilde{\gamma}_2 = \gamma$$

$$\& \ \frac{\sin \hat{\gamma}_1}{\sin \hat{\gamma}_2} = \frac{\sin \gamma_1}{\sin \gamma_2} \longrightarrow \text{This guy is fixed}$$

So γ_1, γ_2 are fixed!!!

$\gamma_1 = \tilde{\gamma}_1$

$\gamma_2 = \tilde{\gamma}_2$

conclusion

details:

So $\overline{\gamma_1} = \gamma - \overline{\gamma_2}$ $\quad \frac{\sin(\gamma - \overline{\gamma_2})}{\sin \overline{\gamma_2}} = \frac{\sin(\gamma - \gamma_2)}{\sin \gamma_2}$

$\gamma_1 = \gamma - \gamma_2$

$$\frac{\sin\gamma \cos\overline{\gamma_2} - \cos\gamma \sin\overline{\gamma_2}}{\sin\overline{\gamma_2}} = \sin\gamma \cot\overline{\gamma_2} - \cos\gamma$$

$$\frac{\sin\gamma\cos\gamma_2 - \cos\gamma\sin\gamma_2}{\sin\gamma_2} = \sin\gamma\cot\gamma_2 - \cos\gamma$$

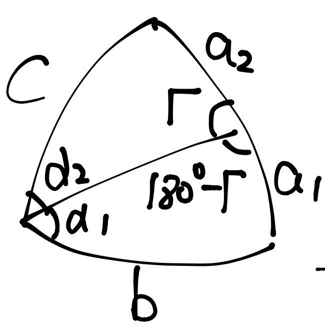
$\Longrightarrow \cot\overline{\gamma_2} = \cot\gamma_2$

So $\gamma_2 = \overline{\gamma_2}$

done!

■

Statement ②



What's $\frac{\sin\alpha_1}{\sin\alpha_2}$?

$\sin\Gamma = \sin(180 - \Gamma)$

$$\frac{\sin\alpha_1}{\sin a_1} = \frac{\sin(180 - \Gamma)}{\sin b}$$

$$\Longrightarrow \sin\alpha_1 = \frac{\sin a_1}{\sin b}\sin\Gamma$$

$$\frac{\sin\alpha_2}{\sin a_2} = \frac{\sin\Gamma}{\sin c}$$

$$\Longrightarrow \sin\alpha_2 = \frac{\sin a_2}{\sin c}\cdot\sin\Gamma$$

$$\text{so } \frac{\sin\alpha_1}{\sin\alpha_2} = \frac{\sin a_1}{\sin a_2}\cdot\frac{\sin c}{\sin b}$$

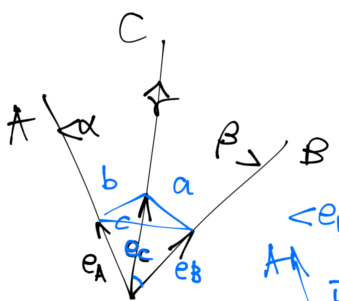
Similarly, $\dfrac{\sin\beta_1}{\sin\beta_2} = \dfrac{\sin b_1}{\sin b_2}\cdot\dfrac{\sin a}{\sin c}$

$$\frac{\sin\gamma_1}{\sin\gamma_2} = \frac{\sin c_1}{\sin c_2}\cdot\frac{\sin b}{\sin a}$$

Then find relation b/w d_1, d_2 & a_1, a_2 ... all cancel out, so 1.

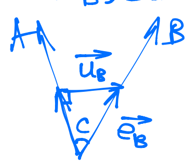
The other direction is also trivial.

<u>Cosine Thm on</u> sph. Geo



$\cos\angle BOC = \cos a$

$\langle e_B, e_C\rangle = \cos a$



project e_B onto OA

$\sin c = |\vec{u_B}|$

$e_B = u_B + \cos c\, e_a$

$\sin b = |\vec{u_c}|$

$e_C = u_C + \cos b\, e_a$

$\langle e_B, e_C\rangle = \cos a$

$\langle u_B + \cos c\cdot e_A,\ u_c + \cos b\, e_A\rangle$

$\langle u_B, e_A\rangle = 0$

$\langle u_C, e_A\rangle = 0$

angle between u_B & u_C = α

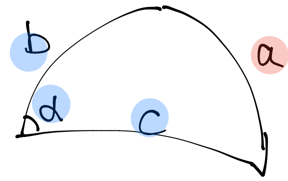
$\langle u_B, u_C\rangle = \sin c\cdot\sin b\cdot\cos\alpha$

$\langle \cos c\, e_A, \cos b\, e_A\rangle = \cos b\cos c$

$$\boxed{\cos a = \cos b\cos c + \sin c\sin b\cdot\cos\alpha}$$

$$\cos\alpha = \frac{\cos a - \cos b\cos c}{\sin c\sin b}$$

There are proofs on wiki both with and without vectors.



