May 16th

Wed 5-6 SS 2135 $\longleftarrow$ Problem Session

Also MAC/TA office hours TBA

$\vec{x}$ is an interior point of $S \iff \exists r>0$, s.t. $B(r,\vec{x}) \subset S$.

$\vec{x}$ is a boundary point $\iff \forall r>0$. $B(r,\vec{x}) \cap S \neq \emptyset$ and $B(r,\vec{x}) \cap S^c \neq \emptyset$

Def: S is called open if it contains none of its boundary points.

Def: S is called closed if it contains all of its boundary points.

Proposition: S is open $\iff$ every point in S is an interior point.
$\iff S = S^{int}$

S is closed $\iff$ none of boundary points are in S^c.

i.e. S^c is open.

i.e. $S^c = (S^c)^{int}$

Claim: if A & B are open, then $A \cap B$ is open.
$A \cup B$ is open. (prove it later)

Proof: If $\vec{x} \in A \cap B \Rightarrow \vec{x} \in A$ and $\vec{x} \in B$.
Therefore $\exists r, r'$, $B(r,\vec{x}) \subset A$
$B(r',\vec{x}) \subset B$ (By the def A, B are open)

Without Loss of Generality: $r < r'$, $B(r,\vec{x}) \subset B(r',\vec{x}) \subset B$
and $B(r,\vec{x}) \subset A$

$\Rightarrow B(r,\vec{x}) \subset A \cap B$

Pop Quiz: Prove $A \cup B$ is open when A, B open.

Def: The closure of S is $\bar{S} = S \cup \partial S$

Why? Cuz S may contain part of its boundary points.

$\vec{x} \in \bar{S} \iff \forall r, B(r,\vec{x}) \cap S \neq \emptyset$

either $B(r,\vec{x}) \subset S$ or $B(r,\vec{x}) \not\subset S \Rightarrow B(r,\vec{x}) \cap S^c \neq \emptyset$

Prop: The closure is closed.

Recall: $\mathbb{R}^n = S^{int} \cup \partial S \cup S^{c\,int}$

$\bar{S}^c = S^{c\,int}$ which is open

$\Rightarrow \bar{S}$ is closed.

Note: * ① interior of a set is open.
② A set is open its complement closed.

Generalizes to $\bigcup_{i=1}^{\infty} A_i$ open

Note! $\bigcap_{i=1}^{\infty} A_i$ not open!

e.g. $A_i = (-\frac{1}{i}, \frac{1}{i})$

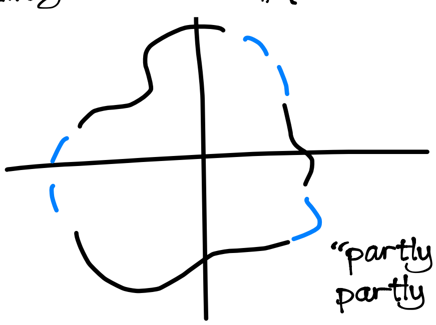
Then $\bigcap_{i=1}^{\infty} A_i = \{0\}$ which is closed.

So, union of arbitrarily many open sets is open.

But, the intersection of arbitrarily many open sets may be not.

Indeed, the intersection of finitely many open sets is open.

Imagine this in $\mathbb{R}^2$



"partly open, partly closed."

similar to $(a, b]$
$[a, b)$
$[a, b]$
(a, b)

On $\mathbb{R}$, an open ball is just like a union of intervals.

Is $\emptyset$ open or closed?

Let $S = \emptyset$

Trivially, contains no boundary points $\Rightarrow$ $\emptyset$ is open

(Note the defn of closed).

$\partial\emptyset = \emptyset$

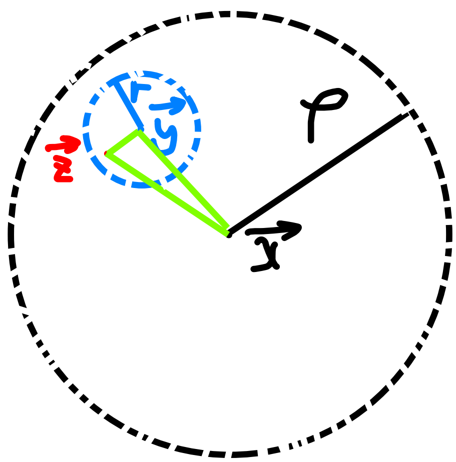
No boundary points $\Rightarrow$ $\emptyset$ has all bound points trivially. $\Rightarrow$ $\emptyset$ is closed.

