March 12th

From now, every thing on $\mathbb{C}$

Def: $T: V \to V$ is triangularizable if there exists a basis α such that $[T]_\alpha$ is upper-triangular

Def: $T: V \to V$ $W \subseteq V$ then W is invariant under T if $T(W) \subset W$

Ex: $T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=0 \right\} \subseteq \mathbb{C}^3$

W is invariant under T $\quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} c \\ b \\ a \end{bmatrix}$

Always true that V is invariant under T.

... $\{0\}$ is ... .

$\left\{ \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} , b \in \mathbb{C} \right\}$ $\qquad T\begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix}$

So $U = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ also invariant under T.

Suppose $T: V \to V$ is trangularizable

so there is basis $\alpha = \{v_1, \cdots, v_n\}$ s.t. $[T]_\alpha = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

$W_i = \text{span}\{v_1, \cdots, v_i\}$ $\quad$ Eg $W_1 = \text{span}\{v_1\}$

Note: $W_1 \subset W_2 \subset W_3 \subset \cdots \subset W_{n-1} \subset W_n = V$

Note: Each W_i is T-invariant

i.e. $T(W_i) \subseteq W_i$

$T(v_1) = \lambda_1 v_1 \Rightarrow T(W_1) \subseteq W_1$

$T(v_2) = \lambda_2 v_2 + a v_1 \in W_2 \Rightarrow T(W_2) \subset W_2$

... $\quad$ ... $T(W_3) \subset W_3$

Prop: $T: V \to V$ T is triangularizable $\Leftrightarrow$ then $\exists$ a sequence of spaces $W_1 \subset W_2 \subset \cdots \subset W_n$ s.t. each W_i is T-invariant and $\dim W_i = i$

In other words, the notation of triangularizability can be phrased in terms of invariant subspaces.

Thm: Every operator is triangularizable.

Proof: A consequence of this Carley-Hamilton Thm:

$T: V \to V$ $p(\lambda)$ characteristic polynomial

$p(T) = 0$

Ex: if $p(\lambda) = \lambda^3 - \lambda + 4$. $p(T) = T^3 - T + 4I$

Ex: $T = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 2 & 1 & -1 \end{bmatrix}$ $\mathbb{C}^3 \longrightarrow \mathbb{C}^3$

$p(\lambda) = -\lambda^3 + 3\lambda^2 + 4\lambda - 6$

$CH \Rightarrow -T^3 + 3T^2 + 4T - 6I = 0$
$\Rightarrow -T^3 + 3T^2 + 4T = 6I$
$\Rightarrow \frac{1}{6}(-T^3 + 3T^2 + 4T) = I$
$\Rightarrow \frac{1}{6}(-T^2 + 3T + 4I)T = I$

$\Rightarrow T^{-1} = \frac{1}{6}(-T^2 + 3T + 4I)$

Proof of CH. Let $\alpha = \{v_1, \cdots, v_n\}$ be a basis s.t. $[T]_\alpha = \begin{bmatrix} \lambda_1 & & & * \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \lambda_n \end{bmatrix}$

so $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$
We want to: $(T - \lambda_1 I)(T - \lambda_2 I)(T - \lambda_3 I) \cdots (T - \lambda_n I) = 0$
i.e. we want to show
$(T - \lambda_1 I) \cdots (T - \lambda_n I) v_i = 0$ for all i.

We'll show by induction on i that $(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)(W_i) = 0$
where $W_i = \text{span}\{v_1, \cdots, v_i\}$

Base case: at $i=1$ the statement is $(T - \lambda_1 I)(W_1) = 0$
$W_1 = \text{span}\{v_1\}$
$Tv_1 = \lambda_1 v_1$
$\Rightarrow (T - \lambda_1 I)(v_1) = 0$
$\Rightarrow (T - \lambda_1 I)(W_1) = 0$

inductive hypothesis: $(T - \lambda_1 I) \cdots (T - \lambda_{i-1} I)(T - \lambda_i I)(W_i) = 0$

