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WIKIPEDIA

Wiki ~~M240~~ 2011F MAT240 Algebra I (with assignments)

TA hours BA 6283 Wednesday 3-4PM and Thursday 10-11 AM

'Question site' LaTex

Sets

A set is a collection of objects. If the set X contains x, we write $x \in X$

eg. ① $\{\text{red}, \text{green}, \text{blue}\} = P$

② $\{0, 1\} = F_2$

③ $\mathbb{Z} = \{\cdots, -1, 0, 1, \cdots\}$

④ $\mathbb{Q} = \{p/q \,.\, p \in \mathbb{Z} \text{ and } q \in \mathbb{Z} - \{0\}\}$

⑤ $\mathbb{R}$ = the set of real numbers

⑥ $\mathbb{C}$ = the complex numbers.

Operations on sets

element

the set of

su... h...

① If X, Y are sets, their union is defined to be $X \cup Y = \{z \mid z \in X \text{ or } z \in Y\}$

eg. $P \cup F_2 = \{0, 1, \text{red}, \text{green}, \text{blue}\}$

② their intersection is defined to be $X \cap Y$

$X \cap Y \doteq \{z \mid z \in X \text{ and } z \in Y\}$

eg $P \cap F_2 = \emptyset$

$\mathbb{Q} \cap \mathbb{Z} = \mathbb{Z}$

Maps between Sets

Definition: A map (or function) from the set X to the set Y assigns to each point in X a unique value in Y

Notation $f: X \to Y$

$x \to f(x)$

e.g. ① map from $\mathbb{R}$ to $\mathbb{R}$ $f: \mathbb{R} \to \mathbb{R}$ is a real valued function on $\mathbb{R}$

e.g. $f(x) = x^2 + x$

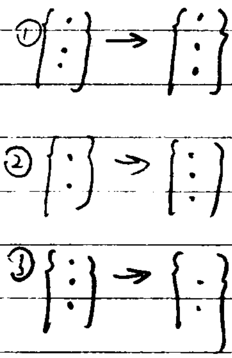
② RGB color is a map $P = \{\text{red}, \text{green}, \text{blue}\} \xrightarrow{c} \{0, 1, 2, \cdots 255\}$

the value $c(\text{red})$ $c(\text{green})$ ... are bright...

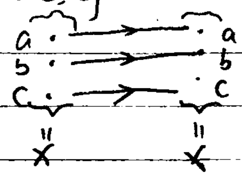
$C: P \to \{0, 1, 2, \cdots 255\}$

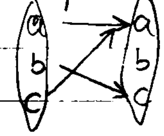
Let's focus on maps between finite sets

e.g.

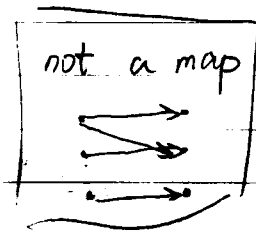


Let $X = \{a, b, c\}$ example of maps $f : X \to X$





is a map
but not an injection



not a unique value

~~Define~~ Definition: A map $f : X \to Y$ is an injection when $f(x) = f(y)$ only when $x = y$.

$$(x = y) \Rightarrow (f(x) = f(y))$$

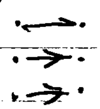
$(x = y) \Leftarrow (f(x) = f(y))$ is ~~true~~ only true for f is injection.

Definition: A map $f : X \to Y$ is an surjection when every possible value in Y is assigned to some element in X.

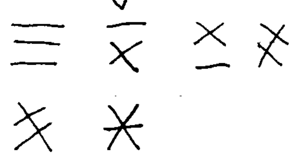
in other words, for all elements $y \in Y$, there exists an element $x \in X$ such that $f(x) = Y$

Definition: $f : X \to Y$ is a bijection, when it's both injection & surjection

e.g.



6 kinds



Bijection $X \to X$ = Permutation

$X = \{a, b, c\}$ $Y = \{d, e\}$ there can be no injections $X \to Y$

not injection
not surjection

surjective
not injective

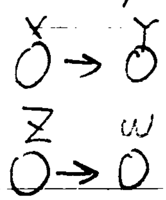
Composition of maps

If $f : Y \to Z$ and $g : X \to Y$ are maps of sets

$$X \xrightarrow{g} Y \xrightarrow{f} Z$$

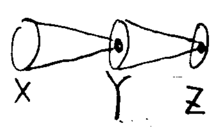
So we define a new map call the composition of f and g:
$f \circ g : X \longrightarrow Z$ defined by $(f \circ g)(x) = f(g(x)) \in Z$

Warning if $f : X \to Y$
$g : Z \to W$



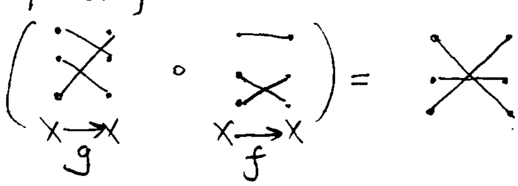
no composition is defined

Need the domain of f must agree ~~the~~ codomain of g.



Example of composition:

$X = \{a, b, c\}$



Theorem: the composition of bijections is a bijection

Proof: Suppose f and g are bijections $X \to X$
I must show $f \circ g$ is also a bijection.
i.e. must show $f \circ g$ is ① injective
② surjective

① must check that if $f(g(x)) = f(g(y))$ then $x = y$.
② ~~must~~ $(f \circ g)$ is surjective

ie. every $y \in X$ is ~~covered~~ covered, i.e. there exists $x \in X$ such that $f(y(x)) = y$

for ① focus on f first. it is an ~~in~~ injection
so $f(g(x)) = f(g(y)) \Rightarrow g(x) = g(y)$ (f inj.)
now focus on g it is an injection
$g(x) = g(y) \Rightarrow x = y$ (g is inj.)

done showing $f \circ g$ inj.

② since f is a surj. $\exists$ some $z \in X$ such that $f(z) = y$. But g also a surjection

so $\exists x \in X$, such that $g(x) = z$ Then $f(g(x)) = y$

~~STA~~ MAT240 Sheldon ~~Alex~~ Axler, Linear algebra done right 2nd ed.

TUT RM 117 Next week

What is a vector space?

A sclar is a numerical quantity in $\mathbb{R}$, $\mathbb{C}$, on any field.

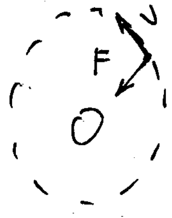
标量

these model mass, length, energy etc.

A vector has "magnitude" and "direction".

矢量

- Force
- Velocity



spin of an electron

"spin state" is a "superposition" of two classical states up & down

$$|\psi\rangle = \alpha|up\rangle + \beta|down\rangle$$

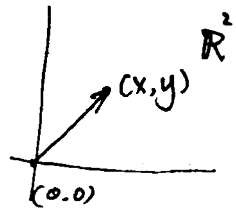
a vector quantity

numbers in $\mathbb{C}$

$$\frac{|\alpha|^2}{|\alpha|^2+|\beta|^2} \quad up$$

Key ~~proputies~~ properties of vectors

e.g. $\mathbb{R}^2 = \{$order pairs (x,y) of real numbers$\}$



(0.0) is call the origin

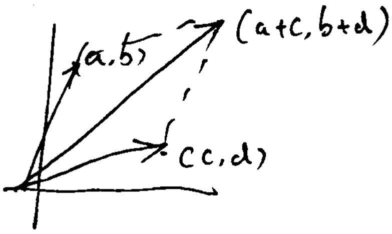
it allows any point (x,y) to be viewed as an arrow $(0,0) \to (x,y)$

$\mathbb{R}^2$ has two operations:

① vector addition : ~~(a,b)+(c,~~

$(a,b)+(c,d) = \cancel{(a+c)}$

$(a+c, b+d)$



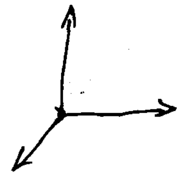
definition of plus operation

② scalar multiplication if $\lambda \in \mathbb{R}$, then

define $\lambda \cdot (a,b) = (\lambda a, \lambda b)$

"list"

example ② $\mathbb{R}^3 = \{ (x, y, z) : x, y, z \in \mathbb{R} \}$



$(a, b, c) + (d, e, f) = (a+d, b+e, c+f)$

$\lambda \cdot (a, \cancel{b} b, c) = (\lambda a, \lambda b, \lambda c)$

③ n-dimentional space $\mathbb{R}^n$

$\mathbb{R}^n = \{ (x_1, x_2 \cdots x_n) : x_i \in \mathbb{R}, \text{ For all } i \}$

n-~~tuple~~ type on list of n#s.

④ If X is any set, define $\mathbb{R}^X = \{\text{function from } X \text{ to } \mathbb{R}\}$

this also has two operations and $\lambda \in \mathbb{R}$

If f, g are functions $X \to \mathbb{R}$ such

then $(f+g)$ ~~(x)~~ is the function that

$$(f+g)(x) = f(x) + g(x) \quad \text{for all } x \in X$$

$\lambda \cdot f$ is the function s.t. $(\lambda f)(x) = \lambda (f(x))$ for all $x \in X$

note: $\mathbb{R}^X$ also has a "zero" or origin

i.e. the special element

$$\left(\begin{array}{l} \underline{0} : X \to \mathbb{R} \\ \underline{0}(x) = 0 \quad \text{for all } x \in X \end{array} \right)$$

Definition: A vector space over $\mathbb{R}$ numbers is: scalar multiplication

a set V and two operations $+$ and $\cdot$

vector addition

$+$ takes a pair $u, v \in V$ and gives $u + v \in V$

$\cdot$ takes a pair $\lambda \in \mathbb{R}$ and $v \in V$ and give $\lambda \cdot v \in V$

SUCH THAT

for $+$

- $u + V = V + u$ (commutative law)
- $(u+V)+\omega = \cancel{\ }\ u + (V + \omega)$ (associative law)
- want a special element $0 \in V$ "zero" s.t. $0 + v = v$ for all $v \in V$
- want opposite of any vector: for any V, there is a W st $V + W = 0$

for $\cdot$: $\begin{cases} (ab)\cdot v = a(b\cdot v) \\ 1\cdot v = v \end{cases}$

Both : $\begin{cases} (a+b)\cdot v = a\cdot v + b\cdot v \\ a\cdot(u+v) = a\cdot u + a\cdot v \end{cases}$

Sept. 20th

1. understand definition
2. linear subspace
3. New fields of numbers

Simple facts about a vector ~~z~~ space

① If $u+v = \omega + v$ then $u = \omega$

② ~~$\vec{0}+v=$~~ the additive identity is unique.

③ the opposite of vector is unique

④ $0\cdot\vec{v} = \vec{0}$

⑤ $(-1)\cdot\vec{v}$ is the opposite of v.

⑥ $\lambda\cdot\vec{0} = \vec{0}$

Proofs:

① If $\vec{u}+\vec{v} = \vec{\omega}+\vec{v}$

take $\boxed{\text{the opposite}}$ $\vec{x}$ of $\vec{v}$ (i.e. $\vec{x}+\vec{v}=\vec{0}$), add it to both sides

$(\vec{u}+\vec{v})+\vec{x} = (\vec{\omega}+\vec{v})+\vec{x}$

use $\boxed{\text{associative law}}$ $\Rightarrow$ $\vec{u}+(\vec{v}+\vec{x}) = \vec{\omega}+(\vec{v}+\vec{x})$

$\vec{u}+\vec{0} = \vec{\omega}+\vec{0}$

$\vec{0}$ is an additive ~~i~~ identity

$\vec{u} = \vec{\omega}$ $\blacksquare$

② Suppose $\vec{0}$ and $\vec{0}'$ are two additive identities.

Since $\vec{0}$ is an $\boxed{\text{additive identity}}$, $\vec{0}+\vec{v}=\vec{v}$

since $\vec{0}'$ is an additive identity, $\vec{0}'+\vec{v}=\vec{v}$

$\Rightarrow$ $\vec{0}+\vec{v} = \vec{0}'+\vec{v}$,

$\overset{①}{\Rightarrow}$ $\vec{0} = \vec{0}'$ $\blacksquare$

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1100256518
011861356047650

③ Suppose $\vec{y}, \vec{y}'$ are two ~~opposites~~ opposites of $\vec{x} \in V$

This means $\vec{y} + \vec{x} = \vec{0}$ and $\vec{y}' + \vec{x} = \vec{0}$

but then $\vec{y} + \vec{x} = \vec{y}' + \vec{x}$

by ① $\Rightarrow \vec{y} = \vec{y}'$

④ Start with $0 \cdot \vec{v}$

$(a+b) \cdot \vec{v} = a\vec{v} + b\vec{v}$

since $0+0=0$

$0\vec{v} = (0+0)\vec{v} = 0 \cdot \vec{v} + 0 \cdot \vec{v}$

(using distributive law)

since $\vec{0}$ is an additive identity

$\vec{0} + 0 \cdot \vec{v} = 0 \cdot \vec{v} = 0 \cdot \vec{v} + 0 \cdot \vec{v}$

use ① to cancel

$\vec{0} = 0 \cdot \vec{v}$

⑤ $(-1) \cdot \vec{v} + 1 \cdot \vec{v} = \vec{0}$

using

by the disitributive law $(-1+1)\vec{v} = \vec{0}$

$\Rightarrow 0 \cdot \vec{v} = \vec{0}$

④ $\Rightarrow 0 \cdot \vec{v} = \vec{0}$

$\Rightarrow (-1+1) \cdot \vec{v} = \vec{0}$

$\xRightarrow{\text{distributive}} (-1) \cdot \vec{v} + 1 \cdot \vec{v} = \vec{0}$

$\Rightarrow (-1) \cdot \vec{v} + \vec{v} = \vec{0}$

SUBSPACE

A subspace U of a vector space V is a subset $U \subset V$ which itself is a ~~version~~ vector space, using the operations from V.

<u>In other words</u> ① $\vec{0}$ must be in U

② if $u, v \in U$ then $u+v \in U$

③ any scalar multiple of ~~the~~ $u \in U$, must lie in U.

The rest of the axioms already hold in V, so they hold in all U.

Subspaces examples

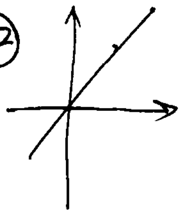
① Any vector ~~space~~ space V has two special spaces $\begin{cases} V \\ \{\vec{0}\} \end{cases}$

② In $\mathbb{R}^2$ $\{(0,0)\}, \mathbb{R}^2$

(2.1) Consider $\mathbb{Z}^2 \subseteq \mathbb{R}^2$
$= \{(m,n)\ m,n \text{ integer}\}$

this satisfies ①&② but not ③ $\boxed{\text{NOT SUBSPACE}}$

(2.2)



All subspaces are either $\{(0,0)\}$, $\mathbb{R}^2$, ~~are~~ or a line through $(0,0)$ (l = line through origin
$= \{(x,y) : ax+by=0\}$

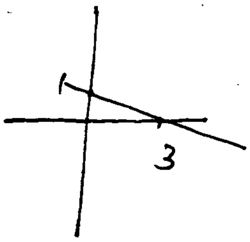
$\Updownarrow$ ~~homogenous~~ homogeneous linear equation

e.g. $a=1, b=0$ $\{x=0\}$
$a=0, b=1$ $\{y=0\}$

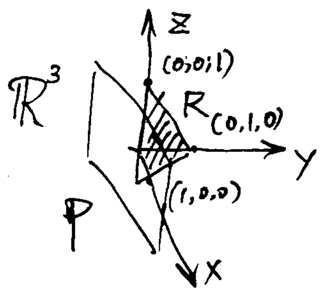
(2.3) what if ~~we~~ ~~(a,b)~~ $\{(x,y) : ax+by=3\}$ inhomogeneous linear equation

$a=1$
$b=3$ $x+3y=3$

$\boxed{\text{NOT SUBSPACE}}$



(3)



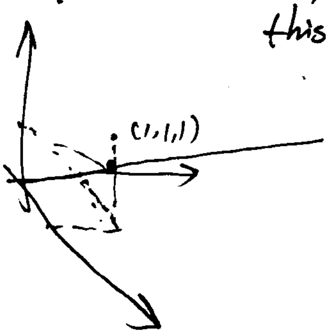
(3.1) $\{(x,y,z) : x+y+z=0\} = P$ is a subspace

(3.2) $\{(x,y,z) : x+y+z=1\} = R$ not a subspace

Remark about ex. 2, 3.
There's a ~~geo~~ geometric way of ~~thinking~~ about linear equations.

~~$\{(x,y,z)\}$~~ $\{(x,y,z) : ax+by+cz=d$, a,b,c,d are real numbers$\}$ defines a plane in $V=\mathbb{R}^3$ and it gives a subspace exactly when $d=0$

(3.3) $\{t\cdot(1,1,1) : t\in\mathbb{R}\}$ and it defines a subspace.
this is a LINE through $\vec{0}$



FACT all subspaces ~~through~~ of $\mathbb{R}^3$ are:

- $\{0\}$
- line through $\vec{0}$
- plane through $\vec{0}$
- $\mathbb{R}^3$

[redacted]

Fields

Definition: A field is a number system "like $\mathbb{R}$"

• 2 operations $+, \times$ such that

	$+$	$\times$
commutative	$a+b=b+a$	$ab=ba$
associative	$(a+b)+c=a+(b+c)$	$(ab)c=a(bc)$
∃ identity elements ~~identities~~ $0,1$	$0+b=b$	$1\times a=a$
inverse	for all a, $\exists b$, $ab=a$;	for all $a\neq 0$, $\exists b$ st. $ab=1$

final axiom distributive law ~~&~~ $a(b+c)=ab+ac$

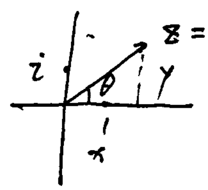
focus on $\mathbb{R}$, $\mathbb{C}$, finite field $\mathbb{F}_p$

Ex 1. $\mathbb{C}=\{a+bi : a,b\in\mathbb{R}\}$ (same as $\mathbb{R}^2$) $\qquad i^2=-1$

operations $\oplus$ $(a+bi)+(c+di)=(a+c)+(b+d)i$

$\otimes$ $(a+bi)(c+di)=\cancel{ac}\ ac-bd+(ad+bc)i$

Tips from ~~trign.~~ trigonometry for $\mathbb{C}$



real part

$z=x+iy \rightarrow$ imaginary part

A complex number $z=x+yi$ has a length $r=\sqrt{x^2+y^2}$ and an argument θ

and $\boxed{z=re^{i\theta}}$ where $\boxed{e^{i\theta}=\cos\theta+i\sin\theta}$

$z=r\cdot\cos\theta+r\cdot i\sin\theta$ ✓

Advantage of writing $z=re^{i\theta}$ is that multiplication is easy:

$$\left.\begin{aligned} z_1&=r_1e^{i\theta_1}\\ z_2&=r_2e^{i\theta_2}\end{aligned}\right\} \Rightarrow z_1z_2=r_1r_2e^{i(\theta_1+\theta_2)}$$ ✓

Fields

~~C, U, R~~ C ∪ R ∪ Q — Recall $\mathbb{F}$ has $+$, $\times$

- commut & associate
- distributive law
- distinct 0, 1 identities
- inverse (for $\times$, 0 not invertible)

[redacted]

smallest field $\mathbb{F}_2 = \{0,1\}$ — a particular field, $1+1=0$

+	0	1
0	0	1
1	1	0

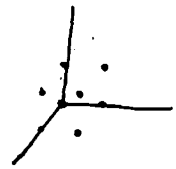
×	0	1
0	0	0
1	0	1

interpretation. 0 means "false" 1 means "true"

$+ \longleftrightarrow$ "XOR"

$\times \longleftrightarrow$ "AND"

What do $\mathbb{F}_2$-vectors look like? / $\mathbb{F}_2^3 \to$



$(\mathbb{F}_2)^2 = \{(a,b) : a \in \mathbb{F}_2, b \in \mathbb{F}_2\}$

Example 1 $(\mathbb{F}_2)^n$ is a vector space with 2^n elements

$\boxed{\ \ |\ \ |\ \cdots|}$ can be decribed by a vector $(a_1, \cdots a_n) \in \mathbb{F}_2^n$

n bits

$(\mathbb{F}_2)^2$

what do subspaces look like?

- $\{(0,0)\}$ zero subspace
- $\{(0,0),(0,1)\}$
- $\{(0,0),(1,0)\}$
- $\{(0,0),(1,1)\}$
- $\mathbb{F}_2^2$ full space

three lines in $\mathbb{F}_2$ — only 5 subspaces

[redacted]

Example 2. $\mathbb{F}_3 = \{0,1,2\}$ "modular arithmetic mod 3"
"throw out mult. of 3"

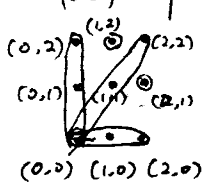
$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\frac{1}{2} = 2$

$$\boxed{\begin{array}{l}\text{vector spaces over } \mathbb{F}_3 \text{ (example)} \\ (\mathbb{F}_3)^n = \{(a_1, \cdots a_n) : a_i \in \mathbb{F}_3 \text{, for all } i\} \\ \lambda \in \mathbb{F}_3 \quad \lambda \cdot (a_1, \cdots a_n) = (\lambda a_1, \cdots \lambda a_n)\end{array}}$$

$(\mathbb{F}_3)^2$ = pairs (x,y), $x, y \in \mathbb{F}_3$



$(\mathbb{F}_3)^2$ has only 9 vectors

Subspaces of $(\mathbb{F}_3)^2$

① $\{(0,0)\}$

"lines" ② $\{(0,0),(1,0),(2,0)\}$

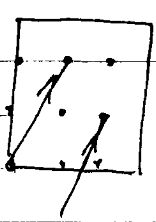
③ $\{(0,0),(1,1),(2,2)\}$

④ $\cancel{\{(0,0),(2,1)\}}$ if $\{(0,0),(2,1)\}$ is a subspace, then $2 \cdot (2,1)$ also in subspace

$\Downarrow$ $2 \cdot (2,1) = (1,2)$

$\{(0,0),(2,1),(1,2)\}$ may not "look" like a line but it is a subspace.

$(1,2),(2,1),(0,0)$ form a line?



⑤ $\{(0,0),(0,1),(0,2)\}$

⑥ $\mathbb{F}_3^2$

Ex 3. $\mathbb{F}_4 = \{0,1,2,3\}$ "mod 4 arithmetic"

~~$+$ 1 2 3~~

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\nearrow$ shows there's no inverse for 2

FAIL

$\Downarrow$

NOT A FIELD

Why does $\mathbb{F}_4$ fail? "4 is not prime" $4 = 2 \times 2 \overset{?}{=} 0$

if 2 has inverse $0 = 2 \cdot 2 \cdot 2^{-1} \Rightarrow 2 = 0$

~~Proposition U_3 is a vector space~~ of $\mathbb{R}^2$

~~Proof: $0 \in U_3$~~ ✓

$\in$ ~~Close~~

Proposition V vector space

$U_1, \cdots U_m \subseteq V$ subspaces

Then $U_1 + \cdots U_n$ also a subspace of V

Proof: $\cdot$ Is the zero vector 0 an element of $U_1, U_2 \cdots U_n$

$$0 = \underbrace{0 + \cdots + 0}_{n}$$

$0 \in U_1, \cdots, U_n$

$\cdot$ closed under addition: $\omega, \theta \in U_1 + \cdots + U_n$

write $\theta = u_1 + \cdots + u_n$ where $u_1 \in U_1, \cdots u_n \in U_n$

$\omega = u_1' + \cdots + u_n'$ $\quad u_1' \in U_1, \cdots u_n' \in U_n$

$\Rightarrow \theta + \omega = (u_1 + u_1') + (u_2 + u_2') + \cdots + (u_n + u_n')$

$\Downarrow$ in U_1 $\quad$ in U_2 $\cdots$ in U_n

$u_1 + u_1' \in U_1$ because U_1 is a subspace of V and $u_1, u_1' \in U_1$

$\Rightarrow \theta + \omega \in U_1 + \cdots + U_n$

<u>Closed under scalar multi</u>

$\lambda \in \mathbb{R}$, $\theta \in U_1 + \cdots + U_n$

$\theta = u_1 + u_2 + \cdots + u_n$, $u_1 \in U_1, u_2 \in U_2 \cdots u_n \in U_n$

$\lambda \theta = (\lambda u_1) + (\lambda u_2) + \cdots + (\lambda u_n)$

$\lambda u_1 \in U_1$, because U_1 is a subspace of V

$\cdots$ etc.

$\lambda \theta \in U_1 + \cdots + U_n$

$\therefore U_1 + U_2 + \cdots + U_n$ is a subspace of V.

MAT240 Sept. 27th

Building Subspaces

Recall for subsets

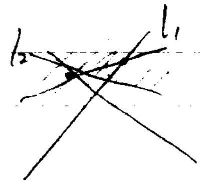
$\left.\begin{array}{l} U_1 \cap U_2 \\ U_1 \cup U_2 \end{array}\right] \Rightarrow$ two new subsets.

For vector spaces

If U_1, U_2 subspaces of V
Then $U_1 \cap U_2$ is a subspace also.
But $U_1 \cup U_2$ is not a subspace in general.

Idea: fix this by defining $U_1 + U_2 = \{u_1 + u_2 : u \in U_1 \text{ and } u_2 \in U_2$

e.g.

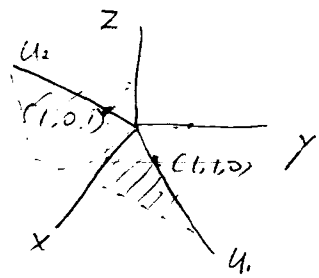


$l_1 + l_2$ This is a subspace

Why? ① 0 is in $U_1 + U_2$
② $(u_1 + u_2) + (u_1' + u_2') = (u_1 + u_1') + (u_2 + u_2') \in U_1 + U_2$
③ sclar multiples

Example of $+, \cap$ on subspace.

① $U_1 = \{\lambda(1,1,0) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^3$
$U_2 = \{\lambda(1,0,1) : \lambda \in \mathbb{R}\} \subset \mathbb{R}^3$

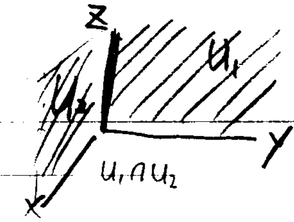


PLANE: $U_1 + U_2$

$U_1 \cap U_2 = \{0\}$

$U_1 + U_2 = \{a(1,1,0) + b(1,0,1) : a, b \in \mathbb{R}\} = \{(a,a,0) + (b,0,b) : a, b \in \mathbb{R}\}$
$= \{(a+b, a, b) : a, b \in \mathbb{R}\}$
$= \{x, y, z : x - y - z = 0\}$

② $U_1 = \{(x,y,z) : x=0\}$
$U_2 = \{(x,y,z) : y=0\}$



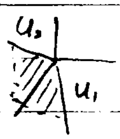
$U_1 \cap U_2 = \{(x,y,z) : x=0, y=0\}$

$U_1 + U_2 = \{(0,y,z) + (s,0,t) : y,z,s,t \in \mathbb{R}\}$

$= \{(s,y,z+t) : s,y,z,t \in \mathbb{R}\} = \mathbb{R}^3$

Direct Sum

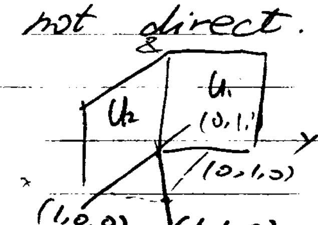
In example ① If $v \in U_1 + U_2$



then $v = u_1 + u_2$, $u_1 \in U_1$, $u_2 \in U_2$
But I claim there is only one possible choice for u_1, u_2 (unique)
Why?
$v = (a+b, a, b)$ for some $a, b \in \mathbb{R}$
want $v = \lambda_1(1,1,0) + \lambda_2(1,0,1)$
has only one solution
$(\lambda_1, \lambda_1, 0) + (\lambda_2, 0, \lambda_2) = (a+b, a, b)$
$\Rightarrow \lambda_1 = a$
$\lambda_2 = b$
$\rightarrow$ there is only one way to choose $u_1 \in U_1$, $u_2 \in U_2$
s.t. $v = u_1 + u_2$

Definition: we say a sum $U_1 + U_2$ is direct when each $v \in U_1 + U_2$ can be written as $v = u_1 + u_2$, $u_1 \in U_1$, $u_2 \in U_2$ but in only one way.

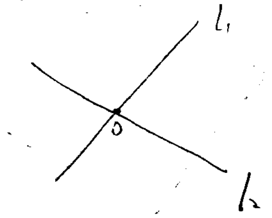
② In example 2, the sum is not direct.
$U_1 = \{(x,y,z), x=0\}$
$U_2 = \{(x,y,z), y=0\}$
take $(1,1,0) \in \mathbb{R}^3$



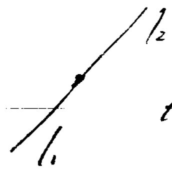
so can write $(1,1,0) = (1,0,0) + (0,1,0)$
but also $(1,1,0) = (0,1,1) + (1,0,-1)$

Rephrasing:

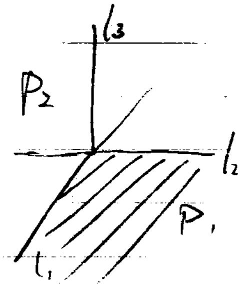
If you are building a subspace by sums.



direct



$l_1 + l_2$ not direct



$l_1 \oplus l_2 =$ plane P_1 direct

$P_1 + l_1 = P_1$ not direct sum

$P_1 \oplus$ any line out of $P_1 = \mathbb{R}^3$ (l_3)

$P_2 = l_1 \oplus l_3$

Direct: (efficient)

$P_1 + P_2$ not a direct sum.

Reformulations of Direct condition:

~~Proposition~~ The sum $U_1 + U_2$ is direct if and only ~~if~~ if $u_1 + u_2 = 0$ is only possible when $u_1 = u_2 = 0$ (*)

Proof: $\Rightarrow$ Suppose $U_1 + U_2$ is direct, ~~then $0 = 0 + 0$ is unique~~ then $\underline{0}$ can be written uniquely as a sum $u_1 + u_2$, $u_1 \in U_1$, $u_2 \in U_2$ but we know $\underline{0} = \underline{0} + \underline{0}$, so $u_1 = \underline{0}$, $u_2 = \underline{0}$

$\Leftarrow$ Suppose (*) is true, must prove $U_1 + U_2$ is direct.

take $v \in U_1 + U_2$, must show $v = u_1 + u_2$ for unique u_1, u_2.

$\Rightarrow$ $v = u_1 + u_2$ for some $u_1 \in U_1$, $u_2 \in U_2$ to show uniqueness, suppose $v = w_1 + w_2$, $w_1 \in U_1$, $w_2 \in U_2$

then $v = u_1 + u_2 = w_1 + w_2 \Rightarrow 0 = (w_1 + w_2) - (u_1 + u_2) = \overbrace{(w_1 - u_1)}^{\in U_1} + \overbrace{(w_2 - u_2)}^{\in U_2}$

$\Rightarrow u_1 = w_1$, $u_2 = w_2$

Prop: The sum $U_1 + U_2$ is direct if and only iff $U_1 \cap U_2 = \{0\}$.

Proof: If sum $U_1 + U_2$ is direct

Suppose $w \in U_1 \cap U_2$

Want to show $w = 0$

but: $0 = w + (-w)$

(should be unique) and $0 = 0 + 0$

So $w = 0$ by directions

$\Leftarrow$ Suppose $U_1 \cap U_2 = \{0\}$, must show $U_1 + U_2$ is direct. take $v \in U_1 + U_2$

$\Rightarrow v = u_1 + u_2$ for $u_1 \in U_1$, $u_2 \in U_2$

Is it unique? If $v = w_1 + w_2$ for $w_1 \in U_1$, $w_2 \in U_2$

then $(u_1 + u_2 = w_1 + w_2) \Rightarrow u_1 - w_1 = w_2 - u_2$

this means $u_1 - w_1 \in U_1 \cap U_2 = \{0\}$

$\Rightarrow u_1 = w_1$

It also means $w_2 - u_2 \in U_1 \cap U_2 \Rightarrow w_2 = u_2$

example: $V = P(\mathbb{R})$

= vector space of polynomials in one variable with real coefficients.

