July 18th

§ 4.4 Change of Variables for Multiple Integrals

Change of variable for 1-dim integral.

If g is 1–1 func of class C^1 then for any continuous function f. g is on interval $[a,b]$. Let F be the antiderivative of f

$$\int_{g(a)}^{g(b)} f(x)\,dx = F(g(b)) - F(g(a))$$

$$F(g(x))' = f(g(x))\,g'(x)$$

$$\int_a^b f(g(x))g'(x)dx = F(g(b)) - F(g(a))$$

$$\Rightarrow \int_{g(a)}^{g(b)} f(x)dx = \int_a^b f(g(x))\,g'(x)\,dx$$

If $g(x)\nearrow$, $g'(x)>0$, $g(a)<g(b) \Rightarrow \int_a^b f(g(x))|g'(x)|\,dx$

$$= \int_{g(a)}^{g(b)} f(x)\,dx = \int_{g([a,b])} f(x)\,dx$$

$g(x)\searrow$, $g'(x)<0$, $g(a)>g(b)$

$$-\int_{g(b)}^{g(a)} f(x)\,dx = -\int_a^b f(g(x))|g'(x)|\,dx \Rightarrow \int_a^b f(g(x))|g'(x)|\,dx = \int_{g(b)}^{g(a)} f(x)dx = \int_{g[a,b]} f x dx$$

Let $I = g([a,b])$, $[a,b] = g^{-1}(I)$

$$\int_I f\,dx = \int_{g^{-1}(I)} f(g(x))\,|g'(x)|\,dx \quad \leftarrow \text{ 1-dim degree of variable formula.}$$

Higher Dimensional

4.37 Thm: Let A be an invertible $n\times n$ matrix, and let $\vec{G}(\vec{u}) = A\vec{u}$ be the corresponding linear transformation

Sps S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $G^{-1}(S) = \{A^{-1}\vec{x} . \vec{x}\in S\}$ is measurable and $f\circ G$ is integrabal on $\vec{G}^{-1}(S)$ and

$$(**)\quad \int\cdots\int_S f(\vec{x})\,d^m\vec{x} = |\det A| \int\cdots\int_{\vec{G}^{-1}(S)} f(A\vec{u})\,d^m\vec{u}$$

Proof: Goal: show $(**)$

Step 1: Refine f on $\mathbb{R}^n$ by $\begin{cases} f(\vec{x}) = \text{old } f(\vec{x}) & \text{if } \vec{x}\in S \\ f(\vec{x}) = 0, & \text{o.w.}\end{cases}$

Step 2: Clain $(**)$ as true when $\vec{G}$ is an "elementrary transformation". Three kinds of "ele .... trans."

(1) $\vec{G}_1(u_1,\cdots,u_k,u_n) = (u_1,\cdots,cu_k,\cdots u_n)$

(2). $\vec{G}_2(u_1,\cdots,u_k,\cdots,u_n) = (u_1,\cdots,u_k+cu_j,\cdots,u_n)$

(3). $\vec{G}_3(u_1,\cdots,u_k,\cdots u_n) = (u_1,\cdots,u_j,\cdots,u_k,\cdots,u_n)$

$\Rightarrow \vec{G}_1$ related to $A \begin{pmatrix} 1 & & 0 \\ & c & \\ 0 & & 1 \end{pmatrix}$ ← (k,k) position

Let $I=g([a,b])$, $[a,b]=g^{-1}(I)$

$(*)\cdots \int_I f\,dx=\int_{g^{-1}(I)} f(g(x))\,|g'(x)|\,dx$ ← 1 dim change of variable formula

$$A_2=\begin{pmatrix}1 & & & 0\\ & 1 & \cdots c \cdots & \\ 0 & & \ddots & \\ & & & 1\end{pmatrix} \quad \downarrow (k,j) \text{ position}$$

$$A_3=\begin{array}{c}\begin{pmatrix}1 & & & & \\ & 0 & \cdots & 1 & \\ & 1 & \cdots & 0 & \\ & & & & \ddots \; 1\end{pmatrix}\begin{array}{l}\\ \leftarrow k\\ \leftarrow w\\ \\ \end{array}\\ \uparrow k \quad \uparrow j\end{array}$$

