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MAT 246 HW8 Rui Qiu 999292509

Q1.

Claim: $\exists f: \mathbb{N} \to \mathbb{Z}$ which is a bijection such that $|\mathbb{Z}| = |\mathbb{N}|$

Proof: Suppose $f: \mathbb{N} \to \mathbb{Z}$

$$f = \begin{cases} \frac{n}{2} & \text{when } n \text{ is even} \\ \frac{-(n+1)}{2} & \text{when } n \text{ is odd} \end{cases}, n \in \mathbb{N}$$

(Injection) Suppose again $f(a) = f(b)$ for $a, b \in \mathbb{N}$

if a and b are both ~~odd~~ even

then $\frac{a}{2} = \frac{b}{2} \Rightarrow a = b$

if a and b are both odd

then $\frac{-(a+1)}{2} = \frac{-(b+1)}{2} \Rightarrow a = b$

Good!

if a is odd, but b is even (opposite situation is similar)

then $\frac{-(a+1)}{2} = \frac{b}{2}$

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$\Rightarrow -(a+1) = b$

Since $a \in \mathbb{N}$ then $-(a+1) < 0$

but $b \in \mathbb{N}$ $b > 0$. contradiction

hence for a, b both even and a, b both odd.

$f(a) = f(b) \Rightarrow a = b$

Thus f is one-to-one.

(Surjection) Suppose m is an arbitrary integer

if m is ~~even~~ positive

then $\exists\ n = 2m$ s.t. $f(n) = f(2m) = \frac{2m}{2} = m. \in \mathbb{Z}$

if m is ~~negative~~ non-positive

then $\exists\ n = -2m-1$ s.t. $f(n) = f(-2m-1) = \frac{-(-2m-1+1)}{2} = m \in \mathbb{Z}$

hence f is onto

Therefore f is a bijection.

By the definition of cardinality, $|\mathbb{Z}| = |\mathbb{N}|$.

Q2.

~~Claim~~ Solution:

~~Suppose $f = \begin{cases} n + \frac{1}{2}(\frac{1}{2} - n) = \frac{1}{2}n + \frac{1}{4} & \text{if } 0 \le n \le \frac{1}{2} \\ n - \frac{1}{2}(n - \frac{1}{2}) = \frac{1}{2}n - \frac{1}{4} & \text{if } \frac{1}{2} < n \le 1 \end{cases}$~~

~~Say $f(m) = f(n)$~~

~~① $m, n \in [0, \frac{1}{2}] \Rightarrow \frac{1}{2}m + \frac{1}{4} = \frac{1}{2}n + \frac{1}{4} \Rightarrow m = n$~~

~~② $m, n \in (\frac{1}{2}, 1] \Rightarrow \frac{1}{2}m - \frac{1}{4} = \frac{1}{2}n - \frac{1}{4} \Rightarrow m = n$~~

~~③ $m \in [0, \frac{1}{2}], n \in (\frac{1}{2}, 1] \Rightarrow \frac{1}{2}m + \frac{1}{4} = \frac{1}{2}n - \frac{1}{4}$~~

Suppose $f = \begin{cases} \frac{1}{2 + \frac{1}{x}} = \frac{x}{2x+1} & \text{if } x = \frac{1}{n+1}, \text{for } n \in \mathbb{Z}, n \ge 1 \\ x & \text{if } x \in (0,1) \text{ but } x \ne \frac{1}{n+1} \text{ for } n \in \mathbb{Z}, n \ge 1. \\ \frac{1}{2} & \text{if } x = 0 \\ \frac{1}{3} & \text{if } x = 1 \end{cases}$

(Injection) Say $f(x) = f(y)$

① When ~~[illegible]~~ $\frac{x}{2x+1} = \frac{y}{2y+1} \Rightarrow x = y$

② $\frac{x}{2x+1} = y \Rightarrow \frac{\frac{1}{n+1}}{\frac{2}{n+1} + 1} = \frac{1}{n+3} = y$ contradicts the condition that $y \ne \frac{1}{n+1}$ for some n.

③ $\frac{x}{2x+1} = \frac{1}{2} \Rightarrow 2x = 2x + 1 \Rightarrow$ no solution

④ $\frac{x}{2x+1} = \frac{1}{3} \Rightarrow 3x = 2x + 1 \Rightarrow x = 1 = y$ ~~contradicts~~

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What do these mean?

~~the condition that $f(x) =$~~ !

