# Problem Set 4

1) $f(x,y) = (x-1)^2 + \sin(\pi y) + x\ln y$ at $(x,y) = (2,1)$

Let $s = x-2$, $t = y-1$

so $f(x,y) = (s+1)^2 + \sin(\pi(t+1)) + (s+2)\ln t+1$

$= 1+2s+s^2 - \sin\pi t + s + 2\left(t - \frac{t^2}{2} + \frac{t^3}{3} - \dots\right)$

$= 1+2s+s^2 - \left(\pi t - \frac{(\pi t)^3}{3!} - \dots\right) + \left(st - \frac{st^2}{2} + \dots\right) + \left(2t - t^2 + \frac{2t^3}{3}\right)$

$= 1+2s+s^2 + (2-\pi)t + \frac{t^3}{3}\left(\frac{\pi^3}{2} + 2\right) + st - \frac{st^2}{2} - t^2 + \dots$

Hence the $3^{rd}$ order Taylor polynomial at $(2,1)$

is $P(x,y) = 1 + 2(x-2) + (x-2)^2 + (2-\pi)(y-1) - (y-1)^2$

$+ (x-2)(y-1) - \frac{1}{2}(x-2)(y-1)^2 + \frac{1}{3}\left(\frac{\pi^3}{2} + 2\right)(y-1)^3$

2) a) $f(x,y) = \frac{1}{1-x-y} = \frac{1}{1-t}$, $t = x+y$.

so as $\frac{1}{1-t} = 1 + t + t^2 + \dots$

$f(x,y) = 1 + (x+y) + (x+y)^2 + \dots$

$P_{0,2}(x,y) = 1 + x + y + x^2 + y^2 + 2xy$

b) $\partial_x f = (1-x-y)^{-2} = \partial_y f$

$\partial_x^2 f = 2(1-x-y)^{-3} = \partial_y^2 f = \partial_x\partial_y f$

so $\sum_{|\alpha|\le 2} \frac{\partial^\alpha f}{\alpha!}$ has $\alpha = (0,0), (1,0), (0,1), (1,1), (2,0), (0,2)$

where $\partial^\alpha f$ were computed above.

So $\displaystyle\sum_{|\alpha|\le 2} \frac{\partial^\alpha f(0,0)}{\alpha!} = f(0,0) + \partial_x f(0)x + \partial_y f(0)y$

$+ \partial_x\partial_y f(0)xy$

$+ \partial_x^2 f(0)\frac{x^2}{2} + \partial_y^2 f(0)\frac{x^2}{2}$

$= 1 + x + y + 2xy + x^2 + y^2$

same as part a)

c) For $|\alpha| = 3$ have $\alpha = (3,0), (2,1), (1,2), (0,3)$

but $\partial^\alpha f(x,y) = 6(1-x-y)^{-4}$ for all such $\alpha$.

For $|x| \le \frac{1}{4}$, $|y| \le \frac{1}{4}$ $\frac{1}{2} \le 1-x-y \le \frac{3}{2}$

so $|\partial^\alpha f(x,y)| \le \dfrac{6}{(\frac{1}{2})^4} = 6 \times 2^4$

d) $|R_{0,2}(x,y)| = \left| \displaystyle\sum_{|\alpha|=3} \frac{\partial^\alpha (f(c_1,c_2)|(x,y)^\alpha}{\alpha!} \right|$

$= \dfrac{6\times 2^4}{3!} \|(x,y)\|^3$ when $\|(x,y)\| = |x| + |y| \le \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

via Cor 2.75

so $|R_{0,2}(x,y)| \le \dfrac{6\times 2^4}{3!} \left|\frac{1}{2}\right|^3 = 2$

3) Consid $F_1(x_1 \dots x_m, u_1 \dots u_n) = 0$

$\vdots$

$F_n(x_1 \dots x_m, u_1 \dots u_n) = 0$

Assume can solve $u_i(x_1 \dots x_m)$ $\forall i \in \{1, \dots n\}$

Taking derivative w.r.t $x_j$:

