March 20th

Countable sets

Def'n: a set S is called countable if $|S| \leq |N|$

any finite set is countable

if S is any infinite set then $|S| \geq |N|$

if S is infinite $\Rightarrow$ pick $s_1 \in S$..

$S \setminus \{s_1\}$ — still infinite

$\Rightarrow$ pick $s_2 \in S \setminus \{s_1\}$

$S \setminus \{s_1, s_2\}$ still infinite

etc.

$\Rightarrow$ can pick $s_1, s_2, \cdots, s_n, \cdots \in S$

$s_i \neq s_j$ for $i \neq j$

this gives a 1-1 map $f: N \to S$

$f(i) = s_i$

$1 - s_1$
$2 - s_2$
$3 - s_3$
$\cdots$

$\Rightarrow |N| \leq |S|$

$\Rightarrow$ S is countable $\Leftrightarrow$ S is either finite or $|S| = |N|$

$\Leftarrow$ obvious

$\Rightarrow$? let S be countable, if S is finite we are done

if S is infinite $\Rightarrow |S| \geq |N|$ & $|S| \leq |N|$ $\Rightarrow |S| = |N|$

EX: $S = \{1, 2, 7, 26\}$ — finite — countable

S = all even natural numbers $= \{2, 4, 6, \cdots\}$ countable

$S = \mathbb{Z}$ all integers countable

$S = N \times N$ countable

$\mathbb{R}$ is not countable

S countable means we can number elements of S by either finite or infinite sequence

Theorem: A union of countably many countable sets is countable

$S = \bigcup_{\alpha \in A} S_\alpha$ A is countable & each S_α is countabe

A is countable then $\bigcup_{\alpha \in A} S_\alpha$ is also countable

A is either finite $\Rightarrow A = \{i_1, \cdots, i_n\}$

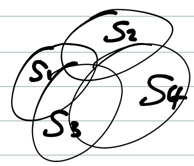
$S = \underbrace{S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_n}}_{\text{all these are countable}}$ $\Rightarrow$ union is also countable or A is infinite $\Rightarrow |A| = |N|$

$A = \{i_1, \cdots, i_n, \cdots\}$

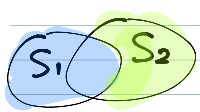
$S = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_n} \cup \cdots$ is also countable

So let $S = \bigcup_{i=1}^{\infty} S_i = S_1 \cup S_2 \cup \cdots$

(we'll prove the case $|A| = |N|$. $|S_i| \leq |N|$ for any $i = 1, 2, 3$)



Put $T_1 = S_1$, $T_2 = S_2 \setminus S_1$ — all elements in S_2 but not in S_1



$S_1 \cup S_2 = T_1 \cup T_2$ note $T_1 \setminus T_2 = \emptyset$

$= S_1 \cup S_2 \setminus S_1$

put $T_3 = S_3 \setminus (S_1 \cup S_2) \Rightarrow$ then again $S_1 \cup S_2 \cup S_3 = T_1 \cup T_2 \cup T_3$

with $T_3 \cap T_1 = \emptyset$, $T_3 \cap T_2 = \emptyset$

put $T_n = S_n \setminus (S_1 \cup \cdots \cup S_{n-1})$

- easy to see by induction

- $\bigcup_{i=1}^{\infty} T_i = \bigcup_{i=1}^{\infty} S_i$ for any n

- $T_i \cap T_j = \emptyset$ if $i \neq j$

- $T_i \leq S_i$, $|T_i| \leq |S_i| \leq |N| \Rightarrow T_i$ is countable

So let $S = \bigcup_{i=1}^{\infty} S = S_1 \cup S_2 \cup \cdots = \bigcup_{i=1}^{\infty} T_i$, T_i's one countable and $T_i \cap T_j = \emptyset$ if $i \neq j$

