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term mark = midterm I (40%) + midterm 2 (40%) + 3 quizzes (20%)
course mark = 60% final exam mark + 40% term mark
or = 40% final exam mark + 60% term mark

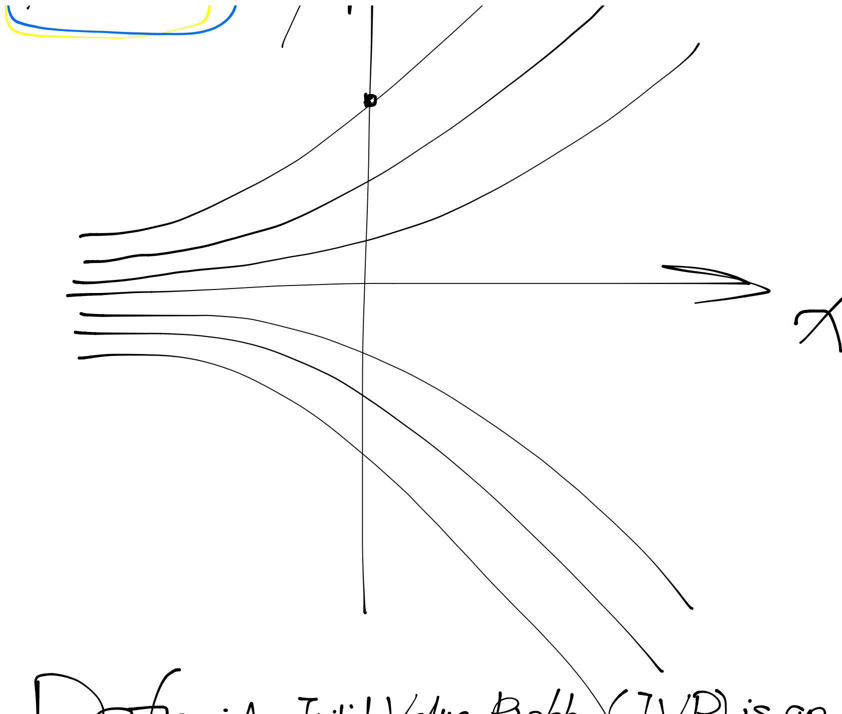
important dates:
Sept. 26th Quiz I
Oct. 10th Midterm I 6:10 pm - 7:00 pm TBA
Oct. 24th Quiz 2
Nov. 7th Midterm 2 6:10 pm - 7:00 pm TBA
Nov. 21st Quiz 3

check the website for course outline!!!

Defn: An ODE is an equation
Given some $g(x): (a,b) \subset \mathbb{R} \to \mathbb{R}$
An ODE is an equation in $x, y(x), y'(x), \dots, y^{(n)}(x)$
In standard Form $y^{(n)}x = f(x, y, y', \dots, y^{(n-1)})$
Ex/ $y'(x) = y\cos x + (y')^2$
x ~ independent variable

Defn: A soln ~~x~~ to an ODE is a specific $y(x)$ such that $y^{(n)}(x) = f(x, y(x), \dots y^{(n-1)}(x))$
Ex' $y' = ky$
$y_1 = 0$,
$y_2 = e^{kx}$
...... inifinite solutions
$y(x) = Ae^{kx}$, A is a constant
Solutions are not unique

$\frac{y'}{y} = k \Rightarrow \int \frac{y'}{y} = \int k\,dx \Rightarrow \ln|y| = kx + C_1$
$|y| = e^{kx+C_1} = e^{C_1}e^{kx} = Ae^{kx}$
$\boxed{y = Ae^{kx}}$



Defn: An Initial Value Problem (IVP) is an ODE with a point
$(x_0, y_0(x_0), y_0'(x_0), \dots, y_0^{(n-1)}(x_0))$

