Lecture 10

* Independence of Multivariate r.v.'s

Thm

Sps $\vec{X} = (\vec{X_1}, \vec{X_2}, \cdots, \vec{X_r})^T$

Then $\vec{X_1}, \vec{X_2}, \cdots, \vec{X_r}$ are independent iff $f(\vec{x_1}, \cdots, \vec{x_r}) = f(\vec{x_1}) \cdots f(\vec{x_r})$ (*)

where $f(\vec{x_i})$ is the density function of $\vec{X_i}$.

Proof: "$\Leftarrow$" if (*) holds then

$$E[H_1(\vec{X_1}) H_2(\vec{X_2}) \cdots H_r(\vec{X_r})]$$

$$= \iint \cdots \int H_1(\vec{x_1}) H_2(\vec{x_2}) \cdots H_r(\vec{x_r}) f(\vec{x_1}, \vec{x_2}, \cdots, \vec{x_r}) d\vec{x_1} d\vec{x_2} \cdots d\vec{x_r}$$

$$= \int \cdots \int H_1(\vec{x_1}) H_2(\vec{x_2}) \cdots H_r(\vec{x_r}) f(\vec{x_1}) f(\vec{x_2}) \cdots f(\vec{x_r}) d\vec{x_1} \cdots d\vec{x_r}$$

$$= \int H_1(\vec{x_1}) f(\vec{x_1}) d\vec{x_1} \cdots \int H_r(\vec{x_r}) f(\vec{x_r}) d\vec{x_r}$$

$$= E[H_1(\vec{X_1})] E[H_2(\vec{X_2})] \cdots E[H_r(\vec{X_r})] \text{ for any } H_i$$

"$\Rightarrow$" If $\vec{X_1}, \vec{X_2}, \cdots, \vec{X_r}$ are independent.

For $\vec{a} = (a_1, \cdots, a_m)^T$
$\vec{b} = (b_1, \cdots, b_m)^T$
define $\vec{a} \le \vec{b}$ by
$a_i \le b_i$ for all $i \in \{1, 2, \cdots, m\}$

$$\Rightarrow P[\vec{X_1} \le \vec{x_1}, \vec{X_2} \le \vec{x_2}, \cdots, \vec{X_r} \le \vec{x_r}]$$

$$= P[\vec{X_1} \le \vec{x_1}] P[\vec{X_2} \le \vec{x_2}] \cdots P[\vec{X_r} \le \vec{x_r}]$$

Take derivative of LHS with respect to $x_1, x_2, \cdots, x_n$ n is the total length of $\vec{X}$

$$\text{LHS} = f(\vec{x_1}, \vec{x_2}, \cdots, \vec{x_r})$$

$$\text{RHS} = f(\vec{x_1}) f(\vec{x_2}) \cdots f(\vec{x_r})$$

$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}$ in this case $n=6$ but $r=3$!

$\vec{x_1} = (1,2)^T = (x_1, x_2)^T$

$\vec{x_2} = (3,4,5)^T = (x_3, x_4, x_5)^T$

...

§ 8.2 Functions of Random Vectors

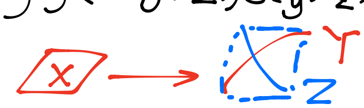
Thm: Sps that vectors $\vec{X}$ & $\vec{Y}$ have dimensions m & r respectively ($r \le m$) and that the transformation $\vec{Y} = a(\vec{X})$ can be completed by a transformation to a vector $\vec{Z} = b(\vec{X})$ of dimension $m-r$.

Such that the combined transformation $\vec{X} \to \begin{pmatrix} \vec{Y} \\ \vec{Z} \end{pmatrix}$ is one to one and prossessed an inverse and a Jacobian

$\hookrightarrow$ tries to see how distorted the transformation is? abs. value of det.

$J = \dfrac{\partial \vec{X}}{\partial(\vec{Y}, \vec{Z})}$ Then $\vec{Y}$ has density $g(\vec{Y}) = \iint f(\vec{X}(\vec{y}, \vec{z})) J(\vec{y}, \vec{z}) d\vec{z}$

note: dim 2 $\to$ dim 1 is not 1-1. think about this.



$\boxed{\vec{Z} \text{ completes the space}}$

Idea (proof): show that

$E[H[\vec{Y}]] = \iint H(\vec{y})\, g(\vec{y})\, d\vec{y}$ for any H

$E[H[\vec{Y}]] = E[H(a(\vec{X}))] = \int \cdots \int H(a(\vec{x})) f(\vec{x})\, d\vec{x}$ ✓

do change of variable.

$\vec{X} \to \begin{pmatrix} \vec{y} \\ \vec{z} \end{pmatrix}$

$$✓ = \int \cdots \int H[\vec{y}] \underbrace{f(\vec{x}(\vec{y}, \vec{z}))\, J(\vec{y}, \vec{z})\, d\vec{z}}_{= g(\vec{y})}\, d\vec{y}$$

$$= \int \cdots \int H[\vec{y}]\, g(\vec{y})\, d\vec{y}$$

Ex: Sps X, Y, iid $N(0,1)$, $Z = X - Y$. Find density of Z.

