Lecture 3

Main Example (Sorgenfrey Line)

Idea: Take $S=\{[a,b):a,b\in\mathbb{R}\}$
and note that this is a basis on $\mathbb{R}$

The topology we get here is called the lower-limit topology, and the topological space is called the Sorgenfrey Line.

Claim: The Sorgenfrey line is weird!

1. Every interval $(0,b)$ is open.
   Notice that $(0,b)=\bigcup_{n\in\mathbb{N}}[\frac{1}{n},b)$ and each $[\frac{1}{n},b)$ is open in the Sorgenfrey line

   Moreover, every interval of the form (a,b) is open.
   So the lower-limit topological refines the usual topology

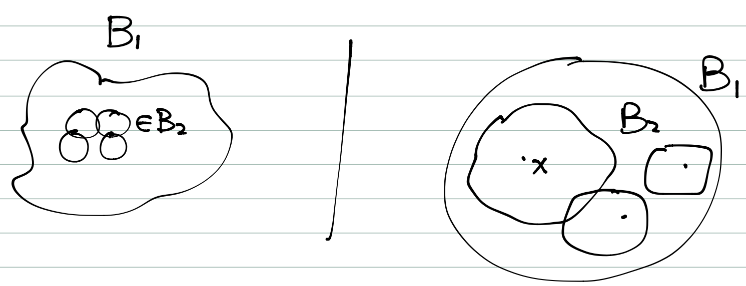
2. $(b,0]$ is not open.
   If any basic open set $[a,c)$ contains 0, then $[a,c)$ also contains elements to the right of 0. So $(b,0]$ can't be written as a union of basic open sets.
3. (so in particular this is not the discrete topology).
4. Notice that $[0,\infty)$ is open, as
   $$[0,\infty)=\bigcup_{n\in\mathbb{N}}[0,n)$$
   Also, $(-\infty,0)$ is open as $(-\infty,0)=\bigcup_{n\in\mathbb{N}}[-n,0)$

   So $\mathbb{R}=(-\infty,0)\cup[0,\infty)$ both sets are open & disjoint.

Proposition: Let $\mathcal{B}_1$ & $\mathcal{B}_2$ be 2 bases on set X, the following are equivalent:

1). $\mathcal{T}_{\mathcal{B}_1}\subseteq\mathcal{T}_{\mathcal{B}_2}$

2). For every $B_1\in\mathcal{B}_1$, and every $x\in B_1$, there is a $B_2\in\mathcal{B}_2$ s.t. $x\in B_2\subseteq B_1$

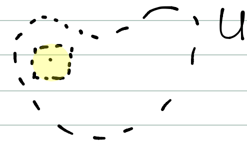


Proof: Stare a while.

Corollary: A basis $\mathcal{B}$ generates a topology $\mathcal{T}$ if and only if for every open set $U\in\mathcal{T}$ and for every $x\in U$, there is a basic open set $B\in\mathcal{B}$ s.t. $x\in B\subseteq U$.

This tells us:

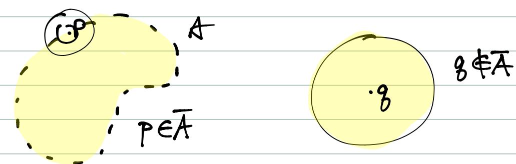
- The usual topology on $\mathbb{R}^2$ is the same as the topology given by open rectangles



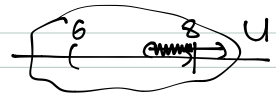
§3. Closed set and Closures

<u>Def'n</u>: Let $(X, \mathcal{T})$ be top. space and let $A \subseteq X$.
For point $p \in X$, we say $p \in \overline{A}$ iff for every open set $U \in \mathcal{T}$ that contains x, we have $U \cap A \neq \emptyset$.
We say $\overline{A}$ is the closure of A (in $\mathcal{T}$)

Pic



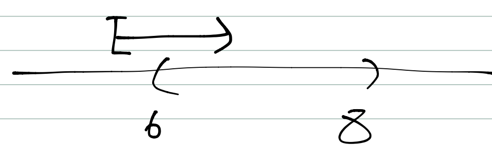
e.g. If $A = (6, 8)$ in the usual topology, then $7 \in \overline{A}$ (Moreover $A \subseteq \overline{A}$)
$8 \in \overline{A}$
Take any open set $U \ni 8$, there is a basic open set $(a, b) \subseteq U$ and $8 \in (a, b)$.
There is a point in A, above a and below 8.
Also $6 \in \overline{A}$ (Similar reason as why $8 \in \overline{A}$)



Q: Is $9 \in \overline{A}$?
No, because $(8.5, 10^{10^{10}})$ is an open set that contains 9, but is disjoint from A.

Same thing, $A = (6, 8)$, but this time in the Sorgenfrey Line.

- $7 \in \overline{A}$ (again $A \subseteq \overline{A}$)
- $6 \in \overline{A}$? Yes! (Any open set contains 6 has a basic open set $[a, b)$ as a subset that contains 6 & something to the right of 6.



- but, $8 \notin \overline{A}$
because $[8, 10)$ is an open set that contains 8 but $A \cap [8, 10) = \emptyset$

Same thing, $A = (6, 8)$, but in discrete topology.

- Again, $A \subseteq \overline{A}$
- $6 \notin \overline{A}$ because $\{6\}$ is open and is disjoint from A
- In fact, if $a \notin A$, then a is open & disjoint from A. So $\overline{A} = A$

Proposition (of possibly false things):

Let $A \subseteq X$, and $(X, \mathcal{T})$ a top space

- $X \setminus \overline{A}$ is open
- $A \subseteq \overline{A}$ $\longrightarrow$ proof next Monday
- $\overline{\emptyset} = \emptyset$, $\overline{X} = X$
- $\overline{\overline{A}} = \overline{A}$ (closure is idempotent)
- $\overline{A \cup B} = \overline{A} \cup \overline{B}$

False things

- For $p \in X$, $\overline{\{p\}} = \{p\}$
- $\overline{A \cap B} = \overline{A} \cap \overline{B}$
- $\overline{A}$ is not open
- If $\{A_n : n \in \mathbb{N}\}$ are all subsets of X, then $\bigcup_{n \in \mathbb{N}} \overline{A_n} = \overline{\bigcup_{n \in \mathbb{N}} A_n}$