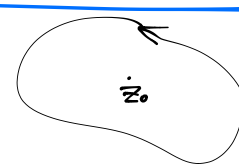
Lecture 14

Residues & Laurent Series

~~Last time~~

$$\mathrm{Res}(f; z_0) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$



γ: simple closed curve going z_0 oriented positively.

If $f(z) = \frac{1}{(z-z_0)^m} g(z)$ $\quad g(z_0) \neq 0$

(i.e. f has a pole of order m at z_0)

then $\mathrm{Res}(f; z_0) = C_{m-1}$ where $g(z) = \sum_{r=0}^{\infty} C_k (z-z_0)^k$

Ex: $f(z) = \frac{\sin z}{z^3}$ at $z_0 = 0$

$\mathrm{Res}(f; z_0) = ?$

$$f(z) = \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = \frac{z}{z^3}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)$$

$$= \frac{1}{z^2}\underbrace{\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots\right)}_{g(z)}$$

$z_0 = 0$ is a pole of order 2

$\mathrm{Res}(f; 0) = C_{m-1} = C_1 = 0$

<u>Another technique</u>:

only in this case

If $f(z) = \frac{P(z)}{Q(z)}$ & z_0 is a pole of order 1 ($Q'(z_0) \neq 0$) then $\mathrm{Res}(f; z_0) = \frac{P(z_0)}{Q'(z_0)}$

<u>Ex</u>: $f(z) = \frac{z^2+3z-1}{z+2}$ $\quad$ Find $\mathrm{Res}(f; -2)$

$z_0 = -2$ is a pole of order 1, so $\mathrm{Res}(f; -2) = \frac{(-2)^2 + 3(-2) - 1}{1} = -3$

$P = z^2 + 3z - 1$

$Q = z + 2$

$Q' = 1$

<u>Ex</u>: $f(z) = \frac{e^z}{z^2 - 2z - 3}$. Find all poles & compute residues.

To look for poles check denom $= 0$.

$z = 3$ or -1

Since $e^z \neq 0$, $z = 3, -1$ are poles of order 1

$P(z) = e^z$

$Q(z) = z^2 - 2z - 3$

$Q'(z) = 2z - 2$

$$\mathrm{Res}(f; 3) = \frac{P(3)}{Q'(3)} = \frac{e^3}{2\cdot 3 - 2} = \frac{e^3}{4}$$

$$\mathrm{Res}(f; -1) = \frac{e^{-1}}{2(-1) - 2} = -\frac{e^{-1}}{4}$$

Laurent Series

If f has a pole of order m at z_0, we can write $f(z)=\frac{1}{(z-z_0)^m}g(z)$ ($g(z)\neq 0$ & g is analytic)

$$=\frac{1}{(z-z_0)^m}\sum_{k=0}^{\infty}C_k(z-z_0)^k$$

$$=\frac{1}{(z-z_0)^m}(C_0+C_1(z-z_0)+C_2(z-z_0)^2+\cdots+C_m(z-z_0)^m+\cdots)$$

$$=\frac{C_0}{(z-z_0)^m}+\frac{C_1}{(z-z_0)^{m-1}}+\cdots+C_m+C_{m+1}(z-z_0)+C_{m+2}(z-z_0)^2+\cdots$$

This is a series that allows both +ve & −ve powers $z-z_0$

LAURENT SERIES (infinite)

- A Laurent series centered at z_0 is a "double-infinite" sum

$$\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$$

$$\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k=\cdots+\frac{a_{-2}}{(z-z_0)^2}+\frac{a_{-1}}{z-z_0}+a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

Ex: $$\sum_{k=-\infty}^{\infty}\frac{z^k}{|k!|}=\cdots+\underbrace{\frac{1}{3!z^3}}_{k=-3}+\underbrace{\frac{1}{2!z^2}}_{k=-2}+\underbrace{\frac{1}{z}}_{k=-1}+\underbrace{1}_{k=0}+\underbrace{\frac{z}{1}}_{k=1}+\underbrace{\frac{z^2}{2!}}_{k=2}+\underbrace{\frac{z^3}{3!}}_{k=3}+\cdots$$

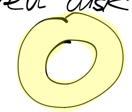
$$=e^{\frac{1}{z}}-1+e^{z}$$

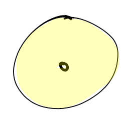
② Every power series is a Laurent series (where $a_k=0$ if $k<0$)

Note: A Laurent series converges in an annulus/punctured disk

$r<|z-z_0|<R$ (annulus)

$0<|z-z_0|<R$ (punctured disk)



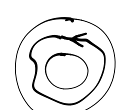


THM: ① Suppose f is analytic in $r<|z-z_0|<R$ (allow $r=0$ for punctured disk) then we can write $f(z)=f_1(z)+f_2(z)$ where f_1 is analytic in $r<|z-z_0|$, f_2 is analytic in $|z-z_0|<R$

② f_1 has a power series in $\frac{1}{z-z_0}$ which converges inside $r<|z-z_0|$.
f_2 has a power series in $z-z_0$ which converges in $|z-z_0|<R$

③ We can write f as a Laurent series using part ① & ②

④ If $f(z)=\sum_{k=-\infty}^{\infty}a_k(z-z_0)^k$ then $a_k=\frac{1}{2\pi i}\int_\gamma\frac{f(z)}{(z-z_0)^{k+1}}dz$
γ simple closed curve inside $r<|z-z_0|<R$.



