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Q1. Definition:

(1) • $\boxed{\text{Strictly}}$: For a stochastic process, its joint distribution is exactly the same as that of the shift version of it.

In formula, we can write:

$$P(X_t \le c_0, X_{t+1} \le c_1, \cdots X_{t+k} \le c_k) = P(X_{t+h} \le c_0, X_{t+1+h} \le c_1, \cdots, X_{t+k+h} \le c_k)$$

• $\boxed{\text{Weakly}}$: A process $\{X_t\}$ is weakly stationary if its mean μ and variance σ^2 are finite and constant, both are independent of time t, and the autocovariance functions are uncorrelated with each other.

(2) ① Plot the graph and decide the preliminary model of the given process.

② Calculate each parameters in such model

③ Check the adequacy of our model. If it's adequate, then we are done; if not, we need to go back to step ①.

(3) a. ① Remove the time trend by differencing.

② conduct a ~~[illegible]~~ inference test with intervening terms

b. We can replace AR(1) process with an ARMA(p,q) or AR(p) process, and thus we can remove the ~~time~~ serial term.

Q2. Solution: $(1-0.6B)(X_t-3)=a_t$, $a_t \sim NID(0,1)$

$X_t - 0.6X_{t-1} + 1.8 - 3 = a_t$

$X_t - 0.6X_{t-1} = a_t + 1.2$

$\cancel{X_t} =$ ~~$X_1 = a_1 + 1.2 + 0.6X_0 = a_1 + 2.2$~~

~~$X_2 = a_2 + 1.2 + 0.6X_1 = a_2 + 1.2 + 0.6a_1 + 1.32 = 0.6a_1 + a_2 + 2.52$~~

$X_3 =$ ~~$a_3 + 1.2 + 0.6X_2 = a_3 + 1.2 + 0.36a_1 + 0.6a_2 + 1.512$~~

~~$= 0.36a_1 + 0.6a_2 + a_3 + 2.712$~~

~~$Var(X_1+X_2+X_3) = \underbrace{E[(X_1+X_2+X_3)^2]}_{①} - \underbrace{E[X_1+X_2+X_3]E[X_1+X_2+X_3]}_{②}$~~

② ~~$E[X_1+X_2+X_3]^2 = E[1.96a_1 + 1.6a_2 + a_3 + 7.432]^2 = 7.432^2 = 55.234624$~~

① ~~$E[X_1^2+X_2^2+X_3^2+2X_1X_2+2X_2X_3+2X_3X_1] = E[X_1^2]+E[X_2^2]+E[X_3^2]+0+0+0$~~

~~$= (1+2.2^2)+(0.6^2+1^2+2.52^2)+(0.36^2+0.6^2+1^2+2.712^2)$~~

$=$

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$a_1 + 1.2$

$= 0.6X_1 + a_2 + 1.2 = 0.6a_1 + a_2 + 1.92$

$_3 = 0.6X_2 + a_3 + 1.2 = 0.36a_1 + 0.6a_2 + a_3 + 2.352$

$(X_1+X_2+X_3) = E[(X_1+X_2+X_3)^2] - E[X_1+X_2+X_3]E[X_1+X_2+X_3]$

$= E[X_1^2]+E[X_2^2]+E[X_3^2]+2E[X_1X_2]+2E[X_2X_3]+2E[X_1X_3]$

$- E[a_1+1.2+0.6a_1+a_2+1.92+0.36a_1+0.6a_2+a_3+2.352]^2$

$= (1^2+1.2^2) + (0.6^2+1^2+1.92^2) + (0.36^2+0.6^2+1^2+2.352^2) + 2(0.6+1.2\times1.92)$

$+2(0.6\times0.36+0.6+1.92\times2.352) + 2(0.36+1.2\times2.352)$

$-(1.2+1.92+2.352)^2$

$= 14.507904 + 5.808 + 10.66368 + 6.3648 - 29.942784$

$= 7.4016$

$\cancel{Var(X_1+X_2+X_3)}$

$Var(X_1+X_2+X_3)$

$= Var(X_1)+Var(X_2)+Var(X_3)+2Cov(X_1,X_2)$

$+2Cov(X_1,X_3)+2Cov(X_2,X_3)$

$= \gamma(0)+\gamma(0)+\gamma(0)+2\gamma(1)$

$+2\gamma(2)+2\gamma(1)$

For AR(1),

$= 3\gamma(0)+4\gamma(1)+2\gamma(2)$

$= 3\gamma(0)+4(0.6)\gamma(0)$

$+2(0.6)^2\gamma(0)$

$= 6.12\gamma(0)$

$= 6.12 \times \frac{1}{0.64}$

$= 9.5625$

Q 3. Solution:

(1). $\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1$

$\cancel{\rho(0) = \phi_1\rho(\ldots) + \phi_2\rho(1-2)}$

$\cancel{\rho(0) = 1 = \phi_1\rho(1)+}$

$X_t = 1.2X_{t-1} - 0.36X_{t-2} + a_t$

$X_t^2 = 1.2X_{t-1}X_t - 0.36X_{t-2}X_t + a_tX_t$ ①

$E(X_t^2) = Var(X_t) = \gamma(0) = 1.2\gamma(1) - 0.36\gamma(2) + 1$

$X_tX_{t-1} = 1.2X_{t-1}^2 - 0.36X_{t-2}X_{t-1} + a_tX_{t-1}$

$E(X_tX_{t-1})$ $\gamma(1) = \gamma(0)\times1.2 - 0.36\gamma(1) + 0$ ②

$X_tX_{t-2} = 1.2X_{t-1}X_{t-2} - 0.36X_{t-2}^2 + a_tX_{t-2}$

$E(X_tX_{t-2}) = \gamma(2) = 1.2\gamma(1) - 0.36\gamma(0) + 0$ ③

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$\gamma(0) = 1.2\gamma(1) - 0.36\gamma(2) + 1$ ①

$\gamma(1) = \gamma(0)\times1.2 - 0.36\gamma(1)$ ②

$\gamma(2) = 1.2\gamma(1) - 0.36\gamma(0)$ ③

so $\begin{cases} \gamma(0) \doteq 5.19 \\ \gamma(1) \doteq 4.58 \\ \gamma(2) \doteq 3.63 \end{cases}$

so $\rho(1) = \frac{\gamma(1)}{\gamma(0)} = 0.88$

$\rho(2) = \frac{\gamma(2)}{\gamma(0)} \doteq 0.70$

3

$b = 1.2a$

$a = \frac{17}{15}b$

$1 = 0.64(a-c)$

$a-c = \frac{25}{16}$

$\frac{17}{15}b - \frac{99}{125}b = \frac{25}{16}$

$b =$

$c = 1.2b - 0.36a$

$= 1.2b - 0.36\cdot\frac{17}{15}b$

$= \frac{99}{125}b$

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$\rho_1 = 0.88$

$$\phi_{22} = \frac{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & \rho_2\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1\\ \rho_1 & 1\end{pmatrix}} = \frac{\rho_2-\rho_1^2}{1-\rho_1^2} = \frac{0.70-0.88^2}{1-0.\cancel{7}^2} = \cancel{-0.15} \quad -0.33$$

$1-0.88^2$

$$\phi_{33} = \frac{\det\begin{pmatrix}1 & \rho_1 & \rho_1\\ \rho_1 & 1 & \rho_2\\ \rho_2 & \rho_1 & \rho_3\end{pmatrix}}{\det\begin{pmatrix}1 & \rho_1 & \rho_2\\ \rho_1 & 1 & \rho_1\\ \rho_2 & \rho_1 & 1\end{pmatrix}} = 0$$

(3). Check stationarity

Write $X_t - 1.2BX_t + 0.36B^2X_t = a_t$

$(1-1.2B+0.36B^2) = 0$

$B = \frac{1.2 \pm \sqrt{1.2^2 - 4 \times 0.36}}{0.72} = \frac{5}{3} > 1$

so it's stationary.

$X_t = \psi(B)a_t$

$\phi(B)X_t = \theta(B)a_t$

$\Rightarrow \psi(B) = \frac{\theta(B)}{\phi(B)}$

$\psi(B)\phi(B) = \theta(B)$ where $\phi(B) = 1-1.2B+0.36B^2$, $\theta(B) = 1$

$(1+\psi_1B+\psi_2B^2+\cdots)(1-1.2B+0.36B^2) = 1$

B^0: 1

B^1: $\psi_1 - 1.2 = 0 \Rightarrow \psi_1 = 1.2$

B^2: $\psi_2 + 0.36 - 1.2\psi_1 = 0 \Rightarrow \psi_2 = 1.2^2 - 0.36 = 1.08$

B^3: $\psi_3 - 1.2\psi_2 + 0.36\psi_1 = 0 \Rightarrow \psi_3 = 1.2\times1.08 - 0.36\times1.2 = 0.864$

B^4: $\psi_4 - 1.2\psi_3 + 0.36\psi_2 = 0 \Rightarrow \psi_4$

$= 1.2\times0.864 - 0.36\times1.08$

$= 0.648.$

So $\{\psi_1, \psi_2, \psi_3, \psi_4\} = \{1.2, 1.08, 0.864, 0.648\}$

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$

$\rho_2 = \phi_1 \rho_1 + \phi_2$

$\rho_1 = \phi_1 + \phi_2 \rho_1$

$\rho_1 = \frac{\hat\gamma(1)}{\hat\gamma(0)} = \frac{1000}{2000} = 0.5$

$\rho_2 = \frac{\hat\gamma(2)}{\hat\gamma(0)} = \frac{100}{2000} \approx \hat{R}(2) = 0.05$

$\phi_{22} = \phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{\frac{100}{2000} - (\frac{1}{2})^2}{1 - \frac{1}{4}} = \frac{-4}{15}$

