Lecture 21

§ 8.4 Series of Functions

Weierstrass M Test:

Sps that $a_k(x)$ is a sequence of functions on $S \subset \mathbb{R}^n$ into $\mathbb{R}^m$, M_k is a sequence of real numbers.
Sps $\exists N \in \mathbb{N}$ s.t. $\|a_k\|_\infty \leq M_k$ for all $k \geq N$
Then if $\sum_{k=1}^\infty M_k$ converges, then $\sum_{k=1}^\infty a_k(x)$ converges uniformly on S

Proof:
Fix $x \in S$, $(a_k(x))$ is a sequence of pts in an Euclidean space which converges absolutely:

$$\sum_{k=1}^\infty \|a_k(x)\| < \sum_{k=1}^\infty \|a_k\|_\infty = \sum_{k=1}^N \|a_k\|_\infty + \sum_{k=N+1}^\infty \|a_k\|_\infty < \sum_{k=1}^N \|a_k\|_\infty + \sum_{k=N+1}^\infty M_k < \infty$$

where N is selected to satisfy the hypo of Thm.

$$\sum_{k=1}^\infty a_k(x) \text{ converges pointwise to } f(x). \; ?$$

Need to show that $\sum_{k=1}^\infty a_k(x)$ converges uniformly $\|f(x) - \sum_{k=1}^l a_k(x)\| = \|\sum_{k=1}^\infty a_k(x) - \sum_{k=1}^l a_k(x)\|$

$$= \|\sum_{k=l+1}^\infty a_k(x)\| \leq \sum_{k=l+1}^\infty \|a_k(x)\| < \sum_{k=l+1}^\infty \|a_k\|_\infty \leq \sum_{k=l+1}^\infty M_k \text{ for all } l \geq N$$

$$\lim_{l\to\infty} \sum_{k=l+1}^\infty M_k = 0$$

So the series converges to $f(x)$ uniformly. ∎

Ex:
$\sum_{n=0}^\infty \frac{x^n}{n!}$, on $[-A, A]$, $A \geq 0$ ← converges uniformly on $[-A, A]$

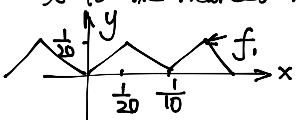
$$\left|\frac{x^n}{n!}\right| \leq \frac{A^n}{n!} := M_n$$

$\sum_{n=0}^\infty \frac{A^n}{n!}$. Ratio Test $a_n = \frac{A^n}{n!}$. $a_{n+1} = \frac{A^{n+1}}{(n+1)!}$

The series does not converge uniformly on $\mathbb{R}$.
Let $\varepsilon = 1$. $\left|\sum_{k=0}^n \frac{x^k}{k!} - f(x)\right| \geq \left|\frac{x^{n+1}}{(n+1)!}\right|$

$$x > \sqrt[n+1]{(n+1)!}$$

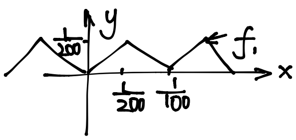
Ex : A simple application of the Weierstrass M-test $f(x)=\{x\}$ denotes the distance from x to the nearest integer.



Let $f_n(x)=\dfrac{\{10^n x\}}{10^n}$

$$f_1(x)=\frac{\{10x\}}{10}$$

$$f_2(x)=\frac{\{10^2x\}}{10^2}$$



$$f(x)=\sum_{n=1}^{\infty}\frac{\{10^n x\}}{10^n}$$

Claim: $f(x)$ is a continuous nowhere differentiable function.

Pf: First we'll show that $f(x)$ is continuous $|f_n(x)|\le\frac{1}{10^n}$

$M_n=\left(\frac{1}{10}\right)^n$

$\sum_{n=1}^{\infty}\frac{1}{10^n}$ geometric series with ratio $r=\frac{1}{10}<1 \Rightarrow$ the series converges $\Rightarrow$

By weierstrass M-test $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuous function

② Now we will show that f is not differentiable at a $\forall a\in\mathbb{R}$.

Enough to consider $0<a\le1$.

$$\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$

Want to show that this limit does not exist.
We will exhibit $\{h_m\}$ s.t. $h_m\to0$ for which $\displaystyle\lim_{m\to\infty}\frac{f(a+h_m)-f(a)}{h_m}$ does not exist.

$a=0.a_1a_2a_3\cdots a_m\cdots$

Let $h_m=10^{-m}$ if $a_m\ne4$, $a_m\ne9$
$h_m=10^{-m}$ if $a_m=4$ or $a_m=9$

$0.1234567\cdots$
$h_1=10^{-1}$
$h_4=-10^{-4}$

$$\frac{f(a+h_m)-f(a)}{h_m} = \sum_{n=1}^{\infty} \frac{\frac{1}{10^n}\{10^n(a+h_m)\} - \frac{\{10^n a\}}{10^n}}{\pm 10^{-m}} = \sum_{n=1}^{\infty} \pm 10^{m-n}(\{10^n(a+h_m)\} - \{10^n a\})$$

This infinite sum is really a finite sum

b/c $\underline{n \geq m}$, $10^n h_m$ is an integer

So $\{10^n(a+h_m)\} - \{10^n a\} = 0$ for $n \geq m$

On the other hand,

for $n<m$,

$10^n a = \text{integer} + 0.a_{n+1}a_{n+2}a_{n+3}\cdots a_m$

$10^n(a+h_m) = \text{integer} + 0.a_{n+1}a_{n+2}\cdots(a_m \pm 1)$

Now suppose that $0.a_{n+1}a_{n+2}\cdots a_m \leq \frac{1}{2}$

Then $0.a_{n+1}a_{n+2}\cdots(a_m \pm 1) \leq \frac{1}{2}$

$\{10^n(a+h_m)\} - \{10^n a\} = \pm 10^{n-m}$

(The same equation true when $a > \frac{1}{2}$)

For $n<m$, $10^{m-n}[\{10^n(a+h_m)\} - \{10^n a\}] = \pm 1$

$\sum_{n=1}^{\infty} \pm 10^{m-n}(\{10^n(a+h_m)\} - \{10^n a\}) = \sum_{n=1}^{m} \pm 1$ does not converge $\Rightarrow f(x)$ is not differentiable.

at a $\forall a \in \mathbb{R}$.

