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Portmanteau Test.

Box & Pierce (1970

$$Q_{BP} = n \sum_1^m \hat{\rho}_k^2 \sim \chi^2_{m-(p+q)}$$

Ljung & Box (1978)

$$Q_{LB} = \sum_1^m \frac{n(n+2)}{n-k} \hat{\rho}_k^2 \sim \chi^2_{m-(p+q)}$$

Model selection

$AIC = -2\log ML + 2k$ (smaller better)

$BIC = -2\log ML + k\log(n)$

Matrix form of a Bivariate $(k=2)$ VAR(1) model:

$$\begin{bmatrix} r_{1,t} \\ r_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{10} \\ \phi_{20} \end{bmatrix} + \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} \begin{bmatrix} r_{1,t-1} \\ r_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}$$

~~How to transform a VAR(p) into~~

Stationarity of VAR(p) process.

① Transform VAR(p) to VAR(1) of $k\times p$ variables

k-variate

let $b_t = [0_{k\times1}, 0_{k\times1}, \cdots, a_t^T]^T$

$X_t = [r_{t-p+1}^T, r_{t-p+2}^T, \cdots, r_t^T]^T$ be a $kxp$ dim t.s.

mean of $b_t$ is 0.

② express $X_t = \Phi^* X_{t-1} + b_t$

where $\Phi^*_{kp\times kp} = \begin{bmatrix} 0 & I_k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_k \\ \Phi_p & \Phi_{p-1} & \cdots & \Phi_1 \end{bmatrix}$

③ Check $k\times p$ eigenvalues of $\Phi^*$ whether $< 1$ in modulus.

Model Building Procedure.

Test VAR(j) v.s. VAR(j-1)

① Estimate model using OLS $r_t = \Phi_0 + \Phi_1 r_{t-1} + \cdots + \Phi_j r_{t-j} + a_t$

② Residuals from VAR(j) are given by $\hat{a}_t^{(j)} = r_t - \hat{\Phi}_1^{(j)} r_{t-1} - \cdots - \hat{\Phi}_j^{(j)} r_{t-j}$

③ Calculate residual covariance matrix

$$\hat{\Sigma}^{(j)} = \frac{1}{T-2j-1} \sum_{t=j+1}^{T} \hat{a}_t (\hat{a}_t)^T, \ j \geq 0$$

④ Calculate test statistics

$$M(j) = -(T-k-j-\frac{3}{2}) \log \frac{|\hat{\Sigma}^{(j)}|}{|\hat{\Sigma}^{(j-1)}|} \sim \chi^2_{k^2}$$

How to use VAR(p) model to determine Granger Causality?

1. VAR(p): write the form in matrix (for simplicity, let $k=2$)

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \sum_{j=1}^{p} \begin{bmatrix} \phi_{j,11} & \phi_{j,12} \\ \phi_{j,21} & \phi_{j,22} \end{bmatrix} \begin{bmatrix} y_{1,t-j} \\ y_{2,t-j} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

2. intuition:

If $y_{2t}$ does not Granger cause $y_{1t}$, then all $\phi_{j,12}$'s must be zeros.

similarly: ...

3. Test procedure (test with and without granger causality)

Obtain ML (or OLS) estimates of the following.

$$y_{1t} = \alpha_1 + \sum_{j=1}^{p} \phi_{j,11} y_{1,t-j} + e_{1t}$$

$$y_{1t} = \alpha_1 + \sum_{j=1}^{p} \phi_{j,11} y_{1,t-j} + \sum_{j=1}^{p} \phi_{j,12} y_{2,t-j} + \varepsilon_{1t}$$

Calculate the values of log likelihood functions in equations above and LR statistic is given by

$$n(\log|\tilde{\Sigma}| - \log|\hat{\Sigma}|) \sim \chi^2_p$$

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Alternative Test for Granger causality (Pierce & Haugh)

Consider 2 causal & invertible t.s $\{X_t\}$ $\{Y_t\}$.

given: $\Phi_x(B)x_t = \Theta_x(B)u_t$, $u_t \sim WN(0, \sigma_u^2)$

$\Phi_y(B)y_t = \Theta_y(B)v_t$, $v_t \sim WN(0, \sigma_v^2)$

where $x_t = X_t - \mu_x$

$y_t = Y_t - \mu_y$

$\Phi_x(B)$, $\Theta_x(B)$, $\Phi_y(B)$, $\Theta_y(B)$ are polynomials of backward shift operator $B$ and satisfy all causal/invertible conditions.

