Lecture 10

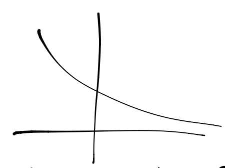
$$\boxed{\begin{aligned}\dot{x}(t)&=Mx(t)-N\alpha(t)\\ x(0)&=x^0\end{aligned}}$$

$x(\cdot):[0,T]\longrightarrow\mathbb{R}^n$
$\alpha(\cdot):[0,T]\longrightarrow[-1,1]^m=:A$
$M$: $n\times n$ matrix, $N$: $n\times m$ matrix
Problem: Given $x^0$, find control $\alpha^*(\cdot)$ such that the corresponding solution $x^*(\cdot)$ reaches $0$ in minimum possible time.

Theorem 1: there exists an optimal control $\alpha(\cdot)$
Idea of the proof:
let $T^*=\inf\{t\geq 0$: there exists some control $\alpha(\cdot)$ steering the system to $0$ in time $t.\}$

Remark: What are we worried about? Consider the much more elementary problem:
Given $f:\mathbb{R}\to\mathbb{R}$. Find a point $x$ that minimizes $f$.

What can go wrong? Sps $f(x)=e^{-x}$



No minimizing pt exists since $f(x)>0\ \forall x$. but $\inf f=0$

Choose a sequence $t_k\searrow T^*$
and control $\alpha_k:[0,t_k]\to[-1,1]^m$ s.t. $\alpha_k$ "steers the system to origin in time $t_k$"

Goal: $\alpha_k$'s converge somehow to a control on $[0,T^*]$
And this control steers $x(\,)$ to $0$ in time $T^*$.

To do this we need:
weak * convergence & Alaoglus Thm (from last lecture). (details in Evans notes)

Topic: How to find a minimizer?

Def: reachable set at time $t$ = $\{$points $x'$ for which $\exists$ a control steer-ing $x(\,)$ from $x^0$ to $x'$ at time $t\}$
Today's
say $K(t,x^0)$

Theorem: (special case of Pontryagin Max Principle)
Sps $\alpha^*$ is an optimal control and $x^*$ is the corresponding solution. Then there is a vector $h$ s.t.

$$\boxed{h^Te^{-Mt}N\alpha(t)=\max\{h^Te^{-Mt}Na,\ a\in A\}\ \forall \text{ all } t.}$$

Proof (sketch)

Claim 1: $0 \in \partial K(t, x^0)$ is <u>closed</u> & <u>convex</u> for every $t$.

<u>convex:</u> sps $x'$ & $\tilde{x}' \in K(t, x^0)$,

Then there are controls $\alpha(\cdot)$ & $\tilde{\alpha}(\cdot)$ such that

$$x' = e^{tM}x^0 + e^{tM}\int_0^t e^{-sM}N\alpha(s)\,ds$$

$$\tilde{x}' = e^{tM}x^0 + e^{tM}\int_0^t e^{-sM}N\tilde{\alpha}(s)\,ds$$

Given $\theta \in (0,1)$, I must find a control steering system to $\theta x' + (1-\theta)\tilde{x}'$ in time $t$:

<u>Choose</u> control $\theta\alpha(\cdot) + (1-\theta)\tilde{\alpha}(\cdot)$

This works:

(multiply 1st eqn by $\theta$, 2nd by $(1-\theta)$ and add)

<u>Closed</u>: consider $y^k \in K(t, x^0)$, $k = 1, 2, \cdots$

such that $y^k \to z$. Must show $z \in K(t, x^0)$.

Idea: for each $k$, find $\alpha^k(\cdot)$ s.t.

$$y^k = e^{tM}x^0 + e^{tM}\int_0^t e^{-sM}N\alpha^k(s)\,ds$$

Show ① $\alpha_k$'s converges to a limit ② the limit "works"

Again, can be done with Alaoglu's thm & "weak-* convergence".

