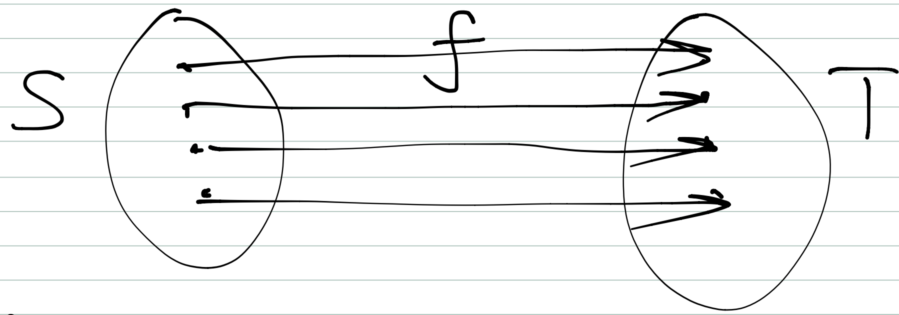
March 13th

Cardinality theory

We say that two sets S & T have the same cardinality ($|S|=|T|$) if $\exists$ $f: S \to T$ 1-1 onto



f is 1-1 means $s_1 \neq s_2 \Rightarrow f(s_1) \neq f(s_2)$
f is onto means for any t in T there is $s \in S$ s.t. $f(s)=t$

(f is a bijection if it's 1-1 & onto)

Ex: $S=\{1,2,4\}$
$T=\{-\frac{1}{5}, \sqrt{2}, 6\}$
can find $f: S \to T$ 1-1 & onto

for example
can force
$f(1)=-\frac{1}{5}$
$f(2)=\sqrt{2}$
$f(4)=6$

f is not unique
can force $f(1)=\sqrt{2}$, $f(2)=6$, $f(4)=-\frac{1}{5}$

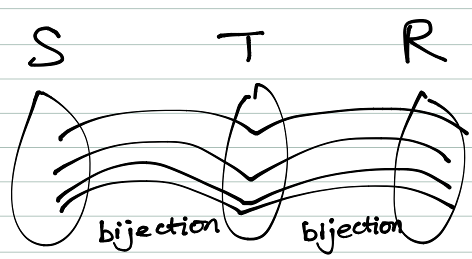
If S & T are finite $\Rightarrow |S|=|T| \Leftrightarrow S$ and T have the same number of elements.
$S=\{1,2,4\}$, $T=\{\frac{1}{5}, \sqrt{2}\}$ no $f: S \to T$ 1-1 & onto exists.

Properties

- $|S|=|T| \Rightarrow |T|=|S|$

if $f: S \to T$ 1-1 & onto
$\Rightarrow f^{-1}: T \to S$ 1-1 & onto

- If $|S|=|T|$ and $|T|=|R|$ then $|S|=|R|$



then $g \circ f: S \to R$ 1-1, onto

ex: $S = N$ - all natural numbers
$T =$ even natural numbers
$= N^{ev}$

$S = \{1, 2, 3, \cdots\}$
$T = \{2, 4, 6, \cdots\}$

Claim: $|S| = |T|$
$|N| = |N^{ev}|$
want $f: N \to N^{ev}$ 1-1, onto

$f(x) = 2x$
$1 \longrightarrow 2$
$2 \longrightarrow 4$
$3 \longrightarrow 6$
$\cdots$

$N \longrightarrow N^{ev}$
1-1 & onto

Claim $|N| = |\mathbb{Z}_{\geq 0}|$

$\mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \cdots\}$

want $f: N \to \mathbb{Z}_{\geq 0}$ 1-1 & onto

$f(x) = x - 1$
$1 \longrightarrow 0$
$2 \longrightarrow 1$
$3 \longrightarrow 2$
$\cdots$

EX: $S = [0, 1] = \{0 \leq x \leq 1\}$
$T = [0, 1) = \{0 \leq x < 1\}$

then $|S| = |T|$

$f: S \to T$

$$f(x) = \begin{cases} x & \text{if } x \neq \frac{1}{n} \\ \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ for some natural } n \end{cases}$$

