Feb. 26th

Gram–Schmidt orthogonalization

V inner product space, $\langle\ ,\ \rangle$

Given $\{u_1, \cdots, u_k\}$ lin. ind.

Want $\{v_1, \cdots, v_k\}$ s.t.

① $\{v_1, \cdots, v_k\}$ is orthogonal

② $\operatorname{span}\{v_1, \cdots, v_k\} = \operatorname{span}\{u_1, \cdots, u_k\}$

To construct $\{v_1, \cdots, v_k\}$

$v_1 = u_1$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle u_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$\vdots$

$$v_k = u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i$$

Claim: ① $\{v_1, \cdots, v_k\}$ is orthog

② $\operatorname{sp}\{v_1, \cdots, v_k\} = \operatorname{sp}\{u_1, \cdots, u_k\}$

Proof: Prove ① by induction on k

base case: $k=1$ nothing to check

inductive hypothesis: Assume G–S process works for a set of $k-1$ vectors

inductive step: we want to prove that if given $\{u_1, \cdots, u_k\}$ then the G–S process works.

By ind. hyp. $\{v_1, \cdots, v_{k-1}\}$ is orthog. Need to show $\langle v_k, v_j\rangle = 0$ for $j = 1, \cdots, k-1$

$$\langle v_k, v_j\rangle = \left\langle u_k - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i ,\ v_j \right\rangle$$

$$= \langle u_k, v_j\rangle - \sum_{i=1}^{k-1} \left\langle \frac{\langle v_k, v_i\rangle}{\langle v_i, v_i\rangle} v_i ,\ v_j \right\rangle$$

$$= \langle u_k, v_j\rangle - \sum_{i=1}^{k-1} \frac{\langle u_k, v_i\rangle}{\langle v_i, v_i\rangle} \langle v_i, v_j\rangle$$

$$= \langle u_k, v_j\rangle - \frac{\langle u_k, v_j\rangle}{\langle v_i, v_j\rangle} \langle v_i, v_j\rangle$$

$$= 0$$

Part ② $\operatorname{sp}\{v_1, \cdots, v_k\} \subseteq \operatorname{sp}\{u_1, \cdots, u_k\}$

We have equality since $\{v_1, \cdots, v_k\}$ ind. b/c they're orthog

(So $\dim \operatorname{sp}\{v_1, \cdots, v_k\} = k$)

Note: to get an orthogonal set of vectors from $\{u_1, \cdots, u_k\}$

① Use GS-process to produce $\{v_1, \cdots, v_k\}$

② $\left\{ \frac{v_1}{\|v_1\|}, \cdots \frac{v_k}{\|v_k\|} \right\}$ $\|v\| = \sqrt{\langle v, v \rangle}$

Reminder on $\oplus$ DIRECT SUM

Def: $W, U \subseteq V$.

$W \oplus U = V$ if

(1) $W \cap U = \{0\}$

$W + U = V$

$\{w+u : w \in W, u \in U\}$

Q1:

$sp\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\} \oplus sp\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\} = \mathbb{R}^3$

$\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}a\\0\\0\end{bmatrix} + \begin{bmatrix}0\\b\\c\end{bmatrix}$ Yes

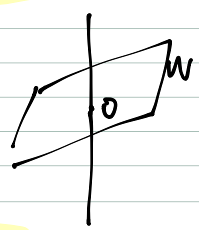
Q2: $sp\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}\right\} \oplus span\left\{\begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = \mathbb{R}^3$?

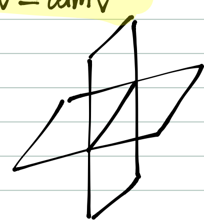
NO

Q3: $V = M_2(\mathbb{C})$, W = upperΔ, U = lowerΔ

$W \oplus U = V$? No, since any diagonal matrix is in $W \cap U$.

Thm: $U \oplus W = V \Rightarrow \dim U + \dim W = \dim V$





Thm: Vips

$W \subseteq V$

$W^{\perp} = \{v \in V : \langle v, w \rangle = 0$ for every $w \in W\}$

$W \oplus W^{\perp} = V$

Proof: Let $v \in W \cap W^{\perp}$. We will show $v = 0$

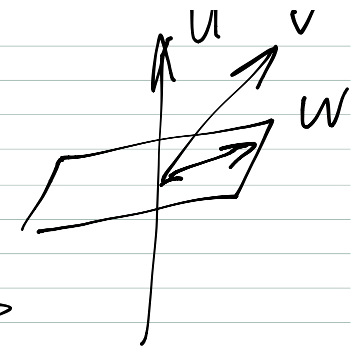
$\langle v, v \rangle = 0 \Rightarrow v = 0$. Thus $W \cap W^{\perp} = \{0\}$

Now we want to show $W + W^{\perp} = V$

Let $v \in V$. Let $\{u_1, \cdots, u_k\}$ be a basis of W.

