Lecture 19 §7.6 continue Fourier Series

$A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$

$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\,dt$

$n \geq 1,\ A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\cos nt\,dt \qquad B_n = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t)\sin nt\,dt$

$\{1, \sqrt{2}\cos n\theta, \sqrt{2}\sin n\theta, n \geq 1\}$ form an orthonormal set in $PC[-\pi, \pi]$

Find the Fourier series for $\cos^3\theta$

$\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

$$\cos^3\theta = \frac{e^{3i\theta} + e^{-3i\theta} + 3e^{i\theta} + 3e^{-i\theta}}{8} = \frac{1}{4}\left(\frac{e^{3i\theta} + e^{-3i\theta}}{2} + 3\,\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$

$$= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

§7.7 Orthogonal Expansions & Hilbert Spaces

recall Projection Thm:

M finite-dimensional $\subset$ inner product space V

P orthogonal projection. range $P = M$

For all $y \in V$, $x \in M$, $\|y - x\|^2 = \|y - Py\|^2 + \|Py - x\|^2$

$Py = \sum_{j=1}^{n} \langle y, e_j \rangle e_j$ for all $y \in V$

$\|y\|^2 \geq \sum_{j=1}^{n} \langle y, e_j \rangle^2$

Bessel's inequality

Let $S \subseteq \mathbb{N}$, let $\{e_n : n \in S\}$ be an orthonormal set in $(V, \langle \cdot, \cdot \rangle)$

For $x \in V$, $\boxed{\sum_{n \in S} |\langle x, e_n \rangle|^2 \leq \|x\|^2}$

If S is finite $\Rightarrow$ done.

Suppose S is infinite, $\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^2$

$\lim_{N \to \infty} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2$

$\sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \leq \|x\|^2$ true for any V

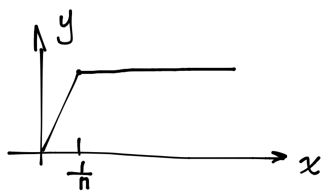
$\lim_{N \to \infty} \sum_{n=1}^{N} |\langle x, e_n \rangle|^2 \leq \|x\|^2 \qquad \blacksquare$

Def: A Hilbert Space is a complete inner product space

Example: $C[-\pi, \pi]$

Take an l_2 norm.

$$f_n(x) = \begin{cases} 0 & -\pi \leq x \leq 0 \\ nx & 0 \leq x \leq \frac{1}{n} \\ 1 & \frac{1}{n} \leq x \leq \pi \end{cases}$$



$\chi_{[0,\pi]} \qquad \|f_n - \chi\|_2^2 = \int_0^{1/n} (n^2x^2 - 2nx + 1)\,dx = \frac{1}{3n} \longrightarrow 0$ as $n \to \infty$

f_n is a Cauchy sequence

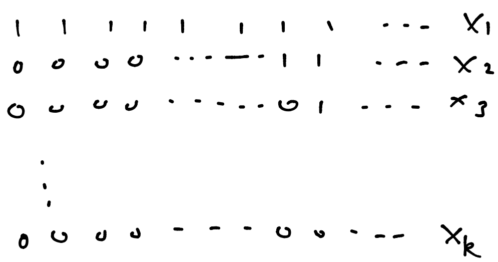
Example: l^2 consists of all sequences $x=(x_n)_{n=1}^{\infty}$ s.t. $\|x\|_2=\left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}<\infty$

$\langle x,y\rangle=\sum_{n=1}^{\infty} x_n y_n$

Theorem:
The inner product space l^2 is complete. i.e.

Proof:
Let $x_k=(x_{k,n})_{n=1}^{\infty}$ be Cauchy.

We would like to show that it converges to a vector $x\in l^2$
We know that $\forall \varepsilon>0, \exists K$ s.t. $|x_{k,n}-x_{l,n}|\le\|x_k-x_l\|<\varepsilon \quad \forall k,l\ge K$



For each coordinate n, (n is fixed) $(x_{k,n})_{k=1}^{\infty}$ this is Cauchy
So $(x_{k,n})_{k=1}^{\infty}$ converges to some point $y_n\in\mathbb{R}$.

