MAT 337.

中国作家协会

sketch of H:

$\limsup_{n\to\infty} \frac{|a_n|}{b_n} = \ell$.

pick $s > \ell$

$\Rightarrow \exists N_1$ s.t.

$\sup\left\{\frac{|a_n|}{b_n}, \frac{|a_{n+1}|}{b_{n+1}}, \dots\right\} < s$

(or $\limsup \geq s \neq \ell$) $\forall n \geq N_1$

$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N_1-1} |a_n| + \sum_{n=N_1}^{\infty} |a_n|$

$< \sum_{n=1}^{N_1-1} |a_n| + s \sum_{n=N_1}^{\infty} b_n$

$(\sum b_n)$ converges

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges

sketch of I:

① $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell < 1$.

$\exists\, \ell < r < 1$.

for N_1 sufficiently large

$\sup\left\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \dots\right\} < r < 1$

(otherwise $\limsup \geq r$)

$\sum_{n=0}^{\infty} a_n = \sum_{n=1}^{N_1-1} a_n + \sum_{N_1}^{\infty} a_n$

for $n \geq N_1$

$\frac{a_{n+1}}{a_n} < r \Rightarrow a_{n+1} < r \cdot a_n$

$\sum_{n}^{\infty} a_n < (a_n + r \cdot a_n + \dots)$

$= \frac{a_n}{1-r}$

$\Rightarrow$ converge

② $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = m > 1$

$\exists\, m > n > 1$ s.t.

for large N_2

$\inf\left\{\frac{a_{n+1}}{a_n}, \dots\right\} > n$

$n \geq N_2$

$\sum |a_n| = \sum_{n=1}^{N_2} |a_n| + \sum_{N_2}^{\infty} |a_n|$

$\geq \sum_{n=1}^{N_2-1} |a_n| + (|a_n| + n \cdot |a_n| + \dots)$

$n > 1$ diverges

I:

P52: D.

a) $a + a^{-1}$

sup: $+\infty$ (no max)

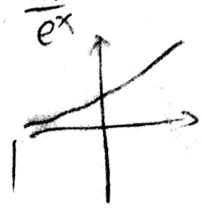
min = inf: 2 $(a = 1)$

b) $a + \frac{1}{2a}$

inf: $\sqrt{2}$ (no min)

sup: 5.1 (also max).

c) $\frac{x}{e^x}$



min: no

inf: $-\infty$.

$\left(\frac{x}{e^x}\right)' = \frac{e^x - xe^x}{(e^x)^2}$

$= \frac{1-x}{e^x}$

$(-\infty, 1)$, 1, $(1, +\infty)$

$+$ $\quad 0 \quad$ $-$

$\nearrow \qquad \searrow$

$x = 1$ $\quad \frac{1}{e}$ max

$\frac{1}{e}$: sup

P18: A, F, G.

a) 0 $\quad$ d) $\frac{1}{2}$

b) 0 $\quad$ e) $\frac{1}{2}$

c) 0 $\left(\frac{x^4}{n!} \to 0\right)$

c) $\frac{3^n}{n!} = \frac{3 \cdot 3 \cdot 3 \cdots 3 \cdot 3}{n(n-1)\cdots 3 \cdot 2 \cdot 1} = \left(\frac{3}{n}\right)\left(\frac{3}{n-1}\right) \cdots \frac{3}{3} \cdot \frac{3}{2} \cdot \frac{3}{1}$

$\frac{1}{n!} \leq \frac{3^n}{n!} \leq \frac{3}{n} \quad \to 0$

a) $\left|\frac{\sin n^2}{\sqrt{n}}\right| < 10^{-6}$ $\quad \Rightarrow n = 10^{12} \cdot \sqrt{\pi}$

$\Rightarrow \left|\frac{1}{\sqrt{n}}\right| < 10^{-6}$

$\frac{1}{\sqrt{n}} < 10^{-6}$

$\sqrt{n} > 10^{6} \Rightarrow n > 10^{12}$

b) $\frac{1}{\ln(\ln n)} < 10^{-6}$ $\quad \ln n > e^{10^6}$

$\ln(\ln n) > 10^{6}$ $\quad n > e^{e^{10^6}}$.

c) $\frac{3^n}{n!} < 10^{-6}$. $\quad n! > 3^4 \cdot 10^6$.

d) $\left|\frac{n^2+2n+1}{2n^2-n+2} - \frac{1}{2}\right| < 10^{-6}$

e) $\left|\sqrt{n^2+n} - n - \frac{1}{2}\right|$

$= \left|\frac{n}{\sqrt{n^2+n}+n} - \frac{1}{2}\right|$

F. $(a_n)_{n=1}^{\infty} = \begin{cases} n, & n \neq k^2 \\ 1, & n = k^2. \end{cases}$

or

G. $\lim_{n\to\infty} a_n = L$ $(L \neq 0)$

$\forall \varepsilon > 0\ \exists N \ni |a_n - L| < \varepsilon \quad \forall n \geq N$

take $\varepsilon = 2|L|$

$\Rightarrow |a_n - L| < 2|L|$ $\quad \exists N_1\ \forall n \geq N_1$

$L > 0$ $\quad |a_n - L| < 2L$ $\quad \Rightarrow a_n \in (L, 3L)$

$L < 0$ $\quad |a_n - L| < -2L$ $\quad \Rightarrow a_n \in (-3L, -L)$ $\quad \exists N_2\ \forall n \geq N_2$

P19: RE.2. $\lim_{x\to 0}\frac{\sin ax}{ax}=1$, $\lim_{x\to 0}\frac{1-\cos ax}{ax}=0$.

B: a) $\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{n\sin^2\frac{1}{n}}=\lim_{n\to\infty}\frac{\frac{2}{n}\cdot\tan\frac{\pi}{n}}{2\cdot\sin^2\frac{1}{n}\frac{2}{n}}$

$=\lim_{n\to\infty}\frac{\frac{2}{n}}{\sin\frac{2}{n}}\cdot\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$

$=\lim_{n\to\infty}\left(\frac{\frac{2}{n}}{\sin\frac{2}{n}}\right)\lim_{n\to\infty}\left(\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}\right)$

$=\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$

$\stackrel{*}{=}\lim_{n\to\infty}\frac{\sec^2\frac{\pi}{n}\cdot(-\frac{\pi}{n^2})}{2\cos\frac{2}{n}\cdot(-\frac{2}{n^2})}$

$n\to\infty$, $\frac{\pi}{n}\to 0$, $\cos\frac{\pi}{n}\to 1$

$\sec^2\frac{\pi}{n}\to 1$

$\stackrel{*}{=}\lim_{n\to\infty}\frac{\pi}{4}=\frac{\pi}{4}$

b) $\lim_{n\to\infty}\frac{2^{100+3n}}{e^{4n-10}}$

$=\lim_{n\to\infty}\frac{2^{100}}{e^{-10}}\cdot\frac{(3^2)^n}{e^{4n}}$

$=\lim_{n\to\infty}\frac{32^n}{e^{4n}}\cdot(2^{100}e^{10})$

$=\lim_{n\to\infty}\left(\frac{32}{e^4}\right)^n\cdot(2^{100}e^{10})$

$\frac{32}{e^4}<1\quad\to 0$.

c) $\lim_{n\to\infty}\left(\frac{\csc\frac{1}{n}}{n}+\frac{2\arctan n}{\log n}\right)$

$\lim_{n\to\infty}\frac{\csc\frac{1}{n}}{n}\stackrel{*}{=}\frac{1}{n\sin\frac{1}{n}}=1$

$\lim_{n\to\infty}n\sin\frac{1}{n}=\lim_{n\to\infty}\frac{\sin(\frac{1}{n})}{(\frac{1}{n})}$

$\lim_{x\to 0}\frac{\sin x}{x}=1$.

$\lim_{n\to\infty}\frac{2\arctan n}{\ln n}\stackrel{*}{=}\lim_{n\to\infty}\frac{\frac{2}{1+n^2}}{\frac{1}{n}}$

$=\lim_{n\to\infty}\frac{2n}{1+n^2}$

$\stackrel{*}{=}\lim_{n\to\infty}\frac{2}{2n}=0$

$\to 1$

E. $\lim_{n\to\infty}\frac{\ln(2+3^n)}{2n}$

$=\lim_{n\to\infty}\frac{\ln[3^n(1+\frac{2}{3^n})]}{2n}$

$=\lim_{n\to\infty}\frac{\ln 3^n+\ln(1+\frac{2}{3^n})}{2n}$

$\lim_{n\to\infty}\ln(1+\frac{2}{3^n})\to 0$

$=\lim_{n\to\infty}\frac{\ln 3^n}{2n}$

$=\lim_{n\to\infty}\frac{n\cdot\ln 3}{n\cdot 2}=\frac{\ln 3}{2}$

I. $\lim_{n\to\infty}\frac{a_1+a_2+\dots+a_n}{n}=L$

? with $\lim_{n\to\infty}a_n=L$

P22.

