July 4th

Today: ① Define smooth curves in the plane $\mathbb{R}^2$.
② Define smooth surfaces in $\mathbb{R}^3$
③ Define smooth curves in $\mathbb{R}^3$

E.g. Consider the problem of solving

$$\begin{cases} x - yu^2 = 0 \\ xy + uv = 0 \end{cases}$$

for u & v as functions of x & y

Let $F = x - yu^2$, $G = xy + uv$ $\Rightarrow$ $\partial_u F = -2yu$, $\partial_v F = 0$, $\partial_u G = v$, $\partial_v G = u$

$$\Rightarrow \frac{\partial(F,G)}{\partial(u,v)} = \det\begin{pmatrix} -2yu & 0 \\ v & u \end{pmatrix} = -2yu^2$$

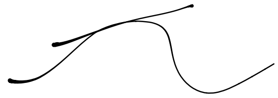
If a point (x_0, y_0, u_0, v_0) satisfies $-2y_0u_0^2 = 0 \Leftrightarrow y_0 = 0$ or $u_0 = 0$

by IFT $\Rightarrow$ There is a local solution near any point (x_0, y_0, u_0, v_0) provided $y_0 \neq 0$ and $u_0 \neq 0$.

---

§ 3.2 Curves in the Plane

Def 1: Smooth curves means the curve possesses a tangent line at each point and the tangent line varies continuously with the point of tangency.



3 common ways of representing smooth curves in plane.

i) as the graph of a function, $y = f(x)$ or $x = f(y)$, where f is of class C^1

ii). as the locus of an equation $F(x,y) = 0$, where F is of class C^1. Here, the locus of an equation $F(\vec{x}) = c$ is $S = \{\vec{x} : F(\vec{x}) = c\}$

iii). parametrically, as the range of a C^1 function $\vec{f} : (a,b) \to \mathbb{R}^2$, $a, b \in \mathbb{R}$, $a < b$

Remark 1: (i) is a special case of the other two

(a). Let $F = y - f(x)$ then $y = f(x)$ is the locus of $F = 0$. and $\nabla F = (-f', 1)$, which implies F is in class of C^1

(b). Let $\vec{f} = (t, f(t))$, and $\vec{f}'(t) = (1, f')$ which implies $\vec{f} \in C^1$

Now

$$\begin{tikzcd} & \text{(ii)} \arrow[dl, bend right, "?"'] \\ \text{(i)} \arrow[ur] \arrow[dr] & \\ & \text{(iii)} \arrow[ul, bend left, "?"] \end{tikzcd}$$

Eg 1: Let $F(x,y) = x^2 + y^2 - C$, where C is a constant, consider $F = 0$.

$C>0$, $\{F=0\}$ is a circle
$=0$ $(0,0)$
<0 empty

when $C=0$, $\nabla F=(2x,2y) \Rightarrow \nabla F(0,0)=\vec{0}$

Eg 2: Let $G(x,y)=x^2-y^2-C$
Consider $G=0$
$\begin{cases} C\neq 0, & \text{hyperbola} \\ C=0, & \text{two lines } \begin{cases} y=x \\ y=-x \end{cases} \end{cases}$

$\nabla G=(2x,-2y)$
$\Rightarrow \nabla G(0,0)=\vec{0}$

Eg 3: Let $H(x,y)=y^3-x^2-C$, consider $H=0$
$\begin{cases} C\neq 0, & \text{smooth curve} \\ C=0, & \text{a curve with sharp cusp at the origin.} \end{cases}$

$\nabla H(x,y)=(-2x, 3y^2)$

Eg 4: $\vec{g}(t)=(\underbrace{\sin^2 t}_{x}, \underbrace{\cos^2 t}_{y})$, $x+y=1 \Rightarrow$ line segment

$t\in(0,\frac{1}{2}\pi)$ point $(0,1)\to(1,0)$
$t\in(\frac{1}{2}\pi,\pi)$ point $(1,0)\to(0,1)$

$\vec{g}'(t)=(2\sin t\cos t, -2\cos t\sin t)$
$\vec{g}'(0)=\vec{0}=\vec{g}'(\frac{\pi}{2})$

