Lecture 4.

Continuous Probability Spaces

- Ω is not countable. (interval, subset of $\mathbb{R}$)
- $\omega \in \mathbb{R}$ (e.g. heights, weights, lifetimes, etc)
- cannot assign probabilities to each outcome
- $\mathcal{F}$ is formed by taking a (countable) number of intersections, unions, and complements of sub-intervals of Ω.

Ex: $\Omega = [0,1]$

$$\mathcal{F} = \{ A = [0, 1/2), B = [1/2, 1], \emptyset, \Omega \}$$

How to define P ?

Idea:
- P should be weighted by the length of the intervals
- $P(\Omega) = 1$
- assign 0 probability to intervals not of interest.

Cdf for Continuous Probability Space:

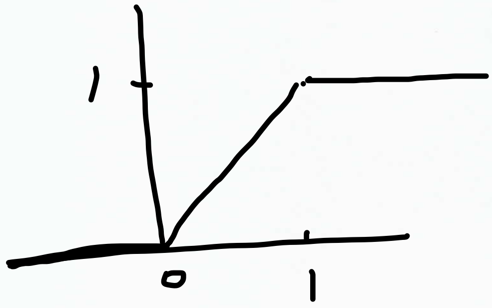
$$F_X(x) = P[(-\infty, x])$$

$$P(\{\omega\}) = P((\omega, \omega]) = F_X(\omega) - F_X(\omega) = 0$$

$$P([a,b]) = P((a,b]) = P([a,b)) = P((a,b))$$

Ex. $X \sim \text{Unif}[0,1]$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



$X \sim \text{Unif}[a,b]$, $b > a$

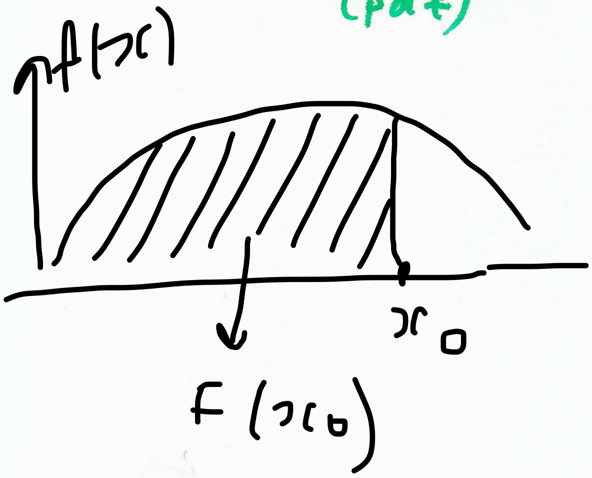
$$F_X(x) = \begin{cases} 0, & x < a \\ \dfrac{x-a}{b-a}, & a \le x \le b \\ 1, & x > b \end{cases}$$

Def. A r.v. X is **continuous** if its distribution function may be written in the form:

$$F_X(x) = \mathbb{P}(X \le x) = \int_{-\infty}^{x} f_X(t)\,dt, \text{ for all } x \in \mathbb{R}$$

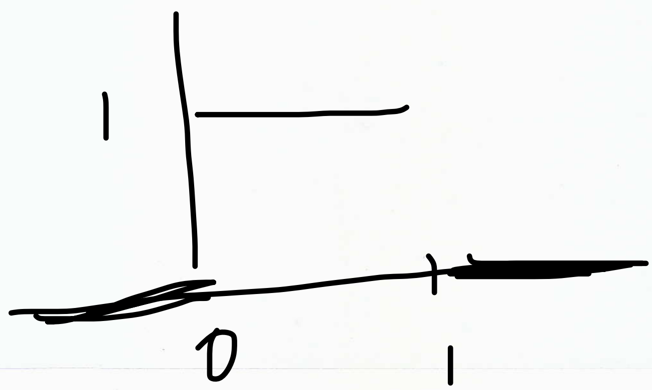
for some non-negative function f.

$f_X(x)$ is called the **(Probability) Density Function** of X.
(pdf)



Ex. $X \sim \text{Unif}[0,1]$.

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0, & \text{OW} \end{cases}$$



Facts and Properties of pdf:

- If X is a continuous r.v. with a well-behaved cdf F then $f_X(x) = \begin{cases} \frac{d}{dx} F_X(x), & \text{if derivative exists at } x \\ 0, & \text{ow} \end{cases}$

- $f_X(x) \geq 0, \ x \in \mathbb{R}$

- $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

Ex. $f_X(x) = \begin{cases} cx^2, & 0 \leq x \leq 2 \\ 0, & \text{ow} \end{cases}$. Find c.

