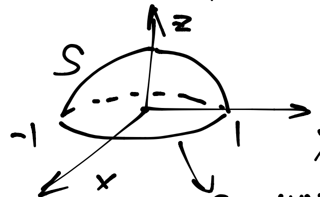
August 6th

Eg. Let $F(x,y,z) = [e^{xz} + e^{x+2y}]\vec{i} + [\log(2+y+z) + 2e^{x+2y}]\vec{j} + 3xyz\vec{k}$

compute $\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA$ where S is the portion of the surface $z = 1 - x^2 - y^2$ above xy-plane oriented with the normal pointing upward



C : unit circle on xy-plane

$\partial S = \partial C$

$$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA = \int_{\partial S}\vec{F}\cdot d\vec{x}$$

$$\| $$

$$\int_{\partial C}\vec{F}\cdot d\vec{x}$$

$$\| $$

$$\iint_C \text{curl}\,\vec{F}\cdot\vec{n}\cdot dA$$

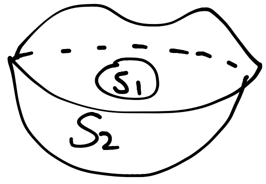
$$\text{curl}\,\vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ e^{xz}+e^{x+2y} & \log(2+y+z)+2e^{x+2y} & 3xyz \end{pmatrix}$$

$= (3xz - \frac{1}{2+y+z})\vec{i} + (xe^{xz} - 3yz)\vec{j} + (2e^{x+2y} - 2e^{x+2y})\vec{k}$ on the unit disk C, $\vec{n} = \vec{k} \Rightarrow \text{curl}\,\vec{F}\cdot\vec{n} = \text{curl}\,\vec{F}\cdot\vec{k} = 0$

$\Rightarrow \iint_C \text{curl}\,\vec{F}\cdot\vec{n}\,dA = 0$

$\Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA = 0$

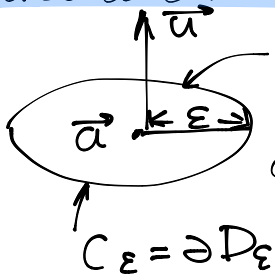
Corollary: If S is a closed surface (surface without boundary) in $\mathbb{R}^3$ with unit outward normal $\vec{n}$ and $\vec{F}$ is a C^1 vector field on S then $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,dA = 0$



Proof: $S = S_1 \cup S_2$

$$\left.\begin{aligned} \iint_{S_1}(\text{curl}\,\vec{F})\cdot\vec{n}\,dA &= \int_C \vec{F}\cdot d\vec{x} \\ \iint_{S_2}(\text{curl}\,\vec{F})\,\vec{n}\,dA &= -\int_C \vec{F}\,d\vec{x} \end{aligned}\right\} \Rightarrow \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dA = 0$$

Geometric meaning of curl



$\vec{u}$ is the positive normal of D_ε when $\varepsilon \to 0$

$$\operatorname{curl} \vec{F}(\vec{a}) \cdot \vec{u} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area } D_\varepsilon} \iint_{D_\varepsilon} \operatorname{curl} \vec{F}(\vec{a}) \vec{u}\, dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \iint_{D_\varepsilon} (\operatorname{curl} \vec{F}(\vec{x})) \vec{u}\, dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\pi \varepsilon^2} \int_{C_\varepsilon} \vec{F} \cdot d\vec{x}$$

$\leftarrow$ The work done by $\vec{F}$ on a particle move around C_ε.

$\Rightarrow$ when $\operatorname{curl}(\vec{F}(\vec{a})) \cdot \vec{u} > 0$, $\operatorname{curl} \vec{F} \cdot \vec{u}$ represents the tendency of the force $\vec{F}$ to push the particle around C_ε counterclockwise.

§ 5.8 Integrating Vector Derivatives

Study $\operatorname{grad} f = \vec{G}$, $\operatorname{curl} \vec{F} = \vec{G}$

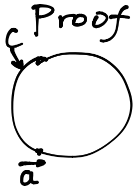
$\nabla f = \vec{G}$

Prop 5.59 Sps $\vec{G}$ is a continuous vector field on an open set R in $\mathbb{R}^n$. The following two conditions are equivalent.

a. If C_1 and C_2 are two oriented curves in R with the same initial and the same final point, then $\int_{C_1} \vec{G} \cdot d\vec{x} = \int_{C_2} \vec{G} \cdot d\vec{x}$.

b. If C is any closed curve in R_2, $\int_C \vec{G} \cdot d\vec{x} = 0$.

