$= e^{1} - e$

§2.4.

1. order of $\frac{\sin z}{z}$.

$f' = \frac{\cos z \cdot z - \sin z}{z^2}$ $\qquad f'(0) \neq 0$

f order is 1 at $z = n\pi$.

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#2. $(e^z-1)^2 = 0 \Rightarrow e^z = 1 \Rightarrow z = 0$

$f' = 2(e^z-1)e^z \Rightarrow$ ~~e~~ ..

~~at z = 0 order = ∞~~

$f'' = 2e^z \cdot e^z + 2(e^z-1)e^z \neq 0$

so $z=0$ order 2.

#9. $z(e^z-1)$ about $z_0 = 0$

$$z \cdot \sum_{n=1}^{\infty} \frac{z^n}{n!} = \sum_{n=1}^{\infty} \frac{z^{n+1}}{n!}$$ valid everywhere

Thm. Suppose that f is analytic in D. z_0 is a point of D, If the disc $\{z : |z-z_0| < R\}$ lies in D. then f has p.s.

$$f(z) = \sum_{k=0}^{\infty} a_k (z-z_0)^k$$ valid

in such disc.

& a_k is given by Cauchy formula.

$$a_k = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{(\xi - z_0)^{k+1}} d\xi$$

#10.

e^z about $z_0 = \pi i$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Thm 1 $\Rightarrow$ Sps f is analytic on domain D,
& at some pt. $z_0 \in D$, $f^{(k)}(z_0) = 0$, $k = 0, 1, \cdots$
Then $f(z) = 0$ $\forall z \in D$.

$f(z) = e^z$ analytic in D, $z_0 = \pi i \in D$.

$$e^z = e^{z_0} e^{z - z_0}$$

$$= e^{\pi i} e^{z - \pi i} = e^{\pi i} \sum \frac{(z - \pi i)^n}{n!}$$

$$= -\sum \frac{(z - \pi i)^n}{n!}$$

$R = 1$.

#11. z^3+6z^2-4z-3 about $z_0=1$

analytic. $|z-1|<R$

$(z-1)^3+9(z-1)^2+(z-1)^3$

$\frac{1}{R}=\frac{1}{9}$ or 9

$\Rightarrow R=9$ or $\frac{1}{9}$

#12. $\frac{z^2}{1-z} = z^2\cdot\frac{1}{1-z} = z^2\sum_{n=0}^{\infty} z^n = \sum_{n=2}^{\infty} z^n$

radius is 1.

#13. $\frac{z+2}{z+3}$ about $z_0=-1$

$f(z)=\frac{(z+1)+1}{(z+1)+2} = (z+1+1)\frac{1}{(z+1)+2}$

$=(z+1+1)\frac{1}{2+(z+1)}$

$=\frac{1}{2}(z+1+1)\frac{1}{1+\frac{z+1}{2}}$

$\boxed{\psi = -\frac{z+1}{2} = \frac{-z-1}{2}}$ $|z+1|<2$

$R=2$

$=\frac{1}{2}(z+2)\sum\left(\frac{-z-1}{2}\right)^n$

cvg if $\left|\frac{-z-1}{2}\right|<1 \Rightarrow |z-1|<2$

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§ 2.4.

