MAT337 HW2 Rui Qiu #999292509

P52 1.

(a) Proof: Since $U: \mathbb{R}^n \to \mathbb{R}^n$ is isometric,

$\|U\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$

So $\langle U\vec{x}, U\vec{x} \rangle = \langle \vec{x}, \vec{x} \rangle$

① $\langle U(\vec{x}+\vec{y}), U(\vec{x}+\vec{y}) \rangle = \langle U\vec{x}+U\vec{y}, U\vec{x}+U\vec{y} \rangle$

$= \langle U\vec{x}, U\vec{x} \rangle + \langle U\vec{x}, U\vec{y} \rangle + \langle U\vec{y}, U\vec{x} \rangle + \langle U\vec{y}, U\vec{y} \rangle$

$= \langle U\vec{x}, U\vec{x} \rangle + 2\langle U\vec{x}, U\vec{y} \rangle + \langle U\vec{y}, U\vec{y} \rangle$

$= \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\langle U\vec{x}, U\vec{y} \rangle$

② $\langle U(\vec{x}+\vec{y}), U(\vec{x}+\vec{y}) \rangle = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle$

$= \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

$= \|\vec{x}\|^2 + \|\vec{y}\|^2 + 2\langle \vec{x}, \vec{y} \rangle$

① = ②, then

① − ② $= 2\langle U\vec{x}, U\vec{y} \rangle - 2\langle \vec{x}, \vec{y} \rangle = 0$

$\langle U\vec{x}, U\vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

Then we are done.

$\blacksquare$

(b) Proof: By (a), $\langle \vec{v_j}, \vec{v_i} \rangle = \langle U\vec{v_j}, U\vec{v_i} \rangle$

since $\{\vec{v_1}, \dots, \vec{v_k}\}$ is an orthonormal basis.

The $\langle \vec{v_j}, \vec{v_i} \rangle = \delta_{ij}$

where $\delta_{ij} = 0$ when $i \neq j$

$\delta_{ii} = 1 \quad \forall i, j$

And we have $\langle U\vec{v_j}, U\vec{v_i} \rangle = \delta_{ij}$

Hence $\{U\vec{v_1}, \dots, U\vec{v_k}\}$ is also orthonormal.

$\blacksquare$

~~Note that v_j, v_i here are not necessarily to be the column vector of U~~

PSS G.

(a) Proof:

Observe: starting from (0,0)

$$x_{n+1} = \sqrt{\frac{x_n^2 + 2y_n^2}{4}}, \quad y_{n+1} = \frac{x_n + y_n + 1}{3}$$

$$V_0 = (0,0)$$

$$V_1 = \left(\sqrt{\frac{0+0}{4}}, \frac{0+0+1}{3}\right) = \left(0, \frac{1}{3}\right)$$

$$V_2 = \left(\sqrt{\frac{0+2\cdot\frac{1}{9}}{4}}, \frac{0+\frac{1}{3}+1}{3}\right) = \left(\frac{\sqrt{2}}{6}, \frac{4}{9}\right)$$

$$V_3 = \left(\sqrt{\frac{\frac{2}{36}+2\times\frac{16}{81}}{4}}, \frac{\frac{\sqrt{2}}{6}+\frac{4}{9}+1}{3}\right) = (\,\cdots\,)$$

Wild Guess: Suppose we have a limit, look for fixed points of the map

Then $V(x,y) = \left(\sqrt{\frac{x^2+2y^2}{4}}, \frac{x+y+1}{3}\right)$

$$\begin{cases} x^2 = \dfrac{x^2+2y^2}{4} \\ y = \dfrac{x+y+1}{3} \end{cases}$$

Solve this we get $y = \frac{1}{10}(6 \pm \sqrt{6})$

$\Rightarrow y = \frac{6+\sqrt{6}}{10}$

Otherwise $x < 0$ which is impossible

$x = \frac{1+\sqrt{6}}{5}$

So the suspected limit is $\left(\frac{1+\sqrt{6}}{5}, \frac{6+\sqrt{6}}{10}\right)$

Now we want to check x_n, y_n are increasing but bounded above.

