

Jan 9th, 2013

Mathematical induction

Let S be a set of natural numbers
such that ① $1 \in S$

② If $n \in S$ then $n+1 \in S$

$$n \geq 1$$

$\Rightarrow S = N$ all natural #

$$S = \{1, 2, 3, \dots\}$$

In problems, you have a formula e.g. $1+2+3+\dots+n = \frac{n(n+1)}{2}$

we want to prove that it holds for all n .

Let $S = \text{set of all natural } \# \text{, for which the formula holds.}$

We want to show $S = N$

Proof: ① $1 \in S$ (i.e. the formula holds for $n=1$)

$$1 = \frac{1(1+1)}{2} = 1 \quad \checkmark$$

② if $n \in S$ (i.e. if the formula holds for $n \Rightarrow$ it holds for $n+1$)

We assume $1+2+\dots+n = \frac{n(n+1)}{2}$ \leftarrow put $n+1$ instead of n

\Rightarrow want to show

$$\underline{1+2+\dots+n+(n+1)} = \frac{(n+1)(n+2)}{2}$$

$$\downarrow \\ = \frac{n(n+1)}{2} + (n+1) = \frac{n^2+n+2n+2}{2} = \frac{(n+1)^2+(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

\Rightarrow we proved the induction step.

\Rightarrow the formula holds for all n .

$$1+2+3+\dots+n = \frac{(n+1)n}{2}$$

$$1^2+2^2+3^2+\dots+n^2 = ?$$

find the formula and prove it by induction.

$$1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Induction:

① Check for $n=1$.

$$\text{LHS} = 1, \text{ RHS} = \frac{1 \cdot 2 \cdot 3}{6} = 1 \quad \checkmark$$

② induction step suppose the formula holds for all n .

$$1^2+2^2+\dots+n^2 = n(n+1)(2n+1)/6$$

$$\text{for } n+1 : \underline{1^2+2^2+\dots+n^2+(n+1)^2} = (n+1)(n+1)+1)(2(n+1)+1)/6 = \frac{(n+1)(n+2)(2n+3)}{6}$$

$$\begin{aligned}
 & \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} = \frac{(n+1)(n(2n+1) + 6(n+1))}{6} = \frac{(n+1)(2n^2+n+6n+6)}{6} \\
 & \text{a. guess: } 1+2+\dots+n = \text{quadratic in } n \\
 & \frac{n(n+1)}{2} = \frac{n^2+n}{2} \quad = \frac{(n+1)(n+2)(2n+3)}{6} \quad v.
 \end{aligned}$$

PPL $1^2+2^2+\dots+n^2$ is cubic in n .

Let's find $a_1, a_2, \dots, a_n \dots$ such that

$$a_1 = 1^3$$

$$a_1+a_2 = 2^3$$

$$a_1+a_2+a_3 = 3^3$$

...

$$a_1+a_2+\dots+a_n = n^3$$

$$a_n = n^3 - (n-1)^3 = n^3 - (n-1)(n^2-2n+1) = n^3 - n^3 + 2n^2 - n + n^3 - 2n + 1 = 3n^2 - 3n + 1$$

$$a_1 = 3 \cdot 1^2 - 3 \cdot 1 + 1 = 1$$

$$a_2 = 3 \cdot 2^2 - 3 \cdot 2 + 1 = 7$$

$$a_3 = 3 \cdot 3^2 - 3 \cdot 3 + 1 = 19$$

...

$$a_1 = 1 = 1^3$$

$$a_1+a_2 = 1+7=8=2^3$$

$$a_1+a_2+a_3 = 1+7+19=27=3^3$$

...

$$a_1+a_2+\dots+a_n = n^3 \quad a_n = 3n^2 - 3n + 1$$

$$(3 \cdot 1^2 - 3 \cdot 1 + 1) + (3 \cdot 2^2 - 3 \cdot 2 + 1) + (3 \cdot 3^2 - 3 \cdot 3 + 1) + \dots + (3n^2 - 3n + 1) = n^3$$

$$3(1^2+2^2+\dots+n^2) - 3(1+2+\dots+n) + n = n^3$$

$$3(1^2+2^2+\dots+n^2) - \underline{3n(n+1)} + n = n^3$$

$$1^2+2^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

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HW: Find $1^3+2^3+\dots+n^3 = ?$

$$\text{find } a_1, \dots, a_n \text{ st. } a_1+a_2+\dots+a_n = n^4$$

$$a_n = n^4 - (n-1)^4$$

Prove $(1+x)^n \geq 1+nx$ if $x \geq -1$
for any $n=1, 2, 3, \dots$

① Check for $n=1$

$$\begin{aligned} \text{LHS} &= 1+x & \text{RHS} &= 1+x \\ 1+x &\geq 1+x & v. \end{aligned}$$

