March 22nd

No lecture next Friday.

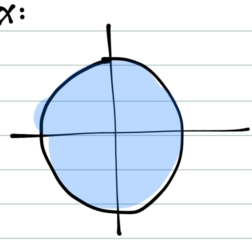
Last time we proved

- union of countablely many countable sets is countable.

- $|\mathbb{R}\times\mathbb{R}| = |\mathbb{R}| \iff |\mathbb{R}^n| = |\mathbb{R}|$ for any $n \geq 1$

- Say set $P(S)$ = set of all subsets of $S$
  - $|P(S)| > |S|$ always
  - $|P(\mathbb{N})| = |\mathbb{R}|$

Ex: $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 1\}$



$|S| = ?$

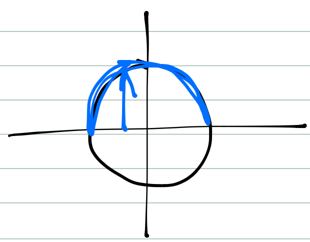
$|S| = |\mathbb{R}|$

$S \subset \mathbb{R}^2 \quad |S| \leq |\mathbb{R}^2| = |\mathbb{R}|$

$S \supset [-1,1] \Rightarrow |S| \geq |[-1,1]| = |\mathbb{R}|$

By S-B thm, implies $|S| = |\mathbb{R}|$

Ex 2: $S = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\}$



$|S| = |\mathbb{R}|$ ?

$S \subseteq \mathbb{R}^2 \Rightarrow |S| \leq |\mathbb{R}^2| = |\mathbb{R}|$

$|S| \overset{?}{\geq} |\mathbb{R}|$

why?

$f: (-1,1) \to S$

$f(x) = (x, \sqrt{1-x^2})$ 1-1

$\Rightarrow |S| \geq |(-1,1)| = |\mathbb{R}|$

By S-B $\Rightarrow |S| = |\mathbb{R}|$

$S$ any set, $P(S) \longmapsto$ functions has $S$ to $\{0,1\}$

$A \subset S \longmapsto \chi_A$ – characteristic function of $A$

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A \\ 0 & \text{if } s \notin A \end{cases}$$

Let $S$ = set of all functions from $\mathbb{R}$ to $\{0,1,2,7\}$

all $f: \mathbb{R} \to \{0,1,2,7\}$

Claim: $|S| = |P(\mathbb{R})|$

Pf: $\underbrace{\{\text{functions } \mathbb{R} \to \{0,1\}\}}_{= P(\mathbb{R})} \subseteq \underbrace{\text{functions } \mathbb{R} \to \{0,1,2,7\}}_{= S}$

$|P(\mathbb{R})| \leq |S|$

need to prove $|P(\mathbb{R})| \geq |S|$

given a function $f: \mathbb{R} \to \{0,1,2,7\}$ or any function $f: \mathbb{R} \to \mathbb{R}$

look at its graph $\Gamma_f \subseteq \mathbb{R}^2$ $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{R}\}$

$f \mapsto \Gamma_f$ is 1-1

$f, g$ - two functions

$f \neq g \Rightarrow \Gamma_f \neq \Gamma_g$

this gives a 1-1 map $\underset{\{f: \mathbb{R} \to \{0,1,2,7\}\}}{\overset{S}{\shortparallel}} \xrightarrow{1-1} P(\mathbb{R}^2)$

$f \longmapsto \Gamma_f$

$\Rightarrow |S| \leq |P(\mathbb{R}^2)| = |P(\mathbb{R})|$ since $|\mathbb{R}| = |\mathbb{R}^2|$

so $|S| \leq |P(\mathbb{R})|$ & $|S| \geq |P(\mathbb{R})| \Rightarrow |S| = |P(\mathbb{R})|$ by S-B

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Let $S$ = set of all sequences of rational #

$S = \{(q_1, q_2, q_3, \cdots)\}$

$q_i \in \mathbb{Q}$

Claim: $|S| = |\mathbb{R}|$ Why?

a sequence of rational numbers is a function from $\mathbb{N}$ to $\mathbb{Q}$

$f: \mathbb{N} \to \mathbb{Q}$

$f(1), f(2), f(3), \cdots$ sequence of rational #.

