March 27th

Correction:

Constructible, not "constructable"

$\overline{0 \qquad 1}$

can construct $\frac{m}{n}$, $m,n \in \mathbb{Z}\,(n \neq 0)$

$\sqrt{2}$ can construct

A set $F \subset \mathbb{R}$ is called a number field if

① $0, 1 \in F$

② if $x, y \in F$ then $x+y$, $x-y$, $x \cdot y$, $x/y\,(y \neq 0)$ are also in F.

We proved that if x, y are constructable then $x+y, x-y, x \cdot y, x/y$ are constructable.

$\Rightarrow$ the set of constructable #s is a number field.

Examples:

① $F = \mathbb{Q}$ — all rationals

② $F = \mathbb{R}$ reals

③ the set S of all constructable #s

④ $F = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ is a number field

$$\frac{1}{1+\sqrt{2}} = a + b\sqrt{2} = \frac{1-\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}$$

$$\frac{a+b\sqrt{2}}{c+d\sqrt{2}} = \frac{ac + bc\sqrt{2} - ad\sqrt{2} - 2bd}{c^2+2d^2}$$

in general if $F_0 \subset \mathbb{R}$ is a number field and $r > 0$ some real #

look at $F_1 = \{a + \sqrt{b}\,r \mid a, b \in F\}$ then F_1 is again a number field

$F_0 \subset F_1$

Ex: $F_0 = \mathbb{Q}$, $r = 2 \Rightarrow F_1 = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$

$F_0 = \mathbb{Q}$, $r = 5 \Rightarrow F_1 = \mathbb{Q}(\sqrt{5}) = \{a + b\sqrt{5} \mid a, b \in \mathbb{Q}\}$

if $r = 4$, $\sqrt{4} = 2 \in \mathbb{Q}$

$\Rightarrow \mathbb{Q}(\sqrt{4}) = \mathbb{Q} = \{a + b\sqrt{4} \mid a, b \in \mathbb{Q}\}$

$F_1 = \mathbb{Q}(\sqrt{2})$, $r = 1+\sqrt{2} \in F_1$

can take $F_2 = F_1(\sqrt{1+\sqrt{2}}) = \{a + b\sqrt{1+\sqrt{2}} \mid a, b \in F_1\}$

$a = a_1 + a_2\sqrt{2}$

$a_1, a_2 \in \mathbb{Q}$

$b = b_1 + b_2\sqrt{2}$

$b_1, b_2 \in \mathbb{Q}$

Tower of fields is

$$\mathbb{Q} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_k$$

where $F_i = F_{i-1}(\sqrt{r_i})$, $r_i \in F_{i-1}$, $r_i > 0$

Ex: $F_0 = \mathbb{Q}$, $F_1 = \mathbb{Q}(\sqrt{2})$, $F_2 = F_1(\sqrt{1+\sqrt{2}})$

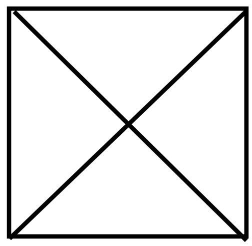
$F_0 \subset F_1 \subset F_2$

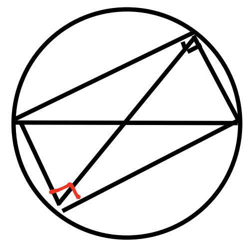
· if $x > 0$ is constructible then $\sqrt{x}$ is also constructible

$\frac{3}{7}$ is rational $\Rightarrow \sqrt{\frac{3}{7}}$ is constructable also. Why?

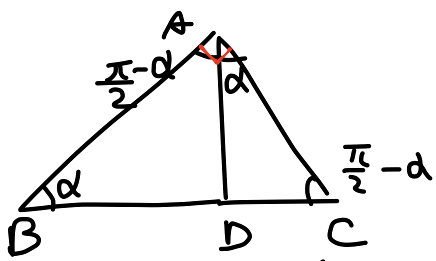
x

which $\sqrt{x}$ is constructable using a straightedge and a compass





How we construct a rectangle



$\triangle ABD \sim \triangle CAD$

proportional

$$\frac{AD}{BD} = \frac{CD}{AD}$$

$$\sqrt{AD} = BD \cdot CD$$

say $AD = x$, $\sqrt{x} = BD \cdot CD$

Defn: a number $r \in \mathbb{R}$ is called surd if $\exists$ a tower of fields $F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$ and $r \in F_k$

$F_0 = \mathbb{Q}$ $\qquad F_i = F_{i-1}(\sqrt{r_i}),\ r_i \in F_{i-1}$

ex: $F_0 = \mathbb{Q},\ F_1 = \mathbb{Q}(\sqrt{3}),\ F_2 = F_1(\sqrt{5+2\sqrt{3}})$

$= a + b\sqrt{5+2\sqrt{3}}$ anything of this form is a surd

$$\underbrace{(3/7 + 6\sqrt{3})}_{a} + \underbrace{\left(\tfrac{2}{11} - \tfrac{1}{17}\sqrt{3}\right)}_{b}\sqrt{5+2\sqrt{3}}$$

we've proved that any surd is constructable

$\sqrt[4]{3} = \sqrt{\sqrt{3}}$

$\sqrt[8]{3} = \sqrt{\sqrt{\sqrt{3}}}$ all constructable

We'll prove that $\boxed{\text{the set of all constructable numbers = set of all surds}}$

$\sqrt[3]{3}$ is it a surd or not?

