Lecture 14 §5.6 IVT

Def'n: Let X be a metric space, a separation of X is a pair U, V of disjoint non-empty, open subset of X whose union is X.

The space X is said to be connected if there does not exist some separation of X.

Theorem: A space X is connected iff the only subsets of X that are both open & closed is X, ϕ.

Theorem: the image of a connected space under a continuous map is connected

Proof: Let $f: X \to Y$ be cont. WLOG assume f is onto. want to show $f(X) = Y$ is connected. Sps $Y = A \cup B$, $A, B \neq \emptyset$, are open $A \cap B = \emptyset$, by continuity of f, $f^{-1}(A)$, $f^{-1}(B)$ are open.

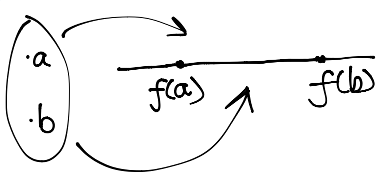
$f^{-1}(A) \cap f^{-1}(B) = \emptyset$

$X = f^{-1}(A) \cup f^{-1}(B)$ contradicts the fact X is connected.

Thm: The real line $\mathbb{R}$ is connected and so are intervals and rays.

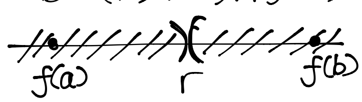
IVT:

Let $f: X \to \mathbb{R}$ be a cont. map where X is a connected metric space, if $a, b \in X$, and $f(a) < r < f(b)$, for some $r \in \mathbb{R}$, $\exists c \in X$ s.t. $f(c) = r$



Proof: $A = (-\infty, r) \cap f(x)$

$B = (r, +\infty) \cap f(x)$



A, B disjoint

$A, B \neq \emptyset$

If for no c, $f(c) = r \Rightarrow A \cup B = f(x)$

$f(x)$ is connected, since X is connected

$A =$ open relative to $f(x)$

$B =$ open w.r.t $f(x) \Rightarrow f(x)$ not connected $\Rightarrow r$ has to be in the range of $f(a), f(b)$.

Ex. $f(x) = x^2$ on $[c, d]$, $f(x)$ is cont. Is it uniformly cont? (Yes)

closed

$[c, d]$: compact. f: cont. $\Rightarrow$ uniformly cont.

Another way:

Let $\varepsilon>0$ be given

$|f(x)-f(a)|=|x^2-a^2|=|x-a||x+a|$

$|x|\leq \max\{|c|,|d|\}$

$|a|\leq \max\{|c|,|d|\}=M$

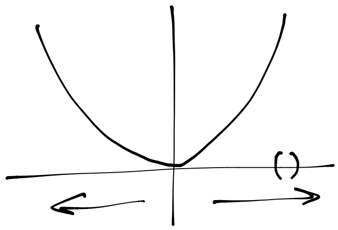
$|x+a|\leq |x|+|a|\leq 2M$

$|x^2-a^2|=|x+a||x-a|<2M\,|x-a|$

Let $\delta=\frac{\varepsilon}{2M} \Rightarrow$ if $|x-a|<\delta$, $|x-a|\cdot 2M<\frac{\varepsilon}{2M}\cdot 2M=\varepsilon$

2). $f(x)=x^2$ uniformly continuous on $\mathbb{R}$? (No)

We will show $\exists\ \varepsilon>0$ s.t. $\forall\ \delta\ \exists$ a pair of point (x,y) & $|x-y|<\delta$ but $|f(x)-f(y)|\geq\varepsilon$



Take $\varepsilon=2$

$|f(k+\frac{1}{k})-f(k)|=|(k+\frac{1}{k})^2-k^2|=|k^2+2+\frac{1}{k^2}-k^2|=|2+\frac{1}{k^2}|>2$

take $x=k+\frac{1}{k}$, $y=k$

$(x-y)=\frac{1}{k}\to 0$ as $k\to\infty$

Proposition: $\forall$ Lipschitz function is uniformly continuous

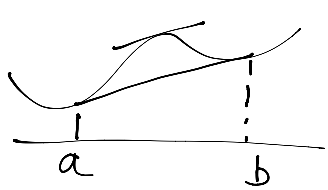
Proof: f is Lipschitz on $S \Rightarrow \exists\ C$ s.t. $\|f(x)-f(y)\|<C\|x-y\|$. Given $\varepsilon>0$ take $\delta=\frac{\varepsilon}{C}$