Let $W = X + Y$

Then $\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} Z \\ W \end{pmatrix}$ is one to one invertible

$$\Rightarrow \begin{pmatrix} X = \frac{Z+W}{2} \\ Y = \frac{W-Z}{2} \end{pmatrix}$$

① $$J = \left| \det \begin{pmatrix} \frac{\partial X}{\partial Z} & \frac{\partial X}{\partial W} \\ \frac{\partial Y}{\partial Z} & \frac{\partial Y}{\partial W} \end{pmatrix} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \right| = \frac{1}{2}$$

② Joint density of X & Y

$$f(x,y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} = \frac{1}{2\pi} e^{-\frac{1}{2}(x^2+y^2)}$$

$$f(z,w) = \frac{1}{2\pi} e^{-\frac{1}{2}\left(\left(\frac{z+w}{2}\right)^2 + \left(\frac{w-z}{2}\right)^2\right)} \underbrace{\frac{1}{2}}_{J}$$

$$f(z,w) = \frac{1}{4\pi} e^{-\frac{1}{4}(z^2+w^2)}$$

③ $$f(z) = \int_{-\infty}^{\infty} f(z,w)\, dw = \int_{-\infty}^{\infty} \frac{1}{4\pi} e^{-\frac{1}{4}(z^2+w^2)}\, dw$$

$$= \frac{1}{4\pi} e^{-\frac{1}{4}z^2} \int_{-\infty}^{\infty} e^{-\frac{1}{4}w^2}\, dw = \frac{1}{4\pi} e^{-\frac{1}{4}z^2}\, 2\sqrt{\pi}$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{1}{4}z^2} \quad -\infty < z < \infty$$

$\boxed{Z \sim N(0,2)}$

Normal density, let $x = \frac{1}{\sqrt{2}} w$

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\, dx = \sqrt{2\pi} = \int_{-\infty}^{\infty} e^{-\frac{1}{4}w^2} \frac{1}{\sqrt{2}}\, dw \Rightarrow \int_{-\infty}^{\infty} e^{-\frac{1}{4}w^2}\, dw = 2\sqrt{\pi}$$

Ex2: Let X, Y iid $U[0,1]$

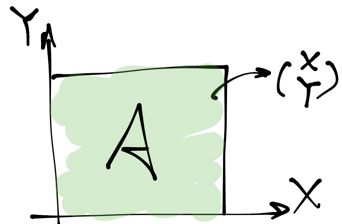
Let $Z = X+Y$

Find density of Z

① $W = X - Y$ $\qquad J = \frac{1}{2}$

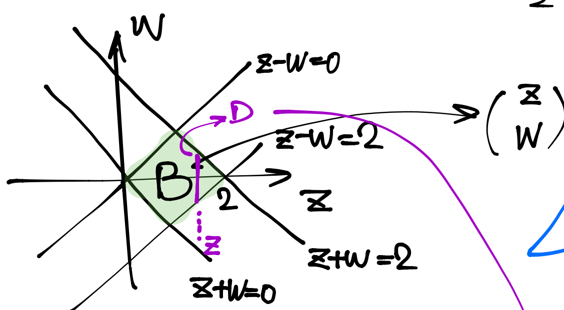
$\begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} Z \\ W \end{pmatrix}$

$X = \frac{Z+W}{2}, \quad Y = \frac{Z-W}{2}$



Region of $\begin{pmatrix} Z \\ W \end{pmatrix}$ is $0 \le \frac{Z+W}{2} \le 1$

$0 \le \frac{Z-W}{2} \le 1$



②

Density of $\begin{pmatrix} X \\ Y \end{pmatrix}$ is $I\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in A\right\}$

Thm Density of $\begin{pmatrix} Z \\ W \end{pmatrix} = \frac{1}{2} I\left\{\begin{pmatrix} z \\ w \end{pmatrix} \in B\right\}$

density of z. $g(z) = \int_0^\infty \frac{1}{2} I\left\{\begin{pmatrix} z \\ w \end{pmatrix} \in B\right\} dw$

③ If $1 \le z \le 2$, then length of $D = 2(2-z)$

$\Rightarrow g(z) = \frac{1}{2} \cdot 2(2-z) = 2-z$

If $0 \le z \le 1$, then length of $D = 2z$

$\Rightarrow g(z) = \frac{1}{2} \cdot 2z = z$

In summary $g(z) = \begin{cases} z & \text{if } 0 \le z \le 1 \\ 2-z & \text{if } 1 \le z \le 2 \\ 0 & \text{o.w.} \end{cases}$

Conditional distribution of Continuous random variable

Def: Sps $\vec{X}, \vec{Y}$ has joint density $f(\vec{x},\vec{y})$

Then the conditional density of $\vec{X}$ given $\vec{Y}$ is defined as

$$f(\vec{x}|\vec{y}) = \begin{cases} \dfrac{f(\vec{x},\vec{y})}{f_Y(\vec{y})} & \text{if } f(\vec{y})>0 \\ 0 & \text{o.w.} \end{cases}$$

$$E[\vec{X}|\vec{Y}=\vec{y}] = \int\cdots\int_{-\infty}^{\infty} \vec{x} f(\vec{x}|\vec{y}) d\vec{x}$$