Hence $S=\emptyset$ is both closed & open : "Clopen"

Another example for "clopen" : $\mathbb{R}^n$ (proof : take complements).

Claim : A ball (open ball) is open.

Proof:



got $B(\rho,\vec{x})$

Take $\vec{y}\in B(\rho,\vec{x})$

Claim: $\vec{y}\in B(\rho,\vec{x})$

$\Rightarrow \exists r\in\mathbb{R}$ s.t. $B(r,\vec{y})\subset B(\rho,\vec{x})$

Then if $\vec{z}\in B(r,\vec{y}) \Rightarrow \vec{z}\in B(\rho,\vec{x})$

Note Triangle xyz

$r<\rho-|\vec{y}-\vec{x}|$

$|\vec{z}-\vec{x}|\leq|\vec{z}-\vec{y}|+|\vec{y}-\vec{x}|$ (Tri. Ineq)

as $\vec{z}\in B(r,\vec{y})$

$\Rightarrow |\vec{z}-\vec{y}|<r<\rho-|\vec{y}-\vec{x}|$

Then $|\vec{z}-\vec{x}|<\rho+|\vec{y}-\vec{x}|-|\vec{y}-\vec{x}|$

Conclusion: $|\vec{z}-\vec{x}|<\rho$

$\Rightarrow \vec{z}\in B(\rho,\vec{x})$

Therefore $\forall\vec{y}\in B(\rho,\vec{x})$,
$\exists$ r's.t. $B(r,\vec{y})\subset B(\rho,\vec{x})$
$\Rightarrow B(\rho,\vec{x})$ is open.

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§ 1.3 LIMIT AND CONTINUITY

Def: A function $\vec{f}:\mathbb{R}^n\to\mathbb{R}^k$ has a limit $\vec{L}\in\mathbb{R}^k$ as $\vec{x}\in\mathbb{R}^n$ goes to $\vec{a}\in\mathbb{R}^n$ and we write

$$\lim_{\vec{x}\to\vec{a}}\vec{f}(\vec{x})=\vec{L}$$

If $\forall\varepsilon>0,\exists\delta>0$ s.t. $|\vec{f}(\vec{x})-\vec{L}|<\varepsilon$ when $0<|\vec{x}-\vec{a}|<\delta$.

Equivalently, $\forall\varepsilon>0,\exists\delta>0$ s.t. $\vec{f}(\vec{x})\in B(\varepsilon,\vec{L})$ when $\vec{x}\in B'(\delta,\vec{a})$

$\vec{0}$

The punctured ball
(without central point)

Recall: $\lim_{x\to a^+} f(x) = \lim_{x\to a,\, a\in S} f(x)$ $\qquad S=\{x \mid x>a\}$

$\lim_{\vec{x}\to\vec{a},\, \vec{x}\in S} \vec{f}(\vec{x})$ "Restrict"

Now we need a little proof of 1.3 on textbook

$\vec{x}=(x_1,\dots,x_n)$. let $M=\max\{|x_1|,\cdots,|x_n|\}$

$$M^2\le x_1^2+x_2^2+\cdots+x_n^2 = |\vec{x}|^2 \le nM^2$$

$$M\le|\vec{x}|\le\sqrt{n}\,M$$

Claim: $\lim_{\vec{x}\to\vec{a}}\vec{f}(\vec{x})=\vec{L} \iff \forall\varepsilon'>0, \exists\,\delta>0$, s.t. $|f_i(\vec{x})-L_i|<\varepsilon'$ when $|\vec{x}-\vec{a}|<\delta$

$f_i(\vec{x})$ is a component of $\vec{f}(\vec{x})$
L_i .... $\vec{L}$

Proof:
($\Leftarrow$) If $|\vec{f}(\vec{x})-\vec{L}|\le\sqrt{\sum_{i=1}^{n}|f_i(\vec{x})-L_i|^2}\le\underbrace{\sqrt{n}\,\varepsilon'}_{=\varepsilon}$ when $0<|\vec{x}-\vec{a}|<\delta$

Choose $\varepsilon'=\varepsilon/\sqrt{n}$, then RHS $\Rightarrow$ LHS.

