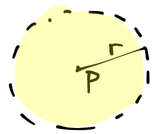
Lecture 3

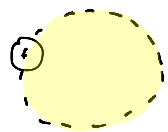
Recall:

Open disks: $D_r(p) = \{ z \in \mathbb{C} \mid |z-p| < r \}$



Open sets: A set is **open** if for every point p there is a small open disk centered at p that is completely contained in S.

Boundary: A pt p is on the boundary of S if every disk centered at p contains pts inside S and pts not in S.

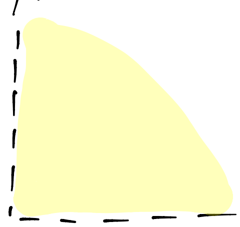


Thm: A set is open if and only if it doesn't contain any of its boundary pts.

Closed sets: A set is **closed** if it contains all its boundary pts

Ex: $\{ z = re^{i\theta} \mid r > 0,\ 0 < \theta < \frac{\pi}{2} \}$

= 1st quadrant

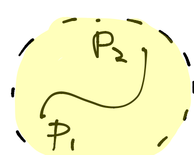


boundary: ∂S = pos x-axis & pos y-axis

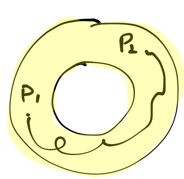
S is open.

Def'n: A set S is **connected** if any two pts in S can be joined by a cts path that stays in S.
(i.e. S is connected if it is not made up of 'separate' parts/pieces.)

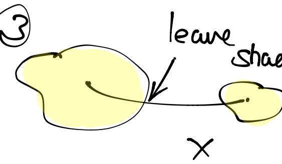
Ex: (1) Disks are connected.



(2) Annuli:



(3)



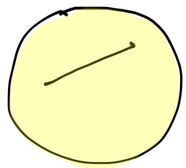
leave shaded region

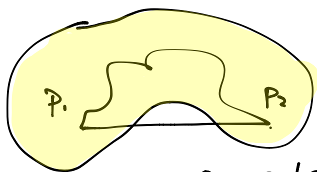
NOT CONNECTED

Def'n: A set D is called a domain if it's both open and connected.

Def'n: A set S is convex if for any 2 pts $p_1, p_2 \in S$, the line segment joining P_1 & P_2 is contained in S.

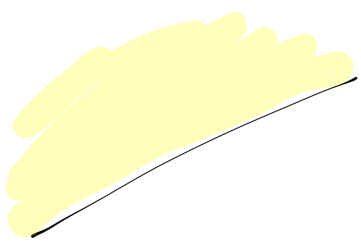
Ex: Disks are convex





connected but not convex

Half-planes:



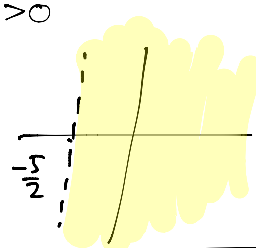
In complex form, a half-plane is given by an inequality

$Re(az+b)>0$, $Re(az+b)<0$ } open half-planes

$Re(az+b)\geq 0$, $Re(az+b)\leq 0$ } closed half-planes

Ex: Sketch the half plane $Re(2z+5)>0$

$$Re(2x+2iy+5)>0$$
$$2x+5>0$$
$$x>-\frac{5}{2}$$



FUNCTIONS

Let $D \subseteq \mathbb{C}$, A function $f: D \to \mathbb{C}$ is a rule that assigns to every element $z \in D$ a unique element $f(z) \in \mathbb{C}$. We call D the domain of f.
The range of f is the set $\{w \in \mathbb{C} \mid w = f(z) \text{ for some } z \in D\}$

