July 30th

Surface Integrals of Scalar Function

Def: $\iint_S f\,dA = \iint_W f(\vec{G}(u,v)) \left| \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v} \right| du\,dv$

for $\vec{x} = \vec{G}(u,v)$ with $(u,v) \in W$ when S is the graph of a function

$z = \varphi(x,y)$, $(x,y) \in W$

$$\iint_S f\,dA = \iint_W f(x,y,\varphi(x,y)) \sqrt{1 + (\partial_x \varphi)^2 + (\partial_y \varphi)^2}\, dx\,dy$$

Surface Integral of Vector Fields

$dA = \left| \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v} \right| du\,dv$

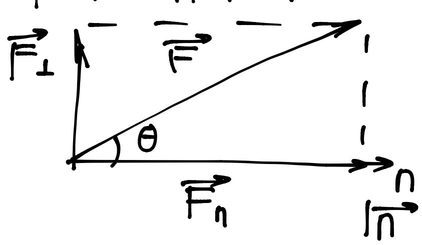
$\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}$ is the same direction as the out normal vector $\vec{n}$.

$\left(\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}\right) du\,dv = \vec{n}\,dA$

Def: the surface integral of $\vec{F}$ over S.

$$\iint_S \vec{F} \cdot \vec{n}\,dA = \iint_W \vec{F}(\vec{G}(u,v)) \cdot \left(\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}\right) du\,dv$$

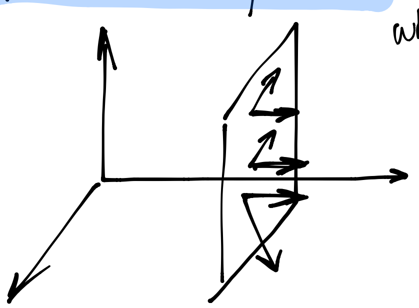
$\vec{F} \cdot \vec{n} = |\vec{F}||\vec{n}| \cos\theta = |\vec{F}| \cdot \cos\theta$



$|\vec{n}| = 1$

$\vec{F}_n = (\vec{F} \cdot \vec{n}) \cdot \vec{n}$

Geometric interpretation



when $z = \varphi(x,y)$, $\vec{G}(u,v) = (x, y, \varphi(x,y))$

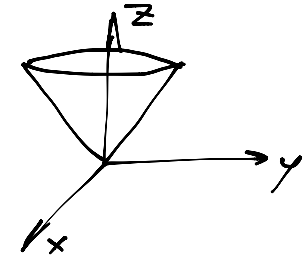
$\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v} = -(\partial_x \varphi)\vec{i} - (\partial_y \varphi)\vec{j} + \vec{k}$

$\vec{n}\,dA = (-\partial_x \varphi\,\vec{i} - \partial_y \varphi\,\vec{j} + \vec{k})\,dx\,dy$

$$\iint_S \vec{F} \cdot \vec{n}\,dA = \iint_W -F_1(\vec{G}(x,y))\frac{\partial \varphi}{\partial x} - F_2(\vec{G}(x,y))\,\partial_y \varphi + F_3(\vec{G}(x,y))\,dx\,dy$$

E.g. Let S be the portion of the cone $x^2+y^2=z^2$ with $0\le z\le 1$, oriented so that the normal points upward

$\vec{F}(x,y,z)=x^2\vec{i}+yz\vec{j}+y\vec{k}$



Sol'n ①

Surface: $\vec{G}(r,\theta)=(r\cos\theta, r\sin\theta, r)$

$\underset{x}{\shortparallel} \quad \underset{y}{\shortparallel} \quad \underset{z}{\shortparallel}=\sqrt{x^2+y^2}=r$

$\partial_r G=(\cos\theta,\sin\theta,1)$

$\partial_\theta G=(-r\sin\theta, r\cos\theta, 0)$

$$\frac{\partial G}{\partial r}\times\frac{\partial G}{\partial \theta}=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos\theta & \sin\theta & 1\\ -r\sin\theta & r\cos\theta & 0\end{pmatrix}=-r\cos\theta\,\vec{i}-r\sin\theta\,\vec{j}+(r\cos^2\theta+r\sin^2\theta)\,\vec{k}$$

$$=-r\cos\theta\,\vec{i}-r\sin\theta\,\vec{j}+r\vec{k}$$

$$\iint_S \vec{F}\cdot\vec{n}\,dA=\int_0^1\int_0^{2\pi}(r\cos\theta)^2(-r\cos\theta)+(r\sin\theta)r(-r\sin\theta)+r\sin\theta\, r\,d\theta\,dr$$

$$=\int_0^1\int_0^{2\pi}-r^3\cos^3\theta-r^3\sin^2\theta+r^2\sin\theta\,d\theta\,dr$$

