GJJ 4.1 (c) $\qquad X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta)$

$$= \theta \exp(-\theta x)$$

$H_0: \theta = \theta_0 \qquad H_1: \theta = \theta_1 > \theta_0$

$$\lambda = \frac{\theta_0^n \exp(-\theta_0 \Sigma x_i)}{\theta_1^n \exp(-\theta_1 \Sigma x_i)} = \left(\frac{\theta_0}{\theta_1}\right)^n \exp(-\theta_0 \Sigma x_i + \theta_1 \Sigma x_i)$$

$$= \left(\frac{\theta_0}{\theta_1}\right)^n \exp([\theta_1 - \theta_0] \Sigma x_i)$$

$$C = \{\lambda \le k\}$$

$$= \left\{ \left(\frac{\theta_0}{\theta_1}\right)^n \exp([\theta_1 - \theta_0] \Sigma x_i) \le k \right\}$$

$$= \{\exp([\theta_1 - \theta_0] \Sigma x_i \le k^*\}$$

$$= \{\overbrace{[\theta_1 - \theta_0]}^{\text{positive}} \Sigma x_i \le k^{**}\}$$

$$= \{\Sigma x_i \le k^{***}\}$$

$X \sim \text{exponential}(\theta)\ ;\ E(x) = 1/\theta$

$\underbrace{\Sigma x_i}_{= Y} \sim \text{Gamma}(n, 1/\theta_0)$

- $MGF(\Sigma x_i) = \left(\frac{1}{1 - t/\theta}\right)^n$

- Determine k^{***} in Small Samples:

$$P(C) = P(Y \leq k^{***}) = \alpha = 0.05$$

- In R: qgamma(0.05, n, $1/\theta_0$)

- In Large Samples use CLT:

$$P(\bar{X} \leq k^{****}) = \alpha = 0.05$$

$$E(\bar{x}) = 1/\theta \quad ; \quad V(\bar{x}) = 1/n\theta^2$$

$$P\left(\frac{\bar{x} - 1/\theta_0}{\sqrt{1/n\theta_0^2}} \leq \frac{k^{****} - 1/\theta_0}{\sqrt{1/n\theta_0}} \right) = \alpha = 0.05$$

$$= P(Z \leq \underset{\underset{-1.64}{\shortparallel}}{k^{*****}}) = 0.05$$

$\therefore$ We reject H_0 if

$$Z \leq -1.64$$

or $$\bar{x} \leq (-1.64)\left(\sqrt{1/n\theta_0^2}\right) + 1/\theta_0$$

d.) $$\left.\begin{aligned} &X_{11}, \dots, X_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2) \\ &X_{12}, \dots, X_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2) \end{aligned}\right\} \sigma_1^2, \sigma_2^2 \text{ are } \underline{\underline{\text{known}}}$$

$$H_0: \mu_2 = \mu_1 \quad vs \quad H_1: \mu_2 = \mu_1 + \delta$$

Where $\delta > 0$

$$\Rightarrow H_0: \mu_2 - \mu_1 = \underbrace{0}_{\delta_0} \quad vs \quad H_1: \mu_2 - \mu_1 = \underbrace{\delta}_{\delta_1}$$

<u>Where $\delta_0 = 0 < \delta_1$</u>

$$\Rightarrow \bar{y} - \bar{x} \sim N\left(\mu_2 - \mu_1, \frac{\sigma_2^2}{n} + \frac{\sigma_1^2}{n}\right)$$

$$E(\bar{y} - \bar{x}) = E(\bar{y}) - E(\bar{x}) = \mu_2 - \mu_1$$

$$V(\bar{y} - \bar{x}) \underset{\uparrow}{=} V(\bar{y}) + V(\bar{x}) = \underbrace{\frac{\sigma_2^2}{n} + \frac{\sigma_1^2}{n}}_{= \gamma^2}$$

