March 5th

Spectral Thm

V: ips $T: V \longrightarrow W$ self-adjoint (i.e. $\langle T(v), w\rangle = \langle v \cdot T(w)\rangle$

Then V has an orthogonal basis of eigenvectors of T. in particular: T is diagonalizable.

Ex: $V = R^4$ with standard inner product.

$$T = \begin{pmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{pmatrix} \quad \text{self-adjoint}$$

eigenvalue of T $\lambda_1 = \frac{7+\sqrt{73}}{2}$, $\lambda_2 = \frac{7-\sqrt{73}}{2}$, $\lambda_3 = 0$

$E\lambda_1 = \text{span}\{v_1\}$ $E\lambda_2 = \text{span}\{v_2\}$ $E\lambda_3 = \text{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

aply GS to $E\lambda_3$ $V_3 = U_3$

$$V_4 = U_4 - \frac{\langle U_4, V_3\rangle}{\langle V_3, V_3\rangle} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix}$$

Ex: $\dim V = 6$ V ips

$T: V \longrightarrow W$ self adjoint

$\alpha = \{v_1, \cdots, v_6\}$ orthog basis of eigenvectors

v_1, v_2, v_3 have eigenvalue λ_1

v_4 v_5 - - - - - λ_2

v_6 —— λ_3

$$[T]_\alpha = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_1 & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \lambda_2 & \\ 0 & & & & & \lambda_3 \end{bmatrix}$$

$E\lambda_i = \lambda_i -$ eigenspace $P_i =$ orthogonal projection onto $E\lambda_i$

Recall $W \subset V$

$V = W \oplus W^\perp$

$P_W(V) = W$, $V = W + W'$

W' $\quad W^\perp$

$E\lambda_1 \oplus E\lambda_1^\perp = V$

$E_{\lambda_1}^\perp = E\lambda_2 \oplus E\lambda_3$

Claim: $E_{\lambda_1}^{\perp} = E_{\lambda_2} \oplus E_{\lambda_3}$

Pf: $E_{\lambda_2} + E_{\lambda_3} \subset E_{\lambda_1}^{\perp}$

$\dim E_{\lambda_1} = 3 \quad \dim E_{\lambda_1}^{\perp} = \dim V - \dim E_{\lambda_1}$
$= 3$

$\dim E_{\lambda_2} \oplus E_{\lambda_3} = 3 \Rightarrow E_{\lambda_2} \oplus E_{\lambda_3} = E_{\lambda_1}$

$[P_1]_\alpha = p_1(v_1) = v_1$
$p_2(v_2) = v_2$
$P_3(v_3) = v_3$
$P_1(v_4) = 0$
$P_1(v_5) = P_1(v_6) = 0$

$$[P_1]_\alpha = \begin{bmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & 0 & \\ 0 & & & & & 0 \end{bmatrix}$$

$$\Rightarrow [T]_\alpha = \begin{bmatrix} \lambda_1 & & & & & 0 \\ & \lambda_1 & & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & & \\ & & & & \lambda_2 & \\ 0 & & & & & \lambda_3 \end{bmatrix} = \lambda_1 [P_1]_\alpha + \lambda_2 [P_2]_\alpha + \lambda_3 [P_3]_\alpha$$

Spectral Theorem: decomposition: V i ips. $T: V \to V$ self-adjoint: let $\lambda_1, \cdots \lambda_k$ distinct eigenvalues & P = orthogonal projection onto E_{λ_i}

Then $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_k P_k$

Ex: Compute special decomposition of $T: \mathbb{R}^3 \to \mathbb{R}^3$

$$T = \begin{matrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{matrix}$$

evalues are $\lambda_1 = -3 \quad \lambda_2 = 3$

$E_{\lambda_1} = sp\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$, $E_{\lambda_2} = sp\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Apply GS to E_{λ_1} result in $E_{\lambda_1} = sp\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1 \\ -1/2 \end{bmatrix} \right\}$

$\alpha = \{v_1, v_2, v_3\}$ orthog basis of evectors

$[T]_\alpha = \begin{bmatrix} -3 & & \\ & -3 & \\ & & -3 \end{bmatrix}$ P_i = orthog proj. onto $[P_{\lambda_1}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$[P_{\lambda_2}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

β: standard basis of $\mathbb{R}^3$

$$[P_1]_\beta = [I]^\beta_\alpha [P_i]_\alpha [I]^\alpha_\beta = \begin{pmatrix} -1 & -1/2 & 1 \\ 0 & 1 & 1 \\ 1 & -1/2 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} - \frac{1}{6}\begin{pmatrix} -3 & 0 & 3 \\ -2 & 4 & -2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Conic Section:

$$Ax^2+2Bxy+Cy^2=1 \qquad Ax^2+2Bxy+Cy^2=[x \ y]\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

spectral thm $\Rightarrow$ there is an orthonormal basis $\{v_1, v_2\}$ of $\mathbb{R}^2$ s.t. $T(v_1)=\lambda_1 v_1$, $T(v_2)=\lambda_2 v_2$

the conic section has formula:

$$[x' \ y']\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}=1$$

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 = 1$$

From columns we know that if λ_1 & λ_2 have same sign this is ellipse λ_1 & λ_2 have different signs then its hyperbola.

