Answer the following questions in the space provided. Give **complete solutions and justify your answers.**

**1.** (a) Find $\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{60}$. Express your answer in $x+iy$ form. (3 pts)

Solution:

$$\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)^{60}=\left(e^{i\frac{7}{6}\pi}\right)^{60} \cdot \left|\left(\frac{\sqrt{3}}{2}\right)^2+\frac{1}{4}\right|$$

$$=e^{i70\pi}$$

$$=e^{i2\pi}$$

$$=e^{i0}$$

$$=1$$

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(b) Solve the equation $e^{iz}=1+i$. Express your answer in $x+iy$ form. (3 pts)

Solution:

$$e^{iz}=1+i$$

$$\log e^{iz} \cancel{\log iz} = \log(1+i)$$

$$iz=\log(1+i)$$

$$iz=\ln|1+i|+i\arg(1+i)$$

$$z=\frac{1}{i}\left(\ln\sqrt{2}+i\left(\frac{\pi}{4}+2k\pi\right)\right) \qquad k=0,\pm1,\pm2,\cdots$$

$$=\left(\frac{\pi}{4}+2k\pi\right)-i\ln\sqrt{2} \qquad k=0,\pm1,\pm2,\cdots$$

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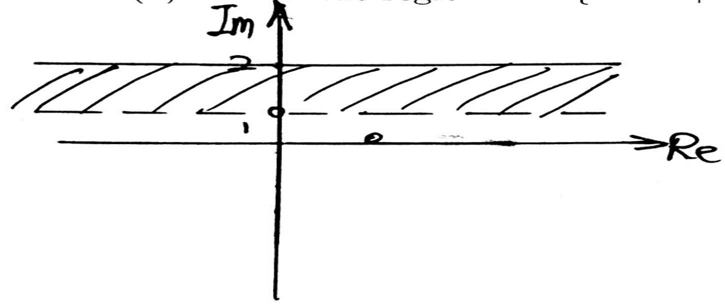
2. Please sketch the following regions in the space provided. Use the convention that a dashed line (— — —–) is used to exclude points on the boundary of the set, and a solid line (——) is used to include points on the boundary of the set, and the interior of the set is shaded.
Use these sketches to determine if the given set is connected and simply-connected.

(a) Sketch the region $D = \{z \in \mathbb{C} \mid 1 < Im(z) \leq 2\}$. (2pts)




~~Connected~~

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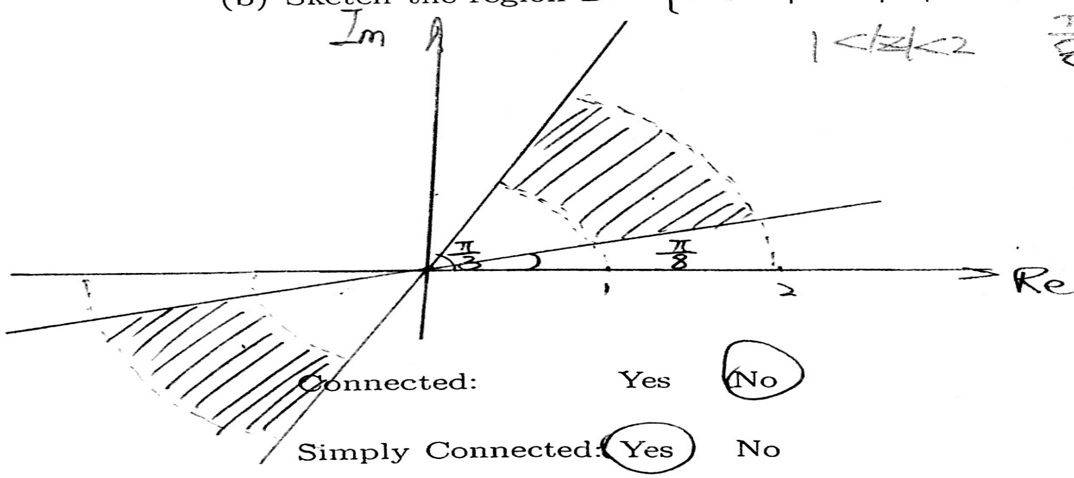
Connected: Yes No