Due to independence

$$L(\delta) = (2\pi \gamma^2)^{-1/2} \exp\left(\frac{1}{2\gamma}\left([\bar{y} - \bar{x}] - \delta\right)^2\right)$$

$$\lambda = \frac{L(\delta_0)}{L(\delta_1)} = \frac{\exp\left(-\frac{1}{2\gamma}\left(\overbrace{[\bar{y}-\bar{x}]}^{\bar{w}} - \delta_0\right)^2\right)}{\exp\left(-\frac{1}{2\gamma}\left(\underbrace{[\bar{y}-\bar{x}]}_{\bar{w}} - \delta_1\right)^2\right)}$$

$$= \exp\Big(-\frac{1}{2\gamma^2}\left[\bar{w}^2 - 2\bar{w}\delta_0 + \delta_0^2\right]$$

$$+ \frac{1}{2\gamma^2}\left[\bar{w}^2 - 2\bar{w}\delta_1 + \delta_1^2\right]\Big)$$

$$= \exp\left(\frac{\bar{w}\delta_0}{\gamma^2} - \frac{\bar{w}\delta_1}{\gamma^2} - \frac{\delta_0^2}{2\gamma^2} + \frac{\delta_1^2}{2\gamma^2}\right)$$

$$= \exp\left(\bar{w}\,\frac{(\delta_0-\delta_1)}{\gamma^2} - \frac{\delta_0^2}{2\gamma^2} + \frac{\delta_1^2}{2\gamma^2}\right)$$

$$C = \left\{\bar{w}\,\frac{\overbrace{(\delta_0-\delta_1)}^{\text{negative}}}{\gamma^2} - \frac{\delta_0^2}{2\gamma^2} + \frac{\delta_1^2}{2\gamma^2} \le k\right\}$$

$$= \left\{\bar{w} > k^*\right\}$$

$$= \left\{\frac{(\bar{y}-\bar{x}) - (\mu_2-\mu_1)}{\sqrt{\sigma_2^2/n + \sigma_1^2/n}} > k^*\right\}$$

$$= \left[\frac{(\bar{y} - \bar{x}) - 0}{\sqrt{\sigma_x^2/n + \sigma_y^2/n}} > k^* \right]$$

$$= \left[z > k^{**} \right]$$

$$\Rightarrow P\left[z > \underset{1.64}{\underset{\sim}{k^{**}}} \right] = \alpha = 0.05$$

$\therefore$ we reject H_0 if

$$z > 1.64$$

or

$$(\bar{y} - \bar{x}) > 1.64 \left(\sqrt{\sigma_x^2/n + \frac{\sigma_y^2}{n}} \right)$$

GJJ Q 4.2

a.) $$C = \left\{ \frac{(\bar{Y} - \bar{X}) - \delta_0}{\sqrt{\sigma_2^2/n + \sigma_1^2/n}} > k \right\}$$

$$= \left\{ \frac{\bar{w} - \delta_0}{\sqrt{\gamma^2}} > k \right\}$$

$$P(C) = P(Z > k) = \alpha = 0.01$$

$$k = 2.33$$

$$\Rightarrow P\left(\frac{\bar{w} - \delta_0}{\sqrt{\gamma^2}} > 2.33 \right)$$

$$= P\left(\frac{\bar{w} - \delta - \delta_0 + \delta}{\sqrt{\gamma^2}} > 2.33 \right)$$

$$= P\left(\frac{\bar{w} - \delta}{\sqrt{\gamma^2}} - \frac{\delta_0 - \delta_1}{\sqrt{\gamma^2}} > 2.33 \right)$$

$$= P\left(\frac{\bar{w} - \delta}{\sqrt{\gamma^2}} > 2.33 + \frac{\delta_0 - \delta_1}{\sqrt{\gamma^2}} \right)$$

• Power is the probability
we reject when H_1 is true.

