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MAT246 HW6 Rui Qiu 999292509

(1). Claim: For a,b are odd integers, $\sqrt{a^2+b^2}$ is irrational.

Proof: Suppose $\sqrt{a^2+b^2}$ is rational.

Then $\sqrt{a^2+b^2} = \frac{p}{q}$, for p,q are integers, gcd(p,q)=1.

So $a^2+b^2 = \frac{p^2}{q^2}$

$(a^2+b^2)q^2 = p^2$

b [redacted] $2^2 | 2^2$ but $2^2 \nmid 2$

Then $a^2+b^2 | p^2 \Rightarrow a^2+b^2 | p$ (at least divides one of its factor)

(*) Assume $a=2m+1$, $b=2n+1$ for some integers m, n.

Then $(2m+1)^2+(2n+1)^2 = 4m^2+4m+1+4n^2+4n+1$

$= 2(2m^2+2m+2n+2n^2+1)$

Since $2(2m^2+2m+2n+2n^2+1) | p$

so $2|p$

Now suppose $p=(a^2+b^2)t$ for some integer t.

Therefore $p^2=((a^2+b^2)t)^2=(a^2+b^2)^2t^2=(a^2+b^2)q^2$

$(a^2+b^2)t^2=q^2$

Then a^2+b^2 divides at least one of the factors of q^2, i.e. q.

So $a^2+b^2 | q$

Similarly by (*) that $2|q$

Then this contradicts that gcd(p,q)=1.

Hence $\sqrt{a^2+b^2}$ is irrational. ∎

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Conclusion: For odd integers a, b, $\sqrt{a^2+b^2}$ is irrational.

(2). Claim: $\sqrt{2}+\sqrt[3]{2}$ is irrational.

Proof: Suppose that $\sqrt{2}+\sqrt[3]{2}=r$ with r a rational number.

Then $\sqrt[3]{2}=r-\sqrt{2}$. $\Rightarrow 2=(r-\sqrt{2})^3$

$(\sqrt[3]{2})^3=(r-\sqrt{2})^3$

$2=(r^2-2\sqrt{2}r+2)(r-\sqrt{2}) = r^3-3r^2\sqrt{2}+6r-2\sqrt{2}$

$= r^3-2\sqrt{2}r^2+2r-\sqrt{2}r^2+4-2\sqrt{2}r$

$= r^3-3\sqrt{2}r^2+(2-2\sqrt{2})r+4 = r^3+6r-\sqrt{2}(3r^2+2)$

Then $3\sqrt{2}r^2+(2\sqrt{2}-2)r=r^3+2$ $\Rightarrow \sqrt{2}=\dfrac{2-(r^3+6r)}{3r^2+2}$

$3\sqrt{2}r+2\sqrt{2}-2=\dfrac{r^3+2}{r}$

$3\sqrt{2}r+2\sqrt{2}=\dfrac{r^3+2r+2}{r}$ which is still rational.

Suppose again $\dfrac{r^3+2r+2}{r}=m$, m is ~~integer~~ rational.

$2\sqrt{2}=m-3\sqrt{2}r$

$(2\sqrt{2})^2=(m-3\sqrt{2}r)^2$

$8=m^2-6\sqrt{2}rm+18r^2$

$6\sqrt{2}rm=m^2+18r^2-8$

$\sqrt{2}=\dfrac{m^2+18r^2-8}{6rm}$

You do not need to introduce this step with m.

See above

Since m, r are ~~integer~~ rational numbers.

So $\dfrac{m^2+18r^2-8}{6rm}$ is rational.

But $\sqrt{2}$ is irrational.

~~contradicts the assumption that $\sqrt{2}+\sqrt[3]{2}$ is irrational rational.~~

So the assumption is wrong that r is not rational.

Hence $\sqrt{2}+\sqrt[3]{2}$ is irrational.

Conclusion $\sqrt{2}+\sqrt[3]{2}$ is irrational.

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(3). Claim: $\forall a<b, a\in\mathbb{R}, b\in\mathbb{R}, \exists c\in\mathbb{R}\setminus\mathbb{Q}$ s.t. $a<c<b$.

Proof (I). We need to first show $\exists r\in\mathbb{Q}$ such that $a<r<b$.

① Suppose that $a>0$.

NOT SURE WHETHER WE CAN USE THIS PROPERTY!

$\boxed{\text{By the Archimedean property that for every } x\in\mathbb{R}, \text{ there exists } n\in\mathbb{N} \text{ such that } n>x}$

then $\exists q\in\mathbb{N}$ such that $q>\frac{1}{b-a}$

Go for it. This is fine.

so that $q(b-a)>1$.