($\Rightarrow$) $|f_i(\vec{x})-L_i|<|\vec{f}(\vec{x})-\vec{L}|<\varepsilon$ when $0<|\vec{x}-\vec{a}|<\delta$

Choose $\varepsilon=\varepsilon'$, then LHS $\Rightarrow$ RHS ∎

Reduce to $f:\mathbb{R}^n\to\mathbb{R}$

Claim: $\lim_{\vec{x}\to\vec{a}} f(\vec{x})=L \iff \forall\varepsilon>0,\exists\,\delta'$ s.t. $|f(\vec{x})-L|<\varepsilon$ when $|x_i-a_i|<\delta'$ $\forall i$, $1\le i\le n$

Proof:
($\Rightarrow$) $|\vec{x}-\vec{a}|<\sqrt{\sum_i|x_i-a_i|^2}\le\sqrt{n}\,\delta'$

Choose $\sqrt{n}\,\delta'=\delta$

($\Leftarrow$) $|x_i-a_i|\le|\vec{x}-\vec{a}|<\delta$ choose $\delta=\delta'$ ∎

Equivalently $\max\{|x_i-a_i|\}\to 0$.

CONTINUITY

Def: So $\vec{f}: S \subset \mathbb{R}^n \to \mathbb{R}^k$ is continuous if

1). $\vec{f}(\vec{a})$ exists $\iff \vec{a} \in S$

2). $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$

Def: $\vec{f}$ is continuous on S if it's continuous $\forall\, \vec{a} \in S$.

Theorem: Composition of continuous functions is continuous.

Let $\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ be continuous on $U \subset \mathbb{R}^n$

$\vec{g}: \mathbb{R}^m \to \mathbb{R}^k$ be continuous on $\vec{f}(U) \subset \mathbb{R}^m$

Claim: $\vec{g}(\vec{f}(\vec{x})) : U \to \vec{g}(\vec{f}(U))$

$U \subset \mathbb{R}^n \quad g(\vec{f}(U)) \subset \mathbb{R}^k$

Proof: Let $\vec{a} \in U$, $\vec{b} = \vec{f}(\vec{a}) \in \vec{f}(U)$

As $\vec{g}$ is continuous, $\forall\, \varepsilon > 0, \exists\, \eta > 0$ s.t. $|\vec{g}(\vec{y}) - \vec{g}(\vec{b})| < \varepsilon$ when $|\vec{y} - \vec{b}| < \eta$

As $\vec{f}$ is continuous, $\forall\, \eta > 0, \exists\, \delta > 0$, s.t. $|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \eta$ when $|\vec{x} - \vec{a}| < \delta$.

So $|\vec{g}(\vec{f}(\vec{x})) - \vec{g}(\vec{f}(\vec{a}))| < \varepsilon$ when $|\vec{x} - \vec{a}| < \delta$

i.e. $\vec{g}(\vec{f}(\vec{x}))$ is continuous.

$\blacksquare$

Now we discuss some typical types in details:

Ex 1: $f(x,y) = C \quad |f(\vec{x}) - f(\vec{a})| = |c - c| = 0 < \varepsilon$

Trivially find a δ s.t. above is TRUE.

Ex 2: $f(x,y) = x$

$|f(x,y) - f(a,b)| = |x - a| < \varepsilon$

Then we let $\delta = \varepsilon$ to prove the above.

Thm: Addition, substraction, multiplication, & division ($\neq 0$) of cont. funcs is cont.

Proof for Addition:

Let $\vec{f}: \mathbb{R}^n \to \mathbb{R}$, $\vec{g}: \mathbb{R}^n \to \mathbb{R}$ be cont.

As $\vec{f}$ is continuous

$|\vec{f}(\vec{x}) - \vec{f}(\vec{a})| < \frac{\varepsilon}{2}$ when $|\vec{x} - \vec{a}| < \delta_f$

As $\vec{g}$ is continuous

$|\vec{g}(\vec{x}) - \vec{g}(\vec{a})| < \frac{\varepsilon}{2}$ when $|\vec{x} - \vec{a}| < \delta_g$

Let $\delta = \min\{\delta_f, \delta_g\}$

then $|(\vec{f}(\vec{x})+\vec{g}(\vec{x}))-(\vec{f}(\vec{a})+\vec{g}(\vec{a}))| \leq |\vec{f}(\vec{x})-\vec{f}(\vec{a})|+|\vec{g}(\vec{x})-\vec{g}(\vec{a})|$

$$= \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$$

$\blacksquare$

Tip for $\varepsilon-\delta$ proof :
Mostly we don't know what "δ", "ε" to use, just use some letter to represent first, then we build their connection with δ or ε.

