Lecture 7

Closed subsets of $\mathbb{R}^n$. (§4.3 continue)

A set $A \subset \mathbb{R}^n$ is closed if it contains all of its limit pts.

If $A, B \subset \mathbb{R}^n$ are closed $\Rightarrow A \cup B$ is closed.

(any finite union is closed)

If $\{A_i : i \in I\}$ is a family of closed subsets $\Rightarrow \bigcap_{i \in I} A_i$ is closed

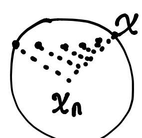
$A \subset \mathbb{R}^n$, $\overline{A}$ that consists of all limit pts of A.

<u>Proposition</u>. Let $A \subset \mathbb{R}^n$, then $\overline{A}$ is the smallest closed set containing A. $\overline{\overline{A}} = \overline{A}$

Proof: Need to prove ① $\overline{A} \supset A$
② $\overline{A}$ is closed
③ smallest

Proof of ②: in order to show $\overline{A}$ is closed.
We need to show that it contains all the limit pts of $\overline{A}$.
We know that. It contains all the limit pts of A.
Let us show that all of the limit pts of $\overline{A}$ are in fact the limit pts of A.

Sps x is a limit pt of $\overline{A}$, then $\exists$ a seq. (x_n) of pts of $\overline{A}$ s.t. $\lim_{n\to\infty} x_n = x$
each x_n is a limit pt of A
for each x_n $\exists$ a seq a_n^k that converges to x_n.



$\overline{A}$

That means that $\forall \varepsilon > 0, \exists N$ s.t. $|x_n - a_n^k| < \varepsilon$ when $k \geq N$

For each x_n let $\varepsilon = \frac{1}{n}$

$|x_n - a_n| < \frac{1}{n}$ $\quad k \geq N$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} x_n + (a_n - x_n) = x$

x is a limit of $A \Rightarrow x \in \overline{A}$.

③ $\overline{A}$ is the smallest closed set containing A.
Sps B is a closed set that contains A
Since B is closed it contains all of its limit pts.
In particular, B contains all of limit pts of $A \Rightarrow B \supset \overline{A}$

$\blacksquare$

<u>Def</u>: The ball about a in $\mathbb{R}^n$ of radius r.
$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$

<u>Def</u>: A subset U of $\mathbb{R}^n$ is open if $\forall a \in U, \exists r = r(a)$ s.t. $B_r(a) \subset U$.

<u>Examples</u>: (1) $\mathbb{R}^n$ open (also closed)
(2) $\emptyset$ open (also closed)
(3). $(a,b) \subset \mathbb{R}$ $\{x \mid a < x < b\}$ open
④ $(0, +\infty)$ open
⑤ $(0, 1]$ neither open nor closed

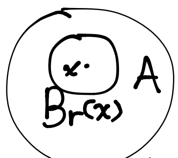
⑥ $B_r(a)$ open

Theorem: A set $A \subset \mathbb{R}^n$ is open iff A^c is closed $\quad A^c = \mathbb{R}^n \setminus A$

Proof: Let A be open. suppose $\lim_{n\to\infty} x_n = x$, where (x_n) is a sequence of pts of A^c. Suppose $x \in A \Rightarrow \exists r>0$, s.t. $B_r(x) \subset A$

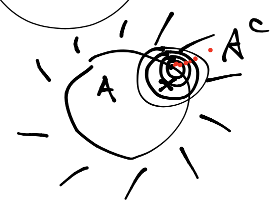
In particular, $x_n \notin B_r(x)$ for any n $\|x - x_n\| \geq r$

So (x_n) does not converge to x, which is a contradiction.



Now sps A is not open $\Rightarrow \exists x \in A$ s.t. for no r, $B_r(x) \subset A$

Let $r = \frac{1}{n}$, then $B_{\frac{1}{n}}(x) \cap A^c \neq \emptyset$



For each n, let $x_n \in B_{\frac{1}{n}}(x) \cap A^c$

$\lim_{x\to\infty} x_n \to x \Rightarrow x$ is a limit pt of A^c

So A^c is not closed. ■

Proposition: If U & V are open in $\mathbb{R}^n \Rightarrow U \cap V$ is open. (Any finite intersection of open sets is open)

If $\{U_i : i \in I\}$ is a family of open sets in $\mathbb{R}^n \Rightarrow \bigcup_{i\in I} U_i$ is open

Let $A \subset \mathbb{R}^n$, Int A is the largest open set s.t. Int $A \subset A$

Ex: $v_0 = (0,0)$.

