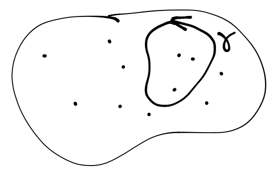
Lecture 16

Residue Theorem & Applications

THM: If f is analytic in D (simply connected) except for poles $z_1, z_2, \cdots, z_N$, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \operatorname{Res}(f : z_k)$$

where γ is a simple closed curve pos. oriented which doesn't pass through any of the poles.



Application: compute real improper integrals

USEFUL FACTS:

① If $p(z)$ is a poly of deg n, and $p(z_0)=0$, then $p(z)=(z-z_0)q(z)$ where q has degree $n-1$.

② $p(z)=0$ has at most n solutions ($n = \deg p$)

③ If $|z|=R$ with R large, then $\frac{1}{2}|a_n|R^n \le |p(z)| \le 2|a_n|R^n$

where $p(z)=a_n z^n + a_{n-1}z^{n-1}+\cdots+a_1 z + a_0$

Proof of ③

$$\lim_{|z|\to\infty}\left|\frac{p(z)}{z^n}\right| = \lim_{|z|\to\infty}\frac{|a_n z^n + a_{n-1}z^{n-1}+\cdots+a_1 z+a_0|}{|z^n|}$$

$$=\lim_{|z|\to\infty}\frac{|a_n z^n|}{|z^n|} = |a_n|$$

$\Rightarrow$ as $|z|\to\infty$, $\left|\frac{p(z)}{z^n}\right| \to |a_n|$

If $|z|=R$ is large enough, $\frac{1}{2}|a_n| \le \left|\frac{p(z)}{z^n}\right| \le 2|a_n|$

$\Rightarrow$ $\frac{1}{2}|a_n||z^n| \le |p(z)| \le 2|a_n|\cdot|z^n|$

$|z|=R$ gives $\frac{1}{2}|a_n|\cdot R^n \le |p(z)| \le 2|a_n|\cdot R^n$

Consider the improper integral

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\,dx \quad \text{where } \deg Q \ge \deg P + 2$$

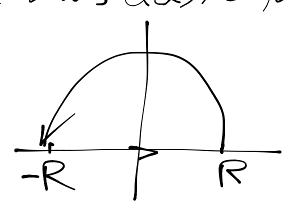
Compare $\frac{P(x)}{Q(x)}$ with $\frac{1}{x^p}$, $p = \deg Q - \deg P$ to see that such an integral should converge. We require that, $Q(x)\neq 0$ for any $x\in\mathbb{R}$.

How to compute this?

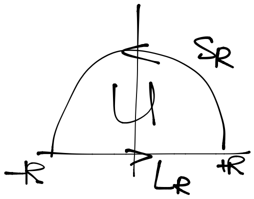
① Consider the function

① Consider the function $f(z)=\frac{P(z)}{Q(z)}$

For $z=x$, real, we get $f(x)=P(x)/Q(x)$



② Consider the curve segment $\gamma_R = L_R \cup S_R$ #semicircle & line



③ Use residues to compute $\int_{\gamma_R} f(z)\,dz$ for R large

④ Let $R \to \infty$ & show $\int_{S_R} f(z)\,dz \to 0$

& $\int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx \longrightarrow \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i \cdot \sum_{z_k \in U} \text{Res}(f : z_k) \quad \text{where } U = \text{upper half plane}}$$

Why?

On S_R: $|z| = R \to z = Re^{it}$, $t \in [0, \pi]$

$dz = Rie^{it}$

$\left| \frac{P(z)}{Q(z)} \right| \le \frac{2|a_n| \cdot R^n}{\frac{1}{2}|b_m| \cdot R^m}$

$P = a_n z^n + \cdots + a_1 z + a_0$

$Q = b_m z^m + \cdots + b_1 z + b_0$

$$\left| \int_{S_R} f(z)\,dz \right| \le \int_{S_R} |f(z)|\,|dz| \le \text{length}(S_R) \cdot \boxed{\max |f(z)|}$$

$$= \frac{\pi \cdot R \cdot 4|a_n|}{|b_m| \cdot R^{m-n}} \longrightarrow 0 \text{ as } R \to \infty$$

since $m - n \ge 2$

On L_R: $z = x$, $-R \le x \le R$

$dz = dx$

$$\int_{L_R} f(z)\,dz = \int_{-R}^{R} \frac{P(x)}{Q(x)}\,dx$$

Put this together:

$$\int_{\gamma_R} f(z)\,dz = \int_{L_R} f(z)\,dz + \int_{S_R} f(z)\,dz = 2\pi i \sum_{z_k \text{ inside } S_R} \text{Res}(f : z_k)$$

$$= 2\pi i \sum_{z_k \in U} \text{Res}(f : z_k)$$ Take R sufficiently large

$$2\pi i \sum_{z_k \in U} \text{Res}(f : z_k) = \int_{L_R} f(z)\,dz + \int_{S_R} f(z)\,dz$$

$R \to \infty$

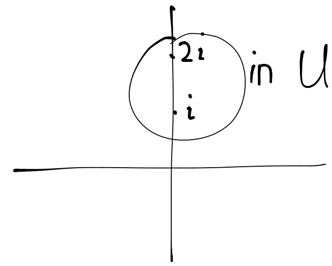
$$2\pi i \sum_{z_k \in U} \text{Res}(f : z_k) = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx + 0$$

Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+4)}\,dx$

$f(z) = \dfrac{z}{(z^2+1)(z^2+4)}$ it has poles of order 1.

at $z = \pm i, \pm 2i$

Only $z = i, 2i$ are in U.



