June 11th

$x^2+y^2=1$

$F(x,y)=1-x^2-y^2$

When $y=f(x)$?
$y=\sqrt{1-x^2}$
$y=-\sqrt{1-x^2}$ > solved small regions

General Question : $F(\underbrace{x_1,\cdots,x_n}_{\text{ind.}},\underbrace{y}_{\text{dep.}})=0$

can I find $y=g(x_1,\cdots,x_n)$?
Answer : Implicit function Thm

§ 2.5

Ex : $F(x,y)=0 \Rightarrow y=g(x)=x$
Trivially true

Ex: $F(x,y)=1-x^2-y^2 \Rightarrow$ 2 solutions : $y=\sqrt{1-x^2}$, $y=-\sqrt{1-x^2}$

Ex : $F(x,y)=x-y-y^5=0$
$x=x(y)=y+y^5$
$y=y(x)$ cannot solve
Qualitatively, $y+y^5$ strictly increasing, range is $(-\infty,\infty) \Rightarrow \forall x$ only one y s.t. $x=y+y^5 \Rightarrow$ there is a sol'n, but can't algebraically solve for it.

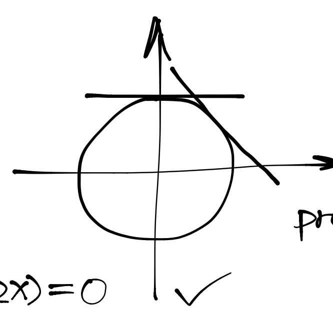
Let us assume, $F(x_1,\cdots,x_n,y)=0$, F diff, can find $y=g(x_1,\cdots,x_n)$

$F(x_1,\cdots,x_n,g(x_1,\cdots,x_n))=0$

$0=\frac{\partial F}{\partial x_j}=\partial_j F+\partial_{n+1}\frac{\partial g}{\partial x_j} \Rightarrow \frac{\partial g}{\partial x_j}=\frac{-\partial_j F}{\partial_{n+1}F}$

Ex: $F(x,y)=x^2+y^2-1=0$

$2x+2y\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$



problem spot $\frac{x}{0}$

$y=\sqrt{1-x^2} \Rightarrow 2x+2\sqrt{1-x^2}\,\frac{1}{2}(1-x^2)^{-1/2}(-2x)=0$ ✓
Same for $y=-\sqrt{1-x^2}$

$\varphi(x_1,\cdots,x_n,y)$ have a constraint $F(x_1,\cdots,x_n,y)=0$
Assume I have $y=g(x_1,\cdots,x_n)$ sub in get

$$w=\varphi(x_1,\cdots,x_n,g(x_1,\cdots,x_n))$$
$$\frac{\partial w}{\partial x_j}=\partial_j\varphi+\partial_{n+1}\varphi\frac{\partial g}{\partial x_j}=\partial_j\varphi+\partial_{n+1}\varphi\left(\frac{-\partial_j F}{\partial_{n+1}F}\right)$$

- For $F(x_1, \cdots, x_n, y) = 0$ $\boxed{\text{no ambiguity}}$

$F(x,y,z)=0$

Ex: $w = xy^2 z$ constraint $x+y+z=0$

First, x, y independent, $w = xy^2(-x-y) = -x^2y^2 - xy^3$ as z is dependent $z=-x-y$

$$\left.\frac{\partial w}{\partial x}\right|_y = -2xy^2 - y^3$$

Second, x, z are ind. $\Rightarrow y$ is dependent, $y=-x-z$

$w = x(x+z)^2 z$ ← only of z ??? why

$$\left.\frac{\partial w}{\partial x}\right|_z = \underline{\qquad\qquad} \neq \left.\frac{\partial w}{\partial x}\right|_y$$

$F(\underbrace{x,y}_{\text{ind.}}, \underbrace{u, v}_{\text{dep.}})=0$ $\quad G(x,y,u,v)=0$

$u = g_1(x,y)$

$v = g_2(x,y)$

Theoristically, want some # of constraints as dep. variables.