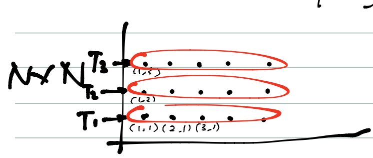
Recall $|N \times N| = |N|$

we'll construct $f: S \to N \times N$ (1-1)

this will imply that $|S| \leq |N \times N| = |N| \Rightarrow S$ is countable

each T is countable $\Rightarrow$

we here maps $f_i: T_i \to N$ (1-1)



define $f: \bigcup_{i=1}^{\infty} T_i \to N \times N$ by the formula

$f(s) = \{(f_i(s), i) \text{ if } s \in T_i\}$

• f is well defined b/c $T_i \cap T_j = \emptyset$ if $i \neq j$

• f is one-to-one if $S_1 \in T_i$, $S_2 \in T_j$, $i \neq j$
$f(S_1) \neq f_2(S_2)$ b/c $f(S_1) = (f(S_1), i) \neq (f(S_2), j) = f(S_2)$

if $S_1, S_2 \in T_i$, $S_1 \neq S_2$
$\Rightarrow f(S_1) = (f_i(S_1), i) \neq (f_2(S_2), i) = f(S_2)$ b/c $f_i : T_i \to N$ (1-1) $\Rightarrow f_i(S_1) \neq f_i(S_2)$

$\Rightarrow f$ is 1-1 $\Rightarrow f: S \to N \times N \Rightarrow |S| \leq |N \times N| = |N|$

Def: a number is called algebraic if it's a root of a polynomial with integer coefficients $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, $a_i \in \mathbb{Z}$

a solves $p(x) = 0$

ex: $\sqrt{2}$ is algebraic
it's a root of $x^2 - 2 = 0$

$\sqrt[3]{2}$ is algebraic $x^3 - 2 = 0$

$\sqrt{2} + \sqrt{3}$ is also algebraic
if $x = \sqrt{2} + \sqrt{3}$
$x^2 = 2 + 3 + 2\sqrt{6}$
$x^2 = 5 + 2\sqrt{6}$
$x^2 - 5 = 2\sqrt{6}$
$(x^2 - 5)^2 = 24$
$(x^2 - 5)^2 = 24$ — polynomial with integer coefficients

$x^4 - 10x^2 + 1 = 0$

any rational number is algebraic $\frac{m}{n}$

$nx - m = 0$ ✓

root is $\frac{m}{n}$

Is there any number not algebraic? Yes.

Theorem: the set of all algebraic numbers is countable ($\Rightarrow$ has cardinality $|N|$)

Recall: we proved $|\mathbb{R}| \neq |N|$.

Cor: $\mathbb{R} \setminus A \neq \emptyset$, non-algebraic numbers are called transcendental
↘ algebraic

Proof: (existence of transcendental #s)

Recall: a nonzero poly. of degree n has at most n roots.
$\Rightarrow f \;\; \deg(f)=n \Rightarrow |Z(f)|=n$

$A = \bigcup Z(f)$
f — nonzero poly. with integer coefficients

$Z(f)$ = set of zero roots of f
$Z(f) = \{x \in \mathbb{R} \mid f(x)=0\}$
$\Rightarrow Z(f)$ is finite ($\Rightarrow$ countable) for any f.

$\boxed{\text{Claim: the set } P \text{ of all poly. with integer coefficients is countable}}$

Once we prove this, this implies that A is countable as a countable union of countable sets.

$P = \bigcup_{n=1,2,3,\dots} P_n$ — countable

P_n = all polynomials with integer coefficients of deg $= n$.
we'll show that each P_n is countable
$P_n = \{a_n x^n + \dots + a_1 x + a_0 \mid a_i \in \mathbb{Z}\} \simeq \mathbb{Z}^{n+1}$

