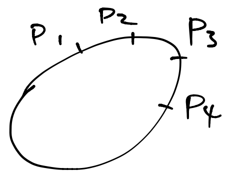
Lecture 9

Conic section

ellipse
hyperbola
parabola

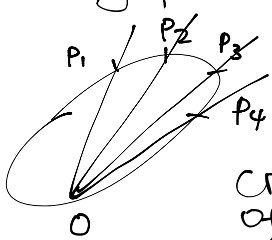
} ⟹ in projective geometry they are all circle

Given 4 pts on a conic



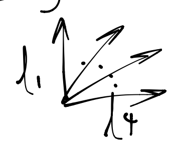
Conic (3 different def's)

① Take any pt on conic



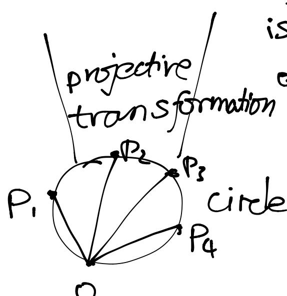
4 lines

Def: cross ratio



$$\frac{\det(l_1, l_3)}{\det(l_1, l_4)} : \frac{\det(l_2, l_3)}{\det(l_2, l_4)}$$

cross ratio of pts here is independent of choice of "O".



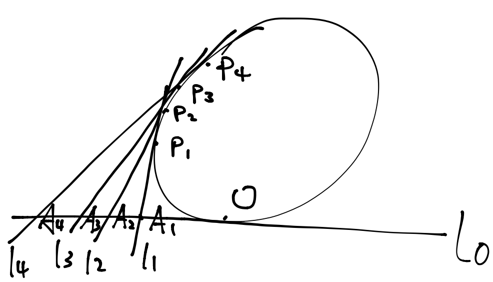
cross-ratio can be transformed to an expression only about angle

So in this case, Cross-ratio is an expression of arcs.

arcs ⟺ angles ⟺ independent choice of O.

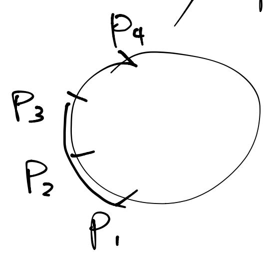
② Given 4 pts, 4 tangent lines,

a point O,
draw tangent at O.
intersecting $l_1, \cdots, l_4$
at $A_1\ A_2\ A_3\ A_4$



Cross ratio $[A_1, A_2, A_3, A_4]$ independent of choice of O (and line l_0)

similarly for

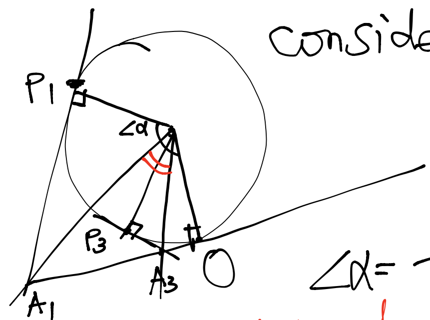


actually be computed this way

$$\frac{\sin\left(\frac{\overset{\frown}{P_1P_3}}{2}\right)}{\sin\left(\frac{\overset{\frown}{P_1P_4}}{2}\right)} : \frac{\sin\left(\frac{\overset{\frown}{P_2P_3}}{2}\right)}{\sin\left(\frac{\overset{\frown}{P_2P_4}}{2}\right)}$$

} arcs e.g. $\overset{\frown}{P_1P_2}$

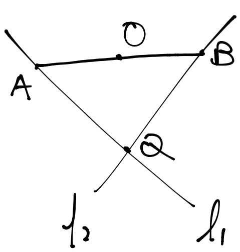
consider P_1 & O only



$$\angle\alpha = \frac{\overset{\frown}{P_1O}}{2}$$

so what angle between A_1 & A_3? that's $\frac{\overset{\frown}{P_1O}}{2} - \frac{\overset{\frown}{P_3O}}{2} = \frac{\overset{\frown}{P_1P_3}}{2}$

$l_1, l_2 \subset \mathbb{P}\mathbb{P}^2$ (projective plane)



$f: l_1 \to l_2$

Proof: f is a projective map $\Longleftrightarrow \forall P_1, P_2, P_3, P_4 \in l_1$, "Preserves Cross-ratio"