<u>Claim 2</u>:

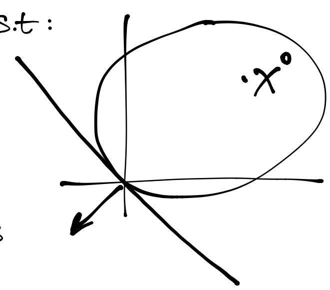
$0 \in \partial K(t, x^0)$

Idea: if it were in interior, "we could have gotten there sooner."

<u>Claim 2</u>: There exists a vector $g$ s.t. $g \cdot x' \leq 0$ for all $x' \in K(t, x^0)$

This says: there is a plane passing through $0$, s.t.:

$K \leq$ "one side of the plane"




This follows from convexity of $K(\tau^*, x^0)$ and the fact that $0 \in \partial K(\tau^*, x^0)$

Now, consider any $x' \in K(\tau^*, x^0)$. There exists a control $\alpha(\cdot)$ such that

$$x' = e^{\tau^* M}x^0 + e^{\tau^* M}\int_0^{\tau^*} e^{-sM}N(s)\alpha(s)\,ds$$

Also, optimal control $\alpha^*$ "steers to origin" so

$$0 = e^{\tau^* M}x^0 + e^{\tau^* M}\int_0^{\tau^*} e^{-sM}N\,\alpha^*(s)\,ds.$$

subtract

$$x' = e^{\tau^* M}\int_0^{\tau^*} e^{-sM}N(\alpha(s) - \alpha^*(s))\,ds$$

Multiply both sides by $g'$ & use "Claim 3"

$$g^T x' = \int_0^{\tau^*} \underbrace{g^T e^{\tau^* M}}_{\text{call this } h^T} e^{-sM} N(\alpha(s) - \alpha^*(s))\,ds$$

$$\| $$

$$g \cdot x' \leq 0$$

This:

$$\boxed{\int_0^{\tau^*} h^T e^{-sM} N(\alpha^*(s) - \alpha(s))\,ds \geq 0 \quad \text{for all } \alpha(\cdot)} \qquad \circledast$$

Claim: $\circledast \Rightarrow h^T e^{-sM} N\alpha^*(s) = \max\{h^T e^{-sM} N a\}$, $a \in A$ for all $s \in [0, \tau^*]$

Idea: if not, we could design a control to violate $\circledast$ ✓

considering ☺ proved, we are to discuss examples now.

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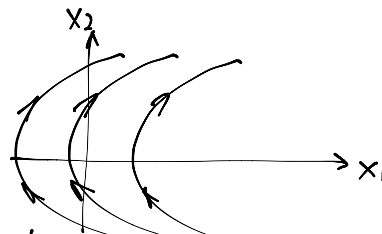
EXAMPLES

① Railroad Rocket Car

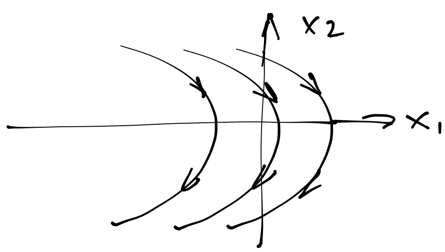
$\dot{x}(t) = Mx(t) + N\alpha(t)$ $\qquad \dot{x}_1 = x_2$, $\dot{x}_2 = \alpha(t)$

$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $\qquad M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$




Recall: we said: If $\alpha = 1$, then solutions stay on parabolas in $x_1$–$x_2$ plane as shown.

If $\alpha = -1$, since for picture




our idea before:
To reach the origin from one family of parabolas to the other.

This can steer us to $O$, but is it optimal?

Pontryagin says: there is $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ s.t. $h^T e^{-Mt} N\alpha^*(t) = \max_{a \in [-1,1]} h^T e^{-Mt} N a$, $\forall$ all $t$ up to $\tau^*$.