Apply GS-process to get $\{v_1, \cdots, v_k\}$ an orthogonal basis.

Let $w = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \cdots + \frac{\langle v, v_k\rangle}{\langle v_k, v_k\rangle} v_k$



Note $w \in W$. Also, $v - w \in W^\perp$

Indeed $\langle v-w, v_i\rangle = \langle v, v_i\rangle - \langle w, v_i\rangle$

$= \langle v, v_i\rangle - \langle v, v_i\rangle$

$= 0$

$$\langle w, v_i\rangle = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle}\langle v_1, v_i\rangle + \cdots + \frac{\langle v, v_k\rangle}{\langle v_k, v_k\rangle}\langle v_k, v_i\rangle$$

$$= \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle}\langle v_i, v_i\rangle$$

$$= \langle v, v_i\rangle$$

$V = W \oplus U \Leftrightarrow$ every $v \in V$ can be written uniquely as $v = w + u$, $w \in W, u \in U$

V ips.

$W \subseteq V$

$P_W : V \to V$ "projection onto W" is given by $P_W(v) = w$, where $v = w + w'$

$w \in W$

$w' \in W^\perp$

$V = W \oplus W^\perp$

$v \in V$ $\quad w = \frac{\langle v, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \cdots + \frac{\langle v, v_k\rangle}{\langle v_k, v_k\rangle} v_k$

$$v = \underset{W}{\underset{\cap}{w}} + \underset{W^\perp}{\underset{\cap}{(v-w)}}$$

$P_W(v) = \sum_{i=1}^{k} \frac{\langle v, v_i\rangle}{\langle v_i, v_i\rangle} v_i$; $\{v_1, \cdots, v_k\}$ orth basis of W

Ex: $V = P_2(\mathbb{R})$

$\langle f, g\rangle = \int_{-1}^{1} fg \, dt$

$W = \text{sp}\{\underset{u_1}{1}, \underset{u_2}{1+t}\}$ $\quad p = at^2 + bt + c$

$P_W(p)$?

① Find orthog basis of W

$v_1 = u_1 = 1$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = 1 + t - \frac{2}{2}\cdot 1 = t$

$\langle v_2, v_1\rangle = \int_{-1}^{1} 1+t \, dt = 2$

$\langle v_1, v_1\rangle = \int_{-1}^{1} dt = 2$

② $P_W(p) = \frac{\langle p, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle p, v_2\rangle}{\langle v_2, v_2\rangle} v_2$

$$= \left(\frac{1}{2}\int_{-1}^{1} p(t)dt\right) v_1 + \left(\frac{\int_{-1}^{1} tp(t)dt}{\int_{-1}^{1} t^2 dt}\right) v_2$$

$$= \left(\frac{2a}{3} + 2c\right) + \left(\frac{2b}{3}\right) t$$

① $P_W(w)=w$; $P_W^2=P_W$; Is P_W diagonalizable? Yes
$v\in W^\perp$ $P_W(v)=0$

$P_W^2=P_W \Rightarrow$ eigenvalues of P_W are $0,1$

$E_1=W$, $E_0=W^\perp$

Since $W\oplus W^\perp=V \Rightarrow \dim W+\dim W^\perp=\dim V.$

$\Rightarrow \dim E_1+\dim E_0=\dim V$

$\Rightarrow P_W$ diagonalizable

$\{v_1,\cdots,v_k\}$ basis of W
$\{v_{k+1},\cdots,v_n\}$ basis of $W^\perp$
$\alpha=\{v_1,\cdots,v_n\}$ basis of V
$[P_W]_\alpha=\cdots$

$$[P_W]_\alpha=\begin{bmatrix} 1 & \cdots & & 0 \\ \vdots & 1 & & \vdots \\ & & 1 & \\ & & & \ddots & \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}$$

Standard inner product on $\mathbb{R}^n$: $\langle v,w\rangle=v^Tw$
— II — $\mathbb{C}^n$: $\langle v,w\rangle=v^*w$

