August 1 st

grad, curl, div
combine them pairwise
curl(grad f), div(curl $\vec{F}$), div(grad f)
curl(curl $\vec{F}$), grad(div $\vec{F}$)

Cor. curl(grad f) $= \vec{0}$ and div(curl $\vec{F}$) $= 0$

Pf: curl(grad f) $= \nabla \times (\partial_1 f, \partial_2 f, \partial_3 f)$

$$= \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ \partial_1 f & \partial_2 f & \partial_3 f \end{pmatrix} = (\partial_2\partial_3 f - \partial_3\partial_2 f)\vec{i} + (\partial_3\partial_1 f - \partial_1\partial_3 f)\vec{j} + (\partial_1\partial_2 f - \partial_2\partial_1 f)\vec{k}$$
$$= \vec{0}$$

$$\text{div}(\text{curl } \vec{F}) = \nabla \cdot \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_1 & \partial_2 & \partial_3 \\ f_1 & f_2 & f_3 \end{pmatrix} = (\partial_1, \partial_2, \partial_3)\cdot((\partial_2 f_3 - \partial_3 f_2)\vec{i} + (\partial_3 f_1 - \partial_1 f_3)\vec{j} + (\partial_1 f_2 - \partial_2 f_1)\vec{k})$$
$$= \partial_1\partial_2 f_3 - \partial_1\partial_3 f_2 + \partial_2\partial_3 f_1 - \partial_2\partial_1 f_3 + \partial_3\partial_1 f_2 - \partial_3\partial_2 f_1$$
$$= 0$$

Eg. Let $\vec{F}(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$

$$\text{curl } \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{pmatrix} = [\partial_y(3xy^2z^2) - \partial_z(2xyz^3)]\vec{i} + [\partial_z(y^2z^3) - \partial_x(3xy^2z^2)]\vec{j} + [\partial_x(2xyz^3) - \partial_y(y^2z^3)]\vec{k}$$

$$= (6xyz^2 - 6xyz^2)\vec{i} + (3y^2z^2 - 3y^2z^2)\vec{j} + (2yz^3 - 2yz^3)\vec{k} = \vec{0}$$

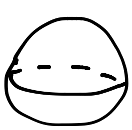
$\nabla f = F$?
$\partial_x f = y^2z^3$, $\partial_y f = 2xyz^3$, $\partial_z f = 3xy^2z^2$
$\Rightarrow f(x,y,z) = xy^2z^3 + h(y,z)$
$\partial_y f = 2xyz^3 + \partial_y h$
$\Rightarrow \partial_y h(y,z) = 0$
$\Rightarrow h(y,z) = g(z)$

$f(x,y,z) = xy^2z^3 + g(z)$
$\partial_z f = 3xy^2z^2 + \partial_z g(z)$
$\partial_z g(z) = 0 \Rightarrow g(z) = C$
$f(x,y,z) = xy^2z^3 + C$

§ 5.5 The Divergence Theorem

Thm 5.34. Suppose R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary ∂R, oriented so that the positive normal points out of R. Suppose also that $\vec{F}$ is a vector field of class C^1 on R, then

$$\iint_{\partial R} \vec{F}\cdot\vec{n}\, dA = \iiint_R \text{div}\vec{F}\, dV \quad (*)$$



Remark: Recall Green's thm: $S \subset \mathbb{R}^2$.

Thm 5.12 $\int_{\partial S} \vec{F}\cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA$

Cor 5.17: $\int_{\partial S} \vec{F}\cdot\vec{n}\,ds = \iint_S \left(\frac{\partial F_1}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\right) dA$

The divergence thm is the 3-dim analogue of Green's thm.

Proof: ① Prove (*) on simple regions

Def R is xy-simple if it has the form

$R = \{(x,y,z) : (x,y) \in W,\ \varphi_1(x,y) \le z \le \varphi_2(x,y)\}$

where W is a regular region in xy-plane and φ_1 & φ_2 are piecewise smooth functions on W.