A: $(a_n)_{n=1}^{\infty}$ divergent

$\Rightarrow\forall R\in\mathbb{R}\ \exists a_n>R$

monotone increasing

$\Rightarrow R<a_n<a_{n+1}<a_{n+2}<\dots$

$\Rightarrow n\geq N.\ a_n>R$.

$\exists\varepsilon\ \forall N>0$.

$|a_n-L|\geq\varepsilon$

$a_n\geq L+\varepsilon$

$R=L+\varepsilon$

$\frac{1}{x}\in(\frac{\pi}{6}+2k\pi,\pi+2k\pi)$

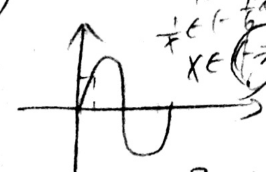
$k=-1$

$(-\frac{11}{6}\pi,-\pi)$.

$\frac{1}{x}\in(-\frac{6}{11\pi}$

$x\in(-\frac{1}{\pi}$

C:



$\frac{1}{x}\in(2k\pi,\frac{\pi}{6}+2k\pi)$

If $\frac{1}{x}\in(0,\frac{\pi}{6})$

$\frac{1}{n}\to 0$. $x\to+\infty$.

$\Rightarrow$ not bdd above

$x\in(-2\pi,-\frac{11\pi}{6})$

$\Rightarrow$ bdd below by $-\frac{6}{11\pi}(=\inf)$

F. $x_0=a$

a) $x_{n+1}=(x_n^{-1}+b)^{-1}$

$=\frac{1}{\frac{1}{x_n}+b}$

$=\frac{x_n}{1+bx_n}$

$x_{n+1}-x_n=\frac{x_n}{1+bx_n}-x_n$

$=x_n\left(\frac{1}{1+bx_n}-1\right)$

$b>0$. $x_n>0$.

$\Rightarrow 1+bx_n>1$

claim $\frac{1}{1+bx_n}<1$.

$\Rightarrow x_{n+1}-x_n<0$.

b) claim bdd below by 0

or $a\geq a_n>a_{n+1}>0$

$n=0$ $x_0=a$ $x_1=\frac{a}{1+ab}<a$

sps $n=k$ (true) i.e. $a\geq a_k>a_{k+1}>0$

$n=k+1$. $a_{k+2}=\frac{a_{k+1}}{1+ba_{k+1}}<a_{k+1}$.

since $a_{k+1}>0\Rightarrow 1+ba_{k+1}>1$

$a_{k+1}>0$, $1, b>0$

$\Rightarrow a_{k+2}>0$

$\Rightarrow$ bdd below & monotone decreasing

$\Rightarrow\inf=0$. & $L=0$.

I a) $(a_n)_{n=1}^{\infty}$ bdd

$\Rightarrow$ sup, inf exist

$b_n=\sup\{a_k: k\geq n\}$

$\Rightarrow b_n$ is a subseq. of $(a_n)_{n=1}^{\infty}$ $\Rightarrow(b_n)$ bdd

claim: b_n monotone decreasing

$\Rightarrow(b_n)$ converges to ~~limsup~~.

b) $c_n=\inf\{a_k: k\geq n\}$.

(c_n) bdd. c_n: monotone increasing

$\Rightarrow c_n$ converges to $\liminf$

c) $\ln a_n$, where $a_n\in\mathbb{Q}$

If $(a_n)_{n=1}^{\infty}$ converges to ℓ $\Rightarrow\forall\varepsilon>0\ \exists N_1\Rightarrow|a_n-\ell|<\varepsilon$ if $n\geq N_1$

$\limsup a_n=\lim_{n\to\infty}\sup\{a_k: k\geq n\}$

for $n\geq N_1$ always have $\sup\{a_k: k\geq n\}<\ell+\varepsilon$

$\Rightarrow\inf\{a_k: k\geq n\}>\ell-\varepsilon$

$\ell-\varepsilon<a_n<\ell+\varepsilon$

$b_n=\sup\{a_k: k\geq N_1\}$, $c_n=\inf\{a_k: k\geq N_1\}$.

$\Rightarrow c_{N_1}\leq c_{N_1+1}\leq\dots\leq b_{N_1+n}\leq\dots\leq b_{N_1}$

$(b_n)\to L$ $(c_n)\to M$.

Nested Value theorem guarantees that $L=M$

(length is $\frac{2\varepsilon}{2^k}$)

$\lim_{n\to\infty}\sup\{a_k: k\geq N_1\}=\ell$

$\ell-\varepsilon<\sup\{a_k: k\geq N_1\}<\ell+\varepsilon$

$\Rightarrow|\sup\{a_k: k\geq N_1\}-\ell|<\varepsilon$.

If not, say ℓ'

$\exists$ a subseq. that converges to ℓ'

a convergent seq.

$\exists\ \ell+\varepsilon>\ell$. s.t. a_{n_k}

中国作家协会

P15. D.

a). $a+\frac{1}{a}$

min: 2 also inf

max: no sup: +∞.

b). $a+\frac{1}{2a}\geq 2\sqrt{a\cdot\frac{1}{2a}}=\sqrt{2}$

$\sqrt{2}$: min & inf (no min) $a\in\mathbb{Q}$

5.1: max & sup

c) $f(x)=\frac{x}{e^x}$

$f'(x)=\frac{e^x-xe^x}{(e^x)^2}$

$=\frac{(1-x)e^x}{(e^x)^2}$

$(-\infty,1)$, 1, $(1,+\infty)$

$+$ 0 $-$

↗ ↘

max: $x=1$

$\frac{1}{e}$: sup/max

inf/min: DNE $(-\infty)$.

P18.:

G. $\lim_{n\to\infty} a_n=L$

$\forall\varepsilon>0\ \exists N$

s.t. $|a_n-L|<\varepsilon$

whenever $n\geq N$

for $\varepsilon=\frac{L}{2}$ $\exists N_1$

s.t. $|a_n-L|<\frac{L}{2}$

$\Rightarrow \frac{L}{2}<a_n<\frac{3L}{2}$

for $\forall n\geq N_1$.

P19.

B. $\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{n\sin^2\frac{2}{n}}=\lim_{n\to\infty}\frac{\frac{2}{n}\tan\frac{\pi}{n}}{2\cdot\sin^2\frac{2}{n}}$

$=\lim_{n\to\infty}\frac{\frac{2}{n}}{\sin\frac{2}{n}}\cdot\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$

$=\left(\lim_{n\to\infty}\frac{\frac{2}{n}}{\sin\frac{2}{n}}\right)\cdot\left(\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}\right)$

$=\lim_{n\to\infty}\frac{\tan\frac{\pi}{n}}{2\sin\frac{2}{n}}$

$\overset{H}{=}\lim_{n\to\infty}\frac{\sec^2(\frac{\pi}{n})\cdot(-\frac{\pi}{n^2})}{2\cos\frac{2}{n}\cdot(-\frac{2}{n^2})}$

$=\lim_{n\to\infty}\frac{\pi\sec^2(\frac{\pi}{n})}{4\cos\frac{2}{n}}$

$n\to\infty$, $\frac{2}{n}\to 0$, $\frac{\pi}{n}\to 0$

$\cos\frac{2}{n}\to 1$. $\sec\frac{\pi}{n}\to 1$

$\to\frac{\pi}{4}$

b) $\lim_{n\to\infty}\frac{2^{\log 3^n}}{e^{4n}\cdot 10}$

$=\lim_{n\to\infty}\frac{(3^2)^n}{(e^4)^n}$. $e^4>9$

$\to 0$.

c) $\lim_{n\to\infty}\frac{\csc\frac{1}{n}}{n}+\frac{2\arctan n}{\log n}$

$\lim_{n\to\infty}\frac{\arctan n}{\log n}\to 0$.

Since $\arctan n\to\frac{\pi}{2}$

$\log n\to\infty$ as $n\to\infty$.

$\csc\frac{1}{n}=\frac{1}{\sin\frac{1}{n}}$

$=\lim_{n\to\infty}\frac{1}{n\sin\frac{1}{n}}$

$=\lim_{n\to\infty}\frac{1}{\frac{\sin\frac{1}{n}}{\frac{1}{n}}}$

$=1$.