$$\partial_j F_1 + \sum_{i=1}^{n} \partial_{m+i} F \partial_j u_i$$

$$* \quad \vdots$$

$$\partial_j F_n + \sum_{i=1}^{n} \partial_{m+i} F \partial_j u_i$$

Setting $\vec{b} = \begin{pmatrix} -\partial_j F_1 \\ \vdots \\ -\partial_j F_n \end{pmatrix}$, $A = \begin{pmatrix} \partial_{m+1} F_1 & \dots & \partial_{m+n} F_1 \\ \vdots & & \vdots \\ \partial_{m+1} F_n & \dots & \partial_{m+n} F_n \end{pmatrix}$

$\vec{x} = \begin{pmatrix} \partial_j u_1 \\ \vdots \\ \partial_j u_n \end{pmatrix}$ So $*$ is $A\vec{x} = \vec{b}$

When $A$ is invertible, can use Cramer's Rule to solve for $\vec{x}$:

$$x_i = \partial_j u_i = \frac{-\det \begin{pmatrix} \partial_{m+1} F_1 \dots & \partial_j F_1 & \dots \partial_{m+n} F_1 \\ \vdots & & \\ \partial_{m+1} F_n \dots & \partial_j F_n & \dots \partial_{m+n} F_n \end{pmatrix}}{\det \begin{pmatrix} & A & \end{pmatrix}}$$

where numerator has $i^{th}$ column replaced with $-b$.

3b) Denote $\frac{dy}{dx} = y'$, $\frac{dz}{dx} = z'$, $\frac{dw}{dx} = w'$, $x$ our independent variable.

Differentiating $F_1$, $F_2$ & $F_3$ with respect to $x$:

$$2y'x^2 + 4xy + z'w + zw' = 0$$
$$\cos(\pi xz)[\pi z + \pi x z'] + y' = 0$$
$$z' + w' = 0$$

at $(1,0,1,1)$ $\Rightarrow$ $2y' + z' + w' = 0$ ①
$$-\pi(1+z') + y' = 0 \quad ②$$
$$z' + w' = 0 \quad ③$$

① & ③ $\Rightarrow y' = 0 \Rightarrow z' = -1 \Rightarrow w' = 1$

so $dy = y'dx = 0$

$dz = z'dx = -1(-0.03) = 0.03$

$dw = w'dx = 1(-0.03) = -0.03$

4) Starting with $f(\vec{a}+\vec{h}) = \sum_{|\alpha| \leq k} \frac{\partial^\alpha f(\vec{a})}{\alpha!} \vec{h}^\alpha + \sum_{|\alpha| = k+1} \frac{\partial^\alpha f(\vec{a}+c\vec{h})}{\alpha!} \vec{h}^\alpha$

some $c \in (0,1)$.

First set $k=0$, i.e. the zeroth order approximation.

$|\alpha| \leq 0 \Rightarrow \alpha = (0,0,\ldots 0)$ so $\partial^\alpha f(\vec{a}) = f(\vec{a})$
$\alpha! = 1$, $\vec{h}^\alpha = 1$

$|\alpha| = 1 \Rightarrow \alpha = (0,\ldots 1, \ldots 0)$ so $\partial^\alpha f(\vec{a}+c\vec{h}) = \partial_i f(\vec{a}+c\vec{h})$
$i^{th}$
$\alpha! = 1$, $\vec{h}^\alpha = h_i$

Putting this together,

$$f(\vec{a}+\vec{h}) = \underbrace{f(\vec{a})1}_{1} + \sum_{i=1}^{n} \underbrace{\partial_i f(\vec{a}+c\vec{h})}_{1} h_i$$

$$= f(\vec{a}) + \nabla f(\vec{a}+c\vec{h}) \cdot \vec{h}$$

some $c \in (0,1)$

Calling $\vec{c} = \vec{a}+c\vec{h}$ where $\vec{c}$ is on the

line between $\vec{a}$ & $\vec{a}+\vec{h}$, we get MVT III

(Also, $C^{k+1}$ on open convex set => diff on line segments

in S)

Now set $n=1$ so that $\alpha = (\alpha_1)$ i.e. $|\alpha| = 1 \Rightarrow \alpha = (1)$

and all vectors become scalars.