To do this, we use induction to prove (twice)

(see next page)

① Bounded above

Suppose $x_n < 1$, $y_n < 1$

Base case: $x_0 = 0 < 1$, $y_0 = 0 < 1$

Suppose now it (I mean $x_n < 1$, $y_n < 1$) is true for n case, check for x_{n+1} & y_{n+1}.

then $x_{n+1} = \sqrt{\frac{x_n^2 + 2y_n^2}{4}} < \sqrt{\frac{1+2}{4}} = \frac{\sqrt{3}}{2} < 1$

$y_{n+1} = \frac{x_n + y_n + 1}{3} < \frac{1+1+1}{3} = 1$

Thus x_n and y_n are bounded above for all n

② Increasing.

Base case: $x_2 - x_1 = \frac{\sqrt{2}}{6} - 0 > 0$

$y_2 - y_1 = \frac{4}{9} - \frac{1}{3} = \frac{1}{9} > 0$

Suppose it's true for case n,

i.e. $x_{n+1} - x_n > 0 \quad \Rightarrow x_{n+1}^2 > x_n^2$

$y_{n+1} - y_n > 0 \quad \Rightarrow y_{n+1}^2 > y_n^2$

Then for case $n+1$.

$x_{n+2}^2 = \frac{x_{n+1}^2 + 2y_{n+1}^2}{4} > \frac{x_n^2 + 2y_n^2}{4} = x_{n+1}^2$

$x_{n+2} > x_{n+1} \Rightarrow x_{n+2} - x_{n+1} > 0$.

$y_{n+2} = \frac{y_{n+1} + x_{n+1} + 1}{3} > \frac{x_n + y_n + 1}{3} = y_{n+1}$

$\Rightarrow y_{n+2} - y_{n+1} > 0$.

Therefore, x_n and y_n are increasing for all n

Combine ① & ②, done.

$\blacksquare$

(b). Note that we have found a suspected limit $\left(\frac{1+\sqrt{6}}{5}, \frac{6+\sqrt{6}}{10}\right)$

since $\lim_{n\to\infty} x_n = \frac{1+\sqrt{6}}{5}$

$\lim_{n\to\infty} y_n = \frac{6+\sqrt{6}}{10}$

Therefore $\lim_{n\to\infty} \vec{v_n} = \left(\frac{1+\sqrt{6}}{5}, \frac{6+\sqrt{6}}{10}\right)$

P60 A: bounded subset of $\mathbb{R}$, show $\sup A$ & $\inf$ belong to $\overline{A}$.

D. Proof:

By definition:

$$\overline{A} = \{x \in \mathbb{R} : a_n \to x \text{ for some sequence } (a_n) \text{ in } A\}$$

and $x \in \overline{A} \iff \forall \varepsilon > 0, \exists a \in A$ s.t. $|x - a| < \varepsilon$.

① If $\sup A$ and $\inf A$ are in A, then we are done, $\sup A$ and $\inf A$ are in $\overline{A}$ as well.

② If not, $\forall \varepsilon > 0, \exists a_k, b_j \in A$ s.t.

$$a_k > \sup A - \varepsilon$$

$$b_j < \inf A + \varepsilon$$

by definition of supremum and infimum

q —— $\sup A$

p —— $\inf A$

Now we can construct a sequence by setting $\varepsilon_n = \frac{1}{n}$.

in details,

$k=1$, $\sup A - 1 < a_1 < \sup A \Rightarrow |a_1 - \sup A| < 1$

$k=2$, $\sup A - \frac{1}{2} < a_2 < \sup A \Rightarrow |a_2 - \sup A| < \frac{1}{2}$

so $k=n$, $\sup A - \frac{1}{n} < a_n < \sup A \Rightarrow |a_n - \sup A| < \frac{1}{n}$