$\forall$ uniformly cont. fcn is Lipschitz? (No) e.g. $f(x)=\sqrt{x}$

Corollary: $\forall$ linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniformly continuous. (because it is Lipschitz)

Any $f:\mathbb{R}\to\mathbb{R}$ that is differentiable with bdd derivative is uniformly continuous

Apply MVT

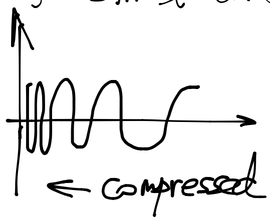


$$\frac{(f(b)-f(a))}{(b-a)}=f'(c)$$

$$\left|\frac{f(b)-f(a)}{b-a}\right|=|f'(c)|\leq M \Rightarrow |f(b)-f(a)|\leq M|b-a|$$

stronger than uniformly continuous

Ex. $f = \sin\frac{1}{x}$ on $(0, 1]$, is it uniformly continuous? (No)



Take $\varepsilon = 1$

$f\left(\frac{1}{(2k+\frac{1}{2})\pi}\right) = \sin\left[(2k+\frac{1}{2})\pi\right] = \sin\frac{\pi}{2} = 1$

$f\left(\frac{1}{(2k-\frac{1}{2})\pi}\right) = \sin\left[(2k-\frac{1}{2})\pi\right] = \sin\left(-\frac{\pi}{2}\right) = -1$

$x_k = \frac{1}{(2k+\frac{1}{2})\pi}$ $\quad y_k = \frac{1}{(2k-\frac{1}{2})\pi}$

$|f(x_k) - f(y_k)| = 2 > \varepsilon$

$|x_k - y_k| = \left|\frac{1}{(2k+\frac{1}{2})\pi} - \frac{1}{(2k-\frac{1}{2})\pi}\right| = \left|\frac{-1}{16k^2-1}\right| \longrightarrow 0$

Ex: $f(x) = \sin x$. Is it uniformly cont? (Yes)

$f'(x) = \cos x$

$|f'(x)| = |\cos x| \leq 1$

$\Rightarrow |\sin x - \sin y| \leq |x - y| \Rightarrow$ uniformly cont.

Ex. $f(x) = x\sin\frac{1}{x}$ on $(0, +\infty)$

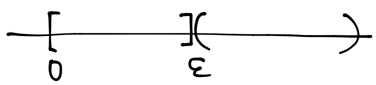
$\lim_{x\to 0} x\sin\frac{1}{x} = 0$

$|x\sin\frac{1}{x}| \leq |x| < \varepsilon$. take $\delta = \varepsilon$

$f(x) = \begin{cases} x\sin\frac{1}{x} & x > 0 \\ 0 & x = 0 \end{cases}$ cont. on $[0, +\infty)$

Is it uniformly continuous? (Yes)

Proof:

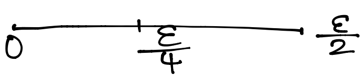


Let $\varepsilon > 0$ be given

? $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{y\to 0}\frac{\sin y}{y} \stackrel{*}{=} 1$

$1 - \frac{1}{x^2} < x\sin\frac{1}{x} < 1$ for $\forall x \geq 1$. Let $0 < \varepsilon < 1$ be given, if $|x| < \frac{\varepsilon}{2}$, $|y| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) - f(y)| = |x\sin\frac{1}{x} - y\sin\frac{1}{y}| \leq |x|\,|\sin\frac{1}{x}| + |y|\,|\sin\frac{1}{y}| < \varepsilon$



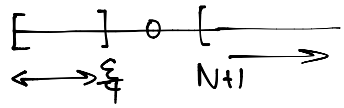
if $x \in [0, \frac{\varepsilon}{4}]$ $\quad |x - y| < \frac{\varepsilon}{4} \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $N > \frac{1}{\varepsilon}$, $x, y > N$

