March 19th

Recall: $N: V \to V$ is nilpotent if there is a $l$ s.t. $N^l = 0$, or equivalently the only ei-genvalue of $N$ is $0$.

$x \neq 0, N^k x = 0, N^{k-1} x \neq 0$

cycle gen. by $x$ is $\{N^{k-1}(x), \cdots, Nx, x\}$ has length $k$. And the cyclic subspace $C(x) = \text{span}\{N^{k-1}(x), \cdots, x\}$

Properties: 1. $N^{k-1}(x)$ is an eigenvalue

2. $\{N^{k-1}(x), \cdots, x\}$ lin. ind. so $\dim C(x) = k$

3. $C(x)$ is invariant under $N$.

4. $[N|_{C(x)}]_\alpha = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix}$

$N: V \to V$, $C(x) \subseteq V$

$N|_{C(x)} : C(x) \to C(x)$

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Suppose we have $\dim V = 8$

& we have 3 cycles $\{N^2 x_1, Nx_1, x_1\}$

$\{N^2 x_2, Nx_2, x_2\}$

$\{Nx_3, x_3\}$

and $\{N^2 x_1, N^2 x_2, Nx_3\}$ are lin. ind.

Claim: $\alpha$ is a basis of $V$.

Where $\alpha = \alpha_1 \cup \alpha_2 \cup \alpha_3$.

Why? $0 = a_1 N^2 x_1 + a_2 N x_1 + a_3 x_1 + b_1 N^2 x_2 + b_2 N x_2 + b_3 x_2 + c_1 N x_3 + c_2 x_3$

$$\begin{array}{c|c|c|c|} 3 & a_1 & a_2 & a_3 \\ 3 & b_1 & b_2 & b_3 \\ 2 & c_1 & c_2 & \end{array} = 0$$

Applying $N^2$ to both sides we obtain

$0 = a_3 N^2 x_1 + b_3 N^2 x_2 \longrightarrow \begin{array}{|c|c|c|} a_3 & 0 & 0 \\ b_3 & 0 & 0 \\ 0 & 0 & \end{array} = 0$

$\Rightarrow a_3 = b_3 = 0 \Rightarrow \begin{array}{|c|c|c|} a_1 & a_2 & 0 \\ b_1 & b_2 & 0 \\ c_1 & c_2 & \end{array} = 0$

Applying $N$ to both sides $\begin{array}{|c|c|c|} a_2 & 0 & 0 \\ b_2 & 0 & 0 \\ c_2 & 0 & \end{array} = 0 \Rightarrow a_2 = b_2 = c_2 = 0$

$\Rightarrow \begin{bmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & 0 & \end{bmatrix} = 0 \Rightarrow a_1 = b_1 = c_1 = 0 \Rightarrow \alpha$ is a basis

What is $[N]_\alpha$ ?

$$[N]_\alpha = \begin{bmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$$

**Thm:** $N: V \longrightarrow V$ nilpotent

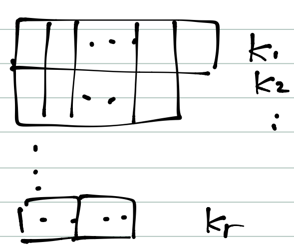
$\alpha_i = \{N^{k_i-1}x_i, N^{k_i-2}x_i, \dots, x_i\}$ is a cycle of length $k_i$, where $1 \leq i \leq r$

and $\{N^{k_1-1}x, \dots, N^{k_r-1}x_r\}$ lin ind.
Then $\alpha = \alpha_1 \cup \cdots \cup \alpha_r$ is independent.

Cycles $\alpha_1, \dots \alpha_r$ satisfying condition of the thm are called non-overlapping and if $\alpha = \alpha_1 \cup \cdots \cup \alpha_r$ is a basis of $V$ then it's called a canonical basis for $N$

**Big Thm:** $N: V \longrightarrow V$ nilpotent, then there exists a canonical basis of $V$ for $N$.

Given $N: V \to V$ nilpotent
Let $\alpha_1, \dots, \alpha_r$ be non-overlapping cycles s.t. $\alpha = \alpha_1 \cup \cdots \cup \alpha_r$ is a basis of $V$, i.e $\alpha$ is a canonical basis for $N$. Let length of $\alpha_i = k_i$, arranged so that $k_1 \geq k_2 \geq \cdots \geq k_r$
The tableau associated to $N$ is




Big Thm Cont'd
The tableau associated to $N$ is unique.
The nilpotent Jordan matrix of size $n$ is $n \times n$ matrix $J_n$ with zero's everywhere except 1's above the diagonal $J_3 = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 6 & 7 \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & 13 \end{bmatrix}$ then $A \oplus B = \left[\begin{array}{c|c} \begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix} & 0 \\ \hline 0 & \begin{matrix} 5 & 6 & 7 \\ \vdots & \vdots & \vdots \\ \cdot & \cdot & 13 \end{matrix} \end{array}\right]$

Thm: If $N: V \to V$ has tableau then with respect to any cannonical basis $\alpha$ of $V$ for $N$

$$[N]_\alpha = J_{k_1} \oplus \cdots \oplus J_{k_r}$$

This is called the Jordan canonical form of $N$.

