July 11th

Linear properties of Integration

4.6 Thm:

a. Sps $a<b<c$, if f is integrable on $[a,b]$ and $[b,c]$, then f is integrable on $[a,c]$ and

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

b. If f and g are integrable on $[a,b]$, then so is $f+g$ and

$$\int_a^b f(x)+g(x)\ dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

Proof: (a) Given $\varepsilon>0$, let P be partition of $[a,b]$,
Q be partition of $[b,c]$
such that $\overline{S}_P f - \underline{S}_P f < \frac{\varepsilon}{2}$, $\overline{S}_Q f - \underline{S}_Q f < \frac{\varepsilon}{2}$ ...... (*)

$\Rightarrow P\cup Q$ is a partition of $[a,b]$ and $\overline{S}_P + \overline{S}_Q = \overline{S}_{P\cup Q}$
$\underline{S}_P + \underline{S}_Q = \underline{S}_{P\cup Q}$

$\Rightarrow \overline{S}_{P\cup Q} - \underline{S}_{P\cup Q} = (\overline{S}_P - \underline{S}_P) + (\overline{S}_Q - \underline{S}_Q) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow f$ is integrable

$<\varepsilon$

$\underline{S}_{P\cup Q} < \int_a^c f\,dx < \overline{S}_{P\cup Q} \quad \Rightarrow \quad \overline{S}_{P\cup Q} - \varepsilon < \int_a^c f\,dx < \underline{S}_{P\cup Q} + \varepsilon$

$\underline{S}_P f + \underline{S}_Q f - \varepsilon < \overline{S}_P f + \overline{S}_Q f - \varepsilon < \int_a^c f(x)dx < \underline{S}_P f + \underline{S}_Q f + \varepsilon < \overline{S}_P f + \overline{S}_Q f + \varepsilon$

$$\xRightarrow{(*)} \begin{cases} \overline{S}_P f - \frac{\varepsilon}{2} < \int_a^b f(x)dx < \underline{S}_P f + \frac{\varepsilon}{2} \\ \overline{S}_Q f - \frac{\varepsilon}{2} < \int_a^b f(x)dx < \underline{S}_Q f + \frac{\varepsilon}{2} \end{cases}$$

$$\begin{aligned}\int_a^b f(x)dx + \int_a^c f(x)dx - 2\varepsilon &< \underline{S}_P f + \underline{S}_Q f + \varepsilon - 2\varepsilon = \underline{S}_P f + \underline{S}_Q f - \varepsilon < \int_a^c f(x)dx \\ &< \overline{S}_P f + \overline{S}_Q f - \varepsilon + 2\varepsilon \\ &< \int_a^b f(x)dx + \int_b^c f(x)dx + 2\varepsilon\end{aligned}$$

Let $\varepsilon \to 0$ we then have

$$\int_a^b f\,dx + \int_b^c f\,dx = \int_a^c f\,dx$$

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(b). Given $\varepsilon>0$, choose P s.t. $\overline{S}_P f - \underline{S}_P f < \frac{\varepsilon}{2}$ on $[a,b]$
Q s.t. $\overline{S}_Q g - \underline{S}_Q g < \frac{\varepsilon}{2}$ on $[a,b]$

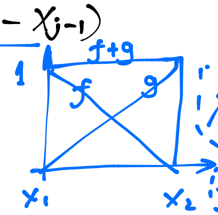
define $R = P\cup Q$

$\overline{S}_P f > \overline{S}_R f$, $\underline{S}_P f < \underline{S}_R f \Rightarrow \overline{S}_P f - \underline{S}_P f > \overline{S}_R f - \underline{S}_R f$

Similarly $\Rightarrow \overline{S}_Q g - \underline{S}_Q g > \overline{S}_R g - \underline{S}_R g$ since R is also a refinement for Q.

$\overline{S}_R(f+g) = \sum M_{f+g}(x_j - x_{j-1}) \le \sum M_f(x_j - x_{j-1}) + \sum M_g(x_j - x_{j-1})$

$= \overline{S}_R f + \overline{S}_R f$ Why?



