June 18th
Mid-Term coverage 1 & 2 - exceptions in pdf
Ch2 > Ch1
List of proofs

Recall:

$\sum_{|\alpha|\leq 2}$ ⟵ sum of all indices of length ≤ 2

$|\alpha|=0$
$|\alpha|=|\alpha_1|+|\alpha_2|+|\alpha_3|=0$, $\alpha=(0,0,0)$
$|\alpha|=1$, $\alpha=(1,0,0)$, $(0,0,1)$, $(0,1,0)$
$|\alpha|=2$, $\alpha=(2,0,0)$, $(0,2,0)$, $(0,0,2)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$

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§ 2.8 Critical Points
Goal: Find local max/min

1). will prove that if it is a local max/min
$\Rightarrow$ critical point
2). will find all critical points to be candidates for max/min.
$\partial_j f(\vec{a})=0,\ \forall j\in\{1,\cdots,n\}$ ⟵ n equations, n unknowns
3). Look at $\underbrace{2^{nd}}_{\text{or higher}}$ order derivatives to determine if a max or min.

Thm: If f has a local max or min at $\vec{a}$, f diff on open set $S\subset\mathbb{R}^n$ then $\nabla f(\vec{a})=\vec{0}$.

Proof: will show all $\boxed{\text{directional derivatives}}$ are 0
Consider $g(t)=f(\vec{a}+t\vec{u})$, $\vec{u}$ as a unit vector.
g has a max at $t=0$, $g:\mathbb{R}\to\mathbb{R}$

$$0=g'(0)=\frac{d}{dt}f(\vec{a}+t\vec{u})\Big|_{t=0}=\partial_{\vec{u}}f(\vec{a})\quad \forall\vec{u}$$

all direction derivatives are 0

$\Rightarrow \partial_j f(\vec{a})=0\ \forall j\in\{1,\cdots,n\}$
$\Rightarrow \nabla f(\vec{a})=\vec{0}$ ∎

Recall: $f(a+h) = f(a) + \cancel{f'(a)h} + \frac{f''(a)h^2}{2} + \underbrace{R_{a,3}(h)}_{\text{small for small } h}$

if $f'(a)=0$

then depends on

$$H(\vec{a}) = \begin{pmatrix} \partial_1^2 f(\vec{a}) & \partial_1\partial_2 f(\vec{a}) & \cdots & \partial_1\partial_n f(\vec{a}) \\ \partial_2\partial_1 f(\vec{a}) & \partial_2^2 f(\vec{a}) & \cdots & \partial_2\partial_n f(\vec{a}) \\ \vdots & & \ddots & \vdots \\ \partial_n\partial_1 f(\vec{a}) & \partial_n\partial_2 f(\vec{a}) & \cdots & \partial_n^2 f(\vec{a}) \end{pmatrix}$$

Hessian of $\vec{a}$

$H_{ij}(\vec{a}) = \partial_i\partial_j f(\vec{a})$

$H_{ij}(\vec{a}) = H_{ji}(\vec{a}) \Rightarrow$ symetric matrix

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$$f(\vec{a}+\vec{k}) = f(\vec{a}) + \sum_{j=1}^{n} \partial_j f(\vec{a}) k_i + \boxed{\frac{1}{2}\sum_{i,j=1}^{n} \underbrace{\partial_i\partial_j f(\vec{a})}_{H_{ij}(\vec{a})} k_i k_j} + R_{\vec{a},2}(\vec{k})$$

$$\Downarrow = \frac{1}{2}\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} H_{ij}k_j\Big)k_i$$

$$= \frac{1}{2}\sum_{i=1}^{n}(H\vec{k})_i k_i = \frac{1}{2}\vec{k}^T H \vec{k}$$

$$(H_{ij})(k_j)$$

Putting together, when a CP. $f(\vec{a}+\vec{h}) - f(\vec{a}) = \underbrace{\frac{1}{2}\vec{k}^T H \vec{k}}_{\text{depends on this term}} + R_{\vec{a},2}(\vec{k})$

For all of this f is C^2 & an open set $S \subset \mathbb{R}^n$

$$H = \begin{pmatrix} \partial_1^2 f(\vec{a}) & & 0 \\ & \ddots & \\ 0 & & \partial_n^2 f(\vec{a}) \end{pmatrix} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$\lambda_j = \partial_j^2 f(\vec{a})$

$$(k_1, \cdots, k_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}\begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix} = (k_1, \cdots, k_n)\begin{pmatrix} \lambda_1 k_1 \\ \vdots \\ \lambda_n k_2 \end{pmatrix} = \sum_{j=1}^{n} \lambda_j k_j^2$$