$\text{Res}(f: i) = \dfrac{i}{4i^3+10i} = \dfrac{1}{6}$

$Q = z^4+5z^2+4$

$Q' = 4z^3+10z$

$\text{Res}(f: 2i) = \dfrac{2i}{-48i+10\cdot 2i} = \dfrac{2i}{-32i+20i} = \dfrac{2i}{-12i} = \dfrac{-1}{6}$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx = 2\pi i\,\left(\text{Res}(f:i)+\text{Res}(f:2i)\right)$$
$$= 2\pi i\left(\tfrac{1}{6}+\left(-\tfrac{1}{6}\right)\right)$$
$$= 2\pi i\cdot 0$$
$$= 0$$

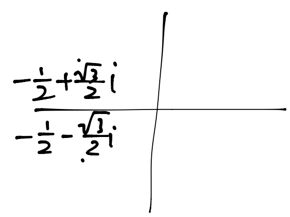
(This agrees the fact that $f(x)$ is odd)

integral $\int_{-\infty}^{\infty} f(x) = 0$

Ex2: Find $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx$

$f(z) = \dfrac{1}{z^2+z+1}$ has poles of order 1 at $z = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

only $z_0 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ is in U



$$\text{Res}\left(f: -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right) = \frac{1}{2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)+1}$$
$$= \frac{1}{-1+\sqrt{3}i+1}$$
$$= \frac{1}{\sqrt{3}i}$$

$$\int_{-\infty}^{\infty} \frac{1}{x^2+x+1}\,dx = 2\pi i\left(\text{Res}\left(f: -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i\right)\right) = 2\pi i\,\frac{1}{\sqrt{3}i} = \frac{2\pi}{\sqrt{3}}$$

TRIG FUNCTIONS

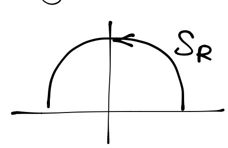
How to calculate $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\cos x\,dx$, or $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\sin x\,dx$?

Idea: same as before, but the "$f(z)$" is more subtle.

We require: Q has no zeros on $\mathbb{R}$ & $\deg Q \geq \deg P + 1$.

USEFUL FACT: JORDAN'S LEMMA

$\left|\int_{S_R} e^{iz}\,dz\right| < \pi$, S_R = semicircle of radius R.

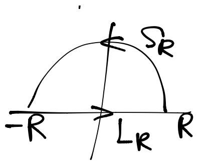


Ex: $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$

$f(z) = \dfrac{e^{iz}}{z^2+1}$ $\leftarrow$ why?

On L_R, $z = x$, $f(x) = \dfrac{e^{ix}}{x^2+1} = \dfrac{\cos x}{x^2+1} + \dfrac{\sin x}{x^2+1}i$

$\gamma_R = S_R \cup L_R$



We see that f has poles of order 1 at $z = \pm i$

By Residue theorem, we get $2\pi i \sum_{z_k \in U} \operatorname{Res}(f : z_k) = \int_{\gamma_R} f(z)\,dz$ for very large R.

On S_R: $\left| \int_{S_R} \frac{e^{iz}}{z^2+i}\,dz \right| \leq \int_{S_R} \left| \frac{e^{iz}}{z^2+1} \right| dz \leq \text{length } S_R \cdot \max \frac{|e^{iz}|}{|z^2+1|}$

$$\leq \pi \cdot R \frac{\max |e^{iz}|}{\frac{1}{2}R^2}$$

$$\boxed{\begin{aligned} |e^{iz}| &= |e^{i(x+iy)}| \\ &= |e^{ix}| \cdot |e^{-y}| \\ &= e^{-y} \\ &= \frac{1}{e^y} \leq 1 \end{aligned}}$$

$$\leq \frac{2\pi R}{R^2} = \frac{2\pi}{R}$$

$\int \frac{e^{iz}}{z^2+1}\,dz \longrightarrow 0$ as $R \to \infty$

On L_R: $z = x$, $dz = dx$ $\qquad -R \leq x \leq R$

$$\int_{L_R} \frac{e^{iz}}{z^2+1}\,dz = \int_{-R}^{R} \frac{e^{ix}}{x^2+1}\,dx = \int_{-R}^{R} \frac{\cos x}{x^2+1}\,dx + i \int_{-R}^{R} \frac{\sin x}{x^2+1}\,dx$$

$$2\pi i \sum_{z_k \in U} \operatorname{Res}(f : z_k) = \int_{L_R} f(z)\,dz + \int_{S_R} f(z)\,dz$$

$$2\pi i \sum_{z_k \in U} \operatorname{Res}(f : z_k) = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx + i \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx + 0$$

Take $\operatorname{Re}(RHS) = \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx$, $\operatorname{Im}(RHS) = \int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx$

$f(z) = \frac{e^{iz}}{z^2+1}$ $\begin{matrix} P \\ Q \end{matrix}$

$\operatorname{Res}(f : i) = \frac{e^{i \cdot i}}{2i} = \frac{e^{-1}}{2i}$

$\operatorname{Res}(f : i) = \frac{P(i)}{Q'(i)}$ b/c pole is order 1

$2\pi i \operatorname{Res}(f : i) = \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos x}{x^2+1}\,dx = \frac{\pi}{e}$, $\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,dx = 0$