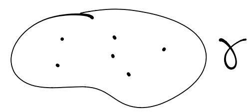
Lecture 19

Recall:

D connected domain in $\mathbb{C}$.

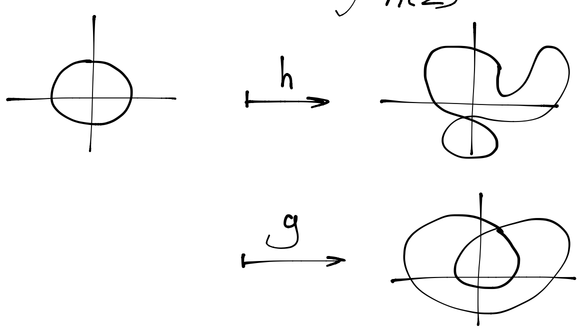
$h: D \to \mathbb{C}$ is holomorphic (meromorphic)

Thm: Let γ be a simple closed curve in D, s.t. h has no zeros or poles on γ.

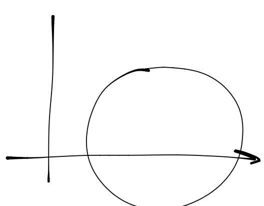


$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)}\,dz = \#\text{ of zeros (counted with multiplicity)} - \#\text{ of poles (counted with multiplicity)}$$

Fact: the integral $\frac{1}{2\pi i}\int \frac{h'(z)}{h(z)}\,dz$ has a geometric intepretation:



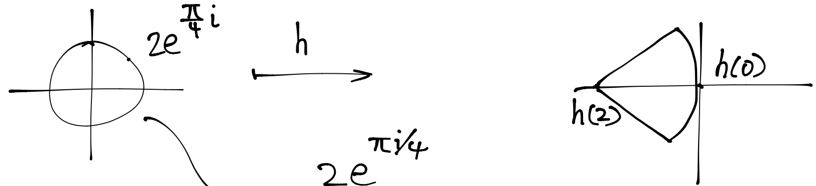
"winding #" := # of times the image curve $h(z)$ winds around the origin in the counterclockwise direction.
(so that each clockwise rotation counts negative)



$\leftarrow$ winding # 0

must be zero:
$h: 0 \to$ could not have zeros or pole inside the unit circle.

$h(z)=\frac{z}{1-z}$ γ = circle of radius 2 about the origin.



$$\frac{2e^{\pi i/4}}{1-2e^{\pi i/4}} \dashrightarrow \frac{a+bi}{c+di} \quad \begin{array}{l} a,b>0 \\ c,d<0 \end{array}$$

$$h(2i)=\frac{2i}{1-2i}=\frac{2i(2i+1)}{5}=\frac{-4+2i}{5}$$

$$h(-2)=\frac{-2}{1+2}=\frac{-2}{3}$$

$$h(-2i)=\frac{-2i}{1+2i}=\frac{-2i(1-2i)}{5}=\frac{-4-2i}{5}$$

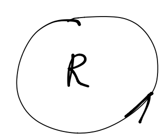
$$h(2e^{\pi i/4})=\frac{2e^{\pi i/4}}{1-2e^{\pi i/4}}=\frac{2e^{i\pi/4}(1+2e^{\pi i/4})}{(\qquad)}$$

$$(1-2e^{\pi i/4})(1+2e^{\pi i/4})=1-4e^{\pi i/2}=1-4i=(\qquad)$$

$$=\frac{2e^{\pi i/4}+4i}{1-4i}=\frac{(2e^{\pi i/4}+4i)(1+4i)}{17}$$
$$=\frac{2e^{\pi i/4}+4i+8ie^{\pi i/4}-16}{17}$$

$e^{\pi i/4}=\frac{2}{\sqrt{2}}(1+i)$ so $h(2e^{\pi i/4})$ is in 2nd quadrant

upshot: if γ bounds a region R, h is a meomorphic function on R which has no zeros or poles on γ.
$h(R)$ is not contained in $h(\gamma)$



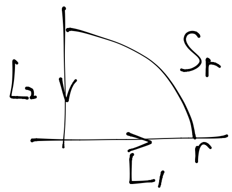
reason: h can have a pole in R & so not continuous.

