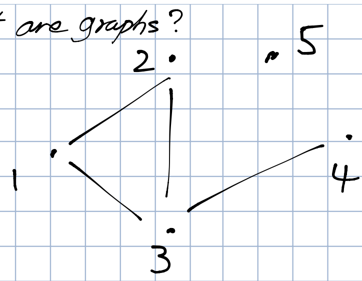
What are graphs?

2015-09-15



"Not a graph, but a representation of a graph"

$V(G)=\{1,2,3,4,5\}$

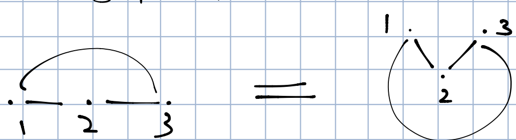
$E(G)=\{\{1,3\},\{1,2\},\{2,3\},\{3,4\}\}$

Def: A graph G is an object consisting of
- a vertex of $V(G)$ which is finite and nonempty
- an edge set $E(G)$ which contains two element subsets of the vertex set

We denote $v=\#$ of vertices, $e=\#$ of edges

Def through example:
- the vertices 1, 2 are adjacent; 2, 4 are not adajacent
- the edge $\{2,3\}$ is incident to each 2 & 3
- 5 is an isolated vertex

Ex: Are these 2 graphs equal?

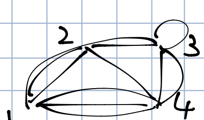


The only thing we cared about is $E(G)$ & $V(G)$

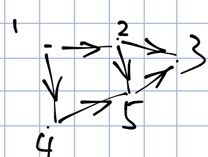
Def: Two graphs G, H are equal iff $V(G)=V(H)$ and $E(G)=E(H)$

Remark: there are 2 other more general notion of graphs:

Multigraphs:



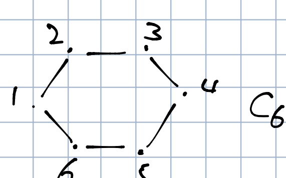
Directed graphs:



$V=\{1,2,3,4\}$

$E=\{$ use tuples $\}$ b/c set cannot have repeated elements

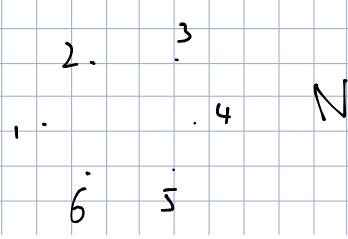
Some important graphs
- The cyclic graph C_v $\quad v\geq 3$



$V(C_v)=\{1,2,\cdots,v\}$

$E(C_v)=\{\{1,2\},\cdots,\{v-1,v\},\{v,1\}\}$

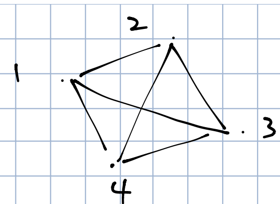
- The null graph N_v



$V(N_v)=\{1,\cdots,v\}$

$E(N_v)=\emptyset$

· The complete graph K_v

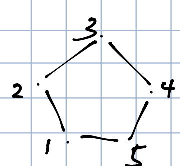


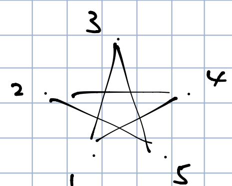
$V(K_v) = \{1, \cdots, v\}$
$E(K_v) = \{\text{all possible edges}\}$

Thm: K_v has $\frac{v}{2}(v-1)$ edges
Proof: Each vertex is connected to $(v-1)$ vertices. Therefore there are $v(v-1)$ "edge-ends". Every edge has two ends. Therefore, divide by 2. ▨

Def: The complement of a graph G, denoted by $\overline{G}$ is the graph with $V(G) = V(\overline{G})$
The edge set of $\overline{G}$ consists of all two element subsets of $V(G)$ that are not in $E(G)$

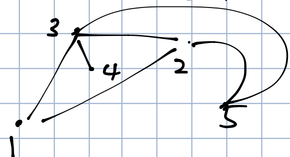
Ex: $\overline{N_v} = K_v$

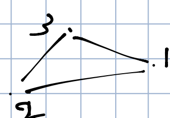




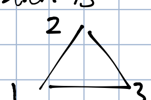
Corollary: $\overline{\overline{G}} = G$

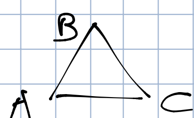
Def: A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$
Ex:





Question is



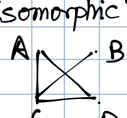
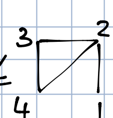


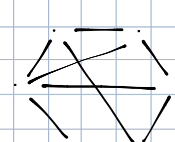
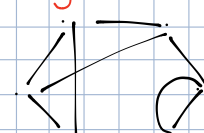
They are not equal, but are they "the same"
We'll use the definition of a bijective map
Def: Two graphs G & H are called isomorphic if there is a ~~one-to-one~~ bijective map $V(G) \to V(H)$ such that two vertices are adjacent in G iff their corresponding vertices in H are adjacent

Ex: In the example above
$1 \to A$
$2 \to B$
$3 \to C$
is such a map inducing an isomorphism

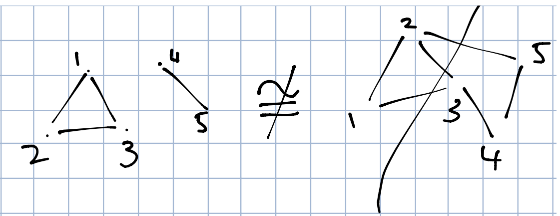
· # of edges
· degree distribution

Are there graphs isomorphic?

