June 13th

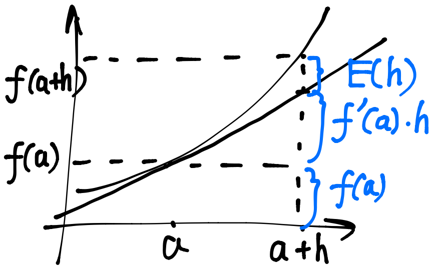
$f(a+h)=f(a)+f'(a)h+E(h)$ $=R_{a,1}(h)$ → increment

↑ pt. expansion is centered-around ← 1st order

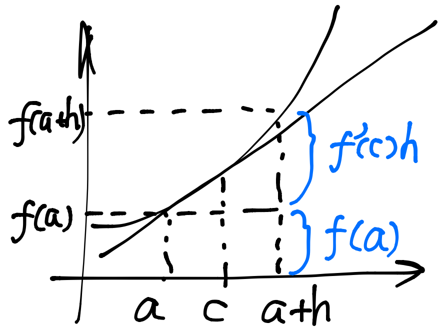
1) came from diff

2). $f(a+h)=f(a)+f'(c)h$ ← $R_{a,0}(h)$

Via MVT $\exists c\in(a,a+h)$



Vs.



Our claim: $f(a+h)=\underbrace{\sum_{j=0}^{k}\frac{f^{(j)}(a)h^j}{j!}}_{P_{a,k}(h)}+\underbrace{\frac{f^{(k+1)}(c)h^{k+1}}{(k+1)!}}_{R_{a,k}(h)}$

"2nd order MVT", $\exists c,\ f(a+h)=f(a)+f'(a)h+\frac{f''(c)h^2}{2}$

We say this is a good approx as. and $|f^{(k+1)}(x)|\leq M$

$$|R_{a,k}(h)|\leq\frac{M|h|^{k+1}}{(k+1)!}\Rightarrow\frac{|R_{a,k}(h)|}{|h|^k}\leq\frac{M|h|}{(k+1)!}\Rightarrow\frac{|R_{a,k}(h)|}{|h|^k}\to 0 \text{ as } h\to 0$$

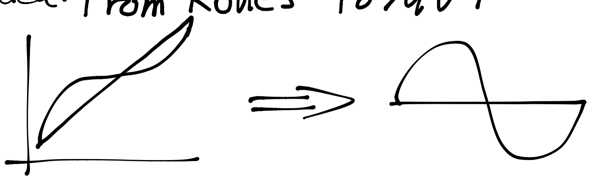
Recall: if f is $k+1$ times diff on $I=[a,b]$

$f(a)=f(b)$, $f^{(j)}(a)=0$, $1\leq j\leq k$ then

$\exists c\in(a,b)$ s.t $f^{(k+1)}(c)=0$

Proved last class

Idea: From Rolle's to MVT



Proof: $\overset{h>0}{g(t)}=R_{a,k}(t)-\frac{R_{a,k}(h)t^{k+1}}{h^{k+1}}=f(a+t)-f(a)-f'(a)t-\cdots\frac{f^{(k)}(a)t^k}{k!}-\frac{R_{a,k}(h)t^{k+1}}{h^{k+1}}$

$g(0)=0$

$g(h)=0$

$g^{(j)}(t)=f^{(j)}(a+t)-0\ldots0-f^{(j)}(a)-\frac{f^{(k)}(a)t^{k-j}}{(k-j)!}-R_{a,k}(h)\,t^{k+1-j}$

$g^{(j)}(0)=0\quad,1\leq j\leq k$

So $\exists\, c \in (0,h)$

$$g^{(k+1)}(c) = f^{k+1}(a+c) - \frac{R_{a,k}(h)(k+1)!}{h^{k+1}}$$

$$\Rightarrow R_{a,k}(h) = \frac{f^{(k+1)}(a+c)\, h^{k+1}}{(k+1)!}$$

$$\boxed{\begin{array}{l} \bullet\ a+c,\ c\in(0,h) \\ \bullet\ \tilde{c},\ \tilde{c}\in(a,a+h) \end{array}}$$

If I give you a $f(x) = \cdots$

process: take derivatives, sub value into formula.

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} = 1 + x + x^2 + \cdots$$

$$\lim_{x\to 0} \frac{x - \sin x}{x^3 e^x} = \frac{x - (x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)}{x^3(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots)} = \frac{\frac{x^3}{3!} - \frac{x^5}{5!} + \cdots}{x^3 + x^4 + \frac{x^5}{2} + \cdots}$$

so taking lowest order terms $\approx \lim_{x\to 0} \frac{x^3/3!}{x^3} = \frac{1}{3!}$

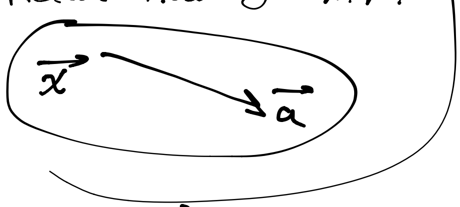
Ex: $f(x) = xe^x$

Find the 2nd order Taylor polynomial

1) use expansion for e^x, multiply term by term

2) Explicitly compute derivatives and sub it in.

