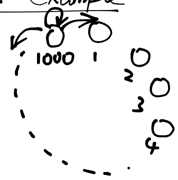
Lecture 1

Markov chain example



1000 lily pads in circle
frog starts at pad 1,000
each min $\frac{1}{3}$ prob stays
$\frac{1}{3}$ jump clockwise 1 pad
$\frac{1}{3}$ jump counter-clockwise 1 pad

$P(\text{at pad\#1 after 1 step}) = \frac{1}{3}$

$P(\text{at pad\#1000 after 1 step}) = \frac{1}{3}$

$P(\text{at pad \# 998 after 2 steps}) = \frac{1}{3} \times \frac{1}{3}$

$P(\text{at pad \# 999 after 2 steps}) = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{2}{9}$

What is $P(\text{frog is at \#428 after 987 steps})$

$\lim_{k \to \infty} P(\text{frog at \# 428 after k steps})$? $= \frac{1}{1000}$

$P(\text{frog will eventually return to \#1,000})$? $= 1$

$P(\text{frog will hit every pad})$? $= 1$

What if we change configuration, numbers, probabilities?

A (discrete time, discrete space, time-homogeneous) Markov chain has 3 ingredients:

→ A state space S (e.g. 1000 lily pads)
  any non-empty finite or countable set.

→ Transition probabilities $\{P_{ij}\}_{i,j \in S}$, where P_{ij} is the prob of jumping to j if you start at i. (1-step)
  (e.g. for frog, $P_{ij} = \begin{cases} \frac{1}{3} & \text{if } i=j \text{ or } |i-j|=1 \text{ or } |i-j| = 999 \\ 0 & \text{o.w.} \end{cases}$)
  where $P_{ij} \geq 0$ & $\sum_{j \in S} P_{ij} = 1$, $i \in S$ (i fixed)

→ initial probabilities $\{\nu_i\}_{i \in S}$
  where ν_i is the probability of starting at i (at time 0)
  (e.g. for frog, $\nu_{1000} = 1$, $\nu_i = 0$ for $i \neq 1000$) where $\nu_i \geq 0$, $\sum_{i \in S} \nu_i = 1$

Let X_n be the Markov chain's state at time n (random variable) (not iid)

In frog example,
$X_0 = 1000$, ie $P(X_0 = 1000) = 1 = \nu_{1000}$
$P(X_0 = 996) = 0 = \nu_{996}$
$P(X_1 = 999) = \frac{1}{3}$, $P(X_1 = 1) = \frac{1}{3}$
$P(X_1 = 2) = 0$, $P(X_2 = 2) = \frac{1}{9}$, etc $P(X_{987} = 428) = ?$

$P(X_{n+1} = j \mid X_n = i) = P_{ij} \quad \forall i,j \in S, \forall n \geq 0$
$P(X_{n+1} = j \mid X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) = P_{ij}$ ⇒ ("Markov Property")

$$P(X_0=i, X_1=j) = P(X_0=i)P(X_1=j \mid X_0=i)$$
$$= \nu_i P_{ij}$$

$P(X_0=i, X_1=j, X_2=k) = \nu_i P_{ij} P_{jk}$

$P(X_0=i_0, X_1=i_1, \cdots, X_n=i_n) = \nu_{i_0} P_{i_0 i_1} \cdot P_{i_1 i_2} \cdots P_{i_{n-1} i_n}$

Defines the prob dist of the $\underbrace{\text{sequence } \{X_n\}}_{\text{"The Markov chain"}} = (X_0, X_1, X_2, \cdots)$ Not independent!

Example: Simple Random Walk

Let $0<p<1$. (e.g. $p=\frac{1}{2}$)

You repeatedly bet \$1 and win with prob p lose with $\overset{\text{prob}}{1-p}$

Let X_n = # dollars won/lost after n bets.

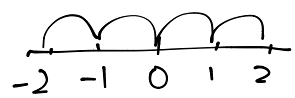
Then $X_0 = 0$

$P(X_1=1) = p$

$P(X_2=2) = p^2$

$P(X_2=0) = 2p(1-p)$

Here $S = \mathbb{Z} = \{\text{all integers}\}$



$$P_{ij} = \begin{cases} 0, & |j-i| \geq 2 \\ 0, & j=i \\ p, & j=i+1 \\ 1-p, & j=i-1 \end{cases}$$

$\nu_0 = 1, \nu_i = 0 \ \forall i \neq 0$

Bernoulli Process:

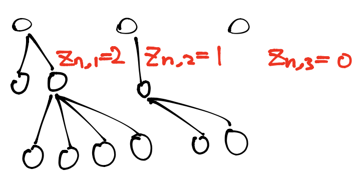
Let $0<p<1$. Repeatedly flip a "p-coin" (ie. prob of heads is p). Then X_n = #heads on first n flips.

