Lecture 10

Duality

also geo trans but different

Vector space L, has dual space L^*.

For all linear functions

$\boxed{f(x) = a_1x_1 + \cdots + a_nx_n}$

$\lambda f(x) = \lambda a_1 x_1 + \cdots + \lambda a_n x_n$

$\mu g(x) = \mu b_1 x_1 + \cdots + \mu b_n x_n$

$(\lambda f + \mu g) = \sum (\lambda a_i + \mu b_i) x_i$

$L \subset (L^*)^*$

$x \in L$

$\forall f, \quad f \to f(x)$

$(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x)$

if $\dim L < \infty$ finite, then $L^{**} = L$

if fix n coeff's the func is fixed

$\dim L^* = \dim L$

$\dim (L^*)^* = \dim L^* = \dim L$

$L^* \xleftrightarrow{\text{interchange}} L$

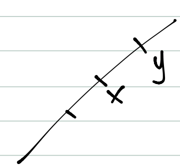
Make geometry out of it: (Projective space).

Say $x \in RP^n \Longleftrightarrow$ line passing close 0 in $\mathbb{R}^{n+1}$)

$\mathbb{R}^{n+1} \setminus \{0\} \ni x, y \quad x \sim y \Longleftrightarrow x = \lambda y$

$PL = L \setminus \{0\}$ up to the $\sim$

$x, y \in$ $\quad x \sim y \Longleftrightarrow x = \lambda y$



PL^*

L say it has a random subspace L_1, $L_1 \subset L$

Def: L_1^* is all linear functions such that $l|_{L_1} = 0$ (on L)

$l(x) = 0 \Rightarrow$ an equation for L_1

l $P_n L_1 = n-1$, $\dim L_1^* = 1$

λl

assume $l_1(x) = 0$, $l_2(x) = 0 \Rightarrow$ equation for L_1

$\lambda_1 l_1 + \mu l_2$ most general function $= 0$ on L_1

$\dim L_1^* = n - \dim L_1$

$\dim L_1 + \dim L_1^* = n$

Parametrization?

P^n, L^{n+1}, say we have subsps L_1 & L_1^* $\quad PL_1$

then $\dim L_1 + \dim L_1^* = \dim L^{n+1} = n+1$

$$\left.\begin{array}{l} PL_1 \text{ has dim } k-1 \\ PL_1^* \text{ has dim } n-k-1 \end{array}\right\} \text{sum } n-2$$

P^2 $\quad l \in \mathbb{R}^2$ $\quad l(x) = 0$, $\lambda l(x)$ $\quad$ line becomes pt. $\quad$ point $\in P^{2*}$

point $\quad \begin{cases} l_1 = 0 \\ l_2 = 0 \end{cases}$

duality on plane transform $\begin{cases} \text{a line} \Rightarrow \text{a point} \\ \text{a point} \Rightarrow \text{a line} \end{cases}$

In a word:

$\boxed{L_1 \subset L_2}$

$f \in L_2^* - f|_{L_2} = 0 + 1$

L_1

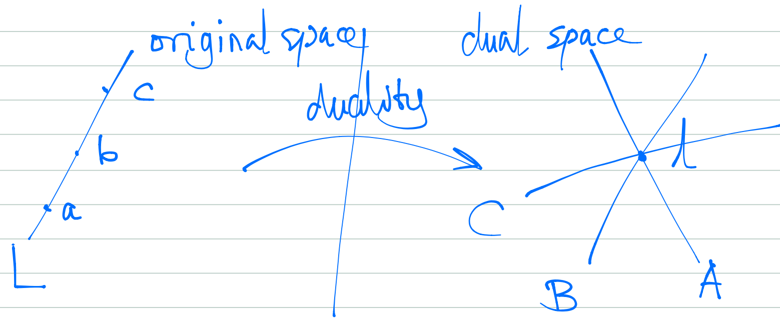
$g \in L_1^*$ $\quad g|_{L_1} \equiv 0$

if $f \in L_2^* \Rightarrow f \in L_1^*$, $\boxed{L_2^* \subset L_1^*}$

In a word, for duality,

smaller one goes to bigger one.

