April 3rd

We proved

X is a cstrtb # $\Longleftrightarrow$ x is a surd
i.e. $x \in F_n$ for some tower of fields.

$$F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_x$$
$F_0 = \mathbb{Q}$

$F_i = F_{i-1}(\sqrt{r_i})$
$r_i > 0$, $r_i \in F_{i-1}$

• Constructible #s are algebraic

Theorem: Let x be a root of a polynomial with constructible coefficients then x is algebraic.

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ if all a_i are constructible. & $P(x)=0$ for some $x \Longrightarrow$ x is algebraic

$P(x) = a_n x^n + \cdots + a_0$
$a_i \in F_k$ for some tower of fields $\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k$ then x is algebraic
induction in k
(1) $k=0$ — base of induction
$P(x) = a_n x^n + \cdots + a_0$, $a_i \in \mathbb{Q}$ — rational
x is a root of $P(x)=0 \Longrightarrow$
$a_i = \frac{p_i}{q_i}$ $\to$ integers

.....
both sides time $q_n q_{n-1} \cdots q_0$ $\Longrightarrow$ get a polynomial with integer coefficients, x is a root $\in F_{i-1}$ $\Longrightarrow$ x is algebraic

Induction step
Suppose we proved it for k, want to prove for $k+1$
$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_{k+1}$
$F_{k+1} = F_k(\sqrt{r_k})$, $r_k \in F_k$

$p(x) = a_n x^n + \cdots + a_0$, $a_i \in F_{k-1}$

$a_i = b_i + c_i\sqrt{r_k}$, $b_i, c_i \in F_k$

$$(b_n + c_n\sqrt{r_k})x^n + (b_{n-1} + c_{n-1}\sqrt{r_k})x^{n-1} + \cdots + (b_0 + c_0\sqrt{r_k}) = 0$$

$$b_n x^n + b_{n-1}x^{n-1} + \cdots + b_0 = -\sqrt{r_k}(c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0)$$ square both sides

$$(b_n x^n + \cdots + b_0)^2 = r_k (c_n x^n + \cdots + c_0)^2, \quad b_i, c_i, r_k \in F_k$$

$\Longrightarrow$ by induction assumption x is algebraic.

Cor: constructible numbers are algebraic.

if x_0 is constructible $\Rightarrow x_0$ is a root of $P(x)=x-x_0 \Rightarrow x_0$ is algebraic by the thm.

Cor: π is not constructible (b/c π is transcendental)

$\Rightarrow$ Can not square a circle

$S=\pi r^2 = a^2$ $\qquad a=\sqrt{\pi}\cdot r$

but $\sqrt{\pi}$ is not constructible if it was then $\sqrt{\pi}\cdot\sqrt{\pi}=\pi$ would be constructible too and it isn't.

$\pi^5+3\pi^2+1$ is not constructible. Why?

Suppose x_0 is constructible $\Rightarrow \pi^5+3\pi^2+1-x_0=0$

$$x^5+3x^2+1+(1-x_0)=0$$

π is a root

$x^5+3x^2+(1-x_0)$ has constructible coefficients

$\Rightarrow \pi$ is algebraic by the theorem this is false.

$\Rightarrow \pi^5+3\pi^2+1$ is not constructible

... $(1+\sqrt{2})\pi^7+(\sqrt{3}-5)\pi^2+2\pi-1$ — is not constructible.

Suppose x_0 is constructible $\Rightarrow (1+\sqrt{2})\pi^7+(\sqrt{3}-5)\pi^2+2\pi-1-x_0=0$

is a root of $(1+\sqrt{2})x^7+(\sqrt{3}-5)x^2+2x+(-1-x_0)=0$

Constructible $\qquad$ $\to$ This is known to be also constbl.

$\Rightarrow$ by the Thm π is algebraic $\Rightarrow$ false $\Rightarrow x_0$ is not constructible

If

Theorem: Let $P(x)=a_3x^3+a_2x^2+a_1x+a_0$, $a_i\in\mathbb{Q}$ - rational

if $P(x)=0$ has a constructible root $\Rightarrow$ it also has a rational root.

Cor: $\sqrt[3]{2}$ is not constructible.

Proof: Sps $\sqrt[3]{2}$ is constructible.

