Lecture 11

Today: lots of practice with the calculus of variations and a little optimal control.

Let $\mathcal{F} = \{ x : [0,1] \to \mathbb{R} : x(\ ) \text{ is } C^2,\ x(0)=x(1)=0 \}$

Consider the problem:

$$\underset{\text{minimize}}{} \quad I[x(\cdot)] = \int_0^1 [\dot{x}(t)^2 + \cos(t)\cdot\cosh(x(t))]\,dt \quad \text{in } \mathcal{F}$$

a). Let $I_1[x(\cdot)] = \int_0^1 \dot{x}(t)^2\,dt$

$$I_2[x(\cdot)] = \int_0^1 \cos(t)\cosh(x(t))\,dt$$

Show that I_1 & I_2 are both convex

Before that, show $\mathcal{F}$ is convex

$\mathcal{F}$ is convex because:
if $x(\cdot)$ & $y(\cdot)$ belong to $\mathcal{F}(\cdot)$ and $\theta \in [0,1]$, then we have to check that $z(\cdot) := \theta x(\cdot) + (1-\theta) y(\cdot)$ belongs to $\mathcal{F}$

In other words, must check that $z(\cdot)$ is C^2 & $z(0)=z(1)=0$

This is clear because:
$x(\cdot), y(\cdot) \in \mathcal{F} \Rightarrow x(\cdot)$ & $y(\cdot)$ both C^2, so linear combination is C^2,
so $z(\cdot)$ is C^2.

Similarly,

$$\left. \begin{aligned} x(\cdot), y(\cdot) \in \mathcal{F} \Rightarrow x(0) = y(0) = 0 \\ x(1) = y(1) = 0 \end{aligned} \right\} \Rightarrow z(0) = z(1) = 0$$

Next: ~~Is I_1 convex?~~

Show that I_1 is convex
Have to check for $x(\cdot)$ & $y(\cdot)$ as above.
$I_1[\theta x(\cdot) + (1-\theta) y(\cdot)] \leq \theta I_1[x(\cdot)] + (1-\theta) I_1[y(\cdot)]$

Thus, we ask:

$$\int_0^1 (\theta \dot{x}(t) + (1-\theta)\dot{y}(t))^2\,dt \overset{?}{\leq} \theta \int_0^1 (\dot{x}(t))^2 + (1-\theta)\int_0^1 (\dot{y}(t))^2\,dt$$

This looks too hard, so let's ask can we use the fact that integrand contains a convex function $(\cdot\cdot\cdot)^2$.

i.e. let $f(x) = x^2$ convex

$f'(x) = 2x \quad f''(x) = 2$

We are trying to show

$$\int_0^1 f(\theta \dot{x}(t) + (1-\theta)\dot{y}(t)) \overset{?}{\leq} \int_0^1 [\theta f(\dot{x}(t)) + (1-\theta) f(\dot{y}(t))]$$

Now this is clear, because for every t, $f(\theta\dot{x}(t)+(1-\theta)\dot{y}(t)) \le \theta f(\dot{x}(t))+(1-\theta)f(\dot{y}(t))$

Integrate both sides from 0 to 1 $\Rightarrow$ ⊛

Next: same for I_2. Fix $x(\cdot)$ and $y(\cdot)$ in F & $\theta\in[0,1]$

want to check that

$I_2[\theta x(\cdot)+(1-\theta)y(\cdot)] \le \theta I_2[x(\cdot)]+(1-\theta)I_2[y(\cdot)]$

Note: $I_2[x(\cdot)] = \int_0^1 \underbrace{\cos(t)}_{\text{positive for } t\in(0,1)} \underbrace{\cosh(x(t))}_{\text{Convex fcn of } x(t)} dt$

Claim: $\cosh(x)$ is convex

$$\frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d^2}{dx^2}\cosh x = \cosh x = \frac{e^x+e^{-x}}{2} > 0$$

$\Longrightarrow$ convex

We want to show that $\int_0^1 \cos(t)\cosh(\theta x(t)+(1-\theta)y(t))\,dt$ ⊛

$$\le \int_0^1 \cos t\,[\theta\cosh(x(t))+(1-\theta)\cosh(y(t))]\,dt$$

True because:

Convexity of cosh $\Rightarrow$ $\cosh(\theta x(t)+(1-\theta)y(t)) \le \theta\cosh(x(t))+(1-\theta)\cosh(y(t))$ for all t. if $\theta\in(0,1)$

and since $\cos t>0$ for $0\le t\le 1$

it follows that

$\cos t\cosh(\theta x(t)+(1-\theta)y(t)) \le \cos t\,[\theta\cosh x(t)+(1-\theta)\cosh y(t)]$ for all $t\in[0,1]$