Simply Connected: Yes No

(b) Sketch the region $D = \left\{z \in \mathbb{C} \mid 1 < |z^2| < 4 \quad \& \quad \frac{\pi}{4} \leq Arg(z^2) \leq \frac{2\pi}{3}\right\}$. (2pts)

$1 < |z| < 2 \quad \frac{\pi}{8} \leq Arg z \leq \frac{1}{3}\pi$

$z = e^{i} \quad -\frac{7}{8}\pi$




2

Connected: Yes No

Simply Connected: Yes No

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**3.** Let $u(x,y) = ax^3 - xy^2 - bx^2y + y^3$. For what $a, b \in \mathbb{R}$ is $u$ the real part of an analytic function $f = u + iv$. Justify your answer.

Solution: $f = u+iv$ analytic

$\Rightarrow$ ~~u & v are harmonic~~ $u$ is harmonic

$\Rightarrow$ $\cancel{\Delta u = \Delta v = 0}$ $\Delta u = 0$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = 3ax^2 - y^2 - 2bxy$$

$$\frac{\partial u}{\partial y} = -2xy - bx^2 + 3y^2$$

$$= 6ax - 2by + -2x + 6y$$

$$= 0$$

so $6 - 2b = 0$, $6a - 2 = 0$

Therefore $b = 3$ and $a = \frac{1}{3}$

so for $b = 3$ & $a = \frac{1}{3}$ s.t. $u$ is a real part of analytic function.

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Solution $\left(a = \frac{1}{3}, b = 3\right)$

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4. Let $\gamma = \gamma_1 \cup \gamma_2$ be the piecewise smooth curve with $\gamma_1$ the line segment joining 0 to $1+i$, and $\gamma_2$ the circle arc joining $1+i$ to $\sqrt{2}i$.

(a) Parametrize $\gamma$. (You can parametrize $\gamma_1, \gamma_2$ separately.) (4 pts)

$\gamma_1$: $\gamma_1(t) = (1-t)\cdot 0 + t\cdot(1+i)$
$= t + ti$ where $t \in [0,1]$

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$\gamma_2$: $\gamma_2(t) = \sqrt{2}e^{it}$ where $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$
$= \cancel{\sqrt{2}(\cos t + i\sin t)}$

(Note: Actually, we'd better choose another parameter in $\gamma_2$ instead of $t$ since $t$ is used in $\gamma_1$, but this does not affect the result.)

(b) Evaluate the line integral $\int_\gamma (\bar{z} + z)dz$. (6pts)

Solution: $\int_\gamma (\bar{z}+z)dz = \int_{\gamma_1}(\bar{z}+z)dz + \int_{\gamma_2}(\bar{z}+z)dz$

note that: For $\gamma_1$: For $\gamma_2$:

$z = t + ti$ $\quad z = \sqrt{2}e^{it}$
$dz = (1+i)dt$ $\quad \bar{z} = \sqrt{2}e^{-it}$
$\bar{z} = t - ti$ $\quad dz = \sqrt{2}ie^{it}dt$

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$\int_{\gamma_1}(\bar{z}+z)dz = \int_0^1 2t\cdot(1+i)dt = \frac{1}{2}(1+i)\cdot 2t^2 \Big|_0^1 = 1+i$

$\int_{\gamma_2}(\bar{z}+z)dz = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2}(e^{it}+e^{-it})\sqrt{2}ie^{it}dt$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2i(e^{2it}+1)dt$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (2ie^{2it} + 2i)dt$

$= e^{2it} + 2it \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = (e^{i\pi} + i\pi) - (e^{\frac{i\pi}{2}} + \frac{i\pi}{2}) = \frac{i\pi}{2} - 1 - i$
$= -1 + i(\frac{\pi}{2} - 1)$

So add up:
$\boxed{\int_\gamma (\bar{z}+z)dz = \frac{i\pi}{2}}$

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Sol:

5. (a) Let $f(z) = \frac{(z-1)^2}{2z}$. Find the power series for $f$ centred at $z_0 = 1$. (7 pts)

