Lecture 2.

Rules of Probability

1. $P(\bar{A}) = 1 - P(A)$, $A \in \mathcal{F}$

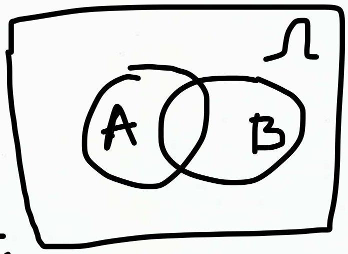
$P(\bar{A} \cup A) = P(\Omega) = 1$, $A \cap \bar{A} = \emptyset$

$P(\bar{A}) + P(A) = 1 \Rightarrow P(\bar{A}) = 1 - P(A)$ ∎

2. $P(\emptyset) = 0$

proved earlier

3. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ (Addition law)



$\Omega = (\overline{A \cup B}) \cup (A \cap B^c) \cup (A \cap B) \cup (B \cap A^c)$

$\overline{A \cup B}$, $A \cap B^c$, $A \cap B$, $B \cap A^c$ are all disjoint

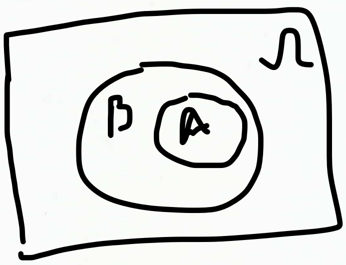
$1 = P(\overline{A \cup B}) + \underbrace{P(A \cap B^c) + P(A \cap B)}_{P(A)} + P(B \cap A^c)$

$P(A \cup B) = \quad P(A) + P(B \cap A^c) \pm P(A \cap B)$

$P(A \cup B) = \quad P(A) + P(B) - P(A \cap B)$

∎

4. If $A \subseteq B$, then $P(A) \leq P(B)$



$B = \underbrace{A \cup (A^c \cap B)}_{\text{disjoint}}$

$P(B) = P(A) + \underbrace{P(A^c \cap B)}_{\geq 0} \Rightarrow P(A) \leq P(B)$ ∎

5. $$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i>j} P(A_i \cap A_j) + \sum_{i>j>k} P(A_i \cap A_j \cap A_k) - \ldots$$

$$\ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) -$$
$$- P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

6. $$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

Proof: By math induction

① $n=2$, $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \le$
$\le P(A_1) + P(A_2)$

② Assume for $n=k$, i.e.

$$P\left(\bigcup_{i=1}^{k} A_i\right) \le \sum_{i=1}^{k} P(A_i)$$

③ Prove for $n=k+1$

$$P(\cup A_i) = P\left[\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right] \underset{①}{\le} P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1})$$

$$\underset{②}{\le} \sum_{i=1}^{k} P(A_i) + P(A_{k+1}) = \sum_{i=1}^{k+1} P(A_i) \quad \blacksquare$$

Ex. In a lottery there are 10 tickets numbered 1, 2, ..., 10. Two numbers are drawn for prizes. You hold tickets 1 and 2. P(you win at least one prize) = ?

Sol'n: A: { #1 is winning }

B: { #2 is winning }

$$\frac{\binom{2}{2}}{\binom{10}{2}} = \frac{1}{45}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{5} + \frac{1}{5} - \frac{1}{5} \cdot \frac{1}{9}$$

$$= \frac{2}{5} - \frac{1}{45} = \frac{17}{45} \quad \blacksquare$$

Conditional Probability.

Suppose we performed a chance experiment, and we don't know the outcome (ω), but we have some partial information (event A) about ω.

Question: Given this partial information what's the probability that the outcome is in some event B?

Ex. Toss a coin 3 times.