Integrate from 0 to 1 to get ⊛

Remark: the general principle is:

If $L(t,x,v)$ is a convex function of (x,v) for every $t\in[a,b]$, then

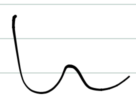
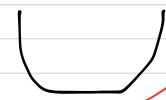
$$I[x(\cdot)] = \int_a^b L(t,x(t),\dot{x}(t))\,dt \text{ is a convex function of } x(\cdot).$$

In fact, $I[\cdot]$ is strictly convex since both $f(v)=v^2$, and $g(t,x)=\cos t\cosh x$ are strictly convex as functions of x and v (for every $t\in[0,1]$)

Conclusion so far: if $0<\theta<1$ then $I[\theta x(\cdot)+(1-\theta)y(\cdot)] = I_1[\cdots]+I_2[\cdots] < \theta I_1[x(\cdot)]+(1-\theta)I_1[y(\cdot)]$

$+\theta I_2[x(\cdot)]+(1-\theta)I_2[y(\cdot)]=\theta I[x(\cdot)]+(1-\theta)I[y(\cdot)]$ (if $x(\cdot)\neq y(\cdot)$)

$\boxed{Q}$: Show that there is at most one minimizer of $I[\cdot]$ in F.

Idea:  not convex  not strictly convex

In 2^{nd} picture, If x, y both minimizers then f at $\theta x+(1-\theta)y \geq \min$ of $f = f(x)$ or $f(y)$
so $f(\theta x+(1-\theta)y) \geq \theta f(x)+(1-\theta)f(y)$ contradicts strict convexity.

So

<u>Answer to question:</u> <u>exactly</u> the same argument works as for strictly convex function on E^n, if $x(\cdot)$ & $y(\cdot)$ are both minimizers of $I[\cdot]$ and $x(\cdot)\neq y(\cdot)$, then

$I[\theta x(\cdot)+(1-\theta)y(\cdot)] \geq \min$ of $I[\cdot] = \theta I[x(\cdot)]+(1-\theta)I[y(\cdot)]$

But this contradicts the strict convexity of $I[\cdot]$.

(Recall: a minimizer $x(\cdot)$ satisfies $I[x(\cdot)] \leq I[z(\cdot)]$ for all $z(\cdot)$ in F)

<u>Next</u> Recall derivation of "Euler Lagrange eqns" = first order necessary conditions

<u>Problem:</u> minimize $I[\cdot]$
subject to some constraints (boundary values, maybe more ...)

suppose $x(\cdot)$ is a minimizer
① Given arbitrary function $y(\cdot)$ with same boundary conditions

Let $i(\tau) = I[\underbrace{x(\cdot)+\tau y(\cdot)}_{\text{satisfying same boundary constraints as } x(\cdot)}]$

② $x(\cdot)$ minimizer $\Rightarrow i'(0)=0$
② compute $i'(0)$
③ Integrate by parts to move all derivatives from $y(\cdot)$ onto $x(\cdot)$.

④ If $\int \underbrace{[\cdots]}_{\text{something involving } x(\cdot)} y(t)\,dt = 0$ for all $y(\cdot)$
then
$[\cdots]=0$

Note: for problems with constraints, need to define
$i(\tau) = I[\underbrace{???}_{\text{something}}]$
containing $x(\cdot)+\tau y(\cdot)$
but modified to satisfy constraints

$\boxed{\text{EX}}$: Let $I[x(\cdot)] = \int_0^1 (x''(t))^2\,dt$

let $F = \left\{ x : [0,1] \Rightarrow \mathbb{R},\ \begin{array}{l} x(0)=a \\ x'(0)=b \\ x(1)=c \\ x'(1)=d \end{array} \right\}$

find minimizer

$\boxed{\text{A}}$ Follow above procedure to find equations for minimizer.