NOTE: $\int_a^b\cos^3\theta\,d\theta=\int_a^b\cos^2\theta\cos\theta\,d\theta=\int_a^b(1-\sin^2\theta)\cos d\theta=\int_a^b(1-u^2)\,du$

and $\sin^2\theta=\frac{1-\cos2\theta}{2}$

$$=\int_0^1\int_0^{2\pi}-r^3\sin^2\theta\,d\theta\,dr=\int_0^1\int_0^{2\pi}-r^3\frac{1-\cos2\theta}{2}\,d\theta\,dr=\int_0^1(-r^3\tfrac{1}{2}\cdot2\pi)\,dr=-\pi\frac{r^4}{4}\Big|_0^1$$

$$=-\frac{\pi}{4}$$

Sol'n ②:

Surface: $\vec{G}(x,y)=(x,y,\sqrt{x^2+y^2})$

$\Rightarrow \varphi(x,y)=\sqrt{x^2+y^2}$

$\Rightarrow \varphi_x=\frac{x}{\sqrt{x^2+y^2}}$, $\varphi_y=\frac{y}{\sqrt{y^2+x^2}}$

$$\iint_S \vec{F}\cdot\vec{n}\,dA=\iint_{x^2+y^2\le1}-x^2\frac{x}{\sqrt{x^2+y^2}}-yz\frac{y}{\sqrt{x^2+y^2}}+y\,dx\,dy$$

$$=\iint_{x^2+y^2\le1}-\frac{x^3+y^2z}{\sqrt{x^2+y^2}}+y\,dx\,dy$$

change to polar coordinate
$x=r\cos\theta, y=r\sin\theta$

$$=\int_0^1\int_0^{2\pi}\left(-\frac{r^3\cos^3\theta+r^3\sin^2\theta}{r}+r\sin\theta\right)r\,dr\,d\theta$$

$$=\int_0^1\int_0^{2\pi}-r^3\cos^3\theta+r^3\sin^2\theta+r^2\sin\theta\,dr\,d\theta$$

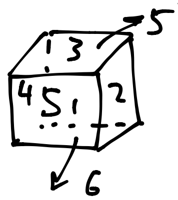
<u>Piecewise smooth Surface S</u> is the union of finitely many pieces $S_1, \dots$ <u>S_k that satisfy</u>

(i). Each S_j admits a smooth parametrization
(ii) The intersections $S_i \cap S_j$ are either empty or finite unions of smooth curves

$$\Rightarrow \iint_S f\,dA = \sum_{j=1}^{k} \iint_{S_j} f\,dA$$

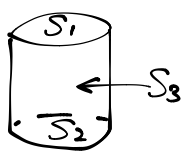
From (ii), surfaces may intersect on edges, but the intersections are all zero content and do not affect integral

E.g. a).



Let S be the surface of a cube
$S = \bigcup_{i=1}^{6} S_i$ where S_i are faces of cube.

b).



S surface of cylindrical solid.

$\{(x,y,z) : x^2+y^2 \leq 1,\ |z| \leq 1\}$

Parametrized form:

Top $S_1 : (x,y) \to (x,y,1)$
Bottom $S_2 : (x,y) \to (x,y,-1)$
Side $S_3 : (\theta, z) \to (\cos\theta, \sin\theta, z)$ $\quad |z| \leq 1, \theta \in [0, 2\pi)$

§ 5.4 Vector Derivatives

Def: ∇ denote the n-tuple of partial differential operators $\partial_j = \partial/\partial x_j$

$\nabla = (\partial_1, \partial_2, \dots, \partial_n) \Rightarrow$ gradient of a C^1 function in $\mathbb{R}^n$
$\text{grad } f = \nabla f = (\partial_1 f, \partial_2 f, \dots, \partial_n f)$

Def: The divergence of $\vec{F}$, where $\vec{F}$ is a C^1 vector field: on an open subset of $\mathbb{R}^n$.