HW: $|[0, 1]| = |[0, 1)|$
$|N| = |\mathbb{Z}|$

$S = (0, 1)$
$T = (0, a)$ $a > 0$ any $\mathbb{R}$

$|S| = |T|$

$f(x) = ax$

$S = (0, a)$
$T = (b, b+a)$
$\Rightarrow |S| = |T|$ $f(x) = x + a$

$S=(0,1)$
$T=(a,b)$ for any $a<b$
$|(0,1)|=|(0,b-a)|=|(a,b)|$
$\Rightarrow |S|=|T|$

$|(0,1)|=|\mathbb{R}|$ $\quad f:(-\frac{\pi}{2},\frac{\pi}{2})\longrightarrow(-\infty,\infty)$ $\quad \tan x:(0,\frac{\pi}{2})\longrightarrow(0,\infty)$

$f(x)=\tan x$ $\quad |(-\frac{\pi}{2},\frac{\pi}{2})|=|(-\infty,\infty)|$ $\quad |(0,\frac{\pi}{2})|\overset{1-1,\ \text{onto}}{=}|(0,\infty)|$

$$\begin{matrix} |(-\frac{\pi}{2},\frac{\pi}{2})| & & |(0,\frac{\pi}{2})| \\ \| & & \| \\ |(0,1)| & & |(0,1)|=|\mathbb{R}| \end{matrix}$$

Counter: $\underset{(-\infty,\infty)}{|\mathbb{R}|}\neq\underset{\{1,2,3,\cdots\}}{|\mathbb{N}|}$

Proof: Claim: there is no map $f:\mathbb{N}\longrightarrow\mathbb{R}$
s.t. f is 1-1 & onto

Suppose not. Sps such f exists.

$f:\mathbb{N}\longrightarrow\mathbb{R}$ 1-1 & onto
$f(1)=x_1.a_{11}a_{12}a_{13}a_{14}\cdots$
integer part $\quad$ digits after the decimal

$f(2)=x_2.a_{21}a_{22}a_{23}a_{24}\cdots$
$f(3)=x_3.a_{31}a_{32}a_{33}a_{34}\cdots$

$0\leq a_{ij}\leq 9$ digits $\quad$ pick $0\leq b_1\leq 9$ s.t. $b_1\neq a_{11}$
pick $0\leq b_2\leq 9$ s.t. $b_2\neq a_{22}$
pick $0\leq b_3\leq 9$ s.t. $b_3\neq a_{33}$
$\vdots$
pick $0\leq b_n\leq 9$ s.t. $b_n\neq a_{nn}$

look at $x=0.b_1b_2b_3b_4\cdots$

Claim: $x\neq f(n)$ for any $n=1,2,3\cdots$
Why? $x\neq f(1)$ b/c x and $f(1)$ have different first digit after the decimal.
$x\neq f(2)$ b/c x and $f(2)$ have different second - - -
$\cdots$
$x\neq f(n)$ b/c nth digit after decimal in x is b_n
" —— " " —— $f(n)$ is a_{nn}

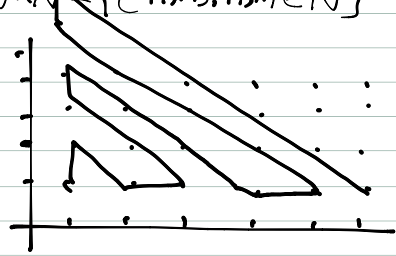
We produced x such that $x\neq f(n)$ for any n
$\Rightarrow f:\mathbb{N}\to\mathbb{R}$ is not onto #
$\Rightarrow |\mathbb{N}|=|\mathbb{R}|$

$f(1) = x_1 . a_{11} a_{12} \cdots$
$f(2) = x_2 . a_{21} a_{22} \cdots$
$f(3) = x_3 . a_{31} a_{32} \cdots$

<u>Claim:</u> $|N \times N| = |N|$

if S, T – sets
$S \times T = \{(s,t) \mid s \in S, t \in T\}$ – produced

$N \times N = \{(n,m), n, m \in N\}$



want $f: N \to N$ 1-1 & onto
$\Rightarrow |N| = |N \times N|$

$|S_1| = |S_2| \quad |T_1| = |T_2|$

$\Rightarrow |S_1 \times T_1| = |T_1 \times T_2|$

$\underset{\text{1-1, onto}}{f: S_1 \to S_2} \quad \underset{\text{1-1, onto}}{g: T_1 \to T_2}$ $\qquad f \times g: S_1 \times T_1 \to S_2 \times T_2$
$(f \times g)(s_1, t_1) = (f(s_1), g(t_1))$