Similarly, we can define yz-simple:

$R = \{(x,y,z) : (y,z) \in W,\ \psi_1(y,z) \le x \le \psi_2(y,z)\}$

xz-simple:

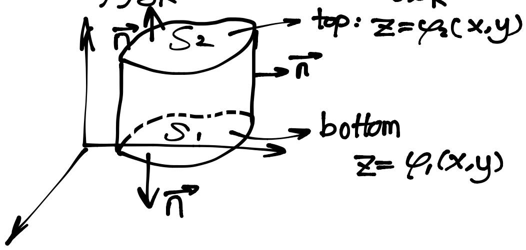
$R = \{(x,y,z) : (x,z) \in W,\ \phi_1(x,z) \le y \le \phi_2(x,z)\}$

We say R is simple if it is xy, xz and yz simple.

Suppose R is simple, Let $\vec{F} = (F_1, F_2, F_3) = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

We will prove the divergence thm for each component of $\vec{F}$. first

Goal: $\iint_{\partial R} F_3\vec{k}\cdot\vec{n}\,dA = \iiint_R \partial_3 F_3\,dV$



$\vec{k} = (0,0,1)$

$\Rightarrow$ on the side : $\vec{n} \perp \vec{k}$

$\Rightarrow \vec{n}\cdot\vec{k} = 0$

On the top: The surface $\vec{G}(x,y) = (x, y, \varphi_2(x,y))$

$dA = |\vec{G_x} \times \vec{G_y}|$, $\vec{G_x} = (1, 0, \partial_x\varphi_2)$

$\vec{G_y} = (0, 1, \partial_y\varphi_2)$

$$\vec{G_x} \times \vec{G_y} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \partial_x\varphi_2 \\ 0 & 1 & \partial_y\varphi_2 \end{pmatrix} = -\partial_x\varphi_2\,\vec{i} - \partial_y\varphi_2\,\vec{j} + \vec{k}$$

$\vec{n} \mathbin{/\!/} \vec{G_x} \times \vec{G_y} \quad \Rightarrow \quad \vec{n}\,dA = \vec{G_x} \times \vec{G_y} = -\partial_x\varphi_2\,\vec{i} - \partial_y\varphi_2\,\vec{j} + \vec{k}$

$\hookrightarrow$ parallel

$$\iint_{S_2} F_3\vec{k}\cdot\vec{n}\,dA = \iint_W F_3\vec{k}\cdot(-\partial_x\varphi_2\,\vec{i} - \partial_y\varphi_2\,\vec{j} + \vec{k})\,dx\,dy$$
$$= \iint_W F_3(x, y, \varphi_2(x,y))\,dx\,dy$$

On the bottom $\vec{G}(x,y) = (x, y, \varphi_1(x,y))$

$\Rightarrow \vec{G_x} \times \vec{G_y} = -\partial_x\varphi_1\,\vec{i} - \partial_y\varphi_1\,\vec{j} + \vec{k}$

$\vec{n}\,dA = -\vec{G_x} \times \vec{G_y}$

$$\iint_{S_1} F_3\vec{k}\cdot\vec{n}\,dA = \iint_W F_3\vec{k}(\partial_x\varphi_1\,\vec{i} + \partial_y\varphi_1\,\vec{j} - \vec{k})\,dx\,dy$$
$$= -\iint_W F_3(x, y, \varphi_1(x,y))\,dx\,dy$$

$$\iint_{\partial R} F_3\,\vec{k}\cdot\vec{n}\,dA = \iint_W F_3(x,y,\varphi_2(x,y)) - F_3(x,y,\varphi_1(x,y))\,dx\,dy$$

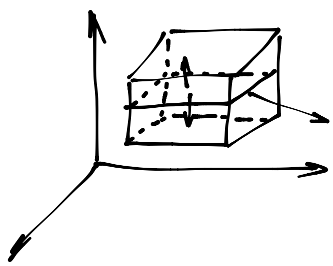
$$= \iint_W \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \frac{\partial}{\partial z} F_3\,dz\,dx\,dy = \iiint_R \frac{\partial}{\partial z} F_3\,dx\,dy\,dz$$