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^2 cx^2\,dx = c\frac{x^3}{3}\Big|_0^2 = c\frac{8}{3}$$

$$\Rightarrow c = \frac{3}{8}$$

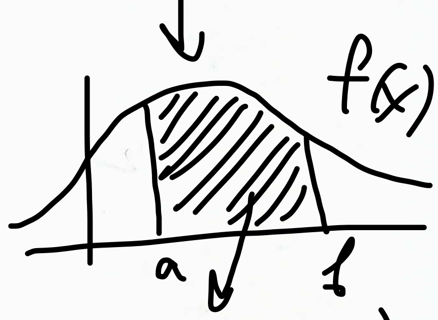
- $P(a \leq X \leq b) = \int_a^b f_X(t)\,dt$

$\hookrightarrow P(X \leq b) - P(X \leq a) = \int_{-\infty}^{b} f\,dx - \int_{-\infty}^{a} f\,dx$

$= \int_a^b f\,dx$

- $P(X = a) = 0$

$\hookrightarrow \int_a^a f\,dx = 0$



$P(a \leq X \leq b)$

The Exponential Distribution.

A r.v. X that counts the waiting time for rare phenomena has Exponential (λ) distribution, where λ = average number of occurrences per unit of time (space).

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x>0 \\ 0, & ow \end{cases}$$

$$\beta = \frac{1}{\lambda} \qquad \frac{1}{\beta} e^{-x/\beta}$$

Is this a valid pdf?

$$\int_{-\infty}^{\infty} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_{0}^{\infty} = 1$$

Memoryless property:

$$P(X > t+s \mid X>t) = P(X>s) \qquad t,s>0$$

$$\| $$

$$\frac{P(X>t+s)}{P(X>t)} = \frac{e^{-\lambda(\cancel{t}+s)}}{e^{-\lambda t}} = e^{-\lambda s}$$

$$F_X(x) = \int_{-\infty}^{x} f(t)\,dt = \int_0^x \lambda e^{-\lambda t} dt = 1-e^{-\lambda x}$$

$$= P(X \le x), \quad P(X>x) = e^{-\lambda x}$$

The Gamma Distribution.

A r.v. X is said to have a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ (or $\lambda = \frac{1}{\beta}$) iff the density function of X is

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-x/\beta}}{\beta^{\alpha}\,\Gamma(\alpha)}, & 0 \le x < \infty \\ 0, & \text{ow} \end{cases}$$

or

$$f_X(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}, & 0 \le x < \infty \\ 0, & \text{ow} \end{cases}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}\, t^{\alpha-1}\, dt \longrightarrow$ Gamma function

Properties of $\Gamma(\alpha)$:

1. $\Gamma(1) = \int_0^{\infty} e^{-t}\, dt = 1$

2. $\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha)$

$$\Gamma(\alpha+1) = \int_0^{\infty} e^{-t}\, t^{\alpha}\, dt = \begin{pmatrix} u = t^{\alpha} & dv = e^{-t}dt \\ du = \alpha t^{\alpha-1}dt & v = -e^{-t} \end{pmatrix}$$

$$= \underbrace{-t^{\alpha} e^{-t}\Big|_0^{\infty}}_{=0} + \alpha \int_0^{\infty} e^{-t}\, t^{\alpha-1}\, dt = \alpha\,\Gamma(\alpha)$$

3. $\Gamma(n) = (n-1)!\,, \quad n \in \mathbb{Z}^+$

The Beta Distribution.

A r.v. X is said to have a Beta distribution with parameters $\alpha>0$ and $\beta>0$ iff the density function of X is

$$f_X(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0\le x\le 1 \\ 0, & \text{ow} \end{cases}$$

Ex. A gasoline wholesale distributor has bulk storage tanks that hold fixed supplies and are filled every Monday. Of interest to the wholesaler is the proportion of this supply that is sold during the week. Over many weeks of observation, the distributor found out that this proportion could be modeled by a Beta distribution with $\alpha=4$ and $\beta=2$. Find the probability that the wholesaler will sell at least 90% of her stock in a given week.

