(1) Prove that $\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}=2-\frac{n+2}{2^n}$

Proof. For $n=1$, LHS $=\frac{1}{2}$, RHS $=2-\frac{1+2}{2^1}=\frac{1}{2}$. $\frac{1}{2}=\frac{1}{2}$. verified

Suppose this formula holds for all natural numbers $n$ then for $n+1$ it also holds.

$$\left(\frac{1}{2}+\frac{2}{2^2}+\frac{3}{2^3}+\cdots+\frac{n}{2^n}\right)+\frac{n+1}{2^{n+1}}=2-\frac{n+2}{2^n}+\frac{n+1}{2^{n+1}}$$

$$=\frac{2^{n+2}-2n-4+n+1}{2^{n+1}}$$

$$=2-\frac{n+3}{2^{n+1}}$$

which formula? $\quad =2-\frac{(n+1)+2}{2^{n+1}}$ verified

Therefore this formula holds for all natural numbers $n$ ■

(2). Prove that

$$1+2q+3q^2+\cdots+nq^{n-1}=\frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}$$

Proof. For $n=1$ and $q\neq 1$ LHS $=1$, RHS $=\frac{1-(1+1)q^1+1\cdot q^{1+1}}{(1-q)^2}$

$$=\frac{1-2q+q^2}{1-2q+q^2}$$

$$=1$$

$1=1$, verified

Suppose this formula holds for all natural numbers $n$, then for $n+1$ and $q\neq 1$ it also holds

$$1+2q+3q^2+\cdots+nq^{n-1}+(n+1)q^n=\frac{1-(n+1)q^n+nq^{n+1}}{(1-q)^2}+(n+1)q^n$$

$$=\frac{1-(n+1)q^n(1-(1-q)^2)+nq^{n+1}}{(1-q)^2}$$

$$=\frac{1-(n+2)q^{n+1}+(n+1)q^{n+2}}{(1-q)^2}$$

$$=\frac{1-((n+1)+1)q^{n+1}+(n+1)q^{(n+1)+1}}{(1-q)^2}$$ verified

Therefore this formula holds for all natural numbers $n$ and $q\neq 1$. ■

(3) The Fibonacci sequence is the sequence of number $F(1), F(2), \ldots$ defined by the following recurrence relations:

$F(1)=1$, $F(2)=1$. $F(n)=F(n-1)+F(n-2)$ for all $n>2$

For example, the first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, \ldots$

Use the induction to prove the identity

$$F(n+1)F(n-1)-F(n)^2=(-1)^n \quad \text{for all natural numbers } n$$

Proof: Proved by induction.

Check for $n=3$. $F(n+1)=F(4)=3$, $F(n-1)=1$, $F(n)=2$

$$3\cdot 1-2^2=-1=(-1)^3 \qquad \text{verified}$$

Suppose this holds for all natural numbers larger than 2, according to the problem. check for $n+1$

$$\begin{aligned}
&F(n+1+1)F(n+1-1)-F(n+1)^2\\
&=(F(n+1)+F(n))(F(n))-F(n+1)^2\\
&=F(n+1)F(n)+F(n)^2-F(n+1)^2\\
&=F(n+1)(F(n)-F(n+1))+(F(n+1)F(n-1)-(-1)^n)\\
&=-F(n+1)F(n-1)+F(n+1)F(n-1)-(-1)^n\\
&=0+(-1)^{n+1}\\
&=(-1)^{n+1}
\end{aligned}$$

verified

Therefore this identity holds for all natural numbers. ∎

(14) Let $x_1>2$. Define $x_n$ by the formula $x_{n+1}=\frac{3x_n+2}{x_n+2}$

Prove that $x_n>2$ for all $n$.

Proof: Proved by induction.

Check for $n=2$. $x_2=\frac{3x_1+2}{x_1+2}=\frac{3x_1+6-4}{x_1+2}=3-\frac{4}{x_1+2}$

since $x_1>2 \Rightarrow x_1+2>4 \Rightarrow \frac{4}{x_1+2}<1$

so $x_2=3-\frac{4}{x_1+2}>3-1=2$ verified.

Suppose this equality holds for all natural numbers $n$. then this must hold also for $n+1$. Check for $n+1$

$$x_{n+1} = \frac{3x_n+2}{x_n+2} = \frac{3x_n+6-4}{x_n+2} = 3 - \frac{4}{x_n+2}$$

Since $x_n > 2 \Rightarrow x_n+2 > 4 \Rightarrow \frac{4}{x_n+2} < 1$

So $x_{n+1} = 3 - \frac{4}{x_n+2} > 3-1 = 2$ verified

Therefore $x_n > 2$ for all $n$. ■

(5). Using the method from class find the formula for the sum
$1^3+2^3+\cdots+n^3$
Then prove the formula you've found by mathematical induction.

