July 9th

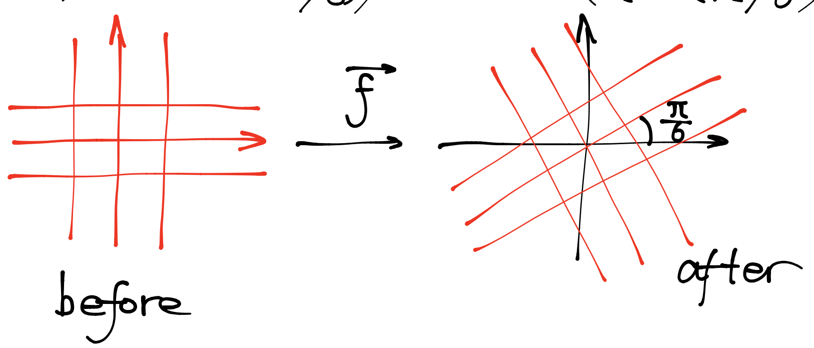
§ 3.4 Transformations and coordinate systems

Def: Sps $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is a map of class C^1, we can regard $\vec{F}$ as a transformation of $\mathbb{R}^n$

Eg 1: Def $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by $\vec{F}(u,v) = \frac{1}{2}(\sqrt{3}u - v, u + \sqrt{3}v)$

$$\vec{F}(u,v) = \left(\frac{\sqrt{3}}{2}u - \frac{1}{2}v, \frac{1}{2}u + \frac{\sqrt{3}}{2}v\right)$$
$$= \left(\cos\pi u - \sin\frac{\pi}{6}v, \sin\frac{\pi}{6}u + \cos\frac{\pi}{6}\cdot v\right)$$
$$= \begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

① $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\vec{F}} \begin{pmatrix} \cos(\pi/6) \\ \sin(\pi/6) \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\vec{F}} \begin{pmatrix} -\sin(\pi/6) \\ \cos(\pi/6) \end{pmatrix}$



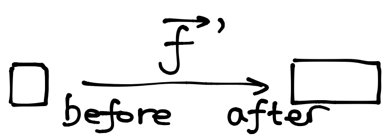
$$\vec{F}(u,v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix}$$

$(u,v) = (r\cos\phi, r\sin\phi)$

$$= \begin{pmatrix} r(\cos\theta\cos\phi - \sin\theta\sin\phi) \\ r(\sin\theta\cos\phi + \cos\theta\sin\phi) \end{pmatrix}$$
$$= \begin{pmatrix} r\cos(\theta+\phi) \\ r\sin(\theta+\phi) \end{pmatrix}$$

Eg2: Def $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$, $\vec{F}(u,v) = (2u, v)$

$(1,0) \xrightarrow{\vec{F}} (2,0)$
$(0,1) \longrightarrow (0,1)$



Matrix form: $\vec{F}(u,v) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

Eg3: Def: $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ by $\vec{F}(u,v) = (u^2 - v^2, 2uv) \equiv (x,y) \Rightarrow \begin{cases} x = u^2 - v^2 \\ y = 2uv \end{cases}$

$$\vec{F}(-u,-v) = ((-u)^2 - (-v)^2, 2(-u)(-v))$$
$$= (u^2 - v^2, 2uv) = \vec{F}(u,v)$$

$\Longrightarrow$ $\vec{f}$ is not one-to-one

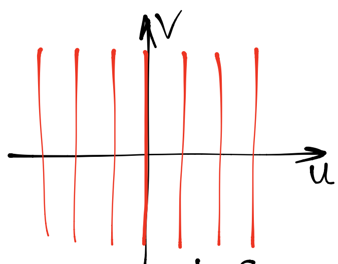
let us restrict attention to the region $u>0$

before / after

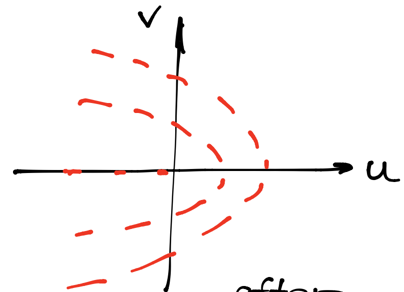
$u=c \xrightarrow{\ \vec{f}\ } \begin{cases} x=c^2-v^2 \\ y=2cv \end{cases} \Longrightarrow$ If $c\neq 0$, $v=y/2c$ and $x=c^2-\frac{y^2}{4c^2}$