Proof. (a) $\Rightarrow$ (b)



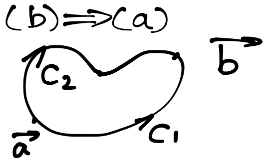
Def: $C_1 = C$ starts and ends at $\vec{a}$

$C_2 : \vec{x}(t) = \vec{a}$, $t \in [0,1]$

$$\Rightarrow \int_C \vec{G} \cdot d\vec{x} = \int_{C_1} \vec{G} \cdot d\vec{x} \underset{\text{by (a)}}{=} \int_{C_2} \vec{G} \cdot d\vec{x} = 0$$

$\vec{x} = \vec{a} \Rightarrow d\vec{x} = 0$

(b) $\Rightarrow$ (a)



Def: C be the closed curve obtained by following C_1 from $\vec{a}$ to $\vec{b}$ and C_2 backwards from $\vec{b}$ to $\vec{a}$.

$$\Rightarrow 0 \underset{(b)}{=} \int_C \vec{G} \cdot d\vec{x} = \int_{C_1} \vec{G}\, d\vec{x} - \int_{C_2} \vec{G} \cdot d\vec{x}$$

$$\Rightarrow \int_{C_1} \vec{G} \cdot d\vec{x} = \int_{C_2} \vec{G} \cdot d\vec{x}$$

Def: A vector field $\vec{G}$ that satisfies (a) & (b) is called **conservative** in the region R

conservative came from that $\int_C \vec{G}\cdot d\vec{x} = 0$, the force does no work on a particle that returns to its starting point.

$\vec{G}$: force

Prop 5.60 A continuous vector field $\vec{G}$ in an open set $R \subset \mathbb{R}^n$ is conservative if and only if G is the gradient of a C' function f on R

Pf: "$\Leftarrow$" If $\vec{G} = \nabla f$, we want to show G is conservative (here we use (b))

Let C be a closed curve parametrized by $\vec{x} = \vec{g}(t)$, $a \le t \le b$

$\Rightarrow \vec{g}(a) = \vec{g}(b)$

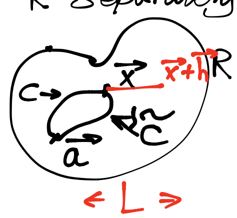
$$\int_C \vec{G}\cdot d\vec{x} = \int_C \nabla f \cdot d\vec{x} \underset{\text{def of line integral}}{=} \int_a^b \nabla f(\vec{g}(t))\cdot \vec{g}\,'(t)\,dt \underset{\text{chain rule}}{=} \int_a^b \frac{\partial f(\vec{g}(t))}{\partial t}\,dt$$

$$\overset{FTC}{=} f(\vec{g}(t))\Big|_a^b = f(\vec{g}(b)) - f(\vec{g}(a)) = 0$$

"$\Rightarrow$" Sps $\vec{G}$ is conservative

Need to find f s.t. $\nabla f = \vec{G}$

Assume R is connected (otherwise we can consider each connected piece of R separately)



Pick $\vec{a} \in R$, $\forall \vec{x} \in R$. Def $f(\vec{x}) = \int_C \vec{G}\,d\vec{x}$, where C is one curve from $\vec{a}$ to $\vec{x}$

If there is another curve $\tilde{C}$ from $\vec{a}$ to $\vec{x}$

$\int_{\tilde{C}} \vec{G}\cdot d\vec{x} = \int_C \vec{G}\cdot d\vec{x}$ by $\vec{G}$ is conservative.

$\Rightarrow f(x)$ does not depend on the curve that we pick

Let $G = (G_1, \cdots, G_n)$, $\nabla f = (\partial_1 f, \cdots \partial_n f)$

Need to show $G_j = \partial_j f$, $\forall j \in \{1, 2, \cdots, n\}$

Let $\vec{h} = (0, 0, \cdots, h, 0, \cdots, 0)$ jth position

Suppose $\vec{h}$ is small enough s.t. the line segment L from $\vec{x}$ to $\vec{x}+\vec{h}$ lies entirely in R Why? b/c R is open, distance is L.

$$f(\vec{x}+\vec{h}) = \int_{C\cup L} \vec{G}\cdot d\vec{x} = \int_C \vec{G}\cdot d\vec{x} + \int_L \vec{G}\cdot d\vec{x}$$

$$\Rightarrow \frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{h} = \frac{1}{h}\int_L \vec{G}\cdot d\vec{x} = \frac{1}{h}\int_0^h G(x_1, \cdots, x_j + t, x_{j+1}, \cdots, x_n)\,dt$$

Let $h \to 0$

$$\partial_j f(x) = \lim_{h\to 0} \frac{f(\vec{x}+\vec{h})-f(\vec{x})}{h} = \lim_{h\to 0} \frac{1}{h}\int_0^h G_j(x_1, x_2, \cdots, x_j+t, x_{j+1}, \cdots, x_n)\,dt$$

$$= G_j(\vec{x})$$

Def: when $\nabla f = \vec{G}$, f is called the potential associated to the conservative vector field $\vec{G}$.