$M_f = 1$
$M_g = 1$
$M_{f+g} = 1 < 2$

Similarly

$\underline{S}_R(f+g) \ge \underline{S}_R f + \underline{S}_R g$

$\overline{S}_R(f+g) \leq \overline{S}_R f + \overline{S}_R g < \underline{S}_R f + \frac{\varepsilon}{2} + \underline{S}_R g + \frac{\varepsilon}{2}$
$= \underline{S}_R f + \underline{S}_R g + \varepsilon$
$\leq \underline{S}_R (f+g) + \varepsilon$

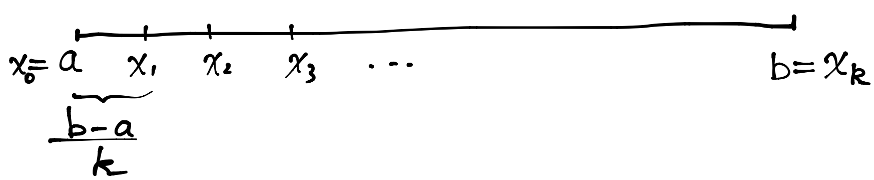
$\Rightarrow \overline{S}_R(f+g) - \underline{S}_R(f+g) < \varepsilon$
$\Rightarrow f+g$ is integrable
Similarly as part (a), we can show $\int_a^b f+g \, dx + \int_a^b f \, dx + \int_a^b g \, dx$
Remark : we define $\int_b^a f(x) = -\int_a^b f(x)\,dx \Rightarrow \int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ is true for any $a, b, c \in \mathbb{R}$

Pf. exercise

More Properties of integrations: when f is integrable on $[a, b]$
a. If $c \in \mathbb{R}$, then cf is integrable on $[a, b]$ and $\int_a^b cf\,dx = c\int_a^b f\,dx$
b. If $[c, d] \subset [a, b]$, then f is integrable on $[a, b]$
c. If g is integrable on $[a, b]$ and $f(x) \leq g(x)$ for $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.
d. $|f|$ is intgrb on $[a, b]$ and $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$.

Pf. exercise. FYI: This is 4.9 Thm

4.10 Thm: If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$
Proof: WLOG, we suppose f is increasing, ow. consider $-f$.
Let P_k is a partition of $[a, b]$ with k equal subintervals



On $[x_{j-1}, x_j]$. Since $f \nearrow$, $m_j = f(x_{j-1})$, $M_j = f(x_j) \Rightarrow \underline{S}_{P_k} = \sum_1^k m_j(x_j - x_{j-1})$
$= \sum_1^k f(x_{j-1}) \frac{b-a}{k}$
$\overline{S}_{P_k} = \cdots = \sum_1^k f(x_j)\frac{b-a}{k} \quad \Rightarrow \overline{S}_{P_k} - \underline{S}_{P_k} = \sum_1^k (f(x_j) - f(x_{j-1}))\frac{b-a}{k}$
$= \frac{b-a}{k}\left(\sum_1^k f(x_j) - \sum_1^k f(x_{j-1})\right)$
$= \frac{b-a}{k}(f(x_k) - f(x_0))$
$= \frac{b-a}{k}(f(b) - f(a))$ ∎

4.11 Thm: If f is continuous on $[a, b]$, then f is integrable on $[a, b]$
Proof: f is continuous $\underset{\text{on } [a,b]}{\Rightarrow}$ f is bounded

$\Rightarrow$ For any partition P, $\overline{S}_P f$, $\underline{S}_P f$ exist f is continuous, $[a, b]$ is compact
$\Rightarrow f$ is uniformly continuous
$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$, for any $|x-y| < \delta$

Let P be any partition of $[a,b]$. s.t. any subintervals $[x_{j-1}, x_j]$ of P have length less than δ.

