Lecture 5

N particles, M cells in space

Ideal: particles distribute uniformly in space and do not have any interac-tion

independence

Def'n: For any $H_k(\ )$

$$E\left[\prod_{k=1}^{M} H_k(X_k)\right] = \prod_{k=1}^{M} E[H_k(X_k)]$$

$\iff X_1, \cdots, X_M$ are independent

$\iff P(X_1 \in A_1, X_2 \in A_2, \cdots, X_m \in A_m) = P(X_1 \in A_1) P(X_2 \in A_2) \cdots \cdot P(X_m \in A_m)$

"stock prices of two days are not indpdt."

But in this class we have to assume lots of stuffs
Are independent.

Discrete Random Variables

① Bernoulli distributions

Counting

Combinations. If we have n objects, we select p out of them and we do not care about the order of them, then the total number of ways to select the p objects is called $\binom{n}{p}$ ← n choose p

where $\binom{n}{p} = \dfrac{n!}{(n-1)!\,p!}$ (proof as a homework)

X is a Bernoulli(p) distribution if
$$\boxed{X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}}$$

w.p. with probability

$\Rightarrow$ all indicator r.v.'s are Bernoulli.

$$I(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

If $X \sim \text{Bernoulli}(p)$,

$$\boxed{\begin{aligned} E[X] &= 1 \times P(X=1) + 0 \times P(X=0) = p \\ Var[X] &= E[X^2] - (E(X))^2 = 1^2 \times P(X=1) + 0^2 \times P(X=0) - p^2 \\ &= p - p^2 \\ &= p(1-p) \end{aligned}}$$

② Binomial distributions

Suppose we have n independent trials and each trial has probability p to succeed. Let X be the total number of successes, then X is called a binomial(n, p) random variable.

$X \sim \text{Binomial}(n, p)$

$P(X = x) = ?$

$x = 0, 1, \cdots, n$ ⋯ $(n+1)$ values to take

Say have 2 successes at trial 2 & 3
then $(1-p)\cdot p\cdot p\cdot(1-p)\cdots(1-p)$
$= p^x(1-p)^{n-x}$

(precondition: each trial is independent to others)

How many choices? COMBINATION! $\binom{n}{x}$

Hence $\boxed{P(X=x)=\binom{n}{x}p^x(1-p)^{n-x}}$

Ex: Sps A has prob. of 0.6 of success in a game. And he plays this game 10 times independently. Calculate $P(X=2)$ where X is the number of winnings.
$P(X=2)=\binom{10}{2}0.6^2\,0.4^8=\cdots$

If $X\sim$ Binomial(n,p)
$EX=EX_1+EX_2+\cdots+EX_n=p+p+\cdots+p=np$ each of them is an expectation of a Bernoulli r.v.

To calculate, need a lemma and a prop.

$Var(X)=V(X_1+X_2+\cdots+X_n)=V(X_1)+\cdots+V(X_n)=np(1-p)$

Write $X=X_1+X_2+\cdots+X_n$ where $X_i=\begin{cases}1 & \text{if ith trial is a success}\\ 0 & \text{if o.w.}\end{cases}$

Lemma: If X & Y are independent, then $Cov(H(X),G(Y))=0$ for any functions H & G.
Proof: $Cov(H(X),G(Y))=E[H(X)G(Y)]-E[H(X)]E[G(Y)]$
By the def'n of independence, this $=0$. done. ∎

Proposition: If $X_1,X_2,\ldots,X_n$ are indpdt, then $Var[a_1X_1+a_2X_2+\cdots+a_nX_n]$
$=a_1^2Var(X_1)+a_2^2Var(X_2)+\cdots+a_n^2Var(X_n)$

Proof: $V[\sum a_iX_i]=E[(\sum a_iX_i-E\sum a_iX_i)^2]$
$=E[(\sum a_i(X_i-EX_i))^2]$
Let $Y_i=X_i-EX_i$
$=E[(\sum a_iY_i)^2]$
$=E[\sum_{i=1}^n\sum_{j=1}^n a_iY_ia_jY_j]$
$E[\sum_{i,j=1}^n a_ia_jY_iY_j]=\sum_{i,j=1}^n a_ia_jE(Y_iY_j)=\sum_{i=1}^n a_i^2Var(X_i)$

$E(Y_iY_j)=\begin{cases}0 & \text{if } i\neq j\\ V(X_i) & \text{if } i=j\end{cases}$ since $E(Y_i)=0$

∎

Ex (Warning!)