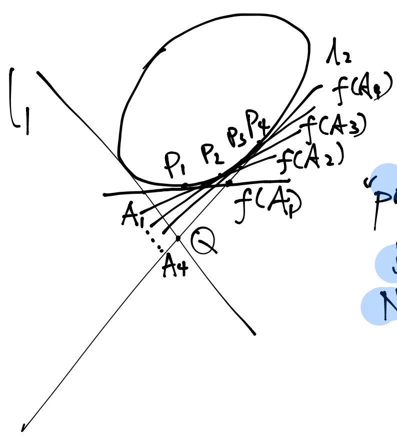
$[P_1, P_2, P_3, P_4] = [f(P_1), f(P_2), f(P_3), f(P_4)]$

$Q = l_1 \cap l_2$

$f(Q) = Q$

Choose any pt O on the plane. (as center of projection)

"takes Q back to itself"



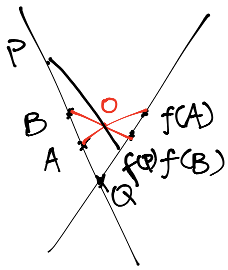
$f(A_i)$ is the image of A_i on l_2. P_i is the tangent point on conic

"preserves cross-ratio"

↳ c-r of A_i's, P_i's & $f(A_i)$'s

but

NOT Takes back Q!



$f: l_1 \to l_2$

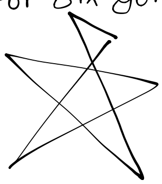
$f(Q) = Q$

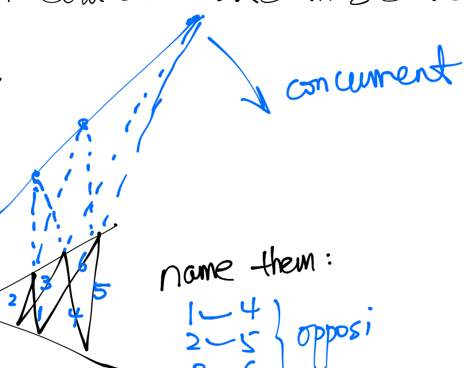
There's only 1 such map

O: unique, preserves c-r.

Before Pascal, Pappus proved a similar theorem.

For six-gon: could be like this (not necessarily be convex)

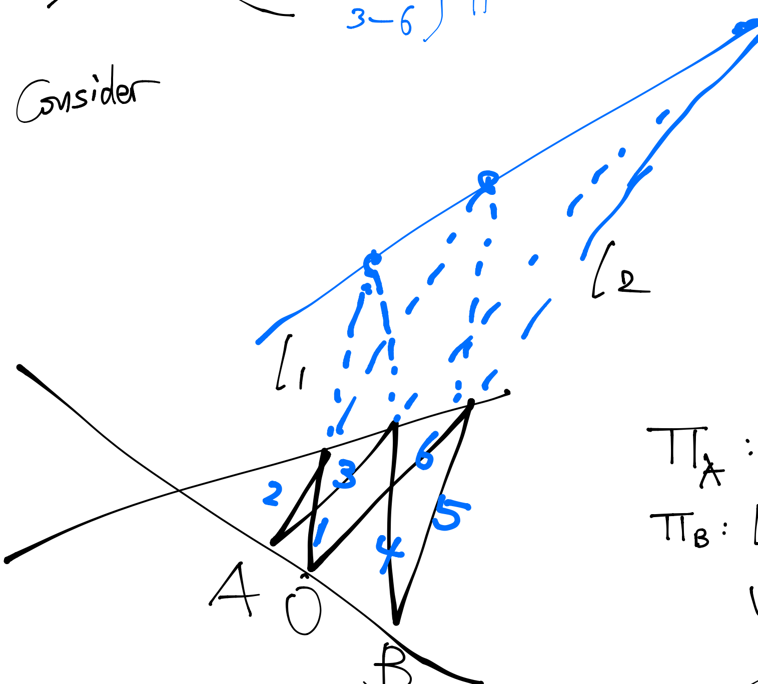


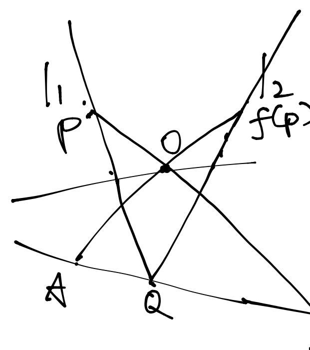


name them:

1 — 4
2 — 5
3 — 6 } opposi

Consider

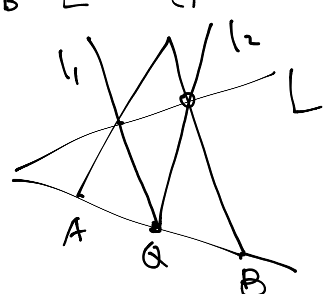




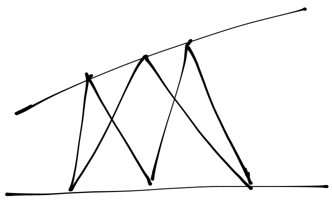
2 projections

$\Pi_A : L \to l_2$

$\Pi_B : L \to l_1$

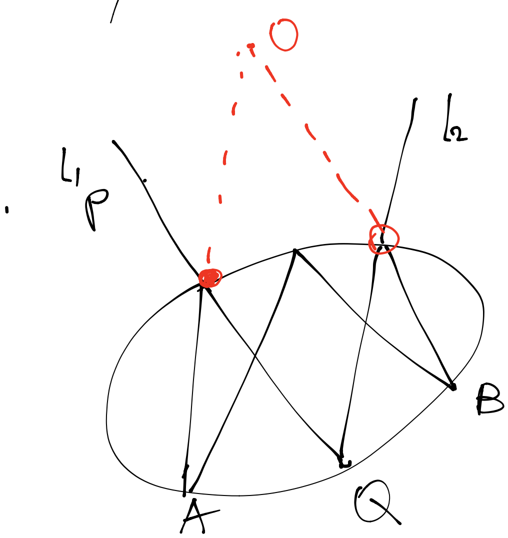


See text for concrete proof

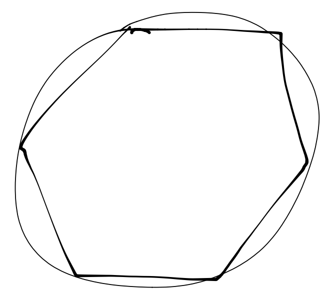


Pascal Thm:

any conic, 6-gon



same as



doesn't matter

project l_1 onto conic, about B, is still on conic, then project to l_2

"maps Q to itself" (2 steps)

"preserves cross-ratio"

actually, l_1 projects to l_2 about conic is just like projection about conic.

$P(x,y,z)=ax^2+bxy+cy^2+dx+ey+f=0$

$ax^2+bxy+cy^2+dxz+eyz+fz^2=0$

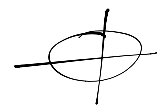
$P(\lambda x,\lambda y,\lambda z)=\lambda^2 P(x,y,z)=0$

$u \quad v \quad w$

$Au^2 + Bv^2 + Cw^2$

A,B,C

1,-1,0

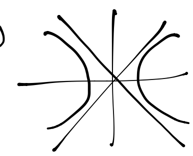


$u^2+v^2+w^2=0 \quad \nearrow \emptyset$

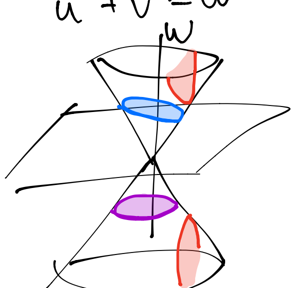
$u^2+v^2-w^2=0$ plug in $w^2=1 \Rightarrow u_1^2+v_1^2=1$ $\quad u_1=\frac{u}{a}$, $v_1=\frac{v}{b}$ $\quad u_1^2+v_1^2=\frac{u^2}{a^2}+\frac{u^2}{b^2}=1$

$u^2-v^2-w^2=0$

$-u^2-v^2-w^2=0$ $\quad v^2=1 \Rightarrow \frac{w^2}{a^2}-\frac{u^2}{b^2}=1 \quad (w^2>u^2)$

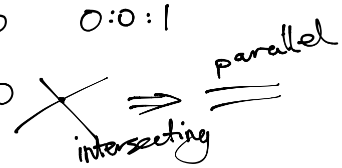


$u^2+v^2=w^2$



$u^2+v^2=0 \quad u=0 \; v=0 \quad 0:0:1$

$u^2-v^2=0 \quad (u+v)(u-v)=0$



$-u^2-v^2=0$

THe only conic passing through 5 pts

$ax^2+bxy+cy^2+dx+ey+f=0$

(2,1)

$4a+2b+c+2d+e+f=0$