Answer the following questions in the space provided. Give **complete solutions** and justify your answers.

then
f g

1. Fill in the blanks in the statements below. No justification is required.

(a) If $\displaystyle\sum_{k=-\infty}^{\infty} \frac{k^2-5}{k^2+1}(z-z_0)^k$ is a Laurent series for f, then $\text{Res}(f : z_0) =$ __-2__.

(b) The function $f(z) = \dfrac{\sin(\pi z)(z^2-1)^4}{(z^4-1)^{10}}$ has a pole of order __6 5__ at $z_0 = 1$.

(c) The function $f(z) = (e^z - 1)(\cos z - 1)$ has a zero of order __2 3__ at $z_0 = 0$.

(d) If f is analytic, and has a zero of order m at z_0 $(m > 2)$, then $\dfrac{(z-z_0)^2}{f(z)}$ has a pole of order __$(m-2)$__ at z_0.

(e) The series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{-2k}$, is a Laurent series, centered at $z_0 = 0$, for the function $f(z) =$ __$\cos(\frac{1}{z})$__.

(a). $\dfrac{(-1)^2-5}{(-1)^2+1} = \dfrac{1-5}{2} = \dfrac{-4}{2} = -2$

$\sin\pi z$ order 1

(b). $\dfrac{\sin\pi z\,(z+1)^4\cancel{(z-1)^4}}{(z^2+1)^{10}(z+1)^{10}\cancel{(z-1)^{10}}}$

(c). $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \dots$

$f = e^z - 1 = z + \frac{z^2}{2!} + \dots$ zero of order 1

$g = \cos z - 1 = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} - 1$ zero of order 2

$f \cdot g \Rightarrow 1 + 2 = 3$

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2. Let $f(z) = \dfrac{e^z - z - 1}{z \sin z}$.

(a) Find all isolated singularities of f, and classify them as either removable, pole, or essential. For any poles, determine their orders. For any removable singularity, determine what the "missing" value $f(z_0)$ should be.

(a). Solution: Set ~~det~~ denominator $z\sin z = 0$,

2 solutions $z=0$ or $z = k\pi$ where $k \in \mathbb{Z}\setminus\{0\}$ +2

① When $z=0$, $e^z - z - 1 = e^0 - 0 - 1 = 0$

$z\sin z = 0$

we have a "$\frac{0}{0}$" case, use L'Hopital rule. +2

$$\lim_{z\to 0} f(z) = \lim_{z\to 0}\frac{e^z - 1}{\sin z + z\cos z} = \lim_{z\to 0}\frac{e^z}{\cos z + \cos z + \cancel{z\cos z} - z\sin z} = \frac{1}{2} < \infty$$

Hence $\lim_{z\to 0}|f(z)| = \frac{1}{2} < \infty$

We have a removable singularity at $z=0$, the "missing" value is $\frac{1}{2}$.

② When $z = k\pi$, $k \in \mathbb{Z}\setminus\{0\}$.

For $z\sin z$, let it be $p(z) = z\sin z$

$p' = \sin z + z\cos z$

plug in $z = k\pi$. $p'(k\pi) = \sin(k\pi) + k\pi \cdot \cos(k\pi)$ +2

$= k\pi\cos(k\pi)$

$= \pm k\pi \neq 0$

So the order of this singularity is 1.

And for $\lim_{z\to k\pi}\left|\dfrac{e^z - z - 1}{z\sin z}\right| = \infty$ since ~~the~~ $e^z - z - 1$ is fixed but $z\sin z \to 0$.

Hence $z = k\pi$ is a pole of order 1.

(Question 2 continues on the next page.)

'r series

(b) For any pole z_k of f, determine $\mathrm{Res}(f : z_k)$.

(b). Solution: $z_k = k\pi$. with order 1.

$g(z) = \frac{e^z - z - 1}{z}$ $\quad$ $f(z) = \frac{g(z)}{z \sin z}$, $\mathrm{Res}(f: z_k) = C_{1-1} = C_0$

$f(z) = \frac{g(z)}{\sin z}$ $\quad$ where C_0 is the coefficient of constant term in power series form of $g(z)$.

$k=0$,

Rewrite. $g(z) = e^z - z - 1 = \sum_{k=0}^{\infty} \frac{z^k}{k!} - z - 1$

$\frac{\frac{z^0}{0!} - z - 1}{z}$

plug in $k=0$ to e^z, where we can find constant term.

$\frac{z^0}{0!} = \frac{1}{1}$

$= -1$

$\mathrm{Res}(f: z_k) = C_{1-1} = C_0 = -1$

But we also have -1 in $g(z)$

Therefore, $\mathrm{Res}(f: z_k) = 0$.

(c) Find $\int_\gamma f(z)dz$, where γ is the unit circle, oriented positively.

(c). Solution:

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_k \in \gamma} \mathrm{Res}(f: z_k)$$

$\times 2$

$$= 2\pi i \cdot 0$$

$$= 0$$

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3. Let $f(z) = \dfrac{1}{z-1} + \dfrac{1}{z}$

(a) Find a Laurent series for f, centred at $z_0 = 0$, which converges in the region $1 < |z| < \infty$?

(a). Solution: $f(z) = z^{-1} + (-\sum_{k=0}^{\infty} z^k)$

$= \frac{1}{z} - 1 - z - z^2 - \cdots - z^k - \cdots$

expand in powers of $\frac{1}{z}$ to ensure convergence.

+1

(b) Find a Laurent series for f, centred at $z_0 = 1$, which converges in the punctured disk $0 < |z-1| < 1$.

(b). Solution: $f(z) = (z-1)^{-1} + \dfrac{1}{1-(-1)(z-1)}$

$= (z-1)^{-1} + \sum_{k=0}^{\infty} [(-1)(z-1)]^k$ ✓

$= \dfrac{1}{z-1} + 1 + (-1)(z-1) + \cdots$

b/c $1-(-1)(z-1) = 1-(-z+1)$
$= 1+z-1$
$= z$ ✓

+4.

(c) Find Res$(f : 1)$.

(c). Solution. Res $(f: 1) =$ coefficient of $(z-1)^{-1}$ term in problem (b)

$= 1$

since we're ~~al~~ already written it ~~it~~ as a Laurent series.

+2

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4. Find $\int_\gamma \frac{e^z}{2z^4+5z^3-3z^2}dz$, where γ is the unit circle, oriented positively.

Solution: $2z^4+5z^3-3\cancel{z}z^2$

$= z^2(2z^2+5z-3)$

$= z^2(2z-1)(z+3) = 0 = 2(z-0)(z-0)(z-\frac{1}{2})(z+3)$

so $z_1=0$, $z_2=\frac{1}{2}$, $z_3=-3$ +2

Ignore z_3 b/c it's outside γ.

1+ Check for location, z_1 is in γ, z_2 is in γ. ~~z_3 not~~

Since for any value $\in$ 0, $\frac{1}{2}$, or -3), z_0 approaches, e^z ~~is constant~~, is finite

what is f? never defined. but $2z^4+5z^3-3z^2$ is approaching to 0, so $\lim_{z\to z_1,z_2,z_3} \left|\frac{e^z}{2z^4+\cdots}\right| = \infty$

So z_1 is a pole of order 2

z_2 is a pole of order 1 -1

z_3 is a pole of order 1. (but we don't need z_3)

z_2 (order 1).

$\text{Res}(f; \frac{1}{2}) = \frac{P(z)}{Q'(z)} = \frac{e^z}{8z^3+15z^2-6z} = \frac{e^{\frac{1}{2}}}{1+\frac{15}{4}-3} = \frac{4}{7}e^{\frac{1}{2}}$ +1

$f(z) = z^{-2}\cdot\frac{e^z}{2(z-\frac{1}{2})(z+3)}$

So $\text{Res}(f: 0)$ = coefficient of the z^1 term in power series form of $\frac{e^z}{2(z-\frac{1}{2})(z+3)}$

$(2z-1)(z+3)$

we suppose this power series is $a_0+a_1z+a_2z^2+\cdots$

then $(2z^2+5z-3)\cdot(a_0+a_1z+a_2z^2+\cdots) = 1+z+\frac{z^2}{2}+\frac{z^3}{3!}+\frac{z^4}{4!}+\cdots$

$2z^2-z+6z-3$

$2z^2+5z-3$

Observe

$z^0: -3a_0 = 1 \Rightarrow a_0 = -\frac{1}{3}$

$z^1: -3a_1 + a_0\cdot 5 = 1$

$-3\times a_1 - \frac{1}{3}\times 5 = 1$

$-3a_1 = 1+\frac{5}{3} = \frac{8}{3}$

$a_1 = \frac{8}{3}\times(-\frac{1}{3}) = -\frac{8}{9}$ +1

So the coeffiicent of z^1 term is $-\frac{8}{9} = \text{Res}(f: 0)$

Therefore $\int_\gamma \frac{e^z}{2z^4+5z^3-3z^2}dz = 2\pi i \sum_{z\in\gamma}\text{Res}(f: z)$

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$= 2\pi i\left(\frac{4}{7}e^{\frac{1}{2}} - \frac{8}{9}\right)$

$= \frac{8}{7}\pi i e^{\frac{1}{2}} - \frac{16}{9}\pi i$

5. Compute $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{x^4+1}\,dx$ using residues.

Solution: Solve for $x^4+1=0$

$$x^4=-1$$
$$(e^{i\theta})^4 = e^{i\pi}$$
$$\theta = \tfrac{\pi}{4},\ \tfrac{\pi}{4}+\tfrac{\pi}{2},\ \tfrac{\pi}{4}+\pi,\ \tfrac{\pi}{4}+\tfrac{3}{2}\pi \quad \text{(and other solutions } +2k\pi \text{, where } k\in\mathbb{Z})$$

So $x = e^{\frac{\pi}{4}i}$ or $e^{\frac{3}{4}\pi i}$ or $e^{\frac{5}{4}\pi i}$ or $e^{\frac{7}{4}\pi i}$

This is $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,dx = \int_{-\infty}^{\infty}\frac{x^2}{(x-e^{\frac{\pi}{4}i})(x-e^{\frac{3}{4}\pi i})(x-e^{\frac{5}{4}\pi i})(x-e^{\frac{7}{4}\pi i})}\,dx$

since, take any 4 ~~xxx~~ values, x^2 is a constant, but the denominator approaches ~~≠~~ 0, so $\lim_{x\to x_0}|f|=\infty$, the 4 singularities are poles of order 1.

So we can use $\frac{P}{Q}$ form. (and we only need to calculate $e^{\frac{\pi}{4}i}$ and $e^{\frac{3}{4}\pi i}$ b/c they are only in upper complex plane)

Let $P(x)=x^2$

$Q(x)=x^4+1$

$Q'(x)=4x^3$

So $\mathrm{Res}(f; e^{\frac{\pi}{4}i}) = \dfrac{e^{\frac{\pi}{2}i}}{4e^{\frac{3}{4}\pi i}}$ $\cancel{= \dfrac{\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}}{4(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi)+1} = \dfrac{i}{4(-\frac{\sqrt2}{2}+i\frac{\sqrt2}{2})+1} = \dfrac{i}{1-2\sqrt2+2\sqrt2 i}}$

$\mathrm{Res}(f; e^{\frac{3}{4}\pi i}) = \dfrac{e^{\frac{3\pi}{2}i}}{4e^{\frac{9}{4}\pi i}}$ $\cancel{= \dfrac{e^{\frac{3\pi}{2}i}}{4e^{\frac{9}{4}\pi i}+1} = \dfrac{\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}}{4(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi)+1} = \dfrac{-i}{4(\frac{\sqrt2}{2}+i\frac{\sqrt2}{2})+1} = \dfrac{-i}{2\sqrt2+1+2\sqrt2 i}}$

~~Res(f~~

$\cancel{\text{Then } \displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,dx = 2\pi i\sum_{x\in U}\mathrm{Res}(f;x_0)}$

$\cancel{= 2\pi i\left(\dfrac{i}{1-2\sqrt2+2\sqrt2 i}+\dfrac{-i}{1+2\sqrt2+2\sqrt2 i}\right)}$

$\cancel{= 2\pi i\,\dfrac{4\sqrt2 i}{4\sqrt2 i-15} = \dfrac{-8\sqrt2\pi}{4\sqrt2 i-15}}$