On each $[x_{j-1}, x_j]$, $|x-y|<\delta$ for any $x, y \in [x_{j-1}, x_j] \Rightarrow |f(x)-f(y)| < \frac{\varepsilon}{b-a}$
$\Rightarrow |M_j - m_j| < \frac{\varepsilon}{b-a}$

$$\overline{S}_P f - \underline{S}_P f = \sum_1^J M_j (x_j - x_{j-1}) - \sum_1^J m_j (x_j - x_{j-1}) = \sum_1^J (M_j - m_j)(x_j - x_{j-1})$$
$$< \sum_1^J \frac{\varepsilon}{b-a}(x_j - x_{j-1})$$
$$= \frac{\varepsilon}{b-a} \sum_1^J (x_j - x_{j-1})$$
$$= \frac{\varepsilon}{b-a}(b-a) \Rightarrow f \text{ is integrable}$$
$$= \varepsilon$$

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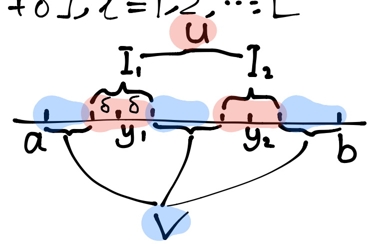
4.12 Thm: If f is bounded on $[a,b]$ and continuous at all points except finitely many points in $[a,b]$, then f is intgrbl on $[a,b]$.

Proof: Let $y_1, \cdots, y_L$ be the points in $[a,b]$ where f is discontinuous.
Given $\delta > 0$, Let $I_l = [a,b] \cap [y_l - \delta, y_l + \delta]$, $l = 1, 2, \cdots, L$
and $U = \bigcup_1^L I_l$, $V = [a,b] \setminus U^{\text{int}}$

Consider the simplest example:



let $m = \inf\{f(x) : a \leq x \leq b\}$
$M = \sup\{f(x) : a \leq x \leq b\}$
let P be any partition of $[a,b]$ that includes the endpoints of the Intervals I_l, $l = 1, 2, \cdots, L$
then $\overline{S}_P f = \overline{S}_P^u f + \overline{S}_P^v f$ $\qquad \underline{S}_P f = \underline{S}_P^u f + \underline{S}_P^v f$

upper sum for intervals in U

For any ε on V, f is continuous on each subinterval in V when P is fine enough $\overline{S}_P^v f - \underline{S}_P^v f < \frac{1}{2}\varepsilon$

On U: $\overline{S}_P^u f - \underline{S}_P^u f = \sum_{[x_{j-1}, x_j] \subset U} (M_j - m_j)(x_j - x_{j-1}) < \sum (M-m)(2\delta) = L(M-m)2\delta$

b/c we can choose any δ we want, let's choose $\delta = \frac{\varepsilon}{2L(M-m)}$

$\Rightarrow \overline{S}_P^u f - \underline{S}_P^u f < \frac{\varepsilon}{4L(M-m)} 2L(M-m) = \frac{\varepsilon}{2}$

$\Rightarrow \overline{S}_P f - \underline{S}_P f < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

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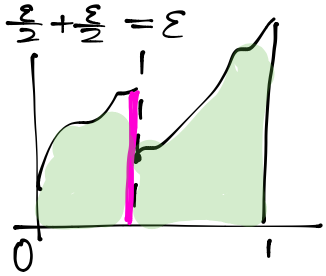
Note: Graph



Idea: use smaller and smaller to "cancel" the effect of discontinu-ity.

Def: A set $Z \in \mathbb{R}$ is said to have zero content if for any $\varepsilon > 0$ there is a finite collection of intervals $I_1, \cdots, I_L$ such that

(i) $Z \subset \bigcup_1^L I_l$

(ii) $\sum_1^L |I_l| < \varepsilon$

4.13 Thm: If f is bounded on $[a, b]$ and the set of points in $[a, b]$ at which f is discontinuous has zero content, then f is integrable on $[a, b]$

Proof: exercise.

Eg. $f(x)$ is discontinuous on the set $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ I_L includes infinitely many points

zero content, but has infinite points

to be specific: $I_1 = [-\frac{1}{40}, \frac{1}{40}]$, $|I_1| = \frac{1}{20}$

$1, \frac{1}{2}, \cdots, \frac{1}{39}$

$I_2 = [1 - \frac{1}{10000}, 1 + \frac{1}{10000}]$, $I_3 = [\frac{1}{2} - \frac{1}{10000}, \frac{1}{2} + \frac{1}{10000}], \cdots$

$I_{40} = [\frac{1}{39} - \frac{1}{10000}, \frac{1}{39} + \frac{1}{10000}]$

$$\sum |I| = \frac{1}{20} + \frac{1}{10000} \times 2 \times 39 < \frac{1}{20} + \frac{1}{5000} \times 40 < \frac{1}{20} + \frac{1}{20} = \frac{1}{10}$$

