MAT 246

Practice Test I.

1(a). What is $\varphi(20^{100})$?

Solution: $\varphi(20^{100}) = \cancel{((2-1)(2-1)(5-1))^{100} = 4^{100} = 2^{200}}$

$$= \varphi(2^{100} \cdot 2^{100} \cdot 5^{100})$$
$$= \varphi(\cancel{4}\,2^{200} \cdot 5^{100})$$
$$= (2^{200} - 2^{199})(5^{100} - 5^{99})$$
$$= 2^{199} \cdot 5^{99}(5-1)$$
$$= 2^{201} \cdot 5^{99}$$

1(b) Find an integer x s.t. $140x = 133 \pmod{301}$. Hint $\gcd(140, 301) = 7$

Solution:

$\cancel{140x - 133y = 301}$

$$140x \equiv 133 \mod 301$$
$$7 \cdot 20x \equiv 7 \cdot 19 \mod (7 \cdot 43)$$
$$20x \equiv 19 \mod 43$$

By Fermat's Theorem

$$20^{42} \equiv 1 \mod 43$$
$$20 \cdot 20^{41} \cdot 19 \equiv 19 \mod 43$$
$$x = 19 \cdot 20^{41}$$

2. (a). Prove by induction, that $1+2+3+\cdots+n = \frac{n(n+1)}{2}$ for every ~~#~~ N.

Proof: Check the formula for $n=1$

$$1 = \frac{1(1+1)}{2}$$

Suppose it is proved for $n \geq 1$

Then $1+2+\cdots+n+(n+1) = \frac{n(n+1)}{2} + (n+1)$

$$= \frac{(n+1)(n+2)}{2}$$

$\blacksquare$

2(b). Prove that for p an odd prime ~~that~~

$$1^p + 2^p + \cdots + (p-1)^p \equiv 0 \mod p$$

Proof: By Fermat's Theorem

$a^{p-1} \equiv 1 \mod p$ p is a prime & a is a natural

$a^p \equiv a \mod p$

$1^p \equiv 1 \mod p$ ...

$1^p + 2^p + \cdots + (p-1)^p \equiv 1 + 2 + \cdots + (p-1) \mod p$

$= \frac{p \cdot (p-1)}{2} \mod p$

$= 0 \mod p.$

3. Prove that for any odd int. a, a & a^{4n+1} have the same last digit for every natural number n.

Proof

If a is odd & divisible by 5 then the last digit is 5. Therefore ~~fore~~ for any a^k its last digit is 5.

Now suppose $(a, 5) = 1$

Since a is odd then $(a, 10) = 1$

By Euler's thm $a^{\varphi(10)} \equiv a^{(2-1)(5-1)} \equiv a^4 \equiv 1 \mod 10$

Therefore $a^{4k} \equiv 1 \mod 10$

Then $a^{4k+1} \equiv a \mod 10$

4. "Perfect square" is a # of the form n^2 where n is a natural #. Show 9120342526523 is not the sum of two perfect squares. Hint: Consider values modulo 4.

Proof: If $a \equiv 0 \bmod 4$
or $a \equiv 2 \bmod 4$
Then $a^2 \equiv 0 \bmod 4$
If $a \equiv 1 \bmod 4$
or $a \equiv 3 \bmod 4$
Then $a^2 \equiv 1 \bmod 4$
Therefore for $a^2+b^2 \equiv 0 \bmod 4$ or
$a^2+b^2 \equiv 1 \bmod 4$ or
$a^2+b^2 \equiv 2 \bmod 4$.
But $9120342526523 \equiv 23 \bmod 4$
$\equiv 3 \bmod 4$.

5. (a). Are there $\mathbb{Q}$ a & b such that
$\sqrt{3} = a + b\sqrt{2}$?

Proof: Sps $a, b \in \mathbb{Q}$.
$3 = (a+b\sqrt{2})^2$
$= a^2 + 2b^2 + 2ab\sqrt{2}$
$3 - a^2 - 2b^2 = 2ab\sqrt{2}$
Cannot have $a = 0$ since
$3 - 2b^2 = 0$
$b = \sqrt{\frac{3}{2}}$
cannot have $b = 0$ since
$3 - a^2 = 0$
$a = \sqrt{3}$
Then $\sqrt{2} = \dfrac{3-a^2-2b^2}{2ab} \in \mathbb{Q}$.

