Lecture 9
Cauchy THEOREM

We've seen $\int_\gamma z^m dz = F(\text{end pt}) - F(\text{Initial pt})$

$$F(z) = \frac{1}{m+1} z^{m+1}$$

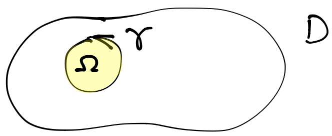
This implies $\int_\gamma z^m dz = 0$ for any closed curve γ.

Recall: If $f = u + iv$ is analytic. u, v satisfy C-R eqns.

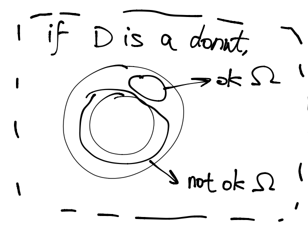
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Green's THEOREM:
$$\int_{\partial\Omega} f(z)dz = i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dxdy$$

THM (Cauchy Theorem) Let f be analytic in D, γ a simple closed curve in D, Ω = inside of γ. & $\Omega \subseteq D$



Then $\int_\gamma f(z)dz = 0$



Proof: By Green's Thm

$$\int_\gamma f(z)dz = i\iint_\Omega \left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)dxdy = i\iint_\Omega \left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]dxdy$$

$$= i\iint_\Omega \left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)i\right]dxdy$$

$$= i\iint_\Omega (0 + 0i)\,dxdy = 0 \quad \text{by C-R equations}$$

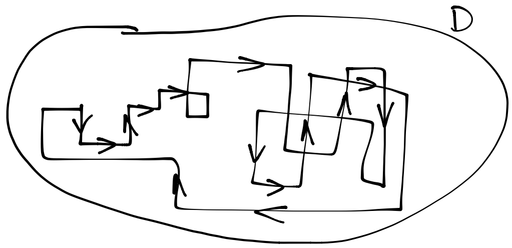
Q: What about antiderivative?
What about non-simple closed curves?

Def'n: We say a set S is **simply connected** if for any simple closed curve γ in S inside of γ is also contained in S. (I.e. S has no holes/punctures)

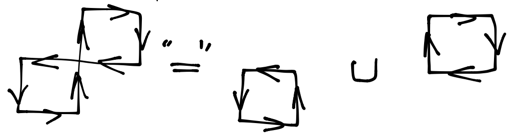
Ex: ① A disk is simply connected.
② A half-plane is simply connected
③ An annulus $r < |z - z_0| < R$ is not simply connected.
④ The set $\mathbb{C}^* = \{z \neq 0\}$ is not simply connected.
⑤ The domain of $f(z) = \frac{1}{z-1}$ is not simply-connected.

$$\text{dom}(f) = \{z \neq 1\}$$

THM Let D be simply connected $f: D \to \mathbb{C}$ analytic. If γ is any closed curve consisting of horizontal & vertical line segments, then $\int_\gamma f(z)dz = 0$.



Proof: Split γ into "simple pieces", then use Cauchy Theorem.



$$\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$$

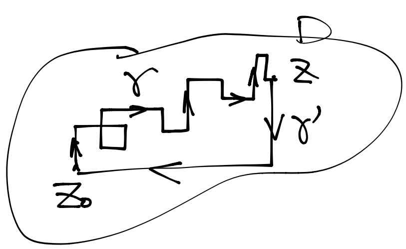
THM: Suppose $f: D \to \mathbb{C}$ is analytic & D is simply connected, then there is an analytic function $F: D \to \mathbb{C}$ s.t. $F' = f$.

($f: \mathbb{R} \to \mathbb{R}$ cts. $F(x) = \int_a^x f(t)dt$)

$$\int_a^b f(x)dx = F(b) - F(a)$$

"Pf": Fix a point $z_0 \in D$

Define $F(z) = \int_\gamma f(w)dw$, where γ is a curve joining z_0 to z consisting of horizontal & vertical line segments.



① To show $\int_\gamma f(z) = \int_{\gamma'} f(z)$

(i.e. F doesn't depend on the choice of γ)

we use previous theorem to $\gamma \cup (-\gamma')$

$$0 = \int_{\gamma \cup (-\gamma')} = \int_\gamma - \int_{\gamma'}$$

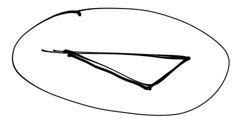
② Show $F'(z) = \lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z)$

skip details

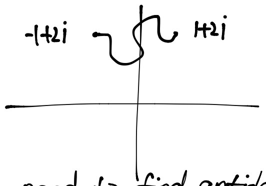
COR: Suppose f is analytic in D, & D is simply connected. If γ is any p.w smooth closed curve, then $\int_\gamma f(z)dz = 0$

Pf: $\int_\gamma f(z)dz = F(\text{end pt}) - F(\text{initial pt})$ where F is antiderivative.

THM (Morera's Theorem). Suppose f is cts in D (simply connected) and that $\int_\gamma f(z)dz = 0$ for all triangles in D then f is analytic.