② Induction step: SPS $(1+x)^n \geq 1+nx$
 $(1+x)^{n+1} \geq 1+(n+1)x$

$$(1+x)^n(1+x) \geq (1+nx)(1+x) = 1+nx+x+\underline{nx^2} \geq 1+(n+1)x \quad v.$$

ex: $x=0.2 \quad n=3$

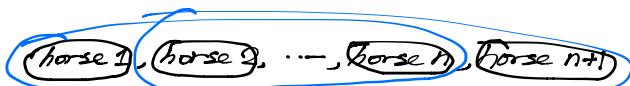
$$(1+0.2)^3 = 1.2^3 \geq 1+3 \times 0.2 = 1.6$$

Claim: All horses have the same color.

Claim: For any $n=1, 2, 3, \dots$ in any collection of n horses have the same color.

① For $n=1 \Rightarrow$ clearly true.

② Induction Step: s.p.s it's true for $n \Rightarrow$ want to prove for $n+1$
 $n+1$ horses



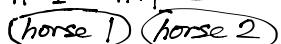
first n horses

\Rightarrow by induction assumption \Rightarrow have the same color.

horse 1 & horse $(n+1)$ have the same color or
 horse 2 & ... -

\Rightarrow all the horses have the same color. ■

$$n=1 \quad n+1=2$$



Induction can start with any integer.

Claim: $n! \geq 2^n$ for $n \geq 4$

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

$$4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24 \geq 16$$

$$2^4 = 16$$

① One of induction is $n=4$

$$4! \geq 2^4 \quad v.$$

② $n! \geq 2^n \quad \forall n \geq 4$

$$\Rightarrow (n+1)! \geq 2^{n+1}$$

$$n!(n+1) \geq 2^n \cdot (n+1) \geq 2^n \cdot 5 \geq 2^n \cdot 2 = 2^{n+1}$$

$\boxed{\text{by induction assumption}}$

$$\text{last time: } a^0 + a^1 + \dots + a^n = \frac{a^{n+1}-1}{a-1} \quad a \neq 1$$

holds for any $n \geq 0$

$$n=0, 1, 2, 3, \dots$$

$$n=0, a^0 = 1, \cancel{a^1} \\ 1 = \frac{a^{0+1}-1}{a-1} = \frac{a-1}{a-1} = 1$$

$$\text{Induction step: if } a^0 + \dots + a^n = \frac{a^{n+1}-1}{a-1} \quad n \geq 0$$

$$\text{then } a^0 + \dots + a^{n+1} = \frac{a^{n+1}-1}{a-1} + a^{n+1} = \frac{a^{n+1}-1+a^{n+1}(a-1)}{a-1} = \frac{a^{n+1}+a^{n+1}a-a^{n+1}-1}{a-1} = \dots$$

Claim: $n^3 + 5n$ is divisible by 3 for any $n \geq 1$

Proof: ① Check for $n=1$: $1^3 + 5 = 6 = 3 \cdot 2$ ✓.
② Suppose holds for $n \Rightarrow$ proof of $(n+1)^3 + 5(n+1)$

$$(n+1)^3 + 5(n+1) = n^3 + 3n^2 + 3n + 1 + 5n + 5 = (n^3 + 5n) + 3n^2 + 3n + 6 = (n^3 + 5n) + 3(n^2 + n + 2) \quad \text{holds for } n+1 \quad \blacksquare$$

Next subject: Prime numbers

Definition: A natural # n is called prime if it can not be written as a product of $n = a \cdot b$ where $a, b > 1$, natural #

$6 = 3 \times 2$ not prime

$2 = 1 \times 2$ prime

$3 = 1 \times 3$ prime

even # not prime

$5 = 1 \times 5$ prime

If n can be written as a product of $n = \underbrace{a \cdot b}_{a, b > 1, a, b \in \mathbb{N}^+}$ then is called composite number.

