Tutorial 10 Solutions

STAT 3013/4027/8027

1. Consider a Poission regression model using the canonical link function (how do we determine the canonical link function?):

$$\begin{aligned} Y_1, \ldots, Y_n &\overset{\text{indep.}}{\sim} \text{Poisson}(\lambda_i) \\ \log(\lambda_i) &= \beta_0 + \beta_1 x_i + \beta_2 x_i^2 \\ &\text{for } i = 1, \ldots, n. \end{aligned}$$

- Data: A sample from a population of 52 female song sparrows was studied over the course of a summer and their reproductive activities were recorded. In particular, the age and number of new offspring were recorded for each sparrow (Arcese et al, 1992). Let $Y =$ fledged (number of offspring), and $X =$ age (age of mother).
- Based on the results from last week, test (through a frequetist approach):

$$H_0 : \beta_1 = 0 \text{ and } \beta_2 = 0 \quad vs. \quad H_1 : \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \ (H_0 \text{ is not true})$$

Ans. The likelihoods under the two models (constrained and unconstrained) are:

$$\begin{aligned} L(\beta_0|\boldsymbol{y}) &= \prod_{i=1}^{n} \left(\frac{e^{-\theta_i}\theta_i^{-y_i}}{y_i!} \right) \\ log(\theta_i) &= \beta_0 \end{aligned}$$

$$\begin{aligned} L(\beta_0, \beta_1, \beta_2, |\boldsymbol{y}) &= \prod_{i=1}^{n} \left(\frac{e^{-\theta_i}\theta_i^{-y_i}}{y_i!} \right) \\ log(\theta_i) &= \beta_0 + \beta_1 x_i + \beta_2 x_i^2 \end{aligned}$$

- Now the likelihood ratio statistic is:

$$\lambda = \frac{L(\hat{\beta}_0|\boldsymbol{y})}{L(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2, |\boldsymbol{y})}$$

- The rejection region asymptotically is:

$$R = \{-2log(\lambda) > \chi^2_{(3-1)=2}\}$$

```
D <- read.table("Data", header=TRUE)
summary(D)
```

```
##     fledged          age       
##  Min.   :0.000   Min.   :1.000  
##  1st Qu.:1.000   1st Qu.:2.000  
##  Median :2.000   Median :3.000  
##  Mean   :2.404   Mean   :3.077  
##  3rd Qu.:3.000   3rd Qu.:4.000  
##  Max.   :7.000   Max.   :6.000  
```

```r
n <- nrow(D)

y <- D[,1]
x <- D[,2]

##
log.lik.h0 <- function(theta){
  beta0 <- theta[1]

  out <- sum(dpois(y, exp(beta0), log=TRUE))
  return(out)
  }

##
theta.start <- 0
out.h0 <- optim(theta.start, log.lik.h0, hessian = TRUE,
                control = list(fnscale=-1), method="BFGS")


##
log.lik.h1 <- function(theta){

  beta0 <- theta[1]
  beta1 <- theta[2]
  beta2 <- theta[3]

  out <- sum(dpois(y, exp(beta0 + beta1*x + beta2*x^2), log=TRUE))
  return(out)
  }

##
theta.start <- c(0,0,0)
out.h1 <- optim(theta.start, log.lik.h1, hessian = TRUE,
                control = list(fnscale=-1), method="BFGS")


##
asym.lrt <- -2*log(exp(out.h0$value)/exp(out.h1$value))
asym.lrt
```

```
## [1] 8.24364
```

```r
qchisq(0.95, 2)
```

```
## [1] 5.991465
```

```r
##
asym.lrt > qchisq(0.95, 2)
```

```
## [1] TRUE
```

As `asym.lrt` > 5.991, we can reject the null hypothesis at the $\alpha = 0.05$ level.

2. SI 7.1, 7.3, 7.10, 7.15. **Ans.: See the handwitten solutions.**

GJJ Q 7.1

a.) $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$

We also have a prior, $p(\theta)$

$\Rightarrow$ The posterior is:

$$P(\theta \mid x_1, \ldots, x_n) = P(\theta \mid \underset{\sim}{x}) \propto P(\underset{\sim}{x} \mid \theta)\, p(\theta)$$

$$\propto \left[\prod_{i=1}^{n} \frac{e^{-\theta}\theta^{x_i}}{x_i!}\right] p(\theta)$$

$$\propto \frac{e^{-n\theta}\theta^{\sum x_i}}{\prod_{i=1}^{n} x_i!}\, p(\theta)$$

$$\propto e^{-n\theta}\theta^{\sum x_i}\, p(\theta)$$

b.) Suppose $Y \sim \text{poisson}(n\theta)$

$$p(\theta \mid y) \propto p(y \mid \theta)\, p(\theta)$$

$$\propto \frac{e^{-n\theta}\theta^{y}}{y!}\, p(\theta)$$

$$\propto e^{-n\theta}\theta^{y}\, p(\theta)$$

c.) If $Y = \Sigma X_i$

$\Rightarrow$ $P(\theta \mid \underset{\sim}{x}) = P(\theta \mid y)$

$\therefore$ By Def 7.1 $Y = \Sigma X_i$ is a sufficient statistic for θ.