Cross correlation of lag $k$ between $u$ and $v$

$$\rho_{uv}(k) = \frac{E(u_t, v_{t+k})}{\sqrt{E(u_t^2)E(v_t^2)}}$$

There are many possible types of causal ~~int~~ intepretation between $\{X_t\}$ and $\{Y_t\}$ which can be characterized by the properties of cross-correlation functions between $\hat{u}_t$ and $\hat{v}_t$

An overall (portmanteau) test for testing Granger causality ($H_0$: $X_t$ does not granger cause $Y_t$) is given by

$$Q_L = n^2 \sum_{k=0}^{L} (n-k)^{-1} r_{uv}^2(k) \sim \chi^2_{L+1}, \quad L \text{ is defined integer.}$$

We will reject null hypothesis if p-value of $Q_L$ < a predetermined significance level.

What's a ~~reg~~ spurious regression?

When a regression model as $y_t = \alpha + \beta X_t + \varepsilon_t$ appears to find a relationship that does not really exist, it's called spurious regression.

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具体案例

How to test the cointegration of 2 I(1) processes using a regression approach?

① use unit root test, e.g. Augmented Dickey-Fuller test, test if the given two t.s $\{X_t\}$ & $\{Y_t\}$ follow I(1) processes

② If both I(1), regress $\{X_t\}$ against $\{Y_t\}$ (or the other way)

Let $\{\varepsilon_t\}$ denote the corresponding regression residuals

③ Apply unit root test again on the regression residuals $\{\varepsilon_t\}$.

If $\{\varepsilon_t\}$ is I(0)/stationary ~~or follows a I(d) process~~, then $\{X_t\}$ and $\{Y_t\}$ are cointegrated.

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Engle-Granger Method to test cointegration

(for simplicity consider 2 I(1), if not, we need to use D-F test method to test I(d) step by step)

① Test with unit-root test whether 2 t.s are I(1).

② If I(1), run regression using the least square method.

③ Collect the residuals of aforesaid regression and test if the residuals are stationary using the unit root test.

④ If residuals do not contain stochastic trend (unit root), we say that the data of interest are cointegrated.

Remarks: There may exist multiple cointegrated relationships among a set of variables.

Why EG?
① straightforward to interpret
② risk-management: minimizes variance is more important
③ natural choice...

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Granger Representation Theorem

ECM?

If t.s. $\{X_t\}$ and $\{Y_t\}$ are cointegrated, an ECM (error correlation mechanism) must be included for their ~~p~~ modeling process.

Specifically, for bivariate model (VAR):

$$\Delta X_t = \alpha_1 + \gamma_1 Z_{t-1} + \sum_i^{m_1} \beta_{1i} \Delta X_{t-i} + \sum_i^{m_2} \beta_{2i} \Delta Y_{t-i} + e_{1t}$$

$$\Delta Y_t = \alpha_2 + \gamma_2 Z_{t-1} + \sum_i^{m_3} \beta_{3i} \Delta X_{t-i} + \sum_i^{m_4} \beta_{4i} \Delta Y_{t-i} + e_{2t}$$

where $(e_{1t}, e_{2t})^T$ follows a bivariate WN, & $Z_t = X_t - \alpha Y_t$ is an I(0) process.

where $(1, -\alpha)$ is a cointegrated vector between $\{X_t\}$ & $\{Y_t\}$

O.w. $\exists$ a ~~cointegrated vec~~ a model misspecification in this VAR model.

Remark ① require $\gamma_1 < 0$, $\gamma_2 > 0$

② ~~cnt~~ cointegration between $\{X_t\}$ & $\{Y_t\}$ $\Longleftrightarrow$ ECM

What is Mean Square Prediction Error (MSPE)?

$$MSPE = \frac{1}{H} \sum_{i=1}^{H} \hat{e}_{t+i}(1)$$

$$= \frac{1}{H} \sum_{i=1}^{H} (X_{t+i+1} - \hat{X}_{t+i}(1))^2$$

---

Granger-Newbold test for forecast accuracy

Consider $x_i = e_{1i} + e_{2i}$

$z_i = e_{1i} - e_{2i}$

$i = 1, \cdots, H$

where $e_{ki}$ stands for the one step ahead forecast error of model $k$, $k=1,2$ at time $t+i$

G&N assumes

① The forecast errors have zero mean and normally distributed

② The forecast errors are serially uncorrelated

Under the above 2 assumptions and under the assumption of equal forecast accuracy ($H_0$), $x_i$ and $z_i$ should be uncorrelated.

(Since $\rho_{xz} = E(xz) = E(e_1^2 - e_2^2) = 0$)

Then we use $r_{xz}$ (sample correlation coefficient between $\{x_i\}$ and $\{z_i\}$ to evaluate the accuracy between model 1 & 2.