$\therefore f(a+h) = f(a) + f'(a+ch)h$ which is MVT I.

(and is likewise diff on S so satisfies assumptions)

5) d) Firstly, Let's prove $(fg)^{(k)} = \sum_{j=0}^{k} \binom{k}{j} f^{j} g^{(k-j)}$

Basis: $(fg)' = f'g + g'f = \sum_{j=0}^{1} \binom{1}{j} f^{j} g^{1-j}$

Induction: Assume true for k.

$$(fg)^{(k+1)} = \frac{d}{dx}(fg)^{(k)} = \frac{d}{dx} \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)}$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left[ f^{j+1} g^{k-j} + f^{j} g^{k+1-j} \right]$$

$$= \sum_{j=0}^{k} \binom{k}{j} \left( f^{j+1} g^{k-j} \right) + \sum_{j=0}^{k} \binom{k}{j} f^{j} g^{k+1-j}$$

$$= f^{(k+1)} g + \sum_{j=0}^{k-1} \binom{k}{j} f^{j+1} g^{k-j} + \sum_{j=1}^{k} \binom{k}{j} f^{j} g^{k+1-j} + f g^{k+1}$$

$$= \quad '' \quad + \sum_{j=1}^{k} \binom{k}{j-1} f^{j} g^{k+1-j} \quad '' \quad + \quad ''$$

$$= f^{k+1} g + \sum_{j=1}^{k} \left[ \binom{k}{j-1} + \binom{k}{j} \right] f^{j} g^{k+1-j} + f g^{k+1}$$

$$= f^{k+1} g + \sum_{j=1}^{k} \binom{k+1}{j} f^{j} g^{k+1-j} + f g^{k+1}$$

$$= \sum_{j=0}^{k+1} \binom{k+1}{j} f^{j} g^{k+1-j} \quad \blacksquare$$

Now consider

$$\partial^{\alpha} fg = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} (fg)$$

Let's do $\partial_n^{\alpha_n}(fg)$ first

$$\partial_n^{\alpha_n}(fg) = \sum_{j=0}^{\alpha_n} \binom{\alpha_n}{j} \partial_n^j f \, \partial_n^{k-j} g$$ via 1 variable claim.

For $\beta = (\beta_1 \ldots \beta_n)$, $\gamma = (\gamma_1 \ldots \gamma_n)$ set $\beta_n = j$, $\gamma_n = \alpha_n - j$
so $\beta_n + \gamma_n = \alpha_n$

above is
$$= \sum_{\beta_n + \gamma_n = \alpha_n} \binom{\alpha_n}{j} \partial_n^{\beta_n} f \, \partial_n^{\gamma_n} g$$

since $\binom{\alpha_n}{j} = \frac{\alpha_n!}{j!(\alpha - j)} = \frac{\alpha_n!}{\beta_n! \gamma_n!}$

so
$$= \sum_{\beta_n + \gamma_n = \alpha_n} \frac{\alpha_n!}{\beta_n! \gamma_n!} \partial_n^{\beta_n} f \, \partial_n^{\gamma_n} g$$

Now apply rest of $\partial_1^{\alpha_1} \ldots \partial_{n-1}^{\alpha_{n-1}}$ similarly.
sum over
when $\beta_i + \gamma_i = \alpha_i \iff \beta + \gamma = \alpha$

So
$$\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n}(fg) = \sum_{\beta_1 + \gamma_1 = \alpha_1} \ldots \sum_{\beta_n + \gamma_n = \alpha_n} \frac{\alpha_1!}{\beta_1! \gamma_1!} \cdots \frac{\alpha_n!}{\beta_n! \gamma_n!} \partial_1^{\beta_1} \ldots \partial_n^{\beta_n} f \, \partial_1^{\gamma_1} \ldots \partial_n^{\gamma_n} g$$
$$= \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta f \, \partial^\gamma g \quad \blacksquare$$

b) For binomial Thm
$$(x_1 + x_2)^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} x_1^j x_2^{k-j} = \sum_{\beta + \gamma = k} \frac{k!}{\beta! \gamma!} x_1^\beta x_2^\gamma$$

