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then $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis for productive
the product topology.

- Product of 2 Hausdorff spaces is Hausdorff.
- Product of 2 separable space is separable.

Q: which are preserved (by taking finite products)?
These properties are called finitely productive property.

Def'n: if $(X_i, \mathcal{T}_i)$ are top. spaces for $1 \le i \le N$, Then the basis for the product topology $X_1 \times X_2 \times \cdots \times X_N$ is $\mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_N$

Remark: To show that a property is finitely productive, sufficient to show it is preserved when taking product of 2 such spaces.
(By induction)

Fact: 2nd ctb is finitely

Q: Is ccc finitely productive?
A: don't know

Thm: (Rich Laver 1970)
if the CH is true, there are 2 top. spaces that are ccc but their product is not ccc

Thm: (Cohen ? 60's)
if assume Martin's Axiom then ccc is finitely productive

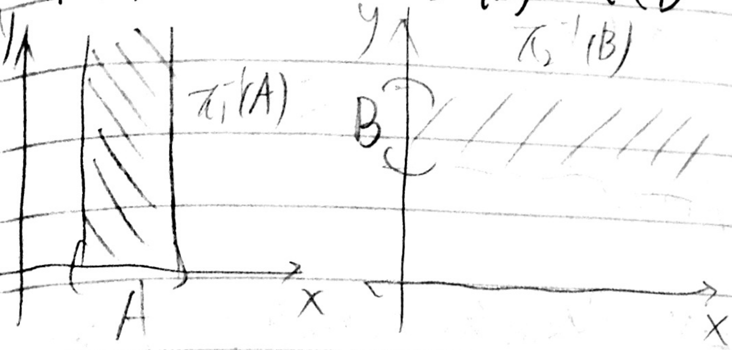
Wed. July 2nd
4-6 p.m. SF3202.

Recall: A basis for the product topology on $X \times Y$ where $(X, \mathcal{T})$, $(Y, \mathcal{U})$ are top. spaces, is $\mathcal{T} \times \mathcal{U} := \{A \times B : A \in \mathcal{T}, B \in \mathcal{U}\}$.

Def'n: For $(X, \mathcal{T})$, $(Y, \mathcal{U})$ top. spaces, define the projection maps:

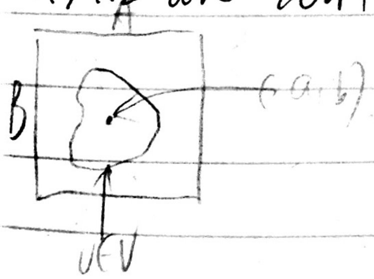
$\pi_1: X\times Y \to X$ by $(x,y)\mapsto x$
and $\pi_2: X\times Y \to Y$ by $(x,y)\mapsto y$.

Identity: For $\forall A\subseteq X$, $B\subseteq Y$.
$\pi_1^{-1}(A) = A\times Y$, $\pi_2^{-1}(B) = X\times B$



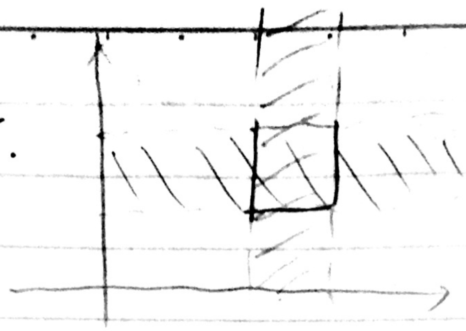
Prop: π_1 & π_2 are conti fcn, moreover, the product top. is the smallest topology (coarest) topology where they are cont.

Proof: Identities give continuity.
Let $\mathcal{V}$ be top. on $X\times Y$ where π_1, π_2 are cont.



Let $(a,b)\in A\times B$

π_1 is cont. $\to \pi_1^{-1}(A) = A\times Y \in \mathcal{V}$
π_2 is cont $\Rightarrow \pi_2^{-1}(B) = X\times B \in \mathcal{V}$
Now $A\times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \mathcal{V}$



Prop. 9 from Sec 2, $\mathcal{V}$ refines the product top.

Prop: $\mathcal{S} := \{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ is a subbasis for the product topology on $X\times Y$.

Q: is $f(x): \mathbb{R}\to\mathbb{R}^2$ defined by $x \mapsto (x^2-2, \arctan x; \pi x)$ cont.?

Prop: Let $(X,\mathcal{T})$, $(Y_1,\mathcal{U}_1)$, $(Y_2,\mathcal{U}_2)$ be top. spaces. Let $f: X\to Y_1\times Y_2$ be a fcn. TFAE:
1) f is cont.
2) $\pi_1\circ f$, $\pi_2\circ f$ are cont.

