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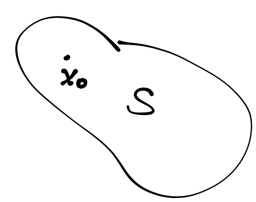
Def:
A point $\vec{a} \in S \subset \mathbb{R}^n$ is a global or absolute maximum of a function (minimum) of a function if $f(\vec{a}) \geq f(\vec{x})\ \forall \vec{x} \in \mathbb{R}^n$
$(f(\vec{a}) \leq f(\vec{x}))$ ← For local, $\forall \vec{x}$ in some open neighbourhood

Recall: If S is compact (i.e. closed & bounded) then f attains a max & min.
— may relax the bounded part.
- S is the closure of an open set.
i.e. we have both interior & boundary points
- want ∂S to be "nice" in some sense, i.e. smooth.

Procedure: 1) Find and classify all CPs
2). Study the boundary (3 methods, 2 learned already)
3) Compare values of f from 1) and 2).

Thm: Let f be continuous on a closed and unbounded set S.
$f(\vec{x}) \to \infty$ when $|\vec{x}| \to \infty$
then it has a minimum but not a maximum

Proof: Clearly no max.
$V = \{\vec{x} \mid f(\vec{x}) \leq f(\vec{x_0})\}$ for some $\vec{x_0} \in S$



Clearly V is not empty as $\vec{x_0} \in V$

Will prove compact.
closed: $(-\infty, f(\vec{x_0})]$ is a closed interval
$f^{-1}((-\infty, f(\vec{x_0})]) = \text{closed} \cap S = \text{closed} = V$
↖ also closed

bdd: $f(\vec{x}) \to \infty$ when $|\vec{x}| \to \infty$
$\forall M > 0, \exists N > 0$, such that $f(x) > M$ when $|x| > N$

as $\exists N$ s.t. for $|\vec{x}| > N$, $f(\vec{x}) > f(\vec{x_0}) \Rightarrow \vec{x} \notin V$ for such $\vec{x}$'s.
so $\vec{x} \in V \Rightarrow |\vec{x}| < N \Rightarrow V$ is bdd. As bdd and closed, V is compact.

As compact, EVT $\Rightarrow$ a minimum at $\vec{a}$ on V
$f(\vec{x}) > f(\vec{x_0}) \geq f(\vec{a})$
$\vec{x} \in S \setminus V \Rightarrow \vec{a}$ minimum on S
Likewise for $f(\vec{x}) \to -\infty$. (a maximum)

■

Thm: f is cont. on a closed & unbounded set S and $f(\vec{x}) \to 0$ as $|\vec{x}| \to \infty$
and $\exists x_0 \in S$ s.t. $f(x_0) > 0$. then we have an abs max.
Proof: $W = \{\vec{x} \in S \mid f(x) \geq f(x_0)\}$
W is closed as it is the inverse image of $[f(x_0), +\infty)$ intersected with S
i.e. closed $\cap$ closed $=$ closed

Have $\forall \varepsilon>0\ \exists N_\varepsilon$ s.t. $|f(\vec{x})|<\varepsilon$ when $|\vec{x}|>N$

So $\varepsilon=\frac{f(x_0)}{2}$ so similarly, W is bdd $\Rightarrow$ compact

$\Rightarrow$ by EVT a max $\vec{a}$ on W

for $\vec{x}\in S\setminus W$, $f(\vec{x})<f(\vec{x_0})\leq f(\vec{a})\Rightarrow \vec{a}$ is a max on S. ■

$S=\{\vec{x}\mid G(\vec{x})=0\}$

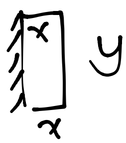
f diff, $G(\vec{x})$ to C^1 & $\nabla G(\vec{x})\neq 0$ on S $\Rightarrow$ can form tangent planes, so "smooth"

Goal is to find Extrema.

3 methods: 1) Solve $x_n=x_n(x_1,\cdots,x_{n-1})$ via $G(\vec{x})=0$

then reduced to $n-1$ variables to find critical points of _?