Let $y=(y_n)_{n=1}^{\infty}$

Claim (1) $y\in l^2$
(2) $\lim_{k\to\infty} x_k=y$

(1) Want to show that $\sum_{n=1}^{\infty}|y_n|^2<\infty$

$$\sum_{n=1}^{\infty}|y_n|^2 = \lim_{N\to\infty}\sum_{n=1}^{N}|y_n|^2$$

want to show that these partial sums are bounded

$\|x_k\|_{k=1}^{\infty}$ is Cauchy

$$\big|\,\|x_k\|-\|x_l\|\,\big|\le\|x_k-x_l\|<\varepsilon \qquad \forall k,l\ge K$$

$$\lim_{k\to\infty}\|x_k\|=L$$

$$\sum_{n=1}^{N}|y_n|^2=\sum_{n=1}^{N}\left|\lim_{k\to\infty}x_{k,n}\right|^2=\lim_{k\to\infty}\sum_{n=1}^{N}|x_{k,n}|^2\le\lim_{k\to\infty}\|x_k\|^2=L^2$$

Therefore $\lim_{N\to\infty}\sum_{n=1}^{N}|y_n|^2\le L^2$

(2) Suppose we are given $\varepsilon>0$ (let K be s.t. $\|x_k-x_l\|<\varepsilon$, $\forall k,l\ge K$

Consider, $\sum_{n=1}^{N}|y_n-x_{k,n}|^2$
(I want $\sum_{n=1}^{\infty}|y_n-x_{k,n}|^2$)

$$\lim_{l\to\infty} \sum_{i=1}^{N} |x_{l,n} - x_{k,n}|^2 \le \lim_{l\to\infty} \|x_l - x_k\|^2 \le \varepsilon^2$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} |y_n - x_{k,n}|^2 \le \varepsilon^2$$

So $x_k \to y$ $\blacksquare$

Def: In a Hilbert space, consider a set of vectors S.

closed span $\overline{\text{span } S}$ = closure of the linear subspace spanned by S.

It is still a subspace

It is a closed subspace of a Hilbert space $\Rightarrow$ it is complete

orthonormal basis of Hilbert space is a set of orthonormal vectors that is maximal as an orthonormal set.

Parseval's Thm:

Let $S \subset \mathbb{N}$ and $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space H

Then the subspace $M = \overline{\text{span } E}$ consists of all vectors.

$x = \sum_{n\in S} a_n e_n$, where $(a_n)_{n=1}^{\infty}$ coefficient sequence belongs to l^2

Also for all $x \in H$, then $x \in M \iff \sum_{n\in S} |\langle x, e_n\rangle|^2 = \|x\|^2$

COR1. Let $E = \{e_n : n \in S\}$ be an orthonormal set in a Hilbert space H. Then there is a continuous linear orthogonal projection P_M of H onto $M = \overline{\text{span } E}$ given by $P_M x = \sum_{n\in S} \langle x, e_n\rangle e_n$

COR2: If $E = \{e_i : i \ge 1\}$ is an orthonormal basis of a Hilbert space H, every vector $x \in H$ may be uniquely expressed as $x = \sum_{i=1}^{\infty} \langle x_i, e_i\rangle e_i$

Chapter 8 Limits of Functions

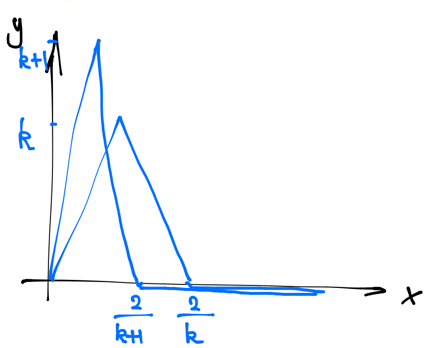
§8.1 Limits of functions

Def: Let (f_n) be a sequence of functions from $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$

This sequence converges pointwise to a function f, if $\lim_{k\to\infty} f_k(x) = f(x)$ $\forall x \in S$

Example:

$$f_k(x) = \begin{cases} kx^2 & 0 \le x \le \frac{1}{k} \\ k^2(\frac{2}{k} - x) & \frac{1}{k} \le x \le \frac{2}{k} \\ 0 & \frac{2}{k} \le x \le 1 \end{cases}$$