Solution:

$$X \sim \text{Beta}(4,2)$$

$$P(X\ge 0.9) = \int_{0.9}^{1} \frac{\Gamma(6)}{\Gamma(4)\Gamma(2)} x^{3}(1-x)\, dx$$

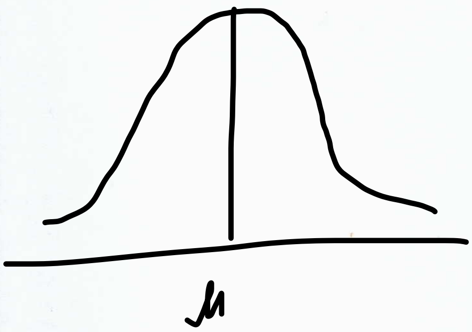
$$= \frac{5!}{3!\cdot 1!}\left[\frac{x^4}{4}\Big|_{0.9}^{1} - \frac{x^5}{5}\Big|_{0.9}^{1}\right]$$

$$= 20\cdot 0.004 = 0.08$$

The Normal Distribution

A r.v. X is said to have a **Normal** distribution iff for $\sigma > 0$ and $-\infty < \mu < \infty$,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$



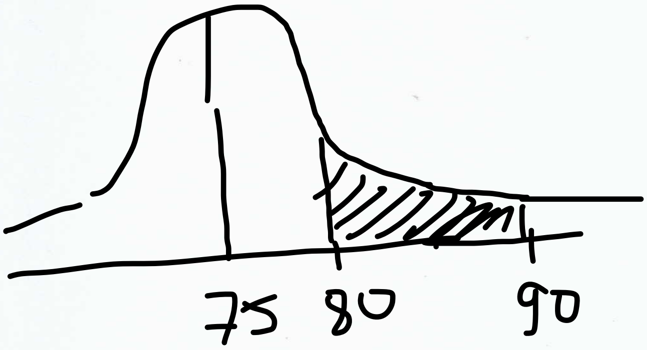
symmetric

$X \sim N(\mu, \sigma)$

$Z = \dfrac{X-\mu}{\sigma} \sim N(0,1)$ standard normal

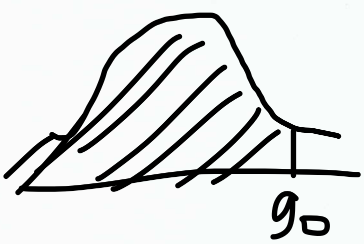
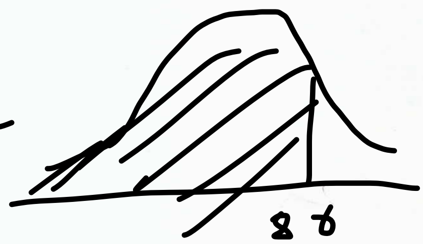
Ex. The achievement scores for a college entrance examination are $N(75, 10)$. What fraction of the scores lies between 80 and 90?

Sol'n:



$P(80 \le X \le 90) = ?$

$= P(X \le 90) - P(X \le 80)$

$=$  $-$ 

$$= P\left(\frac{X-75}{10} \le \frac{90-75}{10}\right) - P\left(\frac{X-75}{10} \le \frac{80-75}{10}\right)$$

$$P(Z \le 1.5) - P(Z \le 0.5)$$

$$= 0.2417$$

Poisson Processes.

- Model for times of occurrences of rare phenomena where λ is an average number of arrivals per time period
  X is a number of arrivals in a time period

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \ldots$$

In t time periods, average # of arrivals is λt

Question: How long do I have to wait until the 1^{st} arrival?

Y = waiting time for 1^{st} arrival

$$P(Y > t) = P(X = 0 \text{ on } (0,t]) = e^{-\lambda t}$$

$$F_Y(t) = P(Y \le t) = 1 - e^{-\lambda t}$$

Claim: The waiting time for the first occurrence of an event when the number of events follows a Poisson distribution is exponentially distributed.

Expectation.

$$E(X),\ E_X,\ \mu,\ \mu_X$$

Discrete case:

For a discrete r.v. X with pmf $p_X(x)$, the expectation is given by $E(X) = \sum_x x\, p(x)$ whenever the sum converges absolutely (i.e. $\sum_x |x|\, p(x) < \infty$)

Ex. (1) Roll a die. Let X = outcome on 1 roll.

$$E(X) = 1\cdot\frac{1}{6} + 2\frac{1}{6} + \ldots + 6\frac{1}{6} = \frac{1}{6}(1 + \ldots + 6) = \underline{\underline{3.5}}$$

$X : 1, 2, \ldots 6$ with $p = \frac{1}{6}$

(2) Bernoulli trials $P(X=1) = p$, $P(X=0) = 1-p$.

$$E(X) = 1\cdot p + 0(1-p) = p$$

(3) $X \sim Bin(n,p)$.

