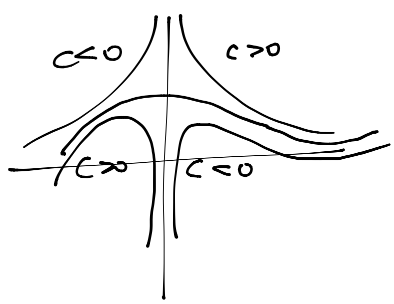
$y' + p(t)y = g(t)$ 1st order linear ODE

Solving this in general using "intergrating factors."

Example 1. $ty' + y = \cos(t)$

$$\Rightarrow ty = \sin(t) + C \Rightarrow y = \frac{\sin(t)}{t} + \frac{C}{t}$$

$$ty' + y = \frac{d}{dt}(ty)$$




$\lim_{t\to 0} \frac{\sin(t)}{t} = 1$

l'Hopital's rule

or Taylor's expansion

$\sin(t) = t - \frac{t^3}{c} \cdots$

2. $ty' + 2y = \cos(t)$

$$t^2y' + 2ty = t\cos(t)$$

$$\frac{d}{dt}(t^2 y) = t\cos(t)$$

$t^2 y = \int t\cos(t)dt$ using integration by parts

$= t.\sin(t) - \int 1 \cdot \sin(t)dt$

$= t\sin(t) + \cos(t) + C$

$\Rightarrow y = \frac{\sin(t)}{t} + \frac{\cos(t)+C}{t^2}$

consider $t \to 0$ limit Note $\frac{\sin(t)}{t} \to 1$, $\cos(t) \to 1$

If $c > -1$, $y(t) \to \infty$

If $c < -1$, $y(t) \to -\infty$

If $c = -1$, $y(t) \to \frac{1}{2}$

General Case: $y' + p(t)y = g(t)$

Multiply by ~~some~~ "intergrating factor" $\mu(t)$ so that LHS becomes $\frac{d}{dt}(\mu \cdot y) = \mu y' + \mu' y$

$y' + py = g \Rightarrow \mu y' + \mu p y = \mu g$

We need: $\mu' = \mu p$ Think of this an ODE for $\mu(t)$ using separation of variable.

$$\frac{d\mu}{dt} = \mu p(t)$$
$$\frac{1}{\mu} d\mu = p(t)dt$$
$$\int \frac{1}{\mu} d\mu = \int p(t)\,dt$$
$$\ln|\mu| = \int p(t)dt + C_1$$
$$|\mu| = e^{\int p(t)dt + C_1}$$
$$\mu = \pm e^{\int p(t)dt + C_1}$$
$$= Ae^{\int p(t)dt}$$

Since we only need an intergrating factor, choose $A=1$.

$\mu(t) = e^{\int p(t)dt}$ is the intergrating factor.

With this choice of $\mu$.

$$\frac{d}{dt}(\mu y) = \mu g$$
$$\mu(t)y = \int \mu(t)\cdot g(t)\,dt + C.$$
$$y = \frac{1}{\mu(t)}\int \mu(t) g(t)\,dt$$

General Solution: $y = \frac{1}{\mu(t)}\int \mu(t) g(t)\,dt + C$

In practice: if it's usually easier to remember the steps. "Method of intergrating factors".

Example: $3y' + y = e^t$

$$3\mu y' + \mu y = \mu \cdot e^t \quad (\mu = \mu(t))$$

Need $\mu = \frac{d}{dt}(3\mu) = 3\mu'$

$$\mu' = \frac{1}{3}\mu \Rightarrow \mu(t) = e^{\frac{1}{3}t}$$
$$\Rightarrow 3\cdot e^{\frac{1}{3}t} y' + e^{\frac{1}{3}t} y = e^{\frac{1}{3}t}\cdot e^t$$
$$\frac{d}{dt}(3e^{\frac{1}{3}t} y) = e^{\frac{4}{3}t}$$
$$y = \frac{1}{4}e^t + \frac{C}{3}e^{-\frac{1}{3}t}$$

~~Self-test~~

1. $\cancel{\lim_{t\to 0}\frac{e^t - 1 - t}{t^2} = \lim_{t\to 0}\frac{e^t - 1}{2t} = \lim_{t\to 0}\frac{e^t}{2} = \frac{1}{2}}$

2. $\cancel{\int xe^x dx = xe^x - 1\cdot e^x}$

3. $\cancel{\int_0^2 x^2 e^x dx = -2e^x + 2}$

~~4. $\frac{d}{dx}\int_1^x \arctan(t)\,dt=\arctan(x)$~~

~~5. Taylor Expansion: $f(x)=\cos(x^3)$~~ ...

