Lecture 18 Continue §7.5 Finite orthonormal Sets

Projection Theorem

$P^2 = P$ (In general, does not have to be orthogonal matrix P does not have to be symmetric).

$(V, \langle \cdot \rangle)$ an inner product space.

$M \subset V$ finite dimensional subspace.

$P =$ the orthogonal projection range $P = M$

(1). $\forall y \in V, x \in M, \|y-x\|^2 = \|y-Py\|^2 + \|Py - x\|^2$

(2). Let $\{e_1, \cdots, e_n\}$ orthonormal basis for $M$.

$Py = \sum_{j=1}^n \langle y, e_j \rangle e_j$ for each $y \in V$

(3). $\|y\|^2 \geq \sum_{j=1}^n \langle y, e_j \rangle^2$

Proof: Let $Q: V \to M$

Linear

orthogonal projection onto $M$

$Qy = \langle y, e_1 \rangle e_1 + \cdots + \langle y, e_n \rangle e_n$

$Q$ is a linear map

$Q: V \to M$

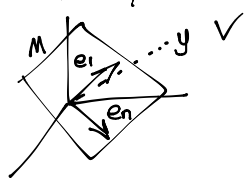
$x \in M \Rightarrow Qx = \langle x, e_1 \rangle e_1 + \cdots + \langle x, e_n \rangle e_n = x$

Projection Theorem:

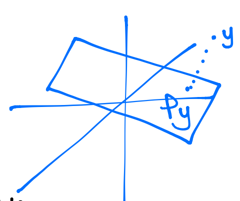
$Q^2 y = Qy \Rightarrow Q$ is a projection. Suppose $x \in M$, suppose $y \in \operatorname{Ker} Q$

$\langle y, e_j \rangle =$

V (infinitely dimensional)




$$\langle x, y \rangle = \sum_{j=1}^n \langle x, e_j \rangle \underbrace{\langle y, e_j \rangle}_{0} = 0$$




$Q$ is an orthogonal projection $Q = P$

$\|Qy\|^2 = \sum_{j=1}^n \beta_j^2$, $\beta_j = \langle y, e_j \rangle$

$\|y\|^2 = \|Qy\|^2 + \|(I-Q)y\|^2 \geq \|Qy\|^2$

$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle$

$x = \sum_{i=1}^n \alpha_i e_i$

$$\sum_{j=1}^n \alpha_j^2 - 2\sum_{j=1}^n \alpha_j \beta_j + \|y\|^2 = \sum_{j=1}^n \alpha_j^2 - 2\sum_{j=1}^n \alpha_j \beta_j + \sum_{j=1}^n \beta_j^2 - \sum_{j=1}^n \beta_j^2 + \|y\|^2$$

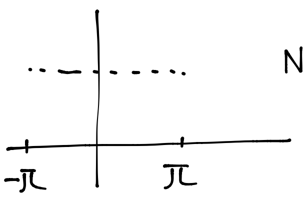
$$\sum_{j=1}^n (\alpha_j - \beta_j)^2 - \|Py\|^2 + \|y\|^2$$

$$\|x - Py\|^2 + \|(I-P)y\|^2 = \|x - Py\|^2 + \|y - Py\|^2$$

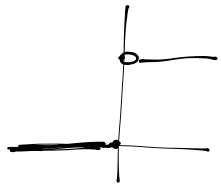
∎

## §7.6 Fourier Series

PC$[-\pi,\pi]$ = the space of piecewise continuous functions
($\exists$ a partition of $[-\pi,\pi]$ s.t. $f$ is continuous on each subinterval)



No



Yes

$f$ is Riemann integrable

$f \cdot g$ is piecewise continuous if $f$ & $g$ are p.c.

unit length.

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)g(t)\,dt$$ dividing by $2\pi$ gives the constant function 1

$$\|f\|_2 := \langle f, f\rangle^{\frac{1}{2}} = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} f^2\,dt\right]^{\frac{1}{2}}$$

Sps $f \equiv 1$, $\|f\|_2 = 1$

**Proposition:** The functions $\{1, \sqrt{2}\cos n\theta, \sqrt{2}\sin n\theta \,;\, n \geq 1\}$ form an orthonormal set in piecewise continuous in PC$[-\pi,\pi]$ with this linear product.

Let $f = \sqrt{2}\cos n\theta$, $g = \sqrt{2}\cos m\theta$

$$\langle f, g\rangle = \frac{2}{2\pi}\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta = \cos n\theta \cos m\theta$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(m+n)\theta + \cos(m-n)\theta$$

Let us assume $n > m$ (WLOG)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} (\cos(m+n)\theta + \cos(m+n)\theta)\,d\theta = \frac{1}{2\pi}\left(\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right)\Bigg|_{-\pi}^{\pi}$$

if $n = m$, $\langle \cdots \rangle = 1$

$$= 0$$

**Def:** A trig polynomial is a finite sum $\boxed{f(\theta) = A_0 + \sum_{k=1}^{N} A_k \cos k\theta + \sum_{k=1}^{N} B_k \sin k\theta}$

It has a degree $N$ if either $A_N$ or $B_N \neq 0$, $\mathbb{TP}$ = the set of all trig polynomial

$\mathbb{TP}_N$ = the set of all trig polynomial of degree at most $N$.

**Def:** The Fourier series of piecewise continuous $f : [-\pi,\pi] \to \mathbb{R}$

$$f \sim A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

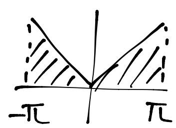
$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

The sequences $(A_n)_{n\geq 0}$ & $(B_n)_{n\geq 1}$ are Fourier coefficients of $f$.

E1: $f(\theta)=|\theta|, -\pi \leq \theta \leq \pi$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |t| dt = \frac{\pi}{2}$$



$B_n = 0$

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |t| \cos nt dt = \frac{2}{\pi}\int_0^{\pi} t\cos nt dt$$

$$= \frac{2t}{\pi}\frac{\sin nt}{n}\Big|_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}\frac{\sin nt}{n} dt$$

$$\frac{2\cos nt}{\pi n^2}\Big|_0^{\pi} = \begin{cases} 0 & n \text{ is even} \\ \frac{-4}{\pi n^2} & n \text{ is odd} \end{cases}$$

$$|\theta| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos(2k+1)\theta}{(2k+1)^2}$$

E2: $f(\theta)=\cos^3\theta$

$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos^3\theta d\theta = \cdots\cdots$$

use $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$