Now use $\beta = (\beta_1 \ldots \beta_n)$, $\gamma = (\gamma_1 \ldots \gamma_n)$

Now
$$(\vec{x}_1 + \vec{x}_2)^\alpha = (x_{11} + x_{21})^{\alpha_1} \ldots (x_{1n} + x_{2n})^{\alpha_n}$$
$$= \left( \sum_{\beta_1 + \gamma_1 = \alpha_1} \frac{\alpha_1!}{\beta_1! \gamma_1!} x_{11}^{\beta_1} x_{21}^{\gamma_1} \right) \ldots \left( \sum_{\beta_n + \gamma_n = \alpha_n} \frac{\alpha_n!}{\beta_n! \gamma_n!} x_{1n}^{\beta_n} x_{2n}^{\gamma_n} \right)$$

$$= \sum_{\beta_1+\gamma_1=\alpha_1} \cdots \sum_{\beta_n+\gamma_n=\alpha_n} \frac{\alpha_1! \cdots \alpha_n!}{\beta_1!\gamma_1! \cdots \beta_n!\gamma_n!} x_{11}^{\beta_1} \cdots x_{1n}^{\beta_n} x_{21}^{\gamma_1} \cdots x_{2n}^{\gamma_n}$$

$$= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} \vec{x}_1^{\beta} \vec{x}_2^{\gamma}$$ as $\beta+\gamma=\alpha \Leftrightarrow \beta_i+\gamma_i=\alpha_i \ \forall i$ ∎

6) $f(x,y)=(y^2-y)x$

$\partial_x f = y^2-y = 0 = y(y-1)$ so $y=0$ or $y=1$

$\partial_y f = (2y-1)x = 0 \Rightarrow x=0$ or $y=\frac{1}{2}$

$y=0 \Rightarrow x=-1$, $y=1 \Rightarrow x=1$, $x=0$ pairs with $y=0, y=1$

$y=\frac{1}{2}$ isn't zero for 1st eqn.

so $(-1,0)$, $(1,1)$, $(0,1)$, $(0,0)$ are all CPs.

$\partial_x^2 f = 0$ $\quad\quad$ $\partial_x\partial_y f = 2y-1 = \partial_y\partial_x f$

$\partial_y^2 f = 2x$

so $H(x,y) = \begin{pmatrix} 0 & 2x \\ 2x & 2y-1 \end{pmatrix}$ so $\det H = -4x^2$

so $D=\det H$ is $<0$ for $(-1,0)$, $(1,1)$ $\rightarrow$ saddles.

$D=\det H$ is $=0$ for $(0,1)$, $(0,0)$ $\Rightarrow$ degenerate.

$f(\vec{a}+\vec{k}) = f(\vec{a}) + \frac{1}{2}\vec{k}^T H k$ at CPs.

$$H(x,y)\vec{k} = \begin{pmatrix} 2xk_2 \\ 2xk_1 + (2y-1)k_2 \end{pmatrix}$$

$$\vec{k}^T H(x,y)\vec{k} = (2xk_2k_1) + (2xk_1k_2 + (2y-1)k_2^2)$$

$$= 4xk_1k_2 + (2y-1)k_2^2$$

At $(-1,0)$ $f(-1+k_1, k_2) = 0 + -4k_1k_2 - k_2^2$ + higher order

at $(1,1)$ $f(1+k_1, 1+k_2) = 0 + 4k_1k_2 + k_2^2$ + higher order

at $(0,1)$ $f(k_1, 1+k_2) = 0 + k_2^2$ + higher order

at $(0,0)$ $f(k_1,k_2) = -k_2^2$