May 30th

Recall: $f(a+h) = \underbrace{f(a) + mh}_{\text{Linear}} + \underbrace{E(h)}_{\text{Error}}$

$f:\mathbb{R}\to\mathbb{R}$ is differentiable if $\exists\, m$ so $\lim\limits_{h\to\infty} \dfrac{E(h)}{h} = 0$

$$m = f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h} + \frac{E(h)}{h}\right) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ exists.}$$

Ex: $(cf+dg)' = cf' + dg'$ — linearity of the derivative

Assume f, g diff at a

Proof: $f(a+h) = f(a) + f'(a)h + E_f(h)$

$g(a+h) = g(a) + g'(a)h + E_g(h)$

$(cf+dg)(a+h) = cf(a+h) + d\,g(a+h)$

$$= [cf(a)+dg(a)] + \underbrace{[cf'(a)+dg'(a)]h}_{\text{linear}} + \underbrace{cE_f(h) + dE_g(h)}_{\text{Error}}$$

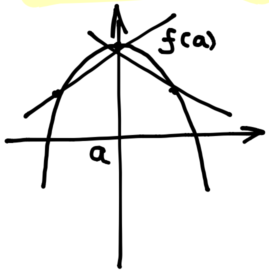
$\lim\limits_{h\to 0}(cE_f(h) + dE_g(h)) = 0$

$$(fg)(a+h) = f(a+h)g(a+h) = f(a)g(a) + h[f'(a)g(a) + g'(a)f(a)] + \underbrace{E_f(h)[g(a) + g'(a)h + E_g(h)] + E_g(h)[f(a)+f'(a)] + f'(a)g'(a)h^2}_{E_{fg}(h)}$$

So $\lim\limits_{h\to 0}\dfrac{E_{fg}(h)}{h} = 0$

so $(fg)'(a) = f'(a)g(a) + g'(a)f(a)$ — Product Rule

Max/Min Thm



Assume $f(a)$ is a local max/min

$\Rightarrow f'(a) = 0$ $\quad$ f is differentiable.

Assume max

$f(a+h) - f(a) \leq 0$

$$\lim_{h\to 0^+}\frac{f(a+h)-f(a)}{h} \leq 0$$

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \geq 0$$

As limit exists, must be equal, so $f'(a)=0$. Likewise for min.

Rolle's Theorem So f is cont. on $[a,b]$ but differentiable on (a,b), and $f(a)=f(b)$, then $\exists\, c \in (a,b)$ s.t. $f'(c)=0$.

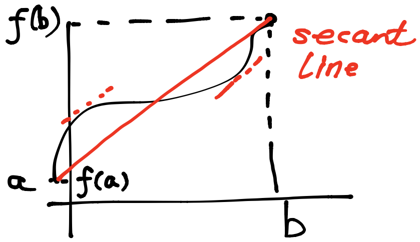
So $f([a,b])$ achieves a max/min by E.V.T (since $[a,b]$ compact)

So suppose max/min are at endpoints $\Rightarrow$ max=min $\Rightarrow$ constant function $\Rightarrow f'(c)=0 \ \forall c \in (a,b)$.

Else, $\exists\, c \in (a,b)$ so $f(c)$ is max or min.
$\Rightarrow f'(c)=0$ by *Max/Min Thm*.

■

Mean Value Thm
Sps f is cont. on $[a,b]$, diff on (a,b), then $\exists\, c$ s.t. $f'(c) = \dfrac{f(b)-f(a)}{b-a}$



$$l(x) = f(a) + \frac{f(b)-f(a)}{b-a}(x-a)$$

$l(a)=f(a)$
$l(b)=f(b)$

$g(x)=f(x)-l(x)$
$g(a)=g(b)=0$
g cont. on $[a,b]$, diff on $(a,b) \Rightarrow g$ satisfies Rolle's Theorem

$\Rightarrow \exists\, c,\ g'(c)=0 \Rightarrow f'(x) = \dfrac{f(b)-f(a)}{b-a}$

■

Summary: – pass local $\Rightarrow$ infintessimal data

– not constructionalist, just existence.

Def: f is incr ↗ (strictly) if $f(a)\leq f(b)$ ($f(a)<f(b)$) when $a<b$.

Thm : f diff on I

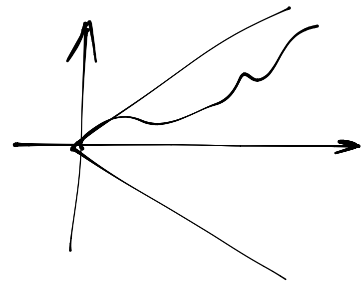
a). $|f'(x)| \leq C \quad \forall x$

Claim: $|f(b)-f(a)| \leq C|b-a|$

Proof: $f(b)-f(a) = f'(c)(b-a)$ $\quad *$

$\leq |f'(c)|\,|b-a|$

$= C|b-a|$



b). If $f'(x)=0 \ \forall x$ then f is constant

$* \Rightarrow f(b)=f(a)\ ,\ \forall a,b$

c). if $f'(x)>0 \ \forall x$, f is increasing From $*$. ■

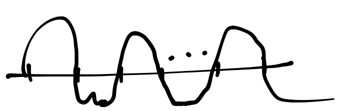
ASIDE

$P_n(x) = a_0 + a_1x + a_2x^2 + \cdots a_nx^n$, $a_i \in \mathbb{R}$

Claim: $P_n(x)$ has at most n roots.

- $P_1(x) = a_0 + a_1x$ has root ⋯ (basis)
- Assume $P_n(x)$ has at most n roots. (inductive step)

via contradiction, assume $P_{n+1}(x)$ has at least $n+2$ roots.
which means it has $n+2$ points :



$\Rightarrow$ Rolle's satisfied for $n+1$ intervals

$\Rightarrow n+1$ points c_i so $P'_{n+1}(c_i) = 0$

However P_{n+1}' is degree n so at most n roots

Here is the contradiction.

$\Rightarrow P_{n+1}$ has at most $n+1$ roots. ■

Now, do $f : \mathbb{R} \to \mathbb{R}^n$ $\leftarrow$ vector-valued function

$\vec{f}(a+h) = \vec{f}(a) + \vec{m}h + \vec{E}(h)$

So f is differentiable at a if $\exists \vec{m}$ so $\lim\limits_{h\to 0} \frac{|\vec{E}(h)|}{h} = 0$

$\Leftrightarrow f'(a) = \lim\limits_{h \to 0} \frac{\vec{f}(a+h) - \vec{f}(a)}{h}$

$\vec{f}$ is diff. $\overset{\text{at } a}{\Longleftrightarrow}$ f_i is diff. at a $\forall i \leq m$

$f_i'(a) = \lim\limits_{h \to 0} \frac{f_i(a+h) - f(a)}{h}$

- φ is scalar function from $\mathbb{R} \to \mathbb{R}$
- $(\varphi \vec{f})' = \varphi' \vec{f} + \varphi \vec{f}'$ $\longleftarrow$ component
- $(\vec{f} \cdot \vec{g})' = \left(\sum_{i=1}^{m} f_i g_i\right)' = \sum_{i=1}^{m} (f_i' g_i + f_i g_i') = \vec{f}' \cdot \vec{g} + \vec{f} \cdot \vec{g}'$
- for $n=3$, $(\vec{f} \times \vec{g}) = \vec{f}' \times \vec{g} + \vec{f} \times \vec{g}'$