Similarly R is yz-simple

$$\iint_{\partial R} F_1\,\vec{i}\cdot\vec{n}\,dA = \iiint_R \partial_1 F_1\,dV$$

R is xz-simple $\quad \iint_{\partial R} F_2\,\vec{j}\cdot\vec{n}\,dA = \iiint_R \partial_2 F_2\,dV$

② For general regions, we cut it into finitely many simple regions $R_1,\cdots,R_k$

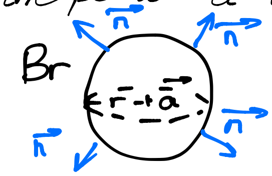


$$\iint_{\partial R} \vec{F}\cdot\vec{n}\,dA = \sum_{i=1}^{k} \iint_{\partial R_i} \vec{F}\cdot\vec{n}\,dA$$

b/c the integrals of $\vec{F}\cdot\vec{n}$ over the portions of boundaries $\partial R_1,\cdots,\partial R_k$ that are not part of ∂R cancel out.

Meaning of div $\vec{F}$

Suppose $\vec{F}$ is a vector field of class C^1 in some open set containing the point $\vec{a}$ when r is small.



$\operatorname{div}\vec{F}(\vec{x}) \simeq \operatorname{div}\vec{F}(\vec{a})$ for all $\vec{x}\in B_r$

$V(B_r) = \frac{4}{3}\pi r^3$

$$\operatorname{div}\vec{F}(\vec{a}) = \frac{1}{V(B_r)}\iiint_{B_r} \operatorname{div}\vec{F}(\vec{a})\,dV \approx \frac{3}{4\pi r^3}\iiint_R \operatorname{div}\vec{F}(\vec{x})\,dV$$

$$\underset{\text{div thm}}{=} \frac{3}{4\pi r^3}\iint_{|\vec{x}-\vec{a}|=r} \vec{F}\cdot\vec{n}\,dA$$

Let $r\to 0$, $\operatorname{div}\vec{F}(\vec{a}) = \lim\limits_{r\to 0}\frac{3}{4\pi r^3}\underbrace{\iint_{|\vec{x}-\vec{a}|=r}\vec{F}\cdot\vec{n}\,dA}_{\text{The flux of }\vec{F}\text{ across }\partial B}$

When $\operatorname{div}\vec{F} > 0$:

There is a net out flow near $\vec{a}$, which means $\vec{F}$ tends to "diverge" from $\vec{a}$.

Since $\iint dA$ is independent of the choice of parameter $\operatorname{div}\vec{F}(\vec{a})$ is independent of the choice of parameter.

Cor 5.37 (Green's Formulas)

Sps R is a regular region in $\mathbb{R}^3$ with piecewise smooth boundary, and f & g are functions of class C^2 on R, then

note: $\nabla^2 = \nabla\cdot\nabla = \Delta$

$$\iint_{\partial R} f\,\nabla g\cdot\vec{n}\,dA = \iiint_R (\nabla f\cdot\nabla g + f\,\nabla^2 g)\,dV$$

$$\iint_{\partial R} (f\nabla g - g\nabla f)\cdot\vec{n}\,dA = \iiint_R (f\,\nabla^2 g - g\,\nabla^2 f)\,dV$$

Proof: $\operatorname{div}(f\vec{G}) = \nabla\cdot(f\cdot\vec{G}) = \nabla f\cdot\vec{G} + f\nabla\cdot\vec{G}$ (5.28)

Let $\vec{G} = \nabla g$

$$\operatorname{div}(f\nabla g) = \nabla f\cdot\nabla g + f\nabla\cdot\nabla g$$
$$= \nabla f\cdot\nabla g + f\cdot\nabla^2 g$$