In particular $\frac{r_{xz}}{\sqrt{\frac{(1-r_{xz})^2}{H-1}}} \sim t_{H-1}$ if ①② holds

Thus if $r_{xz}$ is not zero, reject $H_0$

Specifically, if $r_{xz} > 0$, model 1 has larger MPSE (less accuracy)

if $r_{xz} < 0$, model 2 has larger MPSE

Sample correlation

$$\hat{r}_{xz} = \frac{\sum (x_i - \bar{x})(z_i - \bar{z})}{\sqrt{\sum (x_i - \bar{x})^2}\sqrt{\sum (z_i - \bar{z})^2}}$$

---

Transfer ~~F~~ Function Noise (~~#~~TFN) model building procedure?

① Preliminary identification of the impulse response coefficients $v_i$'s.

(calculate coefficients of $v(B)$)

The regression $Y_t = C + v_0 X_t + v_1 X_{t-1} + \cdots + v_h X_{t-h} + e_t$

① $X_t \sim ARMA$

$$\phi_x(B)(X_t - \mu_x) = \Theta_x(B)\alpha_t$$

$$Y_t - \mu_y = v(B)(X_t - \mu_x) + N_t$$

$$\beta_t = \frac{\phi_x(B)}{\Theta_x(B)} v(B)(X_t - \mu_x) + \frac{\phi_x(B)}{\Theta_x(B)} N_t = v(B)\alpha_t + n_t$$

$$\beta_t \alpha_{t-j} = v(B)\alpha_t\alpha_{t-j} + n_t\alpha_{t-j}$$

$$E(\downarrow) = E(\downarrow)$$

$$Cov(\beta_t, \alpha_{t-j}) = v_j \, Var(\alpha_{t-j})$$

so

$$v_j = \frac{Cov(\beta_t, \alpha_{t-j})}{Var(\alpha_{t-j})} = corr(\beta_t, \alpha_{t-j}) \frac{se(\beta_t)}{se(\alpha_t)}$$

② Specification of noise term $N_t$

③ Specification of Transfer function

to find rational polynomial $\omega(B)$ & $\delta(B)$ as well as parameter $b$

para $s$ para $r$

to approx $v(B)$ best.

$$v(B) = \frac{\omega(B) B^b}{\delta(B)}$$

expand:

$$v_0 + v_1 B + v_2 B^2 + \cdots = \frac{\omega_0 B^b + \omega_1 B^{b+1} + \cdots + \omega_s B^{b+s}}{1 - \delta_1 B - \cdots - \delta_r B^r}$$

待定系数求解 ① $v_j = 0 \; \forall j < b$ if $b > 0$

② ...

④ Estimation of TFN model specified in 2 & 3

⑤ diagnostic checks.

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☆ approximate se of cross-correlation $\rho_{\alpha\beta}$ is

$$se = \frac{1}{\sqrt{\#\text{ of observations}}}$$

Then 95% CI $\hat{\rho}_{\alpha\beta} \pm 1.96 se = \rho_{\alpha\beta} \frac{se(\beta)}{se(\alpha)} \pm 1.96 \frac{1}{\sqrt{\#\text{ of obsv.}}}$

## Dynamic Linear Model?

DLMs arise via state-space formulation of standard t.s models as natural structures for modeling t.s with nonstationary components.

### Basic normal DLMs

normal DLMs for univariate time series of equally spaced observations.

specifically, assume $y_t$ is modeled over time by equation

$$y_t = F_t'\theta_t + v_t$$
$$\theta_t = G_t\theta_{t-1} + \omega_t$$

with following components assumptions:

① $\theta_t = (\theta_{t,1}, \cdots, \theta_{t,p})$ is $p \times 1$ state vector at time $t$

② $F_t$ is the p-dim vector of known constants or regressors at time $t$

③ $v_t$ is the observation noise, with $N(v_t \mid 0, v_t)$

④ $G_t$ is a known $p \times p$ matrix, usually referred to as the state evolution matrix at time $t$

⑤ $\omega_t$ is the state evolution noise, or innovation

⑥ The noise sequences $v_s$ & $\omega_t$ are ind & mutually ind.

E.g.

1. Autoregressions.

AR : $y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + v_t$

$F_t' = (y_{t-1}, \cdots, y_{t-p})$ $\theta_t' = (\phi_{t,1}, \cdots, \phi_{t,p})$, $G_t = I_p$, with $\omega_t = 0 \ \forall t$.