$f(z) = \frac{1}{2}(z-1)^2 \frac{1}{z} = \frac{1}{2}(z-1)^2 \frac{1}{1-(-(z-1))}$

Known $\frac{1}{1-w} = \sum_{n=0}^{\infty} w^n$

So $f(z) = \frac{1}{2}(z-1)^2 \sum_{n=0}^{\infty} (-(z-1))^n$

$= \frac{1}{2}(z-1)^2 \sum_{n=0}^{\infty} (-1)^n (z-1)^n$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2} (z-1)^{n+2}$

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Sol:

(b) Find the radius of convergence of the series $\sum_{n=0}^{\infty} \frac{(4-3i)^{n-1}}{(-1+i)^n} z^n$. (3 pts)

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(4-3i)^n}{(-1+i)^{n+1}} \cdot \frac{(-1+i)^n}{(4-3i)^{n-1}}\right|$

$= \lim_{n\to\infty} \frac{|4-3i|}{|-1+i|}$

$= \frac{5}{\sqrt{2}} \Rightarrow R = \frac{\sqrt{2}}{5}$

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So the radius of convergence is $\frac{\sqrt{2}}{5}$.

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6. Evaluate the line integral $\int_\gamma \frac{e^{iz^2}}{z^{50}} dz$, where $\gamma$ is the circle $|z-1| = 10$ oriented positively.

Solution:

$$\int_\gamma \frac{e^{iz^2}}{z^{50}} dz = 2\pi i a_{49}$$

where $a_{49}$ is the 50th coefficient in our power series $e^{iz^2}$.

Why? Since $e^{iz^2}$, let it be $f(z) = e^{iz^2}$, it's analytic in the domain $|z-1|=10$, because exponential is always differentiable in its domain. Then it guarantees that $e^{iz^2}$ ~~is~~ can be written as a power series.

$$e^{iz^2} = \sum_{n=0}^{\infty} \frac{(iz^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^n (z^{2n})}{n!}$$

$$\text{So } a_n = \frac{i^n}{n!} \Rightarrow a_{49} = \frac{i^{49}}{49!} = \frac{(i^4)^{12} \cdot i}{49!} = \frac{i}{49!}$$

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Therefore, $\int_\gamma \frac{e^{iz^2}}{z^{50}} dz = 2\pi i \cdot \frac{i}{49!} = \frac{-2\pi}{49!}$

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7. Evaluate the integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{3-2\sin\theta}$.

Hint: Under the substitution $z=e^{i\theta}$, $\sin(\theta)=\frac{1}{2i}(z-\frac{1}{z})$ and $d\theta=\frac{dz}{iz}$.

Solution: Let $\gamma$ be a unit circle, so we can parametrize $z=e^{i\theta}$

Rewrite: $$\int_\gamma \frac{1}{iz}\frac{dz}{3-\frac{1}{i}(z-\frac{1}{z})}$$

$$=\int_\gamma \frac{dz}{3iz-z(z-\frac{1}{z})}=\int_\gamma \frac{dz}{3iz-z^2+1}$$

To solve $-z^2+3iz+1$

We calculate:

$$z=\frac{-3i\pm\sqrt{-9+4}}{-2}=\frac{-3i\pm\sqrt{5}i}{-2}$$

$$=\frac{3-\sqrt{5}}{2}i \text{ or } \frac{3+\sqrt{5}}{2}i$$

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Check: $\left|\frac{3-\sqrt{5}}{2}i\right|<1$ $\left|\frac{3+\sqrt{5}}{2}i\right|>1$, $\frac{3-\sqrt{5}}{2}i$ is inside $\gamma$.

So we pick $f(z)=\dfrac{1}{z-(\frac{3+\sqrt{5}}{2}i)}$

$$\int_\gamma \frac{dz}{3iz-z^2+1}=2\pi i\, f\left(\frac{3-\sqrt{5}}{2}i\right)$$ by Cauchy formula

X

$$=2\pi i\,\frac{1}{\frac{3-\sqrt{5}}{2}i-\frac{3+\sqrt{5}}{2}i}$$

$$=2\pi i\,\frac{1}{-\sqrt{5}i}$$

$$=-\frac{2\sqrt{5}}{5}\pi$$

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