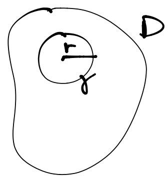
Lecture 11

Recall: If f is analytic in D and $z_0 \in D$ then we can write $f(z)=\sum a_k(z-z_0)^k$, where

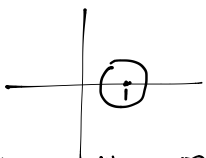


$$a_k=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{k+1}}\,dz$$

γ is simple closed, pos. oriented

We also saw $a_k=\dfrac{f^{(k)}(z_0)}{k!}$

Ex: $\displaystyle\int_\gamma \frac{e^z}{(z-1)^{15}}\,dz$ where γ = circle of radius $\frac{1}{2}$ centered at 1.



We see $z_0=1$, $k+1=15$, $f(z)=e^z$
$(k=14)$

We know $e^w=\sum_{n=0}^{\infty}\frac{w^n}{n!}$

$$e^z=e\cdot e^{z-1}=e\cdot\left(\sum\frac{(z-1)^n}{n!}\right)=\sum\underbrace{\frac{e}{n!}}_{a_n}(z-1)^n$$

So we get $\displaystyle\int_\gamma \frac{e^z}{(z-1)^{15}}=2\pi i\,a_{14}=2\pi i\cdot\frac{e}{(14)!}$

Ex. Let $f(z)=\frac{z+1}{z+3}$ Find power series for f centered at $z_0=1$.

$\boxed{\dfrac{1}{1-w}=\sum_{n=1}^{\infty}w^n \text{ cvgs if } |w|<1}$ known

$f(z)=\dfrac{z+1}{z+3}$ (expand in $z-1$)

$=\dfrac{(z-1)+2}{(z-1)+4}$

$=[(z-1)+2]\dfrac{1}{(z-1)+4}$

Concentrate on: $\displaystyle\frac{1}{(z-1)+4}=\frac{1}{4+(z-1)}=\frac{\frac{1}{4}}{1+\frac{z-1}{4}}=\frac{1}{4}\,\frac{1}{1-\frac{1-z}{4}}$ "w"

$$=\frac{1}{4}\sum\left(\frac{1-z}{4}\right)^n$$

$=\frac{1}{4}\sum\frac{(1-z)^n}{4^{n+1}}$ cvgs if $\left|\frac{1-z}{4}\right|<1 \Rightarrow |1-z|<4$

$$f(z)=((z-1)+2)\cdot\sum\frac{(-1)^n}{4^{n+1}}(z-1)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(z-1)^{n+1}+\sum_{n=0}^{\infty}\frac{2(1-z)^n}{4^{n+1}}$$

The next step is to combine (skipped)

ORDER OF A ZERO

Consider $f(z)=z^2$

We say $z_0=0$ is a zero of order 2 in this case.

$$\left.\begin{array}{l} \text{i.e. } f(0)=0^2=0 \\ f'(0)=2\cdot 0=0 \end{array}\right\} \text{order } 2$$

$$f''(0)=2\neq 0$$

Similarly, $f(z)=z^m$, we get:

$$\left.\begin{array}{l} f(0)=0 \\ \vdots \\ f^{(m-1)}(0)=0 \\ f^{(m)}(0)=m!\neq 0 \end{array}\right\} \text{a zero of order } m$$

In general,

Def'n: We say that a non-constant function f has a zero of order m at z_0 if $f(z_0)=0, f'(z_0)=0, \cdots, f^{(m-1)}(z_0)=0, f^{(m)}(z_0)\neq 0$

Ex: ① $f(z)=e^z-1$, at $z_0=0$

What is the order of $z_0=0$?

$f(0)=0,\ f'(z)=e^z,\ f'(0)=e^0=1\neq 0$

So $z_0=0$ is a zero of order 1.

② $f(z)=(1-z)\cos(\frac{\pi}{2}z)$ What's the order of $z_0=1$?

$f(1)=(1-1)\cos\frac{\pi}{2}=0\cdot 0=0$

$f'(z)=(-1)\cos\frac{\pi}{2}z+(1-z)(-\sin\frac{\pi}{2}z)\frac{\pi}{2}$

$f'(1)=\cdots=0$

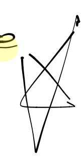
$f''(z)=\frac{\pi}{2}\sin(\frac{\pi}{2}z)+\frac{\pi}{2}\sin(\frac{\pi}{2}z)-(1-z)\frac{\pi^2}{4}\cos(\frac{\pi}{2}z)$

$f''(z)=\pi\neq 0 \Rightarrow z_0=1$ is a zero of order 2.