Here $S = \{0,1,2,3,\cdots\}$

$X_0 = 0$, i.e. $\nu_0 = 1$, $\nu_i = 0$ if $i \neq 0$

$$P_{ij} = \begin{cases} p, & j=i+1 \\ 1-p, & j=i \\ 0, & \text{o.w.} \end{cases}$$

Example: Branching Process



$X_0 = 3$

$X_1 = 3$

$X_2 = 6$

Let φ be any prob dist on $\{0,1,2,\cdots\}$

X_n = size of the population at time n

Each of the X_n items at time n has a random number of offspring at time $n+1$, which are i.i.d. $\sim \varphi(\cdot)$

That is,

$X_{n+1} = Z_{n,1} + Z_{n,2} + \cdots + Z_{n,X_n}$

where $\{Z_{k,l}\}$ are i.i.d $\sim \varphi(\cdot)$.

Here $S = \{0,1,2,\cdots\}$

What about P_{ij}?

Well, $P_{i0} = (\varphi(0))^i$

$P_{0j} = 0 \ \forall j \neq 0$

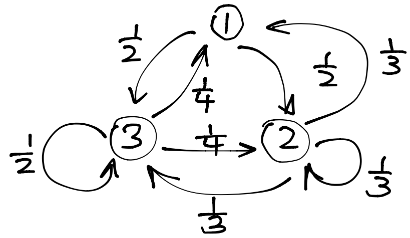
$P_{00} = 1$

$P_{ij} = (\underbrace{\varphi * \varphi * \cdots * \varphi}_{i\text{-fold convolution}})(j)$ Tricky

Is $P(X_n = 0 \text{ for some } n) = 1$?

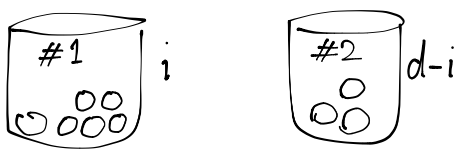
<u>Example</u>: $S = \{1,2,3\}$, $X_0 = 3$.

and $(P_{ij}) = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$



Long run? $\lim_{n\to\infty} P(X_n = 1)$? exist?

Ehrenfest's Urn



d balls in total

At each time, choose one of the d balls uniformly at random and move it to the opposite urn.
Let $X_n = \#$ balls in urn #1 at time n.

Here $S = \{0, 1, \cdots, d\}$

$$P_{ij} = \begin{cases} \frac{d-i}{d} & , j = i+1 \\ \frac{i}{d} & , j = i-1 \\ 0 & , \text{o.w.} \end{cases}$$

Long run?

Simple Queue:

At each time n, one person (if there is one) gets served, and Z_n new people arrive, where $\{Z_n\}$ are iid $\sim \varphi$ "arrival distribution" on $\{0,1,2,\cdots\}$

Let $X_n = \#$ people in the system (i.e. waiting or being served) at time n

$S = \{0,1,2,\cdots\}$ $\quad X_{n+1} = X_n - \min(1, X_n) + Z_n$

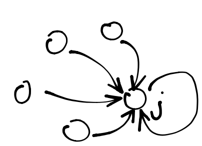
Here ~~$P_{ij} = P(Z_n = j - i + \min(i,1))$~~

$$\begin{aligned} \varphi_{ij} = P(X_{n+1} = j \mid X_n = i) &= P(X_n - \min(X_n, 1) + Z_n = j \mid X_n = i) \\ &= P(i - \min(i,1) + Z_n = j) \\ &= P(Z_n = j - i + \min(i,1)) \\ &= \varphi(j - i + \min(i,1)) \end{aligned}$$

Elementary Computations

Let $\mu_i^{(n)} = P(X_n = i)$ So $\mu_i^{(0)} = P(X_0 = i) = \nu_i$

then $\mu_j^{(1)} = P(X_1 = j) = \sum_{i\in S} P(X_0 = i, X_1 = j) = \sum_{i \in S} \nu_i P_{ij}$



Write $\mu^{(n)} = (\mu_1^{(n)}, \mu_2^{(n)}, \mu_3^{(n)}, \cdots)$ (row vector)

$\nu = (\nu_1, \nu_2, \nu_3, \cdots)$ (row vector)

$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} & \cdots \\ P_{21} & P_{22} & P_{23} & \cdots \\ P_{31} & P_{32} & P_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$ — matrix

Then $\mu^{(1)} = \nu P = (\mu_1^{(1)}, \mu_2^{(1)}, \cdots)$

What about $\mu^{(2)}$?

$$\mu_k^{(2)} = P(X_2=k) = \sum_{i,j\in S} P(X_0=i, X_1=j, X_2=k)$$

$$= \sum_{i,j\in S} \nu_i P_{ij} P_{jk}$$

i.e. $\mu^{(2)} = \nu PP = \nu P^2 = \mu^{(1)} P$

Continuing. $\mu^{(n)} = \nu P^n$

row vector row vector matrix to the n^{th} power

and $\mu^{(n)} = \mu^{(n-1)} P$

n-step transitions: $P_{ij}^{(n)} = P(X_{m+n}=j \mid X_m = i)$

$P_{ij}^{(1)} = P_{ij}$

$P_{ij}^{(2)} = \sum_{k\in S} P_{ik} P_{kj}$

$P_{ij}^{(n)} = ?$ next week