1). $\int xe^x dx = xe^x - e^x + C$

2). $\int_0^1 xe^{x^2}dx$ $\quad x^2=u$, $2x\,dx = du$, $dx = \frac{du}{2\cdot u^{\frac12}}$ $\quad \int u^{\frac12} e^u \cdot \frac{du}{2\cdot u^{\frac12}} = \int \frac12 e^u du = \frac12 e^u = \frac12 e^{x^2} = \frac12 e - \frac12$

3). $\frac{d}{dx}\int_a^x f(t)dt$ $\rightarrow$ $f(x)$

4). Solve $\begin{pmatrix}1&2\\1&3\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ Two ways

① $\begin{matrix}a+2b=0\\a+3b=1\end{matrix}$ $\begin{cases}b=1\\a=-2\end{cases}$ (row reduction)

② $\begin{pmatrix}a\\b\end{pmatrix}=$

5). Find a Taylor expansion of $\frac{\sin x}{x}$

6). $\lim_{t\to0}\frac{e^t-1-t}{t^2}$

5). $\sum_{k=0}^{\infty}\frac{x^{2k+1}(-1)^k}{(2k+1)!} = \sin x \rightarrow \frac{\sin x}{x} = ?$

6). $\frac{e^t-1}{2t} \rightarrow \frac{e^t}{2} \rightarrow \frac12$

迈 礼 华

Quiz#1 Chapter 1 ALL + Direction field, isoclines, integral curves
Chapter 2 2.1, 2.2, 2.6 - integrating factors
This may change, so check the website during the week.

Meinrenken: google it !

Separable equations: $\quad y' = f(x,y)$

$M(x) + N(y)\frac{dy}{dx} = 0$ ①

why not? $\int N(y)dy \overset{?}{=} -\int M(x)dx$ ✗

Let $H_1(x)$, $H_2(y)$ be antiderivitives of $M(x)$, $N(y)$ respectively.

$\frac{d}{dx}H_1(x) = M(x)$, $\frac{d}{dx}H_2(x) = N(y)$

$\frac{d}{dx}H_2(y) = \frac{d}{dy}H_2(y)\frac{dy}{dx} = N(y)\frac{dy}{dx}$

chain rule

② $\frac{d}{dx}H_1(x) + \frac{d}{dx}H_2(y) = 0$

$\frac{d}{dx}(H_1(x) + H_2(y)) = 0$

$\Rightarrow \boxed{H_1(x) + H_2(y) = C}$ ②

specify (x_0, y_0), $y(x_0) = y_0$

$H_1(x) - H_1(x_0) = H_2(y) - H_2(y_0)$

$H_1(x) - H_1(x_0) = \int_{x_0}^{x} M(s)ds$

$H_2(y) - H_2(y_0) = \int_{y_0}^{y} N(s)ds$

$\Rightarrow \int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0$

Ex/ $y' = \frac{2x^2-1}{1+3y^2} \Rightarrow -(2x^2-1) + (1+3y^2)\frac{dy}{dx} = 0$

$(1+3y^2)\frac{dy}{dx} = -\frac{d}{dx}(\quad)$

$\frac{d}{dx}\left(\frac{-2}{3}x^3+x\right) + \underbrace{\frac{d}{dx}(y+y^3)}_{\text{chain rule}} = 0$ ①

$\int(1+3s^2)ds = s+s^3 \rightarrow y+y^3 = H_2(y)$

$\int(-2s^2+1)ds = -\frac{2}{3}s^3+s \Rightarrow -\frac{2}{3}x^3+x = H_1(x)$

① $\Rightarrow \boxed{-\frac{2}{3}x^3+x+y+y^3 = C}$

Eg.2/ $y' = (1-2x)/y \qquad y(1)=2$

$N(y) = y,\ M(x) = -(1-2x)$

$\int_{x_0}^{x} M(s)ds + \int_{y_0}^{y} N(s)ds = 0 \longrightarrow \int_1^x(-1+2s)ds + \int_2^y s\,ds$