Similarly,

$j=1$, $\inf A < b_1 < \inf A + 1 \Rightarrow |b_1 - \inf A| < 1$

$j=2$, $\inf A < b_2 < \inf A + \frac{1}{2} \Rightarrow |b_2 - \inf A| < \frac{1}{2}$

$j=n$, $\inf A < b_n < \inf A + \frac{1}{n} \Rightarrow |b_n - \inf A| < \frac{1}{n}$

Then since $n \to \infty$, $\varepsilon_n = \frac{1}{n} \to 0$, so $\lim_{n\to\infty} a_n = \sup A$

and $\lim_{n\to\infty} b_n = \inf A$

Therefore $\inf A$ and $\sup A$ are two limits of A.

So $\inf A \in \overline{A}$ and $\sup A \in \overline{A}$. $\blacksquare$

4. Proof: $\Rightarrow$ $\boxed{\text{A subset of } \mathbb{R}^n \text{ is complete, then it's closed.}}$

Let $S \subseteq \mathbb{R}^n$ be complete,

so any Cauchy sequence in S converges to a limit in S.

Let $\vec{x}$ be such a limit point in S

we can find a sequence $(\vec{x}_n)$ in S which converges to $\vec{x}$.

$\forall \varepsilon > 0, \exists N$ s.t.

$\|\vec{x}_n - \vec{x}\| < \varepsilon \quad \forall n \geq N$

But convergent sequences are Cauchy & S is complete

Hence $\vec{x}_n$ converges to a point in S

i.e. $\vec{x} \in S$

Therefore S contains all limits.

so S is closed.

$\Leftarrow$ $\boxed{S \text{ is closed then it's complete.}}$

Conversely, let $(\vec{x}_n)$ be a Cauchy sequence in S.

Then $(\vec{x}_n)$ is a Cauchy sequence in $\mathbb{R}^n$

and we know that $\mathbb{R}^n$ is complete.

Then $\vec{x}_n \to \vec{x}$ for some $\vec{x} \in \mathbb{R}^n$.

But $\vec{x}$ must be a limit point of S

So $\vec{x} \in S$

Thus every Cauchy sequence of S converges in S

i.e. S is complete. $\blacksquare$

P66. J.

(a) Solution:

First we know that the removed parts are all open small triangles, let them be a set T.

Then $S^c = T$

T is the union of all non-overlapped open triangles

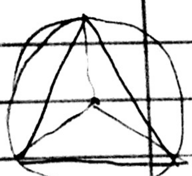
T is also open

i.e. S^c is open

Therefore S is closed.

And obviously S is bounded,

since we can construct a ball centred at the centre of the large triangle with radius $\frac{1}{\sqrt{3}}$ of the side length.



Hence S is closed and bounded, thus compact.

(By Heine-Borel Theorem)

Now we suppose S_1 be the shape at first stage.

S_2 be the second stage, etc.

And we've proved that $S_1, S_2, \cdots$ are compact.

Note that they also satisfy

$S_1 \supset S_2 \supset S_3 \supset \cdots$

S_i are nonempty $\forall i$.

By Cantor's intersection theorem. $S = \bigcap S_i \neq \emptyset$.

Therefore, S is a nonempty compact set.

b. Solution. Suppose S has interior, and $\exists x \in \operatorname{int} S$.

So $x \in S_i \ \forall i$

note that the length of ith stage triangle is 2^{-n}.

(initial length 1)

As $x \in S_i$, $\exists r > 0$, $B_r(x) \subseteq S_i$

since $2^{-n} \to 0$ as $n \to \infty$,

then $r < \lim_{n\to\infty} 2^{-n} = 0$, but $r > 0$, contradiction.

So S has no interior.

Solution:

(d). Let the initial area of S be 1.

① remove $(\frac{1}{4})^1 \times 3^0$

② remove $(\frac{1}{4})^2 \times 3^1$

③ remove $(\frac{1}{4})^3 \times 3^2$

...