$P_n \xrightarrow{\varphi} \mathbb{Z}^{n+1}$ $\qquad \varphi : P_n \to \mathbb{Z}^{n+1}$ is 1-1 & onto
$f(x) = a_n x^n + \dots + a_1 x + a_0 \longmapsto (a_n, a_{n-1}, \dots, a_1, a_0)$
$|P_n| = |\mathbb{Z}^{n+1}| = |\mathbb{Z}| = |\mathbb{N}|$
$P = \bigcup_{n\in\mathbb{N}} P_n$ each P_n is countable $\Rightarrow$
P is countable $\Rightarrow$ A is countable

$\mathbb{Q}$ — set of rational #

$\mathbb{Q} \subseteq A$ (algebraic #s) $\quad |A| = |\mathbb{N}| \Rightarrow |\mathbb{Q}| \le |\mathbb{N}| \Rightarrow \mathbb{Q}$ is countable

$\mathbb{Q} \supseteq \mathbb{N} \Rightarrow |\mathbb{Q}| \ge |\mathbb{N}|$

$\Rightarrow |\mathbb{Q}| = |\mathbb{N}|$

Power Sets

def:
given S any set
$P(S)$ = set of all subsets of S
$P(S) = \{A \mid A \subseteq S\}$

ex: $S = \{1,2,3\}$
$P(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

HW: if S is finite $\Rightarrow |P(S)| = 2^{|S|}$

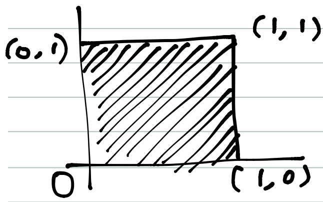
Any sets bigger than $\mathbb{R}$?

<u>Q</u>: can we find a set S s.t. $|S|>|R|$

Claim: $|R\times R|=|R|$

$|(0,1)|=|R|$

we'll prove $|(0,1)\times(0,1)|=|(0,1)|$



We'll construct a map

$f: (0,1)\longrightarrow(0,1)\times(0,1)$ 1-1, onto

given $x\in(0,1)$, $0<x<1$. look at its decimal expre-ssion

$x=0.a_1a_2\cdots \longrightarrow 0.a_1a_3a_5\cdots$

$0.a_2a_4a_6\cdots$

f is 1-1 & onto

$0.23107 9026 \longrightarrow (0.21706\cdots, 0.3092\cdots)$

<u>Claim:</u> $|R\times R|=|R|$

$|R\times R\times R|=|R\times R|=|R|$

etc

$|R^n|=|R|$ for any natural n.

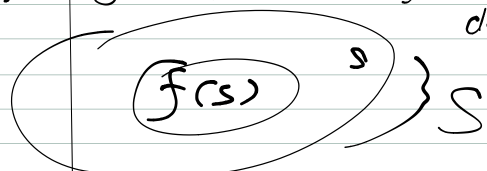
Theorem: $|P(S)|>|S|$ for any set S

<u>$|P(S)|\geq|S|$</u> — obvious

Suppose we can construct a map $f: S\rightarrow P(S)$ 1-1 & onto

for any $s\in S$. look at $f(s)\subset S$ (subset in S)

define $A=\{s\in S \mid$ such that $s\notin f(s)\}$



A is a subset of S, $f: S\rightarrow P(S)$ is onto $\Longrightarrow$

$\exists\ s_0\in S$ s.t. $f(s_0)=A$

Q: does s_0 belong to A or not?

<u>Case 1</u>: $s_0\in A$

$f(s_0)\in A$, $s_0\in A$

This is a contradiction: since if $s_0\in A \Rightarrow s_0\notin f(s_0)=A$

<u>Case 2</u>: $s_0\notin A$. $A=f(s_0)$ $s_0\notin f(s_0)$

$\Rightarrow s_0\in A$ by the definition of A

again a contradiction

$\Rightarrow$ Contradiction $\Rightarrow f$ cannot be a bijection.