Ex: $y(0)=0$
$0 = y(0) = Ae^{k0} = A \Rightarrow A = 0$
$1 = y(0) = Ae^{k0} = A \Rightarrow A = 1$
$y(t)$ = size of Trefer's bank account
$k = 0.01$
$y' = ky \Rightarrow y(t) = Ae^{0.01t}$
$y(0) = 13$
$y(0) = 13 = Ae^{0} \Rightarrow y = 13e^{0.01t}$

1) $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ 2) $e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!}$
define to be equal
3) e^x = the solution of $y' = y$, $y(0) = 1$

- Partial Diff. Eqn $y(x_1, \dots, x_k)$
- Defn: The order of an ODE is the highest degree derivative

Defn / Autonomous 1st order ODE
$y' = f(y)$

Defn A linear ODE:
$a_0(x)y^{(n)} + a_1 y^{(n-1)} + \dots + a_n(x)y(x) = Y(x)$

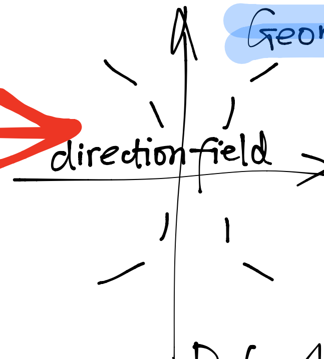
Defn: $Y_1(x), Y_2(x), \dots, Y_n(x)$

Defn: $y_1(x), y_2(x), \dots, y_k(x)$:

$$y_1'(x) = f(x, y_1, y_2, \dots, y_k)$$
$$\vdots$$
$$y_k'(x) = f(x, y_1, y_2, \dots, y_k)$$

Analytic View | **Geometric View**

$y' = f(x,y)$ ⟷ direction field



Defn: A direction field is a correspondence between $(x_0, y_0) \in \mathbb{R}^2$ and a little line segment.

$y(x)$ ⟷ Integral curve

Defn: An Integral curve that its slope corresponds to the direction field. $\forall (x_0, y_0) \in \mathbb{R}^2$

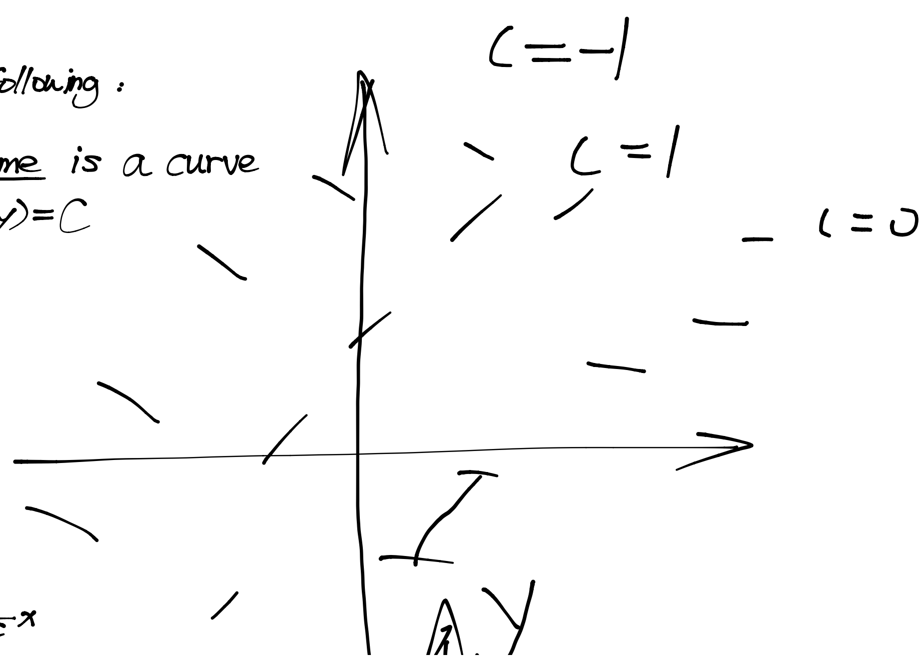
$y(x)$ is a solution to $y' = f(x,y)$ ⟷ Graph of $y(x)$ is an integral curve of the Direction field

