Variation of parameters

$$\underbrace{y'' + py' + qy}_{L[y]} = g$$

y_1, y_2 fundamental set of solutions of homogeneous eqn. $L[y]=0$

Trial solution $Y = v_1 y_1 + v_2 y_2$

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$v_1' = -\frac{1}{W} y_2 g$ where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ Wronskian

$v_2' = \frac{1}{W} y_1 g$

$\Rightarrow v_1(t) = -\int \frac{1}{W(t)} y_2(t) g(t)\, dt$

$v_2(t) = \int \frac{1}{W(t)} y_1(t) g(t)\, dt$

For exam purposes, memorize.

Example: $t^2 y'' - 2y = 3t^2 - 1$ $(t>0)$

$y_1 = t^2, y_2 = t^{-1}$ are fund. set of solutions of $L[y]=0$.

$Y = v_1 y_1 + v_2 y_2$

In our case, $g(t) = \frac{3t^2-1}{t^2}$

$$W = \begin{vmatrix} t^2 & t^{-1} \\ 2t^2 & -t^{-2} \end{vmatrix} = -3$$

$$v_1(t) = \frac{1}{3}\int t^{-1} \frac{3t^2-1}{t^2}\, dt$$

$$= \int \frac{1}{t}\, dt - \frac{1}{3}\int \frac{1}{t^2}\, dt$$

$$= \ln(t) + \frac{1}{6} t^{-2} + C_1$$

$$v_2(t) = -\frac{1}{3}\int \frac{t^2(3t^2-1)}{t^2}\, dt = -\int t^2 dt + \int \frac{1}{3}\, dt = -\frac{t^3}{3} + \frac{t}{3} + C_2$$

May put $C_1 = 0$, $C_2 = 0$

$$Y(t) = \left(\ln(t) + \frac{1}{6t^2}\right)t^2 + \left(-\frac{t^3}{3} + \frac{t}{3}\right)\frac{1}{t} = \underline{\ln(t)t^2 + \frac{1}{2} - \frac{t^2}{3}}$$

solve $L[Y] = g$

Example: $y''-2y'+y=\frac{e^t}{1+t^2}$

Undet. coeff's doesn't apply, but variation of parameters does.

Hom. eqn. $y''-2y'+y=0$ has solutions $y_1(t)=e^t$, $y_2(t)=te^t$

$$W=\begin{vmatrix} e^t & te^t \\ e^t & (1+t)e^t \end{vmatrix} = e^{2t}(1+t)-te^{2t}=e^{2t}$$

$$V_1(t)=-\int \frac{te^t\frac{e^t}{(1+t^2)}}{e^{2t}}\,dt = -\int\frac{t}{1+t^2}\,dt=-\frac{1}{2}\ln(1+t^2)+c_1$$

$$V_2(t)=\int\frac{e^t\frac{e^t}{(1+t^2)}\,dt}{e^{2t}}=\int\frac{dt}{1+t^2}=\arctan(t)+c_2$$

$$\underline{Y(t)=\frac{1}{2}\ln(1+t^2)e^t+\arctan(t)\,te^t}$$

Gen. solution:

$y(t)=Y(t)+c_1y_1(t)+c_2y_2(t)$

Mechanical vibration

 Free, undamped

$my''+ky=0$

k ↑ spring constant

Solution: $y(t)=A\cos(\omega_0 t)+B\sin(\omega_0 t)$ where $\omega_0=\sqrt{\frac{k}{m}}$ char. frequency (eigen frequency)

Free, damped $my''+\gamma y'+ky=0$

If $\gamma^2<4km$, get two complex conjugated roots of char. eqn. $r_1,r_2=-\frac{\gamma}{2m}\pm\frac{i}{2m}\sqrt{4km-\gamma^2}$

$$=-\frac{\gamma}{2m}\pm i\sqrt{\frac{k}{m}-\frac{\gamma^2}{4m^2}}$$

$$=-\frac{\gamma}{2m}\pm i\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}$$

$\Rightarrow$ solution:

$$y(t)=e^{-\frac{\gamma}{2m}t}\left(A\cos\sqrt{\omega_0^2-\left(\frac{\gamma}{2m}\right)^2}\right)+B\sin(\cdots)$$

$$=e^{-\frac{\gamma}{2m}t}\left(R\cos(\sqrt{}\,t-\delta)\right)$$

Forced vibrators

$my''+\gamma y'+ky=F(t)$ ↳ damping force

In principle, can solve this using variation of parameters.

But for $F(t)=F_0\cos(\omega t)$ can also use undetermined coefficients

We'll consider first $F(t)=F_0 e^{i\omega t}$

$(e^{i\theta}=\cos(\theta)+i\sin(\theta))$

Trial solution: $U(t)=Ce^{i\omega t}$

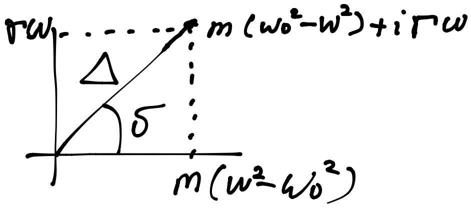
$U'=i\omega Ce^{i\omega t}, \; U''=-\omega^2 Ce^{i\omega t}$

$L[U]=F(t)$ gives

$$Ce^{i\omega t}(-m\omega^2+\Gamma i\omega+k)=F_0 e^{i\omega t}$$

$$C=\frac{F_0}{(k-m\omega^2)+i\gamma\omega}=\frac{F_0}{m(\omega_0^2-\omega^2)+i\gamma\omega}$$

$$m(\omega_0^2-\omega^2)+i\Gamma\omega=\Delta e^{i\delta}=\Delta(\cos(\delta)+i\sin(\delta))$$



$$\Delta=\sqrt{m^2(\omega_0^2-\omega^2)^2+\Gamma^2\omega^2}$$

$$\tan(\delta)=\frac{\Gamma\omega}{m(\omega_0^2-\omega^2)}$$

Note: Δ becomes minimal for $\omega=\omega_0 \Rightarrow$ $\frac{F_0}{\Delta}$ becomes maximal.

$$C=\frac{F_0}{\Delta e^{i\delta}}=\frac{F_0}{\Delta}e^{-i\delta}$$

$$U(t)=\frac{F_0}{\Delta}e^{i(\omega t-\delta)}$$

solves $L[U]=F_0 e^{i\omega t}$

$$Y(t)=\frac{F_0}{\Delta}\cos(\omega t-\delta)$$

solves $L[Y]=F_0\cos(\omega t)$

If $\omega\approx 0$
$\tan(\delta)\approx 0$
$\Rightarrow \delta\approx 0$

If $\omega\approx\omega_0-\varepsilon$
$\tan(\delta)\approx\infty$
$\delta\approx\frac{\pi}{2}$

If $\omega\approx\infty$
$\tan(\delta)\nearrow 0$

$\delta\to\pi$