Ex: Fences problem



$0=2x+y-C \leftarrow G(\vec{x})=0$

$A(x,y)=xy \leftarrow f(x,y)$

$y=C-2x$, $A=xy=x(C-2x)=Cx-2x^2$

$A'=C-4x=0 \Rightarrow x=\frac{C}{4} \Rightarrow y=\frac{C}{2}$

2). Parametiz $n-1$ parameters

$x_j=x_j(t_1,\cdots,t_{n-1})$ again with $n-1$ variables

Ex: polar: $y=\cos t$, $y=\sin t$

Ex: $f(x,y)=2y^2+x^2+2y$ on $S=\{(x,y)\mid x^2+y^2\leq 1\}$ Method ①

① Find CPs on interior

$\partial_x f(x,y)=2x=0\Rightarrow x=0$

$\partial_y f=4y+2=0\Rightarrow y=-\frac{1}{2}$

don't care yet if it is max or min.

on $\partial S=\{(x,y)\mid x^2+y^2=1\}$

$x^2=1-y^2$, $-1\leq y\leq 1$

So $f(y)=2y^2+1-y^2+2y=y^2+2y+1=(y+1)^2$

for $-1\leq y\leq 1$, so max of 4 at $y=1$, $x=0$

min of 0 at $y=-1$, $x=0$

$f(0,-\frac{1}{2})=-\frac{1}{2}$

So global max is 4 at $(0,1)$ and global min is $-\frac{1}{2}$ at $(0,-\frac{1}{2})$

$x=\cos\theta$, $y=\sin\theta$, $f(x,y)=2\sin^2\theta+\cos^2\theta+2\sin\theta=(1+\sin\theta)^2$ Method ②

again $\theta=\frac{\pi}{2} \Rightarrow f(\frac{\pi}{2})=4$

$f(-\frac{\pi}{2})=0$

Method ① *

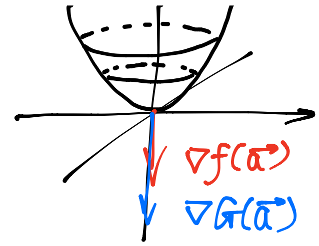
$f(y)=y^2+2y+1$ $f'(y)=2y+2=0\Rightarrow y=-1$

$y\in[-1,1] \leftarrow$ would need to evaluate boundary ...

Method ③

Lagrange's Method

Suppose a diff. f has an extremum at $\vec{a}$



$$S=\{\vec{x}\mid G(\vec{x})=0\}$$

Suppose $\vec{x}=\vec{h}(t)$ a path on our surface

s.t. $\vec{h}(0)=\vec{a}$, $\varphi(t)=f(\vec{h}(t))$ has a extremum at $t=0$

$0=\varphi'(0)=\nabla f(\vec{a})\vec{h}'(0) \Rightarrow \nabla f(\vec{a})$ is orthogonal to $\vec{h}'(0)$.

So true for $\forall$ such $\vec{h} \Rightarrow$ $\boxed{\nabla f(\vec{a}) \text{ is normal}}$ to the tangent plane of our surface.

Also $\boxed{\nabla G(\vec{a}) \text{ is normal}}$

$\Rightarrow$

$$\boxed{\nabla f(\vec{a})=\lambda\nabla G(\vec{a})} \qquad \text{recall } \nabla G(\vec{a})\neq 0$$

So I have n equations $\partial_j f(\vec{a})=\lambda\partial_j G(\vec{a})$
$G(\vec{a})=0$

So n+1 equations, linear in λ

Scalar: $(x_1,\cdots,x_n,\lambda) \longrightarrow$ not important

Ex: $2y^2+x^2+2y=f(x,y)$ on $x^2+y^2-1=G(x,y)=0$

① $2x=\lambda\cdot 2x \quad\Rightarrow x=0$ or $\lambda=1$

② $4y+2=\lambda\cdot 2y$

③ $x^2+y^2=1$

$x=0 \Rightarrow y=\pm1$; $\lambda=1 \Rightarrow 2y=-2 \Rightarrow y=-1 \Rightarrow x=0$

my extrema: $(0,\pm1)$

Generalize: $S=\{\vec{x}\mid G_1(\vec{x})=G_2(\vec{x})=\cdots=G_m(\vec{x})=0\}$

then $\nabla f(\vec{a})=\lambda_1\nabla G_1(\vec{x})+\cdots+\lambda_m\nabla G_m(\vec{x})=0$

So n+m equations $\partial_i f(\vec{a})=\lambda_1\partial_i G_1(\vec{a})+\cdots\lambda_m\partial_i G_m(\vec{a})$

