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1.

Proof:

First, we decide the values of X that it could be. Say they are $1, 2, 3, \cdots$

then $\sum_{n=0}^{\infty} P(X>n) = \sum_{n=0}^{\infty} E[I(X>n)] = E[\sum_{n=0}^{\infty} I(X>n)]$

we can split it up: $E[\sum_{n=0}^{\infty} I(X>n)] = E[I(X>0) + I(X>1) + \cdots + I(X>X-1) + I(X>X) + I(X>X+1) + \cdots]$

$= E[\underbrace{1+1+\cdots+1}_{\text{There are } X \text{ 1's (from 0 to } X-1)} + 0 + 0 + \cdots]$

$= E[X]$

$\blacksquare$

(2) *Solution:*

By (19), $P(S_1=s)=pq^{s-1}$ $(s=1,2,\cdots)$

Then $E(z^{S_1})=\sum_{S_1} P(S_1)z^{S_1}=\sum pq^{S_1-1}z^{S_1}=pq^{-1}\sum(qz)^{S_1}$

$$=\lim_{S_1\to\infty} pq^{-1}\frac{qz(1-(qz)^{S_1})}{1-qz}$$

$$=\frac{pz}{1-qz}$$

since $q=1-p\in[0,1]$ and $|z|\le 1$
hence $(qz)^{S_1}\to 0$ as $S_1\to\infty$

$E(S_1)=\Pi'(1)$

$$\Pi'(z)=\frac{d}{dz}\left(\frac{pz}{1-qz}\right)=\frac{p(1-qz)-pz(-q)}{(1-qz)^2}=\frac{p-pqz+pqz}{(1-qz)^2}=\frac{p}{(1-qz)^2}$$

when $z=1$, $E(S_1)=\frac{p}{(1-q)^2}=\frac{p}{p^2}=\boxed{\frac{1}{p}}$

$Var(S_1)=\Pi''(1)+\mu-\mu^2$ where $\mu=E(X)=qp^{-1}$ (proved in class)

$$\Pi''(z)=\frac{d}{dz}\frac{p}{(1-qz)^2}=\frac{d}{dz}\frac{p}{1-2qz+q^2z^2}=\frac{-p(-2q+2q^2z)}{(1-qz)^4}$$

$$=\frac{2q^2p(1-qz)}{(1-qz)^4}$$

when $z=1$, $Var(S_1)=\Pi''(1)+qp^{-1}-(qp^{-1})^2$

$$=\frac{2q^2p}{p^3}+qp^{-1}-(qp^{-1})^2$$

$$=\frac{2q^2}{p^2}+\frac{qp}{p^2}-\frac{q^2}{p^2}$$

$$=\frac{2q^2+qp-q^2}{p^2}$$

$$=\frac{q(2q+p-q)}{p^2}$$

$$=\frac{q(p+q)}{p^2}=\boxed{\frac{q}{p^2}}$$

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3.

$j=1,2,3\cdots$

N_j are indpdt Poisson variables

$E(N_j)=\lambda_j$.

Show $X=\sum_j jN_j$ has p.g.f $\exp[\sum_j \lambda_j (z^j-1)]$

Proof: $\Pi_X(z)=E(z^X)$ where $X=\sum_j jN_j$, $j=1,2,3,\cdots$

$$=E(z^{\sum jN_j})$$

$$=E(\prod_j z^{jN_j}) \quad \text{since } N_j\text{'s are independent}$$

then $$=\prod_j E(z^{jN_j}) \qquad (\star)$$

As $$E(z^{jN_j})=\sum_{i=0}^{\infty} P(N_j=i)\cdot z^{ij}=\sum_{i=0}^{\infty}\left[\frac{e^{-\lambda_j}(\lambda_j)^i}{i!}\right]z^{ij}$$

$$=e^{-\lambda_j}\sum_{i=0}^{\infty}\frac{(\lambda_j)^i}{i!}z^{ij}$$

$$=e^{-\lambda_j}\sum_{i=0}^{\infty}\frac{(\lambda_j z^j)^i}{i!}$$

since we have $$\sum_{i=0}^{\infty}\frac{a^i}{i!}=e^a$$

Then $E(z^{jN_j})=e^{-\lambda_j}\exp(\lambda_j z^j)=\exp(\lambda_j(z^j-1))$

Therefore for $(\star)$ $\Pi_X(z)=\prod_j \exp(\lambda_j(z^j-1))=\exp\left[\sum_j \lambda_j (z^j-1)\right]$