$\Rightarrow \det A_2 = 1$ $\qquad\qquad$ $\det A_3 = -1$

For $\vec{G_1}$: only involve change in the kth variable

Let $x_k = c u_k$, $dx_k = c\, du_k$

$$\int_{-\infty}^{\infty} f(\dots,x_k,\dots)\,dx_k=\int_{-c^{-1}\infty}^{c^{-1}\infty} f(\dots,cu_k,\dots)\,c\,du_k$$

$$=\begin{cases}\int_{-\infty}^{\infty} f(\dots,cu_k,\dots)\,c\,du_k, & c>0\\ \int_{\infty}^{-\infty} f(\dots,cu_k,\dots)\,c\,du_k & c<0\end{cases} \Rightarrow (**) \text{ is true for } \vec{G_1}$$

For $\vec{G_2}$, let $x_k=u_k+cu_j$, $dx_k=du_k$

$$\int_{-\infty}^{\infty} f(\dots,x_k,\dots)\,dx_k=\int_{-\infty-cu_j}^{\infty-cu_j} f(\dots,u_k,\dots)\,du_k$$

$$=\int_{-\infty}^{\infty} f(\dots,u_k,\dots)\,du_k=|\det A_2|\int_{-\infty}^{\infty} f(\dots,u_k,\dots)\,du_k$$

$\vec{G_3}(u_1,\dots,u_k,\dots,u_j\dots,u_n)=(u_1,\dots,u_k,\dots,u_j,\dots,u_n)$

For $\vec{G_3}$ $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\dots,x_j,\dots,x_k,\dots)\,dx_j\,dx_k=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\dots,u_k,\dots,u_j,\dots)\,du_j\,du_k$

$\Rightarrow (**)$ for $\vec{G_3}$

Step 3, let $\vec{G}(\vec{V})=A\vec{V}$, $\vec{H}(\vec{u})=B\vec{u}$ let two exploitation two elementary linear maps.

$$\iint_S f(\vec{x})\,d(\vec{x}) \overset{(**)}{=} |\det A| \iint_{\vec{G}^{-1}(S)} f(G(\vec{V}))\,dv$$

$= |\det A|\,|\det B| \iint_{H^{-1}(G^{-1}(S))} f(\vec{G}(\vec{H}(\vec{u})))\,d^m\vec{u}$

$= |\det(AB)| \int\cdots\int_{(G\circ H)^{-1}(S)} f(\vec{G}(\vec{H}(\vec{u})))\,d^m\vec{u}$

$\Rightarrow$ the function $G\circ H$ satisfies $(**)$

Step 4: by induction $(**)$ is true for the composition $G_1\circ\cdots\circ G_k$, and any invertible A can be write as the composition of elementary matrix. Thus $(**)$ is true for any invertible A.

Another linear map: (translation)

let $\vec{G}(\vec{u})=\vec{u}+\vec{b}$: $x_j = u_j + b_j \Rightarrow dx_j = du_j$
Thus $\int\cdots\int_S f(\vec{x})\,d^n\vec{x} = \int\cdots\int_{S-\vec{b}} f(\vec{u}+\vec{b})\,d^n\vec{u}$

Nonlinear case : For any $\vec{G}(\vec{u})=\vec{x}$
$\vec{G}(\vec{u}+d\vec{u}) \approx \vec{G}(\vec{u}) + D\vec{G}(\vec{u})\cdot d\vec{u} = \vec{x} + D\vec{G}(\vec{u})\cdot d\vec{u}$

4.41 Thm: Given open sets U and V in $\mathbb{R}^n$, let $\vec{G}: U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\vec{G}(\vec{u})$ is invertible for all $\vec{u}\in U$. Suppose that $T\subset U$, and $S\subset V$ are mble sets s.t. $\vec{G}(T)=S$.
If f is an integrable function on S, then $f\circ\vec{G}$ is integrable on T.
and $\int\cdots\int_S f(\vec{x})\,d^n\vec{x} = \int\cdots\int_{T=\vec{G}^{-1}(S)} f(\vec{G}(\vec{u}))\,|\det D\vec{G}(\vec{u})|\,d^n\vec{u}$

$\| \; |\det A|$ in (**)

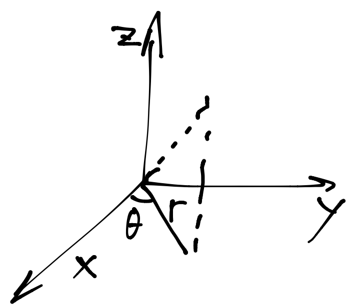
$\vec{G}=A\vec{u} \Rightarrow D\vec{G}=A$

No proof

Eg ① Cylindrical coordination

like "polar in $\mathbb{R}^3$"

