Jan 22nd

Today: linear Transformations & matrices **Transf**

defn: let V, W be vector spaces over a field F.
a map: $T: V \to W$ is called a linear transformation if
1). $T(v_1+v_2) = T(v_1) + T(v_2)$
2). $T(\alpha v) = \alpha T(v)$, for $\alpha \in F$.

ex: if $V=W$, $\dim(V)=1$
then there is a number $\lambda \in F$ such that $T(v)=\lambda v$ for all v.
Pf: $V=Fv$, $W=Fw$
$T(v)=\lambda w$ (since every element of W is of this form
now, take an arbitrary $e \in V$ then $e=\alpha \cdot v$
$\Rightarrow T(e)=T(\alpha \cdot v)=\alpha T(v)=\alpha \lambda v=\lambda \alpha \cdot v=\lambda e$

Suppose you know $T(v)=\lambda v$ for all $v \in V$
then: $T(v_1+v_2)=\lambda(v_1+v_2)=\lambda v_1+\lambda v_2=T(v_1)+T(v_2)$

Higher dimension:
Now suppose $T: V \to W$ and $\dim(V)=k$, $\dim(W)=l$, $(k, l < \infty)$
and T is a linear transformation

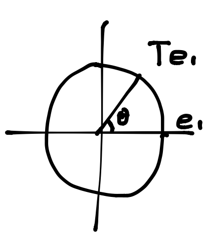
FACT: there exists a basis $\{v_1, \cdots, v_k\}$ of V and $\{w_1, \cdots, w_l\}$ of W
(furthermore, all bases of V have size k, all bases of W have size l)

($\Rightarrow$ any $v \in V$ can be written uniquely as $v=a_1v_1+a_2v_2+\cdots+a_kv_k$, $a_i \in F$)

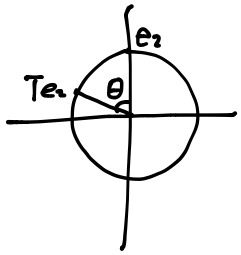
Now let's write
$T(v_i)=a_{1i}w_1+a_{2i}w_2+\cdots+a_{li}w_l$
for unique $\{a_{1i}, a_{2i}, \cdots, a_{li}\}$.
So this gives collection of numbers $\{a_{ji}\}_{\substack{1\le j\le k \\ 1\le i\le l}}$ (this is $k \cdot l$ numbers)

ex: $V=W=\mathbb{R}^2$
Take T to be the map "rotation by θ" $\theta \in [0, 2\pi)$. (counter-clockwise rotation)
To make a matrix: need to choose
bases for V & W" in this case, just choose basis for $\mathbb{R}^2$.
if write $\mathbb{R}^2=\{(a,b) \mid a, b \in \mathbb{R}\}$ so as a basis, we may choose
$e_1=(1,0)$, & $(0,1)=e_2$

next, evaluate Te_1 & Te_2



$\Rightarrow Te_1=(\cos\theta, \sin\theta)=\cos\theta e_1+\sin\theta e_2 (=a_{11}e_1+a_{21}e_1)$



$\Rightarrow Te_2 = (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2}))$

$\cos(\theta + \frac{\pi}{2}) = -\sin\theta \qquad \sin(\theta + \frac{\pi}{2}) = \cos\theta$

$Te_2 = (-\sin\theta, \cos\theta) = -\sin\theta e_1 + \cos\theta e_2$
$(= a_{21}e_2 + a_{22}e_2)$.

Matrix

def'n: an $l \times k$ matrix is an array of numbers which has l rows and k columns.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdots & a_{2k} \\ \vdots & & & & \vdots \\ a_{l1} & \cdot & \cdot & \cdot & a_{lk} \end{pmatrix}$$

Therefore, if $T: V \to W$ is a linear transformation, ($\dim V = k$, $\dim W = l$)
we associate to it the matrix where entries are determined by
$T(v_i) = \underbrace{a_{1i}w_1 + a_{2i}w_2 + \cdots + a_{li}w_l}$)
↳ the x #'s are the i^{th} columns of the matrix.

