Lecture 3 (continue §2.8)

Homework & recommended problem will be posted soon. (tmr)

Def of Cauchy:

Let $(a_n)_{n=1}^{\infty}$ be a sequence, it is Cauchy if $\forall \varepsilon>0, \exists N$ s.t. $|a_n-a_m|<\varepsilon$ $\forall n,m \geq N$

Completeness Thm: Every Cauchy sequence of real numbers converges

Proof: Every Cauchy seq. is bounded.

Every bounded seq. has a convergent subsequence

$$\lim_{k\to\infty} a_{n_k} = L \in \mathbb{R}$$

Suppose we are given $\varepsilon>0$.

$$|a_n-L| = |a_n-a_{n_k}+a_{n_k}-L| \leq |a_n-a_{n_k}|+|a_{n_k}-L| \quad (*)$$

$\exists N$ s.t. $|a_{n_k}-L|<\frac{\varepsilon}{2}$ for all $n_k \geq N$

$\exists M$ s.t. $|a_n-a_{n_k}|<\frac{\varepsilon}{2}$ for all $n, n_k \geq M$

So $(*) < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

$\blacksquare$

Ex: $\mathbb{Q} \subset \mathbb{R}$, not complete.

$\sqrt{2} = 1.41421\cdots$

$a_1 = 1$

$a_2 = 1.4$ $\quad\quad |a_n-a_m|<10^{-N}$ for all $m,n \geq N$

$a_3 = 1.41$

$a_4 = 1.414$

$\vdots$ $\quad\quad$ Cauchy sequence but not convergent

Ex: $\alpha \in \mathbb{R}$

$a_n = \frac{[n\alpha]}{n}$, show it's Cauchy

Note: $[x]$ = the nearest integer to x: $[1.2]=1$, $[1.8]=2$, $[1.5]=2$ (choose nearest even number)

Claim: $|[n\alpha]-n\alpha| \leq \frac{1}{2}$

Then $|a_n-\alpha| = \left|\frac{[n\alpha]}{n}-\alpha\right| = \left|\frac{[n\alpha]-n\alpha}{n}\right| \leq \frac{1}{2n}$

$\lim_{n\to\infty} a_n = \alpha \Rightarrow$ the sequence is Cauchy.

$\blacksquare$

Ex: $\dfrac{1}{2+\dfrac{1}{2+\dfrac{1}{2+\cdots}}}$ continued function

$a_1 = \frac{1}{2}$, $a_2 = \frac{1}{2+\frac{1}{2}} = \frac{2}{5}, \dots, a_{n+1} = \frac{1}{2+a_n}$

$\lim_{n\to\infty} a_n = L \qquad L = \frac{1}{2+L}$ Solve it.

But we have to make sure that Limit exists.

Claim: (a_n) is Cauchy

$a_1 > 0 \Rightarrow a_n > 0,\ \forall n$
$(2+a_n) > 2$
$(2+a_{n+1}) > 2$

$$a_{n+1} - a_{n+2} = \frac{1}{2+a_n} - \frac{1}{2+a_{n+1}} = \frac{a_{n+1}-a_n}{(2+a_n)(2+a_{n+1})}$$

$$|a_{n+1} - a_{n+2}| = \left|\frac{a_{n+1}-a_n}{(2+a_n)(2+a_{n+1})}\right| < \frac{|a_n - a_{n+1}|}{4}$$

$$|a_1 - a_2| = \left|\frac{1}{2} - \frac{2}{5}\right| = \frac{1}{10}$$

$$|a_2 - a_3| < \frac{1}{10}\cdot\frac{1}{4}$$

$$|a_3 - a_4| < \frac{1}{10}\cdot\frac{1}{4^2}$$

$$|a_n - a_{n+1}| < \frac{1}{10\cdot 4^{n-1}} = \frac{2}{5\cdot 4^n}$$

$$|a_m - a_n| = |a_m - a_{m+1} + a_{m+1} - a_{m+2} + \cdots + a_{n-1} + a_n| <$$

$$< \frac{2}{5}\left(4^{-m} + \cdots + 4^{-m-1} + \cdots 4^{1-n}\right)$$

$$= \frac{2}{5}\left(\frac{1}{4^m} + \cdots + \frac{1}{4^{n-1}}\right) = \frac{2}{5}\cdot\frac{1}{4^m}\left(1 + \frac{1}{4} + \cdots + \frac{1}{4^{n-m-1}}\right)$$

$$< \frac{2}{5}\cdot\frac{1}{4^m}\cdot\frac{1}{1-\frac{1}{4}}$$

$$= \frac{2}{5}\cdot\frac{1}{4^m}\cdot\frac{1}{3/4} = \frac{8}{15}\cdot\frac{1}{4^m} < \frac{1}{4^m}$$