Here $e^{-Mt} = I + (tM) + \frac{1}{2}(-tM)^2 + \cdots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

So $e^{-tM} N = \begin{pmatrix} t \\ 1 \end{pmatrix}$

So $h^T e^{-tM} N = -h_1 t + h_2$

So $(-h_1 t + h_2)\,\alpha^*(t) = \max_{|a| \leq 1} (-h_1 t + h_2)\,a$

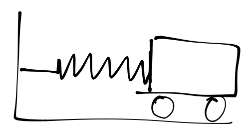
So $\alpha^*(t) = \begin{cases} +1 & \text{if } -h_1 t + h_2 > 0 \\ -1 & \text{if } -h_1 t + h_2 < 0 \end{cases}$

<u>Conclusions</u>: ① should always take $\alpha^* \in \{\pm 1\}$ ("bang-bang" control)
② should change the sign of $\alpha^*(\cdot)$ at most once.
(since $-h_1 t + h_2$ changes sign at most once)
So our earlier idea <u>was</u> in fact optimal.

<u>Example 2</u>
controlled oscillator:
$\ddot{x}(t) = -x(t) + \alpha(t)$
① Rewrite as a system:
$\dot{x}_1^*(t) = x_2(t)$
$\dot{x}_2^*(t) = -x_1(t) + \alpha(t)$

$x$
$\shortparallel$
$x_1$ = displacement
$x_2$ = velocity

$\dot{x}(t) = Mx(t) + N\alpha(t)$
$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



<u>Recall</u>:
$$e^{tM} = \cos t\, I + \sin t\, M = \begin{pmatrix} \cos t & \sin t \\ \sin t & \cos t \end{pmatrix}$$

$$e^{-tM} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

so: $e^{-tM} N = \begin{pmatrix} \cos & -\sin \\ \sin & \cos \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$

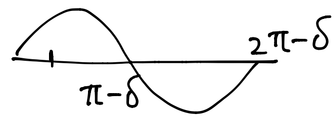
For $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, $h^T e^{-tM} N = -h_1 \sin t + h_2 \cos t$

Pontryagin: If $\alpha^*$ is optimal, $(-h_1 \sin t + h_2 \cos t)\alpha^*(t) = \max\limits_{|a| \le 1} \{(-h_1 \sin t + h_2 \cos t) a\}$

conclusion:
$\alpha^*(t) = \begin{cases} 1 & \text{if } -h_1 \sin t + h_2 \cos t > 0 \\ -1 & \text{if } \text{------} < 0 \end{cases}$
so "bang-bang" control

Next, since vector $h \neq 0$. we can arrange so that $h_1^2 + h_2^2 = 1$

so there is an angle $\delta$ s.t.
$-h_1 = \cos\delta$
$h_2 = \sin\delta$
Then $-h_1 \sin t + h_2 \cos t = \cos\delta \sin t + \sin\delta \cos t = \sin(t + \delta)$
Further conclusion . . . .
so $\alpha^*(t) = \begin{cases} +1 & \text{if } \sin(t+\delta) > 0 \\ -1 & \text{if } \sin(t+\delta) < 0 \end{cases}$
i.e. $\alpha^*$ changes sign every "$\pi$" units of time.



## Geometric analysis

If $\alpha = 1$. $\dot{x}_1 = x_2$

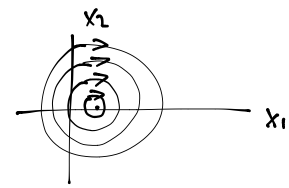
$\dot{x}_2 = -x_1 + 1$

Note that

$$\frac{d}{dt}\left((x_1(t)-1)^2 + x_2(t)^2\right) = 2(x_1(t)-1)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t) = 2[(x_1-1)x_2 + x_2(1-x_1)] = 0$$

So solutions stay on curves

$(x_1-1)^2 + x_2^2 = \text{const} =$ circles centered as $(1,0)$




In fact, once we know this, we can check sol'n has the form

$x_1(t) = r\cos(\theta - t) + 1$

$x_2(t) = r\sin(\theta - t)$

for some $r, \theta$

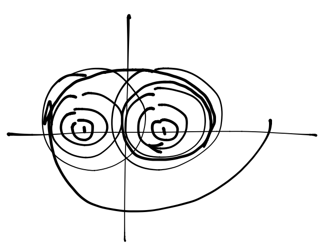
Similarly if $\alpha = -1$

sol'n stays on curves $(x_1+1)^2 + x_2^2 = \text{const} =$ circles centered at $(-1,0)$

And in fact

$x_1(t) = r\cos(\theta - t) - 1$

$x_2(t) = r\sin(\theta - t)$ for some $\theta, r$



$\pi$ units of time $= \frac{1}{2}$ revolution around a circle.