Hence no such a, b.

(b). Prove $\frac{\sqrt{5}}{\sqrt{2}+\sqrt{11}}$ is irrational.

Pf. Suppose $q = \frac{\sqrt{5}}{\sqrt{2}+\sqrt{11}}$ is rational.

$$\sqrt{5} = q(\sqrt{2}+\sqrt{11}) \quad (q \neq 0)$$

$$5 = q^2 \cdot (2+11+2\sqrt{22})$$
$$= 13q^2 + 2\sqrt{22}\,q^2$$
$$\sqrt{22} = \frac{5-13q^2}{2q^2}$$

irrational!

6. (a). What is the cardinality of the set of roots of polynomial with constructible coefficients.

Solution: S be such a set.

$|S| \geq |N|$ obviously.

It was proved that a root of a polynomial with constructible coefficients is also a root of a polynomial with rational coefficients.

~~Therefore~~ Therefore all elements in S are algebraic

hence $|S| \leq |N|$

$\Rightarrow |S| = |N|$.

~~6~~6(b). What is the cardinality of the set S of all functions from $\mathbb{N}$ to $\{1,3,5\}$?

Pf: ~~Let~~
First observe that any such function corresponds to a sequence $a_1, a_2, a_3, \dots$ where each a_i is equal either $1, 3$ or 5.

Consider the map. $f: S \to \mathbb{R}$ given by
$f(a_1, a_2, a_3, \dots) = 0.a_1 a_2 a_3 \dots$
Clearly it is 1-1 $\Rightarrow$ $|S| \le |\mathbb{R}|$
On the other hand recall that $|\mathbb{R}| = |P(\mathbb{N})|$ and
$|P(\mathbb{N})| = |$the set of functions from $\mathbb{N}$ to $\{0,1\}|$.
Since $|\{0,1\}| \le |\{1,3,5\}|$
then $|\mathbb{R}| \le |S|$
$\Rightarrow |S| = |\mathbb{R}|$

7. θ is between 0 & 90°. Suppose $\cos\theta = \frac{3}{4}$. Prove $\frac{\theta}{3}$ is not a constructible angle.

Proof: $x = \cos\frac{\theta}{3}$, suppose ~~x~~ x is constructible.

$$\cos(\theta) = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$$
$$4x^3 - 3x = \frac{3}{4}$$
$$16x^3 - 12x - 3 = 0$$

If it is constructible then
$y = 2x$ is constructible
$$2y^3 - 6y - 3 = 0$$
It has a constructible root if it has a rational root.
By rational root theorem. Suppose $\frac{p}{q}$ is 2^a rational root.
Then $p \mid 3$, $q \mid 2$
$p = \pm 1, \pm 3$
$q = \pm 1, \pm 2$
$\frac{p}{q} = \pm 1, \pm \frac{1}{2}, \pm 3, \pm \frac{3}{2}$
none of them are roots of $2y^3 - 6y - 3 = 0$

Hence, contradiction. x is not constructible.

8. constructible or not?

(a). $\cos\theta$, where $\frac{\theta}{3}$ is constructible

Yes. $\cos\theta = 4\cos^3\frac{\theta}{3} - 3\cos\frac{\theta}{3}$

(b). $\sqrt[3]{\frac{25}{8}}$ no. $\frac{\sqrt[3]{25}}{2}$ => By Rational Root Thm

(c). $\sqrt{7+\sqrt{5}}$ belongs to F_2 for the tower of $\mathbb{Q} = F_0 \subset F_1 = F_0(\sqrt{5}) \subset F_2 = F_1(\sqrt{7+\sqrt{5}})$.

It is constructible

(d). $(0.029)^{\frac{1}{3}}$ no. same as (b).

(e). $\tan 22.5° = \frac{90°}{4}$ Yes.