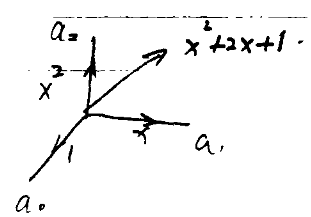
$= \{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m : a_i \in \mathbb{R}^{\forall}, m = \{0, 1, 2, \cdots\}\}$

Subspaces of V:

$P_k(\mathbb{R}) = \{$polynomials of degree $\le k\}$

$P_2(\mathbb{R}) = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$

quadratic polynomials



take $U_1 = P_2(\mathbb{R})$, $U_2 = P_3(\mathbb{R})$ two spaces in $P(\mathbb{R})$

Q: what is $U_1 + U_2 = \{a_0 + a_1 x + a_2 x^2 + b_0 + b_1 x + b_2 x^2 + b_3 x^3 ; a_i, b_j \in \mathbb{R}, \forall i, j\}$

$= \{a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + b_3 x^3\}$

$= P_3(\mathbb{R}) = U_2$

Q: What is $U_1 \cap U_2$

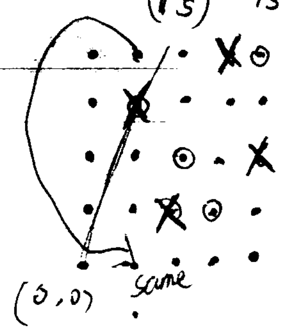
A: $U_1 \cap U_2 = \{$polynomials of $f \in U_1$, and $f \in U_2\}$

$= \{f = a_0 + a_1 x + a_2 x^2\} = U_1$

Note: $U_1 + U_2$ is not direct.

modular arithmetic mod q only gives a field when q is prime.
we call these $\mathbb{F}_p$ (p: prime)
the most common finite field.
example:

$(\mathbb{F}_5)^2$ is a vector space over $\mathbb{F}_5$



example of a line in $(\mathbb{F}_5)^2$

$\{(0,0), (1,3), (2,1), (3,4), (4,2)\}$

$3+3=?$

these subspaces of $(\mathbb{F}_p)^n$ are used in "error correcting codes".

idea: tell before hand message lies in
a subspace $\mathcal{C} \subset (\mathbb{F}_p)^n$ if the message is garbled,
take "closest" point in $\mathcal{C}$ allows correction.

example: Hamming code is a subspace $\mathcal{C}$ in $(\mathbb{F}_2)^7$ (has $128 = 2^7$ elements)
which has 16 elements,

16

MAT 240 TUT
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Def. A field consists of a set F, and two maps $+, \times : F \times F \to F_2$, satisfying several properties (computibily conditions)

$\mathbb{F} = \{0, 1\}$

$0+0=0$

$1+0=1$

$0+1=1$

$1+1=0$

ABOUT Exerzitien 1

Excercise 3.

#3

Set $\mathbb{R}^2$

Candidate addition : $(x+y)+(u.v) = (x+u.0)$

Candidate Scalar Multi: $\lambda(x,y) = (\lambda x, 0)$. $\lambda \in \mathbb{R}$

Q Is this a vector over $\mathbb{R}$?

$(1,1)+(x,y) = (x+1, 0)$ $\forall x, y \in \mathbb{R}$ for all

$\neq (1,1)$

$\forall x, y \in \mathbb{R}, (1,1)+(x,y) \neq (1,1)$

$\Rightarrow$ There does not exist a zero ~~factor~~ vector

Conclusion: There exists an element of $\mathbb{R}^2$: V, such that $V+W \neq V$ for any $W \in \mathbb{R}^2$ (contradiction)

Another way: $1 \cdot (1,1) = (1,0) \neq (1,1)$

Axiom: $\exists\, 0 \in V$, st. $\forall v \in V$, $v+0=v$

$\exists\, 0 \in V$ st. $\forall w \in V$, $v+w \neq v$

another is: $\forall v \in V$, $1 \cdot v = v$

negate $\exists\, v \in V$, st. $1 \cdot v \neq v$

On the other hand

$Q_3(\mathbb{R})$ = polynomials of the form $\{0+0x+0x^2+a_3x^3+\cdots+a_mx^m\}$

Q: Is Q_3 a subspace? Yes! (3 steps to check)

Q: What is $Q_3^{(\mathbb{R})} \cap P_2^{(\mathbb{R})}$? $=\{0\}$

Prop $\Rightarrow$ This means $Q_3(\mathbb{R})+P_2(\mathbb{R})$ is a **direct sum**.

OK, $Q_3(\mathbb{R})+P_2(\mathbb{R})$ is direct

Is it true that $\underbrace{Q_3(\mathbb{R})\oplus P_2(\mathbb{R})}_{\text{this direct sum}} = P(\mathbb{R})$?

To prove $Q_3(\mathbb{R})+P_2(\mathbb{R}) = P(\mathbb{R})$

we'll show $LHS \subseteq RHS$, $RHS \subseteq LHS$

$LHS \subseteq RHS$ is obvious from definition of sum of subspaces (it's a subspace)

$RHS \subseteq LHS$: Take any polynomial $f(x) = a_0 + a_1x + \cdots + a_mx^m$

If we split it:

$f(x) = P_2 + Q_3 = f \in LHS$

[redacted]

Mat240 Sept. 29th

$U_1, U_2 \subset V$ subspaces

① Intersection of U_1 with U_2 $U_1 \cap U_2 = \{v \in V : v \in U_1, v \in U_2\}$

② Sum of U_1 and U_2 : $U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}$

We say $U_1 + U_2$ is <u>direct</u> when every element of $U_1 + U_2$ has a unique decomposition as a sum of two elements $u_1 \in U_1$, $u_2 \in U_2$

If $v \in U_1 + U_2$, $v = u_1 + u_2 = u_1' + u_2'$

then $u_1 = u_1'$, $u_2 = u_2'$

Prop: The sum $U_1 + U_2$ is direct if and only if $U_1 \cap U_2 = \{\underline{0}\}$

(notation for direct sum. $U_1 \oplus U_2$)

Span

Definition: The span of a vector is

$$\text{Span}(v) = \{av : a \in \mathbb{F}\}$$

Example. In $V = \mathbb{R}^3$, if $v = (a, b, c) \neq (0,0,0)$

then Span(v) is the line through (a,b,c) and $(0,0,0)$

If $v = (0,0,0)$ $\text{Span}(v) = \{\underline{0}\}$

Span of 2 vectors:

$$\text{Span}(v_1, v_2) = \underbrace{\{av_1 + bv_2 / a, b \in \mathbb{F}\}}_{\text{all "linear combinations" of } v_1, v_2}$$

$\text{Span}(v_1, v_2) = \text{Span}(v_1) + \text{Span}(v_2)$

$\text{Span}(v_1, v_1) = \text{Span}(v_1) + \text{Span}(v_1) = \text{Span}(v_1)$ (not direct!)

$V = \mathbb{R}^3$ $\text{Span}(v_1, v_2)$

- If $v_1, v_2 = 0$, then $\text{Span}(v_1, v_2) = \{\underline{0}\}$ // Span (v_2) if $v_2 \neq 0$
- If $v_1 \neq 0$, and $v_2 \in \text{Span}(v_1)$. $\text{Span}(v_1, v_2) = \text{Span}(v_1)$ = line through v_1 and $\underline{0}$.
- If $v_1 \neq 0$, and $v_2 \notin \text{span}(v_1)$, $\text{Span}(v_1, v_2)$ is a plane containing $\underline{0}$.

$$\text{Span}(v_1, \dots, v_m) = \{a_1 v_1 + a_2 v_2 + \dots + a_m v_m : a_1, \dots a_m \in \mathbb{F}\}$$
$$= \text{span}(v_1) + \text{span}(v_2) + \dots + \text{spain}(v_m)$$

$$\boxed{\text{Span}(\ ) = \{\underline{0}\}}$$

vectors $v_1, \dots v_m$

Definition: A vector space V is a finite-dimension when $V = \mathrm{Span}(v_1, \dots v_m)$ for some

$\forall = \{f(x) = a_n x^n + \dots + a_1 x + a_0 \mid a_0, \dots a_n \in \mathbb{F}, n > 0\}$
$= \mathrm{span}\{1, x, x^2, x^3, \dots\}$ not finite-demisional

Definition: $v_1, \dots v_m$ are linearly dependent if $\exists$ sclars $a_1, \dots a_m$ not all zero, s.t.

$$a_1 v_1 + \dots + a_m v_m = 0$$

Suppose $a_i \neq 0$ $\quad a_i v_i = -\sum_{k \neq i} a_k b_k$

$$v_i = \left(-\sum_{k \neq i} a_k v_k\right) / a_i$$

$$v_i = \mathrm{Span}\underbrace{(v_1, v_2 \cdots v_m)}_{\text{no } v_i}$$

Consider $(1,0), (0,1), (2,1) \in \mathbb{R}^3$

linear dependent

$$2(1,0) - (2,1) + (0,1) = (0,0)$$

$(v_1 \cdots v_m, \underline{0})$

$$100(\underline{0}) + 0v_1 + 0v_2 + \dots 0v_m = \underline{0}$$

Example 3:

$\underbrace{(1,3,5)(-2,1,3)}_{\text{linearly independent}}, (-4,9,1) \in \mathbb{R}^3$

$$a\begin{pmatrix}1\\3\\5\end{pmatrix} + b\begin{pmatrix}-2\\1\\3\end{pmatrix} = \begin{pmatrix}-4\\9\\1\end{pmatrix}$$

$$\left.\begin{aligned} a - 2b &= -4 \\ 3a + b &= 9 \\ 5a + 3b &= 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a &= 2 \\ b &= 3 \end{aligned}$$

$$2(1,3,5) + 3(-2,1,3) = (4,9,1)$$

$$2(1,3,5) + 3(-2,1,-3) - (-4,9,1) = (0,0,0)$$

$\therefore$ List is dependent.

$(v_1, \cdots v_m)$ is linearly independent when the only linear solution
$a_1v_1+a_2v_2+\cdots+a_mv_m=0$ is the one with $a_1=\cdots=a_m=0$

MAT240 TUT Sept 29th

~~Fixed~~ Fix a field $\mathbb{F}$.

Def: A F-vector space (vector space F) is a set V, together with two maps

In $\mathbb{F}_2$ one sometimes writes $1+1=0$

1 means "the set containing 1"

0 means "the set containing 0"

$\mathbb{F}_7$: $5+6$

$5+6=11$ is in the set containing 4 $\quad 5+6=4$ in $\mathbb{F}_7$

Assignment 2#

Exercise 3

$\boxed{\mathbb{F}_2^3}$

Suppose that $S \subseteq \mathbb{F}_2^3$ that is a subspace. contains at least 3 ~~least.~~ elements. Then S contains two ~~distant~~ distinct non-zero elements, v and w.

Note that $v+w \in S$ as well. Since S is subspace

Observe that $v+w \neq 0$, $v+w \neq v$, $v+w \neq w$.

Mark: We know that $v+v=0$

$(x,y,z)+(x,y,z)=(x+x, y+y, z+z)$ $x,y,z \in \mathbb{F}_2$

So, if $v+w=0$, then $v+w=v+v \Rightarrow v=w$ contradiction

$\therefore$ S contains at least 4 elements.

It suffices to show that if S has at least 5 elements ~~that~~ then it must have 8 elements.

Suppose S has at least 5 elements

Let v, w be as before

Recall that $\{0, v, w, v+w\}$ is a 4-element subset of S

Choose $\alpha \in S$ s.t. $\alpha \neq 0$, v, w or $v+w$

Consider the below elements of S (noting that S is a subspace)

α, $\alpha+v$, $\alpha+w$, $\alpha+v+w$

These are 4 ~~distant~~ distinct elements of S

(If and ~~only if~~ any two of these were equal, then, by cancelling α's, we would contradict the fact that $0, v, w$ and $v+w$ are all distinct.

To show that S has 8 elements, it remains to prove that

$\{\alpha, \alpha+v, \alpha+w, \alpha+v+w\} \cap \{0, v, w, v+w\} = \emptyset$

~~① $\alpha \neq 0, v, w, v+w$ (by hypothesis)~~

② Notice that $\cancel{\alpha =}$ $= \{\alpha + s : s \in \_\_\}$

If the intersection were ~~not~~ non-empty, then $\alpha + s_1 = s_2$ for some $s_1, s_2 \in A$

$\Rightarrow \alpha = s_2 - s$

MAT240 Oct. 4th

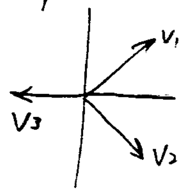
LAST TIME

Span of a list of vectors

$\text{Span}(\underset{\text{empty}}{\ }) = \{0\}$

$\text{Span}(v_1, \cdots v_n) = \{a_1 v_1 + \cdots + a_n v_n : a_i \in \mathbb{F}.\ V_i \in V\}$

Ex: $\text{Span}(v_1, v_2, v_3)$ $\quad v_i \in \mathbb{R}^2$



Def: A list is linearly dependent when it's not empty and there is a linear relation

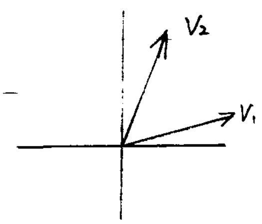
$a_1 v_1 + \cdots + a_n v_n = 0$

with not all $a_i = 0$

Def. A list is linearly independent when it's not linearly dependent.

i.e. list is either empty or the only linear relation $\underbrace{\ }_{a_1 v_1 + \cdots + a_n v_n = 0}$ is the zero relation $a_i = 0,\ \forall i$

TODAY: What is dimension?



Idea: dimension of V is related to the length of a spanning set

Warning: this breaks down if the list is linearly dependent.

Throwing away redundent vector

Lemma: Suppose $(v_1, \cdots v_m)$ is linearly dependent and $v_1 \neq 0$ (otherwise, throw out)

Then there is a vector in the list, which is in $\text{Span}(v_1, \cdots v_{j-1})$

and if we throw out V_j, Span of remaining list is unchanged

Proof: Since this list is linearly dependent there are $a_i \in \mathbb{F}$, not all zero, with

$a_1 v_1 + \cdots + a_m v_m = 0$

[redacted]

Let a_j be the last non-zero constant
then we can solve for v_j:

$$a_j v_j = -a_1 v_1 - \cdots - a_{j-1} v_{j-1}$$

$$v_j = -\frac{1}{a_j}(a_1 v_1 + \cdots + a_{j-1} v_{j-1}) \in \text{Span}(v_1, \cdots, v_{j-1})$$

for second part

hat means v_j is deleted

$$\text{Span}(v_1, \cdots, v_m) = \text{Span}(v_1, \cdots, \hat{v_j}, \cdots, v_m)$$

It's clear that this $\supseteq$ holds

To show $\subseteq$: take $w \in \text{Span}(v_1, \cdots v_m)$

then $w = c_1 v_1 + \cdots + c_m v_m$

then plug in expression for v_j

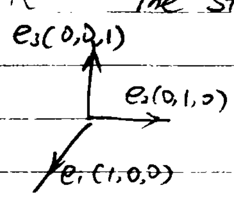
$$w = \left(c_1 - \frac{c_j}{a_j} a_1\right) + \cdots + \cdots (\text{no } v_j \text{ terms})$$

Definition: A **BASIS** for the finite dimensional space V is a linearly independent list $(v_1, \cdots v_m)$ which spans V.

Note: the empty list is basis for $\{0\}$

e.g. ① The standard basis of $\mathbb{F}^n$ is the list
$((1,0,0,\cdots,0), (0,1,0,\cdots,0), (0,0,1,\cdots,0), \cdots (0,0,0,\cdots 1))$

$\mathbb{R}^3$ The standard basis for $\mathbb{R}^3$ is (e_1, e_2, e_3)



Why is it a basis?

Check 2 things ① Span $= V$ ② lin. indep

① $\text{Span}(e_1, \cdots e_n) = \{a_1 e_1 + a_2 e_2 + \cdots + a_n e_n \mid a_i \in \mathbb{F}\}$
$= \{(a_1, a_2, \cdots a_n) : a_i \in \mathbb{F}\}$
$= \mathbb{F}^n$

② to show lin. indep.
check that any linear relation must be zero.
Suppose there is a linear relation

$c_1 e_1 + \cdots + c_n e_n = 0$

This means $(c_1, \cdots c_n) = (0, \cdots, 0)$

this means $c_i = 0$, $\forall i$, zero relation ■

warning: in general, a vector space will not have a special basis. It has many bases, no special one.

non-standard basis for $\mathbb{R}^2$

$((1,1),(0,1))$ is also a basis for $\mathbb{R}^2$

why? Span $= \mathbb{R}^2$ lin. indep

to prove ~~that~~ this span, take $(a,b) \in \mathbb{R}^2$

write $(a,b) = c_1(1,1) + c_2(0,1)$

$a = c_1$

$b = c_1 + c_2$

solution $c_1 = a$

$c_2 = b - a$.

$(a,b) = a(1,1) + (b-a)(0,1)$

② lin. indep. suppose

$\lambda_1(1,1) + \lambda_2(0,1) = 0 \iff \lambda_1 = 0,\ \lambda_1 + \lambda_2 = 0$ solution: $\lambda_1 = \lambda_2 = 0$ □

Thm: Every finite dimentional v space has a basis.

Proof: Recall "finite dimentional" means V is spanned by a list $(v_1, \cdots v_n)$

(If $V = \{0\}$, the empty list $(\ )$ is a basis)

Idea: We'll start with spanning list use lemma to throw out redundant vectors.

If $(v_1, \cdots, v_n)$ is lin. independent, done.

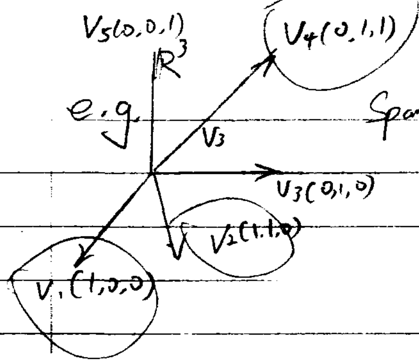
If ... is lin. dependent, need to throw out stuff.

First, throw out any zero vectors

for all $i = 2, \cdots, n$ If $v_i \in \mathrm{Span}(v_1, \cdots, v_{j-1})$, remove it.

Claim: the resulting list is lin. indep. and spans V.

e.g.



$\mathrm{Span}(v_1, v_2, \cancel{v_3}, v_4, \cancel{v_5}) = \mathbb{R}^3$

$v_2 \notin \mathrm{span}\, v_1$ leave v_2 alone

$v_3 = -v_1 + v_2$, $v_3 \in \mathrm{span}(v_1, v_2)$, delete it

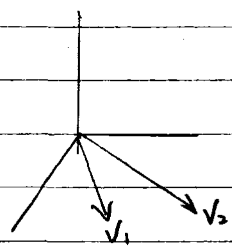
$v_4 \notin \mathrm{span}(v_1, v_2, v_3)$ leave it alone

$v_5 = -v_1 + v_2 + v_4$, $v_5 \in \mathrm{span}(v_1, v_2, v_3, v_4)$, delete it

Result (v_1, v_2, v_4)

Theorem: Any lin. independent can be extended to a basis of the finite dimentioned V.

e.g.



given (v_1, v_2) lin. indep.

theorems says can extend

to (v_1, v_2, v_3) basis

Proof: Suppose $(v_1, \cdots, v_n)$ is lin. indep. want to extend it to a basis.

Since V is finite dimentional, so it has a spanning list

$V = \mathrm{Span}(w_1, \cdots, w_n)$

$(w_1, \cdots w_n)$ spans V

Step 1 Check if $w_1 \in \mathrm{span}(v_1, \cdots v_m)$, if it is, don't add w_1 to $(v_1, \cdots v_m)$

If not, extend: $(v_1, \cdots v_n, w_1)$

Step 2 If w_j is in Span of the list so far, don't add it, if not, extend list

....

After n steps, get a list $(v_1, v_2 \cdots v_m, w_1, \cdots w_k)$ which is a BASIS

Check the final list is a basis:

① It spans V since each w_i is in Span of final list.

② It is linearly independent by lemma

Remark: this theorem justifies idea that can build a basis as follow.

①pick $v_1 \neq 0$

②pick $v_2 \notin \mathrm{span}(v_1)$

③ pick $v_3 \notin \mathrm{span}(v_1, v_2)$

continue

Mat240 Oct. 6^{th}

Dimension:

Definition: A Basis for a finite ~~...~~ dimensional vector space is a list $(v_1, \cdots v_m)$ of vectors which
- span V
- ~~...~~ linearly independent

Want to define dimension

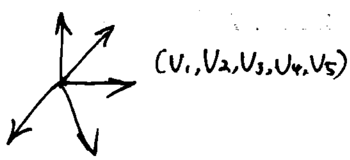
$\dim V$ = length of a basis for V.

before we do this, we must check all bases have same length.

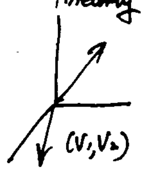
Theorem: length of a linearly independent list is always $\leq$ length of a spanning list.

Intuition: "linearly indep. lists are efficient."

spanning list



linearly indep. lists



Proof: let $(u_1, \cdots u_m)$ be lin. indept. ~~...~~

$(w_1, \cdots, w_n)$ be a spanning list

Strategy: pair each u_i with a unique w_j, with some w_j's left over (possibly)

Step 1 $\text{Span}(w_1, \cdots, w_n) = V$

$\Rightarrow u_1 \in \text{Span}(w_1, \cdots w_n)$

$\Rightarrow (u_1, w_1, w_2, w_3, \cdots w_n)$ is linearly dependent

lemma $\Rightarrow$ there must be a ~~...~~ w_{i_1} in the span of previous vectors.

So we will pair $u_1 \leftrightarrow w_{i_1}$

remove w_{i_1} from the list. span unchanged.

Step 2 $\text{Span}(u_1, w_1, \hat{w}_{i_1}, \cdots, w_n) = V$

$\Rightarrow u_2 \in \text{Span}(u_1, w_1, \hat{w}_{i_1}, \cdots, w_n)$

$\Rightarrow (u_1, u_2, w_1, \cdots \hat{w}_{i_1} \cdots w_n)$ linearly dependent

lemma $\Rightarrow$ there must be a vector in the span of previous

but it's not u_1, u_2 since these are lin. independent

So, must be a w_{i_2}, ~~...~~ pair it with u_2, throw it out, continue.

Step j. $\mathrm{Span}(u_1, u_2, \cdots, w_1, \cdots, \hat{w}_{i_1}, \hat{w}_{i_2}, \cdots w_n) = V$

$u_{j+1} \in \mathrm{span}(\quad)$

$\Rightarrow$ $(u_1, \cdots u_j, u_{j+1}, w_1 \cdots, \hat{w}_{i_1}, \hat{w}_{i_2}, \cdots \hat{w}_{i_j}, \cdots w_n)$ lin. dependant

$\Rightarrow$ must be a vector in span of prev. not u. must be $w_{i_{j+1}}$

Terminate after m steps.

each u_i corresponds to a unique w_j $\Rightarrow \boxed{m \le n}$ ∎

Corollary: Any two basis of V have same length.

Proof: If $(v_1, \cdots, v_m)$, $(w_1, \cdots, w_n)$ are bases for V.

then $(v_1, \cdots v_m)$ lin. indep. and $(w_1, \cdots w_n)$ spans V.

and $(v_1, \cdots v_m)$ spans V and $(w_1, \cdots w_n)$ is linear indept. $n \le m$.

$\Rightarrow \boxed{m = n}$

Def: $\mathrm{Dim}(V) = \mathrm{length}(\mathrm{basis})$

Example: fancy basis for polynomials. (Lagrange interpolation)

$V = P_n(\mathbb{F})$ = ~~deg~~ polynomials of ~~deg~~ degree $\le n$.

V has basis $(1, x, x^2, x^3, \cdots, x^n) \Rightarrow \dim V = n+1$

new basis: pick number $c_0, c_1, \cdots, c_n$.

Define a list of polynomials $(f_0, f_1, \cdots f_n)$

where $f_i(x) = \prod_{\substack{0 \le k \le n \\ k \ne i}} \left(\frac{x - c_k}{c_i - c_k}\right) = \underbrace{\left(\frac{x - c_0}{c_i - c_0}\right) \cdot \left(\frac{x - c_1}{c_i - c_1}\right) \cdots \left(\frac{x - c_{i-1}}{c_i - c_{i-1}}\right) \cdot \left(\frac{x - c_{i+1}}{c_i - c_{i+1}}\right) \cdots \left(\frac{x - c_n}{c_i - c_n}\right)}_{n \text{ terms} \Rightarrow f_i \text{ degree} \le n}$

Claim this is a basis

key property $f_i(c_j) = \begin{cases} 0 & j \ne i \\ 1 & j = i \end{cases}$

check linearly ~~independent~~ independence.

Suppose have a linear relation

$a_0 f_0 + a_1 f_1 + \cdots + a_n f_n = 0$

must show all $a_i = 0$

to do it ① evaluate this combo on c_0, get $\boxed{a_0=0}$
② evaluate on c_1, get $a_1=0$
⋮
(n+1) evaluate on c_n, get $a_n=0$.
⟹ linearly independent

but $\dim V = n+1$
So we have a spanning list length $n+1$ $(1, x, x^2, x^3, \cdots x^n)$
If it didn't span, we could add a vector to list, not in span.
get lin. indep. list. longer than $n+1$. contradiction Theorem.

MAT240 October 11th

Recall V finite dimensional

Thm. V has a basis

Thm. Every linearly independent list can be extended to a basis

Thm. Linearly independent list is shorter than any spanning list

Thm. Bases for V have same length.

Def: $\dim V$ = length (basis)

Corollary: ① Any spanning list with $\dim V$ elements, it is a basis.

② ~~Any~~ lin. independent list with length = ~~dimV~~ , it is a basis.

Examples of Bases (again)

① $\mathbb{F}^n$ has bases $((1,0,\cdots,0),(0,1,0,\cdots,0),\cdots(0,0,\cdots 0,1))$ of n.

$\Rightarrow \dim \mathbb{F}^n = n$

$\boxed{② \dim P_n(\mathbb{F}) = n+1 \quad (1, x, x^2, \cdots, x^n)}$ with basis

Alternate basis for $P_n(\mathbb{F})$ adapted to $n+1$ distinct numbers $c_0, \cdots c_n \in \mathbb{F}$

defined $(f_0, \cdots f_n)$ $\quad f_i = \prod_{k \neq i} \frac{x - c_k}{c_i - c_k}$ $\quad$ key prop: $f_i(c_j) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

To check it is a basis:

① check lin. independent. If $a_0 f_0 + \cdots + a_n f_n = 0$

then $\underbrace{(a_0 f_0 + \cdots + a_n f_n)(c_i)}_{= a_i} = 0$

i.e. $a_i = 0 \ \forall i$

② length = dim, done. it's a basis.

How to use this basis.

Q: ~~to~~ find a poly of deg. 2 with value 2, 1, 3 @ pts. -1, 1, 2

A: we use basis adapted to $(-1, 1, 2) = (c_0, c_1, c_2)$

$$f_0 = \frac{(x-1)(x-2)}{(-1-1)(-1-2)} = \frac{1}{6}(x^2 - 3x + 2)$$

$$f_1 = \frac{(x-(-1))(x-2)}{(1-(-1))(1-2)} = \frac{1}{2}(x^2-x-2)$$

$$f_2 = \frac{(x-(-1))(x-1)}{(2-(-1))(2-1)} = \frac{1}{3}(x^2-1)$$

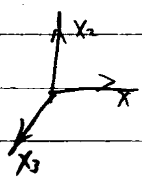
(Lagrange interpolation)

Take $f_0 + 1\cdot f_1 + 3 f_2$ has disired behaviors

$-(\frac{1}{3}-\frac{1}{2}+1)x^2+(-1+\frac{1}{2})x+\frac{2}{3}(-1+1)$

$= \frac{1}{6}(5x^3-3x+4)$

Rem. $\dim P_2(\mathbb{F}) = 3$



total

Ex: $\mathbb{F}$ field $\mathbb{F}=\mathbb{F}^1$ lists one numbers $\mathbb{F}$ is a a.v. spa over $\mathbb{F}_3$

$\dim(\mathbb{F})=1$ because it has basis of length $\neq$ than 1.

e.g. $1 \in \mathbb{F}$ g pride basis

$\forall a \in \mathbb{F},\ a = a\cdot 1$

in particular, $\mathbb{C}$ is 1 dim. $\mathbb{C}$ vector space but $\mathbb{C}$ is ~~att~~ also a v.space over $\mathbb{R}$.

(i.e.) this means, schedule mult, only by seminar $\mathbb{R}$

Why is $(1, i)$ a basis?

① Is it span? any $z \in \mathbb{C}$, $z = x+\overset{\mathbb{R}}{y}i = x+y\cdot i$

② Is it linearly independent if $a\,1+b\,i=0$, $a,b\in\mathbb{R}$

$\Rightarrow a+bi=0 \Leftrightarrow a=b=0$

Gaussian Elimination

① is list $(v_1, \cdots v_m)$ linearly indep?

② What is the dimension of its span?

use a basis to investigate systematically

Algorithm: sequence of moves on list (modify list)

each move does not affect span, in. indep. of list

Moves: ① Swith $v_i \leftrightarrow v_j$ $\quad (v_1, \cdots v_m)$

$$\downarrow$$

$$(v_1, \cdots, v_{i-1}, v_j, \cdots, v_{j-1}, v_i, \cdots v_n)$$

② nonzero rescaling: replace v with λv, $\lambda \neq 0$

③ replace v_i with $v_i + \lambda v_j$, leave v_j alone

e.g. $(v_1, v_2, v_3) \longrightarrow (v_1, v_2 + 3v_1, v_3)$

Claim: $\text{span}(v_1, \cdots v_n) = \text{span}(v_1, \cdots, v_{i-1}, v_i + \lambda v_j, \cdots, v_n)$

$\supseteq$ clear since each vector is a combo of $(v_1, \cdots, v_n)$

$\subseteq$

$v_i = 1 \cdot (v_i + \lambda v_j) + (-\lambda) v_j$

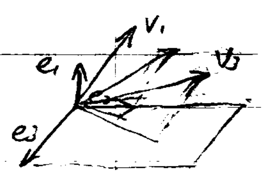
$v_k = 1 \cdot v_k \quad \forall k \neq 1$

Check that $(v_1, \cdots v_m)$ lin. indep. iff $(v_1, \cdots v_{i-1}, v_i + \lambda v_j, \cdots, v_m)$ is lin. indep.

Question: Given a list, we use moves to "simplify" the list until obvious if lin. indep. or not, what dim span is.

Basic idea of simplification (use basis)

$v_1 = ae_1 + be_2 + ce_3 \quad a, b, c \in \mathbb{F}$



rescale v_1 so that component in e_1 direction is $= 1$

move v_2 into plane by adding multiple of v_1.

$v_3 \cdots$

computationally. $\text{Basis} = (e_1, \cdots, e_n)$

$v_1 = a_{11}e_1 + a_{12}e_2 + \cdots + a_{1n}e_n$

$v_2 = a_{21}e_1 + a_{22}e_2 + \cdots + a_{2n}e_n$

$v_k = a_{k1}e_1 + a_{k2}e_2 + \cdots + a_{kn}e_n$

a_{ij} = coordinates of v_i in the basis

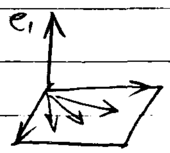
used basis to convent list $(v_1, \cdots v_k)$
to a list $((a_{11}, \cdots, a_{1n}),$
$(a_{21}, \cdots a_{2n}),$
$\vdots$
$(a_{k1}, \cdots, a_{kn}))$.

$k \times n$ matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$

moves become:
- switching 2 rows
- rescaling $(\neq 0)$ row
- adding a mult. of a row to another row.

"row operation"



First step: If the first column is all zero
it means all the vectors are in ~~the~~ span$(e_2, e_3, e_4, \dots e_n)$
hence vectors donot span V and we can ignore first column
work on the rest of the matrix
~~if second column is zero etc.~~
Same thing goes ~~for any~~

Second step: consider first column where not all$=0$

e.g. $\begin{bmatrix} 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 3 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 3 & 8 & 0 \end{bmatrix}$

take ~~the~~ a row with nonzero entry in that column.
switch it with 1st row

Third Step And rescale this row so it starts with 1.

Forth Step. Add multiple of 1st row to any lower row w/nonzero coordinate in the column to make this $= 0$

$$\longrightarrow \begin{bmatrix} 0 & 0 & 1 & \frac{7}{2} & 0 \\ 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & (8-\frac{3 \times 7}{2}) & 0 \end{bmatrix}$$

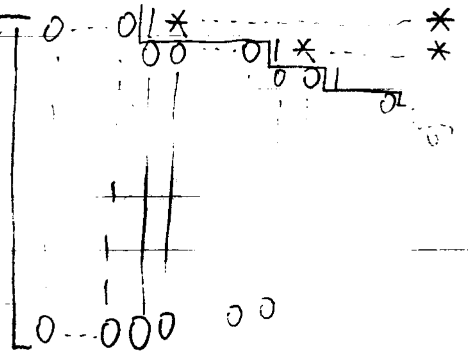
Recursion step

As a result, maatrix is in form:

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & * & \cdots & * \end{bmatrix}$$

now focus on this area and repeat steps

$$\begin{bmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & * \\ \vdots & & 0 & 0 & 0 & \cdots & 1 & * & * \\ \vdots & & \vdots & \vdots & \vdots & & 0 & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 \cdots & 0 & * & * \end{bmatrix}$$

continue ~~for~~ all ~~submatrix~~ submatrices get 

Row Echelon form ~~with~~ algorithm
(end of Gaussian elimination ~~algorithm~~)

<u>Claim:</u> this is simple form where linearly indep. and dim(span) are obvious

$[0] \to$ Row Echelon $\Rightarrow$ all $v_i = 0$

$$\begin{array}{c} w_1 \\ w_2 \\ w_3 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$ not lin. indep. (but original 3 vectors could all be nonzero)

RE $\Rightarrow$ original vectors are lin. dependent

Claim w_1, w_2 are linearly independent $\quad aw_1 + bw_2 = 0$

$$\begin{array}{l} (a \;\; 2a \;\; 4a) \\ (0 \;\; b \;\; 3b) \\ \hline (a, 2a+b, 3b+4a) = 0 \end{array}$$

$a = b = 0$

in general, the nonzero rows of RE form must be lin. indep. and so # of these gives dim span $(v_1, \cdots, v_k)$

MAT240 October 13th

LAST TIME

given list $(v_1, \dots, v_k)$ in a n-dimensional space V

choose basis $(e_1, \dots, e_n)$ for V, and write

$v_1 = a_{11}e_1 + \cdots + a_{1n}e_n$

$\vdots$

$v_k = a_{k1}e_1 + \cdots + a_{kn}e_n$

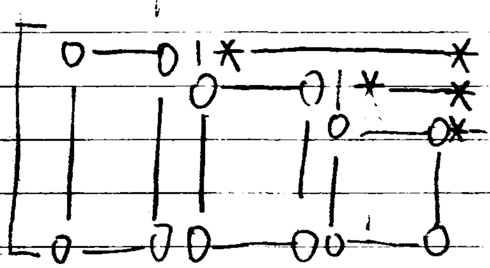
information is $k \times n$ matrix

$$\begin{array}{c} \xleftarrow{\quad n \quad} \\ k \left(\begin{matrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{matrix} \right) \end{array}$$

idea: Simplify using moves → don't change span

① $v_i \leftrightarrow v_j$ ② $v_i \mapsto \lambda v_i$, $\lambda \neq 0$ ③ $v_i \mapsto v_i + \lambda v_j$

(Row ops) Until



RE: good since nonzero rows give a basis for span $(v_1, \dots, v_n)$

This algorithm is used to
- determine basis for span $(v_1, \dots, v_k)$
- solve a system of linear equations
- (later) inverting matrices
- ...

