Lecture 8

§2.2 Power series

① Test on June 24th
② HW2 due on June 12th

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ where $a_n \in \mathbb{C}$ (coeff's), is a power series centered at z_0

$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$

Ex: ① $\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \cdots$
② $\sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$
③ $p(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots + a_n(z-z_0)^n$ ($a_k = 0$ if $k > n$)

Radius of Convergence

- Given a power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, exactly one of the following occurs:
  ① ($R=0$) $\sum a_n(z-z_0)^n$ converges only for $z=z_0$
  ② ($R=\infty$) $\sum a_n(z-z_0)^n$ converges everywhere.
  ③ There is an $R>0$ so that $\sum a_n(z-z_0)^n$ converges for all $|z-z_0|<R$, & diverges if $|z-z_0|>R$. (We don't study boundary here: maybe does, maybe not.)

only disk radius R centered at z_0

Ex: Recall: $\frac{1-z^{n+1}}{1-z} = 1 + z + z^2 + \cdots + z^n$ ($z \neq 1$)

If $|z|<1$, then $\lim_{n\to\infty} |z^{n+1}| = 0$ $\Rightarrow$ as $n\to\infty$, $\frac{1-z^{n+1}}{1-z} \to \frac{1}{1-z} = 1 + z + z^2 + \cdots$

Easy to show that if $|z|>1$, then $|z^{n+1}|$ does not converge as $n \to \infty$.
So $\sum z^n$ converges in $|z|<1$ & diverges in $|z|>1$, so $R=1$.

Theorem: ① If $\sum a_n(z-z_0)$ has radius R, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$

② Similarly, $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R}$

Ex: $\sum \underbrace{\frac{3^n}{2^{n+1}}}_{a_n} (z-5)^n$ Find R.

$\lim_{n\to\infty} \frac{3^{n+1}}{2^{n+2}} \cdot \frac{2^{n+1}}{3^n} = \frac{3}{2} = \frac{1}{R}$ $R = \frac{2}{3} < 1$

Power Series as Functions

Suppose $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges in a disk D, then we can define a function $f: D \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$

Theorem: If $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$ cvgs for $|z-z_0|<R$ (R=rad of cvg) then f is analytic in $|z-z_0|<R$ and $f'(z) = \sum n a_n (z-z_0)^{n-1}$
Note: $\sum n a_n(z-z_0)^{n-1}$ has the same radius of convergence.

Ex: $f(z)=\sum_{n=0}^{\infty} z^n$ converges in $|z|<1$

$f'(z)=\sum n z^{n-1}$ converges in $|z|<1$

$$\left(\frac{1}{1-z}\right)'=\frac{1}{(1-z)^2}=1+2z+3z^2+4z^3+\cdots$$

$$\boxed{((1-z)^{-1})'=-(1-z)^{-2}\cdot(-1)=(1-z)^{-2}}$$

$$1+z+z^2+\cdots=\frac{1}{1-z}$$

Cor: If $f(z)=\sum a_n(z-z_0)^n$ (& converges in D) then f is infinitely differentiable.

$$f^{(k)}(z)=\sum n(n-1)(n-2)\cdots(n-k+1)(z-z_0)^{n-k}$$

$f^{(k)}(z_0)=$ constant term $=a_n n(n-1)\cdots(n-k+1)$

(Constant term is when $n=k$)

$$=a_n\, n(n-1)\cdots(n-n+1)$$
$$=a_n\cdot n!$$

So $f^{(n)}(z_0)=a_n\cdot n!$

$$\boxed{a_n=\frac{f^{(n)}(z_0)}{n!}}$$

Ex: $f(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$

$$f'(z)=\sum\frac{nz^{n-1}}{n!}=\sum\frac{z^{n-1}}{(n-1)!}=f(z)$$

Let's use this to check $f(z)=e^z$.

Consider $g(z)=e^{-z}f(z)$

Then $g'(z)=-e^{-z}f(z)+e^{-z}f'(z)$

$=-e^{-z}f(z)+e^{-z}f(z)$

$=f(z)(e^{-z}-e^{-z})$

$=0$

$\Rightarrow g'(z)=0\Rightarrow g(z)=$ constant

To find constant, $g(0)=e^{-0}f(0)=1$, $1=1=$ const.

$\Rightarrow 1=e^{-z}f(z)\Rightarrow f(z)=e^z=\sum\frac{z^n}{n!}$

Ex: Use power series for e^z & def'n of $\cos z$, $\sin z$ to check that.

$$\sin z=\sum\frac{(-1)^k z^{2k+1}}{(2k+1)!} \qquad \cos z=\frac{1}{2}(e^{iz}+e^{-iz})$$

$$\cos z=\sum\frac{(-1)^k z^{2k}}{(2k)!} \qquad \sin z=\frac{1}{2i}(e^{iz}-e^{-iz})$$

just check by calculation.