Lecture 23

CONFORMAL MAPS

Def'n: An analytic map f is conformal at z_0 if it preserves angles.

THM1: If f analytic at z_0, & $f'(z_0)\neq 0$, then f is conformal.

THM2: If f is analytic & injective (one-to-one) then it's conformal.

THM3 If f & g are conformal then $f\circ g$ is conformal

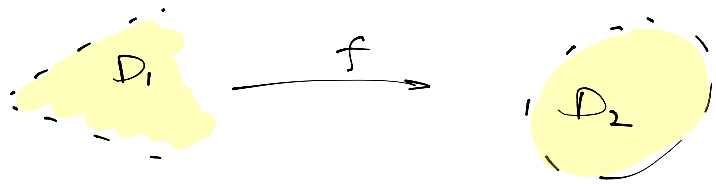
Ex: $f(z)=z^n$ is conformal for $z\neq 0$.

$f(z)=e^z$ is conformal for all z.

$T(z)=\frac{az+b}{cz+d}$ is conformal

Riemman Mapping Thm

D_1, D_2 are simply-connected domains in $\mathbb{C}$, then there exists a conformal bijection $f: D_1 \to D_2$

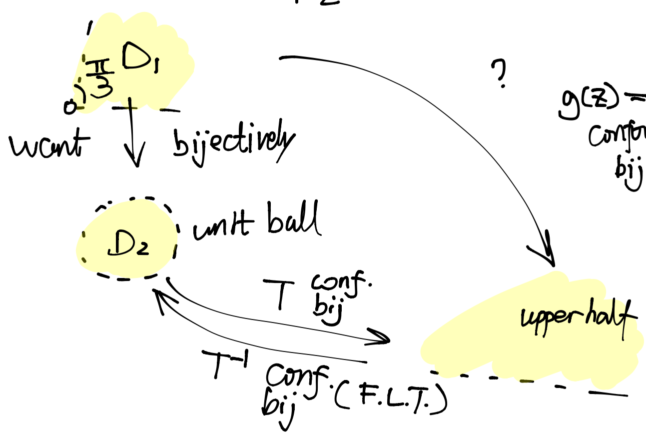


Ex: find a conformal bijection $z\neq 0$

$f: D_1 \to D_2$, $D_1=\{z \mid 0\leq \operatorname{Arg} z \leq \frac{\pi}{3}\}$

$D_2=\{z \mid |z|<1\}$

Hint: $T(z)= i\,\frac{1+z}{1-z}$ maps $D_2 \to U$ bijectively (& conformal



?

$g(z)=z^3$

conformal b/c $z=0$, $0\notin D$

bij? $\Rightarrow$ yes!

$f(z)=f(w) \Rightarrow z=w$

$z^3=w^3$

$z^3-w^3=0$

$(z-w)(z^2+zw+w^2)=0$

if $z,w\in D$, then $z^2+zw+w^2\neq 0$

So $z=w$

Sol'n: The function $g(z)=z^3$ is a conformal map $D_1 \to U$

It's also bijective.

Why?: If $z,w\in D_1$ & $z^3=w^3$

$|z|e^{i3t}=|w|e^{i3\theta}$

$\Rightarrow |z|=|w|$

$3t=3\theta+2\pi k$ since $t,\theta\in[0,\pi/3]$, $\theta=t$

so $z=w$

Let $T=i\,\frac{z+1}{1-z}$, T^{-1} is a bij conf. map.

$U \to D_2$

Need $T^{-1}(z)=\,??$

$M_T=\begin{pmatrix} i & i \\ -1 & 1\end{pmatrix}$ $\quad M_T^{-1}=\frac{1}{\det M_T}\begin{pmatrix} 1 & -i \\ 1 & i\end{pmatrix}$ $\quad T^{-1}=\frac{z-i}{z+i}$