$\Updownarrow$

$y_1' = f(x, y_1)$ | $y_1'(x) = f(x, y_1)$

A human does the following:

Defn. An <u>issoclime</u> is a curve $y' = f(x,y) = C$

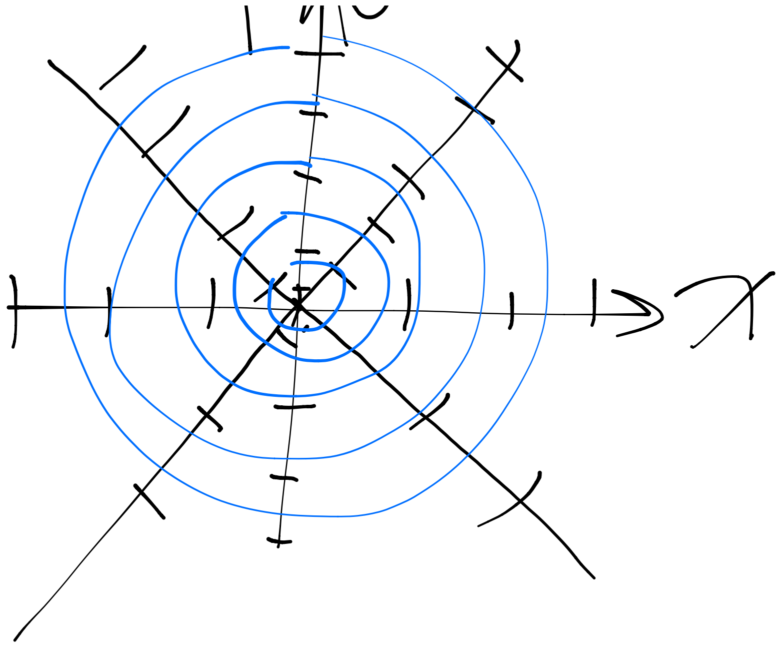


$y' = -\frac{x'}{y} = C \Rightarrow y = -\frac{1}{c}x$

$c=1$
$y=-x$
$c=-1$
$y=x$

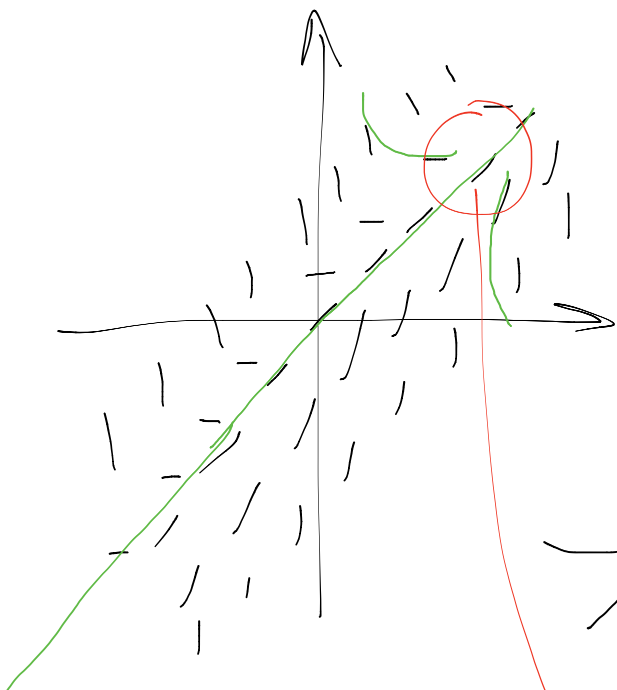
$C=0$
$-x=cy=0$
$C=\text{“}\infty\text{”}$

$$\boxed{\begin{aligned}\int y'y&=\int x\\ y^2&=x^2+C\end{aligned}}$$



$y'=1+x-y=c \Rightarrow -y=c-1-x$

$y=x+(1-c)$



$C=0$
$C=1,\ y=x$
$C=2,\ y=x-1$

$$\boxed{\begin{aligned}&\text{X true crosses cannot happen}\\ &y_1'=f(x,y_1)\\ &y_2'=f(x,y_2)\end{aligned}}$$

Could this happen? No.

