Lecture 11

Ex: Sps $X_1, X_2, \cdots, X_n$ r.v.'s are i.i.d. Exp(θ)

Prove that $\min(X_1, X_2, \cdots, X_n) \xrightarrow{P} 0$

Let $U_n = \min(X_1, \cdots, X_n)$ need to show the following:

$P[|U_n - 0| > \varepsilon] \to 0$ as $n \to \infty$ for any positive ε.

$P[|U_n| > \varepsilon] = P[U_n > \varepsilon] = P[X_1 > \varepsilon, X_2 > \varepsilon, \cdots, X_n > \varepsilon] = P[X_1 > \varepsilon] P[X_2 > \varepsilon] \cdots P[X_n > \varepsilon]$

$$= \underbrace{e^{-\varepsilon/\theta} \cdots e^{-\varepsilon/\theta}}_{n \text{ of them}} = e^{-\varepsilon n/\theta} \longrightarrow 0 \quad \text{as } n \to \infty$$

HW: Will $U_n \xrightarrow{a.s.} 0$?

Properties:
1. $X_n \xrightarrow{P} X \Rightarrow cX_n \xrightarrow{P} cX$, $\forall$ constant C
2. If $X_n \xrightarrow{P} X$, $Y_n \xrightarrow{P} Y$, Then $X_n + Y_n \xrightarrow{P} X + Y$
3. If $X_n \xrightarrow{P} X$ & $Y_n \xrightarrow{P} Y \Rightarrow X_n Y_n \xrightarrow{P} XY$
4. If $X_n \xrightarrow{P} X$ & $Y_n \xrightarrow{P} Y$ & $P(Y \neq 0) = 1$, then $X_n / Y_n \xrightarrow{P} X/Y$

Note: All P's can be changed into a.s. in ①–④

Choose ② to prove.

Proof: First we prove the a.s. version.

If $X_n \xrightarrow{a.s.} X$ & $Y_n \xrightarrow{a.s.} Y$

Let $A = \{\omega : X_n(\omega) \to X(\omega)\}$

we know $P(A) = 1$

Let $B = \{\omega : Y_n(\omega) \to Y(\omega)\}$

we know $P(B) = 1$

Let $C = A \cap B \Rightarrow P(C) = 1$

For $\forall \omega \in C$, $\left.\begin{array}{l} X_n(\omega) \to X(\omega) \\ Y_n(\omega) \to Y(\omega) \end{array}\right\} \to X_n(\omega) + Y_n(\omega) \to X(\omega) + Y(\omega)$

Let $D = \{\omega : X_n(\omega) + Y_n(\omega) \to X(\omega) + Y(\omega)\} \Rightarrow C \subseteq D \Rightarrow P(D) = 1$

Now we prove the P version

$X_n \xrightarrow{P} X$ & $Y_n \xrightarrow{P} Y \Rightarrow X_n + Y_n \xrightarrow{P} X + Y$

We need to show for $\boxed{\forall \varepsilon > 0,\ P(|X_n + Y_n - X - Y| > \varepsilon) \longrightarrow 0}$

Remark: if we change + to · then ③ is proved.

Wanted !

$\boxed{\text{Observation:}}$ i.e. $P(|\underbrace{X_n - X}_{\text{call it } Z_n} + \underbrace{Y_n - Y}_{\text{call it } W_n}| > \varepsilon) \to 0$ wanted!

- At least one of Z_n & W_n will be greater than $\varepsilon/2$ if $Z_n + W_n > \varepsilon$.

$\Rightarrow P(Z_n + W_n > \varepsilon) \leq P(Z_n > \varepsilon/2) + P(W_n > \varepsilon/2)$

Similarly, $P(Z_n+W_n<-\varepsilon)\leq P(Z_n<-\varepsilon/2)+P(W_n<-\varepsilon/2)$

Because $Z_n \xrightarrow{P} 0$, $W_n \xrightarrow{P} 0$
We find that $P(Z_n>\varepsilon/2)$, $P(Z_n<-\varepsilon/2)$, $P(W_n>\varepsilon/2)$, $P(W_n<-\varepsilon/2)$
All of them converge to 0.
This implies ② as desired. ∎

Proposition: Suppose $r>0$ is a constant and $E|X_n-X|^r \to 0$ as $n\to\infty$ then $X_n \xrightarrow{P} X$

It's useful since you can choose r as whatever you want.