$S = \{f: \mathbb{N} \to \mathbb{Q}\} \supseteq \{f: \mathbb{N} \to \{0,1\}\} = P(\mathbb{N})$

$|S| = ?$

$\Rightarrow |S| \geq |P(\mathbb{N})| = |\mathbb{R}|$

$\Rightarrow |S| \geq |\mathbb{R}|$

given $f: \mathbb{N} \to \mathbb{Q}$ look at $\Gamma_f$ - graph of $f$

$\Gamma_f \subseteq \mathbb{N} \times \mathbb{Q}$

$\Gamma_f = \{(x,y) \in \mathbb{N} \times \mathbb{Q} \mid y = f(x)\}$

$f \longmapsto \Gamma_f$ 1-1

$\Rightarrow$ get a 1-1 map

$S \to P(\mathbb{N} \times \mathbb{Q})$ $|P(\mathbb{N} \times \mathbb{Q})| = |P(\mathbb{N})| = |\mathbb{R}|$

$f \longmapsto \Gamma_f$

$\left.\begin{array}{l} |\mathbb{N}| = |\mathbb{N}| \\ |\mathbb{Q}| = |\mathbb{N}| \end{array}\right\} \Rightarrow |\mathbb{N} \times \mathbb{N}| = |\mathbb{N} \times \mathbb{Q}| = |\mathbb{N}|$

$|S| \leq |P(\mathbb{N} \times \mathbb{Q})| = |P(\mathbb{N})| = |\mathbb{R}|$

By S-B $|S| = |\mathbb{R}|$

$\boxed{f: A \to B \quad \Gamma_f \subset A \times B \quad \Gamma_f = \{(a, f(a)) \mid a \in A\}}$

Theorem: if $S$ is infinite set and $A$ is countable then $|S \cup A| = |S|$

for ex: $S$ infinite $\Rightarrow |S \cup \{x\}| = |S|$

$$\left.\begin{array}{l} S = [0,1) \\ A = \{1\} \end{array}\right\} \Rightarrow |S \cup A| = |[0,1]| = |[0,1)|$$

Proof: $S$ is infinite. pick some $s_1 \in S$ possible since $S \setminus \{s_1\}$ is still infinite
pick some $s_2 \neq s_1$
$S \setminus \{s_1, s_2\}$ is still infintin
pick $s_3$ etc

can find $s_1, s_2, \cdots, s_n, \cdots \in S$
$s_i \neq s_j$ for $i \neq j$

Let $T = \{s_1, s_2, \cdots, s_n, \cdots\} \subseteq S$

- countable $|T| = |N|$

$T \cup A$ — countable as a union of a countably many countable sets
both are countable

$$\left.\begin{array}{l} \Rightarrow |T \cup A| = |N| \quad T \cup A \supset T \\ \quad |T \cup A| \geq |T| = |N| \end{array}\right\} \Rightarrow |T \cup A| = |N|$$

$|T| = |N|$
$|T \cup A| = |N|$
So $|T| = |T \cup A|$
$\exists f: T \to T \cup A$ 1-1 & onto

Define $h: S \to S \cup A$ by formula:

$$h(s) = \begin{cases} s & \text{if } s \notin T \\ f(s) & \text{if } s \in T \end{cases}$$

$\Rightarrow h$ is 1-1 & onto

Cor: Set of transcendental. #s has cardinality $|R|$.
We proved $|A| = |N|$
$\hookrightarrow$ algebraic #s

① $T$ = set of all trans #s $T = R \setminus A$, $R = T \cup A$
$T$ is infinite (if $T$ were finite $\Rightarrow R = \underbrace{T \cup A}_{\text{countable}}$ false!)

$\Rightarrow T$ is infinite
$\Rightarrow A$ is countable

$|T \cup A| = |T|$
$\Downarrow$
$\mathbb{R}$
$\Downarrow$
$|\mathbb{R}|$

Cor: $\forall a < b$, the interval $(a,b)$ contains a trans #

Sps not $\Rightarrow (a,b) \subseteq \underset{\text{countable}}{A}$ $\Rightarrow (a,b)$ ctbl $\Rightarrow$ we know this is false

Cor: $\forall a<b$, $\exists$ a irrational # $c$ s.t. $a<c<b$.

Proof: proved $\exists$ a trans $c$ s.t. $a<c<b$ (previously)

But all trans are irrational b/c $\forall$ $\mathbb{Q}$ # are algebraic
if $x = \frac{m}{n} \Rightarrow$ root of $nx - m = 0$