July 13th

CHAPTER 3
The Implicit Function Theorem and Its Applications

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Outline
Chapter 3
Define curves and surfaces
Implicit Function Theorem (Today)
Chapter 4 Integral Calculus
Define the integration on the line
How to compute integration
change of variable ← related to chain rule
Iterated Integrals
Chapter 5
Integrals over curves, surfaces and vector fields.
The relations between those integration $\left\{\begin{array}{l}\text{Green's thm}\\ \text{The Divergence thm}\\ \text{The Stoke's thm}\end{array}\right.$

40 first half
out of 120 final, so 80 second half.

- No problem section tmr.
- Quiz & Problem Set next week.

§3.1 The Implicit Function Thm

Q: Given equation $F(x_1,\cdots,x_n)=0$, can we solve it for one of the variables. x_j as a function of the remaining $n-1$ variables?

or can we solve a system of k equations for k of the variables as functions of the remaining $n-k$ variables?

Eg. 1: consider $k=2$
can we get $\begin{cases} x_1=f(x_3,x_4,\cdots,x_n) \\ x_2=g(x_3,x_4,\cdots,x_n)\end{cases}$?

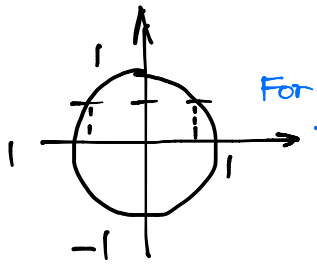
Eg. 2: consider $n=2$, $F(x,y)=0$
consider the set $S=\{(x,y).F(x,y)=0\}$

Q: when can S be represented as the graph of a function $y=f(x)$ or $x=f(y)$

Eg 3: consider $n=3$, $F(x,y,z)=0$
The set $S=\{(x,y,z): F(x,y,z)=0\}$ is usually a surface.

Q: when this surface can be represented as the graph of a function $z=f(x,y)$, $y=g(x,z)$ or $x=h(y,z)$.

Eg 4. Consider $F(x,y)=x^2+y^2-1$
The set $S=\{(x,y): F(\vec{x})=0\}$ is a circle.



For each y, you have two x's
$\cdots x, \cdots$ y's

Here, $f(x)=y=\begin{cases}\sqrt{1-x^2} & \text{upper semicircle} \\ -\sqrt{1-x^2} & \text{lower}\end{cases}$

$\Rightarrow$ we can represent a piece of S in the neighbourhood of a given point $\vec{a}\in S$ as a graph.

Since we cannot put the conclusion in the whole S.

Precise analytical statement of the problem:
Given a function $F(\vec{x},y)$ of class C^1 and a point $(\vec{a},b)$ satisfying $F(\vec{a},\vec{b})=0$, when is there
i). a function $f(\vec{x})$, defined in some open set $S\subset\mathbb{R}^n$ containing $\vec{a}$ and
ii). an open set $U\subset\mathbb{R}^{n+1}$ containing $(\vec{a},b)$ s.t. for $(\vec{x},y)\subset U$,
$F(\vec{x},y)=0 \iff y=f(\vec{x})$

Notation: $\begin{cases}\vec{x}=(x_1,\cdots,x_n) \\ y=x_{n+1}\end{cases}$

Eg. Consider the $\boxed{\text{linear case}}$
$L(x_1,\cdots,x_n,y)=\alpha_1x_1+\cdots+\alpha_nx_n+\beta y+c$
The solution is obvious: The equation $L(\vec{x},y)=0$ can be solved for y $\iff \beta\neq 0$.
Consider near a given point $(\vec{a},b)$ Non-linear case
every diff. function $F(\vec{x},y)$ can be approximated by a linear function.
in fact, $F(\vec{a},b)=0$
By Taylor's thm
$$F(\vec{x},y)=[\partial_1F(\vec{a},b)](x_1-a_1)+[\partial_2F(\vec{a},b)](x_2-a_2)+\cdots+[\partial_nF(\vec{a},b)](x_n-a_n)+[\partial_yF(\vec{a},b)](y-b)+\text{small error}$$

Conjecture: $\partial_yF(\vec{a},b)\neq 0 \Rightarrow F(\vec{x},y)$ could be solved for y precisely (if the "small error" were not there.

3.1 Thm (The Implicit Function Thm for a single equation)
Let $F(\vec{x}, y)$ be a function of class C^1 on some neighbourhood of a point $(\vec{a}, b) \in \mathbb{R}^{n+1}$. Suppose that $F(\vec{a}, b) = 0$ and $\partial_y F(\vec{a}, b) \neq 0$. Then there exists positive number r_0, r_1 s.t. the following conclusions are valid:

a. for each $\vec{x}$ in the ball $|\vec{x} - \vec{a}| < r_0$ there is a unique y s.t. $|y - b| < r_1$ and $F(\vec{x}, y) = 0$. We denote this y by $f(x)$; in particular, $f(\vec{a}) = b$.

