Lecture 11.

Convergence in Distribution.

Recall: $X_n \xrightarrow{P} X$ if $\lim\limits_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0$

Def. Let $X_1, X_2, \dots$ be a sequence of random variables with cdf's $F_{X_1}, F_{X_2}, \dots$ and let X be a random variable with cdf $F_X(x)$. We say that the sequence $\{X_n\}$ converges in distribution to X ($X_n \xrightarrow{d} X$) if $\lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x)$ at every point x in which F is continuous.

$$\lim_{n\to\infty} P(X_n \leq x) = P(X \leq x) \quad \leftarrow \text{weak convergence}$$

Ex. Let X_n be a r.v. s.t. for every $n \geq 2$

$$P(X_n = 0) = \frac{1}{2} - \underbrace{\frac{1}{n}}_{\to 0}, \quad P(X_n = 1) = \frac{1}{2} + \underbrace{\frac{1}{n}}_{\to 0}.$$

$X_n \xrightarrow{d} X$, where $X \sim \text{Bernoulli}(1/2)$

X:	0	1
p:	$\frac{1}{2}$	$\frac{1}{2}$

<u>Fact</u>: $X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{d} X$

<u>Pf</u>:

<u>Lemma</u>. Let X, Y be r.v's, $c \in \mathbb{R}$, $\varepsilon > 0$, then

$$P(Y \le c) \le P(X \le c+\varepsilon) + P(|Y-X| > \varepsilon)$$

<u>Pf of lemma</u>: $P(Y \le c) = P(Y \le c, X \le c+\varepsilon) + P(Y \le c, X > c+\varepsilon)$

$$\le P(X \le c+\varepsilon) + P(Y \le c \mid X > c+\varepsilon) P(X > c+\varepsilon)$$

$$-X < -c-\varepsilon$$

$$\le P(X \le c+\varepsilon) + P(Y - X < -\varepsilon) \le$$

$$P(Y-X < -\varepsilon) \le P(|Y-X| > \varepsilon) =$$

$$= P(Y-X < -\varepsilon) + P(Y-X > \varepsilon)$$

$$P(Y \le c) \le P(X \le c+\varepsilon) + P(|Y-X| > \varepsilon) \quad \blacksquare$$

$$F_{X_n}(x) = P(X_n \le x) \underset{\text{lemma}}{\le} P(X \le x+\varepsilon) + P(|X_n - X| > \varepsilon)$$

$$* \quad P(X \le x-\varepsilon) \le P(X_n \le x) + P(|X_n - X| > \varepsilon)$$

$$P(X \le x-\varepsilon) - \underbrace{P(|X_n-X|>\varepsilon)}_{\to 0} \le P(X_n \le x) \le P(X \le x+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{\to 0}$$

As $n \to \infty$ $P(|X_n - X| > \varepsilon) \to 0$

$$P(X \le x-\varepsilon) \le \lim P(X_n \le x) \le P(X \le x+\varepsilon)$$

$$F_X(x-\varepsilon) \le \lim F_{X_n}(x) \le F_X(x+\varepsilon)$$

take limit for $\varepsilon \to 0^+ \Rightarrow \lim_{n\to\infty} F_{X_n}(x) = F_X(x) \quad \blacksquare$

Ex.

Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Define X_n and X s.t.

$$X_n(\omega_1) = X_n(\omega_2) = 1, \quad X_n(\omega_3) = X_n(\omega_4) = 0 \quad \text{for all } n$$

$$X(\omega_1) = X(\omega_2) = 0, \quad X(\omega_3) = X(\omega_4) = 1 \quad \text{for all } n$$

Assign equal probabilities to each.

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases} \qquad F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$$F_{X_n}(x) = F_X \text{ for all } n \Rightarrow X_n \xrightarrow{d} X$$

$$\lim_{n \to \infty} P\left[\omega : |X_n(\omega) - X(\omega)| \ge \tfrac{1}{2}\right] = 1$$

$$|X_n(\omega) - X(\omega)| = 1 \quad \text{for all } n \text{ and } \omega$$

$$X_n \overset{P}{\not\to} X.$$

Fact. $X_n \xrightarrow{d} c$, a constant, then $X_n \xrightarrow{P} c$.

