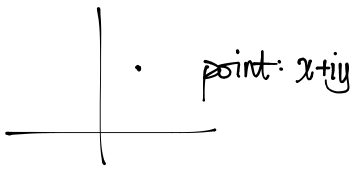
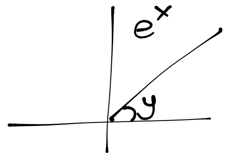
Lecture 5

Exponential, Logarithm, Trig Function:

Last time: $z=x+iy$, $e^z=e^x\cdot e^{iy}$





$w=e^{x+iy}=e^x\cdot e^{iy}$

- $e^{z+2\pi in}=e^z$ for any $n\in\mathbb{Z}$
- $e^z=w$ has infinitely many sol'n $(w\neq 0)$
- $e^z=e^{x+iy}=e^x\cdot e^{iy}=re^{i\theta}$, $w=re^{i\theta}$ in polar form

$e^x=r \Rightarrow x=\ln r$

$e^{iy}=e^{i\theta} \Rightarrow y=\theta+2\pi n$

$e^z=w$ has solutions with the form

$$\boxed{\begin{aligned} z&=x+iy \\ &=\ln|w|+i\arg(w) \end{aligned}}$$ ← all arguments of w

Solutions of $e^z=w$ with $w\neq 0$

LOGARITHMS

Def'n: For a complex number z, we define a logarithm of z to any complex number w, so that $z=e^w$, we usually write $w=\log z$.

We've seen there are infinitely many w's satisfying $e^w=z$. We also saw that w has the form

$$\boxed{\begin{aligned} &w=\ln|z|+i\arg z \\ &\log z=w=\ln|z|+i\arg z \\ &z\longmapsto \log z \text{ is NOT a function.} \end{aligned}}$$

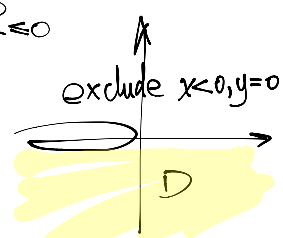
Recall: $\mathrm{Arg}(z)$ is the unique angle $\theta\in[-\pi,\pi)$, so that $z=re^{i\theta}$

We define $\mathrm{Log}:\mathbb{C}\setminus\{0\}\to\mathbb{C}$ by

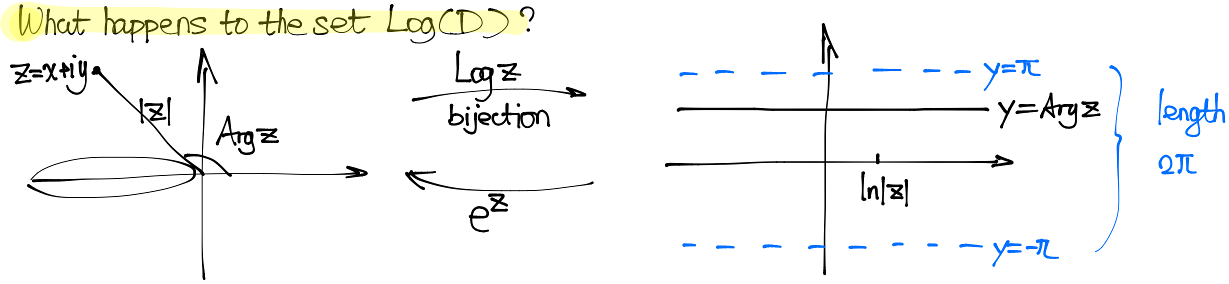
$$\mathrm{Log}\, z=\ln|z|+i\,\mathrm{Arg}\, z$$

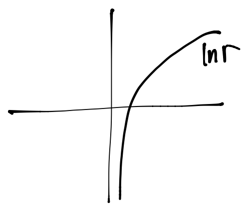
Note: Log is a function, this is not a continuous function, because $\mathrm{Arg}\, z$ is not continuous

If we restrict the domain at Log to $D=\mathbb{C}-\mathbb{R}_{\leq 0}$
then $\mathrm{Log}: D\to\mathbb{C}$



What happens to the set Log(D)?





