Feb 15th

Last topic covered in term test — irrational numbers

- no aids
- 1 hour 50 min.
- a homework
- past tests posted (1 hour previously)

Irrational numbers

q is rational if q can be written as $q = m/n$, m, n integers
if q is not rational $\Rightarrow$ it's called irrational

ex: $\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}$ are all irrational

Claim $q_1\sqrt{2}+q_2\sqrt{3}$ is irrational unless $q_1 = q_2 = 0$
q_1, q_2 are rational

Proof:

Let $x = q_1\sqrt{2}+q_2\sqrt{3}$
sps x is rational
$\Rightarrow x^2 = q_1^2 \cdot 2 + 2q_1q_2\sqrt{6} + q_2^2 \cdot 3$

$x^2 - 2q_1^2 - 3q_2^2 = 2q_1q_2\sqrt{6}$

$\sqrt{6} = \dfrac{x^2 - 2q_1^2 - 3q_2^2}{2q_1q_2}$ rational because x, q_1, q_2 are rational

$\Rightarrow \sqrt{6}$ is rational $\Rightarrow$ we know that's false.

Contradiction $\Rightarrow x$ is irrational
$\Rightarrow q_1\sqrt{2}+q_2\sqrt{3}$ is irrational

Case 1: if $q_1, q_2 \neq 0$ (i.e. $q_1 \neq 0$ & $q_2 \neq 0 \Rightarrow$ can divide
$\Rightarrow$ can divide by $2q_1q_2 \Rightarrow$ the argument works

Case 2: $q_1 = 0, q_2 \neq 0$

Case 3: $q_1 \neq 0, q_2 = 0$

Case 4: $q_1, q_2 = 0$

Case 2: $q_1 = 0, q_2 \neq 0$
$x = q_2\sqrt{3}$ can not be rational
if x is rational $\frac{x}{q_2} = \sqrt{3}$ rational
we know this is false

...

Rational root Theorem (proved last time)

$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$ a_i: integers

if $x = \frac{p}{q}$ is a root and $\gcd(p, q) = 1$

then $p | a_0$, and $q | a_n$

$x^5 + 2x - 3 = 0$ Q: does it have any rational numbers?

if $x = \frac{p}{q}$ is a root $\gcd(p, q) = 1$

then $p | -3$, $q | 1 \Rightarrow p = \pm 1, \pm 3$, $q = \pm 1$

$\Rightarrow \frac{p}{q} = \pm 1, \pm 3$

plug in $x = 1$, $x = -1$, $x = 3$, $x = -3$

Check if it solves

$x = 1 \Rightarrow$ solution $1^5 + 2 \cdot 1 - 3 = 0$

$x = -1 \Rightarrow -1^5 - 2 - 3 \neq 0$

$x = 3 \Rightarrow 3^5 + 2 \times 3 - 3 \neq 0$

$x = -3$ also not a solution

ex: if $x = \sqrt{2} \Rightarrow x^2 = 2$

$x^2 - 2 = 0$

if $x = p/q$ is a rational root

$(p, q) = 1 \Rightarrow p | -2$ & $q | 1 \Rightarrow$ $p = \pm 1, \pm 2$, $q = \pm 1$

$\Rightarrow p/q = \pm 1, \pm 2$

$x = 1 \Rightarrow 1^2 - 2 = -1 \neq 0$

$x = -1 \Rightarrow \cdots \neq 0$

$x = 2 \Rightarrow \cdots \neq 0$

$x = -2 \Rightarrow \cdots \neq 0$

$\Rightarrow x^2 - 2 = 0$ has no rational solutions

$\Rightarrow \sqrt{2}$ is irrational

Cor: $\sqrt{n}$ is rational iff n is a complete square. (i.e. $n = m^2$ for some m - natural number)

$n \in \mathbb{N}$

Proof: if $n = m^2 \Rightarrow \sqrt{n} = m$ rational

$\sqrt{9} = 3$ rational, $\sqrt{25} = 5$ rational

Sps $x = \sqrt{n}$ is rational

then $x^2 = n$, $x^2 - n = 0 \Rightarrow x = \sqrt{n}$ is a rational

$\Rightarrow \sqrt{n} = p/q \Rightarrow p | -n$ by the rational root thm

$(p, q) = 1$ $q | 1$

$\Rightarrow q = \pm 1$

$\Rightarrow \sqrt{n} = p/q = p/\pm 1 = m$ (integer)

$\Rightarrow n = m^2$ is a complete square.

Same for other roots

$\sqrt[5]{n}$ is rational $\Leftrightarrow n = m^5$ for m integer if $x = \sqrt[5]{n}$ is rational
$\Rightarrow x^5 = n$, $x^5 - n = 0$

$\Rightarrow x$ is a rational root of $x^5 - n = 0$

$x = p/q \Rightarrow p \mid -n$
$(p,q) = 1$ $q \mid -1 \Rightarrow q = \pm 1 \Rightarrow x = p/q$ is an integer.

$x = \sqrt{2} + \sqrt{3}$

$x^2 = 2 + 3 + 2\sqrt{6} = 5 + 2\sqrt{6}$

$x^2 - 5 = 2\sqrt{6}$

$x^4 - 10x^2 + 25 = 24$

$x^4 - 10x^2 + 1 = 0$

use Rational Root Thm to solve and check it's irrational.

Definition : a number x is called algebraic if it solves some polynomial equation with integer coefficients as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0$$

all transcendental numbers are irrational

Fact: π, e — transcendental

if x is not algebraic
$\Rightarrow$ it's called transcendental number

— any rational # is algebraic

$\frac{m}{n}$ solves $nx - m = 0$