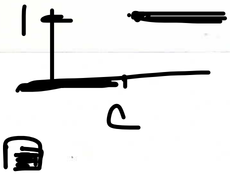
Pf. Let $\varepsilon > 0$

$$P(|X_n - c| \ge \varepsilon) = P(X_n \le c - \varepsilon) + P(X_n \ge c + \varepsilon) \le$$

$$\le P(X_n \le c - \varepsilon) + P\left(X_n > c + \tfrac{\varepsilon}{2}\right) = F_{X_n}(c - \varepsilon) + 1 - F_{X_n}\left(c + \tfrac{\varepsilon}{2}\right)$$

$$\xrightarrow[n \to \infty]{} F_c(c - \varepsilon) + 1 - F_c\left(c + \tfrac{\varepsilon}{2}\right)$$

$$= 0 + 1 - 1 = 0 \Rightarrow X_n \xrightarrow{P} c \quad \blacksquare$$



More facts about convergence in probability and in distribution.

Suppose $X_n \xrightarrow{P} a$, $Y_n \xrightarrow{P} b$, $a, b, c \in \mathbb{R}$

Then,

(i) $cX_n \xrightarrow{P} ca$

(ii) $X_n + Y_n \xrightarrow{P} a+b$

(iii) $X_n Y_n \xrightarrow{P} ab$

(iv) $\dfrac{X_n}{Y_n} \xrightarrow{P} \dfrac{a}{b}$, $b \neq 0$

Pf. (ii) $P(|(X_n+Y_n)-(a+b)| \geq \varepsilon)$

$$= P(|(X_n-a)+(Y_n-b)| \geq \varepsilon)$$

$$\leq P(|X_n-a|+|Y_n-b| \geq \varepsilon)$$

$$\leq P\left(|X_n-a| \geq \frac{\varepsilon}{2} \text{ or } |Y_n-b| \geq \frac{\varepsilon}{2}\right)$$

$$\leq \underbrace{P\left(|X_n-a| \geq \frac{\varepsilon}{2}\right)}_{\to 0} + \underbrace{P\left(|Y_n-b| \geq \frac{\varepsilon}{2}\right)}_{\to 0}$$

$$\xrightarrow[n\to\infty]{} 0 \Rightarrow X_n + Y_n \xrightarrow{P} a+b$$

∎

Slutsky's Theorem. Suppose that $X_n \xrightarrow{P} c$, a constant, and let $h(\cdot)$ be a continuous function at c. Then, $h(X_n) \xrightarrow{P} h(c)$.

Pf: By continuity of $h(\cdot)$, for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ s.t.

$$|x-c| < \delta_\varepsilon \Rightarrow |h(x) - h(c)| < \varepsilon$$

$$\{\omega : |h(X_n(\omega)) - h(c)| < \varepsilon\} \supset \{\omega : |X_n(\omega) - c| < \delta_\varepsilon\}$$

$$P(|h(X_n) - h(c)| < \varepsilon) \geq P(|X_n - c| < \delta_\varepsilon) \to 1$$

$$\Rightarrow P(|h(X_n) - h(c)| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0 \Rightarrow h(X_n) \xrightarrow{P} h(c)$$

Cramer Convergence Theorem. Suppose that $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} c$.

Then,

(i) $X_n + Y_n \xrightarrow{d} X + c$

(ii) $X_n Y_n \xrightarrow{d} cX$

(iii) $\frac{X_n}{Y_n} \xrightarrow{d} \frac{X}{c}$, $c \neq 0$

Note: $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{d} Y \not\Rightarrow X_n + Y_n \xrightarrow{d} X + Y$

$P(X_n = 0) = P(X_n = 1) = \frac{1}{2}$, $P(Y_n = 0) = P(Y = 1) = \frac{1}{2}$

$P(Z = 0) = P(Z = 1) = \frac{1}{2}$ $\qquad W \neq 2Z$

$X_n \xrightarrow{d} Z$, $Y_n \xrightarrow{d} Z$

$X_n + Y_n \to W$ s.t. $P(W = 0) = P(W = 2) = \frac{1}{4}$, $P(W = 1) = \frac{1}{2}$

Theorem. Suppose $X_n \xrightarrow{d} X$, and let $h(\cdot)$ be a function continuous on a set A s.t. $P(X \in A) = 1$. Then $h(X_n) \xrightarrow{d} h(X)$.

$$X_n \xrightarrow{d} X \Rightarrow X_n^2 \longrightarrow X^2$$

$$X_n \xrightarrow{d} N(0,1)$$

$$X_n^2 \xrightarrow{d} \chi^2_{(1)}$$

Ex. Let $X_1, \dots, X_n \sim$ iid $Exp(1)$, $Y_n = X_{(n)} - \ln(n)$.

