Convolution.

Suppose X, Y jointly distributed r.v's. We want to find the probability/density function of $Z = X + Y$.

Discrete case:

Let X, Y have joint probability function $p_{X,Y}(x,y)$.

$Z = z$ whenever $X = x$ and $Y = z - x$

$$p_Z(z) = P(Z = z) = \sum_x p_{X,Y}(x, z-x)$$

If X, Y are independent,

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z-x) \leftarrow p_Z = p_X \circ p_Y$$

convolution

Ex. $X \sim \text{Poisson}(\lambda_1)$, $Y \sim \text{Poisson}(\lambda_2)$, X and Y are independent. Find the distribution of $X+Y$.

$$(a+b)^n = \sum_{k=1}^{n} \binom{n}{k} a^k b^{n-k}$$

Sol'n: $Z = X + Y$

$$p_Z(z) = \sum_x p_X(x)\, p_Y(z-x)$$

$$= \sum_x \frac{\lambda_1^x e^{-\lambda_1}}{x!} \frac{\lambda_2^{z-x} e^{-\lambda_2}}{(z-x)!}$$

$$= e^{-\lambda_1-\lambda_2} \sum_x \frac{\lambda_1^x \lambda_2^{z-x}}{x!\,(z-x)!} \frac{z!}{z!}$$

$$= \frac{e^{-\lambda_1-\lambda_2}}{z!} \sum_x \binom{z}{x} \lambda_1^x \lambda_2^{z-x}$$

$$= \frac{e^{-\lambda_1-\lambda_2}}{z!} (\lambda_1+\lambda_2)^z \sim \text{Poisson}(\lambda_1+\lambda_2)$$

Continuous case:

Suppose X, Y are r.v.'s with joint density function $f_{X,Y}(x,y)$. We want to find the density function of $Z = X+Y$.

$$F_Z(z) = P(Z \le z) = P(\underline{X+Y \le z})$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z-x} f_{X,Y}(x,y)\, dy\, dx$$

$$= \begin{pmatrix} y = u-x \\ dy = du \end{pmatrix} = \int_{-\infty}^{\infty} \int_{-\infty}^{z} f_{X,Y}(x, u-x)\, du\, dx$$

$$= \int_{-\infty}^{z} \underline{\int_{-\infty}^{\infty} f_{X,Y}(x, u-x)\, dx}\, du$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx$$

if $\int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx$ is continuous at z

If X, Y are independent,

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= f_X \circ f_Y \rightarrow \text{convolution}$$

Ex. $X, Y \sim$ iid $Exp(\lambda)$. Find the density of $Z = X+Y$.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & ow \end{cases}, \quad f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0 \\ 0, & ow \end{cases}$$

$$f_Z(z) = \int_0^\infty f_X(x) f_Y(z-x)\, dx$$

$$= \int_0^z \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda(z-x)}\, dx = \lambda^2 e^{-\lambda z} \int_0^z dx$$

$$= \begin{cases} \lambda^2 e^{-\lambda z} z, & z \geq 0 \\ 0, & z < 0 \end{cases} \Rightarrow Z \sim Gamma(2, \lambda)$$

More on Normal Distribution.

If $Z \sim N(0,1)$, then

$$\varphi_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

$$X = \sigma Z + \mu \Rightarrow X \sim N(\mu, \sigma^2)$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

$$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$$

Let $X, Y \sim \text{iid } N(0,1)$, $Z = X+Y$

$$f_Z(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-x)^2}{2}} dx \ \ominus$$

$$-\frac{x^2}{2} - \frac{(z-x)^2}{2} = -\frac{x^2}{2} - \frac{z^2}{2} + zx - \frac{x^2}{2}$$

$$= -\frac{z^2}{4} - \frac{z^2}{4} + zx - x^2 = -\frac{z^2}{4} - \left(x - \frac{z}{2}\right)^2$$

$$\ominus \ \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2} dx \cdot \frac{\sqrt{\pi}}{\sqrt{\pi}}$$