If $c=0$, $y=0$, $x=-v^2<0$

parametric form of a curve



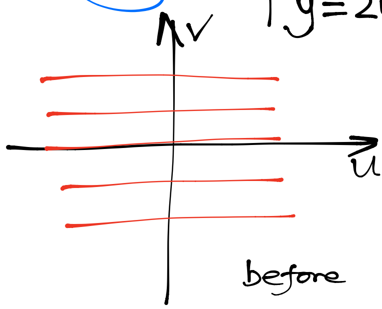
before



after

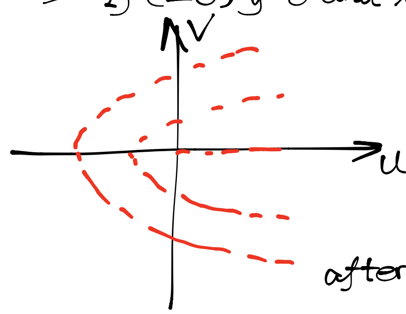
line $v=c \xrightarrow{\vec{f}} \begin{cases} x=u^2-c^2 \\ y=2uc \end{cases} \Longrightarrow$ If $c\neq 0$, $u=y/2c$, $x=y^2/4c^2-c^2$

If $c=0$, $y=0$ and $x=u^2>0$



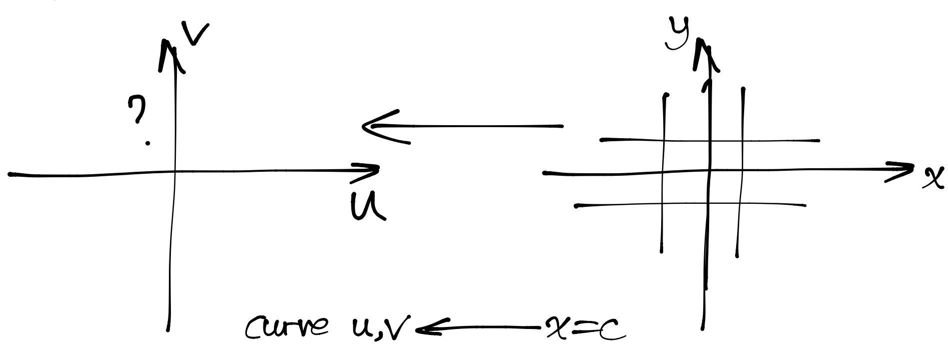
before

$\xrightarrow{\vec{f}}$



after

Q: Reverse Curve:



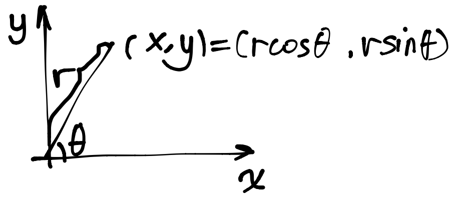
curve $u,v \longleftarrow x=c$

Exercise

curve $u,v \longleftarrow y=c$

Interpretation:
$\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$ is a coordinate system on $\mathbb{R}^n$, where $\vec{f}$ is a C^1 function.

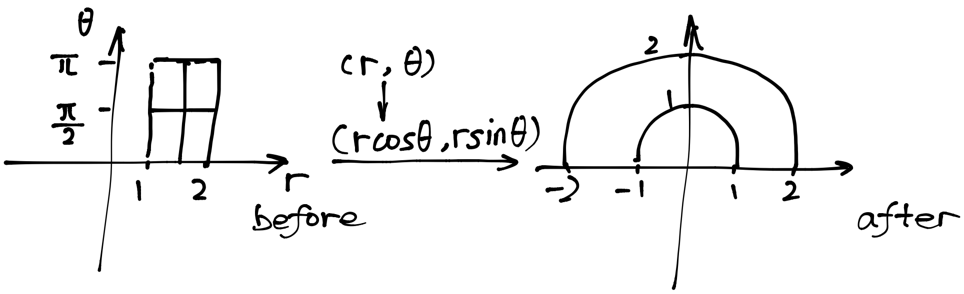
Eg: $f(r,\theta) = (r\cos\theta, r\sin\theta)$



Remark:
Coordinate: points stay the same but "label" changes

Transformation: the "label" stays the same, but points moves.