Solution:

A guess: $1^2+2^2+3^2+\cdots+n^2$ is cubic, probably $1^3+2^3+3^3+\cdots+n^3$ is quartic in $n$.

Let $a_1 = 1^4$

$a_1+a_2 = 2^4$

$a_1+a_2+a_3 = 3^4$

$\cdots$

$a_1+a_2+a_3+\cdots+a_n = n^4$

So $a_n = n^4-(n-1)^4 = n^4-(n^4-2n^3+n^2-2n^3+4n^2-2n+n^2-2n+1)$

$= 2n^3-n^2+2n^3-4n^2+2n-n^2+2n-1$

$= 4n^3-6n^2+4n-1$

Then $a_1 = 4\cdot1^3-6\cdot1^2+4\cdot1-1 = 1$

$a_2 = 4\cdot2^3-6\cdot2^2+4\cdot2-1 = 15$

$a_3 = 4\cdot3^3-6\cdot3^2+4\cdot3-1 = 65$

$\cdots$

And $a_1 = 1 = 1^4$

$a_1+a_2 = 1+15 = 16 = 2^4$

$a_1+a_2+a_3 = 1+15+65 = 81 = 3^4$

$\cdots$

$a_1+a_2+a_3+\cdots+a_n = n^4$ $\quad a_n = 4n^3-6n^2+4n-1$

Therefore $(4\cdot1^3-6\cdot1^2+4\cdot1-1)+(4\cdot2^3-6\cdot2^2+4\cdot2-1)+(4\cdot3^3-6\cdot3^2+4\cdot3-1)$
$+\cdots+(4\cdot n^3-6\cdot n^2+4\cdot n-1)=n^4$

$$4(1^3+2^3+3^3+\cdots+n^3)-6(1^2+2^2+3^2+\cdots+n^2)+4(1+2+3+\cdots+n)-n=n^4$$

$$4(1^3+2^3+3^3+\cdots+n^3)-6\,\frac{n(n+1)(2n+1)}{6}+4\cdot\frac{n(n+1)}{2}-n=n^4$$

$$4(1^3+2^3+3^3+\cdots+n^3)-n(n+1)(2n+1)+2n(n+1)-n=n^4$$

$$4(1^3+2^3+3^3+\cdots+n^3)=n^2(n+1)^2$$

Hence $$1^3+2^3+3^3+\cdots+n^3=\frac{n^2(n+1)^2}{4} \qquad (*)$$

Proof: Proved by induction:
Check for $n=1$, $1^3=\frac{1}{4}\cdot1^2\cdot(1+1)^2=1$ verified.

Suppose this $(*)$ holds for all $n$, then it also holds for $n+1$. then check for $n+1$.

$$(1^3+2^3+3^3+\cdots+n^3)+(n+1)^3=\frac{n^2(n+1)^2}{4}+(n+1)^3$$
$$=\frac{(n+1)^2(n^2+4(n+1))}{4}$$
$$=\frac{(n+1)^2(n+2)^2}{4} \quad \text{verified.}$$

Therefore $(*)$ holds for all natural numbers $n$ ∎

(6) Find a mistake in the following proof:

Claim. Any two natural numbers are equal.

We'll prove the following statement by induction in $n$.

Any two natural numbers $\le n$ are equal.

We prove it by induction in $n$.

a). The statement is trivially true for $n=1$.

b). Suppose it's true for $n \ge 1$. Let $a, b$ be two natural numbers $\le n+1$. Then $a-1 \le n$ and $b-1 \le n$. There, by the induction assumption

$$a-1 = b-1$$

Adding 1 to both sides of the above equality we get that $a=b$. Thus the statement is true for $n+1$. By the principle of mathematical induction this means that it's true for all natural $n$. ∎

Answers

The inductive step is invalid because it is wrong for $n=1$:

If $n=1$, $a \le n+1 = 2$, $b \le n+1 = 2$,

when $a=1, b=2$, $a \ne b$

So the step from $n$ to $n+1$ is invalid.

Hence the proof is wrong.

$\frac{2.5}{5}$

This shows the proof has an error but does not say what it is.

(7) Using the method from class write a table of all prime numbers $\leq 100$. Explain why you only need to cross out the numbers divisible by 2, 3, 5 and 7.

