Feb 5th

Last time

DEF: $T: V \to W$ is invertible if there is $S: W \to V$ such that $ST = I_V$, $TS = I_W$

Exs: ① $I: V \to V$ is invertible and $I^{-1} = I$

② $0: V \to V$ not invertible

③ $R_\theta: \mathbb{R}^2 \to \mathbb{R}^2$ is invertible $R_\theta^{-1} = R_{-\theta}$

④ $T: V \to F^n$ (V vector space over F)

$T(v) = [v]_\alpha$ is invertible

and $T^{-1}\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_1v_1 + \cdots + a_nv_n$

⑤ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}: \mathbb{R}^2 \to \mathbb{R}^2$ when is A invertible? when $\det A \neq 0$.

DEF: 1. $T: V \to W$ is an isomorphism of T is invertible

2. V, W are isomorphic if there exists some isomorphism $T: V \to W$ intuition "isomorphism" = "the same"

Ex: 1. Is $P_{n-1}(\mathbb{R})$ isomorphic to $\mathbb{R}^n$?

$T: \mathbb{R}^n \to P_{n-1}(\mathbb{R})$

$T\left(\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}\right) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1}$

$S(a_0 + a_1x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$

$ST = I_{\mathbb{R}^n}$, $TS = I_{P_{n-1}}$

Ex: $SM_3(\mathbb{R})$ vector space of 3×3 symm matrices $A^T = A$

Q: Is SM_3 isom to $\mathbb{R}^n$ for some n?

What's $\dim SM_3$?

$\dim SM_3 = 6$

Claim: SM_3 isomorphic to $\mathbb{R}^6$

Pf: $T: \mathbb{R}^6 \to SM_3$

$T\left(\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix}\right) = A$

TM: V, W f d vector spaces over F. Then V iso to W iff $\dim V = \dim W$.

Proof: $\Rightarrow$ suppose V iso to W

Then Let $T: V \to W$ be an invertible linear transformation

Therefore T is a bijection

so T is onto $\Rightarrow \dim V \geq \dim W$

T is one to one $\Rightarrow \dim W \geq \dim V$

so $\dim V = \dim W$

(Although we could also show that T invertible $\Rightarrow T(\text{basis of } V) = (\text{basis of } W)$)

$\Leftarrow \dim W = \dim V = n$

let $\{v_1, \cdots, v_n\}$ be a basis of V

$\{w_1, \cdots, w_n\}$ be a basis of W

Define $T: V \to W$ by $T(v_i) = w_i$

linear trans $V \to W$ it invertible iff its basis of V match basis of W

$T: V \to W$ invertible $\quad \alpha$ basis of $V \quad \beta$ basis of W

$$[T^{-1}]_\beta^\alpha = ([T]_\alpha^\beta)^{-1}$$

Proof: need to check that $[T]_\alpha^\beta [T^{-1}]_\beta^\alpha = I_n$

$$[T]_\alpha^\beta [T^{-1}]_\beta^\alpha = [TT^{-1}]_\beta^\beta = [I_W]_\beta^\beta$$

$[I_W]_\beta^\beta$

$I_W(w_j) = w_j$

$\Rightarrow [I_W]_\beta^\beta = I_n$

Change of Basis

V f.d. vector space of dim n

α, β two basis

Want: A $n \times n$ matrix s.t. for $v \in V$

$A[v]_\alpha = [v]_\beta$

A is called a change of basis matrix what's A?

Recall: $T: V \to V$

$[T]_\alpha^\beta([v]_\alpha) = [T(v)]_\beta$

take $T = I$ then $[I]_\alpha^\beta([v]_\alpha) = [v]_\beta$

Ex: $V = \mathbb{R}^2$, $\alpha = \left\{\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$

$\beta = \left\{\begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\end{bmatrix}\right\}$

$[I]_\alpha^\beta \quad \begin{bmatrix}1\\-1\end{bmatrix} = 2\begin{bmatrix}2\\1\end{bmatrix} + 3\begin{bmatrix}-1\\-1\end{bmatrix}$

$\begin{bmatrix}1\\1\end{bmatrix} = -\begin{bmatrix}-1\\-1\end{bmatrix} \Rightarrow [I]_\alpha^\beta = \begin{bmatrix}2 & 0\\3 & -1\end{bmatrix}$

$v = \begin{bmatrix}5\\3\end{bmatrix} \in \mathbb{R}^2 \quad [v]_\alpha = \begin{bmatrix}1\\4\end{bmatrix}$

$[V]_\beta = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Ex: $V = \mathbb{R}^n$, $\alpha = \{v_1, \cdots, v_n\}$

$\beta = \{e_1, \cdots, e_n\}$ st. basis

$V = R^2$ $\alpha = \left\{\begin{bmatrix} 6 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\}$

$[V]_\alpha = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, what's v?

$v = 2\begin{bmatrix} 6 \\ 2 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -2 \end{bmatrix}$

Prop: $T: V \to V$, α, β basis

$A[T]_\alpha^\alpha A^{-1} = [T]_\beta^\beta$

where $A = [I]_\alpha^\beta$

proof: $A[T]_\alpha^\alpha A^{-1} = [I]_\alpha^\beta [T]_\alpha^\alpha ([T]_\alpha^\beta)^{-1} = [I]_\alpha^\beta [T]_\alpha^\alpha [I]_\beta^\alpha = [T]_\beta^\beta$

Ex: $T: R^2 \to R^2$ has matrix
$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ relative to basis $\alpha = \left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}\right\}$

What's matrix of T in standard basis β?

$[I]_\alpha^\beta = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ is from $\alpha \to$ standard basis

$[I]_\beta^\alpha = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$\Rightarrow [T]_\beta^\beta = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} =$

Ex: when $V = R^n$, α β basis of V
want: $[I]_\alpha^\beta$

Method: $\gamma = \{e_1, \cdots, e_n\}$ standard basis

$[I]_\alpha^\beta = [I]_\gamma^\beta [I]_\alpha^\gamma$

DEF: A, B $n \times n$ matrices
A is similar to B if there is an invertible X s.t.
$XAX^{-1} = B$

Ex: $[T]_\alpha^\alpha$ similar to $[T]_\beta^\beta$

Prop: 1. Is A similar to itself?
$IAI^{-1} = A$ so yes

2. If A sim to B, is B sim to A?
A sim to B then $XAX^{-1} = B \Rightarrow X^{-1}BX = A \Rightarrow B$ sim to A

3. If A sim to B and B sim to C is A sim to C?
$XAX^{-1} = B \quad YBY^{-1} = C$

$$YXAX^{-1}Y^{-1} = YBY^{-1} = C$$

DEF: $T: V \to V$
If $T(v) = \lambda v$ and $\vec{v} \neq \vec{0}$ then we say λ is a eigenvalue and v is a eigenvector

EXs: 1. $V = C(\mathbb{R})$ $T = \frac{d}{dx}$, $\lambda \in \mathbb{R}$

want eigenvector $f \in V$ of eigenvalue λ.
$T(f) = \lambda f$
$f' = \lambda f \quad f = e^{\lambda x} + c$

2. $V = (\mathbb{Z}_3)^2$

$T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow T(v) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and so v is an eigenvector with eigenvalue $0 \in \mathbb{Z}_3$

3. Does $R_{\pi/2}$ have an e vector?
No, if we are over $\mathbb{R}$, but on $\mathbb{C}$, yes!