E. $\log(2+3^n)$

$=\log 3^n+\log\frac{2+3^n}{3^n}$

$=n\log 3+\log\frac{2+3^n}{3^n}$

$\log\frac{2+3^n}{3^n}=\log(1+\frac{2}{3^n})\to 0$.

as $n\to\infty$

$\Rightarrow\frac{\log 3}{2}$

I. $\lim_{n\to\infty}a_n=L$

$\Rightarrow\forall\varepsilon>0\ \exists N_0$

s.t. $|a_n-L|<\frac{\varepsilon}{2}$

whenever $n\geq N_0$

$\left|\frac{a_1+\dots+a_n}{n}-L\right|$

$=\left|\frac{a_1+\dots+a_n-nL}{n}\right|$

$\leq\frac{|a_1+\dots+a_{N_0}-N_0L|}{n}+\frac{|a_{N_0+1}-L|}{n}+\dots+\frac{|a_n-L|}{n}$

$<\frac{M}{n}+\frac{(n-N_0)\cdot\frac{\varepsilon}{2}}{n}$

P24. A. sps not

sps $\exists R$ s.t. $\exists N$

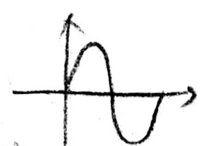
$a_n\leq R$ for all $n\geq N$

(a_n) monotone increasing

$\Rightarrow(a_n)$ is bdd by (a_1, R)

$\Rightarrow a_n$ converges

C. $0<\sin\frac{1}{x}<\frac{1}{2}$.



$\frac{1}{x}\in(2k\pi,\frac{\pi}{6}+2k\pi)$

$\cup(\frac{5\pi}{6}+2k\pi,\pi+2k\pi)$

clearly ~~not bdd above~~

consider $\frac{1}{x}\in(0,\frac{\pi}{6})$

~~gives~~ us. $x>\frac{6}{\pi}$.

$k=-1$ $\frac{1}{x}\in(-2\pi,-\frac{11\pi}{6})$

$\cup(-\frac{7\pi}{6},-\pi)$

$x\in(-\frac{6}{11\pi},-\frac{1}{2\pi})$

$\cup(-\frac{1}{\pi},-\frac{6}{7\pi})$

$-\frac{1}{\pi}$ $-\frac{6}{7\pi}$ 0

bdd below by $-\frac{1}{\pi}$

F. a) $0<x_k\leq a$

$0<x_1=\frac{1}{\frac{1}{a}+b}=\frac{a}{ab+1}<a$

since a,b positive

② sps $x_k>x_{k+1}>0$.

③ $n=k+1$

$x_{k+2}=\frac{1}{\frac{1}{x_{k+1}}+b}=\frac{x_{k+1}}{bx_{k+1}+1}$

$x_{k+1}>0$, $b>0$

$\Rightarrow x_{k+2}>0$.

$0<x_n\leq a$.

$\forall n$

$L=\frac{1}{\frac{1}{L}+b}$

$L=\frac{L}{1+bL}$ $1+bL=1$

$bL=0$

$b>0$

$\Rightarrow L=0$. $\Rightarrow L=0$

I all a_n bdd by inf a_n & sup a_n.

know (b_n) monotone decreasing

$\Rightarrow\forall b_n\leq\sup a_n$

$\forall b_n\geq\inf a_n\Rightarrow$ bdd.

$b_1=\sup\{a_k: k\geq 1\}$

$b_2=\sup\{a_k: k\geq 2\}$

$\Rightarrow b_1\geq b_2$

$\Rightarrow$ monotone decreasing

J $\lim\sup a_n = \lim\inf a_n = L$

iff $a_n \to L$

"⇒" if $a_n \to L$,

$\forall \varepsilon > 0\ \exists N$

s.t. $|a_n - L| < \varepsilon$ whenever $n \ge N$

$\Rightarrow L-\varepsilon < a_n < L+\varepsilon\ \forall n \ge N$

$\Rightarrow \sup\{a_k : k \ge N\} \le L+\varepsilon$

$\sup\{a_k : k \ge N\} > L-\varepsilon$

$\Rightarrow |\sup - L| < \varepsilon$

"⇐" $\lim\sup a_n = \lim\inf a_n = L$.

$\forall \varepsilon > 0$. $|\sup a_n - L| < \varepsilon$

$n \ge N_1$

$|\inf a_n - L| < \varepsilon$

$n \ge N_2$

$N = \max\{N_1, N_2\}$

$\Rightarrow L-\varepsilon < \inf a_n < \sup a_n < L+\varepsilon$

However $n \ge N$.

$\inf a_n \le a_n \le \sup a_n$

$\Rightarrow L-\varepsilon < a_n < L+\varepsilon$

$\Rightarrow |a_n - L| < \varepsilon$

B1 C

Cauchy criterion

$\lim_{n\to\infty} \sum_{k=1}^{n} |a_n - a_{n+1}| = L$

$\forall \varepsilon > 0\ \exists N$

$\sum_{n=m}^{n} |a_k - a_{k+1}| < \varepsilon$

whenever $m, n \ge N$ $(m < n)$.

$|a_m - a_n|$

$= |a_m - a_{m+1} + a_{m+1} - a_{m+2} + \dots + a_{n-1} - a_n|$

$\le \sum |a_m - a_{m+1}| + \dots + |a_{n-1} - a_n|$

$< \varepsilon$

$\Rightarrow$ Cauchy.

P36 A $\sum \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$

$= \sum \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} \dots\right)$

$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$

B $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3}\right)$

$+ \sum_{n=1}^{\infty}\left(\frac{1}{n+3} - \frac{1}{n+4}\right)$

$\to 1 \quad - \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5}\right)$

$- \frac{1}{6}$

$+ \frac{1}{4} = \frac{12 - 10 + 3}{12} = \frac{5}{12}$

$\frac{1}{12}$.

C. $\sum_{k=1}^{\infty} t_k \to L$

$\left(\sum_{k=1}^{\infty} t_k\right)^p \to L^p$

$\sum_{k=1}^{\infty} t_k^p \le \left(\sum_{k=1}^{\infty} t_k\right)^p$

since t_k non-negative

$\sum_{k=n+1}^{m} t_k < (\varepsilon)^{\frac{1}{p}} = \varepsilon_1$

$\left(\sum_{k=n+1}^{m} t_k\right)^p < \varepsilon_1^p = \varepsilon$

$\Rightarrow \sum_{k=n+1}^{m} t_k^p \le \left(\sum_{k=n+1}^{m} t_k\right)^p < \varepsilon$

P42. P.

a) $\frac{3n}{n^3+1} < \frac{3n}{n^3} \Rightarrow$ cvg

b) $\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n} \to \frac{1}{2} \Rightarrow$ cvg. ratio test

c) $\left(\frac{\ln x}{x}\right)' = \frac{\frac{1}{x} x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

Alternating series test

$\Rightarrow$ cvg.

d): div since $\sum_{n=1}^{\infty}(\sqrt{n+1} - \sqrt{n})$

$= \sqrt{n+1} - 1 \to +\infty$.

e) nth root test

$(e^{-n^2})^{\frac{1}{n}} = e^{-n} \to 0$

$\Rightarrow$ cvg.

f) $\sin\frac{n\pi}{4}$

div.

since $\not\to 0$.

g) $\sum_{n=1} \sin\frac{1}{n}$

$\sin\frac{1}{n} < \frac{1}{n}$

$\Rightarrow$ cvg

h)

$\sim \frac{1}{n^2}$

$\Rightarrow$ cvg

i) $n^{\frac{1}{n}} - 1 \to 0$

nth-root test

$\Rightarrow$ cvg.

j): $\frac{1}{n(\sqrt{n+1} + \sqrt{n})}$

$\sim \frac{1}{2n^{\frac{3}{2}}}$

$\Rightarrow$ cvg.

k). Alternating series test

$\Rightarrow$ cvg.

l). $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt[n]{n}}$

$= (-1)^n \not\to 0$.

$\Rightarrow$ div.

m) $\sum \frac{1}{(\ln n)^k}$

$\ln n < n$

$\Rightarrow \frac{1}{\ln n} > \frac{1}{n}$

if $k \le 1$

$\Rightarrow$ div.

$k > 1$?

n) ratio test

$\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = (n+1)\frac{n^n}{(n+1)^{n+1}}$

$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n$

$= \left(1 - \frac{1}{n+1}\right)^n = \left(1 - \frac{1}{n+1}\right)^{n+1-1}$

$= e^{-1} < 1$.