$= \vec{k}^T H \vec{k}$

$= (H\vec{k}) \cdot \vec{k}$

$$f(\vec{a}+\vec{k})-f(\vec{a})=\frac{1}{2}\sum_{j=1}^{n}\lambda_j k_j^2+R_{\vec{a},2}(\vec{k})$$

- Sps $\lambda_j>0$, $\forall j\in\{1,\cdots,n\}$. then for small enough $\vec{k}$, $f(\vec{a}+\vec{k})-f(\vec{a})>0$
$\Rightarrow f(\vec{a}+\vec{k})>f(\vec{a})\Rightarrow f(\vec{a})$ is a local min

- Sps $\lambda_j<0$ $\forall j\in\{1,\cdots n\}$, then $f(\vec{a}+\vec{k})-f(\vec{a})<0\Rightarrow f(\vec{a})$ is a local max

- some $\lambda_j>0$, others <0

Note, $\vec{u_1}=(1,0,\cdots,0)$, $\vec{u_2}=(0,1,0,\cdots,0)\cdots$
$\vec{u_i}=(0,\cdots,1,\cdots,0)$ ith index is 1

Ex: $\lambda_1<0$, $\lambda_2>0$
choose $\vec{k}=(t,0,\cdots,0)$. then $f(\vec{a}+\vec{k})-f(\vec{a})=\frac{1}{2}\lambda_1 t^2+R_{\vec{a},2}(t,0,\cdots,0)$
$f(\vec{a}+\vec{k})-f(\vec{a})>0$ for t small and this k.
choose $\vec{k}=(0,t,0,\cdots,0)=t\vec{u_2}$, $f(\vec{a}+\vec{k})-f(\vec{a})=\frac{1}{2}\lambda_2 t^2+R_{\vec{a},2}(\vec{k})$ for small t
$f(\vec{a}+\vec{k})-f(\vec{a})<0\Rightarrow$ neither a max nor min.

- suppose some $\lambda_j=0$
choose $\vec{k}=(0,\cdots,\underset{\hookrightarrow j\text{th}}{t},\cdots,0)$

$f(\vec{a}+\vec{h})-f(\vec{a})=0+R_{\vec{a},2}(\vec{k})$
would then looked at higher order

$H_{ijk}=\partial_i\partial_j\partial_k f(\vec{a})$

For a special case of $H=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$

General case: $\vec{k}^T H\vec{k}$
can be transformed to a special case by a change of basis.

Spectral Thm For a symmetric matrix A can diagonalize, $A=P^{-1}DP$
$=P^TDP$
where $D=\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ λ_j are eigenvalues and P formed
by orthonormal eigenvectors

$$k^TAk=k^TP^TDPk=(Pk)^TD(Pk)$$

Thm: If f is C^2 on an open subset $S \subset \mathbb{R}^n$, and $\nabla f(\vec{a}) = \vec{0}$, $\vec{a} \in S$. Then for $f(\vec{a})$ to be a local min it is necessary that all eigenvalues of $H(\vec{a})$ are nonnegative & sufficient that all eigenvalues are strictly positive.

Also, max, necessary nonpositive, sufficient strictly negative.

PROOF: Let $u_1, \cdots, u_n$ be our orthonormal basis of eigenvectors with corresponding $\lambda_1, \cdots, \lambda_n$

$$\vec{k} = c_1\vec{u_1} + \cdots + c_n\vec{u_n}$$

$$\vec{k}^T H \vec{k} = \sum_{j=1}^{n} \lambda_j c_j^2 \geq \lambda \sum_{j=1}^{n} c_j^2 = \lambda\, \vec{k}^T\vec{k} = \lambda\, \vec{k}\cdot\vec{k} = \lambda\, |\vec{k}|^2$$

$H\vec{u_i} = \lambda_i \vec{u_i}$

$\boxed{\text{Assume } \lambda_j > 0 \ \forall j \in \{1, \cdots, n\}.\ \exists \lambda \text{ the least positive eigenvalue}}$

$$\frac{|R_{\vec{a},2}(\vec{k})|}{|\vec{k}|^2} < \varepsilon \quad \text{for } |\vec{k}| \text{ small enough}$$

choose $\varepsilon = \dfrac{\lambda}{4} \Rightarrow |R_{\vec{a},2}(k)| \leq \dfrac{\lambda|\vec{k}|^2}{4}$