Q ① $(1,0,2,3)$, $(0,2,1,-1)$, $(0,1,1,0)$, $(1,1,1,1)$ vectors in $\mathbb{R}^4$ linearly indep? If not give a nonzero relation.

$$\begin{pmatrix} 1,0,2,3 \\ 0,2,1,-1 \\ 0,1,1,0 \\ 1,1,1,1 \end{pmatrix} \xrightarrow{V_4 \to V_4 - V_1} \begin{pmatrix} 1,0,2,3 \\ 0,2,1,-1 \\ 0,1,1,0 \\ 0,1,-1,-2 \end{pmatrix} \xleftrightarrow{V_3 \leftrightarrow V_2} \begin{pmatrix} 1,0,2,3 \\ 0\ 1\ 1\ 0 \\ 0\ 2\ 1\ -1 \\ 0\ 1\ -1\ -2 \end{pmatrix}$$

$$\xrightarrow{V_3 \mapsto V_3 - 2V_2} \begin{pmatrix} 1,0,2,3 \\ 0\ 1\ 1\ 0 \\ 0\ 0\ -1\ -1 \\ 0\ 1\ -1\ -2 \end{pmatrix} \xrightarrow{V_4 \mapsto V_4 - V_2} \begin{pmatrix} 1,0\ 2\ 3 \\ 0\ 1\ 1\ 0 \\ 0\ 0\ -1\ -1 \\ 0\ 0\ -2\ -2 \end{pmatrix}$$

$$\xrightarrow{V_3 \mapsto (-1)V_3} \begin{pmatrix} 1\ 0\ 2\ 3 \\ 0\ 1\ 1\ 0 \\ 0\ 0\ 1\ 1 \\ 0\ 0\ -2\ -2 \end{pmatrix} \xrightarrow{V_4 \mapsto V_4 + 2V_3} \begin{pmatrix} 1\ 0\ 2\ 3 \\ 0\ 1\ 1\ 0 \\ 0\ 0\ 1\ 1 \\ 0\ 0\ 0\ 0 \end{pmatrix}$$

Conclusion: $\mathrm{Span}(v_1, v_2, v_3, v_4)$ has a basis of length 3

i.e. $\dim \mathrm{span}(v_1, v_2, v_3, v_4) = 3$

(v_1, v_2, v_3, v_4) not linearly independent

At the end, 4th row is $0 = (0,0,0,0)$

$(0,0,0,0) = v_4 + 2v_3$ in step

$= v_4 + 2(-v_3)$

$= (v_4 - v_2) + 2(-v_3)$

$= (v_4 - v_2) + 2(-(v_3 - 2v_2))$

$= v_4 - v_3 - 2(v_2 - 2v_3)$

$= (v_4 - v_1) - v_3 - 2(v_2 - 2v_3)$

$= -v_1 - 2v_2 + 3v_3 + v_4 = 0$

Check

$$\begin{array}{rl} & -1,0\ -2\ -3 \\ & 0\ -4\ -2\ 2 \\ & 0\ 3\ 3\ 0 \\ + & 1\ 1\ 1\ 1 \\ \hline & 0\ 0\ 0\ 0 \end{array}$$

② When is $\begin{pmatrix} (-1,0,r) \\ (0,1,s) \\ (-t,1,1) \end{pmatrix}$ vectors in $\mathbb{Q}^3$ linearly independent?

algorithm: $\begin{pmatrix} -1 & 0 & r \\ 0 & 1 & s \\ -t & 1 & 1 \end{pmatrix} \xrightarrow{r_1 \mapsto -r_1} \begin{pmatrix} 1, & 0 & -r \\ 0 & 1 & s \\ -t & 1 & 1 \end{pmatrix} \xrightarrow{r_3 \mapsto r_3 + (-1)r_2} \begin{pmatrix} 1, & 0, & -r \\ 0 & 1 & s \\ 0 & 0 & (1-tr-s) \end{pmatrix} \rightarrow$

last step

$$\begin{pmatrix} 1, & 0, & -r \\ 0, & 1, & s \\ 0, & 0, & 1-tr-s \end{pmatrix} \longrightarrow \text{If } 1-tr-s \neq 0 \quad r_3 \mapsto \left(\frac{1}{1-tr-s}\right) r_3 \quad \begin{pmatrix} 1, & 0, & -r \\ 0, & 1, & s \\ 0, & 0, & 1 \end{pmatrix}$$

lin indep.

otherwise

lin. dep.

A: Condition is $\boxed{1-tr-s \neq 0}$

Example 3 If given a system of (in)homogeneous linear equations

1st eq. $a_{11}x_1 + \cdots + a_{1n}x_n = b_1$

$\vdots$

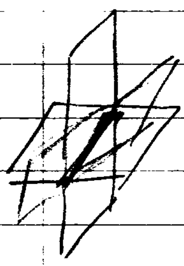
k^{th} eq. $a_{k1}x_1 + \cdots + a_{kn}x_n = b_k$

$a_{ij}, b_i \in \mathbb{F}$

"system of k linear equations in n unknowns"

Solve for $x_1, \cdots x_k$

(note each equation defines a $(n-1)$-dimensional plane in $\mathbb{F}^n$
if RHS$=0$, plane contains $(0, 0, \cdots, 0)$)



intersection of k planes

WARNING: IF equations are homogeneous $\underline{0}$ is always a solution.
IF inhomogeneous, may not be any solution.

MAT240 October 20th

LINEAR MAPS (aka. linear transformation)

Let V and W are vecter spaces over $\mathbb{F}$.

A linear map T from V to W $\Rightarrow$ $T: V \to W$ is a map

st. ① $T(v_1+v_2) = T(v_1) + T(v_2)$ $\forall v_1, v_2 \in V$

② for all $\lambda \in \mathbb{F}$

$$\underbrace{T(\lambda v)}_{V} = \underbrace{\lambda T(v)}_{W}, \quad \forall v \in V$$

combine these: $T(\lambda v_1 + v_2) = \lambda T(v_1) + T(v_2)$

e.g. $\mathbb{R} \xrightarrow{f} \mathbb{R}$

linear means $f(ax+b) = af(x) + f(b)$

$f(x) = x$

note: $f(0) = 0$ for linear map

$f(x) = ax+b$ only linear when $b=0$ $\mathbb{R} \to \mathbb{R}$

✱ Example 1 $V \xrightarrow{f} W$

zero map $f(v) = 0$, $\forall v$

✱ Example 2 Identity map $V \to V$

$I_V(u) = u$, $I(u_1+u_2) = u_1+u_2 = I(u_1)+I(u_2)$

$I(\lambda u) = \lambda u = \lambda I(u)$

Example 3 linear maps $P(\mathbb{F}) \to P(\mathbb{F})$

3.1 differentiation

$D(f) = f'$

e.g. $D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$

D is linear because $(f+g)' = f' + g'$

$(\lambda f)' = \lambda \cdot f'$

3.2 If we fix a polynomial P_0 (e.g. $P_0 = x^2$), we get a linear map $T_{P_0}: P(\mathbb{F}) \to P(\mathbb{F})$

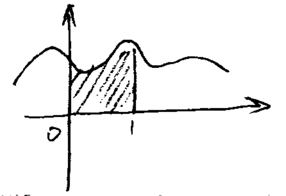
$T_{P_0}(f) = P_0 \cdot f$

e.g. for $P_0 = x^2$

$T_{x^2}(x+1) = x^2(x+1)$

Why linear? $P_0(f_1+f_2) = P_0 f_1 + P_0 f_2$

$P_0(\lambda f) = \lambda P_0 f$



Example 4

linear map: $P(\mathbb{R}) \to \mathbb{R}$ $\quad T(f) = \int_0^1 f(x)\,dx$

∞-dim $\quad$ 1-dim

vsp/$\mathbb{R}$ $\quad$ vsp/$\mathbb{R}$ $\quad$ Linear? $\quad T(f_1+f_2) = \int_0^1 (f_1+f_2)(x)\,dx$

$= \int_0^1 f_1(x)\,dx + \int_0^1 f_2(x)\,dx$

$= T(f_1) + T_2(f_2)$

Example 5 Shift operators $\mathbb{F}^\infty \to \mathbb{F}^\infty$

$\{(a_1, a_2, \cdots, \cdots) : a_i \in \mathbb{F}\}$

l, r defined as:

$l(a_1, a_2, a_3, \cdots) = (a_2, a_3, a_4, \cdots)$

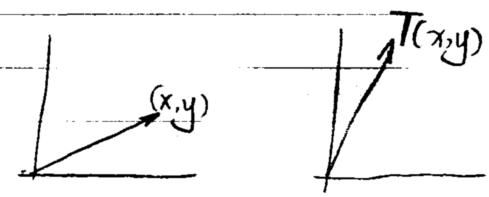
$r(a_1, a_2, a_3, \cdots) = (0, a_1, a_2, \cdots)$

$l((a_i)_{i\in N} + (b_i)_{i\in N}) = l((a_i + b_i)_{i\in N}) = (a_2+b_2, a_3+b_3, \cdots)$

Example 6 $\quad T : \mathbb{R}^2 \to \mathbb{R}^2$

$T(x,y) = (a_{11}x + a_{12}y, a_{21}x + a_{22}y)$

$\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \in \mathbb{R}$

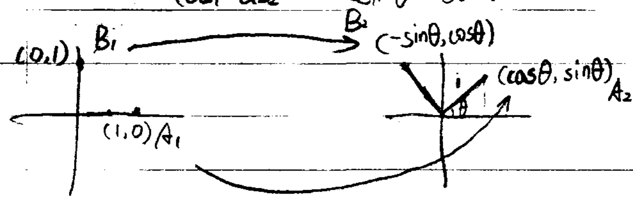


Special cases:

(i) $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ get. zero map

(ii) $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ get $T(x,y) = (x,y)$ identity map

(iii) $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ for fixed $\theta \in \mathbb{R}$



Proposition: $L(V,W)$ is a vector space

Definition: Let $L(V,W)$ be the set of all linear maps from V to W.
This has two operations $(T_1 + T_2)(v) = T_1(v) + T_2(v)$
$(\lambda T)(v) = \lambda T(v)$

MAT240 Oct. 25^{th}

Linear Maps $\quad V, W$ vsp over $\mathbb{F}$

$L(V,W)$ = space of linear maps $T: V \to W$

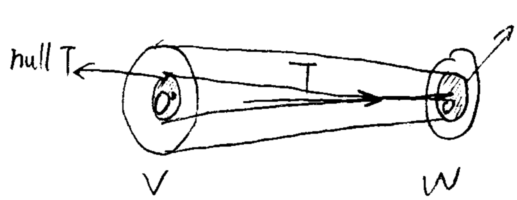
is a vector space over $\mathbb{F}$.

For a fixed $T: V \to W$

There are two special subspaces

null T "kernel" $= \{v \in V,\ s.t.\ T(v) = 0\}$

range T "image" $= \{T(v);\ v \in V\}$



T takes 0 to 0, but there ~~might be more taken to 0.~~

Something out of range T is out of linear map.

Why are null T, range T subspaces of V, W.

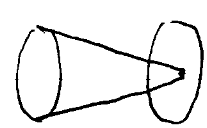
null T $\quad$ If $v_1, v_2 \in$ null T, $\lambda \in \mathbb{F}$ $\quad T(\lambda v_1 + v_2) = \lambda T v_1 + T v_2 = 0 \Rightarrow$ null T is a subspace.

range T $\quad$ If $w_1, w_2 \in$ range T, this means $\exists\, v_1, v_2 \in V$

$w_1 = T v_1$, $w_2 = T v_2$ $\Rightarrow \lambda w_1 + w_2 = \lambda T v_1 + T v_2 = T(\lambda v_1 + v_2)$

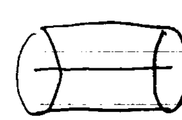
extreme examples

① zero map $\quad V \xrightarrow{T} W \quad$ range $T = \{0\}$, null $T = V$



NOT INJ
NOT SUR

② identity map $\quad I(v) = v \quad$ range $T = V$, null $T = \{0\}$



NOT INJ

③ $(v_1, v_2, \dots v_k)$ List of vectors in V define a linear map

$\mathbb{F}^k \xrightarrow{T} V$

$T(a_1, a_2, \dots a_k) = a_1 v_1 + \dots + a_k v_k$

range $T = \mathrm{span}(v_1, v_2, \dots v_k)$

T surjective $\Leftrightarrow (v_1, \dots v_k)$ spans V

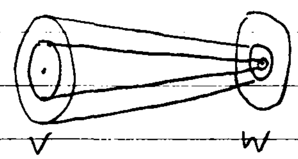
T injective $\Leftrightarrow (v_1, \dots v_k)$ linearly independent

T is inj + surj iff $(v_1, \dots v_k)$ is a basis.

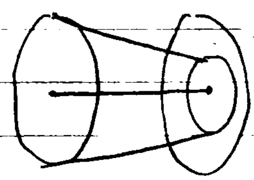
Def: T is called injective when null $T = \{0\}$
~~surj~~ surjective when range $= W$

note: injective implies that $Tv_1 = Tv_2$ then $v_1 = v_2$
Why? ($T(v_1 - v_2) = 0$, $v_1 - v_2$ is a null space so $v_1 = v_2$, $v_1 - v_2 = 0$)

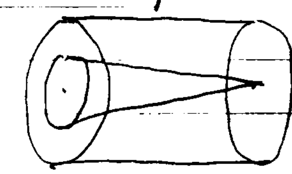
general map

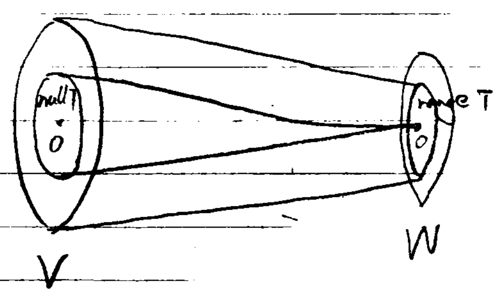


injective map



surjective map





Thm: If V is finite dimesional, then rank-nullity theorem

$$\boxed{\underbrace{\dim \text{null}\, T}_{\text{"nullity"}} + \underbrace{\dim \text{range}\, T}_{\text{"rank"}} = \dim V}$$

Sketch of proof.
Idea: find a basis for V related to basis for null T & basis for range T.
Since null $T \subset V$, it is finite dimensional
choose a basis $(u_1, \cdots u_k)$ for null T
using a previous result, extent to a basis of V.
$(u_1 \cdots u_k, w_1 \cdots w_m)$

Claim $(Tw_1, Tw_2, \cdots, Tw_m)$ is a basis for range T.
$\dim V = k + m$
$= \dim \text{null}\, T + \dim \text{range}\, T$

Corollary

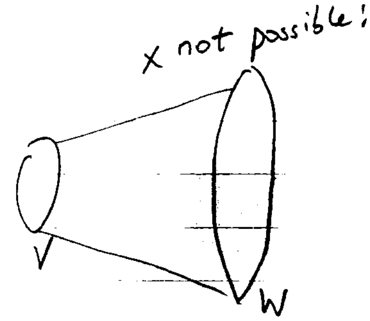
① If $\dim V < \dim W$, no surject $\overset{\text{linear}}{\text{maps}}$, $V \to W$.

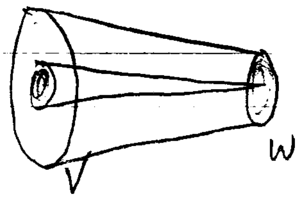
② If $\dim V > \dim W$

null T forced to be nonzero

no injective linear maps $V \to W$

(why? .....)





Composition.

If $T : U \to V$ $\quad$ U, V, W

$S : V \to W$ $\quad$ vsp over $\mathbb{F}$

The composition of these is $ST : U \to W$

defined by $ST(u) = S(Tu)$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & V \arrow[r, "S"] & W \end{tikzcd}$$

(apply T first, then apply S)

~~note~~ warning: ① The composition only makes sense in one order

② first act the map on RHS, like ST instead of TS

(reversal in notation)

③ If $U = V = W$ then composition does make sense in both orders.

ST $\quad$ TU $\quad$ both $U \to U$

$$\begin{tikzcd} U \arrow[r, "T"] \arrow[rr, bend right, "ST"'] & U \arrow[r, "S"] \arrow[rr, bend right, "TS"'] & U \arrow[r, "T"] & U \end{tikzcd}$$

but ST need not equal to TS. $\quad$ (non-commutative)

$S : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $S(x, y) = (x, 0)$

$T : \mathbb{R}^2 \to \mathbb{R}^2$ $\quad$ $T(x, y) = (y, x)$

$ST(x y) = (y, 0)$

$TS(x y) = (0, x)$

[redacted]

Matrices

Decribe a linear map $T: V \to W$
using a basis $\beta = (v_1, \cdots v_n)$ for V
$\gamma = (w_1, \cdots w_k)$ for W

① any vector $x \in V$ is $x = \sum_{i=1}^{n} x_i v_i$ → coordinate of x "w.r.t. β"

as a matrix.

$$[x]^\beta = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$$

② The linear map T is determined by ~~its~~ the action on β.

$$T(v_j) = \sum_{i=1}^{k} a_{ij} w_i \quad \text{in } \mathbb{F}$$

as a matrix we write

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \boxed{k \times n} \text{ matrix} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$ ← jth column is result of $T(v_j)$ expanded in γ basis.

③ How to figure out Tx if you know $[x]^\beta$ and $[T]_\beta^\gamma$
i.e. what is matrix of (Tv) in basis γ?
so we define

$$[Tx]^\gamma = \underbrace{[T]_\beta^\gamma [x]^\beta}_{\text{definition of } \cancel{\text{this}} \text{ operation is}}$$

$$[T]_\beta^\gamma \cdot [x]^\beta = [Tx]^\gamma$$
$$= [T(\sum_{j=1}^{n} x_j v_j)]$$
$$= \left[\sum_{j=1}^{n} x_j \sum_{i=1}^{n} a_{ij} w_i\right]^\gamma = \begin{bmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \vdots \\ \sum_{j=1}^{n} a_{kj} x_j \end{bmatrix}$$

$k \times 1$ matrix

example:

$$P_2(\mathbb{Q}) \xrightarrow[\text{derivative}]{D} P_1(\mathbb{Q})$$

$\dim P_2 = 3$ $\qquad$ $\dim P_1 = 2$

$\beta = (1, x, x^2)$ $\qquad$ $\gamma = (1, x)$

$D(1) = 0 \times 1 + 0x$
$D(x) = 1 + 0x$
$D(x^2) = 0 + 2x$

$$[D]_\beta^\gamma = \left(\quad 2\times 3 \quad \right)$$

$$[D]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$D: P_n \to P_{n-1}$

$\dim P_n = n+1$ $\qquad$ $\dim P_n = n$

$\beta = (1, x, \cdots x^n)$ $\qquad$ $\gamma = (1, x, \cdots x^{n-1})$

$$[D]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & n\times(n+1) & \\ \vdots & \vdots & \vdots & \vdots & & \text{matrix} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}$$

matrix multiplication

$$U \xrightarrow{S} V \xrightarrow{T} W$$

$\alpha = (u_1, u_2, \cdots u_m)$ basis for U
$\beta = (v_1, v_2, \cdots v_n)$ $\quad$ - - $\quad V$
$\gamma = (w_1, w_2, \cdots w_k)$ $\quad$ - - - $\quad W$

S has matrix $[S]_\alpha^\beta$, T has matrix $[T]_\beta^\gamma$

Def: matrix mult. $[T]_\beta^\gamma \cdot [S]_\alpha^\beta$
is the matrix $[T\cdot S]_\alpha^\gamma$ of TS

work out details

$$[TS]_\alpha^\gamma = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{km} \end{bmatrix} \qquad [T]_\beta^\gamma = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix}$$

$$[S]_\alpha^\beta = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$(TS)(u_j) = \sum_{i=j}^{k} c_{ij} w_i$$

$$(TS)(u_j) = T(Su_j) = T\left(\sum_p b_{pj} v_p\right) = \sum_p T b_{pj} v_p = \sum_{p=1}^{n} b_{pj}\left(\sum_{i=1}^{k} a_{ip} w_i\right) = \sum_{i=1}^{k}\left(\sum_{p=1}^{n} a_{ip} b_{pj}\right) w_i$$

$$(TS)(u_j) = \sum (c_{ij})(w_i)$$

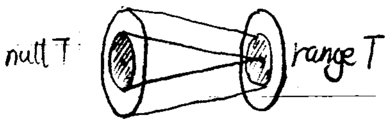
$$c_{ij} = \sum_{p=1}^{n} a_{ip} b_{pj}$$

$$\begin{array}{ccc} [T] & [S] & [TS] \\ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} & \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} = & \begin{pmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{k1} & \cdots & c_{km} \end{pmatrix} \end{array}$$

MAT240 October 27th

Recall

$V \xrightarrow{T} W$



If V is finite dimensional

$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$

Matrix

$V \xrightarrow{T} W$

$\beta = (v_1, v_2, \cdots, v_n)$ $\quad \gamma = (w_1, w_2, \cdots, w_k)$ $\quad$ choose bases

$$[T]_\beta^\gamma = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{bmatrix}$$

columns are the coefficients of Tv_j in basis γ.

vector $x \in V$ $\quad [x]^\beta \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

to find coordinates of Tx in basis γ

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \circledast & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbb{F}^n} = \underbrace{\begin{bmatrix} \vdots \\ \circledast \\ \vdots \end{bmatrix}}_{\mathbb{F}^k} \leftarrow [Tx]^\gamma$$

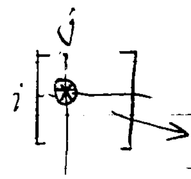
The matrix of a composition

$U \xrightarrow{S} V \xrightarrow{T} W$

domain $\quad \alpha \quad \beta \quad \gamma$

dim: $\quad m \quad n \quad k$

matrix $[TS]_\alpha^\gamma = \underbrace{\begin{bmatrix} & & \\ & T & \\ & & \end{bmatrix}_\beta^\gamma}_{k\times n} \underbrace{\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix}_\alpha^\beta}_{n \times m}$



$c_{ij} = \sum_p a_{ip} b_{pj}$

$\mathbb{F}^{k\times n}$ means $k\times n$ matrices entries in $\mathbb{F}$.

① Remark:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix}$$ defines a linear map $\mathbb{F}^? \longrightarrow \mathbb{F}^?$

$\mathbb{F}^n \longrightarrow \mathbb{F}^k$

(n columns)

define a linear map $\mathbb{F}^n \xrightarrow{A} \mathbb{F}^k$

the range of A is a subspace of $\mathbb{F}^k$ spanned by $(Ae_1, \cdots, Ae_n) = \operatorname{span}(\operatorname{column}(A))$

column of A = "the column space"

② The null space of A is the set of $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ s.t. $[A]\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

i.e: $= \{x \in \mathbb{F}^n ; Ax = 0 \text{ in } \mathbb{F}^k\}$

but notice $Ax=0$ is: a system of k linear equations:

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \Rightarrow \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n = 0 \end{cases}$$

So, finding null(A) = same as solving system of k linear equations in n unknowns.

EXAMPLE

$\mathbb{R}^2 \underset{S}{\overset{T}{\rightleftarrows}} \mathbb{R}^3$ linear maps

$$[T]_e^e = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad [S]_e^e = \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix}$$

can produce many linear maps:

$ST: \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \rightarrow ST = \begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} ga+hc+ie, & gb+hd+if \\ ja+kc+le, & jb+kd+lf \end{bmatrix}$

$TS: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$STS\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$TST\ \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

...

$TS = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} g & h & i \\ j & k & l \end{bmatrix} = \begin{bmatrix} ag+bj, & ah+bk, & ai+bl \\ cg+dj, & ch+dk, & ci+dl \\ eg+fj, & eh+fk, & ei+fl \end{bmatrix}$

How to change basis?

$V \xrightarrow{T} W$

$[T]_\beta^\gamma$ want to find $[T]_{\beta'}^{\gamma'}$

old bases β, γ new bases β', γ'

$$[T]_{\beta'}^{\gamma'} = [I_W T I_V]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'} [T]_\beta^\gamma [I_V]_{\beta'}^{\beta}$$

change of basis matrices

MAT240 Nov. 1st

Linear Maps, Bases and Inverses.

- Linear maps between vec. spaces can be represented by matrices:
  if V has dim m and W has dim n and $T: V \to W$ is a linear map, then matrix representation of T has size $n \times m$

$$\underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}}_{\text{n rows},\ \text{m columns}} \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{v \in V} \Big\}m = \begin{bmatrix} \\ \\ \end{bmatrix} \Big\}n$$

$[T]_\beta^\gamma$ ($\gamma \to$ basis for W, $\beta \to$ basis for V) $\longrightarrow$ jth column is coords at $T(v_j)$ in the basis γ, where $v_j \in \beta$

$$\begin{tikzcd} V \arrow[r, "I_V"] \arrow[rrr, bend right, "T"'] & V \arrow[r, "T"] & W \arrow[r, "I_W"] & W \\ \beta' & \beta & \gamma & \gamma' \end{tikzcd}$$

$$\boxed{[T]_{\beta'}^{\gamma'} = [I_W]_\gamma^{\gamma'}[T]_\beta^\gamma[I_V]_{\beta'}^\beta}$$ change of basis

$[I_V]_{\beta'}^\beta \to$ each column is the coords of a β' vector written as a lin. comb. of β vectors.

$[I_V]_\beta^\beta = I_{\dim V}$

E.g. $D: P_2(\mathbb{R}) \to P_1(\mathbb{R})$ taking polys of degrees ≤ 2 (real coefficients) to ones at deg ≤ 1 (by differentiation)

Basis of P_2 is $\beta = (1, x, x^2)$ ~~at~~ of P_1 is $\gamma = (1, x)$

$D(1) = 0 = 0 \cdot 1 + 0 \cdot x$

$D(x) = 1 = 1 \cdot 1 + 0 \cdot x$

$D(x^2) = 2x = 0 \cdot 1 + 2 \cdot x$

$$[D]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

! if $\beta = (x, 1, x^2)$, $\gamma = (x, 1)$

$$[D]_\beta^\gamma = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}$$

$1 = 1 \cdot 1 + 0 \cdot (x-1)$

$x = (x-1) + 1$
$= 1 \cdot 1 + 1 \cdot (x-1)$

$1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$(x-2) = -2 \cdot 1 + 1 \cdot x + 0 \cdot x^2$

$(x-2)^2 = x^2 - 4x + 4$
$= 4 \cdot 1 - 4 \cdot x + 1 \cdot x^2$

~~if~~ $\beta' = ((x-2)^2, (x-2), 1)$ $\gamma' = (1, (x-1))$

$$\cancel{[D]_\beta^{\gamma'} = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 1 & 0 \end{pmatrix}}$$

$$[I_P]_{\beta'}^\beta = \begin{bmatrix} 4 & -2 & 1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$[I_P]_\gamma^{\gamma'} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$[D]_{\beta'}^{\gamma'} = [I_{P_1}]_{\gamma}^{\gamma'} [D]_{\beta}^{\gamma} [I_{P_2}]_{\beta'}^{\beta}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 & -2 & 1 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Invertible Maps

Definition: A linear map $T: V \to W$ is invertible when $\exists$ another linear map $S: W \to V$ s.t. $ST = I_V$, $TS = I_W$. (different 2 identity maps)

• Is I_V invertible?

Yes $I_V^2 = I_V$

$I_V \cdot I_V = I_V$ $\quad S = I_V,\ T = I_V$

So I_V is invertible

• Is 0 map invertible?

$0: V \to W$

Suppose it is, then $\exists\, S: W \to V$ such that

$S \cdot 0 = I_V$ & $0 \cdot S = I_W$

$0 = I_V$ $\quad 0 = I_W$

this implies V & W are just (0 vectors) $\{\vec{0}\}$.

if not, not invertible.

* 0 invertible when V, W just both 0.

• If $\exists\, S: W \to V$ st. $TS = I_W$, then $ST = I_V$ $\quad$ T / F

Counterexample. $T: \mathbb{R}^2 \to \mathbb{R}^1$ represented by $[3\ \ 4]$.

$S: \mathbb{R}^1 \to \mathbb{R}^2$ represented by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$TS = [3\ \ 4]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = [1] = I_{\mathbb{R}^1}$$

$$ST = \begin{bmatrix} -1 \\ 1 \end{bmatrix}[3, 4] = \begin{bmatrix} -3 & -4 \\ 3 & 4 \end{bmatrix} \neq I_{\mathbb{R}^2}$$

Can have T have 2 distinct inverse?

Say S, S' both satisfy the conditions.

$$ST = I$$

$$\left.\begin{array}{l}(ST)S' = IS' \\ S(TS') = S \cdot I\end{array}\right\} \Rightarrow S = S'$$

The the INVERSE is always UNIQUE.

$\therefore$ We may speak of the inverse of T should it exist denote it T^{-1}

THEOREM:

T is invertible iff T injective + surjective ("1:1", "onto")

Proof: $\Rightarrow$ T^{-1} exists Let $v \in \text{null}(T)$

$$v = I_V v = T^{-1} \cdot T v = T^{-1} \cdot 0 = 0$$

Therefore null (T) is $\{0\}$

$\Rightarrow$ T is injective

one direction

$w \in W$, then $T^{-1} w \in V$

let this vector be $u \in V$, $u = T^{-1} w$

so $Tu = T(T^{-1} \cdot w) = (TT^{-1}) w = I w = w$

so for each $w \in W$ $\exists$ something in V that goes to it under T, i.e. T is surjective.

The other direction

$\Leftarrow$ T 1:1, onto. want to construct T^{-1}. for any $w \in W$, the surjectivity of T means $\exists\, u \in V$ s.t. $Tu = w$

u is, furthermore, necessarily unique (by injectivity of T).

So define $Sw = u$ $(S: W \to V)$

$ST = I_V$? $STu = Sw = u$ ST sends u to u $\forall u \in V$.

$ST = I_V$ ✓

$TS = I_W$? $TSw = Tu = w$ TS sends w to w $\forall w \in W$

$TS = I_W$ ✓

Check: S is linear, Then declare S to be T^{-1} $\Rightarrow$ it satisfies everything.

Done.

[redacted]

Definition: Two vector spaces are called isomorphic when there is an invertible linear map between them.

Theorem: Two finite-dim ~~vectors~~ vector spaces are iso iff they have same dim.
(over same $\mathbb{F}$)

Proof: $\Rightarrow$ Suppose T^{-1} ~~exist~~ exists
Take $(v_1, \cdots, v_n)$ basis of V (Let $n = \dim V$, $m = \dim W$)
Claim $(Tv_1, \cdots, Tv_n)$ are linear, indep. in W.
$a_1 Tv_1 + a_2 Tv_2 + \cdots + a_n Tv_n = 0_W$
$T(a_1 v_1 + \cdots + a_n v_n) = 0_W$
But T^{-1} implies that T is injective
so $a_1 v_1 + \cdots + a_n v_n = 0_V$ cuz $\operatorname{null}(T) = \{0\}$
But $(v_1, \cdots v_n)$ are lin. indep. in $V \Rightarrow a_1 = \cdots = a_n = 0$.
$\therefore (Tv_1, \cdots Tv_n)$ are lin. indep. in W.
$\Rightarrow \dim W \geq n = \dim V$

On the other hand, T is surjective, so if $(w_1, \cdots, w_n)$ is basis of W, $(T^{-1}w_1, \cdots T^{-1}w_n)$ are vectors in V.
Claim $(T^{-1}w_1, T^{-1}w_2, \cdots, T^{-1}w_n)$ lin. indep. in V.
Same argument, just need to know first that T^{-1} is injective. (T^{-1} is invertible: T is its inverse!
$\therefore T^{-1}$ injective (previous theorem)
$\therefore \dim V \geq m = \dim W$

$m = n \Rightarrow \dim V = \dim W$

$\Leftarrow$ Suppose $\dim W = \dim V$ Need to construct invertible $T: V \to W$.
Choose bases $\alpha = (v_1, \cdots, v_n)$ for V, $\beta = (w_1, \cdots, w_n)$ for W
Define $T: \alpha \to \beta$, by $Tv_j = w_j$
This determines a unique linear map $T: V \to W$
by $T(x) = T(a_1 v_1 + \cdots + a_n v_n) = a_1 Tv_1 + a_2 Tv_2 + \cdots + a_n Tv_n$
$= a_1 w_1 + a_2 w_2 + \cdots a_n w_n$
So $T: V \to W$ determined by what $T: \alpha \to \beta$ does!
Is T 1:1? Let $x \in \operatorname{null}(T)$ $x = a_1 v_1 + \cdots + a_n v_n$
$Tx = 0$ $a_1 Tv_1 + \cdots + a_n Tv_n = a_1 w_1 + \cdots + a_n w_n = 0$

But $w_1, \ldots, w_n$ lin. indep.