$W_i = \text{span}\{v_1, v_2, \cdots, v_i\}$ so want $(T - \lambda_1 I) \cdots (T - \lambda_i I)(v_j) = 0$
for $j = 1, 2, \cdots$

if $j = 1, \cdots, j-1$ then $(T - \lambda_i I)(v_j) \in W_{i-1}$ since $T(v_j) \in W_{i-1}$ and
$\lambda_i v_j \in W_{i-1}$

so let $y_j = (T - \lambda_i)(v_j) \in W_{i-1}$
$(T - \lambda_1 I) \cdots (T - \lambda_i I)(v_j) = (T - \lambda_1 I) \cdots (T - \lambda_{i-1} I)(y_j) \underset{\uparrow}{=} 0$
by inductive hypothesis

If $j = i$, $T(v_i) = \lambda_i v_i + x_i$ where $x_i \in W_{i-1}$
$(T - \lambda_i I)(v_i) = x_i \in W_{i-1}$
so $(T - \lambda_1 I) \cdots (T - \lambda_i I)(v_i) = \underbrace{(T - \lambda_1 I) \cdots (T - \lambda_{i-1} I)(x_i) = 0}_{\text{by ind hyp.}}$

LA F-C A35

Nilpotent matrices

Suppose $T: V \to V$ whose only eigenvalue is 0.
So there is a basis α s.t.

$$[T]_\alpha = \begin{bmatrix} 0 & * & * & \cdots & * \\ 0 & 0 & * & & \vdots \\ \vdots & \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} \text{ "strictly upper triangle"}$$

$$p(\lambda) = \lambda^n$$

e.g. $[T]_\alpha = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$$p(\lambda) = \begin{vmatrix} \lambda & -1 & -2 \\ 0 & \lambda & -3 \\ 0 & 0 & \lambda \end{vmatrix} = \lambda^3$$

$CH \Rightarrow T^n = 0$

Conversely, sps $T^k = 0$. Let λ be an evalue of T. $T(v) = \lambda v$ for some $v \neq 0$.

Easy proof:
$T(v) = \lambda v \Rightarrow T^k(v) = \lambda^k v$
$\Rightarrow \lambda^k v = 0 \Rightarrow \lambda^k = 0$
$\Rightarrow \lambda = 0$

We proved: $T: V \to V$. Then the only eigenvalue is $0 \Leftrightarrow T^k = 0$ for some k

Def: $T: V \to V$ is nilpotent if $T^k = 0$ for some k or equivalently if the only evalue of T is 0.

Eg. $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} : \mathbb{C}^3 \to \mathbb{C}^3$

$N^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^3 = 0$ so N is nilpotent

$N: V \to V$ nilpotent: $\Rightarrow N^k = 0$
$v \neq 0 \quad N^k(v) = 0$
$\underset{\neq 0}{v}, N(v), N^2(v), \cdots, \underset{\neq 0}{N^k(v)}$

so $\exists\, j$ s.t. $N^{j-1}(v) \neq 0$ while $N^j(v) = 0$

in Ex $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad v = e_2$

$v = e_2, N(v) = e_1, N^2(v) = 0, N^3(v) = 0, \cdots$
So $N^2(v) = 0$ while $N(v) \neq 0$.

if $v = e_3$ then we look at e_3, $N(e_3) = e_2$. $N^2(e_3) = e_1$, $N^3(e_3) = 0$.
so $j = 3$ here.

Def: $N: V \to V$ nilpotent $v \neq 0$
Choose j s.t. $N^{j-1}(v) \neq 0$, $N^j(v) = 0$.
$\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$ is called the cycle associated to N and v; j is the length of the cycle and $C(v) = \text{span}\{N^{j-1}(v), N^{j-2}(v), \cdots, v\}$ is called the cyclic-subspace associated to N and v.

Eg: $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ the cyclic subspace associated to e_2 is $sp\{e_1, e_2\}$

- - - ——— - - e_3 $sp\{e_1, e_2, e_3\}$

Prop: $N: V \to V$ nilpotent
$v \neq 0$ Then
$\dim C(v) =$ length of cycle associated to N and v. $= k$