SCHOOL OF FINANCE AND APPLIED STATISTICS

STATISTICAL INFERENCE
(STAT3013/STAT8027)
TUTORIAL 0 - REVISION EXERCISES

1. Let X and Y be discrete random variables each taking values in the sample space $S = \{0, 1, 2\}$ and having a joint probability mass function *(pmf)* given by the following table:

Probability of all possible (x, y) pairs:

Values of X:	Values of Y: 0	1	2
0	0.1	0.1	0.2
1	0.25	0	0.2
2	0.05	0.05	0.05

   (a) Find the *pmf* and cumulative distribution function *(CDF)* of $U = X + Y$.
   (b) Find the marginal *pmf*s of both X and Y. Are X and Y independent?
   (c) Let X_1 be a discrete random variable having a *pmf* equal to the marginal *pmf* of X calculated in part *(b)*. Similarly, let Y_1 be a discrete random variable having a *pmf* equal to the marginal *pmf* of Y calculated in part *(b)*. Also, let X_1 and Y_1 be independent. Construct a table similar to the one above giving the joint *pmf* of X_1 and Y_1.
   (d) Using your result from part *(c)*, calculate the *pmf* of the random variable $U_1 = X_1 + Y_1$. Compare this *pmf* with the one you calculated in part *(a)*.
   (e) Compute $E(X)$ and $Var(Y)$.
   (f) Calculate the *pmf* of the random variable $E(X|Y)$. In other words, find $E(X|Y = y)$ and calculate $Pr\{E(X|Y) = E(X|Y = y)\}$ for each of $y = 0, 1, 2$. Verify the identity $E(X) = E\{E(X|Y)\}$ for these two random variables.

2. Let X be normally distributed with mean μ and variance σ^2. Find the moment generating function (mgf) of X, $m_X(t) = E(e^{tX})$. [HINT: Write the integral definition of the required expectation and then "complete-the-square" in the exponent. Recall that

$$\int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} \exp\left\{-\frac{1}{2b^2}(x-a)^2\right\} dx = 1$$

for any values a and $b > 0$, since this is just the integral of the normal density with mean a and variance b^2 over its entire range.]

3. Let X be a continuous random variable having a *pdf* of the form

$$f(x) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}x^2\right), \qquad x > 0.$$

The distribution having this density is often referred to as the "folded normal".
   (a) Let $Y = X^2$. Find the *pdf* of Y.
   (b) Do you recognise the density you found in part *(a)*?

4. Suppose that X_1 and X_2 are two continuous random variables with joint *CDF* $F_{X_1X_2}(x_1, x_2)$ and joint *pdf* $f_{X_1X_2}(x_1, x_2)$. Let $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ be two new continuous random variables such that the functions $g_1(\cdot,\cdot)$ and $g_2(\cdot,\cdot)$ are differentiable and "invertible"; that is, there exist two differentiable functions $h_1(\cdot,\cdot)$ and $h_2(\cdot,\cdot)$ such that $X_1 = h_1(Y_1, Y_2)$

(a).

Q1. $P(U=0)=0.1$ $\qquad$ $P(U<0)=0$

$P(U=1)=0.1+0.25=0.35$ $\qquad$ $0\le U<1 \quad 0.1$

$P(U=2)=0.05+0.2=0.25$ $\qquad$ $1\le U<2 \quad 0.45$

$P(U=3)=0.05+0.2=0.25$ $\qquad$ $2\le U<3 \quad 0.7$

$P(U=4)=0.05$ $\qquad$ $3\le U<4 \quad 0.95$

(b). $\qquad$ $U\ge 4 \quad 1$

$$P(X)=\sum_y P(X,y)=\begin{cases}0.1+0.1+0.2=0.4 & X=0\\ 0.25+0+0.2=0.45 & X=1\\ 0.05+0.05+0.05=0.15 & X=2\end{cases}$$

$$P(Y)=\sum_x P(x,Y)=\begin{cases}0.4 & Y=0\\ 0.15 & Y=1\\ 0.45 & Y=2\end{cases}$$