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4. A B C

a, b, c, d 4 people

Solution:

(a). $P(A=2,B=1,C=1)=\frac{4!}{2!1!1!}(\frac{1}{3})^2(\frac{1}{3})(\frac{1}{3})=\frac{24}{2}\times\frac{1}{81}=\boxed{\frac{4}{27}}$

(b). $P(A=4)+P(B=4)+P(C=4)=(\frac{1}{3})^4\times 3=\frac{3}{81}=\boxed{\frac{1}{27}}$

(c). All gates are used $\Rightarrow$ there is only 1 gate that used twice

$\Rightarrow$ this 1 gate could be A, B or C.

$\Rightarrow P(\text{all gates used})= P(A=2,B=1,C=1)$

$+P(A=1,B=2,C=1)$

$+P(A=1,B=1,C=2)$

$=3\times 4/27=\boxed{\frac{4}{9}}$

5. Solution:

$E(X_1+4X_2) = E(X_1)+4E(X_2) = 10\times 10\% + 4\times 10\times 5\% = 1+2 = \boxed{3}$

$V(X_1+4X_2) = VX_1 + 16VX_2 + 8Cov(X_1, X_2)$

since X_i's are binomial r.v.'s

So $VX_1 = 0.1\times(1-0.1)\times 10 = 0.09\times 10 = 0.9$

$VX_2 = 0.05\times(1-0.05)\times 10 = 0.475$

$Cov(X_1, X_2) = -10\times 0.1\times 0.05$

$= -0.05$

$V(X_1+4X_2) = VX_1 + 16VX_2 - 8Cov(X_1, X_2) = \boxed{8.1}$

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6. Proof:

Suppose $X_i \overset{iid}{\sim}$ Binomial (p)

$M_{X_1+\cdots+X_n}(t) = E(e^{(X_1+\cdots+X_n)t}) = E(e^{X_1 t})E(e^{X_2 t})\cdots E(e^{X_n t})$

$= M_{X_1}(t)\cdots M_{X_n}(t)$

Note that $X_i = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q = 1-p \end{cases}$

So $M_{X_i}(t) = E(e^{X_i t}) = \sum_j P(X_i = j)\cdot e^{jt} = P(X_i = 0)\cdot e^{0\cdot t} + P(X_i = 1)\cdot e^t$

$= q\cdot 1 + p\cdot e^t$

$= p\cdot e^t + q$

Therefore $M(t) = M_{X_1+\cdots+X_n}(t) = \prod_n (p\cdot e^t + q) = (p\cdot e^t + q)^n$

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7. Solution: let $q = 1-p$

$$M(t) = \sum_{j=0}^{\infty} \binom{j+r-1}{r-1} p^r q^j e^{jt} = \sum_{j=0}^{\infty} e^{jt} \frac{(j+r-1)!}{(r-1)!\,j!} p^r q^j$$

$$= \sum_{j=0}^{\infty} e^{t(j+r)} \frac{(r+j-1)!}{(r-1)!\,j!} p^r q^j$$

$$= \frac{e^{tr}p^r}{(r-1)!} \sum_{j=0}^{\infty} \frac{(r+j-1)!}{j!} e^{tj} q^j$$

$$= \frac{e^{tr}p^r}{(r-1)!} \sum_{j=0}^{\infty} \frac{d^{r-1}}{dx^{r-1}} x^{r+j-1} \quad \text{where } x = e^t q$$

$$= \frac{e^{tr}p^r}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} \sum_{j=0}^{\infty} x^{r+j-1}$$

$$= \frac{e^{tr}p^r}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} \sum_{j=0}^{\infty} x^j$$

(since n-1th derivative will kill any term x^k where $k < n-1$.)

$$= \frac{e^{tr}p^r}{(r-1)!} \frac{d^{r-1}}{dx^{r-1}} \frac{1}{1-x}$$

$$= \frac{e^{tr}p^r}{(r-1)!} \frac{(r-1)!}{(1-x)^r}$$

$$= \frac{e^{tr}p^r}{(1-e^t q)^r} \quad (*)$$

* Note that the m.g.f derived here and the expectation/variance derived below are different from what we learned in the class though. I checked other references and founded that the negative binomial distribution actually has two forms of E(X) and Var(X). I'm not sure about why I got the alternate form in this question.