For positive real number qa,

by the Archimedean property again,

$\exists n\in\mathbb{N}$ such that $n>qa$

Then $S=\{n\in\mathbb{N}: n>qa\}$ is a non-empty set of natural numbers with at least one element p.

Now claim that $p-1\leq qa$.

Note that if $p=1$, then $p-1=1-1=0<qa$.

else $p\neq 1$, then $p-1>qa$ would contradict the definition of p.

Now we have $qa<p=(p-1)+1<qa+q(b-a)=qb$

then

$$a<\frac{p}{q}<b$$

② Now suppose that $a\leq 0$. Use Archimedean property again

$\exists k\in\mathbb{N}$ such that $k>-a$,

then $k+a>0$.

Then $\exists s\in\mathbb{Q}$ such that $a+k<s<b+k$ such that

$a<s-k<b$

Obviously $s-k=r\in\mathbb{Q}$.

(II) We now need to show $\exists\, c \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < c < b$.
By (I), $\exists\, r_1, r_2 \in \mathbb{Q}$ such that $a < r_1 < r_2 < b$.

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Then the number
$$c = r_1 + \frac{r_2 - r_1}{\sqrt{2}}$$ is irrational as desired.

check this is irrational and $a < c < b$

Hence $\forall a, b \in \mathbb{R}$, $a < b$, $\exists\, c \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < c < b$. ∎

(4). ~~Solution~~:

Claim: The equation $3x^3 + 2x^2 - 5x - 2 = 0$ has no rational solutions.

Proof: Assume that the equation has a rational root $\frac{p}{q}$ for $p, q \in \mathbb{Z}$ and $\gcd(p, q) = 1$.
By the Rational Roots Theorem that
$$p \mid -2, \quad q \mid 3$$
so $p = \pm 1$ or ± 2
$q = \pm 1$ or ± 3

Thus $\frac{p}{q}$ could be $\pm 1, \pm 2, \pm \frac{1}{3}, \pm \frac{2}{3}$.

when $\frac{p}{q} = 1$, $3 + 2 - 5 - 2 \neq 0$

$\frac{p}{q} = -1$, $-3 + 2 + 5 - 2 \neq 0$

$\frac{p}{q} = 2$, $24 + 8 - 10 - 2 \neq 0$

$\frac{p}{q} = -2$, $-24 + 8 + 10 - 2 \neq 0$

$\frac{p}{q} = \frac{1}{3}$, $\frac{1}{9} + \frac{2}{9} - \frac{5}{3} - 2 \neq 0$

$\frac{p}{q} = -\frac{1}{3}$, $-\frac{1}{9} + \frac{2}{9} + \frac{5}{3} - 2 \neq 0$

$\frac{p}{q} = \frac{2}{3}$, $\frac{24}{27} + \frac{8}{9} - \frac{10}{3} - 2 \neq 0$

$\frac{p}{q} = -\frac{2}{3}$, $\frac{-24}{27} + \frac{8}{9} + \frac{10}{3} - 2 \neq 0$

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Therefore it's a contradiction.

Thus $3x^3+2x^2-5x-2=0$ doesn't have a rational root. $\blacksquare$

(5). Claim: For any $z_1, z_2 \in \mathbb{C}$ such that $z_2 \neq 0$ we have

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Proof: Suppose $z_1 = a+bi$, $z_2 = c+di$. for $a,b,c,d \in \mathbb{R}$ but $c \neq 0$ and $d \neq 0$.

Then
$$\left|\frac{z_1}{z_2}\right| = \left|\frac{a+bi}{c+di}\right| = \left|\frac{(a+bi)(c-di)}{(c+di)(c-di)}\right| = \left|\frac{(bd+ac)+(bc-ad)i}{c^2+d^2}\right|$$

$$= \sqrt{\frac{(bd+ac)^2}{(c^2+d^2)^2} + \frac{(bc-ad)^2}{(c^2+d^2)^2}}$$

$$= \frac{\sqrt{b^2d^2+a^2c^2+b^2c^2+a^2d^2}}{c^2+d^2}$$

$$= \sqrt{\frac{b^2}{c^2+d^2} + \frac{a^2}{c^2+d^2}}$$

$$= \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

And
$$\frac{|z_1|}{|z_2|} = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

Therefore $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$. $\blacksquare$

(6). Claim: For $z_1, z_2, z_3 \in \mathbb{C}$ we have $(z_1 z_2)z_3 = z_1(z_2 z_3)$.

