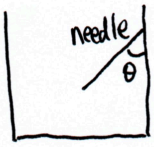
1: Solution: Whether the needle hits the crack or not depend on the angle between the needle and the crack.



the occurence would occur only if $X \le \frac{L}{2}\sin\theta$.

Moreover, $\theta \in [0, \frac{\pi}{2}]$ and the density of X is $\frac{2}{D}$.

$$P(\text{needle intersects crack}) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{L}{2}\sin\theta} \frac{4}{\pi D}\, dx\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{4}{\pi D} \cdot \frac{L}{2}\sin\theta\, d\theta = \frac{2L}{\pi D} \qquad \text{QED}.$$

2: Solution:

Let X_1 denote the random variable that the particle makes a collision. X_2 denote the random variable that the particle. get out of the chamber.

$$P(X_1 \le x_2) = \int_0^{+\infty} \int_0^{x_2} \alpha e^{-\alpha x_1} \cdot \beta e^{-\beta x_2}\, dx_1\, dx_2$$

$$= \alpha \cdot \beta \cdot \int_0^{+\infty} e^{-\beta x_2} (-\tfrac{1}{\alpha})(e^{-\alpha x_2} - 1)\, dx_2$$

$$= -\beta \cdot \int_0^{+\infty} e^{-(\alpha+\beta)x_2} - e^{-\beta x_2}\, dx_2$$

$$= 1 - \frac{\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}.$$

3: Solution: the mapping from $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ to $\begin{pmatrix} X_1 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} X_1 \\ Y \end{pmatrix}$ is one-to-one

$$\begin{cases} X_1 = x_1 \\ x_2 = y - x_1 \end{cases} \Rightarrow J(X, Y) = \det\left|\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right| = 1 \quad \text{so } g(y) = \int f(x_1, y - x_1)\, dx_1$$

Part 2:

the characteristic function of normal is $\frac{1}{\sqrt{2\pi V}} \cdot e^{-X_j^2/2V}$

$$\phi_Y(\theta) = E(e^{i\theta Y}) = E(\tfrac{i\theta X_1^2}{2})\, E(\tfrac{i\theta X_2^2}{2}) \cdots E(\tfrac{i\theta X_n^2}{2}) \text{ for } X_i\text{'s}.$$

$$E(e^{i\theta \frac{X_i^2}{2}}) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{\frac{i\theta X_i^2}{2}} e^{-\frac{X_i^2}{2V}}\, dx = \frac{1}{\sqrt{2\pi V}} \int_0^{\infty} e^{-\frac{X_i^2}{2V}}\, dz$$

$$= (1 - i\theta V)^{-1/2}.$$

then $\phi_Y(\theta) = (1 - i\theta V)^{-\frac{m}{2}}$.

4: Solution: we first divide the interval into $[-\pi, 0] \cup [0, \pi]$ both with the same density $\frac{1}{2\pi}$ for $x \in [-\pi, 0]$:

$$f(x) = \frac{1}{2\pi} \quad \text{and} \quad \left|\frac{dx}{dy}\right| = \left|\frac{d\arccos\frac{y}{a}}{dy}\right| = \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}$$

$$g(y) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}.$$

Similarly for $x \in [0, \pi]$ $g(y) = \frac{1}{2\pi} \cdot \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}$ $y \in [-a, a]$.

$$\int_{-a}^{a} \frac{1}{2\pi a} \cdot \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}\, dy + \int_{-a}^{a} \frac{1}{2\pi a} \cdot \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}\, dy = 1$$

$$\Rightarrow \int_{-a}^{a} \frac{1}{2\pi a} \cdot \frac{1}{\sqrt{1-(\frac{y}{a})^2}} \cdot \frac{1}{a}\, dy = \frac{1}{2}.$$

$$\Rightarrow \int_{-a}^{a} \frac{a}{\pi a\sqrt{a^2-y^2}}\, dy = 1. \Rightarrow \text{the density of } Y \text{ is}$$

$$\frac{1}{\pi\sqrt{a^2-y^2}}$$

5: Solution:

$$\boxed{\begin{aligned}&\text{Let the distribution function of } X \text{ be } F.\\ &P(Y \leq y) = P(Y=1) + \cdots + P(Y = \lfloor y \rfloor). + P(Y=0)\\ &\quad = P(\lfloor X \rfloor = 0) +\end{aligned}}$$ neglect this part

