June 5th

① $f: S \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $g: f(S) \subseteq \mathbb{R}^m \to \mathbb{R}^k$. Sps f, g are uniformly continuous
$(g \circ f)(x) = g(f(x))$

Proof: Fix $\varepsilon > 0$. Need to show that $\exists\, \delta > 0$ s.t. for all $x, y \in S$ we have $|x-y| < \delta \Rightarrow |g(f(x)) - g(f(y))| < \varepsilon$. Since g is unif. cts, $\exists\, \delta_1 > 0$ s.t. $\forall u, v \in f(S) =$ (domain of g) we have
$|u-v| < \delta_1 \Rightarrow |g(u) - g(v)| < \varepsilon$ ①
Since f is unif cts, $\exists\, \delta_2 > 0$ s.t. $\forall x, y \in S$, ② $|x-y| < \delta_2 \Rightarrow |f(x) - f(y)| < \delta_1$

Taking $\varepsilon = \delta_1$ in the def of unif cts for f.

Claim: $\delta = \delta_2$ works

Check: Sps $x, y \in S$ and $|x-y| < \delta = \delta_2$. By (2)
$|f(x) - f(y)| < \delta_1$ ③
Let $u = f(x)$, $v = f(y)$, eqn ③ becomes $|u - v| < \delta_1$
$\Rightarrow$ By ①, $|g(u) - g(v)| < \varepsilon$

$$|g(u) - g(v)| = |g(f(x)) - g(f(y))|$$

Def: $f: \mathbb{R}^n \to \mathbb{R}$. $f(\vec{x})$ is diff at $\vec{a}$ if $\exists\, \vec{c}$ such that $f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{c} \cdot \vec{h} + E(\vec{h})$ s.t.

$$\lim_{\vec{h} \to \vec{0}} \frac{E(\vec{h})}{|\vec{h}|} = 0$$

$\Leftrightarrow f(\vec{a} + \vec{h}) - f(\vec{a}) - \vec{c} \cdot \vec{h} = E(\vec{h})$

② $f(x) = \sqrt{x}$, prove $f(x)$ is diff. at $x = 4$
(we know that $f'(x) = \frac{1}{2}(\sqrt{x})^{-1}$, $f'(4) = \frac{1}{4}$, so c should be $\frac{1}{4}$.)

Idea: Calculate $f(4+h) - f(4) - \frac{1}{4}h = E(h)$

Then to prove that f is diff we just need to check that the condition

$$\lim_{h \to 0} \frac{E(h)}{|h|} = 0.$$

$$E(h) = f(4+h) - f(4) - \frac{1}{4}h = \frac{(4+h) - 4}{\sqrt{4+h} + 2} - \frac{1}{4}h$$

$$\Rightarrow E(h) = h\left(\frac{1}{\sqrt{4+h} + 2} - \frac{1}{4}\right)$$

$$\lim_{h \to 0^+} \frac{E(h)}{h} = \lim\left(\frac{1}{\sqrt{4+h} + 2} - \frac{1}{4}\right) = 0$$

Similarly for $h \to 0^-$, $\lim = 0$.

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③ $f(x)=\sin x$, $x=\pi/3$, $f'(x)=\cos x$, $\cos(\pi/3)=\frac{1}{2}$ ← should be c.

Calculate:

$$\sin(\tfrac{\pi}{3}+h)-\sin(\pi/3)-\tfrac{1}{2}h$$
$$=\sin(\tfrac{\pi}{3})\cos h+\sin h\cos(\tfrac{\pi}{3})-\tfrac{\sqrt{3}}{2}-\tfrac{1}{2}h$$
$$=\tfrac{\sqrt{3}}{2}(\cos(h)-1)+\tfrac{1}{2}(\sin(h)-h)$$

$$\frac{E(h)}{|h|}=\frac{\sqrt{3}}{2}\left(\frac{\cos(h)-1}{|h|}\right)+\frac{1}{2}\left(\frac{\sinh-h}{|h|}\right)$$