Eg. Consider $\vec{f}(r,\theta) = (r\cos\theta, r\sin\theta)$ as transformation



Q: Given $\vec{f}: U \subset \mathbb{R}^n \to V \subset \mathbb{R}^n$, C^1
Can we find an inverse mapping $\vec{f}^{-1}: V \to U$, C^1 ?

3.18 (The Inverse Mapping Theorem)
Let U and V be open sets in R^n, **a** in U and **b**=**f**(**a**). Suppose that **f**: U -> V is a mapping of class C1 and the Fréchet derivative D**f**(**a**) is inventively (that is, the Jacobian det D**f**(**a**) is nonzero). Then there exist neighborhoods M in U and N in V of **a** and **b**, respectively, so that **f** is a one-to-one map from M onto N, and the inverse map **f^-1** from N to M is also of class C1. Moreover, if **y**=**f**(**x**) $\in N$,
$D(\vec{f}^{-1})(\vec{y}) = [D\vec{f}(\vec{x})]^{-1}$

Proof: Let $\vec{F} = (\vec{x}, \vec{y}) = \vec{f}(\vec{x}) - \vec{y}$
Then the derivative of $\vec{F}$ as a function of $\vec{x}$

IFT $D_{\vec{x}}\vec{F} = D\vec{f}$ is invertible
$\Rightarrow$ The equation $\vec{F}(\vec{x}, \vec{y}) = \vec{f}(\vec{x}) - \vec{y} = \vec{0}$ can be solved for $\vec{x}$ as a function of $\vec{y}$ near $(\vec{a}, \vec{b})$, say $\vec{a} \in M \subset U$, $\vec{b} \in N \subset V$
call this function as $\vec{x} = \vec{f}^{-1}(\vec{y}) \Rightarrow \vec{x} = \vec{f}^{-1}(\vec{f}(\vec{x}))$
take derivative w.r.t. $\vec{x}$ $\Rightarrow$ by Chain Rule
$I = D\vec{f}^{-1}(\vec{f}(\vec{x})) \cdot D\vec{f}(\vec{x})$
$\Rightarrow [D\vec{f}(\vec{x})]^{-1} = D\vec{f}^{-1}(\vec{f}(\vec{x})) \cdot D\vec{f}(\vec{x}) \cdot (D\vec{f}(\vec{x}))^{-1} = D\vec{f}^{-1}(\vec{f}(\vec{x})) = D\vec{f}^{-1}(\vec{y})$ ∎

Remark: 1. IMT is a local property

2. Is IMT Global?

i.e. sps $\vec{f}: U \to V$ is of class C^1 and that $D\vec{f}(\vec{x})$ is invertible for every $\vec{x} \in U$, Is $\vec{f}$ one-to-one on U?

A: when dim $= 1$. Yes.

Proof: In this case $f'(x) \neq 0$ for all $x \in U$ and f' is continuous

$\Rightarrow$ $\underline{f' > 0 \text{ for } x \in U}$ or $\underline{f' < 0 \text{ for all } x \in U}$

$\downarrow$ strictly $\nearrow$ strictly $\downarrow$

$\searrow$ $\swarrow$

f is one-to-one

when dim > 1. No.

Eg: $\vec{f}(r, \theta) = (r\cos\theta, r\sin\theta)$, $\{(r, \theta) : r > 0\}$

$$D\vec{f} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det D\vec{f} = r\cos^2\theta + r\sin^2\theta = r > 0$$

$\Rightarrow$ IMT, $\vec{f}(r, \theta)$ is locally one-to-one and invertible but $\vec{f}(r, \theta) = (r\cos\theta, r\sin\theta)$
$= (r\cos(\theta + 2k\pi), r\sin(\theta + 2k\pi)), k \in \mathbb{Z}$.