Summerize: In order to get smooth curve, we need to impose
a) $\nabla F\neq\vec{0}$ on the set $F=0$ for definition (ii)
b) $\vec{f}'(0)\neq\vec{0}$ in the definition (iii)

3.11 Thm:
a). Let F be a real valued function of class C^1 on an open set in $\mathbb{R}^2$, and let $S=\{(x,y): F(x,y)=0\}$. If $\vec{a}\in S$ and $\nabla F(\vec{a})\neq\vec{0}$, there is a neighbourhood N of $\vec{a}$ in $\mathbb{R}^2$ such that $S\cap N$ is the graph of a C^1 function f (either $y=f(x)$ or $x=f(y)$).
b). Let $\vec{f}:(a,b)\to\mathbb{R}^2$ be a function of class C^1. If $\vec{f}'(t_0)\neq\vec{0}$, there is an open interval I containing t_0 such that the set $\{\vec{f}(t): t\in I\}$ is the graph of a C^1 function f (either $y=f(x)$ or $x=f(y)$).

Remark 2: From (a), we know a curve given by (ii) can be presented in the form (i)
From (b), we know a curve given by (iii) can be presented in the form (i).

Pf: (a). This is Cor 3.3
(b). Let $\vec{f}(t)=(\varphi(t),\psi(t))$, $t\in(a,b)$
If $\vec{f}'(t_0)\neq\vec{0}$, this either $\varphi'(t_0)\neq 0$ or $\psi'(t_0)\neq 0$
WLOG, assuming $\varphi'(t_0)\neq 0$ (otherwise, switch x and y)

Let $F(x,t)=x-\varphi(t)$ and $x_0=\varphi(t_0)$

Then $F(x_0,t_0)=x_0-\varphi(t_0)=0$

$\partial_t F(x_0,t_0)=-\varphi'(t_0)\neq 0$

$\Rightarrow$ by IFT $F(x,t)$ can be solved for t as a C' function of x, denote -d as $t=\omega(x)$, near (x_0,t_0) $\Rightarrow$

$$\begin{aligned}\vec{f}=(\varphi(t),\psi(t))&=(x,\psi(t)) \longleftarrow \text{by } F(x,t)=0\\ &=(x,\psi(\omega(x))) \longleftarrow \text{by } t=\omega(x)\\ &=(x,f(x)), \text{ where } f=\psi\circ\omega\end{aligned}$$

Thus $\{\vec{f}(t): t\in I\}$ is the graph of a C' function $f=\psi\circ\omega$ $\blacksquare$

Def 2 (formal definition of smooth curve)

A set $S\subset\mathbb{R}^2$ is a smooth curve if

(a) S is connected

(b). every $\vec{a}\in S$ has a neighbourhood N such that $S\cap N$ is the graph of C' function.

Ex: Check this def agree with def 1

Q: True or False

Given a smooth curve

(a) $\boxed{\nabla F(\vec{a})\neq\vec{0}}$ FALSE, there are some smooth curves with a point $\vec{a}$ on the curve satisfying $\nabla F(\vec{a})=\vec{0}$

E.g. let $G(x,y)$ is a C' function and $\nabla G(x,y)\neq 0$. let $F=G^2$

$\Rightarrow S=\{(x,y): F(x,y)=0\}=\{(x,y): G(x,y)=0\}$

but $\nabla F=\nabla(G^2)=2G\nabla G=0$ on S

(b) $\boxed{\vec{f}'(t_0)\neq\vec{0}}$ FALSE

counter e.g:

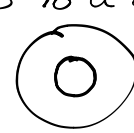
$t\in(-1,1)$, $\vec{f}(t)=(t,t)$

$\vec{g}(t)=\vec{f}(t^5)=(t^5,t^5)$, but $\vec{g}'(t)=(5t^4,5t^4)\Rightarrow \vec{g}'(0)=\vec{0}$

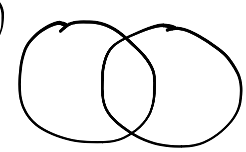
Thus: $\nabla F\neq 0$ and $\vec{f}'\neq 0$ are sufficient condition for the smoothness of a curve, but they are not necessary condition.

