Tutorial 11

STAT 3013/4027/8027

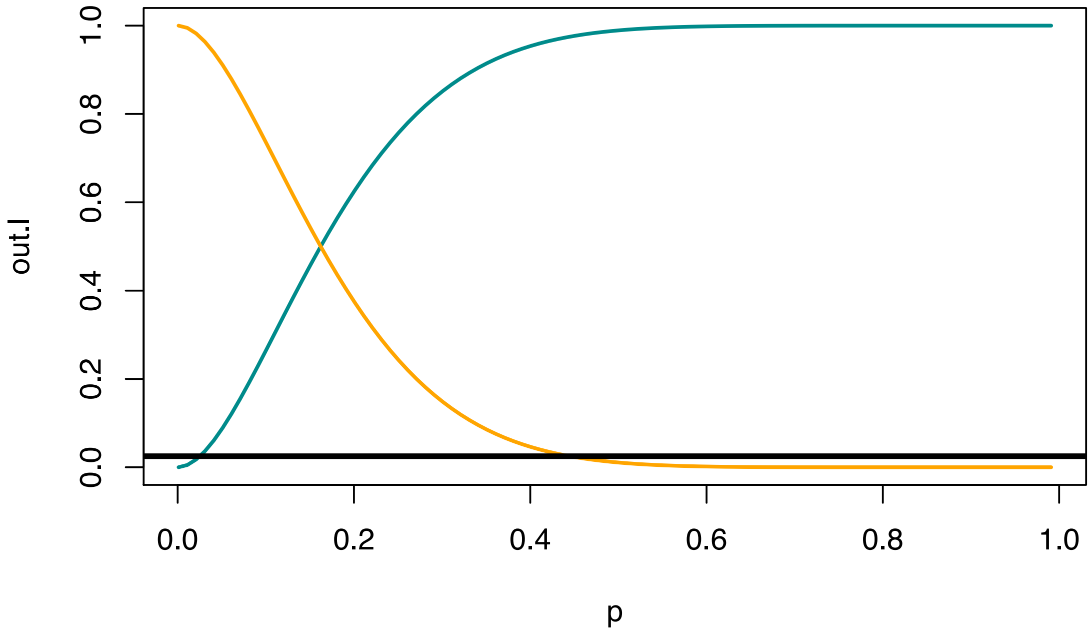
1. SI 5.10 **Solution:**

```
p <- seq(0.001, 0.999, by=0.01)


out.l <- pbinom(1, 10, p, lower.tail=FALSE)
out.u <- pbinom(1, 10, p)

plot(p, out.l, type="l", lwd=2, col="cyan4")
lines(p, out.u, lwd=2, col="orange")

abline(h=0.025, col="black", lwd=3)
```




```
max(p[out.l<0.025])
```

```
## [1] 0.021
```

```
min(p[out.u<0.025])
```

```
## [1] 0.451
```

2. SI 5.4, 6.2, 6.8 (**See handwritten solutions below**).

GJJ Q 5.4

$$X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta) = \theta x^{\theta-1}$$

$$0 < x < 1 \; ; \; \theta > 0$$

$$L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1}$$

$$\ell(\theta) = \sum_{i=1}^{n} \left[\log(\theta) + (\theta-1)\log(x_i)\right]$$

$$= n\log(\theta) + (\theta-1)\sum\log(x_i)$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum\log(x_i) = 0$$

$$\therefore \hat{\theta} = \frac{n}{-\sum\log(x_i)}$$

$$\Rightarrow \frac{\partial^2 \ell}{\partial \theta^2} = \frac{-n}{\theta^2} \qquad \therefore I(\theta) = -E\left(\frac{-n}{\theta^2}\right) = \frac{n}{\theta^2}$$

$$\therefore \hat{\theta} \;\dot{\sim}\; N(\theta, I(\theta)^{-1}) = N(\theta, \theta^2/n)$$

approx

- An approximate 95% CI for θ is:

$$P\left(z_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{\theta / \sqrt{n}} \leq z_{1-\alpha/2} \right) = 0.95$$

use the estimator here.

$$P\left(z_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{\hat{\theta} / \sqrt{n}} \leq z_{1-\alpha/2} \right) = 0.95$$

$$P\left(z_{\alpha/2}\, \hat{\theta}/\sqrt{n} \leq \hat{\theta} - \theta \leq z_{1-\alpha/2}\, \hat{\theta}/\sqrt{n} \right) = 0.95$$

$$P\left(z_{\alpha/2}\, \hat{\theta}/\sqrt{n} - \hat{\theta} \leq -\theta \leq z_{1-\alpha/2}\, \hat{\theta}/\sqrt{n} - \hat{\theta} \right) = 0.95$$

$$P\left(\hat{\theta} - z_{1-\alpha/2}\, \hat{\theta}/\sqrt{n} \leq \theta \leq \hat{\theta} - z_{\alpha/2} \cdot \hat{\theta}/\sqrt{n} \right) = 0.95$$

$$\therefore \quad \hat{\theta} \pm z_{\alpha/2}\, \hat{\theta}/\sqrt{n}$$

GJJ Q 5.10

$X = 1$ success out of n trials

$n = 10$

- Based on the simple null hypothesis:

$$H_0 : p = p_0 \quad vs \quad H_1 : p \neq p_0$$

- Based on the Binomial distribution (under H_0) we reject H_0 if:

$$P(X \leq x \mid p_0) < \alpha/2$$

or

$$P(X \geq x \mid p_0) > \alpha/2$$

$\therefore$ we accept H_0 if:

$$\sum_{i=x}^{n} \binom{n}{i} p_L^{\,i} (1 - p_L)^{n-i} = \alpha/2$$

and

$$\sum_{i=0}^{x} \binom{n}{i} p_U^{\,i} (1 - p_U)^{n-i} = \alpha/2$$

lower

$$\Rightarrow \sum_{i=1}^{10} \binom{10}{i} p_L^{i} (1-p_L)^{10-i} = 0.025$$

upper

$$\Rightarrow \binom{10}{0} p_U^{0} (1-p_U)^{10} + \binom{10}{1} p_U^{1} (1-p_U)^{9} = 0.025$$

$\rightarrow$ See the R-code.

$$p_L = 0.021, \quad p_U = 0.457$$

GJJ Q 6.2

- Suppose that δ^* is a Bayes procedure with respect to the prior distribution $p_1, p_2, \ldots, p_K$.

- Suppose that δ^* is inadmissible.

$$\Rightarrow \quad R(\theta_j, \delta') \leq R(\theta_j, \delta^*) \quad \forall j$$

$$R(\theta_j, \delta') < R(\theta_j, \delta^*) \quad \text{for some } j$$

$\Rightarrow$ As all the $p_j > 0$:

$$\sum_{j=1}^{K} p_j R(\theta_j, \delta') < \sum_{j=1}^{K} p_j R(\theta_j, \delta^*)$$

This means δ^* was not a Bayes procedure w.r.t $\underline{p}$ $\therefore$ we have a contradiction.

GJJ Q 6.8

$$f_X(x) = (x-1)\,\theta^2 (1-\theta)^{x-2} \quad ; x = 2, 3, 4, \dots$$

$$p(\theta) = \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)}\,\theta^3 (1-\theta)^2$$

a.) $p(\theta \mid x) \propto p(x \mid \theta)\, p(\theta)$

$$\propto (x-1)\,\theta^2 (1-\theta)^{x-2} \frac{\Gamma(7)}{\Gamma(4)\Gamma(7)}\,\theta^3 (1-\theta)^2$$

$$\propto \theta^5 (1-\theta)^x = \theta^{6-1} (1-\theta)^{x+1-1}$$

$$\therefore\ p(\theta \mid x) = \text{beta}(\underbrace{6}_{=a}, \underbrace{x+1}_{=b})$$

b.) Under squared error loss the Bayes' estimator is the mean $\therefore$

$$\hat{\theta} = \frac{a}{a+b} = \frac{6}{7+x}$$

c.) To Determine the Bayes risk

$$\text{Bayes Risk} = \iint L_s(\theta, \delta(x))\, L(\theta \mid x)\, p(\theta)\, dx\, d\theta$$

$$= \int \underbrace{h(x)}_{\text{marginal of } x} \underbrace{\int L_s(\theta, \delta(x))\, p(\theta \mid x)\, d\theta}_{E_{\theta \mid x}\left[L_s(\theta, d(x))\right]}\, dx$$

$$\Rightarrow \int h(x) \frac{6(x+1)}{(x+7)^2 (x+8)} \, dx$$

we need to know $h(x)$

$$\underbrace{(\hat{\theta} - \bar{\theta})}_{\text{minimized when}}\overset{0}{} + V(\theta) = V(\theta)$$

$\hat{\theta} = \bar{\theta}$ mean of the posterior

$$V(\theta) = \frac{ab}{(a+b)^2 (a+b+1)}$$

$$= \frac{6(x+1)}{(x+7)^2 (x+8)}$$

$$h(x) = \int p(x \mid \theta) \, p(\theta) \, d\theta$$

$$= \int (x-1) \, \theta^2 (1-\theta)^{x-2} \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \theta^3 (1-\theta)^2 \, d\theta$$

$$= (x-1) \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \int \theta^{5+1-1} (1-\theta)^{x+1-1} \, d\theta$$

$$= (x-1) \frac{\Gamma(7)}{\Gamma(4)\Gamma(3)} \frac{\Gamma(6)\Gamma(x+1)}{\Gamma(7+x)} \underbrace{\int \frac{\Gamma(7+x)}{\Gamma(6)\Gamma(x+1)} \theta^{6-1} (1-\theta)^{7-1} \, d\theta}_{=1}$$

$$\therefore \text{Bayes' risk} = \sum_{i=2}^{\infty} \frac{(x-1)\,\Gamma(7)\,\Gamma(6)\,\Gamma(x+1)}{\Gamma(4)\,\Gamma(3)\,\Gamma(7+x)} \frac{6(x+1)}{(x+7)^2 (x+8)}$$

$$\approx 0.00005$$