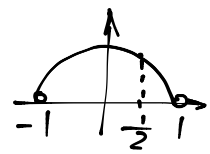
b. The function f thus defined for $|\vec{x} - \vec{a}| < r_0$ is of class C^1, and its partial derivatives are given by

$$\partial_j f(\vec{x}) = -\frac{\partial_j F(\vec{x}, f(\vec{x}))}{\partial_y F(\vec{x}, f(\vec{x}))} \qquad (*)$$

Remarks:

i. The number r_0 may be very small, and there is no way to estimate its size without further hypotheses on F.

Eg. a). linear fun $F(x, y) = x + y = 0$, r_0 is any real number
b). $F(x, y) = x^2 + y^2 - 1 = 0$, $y > 0$ if $a = 0$, $r_0 = 1$; $a = \frac{1}{2}$, $r_0 = \frac{1}{2}$



ii. The formula $(*)$ for $\partial_j f$ is, of course, the one obtained via the chain rule by differentiating the equation $F(\vec{x}, f(\vec{x})) = 0$.

$$\Rightarrow \underbrace{\frac{\partial}{\partial x_j} F(\vec{x}, f(\vec{x}))}_{=0} = \partial_j F(\vec{x}, f(\vec{x})) + \partial_y F(\vec{x}, f(\vec{x})) \cdot \partial_j f(\vec{x})$$

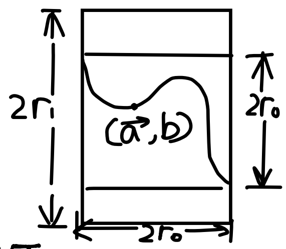
$\Rightarrow (*)$

Proof:
(a). WLOG, assume $\partial_y F(\vec{a}, b) > 0$. (otherwise, we consider $-F$)
Since $\partial_y F$ is continuous, $\exists\, r_1 > 0$. s.t. $\partial_y F(\vec{x}, y) > 0$ for $|\vec{x} - \vec{a}| < r_1$ and $|y - b| < r_1$
$\Rightarrow$ on $|\vec{x} - \vec{a}| < r_1$ and $|y - b| < r_1$ (it's a set)
For each fixed $\vec{x}$, $F(\vec{x}, y)$ is strictly increasing.

$$\Rightarrow \begin{cases} F(\vec{a}, b) = 0 \\ F(\vec{a}, b + r_1) > 0 \\ F(\vec{a}, b - r_1) < 0 \end{cases}$$

Since F is continuous w.r.t. to $\vec{x}$, $\exists\, r_0 \leq r_1$ s.t.
$F(\vec{x}, b + r_1) > 0$ and $F(\vec{x}, b - r_1) < 0$ for $|\vec{x} - \vec{a}| < r_0$
Thus, for each $\vec{x}$ in the ball $B = \{\vec{x} : |\vec{x} - \vec{a}| < r_0\}$ we have

$$\begin{cases} F(\vec{x}, b - r_1) < 0 \\ F(\vec{x}, b + r_1) > 0 \end{cases}$$ and $F(\vec{x}, y)$ is strictly increasing w.r.t. y for $|y - b| < r_1$



The geometry of the implicit function thm. $\partial_y F>0$ in the box, $F>0$ on the top side, $F<0$ on the bottom side, and $F=0$ on the curve.

by Intermediate Value thm $\overset{\text{IVT}}{\Longrightarrow} \exists$ a unique y for each $\vec{x}\in B$ s.t. $F(\vec{x},y)=0$ and $|y-b|<r$.

(b) ① Goal: $y=f(\vec{x})$ is continuous for all $\vec{x}$ in B:

$\forall \varepsilon>0, \exists \delta>0$ s.t. $|f(\vec{x})-f(\vec{x}_0)|<\varepsilon$ for $\forall\, |\vec{x}-\vec{x}_0|<\delta$

Now given $\varepsilon>0$ Redo the argument in Part (a) and get $\exists \delta>0$, s.t. if $|\vec{x}-\vec{x}_0|<\delta$ we have

$$\begin{cases} F(\vec{x}, y_0-\varepsilon)<0 \\ F(\vec{x}, y_0+\varepsilon)>0 \\ y_0=f(\vec{x}_0) \end{cases}$$

$\Rightarrow$ for each $\vec{x}$, $\exists$ a unique y s.t. $F(\vec{x},y)=0$, $y=f(\vec{x})$ and $|y-y_0|<\varepsilon \Rightarrow f(\vec{x})$ is continuous.