9. Find all complex solutions of $z^6 + z^3 + 1 = 0$.

Let $x = z^3$

$x^2 + x + 1 = 0$

$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}i}{2}$

① if $x = \frac{-1+\sqrt{3}i}{2} = \cos(\frac{2}{3}\pi) + i\sin(\frac{2\pi}{3})$

Solving $z^3 = x = \cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi$

we get $z = \cos(\frac{2}{9}\pi + \frac{2\pi k}{3}) + i\sin(\frac{2}{9}\pi + \frac{2\pi k}{3})$ where $k = 0, 1, 2$

Then we have 3 solutions.

When $k=0$, $z_1 = \cos(\frac{2}{9}\pi) + i\sin(\frac{2}{9}\pi)$

$k=1$, $z_2 = \cos(\cancel{\frac{2}{9}\pi + \frac{2\pi}{3}}\ \frac{8}{9}\pi) + i\sin(\frac{8}{9}\pi)$

$k=2$, $z_3 = \cos(\frac{14\pi}{9}) + i\sin(\frac{14\pi}{9})$

② if $x = \frac{-1-\sqrt{3}i}{2} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$.

(similarly), - - -

10. $p=3,\ q=11,\ E=7,\ N=3\cdot 11=33.$

$M \longrightarrow R=6.$

$M=?$

$\cancel{M \equiv R^D}$

$\cancel{R \equiv M^E \bmod N}$

$\cancel{6 \equiv M^7 \bmod 33}$

$$DE \equiv 1 \bmod \varphi(N)$$
$$\equiv 1 \bmod 2\cdot 10$$
$$\equiv 1 \bmod 20$$
$$7D \equiv 1 \bmod 20$$
$$D \equiv 3$$

$$R^D \equiv M \bmod N$$
$$6^3 \equiv M \bmod 33$$
$$\equiv 3\cdot 6 \bmod 33$$
$$\equiv 18 \bmod 33$$

$M=18$

11. Construct a polynomial with integer coefficients which has $\sqrt{2}+\sqrt{5}$ as a root.

Solution: $x=\sqrt{2}+\sqrt{5}$

$$x-\sqrt{2}=\sqrt{5}$$
$$x^2-2\sqrt{2}x+2=5$$
$$x^2-3=2\sqrt{2}x$$
$$(x^2-3)^2=8x^2$$
$$x^4-6x^2+9=8x^2$$
$$x^4-14x^2+9=0.$$

MAT246 Practice Test 2

1. Induction prove:

$$1^2+3^2+\cdots+(2n-1)^2=\frac{(n+1)(2n+1)(2n+3)}{3}$$

2. $a,b,c\in\mathbb{N}$

(a). Show $ax+by=c$ has a solution iff $(a,b)|c$.

Proof: ($\Rightarrow$) Suppose $ax+by=c$ has solutions x & y.
If $d|a$, $d|b$ then obviously

$$d\,|\,ax+by=c$$

In particular if $(a,b)|c$.

($\Leftarrow$) Suppose $(a,b)|c$ then $c=d(a,b)$
$ax+by=(a,b)$ has an integer solution.
$a(dx)+b(dy)=d(a,b)=c$

(b). find all int. solutions of $6x+15y=9$

Solution: $2x+5y=3$.
Since $(2,5)=1$.
Euclidean algorithm,

$$2\cdot(-2)+5\cdot 1=1$$
$$2\cdot(-6)+5\cdot(3)=3$$
$$x_0=-6,\ y_0=3$$

It's easy to see that $x=-6-5k$, $y=3+2k$
is a solution of $2x+5y=3$ for any k

3. last digit of the sum

$$2(1+3+3^2+\cdots+3^{309})$$

$$= 2 \cdot \frac{3^{310}-1}{3-1}$$

$$= 3^{310} - 1$$

$$\varphi(10) = (2-1)(5-1) = 4$$

$$3^4 \equiv 1 \mod (10)$$

$$310 = 308 + 2 = 77 \times 4 + 2$$

$$3^{310} \equiv 3^2 \mod (10)$$

$$= 9 \mod 10.$$

$$3^{310} - 1 \equiv 8 \mod 10.$$

4. S be infinite & $A \subset S$ be finite.
Prove $|S| = |S \setminus A|$

use induction then ...