(c). $\quad P(1,1)=0\ne P_X(1)P_Y(1) \Rightarrow X \not\perp Y$

$\searrow$

0.16	0.06	0.18
0.18	0.0675	0.2025
0.06	0.0225	0.0675

$\sum=1$

(d). ✓

(e). $E(X)=0\times 0.4+1\times 0.45+2\times 0.15=0.75$

$V(Y)=E(Y^2)-(E(Y))^2$

$= \ - - -$

(f). $P(x=0|y=0)=\frac{P(x=0,y=0)}{P(y=0)}= , \cdots ,$ $\quad E(x|y=0)=0\,P(x=0|y=0)+1\cdots +2\cdots \ , - - -$

Q2. $M_x(t)=\int_{-\infty}^{\infty} e^{tx}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)dx$

$\quad \parallel$

$E(e^{tX})$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}(x^2-2\mu x+\mu^2-2\sigma^2 tx)\right)dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}(x^2-2(\mu+\sigma^2 t)x+(\mu+\sigma^2 t)^2-2\mu\sigma^2 t-\sigma^4 t^2)\right)dx$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\left[(x-(\mu+\sigma^2 t))^2-2\mu\sigma^2 t-\sigma^4 t^2\right]\right)dx$$

$$=e^{\mu t+\frac{1}{2}\sigma^2 t^2}\underbrace{\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{1}{2\sigma^2}\left[x-(\mu+\sigma^2 t)\right]^2\right)dx}_{N(\mu+\sigma^2 t,\ \sigma^2)}$$

$$=e^{\mu t+\frac{1}{2}\sigma^2 t^2}\cdot 1$$

Q3 $f(x) = \sqrt{\frac{2}{\pi}} \exp(-\frac{1}{2}x^2)$, $x > 0$

$$Y = X^2 \Rightarrow X = \sqrt{Y}$$

$$dx = \frac{1}{2} y^{-\frac{1}{2}} dy$$

$$\Rightarrow |J| = \left|\frac{dx}{dy}\right| = \left|\frac{1}{2} y^{-\frac{1}{2}}\right|$$

$$\therefore f_Y(y) = f_X[x(y)]\,|J|, \quad y > 0$$

$$= \sqrt{\frac{2}{\pi}} \exp\{-\frac{1}{2}(\sqrt{y})^2\} \frac{1}{2} y^{-\frac{1}{2}}$$

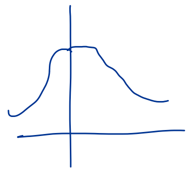
$$= \frac{\sqrt{2}}{\sqrt{\pi}\sqrt{y}} \cdot \frac{1}{\sqrt{2}\cdot\sqrt{2}} \exp\{-\frac{1}{2}y\}$$

$$= \frac{1}{\sqrt{2y}\cdot\sqrt{\frac{1}{2}}} \exp\{-\frac{1}{2}y\}$$

$$= \text{Gamma}(\tfrac{1}{2}, \tfrac{1}{2})$$

Sps. $X \sim N(0, 1)$

$Y = |X|^2 = X^2 \sim \chi^2_1$



$$\boxed{\begin{array}{l} \text{MLE}: X \sim U(0, \theta) \\ \quad \text{MLE of } \theta\,? \\ \quad \hat{\theta}_{MLE} = \,? \end{array}}$$

the order statistics
from $U(0, 1)$ follow
a Beta dist.