Suppose f has a zero of order m at z_0. If f is analytic, we can write

$f(z)=\sum a_n(z-z_0)^n$

$$a_n=\frac{f^{(n)}(z_0)}{n!} \Rightarrow \left\{\begin{array}{l} a_n=0,\ n=1,2,\cdots,m-1 \\ a_m\neq 0. \end{array}\right.$$



So $f(z)=a_m(z-z_0)^m+a_{m+1}(z-z_0)^{m+1}+\cdots$ (lower coefficients are 0's)

$$=(z-z_0)^m\underbrace{(a_m+a_{m+1}(z-z_0)+a_{m+2}(z-z_0)^2+\cdots)}_{\text{where } g(z_0)\neq 0}$$

$$=(z-z_0)^m\cdot g(z)$$

$z^3-3z^2+3z-1=(z-1)(z^2-2z+1)=(z-1)^2(z-1)=(z-1)^3$

__Ex__: Find the order of the zeroes of $f(z)=z^3\sin(z^2)$

$$z^2=\pi k \Rightarrow z_k=\sqrt{\pi k}\,,\ k=0,1,2,\cdots$$
$$-1,-2,-3,\cdots$$

$k=0$: $f(z)=z^3\sin(z^2)=z^3\sum\frac{(-1)^n}{(2n+1)!}(z^2)^{2n+1}=z^3\left(z^2-\frac{1}{3!}z^6+\frac{1}{5!}z^{10}+\cdots\right)$

$$=z^5\left(1-\frac{1}{3!}z^4+\frac{1}{5!}z^8-\cdots\right)=1 \text{ when } z=0$$

$\Rightarrow$ __order is 5__ __(easy way)__

What about other k?
It is easiest to use the following fact:
If f has order m at z_0,
g has order n at z_0
then $f\cdot g$ has order $m+n$ at z_0
If $f(z_0)\neq 0$, we say it's order is 0.

So we only need to worry about the term $\sin(z^2)$ for $k\neq 0$.
Try differentiating (power series not so easy)
$g(z)=\sin(z^2)$
$g'(z)=2z\cos(z^2)$
$g'(z_k)=2z_k\cdot\cos(z_k^2)=2\sqrt{\pi k}\cdot\cos\pi k=\pm 2\sqrt{\pi k}\neq 0$

$\Rightarrow$ order of $z_k=\sqrt{\pi k}$, $k\neq 0$ is 1.

__Liouville's Thm:__

If f is bdd and entire, then it's a constant.

__Sketch of Proof__: Suppose f is bdd & entire, $|f(z)|\leq M$ for all $z\in\mathbb{C}$

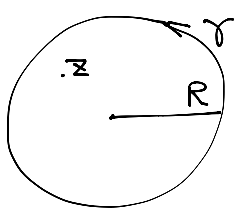
Now consider the function $g(z)=\begin{cases}\frac{f(z)-f(0)}{z} & z\neq 0\\ f'(0) & z=0\end{cases}$

So g is differentiable.

For $|z|=R$, we get: $|g(z)|=\left|\frac{f(z)-f(0)}{z}\right|=\frac{|f(z)-f(0)|}{|z|}=\frac{|f(z)-f(0)|}{R}\leq\frac{|f(z)|+|f(0)|}{R}$

$$\leq\frac{M+M}{R}=\frac{2M}{R}$$

$\Rightarrow |g(z)|\to 0$ as $R\to\infty$



$$g(z)=\frac{1}{2\pi i}\int_\gamma\frac{g(z)}{z-p}dz$$

$$|g(p)|=\frac{1}{2\pi}\left|\int_\gamma\frac{g(z)}{z-p}dz\right|\leq\frac{1}{2\pi}\text{leng}\,\gamma\cdot\max\frac{|g(z)|}{|z-p|}$$

$$=\frac{1}{2\pi}2\pi R\cdot\frac{2M}{R}\cdot\max\frac{1}{|z-p|}\leq\frac{2M}{R-|p|}$$

$$|z-p|\geq|z|-|p| \Rightarrow \frac{1}{|z-p|}\leq\frac{1}{|z|-|p|}$$

So $|g(p)| \leq \frac{2M}{R-|p|}$ $\forall$ all R suff large. as $n \to \infty$ $|g(p)| \leq 0 \Rightarrow g(p)=0$, $\forall p$

$z \neq 0$, $g(z) = \frac{f(z)-f(0)}{z} = 0 \Rightarrow f(z)-f(0)=0 \Rightarrow f(z)=f(0)$ for all z.

$\blacksquare$