Lecture 3.

Random Variables.

Def. A random variable (r.v.) X on the probability space $(\Omega, \mathcal{F}, P)$ is a mapping $X : \Omega \to \mathbb{R}$ s.t.
$\underline{\{\omega \in \Omega : X(\omega) = x\} \in \mathcal{F}}$ for all $x \in \mathbb{R}$.

Similarly, vector $\begin{pmatrix} X \\ Y \end{pmatrix} : \Omega \to \mathbb{R}^2$ is a random vector.

Ex. Toss a coin.

$$\Omega = \{\underset{\omega_1}{H}, \underset{\omega_2}{T}\}$$

$$X = \{\#\text{ of heads}\} \in \{0, 1\}$$

$$X(\omega_1) = X(H) = 1, \quad X(\omega_2) = X(T) = 0$$

Toss a coin 10 times

$$X \in \{0, 1, \dots, 10\}$$

Ex. X = mark for this course

$$\Omega = [0, 100], \quad X(\omega) = \omega \in [0, 100]$$

Discrete Random Variable.

Recall: Ω is finite, then for any $A \in \mathcal{F}$

$$A = \bigcup_{w_i \in A} \{w_i\}, \quad P(A) = \sum_{w \in A} P(\{w\})$$

Ex. Roll a die 6 times. $A = \{$all outcomes same$\}$

$$P(A) = P(\{111111\}) + P(\{222222\}) + \dots$$

Formal definition of a Discrete Probability Space $(\Omega, \mathcal{F}, P)$:

Ω - countable

$p: \Omega \to [0;1]$: $p(w) \geq 0$, $w \in \Omega$

$$\sum_{w \in \Omega} p(w) = 1$$

$\to$ probability mass function

Def. A r.v. X is said to be discrete if it can take only a finite or countably infinite number of distinct values.

$X: \Omega \to$ a countable set

$$p_X(x) = P(X = x) \geq 0, \quad \sum_x p_X(x) = 1$$

Def. The probability distribution of a discrete r.v. X is represented by a formula, a table or a graph which provides the list of all possible values that X can take and the pmf for each value.

Ex. we roll a die.

$X =$ # comes up , $X(\omega) = \omega \in \{1, \dots 6\}$

$$P(X=1) = P(X=2) = \dots = P(X=6) = \frac{1}{6}$$

$$p_X(x) = \frac{1}{6}, \quad x = 1, 2, \dots 6$$

X has a uniform dist'n.

Bernoulli Distribution.

Ex. Roll a die. $X = \begin{cases} 1, & 6 \text{ comes up} \\ 0, & \text{ow} \end{cases}$

$$P(X=1) = \frac{1}{6}, \quad P(X=0) = \frac{5}{6}$$

pmf : $p_X(x) = \begin{cases} \frac{1}{6}, & x = 1 \\ \frac{5}{6}, & x = 0 \end{cases}$

x	p
0	5/6
1	1/6

$X = \begin{cases} 1, & \text{with prob. } p \\ 0, & \text{with prob. } q \end{cases}$ ~ Bernoulli Dist'n with parameter p

$q = 1 - p$

Binomial Distribution.

Ex. Roll a die n times and count the number of times 6 came up.

$X = \#$ of times we have '6'

$X \in \{0, 1, 2, \dots, n\}$

$$P(X=3) = \binom{n}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{n-3}$$

In general, if identical Bernoulli trials are repeated n times independently and X is a r.v. that counts the number of successes in the n trials then

$$\boxed{p_X(x)} = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x \in \{0, 1, \dots, n\}$$

p = prob. of success

$X \sim$ Binomial dist'n with parameters n and p. $X \sim \text{Bin}(n, p)$

Q: Is $p_X(x)$ a valid pmf?

$p_X(x) \geq 0$

$$\sum_{x=0}^{n} p_X(x) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x}$$

$$\underset{\text{binomial formula}}{=} (p + (1-p))^n = 1$$

Geometric Distribution.

Ex. we roll a die untill the first 6 comes up.

