May 23rd
Test in class next Thursday.

Def: A subset $S \subset \mathbb{R}^n$ is complete if every Cauchy sequence converges to a limit in S.
Aside general: metric space i.e. has a notion of distance.

Aside One can construct $\mathbb{R}$ by looking at Cauchy sequences of rational numbers.

—What subsets of $\mathbb{R}^n$ are complete?

Thm: S is complete iff S is closed.

Proof: ($\Leftarrow$) If $\{x_k\}$ cauchy in S it converges to $\vec{x} \in \mathbb{R}^n$. via completeness.
$\Rightarrow \vec{x} \in \bar{S} = S$ as S is closed
i.e. $\vec{x} \in S$ so complete.

($\Rightarrow$) So take $\{x_k\}$ a sequence converging to $\vec{x}$. So $\vec{x} \in S$, so closed.

Def: A set S is compact if it is closed and bounded.

Ex: $[0,1]$

Thm: Bolzano-Weierstrauss Version 3 for $\mathbb{R}^n$, $S \subset \mathbb{R}^n$ T.F.A.E.
a) S is compact
b) Every $\{x_k\}$ in S has convergent subsequence with limit in S.

Proof: a) $\Rightarrow$ (b)
S compact $\Rightarrow \{x_k\}$ is bounded $\xRightarrow{\text{B.W.C. } \mathbb{R}^n} \exists \{x_{k_j}\}$ s.t. $x_{k_j} \to x$.

However S compact $\Rightarrow S$ closed $\Rightarrow$ contains all limit points.

So $x \in S$

(b $\Rightarrow$ a) contrapositive i.e. assume S is not compact. if S is not bounded.

if S is not bounded then construct sequence $\{x_k\}$ diverging to ∞ by choosing x_k so $|x_k| > k$.

So all $\{x_{k_j}\}$ also necessarily diverge to ∞
— if S is not closed. $\exists x \in \bar{S} \setminus S$
So $\exists \{x_k\} \to x$, $x_k \in S$, $x \notin S$.

any subsequence all has to converge to $x \notin S$.

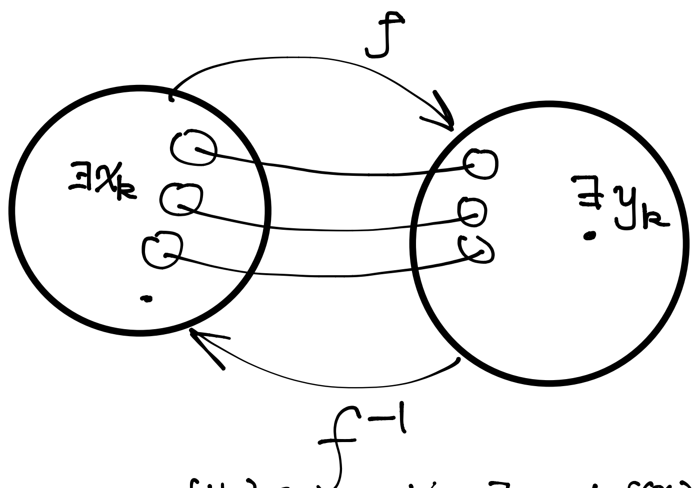
∎

Ex: Finite Sets. a). clearly closed & bounded.
(b). Every sequence hits at least on $x \in F$, ∞ many times. Take subseq just those hitting x.
$\{x_{k_j}\}$ $x_{k_j} \to x \in F$

Thm: If $S \subseteq \mathbb{R}^n$ is compact, f is continuous, then $f(S) := \{f(x) \mid x \in S\}$ ← image is complete.

Note: $f^{-1}(\text{compact}) \neq \text{compact}$, had $f^{-1}(\text{closed}) = \text{closed}$. but $f^{-1}(\text{bounded}) \neq \text{bounded}$
ex: $f(x) = 1, x \in \mathbb{R}$.

Proof:



Let S compact. Take $\{y_k\} \in f(S)$. $\forall y_k$ $\exists x_k$ st. $f(x_k) = f(y_k)$. But S is compact,

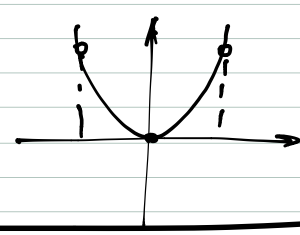
$\{x_k\}$ has a convergent subsequence $\{x_{k_j}\} \to x \in S$.