Let $x(\cdot)$ = minimizer

Let $y(\cdot)$ = any function such that

$y(0)=y'(0)=y(1)=y'(1)=0$

Then $x(\cdot)+\tau y(\cdot)$ satisfies same boundary conditions as $x(\cdot)$

Let $i(\tau) = I[x(\cdot)+\tau y(\cdot)]$

② $x(\cdot)$ minimizer $\Rightarrow i'(0)=0$

$$i'(\tau) = \frac{d}{dt}\int_0^1 (x''(t)+\tau y''(t))^2\,dt = \int_0^1 2(x''(t)+\tau y''(t))\,y''(t)\,dt$$

So $i'(0) = 2\int_0^1 x''(t)\,y''(t)\,dt$

③ Intregrate by parts

$$i'(0) = 2\int_0^1 x''(t)\,y''(t)\,dt = \underbrace{uv\Big|_0^1}_{=0} - \int_0^1 y'(t)\,x'''(t)\,dt$$

$\boxed{\begin{array}{l} u = x''(t) \\ v = y''(t) \end{array}}$

repeat $= \underbrace{y(t)\,x'''(t)\Big|_0^1}_{=0} + \int_0^1 x''''(t)\,y(t)\,dt$

So $i'(0) = 2\int_0^1 x''''(t)\,y(t)\,dt$

④ Since this holds for all $y(\cdot)$, it follows that $\underbrace{\boxed{x''''(t)=0}}_{\text{ODE}}$ $\quad y(0)=y'(0)=y(1)=y'(1)$

Solution of ODE is $\boxed{x(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3}$

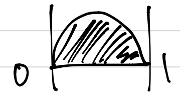
To finish problem, choose c_1, c_2, c_3, c_4 to satisfy boundary conditions: $\left.\begin{array}{l} x(0)=a \\ x'(0)=b \\ x(1)=c \\ x'(1)=d \end{array}\right\}$

$\boxed{\text{EX}}$: Minimize $I[x(\cdot)] = -\int_0^1 x(t)\,dt$

subject to $x(0)=x(1)=0$ <u>and</u> $G[x(\cdot)] = \int_0^1 \sqrt{1+x'^2(t)}\,dt = \pi/2$

$\downarrow$ Some # > 1

i.e. maximize "area under the graph" with constraint: arc length of curve $= \frac{\pi}{2}$



(1) Compute the Euler-Lagrange equations

Let $x(\cdot)$ be a minimizer

Fix any $y(\cdot)$ s.t. $y(0)=y(1)=0$

define $i(\tau)=I\left[\frac{x(\cdot)+\tau y(\cdot)}{\alpha(\tau)}\right]$ $\underline{\underline{\text{where}}}$ $\alpha(\tau)$ is defined by requiring that

$$G\left[\frac{x(\cdot)+\tau y(\cdot)}{\alpha(\tau)}\right]=\pi/2$$

So $\frac{x(\cdot)+\tau y(\cdot)}{\alpha(\tau)}$ satisfies all constraints

so $i(\tau)=I[\quad]\geqslant I[x(\cdot)]=i(0)$

Hence $i'(0)=0$

$$\text{So } 0=i'(0)=-\frac{d}{dt}\left[\int_0^1\frac{1}{\alpha(\tau)}(x(t)+\tau y(t))\,dt\right]\Big|_{\tau=0}$$

$$=-\frac{d}{dt}\left[\alpha(\tau)^{-1}\int_0^1(x(t)+\tau y(t))\,dt\right]\Big|_{\tau=0}$$

$$=-\left[-\alpha(\tau)^{-2}\alpha'(0)\int_0^1(x(t)+\tau y(t))\,dt+\alpha(\tau)^{-1}\int_0^1 y(t)\,dt\right]$$

Note $\alpha(0)=1$, since $G\left[\frac{x(\cdot)}{1}\right]=\frac{\pi}{2}$ $=\alpha'(0)\cdot\int_0^1 x(t)\,dt-\int_0^1 y(t)\,dt$ (1)

How to find $\alpha'(\tau)$

Only hope: look at $\frac{d}{dt}G\left[\frac{x(\cdot)+\tau y(\cdot)}{\alpha(\tau)}\right]=0$

$$\frac{d}{dt}\int_0^1\sqrt{1+\left(\frac{\dot x(t)+\tau\dot y(t)}{\alpha(\tau)}\right)^2}\,dt\ \Big|_{\tau=0}$$

$$=\int_0^1\frac{1}{\cancel{2}}(\cdots)^{-1/2}\cancel{2}\left(\frac{\dot y(t)\alpha(\tau)-(\dot x(t)+\tau\dot y(t))\alpha'(\tau)}{\alpha(\tau)^2}\right)\times\left(\frac{\dot x(t)+\tau\dot y(t)}{\alpha(\tau)}\right)$$

set $t=0$, recall $\alpha(0)=1$

$$0=\int\frac{\dot x(t)\dot y(t)}{\sqrt{1+\dot x(t)^2}}-\alpha'(\tau)\int\frac{\dot x^2(t)}{\sqrt{1+\dot x^2(t)}}\,dt \qquad (2)$$

