Lecture 8

Recall : $f(X, \mathcal{T}) \to (Y, \mathcal{U})$

- is continuous if $f^{-1}(V) \in \mathcal{T}$ for all $V \in \mathcal{U}$ (preimage of open is open).
- is open if $f(U) \in \mathcal{U}$ for all $U \in \mathcal{T}$ (image of open is open)

Fact: $f: X \to Y$ be a 1-1 function with $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be top spaces.
- f is cts iff f^{-1} is open
- f is open iff f^{-1} is continuous.

(Add. comment.
def'n: $f: X \to Y$ is a closed function if $f(C)$ is closed in Y for all C closed in X).

Now we'll describe when two top spaces are the same, but first some boring facts.
Fact: Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be topspaces
① If $f: X \to Y$ maps each $x \in X$ to a fixed constant $y_0 \in Y$, then f is cts.
② If $f: X \to Y$, and $g: Y \to Z$ are cts, then $g \circ f: X \to Z$ is continuous.

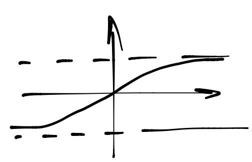
Now, we'll describe when 2 top spaces "are topologically the same"

def'n: Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be top spaces, and let $f: X \to Y$ be a bijection, we say that f is a homeomorphism between $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ if f is cts and f^{-1} is cts.
In this case we say $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ are homeomorphic

$X \approx Y$ or $X \approx_{homeo} Y$

Example: $f: \mathbb{R}_{usual} \to (-\frac{\pi}{2}, \frac{\pi}{2})_{usual}$ where $f(x) = \arctan(x)$ is a homeomorphism.

From 1st year we know.
- arctan is cts
- $\arctan^{-1}(x) = \tan x$
- $\tan x$ is cts
- $\arctan x$ is a bijection.



Example 2: $f: X_{discrete} \to X_{discrete}$ is a homeomorphism iff f is a bijection

Cor: Regardless of topology, $\mathbb{Q}$ is never a homeomorphic to $\mathbb{R}$.

Some boring facts
Let $(X, \mathcal{T}_1)$, $(Y, \mathcal{T}_2)$ and $(Z, \mathcal{T}_3)$ be top spaces.
1). The identity function $id_X : X \to X$ where $id_X(a) = a$ for all $a \in X$, is a homeomorphism
2). If $f: X \to Y$ is a homeo, then $f^{-1}: Y \to X$ is a homeo
3). If $f: X \to Y$ is a homeo, then $f^{-1} \circ f : X \to X$ is id_X
4). If $f: X \to Y$ is a homeo & $g: Y \to Z$ is a homeo, then $g \circ f : X \to Z$ is a homeo.

Followup : Define: $\mathcal{H}(X) = Homeo(X) := \{f: X \to X \mid f \text{ is a homeo}\}$
Note $\mathcal{H}(X)$ is a group.

Ex: $f: \mathbb{R}_{sorgen} \to \mathbb{R}_{sorgen}$ given by $f(x) = x + 70$ is a homeomorphism

Q: Is $f:(\mathbb{R},\mathcal{T})\to(\mathbb{R},\mathcal{T})$ given by $f(x)=x+70$ always a homeo?

Prop: Let $(X,\mathcal{T})$, $(Y,\mathcal{U})$ be top spaces. and let $f:X\to Y$ be a bijection TFAE.
① f is a homeo.
② f is cts & f is open.
③ f is cts & f is closed.

Big question. When are two top spaces the same? When are they different?

We will know that $X \not\cong Y$ if
- $|X| \neq |Y|$
- X has a point which is open but Y doesn't.

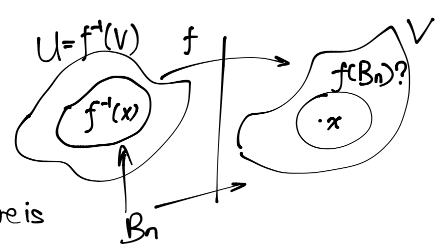
In general, we will look for properties of topological spaces $(X,\mathcal{T})$ and $(Y,\mathcal{U})$ s.t. the property is preserved when taking a homeo. ie. if $f:X\to Y$ is a homeo, then $(X,\mathcal{T})$ has the property iff $(Y,\mathcal{U})$ has the property.

Such properties are called topological invariants.

Here are some properties that are top. invariants.
- X is finite
- X is ctble
- X is unctble
- X is a Hausdorff space
- X is 1st ctble
- X is 2nd ctble
- X is separable
- X is ccc
- X has a finite open set.

Proof that second countable is a top inv.

Let: $f:X\to Y$ be a homeo and assume X is second countable. Let $\mathcal{B}=\{B_n : n\in\mathbb{N}\}$ be a ctble basis for X.



Claim: $\mathcal{C}=\{f(B_n) : n\in\mathbb{N}\}$ is a basis for Y.
Note, $\mathcal{C}$ is countable.
Also since f is open, each $f(B_n)$ is open. Fix $x\in V$ open in Y.