Consider $\vec{F} = f\nabla g$ in (*)

$$\iint_{\partial R} f\nabla g\cdot\vec{n}\,dA \underset{(*)}{=} \iiint_R \operatorname{div}(f\nabla g)\,dV = \iiint_R \nabla f\cdot\nabla g + f\nabla^2 g\,dV$$

from ① $\quad \iint_{\partial R} g\,\nabla f\cdot\vec{n}\,dA = \iiint_R \nabla g\cdot\nabla f + g\,\nabla^2 f\,dV$

$$\Rightarrow \iint_{\partial R}(f\nabla g - g\nabla f)\cdot\vec{n}\,dA = \iiint_R f\nabla^2 g - g\nabla^2 f\,dA$$

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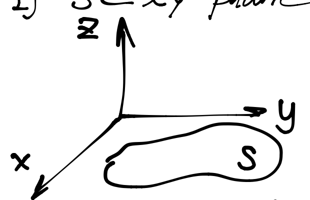
§ 5.7 Stoke's Thm

This is Green's Thm on the curved surface
Orientation walk around ∂S in the positive direction, standing on the positive side of S. (usually out normal vector as positive side) then S is on the left.

Thm 5.82 (Stoke's thm)
Let S and ∂S be as described above, and let $\vec{F}$ be a C^1 vector field defined on some neighbourhood of S in $\mathbb{R}^3$
Then $\int_{\partial S}\vec{F}\cdot d\vec{x} = \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dA$ (**)

Remark: Relation between Green's thm & Stoke's thm

If $S\subset xy$-plane, then $\vec{n} = \vec{k} = (0,0,1)$



Let $F = (F_1, F_2, F_3)$

$$\int_{\partial S}\vec{F}\cdot d\vec{x} = \int_{\partial S} F_1\,dx + F_2\,dy + \cancel{F_3\,dz} \quad (\text{by } dz = 0)$$

$$\operatorname{curl}\vec{F} = \det\Big(\qquad\Big) = (\partial_2F_3 - \partial_3F_2)\vec{i} + (\partial_3F_1 - \partial_1F_3)\vec{j} + (\partial_1F_2 - \partial_2F_1)\vec{k}$$

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dA = \iint_S \partial_1F_2 - \partial_2F_1\,dA$$

Green's thm

From (**) $\quad \int_{\partial S} F_1\,dx + F_2\,dy = \iint_S \partial_1F_2 - \partial_2F_1\,dA$

Proof: parametrization S as $\vec{x} = \vec{G}(u,v)$ where $(u,v)\in W\subset uv$-plane
Idea: pull back the integration over S and ∂S to W and ∂W, then apply Green's thm for uv
We consider the components of $\vec{F}$ separately. Let $\vec{F} = (F, G, H) = F\vec{i} + G\vec{j} + H\vec{k}$
We need to show $\quad \int_{\partial S} F\,dx = \iint_S (\partial_3 F\vec{j} - \partial_2 F\vec{k})\vec{n}\,dA$

$$\vec{n}\,dA = \frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v} \qquad = \iint_S (\partial_3 F \vec{j} - \partial_2 F \vec{k}) \vec{n}\, dA \quad (\star)$$

$$= \iint_S (\partial_3 F \vec{j} - \partial_2 F \vec{k}) \cdot \left(\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v}\right) dA$$

Since $\vec{G} = (x(u,v), y(u,v), z(u,v))$

$\frac{\partial \vec{G}}{\partial u} = (\partial_u x, \partial_u y, \partial_u z)$

$\frac{\partial \vec{G}}{\partial v} = (\partial_v x, \partial_v y, \partial_v z)$

$$\frac{\partial \vec{G}}{\partial u} \times \frac{\partial \vec{G}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_u x & \partial_u y & \partial_u z \\ \partial_v x & \partial_v y & \partial_v z \end{vmatrix}$$

$$= \frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{j} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k}$$

$$(\star) = \iint_W \partial_3 F \frac{\partial(z,x)}{\partial(u,v)} - \partial_2 F \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