$\Rightarrow |P(\mathbb{R})| > |\mathbb{R}|$

$|P(\mathbb{N})| > |\mathbb{N}|$

We know that $|\mathbb{R}| > |\mathbb{N}|$ also $|P(\mathbb{N})| > |\mathbb{N}|$

Theorem: $|P(\mathbb{N})| = |\mathbb{R}|$

HW: $\left.\begin{array}{l} |\mathbb{R}| \geq |P(\mathbb{N})| \\ |\mathbb{R}| \leq |P(\mathbb{N})| \end{array}\right\}$

We'll prove this:

construct $f : (1,2) \longrightarrow P(\mathbb{N})$ 1–1

$$\underset{\substack{\shortparallel \\ |\mathbb{R}|}}{|(1,2)|} \leq |P(\mathbb{N})|$$

Let $1 < x < 2$, $x = 1.a_1 a_2 a_3 \cdots$ decimal expression

$x \rightarrow \{1, 1a_1, 1a_1a_2, \cdots\}$

if $x = 1.230796 \longrightarrow \{1, 12, 123, \cdots\}$

this gives $f : (1,2) \xrightarrow[1-1]{} P(\mathbb{N})$ $\Rightarrow |P(\mathbb{N})| \geq |(1,2)| = |\mathbb{R}|$

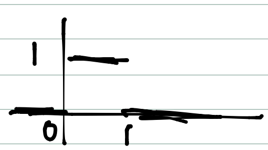
$|P(S)| = \{$set of all functions $f : S \rightarrow \{0,1\}\}$

for $A \subseteq S$ (subset) defines χ_A — characteristic function of A ....

$$\chi_A(s) = \begin{cases} 1 & \text{if } s \in A, \\ 0 & \text{if } s \in A \end{cases}$$

ex: $S = \mathbb{R}$, $A = [0,1]$

$$\chi_A(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{o.w.} \end{cases}$$



$P(S) \xrightarrow{\varphi}$ set of all funcs $S \rightarrow \{0,1\}$

$\varphi(A) = \chi_A$

$A \subseteq S$

φ is 1–1 $\quad \left.\begin{array}{l} A, B \subseteq S \\ A \neq B \end{array}\right\} \Rightarrow \chi_A \neq \chi_B$

φ is also onto if $f : S \rightarrow \{0,1\}$

take $A = \{s \in S$, such that $f(s) = 1\}$

then $f = \chi_A = \varphi(A)$

Ex: S = all functions from R to R = $\{f: \mathbb{R} \to \mathbb{R}\}$

Claim: $|S| = |P(R)|$

$S \supset \{\text{functions } f: S \to \{0, 1\}\}$

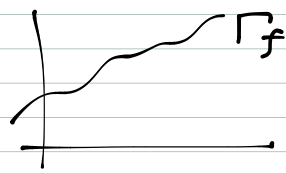
$\Leftrightarrow |S| \geq |P(R)|$ we'll also have $|S| \leq |P(R)|$

given $f: R \to R$ look at $\Gamma_f \subseteq \mathbb{R} \times R$

Γ_f – graph of f

$\Gamma_f = \{(x, f(x)) \mid x \in R\}$

$f_1 \neq f_2 \Rightarrow \Gamma_{f_1} \neq \Gamma_{f_2}$



we have a map $S \to P(R \times R)$

$f \mapsto \Gamma_f$ – graph of f this map is 1–1

$\Rightarrow |S| \leq |P(R \times R)| = |P(R)|$ b/c $|R| = |R \times R|$

same trick can be used to show sets of all functions $f: N \to N$ has cardinality $|P(N)| = |R|$

R – all real #

A – algebraic #

T – $R \setminus A$ trascendental #

$|A| = |N|$
$|R| \neq |N|$ $\Rightarrow T \neq \emptyset$ but what is $|T| = ?$

Claim: $|T| = |R|$

Theorem: if T is infinite and S countable then $|T \cup S| = |T|$

in our case: $T = R \setminus A$ – all transcendental #

$S = A$ – algebraic # – countable

$\Rightarrow$ the theorem applies

$$|T| = |\underbrace{T \cup A}_{= R}| = |R|$$

The proof of this theorem will be covered next time