$\rho(0) = \frac{\hat\gamma(0)}{\hat\gamma(0)} = 1$

$\frac{1}{2} = \phi_1 + (-\frac{4}{15}) \times \frac{1}{2}$

$\Rightarrow \phi_1 = \frac{19}{30}$

$\sigma^2 = \gamma(0) - \phi_1 \gamma(1) - \phi_2 \gamma(2)$

$= 2000 - \frac{19}{30} \times 1000 + \frac{4}{15} \times 100$

$= 1393.33$

$\hat\sigma^2 = Var(X_t - \phi_1 X_{t-1} - \phi_{22} X_{t-2})$

$= \hat\gamma(0) + \phi_1 \hat\gamma(0) + \phi_{22} \hat\gamma(0)$

$- 2\phi_1 \hat\gamma(1) - 2\phi_{22} \hat\gamma(2) + 2\phi_1\phi_{22} \hat\gamma(1)$

$= (1 + \phi_1 + \phi_{22}) \hat\gamma(0) - (2\phi_1 - 2\phi_1\phi_{22}) \hat\gamma(1)$

$- 2\phi_{22} \hat\gamma(2)$

$= 2466 - 1267 + 2 \times 0.267 \times 100$

$= 1252.4$

$\phi_1 = \rho_1 = 0.5$

$\phi_{22} = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2} = \frac{0.05 - 0.5^2}{1 - 0.5^2} = \frac{-0.2}{0.75} \doteq -0.267$

Q5:

(1). $(1 - 0.5B)(X_t - 3) = X_t - 0.5X_{t-1} + 1.5 - 3 = a_t + 0.4a_{t-1}$

$X_t - 0.5X_{t-1} = a_t + 0.4a_{t-1} + 1.5$

$X_t = 0.5X_{t-1} + a_t + 0.4a_{t-1} + 1.5$

$\hat X_{t+l} = 0.5X_{t+l-1} + (X_{t+l} - \hat X_{t+l}) + 0.4(X_{t+l-1} - \hat X_{t+l-1}) + 1.5$

$2\hat X_{t+l} = 0.9X_{t+l-1} + X_{t+l} - 0.4\hat X_{t+l-1} + 1.5$

$\hat X_{t+l} = (0.9X_{t+l-1} + X_{t+l} - 0.4\hat X_{t+l-1} + 1.5) \times \frac{1}{2}$

$= \frac{9}{20} X_{t+l-1} + \frac{1}{2} X_{t+l} - \frac{1}{5} \hat X_{t+l-1} + \frac{3}{4}$

rewrite: $\hat X_t(l) = \frac{9}{20} X_{t-1}(l) + \frac{1}{2} X_t(l) - \frac{1}{5} \hat X_{t-1}(l) + \frac{3}{4}$

(2). $\hat X_{100}(1) = \frac{9}{20} X_{99}(1) + \frac{1}{2} X_{100}(1) - \frac{1}{5} \hat X_{99}(1) + \frac{3}{4} = \frac{9}{20} \times 3.5 + 0.5 \times 2.3 - \frac{1}{5} \times 3.1 + 0.75$

$= 2.855$

$\hat X_{100}(2) = \frac{9}{20} X_{99}(2) + \frac{1}{2} X_{100}(2) - \frac{1}{5} \hat X_{99}(2) + \frac{3}{4} = \frac{9}{20} \times 2.3 + 0.5 \times 3.7 - \frac{1}{5} \times 2.855 + 0.75$

$= 3.064$

$\hat X_{100}(3) = \frac{9}{20} X_{99}(3) + \frac{1}{2} X_{100}(3) - \frac{1}{5} \hat X_{99}(3) + \frac{3}{4} = \frac{9}{20} \times 3.7 + 0.5 \times 5.8 - \frac{1}{5} \times 3.064 + 0.75$

$= 4.7022$

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$(1-0.5B)X_t=(1+0.4B)a_t+1.5$

Ignore the constant.

$\Phi(B)=1-0.5B$

$\Theta(B)=1+0.4B$

$\psi(B)a_t=X_t$

$\psi(B)=\frac{\Theta(B)}{\Phi(B)}$

$(1-0.5B)(1+\psi_1B+\psi_2B^2+\psi_3B^3+\cdots)=1+0.4B$

B^0: 1

B^1: $-0.5+\psi_1=0.4$ $\quad\psi_1=0.9$

B^2: $-0.5\psi_1+\psi_2=0$ $\quad\psi_2=0.5\psi_1=0.45$

B^3: $-0.5\psi_2+\psi_3=0$ $\quad\psi_3=0.5\psi_2=0.225$

$Var(e_{100}(1))=\sigma^2(1+0.9^2)=1(1+0.81)=1.81$

$Var(e_{100}(2))=\sigma^2(1+0.9^2+0.45^2)=2.0125$

$Var(e_{100}(3))=\sigma^2(1+0.9^2+0.45^2+0.225^2)\doteq 2.0631$

(4). $X_{101}+X_{102}+X_{103}$ best linear forecast

is $\hat{X}_{100}(1)+\hat{X}_{100}(2)+\hat{X}_{100}(3)=2.855+3.064+4.7022$

$=10.6212$