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MAT246 HW7 Rui Qiu # 999292509

#1 Solve the equation

$$z^2+(1+i)z+i=0$$

Solution:

$$z=\frac{-(1+i)\pm\sqrt{(1+i)^2-4\cdot i}}{2}$$

$$=\frac{-(1+i)\pm(1-i)}{2}$$

$$=\frac{-1-i\pm(1-i)}{2}$$

$$=-i \text{ or } -1$$

So the solutions are $z_1=-i$, $z_2=-1$.

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#2. Claim: $P(z)$ be a polynomial with real coefficients.
Then if z_0 is a root of $P(z)$ then $\overline{z_0}$ is also a root of $P(z)$.

Proof: Let $P(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0$.

We have $p(z_0)=a_n z_0^n+a_{n-1}z_0^{n-1}+\cdots+a_1 z_0+a_0$

Since $\overline{z_1}+\overline{z_2}=\overline{z_1+z_2}$

and $\overline{z_1}\cdot\overline{z_2}=\overline{z_1 z_2}$ for all $z_1,z_2\in\mathbb{C}$ < and $r\in\mathbb{R}$, $\bar{r}=r$

Then

$$\overline{p(z_0)}=\overline{a_n z_0^n+a_{n-1}z_0^{n-1}+\cdots+a_1 z_0+a_0}$$

$$=\overline{a_n z_0^n}+\overline{a_{n-1}z_0^{n-1}}+\cdots+\overline{a_1 z_0}+\overline{a_0}$$

$$=a_n\overline{z_0^n}+a_{n-1}\overline{z_0^{n-1}}+\cdots+a_1\overline{z_0}+a_0$$

$$=p(\overline{z_0})$$

As for all a_j above are real, i.e. $a_j=\overline{a_j}$, $j=0,1,\cdots,n$.

Thus $p(z_0)=0$ implies $p(\overline{z_0})=0$

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#3. Claim: $P(z), Q(z)$ two polynomials with complex coefficients such that $P(\frac{z}{n}) = Q(\frac{z}{n})$ for ~~all z~~ natural n. then $P(z)=Q(z)$ for all z.

Proof: $P(z) - Q(z) = a(z-1)(z-2)(z-3)\cdots(z-n)\cdot\cancel{(z-k)} = 0$
Since $P(z), Q(z)$ are ~~fin~~ polynomials with finite degrees, say it is k.
But $P(z), Q(z)$ agree at infinitely many points which more than k, ~~so~~
So $p(z) = Q(z)$
i.e. $a=0$, $p(z), Q(z)$ are zero polynomials.
Thus $P(z) = Q(z)$ for all $z \in \mathbb{C}$. ∎

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~~Suppose $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$~~
~~$Q(z) = b_0 +$~~

Suppose $P(n) = a_0 + a_1 n + a_2 n^2 + a_3 n^3 + \cdots$
$Q(n) = b_0 + b_1 n + b_2 n^2 + b_3 n^3 + \cdots$ for all $a_i, b_i \in \mathbb{C}$, $i \in [0, +\infty)$, $i \in \mathbb{Z}$
Then $(P-Q)(n) = (a_0 - b_0) + (a_1 - b_1)n + (a_2 - b_2)n^2 + (a_3 - b_3)n^3 + \cdots = 0$ for all $n \in \mathbb{N}$.
Here $P(n), Q(n)$ agree at infinitely many points
therefore $a_0 - b_0 = a_1 - b_1 = a_2 - b_2 = a_3 - b_3 = \cdots = 0$

why?

This means "all polynomials of all degrees are equal everywhere" on the naturals.

Now, consider $z \in \mathbb{C}$.
Since the coefficients of $P-Q$ ~~is~~ ~~are all zeroes~~ zeros.
then $(P-Q)(z) = (a_0 - b_0) + (a_1 - b_1)z + (a_2 - b_2)z^2 + (a_3 - b_3)z^3 + \cdots$
$= 0 + 0z + 0z^2 + \cdots$
$= 0$
$= P(z) - Q(z)$
hence $P(z) = Q(z)$.

Certainly that is more than you want to assume. ∎

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#4. Solution:

$$\frac{(3-\sqrt{3}i)^{71}}{(1-i)^{53}}=\frac{\left[\sqrt{3^2+3}\left(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})\right)\right]^{71}}{\left[\sqrt{1^2+1^2}\left(\cos(-\frac{\pi}{4})+i\sin(-\frac{\pi}{4})\right)\right]^{53}}$$

$$=\frac{(\sqrt{12})^{71}\left[\cos(-\frac{71}{6}\pi)+i\sin(-\frac{71}{6}\pi)\right]}{(\sqrt{2})^{53}\left[\cos(-\frac{53}{4}\pi)+i\sin(-\frac{53}{4}\pi)\right]}$$

$$=\frac{(\sqrt{12})^{71}\left[\cos\frac{1}{6}\pi+i\sin\frac{1}{6}\pi\right]}{(\sqrt{2})^{53}\left[\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right]}$$