$B = \{$ 2 or more heads $\} = \{HHH, HHT, HTH, THH\}$

$A = \{$ 1st outcome is $H\} = \{HHH, HHT, HTH, HTT\}$

$$P(B|A) = \frac{3}{4} = \frac{3/8}{4/8} = \frac{P(A \cap B)}{P(A)}$$

$A \cap B = \{HHH, HHT, HTH\}$

Def. Given a probability space $(\Omega, \mathcal{F}, P)$ and events $A, B \in \mathcal{F}$, $P(A) > 0$. The conditional probability of B given the information that A has occurred is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

2.4

Ex. We toss a die. P(observing 6 | outcome is even) = ?

$$A = \{\text{even}\}, \quad B = \{6\}$$

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}$$

Theorem. If $A \in \mathcal{F}$, $P(A) > 0$, then $(\Omega, \mathcal{F}, Q)$ is a probability space, where $Q : \mathcal{F} \to [0,1]$ is defined by $Q(B) = P(B|A)$.

$$P(\overline{B}|A) = 1 - P(B|A), \quad A, B \in \mathcal{F}, \ P(A) > 0$$

$$P(B_1 \cup B_2 | A) = P(B_1|A) + P(B_2|A) - P(B_1 \cap B_2 | A)$$

Multiplication rule.

- For any two events A and B, $P(A\cap B) = P(B|A)P(A)$
- For any 3 events A, B, and C, $P(A\cap B\cap C) = P(A)P(B|A)P(C|A\cap B)$
- In general,

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)P(A_2|A_1)P(A_3|A_1\cap A_2)\cdots P\left(A_n \,\Big|\, \bigcap_{i=1}^{n-1} A_i\right)$$

Ex. An urn contains 10 balls, 3 blue and 7 white. We draw a ball and note its color. Then we replace it and add one more of the same color. We repeat this process 3 times. What is the probability that the first 2 balls are blue and the third one is white?

Sol'n:

$A_1 = \{1^{st}$ ball is blue$\}$

$A_2 = \{2^{nd}$ ball is blue$\}$

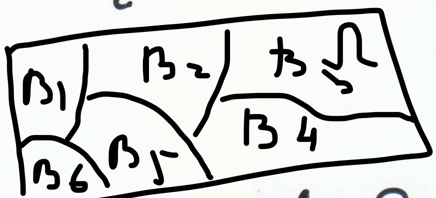
$A_3 = \{3^{rd}$ is white$\}$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 \cap A_2)$$
$$= \frac{3}{10} \cdot \frac{4}{11} \cdot \frac{7}{12} = \frac{7}{110}$$

Law of Total Probability.

Def. For a probability space $(\Omega, \mathcal{F}, P)$, a partition of Ω is a countable collection $\{B_i\}$ of events such that

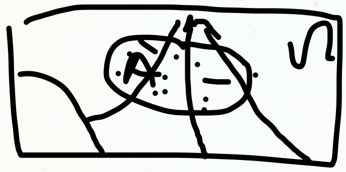
$$B_i \in \mathcal{F}, \quad B_i \cap B_j = \emptyset_{\,i \neq j} \quad \text{and} \quad \bigcup_i B_i = \Omega$$



$$\Omega = \bigcup_{i=1}^{6} B_i$$

Theorem. If $\{B_1, B_2, \ldots, B_k\}$ is a partition of Ω such that $P(B_i) > 0$ $\forall i = \overline{1,k}$, then $\forall$ for all $\forall A \in \mathcal{F}$

$$P(A) = \sum_{i=1}^{k} P(A|B_i) P(B_i)$$



Proof: $A \in \mathcal{F}$, $A = A \cap \Omega$

$$\Omega = \bigcup_{i=1}^{k} B_i, \quad A = A \cap (\cup B_i)$$
$$= (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_k)$$

disjoint

$$P(A) = P(A \cap B_1) + \ldots + P(A \cap B_k)$$
$$= P(A|B_1) P(B_1) + \ldots + P(A|B_k) P(B_k) = \sum_{i=1}^{k} P(A|B_i) P(B_i)$$

Bayes' Rule.