 $\cong$ 

 $\cong$ 

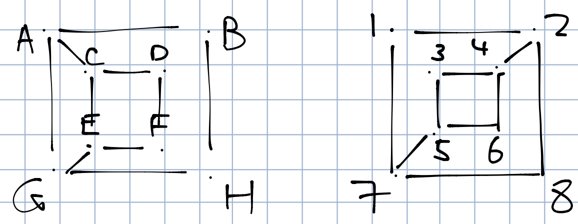
Remark: A graph isomorphism preserves
(Thm)
- number of vertices
- number of edges
- number of connected components
- "degree distribution"



Def: The degree of a vertex is the number of edges incident to it

Remark*: Even if all conditions as above are met, the graphs might not be isomorphic

Ex:



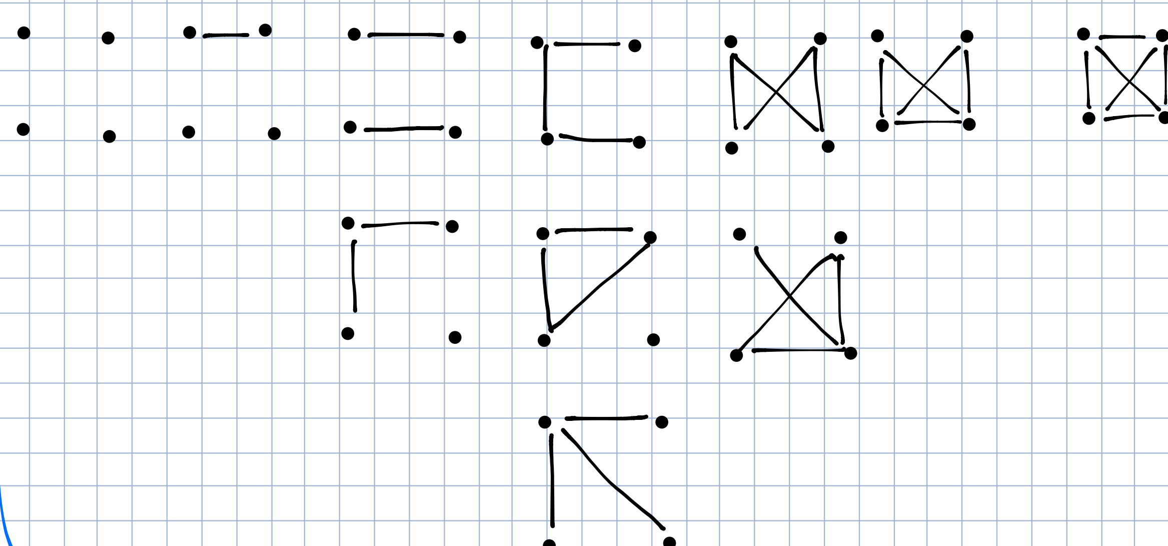
d	#
0	8
1	0
2	4
3	4

No vertices of deg 2 that are adjcent here

Remark: from now on, we'll distinguish graphs only up to isomorphism
This is why we will omit labels in most cases.

How many graphs are there with a given # of vertices? Starting from v=4

e=0 e=1 e=2 e=3 e=4 e=5 e=6



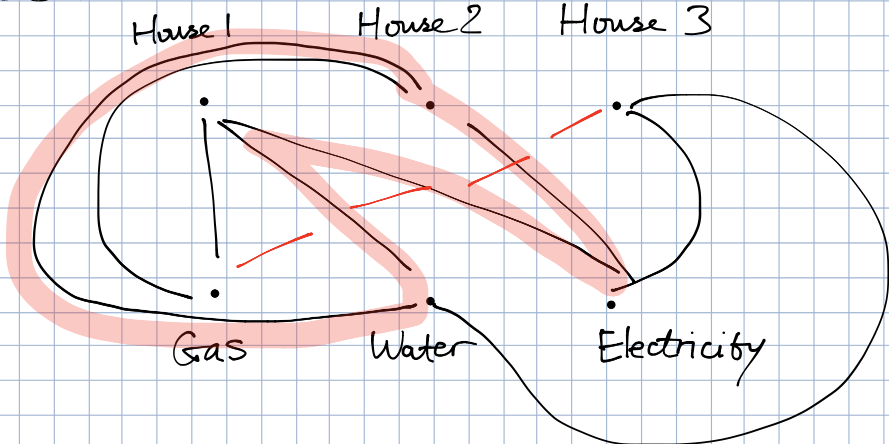
Remark: The # of graphs with a given # of vertices grows fast

# of vertices	# of graphs
1	1
2	2
3	4
4	11
5	34
6	156
7	1044
8	12346
9	308708

Outlook: Planar graphs

Def: A graph is planar if it is isomorphic to graph that can be drawn in the plane without edge crossings. Otherwise it is nonplanar.

Ex: The "utility graph"



The study of "planar" graphs involves an important seemingly trial theorem

Thm (Jordan Curve Thm): If C is a continuous simple closed curve in a plane, then the curve C divides the rest of the plane into two regions.
(not self-intersecting)

If a point P in one of these regions is joint to a point Q in the other region by a cts. curve L in this plane, then L intersects C.