

STA 257S - Summer, 1996

Test #1

July 15, 1996

INSTRUCTIONS:

- Time: 50 minutes
- No aids allowed.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total points: 50

NAME: SOLUTIONS.

STUDENT NUMBER: _____

TUTOR: _____

1. The following questions involve a coin being tossed repeatedly. Assume that, on each toss, there are two equally likely outcomes (Heads and Tails).

- (a) (6 points) The coin is tossed three times. The outcome of every toss is of interest. Describe the probability space.

S - sample space (set of possible outcomes)

$$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$$

\mathcal{F} - event space.

- set of all possible subsets of S (2^8 elements)

P - probability measure

$$P(w) = \frac{1}{8} \quad \text{for all } w \in S$$

- (b) (6 points) Let the random variable X be the number of Heads minus the number of Tails in the first four tosses of the coin. What is the probability mass function for X ?

Possible values for X : $\{-4, -2, 0, 2, 4\}$

$$P(X=4) = P(X=-4) = \frac{1}{2^4}$$

$$P(X=2) = P(X=-2) = \binom{4}{1} \frac{1}{2^4}$$

$$P(X=0) = \binom{4}{2} \frac{1}{2^4}$$

- (c) Suppose the coin is tossed until, for the first time, the same result appears two times in succession. Let the random variable Y be the total number of tosses required.

- i. (3 points) What is the probability mass function for Y ?

$$P(Y=y) = \frac{1}{2^{y-1}}, \quad y = 2, 3, 4, \dots \\ (0 \text{ otherwise})$$

- ii. (5 points) Find the probability that an odd number of tosses is required.

$$\begin{aligned} P(Y \text{ odd}) &= P(Y=3) + P(Y=5) + P(Y=7) + \dots \\ &= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots \\ &= \frac{\frac{1}{4}}{1 - \frac{1}{4}} \\ &= \frac{1}{3} \end{aligned}$$

2. (5 points) A random variable X takes the values $0, 1, 2, \dots$ with positive probability and $P(X \geq k) = \left(\frac{1}{3}\right)^k$ for $k = 0, 1, 2, \dots$. Identify the distribution of X , including the value of any parameters.

$$\begin{aligned} P(X=k) &= P(X \geq k) - P(X \geq k+1) \\ &= \left(\frac{1}{3}\right)^k - \left(\frac{1}{3}\right)^{k+1} \\ &= \left(\frac{1}{3}\right)^k \left(1 - \frac{1}{3}\right) \\ &= \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right) \end{aligned}$$

X has a Geometric distribution with parameter $\frac{2}{3}$

3. (7 points) $X \sim \text{Exponential}(\lambda = 2)$. Give the density function of X and find $P(2X^2 + 5 > 55)$.

Density: $f(x) = 2e^{-2x}$ for $x > 0$ (0 otherwise)

$$\begin{aligned} P(2X^2 + 5 > 55) &= P(X > 5) \\ &= 1 - \int_0^5 2e^{-2x} dx \\ &= e^{-10} \end{aligned}$$

4. (6 points) Suppose $X \sim \text{Poisson}(\lambda)$ (i.e. X has probability mass function $p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, 3, \dots$) Find $E\left(\frac{1}{1+x}\right)$.

$$\begin{aligned} E\left(\frac{1}{1+x}\right) &= \sum_{x=0}^{\infty} \frac{1}{1+x} \frac{\lambda^x e^{-\lambda}}{x!} \\ &= \frac{1}{\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x+1} e^{-\lambda}}{(x+1)!} \\ &= \frac{1}{\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\ &= \frac{1}{\lambda} [1 - e^{-\lambda}] \end{aligned}$$

5. Suppose that the density function for the length L of a telephone call is

$$f(x) = \begin{cases} e^{-x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

The cost of a call is

$$C(L) = 2 + 3L \quad \text{if } L > 0$$

Find (you may not assume the values for the expectation and variance of an exponential random variable developed in class are known):

(a) (6 points) the mean cost of a call

$$\begin{aligned} E(L) &= \int_0^\infty x e^{-x} dx \\ &= -x e^{-x} \Big|_0^\infty + \int_0^\infty e^{-x} dx \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{Mean cost: } E(C) &= E(2+3L) \\ &= 2 + 3(1) = 5 \end{aligned}$$

(b) (6 points) the variance of the cost of a call

$$\begin{aligned} E(L^2) &= \int_0^\infty x^2 e^{-x} dx \\ &= -x^2 e^{-x} \Big|_0^\infty + 2 \underbrace{\int_0^\infty x e^{-x} dx}_{=1 \text{ from (a)}} \\ &= 2 \end{aligned}$$

$$\text{So } V(L) = 2 - 1 = 1$$

$$\text{Var of cost} = V(2+3L) = 9 V(L) = 9$$