Suppose there are 30 black balls and 70 white balls in a box. Select 10 balls randomly from the box. Let X be then number of black ball.
$X\not\sim$ Binomial$(10, 0.3)$ 10 draws are not independent

③ Geometric distributions

X is a Geometric random variable with success probability p, if X represents the number of independent trials prior to the first success (p is the success prob for each single trial)

$$P(X=x) = P(\underbrace{FF\cdots F}_{\text{failures}}S) = \underbrace{P(F)P(F)\cdots P(F)}_{x \text{ failures}}P(S) = (1-p)^x p \qquad x=0,1,\cdots$$

$\uparrow$ Geometric (p)

$$EX = \sum_{x=0}^{\infty} x P(X=x) = \sum_{x=1}^{\infty} x(1-p)^x p = p\sum_{x=0}^{\infty} x q^x = p\frac{q}{(1-q)^2} = p\cdot\frac{q}{p^2} = \frac{q}{p}$$

$\rightarrow q = 1-p$

$$\sum_{j=0}^{\infty} q^j = \frac{1}{1-q} \qquad \sum_{j=0}^{\infty} j q^{j-1} = \frac{1}{(1-q)^2}$$

$$\Rightarrow \sum_{j=0}^{\infty} j q^j = \frac{q}{(1-q)^2}$$

$$Var[X] = EX^2 - (EX)^2 = EX^2 - q^2/p^2 = \sum_{x=0} x q^x p - q^2/p^2 = p\left(\sum_{x=0}^{\infty} x^2 q^x\right) - q^2/p^2$$

$= \cdots$ (rest as HW)

Variance of Geometric Distribution - ProofWiki

Proof 1

From the definition of Variance as Expectation of Square minus Square of Expectation:

$$\text{var}(X) = E(X^2) - (E(X))^2$$

From Expectation of Function of Discrete Random Variable:

$$E(X^2) = \sum_{x \in \Omega_X} x^2 \Pr(X = x)$$

To simplify the algebra a bit, let $q = 1 - p$, so $p + q = 1$.

Thus:

$$\begin{aligned} E(X^2) &= \sum_{k \geq 1} k^2 q p^k && \text{Definition of geometric distribution, with } p + q = 1 \\ &= p \sum_{k \geq 1} k^2 q p^{k-1} \\ &= p\left(\frac{2}{q^2} - \frac{1}{q}\right) && \text{from Proof 1 of Variance of Shifted Geometric Distribution} \end{aligned}$$

Then:

$$\begin{aligned} \text{var}(X) &= E(X^2) - (E(X))^2 \\ &= p\left(\frac{2}{(1-p)^2} - \frac{1}{1-p}\right) - \frac{p^2}{(1-p)^2} && \text{Expectation of Geometric Distribution: } E(X) = \frac{p}{1-p} \\ &= \frac{p}{(1-p)^2} && \text{after some algebra} \end{aligned}$$

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Proof 2

From Variance of Discrete Random Variable from PGF, we have:

$$\text{var}(X) = \Pi''_X(1) + \mu - \mu^2$$

where $\mu = E(x)$ is the expectation of X.

From the Probability Generating Function of Geometric Distribution, we have:

$$\Pi_X(s) = \frac{q}{1 - ps}$$

where $q = 1 - p$.

From Expectation of Geometric Distribution, we have:

$$\mu = \frac{p}{q}$$

From Derivatives of PGF of Geometric Distribution, we have:

$$\Pi''_X(s) = \frac{2qp^2}{(1-ps)^3}$$

Putting $s = 1$ using the formula $\Pi''_X(1) + \mu - \mu^2$:

$$\text{var}(X) = \frac{2qp^2}{(1-p)^3} + \frac{p}{q} - \left(\frac{p}{q}\right)^2$$

and hence the result, after some algebra.

