May 21st

"Two different ways to define CONTINUITY"? (ε-δ & topological def)

§1.4 Sequences.

Def: A sequence is a function $f: \mathbb{Z}^+ \to S$.

Ex: $1, -1, 1, -1, \dots$

Ex: $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}, \dots$

Ex: $\{\vec{x}_k\}_{k=1}^{\infty}$, $\vec{x}_k \in \mathbb{R}^n$

Ex: $\{\vec{x}_k\}_{k=1}^{n}$ finite sequence

Ex: $\{\vec{x}_k\}_{k=-\infty}^{\infty}$ doubly infinite sequence

Def: A sequence $\{\vec{x}_k\}$ converges to L if $\forall \varepsilon > 0, \exists K$ s.t. $|\vec{x}_k - \vec{L}| < \varepsilon$ when $k > K$

tail of the sequence

Else it diverges.

Def: $\lim\limits_{k\to\infty} x_k = \infty$ (diverges to ∞), $\forall C > 0, \exists K > 0$ s.t. $|x_k| > C$ when $k > K$.

Ex: $\left\{\frac{1}{k}\right\}$, show $\left|\frac{1}{k} - 0\right| = \frac{1}{k} < \varepsilon$, $K = \frac{1}{\varepsilon}$ think $k > K \Rightarrow \frac{1}{K} > \frac{1}{k}$

so $\frac{1}{k} \to 0$

Ex: $\left\{1 + \frac{1}{2^n}\right\}$ Guess $L = 1$

$|a_k - L| = \left|1 + \frac{1}{2^k} - 1\right| = \frac{1}{2^k} < \frac{1}{k}$

claim

Induction: Basis (base case) $2 > 1$, if $2^n > n$, $2^{n+1} = 2 \cdot 2^n > 2n > n+1$ ■

Ex: $\{2^k\}$ diverges to ∞.

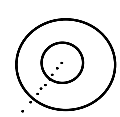
Theorem: $x \in \bar{S}$ iff $\exists x_k \in S$ s.t. $\vec{x}_k$ converges to $\vec{x}$.

Proof:

($\Leftarrow$) $\forall \varepsilon > 0, \exists K$ s.t. $|\vec{x}_k - \vec{x}| < \varepsilon$ when $k > K$

$\Rightarrow B(\varepsilon, \vec{x}) \cap S \neq \emptyset \Rightarrow \vec{x} \in \bar{S}$

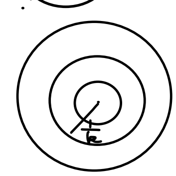
$\vec{x}_k$



($\Rightarrow$) $\vec{x} \in \bar{S}$

take nested sequence of balls of radius $\frac{1}{k}$.

$B(\frac{1}{k}, \vec{x}) \cap S \neq \emptyset$ so choose $x_k \in B(\frac{1}{k}, \vec{x}) \cap S$



Need $\vec{x}_k \to \vec{x}$, $|\vec{x}_k - \vec{x}| < \frac{1}{k}$ so as $\{\frac{1}{k}\} \to 0$, $\vec{x}_k \to \vec{x}$

$\blacksquare$

Thm: $S \subset \mathbb{R}^n$, $\vec{a} \in S$, $f: S \to \mathbb{R}^m$, then the following are equivalent:

a) f is continuous
b) $\forall \{\vec{x}_k\}$ such that $\vec{x}_k \to \vec{a}$, $f(\vec{x}_k) \to f(\vec{a})$

Proof:
$(a \Rightarrow b)$. As f is continuous, $\forall \varepsilon > 0$, $|f(\vec{x}_k) - f(\vec{a})| < \varepsilon$ when $|\vec{x}_k - \vec{a}| < \delta$
As $\vec{x}_k \to \vec{a} \Rightarrow \exists K$ s.t. $|\vec{x}_k - \vec{a}| < \delta$ for $k > K$

Then $\forall \varepsilon > 0$ $\exists K$, s.t. $|f(\vec{x}_k) - f(\vec{a})| < \varepsilon$ when $k > K$.

$(a \Leftarrow b)$ Using contrapositive! $(\neg a \Rightarrow \neg b)$
Assume f is not continuous, i.e. $\exists \varepsilon > 0$, $\forall \delta > 0$, $\exists \vec{x} \in S$
s.t. $|f(\vec{x}) - f(\vec{a})| \geq \varepsilon$ when $|\vec{x} - \vec{a}| < \delta$.

