Lecture 7

§4.5 The St. Petersburg Paradox

Martingale Fair Game

If the expected net win of a game per round is always 0, then game is called a fair game.

A player plays a fair game and he follows the following rules

① He will withdraw at his first win

② He starts with gambling a unit stake

③ each time he loses. he will double the stake for the next game.

For simplicity, assume each time he has prob $\frac{1}{2}$ of winning.

Suppose he wins at the nth game, then he needs to pay.

$1+2+2^2+\cdots 2^{n-1}=2^n-1$

However, at the nth game, he will win 2^n stakes.

$\Rightarrow$ his net win is 1

* Will this game always end in finite many plays?

Answer: Yes.

Sps the plays are independent, then let $A_i=\{\text{Game ends at } i\text{th play}\}$

$P(A_i)=\frac{1}{2^i}$ $i=1,2,\cdots$

Let $B_i=\{\text{Game will last after } i\text{th game}\}$

$P(B_i)=\frac{1}{2^i}$ $i=1,2,\cdots$

Consider the event $B=\bigcap_{i=1}^{\infty}B_i$

$B=\{\text{Game never ends}\}$

$P(B)=\lim_{i\to\infty}P(B_i)=\lim_{i\to\infty}(\frac{1}{2})^i=0$

So it will end.

Let's ~~take~~ a look at the expected amount of stakes he needs to in order to prepare

in order to carry out this strategy. We now at if he wins the ith game he needs to prepare 2^i-1 stakes.

$P(A_i)=(\frac{1}{2})^i$

On average, he needs to prepare

$$\sum_{i=1}^{\infty}(\frac{1}{2})^i[2^i-1]=\infty \text{ Stakes}$$

Think about a discount (interest) problem. Suppose that at each round, the interest rate is fixed at β. We always calculate according to the value of stake at the first round.

In general, one stake at ith game is worth $\frac{1}{(1+\beta)^{i-1}}$ stakes for the first round.

Suppose he wins at the ith game, he needs to pay ? stakes.

$$1+\frac{2}{1+\beta}+\frac{4}{(1+\beta)^2}+\cdots+\frac{2^{i-2}}{(1+\beta)^{i-2}}=\sum_{j=0}^{i-2}\alpha^j=\begin{cases}\frac{\alpha^i-1}{\alpha-1} & \text{if } \alpha\neq 1\\ i & \text{if } \alpha=1\end{cases}$$

Let $\frac{2}{1+\beta}=\alpha$

How much does he win at the ith game?
He wins $\frac{2^{i-1}}{(1+\beta)^{i-1}}$ stakes.

The expected winning $=\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i\frac{2^{i-1}}{(1+\beta)^{i-1}}=\frac{1}{2}\sum_{i=1}^{\infty}\left(\frac{1}{1+\beta}\right)^{i-1}=\frac{1}{2}\frac{1}{1-\frac{1}{1+\beta}}=\frac{1}{2}\frac{1+\beta}{\beta}$

On the other hand, the average payment is

$$=\sum_{i=1}^{\infty}\left(\frac{\alpha^{i-1}-1}{\alpha-1}\right)\left(\frac{1}{2}\right)^i$$

$$=\sum_{i=1}^{\infty}\frac{\alpha^{i-1}\left(\frac{1}{2}\right)^i}{\alpha-1}-\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\right)^i}{\alpha-1}$$

① = first sum, ② = second sum

$$① = \frac{1}{\alpha-1}\sum_{i=1}^{\infty}\frac{1}{(1+\beta)^{i-1}}\cdot\frac{1}{2}=\frac{1}{\alpha-1}\frac{1}{2}\frac{1}{1-\frac{1}{1+\beta}}$$

$$② = \frac{1}{\alpha-1}\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i=\frac{1}{\alpha-1}$$

$$①-② = \frac{1}{2}\frac{1}{\alpha-1}\frac{1+\beta}{\beta}-\frac{1}{\alpha-1}=\frac{1}{\alpha-1}\left[\frac{1+\beta}{2\beta}-\frac{2\beta}{2\beta}\right]$$

$$=\frac{1-\beta}{2\beta}\cdot\frac{1}{\frac{2}{1+\beta}-1}$$

$$=\frac{1-\beta}{2\beta}\frac{1+\beta}{1-\beta}=\frac{1}{2}\frac{1+\beta}{\beta}$$

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§4.6 (Reading)
§4.7 (Conditioning, will talk about it in Chapter 5).