5. $S = [0,1]$, $T = [0,2)$
$f: S \to T$ be given by $f(x) = x$
$g: T \to S$ be given by $g(x) = \frac{x}{2}$

(a). S_S, S_T, S_∞?
(b). $h = ?$ then h is 1-1 & onto

6. $n=2p$, p is an odd prime.
find remainder $\varphi(n)!$ mod n.

$\varphi(n)=(2-1)(p-1)=(p-1)$

$(p-1)!$ mod $n \equiv (p-1)!$ mod $2p$

$(p-1)!$ mod $p \equiv -1$
$\equiv (p-1)$ mod p

So $p \mid (p-1)!-(p-1)$
Since p is odd
$p-1$ is even
therefore $2 \mid (p-1)!-(p-1)$

also $(2,p)=1$.
$\Rightarrow 2p \mid (p-1)!-(p-1)$

$\Rightarrow n \mid (p-1)!-(p-1)$

$\Rightarrow (p-1)! \equiv (p-1)$ mod $2p$

remainder is $p-1$.

7. Prove that $q_1\sqrt{3}+q_2\sqrt{5} \neq q_1'\sqrt{3}+q_2'\sqrt{5}$ for any $q_1, q_2, q_1', q_2' \in \mathbb{Q}$.
unless $q_1=q_1'$, $q_2=q_2'$

$(q_1-q_1')\sqrt{3}+(q_2-q_2')\sqrt{5}=0$
$a\sqrt{3}+b\sqrt{5}=0$.
show $a=b=0$.
If $a\neq 0$ then $b= \cdots$
If $b\neq 0$, then $a \cdots$

Therefore $\cdots$

8. a be a root of $x^5 - 6x^3 + 2x^2 + 5x - 1 = 0$.
construct a polynomial with integer coefficients which has a^2 as a root.

Solution: $x^5 - 6x^3 + 5x = 1 - 2x^2$

$\cancel{(x^5 - 6x^3 + 5x)^2 = (1-2x^2)^2}$

$x(x^4 - 6x^2 + 5) = 1 - 2x^2$

$x^2(x^4 - 6x^2 + 5)^2 = (1 - 2x^2)^2$

let $y = x^2$

$y(y^2 - 6y + 5)^2 = (1 - 2y)^2$

9. Find roots of $x^6 + 7x^3 - 8 = 0$.

$z = x^3$.

$z^2 + 7z - 8 = 0$

$z = \frac{-7 \pm \sqrt{49 + 32}}{2} = \frac{-7 \pm 9}{2} = 1$ or -8

$x^3 = 1$ & $x^3 = -8$

$x = 1$, $x = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

~~x = cos~~

$x = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$

$x^3 = -8$.

$x = -2$

$x = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$.

$x = 1 - i\sqrt{3}$.

10. Represent $\sin(5\theta)$ ~~a~~ as a poly in $\sin(\theta)$

$$\sin 5\theta = \sin(2\theta+3\theta)$$
$$= \sin 2\theta\cos 3\theta + \cos 2\theta \sin 3\theta$$
$$= 2\sin\theta\cos\theta\cos(2\theta+\theta) + (\cos^2\theta - \sin^2\theta)\sin(2\theta+\theta)$$
$$= 2\sin\theta\cos\theta(\cos 2\theta\cos\theta - \sin 2\theta\sin\theta) + (\cos^2\theta-\sin^2\theta)\sin 2\theta\cos\theta$$
$$= 2\sin\theta\cos\theta\cos\theta(\cos^2\theta-\sin^2\theta) - 2\sin\theta\sin\theta\cdot\cos\theta\cdot 2\sin\theta\cos\theta$$
$$+ 2\sin\theta\cos^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta\cdot\sin^2\theta$$