Choose $\delta_k = \{1, \frac{1}{2}, \frac{1}{3}, \cdots\}$ Then for each δ_k $\exists x_k$

$|x_k - a| < \delta_k = \frac{1}{k}$ so converges, i.e. $\vec{x}_k \to \vec{a}$.
But $|f(\vec{x}_k) - f(\vec{a})| \geq \varepsilon$. CONTRADICTION.

$\blacksquare$

Claim: TFAE (The following are equivalent)
a). $\to$ $\varepsilon - \delta$
b). $\to$ sequence
c). $f^{-1}(\text{open}) = \text{open}$ Topological def! (we proved $a \Leftrightarrow c$)

Can define continuity in either way.

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§ 1.5 Completeness

different notations
lub, sup

Def: A non-empty subset of $\mathbb{R}$, S has a least upper bound l,
if $x \leq l$ $\forall x \in S$, and if l' is an upper bound then $l \leq l'$

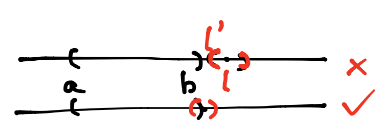
Axiom for $\mathbb{R}$. Every non-empty set with an upper bound, has a least upper bound.

Ex: $\text{lub}\{(0,1)\} = 1$

Analagously, inf, glb

For $\mathbb{R}^n$, no canonical ordering

Equivalently: $\forall \varepsilon > 0$, $B(\varepsilon, l) \cap S \neq \emptyset$



Def: A sequence is bdd (bounded) in $\mathbb{R}$. if $\{x_k\}$ all in a bounded interval.

Def: A sequence is increasing if $x_n \le x_m$ when $n \le m$.

Def: A sequence is decreasing if $x_n \ge x_m$ when $n \le m$.

Def: A sequence is monotone if either increasing or decreasing.

Thm (Bounded monotone sequence thm)
if $\{x_k\}$ is bdd & monotone, then it converges.

points cluster:

closer, more

Proof: For increasing, Consider set of x_k S = range of $\{x_k\}$ if has a least upper bound via completeness. $\Rightarrow x_k \le l$ (~~upper, not least, not proved yet~~)

as it's bounded

$x_k > l - \varepsilon, \forall k > K$ as it's least

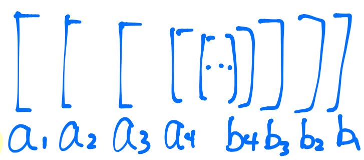
$l - \varepsilon < x_k < l, \forall k > K \Rightarrow |x_k - l| < \varepsilon . \forall k > K \Rightarrow x_k$ converges to l.

Similarly, for decreasing.

Translated a notion of completeness to one about convergence.

a bounded set $\Rightarrow$ a bounded sequence.

Theorem (Lemma / Nested interval theorem)



Let $I_k = [a_k, b_k]$, $I_{k+1} \subseteq I_k$ and $b_k - a_k \to 0$
Then $\exists$ exactly one $x \in I_k$ $\forall k$.

basic idea: "a_i is an increasing seq. L,
b_i is a decreasing seq. L'
$|L - L'| \to 0$, $L = L'$ the same"

Proof: $\{a_k\}$ is increasing, $a_k \ge a_{k-1}$ & $a_1 \le a_k \le b_1$ so bounded.

By Marginal Convergence Theorem $\Rightarrow$ lub l.

likewise $\{b_k\}$ decreasing, & $a_1 \le b_k \le b_1$

By M.C.T. $\Rightarrow$ glb g

$b_k - a_k \to 0$

also know $b_k - a_k \to g - l = 0 \Rightarrow g = l \in [a_k, b_k] = I_k$ $\forall k$. (why? since $b_k \to g$, $a_k \to l$) at least 1

Suppose l', $|l - l'| < \varepsilon$ so have K s.t. $|b_k - a_k| < \varepsilon$ $\forall k > K$.
at most 1 contradiction

$\therefore$ $\boxed{\text{Exactly one}}$ point.

Subsequence

Def a subsequence of a sequence $\{x_k\}$ is denoted $\{x_{k_j}\}_{j=1}^{\infty}$ is defined by $f: \mathbb{Z}^+ \to \mathbb{Z}^+ \to S$, namely f is one-to-one/injective , and increasing.

