Lecture 2

§3.3 Types of orbits

①

Fixed points $F(x_0)=x_0$

orbit: $x_0, x_0, x_0, \cdots$

Ex: let $F(x)=x^2$

then 0 & 1 are fixed points

②

Periodic Orbit or Cycle

The point x_0 is periodic if $F^n(x_0)=x_0$ for some natural number $n\in\mathbb{N}$

The least such n is called the prime period of the orbit.

The orbit of a cycle of prime period n (n-cycle) is:

$x_0, F(x_0), F^2(x_0), \cdots, F^{n-1}(x_0), x_0, F(x_0), \cdots$

Ex: let $F(x)=x^2-1$

The point $0,-1$ form a 2-cycle.

The orbit is $0,-1,0,-1,\cdots$

or $-1,0,-1,0,\cdots$

Ex: • Find fixed points for $F(x)$

$$F(x)=x \iff x^2-1=x \iff x^2-x-1=0 \iff x=\frac{1\pm\sqrt{1+4}}{2}=\frac{1\pm\sqrt{5}}{2}$$

• Find 2-cycles

$$F^2(x)=x \iff (x^2-1)^2-1=x \iff x^4-2x^2-x=0$$
$$\iff x(x^3-2x-1)=0$$
$$\iff (x^2-x-1)(x+1)x=0$$

4 solutions
one is zero
two are the fixed point
there's another one left

We have 4 solutions: 2 fixed pts + 2 2-cycles.

• Find 3-cycles:

$$F^3(x)=x \iff (x^4-2x^2)^2-1=x$$

$$\iff x^8-4x^6+4x^4-x-1=0$$

$$\iff \underbrace{(x^2-x-1)}_{\text{fixed pts}}\underbrace{(x^6+x^5-2x^4-x^3+x^2+1)}_{\text{this poly has no real roots}}=0$$

There're no 3-cycles

3-cycle is special.

· To find n-cycles, you have to find the roots of a polynomial of order 2^n.

Eventually fixed Point or Eventually periodic

def: A point x_0 is eventually fixed or eventually periodic if x_0 itself is not fixed or periodic, but some point on its orbit is fixed or periodic.

Ex: Let $F(x)=x^2-1$

The orbit of $x_0=1$ is:

$(\sqrt{2},)\ 1, 0, -1, 0, -1, \cdots$

So 1 is eventually periodic.

Ex: let $F(x)=x^2$

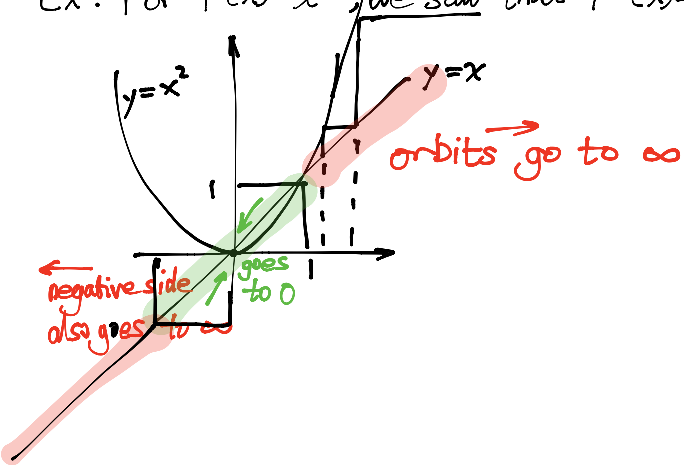
The orbit of $x_0=-1$ is:

$-1, 1, 1, \cdots$

So -1 is eventually fixed.

The Typical orbit is neither fixed nor periodic.

Ex: For $F(x)=x^2$, we saw that $F^n(x)=x^{2^n}$



$F(x)=x^2-2$

$0, -2, 2, 2, \cdots$