$\cos^2\theta = 1-\sin^2\theta$

$$= 2\sin\theta\cos^4\theta - 2\sin^3\theta\cos^2\theta - 4\sin^3\theta\cos^2\theta + 2\sin\theta\cos^4\theta - 2\sin^3\theta\cos^2\theta$$
$$= 4\sin\theta\cos^4\theta - \cancel{4\sin^3\theta\cos^2\theta - 4\sin}$$
$$-8\sin^3\theta\cos^2\theta$$
$$= 4\sin\theta(1-\sin^2\theta)^2 - 8\sin^3\theta(1-\sin^2\theta) = 4\sin\theta(1-2\sin^2\theta+\sin^4\theta) - 8\sin^3\theta + 8\sin^5\theta$$

or $(\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta)$

$$= (\cos\theta+i\sin\theta)^2(\cos\theta+i\sin\theta)^3$$
$$\Rightarrow$$
$$(\cos\theta+i\sin\theta)^2 = (\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)$$
$$(\cos\theta+i\sin\theta)^3 = (\cos^2\theta+2i\sin\theta\cos\theta-\sin^2\theta)(\cos\theta+i\sin\theta)$$
$$= \cos^3\theta + 2i\sin^2\theta\cos\theta - \sin^2\theta\cos\theta +$$
$$i\sin\theta\cos^2\theta - 2\sin^2\theta\cos\theta - i\sin^3\theta$$

11. Is $\dfrac{\sqrt[6]{5}-\sqrt{5}}{1+2\sqrt{7}}$ constructible?

Sps it is, then since $\sqrt{5}, \sqrt{7}$ are constructible
so this $\Rightarrow \sqrt[6]{5}$ is constructible
Then $(\sqrt[6]{5})^2 = \sqrt[3]{5}$ is constructible.
~~Thus $x^3-5=0$ has constructible solution~~
$\sqrt[3]{5}$ is a root of $x^3-5=0$ which is a cubic equation ~~a~~ with integer coefficients. By a theorem from class if it has a constructible --- rational root.

say the rational root is $\frac{p}{q}$,
then by Rational root theorem
$p \mid 5,\ q \mid 1$

$p = \pm 5, \pm 1$
$q = \pm 1$

$\frac{p}{q} = \pm 5, \pm 1.$ but none of those is solution contradiction.

... not constructible.

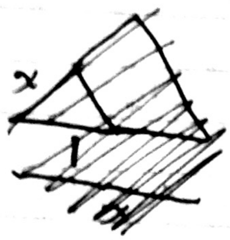
12.

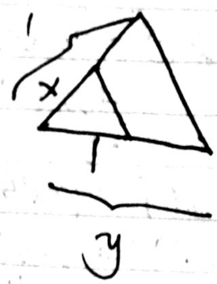
a). If $\frac{x}{y}$ constructible then both x & y constructible.

No. ~~x~~ $x=\pi$, $y=\pi$.

b). If x is cons then $\frac{1}{x}$ is cons.

Yes. $\frac{x}{1}=\frac{1}{y}$ $y=\frac{1}{x}$.





c). There is θ that $\cos\theta$ is constructible but $\sin\theta$ is not.

No. $\sin^2\theta = 1-\cos^2\theta$.

$\sin\theta = \pm\sqrt{1-\cos^2\theta}$.

d). $\sqrt[3]{\frac{10}{27}}$ is cons.

No. If it is then $\frac{\sqrt[3]{10}}{3}$ is.

$\frac{\sqrt[3]{10}}{3}$ is a root of ~~[illegible]~~ $x^3=\frac{10}{27}$

$27x^3-10=0$.

... Cons root $\Rightarrow$ rational root $\frac{p}{q}$

RRT: $p|10$, $q|27$

$p=\pm1, \pm2, \pm5, \pm10$

$q=\pm1, \pm3, \pm9, \pm27$.

None of it is.

So not constructible

13. Prove $(1+x^{19})^3+(1+x^{19})^2-3=0$ has no cons. roots.

Let $1+x^{19}=y$

if x is cons.

then y is cons.

$y^3+y^2-3=0$.

$\Rightarrow$ has a rational root $\frac{p}{q}$.

$p|3$, $q|1$

$p=\pm 3, \pm 1$

$q=\pm 1$

$\frac{p}{q}=\pm 3, \pm 1$.

$27+9-3\neq 0$

$-27+9-3\neq 0$

$1+1-3\neq 0$

$-1+1-3\neq 0$

none. no cons. solution. done.

