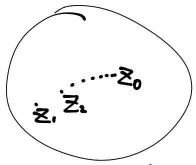
Lecture 18

ZEROS OF ANALYTIC FUNCTIONS

THM: Suppose f is analytic in D (a domain) and $z_1, z_2, z_3, \dots$ is a sequence of zeros of f in D. If $\lim_{n\to\infty} z_n = z_0 \in D$, then $f(z)=0$ for all z



This THM tells us that the zeros are "isolated" i.e. they don't "bunch up".

In particular, this will allow us to count the # of zeros of an analytic functions inside a curve.

(skip the proof, in text book)

In particular, there are only finitely many inside a simple closed curve γ.

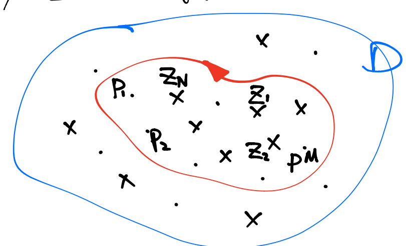
Argument Principle (Chapter 8)

Let h be analytic in a domain D, except at poles $P_1, P_2, P_3, \dots, P_m$

Let γ be a simple closed curve in D.

Let $z_1, z_2, \dots, z_N$ be the zeros inside γ

Let $P_1, P_2, \dots P_M$ be the poles inside γ.



Let's look at $\frac{h'(z)}{h(z)}$ close to a zero z_k.

$h(z) = (z-z_k)^{n_k} g_k(z)$, $g(z_k) \neq 0$

$h'(z)\ n_k(z-z_k)^{n_k-1} g_k(z) + (z-z_k)^{n_k} g_k'(z)$

$$\frac{h'(z)}{h(z)} = \frac{n_k}{(z-z_k)} + \frac{g_k'(z)}{g_k(z)} \longrightarrow \text{is analytic near } z_k$$

$$\mathrm{Res}\left(\frac{h'}{h} : z_k\right) = n_k$$

• Now let's look $\boxed{\text{near a pole of } h}$

Close to P_k: $h(z) = \frac{1}{(z-P_k)^{m_k}} g_k(z)$, $\boxed{\text{again } (g_k(P_k) \neq 0)}$

$$h'(z) = \frac{-m_k}{(z-P_k)^{m_k+1}} g_k(z) + \frac{1}{(z-P_k)^{m_k}} g_k'(z)$$

$$\frac{h'(z)}{h(z)} = \frac{-m_k}{z-P_k} + \frac{g_k'(z)}{g_k(z)} \longrightarrow \text{analytic near } P_k$$

$$\Rightarrow \mathrm{Res}\left(\frac{h'}{h} : P_k\right) = -m_k$$

By Residue Thm:

$$\int_\gamma \frac{h'(z)}{h(z)}\,dz = 2\pi i\Big[\sum_{z_k \text{ inside } \gamma} \mathrm{Res}\Big(\frac{h'}{h}; z_k\Big) + \sum_{p_k \text{ inside } \gamma} \mathrm{Res}\Big(\frac{h'}{h}; p_k\Big)\Big]$$

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \sum_{z_k \text{ inside } \gamma} n_k - \sum_{p_j \text{ inside } \gamma} m_j$$

$$= \#\text{ of zeros of } h \text{ inside } \gamma - \#\text{ of poles of } h \text{ inside } \gamma.$$

(but we count the zeros and poles "with multiplicity")

When "counting with multiplicity", a zero/pole of order n, gets counted n times.

Ex: $f(z) = z^2$ has a zero w/ multiplicity 2 at $z=0$.

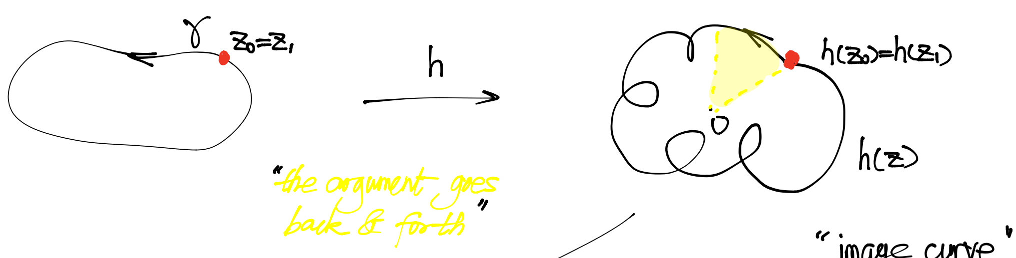
THM h analytic except at poles p_k γ simple closed curve.

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \#\text{zeros inside } \gamma - \#\text{poles inside } \gamma$$

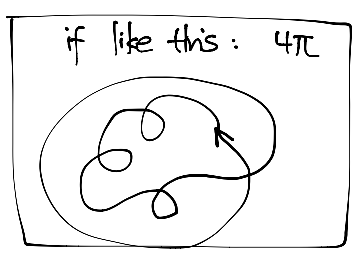
where we count w/ multiplicity.

Q: What does $\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz$ represent?

A: It measures the net change in $\arg h(z)$ as z goes along γ.



arg $h(z)$ changes by 2π.



$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \#\text{ of times } h(z) \text{ goes around } 0 \text{ as } z \text{ goes around } \gamma.$$

THM (Argument Principle)

$$\frac{1}{2\pi}(\text{net change in } \arg h(z) \text{ as } z \text{ goes around } \gamma) = \#\text{ zeros inside } \gamma - \#\text{poles inside } \gamma.$$

If h has no poles, the $\frac{1}{2\pi}(\text{net change in } \arg h) = \#\text{ zeros inside } \gamma$

Why does $\frac{1}{2\pi i}\int_\gamma \frac{h'}{h}$ represent winding number of image curve?

Let's look at an example.

<u>Ex</u> $h(z)=z^2$, γ the unit circle.

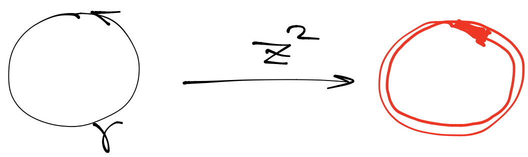
$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}dz = \frac{1}{2\pi i}\int_\gamma \frac{2z}{z^2}dz = \frac{1}{\pi i}\int_\gamma \frac{1}{z}dz = \frac{1}{\pi i}2\pi i = 2$$

Take a look at the pic.

$\gamma = e^{it}$, $0\le t\le 2\pi$

$h(z)=z^2$ so image curve is $(e^{it})^2 = e^{i2t}$, $0\le t\le 2\pi$

$= e^{i\theta}$, $0\le\theta\le 4\pi$



"2 times around unit circle"

$\arg h(z)$ goes from $0\to 4\pi$.

so $\frac{1}{2\pi}$(net change in $\arg h$) $= \frac{1}{2\pi}\cdot 4\pi = 2 = \frac{1}{2\pi i}\int_\gamma \frac{h'}{h}$

This works

& $2 = \#$ zeros of h inside γ w/ multiplicity

"actually one zero with order 2"