On the other hand

$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv$

consider $x(t) = x(u(t), v(t))$

$\frac{dx}{dt} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt}$

$$\int_{\partial S} F\,dx = \int_{\partial W} F \frac{\partial x}{\partial u} du + F \frac{\partial x}{\partial v} dv$$

<u>Green's Thm</u> $\iint_W \frac{\partial}{\partial u}\left(F \frac{\partial x}{\partial v}\right) - \frac{\partial}{\partial v}\left(F\frac{\partial x}{\partial u}\right) du\, dv$

$$= \iint_W \frac{\partial F}{\partial u}\frac{\partial x}{\partial v} + F\frac{\partial^2 x}{\partial u \partial v} - \frac{\partial F}{\partial v}\frac{\partial x}{\partial u} - F\frac{\partial^2 x}{\partial v \partial u}\, du\, dv$$

$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial v}$$

$$\Rightarrow \int_{\partial S} F\,dx = \iint_W \frac{\partial F}{\partial x}\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}\frac{\partial z}{\partial v}$$

$$- \frac{\partial F}{\partial x}\frac{\partial x}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial F}{\partial y}\frac{\partial y}{\partial v}\frac{\partial x}{\partial u} - \frac{\partial F}{\partial z}\frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\, du\, dv$$

$$= \iint_W \frac{\partial F}{\partial y}\left(\frac{\partial y}{\partial u}\frac{\partial x}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial x}{\partial u}\right) + \frac{\partial F}{\partial z}\left(\frac{\partial z}{\partial u}\cdot\frac{\partial x}{\partial v} - \frac{\partial z}{\partial v}\frac{\partial x}{\partial u}\right) du\, dv$$

$$= \iint_W \frac{\partial F}{\partial y}\cdot\frac{\partial(y,x)}{\partial(u,v)} + \frac{\partial F}{\partial z}\cdot\frac{\partial(z,x)}{\partial(u,v)}\, du\, dv$$

$$\Rightarrow \int_{\partial S} F\,dx = \iint_S (\partial_3 F \vec{j} - \partial_2 F \vec{k}) \cdot \vec{n}\, dA$$

Similarly, $\int_{\partial S} G\,dy = \iint_S (\partial_x G \vec{k} - \partial_z G \vec{i}) \vec{n}\, dA$

$$\int_{\partial S} H\,dz = \iint_S (\partial_y H \vec{i} - \partial_x H \vec{j}) \cdot \vec{n}\, dA$$

Eg. $\vec{F}(x,y,z) = \sqrt{x^2+1}\,\vec{i} + x\vec{j} + 2y\vec{k}$

Curve C is the intersection of the surface $z = xy$ and $x^2+y^2=1$

Evaluate $\int_C \vec{F}\cdot d\vec{x} = \iint_{x^2+y^2\le 1} (\text{curl } \vec{F})\cdot\vec{n}\, dA$

$$\text{curl } \vec{F} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ \sqrt{x^2+1} & x & 2y \end{pmatrix} = 2\vec{i} + 0\vec{j} + \vec{k} = 2\vec{i} + \vec{k}$$

Surface $\vec{G}(x,y) = (x, y, xy)$, $x^2+y^2 \le 1$

$\vec{n}\, dA = \vec{G_x} \times \vec{G_y}$ $\qquad \vec{G_x} = (1, 0, y)$, $\vec{G_y} = (0, 1, x)$

$$= \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{pmatrix} = -y\vec{i} - x\vec{j} + \vec{k}$$

$\vec{G_x} \times \vec{G_y} = -\varphi_x \vec{i} - \varphi_y \vec{j} + \vec{k}$

$$\iint_{x^2+y^2\le 1} (2\vec{i} + \vec{k})\cdot(-y\vec{i} - x\vec{j} + \vec{k})\, dxdy$$

$$= \iint_{x^2+y^2\le 1} -2y + 1\, dxdy$$

$$= \iint_{x^2+y^2\le 1} dxdy$$

$$= \pi$$