$T^{-1} = \dfrac{z-i}{z+i}$

So then $f(z) = T^{-1} \circ g(z) = \dfrac{z^3 - i}{z^3 + i}$

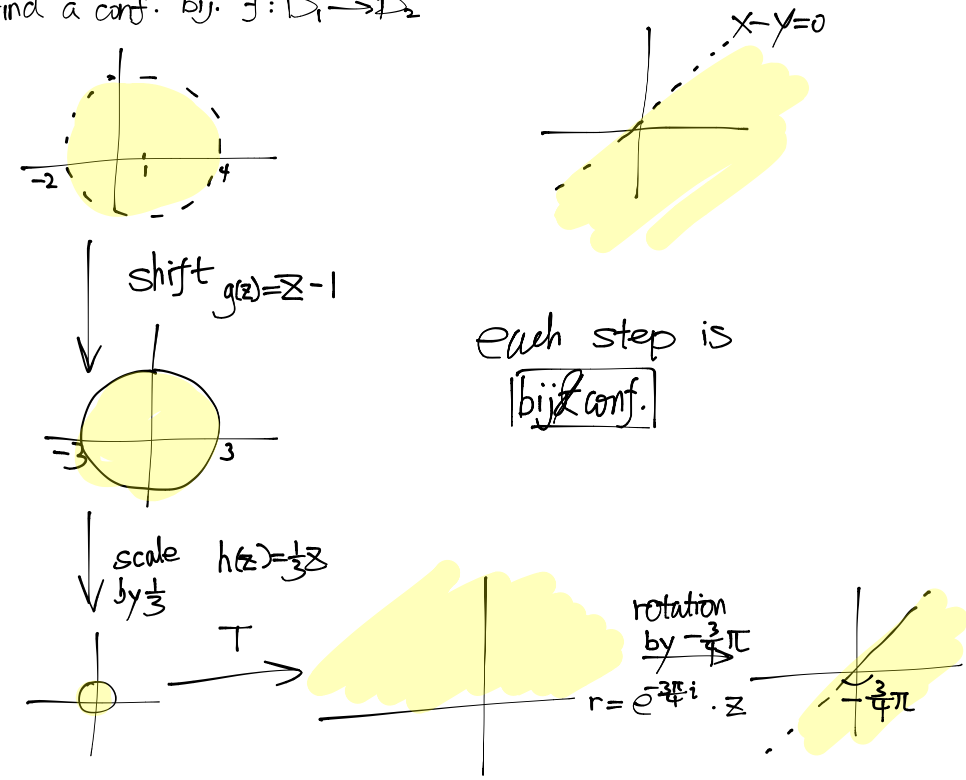
is a conformal bij. $D_1 \to D_2$, since both g, T^{-1} are conf. & bij.

Ex: 2

$D_1 = \{z \mid |z-1| < 3\}$

$D_2 = \{z \mid \mathrm{Re}((1+i)z) < 0\}$

Find a conf. bij. $f: D_1 \to D_2$

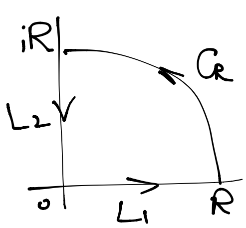


Review

$f(z) = z^2 + iz + 2 + i$ (zeros in 1st Quad)

Argument Principle

$\#\text{zeros} = \dfrac{1}{2\pi}\left(\begin{array}{l}\text{net change in } \arg f(z) \\ \text{as } z \text{ varies over } \gamma\end{array}\right)$



On L_1: $z = x \quad 0 \le x \le R$

$f(z) = f(x) = x^2 + ix + 2 + i$

$= \underset{+}{(x^2+2)} + i\underset{+}{(x+1)}$ b/c $x \ge 0$

$f(z)$ is always in the 1st quad on L_1

On C_R: (R is big) $|z| = R \rightsquigarrow f(z) \simeq z^2$ net change in $\arg f(z) \approx 2 \cdot \frac{\pi}{2} = \pi$

$(0, \frac{\pi}{2}) \to (0, \pi)$

At $z = iR$, $R \ge 0$ large: $f(iR) = (iR)^2 + i(iR) + 2 + i = \underset{-}{(-R^2 - R + 2)} + \underset{+}{i}$

Over C_R: $f(z)$ goes from 1st Quad to 2nd

<u>On L_2</u>: $z = iy$, $R \geq y \geq 0$

$$f(z) = f(iy) = (-y^2 - y + 2) + i = -(y+2)(y-1) + i$$

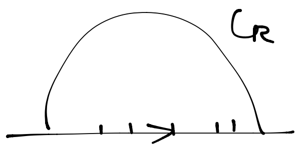
When pass $y=1$, moving to 1st quad.

For $R \geq y > 1$, $f(iy)$ is in 2nd quad, when $0 \leq y \leq 2$

$f(i,y)$ Is in 1st quad

Never pass through all 4 quads

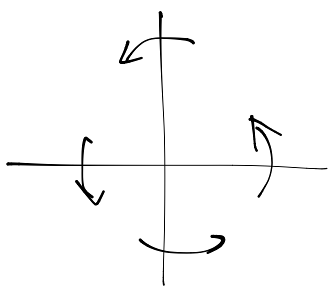
$\Rightarrow$ net change $= 0 \Rightarrow$ # zeros $= 0$



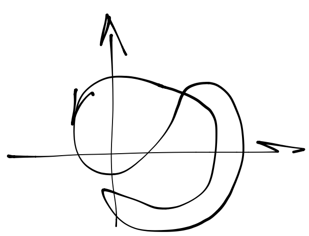
On L: $f(y) = (p_1(x)) + (p_2(x))i$

Solve $p_1(x) = 0$

$p_2(x) = 0$



$Im(z) = 0$



$1 \to 4 \to 3 \to 4 \to 1 \to 2 \to 3 \to 4 \to 1$

3.3.5(a)