Recall: had gen MVT



Let take a C^{k+1} f on an open convex set $S \subset \mathbb{R}^n$.

$$g(t) = f(\vec{a} + t(\vec{x} - \vec{a})) = f(\vec{a} + t\vec{h})$$

$$g'(t) = \nabla f(\vec{a} + t\vec{h}) \cdot \vec{h} = \vec{h} \cdot \nabla f(\vec{a} + t\vec{h})$$

$$g^{(j)}(t) = (\vec{h} \cdot \nabla)^j f(\vec{a} + t\vec{h})$$

$\vec{h} \cdot \nabla = h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}$ $\leftarrow$ diff. operator

$$(\vec{h} \cdot \nabla)^j = \left(h_1 \frac{\partial}{\partial x_1} + \cdots + h_n \frac{\partial}{\partial x_n}\right)^j$$

Let's do Taylor's Thm on g about 0, with $h = 1$

$$g(1) = \sum_{j=0}^{k} \frac{g^{(j)}(0)\, 1^j}{j!} + \text{Remainder}$$

$$\| $$

$$f(a+h) = \sum_{j=0}^{k} \frac{(h\cdot\nabla)^j f(\vec{a})}{j!} + R_{\vec{a},k}(\vec{h})$$

kth order approx

$$R_{\vec{a},k}(\vec{h}) = \frac{g^{(k+1)}(0+c)}{(k+1)!} = \frac{(h\cdot\nabla)^{k+1} f(\vec{a} + c\vec{h})}{(k+1)!}$$

remainder

$$(\vec{h}\cdot\nabla)^{k+1}=\left(h_1\frac{\partial}{\partial x_1}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^{k+1}=\sum_{|\alpha|=k+1}\frac{(k+1)!}{\alpha!}\vec{h}^{\alpha}\partial^{\alpha}$$

multinomial theorem

$\vec{h}^{\alpha}=h_1^{\alpha_1}\cdots h_n^{\alpha_n}$

$\partial^{\alpha}=\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$

$$\Rightarrow f(\vec{a}+\vec{h})=\underbrace{\sum_{|\alpha|\le k}\frac{\partial^{\alpha}f(\vec{a})\vec{h}^{\alpha}}{\alpha!}}_{\text{approx}}+\underbrace{\sum_{|\alpha|=k+1}\frac{\partial^{\alpha}f(\vec{a}+c\vec{h})\vec{h}^{\alpha}}{\alpha!}}_{\text{remainder}}$$

$$(k+1)\sum_{|\alpha|=k+1}\frac{\vec{h}^{\alpha}}{\alpha!}\int_0^1(1-t)^k\partial^{\alpha}f(\vec{a}+t\vec{h})dt$$

$$P_{\vec{a},2}(\vec{h})=\underbrace{f(\vec{a})}_{0^{th}\text{ order approx.}}+\underbrace{\sum_{j=1}^{n}\frac{\partial f(\vec{a})}{\partial x_j}h_j}_{1^{st}\text{ order term}}+\underbrace{\frac{1}{2}\sum_{i,j=1}^{n}\partial_i\partial_j f(\vec{a})h_ih_j}_{2nd\text{ order term}}$$

2nd order: $\frac{1}{2}\sum_{j=1}^{n}\partial_j^2 f(\vec{a})h_j^2+\sum_{1\le i<j\le n}\partial_i\partial_j f(\vec{a})h_ih_j$

as $\partial_i\partial_j f(\vec{a})=\partial_j\partial_i f(\vec{a})$

$|\vec{h}|=\sqrt{h_1^2+\cdots+h_n^2}$

$\|\vec{h}\|_1=|h_1|+\cdots+|h_n|$ $\Leftarrow$ taxicab norm or L^1 norm

not in the book

Let $M=\max\{|h_1|,\cdots,|h_n|\}$

$M\le|\vec{h}|\le\sqrt{n}M$

$M\le\|\vec{h}\|_1\le nM$

$\left.\begin{aligned}\|\vec{h}\|_1&\le nM\le n|\vec{h}|\\|\vec{h}|&\le\sqrt{n}M\le\sqrt{n}\|h\|_1\end{aligned}\right\}$ $\|\vec{h}\|_1$ is "equivalent" to $|\vec{h}|$

Assume $|\partial^{\alpha}f(\vec{x})|<M$ $\forall x\in S$

$$|R_{\vec{a},k}(\vec{h})| \leq M \sum_{|\alpha|=k+1} \frac{|\vec{h}|^{\alpha}}{\alpha!} \leq \frac{M}{(k+1)!} \left| (h_1 + \cdots + h_n)^{k+1} \right|$$

multinomial Thm

via triangle

$$\leq \frac{M}{(k+1)!} \|\vec{h}\|_1^{k+1} \Rightarrow \frac{|R_{\vec{a},k}(\vec{h})|}{\|\vec{h}\|_1^{k}} \leq \frac{M}{(k+1)!} \|\vec{h}\|_1 \Rightarrow \frac{|R_{\vec{a},k}(\vec{h})|}{\|\vec{h}\|_1^{k}} \to 0$$

as $\|\vec{h}\|_1 \to 0$

$$\Rightarrow \frac{|R_{\vec{a},k}(\vec{h})|}{|\vec{h}|^{k}} \to 0 \text{ as } |\vec{h}| \to 0$$

