Lecture 6

X is separable (ctble dense subset)

X is second ctble (ctble basis)

Examples from last time:
① $\mathbb{R}_{usual}$ is separable (e.g. $\mathbb{Q}$ are ctble & dense)
② $\mathbb{R}_{discrete}$ is not separable

Last time we actually saw
1. $\mathbb{R}_{usual}$ is second ctble
2. $\mathbb{R}_{discrete}$ is not second ctble

Other examples
3. The Sorgenfrey Line is separable (e.g. $\mathbb{Q}$ is dense here)
4. $\mathbb{R}_{co\text{-}countable}$ is not separable.

recall: $\mathbb{R}_{co\text{-}countable}$ is the topology on $\mathbb{R}$ where U is open iff $U=\emptyset$ or $\mathbb{R}\setminus U$ is ctble.

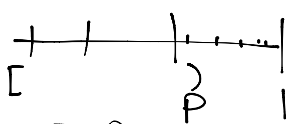
Let's see that $\mathbb{R}_{co\text{-}countable}$ is not separable.
Pf: Fix $C\subseteq\mathbb{R}$ which is ctble.
Q: is $\overline{C}=\mathbb{R}$?
No (not at all!) as $\mathbb{R}\setminus C$ is an open set disjoint from C.

§5. Covergence & 1st Countable spaces.

In 1st year calc we laboured over this arrow: $x_n \longrightarrow x$
1st year's def'n: $x_n\to x$ if $\forall\varepsilon>0\ \exists N$ s.t. $n\geq N \implies |x-x_n|<\varepsilon$.
iff $\forall\varepsilon>0, \exists N$ s.t. $n\geq N\Rightarrow x_n\in B_\varepsilon(x)$
iff $\forall B_\varepsilon(x)\ \exists N$ st $n\geq N\Rightarrow x_n\in B_\varepsilon(x)$
iff $\forall$ basic open B, $\exists N$ s.t. $n\geq N\Rightarrow x_n\in B$
(containing x)
iff $\forall$ open $U\ni x, \exists N$ s.t. $n\geq N\Rightarrow x_n\in U$

def'n: Let $(X,\mathcal{T})$ be a top space, let $\langle x_n\rangle$ be a sequence in X and let p be a point in X
We say $x_n\to x$ if for all open sets U containing x, there is an $N\in\mathbb{N}$ st. $n\geq N\Rightarrow x_n\in U$

e.g. 1. In $\mathbb{R}_{usual}$, $\langle\frac{1}{n}\rangle$ converges to 0,
2. In the SL $\langle\frac{1}{n}\rangle$ also converges to 0, but $\langle\frac{n}{n+1}\rangle$ does not converge to 1 in the SL! (e.g. $[1,7)$ is an open set containing 1, but not $\frac{n}{n+1}$)



3. In $\mathbb{R}_{discrete}$, what sequences converge?
Does $\langle\frac{1}{n}\rangle$ converge to anything?
No. Because if $\frac{1}{n}\to p$ here, well, $\{p\}$ is an open set that would have to contain $\frac{1}{n}$ for $n\geq$ some N.
Note that $0,1,2,7,p,p,p,\cdots$ does converge to p.
(These sequences are called eventually constant)

4. Look at $\mathbb{R}_{\text{indiscrete}}$ topology.

Claim: Every sequence converges to every point. (SO WEIRD)

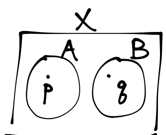
Pf: let $\langle x_n \rangle$ be a sequence, and $p \in \mathbb{R}$ indiscrete.
Take U open containing p. So $U = \mathbb{R}$. so for $N=1$, and $n \geq N$, $x_n \in U$

5. In $\mathbb{R}_{\text{cocountable}}$, does the sequence $\langle \frac{1}{n} \rangle$ converge to 0?
No, as $C = \{\frac{1}{n} : n \in \mathbb{N}\}$ is ctble, and $0 \in \mathbb{R} \setminus C$ is open, and $\mathbb{R} \setminus C$ is disjoint from $\{\frac{1}{n} : n \in \mathbb{N}\}$

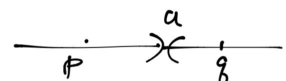
It isn't nice when a sequence converges to more than one point.
So we would like to find a property that a topological space could have where a sequence can only converge to a single point (if it does converge).

def'n: A top space $(X, \mathcal{T})$ is a Hausdorff space (or T_2)

If for distinct points, $p \neq q$, in X, there are disjoint open sets A, B such that $A \ni p$ and $B \ni q$

pic 

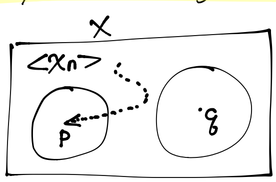
"You can separate points by disjoint open sets"



e.g. 1. $\mathbb{R}_{\text{usual}}$ (Take $p \neq q$. let $a = \frac{p+q}{2}$. Note $p \in (-\infty, a)$ and $q \in (a, \infty)$, wlog $p < q$.

2. $\mathbb{R}_{\text{discrete}}$ is Hausdorff.
3. $\mathbb{R}_{\text{discrete}}$ is NOT Hausdorff.
4. X indiscrete is NOT Hausdorff (unless in the silly case where X only has 1 element.)
5. $X = \{a, b, c\}$, take $\mathcal{T} = \{\emptyset, X, \{a, b\}\}$ is not Hausdorff.

