Lecture 2

- *Matrix form*: $P^{(2)} = (p_{ij}^{(2)}) = PP = P^2$
- By induction, $P^{(n)} = P^n$, i.e. to compute probabilities of n-step jumps, you can take nth powers of the transition matrix P.
- Convention: $P^{(0)} = I$
- Observe: $p_{ij}^{(m+n)} = P(X_{m+n}=j \mid X_0=i) = \sum_{k\in S} P(X_{m+n}=j, X_m=k \mid X_0=i) = \sum_{k\in S} p_{ik}^{(m)} p_{kj}^{(n)}$
  matrix form: $P^{(m+n)} = P^{(m)}P^{(n)}$

Classification of States:

- Shorthand: write $P_i(\cdots)$ for $P(\cdots \mid X_0=i)$ And write $E_i(\cdots)$ for $E(\cdots \mid X_0=i)$
- Def'n: a state i of a Markov chain is <u>recurrent</u> (or, <u>persistent</u>) if $P_i(X_n=i \text{ for some } n\geq 1) = 1$. Otherwise, i is <u>transient</u>.
- Let $N(i) = \#\{n\geq 1 : X_n = i\}$ = total # times the chain hits i. (R.v.; could be ∞)
- Recurrence theorem
  - i recurrent iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$ iff $P_i(N(i)=\infty)=1$
  - And, i transient iff $\sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ iff $P_i(N(i)=\infty)=0$

- To prove this let $f_{ij} = P_i(X_n = j \text{ for some } n\geq 1)$ prob of starting from i, and eventually hitting j

- Then i <u>recurrent</u> iff $f_{ii}=1$
- And, i <u>transient</u> iff $f_{ii}<1$

- Also $P_i(N(i)\geq 1) = f_{ii}$ and $P_i(N(i)\geq 2) = (f_{ii})^2$, etc.
  $P_i(N(i)\geq k) = (f_{ii})^k$

Also, recall that if Z is any non-negative-integer-valued r.v. then $\sum_{k=1}^{\infty} P(Z\geq k) = E(Z)$

PROOF OF RECURRENCE THM: First, by continuity of probabilities

$$P_i(N(i)=\infty) = \lim_{k\to\infty} P_i(N(i)\geq k) = \lim_{k\to\infty} (f_{ii})^k = \begin{cases} 1, & f_{ii}=1 \\ 0, & f_{ii}=0 \end{cases}$$

Second, using countable linearity,

$$\sum_{n=1}^{\infty} p_{ii}^{(n)} = \sum_{n=1}^{\infty} P_i(X_n=i) = \sum_{n=1}^{\infty} E_i(1_{X_n=i})$$
$$= E_i\left(\sum_{n=1}^{\infty} 1_{X_n=i}\right) = E_i(N(i)) = \underline{\sum_{k=1}^{\infty} P_i(N(i)\geq k)}$$
$$= \sum_{k=1}^{\infty} (f_{ii})^k = \begin{cases} \infty, & f_{ii}=1 \\ \dfrac{f_{ii}}{1-f_{ii}} < \infty, & f_{ii}<1 \end{cases}$$

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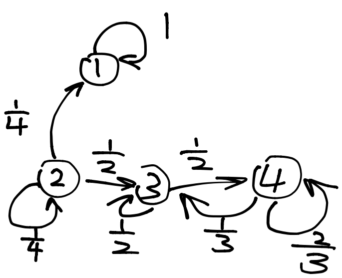
Example:

$S = \{1,2,3,4\}$

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$



$f_{11} = 1$

$f_{22} = \frac{1}{4}$ not P_{22} !

$f_{33} = 1$

$f_{44} = 1$ $f_{43} = 1$

$f_{21} = \frac{1}{3}$ Conditioned on "leaving 2"

$f_{21} = p_{21} + p_{22}f_{21} + p_{23}f_{31} + p_{24}f_{41} = \frac{1}{4} + \frac{1}{4}f_{21} + 0 + 0 \Rightarrow f_{21} = \frac{1}{3}$ so $f_{23} = \frac{2}{3}$

$f_{24} = \frac{2}{3}$

· Example : Simple Random Walk (s.r.w.) ($S=\mathbb{Z}$, and $P_{i,i+1}=p$, and $P_{i,i-1}=1-p$)

· Is the state 0 recurrent ?

If n odd, $P_{00}^{(n)}=0$ $\qquad \sum_{n=1}^{\infty} P^{(n)}$ n even

$P_{00}^{(2)}=p(1-p)+(1-p)p=2p(1-p)$

$P_{00}^{(4)}=pp(1-p)(1-p)+\cdots=\binom{4}{2}p^2(1-p)^2$

$P_{00}^{(n)}=P(n/2 \text{ heads } \& \ n/2 \text{ tails on first } n \text{ tosses})=\binom{n}{n/2}p^{n/2}(1-p)^{n/2}=\dfrac{n!}{[(\frac{n}{2})!]^2}p^{n/2}(1-p)^{n/2}$

$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2\pi n}$ Stirling Approx

So n large & even

$$P_{00}^{(n)} \approx \frac{(n/e)^n\sqrt{2\pi n}}{[(n/2e)^{n/2}\sqrt{2\pi n/2}]^2}p^{n/2}(1-p)^{n/2}=[4p(1-p)]^{n/2}\sqrt{2/\pi n}$$

Now, if $p=\frac{1}{2}$, then $4p(1-p)=1$, so $\sum_{n=1}^{\infty}P_{00}^{(n)} \approx \sum_{n=2,4,6,\dots}\sqrt{2/\pi n}=\infty$, so state 0 is recurrent.