Show that $F_{Y_n}(y) \longrightarrow e^{-e^{-y}}$, $y \in \mathbb{R}$.

$$F_{Y_n}(y) = P(Y_n \le y) = P(X_{(n)} - \ln(n) \le y)$$

$$= P(X_{(n)} \le y + \ln(n)) = F_{X_{(n)}}(y + \ln(n))$$

$$= [F_X(y + \ln(n))]^n = [1 - e^{-(y+\ln(n))}]^n$$

$$= \left[1 - \frac{e^{-y}}{n}\right]^n \xrightarrow[n\to\infty]{} e^{-e^{-y}} \quad \left[\left(1 - \frac{x}{n}\right)^n \xrightarrow[n\to\infty]{} e^{-x}\right]$$

Ex. Let $X_n \sim F_n(x) = 1 - \frac{1}{x^n}$, $Y_n = nX_n - n$.

$$F_{Y_n}(y) = P(Y_n \le y) = P(nX_n - n \le y)$$

$$= P\left(X_n \le 1 + \frac{y}{n}\right)$$

$$= F_{X_n}\left(1 + \frac{y}{n}\right) = 1 - \frac{1}{\left(1 + \frac{y}{n}\right)^n} \qquad Y_n \xrightarrow{d} Y$$

$$\xrightarrow[n\to\infty]{} 1 - \frac{1}{e^y} = 1 - e^{-y} \Rightarrow Y \sim Exp(1)$$

Continuity Theorem.

Let X be a r.v. such that for some $t_0 > 0$ we have $m_X(t) < \infty$ for $t \in (-t_0, t_0)$. Further, if $X_1, X_2, \dots$ is a sequence of r.v.s with $m_{X_n}(t) < \infty$ and

$$\lim_{n \to \infty} m_{X_n}(t) = m_X(t) \quad \text{for all } t \in (-t_0, t_0),$$

then $X_n \xrightarrow{d} X$.

$$m_{X_n}(t) \to m_X(t) \Rightarrow F_{X_n}(x) \to F_X(x)$$

Ex. Poisson distribution can be approximated by a Normal distribution.

Let $\lambda_1, \lambda_2, \dots$ be an increasing sequence with $\lambda_n \to \infty$ as $n \to \infty$ and let $\{X_n\}$ be s.t. $X_n \sim \text{Poisson}(\lambda_n)$. Then $E(X_n) = \lambda_n = Var(X_n)$

$$\text{Let } Z_n = \frac{X_n - E(X_n)}{\sqrt{Var(X_n)}} = \frac{X_n - \lambda_n}{\sqrt{\lambda_n}}$$

$$\Rightarrow E(Z_n) = 0, \quad Var(Z_n) = 1$$

Show: $Z_n \xrightarrow{d} Z \sim N(0,1)$

$$m_{X_n}(t) = e^{\lambda_n(e^t - 1)}$$

$$Z_n = \frac{X_n}{\sqrt{\lambda_n}} - \sqrt{\lambda_n}$$

$$m_{Z_n}(t) = E(e^{Z_n t}) = E\left(e^{\left(\frac{X_n}{\lambda_n} - \sqrt{\lambda_n}\right)t}\right) = e^{-t\sqrt{\lambda_n}}\, m_{X_n}\left(\frac{t}{\lambda_n}\right)$$

$$= e^{-t\sqrt{\lambda_n}}\, e^{\lambda_n\left(e^{t/\sqrt{\lambda_n}} - 1\right)}$$

$$\ln(m_{Z_n}(t)) = -t\sqrt{\lambda_n} + \lambda_n\left(e^{t/\sqrt{\lambda_n}} - 1\right) = \frac{t^2}{2} + \underbrace{\frac{t^3/\lambda_n^{3/2}}{3!}\lambda_n + \cdots}_{\xrightarrow[n\to\infty]{} 0 \text{ because } \lambda_n \to \infty}$$

$$e^{t/\sqrt{\lambda_n}} = 1 + \frac{t/\sqrt{\lambda_n}}{1!} + \frac{(t/\sqrt{\lambda_n})^2}{2!} + \frac{(t/\sqrt{\lambda_n})^3}{3!} + \cdots$$

Thus, $\ln(m_{Z_n}(t)) \longrightarrow \frac{t^2}{2}$

$$m_{Z_n}(t) \to e^{t^2/2} \to \text{mgf of } N(0,1)$$

$$\Rightarrow \quad Z_n \xrightarrow{d} Z \sim N(0,1).$$

Ex. $X \sim \text{Poisson}(900)$. Find $P(X > 950)$

$$P(X > 950) = P\left(\frac{X-900}{\sqrt{900}} > \frac{950-900}{\sqrt{900}}\right)$$

$$= 1 - P\left(\frac{X-900}{\sqrt{900}} \le \frac{5}{3}\right)$$

$$\approx 1 - \Phi\left(\frac{5}{3}\right) = 0.04779$$

Software: 0.04712 ← exact value.