$$=\frac{(\sqrt{12})^{71}}{(\sqrt{2})^{53}}\cdot\frac{\frac{\sqrt{3}}{2}+\frac{1}{2}i}{-\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i}$$

$$=\frac{(\sqrt{12})^{71}}{(\sqrt{2})^{53}}\cdot\frac{\sqrt{3}+i}{-\sqrt{2}+\sqrt{2}i}\cdot\frac{(-\sqrt{2}-\sqrt{2}i)}{(-\sqrt{2}-\sqrt{2}i)}$$

$$=\frac{(\sqrt{12})^{71}}{(\sqrt{2})^{53}}\cdot\frac{-\sqrt{6}-\sqrt{2}i-\sqrt{6}i+\sqrt{2}}{2+2}$$

$$=(\sqrt{2})^{14}(\sqrt{6})^{71}\left[(\sqrt{2}-\sqrt{6})-(\sqrt{2}+\sqrt{6})i\right]$$

$$=\left[(\sqrt{2})^{15}(\sqrt{6})^{71}-(\sqrt{2})^{14}(\sqrt{6})^{72}\right]-\left[(\sqrt{2})^{15}(\sqrt{6})^{71}+(\sqrt{2})^{14}(\sqrt{6})^{72}\right]i$$

$$=\left[2^{43}3^{35}\sqrt{3}-2^{43}3^{36}\right]-\left[2^{43}3^{35}\sqrt{3}+2^{43}3^{36}\right]i$$

#5. Solution:

$x^6+7x^3-8=0$

Let $y=x^3$. then the equation turns to be

$y^2+7y-8=0$

Solve for solutions we get

$$y=\frac{-7\pm\sqrt{7^2+4\times 8}}{2}=\frac{-7\pm 9}{2}=1 \text{ or } -8$$

① For $y=x^3=1$

$$1=\sqrt{1^2+0^2}\,(\cos 0+i\sin 0)$$

$$x_k=\cos\left(\frac{0+2\pi k}{3}\right)+i\sin\left(\frac{0+2\pi k}{3}\right) \qquad k=0,1,2$$

$k=0$, $x_0=\cos 0+i\sin 0=1$

$k=1$, $x_1=\cos\frac{2}{3}\pi+i\sin\frac{2}{3}\pi=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$k=2$, $x_2=\cos\frac{4}{3}\pi+i\sin\frac{4}{3}\pi=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$

② For $y=x^3=-8$

$$-8=8(\cos\pi+i\sin\pi)$$

$$x_{k+3}=\left[\cos\left(\frac{\pi+2\pi k}{3}\right)+i\sin\left(\frac{\pi+2k\pi}{3}\right)\right]\cdot\sqrt[3]{8},\ k=0,1,2$$

$k=0$, $x_3=2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=1+\sqrt{3}i$

$k=1$, $x_4=2(\cos\pi+i\sin\pi)=2(-1)=-2$

$k=2$, $x_5=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5}{3}\pi\right)=2\left(\frac{1}{2}+-\frac{\sqrt{3}}{2}i\right)=1-\sqrt{3}i$

Therefore all the complex solutions of $x^6+7x^3-8=0$ are

$x_0=1$, $x_1=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$, $x_2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$,

$x_3=1+\sqrt{3}i$, $x_4=-2$, $x_5=1-\sqrt{3}i$

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Problems from Textbook

P85.

#4b. Find the cube roots of the following numbers:

(b). $8\sqrt{3} + 8i$

Solution: $(8\sqrt{3} + 8i) = \sqrt{(8\sqrt{3})^2 + 8^2}\,(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

$= 16(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

~~Then~~ $\cancel{\sqrt[3]{8\sqrt{3}+8i} = (8\sqrt{3}+8i)^{\frac{1}{3}} = 16^{\frac{1}{3}}(\cos\frac{\pi}{6\cdot 3} + i\sin\frac{\pi}{6\cdot 3})}$

Suppose $z = r(\cos\theta + i\sin\theta)$ and $z^3 = 8\sqrt{3} + 8i$

Then $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

Therefore $r^3 = 16$, so $r = \sqrt[3]{16} = 2^{\frac{4}{3}}$

and $3\theta = \frac{\pi}{6}$ or $\frac{\pi}{6} + 2\pi$ or $\frac{\pi}{6} + 4\pi$

i.e. $\theta = \frac{\pi}{18}$ or $\frac{13}{18}\pi$ or $\frac{25}{18}\pi$

Therefore $\sqrt[3]{8\sqrt{3}+8i} = 2^{\frac{4}{3}}(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})$ or $2^{\frac{4}{3}}(\cos\frac{13}{18}\pi + i\sin\frac{13\pi}{18})$

or $2^{\frac{4}{3}}(\cos\frac{25}{18}\pi + i\sin\frac{25}{18}\pi)$.

#6. Find all the complex roots of $z^6 + z^3 + 1 = 0$.