$v_n = (x_n, y_n)$

$x_{n+1} = \dfrac{x_n + y_n + 1}{2} \qquad y_{n+1} = \dfrac{x_n - y_n + 1}{2}$

Suppose the limit exists, how do we find it?

$L = (x,y)$. It is a fixed pt of $T(x,y) = \left(\frac{x+y+1}{2}, \frac{x-y+1}{2}\right)$

$x = \frac{x+y+1}{2}$, $y = \frac{x-y+1}{2} \Rightarrow y = 1, x = 2$

$$\|v_{n+1} - L\|^2 = \left\|\left(\frac{x_n+y_n+1}{2} - 2, \frac{x_n - y_n + 1}{2} - 1\right)\right\|^2 = \left\|\left(\frac{x_n+y_n-3}{2}, \frac{x_n-y_n-1}{2}\right)\right\|^2$$

$$= \frac{(x_n+y_n-3)^2 + (x_n-y_n-1)^2}{4}$$

$$= \frac{2x_n^2 + 2y_n^2 - 8x_n - 4y_n + 10}{4} = \frac{(x_n-2)^2 + (y_n-1)^2}{2}$$

$$= \frac{1}{2}\|v_n - L\|^2$$

$$\|v_{n+1} - L\| = \frac{1}{\sqrt{2}}\|v_n - L\|$$

$$\underbrace{\|v_n - L\|}_{<\varepsilon \text{ for large } n} = \frac{1}{(\sqrt{2})^n}\|v_0 - L\| = \frac{\sqrt{5}}{(\sqrt{2})^n}$$

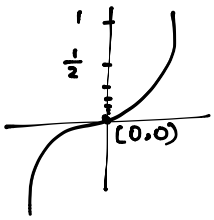
Example: $A \subset \mathbb{R}^2$

$A = \{(x,y) : \underline{x \in \mathbb{Q}}, y > x^3\}$

not open,
not closed

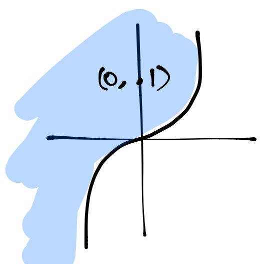
$\text{int} A = \emptyset$
$\overline{A} = \{(x,y) : y \geq x^3\}$

① A is not closed $(0, \frac{1}{n})$

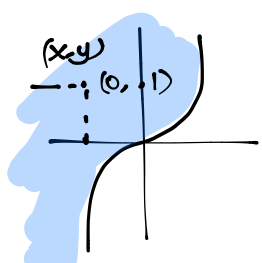


② A is not open

We can show that by showing that A^c is not closed.
Let $x = (0,1) \in A$, $(\frac{\sqrt{2}}{n}, 1) \to (0,1)$



③ $\overline{A} = \{(x,y) : y \geq x^3\}$
Let (x,y), be s.t. $y \geq x^3$. take $x_n \in \mathbb{Q}$, x_n is increasing
$y_n = y + \frac{1}{n}$

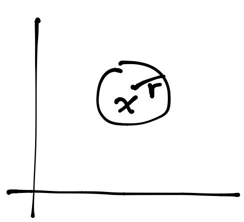


Want to show that $(x_n, y_n) \in A$
$y_n > y \geq x^3 > x_n^3$ ∎

Sps $\lim_{n\to\infty} a_n = (x,y)$
$a_n = (x_n, y_n) \in A$
Want to show (x,y) satisfies $y \geq x^3$.
Example : $y = \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n^3 = x^3$

∎

$\text{Int} A = \emptyset$



Compact sets §4.4 Compact Sets & the Heine-Borel theorem

Def A subset A of $\mathbb{R}^n$ is compact if every sequence $(a_k)_{k=1}^{\infty}$ of pts of A has a convergent subsequence $(a_{k_i})_{i=1}^{\infty}$ s.t. $\lim_{i\to\infty} a_{k_i} \in A$

Thm: Every closed and bounded subset of $\mathbb{R}$ is compact.

Proof: Let A be a closed & bounded subset of $\mathbb{R}$. Let (a_n) be a seq. of pts of A. Since A is bounded.