Recall:

If $z=x>0$, then $\operatorname{Log} z = \ln x$

Exponentials with other bases:

For real #'s: $a^x = (e^{\ln a})^x \qquad a>0$

$a^x = e^{(\ln a)x}$

For complex #'s, $a \neq 0$, define a^z by:

$a^z = e^{(\log a)z}$ where $\log a$ is a logarithm of a

Ex: Find all values of $(i)^{1+i}$

(Let $a=i$, $z=1+i$)

$$\log i = \ln|i| + i\arg(i)$$
$$= \ln 1 + i\left(\tfrac{\pi}{2}+2\pi n\right)$$
$$= i\left(\tfrac{\pi}{2}+2\pi n\right)$$

$$(i)^{1+i} = e^{(\log i)(1+i)} = e^{i(\frac{\pi}{2}+2\pi n)(1+i)} = e^{i(\frac{\pi}{2}+2\pi n+\frac{\pi}{2}i+2\pi n i)}$$
$$= e^{i\pi/2}\cdot e^{2\pi n i}\cdot e^{-\frac{\pi}{2}}\cdot e^{-2\pi n} = i e^{-\pi/2}\cdot e^{-2\pi n},\ n\in\mathbb{Z}$$

TRIG FUNCTION

$$\boxed{\begin{aligned}\cos z &= \tfrac{1}{2}(e^{iz}+e^{-iz})\\ \sin z &= \tfrac{1}{2i}(e^{iz}-e^{-iz})\end{aligned}}$$

Let's start check that if $z=x\in\mathbb{R}$, then $\cos x$ agrees with usual <u>def'n</u>.

$$\cos x = \tfrac{1}{2}(e^{ix}+e^{-ix})$$
$$= \tfrac{1}{2}(\cos x + i\sin x + \cos(-x) + i\sin(-x))$$
$$= \tfrac{1}{2}(\cos x + i\sin x + \cos x - i\sin x)$$
$$= \underline{\cos x} \quad \text{usual def'n}$$

proof of sin is similar

Note: $\cos(z+2\pi) = \cos z$

$\sin(z+2\pi) = \sin z$

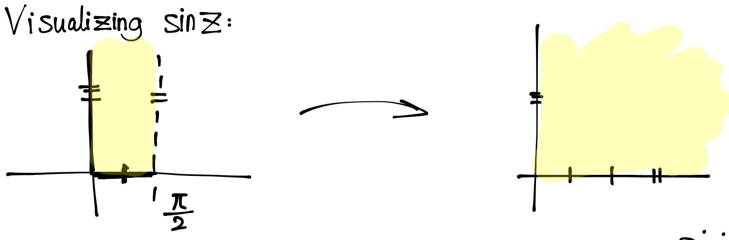
In fact, if $\cos(z+\alpha) = \cos z$ for all z, then $\alpha = 2\pi n$

(no new period for sin, cos)

As usual, $\tan z = \dfrac{\sin z}{\cos z}$, $\sec z = \dfrac{1}{\cos z}$

$\csc z = \dfrac{1}{\sin z}$, $\cot z = \dfrac{\cos z}{\sin z}$

Visualizing $\sin z$:



similar for $\cos z$

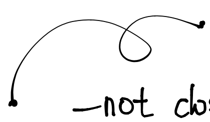
Skip inverse strict functions

Line Integrals

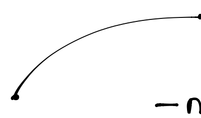
Curves: A curve γ is a cts function $\gamma : [a,b] \to \mathbb{C}$

A curve is closed if $\gamma(a) = \gamma(b)$

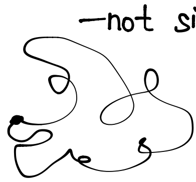
A curve is simple if $\gamma(t_1) = \gamma(t_2) \Longleftrightarrow t_1 = t_2$



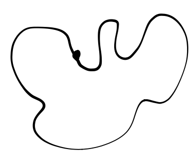
– not closed
– not simple



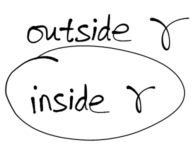
– not closed
– it is simple



– closed
– not simple



– closed
– simple



FACT: A simple closed γ, it splits the plane into two disjoint pieces, "the inside of γ" & "the outside of γ"