$a_1 = \cdots = a_n = 0$

So $x = 0$, i.e. $null(T) = 0$

i.e. T is 1:1

Is T onto? take $y \in W$, $y = b_1 w_1 + \cdots + b_n w_n$

$T(\underbrace{b_1 v_1 + \cdots + b_n v_n}_{\in V}) = b_1 T v_1 + \cdots + b_n T v_n$

$= b_1 w_1 + \cdots b_n w_n$

So we found ~~that~~ sth. in V ~~that~~ goes to y!

T is surjective.

T^{-1} exists!

Done.

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Def: A linear map $T: U \to V$ is called invertible when $\exists\ S: V \to U$
st. $ST = I_U$
$TS = I_V$

Thm: A linear map $T: U \to V$ is invertible $\Leftrightarrow$ it is surjective & injective.
(injective $\Leftrightarrow$ $Null(T) = \{0\}$)
(surjective $\Leftrightarrow$ $Range(T) = V$)
$u \leftrightarrow v = Tu$

Def: $T: U \to V$ an invertible linear map is called an (vector space) isomorphism. U and V are isomorphic. (~~two~~ an invertible lineaper map between the two)

Def: An isomorphism $T: V \to V$ is called an automorphism.

$T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T(x,y,z) = (x+y,\ y+z,\ z+x)$
T is an isomorphism. but the same map is not ~~any~~ an isomorphism over $(\mathbb{F}_2)^3$
$e_1 = (1,0,0),\ e_2 = (0,1,0),\ e_3 = (0,0,1)$
$T(e_1) = e_1 + e_3$
$T(e_2) = e_2 + e_1$
$T(e_3) = e_3 + e_2$
if $T(x,y,z) = (0,0,0) \Rightarrow x+y = y+z = z+x = 0 \Rightarrow x = y = z = 0$
$Null(T) = \{0\}$
over $\mathbb{F}_2$ $\quad T(1,1,1) = (1+1,\ 1+1,\ 1+1) = (0,0,0)$
$Null(T) \neq \{0\}$

Prop: $T: V \to W$ is linear and $\dim V = \dim W$, $W < \infty$
Then 1). T injective iff invertible
2). T surjective iff invertible

Proof: $\dim V = \dim Null\, T + \dim Range\, T$
If T is injective, $n = \dim range\, T + 0 \Rightarrow \dim Range\, T = n = \dim W$
$\Rightarrow range\, T = W$
Similarly if T is surjective, then the $\dim Range\, T = \dim W = n$.
$\Rightarrow \dim null(T) = 0 \Rightarrow$ ~~dim~~ $Null(T) = \{0\}$ $\quad \square$

$T: V \to W$

V and W finite dimensional

$\dim V = \dim W$

T surjective — T injective

Thm: If $\dim V = \dim W = n$

$A = [T]_\beta^\gamma$ is $n \times n$

A invertible <=> col.s of A are lin. indep.

<=> Rows of A are linearly indep.

Proof: cols of A span the range

$A: \mathbb{F}^n \to \mathbb{F}^n$

A invertible <=> injective + surjective

which means columns of A are lin. ind.

$A\vec{x} = \vec{0}$ gives a system of n linear equations

Rows of A independent

RREF A gives a unique solution — $\vec{x} = \vec{0}$

A injective

Invertible <=> injective <=> surjective

$\Updownarrow$ independent rows

$\Updownarrow$ independent columns

Finding A^{-1}

Doing GE to A we get Id Matrix.

Each operation corresponds to an elementary matrix

$$\xrightarrow{R_3-R_2} \begin{pmatrix}1&1&0\\0&1&-1\\0&0&2\end{pmatrix} \begin{pmatrix}1&0&0\\0&1&0\\0&-1&1\end{pmatrix}$$

$$R_1-R_2 \begin{pmatrix}1&0&1\\0&1&-1\\0&0&2\end{pmatrix} \begin{pmatrix}1&-1&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\tfrac{1}{2}R_3 \begin{pmatrix}1&0&1\\0&1&-1\\0&0&1\end{pmatrix} \begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{pmatrix}$$

$$R_2+R_3 \begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix} \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

$$R_1-R_3 \quad Id \begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}$$

$$A^{-1} \begin{pmatrix}1&0&-1\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{1}{2}\end{pmatrix} \cdots$$

$$A = \left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\1&0&1&0&1&0\\0&1&1&0&0&1\end{array}\right) \qquad -R_2 \left(\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&-1&1&-1&0\\0&0&2&-1&1&1\end{array}\right)$$

$$R_2-R_1 \left(\begin{array}{ccc|ccc}1&1&0&1&0&0\\0&-1&1&-1&1&0\\0&1&1&0&0&1\end{array}\right) \qquad \tfrac{1}{2}R_3 \left(\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&1&-1&1&-1&0\\0&0&1&-\frac{1}{2}&\frac{1}{2}&\frac{1}{2}\end{array}\right)$$

$$\begin{matrix}R_1+R_2\\ \\ R_3+R_2\end{matrix} \left(\begin{array}{ccc|ccc}1&0&1&0&1&0\\0&-1&1&-1&1&0\\0&0&2&-1&1&1\end{array}\right) \qquad \begin{matrix}R_2+R_3\\R_1-R_3\end{matrix} \left(\begin{array}{c|ccc} & \frac{1}{2} & 1-\frac{1}{2} & -\frac{1}{2}\\ Id & 1-\frac{1}{2} & -1+\frac{1}{2} & \frac{1}{2}\\ & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2}\end{array}\right) A^{-1}$$

MAT240 Nov ~~Oct~~. 10th

V finite dim v.s.p $\dim V = n$

$L(V)$ = linear maps $T: V \to V$ "operators".

$= L(V, V)$ $\dim L(V) = n^2$. (square $n \times n$ matrices)

Remark: When representing one of the operators $T \in L(V)$ as a matrix, only need one basis $\beta = (v_1, \dots, v_n)$ for V.

$[T]_\beta^\beta$

If we ~~to~~ change basis for β'

$[T]_{\beta'}^{\beta'} = [I]_\beta^{\beta'} [T]_\beta^\beta \cdot [I]_{\beta'}^\beta$

These are inverses of each other.

$\boxed{\text{"} A' = P \cdot A \cdot P^{-1} \text{"}}$

Terminology: When $A, A' \in \mathbb{F}^{n \times n}$ are related by $A' = PAP^{-1}$, we say A and A' are SIMILAR.

Q: What kinds of linear operators are there? "Classification"

A: Jordan Canonical form

Simplest case:

① zero operator $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

② identity operator $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

③ λI, $\lambda \in \mathbb{F}$

rescaling by λ

$[\lambda I]_\beta^\beta = \begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}$

④ diagonal matrix $\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ rescaling each basis elt. by its own factor.

warning for ④ a diagonal matrix need not be diagonal in a different basis.

$$\begin{pmatrix} \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{-1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt2} & \frac{-1}{\sqrt2} \\ \frac{1}{\sqrt2} & \frac{1}{\sqrt2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

change of basis

Question for ④: Given an operator T,
is there a basis for which $[T]_\beta^\beta$ is diagonal?
"Is T diagonalizable?"

not ~~always~~?

The answer is that not always possible.
But over $\mathbb{C}$ numbers, almost all operators are diagonalizable but NOT ALL!

Defn: ① ~~a~~ $\lambda \in \mathbb{F}$ is an EIGENVALUE of $T \in L(V)$
when there is a ~~nonzero~~ nonzero $v \in V$ s.t. $Tv = \lambda v$

In other words: $\dim \text{null}(T - \lambda \cdot I) \cancel{=0} > 0$

② If λ is eigenvalue, then any ~~vectors~~ vector in ~~all~~ $\text{null}(T-\lambda I)$ is called an eigenvector
i.e. any vector u s.t. $Tu = \lambda u$

eigenspace

Remark: The purpose of "eigen" stuff is to try to find a basis in which T is diagonal.

$$[T]_\beta^\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$Tv_1 = \lambda_1 v_1$
$Tv_2 = \lambda_2 v_2$
$\vdots$
$Tv_n = \lambda_n v_n$

the basis is eigenvector

e.g. $\lambda \cdot I$ has eigenvalue λ
why? because $u \neq 0$ satisfies
$Tu = \lambda u$
Then every vector in V is an eigenvector.

* The two obstacles preventing diagonalization?

① operator has no eigenvalue at all.

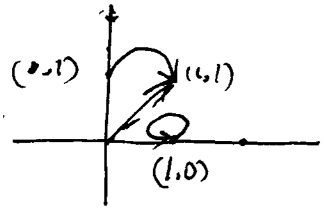
e.g. $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad 0<\theta<\pi$

as a map $\mathbb{R}^2 \to \mathbb{R}^2$

note: over $\mathbb{C}$ this is not a problem.

② "Cycling"

Look at this matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$



$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

not sent to itself
but itself plus another vector of the previous ~~basis~~ matrix.

A Shield Transformation.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

← eigenvector

↑ eigenvalue

$$T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

What we will see:

Over $\mathbb{C}$, always possible to find a basis where $[T]_\beta^\beta$ has the following shape.

Jordan Block

CYCLING

$$\begin{pmatrix} \boxed{\begin{matrix} \lambda_1 & 1 & & \\ & \lambda_1 & 1 & \\ & & \lambda_1 & 1 \\ & & & \lambda_1 \end{matrix}} & & \\ & \boxed{\begin{matrix} \lambda_2 & 1 & & \\ & \lambda_2 & 1 & \\ & & \ddots & \\ & & & \lambda_2 \end{matrix}} & \\ & & \boxed{\begin{matrix} \lambda_3 & 1 \\ & \lambda_3 \end{matrix}} \cdots \end{pmatrix}$$

Warning

Blocks could be 1×1, 2×2, $\cdots k\times k$ ...

$$\begin{matrix} \boxed{\lambda_1} & & & & \\ & \boxed{\lambda_2} & & & \\ & & \boxed{\lambda_3} & & \\ & & & \boxed{\lambda_4} & \\ & & & & \ddots \end{matrix}$$

1×1

Warning Some λ_i may coincide

$$I = \begin{pmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{pmatrix}$$

MAT 240 Nov. 15th

eigenvalues & eigenvectors

Recall: $\lambda \in \mathbb{F}$ is eigenvalue when ~~nt~~ null $(T-\lambda) \neq \{0\}$. (i.e. $Tu = \lambda u$ for some $u \neq 0$).

null $(T-\lambda)$ is eigenspace corresponding to λ.

$\parallel$

V eigenvector.

Warning: may not be any eigenvalue in $\mathbb{F}$

(over $\mathbb{C}$ there's always one)

-even with eigenvector, may not be a basis of eigenvector.

$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $0<\theta<\pi$ no eigenvalues at all in $\mathbb{R}$.

(but one in $\mathbb{C}$).

Shield or Cycling

$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix} = A$

$\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

eigenvalue of A

null $(A-\lambda) \neq \{0\}$ 2 is an eigenvalue

$\begin{pmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{pmatrix}$ suppose $\lambda \neq 2$

$2-\lambda \neq 0$

this is invertible

~~nt~~ null $= \{0\}$

$\Rightarrow$ so only have eigenvalue 2.

Just because $\lambda = 2$ is only eigenvalue

there still may be many ~~eigenvalues~~ eigenvectors

$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\leftarrow$ has a basis of eigenvector

null $(A-2)$

null $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\overbrace{\dim \text{null}(A-2)}^{\text{span}\begin{pmatrix}1\\0\\0\end{pmatrix}} = 1$

$\Rightarrow A$ has one eigenvalue 2

dim eigenspace $\text{span}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$$\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix} \quad \begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix} \quad \begin{pmatrix}2&0&0\\0&2&1\\0&0&2\end{pmatrix} \to \text{null}(A-2)\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} \to \text{2-di null space}$$

dimNull$(A-2)$: $3 \qquad 1 \qquad 2$

Thm: If $\lambda_1, \cdots \lambda_m$ are distinct eigenvalue of $T \in L(V)$ with nonzero $(v_1, \cdots, v_m)$ eigenvectors $Tv_i = \lambda_i v_i$

Then they are linearly independent.

Cor: maximum of $\dim V$ eigenvalues.

Proof: suppose $a_1 v_1 + \cdots + a_m v_m = 0$

take v_{k+1} the first vector which is ~~redundant~~ redundant,

i.e. $v_{k+1} = a_1 v_1 + \cdots + a_k v_k$

$$Tv_{k+1} = \lambda_{k+1} v_{k+1} = a_1 \lambda_1 v_1 + \cdots + a_k \lambda_k v_k \quad ①$$

On the other hand, $\lambda_{k+1} v_{k+1} = a_1 \lambda_{k+1} v_1 + \cdots + a_k \lambda_{k+1} v_k \quad ②$

$① - ②$: $a_1 \underbrace{(\lambda_1 - \lambda_{k+1})}_{\text{non-zero}} v_1 + a_2 (\lambda_2 - \lambda_{k+1}) v_2 + \cdots + a_k (\lambda_k - \lambda_{k+1}) v_k = 0$

$$(\lambda_1 - \lambda_{k+1})(a_1 v_1 + \cdots + a_k v_k) = 0$$

$\Rightarrow a_1 = a_2 = \cdots = a_k = 0 \Rightarrow v_{k+1} = 0$

contradiction.

Thm: every $T \in L(V)$ V finite dim over $\mathbb{C}$, has at least 1 eigenvalue.

Proof: Take $v \in V$, $v \neq 0$

Suppose $\dim V = n$

form list $(v, Tv, T^2 v, \cdots, T^n v)$ has length $n+1$.

$\Rightarrow$ list is lin. dep.

$\exists\, a_i \in \mathbb{C}$, not all 0.

s.t. $a_0 v + a_1 Tv + \cdots + a_n T^n v = 0$

$(\cdots + \cdots + a_n T^n) v = 0$ polynomial in 1 variable $(a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n)$

Let a_m be the ~~the~~ last non-zero of a_i's

$\Rightarrow p(z)$ has degree m

because we are over $\mathbb{C}$ the polynomial factors ... into linear factors.

$p(z) = c(z-\lambda_1)\cdots(z-\lambda_m)$ $\left.\begin{matrix} c \neq 0 \\ \lambda_i \end{matrix}\right\}$ numbers in $\mathbb{C}$

$\Rightarrow c(T-\lambda_1)\cdots(T-\lambda_m)v = 0$

λ dead so one of the $(T-\lambda_i)$ killed it

$\Rightarrow$ one of the λ_i is eigenvalue.

Comment $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $0<\theta<\pi$ $\rightarrow$ no ~~$\exists$~~ eigenvalue

The interpretation of this

~~this~~ that we ~~use~~ would get a quadratic real polynomial

no real solutions $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Motivation:

$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$ ~~degree~~ eigenvalue λ null $(T-\lambda)$ dim $=1$

$(T-\lambda) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $(T-\lambda)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Similarly

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix} - \lambda = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \qquad (T-\lambda)^4 = 0$$

Def^n If λ eigenvalue of $T \in L(V)$

~~the~~ null $(T-\lambda)^k$ is the generalized eigen~~vector~~

v ~~generali~~ generalized eigenvector for λ.

e.g $D = \frac{d}{dt}$ ~~adds also~~ acts on diff. functions on $\mathbb{R}$ $f(t)$

$De^{\lambda t} = \lambda e^{\lambda t}$ for any $\lambda \in \mathbb{R}$

exponential f^n, $e^{\lambda t}$ is eigenvector for eig. λ

actually only eigenvector for λ (up to rescaling)

What about generalized eigenvectors?

$$\text{null } (D-\lambda)^2 = \{f : (D-\lambda)^2 f = 0\}$$

$$= \left\{f : \frac{d^2f}{dt^2} - 2\lambda \frac{df}{dt} + \lambda^2 f = 0\right\}$$

or $(D-\lambda)(\underbrace{D-\lambda)f}) = 0$

$\Rightarrow (D-\lambda)f = Ce^{\lambda t}$ (because that's the only eigenvector)

$\Rightarrow Df - \lambda f = Ce^{\lambda t}$

$$\frac{df}{dt} - \lambda f = Ce^{\lambda t}$$

$$\frac{d}{dt}(fe^{-\lambda t}) = \left(\frac{df}{dt}\right)e^{-\lambda t} - \lambda f e^{-\lambda t} = \left(\frac{d}{dt}f - \lambda f\right)e^{-\lambda t}$$

$$\frac{d}{dt}(fe^{-\lambda t}) = C$$

$fe^{-\lambda t} = Ct + B$ for some constant C, B.

$\Rightarrow \boxed{f = Cte^{\lambda t} + Be^{\lambda t}}$

$Cte^{\lambda t}$ → generalized eigenvector; $Be^{\lambda t}$ → eigenvector

write D as a matrix on span $(e^{\lambda t}, te^{\lambda t})$

$D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ derivative ~~of~~ always has plenty of Jordan blocks

Thm: Decomposition thm:

if $(\lambda_1, \ldots, \lambda_m)$ distinct numbers

then ~~$\text{null}((T-\lambda_1)^{k_1}\cdots(T-\lambda_m)^{k_m}) = \text{null}(T-\lambda_1)^{k_1}$~~

the sum $\text{null}(T-\lambda_1)^{k_1} \oplus \cdots \oplus \text{null}(T-\lambda_m)^{k_m}$ is direct.

and equals $= \text{null}((T-\lambda_1)^{k_1}\cdots(T-\lambda_m)^{k_m})$

Remark: special case $k_i = 1$, $\forall i$

factor polynomials ...

Application.

diff. eqn $\quad y''' - 4y'' + 5y' - 2y = 0$

general solution?

$D = \frac{d}{dt} \qquad (D^3 - 4D^2 - 5D - 2)y = 0$

$(D-1)^2(D-2)\,y = 0$

$\text{null}\left((D-1)^2(D-2)\right) = \text{null}\,(D-1)^2 \oplus \text{null}\,(D-2)$

$Ce^t + Dte^t$

~~gen~~ $\quad C_1e^t + C_2te^t + C_3e^{2t}$

eigenvector $\quad$ gen... eigenvector

$\lambda = 1 \qquad \lambda = 1$

$\Rightarrow$ get a 3-dim space of solution.

MAT240 November 17th

Last time

① If $T \in L(V)$ with $n = \dim V$ different eigenvalues $\lambda_1, \dots \lambda_n$.
Then $(v_1, \cdots, v_n)$ is a basis.

(v_i, nonzero eigenvector for λ_i)

i.e. $V = \underbrace{\text{null}(T-\lambda_1)}_{V_1} \oplus \underbrace{\text{null}(T-\lambda_2)}_{V_2} \oplus \cdots \oplus \underbrace{\text{null}(T-\lambda_i)}_{V_n}$

② If V is complex vector space
Then T has at least $\underline{\underline{1}}$ eigenvalue.

$(v, Tv, T^2v, \cdots T^n v)$

$p(T)v = 0$

$c(z-\lambda_1) \cdots (z-\lambda_m)$

$T \in L(V) \cong \mathbb{F}^{n^2}$

spectrum $\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n \quad \mathbb{F}$
(set of eigenvalues)

$T \in L(\mathbb{R}^n) \subset L(\mathbb{C}^n)$

$n \times n$ real matrix

$n \times n$ complex matrix

$\boxed{\text{Polynomials applied to operator}}$

Look for polynomials $p(z)$ s.t. $p(T) = 0$

e.g. ① $T = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$ $\quad T - I = 0$

$p(z) = z - 1$

$p(z)$ s.t. $p(T) = 0$
~~quadr~~ quadratic polynomial

② $T = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$ $\quad (z-\lambda_1)(z-\lambda_2)$

$\begin{pmatrix} 0 & 0 \\ 0 & \lambda_2-\lambda_1 \end{pmatrix} \begin{pmatrix} \lambda_1-\lambda_2 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$p(z) = (z-\lambda_1) \cdots (z-\lambda_n)$
degree-n polynomial

Suppose that you know ~~p(z)~~ $p(T) = 0$ for $p(z) = (z-\lambda_1)(z-\lambda_2) \cdots (z-\lambda_n)$

$n = \dim V \qquad \lambda_i$ different [redacted]

$p(T)v = 0 \quad \forall v$

③ $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad (T-2I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$(T-2I)^3 = 0$ i.e. polynomials

$(z-2)^3$ cubic minimal polynomial

Terminology: ① ~~Minimum~~ Minimal polynomial of T

② Characteristic polynomial of T

It's the monic polynomial of smallest degree $p(T)=0$

monic: $z^n + az^{n-1} + \cdots$ with shortest coeff. $=1$

3x3	min.	characteristic
$\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$	$z-2$ largest	$(z-2)^3$ 3x3 ✓
$\begin{pmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{pmatrix}$	$(z-2)^2$	$(z-2)^3$
$\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$	$(z-2)^3$	$(z-2)^3$

dimension of generalized eigenspace null $(T-\lambda)^k$

It has multiplicity

④ $\begin{pmatrix} 2 & 1 & & & \\ & 2 & 1 & & \\ & & 2 & & \\ & & & 2 & \\ & & & & 3 & 1 \\ & & & & & 3 \end{pmatrix}$

eigenvalues 2, 3

3 ~~Jordan blocks~~ size (3, 1) $\lambda=2$, size 2 $\lambda=3$

Characteristic : $(z-2)^4(z-3)^2$ total degree 6

minimal : $(z-2)^3(z-3)^2$

$\begin{pmatrix} 3 & 1 & & & \\ & 3 & & & \\ & & 3 & 1 & \\ & & & 3 & \\ & & & & 3 \end{pmatrix}$

Remark minimal polynomial tells you the size of the largest Jordan block for each of ~~the value~~ eigenvalue.

$(z-3)^2$

For example : minimal poly: $(z-1)z^2$, so it is : $\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$

If min. poly is $(z-1)z^2$ and char. poly. is $(z-1)^3 z^2$

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \quad \times$$

SHOULD BE multiplicity $=3$

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \checkmark$$

Prop: the minimal polynomial p divides any other polynomial q s.t. $q(T)=0$.

Reason: $q(z) = p(z)\cdot a(z) + r(z)$ $\deg r < \deg p$

apply to T $q(T)=0 = p(T)\cdot a(T) + r(T)$

$\Rightarrow$ s.t. $r(T)=0$ contradiction

So $r=0$

Thus p actually divides q.

Why must there be any polynomials killing T?
The answer

Thm: recall $\dim L(V) = n^2$ $n = \dim V$

$(I, T, T^2, T^3, \cdots T^{n^2})$

This must be linearly dependent
which means that

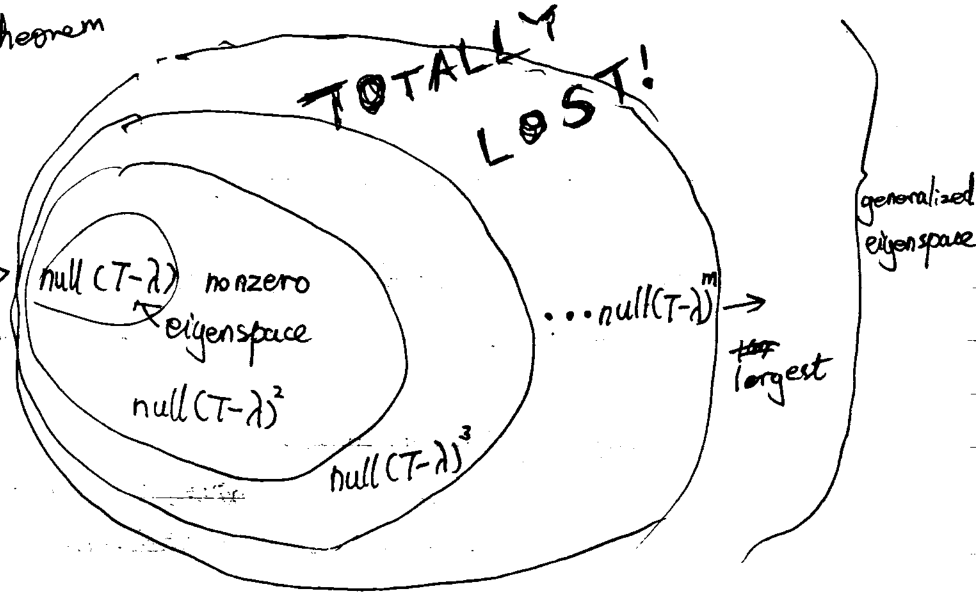
$$a_0 I + a_1 T + a_2 T^2 + \cdots + a_{n^2} T^{n^2} = 0$$

Decomposition theorem

MAT240 Nov. 22nd

Recall :

λ eigenvalue $\Leftrightarrow$ null $(T-\lambda)$ nonzero



Main Tool: <u>Decomposition Thm.</u>

Thm: If $P(T)=0$ for $p=(z-\lambda_1)^{k_1}\cdots(z-\lambda_m)^{k_m}$, λ_i distinct

<u>then</u> $V=\text{null}\,(T-\lambda_1)^{k_1}\oplus\cdots\oplus\text{null}\,(T-\lambda_m)^{k_m}$

(V is a direct sum of the generalized eigenspaces of T)

<u>Step 1</u> : RHS is a <u>direct sum</u>

Suppose we have $v_1+v_2+\cdots+v_m=0$. v_i is in null $(T-\lambda_i)^{k_i}$

Want to show that $v_i=0$ $\forall i$.

$(T-\lambda_1)^{k_1}$ kills null $(T-\lambda_1)^{k_1}$

but it is invertible on null $(T-\lambda_2)^{k_2}$ or $\cdots$ or null $(T-\lambda_m)^{k_m}$

<u>Lemma</u>: ① If $T^k=0$ then $I-T$ is invertible.

Proof: $(I-T)^{-1}=I+T+T^2+\cdots T^{k-1}$

$$\underbrace{(I+T+T^2+\cdots+T^{k-1})}_{\text{This is the Inverse.}}(I-T)=I$$

② If $(T-a)^k=0$ for some a. then $(T-b)$ is invertible when $b\neq a$.

$$(T-b)^{-1}=\frac{-1}{b-a}\left(I+\left(\frac{T-a}{b-a}\right)+\left(\frac{T-a}{b-a}\right)^2+\cdots+\left(\frac{T-a}{b-a}\right)^{k-1}\right)$$

(<u>Lemma</u>: on subspace null $(T-\lambda_2)^{k_2}=V(\lambda_2)$

we have $(T-\lambda_2)^{k_2}=0$ and $(T-\lambda_i)^{k_i}$ invertible.

Apply $\prod_{i\neq j}(T-\lambda_i)^{k_i}$ to $v_1+\cdots+v_m$:

$\prod$: product

get $\prod_{i\neq j}(T-\lambda_i)^{k_i}v_j=0$

but each of the $(T-\lambda_i)^{k_i}$, $i\neq j$ are invertible on $V(\lambda_j) = \text{null}\,(T-\lambda_j)^{k_j} \Rightarrow \boxed{v_j = 0}$

Step 2 : Show that the ~~rig~~ RHS $= V$ (not a proper subspace)
must show every $v \in V$ can be decomposed as $v = v_1 + \cdots + v_m$
$v_i = \text{null}\,(T-\lambda_i)^{k_i}$

Induction: ① $V(\lambda_1) = \text{null}\,(T-\lambda_1)^{k_1}$

② $V(\lambda_1) \oplus V(\lambda_2) = \cancel{\text{null}\,(T-\lambda_1)^{k_1}\,\text{null}\,(T-\lambda_2)^{k_2}}\ \text{null}\left((T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}\right)$

③ $V(\lambda_1) \oplus V(\lambda_2) \oplus V(\lambda_3) = \text{null}\left((T-\lambda_1)^{k_1}\cdot(T-\lambda_2)^{k_2}\cdot(T-\lambda_3)^{k_3}\right)$

$\vdots$

end $V(\lambda_1) \oplus V(\lambda_2) \oplus \cdots \oplus V(\lambda_m) = \text{null}\left((T-\lambda_1)^{k_1} \cdots (T-\lambda_m)^{k_m}\right) = V$

first step:

$V(\lambda_1) \oplus V(\lambda_2) \subseteq \text{null}\left((T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}\right)$

OK

$u + v \qquad (T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}(u+v)$

$(T-\lambda_2)^{k_2}(T-\lambda_1)^{k_1} u + (T-\lambda_2)^{k_2}(T-\lambda_1)^{k_1} v$

$0 + 0 = 0$

To show $\supseteq$: $V(\lambda_1) \oplus V(\lambda_2) \supseteq \text{null}\left((T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}\right)$

let $v \in \text{null}\left((T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}\right)$

$(T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2} v = 0$

$\Rightarrow (T-\lambda_2)^{k_2} v \in \text{null}\,(T-\lambda_1)^{k_1}$

goal:
$v = v_1 + v_2$
$v_1 \in V(\lambda_1)$, $v_2 \in V(\lambda_2)$

now $(T-\lambda_2)^{k_2}$ is invertible on $\text{null}\,(T-\lambda_1)^{k_1}$
(Lemma)

$\boxed{\text{Let } w_1 = (T-\lambda_2)^{k_2} v_1}$ (inverse only on $V(\lambda_1)$)

$v - w_1 \in \text{null}\,(T-\lambda_2)^{k_2}$

$(T-\lambda_2)^{k_2}(v-w_1) = v_1 - v_1 = 0$

$\Rightarrow$ define $w_2 = v - w_1$

get $v = w_1 + w_2$, $w_1 \in V(\lambda_1)$, $w_2 \in V(\lambda_2)$

rest of induction uses the same principle

$(T-\lambda_1)^{k_1} \cdots (T-\lambda_p)^{k_p} = 0$

$\Rightarrow \prod_{\substack{i=1 \\ i \neq j}}^{p} (T-\lambda_i) v \in \text{null}\,(T-\lambda_j)^{k_j}$

v_j

apply inverse of to v_j get w_j.

$\Gamma\ (T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}=0$

$V = V(\lambda_1)\oplus V(\lambda_2)$

$v = w_1 \quad u_2$

$(T-\lambda_1)^{k_1}(T-\lambda_2)^{k_2}v=0$ Find them

apply $(T-\lambda_2)^{k_2}v = (T-\lambda_1)^{k_1}_{w} + 0$

to solve for w_1 write $w_1 = (T-\lambda_2)^{k_2} v_1$ inverse on $V(\lambda_1)$

Example of application of this

① $y''' - 4y'' + 5y' - 2y = 0$ homogeneous linear ~~ODE~~ ODE

$$(D^3 - 4D^2 + 5D - 2)y = 0$$

$$(D-1)^2(D-2)y = 0$$

find null $(D-1)^2(D-2) = V$

Thm $\Rightarrow V = \text{null}(D-1)^2 \oplus \text{null}(D-2)$

$$\text{span}(e^t, te^t) \qquad \text{span}(e^{2t})$$

general solution $Ae^t + Bte^t + Ce^{2t}$ $\quad A, B, C \in \mathbb{R}$

EX_2: $y''' - 4y'' + 5y' - 2y = e^{3t}$ inhomogeneous

$(D-1)^2(D-2)y = e^{3t}$ notice in null $(D-3)$

$(D-1)^2(D-2)(D-3)y = (D-3)e^{3t} = 0$ ④

null $(D-1)^2(D-2)(D-3) = \text{span}(e^t, te^t, e^{2t}, e^{3t})$

general solution:

$\boxed{y = Ae^t + Bte^t + Ce^{2t} + De^{3t}}$

null $(T-1)^2$ $\quad$ null $(T-2)$ $\quad$ null $(T-3)$

must solve ④

Apply $(D-1)^2(D-2)$ get $(D-1)^2(D-2)Fe^{3t} = e^{3t}$

null $(D-3)$

but $(D-2)e^{3t} = 3e^{3t} - 2e^{3t} = 1\cdot e^{3t}$

$(D-1)e^{3t} = (3-1)e^{3t} = 2e^{3t}$ $\quad\Rightarrow\quad 4Fe^{3t} = e^{3t}$

$F = \frac{1}{4}$

idea. $T = D$

ODE $\leftrightarrow P(T)y = 0 \quad P = T^3 + 2T^2 + \cdots$

Sol^{ns} = vectorspace V on which $P(T) = 0$

Thm If you factor $p = (z-\lambda_1)^{k_1} \cdots (z-\lambda_m)^{k_m}$

$$V = null(T-\lambda_1)^{k_1} \oplus \cdots \oplus null(T-\lambda_m)^{k_m}$$

spans $\quad span\,(e^{\lambda t}, te^{3t}, t^2e^{3t})$

How to find a Jordan Basis?