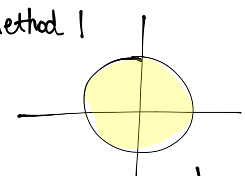
D_1 : $|z| = 1$

D_2 : $Re((1+i)w) = 0$

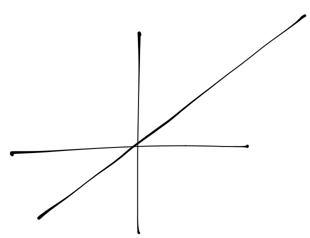
$Re((1+i)(x+iy)) = Re(x - y + ix + iy) = 0$

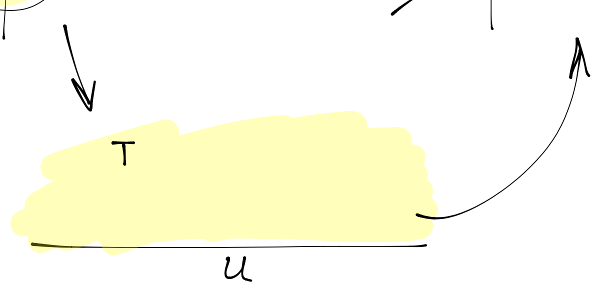
$y = x$

Method 1



$F = r \circ T$



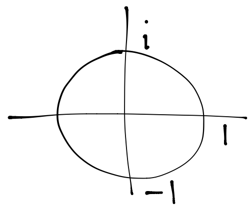


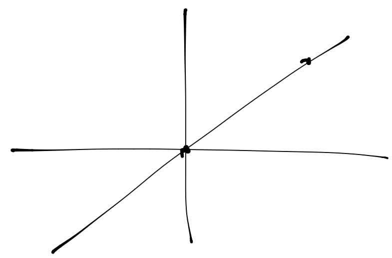
$r(z) = e^{iu} \cdot z \leftarrow$ F.L.T

$\frac{az+0}{0z+d}$ $\quad$ $\frac{a = e^{iu}}{d = 1}$

$T = i\frac{z+1}{1-z}$

method 2:





$i \to \infty$

$1 \to 0$

$-1 \to 1+i$

$1 = z_1 \to 0 \leftarrow w_1$

$4 = z_2 \to 1 \leftarrow w_2$

$\infty\ z_3 \to \infty \leftarrow w_3$

$T(z) = \frac{az+b}{cz+d}$.

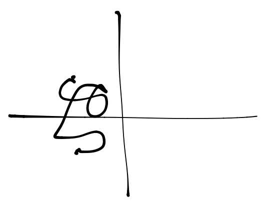
$T(-\frac{d}{c}) = \infty$

$T(\infty) = \frac{a}{c} \to c = 0$

$T(1) = 0 = a+b$

$T(4) = 1 = 4a+b$

Ex Find $\int_\gamma \frac{1}{z}\,dz$, where γ is any curve joining $z_0 = -1-i$ to $z_1 = -1+i$ contained in the half plane $Re(z) < 0$.

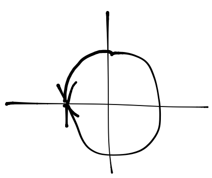


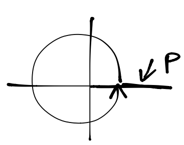
$\int_\gamma f(z)dz = F(\text{end pt}) - F(\text{initial pt})$

provided f is analytic in D

F must also be analytic in D !

Let $\arg(z) \in [0, 2\pi)$





Then $\log(z) = \ln|z| + i\arg z$ is analytic in $Re(z) < 0$

Then since $\frac{d}{dz}(\log z) = \frac{1}{z}$ in $Re(z) > 0$

We can evaluate the line integral

$\int_\gamma \frac{1}{z}\,dz = \log(-1+i) - \log(-1-i) = \ln|-1+i| + i\arg(-1+i) - (\ln|-1-i| + i\arg(-1-i))$

$= \ln\sqrt{2} + i\frac{3\pi}{4} - (\ln\sqrt{2} + i(\frac{5}{4}\pi))$

$= -\frac{\pi}{2}i$

2?

$f(t) = (1-t)(-1-i) + t(-1+i)$

$= -1-i+t+it - t + ti$

$= 2ti - 1 - i$

$f'(t) = 2i \cdot dt$

$\frac{3}{4}\pi i - (-\frac{3}{4}\pi)i = \frac{3}{2}\pi i$

$\int_\gamma \frac{1}{z}\,dz = \int_0^1 \frac{1}{2ti-1-i}\, 2i\,dt = 2i[\log(2ti-1-i)]_0^1 = 2i$