Thm (Uniqueness of Taylor polynomial)

① $f(\vec{a}+\vec{h}) = P_{\vec{a},k}(\vec{h}) + R_{\vec{a},k}(\vec{h})$ ← exists via Taylor

Suppose: ② $f(\vec{a}+\vec{h}) = \underbrace{Q(\vec{h})}_{\text{polynomial}} + \underbrace{E(\vec{h})}_{\frac{E(\vec{h})}{|\vec{h}|^k} \to 0 \text{ as } \vec{h} \to \vec{0}}$

$\Rightarrow P_{\vec{a},k}(h) = Q(\vec{h})$

Lemma: If P is a polynomial of degree $\leq k$ and $\frac{P(\vec{h})}{|\vec{h}|^k} \to 0$ as $|\vec{h}| \to 0$ then $P \equiv 0$.

Proof: $\frac{P(t\vec{h})}{|t|^k} \to 0$ as $t \to 0$

$P(\vec{h}) = P_0 + P_1(\vec{h}) + \cdots + P_k(\vec{h})$ where $P_j(\vec{h})$ = sum of terms of order's $= \sum_{|\alpha|=j} C_\alpha h^\alpha$ where C_α chosen so it's in $P(\vec{h})$

$P(t\vec{h}) = P_0 + tP_1(\vec{h}) + \cdots + t^k P_k(\vec{h})$

- as $P(t\vec{h})$ as $t \to 0 \Rightarrow 0 = P_0 + \cdots + 0 = P_0$
- $\frac{P(t\vec{h})}{t} = P_1(\vec{h}) + tP_2(\vec{h}) + \cdots + t^{k-1}P_k(\vec{h})$

as $\frac{P(t\vec{h})}{t} \to 0$, $0 = P_1(\vec{h}) + \underline{\quad 0 \quad}$

Inductively, $P_j(\vec{h}) = 0$, $\forall j \in \{0, \cdots, k\}$ ∎

Proof of Uniqueness

① − ② $= 0 \Rightarrow \frac{P_{a,k}(\vec{h}) - Q(\vec{h})}{|\vec{h}|^k} = \frac{E(\vec{h}) - R_{\vec{a},k}(\vec{h})}{|\vec{h}|^k}$ divided by $|\vec{h}|^k$

so RHS $\to 0$ as $\vec{h} \to \vec{0}$ by assumption

$\therefore$ LHS $\to 0$ as $\vec{h} \to \vec{0}$

and since $P_{\vec{a},k}(\vec{h})-Q(\vec{h})$ is a polynomial of degree less than k

by lemma $\rightarrow P_{\vec{a},k}(\vec{h})-Q(\vec{h})\equiv 0 \Rightarrow P_{\vec{a},k}(\vec{h})\equiv Q(\vec{h})$

Ex: $x\sin(2x+y^2)=x\left((2x+y^2)-\frac{(2x+y^2)^3}{3!}+\frac{(2x+y^2)^5}{5!}-\cdots\right)$

$$\underbrace{=2x^2+xy^2-\frac{1}{3!}\left(x(2x)^3+3x(2x)^2y^2+\text{higher order}\right)}_{\text{is the Taylor 5th order approximation}}+\text{higher order}$$

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§2.8 Critical Points

Goal: find max/min of multivariable functions

For $f: S\subset\mathbb{R}^n\to\mathbb{R}$. S is open, diff

Def: A local max/min is a point $\vec{a}\in S$ $\left.\begin{array}{l} f(\vec{a}+\vec{h})\le f(\vec{a}) \text{ or} \\ f(\vec{a}+\vec{h})\ge f(\vec{a})\end{array}\right\}$ for some neighbourhood of $\vec{a}$.

Def: A critical point $\vec{a}\in S$ has $\nabla f(\vec{a})=0$.

To find critical points $\frac{\partial f(\vec{a})}{\partial x_j}=0\ \ \forall j\in\{1,\dots,n\}$

– n equations, – n components

– For 1 dim, $\vec{a}$ being a max/min $\Rightarrow f'(a)=0$

However, $f'(a)=0$ doesn't always implies a is a max/min

e.g. $f(x)=x^3$

Then 2nd derivative test, 1) $f''(a)>0$, $\{$ min
2) $f''(a)<0$, $\{$ max
3) $f''(a)=0$, indeterminate why?

$f(a+h)-f(a)=f'(a)\cdot h+f''(a)\,h^2/2+R_{a,2}(h)$
say $f''(a)=0$, then $f(a+h)-f(a)=\frac{f'''(a)h^3}{3!}+R_{a,3}(h)$
sign changes deg. on $h>0$ or $h<0$.

1). Goal: Find max/min
2). Find CP's for candidates
3). Study the higher order behaviour of CP.