$$0 = \frac{\partial F}{\partial x} = \partial_x F + \partial_u F \frac{\partial u}{\partial x} + \partial_v F \frac{\partial v}{\partial x}$$

$$0 = \frac{\partial G}{\partial x} = \partial_x G + \partial_u G \frac{\partial u}{\partial x} + \partial_v G \frac{\partial v}{\partial x}$$

$$A\vec{x} = \vec{b},\ A = \begin{pmatrix} \partial_u F & \partial_v F \\ \partial_u G & \partial_v G \end{pmatrix},\ \vec{x} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \end{pmatrix},\ \vec{b} = \begin{pmatrix} \partial_x F \\ \partial_x G \end{pmatrix}$$

!!! Trefor's wrong here

Via Cramer's Rule (pg. 76 on txtbk)

$$\frac{\partial u}{\partial x} = \frac{\det\begin{pmatrix} \partial_x F & \partial_v F \\ \partial_x G & \partial_v G \end{pmatrix}}{\det A}, \quad \frac{\partial v}{\partial x} = \frac{\det\begin{pmatrix} \partial_u F & \partial_x F \\ \partial_u G & \partial_x G \end{pmatrix}}{\det A}$$

- generalizes to n-dimensions with n-dim Cramer's Rule.
- Of course, do $\frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$

Ex: $xyz = 6$, $x+y+z=0$ $\qquad$ say $(x,y,z) = (1,2,3)$

How does y, z change as x changes?

$\frac{dy}{dx} = y'$, $\frac{dz}{dx} = z'$

$0 = yz + xzy' + xyz'$ & $1 + y' + z' = 0$ $\qquad$ This example is wrong.

$0 = 6 + 3y' + 2z'$, $y' = -1 - z'$

So $z' = 3$, $y' = -4$

$dy = y'dx = -0.4$, $dz = z'dx = 0.3$ $\qquad$ as $dx = 0.1$

Of course (1,2,3) doesn't satisfy $x+y+z=0$
so Imagine other (a,b,c) ...

§ 2.6 Higher-Order Partial Derivatives

f is diff on $S\subset\mathbb{R}^n$, $\frac{\partial f}{\partial x}: S\to\mathbb{R}$ Ask $\frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$

Equivalent notation: $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $\partial x_i \partial_j f$, $\partial_i\partial_j f$, f_{ji}, $f_{x_j x_i}$

done like composition   read like English

Def: A fun is C^k on I if $\partial x_{i_1}\cdots\partial x_{i_k} f$ exists and is continuous function, i.e. all $(i_1,\cdots,i_k)$, $i_j\in\{1,\cdots,n\}$

Ex: $f(x,y)=e^{x+y^2}$
$f_x=e^{x+y^2}$ $f_y=2ye^{x+y^2}$

$f_{yx}=\frac{\partial}{\partial x}f_y=2ye^{x+y^2}$

$f_{xy}=\frac{\partial}{\partial y}f_x=2ye^{x+y^2}$ $\boxed{f_{xy}=f_{yx}}$ in this case, so no ambiguity.

But is this always true?
Yes, but under some certain circumstances:
Thm:
Let f be defined $S\subset\mathbb{R}^n$, if $\partial_i f$, $\partial_j f$, $\partial_i\partial_j f$, $\partial_j\partial_i f$ all exist and $\partial_i\partial_j f$ and $\partial_j\partial_i f$ are continuous at $\vec{a}$, then $\partial_i\partial_j f(\vec{a})=\partial_j\partial_i f(\vec{a})$, $i,j\in\{1,2,\cdots,n\}$.

"difference of 2 differences"

Proof: Just do $n=2$. $\vec{x}=(x,y)$, $\vec{a}=(a,b)$
Set $D=[f(a+h,b+h)-f(a+h,b)]-[f(a,b+h)-f(a,b)]$
$=[f(a+h,b+h)-f(a,b+h)]-[f(a+h,b)-f(a,b)]$

Set $\boxed{\begin{array}{l}\varphi(t)=f(a+h,b+t)-f(a,b+t)\\ \psi(t)=f(a+t,b+h)-f(a+t,b)\end{array}}$

then $D=\varphi(h)-\varphi(0)=\psi(h)-\psi(0)$
By MVT, $\exists v\in(0,h)$. $D=\varphi(h)-\varphi(0)=\varphi'(v)\,h$
$=[\partial_y f(a+h,b+v)-\partial_y f(a,b+v)]h$
Apply MVT here again $\exists u\in(0,h)$ $=\partial_x\partial_y f(a+u,b+v)h^2$ ~~where $u\in(0,h)$~~