$X = \#$ of rolls untill we get 1^{st} 6.

$$P(X=4) = \left(\frac{5}{6}\right)^3 \frac{1}{6} =$$

$$x = 1, 2, 3, \ldots$$

In general, X is a r.v. that counts the # of trials until the 1^{st} success

$$p_X(x) = \underline{P}(X=x) = (1-p)^{x-1} p$$

$$X \sim \text{Geom}(p)$$

Question: Is this a valid pmf?

$$\sum_x p_X(x) = \sum_{x=1}^{\infty} (1-p)^{x-1} p = p + qp + q^2p + \ldots \quad 1-p = q$$

$$= p(1+q+q^2+\ldots) = \frac{p}{1-q} = \frac{p}{p} = 1$$

Calculate $\underline{P}(X>k)$

$$\underline{P}(X>k) = \underline{P}(X=k+1) + P(X=k+2) + \ldots$$

$$= q^k p + q^{k+1} p + \ldots = pq^k(1+q+q^2+\ldots)$$

$$= pq^k \frac{1}{1-q} = q^k$$

Memoryless property of geometric r.v:

$$P[X>m+n \mid X>m] = P[X>n], \quad m,n \in \mathbb{Z}^+$$

$$\hookrightarrow = \frac{P[X>m+n \text{ and } X>m]}{P[X>m]} = \frac{P[X>m+n]}{P[X>m]}$$

$$= \frac{q^{m+n}}{q^m} = q^n = P[X>n] \qquad \blacksquare$$

Negative Binomial Distribution.

Ex. We roll a die until the second 6 comes up

$$P(X=4) = 3\cdot\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2$$

X = # of roll until we get two '6'

In general, X = total # of experiments when waiting for r^{th} success

$$P_X(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \quad x = r, r+1, \ldots$$

$X \sim$ neg. bin (p, r)

Q: Is $p_X(x)$ valid?

Show: $\displaystyle\sum_{x=r}^{\infty} \binom{x-1}{r-1} p^r (1-p)^{x-r} = 1$

Hint: $\displaystyle\sum_{k=0}^{\infty} \binom{k+r-1}{r-1} w^k = (1-w)^{-r}$

Hypergeometric Distribution.

Ex. A hat contains 12 tickets : 7 black and 5 white.
Three tickets are drawn at random.
let X = # of black tickets drawn.

$X \in \{0, 1, 2, 3\}$

$$P(X=2) = \frac{\binom{7}{2}\binom{5}{1}}{\binom{12}{3}}$$

In general, N = # of balls = 12
r = # of black balls = 7
$N-r$ = # of white balls
x = # of black balls drawn when taking n ball without replacement.

$$P(X=k) = \frac{\binom{r}{k}\binom{N-r}{n-k}}{\binom{N}{n}}$$

Poisson Distribution.

- Model for the number of events occuring in a time (or space) interval where λ (a parameter of the distribution) is the rate of the occurrence of the events per one unit of time (or space).

$$X \sim \text{Poisson}(\lambda)$$

$$p_X(x) = P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0, 1, 2, \dots$$

X = # of events per one unit of time

Q: Is $p_X(x)$ valid?

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} \boxed{\sum_{x=1}^{\infty} \frac{\lambda^{x}}{x!}} = e^{-\lambda} e^{\lambda} = 1$$

$$e^{\lambda}$$

Ex. The number of cars that cross Spadina and Bloor intersection is a Poisson r.v. with $\lambda = 15$ cars per minute. What is the probability that in a given minute exactly 8 cars will cross the intersection? less than 8?

$$P(X=8) = \frac{e^{-15} \cdot 15^{8}}{8!} =$$

$$P(X<8) = p(0) + p(1) + \dots + p(7)$$

$$= \frac{e^{-15}\, 15^{0}}{0!} + \dots + \frac{e^{-15}\, 15^{7}}{7!}$$

$$P(X>8) = 1 - P(X \le 8)$$

Distribution Function of Random Variables.