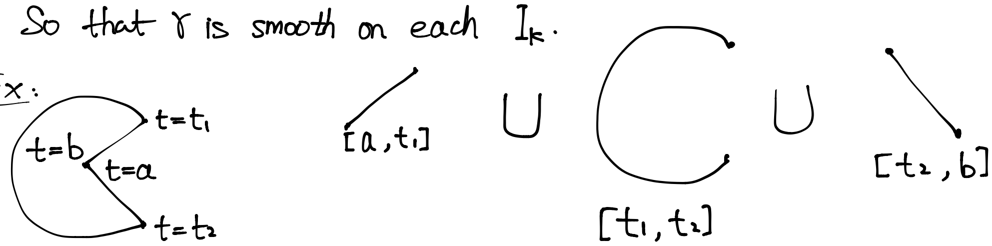
If we write $\gamma(t) = x(t) + iy(t)$, then we say γ is a smooth curve if x, y are differentiable & x', y' are cts.

A curve γ is called piecewise smooth if we can subdivide $[a,b] = I_1 \cup I_2 \cup \cdots \cup I_n$
$= [a,t_1] \cup \cdots \cup [t_{n-1}, b]$

$(t_n = b)$

So that γ is smooth on each I_k.

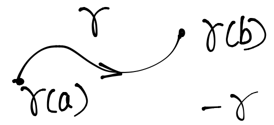
Ex:



If γ is piecewise smooth we write

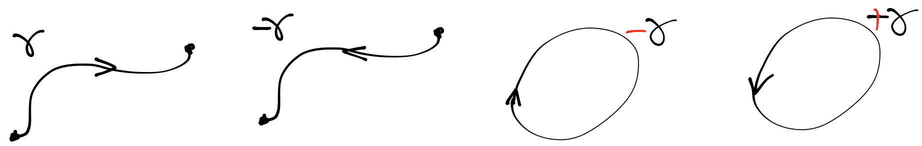
$$\gamma = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n \ , \ \gamma_k = \gamma \text{ on } I_k$$

A curve comes with a direction or orientation.

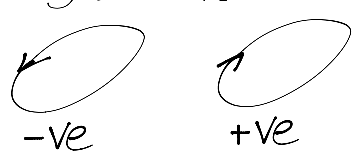


Every curve has an "opposite" orientation, we denote $-\gamma$.

$-\gamma$ is given by travelling along γ in the opposite direction.



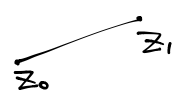
A simple closed curve γ is positively oriented if the inside of γ is to the left as we travel along the curve.



Parametrizations

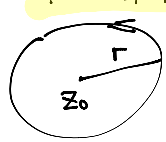
A function $\gamma : [a,b] \to \mathbb{C}$ $\quad \gamma(t) = x(t) + iy(t)$ is a parametrization.

Basic Parametrizations



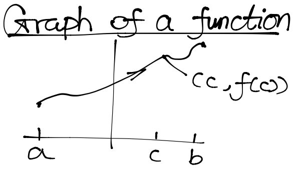
- Line segment
  joining z_0 to z_1 $\quad \gamma(t) = (1-t)z_0 + tz_1$
  $\gamma : [0,1] \to \mathbb{C}$

- Circle/Arc of circle
  $|z - z_0| = r \rightsquigarrow \gamma(t) = z_0 + re^{it}$



circle of rad r, centered at z_0
$\gamma : [0, 2\pi] \to \mathbb{C}$
arcs $\to$ restricting t values

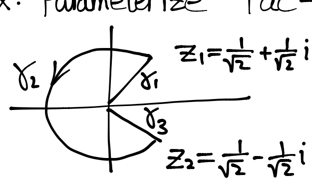
Graph of a function $\quad y = f(x) \quad x \in [a,b]$



$$\gamma(t) = x(t) + iy(t) = t + if(t)$$

* To parameterize a piecewise curve, you just need to parameterize each piece.

Ex: Parameterize "Pac-man"



$\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3$ is line segment joining 0 to z_1

$\gamma_1 = [0,1] \to \mathbb{C}$, $\gamma_1(t) = (1-t)0 + tz_1$

$\gamma_1(t) = tz_1$

$\gamma_2(t) = e^{iz}$, $\gamma : [\pi/4, 7\pi/4] \to \mathbb{C}$

$\gamma_3(t) = (1-t)z_2 + t \cdot 0 = (1-t)z_2$