$x^3-2=0$ $\qquad \sqrt[3]{2}$ is a root

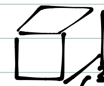
$\Rightarrow$ by the thm $x^3-2=0$ has a rational root

if $\frac{p}{q}$ $\quad gcd(p,q)=1$ $\quad$ is a root of $x^3-2=0 \Rightarrow$ by the rational root theorem $p|-2$, $q|1$. $\quad p=\pm1,\pm2$, $q=\pm1$

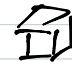
$\frac{p}{q}=\pm1,\pm2$, none of these are roots of $x^3-2=0$

Hence, $\sqrt[3]{2}$ is not constructible.

$\Rightarrow$ cannot double a cube



$(\sqrt[3]{2})^3$



$(1)^3$

$\sqrt[6]{2}$? Not cons. Why? If cons. $\Rightarrow \sqrt[6]{2}\cdot\sqrt[6]{2} = \sqrt[3]{2}$ would also be cons. and it's not!

Theorem: Let $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, $a_i \in \mathbb{Q}$

if $P(x) = 0$ has a constructible root $\Rightarrow$ also a rational one.

— observation: Let P_1, P_2 be polynomials with coefficients in a number field F. divide P_1, by P_2 with remainder.

$P_1(x) = Q(x)P_2(x) + R(x)$ $\quad \deg R < \deg P_2$
then $Q(x)$ and $R(x)$ have coefficients in F.

We'll prove by induction in k that if we have a tower of fields.

$$\mathbb{Q} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \quad , F_i = F_{i-1}(\sqrt{r_i})$$

$P(x)$ has a root in $F_k \Rightarrow$ has a root in $\mathbb{Q}$

① base of induction: $k=0$, $F_k = F_0 = \mathbb{Q}$
② induction step: sps we proved it for F_k, want to prove for F_{k+1}

$F_0 \subseteq F_1 \subseteq \cdots \subseteq F_k \subseteq F_{k+1}$

$F_{k+1} = F_k(\sqrt{r_k})$

$r_k \in F_k$

$P(x) = a_3x^3 + \cdots + a_0 = 0$ has a root in F_{k+1}

Let $x_0 \in F_{k+1}$ be a root of $P(x) = 0$

$x_0 = a + b\sqrt{r_k}$, $a, b \in F_k$

if $x_0 \in F_k \Rightarrow$ we are done by the induction assumption

So suppose $x_0 \notin F_k$ $(b \neq 0)$

put $P_2(x) = (x - a - b\sqrt{r_k})(x - a + b\sqrt{r_k}) = (x-a)^2 - (b\sqrt{r_k})^2$

$= x^2 - 2ax + a^2 - b^2 r_k$ $\longrightarrow$ quadratic poly with coeffi in F_k

$P_2(x_0) = 0$

$P = a_3x^3 + a_2x^2 + a_1x + a_0 = 0$ has coefficients $\in \mathbb{Q} \subseteq F_k$

divide P by P_2 with remainder

$P(x) = Q(x)P_2(x) + R(x)$

Q: deg 1 $\qquad \deg R < 2$: 1 or 0

Q & R have coeff. in F_k

$R(x) = \lambda_1 x + \lambda_2$, $\lambda_1, \lambda_2 \in F_k$

$P(x)=Q(x)P_2(x)+\lambda_1 x+\lambda_2$. Plug in $x=x_0=a+b\sqrt{r_k}$

$P(x_0)=0$

$P_2(x_0)=0$

$0=0+\lambda_1 x_0+\lambda_2 \quad \Rightarrow$ if $\lambda_1\neq 0 \quad x_0=\frac{-\lambda_2}{\lambda_1}\in F_k$ but $x_0\notin F_k$

$\Rightarrow \lambda_1=0,\ \lambda_2=0 \quad \Rightarrow 0=0+0\cdot x_0+0$

$\underset{3}{P(x)}=\underset{\text{deg } 1}{Q(x)}\overset{2}{P_2(x)}+0 \qquad Q(x)=c_1x+c_2$ has a root, $c_1,c_2\in F_k,\ c_1\neq 0$

$\Rightarrow x_1=\frac{-c_2}{c_1}\in F_k$

$P(x)=\underset{\overset{\parallel}{Q}}{(c_1x+c_2)}P_2(x) \quad \Rightarrow x=-\frac{c_2}{c_1}$ is a root of $P(x)$ and $x_1\in F_k$

$\Rightarrow$ by the induction assumption $P(x)$ has a rational root.