• the set of all surds is a number field

x, y - surds $\Rightarrow x+y,\ x-y,\ x\cdot y,\ x/y$ are also surds

x is a surd $\Rightarrow$ there is some tower of fields

$$F_0 \subset F_1 \subset \cdots \subset F_k \quad F_i = F_{i-1}(\sqrt{r_i}),\ r_i \in F_{i-1}$$

y is a surd $\Rightarrow \exists$ another tower of fields

$$\underset{\overset{\shortparallel}{\mathbb{Q}}}{F_0'} \subseteq F_1' \subseteq \cdots \subseteq F_l' \to y \quad , y \in F_l'$$

$$F_j' = F_{j-1}(\sqrt{r_j'}) \quad F_j \in F_{j-1}'$$

ex: $F_0 = \mathbb{Q} \subseteq F_1 = \mathbb{Q}(\sqrt{2}) \subseteq F_2 = F_1(\sqrt{1+\sqrt{2}}) \subseteq F_3 = F_2(\sqrt{3}) \subset F_4 = F_3(\sqrt{\tfrac{2}{11}+\sqrt{3}})$

$$x = 2 + \frac{3}{7}\sqrt{2} + \frac{1}{11}\sqrt{1+\sqrt{2}}$$

$$F_0' = \mathbb{Q} \subseteq F_1' = \mathbb{Q}(\sqrt{3}) \subseteq F_2' = F_1(\sqrt{\tfrac{2}{11}+\sqrt{3}})$$

$$y = (1+\sqrt{3}) + (5-2\sqrt{3})\sqrt{\tfrac{2}{11}+\sqrt{3}}$$

$$F_0 = \mathbb{Q} \subset F_1 \subset \cdots \subseteq F_k \subseteq F_{k-1}$$

$$F_0' = \mathbb{Q} \subseteq \underset{\underset{F_0(\sqrt{r_1'})}{\shortparallel}}{F_1'} \subseteq \cdots \subseteq \underset{\underset{F_{l-1}'(\sqrt{r_l'})}{\shortparallel}}{F_l'}$$

past $F_{k+1} = F_k\sqrt{r_1'} \subseteq F_{k+2} = F_{k+1}(\sqrt{r_2'}) \subseteq \cdots$

$$F_{k+l} = F_{k+l-1}\sqrt{r_l'} \quad \cdots$$

$\Rightarrow$ tower of fields

$x, y \in F_{k+l}$ — a number field

$\Rightarrow x+y,\ x-y,\ x\cdot y,\ x/y \in F_{k+l} \Rightarrow$ they are all surds.

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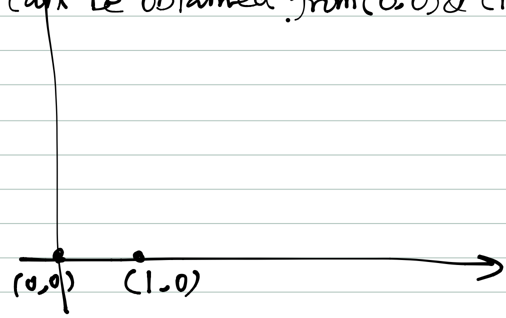
here points

$(0,0), (1,0)$ on $\mathbb{R}^2$ a point (x,y) is called constructible if it can be obtained from $(0,0)$ & $(1,0)$ using ruler/compass operations



• (x,y) is constructible $\Leftrightarrow x, y$ are constructible numbers

• if x, y are surds $\Rightarrow$ they are const.

$\Rightarrow (x,y)$ is a constructible point

Claim: if (x,y) is a constructible pt $\Rightarrow x, y$ are surds

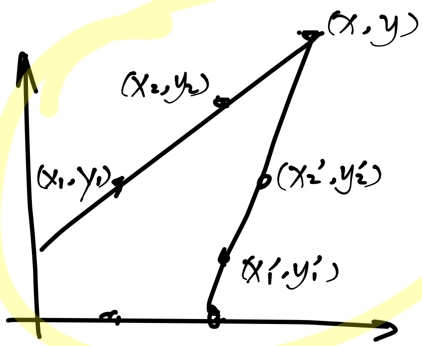
Pf: induction on the number of ruler and compass operations needed to produce (x, y)

Base of induction: n=0 0 steps no operations, only have (0,0) and (1,0) they are surds

Induction step: Suppose we have n>=0 and for any point (x,y) that can be constructed using at most n ruler and compass operations
We know that both x and y must be surds

Let (x,y) be a point constructed using n+1 operations
We want to show x,y are surds

How do you produce new points? In one of the following ways:

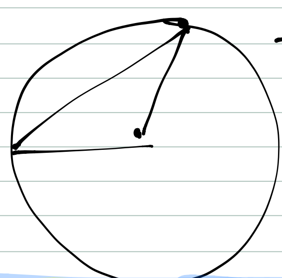


① $(x_1,y_1), (x_2,y_2), (x_1',y_1'), (x_2',y_2')$ already constructed

$\Rightarrow$ draw l_1, l_2

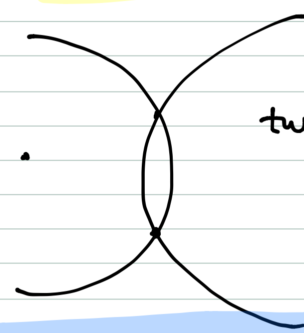
$P = l_1 \cap l_2 = (x,y)$ new point

② circle - line



two new points

③ circle - circle



two new points

CASE ①

(x_1,y_1) (x_2,y_2) equation of the line?

$$\frac{x-x_1}{y-y_1} = \frac{x_2-x_1}{y_2-y_1}$$

$$y = y_1 + \frac{y_2-y_1}{x_2-x_1}(x-x_1) \qquad k = \frac{y_2-y_1}{x_2-x_1}$$

$$= kx + (y_1 - kx_1)$$

if x_1, y_1, x_2, y_2 are already constructible $\Rightarrow$ they are surds by the induction assumption

$k=\frac{y_2-y_1}{x_2-x_1}$ is also a surd b/c surds form a number field.

$l_1: y=k_1x+b_1$ $\quad$ k_1, k_2, b_1, b_2 are surds

$l_2: y=k_2x+b_2$ $\quad$ $k_1x-k_2x=b_2-b_1$

$$x(k_1-k_2)=b_2-b_1$$

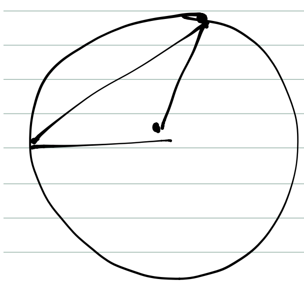
$$x=\frac{b_2-b_1}{k_1-k_2} \qquad y=k_1x+b_1$$

this is again a surd

CASE ②

$l: y=kx+b$

k, b are surds by the assumption



$(x_1'-x_2')^2+(y_1'-y_2')^2=R^2$ circle — surd

$y=kx+b$

$k, b, x_1', x_2', y_1', y_2'$ all surds

$$\begin{cases}(x-x_1')^2+(y-y_1')^2=R^2\\ y=kx+b\end{cases}$$

$x^2-2xx_1'+x_1'^2+(kx+b-y_1')^2=R^2$

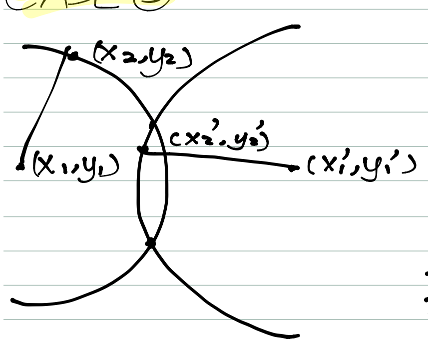
$x^2-2xx_1'+(x_1')^2+k^2x^2+(b-y_1')^2+2kx(b-y_1')-R^2=0$

$Ax^2+Bx+C=0$

A, B, C all surds

$x=\frac{-B\pm\sqrt{B^2-4AC}}{2A}$ $\quad$ this is also a surd

CASE ③



$$\begin{cases}(x-x_1)^2+(y-y_1)^2=R_1^2\\ (x-x_2)^2+(y-y_2)^2=R_2^2\end{cases}$$

$x^2-2xx_1+x_1^2+y^2-2yy_1+y_1^2=R_1^2$

$-$ $\quad x^2-2xx_1'+x_1'^2+y^2-2yy_1'+y_1'^2=R_2^2$

$x(2x_1'-2x_1)+y(2y_1'-2y_1)+x_1^2-(x_1')^2+y_1^2-(y_1')^2-R_1^2+R_2^2=0$

$y=kx+b$ $\quad$ k, b are surds again

$\Rightarrow$ constructible numbers = surds

Q: given a real number can we decide if it's a surd or not?

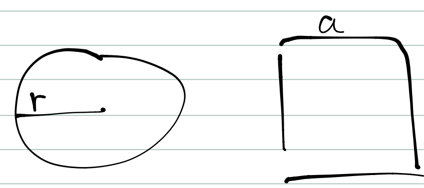
Claim: all surds are algebraic. ($\Rightarrow$ all constructible #s are algebraic)

$\Rightarrow$ if x is transcendental $\Rightarrow$ it's not constructible

For ex: π is transcendental $\Rightarrow$ not constructible

$\Rightarrow \sqrt{\pi}$ is also constructible (if $r = \sqrt{\pi}$ is constructible then $r \cdot r = \pi$ also)

$\Rightarrow$ Squaring a circle is impossible



want to construct a square of the same area

$\pi r^2 = a^2$

Squaring a circle is equivalent to construct $\sqrt{\pi}$ but π is transcendental $\Rightarrow$ not constructible $\Rightarrow \sqrt{\pi}$ is also not constructible.