$$\cos a = \cos b \cos c + \sin b \sin c \cdot \cos \alpha$$

Cor: assume $\alpha = 90°$

$$\boxed{\cos a = \cos b \cos c}$$ Is this Pythagorean ?

we use Taylor expansion:

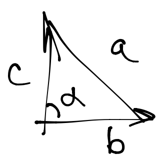
$$1 - \frac{a^2}{2} = (1 - \frac{b^2}{2})(1 - \frac{c^2}{2})$$

$$= 1 - \frac{b^2+c^2}{2} + \cancel{\frac{b^2c^2}{4}} \text{ omit}$$

$$a^2 \approx b^2 + c^2$$

If α not $90°$, we have usual expression $a^2 \approx b^2 + c^2 - 2bc\cos\alpha$

Triangle Inequality in Sph. Geo



want $a \uparrow$

so $\cos\alpha \downarrow$ $\quad -1$, $\alpha = 180°$

so, then cosine thm goes with:

$$\cos a = \cos b \cos c - \sin b \sin c$$

$$= \cos(b+c)$$

$$a = b + c$$

actually $\max a = b + c$

so $\quad a \leq b + c$

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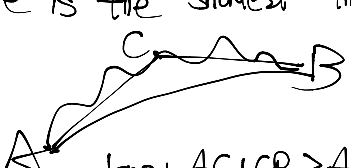
great circle is the "shortest" line on sphere! between 2 pts

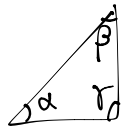


know $AC + CB \geq AB$, $\triangle ABC$ is a $\triangle$ on sphere.

consider as broken lines

. . . . use triangle inequality step by step.

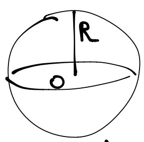
Formula of area of spherical triangle?



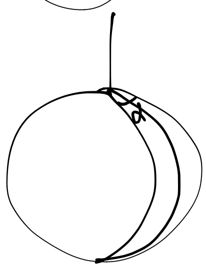
$\alpha+\beta+\gamma=\pi$

But for spherical $\triangle$, $\alpha+\beta+\gamma>\pi$

& $(\alpha+\beta+\gamma-\pi)=R^2 \text{Area}\triangle$



$4\pi r^2=$ Area Sphere



$S_{2\pi}=4\pi R^2=2\cdot 2\pi R^2$

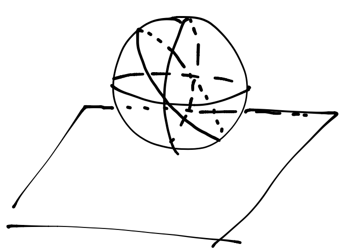
$\boxed{S_\alpha=2R^2\alpha}$

assume $R=1$

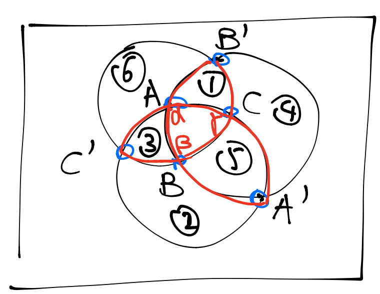
$S_\alpha=2\alpha$

2-gon on plane is like a segment

2-gon on sphere is like 



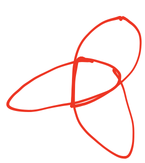
→ projection is



8 pieces

① – ② are spherical symmetries

3 sectors



S_α
S_β
S_γ

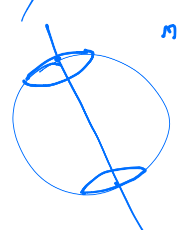
$S_\beta=\text{Area}\triangle\alpha\beta\gamma+\text{Area}\triangle C'A'B$

public $\triangle$ Counted 3 times in total)

which is equal to $\triangle A B' C$

$S_\alpha+S_\beta+S_\gamma=\frac{1}{2}\text{Area of sphere}+2\triangle$

spherical symmetries like



north pole

equal in area & shape

south pole

$$2\alpha+2\beta+2\gamma=\frac{4\pi}{2}+2\cdot x$$

~~comm~~ public $\angle$

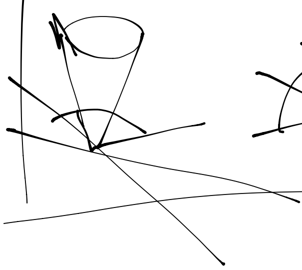
$R=1$

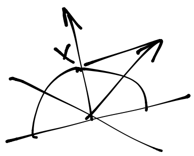
$$x=(\alpha+\beta+\gamma)-\pi$$

<u>done</u>

won't be included

"The second cosine law"





?

<u>dual cone</u>

$C\subset\mathbb{R}^3$

$C^{\perp}$, $\forall x\in C$, $\forall y\in C^{\perp}$, s.t. $\langle y,x\rangle>0$