$\Rightarrow$ cvg

o). decreasing

$\Rightarrow$ A S.T. $\Rightarrow$ cvg.

p) $\frac{(-1)^n}{\sqrt{n} + (-1)^n}$. ?

q) $e^{\frac{1}{x}} - 1$ decreasing

Alternating series test $\Rightarrow$ cvg

r). $\frac{(-1)^{n+1}}{(-1)^n} \cdot \frac{(n+1)^{42}}{n^{42}} \cdot \frac{n!}{(n+1)!}$

$= (-1) \cdot \left(\frac{n+1}{n}\right)^{42} \cdot \frac{1}{n+1}$

$\to 0$. $\Rightarrow$ cvg.

s) $\sim \frac{1}{n^2} \Rightarrow$ cvg.

t) $\ln(e^n + e^{-n}) < \ln(e^n + e^n)$

$\frac{1}{\ln(e^n + e^{-n})} > \frac{1}{\ln 2e^n}$

$= \frac{1}{\ln 2 + n}$

$\Rightarrow$ div.

v) $\frac{(n+1)^{10} \cdot 10^n}{n^{10} \cdot 10^{n+1}} = \frac{1}{10}\left(\frac{n+1}{n}\right)^{10}$

$\to \frac{1}{10} < 1$

w). nth-root test:

$\frac{1}{\ln n} < 1$ $(\to 0.)$

x) integral test.

$\int_2^{+\infty} \frac{1}{x \ln x}\,dx = \int \frac{1}{\ln x}\,d(\ln x)$

$= \ln(\ln x)\Big|_2^{+\infty}$

P47. B.

a) $\frac{(-1)^n}{n\ln(n+1)}$ $\quad \frac{1}{n\ln(n+1)} > \frac{1}{(n+1)\ln(n+1)}$

div by integral test

; cvg by alternating series test

b) $\frac{1}{[2+(-1)^n]n} > \frac{1}{3n}$.

$\Rightarrow$ div.

中国作家协会

$\Rightarrow (a_{n_k}) \to \vec{x} \notin A$.
$\to \vec{a} \in A$
at the same time X

2. $(b_n) = (\text{some pts in } (a_n)) \cup \{\vec{a}\}$.
$(b_n) = (a_{n_1}, a_{n_2} \ldots, \quad \vec{a}) \to \vec{x} \notin A$
$\Rightarrow \|a_{n_k} - x\| < \varepsilon, \ \forall \varepsilon > 0 \ n_k \geq N$
however can show
$b_n \to \vec{a}$.

N. a) $\lim_{n\to\infty} a_n = \vec{x} \to$ limit point
① $\vec{x} \in A$. ✓
② $\vec{x} \notin A \Rightarrow$ cluster pt
b). closed: contains all limit pt
in particular contains all cluster pts, since {cluster pts} $\subseteq$ {limit pts}.
c) $\mathbb{Q}$: irrational pts.
$\mathbb{Z}$: no limit pts.
$(0,1)$: 0, 1.

4.4

I. $d(A,B) = \inf\{\|a-b\| : a \in A, b \in B\}$
a). A B singleton
if $\exists B$ s.t. $d(A,B) = 0$.
$\Rightarrow \inf\{\|a-b\| : a \in A, b \in B\} = 0$
(defn of g(b))
take $\varepsilon_n = \frac{1}{n}$
$\varepsilon_1 = 1 \quad \exists b_1 \in B$ s.t. $\|a - b_1\| < 1$
$\varepsilon_2 = \frac{1}{2} \quad \exists b_2 \in B$ s.t. $\|a - b_2\| < \frac{1}{2}$
$\ldots \quad \exists b_n \in B$ s.t. $\|a - b_n\| < \frac{1}{n}$
$\varepsilon_n = \frac{1}{n}$
i.e. can find a seq. $(b_n) \in B$
s.t. $b_n \to a$

$a \in A$. ~~A, B disjoint~~
$\Rightarrow a \notin B \Rightarrow B$ not closed.

b) A: compact
$\exists B$ s.t.
$d(A,B) = 0$.
i.e. $\inf\{\|a-b\| : a \in A, b \in B\} = 0$ (defn of inf)
$\forall \varepsilon > 0$
$\exists$ a sequence (d_n).
$\{d_n = \|a_n - b_n\| : a_n \in A, b_n \in B\}$
s.t. $d_n \to 0$.
$\Rightarrow \lim_{n\to\infty} \|a_n - b_n\| = \lim_{n\to\infty} d_n = 0$.

9.1

B. open: $\forall X \in X$ subset of discrete open space

$B_{\frac{1}{2}}(X) = \{X: \rho(X,Y) < \frac{1}{2}\}$ $0 < r < 1$

$= X$

closed:

$\forall$ subset is open

$\Rightarrow X \setminus U$, a subset of X is open $(U \subseteq X)$

so its complement $U \subseteq X$ is closed

C. U open.

sps $(X_n) \in X \setminus U$

& $X_n \to X$ ($\lim_{n\to\infty} \rho(X_n, X) = 0$).

but $X \in U$

$\Rightarrow \exists r$ s.t. $B_r(X) \subseteq U$

$\Rightarrow \rho(X_n, X) \geq r$

U not open.

$\exists X \in U$ s.t. $\exists r$.

$B_r(X) \not\subseteq U$

$\Rightarrow r = \varepsilon_n = \frac{1}{n}$

$B_{\frac{1}{n}}(X) \cap U^c \neq \phi$

$X_1 \in (B_{\frac{1}{1}} \cap U^c)$

$\Rightarrow \rho(X_1, X) < 1$.

$\rho(X_2, X) < \frac{1}{2}$

$\vdots$

$\rho(X_n, X) < \frac{1}{n}$

$\Rightarrow \lim_{n\to\infty} \rho(X_n, X) = 0$

$\Rightarrow (X_n) \to X \in U$

$\Rightarrow U^c$ not closed

E. a) see sample mid.

b) "$\Rightarrow$" $\lim_{n\to\infty} X_n = X$ in (X, ρ)

$\forall \varepsilon > 0$, $\exists N$ s.t.

$\Rightarrow \rho(X_n, X) < \varepsilon$

whenever $n \geq N$

$\Rightarrow$ for $\forall \varepsilon < 1$

$\sigma(X_n, X) = \min\{\rho(X, X), 1\}$

$= \rho(X_n, X) < \varepsilon$.

$\Rightarrow \forall \varepsilon < 1$. $\exists N$

s.t. $\sigma(X_n, X) < \varepsilon$ whenever $n \geq N$.

but for $\varepsilon > 1$

$\sigma(X_n, X) = \min\{\rho(X,Y), 1\} = 1 < \varepsilon$.

$\Rightarrow \forall \varepsilon > 0$. $\exists N$ s.t. $\sigma(X_n, X) < \varepsilon$ whenever $n \geq N$.

"$\Leftarrow$" if $\lim_{n\to\infty} X_n = X$ in (X, σ).

$\Rightarrow \forall \varepsilon > 0$ $\exists N$ s.t. $\sigma(X_n, X) < \varepsilon$ whenever $n \geq N$.

clearly, $\sigma(X,Y) \leq 1$. by def'n.

$\sigma(X_n, X) = \min\{\rho(X,Y), 1\} < \varepsilon$.

So if $\varepsilon < 1$

$\exists N$ s.t. $\sigma(X_n, X) = \rho(X_n, X) < \varepsilon$. whenever $n \geq N$.

if $\varepsilon \geq 1$

since we have $\forall \varepsilon < 1$. $\exists N$.

it definitely satisfies $\exists N$ $\rho(X_n, X) < \varepsilon$

when $\varepsilon \geq 1$.

中国作家协会

2.8 C.

$\lim_{n\to\infty}\sum|a_n-a_{n+1}|<\infty$.

(a_n) is Cauchy

$\lim_{n\to\infty}\sum_{N\to\infty}|a_n-a_{n+1}|=L$

$\forall\varepsilon>0\ \exists N>0 \Rightarrow$

$|a_m-a_n|<\varepsilon$ whenever $m,n\geq N$.

$|a_1-a_2|+|a_2-a_3|$

$+\dots+|a_n-a_{n+1}|$

$b_n=|a_n-a_{n+1}|,\ n=1,2,\dots$

$S_n=\sum b_n$ series converges

Cauchy criteria:

$\exists N$ s.t. $|\sum_{n+1} b_n|<\varepsilon$

$|b_{n+1}+b_{n+2}+\dots+b_m|<\varepsilon$

(since $b_n=|a_n-a_{n+1}|\geq0$)

$\Rightarrow \underbrace{|a_{n+1}-a_{n+2}|+|a_{n+2}-a_{n+3}|+\dots+|a_m-a_{m+1}|}_{\text{triangle inequality}}<\varepsilon$

$\Rightarrow |a_{n+1}-a_{m+1}|\leq(\quad)<\varepsilon$

$\Rightarrow$ Cauchy.

P38: A

$\frac{1}{n(n+2)}=\frac12\left(\frac1n-\frac{1}{n+2}\right)$

$\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n(n+2)}=\sum\frac12\left(\frac1n-\frac{1}{n+2}\right)$

$=\frac12\left[\left(1-\frac13\right)+\left(\frac12-\frac14\right)+\left(\frac13-\frac15\right)\right.$

$\left.+\dots+\left(\frac{1}{n-1}-\frac{1}{n+1}\right)+\left(\frac1n-\frac{1}{n+2}\right)\right]$

$=\frac12\left(1+\frac12-\frac{1}{n+1}-\frac{1}{n+2}\right)$

$\to\frac34$.