~~6. $\begin{pmatrix}1&2\\3&4\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$~~

Method: 1. Find $\mu(t)$ s.t. the LHS becomes $\frac{d}{dt}(\mu\cdot y)=\mu y'+\mu' y$

(Either guess $\mu(t)$, or use formula $\mu(t)=\exp(\int p(t)\,dt)$ )

2. Integrate! Get formula

$$y=\frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt+C\right)$$

Example 1: $y'+\cot(t)y=\sin(t)$

① $\mu(t)=\sin(t)$ is integ. factor

$\frac{d}{dt}(\sin(t)y)=\sin t\, y'+\cos(t)y=\sin^2(t)$

② $\sin(t)y=\int \sin^2(t)\,dt=\int \frac{1-\cos(2t)}{2}dt=\frac{t}{2}-\frac{1}{4}\sin(2t)+C$

Example 2. $y'+\tan(t)y=\sin(t)$

① Hard to guess integ. factor (cos doesn't work)

Use formula!

$\mu(t)=\exp(\int \tan(t)\,dt)$

$=\exp(-\ln|\cos t|+c_1)$

$=\pm e^{c_1}\frac{1}{|\cos t|}$

choose: $\frac{1}{\cos t}$

$\int \frac{\sin(t)}{\cos(t)}dt$ $\quad u=\cos(t)$, $du=-\sin t\,dt$

$dt=\frac{du}{-\sin u}$

$\int \frac{\sin t}{u}\cdot\frac{du}{-\sin t}=\int -\frac{1}{u}du$

$=-\ln|u|+C$

$=-\ln|\cos t|+C$

$\frac{d}{dt}\left(\frac{1}{\cos(t)}y\right)=\frac{1}{\cos t}y'+\frac{\sin(t)}{\cos^2(t)}y=\frac{\sin t}{\cos t}$

② $\frac{y}{\cos(t)}=\int\frac{\sin(t)}{\cos(t)}dt=-\ln|\cos t|+C$

$y=\cos(t)(-\ln|\cos t|+C)$

Theorem: Existence & uniqueness for 1st order linear ODE's

Suppose $p(t), g(t)$ are continuous for $t\in(\alpha,\beta)$. For any $t_0\in(a,b)$

The initial value problem $y'+p(t)y=g(t)$, $y(t_0)=y_0$ (given: has a unique solution $y(t)$

defined for $t\in(a,b)$).

The solution is:

$$y(t)=\frac{1}{\mu(t)}\int_{t_0}^{t}\mu(s)g(s)ds+y_0)$$

$$\mu(t)=\exp\left(\int_{t_0}^{t}p(s)ds\right) \qquad (\text{note}: \mu(t_0)=1)$$

Separable equation:

A 1st order ODE $y'=f(t,y)$ is called separable is a product of a function of $t$ and a function of $y$:

$$f(t,y)=\varphi(t)\psi(y)$$

Example: $y'=\frac{t^2}{y^2}=t^3\cdot\frac{1}{y^3}$

$$y'=\varphi(t)+\psi(y)$$

$$\frac{1}{\psi(y)}\frac{dy}{dt}=\varphi(t)\Rightarrow\frac{1}{\psi(y)}dy=\varphi(t)\,dt\Rightarrow\int\cdots=\int\cdots$$

