Lecture 21

FRACTIONAL LINEAR TRANSFORMATION

A function $T(z)=\dfrac{az+b}{cz+d}$ with $ad-bc\neq 0$ is called a fractional linear transformation (F.L.T.)

Recall: The condition that $ad-bc\neq 0$ implies $T'(z)=\dfrac{ad-bc}{(cz+d)^2}\neq 0$

Note: T is one-to-one (injective)

Reminder: f is injective if: $z_1\neq z_2 \Rightarrow f(z_1)\neq f(z_2)$ or $f(z_1)=f(z_2)\Rightarrow z_1=z_2$

To see T is one-to-one.

Suppose $T(z_1)=T(z_2)$

$$\frac{az_1+b}{cz_1+d}=\frac{az_2+b}{cz_2+d}$$

$$(az_1+b)(cz_2+d)=(az_2+b)(cz_1+d)$$
$$acz_1z_2+adz_1+bcz_2+bd=acz_1z_2+adz_2+bcz_1+bd$$
$$(ad-bc)z_1=(ad-bc)z_2$$
since $ad-bc\neq 0 \Rightarrow z_1=z_2$

Since T is one-to-one, it has an inverse (i.e. a function S, satisfying $T\circ S(z)=z=S\circ T(z)$)

Note: ① If T & S are F.L.T's then so is $S\circ T$.
Hence if T is an F.L.T then T^{-1} is an FLT.

② Each $T(z)=\dfrac{az+b}{cz+d}$ can be associated with a 2×2 matrix

$$M_T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$ ($\det M_T=ad-bc\neq 0 \Rightarrow M_T$ has an inverse)

③ The "identity function" $T(z)=z$ is an F.L.T. $T(z)=\dfrac{1\cdot z+0}{0\cdot z+1}$

Q: How do $T, S, M_T, M_S, S\circ T, M_{S\circ T}$ relate?

A: $M_{S\circ T}=M_S\cdot M_T$

$T=\dfrac{az+b}{cz+d}$, $S=\dfrac{mz+n}{pz+q}$

$$S\circ T(z)=S\left(\frac{az+b}{cz+d}\right)=\frac{m\frac{az+b}{cz+d}+n}{p\frac{az+b}{cz+d}+q}$$
$$=\frac{m(az+b)+n(cz+d)}{p(az+b)+q(cz+d)}$$
$$=\frac{(ma+nc)z+(mb+nd)}{(pa+qc)z+(pb+qd)}$$

$M_S=\begin{pmatrix} m & n\\ p & q\end{pmatrix}$ $M_T=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ $M_S\cdot M_T=\begin{pmatrix} m & n\\ p & q\end{pmatrix}\begin{pmatrix} a & b\\ c & d\end{pmatrix}=\begin{pmatrix} am+cn & bm+dn\\ ap+cq & bp+dq\end{pmatrix}$

$=M_{S\circ T}$

$\Rightarrow$ Given $T(z)=\frac{az+b}{cz+d}$, to find T^{-1}, we can use matrices

Ex: $T(z)=\frac{2z+3}{5z-1}$, Find T^{-1}

$T \to M_T = \begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}$

Find $(M_T)^{-1}$

$$\begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}^{-1} = \frac{1}{-17}\begin{pmatrix} -1 & -3 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{17} & \frac{3}{17} \\ \frac{5}{17} & \frac{-2}{17} \end{pmatrix}$$

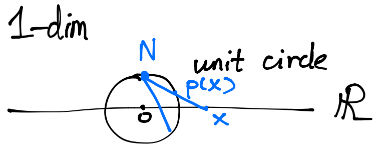
$$T^{-1}(z) = \frac{\frac{1}{17}z+\frac{3}{17}}{\frac{5}{17}z-\frac{2}{17}} = \frac{\frac{1}{17}(z+3)}{\frac{1}{17}(5z-2)} = \frac{z+3}{5z-2} = \frac{-z-3}{-5z+2}$$

$$M_T=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (M_T)^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$T^{-1} = \frac{dz-b}{-cz+a}$$

THE POINT AT INFINITY

1-dim



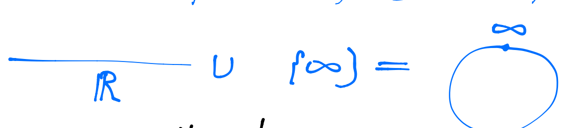
Every pt $x \in \mathbb{R}$ is uniquely identified with a point $p(x)$ on unit circle. Only pt on unit circle w/o an "x" is N, the north pole.

