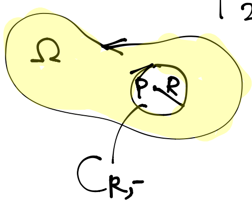
Lecture 7

Ex:

$$\int_\gamma \frac{1}{z-p}\,dz \ , \ \gamma: \text{pw smooth, simple, closed, oriented pos.}$$

$$= \begin{cases} 0 & \text{if } p \text{ is outside } \gamma \\ 2\pi i & \text{if } p \text{ inside } \gamma. \end{cases}$$



We saw using Green's Thm:

$$\int_\gamma \frac{1}{z-p}\,dz = \int_{C_{R,+}} \frac{1}{z-p}\,dz \quad \leftarrow \text{counterclockwise}$$

Let's calculate $\displaystyle\int_{C_{R,+}} \frac{1}{z-p}\,dz$

We can parametrize $C_{R,+}$.

$$C(t) = Re^{it} + p$$
$$C'(t) = iRe^{it}$$

$$\int_{C_{R,+}} \frac{1}{z-p}\,dz = \int_0^{2\pi} \frac{1}{\underbrace{p+Re^{it}}_{z}-p}\,\underbrace{iRe^{it}dt}_{dz}$$

$$= \int_0^{2\pi} \frac{iRe^{it}}{Re^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Chapter 2 §2.1

Differentiability & Analytic Functions

Similarly to real function we define the derivative of a $\mathbb{C}$-function as a limit:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

Recall: For this limit to exist, we must get the same answer, no matter how $h \to 0$.

- If $f'(z)$ exists $\forall z \in D$, we say f is analytic on D.
- If f is analytic on $\mathbb{C}$, then X is entire.

Ex:

$f(z) = z^2$

$$f(z) = \lim_{h\to 0} \frac{(z+h)^2-h^2}{h} = \lim_{h\to 0} \frac{z^2+2zh+h^2-z^2}{h} = \lim_{h\to 0} \frac{2zh+h^2}{h} = 2z$$

Similarly. $f(z), f(z) = z^m$ is entire.

"analytic" stronger than differentiable
means $f(z)$ admits a power series expansion.

Properties:

1. $(f \pm g)' = f' \pm g'$
2. $(cf)' = cf'$
3. $(f\cdot g)' = f'\cdot g + f\cdot g'$
4. $(f/g)' = (f'g - fg')/g^2$
5. $(f\circ g)'(z) = f'(g(x))$.

Replace x's by z's in 1st year book.

given a power series, can have:

converge
- everywhere
- in a disk
- no where

Cor: Poly' are entire
Rational functions are analytic wherever, they are defined.

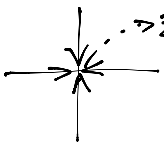
Cauchy–Riemman Equations

If $f=u+iv$, we might guess that to show f is diff. it's enough to check u,v are diff'ble

But we would been wrong!

Let's assume $f=u+iv$ is diff at $z_0=x_0+iy_0$ at $z_0=x_0+iy_0$.

Then $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ doesn't, depends on how $h\to 0$.



First let's

First let's let $h \Rightarrow 0$, with h real.

$$f'(z_0)=\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h} \qquad h\in\mathbb{R}$$

$$= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\left(\frac{v(x_0+h,y_0),v(x_0,y_0)}{h}\right)$$

$$\boxed{\begin{array}{l} h=0+it \\ z_0=x_0+iy_0 \\ z_0+h=x_0+(y_0-t)i \end{array}}$$

$$= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}} \frac{u(x_0+h,y_0)-u(x_0,y_0)}{h} + i\frac{(v(x_0+h,y_0)-v(x_0,y_0))}{h}$$

$$= \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$

Now, let $h\to 0$, with h imaginary

$$\lim_{h\to 0,\, h\in \mathrm{Im}} \frac{f(z_0+h)-f(z_0)}{h}, \quad h=it$$

$$= \lim_{\substack{t\to 0\\ t\in\mathbb{R}}} \frac{u(x_0,y_0+t)+iv(x_0,x_0+t)-(u(x_0,y_0)+iv(x_0,y_0))}{it}$$

$$= \lim_{t\to 0,\, t\in\mathbb{R}} \frac{1}{i}\frac{u(x_0,y_0+t)-u(y_0+y_0)}{t} + \frac{[v(x_0,y_0+t)-v(x_0,y_0)]}{t}\frac{i}{i}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0)$$

$$= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0)$$

Since both limits are. the same we get.

$$\frac{\partial u}{\partial x}=(x_0,y_0) = \frac{\partial v}{\partial x}(x_0,y_0)$$

$$\frac{\partial u}{\partial y}(x_0,y_0) = \frac{-\partial v}{\partial x}(x_0,y_0)$$

We call the partial diff. eqns we get. the Cauchy Riemann eq'ns

$$\boxed{\begin{array}{c} \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y}=\dfrac{-\partial v}{\partial x} \end{array}}$$

$f=u+iv$

analytic $\Longleftrightarrow$ cauchy Riemann

Thm: Suppose $f=u+iv$, f is analytic on D iff u,v satisfy C-R eqn s at every point in D.