$\vec{G}_{cyl}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$

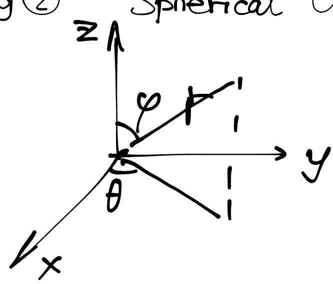


$$D\vec{G}(r,\theta,z) = \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$|\det D\vec{G}| = |r\cos^2\theta + r\sin^2\theta| = |r| = r$

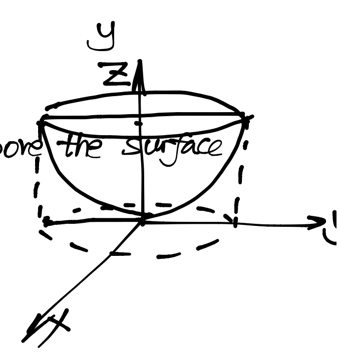
(***) $\Longrightarrow \iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_{\vec{G}^{-1}_{cyl}(S)} f(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz$

Eg ② Spherical Coordinates



$Sph(r,\varphi,\theta) = (r\sin\varphi\cos\theta, r\sin\varphi\sin\theta, r\cos\varphi)$

$\det D\vec{S}_{sph}(r,\varphi,\theta) = r^2\sin\varphi$



Eg. Find the volume & the centroid of the region S above the surface $z = x^2 + {}^2$ and below $z=4$.

$V = \iint_R (4-z)\,dx\,dy$
$= \int_0^2\int_0^{2\pi}(4-r^2)\,r\,d\theta\,dr$

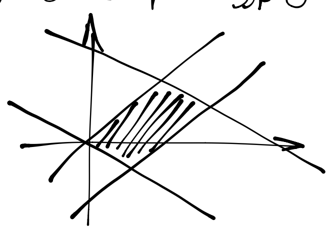
$$= 2\pi\left[2r^2 - \tfrac{1}{4}r^4\right]_0^2$$
$$= 8\pi$$

$$\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ z = x^2+y^2 \\ \quad = r \end{array}}$$

$(\bar{x}, \bar{y}, \bar{z})$ by symmetry. $\bar{x} = \bar{y} = 0$

$$\bar{z} = \frac{1}{V}\iiint_S z\, d^3V = \frac{1}{V}\int_0^2\int_0^{2\pi}\int_{r^2}^{4} z\, r\, dz\, d\theta\, dr$$
$$= \frac{1}{8\pi}\int_0^2\int_0^{2\pi}\left(\tfrac{1}{2}z^2 r\Big|_{r^2}^{4}\right)d\theta\, dr$$
$$= \frac{1}{8\pi}\int_0^2\int_0^{2\pi}\left(8r - \tfrac{1}{2}r^5\right)d\theta\, dr$$
$$= \frac{1}{4}\int_0^2\left(8r - \tfrac{1}{2}r^5\right)dr = \frac{8}{3}$$

Eg. Let P be the parallelogram bounded by $x-y=0$, $x+2y=0$, $x-y=1$ and $x+2y=6$. Compute $\iint_P xy\, dA$



let $u = x-y$ ··· (1)
$v = x+2y$ ··· (2)
$\Rightarrow 0 \le u \le 1,\ 0 \le v \le 6$

$2(1)+(2) \Rightarrow 2u+v = 3x \Rightarrow x = \frac{2u+v}{3}$

$(1) \Rightarrow y = x-u = \frac{2u+v}{3} - u = \frac{v-u}{3}$

$|\det DG| = \left|\det\begin{pmatrix} 2/3 & 1/3 \\ -1/3 & 1/3 \end{pmatrix}\right| = \frac{1}{3}$

$$\iint_P xy\, dA = \int_0^1\int_0^6 \left(\frac{2u+v}{3}\right)\left(\frac{v-u}{3}\right)\frac{1}{3}\, du\, dv = \frac{77}{27}$$

§ 4.5 Functions Defined by integrals

Let $F(\vec{x}) = \int\cdots\int_S f(\vec{x}, \vec{y})\, d^n\vec{y}$

if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}, \vec{y}) = g(\vec{y})$

Is $\lim_{\vec{x}\to\vec{a}} F(\vec{x}) = \lim_{\vec{x}\to\vec{a}} \int\cdots\int_S f(\vec{x}, \vec{y})\, d^n\vec{y} = \int\cdots\int_S \lim_{\vec{x}\to\vec{a}} f(\vec{x}, \vec{y})\, d^n\vec{y}$
$= \int\cdots\int g(\vec{y})\, d^n\vec{y}$