Proof: $P[|X_n-X|>\varepsilon]=P[|X_n-X|^r>\varepsilon^r] \leq \dfrac{E[|X_n-X|^r]}{\varepsilon^r} \longrightarrow 0$

markov

Ex: A version of Law of Large #s. (LLN)

Sps $X_1,\dots,X_n$ i.i.d. with $EX_i=\mu$ and $V[X_i]=\sigma^2<\infty$
Then $\overline{X}_n \xrightarrow{P} \mu$
Proof: $E|\overline{X}_n-\mu|^2$ (i.e. choose $r=2$)

$$=E\left[\left\{\frac{(X_1-\mu)+(X_2-\mu)+\cdots+(X_n-\mu)}{n}\right\}^2\right]$$

$$=\frac{1}{2}\sum_{i=1}^{n}E|X_i-\mu|^2=\frac{1}{n^2}\cdot n\sigma^2=\frac{\sigma^2}{n}\to 0 \text{ as } n\to\infty$$

According to proposition $\overline{X}_n \xrightarrow{P} \mu$

Theorem: if $X_n \xrightarrow{P} X$ and h is a continuous function, then $h(X_n) \xrightarrow{P} h(X)$
Proof: First, let's assume h is uniformly continuous.
$P[|h(X_n)-h(X)|>\varepsilon]$ continuous
Because uniform continuity, we can find a δ s.t.
$|h(x)-h(y)|\leq\varepsilon$ if $|x-y|\leq\delta$
$\Rightarrow P[|h(X_n)-h(X)|>\varepsilon]\leq P(|X_n-X|>\delta)$
$\longrightarrow 0$ as $n\to\infty$
Proof of general h is HW.

CONVERGE IN DISTRIBUTION

Def: $X_n \xrightarrow{D} X$ if either 1° or 2° holds:
1°. For any bounded continuous function h. $E[h(X_n)]\to E[h(X)]$

2° $F_n(x)\to F(x)$ for $\forall x$ such that the function F is continuous at x, where F_n is the CDF of X_n, F is the CDF of X.

Properties & facts:

① If $X_n \xrightarrow{P} X$, then $X_n \xrightarrow{D} X$ (the weakest among 3 kinds of convergence)

Proof: $\forall x$, s.t. F is cont. at x,

[Want]: $\big|P[X_n \le x] - P[X \le x]\big| \to 0$

We know that

$P[|X_n - X| > \varepsilon] \to 0$ for $\forall \varepsilon$ $\Rightarrow$ the chance that $X_n \le x$ but $X \ge x+\varepsilon$ is small.

$\Rightarrow \big|P(X_n \le \varepsilon) - P(X \le x+\delta)\big| \to 0$ for $\forall \delta > 0$

Hint: $I\{X_n \le x\} - I\{X \le x+\delta\}$
Find a bound for this, then take expectation. $\boxed{HW}$

(Using the hint) Then $\big|P[X_n \le x] - P[X \le x]\big| \le \overset{\to 0}{\big|P[X_n \le x] - P[X \le x+\delta]\big|}$

$+ \underbrace{\big|P[X \le x+\delta] - P[X \le x]\big|}_{(*)}$

$(*) \to 0$ as $\delta \to 0$

$\Rightarrow \big|P[X_n \le x] - P[X \le x]\big| \to 0$ (done)

② Slutsky's Theorem

If $X_n \xrightarrow{D} X$ & $Y_n \xrightarrow{D} C$ where $\underline{C \text{ is a constant}}$ then

if this is subbed by Y, this doesn't hold!

