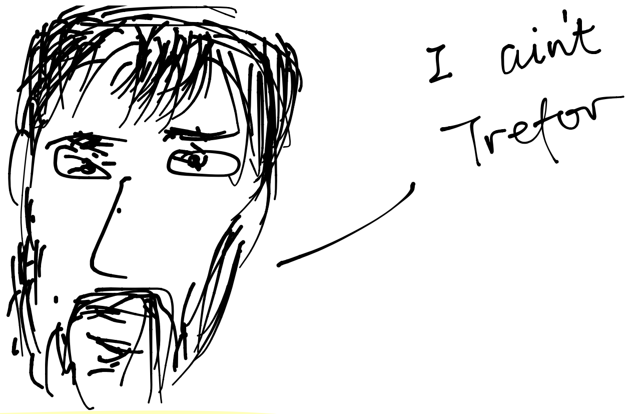
May 28th



Thm: If S is connected then $\bar{S}$ is connected.

Proof: Contrapositive i.e. Assume $\bar{S}$ is disconnected $\Rightarrow \exists$ a disconnection (T_1, T_2)

$S_1 = T_1 \cap S$

$S_2 = T_2 \cap S$

check: a) $S_1 \cup S_2 = S$

b). $\bar{S_1} \cap S_2 = \emptyset$, $S_1 \cap \bar{S_2} = \emptyset$

b/c $\bar{T_1} \cap T_2 = \emptyset$, $T_1 \cap \bar{T_2} = \emptyset$

c). Assume WLOG $S_1 = \emptyset$

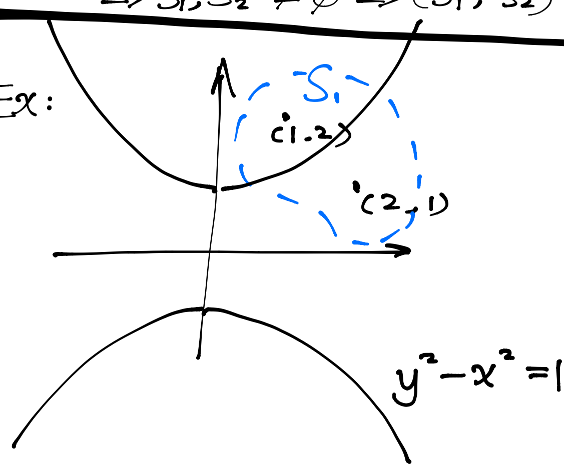
$S \subset S_2 \subset T_2 \Rightarrow \bar{S} \subset \bar{T_2}$

$T_1 = T_1 \cap \bar{S} = T_1 \cap \bar{T_2} = \emptyset$ as T_1, T_2 are a disconnection.

contradiction as $T_1 \neq \emptyset$.

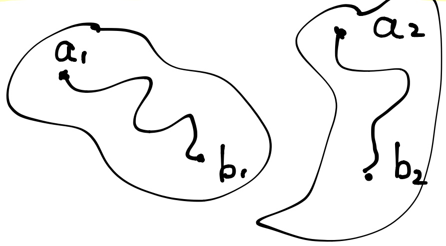
$\Rightarrow S_1, S_2 \neq \emptyset \Rightarrow (S_1, S_2)$ is a disconnection on S. ∎

Ex:



prove that a connected set S containing $(1,2)$ and $(2,1)$
Then $\exists$ a point on hyperbola intersecting S. Take $f(x,y) = y^2 - x^2 - 1$
(zeros of f are points on hyperbola)
f is connected, an interval
$f(1,2) = 2$, $f(2,1) = -4$
so $-4 < 0 < 2$
By IVT. $\exists (c_1, c_2)$ s.t. $f(c_1, c_2) = 0$. ∎

"PATH CONNECTEDNESS"



Def: A **path** is a function, $f:[0,1] \to \mathbb{R}^n$ that is continuous.

Def: A set $S \subseteq \mathbb{R}^n$ is $\underbrace{\text{pathwise}}_{\text{arcwise}}$ **connected** if $\forall a,b \in S$

$\exists\ f:[0,1] \in S$ such that $f(0)=a$, $f(1)=b$

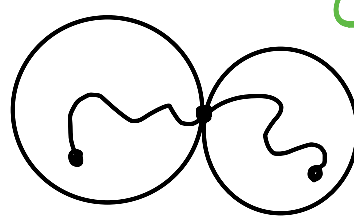
Suppose f is a path $a \to b$, g is a path $b \to c$, form $h: a \to c$

$$h(t): \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$

Composition Trick

* will be used later.