Practice Test 3

Q1. Fibonacci Induction:

Q2(a). Find the remainder when $7^{3^{100}}$ is divided by 20.

$\gcd(7,20)=1 \quad \varphi(20)= \cancel{2^2\times} (2^2-2)(5-1) = 2\times 4 = 8$

$\cancel{\gcd(3,8)=1 \quad \varphi(8) = 2^3-2^2} = 4$

$3^{100} \bmod 8 \equiv (3^2)^{50} \bmod 8 \equiv 1^{50} \bmod 8 \equiv 1 \bmod 8$

$\cancel{3^{100}}$

$3^{100} \cancel{\bmod 8 = (3^2)^{50} \bmod 4 = 1^{50} \bmod 4 = 1 \bmod} \ 4$

$\cancel{7^1 \bmod 20}$

$7^{3^{100}} = 7^{8k+1} = (7^8)^k \cdot 7 \equiv 1^k \cdot 7 \equiv 7 \bmod 20$

(b). Find $2^{p!} \pmod p$ where p is an odd prime.

By Fermat $2^{p-1} \equiv 1 \bmod p$

Since $p-1 \mid p!$ that

$2^{p!} \equiv 1 \bmod p$

3. Prove $q_1\sqrt{2}+q_2\sqrt{6}$ is irrational for any $q_1, q_2 \in \mathbb{Q}$, unless $q_1=q_2=0$

Pf: Sps ~~both Q &~~ at least one of them is not zero.

Q

① $q_1=0,\ q_2\neq 0$.

$x = q_2\sqrt{6}$ is rational ; $\cancel{q_2 = x}$ $\sqrt{6} = \frac{x}{q_2}$ is rational too. $\times$

② $q_2=0,\ q_1\neq 0$.

$x = q_1\sqrt{2}$, $\sqrt{2} = \frac{x}{q_1}$ is rational too. $\times$

③ $q_1\neq 0,\ q_2\neq 0$.

(Square it) ...

(4.) $(\varphi(m), m)=1$. $m \in \mathbb{N}$. Prove $\sqrt{m}$ is irrational.

~~Proo~~ Pf. Let $m = p_1^{k_1} \cdots p_l^{k_l}$

$$\varphi(m) = (p_1^{k_1} - p_1^{k_1-1}) \cdots (p_l^{k_l} - p_l^{k_l-1})$$

If any $k_i > 1$ this formula implies that p_i divides $\varphi(m)$ & $\Rightarrow \gcd(\varphi(m), m) \neq 1$.

Thus this holds when all $k_i = 1$.
Therefore $m = p_1 \cdots p_k$ is not a complete square & hence $\sqrt{m}$ is irrational.

5. $p=11, q=5, E=23, N=11 \cdot 5 = 55$.
$R=2$. $M=?$

$$DE \equiv 1 \bmod \varphi(N)$$
$$23D \equiv 1 \bmod 10 \cdot 4 \equiv 1 \bmod 40$$
$$D = 7$$

$$R^D \equiv M \bmod N$$
$$2^7 \equiv M \bmod 55$$
$$2 \cdot (64 \bmod 55) \equiv 18 \bmod 55$$
$$M = 18.$$

6. (a) Find all complex roots of $z^6 + (1-i)z^3 - i = 0$.

$$y = z^3$$

$$y^2 + (1-i)y - i = 0$$

$$y = \frac{(i-1) \pm \sqrt{(1-i)^2 + 4i}}{2} = \frac{(i-1) \pm (i+1)}{2} = i \text{ or } -1$$

$$y_1 = i, \ y_2 = -1.$$

$z^3 = i$ & $z^3 = -1$

$z^3 = i = 1\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$

$z = \cancel{\cos} \sqrt[3]{1}\left(\cos\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{3}\right)\right)$

$= \cos\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right) + i\sin\left(\frac{\pi}{6} + \frac{2}{3}k\pi\right)$

$k=0,$

$k=1,$

$k=2.$

$z^3 = -1 = \cancel{1(\cos} \cos\pi + i\sin\pi$

$z = \cos\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right) + i\sin\left(\frac{\pi}{3} + \frac{2\pi k}{3}\right)$