Conclusion: combine (1) & (2) to find

$$\alpha'(0)\leftarrow\left(\frac{\int\frac{\dot x(t)\dot y(t)}{\sqrt{1+\dot x(t)^2}}}{\int\frac{\dot x(t)^2\,dt}{1+\dot x^2(t)}}\right.\int_0^1 x(t)\,dt-\int_0^1 y(t)\,dt=0$$

This says:

$$i'(0)=\int\left[\lambda\frac{\dot x(t)}{\sqrt{1+\dot x^2(t)}}\dot y(t)-y(t)\right]dt=0$$

$$\text{for } \lambda=\underbrace{\left(\frac{\int_0^1 x(t)\,dt}{\int_0^1\frac{x(t)^2}{\sqrt{1+x^2(t)}}}\right)}_{\text{independent of } y.}$$

Now: integrate by parts: $i'(0) = \int_0^1 \lambda \frac{d}{dt}\left(\frac{\dot{x}(t)}{\sqrt{1+\dot{x}(t)^2}}\right) - 1 \,]\, y(t)\,dt = 0$

E–L eqns: this $= 0$

i.e. $\boxed{\frac{d}{dt}\left(\frac{\dot{x}(t)}{1+\dot{x}(t)^2}\right) = \frac{1}{\lambda} = \text{constant}}$

We can check that $x(t) = \sqrt{\frac{1}{4} - (t-\frac{1}{2})^2}$ satisfies the equation (graph is semi-circle) set $t=0$, recall $\alpha(0)=1$

Optimal control problems:

$\dot{x}(t) = Mx(t) + N\alpha(t)$ $\qquad\qquad \alpha(t) \in [-1, 1]$ for all t

$M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

Let $C(t) = \{x^0$: there is a contro $\alpha(\cdot)$ steering the system from x to origin in time $t\}$

$C = \bigcup_{t>0} C(t)$

[Q] is 0 in interior of C ?

[A]: Let $G = [N, MN, M^2N]$

check rank of G

$MN = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad M^2N = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \qquad \text{rank}\, G = 3$

So theorem says yes, $0 \in C^0$

[Q] Is $\mathbb{R}^3 = C$?

Thm says ok if $\text{Re}\,\lambda_i \leq 0$ for all eigenvalues λ_i of M.

eigenvalues of $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are 1 & 0

(For upper trianguler matrix, eigenvalues are the diagonal entries)

The thm does not say

[Q]: Assume that $x^0 \in C$ & that $\alpha^*(\cdot)$ is an optimal control for reaching the origin in minimal time.

What can we say about $\alpha^*(\cdot)$?

i.e. what does Pontryagin Max Principle tell us?

[A] Thm says: $\exists$ a vector $h \in \mathbb{R}^3$ s.t.

$$h^T e^{-Mt} N \alpha^*(t) = \min_{a \in [-1,1]} (h^T e^{-Mt} N a) \qquad \forall t \in [0, \tau^*]$$

$$e^{Mt} = ? = I + M + \frac{(tM)^2}{2!} + \frac{(tM)^3}{3!} + \cdots$$

$$M^2 = \begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}\begin{pmatrix} 0&1&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix} = \begin{pmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&1 \end{pmatrix}$$

$$M^k = M^2 = \nearrow \qquad \forall k \geq 2$$

$$e^{tM} = \begin{pmatrix} 1 && \\ &1& \\ &&1 \end{pmatrix} + \begin{pmatrix} &t& \\ && \\ &&t \end{pmatrix} + \begin{pmatrix} && \\ && \\ &&t^2/2 \end{pmatrix} + \begin{pmatrix} && \\ && \\ &&t^3/6 \end{pmatrix} = \begin{pmatrix} 1&t&0 \\ 0&1&0 \\ 0&0&1+t+\frac{t^2}{2}+\frac{t^3}{3!}+\cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1&t&0 \\ 0&1&0 \\ 0&0&e^t \end{pmatrix}$$

so thm says

$$(h_1 \quad h_2 \quad h_3)\begin{pmatrix} 1&t&0 \\ 0&1&0 \\ 0&0&e^t \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\alpha(t) = \max_{|a|\leq 1} (\text{same})$$

$$(h_1 t + h_2 + h_3 e^t)\alpha(t) = \max_{|a|\leq 1}\left((h_1 t + h_2 + h_3 e^t)a\right)$$

Conclusion: $\alpha(t) = \begin{cases} 1 & \text{if } h_1 t + h_2 + h_3 e^t > 0 \\ -1 & \text{if } h_1 t + h_2 + h_3 e^t < 0 \end{cases}$

change sign at most twice.