Lecture 13

1. Tests will be returned Tuesday.
2. Usual office hours resume next week.
3. Next week's quiz will cover up to today's class.

Singularities

Three Types:

① Removable: $|f(z)|$ is bdd as $z \to z_0$

② Pole: $\lim_{z \to z_0} |f(z)| = \infty$

③ Essential: Neither ① nor ② hold

Basic examples:

i. $f(z) = \frac{z^2-1}{z-1}$ $z_0 = 1$ is removable

ii. $f(z) = \frac{1}{z-1}$ $z_0 = 1$ is a pole

iii. $f(z) = e^{\frac{1}{z-1}}$ $z_0 = 1$ is an essential singularity.

Case (i). There is an analytic F s.t. $F = f$ except at z_0 but $F(z_0)$ is defined.

Ex: $f(z) = \frac{z^2-1}{z-1}$, $F(z) = z+1$

$$f(z) = \frac{z^2-1}{z-1} = z+1 = F(z) \text{ if } z \neq 1$$

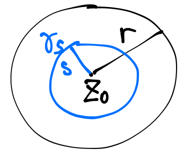
Case (ii) If f has a pole at z_0 then we can write $f(z) = \frac{1}{(z-z_0)^m} g(z)$ with g analytic & $g(z_0) \neq 0$. m is unique & we call it the order of the pole, $\operatorname{ord}(f : z_0) = m$

More about Poles.

Residues: Suppose f is analytic in a punctured disk $0 < |z - z_0| < r$.

Then $\frac{1}{2\pi i} \int_{\gamma_s} f(z)dz$ does not depend on the curve γ_s.

γ_s is a circle of radius s centered at z_0. $(s < r)$



We call $\frac{1}{2\pi i} \int_\gamma f(z)dz = \operatorname{Res}(f : z_0)$ the residue of f at z_0.

(γ is any simple closed curve)

- Why does $\int_{\gamma_s} f(z)dz$ not depend on s? Residue doesn't depend on γ (or say radius of γ : s)?

A:

Let γ_1, γ_2 be 2 different circles.

Split the region between γ_1 & γ_2 into 4 regions R_1, R_2, R_3 & R_4 as in the picture.

- f is analytic on R_1, R_2, R_3, R_4. We're done by Cauchy's Theorem.

$$\frac{1}{2\pi i}\left(\int_{\partial R_1} f(z)dz+\int_{\partial R_2} f(z)dz + \int_{\partial R_3} f(z)dz + \int_{\partial R_4} f(z)dz\right)=0+0+0+0$$

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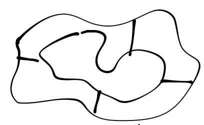
& actually we integrate γ_1 & γ_2 (opposite direction). & the line segments 2 times each with opposite directions so line segments canceled out.

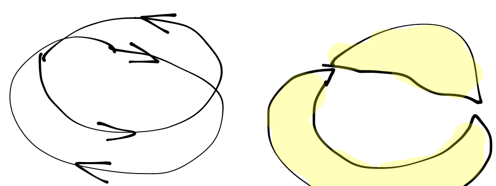
In fact: the previous formula $= \frac{1}{2\pi i}\int_{\gamma_1} f(z)dz - \frac{1}{2\pi i}\int_{\gamma_2} f(z)dz = 0$

$$\Rightarrow \frac{1}{2\pi i}\int_{\gamma_1} f(z)dz = \frac{1}{2\pi i}\int_{\gamma_2} f(z)dz.$$

so γ_1 & γ_2 are the same. i.e. "does not depend on S"

- This holds for 

and (a little bit harder but still holds)



Recall: ① $\boxed{\frac{1}{2\pi i}\int_\gamma \frac{1}{(z-z_0)^n}dz = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{otherwise}\end{cases}}$

② If f has a pole of order m at z_0, we can write $f(z)=\frac{1}{(z-z_0)^m}g(z)$ where $g(z)$ is analytic & $g(z_0)\neq 0$.

Write g as a power series

$$g(z)=\sum C_k (z-z_0)^k$$

$$\Rightarrow f(z) = \frac{1}{(z-z_0)^m}\cdot\left(\sum_{k=0}^{\infty} C_k (z-z_0)^k\right) = \sum_{k=0}^{\infty} C_k (z-z_0)^{k-m}$$ ← converges absolutely for $0<|z-z_0|<r$

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \frac{1}{2\pi i}\int_\gamma \sum_{k=0}^{\infty} C_k (z-z_0)^{k-m}dz$$

$$= \frac{1}{2\pi i}\sum_{k=0}^{\infty} C_k \int_\gamma (z-z_0)^{k-m}dz$$

By ① $\int (z-z_0)^{k-m}dz = 0$ unless $k-m=-1$, $k=m-1$

$$= \frac{1}{2\pi i} C_{m-1}\, 2\pi i$$

$$= C_{m-1}$$

$\Rightarrow$ Residue$(f : z_0) = C_{m-1}$ where $g(z)=\sum_{k=0}^{\infty} C_k (z-z_0)^k$ & z_0 is a pole of order m.

C is not coefficient for power series of f but power series of g

Warning: Don't use P.S. for f to find Res$(f : z_0)$!

Ex: Find Res$(f: -2)$ for $f(z) = \dfrac{z^2+3z+1}{z+2} = \dfrac{1}{\underset{\downarrow\, m}{(z+2)^1}} \underbrace{(z^2+3z+1)}_{g}$

We need $g(z) = z^2+3z+1$ expanded in power series of $(z+2)$

$g(z) = (z+2)^2 - (z+2) - 3 = -3 - (z+2) + (z+2)^2$

$C_{m-1} = C_{1-1} = C_0 = -3 = \text{Res}(f: -2)$

Ex2: Find Res$(f: 1)$ where $f(z) = \dfrac{e^z}{(z-1)^3}$

$f(z) = \dfrac{1}{(z-1)^3} e^z$

$g(z) = e^z$, $m = 3$

want to expand e^z in powers of $(z-1)$.

$e^z = e \cdot e^{z-1} = e \sum \dfrac{(z-1)^k}{k!} = \sum \dfrac{e}{k!}(z-1)^k$

$\Rightarrow C_{m-1} = C_{3-1} = C_2 = \dfrac{e}{2!} = \dfrac{e}{2}$