Sps h has no poles in R.
then: h is a continuous function on R and then $h(R)$ contains origin $\Longleftrightarrow$ h has a zero in R.
(There is an analogue for poles, won't state today, inovling "Riemann sphere")

Applications

$z^3 - 2z^2 + 4$

Q: how many zeros in 1st quadrant ?



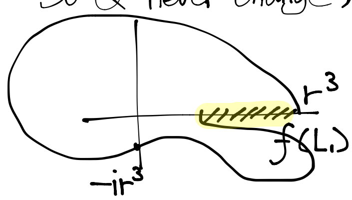
for r sufficiently large, all the zeros will live in this region.

f on L_1: $x^3 - 2x^2 + 4$ for $x \in [0, r]$

$f(0) = 4$

$f(x) \geq 2 \qquad \forall x \in [0, r]$ (from Calculus)

So Q never change,



$f(z) = z^3 \left(1 - \frac{2}{z} + \frac{4}{z^3}\right) \longrightarrow z^3$ as $r \uparrow \uparrow \uparrow$

z^3 on S_r :

$z^3(r) = r^3$

$z^3(ri) = (ri)^3 = -ir^3$

$f(iy) \quad y \in [r, 0]$

$f(iy) = -iy^3 + 2y^2 + 4$, with

real part $= 2y^2 + 4 > 0$

Im part $= -iy^3 < 0$

Wind #

zeros is 1

Rouchés' Thm

f, g holomorphic (no poles) in D, γ simple closed in D.

Sps that $|f(z) + g(z)| < |f(z)|$ on γ. then f & g have the same number of zeros inside γ (counted with multiplicity)

Pf: ① neither f nor g has a zero on γ if $f(z_0) = 0 \quad (z_0 \in \gamma)$

$|f(z_0) + g(z_0)| < |f(z_0)| = 0$? $\quad \lightning$

if $g(z_0) = 0$

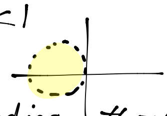
$|f(z_0)| = |f(z_0) + g(z_0)| < |f(z_0)| \quad \lightning$

② set $h(z) = \dfrac{g(z)}{f(z)}$, meomorphic in D

By ①, $h(z)$ has no zeros or poles in γ

winding # of h : $|1 + h(z)| < 1$

$\{z \mid |1+z| < 1\} :=$



so $h(\gamma)$ misses origin; winding # $= 0$

$0 = \underbrace{\text{\# zeros of } h}_{\text{zeros of } g} - \underbrace{\text{\# of poles of } h}_{\text{zeros of } f}$ (by argument principle).

So # zeros of g & f agree.

Thm: (Fundamental thm of Algebra)
A polynomial p of degree n has n zeros (w/ multiplicity)
In particular, p has a zero.
N.B. over real #'s, this is false, e.g: x^2+1

Pf: $\deg(p)=n$
$$p(z)=a_n z^n + a_{n-1}z^{n-1}+\cdots+a_1 z+a_0$$
$$(a_n \neq 0)$$

$\frac{1}{a_n}p(z)$ has the zeros
so for convenience, sps $a_n=1$
$$p(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1 z_1+z_0$$

Strategy: compare $p(z)$ and z^n
$$\left|\frac{p(z)-z^n}{z^n}\right| = \left|\sum_{i=0}^{n-1} a_i z^i \big/ z^n\right| = \left|\frac{a_{n-1}}{z}+\frac{a_{n-2}}{z^2}+\cdots+\frac{a_1}{z^{n-1}}+\frac{a_0}{z^n}\right|$$
$$= \frac{1}{|z|}\left|a_{n-1}+\frac{a_{n-2}}{z}+\cdots+\frac{a_1}{z^{n-2}}+\frac{a_0}{z^{n-1}}\right|$$
sps $|z|=r$, use Δ ineq, we get.
$$\leq \frac{1}{r}\left(|a_{n-1}|+|a_{n-2}|/r+|a_{n-3}|/r^2+\cdots+|a_1|/r^{n-2}+|a_0|/r^{n-1}\right)$$
$$\leq \frac{1}{2} \quad \text{if } r \longrightarrow \infty$$
arbitrarily taken

$|p(z)-z^n| \leq \frac{1}{2}|z^n| < |z^n|$ on S_r for some large r

So Rouché's $\Rightarrow p(z)$ & z^n have the same # of zeros inside S_r.
$\longrightarrow z^n$ has n zeros.

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