Unit circle $= \mathbb{R} \cup N$

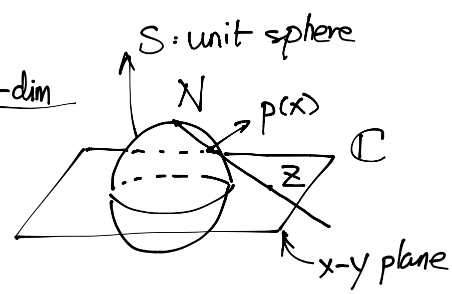
As $x \to \infty$ $\quad p(x) \to N$

$x \to -\infty$ $\quad p(x) \to N$

So we call N the point at infinity & usually denote it by ∞



2-dim



Every complex # z is uniquely identified with a point $p(z)$ on the unit sphere. The only point on unit sphere w/o a "z" is the north pole N.

As $|z| \to \infty$, $p(z) \to N$

Call N point at infinity & denote it by ∞.

So we get $S = \mathbb{C} \cup \{\infty\}$

We can now extend the domain & target space of a F.L.T.

$T: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$

i.e. $T: S \to S$ (S=sphere)

How to extend?

We use $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$

What about $\frac{\infty}{\infty}$? Use limits.

e.g. $T(z) = \frac{2z+3}{5z-6}$

$T(\infty) = \text{"}\frac{\infty}{\infty}\text{"}$, $\lim_{z\to\infty} \frac{2z+3}{5z-6} = \frac{2}{5} = T(\infty)$

wrong!

So we get $T: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is given by: $T(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } z \neq \infty, z \neq \frac{-d}{c} \\ \infty, & \text{if } z = \frac{-d}{c} \\ \frac{a}{c} & \text{if } z = \infty \end{cases}$

FACT: A F.L.T. is uniquely determined by prescribing the value of 3 distinct points.

I.e. the conditions $z_1 \to w_1$

$z_2 \to w_2$

$z_3 \to w_3$

Q: Given z_1, z_2, z_3 & w_1, w_2, w_3. how do we find T?

1st step: The FLT

$$T(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1}$$

has the property that $z_1 \to 0$

$z_2 \to 1$

$z_3 \to \infty$

$$z_1 \xrightarrow{T} 0 \underset{S^{-1}}{\overset{S}{\rightleftarrows}} w_1$$

$$z_2 \xrightarrow{T} 1 \underset{S^{-1}}{\overset{S}{\rightleftarrows}} w_2$$

$$z_3 \xrightarrow{T} \infty \underset{S^{-1}}{\overset{S}{\rightleftarrows}} w_3$$

$S^{-1} \circ T$ has the property that $z_1 \to w_1$, $z_2 \to w_2$, $z_3 \to w_3$.

Ex: Find a FLT that takes $0, 1, i$ to $1, i, -i$.

$z_1 = 0 \to 0 \leftarrow 1 = w_1$

$z_2 = 1 \to 1 \leftarrow i = w_2$

$z_3 = i \to \infty \leftarrow -i = w_3$

$\quad T \qquad S$

$T(z) = \frac{z-0}{z-i} \cdot \frac{1-i}{1-0} = \frac{(1-i)z}{z-i}$

$M_T = \begin{pmatrix} 1-i & 0 \\ 1 & -i \end{pmatrix}$

$S = \frac{z-1}{z+i} \cdot \frac{2i}{i-1} = \frac{2iz-2i}{(-1+i)z-1-i}$

$M_S = \begin{pmatrix} 2i & -2i \\ -1+1 & -1-i \end{pmatrix}$

We need $M_S^{-1} = \cancel{\frac{1}{det}} \begin{pmatrix} -1-i & 2i \\ 1-i & 2i \end{pmatrix}$

$M_S^{-1} \cdot M_T = \begin{pmatrix} -1-i & 2i \\ 1-i & 2i \end{pmatrix} \begin{pmatrix} 1-i & 0 \\ 1 & -i \end{pmatrix} = \begin{pmatrix} -1-i+i-1+2i & 2 \\ 1-2i-1+2i & 2 \end{pmatrix} = \begin{pmatrix} -2+2i & 2 \\ 0 & 2 \end{pmatrix}$

$\longrightarrow F = \frac{(-2+2i)z+2}{2} = (-1+i)z+1$

LINES & CIRCLES

FACT: If T is an FLT, then T maps a line to either a line or a circle. and T maps a circle to either a circle or a line.

Why? A F.L.T. is the composition of 3 types of maps ($T = M \circ I \circ L$)

① A linear map: $z \xrightarrow{L} cz+d$ (line → line, circle → circle)

② Inversion: $w \xrightarrow{I} \frac{1}{w}$ (circle through 0 → line
circle not through 0 rad R
→ circle not through 0 rad $\frac{1}{R}$
line through 0 → line
not through 0
line not through 0 → circle)

③ A linear map: $u \xrightarrow{M} \frac{1}{c}((ad-bc)u+a)$
(circle → circle)
(line → line)

Projective geometry