# Lecture 1

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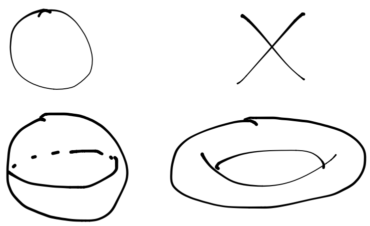
$X_n \to X$

$\lim_{n\to\infty} x_n = x$

Is $(-\infty, 0)$ close to $(0,\infty)$? Yes, as close as you want it to be.
Is $(-\infty, 0)$, $[0,\infty)$ close?

closeness

How to tell the difference between 2 shapes mathematically?



distinguishing stuffs

$\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $[0,1]$, $[0,\infty)$ ...

$\mathbb{N}$ → small b/c countable but also infinite, so "big"

$\mathbb{R}$ → big

$[0,1]$ → uncountable → big but also small

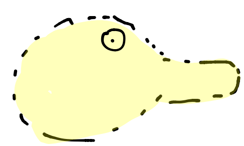
$[0,\infty)$ → big

big/small

## §1 – Topological Spaces

<u>def'n</u>:
Given $\varepsilon > 0$, $x \in \mathbb{R}^2$,
$B_\varepsilon(x) = \{y \in \mathbb{R}^2, d(x,y) < \varepsilon\}$

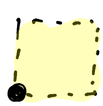
pic:



$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ $\quad x=(x_1,x_2)$, $y=(y_1,y_2)$

<u>def'n</u>: $U \subseteq \mathbb{R}^2$ is open if, $\forall x \in U$, $\exists \varepsilon > 0$ s.t $B_\varepsilon(x) \subseteq U$

e.g. $(0,1)\times(0,1) \cup \{0\}$ is not open.



Prop: i). $\emptyset$, $\mathbb{R}^2$ are open
ii) If $U_1, U_2, \cdots, U_N$ are finitely many open sets then $U_1 \cap U_2 \cap \cdots \cap U_N$ is open.
iii). If $\{U_\alpha : \alpha \in I\}$ is a collection of open sets indexed by $I$, then $\bigcup_{\alpha \in I} U_\alpha$ is open.

<u>def'n</u>: Let $X$ be a nonempty set and $\mathcal{T} \subseteq P(X)$ (power set of $X$, i.e. the collection of all subsets of $X$). We say that $\mathcal{T}$ is a topology on $X$, provide that:

i). $\emptyset, X \in \mathcal{T}$
ii). If $U_1, \dots, U_N \in \mathcal{T}$ then $U_1 \cap U_2 \cap \cdots \cap U_N \in \mathcal{T}$ (closed under fin. intersection)
iii). If $\{U_\alpha : \alpha \in I\}$ is a indexed collection of sets in $\mathcal{T}$, then $\bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$

We call $(X, \mathcal{T})$ a topological space, and we call any $U \in \mathcal{T}$ open or "open relative to $\mathcal{T}$".

e.g. $(\mathbb{R}^2, \{\text{open as in the defn}\})$ $\to \mathcal{T}$
is a topological space, we often denote it as $\mathbb{R}^2_{usual}$.

e.g.2 $(\mathbb{R}^2, P(\mathbb{R}^2))$
in other words, every subset of $\mathbb{R}^2$ is open. is called the discrete topology

e.g.3 $(\mathbb{R}^2, \{\emptyset, \mathbb{R}^2\})$
Here $\mathcal{T} = \{\emptyset, \mathbb{R}^2\}$
is called the indiscrete space. (useless)

e.g.4 Let $X$ be an infinite set, and let $\mathcal{T} = \{A \subseteq X : A = \emptyset$ or $X \setminus A$ is finite$\}$
This is called the co-finite topology on $X$.

e.g. If $X = \mathbb{N}$, then $\{2,3,4,5,6,\dots,\} \in \mathcal{T}_{cofinite}$ because $\mathbb{N} \setminus \{2,3,4,5,\dots\} = \{1\}$

Also, $\{7, 10, 100, 1001, 1002, 1003, \dots\} \in \mathcal{T}_{cofinite}$
Also $\emptyset \in \mathcal{T}_{cofinite}$ (by definition)

e.g.5 $\mathbb{R}^2_{cofinite}$ ($X = \mathbb{R}^2$ from e.g.4)
$\emptyset$ is open here.
$\mathbb{R}^2 \setminus \{\bar{0}\}$ is open here. (and in discrete topology, also in usual topology)
$\mathbb{R}^2 \setminus \{\bar{0}, (\pi,\pi), (e,-\pi)\}$ is open in usual topology.

Fact here: every set that is open in $\mathbb{R}^2$ cofinite is also open in $\mathbb{R}^2$ usual.

We write $\mathcal{T}_{cofinite} \subseteq \mathcal{T}_{usual}$ (where these are topologies on $\mathbb{R}^2$)

Working in $\mathbb{R}^2$, lets organize the topologies $\mathcal{T}_{disc}, \mathcal{T}_{indisc}, \mathcal{T}_{cofin}, \mathcal{T}_{usual}$

$$\boxed{\mathcal{T}_{indisc} \subseteq \mathcal{T}_{cofin} \subseteq \mathcal{T}_{usual} \subseteq \mathcal{T}_{disc}}$$

There are examples of topologies on $\mathbb{R}^2$ that don't fit into this picture.
Find one.

defn: Let $X$ be a set with topologies $\mathcal{T}$ and $\Gamma$. If $\mathcal{T} \subseteq \Gamma$, we say $\Gamma$ refines $\mathcal{T}$, or we say $\mathcal{T}$ is refined by $\Gamma$.

e.g. On $\mathbb{N}$, $\mathcal{T}_{cofinite}$ refines $\mathcal{T}_{indiscrete}$.

Secret example.
Let $X = \{\text{students sitting in this class right now}\}$. We say $U \in \mathcal{T}$, if $\forall x \in U$, if $y$ is sitting to the right of $x$, then $y \in U$.

For example, all people in the 1st row ...
the 2nd row ... (& of course, 1st & 2nd row

## §2. Basis

Idea: describe a topological space by only describing some of the open sets

e.g. Say that $(X, \mathcal{T})$ is a topological space and you know that $\{x\} \in \mathcal{T}$ for all $x \in X$.
What topological space is it?
$(\bigcup_{n\in\mathbb{N}} \{n\} = \mathbb{N})$ This is the discrete topology

Idea: $\{\{x\}, x \in X\}$ gives us enough info to decide the topology.
e.g. 2 Let $X = \mathbb{N}$, and we know $\mathbb{N} \setminus \{n\}$ for $n \in \mathbb{N}$ is open.
Any such topology must contain all of the cofinite sets.

i.e. it refines the cofinite topology.

We want to describe a collection of sets that when we take unions, we get a topology.
What properties should this collection have?
"defn": Let $X$ be a set, Let $\mathcal{B}$ be a collection of subsets of $X$ such that

i)

ii)

We want it to be the case that taking unions & intersection of sets in $\mathcal{B}$ gives us a topology on $X$.