Look at $f^{-1}(x)\in f^{-1}(V)$ which is open since f is cts. So there is a $B_n\in\mathcal{B}$ s.t. $f^{-1}(x)\in B_n\subseteq U$
So $x\in f(B_n)\subseteq V$ ∎

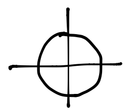
Prop: $\mathbb{R}_{usual}$ is not homeo to $\mathbb{R}_{indiscrete}$, $\mathbb{R}_{cofinite}$ & $\mathbb{R}_{cocountable}$.
$\mathbb{R}_{usual}$ is Hausdorff, but the others are not.
$\mathbb{R}_{ind}$ is not homeo to $\mathbb{R}_{cof}$ or $\mathbb{R}_{cctble}$ as $\mathbb{R}_{ind}$ is 2nd ctble but the others aren't
$\mathbb{R}_{cof}$ is different from $\mathbb{R}_{cctble}$.
↓ is separable ↓ is not separable

Q Is $\mathbb{R}_{usual} \approx \mathbb{R}^2_{usual}$? NO.
Is $\mathbb{N}_{discrete} \approx \mathbb{N}\times\mathbb{N}_{discrete}$? YES
Is $\mathbb{R} \approx (0,1)$? YES
Is $\mathbb{R}_{usual} \approx [0,1]$? Use EVT to show they are not homeo.

§ 7. Subspaces.

Q: How do we get a new topological spaces ?
Motivation: $S^1 = \{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$ (the circle)
Q What topology should we put on it ?

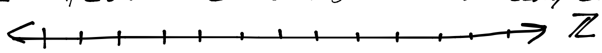
take all possible open arcs



def'n: For $(X,\mathcal{T})$ a top space, and $Y\subseteq X$. We define $\mathcal{T}_{subspace}$ on Y as $\mathcal{T}_{subspace} = \{U\cap Y : U\in\mathcal{T}\}$ which is called the subspace top. on Y (inherited from $(X,\mathcal{T})$)

We sometimes write $\mathcal{T}_Y$.

eg 1. Let $\mathbb{Z}\subseteq\mathbb{R}$, take $\mathcal{T}=\mathcal{T}_{usual}$. What does $\mathcal{T}_{subspace}$ look like?

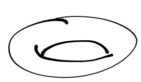


$\mathcal{T}_{subspace} \supseteq \{(n-\frac{1}{2}, n+\frac{1}{2})\cap\mathbb{Z} : n\in\mathbb{Z}\} = \{\{n\} : n\in\mathbb{Z}\}$
So $\mathcal{T}_{subspace}$ is the discrete topology on $\mathbb{Z}$.

eg. 2 For $[0,1]\subseteq\mathbb{R}$ with Sorgenfrey top,
Note $\{1\} = \underbrace{[1,7)}_{U}\cap\underbrace{[0,1]}_{Y}$
so $\{1\}$ is open in the subspace
so for example $[0,1]$ subspace of sorgen $\not\approx \mathbb{R}_{sorgenfrey}$

In general, subspaces of $\mathbb{R}^n_{usual}$ will be very interesting.

e.g. · $S^1\subseteq\mathbb{R}^2$ (the circle)
· $S^1\times S^1\subseteq\mathbb{R}^3$ (Torus)



· Disc $\times S^1\subseteq\mathbb{R}^3$ (Solid Torus)
· $S^n := \{(x_1,x_2,\cdots,x_n)\in\mathbb{R}^{n+1} : x_1^2+x_2^2+\cdots+x_n^2=1\}$
This is the n-dimensional sphere in $\mathbb{R}^{n+1}$

Take $GL(2,\mathbb{R}) = \left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix} : a,b,c,d\in\mathbb{R}, \det\begin{bmatrix} a & b\\ c & c\end{bmatrix}\neq 0\right\}$

We think about $GL(2,\mathbb{R}) \hookrightarrow \mathbb{R}^4$
by the map $f(\begin{bmatrix} a & b\\ c & d\end{bmatrix}) = (a,b,c,d)$
i.e. $GL(2,\mathbb{R}) = \{(a,b,c,d)\in\mathbb{R}^4 : ad-bc\neq 0\}$

$\det : \mathbb{R}^4 \to \mathbb{R}$ is a cts map.

Prop: Let (X,τ) be a top space with basis $\mathcal{B}$, and let $Y \subseteq X$.
The collection

$$\mathcal{B}_Y := \{B \cap Y : B \in \mathcal{B}\}$$ is a basis for the subspace top on Y.

Hereditary Properties.

def'n: Let Φ be a property. We say that Φ is hereditary if whenever (X,τ) has the property Φ, then all of its subspaces have property Φ.

examples:

- 2nd ctble.
- 1st ctble
- Hausdorff
- is finite
- Discrete

Nonexamples:

- ccc
- separable
- is infinite