Lecture 12

Continue §5.1 Limits and Continuity

Def: A function $f: S \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called a Lipschitz function if there is a constant C s.t.

$\| f(x) - f(y) \| \leq C \| x - y \|$ for all $x, y \in S$

The Lipschitz constant of f is the smallest C for which this condition holds.

Proposition: Every Lipschitz function is continuous

Proof: Let $\varepsilon > 0$ be given. Take $\delta = \frac{\varepsilon}{C}$, then if $\|x-y\| < \delta$,

$$\| f(x) - f(y) \| \leq C \| x - y \| < C\delta = \varepsilon$$

■

Thm: Every linear map $A: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz, and therefore is continuous

Proof: $A(x_1, x_2, \cdots, x_n) = \left(\sum_{j=1}^n a_{1j} x_j, \cdots, \sum_{j=1}^n a_{mj} x_j \right)$

$$A = (a_{ij}) \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\| Ax - Ay \| = \| A(x-y) \| = \left(\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} (x_j - y_j) \right)^2 \right)^{\frac{1}{2}}$$

$|\langle x, y \rangle| \leq \|x\| \|y\|$

$\langle u, v \rangle^2 \leq \|u\|^2 \|v\|^2$

Let $u = (a_{i1}, \cdots, a_{in})$, $v = (x_1 - y_1, \cdots, x_n - y_n)$

$\left| \sum_{j=1}^n a_{ij} (x_j - y_j) \right|^2 \leq \sum_{j=1}^n |a_{ij}|^2 \sum_{j=1}^n |x_j - y_j|^2$ by the Schwartz inequality

Let $C = \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}}$

$\| Ax - Ay \| = \| A(x-y) \|$

Note $\sum_{j=1}^n (x_j - y_j)^2 = \| x - y |^2$

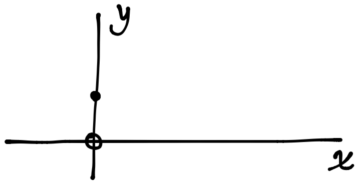
$$\| A(x-y) \| = \left(\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} (x_j - y_j) \right)^2 \right)^{\frac{1}{2}}$$
$$\leq \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \|x-y\|^2 \right)^{\frac{1}{2}} = \| x - y \| \left(\sum_{i=1}^m \sum_{j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} = C \|x-y\|$$

Ex:

$\pi(x_1, \cdots, x_n) = x_j$ coordinate functions

Linear $\Rightarrow$ Lipschitz $\Rightarrow$ continuous

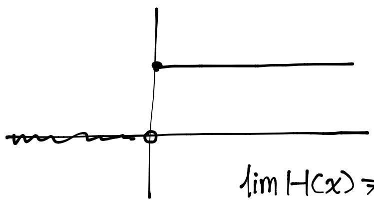
§ 5.2 Discontinuous functions



$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

$f: \mathbb{R} \to \mathbb{R}$
removable singularity

The Heaviside function



$$\begin{cases} H(x) = 0 & x < 0 \\ H(x) = 1 & x \geq 0 \end{cases}$$

Jump singularity

$$\lim_{x \to 0^-} H(x) \neq \lim_{x \to 0^+} H(x)$$

$$\lim_{x \to a^-} f(x) = f(a^-)$$

$$\lim_{x \to a^+} f(x) = f(a^+)$$

Ex: $f: \mathbb{R}^n \setminus \{0\}$ $\quad f(x) = \frac{1}{\|x\|}$

Def: The limit of a function $f(x)$ as $x \to a$ is infinity if $\forall N \in \mathbb{N}$ $\exists r > 0$ s.t. $f(x) > N$ for all $0 < |x-a| < r$.