$f(2 \cdot \frac{1}{2}) = f(1) = b$
$f(2 \cdot \frac{1}{2} - 1) = g(0) = b$

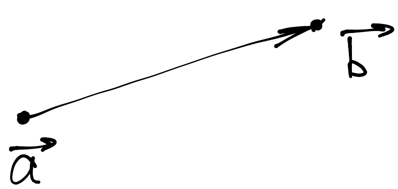


– So if f is a path from a to b then $g(t) = f(1-t)$ is a path from b to a.

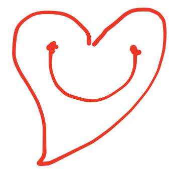
– Suppose $f:[a,b] \to S$, f cont.

$g(t) = f(a + t(b-a))$ gives us a corresponding path.

Note: $\vec{a} + t(\vec{b} - \vec{a})$ is the "straight path".



Claim: $\mathbb{R}^n$ is path connected



Choose $\vec{a}, \vec{b} \in \mathbb{R}^n$

$f(t) = \vec{a} + t(\vec{b} - \vec{a})$

- is continuous
- $f(t) \in \mathbb{R}^n \; \forall t \in [0,1]$
- $f(0) = a, f(1) = b$ $\quad \Rightarrow f(t)$ is a path

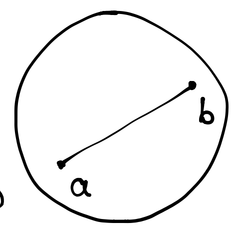
$\Rightarrow \mathbb{R}^n$ is path connected $\blacksquare$

Claim: $B(r,x)$ is connected.

$$|x - f(t)| = \left| tx + (1-t)x - (a + t(b-a)) \right|$$

$$= |(1-t)(x-a) + t(x-b)|$$

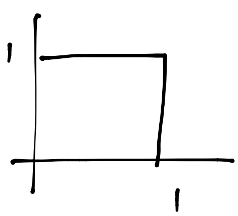
$$< (1-t)r + t(r) = r \Rightarrow f(t) \in B(r,x) \quad \forall t \in [0,1]$$



* show this guy inside the ball.

$\Rightarrow B(r,x)$ is path connected.

Exercise:

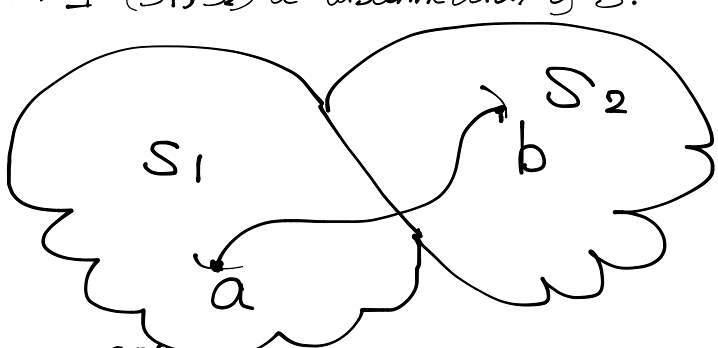


prove closed square is path connected.

Theorem: If S is path connected, it is connected.

Proof: by contradiction. Assume is path connected but not connected.

$\Rightarrow \exists \; (S_1, S_2)$ a disconnection of S.



As f is $\overset{\text{cont.}}{\text{connected}}$, so is $f([0,1]) = V$ is connected.

a). $V = \underbrace{(V \cap S_1)}_{V_1} \cup \underbrace{(V \cap S_2)}_{V_2}$

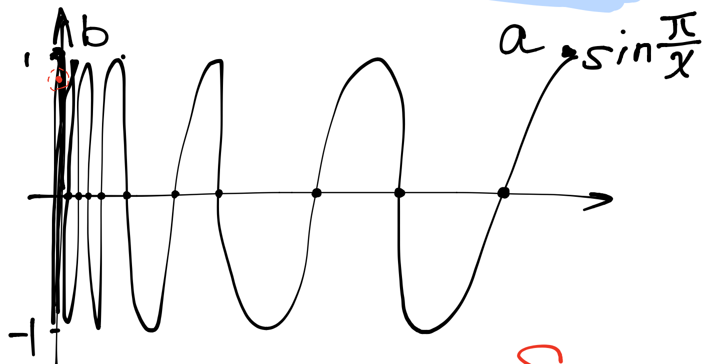
b). $\overline{V_1}\cap V_2=\emptyset=V_1\cap\overline{V_2}$ as $\overline{S_1}\cap S_2=\emptyset=S_1\cap\overline{S_2}$