B. $\frac{1}{n(n+1)(n+3)(n+4)}$

$=\frac13\left[\frac{1}{n(n+4)}-\frac{1}{(n+1)(n+3)}\right]$

$=\frac13\left[\frac14\left(\frac1n-\frac{1}{n+4}\right)-\frac12\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\right]$

$=\frac{1}{12}\frac1n-\frac16\frac{1}{n+1}+\frac16\frac{1}{n+3}-\frac{1}{12}\frac{1}{n+4}$

$=\frac{1}{12}\left(\frac1n-\frac{2}{n+1}+\frac{2}{n+3}-\frac{1}{n+4}\right)$

$=\frac{1}{12}\left[\left(\frac1n-\frac{1}{n+1}\right)-\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)\right]$

$=\frac{1}{12}\left(\frac1n-\frac{1}{n+1}\right)-\frac{1}{12}\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\frac{1}{12}\left(\frac{1}{n+3}-\frac{1}{n+4}\right)$

$=\frac{1}{12}\sum\left(\frac1n-\frac{1}{n+1}\right)-\frac{1}{12}\sum\left(\frac{1}{n+1}-\frac{1}{n+3}\right)+\frac{1}{12}\sum\left(\frac{1}{n+3}-\frac{1}{n+4}\right)$

$=\frac{1}{12}\sum\left(1-\frac12+\frac12-\frac13+\dots+\frac1n-\frac{1}{n+1}\right)-\frac{1}{12}\sum\left(\frac12-\frac14+\frac13-\frac15+\dots+\frac1n-\frac{1}{n+2}+\frac{1}{n+1}-\frac{1}{n+3}\right)$

$+\frac{1}{12}\sum\left(\frac14-\frac15+\dots\right)$

$=\frac{1}{12}\cdot1-\frac{1}{12}\left(\frac12+\frac13\right)+\frac{1}{12}\cdot\frac14$

$=\frac{1}{12}\left(1+\frac14-\frac12-\frac13\right)$

$=\frac{1}{12}\left(\frac14+\frac16\right)$

$=\frac{1}{12}\cdot\frac{5}{12}=\frac{5}{144}$

C. $\sum_{k=1}^{\infty}t_k$: convergent series

$\forall\varepsilon>0\ \exists N$

$\Rightarrow t_k\to0$

$|\sum t_k|<\varepsilon$ whenever $k>N$

take $\varepsilon=(\varepsilon_1)^{\frac1p}$

$|\sum t_k|^p<\varepsilon^p=((\varepsilon_1)^{\frac1p})^p=\varepsilon_1$

$(\sum t_k)^p\geq\sum t_k^p\quad(p>1)$

since (t_k) non-negative

P. a) $\sum_{k=2}^{\infty}\frac{3n}{n^3+1}$

consider $\frac{3n}{n^3}\geq\frac{3n}{n^3+1}$.

$\frac{a_{n+1}}{a_n}=$ $\sum_{n=2}^{\infty}\frac{3}{n^2}$ converges

since $\frac{3}{n^2}<\frac{3}{n(n-1)}\to3$.

d) $\sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}$.

$=(\sqrt2-1)+(\sqrt3-\sqrt2)$

$+\dots+(\sqrt{n+1}-\sqrt{n})$

$=\sqrt{n+1}-1\to\infty$

diverge

b) $\sum_{n=1}^{\infty}\frac{n}{2^n}$

$\left(\frac{n}{2^n}\right)^{\frac1n}=\frac{n^{\frac1n}}{2}\to\frac12$

$(n^{\frac1n}\to1.)$

converges

c). $\sum_{n=2}^{\infty}\frac{(-1)^n\log n}{n}$ converges to 0

since $\frac{\ln n}{n}\to0$.

e) $\sum_{n=1}^{\infty}e^{-n^2}$ $(e^{-n^2})^{\frac1n}=e^{-n}\to0<1$

$\Rightarrow$ converge

f) $\sin\frac{n\pi}{4}\not\to$ always 0.

$\Rightarrow$ diverge

g) As $n\to\infty$ $\sin\frac1n<\frac1n$

$\sum(-1)^n\frac1n$ converges (Alternating series test)

$\Rightarrow$ converges

h) $\frac{1}{\sqrt{n^3+4}} \sim \frac{1}{\sqrt{n^3}}$

which converges

i) $(\sqrt[n]{n}-1)^n$

nth root test

$= n^{\frac{1}{n}}-1 \to 0$

$\to$ converges

j $\frac{1}{n(\sqrt{n+1}+\sqrt{n})}$

behaves like

$\frac{1}{2n^{\frac{3}{2}}} \Rightarrow$ converges

$\frac{(-1)^n}{\sqrt{n}+(-1)^n} < \frac{(-1)^n}{\sqrt{n}-1}$

$\frac{(-1)^n}{\sqrt{n}-1}$ converges

7) $e^{\frac{1}{n}}-1$ decreasing

since $\frac{1}{x}$ decreasing, e^x increasing

$\Rightarrow$ converge

r) $\dfrac{(-1)^{n+1}\frac{(n+1)^{42}}{(n+2)!}}{(-1)^n\cdot\frac{n^{42}}{(n+1)!}} = \left(\frac{n+1}{n}\right)^{42}\cdot\frac{1}{n+2}\cdot(-1)$

$= \left(1+\frac{1}{n}\right)^{42}\cdot\frac{1}{n+2}\cdot(-1)$

$\to 0 < 1$

$= \ln(\ln x)\Big|_2^{+\infty}$

$\Rightarrow$ diverge

u) & m)

m) $\ln n < n$?

$\Rightarrow \frac{1}{(\ln n)^k} > \frac{1}{n^k}$

$\Rightarrow k \le 1$ diverge

Now check for $k > 1$

k. $\frac{1}{n^{\frac{1}{2}}\ln n}$ decreasing & $\to 0$ as $n \to +\infty$ $\Rightarrow$ converges

Alternating series Test

$\Rightarrow$ converges

l. $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^{\frac{1}{n}}}$

$n^{\frac{1}{n}} \to 1 \Rightarrow \frac{(-1)^n}{n^{\frac{1}{n}}} \to 1$ or -1

$\Rightarrow$ diverge

m) $\sum_{n=2}^{\infty}\frac{1}{(\ln n)^k}$

s) $\frac{1}{1+n^2} \sim \frac{1}{n^2}$ converges

t) $\frac{1}{\ln(e^n+e^{-n})} =$

$e^n > e^{-n}\ \forall n \ge 1$

$\Rightarrow \frac{1}{\ln(e^n+e^{-n})} > \frac{1}{\ln(e^n+e^n)}$

$= \frac{1}{\ln 2\cdot e^n} = \frac{1}{\ln 2+n} \sim \frac{1}{n}$

$\to$ diverge.

n) $\sum\frac{n!}{n^n} \to 0$ as $n \to \infty$.

$\dfrac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)\cdot n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n$

$= \left(1-\frac{1}{n+1}\right)^{n+1-1}$

$= e^{-1}\cdot\left(1-\frac{1}{n+1}\right)^{-1}$

$= \frac{1}{e} < 1 \Rightarrow$ converge

v) $\dfrac{\frac{(n+1)^{10}}{10^{n+1}}}{\frac{n^{10}}{10^n}} = \left(\frac{n+1}{n}\right)^{10}\cdot\frac{1}{10}$

$= \left(1+\frac{1}{n}\right)^{10}\cdot\frac{1}{10} \to \frac{1}{10}$

as $n \to \infty$

$\Rightarrow$ converge

o) $\frac{\arctan n}{n}$ decreasing

$\to 0$

$\Rightarrow$ converge

w) $\frac{1}{(\ln n)^n}$ $\left(\frac{1}{(\ln n)^n}\right)^{\frac{1}{n}} = \frac{1}{\ln n} \to 0 < 1$

$\Rightarrow$ converge

p) $\frac{(-1)^n}{\sqrt{n}+(-1)^n} = \sum_{n=2k}^{\infty}\frac{1}{\sqrt{n}+1} - \sum_{2k+1}^{\infty}\frac{1}{\sqrt{n}-1}$

x) $\frac{1}{n\ln n}$ Integral test

$\int_2^{\infty}\frac{1}{x\ln x}dx = \int\frac{1}{\ln x}d(\ln x)$

中国作家协会

P47.

B. a) $\frac{(-1)^n}{n\ln(n+1)}$

alternating test

$\frac{1}{n\ln(n+1)} > \frac{1}{(n+1)\ln(n+1)}$

diverge

$\Rightarrow$ conditional cvg

b) $\to 0$.

