GJJ Q 3.17

$$f(x) = \frac{\beta^{\alpha} x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)}$$

Note: α is known.

a.) $$L(\beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha} x_i^{\alpha-1} \exp(-\beta x_i)}{\Gamma(\alpha)}$$

$$= \frac{\beta^{n\alpha}}{\Gamma(\alpha)^n} \left[\prod_{i=1}^{n} x_i^{\alpha-1} \right] \exp(-\beta \Sigma x_i)$$

$$\ell(\beta) = n\alpha \log(\beta) - n \log(\Gamma(\alpha)) + (\alpha-1) \sum_{i=1}^{n} \log(x_i) - \beta \Sigma x_i$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n\alpha}{\beta} - \Sigma x_i = 0$$

$$\hat{\beta} = \frac{n\alpha}{\Sigma x_i} = \frac{\alpha}{\overline{X}}$$ $\Rightarrow$ By the invarience property of MLEs

$\Rightarrow$ MLE for $1/\beta$ is

$$\hat{\theta} = 1/\hat{\beta} = \overline{X}/\alpha$$

- Let's calculate $E(\hat{\theta})$:

$$E(\hat{\theta}) = E\left(1/\hat{\beta}\right) = E\left(\bar{x}/\alpha\right) = 1/\alpha \, E(x)$$

$$= (1/\alpha)\, \alpha/\beta = 1/\beta$$

$\therefore$ $\hat{\theta}$ is an unbiased estimator of $1/\beta$.

- Let's calculate $E(\hat{\beta})$:

$$E(\hat{\beta}) = E\left(\alpha/\bar{x}\right) = \alpha n E\left(\frac{1}{\sum x_i}\right)$$

Let $W = \sum x_i \Rightarrow W \sim \text{gamma}(n\alpha, \beta)$

Use MGF to show this!

Let $Y = \frac{1}{W} \Rightarrow w = 1/y$

$$\frac{dw}{dy} = -\frac{1}{y^2}$$

$$f_Y = f_W(w = 1/y)\left|-\frac{1}{y^2}\right|$$

$$f_W(w) = \frac{\beta^{n\alpha} w^{n\alpha - 1} \exp(-\beta w)}{\Gamma(n\alpha)}$$

$$\Rightarrow \quad f_Y(y) = \frac{\beta^{n\alpha} \left(1/y\right)^{n\alpha-1} \exp(-\beta/y)}{\Gamma(n\alpha)} \left| -\frac{1}{y^2} \right|$$

$$= \frac{\beta^{n\alpha} \left(\frac{1}{y}\right)^{n\alpha+1} \exp(-\beta/y)}{\Gamma(n\alpha)}$$

$$= \frac{\beta^{n\alpha} \, y^{-(n\alpha+1)} \exp(-\beta/y)}{\Gamma(n\alpha)}$$

$\therefore$ $Y \sim$ inv-gamma$(n\alpha, \beta)$

$\Rightarrow$ Back to $E(\hat{\beta})$

$$E(\hat{\beta}) = \alpha n \, E\left(\frac{1}{\Sigma x_i}\right) = \alpha n E(Y)$$

$$= \alpha n \left(\frac{\beta}{\alpha n - 1}\right) = \left(\frac{\alpha n}{\alpha n - 1}\right) \beta$$

$\therefore$ an unbiased estimater of β is

$$\hat{\beta}_U = \left(\frac{\alpha n - 1}{\alpha n}\right) \left(\frac{\alpha}{\bar{X}}\right) = (\alpha n - 1) \left(\frac{1}{\Sigma x_i}\right)$$

• Let's calculate $V(\hat{\theta})$:

$$V(\hat{\theta}) = V\left(\frac{\Sigma x_i}{n\alpha}\right) = \frac{1}{n^2\alpha^2} V(\Sigma x_i)$$

$$= \frac{1}{n\alpha^2} V(x) = \frac{1}{n\alpha^2}\left(\frac{\alpha}{\beta^2}\right)$$

$$= \frac{1}{n\alpha\beta^2}$$

• Let's calculate the $V(\hat{\beta}_v) = V\left[(\alpha n - 1) V\left(\frac{1}{\Sigma x_i}\right)\right]$