But if $p\neq\frac{1}{2}$, then $4p(1-p)<1$, so $\sum_{n=1}^{\infty}P_{00}^{(n)} \approx \sum_{n=2,4,6,\dots}[4p(1-p)]^{n/2}\sqrt{2/\pi n}<\infty$ so state 0 is transient

- similarly true for all other states besides 0, too.

Communicating States :

· Say that state i communicates with state j, $i\to j$ if $f_{ij}>0$, i.e. if it is possible to get from i to j i.e. $\exists m\geq 1$ with $P_{ij}^{(m)}>0$.

· $i \leftrightarrow j$

· Say a MC is irreducible if $i\to j$ for all $i,j\in S$.

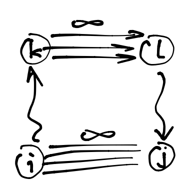
CASES THEOREM : For an irreducible MC, either

(a). $\sum_{n=1}^{\infty}P_{ij}^{(n)}=\infty$ for all $i,j\in S$, so all states are recurrent. "Recurrent MC"

(b). $\sum_{n=1}^{\infty}P_{ij}^{(n)}<\infty$ for all $i,j\in S$, so all states are transient. "transient MC"

· It follows from

SUM LEMMA : if $i\to k$, and $l\to j$, and $\sum_{n=1}^{\infty}P_{kl}^{(n)}=\infty$, then $\sum_{n=1}^{\infty}P_{ij}^{(n)}=\infty$.



Proof of Cases Thm from Sum Lemma

Either (a) $\exists\, k,l\in S$ with $\sum_n P_{kl}^{(n)}=\infty$, then $\sum_n P_{ij}^{(n)}=\infty, \forall i,j$

(b) or $\nexists\, k,l\in S$ with $\sum_n P_{kl}^{(n)}=\infty$, so $\sum_n P_{ij}^{(n)}<\infty \ \forall i,j$

Proof of SUM LEMMA

Find $m,r\geq 1$ with $P_{ik}^{(m)}>0$ $P_{lj}^{(r)}>0$

Note that $P_{ij}^{(m+s+r)}\geq P_{ik}^{(m)}P_{kl}^{(s)}P_{lj}^{(r)}$. Hence

$$\sum_{n=1}^{\infty}P_{ij}^{(n)}\geq\sum_{n=m+r+1}^{\infty}P_{ij}^{(n)}=\sum_{s=1}^{\infty}P_{ij}^{(m+s+r)}\geq\sum_{s=1}^{\infty}P_{ik}^{(m)}P_{kl}^{(s)}P_{lj}^{(r)}=P_{ik}^{(m)}P_{lj}^{(r)}\sum_{s=1}^{\infty}P_{kl}^{(s)}$$
$$=(+)(+)(\infty)=\infty$$

Simple random walk. Irreducible.

$p=\frac{1}{2}$ case (a)

$p\neq\frac{1}{2}$, case (b)

Finite SPACE THM : an irreducible MC on a finite state space always falls into case (a), i.e. $\sum_{n=1}^{\infty}P_{ij}^{(n)}=\infty$ for all $i,j\in S$, and all states are recurrent

· PROOF OF FINITE SPACE THM:

· Choose any state $i\in S$, then $\sum_{j\in S}\sum_{n=1}^{\infty}P_{ij}^{(n)}=\sum_{n=1}^{\infty}\sum_{j\in S}P_{ij}^{(n)}=\sum_{n=1}^{\infty}1=\infty$

Since S is finite, $\exists$ at least one $j \in S$ with $\sum_{n=1}^{\infty} P_{ij}^{(n)} = \infty$, so must be case (a)

F-LEMMA: if $j \to i$ & $f_{jj} = 1$, then $f_{ij} = 1$

possibly j to i — eventually in j — so $i \to j$ eventually

PROOF OF F-LEMMA:

- Assume $i \neq j$ (otherwise trivial)

Let $T_i = \min\{n \geq 1 : X_n = i\}$ ($T_i = \infty$ if never hit i)

Since $j \to i$, $P_j(T_i < \infty) > 0$

In fact, must have $P_j(T_i < T_j) > 0$ (o.w. would always return to j before hitting i, so would never reach i from j)

But $1 - f_{jj} = P_j(T_j = \infty) \geq P_j(T_i < T_j) P_i(T_j = \infty) = P_j(T_i < T_j)(1 - f_{ij})$

If $f_{jj} = 1$, then $1 - f_{jj} = 0$, so must have $1 - f_{ij} = 0$, i.e. $f_{ij} = 1$.