March 26th

<u>Basic idea</u> : V complex vector space

Given $T: V \to V$, we want to decompose $V = W_1 \oplus \cdots \oplus W_k$ s.t.

① Each W_i is T-stable

② $T|_{W_i} : W_i \to W_i$ has only one eigenvalue.

<u>Note:</u> If T is diaganolizable then this is easy since we could take a basis of eigenvalues $\{v_1, \cdots, v_n\}$ $T(v_i) = \lambda_i v_i$ and define $W_i = sp\{v_i\}$

Fix $T: V \to V$ let $p(x)$ char. poly.

Carley – Hamilton thm says $p(T) = 0$

Since we're working over $\mathbb{C}$ we know $p(x) = (x-\lambda_1)^{r_1}(x-\lambda_2)^{r_2} \cdots (x-\lambda_k)^{r_k}$

Define : $f_i(x) = \frac{p(x)}{(x-\lambda_i)^{r_i}}$ Eg. If $p(x) = (x-1)^2(x-2)^3(x-4)$ then $f_1(x) = (x-2)^3(x-4)$

From basis number theory :

$n_1, \cdots, n_k$

are relatively prime $\iff \exists\, d_1, \cdots, d_k \in \mathbb{Z}$ s.t. $d_1 n_1 + \cdots + d_k n_k = 1$

The poly $f_1, \cdots, f_k$ are "relatively prime" i.e. they have no common factors

E.g. $f_1(x) = (x-2)^3(x-4)$, $f_2(x) = (x-1)^3(x-4)$

$f_3(x) = (x-1)^3(x-2)^3$

A factor from basic algebra $\exists\, g_1, \cdots, g_k$ s.t.

$g_1 f_1 + \cdots + g_k f_k = 1$

Now define $E_i : V \to V$ by $E_i = g_i(T) f_i(T)$

① $E_1 + \cdots + E_k = I$

② If $i \neq j$, then $E_i E_j = 0$

Why? $E_i E_j = g_i(T) f_i(T) g_j(T) f_j(T) = g_i(T) g_j(T) f_i(T) f_j(T)$

$f_i(x) f_j(x) = \underline{\quad ? \quad}\, p(x)$

<u>Back to ex</u> : $f_1(x) f_2(x) = (x-1)^3 (x-2)^3 (x-4)^2 = p(x) \cdot (x-4)$

Let $W_i = E_i(V)$ i.e. $im(E_i)$

Claim: $V = W_1 \oplus \cdots \oplus W_k$

<u>Proof</u> : need to show $W_1 + \cdots + W_k = V$

and $W_i \cap W_j = \{0\}$ for $i \neq j$

By condition ① for $v \in V$ we have $v = E_1(v) + \cdots + E_k(v)$

of course $E_i(v) \in W_i \Rightarrow V = W_1 + \cdots + W_k$

Suppose $v \in W_i \cap W_j$. Then

$v = E_i(v_i)$ and $v = E_j(v_j)$

Also have $E_1(v_i) + \cdots + E_k(v_i) = v_i$

Apply E_i to both sides of this equation

$E_iE_1(v_i) + E_iE_2(v_i) + \cdots + E_iE_k(v_i) \underset{②}{\Rightarrow} E_i^2(v_i) = E_i(v_i) \Rightarrow E_i(v) = v$

Similarly, have $E_j(v) = v$. So applying E_j to both sides of $E_i(v) = v$

get $E_jE_i(v) = E_j(v) \Rightarrow 0 = v \Rightarrow W_i \cap W_j = \{0\}$

$\blacksquare$

$p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

Claim: $W_i = \ker((T-\lambda_i I)^{r_i})$

Pf: First show $W_i \subseteq \ker((T-\lambda_i)^{r_i})$.

Take $w \in W_i = \mathrm{im}(E_i)$, so $\exists\, v$ s.t. $w = E_i(v) = (g_i(T)f_i(T))(v)$

$(T-\lambda_i I)^{r_i}(w) = (T-\lambda_i I)^{r_i} g_i(T) f_i(T)(v)$

$((T-\lambda_i I)^{r_i} f_i(T) = p(T) = 0))$

$= g_i(T)(T-\lambda_i I)^{r_i} f_i(T)(v)$

$= g_i(T)p(T)(v) = 0$

Now we show $\ker((T-\lambda_i I)^{r_i}) \subseteq W_i$

Let $w \in \ker((T-\lambda_i I)^{r_i})$. For $j \neq i$

$E_j(w) = g_j(T) f_j(T)(w) = 0$ since $f_j(x) = \underline{\quad ? \quad}(x-\lambda_i)^{r_i}$

$\blacksquare$

$p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

Claim: W_i is T-invariant

Pf: $w \in W_i = \ker((T-\lambda_i I)^{r_i})$.