① V decomposes $V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_m)$

so I only need to explain how to do it for $V(\lambda_1)$

work in ~~$V(\lambda_1)$~~

$$V(\lambda) = null\,(T-\lambda)^k$$

② It may be very easy, if you know $\dim V(\lambda)$

know minimum polynomial

e.g. $P_{min}(T) = (T-\lambda)^2$ and $\dim V = 2$

$\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} \Rightarrow$ we must find : 1 eigenvector $Tu = \lambda u$

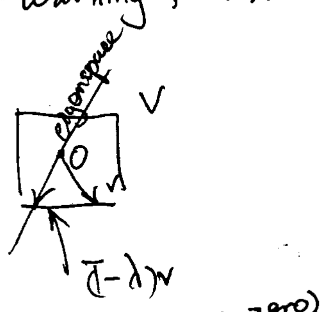
1 cyclic vector v s.t.

v satifies $\left.\begin{array}{l} (T-\lambda)^2 v = 0 \\ \cancel{(T-\lambda)^2 v = u} \\ (T-\lambda)v = u \end{array}\right\}$ this is what must be solved

why not pick any $v \in V$ (know $(T-\lambda)^2 v = 0$)

then define $u = (T-\lambda)v$, it must be an eigenvector

warning; must chose $v \in V$ not in eigenspace



computationally,

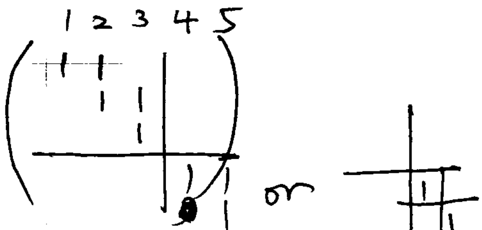
how to pick v not in null $(T-\lambda)$?

first choose an nonzero eigenvector u

extend to a basis (u, v)

then $((T-\lambda)v, v)$ is Jordan B.

Ex (easy): $P_{min} = (z-1)^3$ $\dim V = 5$



2×2 or 2 "1×1"

for this $\dim(\text{null}(T-1)) = 3$ } 2 cases.

for that $\dim \text{null}((T-1)) = 2$

<u>idea</u>: to find the shape of Jordan form we need to know the ~~def dem~~ $\dim\text{null}(T-\lambda)^k$ for $k=1,2,3,\dots$

$(t_1, t_2, t_3, \dots)$ $t_i = \dim \text{null}(T-\lambda)^i$.

Sequence $\vec{t} = (2, 4, 5, 5, \dots)$

2 eigenvectors

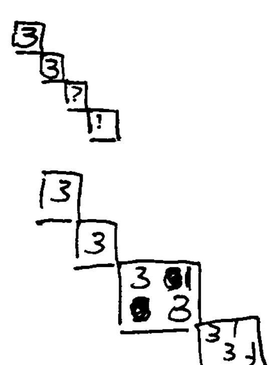
sequence: $\vec{t} = (3, 4, 5, 5, \dots)$

Q: suppose eigenvalue $= \{3\}$
and $\vec{t} = (4, \overset{+2}{6}, \overset{+1}{7}, 7, 7, \dots)$
What is the Jordam form?

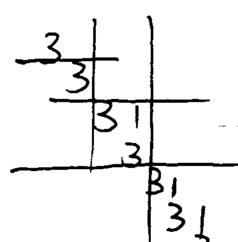
~~7-dim~~ 7-d space

4 Jordan blocks, $6-4=2$ of them are larger than 1×1.

$7-6=1$ 1 of them is larger than 2×2



$\Rightarrow$



MAT240 Nov. 24th

Finding a Jordan basis (partial explain)

T operator V f.d. $\mathbb{C}$ v.sp

① If min poly. $p(z)$. factor $p(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_m)^{k_m}$

$V(\lambda_i) = \text{null}(T-\lambda_i)^{\dim V}$ by Decomposition thm. $V = V(\lambda_1) \oplus \cdots \oplus V(\lambda_m)$

($V(\lambda_i)$ generalized eigenvector for eigenvalue λ_i)

So just work in each $V(\lambda_i)$ separately

find Jordan Basis for $V(\lambda_1)$, $V(\lambda_2)$, ..., put them together

② now suppose T has $p_{min}(z) = (z-\lambda)^k$ ($k < \dim V$)

must find Jordan basis

③ figure out shape, $\vec{t}\,(t_1, t_2, \cdots)$

$t_i = \dim \text{ null } (T-\lambda)^i$

$t_1 = \dim(\text{eigenspace}) = \text{total \# of Jordan blocks (including } 1\times 1)$

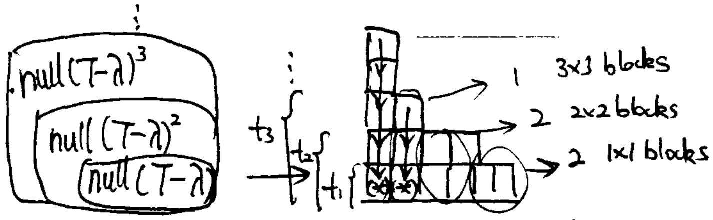
(Since each Jordan block has exactly 1 eigen-vector)

$t_2 \geq t_1$ and $t_2 - t_1 = \#$ of J.B of size > 1

since $\text{null}(T-\lambda) \subset \text{null}(T-\lambda)^2$

$t_3 \geq t_2$ $\quad t_3 - t_2 = \#$ J.B size > 2

"KILLS"



The corner vectors generate everything down by apply $(T-\lambda)$.

If $v_1 \in \text{null}(T-\lambda)^2$ but not in $\text{null}(T-\lambda)$

$(T-\lambda)v_1 \in \text{null}(T-\lambda)$ this is sent to (*) (to a previous one)

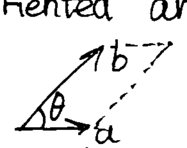
MAT240 Nov. 29 th

Determinants

The determinant of operator $T \in L(V)$ is a number $\in \mathbb{F}$
nonzero $\Leftrightarrow$ T invertible

model for oriented volume
if $T \in \mathbb{R}^{n\times n}$, then $\det T$ = oriented volume of Euclidean box w/edges = columns

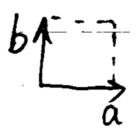
Oriented area in plane



angle θ in $[0,\pi]$

Area is given by:
$\text{Area} = |a|\cdot|b|\cdot\sin\theta$

Oriented area
$A(a,b) = \pm|a||b|\sin\theta$
+ if θ from a to b counterclockwise
− otherwise

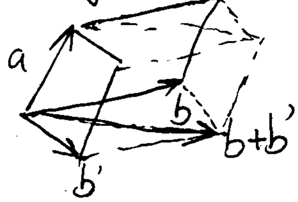


$A(a,b)=1$

$A(a,b)=-1$

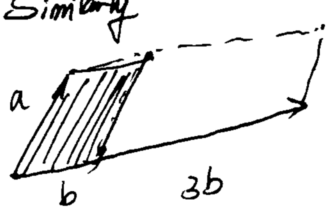
essentially, $A(a,b)$ gives −Area
− comparison orientation (a,b) and (e_1,e_2)

good property:



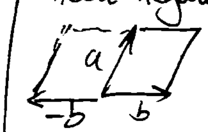
$A(a,b)+A(a,b') = A(a,b+b')$
Area is linear in each of the two argument a, b. "bilinear".

Similarly



$A(a,3b)=3A(a,b)$
$\lambda\in\mathbb{R}$, e.g. 3

need negative areas so that $A(a,-b)$



$A(a,-b) = -A(a,b)$
makes sense

[redacted]

Exterior Algebra V vectorspace / $\mathbb{F}$ $\mathbb{F} = \mathbb{R}, \mathbb{C}$

Create new vector spaces: $\boxed{V}$, $V \wedge V = \Lambda^2 V$

$V \wedge V \wedge V = \Lambda^3 V$

$\boxed{\Lambda^n V} \Rightarrow n = \dim V$

$\boxed{\Lambda^0 V = \mathbb{F}}$

⓪ $\Lambda^0 V = \mathbb{F}$ ✓

① $\Lambda^1 V = V$ ✓

② $\Lambda^2 V =$ "formal area" $\longrightarrow$ vectorspace of formal linear combos of expressions $u \wedge v$ where $u, v \in V$

i.e. $\Lambda^2 V = \{u_1 \wedge v_1 + u_2 \wedge u_2 + \cdots + u_k \wedge v_k\}$

Such that • $u \wedge (v_1 + v_2) = u \wedge v_1 + u \wedge v_2$

$\uparrow$ sum in $\wedge V$ $\uparrow$ sum in $\Lambda^2 V$

• $u \wedge (\lambda v) = \lambda (u \wedge v)$

• $u \wedge v = -v \wedge u$

"$\wedge$" doesn't mean an operation, it's a new vector space ?

again: ① $u \wedge (v_1 + v_2) = u \wedge v_1 + u \wedge v_2$

② $u \wedge (\lambda v) = \lambda (u \wedge v)$

③ $u \wedge v = -v \wedge u$

(SAME PROPERTIES OF Area(u, v) **!**) $\Rightarrow$ $u \wedge v$ is bilinear

$(u_1 + u_2) \wedge v = u_1 \wedge v + u_2 \wedge v$

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e.g. $\mathbb{R}^2$ has a standard basis (e_1, e_2)

$\Lambda^2(\mathbb{R}^2) = \{u_1 \wedge v_1 + u_2 \wedge v_2 + \cdots + u_k \wedge v_k\}$

~~$(ae_1 + be_2) \times (ce_1 + de_2)$~~

$(ae_1 + be_2) \wedge (ce_1 + de_2)$

$\parallel$ rule ①

$(ae_1 + be_2) \wedge ce_1 + (ae_1 + be_2) \wedge de_2$

$\parallel$ rule ①③②

$ac(e_1 \wedge e_1) + bc(e_2 \wedge e_1) + ad(e_1 \wedge e_2) + bd(e_2 \wedge e_2)$

Idea: "wedge" understands linear ~~dependent~~ dependence.

$\parallel$

$0 + bc(e_2 \wedge e_1) + ad(e_1 \wedge e_2) + 0$

$= ad(e_1 \wedge e_2) - bc(e_1 \wedge e_2)$

$= (ad - bc)(e_1 \wedge e_2)$

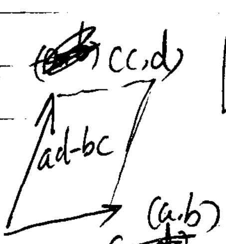
$\parallel$

$A(\vec{a}, \vec{b})\, e_1 \wedge e_2 = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$

$\vec{a} : (ae_1 + be_2)$

$\vec{b} : (ce_1 + de_2)$

notice: If $u \wedge v = -v \wedge u$
then $u \wedge u = -u \wedge u$
so $u \wedge u = 0$



Remark: if you replace $\vec{a}$ with $\vec{a}+\vec{b}$ then $(\vec{a}+\vec{b})\wedge\vec{b}$

$$\overset{\parallel}{\vec{a}\wedge\vec{b}+\vec{b}\wedge\vec{b}}$$
$$\overset{\parallel}{\vec{a}\wedge\vec{b}+0}=\vec{a}\wedge\vec{b}$$

Conclusion (dim 2)

Def: ① If e_1, e_2 basis $(Ae_1)\wedge(Ae_2) = (\det A)(e_1\wedge e_2)$

(number defined to be $\det A$)

② If $T \in L(V)$, then it defines a linear operator on $\Lambda^2 V$ "$\Lambda^2 T$"

by $(\Lambda^2 T)(u\wedge v) = (Tu)\wedge(Tv)$

- If V has dim 2, $\Lambda^2 V$ is 1-dimensional (spanned by $e_1\wedge e_2$)
  so ($\Lambda^2 T$) must be scalar multiplication by a <u>number</u> $\in \mathbb{F}$
  this number <u>is</u> the determinant by defn.
  $(\Lambda^2 T) = (\det T)\cdot I$

<u>Increase dim n</u> $V=\mathbb{R}^3$

$(a_1e_1+a_2e_2+a_3e_3)\wedge(b_1e_1+b_2e_2+b_3e_3) \in \Lambda^2 V$

$\binom{3}{2}$ terms: $= (a_1b_2-a_2b_1)\, e_1\wedge e_2 +$
$(a_1b_3-a_3b_1)\, e_1\wedge e_3 +$
$(a_2b_3-a_3b_2)\, e_2\wedge e_3$

(compare with $\times$)

we started with $\dim V=3$, got $\dim \Lambda^2 V = \binom{3}{2} = 3$

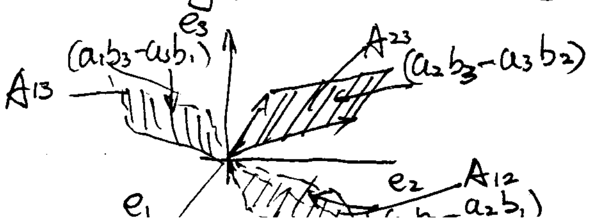
So If $V=\mathbb{R}^3$, we can define isomorphism $\Lambda^2\mathbb{R}^3 \to \mathbb{R}^3$

$$e_1\wedge e_2 \mapsto e_3$$
$$e_2\wedge e_3 \mapsto e_1$$
$$e_3\wedge e_1 \mapsto e_2$$

So $\varphi\big((a_1e_1+a_2e_2+a_3e_3)\wedge(b_1e_1+b_2e_2+b_3e_3)\big)$

$= (a_1b_2-a_2b_1)e_3 + (a_2b_3-a_3b_2)e_1 + (a_3b_1-a_1b_3)e_2$
$= (a_1e_1+a_2e_2+a_3e_3)\times(b_1e_1+b_2e_2+b_3e_3)$

Insight: 2-d area in 3 dimension is a <u>vector</u>.



actual area
$= \sqrt{A_{12}^2 + A_{23}^2 + A_{13}^2}$

Volumn in $\mathbb{R}^3$: volumn is trilinear, so we need $\Lambda^3 V$

$$\Lambda^3 V = \{u_1\wedge v_1\wedge w_1 + u_2\wedge v_2\wedge w_2 + \cdots + u_k\wedge v_k\wedge w_k\}, \quad u_i, v_i, w_i \in V$$

rules: • distributes

• any pair $u\wedge v = -v\wedge u$

in $\mathbb{R}^3$ $\quad a = a_1e_1 + a_2e_2 + a_3e_3$

$b = b_1e_1 + b_2e_2 + b_3e_3$

$c = c_1e_1 + c_2e_2 + c_3e_3$

$$\begin{aligned}
a\wedge b\wedge c = {} & a_1b_2c_3(e_1\wedge e_2\wedge e_3) + \\
& a_1b_3c_2(e_1\wedge e_3\wedge e_2) + && = -a_1b_3c_2(e_1\wedge e_2\wedge e_3) \\
& a_2b_1c_3(e_2\wedge e_1\wedge e_3) + && = -a_2b_1c_3(e_1\wedge e_2\wedge e_3) \\
& (a_2b_3c_1)(e_2\wedge e_3\wedge e_1) + && = a_2b_3c_1(e_1\wedge e_2\wedge e_3) \\
& (a_3b_1c_2)(e_3\wedge e_1\wedge e_2) + && = +a_3b_1c_2(e_1\wedge e_2\wedge e_3) \\
& (a_3b_2c_1)(e_3\wedge e_2\wedge e_1) && = -a_3b_2c_1(e_1\wedge e_2\wedge e_3)
\end{aligned}$$

$$= \boxed{(a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1)}(e_1\wedge e_2\wedge e_3)$$

$$\| \quad \text{volume}(a,b,c)$$

$$\| \quad \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

If $\dim V = 3$, $\Lambda^3 V$ 1-dimensional

i.e. if $V = \mathrm{span}(e_1, e_2, e_3)$, $\Lambda^3 V = \mathrm{Span}(e_1\wedge e_2\wedge e_3)$

and $(Te_1)\wedge(Te_2)\wedge(Te_3) = \boxed{\det T}\, e_1\wedge e_2\wedge e_3$

Recap: In dim 3 Area has 3 components $\Leftrightarrow \dim(\Lambda^2 V) = 3$

Volumn has 1 compoment $\Leftrightarrow \dim(\Lambda^3 V) = 1$

~~Comer~~ Comment. Expansion along column:

$$\begin{pmatrix} a_1e_1 \\ + \\ a_2e_2 \\ + \\ a_3e_3 \end{pmatrix} \wedge \begin{pmatrix} b_1e_1 \\ + \\ b_2e_2 \\ + \\ b_3e_3 \end{pmatrix} \wedge \begin{pmatrix} c_1e_1 \\ + \\ c_2e_2 \\ + \\ c_3e_3 \end{pmatrix} = (a_1e_1) \wedge \boxed{\begin{pmatrix} \cancel{b_1e_1} \\ b_2e_2 \\ b_3e_3 \end{pmatrix} \wedge \begin{pmatrix} \cancel{c_1e_1} \\ c_2e_2 \\ c_3e_3 \end{pmatrix}}$$

$$+ (a_2e_2) \wedge \boxed{\begin{pmatrix} b_1e_1 \\ \cancel{b_2e_2} \\ b_3e_3 \end{pmatrix} \wedge \begin{pmatrix} c_1e_1 \\ \cancel{c_2e_2} \\ c_3e_3 \end{pmatrix}}$$

$$+ (a_3e_3) \wedge \begin{pmatrix} b_1e_1 \\ b_2e_2 \\ \cancel{b_3e_3} \end{pmatrix} \wedge \begin{pmatrix} c_1e_1 \\ c_2e_2 \\ \cancel{c_3e_3} \end{pmatrix}.$$

$$= a_1 \det\begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix} \cdot e_1\wedge e_2\wedge e_3$$

$$+$$

$$a_2 \det\begin{pmatrix} b_1 & c_1 \\ b_3 & c_3 \end{pmatrix} \cdot e_2\wedge e_1\wedge e_3$$

$$+$$

$$a_3 \det\begin{pmatrix} b_1 & c_1 \\ b_2 & c_2 \end{pmatrix} \cdot e_3\wedge e_1\wedge e_2$$

$$= \cancel{a_1 \det\begin{pmatrix} b_2 & c_2 \\ b_3 & c_3 \end{pmatrix}}\ a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= \det\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

<u>Ingeneral</u>

If $\dim V = n$

an operator $T: V \to V$

agrees a map $\Lambda^n V \to \Lambda^n V$

defined by $u_1 \wedge \cdots \wedge u_n \longmapsto Tu_1 \wedge Tu_2 \wedge \cdots \wedge Tu_n$

This map must be scalar multiplication by a constant, this constant $= \det T$

Why? because $\Lambda^n V$ has dimension 1

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MAT240 Exerzitien 1

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Exercise 1

1. Solution: $(a+bi)^{-1}=c+di$

so $(a+bi)(c+di)=1$

$ac+(bc+ad)i-bd=1$

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$$\begin{cases} ac-bd=1 \\ bc+ad=0 \end{cases}$$

solve these two equations about c, d

so we can get

$$\left.\begin{aligned} c&=\frac{a}{a^2+b^2} \\ d&=\frac{-b}{a^2+b^2} \end{aligned}\right\}$$

Why?

2. Proof: The cube of $\frac{-1+i\sqrt{3}}{2}$ is $\left(\frac{-1+i\sqrt{3}}{2}\right)^3$.

which is equal to:

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^3=\left(\frac{1-i2\sqrt{3}+(-3)}{4}\right)\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$=\left(\frac{-1-i\sqrt{3}}{2}\right)\left(\frac{-1+i\sqrt{3}}{2}\right)$$

$$=1$$

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End of the proof.

3. Solution: The elements in S are $S=\{1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}\}$

The elements in T are $T=\{-1, 1, i, -i\}$ } Why?

The graph is:

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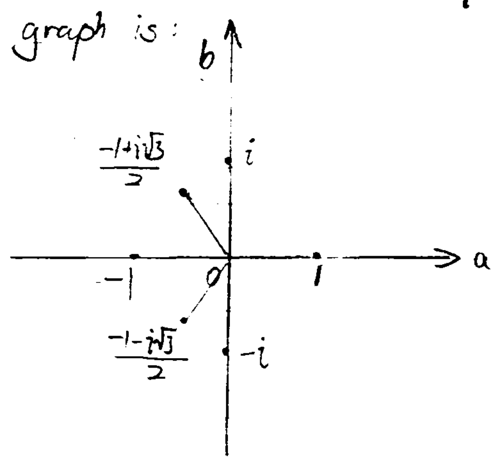
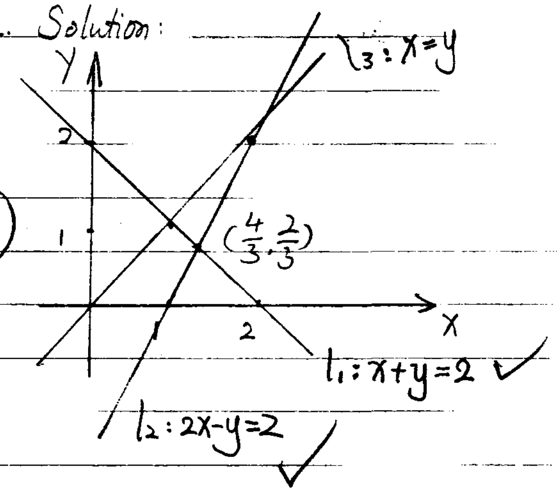


Exercise 2

1. Solution:



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$\begin{cases} y=2-x \\ y=2x-2 \end{cases}$

so $\begin{cases} x=\frac{4}{3} \\ y=\frac{2}{3} \end{cases}$

thus the intersection of l_1 and l_2 is point $(\frac{4}{3}, \frac{2}{3})$ ✓

2. Solution:

3

$\left.\begin{array}{l} y=x \\ y=2-x \end{array}\right\} \Rightarrow x=1, y=1$

$\left.\begin{array}{l} y=x \\ y=2x-2 \end{array}\right\} \Rightarrow x=2, y=2$

$\Rightarrow$ The intersections of $l_1 \cap l_3$ and $l_2 \cap l_3$ are $(1,1)$ and $(2,2)$ ✓

and there is the intersection of all three lines $l_1 \cap l_2 \cap l_3$ is $\emptyset$.

Why?

3. l_3 is linear subspace

Proof: As l_3 is the only line which goes through the origin $(0,0)$

if we name the line 3 as a vector space U.

Then $0 \in U$

We chose two points on the l_3 $u=(x_1, y_1)$ and $v=(x_2, y_2)$

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which satisfy $u, v \in U$

and $u+v = (x_1+x_2, y_1+y_2)$

Since the points on l_3 has the property that $x=y$

So $u+V = (x_1+x_2, y_1+y_2) = (x_1+x_2, x_1+x_2) \in U$

Using the point $u=(x_1, y_1)$ above, multiply it by $a \in \mathbb{R}$.

we will have $au = a(x_1+y_1) \in U$ Why? $au = (ax_1, ay_1) = (ax_1, ax_1) \in U$.

In a word, l_3 is a linear subspace.

End of the proof. ✓

Rui Qiu

Exercise 3:

Solution: No, V is not a vector space.

According to the operation mentioned above,

$$c \cdot (x, y) = c(x, 0)$$

Let's suppose $c = 1$, thus

$$1 \cdot (x, y) = (x, 0)$$

since $y = 0$ is not certain,

Therefore V does not ~~have~~ satisfy the property of a vector space.

which means, V is not a vecter space.

The end of the proof.

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You should mention that the axiom requires that $1 \cdot v = v \ \forall v \in V$, and then give an instance in which this is violated.

ie $1 \cdot (1,1) = (1,0) \neq (1,1)$.

MAT240 Dec. 6th

FOCUS ON THE ASSIGNMENTS! MAN!

Review

1. Vector spaces

Fields : $\mathbb{F}_p$ finite ~~finite~~ with ~~point~~ prime # ? $\mathbb{Q}, \mathbb{R}, \mathbb{C}$.

- $\mathbb{F}^n$ n-dim' space most important because ← $\boxed{\text{Any finite-dim' v.space over } \mathbb{F} \text{ is isomorphic to } \mathbb{F}^n}$
- $\mathbb{F}^X = \{\text{maps } X \to \mathbb{F}\}$ ↖ a set

$\Updownarrow$

$\boxed{V \text{ has a basis}}$ T linear map $V \to \mathbb{F}^n$ isomorphism $\longleftrightarrow$ invertible $\longleftrightarrow$ injective & surjective $\longleftrightarrow$ null $T = \{0\}$, range $T = \mathbb{F}^n$

- sequences $(x_1, x_2, \cdots)$ $x_i \in \mathbb{F}$
- V, W v spaces $L(V,W)$ v space V^* v space

To understand $\boxed{\text{basis}}$ need
- linear (in)dependence
- span

} Gaussian elim. algor, row reduction

Def. A list $(v_1, \cdots, v_n)$ is lin independent when the only lin. relation $a_1v_1 + \cdots + a_nv_n = 0$ is when $a_1 = \cdots = a_n = 0$

Basis $(u_1, \cdots, u_n)$

lin. indept. : $(v_1, \cdots, v_k)$ lin. indept $\Leftrightarrow v_1 \wedge \cdots \wedge v_k \neq 0$ (wedge product)

Ex : $(1, x, y)$, $(1, t, 0)$, $(0, 1, 1)$ are they lin. indep?

$\Downarrow$ ROW REDUCTION

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 1 \\ 0 & 0 & 1-y-t+x \end{pmatrix}$$

~~1-y-t+x = 1~~

lin. indep. $\Updownarrow$ $1-y-t+x \neq 0$

lin dept. $\Leftrightarrow 1-y-t+x = 0$

Given $u_1, \cdots, u_k$, find a basis for span of the list.

Linear equations

$$\underset{\mathbb{F}^{k\times n} \text{ (matrix)}}{A}\underset{\mathbb{F}^n}{x} = \underset{\mathbb{F}^k}{b}$$

$$\begin{bmatrix} k \text{ given } n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_k \end{bmatrix} \text{(given)}$$

Solve that? Exist solution? $\Updownarrow$ is be in range (A)?

Matrix convention

matrix of operator $\left[A\ \overset{n}{\text{———}}\right]$
from $\mathbb{F}^{n}$ to $\mathbb{F}^{k}$ has columns $\begin{bmatrix} Ae_1 & Ae_2 & Ae_3 & \cdots & Ae_n \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{bmatrix}$

~~Solve~~

$$\left[\ A\ \right]\cdot\begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix} = 1^{st}\text{ column}$$

suppose it is, $\exists\, x_0$ with $Ax_0 = b$

Are there any other solutions?

Yes! if A is not injective $\Leftrightarrow$ null $A \neq \{0\}$

$x \in \text{null}(A)$ $\quad A(x_0 + x) = Ax_0 = b$

Therefore, It has solutions iff b is given in the range of A.

The number of solutions depends on ~~the~~ whether null A is $\{0\}$.

How to ~~implend~~ implement?

$\cancel{Ax=b}$ $Ax = b$

$E_m \cdots E_2 E_1 A x = E_m \cdots E_2 b'$

$$\begin{pmatrix} 1 & * & \cdots & 0 & & & \\ & & & 1 & * & \cdots & 0 \\ & & & & & & 1 & \cdots \\ 0 & & & & & & & 0 \end{pmatrix}\begin{pmatrix}x_1\\ \vdots\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}\tilde{b}_1\\ \vdots\\ \vdots\\ \tilde{b}_k\end{pmatrix}$$

For instance:

$$\begin{pmatrix} 1 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}1\\2\\3\\4\end{pmatrix}$$

no solution by looking at the last row

$0\cdot x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 = 4$

lin. indep. columns. span the range.

$\Downarrow$

range is 3-dim!

$$\begin{pmatrix}1\\2\\3\\0\end{pmatrix}$$

$x_1 + 2x_3 + 2x_4 + 3x_5 = 1$

$x_2 + 4x_5 = 2$

$x_4 + 5x_5 = 3$

take $\boxed{\begin{matrix} x_5 = t \\ x_3 = s \end{matrix}}$, $x_4 = 3-5t$, $x_2 = 2-4t$, $x_1 = 1-2s-6+10t-3t$

$= -5+5t$

dim range = 3

dim null = 2

rank nullity theorem!

just take $t = s = 0$.

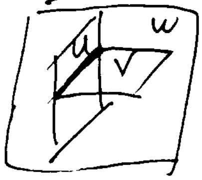
then $x_1 = -5, x_2 = 2, x_3 = 0, x_4 = 3, x_5 = 0$

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}-5\\2\\0\\3\\0\end{pmatrix} + s\begin{pmatrix}-2\\0\\1\\0\\0\end{pmatrix} + t\begin{pmatrix}7\\-4\\0\\-5\\1\end{pmatrix}$$

two-dim'l null space

particular solution.

~~Interse~~ Intersections

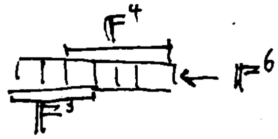


$\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

$\underbrace{U, V}_{\text{subspaces}}$

$\dim(U+V) = \dim U + \dim V$ only ~~if~~ when $\cancel{U\oplus V}$ $U+V$ is direct

e.x ①



$\dim U \cap V = \begin{cases} \cancel{0} & \text{can't be zero} \\ 1 \\ 2 \\ 3 \end{cases}$

e.x ② U, V planes in $\mathbb{R}^4$

$\dim U \cap V \begin{cases} 0 \\ 1 \\ 2 \end{cases}$

$\dim U \cap V = \dim U + \dim V - \dim(U+V)$

$\geq \dim U + \dim V - \dim W$

$\dim(U \cap V) \leq \min(\dim U, \dim V)$

Eigenvalue of operators

Def. $\lambda \leftarrow \mathbb{F}$ eigenvalue when $\text{null}(T-\lambda)$ not $\{0\}$

Def. eigenvector $v \in$ ~~eigenvalue~~ eigenspace.

~~No eigenvalues~~ Warning: T needn't have eigenvalue, ~~but~~ over $\mathbb{C}$ has at least 1

Why? · polynomial $p(z)$ s.t. $p(T)=0$

· factor it $p(z) = (z-\lambda_1)\cdots(z-\lambda_m)$ (only possible over $\mathbb{C}$)

e.g for $\begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{pmatrix}$ $p(z) = z$ for $\begin{pmatrix} 1 & 1 & \\ & 1 & \\ & & 0 \end{pmatrix}$ $p(z) = z(z-1)^2$

$(T-\lambda_1)\cdots(T-\lambda_m) = 0$

If $T-\lambda_i$ had $\{0\}$ nullspace $\forall i$,

then $(T-\lambda_i)$ invertible $\forall i$ $\Rightarrow$ one must have nonzero null space ∎.

Def. a generalized eigenspace for eigenvalue λ.
is $\text{null}(T-\lambda)^{\dim V}$

basis of eigenvectors

T has 1 eigenvalue, is it true that T not diagonalisable? ($T \in \mathbb{R}^3$)

No. it could be diagonalisable.

What if T has 2 distinct eigenvalues? $\lambda_1, \lambda_2 \in \mathbb{R}$, is T necessarily diagonalisable? No.

could be like this

$$\left(\begin{array}{c|c} \lambda_1\ 1 & \\ \ \ \lambda_1 & \\ \hline & \lambda_2 \end{array}\right) \quad \& \quad \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_2 \end{pmatrix}$$

① find eigenvalue

diagonalize : $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \begin{pmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

find λ s.t. null $\begin{pmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{pmatrix}$ non zero

$$\begin{pmatrix} 1-\lambda & 2 \\ 3 & 5-\lambda \end{pmatrix} \to \begin{pmatrix} 1-\lambda & 2 \\ 1 & \frac{5-\lambda}{3} \end{pmatrix} \to \begin{pmatrix} 1 & \frac{5-\lambda}{3} \\ 0 & 2-(1-\lambda)\frac{5-\lambda}{3} \end{pmatrix}$$

nonzero nullspace $\iff (3 \times 2) - (1-\lambda)(5-\lambda) = 0 \iff -\lambda^2 + 6\lambda + 1 = 0$

$\lambda^2 - 6\lambda - 1 = 0$ two distinct real solution

diagonalysable over $\mathbb{R}$

~~$(\lambda-3)(\lambda+1)$~~

② find eigenvectors $\beta = (u_1, u_2)$

solve system with 2 values of λ.

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} = [T]_e^e = PDP^{-1} = [I]_\beta^e [T]_\beta^\beta [I]_e^\beta$$

standard basis

diagonal

colums are β in term of e.

invert

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 0 & -1 & -3 & 1 \\ 1 & 0 & -5 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 0 & 1 & 3 & -1 \\ 1 & 0 & -5 & 2 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right) \to \text{got it.}$$

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Exerzitien II

Exercise 1

1. Proof: Let U_3 be the intersection of $U_1 \cap U_2$

Assume $u, v \in U_3$,

thus $u, v \in U_1$, $u, v \in U_2$

since U_1, U_2 are two subspaces

so $u+v \in U_1$, $u+v \in U_2$ ✓

$\Rightarrow$ $u+v \in U_3 \Rightarrow U_3$ is indeed closed under addition.

Assume $x \in U_3$, $a \in \mathbb{F}$.

As $U_3 \subset U_1$, U_1 is a subspace of V which is closed under scalar multiplication

Therefore $ax \in U_1$ ✓

Similarly $ax \in U_2$ ✓

Thus $ax \in U_3 \Rightarrow U_3$ is closed under multiplication ✓

3

As U_1, U_2 are two subspaces of V

so $0 \in U_1$, $0 \in U_2$

$\Rightarrow 0 \in U_3 \Rightarrow U_3$ has additive identity ✓

✓

In a word, the intersection $U_1 \cap U_2$ is also a subspace of V. ■

2. Proof: Assume $u, v \in \bigcap_{n \in \mathbb{N}} U_n$

As $U_1 \supset \bigcap_{n \in \mathbb{N}} U_n$

so $u, v \in U_1$ ✓ ✓

since U_1 is a subspace of V

so $u+v \in U_1$ ✓

Similarly we can prove $u+v \in U_2, U_3 \cdots U_n$ ✓

Thus $u+v \in \bigcap_{n \in \mathbb{N}} U_n$ ✓

Assume $x \in \bigcap_{n \in \mathbb{N}} U_n$, $a \in \mathbb{F}$

As $U_1 \supset \bigcap_{n \in \mathbb{N}} U_n$, U_1 is a subspace of V

so $ax \in U_1$

Similarly we can prove $ax \in U_2, U_3 \cdots U_n$

Thus $ax \in \bigcap_{n \in \mathbb{N}} U_n$ ✓

As $U_1, U_2, U_3 \cdots U_n$ are subspaces of V
so $0 \in U_1, U_2, \cdots U_n$
hence $0 \in \bigcap_{n \in \mathbb{N}} U_n$

4

Therefore, $\bigcap_{n \in \mathbb{N}} U_n$ is a subspace of V.