"push forward" to get $f(x_{k_j})$ a subseq. of $\{y_k\}$ in $f(S)$ then by continuity, x_{k_j}
$\Rightarrow x \Rightarrow f(x_{k_j}) \to f(x) \in f(S) \Rightarrow f(S)$ compact. ∎

Thm / Extremum Value Theorem
For $f: S \subset \mathbb{R}^n \to \mathbb{R}$. S compact, f is continuous. $\exists a, b \in S$ so $f(a) \leq f(x) \leq f(b)$. $\forall x \in S$ so $f(S)$ is a compact set $\Rightarrow$ bounded so $\Rightarrow \exists \text{lub} f(S)$, $\exists \text{glb}(f(S))$
as $f(S)$ is closed $\Rightarrow \text{lub}(f(S)) \in f(S)$
$\text{glb}(f(S)) \in f(S)$

∎

Ex: x^2 on $(-r, r)$
attains its min but not its max



Thm (Nested Compact Set)
$\{S_k\}$ ← sequence of compact sets that are nested $S_1 \supset S_2 \supset S_3 \supset \dots$
Then $\exists x \in S_k \ \forall k$.

Proof: Create sequence $\{x_k\}$ s.t. $x_k \in S_k$, $\{x_k\}$ is bounded as it's in S.
$\Rightarrow$ (By BW) $\exists$ convergent subsequence $\{x_{k_j}\} \to x$ so $\{x_j\} \in S_k$ $\forall$ sufficient-ly large j. $\Rightarrow x \in S_k \ \forall k$ $\blacksquare$

Def: An open cover of S is a collection of open sets U_i, s.t. $\bigcup_i U_i \supset S$.

Thm / Heine-Borel : TFAE (a $\Leftrightarrow$ b)

a). compact

b). Every open cover has a finite subcover.

Aside:
*Don't need to understand the proof.

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§ 1.7 Connectedness

$$X = A \cup B,\ A \cap B = \emptyset,\ A \neq \emptyset \ \& \ B \neq \emptyset$$

Note $[_a \quad)[_c \quad]_b$ this is CONNECTED
(need to exclude this possibility)

Def: S is disconnected if $\exists (S_1, S_2) \leftarrow$ disconnection s.t. $S_1 \neq \emptyset \neq S_2$
$\boxed{S_1 \cap \overline{S_2} = \emptyset,\ \overline{S_1} \cap S_2 = \emptyset}$

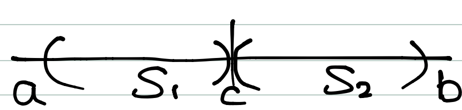
Def: S is connected if not disconnected.

Note: $\overline{S_1} \cap \overline{S_2} \neq \emptyset$
maybe ?

But $(0,0) \in \overline{B(1,(-1,0))} \cap \overline{B(1,(1,0))}$
But $B(1,(-1,0)) \cup B(1,(1,0))$ is still disconnected

Def: $I \subset \mathbb{R}$ is an interval if $\forall a, b \in I$ s.t. $a < c < b$ then $c \in I$

Theorem : $S \subset \mathbb{R}$, S is connected $\Leftrightarrow$ S is an interval.
Proof : ($\Rightarrow$) Contrapositive i.e. if S is not an interval, then S is not connected.



As S is not an interval, $\exists c \notin S$ s.t. $a<c<b$ for $a,b\in S$.
So $S_1=(-\infty,c)\cap S$
$S_2=(c,+\infty)\cap S$
Clearly $S_1\cup S_2=S$
$S_1,S_2\neq\emptyset$ (since, we have a,b in S_1, S_2)
$(-\infty,c]\cap(c,+\infty)=\emptyset \Rightarrow \overline{S_1}\cap S_2=\emptyset$ Likewise $S_1\cap\overline{S_2}=\emptyset$
$\Rightarrow (S_1,S_2)$ is a disconnection.

$(\Leftarrow)$ By Contradiction Assume it is an interval & S is disconnected.

