Lecture 17 (§7.3 continued)

Thm: Let $(V, \|\cdot\|)$ be normed vector space, let W be a finite dimensional subspace of V. Then for $\forall v \in V$, $\exists$ at least closest pt $w^* \in W$ to v.
(That is $\exists\ w^* \in W$ st. $\|v-w^*\| = \inf\{\|v-w\| : w \in W\}$)

Proof: W is a subspace of V. Therefore, $0 \in W$.

$\|v-0\| = \|v\|$

$\inf\{\|v-w\| \le \|v\| = M : w \in W\}$

Sps W satisfies $\|v-w\| < \|v\|$

$\|w\| \le \|w-v\| + \|v\| \le M+M = 2M$

Then $\inf.\{\|v-w\| : w \in K\} = \inf\{\|v-w\| : w \in W\}$

If we define ($K = \{w \in W : \|w\| \le 2M\}$)

Claim: K is compact. K is bdd (by $2M$)

Let v_k be a cvgt. seq of pts in K.

$v_k \to v$

$\|v\| \le \|v - v_k\| + \|v_k\| \le \varepsilon + \|v_k\| \le \cancel{\varepsilon + \|v_k\|} \le \varepsilon + 2M$ for $\forall \varepsilon$

$\Rightarrow \|v\| \le 2M$, $v \in W$. $v \in K \Rightarrow K$ is closed

$f(w) = \|v-w\|$

Lipschitz with constant $\quad \|f(w) - f(x)\| = |\ \|v-w\| - \|v-x\|\ | \le |\ \|w-x\|\ |$

By EVT, $\exists$ min $w^* \in K$, so w^* is a closest pt to v.

$\hookrightarrow$ and it is not always unique.

§7.4 Inner Product Spaces

Def'n: An inner product on a vector space V is a function $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$
s.t.

① positive definite

$\langle x, x\rangle \ge 0 \quad \forall x \in V$

$\langle x, x\rangle = 0$ iff $x = 0$

② symmetric $\langle x, y\rangle = \langle y, x\rangle$, $\forall x, y \in V$

③ bi linearity:

$\forall x, y, z \in V$, $\alpha, \beta \in \mathbb{R}$

$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

$\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$

An inner product space is a vector space,

inner product space

$\Downarrow$

norm $\qquad\qquad \|x\| = \langle x, x\rangle^{\frac{1}{2}}$

$\Downarrow$

metric

Ex: $C[a, b]$

$\langle f, g\rangle = \int_a^b f(x) \cdot g(x)\, dx$

$\|f\| = [\int_a^b f^2 dx]^{1/2}$ — L_2 norm

Ex: $A: \mathbb{R}^n \to \mathbb{R}^n$ A is a matrix $A=[a_{ij}]$

$\langle x,y\rangle_A = \langle Ax, y\rangle$

A has positive eigenvalues, A must be symmetric.

7.4.4 Cauchy-Schwarz Inequality.

For all vectors x,y in the inner product space

$\boxed{|\langle x,y\rangle| \leq \|x\|\cdot\|y\|}$ ... (*)

equality holds iff x,y are co-linear

Proof: ① if either x or $y=0$.

(*) is equality. x,y are colinear

② neither is 0

$x-ty,\ t\in\mathbb{R}$

$\langle x-ty, x-ty\rangle \geq 0$

$\langle x,x\rangle - 2t\langle x,y\rangle + t^2\langle y,y\rangle = \|x\|^2 - 2t\langle x,y\rangle + t^2\|y\|^2$

take $t = \dfrac{\langle x,y\rangle}{\|y\|^2}$

$$0 \leq \|x\|^2 - 2\cdot\frac{\langle x,y\rangle}{\|y\|^2}\cdot\langle x,y\rangle + \frac{\langle x,y\rangle^2}{\|y\|^2} = \|x\|^2 - \frac{\langle x,y\rangle^2}{\|y\|^2}$$

$\langle x,y\rangle^2 \leq \|x\|^2\cdot\|y\|^2$

$|\langle x,y\rangle| \leq \|x\|\cdot\|y\|$

If equality holds $|\langle x,y\rangle| = \|x\|\cdot\|y\|$

Let $t = \dfrac{\|x\|}{\|y\|}$, in this case $\|x-ty\|=0$

$x-ty=0 \Rightarrow x=ty$

If x,y is colinear $\Rightarrow x=ty$

$|\langle x,y\rangle| = |\langle ty,y\rangle|$

$\langle x-ty, x-ty\rangle \leftarrow$ compute this & get equality.