2. ARMA (zero mean)

$y_t = \sum_{j=1}^{p} \phi_j y_{t-j} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t$ with $N(\varepsilon_t \mid 0, w_t)$

set $m = \max(p, q+1)$

extend ARMA coeff to $\phi_j = 0$ for $j > p$, $\theta_j = 0$ for $j > q$.

write $u = (1, \theta_1, \cdots, \theta_{m-1})'$

Then DLM holds for $F' = (1, 0, \cdots, 0)$, $v_t = 0$, $\omega_t = w_t u u'$ and

$$G_p = \begin{bmatrix} \phi_1 & 1 & 0 & \cdots & 0 \\ \phi_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{p-1} & 0 & 0 & \cdots & 1 \\ \phi_p & 0 & 0 & \cdots & 0 \end{bmatrix}_{p \times p}$$

and

$$G = \begin{bmatrix} G_p & 0 & \cdots & 0 \\ \phi_{p+1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_m & 0 & \cdots & 0 \end{bmatrix}_{m \times m}$$

---

Population spectrum/spectral density function (for MA($\infty$))

$$S_Y(\omega) = \frac{1}{2\pi}\sigma^2 \psi(e^{-i\omega})\psi(e^{i\omega})$$

☆

Check Stationality of VAR(2) model.

$k=2$
$p=2$

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} \phi_{1,11} & \phi_{1,12} \\ \phi_{1,21} & \phi_{1,22} \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \phi_{2,11} & \phi_{2,12} \\ \phi_{2,21} & \phi_{2,22} \end{bmatrix} \begin{bmatrix} X_{1,t-2} \\ X_{2,t-2} \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix}$$

$$b_t = \begin{pmatrix} 0 \\ 0 \\ a_{1,t} \\ a_{2,t} \end{pmatrix} \qquad X_t = \begin{pmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{1,t} \\ X_{2,t} \end{pmatrix}$$

$$\Phi^*_{(2\times2)\times(2\times2)} = \begin{pmatrix} \vec{0} & I_2 \\ \vec{0} & \\ \Phi_2 & \Phi_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \phi_{2,11} & \phi_{2,12} & \phi_{1,11} & \phi_{1,12} \\ \phi_{2,21} & \phi_{2,22} & \phi_{1,21} & \phi_{1,22} \end{pmatrix}$$

$$X_t = X_{t-1} \cdot \Phi^* + b_t$$

A forecast problem Recap:

Forecast ARMA(1,1) model:

$$X_t - 0.5X_{t-1} = a_t + 0.25a_{t-1}, \quad a_t \sim NID(0,1)$$

Sps $X_{97} = -0.7$

$X_{98} = -1$

$X_{99} = -0.8$

$X_{100} = -0.4$

a). $\hat{X}_t(1) = 0.5X_t + 0.25a_t$

$= 0.5X_t + 0.25(X_t - \hat{X}_{t-1}(1))$

$h>1$, $\hat{X}_t(h) = 0.5\hat{X}_t(h-1)$

b). BLF of $X_{101} + X_{102} + X_{103}$

$\hat{X}_{100}(1) + \hat{X}_{100}(2) + \hat{X}_{100}(3)$

For simplicity, assume $a_t = 0$

$\hat{X}_{100}(1) = 0.5X_{100} + 0.25(X_{100} - \hat{X}_{99}(1))$

$= 0.5X_{100} = -0.2$

$\hat{X}_{100}(2) = -0.1$

$\hat{X}_{100}(3) = -0.05$

So BLF $= -0.35$

c). 95% CI ?

$se = \sqrt{var(e_t(1) + e_t(2) + e_t(3))}$

$e_{100}(1) = X_{101} - \hat{X}_{100}(1) = a_{101}$

$e_{100}(2) = X_{102} - \hat{X}_{100}(2) = a_{102} \ldots$

$e_{100}(3) = \ldots$

Another EX:

ARMA(1,1)

$(1-0.5B)(X_t-4)=(1+0.5B)a_t,\quad a_t \sim NID(0,1)$

$$X_t - 0.5X_{t-1} - 4 + 2 = a_t + 0.5a_{t-1}$$

$$X_t = 0.5X_{t-1} + a_t + 0.5a_{t-1} + 2$$

$$X_t(l) = 0.5X_{t+l-1} + a_{t+l} + 0.5a_{t+l-1} + 2$$

$$\hat{X}_t(l) = 0.5X_{t+l-1} + 0 + 0.5(X_{t+l} - \hat{X}_{t+l-1}(1)) + 2$$

so 1-step ahead: $\hat{X}_t(1) = 0.5X_{t-1} + 0.5(X_t - \hat{X}_{t-1}(1)) + 2$

For $l>1$: $\hat{X}_t(l) = 0.5X_{t-l} + 2$

- - - -

forecast error variance:

$$\cancel{(1-\phi B)(1+\psi_1 B+\cdots)=1-\theta B}$$

$$\Phi(B)X_t = \Theta(B)a_t$$

$$X_t = \psi(B)a_t$$

$$\text{so } \psi(B) = \frac{\Theta(B)}{\Phi(B)}$$

Solve for $\psi_1, \psi_2, \ldots$

then $var(e_t(l)) = \sigma^2(1+\psi_1^2+\cdots+\psi_{l-1}^2)$