$\frac{1}{[2+(-1)^n]n} > \frac{1}{3n}$ diverge

$\frac{(-1)^n}{[2+(-1)^n]n} < \frac{(-1)^n}{n} \to$ cvg

$\Rightarrow$ cond. cvg

c) $\to 0$

$\frac{\sin\frac{1}{n}}{n} \nearrow$ monotone decreasing

$(-1)^n \frac{\sin\frac{1}{n}}{n}$ cvg

$\frac{\sin\frac{1}{n}}{n} < \frac{1}{n^2}$ cvg

since $\sin\frac{1}{n} < \frac{1}{n}$.

$\Rightarrow$ abs. cvg

C. $\sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)}$

$\frac{1}{n^2(2n-1)} = \frac{1}{2n(2n-1)} - \frac{1}{n^2}$

$\sum_{n=1}^{\infty}\left[\frac{1}{2n(2n-1)} - \frac{1}{n^2}\right]$

$= \sum_{n=1}^{\infty} \frac{4}{2n(2n-1)} - \sum_{n=1}^{\infty}\frac{1}{n^2}$

$4\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$

$= 4(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots)$

B the taylor series

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$= \sum \frac{(-1)^{k+1}x^k}{k}$

when $x=1$

$\Rightarrow 4\ln 2 - \frac{\pi^2}{6}$

P12: $\|x+y\|^2 =$

A. $\langle x+y, x+y\rangle$

$= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle$

$= \|x\|^2 + \|y\|^2$

$\Rightarrow \|x+y\| = [\|x\|^2 + \|y\|^2]^{\frac{1}{2}}$

H. a) $\langle x+y, x+y\rangle = \|x+y\|^2$

$= \|x\|^2 + \|y\|^2 + 2\langle x,y\rangle$

$= \|x\|^2 + \|y\|^2 + 2\|x\|\|y\|\cos\theta$

b) $\langle x,y\rangle = \|x\|\|y\|\cos\theta$

$\cos\theta = \frac{\|x+y\|^2 - (\|x\|^2 + \|y\|^2)}{2\|x\|\|y\|}$

$\Rightarrow \langle x,y\rangle = \frac{\|x+y\|^2 - \|x\|^2 + \|y\|^2}{2}$

K. a) M: basis $\alpha = \{\vec v_1 \ldots \vec v_k\}$

$P\vec x = \sum_{i=1}^{k}\langle \vec x, \vec v_i\rangle \cdot \vec v_i$

① for $\forall \vec x$, $P\vec x$ can be written as a linear combination of basis α

$\Rightarrow P\vec x \in M$

② sps $\vec y = a_1\vec v_1 + \cdots + a_k\vec v_k$

$P\vec y = \sum_{i=1}\langle \vec y, \vec v_i\rangle \vec v_i$

$\langle \vec y, \vec v_i\rangle = \langle a_1\vec v_1 + \cdots + a_k\vec v_k, \vec v_i\rangle$ orthonormal

$= \langle a_i\vec v_i, \vec v_i\rangle = a_i\langle \vec v_i, \vec v_i\rangle$

$P\vec y = \sum\langle \vec y, \vec v_i\rangle \vec v_i = \sum a_i v_i = \vec y$

③ $P\vec y = \vec y$

$\forall \vec x \in R^n$

$P^2\vec x = P(P\vec x)$

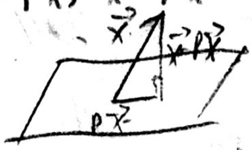
sps $P\vec x = \vec y \in M$

so $P\vec y = \vec y$

so $P^2\vec x = P(P\vec x) = P\vec y = \vec y$

$P\vec x = \vec y$

b) $\langle P\vec x, \vec x - P\vec x\rangle$



$= \langle P\vec x, \vec x\rangle - \langle P\vec x, P\vec x\rangle$

$= \langle P\vec x, \vec x\rangle - \underbrace{\langle \sum_{i=1}^{n}\langle \vec v_i, \vec x\rangle \vec v_i, \sum_{j=1}^{n}\langle \vec x, \vec v_j\rangle \vec v_j\rangle}_{= \begin{cases}\langle \vec v_i, \vec x\rangle\langle \vec v_i, \vec x\rangle & i=j \\ 0 & i\neq j\end{cases}}$

$= \langle P\vec x, \vec x\rangle - \ldots$

$= \langle \underbrace{\sum_{i=1}^{k}\langle v_i, \vec x\rangle \cdot v_i}, \vec x\rangle -$

$= \sum\langle v_i, \vec x\rangle \cdot \langle v_i, \vec x\rangle$

$+ \langle v_2, \vec x\rangle \cdot \langle v_2, \vec x\rangle$

$+ \cdots + \langle v_k, \vec x\rangle \cdot \langle v_k, \vec x\rangle$

$= \langle \vec v_i, \vec x\rangle ; \langle \vec v_i, \vec x\rangle$

c) $\vec x = (\vec x - P\vec x) + P\vec x$

$\langle \vec x, \vec x\rangle = \langle \vec x - P\vec x + P\vec x, \vec x - P\vec x + P\vec x\rangle$

$= \|\vec x - P\vec x\|^2 + \|P\vec x\|^2 + 2\langle \vec x - P\vec x, P\vec x\rangle$ 0.

d) $\vec x - \vec y = \vec x - P\vec x + P\vec x - \vec y$

$(P\vec x - \vec y) \in M$

$\langle \vec x - P\vec x, \vec v_i\rangle$

$= \langle \vec x, \vec v_i\rangle - \langle P\vec x, \vec v_i\rangle$

$= \langle \vec x, \vec v_i\rangle - \langle \sum\langle \vec x, \vec v_j\rangle \vec v_j, \vec v_i\rangle$

$\langle \sum\langle \vec x, \vec v_j\rangle \vec v_j, \vec v_i\rangle$

$= \langle\langle \vec x, v_i\rangle \vec v_i, \vec v_i\rangle = \langle \vec x, \vec v_i\rangle$

$= 0$.

$(\vec{x}-P\vec{x})$ is orthogonal to $\forall$ basis vector of M

$\Rightarrow (\vec{x}-P\vec{x}) \in M^{\perp}$

e) $\|\vec{y}-\vec{x}\|^2 = \|\vec{y}-P\vec{x}\|^2 + \|x-Px\|^2$

$\geq \|x-Px\|^2$

eq'n iff $\vec{y}=P\vec{x}$.

P5:

F. a)

① $n=0$.

$V_1=(X_1,Y_1)=(\sqrt{X_0Y_0}, \frac{X_0+Y_0}{2})$

$X_1=\sqrt{X_0Y_0}$, $X_0<Y_0$

$\Rightarrow X_1=\sqrt{X_0Y_0}>\sqrt{X_0X_0}=X_0$.

$\sqrt{X_0Y_0}\leq \frac{X_0+Y_0}{2}$ since $X_0<Y_0$

since $(X_0-Y_0)^2>0$.

$Y_1=\frac{X_0+Y_0}{2}<\frac{Y_0+Y_0}{2}=Y_0$

$\Rightarrow 0<X_0<X_1<Y_1<Y_0$.

② sps true for $n=k$

i.e $0\leq X_k<X_{k+1}<Y_{k+1}<Y_k$.

③ for $n=k+1$

$X_{k+2}=\sqrt{X_{k+1}Y_{k+1}}>\sqrt{X_{k+1}X_{k+1}}=X_{k+1}$

$X_{k+2}=\sqrt{X_{k+1}Y_{k+1}}$

$Y_{k+2}=\frac{X_{k+1}+Y_{k+1}}{2}$

$Y_{k+2}-X_{k+2}=\frac{1}{2}(X_{k+1}-2\sqrt{X_{k+1}Y_{k+1}}+Y_{k+1})$

$=\frac{1}{2}(\sqrt{X_{k+1}}-\sqrt{Y_{k+1}})^2>0$.

$Y_{k+2}=\frac{X_{k+1}+Y_{k+1}}{2}<\frac{Y_{k+1}+Y_{k+1}}{2}=Y_{k+1}$

b)

c) $Y_{n+1}-X_{n+1}<Y_n-X_n$

$\Rightarrow \lim_{n\to\infty}(Y_{n+1}-X_{n+1})\to 0$

$X_0<X_1<\cdots<X_n<X_{n+1}<$

$Y_{n+1}<\cdots<Y_1<Y_0$

(X_n) monotone increasing

(Y_n) monotone decreasing

both bdd

$\Rightarrow X_n\to L$, $Y_n\to M$

however $\lim_{n\to\infty}(Y_n-X_n)\to 0$

$\Rightarrow L=M$.