Let the distribution function of X be F.

$$P(Y \leq y) = P(\lfloor X \rfloor = 0, Y \leq y) + P(\lfloor X \rfloor = 1, Y \leq y) + \cdots \quad (\text{which is } \sum_{i=0}^{\infty} P(\lfloor X \rfloor = i, Y \leq y))$$

$$= [F(0+y) - F(0)] + (F(1+y) - F(1)] + \cdots$$

$$= \sum_{j=0}^{\infty} (F(j+y) - F(j)) \qquad \text{QED.}$$

6: (2) Solution:

$$Cov(V, W) = E((V-E(V))(W-E(W))) \text{ and } \quad ①$$

$$Cov(W, E(V|W)) = E((W-E(W))(E(V|W) - E(E(V|W))))$$
$$= E((W-E(W))(E(V|W) - E(V))) \quad ②$$

use ① - ② we have $E((W-E(W))(V-E(V)+E(V)-E(V|W)))$
$$= E((V-E(V|W))(W-E(W)))$$

note that $W-E(W)$ is a function of W.

By the definition of conditional expectation, this last formula is 0.

Hence $Cov(V, W) = Cov(W, E(V|W))$.

(7)
Solution: we should analyze $E((E(XZ|Y) - X E(Z|Y))^2)$

expand we have $E((E(XZ|Y) - XZ + XZ - X \cdot E(Z|Y))^2)$
$$= E((E(XZ|Y) - XZ)^2) + E((XZ - XE(Z|Y))^2)$$
$$+ 2E((E(XZ|Y) - XZ) \cdot (XZ - X \cdot E(Z|Y)))$$

we analyze this term by term.

① For $E((XZ - E(XZ|Y))^2)$: as X is a function of y. $E(XZ|Y)$ is a function of y. Hence this is simply $E((XZ - E(XZ|Y) \cdot H(Y)) = 0$.

② For $E((E(XZ|Y) - XZ) \cdot X \cdot (Z - E(Z|Y)))$. $X = f_1(y)$ and. $E(XZ|Y) \cdot XZ = H(y)$

So similar to ①, it is again 0.

③ similar as above, it is again 0.

Hence, $E(XZ|Y) = X \cdot E(Z|Y)$ a.s

7: Solution: For the case where none of $E(X^2|Z)$, $E(Y^2|Z)$ equals to 0:

$$(X-tY)^2 \geq 0 \Rightarrow E[(X-tY)^2|Z] \geq 0. \Rightarrow E[(X^2-2tXY+t^2Y^2|Z] \geq 0$$

$$\Rightarrow E(X^2|Z) - 2t\, E(XY|Z) + t^2\, E(Y^2|Z) \geq 0.$$

$$\Rightarrow \frac{4E(Y^2|Z)E(X^2|Z) - 4(E(XY|Z))^2}{4E(Y^2|Z)} \geq 0.$$

$$\Rightarrow. \quad E((XY|Z))^2 \leq 4E(Y^2|Z)\cdot E(X^2|Z)$$

If one of $E(X^2|Z)$ or $E(Y^2|Z)=0$:

the RHS $=0$. Without loss of generality, we assume $E(X^2|Z)=0$.

$$\Rightarrow E[E(X^2|Z)] = E(X^2) = 0 \Rightarrow X=0 \quad a.s.$$

Suppose $E[XY|Z] \neq 0$. $\exists$ a Borel set $S \subseteq \mathbb{R}$ $E[E(XY|Z) I_{Z\in S}] \neq 0$.

$\Leftrightarrow E[XY I_{Z\in S}] \neq 0$. As $E(Y^2) < \infty$ and $X=0$ a.s. this is impossible.

Hence $E[XY|Z]=0 \Rightarrow$ Shows the equality.