Eg. Let $f(x,y) = \frac{x^2 y}{(x^2+y^2)^2}$, $(x,y) \neq (0,0)$. $f(0,0) = 0$

For each y, $\lim_{x\to 0} f(x,y) = 0 \Rightarrow \int_0^1 \lim_{x\to 0} f(x,y)\, dy = 0$

$\lim_{x\to 0}\int_0^1 \frac{x^2 y}{(x^2+y^2)^2}\, dy = \lim_{x\to 0}\left[-\frac{1}{2}\frac{x^2}{(x^2+y^2)}\Big|_0^1\right] = \lim_{x\to 0}\left(-\frac{1}{2}\right)\left[\frac{x^2}{x^2+1} - 1\right] = \frac{1}{2}$

NOT EQUAL
cannot switch operations as you like

4.46 Thm: Sps S is compact and T is open subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively and S is mble. If $(\vec{x}, \vec{y})$ is continuous on the set $T \times S$, then the function F defined in ☆ is continuous on T.

Proof: Given $\varepsilon > 0$. Goal: find $\delta > 0$, st. $|F(\vec{x}) - F(\vec{x}')| < \varepsilon$ when $|\vec{x} - \vec{x}'| < \delta$

Let $|S|$ be n-dimensional volume of S.

For any $\vec{x}' \in T$, T is open. $\exists r > 0$, Ball $B(r, \vec{x}') \subset T$

$\Rightarrow \overline{B(\frac{r}{2}, \vec{x}')} \subset T \Rightarrow B \times S$ is compact

Since f is continuous on $B \times S \Rightarrow f$ is uniformlly continuous on $B \times S$

$\Rightarrow \exists \delta > 0$ and $\delta < r/2$, s.t. $|f(\vec{x}, \vec{y}) - f(\vec{x}' - \vec{y}')| < \varepsilon / |\delta|$ when $|\vec{x} - \vec{x}'| < \delta$

$$\Rightarrow |F(\vec{x}) - F(\vec{x}')| = \left| \int \cdots \int_S f(\vec{x}, \vec{y}) - f(\vec{x}', \vec{y}') \, d^n\vec{y} \right|$$

$$\leq \int \cdots \int_S |f(\vec{x}, \vec{y}) - f(\vec{x}' - \vec{y}')| \, d^n\vec{y}$$

$$\leq \int \cdots \int_S \frac{\varepsilon}{|S|} d^n\vec{y} = \frac{\varepsilon}{|S|} |S| = \varepsilon$$

■

Q2: $$\frac{\partial}{\partial y_j} F = \frac{\partial}{\partial y_j} \int \cdots \int_S f(x,y) \, d^m\vec{x} \overset{?}{=} \int \cdots \int_S \frac{\partial}{\partial y_j} f(x,y) \, d^m\vec{x}$$

in general, not true

4.47 Thm: Sps $S \subset \mathbb{R}^n$ is compact and mble, $T \subset \mathbb{R}^m$ is open. If f is a cont. function on $T \times S$ that is of class C^1 as a function of $\vec{x} \in T$ for each $\vec{y} \in S$, then the function F (☆) is of class C^1 on T and

$$\frac{\partial F}{\partial x_j}(\vec{x}) = \int \cdots \int_S \frac{\partial f}{\partial x_j}(\vec{x}, \vec{y}) \, d^n\vec{y}, \ (\vec{x} \in T)$$

Proof: $\forall \vec{x_0} \in T$, T is open $\Rightarrow \exists r > 0$, s.t. $\vec{x} \in T$, when $|\vec{x} - \vec{x_0}| \leq 2r$ $(\vec{x_0} \in \overline{B(2r, r_0)} \subset T)$

Goal: Show F is C^1 on $B(r, \vec{x_0})$, since $\vec{x_0}$ is arbitrary in $T \Rightarrow F$ is C^1 on T.

Since we only consider one variable derivative, we use $n=1, m=1$, so $\vec{x} \Rightarrow x$, $\vec{y} = y$.

For $0 < |h| \leq r$, $|x - x_0| < r$

$$\frac{F(x+h) - F(x)}{h} = \int_S \frac{f(x+h,y) - f(x,y)}{h} dy$$

$$= \int_S \frac{\cancel{h} \, \partial_x f(x+th, y)}{\cancel{h}} dy, \text{ by MVT and } t \in (0,1)$$

$$\left| \frac{F(x+h) - F(x)}{h} - \int_S \partial_x f(x,y) \, dy \right| = \left| \int_S \partial_x f(x+th, y) - \partial_x f(x,y) \, dy \right|$$

Since $\partial_x f$ is cont. on $\overline{B(r, x_0)} \times S$, which is compact

$\Rightarrow \partial_x f$ is uniformly continuous $\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$, st.