$= (-s+s^2)\Big|_1^x + \left(\frac{s^2}{2}\right)\Big|_2^y$

$\Rightarrow \boxed{[(-x+x^2)-(-1+1^2)] + \left[\frac{y^2}{2}-\frac{2^2}{2}\right] = 0}$

Homogenous Equations/

$y' = f(x,y) = f\left(\frac{y}{x}\right)$

Trick: Homogenous $\Rightarrow$ separable

Let $v = \frac{y}{x} \Rightarrow vx = y,\ \frac{dy}{dx} = \frac{d}{dx}(v(x)x) = v\cdot 1 + x\frac{dv}{dx}$

$\boxed{v + x\frac{dv}{dx} = f(v)}$ is separable

$\rightarrow x\frac{dv}{dx} = f(v) - v \Rightarrow \boxed{\frac{dv}{dx} = \frac{f(v)-v}{x}} \Rightarrow \begin{matrix} v = v(x) \\ y = v(x)x \end{matrix}$

$\frac{dy}{dx} = \frac{4y-3x}{2x-y}$

$= \frac{4\frac{y}{x}-3}{2-\frac{y}{x}} = \frac{4v-3}{2-v}$

$v = \frac{y}{x} \Rightarrow v + x\frac{dv}{dx} = \frac{4v-3}{2-v} \Rightarrow x\frac{dv}{dx} = \frac{4v-3}{2-v} - v\frac{(2-v)}{(2-v)} = \frac{2v-3+v^2}{2-v}$

$\Rightarrow \frac{2-v}{(v+3)(v-1)}dv = \frac{1}{x}dx$

$\Rightarrow \int\frac{2-v}{(v+3)(v-1)}dv = \int\frac{1}{x}dx$

Integration by partial fractions.

$\frac{2-v}{(v+3)(v-1)} = \frac{A}{(v+3)} + \frac{B}{(v-1)} = \frac{Av-A+Bv+3B}{(v+3)(v-1)} \Rightarrow \begin{cases} 3B-A=2 \\ A+B=-1 \end{cases} \quad \begin{matrix} A = -\frac{5}{4} \\ B = \frac{1}{4} \end{matrix}$

$LHS = \int\left(\frac{-5}{4}\frac{1}{(v+3)} + \frac{1}{4}\frac{1}{(v-1)}\right)dv = RHS$

$\Rightarrow -\frac{5}{4}\ln|v+3| + \frac{1}{4}\ln|v-1| = \ln|x| + C$

$\Rightarrow |v+3|^{-\frac{5}{4}} |v-1|^{\frac{1}{4}} = C_1|x|$

$\left|\frac{y}{x}+3\right|^{-\frac{5}{4}} \left|\frac{y}{x}-1\right|^{\frac{1}{4}} = C_1|x|$

2.1, 2.2 end

2.6 now

Ex / $\boxed{2x+2xy} + \boxed{x^2} y' = 0$

$\psi(x,y) = x^2 + yx^2$, $\psi_x(x,y) = \frac{\partial}{\partial x}\psi(x,y)$

$= \boxed{2x+2xy}$

$\psi_y = \boxed{x^2}$

$\frac{d}{dx}\psi(x,y) = \frac{\partial\psi}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial\psi(x,y)}{\partial y}\frac{dy}{dx}$

$= \psi_x + \psi_y \frac{dy}{dx}$

multivariable chain rule

$\Rightarrow \frac{d}{dx}\psi(x,y) = 0 \Rightarrow \psi(x,y) = C$

$\hookrightarrow y = \phi(x)$ that solves our ODE

Exact Equations

$M(x,y) + N(x,y)\,y' = 0$ ①

If you can find a $\psi(x,y)$ such that

1). $\psi_x(x,y) = M(x,y)$ 2) $\psi_y(x,y) = N(x,y)$

3). $\psi(x,y) = c \Rightarrow y = \phi(x)$

then ① is exact ("bad condition")