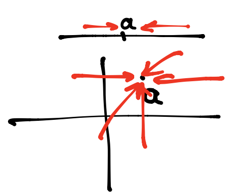
Equivalently, prove $f(x,y) = x+y$ is continuous & we use compositions to prove.

Ex : $f(x,y) = \dfrac{xy^3}{x^2+y^6}$, $f(0,0)=0$.

Continuity ?

For $\mathbb{R}$

For $\mathbb{R}^2$



$f(x,0) \to 0$ as $x \to 0$

① $y=cx$ $\quad \dfrac{c^3x^4}{x^2+c^6x^6} = \dfrac{c^3x^2}{1+c^6x^4} \to 0$ as $x \to 0$

② $y = \sqrt[3]{cx}$

$f(x, \sqrt[3]{cx}) = \dfrac{cx^2}{x^2+c^2x^4} = \dfrac{c}{1+c^2} \neq 0$ not zero necessarily

e.g. $c=1$, $f(x, \sqrt[3]{cx}) = \frac{1}{2}$

$\Rightarrow$ Therefore limit DOES NOT EXIST.
i.e. not continuous

NOTE : Don't use ε-δ here.

Ex: $f(x,y) = \dfrac{x^3-xy^2}{x^2+y^2}$, 0 at $(0,0)$

so limit exists away from $\vec{0}$.

$|f(x,y)| = \left|\dfrac{x^3-xy^2}{x^2+y^2}\right|$

$|x^3-xy^2| \leq |x||x^2-y^2| \leq |x|(x^2+y^2) \leq |x|$

so $f \to 0$ as $\vec{x} \to \vec{0}$

$\Rightarrow L$ at $\vec{0}$ is 0.

INVERSE IMAGE

Let $f: \mathbb{R}^n \to \mathbb{R}^k$. let $U \subset \mathbb{R}^k$

$f^{-1}(U) = \{x \in \mathbb{R}^n \mid f(x) \in U\} = S$

f^{-1} is not a function, it's called "Inverse image".

Claim: $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$

Proof: $x \in f^{-1}(U \cup V) \iff f(x) \in U \cup V$

$\iff f(x) \in U$ or $f(x) \in V$

$\iff x \in f^{-1}(U)$ or $x \in f^{-1}(V)$

$\iff x \in f^{-1}(U) \cup f^{-1}(V)$

So $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ ∎

Analegously, for intersection, compliment, subset...

* S open, f cont: $f(S)$ not necessarily open

Ex: $f(x) = 1$.

Thm: If U is open, then $f^{-1}(U) = S$ is open, when f is continuous.

Proof:

($\Leftarrow$) Let f cont., U be open.

$\Rightarrow x \in U, \exists \varepsilon > 0, \exists B(\varepsilon, f(x)) \subset U$

As f is cont.

$\exists \delta > 0, |x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$

$f^{-1}(B(\varepsilon, f(a)) \subset S$

($\Rightarrow$) skip it ...

∎

Claim: U is closed then $f^{-1}(U) = S$ is closed likewise.

Proof:

U is closed $\iff U^c$ is open.

$\iff (f^{-1}(U))^c$ is open

$(f^{-1}(U))^c = f^{-1}(U^c)$

Then $f^{-1}(U)$ is closed.

∎

Define (topologically) f is continuous via $f^{-1}(\text{open}) = \text{open}$.

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§ 1.4 Sequences (A bite of)

Intuitive def: A sequence is a list of mathematical objects indexed by positive int. $\mathbb{Z}^+$.

$\{x_k\}_{k=1}^{\infty}$, Ex: $1, -1, 1, -1, \cdots$

Rigorous def: A sequence is a function $f: \mathbb{Z}^+ \to S$ where $S = \{x_1, x_2, x_3, \cdots\}$

$$\begin{aligned} 1 &\longrightarrow x_1 \\ 2 &\longrightarrow x_2 \\ 3 &\longrightarrow x_3 \\ \cdots & \quad\ \ \cdots \end{aligned}$$