Ex: ① $f(z) = z^3 + iz^2 + 2z + 3 + i$

$f: \mathbb{C} \to \mathbb{C}$

② $f(z) = \dfrac{z^3-1}{z+2}$ domain $= \{z \mid z \neq -2\}$

$f: \mathbb{C} \setminus \{-2\} \to \mathbb{C}$

③ $f(z) = \frac{z+2}{z^3-1}$

$z^3 - 1 \neq 0$

$z^3 = 1 \Rightarrow e^{i2\pi}$ or $e^{i\frac{2\pi}{3}}$ or $e^{i\frac{4\pi}{3}} = z$

so $f: \mathbb{C} \setminus \{e^{i2\pi}, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\} \longrightarrow \mathbb{C}$

Recall: We can identify $\mathbb{C}$ with $\mathbb{R}^2$. $z = x+iy \longrightarrow (x,y)$

So any function $f: \mathbb{C} \to \mathbb{C}$ can be viewed as a function $\mathbb{R}^2 \to \mathbb{R}^2$
$f(x,y) = (u(x,y), v(x,y))$

We can write $f(z) = u(x,y) + iv(x,y)$.

Ex: $f(z) = iz + (2+i)$. Find u & v s.t. $f(z) = (u(x,y), v(x,y))$

$f(z) = i(x+iy) + 2 + i = ix - y + 2 + i = (2-y) + i(x+1)$

so $u(x,y) = 2-y$
$v(x,y) = x+1$

Ex ②: $f(z) = z^2$. Find u, v

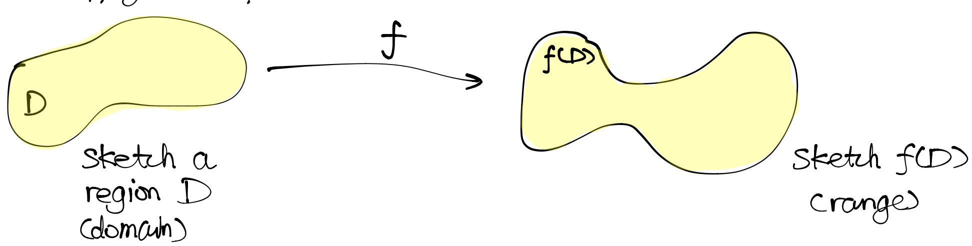
$f(x+iy) = (x+iy)^2 = \underbrace{x^2-y^2}_{u} + \underbrace{2xy}_{v}i$

Visualizing Functions

Graphing $f: D \to \mathbb{C}$ is hard. Why?

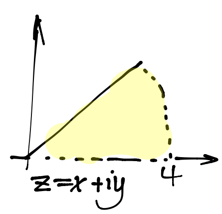
$\text{graph}(f) \subseteq \mathbb{C} \times \mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2 \leftarrow$ 4-dim

Instead, we sometimes function as follows: We study what happens to a specific region when we apply the map.



Ex: Consider $f(z) = z^2$

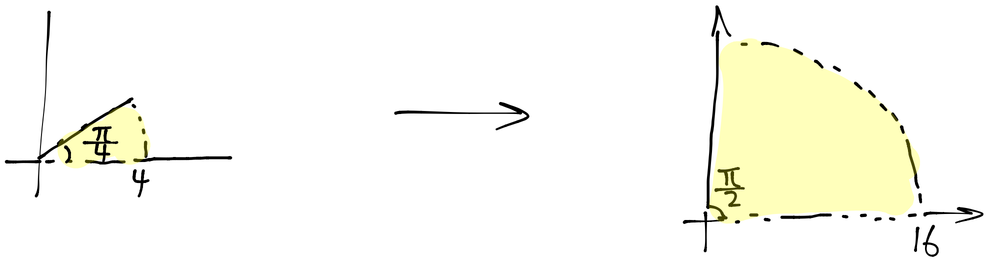
$D = \{z = re^{i\theta} \mid 0 < r < 4,\ 0 < \theta \leq \frac{\pi}{4}\}$



$f(D) = \{w \in \mathbb{C} \mid w = z^2,\ z \in D\}$

$z^2 = (re^{i\theta})^2 = r^2 e^{i2\theta} = w$

$f(D) = \{w \in Re^{i\phi} \mid 0 < R < 16,\ 0 < \phi \leq \frac{\pi}{2}\}$

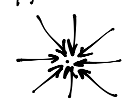


Limits & continuity

Def'n: We say $\lim_{z\to p} f(z)=L$ if $f(z)$ gets closer & closer to L as z gets closer & closer to p.