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}^*=[\,\bar{z}_1,\cdots \bar{z}_n\,] \quad ; \overline{x+iy}=x-iy$$

<u>observe</u> $A\in M_n(\mathbb{R})$, A symmetric $\Leftrightarrow \langle Av,w\rangle=\langle v,Aw\rangle$

<u>Why?</u> A symm $\Rightarrow A=A^T$

$\langle Av,w\rangle=(Av)^Tw=v^TA^Tw=\langle v,A^Tw\rangle=\langle v,Aw\rangle$

<u>Def:</u> V ips $T:V\to V$ is symmetric or self-adjoint if for any $v,w\in V$
$\langle Tv,w\rangle=\langle v,Tw\rangle$

<u>Ex</u>: What does it mean for $A\in M_n(\mathbb{C})$ to be self-adjoint? In order for A to be self-adjoint
$\langle Av,w\rangle\overset{?}{=}\langle v,Aw\rangle \Leftrightarrow (Av)^*w=v^*Aw$
$\Leftrightarrow v^*A^*w=v^*Aw$

where $A^*=\bar{A}^T$
$\Leftrightarrow A^*=A$

<u>Ex:</u> $\begin{bmatrix} 1 & 2+i \\ 2-i & 2 \end{bmatrix}$ is self-adjoint

self-adjoint for $M_n(\mathbb{C})$ is also called Hermitian

Q: V ips, $W \subseteq V$.
is P_W self-adjoint?

i.e. is it true that for all $v_1, v_2 \in V$
$\langle P_W(v_1), v_2 \rangle = \langle v_1, P_W(v_2) \rangle$?
$v_1 = w_1 + w_1' : w_1 \in W, w_1' \in W^\perp$
$v_2 = w_2 + w_2' :$. - -

$\langle P_W(v_1), v_2 \rangle = \langle w_1, v_2 \rangle = \langle w_1, w_2 + w_2' \rangle = \langle w_1, w_2 \rangle + \langle w_1, w_2' \rangle$
$= \langle w_1, w_2 \rangle$

$\langle v_1, P_W(v_2) \rangle = \langle w_1 + w_1', w_2 \rangle$
$= \langle w_1, w_2 \rangle + \langle w_1', w_2 \rangle = \langle w_1, w_2 \rangle$

so P_W self-adjoint $\quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Thm: V ips. $T: V \to V$ self-adjoint
Then eigenvalues of T are real.

Proof: λ eigenvalue, Write $\lambda = x + iy$:
$T - \lambda I$ is not invertible
so $(T - \bar{\lambda} I)(T - \lambda I)$ also not invertible

$$(T - \bar{\lambda} I)(T - \lambda I) = T^2 - (\lambda + \bar{\lambda})T + \lambda\bar{\lambda} I$$
$$= T^2 - 2xT + (x^2 + y^2) I$$
$$= (T^2 - 2xT + x^2 I) + y^2 I$$
$$= (T - xI)^2 + y^2 I$$

Take $v \neq 0$ s.t. $((T - xI)^2 + y^2 I)(v) = 0$
$0 = \langle (T - xI)^2(v) + y^2 v, v \rangle$
$= \langle (T - xI)^2 v, v \rangle + y^2 \langle v, v \rangle$
since T self-adjoint so is $T - xI$ so:
$= \underbrace{\langle (T - xI)v, (T - xI)v \rangle}_{\geq 0} + \underbrace{y^2 \langle v, v \rangle}_{\geq 0}$

$\Rightarrow (T - xI)v = 0$ and $y^2 \langle v, v \rangle = 0$

Since $v \neq 0 \Rightarrow y^2 = 0 \Rightarrow y = 0$

Therefore $\lambda = x \in \mathbb{R}$ $\blacksquare$

Thm 2: V ips $T: V \to V$ self-adjoint
If x_1 an eigenvector of T with eigenvalue λ_1
x_2 —— λ_2

and $\lambda_1 \neq \lambda_2$, then x_1 and x_2 orthog

Proof: $\langle Tx_1, x_2 \rangle = \langle \lambda_1 x_1, x_2 \rangle = \lambda_1 \langle x_1, x_2 \rangle$

$\quad\quad \shortparallel$

$\langle x_1, Tx_2 \rangle = \langle x_1, \lambda_2 x_2 \rangle = \lambda_2 \langle x_1, x_2 \rangle$

since $\lambda_1 \neq \lambda_2$, $\Rightarrow \langle x_1, x_2 \rangle = 0$ ∎