$$P\left(\frac{\bar{D} - \delta_1}{\sqrt{\sigma^2}} > 2.33 - \frac{\delta_1 - \delta_0}{\sqrt{\sigma^2}} \right)$$

$$= P\left(Z > 2.33 - \frac{(1-0)}{\sqrt{1/10 + 1/10}} \right)$$

$$= P(Z > 2.33 - 2.24)$$

$$= P(Z > 0.09) = 0.464$$

b.) n? for power ≥ 0.95
$n = n_1 = n_2$

$$P\Big(Z > \underbrace{2.33 - 1 / \sqrt{1/n + 1/n}}_{x^*} \Big) = 0.95$$

$$1 - P(Z < x^*) = 0.95$$

$$P(Z < x^*) = 0.05$$

$$x^* = -1.64$$

$\Rightarrow \quad -1.64 = 2.33 - 1/\sqrt{2/n}$

$-3.97 = -1/\sqrt{2/n} \Rightarrow 3.97 = 1/\sqrt{2/n}$

$x = 31.52 \Rightarrow n = 32$ units

for n_1, n_2.

GJJ Q 4.3

$$X_1, \dots, X_n \overset{iid}{\sim} f(x)$$

$$f(x) = \frac{\theta^\lambda}{\Gamma(\lambda)} x^{\lambda-1} \exp(-\theta x)$$

Where $\lambda > 0$ is known.

$$H_0: \theta = \theta_0 \quad vs \quad H_1: \theta = \theta_1 > \theta_0$$

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^\lambda}{\Gamma(\lambda)} x_i^{\lambda-1} \exp(-\theta x_i)$$

$$= \left[\frac{\theta^\lambda}{\Gamma(\lambda)}\right]^n \left[\prod_{i=1}^{n} x_i^{\lambda-1}\right] \left[\exp\left(-\theta \sum x_i\right)\right]$$

$$\Rightarrow \lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\left[\frac{\theta_0^\lambda}{\Gamma(\lambda)}\right]^n \left[\prod x_i^{\lambda-1}\right] \left[\exp\left(-\theta_0 \sum x_i\right)\right]}{\left[\frac{\theta_1^\lambda}{\Gamma(\lambda)}\right]^n \left[\prod x_i^{\lambda-1}\right] \left[\exp\left(-\theta_1 \sum x_i\right)\right]}$$

$$= A \exp\left(-\theta_0 \sum x_i + \theta_1 \sum x_i\right)$$

$$= A \exp\left((\theta_1 - \theta_0) \sum x_i\right)$$

$$\Rightarrow \quad C = \left\{ A \exp\left((\theta_1 - \theta_0) \sum x_i \le k \right\}\right.$$

$$= \left\{ \exp\left((\theta_1 - \theta_0) \sum x_i \le k^* \right\}\right.$$

$$= \left\{ \underbrace{(\theta_1 - \theta_0)}_{\text{positive}} \sum x_i \le k^{**} \right\}$$

$$= \left\{ \sum x_i \le k^{***} \right\}$$

$$P\left(\sum x_i \le k^{***}\right) = \alpha$$

b.) When $n = 1$ we have:

$$P(X \le k^{***}) = \alpha$$

Under H_0 $X \sim \text{Gamma}(\lambda, \theta_0)$

$\Rightarrow$ when $\lambda = 1 \Rightarrow X \sim \text{Exponential}(\theta_0)$

$$P(X < k^{***}) = \alpha$$

$$\int_0^{k^{***}} \theta_0 \exp(-\theta_0 x)\, dx = \alpha$$

$$\Rightarrow \quad 1 - \exp(-k^{**}\theta_0] = \alpha$$

$$k^{***} = -\log(1-\alpha)/\theta_0$$

$$\Rightarrow \text{Power} = P(X < -\log(1-\alpha)/\theta_0 \mid \theta = \theta_1)$$

$$\Rightarrow \int_0^{-\log(1-\alpha)/\theta_0} \theta_1 \exp(-\theta_1 x)\, dx$$

$$= 1 - \exp\left(-\left(-\log(1-\alpha)\right)\theta_1/\theta_0\right)$$

$$= 1 - (1-\alpha)^{\theta_1/\theta_0}$$

Note: $a^b = \exp(b \log(a))$.