For (iii) and (ii), if $\nabla F\neq 0$ or $\vec{f}'\neq 0$, then S is a smooth curve locally but this is not true globally!

E.g. 5: $F(x,y)=(x^2+y^2-1)(x^2+y^2-2)=0$

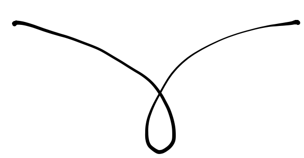


Eg 6: $F(x,y)=(x^2+y^2-1)(x^2+y^2-2x)=0$



Eg 7: Let $\vec{f}(t)=(t^3-t,t^2)$

$\vec{f}'(t)=(3t^2-1,2t)\neq 0$



§ 3.3 Surfaces and Curves in Space

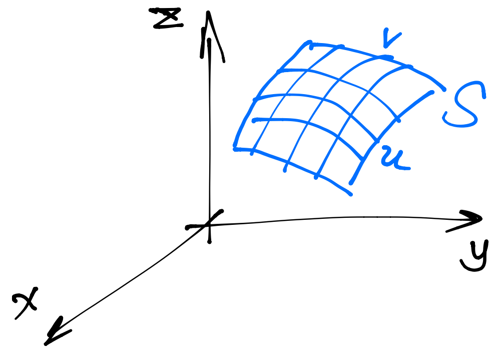
Three standard ways:

i) as the graph of a function z=f(x, y) (or y=f(x, z) or x=f(y, z)) where f is of class C1.

ii) as the locus of an equation F(x, y, z)=0, where F is of class C1.

iii) parametric ally, as the range of C1 function f: R^2-> R^3 say (u, v) to (x, y, z)

Remark: we can see(iii) as a coordinate system on the surface S.



Relations:

a). (i) is a special case of (ii) and (iii)
with $F = z - f(x,y)$ and $\vec{f}(u,v) = (u, v, f(u,v))$

b). In case (ii), we need impose $\nabla F \neq 0$ to guarantee the smoothness of the surface

c). In case (iii) we need to impose that the vectors

$$\frac{\partial \vec{f}}{\partial u}(u,v) \text{ and } \frac{\partial \vec{f}}{\partial v}(u,v)$$

are linearly indpt. at each u, v. (*)

Eg. consider $\vec{f}(u,v) = u\vec{a} + v\vec{b} + \vec{c}$, $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$
usually this is a plane for instance
if $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

then $\vec{f}(u,v) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$, which is x-y plane.

but if $\vec{a}$ and $\vec{b}$ are dependent, i.e. $\vec{b} = \lambda \vec{a}$, $\lambda \neq 0$
$\vec{f}(u,v) = u\vec{a} + \lambda v \vec{a} + \vec{c} = (u + \lambda v)\vec{a} + \vec{c}$, which is a line when $\vec{a} \neq \vec{0}$.

Here $\frac{\partial \vec{f}}{\partial u} = \vec{a}$, $\frac{\partial \vec{f}}{\partial v} = \vec{b}$

Thus when $\partial \vec{f}/\partial u$ is linearly independent with $\partial \vec{f}/\partial v$, $\vec{f}(u,v)$ is not a dimensional surface.

Remark : (*) $\iff \dfrac{\partial \vec{f}}{\partial u} \times \dfrac{\partial \vec{f}}{\partial v} \neq 0$ at each $(u,v) \in U \leftarrow$ open set in $\mathbb{R}^2$

Thm:
a). Let F be a real-valued function of class C^1 on an open set in $\mathbb{R}^3$, and let $S=\{(x,y,z): F(x,y,z)=0\}$. If $\vec{a} \in S$ and $\nabla F(\vec{a}) \neq \vec{0}$, there is a neighbourhood N of $\vec{a}$ in $\mathbb{R}^3$ s.t. $S \cap N$ is the graph of a C^1 function f (either $z=f(x,y)$, $y=f(x,z)$ or $x=f(y,z)$).

b). Let $\vec{f}$ be a C^1 mapping from an open set in $\mathbb{R}^3$. If $[\partial_u \vec{f} \times \partial_v \vec{f}](u_0,v_0) \neq \vec{0}$, there is a neighbourhood N of (u_0,v_0) in $\mathbb{R}^2$ s.t. the set $\{\vec{f}(u,v): (u,v) \in N\}$ is the graph of a C^1 function.