Eg. If $\dim V = 6$ &

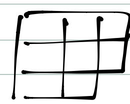
?

HALF-CLASS

Questions?

① How do we prove Big Thm?
② How do we find JCF of $N$?
③ How — — ··a canonical basis of $V$ for $N$?

Constructing tableau of $N$:

Sps: $N: \mathbb{C}^8 \to \mathbb{C}^8$ has tableau 

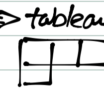
$\dim \ker N = 3$
$\dim \ker N^2 = 6$
$\dim \ker N^3 = 8$
$\dim \ker N^4 = 8$

Upshot: $\dim \ker N =$ # boxes in first column
·· $N^2 =$ ·· # ·:·· first two columns

Ex: $N: \mathbb{C}^4 \to \mathbb{C}^4$

$$N = \begin{bmatrix} 4 & 1 & -1 & 2 \\ -4 & -1 & 2 & -1 \\ 4 & 1 & -1 & 2 \\ -4 & -1 & 1 & -2 \end{bmatrix} \qquad N^2 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

$N^3 = 0$

$$N \sim \begin{bmatrix} 4 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ $\Rightarrow \dim \ker N = 2$, $\dim \ker N^2 = 3$
$\dim \ker N^3 = 4 \Rightarrow$ tableau of $N$ is 

So then JCF of $N$ is $J_3 \oplus J_1$ (btw $J_1 = 0$, $J_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $J_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ ...)

$$J_3 \oplus J_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Want: canonical basis of $\mathbb{C}^4$ for $N$.

i.e. we want non-overlapping cycles

$\alpha_1 = \{N^2 x_1, N x_1, x_1\}$

$\alpha_2 = \{x_2\}$

Let $y = N^2 x_1$. Note: $y \in \ker N \cap \operatorname{im} N^2$

$\operatorname{im} N^2 = sp\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right\}$ choose $y = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$

$N^2 x_1 = y$. choose $x_1 = e_3$

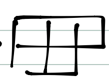
$$\alpha_1 = \left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Want to find $x_2$
$$\begin{cases} x_2 \in \ker N \\ \left\{x_2, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}\right\} \text{ ind} \end{cases}$$

$x_2 = \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \end{bmatrix}$ works. so $\left\{\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 0 \\ 0 \end{bmatrix}\right\}$ a canonical basis

$N : \mathbb{C}^5 \to \mathbb{C}^5$

$$N = \begin{bmatrix} 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad N^2 = \begin{bmatrix} & & & & 3 \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}, \quad N^3 = 0$$

$\dim \ker N = 2$
$\dim \ker N^2 = 4$
$\dim \ker N^3 = 5$
$\Rightarrow$  $\Rightarrow$

JCF of $N$ is
$$\begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & \multicolumn{2}{c}{0} \\ 0 & 0 & 0 & & \\ & 0 & & 0 & 1 \\ & & & 0 & 0 \end{bmatrix}$$

want $\alpha_1 = \{N^2 x_1, N x_1, x_1\}$ non-overlapping cycles

$\alpha_2 = \{N x_2, x_2\}$

$y = N^2 x_1$, $y \in \ker N \cap \operatorname{im} N^2$ $\quad \operatorname{im} N^2 = sp\{e_1\}$

Choose $y = e_1$ Solve for $x_1 : N^2 x_1 = e_1$

$x_1 = \frac{1}{3} e_5$. so $N x_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha_1 = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}\right\}$

want: $\{Nx_2, x_2\} = \alpha_2$

$$z = Nx_2 \begin{cases} z \in \ker N \cap \operatorname{im} N \\ \{z, e_1\} \text{ ind.} \end{cases}$$

$\operatorname{im} N = sp\{e_1, e_2, e_3\}$

$\ker N = sp\{e_1, e_2\}$

$\ker N \cap \operatorname{im} N = sp\{e_1, e_2\}$

choose $z = e_2$

Solve $Nx_2 = e_2$

$x_2 = \frac{1}{2}e_4 - 2e_3 \Rightarrow x_2 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1/2 \\ 0 \end{bmatrix} \right\}$

A canonical basis of $V$ for $N$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1/2 \\ 0 \end{bmatrix} \right\}$$