Exact ODE's

Review of partial derivatives

Suppose f is a function of several variables x_1, x_2, x_3 (or u, v, or x, y, z ...)

The partial derivative with respect to one of these variables is defined as the usual derivative with that variable, keeping the others constant.

E.g.: $f(x_1, x_2, x_3) \rightarrow \frac{\partial f}{\partial x_2} = \lim_{h \to 0} \frac{f(x_1, x_2+h, x_3)}{h}$

Can iterate:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

Example: $f(x,y) = x^2 \sin(y)$

$$\frac{\partial f}{\partial x} = 2x\sin(y)$$

$$\frac{\partial f}{\partial y} = x^2 \cos y$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x \partial x} = 2\sin(y)$$

$$\frac{\partial^2 f}{\partial y^2} = -x^2 \sin(y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x\cos(y)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 2x\cos(y)$$

General fact:

Equality of mixed partials

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Similarly for higher order derivatives

Suppose x_1, x_2, x_3 are themselves function of t.

Then $f(x_1, x_2, x_3)$ becomes a function of t: $f(x_1(t), x_2(t), x_3(t))$. Its t-derivative is given by chain rule.

$$\frac{d}{dt} f(x_1(t), x_2(t), x_3(t)) = \frac{\partial f}{\partial x_1}(x_1(t), x_2(t), x_3(t))\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}(\ldots)\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}(\ldots)\frac{dx_3}{dt}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1}\cdot\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dt} + \frac{\partial f}{\partial x_3}\cdot\frac{dx_3}{dt}$$

Example: $f(x_1, x_2) = x_1 x_2$: $\frac{\partial f}{\partial x_1} = x_2$, $\frac{\partial f}{\partial x_2} = x_1$, $\frac{\partial^2 f}{\partial x_1^2} = 0$, $\frac{\partial^2 f}{\partial x_2^2} = 0$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 1$

$\varphi(t) = \sin(e^t \cos t^2)$

Can view this as $f(x_1(t), x_2(t))$ where $f(x_1, x_2) = \sin(x_1)\cos(x_2)$

and $x_1(t) = e^t$, $x_2(t) = t^2$

$$\frac{d\varphi}{dt} = \frac{df}{dt} = \cos(x_1)\cos(x_2)\frac{dx_1}{dt} + \sin(x_1)(-\sin(x_2))\frac{dx_2}{dt}$$

$$= \underline{\cos(e^t)\cos(t^2)e^t} \underline{-\sin(e^t)\sin(t^2) - 2t}$$

Defn: An exact 1st ODE is an ODE of the form

$M(t,y) + N(t,y)\cdot y' = 0$

such that there exists a function $\psi(t,y)$ with $N = \frac{d\psi}{dx}$, $M = \frac{d\psi}{dt}$

Prefer to write this in "differential form"

$M(t,y)dt + N(t,y)\,dy = 0$

Condition: $M = \frac{d\psi}{dt}$, $N = \frac{d\psi}{dy}$ for some function ψ.

In this case, the general solution of this ODE is given by $\psi(t,y) = C$.

(y implicitly given as function of t.)

Check: Take t-derivative of both-sides

total

$$\underbrace{\frac{d}{dt}\psi(t,y)}_{\text{a function of } t} = \frac{d\psi}{dt}\frac{dt}{dt} + \frac{d\psi}{dy}\frac{dy}{dt}$$

$$= M(t,y) + N(t,y)\cdot\frac{dy}{dt} = 0 \leftarrow \frac{d}{dt}(C)$$

Remark: If $M = \frac{d\psi}{dt}$ $N = \frac{d\psi}{dy}$

Then $\frac{dM}{dy} = \frac{d^2\psi}{dy\,dt} = \frac{d^2\psi}{dt\,dy} = \frac{dN}{dt}$

Theorem: The ODE

$M(t,y)dt + N(t,y)dy = 0$ is exact $\Leftrightarrow \frac{dM}{dy} = \frac{dN}{dt}$

Thus, if this condition holds, can find $\psi(t,y)$ with $M = \frac{d\psi}{dt}$, $N = \frac{d\psi}{dy}$

and then $\psi(t,y) = C$ is the general solution of the ODE.

Example: $\underbrace{(2t+y^2)}_{M} + \underbrace{2ty}_{N}\,dy = 0$

$\frac{dM}{dy} = 2y$ $\quad \frac{dN}{dt} = 2y$

Since they're equal, can find $\psi(t,y)$ with $\frac{d\psi}{dt} = M$, $\frac{d\psi}{dy} = N$

$\frac{d\psi}{dt} = M(t,y) = 2t + y^2$

Integrate with respect of t:

$$\psi(t,y) = t^2 + ty^2 + C(y) \quad (y \text{ is constant here})$$

"the constant of integration" can still depend on y.

$$\frac{d\psi}{dy} = 2ty + C'(y) \overset{!}{=} N(t,y) = 2ty$$

Thus we get $2ty + C'(y) = 2ty$

thus $C(y) = A$ really constant.

$\Rightarrow \psi(t,y) = t^2 + ty^2$ satisfies $\frac{d\psi}{dt} = M$, $\frac{d\psi}{dy} = N$ ✓

$\Rightarrow$ general solution of our ODE is $t^2 + ty^2 = B$.