§8.5 Power series

$$\sum_{n=1}^{\infty} a_n x^n \quad \text{or} \quad \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

Hardamard's Thm

Given a power series $\sum_{n=0}^{\infty} a_n x^n$, there is $R \in [0, +\infty)$ (or possibly ∞), s.t. the series converges for all x with $|x|<R$ and diverges for all x with $|x|>R$

Moreover, the series converges uniformly on each closed interval.

$[a,b] \subset (-R, R)$

if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \alpha \Rightarrow R = \begin{cases} +\infty & \alpha = 0 \\ 0 & \alpha = +\infty \\ \frac{1}{\alpha} & \alpha \in (0, +\infty) \end{cases}$

R = the radius of convergence

What happens when $x = \pm R$?

Ex 1: $\sum_{n=1}^{\infty} \frac{x^n}{2^n \cdot n^2}$

$a_n = \frac{1}{2^n \cdot n^2}$

$\sqrt[n]{(a_n)} = \frac{1}{2(\sqrt[n]{n})^2}$

$\lim_{n\to\infty} \frac{1}{2(\sqrt[n]{n})^2} = \frac{1}{2}$, $R = \frac{1}{\frac{1}{2}} = 2$

Rad of convergence $= 2$

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty}\frac{|x^{n+1}|2^n n^2}{2^{n+1}(n+1)^2|x|^n} = \lim_{n\to\infty}\frac{|x|}{2}\left(\frac{n}{n+1}\right)^2 = \frac{|x|}{2} < 1 \Rightarrow |x| < 2$$

$x=-2$, $\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges by the AST.

$x=2$, $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges p-series, $p=2$

$\Rightarrow [-2,2]$

Ex 2 $\sum_{n=1}^{\infty}\frac{x^n}{2^n\cdot n}$

Find the interval of convergence

$a_n = \frac{1}{2^n\cdot n}$

$\sqrt[n]{a_n} = \frac{1}{2\sqrt[n]{n}}$

$\lim_{n\to\infty}\frac{1}{2\sqrt[n]{n}} = \frac{1}{2}$ $\quad R = \frac{1}{1/2} = 2$

$x=-2$ $\sum_{n=1}^{\infty}\frac{(-2)^n}{2^n\cdot n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$

Alternating Harmonic series, converges

$x=2$, $\sum_{n=1}^{\infty}\frac{1}{n}$ Harmonic series, diverges

so $I = [-2, 2)$

Term-by-term operations on series

Thm: If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has rad of convergence $R > 0$, then $\sum_{n=1}^{\infty} n a_n x^{n-1}$ has rad of convergence R, f is dif on $(-R,R)$, and for $x\in(-R,R)$, $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

Also, $\sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$ has rad of convergence R, for $x\in(-R,R)$

$$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty}\frac{a_n}{n+1}x^{n+1}$$

$\sum_{n=1}^{\infty}\frac{n^2}{2^n} = \ ?$

Ex $\quad \sum_{n=1}^{\infty} n^2 x^n$

$x=-1$? $\quad$ radius of convergence $=1$

$x=1$? $\quad$ Interval $(-1,1)$

$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = g(x) = (1-x)^{-1}$, $R=1$

$g'(x) = \frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$, $R=1$

$xg'(x) = \frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n \qquad R=1$

$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{(1+x)}{(1-x)^3}$

$\sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$

$x = \frac{1}{2}, \ \sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2}(1+\frac{1}{2})}{(\frac{1}{2})^3} = 4 \cdot \frac{3}{2} = 6$

Sets of functions in $C(K)$

§8.6 Compactness of subsets of $C(K)$, $F \subseteq C(K)$ is compact

if every sequence (f_n) of functions of F has a subsequence f_{n_i} that converges uniformly to $f \in F$.

The Heine-Borel Thm.

A subset of $\underline{\underline{\mathbb{R}}}^n$ is compact if it is closed and bounded.

If $F \subset C(K)$ is compact $\Rightarrow$ closed & bounded

<u>Ex</u>: $F = \{ f_n(x) = x^n, n \geq 1 \}$ on $[0,1]$

not closed $\Rightarrow \exists$ a seq $(f_n) \in F$ that converges to $f \notin F$ uniformly

$f_{n_i} \to f \notin F \Rightarrow$ not compact

not bdd $\Rightarrow \exists$ a seq f_n s.t. $\|f_n\| > n \ \forall n \geq 1$

$\|f_{n_i}\| > n_i$ so f_{n_i} does not converge.

$\|f_n(x)\|_\infty \leq 1$

$f_{n_i} \qquad f_n \to f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$

closed but F not <u>compact</u>

(f_n) does not have a convergent subsequence.

Ex: Let (g_n) of continuous functions on K, $\lim_{n \to \infty} g_n = g$, g_n converges to g uniformly.

$G = \{ g_n : n \geq 1 \} \cup \{ g \}$

Let $(f_k)_{k=1}^{\infty}$ be a sequence of functions of G

Either, there is a constant subsequence that converges in G.

or $\exists f_{n_i} \to g \in G$.

G is compact.