First do for compact interval: Let (S_1,S_2) be a disconnection.
WLOG, $a\in S_1$, $b\in S_2$, take $\sup S_1 = c$ $\leftarrow$ exists as compact

$\Rightarrow c\in\overline{S_1} \Rightarrow c\notin S_2$ (as $\overline{S_1}\cap S_2=\emptyset$) $\Rightarrow c\in S_1$

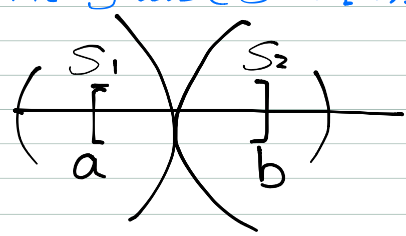
for $(c,b]$, $x\in(c,b]$, $x>y$ $\forall y\in S_1 \Rightarrow (c,b]\subset S_2 \Rightarrow c\in\overline{S_2} \Rightarrow c\notin S_1$

Contradiction.

SPECIAL

Just showed that a compact interval is connected.

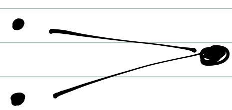
Now, let S be an arbitary interval, Assume S is disconnected with disconnection (S_1,S_2). Pick any $a,b\in S \Rightarrow [a,b]\subset S$



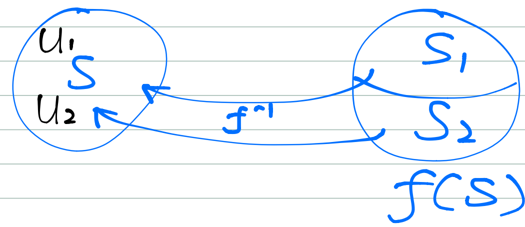
WLOG, $a\in S_1$, $b\in S_2$
$T_1=[a,b]\cap S_1$
$T_2=[a,b]\cap S_2$
$a\in T_1$, $b\in T_2$, so $T_1\neq\emptyset$, $T_2\neq\emptyset$
$T_1\subset S_1$, $T_2\subset S_2$
$\overline{T_1}\cap T_2=\emptyset$ as $\overline{S_1}\cap S_2=\emptyset$
$T_1\cap\overline{T_2}=\emptyset$ as $S_1\cap\overline{S_2}=\emptyset$
$\Rightarrow (T_1,T_2)$ is a disconnection of $[a,b]$. Contradiction as had just shown that compact intervals are connected. $\Rightarrow$ intervals are connected.

GENERAL $\blacksquare$

Theorem: if f is continuous, if S is connected then $f(S)$ is connected.

Note:  so $f^{-1}(\text{connected})\neq$ connected

Proof : By contradiction.
Assume $f(S)$ is disconnected $\Rightarrow \exists (U_1, U_2)$, a disconnection of $f(S)$.



$S_1 = f^{-1}(U_1)$ as $U_1, U_2 \neq \emptyset \Rightarrow S_1, S_2 \neq \emptyset$
$S_2 = f^{-1}(U_2)$

$S = S_1 \cup S_2 \longleftarrow$ " $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$ " previously

*
if $x \in S_1$ and $x \in \overline{S_2}$, i.e. $x \in S_1 \cap \overline{S_2} \neq \emptyset$
$x \in \overline{S_2} \Rightarrow x$ is the limit of a sequence in $S_2 \Rightarrow f(x) \in U_1$
$f(x_k) \in U_2$

by continuity, $f(x)$ is in $\overline{U_2} \Rightarrow U_1 \cap \overline{U_2} \neq \emptyset \Rightarrow$ contradiction

Likewise for $x \in \overline{S_1}$ and $x \in S_2$ . . . (or say WLOG at *)

$\blacksquare$

Corollary / Intermediate Value Theorem
$f: S \subset \mathbb{R}^n \to \mathbb{R}$, f is continuous, S is connected, then $\forall a, b \in S$, WLOG, $f(a) \leq f(b)$
then $\forall t$, $f(a) \leq t \leq f(b)$, $\exists c \in S$, $f(c) = t$.

Proof: $f(S)$ is connected as S is connected & f is continuous $\Rightarrow f(S)$ is an interval $\Rightarrow f(a) \leq t \leq f(b)$ then $t \in f(S)$ as $f(S)$ an interval $\Rightarrow \exists c$ s.t. $f(c) = t$.

$\blacksquare$