Ex: $f(z)=\frac{1}{z^2+z}$ is analytic in the punctured disk $0<|z|<1$, $f(z)=\frac{1}{z^2+z}=\frac{1}{z(z+1)}$

$$=\frac{A}{z}+\frac{B}{z+1}=\underbrace{\frac{1}{z}}_{f_1}+\underbrace{\frac{-1}{z+1}}_{f_2}$$

Expand: $\frac{1}{z}$ as power series in $\frac{1}{z}$ (✓, already done)

Expand $\frac{-1}{z+1}$ in power z.

Expand $\frac{-1}{z+1}$ in power z

note: $\frac{1}{1-w} = \sum w^n$ cvgs when $|w|<1$

$$\frac{-1}{z+1} = -\frac{1}{1+z} = -\sum(-z)^n = \sum(-1)^{n+1}z^n \leftarrow \text{cvgs if } |z|<1$$

$$\Rightarrow f(z) = \frac{1}{z^2+z} = \frac{1}{z} + \underbrace{\sum_{n=0}^{\infty}(-1)^{n+1}z^n}_{\text{Laurent series for } f}$$

This series cvgs if $0<|z|<1$

Ex: $f(z) = \frac{1}{z^2+z}$ also analytic in $|z|>1$.

Let's find a Laurent series centered at $z_0=-1$. (i.e. expand in powers of $z+1$)

$$f(z) = \frac{1}{z^2+z} = \underbrace{\frac{1}{z}}_{f_2} - \underbrace{\frac{1}{z+1}}_{f_1}$$

Want to expand $\frac{1}{z}$ in powers of $z+1$:

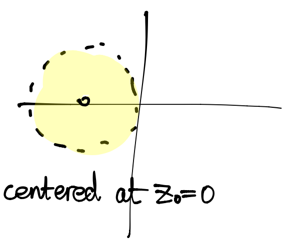
$$\frac{1}{z} = \frac{1}{z+1-1} = -\frac{1}{1-(z+1)} \leftarrow w$$

$$= -\sum(z+1)^n \leftarrow \text{cvgs if } |z+1|<1$$

$$\Rightarrow f(z) = \frac{1}{z} - \frac{1}{z+1} = -\frac{1}{z+1} - \sum_{n=0}^{\infty}(z+1)^n$$

$-\frac{1}{z+1}$: cvgs if $|z+1|>0$ $\qquad$ $\sum_{n=0}^{\infty}(z+1)^n$: cvgs if $|z+1|<1$

so the Laurent series cvges in $0<|z+1|<1$.



Ex: Let $f(z) = \frac{\sin z}{z^3}$. Find the Laurent series for f, centered at $z_0=0$

$$f(z) = \frac{1}{z^3}\left(\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right)$$

$$= \frac{1}{z^3}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right)$$

$$= \frac{1}{z^2} - \underbrace{\frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots}_{\text{cvgs everywhere}}$$

$\frac{1}{z^2}$: cvgs $|z|>0$

The Laurent series cvgs $|z|>0$

Recall: $\frac{\sin z}{z^3} = \frac{1}{z^2}\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots\right)$

↑ coefficients of z gives Res$(f;0)$

$$= \frac{1}{z^2} + \frac{0}{z} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \cdots)$$

THM: If $f(z)=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k$ then $\text{Res}(f:z_0)=a_{-1}=$ coefficient of $\frac{1}{z-z_0}$

Pf: $f(z)=\frac{1}{(z-z_0)^m}g(z)=\frac{1}{(z-z_0)^m}\left(\sum_{k=0}^{\infty} c_k(z-z_0)^k\right)$

$$=\frac{1}{(z-z_0)^m}\left(c_0+c_1(z-z_0)+c_2(z-z_0)^2+\cdots+c_{m-1}(z-z_0)^{m-1}+c_m(z-z_0)^m+\cdots\right)$$

$$=\frac{c_0}{(z-z_0)^m}+\frac{c_1}{(z-z_0)^{m-1}}+\cdots+\underbrace{\frac{c_{m-1}}{z-z_0}}_{\text{residue}}+c_m+c_{m+1}(z-z_0)+\cdots$$

$=\sum_{k=-\infty}^{\infty} a_k(z-z_0)^k \leftarrow$ Laurent Series for f

$\Rightarrow a_k=0$ for $k<-m$

$a_{-m}=c_0$

$a_{-m+1}=c_1$

$a_{-m+k}=c_k$

$\Rightarrow a_{-1}=c_{m-1}=\text{Res}(f:z_0)$

Ex: Suppose $f(z)=\sum_{k=-\infty}^{\infty}\frac{z^k}{|k|!}$

$\text{Res}(f:0)=a_{-1}=\frac{1}{|-1|!}=\frac{1}{1}=1$

Ex: $f(z)=\frac{1}{z(z+1)}=\frac{1}{z}+\sum(-1)^{n+1}(z+1)^n$

$\text{Res}(f:0)=a_{-1}=1$

Ex: Find $\text{Res}(\cot z:0)$

$f(z)=\frac{\cos z}{\sin z}$ has a pole of order 1 at $z_0=0$.

$\cot z=\frac{1}{z}g(z)=\frac{1}{z}(c_0+c_1z+c_2z^2+\cdots)=\frac{c_0}{z}+c_1+c_2z+c_3z^2+\cdots$

$\text{Res}(\cot z:0)=c_0$

$\cot z=\sum c_k z^k=\frac{\cos z}{\sin z}$

$\sin z\cdot(\sum c_k z^k)=\cos z$

$\left(\sum\frac{(-1)^n z^{2n+1}}{(2n+1)!}\right)\left(\sum c_k z^k\right)=\sum\frac{(-1)^n(z^{2n})}{(2n)!}$

expand & compare coefficients to find c_k's.

$\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(\frac{c_{-1}}{z}+c_0+c_1z+\cdots\right)=1-\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots$

$\ldots$ expand