Lecture 11

$(X, \leq)$ is a partial order if
- (Transitive) $a \leq b$ & $b \leq c \Rightarrow a \leq c$, $\forall a, b, c \in X$
- (Reflexive). $a \leq a$, $\forall a \in X$
- (Antisymmetric) $a \leq b$ & $b \leq a \Rightarrow a = b$, $\forall a, b \in X$.

e.g. $(\mathbb{R}, \leq)$
e.g. $\mathbb{N}$, $a \preccurlyeq b$ iff $a | b$ | e.g. $7 \preccurlyeq 56$ but $3 \not\preccurlyeq 5$

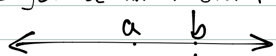
$(X, \leq)$ is a linear order if it is a partial order and:
- $a \leq b$ or $b \leq a$, $\forall a, b \in X$

e.g. $(\mathbb{R}, \leq)$ is a linear order
$(\mathbb{N}, |)$ is not a linear order as $3 \not\preccurlyeq 5$ & $5 \not\preccurlyeq 3$

$(X, \leq)$ is a well-order if it is a partial order and
- $\forall$ nonempty $S \subseteq X$, there is an element $m \in S$ s.t. $m \leq x$, $\forall x \in S$

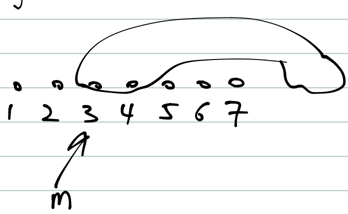
e.g. $(\mathbb{N}, \leq)$ is a well-order

Pic for a linear order



if $a \leq b$

Idea :



Fact: Every well-order is a linear order.
Take $x \neq y$ in X.
Set $S = \{x, y\}$.
The element $m \in S$ that the well-order gives you will be $\leq$ the other

Recall: We got a subbasis for $\mathbb{R}_{usual}$ by taking $S = \{(-\infty, a), a \in \mathbb{R}\} \cup \{(b, \infty) : b \in \mathbb{R}\}$
where $(-\infty, a) := \{x \in \mathbb{R} : x < a\}$ and $(b, \infty) = \{x \in \mathbb{R} : b < x\}$

Def'n: If $(L, \leq)$ is a linear order, then we define the order topology by defining the order subbasis as $S = \{(-\infty, a) : a \in L\} \cup \{(b, \infty) : b \in L\}$. Assume $|L| \geq 2$
Check that this forms a subbasis.

e.g. $(\mathbb{N}, \leq)$ usual order.

Take $n \in \mathbb{N}$, notice $\{n\} = (-\infty, n+1) \cap (n-1, \infty)$
Oops, if $n = 1$, $\{1\} = (-\infty, 2)$
So the order top is the discrete topology on $\mathbb{N}$.

Example: Let $X = \mathbb{N} \cup \{w\}$ where "w" is just a thing not in $\mathbb{N}$

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot$
$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \cdots w$

Define: $a \leq b$ on X by $a \leq_{usual} b$, and $a, b \in \mathbb{N}$ and say $a \leq \omega$, $\forall a \in X$.

(This looks like $\{-\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$)

Claim: This is a well-order

Take $S \subseteq X$ nonempty

Case 1: $S = \{\omega\}$

Here ω is the minimal element of S.

Case 2: $S \cap \mathbb{N} \neq \emptyset$

In this case S contains a minimal element since $\mathbb{N}$ are well-ordered.

Claim 1: each $\{n\}$ is open

Claim 2: $\{\omega\}$ is not open

Claim 3: $n \longrightarrow \omega$

(ω is a limit pt of $\langle n \rangle_{n \in \mathbb{N}}$)

2 If U is an open set that contains ω then there is an $n \in \mathbb{N}$ s.t. $(n, +\infty) \subseteq U$

So $\langle n \rangle \longrightarrow \omega$.

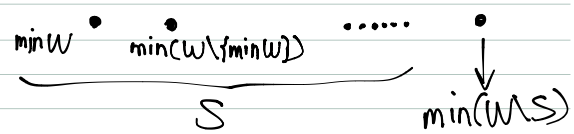
Prop: If $(L, \leq)$ is a linear order, then its order top. is T_2, T_3, & T_4.

(Moral: Linear orders give nice topologies)

Facts about well orders $(W, \leq)$

- $(W, \leq)$ is a linear order)
- W has a minimal element.
- If W has two elements, the non-min element is just above the minimal element.
- If W is infinite, it "looks like $\mathbb{N}$ to start"
- If W is uncountable, it "looks like" $\omega+1$ to start

$(W, \leq)$ Picture



Q: Is there an example of uncountable well-ordered set?

Note: $\mathbb{R}$ is NOT!

YES!

A: Axiom of well-orders: Every set X has a well-order $\leq$.

Take your fav. unctble set ($P(\mathbb{N})$ or $P(P(\mathbb{N}))$ or $\mathbb{R}$), call it X.

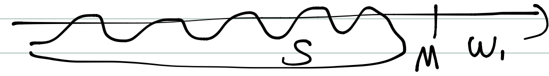
The axiom says there is a well-order $\leq$ on X.

$\omega_1 := \{x \in X : \text{There are only ctbly many elements } y \in X \text{ s.t } y \leq x\}$

Facts about ω_1

- infinite
- well-ordered (since it's a subset of a well-ordered set)
- If $\alpha \in \omega_1$, then the initial segment $\{x \leq \omega_1, x \leq \alpha\}$ is ctble
- ω_1 is unctble (see the proof in the notes)
- "ctble subsets of ω_1 are bounded above".

If $S \subseteq \omega_1$ is ctble, there is an $M \in \omega_1$, s.t. $x \leq M$ for all $x \in S$.



"ω_1 is very long"

We'll look at ω_1 with the order topology

Fact: ω_1 is not separable

Take $D \subseteq \omega_1$, a ctble set.

Find $M \in \omega_1$ that bounds D above

Notice (M, ∞) is disjoint from D.
So D is not dense.

$\blacksquare$