$$\text{div } \vec{F} = \nabla \cdot \vec{F} = \sum_{i=1}^{n} \partial_i f_i = \partial_1 f_1 + \partial_2 f_2 + \cdots + \partial_n f_n$$

The curl of $\vec{F}$ in $\mathbb{R}^3$
$$\text{curl } \vec{F} = \nabla \times \vec{F} = (\partial_2 f_3 - \partial_3 f_2)\vec{i} + (\partial_3 f_1 - \partial_1 f_3)\vec{j} + (\partial_1 f_2 + \partial_2 f_1)\vec{k}$$

or $\text{curl } \vec{F} = \nabla \times \vec{F} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{pmatrix}$

$$\text{scalar function} \xrightarrow{\text{grad}} \text{vector field} \xrightarrow{\text{curl}} \text{vector field} \xrightarrow{\text{div}} \text{scalar function}$$

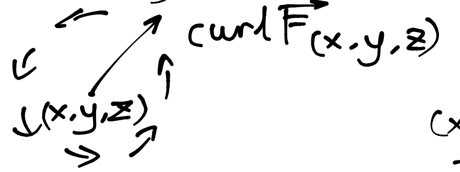
Eg. If $\vec{F}(x,y,z) = xz\,\vec{i} + xyz\,\vec{j} - y^2\,\vec{k}$, find curl $\vec{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & xyz & -y^2 \end{pmatrix}$$

$$= \left(\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right)\vec{i} + \left(\frac{\partial}{\partial z}(xz) - \frac{\partial}{\partial x}(-y^2)\right)\vec{j} + \left(\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right)\cdot\vec{k}$$

$$= (-2y - xy)\,\vec{i} + x\,\vec{j} + yz\cdot\vec{k}$$

The name of curl is associated with rotations



In fluid dynamic particles near (x,y,z) in the fluid rotate about the axis that points in the direction of curl $\vec{F}$.

Useful formulas

3 dim

(1). $\text{grad}(fg) = f\,\text{grad}\,g + g\,\text{grad}\,f$

(2). $\text{grad}(\vec{F}\cdot\vec{G}) = (\vec{F}\cdot\nabla)\vec{G} + \vec{F}\times(\text{curl}\,\vec{G}) + (\vec{G}\cdot\nabla)\vec{F} + \vec{G}\times(\text{curl}\,\vec{F})$

(3). $\text{curl}(f\vec{G}) = f\,\text{curl}\,\vec{G} + (\text{grad} f)\vec{G}$

(4). $\text{curl}(\vec{F}\times\vec{G}) = (\vec{G}\cdot\nabla)\vec{F} + (\text{div}\,\vec{G})\vec{F} - (\vec{F}\cdot\nabla)\vec{G} - (\text{div}\,\vec{F})\vec{G}$

(5). $\text{div}(f\vec{G}) = f\,\text{div}\,\vec{G} + (\text{grad} f)\,\vec{G} \longrightarrow$ n-dim

(6). $\text{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot(\text{curl}\,\vec{F}) - \vec{F}\cdot(\text{curl}\,\vec{G}) \longrightarrow$ 3-dim

Pf: (1) $\text{grad}(fg) = \left(\frac{\partial fg}{\partial x_1}, \frac{\partial fg}{\partial x_2}, \cdots \frac{\partial fg}{\partial x_n}\right)$

For each $i = 1, \cdots, n$

$$\frac{\partial(fg)}{\partial x_i} = \partial_i f\cdot g + f\partial_i g$$

$$\nabla g = (\partial_1 f\cdot g + f\partial_1 g, \partial_2 f\cdot g + f\partial_2 g, \cdots, \partial_n f\cdot g + f\partial_n g)$$

$$= g(\partial_1 f, \cdots, \partial_n f) + f(\partial_1 g, \cdots, \partial_n g)$$

$$= g\nabla f + f\nabla g$$

(3). $\text{curl}(f\cdot\vec{G}) = \nabla\times(fg_1, fg_2, fg_3) = \begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ fg_1 & fg_2 & fg_3 \end{pmatrix}$

$$= (\partial_2(fg_3) - \partial_3(fg_2))\vec{i} + (\partial_3(fg_1) - \partial_1(fg_3))\vec{j} + (\partial_1(fg_2) - \partial_2(fg_1))\vec{k}$$