Corollary: for $f,g\in C[a,b]$

$$\left|\int_a^b f\cdot g\,dx\right| \leq \left(\int_a^b f^2\,dx\right)^{1/2}\left(\int_a^b g^2\,dx\right)^{1/2}$$

Corollary: triangle inequality. $\|x+y\| \leq \|x\|+\|y\|$

Moreover, if equality holds, x,y are colinear.

$$\begin{aligned}\langle x+y, x+y\rangle = \|x+y\|^2 &= \langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle \\ &\leq \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \\ &\leq \langle x,x\rangle + 2|\langle x,y\rangle| + \langle y,y\rangle \\ &= (\|x\|+\|y\|)^2\end{aligned}$$

by Cauchy-Schwarz

§7.5 Finite Orthonormal Sets

Def'n: 2 vectors, x,y are orthogonal if $\langle x,y\rangle = 0$. A collection of vectors $\{e_i : i\in S\}$ in V:

$\langle e_i, e_j\rangle = 0,\ i\neq j$

$\|e_i\| = 1,\ \forall i\in S$ Orthonormal

Ex: $\mathbb{R}^n$: $e_1=(1,0,\cdots,0)$, $e_2=(0,1,0,\cdots,0)$, $\cdots$, $e_n=(0,0,\cdots,1)$

Lemma: let $\{e_1, e_2, \cdots, e_n\}$ be a finite orthonormal set & $\alpha_1, \cdots, \alpha_n$ be real numbers
If w is $\sum_{i=1}^{n} \alpha_i e_i \Rightarrow \alpha_i = \langle w, e_i \rangle$
Also for $\forall v$, $v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i$ is orthogonal to each e_j, $\forall j = \{1, 2, \cdots, n\}$

Proof: $\langle v - \sum_{i=1}^{n} \langle v, e_i \rangle e_i, e_j \rangle$

$= \langle v, e_j \rangle - \sum_{i=1}^{n} \langle v, e_i \rangle \langle e_i, e_j \rangle$

$= \langle v, e_j \rangle - \langle v, e_j \rangle$

$= 0$

So $\langle e_i, e_j \rangle = 1$ only when $i = j$.

(?) j is an index, not variable

Corollary: $\{e_1, \cdots, e_n\}$ be finite orthonormal set. If x is in span $\{e_1, \cdots, e_n\}$, then x can be uniquely written as $\sum_{i=1}^{n} \langle x, e_i \rangle e_i$. Also, all orthonormal sets are L.I.

Cor:
In a finite-dimensional vector space, an orthonormal set is a basis iff it is maximized w.r.t. being a orthonormal set.

Gram-Schmidt Process.
L.I. set $\{x_1, \cdots, x_n\} \Rightarrow$ orthonormal basis.

$f_1 = \frac{x_1}{\|x_1\|}$

$\vdots$

Sps we have $f_1, \cdots, f_k$ orthonormal vectors.

$y_{k+1} = x_{k+1} - \sum_{i=1}^{k} \langle x_{k+1}, f_i \rangle f_i$.

$f_{k+1} = \frac{y_{k+1}}{\|y_{k+1}\|}$

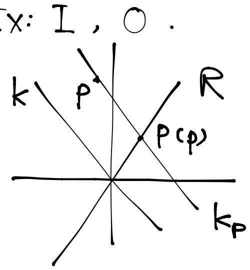
Lemma:
Let $\{e_1, \cdots, e_n\}$ be an orthonormal set in ~~$(V, \langle , \rangle)$~~ an inner product space V. If $x = \sum_{j=1}^{n} \alpha_j e_j$, $y = \sum_{j=1}^{n} \beta_j e_j$ $(\langle y, e_j \rangle = \beta_j)$ then $\langle x, y \rangle = \sum_{j=1}^{n} \alpha_j \beta_j$

$\|x\|^2 = \sum_{j=1}^{n} \alpha_j^2$

Cor:
If $\langle V, \langle , \rangle \rangle$ of dimension n, then it has an orthonormal basis with n elements & the inner product is given by $\langle \sum_{i=1}^{n} \alpha_i e_i, \sum_{j=1}^{n} \beta_j e_j \rangle = \sum_{i=1}^{n} \alpha_i \beta_i$

Def'n: A projection is a linear map $P: V \to V$, $P^2 = P$

Ex: I, O.