$\Rightarrow$ Cor: $|N^n| = |N|$ for any n – natural number

$N^n = \underbrace{N \times N \times \cdots \times N}_{n \text{ times}} = \{m_1, \cdots, m_n\}$ $\quad m_i \in N$

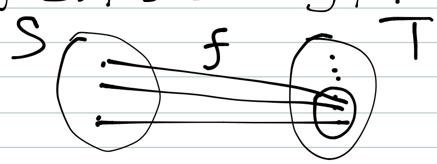
<u>Pf:</u> induction: ① $n = 1$ – obvious $|N| = |N|$

ind. step if $|N^n| = |N|$ by induction assumption
$N^{n+1} = N^n \times N$ $\qquad |N^n| = |N| \Rightarrow$
$|N^{n+1}| = |N^n \times N| = |N \times N| = |N|$
↳ we just proved

$|N| = |Z| \Rightarrow$ same for Z

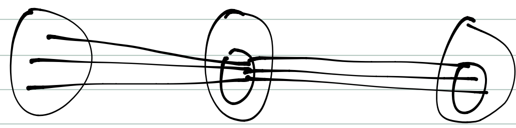
$|Z^n| = |Z| = |N| = |N^n|$

If S, T sets we say $|S| \leq |T|$ if $\exists$ 1-1 map $f: S \to T$
(need not be onto)
(i.e. $|S| = |f(S)|$)



Properties $|S| \le |T|$, $|T| \le |R| \Rightarrow |S| \le |R|$

S T R



$g \circ f : S \to R$
$g \circ f$ is 1-1

if $S \subseteq T \longrightarrow$ always $|S| \le |T|$
S is a subset of T

ex: $|\mathbb{N}| \le |\mathbb{R}|$ $|[0,1)| \le |[0,2]|$

$S = [0,1]$, $T = [0,1)$
$|S| \le |T|$ want $f : [0,1] \to [0,1)$ 1-1,
ex: $f(x) = \frac{x}{2}$ 1-1 $f([0,1]) = [0, \frac{1}{2}] \subset [0,1)$.

for numbers
a, b real numbers
$a \le b$ and $b \le a \Rightarrow a = b$

Q: is this true for cardinalities?
S, T two sets
$|S| \le |T|$?
$|T| \le |S| \Longrightarrow |S| = |T|$

ex: $S = [0,1]$
$T = [0,1)$

$T \subseteq S \Rightarrow |T| \le |S|$
$f(x) = x$
$T \to S$ 1-1

$|S| \le |T|$ $g(x) = \frac{x}{2}$
$[0,1] \to [0,1)$ 1-1

Theorem (The Cantor–Bernstein Theorem) If $|S| \le |T|$ & $|T| \le |S|$ then $|S| = |T|$
Schroeder

3 possibilities
given $t \in T \Rightarrow$ look at its ancestoral chain
- if it's infinite we say that $t \in T_\infty$
- it terminates in $S \Rightarrow$ we say $t \in T_S$
- it terminates in $T \Rightarrow$ we say $t \in T_T$

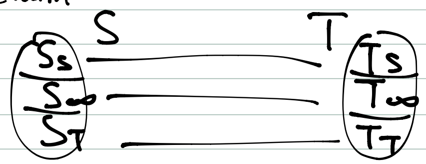
$T = T_\infty \cup T_S \cup T_T$

None of them intersect

$T_\infty = \{ t \in T \mid$ ancestoral chain of t is $\infty \}$
$T_T = \{ t \in T \mid$ last ancestor is in $T \}$
$T_S = \ldots$ $S \}$

$S = S_\infty \cup S_S \cup S_T$

Claim:



$f: S_S — T_S$ 1-1, onto
$f: S_\infty — T_\infty$ 1-1, onto
$f: S_T — T_T$ 1-1, onto

How to prove $f: S_S - T_S$ bijection?
given the claim:
$\Rightarrow$ set $h: S — T$ by the formula