Lecture 2

Example: Suppose we toss a coin many times (may not be indpdt)

Suppose that $X_n \to p$ as $n \to \infty$. Here X_n is the proportion of heads in the first n tosses. $p \in (0, 1)$

Prove that $Var(X_n) \to 0$

Scenario of non-independent coin tosses

Rule: If we have a head, then next time we will toss a coin w. prob of landing heads $\frac{1}{4}$. If $\cdots$ tail $\cdots$ $\frac{3}{4}$. (it's like a balancing game)

Proof: Since $X_n \to p$

Therefore $X_n^2 \to p^2$

⊛ Now note that $|X_n| \leq 1$ $E(1) = 1 < \infty$

Here by the dominated convergence Thm

$E(X_n) \to p$.

$X_n^2 \to p^2$ & $|X_n^2| \leq 1$, $E(1) = 1 < \infty$

Here by DCT

$E(X_n^2) \to p^2$

Therefore $Var(X_n) = E(X_n^2) - (E(X_n))^2 = p^2 - p^2 = 0$

Motivations of Expectation

⊛ Discrete Ω

Example: Suppose Ω is composed of N individuals. For the individuals in Ω there

n_1 $\cdots$ ω_1's

n_2 $\cdots$ ω_2's

$\vdots$

n_k $\cdots$ ω_k's

Suppose $X : \Omega \to \mathbb{R}$ is a r.v.

& each individual is equally likely to be choosen in an experiment.

$$E[X] = \frac{1}{N} \sum_{i=1}^{k} n_i X(\omega_i) = \sum_{i=1}^{k} P_i X(\omega_i) \text{ where } P_i = \frac{n_i}{N} \quad \checkmark\checkmark$$

Thm: Sps Ω is discrete. Then individuals are confined to a set $\{\omega_1, \omega_2, \cdots \omega_k\}$ iff the expectation operator takes the form

$$E[X] = \sum_{i=1}^{k} P_i X(\omega_i) \text{ with } P_i \geq 0 \text{ \& } \sum_{i=1}^{k} P_i = 1$$

Proof: "$\Rightarrow$"

Note that

$$\checkmark \quad X(\omega) = \sum_{i=1}^{k} I\{\omega = \omega_i\} X(\omega_i)$$

Reason: If $\omega = \omega_j$ for some $j \in \{1, 2, \cdots, k\}$. Then LHS $= X(\omega_j)$ and RHS $= X(\omega_j)$

Take expectation of ✓

LHS $= E(X)$

RHS $= \sum_{i=1}^{k} E[\, I\{\omega=\omega_i\}\, X(\omega_i) = \sum_{i=1}^{k} \underbrace{P[\omega=\omega_i]}_{P_i} X(\omega_i)$

Then $P_i \geq 0$ & $\sum_{i=1}^{k} P_i = 1$

"$\Leftarrow$" Let $X(\omega) = I\{\omega=\omega_i\}$

LHS of ✓✓ $= P[\omega=\omega_i]$

RHS of ✓✓ note $X(\omega_i) \leftarrow$ is fixed

$$X(\omega_i) = I\{\omega_1=\omega_i\} = \begin{cases} 1 & \text{if } i=1 \\ 0 & \text{if } i\neq 1 \end{cases}$$

$\Rightarrow$ RHS $= P_i$

Therefore $\boxed{P[\omega=\omega_i] = P_i}$

$P[\omega=\omega_i]$ is a r.v. (random); P_i is a fixed value

because $\sum_{i=1}^{k} P_i = 1$

$\Rightarrow \sum_{i=1}^{k} P(\omega=\omega_i) = 1$

$\Rightarrow$ conclusion holds

Example: If we throw 2 dies. Let X be the sum of the two dies

Find $E(X)$

Task. find P_i's.

