Jan 9-th, 2013

Mathematical induction

Let S be a set of natural numbers such that ① $1 \in S$

② If $n \in S$ then $n+1 \in S$
$n \geq 1$

$\Rightarrow S = N$ all natural #
$S = \{1, 2, 3, \cdots\}$

In problems, you have a formula e.g. $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

we want to prove that it holds for all n.
Let S = set of all natural #, for which the formula holds.
We want to show $S = N$
Proof: ① $1 \in S$ (i.e. the formula holds for $n=1$)
$1 = \frac{1(1+1)}{2} = 1$ ✓

② if $n \in S$ (i.e. if the formula holds for $n \Rightarrow$ it holds for $n+1$

We assume $1+2+\cdots+n = \frac{n(n+1)}{2}$ ⟵ put $n+1$ instead of n

$\Rightarrow$ want to show
$\underline{1+2+\cdots+n+(n+1)} = \frac{(n+1)(n+2)}{2}$
$= \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2n+2}{2} = \frac{(n+1)^2+(n+1)}{2} = \frac{(n+1)(n+2)}{2}$

$\Rightarrow$ we proved the induction step.
$\Rightarrow$ the formula holds for all n.

$1+2+3+\cdots+n = \frac{(n+1)n}{2}$

$1^2+2^2+3^2+\cdots+n^2 = ?$
find the formula and prove it by induction.

$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

Induction:
① Check for $n=1$.
$LHS = 1$, $RHS = \frac{1\cdot 2\cdot 3}{6} = 1$ v.

② induction step suppose the formula holds for all n.
$1^2+2^2+\cdots+n^2 = n(n+1)(2n+1)/6$

for $n+1$: $\underline{1^2+2^2+\cdots+n^2}+(n+1)^2 = (n+1)((n+1)+1)(2(n+1)+1)/6 = \underline{(n+1)(n+2)(2n+3)}$

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$$\frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} = \frac{(n+1)(n(2n+1)+6(n+1))}{6} = \frac{(n+1)(2n^2+n+6n+6)}{6}$$

a guess: $1+2+\cdots+n$ = quadratic in n

$$\frac{n(n+1)}{2} = \frac{n^2+1}{2} \qquad = \frac{(n+1)(n+2)(2n+3)}{6} \quad \text{v.}$$

ppl $1^2+2^2+\cdots+n^2$ is cubic in n.

Let's find $a_1, a_2, \ldots, a_n \ldots$ such that

$a_1 = 1^3$

$a_1+a_2 = 2^3$

$a_1+a_2+a_3 = 3^3$

...

$a_1+a_2+a_3+\cdots+a_n = n^3$

$$a_n = n^3-(n-1)^3 = n^3-(n-1)(n^2-2n+1) = n^3-n^3+2n^2-n+n^2-2n+1 = 3n^2-3n+1$$

$a_1 = 3\cdot1^2-3\cdot1+1 = 1$

$a_2 = 3\cdot2^2-3\cdot2+1 = 7$

$a_3 = 3\cdot3^2-3\cdot3+1 = 19$

...

$a_1 = 1 = 1^3$

$a_1+a_2 = 1+7 = 8 = 2^3$

$a_1+a_2+a_3 = 1+7+19 = 27 = 3^3$

...

$a_1+a_2+\cdots+a_n = n^3 \qquad a_n = 3n^2-3n+1$

$(3\cdot1^2-3\cdot1+1)+(3\cdot2^2-3\cdot2+1)+(3\cdot3^2-3\cdot3+1)+\cdots+(3n^2-3n+1) = n^3$

$3(1^2+2^2+3^2+\cdots+n^2)-3(1+2+\cdots+n)+n = n^3$

$3(1^2+2^2+\cdots+n^2) - \frac{3n(n+1)}{2} + n = n^3$

$$1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$\blacksquare$

HW: Find $1^3+2^3+\cdots+n^3 = ?$

find $a_1, \ldots, a_n$ s.t. $a_1+a_2+\cdots+a_n = n^4$

$a_n = n^4-(n-1)^4$

Prove $(1+x)^n \geq 1+nx$ if $x \geq -1$

for any $n = 1, 2, 3, \ldots$

① Check for $n=1$

LHS $= 1+x$ RHS $= 1+x$

$1+x \geq 1+x$ v.