$X_n + Y_n \xrightarrow{D} X + C$

$X_n Y_n \xrightarrow{D} XC$

If $C \ne 0$ then $X_n / Y_n \xrightarrow{D} X/C$

Only prove that $X_n + Y_n \xrightarrow{D} X + C$

Proof: $\big|P[X_n + Y_n \le x] - P[X + C \le x]\big| \to 0$ $\forall x$ s.t.

$x - c$ is a continuous point of F

Proposition: If $Y_n \xrightarrow{D} C$ & C is a constant. Then $Y_n \xrightarrow{P} C$

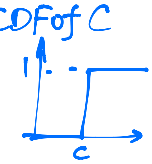
Note: the condition is C as a constant.

Proof: $P[|Y_n - C| > \varepsilon] = \overset{(1)}{P[Y_n > C+\varepsilon]} + \overset{(2)}{P[Y_n < C-\varepsilon]}$

as $n \to \infty$, $P[Y_n > C+\varepsilon] \to P[C > C+\varepsilon]$ $\forall \varepsilon > 0$

$\Rightarrow P(Y_n \le x) \to P(C \le x)$

$\Leftrightarrow P(Y_n \le x) \to \begin{cases} 1 & \text{if } x > c \\ 0 & \text{if } x < c \end{cases}$ CDF of C



Hence (1) $\to 0$, (2) $\to 0$, Done.

Back to the proof (we want to show "those" prob. are close.

$$I\{X_n+Y_n\le x\}-I\{X+c\le x\}=\{I\{X_n+Y_n\le x\}-I\{X+c\le x\}\}\cdot I\{|Y_n-C|>\varepsilon\} \quad ①$$

$$+\{I\{X_n+Y_n\le x\}-I\{X+c\le x\}\}\cdot I\{|Y_n-C|<\varepsilon\} \quad ②$$

$|①| \le I\{|Y_n-c|>\varepsilon\}$ (since the common $I\{\cdots\}$ is 0, or -1 or 1)

$|②| \le I\{X_n+C-\varepsilon\le x\}-I\{X+c\le x\}$ Take the smallest Y_n can be

Similarly $② \ge I\{X_n+C+\varepsilon\le x\}-I\{X+C\le x\}$

so $② = I\{X_n+C+\varepsilon\le x\}-I\{X+C\le x\}$

$\Rightarrow P(X_n+Y_n\le x)-P(X_n+C\le x)$

$$\le \underbrace{P[|Y_n-c|<\varepsilon]}_{\to 0}+\underbrace{P[X_n+c-\varepsilon\le x]-P[X+c\le x]}_{\to 0}$$

On the other hand, $P(X_n+Y_n\le x)-P(X_n+C\le x)\ge \overbrace{-P(|Y_n-c|>\varepsilon)}^{\to 0}+P(X_n+C+\varepsilon\le x)$

$P[X+c\le x]$

$\searrow 0$

Both upper & lower bounds $\to 0$

$\Rightarrow X_n+Y_n \xrightarrow{D} X+C$

■

C.f. (Characteristic function)

$\phi(\theta)=E[\exp(i\theta X)]$ is called the c.f. of X where $i^2=-1$

Recall: $\exp(ix)=\cos x+i\sin x$

(P126 lists c.f. of some commonly used distributions.)

Distb	C.f.
Degenerate	
Binomial	
Poisson	
Negative Bino	
Re	

Properties:

① $\phi_{X+Y}(\theta)=\phi_X(\theta)\,\phi_Y(\theta)$ if $X \perp\!\!\!\perp Y$

② $\phi_{a+bX}(\theta)=e^{ia\theta}\phi(b\theta)$

Proposition: (no need to prove)

① X, Y have the same distribution $\Longleftrightarrow \phi_X(\theta)=\phi_Y(\theta)$ for $\forall\theta\in\mathbb{R}$

② $X_n \xrightarrow{D} X$ iff $\phi_{X_n}(\theta)\longrightarrow\phi_X(\theta)$ $\forall\theta\in\mathbb{R}$

See chapter 19.5 for proofs.