Solution: Let $y = z^3$. Then $z^6 + z^3 + 1 = y^2 + y + 1 = 0$

$$y = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$$

so $y_1 = \frac{-1+\sqrt{3}i}{2}$, $y_2 = \frac{-1-\sqrt{3}i}{2}$

① For $y = z^3 = \frac{-1+\sqrt{3}i}{2}$

$$\frac{-1}{2}+\frac{\sqrt{3}}{2}i = \sqrt{\frac{1}{4}+\frac{3}{4}}\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$$

say $z = r(\cos\theta + i\sin\theta)$
$z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

So $z_k = \cos\left(\frac{\frac{2}{3}\pi+2k\pi}{3}\right) + i\sin\left(\frac{\frac{2}{3}\pi+2k\pi}{3}\right)$, $k=0,1,2$

$k=0$, $z_0 = \cos\frac{2}{9}\pi + i\sin\frac{2}{9}\pi$
$k=1$, $z_1 = \cos\frac{8}{9}\pi + i\sin\frac{8}{9}\pi$
$k=2$, $z_2 = \cos\frac{14}{9}\pi + i\sin\frac{14}{9}\pi$

② For $y = z^3 = \frac{-1}{2} - \frac{\sqrt{3}}{2}i$

$$\frac{-1}{2}-\frac{\sqrt{3}}{2}i = \sqrt{\frac{1}{4}+\frac{3}{4}}\left(\cancel{\cos\frac{3}{4}}\ \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right)$$

$$= \cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi$$

So $z_{k+3} = \cos\left(\frac{\frac{4}{3}\pi+2k\pi}{3}\right) + i\sin\left(\frac{\frac{4}{3}\pi+2k\pi}{3}\right)$, $k=0,1,2$

$k=0$, $z_3 = \cos\frac{4}{9}\pi + i\sin\frac{4}{9}\pi$
$k=1$, $z_4 = \cos\frac{10}{9}\pi + i\sin\frac{10}{9}\pi$
$k=2$, $z_5 = \cos\frac{16}{9}\pi + i\sin\frac{16}{9}\pi$

Therefore the original equation $z^6+z^3+1=0$ has six complex roots, which are:

$z_0 = \cos\frac{2}{9}\pi + i\sin\frac{2}{9}\pi$, $z_1 = \cos\frac{8}{9}\pi + i\sin\frac{8}{9}\pi$
$z_2 = \cos\frac{14}{9}\pi + i\sin\frac{14}{9}\pi$, $z_3 = \cos\frac{4}{9}\pi + i\sin\frac{4}{9}\pi$
$z_4 = \cos\frac{10}{9}\pi + i\sin\frac{10}{9}\pi$, $z_5 = \cos\frac{16}{9}\pi + i\sin\frac{16}{9}\pi$.

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#8. Solution: The complex solutions are $\{2-i, 2+i, 7\}$

Then it must have these factors

$(z-(2-i))$, $(z-(2+i))$ and $(z-7)$

for a polynomial $p(z)=0$.

Let's assume $p(z)$ is constructed by these ~~two~~ three simple factors directly.

So

$$p(z) = (z-2+i)(z-2-i)(z-7)$$
$$= (z^2-2z+iz-2z+4-2i-iz+2i+1)(z-7)$$
$$= (z^2-4z+5)(z-7)$$
$$= z^3-4z^2+5z-7z^2+28z-35$$
$$= z^3-11z^2+33z-35$$

Hence $z^3-11z^2+33z-35$ is such a polynomial with complex roots $\{2-i, 2+i, 7\}$.

#9. Solution:

$$\frac{z^3+1}{z^3-1} = i \Rightarrow z^3+1 = iz^3-i$$
$$\Rightarrow z^3(1-i) = -i-1$$
$$\Rightarrow z^3 = \frac{-i-1}{1-i} = \frac{(-i-1)(1+i)}{(1-i)(1+i)} = \frac{-i-1+1-i}{1+1} = \frac{-2i}{2} = -i$$
$$\Rightarrow z^3 = -i = \sqrt{0+(-1)^2}\left(\cos\tfrac{3}{2}\pi + i\sin\tfrac{3}{2}\pi\right)$$

Let $z = r(\cos\theta + i\sin\theta)$

then $z^3 = r^3(\cos 3\theta + i\sin 3\theta) = -i$

so $r=1$, $3\theta = \frac{3}{2}\pi$ or $\frac{3}{2}\pi+2\pi$ or $\frac{3}{2}\pi+4\pi$

then $\theta = \frac{1}{2}\pi$ or $\frac{7}{6}\pi$ or $\frac{11}{6}\pi$

Therefore $z_1 = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi = i$, $z_2 = \cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$ next page $\hookrightarrow$

$Z_3 = \cos\frac{11}{6}\pi + i\sin\frac{11}{6}\pi = \frac{\sqrt{3}}{2} + i\,\frac{-1}{2} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Hence the complex roots of $\frac{z^3+1}{z^3-1} = i$ are $\left\{ i, -\frac{\sqrt{3}}{2} + \frac{-1}{2}i, \frac{\sqrt{3}}{2} - \frac{1}{2}i \right\}$.