## General Pontryagin max Pinciple

Setup:

$\dot{x}(t) = f(x(t), \alpha(t))$, $f: \mathbb{R}^n \times A \to \mathbb{R}^n$

$x(t) \in \mathbb{R}^n$ for all $t$

$\alpha(t) \in A =$ "admissible set $\subset \mathbb{R}^m$"

$$\boxed{\text{Goal: maximizer } P[\alpha(\cdot)] = \int_0^T r(x(t), \alpha(t))\,dt + g(x(T))}$$

"pay off"

for some functions $r: \mathbb{R}^n \times A \to \mathbb{R}$

and $g: \mathbb{R}^n \to \mathbb{R}$

Theorem: Suppose $\alpha^*$ is an optimal control & $x^*$ is the associated trajectory

THEN Define $H(x,p,a) = f(x,a)p + r(x,a)$ = "control theory Hamiltonian"

there exists $p^* : [0,T] \to \mathbb{R}^n$ such that

(1). $\dot{x}^*(t) = f(x^*(t), \alpha^*(t)) = \nabla_p H(x^*, p^*, \alpha^*)$

(2). $\dot{p}^*(t) = -\nabla_x H(x^*, p^*, \alpha^*)$ ⟵ (adjoint eqn)

(3). $H(x^*(t), p^*(t), \alpha^*(t)) = \max_{a \in A} H(x^*(t), p^*(t), a)$ for all $t$

(4). $p^*(T) = -\nabla g(x^*(T))$

Example: $x(t)$ = output at time $t$

$\alpha(t)$ = fraction of output reinvested

$P[\alpha(\cdot)]$ = amount consumed to time $T = \int_0^T (1-\alpha(t))x(t)\,dt$

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ODE: $\dot{x}(t) = \alpha(t)x(t)$

Here $f(x,a) = xa$

$r(x,a) = x(1-a)$

$g(x) = 0$

$H(x,p,a) = pxa + x(1-a)$

So: eqns are (omitting $*$'s for simplicity)

(1) $\dot{x}(t) = \alpha(t)\,x(t)$

(2) $\dot{p}(t) = -[p(t)\alpha(t) + 1 - \alpha(t)]$

(3). $H(x(t), p(t), \alpha(t)) = \max_{a \in [0,1]} H(x(t), p(t), a)$

(4) $p^*(T) = 0$

Consider (3)

$$\max_{a\in[0,1]} H(x(t), p(t), a) = x + \max_{a \in [0,1]} \{a(p-1)x\}$$

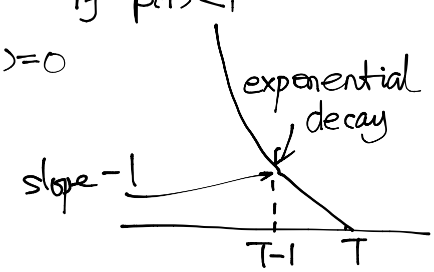
if $x \geq 0$, then best choice is

$$\alpha(t) = \begin{cases} 1 & \text{if } p(t) > 1 \\ 0 & \text{if } p(t) < 1 \end{cases}$$

Now consider (2):

$$\dot{p}(t) = \begin{cases} -p(t) & \text{if } p(t) > 1 \\ -1 & \text{if } p(t) < 1 \end{cases}$$

Recall (4): $p(T) = 0$




So $$p(t) = \begin{cases} (T-t) & \text{if } T-1 \leq t \leq T \\ e^{-(t-T+1)} & \text{if } t \leq T-1 \end{cases}$$

Conclusions: ① I know $p^*(t)$ and $d^*(t)$, can solve for $x^*(t)$ and $P[d^*(\cdot)]$
But if all I care about is optimal strategy, I already know it:

$$d^*(t) = \begin{cases} 1 & \text{if } t < T-1 \\ 0 & \text{if } T-1 < t < T \end{cases}$$