③ If $E|X|^r < \infty$, where r is a positive integer that $\phi_X(\theta)$ is r-times differentiable and has the following Taylor expansion.

$$\phi_X(\theta) = \sum_{j=0}^{r} \frac{(i\theta)^j}{j!} E[X^j] + o(\theta^r) \qquad (*)$$

$\longrightarrow$ 小量?

Properties:

$f(\theta) = o(\theta^r)$

means $\frac{f(\theta)}{\theta^r} \longrightarrow 0$ as $\theta \to 0$

Taylor expansion

$$f(x) = \sum_{k=0}^{m} \frac{(x-x_0)^k}{k!} f^{(k)}(x_0) + R_m$$

$$R_m = \int_{x_0}^{x} f^{(m+1)}(t) \frac{(x-t)^m}{m!} dt$$

The rest of the proof can be found on page 127 of the textbook.

c.f. usage: ① prove convergence in distribution.
② decide CDF.

Thm (WLLN) Weak Law of Large Number
$X_1, \cdots, X_n$ iid. $E[X_i] = \mu < \infty$. Then $\bar{X}_n \xrightarrow{P} \mu$

Proof:

$$\phi_{\bar{X}_n}(\theta) = \phi_{\frac{X_1}{n}}(\theta)\, \phi_{\frac{X_2}{n}}(\theta) \cdots \phi_{\frac{X_n}{n}}(\theta) \quad \text{since iid}$$

$$= \phi_{X_1}\left(\frac{\theta}{n}\right) \cdots \phi_{X_n}\left(\frac{\theta}{n}\right)$$

$$= \left[\phi_{X_1}\left(\frac{\theta}{n}\right)\right]^n \overset{(*)}{=\!=} \left[1 + i\frac{\theta}{n} + o\left(\frac{\theta}{n}\right)\right]^n$$

$$= \left[1 + \frac{i\theta\mu}{n} + o\left(\frac{\theta}{n}\right)\right]^n \longrightarrow e^{i\theta}$$

Note that $e^{i\theta\mu}$ is the c.f. of a constant μ.
Therefore, $\bar{X}_n \xrightarrow{D} \mu \Rightarrow \bar{X}_n \xrightarrow{P} \mu$

Thm (Central Limit Thm)
If $X_1, \cdots, X_n$ are iid w/ $E(X_i) = \mu$ & $V[X_i] = \sigma^2 < \infty$
Then $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{D} N(0, \sigma^2)$

Proof: Since $\phi_{aX+b}(\theta) = \phi(b\theta) \exp(ia\theta)$

we only need to show the case where $\mu = 0$ & $\sigma^2 = 1$.

$$\phi_{\sqrt{n}\,\overline{X}_n} = \phi_{X_1/\sqrt{n}}(\theta)\,\phi_{X_2/\sqrt{n}}(\theta)\cdots\phi_{X_n/\sqrt{n}}(\theta)$$

$$= \phi_{X_1}\left(\frac{\theta}{\sqrt{n}}\right)\cdots\phi_{X_n}\left(\frac{\theta}{\sqrt{n}}\right)$$

$$= \left[\phi_{X_1}\left(\frac{\theta}{\sqrt{n}}\right)\right]^n \overset{(*)}{=\!=} \left[1 + \frac{i\left(\frac{\theta}{\sqrt{n}}\right)0}{1!} - \frac{\left(\frac{\theta}{\sqrt{n}}\right)^2 \cdot 1}{2!} + o\left(\left(\frac{\theta}{\sqrt{n}}\right)^2\right)\right]^n$$

$$= \left[1 - \frac{\theta^2}{2n} + o\left(\frac{\theta^2}{n}\right)\right]^n \to e^{-\theta^2/2} \quad \text{as } n \to \infty$$

Note the c.f. of $N(0,1)$ is $e^{-\theta^2/2}$. Hence $\sqrt{n}\,\overline{X}_n \xrightarrow{D} N(0,1)$

∎