(a_n) is bounded. By Bolzano-Weierstrass theorem, $\exists\, a_{n_k}$, $\lim_{k\to\infty} a_{n_k} = a$

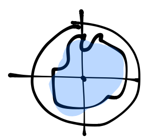
(a_{n_k}) is a sequence of pts of A. $\Rightarrow a$ is a limit point of A. Since A is closed, $a \in A \Rightarrow A$ is compact

Ex: $S = (0,1]$ not compact

$\{\frac{1}{n} : n \in \mathbb{N}\}$

$\mathbb{N}$ not compact

Def: A subset $S \subset \mathbb{R}^n$ is bounded if $\exists R \in \mathbb{R}$ s.t. $S \subset B_R(0) \Leftrightarrow \sup_{x\in S} \|x\| < \infty$



Lemma: A compact subset of $\mathbb{R}^n$ is closed & bounded

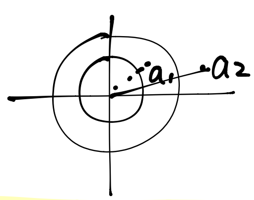
Proof: (1). A is compact $\Rightarrow A$ is closed

Let a be a limit pt of $A \Rightarrow \exists\, (x_n)$ a sequence of pts of A that converges to a $\lim_{n\to\infty} x_n = a$

Since A is compact, $\exists\, x_{n_k}$ s.t. $\lim_{k\to\infty} x_{n_k} \in A$, $a \in A$ ∎

(2). A is compact $\Rightarrow A$ is bounded

Sps not, let a_n be a seq. of pts of A s.t. $\|a_n\| > n$, $\forall n \in \mathbb{N}$



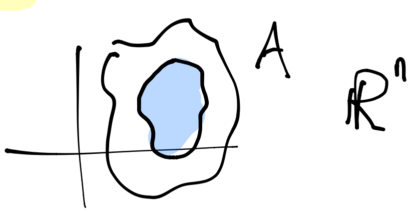
Sps (a_n) has a conv. subseq. $a_{n_i} \to a$

$\|a\| = \lim_{i\to\infty} \|a_{n_i}\| \geq \lim_{i\to\infty} n_i = +\infty$

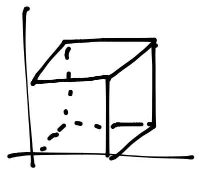
a contradiction ∎

Lemma. If C is a closed subset of a compact subset of $\mathbb{R}^n$, then C is compact.

Pf:



Let (x_n) be a seq of pts of C. This is also a seq of pts of A, which is compact, so it has a subseq. (x_{n_k}) of pts C that converges to a pt $a \in A$. But a is a limit point of $C \Rightarrow$ Since C is closed, $a \in C$, $x_{n_k} \to a \in C \Rightarrow C$ is compact. ∎



$[a,b]^n = [a,b] \times [a,b] \times \cdots \times [a,b]$?

Thm: The cube $[a,b]^n$ is a compact subset of $\mathbb{R}^n$

Proof: Let $x_k = (x_{k,1}, \dots, x_{k,n})$, $k \geq 1$, $a \leq x_{k,i} \leq b$

Consider $(x_{k,1})_{k=1}^{\infty} \in [a,b]$
It has a convergent subsequence. $(x_{k_j,1})_{j=1}^{\infty} \to z_1$

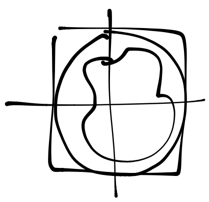
Consider x_{k_j}

$(x_{k_j,2}) \in [a,b]$ It must have a conv. subseq. $\to z_2$

$(x_{k_{j_l},2}) \to z_2$
after n steps we will stop by obtaining a subseq. that conv. to $z = (z_1, \dots, z_n)$

The Heine-Borel Thm: A subset $S \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ it is closed & bdd

Proof: If C is compact $\Rightarrow$ it's closed & bounded. Suppose C is closed & bounded
If C is bounded $\Rightarrow C \subset B_R(0)$ for R that is large enough. $C \subset B_R(0) \subset [-R,R]^n$
C is a closed subset of a compact set. So it is compact.



The Cantor's intersection thm.
If $A_1 \supset A_2 \supset A_3 \supset \cdots$ is a decreasing sequence of nonempty compact subsets of $\mathbb{R}^n \Rightarrow \bigcap_{k \geq 1} A_k \neq \emptyset$
Sps $A_1 \supset A_2 \supset \dots$, $\emptyset \neq A_i$ is closed

$\bigcap_n [n, +\infty) = \emptyset$

Proof. $A_n \neq \emptyset \Rightarrow \exists a_n \in A_n$, $n \geq 1$, $(a_n)_{n=1}^{\infty}$ is a seq. of pts of $A_1 \Rightarrow \exists a_{n_k} \to a \in A$

$\underbrace{\cancel{a_1}, a_2, a_3, a_4, \cdots}_{A_2}$ (A_1)

$\exists$ a conv. subseq. $\to a \in A_2$
For any n remove the 1st n terms obtain a seq. in A_n
Must have a subsequence that converges to $a \in A_n$. For all n, $a \in A_n$

$a \in \bigcap_{n \geq 1} A_n$ ∎