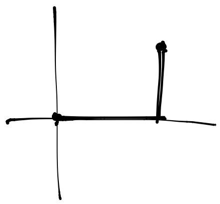
Lecture 15

Chapter 7

Norms & Inner Products

$\|x\|_1 = \|(x_1, \cdots, x_n)\| = \sum_{i=1}^{n} |x_i|$



$\|x\|_\infty = \|(x_1, \cdots, x_n)\|_\infty = \max_{1 \le i \le n} |x_i|$

Ex: Let K be a compact subset of $\mathbb{R}^n$, let $C(K)$ denote the vector space of all continuous functions

$f: K \to \mathbb{R}$

$(f+g)(x) = f(x) + g(x)$

$(\alpha f)(x) = \alpha(f(x))$

Possible norms on $C(K)$ is uniform norm

$\|f\|_\infty = \sup_{x \in K} |f(x)|$

$\|f\|_\infty = f(x_0) < \infty$ By the EVT if $f \mapsto \|f\|_\infty$ is nonnegative $\|f\|_\infty = 0 \Rightarrow f \equiv 0$

$\|\alpha f\|_\infty = \sup |\alpha f| = \sup |\alpha||f| = |\alpha| \sup |f| = |\alpha| \, \|f\|_\infty$

Triangle inequality

$\|f+g\|_\infty = \sup_{x \in K} |f(x) + g(x)|$

Triangle inequality
Euclidean space

$\le \sup(|f(x)| + |g(x)|) \le \sup_{x \in K} |f(x)| + \sup_{x \in K} |g(x)|$

$= \|f\|_\infty + \|g\|_\infty$

Ex:
Let $C^3[a,b]$ be a vector space of all functions $f: [a,b] \to \mathbb{R}$ s.t. f, f', f'', f''' are defined & continuous

$\|f\|_{C^3} = \max_{0 \le j \le 3} \|f^{(j)}\|_\infty$

L^p norms

$L^p[a,b]$ is defined on $C[a,b]$. fix $p \in [1, \infty)$

$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx \right)^{1/p}$

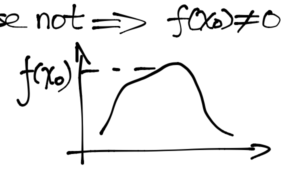
(1) $\|f\|_p \ge 0$

(2) Suppose $\|f\|_p = 0$

I want to show that $f \equiv 0$ on $[a,b]$ Suppose not $\Rightarrow$ $f(x_0) \ne 0$ at some x_0.

Take $\varepsilon = \frac{|f(x_0)|}{2}$, $\exists \, \delta > 0$ s.t.

$|f(x) - f(x_0)| < |f(x_0)|/2$ if $|x - x_0| < \delta$



$$|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)|/2$$

$$\left[\int_{x_0-\delta}^{x_0+\delta} |f(x)|^p dx\right]^{1/p} > \left[\int_{x_0-\delta}^{x_0+\delta} \left(\frac{|f(x_0)|}{2}\right)^p dx\right]^{1/p}$$

$$= (2\delta)^{1/p} \frac{|f(x_0)|}{2} > 0$$

Homogeneity

Triangle inequality ?

① L_1-norm

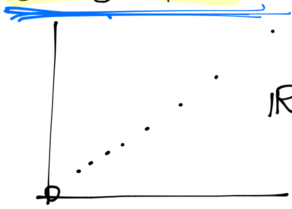
$$\|f+g\|_1 = \int_a^b |f(x)+g(x)| dx \leq \int_a^b (|f(x)| + |g(x)|) dx = \int_a^b |f(x)| dx + \int_a^b |g(x)| dx$$

$$= \|f\|_1 + \|g\|_1$$

§7.2 Topology in Normed Spaces

Def: In a normed v space $(V, \|\cdot\|)$, a sequence $(v_n)_{n=1}^{\infty}$ converges to $v \in V$ if $\lim_{n\to\infty} \|v_n - v\| = 0$. Equivalently, $\forall \varepsilon > 0, \exists N > 0$ s.t. $\|v_n - v\| < \varepsilon \ \forall n \geq N$. Written as $\lim_{n\to\infty} v_n = v$.

Cauchy sequences :



$\mathbb{R}^2 - \{0\}$

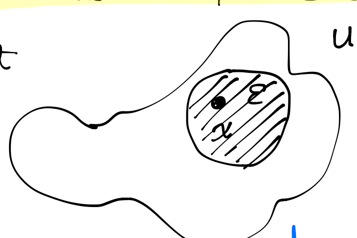
$\{(\frac{1}{n}, 0), n \in \mathbb{N}\}$

$(v_n)_{n=1}^{\infty}$ is Cauchy if for every $\varepsilon > 0, \exists N > 0$ s.t. $\|v_n - v_m\| < \varepsilon, \forall n, m \geq N$.

Def $(V, \|\cdot\|)$ is complete if every Cauchy sequence in V converges to some $v \in V$

A complete normed space is called a Banach space

open set



Def: For a normed vector space $(V, \|\cdot\|)$, we define the open ball with center $a \in V$ and radius $r > 0$ to be $B_r(a) = \{v \in V : \|v - a\| < r\}$.

A subset U of V is open if $\forall a \in U, \exists r > 0$ s.t. $B_r(a) \subset U$.

- - - - - C . . . . . closed if it contains all of its limit pts. i.e. whenever (x_n) is a seq. in C & $x = \lim_{n\to\infty} x_n$, $x \in C$. in a normed space

Proposition. A sequence $(v_n)_{n=1}^{\infty}$ in $(V, \|\cdot\|)$ converges to v iff for each open set U containing v, $\exists N$ s.t. $v_n \in U$, $\forall n \geq N$

(skipped 7.2.5 Definition)

$f: X \to Y$ one-to-one onto continuously inverse continuous (homeomorphism)

§ 7.3 Finite-Dimensional Normed Spaces

Prop ~~Lemma~~: If $\{v_1, v_2, \cdots, v_n\}$ is a linearly independent set in a normed vector space $(V, \|\cdot\|)$, then there exists positive constants $0 < c < C$ s.t. for all $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$, we have $c\|a\|_2 \overset{(2)}{\leq} \|\sum_{i=1}^n a_i v_i\| \overset{(1)}{\leq} C\|a\|_2$

Proof: $\|\sum_{i=1}^n a_i v_i\| = \|a_1v_1 + \cdots + a_nv_n\| \leq \|a_1v_1\| + \cdots + \|a_nv_n\| = \sum_{i=1}^n |a_i| \|v_i\|$

$$\leq \left(\sum_{i=1}^n a_i^2\right)^{1/2} \left(\sum_{i=1}^n \|v_i\|^2\right)^{1/2}$$

$\|a\|_2 \cdot C$ where C is