Theorem: The general solution of a sp. ODE

$$y'=\varphi(t)+\psi(y)$$

is $\int\frac{1}{\psi(y)}dy=\int\varphi(t)\,dt$

The solution of the initial value problem (IVP)

$$y'=\varphi(t)\psi(y),\ y(t_0)=y_0$$

is given by $\int_{y_0}^{y}\frac{1}{\psi(s)}ds=\int_{t_0}^{t}\varphi(u)du.$

check $y_0, t_0$ both sides are zero

Example: 1. $\frac{dy}{dt}=(t-1)e^{-y}$

$$e^{y}dy=(t-1)dt$$

$$\int e^{y}dy=\int(t-1)dt$$

$$e^{y}=\frac{1}{2}t^2-t+C$$

$$y=\ln\left(\frac{1}{2}t^2-t+C\right)$$

2. $\frac{dy}{dt}=y^2$

$$\frac{1}{y^2}dy=dt$$

$$\int\frac{1}{y^2}dy=\int dt$$

$$-\frac{1}{y}=t+C$$

$$y=-\frac{1}{t+C}$$

Example: a) $\frac{dy}{dt} = \frac{-t}{y}$

$y\,dy = -t\,dt$

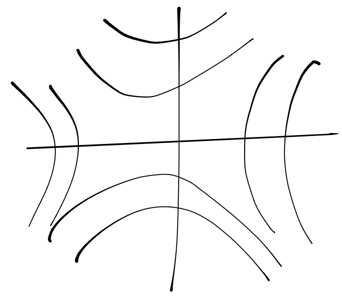
$\frac{y^2}{2} = \frac{-t^2}{2} + C_1$

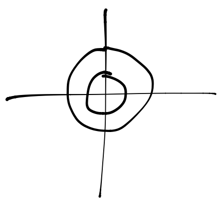
$y^2 + t^2 = C = 2C_1$

b. $\frac{dy}{dt} = \frac{t}{y}$

$\vdots$

$y^2 - t^2 = C$





Example: $\frac{dy}{dt} + ty = t$

$\frac{dy}{dt} = t(1-y)$

$\frac{1}{1-y}\,dy = t\,dt$

$-\ln|1-y| = \frac{t^2}{2} + C$

$|1-y| = \exp\left(\frac{-t^2}{2} - C\right)$

$\frac{dy}{dt} = \frac{t+e^t}{y+e^y}$

$y + e^y\,dy = (t+e^t)\,dt$

$\frac{y^2}{2} + e^y = \frac{t^2}{2} + e^t + C$

Can not solve it.

Solve the IVP.

$2t(y+1) - y\frac{dy}{dt} = 0$, $y(0) = -2$

$y\frac{dy}{dt} = 2t(y+1)$

$\frac{y}{y+1}\,dy = 2t\,dt$

$\int_{-2}^{y} \frac{s}{s+1}\,ds = \int_0^t 2u\,du$ $\qquad$ $\frac{s}{s+1} = \frac{s+1-1}{s+1} = 1 - \frac{1}{s+1}$

thus $\int_2^y \left(1 - \frac{1}{s+1}\right) ds = \left[s - \ln|1+s|\right]_{-2}^{y} = y - \ln|1+y| + 2 + \ln 1$

Hence $y - \ln|1+y| + 2 = t^2$

HOMOGENEOUS EQUATION

There are ODE's of form $y' = \sigma\left(\frac{y}{2}\right)$, $\sigma$ "sigma"

Task : put $v = \frac{y}{t}$

Then $v' = \frac{y'}{t} - \frac{y}{t^2} = \frac{1}{t}\left(y' - \frac{y}{t}\right) = \frac{1}{t}\left(\sigma\left(\frac{y}{t}\right) - v\right)$

Example $y' = \exp\left(\frac{y}{t}\right)$

$v = \frac{y}{t}$

$v' = \frac{1}{t}\left(\exp\left(\frac{y}{t}\right) - \frac{y}{t}\right) = \frac{1}{t}(\exp v - v)$

$\frac{1}{\exp v - v}\, dv = \frac{1}{t}\, dt$

$\int \frac{1}{\exp(v) - v}\, dv = \ln|t| + C$

This gives $v$ which is $\frac{y}{t}$ implicity as function of $t$.

Note this applies to $y' = \frac{3y+2t}{5y+7t}$