Likewise, $D=\psi'(\tilde{u})h=[\partial_x f(a+\tilde{u},b+h)-\partial_x f(a+\tilde{u},b)]h$
$\exists\tilde{u},\tilde{v}\in(0,h)$ $=\partial_y\partial_x f(a+\tilde{u},b+\tilde{v})\,h^2$

MVT twice

as $u,v,\tilde{u},\tilde{v}<h$, they $\to 0$ as $h\to 0$, by continuity, $\partial_y\partial_x f(a,b)=\partial_x\partial_y f(a,b)$. ■

Corollary: If f is of class C^k on S, then $\partial_i\partial_j f = \partial_j\partial_i f \ \forall\, i,j$.
– generalizes $\partial_{i_1}\cdots\partial_{i_k} f = \partial_{j_1}\cdots\partial_{j_k} f$ if C^k

reordering

Multi-index notation

$\alpha = (\alpha_1,\dots,\alpha_n)$
$\uparrow$
index, $\alpha_i \in \mathbb{Z}^+$

$|\alpha| := \alpha_1+\alpha_2+\cdots+\alpha_n$
$\alpha! := \alpha_1!\,\alpha_2!\cdots\alpha_n!$
$\vec{x}^{\alpha} := x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$
$\partial^{\alpha} f = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n} f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}}$

Ex:
$f = xy^4$, $\quad \partial^{(1,3)} f = \partial_1\partial_2^3 f = \partial_1(4\cdot3\cdot2\,xy) = 4\cdot3\cdot2y = 24y$

Binomial Theorem: $(x_1+x_2)^k = \sum_{j=0}^{k} \underbrace{\dfrac{k!}{j!(k-j)!}}_{\binom{k}{j}} x_1^{k-j}x_2^{j}$ Proved by induction

Trefor's wrong here

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$(x+y)^4 = x^4+4xy^3+6x^2y^2+4x^3y+y^4$

Multinomial Thm: For any $\vec{x}=(x_1,\dots,x_n)\in\mathbb{R}^n$

$$(x_1+x_2+\cdots+x_n)^k = \sum_{|\alpha|=k}\frac{k!}{\alpha!}\vec{x}^{\alpha}$$

Basis $(x_1+x_2)^k$ use $\alpha=(j,k-j)$
$= \sum_{\alpha_1+\alpha_2} \dfrac{k!}{\alpha!}\vec{x}^{\alpha}$

Sps true for $n<N$
$(x_1+\cdots+x_N)^k = (\underbrace{x_1+\cdots+x_{N-1}}_{\text{as a whole}} + x_N)^k$; understand via assumption

$$= \sum_{i+j=k}\frac{k!}{i!\,j!}(x_1+\cdots+x_{N-1})^i x_N^j$$

$$= \sum_{i+j=k}\frac{k!}{i!\,j!}\sum_{|\beta|=i}\frac{i!}{\beta!}\tilde{\vec{x}}^{\beta}x_N^j$$

$\tilde{\vec{x}} = (x_1,\dots,x_{N-1})$

$\vec{x} = (x_1,\dots,x_N)$

$\alpha = (\beta_1,\dots,\beta_{N-1})$ s.t. $\beta!\,j! = \alpha!$ and $\tilde{\vec{x}}^{\beta}x_N^j = \vec{x}^{\alpha}$
$|\alpha| = |\beta|+j = i+j = k$ $= \sum_{|\alpha|=k}\dfrac{k!}{\alpha!}\vec{x}^{\alpha}$ ∎

§ 2.7 Taylor's Theorem

<u>Diff</u> : $f(a+h) = f(a) + \underbrace{mh}_{\text{linear}} + \underbrace{E(h)}_{\text{error}}$

can we improve on our error?