Ex: $\lim_{z\to 1} \frac{z^2-1}{z-1} = \lim_{z\to 1} z+1 = 2$

~~infinitely~~ many ways to approach a point



But we can still disprove a limit.

Roughly speaking, limits work like they did before.

$\lim_{z\to p} f(z)=L$, $\lim_{z\to p} g(z)=M$

then ① $\lim_{z\to p}(f\pm g)(z) = L\pm M$

② $\lim_{z\to p}(f\cdot g)(z) = L\cdot M$

③ $\lim_{z\to p}(f/g)(z) = L/M$ provided $M\neq 0$

④ $\lim_{z\to p}(\lambda f)(z) = \lambda L$ $(\lambda\in\mathbb{C})$

Def'n: We say f is cts at p if $\lim_{z\to p} f(z)=f(p)$

Ex: ① $f(z)=z^2$ is cts everywhere

② $f(z)=|z|^2+iz$ is cts everywhere

③ $f(z)=\frac{1}{z-1}$ is cts wherever it's defined.

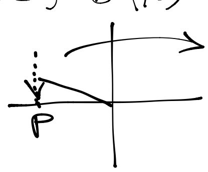
Note: If f,g are cts then $f\pm g$, $f\cdot g$, f/g are cts whenever they are defined.

Recall: We can write $f=u+iv$ where $u,v:\mathbb{R}^2\to\mathbb{R}$

Proposition: $f=u+iv$ is cts if and only if u,v are cts.

Note: Polynomials, rational functions are cts wherever they are defined.

Ex: Define $f:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ by $f(z)=\mathrm{Arg}(z)$, $\mathrm{Arg}\, z\in[-\pi,\pi)$



approaching this way, $\lim_{z\to p}(z)=\pi$

but actually $f(p)=-\pi$!

f is continuous except for z on negative real axis.

$\lim_{z\to p} f(z)$ DIVE!

$\Rightarrow f$ is not cts at pts on neg. real axis

Sequences & Series

A sequence is an infinite list of complex numbers $\{a_1, a_2, a_3, \cdots\}$

We often define a sequence by writing a rule for the n^{th} term.

Ex: ① $z_n = \frac{1+i}{n}$ $\quad \{1+i, \frac{1}{2}+\frac{1}{2}i, \frac{1}{3}+\frac{1}{3}i, \cdots\}$ $\longrightarrow 0$

② $z_n = e^{i\frac{\pi}{n}}$ $\quad \{-1, i, \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i, \cdots\}$ $\longrightarrow 1$

③ $z_n = i^n$ $\quad \{i, -1, -i, 1, \cdots\}$ $\longrightarrow$ DNE

④ $z_n = \frac{n+i}{n}$ $\quad \{1+i, 1+\frac{i}{2}, 1+\frac{i}{3}, \cdots\}$ $\longrightarrow 1$

Defn: We say $z_n \to L$ ($\lim_{n\to\infty} z_n = L$) if z_n gets closer & closer to L as n gets larger & larger.

Properties if $z_n \to L$, $w_n \to M$

① $z_n \pm w_n \to L \pm M$

② $z_n \cdot w_n \to LM$

③ $\frac{z_n}{w_n} \to \frac{L}{M}$, $M \neq 0$

④ $\lambda z_n \to \lambda L$

Write $z_n = x_n + iy_n$, $L = x+iy$

Then $z_n \to L$ if and only if $x_n \to x$, $y_n \to y$

Allows us to use what we know about real sequences (conv. tests, limit thm sets).