Proof: 1)$\Rightarrow$2) obvious.
2)$\Rightarrow$1). check that pre-image of a subbasic open set is open.
Let $U\times Y_2$ be a subbasic open set.
$f^{-1}(U\times Y_2) = f^{-1}(\pi_1^{-1}(U)) = (\pi_1\circ f)^{-1}(U)$ open by 2).
So our previous Ex. is cont.

as $\left.\begin{array}{l}\pi_1 \circ f(x) = x^2 - 2, \\ \pi_2 \circ f(x) = \arctan x \\ \pi_3 \circ f(x) = \sqrt{|x|}\end{array}\right\}$ cont.

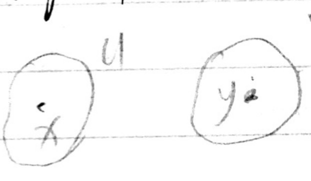
Product things

- $\mathbb{R}^2 \not\cong \mathbb{R}$.
- (Cantor) For each $n \in \mathbb{N}$, $\mathbb{Q}^n \cong \mathbb{Q}$.
- $\forall n \in \mathbb{N}$: $(\mathbb{R}\setminus\mathbb{Q})^n \cong \mathbb{R}\setminus\mathbb{Q}$.

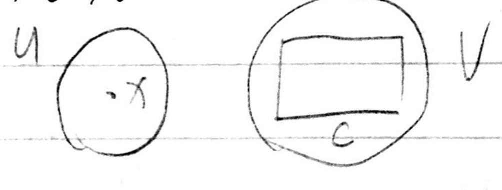
(all with usual subspace topology)

Sec 09: Separation Axiom

Defn: (X,τ) is Hausdorff if $\forall x \neq y \in X$, $\exists U \ni x$, $V \ni y$, U, V open and disjoint.

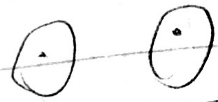


Defn: A space (X,τ) is Regular if $\forall$ closed $C \subseteq X$, $\forall x \in X\setminus C$, $\exists U, V \in \tau$ s.t. $x \in U$, $C \subseteq V$, U, V are disjoint. (T_3 in Ass 4)

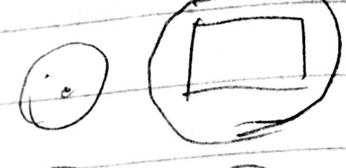


Defn: (X,τ) is Normal if $\forall$ closed, disjoint sets $C, D \subseteq X$, $\exists$ disjoint $U, V \in \tau$, s.t. $C \subseteq U$, $D \subseteq V$

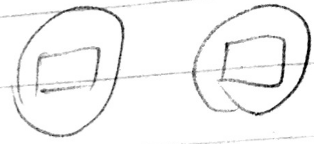
Hausdorff:



Regular



Normal:

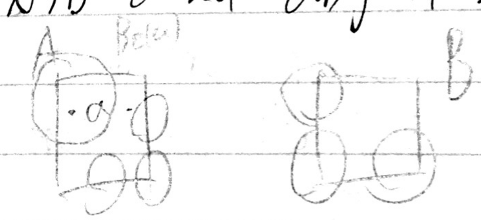


Ex. $\mathbb{R}$ discrete: Hausdorff, Regular, Normal.

Ex. $\mathbb{R}^n$ usual: H ✓, R ✓, N ✓.

Let's check that $\mathbb{R}^n$ usual is normal.

A, B closed disjoint set in $\mathbb{R}^n$.



Take $a \in A$, $X\setminus B$ is open and $a \in X\setminus B$.

$\exists \varepsilon_a$ s.t. $B_{\varepsilon_a}(a) \subseteq X\setminus B$.

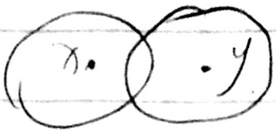
Find $\varepsilon_b > 0$, $\forall b \in B$

Want to take $A' = \bigcup_{a\in A} B_{\varepsilon_a}(a)$ and $B' = \bigcup_{b\in B} B_{\varepsilon_b}(b)$, but might not be disjoint.

(triangle inequality).

So $A' = \bigcup_{a\in A} B_{\frac{\varepsilon_a}{2}}(a)$

$B' = \bigcup_{b\in B} B_{\frac{\varepsilon_b}{2}}(b)$. instead



A horrible,

Indiscrete is not Hausdorff but is (vacuously) regular & Normal (No non-trivial disjoint closed set).

Normal $\Rightarrow$ regular $\Rightarrow$ Hausdorff

Prop: if $\{x\}$ is closed, $\forall x \in X$ then X is normal $\Rightarrow$ X is regular $\Rightarrow$ X is Hausdorff.

Def'n: X is T_2 if Hausdorff & points are closed.; X is T_3 if regular & pts are closed; X is T_4 if normal & pts are closed

Def'n: X is T_1 if $\forall x \neq y \in X$, $\exists U, V$ open s.t. $x \in U$, $y \in V$ (not necessarily disjoint). $x \notin V$, $y \notin U$.