$$N\Big(\underset{\mu}{\frac{z}{2}}, \underset{\sigma^2}{\frac{1}{2}}\Big)$$

$$= \frac{1}{2\sqrt{\pi}} e^{-z^2/4} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi \cdot \frac{1}{2}}} e^{-\frac{(x-z/2)^2}{2 \cdot 1/2}} dx}_{=1}$$

$$= \frac{1}{\sqrt{4\pi}} e^{-z^2/4}$$

$$Z \sim N\big(\underset{\mu}{0}, \underset{\sigma^2}{2}\big)$$

In general,

$X_1, \dots, X_n \sim \text{iid } N(0,1)$

$X_1 + \dots + X_n \sim N(0, n)$

Ex. $X_1, X_2 \sim \chi^2_{(1)}$, $Y = X_1 + X_2$. Find density of Y. 9.5

$$f_{X_i}(x) = \frac{x^{-1/2} e^{-x/2} (1/2)^{1/2}}{\Gamma(1/2)}$$

$$f_Y(y) = \int_0^y \frac{x^{-1/2} e^{-x/2}}{\sqrt{2}\sqrt{\pi}} \frac{(y-x)^{-1/2} e^{-(y-x)/2}}{\sqrt{2}\sqrt{\pi}}\, dx$$

$$= \frac{1}{2\pi} e^{-y/2} \int_0^y x^{-1/2}(y-x)^{-1/2}\, dx$$

$$u = \frac{x}{y} \Rightarrow y\,du = dx$$

$$= \frac{1}{2\pi} e^{-y/2} \int_0^1 (uy)^{-1/2} (y-uy)^{-1/2} y\, du$$

$$= \frac{1}{2\pi} e^{-y/2} \int_0^1 u^{-1/2} (1-u)^{-1/2}\, du \; =$$

$$\text{Beta}(\alpha, \beta) = \int_0^1 u^{\beta-1}(1-u)^{\alpha-1}\, du = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$= \frac{1}{2\pi} e^{-y/2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})} = \frac{1}{2\pi} e^{-y/2} \frac{\sqrt{\pi}\sqrt{\pi}}{1}$$

$$= \frac{1}{2} e^{-y/2} \Rightarrow Y \sim \text{Gamma}(1, \tfrac{1}{2})$$

Cauchy Distribution: The standard Cauchy distribution can be expressed as the ratio of two standard Normal r.v's.

Let $X, Y \sim$ iid $N(0,1)$. Then $Z = \frac{X}{Y} \sim$ Cauchy.

What's the density function of Z?

Multivariable Transformations.

Recall: If X is a r.v. with $f_X(x)$ and $U = h(X)$, where h is either $\nearrow$ or $\searrow$

Then $f_U(u) = f_X(h^{-1}(u)) \left| \frac{d h^{-1}(u)}{du} \right|$

Suppose X and Y are jointly continuous r.v's, and $U_1 = X + Y$, $U_2 = X - Y$

$f_{U_1, U_2} = ?$

<u>Theorem</u>. Suppose X, Y are continuous r.v's with joint df $f_{X,Y}(x,y)$ and that for all (x,y) s.t. $f_{X,Y}(x,y) > 0$

$u_1 = h_1(x,y)$ and $u_2 = h_2(x,y)$

is a 1-1 transformation from (x,y) to (u_1, u_2) with inverse

$x = h_1^{-1}(u_1, u_2)$ and $y = h_2^{-1}(u_1, u_2)$.

Then $f_{U_1,U_2}(u_1,u_2) = f_{X,Y}\left(h_1^{-1}(u_1,u_2), h_2^{-1}(u_1,u_2)\right)|J|$.

where $J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} \rightarrow$ Jacobian