example: 1) if $V = W$, $\dim V = 1$
$T \longmapsto (\lambda)$.
2) for $V = W = \mathbb{R}^2$, $T = \text{rot}\,\theta$,
$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdots & a_{2k} \\ \vdots & & & & \vdots \\ a_{l1} & \cdot & \cdot & \cdot & a_{lk} \end{pmatrix}$$

$\boxed{l \times k \text{ matrix}}$

matrix multiplication

matrix mult. takes an $l \times k$ matrix and a $k \times n$ matrix
$\longrightarrow$ a $l \times n$ matrix

Key case for us: case where $n = 1$

a $k \times 1$ matrix $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix}$ is a column vector of length k.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdots & a_{2k} \\ \vdots & & & & \vdots \\ a_{l1} & \cdot & \cdot & \cdot & a_{lk} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{pmatrix} = \begin{pmatrix} a_1 a_{11} + a_2 a_{12} + \cdots + a_k \cdot a_{1k} \\ a_1 a_{21} + a_2 a_{22} + \cdots + a_k a_{2k} \\ \vdots \\ a_1 a_{l1} + a_2 a_{l2} + \cdots a_k a_{lk} \end{pmatrix}$$

$\boxed{l \times k \text{ matrix}}$ $\boxed{k \times 1 \text{ matrix}}$ $\boxed{l \times 1 \text{ matrix}}$

Now, let's define a linear transformation

$S: V \to W$ out of the matrix

note that choosing a basis $(v_1, \cdots, v_k)$ of V

Let's take $\begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix} \to b_1 v_1 + b_2 v_2 + \cdots + b_k v_k \in V$

define $S(b_1 v_1 + b_2 v_2 + \cdots + b_k v_k) : A \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}$ where A is my matrix $= \begin{pmatrix} c_1 \\ \vdots \\ c_l \end{pmatrix}$

and then $S(b_1 v_1 + \cdots + b_k v_k) = c_1 w_1 + c_2 w_2 + \cdots + c_l w_l$

To show $S = T$, all we need is $A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith spot $= T(v_i)$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1k} \\ a_{21} & a_{22} & & \cdots & a_{2k} \\ \vdots & & & & \vdots \\ a_{l1} & \cdot & \cdot & \cdot & a_{lk} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{li} \end{pmatrix}$$

$a_{1i} w_1 + a_{2i} w_2 + \cdots + a_{li} w_l = T(v_i)$

N.B. each matrix does indeed give n linear transformation by the rule.

Kernels & images

Let $T: V \to W$ be a linear transformation.

~~Let the~~ Kernel of T

$Ker(T) = \{v \in V \mid T(v) = 0\}$ (also called "null space")

Claim: $Ker(T)$ is a linear subspace of V.

1). $v_1, v_2 \in Ker(T) \overset{?}{\Longrightarrow} v_1 + v_2 \in Ker(T)$

If $Tv_1 = 0$ $Tv_2 = 0$

Then $T(v_1 + v_2) = Tv_1 + Tv_2 = 0 + 0 = 0$ ✓

2). if $v \in Ker(T) \overset{?}{\Longrightarrow} \alpha v \in Ker(T)$

for any $\alpha \in F$

$T(\alpha v) = \alpha\, T(v) = \alpha \cdot 0 = 0$ ✓

3). NOTE: $0 \in Ker(T)$ is automatic
i.e. Kernel is not empty all the time.

ex: $T(v) = \lambda v \quad (T: V \to V)$

$$Ker(T) = \begin{cases} 0 & \text{if } \lambda \neq 0 \\ V & \text{if } \lambda = 0 \end{cases}$$

Ex: $V = \mathbb{R}^n$, $\langle\, ,\, \rangle \leftarrow$ dot product
fix $a \in V \setminus \{0\}$

$$P_a(V) \overset{?}{=} \frac{\langle a, v\rangle}{\langle a, a\rangle} \cdot a$$

$P_a(V) = 0 \Rightarrow \langle a, v\rangle = 0 \Rightarrow \{v \in V \mid \langle a, v\rangle = 0\}$ is a subspace

<u>Claim:</u> this space has dim $n-1$

$T: V \to W$
def'n: the image of T, $Im(T) = \{w \in W \mid w = T(v) \text{ for some } v\}$

Image is a subspace