Given $\varepsilon > 0$, choose N to be large enough so that $4^{-N} < \varepsilon$

$|a_m - a_n| < 4^{-m} \leq 4^{-N} < \varepsilon,\ \forall m, n \geq N$

so (a_n) is Cauchy $\Rightarrow$ converges.

$\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} a_{n+1} = L$, $L = \frac{1}{2+L}$, $L = -1 \pm \sqrt{2}$, since $L > 0$,
$L = -1 + \sqrt{2}$

Suppose $\forall \varepsilon > 0, \exists N$ s.t. $|a_{n+1} - a_n| < \varepsilon,\ \forall n \geq N$. Is it true that (a_n) converges?

$\boxed{\text{No.}}$

Let $a_n = \sqrt{n}$ diverges

$$|\sqrt{n+1} - \sqrt{n}| = \left|\frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{\sqrt{n+1}+\sqrt{n}}\right| = \left|\frac{1}{\sqrt{n+1}+\sqrt{n}}\right| < \frac{1}{2\sqrt{n}} \to 0$$

This is a counter example.

Example: Sps that $0<\alpha<1$ and a seq. (x_n) s.t. $|x_{n+1}-x_n|\le\alpha^n$, $n=1,2,\cdots$
Then (x_n) is Cauchy.

Proof: $|x_m-x_n| = |x_m-x_{m-1}+x_{m-1}-\cdots+x_{n-1}-x_n|$

$$\begin{aligned}
&\le |x_m-x_{m-1}|+\cdots+|x_{n+1}-x_n| \\
&\le \alpha^{m-1}+\cdots+\alpha^n \\
&= \alpha^n(1+\cdots+\alpha^{m-n-1}) \\
&< \frac{\alpha^n}{1-\alpha} \to 0 \text{ as } n\to\infty
\end{aligned}$$

So (x_n) is Cauchy.

Example:
(x_n) $\quad x_1=a,\ x_2=b,\ x_{n+2}=\frac{1}{2}(x_{n+1}+x_n)$, $n=1,2,3,\cdots$

Claim (x_n) converges

Proof: $x_{n+2}-x_{n+1} = \frac{1}{2}(x_{n+1}+x_n)-x_{n+1} = \frac{1}{2}(x_n-x_{n+1})$

$$\begin{aligned}
|x_{n+2}-x_{n+1}| &= \frac{1}{2}|x_n-x_{n+1}| \\
&= \frac{1}{2^2}|x_{n-1}-x_n| \\
&= \cdots = \frac{1}{2^n}|b-a|
\end{aligned}$$

by the previous statement (x_n) is Cauchy ∎

Series

§3.1 convergent series

Def: If $(a_n)_{n=1}^{\infty}$ is a sequence of numbers then the infinite series is

$$\sum_{n=1}^{\infty} a_n$$

$(S_n)_{n=1}^{\infty}$, $S_n=\sum_{k=1}^{n} a_k = a_1+\cdots+a_n$ partial sum

$\lim_{n\to\infty} S_n = \sum_{n=1}^{\infty} a_k$ series converges

otherwise series diverges.