Proof: Suppose
$$z_1 = a_1 + b_1 i$$
$$z_2 = a_2 + b_2 i$$
$$z_3 = a_3 + b_3 i$$
for $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$

Then
$$\begin{aligned}(z_1 z_2) z_3 &= ((a_1+b_1 i)(a_2+b_2 i))(a_3+b_3 i)\\ &= ((a_1a_2 - b_1b_2) + (a_2b_1 + a_1b_2)i)(a_3+b_3 i)\\ &= ((a_1a_2a_3 - a_3b_1b_2) - a_2b_1b_3 - a_1b_2b_3)\\ &\quad + (a_2a_3b_1 + a_1a_3b_2 + a_1a_2b_3 - b_1b_2b_3)i\end{aligned}$$

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As
$$\begin{aligned}z_1(z_2 z_3) &= (a_1+b_1 i)((a_2+b_2 i)(a_3+b_3 i))\\ &= (a_1+b_1 i)(a_2a_3 - b_2b_3 + (a_3b_2 + a_2b_3)i)\\ &= (a_1a_2a_3 - a_1b_2b_3 - a_3b_1b_2 - a_2b_1b_3)\\ &\quad + (a_2a_3b_1 - b_1b_2b_3 + a_1a_3b_2 + a_1a_2a_3)i\end{aligned}$$

Thus $(z_1 z_2) z_3 = z_1(z_2 z_3)$

(7). Solution:

$$\begin{aligned}z = \left(\frac{\overline{(2+3i)\cdot|1-3i|}}{1+\sqrt{2}i}\right)^2 &= \left(\frac{\overline{(2+3i)\sqrt{1+9}}}{1+\sqrt{2}i}\right)^2\\ &= \left(\frac{\overline{2\sqrt{10}+3\sqrt{10}i}}{1+\sqrt{2}i}\right)^2\\ &= \left(\frac{(2\sqrt{10}-3\sqrt{10}i)(1-\sqrt{2}i)}{(1+\sqrt{2}i)(1-\sqrt{2}i)}\right)^2\\ &= \left(\frac{2\sqrt{10}-3\sqrt{20}-3\sqrt{10}i-2\sqrt{20}i}{1+2}\right)^2\\ &= \left(\left(\frac{2}{3}\sqrt{10}-2\sqrt{5}\right)-\left(\sqrt{10}+\frac{4\sqrt{5}}{3}\right)i\right)^2\\ &=\end{aligned}$$

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$$Z = \left(\frac{\overline{(2+3i)|1-3i|}}{1+\sqrt{2}i}\right)^2 = \left(\frac{\overline{2\sqrt{10}+3\sqrt{10}i}}{1+\sqrt{2}i}\right)^2 = \frac{(2\sqrt{10}-3\sqrt{10}i)^2}{(1+\sqrt{2}i)^2}$$

$$= \frac{(4\times10-9\times10-120i)^{\cancel{2}}}{(1+2\sqrt{2}i-2)^{\cancel{2}}}$$

$$= \frac{(50+120i)^{\cancel{2}}}{(1-2\sqrt{2}i)^{\cancel{2}}}$$

$$= \frac{(50+120i)(1+2\sqrt{2}i)}{(1-2\sqrt{2}i)(1+2\sqrt{2}i)}$$

$$= \frac{50-240\sqrt{2}+(120+100\sqrt{2})i}{9}$$

$$= \frac{50-240\sqrt{2}}{9} + \frac{120+100\sqrt{2}}{9}i$$

(3). Solution: $\because Z = \dfrac{(1+3i)^{150}}{(2+2i)^{50}(3+4i)^{75}}$ and $\left|\dfrac{Z_1}{Z_2}\right| = \dfrac{|Z_1|}{|Z_2|}$, $|Z_1Z_2| = |Z_1||Z_2|$ for $Z_1, Z_2 \in \mathbb{C}$, $Z_2 \neq 0$.

$$\therefore |Z| = \frac{|1+3i|^{150}}{|2+2i|^{50}|3+4i|^{75}}$$

$$= \frac{(\sqrt{1^2+3^2})^{150}}{(\sqrt{2^2+2^2})^{50}(\sqrt{3^2+4^2})^{75}}$$

$$= \frac{(\sqrt{10})^{150}}{(2\sqrt{2})^{50}\,5^{75}}$$

$$= \frac{(\sqrt{2})^{150}\cdot(\sqrt{5})^{150}}{2^{50}\cdot(\sqrt{2})^{50}\cdot(\sqrt{5})^{150}}$$

$$= \frac{(\sqrt{2})^{100}}{(\sqrt{2})^{2\cdot50}}$$

$$= 1$$

It seems that this problem intends to test our understanding of De Moivre Theorem, but if then we have to calculate some angles like $\cos^{-1}\frac{1}{\sqrt{10}}$ etc. To simplify this, I just calculate the norm of $|Z|$ directly.