§4.8 Continuous R.V.'s

Def: X is called a continuous r.v. if there exist a function $f(x)$ such that

density function

① $f(x)\geq 0$
② $\int_{-\infty}^{\infty}f(x)dx=1$
③ $P(X\in[a,b])=\int_a^b f(x)dx$

CDF (Cumulative distribution function)
$F_X(x)=P[X\leq x]=\int_{-\infty}^{x}f(a)da$ ⊛

According to ⊛ we have $f(x) = \frac{dF(x)}{dx}$

Notes: $P[X=a]=0$ for any $a \in \mathbb{R}$

$$P[X\in(a,b)] = P[X\in(a,b]] = P[X\in[a,b)] = P[X\in[a,b]]$$

Def: Expectation of a continuous random variable X such that $\int_{-\infty}^{\infty} |x| f(x)dx < \infty$ is defined as $EX = \int_{-\infty}^{\infty} x f(x)dx$

Proposition: If $X \geq 0$, then $E[X] = \int_0^\infty P[X\geq a]da$ (**)

Proof: RHS of (**) $= \int_0^\infty \left[\int_a^\infty f(x)dx\right]da$ Fubini's thm

$$= \int_0^\infty \left[\int_0^\infty f(x)\, I\{x\geq a\}dx\right]da$$

$$= \int_0^\infty \left(\int_0^\infty f(x) I\{x\geq a\}da\right)dx \longrightarrow \text{check function}$$

Note: if $a \leq x$, then $I\{x\geq a\}=1$
$a > x$, $I\{x\geq a\}=0$

$\Rightarrow$ check function $= \int_0^x f(x)da = f(x)\int_0^x da = xf(x)$

$$= \int_0^\infty x f(x)dx = EX$$

Thm: If X is continuous with density $f(x)$. Then for any H.
$E[H(x)] = \int_{-\infty}^{\infty} H(x)f(x)\,dx$

1. If $H(\ ) \geq 0$, according to proposition
$E[H(x)] = \int_0^\infty P[H(X)\geq a]da$

For each a, define the set $g_a = \{y : H(y) \geq a\}$

$$= \int_0^\infty P[X\in g_a]da$$

$$= \int_0^\infty \left[\int_{g_a} f(x)dx\right]da$$

$$= \int_0^\infty \int_{-\infty}^{\infty} f(x) I\{x\in g_a\}dx\,da$$

Fubini $\Rightarrow$ $= \int_{-\infty}^{\infty} \boxed{\int_0^\infty f(x) I\{x\in g_a\}da}\,dx \longrightarrow$ ② $= f(x)\int_0^\infty I\{x\in g_a\}da$

Note $I\{x\in g_a\}=1 \iff H(x)\geq a$

$= f(x)\int_0^{H(x)} da = H(x)f(x).$

2. For any $H(\ )$, write $H(x) = H^+(x) - H^-(x)$
where $H^+ = \max\{H(x), 0\}$
$H^- = \max\{-H(x), 0\}$

HW: $H(X) = H^+(x) - H^-(x)$ ← show
$H^+(x)$ & $H^-(x) \geq 0$

Hence $EH^+(X) = \int_{-\infty}^{\infty} H^+(x) f(x) dx$

$EH^-(x) = \int_{-\infty}^{\infty} H^-(x) f(x) dx$

$$\Rightarrow EH(X) = EH^+(X) - EH^-(X) = \int_{-\infty}^{\infty} H^+(x) f(x) dx - \int_{-\infty}^{\infty} H^-(x) f(x) dx$$
$$= \int_{-\infty}^{\infty} [H^+(x) - H^-(x)] f(x) dx$$
$$= \int_{-\infty}^{\infty} H(x) f(x) dx$$

Ex: Uniform R.v.'s
$U \sim \text{Unif}[a,b]$ r.v. if $f_U(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{o.w.} \end{cases}$

$$EU = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$$

$$EU^2 \overset{\text{thm}}{=\!=} \int_a^b x^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \frac{1}{3}(b^3 - a^3) = \frac{1}{3}(a^2 + ab + b^2)$$

$$V[U] = EU^2 - (EU)^2 = \frac{1}{3}(a^2 + ab + b^2) - [\tfrac{1}{2}(a+b)]^2 = \frac{1}{12}(b-a)^2$$

Exponential r.v.'s
$X \sim \text{Exp}(\lambda)$ if $f_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{\lambda} e^{-x/\lambda} & \text{if } x \geq 0 \end{cases}$

HW: $EX = \lambda$, $VX = \lambda^2$

note $M_X(t) < \infty$ iff $\frac{1}{\lambda} > t$

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx} \frac{1}{\lambda} e^{-x/\lambda} dx = \frac{1}{\lambda} \int_0^\infty e^{-(\frac{1}{\lambda} - t)x} dx$$

Moment generating function of exp r.v.

$$= \frac{1}{\lambda} \cdot \frac{1}{-(\frac{1}{\lambda} - t)} e^{-(\frac{1}{\lambda} - t)x} \Big|_0^\infty$$

$$= \frac{1}{\lambda} \frac{1}{\frac{1}{\lambda} - t} = \frac{1}{1 - \lambda t}$$