3. Proof: If neither U_1 nor U_2 is contained in the other, then
$\exists u_1 \in U_1, u_2 \in U_2$ such that
$u_1 \notin U_2, u_2 \notin U_1$
Since U_1 is closed under addition
$u_1 \in U_1$ and $u_2 = (u_1 + u_2) - u_1$
so $u_1 + u_2 \notin U_1$
Similarly $u_1 + u_2 \notin U_2$
Hence $U_1 \cup U_2$ is not closed under addition
and $U_1 \cup U_2$ is not a subspace of V.
But if $U_1 \subseteq U_2$ or $U_2 \supseteq U_1$, then we can
get $U_1 \cup U_2 = U_2$ or $U_1 \cup U_2 = U_1$ is a subspace of V. Yes

3

You really should provide a specific counter-example.

Exercise 2. [redacted]
1. Proof: Let f and g be two functions of V_e and $c \in \mathbb{R}$.
Then $(f + cg)(-x) = f(-x) + c \cdot g(-x) = f(x) + c \cdot g(x) = (f + cg)(x)$
Similarly, if f and g are two functions of V_o and $c \in \mathbb{R}$
Then $(f + cg)(-x) = f(-x) + c \cdot g(-x) = -f(x) - c \cdot g(x) = -(f(x) + c \cdot g(x))$
$= -(f + cg)(x)$
Thus both V_o and V_e are closed under addition and sclar multiplication.
So V_o and V_e are subspaces
Does each of V_e, V_o contain 0?

2

2. Proof: Suppose there exist two functions g and h, such that
$g \in V_e$, $h \in V_o$
$\therefore g = \frac{1}{2}[f(x) + f(-x)]$
$h = \frac{1}{2}[f(x) - f(-x)]$
$\therefore f(x) = g + h \Rightarrow V_e + V_o \in V$
vice versa
$\therefore V = V_e + V_o$

3

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3. Proof: Suppose that $g=h$ such that

$$\frac{1}{2}[f(x)+f(-x)] = \frac{1}{2}[f(x)-f(-x)]$$

$$f(x)+f(-x) = f(x)-f(-x)$$

$$2f(-x)=0$$

$$f(-x)=0$$

?

For an odd function when $f(-x)=f(x)=0$

such that $-x=x=0$

So $V_e \cap V_o = \{0\}$

1

Then suppose that

$0 = v_e + v_o$ for $v_e \in V_e$, $v_o \in V_o$

so $v_e = -v_o \in V_o$

Thus $v_e \in V_e \cap V_o$

hence $v_e = 0$

so $v_o = v_e = 0$

Therefore $V = V_e \oplus V_o$ ∎

It suffices to just prove $V_e \cap V_o = \{0\}$.

4. Solution: Suppose $f_e(x) + f_o(x) = e^x$ ……①

$f_e(-x) - f_o(-x) = e^x$

$\therefore -f_o(x) + f_e(x) = e^{-x}$ ……②

① − ② $= 2f_o(x) = e^x - e^{-x}$

$\therefore f_o(x) = \frac{e^x - e^{-x}}{2}$ ✓

Then $f_e(x) = e^x - \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} = 0$

No, $f_e(x) = \frac{1}{2}(e^x + e^{-x})$

1

+	0	1
0	0	1
1	1	0

Exercise 3

1\. Proof: As the set of triples (x, y, z) only has 2 possible numbers, 1 and 0. ? elements?

So the possible sets are (0,0,0), (0,0,1), (0,1,0), (1,0,0), (0,1,1), (1,0,1), (1,1,0), (1,1,1).

(1)

If the sets can form a subspace of V, it should contain the vector 0. ✓ element

thus we should have at least one set (0,0,0)

And the if the subspace has other elements their sum should also be in V (closed under addition) Why?

so we could have subspaces with (0,0,0) and one other element

In this method we can have 4 elements subspaces and of course the largest subspace, V itself (with 8 elements).

PS: Do we really have to Lagrange's Theorem or other unknown method to solve this? No.

2\. Solution:

(2)

One: {(0,0,0)} ✓

Two: {(0,0,0), (0,0,1)}, {(0,0,0), (0,1,0)}, {(0,0,0), (1,0,0)}, {(0,0,0), (1,1,0)}, {(0,0,0), (0,1,1)}, {(0,0,0), (1,1,1)} missing one.

Four: {(0,0,0), (0,0,1), (0,1,0), (0,1,1)} {(0,0,0), (0,0,1), (1,0,0), (1,0,1)} {(0,0,0), (1,0,0), (0,1,0), (1,1,0)} {(0,0,0), (1,0,0), (0,1,1), (1,1,1)} {(0,0,0), (0,1,0), (1,0,1), (1,1,1)}, {(0,0,0), (0,0,1), (1,1,0), (1,1,1)} missing one.

Eight: {(0,0,0), (0,1,0), (0,0,1), (1,0,0), (1,1,0), (0,1,1), (1,0,1), (1,1,1)}

? 3. Solution: $x = y = z$ Why?

lse Span?

(2)

[redacted]

[redacted]

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Mat240 Exerzition III

19/40

Exercise 1

4

1. Proof: $\because (v_1, v_2, v_3, v_4)$ spans V

$\therefore V = \mathrm{span}(v_1, v_2, v_3, v_4)$

$= \{a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 : a_1, a_2, a_3, a_4 \in \mathbb{F}\}$

$\because a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = a_1(v_1-v_2) + (a_1+a_2)(v_2-v_3) + (a_1+a_2+a_3)(v_3-v_4) + (a_1+a_2+a_3+a_4)v_4$

$\therefore \mathrm{span}(v_1-v_2, v_2-v_3, v_3-v_4, v_4) = V$

$\therefore (v_1-v_2, v_2-v_3, v_3-v_4, v_4)$ spans V. ■

2. Proof: $\because (v_1, v_2, v_3, v_4)$ is linearly independent.

$\therefore$ only $a_1 = a_2 = a_3 = a_4 = 0$ makes $V = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$

$\therefore V = a_1(v_1-v_2) + (a_1+a_2)(v_2-v_3) + (a_1+a_2+a_3)(v_3-v_4) + (a_1+a_2+a_3+a_4)v_4 = 0$

2 This does not seem to follow

Suppose that $\exists$ four coefficients b_1, b_2, b_3, b_4, s.t. $V = b_1(v_1-v_2) + b_2(v_2-v_3) + b_3(v_3-v_4) + b_4v_4$

Thus $a_1 - b_1 = (a_1+a_2) - b_2 = (a_1+a_2+a_3) - b_3 = (a_1+a_2+a_3+a_4) - b_4 = 0$??

So $a_1 = b_1$, $a_1+a_2 = b_2$, $a_1+a_2+a_3 = b_3$, $a_1+a_2+a_3+a_4 = b_4$

contradiction

Therefore. $\mathrm{span}(v_1-v_2, v_2-v_3, v_3-v_4, v_4)$ has only one representation as a linear combination of $(v_1-v_2, v_2-v_3, v_3-v_4, v_4)$

Hence, $(v_1-v_2, v_2-v_3, v_3-v_4, v_4)$ is linearly independent. ■

3. Proof: As the list of vectors $(v_1-v_2, v_2-v_3, v_3-v_4, v_4-v_1)$ satisfy

2

$1\cdot(v_1-v_2) + 1\cdot(v_2-v_3) + 1\cdot(v_3-v_4) + 1\cdot(v_4-v_1) = 0$

such that $a_1 = a_2 = a_3 = a_4 = 1 \neq 0$

Therefore $(v_1-v_2, v_2-v_3, v_3-v_4, v_4-v_1)$ is linearly dependent. ■

Exercise 2

1. Solution: Setting equations that ~~Suppose it is linearly independent.~~

2

$a(-1,1,1,1) + b(1,-1,1,1) + c(1,1,-1,1) + d(1,1,1,-1) = 0$, s.t.

$$\left.\begin{array}{l} -a+b+c+d=0 \\ a-b+c+d=0 \\ a+b-c+d=0 \\ a+b+c-d=0 \end{array}\right\} \Rightarrow 2a+2b+2c+2d=0 \Rightarrow a+b+c+d=0 \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \\ d=0 \end{cases}$$

Explain

Thus the sequence is linearly independent.

2. Solution: Setting equations that:

$a(1,0)+b(i,0)+c(0,1)+d(0,i)=0$

$\left.\begin{array}{l} a+bi=0 \\ c+di=0 \end{array}\right\} \Rightarrow a=b=c=d=0$

Therefore the sequence is linearly independent.

0

X

3. Solution: Setting equations that

$ax^2+b(x^2+1)+c(x^2+2)=0$

$(a+b+c)x^2+b+2c=0$

$\therefore\ b=-2c\ ,\ a+b+c=a-2c+c=a-c=0$

$\therefore\ a=c$

As this equation above could have more than one set of solutions.

thus the sequence is not linearly independent.

2

4. Solution: Setting equations that

$ax^2+b(x+1)^2+c(x+2)^2=0$

$(a+b+c)x^2+(2b+4c)x+(b+4c)=0$

As $2b+4c=0$ and $b+4c=0$ could not be satisfied at the same time

So the sequence is not linearly independent.

0

No, it just means $b=0$.

$+$	0	1
0	0	1
1	1	0

$\times$	0	1
0	0	0
1	0	1

5. Solution Setting equations that

$x(1,1,0)+y(1,0,1)+z(0,1,1)=0$

$(x+y)+(x+z)+(y+z)=0$

As the solutions could be $x=y=z=1$ or $x=y=z=0$

So the sequence is not linearly independent

2

Exercise 3

Solution: In $(\mathbb{F}_2)^5$, we want to make a list of vectors linearly independent, so

Relevance $a_1v_1+a_2v_2+av_3=0$, $v_1=(x_1,x_2,x_3,x_4,x_5)$, $v_2=(y_1,y_2,y_3,y_4,y_5)$, $v_3=(z_1,z_2,z_3,z_4,z_5)$, $x,y,z\in\{0,1\}$

thus for each first elements in v_1,v_2,v_3 the only occasions for $x+y+z\neq 0$ are

$1+1+1=1$, $1+0+0=1$, $0+1+0=1$, $0+0+1=1$

And there are $2^3=8$ possible combinations available.

So the possibility is $P_1=\left(\frac{4}{8}\right)^5=\frac{1}{2^5}=\frac{1}{32}$ (other elements will be the same as first element)

X

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Similarly, in $(\mathbb{F}_3)^5$ and $(\mathbb{F}_5)^5$:

there are 3^5 and 5^5 possible combinations.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

3

① for $(\mathbb{F}_3)^5$ the possible combinations that make $x_1+y_1+z_1\neq 0$ are:

$0+0+1=1$, $0+0+2=2$, $0+1+1=2$, $0+2+2=1$, $1+1+2=1$, $2+2+1=2$

$0+1+0=1$, $0+2+0=2$, $1+0+1=2$, $2+0+2=1$, $1+2+1=1$, $2+1+2=2$

$1+0+0=1$, $2+0+0=2$, $1+1+0=2$, $2+2+0=1$, $2+1+1=1$, $1+2+2=2$

So $P_2=\left(\frac{3\times 6}{3^3}\right)^5=\left(\frac{2}{3}\right)^5$

② for $(\mathbb{F}_5)^5$ the possible combinations that make $x_1+y_1+z_1\neq 0$ are:

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$0+0+1=1$ 010   020   030   040

$0+0+2=2$ 011   021   031   042

$0+0+3=3$ 012   022   033   043

$0+0+4=4$ 013   024   034   044

$\Big\}\ 5\times 4=20$

$20\times 5=100$

So $P_3=\left(\frac{100}{5^3}\right)^5=\left(\frac{4}{5}\right)^5$

Thus the possibility is increasing

Exercise 4

Solution: $f(x)=a_4x^4+a_3x^3+a_2x^2+a_1x^1+a_0x^0$

If $\exists\lambda\in\mathbb{R}$, then $\lambda f(x)=\lambda a_4x^4+\lambda a_3x^3+\lambda a_2x^2+\lambda a_1x+\lambda a_0\in V$ } Why?

So we know that it satisfies scalar multiplication.

If $f'(x)=a_4'x^4+a_3'x^3+a_2'x^2+a_1'x+a_0'$

then $f(x)+f'(x)=(a_4+a_4')x^4+(a_3+a_3')x^3+(a_2+a_2')x^2+(a_1+a_1')x+(a_0+a_0')\in V$ } Why?

So we know that it satisfies vector addition

3

And we also know that $f(0)=0$

Thus it has additive identity.

Thus f is a subspace of V.

So $f=\operatorname{span}(x^0,x^1,x^2,x^3,x^4)$ X

Notice that $x^4\notin$ the subspace

As $f(1)=f(0)=0$

So $a_0x^0+a_1x^1+a_2x^2+a_3x^3+a_4x^4=0$

Suppose there exists another set of constant numbers, such that

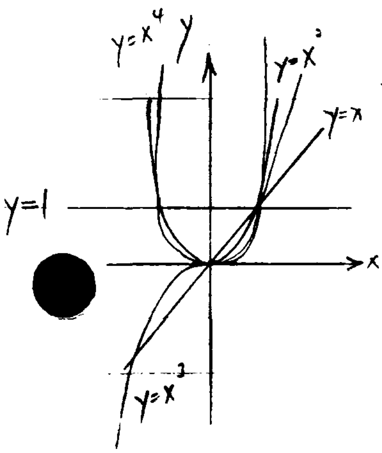
$b_0x^0+b_1x^1+b_2x^2+b_3x^3+b_4x^4=0$

So $(a_0-b_0)x^0+(a_1-b_1)x^1+(a_2-b_2)x^2+(a_3-b_3)x^3+(a_4-b_4)x^4=0$

Thus $a_0=b_0$, $a_1=b_1$, $a_2=b_2$, $a_3=b_3$, $a_4=b_4$

There is only one set of coefficients for f. (linearly independent)

Therefore, $(1,x,x^2,x^3,x^4)$ is a basis for f.



MAT240 TUT October 6th

Exerzitien II.

Exercise 3.

$2^5-1=31$

$(V_1, V_2, V_3) \in (\mathbb{F}_2)^5 \times (\mathbb{F}_2)^5 \times (\mathbb{F}_2)^5$ Total: $2^5 \times 2^5 \times 2^5$

$\Leftrightarrow V_1 \neq 0,\ V_2 \notin span\{v\}, \ldots\ V_3 \notin span\{v_1, v_2\}$

$\Rightarrow$ is immediate

$V_1 \neq 0$ & $V_2 \in span\{v_1\} \Rightarrow V_2 = aV_1$ for some a

$aV_1 - V_2 + 0 \cdot V_3 = 0 \Rightarrow V_2 \notin Span\{V_1\}$

if $V_3 \in span\{v_1, v_2\} \Rightarrow V_3 = a_1 v_1 + a_2 v_2 \Rightarrow a_1 v_1 + a_2 v_2 - V_3 = 0 \Rightarrow V_3 \notin span\{v_1, v_2\}$

Suppose $V_1 \neq 0$, & $V_2 \notin span\{V_1\}$, $V_3 \notin span\{V_1 V_2\}$

$a_1 v_1 + a_2 V_2 + a_3 V_3 = 0$

if $a_3 \neq 0$ ~~If~~ $a_3 V_3 = -a_1 v_1 - a_2 V_2$

$\Rightarrow V_3 = -\frac{a_1}{a_3} V_1 - \frac{a_2}{a_3} V_2$ (contradiction)

$\Rightarrow a_3 = 0$, $a_1 V_1 + a_2 V_2 = 0$. if $a_2 \neq 0$, $V_2 = \frac{-a_1}{a_2} V_1 \Rightarrow V_2 \in span\{v_1\}$ (contradiction)

$a_3 = a_2 = 0 \Rightarrow a_1 V_1 = 0 \Rightarrow a_1 = 0$

$(2^5-1)(2^5-2)(2^5-4) = (31 \times 30 \times 28)/(2^5)^3 =$

$$\frac{(p^5-1)(p^5-p)(p^5-p^2)}{(p^5)^3} \to 1$$

Exercise 4

$U = \{f \in P_4(\mathbb{R}) : f(0) = f(1) = 0\}$

$0 \in U$ ($h(x) = 0$, $h(x) \in P_4(\mathbb{R})$, $h(0) = 0 = h(1)$)

Suppose $f, g \in U \Rightarrow (f+g)(0) = f(0) + g(0) = 0 + 0 = 0$

$(f+g)(1) = 0$

$\Uparrow af \in U$

$a \in \mathbb{R}, f \in U$, $(af)(1) = af(1) = a \cdot 0 = 0$, $(af)(0) = af(0) = a \cdot 0 = 0$

$f(a) = 0$, iff $(x-a)$ is a divisor of f.

$f(x) = (x-a)p(x)$

$g \in P_2(\mathbb{R})$

$f(-1) = 0$

$f \in U$, $f(0) = 0 = f(1) \Rightarrow x$ & $x-1$ both ~~div~~ divide $f(x)$

$f(x) = x(x-1)g(x)$

[redacted]

$$f(x) = x(x-1)(ax^2+bx+c)$$
$$= cx(x-1) + bx^2(x-1) + ax^3(x-1)$$
$$\{x(x-1),\ x^2(x-1),\ x^3(x-1)\}$$
$$0 = cx(x-1) + bx^2(x-1) + ax^3(x-1) = cx^2 - cx + bx^3 - bx^2 + ax^4 - ax^3$$
$$\cancel{= ax^3 - ax + bx^3 - bx^2}$$
$$= ax^4 + (b-a)x^3 + (c-b)x^2 - cx = 0$$
$$\begin{cases} a=0 \\ b=c=0 \end{cases}$$

[redacted]

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MAT240

Exerzition IV

31/40

Exercise 1

Solution: Suppose $V=(v_1, v_2, v_3, v_4)$.

then $a_1v_1+a_2v_2+a_3v_3+a_4v_4=0$, that is

Why?

$a_1(2,3,0,0)+a_2(0,0,1,-1)+a_3(1,0,0,4)+a_4(0,0,0,2)=0$

so $a_1=a_2=a_3=a_4=0$ is the only choice to satisfy this equation

thus V is linearly independent.

As $\mathbb{R}^4$ is finite dimensional, using the proposition that every linearly independent list of vectors in $\mathbb{R}^4$ with length $\dim \mathbb{R}^4$ (which is 4) is a basis of $\mathbb{R}^4$

So (v_1, v_2, v_3, v_4) is a basis for $\mathbb{R}^4$.

Then the coordinates of standard basis vectors in (v_1, v_2, v_3, v_4) satisfy that. $(x_i, y_i, z_i, m_i \in \mathbb{R}, i \in (1,2,3,4))$

8

$$(1,0,0,0)=x_1(2,3,0,0)+x_2(0,0,1,-1)+x_3(1,0,0,4)+x_4(0,0,0,2)$$
$$(0,1,0,0)=y_1(2,3,0,0)+y_2(0,0,1,-1)+y_3(1,0,0,4)+y_4(0,0,0,2)$$
$$(0,0,1,0)=z_1(2,3,0,0)+z_2(0,0,1,-1)+z_3(1,0,0,4)+z_4(0,0,0,2)$$
$$(0,0,0,1)=m_1(2,3,0,0)+m_2(0,0,1,-1)+m_3(1,0,0,4)+m_4(0,0,0,2)$$

hence
$$\begin{cases}2x_1+x_3=1\\3x_1=0\\x_2=0\\-x_2+4x_3+2x_4=0\end{cases}\quad\begin{cases}2y_1+y_3=0\\3y_1=1\\y_2=0\\-y_2+4y_3+2y_4=0\end{cases}\quad\begin{cases}2z_1+z_3=0\\3z_1=0\\z_2=1\\-z_2+4z_3+2z_4=0\end{cases}\quad\begin{cases}2m_1+m_3=0\\3m_1=0\\m_2=0\\-m_2+4m_3+2m_4=1\end{cases}$$

Solving these equations we can get

$x_1=0, x_2=0, x_3=1, x_4=-2$

$y_1=\frac{1}{3}, y_2=0, y_3=-\frac{2}{3}, y_4=\frac{4}{3}$

$z_1=0, z_2=1, z_3=0, z_4=\frac{1}{2}$

$m_1=0, m_2=0, m_3=0, m_4=\frac{1}{2}$

Therefore, the coordinates of each of standard basis vectors are: $0,0,1,-2$ for $(1,0,0,0)$; $\frac{1}{3},0,-\frac{2}{3},\frac{4}{3}$ for $(0,1,0,0)$; $0,1,0,\frac{1}{2}$ for $(0,0,1,0)$; and $0,0,0,\frac{1}{2}$ for $(0,0,0,1)$

Exercise 2

1. Proof: Suppose vectors $v_1, v_2 \in V$ such that $f(v_1)=f(v_2)=0$

So $v_1, v_2 \in H_f$

4

Since $f(v_1+0)=f(v_1)+f(0)$

So $f(0)=0 \Rightarrow 0 \in H_f$ (additive identity)

Besides $v_1, v_2 \in V$, $f(v_1+v_2)=f(v_1)+f(v_2)=0+0=0 \Rightarrow v_1+v_2 \in H_f$ (closed under addition)

$\lambda \in \mathbb{F}$, $f(\lambda v_1)=\lambda f(v_1)=\lambda \cdot 0=0 \Rightarrow \lambda v_1 \in H_f$ = (closed under scalar multiplication)

Therefore, H_f is a linear subspace of V. ■

2. Proof: With the problem 1 above, we can take $(x_1, x_2, \cdots x_n)$ as the vector v, $a_1x_1+a_2x_2+\cdots+a_nx_n$ as the function f about $(x_1, x_2, \cdots x_n)$.

3

Suppose vectors $v_1, v_2 \in \mathbb{F}^n$ such that $f(v_1)=f(v_2)=a_1x_1+\cdots+a_nx_n=a_1x_1'+\cdots+a_nx_n'=0$

$(v_1, v_2 \in H(a_1, \cdots a_n))$ $f(v_1+v_2)$

Since $f(v_1)+f(v_2)= a_1x_1+\cdots+a_nx_n+a_1x_1'+\cdots+a_nx_n'=a_1(x_1+x_1')+\cdots a_n(x_n+x_n')=0+0=0$

So $(x_1+x_1', x_2+x_2', \cdots, x_n+x_n') \in H(a_1, \cdots a_n)$ (closed under addition)

Since $\lambda \in \mathbb{F}$, $f(\lambda v_1)= a_1\lambda x_1+\cdots+a_n\lambda x_n=\lambda(a_1x_1+\cdots+a_nx_n)=\lambda f(v_1)$

So $(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) \in H(a_1, \cdots a_n)$ (closed under scalar multiplication)

Besides for any $(x_1, x_2 \cdots x_n)=0$, $a_1x_1+\cdots+a_nx_n=0$ is true

Hence $0 \in H(a_1, \cdots a_n)$ (additive identity)

Therefore $H(a_1, \cdots a_n)$ is a linear subspace of $\mathbb{F}^n$ ■

You have basically shown f to be linear. Therefore, you may invoke the previous exercise.

3. Solution:

As $(a_1, a_2, a_3)=1,2,3$, we can get

$H(1,2,3)=\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1+2x_2+3x_3=0\}$

Thus the basis could be $((1,-\frac{1}{2},0),(1,0,-\frac{1}{3}))$

as the only choice of $b_1=b_2=0$ that makes

3

$b_1(1,-\frac{1}{2},0)+b_2(1,0,-\frac{1}{3})=0$ $\Rightarrow$ The basis is linearly independent

As x_3 in (x_1, x_2, x_3) can be written as $x_3=\frac{-x_1-2x_2}{3}$

So $m(1,-\frac{1}{2},0)+n(1,0,-\frac{1}{3})=(m+n, -\frac{1}{2}m, -\frac{1}{3}n)$

it also satisfies $-\frac{1}{3}n=\frac{-m-n+m}{3}$,

Therefore $\mathrm{Span}((1,-\frac{1}{2},0),(1,0,-\frac{1}{3}))=\mathbb{R}^3$ $H(1,2,3)$

Hence, $((1,-\frac{1}{2},0),(1,0,-\frac{1}{3}))$ is a basis of $H(1,2,3)$, and $\dim H(1,2,3)=2$

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Exercise 3

1. Proof. Let $f, g \in V^*$ be arbitrary elements

Need to show $f+g \in V^*$

1

(i) As the conditions in Exercise 2 says, for $v_1, v_2 \in V$

$f(v_1+v_2) = f(v_1)+f(v_2)$, $g(v_1+v_2) = g(v_1)+g(v_2)$

Since $f(v_1+v_2)+g(v_1+v_2) = f+g\,(v_1+v_2)$

So $f+g\,(v_1+v_2) = f(v_1)+g(v_1)+f(v_2)+g(v_2)$

$= f+g\,(v_1)+f+g\,(v_2)$ (closed under addition)

~~what~~ ~~about~~ ($\cancel{f+g \in V^*}$)

Actually you have just shown that f+g is linear. You must also prove that λf is linear.

(ii) Similarly, in V we have $f(\lambda v_1) = \lambda f(v_1)$, $g(\lambda v_1) = \lambda g(v_1)$ for $\lambda \in \mathbb{F}$

So $f+g\,(\lambda \cdot v) = f(\lambda v)+g(\lambda v) = \lambda f(v)+\lambda g(v) = \lambda f+g\,(v)$

Closed under scalar multiplication

(iii) Finally as $f(v_1+0) = f(v_1)+f(0)$

So $f(0)=0$, $0 \in V^*$ (additive identity) ?

Therefore V^* is a linear subspace of the vector space $\mathbb{F}^V$ of all functions from V to $\mathbb{F}$.

■

2. Proof: If we want to prove f_i is a linear function, we have to show

$f_i(v_1+v_2) = f_i(v_1)+f_i(v_2)$ and $f_i(\lambda v_1) = \lambda f_i(v_1)$ for all $\lambda \in \mathbb{F}$, $v_1, v_2 \in V$. $v_1 = x_1e_1+\dots+x_ne_n$

$v_2 = x_1'e_1+\dots+x_n'e_n$

3

(i). $f_i(\lambda v_1) = f_i(\lambda\cdot(x_1e_1+\dots+x_ne_n)) = f_i(\lambda x_1e_1+\dots+\lambda x_ne_n)$

As $f_i(v) = x_i$

So $f_i(\lambda v_1) = f_i(\lambda x_1e_1+\dots+\lambda x_ne_n) = \lambda\cdot x_i$ ①

$\lambda f_i(v_1) = \lambda\cdot x_i$ ②

According to ① & ② we get $f_i(\lambda v_1) = \lambda f_i(v_1)$ ✓

(ii) $f_i(v_1+v_2) = f_i(x_1e_1+\dots x_ne_n+x_1'e_1+\dots+x_n'e_n) = x_i+x_i'$ ③

$f_i(v_1)+f_i(v_2) = f_i(x_1e_1+\dots+x_ne_n)+f_i(x_1'e_1+\dots+x_n'e_n) = x_i+x_i'$ ④

According to ③ & ④ we get $f_i(v_1+v_2) = f_i(v_1)+f_i(v_2)$

Therefore f_i is a linear function. ✓ ■

3. Proof: Suppose V is finite dimensional, $\dim V = n$, such that ∃ a basis $(e_1, \dots e_n)$

1

And suppose that $(f_1, f_2, \dots f_n)$ is a list of elements in V^*

As for $\forall\, v \in V$ can be written as $v = x_1e_1+\dots+x_ne_n$, $x_i \in \mathbb{F}$

and $(e_1, \dots, e_n)$ are linearly independent

Why? As for $\forall\, v^* \in V^*$ can be written as $v^* = a_1f_1+a_2f_2+\dots+a_nf_n$ in a unique way, $a_i \in \mathbb{F} \Rightarrow \operatorname{span}(f_1, \dots f_n) = V^*$

Exercise 4

8

Proof: Suppose X_m is the element of $\mathbb{F}^\infty$, such that

$X_m = (0, \cdots, 0, 1, 0, \cdots)$, for is a positive integer.

"1" is the mth coordinate of X_m, others are 0s.

So the list $(X_1, X_2, \cdots X_m)$ is linearly independent

? because the only choice of $a_1, \cdots, a_m \in \mathbb{F}$ that makes

$a_1 X_1 + \cdots + a_m X_m = 0$ is $a_1 = \cdots = a_m = 0$.

Hence the length of list $(X_1, \cdots X_m)$ is m

linearly independent? Since in this infinite sequences, there always exists a spanning list with a larger length, e.g. $(X_1, \cdots X_{m+1})$ whose length is $m+1 > m$

You have not shown that $\exists$ a spanning list shorter than your linearly independent list

This contradicts the theorem that in a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Therefore, F^∞ is infinite dimensional.

Followings are Exercise 3.3:

but your approach is promising

($\Downarrow$) Besides the only choice to show that $a_1 f_1 + a_2 f_2 + \cdots + a_n f_n = 0$ is that $a_1 = a_2 = \cdots = a_n = 0 \Rightarrow (f_1, \cdots f_n)$ is linearly independent

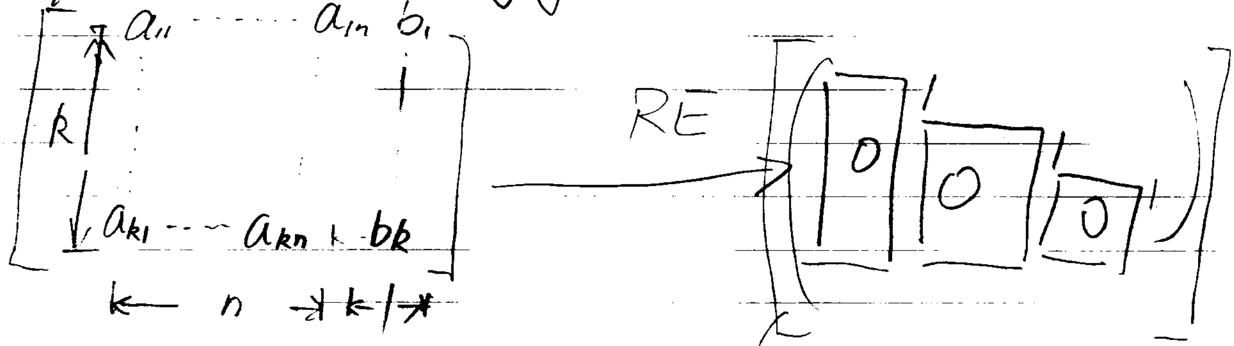
Hence $(f_1, \cdots f_n)$ both spans V^* and it is linearly independent

$\Rightarrow (f_1, \cdots f_n)$ is a basis of V^*

Thus V^* is finite dimensional and $\dim V^* = \dim V = n$

TUT

Key ideas You can do the 3 moves on the equations without changing the solution.



get a new system of k equations
Simpler to analyze
<u>and</u> same solution set as original system.

MAT240 TUT October 13th

a_1 $(3,2,0,1)$
a_2 $(-1,1,2,0)$
a_3 $(0,1,1,0)$
a_4 $(1,1,1,1)$

$$\begin{pmatrix} 3,2,0,1 \\ -1,1,2,0 \\ 0,1,1,0 \\ 1,1,1,1 \end{pmatrix} \to \begin{pmatrix} 1,1,1,1 \\ -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1, & 1, & 1, & 1 \\ 0 & 2 & 3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -3 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -3 & -3 \end{pmatrix} \quad \begin{matrix} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix}$$

Assignment 3

$(1,0),(i,0),(0,1),(0,i)$ lin. dep or lin. indep.

$a+bi=0$ $a=1$ $b=-i$
$c+di=0$ $c=1$, $d=-i$

$1(1,0)+i(i,0)+0(0,1)+i(0,i)=0$ lin. dep.

Assignment 4.4

Prop: If V is an n-dimensional vector space then any list of vectors on V of length $>n$ is linearly dependent

Proof: Assume that $\mathbb{F}^\infty$ is finite-dimensional and of dimension n.
Consider the list
$(1,0,0\cdots),(0,1,0\cdots),(0,0,1,\cdots),\cdots \underbrace{(0,0,0,\cdots 0 1,0\cdots)}_{n+1}$
This list is linearly independent. However proposition implies that it is linearly dependent
This is a contradiction. ∎

Exercise 3.3

Def: Let V be a vector space over a field $\mathbb{F}$
The dual of V, V^* is defined by
$V^* = \{ f : V \to \mathbb{F} \mid f \text{ is linear} \}$ (as a set)

Addition: $(f+g)(v) = f(v) + g(v)$
Scalar Mult: $(\lambda f)(v) = \lambda f(v)$

Let $(e_1, \cdots e_n)$ be a basis of V
$f_i(v) =$ the coefficient of e_i ~~in the~~ rep. of V ~~is~~ in term of $e_1 \cdots e_n$

Claim: $f_1, \cdots, f_n$ is a basis of V^*

Proof: Spanning set: $\varphi \in V^*$

$\Rightarrow \varphi = \varphi(e_1) f_1 + \cdots + \varphi(e_n) f_n$

why?