$\boxed{\text{Converges} = \text{Summable}}$

Ex: $\sum_{k=1}^{\infty}\frac{1}{k} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges. (harmonic)

$$\begin{aligned}
S_n = S_{2^k} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\cdots+\left(\frac{1}{2^{k-1}}+\cdots+\frac{1}{2^k}\right) \\
&\ge 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots+\frac{1}{2} = 1+\frac{k}{2}, \quad k\to\infty,\quad S_{2^k}\to\infty.
\end{aligned}$$

diverge

Example :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)}$$

$$\frac{1}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3} = \frac{A(n+3)+Bn}{n(n+3)} = \frac{An+Bn+3A}{n(n+3)} = \frac{1}{n(n+3)}$$

$$\left.\begin{array}{l} 3A=1, \\ A+B=0 \end{array}\right\} \Rightarrow A=\frac{1}{3}, B=-\frac{1}{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+3)} = \frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

$$3S_n = 1 - \frac{1}{4} + \frac{1}{2} - \frac{1}{5} + \frac{1}{3} - \frac{1}{6} + \frac{1}{4} - \frac{1}{7} + \cdots + \left(\frac{1}{n} - \frac{1}{n+3}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3}$$

$\lim_{n\to\infty} 3S_n = \frac{11}{6}$ $\quad \lim_{n\to\infty} S_n = \frac{11}{18}$ This is a direct computation

Theorem: If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$

Proof: If the series converges $\Rightarrow \lim_{n\to 1} S_n = L$ $\boxed{\text{(nth term test)}}$

$$\lim_{n\to\infty} S_{n+1} = L$$

$$S_n - S_{n-1} = a_n$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = L - L = 0$$

∎

The following are equivalent for a series $\sum a_n$

(a). The series converges

(b). $\forall \varepsilon > 0 \, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\left|\sum_{k=n+1}^{\infty} a_k\right| < \varepsilon$

(c). $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. if $n, m \geq N$, $\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon$

Cauchy Criterion for Series

§ 3.2 Convergence Tests for Series

Proposition: if $a_k \geq 0$ for $k \geq 1$ and $S_n = \sum_{k=1}^{\infty} a_k$, then either

(1) $(S_n)_{n=1}^{\infty}$ is bdd above, in which ~~cases~~ case $\sum_{n=1}^{\infty} a_n$ converges

or

(2). $(S_n)_{n=1}^{\infty}$ is unbounded $\Rightarrow \sum_{n=1}^{\infty} a_n$ diverges

Example: Geometric series.

$(a_n)_{n=0}^{\infty}$

$a_{n+1} = r a_n$, $\forall n \geq 0$
$a_n = a_0 r^n$
A geometric series converges when $|r|<1$
$\sum_{n=0}^{\infty} a_0 r^n = \frac{a_0}{1-r}$, $a \neq 0$. $|r| \geq 1 \longrightarrow$ diverges by $a_n \not\to 0$
nth term test.

$\boxed{\text{The Comparison Test}}$

Consider two sequence of real numbers $(a_n), (b_n)$ with $|a_n| \leq b_n$, $\forall n \geq 1$.

- If (b_n) is summable (i.e. $\sum_{n=1}^{\infty} b_n$ converges), then (a_n) is summable and the following is true: $|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} b_n$.
- If (a_n) is not summable, then (b_n) is not summable.

Proof:

- Let $\varepsilon > 0$ be given since (b_n) is summable, $\exists N > 0$ s.t.

$$\sum_{k=n+1}^{m} b_k < \varepsilon \quad , \forall m \geq n \geq N$$

$$\left|\sum_{k=n+1}^{m} a_k\right| \leq \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} b_k < \varepsilon \text{ when } m \geq n \geq N$$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ converges

- Sps $\sum_{n=1}^{\infty} b_n$ does not diverge. $\Rightarrow \sum_{n=1}^{\infty} a_n$ converges as well.
(contrapositive)

$\boxed{\text{The n-th root test}}$

Suppose $a_n \geq 0$, for all n.
let $\ell = \limsup \sqrt[n]{a_n}$. if $\ell < 1$, then $\sum_{n=1}^{\infty} a_n$ converges
if $\ell > 1$, then $\cdots$ diverges
if $\ell = 1$, nothing can be determined

Proposition: let (x_n) be a bdd sequence, let L be the set of all real #s which are the limit of some subsequence of (x_n). $\Rightarrow L$ has maximum & minimum
(proof comes later in this course)

$\max L = \bar{\ell} =$ limit superior $= \limsup$
$\min L = \underline{\ell} = \liminf$

Ex: ① $(\frac{1}{n})_{n=1}^{\infty}$, $\limsup = 0 = \liminf$ since $L = \{0\}$
② $((-1)^n)$, $\limsup = 1$, $\liminf = -1$