Recall: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ This gives a hyperbola. "vertices" $(\pm a, 0)$

asymptotes $= y = \pm(b/a)\cdot x$

$$x^2+4xy+y^2=1 \qquad [x \ y]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=1$$

$$\lambda_1=3,\ \lambda_2=-1 \qquad v_1=\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2=\frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\alpha = \{v_1, v_2\}$$

In α-coords the conic section takes the form

$$[x' \ y']\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}=1$$

i.e. $3(x')^2-(y')^2=1$

standard inner product $\mathbb{C}^n$

$$\left\langle \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$$

Def: A Hermition inner product on a complex vector space V is $\langle \ , \ \rangle : V\times V \to \mathbb{C}$ s.t. (1) $\langle v_1+v_2, v_3\rangle = \langle v_1, v_3\rangle + \langle v_2, v_3\rangle$

(2). $\langle v, w\rangle = \overline{\langle w, v\rangle}$

(3). $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v=0$

Remark: ① $\langle v_1, z v_2 + v_3\rangle = \overline{\langle z v_2 + v_3, v_1\rangle} = \overline{z\langle v_2, v_1\rangle + \langle v_3, v_1\rangle}$
$= \bar{z}\cdot\overline{\langle v_2, v_1\rangle} + \overline{\langle v_3, v_1\rangle}$
$= \bar{z}\langle v_1, v_2\rangle + \langle v_1, v_3\rangle$

② $\langle v, v\rangle \in \mathbb{R}$ why? $\langle v, v\rangle = \overline{\langle v, v\rangle}$

Def: V $\mathbb{C}$-vector space with Hermitian inner product $T: V \to V$

The adjoint of T is the operator $T^*: V \to V$
satisfy $\langle T(v), w\rangle = \langle v, T^*(w)\rangle$

$\alpha = \{v_1, \cdots, v_n\}$ orthonormal basis of V

$[T]_\alpha = [a_{ij}]$

$T(v_j) = a_{1j}v_1 + a_{2j}v_2 + \cdots + a_{nj}v_n$

$\langle T(v_j), v_i\rangle = \langle \sum_{k=1}^{n} a_{kj}v_k, v_i\rangle = \sum_{k=1}^{n} a_{kj}\langle v_k, v_i\rangle = a_{ij}$

Question: What's $[T^*]_\alpha$?
$[T^*]_\alpha = [b_{ij}]$
$b_{ij} = \langle T^*(v_j), v_i\rangle$
$= \overline{\langle v_i, T^*(v_j)\rangle}$
$= \overline{\langle T(v_i), v_j\rangle}$
$= \overline{a_{ji}}$

So this says that:

$[T^*]_\alpha = [T]_\alpha^*$ ← Conjugate-transpose

Q: How much can spectral theorem be generalized?

Def: $T: V \to V$ normal if $TT^* = T^*T$ if T self-adjoint T is normal since $T = T^*$

Ex: $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ normal but not self-adjoint. this is normal since

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

Claim: T normal then eigenvectors corresponding to distinct eigenvalues are orthogonal

Lemma ①: $(T^*)^* = T$

Proof: $\langle T^*(v), w\rangle = \langle v, (T^*)^*(w)\rangle$

want: $\langle T^*(v), w\rangle = \langle v, T(w)\rangle$
$\langle T^*(v), w\rangle = \overline{\langle w, T^*(v)\rangle} = \langle v, T(w)\rangle$

Lemma ② T normal

$\operatorname{Ker} T = \operatorname{Ker} T^*$

Proof: $v \in \operatorname{Ker} T \Rightarrow T(v) = 0$
$\Rightarrow \langle T(v), T(v) \rangle = 0$
$\Rightarrow \langle v, T^* T(v) \rangle = 0$
$\Rightarrow \langle v, TT^*(v) \rangle = 0$
$\Rightarrow \overline{\langle TT^*(v), v \rangle} = 0$
$\Rightarrow \langle T^*(v), T^*(v) \rangle = 0$
$\Rightarrow T^*(v) = 0$
$\Rightarrow v \in \operatorname{Ker} T^*$ ∎

Lemma ③: T normal, $\lambda \in \mathbb{C}$ then $T - \lambda I$ also normal
Lemma ④: T normal $Tv = \lambda v$ then $T^* v = \bar{\lambda} v$

Proof: $\operatorname{Ker}(T - \lambda I) = \operatorname{Ker}((T - \lambda I)^*)$

$(T - \lambda I)^* = T^* - \bar{\lambda} I$

so if $v \in \operatorname{Ker}(T - \lambda I)$ then $v \in \operatorname{Ker}(T^* - \bar{\lambda} I)$ ∎

Proof: Claim:
$T(v) = \lambda v$, $T(w) = \mu w$, $\lambda \neq \mu$

want $\langle v, w \rangle = 0$

$\mu \langle v, w \rangle = \langle v, \bar{\mu} w \rangle$
$= \langle v, T^*(w) \rangle$
$= \langle T(v), w \rangle$
$= \langle \lambda v, w \rangle$
$= \lambda \langle v, w \rangle$
$= (\lambda - \mu) \langle v, w \rangle = 0 \Rightarrow \langle v, w \rangle = 0$