$\mathrm{Res}(f; e^{\frac{\pi}{4}i}) = \frac14 e^{(\frac{\pi}{2}i-\frac{3}{4}\pi i)}$

$= \frac14 e^{-\frac14\pi i} = \frac14[\cos(-\frac14\pi)+i\sin(-\frac14\pi)] = \frac14[\frac{\sqrt2}{2}-\frac{\sqrt2}{2}i]$

$\mathrm{Res}(f; e^{\frac{3}{4}\pi i}) = \frac14 e^{(\frac{3\pi}{2}i-\frac{9}{4}\pi i)} = \frac14 e^{\frac{9}{4}\pi i} = \frac14[\frac{-\sqrt2}{2}-\frac{\sqrt2}{2}i]$

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Then $\displaystyle\int_{-\infty}^{\infty}\frac{x^2}{x^4+1}\,dx = 2\pi i\sum_{x\in U}\mathrm{Res}(f;x_0) = 2\pi i\left[\frac{\sqrt2}{8}-\frac{\sqrt2}{8}i-\frac{\sqrt2}{8}-\frac{\sqrt2}{8}i\right] = 2\pi i\left(-\frac{\sqrt2}{4}i\right) = \frac{\sqrt2}{2}\pi$

6. Compute $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4} dx$ using residues.

Solution: $x^2+4 = (x-2i)(x+2i)$

So $x_1 = 2i$, $x_2 = -2i$

use z for complex coordinat x is real. +1

Let $f(x) = \frac{xe^{ix}}{x^2+4} = \frac{x\cos x}{x^2+4} + i\frac{x\sin x}{x^2+4}$

Then $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+4} dx = \operatorname{Im}(2\pi i \sum \operatorname{Res}(f; x_i))$

this is because of the proof of Jordan lemma.

$\frac{x\sin x}{(x+2i)(x-2i)} \Rightarrow \lim_{x\to x_i} \left|\frac{x \sin x}{(x+2i)(x-2i)}\right| = \infty$, so $x_1 = 2i$ and $x_2 = -2i$ are both pole of order 1, and only $x_1 = 2i$ on upperplane of complex plane.

So $\operatorname{Res}(f; 2i) = \frac{P(2i)}{Q'(2i)} = \frac{2i\sin(2i)}{4i}$

$P(x) = \cancel{x\sin x} = ze^z$

$Q(x) = x^2+4$

$Q'(x) = 2x$

$f = \frac{xe^{ix}}{\dots} \Rightarrow \operatorname{Res} = \frac{1}{2}e^{-2} = \frac{1}{2}\cancel{\sin(2i)}$

~~$\operatorname{Res}(f; -2i) = \frac{P(-2i)}{Q'(-2i)} = \frac{-2i\sin(-2i)}{-4i} = \frac{1}{2}\sin(2i)$~~

Then $\sum \operatorname{Res}(f; x_i) = \frac{1}{2}\sin(2i)$

Then $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+4} dx = \operatorname{Im}(2\pi i \cdot \sum \operatorname{Res}(f, x_i))$

$= \operatorname{Im}(2\pi i \cdot \frac{1}{2}\sin(2i))$

$= \operatorname{Im}(\pi i \sin(2i))$

let's calculate $\sin(2i)$

Since $\sin z = \frac{e^z - e^{-z}}{2i}$

$\cos z = \frac{e^z + e^{-z}}{2}$

$\sin(2i) = \frac{e^{2i} - e^{-2i}}{2i}$

$\pi i \sin(2i) = \frac{\pi}{2}(e^{2i} - e^{-2i}) = \frac{\pi}{2}[(\cos(2) + i\sin(2)) - (\cos(-2) + i\sin(-2))]$

Note. $\left(e^{\theta i} = e^{2i},\ \theta = 2\right)$

$= \frac{\pi}{2}[\cos(2) - \cos(-2) + i\sin 2 - i\sin(-2)]$

$= \frac{\pi}{2}[0 + 2i\sin(2)]$

$= \pi i \sin(2)$

Therefore $\operatorname{Im}(\pi i \sin(2)) = \pi\sin(2)$

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