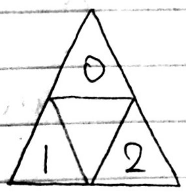
So area of all removed is

$$\frac{1}{4}+\frac{3}{16}+\frac{9}{64}+\cdots = \frac{\frac{1}{4}-(\frac{3}{4})^n}{1-\frac{3}{4}} = 1-\frac{3^n}{4^{n-1}}$$

as $\lim_{n\to\infty}\frac{3^n}{4^{n-1}} = \lim_{n\to\infty}(\frac{3}{4})^{n-1}\cdot 3 = 0$

So the area removed is convergent to 1, which means S has zero area left.

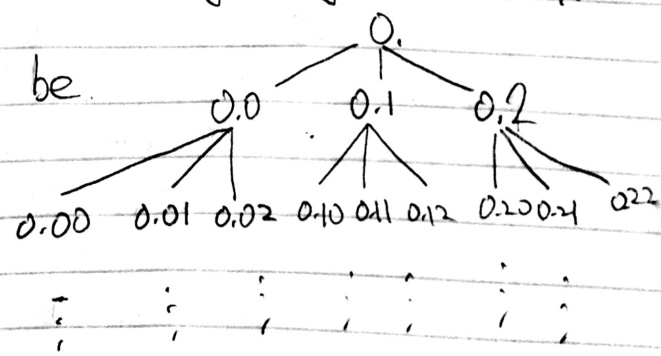
(e) Solution.



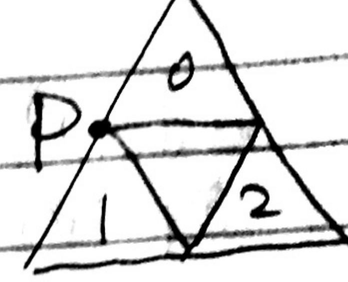
We define a way to represent every point in S by this simple rule:
If the point falls in the upper triangle, the digit is 0.
the left triangle represents 1,
the right triangle represents 2.
For example, if it's in 1, then we write 0.1
We again subdivide $\triangle 1$ into 3 smaller triangles,

and decide the second digit by its location.

The decision tree would be.



But the problem is, for points like P,



it can either be written as

0.01111.... or 0.10000...

But we can add a rule that for each point we only choose its representation of small first different digit. i.e. P choose: 0.01111... instead of 0.10000...

Then the decision tree exactly corresponds to one point.

Now we need to show S is uncountable.

$\forall x \in S$, $x = 0.x_1 x_2 x_3 \cdots$ for $x_i \in \{0.1.2\}$,

so $x \in [0,1]$

$\Rightarrow S \subseteq [0,1]$

$|S| \leq |[0,1]|$

So we construct one injection from S to $[0,1]$

by $x = (0.x_1 x_2 \cdots)_{base 3} = \sum_{k \geq 1} 3^{-k} x_k$, for $x_i \in \{0,1,2\}$.

By Schroeder-Bernstein Thm

$|S| = |[0,1]|$

Since $|[0,1]| = C$, which is uncountable.

Hence S is uncountable.

P178.

A. Solution:

① positive definiteness:

$$\rho(x,y)=|e^x-e^y|=0$$
$$e^x=e^y$$
$$x=y$$

And it's the only solution.

② symmetry:

$$\rho(x,y)=|e^x-e^y|=|e^y-e^x|=\rho(x,y)$$

③ triangle inequality:

$$\rho(x,z)=|e^x-e^z|$$

$$\rho(x,y)+\rho(y,z)=|e^x-e^y|+|e^y-e^z|$$
$$\geq|e^x-e^y+e^y-e^z|$$
$$=|e^x-e^z|$$

So $\rho(x,z)\leq\rho(x,y)+\rho(y,z)$

Therefore ρ_1 is a metric on $\mathbb{R}$