Lecture 6.

§ 2.8 C. One of suggested question.

Let $(a_n)$ be a sequence s.t. $\lim_{n\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| < \infty$ Show that $(a_n)$ is Cauchy.

Cauchy Criterion for series

$\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $|\sum_{k=n+1}^{m} a_k| < \varepsilon$

$\lim_{N\to\infty} \sum_{n=1}^{N} |a_n - a_{n+1}| = L < \infty$

$\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges $\Rightarrow$ the sequence of partial sums is Cauchy.

$\forall \varepsilon > 0$, $\exists N$ s.t. $\sum_{k=n+1}^{m} |a_k - a_{k+1}| < \varepsilon$, $\forall m, n \geq N$.

Now consider $|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} + \cdots + a_{n+1} - a_n| \leq |a_m - a_{m-1}| + \cdots + |a_{n+1} - a_n|$

$= \sum_{k=n}^{m-1} |a_k - a_{k+1}| < \varepsilon$

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Continuing Topology of $\mathbb{R}^n$

Continue §4.2 Convergence and Completeness in $\mathbb{R}^n$

A seq of points $(x_k)$ in $\mathbb{R}^n$

Lemma: Let $(x_k)$ be a seq of point in $\mathbb{R}^n$. Then $\lim_{k\to\infty} x_k = a$ iff $\lim_{k\to\infty} \|x_k - a\| = 0$

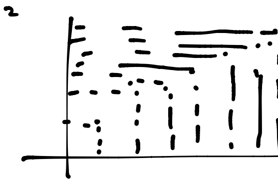
Proof: (1) Suppose $\lim_{k\to\infty} x_k = a$, want to show $\lim_{k\to\infty} \|x_k - a\| = 0$

We know that for every $\varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\|x_k - a\| < \varepsilon$ when $k \geq N$.

Suppose $\varepsilon > 0$ is given we want to show that $| \|x_k - a\| - 0| = \|x_k - a\| < \varepsilon$

Definitely $\exists N$ by our assumption. (skip the other direction)

Lemma. A seq. $(x_k)$, where $x_k = (x_{k,1}, x_{k,2}, \cdots, x_{k,n})$ in $\mathbb{R}^n$ converges to a point $a = (a_1, \cdots, a_n)$ iff each coefficient converges, $\lim_{k\to\infty} x_k = a \iff \lim_{k\to\infty} x_{k,i} = a_i$ for $1 \leq i \leq n$




Proof: ($\Rightarrow$) Suppose $\lim_{k\to\infty} x_k = a$, given $\varepsilon > 0, \exists N$ s.t. $\|x_k - a\| < \varepsilon$ when $k \geq N$

$|x_{k,i} - a_i| \leq \|x_k - a\| < \varepsilon$ when $k \geq N$

($\Leftarrow$) Suppose $x_{k,i} \longrightarrow a_i \ \forall\ 1 \leq i \leq n$

We want to make $\|x_k - a\| < \varepsilon$

Let $\varepsilon > 0$ be given, we know that $\exists N_i$ s.t. $|x_{k,i} - a_i| < \frac{\varepsilon}{n}$ for all $k \geq N_i$

Let $N = \max\{N_1, \cdots, N_n\}$

$$\|x_k - a\| = \sqrt{\|x_{k,1}-a_1\|^2 + \cdots + \|x_{k,n} - a_n\|^2} < \sqrt{n \cdot \frac{\varepsilon^2}{n^2}} = \frac{\varepsilon}{\sqrt{n}} < \varepsilon, \quad k \geq N$$ ■

**Def :**

A seq $x_k$ in $\mathbb{R}^n$ is Cauchy if $\forall \varepsilon > 0, \exists N$ s.t $\|x_k - x_l\| < \varepsilon$ for all $k, l \geq N$

**Def:** A set $S \subset \mathbb{R}^n$ is complete if every Cauchy sequence of pts in $S$ converge to a pt in $S$.

**Thm:** Completeness Theorem for $\mathbb{R}^n$ (Compare with $\mathbb{R}$)
every Cauchy sequence in $\mathbb{R}^n$ converges. Thus $\mathbb{R}^n$ is complete.

Proof:
Let $x_k = (x_{k,1}, \cdots, x_{k,n})$ be a sequence of pts in $\mathbb{R}^n$
Let $\varepsilon > 0$ be given, $|x_{k,i} - x_{l,i}| \leq \|x_k - x_l\| < \varepsilon$.
$\forall k, l \geq N$
$\Rightarrow (x_{k,i})$ is Cauchy $\forall\ 1 \leq i \leq n$
$\Rightarrow (x_{k,i})$ converges $\forall\ 1 \leq i \leq n$ to $a_i \in \mathbb{R}$
$\Rightarrow (x_k)$ converges to $\vec{a} = (a_1, \cdots, a_n) \in \mathbb{R}^n$. ■

## §4.3

### Closed & open subsets of $\mathbb{R}^n$

**Def:** A pt $x$ is a limit point of a subset $A$ of $\mathbb{R}^n$ if $\exists$ a sequence $(a_n)_{n=1}^{\infty}$, $a_n \in A$ s.t. $\lim_{n\to\infty} a_n = x$.

A set $A$ is closed if it contains all of its limit pt.
① $[a,b] \subset \mathbb{R}$ closed
② $\emptyset$ closed, $\mathbb{R}^n$ closed
③ $[0, +\infty)$ closed in $\mathbb{R}$
$(0, +\infty)$ not closed
$(0, 1]$ not closed

**Proposition:** If $A, B \subset \mathbb{R}^n$, are closed $\Rightarrow A \cup B \subset \mathbb{R}^n$ is closed. If $\{A_i : i \in I\}$ is a family of closed sets in $\mathbb{R}^n$, then $\bigcap_{i \in I} A_i$ is closed.

Proof: (1) We want to show $A \cup B$ is closed
Sps $(x_n)_{n=1}^{\infty}$ is a seq of pts in $A \cup B$ s.t. $\lim_{n\to\infty} x_n = x$
Sps WLOG, that for infinitely many
$x_n \in A \Rightarrow \exists$ a subseq $(x_{n_k})$ in $A$
$x_{n_k} \to x$, since $A$ is closed $x \in A \Rightarrow x \in A \cup B$ ■

(2) exercise

**Def:** if $A$ is a subset of $\mathbb{R}^n$, the closure of $A$, $\overline{A}$ is the smallest set that contains all limit pts of $A$.

**Proposition:** Let $A \subset \mathbb{R}^n$, then $\overline{A}$ is the smallest closed set containing $A$. In particular, $\overline{\overline{A}} = \overline{A}$

Proof: Need to prove ① $\overline{A} \supset A$
② $\overline{A}$ is closed
③ smallest such set.

Part ① : Let $a \in A$, let $x_n = a, \forall n$. $x_n \to a \Rightarrow a$ is a limit pt of $A \Rightarrow a \in \overline{A}$

Part ②: We would like to show that if $x$ is a limit point of pts of $\bar{A} \Rightarrow x \in \bar{A}$.

We will show that if $x$ is a limit point of points of $\bar{A} \Rightarrow$ it is a limit pt of points of $A$.

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