GJJ Q 7.3

$$f(x|\theta) = \frac{\theta^2}{\theta+1}(x+1)\exp(-x\theta)$$

for $x>0$, $\theta>0$.

- To show this a pdf it should integrate to 1 when integrating by x.

$$\Rightarrow \int \frac{\theta^2}{\theta+1}(x+1)\exp(-x\theta)\,dx$$

$$= \frac{\theta^2}{\theta+1}\int (x+1)\exp(-x\theta)\,dx$$

$$= \frac{\theta^2}{\theta+1}\left[\int x\exp(-x\theta)\,dx + \int \exp(-x\theta)\,dx\right]$$

$$= \frac{\theta^2}{\theta+1}\left[\int \underbrace{x^{2-1}\exp(-x\theta)}_{\text{Kernel for a gamma distribution}}dx + \int \exp(-x\theta)\,dx\right]$$

$$= \frac{\theta^2}{\theta+1}\left[\frac{\Gamma(2)}{\theta^2}\underbrace{\int \frac{\theta^2}{\Gamma(2)}x^{1-1}\exp(-x\theta)\,dx}_{=1} + \int \exp(-x\theta)\,dx\right]$$

$$\Gamma(n) = (n-1)! \Rightarrow \Gamma(2) = \underline{1!} = \underline{1}$$

$$= \frac{\theta^2}{\theta+1}\left[1 + \underbrace{\int \exp(-x\theta)\,dx}_{=\frac{1}{\theta}}\right]$$

Note, we know the integrating constant of $\int c\exp(-x\theta) = 1 \Rightarrow c = \theta$

$$\therefore \int \theta \exp(-x\theta) = 1$$

$$\therefore \int \exp(-x\theta) = \frac{1}{\theta}.$$

$$) = \frac{\theta^2}{\theta+1}\left[\frac{1}{\theta^2} + \frac{1}{\theta}\right]$$

$$= \frac{\theta^2}{\theta+1}\left[\frac{1}{\theta^2} + \frac{\theta}{\theta^2}\right]$$

$$= \frac{\theta^2}{\theta+1}\left[\frac{1+\theta}{\theta^2}\right] = 1 \checkmark$$

$\therefore$ $f(x|\theta)$ is a pdf.

a.) $L(\theta) = \prod_{i=1}^{n} \frac{\theta^2}{\theta+1} (x_i+1) \exp(-x\theta)$

$$= \frac{\theta^{2n}}{(\theta+1)^n} \left[\prod_{i=1}^{n} (x_i+1)\right] \exp(-\theta \sum x_i)$$

- In Bayes land, when we ignore constants of proportionality we mean the $\underset{\sim}{x}$.

$$L(\theta) \propto \frac{\theta^{2n}}{(\theta+1)^n} \exp(-\theta \sum x_i)$$

$$\propto \left(\frac{\theta}{\theta+1}\right)^n \exp(-\theta \sum x_i)$$

b.) $L(\theta) = \underbrace{\left(\frac{\theta^2}{\theta+1}\right)^n \exp(-\theta \sum x_i)}_{K_1[t \mid \theta]} \underbrace{\left[\prod_{i=1}^{n} (x_i+1)\right]}_{K_2[\underset{\sim}{x}]}$

$T = \sum x_i$. From Q7.1, we know T is a sufficient statistic.

c.) We want a conjugate prior for θ. Based on the likelihood replace $n, \Sigma x_i$ with α_1, α_2:

$$P(\theta) \propto \left(\frac{\theta^2}{\theta+1}\right)^{\alpha_1} \exp\left(-\theta \alpha_2\right)$$

$\Rightarrow$ Check that this is conjugate:

$$P(\theta|y) \propto p(y|\theta)\, p(\theta)$$

$$\propto \left(\frac{\theta^2}{\theta+1}\right)^{n} \exp\left(-\theta \Sigma x_i\right) \left(\frac{\theta^2}{\theta+1}\right)^{\alpha_1} \exp\left(-\theta \alpha_2\right)$$

$$\propto \left(\frac{\theta^2}{\theta+1}\right)^{n+\alpha_1} \exp\left(-\theta\left(\Sigma x_i + \alpha_2\right)\right)$$

$\therefore$ posterior and prior have the same form.

- Note, we just need the integrating constant. Or consider approaches to sample from the posterior without the integrating constant (eg Metropolis - Hastings)

GJJ Q 7.10 $\quad X_1, \ldots, x_n \overset{iid}{\sim} \text{Poisson}(\theta)$

a.) $L(\theta) = \prod_{i=1}^{n} \frac{e^{-\theta} \theta^{x_i}}{x_i!} = \frac{e^{-n\theta} \theta^{\Sigma x_i}}{[\Pi x_i!]}$

$\propto \underbrace{e^{-n\theta} \theta^{\Sigma x_i}}$

In terms of θ, this is a kernel for a gamma distribution $\therefore$ the conjugate prior is a gamma distribution.

$\theta \sim \text{gamma}(\alpha, \beta)$

$$p(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\theta\beta)$$

$$E(\theta) = \frac{\alpha}{\beta} \; ; \; Var(\theta) = \frac{\alpha}{\beta^2}$$

We want $\frac{\alpha}{\beta} = 1$ & $\frac{\alpha}{\beta^2} = 1$

$\Rightarrow \alpha = \beta \Rightarrow \frac{\beta}{\beta^2} = 1 \Rightarrow \frac{1}{\beta} = 1$

$\therefore \alpha = \beta = 1$

$$\therefore \quad p(\theta) = \frac{1^{1}}{\Gamma(1)} \theta^{1-1} \exp(-\theta(1))$$

$$= \exp(-\theta)$$

b.) $p(\theta \mid \underset{\sim}{x}) \propto p(\underset{\sim}{x} \mid \theta)\, p(\theta)$

$$\propto \left[\theta^{\Sigma x_i} \exp(-\theta n) \right] \exp(-\theta)$$

$$\propto \theta^{\Sigma x_i} \exp(-\theta(n+1))$$

$$\propto \underbrace{\theta^{\Sigma x_i + 1 - 1} \exp(-\theta(n+1))}$$

Kernel for a gamma
(Recall we used a
conjugate prior so
this must be true!)

$$[\theta \mid x] \sim \text{gamma}(a = \Sigma x_i + 1,\ b = n+1)$$

c.) $2b[\theta \mid x] \sim \chi^2_{2a}$

$$P\left(\chi^2_{\alpha/2, 2a} \leq 2b[\theta \mid x] \leq \chi^2_{1-\alpha/2, 2a} \right) = 1 - \alpha$$

$$\Rightarrow P\left(\frac{\chi^2_{\alpha/2,\,2a}}{2b} \leq [\theta \mid x] \leq \frac{\chi^2_{1-\alpha/2,\,2a}}{2b}\right) 1-\alpha$$

GJJ Q 7.15

$$X_1, \ldots, X_n \overset{iid}{\sim} \exp(\theta) \; ; \; E(X) = 1/\theta$$

Test $H_0: \theta = 1$ vs $H_1: \theta \neq 1$

Note: $P(H_0) = p$; $P(H_1) = 1 - p$

$$P(H_0 \mid \underset{\sim}{x}) = \frac{P(\underset{\sim}{x} \mid H_0)\, P(H_0)}{P(\underset{\sim}{x})}$$

$$P(H_1 \mid \underset{\sim}{x}) = \frac{P(\underset{\sim}{x} \mid H_1)\, P(H_1)}{P(\underset{\sim}{x})}$$

$$Q^* = \underbrace{\frac{P(\underset{\sim}{x} \mid H_0)}{P(\underset{\sim}{x} \mid H_1)}}_{\text{Bayes Factor}} \times \frac{P(H_0)}{P(H_1)}$$

under $H_0: \theta = 1$

$= \exp(-\Sigma x_i)$

$$BF = \frac{\theta_0^n \exp(-\theta_0 \Sigma x_i)}{\int_{H_1} \theta^n \exp(-\theta \Sigma x_i)\, p(\theta \mid H_1)}$$

$A = \left\{ \int_{H_1} \theta^n \exp(-\theta\, \Sigma x_i)\, p(\theta \mid H_1) \right.$

$$[\theta \mid H_1] \sim \text{gamma}(\alpha, \beta)$$

$$E(\theta \mid H_1) = \alpha/\beta$$

$$A = \int_{H_1} \theta^{n} \exp(-\theta \Sigma x_i) \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} \exp(-\theta \beta) \, d\theta$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{H_1} \theta^{\overbrace{n+\alpha-1}^{a}} \exp\left(-\theta \left(\overbrace{\Sigma x_i + \beta}^{b}\right)\right) d\theta$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_{H_1} \frac{\Gamma(a)}{b^{a}} \frac{b^{a}}{\Gamma(a)} \theta^{a-1} \exp(-\theta b) \, d\theta$$

$$= \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{\Gamma(a)}{b^{a}} \underbrace{\int_{H_1} \frac{b^{a}}{\Gamma(a)} \theta^{a-1} \exp(-\theta b) \, d\theta}_{=1}$$

$$BF = \frac{\exp(-\Sigma x_i) \, \Gamma(\alpha) \, b^{a}}{\beta^{\alpha} \, \Gamma(a)}$$

$$Q^{*} = BF \frac{P(H_0)}{P(H_1)} = \frac{\exp(-\Sigma x_i) \, \Gamma(\alpha) \, b^{a}}{\beta^{\alpha} \, \Gamma(a)} \left[\frac{p}{1-p}\right]$$