$\hat{\theta}_{MLE} = X_{(n)} \sim$ Beta.

and $X_2 = h_2(Y_1, Y_2)$. Furthermore, the change of variable formula for double integrals from multi-variable calculus states that for any set $A \in \mathbb{R}^2$:

$$\iint_A f_{X_1X_2}(u_1, u_2)du_1du_2 = \iint_{A_h} f_{X_1X_2}\{h_1(v_1, v_2), h_2(v_1, v_2)\}|J(v_1, v_2)|dv_1dv_2,$$

where

$$A_h = [(v_1, v_2) : \ \{h_1(v_1, v_2), h_2(v_1, v_2)\} \in A],$$

and $|J(y_1, y_2)|$ is the absolute value of the determinant of the Jacobian matrix

$$J(y_1, y_2) = \left\{ \begin{array}{cc} \frac{\partial}{\partial y_1}h_1(y_1, y_2) & \frac{\partial}{\partial y_1}h_2(y_1, y_2) \\ \frac{\partial}{\partial y_2}h_1(y_1, y_2) & \frac{\partial}{\partial y_2}h_2(y_1, y_2) \end{array} \right\}.$$

Use this fact to show that the joint density function for Y_1 and Y_2 is given by:

$$f_{Y_1Y_2}(y_1, y_2) = |J(y_1, y_2)|f_{X_1X_2}\{h_1(y_1, y_2), h_2(y_1, y_2)\}.$$

[HINT: Let $A = \{(u_1, u_2) : \ g_1(u_1, u_2) \le y_1, \ g_2(u_1, u_2) \le y_2\}$ and note that

$$g_1\{h_1(v_1, v_2), h_2(v_1, v_2)\} = v_1, \quad g_2\{h_1(v_1, v_2), h_2(v_1, v_2)\} = v_2$$

by the defining relationship between the g functions and the h functions.]

5. Suppose that X and Y are two independent normal random variables with mean μ and variance σ^2 and define the two random variables $R = \sqrt{X^2 + Y^2}$ and $\Theta = \tan^{-1}\left(\frac{Y}{X}\right)$, $0 \le R < \infty$ and $-\pi \le \Theta < \pi$.
   (a) Show that $X = R\cos\Theta$ and $Y = R\sin\Theta$. [HINT: Recall that $\cos^2\theta + \sin^2\theta = 1$ for all θ.]
   (b) Use the identity from Question 4 to find the joint *pdf* of R and Θ, $f_{R\Theta}(r, \theta)$.
   (c) When $\mu = 0$, show that Θ is independent of R and is uniformly distributed on the interval $(-\pi, \pi)$.
   (d) When $\mu \ne 0$, show that, for any given value of r, the joint density $f_{R\Theta}(r, \theta)$ is maximised at $\theta = \frac{\pi}{4}$ for $\mu > 0$ and at $\theta = -\frac{3\pi}{4}$ for $\mu < 0$.
   (e) Interpret the results of parts (*c*) and (*d*). [HINT: Think polar co-ordinates.]

6. Let X be a continuous random variable with *CDF* $F(x)$. Show that the new random variable $U = F(X)$ has a uniform distribution on the interval $(0, 1)$. Furthermore, suppose that we can generate a random variable, U, from a uniform distribution on $(0, 1)$. Show how we can use this capability to generate a random variable, X, from an exponential distribution with mean parameter μ. [HINT: Find the *CDF* of U and recall that by the definition of an inverse function, $F^{-1}\{F(x)\} = x$ and $F\{F^{-1}(x)\} = x$. Also, note that the inverse function $F^{-1}(x)$ will exist in this case since we have assumed that X is continuous.]

$X \sim f_X(x)$; $Y = F(X) \leftarrow$ CDF $\in [0,1]$

what the distribution of Y is ?

$P(Y \le y) = P(F(X) \le y)$
$= P(F^{-1}(F(X)) \le F^{-1}(y))$
$= P(X \le F^{-1}(y)) \rightarrow$ itself is a cdf of CDF
$= F(F^{-1}(y))$
$= y \qquad \therefore y \sim Unif(0,1)$

$U \sim Unif(0,1) \Rightarrow U = F(X) \Rightarrow X = F^{-1}(U)$

Q5.