Lecture 15

Recall: If f has a pole of order m at z_0, then we can write $f(z)=\frac{1}{(z-z_0)^m}g(z)$ where g is analytic & $g(z_0)\neq 0$.

$$\text{Res}(f:z_0)=\frac{1}{2\pi i}\int_\gamma f(z)dz=\frac{1}{2\pi i}\int_\gamma \frac{g(z)}{(z-z_0)^m}dz=C_{m-1} \quad \text{where } g(z)=\sum_{k=0}^{\infty}C_k(z-z_0)^k$$

We can get a Laurent series for f by writing $f(z)=\frac{1}{(z-z_0)^m}g(z)=\frac{1}{(z-z_0)^m}\sum C_k(z-z_0)^k$

$$=\sum_{k=0}^{\infty}C_k(z-z_0)^{k-m}$$

If $f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$ is a Laurent series, then $\text{Res}(f:z_0)=a_{-1}$

$$=\sum_{n=-m}^{\infty}C_{n+m}(z-z_0)^n$$

index starting from negative. Looks more like a Laurent series

Ex: Find a L.S. for $f(z)=\frac{\cos z}{z^4}$ centered at $z_0=0$.

$$f(z)=\frac{1}{z^4}\left(\sum\frac{(-1)^k z^{2k}}{(2k)!}\right)$$

$$=\frac{1}{z^4}\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)$$

$$=\frac{1}{z^4}-\frac{1}{2z^2}+\frac{1}{4!}-\frac{z^2}{6!}+\cdots$$

$$=\sum\frac{(-1)^k z^{2k-4}}{(2k)!} \implies \text{Res}(f:0)=a_{-1}=0=\text{coefficient of } \tfrac{1}{z} \text{ term}$$

Ex: Find $\text{Res}(f:0)$ where $f(z)=\cot z=\frac{\cos z}{\sin z}$

Trick: use pole of order 1.

Today we show another trick:

Note: $z_0=0$ is a pole of order 1, $\cot z=\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+\cdots$

want to find $a_{-1}=\text{Res}(\cot z:0)$

$\sin z\cdot\cot z=\cos z$

$$\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(\frac{a_{-1}}{z}+a_0+a_1z+a_2z^2+\cdots\right)=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots$$

expand this: $a_{-1}+a_0z+\left(a_1-\frac{a_{-1}}{3!}\right)z^2+\left(a_2-\frac{a_0}{3!}\right)z^3+\cdots=1-\frac{z^2}{2!}+\frac{z^4}{4!}-$

compare

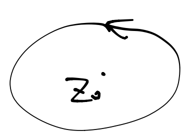
comparing coefficients:

constant	$a_{-1}=1$
z	$a_0=0$
z^2	$a_1-\frac{a_{-1}}{6}=-\frac{1}{2} \implies a_1=\frac{a_{-1}}{6}-\frac{1}{2}=-\frac{1}{3}$
z^3	$a_2-\frac{a_0}{6}=0 \implies a_2=\frac{a_0}{6}=0$
$\cdots$	$\cdots$

Hence $\text{Res}(f:0)=a_{-1}=1$

Residue Theorem:

Cauchy Formula: $\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{z-z_0}dz=g(z_0)$



(f analytic)

$$\boxed{\frac{1}{2\pi i}\int_\gamma\frac{g(z)}{(z-z_0)^{m+1}}dz=C_m=\frac{g^{(m)}(z_0)}{m!}}$$

more like a generalization of Cauchy Formula

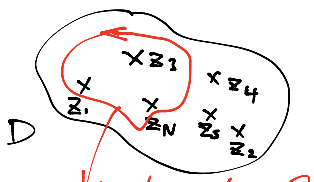
What if we want to integrate $\int_\gamma\frac{g(z)}{(z-z_0)(z-z_1)}dz$ where both z_0, z_1 lie inside γ.

Note: $\frac{1}{2\pi i}\int_\gamma \frac{g(z)}{(z-z_0)^{m+1}}\,dz = \operatorname{Res}\left(\frac{g(z)}{(z-z_0)^{m+1}} : z_0\right)$

THM (Residue THM): Suppose f is analytic in a simply-connected region D except for poles $z_1, \cdots, z_N$. If γ is a simple closed curve which does not pass through the poles, then (+ve orient.)

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f : z_k)$$

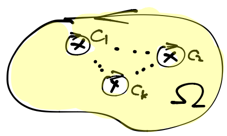
add up residues of all poles



add up these 3.

Note: When there is only 1 pole, we get Cauchy Formula.

poles outside curve contribute nothing



circle 1
circle 2
⋮
circle k

Put small circles C_k around the pole z_k & take Ω to be the region between γ and the circles.
f is analytic in Ω, so by Green's Thm & C-R equations we get

$$\int_{\partial\Omega} f(z)\,dz = 0 = \frac{1}{i}\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dx\,dy$$

$$0 = \int_{\partial\Omega} f(z)\,dz = \int_\gamma f(z)\,dz + \sum_{z_k} \int_{-C_k} f(z)\,dz$$

$$= \int_\gamma f(z)\,dz - \sum_{z_k} \int_{C_k} f(z)\,dz \quad = 2\pi i \operatorname{Res}(f : z_k)$$

$$= \int_\gamma f(z)\,dz - 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f : z_k)$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f : z_k)$$

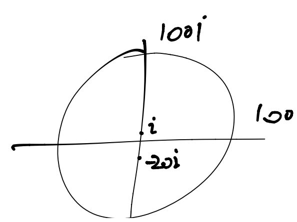
we are done. ∎

Ex: Let γ be the circle of radius 100 centered at 0, find $\int_\gamma \frac{e^{iz}}{z^2+19zi+20}\,dz$

· If a removable singularity inside γ, you don't need to add it up b/c the residue of it is always 0.
(think about Laurent Series with only removable singularities? Just another power series.)

Idea: ① Find poles (solve denom $=0$)
② Determine which are inside γ.
③ Compute Residues
④ Use Residue Thm

$z^2+19iz+20 = (z+20i)(z-i)$
Poles of order at $z=i$, $z=-20i$
Both poles inside γ.
(So calculate both residues)



$$\operatorname{Res}(f: i) = \frac{P(i)}{Q'(i)} = \frac{e^{ii}}{21i}$$

$$\operatorname{Res}(f: 19i) = \frac{P(i)}{Q'(i)} = \frac{e^{i(-20i)}}{2\cdot(-20i)+19i} = \frac{e^{20}}{-21i}$$

$$\int_\gamma f(z)\,dz = 2\pi i\left(\frac{e^{-1}}{21i} - \frac{e^{20}}{21i}\right) = \frac{2\pi}{21}\left(e^{-1} - e^{20}\right)$$