$1-\varepsilon < 1-\frac{1}{C^2} ? < f(x) < 1$

$|f(x)-1| < \varepsilon$, $|f(y)-1| < \varepsilon$

$|f(x)-f(y)| = |f(x)-1+1-f(y)| \le |f(x)-1| + |f(y)-1| < \varepsilon+\varepsilon = 2\varepsilon$



Sps $x, y \in [\varepsilon/4, N+1]$

$f'(x) = \sin\frac{1}{x} - \frac{1}{x}\cos x$

$f'(x) \le 1+x \le 1+4/\varepsilon = M$

By MVT, $|f(x)-f(y)| \le M|x-y|$

Take $\delta = \frac{\varepsilon}{M}$

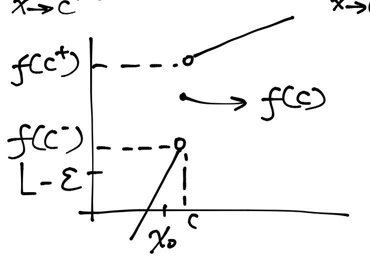
§5.7

Monotone Functions

Monotone: increasing or decreasing strictly

Prop: if f is an increasing function on (a,b), then one-side-limit of f exists at each point $c \in (a,b)$.

$\lim_{x\to c^-} f(x) \le f(c) \le \lim_{x\to c^+} f(x)$



Proof: $F = \{f(x) : a < x < c\}$ non-empty set of real numbers bdd above by $f(c) \Rightarrow \exists \sup$ ($\sup F = L$)

Sps given $\varepsilon > 0$.

Consider $L-\varepsilon$, $\exists x_0$ s.t. $a < x_0 < c$ st. $L-\varepsilon < f(x_0) < L$

For $x_0 < x < c$, $L-\varepsilon < f(x_0) \le f(x) < L$

Take $\delta = c - x_0$

so $c-x < \delta \Rightarrow |f(x)-L| < \varepsilon \Rightarrow \lim_{x\to c^-} f(x) = L$

$L \le f(c)$ since $f(c)$ is an upper bound, but L is sup.

Corollary: The only discontinuity that a monotone function can have is jump discontinuity.

Corollary: A monotone function on $[a,b]$ has at most countably many discontinuities.

Chapter 7 Norms & Inner Products

§ 7.1 Normed Vector Spaces

Normed vector space

Def'n: Let V be a vector space on $\mathbb{R}$; A norm on V is a function $\|\cdot\|$ on V taking values on $[0,+\infty)$ satisfying the following properties:

① positive definite $\|x\|=0$ iff $x=0$

② homogeneous

$\|\alpha x\| = |\alpha|\,\|x\| \quad \forall x \in V,\ \alpha \in \mathbb{R}$

③ triangle inequality

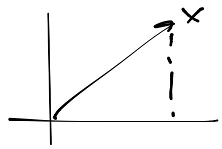
$\|x+y\| \leq \|x\| + \|y\|$

Example of norm

1). Euclidean norm

2). $\mathbb{R}^n$

$$\|x\|_1 = \|(x_1, \cdots, x_n)\| = \sum_{i=1}^{n} |x_i|$$



the pair $(V, \|\cdot\|)$ is a normed vectorspace

3). $\|X\|_\infty = \max\limits_{1\leq i\leq n} |X_i|$

Claim: $\|X\|_1$, $\|X\|_\infty$ are norms

Check triangle inequality:

$$\|X+Y\|_1 = \sum_{i=1}^{n} |X_i + y_i| \leq \sum_{i=1}^{n} (|X_i| + |y_i|) = \sum_{i=1}^{n} |X_i| + \sum_{i=1}^{n} |y_i| = \|X\|_1 + \|y\|_1$$

$$\|x+y\|_\infty = \max_{1\leq i\leq n} |x_i + y_i| \leq \max_{1\leq i\leq n} |x_i| + |y_i| \leq \max_{1\leq i\leq n} |x_i| + \max_{1\leq i\leq n} |y_i| = |x|_\infty + |y|_\infty$$

Balls of radius 1 centered at 0

$\{X : |X| < 1\}$

Euc. norm