Lecture 2 (continue §2.6)

Nested Interval Lemma

Suppose that $I_n=[a_n,b_n]$, $a_n<b_n$ and $I_{n+1}\subseteq I_n$, $\forall n\geq 1$.
Then $\bigcap_{n\geq 1} I_n\neq\emptyset$

e.g. $[0,1]=I_1$

$[0,\frac{1}{2}]=I_2$

$\vdots$

$[0,\frac{1}{k}]=I_k$

Proof: $a_n\leq a_{n+1}\leq b_{n+1}\leq b_n$
(a_n) is monotone increasing bounded from above, coverges to some point a.
$(a_n\leq a\leq b_n, \forall n)$
(b_n) is $\cdots$ decreasing, bounded from below, converges to some point b.
$(a_n\leq b\leq b_n, \forall n)$
so $a, b\in\bigcap_{n\geq 1} I_n$

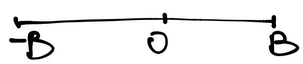
§ 2.7 Subsequences

Def: A subsequence $(a_n)_{n=1}^{\infty}$ is a sequence $(a_{n_k})_{k=1}^{\infty}$ where $n_1<n_2<\cdots$

$a_1, a_2, a_3, a_4, \cdots$

Bolzano–Weierstrass Thm:
Every bdd sequence of real numbers has a convergent subsequence.

Proof: Let (a_n) be bounded by B.
$[-B,B]$ contains $\{a_n : n\in\mathbb{N}\}=S$
Let's assume S is infinite.



$[-B,B]=[-B,0]\cup[0,B]$

Let I_1 be a subset that contains infinitely many pts of S
$I_1=T\cup K$
either T or K contains infinitely many pt of S, call it I_2.
$[-B,B]=I\supset I_1\supset I_2\supset\cdots\supset I_k\supset\cdots$
By the Nested Interval Lemma, $\bigcap_{k\geq 1} I_k\neq\emptyset$, contains $L\in\bigcap_{k\geq 1} I_k$.

Choosing increasing sequence n_k
$a_{n_k}\in I_k$, $\lim a_{n_k}=L$

$$|a_{n_k}-L|\leq |I_k|=\frac{2B}{2^k}\longrightarrow 0 \quad \text{as } k\to\infty$$

Thm: Every sequence has a monotone subsequence.
Proof: Let (x_n) be any set of real numbers.
We will construct a sequence that is either increasing or decreasing.
(i). every set $\{x_n : n>N\}$ has a max.

$x_{n_1}=\max_{n>1} x_n$

$x_{n_2}=\max_{n>n_1} x_n$

$x_{n_3}=\max_{n>n_2} x_n \quad \cdots \quad x_{n_k}=\max_{n>n_{k-1}} x_n$

So we obtain a decreasing subsequence

(ii). Suppose this is not true for some N_1 the set $\{x_n : n > N_1\}$ has no max
for any x_m with $m > N_1$, $\exists$ x_n following x_m s.t. $x_n > x_m$.
If this is not true $\Rightarrow$ the biggest $x_{N_1+1}, \cdots, x_m$ would be a max.

$$x_{n_1} = x_{N_1+1}$$

x_{n_2} $\cdots$ the first term after x_{n_1} s.t. $x_{n_2} > x_{n_1}$ etc.

we will have an increasing sequence

Proof of Bolzano-Weierstrass:

Every seq. has a monotone subsequence. But every bounded monotone subsequence converges.

Ex: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \cdots$

$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \cdots$

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \cdots$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \cdots$

§2.8 Cauchy sequences

Completeness:

Proposition: Let (a_n) be a sequence that converges to L. Then $\forall \varepsilon > 0$, $\exists N \in \mathbb{Z}_+$ s.t. $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

Pf: Let $\varepsilon > 0$ be given

$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Choose N s.t. $|a_n - L| < \frac{\varepsilon}{2}$, $\forall n \geq N$

Def: A sequence is called Cauchy if $\forall \varepsilon > 0$ $\exists N$ s.t. $|a_n - a_m| < \varepsilon$, $\forall n, m \geq N$.

Ex: $X = (0, 1)$

$(\frac{1}{n})$

Prop: Every Cauchy sequence is bdd.

Proof: Let (a_n) be Cauchy and let $\varepsilon = 1$.

$\exists$ N s.t. $|a_n - a_m| < 1$, $\forall n, m \geq N$, in particular $|a_n - a_N| < 1$, $\forall n \geq N$

Remark: $|c| - |d| \leq |c - d|$ $\Downarrow$

$|a_n| - |a_N| \leq |a_n - a_N| < 1$

$|a_n| < 1 + |a_N|$, $\forall n \geq N$

Let $B = \max\{|a_1|, |a_2|, \cdots, |a_{N-1}|, 1 + |a_N|\} \Rightarrow (a_n)$ is bounded

Def: A subset $S \subset \mathbb{R}$ is complete if every Cauchy sequence in S converges to a point in S.