② Goal: $\partial_j f$ exists, continuous and (*)

Given $\vec{x}\in B$, let $y=f(\vec{x})$ and $\underbrace{k=f(\vec{x}+\vec{h})-f(\vec{x})}_{(**)}$ where $\vec{h}=(0,0,\cdots,0,\underset{j\text{ th place}}{h},0,\cdots,0)$

Then $y+k=f(\vec{x}+\vec{h})$ so $F(\vec{x}+\vec{h}, y+k)=F(\vec{x}+\vec{h}, f(\vec{x})+k)$

By (**), $=F(\vec{x}+\vec{h}, f(\vec{x}+\vec{h}))=0$

By MVT, $0=F(\vec{x}+\vec{h}, y+k)-F(\vec{x},y)=h\,\partial_j F(\vec{x}+t\vec{h}, y+tk)$

$+k\,\partial_y F(\vec{x}+t\vec{h}, y+tk)$ for some t in $(0,1)$

Then
$$\frac{k}{h}=-\frac{\partial_j F(\vec{x}+t\vec{h}, y+tk)}{\partial_y F(\vec{x}+t\vec{h}, y+tk)}$$

And by (**) $\dfrac{f(\vec{x}+\vec{h})-f(\vec{x})}{h}=\dfrac{k}{h}$

$$\Rightarrow \frac{f(\vec{x}+\vec{h})-f(\vec{x})}{h}=-\frac{\partial_j F(\vec{x}+t\vec{h}, y+tk)}{\partial_y F(\vec{x}+t\vec{h}, y+tk)}$$

let $h\to 0$ & $k\to 0$

$$\Rightarrow \frac{\partial}{\partial x_j} f(\vec{x})=\lim_{h\to 0}\frac{f(\vec{x}+\vec{h})-f(\vec{x})}{h}=\lim_{\substack{h\to 0\\ k\to 0}}\frac{\partial_j F(\vec{x}+t\vec{h}, y+tk)}{\partial_y F(\vec{x}+t\vec{h}, y+tk)} \overset{\substack{F\in C^1\\ \partial_y F(\vec{x},y)\neq 0}}{=} -\frac{\partial_j F(\vec{x},y)}{\partial_y F(\vec{x},y)}$$

∎

3.3 **Corollary**: Let F be a function of class C^1 on $\mathbb{R}^n$, and let $S=\{\vec{x}: F(\vec{x})=0\}$. For every $\vec{a}\in S$ such that $\nabla F(\vec{a})\neq 0$ there is a neighbourhood N of $\vec{a}$ s.t. $S\cap N$ is the graph of a C^1 function.

Proof:

$\nabla F(\vec{a})\neq 0$

$\Rightarrow (\partial_1 F(\vec{a}), \cdots, \partial_n F(\vec{a}))\neq 0$

$\Rightarrow \partial_j F(\vec{a})\neq 0$. for some j.

$\Rightarrow$ by IFT : $F=0$ can be solved to yield $\overbrace{x_j}^{\text{i.e. } x_j = g(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n)}$ as a C^1 function of the remaining variables near $\vec{a}$.

∎

Eg. 6 Consider $F(x,y)=x-y^2-1$

$$\Rightarrow \begin{cases} \partial_x F(x,y)=1 \\ \partial_y F(x,y)=-2y \end{cases}$$

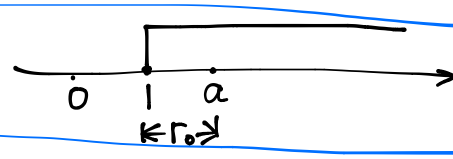
① $\partial_x F\neq 0$, by IFT $\Rightarrow$ $F(x,y)=0$ can be solved for x locally near any point (a,b) s.t. $F(a,b)=0$.

Indeed : $x=y^2+1$ (and this solution is valid not just locally but globally.

What is r_0 ?

$x=y^2+1\geq 1 \Rightarrow x\in[1,+\infty)$

so $r_0=a-1$

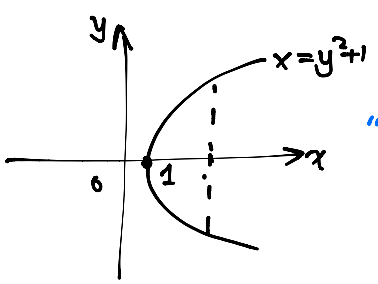


② $\partial_y F(a,b)=-2b=0 \Leftrightarrow b=0$

$\Rightarrow$ $F(x,y)$ can be solved uniquely for y near any point (a,b) s.t. $F(a,b)=0$ and $b\neq 0$.