4.14 Prop: Sps f and g are intgrb on $[a, b]$ and $f(x) = g(x)$ for all except finite many points, then $\int_a^b f(x) dx = \int_a^b g(x) dx$

Proof: let $h = f - g$

then $h = 0$, except finite many points $\overset{\text{exercise}}{\Longrightarrow} \int_a^b h \, dx = 0$ by computing the $\overline{S}h$ and $\underline{S}h$. ∎

4.15 Thm (The Fundamental Thm of Calculus)

a). Let f be an integrable function on $[a, b]$, For $x \in [a, b]$. Let $F(x) = \int_a^x f(t) dt$ then F is continuous on $[a, b]$. Moreover $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous.

b.) Let F be a continuous function on $[a, b]$, and F is differentiable except perhaps at finite many points in $[a, b]$ and let f be a function on $[a, b]$ that agrees with F' at all points where F' is defined. If f is intgrb on $[a, b]$ then

$$\int_a^b f(t) dt = F(b) - F(a) \quad (*)$$

Remark: a) $\Rightarrow [\int_a^x f(x) dx]' = f(x)$

b). $\Rightarrow \int_a^b F'(x) dx = F(b) - F(a)$

Proof: (a) If $x, y \in [a, b]$

$$|F(y) - F(x)| = |\int_a^y f(t) dt - \int_a^x f(t) dt| = |\int_x^y f(t) dt| \le \int_x^y |f(t)| dt$$
$$\le \int_x^y c \, dt, \text{ where } C = \sup\{|f(t)|\}$$
$$= c \int_x^y dt = c|y - x|$$

When $y \to x$, $F(y) \to F(x) \Rightarrow F(x)$ is continuous

Fix x, where $f(x)$ is continuous

$\forall \varepsilon > 0, \exists \delta > 0$, st. $|f(y) - f(x)| < \varepsilon$, when $|y - x| < \delta \Rightarrow \frac{F(x) - F(y)}{y - x} - f(x)$

$$= \frac{\int_x^y f(t) dt}{y - x} - f(x) \frac{1}{y - x} \int_x^y dx = \frac{\int_x^y f(t) dt}{y - x} - \frac{1}{y - x} \int_x^y f(x) dt$$

$$= \frac{1}{y-x}\left(\int_x^y f(t)\,dt - \int_x^y f(x)\,dt\right) = \frac{1}{y-x}\int_x^y f(t) - f(x)\,dt$$

$$\left|\frac{F(y)-F(x)}{y-x} - f(x)\right| = \frac{1}{y-x}\left|\int_x^y f(t)-f(x)\,dt\right| \le \frac{1}{y-x}\int_x^y |f(t)-f(x)|\,dt$$

$$< \frac{1}{y-x}\int_x^y \varepsilon\,dt, \text{ when } |y-x| < \delta \qquad x\overbrace{\quad t \quad}^{\le\delta}y$$

$$= \frac{1}{y-x}\,\varepsilon\cdot(y-x) = \varepsilon$$

Let $\varepsilon \to 0$, $\lim_{y\to x}\frac{F(y)-F(x)}{y-x} = f(x) \Rightarrow F'(x) = f(x)$. $\blacksquare$

(b). Let $P=\{x_0,\cdots,x_J\}$ be a partition of $[a,b]$ we assume that all points where F is not diff. are among the subdivision points.
On each interval $F \in C'$

MVT
$\Longrightarrow F(x_j)-F(x_{j-1}) = F'(t_j)(x_j - x_{j-1})$ where $t_j \in (x_j, x_{j-1})$

$$F(b)-F(a) = \sum_1^J F'(t_j)(x_j-x_{j-1}) = \sum_1 f(t_j)(x_j-x_{j-1}) \quad \text{Riemann Sum}$$

$$\underline{S}_P f \le \sum_1^J f(t_j)(x_j - x_{j-1}) \le \overline{S}_P f$$

$\Rightarrow \underline{S}_P f \le F(b)-F(a) \le \overline{S}_P f$ for any partition P
Let P finer and finer.