$|\partial_x f(x+th, y) - \partial_x f(x,y)| < \varepsilon / |S|$ when $|h| < \delta$

$$\Rightarrow \left| \frac{F(x+h)-F(x)}{h} - \int_S \partial_x f(x,y)\,dy \right| < \left| \int_S \frac{\varepsilon}{|S|}\,dy \right| = \varepsilon$$

when $|h|<\delta$ $\Rightarrow \lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = \int_S \partial_x f(x,y)\,dy$ $\blacksquare$

Eg. $F(x)=\int_0^{\pi} y^{-1} e^{xy} \sin y\,dy$. Find $F'(x)$

Solution: $F'(x)=\int_0^{\pi} \frac{\partial}{\partial x}(y^{-1}e^{xy}\sin y)\,dy = \int_0^{\pi} e^{xy}\sin y\,dy$

Integration by parts $(fg)'=f'g+fg'$

$$\int_a^b (fg)'\,dx = \int_a^b f'g\,dx + \int_a^b fg'\,dx$$

$\int_a^b (fg)'\,dx = fg\big|_a^b$

$$* \quad \int_a^b f'g\,dx = fg\big|_a^b - \int_a^b fg'\,dx$$

so $\int_0^{\pi} e^{xy}\sin y\,dy = \int_0^{\pi} e^{xy}(-\cos y)\,dy = e^{xy}(-\cos y)\big|_0^{\pi} - \int_0^{\pi} xe^{xy}(-\cos y)\,dy$

$= e^{\pi x}+1 + x\int_0^{\pi} e^{xy}\cos y\,dy$

$= e^{\pi x}+1+x\int_0^{\pi} e^{xy}(\sin y)'\,dy = e^{\pi x}+1+x[e^{xy}\sin y\big|_0^{\pi} - \int_0^{\pi} xe^{xy}\sin y\,dy]$

$= e^{\pi x}+1-x^2\int_0^{\pi} e^{xy}\sin y\,dy$

$\Rightarrow (1+x^2)\int_0^{\pi} e^{xy}\sin y\,dy = e^{\pi x}+1$

$\Rightarrow \int_0^{\pi} e^{xy}\sin y\,dy = \frac{e^{\pi x}+1}{1+x^2}$

General formula : $F(x)=\int_{\psi(x)}^{\varphi(x)} f(x,y)\,dy$

$F'(x)$: let $u=\varphi(x)$, $v=\psi(x)$

$F'(x)= \frac{\partial}{\partial x}\int_v^u f(x,y)\,dy$

$= \int_v^u \frac{\partial f}{\partial x}(x,y)\,dy + f(u,y)\frac{\partial u}{\partial x} - f(v,y)\frac{\partial v}{\partial x}$

$$\boxed{= \int_{\psi(x)}^{\varphi(x)} \frac{\partial f}{\partial x}(x,y)\,dy + f(\varphi(x),y)\,\varphi' - f(\psi(x),y)\,\psi'}$$

4.52 Thm (The Bounded Convergence Thm)

Let S be a mble subset of $\mathbb{R}^n$ and $\{f_j\}$ a seq of integrable func s on S. Sps that $f_j(\vec{y}) \to f(\vec{y})$ for each $\vec{y}\in S$, where f is an integ-rable function on S, and there is a constant C s.t. $|f_j(\vec{y})| \le C$

$\forall j, \forall \vec{y} \in S$. Then $\lim_{j\to\infty} \int \cdots \int_S f_j(\vec{y})\, d^n\vec{y} = \int \cdots \int f(\vec{y})\, d^n(\vec{y})$

(No proof needed, it's a real analysis)

4.53 Cor:
Let S be a mble subset of $\mathbb{R}^n$ & $T \subset \mathbb{R}^m$. Sps $f(\vec{x}, \vec{y})$ is a func. on $T \times S$ that is intgrb as a func. of $\vec{y} \in S$ for $\vec{x} \in T$ and let F as $(\star)$.

(a). If f is cont. as a func. of $\vec{x} \in T$ $\forall \vec{y} \in S$. and $\exists C > 0$ s.t. $|f| \leq C$ for all $x \in T$, $y \in S$ then F is cont. on T

(b). Sps T is open, If $f(\vec{x}, \vec{y})$ is C^1 of $\vec{x} \in T$ $\forall \vec{y} \in S$, $\exists C > 0$, s.t. $|\nabla_{\vec{x}} f(\vec{x}, \vec{y})| \leq C$ for all $\vec{x} \in T$ and $\vec{y} \in S$ then F is C^1 on T.