⑤ $x = \frac{1}{2}$ contradicts that $x \ne \frac{1}{n+1}$ for some n

⑥ $x = \frac{1}{3}$ contradicts that $x \ne \frac{1}{n+1}$ for some n.

⑦ $x = y \Rightarrow x = y$ automatically

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⑧ $\frac{1}{2} = \frac{1}{3}$? impossible.

To conclude. $f(x) = f(y) \Rightarrow x = y$.

Thus f is one-to-one.

(Surjection).

Suppose m is an arbitrary number in $(0,1)$.

if $m = \frac{1}{n+3}$ for some $n \in \mathbb{Z}, n \geq 1$

$$f\left(\frac{1}{n+1}\right) = \frac{\frac{1}{n+1}}{\frac{2}{n+1}+1} = \frac{1}{n+3} = m$$

if $m = \frac{1}{2}$, $f(0) = \frac{1}{2} = m$

if $m = \frac{1}{3}$, $f(1) = \frac{1}{3} = m$

if $m \neq \frac{1}{n+3}$ for some $n \in \mathbb{Z}, n \geq 1$

$f(m) = m$

Thus f is onto.

Hence f is a bijection that $f: [0,1] \to (0,1)$.

Q3.

Claim: $|P(A)| = 2^{|A|}$

Proof: Let $A = \{x_1, \cdots, x_n\}$. Say a subset S of A is represented by a sequence of 0s & 1s of length n such that the ith element in the sequence is 1 if $x_i \in S$ and 0 is if $x_i \notin S$.

~~ie~~ e.g. if $S = \{x_1, x_2\}$, we writing $(1,1,0,0,\cdots 0)$ in total n elements.

if $S = \emptyset$, we writing $(0,0,\cdots,0)$ in total n elements.

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Then we look at how many sequenes can we write to represent all subsets of A.

Since A has n elements,

i.e. the sequence we write has n elements (n digits).

and each element has two possibilities, namely 1 and 0.

hence, ~~by~~ we can write 2^n sequences.

Thus the cardinality of $P(A)$ is 2^n.

i.e. $|P(A)| = 2^n$

Since $|A| = n$

Therefore $|P(A)| = 2^{|A|}$

[redacted]

Construct $f: P(S) \to \{0,1\}^{|S|}$ and show that it is bijective. Then we get:

$$|P(S)| = |\{0,1\}^{|S|}| = |\{0,1\}|^{|S|} = 2^{|S|}.$$

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Q4.

Proof (Part of the Shrader -- Berenstein Theorem):

We've already have $f: S \to T$ and $g: T \to S$ one-to-one.
And we're devided S into three subsets S_∞, S_S & S_T.
similarly T is devided into T_∞, T_S, T_T.

Define $h: S \to T$

$$h(s) = \begin{cases} f(x) \text{ if } s \in S_S \cup S_\infty \\ g^{-1}(x) \text{ if } s \in S_T \end{cases}$$

Proved $h: S_S \to T_S$ is 1-1 and onto.

want to prove $h: S_T \to T_T$ and $h: S_\infty \to T_\infty$ also 1-1 and onto.

(Injection) ① s_1, s_2 are in the same subset (i.e. $S_S \cup S_\infty$ or S_T)

Suppose $h(s_1) = h(s_2)$ for $s_1, s_2 \in S$.

if both $s_1, s_2 \in S_S \cup S_\infty$, then $h(s_1) = f(s_1)$ and $h(s_2) = f(s_2)$

then $f(s_1) = f(s_2)$

since f is one-to-one

then $s_1 = s_2$

if both $s_1, s_2 \in S_T$

then $h(s_1) = g^{-1}(s_1)$ and $h(s_2) = g^{-1}(s_2)$

then $g^{-1}(s_1) = g^{-1}(s_2)$

apply g to both sides

$g(g^{-1}(s_1)) = g(g^{-1}(s_2))$

so $s_1 = s_2$

Therefore $s_1 = s_2$ in this case.

② $s_1 \in S_\infty \cup S_S$ and $s_2 \in S_T$.

then $h(s_1) = f(s_1)$ and $h(s_2) = g^{-1}(s_2)$

therefore $f(s_1) = g^{-1}(s_2)$

We need to show a contradiction here.

For $f(s_1) = g^{-1}(s_2) \Rightarrow s_1$ is an immediate ancestor of $g^{-1}(s_2)$.