Def. A cumulative distribution function (cdf) of a r.v. X is a mapping $F : \mathbb{R} \to [0,1]$ defined by

$$F_X(x) = P(\{\omega \in \Omega : X(\omega) \le x\}) = P(X \le x)$$

X is discrete with pmf $p_X(x)$, $x = 0, 1, 2 \ldots$

$$F_X(s) = \begin{cases} 0, & s < 0 \\ p_X(0) + p_X(1) + \ldots + p_X(\lfloor s \rfloor), & s \ge 0 \end{cases}$$

$\lfloor s \rfloor$ = greatest integer $\le s$

Ex. $X \sim Bin(2, \frac{1}{2})$. Find $F_X(x)$.

$$p_X(x) = \binom{2}{x} \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{2-x}, \quad x = 0, 1, 2$$

$$p_X(0) = \frac{1}{4}, \quad p_X(1) = \frac{1}{2}, \quad p_X(2) = \frac{1}{4}$$

$$F(-2) = P(X \le -2) = 0$$

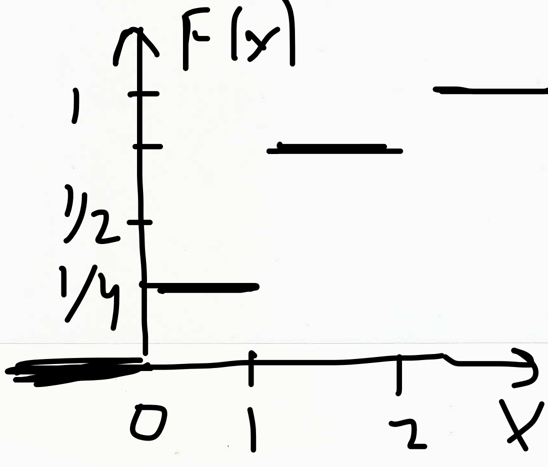
$$F(0.5) = P(X \le 0.5) = p_X(0) = \frac{1}{4}$$

$$F(1.5) = P(X \le 1.5) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$$

$$F(3) = P(X \le 3) = \sum_x p_X(x) = 1$$

$$F_X(x) = \begin{cases} 0, & x < 0 \\ 1/4, & 0 \le x < 1 \\ 3/4, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases}$$

Step function



Properties of cdf:

- F is monotone, non-decreasing
- $F(x) \to 0$ as $x \to -\infty$
- $F(x) \to 1$ as $x \to \infty$
- $F(x)$ is continuous from the right
- For $a < b$ $\quad P(a < X \leq b) = F_X(b) - F_X(a)$

why?

$$P(a < X \leq b) = P(\{a < X\} \cap \{X \leq b\})$$

$$= P(a < X) + P(X \leq b) - P(\underbrace{\{a < X\} \cup \{X \leq b\}}_{= R})$$

$$= P(a < X) + P(X \leq b) - 1$$

$$= P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a)$$

Relation between Binomial and Poisson Distributions,

$$p_X(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

p – prob. of succes $\to 0$

$n \to \infty$

Choose these limits so that

$$pn = \lambda$$

$$\lim_{n \to \infty} \binom{n}{x} p^x (1-p)^{n-x} = \frac{\lambda^x}{x!} e^{-\lambda}$$

$$\lambda = pn$$

$$\lim_{n\to\infty} \binom{n}{x} p^{x} (1-p)^{n-x} =$$

$$= \lim_{\substack{n\to\infty \\ np=\lambda}} \frac{n(n-1)\cdot\ldots\cdot(n-x+1)}{x!} \left(\frac{\lambda}{n}\right)^{x} \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \frac{\lambda^{x}}{x!} \lim \frac{n(n-1)\cdot\ldots\cdot(n-x+1)}{n^{x}} \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^{x}}{x!} \lim \left(\frac{n}{n}\right)\cdot\left(\frac{n-1}{n}\right)\cdot\ldots\cdot\left(\frac{n-x+1}{n}\right) \left(1-\frac{\lambda}{n}\right)^{n} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= \frac{\lambda^{x}}{x!} \lim_{n\to\infty} 1\cdot\left(1-\underbrace{\frac{1}{n}}_{\to 0}\right)\cdot\ldots\cdot\left(1-\underbrace{\frac{x-1}{n}}_{\to 0}\right) \left(\quad\right)^{n} \left(\quad\right)^{-x}$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{n} = e^{-\lambda}$$

$$= \frac{\lambda^{x}}{x!} e^{-\lambda} - \text{poisson}(\lambda)$$

$$n\cdot(n-1)(n-2)\cdot\ldots\cdot(n-x+1)$$

$$= (n-0)(n-1)\ldots(n-(x-1))$$