$$\Rightarrow \begin{array}{l}\underline{S}_P f \nearrow \underline{\int_a^b} f\,dx \\ \overline{S}_P f \searrow \overline{\int_a^b} f(x)\end{array}$$

$\Rightarrow \underline{\int_a^b} f(x)dx \le F(b)-F(a) \le \overline{\int_a^b} f(x)dx \Rightarrow (*)$

$\blacksquare$

4.16 Prop: Sps f is intgrb on $[a,b]$. Given $\varepsilon>0$, $\exists\,\delta>0$ s.t. if $P=\{x_0,\cdots,x_J\}$ is any partition of $[a,b]$ satisfying

$$\max_{1\le j\le J}(x_j - x_{j-1}) < \delta$$

the sum $\underline{S}_P f$ and $\overline{S}_P f$ differ from $\int_a^b f(x)dx$ by at most ε.

Proof not required.

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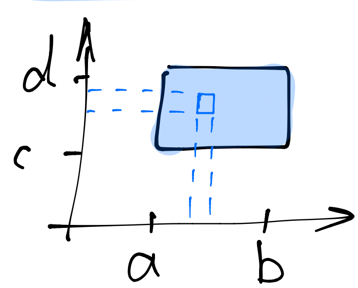
§4.2 Integrals in Higher dimensions ($n=2$, $n=3$)

Double Integrals ($\mathbb{R}^2$)
Step 1: define integration on rectangle
$R=[a,b]\times[c,d]$



Partition of R, $P=\{x_0,\cdots,x_J : y_0,\cdots,y_K\}$
where $a=x_0<x_1<\cdots<x_J=b$
$c=y_0<y_1<\cdots<y_K=d$
$R_{jk}=[x_{j-1},x_j]\times[y_{k-1},y_k]$

$$R=\bigcup_{j,k} R_{jk} \qquad \Delta A_{jk} = \text{Area } R_{jk} = (x_{j-1}-x_j)(y_k - y_{k-1})$$

Let $m_{jk} = \inf\{f(x,y) : (x,y) \in R_{jk}\}$
$M_{jk} = \sup\{f(x,y) : (x,y) \in R_{jk}\}$

Lower Riemann sum : $\underline{S}_R f = \sum_{j,k} m_{jk} \Delta A_{jk}$

upper ... $\overline{S}_R f = \sum_{j,k} M_{jk} \Delta A_{jk}$

lower integral : $\underline{I}_R f = \sup \underline{S}_R f$
$\overline{I}_R f = \inf \overline{S}_R f$

when $\underline{I}_R f = \overline{I}_R f$ we say f is intgrb and

$$\iint_R f \, dA = \underline{I}_R f = \overline{I}_R f$$

Step 2 : Integration on any subset S

Define a characteristic function

$$\phi_S(\vec{x}) = \begin{cases} 1 & \text{if } \vec{x} \in S \\ 0 & \text{o.w.} \end{cases}$$

Consider a rectangle R, set $S \subset R$, f is integrable on S if $f\phi_S$ is intgrb on R and $\iint_S f \, dA = \iint_R f \phi_S \, dA$

Remark : when $\vec{x} \in S$, $f\phi_S = f$ when $\vec{x} \notin S$, $f\phi_S = 0$

Properties of integrals

4.17 Thm : a). If f_1 and f_2 are intgrb. on bounded set S, and $C_1, C_2 \in \mathbb{R}$, then $C_1 f_1 + C_2 f_2$ is intgrb on S. and

$$\iint_S C_1 f_1 + C_2 f_2 \, dA = C_1 \iint_S f_1 dA + C_2 \iint_S f_2 dA$$

b). Let S_1 and S_2 be bounded sets with no points in common and let f be a bounded function. If f is intgrb on S_1 and S_2, then f is intgrb. on $S_1 \cup S_2$ and

$$\iint_{S_1 \cup S_2} f \, dA = \iint_{S_1} f \, dA + \iint_{S_2} f \, dA$$

c). If f and g are integrable on S and $f(\vec{x}) \le g(\vec{x})$ for $\vec{x} \in S$ then $\iint_S f \, dA \le \iint_S g \, dA$

d). If f is intgrb. on S, then $|f|$ is also intgrb and

$$\left| \iint_S f \, dA \right| \le \iint_S |f| \, dA$$