Write $h(\vec{y}) = E[\vec{X}|\vec{Y}=\vec{y}]$

Proposition: $h(\vec{Y})$ is the conditional expectation $E[\vec{X}|\vec{Y}]$

Proof: Just need to check

$E([\vec{X}-h(\vec{Y})]H(\vec{Y}))=0$ for any H

might appear in final

Ex: Suppose (X,Y) has the joint density $f(x,y)=\begin{cases}(x+y) & \text{if } 0\le x\le 1, 0\le y\le 1 \\ 0 & \text{o.w.}\end{cases}$

Find $E[X|Y=0.5]$

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{(x+y)I\left(\begin{smallmatrix}0\le x\le 1\\ 0\le y\le 1\end{smallmatrix}\right)}{y+\frac{1}{2}}$$

$$f_Y(y) = \int_0^1 f(x,y)dx = \int_0^1 (x+y)dx = y+\frac{1}{2}$$

$$E[X|y=\tfrac{1}{2}] = \int_0^1 x f(x|y=\tfrac{1}{2})dx = \int_0^1 \frac{x(x+\frac{1}{2})}{\frac{1}{2}+\frac{1}{2}}dx = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$$

CHAPTER 7

Def: Convergence a.s.

We say a sequence of random variables X_n's convergence almost surely to X if

$P[\omega : X_n(\omega) \longrightarrow X(\omega)] = 1$ Notation: $X_n \xrightarrow{a.s.} X$

Def: $X_n(\omega)$ is called converging in probability to $X(\omega)$ if

$P[|X_n(\omega) - X(\omega)| > \varepsilon] \longrightarrow 0$ for any $\varepsilon > 0$.

$X_n \xrightarrow{P} X$

To find which one is stronger, we're going to look at some sufficient/necessary conditions for those 2 to hold.

Prop: $X_n(\omega) \xrightarrow{a.s.} X(\omega)$ iff for all $\varepsilon > 0$, we have

$$\lim_{n\to\infty} P[\sup_{m\geq n} |X_m(\omega) - X(\omega)| > \varepsilon] = 0$$

Converging a.s. is stronger. WHY?

↓

The supreme $> \varepsilon$, so each single part $> \varepsilon$.

Corollary: If $X_n(\omega) \xrightarrow{a.s.} X(\omega)$, then $X_n(\omega) \xrightarrow{P} X(\omega)$

Sketch of proof (proof not required)

"$\Longrightarrow$" Consider the event

$$\bigcap_{n=1}^{\infty} \underbrace{\{\sup_{m\geq n} |X_n(\omega) \to X(\omega)| > \varepsilon\}}_{\text{call it } \Lambda_n}$$

Λ_n is a decreasing event ↓

Claim: $P[\bigcap_{n=1}^{\infty} \Lambda_n] = 0$

$\Rightarrow \lim_{n\to\infty} P(\Lambda_n) = 0$ by monotonic convergence thm

"$\Longleftarrow$" Consider $A = \bigcup_{i=1}^{\infty} \bigcap_{n=1}^{\infty} \{\omega : \sup_{m\geq n} |X_m(\omega) - X(\omega)| > \frac{1}{2^i}\}$

Claim: $P(A) = 0$

Second claim: for any $\omega \notin A$, $X_n(\omega) \to X(\omega)$

$\Rightarrow \lim_{n\to\infty} P(\Lambda_n) = 0$

Counter example that converging in $P \not\to$ convergence a.s.

Let $U_1, U_2, \cdots, U_n, \cdots$ be iid $U[0,1]$

Define $X_n = \begin{cases} 0 & \text{if } U_n \in [\frac{1}{n}, 1] \\ n & \text{o.w.} \end{cases}$ $\quad n = 1, 2, 3, \cdots$

For $\forall \varepsilon > 0$. $P[|X_n(\omega) - 0| > \varepsilon] = \frac{1}{n}$ if n is large enough

$$\lim_{n\to\infty} P[\text{---}] = 0 \quad \Rightarrow X_n \xrightarrow{P} 0$$

Then show $X_n \overset{a.s.}{\not\to} 0$ $(\Rightarrow \Leftarrow)$

$$P[\sup_{m\geq n} |X_m - 0| \leq \varepsilon] = P[\sup_{m\geq n} X_m \leq \varepsilon] = P[X_n \leq \varepsilon, X_{n+1} \leq \varepsilon, \cdots, X_{n+m} \leq \varepsilon]$$

$$= P(X_n \leq \varepsilon) P(X_{n+1} \leq \varepsilon) \cdots$$

$$= \frac{n-1}{n} \cdot \frac{n}{n+1} \cdot \frac{n+1}{n+2} \cdots$$

$$= \frac{n-1}{\infty} \to 0$$

$$\Rightarrow P[\sup_{m\geq n} |X_n - X| > \varepsilon] = 1 \Rightarrow X_n(\omega) \overset{a.s.}{\not\to} 0$$

∎