$$= (\partial_2 f\cdot g_3 + f\partial_2 g_3 - \partial_3 f g_2 - f\partial_3 g_2)\vec{i} + (\partial_3 f\cdot g_1 + f\partial_3 g_1 - \partial_1 f g_3 - f\partial_1 g_3)\vec{j}$$
$$+ (\partial_1 f g_2 + f\partial_1 g_2 - \partial_2 f g_1 - f\partial_2 g_1)\vec{k} = f(\partial_3 g_1 - \partial_1 g_3)\vec{j} + (\partial_3 f g_1 - \partial_1 f g_3)\vec{j}$$
$$+ f(\partial_2 g_3 - \partial_3 g_2)\vec{i} + (\partial_2 f g_3 - \partial_3 f g_2)\vec{i} + f(\partial_1 g_2 - \partial_2 g_1)\vec{k} + (\partial_1 f g_2 - \partial_2 f g_1)\vec{k}$$

$$f\ \text{curl}\,\vec{G} = f\begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ g_1 & g_2 & g_3 \end{pmatrix} = f((\partial_2 g_3 - \partial_3 g_2)\vec{i} + (\partial_3 g_1 - \partial_1 g_3)\vec{j} + (\partial_1 g_2 - \partial_2 g_1)\vec{k}$$

$$\operatorname{grad} f \times \vec{G} = \det\begin{pmatrix} i & j & k \\ \partial_1 f & \partial_2 f & \partial_3 f \\ g_1 & g_2 & g_3 \end{pmatrix} = \cdots$$

$$(\vec{F}\cdot\nabla)\vec{G} = \Big(\sum_{i=1}^{3} f_i\partial_i\Big)\vec{G} = f_1\partial_1 g_1 + f_2\partial_2 g_1 + f_3\partial_3 g_1,\ f_1\partial_1 g_2 + f_2\partial_2 g_2$$
$$+ f_3\partial_3 g_2,\ f_1\partial_1 g_3 + f_2\partial_2 g_3 + f_3\partial_3 g_3)$$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$, have $\vec{F}\cdot\nabla \neq \nabla\cdot\vec{F}$

$$\vec{F}\times\operatorname{curl}\vec{G} = \vec{F}\times\big((\partial_2 g_3 - \partial_3 g_2)\vec{i} + (\partial_3 g_1 - \partial_1 g_3)\vec{j} + (\partial_1 g_2 - \partial_2 g_1)\vec{k}\big)$$
$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1 & f_2 & f_3 \\ \partial_2 g_3 - \partial_3 g_2 & \partial_3 g_1 - \partial_1 g_3 & \partial_1 g_2 - \partial_2 g_1 \end{pmatrix}$$

$$= (f_2\partial_1 g_2 - f_2\partial_2 g_1 - f_3\partial_3 g_1 + f_3\partial_1 g_3)\vec{i} + (f_3\partial_2 g_3 - f_3\partial_3 g_2 - f_1\partial_1 g_2 +$$
$$(f_1\partial_2 g_3)\vec{j} + (f_1\partial_3 g_1 - f_1\partial_1 g_3 - f_2\partial_2 g_3 + f_2\partial_3 g_2)\vec{k}$$

$$\nabla(\vec{F}\cdot\vec{G}) = \nabla(f_1 g_1 + f_2 g_2 + f_3 g_3)$$
$$= (\partial_1 f_1 g_1 + f_1\partial_1 g_1 + \partial_1 f_2 g_2 + f_2\partial_1 g_2 + \partial_1 f_3 g_3 + f_3\partial_1 g_3)\vec{i}$$
$$+ (\partial_2 f_1\cdot g_1 + f_1\partial_2 g_1 + \partial_2 f_2 g_2 + f\partial_2 g_2 + \partial_2 f_3 g_3 + f_3\partial_2 g_3)\vec{j}$$
$$+ (\partial_3 f_1 g_1 + f_1\partial_3 g_1 + \partial_3 f_2 g_2 + f\partial_3 g_2 + \partial_3 f_3 g_3 + f_3\cdot\partial_3 g_3)\vec{k}$$

$$\vec{F}\times(\operatorname{curl}\vec{G}) = \vec{F}\times(\nabla\times\vec{G}) = \nabla(\vec{F}\cdot\vec{G}) - \vec{G}(\vec{F}\cdot\nabla)$$