Ex. Sps $f=u+iv$, where $u=x^2-y^2$, $v=2xy$

Is it analytic ?

We check C-R equations

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \;,\; \frac{\partial v}{\partial x} = 2y \Rightarrow \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

So f is analytic. ($f(z)=z^2$)

Ex: $f(z) = e^z$ is entire.

$$= e^x e^{iy}$$
$$= e^x \cos y + i e^x \sin y$$

$u = e^x \cos y$, $v = e^x \sin y$

Need to compute $\frac{\partial u}{\partial x} = e^x \cos y$ $\;\frac{\partial v}{\partial y} = e^x \cos y$ $\;\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

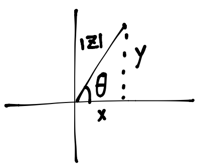
$\frac{\partial u}{\partial y} = -e^x \sin y$ $\;\frac{\partial v}{\partial x} = e^x \sin y$ $\;\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

This gives: $f(z) = \sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ are entire.

Ex: $\mathrm{Log}(z)$ on $D = \mathbb{C} - \mathbb{R}_{\leq 0}$

$\mathrm{Log}(z) = \ln|z| + i\,\mathrm{Arg}(z)$



$\tan\theta = \frac{y}{x}$

$\theta = \arctan\frac{y}{x}$

$= \mathrm{Arg}\, z$ (in this case)

In 1st quadrant: $u = \ln|z| = \ln(\sqrt{x^2+y^2})$

$v = \mathrm{Arg}\, z = \arctan(\frac{y}{x})$

check u, v satisfy Cauchy-Riemann.

For other quads: $\cdots$ (slightly different)

Suppose $f = u + iv$ is analytic. Then u, v satisfy C-R:

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}\right\} \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{-\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} = 0$$

(Provided 2nd order partials of u, v are cts).

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \iff u$ is harmonic

similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

Defn: $u: \mathbb{R}^2 \to \mathbb{R}$ satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, we say u is harmonic

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ — Laplacian   we write $\Delta u = 0$ as shorthand

THM: If $f=u+iv$ is analytic $\Rightarrow$ then u & v are both harmonic. (Not "iff")

Q: Given u, harmonic can we find harmonic function v, so that $f=u+iv$ is analytic?

If u,v are harmonic & $f=u+iv$ is analytic we say u & v are conjugate harmonic.

A: Yes, given u harmonic, we can find v by solving C-R eqn's.

Ex: Is $u=x^3-2xy^2$ the real part of an analytic function, if yes, find conjugate v. If no, explain why not.

Check harmonic (if no $\Rightarrow$ definitely not analytic,
if yes $\Rightarrow$ could be)

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}= 6x-4x=2x\neq 0$$

Since $\Delta u\neq 0$, u is not the real part of an analytic function.

Ex Is $u=x^3-3xy^2$ the real part of an analytic function? Yes find v, no explain.

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=6x-6x=0 \qquad u \text{ is harmonic}$$

and is the real part of analytic function.

To find v, solve C-R

$$\left.\begin{aligned}\frac{\partial u}{\partial x}&=3x^2-3y^2=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}&=-6xy=\frac{-\partial v}{\partial x}\end{aligned}\right\} \quad \frac{\partial v}{\partial y}=3x^2-3y^2 \Rightarrow v=\int(3x^2-3y^2)dy=3x^2y-y^3+g(x)$$

$$\frac{\partial v}{\partial x}=6xy \qquad \Rightarrow \frac{\partial v}{\partial x}=6xy+g'(x)$$

We can do the same thing given $v=Im(f)$ to find $u=Re(f)$

$$\Rightarrow g'(x)=0 \Rightarrow g(x)=k \text{ constant}$$

$$\Rightarrow v=3x^2y-y^3+k \quad (k\in\mathbb{R})$$

THM Suppose $f=u+iv$ analytic on D & u^2+v^2, u,v are constant on D, then f is constant.

Ex: $f=2xy+3i$ cannot be analytic

Pf: If $u=$const, then $\frac{\partial u}{\partial x}=0=\frac{\partial u}{\partial y}$

By C-R we get $\frac{\partial v}{\partial y}=0=\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial x}=0=-\frac{\partial u}{\partial y}$

So $v=$constant

Same proof for $v=$constant

Suppose $u^2+v^2=const=|f|^2=f\cdot\bar{f}$

If $u^2+v^2=0$, then $|f(z)|^2=0$ for all $z\in D$

So $f(z)=0=$const

If $u^2+v^2\neq 0$, scale f to get $u^2+v^2=1$

So we have $f(z)\cdot\overline{f(z)}=1$

$\Rightarrow \overline{f(z)}=\frac{1}{f(z)}$ is analytic (quotient rule)

$F(z)=f(z)+\overline{f(z)}=u+iv+u-iv=\underbrace{2u}_{U}+\underbrace{0i}_{V} \Rightarrow F=constant \Rightarrow f(z)$ const