$k=0,$

$k=1,$

$k=2.$

(b). $\dfrac{6^{100}}{(3+\sqrt{3}i)^{103}}$

$(2\sqrt{3})^{103}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)^{103} = (2\sqrt{3})^{103} \cdot \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^{103}$

$= (2\sqrt{3})^{103} \cdot \left(\cos\left(\frac{103}{6}\pi\right) + i\sin\left(\frac{103}{6}\pi\right)\right)$

$\dfrac{6^{100}}{(2\sqrt{3})^{103}} = \dfrac{3^{100}}{2^3 \cdot 3^{51}\sqrt{3}} = \dfrac{3^{49}}{2^3\sqrt{3}}\left(\cos\frac{103}{6}\pi - i\sin\frac{103}{6}\pi\right)$

$= \dfrac{3^{49}}{2^3\sqrt{3}}\left(\cos\frac{7}{6}\pi - i\sin\frac{7}{6}\pi\right)$

$= \dfrac{3^{49}}{2^3\sqrt{3}}\left(\cos\left(-\frac{1}{6}\pi\right) - i\sin\left(-\frac{1}{6}\pi\right)\right)$

$= \dfrac{3^{49}}{2^3\sqrt{3}}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) = \dfrac{-3^{49}}{2^4} + i\dfrac{3^{49}}{2^4\sqrt{3}}$

! (7)

Prove the set of algebraic numbers is countable.

Pf. For a polynomial f let us denote by Z_f the set of roots of f.

The set of algebraic numbers A is equal to $\bigcup_{f \in P} Z_f$ where P is the set of all nonezero polynomials.

Since a union of ~~a~~ countably many countable sets is countable

So it's therefore enough to prove that P is countable.

$P = \bigcup_{n \in N} P_n$ where P_n is the set of nonzero polynomials of degree n.

$f(x) = a_n x^n + \cdots + a_1 x + a_0$.

$f \to$ ~~[illegible]~~ $(a_n, a_{n-1}, \cdots, a_1, a_0)$

gives an injective map $P_n \to Z^{n+1}$

and since $|Z^{n+1}| = |N^{n+1}| = |N|$

P_n is countable. $\blacksquare$

8. $\cos\alpha = \frac{1}{6}$, $0<\alpha<\frac{\pi}{2}$.
Prove it cannot be trisected.

$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ for any θ.

$x = \cos\frac{\alpha}{3}$

$4x^3 - 3x = \frac{1}{6}$

$24x^3 - 18x - 1 = 0$

~~$\frac{p}{q}$ $p|-1$, $q|$~~

let $2x = y$.

$3y^3 - 9y - 1 = 0$.

$\frac{p}{q}$. $p|-1$, $q|3$

$p = \pm 1$, $q = \pm 1, \pm 3$

$\frac{p}{q} = \pm 1, \pm\frac{1}{3}$

... has no rational roots.

.... contradiction.

9. S be the set of all functions $f: \mathbb{R} \to \mathbb{R}$
Prove that $|S| > |\mathbb{R}|$.

Pf: ~~Suppose~~

$S \supseteq T$ where $T = \{f : \mathbb{R} \to \{0,1\}\}$.

$\therefore |S| \geq |T|$.

However T is bijective to $\mathcal{P}(\mathbb{R})$ and $|\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$ by Cantor's Thm.

$\therefore |S| \geq |T| = |\mathcal{P}(\mathbb{R})| > |\mathbb{R}|$

$\therefore |S| > |\mathbb{R}|$

10. ① $\sqrt{\frac{\sqrt{5}}{\sqrt[3]{12}+\sqrt{11}}}$ not con.