Can never hit the straight line.

Given a 1^{st} order IVP
$\exists$ one and only one solution to the IVP
called "The existence & uniqueness" Theorem
the f is continuous $f_y(x,y)$ is continuous

Method of integrating factors

$y' + p(x)y = g(x)$

Ex: $y' - (\tan x)y = x$

$\mu(x)y' - \mu(x)(\tan x)y = \mu(x)x$

$\Rightarrow \cos x\, y' - \sin x\, y = x\cos x$

$\Downarrow$

$\int [\cos x\, y(x)]' = \int [\cos x\, y' - \sin x\, y(x)] = \int x\cos x$

$\Rightarrow \cos x\, y(x) = \int x\cos x + C = -x\sin x - \int -\sin x\, dx$

$y(x) = -x\tan x - 1 + C'$

$y' + p(x)y = g(x)$ ①

$u(x)y' + u(x)p(x)y = u(x)g(x)$ ②

want: $[u(x)y(x)]' = u(x)y'(x) + u(x)p(x)y(x)$ $\Rightarrow u'(x) = u(x)p(x)$

$\qquad \| $

$u'(x)y(x) + u(x)y'(x)$

$\int \frac{u'(x)}{u(x)}dx = \int p(x)dx$

$\Rightarrow \ln|u(x)| = \int p(x)dx$

$\Rightarrow |u(x)| = e^{\int p(x)dx}$

$u(x) = e^{\int p(x)dx}$

Integrating factor

② $\Rightarrow$ $[u(x)y(x)]' = \mu(x)g(x)$

$\int (u(x)y(x))'dx = \int \mu(x)g(x)dx$

$\mu(x)y(x) = \int \mu(x)g(x)dx + C$

$\Rightarrow \boxed{y(x) = \frac{1}{\mu(x)}\int_{x_0}^{x} \mu(\tau)g(\tau)d\tau + \frac{C}{\mu(x)}}$

Ex/ $y' + \frac{2}{t}y = \frac{-\sin t}{t^2}$ ①

In this case $p(t) = \frac{2}{t}$, $g(t) = \frac{-\sin t}{t^2}$, $u(x) = e^{\int \frac{2}{t}dt} = e^{2\ln|t|} = e^{\ln|t|^2} = t^2$

$\Rightarrow t^2y' + 2ty = -\sin t$ ② = ① $\times t^2 = \mu(t)$

$(t^2y)' = -\sin t \Rightarrow t^2y = -\int \sin t \Rightarrow t^2y = \cos t + c \Rightarrow y(t) = \frac{\cos t + c}{t^2}$

Ex/ $y' - 2y = e^{2t}$ $y(0) = 1$

$u(t) = e^{-2t}$

$e^{-2t}y'(t) - 2y(t)e^{-2t} = 1$

$\frac{d}{dt}(y(t)e^{-2t}) = 1 \Rightarrow y(t)e^{-2t} = t + c \Rightarrow y(t) = (t+c)e^{2t}$

Using our initial condition

$1 = y(0) = (0+c)e^{2\cdot 0} = c$

$$\boxed{y(t)=(t+1)e^{2t}}$$

Seperable Equation

$y'(t)=Q(y)R(t)$

$\parallel$

$\frac{dy}{dt} \Rightarrow \quad \frac{dy}{Q(y)}=R(t)dt \Rightarrow \int\frac{dy}{Q(y)}=\int R(t)dt$ $\Rightarrow$ implicit equation in y, t and u constant.

$y'=\frac{x}{y}$ $\quad Q(y)=\frac{1}{y}$, $R(t)=-x$

$\int dy\cdot y=\int -x\,dx$

$\frac{1}{2}y^2=-\frac{x^2}{2}+C \Rightarrow$ equation of a circle of radius $\sqrt{2C}$