To show $T(w) \in W_i$ suffices to show $(T-\lambda_i I)^{r_i}(T(w)) = 0$.

$(T-\lambda_i I)^{r_i}(T(w)) = T((T-\lambda_i I)^{r_i}(w)) = T(0) = 0$. $\blacksquare$

Upshot: Given $T: V \to V$ with

$p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$ then letting $W_i = \ker((T-\lambda_i)^{r_i})$ we have

① $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$

② Each W_i is T-invariant

③ T has one eigenvalue on W_i

This is called the "primary decomposition" of V relative to T. W_i is called the generaliz-ed λ_i eigenspace.

Note: The λ_i-eigenspace is $\ker(T-\lambda_i I)$ so λ_i-eigenspace $\subseteq$ λ_i-generalized eigenspace.

HALF-CLASS

Strategy: Given $T: V \to V$

① compute $p(x) = (x-\lambda_1)^{r_1} \cdots (x-\lambda_k)^{r_k}$

② $W_i = \ker((T-\lambda_i I)^{r_i})$, there are T-invariant subspace and $T|_{W_i}$ has only 1 eigenvalue which is equal to λ_i.

③ Apply JCF thm to each $T|_{W_i}$, this obtaining a basis γ_i of W_i s.t. $[T|_{W_i}]_{\gamma_i} =$
$J_{n_1}(\lambda_i) \oplus \cdots \oplus J_{n_s}(\lambda_i)$

④ Put all together; $\gamma = \gamma_1 \cup \cdots \cup \gamma_k$
$[T]_\gamma = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_1) \oplus \cdots \oplus J_{m_1}(\lambda_k) \oplus \cdots \oplus J_{m_t}(\lambda_k)$

Thm: $T: V \to V$ there is a "canonical basis" γ of V s.t.
$[T]_\gamma = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_s}(\lambda_k)$.
Moreover, the $n_1, \ldots, n_s$ are unique (up to ordering)

LAST PART OF THIS COURSE

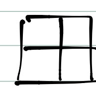
Ex: $T: \mathbb{C}^8 \to \mathbb{C}^8$ and $p(x) = (x-1)^4 (x-2)^4$ and $\dim\ker(T-I) = 2$ $\dim\ker((T-I)^2) = 4$
$\dim\ker(T-2I) = 2$, $\dim\ker((T-2I)^2) = 3$, $\dim\ker((T-2I)^3) = 4$. What's JCF of T?

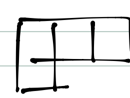
$\mathbb{C}^8 = W_1 \oplus W_2$
$W_1 = \ker((T-I)^4)$
$W_2 = \ker((T-2I)^4)$
Consider, $T|_{W_1}: W_1 \to W_1$ with one evalue.
Set $N_1 = (T-I)|_{W_1}$ nilpotent operator on W_1
$\dim\ker N_1 = \dim\ker((T-I)|_{W_1})$
Observe: $\ker((T-I)|_{W_1}^j) = \ker((T-W_1)^j)$

for $j = 1, \cdots, r_1$
$\ker((T-I)^j) \subset \ker((T-I)^{r_1}) = W_1$

$\Rightarrow \ker($. - — .

So $\dim\ker N_1 = 2$
$\dim\ker N_1^2 = 4$

$\dim W_1 = 4$, i.e. $\ker N_1^2 = W_1$. So tableau of N_1 

So JCF of N_1 is $J_2(0) \oplus J_2(0)$
- - - - - - of $T|_{W_1}$ is $J_2(1) \oplus J_2(1)$
Now consider $T|_{W_2}$ Let $N_2 = (T-2I)|_{W_2}$ has tableau  so the JCF of $T|_{W_2}$ is $J_3(2) \oplus J_1(2)$

So JCF of T is $J_2(1) \oplus J_2(1) \oplus J_3(2) \oplus J_1(2)$

$$\begin{bmatrix} \boxed{\begin{matrix}1&1\\0&1\end{matrix}} & & & O \\ & \boxed{\begin{matrix}1&1\\0&1\end{matrix}} & & \\ & & \boxed{\begin{matrix}2&1&0\\0&2&1\\0&0&2\end{matrix}} & \\ O & & & \boxed{2} \end{bmatrix}$$