$V = a_1 e_1 + \cdots + a_n e_n \in V$
$= f_1(v) e_1 + \cdots + f_n(v) e_n$
$\varphi(v) = f_1(v)\varphi(e_1) + \cdots + f_n(v)\varphi(e_n)$ (induction)

$\Rightarrow (f_1, \cdots f_n)$ spans V^*

linearly independent:
$a_1 f_1 + \cdots + a_n f_n = 0$ (the zero-function)

$0 = (a_1 f_1 + \cdots + a_n f_n)(e_j)$, $j = 1, \cdots, n$
$\overset{\cap}{\mathbb{F}}$ $= a_1 f_1(e_j) + \cdots + a_n f_n(e_j)$ $\quad e_j = 0 \cdot e_1 + \cdots + 0 \cdot e_{j-1} + 1 \cdot e_j + \cdots + 0 \cdot e_n$
$= a_j \quad j = 1, \cdots, n$

$\Rightarrow$ Linearly independent

So ... proved.

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Exerzitien V

Exercise 1.

a) Writing in Row Echelon form:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 2 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & \frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \end{array}\right]$$

so the linear system has solutions.

$$\begin{cases} x_1 = x_2 - 2x_3 + 1 = 0 - 2 \times \frac{1}{2} + 1 = 0 \\ x_2 = 0 \\ x_3 = \frac{1}{2} \end{cases}$$

The solution set is $\{(0, 0, \frac{1}{2})\}$

b). Writing in Row Echelon form:

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & 2 & 1 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 1 & -2 & 1 & 2 & 1 \\ 1 & 7 & -5 & -1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & -6 & 4 & 2 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & 2 \\ 0 & 3 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

so the linear system has no solutions.

c). Writing in Row Echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 2 & -2 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 1 & -1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & -2 \\ 0 & 1 & -1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{3}{2} \end{array}\right]$$

so the linear system has no solutions.

d). Writing in Row Echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 6 & -1 \\ 2 & 1 & 1 & 8 \\ 3 & 1 & -1 & 15 \\ 1 & 3 & 10 & -5 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & -1 \\ 0 & 3 & 11 & -10 \\ 0 & 5 & 19 & -18 \\ 0 & -1 & -4 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & 3 & 11 & -10 \\ 0 & 5 & 19 & -18 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 6 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

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$$\to \left[\begin{array}{ccc|c} 1 & 2 & 6 & -1 \\ 0 & 1 & 4 & -4 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the linear system has solutions

$$\begin{cases} x_1 = -2x_2 - 6x_3 - 1 = -2\times 4 - 6\times(-2) - 1 = -8+12-1 = 3 \\ x_2 = -4x_3 - 4 = -4\times(-2) - 4 = 4 \\ x_3 = -2 \end{cases}$$

The solution set is $\{(3,4,-2)\}$.

Exercise 2

1. Proof: Suppose $v \in V$

$P^2 v = Pv \qquad Id_V v = v$

then $Qv = v - Pv$

$Q^2 v = Q(v-Pv) = v - Pv - P(v-Pv)$

$= v - Pv - Pv + P^2 v$

$= v - Pv$

Thus $Q^2 = Q$

2

2. Proof: Suppose v is a vector in V, such that

$PQv = P(v-Pv) = Pv - P^2v = 0$

$QPv = Pv - P^2 v = 0$

So $PQ = QP$

2

3. Proof: As P is a linear map from V to V.

Suppose $\exists u \in V$, s.t. $u = Pv$

As $P^2 v = P(Pv) = Pu$

$Pv = u$

So $Pu = u \Rightarrow P$ is identity map or zero map ?

1

① If P is Id_V, thus $N = \{0\}$ (as 0 is the only one to make $P(0) = 0$) ?

Then $R \neq \emptyset$

So $N \cap R = \{0\}$

② If P is zero map, thus $R = \{0\}$, then $N \neq \emptyset$ So $N \cap R = \{0\}$

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4. Proof: According to Problem 2.1 and 2.3

$Pv = v$, $Qv = Id_v v - Pv$

So $Pv + Qv = Pv + Id_v v - Pv = Id_v v = v$

2

5. Proof: ① If P is identity map such that $P(v) = v$, only $P(0) = 0$

$\therefore N = \{0\}$?

$\because R = V$,

$\therefore R + N = V$

$\because N \cap R = \{0\}$

$\therefore N \oplus R = V$?

② If P is zero map, such that $P(v) = 0$

$\therefore R = \{0\}$

$\because N = V$ $\therefore R+N = V$ $\because N \cap R = \{0\}$ $\therefore N \oplus R = V$

1

Exercise 3.

1. Proof: Suppose the two components of $F^n(v_0)$ are x_n, y_n.

$F(v_0) = (1, 1+0) = (1,1)$ $x_1 = 1$

$F^2(v_0) = (1, 1+1) = (1, 2)$ $x_2 = 1$

$F^3(v_0) = (2, 1+2) = (2,3)$ $x_3 = 2$

....

Let $F^{n-2}(v_0) = (a, b)$ $x_{n-2} = a$

$F^{n-1}(v_0) = (b, a+b)$ $x_{n-1} = b$

$F^n(v_0) = (a+b, a+2b)$ $x_n = a+b$

3

So we find $x_n = x_{n-2} + x_{n-1}$, which is in the Fibonacci sequence.
Therefore nth number is given by the first component of $F^n(v_0)$

2. Solution: Suppose the matrix is $\begin{bmatrix} x & y \\ z & m \end{bmatrix}$

so $\begin{bmatrix} x & y \\ z & m \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ a+b \end{bmatrix}$ ($v = av_1 + bv_2 = (a, b)$ as $v_1 = (1,0)$ $v_2 = (0, 1)$)

$\begin{cases} xa + yb = b \\ za + mb = a+b \end{cases} = \begin{cases} x = 0 & y = 1 \\ z = 1 & m = 1 \end{cases}$

So $A = [F]_e^e = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

2

Then $A^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $A^3 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $A^4 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ $A^5 = \begin{bmatrix} 3 & 5 \\ 5 & 8 \end{bmatrix}$

[redacted]

3. Solution: $F(1, \frac{1+\sqrt{5}}{2}) = (\frac{1+\sqrt{5}}{2}, 1+\frac{1+\sqrt{5}}{2})$

$F(-\frac{1+\sqrt{5}}{2}, 1) = (1, 1-\frac{1+\sqrt{5}}{2})$

Suppose that $\exists$ a matrix $\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$

$$\begin{cases} x - \frac{1+\sqrt{5}}{2} y = \frac{1+\sqrt{5}}{2} \\ \frac{1+\sqrt{5}}{2} x + y = 1 + \frac{1+\sqrt{5}}{2} \end{cases} \Rightarrow \begin{cases} x = \frac{1+\sqrt{5}}{2} \\ y = 0 \end{cases}$$

1

So $B = [F]_b^b = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$ $\frac{1-\sqrt{5}}{2}$

Hence $B^k = \begin{bmatrix} (\frac{1+\sqrt{5}}{2})^k, & 0 \\ 0, & (\frac{1+\sqrt{5}}{2})^k \end{bmatrix}$

4. Solution: as ~~b = {}~~ $V_1, V_2 \in b$, $v_1 = (1, \frac{1+\sqrt{5}}{2})$, $V_2 = (-\frac{1+\sqrt{5}}{2}, 1)$

so $V_0 = aV_1 + bV_2 = (a, \frac{1+\sqrt{5}}{2}a) + (-\frac{1+\sqrt{5}}{2}b, b) = (0, 1)$

0

~~so~~ then $a =$

?

MAT240 TUT Oct. 27th

Exercise 3

part ③ " The Fibonacci one"

V, W are vector spaces over a ~~field~~ field $\mathbb{F}$.

$\{v_1, \cdots v_n\}$ basis for V.

$\{w_1, \cdots w_m\}$ basis for W.

data are made in order to represent linear maps $V - W$ as matrices.

Suppose $F: V - W$ is linear.

$$F(v_1) = a_{11}w_1 + a_{12}w_2 + \cdots + a_{1m}w_m$$

$$\vdots$$

$$F(v_n) = a_{n1}w_1 + a_{n2}w_2 + \cdots + a_{nm}w_m$$

In general

$$F(v_j) = \sum_{i=1}^{m} a_{ij}w_i \quad , a_{ij} \in \mathbb{F}$$

We have assembled an $m \times n$ matrix with entries in $\mathbb{F}$.

$B = \{v_1, \cdots v_n\}$ $C = \{w_1, \cdots w_m\}$

The matrix $[F]_B^C$ is given by

$$[F]_B^C = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Upshot: To evaluate F on a given $v \in V$, find the coordinates of V with respect to the basis B of V.

$$v = b_1v_1 + \cdots + b_nv_n \quad \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Evaluate $[F]_B^C \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

note that $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is the column of coordinates $F(v)$ with respect to the basis C of W. Namely, $F(v) = c_1w_1 + \cdots + c_mw_m$

For ③ We consider the basis

$b = \{(1, \frac{1+\sqrt{5}}{2}), (-\frac{1+\sqrt{5}}{2}, 1)\}$

we ~~want~~ want to find $[F]_b^b$

$$F(1, \tfrac{1+\sqrt{5}}{2}) = (\tfrac{1+\sqrt{5}}{2}, \tfrac{3+\sqrt{5}}{2}) = \underbrace{\boxed{}}_{\frac{1+\sqrt{5}}{2}\ (a_{11})}(1, \tfrac{1+\sqrt{5}}{2}) + \underbrace{\boxed{}}_{0\ (a_{21})}(-\tfrac{1+\sqrt{5}}{2}, 1)$$

$$F(-\tfrac{1+\sqrt{5}}{2}, 1) = (1, \tfrac{1-\sqrt{5}}{2}) = \underbrace{\boxed{}}_{0\ (a_{12})}(1, \tfrac{1+\sqrt{5}}{2}) + \underbrace{\boxed{}}_{\frac{1-\sqrt{5}}{2}\ (a_{22})}(-\tfrac{1+\sqrt{5}}{2}, 1)$$

$$B = [F]_b^b = \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}$$

Observation $\begin{pmatrix} a, 0 \\ 0, b \end{pmatrix}^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \quad \forall k \in N$

So . . . .

④ $V_0 = (0,1)$

$(0,1) = \square \left(1 + \frac{1+\sqrt{5}}{2}\right) + \square \left(-\frac{1+\sqrt{5}}{2}, 1\right)$

$= a_1(1, \omega) + a_2(-\omega, 1)$

$\Rightarrow a_1 - a_2 \omega = 0 \qquad a_1 = \frac{\omega}{\omega^2+1}, \quad a_2 = \frac{1}{\omega^2+1}$

$a_1 \omega + a_2 = 1$

V_0 ~ coordinate

$F^n(v_0)$ is represented by the coordinate vector $B^n k$ with . . . to b.

$$k = \begin{pmatrix} \frac{\omega}{\omega^2+1} \\ \frac{1}{\omega^2+1} \end{pmatrix}$$

$$B^n k = \begin{pmatrix} \omega^n & 0 \\ 0 & (1-\omega)^n \end{pmatrix} \begin{pmatrix} \frac{\omega}{\omega^2+1} \\ \frac{1}{\omega^2+1} \end{pmatrix} = \begin{pmatrix} \frac{\omega^{n+1}}{\omega^2+1} \\ \frac{(1-\omega)^n}{\omega^2+1} \end{pmatrix}$$

This is the coord vector for $F^n(v_0)$ in terms of b.

$\Rightarrow F^n(v_0) = \frac{\omega^{n+1}}{\omega^2+1}(1, \omega) + \left(\frac{(1-\omega)^n}{\omega^2+1}\right)(-\omega, 1)$

$$= \left(\frac{\omega^{n+1} - \omega(1-\omega)^n}{\omega^2+1}, \frac{\omega^{n+2} + (1-\omega)^n}{\omega^2+1} \right)$$

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Exerzitien VI

Exercise 1.

1. Solution: As $[x_1\ x_2\ x_3\ x_4\ x_5][L] = [x_2\ x_3\ x_4\ x_5\ 0]$

Then
$$x_1\cdot 0 + x_2\cdot 1 + x_3\cdot 0 + x_4\cdot 0 + x_5\cdot 0 = x_2$$
$$x_1\cdot 0 + x_2\cdot 0 + x_3\cdot 1 + x_4\cdot 0 + x_5\cdot 0 = x_3$$
$$x_1\cdot 0 + x_2\cdot 0 + x_3\cdot 0 + x_4\cdot 1 + x_5\cdot 0 = x_4$$
$$x_1\cdot 0 + x_2\cdot 0 + x_3\cdot 0 + x_4\cdot 0 + x_5\cdot 1 = x_5$$
$$x_1\cdot 0 + x_2\cdot 0 + x_3\cdot 0 + x_4\cdot 0 + x_5\cdot 0 = 0$$

2

So $[L] = \begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix}$ — almost, but not quite right

2. Solution

$$[L^2] = [x_2\ x_3\ x_4\ x_5\ 0]\begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = [x_3\ x_4\ x_5\ 0\ 0]$$

$$[L^3] = [x_3\ x_4\ x_5\ 0\ 0]\begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = [x_4\ x_5\ 0\ 0\ 0]$$?

1

$$[L^4] = [x_4\ x_5\ 0\ 0\ 0]\begin{bmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \end{bmatrix} = [x_5\ 0\ 0\ 0\ 0]$$

$[L^5] = [x_5\ 0\ 0\ 0\ 0]\,[L] = [0\ 0\ 0\ 0\ 0]$?

...

$[L^n] = [0\ 0\ 0\ 0\ 0]\,[L] = [0\ 0\ 0\ 0\ 0]$ for all $n \in \mathbb{N}$, $n > 5$

3. Solution: For $k=1$

as $[L]\cdot[L^4] = [0\ 0\ 0\ 0\ 0]$

null(L) can be written as $(a,0,0,0,0) = a(1,0,0,0,0)$

so dim null$(L) = 1$

By ~~the~~ theorem that dim range(L) = dim $\mathbb{R}^5$ − dim null$(L) = 5-1=4$

Similarly, for $k=2$, as $[L^2]\cdot[L^3] = [0\ 0\ 0\ 0\ 0]$

null(L^2) can be written as $(a,b,0,0,0) = a(1,0,0,0,0) + b(0,1,0,0,0)$

so dim null$(L) = 2$

By theorem that dim range(L) = dim $\mathbb{R}^5$ − dim [illegible] $= 5-2=3$

Exercise 2

1. Proof. Suppose that $u_1, u_2 \in U$ and $Su_1 = Su_2$

Then $u_1 = Iu_1 = T(Su_1) = T(Su_2) = Iu_2 = u_2$

so $u_1 = u_2$

Hence S is injective.

Suppose now that $u_3 \in U$.

Then $u_3 = T(Su_3) = Iu_3 = u_3$

This implies that u_3 is in the range of T is

Thus range $T = U$

Hence T is surjective.

3

For $k=3$, $[L^3][L^2] = [0\,0\,0\,0\,0]$

null(L^3) can be written as $(a,b,c,0,0)$

which is $a(1,0,0,0,0)+b(0,1,0,0,0)+c(0,0,1,0,0)$

so dim null $(L^3) = 3$

and dim range $(L^3) = \dim \mathbb{R}^5 - \dim \text{null}(L^3)$

$= 5-3 = 2$

For $k=4$, $[L^4]\cdot[L] = [0\,0\,0\,0\,0]$

null(L^4) can be written as $(a,b,c,d,0)$

which is $a(1,0,0,0,0)+b(0,1,0,0,0)+c(0,0,1,0,0)$

$+d(0,0,0,1,0)$

so dim null $(L^4) = 4$

and dim range$(L^4) = \dim \mathbb{R}^5 - \dim \text{null}(L^4)$

$= 5-4 = 1$

For $k \geq 5$, $[L^k]\cdot[L] = [0\,0\,0\,0\,0]$

null(L^k) can be written as (a,b,c,d,e)

$= a(1,0,0,0,0)+b(0,1,0,0,0)+c(0,0,1,0,0)+$

$d(0,0,0,1,0)+e(0,0,0,0,1)$

so dim null $(L^5) = 5$

and dim range $(L^k) = \dim \mathbb{R}^5 - \dim \text{null}(L^k)$

$= 5-5 = 0$

4

2. Proof: Want to show $ST = I_V$

Suppose $S: \mathbb{R}^2 \to \mathbb{R}^1$, such as $[3\ 4]$

$T: \mathbb{R}^1 \to \mathbb{R}^2$, such as $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

So $TS = [3\ 4]\begin{bmatrix} -1 \\ 1 \end{bmatrix} = [1] = I_{\mathbb{R}^1}$

$ST = \begin{bmatrix} -1 \\ 1 \end{bmatrix}[3\ 4] = \begin{bmatrix} -3 & -4 \\ 3 & 4 \end{bmatrix} \neq I_{\mathbb{R}^2}$

Therefore S, T are not invertible

So S, T are not isomorphisms.

2

3. Proof: Suppose $\exists\, v$, s.t. $v \in \text{range}(S)$, $v \in \text{null}(T)$

and $v = Su$ for some $u \in U$.

and $Tv = 0$ since $v \in \text{null}(T)$.

then $Tv = TSu = Iu = u = 0$

$v = Su \Rightarrow v = S0 \Rightarrow v = 0$ (S is injective)

Thus $\text{Null}(T) \cap \text{range}(S) = \{0\}$ ①

Let $v \in V$, then $0 = STv - STv$

$= ST\,I_V - ST\,T_V$

$= ST\,I_V - STSTv$

$= ST(I_V - STv)$

So $I_V - STv$ is in null (ST)

As S is injective, $STv = 0 \Rightarrow Tv = 0$

$\Rightarrow \text{null}(ST) = \text{null}(T)$

As T is surjective, so $Tv = u \Rightarrow STv$ is in range (S)

Thus $STv + I_V - STv = I_V = V \Rightarrow \text{range}(S) + \text{null}(T) = V$

According to ① & ②, therefore $\text{range}(S) \oplus \text{null}(T) = V$.

3

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2.4. Proof: $\because$ TS, ST are injective and surjective by the definition of isomorphism.

Suppose $Su=0$, for $u \in U$.

$T_0 = 0$

So $To = TSu = 0$ but ST and TS are injective

then $u=0$, S is injective. ①

Similarly let $Tv=0$, for $v \in V$

$S_0 = 0$

$So = STv = 0$ but ST and TS are injective

then $v=0$, T is injective ②

Since TS is surjective, we can say

$\exists u_1 \in U$ such that $TSu_1 = u_2$ for $\forall u_2 \in U$.

Then $Su_1 = v$ for some $v \in V$

Thus $TSu_1 = Tv = u_2$

So T is surjective. ③

2

Similarly ST is surjective, we can say

$\exists v_1 \in V$ such that $STv_1 = v_2$ for $\forall v_2 \in V$

Then $Tv_1 = u$ for some $u \in U$

Thus $STv_1 = Su = v_2$

So S is surjective. ④

According to ①②③④ we have S and T are both injective and surjective.

Therefore, S and T are themselves isomorphisms.

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Exercise 3.

1. Proof. As $T: Mat(2,2,\mathbb{Q}) \to (2,2,\mathbb{Q})$ via $T(X) = AX - XA$

Then we have

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix} - \begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix}$$

$$= \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$$

Suppose $[Y] = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then $T(Y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} - \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} e+g & f+h \\ g & h \end{pmatrix} - \begin{pmatrix} e & e+f \\ g & g+h \end{pmatrix} = \begin{pmatrix} g & h-e \\ 0 & -g \end{pmatrix}$$

so $T(X+Y) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} - \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a+e+c+g & b+f+d+h \\ c+g & d+h \end{pmatrix} - \begin{pmatrix} a+e & a+e+b+f \\ c+g & c+g+d+h \end{pmatrix}$$

$$= \begin{pmatrix} c+g & d+h-a-e \\ 0 & -c-g \end{pmatrix}$$

4

$T(X) + T(Y) = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix} + \begin{pmatrix} g & h-e \\ 0 & -g \end{pmatrix} = \begin{pmatrix} c+g & d+h-e-a \\ 0 & -c-g \end{pmatrix}$

Thus $T(X+Y) = T(X) + T(Y)$

Then suppose $\lambda \in \mathbb{F}$. such that $T(\lambda x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix} - \begin{pmatrix} \lambda a & \lambda b \\ \lambda c & \lambda d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \lambda c & \lambda(d-a) \\ 0 & -\lambda c \end{pmatrix}$$

Thus $T(\lambda X) = \lambda T(X)$

Therefore T is a linear map.

2. Solution: Want to find $null(T) = 0$

suppose $T(X) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $[X] = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So we have $c = 0$, $d - a = 0$,

Thus $[X] = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

as $a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ only when $a = b = 0$.

so the basis is $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

[redacted]

And $\text{range}(T) = \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}$ by part 1.

So $\text{range}(T) = c\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + (d-a)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

So the basis is $\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$

1

Exercise 4.

1. Proof: $\because S^n = 0$

$\therefore I - S^n = I - 0 = I$

$\therefore I - S^n = (I-S)(I^{n-1} + I^{n-2}S + I^{n-3}S^2 + \cdots + IS^{n-2} + S^{n-1})$

$= (I-S)(I + S + S^2 + \cdots + S^{n-1})$

$= (S-I)(-I - S - S^2 - \cdots - S^{n-1})$

$= I$

3

$\therefore (S-I)^{-1} = -I - S - S^2 - \cdots - S^{n-1}$ ← You must also establish that $(-I - S - \ldots - S^{n-1})(S-I) = I$

This $S-I$ is an isomorphism.

2. Solution: According to 4.1, we have

$(S-I)^{-1} = -I - S - S^2 - \cdots^{n-1}$ for $S: W \to W$, $S^n = 0$

As $L: \mathbb{R}^5 \to \mathbb{R}^5$, $L^k = 0$ for $k \geq 5$

so $(L-I)^{-1} = -I - L - L^2 - \cdots - L^{k-1}$

2

For $k = 1$, $(L-I)^{-1} = -I = (-x_1 \quad -x_2 \quad -x_3 \quad -x_4 \quad -x_5)$

$k = 2$, $(L-I)^{-1} = -I - L = (-x_1 - x_2 \quad -x_2 - x_3 \quad -x_3 - x_4 \quad -x_4 - x_5 \quad -x_5)$

$k = 3$, $(L-I)^{-1} = -I - L - L^2 = (-x_1 - x_2 - x_3 \quad -x_2 - x_3 - x_4 \quad -x_3 - x_4 - x_5 \quad -x_4 - x_5 \quad -x_5)$

$k = 4$, $(L-I)^{-1} = -I - L - L^2 - L^3 = (-x_1 - x_2 - x_3 - x_4 \quad -x_2 - x_3 - x_4 - x_5 \quad -x_3 - x_4 - x_5 \quad -x_4 - x_5 \quad -x_5)$

$k \geq 5$, $(L-I)^{-1} = -I - L - L^2 - L^3 - L^4 - \cdots - L^{k-1} = -I - L - L^2 - L^3 - L^4$

$= (-x_1 - x_2 - x_3 - x_4 - x_5 \quad -x_2 - x_3 - x_4 - x_5 \quad -x_3 - x_4 - x_5 \quad -x_4 - x_5 \quad -x_5)$

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3. Proof: Want to show $T-bI$ invertible $\Leftrightarrow a \neq b$

$\Rightarrow$ $T-bI$ is invertible, $\exists$ an inverse W such that

$W\cdot(T-bI)=I$ ~~Why is it true that~~ $\left(W\cancel{(T-bI)}\right)^n = W^n(T-bI)^n$?

$\because I^n = I$

$\therefore W^n(T-bI)^n = I^n = I$

Suppose now $a=b$, then $W^n(T-bI)^n = W^n\cdot(T-aI)^n = W^n\cdot 0 = 0 \neq I$

Therefore $a \neq b$

$\Leftarrow$ $a \neq b$, $(T-aI)^n = 0$, then we have:

$$\left(\frac{1}{b-a}(T-aI)\right)^n = 0$$

Let $S = \frac{1}{b-a}\cdot(T-aI)$, $S^n = 0$

So $S - I = \frac{1}{b-a}\cdot(T-aI) - I = \frac{T-aI}{b-a} - \frac{I(b-a)}{b-a} = \frac{T-bI}{b-a} = \frac{1}{b-a}(T-bI)$

by Question 4.1 we can know that for $S^n=0$.

$I - S^n = I - 0 = I$

Then $I - S^n = (I-S)(I^{n-1} + I^{n-2}S + \cdots + S^{n-1})$

$= (S-I)(-I-S-\cdots-S^{n-1})$

$= I$

2

Thus $(S-I)$ is invertible or $\frac{1}{b-a}(T-bI)$ is invertible

As $\left(\frac{1}{b-a}(T-bI)\right)\cdot\left((-I-S-\cdots-S^{n-1})\cdot\cancel{(b-a)}\right) = (T-bI)\cdot\left(\frac{1}{b-a}\cancel{(b-a)}\cdot(-I-S-\cdots-S^{n-1})\right)$

$= I$

So $T-bI$ is invertible

Therefore $T-bI$ is invartible if and only if $a \neq b$.

You should also mention that $\left(\frac{1}{b-a}(-I+S+\cdots+S^{n-1})\right)(T-bI) = I$

[redacted]

MAT240 TUT ~~Oct~~ Nov. 10th

$\theta \in (0, \pi)$

$$T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1. Does there exist a non-zero vector $u \in \mathbb{R}^2$ s.t. $T(u) = \lambda u$ for some $\lambda \in \mathbb{R}$?

(Is there a vector $u \in \mathbb{R}^2$ such that $Tu = \lambda u$, for some nonzero $\lambda \in \mathbb{R}$?)

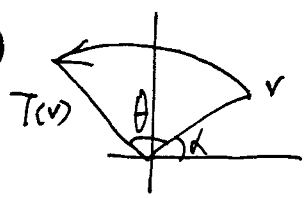
Yes. for each non-zero λ, note that $T(0) = 0 = \lambda \cdot 0$.

Claim: The only vector in $\mathbb{R}^2$ ~~that~~ T scales is zero vector.

Proof: Suppose that $v \in \mathbb{R}^2$ is non-zero, ~~the~~ $\alpha \in (0, \pi)$

Observation: $v = (R\cos\alpha, R\sin\alpha)$ for some $R > 0$ and ~~$\alpha \in [0, 2\pi)$~~

$$T(v) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} R\cos\alpha \\ R\sin\alpha \end{pmatrix} = \begin{pmatrix} R\cos(\alpha+\theta) \\ R\sin(\alpha+\theta) \end{pmatrix}$$



2. For $\lambda \in \mathbb{C}$, solve the eq'n $T(v) = \lambda v$ for $v \in \mathbb{C}^2$

(Recall: T scales v on V $\iff$ $T(v) = \lambda v$ for some $\lambda \in \mathbb{C}$)

$$T(v) = \lambda v \iff T(v) - \lambda v = 0$$
$$\iff T(v) - \lambda \cdot I(v) = 0$$
$$\iff (T - \lambda I)(v) = 0$$

The set of solutions to $T(v) = \lambda v$ is $\underline{\text{null}(T - \lambda I)}$

$$T - \lambda I = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix}$$

Find the null space of $(T - \lambda I)$ matrix. $\nearrow$ $\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Solve it we get ① $x = y = 0$

② $\{x(1, -i) : x \in \mathbb{C}\}$

$\{x(1, i) : x \in \mathbb{C}\}$

~~Use Row Reduction~~

~~Note Since $\theta \in (0, \pi)$ $\sin\theta \neq 0$~~

~~$\begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} \to \begin{pmatrix} \sin\theta & \cos\theta - \lambda \\ \cos\theta - \lambda & -\sin\theta \end{pmatrix} \to \begin{pmatrix} 1 & \frac{\cos\theta - \lambda}{\sin\theta} \\ \cos\theta - \lambda & -\sin\theta \end{pmatrix} \to \begin{pmatrix} 1 & \frac{\cos\theta - \lambda}{\sin\theta} \\ 0 & -\sin\theta - \frac{(\cos\theta - \lambda)^2}{\sin\theta} \end{pmatrix}$~~

We have shown ~~that~~

① If $\lambda \neq \cos\theta + i\sin\theta$ or $\cos\theta - i\sin\theta$, then the only solution to $T(v) = \lambda v$ is the zero vector.

② If $\lambda = \cos\theta + i\sin\theta$ then the solutions to $T(v) = \lambda v$ are
$\{x \cdot (1, i) : x \in \mathbb{C}\} = \text{span}(1, i)$

③ If $\lambda = \cos\theta - i\sin\theta$, then the solutions to $T(v) = \lambda v$ are
$\{x \cdot (1, -i) : x \in \mathbb{C}\} = \text{span}(1, -i)$

$\Rightarrow$ ~~scales~~ $v \in \mathbb{C}^2$ scaled by $T \Leftrightarrow v$ belongs to the set
$\{x(1, i) : x \in \mathbb{C}\} \cup \{x(1, -i) : x \in \mathbb{C}\}$

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Exerzitien VII

Exercise 1.

1\. Proof: We have

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda (a, b)$$

2

then $\cos\theta \cdot a - \sin\theta b = \lambda a$
$\sin\theta \cdot a + \cos\theta b = \lambda b$ $\Rightarrow$ $\lambda = \cos\theta$ has a solution when $a=0, b=0$

Therefore, if $u = (0,0)$, $Tu = \lambda u$.

Exercise 2:

1. Proof $Tr(cA+B) = \sum_{k=1}^{n}(c a_{kk} + b_{kk}) = \sum_{k=1}^{n} c a_{kk} + \sum_{k=1}^{n} b_{kk}$

$$= c\sum_{k=1}^{n} a_{kk} + \sum_{k=1}^{n} b_{kk}$$

$$= c\,Tr(A) + Tr(B)$$

3

Therefore Tr is a linear function. ∎

2. Proof: $Tr(AB) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki}$

$Tr(BA) = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki} b_{ik}$.

3

But $\sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik} b_{ki} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki} b_{ik}$ Why?

Therefore $Tr(AB) = Tr(BA)$ ∎

3. Proof: From part 2 we know that $Tr(AB) = Tr(BA)$

Let $PA = C$, $P^{-1} = D$, then we have

$Tr(CD) = Tr(DC)$

Thus $Tr(PAP^{-1}) = Tr(P^{-1}PA)$

As $I = P^{-1}P$

$Tr(P^{-1}PA) = Tr(IA) = Tr(A)$

Therefore $Tr(PAP^{-1}) = Tr(A)$ ∎

3

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Exercise 3.

Solution:

Name four matrices T_1, T_2, T_3, T_4.

T_1:
$$\left[\begin{array}{cc|cc}1&2&1&0\\1&1&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cc|cc}1&1&0&1\\1&2&1&0\end{array}\right]\Rightarrow\left[\begin{array}{cc|cc}1&1&0&1\\0&1&1&-1\end{array}\right]\Rightarrow\left[\begin{array}{cc|cc}1&0&-1&2\\0&1&1&-1\end{array}\right]$$

rank = dim range $T_1 = 2$. $\quad T_1^{-1}=\begin{bmatrix}-1&2\\1&-1\end{bmatrix}$

T_2:
$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\1&3&4&0&1&0\\2&3&-1&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&-1&-3&1&-1&0\\0&1&3&2&0&-1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&-1&1&0\\0&1&3&2&0&-1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&3&-1&1&0\\0&0&0&-3&1&1\end{array}\right]$$

rank = dim range $T_2 = 2$. $\quad T_2^{-1}$ does not exist.

T_3:
$$\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\-1&1&2&0&1&0\\1&0&1&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&3&3&1&1&0\\0&2&0&1&0&-1\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&1&\frac13&\frac13&0\\0&1&0&\frac12&0&-\frac12\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{ccc|ccc}1&2&1&1&0&0\\0&1&0&\frac12&0&-\frac12\\0&0&1&-\frac16&\frac13&\frac12\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&2&0&\frac76&-\frac13&-\frac12\\0&1&0&\frac12&0&-\frac12\\0&0&1&-\frac16&\frac13&\frac12\end{array}\right]\Rightarrow\left[\begin{array}{ccc|ccc}1&0&0&\frac16&-\frac13&\frac12\\0&1&0&\frac12&0&-\frac12\\0&0&1&-\frac16&\frac13&\frac12\end{array}\right]$$

rank = dim range $T_3 = 3$ $\quad T_3^{-1}=\begin{bmatrix}\frac16&-\frac13&\frac12\\\frac12&0&-\frac12\\-\frac16&\frac13&\frac12\end{bmatrix}$

T_4:
$$\left[\begin{array}{cccc|cccc}1&2&1&0&1&0&0&0\\2&5&5&1&0&1&0&0\\-2&-3&0&3&0&0&1&0\\3&4&-2&-3&0&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&2&1&0&1&0&0&0\\0&1&3&1&-2&1&0&0\\0&1&2&3&2&0&1&0\\0&-2&-5&-3&-3&0&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&3&1&-2&1&0&0\\0&1&2&3&2&0&1&0\\0&-2&-5&-3&-3&0&0&1\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&3&1&-2&1&0&0\\0&0&-1&3&1&0&2&1\\0&0&1&-1&-7&2&0&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&3&1&-2&1&0&0\\0&0&-1&3&1&0&2&1\\0&0&0&2&-6&2&2&2\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&3&1&-2&1&0&0\\0&0&-1&-3&-1&0&-2&-1\\0&0&0&1&-3&1&1&1\end{array}\right]$$

$$\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&3&1&-2&1&0&0\\0&0&1&0&-10&3&1&2\\0&0&0&1&-3&1&1&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&0&-4&-3&-2&0&0&1\\0&1&0&0&31&-9&-4&-7\\0&0&1&0&-10&3&1&2\\0&0&0&1&-3&1&1&1\end{array}\right]\Rightarrow\left[\begin{array}{cccc|cccc}1&0&0&0&-51&15&7&12\\0&1&0&0&31&-9&-4&-7\\0&0&1&0&-10&3&1&2\\0&0&0&1&-3&1&1&1\end{array}\right]$$

rank = dim range $T_4 = 4$

$$T_4^{-1} = \begin{bmatrix} -51 & 15 & 7 & 12 \\ 31 & -9 & -4 & -7 \\ -10 & 3 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{bmatrix}$$

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MAT240 Exerzitien VII

Exercise 1.

