STA 257H1F - Summer, 2003
Test 1
May 26, 2003

NAME: SOLUTIONS

STUDENT NUMBER:

TUTORIAL: (circle one)

Tutorial A (A-G)	Tutorial B (H-Lin)	Tutorial C (Liu-Sr)	Tutorial D (St-Z)
Mohammad (SS 2118)	Zengxin (SS 1084)	Hanna (SS 2106)	Xiaobin (SS 1085)

INSTRUCTIONS:
- Time: 50 minutes
- No aids allowed.
- Answers that are algebraic expressions should be simplified. Series and integrals should be evaluated. Numerical answers need not be expressed in decimal form.
- Total points: 30

Some Important Discrete Probability Distributions

Distribution	Probability Mass Function
Binomial(n, p)	$p(x) = \binom{n}{x} p^x(1-p)^{n-x}$ for $x = 0, 1, 2, \ldots, n$
Bernoulli(p)	same as Binomial$(1, p)$
Poisson(λ)	$p(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, \ldots$
Geometric(p)	$p(x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, \ldots$
Negative Binomial(p, r)	$p(x) = \binom{x-1}{r-1} p^r(1-p)^{x-r}$ for $x = r, r+1, \ldots$

1	2	3	4	5	6

1. (3 points) Let P be a probability measure defined on a sample space Ω. For any event A define $Q(A) = [P(A)]^2$. Is Q a probability measure on Ω? Why or why not?

No - $Q(A)$ doesn't satisfy the condition of additivity of the union of disjoint events

Let A, B be disjoint events with $P(A) > 0$, $P(B) > 0$

$$Q(A \cup B) = [P(A \cup B)]^2 = [P(A) + P(B)]^2$$
$$= [P(A)]^2 + 2P(A)P(B) + [P(B)]^2$$
$$\neq Q(A) + Q(B)$$

2. (4 points) Show that if A, B, C are independent events, then A and $B \cup C$ are independent.

$$P(A \cap (B \cup C)) = P(AB \cup AC)$$
$$= P(AB) + P(AC) - P(AB \cap AC)$$
$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$

since A, B, C indept.

$$= P(A)[P(B) + P(C) - P(BC)]$$
$$= P(A)\, P(B \cup C)$$

3. (4 points) Suppose A, B, and C are events such that $P(A|C) \geq P(B|C)$ and $P(A|C^c) \geq P(B|C^c)$. Prove that $P(A) \geq P(B)$.

If $P(A|C) \geq P(B|C)$, $\frac{P(AC)}{P(C)} \geq \frac{P(BC)}{P(C)}$

so $P(AC) \geq P(BC)$

Similarly, $P(AC^c) \geq P(BC^c)$

since $P(A|C^c) \geq P(B|C^c)$

Adding: $P(AC) + P(AC^c) \geq P(BC) + P(BC^c)$

so $P(A) \geq P(B)$

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4. (3 points) Suppose A and B are events such that $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$. Can A and B be disjoint? Explain your answer.

If A, B were disjoint,

$P(A \cup B) = P(A) + P(B)$

But $P(A) = \frac{1}{3}$ and $P(B) = 3/4$

so $P(A) + P(B) \geq 1$

so A, B can't be disjoint

5. (6 points) For each of the following, state whether the function $p(\omega)$ is a valid probability mass function. If not find a constant a, if possible, so that $ap(\omega)$ is a probability mass function.

(a) $p(\omega) = \frac{1}{3}\left(\frac{2}{3}\right)^{\omega} \quad \omega = 3, 4, 5, \ldots$

$p(w) \geq 0 \quad , w = 3, 4, 5, \ldots$

$\sum_{w=3}^{\infty} p(w) = \frac{1}{3}\left(\frac{2}{3}\right)^3 \frac{1}{1-\frac{2}{3}} = \frac{8}{27}$

So $a p(w)$ is a valid pmf for $a = \frac{27}{8}$

(b) $p(\omega) = \frac{1}{3}(\omega - 2) \quad \omega = 0, 1, 2, 3, 4, 5$

Not a valid pmf
since $p(0) < 0$
And there is no a

6. (4 points) A coin has two sides: Heads and Tails. Assume that when tossed, it is equally likely to come up either side. Suppose the coin is tossed until the second Head appears. Find the probabilities that more than 4 Tails come up before the second Head.

Total number of tosses, X, has a negative binomial distribution with $p=\frac{1}{2}$, $r=2$

We want

$$P(X>6) = \sum_{x=7}^{\infty} (x-1)\left(\frac{1}{2}\right)^x$$

$$= \frac{1}{2^2}\left[\frac{1}{(1-\frac{1}{2})^2} - \left(1+2\left(\tfrac{1}{2}\right)+3\left(\tfrac{1}{2}\right)^2+4\left(\tfrac{1}{2}\right)^3+5\left(\tfrac{1}{2}\right)^4\right)\right]$$

7. (6 points) An urn contains 5 balls. When the balls were put in the urn, they were each coloured red or blue with probability 1/2. Suppose 3 balls are randomly drawn from the urn in succession with replacement, i.e. one ball is drawn then returned to the urn, and then the next ball is drawn, etc. If no red balls appear in the balls drawn, what is the probability that the urn contains no red balls?

Define the r.v.'s $U = \#$ of red balls in urn
$D = \#$ of red balls drawn

$$P(U=0|D=0) = \frac{P(D=0|U=0)P(U=0)}{P(D=0|U=0)P(U=0)+P(D=0|U=1)P(U=1)+P(D=0|U=2)P(U=2)+P(D=0|U=3)P(U=3)+P(D=0|U=4)P(U=4)+P(D=0|U=5)\cdot P(U=5)}$$

$$= \frac{5\ 1\left(\frac{1}{2^5}\right)}{\frac{1}{2^5}+\left(\frac{4}{5}\right)^3 5\left(\frac{1}{2}\right)^5+\left(\frac{3}{5}\right)^3\binom{5}{2}\left(\frac{1}{2}\right)^5+\left(\frac{2}{5}\right)^3\binom{5}{3}\left(\frac{1}{2}\right)^5+\left(\frac{1}{5}\right)^3\binom{5}{4}\left(\frac{1}{2}\right)^5+0}$$