$$f(a+h) = \underbrace{f(a) + f'(a)h + \frac{f''(a)h^2}{2!} + \sum_{j=3}^{k} \frac{f^{(j)}(a)h^j}{j!}}_{k^{th}\text{ order linear approx}} + \underbrace{R_{a,k}(h)}_{\text{remainder}}$$

will show $R_{a,k}(h)$ is a very good approximation

$$\frac{R_{a,k}(h)}{h} \longrightarrow 0 \quad \text{as} \quad h \to 0$$

$$\boxed{f(a+h) - P_{a,k}(h) = R_{a,k}(h) \quad \text{, where} \quad P_{a,k}(h) = \sum_{j=0}^{k} \frac{f^{(j)}(a)h^j}{j!}}$$

Ex: $f(x) = \sin x$

$f(0) = 0$

$f'(0) = \cos 0 = 1$

$f''(0) = -\sin 0 = 0$

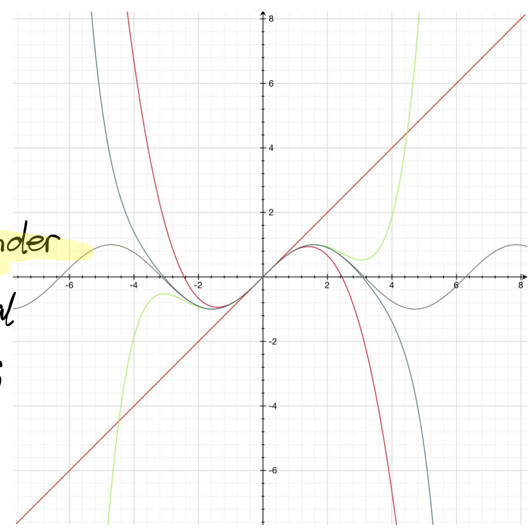
$f'''(0) = -\cos 0 = -1$

$f^{(4)}(0) = \sin 0 = 0$

$f^{(2j)}(0) = 0$

$f^{(2j+1)}(0) = (-1)^j$

$$\sin(h) = h - \frac{h^3}{3!} + \frac{h^5}{5!} - \frac{h^7}{7!}$$



Tayler Thm with Lagrange's Remainder

Sps f is $k+1$ times <u>diff</u> on an interval $I \subset \mathbb{R}$, and $a \in I$. For each $h \in \mathbb{R}$ such that $a+h \in I$ there is a point c between 0 and h such that

$$R_{a,k}(h) = f^{(k+1)}(a+c)\frac{h^{k+1}}{(k+1)!}$$

Change to Class C^{k+1} on I

$$R_{a,k}(h) = \frac{h^{k+1}}{(k-1)!}\int_0^1 (1-t)^{k+1} f^{(k)}(a+th)\,dt$$

Pretty sure we have a mistake here since it matches not a single theorem on textbook (pg 85-87)

Corollary: have conditions, $f^{(k+1)}(x) \leq M$

$$|R_{a,k}(h)| \leq M\frac{|h|^{k+1}}{(k+1)!} \Rightarrow \frac{|R_{a,k}(h)|}{|h|^k} \leq \frac{M|h|}{(k+1)!} \Rightarrow \frac{|R_{a,k}(h)|}{|h|} \to 0 \text{ as } h \to 0$$

- i.e. our remainder is very good
- $k=1 \Leftrightarrow$ differentiability
- Extends MVT
- It is based on an extension of Rolle's

$\boxed{\begin{array}{l}\text{Generalized Rolle's}: g \text{ be } k+1 \text{ diff on } I=[a,b] \\ g(a)=g(b)\ \&\ g^{(j)}(a)=0,\ \forall j \in \{1,\cdots,k\} \\ \text{then } \exists c \in (a,b),\ g^{(k+1)}(c)=0\end{array}}$

Proof: By ordinary Rolle's, $\exists c \in (a,b)$. s.t. $g'(c)=0$

So $g'(a)=g'(c)=0$, diff on $[a,c]$ satisfies Rolles $\Rightarrow \exists c_2 \in (a,c_1)$ $g''(c_2)=0$

Inductively, $\exists c_j \in (a, c_{j-1})$, $g^{(j)}(c_j)=0$

$1 \leq j \leq k+1$ our theorem is the $c := c_{k+1}$ case.