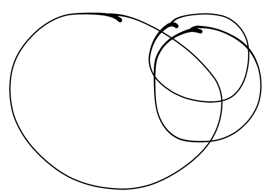
Lecture 11

Borel-Lebesgue Theorem (4) ⟹ (1)

If X is complete and totally bounded, then X is compact.

Proof: Suppose there is a cover $\mathcal{U}$ by open sets that does not have a finite subcover.

$\varepsilon = \frac{1}{k}$ X_k (the set of radii of metric balls of rad $\frac{1}{k}$ that cover X) finite for all k.



$u \in \mathcal{U}$

?? picture

For each k, $\underset{\text{finite}}{X_k} \subset X$ s.t. $\bigcup_{x \in X_k} B_{\frac{1}{k}}(u)$ covers X

Recursively, define $y_k \in X_k$ s.t. $\bigcap_{j=1}^{k} \overline{B_{\frac{1}{j}}(y_j)}$ does not have a finite subcover from $\mathcal{U}$.

$k=1$, $\exists\ \overline{B_1(x)}$ that cannot be covered by finitely many elements of $\mathcal{U}$ (If every ball $B_1(x_i)$ where $x_i \in X_1$ can be covered by finitely many elements

$U_1^i, \cdots, U_{k_i}^i$

Then take union of all of these sets over X_1, it will be a finite subcover of X)

Suppose we have defined $y_1, \cdots, y_{k-1}$

Let $S = \bigcap_{j=1}^{k-1} \overline{B_{\frac{1}{j}}(y_j)}$

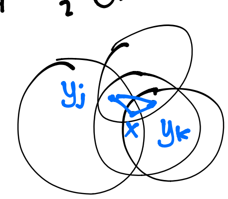
Let $y_k \in X_t$ be st. $S \cap B_{\frac{1}{k}}(x)$ has no finite subcover

(y_k)

Claim y_k is a Cauchy sequence.

Suppose $k, j \in \mathbb{N}$ WLOG assume $k \geq j$, $\bigcap_{i=1}^{k} \overline{B_{\frac{1}{i}}(y_i)} \neq \emptyset$ (otherwise there would be a finite subcover)

Take $x \in \overline{B_{\frac{1}{j}}(y_j)} \cap \overline{B_{\frac{1}{k}}(y_k)}$



$S(y_j, y_k) \leq S(y_j, x) + S(y_k, x) \leq \frac{1}{j} + \frac{1}{k}$

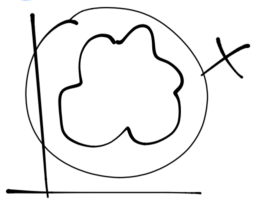
Take $N > \frac{2}{\varepsilon}$ $\qquad s, k \geq N \Rightarrow S(y_i, y_k) < \varepsilon$.

X is complete $\Rightarrow y_k$ converges to $y \in X$.
$\exists$ open set $U \in \mathcal{U}$ s.t. $y \in \mathcal{U}$
Since $\mathcal{U}$ is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(y) \subset \mathcal{U}$
Choose $k > 2/\varepsilon$
$S(y_k, y) < \frac{\varepsilon}{2}$, $\frac{1}{k} < \frac{\varepsilon}{2}$

$$\cap_{i=1}^{k} \overline{B_{\frac{1}{i}}(y_i)} \subset \overline{B_{\frac{1}{k}}(y_k)} \subset B_\varepsilon(y) \subset \mathcal{U}$$

contradiction. ■

The Lebesgue number lemma



If the metric space (X, d) is compact and an open cover of X is given, then there exists a number $\delta > 0$ such that every subset of X having diameter less than δ is contained in some member of the cover.

Proof: Let $\{A_1, \cdots, A_n\}$ be a finite subcover of X by elements of A for each A_i, consider $C_i = X - A_i$

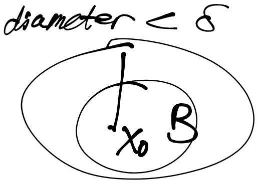
Consider an average $f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i)$

Claim: $f(x) > 0$, $\forall x \in X$

Reason: for each x $\exists A_i$ s.t. $x \in A_i$ (is open), choose $\varepsilon > 0$ s.t. $B_\varepsilon(x) \subset A_i$, $d(x, C_i) \geq \varepsilon$

By EVT, f has a minimum value δ (This δ is our Lebsgue number)

Let B have diameter $< \delta$



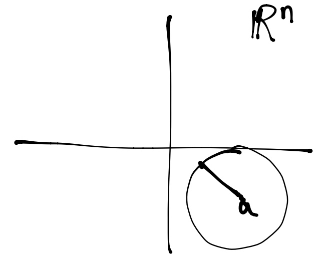
Take $x_0 \in B$, Consider $B_\delta(x_0) \Rightarrow B \subset B_\delta(x_0)$
Proof: $\delta \leq f(x_0) \leq d(x_0, C_m)$ where $d(x_0, C_m)$ is the largest of the numbe-s $d(x_0, C_i) \Rightarrow B_\delta(x_0) \subset A_m \Rightarrow B \subset A_m$

■

$S \subset \mathbb{R}^n$ and let $f: S \to \mathbb{R}^m$. If a is a limit point of $S \setminus \{a\}$, then $v \in \mathbb{R}^m$ is the limit of f at a if for any $\varepsilon > 0$ $\exists$ $v > 0$ s.t. $\| f(x) - v \| < \varepsilon$ whenever $0 < \| x - a \| < \delta$

§5.1
Limit and continuity of functions

Def: Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}^m$
f is continuous at $a \in S$ if $\forall \varepsilon > 0$, $\exists r > 0$, s.t. $\forall x \in S$,
$\|x-a\| < r \Rightarrow \|f(x) - f(a)\| < \varepsilon$
f is continuous on S if it is continuous $\forall a \in S$.



Ex: $f(x) = \dfrac{1}{\|x\|}$ (cont. except the origin)

Claim: $f(x)$ is continuous at a $\forall a \in \mathbb{R}^n - \{0\}$

Proof: $\|f(x) - f(a)\| = \left| \dfrac{1}{\|x\|} - \dfrac{1}{\|a\|} \right| = \left| \dfrac{\|a\| - \|x\|}{\|x\|\|a\|} \right|$

$$\|x\| = \|x - a + a\| \leq \|x - a\| + \|a\|$$

$$\|a\| = \|a - x + x\| \leq \|x - a\| + \|x\|$$

$$\begin{cases} \|x\| - \|a\| \leq \|x - a\| \\ \|a\| - \|x\| \leq \|x - a\| \\ \big| \|x\| - \|a\| \big| \leq \|x - a\| \end{cases}$$

If $\|a - x\| < \dfrac{\|a\|}{2}$

Then $\|x\| \geq \|a\| - \|x - a\| > \dfrac{\|a\|}{2}$

$$|f(x) - f(a)| = \frac{\big| \|a\| - \|x\| \big|}{\|x\| \, \|a\|} \leq \frac{\|x - a\|}{\|x\| \, \|a\|} < \frac{2\|x - a\|}{\|a\|^2}$$ would like this to be $< \varepsilon$

Let $r = \min \left\{ \dfrac{\varepsilon \|a\|^2}{2}, \dfrac{\|a\|}{2} \right\}$