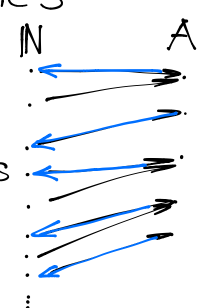
# Lecture 5

## Recall

- $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are ctble
- $A \times B$ is ctble (if $A, B$ are countable)
- $A \cup B$ is ctble (if $A, B$ are ctble)
- $\bigcup_{n \in \mathbb{N}} A_n$ is ctble (if $A_n$ is ctble, for all $n$)
- Infinite subsets of ctble sets are ctble

Thm: For an infinite set $A$, TFAE
1. $A$ is ctble
2. There is an injection $f: A \to \mathbb{N}$
3. There is a surjective $g: \mathbb{N} \to A$

pic 3




take injection from $A$ back to $\mathbb{N}$

so that countable

Def'n: Let $A, B$ be sets. We say $|A| \leq |B|$ if there is an injection $f: A \to B$

Facts: If $|A| \leq |B|$ & $|B| \leq |C|$, then $|A| \leq |C|$
proof by composition of injections is injection.

Thm (Cantor – Bernstein): If $|A| \leq |B|$ & $|B| \leq |A|$, then $|A| = |B|$

Def'n: A set $X$ is uncountable if it not at most countable, i.e. it is not finite & not countably infinite,
i.e. There's no surjection $f: \mathbb{N} \to X$
i.e. You can't list out the elements of $X$.

Thm (Cantor's ~1880)
The interval $(0,1) \subseteq \mathbb{R}$ is unctble.

Pf: Suppose for the sake of contradiction that $f: \mathbb{N} \to (0,1)$ is such a surjection.

$f(1)$ $\quad 0.\boxed{x_{1,1}}\, x_{1,2}\, x_{1,3}\, x_{1,4} \cdots$ (decimal expansion)
$f(2)$ $\quad 0.x_{2,1}\,\boxed{x_{2,2}}\, x_{2,3}\, x_{2,4} \cdots$
$f(3)$ $\quad 0.x_{3,1}\, x_{3,2}\,\boxed{x_{3,3}}\, x_{3,4} \cdots$
$f(4)$ $\quad 0.x_{4,1}\, x_{4,2}\, x_{4,3}\,\boxed{x_{4,4}} \cdots$
$\vdots$

"The Diagonalization Argument"
(combine "countably many" with sth. new)

Let's find a number $y \in (0,1)$ not on this list
Pick $y_1$ a number in $\{0, \cdots, 9\}$ different from $x_{1,1}$
Pick $y_2$ a number in $\{0, \cdots, 9\}$ diff from $x_{2,2}$

In general, pick $y_n$ to be different from $x_{n,n}$

$y = 0.y_1 y_2 y_3 \cdots$
Q: Is $y$ on our list?
No: b/c the differs from $f(i)$ in the $i$th digit. So $f$ is not a surjection. ∎

Corollary: $\mathbb{R}$ is uncountable. (as $(0,1) \subseteq \mathbb{R}$)

$\mathbb{R}^2, \mathbb{R}^3, \dots, \mathbb{R}^n$ are uncountable

Temptation: $X$ is uncountable iff $|X| = |\mathbb{R}|$ ?

So, we are going to find a "bigger" size of uncountability.

Recall: For $X$ a set, $P(X) = \{A : A \subseteq X\}$

Fact: $|X| < |P(X)|$

Pf: (Think about $X = \mathbb{N}$)

Let $f : X \longrightarrow P(X)$ be a function

(We will show that it is <u>not</u> a surjection)

(Russel's Paradox)

Let $R = \{a \in X : a \notin f(a)\} \subseteq X$

Q: Is there an $r \in X$ s.t. $f(r) = R$ ?

Suppose yes, i.e. suppose $f(r) = R$

Q: Is $r \in f(r)$ ?

If Yes: Then $r \in f(r) = R$, so it satisfies the prop of $R$, ie. $r \notin f(r)$. $\Rightarrow\!\Leftarrow$

If No: Then $r \notin f(r)$. So $r \in R = f(r)$, so $r \in f(r)$ $\Rightarrow\!\Leftarrow$

We get a contradiction either way.

So there is no $R$ s.t. $f(r) = R$, i.e. $f$ is not a surjection. $\blacksquare$

def'n: Let $(X, \mathcal{T})$ be a top space

- If $X$ has a countable dense set $D \subseteq X$, then we say $X$ is <u>separable</u>
- If $X$ has a countable basis $\mathcal{B}$, the $X$ is <u>second-countable</u>

$\downarrow$ space is not very wide

$\downarrow$ You don't need a lot of information to describe an open set.

e.g. 1 $\mathbb{R}_{usual}$ is second countable, as $\{(a,b) : a, b \in \mathbb{Q}\}$ is a basis and $\mathbb{Q} \times \mathbb{Q}$ is ctble.

e.g. 2 $\mathbb{R}_{discrete}$ is <u>not</u> 2nd ctble

Proof: Let $\mathcal{B}$ be a basis for $\mathbb{R}_{discrete}$.

For each $x \in \mathbb{R}$, take $U_x = \{x\}$. So there is $B_x \in \mathcal{B}$ s.t. $x \in B_x \subseteq \{x\}$

So $B_x$ is just $\{x\}$.

so this says $\{\{x\} : x \in \mathbb{R}\} \subseteq \mathcal{B}$

This unctble so $\mathcal{B}$ is unctble