$\boxed{\text{remark}: \vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})}$ not true if one of $\vec{a}, \vec{b}, \vec{c}$ is ∇.

$$\Rightarrow \nabla(\vec{F}\cdot\vec{G}) = \vec{F}\times(\operatorname{curl}\vec{G}) + \vec{G}(\vec{F}\cdot\nabla)$$
$$\| $$
$$\nabla(\vec{G}\cdot\vec{F})$$

$$\vec{G}\times(\operatorname{curl}\vec{F}) = \vec{G}\times(\nabla\times\vec{F}) = \nabla(\vec{G}\cdot\vec{F}) - \vec{F}(\vec{G}\cdot\nabla)$$

$$\Rightarrow \nabla(\vec{G}\cdot\vec{F}) = \vec{G}\times(\operatorname{curl}\vec{F}) + \vec{F}(\vec{G}\nabla)$$

(5). $\operatorname{div}(f\vec{G}) = f\operatorname{div}\vec{G} + (\operatorname{grad} f)\cdot\vec{G}$

$$\nabla\cdot(f\vec{G}) = \sum_{i=1}^{n}\partial_i(f g_i) = \sum_{i=1}^{n}(\partial_i f g_i + f\partial_i g_i)$$
$$= \nabla f\cdot\vec{G} + f\sum_{i=1}^{n}\partial_i g_i$$
$$= \nabla f\cdot\vec{G} + f\operatorname{div}\vec{G}$$

(6). $\operatorname{div}(\vec{F}\times\vec{G}) = \vec{G}(\operatorname{curl}\vec{F}) - \vec{F}(\operatorname{curl}\vec{G})$

$$\vec{F}\times\vec{G} = \begin{pmatrix} i & j & k \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix} = (f_2 g_3 - f_3 g_2)\vec{i} + (f_3 g_1 - f_1 g_3)\vec{j} + (f_1 g_2 - f_2 g_1)\vec{k}$$

$$\operatorname{div}(\vec{F}\times\vec{G}) = \partial_1(f_2 g_3 - f_3 g_2) + \partial_2(f_3 g_1 - f_1 g_3) + \partial_3(f_1 g_2 - f_2 g_1)$$
$$= \partial_1 f_2 g_3 + f_2\partial_1 g_3 - \partial_1 f_3 g_2 - f_3\partial_1 g_2 + \partial_2 f_3 g_1 - f_3\partial_2 g_1 - \partial_2 f_1 g_3$$

$$-f_1\partial_2 g_3 + \partial_3 f_1 g_2 + f_1\partial_3 g_2 - \partial_3 f_2 g_1 - f_2\partial_3 g_1$$
$$= g_1(\partial_2 f_3 - \partial_3 f_2) + g_2(\partial_3 f_1 - \partial_1 f_3) + g_3(\partial_1 f_2 - \partial_2 f_1)$$
$$+ f_1(\partial_3 g_2 - \partial_2 g_3) + f_2(\partial_1 g_3 - \partial_3 g_1) + f_3(\partial_2 g_1 - \partial_1 g_2)$$

$$\text{Curl } \vec{F} = \begin{pmatrix} i & j & k \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{pmatrix} = (\partial_2 f_3 - \partial_3 f_2)\vec{i} + (\partial_3 f_1 - \partial_1 f_3)\vec{j} + (\partial_1 f_2 - \partial_2 f_1)\vec{k}$$

$$\text{curl}(\vec{F} \times \vec{G}) = \nabla \times \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix} = \nabla \times \begin{pmatrix} (f_2 g_3 - f_3 g_2)\vec{i} \\ + (f_3 g_1 - f_1 g_3)\vec{j} \\ + (f_1 g_2 - f_2 g_1)\vec{k} \end{pmatrix}$$

$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_2 g_3 - f_3 g_2 & f_3 g_1 - f_1 g_3 & f_1 g_2 - f_2 g_1 \end{pmatrix} = \cdot - \vec{i} + \cdots \vec{j} + \cdots \vec{k}$$

$= (\partial_2 f_1 g_1 - \partial_2 f_2 g_1 - \partial_3 f_3 g_1 + \partial_3 f_1 g_3)\vec{i}$ $\longrightarrow$ the previous terms with f & g switched.
$+ (\cdot\ --)\ \vec{j} + (\cdots)\vec{k}$