∎

ex: $\sqrt[3]{2}$ is not constbl.

$\sqrt[3]{3/5}$ is not cstrctb

<u>why?</u> sps $\sqrt[3]{\frac{3}{5}}$ is cons.

$x^3=\frac{3}{5}$, $5x_0^3=3$, x_0 is a root of $5x^3-3=0$ has rational coeff.

$\Rightarrow 5x^3-3=0$ has a rational root

$\frac{p}{q}$, $(p,q)=1$

$\Rightarrow$ by the rational root theorem, $p|3$, $q|5$, $p=\pm1,\pm3$, $q=\pm1,\pm5$

$\frac{p}{q}=\pm1,\pm3,\pm\frac{1}{5},\pm\frac{3}{5}$ plug these numbers into $5x^3-3=0 \Rightarrow$ none of these fits

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$x^3-6x+2\sqrt{2}=0$

Claim: this has no constructible solutions

Cannot use the theorem directly because $\sqrt{2}$ is not rational

Suppose this has a constructible root x.

then $y=\frac{x}{\sqrt{2}}$ is also constructible.

$x=\sqrt{2}y$

$(\sqrt{2}y)^3-6(\sqrt{2}y)+2\sqrt{2}=0$

$2\sqrt{2}y^3-6\sqrt{2}y+2\sqrt{2}=0 \ /2\sqrt{2}$

$y^3-3y+1=0$ cubic with rational coefficients

$y^3-3y+1=0$ has a constructible root $\Rightarrow$ also has a rational root.

if p/q, $\gcd(p,q)=1 \Rightarrow p \mid 1,\ q \mid 1,\ p=\pm1, q=\pm1$

$\frac{p}{q}=\pm1$, plug in. $1^3-3\cdot1+1=-1\neq0$

...

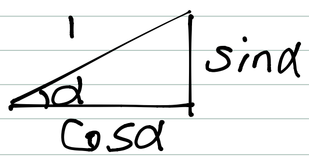
none of those are roots ...

$\Rightarrow x$ is not constructible.

Trisecting an angle

want to divide α into 3 equal parts using a straight edge and a compass

- an angle α is constructible iff $\cos\alpha$ is a constructible number.
- if $\cos\alpha$ is constructible number, then $\sin\alpha=\sqrt{1-\cos^2\alpha}$ is also constructible



Sps we could trisect any angle $\Rightarrow$ in particular one could then trisect the angle $\alpha=60^\circ$ $\frac{\pi}{3}$

$\Rightarrow 20^\circ$ would be constructible.

Now we want to show 20° is not constructible.
$\cos 20^\circ$ is not constructible.

$(a+b)^3=(a+b)^2(a+b)=(a^2+2ab+b^2)(a+b)=a^3+3a^2b+3ab^2+b^3$

$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$

$(\cos\theta+i\sin\theta)^3=\cos3\theta+i\sin3\theta=\cos^3\theta+3\cos^2\theta(i\sin\theta)+3\cos\theta(i\sin\theta)^2+(i\sin\theta)^3$

$=\cos^3\theta-3\cos\theta\sin^2\theta+(3\cos^2\theta\sin\theta-\sin^3\theta)i$

$=\cos3\theta+i\sin3\theta$

$\cos3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$

$=4\cos^3\theta-3\cos\theta$

put $\theta=20^\circ$ $\quad 3\theta=60^\circ$

$\frac{1}{2}=4\cos^3 20^\circ-3\cos20^\circ$

put $x_0=\cos20^\circ$ $\Rightarrow 1=8x_0^3-6x_0 \Rightarrow 8x_0^3-6x_0-1=0$

Claim: $x_0=\cos20^\circ$ is not constructible.

Suppose not i.e. Sps $x_0 = \cos 20^\circ$ is constructible

$y_0 = 2x_0$ is also constructible.

$y_0^3 - 3y_0 - 1 = 0$

$\Rightarrow y_0^3 - 3y_0 - 1 = 0$ has a constructible root

Cubic with rational coeff. $\Big\} \Rightarrow$ also has a rational root

but $y^3 - 3y - 1 = 0$ has no rational roots by rational root theorem

Therefore, y_0 is not constructible $\Rightarrow x_0$ is not constructible $\Rightarrow$ not every angle can be trisected.

Then 1° angle is not constructible!

if so, every angle is constructible!