Remark: If $\nabla f = \vec{G}$ then $\nabla(f + \underbrace{C}_{\text{constant}}) = \vec{G}$

$\Rightarrow f$ can only be determined up to a constant.

Determine if $\vec{G}$ is conserved

when $\vec{G} = \nabla f \Rightarrow \partial_j f = G_j$, $\partial_k f = G_k$ for any j, k

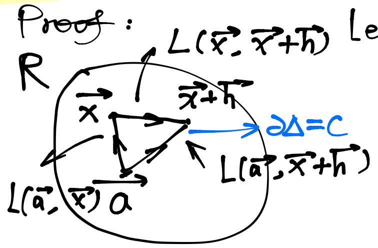
$\Rightarrow \partial_k G_j = \partial_k \partial_j f = \partial_j \partial_k f = \partial_j G_k$

$\Rightarrow \partial_k G_j - \partial_j G_k = 0$

when $n=3$. This is curl $\vec{G} = 0$.

Thm 5.62 Sps R is a convex open set in $\mathbb{R}^n$ and $\vec{a}$ is a C^1 vector field on R. If $\vec{G}$ satisfies $\partial_k G_j - \partial_j G_k = 0$ for any k, j. $(*)$ then $\vec{G}$ is the gradient of a C^2 function on R.

Proof:



Let $\vec{h} = (h, 0, 0, \cdots, 0)$

when $n=2$, $\vec{h} = (h, 0)$, $k=1$, $j=2$

$(*) \Rightarrow \partial_1 G_2 - \partial_2 G_1 = 0$

Def $f(\vec{x}) = \int_{L(\vec{a}, \vec{x})} \vec{G} \cdot d\vec{x}$

$$f(\vec{x}+\vec{h}) = \int_{L(\vec{a}, \vec{x}+\vec{h})} \vec{G} \cdot d\vec{x}$$

$$\int_C \vec{G} \cdot d\vec{x} \underset{\text{Green's thm}}{=} \iint_\Delta \partial_1 G_2 - \partial_2 G_1\, dA \overset{(*)}{=} 0$$

$\partial\Delta$, oriented counterclockwised

$$\left(\left(\int_{L(\vec{a}, \vec{x}+\vec{h})} \vec{a} \cdot d\vec{x} - \int_{L(\vec{x}, \vec{x}+\vec{h})} \vec{G} \cdot d\vec{x} - \int_{L(\vec{a}, \vec{x})} \vec{G} \cdot d\vec{x} \right.\right.$$

$$f(\vec{x}+\vec{h}) - f(\vec{x}) = \int_{L(\vec{x}, \vec{x}+\vec{h})} \vec{G} \cdot d\vec{x}$$

$$\Rightarrow \frac{f(\vec{x}+\vec{h}) - f(\vec{x})}{h} = \int_{L(\vec{x}, \vec{x}+\vec{h})} \vec{G} \cdot d\vec{x}$$

$$\left.\begin{array}{l} \text{Let } h \to 0,\ \partial_1 f = G_1(\vec{x}) \\ \text{Similarly } \partial_2 f = G_2(\vec{x}) \end{array}\right\} \Rightarrow \nabla f = \vec{G}$$

When $n=3$

$$\int_{L(\vec{a},\,\vec{x}+\vec{h})} \vec{a}\, dx - \int_{L(\vec{x},\,\vec{x}+\vec{h})} \vec{G}\cdot d\vec{x} - \int_{L(\vec{a},\,\vec{x})} \vec{G}\cdot d\vec{x} = \int_C \vec{G}\cdot d\vec{x}$$

Stoke's thm $\int_{\Delta} (\text{curl}\,\vec{G})\cdot\vec{n}\, dA = 0$

Similarly $\partial_j f = G \Rightarrow \nabla f = G$

NOT REQUIRED

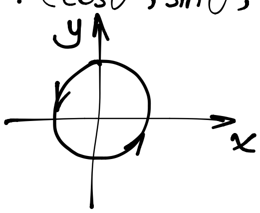
(For $n \geq 3$, we use stoke's thm in higher dim can be used)

Remark : Convex is very important, since we use stoke's or Green's thm, and there is no 'hole' in the triangle in R
→ where G is not defined

Eg. Let R be the complement of z-axis in $\mathbb{R}^3$
Let $\vec{a}(x,y,z) = \dfrac{-y\vec{i} + x\vec{j}}{x^2+y^2}$

$$\text{curl}\,\vec{G} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix} = \vec{0}$$

but consider C is the unit circle on xy-plane
$C: (\cos\theta, \sin\theta, 0)$. $\theta \in [0, 2\pi]$



$$\int_C \vec{G}\cdot d\vec{x} = \int_0^{2\pi} \left(\frac{-\sin\theta}{\sin^2\theta+\cos^2\theta}, \frac{\cos\theta}{1}, 0\right)(-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} (-\sin\theta, \cos\theta, 0)(-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta\, d\theta = \int_0^{2\pi} d\theta = 2\pi$$

$\Rightarrow \vec{G}$ is not conservative.