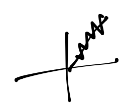
$$\lim_{x \to 0} \frac{1}{\|x\|} = +\infty \qquad f: \mathbb{R}^n \to \mathbb{R}$$

For any $N \in \mathbb{N}$, take $r = \frac{1}{N}$, then $\left|\frac{1}{\|x\|}\right| = \frac{1}{\|x\|} > N$ if $0 < \|x\| < r$

Ex:

$$f(x,y) = \begin{cases} \frac{x^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

$y = r\sin\theta$, $x = r\cos\theta$ $\quad$, $f(x,y) = \frac{r^2\cos^2\theta}{r^2(1)} = \cos^2\theta$

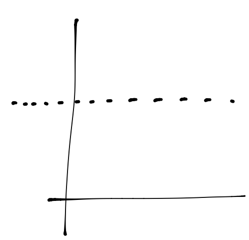


Ex: For any subset A of $\mathbb{R}^n$, the characteristic function of A is

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

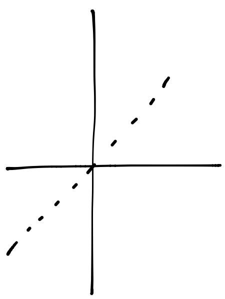
Take $A = \mathbb{Q} \subset \mathbb{R}$

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{o.w.} \end{cases}$$



Not continuous: $\forall r > 0$, $\exists x$ s.t. $|x-a| < r$, $|f(x) - f(a)| = 1$

$$x\chi_{\mathbb{Q}}(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & \text{o.w.} \end{cases}$$



Continuous ONLY at (0,0)

why? $\lim_{x\to 0} x\chi_{\mathbb{Q}} = 0 = f(0)$

$|f(x)| \le |x| < \varepsilon$

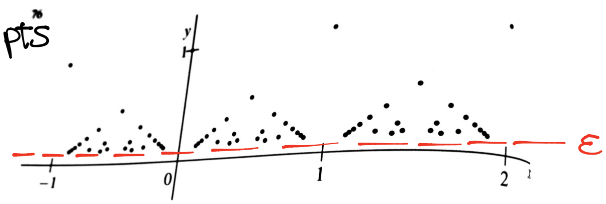
Take $r = \varepsilon$

Ex: $f:[0,1] \longrightarrow \mathbb{R}$

$f(0) = f(1) = 1$

$f(x) = \begin{cases} 0 & x \notin \mathbb{Q} \\ \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms}, \ q > 0 \end{cases}$

Continuous at every irrational pts



Claim: $f(x)$ is continuous at every irrational pt on $[0,1]$

In fact, $\lim_{x\to a} f(x) = 0$ for all $a \in [0,1]$

We need to show that given $\varepsilon > 0$ $\exists$ $\delta > 0$ s.t.

$|f(x)| < \varepsilon$ for all $0 < |x-a| < \delta$

Consider any $\varepsilon > 0$, Let $n \in \mathbb{N}$ be large enough s.t. $\frac{1}{n} \le \varepsilon$

$|f(x)| \ge \varepsilon$? $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \cdots, \frac{n-1}{n}$ (have to stop here, so a finite list)

$\frac{1}{n+1}$, $f(\frac{1}{n+1}) = \frac{1}{n+1} < \varepsilon$

$f(\frac{2}{n+1}) = \frac{1}{n+1} < \varepsilon$

If a is rational and is a point on this list, simply remove it.

$|a - y_i|$ where y_i is on the list

Let $\delta = \min_i |a - y_i|$

Let $a = \frac{1}{10}$, Let $\varepsilon = \frac{1}{5}$, choose $n=5$

$\underbrace{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}}_{\text{potentially "bad pts"}}$. that's it

pick $\frac{1}{5}$, $|\frac{1}{5} - \frac{1}{10}| = \frac{1}{10}$

take $\delta = \frac{1}{10}$, $|x - \frac{1}{10}| < \frac{1}{10}$

§ 5.3 Properties of Continuous Functions

Def: A function $f : (X, \rho) \to (Y, \sigma)$ is continuous if for all $x_0 \in X$ & $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\sigma(f(x), f(x_0)) < \varepsilon$ whenever $\rho(x, x_0) < \delta$. (X, ρ), (Y, σ) are metric spaces

Theorem: let $f : (X, \rho) \to (Y, \sigma)$ be a function between two metric spaces. Then the following are equivalent

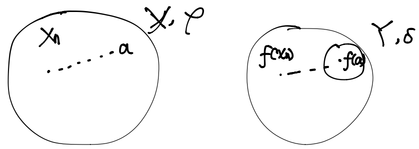
(1) f is continuous on X

(2) For every sequence (x_n) with $\lim_{n\to\infty} x_n = a \in X$, $\lim_{n\to\infty} f(x_n) = f(a)$

(3) $f^{-1}(U) = \{x \in X : f(x) \in U\}$ is open in X for every open set U in Y.

