Lecture 1

Complex numbers

Def'n: A complex number is an expression of the form $a+bi$ where $a,b\in\mathbb{R}$ and i satisfies $i^2=-1$.

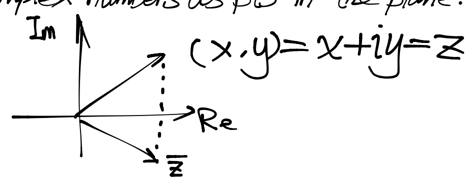
We denote the set of complex numbers by $\mathbb{C}$.

Given a complex $z=x+iy$
We call $\mathrm{Re}(z)=x$ the real part of z & we call $\mathrm{Im}(z)=y$ the imaginary part of z.

Ex: $1+3i$, $\sqrt{2}-\frac{\pi}{2}i$, $5i$, 2 are all complex numbers

$\mathrm{Re}(\sqrt{2}-\frac{\pi}{2}i)=\sqrt{2}$
$\mathrm{Im}(\sqrt{2}-\frac{\pi}{2}i)=-\frac{\pi}{2}$

We can visualize complex numbers as pts in the plane.
$z=x+iy \longrightarrow$



We denote the modulus or length of $z=x+iy$ by $|z|=\sqrt{x^2+y^2}$

We see that $|x|\le|z|$
$|y|\le|z|$
$|z|\le|x|+|y|$

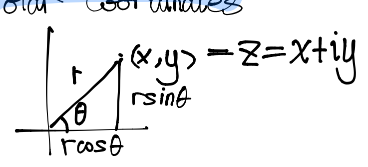
Every complex number $z=x+iy$ has a complex conjugate $\bar{z}=x-iy$

Just like other numbers, we can do arithmetic with complex numbers:
Let $z=x+iy$, $w=a+bi$
Addition: $z+w=(x+a)+i(y+b)$
Subtration: Same pattern
Multiplication: $z\cdot w=(xa-yb)+i(xb+ya)$
Division: $\frac{1}{w}=\frac{1}{a+bi}=\frac{a-bi}{a^2+b^2}=\frac{a}{a^2+b^2}-\frac{b}{a^2+b^2}i=\frac{\bar{w}}{|w|^2}$

<u>Ex:</u> $z=1+i$, $w=2-3i$, $\qquad w\cdot\bar{w}=|w|^2$
$z\cdot w=(1+i)(2-3i)=5-i$

$\frac{z}{w}=\frac{z\cdot\bar{w}}{|w|^2}=\frac{1}{13}+\frac{5}{13}i$

Polar Coordinates



recall: A pt in the plane can be specified by a pair (r,θ) where $r=\sqrt{x^2+y^2}$
$\theta=$ angle between (x,y) & x-axis

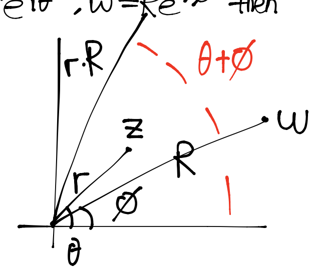
$z \rightsquigarrow |z|=r$

We can actually write $z = |z|\cos\theta + |z|\sin\theta\, i$
$= |z| e^{i\theta}$

$(e^{i\theta} = \cos\theta + i\sin\theta)$

Multiplying in Polar Form

$z = re^{i\theta}$, $w = Re^{i\phi}$ then $z \cdot w = (re^{i\theta})(Re^{i\phi}) = rRe^{i(\theta+\phi)}$



Multiplying by z is scaling by $|z|$ & rotating by angle of z.

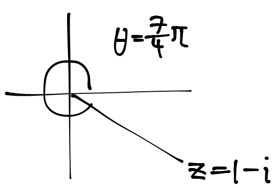
$\frac{z}{w} = \frac{re^{i\theta}}{Re^{i\phi}} = \frac{r}{R} e^{i(\theta-\phi)}$

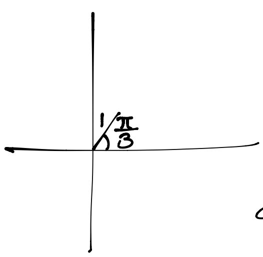
Ex: $z = 1-i$, $w = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

Find z, w in polar form & compute $z \cdot w$, z/w

$|z| = \sqrt{1^2+1^2} = \sqrt{2}$

$z = \sqrt{2}\, e^{i\frac{7\pi}{4}}$





$|w| = 1$
$w = e^{i\frac{\pi}{3}}$

So $z \cdot w = \sqrt{2} \cdot 1\; e^{i(\frac{7\pi}{4} + \frac{\pi}{3})}$
$= \sqrt{2}\; e^{i\frac{25}{12}\pi}$
$= \sqrt{2} e^{i\frac{\pi}{12}}$

$\frac{z}{w} = \sqrt{2} e^{i(\frac{17}{12}\pi)}$

DeMoivre's Thm

$(e^{i\theta})^n = e^{in\theta}$

Ex: Let $z = 1-i$ & find z^4. $z = \sqrt{2}\, e^{i\,7\pi/4}$, so, $z^4 = 4e^{i\pi} = 4(-1) = -4$

$$e^{i\pi}+1=0$$

Argument

Recall: If $z=re^{i\theta}$, then $z=re^{i(\theta+2\pi)}$

So the angle θ is only uniquely defined up to multiples of 2π.

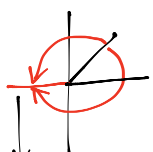
Def'n: An argument for a complex number $z\neq 0$ is any angle θ so that $z=re^{i\theta}$

Ex: Let $z=1+i$, we can take $\theta=\pi/4$, but can also take $\theta=\frac{9\pi}{4}, \frac{17}{4}\pi, \frac{25}{4}\pi \cdots$

We denote argument by $\arg z$.

Ex: $z=1+i$, we have $\arg z=\frac{\pi}{4}, \frac{9\pi}{4}, \cdots$

To remove ambiguity in $\arg z$, define the principal argument $\operatorname{Arg} z$ to be the unique angle $\theta\in[-\pi,\pi)$ so that $z=re^{i\theta}$



"reseting the clock"

points on it $\Rightarrow$ angle $=-\pi$, particularly not π !

We know that $z\cdot w=(re^{i\theta})(Re^{i\phi})=rRe^{i(\theta+\phi)}$

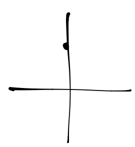
$\rightarrow \arg(z\cdot w)=\arg z+\arg w$

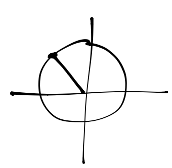
Be careful with $\operatorname{Arg}(z\cdot w)$!

$\operatorname{Arg}(z\cdot w)\neq \operatorname{Arg}(z)+\operatorname{Arg}(w)$

$\operatorname{Arg}(z\cdot w)=\operatorname{Arg} z+\operatorname{Arg} z$ after "reducing"

Ex: $z=i$, $z=e^{i\frac{\pi}{2}}$ $\qquad w=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$ $=e^{i3\pi/4}$





$$z\cdot w=e^{i(\frac{\pi}{2}+\frac{3}{4}\pi)}=e^{i(\frac{5}{4}\pi)}=e^{i(-\frac{3}{4}\pi)}$$

so $\operatorname{Arg}(z\cdot w)=-\frac{3}{4}\pi$