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④

The negative binomial distributions

X is called a negative binomial random variable with parameters p and r, if X represents the number of fails prior to the rth success, and each single trial is independent of one another and has probability p of success.

$P(X=x)$

$\underbrace{FFS\,FFS\,FF\cdots S\,\cdot F\cdots}_{x+r-1 \text{ total trials}}\,S \longrightarrow$ r−1 successes before the last success

$\binom{x+r-1}{r-1} p^{r-1} q^{x}\, p \rightarrow$ don't forget

$$\Rightarrow \boxed{P(X=x)=\binom{x+r-1}{r-1} p^{r} q^{x}}$$

$E[X]=?$

$Var[X]=?$

"Think : negative bino = sum of geometrics"

"segments"

Geo(p) r.v.'s

Y_i's are indepdent

r segments in total

$X=Y_1+Y_2+\cdots+Y_r$

where Y_i is the # of trials between ith & (i−1)th success.

Let the 0th success = 0

Hence

$$\boxed{\begin{aligned} E[X] &= E(Y_1)+E(Y_2)+\cdots+E(Y_r)=\frac{rq}{p} \\ Var[X] &= V[Y_1]+\cdots+V[Y_r]=r\frac{q}{p^2} \end{aligned}}$$

Note: $\dfrac{V[X]}{E[X]}=\dfrac{rq/p^2}{rq/p}=\dfrac{1}{p}>1$

∗ for Poisson, the ratio is 1

In statistics, if a data set shows $V[X]/E[X]>1$, then one should consider a negative binomial model for the counts.

⑤

Poisson random variables

Ex suppose we want to model the number of car accidents in a week time. What we can do is that we divide this week time into n equal sized time blocks.

n is large that there is negligible chance that more than one accident will happen in each block.

Then X, the number of accidents equals $X_1+\cdots+X_n$ where X_i is # of accidents in the block. According to set up

$$X_i=\begin{cases} 1 & \text{w.p. } P_n \\ 0 & \text{w.p. } 1-P_n \end{cases}$$

Assume X_i's are independent, then $X \sim Binomial(n, P_n)$

$\boxed{EX=nP_n=\lambda}$ constant

$$P(X=x) = \binom{n}{x} p_n^x (1-p_n)^{n-x} = \frac{n!}{(n-x)!\,x!} p_n^x (1-p_n)^{n-x}$$

$$= \frac{1}{x!}\left[n(n-1)(n-2)\cdots(n-x+1)\right] p_n^x (1-p_n)^{n-x}$$

Write $p_n = \lambda/n$ since $np_n = \lambda$

$$= \frac{1}{x!}\left[n(n-1)\cdots(n-x+1)\right]\left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^x}{x!}\underbrace{\left[\frac{n(n-1)\cdots(n-x+1)}{n^x}\right]}_{A}\underbrace{\left(1-\frac{\lambda}{n}\right)^n}_{B} \Big/ \underbrace{\left(1-\frac{\lambda}{n}\right)^x}_{C}$$

as $n \to \infty$, $B \to e^{-\lambda}$

Recall: $(1-\frac{a}{n})^n \to e^a$

$A = \frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-x+1}{n}$

$C \to 1$

$A \to 1$

$$\boxed{P(X=x) = \frac{\lambda^x}{x!} e^{-\lambda}}$$

$\lambda > 0$

A Poisson r.v.with rate lambda has probability distribution … Lambda must be positive.

Poisson distributions are frequently used to model counts.

HW: Show that if $X \sim \text{Poisson}(\lambda)$, then

① $E(X) = V(X) = \lambda$

② if X, Y are indpdt $\text{Pois}(\lambda)$, then $X+Y \sim \text{Pois}(2\lambda)$

Midterm coverage: chapter 1 to chapter 4.2 except multinomial distributions but includes materials covered in the lectures but not in the textbooks, like dominant convergence theorem.

CALCULATOR!!!