Thus s_1 is an ancestor of s_2, the ultimate ancestor of s_2

is also the ultimate ancestor of ~~S_2~~ S_1.
But $S_1 \in S_\infty \cup S_S$ which means S_1 either has an ultimate ancestor in ~~S~~ S or it does not have an ultimate ancestor. THIS contradicts the fact that S_2 has an ultimate ancestor in T.
Contradiction shown.
So far, we proved h is one-to-one.
then $h: S_\infty \to T_\infty$ and $h: S_T \to T_T$ are one-to-one.

(surjection): Note that already proved
$h: S_S \to T_S$ is onto.

① Now consider any $t \in T_\infty$.
For $t \in T_\infty$, t has an immediate ancestor in S, $f^{-1}(t)$.
Since the ancestors of $f^{-1}(t)$ are the same ancestors of t,
~~$\in$~~ t has no ultimate ancestors.
then $f^{-1}(t)$ has no ultimate ancestors as well.
Thus $f^{-1}(t) \in S_\infty$.
The function h was defined to be the function f on S_∞.
So $h(f^{-1}(t)) = f(f^{-1}(t)) = t$.
Therefore $Im(h) \supseteq T_\infty$.

② Consider any $t \in T_T$.
Let $s = g(t)$, so t is the immediate ancestor of s.
Then t and s share the same ultimate ancestor
Since the ~~at~~ ultimate ancestor of t lies in T,
the ultimate ancestor of s also is in T.
i.e. $s \in S_T$
So $h(s) = g^{-1}(s)$
Since $s = g(t)$
$h(s) = g^{-1}(g(t)) = t$
Therefore $Im(h) \supseteq T_T$

Hence h is onto.

Thus h is a bijection from S to T.

Then $|S| \geq |T|$ & $|S| \leq |T| \Rightarrow |S| = |T|$

Q5. Claim: S is infinite and $A \subset S$ be finite, then
$|S| = |S \setminus A|$

Proof: Let $A = \{s_1, \cdots, s_n\}$.
Since S is infinite the set $S \setminus A$ is non-empty.
Pick any $s_{n+1} \in S \setminus A = S \setminus \{s_1, \cdots, s_n\}$
Since $S \setminus \{s_1, \cdots, s_n\}$ is non-empty
we can pick another $s_{n+2} \in S \setminus \{s_1, \cdots, s_n, s_{n+1}\}$.

$s_{n+k} \in S \setminus \{s_1 \cdots s_n\}$ Continuing this by induction we construct
for k is some natural number which is larger than n.
So now we can define a map $f: S \to S \setminus A$ by
$f(s_i) = s_{i+n}$ for any i $s_i \in A$
and $f(x) = x$ if $x \in S \setminus \{s_1, s_2, \cdots s_n\}$.

$S \to$

Now we only need to prove f is a bijection. $s_i = s_{i+n} = s_{i+2n} = s_{i+in}$

① Suppose $s_p, s_q \in A$.
say $f(s_p) = f(s_q)$
then $s_{p+n} = s_{q+n}$
hence $p+n = q+n$
$p = q$
therefore $s_p = s_q$

② Suppose $s_p, s_q \in S \setminus A$
say $f(s_p) = f(s_q)$
then $s_p = s_q$

③ Suppose $s_p \in A$, $s_q \in S \setminus A$
say $f(s_p) = f(s_q)$ Impossible
$s_{p+n} = s_q = s_{q+n}$
so $s_p = s_q$

Hence f is an injection $\Rightarrow |S| \leq |S \setminus A|$

And since $S \setminus A$ is ~~at~~ a subset of S

$S \setminus A \subseteq S$

so $|S| \geq |S \setminus A|$

By Shroeder-Berenstein Theorem:

$|S| = |S \setminus A|$

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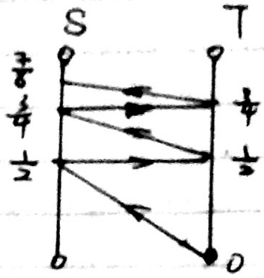
Q6. $S=(0,1)$, $T=[0,1)$

$f: S \to T \quad f(x)=x$

$g: T \to S \quad g(x)=\frac{x+1}{2}$



(a). Solution:

Notice that for $[\frac{1}{2}, 1)$

$f(1-\frac{1}{2^n}) = 1-\frac{1}{2^n}$ for any $n \geq 1$ and

$g(1-\frac{1}{2^n}) = 1-\frac{1}{2^{n+1}}$ for any $n \geq 0$

So, here we have $g(0)=\frac{1}{2}$, $f(\frac{1}{2})=\frac{1}{2}$, $g(\frac{1}{2})=\frac{3}{4}$, $f(\frac{3}{4})=\frac{3}{4}$, $g(\frac{3}{4})=\frac{7}{8}$ etc. (as the graph shows)

~~Now we use induction to prove.~~

① Consider that $1-\frac{1}{2^n} \in S$ for any $n \geq 1$ and $1-\frac{1}{2^n} \in T$ for any $n \geq 0$ share the same last ancestor $y=0 \in T$ (note that $\{0\} \notin Im(f)$).

Thus $\{1-\frac{1}{2^n} \mid \text{for } n \geq 1\} \subset S_T$

$\{1-\frac{1}{2^n} \mid \text{for } n \geq 0\} \subset T_T$

② Claim that $\forall n \geq 0$ the interval $(1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$ is contained in S_S. Prove this by induction.

(Base case) When $n=0$, the corresponding interval is $(0, \frac{1}{2})$.

Since $g([0,1)) = [\frac{1}{2}, 1)$

hence if $x \in (0, \frac{1}{2})$, it doesn't have any ancestors

therefore $(0, \frac{1}{2}) \subset S_S$ by definition.

(Inductive Step). Suppose we proved that $(1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$ is contained in S_S for some $n \geq 0$

(proof of inductive step)

Suppose $x \in (1-\frac{1}{2^{n+1}}, 1-\frac{1}{2^{n+2}})$

Then $x = g(y)$ with $y = 2x-1$ and $y \in (1-\frac{1}{2^n}, 1-\frac{1}{2^{n+1}})$

so $y \in T$ is the immediate ancestor of x.

Then $y = f(y) \in S$ is the second ancestor of x.

And we know that the last ancestor of x is the same as the last ancestor of y.

But $y \in S_S$ by the ~~inductive~~ assumption

thus $x \in S_S$.

So far, we proved that all $x \neq 1-\frac{1}{2^n}$ is contained in S_S.

And note that $S_\infty = T_\infty = \emptyset$.

$S_T = \{1-\frac{1}{2^n} \mid \text{for } n \geq 1\}$

$S_S = (0,1) \setminus S_T$.

$T_T = \{1-\frac{1}{2^n} \mid \text{for } n \geq 1\} \cup \{0\}$

$T_S = [0,1) \setminus T_T$.

(b). Solution: By the proof of S-B theorem and some conclusions in (a), define $f: S \to T$ by the formula:

$$h(x) = \begin{cases} f(x) = x & \text{if } x \in S_S \\ g^{-1}(x) = 2x-1 & \text{if } x \in S_T \end{cases}$$

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Q 6.

Claim: $S = P(\mathbb{N})$, $|S| \leq |\mathbb{R}|$

~~$|S| < |\mathbb{R}|$~~

Proof: ~~Since $|\mathbb{R}| = |(0,1)|$~~

Note that $|\mathbb{R}| = |(0,1)|$

First we make a rule that all real numbers in $(0,1)$ can be written as a binary decimal form:

$x = 0.b_1 b_2 b_3 \cdots$ for $b_j = 0$ or 1.

e.g:

$x = 0.1010\cdots_{(2)} = \frac{1}{2} + \frac{0}{2^2} + \frac{1}{2^3} + \frac{0}{2^4}_{(10)} + \cdots = \frac{5}{8}$

Now we associate these binary decimal form real numbers with the subset of natural numbers consisting of elements j where $b_j = 1$ as shown previously.

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e.g:

~~0.1010~~ $0.1010\cdots \longrightarrow \{1, 3, \cdots\}$

$0.111010\cdots \longrightarrow \{1, 2, 3, 5, \cdots\}$

$0.001011\cdots \longrightarrow \{3, 5, 6, \cdots\}$

Hence have an ~~one to one~~ surjective map which is $f: (0,1) \to P(\mathbb{N})$ (This is True b/c all the elements in power set of $\mathbb{N}$ can be written as $0.b_1 b_2 \cdots$) obviously

Thus $|(0,1)| \geq |P(\mathbb{N})|$

Since $|(0,1)| = |\mathbb{R}|$ and $|S| = |P(\mathbb{N})|$

Therefore $|S| \leq |\mathbb{R}|$.

Why is this well defined?

$\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$