$\blacksquare$

3. The invertible $D\vec{f}(\vec{a})$ is not a necessary condition for the invertibility of the function $\vec{f}$

i.e. $D\vec{f}(\vec{a})$ invertible $\not\Leftarrow$ $\vec{f}$ invertible

Eg. Let $f(x) = x^3$

$f'(x) = x^2 \Rightarrow f'(0) = 0$ but $f^{-1}(x) = x^{\frac{1}{3}}$

Chapter 4

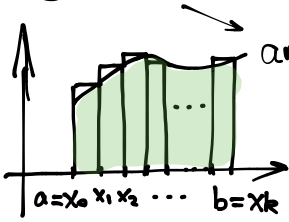
Integral Calculus

§4.1 — 4.5 No improper integration

No Lebesgue integration

Integration $\to$ antiderivative

$\searrow$ area under the curve



$$\int_a^b f(x)\,dx \approx \sum f(x_i)\Delta x_i$$

§4.1 Integration on the line

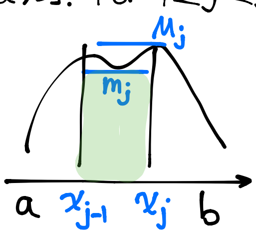
Def: A partition P of the interval $[a,b]$ is a subdivision of $[a,b]$ into nonover-lapping subintervals. In symbols, we shall write:

$$P=\{x_0, x_1, \cdots, \cdots, x_J\} \text{ with } a=x_0<x_1<x_2<\cdots<x_J=b$$

If P and P' are partitions of $[a,b]$, we say P' is a refinement of P if P' is obtain-ed from P by adding more subdivision points, that is $P\subset P'$.

Notes: If P and Q are both partitions of $[a,b]$ then $P\cup Q$ is a refinement of both P and Q.

Let f be a bounded real-valued function on $[a,b]$, $P=\{x_0, x_1, \cdots x_J\}$ is a partition of $[a,b]$. For $1\le j\le J$
$$m_j=\inf\{f(x): x_{j-1}\le x\le x_j\}$$
$$M_j=\sup\{f(x): x_{j-1}\le x\le x_j\}$$



Def: Lower Riemann Sum: $\underline{S}_P=\sum_1^J m_j(x_j-x_{j-1})$

upper Riemann Sum: $\overline{S}_P=\sum_1^J M_j(x_j-x_{j-1})$

If $m=\inf\{m_j\}$, $M=\sup\{M_j\}$

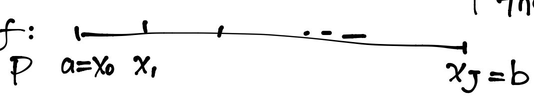
Then $\underline{S}_P=\sum_1^J m_j(x_j-x_{j-1})\ge\sum_1^J m(x_j-x_{j-1})=m\sum_1^J(x_j-x_{j-1})=m(b-a)$

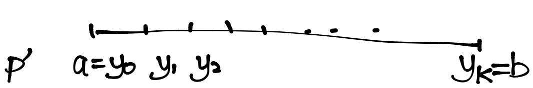
$\overline{S}_P=\cdots\cdots=M(b-a)$

Lemma 4.3: If P' is a refinement of P, then $\underline{S}_{P'}f\ge\underline{S}_Pf$ and $\overline{S}_{P'}f\le\overline{S}_Pf$

i.e. If we refine the partition $\begin{cases}\text{the lower sum } \nearrow \\ \text{the upper sum } \searrow\end{cases}$

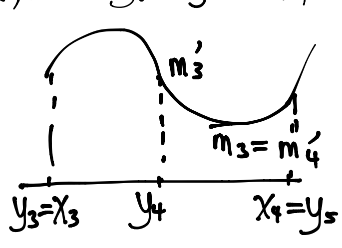
Proof:





$K\ge J$, for each x_j, $\exists$ $i\in\{0,\cdots,K\}$, s.t. $y_i=x_j$ then $P\subset P'$

Let $m_j=\inf\{f(x): x_{j-1}\le x\le x_j\}$
$m_j'=\inf\{f(y): y_{i-1}\le y\le y_i\}$



From this eg on the same interval from P, $m_i'\ge m_j$

$\Rightarrow \underline{S}_{P'}f = \sum_1^K m_i'(y_i - y_{i-1}) \geq \sum_1^J m_j(m_j - m_{j-1}) = \underline{S}_P f$