H. a) ① $X_1=\begin{bmatrix}\frac{7}{4} & -\frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix}$

$=(\frac{7}{4},\frac{3}{4})$

② $n=k$. $X_k=(\frac{3-2^{-k}}{2}, \frac{3(1-2^{-k})}{2})$

$X_{k+1}=\begin{bmatrix}\frac{7}{4} & -\frac{1}{4}\\ \frac{3}{4} & \frac{1}{4}\end{bmatrix}\begin{pmatrix}\frac{3-2^{-k}}{2}\\ \frac{3(1-2^{-k})}{2}\end{pmatrix}$

$=\begin{pmatrix}\frac{7(3-2^{-k})}{8}-\frac{3(1-2^{-k})}{8}\\ \frac{9-3\cdot2^{-k}+3(1-2^{-k})}{8}\end{pmatrix}$

$=\begin{pmatrix}\frac{21-7\cdot2^{-k}-3+3\cdot2^{-k}}{8}\\ \frac{12-6\cdot2^{-k}}{8}\end{pmatrix}$

$=\begin{pmatrix}\frac{18-4\cdot2^{-k}}{8}\\ \frac{12-12\cdot2^{-(k+1)}}{8}\end{pmatrix}$

$=\begin{pmatrix}\frac{18-4\cdot2^{-k}}{8}\\ \frac{3(1-2^{-(k+1)})}{2}\end{pmatrix}$

$=\begin{pmatrix}\frac{3-2^{-(k+1)}}{2}\\ \frac{3(1-2^{-(k+1)})}{2}\end{pmatrix}$

b) $\lim_{n\to\infty}\frac{3-2^{-n}}{2}=\frac{3}{2}$

$\lim_{n\to\infty}\frac{3(1-2^{-n})}{2}=\frac{3}{2}$

$\|X_n-Y_n\|^2$

$=(\frac{-2^{-n}}{2})^2+(\frac{3-(1-2^{-n})}{2})^2=$

$=(-\frac{1}{2^{n+1}})^2+(-\frac{3}{2^{n+1}})^2$

$=\frac{10}{2^{2n+2}}$.

$\|X_n-Y\|=\frac{\sqrt{10}}{2^{n+1}}<\frac{1}{2}10^{-100}$

$\Rightarrow \frac{\sqrt{10}}{2^n}<10^{-100}$

$2^n>10^{100}\cdot\sqrt{10}$.

P6:

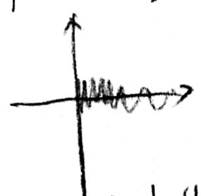
$1.4, 1.41, 1.414, \ldots\to\sqrt{2}$

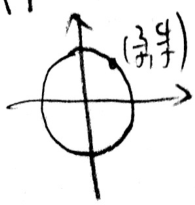
A. a) R $\quad (1-\frac{1}{n}, 1-\frac{1}{n})\to(1,1)$

b) $xy\leq 1$

c) $\{(x,\sin\frac{1}{x}): x>0\}\cup\{(0,0)\}$

d) $\{x^2+y^2\leq 1\}$.





$(\frac{3}{5}-\frac{1}{n}, \frac{4}{5}-\frac{1}{n})\to(\frac{3}{5},\frac{4}{5})$

$\Rightarrow$ conclude $x^2+y^2=1$

$\exists$ a seq. $\to(\frac{1}{2},\frac{\sqrt{3}}{2})$

say $(\frac{1}{2}, 0.\widetilde{\ \ })$ as digit increases)

B. $\lim_{n\to\infty}\vec{a}_n=\vec{a}$, $A=\{\vec{a}_n: n\geq 1\}\cup\{\vec{a}\}$

by contradiction

sps not. $\exists\vec{x}$ a limit pt of the set A but $\vec{x}\notin A$ (i.e $\vec{x}\neq\vec{a}_n$, $\vec{x}\neq\vec{a}$)

$\exists$ a seq. $(b_n)\to\vec{x}$. $(b_n)\in A$

$(\vec{b}_n)\in A$

$a_n\to\vec{a}$ means $\forall\varepsilon_2$ $\exists N_2$ $\|\vec{a}_n-\vec{a}\|<\varepsilon_2$ whenever $n\geq N_2$

consider 2 cases:

1. $b_n\in\{a_n: n\geq 1\}$ $\quad \tilde{N}=\max\{N_1,N_2\}$.

$\Rightarrow (b_n)$ a subseq. of (a_n)

if $b_n\to\vec{x}\notin A$. $\quad (b_n)$ re written as a_{n_k} $\Rightarrow n_k\geq\tilde{N}$

$\forall\varepsilon_1>0$ $\exists N_1$ s.t.

$\|b_n-x\|<\varepsilon_1$ whenever $\quad \|a_{n_k}-\vec{x}\|<\varepsilon_1$

a_{n_k} $\leftarrow n_k\geq N_1$. $\quad \|a_{n_k}-\vec{a}\|<\varepsilon_2$

Sample Test — Mid Term:

中国作家协会

1\. a) $a_n = \frac{2^{n^2}}{n!}$

$a_{n+1} = \frac{2^{(n+1)^2}}{(n+1)!}$

$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{(n+1)^2}}{(n+1)!}}{\frac{2^{n^2}}{n!}}$

$= \frac{2^{2n+1}}{n+1}$

$= \frac{2^{n+1}\cdot 2^{n}}{n+1}$

clearly $2^{n+1} > n+1$ as n gets sufficiently large

$\Rightarrow \frac{a_{n+1}}{a_n} > 1$ (& the ratio $\to +\infty$ as $n \to +\infty$.)

i.e. (a_n) is a increasing sequence & unbdd.

$\Rightarrow a_n \to \infty$

$\Rightarrow$ diverge

b) claim: (b_n) increasing & bdd $\quad b_{n+1} = 4b_n$

show $b_1 < b_2 < \cdots b_n < \cdots < 2$

① $n=1$. $b_1 = 2^{\frac{1}{2}}$, $b_2 = 2^{\frac{3}{8}}$

$\Rightarrow b_1 < b_2 < 2$

② $n=k$ true i.e. $b_k < b_{k+1} < 2$

③ $n=k+1$ $\quad b_{k+2} = \sqrt{2\sqrt{b_{k+1}}}$

$b_{k+1} < 2 \Rightarrow b_{k+2} < \sqrt{2\cdot\sqrt{2}} = 2^{\frac{3}{4}} < 2$!

$b_{k+2} = \sqrt{2\sqrt{b_{k+1}}} > \sqrt{2\sqrt{b_k}} = b_{k+1}$

$\Rightarrow (b_n)$: bdd ($\forall n$: $\sqrt{2} \le b_n < 2$)

b_n increasing as well

$\Rightarrow b_n \to L$

$\lim_{n\to\infty} b_n = L$, $\lim_{n\to\infty} b_{n+1} = L$

$\Rightarrow L = \sqrt{2\sqrt{L}}$

$L^2 = 2\sqrt{L}$

$L^4 = 4L$

$\Rightarrow L(L^3 - 4) = 0$

$\Rightarrow L = 2^{\frac{2}{3}}$ or 0

$\forall b_n \ge \sqrt{2}$ $\quad 2^{\frac{2}{3}}$.

$\Rightarrow \lim_{n\to\infty} b_n = 2^{\frac{2}{3}}$

2\. a) S bdd above

$S_0 \subset S$

$\sup S_0 \le \sup S$

$\sup S_0 = x$,

$\sup S = y$

by contradiction: sps $x > y$.

$\sup S = y$

$\Rightarrow \forall a \in S$ we have $a \le y$

$\sup S_0 = x$ $\Rightarrow \forall b \in S_0$, we have $b \le x$.

Also, by defn of sup, $\forall \varepsilon > 0$ $\exists b' \in S_0$ st. $b' > x - \varepsilon$

take $\varepsilon = \frac{x-y}{2}$

$\Rightarrow \exists b' \in S_0$ s.t. $b' > x - \frac{x-y}{2}$

$= \frac{x+y}{2}$

$b' \in S_0$, $S_0 \subset S$.

$\Rightarrow b' \in S$ as well

$\Rightarrow \exists b' \in S$ st. $b' > \frac{x+y}{2}$.

$x > y$ tells us $\frac{x+y}{2} > y$

$\Rightarrow y$ not a upper bound

$\Rightarrow \Leftarrow$.

b) wts $\sup_{x\in S}(x+\alpha) = \sup_{x\in S} x + \alpha$.

by a) $\sup_{x\in S} x = y$

$\sup_{x\in S} x = y+\alpha$

$\Rightarrow$ have to show $\sup_{x\in S}(x+\alpha) = y+\alpha$

$\sup_{x\in S} x = y$

$\Rightarrow \forall x \in S$, $x \le y$

so $\forall x \in S$, $x+\alpha \le y+\alpha$

$\Rightarrow y+\alpha$ is an upper bound for $S_1 = \{x+\alpha : x \in S\}$

sps $z = \sup_{x\in S}(x+\alpha)$ & $z < y+\alpha$

y is lub of S.