Theorem: If $(X, \mathcal{T})$ is a Hausdorff top-space, and $\langle x_n \rangle$ is a sequence X, and $p \neq q \in X$, and $x_n \longrightarrow p$, then $x_n \not\longrightarrow q$ ("In T_2 spaces, sequences converge to at most 1 point")

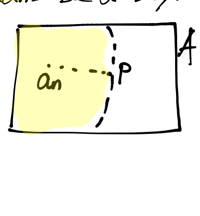
Pic 

"mostly inside $B_\varepsilon(p)$"

Pf: Suppose $x_n \longrightarrow p$. Since $(X, \mathcal{T})$ is Hausdorff, find U, V disjoint open with $p \in U$ and $q \in V$
There is a $N \in \mathbb{N}$ s.t. $(n \geq N \Rightarrow x_n \in U$

In particular, $x_n \notin V$ for $n \geq N$, so $x_n \not\longrightarrow q$ ∎

Theorem: Let $(X, \mathcal{T})$ be a topological space, let $A \subseteq X$ and $p \in X$.
Let $\langle a_n \rangle$ be a sequence in A. If $a_n \longrightarrow p$, then $p \in \overline{A}$.

Pic: 

Pf: Exercise.

Natural question: "Is the converse true?"

No, the converse can be false. (In analysis, it's true)

For example, $\mathbb{R}_{cocountable}$, let $D = \{x : x > 0\}$

Claim: $-2 \in \overline{D}$, but no sequence from D converges to -2

In this space, D is dense as any open set $U \ni -2$, has a ctble complement, so it can't miss all the positive numbers.

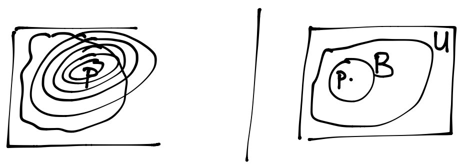
So $-2 \in \overline{D}$, but take $\langle x_n \rangle$ a sequence in D. Does $x_n \to -2$?

No, as the only seqs that converge to -2 are eventually equal to -2. Here $x_n > 0$ $\forall n$, so $x_n \not\to -2$.

def'n: Let $(X, \mathcal{T})$ be a top space and let $p \in X$. A local basis at p is a collection $\mathcal{B}_p$ for open sets such that

1. $p \in B$ $\forall B \in \mathcal{B}_x$
2. If U is open, and $p \in U$, then there is a $B \in \mathcal{B}_x$ s.t. $B \subseteq U$

Pic



e.g. In the sorgenfrey line.

$\mathcal{B}_7 = \{[7, x) : x > 7\}$ is a local basis at 7

e.g.2 In $\mathbb{N}_{discrete}$ $\mathcal{B}_{10} = \{\{10\}\}$ is a local basis at 10.

e.g.3 In $\mathbb{R}_{usual}$ $\mathcal{B}_x = \{B_{\frac{1}{n}}(x) : n \in \mathbb{N}\}$ is a local basis at x.

def'n A topological space $(X, \mathcal{T})$ is 1st ctble, if every point $p \in X$, has a ctble local basis $\mathcal{B}_p$ at p.

e.g. Discrete topologies
Second ctble topologies (important!)
$\mathbb{R}_{usual}$
} all 1-st ctble

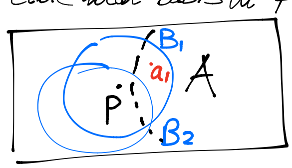
non eg. $\cdot$ $\mathbb{R}_{cocountable}$
$\mathbb{R}_{cofinite}$ are not 1st countable (exercise)

Theorem: Let $(X, \mathcal{T})$ be a first ctble space, let $A \subseteq X$, $p \in X$.
$p \in \overline{A} \iff \exists \langle a_n \rangle$ in A such that $a_n \to p$.

Proof: ($\Leftarrow$) Already proved
($\Rightarrow$) Suppose $p \in \overline{A}$
Since $(X, \mathcal{T})$ is 1st ctble, let $\mathcal{B}_p$ be a ctble local basis at p, so $\mathcal{B}_p = \{B_n : n \in \mathbb{N}\}$

Find a seq $\langle a_n \rangle$ in A, s.t. $a_n \to p$

Since $p \in B_1$ is open, $\exists a_1 \in A \cap B_1$



$\boxed{\text{For each } n \in \mathbb{N}, \text{ find } a_n \in B_n \cap A \quad \text{Think about how this could fail}}$

For each $n \in \mathbb{N}$, find $a_n \in (B_1 \cap \cdots \cap B_n) \cap A$

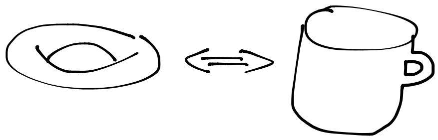
Claim: $a_n \to p$

Take U an open set containing p, by (2), $\exists N \in \mathbb{N}$ s.t. $B_N \subseteq U$

Subclaim: $\forall n \geq N$, $a_n \in B_N \subseteq U$.

Notice $B_1 \cap B_2 \cap \cdots \cap B_N \subseteq U$
so also $B_1 \cap \cdots \cap B_n \subseteq U$
So $a_n \in B_1 \cap \cdots \cap B_n \subseteq U$

§6. Continuity & Homeomorphisms



Topologically the same

<u>Homeomorphic</u>

Which letters are top. the same? (use stretching, squishing, reflecting, rotating but no cutting or glueing).

e.g. $L \approx M$

but $E \not\approx L$