$f(\vec{a}+\vec{k}) - f(\vec{a}) \geq \lambda|\vec{k}|^2 - \frac{\lambda}{4}|\vec{k}|^2 > 0$ since λ is possitive

$\Rightarrow f(\vec{a}+\vec{k}) > f(\vec{a}) \Rightarrow \vec{a}$ is a local min (here, done for sufficient case)

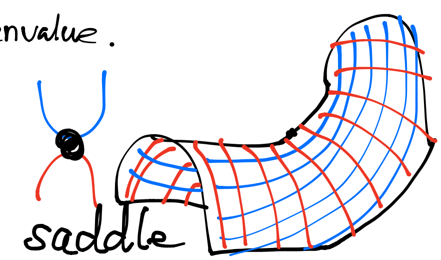
Now suppose $\lambda_j < 0$, $f(\vec{a}+t\vec{u_j}) - f(\vec{a}) = \lambda_j t^2 + R_{\vec{a},2}(t\vec{u_j})$

for small t, get a contradiction, as $\lambda_j t^2 < 0$.
likewise, for $\lambda_j = 0$. so choose $\vec{k} = t\vec{u_j}$, get $f(\vec{a}+\vec{k}) - f(\vec{a}) = 0 + R_{\vec{a},2}(\vec{k})$

1). Find CP's
2). Compute H
3). find eigenvalues of H
4). Use Thm to determine a max/min

Defn: A saddle point. has both a possitive & negative eigenvalue

Def'n: A degenerate point has a zero eigenvalue.



Thm: $f: \mathbb{R}^2 \to \mathbb{R}$, C^2 on open S

$$D = \det|H(\vec{a})| = \begin{vmatrix} \alpha & \beta \\ \beta & \gamma \end{vmatrix} = \alpha\gamma - \beta^2 \qquad \alpha = \partial_1^2 f(\vec{a}),\ \beta = \partial_1\partial_2 f(\vec{a}),\ \gamma = \partial_2^2 f(\vec{a})$$

then
① $D < 0$ saddle.
② $D > 0$ and $\alpha > 0$ local min.
③ $D > 0$ & $\alpha < 0$ local max
④ $D = 0$ degenerate

Proof: $\det(H(\vec{a})) = \lambda_1\lambda_2$

① $D<0$, $\Rightarrow \lambda_1, \lambda_2$ have different sign $\Rightarrow$ saddle by previous thm.

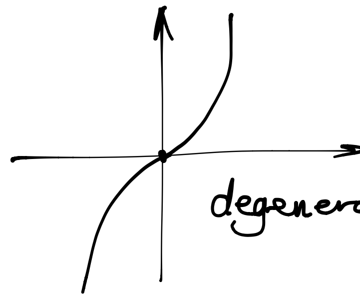
② $D>0 \Rightarrow \lambda_1, \lambda_2$ same.

③ $\alpha = \vec{u}^T H \vec{u}$ for $\vec{u}=(1,0)$

by possible change of basis

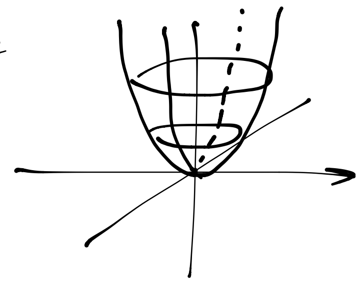
$\alpha>0 \Rightarrow \lambda_1>0 \Rightarrow \lambda_2>0 \Rightarrow$ local min

$\alpha<0 \Rightarrow \lambda_1<0 \Rightarrow \lambda_2<0 \Rightarrow$ local max



degenerate

④ $D=0 \Rightarrow$ a 0 eigenvalue $\Rightarrow$ degenerate



Ex: $z=f(x,y)=x^2+y^2$

1) find CPs

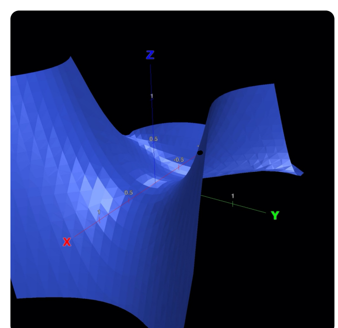
$\nabla f(\vec{x})=\vec{0} \Rightarrow \partial_x f(\vec{x})=\partial x=0$