$P(p) = p$

so $P^2(p) = P(p) = p$

Def'n: P is an orthogonal projection if $\ker P$ is orthogonal to $\operatorname{Im} P$.
$\ker P = \{v \in V : Pv = 0\}$

Prop:
If P is a projection on a normal vector space then
1) $\ker P = \operatorname{Range}(I-P)$
If, V is an inner product space and P is an orthogonal projection, then
2). $\forall x \in V$, $\|x\|^2 = \|Px\|^2 + \|(I-P)x\|^2$
3). P is uniquely determined by its range

Proof: 1) Sps $x \in \ker P \Rightarrow Px = 0$
Then $(I-P)x = Ix - Px = x \Rightarrow x \in \operatorname{Range}(I-P)$
Sps $x \in \operatorname{Im}(I-P)$
$x = (I-P)y$
$\Rightarrow Px = (P-P^2)y = 0$
$((I-P)x = Ix - Px = x - Px = x \Rightarrow Px = 0 \Rightarrow x \in \ker P)$

2). By our assumption, $\underbrace{\text{range}(I-P)}_{=\ker P}$ & range P are orthogonal

$x = Px + (I-P)x$
$\|x\|^2 = \langle x, x\rangle$
$= \langle Px + (I-P)x, Px + (I-P)x\rangle$
$= \langle Px, Px\rangle + 2\langle Px, (I-P)x\rangle + \langle (I-P)x, (I-P)x\rangle$
$= \|Px\|^2 + \|(I-P)x\|^2$
Proof of (3) as an exercise
Projection Thm:
Let M be a finite-dimensional subspace of an inner product space V and P. The orthogonal projection with $P = M$. Then for all $y \in V$, all $x \in M$.
$\|y-x\|^2 = \|y-Py\|^2 + \|Py-x\|^2$
In particular, Py is the closest vector in M to y. If $\{e_1, \cdots, e_n\}$ is an orthonormal basis for M, Then $Py = \sum_{j=1}^n \langle y, e_j\rangle e_j$ for $\forall y \in V$
Further, $\|y\|^2 \geq \sum_{j=1}^n \langle y, e_j\rangle^2$

7.1. 1)
$$f(x,y) = \begin{cases} (1+xy)^{1/x} & , x \neq 0 \\ e^y & , x = 0 \end{cases}$$

$(0, y_0)$
$f(x,y) = e^{g(x,y)}$
$$g(x,y) = \begin{cases} \dfrac{\ln(1+xy)}{x} & x \neq 0 \\ y & x = 0 \end{cases}$$

When $xy > -1$, $|x| < \dfrac{1}{|y|+1}$, enough to show $g(x,y)$ is cont.

wts $g(x,y) - g(0,y_0)$ can be made arbitrarily small by making (x,y) close to $(0,y_0)$

$$g(x,y)-g(0,y_0)=\frac{\ln(1+xy)}{x}-y_0$$
$$=\frac{\ln(1+xy)}{x}-y+y-y_0$$
$$=\frac{\ln(1+xy)-xy}{x}+y-y_0$$
$$=y\cdot\frac{\ln(1+xy)-xy}{xy}+y-y_0$$

if $xy\to 0$, $\frac{\ln(1+xy)-xy}{xy}\to 0$

Can assume by making $|y-y_0|<\frac{|y_0|}{2}$, $|y|<\frac{3|y_0|}{2}$

Take x much smaller compared to $\frac{3|y_0|}{2}$

$\frac{\ln(1+xy)-xy}{xy}\longrightarrow 0$, $y-y_0\longrightarrow 0$ $\Rightarrow$ function is continuous.