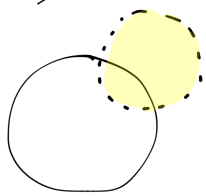
Lecture 9

If (X,τ) is a top space & $Y \subseteq X$
$\tau_Y = \{U \cap Y : U \in \tau\}$

e.g. $S^1 \subseteq \mathbb{R}^2_{usual}$



ex: There is a space (X,τ) that is ccc & separable, but has a non-ccc, non-separable subspace (i.e. ccc & separable are not hereditary)

Take $X=\mathbb{R}$, $\tau = \{U \subseteq \mathbb{R},\ 7 \in U, \text{or } \emptyset = U\}$
This is the particular point topology.

Claim: (X,τ) is separable
$\{7\}$ is a dense set (convince yourself)

Claim 2: (X,τ) is ccc.
You can't even have 2 disjoint non-empty set!

Take $Y = \mathbb{R}\setminus\{7\}$. Then τ_Y is the discrete topology on $\mathbb{R}\setminus\{7\}$

Claim 3: (Y,τ_Y) is not separable
$\mathbb{R}\setminus\{7\}$ is unctble, and unctble discrete spaces are not separable.

Claim 4: (Y,τ_Y) is not ccc.
$\{\{x\} : x \in Y\}$ is an unctble collection of disjoint non-empty open sets. ■

Prop: If (X,τ) is separable and U is an open subset of X. Then (U,τ_U) is separable.

Proof: Let $D \subseteq X$ be dense in X, and ctble.
Take $D \cap U$ (which is ctble)
We will show $D \cap U$ is dense.
Actually you showed this in the previous assignments! (A3 in HW2) ■

ex: Check that if you replace "open" by "closed" the prop is false. (Try the part. pt. top.)

Prop: Every subspace of $\mathbb{R}^n$ ($\mathbb{R}^n_{usual}$) is separable (This is called "$\mathbb{R}^n$ is hereditarily separable")
The proof will be shown in the 2nd part of this course.

Boring facts:

Prop (Restriction of domain):
If $f: X \to Y$ is a cts function & A is a subspace of X, then the restricted function $f|A: A \to Y$ is cts.

Prop (Expansion of range):
Let B be a subspace of Y: If $f: X \to B$ is cts, then $f: X \to Y$ is cts.

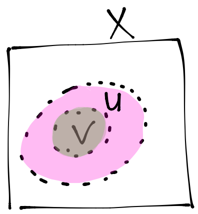
Prop (Inclusion) Let A be a subspace of X.
If $f: A \to X$ is the inclusion map ($f(a)=a$, $\forall a \in A$) then f is cts.

Prop: Let A be a subspace of X. For any $B \subseteq A$, the closure of B (in A) is given by $\overline{B}^A = A \cap \overline{B}^X$

ex: If $X = \mathbb{R}_{usual}$ and $A = (0,1)$ with $B = (0, 1/2)$
Then $\overline{B}^A = A \cap [0, 1/2] = (0, \frac{1}{2}]$

Useful (but still boring) props

Prop: Let A be a closed subset of X. If B is closed in A, then B is closed in X.
Prop: Let U be an open subset of X. If V is open in U, then V is open in X.



Pf: Let $(X, \mathcal{T})$ be a top. space.
Let $U \in \mathcal{T}$
Let $V \in \mathcal{T}_U$
So there is a set $W \in \mathcal{T}$ s.t. $V = W \cap U \in \mathcal{T}$.

$\blacksquare$

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§ 8. Finite Products

Recall: If $(X, \mathcal{T})$ & $(Y, \mathcal{U})$ are top. spaces. Then a basis for the product top. on $X \times Y$ is $\mathcal{T} \times \mathcal{U}$.
If $\mathcal{B}_1$ is a basis for $(X, \mathcal{T})$ and $\mathcal{B}_2$ is a basis for $(Y, \mathcal{U})$ then $\mathcal{B}_1 \times \mathcal{B}_2$ is a basis for the product topology.

- Product of two Hausdorff spaces is Hausdorff.
- Product of two separable spaces is separable
- ... Q: Which property are preserved by taking finite products?

These properties are called "finitely productive properties"

Def'n: If $(X_i, \mathcal{T}_i)$ are top spaces for $1 \le i \le N$
Then the basis for the product top on $X_1 \times X_2 \times \cdots \times X_N$ is $\mathcal{T}_1 \times \mathcal{T}_2 \times \cdots \times \mathcal{T}_N$

Remark: To show that a property is finite product it is enough that it preserved when taking the product of two such spaces. (By induction)

Fact: Senond ctble is finitely productive.

Q: Is ccc finitely productive?
A: We can't know!

Thm (Rich Laver, 1970s) If the CH is true, there are two top spaces that are ccc but their product is not ccc.

Thm (Cohen? 60s)

If you assume Martin's Axiom ($\aleph_1$) Then ccc is finitely productive.