$\Rightarrow \forall \varepsilon > 0$ $\exists c \in S$. s.t. $c > y - \varepsilon$

take $\varepsilon = y+\alpha-z$

so $\exists c > y - (y+\alpha-z)$

$= z - \alpha$

$\in S$.

$\Rightarrow c+\alpha \in S_1$

$\Rightarrow c+\alpha > z - \alpha + \alpha$

$> z$

$\Rightarrow z$ not an upper bound

$\Rightarrow \sup_{x\in S}(x+\alpha) = y+\alpha$

$= \sup_{x\in S} x + \alpha$

3\. a) $|a_n| = \left|\frac{1}{n\ln(n+1)}\right|$

$n\ln(n+1) < (n+1)\ln(n+1)$

$n \ge 1$. $\ln(n+1) > 0$.

$\Rightarrow 0 < n\ln(n+1) < (n+1)\ln(n+1)$

$\Rightarrow \frac{1}{n\ln(n+1)} > \frac{1}{(n+1)\ln(n+1)}$

$\sum_{n=1}^{\infty} \frac{1}{(n+1)\ln(n+1)}$ is decreasing, cont. and positive

so apply integral test:

$\int_1^{+\infty} \frac{1}{x+1}\frac{1}{\ln(x+1)}\,dx$

$= \int_1^{+\infty} \frac{1}{\ln(x+1)}\,d[\ln(x+1)]$

$= \ln|\ln(x+1)|\Big|_1^{+\infty}$

$\to +\infty$

$\Rightarrow$ not abs. cvg.

However, if apply alternating series test

$\frac{1}{n\ln(n+1)} \to 0$

$\Rightarrow$ cvg conditionally

b) $\frac{(-1)^n \sin\frac{1}{n}}{n}$

$|b_n| = \frac{\sin\frac{1}{n}}{n}$

for $x \in (0, \frac{\pi}{2})$

we have $\sin x < x$

This can be shown by $\triangle OAC \le$ arc OAC

$\triangle OAC = \frac{1}{2} OC \cdot AB$

$= \frac{1}{2}\sin x$

"arc" OAC

$= \frac{1}{2} \ell r$

$= \frac{1}{2} x \cdot 1 = \frac{1}{2} x$

$\Rightarrow \frac{1}{2}\sin x < \frac{1}{2}x$

$\Rightarrow \sin x < x$

$\frac{1}{n} \in (0,1)$

for $\forall n \in N$

$\Rightarrow \frac{\sin\frac{1}{n}}{n} < \frac{\frac{1}{n}}{n} = \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ cvg

$\Rightarrow b_n$ cvg abstly.

4. a). perfect set

sps X is a isolated pt.

X locate in S_n.

$\Rightarrow X$ locate in S_n

if X is not right edpt of a small interval in S_n.

the distance from X to right edpt is thus smaller than the length of the small interval

$\frac{1}{3^n}$, $n\to\infty$, $\frac{1}{3^n}\to 0$.

if X: right edpt consider left edpt

$|X-y| = \frac{1}{3^n} \to 0$

as $n\to\infty$

$\Rightarrow X$ not an isolated pt

$\Rightarrow$ Perfect set

b). $(\frac{1}{3} \ldots$ length removed from S_0.

$\frac{2}{3} \cdot \frac{1}{3}$,

$(\frac{2}{3})^2 \frac{1}{3}$

$\sum_{i=0}^{\infty} (\frac{2}{3})^n \cdot \frac{1}{3}$

$= \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$

so total length removed is 1

null set: $\forall \varepsilon>0$ the set can be covered by a sequence of intervals whose length is at most ε

$n\to\infty$ length of $S \to 0$

, smaller than $\forall \varepsilon > 0$

c). $X = \sum_{i=1}^{\infty} \frac{1}{3^i}$.

$= \frac{\frac{1}{3}}{1-\frac{1}{3}} = \frac{1}{2}$ $\Rightarrow$

$\notin C$.

d). sps $X \in (\overline{C})^{int}$

$\exists B_r(X) \subset (\overline{C})^{int}$

for some r

i.e. $\exists$ interval

$(X-r, X+r) \subset (\overline{C})^{int}$

claim $\overline{C} = C$

① show C is closed

$C = R \setminus (-\infty, 0) \cup (1, +\infty)$

$\cup (\ )$ open intervals

$\Rightarrow C$ closed.

② for closed interval C,

$C = \overline{C}$

$C \subseteq \overline{C}$.

C: closed $\Rightarrow$ contain all of its limit pt;

$\overline{C}$: smallest contain all limit pt of C

$\Rightarrow \overline{C} \subset C$

$(\overline{C})^{int} = C^{int}$.

if $(X-r, X-1) \in C^{int}$

if $\in S_n$ $\forall n$

the length of small interval

length $< \frac{1}{3^n} \to 0$.

5. ① positive definite

a)

$d(X,Y) = 0$

$\overline{d}(X,Y) = \min\{d(X,Y),1\}$

if $\overline{d}(X,Y) = 0$

$\Rightarrow d(X,Y) = 0$ $(1 \ne 0)$

d metric

$\Rightarrow X = Y$.

if $X=Y$ $d(X,Y)=0$.

$\overline{d}(X,Y) = \min\{0,1\} = 0$.

② symmetry.

$\overline{d}(Y,X)$

$= \min\{d(Y,X),1\}$.

$d(Y,X) = d(X,Y)$

$\Rightarrow \min\{d(Y,X),1\}$

$= \min\{d(X,Y),1\}$

$= \overline{d}(X,Y)$

③ triangle inequality

$\overline{d}(X,Y) = \min\{d(X,Y),1\}$

$\overline{d}(X,Z) = \min\{d(X,Z),1\}$

$\overline{d}(Z,Y) = \min\{d(Z,Y),1\}$

$\overline{d}(X,Z) = \min\{d(X,Z),1\}$

$\Rightarrow \overline{d}(X,Z) \le 1$

$\overline{d}(X,Y) + \overline{d}(Y,Z)$

$= \min\{d(X,Y),1\} + \min\{d(Y,Z),1\}$

① if one of $d(X,Y), d(Y,Z) \ge 1$

we have $\overline{d}(X,Y) + \overline{d}(Y,Z) \ge 1$

combine $\overline{d}(X,Z) \le 1$. done

② if both $d(X,Y), d(Y,Z) < 1$

$\Rightarrow \overline{d}(X,Y) + \overline{d}(Y,Z)$

$= d(X,Y) + d(Y,Z)$

$\ge d(X,Z) \ge \overline{d}(X,Z)$.

Since $\overline{d}(X,Z)$ is the smaller element of $\{d(X,Z),1\}$

b) show $\forall$ subset of X is bdd

6. Cauchy: $\forall \varepsilon > 0$ $\exists N > 0$

s.t. $\rho(X_m, X_n) < \varepsilon$

whenever $m, n \ge N$

sps $\varepsilon = 1$ then $\exists N_1$

st. $\rho(X_m, X_n) < 1$

whenever $m, n \ge N_1$

let. $n = N_1$ $m > N_1$

$\Rightarrow \rho(X_m, X_{N_1}) < 1$

Find the largest distance of X_{N_1} to the first $(N_1 - 1)$ terms

$\Rightarrow d = \max\{\rho(X_{N_1}, X_1), \ldots, \rho(X_{N_1}, X_{N-1})\}$

So $D = \max\{d, 1\}$

$\Rightarrow \forall X_n$, we have

$\rho(X_n, X_{N_1}) \le d$

$\Rightarrow$ bdd.

中 国 作 家 协 会

7. a) $A = \{(x,y) : (x,y \in \mathbb{Q}\}$

not complete

since $\exists$ seq

$(1,1), (1.4, 1.4), (1.41, 1.41)$

$(1.414, 1.414) \ldots \to (\sqrt{2}, \sqrt{2})$

but $(\sqrt{2}, \sqrt{2}) \notin A$

b) $B_r(0)\{(x,y) \mid x^2+y^2 < r\}$

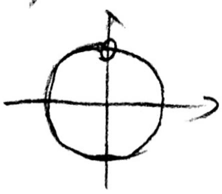
not closed

$\to$ not ~~cdd~~

(subset of Euclidean space is complete iff closed).

c). closed $\Rightarrow$ complete.

b). consider seq $(0, \sqrt{r} - \frac{1}{n})$



$\to (0, \sqrt{r})$

but $(0, \sqrt{r}) \notin B$.

contain 2^n ↑ interval

each length $\frac{1}{3^n}$.

$\lim_{n\to\infty} \frac{2^n}{3^n} = 0$.