① $\Rightarrow \psi_x + \psi_y \frac{dy}{dx} = 0$) chain rule

$\Rightarrow \frac{dy}{dx}(\psi(x,y)) = 0$

$\Rightarrow \psi(x,y) = C$ solves ①

Thm / Given a rectangle R : $\alpha < x < \beta$, $\gamma < y < \delta$

and M, N, M_y, N_x to be continuous

Then ① is exact iff $M_y(x,y) = N_x(x,y)$

Proof: ($\Rightarrow$) $M_y = \psi_{xy}$ (by defn of exact, then $\frac{\partial}{\partial y}$

$\overset{?}{=}$ $\quad\quad$ $\parallel$

$N_x = \psi_{yx}$ equality is because of continuity ◢

($\Leftarrow$) $\psi_x = M$

$\xrightarrow{\text{integrate}}$ $\boxed{\psi(x,y) = Q(x,y) + h(y)}$ ② $\quad Q(x,y) = \int_{x_0}^{x} M(s)\,ds$

integration constant function only of y.

$\Rightarrow \psi_y(x,y) = \boxed{\frac{\partial}{\partial y}Q(x,y) + h'(y) = N}$

$h'(y) = N(x,y) - \frac{\partial}{\partial y} Q(x,y)$

only solvable if the right hand side is only a function of y.

$\Rightarrow h(y) = \int RHS$

$\Rightarrow$ 2 is defined.

$\frac{\partial}{\partial x}\left[N(x,y) - \frac{\partial}{\partial y} Q(x,y)\right] = 0 \Rightarrow RHS(y)$

$N_x(x,y) - \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y} Q(x,y) \stackrel{?}{=} 0$

$N_x(x,y) - \frac{\partial}{\partial y} \boxed{\frac{\partial}{\partial x} Q(x,y)} = N_x(x,y) - M_y(x,y) = 0$ (by assumption)

We have a method to find $\psi(x,y)$

Given our initial ODE, solutions are given by $\boxed{\psi(x,y) = C}$

Ex) 1: $\underbrace{(2xy^2+2y)}_{M(x,y)} + \underbrace{(2x^2y+2x)}_{N(x,y)}y' = 0$

$M_y(x,y) = 4xy+2 \qquad N_x(x,y) = 4xy+2$ ✓

so exact (step 1)

$\psi_x(x,y) = M = 2xy^2 + 2y$ (A)

$\psi_y(x,y) = N = 2x^2y + 2x$ (B)

integrate (A) $\Rightarrow \psi(x,y) = x^2y^2 + 2xy + h(y)$

$\frac{\partial}{\partial y}\psi(x,y) = 2x^2y + 2x + h'(y) = N = 2x^2y + 2x \Rightarrow h'(y) = 0$

$h(y) = C$ (make it $C = 0$!)

$\Rightarrow \psi(x,y) = x^2y^2 + 2xy + C = \tilde{C}$

$\Rightarrow \boxed{x^2y^2 + 2xy = \tilde{C}}$ where $\tilde{C}$ is determined by initial conditions.

Convert from a special nonexact $\Rightarrow$ exact

$M + N\frac{dy}{dx} = 0$ was not exact

$\mu(x,y)M(x,y) + \mu(x,y)N(x,y)\frac{dy}{dx} = 0$, μ an integrating factor

want $(\mu M)_y = (\mu N)_x \Rightarrow \mu_y M + \mu M_y = \mu_x N + \mu N_x$

if $\mu(x,y) = \mu(x)$

$\Rightarrow \mu_y = 0$

$\mu_x = \frac{\mu(N_x - M_y)}{N}$ solvable by separation of variable IF $\left(\frac{N_x - M_y}{N}\right)$ is only a function of x.

if $\mu(x,y) = \mu(y)$

$$\Rightarrow \mu_y M = \mu(N_x - M_y)$$

$$\Rightarrow \mu_y = \frac{\mu(N_x - M_y)}{M} \longrightarrow \text{needs to be a function of } \underline{y}.$$