Next curl $\vec{F} = \vec{G}$
Since $\text{div}(\text{curl}\,\vec{F}) = 0 \Rightarrow \text{div}(\vec{G}) = 0$

5.63 Thm Sps R is a convex open set in $\mathbb{R}^3$ and $\vec{G}$ is a C^1 vector field on R. If $\vec{G}$ satisfies $\text{div}\,\vec{G} = 0$ on R, then $\vec{G}$ is the curl of a C^2 vector field on R.

Proof: Since $\text{curl}(\nabla f) = 0$ for any $f \in C^2$ if $\text{curl}\,\vec{F} = \vec{G}$ then $\text{curl}(\vec{F} + \nabla f) = \text{curl}\,\vec{F} = \vec{G}$
Let $f = -\int F_3\, dz \Rightarrow \partial_z f = -F_3 \Rightarrow F_3 + \partial_z f = 0$

i.e. Def $\vec{F}_{New} = \vec{F} + \nabla f$, the z component of $\vec{F}_{New} = 0$ and $\text{curl}\,\vec{F}_{New} = \vec{G}$
$\Rightarrow$ we can assume $\vec{F} = F_1\vec{i} + F_2\vec{j}$.

$$\text{curl } \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & 0 \end{pmatrix} = -\partial_3 F_2 \vec{i} + \partial_3 F_1 \vec{j} + (\partial_1 F_2 - \partial_2 F_1)\vec{k}$$

$$= G_1 \vec{i} + G_2 \vec{j} + G_3 \vec{k}$$

$$\Rightarrow G_1 = -\partial_3 F_2 \Rightarrow F_2 = -\int_C^z G_1(x,y,t)\,dt + \varphi(x,y)$$

$$G_2 = \partial_3 F_1 \Rightarrow F_1 = \int_C^z G_2(x,y,t)\,dt + \psi(x,y)$$

$$G_3 = \partial_1 F_2 - \partial_2 F_1 = -\int_C^z \partial_x G_1(x,y,t)\,dt + \varphi(x,y) \quad - \int_C^z \partial_y G_2(x,y,t)\,dt - \psi(x,y)$$

$$= -\int_C^z \partial_x G_1(x,y,t) + \partial_y G_2(x,y,t)\,dt + \varphi(x,y) - \psi(x,y)$$

$\Uparrow$ plug in

since $\operatorname{div} \vec{G} = 0$

$$\partial_x G_1(x,y,t) + \partial_y G_2(x,y,t) + \partial_z G_3(x,y,z) = 0$$

$$\Rightarrow G_3 = \int_C^z \partial_z G_3(x,y,t)\,dt + \varphi(x,y) - \psi(x,y) = G_3(x,y,z) - G_3(x,y,c) + \varphi(x,y) - \psi(x,y)$$

$$\Rightarrow \varphi_x(x,y) - \psi_y(x,y) = G_3(x,y,c)$$

Let $\psi = 0 \Rightarrow \varphi_x(x,y) = G_3(x,y,c)$

$$\Rightarrow \varphi(x,y) = \int_a^x G_3(t,y,c)\,dt$$

$\blacksquare$

Eg. $\vec{G}(x,y,z) = (6xz + x^3)\vec{i} - (3x^2y + y^2)\vec{j} + (4x + 2yz - 3z^2)\vec{k}$

Find $\vec{F}$.

Let $\vec{F} = F_1 \vec{j} + F_2 \vec{j}$

$\partial_z F_2 = -G_1 = -6xz - x^3 \quad \Rightarrow F_2 = -3xz^2 - x^3 z + \varphi(x,y)$

$\partial_z F_1 = -3x^2y - y^2 \quad \Rightarrow F_1 = -3x^2yz - y^2 z + \psi(x,y)$

$\partial_x F_2 - \partial_y F_1 = 4x + 2yz - 3z^2 \quad \Rightarrow \partial_x \varphi - \partial_y \psi = 4x$

Therefore one soln is $(\varphi = 2x^2,\ \psi = 0)$

$$\vec{F_0} = -(3x^2yz + y^2z)\vec{i} + (2x^2 - 3xz^3 - x^3z)\vec{j}$$

The general soln is $\vec{F} = \vec{F_0} + \nabla f$ where f is an arbitrary C^1 func.