$\Rightarrow \lim_{h\to 0}\frac{E(h)}{h}=0$ by trig limits from MAT137

$\lim_{h\to 0}\frac{\sin(h)}{h}=1$, $\lim_{h\to 0}\frac{\cos(h)-1}{h}=0$

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④ $f(x,y,z)=3x^2yz^3-xy^2$, show f is diff at $\vec{a}=(1,2,3)$

Let $\vec{h}=(h_1,h_2,h_3)$

$f(\vec{a}+\vec{h})-f(\vec{a})=320h_1+77h_2+162h_3+\cdots$

Claim: $\vec{c}=(320,77,162)$

$E(\vec{h})=f(\vec{a}+\vec{h})-f(\vec{a})-\vec{c}\cdot\vec{h}=27h_1^2h_2h_3^2+324h_1h_3+\cdots$

Check that $\lim \boxed{\ } = 0$

I'll do this 1 term, same argument works for others.

look at $\lim\left|\frac{324h_1h_3}{|\vec{h}|}\right|=324\lim\frac{|h_1h_3|}{|\vec{h}_1|}$

But $|h_1|\le|\vec{h}|=\sqrt{h_1^2+h_2^2+h_3^2}$ $(h_2^2, h_3^2\ge 0)$
$|h_3|\le|\vec{h}|$

$\Rightarrow 324\lim\frac{|h_1h_3|}{|\vec{h}|}\le 324\lim\frac{|\vec{h}|^2}{|\vec{h}|}=324\lim_{h\to 0}|\vec{h}|=0$

By squeeze thm, $\lim \boxed{\ } = 0$

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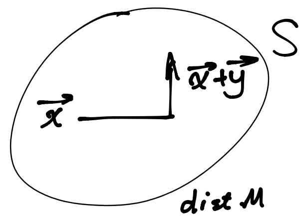
⑥ Let $f: S\to\mathbb{R}^n\to\mathbb{R}$, S is an open set. If $\partial_j f$ $j=1,\dots,n$ exist and are <u>bounded</u> on S then f is cts on S.

Pf: Let $M \geq |\partial_j f(x)|$. $\forall x \in S$, $j = 1, \cdots, n$.



Main Idea derives from Thm 2.19

Fix $\varepsilon > 0$. $|f(\vec{x}+\vec{y}) - f(\vec{x})| = |f(x_1+y_1, \cdots, x_n+y_n) - f(x_1, \cdots, x_n)|$

$$
\begin{aligned}
&① \quad = |f(x_1+y_1, \cdots, x_n+y_n) - f(x_1+y_1, \cdots, x_{n-1}+y_{n-1}, x_n) \\
&② \quad + (f(x_1+y_1, \cdots, x_{n-1}+y_{n-1}, x_n) - f(x_1+y_1, \cdots, x_{n-2}+y_{n-2}, x_{n-1}, x_n) \\
&③ \quad + f(x_1+y_1, \cdots, x_{n-2}+y_{n-2}, x_{n-1}, x_n) - f(x_1+y_1, \cdots, x_{n-3}+y_{n-3}, x_{n-2}, x_{n-1}, x_n) \\
&\quad\quad + \cdots \\
&ⓝ \quad + f(x_1+y_1, \cdots, x_{n-1}, x_n) - f(x_1, \cdots, x_{n-1}, x_n)| \\
&\quad\quad \leq |①| + |②| + \cdots + |ⓝ| \quad \text{by triangle Ineq'}
\end{aligned}
$$

Use MVT, $f(x_1+y_1, \cdots, x_n+y_n) - f(x_1+y_1, \cdots, x_n) \leq M|y_n| \leq M|\vec{y}|$

$\Rightarrow |f(\vec{x}+\vec{y}) - f(\vec{x})| \leq M \cdot n |\vec{y}| \cdots$ (*)

Take $\delta = \frac{\varepsilon}{Mn}$, then when $|\vec{y}| < \delta = \varepsilon / M \cdot n$

(*) $\Rightarrow\ < Mn \cdot \varepsilon / M \cdot n = \varepsilon$. ∎