Ex. $X, Y \sim \text{iid } N(0,1)$

$U_1 = X+Y$, $U_2 = X-Y$

$f_{U_1,U_2} = ?$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \quad \underline{f_{X,Y} = \frac{1}{2\pi} e^{-\frac{x^2}{2} - \frac{y^2}{2}}}$$

$$u_1 = x+y = h_1(x,y), \quad u_2 = x-y = h_2(x,y)$$

$$x = \frac{u_1+u_2}{2} = h_1^{-1}(u_1,u_2), \quad y = \frac{u_1-u_2}{2} = h_2^{-1}(u_1,u_2)$$

$$J = \det \begin{bmatrix} \frac{\partial h_1^{-1}}{\partial u_1} & \frac{\partial h_1^{-1}}{\partial u_2} \\ \frac{\partial h_2^{-1}}{\partial u_1} & \frac{\partial h_2^{-1}}{\partial u_2} \end{bmatrix} = \det \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \frac{1}{2}\left(-\frac{1}{2}\right) - \frac{1}{2} \cdot \frac{1}{2} = \frac{-1}{2}$$

$$f_{U_1,U_2}(u_1,u_2) = f_{X,Y}(h_1^{-1}, h_2^{-1}) \, |J|$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2}\left(\frac{u_1+u_2}{2}\right)^2} e^{-\frac{1}{2}\left(\frac{u_1-u_2}{2}\right)^2} \left|-\frac{1}{2}\right|$$

$$= \frac{1}{4\pi} e^{-\frac{u_1^2}{4} - \frac{u_2^2}{4}} = \underbrace{\frac{e^{-u_1^2/4}}{\sqrt{4\pi}}}_{f_{U_1}(u_1)} \cdot \underbrace{\frac{e^{-u_2^2/4}}{\sqrt{4\pi}}}_{f_{U_2}(u_1)}$$

$U_1, U_2 \sim N(0, 2)$

Density of Quotient.

Suppose X, Y are independent continuous r.v's and we are interested in the density of

$$Z = \frac{X}{Y}.$$

$$U_1 = \frac{X}{Y}, \quad U_2 = Y$$

$$h_1(x,y) = \frac{x}{y}, \quad h_2(x,y) = y$$

$$x = u_1 u_2 = h_1^{-1}(u_1, u_2), \quad y = u_2 = h_2^{-1}(u_1, u_2)$$

$$J = \det \begin{bmatrix} u_2 & u_1 \\ 0 & 1 \end{bmatrix} = u_2 \cdot 1 - 0 \cdot u_1 = u_2$$

$$f_{U_1,U_2}(u_1,u_2) = f_{X,Y}\left(h_1^{-1}(u_1,u_2), h_2^{-1}(u_1,u_2)\right)|J|$$

$$= f_{X,Y}(u_1 u_2, u_2)|u_2|$$

$$= f_X(u_1 u_2) f_Y(u_2) |u_2|$$

$$\boxed{f_{U_1}(u_1) = \int_{-\infty}^{\infty} f_X(u_1 u_2) f_Y(u_2) |u_2| \, du_2}$$

$$X, Y \sim \text{iid } N(0,1)$$

$$Z = \frac{X}{Y}$$

$$U_1 = \frac{X}{Y}, \quad U_2 = Y$$

$$f_{U_1,U_2}(u_1,u_2) = f_X(u_1 u_2)\, f_Y(u_2)\, |u_2|$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(u_1 u_2)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u_2^2}{2}} |u_2|$$

$$f_{U_1}(u_1) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{2\pi} e^{-\frac{u_2^2}{2}(u_1^2+1)} |u_2|}_{\text{even}}\, du_2$$

$$= \cancel{2} \int_0^{\infty} \frac{1}{\cancel{2}\pi} e^{-\frac{u_2^2}{2}(u_1^2+1)} u_2\, du_2$$

$$w = \frac{u_2^2}{2}(u_1^2+1)$$

$$\frac{1}{u_1^2+1} dw = u_2\, du_2$$

$$= \frac{1}{\pi} \int_0^{\infty} \frac{1}{u_1^2+1} e^{-w}\, dw = \frac{1}{\pi} \frac{1}{u_1^2+1} \underbrace{\int_0^{\infty} e^{-w}\, dw}_{1}$$

$$Z = \frac{X}{Y} \Rightarrow f_Z(z) = \frac{1}{\pi} \frac{1}{z^2+1}$$ ← Cauchy df

t- distribution.

Beta Distribution.