1. Solution: We want to show that a.b.c.d holds a condition

$\Leftrightarrow$ T is invertible which means $\exists T^{-1}$ such that $TT^{-1}=I$.

As $T \in \mathbb{F}^{2\times2}$, so $I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Why is $a\neq 0$?

Using row reduction we can get T^{-1}

3

$$\left[\begin{array}{cc|cc}a&b&1&0\\c&d&0&1\end{array}\right]\xrightarrow{r_1\times\frac{1}{a}}\left[\begin{array}{cc|cc}1&\frac{b}{a}&\frac{1}{a}&0\\c&d&0&1\end{array}\right]\xrightarrow{r_2-r_1\times c}\left[\begin{array}{cc|cc}1&\frac{b}{a}&\frac{1}{a}&0\\0&d-\frac{bc}{a}&-\frac{c}{a}&1\end{array}\right]\xrightarrow{r_2\times\frac{a}{ad-bc}}\left[\begin{array}{cc|cc}1&\frac{b}{a}&\frac{1}{a}&0\\0&1&\frac{c}{bc-ad}&\frac{a}{ad-bc}\end{array}\right]$$

$$\xrightarrow{r_1\times\frac{a}{b}-r_2}\left[\begin{array}{cc|cc}\frac{a}{b}&0&\frac{1}{b}-\frac{c}{bc-ad}&\frac{a}{bc-ad}\\0&1&\frac{c}{bc-ad}&\frac{a}{ad-bc}\end{array}\right]\xrightarrow{r_1\times\frac{b}{a}}\left[\begin{array}{cc|cc}1&0&\frac{d}{ad-bc}&\frac{-b}{ad-bc}\\0&1&\frac{-c}{ad-bc}&\frac{a}{ad-bc}\end{array}\right]$$

What to show that T^{-1} exists, so $bc-ad\neq 0$ as a vital condition.

[SEE (*) BELOW

2. According to Question 1.1, $T^{-1}=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$. 2

3. Solution: $\Rightarrow$ Suppose that T has an eigenvalue λ.

So $Tv=\lambda v \Rightarrow (T-\lambda I)v=0$

As v is a non-zero vector, so $T-\lambda=\begin{pmatrix}a&b\\c&d\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}=\begin{pmatrix}a-\lambda&b\\c&d-\lambda\end{pmatrix}=0$

Then $(a-\lambda)(d-\lambda)-bc=\lambda^2-(a+d)\lambda+ad-bc=0$

As T has eigenvalue values

hence $(a+d)^2-4(ad-bc)=(a-d)^2+4bc\geq 0$

So if T has eigenvalues, then $(a-d)^2+4bc\geq 0$

4

$\Leftarrow$ As $(a-d)^2-4bc\geq 0$, so $(a+d)^2-4(ad-bc)\geq 0$

Therefore in the equation $\alpha^2-(a+d)\alpha+(ad-bc)=(a-\alpha)(d-\alpha)-bc\geq 0$

has one solution. for real numbers

Then we suppose that $T=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, $\alpha I=\begin{pmatrix}\alpha&0\\0&\alpha\end{pmatrix}$

obviously we see that $T-\alpha I=0 \Rightarrow T=\alpha I$

Both sides multiply a non-zero vector v, such that

$Tv=\alpha Iv=\alpha v$

Therefore, T has an eigenvalue α.

Example for a matrix which doesn't have eigenvalue: $\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$ $0<\theta<\pi$

[* PART] We have to prove this from the other direction.

This is not T^{-1} Suppose that $T=\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ counter operation of

If $ad \neq bc$ then $\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists, by row reduction we can get the conclusion ~~that~~ $T \cdot T^{-1} = I$, completing the proof.

Exercise 2

1. Solution: Written in a matrix form:

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \frac{dx_3}{dt} \end{pmatrix} = \frac{dx}{dt} = AX = A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1+x_2 \\ x_1+3x_2+x_3 \\ x_2+2x_3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

2

$Y = PX$

then $P^{-1}PX = P^{-1}Y$

$X = P^{-1}Y$

$X' = AX$

$(P^{-1}Y)' = A(P^{-1}Y)$

$P^{-1}Y' = AP^{-1}Y$

$PP^{-1}Y' = PAP^{-1}Y$

$Y' = PAP^{-1}Y$

$= (P^{-1})^{-1}AP^{-1}Y$

Let $(P^{-1})^{-1}AP^{-1} = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix}$

$P^{-1}PAP^{-1} = P^{-1}D$

$AP^{-1} = P^{-1}D$

2. Solution. ? (~~As PAP^{-1} is diagonal, A is similar with PAP^{-1} thus A has an eigenvalue λ, such that~~)

$A - \lambda I = 0 \Rightarrow \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & 1 \\ 0 & 1 & 2-\lambda \end{pmatrix} = 0$ Suppose $V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$ and $A(v) - \lambda(v) = 0$

So $(2-\lambda)\begin{pmatrix} 3-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} - 1 \times \begin{pmatrix} 1 & 0 \\ 1 & 2-\lambda \end{pmatrix} + 0 \times \begin{pmatrix} 1 & 0 \\ 3-\lambda & 1 \end{pmatrix}$

$= (2-\lambda)((3-\lambda)(2-\lambda)-1) - (2-\lambda)$

$= (2-\lambda)(6-5\lambda+\lambda^2-2)$

$= (2-\lambda)(\lambda-1)(\lambda-4) = 0$

Therefore λ equals to $1, 2$ or 4.

① Take $\lambda = 1$ into $A - \lambda I = \begin{pmatrix} 2-1 & 1 & 0 \\ 1 & 3-1 & 1 \\ 0 & 1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

Using Row Reduction we get $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Suppose $V = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$ Then we have $\begin{cases} V_1 + V_2 = 0 \Rightarrow V_1 = -V_2 \\ V_2 + V_3 = 0 \Rightarrow V_2 = -V_3 \end{cases}$

Let $V_1 = t \in F$ so $v = \begin{pmatrix} t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

Thus $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is an eigenvector.

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② Take ~~λ~~ $\lambda=2$ into $A-\lambda I=\begin{pmatrix}2-2 & 1 & 0\\ 1 & 3-2 & 1\\ 0 & 1 & 2-2\end{pmatrix}=\begin{pmatrix}0&1&0\\1&1&1\\0&1&0\end{pmatrix}$

Using Row Reduction we get $\begin{pmatrix}0&1&0\\1&1&1\\0&1&0\end{pmatrix}=\begin{pmatrix}1&1&1\\0&1&0\\0&0&0\end{pmatrix}$

so, similarly we have $\begin{cases}V_1+V_2+V_3=0 \Rightarrow V_1=-V_3\\ V_2=0\end{cases}$

Let $V_1=t\in\mathbb{F}$ so $V=\begin{pmatrix}t\\0\\-t\end{pmatrix}=t\begin{pmatrix}1\\0\\-1\end{pmatrix}$

Thus $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ is an eigenvector.

③ Take $\lambda=4$ into $A-\lambda I=\begin{pmatrix}2-4&1&0\\1&3-4&1\\0&1&2-4\end{pmatrix}=\begin{pmatrix}-2&1&0\\1&-1&1\\0&1&-2\end{pmatrix}$

Using Row Reduction we get $\begin{pmatrix}-2&1&0\\1&-1&1\\0&1&-2\end{pmatrix}=\begin{pmatrix}1&-1&1\\0&-1&2\\0&1&-2\end{pmatrix}=\begin{pmatrix}1&-1&1\\0&-1&2\\0&0&0\end{pmatrix}$

Similarly we have $\begin{cases}V_1-V_2+V_3=0\\-V_2+2V_3=0\end{cases}\Rightarrow\begin{cases}V_1=V_3\\V_2=2V_1\end{cases}$

Let $V_1=t\in\mathbb{F}$ so $V=\begin{pmatrix}t\\2t\\t\end{pmatrix}=t\begin{pmatrix}1\\2\\1\end{pmatrix}$

Thus $\begin{pmatrix}1\\2\\1\end{pmatrix}$ is an eigenvector.

Combining ①②③, the three eigenvectors from a matrix P^{-1},

which is $P^{-1}=\begin{pmatrix}1&1&1\\-1&0&2\\1&-1&1\end{pmatrix}$

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3. Solution: As $P^{-1}=\begin{pmatrix}1&1&1\\-1&0&2\\1&-1&1\end{pmatrix}$, $P=\begin{pmatrix}\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{6}&\frac{1}{3}&\frac{1}{6}\end{pmatrix}$ (by Row Reduction; procedure omitted).

Then $\frac{dY}{dt}=(PAP^{-1})Y=\begin{pmatrix}\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{6}&\frac{1}{3}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}2&1&0\\1&3&1\\0&1&2\end{pmatrix}\begin{pmatrix}1&1&1\\-1&0&2\\1&-1&1\end{pmatrix}Y=\begin{pmatrix}\frac{1}{3}&-\frac{1}{3}&\frac{1}{3}\\\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{6}&\frac{1}{3}&\frac{1}{6}\end{pmatrix}\begin{pmatrix}1&2&4\\-1&0&8\\1&-2&4\end{pmatrix}Y$

$=\begin{pmatrix}1&0&0\\0&2&0\end{pmatrix}Y$

~~As (PAP^{-1}) is diagonal, so we should do Row Reduction to~~

$$\cancel{\begin{pmatrix} 4 & 0 & -1 \\ \frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 0 \\ \frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & -4 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Thus we have $\frac{dY}{dt} = \begin{pmatrix} Y_1' \\ Y_2' \\ Y_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} Y_1 \\ 2Y_2 \\ 4Y_3 \end{pmatrix}$

Hence $Y_1 = C_1e^t$, $Y_2 = C_2e^{2t}$ $Y_3 = C_3e^{4t}$

Then $Y' = \begin{pmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{4t} \end{pmatrix}$

Because $Y = PX \Rightarrow P^{-1}Y = P^{-1}PX = (P^{-1}P)X = IX = X$

$\Rightarrow X = P^{-1}Y = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 2 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} C_1e^t \\ C_2e^{2t} \\ C_3e^{4t} \end{pmatrix} = \begin{pmatrix} C_1e^t + C_2e^{2t} + C_3e^{4t} \\ -C_1e^t + 2C_3e^{4t} \\ C_1e^t - C_2e^{2t} + C_3e^{4t} \end{pmatrix}$

Therefore the solutions are: $x_1 = C_1e^t + C_2e^{2t} + C_3e^{4t}$
$x_2 = -C_1e^t + 2C_3e^{4t}$
$x_3 = C_1e^t - C_2e^{2t} + C_3e^{4t}$ ✓

4

Exercise 3

1. Solution: According to question,

$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \lambda & 1 & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & & \lambda \end{pmatrix}$

$J_k(\lambda) - \lambda I = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \lambda & 1 & \vdots \\ 0 & \cdots & & \lambda \end{pmatrix} - \begin{pmatrix} \lambda & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & 1 & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & & 0 \end{pmatrix}$

You should prove

As we can see, the number of independent column is $k-n$, and dim null is total number k minus independent column

Therefore, $k-(k-n)=n$

Thus ~~≠~~ $\text{Dim null}(J_k(\lambda)-\lambda I)^n = n$

What happens if $k=2$ and $n=10$?

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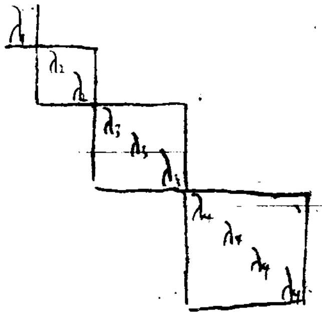
2. Solution:



The property of Jordan Block shows that Total size of this matrix is

$i_1\times 1 + i_2\times 2 + \cdots + i_m \times m$

And number of independent column of $(A-\lambda I)^n$ is:

$(1-n)i_1 + (2-n)i_2 + \cdots + (m-n)i_m$

if $J-n<0$, then let $J-n=0$, for $\forall J \in (1,\cdots m)$

Therefore, $\dim \text{Null}(A-\lambda I)^n$ = total size − independent column

$= \sum_1^k i_k \cdot k - \sum_1^k (k-n) I_k \Rightarrow$ for every $k-n<0$, $k-n=0$

This does not make sense.

Your answer is not stated properly.

4

Exercise 4.

Proof: Suppose λ is the eigenvalue of ST, eigenvector is v, such that $S(T(v)) = \lambda v$. Thus

$T(S(T(v))) = \lambda T(v)$

and so λ is also the eigenvalue of TS, unless $T(v)=0$.

If $T(v)=0$, then $\text{null}(T) \neq \{0\}$, therefore $\lambda = 0$.

because $0 = S(T(v)) = \lambda v$ and $v \neq 0$)

As V is finite dimensional

$\dim V = \dim \text{null}(T) + \text{rank}(T) = \dim \text{null}(TS) + \text{rank}(TS)$

As $\text{range}(TS) \subseteq \text{range}(T)$, we can get that

$\dim \text{null}(TS) \geq \dim \text{null}(T) > 0$

So $\text{null}(TS)$ is nontrivial

Thus the eigenvalue of TS is $\lambda = 0$.

(If we switch T and S, the proof will be similar.) ■

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Exerzitien IX

Exercise 1

$n \times n$ stochastic

1. Proof. Suppose that there exist 2 matrices which are $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ and $\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix}$

then $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11}+\cdots+a_{1n}b_{n1} & \cdots & a_{11}b_{1n}+\cdots+a_{1n}b_{nn} \\ \vdots & & \vdots \\ a_{n1}b_{11}+\cdots+a_{nn}b_{n1} & \cdots & a_{n1}b_{1n}+\cdots+a_{nn}b_{nn} \end{bmatrix}$

Hence the sums of of the entries in two columns of the product satisfy.

2

$(a_{11}b_{11}+\cdots a_{1n}b_{n1})+\cdots+(a_{n1}b_{11}+\cdots+a_{nn}b_{n1}) = b_{11}(a_{11}+\cdots+a_{n1})+\cdots+b_{n1}(a_{1n}+\cdots+a_{nn})$ ①

$(a_{11}b_{1n}+\cdots+a_{1n}b_{nn})+\cdots+(a_{n1}b_{1n}+\cdots+a_{nn}b_{nn}) = b_{1n}(a_{11}+\cdots+a_{n1})+\cdots+b_{nn}(a_{1n}+\cdots+a_{nn})$ ②

By the definition of stochastic matrix that,

$a_{11}+\cdots+a_{n1} = \cdots = a_{1n}+\cdots+a_{nn} = b_{11}+\cdots+b_{n1} = \cdots = b_{1n}+\cdots+b_{nn} = 1$

This idea is more easily communicated via indices and sums.

Therefore ① & ② can be written as

$b_{11}(a_{11}+\cdots+a_{n1})+\cdots+b_{n1}(a_{1n}+\cdots+a_{nn}) = b_{11}+\cdots+b_{n1} = 1$

$b_{1n}(a_{11}+\cdots+a_{n1})+\cdots+b_{nn}(a_{1n}+\cdots+a_{nn}) = b_{1n}+\cdots+b_{nn} = 1$

So the product of two stochastic matrices is also stochastic. ■

Also, why are the entries of AB non-negative?

2. Proof. Suppose $\exists$ a vector $\left.\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}\right\}n$ in the image of $(A-I)$. Suppose $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

Then the transpose of A, A^T is a row-stochastic.

$A^T = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix}$

3

Thus $A^T \cdot \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{1n} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{11}+\cdots+a_{n1} \\ \vdots \\ a_{1n}+\cdots+a_{nn} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Name vector $\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ u. then we have $A^T u = 1 \cdot u$

So the eigenvalue of A^T is $\lambda = 1$

Since A and A^T share the same eigenvalue

Thus the eigenvalue of A is 1. ■

[Don't know whether we can use ~~the~~ property of transpose to prove this]

So the other way to prove, I ~~think~~, is following

[redacted]

Proof: Suppose that $A=[a_{ij}]$ is a column stochastic matrix, which is $R^{n\times n}$

Then $A-I=\begin{bmatrix} a_{11}-1 & \cdots & & a_{1n} \\ \vdots & a_{22}-1 & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn}-1 \end{bmatrix}=[b_{ij}]$

Good!

$\because \sum_{i=1}^{n} a_{ij} = 1$

$\therefore \sum_{i=1}^{n} b_{ij} = (a_{11}-1)+a_{21}+a_{31}+\cdots+a_{n1} = 1-1=0$

$\therefore$ The matrix $(A-I)$ is column dependent

$\therefore$ $(A-I)$ is not surjective, as it is a square matrix, this implies that it is not injective, either.

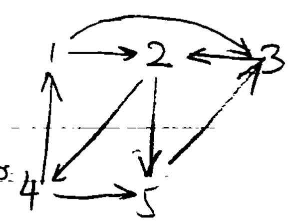
$A-I$

$\therefore null(A) \neq \{0\}$

$\therefore$ There exists some eigenvector for A

$\therefore$ 1 is an eigenvalue.

3 Solution. Let the time staying with each bureaucrats be x_1, x_2, x_3, x_4, x_5.
then we have the relations among 5 letters.



$x_1 = \frac{1}{2}x_4$

$x_2 = \frac{1}{2}x_1 + x_3$

$x_3 = \frac{1}{2}x_1 + x_5$

$x_4 = \frac{1}{2}x_2$

$x_5 = \frac{1}{2}x_4 + \frac{1}{2}x_2$

$$\overset{A}{\begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}} \overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}} = \overset{X}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}}$$

4

So $AX = 1\cdot X \Rightarrow (A-I)X=0$

Solve this we get $X=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2k \\ 8k \\ 7k \\ 4k \\ 6k \end{pmatrix}$ $(k\in\mathbb{R})$

Ranking x_1 to x_5 we get $x_2 > x_3 > x_5 > x_4 > x_1$

So the 5 bureaucrats from most time-wasting to the least are 2, 3, 5, 4, 1

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Exercise 2.

Solution: The matrix $A=\begin{bmatrix}2&1&1\\0&2&1\\0&0&1\end{bmatrix}$ with $\lambda_1=2, \lambda_2=1$

$$A-2I=\begin{bmatrix}2-2&1&1\\0&2-2&1\\0&0&1-2\end{bmatrix}=\begin{bmatrix}0&1&1\\0&0&1\\0&0&-1\end{bmatrix}\xrightarrow{\text{Row Reduction}}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix} \quad (\text{Rank}=2)$$

then $\dim null(A-2I)=3-2=1$

thus $\exists$ one Jordan Block of size 2 for the eigenvalue $\lambda_1=2$

$\begin{pmatrix}2&0&0\\0&2&0\\0&0&1\end{pmatrix}$ So the minimal polynomial of matrix A is

$p(A)=\cancel{(A-2I)^2(A-I)}(A-2)^2(A-1)$

We have $PAP^{-1}=\begin{bmatrix}2&1&0\\0&2&0\\0&0&1\end{bmatrix}$ let $P^{-1}=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$

$\therefore P^{-1}(PAP^{-1})=P^{-1}\begin{bmatrix}2&1&0\\0&2&0\\0&0&1\end{bmatrix}$

$AP^{-1}=P^{-1}\begin{bmatrix}2&1&0\\0&2&0\\0&0&1\end{bmatrix}$

$$\begin{pmatrix}2&1&1\\0&2&1\\0&0&1\end{pmatrix}\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\begin{pmatrix}2&1&0\\0&2&0\\0&0&1\end{pmatrix}$$

$$\begin{pmatrix}2a+d+g&2b+e+h&2c+f+i\\2d+g&2e+h&2f+i\\g&h&i\end{pmatrix}=\begin{pmatrix}2a&a+2b&c\\2d&d+2e&f\\2g&g+2h&i\end{pmatrix}$$

$\Rightarrow \begin{cases}g=2g \Rightarrow g=0\\ g+2h=h \Rightarrow h=0\\ 2e+h=d+2e \Rightarrow d=0\end{cases} \Rightarrow \begin{pmatrix}2a&2b+e&2c+f+i\\0&2e&2f+i\\0&0&i\end{pmatrix}=\begin{pmatrix}2a&a+2b&c\\0&2e&f\\0&0&i\end{pmatrix}$

$\Rightarrow \begin{cases}2b+e=a+2b \Rightarrow a=e\\ \left.\begin{matrix}2c+f+i=c\\2f+i=f\end{matrix}\right\} \Rightarrow \begin{matrix}c=0\\f+i=0\end{matrix}\end{cases} \Rightarrow \begin{pmatrix}2a&2b+a&0\\0&2a&f\\0&0&i\end{pmatrix}=\begin{pmatrix}2a&a+2b&0\\0&2a&f\\0&0&i\end{pmatrix}$

So let $\begin{cases}a=e=1\\b=1\\g=h=d=c=0\\f=1\\i=-1\end{cases}$ then we get $P^{-1}=\begin{pmatrix}1&1&0\\0&1&1\\0&0&-1\end{pmatrix}$

[redacted]

Finally we use Row Reduction to Calculate P:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{r_3 = r_3 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right) \xrightarrow{r_2 = r_2 - r_3} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

$$\xrightarrow{r_1 = r_1 - r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)$$

Thus $P = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$

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Exerzitien X

Exercise 1.

1. Solution: For every vector u that makes $S(Tu)=0$, there are two possible situations: ① $Tu=0$, $u\in null(T)$, then $S(Tu)=S(0)=0$.

② $Tu\neq 0$, then $Tu\in null\, S$, so $S(Tu)=0$.

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Notice that in ② $Tu\in range(T)$

By Rank-Nullity Theoren that,

$\dim(null(ST)) = \dim(null(T|null(ST)) + \dim(range(T|null(ST)))$

As $null(T|null(ST)) = null\,T$, $range(T|null(ST)) = null(S)\cap range(T)$

Therefore $\dim(null(ST)) = \dim(null(T)) + \dim(null(S)\cap range(T))$

2. Proof: Suppose there exists a linear map T' such that $null\, ST \to V$ by $T'u = Tu$

If $u\in null(ST)$, then $S(Tu)=0$

So $Tu\in null S$

So $range\, T' \subset null\, S$

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Now we have $\dim null(ST) = \dim null(T') + \dim range(T')$ (by Theorem 3.4)

$\leq \dim null(T') + \dim null(S)$

$\leq \dim null(T) + \dim null(S)$

According to Question 1.1 that $\dim null(ST) = \dim(T) + \dim(null(S)\cap range(T))$

so we have $\dim(null(S)\cap range(T)) \leq \dim null(S)$

thus $null(S) \subseteq range(T)$

■

3. Proof: Since T is injective, if $Tu = Tv$, then $u=v$.

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Set $a=Tu$, $b=Tv$, Since S is injective,

So $Sa = S(Tu) = S(Tv) = Sb$ for $a=Tu=Tv=b$

Therefore, ST is also injective.

Give an example!

Now suppose that $\dim U=3$, $\dim V = \dim W = 2$.

Since T is surjective, so $\dim range(T) = \dim V = 2$, $\dim null(U) = 3-2 = 1$

then $\dim range(T) = \dim V = \dim W = 2$, satisfies that S is injective.

But $\dim U \neq \dim W$, which means ST between V and W is not one-to-one.

Thus ST is not necessarily injective. ■

Claim: $(e^x, xe^x, \cdots, x^n e^x)$ spans null $((D-I)^{n+1})$

4. Proof: Since span(e^x) = null$(D-I)$

Suppose $a_0 \in \mathbb{F}$, then span $(e^x) = \{a_0 e^x\}$

so $(D-I)(a_0 e^x) = a_0 e^x - a_0 e^x = 0$.

Thus for $n=0$, the hypothesis is true. ✓

We want to prove the the hypothesis is also true for $n=k+1$.

If $n=k+1$, then span $(e^x, xe^x, \cdots x^{k+1} e^x) = \{a_0 e^x + a_1 xe^x + a_2 x^2 e^x + \cdots + a_{k+1} x^{k+1} e^x\}$

Then we add an operator $(D-I)$ to this span, we get.

$$(D-I)(a_0 e^x + a_1 xe^x + a_2 x^2 e^x + \cdots + a_{k+1} x^{k+1} e^x)$$
$$= (a_0 e^x + a_1 e^x + a_1 xe^x + 2a_2 xe^x + a_2 x^2 e^x + \cdots + (k+1) a_{k+1} x^k e^x + a_{k+1} x^{k+1} e^x)$$
$$- (a_0 e^x + a_1 xe^x + a_2 x^2 e^x + \cdots + a_{k+1} x^{k+1} e^x)$$
$$= a_1 e^x + 2a_2 xe^x + 3a_3 x^2 e^x + \cdots + (k+1) a_{k+1} x^k e^x$$

So we eliminate the first term with a_0.

Then we continute to operate on $(D-I)$:

$$(D-I)(a_1 e^x + 2a_2 xe^x + 3a_3 x^2 e^x + \cdots + (k+1) a_{k+1} x^k e^x)$$
$$= (a_1 e^x + 2a_2 e^x + 2a_2 xe^x + 6a_3 x e^x + 3a_3 x^2 e^x + \cdots + k(k+1) a_{k+1} x^{k-1} e^x + (k+1) a_{k+1} x^k e^x)$$
$$- (a_1 e^x + 2a_2 xe^x + 3a_3 x^2 e^x + \cdots + (k+1) a_{k+1} x^k e^x)$$
$$= 2a_2 e^x + 6a_3 xe^x + \cdots + k(k+1) a_{k+1} x^{k-1} e^x$$

So we eliminate the second term with a_1

Thus by repeating $(D-I)$ $(k+2)$ times, we can eliminate all $(k+2)$ terms.

Therefore, $(D-I)^{k+2}(a_0 e^x + a_1 xe^x + a_2 x^2 e^x + \cdots + a_{k+1} x^{k+1} e^x) = 0$ (for $n=k+1$, it's true)

Then when $n=k+2$, span $(e^x, xe^x, \cdots, x^{k+1} e^x, x^{k+2} e^x) = \{a_0 e^x + a_1 xe^x + \cdots + a_{k+1} x^{k+1} e^x + a_{k+2} x^{k+2} e^x\}$

So $(D-I)^{k+3}(a_0 e^x + a_1 xe^x + \cdots + a_{k+1} x^{k+1} e^x + a_{k+2} x^{k+2} e^x)$
$$= (D-I)^{k+3}(a_0 e^x + a_1 xe^x + \cdots + a_{k+1} x^{k+1} e^x) + (D-I)^{k+3} a_{k+2} x^{k+2} e^x$$
$$= (D-I) \cdot (D-I)^{k+2}(a_0 e^x + a_1 xe^x + \cdots + a_{k+1} x^{k+1} e^x) + 0$$
$$= 0 + 0$$
$$= 0$$

Hence hypothesis is also true when $n=k+2$

Proved.

This just proves that null $((D-I)^{k+3})$ contains the span.

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Exercise 2.

1. Proof: Since $(L-\lambda I)(S_1, S_2, \cdots) = (S_2 - \lambda S_1, S_3 - \lambda S_2, \cdots)$
$= 0$

So $\lambda S_1 = S_2$ $\lambda^2 S_1 = S_3$, ...

Sequence S can be written as $(S_1, \lambda S_1, \lambda^2 S_1, \cdots)$ and λ can be any number.

When $\lambda = 1$, $S = (S_1, S_1, S_1, \cdots) = S_1(1, 1, 1, \cdots)$

as S_1 is a constant, then (1) spans eigenspace.

Then we have.

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$(L-I)(n) = (2, 3, 4, \cdots) - (1, 2, 3, \cdots) = (1, 1, 1, \cdots)$

$(L-I)(n^2) = L(n^2) - I(n^2) = (n+1)^2 - n^2 = 2(n) + (1)$

$(L-I)^2(n^2) = (L-I)(L-I)(n^2) = (L-I)(2(n) + (1))$

$= 2(L-I)(n) + (L-I)(1)$

$= 2(1) + 0$

Why is ... a generalized eigenvector?

but $(L-I)(2(1)) = 2(1-1) + 0 = 0$

Therefore $(L-I)^3(n^2) = (L-I)(L-I)^2(n^2) = (L-I)(0) = 0$

Thus (n^2) is in null $(L-I)^3$.

2. ~~Solution~~ $\cancel{\because t_n = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2}$

$\cancel{\therefore t = \left(\left(\frac{n(n+1)}{2}\right)^2\right) = \left(\frac{1}{4}n^2(n^2+2n+1)\right) = \left(\frac{1}{4}n^4 + \frac{1}{4}2n^3 + \frac{1}{4}n^2\right)}$

$\cancel{= \left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right)}$

2. Solution: $\because t_1 = \left(\frac{1(1+1)}{2}\right)^2 = 1^3$

Assume that $\left(\frac{n(n+1)}{2}\right)^2$ is the formula we want to get.

have to prove $n+1$ is true.

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$\therefore \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3 = \frac{n^2(n+1)^2}{4} + \frac{4(n+1)^3}{4} = \frac{(n+1)^2(n^2+4(n+1))}{4}$

$= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$. it's true.

$\therefore t_n = 1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

$\therefore t = \left(\left(\frac{n(n+1)}{2}\right)^2\right) = \left(\frac{1}{4}n^2(n^2+2n+1)\right) = \left(\frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2\right)$

Exercise 3. $|V| = \prod_{0\le j<i\le n} (c_i - c_j)$ Rui Qiu 999292509

Claim: ~~$|V| = \prod (c_j - c_i)$~~ $(*)$

Proof:

By induction, when $n=1$, we have

$\det V = \begin{vmatrix} 1 & c_0 \\ 1 & c_1 \end{vmatrix} = c_1 - c_0$, ~~$(*)$ is true~~

$$V = \begin{pmatrix} 1 & c_0 & c_0^2 & \cdots & c_0^n \\ 1 & c_1 & c_1^2 & \cdots & c_1^n \\ 1 & c_2 & c_2^2 & \cdots & c_2^n \\ \vdots & \vdots & \vdots & & \\ 1 & c_n & c_n^2 & \cdots & c_n^n \end{pmatrix}$$

We want to prove $(*)$ is true for any $n \in N^*$.

Let $n = k+1$

Let the $(k+2)$th column minus c_0 times $(k+1)$th column,

the $(k+1)$th column minus c_0 times k th column.

...

that is, each column minus c_0 times its left-side column from right to left.

So we can get:

$$|V| = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^{k+1} - c_0 c_1^k \\ 1 & c_2 - c_0 & c_2^2 - c_0 c_2 & \cdots & c_2^{k+1} - c_0 c_2^k \\ \vdots & \vdots & \vdots & & \\ 1 & c_{k+1} - c_0 & c_{k+1}^2 - c_0 c_{k+1} & \cdots & c_{k+1}^{k+1} - c_0 c_{k+1}^k \end{vmatrix} = \begin{vmatrix} c_1 - c_0 & c_1^2 - c_0 c_1 & \cdots & c_1^{k+1} - c_0 c_1^k \\ c_2 - c_0 & c_2^2 - c_0 c_2 & \cdots & c_2^{k+1} - c_0 c_2^k \\ \vdots & \vdots & & \vdots \\ c_{k+1} - c_0 & c_{k+1}^2 - c_0 c_{k+1} & \cdots & c_{k+1}^{k+1} - c_0 c_{k+1}^k \end{vmatrix}$$

$$= (c_1 - c_0)(c_2 - c_0) \cdots (c_{k+1} - c_0) \begin{vmatrix} 1 & c_1 & \cdots & c_1^k \\ 1 & c_2 & \cdots & c_2^k \\ \vdots & \vdots & & \vdots \\ 1 & c_{k+1} & \cdots & c_{k+1}^k \end{vmatrix}$$

Since we get a new $\underset{(k+1)\times(k+1)}{n \times n}$ matrix which is similar to previous V matrix

Notice that $(c_i^k - c_0^k) - c_0(c_i^{k-1} - c_0^{k-1}) = (c_i - c_0) c_i^{k-1}$. So we can extract all these factors, we get:

$$|V|_{k+1} = \prod_{i=1}^{k+1} (c_i - c_0) \begin{vmatrix} 1 & c_1 & \cdots & c_1^k \\ 1 & c_2 & \cdots & c_2^k \\ \vdots & \vdots & & \\ 1 & c_{k+1} & \cdots & c_{k+1}^k \end{vmatrix} = \prod_{i=1}^{k+1} (c_i - c_0) \cdot |V|_k$$

Therefore $|V|_n = \prod_{i=1}^{n} (c_i - c_0) \cdot \prod_{1\le j<i\le n} (c_i - c_j) = \prod_{0\le j<i\le n} (c_i - c_j)$

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