And, $f(0)=a, f(1)=b \Rightarrow a,b\in V$ by choosing above that $a\in S_1$, $b\in S_2$
$\Rightarrow a\in V_1$, $b\in V_2$ so $V_1, V_2\neq\emptyset$

so (V_1, V_2) is a disconnection of V. Contradiction as $V=f(\text{connected})$ is connected. ■

Connectedness doesn't necessarily imply path connectedness

Closed Topologist's sin-curve



$S=\underbrace{\{(x,y)\in\mathbb{R}^2 \mid y=\sin\frac{\pi}{x},\ 0<x\le 2\}}_{S_1}\cup\underbrace{\{(0,y)\in\mathbb{R}^2\mid y\in[-1,1]\}}_{S_2}$

i.e. graph of $y=\sin\frac{\pi}{x}$ $\cup$ y-axis b/w -1 & 1

for $0\le x\le 2$, $\sin\frac{\pi}{x}$ provides our path for any $a,b\in S$.

$[a,b]\to S$ gives a path between a, b $\Rightarrow$ S_1 is connected
$\uparrow$
$[0,1]$ — Now $\overline{S_1}$ is going to be S, for $(0,y)\in S$

$B(\varepsilon,(0,y))\cap S_1\neq\emptyset\ \forall\varepsilon>0$

$\Rightarrow S$ is connected by previous Thm.

Show S is not path connected.

Assume it is. i.e. assume $\exists$ a path from a to b
$\Rightarrow f_1$ is continuous
f_2 is continuous as f_1 starts at 2 and goes to 0
$\Rightarrow$ by IVT attains all value in the middle, in particular $\frac{1}{2k}$

$\Rightarrow\exists$ a sequence $\{t_k\}$ s.t. $f(t_k)=(\frac{1}{2k},0)$ as $[0,2]$ is compact
BW $\Rightarrow\exists\{t_{k_j}\}$ convergent (subsequence) to t_0

B/w each t_k and t_j, achieve all values between -1 & 1
However, for $B(\varepsilon, t_0)$ get under f values widely apart.
Contradiction (continuity).
$\Rightarrow S$ is not path connected but it's connected. ∎

path connected is "stronger"

Thm: if $S \subset \mathbb{R}^n$ is open and connected $\Rightarrow$ path connected.

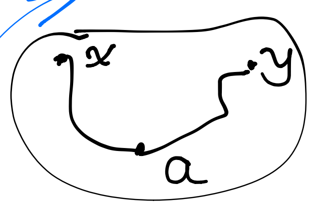
Proof: (directly)

Let $a \in S$. $S_1 = \{x \in S \mid \exists$ path between a and $x\}$
$S_2 = \{x \in S \mid x \notin S_1\}$

a). $S_1 \cup S_2 = S$ ✓ (trivially)
b). $S_1 \neq \emptyset$, $a \in S_1$ via $p = [0,1] \to \{a\}$ ✓ (trivial path)
c). $\overline{S_1} \cap S_2 = \emptyset$
d). $S_1 \cap \overline{S_2} = \emptyset$
e). $S_2 \neq \emptyset$

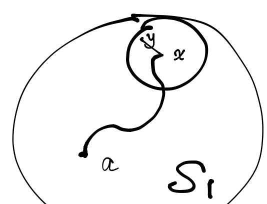
we'll prove (c) ✓ true
(d) ✓ true
$\Rightarrow$ (e) ✗ false $\Rightarrow S_2 = \emptyset$

$\Rightarrow S_1 = S \Rightarrow \forall x, y\ \exists$ paths s.t.
Use the "composition trick"
so composition $x \to a$ then inverse
$a \to y$ gives $x \to y$



$\Rightarrow$ path connected

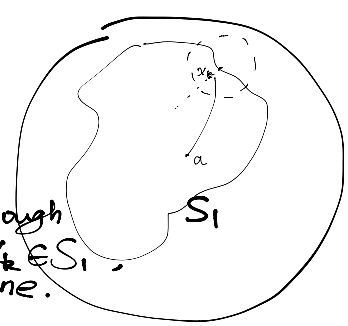
To be specific (proof of (d))
d).
Let $x \in S_1 \Rightarrow \exists$ a path from $a \to x$.
as S is open $\Rightarrow \exists B(r, x) \subset S$
as $B(r,x)$ is path connected $\Rightarrow \exists$ a path $x \to y$
$\Rightarrow$ the composition $\Rightarrow \exists$ a path from $a \to y$
$\Rightarrow B(r,x) \subset S_1 \Rightarrow S_1 \cap \overline{S_2} = \emptyset$



(proof of (c))

Let $x \in \overline{S_1} \Rightarrow \exists \{x_k\} \in S_1$
$x_k \to x$

so for our $B(r,x)$, for k large enough
$x_k \in B(r,x) \Rightarrow \exists$ a path $a \to x_k$ as $x_k \in S_1$,
$x_k \to x$ as in a ball so use straight line.
$\Rightarrow x \in S_1 \Rightarrow x \notin S_2$.



