Lecture 4

Prop: Let X be a top space

i). $A \subseteq \overline{A}$

ii). $\overline{\emptyset} = \emptyset$, $\overline{\mathbb{R}} = \mathbb{R}$

iii) If $X \setminus A$ is open, then $\overline{A} = A$

iv). $\overline{\overline{A}} = \overline{A}$

v) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

proof of iii)

Assume $X \setminus A$ is open. Want to show $\overline{A} = A$, so need $A \subseteq \overline{A}$ & $\overline{A} \subseteq A$

by i) $A \subseteq \overline{A}$, so we only need to show $\overline{A} \subseteq A$.

It is enough to show that $\overline{A} \setminus A = \emptyset$

Let $a \in \overline{A} \setminus A$, so $a \notin A$. i.e. $a \in X \setminus A$ which is open, and disjoint from A, so $a \notin \overline{A}$. $\Rightarrow\!\Leftarrow$ ■

Exercise: Prove this without using contradiction (Hint: you can do this by changing only 1 letter.)

Things that aren't true:

1. $\overline{A \cap B} = \overline{A} \cap \overline{B}$

e.g. 1 $A = (0,1)$, $B = (1,2)$

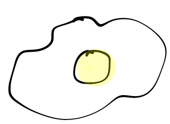
e.g. 2 $A = \mathbb{Q}$, $B = \mathbb{R} \setminus \mathbb{Q}$

Intuitively, $p \in \overline{A}$, p is very close to A

Prop: Let $\mathcal{B}$ be a basis for $(X, \mathcal{T})$ and $A \subseteq X$, then $p \in \overline{A}$ iff for all basic open sets, $B \in \mathcal{B}$ that contain p, we have $B \cap A \neq \emptyset$

Proof. Exercise

(Pic of fried egg)

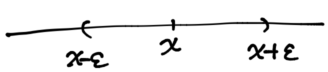


def'n: Let $A \subseteq X$, a top space. If $\overline{A} = X$, we say that A is dense (in X).

e.g. 1 $\mathbb{Q}$ is a dense subset of $\mathbb{R}_{usual}$
A $\quad$ X

Proof: Let $x \in \mathbb{R}$, Let $B_\varepsilon(x)$ be a basic open set,

Note $B_\varepsilon(x)$ contains a rational number.



(Analysis density of $\mathbb{Q}$ in $\mathbb{R}$: If $a<b$, then $\exists p \in \mathbb{Q}$ s.t. $a<p<b$)

e.g. 2 $\mathbb{Q}$ is dense in the Sorgenfrey Line.

Let $x \in \mathbb{R}$, note that any basic element $[a,b)$ which contains x, also contains (a,b) which contains a rational number (by part i)

e.g.3 $\mathbb{Q}$ is dense in $\mathbb{R}_{cofinite}$
Let $x \in \mathbb{R}$, let U be open, containing x. Does $U \cap \mathbb{Q} \neq \emptyset$? Recall $U = \mathbb{R} \setminus F$ where F is finite.
Yes $U \cap \mathbb{Q} \neq \emptyset$, because $\mathbb{Q}$ are infinite and $\mathbb{Q} \subseteq F$.
* actually, any infinite sets work in $\mathbb{R}_{usual}$

e.g.4 $\mathbb{Q}$ is dense in $\mathbb{R}_{indiscrete}$
In fact, every nonempty set is dense in $\mathbb{R}_{indisc}$. In particular, $\{\pi\}$ is dense here.

Non-example: Note $\mathbb{Q}$ is not dense in $\mathbb{R}_{discrete}$
$\overline{\mathbb{Q}} = \mathbb{Q}$

Now we have 2 ways of particularly describing a top space.

1). Give a dense subset.
2). Describe a basis

These will be especially interesting when we only need a "small" dense set or a "small" basis.

For us, "small" = countable

defn: If $A \subseteq X$, a top space, then A is closed iff $X \setminus A$ is open.

e.g. In $\mathbb{R}_{usual}$ $[0,8]$ is closed because $\mathbb{R} \setminus [0,8] = (-\infty, 0) \cup (8, +\infty)$ is open

e.g.2 In $\mathbb{R}_{sorgenfrey\ Line}$, $[0,8]$ is closed, as $\mathbb{R} \setminus [0,8] = (-\infty, 0) \cup (8, \infty)$ which is still open in Sorgen. Line.

weird thing: $[0,6)$ is open in Sorgen. Line.
it's also closed in Sorgen Line, because $\mathbb{R} \setminus [0,6) = (-\infty, 0) \cup [6, +\infty)$ which is open in Sorgen. Line

Note: every basic open sets $[a,b)$ in the Sorgenfrey Line is clopen (i.e. closed & open)
This is called being Zero-dimensional.
Notice that $(0,1)$ is open in SL, but it's not closed.