Ex Compute $\int_\gamma \frac{z}{z+1}dz$ where γ is any curve joining $1+2i$ to $-1+2i$ in upper half-plane $\text{Im}(z)>0$



D is the upper half-plane $\text{Im}(z)>0$

$D = \{z \mid \text{Im}(z)>0\}$ is simply connected f is analytic in D

We need to find antiderivative for $f(z) = \frac{z}{z+1} = \frac{z+1-1}{z+1} = 1 - \frac{1}{z+1}$

$$F(z) = z - \text{Log}(z+1)$$

$$\int_\gamma \frac{z}{z+1}dz = F(z)\Big|_{1+2i}^{-1+2i} = z - \text{Log}(z+1)\Big|_{1+2i}^{-1+2i}$$

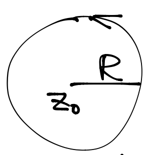
Cauchy Formula

Recall: ① $\left|\int_\gamma f(z)dz\right| \leq \text{length}\,\gamma \cdot \max|f(z)|$

② If f is cts at z_0 then $|f(z)-f(z_0)| \longrightarrow 0$ as $z \to z_0$

Lemma: Suppose f is cts then $\frac{1}{2\pi i}\int_{C_R(z_0)} \frac{f(z)}{z-z_0}dz \longrightarrow f(z_0)$ as $R \to 0$

$C_R(z_0)$ = circle of rad R, centered at z_0.



Take a close look at: $\left|\frac{1}{2\pi i}\int_{C_R} \frac{f(z)}{z-z_0}dz - f(z_0)\right|$

$$= \left|\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}dz - \frac{f(z_0)}{2\pi i}\int_{C_R(z_0)}\frac{1}{z-z_0}dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_{C_R}\frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_R}\frac{f(z_0)}{z-z_0}dz\right|$$

$$= \left|\frac{1}{2\pi i}\int_{C_R}\frac{f(z)-f(z_0)}{z-z_0}dz\right|$$

$$\leq \frac{1}{2\pi}\,\text{length of } |C_R| \cdot \max_{z\in C_R}\left|\frac{f(z)-f(z_0)}{z-z_0}\right|$$

$$= \frac{1}{2\pi}\cdot 2\pi R \cdot \max_{z\in C_R}\frac{|f(z)-f(z_0)|}{|z-z_0|}$$

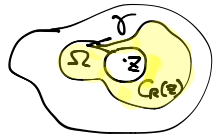
$$= R\,\frac{\max|f(z)-f(z_0)|}{R} = \max|f(z)-f(z_0)|$$

($R\to 0$, $z \to z_0$ & cts f. $\Rightarrow f(z) \to f(z_0)$)

THM (Cauchy Formula)
If f is analytic in D, γ any closed curve whose inside is contained in D, then

$$f(z)=\frac{1}{2\pi i}\int_{\gamma}\frac{f(\xi)}{\xi-z}\,d\xi$$

where z is any point inside γ.



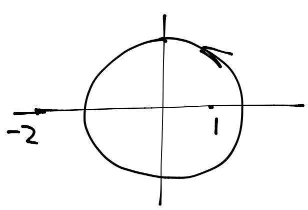
Proof: Take a circle.
By Green's Theorem & the fact f is analytic we get:

$$\int_{\gamma}\frac{f(\xi)}{\xi-z}\,d\xi=\int_{C_R(z)}\frac{f(\xi)}{\xi-z}\,d\xi$$

$$\downarrow \qquad\qquad \downarrow$$

$$\int\frac{f(\xi)}{\xi-z}\,d\xi = 2\pi i f(z)$$

Ex: Find $\int_{\gamma}\frac{e^z}{(z-1)(z+2)}\,dz$ where γ is the circle $|z|=\frac{3}{2}$ oriented positively.

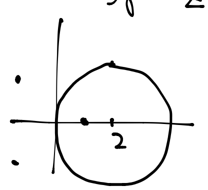


$$\int_{\gamma}\frac{e^z}{(z-1)(z+2)} = 2\pi i f(1) = 2\pi i\,\frac{e^1}{3}$$
$$= \frac{2}{3}\pi e i$$

$f(z)$

pt inside γ

Ex $\int_{\gamma}\frac{\cos(2\pi z)}{z^3-1}\,dz$ where γ is the circle of radius 2 centered at 2.



Need to know sol'ns of $z^3-1=0$, $z^3=1$
$z=1, e^{i2\pi/3}, e^{i4\pi/3}$
(1 in γ, 2 not in γ)

$z^3-1=(z-1)(z^2+z+1)$

f

$$\int_{\gamma}\frac{\cos(2\pi z)}{z^3-1}\,dz=\int\frac{\cos(2\pi z)}{(z-1)(z^2+z+1)}\,dz$$

analytic inside γ

$$= 2\pi i f(1)$$
$$= \frac{2\pi i}{3}$$