"inclusion changes sign"



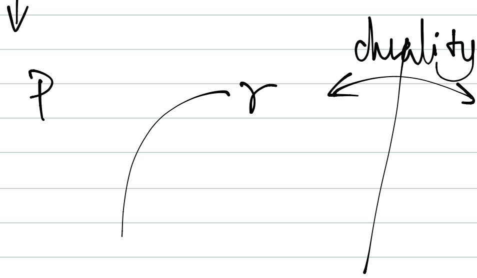
This is the rough idea of duality

And similarly, conversely

and, quadrics $\Longleftrightarrow$ quadrics (special case)

curves $\Longleftrightarrow$ curves ? (general case)

$\downarrow$



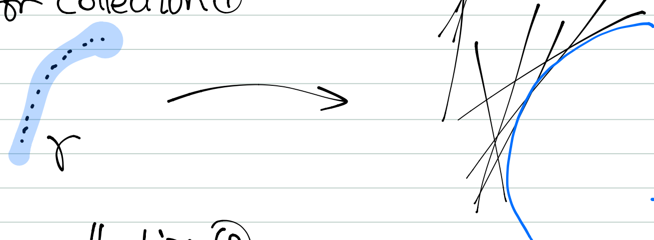
$\gamma^* \in P^*$

WHAT DOES γ^* LOOK LIKE?

What's a curve?
A collection of pts.
or
A collection of tangent lines

2 viewpoints

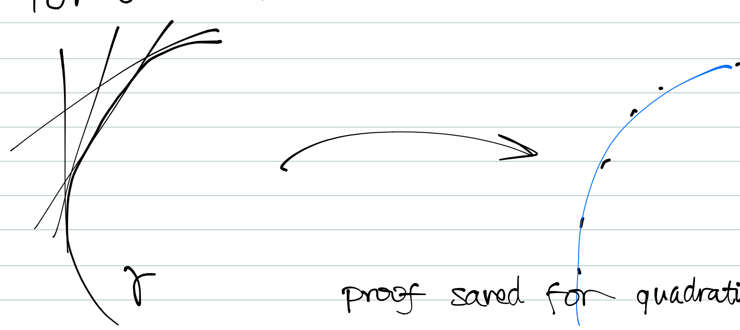
For collection ①



draw a line tangent to all of them

That's γ^*

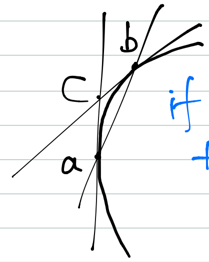
For collection ②



Connect them smoothly, that's γ^*

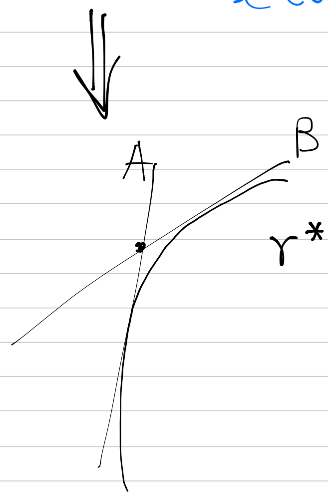
proof saved for quadratics

Consider:



if a closer to b
then the curve C not
passing will
be closer to C.

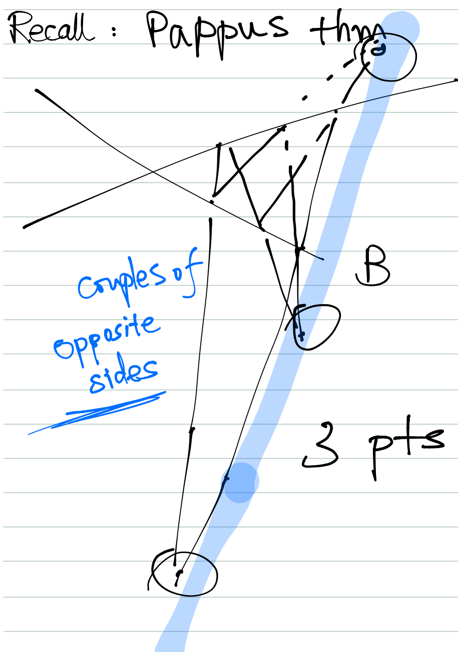
The hidden pattern
inside duality



Previously in mat402, whenever prof referred 'DeWalitti' I thought he was talking about a French or Italian mathematician who lived in 17th century, dedicated his whole life proving a gorgeous theorem, that I had not heard of. Well, it turns out that he was actually talking about duality*. ORZ