$$E(X) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} = \sum_{x=1}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$$

$$= pn \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} (1-p)^{n-x} = (z = x-1)$$

$$= pn \sum_{z=0}^{n-1} \underbrace{\frac{(n-1)!}{z!(n-1-z)!} p^z (1-p)^{n-1-z}}_{\text{pmf of } Bin(n-1,p)} = pn \cdot 1 = np$$

(4) $X \sim Geom(p)$.

$$E(X) = \sum_{x=1}^{\infty} x\, p(1-p)^{x-1} = p \sum_{x=1}^{\infty} x\, q^{x-1} = p \frac{d}{dq} \sum_{x=1}^{\infty} q^x$$

$$= p \frac{d}{dq}\left(\frac{q}{1-q}\right) = p \frac{1}{(1-q)^2} = \frac{1}{p}$$

(5) $X \sim \text{Poisson}(\lambda)$.

$$E(X) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!} = \lambda \sum_{x=1}^{\infty} \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!} \underset{z=x-1}{=}$$

$$= \lambda \underbrace{\sum_{z=0}^{\infty} \frac{e^{-\lambda}\lambda^{z}}{z!}}_{=1} = \lambda$$

Continuous case:

For a continuous r.v. X with density $f_X(x)$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx$$

whenever the integral converges absolutely,

i.e. $\int_{-\infty}^{\infty} |x| f_X(x)\,dx < \infty$

Ex. (1) $X \sim \text{Unif}(a, b)$ $\quad f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a, b] \\ 0, & ow \end{cases}$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b \frac{1}{b-a} x\,dx = \frac{1}{b-a} \frac{x^2}{2}\Big|_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

(2) $X \sim \text{Exp}(\lambda)$

$$E(X) = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv} = \begin{bmatrix} du = dx \\ v = -e^{-\lambda x} \end{bmatrix}$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & ow \end{cases}$$

$$= \underbrace{-xe^{-\lambda x}\Big|_0^{\infty}}_{=0} + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty} = \frac{1}{\lambda}$$

4.11

(3) X is a r.v. with $f_X(x) = \begin{cases} \frac{1}{x^2}, & x>1 \\ 0, & ow \end{cases}$

(a) Check if this is a valid density

(b) Find $E(X)$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_1^{\infty} \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^{\infty} = 1$$

$$E(X) = \int_1^{\infty} x \cdot \frac{1}{x^2}\,dx = \int_1^{\infty} \frac{1}{x}\,dx = \ln x\Big|_1^{\infty}$$

(4) $X \sim \text{Gamma}(\alpha, \lambda)$ $\quad f_X(x) = \frac{x^{\alpha-1} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}, \; x>0$

$$E(X) = \int_0^{\infty} \frac{x^{\alpha} e^{-\lambda x} \lambda^{\alpha}}{\Gamma(\alpha)}\,dx = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha} e^{-\lambda x}\,dx$$

$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \int_0^{\infty} \underbrace{\frac{x^{\alpha} e^{-\lambda x} \lambda^{\alpha+1}}{\Gamma(\alpha+1)}}_{=1} dx \quad \sim \text{Gamma}(\alpha+1, \lambda)$$

$$= \frac{\cancel{\lambda^{\alpha}}}{\cancel{\Gamma(\alpha)}} \cdot \frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\lambda^{\alpha+1}}}$$

$$= \frac{\alpha}{\lambda} \qquad \frac{1}{\lambda}$$

$X \sim \text{Gamma}(\alpha, \beta)$

$E(X) = \alpha\beta$

(5) $X \sim \text{Beta}(\alpha, \beta)$

$E(X) =$

$$f_X(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}$$

$0 \le x \le 1$

$$E(X) = \int_0^1 x \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 x^{\alpha}(1-x)^{\beta-1}\,dx$$

$$= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \cdot \frac{\Gamma(\alpha+1)\Gamma(\beta)}{\Gamma(\alpha+1+\beta)} \underbrace{\int_0^1 \frac{\Gamma(\alpha+1+\beta)}{\Gamma(\alpha+1)\Gamma(\beta)} x^{\alpha}(1-x)^{\beta-1}\,dx}_{\mathrm{Beta}(\alpha+1,\beta) \; = 1}$$

$$= \frac{\cancel{\Gamma(\alpha+\beta)}}{\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}} \cdot \frac{\alpha\cancel{\Gamma(\alpha)}\cancel{\Gamma(\beta)}}{(\alpha+\beta)\cancel{\Gamma(\alpha+\beta)}}$$

$$= \frac{\alpha}{\alpha+\beta}$$