Indeed : $y=\begin{cases}\sqrt{x-1} & \text{, if } b>0 \\ -\sqrt{x-1} & \text{, if } b<0\end{cases}$

when $b=0$, we have $x-b^2-1=0 \Rightarrow x=1 \Rightarrow$ the equation cannot be solved uniquely for y in any neighbourhood of $(1,0)$.



"each x, has two y's"

Eg. Consider $G(x,y)=x-e^{1-x}-y^3$

① $\partial_x G=1+e^{1-x}>1$ for all (x,y)

$\Rightarrow$ $G(x,y)=0$ can be solved for x locally near any point (a,b) s.t. $G(a,b)=0$.

but we cannot get the explicit form of $x=g(y)$

② $\partial_y G = -3y^2 = 0 \Rightarrow y = 0$

$\Rightarrow G(x,y)$ can be solved uniquely for y locally near any point (a,b) s.t. $G(a,b)=0$ and $b \neq 0$

Indeed $y = (x - e^{1-x})^{\frac{1}{3}}$

when $b=0$, $y=0 \Rightarrow x - e^{1-x} = 0 \quad \Rightarrow x = 1$

$\Rightarrow y' = \frac{1}{3}(x - e^{1-x})^{-2/3}(1 + e^{1-x})$ not defined at $x=1$

Thus the solution y is globally uniquely defined but fails to be diff at the point where $y=0$ (i.e., $x=1$).

General Problem: Solving several eqns simutaneously for some of variables occuring in them:

Precise description:
Consider k functions $F_1, \cdots, F_k$ of $n+k$ variables $x_1, \cdots, x_n, y_1, \cdots, y_k$ and ask when we can solve the eqns:

$$F_1(x_1, \cdots, x_n, y_1, \cdots, y_k) = 0$$
$$\vdots$$
$$F_k(x_1, \cdots, x_n, y_1, \cdots, y_k) = 0$$

More abbreviate notation:

$$\vec{F}(\vec{x}, \vec{y}) = 0 \qquad (***)$$

we assume $\vec{F} \in C'$ near $(\vec{a}, \vec{b})$ s.t. $\vec{F}(\vec{a}, \vec{b}) = 0$.

Q: when $(***)$ determine $\vec{y}$ as a C' function of $\vec{x}$ in some neighbourhood of $(\vec{a}, \vec{b})$

Consider the linear case:

$$\vec{F}(\vec{x}, \vec{y}) = A\vec{x} + B\vec{y} + \vec{c} = \vec{0}$$

where A is a $k \times n$ matrix
B is a $k \times k$ matrix
and $\vec{c} \in \mathbb{R}^k$

$\vec{F} = \vec{0}$ is solvable $\Leftrightarrow B$ is invertible

$\Rightarrow \vec{y} = -B^{-1}(A\vec{x} + \vec{c})$

In general, let B be the Frechet Derivative of $\vec{F}$ w.r.t. variables $\vec{y}$, evaluated at $(\vec{a}, \vec{b})$

$$B_{ij} = \frac{\partial F_i}{\partial y_j}(\vec{a}, \vec{b}) \qquad (1 \leq i, j \leq k)$$

$\Rightarrow$ Solvability related to $\det B \overset{?}{=} 0$

3.9 Thm (The Implicit Function Thm for a System of eqns).
Let $\vec{F}(\vec{x}, \vec{y})$ be an $\mathbb{R}^k$-valued function of class C' on some neighbourhood of a point $(\vec{a}, \vec{b}) \in \mathbb{R}^{n+k}$ and let $B_{ij} = (\partial F_i / \partial F_{y_j})(\vec{a}, \vec{b})$. Sps that $\vec{F}(\vec{a}, \vec{b}) = \vec{0}$ and $\det B \neq 0$. Then $\exists\ r_0, r_1 > 0$ s.t. the following conclusions are valid.

a. For each $\vec{x}$ in the ball $|\vec{x}-\vec{a}|<r_0$, $\exists$ a unique $\vec{y}$ s.t. $|\vec{y}-\vec{b}|<r_1$ and $\vec{F}(\vec{x},\vec{y})=0$. We denote this $\vec{y}$ by $\vec{f}(\vec{x})$; in particular, $\vec{f}(\vec{a})=\vec{b}$.

b. The function $\vec{f}$ thus defined for $|\vec{x}-\vec{a}|<r_0$ is of class C^1, and its partial derivatives $\partial_{x_j}\vec{f}$ can be computed by differentiating the eqns $\vec{F}(\vec{x},\vec{f}(\vec{x}))=\vec{0}$ w.r.t. x_j & solving the resulting linear system of eqns for $\partial_{x_j}f_1, \cdots, \partial_{x_j}f_k$.