② x not con $\Rightarrow$ $\sqrt{x}$ not con. ✓

③ x con $\Rightarrow$ $\sqrt[8]{x}$ ~~is~~ con.

$$\sqrt[8]{x} = \sqrt{\sqrt[4]{x}} = \sqrt{\sqrt{\sqrt{x}}}$$ ✓

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MAT246

1. $\frac{(3(n+1))!}{(3!)^{n+1}} = \frac{(3n+3)(3n+2)\cdots 3\cdot 2\cdot 1}{\underbrace{3\cdot 2\cdot 1}_{\text{1st } 3\cdot 2\cdot 1}\cdot 3\cdot 2\cdot 1 \cdots \underbrace{3\cdot 2\cdot 1}_{(n+1)\text{th } 3\cdot 2\cdot 1}}$ $\rightarrow$ $3n+3$ numbers, $(n+1)$ ~~[illegible]~~ numbers divisible by 3

$= \frac{(3n)!\,(3n+1)(3n+2)(3n+3)}{(3!)^n\,(3!)}$

$= p\,\frac{(3n+1)(3n+2)(3n+3)}{3\cdot 2\cdot 1} = \frac{(9n^2+5n+2)(3n+3)}{6}$

$= \frac{27n^3+15n^2+6n+27n^2+15n+6}{6}$

$= \frac{27n^3+42n^2+21n+6}{6}$

$= \frac{9}{2}n^3+7n^2+\frac{7}{2}n+1$

since $p = \frac{(3n)!}{(3!)^n} = \frac{1\cdot 2\cdot 3\cdot 4\cdots 3n}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdots 1\cdot 2\cdot 3}$ must contain 2 so.

$p\frac{(\quad)}{(\quad)}$ .. is a natural

2. $8 \mid a^2+3b^2$

$a^2+3b^2 = 8m$

$(8k+1)^2 \bmod 8 = 1$
$(8k+2)^2 \bmod 8 \equiv 4$
$3^2 \equiv 1$
$4^2 \equiv 0$
$5^2 \equiv 1$
$6^2 \equiv 4$
$7^2 \equiv 1$
$\cancel{8^2} \equiv 0$

$0+0 \bmod 8 \equiv 0$
or
$4+4 \bmod 8 \equiv 0$
hence $a^2 = (8k+4)^2$ or $(8k)^2$
both even
$3b^2 = 8k$ ...
similarly ..

3. $n \in N$

$5 \mid 4^n - 3^n$ iff $4 \mid n$

① if $4 \mid n$ then $n = 4p$

$4^{4p} - 3^{4p}$

$4^4 \equiv 1 \mod 5$

$4^{4p} \mod 5 \equiv 1^p \mod 5 \equiv 1 \mod 5$

$3^{4p} \mod 5 \equiv 1 \mod 5$

$4^{4p} - 3^{4p} \equiv 1 - 1 \equiv 0 \mod 5$

4. $\gcd(a,b) = 1$ $a, b \in \mathbb{Z} \neq 0$.

Prove if $p^2 \mid ab \Rightarrow p^2 \mid a$ or $p^2 \mid b$

Spse $p^2 \mid ab$ but $p^2 \nmid a$ nor $p^2 \nmid b$.

Let $a = p_1 p_2 \cdots p_n$

$b = q_1 q_2 \cdots q_m$

say $p^2 \mid ab$ then

$p^2 \cdot t = p_1 p_2 \cdots p_n q_1 \cdots q_m$

so $p \mid ab$

but if $p^2 \nmid a$ nor $p^2 \nmid b$ then $p \mid a$ & $p \mid b$.

contradicts that $\gcd(a,b) = 1$, p is a prime $\neq 1$.

Hence

$p^2 \mid ab \Rightarrow p^2 \mid a$ or $p^2 \mid b$.

5. a). $n \in \mathbb{Z}$, $\gcd(6n-1, 2n-4) = 1$ or 11.

$(6n-1)-(2n-4) = 4n+3$

$\cancel{4n+3 < 6n-1 \Rightarrow 2n > 4 \Rightarrow n > 2}$

$\cancel{4n+3 < 2n-4 \Rightarrow 2n < -7 \Rightarrow}$

$\cancel{6n-1 \bmod k = 2n-4 \bmod k}$

$\cancel{(6n-1)x + (2n-4)y}$

$\cancel{= (6x+2y)n - (x+4y)}$

$\cancel{6n}$

$6n-1 = 3(2n-4) + 11$

$\cancel{2n-4 \mid}$ $2n-4 \mid 11$ or relatively prime