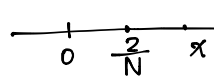


$\lim_{k\to\infty} f_k(x) = 0$ $\forall\, 0 \le x \le 1$

At $x=0$, $f_k(0) = 0$ $\forall k \ge 1$

$x > 0 \Rightarrow \exists N > 0$ s.t. $x > \frac{2}{N}$

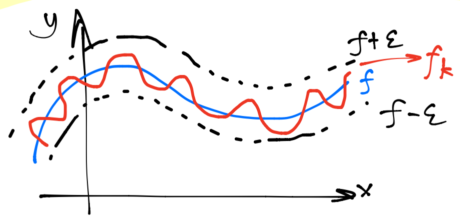
If $k > N \Rightarrow f_k(x) = 0$ done.



How about integrals of f_k $\quad \int_0^1 f_k\,dx = \frac{1}{2}\cdot\frac{2}{k}\cdot k = 1$

$f_k \to f \equiv 0, \quad \int_0^1 f = 0$

Def: Let (f_k) be a sequence of functions from $S \subset \mathbb{R}^n \to \mathbb{R}^m$. This sequence converges uniformly to f if $\forall \varepsilon > 0, \exists N$ s.t. $\|f_k(x) - f(x)\| < \varepsilon, \forall x \in S, k \geq N$.



uniform convergence $\Rightarrow$ pointwise convergence

Define a norm on the space $C(K)$ of all real-valued continuous functions on K by

Let K be a compact subset of $\mathbb{R}^n$

$\|f\|_\infty = \sup_{x \in K} |f(x)|$

Can instead consider $\quad C_b(S, \mathbb{R}^m) =$ the set of bounded continuous functions $f: S \to \mathbb{R}^m$

$\|f\|_\infty = \sup_{x \in S} \|f(x)\|_2$

have following

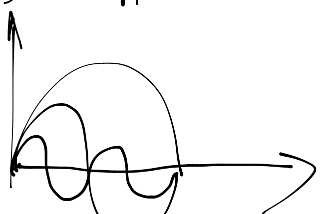
Thm: Given $S \subset \mathbb{R}^n$, and a sequence $(f_k)_{k=1}^\infty \in C(S, \mathbb{R}^m)$, f_k converges uniformly to f iff $f_k - f \in C_b(S, \mathbb{R}^m)$ for all k sufficiently large & $\lim_{k\to\infty} \|f_k - f\|_\infty = 0$

Ex: $f_k(x) = x^k, x \in [0,1]$

$\lim_{k\to\infty} f_k(x) = \begin{cases} 0 & \text{otherwise} \\ 1 & x = 1 \end{cases}$ pointwise

$f_k(x)$ does not converge uniformly.

Ex: $f_n(x) = \frac{1}{n}\sin nx \quad [0, \pi]$ $\qquad$ $f_n(x)$ converges uniformly to 0.



$0 \leq |\frac{1}{n}\sin nx| \leq \frac{1}{n}$

$\frac{1}{n} \to 0$ as $n \to \infty$

§ 8.2 Uniform Convergence and continuity.

Theorem: Let (f_k) be a sequence of continuous functions $f_k: S \subset \mathbb{R}^n \to \mathbb{R}^m$ that converges uniformly to f. Then f is continuous.

Proof: Fix $a \in S$ and let $\varepsilon > 0$ be given

$$\|f(x) - f(a)\| = \|f(x) - f_k(x) + f_k(x) - f_k(a) + f_k(a) - f(a)\| \leq \|f(x) - f_k(x)\| + \|f_k(x) - f_k(a)\| + \|f_k(a) - f(a)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

(I know $\exists K$ s.t.: $\|f_k(x) - f(x)\| < \varepsilon/3$ for all $k \geq K$)

By consisting of f_k $\exists \delta > 0$ s.t. $\|f_k(x) - f_k(a)\| < \frac{\varepsilon}{3}$

$x^k \quad f = \begin{cases} 0 & x \neq 1 \\ 1 & x = 1 \end{cases}$

Completeness Theorem for $C(K, \mathbb{R}^m)$.

If $K \subset \mathbb{R}^n$ is a compact set, the space of all continuous $\mathbb{R}^m$-valued functions on K with the sup norm is complete $C[-\pi, \pi]$. Then f is continuous.