Pf: (a). Cor. 3.3
(b). Let $\vec{f}(u,v) = (\varphi(u,v), \psi(u,v), \theta(u,v)) = (x,y,z)$

$$\frac{\partial \vec{f}}{\partial u} \times \frac{\partial \vec{f}}{\partial v} = \left(\frac{\partial(\varphi,\psi)}{\partial(u,v)}, \frac{\partial(\psi,\theta)}{\partial(u,v)}, \frac{\partial(\theta,\varphi)}{\partial(u,v)} \right) \neq 0$$

$\Rightarrow$ at least one of those Jacobians are not zero
Assuming $\partial(\varphi,\psi)/\partial(u,v) \neq 0$
$\Rightarrow$ By IFT, the equation $\varphi(u,v)=x$, $\psi(u,v)=y$ can be solved for $u=\alpha(x,y)$, $v=\beta(x,y)$, $\alpha, \beta \in C^1$
near $u=u_0$, $v=v_0$, $x_0=\varphi(u_0,v_0)$ and $y_0=\psi(u_0,v_0)$
$\Rightarrow z=\theta(u,v)=\theta(\alpha(x,y), \beta(x,y))$
$\Rightarrow \vec{f}(u,v)=(x, y, \theta(\alpha(x,y), \beta(x,y)))$
$\Rightarrow$ the set S is the graph of a C^1 function.

$\blacksquare$

Again, those properties are only local properties
Eg 8. Let $\vec{f}(u,v)=((u+v)\cos(u-v), (u+v)\sin(u-v), u+v)$
define $u+v=\theta$, $u-v=\phi$

$$\vec{f}(u,v) = (\theta\cos\phi, \theta\sin\phi, \theta) = (x,y,z)$$

$\Rightarrow x^2+y^2=\theta^2\cos^2\phi+\theta^2\sin^2\phi=\theta^2=z^2$

$$\frac{\partial \vec{f}}{\partial u}=(\cos(u-v)-(u+v)\sin(u-v), \sin(u-v)+(u+v)\cos(u-v), 1)$$

$$\frac{\partial \vec{f}}{\partial v}=(\cos(u-v)+(u+v)\sin(u-v), \sin(u-v)-(u+v)\cos(u-v), 1)$$

when $u+v=0$

$$\frac{\partial \vec{f}}{\partial u}=(\cos(u-v), \sin(u-v), 1)=\frac{\partial \vec{f}}{\partial v}$$

$$\vec{f}(u,v)=\vec{0}$$

Curves in $\mathbb{R}^3$
i) as a graph, y=f(x) and z=g(x) (or similar expressions with the coordinates permitted), where f and g are C1 functions.
ii) as the locus of two equations F(x, y, z)=G(x, y, z)=0, where F and G are C1 functions.

iii) parametrically, as the range of a C1 function $\vec{f}: \mathbb{R} \to \mathbb{R}^3$.

Relations:

a). (i) $\Rightarrow$ (ii) by $F = y - f(x)$ and $G = z - g(x)$

(i) $\Rightarrow$ (iii) by $\vec{f} = (t, f(t), g(t))$

b). (ii) $\Rightarrow$ (i) when $\nabla F(\vec{x})$ and $\nabla G(\vec{x})$ are linearly independent where $F(\vec{x}) = G(\vec{x}) = 0$.

c). (iii) $\Rightarrow$ (i), when $\vec{f}'(t) \neq 0$.

Pf: exercise.

For Surface: we know the tangent plane at $\vec{a}$ is $\vec{n} \cdot (\vec{x} - \vec{a}) = 0$, where $\vec{n}$ is the normal vector.

① $\nabla F(\vec{a}) = \vec{n}$

② $[\partial_u \vec{f} \times \partial_v \vec{f}](\vec{a}) = \vec{n}$.

For Curve:

① $\vec{f}'(t)$ is the tangent vector of the curve.

② tangent vector $= \nabla F \times \nabla G$