$G_1(\vec{a})=0$
$\vdots$
$G_m(\vec{a})=0$

can be solved for n+m variables, $(x_1,\cdots,x_n,\lambda_1,\cdots,\lambda_n)$. yielding the points where local extrema can occur

Ex: Often want to minimize distance between point and a set.

$$d^2=f(x,y,z)=x^2+y^2+z^2$$

$G_1(x,y,z)=x+y+z-1=0$
$G_2(x,y,z)=x+y-z=0$ > define planes (intersects in a line)

① $2x=\lambda_1+\lambda_2$

② $2y=\lambda_1+\lambda_2 \quad \Leftarrow \partial_y f=\lambda_1\partial_y G_1+\lambda_2\partial_y G_2$

③ $2z=\lambda_1-\lambda_2$

④ $x+y+z=1 \qquad \Rightarrow ①-② \Rightarrow x=y$

⑤ $x+y-z=0 \qquad \Rightarrow ④+⑤=4x=1 \Rightarrow x=\frac{1}{4}=y \Rightarrow z=\frac{1}{2}$

$f(-\frac{1}{4}, -\frac{1}{4}, \frac{1}{2}) = \_\_\_$ a min, geometric consideration

§ 2.10 Vector-Valued Functions and their derivatives.

$f: \mathbb{R}^n \to \mathbb{R}^m$

Ex: paramatized points, $\vec{h}(t): \mathbb{R} \to \mathbb{R}^n$

Ex: Linear Transformations

$T(a\vec{x} + b\vec{y}) = aT(\vec{x}) + bT(\vec{y})$

A m×n matrix

$$\begin{array}{cc} & n \\ m & \begin{pmatrix} & & \end{pmatrix} \end{array} \begin{pmatrix} n \end{pmatrix} = \begin{pmatrix} m \end{pmatrix} \qquad T_j(\vec{x}) = \sum_{k=1}^{h} A_{jk} x_k$$

Had a correspondence between T and matrices.

$\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$ is diff if $\exists$ an m×n matrix L s.t.

$$\lim_{\vec{h} \to \vec{0}} \frac{|f(\vec{a} + \vec{h}) - f(\vec{a}) - L \cdot \vec{h}|}{|\vec{h}|} = 0$$

$L = D\vec{f}(\vec{a}) = \vec{f}'(\vec{a})$ "Frechet Derivative"

Equivalently, $\vec{f}(\vec{a} + \vec{h}) = f(\vec{a}) + L \cdot \vec{h} + \vec{E}(\vec{h})$ where $\displaystyle\lim_{\vec{h} \to \vec{0}} \frac{|\vec{E}(\vec{h})|}{|\vec{h}|} = 0$

* can prove Frechet Derivative is unique

* Let me say $L^j = j$th row of L.

diff $\iff \displaystyle\lim_{\vec{h} \to \vec{0}} \frac{|f_j(\vec{a} + \vec{h}) - f_j(\vec{a}) - L^j \cdot \vec{h}|}{|\vec{h}|} = 0$

$\forall j \in \{1, \cdots, m\} \iff f_j: \mathbb{R}^n \to \mathbb{R}$ is diff $\forall j \in \{1, \cdots, m\}$

$$\Rightarrow L^j = \nabla f_j(\vec{a}) \text{ so } Df(\vec{a}) = \begin{pmatrix} \partial_1 f_1(\vec{a}) & \cdots & \partial_n f_1(\vec{a}) \\ \vdots & & \vdots \\ \partial_1 f_m(\vec{a}) & \cdots & \partial_n f_m(\vec{a}) \end{pmatrix}$$

Chain Rule III (No Proof): Suppose $\vec{g}: \mathbb{R}^k \to \mathbb{R}^n$ diff at $\vec{a} \in \mathbb{R}^k$

$\vec{f}: \mathbb{R}^n \to \mathbb{R}^m$, diff at $\vec{g}(\vec{a}) \in \mathbb{R}^n$.

Then $\vec{H}(\vec{a}) = \vec{f} \circ \vec{g}(\vec{a})$ then

$D\vec{H}(\vec{a}) = D\vec{f}(\vec{g}(\vec{a})) \cdot D\vec{g}(\vec{a})$

matrix mult.

Ex: