DERIVATIONS WITH AND, OR AND BICONDITIONAL
NATURAL DEDUCTION
Part 2

3.11 Derivations with AND, OR and BICONDITIONAL

So far we've learned how to derive any sentence from any set of sentences that entails it, provided that the sentences are symbolized using the conditional and the negation signs. But often it is easier to symbolize sentences with the other three logical connectives – and, or, biconditional.

We need to extend the derivation system to include rules to cope with these new connectives.

The Extended Derivation System

We will continue to use the three basic forms of derivation and subderivation:

Direct Derivation

Conditional Derivation

Indirect Derivation

We will also continue to use the rules of inference that we have already learned:

Modus Ponens (MP or mp)

$$\frac{(\phi \rightarrow \psi) \qquad \phi}{\psi}$$

Modus Tollens (MT or mt)

$$\frac{(\phi \rightarrow \psi) \qquad \sim\psi}{\sim\phi}$$

Double Negation (DN or dn)

$$\frac{\phi}{\sim\sim\phi} \qquad \frac{\sim\sim\phi}{\phi}$$

Repetition (R or r)

$$\frac{\phi}{\phi}$$

We will also get a hundred new theorems (out of the infinitely many)!

But we need some new rules as well . . .

Rules for $\wedge$, $\vee$ and $\leftrightarrow$

The new rules come in pairs – an elimination rule and an introduction rule for each new connective. The Greek letters ϕ (phi) and ψ (psi) can represent any sentence, whether atomic or molecular.

Simplification (S or s; SL/SR or sl/sr)

$$\frac{\phi \wedge \psi}{\phi} \qquad \frac{\phi \wedge \psi}{\psi}$$

This rule allows us to infer either conjunct from a sentence whose main logical connective is a conjunction. Although you can just use the justification 'S' to simplify to either conjunct; 'SL' is an alternate justification for simplifying to the left conjunct; 'SR' to the right conjunct.

Adjunction (ADJ or adj)

$$\frac{\begin{array}{l}\phi \\ \psi\end{array}}{\phi \wedge \psi}$$

This rule allows us to infer a conjunction sentence from the two conjuncts.

Modus Tollendo Ponens (MTP or mtp)

$$\frac{\begin{array}{l}\phi \vee \psi \\ \sim \phi\end{array}}{\psi} \qquad \frac{\begin{array}{l}\phi \vee \psi \\ \sim \psi\end{array}}{\phi}$$

This rule allows us to infer one disjunct from a disjunction and the negation of the other disjunct. This makes sense: if the butler or the gardener did it, and the gardener didn't do it, then it must have been the butler!

'Modus tollendo ponens' is a Latin term which means, 'The mode of argument that asserts by denying.' By denying one disjunct of a disjunction, you can assert the other disjunct.

Addition (ADD or add)

$$\frac{\phi}{\phi \vee \psi} \qquad \frac{\psi}{\phi \vee \psi}$$

This rule allows us to take a sentence and infer from it a disjunction with the sentence as one disjunct. This makes sense: if it will rain, then it will rain OR it will snow.

You can use ADD to join *any* sentence to an available sentence creating any number of new sentences. Sounds powerful! But, be careful, although it can be useful, ADD doesn't really get you anything you didn't have in the first place.

Biconditional-Conditional (BC or bc)

$$\frac{\phi \leftrightarrow \psi}{\phi \rightarrow \psi} \qquad \frac{\phi \leftrightarrow \psi}{\psi \rightarrow \phi}$$

This rule allows us to infer either conditional from a biconditional. This makes sense; the biconditional is really a conjunction of two conditionals.

Conditional-Biconditional (CB or cb)

$$\frac{\begin{array}{l}\phi \rightarrow \psi \\ \psi \rightarrow \phi\end{array}}{\phi \leftrightarrow \psi}$$

This rule allows us to infer a biconditional from two conditionals. The two conditionals must be such that the antecedent of the first is the consequent in the other and the consequent of the first is the antecedent in the other.

Like the original rules of inference, these new rules are valid no matter how simple or complex the sentences being inferred are (how complex ϕ and ψ are in the rules above.) What matters is the logical form of the inference.

A few things to remember:

Justifications:

The justification consists of line numbers and/or premise numbers and the rule. For S, ADD and BC you cite one line number. For ADJ, MTP and CB you must cite two line numbers.

Main Logical Connective:

To use any rule, you need to focus on the main logical connective – that is the connective that the elimination rule acts on and the connective that the introduction rule introduces.

So, watch the main connective especially when you have been using informal notations and leaving out the parentheses.

Suppose you have the sentence: $P \rightarrow Q \wedge R$.

It looks like you can use Simplification to get $P \rightarrow Q$ … BUT IT'S A MISTAKE!.

$P \rightarrow Q \wedge R$ is a more informal notation of $P \rightarrow (Q \wedge R)$.

The main connective is "$\rightarrow$" not "$\wedge$". One cannot use S with that sentence.

3.11 E1

Which inference rule if any justifies the following arguments?
(S, ADJ, ADD, MTP, BC, CB or none)

a) $R \rightarrow (P \vee \sim S)$
$(P \vee \sim S) \rightarrow R$
$\therefore (P \vee \sim S) \leftrightarrow R$

CB

b) $(P \vee Q) \wedge (S \rightarrow T)$
$\therefore S \rightarrow T$

S

c) $(P \rightarrow (S \rightarrow \sim Q))$
$\therefore \sim P \vee (P \rightarrow (S \rightarrow \sim Q))$

ADD

d) $P \rightarrow R$
S
$\therefore P \rightarrow R \wedge S$

~~ADJ~~ none

e) $Q \vee \sim(S \rightarrow P)$
$(S \rightarrow P)$
$\therefore Q$

~~MTP~~

none; (2 steps: pr 2 dn, pr 1 mtp

f) $(V \leftrightarrow Z) \vee (\sim W \wedge Y)$
$\sim(\sim W \wedge Y)$
$\therefore V \leftrightarrow Z$

MTP

g) $S \rightarrow \sim R$
$S \leftarrow \sim R$
$\therefore S \leftrightarrow \sim R$

~~CB~~

← is not a symbol of SL

h) P
$P \wedge R$
$\therefore P \wedge R \wedge P$

ADJ

Derivation hints for the new rules:

Simplification (S/SL/SR or s/sl/sr)

- Make sure that conjunction (∧) is the main connective.
- You can Simplify the same conjunction more than once, deriving each conjunct.
- You can Simplify to either conjunct.
- You can use ‘S’ to justify deriving either conjunct. Alternatively, use ‘SL’ (simplification left) to derive the left conjunct or ‘SR’ (simplification right) to derive the right conjunct. This is especially important for clarity when you use abbreviated proofs.
- When you’re stuck in a derivation, use S where you can – it can make the solution easier to see.

Adjunction (ADJ or adj)

- If the main connective of your goal sentence is conjunction (∧), you might want to derive each conjunct separately and use ADJ.
- Make sure you put the two conjuncts in the order you want them in when using ADJ.

Addition (ADD or add)

- Don’t use ADD if you don’t know how you are going to use the resulting disjunction.
- If the main connective of your goal sentence is a disjunction (∨), consider whether it is possible to derive one of the disjuncts and use ADD. (Many times it will not be. Often you ***must*** derive the disjunction some other way!)

Modus Tollendo Ponens (MTP or mtp)

- If you have a sentence whose main connective is disjunction (∨), always be on the lookout for the negation of one disjunct. You can use MTP and derive the other disjunct.
- Like S, MP and MT, MTP is a good one to use if you can when you are stuck in a derivation. It helps to make things a little clearer to derive simpler sentences.

Biconditional-Conditional (BC or bc)

- If the main connective of a sentence is a biconditional (↔), use BC to get either conditional.
- Make sure you are deriving the conditional that you need (think about which side you want as the antecedent.)

Conditional-Biconditional (CB or cb)

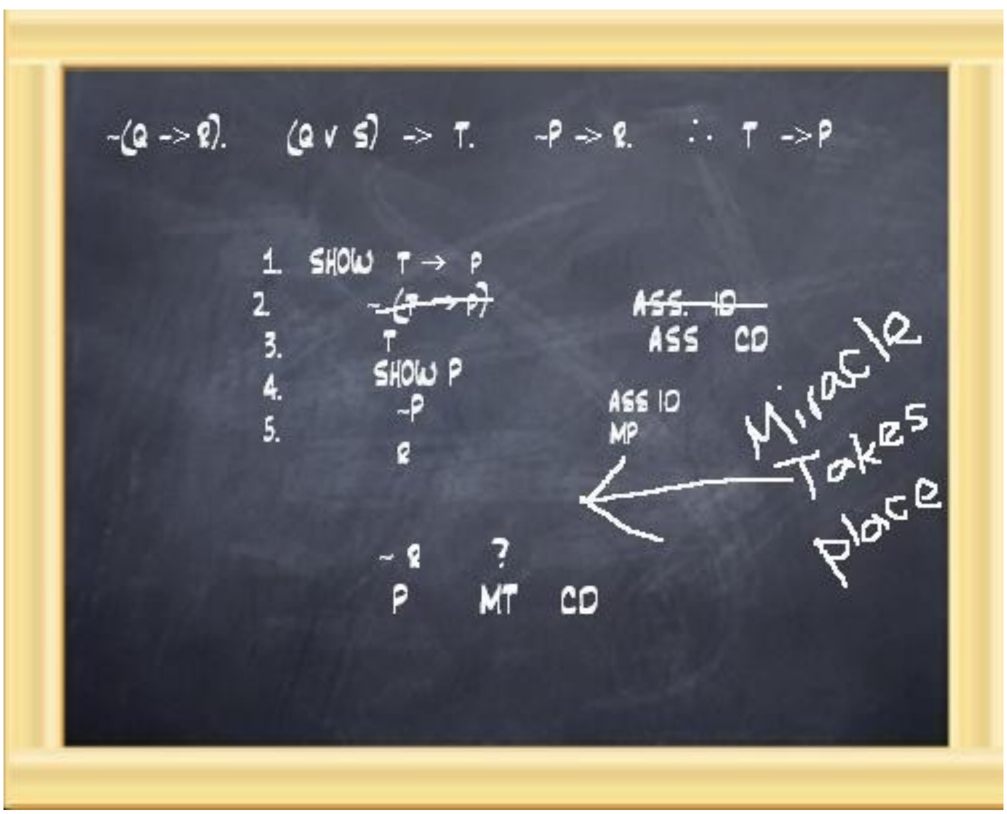
- If your main goal is a biconditional (↔), derive each conditional separately and use CB. You may want to do two conditional proofs.

3.11 EG1 Let's try out the new rules

$P \wedge Q$. $(R \vee P) \rightarrow \sim S$. $S \vee T$. $(V \rightarrow W) \leftrightarrow Q$. $W \rightarrow V$. $\therefore T \wedge (V \leftrightarrow W)$

1 Show $T \wedge (V \leftrightarrow W)$
2
3
4
5
6
7
8
9
10

Answer see next page



Here's the completed derivation. It uses each of the new rules!

P ∧ Q. (R ∨ P) → ~S . S ∨ T. (V → W) ↔ Q. W →V. ∴ T ∧ (V ↔ W)

1	~~Show~~ T ∧ (V ↔ W)		
2	P	pr1 SL	
3	Q	pr1 SR	← Simplification
4	R ∨ P	2 ADD	← Addition
5	~S	4 pr2 MP	
6	T	5 pr3 MTP	← Modus Tollendo Ponens
7	Q → (V →W)	pr4 BC	← Biconditional to Conditional
8	V→W	3 7 MP	
9	V ↔W	pr5 8 CB	← Conditional to Biconditional
10	T ∧ (V ↔W)	6 9 ADJ	← Adjunction
11		10 DD	

3.11 EG2 Let's try another:

S →(W ∧ (S →P)). V ∧ S. X ↔ (V ∧ W). (X ∨ Q) → (P→S). ~(P ↔S) ∨ (Z ∧R). ∴Z ∧ R

1	~~Show~~ Z ∧ R
2	S
3	W∧(S→P)
4	W
5	V
6	V∧W
7	(V∧W)→X
8	X
9	X∨Q
10	P→S
11	S→P
12	P↔S
13	~~(P↔S)
14	Z∧R
15	

S	PR2 S
W∧(S→P)	2 PR1 MP
S→P	3 S
P	2 4 MP
V	PR2 S
W	3 S
V∧W	6 7 ADJ
(V∧W)→X	PR3 BC
X	8 9 MP
X∨Q	10 ADD
P→S	11 PR4 MP
P↔S	4 12 CB
~~(P↔S)	13 DN
Z∧R	PR5 14 MTP DD

Of course the new rules also work with CD and ID.

3.11 EG3 Let's try this one:

$(P \vee W) \rightarrow \sim R. \quad \sim Q \vee S. \quad (S \rightarrow R) \wedge (S \leftrightarrow \sim Q). \quad \therefore (P \leftrightarrow Q)$

1	~~Show~~ $P \leftrightarrow Q$	Show conditional (so it can be used for CB)
2	~~Show~~ $P \rightarrow Q$	
3	P	ASS CD
4	$P \vee W$	3 ADD
5	$\sim R$	4 PR1 MP
6	$S \rightarrow R$	PR3 S
7	$\sim S$	5 6 MT
8	$S \leftrightarrow \sim Q$	PR3 S
9	$\sim Q \rightarrow S$	8 BC
10	$\sim\sim Q$	7 9 MT
11	Q	10 DN CD
12	~~Show~~ $Q \rightarrow P$	Show conditional
13	Q	ASS CD
14	~~Show~~ P	show cons. (consequent of 12)
15	$\sim P$	ASS ID
16	$\sim\sim Q$	13 DN
17	S	16 PR2 ADJ
18	$S \leftrightarrow \sim Q$	PR 3 S
19	$S \rightarrow \sim Q$	18 BC
20	$\sim Q$	17 19 MP 16 ID
21		14 CD
22	$P \leftrightarrow Q$	2 12 CB DD

You may have to try out a few different strategies. If one isn't working, don't panic, just try something else! In the above example, in showing '$P \rightarrow Q$', premise 2 could lead you astray:

$(P \vee W) \rightarrow \sim R. \quad \sim Q \vee S. \quad (S \rightarrow R) \wedge (S \leftrightarrow \sim Q). \quad \therefore (P \leftrightarrow Q)$

2	Show $P \rightarrow Q$	
3	P	Ass CD
4	$P \vee W$	3 ADD
5	$\sim R$	pr1 4 MP
6	$S \rightarrow R$	pr3 S
7	$\sim S$	5 6 MP
8	$\sim Q$	pr2 7 MTP **The negation of what you want … but DON'T PANIC!**
		Check if there is another way to get Q! (hint: use $S \leftrightarrow \sim Q$)

3.11 EG4 Let's try this one:

(P ∧ ~R) → T. (S ↔ T) ∧ ~(P ∧ S) . ~ P ∨ ~R ∴ Z ∨ (P → W)

1	Show Z ∨ (P → W)	
2	Show P → W	show disjunct (for ADD)
3	P	ASS CD
4	Show W	Show cons
5	~W	ASS ID
6	~~P	3 DN
7	~R	6 PR3 MTP
8	P∧~R	3 7 ADJ
9	T	8 PR1 MP
10	S↔T	PR2 S
11	T→S	10 BC
12	S	11 9 MP
13	P∧S	3 12 ADJ
14	~(P∧S)	PR2 SR 13 ID
15		4 CD
16	Z∨(P→W)	2 ADD DD
17		
18		
19		

Now we can derive the conclusion of any valid argument. Let's try the ones that we symbolized at the end of Unit 2:

3.11 E1

(a)

I realized, as I lay in bed thinking, that we are not responsible for what we do. This is because either determinism or indeterminism must be true. Provided that determinism is true, we cannot do other than we do. If so, we are but puppets on strings – our actions are not free. If indeterminism is true, then human actions are random, and hence not free. If our actions are not free, it must be conceded that we are not responsible for what we do.

P ∨ Q. P → ~R. ~R → ~S. Q → T. T → ~S. ~S → ~U ∴ ~ U

(b)

In our world, there are conscious experiences. Yet, there is a logically possible world physically identical to ours, and in that world there are no conscious experiences. If there are conscious experiences in our world, but not in a physically identical world, then facts about consciousness are further facts about our world, over and above the physical facts. If this is so, not all facts are physical facts. It follows, then, that materialism is false. For, in virtue of the meaning of materialism, materialism is true only if all facts are physical facts.

(Based on David Chalmers, *The Conscious Mind: In Search of a Fundamental Theory*. Oxford: Oxford University Press, 1996. 123-129.)

Illustration: Robert Fludd (1598-1637) Fludd was an English physician, mathematician, cosmologist. interested in occult philosophy.[1]

P. Q. $P \wedge Q \rightarrow R$. $R \rightarrow \sim S$. $T \rightarrow S$. $\therefore \sim T$

(c)

Next we must consider what virtue is. Since things that are found in the soul are of three kinds —passions, faculties, states of character — virtue must be one of these. We are not called good or bad on the ground of our passions, but are so called on the ground of our virtues. And if we are called good or bad on the grounds of the one, but not the other, then virtues cannot be passions. Likewise, virtues are faculties only if we are called good or bad on the grounds of our faculties as we are so called on the grounds of our virtues. If we have the faculties by nature (which we do) but we are not made good or bad by nature (which we are not) then we cannot be called good or bad on the grounds of our faculties. And since this shows that the virtues are neither passions nor faculties, all that remains is that they should be states of character.

(Based on Aristotle, *Nicomachean Ethics*, Book 1, Ch. V)

$P \vee Q \vee R$. $\sim S$. U. $U \wedge \sim S \rightarrow \sim P$. $Q \rightarrow T$. $V \wedge \sim W \rightarrow \sim T$. V. $\sim W$. $\therefore R$

[1] Illustration and information from Wikipedia article on Robert Fludd: http://en.wikipedia.org/wiki/Robert_Fludd

3.11 E2

Euathlus wanted to become a lawyer; however, the sophists who taught the art of rhetoric, demanded higher fees for their teaching than Euathlus wanted to pay. But the sophist, Protagoras, agreed to take Euathlus on as a student, agreeing that Euathlus would not have to pay him until he won his first case.

Protagoras taught Euathlus well, and soon he had mastered the art of rhetoric. Yet, years passed and, despite his great skill in argument, Euathlus refused to practice law.
He reasoned:

(a) I have to pay Protagoras only if I win my first case. Either I won't take any cases and will not have to pay, or Protagoras will sue me. Of course, if he does that then I will defend myself. But if I defend myself and win this case, then I won't have to pay him (by judgment of the court.) If I don't win then I won't have won my first case. Thus, I will not have to pay Protagoras.

Well, Euathlus refuses to practice law, and eventually Protagoras sues him. Euathlus presents his argument; but Protagoras argues to the court:

(b) Euathlus must pay me, since he has to pay me if he wins his first case. I am suing Protagoras and he is defending himself. And if he defends himself, then if he wins this case then he will have won his first case. But if I am suing Protagoras and Euathlus doesn't win, then he must pay me (by judgment of the court).

The court hears both sides of the argument and reasons:

(c) Euathlus has to pay Protagoras if and only if he wins his first case; and he wins his first case only if he wins this case. But if we make a ruling and Euathlus wins this case then he doesn't have to pay Protagoras (by our judgment). On the other hand, if Euathlus doesn't have to pay Protagoras if he doesn't win this case, then he does have to pay Protagoras if we make a ruling. If we do not make a ruling then we shall adjourn the court for an indefinite length of time. Therefore, we shall do exactly that.

Symbolize each of the three arguments (a, b and c) using the abbreviation scheme below. Show that the arguments are valid by deriving the conclusion from the premises.

P: Euathlus wins his first case.
Q: Euathlus wins this case (Protagoras' suit against him.)
R: Euathlus has to pay Protagoras.
S: Protagoras sues Euathlus.
T: Euathlus doesn't take any cases.
U: Euathlus defends himself.
V: The court makes a ruling.
W: The court adjourns for an indefinite length of time.

(d) Suppose the court had ruled in favor of Protagoras. Do you think Euathlus should appeal in this case? What argument should he give? (And of course, symbolize it and derive the conclusion!)

Based on: Auleus Gellius (~125AD – 180AD). *Attic Nights*, 5.10.

3.12 Theorems with $\wedge$, $\vee$ and $\leftrightarrow$

We can also use the new rules to derive theorems:

Theorems are sentences derivable from the empty set – from nothing. In the last unit, we looked at 23 numbered theorems – but of course there are an infinite number of sentences that can be derived from the empty set (and thus are theorems). In the exercises, you may have derived other theorems.

With the new symbols we have about 100 new numbered theorems. We will derive a few of the new theorems – but it is a good exercise to do more on your own. And once you have derived a theorem (proven it to be true), you are entitled to use it in any other derivation. That could make your effort pay off and save you time later!

3.12 EG1 Let's try one:

T27 $((P \wedge Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$

1	~~Show~~ $((P \wedge Q) \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$	Show conc
2	~~Show~~ $(P \wedge Q) \rightarrow R) \rightarrow (P \rightarrow (Q \rightarrow R))$	Show cond
3	$P \wedge Q \rightarrow R$	ass cd
4	~~Show~~ $P \rightarrow (Q \rightarrow R)$	Show cons
5	P	ass cd
6	~~Show~~ $Q \rightarrow R$	Show cons
7	Q	ass cd
8	$P \wedge Q$	5 7 adj
9	R	8 3 mp
10		9 cd
11		6 cd
12		4 cd
13	~~Show~~ $(P \rightarrow (Q \rightarrow R)) \rightarrow (P \wedge Q \rightarrow R)$	show cond
14	$P \rightarrow (Q \rightarrow R)$	ass cd
15	~~Show~~ $P \wedge Q \rightarrow R$	show cons
16	$P \wedge Q$	ass cd
17	P	16 s
18	$Q \rightarrow R$	14 17 mp
19	Q	16 s
20	R	18 19 mp
21		20 cd
22		15 cd
23	$(P \wedge Q \rightarrow R) \leftrightarrow (P \rightarrow (Q \rightarrow R))$	2 13 cb
24		23 dd

3.12 EG2 Let's try another one:

T 63 P ∧ Q ↔ ~(~P ∨ ~Q)

1	~~Show P~~ ∧ Q ↔ ~(~P ∨ ~Q)	show conc
2	~~Show~~ P∧Q → ~(~P∨~Q)	show cond
3	P∧Q	ass cd
4	~~Show~~ ~(~P∨~Q)	show cons
5	~P∨~Q	ass id
6	~~P	3 s dn
7	~Q	5 6 mtp
8	Q	3 s
9		7 8 id
10		4 cd
11	~~Show~~ ~(~P∨~Q) → P∧Q	show cond
12	~(~P∨~Q)	ass cd
13	~~Show~~ P	
14	~P	ass id
15	~P∨~Q	14 add
16	~(~P∨~Q)	12 r
17		15 16 id
18	~~Show~~ Q	
19	~Q	ass id
20	~P∨~Q	19 add
21	~(~P∨~Q)	12 r
22		20 21 id
23	P∧Q	13 18 adj
24		23 cd
25	P∧Q ↔ ~(~P∨~Q)	2 11 cb
26		25 dd

Using Theorems in Derivations:

If you can prove a theorem is true by deriving it without the use of premises, you can use the theorem in another derivation! Why not? Why should you do the work again? After all, one of the reasons for learning symbolic logic is to recognize argument forms – including tautologies. Suppose you have derived the theorem: $P \to (\sim P \to Q)$. Once you can derive this, you know that from any sentence, ψ, you can derive the conditional that has the negation of that sentence as the antecedent: $\sim\psi \to \phi$. If you've done it once, you can do it again.

Suppose, in a new derivation, you have derived 'S' but you need '$\sim S \to T$' to finish the proof. The same reasoning that got you $\sim P \to Q$ from P applies here. Of course you could just do it again… this time, deriving '$S \to (\sim S \to T)$'. But is this really necessary?

No! We can introduce a theorem *provided that we have previously proved it* and making the necessary alterations (replacing atomic sentences with other sentences consistently and when necessary.) This would be an 'instance' of the theorem, which follows the same pattern of reasoning. NOTE: you need to make sure you are replacing each occurrence of an atomic sentence with the same sentence… otherwise you would be changing the pattern of reasoning. For instance, in this case, to use the theorem $P \to (\sim P \to Q)$, we would have to replace P with S and Q with T.

Using a Theorem: We can introduce an instance of a theorem on any line of a derivation provided that we have already proved it with a valid derivation. The justification consists of the name/number of the theorem. Further annotation might include the substitution scheme…

$S \to (\sim S \to T)$. T17 S/P, T/Q. It's theorem 17; substituting S for P and T for Q.

format

Here's another theorem... T7 or commutation: $(P \to (Q \to R)) \to (Q \to (P \to R))$

This gives us the pattern: $(\phi \to (\psi \to \chi)) \to (\psi \to (\phi \to \chi))$

But ϕ, ψ, and χ can be atomic or molecular sentences, giving us such instances as:

$(S \to (T \to W)) \to (T \to (S \to W))$	T7	S/P; T/Q; W/R
$((S \to R) \to (W \to \sim X)) \to (W \to ((S \to R) \to \sim X))$	T7	$(S \to R)$/P; W/Q; $\sim X$/R

Let's give it a try:

$S \to (T \to W)$. T. $\therefore$ $S \to W$

1	~~Show~~ $S \to W$	
2	$(S \to (T \to W)) \to (T \to (S \to W))$	T7
3	$(T \to (S \to W))$	2 pr1 mp
4	$S \to W$	3 pr2 mp dd

MP + Theorems ~~DD&CD&ID~~ → many ...

Using Theorems as Rules

The system we are using is complete with just the three methods of derivation (DD, CD and ID) and one rule of derivation (MP). Thus, it is not surprising that many of the theorems are closely connected with the rules for derivation – we should be able to derive from the empty set any expression of a legitimate rule of inference.

So no MT, dn, ...

Let's look again some of the theorems from the last unit:

T1 $P \rightarrow P$

If we have ϕ we can use T1 & MP to infer ϕ. That sounds like Repetition.

T3 $P \rightarrow ((P \rightarrow Q) \rightarrow Q)$

If we have ϕ, then if we have $\phi \rightarrow \psi$, we can use T3 and MP to infer ψ. That's MP.

T11 $\sim\sim P \rightarrow P$

T12 $P \rightarrow \sim\sim P$

If we have $\sim\sim\phi$ or ϕ, we can use T11/T12 and MP to infer ϕ or $\sim\sim\phi$ respectively. That's DN.

T13 $(P \rightarrow Q) \rightarrow (\sim Q \rightarrow \sim P)$

If we have $\phi \rightarrow \psi$, then if we have $\sim\psi$, we can use T13 and MP to infer $\sim\phi$. That's MT.

If rules are basically theorems then we should be able to use theorems as rules… and we can!

Using Theorems as Rules

Let's look at one of our new theorems –

T61. $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$

This is 'distribution'. By deriving the theorem, we show that it is always valid to use it in our reasoning.

Let's work through this proof, putting in the justifications and thinking it through …

3.12 EG3

T61. $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$

1	~~Show~~ $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$	
2	~~Show~~ $P \wedge (Q \vee R) \rightarrow (P \wedge Q) \vee (P \wedge R)$	
3	$P \wedge (Q \vee R)$	ass cd
4	~~Show~~ $(P \wedge Q) \vee (P \wedge R)$	show cons
5	$\sim[(P \wedge Q) \vee (P \wedge R)]$	ass id
6	P	3 s
7	~~Show~~ $\sim R$	
8	R	ass id
9	$P \wedge R$	6 9 adj
10	$(P \wedge Q) \vee (P \wedge R)$	9 add
11	$\sim[(P \wedge Q) \vee (P \wedge R)]$	5 r 10 id
12	$Q \vee R$	3 s
13	Q	7 12 mtp
14	$P \wedge Q$	6 13 adj
15	$(P \wedge Q) \vee (P \wedge R)$	14 add
16	$\sim[(P \wedge Q) \vee (P \wedge R)]$	5 r 15 id
17		4 cd
18		

… now the other way on the next page

1	Show $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$	
2	~~Show~~ $P \wedge (Q \vee R) \rightarrow (P \wedge Q) \vee (P \wedge R)$	
…		
18	~~Show~~ $(P \wedge Q) \vee (P \wedge R) \rightarrow P \wedge (Q \vee R)$	
19	$(P \wedge Q) \vee (P \wedge R)$	ass cd
20	~~Show~~ $P \wedge (Q \vee R)$	show cons
21	$\sim(P \wedge (Q \vee R))$	ass id
22	~~Show~~ $\sim(P \wedge Q)$	
23	$P \wedge Q$	ass id
24	Q	23 s
25	$Q \vee R$	24 add
26	P	23 s
27	$P \wedge (Q \vee R)$	25 26 adj
28	$\sim(P \wedge (Q \vee R))$	21 r 27 id
29	$P \wedge R$	22 19 mtp
30	R	29 s
31	$Q \vee R$	30 add
32	P	29 s
33	$P \wedge (Q \vee R)$	31 32 adj
34	$\sim(P \wedge (Q \vee R))$	21 r 33 id
35		20 cd
35		2 18 cb dd

Now that we have proved the theorem once, we shouldn't have to do it again.
If only we had a rule of distribution that would save a lot of work. It might look like this:

My Rule of Distribution.

$$\frac{\phi \wedge (\psi \vee \chi)}{(\phi \wedge \psi) \vee (\phi \wedge \chi)} \qquad \frac{(\phi \wedge \psi) \vee (\phi \wedge \chi)}{\phi \wedge (\psi \vee \chi)}$$

Although distribution isn't explicitly a rule in our system, we *can* use the theorem as a rule.

Anytime you have derived a theorem which has as its main connective a conditional (or biconditional), it can be used as a rule that allows you derive the consequent of an instance of the conditional (or one side of the biconditional) from the antecedent of the same instance of the conditional (or other side of the biconditional.)

Using Theorems as Rules:

- Justification is line or premise number 'RT #'.
  '#' is the theorem number.
  RT# = Rule from Theorem #
- To use RT you must have previously proven that theorem.
- The theorem must be a conditional or biconditional.
- If the theorem is a conditional, from the antecedent of an instance of the theorem, you are entitled to derive the consequent of that instance.
- If the theorem is a biconditional, from one side of an instance of the theorem, you are entitled to derive the other side of that instance.
- If the theorem has two or more conjuncts in the antecedent (or on one side of the biconditional), the inference can be made directly from the distinct sentences/premises, rather than using ADJ to conjoin the sentences first.

Let's try it:

T61. $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$ We've proved it earlier, now we can use it!

$\sim(S \wedge T)$. $S \wedge (T \vee U)$. $\therefore U$

1	~~Show~~ U		First line is always the show line.
2	$(S \wedge T) \vee (S \wedge U)$	pr2 RT61	This is where we make use of the theorem. We use T61 as a rule that lets us derive a sentence of this form: $(\phi \wedge \psi) \vee (\phi \wedge \chi)$ from a sentence of this form: $\phi \wedge (\psi \vee \chi)$. That's what we are doing here – since this line and premise 2 are of those forms. The justification is the line or premise number followed by 'RT' and the theorem number.
3	$S \wedge U$	pr1 2 MTP	
4	U	3 S DD	

You will find that some theorems are particularly useful as rules…

Transposition

If you think about how modus tollens works, you can see why these hold. They allow you to transpose a conditional: taking the opposite of the antecedent and moving it to the consequent position; and taking the opposite of the consequent and moving it to the antecedent position.

T13	$(P \to Q) \to (\sim Q \to \sim P)$	Transposition
T14	$(P \to \sim Q) \to (Q \to \sim P)$	Transposition
T15	$(\sim P \to Q) \to (\sim Q \to P)$	Transposition
T16	$(\sim P \to \sim Q) \to (Q \to P)$	Transposition

Here's the proof for T13. You can see how it relates to MT. Try the rest on your own!

1	~~Show~~ $(P \to Q) \to (\sim Q \to \sim P)$		T13 is our goal.
2	$P \to Q$	ASS CD	Assume the antecedent of T13.
3	~~Show~~ $\sim Q \to \sim P$		Our new goal – the consequent.
4	$\sim Q$	ASS CD	Assume the antecedent of line 3.
5	$\sim P$	2 4 MT	Our new goal, the consequent, is easy to derive!
6		5 CD	
7		2 CD	

Laws of Commutation for $\wedge$ and $\vee$:

T24	$P \wedge Q \leftrightarrow Q \wedge P$	Commutation (conjunction)
T53	$P \vee Q \leftrightarrow Q \vee P$	Commutation (disjunction)

Laws of Association for $\wedge$ and $\vee$:

T25	$P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$	Association (conjunction)
T54	$P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$	Association (disjunction)

Exportation:

T27	$((P \wedge Q) \to R) \leftrightarrow (P \to (Q \to R))$	Exportation.

Hypothetical Syllogism:

T26	$[(P \to Q) \wedge (Q \to R)] \to (P \to R)$	Hypothetical Syllogism

Hypothetical syllogism is a very useful theorem but it seems a little awkward to use as a rule because you have to use ADJ to join the two conditionals.

$(U \vee R) \to Z$. $V \to (U \vee R)$. $\therefore V \to Z$.

1	~~Show~~ $V \to Z$	
2	$(V \to (U \vee R)) \wedge ((U \vee R) \to Z)$	pr1 pr2 ADJ — Do we really need this line?
3	$V \to Z$	2 RT26 DD

We don't really need line 2. Line 3 follows from the two premises if we use T26 as a rule. We can do it like this:

$(U \vee R) \to Z$. $V \to (U \vee R)$. $\therefore V \to Z$.

1	~~Show~~ $V \to Z$	
3	$V \to Z$	pr1 pr2 RT26 DD

So can also use theorems as rules in which the antecedent is a conjunction if we cite in our justifications an available line for each of the conjuncts.

USING THEOREMS AS RULES:

You can use any theorem as a rule provided you have proved it first.
A theorem validates the related rule. For example, T13 validates RT13.

- The theorem must be a conditional or a biconditional.
- The justification for the related rule consists of the line or premise numbers and "RT#".
- If the theorem has a conjunction in the antecedent or on one side of the biconditional, the theorem validates a rule that may have multiple premises; otherwise the theorem validates a rule with a single premise.

3.12 E1 Try the following using the suggested theorems as rules

T27	$(P \wedge Q \to R) \leftrightarrow (P \to (Q \to R))$
T29	$(P \to Q \wedge R) \leftrightarrow (P \to Q) \wedge (P \to R)$
T24	$P \wedge Q \leftrightarrow Q \wedge P$
T61	$P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$
T62	$P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$

a) $(S \wedge T) \to (V \wedge W)$. $\therefore S \to (T \to V)$

b) $\sim (T \wedge U)$. $(S \vee U) \wedge (S \vee T)$. $\therefore S$

c) $R \wedge (S \vee T)$. $R \to (T \to U)$. $\therefore \sim(R \wedge S) \to U$

3.13 Derived Rules

We have already looked at lot of theorems. At the end of this unit, there is a list of over a hundred numbered theorems. That's a lot of theorems to keep track of. It is customary to take a few of the theorems that are most useful as rules and give them special names – these are what we call 'derived rules'.

The derived rules that we will be using are based on the following theorems. It is a good idea to derive each of these theorems so that you are justified in using the derived rules!

3.13 E1 Provide a derivation that shows that the following theorems are valid:

T40	$\sim(P \rightarrow Q) \leftrightarrow (P \wedge \sim Q)$	Negation of Conditional
T45	$P \vee Q \leftrightarrow (\sim P \rightarrow Q)$	Conditional as Disjunction
T46	$(P \rightarrow Q) \leftrightarrow \sim P \vee Q$	Conditional as Disjunction
T33	$(P \rightarrow Q) \wedge (\sim P \rightarrow Q) \rightarrow Q$	Separation of Cases
T49	$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$	Separation of Cases
T63	$P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$	De Morgan's
T64	$P \vee Q \leftrightarrow \sim(\sim P \wedge \sim Q)$	De Morgan's
T65	$\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$	De Morgan's
T66	$\sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$	De Morgan's
T90	$\sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$	Negation of Biconditional

Negation of Conditional

The first is Negation of Conditional (NC or nc) which is T40 used as a rule.

T40 $\sim(P \rightarrow Q) \leftrightarrow (P \wedge \sim Q)$ Negation of conditional.

This theorem is so useful as a rule that we can cite it by rule name rather than RT40. It makes sense, since a conditional is false only if the antecedent is true and the consequent is false.

Negation of Conditional
NC or nc

$$\frac{\sim(\phi \rightarrow \psi)}{\phi \wedge \sim\psi} \qquad\qquad \frac{\phi \wedge \sim\psi}{\sim(\phi \rightarrow \psi)}$$

Let's see it in action: $(S \to R) \to T$. $\sim T$. $\therefore \sim R$

1	~~Show~~ $\sim R$		Our goal sentence.
2	$\sim (S \to R)$	pr1 pr2 MT	We derive the negation of a conjunction.
3	$S \wedge \sim R$	2 NC	Now we can turn it into a conjunction.
4	$\sim R$	3 S DD	

It is particularly useful when you need to derive a conditional but don't see how to do it. If you do an indirect derivation, assuming the negation of the conditional, you can use NC to turn it into a conjunction. The result is two simpler sentential components that are easier to work with.

Without NC, you might need an indirect subderivation inside a conditional derivation to prove that this argument is valid:

$$\begin{array}{l} P \to Q \\ P \vee R \\ R \to P \\ \hline T \to Q \end{array}$$

It is quite straightforward to prove using NC.

$P \to Q$. $P \vee R$. $R \to P$. $\therefore T \to Q$.

1	~~Show~~ $T \to Q$		
2	$\sim(T \to Q)$	ASS ID	
3	$T \wedge \sim Q$	2 NC	Using NC on line 2 gives us two simpler components. It is now easy to derive a contradiction.
4	$\sim Q$	3 S	
5	$\sim P$	pr1 4 MT	
6	R	pr2 5 MTP	
7	P	pr3 6 MP	
8		5 7 ID	

Conditional as Disjunction

The second derived rule is based on the equivalence between conditionals and disjunctions, expressed in theorems T45 and T46. It allows us to derive one sentence from the equivalent one**.** It makes sense, especially if you think about the different ways we symbolized 'unless'.

T45 $P \vee Q \leftrightarrow (\sim P \rightarrow Q)$

T46 $(P \rightarrow Q) \leftrightarrow \sim P \vee Q$

Conditional as disjunction

CDJ or cdj

$\dfrac{\phi \rightarrow \psi}{\sim\phi \vee \psi}$	$\dfrac{\sim\phi \vee \psi}{\phi \rightarrow \psi}$
$\dfrac{\sim\phi \rightarrow \psi}{\phi \vee \psi}$	$\dfrac{\phi \vee \psi}{\sim\phi \rightarrow \psi}$

You might want to use this rule if you prefer conditional sentences to disjunctions. If you have a disjunction, but don't know how to make use of it, change it into a conditional! Often it is easier to see what to do with it when it is in the form of a conditional.

Or, suppose you are trying to derive a disjunction, but the two disjuncts are tied up with one another (so that you can't derive either disjunct by itself.) It might be easy to see the solution if you do a conditional derivation: assume the negation of one disjunct and derive the other disjunct. Then, once you have derived the conditional, you can turn it back into the desired disjunction using CDJ.

Let's try it …

$(R \rightarrow S) \rightarrow P$. $R \leftrightarrow T$. $(T \wedge \sim S) \rightarrow Q$. $\therefore P \vee Q$

Derivation tip:

It can help to think a derivation through from the bottom up to create a rough sketch of the proof.

1	Show $P \vee Q$		We want to show a disjunction.
2			
	somewhere further down the page		
	$P \vee Q$	2 CDJ	We know we want the last line to be achieved through CDJ. Thus, we need to derive the equivalent conditional in between.

Now we fill in a bit more …

1	Show $P \vee Q$		The equivalent conditional is $\sim P \rightarrow Q$ That's our new goal.
2	Show $\sim P \rightarrow Q$		We state the new goal. It's a conditional, so we want a conditional derivation.
	$\sim P$	ASS CD	Assume the antecedent of the new show line.
	somewhere further down the page		
	Q		We want to derive the consequent.
	$P \vee Q$	2 CDJ	

Now we have a rough sketch of the outline of the derivation. It is just a matter of filling in the missing lines.

$(R \rightarrow S) \rightarrow P$. $R \leftrightarrow T$. $(T \wedge \sim S) \rightarrow Q$. $\therefore P \vee Q$

1	~~Show~~ $P \vee Q$		The main goal is a disjunction. It's not going to be easy to derive P by itself or Q by itself. By deriving the equivalent conditional, we can use CDJ to turn it back into the disjunction we want.
2	~~Show~~ $\sim P \rightarrow Q$		We want to show the conditional equivalent to the disjunction which is the goal. The negation of one disjunct is the antecedent of the new show sentence, and the other disjunct is the consequent.
3	$\sim P$	ASS CD	
4	$\sim(R \rightarrow S)$	3 pr1 MT	
5	$R \wedge \sim S$	4 NC	
6	R	5 S	
7	$R \rightarrow T$	pr2 BC	
8	T	6 7 MP	
9	$\sim S$	5 S	
10	$T \wedge \sim S$	8 9 ADJ	
11	Q	10 pr3 MP CD	
12	$P \vee Q$	2 CDJ DD	

Separation of Cases:

The third derived rule comes from theorems T33 and T49. It allows you to reason from a disjunction to another sentence on the grounds that both disjuncts logically imply that sentence.

T33 $(P \rightarrow Q) \wedge (\sim P \rightarrow Q) \rightarrow Q$

T49 $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \rightarrow R$

This makes sense: if there are two roads and you have to take one or the other, and both roads lead to Rome, then you will end up in Rome!

Separation of Cases
SC or sc

$\phi \vee \psi$	
$\phi \rightarrow \chi$	$\phi \rightarrow \chi$
$\psi \rightarrow \chi$	$\sim\phi \rightarrow \chi$
χ	χ

The form on the right is a variant which in its full form looks like this:

$$\frac{\begin{array}{l}\phi \vee \sim\phi \\ \phi \rightarrow \chi \\ \sim\phi \rightarrow \chi\end{array}}{\chi}$$

This is clearly an instance of the form on the left ($\sim\phi = \psi$). The first line is T59 $(P \vee \sim P)$ and thus we don't have to include it. After all, every sentence is true or false. Thus, if another sentence follows from both a sentence and it's negation, then that further sentence must be true!

Note that the justification for this will often require citation of three lines or premises.

This is a very useful rule when you have an available disjunction and your other attempts at deriving your conclusion aren't working. You can try to derive the conclusion from each disjunct separately.

It may not seem obvious how else to prove that this argument is valid:

$P \vee Q$
$P \rightarrow Z$
$\sim W \rightarrow \sim Q$
$\therefore\ W \vee Z$

$P \vee Q.\quad P \rightarrow Z.\quad \sim W \rightarrow \sim Q.\quad \therefore\ W \vee Z$

Again, a rough sketch makes the argument clearer.

1	Show W ∨ Z		We want to show this follows from both of the disjuncts in the first premise: P ∨ Q
2	Show P → (W ∨ Z)		
	somewhere further down the page		
	Show Q → (W ∨ Z)		
	somewhere further down the page		
	W ∨ Z	pr1 # # SC DD	We know the last step is SC.

Now we fill in the details:

P ∨ Q.   P → Z.   ~W → ~Q.   ∴ W ∨ Z

1	~~Show~~ W ∨ Z	
2	~~Show~~ P → (W ∨ Z)	
3	P	ASS CD
4	Z	3 pr2 MP
5	W ∨ Z	4 ADD CD
6	~~Show~~ Q → (W ∨ Z)	
7	Q	ASS CD
8	~~Q	7 DN
9	~~W	8 pr3 MT
10	W	9 DN
11	W ∨ Z	10 ADD CD
12	W ∨ Z	pr1 2 6 SC DD

De Morgan's Law

De Morgan's is a derived rule that is based on T63-T66. One of the most useful derived rule!

T63	$P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$	T64	$P \vee Q \leftrightarrow \sim(\sim P \wedge \sim Q)$
T65	$\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$	T66	$\sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$

De Morgan's has eight forms. It sounds daunting but they are based on two similar equivalences that should be familiar from symbolizations:

- The negation of a disjunction is equivalent to the conjunction of the negations of its parts.
  'neither P nor Q' is equivalent to 'both not P and not Q'
- The negation of a conjunction is equivalent to the disjunction of the negations of its parts.
  'not both P and Q' is equivalent to 'either not P or not Q'.

De Morgan's dm or DM			
$\sim(\phi \vee \psi)$ ——— $\sim\phi \wedge \sim\psi$	$\sim(\sim\phi \vee \sim\psi)$ ——— $\phi \wedge \psi$	$\sim(\phi \wedge \psi)$ ——— $\sim\phi \vee \sim\psi$	$\sim(\sim\phi \wedge \sim\psi)$ ——— $\phi \vee \psi$
$\sim\phi \wedge \sim\psi$ ——— $\sim(\phi \vee \psi)$	$\phi \wedge \psi$ ——— $\sim(\sim\phi \vee \sim\psi)$	$\sim\phi \vee \sim\psi$ ——— $\sim(\phi \wedge \psi)$	$\phi \vee \psi$ ——— $\sim(\sim\phi \wedge \sim\psi)$

De Morgan's Law is one of the most useful of our derived rules. Use it to transform sentences into forms that you can use more easily with other available lines. Use it to derive a disjunction – begin an indirect derivation by assuming the negation of the disjunction, then use De Morgan's Law to transform it into a conjunction of negations and then you have useful sentential components.

$R \rightarrow S.\quad R \vee T.\quad T \rightarrow W.\quad \therefore S \vee W$

Augustus De Morgan

Augustus De Morgan (1806-1871)

1	~~Show~~ $S \vee W$	
2	$\sim(S \vee W)$	ASS ID
3	$\sim S \wedge \sim W$	2 DM
4	$\sim S$	3 S
5	$\sim R$	pr1 4 MT
6	$\sim W$	3 S
7	$\sim T$	pr3 6 MT
8	R	pr2 7 MTP 5 ID

Negation of Biconditional

The last derived rule is the negation of a biconditional, based on theorem 90:

T90: $\sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$

This makes sense, especially if you think of the biconditional as equivalence. If it is not the case that two sentences are equivalent (have the same truth-value); then one sentence is true if and only if the other sentence is false. Or, consider the different ways to symbolize one or the other but not both.

Negation of Biconditional
nb or NB

$$\frac{\sim(\phi \leftrightarrow \psi)}{\phi \leftrightarrow \sim\psi} \qquad \frac{\phi \leftrightarrow \sim\psi}{\sim(\phi \leftrightarrow \psi)}$$

If you ever derive or need to derive the negation of a biconditional, this rule is very useful.

$P \rightarrow R.$ $Q \rightarrow \sim R.$ $Q \vee P.$ $\therefore \sim(P \leftrightarrow Q)$

One could just try to derive this through an indirect derivation, but it is faster to prove $P \leftrightarrow \sim Q$ and use NB to convert it to the equivalent sentence that we want.

1	~~Show~~ $\sim(P \leftrightarrow Q)$		
2	$R \rightarrow \sim Q$	pr2 RT14	Transpose the second premise.
3	$P \rightarrow \sim Q$	pr1 2 T26	This is hypothetical syllogism.
4	$\sim Q \rightarrow P$	pr3 CDJ	
5	$P \leftrightarrow \sim Q$	3 4 CB	
6	$\sim(P \leftrightarrow Q)$	5 NB DD	

You can also construct your own rules…

Prove a theorem with a conditional or biconditional as the main connective. Use it as a rule.

These are the derived rules we will be using:

<table>
<tr><th>Derived rule</th><th colspan="2"></th><th colspan="2"></th><th>Derived from theorems…</th></tr>
<tr><td>Negation of Conditional
nc or NC</td><td colspan="2">$\dfrac{\sim(\phi \to \psi)}{\phi \land \sim\psi}$</td><td colspan="2">$\dfrac{\phi \land \sim\psi}{\sim(\phi \to \psi)}$</td><td>T40</td></tr>
<tr><td rowspan="2">Conditional as Disjunction
cdj or CDJ</td><td colspan="2">$\dfrac{\phi \to \psi}{\sim\phi \lor \psi}$</td><td colspan="2">$\dfrac{\sim\phi \lor \psi}{\phi \to \psi}$</td><td rowspan="2">T45, T46</td></tr>
<tr><td colspan="2">$\dfrac{\sim\phi \to \psi}{\phi \lor \psi}$</td><td colspan="2">$\dfrac{\phi \lor \psi}{\sim\phi \to \psi}$</td></tr>
<tr><td>Separation of Cases
sc or SC</td><td colspan="2">$\dfrac{\begin{array}{l}\phi \lor \psi \\ \phi \to \chi \\ \psi \to \chi\end{array}}{\chi}$</td><td colspan="2">$\dfrac{\begin{array}{l}\phi \to \chi \\ \sim\phi \to \chi\end{array}}{\chi}$</td><td>T33, T49</td></tr>
<tr><td rowspan="2">De Morgan's
dm or DM</td><td>$\dfrac{\sim(\phi \lor \psi)}{\sim\phi \land \sim\psi}$</td><td>$\dfrac{\sim(\sim\phi \lor \sim\psi)}{\phi \land \psi}$</td><td>$\dfrac{\sim(\phi \land \psi)}{\sim\phi \lor \sim\psi}$</td><td>$\dfrac{\sim(\sim\phi \land \sim\psi)}{\phi \lor \psi}$</td><td rowspan="2">T63-66</td></tr>
<tr><td>$\dfrac{\sim\phi \land \sim\psi}{\sim(\phi \lor \psi)}$</td><td>$\dfrac{\phi \land \psi}{\sim(\sim\phi \lor \sim\psi)}$</td><td>$\dfrac{\sim\phi \lor \sim\psi}{\sim(\phi \land \psi)}$</td><td>$\dfrac{\phi \lor \psi}{\sim(\sim\phi \land \sim\psi)}$</td></tr>
<tr><td>Negation of Biconditional
nb or NB</td><td colspan="2">$\dfrac{\sim(\phi \leftrightarrow \psi)}{\phi \leftrightarrow \sim\psi}$</td><td colspan="2">$\dfrac{\phi \leftrightarrow \sim\psi}{\sim(\phi \leftrightarrow \psi)}$</td><td>T90</td></tr>
</table>

3.13 E2 Use the derived rules in the following derivations.

(a) ~(P ∨ Q). R → Q. ~(~R ↔ S). ∴~S

(b) W → ~(S ∧ T). ~S → ~Z. ~T → ~Z. ∴ (~W ∨ ~Z)

(c) ~(R → S). ~(T ∨ W ↔ R). V → T. ~V → (S ∨ W). ∴P

(a). 1 ~~Show~~ ~S show conc
2 ~P∧~Q pr1 dm
3 ~P 2 sl
4 ~Q 2 sr
5 ~R 4 pr2 mt
6 ~R↔~S pr3 nb
7 ~R→~S 6 bc
8 ~S 5 7 mp
9 8 dd

(b). 1. ~~Show~~ ~W V~Z show conc
2. ~(~WV~Z) ass id
3. W∧Z 2 dm
4. W 3 sl
5. ~(S∧T) 4 pr1 mp
6. ~SV~T 5 dm
7. ~~Z 3 sr dn
8. ~~T 7 pr3 mt
9. ~S 8 6 mtp
10. ~Z pr2 9 mp
11 7 10 id

1. ~~Show~~ P show conc
2. ~P ass id
3. R∧~S pr1 nc
4. R 3 sl
5. ~S 3 sr
6. TVW↔~R pr2 nb
7. TVW→~R 6 bc
8. ~(TVW) 4 dn 7 mt
9. ~T∧~W 8 dm
10. ~T 9 sl
11. ~W 9 sr
12. ~S∧~W 5 11 adj
13. ~(SVW) 12 dm
14. ~~V 13 pr4 mt
15. T 14 dn pr3 mp
16. 10 15 id

3.14 Another Shortcut: Abbreviated Derivations

Often it is handy to be able to combine two or even more steps into one.

$R \wedge (P \rightarrow \sim R)$. $\sim\sim P \vee Q$. $\therefore Q$

1	~~Show~~ Q		
2	R	pr1 S	
3	$P \rightarrow \sim R$	pr1 S	
4	$\sim\sim R$	2 DN	We need to use DN with line 2 before we can use MT with line 3.
5	$\sim P$	3 4 MT	
6	$\sim\sim\sim P$	5 DN	We need to use DN with line 5 before we can use MTP with pr3.
7	Q	pr3 6 MTP DD	

We could shorten the proof a little by combining the DN steps with other steps.

1	~~Show~~ Q		
2	R	pr1 S	
3	$P \rightarrow \sim R$	pr1 S	
4	$\sim P$	2 DN 3 MT	This means use DN on 2 and then use the result with line 3 to derive ~P using MT.
5	Q	4 DN pr3 MTP DD	This means use DN on line 4 and then use the result with pr3 to derive Q using MTP.

And we can shorten it even more.

$R \wedge (P \rightarrow \sim R)$. $\sim\sim P \vee Q$. $\therefore Q$

1	~~Show~~ Q		
2	$\sim P$	pr1 SL DN pr1 SR MT	This means use S one pr1 then DN and use the result with the result of using S on pr1 to derive ~P through MT. NOTE: I now use the justifications SL (simplification to the left conjunct) and SR (simplification to the right conjunct). Now the justification is unambiguous.
3	Q	2 DN pr3 MTP DD	This means use DN on line 4 and then use the result with premise 3 to derive Q using MTP.

Caution: In using abbreviated derivations you need to be careful that your proof is still interpretable!

Keep your abbreviated derivations clear by combining simple rules like DN, S, ADD and BC that act on a single sentence with more complex rules that act on two (or three) sentences.

Use SL and SR in abbreviated steps, rather than S, making it clear which conjunct you are working with.

You 'read' the abbreviated justification as follows:

> Start at the left and move towards the right.
>
> The line or premise number gives you the sentence on that line.
>
> The rule gives you the sentence that follows by the rule from the line cited.
>
> A sentence followed by 'DN', 'S', 'ADD' and 'BC' gives you the result of applying that rule to that sentence. (You need to use your judgment to decipher the resulting sentence.)
>
> Two sentences followed by 'MP', 'MT', 'ADJ' or 'CB' gives you the result of applying that rule to those sentences.

Some common abbreviated steps:

1.	$P \wedge Q$	
2.	$P \rightarrow S$	
3.	S	1 SL 2 MP

1.	$P \vee Q \rightarrow R$	
2.	P	
3.	R	2 ADD 1 MP

1.	$\sim P \rightarrow \sim Q$	
2.	Q	
3.	P	2 DN 1 MT DN

1.	P	
2.	$P \rightarrow Q$	
3.	$Q \rightarrow R$	
4.	R	1 2 MP 3 MP

3.14 E1

Try the following derivations using derived rules, abbreviated proofs and/or theorems as rules:

a) $V \wedge W \rightarrow Y$. $\sim W \rightarrow Z$. $(Y \vee Z) \rightarrow \sim U$. $\therefore U \rightarrow \sim V$

b) $P \vee Q$. $Q \rightarrow S$. $U \vee \sim S$. $P \vee S \rightarrow R$. $R \rightarrow U$. $\therefore U$

c) $R \wedge (S \vee T)$. $R \rightarrow (T \rightarrow U)$. $\therefore \sim(R \wedge S) \rightarrow U$

3.15 The Complete Derivation System for Sentential Logic

The basic rules for conditional and negation:

Modus Ponens (MP or mp) $(\phi \to \psi)$ ϕ ——— ψ	Modus Tollens (MT or mt) $(\phi \to \psi)$ $\sim\psi$ ——— $\sim\phi$
Double Negation (DN or dn) ϕ / ——— / $\sim\sim\phi$ $\sim\sim\phi$ / ——— / ϕ	Repetition (R or r) ϕ ——— ϕ

The basic rules for conjunction, disjunction and biconditional.

Simplification (S/SL/SR or s/sl/sr) $\phi \wedge \psi$ / ——— / ϕ $\phi \wedge \psi$ / ——— / ψ	Adjunction (ADJ or adj) ϕ ψ ——— $\phi \wedge \psi$
Addition (ADD or add) ϕ / ——— / $\phi \vee \psi$ ψ / ——— / $\phi \vee \psi$	Modus Tollendo Ponens (MTP or mtp) $\phi \vee \psi$, $\sim\phi$ / ——— / ψ $\phi \vee \psi$, $\sim\psi$ / ——— / ϕ
Biconditional-Conditional (BC or bc) $\phi \leftrightarrow \psi$ / ——— / $\phi \to \psi$ $\phi \leftrightarrow \psi$ / ——— / $\psi \to \phi$	Conditional-Biconditional (CB or cb) $\phi \to \psi$ $\psi \to \phi$ ——— $\phi \leftrightarrow \psi$

The derived rules:

Negation of Conditional nc or NC	$\dfrac{\sim(\phi \to \psi)}{\phi \wedge \sim\psi}$		$\dfrac{\phi \wedge \sim\psi}{\sim(\phi \to \psi)}$	
Conditional as Disjunction cdj or CDJ	$\dfrac{\phi \to \psi}{\sim\phi \vee \psi}$		$\dfrac{\sim\phi \vee \psi}{\phi \to \psi}$	
	$\dfrac{\sim\phi \to \psi}{\phi \vee \psi}$		$\dfrac{\phi \vee \psi}{\sim\phi \to \psi}$	
Separation of Cases sc or SC	$\begin{array}{l} \phi \vee \psi \\ \phi \to \chi \\ \psi \to \chi \\ \hline \chi \end{array}$		$\begin{array}{l} \phi \to \chi \\ \sim\phi \to \chi \\ \hline \chi \end{array}$	
De Morgan's dm or DM	$\dfrac{\sim(\phi \vee \psi)}{\sim\phi \wedge \sim\psi}$	$\dfrac{\sim(\sim\phi \vee \sim\psi)}{\phi \wedge \psi}$	$\dfrac{\sim(\phi \wedge \psi)}{\sim\phi \vee \sim\psi}$	$\dfrac{\sim(\sim\phi \wedge \sim\psi)}{\phi \vee \psi}$
	$\dfrac{\sim\phi \wedge \sim\psi}{\sim(\phi \vee \psi)}$	$\dfrac{\phi \wedge \psi}{\sim(\sim\phi \vee \sim\psi)}$	$\dfrac{\sim\phi \vee \sim\psi}{\sim(\phi \wedge \psi)}$	$\dfrac{\phi \vee \psi}{\sim(\sim\phi \wedge \sim\psi)}$
Negation of Biconditional nb or NB	$\dfrac{\sim(\phi \leftrightarrow \psi)}{\phi \leftrightarrow \sim\psi}$		$\dfrac{\phi \leftrightarrow \sim\psi}{\sim(\phi \leftrightarrow \psi)}$	

Theorems as Rules:

A theorem of conditional form: $\therefore\ \phi \wedge \psi \to \chi$ Justifies a rule of the form: $\begin{array}{l} \phi \\ \psi \\ \hline \therefore\ \chi \end{array}$	A theorem of biconditional form $\therefore \phi \leftrightarrow \psi$ Justifies rules of the form: $\dfrac{\phi}{\therefore\ \psi}$ $\quad$ $\dfrac{\psi}{\therefore\ \phi}$

And here are all the theorems…[2]

T1 $P \to P$

T2 $Q \to (P \to Q)$

T3 $P \to ((P \to Q) \to Q)$

T4 $(P \to Q) \to ((Q \to R) \to (P \to R))$

T5 $(Q \to R) \to ((P \to Q) \to (P \to R))$

T6 $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

T7 $((P \to Q) \to (P \to R)) \to (P \to (Q \to R))$

T8 $P \to (Q \to R)) \to (Q \to (P \to R))$

T9 $(P \to (P \to Q)) \to (P \to Q)$

T10 $((P \to Q) \to Q) \to ((Q \to P) \to P)$

T11 $\sim\sim P \to P$

T12 $P \to \sim\sim P$

T13 $(P \to Q) \to (\sim Q \to \sim P)$

T14 $(P \to \sim Q) \to (Q \to \sim P)$

T15 $(\sim P \to Q) \to (\sim Q \to P)$

T16 $(\sim P \to \sim Q) \to (Q \to P)$

T17 $P \to (\sim P \to Q)$

T18 $\sim P \to (P \to Q)$

T19 $(\sim P \to P) \to P$

T20 $(P \to \sim P) \to \sim P$

T21 $\sim(P \to Q) \to P$

T22 $\sim(P \to Q) \to \sim Q$

T23 $((P \to Q) \to P) \to P$

T24 $P \wedge Q \leftrightarrow Q \wedge P$

T25 $P \wedge (Q \wedge R) \leftrightarrow (P \wedge Q) \wedge R$

T26 $(P \to Q) \wedge (Q \to R) \to (P \to R)$

T27 $(P \wedge Q \to R) \leftrightarrow (P \to (Q \to R))$

T28 $(P \wedge Q \to R) \leftrightarrow (P \wedge \sim R \to \sim Q)$

THEOREMS

Theorems are general patterns of reasoning, just like rules of inference are. Indeed, any theorem that has a conditional or biconditional as the main connective can be used as a rule of inference (3.12)

Think of P, Q and R as general terms that stand for any three sentences – molecular or atomic.

For instance, consider theorem 13.

$(P \to Q) \to (\sim Q \to \sim P)$

If we replace "P" with "R" and "Q" with "$T \vee S$" we would get the following sentence:

$(R \to T \vee S) \to (\sim(T \vee S) \to \sim R)$

This sentence is also a theorem.

Since these theorems do express regular patterns of reasoning, many of them have traditional names.

T4, T5: Hypothetical Syllogism

T11, T12: Double Negation

T13, T14, T15, T16: Transposition

T24, 53: commutation

T25, 54: association

[2] The number of possible theorems is infinite. Thus, the numbering is arbitrary. This list of theorems is from: Kalish, Montague, and Mar. *Logic: Techniques of Formal Reasoning*. Oxford: 1980. Pg. 107

T29 $(P \to Q \wedge R) \leftrightarrow (P \to Q) \wedge (P \to R)$

T30 $(P \to Q) \to (R \wedge P \to R \wedge Q)$

T31 $(P \to Q) \to (P \wedge R \to Q \wedge R)$

T32 $(P \to R) \wedge (Q \to S) \to (P \wedge Q \to R \wedge S)$

T33 $(P \to Q) \wedge (\sim P \to Q) \to Q$

T34 $(P \to Q) \wedge (P \to \sim Q) \to \sim P$

T35 $(\sim P \to R) \wedge (Q \to R) \leftrightarrow ((P \to Q) \to R)$

T36 $\sim(P \wedge \sim P)$

T37 $(P \to Q) \leftrightarrow \sim(P \wedge \sim Q)$

T38 $P \wedge Q \leftrightarrow \sim(P \to \sim Q)$

T39 $\sim(P \wedge Q) \leftrightarrow (P \to \sim Q)$

T40 $\sim(P \to Q) \leftrightarrow P \wedge \sim Q$

T41 $P \leftrightarrow P \wedge P$

T42 $P \wedge \sim Q \to \sim(P \to Q)$

T43 $\sim P \to \sim(P \wedge Q)$

T44 $\sim Q \to \sim(P \wedge Q)$

T45 $P \vee Q \leftrightarrow (\sim P \to Q)$

T46 $(P \to Q) \leftrightarrow \sim P \vee Q$

T47 $P \leftrightarrow P \vee P$

T48 $(P \vee Q) \wedge (P \to R) \wedge (Q \to S) \to R \vee S$

T49 $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \to R$

T50 $(P \to R) \wedge (Q \to R) \leftrightarrow (P \vee Q \to R)$

T51 $(P \vee Q) \wedge (P \to R) \wedge (\sim P \wedge Q \to R) \to R$

T52 $(P \to R) \wedge (\sim P \wedge Q \to R) \leftrightarrow (P \vee Q \to R)$

T53 $P \vee Q \leftrightarrow Q \vee P$

T54 $P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$

T55 $(P \to Q \vee R) \leftrightarrow (P \to Q) \vee (P \to R)$

T56 $(P \to Q) \to (R \vee P \to R \vee Q)$

T57 $(P \to Q) \to (P \vee R \to Q \vee R)$

T58 $(P \to Q) \vee (Q \to R)$

T59 $P \vee \sim P$

T60 $(P \to R) \vee (Q \to R) \leftrightarrow (P \wedge Q \to R)$

T61 $P \wedge (Q \vee R) \leftrightarrow (P \wedge Q) \vee (P \wedge R)$

T62 $P \vee (Q \wedge R) \leftrightarrow (P \vee Q) \wedge (P \vee R)$

T63 $P \wedge Q \leftrightarrow \sim(\sim P \vee \sim Q)$

T64 $P \vee Q \leftrightarrow \sim(\sim P \wedge \sim Q)$

T65 $\sim(P \wedge Q) \leftrightarrow \sim P \vee \sim Q$

T66 $\sim(P \vee Q) \leftrightarrow \sim P \wedge \sim Q$

T67 $\sim P \wedge \sim Q \rightarrow \sim(P \vee Q)$

T68 $P \leftrightarrow (P \wedge Q) \vee (P \wedge \sim Q)$

T69 $P \leftrightarrow (P \vee Q) \wedge (P \vee \sim Q)$

T70 $Q \rightarrow (P \wedge Q \leftrightarrow P)$

T71 $\sim Q \rightarrow (P \vee Q \leftrightarrow P)$

T72 $(P \rightarrow Q) \leftrightarrow (P \wedge Q \leftrightarrow P)$

T73 $(P \rightarrow Q) \leftrightarrow (P \vee Q \leftrightarrow Q)$

T74 $(P \leftrightarrow Q) \wedge P \rightarrow Q$

T75 $(P \leftrightarrow Q) \wedge Q \rightarrow P$

T76 $(P \leftrightarrow Q) \wedge \sim P \rightarrow \sim Q$

T77 $(P \leftrightarrow Q) \wedge \sim Q \rightarrow \sim P$

T78 $(P \rightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \rightarrow Q) \leftrightarrow (P \rightarrow R))$

T79 $(P \rightarrow (Q \leftrightarrow R)) \leftrightarrow (P \wedge Q \leftrightarrow P \wedge R)$

T80 $(P \leftrightarrow Q) \vee (P \leftrightarrow \sim Q)$

T81 $(P \leftrightarrow Q) \leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

T82 $(P \leftrightarrow Q) \leftrightarrow \sim((P \rightarrow Q) \rightarrow \sim(Q \rightarrow P))$

T83 $(P \leftrightarrow Q) \leftrightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$

T84 $P \wedge Q \rightarrow (P \leftrightarrow Q)$

T86 $((P \leftrightarrow Q) \rightarrow R) \leftrightarrow (P \wedge Q \rightarrow R) \wedge (\sim P \wedge \sim Q \rightarrow R)$

T87 $\sim(P \leftrightarrow Q) \leftrightarrow (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

T88 $P \wedge \sim Q \rightarrow \sim(P \leftrightarrow Q)$

T89 $\sim P \wedge Q \rightarrow \sim(P \leftrightarrow Q)$

T90 $\sim(P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow \sim Q)$

T91 $P \leftrightarrow P$

T92 $(P \leftrightarrow Q) \leftrightarrow (Q \leftrightarrow P)$

T93 $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \rightarrow (P \leftrightarrow R)$

T94 $(P \leftrightarrow (Q \leftrightarrow R)) \leftrightarrow ((P \leftrightarrow Q) \leftrightarrow R)$

T95 $(P \rightarrow Q) \leftrightarrow ((P \leftrightarrow R) \leftrightarrow (Q \leftrightarrow R))$

T96 $(P \leftrightarrow Q) \leftrightarrow (\sim P \leftrightarrow \sim Q)$

T97 $(P \leftrightarrow R) \wedge (Q \leftrightarrow S) \rightarrow ((P \rightarrow Q) \leftrightarrow (R \rightarrow S))$

T98 $(P \leftrightarrow R) \wedge (Q \leftrightarrow S) \rightarrow (P \wedge Q \leftrightarrow R \wedge S)$

T99 $(P \leftrightarrow R) \wedge (Q \leftrightarrow S) \rightarrow (P \vee R \leftrightarrow Q \vee S)$

T100 $(P \leftrightarrow R) \wedge (Q \leftrightarrow S) \rightarrow ((P \leftrightarrow Q) \leftrightarrow (R \leftrightarrow S))$

T101 $(Q \leftrightarrow S) \rightarrow ((P \rightarrow Q) \leftrightarrow (P \rightarrow S)) \wedge ((Q \rightarrow P) \leftrightarrow (S \rightarrow P))$

T102 $(Q \leftrightarrow S) \rightarrow (P \wedge Q \leftrightarrow P \wedge S)$

T103 $(Q \leftrightarrow S) \rightarrow (P \vee Q \leftrightarrow P \vee S)$

T104 $(Q \leftrightarrow S) \rightarrow ((P \leftrightarrow Q) \leftrightarrow (P \leftrightarrow S))$

T105 $P \wedge (Q \leftrightarrow R) \rightarrow (P \wedge Q \leftrightarrow R)$

T106 $(P \rightarrow (Q \rightarrow R)) \leftrightarrow ((P \rightarrow Q) \rightarrow (P \rightarrow R))$

T107 $(P \rightarrow (Q \rightarrow R)) \leftrightarrow (Q \rightarrow (P \rightarrow R))$

T108 $(P \rightarrow (P \rightarrow Q)) \leftrightarrow (P \rightarrow Q)$

T109 $((P \rightarrow Q) \rightarrow Q) \leftrightarrow ((Q \rightarrow P) \rightarrow P)$

T110 $P \leftrightarrow \sim\sim P$

T111 $(P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow \sim P)$

T112 $(P \rightarrow \sim Q) \leftrightarrow (Q \rightarrow \sim P)$

T113 $(\sim P \rightarrow Q) \leftrightarrow (\sim Q \rightarrow P)$

T114 $(\sim P \rightarrow P) \leftrightarrow P$

T115 $(P \rightarrow \sim P) \leftrightarrow \sim P$

T116 $(P \wedge Q) \vee (R \wedge S) \leftrightarrow (P \vee R) \wedge (P \vee S) \wedge (Q \vee R) \wedge (Q \vee S)$

T117 $(P \vee Q) \wedge (R \vee S) \leftrightarrow (P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$

T118 $(P \rightarrow Q) \wedge (R \rightarrow S) \leftrightarrow (\sim P \wedge \sim R) \vee (\sim P \wedge S) \vee (Q \wedge \sim R) \vee (Q \wedge S)$

T119 $(P \vee \sim P) \wedge Q \leftrightarrow Q$

T120 $(P \wedge \sim P) \vee Q \leftrightarrow Q$

T121 $P \vee (\sim P \wedge Q) \leftrightarrow P \vee Q$

T122 $P \wedge (\sim P \vee Q) \leftrightarrow P \wedge Q$

T123 $P \leftrightarrow P \vee (P \wedge Q)$

T124 $P \leftrightarrow P \wedge (P \vee Q)$

T125 $(P \rightarrow Q \wedge R) \rightarrow (P \wedge Q \leftrightarrow P \wedge R)$

3.16 MORE DERIVATION FAQ's

With all those new rules, how do I begin?

- .The earlier suggestions still apply:
    - Your first line is a show line. Show: the conclusion of the argument – your main goal.
    - Think through the argument: Does that suggest a strategy?
    - Analyze the goal sentence: What is the main logical connective? Do any sentential components (negated or unnegated) appear in the premises?
    - Analyze the premises: Do any sentential components (negated or unnegated) occur in more than one premise? Can you use them together? Are there any obvious contradictions?
    - If the goal sentence is completely unrelated to the premises, look for a contradiction in the premises. Any conclusion follows from a set of inconsistent sentences.

- If you are still stuck, just work with what you've got to make things a bit simpler. Then, when you are done, look at what you've got and see if you can work out a strategy.
    - Use S (simplification) to eliminate all the conjunctions and free the conjuncts.
    - Use MP (modus ponens) and MT (modus tollens) and MTP (modus tollendo ponens) if you can.
    - Will a quick use of DN, ADJ (adjunction) or ADD (addition) make MP, MT or MTP possible?
    - Use NB (negation of a biconditional) to turn negated biconditionals into unnegated ones.

- Examine all of your disjunction sentences:
    - Think of each as a conditional. Would that make it easier to use with other sentences? If so use CDJ (conditional as disjunction) and keep going.
    - Can you derive the same goal from each disjunct by itself? If so, you want to derive the two conditional sentences that show that, then use SC (separation of cases) and keep going. (For instance, if your disjunction is $(\phi \vee \psi)$ and your goal is χ, then derive $(\phi \rightarrow \chi)$ and $(\psi \rightarrow \chi)$ and use SC to get χ.)
    - Think of the disjunction as the negation of a conjunction. Would that make it easier to use with other sentences (for instance if the conjunction occurs in the consequent position of a conditional)? If so, use DM (de Morgan's) and keep going.

What if I'm stuck in the middle of a derivation?

- Keep track of your available lines. Stop when you are stuck and determine exactly which lines are available. Don't forget that cancelled show lines and premises are available (but uncancelled show lines and boxed lines are not!) Then, after you have done that, think through the argument again trying to see how the conclusion follows from the available lines. Does it suggest a strategy?

- If you are still stuck (especially if you are midway through the derivation and you've tried doing everything else), check if there is a conditional sentence that you haven't used yet. Does the consequent look like it might be useful? If so consider trying to show that the antecedent is true through an indirect proof, then you can use MP to derive the consequent. (Suppose the conditional is $\phi \rightarrow \psi$ and you'd really like to have ψ. Put in a show line: show ϕ. Then assume $\sim\phi$. Try to derive a contradiction. If successful, then you can cancel the show line, and use MP to get ψ**.**

How do I know when to put a show line for a subderivation?

- Use a show line whenever you can't get what you need through a direct derivation. All the subderivations will be conditional or indirect derivations – after all, if you can derive it directly, just using the rules of inference, you don't need a subderivation. (Occasionally, you will put a show line in for something that you can derive directly or already have – that's not a problem. Just use DD as soon as you can, box and cancel, and keep going.)
- Look at your working goal (the show line you are working with). Thinking about all the other available lines. Ask yourself what you need to achieve that goal? (For instance, if you have $\phi \rightarrow \psi$ on an available line, and your main goal is ψ, then what you need to achieve it is: ϕ.) Once you know what you need, make it a show line.

How do I know what sentence to assume?

- If you know ***why*** you are starting a subderivation then you will know what assumption to make. Once you have your show line, ask yourself how you want to show it? Then it should be obvious what assumption to make.
  - Is it a conditional? Then you probably will do a conditional derivation. Assume the antecedent. Your new goal is the consequent.
  - If it isn't a conditional, then you want to do an indirect derivation. Assume the negated (or unnegated) sentence. Your new goal is a contradiction.
  - Occasionally when you want to show a conditional the best way is with an indirect derivation. Assume the negated conditional. Your new goal is a contradiction. Remember, a negated conditional can be easily turned into a conjunction.

See the chart on the next page: Determining your goals for subderivations.

Determining your goals for subderivations (new show lines).

What you want to show	**You have available OR You can derive…**	**If so, try this strategy:** If you need to derive a sentence think about whether you can get it directly through the rules and available lines, or if you want to use ID or CD to derive it.
A conditional: $\phi \rightarrow \psi$	ψ	Show $\phi \rightarrow \psi$. Assume ϕ. Derive or reiterate ψ. 'CD', box and cancel. (Uniform derivation)
A conditional: $\phi \rightarrow \psi$	$\sim\phi$	Show $\phi \rightarrow \psi$. Assume ϕ. Derive or reiterate $\sim\phi$. 'ID', box and cancel. (Mixed derivation, CD/ID)
A disjunction: $\phi \vee \psi$	ψ (or ϕ)	Derive ψ (or ϕ) Use ADD to get $(\phi \vee \psi)$
A disjunction: $\phi \vee \psi$	$(\sim\phi \rightarrow \psi)$ OR $(\sim\psi \rightarrow \phi)$	Derive $(\sim\phi \rightarrow \psi)$ OR $(\sim\psi \rightarrow \phi)$ Use CDJ to get $(\phi \vee \psi)$ Remember you might need to use RT53 to get the disjuncts in the right order or RT13-16 (transposition) to get the conditional in the correct form.
A conjunction: $\phi \wedge \psi$	ϕ ψ	Derive ϕ Derive ψ Use ADJ to get $(\phi \wedge \psi)$ Remember you might need to use ID or CD to derive ϕ and/or ψ.
A biconditional: $\phi \leftrightarrow \psi$	$\phi \rightarrow \psi$ $\psi \rightarrow \phi$	Derive: $\phi \rightarrow \psi$ Derive: $\psi \rightarrow \phi$ Use CB to get $(\phi \leftrightarrow \psi)$. Remember you can use CDJ to get $(\phi \rightarrow \psi)$ from $(\sim\phi \vee \psi)$.
The negation of a conjunction: $\sim(\phi \wedge \psi)$		Show $\sim(\phi \wedge \psi)$. Assume $(\phi \wedge \psi)$ for an indirect proof. Use S to get ϕ and ψ and derive a contradiction. 'ID', box and cancel.
The negation of a disjunction $\sim(\phi \vee \psi)$	$\sim\phi$ AND $\sim\psi$	Derive $\sim\phi$ AND $\sim\psi$ Use ADJ to get $(\sim\phi \wedge \sim\psi)$ Use DM to get $\sim(\phi \vee \psi)$