$$E(X)=M'(0)=\left.\frac{\left(\frac{e^tp}{1-e^tq}\right)^r r\left(\frac{e^tp}{1-e^tq}-\frac{e^tp(-qe^t)}{(1-q)^2}\right)(1-e^tq)}{e^tp}\right|_{t=0}$$

$$=\left.\frac{e^{-t}r\left(\frac{p}{p-1+e^{-t}}\right)^r}{p-1+e^{-t}}\right|_{t=0}$$

$$=\frac{r}{p}$$

Then $$E(X^2)=M''(0)=\frac{\left(\frac{e^tp}{1-e^tq}\right)^r r^2\left(\frac{e^tp}{1-e^tq}-\frac{e^tp(-qe^t)}{(1-e^tq)^2}\right)^2(1-e^tq)^2}{(e^t)^2p^2} \quad ①$$

$$+\frac{\left(\frac{e^tp}{1-qe^t}\right)^r r\left(\frac{e^tp}{1-e^tq}-3\frac{e^tp(-qe^t)}{(1-e^tq)^2}+2\frac{e^tp(-qe^t)^2}{(1-e^tq)^3}\right)(1-e^tq)}{e^tp} \quad ②$$

$$+\frac{\left(\frac{e^tp}{1-e^tq}\right)^r r\left(\frac{e^tp}{1-e^tq}-\frac{e^t(-qe^t)}{(1-e^tq)^2}\right)(1-e^tq)}{e^tp} \quad ③$$

$$+\frac{\left(\frac{e^tp}{1-e^tq}\right)^r r\left(\frac{e^tp}{1-e^tq}-\frac{e^tp(-qe^t)}{(1-e^tq)^2}\right)(1-e^tq)}{e^tp} \quad ④$$

It takes 20 min to simplify the sum of ①+②+③+④, but

$$M''(0)=\frac{r(q+r)}{p^2}=E(X^2) \qquad (*)$$

Hence, $$Var(X)=E(X^2)-E(X)^2=\frac{r(q+r)}{p^2}-\frac{r^2}{p^2}=\boxed{\frac{rq}{p^2}}$$

⑦ Proof:

mgf of X is $M_X(t)=E(e^{tX})=\sum_n P(X=n)e^{tn}$

$$=\sum_{n=0}^{\infty}\left[\binom{n+r-1}{r-1}p^r q^n\right]e^{tn}$$

since $\binom{n+r-1}{r-1}=\binom{n+r-1}{n}$, $\quad =p^r\sum_{n=0}^{\infty}\binom{n+r-1}{n}(qe^t)^n$

since $\sum_{n=0}^{\infty}\binom{n+r-1}{n}a^n=\frac{1}{(1-a)^r}$, $\quad =p^r\frac{1}{(1-qe^t)^r}$

$$=\boxed{\left(\frac{p}{1-qe^t}\right)^r}$$

$$E(X)=M_X'(0)=p^r(-r)(1-qe^t)^{-r-1}(-qe^t)\Big|_{t=0}$$

$$=rqp^re^t(1-qe^t)^{-r-1}\Big|_{t=0}$$

$$=rqp^r(1-q)^{-r-1}$$

$$=\boxed{\frac{rq}{p}}$$

$$E(X^2)=M_X''(0)=rqp^r\cdot\left[e^t(1-qe^t)^{-r-1}+e^t(-r-1)(1-qe^t)^{-r-2}(-qe^t)\right]\Big|_{t=0}$$

$$=rqp^re^t(1-qe^t)^{-r-2}\left[(1-qe^t)+(r+1)(qe^t)\right]\Big|_{t=0}$$

$$=rqp^re^t(1-qe^t)^{-r-2}(1+rqe^t)\Big|_{t=0}$$

$$=rqp^rp^{-r-2}(1+rq)$$

$$=\frac{rq(1+rq)}{p^2}$$

Then $Var(X)=E(X^2)-E(X)^2=\frac{rq(1+rq)}{p^2}-\frac{(rq)^2}{p^2}=\boxed{\frac{rq}{p^2}}$

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