Central Limit Theorem.

Let $X_1, X_2, \ldots$ be iid r.v's with mean μ and st. dev. σ.

$$S_n = \sum_{i=1}^{n} X_i$$

$\Rightarrow$ by WLLN, $\frac{S_n}{n} \xrightarrow{P} \mu$

$E(S_n) = n\mu$, $Var(S_n) = n\sigma^2$

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}, \quad E(Z_n) = 0, \; Var(Z_n) = 1$$

Theorem. Let $X_1, X_2, \ldots$ be a sequence of iid r.v's with $E(X_i) = \mu < \infty$ and $Var(X_i) = \sigma^2 < \infty$. Suppose the common distribution function $F_X(x)$ and the common moment generating function $m_X(t)$ are defined in a neighborhood of zero. Let

$$S_n = \sum_{i=1}^{n} X_i$$

Then $\lim P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le x\right) = \Phi(x), \quad -\infty < x < \infty$

where $\Phi(x)$ is the cdf for $N(0,1)$.

$$Z_n = \frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

$$\bar{X}_n = \frac{1}{n} S_n$$

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

Proof: $\mu = 0$

Let $Z_n = \frac{S_n}{\sigma\sqrt{n}}$

$$m_{S_n}(t) = E\left(e^{t\sum_i X_i}\right) = \left[m_X(t)\right]^n$$

$$m_{Z_n}(t) = \left[m_X\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$m_X(t) = m_X(0) + t\,m_X'(0) + \frac{1}{2}t^2 m_X''(0) + \varepsilon_t$$

$\longrightarrow$ Taylor series expansion about zero

$$\frac{\varepsilon_t}{t^2} \xrightarrow[t\to 0]{} 0$$

$$E(X) = 0 \Rightarrow m_X'(0) = 0, \quad m_X''(0) = \sigma^2$$

$$\frac{t}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

$$m_X\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + t\cdot 0 + \frac{1}{2}\frac{t^2}{\cancel{\sigma^2} n}\cdot\cancel{\sigma^2} + \varepsilon_n$$

$$\frac{\varepsilon_n}{t^2/\sigma^2 n} \xrightarrow[n\to\infty]{} 0$$

$$\Rightarrow m_{Z_n}(t) = \left(1 + \frac{t^2/2}{n} + \varepsilon_n\right)^n \xrightarrow[n\to\infty]{}$$

$$\longrightarrow e^{t^2/2} \text{ — mgf for } N(0,1)$$

$$\Rightarrow Z_n \xrightarrow{d} N(0,1)$$

∎

Ex.

Let $X_1, X_2, \ldots \sim$ iid Bernoulli(p)

$E(X_i) = p$, $Var(X_i) = pq$

$$P\left(\frac{X_1 + \ldots + X_n - np}{\sqrt{npq}} \le x\right) \xrightarrow{d} \Phi(x)$$

by CLT

$Y_n = X_1 + \ldots + X_n \Rightarrow Y_n \sim Bin(n, p)$

$E(Y_n) = np$, $Var(Y_n) = npq$

$$P(Y_n \le y) = P\left(\frac{Y_n - np}{\sqrt{npq}} \le \frac{y - np}{\sqrt{npq}}\right) \approx \Phi\left(\frac{y-np}{\sqrt{npq}}\right)$$

Suppose a coin is tossed 100 times and we observe 60 heads. Is the coin fair?

$X = \#$ of heads, then if coin is fair

$\Rightarrow X \sim Bin(100, \frac{1}{2})$

$E(X) = np = 50$, $Var(X) = npq = 25$

$\sigma_X = \sqrt{25} = 5$

$P(X \ge 60 \mid \text{coin is fair})$

$= P(X \ge 60 \mid X \sim Bin(100, \frac{1}{2}))$

$= P\left(\frac{X - 50}{5} \ge \frac{60 - 50}{5}\right) = P(Z \ge 2)$

$\approx 1 - \Phi(2) = 0.0228$

$\Rightarrow$ coin might not be fair..