#5 & #7

#5. $z^2(1-\cos z) = z^2 - z^2\cos z$

~~$f' = 2z(1-\cos z) + z^2\sin z$~~

~~$= 2z - 2z\cos z + z^2\sin z$~~

~~$f'(0) = 0$~~

~~order is 1.~~

$f' = 2z - (2z\cos z - z^2\sin z)$

$= 2z - 2z\cos z + z^2\sin z$

~~#7. $e^{2z} - 3e^z + 4$~~

~~$f' = 2e^{2z} - 3e^z$~~

~~$f'(0) = 2 - 3 = -1$~~

~~$f'' = 4e^{2z} - 3e^z$~~

~~$f''(0) \neq 0$~~

if $z=0$,

$f'(0)=0$

$f''(0) = 2 - 2\cos z + 2z\sin z + 2z\sin z + z^2\cos z$

$= 0$

$f'''(0) = 2\sin z + 2\sin z + 2z\cos z + 2\sin z + 2z\cos z$

$+ 2z\cos z + z^2(-\sin z)$

$= 0$

$f''''(0) = (4\sin z + 2\sin z + 4z\cos z + 2z\cos z - z^2\sin z)'$

$= (6\sin z + 6z\cos z - z^2\sin z)'$

$= 6\cos z + 6\cos z + 6z(-\sin z)$

$-(2z\sin z + z^2\cos z)$

$= 12\cos z - 6z\sin z - 2z\sin z - z^2\cos z$

$= 12\cos z - 8z\sin z - z^2\cos z$

$= 12 \neq 0$ ✓

order is 4.

if $z = 2\pi n$, $n = \pm 1, \pm 2, \pm 3$,

order is 2.

~~proof~~ detail skipped.

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#7. $e^{2z}-3e^{z}-4$

$f' = 2e^{2z}-3e^{z}$

when ~~$z=2\pi$~~ ~~$2\pi ni$~~ $2z = 2\pi ni$, $z = 2\pi ni$ $\Rightarrow$ $z=(2n+1)\pi i$, $n=0,\pm1,\pm2,\cdots$

i.e. $z=$ 奇数个 πi.

then $f'(\omega)=0$.

also. ~~$e^{z}(e^{z}-3)=4$~~ ~~$\log(2e^{2z}-3e^{z})=\log 4+i\arg$~~

$z=\log a$

~~$2a^2-3a=\log 4$~~

~~$2z-3z=\log 4$~~

$z=\log 4+2m\pi i$, $m\in\mathbb{Z}$.

$2e^{2z}=2\times(4\cdot4)=16$

$3e^{z}=12$

$16-12-4=0$. ✓.

§2.5 #1, #3, #5 (before quiz 2)

#1. $\frac{e^{z}-1}{z}$ ($z_0=0$)

$$\frac{e^{z}-1}{z}=\left(\sum_{k=0}^{\infty}\frac{z^k}{k!}-1\right)\frac{1}{z}=\sum_{k=1}^{\infty}\frac{z^{k-1}}{k!}$$

Test by Ratio Test, the series converges absolutely.

so. $|f(z)|$ bounded as $z\to z_0$.

so it's a removable singularity.

#3. ~~$\frac{z^2}{\sin z}$~~ as $z\to0$, $\lim=1$. by L'Hopital.

$$\frac{z^4-2z^2+1}{(z-1)^2}=\frac{(z^2-1)^2}{(z-1)^2}=\frac{(z+1)(z-1)}{(z-1)(z-1)}\qquad z_0=1.$$

removable.

$\lim=4$ by L'Hopital

~~$\lim_{z\to z_0} f(z)=\left|\frac{z+1}{z-1}\right|\left|\frac{z+1+2}{z-1}\right|\left|\frac{2}{z-1}\right|$~~

$\left(\sum_{k=0}^{\infty} z^k\right)-(1+z)$ converge. when $|z-z_0|<1$

$|f(z)|$ bdd. b/c P.S. bdd & $-(1+z)$ bdd. as $z\to z_0$

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#5. $\frac{2z+1}{z+2}$ $\lim_{z \to z_0} |f(z)| = \lim \left|\frac{2z+4-3}{z+2}\right| = \lim \left|2 - \frac{3}{z+2}\right| = \infty$

pole

$$\boxed{\begin{aligned} &z_0 = -2. \\ &g(z) = \frac{1}{f(z)} = \frac{z+2}{2z+1} = (z+2)^m h(z) \\ &\quad \text{where } z_0 = -2, \\ &\quad m = 1, \\ &\quad h(z) = \frac{1}{2z+1} \end{aligned}}$$

$\therefore$ $f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^m} \frac{1}{h(z)} = \frac{H(z)}{(z-z_0)^m}$

$$H(z) = \frac{1}{h(z)} = 2z+1$$

$$z+2 = z - z_0$$
$$m = 1$$

✓