Solution:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| ~~1~~ | 2 | 3 | ~~4~~ | 5 | ~~6~~ | 7 | ~~8~~ | ~~9~~ | ~~10~~ |
| 11 | ~~12~~ | 13 | ~~14~~ | ~~15~~ | ~~16~~ | 17 | ~~18~~ | 19 | ~~20~~ |
| ~~21~~ | ~~22~~ | 23 | ~~24~~ | ~~25~~ | ~~26~~ | ~~27~~ | ~~28~~ | 29 | ~~30~~ |
| 31 | ~~32~~ | ~~33~~ | ~~34~~ | ~~35~~ | ~~36~~ | 37 | ~~38~~ | ~~39~~ | ~~40~~ |
| 41 | ~~42~~ | 43 | ~~44~~ | ~~45~~ | ~~46~~ | 47 | ~~48~~ | ~~49~~ | ~~50~~ |
| ~~51~~ | ~~52~~ | 53 | ~~54~~ | ~~55~~ | ~~56~~ | ~~57~~ | ~~58~~ | 59 | ~~60~~ |
| 61 | ~~62~~ | ~~63~~ | ~~64~~ | ~~65~~ | ~~66~~ | 67 | ~~68~~ | ~~69~~ | ~~70~~ |
| 71 | ~~72~~ | 73 | ~~74~~ | ~~75~~ | ~~76~~ | ~~77~~ | ~~78~~ | 79 | ~~80~~ |
| ~~81~~ | ~~82~~ | 83 | ~~84~~ | ~~85~~ | ~~86~~ | ~~87~~ | ~~88~~ | 89 | ~~90~~ |
| ~~91~~ | ~~92~~ | ~~93~~ | ~~94~~ | ~~95~~ | ~~96~~ | 97 | ~~98~~ | ~~99~~ | ~~100~~ |

Note: All the composites can be written as the products of prime numbers. The composite with prime factors large than 7 is $121 = 11 \times 11$ which is larger than 100, so all the composites within 100 are divisible by 2, 3, 5, 7.

If $n$ composite then $n$ has a prime factor $p$ s.t. $p \leq \sqrt{n}$

Textbook problem

P.7

#6 Prove that for every natural number, there are $n$ consecutive composite numbers. [Hint $(n+1)!+2$ is a composite number.]

Proof: Consider the $n$ consecutive numbers starting from $(n+1)!+2$.

$(n+1)!+2 = 2(3\cdot4\cdots(n+1)+1)$ is divisible by 2

$(n+1)!+3 = 3(2\cdot4\cdots(n+1)+1)$ is divisible by 3

...

$(n+1)!+(n+1) = (n+1)(2\cdot3\cdots n+1)$ is divisible by $n$

Hence the claim is proved.

P23

#6.

Prove by induction that, for every natural number $n$,

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}<2\sqrt{n}$$

Proof: Proved by induction

Check for $n=1$, $1<2\sqrt{1}=2$

Suppose the inequality holds for all natural numbers $n$.

then it holds for $n+1$

$$\left(1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\right)+\frac{1}{\sqrt{n+1}}<2\sqrt{n}+\frac{1}{\sqrt{n+1}}=\frac{2\sqrt{n(n+1)}+1}{\sqrt{n+1}}$$

Since
$$4n^2+4n+1>4n^2+4n$$
$$(2n+1)^2>4n(n+1)$$
$$2n+1>2\sqrt{n(n+1)}$$
$$2(n+1)>1+2\sqrt{n(n+1)}$$

So
$$\frac{1+2\sqrt{n(n+1)}}{\sqrt{n+1}}<2\sqrt{n+1}$$

Therefore $\left(1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{n}}\right)+\frac{1}{\sqrt{n+1}}<2\sqrt{n+1}$

Hence the inequality holds for all natural numbers $n$. ■

#13.

Define the $n^{th}$ Fermat number, $F_n=2^{2^n}+1$ for $n\in N$. The first few Fermat numbers are $F_0=3$, $F_1=5$, $F_2=17$, $F_3=257$.

Prove by induction that $F_0\cdot F_1\cdots F_{n-1}+2=F_n$ for $n\geq 1$.

Proof: Check for $n=1$, $F_1=F_0+2=3+2=5$ verified.

Suppose this formula is true for all $n\geq 1$. then it is true for

$$F_0\cdot F_1\cdots F_{n-1}F_n+2=F_{n+1}.$$

$$LHS=F_0\cdot F_1\cdots F_{n-1}F_n+2=(F_n-2)\cdot F_n+2=(2^{2^n}+1-2)(2^{2^n}+1)+2$$

$$RHS=2^{2^{n+1}}+1$$

$$=(2^{2^n})^2-1+2$$
$$=2^{2^n\cdot 2}+1$$
$$=2^{2^{n+1}}+1$$

RHS=LHS verified.

Therefore the formula is true for all $n\geq 1$. ■