Cut off everything including statement but not the proof.

---

§ 1.8 Uniform Continuity

Recall "normal continuity" for $f: S \subset \mathbb{R}^n \to \mathbb{R}^k$ at all $x \in S$ has
$\forall \varepsilon > 0, \forall x \in S \ \exists \delta > 0$ s.t. $\forall y \in S$ when $|x-y| < \delta$, $|f(x)-f(y)| < \varepsilon$

Ex: $f(x) = x$, $|f(x)-f(y)| = |x-y|$
choose $\delta = \varepsilon$ then $|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$.

Note that δ works for all x, y.

Ex: $f(x) = x^2$, choose $a, a+h$, $a>0, h>0$

$|f(a+h)-f(a)| = |(a+h)^2 - a^2| = 2ah + h^2 < \varepsilon$

$|a+h-a| = |h| < \delta$ $\boxed{\text{"}\delta = \frac{\varepsilon}{2a}\text{" sth. like this!}}$ IDEA

$\boxed{\delta = \min(1, \frac{\varepsilon}{2a+1})}$?

can have δ as a function of a

THIS IS INCOMPLETE

Def: Uniform Continuity has $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x, y \in S$ then
$|x-y| < \delta \Rightarrow |f(x)-f(y)| < \varepsilon$

The bridge between continuity & uniform continuity is compactness.

Thm: Sps S is compact, and f is a continuous function on S, then f is uniformly continuous.

Proof: Suppose not uniformly continuous
$\exists \varepsilon > 0, \forall \delta > 0, \exists x, y \in S$ s.t. if $|x-y| < \delta$, then $|f(x)-f(y)| \geq \varepsilon$.

Let $\delta_k = \frac{1}{k}$, then we get $\{x_k\}$ and $\{y_k\}$ s.t. $|x_k - y_k| < \delta_k = \frac{1}{k}$

$\Rightarrow |f(x_k) - f(y_k)| \geq \varepsilon$.

As S is compact, then $\exists \{x_{k_j}\}$ so $x_{k_j} \longrightarrow a \in S$ by BW

$|y_{k_j} - a| = |y_{k_j} - x_{k_j} + x_{k_j} - a| \leq |y_{k_j} - x_{k_j}| + |x_{k_j} - a| \longrightarrow 0$ since $| \quad | \to 0$, $| \quad | \to 0$

$\Rightarrow y_{k_j} \longrightarrow a$

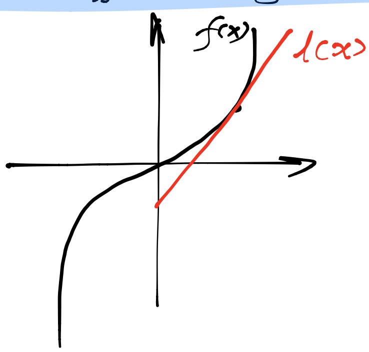
So $|f(x_{k_j}) - f(y_{k_j})| \longrightarrow |f(a) - f(a)| = 0$ contradiction

$\Rightarrow$ Assumption false $\Rightarrow f$ is uniformly continuously. ∎

— respects linearity.

CHAPTER 2
Differential Calculus

§2.1 Differentiability in one variable



Geometrically the tangent line at a point.

$l(x) = mx + b$

Analytically, m is the slope

$f(a) = l(a) = ma + b \Rightarrow b = f(a) - ma$

$\Rightarrow l(x) = m(x-a) + f(a)$

$\lim \frac{f(x) - l(x)}{x-a} = 0$ So $f(x)$ approximates $l(x)$ faster then x approaches a.

$h = x - a$

$f(x) - f(l) = f(a+h) - l(a+h) = f(a+h) - f(a) - mh$

$= E(h)$ "Error"

$\Rightarrow f(a+h) = \underbrace{f(a) + mh}_{\text{linear}} + \underbrace{E(h)}_{\text{error}}$

Def: f is **differentiable** at a if $\exists$ m s.t. above is true and $\lim_{h \to 0} \frac{E(h)}{h} \to 0$

denoted: $m = f'(a)$

So if cannot find a derivitive $\Rightarrow$ can get a linear approximation.