Jan 8th, 2013
MAT224H1 S
Instructor: Oded Yacobi

Week 1 beginning January 7.
Lecture: Complex Numbers, Fields, Vector Spaces over a Field.
Section 5.1
Section 5.2
Note: It is highly recommended that you review **Chapter 1** of the textbook (Vector Spaces) and famil-iarize yourself with all the topics involved. This material was covered in MAT223 and is essential materialfor understanding Sections 5.1, 5.2.

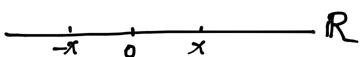
Homework 20% + Term Test 30% + Final 50%

Section 5.1

Complex numbers & complex vector space (VS)

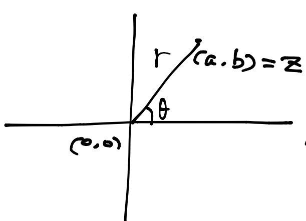
Introduction: $\mathbb{R}$ = real numbers
$\mathbb{C}$ = complex numbers

Geometric Construction of $\mathbb{C}$:



additive identity $x \in \mathbb{R}$. $x+0=x$

additive inverse, $\forall x \in \mathbb{R}$, x admits an additive inverse $-x$.



- The absolute value of z is $|z| = r = \sqrt{a^2+b^2}$
- Polar angle of z is the angle between segment $\overline{(0,0)\,(a,b)}$ and the positive x-coordinate

The polar angle is unique up to 2π.
Notice that $\boxed{\cos\theta = \frac{a}{r}}$, i.e. $a = r\cos\theta$

$\boxed{\sin\theta = \frac{b}{r}}$, $b = r\sin\theta$

z can be rewritten as $z = (a, b) = (r\cos\theta, r\sin\theta) = |z|(\cos\theta, \sin\theta)$

- We are going to represent z algebraically as
$z = a + bi$
and define $Re(z) = a$, $Im(z) = b$
$z = Re(z) + i\, Im(z)$

$z = |z|\cos\theta + (|z|\sin\theta)\, i$

examples:

$(1,0) = 1 + 0i \longleftarrow \theta = 0$

$(0,-1) = 0 - 1i \longleftarrow \theta = \frac{3}{2}\pi$

$(1,1) = 1 + i \longleftarrow \theta = \frac{\pi}{4}$

$(1,-1) = 1 - 1i \longleftarrow \theta = -\frac{\pi}{4}$

$\boxed{\tan\theta = \frac{b}{a}}$

$(100\sqrt{3}, 100) = 200\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \longrightarrow \theta = \frac{\pi}{6}$

or $\tan\theta = \frac{100}{100\sqrt{3}} = \frac{\sqrt{3}}{3}$ $\quad \theta = \arctan\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

Addition and Multiplication of Complex numbers:

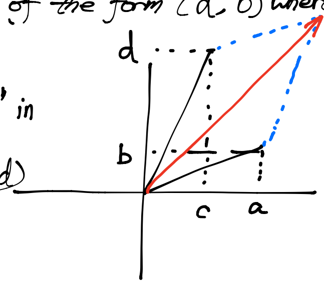
We wanted to "extend" to addition and multiplication of real numbers to the object $\mathbb{C}$ that we constructed geometrically via the plane $\mathbb{R}^2$.

The wish in our head was that our operations match the ones we know over $\mathbb{R}$ when we restrict ourselves to couples of the form $(d, 0)$ where $d \in \mathbb{R}$.

Definition:

Define the addition of "numbers" in $\mathbb{C}$ as:

$(a,b) + (c,d)$ as $(a+c, b+d)$



$(a+bi) + (c+di)$ is the new complex number $(a+c) + (b+d)i$

& the multiplication as

$(a,b)\cdot(c,d) = (ac-bd, ad+bc)$

Let's test: $i \cdot i = (0,1)\cdot(0,1) = (0-1, 0+0) = (-1,0) = -1$

Facts:

Fix two complex numbers z_1 and z_2 such that

$z_1 = a+bi \qquad\qquad z_2 = c+di$

$\downarrow \qquad\qquad\qquad \downarrow$

$|z_1|\cos\theta + (|z_1|\sin\theta)i \qquad |z_2|\cos\theta + (|z_2|\sin\theta)i$

Fact ① $|z_1 \cdot z_2| = |z_1||z_2|$

Fact ② : $\overbrace{\text{Polar angle}}^{\text{argument}}$ of $z_1 \cdot z_2$ = polar angle (z_1) + polar angle (z_2)

Example(s):

compute $\mathbb{C}$

1). $(\sqrt{3}+i)(\frac{3}{2}+\frac{3\sqrt{3}}{2}i)$ in 2 ways

i). Geometric with facts ① & ②

ii). Algebraic.

$|z_1| = \sqrt{a^2+b^2} = \sqrt{3+1} = 2$

$\theta_1 = ?$

$\tan\theta_1 = \frac{b}{a} = \frac{1}{\sqrt{3}}$

$\cos\theta_1 = \frac{a}{|z_1|} = \frac{\sqrt{3}}{2}$ $\quad \theta_1 = \frac{\pi}{6}$

$|z_2| = \sqrt{\frac{9}{4}+\frac{9\times3}{4}} = \frac{6}{2} = 3$

$\theta_2 = ?$

$\cos\theta_2 = \frac{a}{r} = \frac{3/2}{3} = \frac{1}{2}$

$\sin\theta_2 = \frac{b}{r} = \frac{3\sqrt{3}/2}{3} = \frac{\sqrt{3}}{2}$

$\theta_2 = \frac{\pi}{3}$

Polar angle θ

↓

$z = z_1 \cdot z_2 = |z_1 z_2|\cos\theta + (|z_1 z_2|\sin\theta)i$

$|z_1 z_2| = |z_1||z_2| = 2\times3 = 6$

$\theta = \theta_1+\theta_2 = \frac{\pi}{6}+\frac{\pi}{3} = \frac{\pi}{2}$

$$\boxed{z_1 \cdot z_2 = 6\cdot0 + (6\cdot\sin\frac{\pi}{2})i = 6i}$$

* Algebraic method is easy

$(\sqrt{3}+i)(\frac{3}{2}+\frac{3\sqrt{3}}{2}i) = \frac{3\sqrt{3}}{2}+\frac{3}{2}i+\frac{9}{2}i-\frac{3\sqrt{3}}{2} = 6i$

$z = \frac{1}{1+i}$

$(1+i)$ is not the zero complex number. Thus it admits a multiplicative inverse denoted by $z = \frac{1}{1+i}$

$\frac{1}{1+i} = \frac{1}{1+i}\cdot\frac{1-i}{1-i} = \frac{1-i}{2}$