Proof: (1) ⇒ (2)

Suppose f is continuous



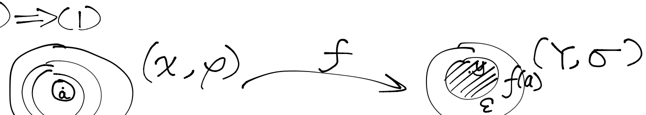
Let $\varepsilon > 0$ be given

(A) $\exists\, \delta > 0$, s.t. $\sigma(f(x), f(a)) < \varepsilon$ whenever $\rho(x, a) < \delta$ (by definition of continuity of f)

(B) $\exists\, N$ s.t. $\forall\, n \geq N$, $\rho(x_n, a) < \delta$

By combining A & B, if $n \geq N \Rightarrow \rho(x_n, a) < \delta \Rightarrow \sigma(f(x_n), f(a)) < \varepsilon$

(2) ⇒ (1)



Suppose f is not continuous $\Rightarrow \exists f(a) = Y$, $\varepsilon > 0$ such that

$\forall\, \delta$, $\rho(x, a) < \delta$, but $\sigma(f(x), f(a)) \geq \varepsilon$

$B_1(a)$ define a sequence of pts of X by taking $x_1 \in B_1(a)$, $x_n \in B_{\frac{1}{n}}(a)$,

$x_2 \in B_{\frac{1}{2}}(a) \cdots$

$B_1(a)$

$B_{\frac{1}{2}}(a)$

$B_{\frac{1}{3}}(a)$

$\vdots$

$\lim_{n\to\infty} x_n = a$

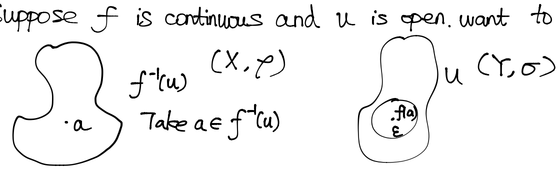
$\lim_{n\to\infty} f(x_n) \neq f(a)$ a contradiction

$\sigma(f(x_n), f(a)) \geq \varepsilon$, $\forall n$ ∎

Proof:

(1) ⇒ (3)

Suppose f is continuous and u is open. want to show that $f^{-1}(u)$ is open



U is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(f(a)) \subset U$

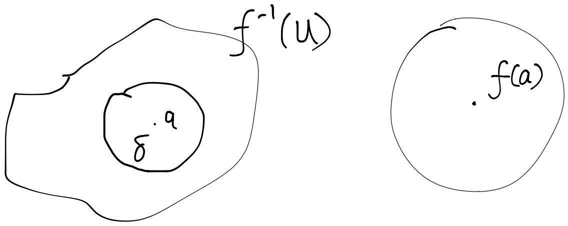
By continuity of f, $\exists\, \delta > 0$, s.t. if $\rho(a, x) < \delta \Rightarrow$

$\sigma(f(a), f(x)) < \varepsilon$

$\Rightarrow B_\delta(a) < f^{-1}(U) \Rightarrow f^{-1}(U)$ is open

$(3) \Rightarrow (1)$

Suppose (3) holds. Let $a \in X$ and $\varepsilon > 0$ be given



Let $U = B_\varepsilon(f(a))$

By our hypothesis, $f^{-1}(U)$ is open, so $\exists\, \delta > 0$ s.t. $B_\delta(a) \subset f^{-1}(U) \Rightarrow \rho(x,a) < \delta$ implies $\sigma(f(x), f(a)) < \varepsilon$

∎

Thm: Let $f: X \to Y$ be a continuous map between metric spaces. If $C \subseteq X$ is compact, then $f(C)$ is compact.