② Induction step: sps $(1+x)^n \geq 1+nx$

$(1+x)^{n+1} \overset{?}{\geq} 1+(n+1)x$

$(1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx+x+\underline{nx^2} \geq 1+(n+1)x$ ✓.

ex: $x=0.2$ $n=3$

$(1+0.2)^3 = 1.2^3 \geq 1+3\times 0.2 = 1.6$

Claim: All horses have the same color.

Claim: For any $n=1,2,3,\cdots$ in any collection of n horses have the same color.

① For $n=1$ $\Rightarrow$ clearly true.

② Induction step: sps it's true for $n \Rightarrow$ want to prove for $n+1$

$n+1$ horses

horse 1, horse 2, $\cdots$, horse n, horse $n+1$

first n horses

$\Rightarrow$ by induction assumption $\Rightarrow$ have the same color.

horse 1 & horse$(n+1)$ have the same color or

horse. 2 & $\cdots$ –

$\Rightarrow$ all the horses have the same color. $\blacksquare$

$n=1$ $n+1=2$

horse 1 horse 2

Induction can start with any integer.

Claim: $n! \geq 2^n$ for $n \geq 4$

$n! = 1\cdot 2\cdot 3\cdot \cdots n$

$4! = 1\cdot 2\cdot 3\cdot 4 = 24 \geq 16$

$2^4 = 16$

① One of induction is $n=4$

$4! \geq 2^4$ ✓

② $n! \geq 2^n$ $\forall n \geq 4$

$\Rightarrow (n+1)! \geq 2^{n+1}$

$\underbrace{n!(n+1) \geq 2^n\cdot(n+1)}_{\text{by induction assumption}} \geq 2^n\cdot 5 \geq 2^n\cdot 2 = 2^{n+1}$

last time: $a^0+a^1+a^2+\cdots+a^n = \frac{a^{n+1}-1}{a-1}$ $a\neq 1$

holds for any $n \geq 0$

$n=0,1,2,3,\cdots$

$n=0$, $a^0=1$, ✓

$1 = \frac{a^{0+1}-1}{a-1} = \frac{a-1}{a-1} = 1$

Induction step: if $a^0+\cdots+a^n = \frac{a^{n+1}-1}{a-1}$ $n\geq 0$

then $a^0+\cdots+a^{n+1} = \frac{a^{n+1}-1}{a-1} + a^{n+1} = \frac{a^{n+1}-1+a^{n+1}(a-1)}{a-1} = \frac{a^{n+1}+a^{n+2}-a^{n+1}-1}{a-1} = \cdots$

Claim: n^3+5n is divisible by 3 for any $n \geq 1$

Proof: ① Check for $n=1$ $1^3+5 = 6 = 3\cdot 2$ ✓.

② Sps holds for n $\Rightarrow$ proof of $(n+1)^3+5(n+1)$

$$(n+1)^3+5(n+1) = n^3+3n^2+3n+1+5n+5 = (n^3+5n)+3n^2+3n+6 = (n^3+5n)+3(n^2+n+2) \quad ✓.$$

holds for $n+1$

Next subject: Prime numbers

Definition: a natural # n is called prime if it can not be written as a product of $n = a\cdot b$ where $a, b > 1$, natural #

$6 = 3\times 2$ not prime

$2 = 1\times 2$ prime

$3 = 1\times 3$ prime

even # not prime

$5 = 1\times 5$ prime

If n can be written as a product of $n = \underbrace{a\cdot b}_{a,b>1,\ a,b\in N^+}$ then is called composite number.