$$= (\alpha n - 1)^2 V(Y)$$

$$= (\alpha n - 1)^2 \frac{\beta^2}{(\alpha n - 1)^2 (\alpha n - 2)}$$

$$= \frac{\beta^2}{(\alpha n - 2)}$$

- CRLB for an unbiased estimator of β:

$$\frac{\partial^2 \ell}{\partial \beta^2} = -\frac{n\alpha}{\beta^2}$$

$$\Rightarrow \quad I(\beta) = -E\left(-\frac{n\alpha}{\beta^2}\right) = \frac{n\alpha}{\beta^2}$$

$$\therefore \text{CRLB} = \frac{\beta^2}{n\alpha}$$

$\therefore$ $\hat{\beta}$ does not attain the CRLB

- CRLB for $\frac{1}{\beta}$

$$V_{1/\beta} \geq \frac{(g'(\beta))^2}{I^{-1}} = \frac{\left(-\frac{1}{\beta^2}\right)^2}{\frac{n\alpha}{\beta^2}} = \frac{\frac{1}{\beta^4}}{\frac{n\alpha}{\beta^2}}$$

$$= \frac{1}{n\alpha\beta^2}$$

$\therefore$ $\hat{\theta}$ does attain the CRLB.

b.) Let's first check whether $X \sim \text{gamma}(\alpha, \beta)$ is a member of an exponential family:

$$f_X(x) = \frac{\beta^{\alpha} x^{\alpha-1} \exp(-\beta x)}{\Gamma(\alpha)}$$

$$= \exp\Big(-\beta x + \alpha \log(\beta) + (\alpha-1)\log(x) - \log(\Gamma(\alpha))\Big)$$

$A(\theta) = -\beta \qquad B(x) = x$

$C(x) = 0 \qquad D(\theta) = \alpha \log(\beta) - \log(\Gamma(\alpha))$

$\therefore \sum X_i$ is a sufficient statistic for β. $\therefore \frac{\alpha}{\bar{X}}$ is a sufficient statistic for β (1-1 transform).

$\therefore \hat{\beta}_U$ is also a sufficient statistic for β!

- Let $\Theta = 1/\beta \Rightarrow \beta = 1/\Theta$

$\Rightarrow$ we can the plug into the density above:

$$\Rightarrow \exp\Big(-\left(\tfrac{1}{\Theta}\right)x + \alpha\log\left(1/\Theta\right) + (\alpha-1)\log(x) - \log(\Gamma(\alpha))\Big)$$

$\therefore$ Σx_i is sufficient for $(1/\beta)$

$\therefore$ $\bar{x}/\alpha$ is sufficient for $(1/\beta)$

GJJ Q 3.23

- We want to $P(A)$, $P(B)$, $P(C)$ in terms θ (the probability of yes).

$$P(A) = \overbrace{\tfrac{1}{6}(1-\theta)}^{= P(A \text{ and } 2) = P(2)P(A|2)} + \overbrace{\tfrac{1}{6}(1-\theta)}^{= P(3)P(A|3)}$$

$$+ \underbrace{\tfrac{1}{6}\theta}_{= P(4)P(A|4)} + \tfrac{1}{6}(1-\theta) + \tfrac{1}{6}\theta$$

$$= \tfrac{3}{6}(1-\theta) + \tfrac{2}{6}\theta = \underline{\underline{\frac{1}{6}(3-\theta)}}$$

$$P(B) = \tfrac{1}{6}(1-\theta) + \tfrac{1}{6}\theta + \tfrac{1}{6}(1-\theta)$$

$$= \tfrac{2}{6}(1-\theta) + \tfrac{1}{6}\theta = \underline{\underline{\frac{1}{6}(2-\theta)}}$$

$$P(C) = \tfrac{1}{6}\theta + \tfrac{1}{6}(1-\theta) + \tfrac{1}{6}\theta + \tfrac{1}{6}\theta$$

$$= \tfrac{3}{6}\theta + \tfrac{1}{6}(1-\theta) = \underline{\underline{\frac{1}{6}(2\theta+1)}}$$

- As there are thee categories we have a multinomial distribution:

$$L(\theta) = \frac{n!}{n_A!\,n_B!\,n_C!} P(A)^{n_A} P(B)^{n_B} P(C)^{n_C}$$

Where $n = n_A + n_B + n_C$; Note $P(A)+P(B)+P(C) = \underline{1}$.