Prop: X is T_1 iff pts are closed in X

"$\Rightarrow$" fix $x \in X$. $\forall y \neq x$. find an open set U_y, $x \notin U_y$. So $x \in \bigcap_{y \in X\setminus\{x\}} (X \setminus U_y)$.

In fact, $\{x\} = \bigcap_{y\in X\setminus\{x\}} (X\setminus U_y)$ So $\{x\}$ is closed.

"$\Leftarrow$" $x \neq y$. $x \in X\setminus\{y\}$, $y \in X\setminus\{x\}$.

Note: $T_4 + T_1 \Rightarrow T_3 + T_1 \Rightarrow T_2 + T_1$

normal $+T_1 \Rightarrow$ regular $+T_1 \Rightarrow$ Hausdorff $+T_1$

Note: X is $T_2 \Rightarrow X$ is T_1.

Prop: Let X be a T_1 space. X is regular iff $\forall x$, U open $U \ni x$ $\exists V$ open s.t. $x \in V \subseteq \bar{V} \subseteq U$.

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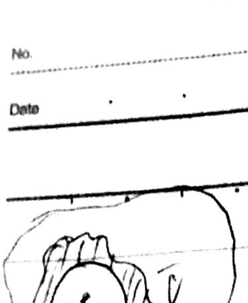
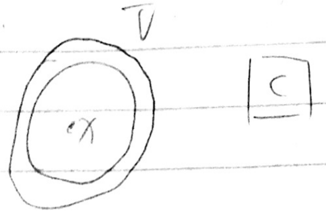


"⇐" C closed and take $x \in X \setminus C$. Find V s.t. $x \in V \subseteq \overline{V} \subseteq X \setminus C$.

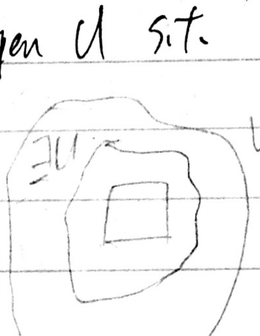


Since V is open, find a U s.t. $x \in U \subseteq \overline{U} \subseteq V$.

So $x \in U$ open, $C \subseteq X \setminus \overline{U}$.

C and $(X \setminus \overline{U})$ are disjoint.

Prop. X be T_1. X is normal iff $\forall$ closed C, open V, $C \subseteq V$ $\exists$ open U s.t. $C \subseteq U \subseteq \overline{U} \subseteq V$.



Fact: each T_i axioms are topological invariants.

Fact: T_1, T_2, T_3 are Hereditary but T_4 not.

Fact: If X is T_4, and $C \subseteq X$, closed, then (C, τ_{sub}) is T_4.

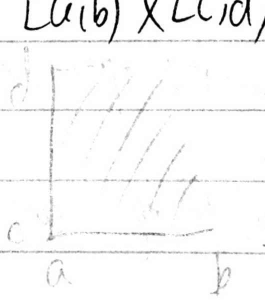
Proof: straightforward

Finitely Productive?

Fact: T_2, T_3 are productive (see previous PS). but, T_4 need not be productive

Ex. Take X the Sorgenfrey line, X is normal (Exercise) but $X \times X$ is not normal.

basic open sets look like $[a,b) \times [c,d)$.



Take $D = \{(x,-x) : x \in \mathbb{R}\}$ the anti-diagonal

claim: D forms a discrete subspace.

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for $(x,-x)\in D$. take $A_x :=$
$[x, x+1)\times[-x,-x+1)$.
Note $A_x\cap D=\{x\}$,
So any subset of D is closed.

Claim: $C=\{(x,-x): x\in \mathbb{Q}\}$.
$D=\{(x,-x): x\in \mathbb{R}\setminus\mathbb{Q}\}$. They are disjoint, closed and can't separate by open sets. (Munkres).

Prop: If X is T_3 and <u>2nd ctb</u>
X is T_4.
Proof: Let A, B be disjoint closed set.
$\forall a\in A$. find an open U_a
s.t. $a\in U_a$
$\overline{U_a}\cap B=\phi$.

Assume also that U_a is from $\mathcal{B}$, a fixed ctb basis.
Repeat finding V_b s.t. $\overline{V_b}\cap A=\phi$.
Assume $\bigcup_{b\in B} V_b=\bigcup_{n\in\mathbb{N}} V_n$, and
$\bigcup_{a\in A} U_a=\bigcup_{n\in\mathbb{N}} U_n$.

define $U_n' := U_n\setminus\left(\bigcup_{i=1}^{n}\overline{V_i}\right)$.

$V_n' := V_n\setminus\left(\bigcup_{i=1}^{n}\overline{U_i}\right)$.
check $A\subseteq\bigcup U_n'$, $B\subseteq\bigcup V_n'$.
and disjoint.