what a mess!
anyway, sth.
like
this

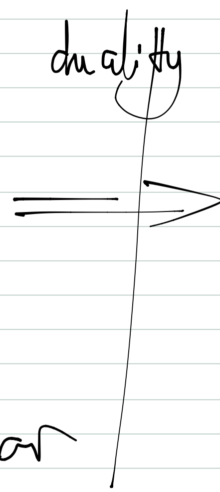
Recall: Pappus thm

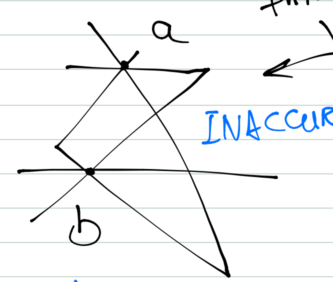


couples of
opposite
sides

3 pts colinear

duality





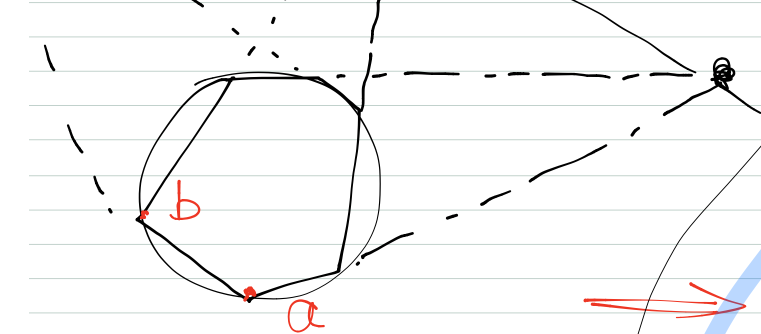
INACCURATE

couples of
opposite vertices

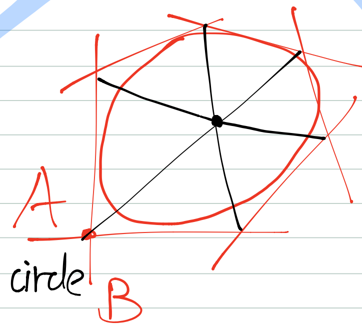
so we have
diagonals instead
of intersections

Dual Quadric space. (later)

Recall Pascal's thm,



dual



inscribed sixgon in a conic $\Longrightarrow$ inscribed circle in a sixgon

& intersections $\Longrightarrow$ diagonals

i.e.

3 pts on the same line $\Longrightarrow$ 3 lines passing the same pt

dual statement of Pascal Thm: Brianchon's Theorem

The proof is "obvious" if you go over the proof of Pascal.

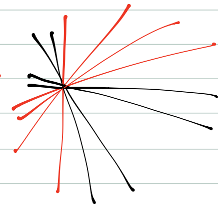
Go on for (5 pts decides 1 unique conic w/ no 3 on a line!)

$\Downarrow$ dual

5 lines (in general position, no 3 lines pass 1pt) decides a conic

construct cross-ratio

dual $\Rightarrow$

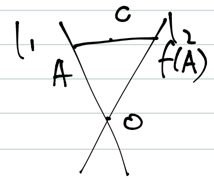


pick 4 define cross-ratio

projective map: (that preserves cross-ratio)

$(A,B,C,D) = (f(A), f(B), f(C), f(D))$

assume 2 lines.

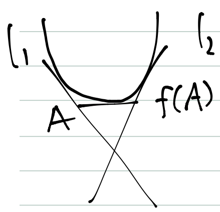


$f: l_1 \to l_2$

① $f(O)=O$

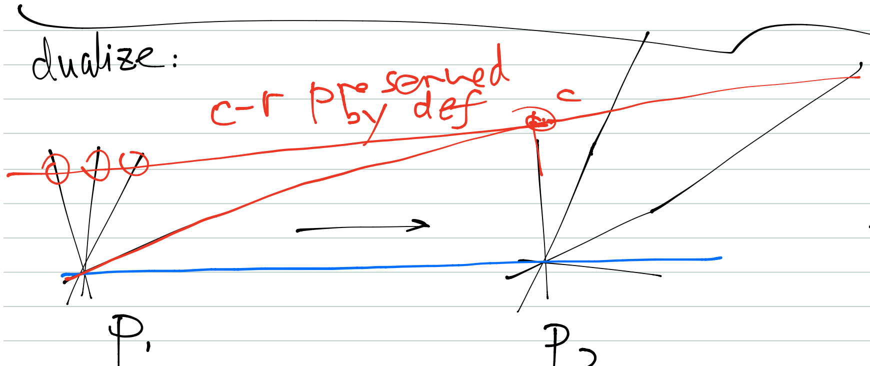
$O \in l_1$, $O \in l_2$

$\exists c$, f: projection centered at f_0



②

dualize:



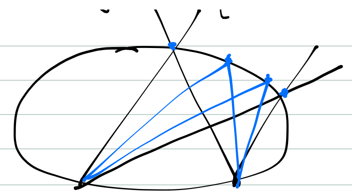
$f: P_1 \to P_2$

$O \in P$, $O \in P_2$

$f(O)=O$

in dual pic: common line maps to common line.

f(l), . l

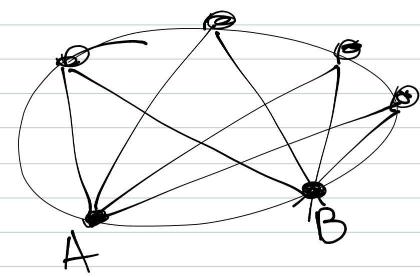


this preserves c-r

b/c firstly ⇒ become circle

⇒ angle preserves ⇒ c-r preserves.

So this is a projective transformation

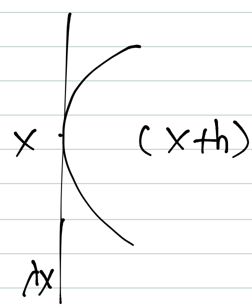


P^n

$x \in \mathbb{R}^{n+1}$

$\langle Ax, x\rangle = 0$

$x+h$



$x+h$ $\quad \langle A(x+h), x+h\rangle = 0$ $\quad \langle Ax, x\rangle + \langle Ah, x\rangle + \langle Ax, h\rangle + \langle Ah, h\rangle = 0$

$A = \{a_{ij}\}$ $\quad a_{ji} = a_{ij}$

$\langle Ax, y\rangle = \langle x, Ay\rangle$ $\qquad 2\langle Ax, h\rangle = 0$

$\underset{\substack{\parallel \\ 0}}{\langle Ax, x\rangle} + 2\langle Ax, h\rangle + \langle h, h\rangle = 0$ $\qquad \langle Ax, h\rangle = 0$

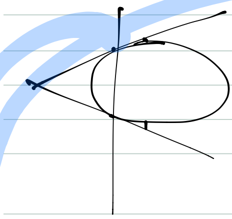
$x \in \langle Ax, x\rangle \Rightarrow$ tangent plane. $\langle Ax, h\rangle = 0$

$\langle a, x\rangle$ $\qquad x \to (Ax)$ $\quad x \to u$

$\langle Ax, x\rangle = 0$ $\qquad u = Ax$

$x = A^{-1}u$

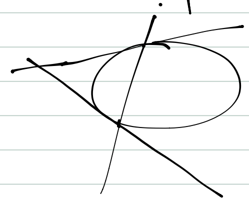
$\langle u, A^{-1}u\rangle = 0$, $\qquad (A^{-1})^{-1} = A$

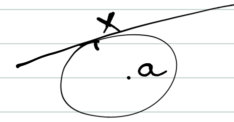


$a \longrightarrow T_a$

polar line

http://en.wikipedia.org/wiki/Pole_and_polar





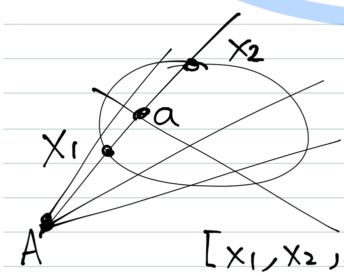
$\langle Ax, x\rangle = 0$

$\vec{X} \longrightarrow \boxed{f_X(h) = \langle Ax, h\rangle}$

$\langle Ax, h\rangle = 0$

http://en.wikipedia.org/wiki/Duality_(projective_geometry)

Construct polar plane:

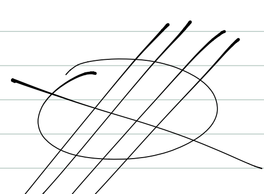


$[x_1, x_2, A, a] = -1$ $\boxed{\langle Ax, h\rangle = 0}$

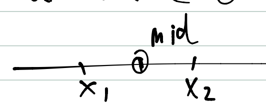
proof:

Consider A at ∞.

we have family of parallels

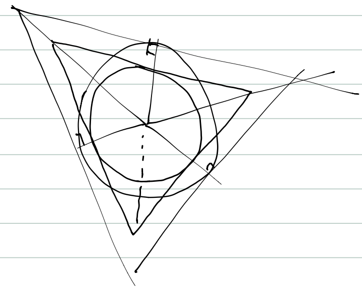


$ax^2 + bx + c = 0$



$x_1 + x_2 = \frac{-b}{a}$

so mid $= \frac{-b}{2a}$



Text