Prop: $A \subseteq X$ a top space.
i) A is closed iff $\overline{A} = A$.
ii). A is closed iff $\overline{A} \subseteq A$
iii). $\overline{A}$ is closed.

Prop: Let (X, T) be a top space.
i). $\emptyset$ & X are closed
ii). if $C_1, C_2, \cdots, C_N$ are closed sets, then $C_1 \cup C_2 \cup \cdots \cup C_N$ is closed
iii). if C_α where $\alpha \in I$ is closed, then $\bigcap_{\alpha \in I} C_\alpha$ is closed.

Proof: DeMorgan's Laws $(X \setminus (A \cup B)) = (X \setminus A) \cap (X \setminus B)$

Secret Topologies

$X = \{\text{students in class}\}$
U is open iff when $x \in U$, and y is to the right of x, then $y \in U$.

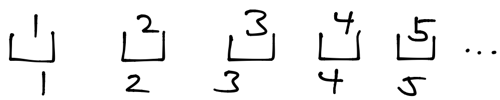
Q: What is the lecture of the set of women in the class?

§4 Countability

Idea: is to capture "equivalence of size" with bijections.

def'n: For sets A, B we say $|A| = |B|$ if there is a bijection $f: A \to B$ we say "A and B have the same cardinality".

Let's think about a soccer team where there is a player for each # $n \in \mathbb{N}$. (Recall $0 \notin \mathbb{N}$)



Now seat 1 is broken, everyone shifts to the right 1 seat.

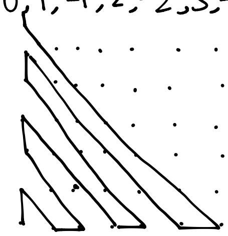
$f(n) = n+1$ bijection

or $|\mathbb{N}| = |\mathbb{N} \setminus \{1\}|$

other facts

- $|\mathbb{Z}| = |\mathbb{N}|$
- $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & \cdots \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ 0, & 1, & -1, & 2, & -2, & 3, & -3 & \cdots \end{array}$$

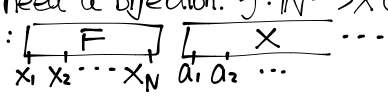


- $|\mathbb{Q}| = |\mathbb{N}|$

$$\left\{\frac{a}{b}, a \in \mathbb{Z}, b \in \mathbb{N}\right\} \approx \{(a,b)\ a \in \mathbb{Z}, b \in \mathbb{N}\} = \mathbb{Z} \times \mathbb{N}$$

def'n: For a set X, we say that X is countable iff there is a bijection $f: \mathbb{N} \to X$, i.e. if $|X| = |\mathbb{N}|$.
(countably infinite)
If there is a surjection $f: \mathbb{N} \to X$ we say X is at most countable.
Proposition: If X is countable and F is finite, then $X \cup F$ is countable)

Proof: We need a bijection. $f: \mathbb{N} \to X \cup F$.
idea:



since X is countable, find $g: \mathbb{N} \to X$
write $F = \{x_1, x_2, \cdots, \cdots, x_N\}$
define $f: \mathbb{N} \to X \cup F$ by

$$f(i) = \begin{cases} x_i & \text{if } 1 \leq i \leq N \\ a_{i-N} & \text{if } i > N \end{cases}$$

Fact: If A is ctble & B is ctble, then $A \cup B$ is ctble.

The technical fix for taking care of duplicates is to use $A \cup B = A \sqcup (B \setminus A)$
$\sqcup$ ↓ disjoint union

Prop: If A is a ctble set & $B \subseteq A$, then B is finite or ctbly infinite

idea: $a_1 \quad a_2 \quad \underset{\overset{\uparrow}{1}}{\overset{\in B}{a_3}} \quad a_4 \quad a_5 \quad \underset{\overset{\uparrow}{2}}{\overset{\in B}{a_6}} \quad a_7 \quad a_8 \quad \underset{\overset{\uparrow}{3}}{\overset{\in B}{a_9}} \quad a_{10} \cdots$

<u>Recap</u>:
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N} \times \mathbb{N}$, $\mathbb{Q} \times \mathbb{Q}$ are ctble
- Unions of ∞ ctble sets are ctble
- Products of 2 ctble sets are ctble. $(A \times B)$
- Subsets of ctble sets are ctble.

Prop: If $A_1, A_2, A_3, \cdots$ are all countable, then $\bigcup_{n \in \mathbb{N}} A_n$ is ctble.

Idea:
$$\begin{array}{cccc} \vdots & \vdots & \vdots & \\ a_{13} & a_{23} & a_{33} & \\ a_{12} & a_{22} & a_{32} & \\ a_{11} & a_{21} & a_{31} & \\ A_1 & A_2 & A_3 & \cdots \end{array}$$

There is a bijection between $\bigcup_{n \in \mathbb{N}} A_n$ and $\mathbb{N} \times \mathbb{N}$