$\partial_y f(\vec{x})=\partial y=0$

when $(x,y)=(0,0)$

$H(\vec{x})=\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $\lambda_1=0=\lambda_2$ or $D=4$, $\alpha=2 \Rightarrow$ local min

Ex: $z=f(x,y)=(x+1)(y^2-x^2)$



$\partial_x f = (y^2-x^2)-2x(x+1)=y^3-3x^2-2x=0$

$\partial_y f = 2y(x+1)=0 \Rightarrow y=0$ or $x=-1$

if $y=0 \Rightarrow -3x^2-2x=0$

$\Rightarrow x=0$ or $x=-2/3$

if $x=-1$, $y^2=1 \Rightarrow y=\pm 1$

CPs are $(0,0,0)$, $(-\frac{2}{3}, 0, -\frac{4}{27})$, $(-1,1,0)$ and $(-1,-1,0)$

$\partial_x^2 f=-6x-2$

$\partial_y^2 f=2(x+1)$

$\partial_x\partial_y f=2y=\partial_y\partial_x f$

$f(x,y)$ is C^∞

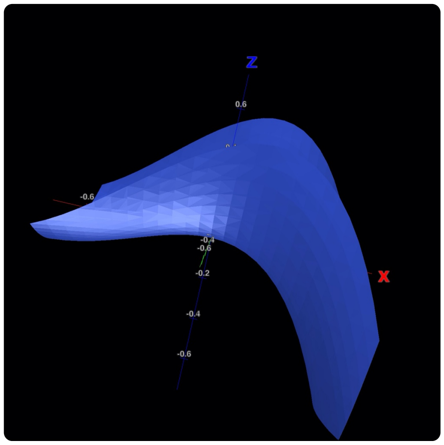
$H(\vec{x})=\begin{pmatrix} -2(3x+1) & 2y \\ 2y & 2(x+1) \end{pmatrix}$

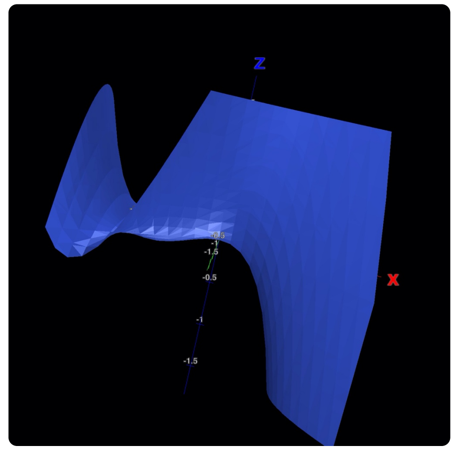
$D=-4(3x+1)(x+1)-4y^2$

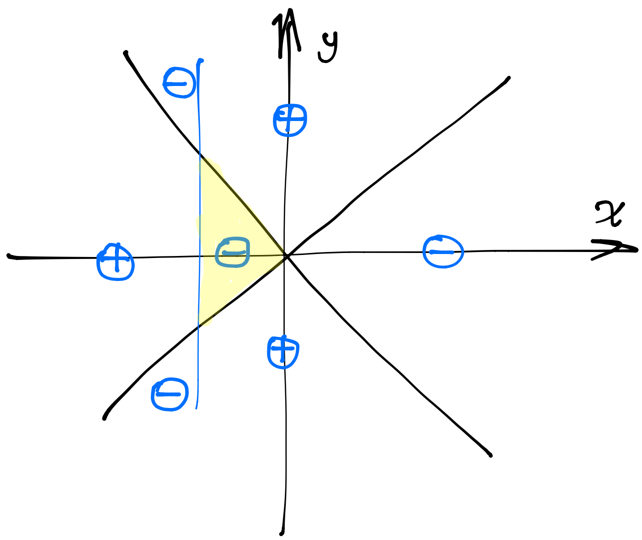
$D|_{(0,0)}=-4 \Rightarrow$ saddle

$D|_{(-1,1)}=-4=D|_{(-1,1)}$ two saddles

$D|_{(-\frac{2}{3},0)}=\frac{4}{3}>0$, $\alpha>0 \Rightarrow$ local min


z
x
0.6
-0.6
-0.2
-0.4
-0.6


z
x
-0.5
-1
-1.5


y
x