Let $\{B_1, B_2, \dots, B_k\}$ be a partition of Ω such that $P(B_i) > 0$, $i = \overline{1,k}$. Then

$$\underline{P}(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} P(A|B_i)P(B_i)}, \quad A \in \mathcal{F}$$

Proof:

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{i=1}^{k} (A|B_i)P(B_i)}$$

Ex. There are 2 lists of applicants for jobs. List 1 contains the names of 5 women and 2 men. List 2 contains the names of 2 women and 6 men. A name is randomly selected from list 1 and added to list 2. A name is then randomly selected from list 2. Given that the name selected is that of a man, what is the probability that a woman's name was originally selected from list 1?

Sol'n: A = man's name selected from list 2

B = woman's name ... list 1

$P(B|A) = ?$ $\quad \Omega = B \cup \overline{B}$, $B_1 = B$, $B_2 = \overline{B}$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|\overline{B})P(\overline{B})}$$

$$= \frac{2}{3} \cdot \frac{5}{7} \Big/ \left(\frac{2}{3} \cdot \frac{5}{7} + \frac{7}{9} \cdot \frac{2}{7}\right) = \frac{15}{22}$$

Independence.

Ex. Roll a die twice.

A: 3 or less on first roll $= \{(3,\cdot),(2,\cdot),(1,\cdot)\}$

B: sum is odd $= \{(1,2),(1,4),(1,6),\ \dots\ \}$

$P(A) = \frac{18}{36} = \frac{1}{2} = P(B)$, $A \cap B = \left(\begin{array}{l}(3,2),(3,4)\\(3,6)\dots\end{array}\right)$

$P(A \cap B) = \frac{1}{4}$ $\quad$,, $P(A) = P(B)$

$P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{1}{2} = \frac{P(A\cap B)}{P(B)} = P(A|B)$

- If occurrence of one event does not affect the probability that the other occurs, then these events are independent.

Def. Events A and B are independent if

$P(A\cap B) = P(A)P(B) \iff P(A|B) = P(A)$

$P(B|A) = P(B)$

Note: Independence $\neq$ dis-joint !!!

Ex. Toss a die

A: observe an odd number

B: observe an even number

C: observe a 1 or 2

$P(A) = P(B) = \frac{1}{2}$

$A \cap B = \emptyset$

A, B - disjoint

$$P(A\cap B) = 0 \neq P(A)P(B) = \frac{1}{4}$$

$\Rightarrow$ not independent

$P(C) = \frac{1}{3}$, $A \cap C = \{1\}$, $P(A\cap C) = \frac{1}{6}$

$= P(A)\cdot P(C) = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$

In general,
A collection of events $\{A_1, A_2, \ldots, A_n\}$ is (mutually) independent if for any subcollection $\{A_{i_1}, A_{i_2}, \ldots, A_{i_m}\}$

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_m}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_m})$$

Note: Pairwise independence does not guarantee mutual independence.

Ex. Roll a die twice

A : 1st die odd

B : 2nd die odd

C : sum is odd

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(A \cap B) = \frac{1}{4} = P(A) P(B)$$

$$P(A \cap C) = \frac{1}{4} = P(A) P(C)$$

$$P(B \cap C) = \frac{1}{4} = P(B) P(C)$$

$\Rightarrow$ A, B, C are pairwize independent

$$P(A \cap B \cap C) = P(\emptyset) = 0 \neq P(A) P(B) P(C)$$

Claim: If A, B are independent so are $A, \bar{B}$ and $\bar{A}, B$, and $\bar{A}, \bar{B}$

Proof:

Indicator function : $I_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$

Fact: $A \subset B \iff I_A(x) \le I_B(x)$
iff

Proof: $\Rightarrow$ $A \subset B$

let $x \in A \Rightarrow x \in B$

$$I_A(x) = 1 \le I_B(x) = 1$$

$$x \notin A \Rightarrow I_A(x) = 0$$

$$I_B(x) \geq 0 = I_A(x)$$

$$\Rightarrow I_A(x) \le I_B(x) \quad \forall x$$

$\Leftarrow$ $I_A(x) \le I_B(x)$

let $x \in A$, need: $x \in B$

suppose $x \notin B \Rightarrow I_B(x) = 0$

$0 = I_B(x) \geq I_A(x) = 1 \rightarrow \leftarrow \Rightarrow A \subset B$

$\blacksquare$