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

$X(\omega_1) = 2$

$X(\omega_2) = 3$

$X(\omega_3) = 4$

$\vdots$

$X(\omega_{36}) = 12$

$$E[X] = \sum_{i=1}^{36} P_i X(\omega_i) = \frac{1}{36}\sum_{i=1}^{36} X(\omega_i) = \frac{1}{36}(2+2\times 3+3\times 4 + \cdots) = 7$$

Continuous R.V's

Suppose Ω is the real line

$X: \Omega \to \mathbb{R}$

Define: $E[X] = \int_{-\infty}^{\infty} X(\omega) f(\omega)\, d\omega$ (∞)

Here $f(\omega) \geq 0$ $\quad \int_{-\infty}^{\infty} f(\omega)\, d\omega = 1$

Because of Axiom 1

$E[X] \geq 0$ for any X. If $f(\omega) < 0$ for some ω. We can easily design a non-negative r.v. X s.t. $E[X] < 0$.

Implications:
Let $X = I\{\omega \in A\}$ where A is a subset of $\mathbb{R}$, plug into ⓞⓞ
$LHS = E\,I\{\omega \in A\} = P\{\omega \in A\} = P(A)$

$$RHS = \int_{-\infty}^{\infty} I\{\omega \in A\} f(\omega)\,d\omega = \int_A f(\omega)\,d\omega$$

2.5 MOMENTS

For any $j \geq 0$ the jth moment of a r.v. X is defined as $\mu_j' = E[X^j]$

2 important moments: 1st & 2nd moments.

Continuous R.V.'s
$E[(X - E(X))^2] = E(X^2) - (E(X))^2 = \mu_2 - \mu_1^2$ is called the variance of X or $V(X)$.

Ex: Suppose
$$f(x) = \begin{cases} \frac{1}{2}x & 0 \leq x \leq 2 \\ 0 & \text{o.w.} \end{cases}$$

Find $E(X)$ & $Var(X)$

$$E(X) = \int_0^2 x f(x)\,dx = \int_0^2 \frac{1}{2}x^2\,dx = \frac{1}{6}x^3\Big|_0^2 = \frac{1}{6}2^3 = \frac{4}{3}$$

$$E(X^2) = \int_0^2 \frac{1}{2}x \cdot x^2\,dx = \frac{1}{8}x^4\Big|_0^2 = 2$$

$$V[X] = \mu_2 \cdot \mu_1^2 = 2\,(4/3)^2 = 2/9$$

Setup:
- X: daily demand of newspaper
- N: Daily Stock of newspaper
- a: profit of selling one paper
- b: loss of one unsold paper
- c: loss of each unsatisfied demand

Only 1 Variable

If $X \leq N$
$$aX - b(N - X) = (a+b)X - bN$$
If $X > N$
$$aN - c(X - N) = (a+c)N - cX$$
call the profit $g_N(X)$

Goal is to chose an N
Such that $E(g_N(X))$ is maximized.

Difference: $E(g_{N+1}(X)) - E(g_N(X))$

Find an N such that the difference is the smallest in absolute value.

$g_{N+1}(x) - g_N(x)$

1° if $X \leq N$ $\quad g_N(x) = (a+b)X - bN$

$$g_{N+1}(x) = (a+b)X - b(N+1)$$

$$g_{N+1}(x) - g_N(x) = -b$$

2°

If $X > N$, $g_N(x) = (a+c)N - cX$

$$g_{N+1}(X) = \begin{cases} (a+b)X - b(N+1) & \text{If } X = N+1 \\ (a+c)(N+1) - cX & \text{if } X > N+1 \end{cases}$$

$$g_{N+1}(x) - g_N(x) = \begin{cases} (a+b+c)X - (a+b+c)N - b & \text{if } X = N+1 \\ a+c & \text{if } X > N+1 \end{cases}$$

$$E[g_{N+1}(X) - g_N(X)] = E[-bI\{X \leq N\} + (a+c)I\{X > N\}]$$

$$= -bP[X \leq N] + (a+c)P[X > N]$$

if $E[g_{N+1}(X)] - E(g_N(X)) = 0$ then

$bP[X \leq N] = (a+c)P[X > N] \Rightarrow b[1 - P[X > N]] = (a+c)P[X > N]$

$\Rightarrow \boxed{P[X > N] = \frac{b}{a+b+c}}$

This suggests that we choose an N s.t. $P(X > N) \approx \frac{b}{a+b+c}$