Check this!

$$\ell(\theta) = \log\left(\frac{n!}{n_A!\, n_B!\, n_C!}\right) = n_A \log(P(A))$$
$$+ n_B \log(P(B))$$
$$+ n_C \log(P(C))$$

$$= n_A \left[\log(1/6) + \log(3-\theta)\right]$$
$$+ n_B \left[\log(1/6) + \log(2-\theta)\right]$$
$$+ n_C \left[\log(1/6) + \log(2\theta+1)\right] + Con$$

$$= n_A \log(3-\theta) + n_B \log(2-\theta) + n_C \log(2\theta+1)$$
$$+ Con$$

$$= 440 \log(3-\theta) + 310 \log(2-\theta) + 250 \log(2\theta+1)$$
$$+ Con$$

$$\Rightarrow \frac{\ell(\theta)}{\partial\theta} = -\frac{440}{3-\theta} - \frac{310}{2-\theta} + \frac{250}{2\theta+1}(2) = 0$$

- Solving this we have:

$$\hat{\theta} = \frac{1}{400}\left(537 - \sqrt{193769}\right)$$
$$= 0.2437$$

$$\Rightarrow \frac{\ell^{2}(\theta)}{\partial\theta} = -\frac{n_A}{(3-\theta)^2} - \frac{n_B}{(2-\theta)^2} - \frac{4(n_c)}{(2\theta+1)^2}$$

$$\Rightarrow I(\theta) = -E\left(-\frac{n_A}{(3-\theta)^2} \quad \frac{n_B}{(2-\theta)^2} - \frac{4n_c}{(2\theta+1)^2}\right)$$

$$= \frac{E(n_A)}{(3-\theta)^2} + \frac{E(n_B)}{(2-\theta)^2} + \frac{4E(n_c)}{(2\theta+1)^2}$$

$$= \frac{n(1/6)(3-\theta)}{(3-\theta)^2} + \frac{n(1/6)(2-\theta)}{(2-\theta)} + \frac{4n(1/6)(2\theta+1)}{(2\theta+1)^2}$$

$$= \frac{n/6}{(3-\theta)} + \frac{n/6}{(2-\theta)} + \frac{4n/6}{(2\theta+1)}$$

$$V(\hat{\theta}) = I(\theta)^{-1}$$ (This is the asymptotic variance)

$\Rightarrow$ Plug in the values for n and $\hat{\theta}$ and invert!

$$\Rightarrow V(\hat{\theta}) = 0.00166$$

$$\therefore \hat{\theta} \overset{\cdot}{\sim} \text{Normal}(\mu = 0.2437,\ V = 0.00166)$$

asymptotically

GJJ Q 3.28

a.) $f(x) = \theta(1-\theta)^{x-1}$; $x = 1, 2, \ldots$

from the table

$E(x) = \frac{1}{\theta}$ $\quad\quad m(x) = \bar{x}$

For the MoM set $\frac{1}{\theta} = \bar{x}$ and Solve for θ

$\Rightarrow \tilde{\theta} = \frac{1}{\bar{x}}$

b.) $X \sim \text{Unif}(\overset{\text{“}a}{-\theta/2}, \overset{\text{“}b}{\theta/2})$

from the table

$E(x) = \frac{b+a}{2} = \frac{\theta/2 + \theta/2}{2} = \frac{2\theta/2}{2} = \theta/2$

$\Rightarrow \frac{\theta}{2} = \bar{x} \quad \Rightarrow \tilde{\theta} = 2\bar{x}$

c.) $f(x) = \frac{1}{\theta^2} x \exp(-x/\theta)$; $\alpha = 2$

$E(x) = 2\theta \quad \Rightarrow 2\theta = \bar{x}$

from the table $\quad \Rightarrow \tilde{\theta} = \bar{x}/2$

d.) $X \sim \text{Poisson}(\lambda)$

$E(x) = \lambda \quad \Rightarrow \tilde{\lambda} = \bar{x}$

from the table.

- While the expected values were taken from the table, you may be given distributions not on the table or asked to derive the expected values even if they are on the table.