Similarly, $\overline{S}_{P'}f \leq \overline{S}_P f$ (exercise) $\blacksquare$

Lemma 4.4: If P and Q are partitions of $[a,b]$ then $\underline{S}_P f \leq \overline{S}_Q f$

Proof: Consider $P \cup Q$, $P \subset P \cup Q$, $Q \subset P \cup Q$

$$\underline{S}_P f \leq \underline{S}_{P\cup Q} f \leq \overline{S}_{P\cup Q} f \leq \overline{S}_Q f$$

$\blacksquare$

Def: Lower integrals of f on $[a,b]$ $\quad \underline{I}_a^b(f) = \sup_P \underline{S}_P f$

Upper integrals of f on $[a,b]$ $\quad \overline{I}_a^b(f) = \inf_P \overline{S}_P f$

by Lemma 4.4

$$\underline{I}_a^b(f) \leq \overline{I}_a^b(f)$$

If $=$, f is called Riemann integrable. The common value of $I_a^b f$ is the Riemann integral.

Criterion of integrability

Lemma 4.5:

If f is a bounded function $[a,b]$, the following conditions are equivalent.

a. f is integrable on $[a,b]$.

b $\forall \varepsilon > 0, \exists$ a partition P of $[a,b]$ s.t. $\overline{S}_P f - \underline{S}_P f < \varepsilon$.

Proof:

$b \Rightarrow a$. $\forall \varepsilon$, $\overline{S}_P f - \underline{S}_P f < \varepsilon$

$$\underline{S}_P f \leq \underline{I}_a^b f \leq \overline{I}_a^b f \leq \overline{S}_P f$$

$\Rightarrow \overline{I}_a^b f - \underline{I}_a^b f < \overline{S}_P f - \underline{S}_P f < \varepsilon$

Let $\varepsilon \to 0 \Rightarrow \overline{I}_a^b f = \underline{I}_a^b f \Rightarrow f$ is integrable $\to$ Riemann integrable

$a \Rightarrow b$. $\forall \varepsilon$ Since $\overline{I}_a^b f = \inf_P \overline{S}_P f$ we can find a partition Q s.t. $\overline{S}_Q f < \overline{I}_a^b f + \frac{\varepsilon}{2}$

$\underline{I}_a^b(f) = \sup_P \underline{S}_P f$, we can find a partition Q', s.t. $\underline{S}_{Q'} f > \underline{I}_a^b f - \frac{\varepsilon}{2}$

Let $P = Q \cup Q'$, $\underline{S}_{Q'} f \leq \underline{S}_P f \leq \overline{S}_P f \leq \overline{S}_Q f$

$\overline{S}_P f - \underline{S}_P f \leq \overline{S}_Q f - \underline{S}_{Q'} f$ (3)

From (1), $\overline{S}_Q f - \frac{\varepsilon}{2} < \overline{I}_a^b f$

(2), $\underline{S}_{Q'} f + \frac{\varepsilon}{2} > \underline{I}_a^b f$

since f is integrable, $\overline{I}_a^b f = \underline{I}_a^b f$

$\overline{S}_Q f - \frac{\varepsilon}{2} < \underline{S}_{Q'} f + \frac{\varepsilon}{2}$

$\overline{S}_Q f - \underline{S}_{Q'} f < \varepsilon$ From (3) $\overline{S}_P f - \underline{S}_P f < \varepsilon$.

$\blacksquare$

Remark : From this lemma, we can use $\underline{S}_P f$ and $\overline{S}_P f$ to approximate the in-tegration since $\underline{S}_P f \le \int_a^b f(x)dx \le \overline{S}_P f$

If $\overline{S}_P f - \underline{S}_P f < \varepsilon \Rightarrow \begin{cases} \int_a^b f(x)dx - \underline{S}_P f < \varepsilon \\ \overline{S}_P f - \int_a^b f(x)dx < \varepsilon \end{cases}$

$\overline{S}_P f - \varepsilon \le \int_a^b f\,dx \le \underline{S}_P f + \varepsilon$

$\Rightarrow \int_a^b f\,dx \approx \overline{S}_P f \approx \underline{S}_P f$ when ε is small.