Ex:
Seq: $\{x_1, x_2, \cdots\}$
subseq $\{x_2, x_4, x_{77}, \cdots\}$

$$\boxed{\begin{array}{l} \text{Seq. converges} \Rightarrow \text{subseq converges} \\ \text{subseq. converges} \ \underrightarrow{\text{not necessarily}} \ \text{seq. converges} \end{array}}$$

NOTE:
$\{\vec{x_j}\}$. $\{\vec{x}_{jm}\}$ $\leftarrow$ m-th component of j-th vector

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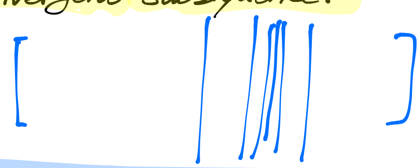
Thm (Bolzano-Weieistrass for $\mathbb{R}$)

[weaker condition, weaker result]

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

"half-half infinite"



Inductively, we will bisect I_k into two halves.
Choose I_{k+1} such that it is the half with ∞ many points of $\{x_k\}$ in it.
(or left if both). Then construct our sequence of $\{x_{k_j}\}$'s by choosing $x_{k_j} \in I_k$

By Nested Interval Theorem $\Rightarrow \exists$ exactly one $l \in I_k \ \forall k$

$$|x_{k_j} - l| \leq 2^{-k}|b-a| \to 0$$

So $x_{k_j} \to l$, i.e. constructed a convergent subsequence.

Thm (Bolzano-W for $\mathbb{R}^n$)

Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence in $\mathbb{R}^n$.

Proof: $|\vec{x_k}| < B \ \forall k$, $\vec{x}_{km} \in [-B, B]$

inductively: $\overset{\text{1 st component}}{\vec{x}_{k,1}} \leftarrow$ real , bounded $\Rightarrow$ by B-W for $\mathbb{R}$, $\exists$ a convergent subseq.
$\{\vec{x}_{k_j}\}$ ($\vec{x}_{k_j,1}$ converges)

For 2nd component get a new subsequence that converges.
...
nth component have a subsequence of $\{\vec{x_k}\}$ that converges for each component.

$\Rightarrow \cdots\cdots$ on $\mathbb{R}^n$.

(basically, this is like. 1st component converges, a new subseq. then 2nd component in this subseq. also converges, forming a new subseq, then 3rd component ...) all components converge, the whole vector converges!

Result from completeness extends to $\mathbb{R}^n$.

Cauchy Sequence

(instead of saying each terms in seq. goes to a limit, Cauchy means all terms are getting closer!!!)

Def: A sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ is Cauchy if $\exists K$. $|\vec{x}_k - \vec{x}_j| < \varepsilon$ when $k, j > K$

Thm: A sequence $\{\vec{x}_k\}$ in $\mathbb{R}^n$ is Cauchy iff it converges.

Proof:

($\Leftarrow$) Assume $\vec{x}_k \to \vec{L}$, $|\vec{x}_k - \vec{x}_j| \leq |\vec{x}_k - \vec{L}| + |\vec{x}_j - \vec{L}|$

Since $\vec{x}_k - \vec{L} \to \vec{0}$, $\vec{x}_j - \vec{L} \to \vec{0}$ then $|\vec{x}_k - \vec{x}_j| \to 0$

($\Rightarrow$) Let $\{\vec{x}_k\}$ be Cauchy. Choose $\varepsilon = 1$, so can find K s.t. $|\vec{x}_k - \vec{x}_j| < 1$ when $k, j > K$.

$|\vec{x}_k| < |\vec{x}_{k+1}| + 1$ so bounded. (Cauchy seq is bdd)

By B-W $\Rightarrow \exists$ a convergent subseq. $\{\vec{x}_{k_j}\} \to \vec{L}$

$|\vec{x}_k - \vec{L}| = |\vec{x}_k - \vec{L} + \vec{x}_{k_j} - \vec{x}_{k_j}| \leq |\vec{x}_k - \vec{x}_{k_j}| + |\vec{x}_{k_j} - \vec{L}|$

pick $\forall j > J$, so $k_j > K$, then

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ when } k > K$$

Def (Alternate):

A space is complete if every Cauchy Sequence converges.

Ex: $\mathbb{R}^n$ is complete $\leftarrow$ no ordering

Ex: $\mathbb{Q}$ is not complete in a normal sense.