# Tutorial 8

*Rui Qiu*

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### Q4.1

Use the Neyman-Pearson lemma to find the form of the critical region for the best test of $H_0$ against $H_1$ when

#### (c)

$X_1, X_2, \ldots, X_n$ are a random sample from the exponential distribution with p.d.f. $f(x;\theta) = \theta e^{-\theta x},\ x > 0$ and $H_0 : \theta = \theta_0, H_1 : \theta = \theta_1,\ \theta_1 > \theta_0$.

**Solution:**

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} \theta \cdot e^{-\theta x_i} \\
&= \theta^n \cdot e^{-\theta \sum_{i=1}^{n} x_i} \\
\lambda &= \frac{L(\theta_0|\mathbf{x})}{L(\theta_1|\mathbf{x})} \\
&= \frac{\theta_0^n \cdot e^{-\theta_0 \sum_{i=1}^{n} x_i}}{\theta_1^n \cdot e^{-\theta_1 \sum_{i=1}^{n} x_i}} \\
&= \exp((\theta_1 - \theta_0) \sum x_i) \cdot \left(\frac{\theta_0}{\theta_1}\right)^n \\
C &= \{\lambda \leq k\} \\
&= \{(\theta_1 - \theta_0) \cdot \sum x_i + n \log\left(\frac{\theta_0}{\theta_1}\right) \leq \log(k)\} \\
&= \left\{ \sum x_i \leq \frac{\log(k) + n \log\left(\frac{\theta_0}{\theta_1}\right)}{\theta_1 - \theta_0} \right\}
\end{aligned}
$$

#### (d)

$X_{11}, X_{12}, \ldots, X_{1n_1} \sim N(\mu_1, \sigma_1^2),\ X_{21}, X_{22}, \ldots, X_{2n_2} \sim N(\mu_2, \sigma_2^2)$, all $X_{ij}$ are independent of each other, $\sigma_1^2, \sigma_2^2$ are known, and $H_0 : \mu_2 = \mu_1, H_1 : \mu_2 = \mu_1 + \delta$ with $\delta > 0$ ($\mu_1$ and $\delta$ are both known constants).

**Solution:**

The following questions are written by hands, apologies for the inconsistency.

Tutorial 8.

Q4.1 (c).

$$L(\theta) = \prod_{i=1}^{n} \theta \cdot e^{-\theta x_i} = \theta^n \cdot e^{-\theta \sum_{i=1}^{n} x_i}$$

$$\lambda = \frac{L(\theta_0)}{L(\theta_1)} = \frac{\theta_0^n \cdot e^{-\theta_0 \sum x_i}}{\theta_1^n \cdot e^{-\theta_1 \sum x_i}} = \exp\left( (\theta_1 - \theta_0) \cdot \sum x_i \right) \cdot \left( \frac{\theta_0}{\theta_1} \right)^n$$

$$C = \{ \lambda \leq k \}$$

$$= \{ (\theta_1 - \theta_0) \cdot \sum x_i + n \log\left(\frac{\theta_0}{\theta_1}\right) \leq \log k \}$$

$$= \left\{ \sum x_i \leq \frac{\log k + n \log\left(\frac{\theta_0}{\theta_1}\right)}{\theta_1 - \theta_0} \right\}$$

Q4.1 (d).

$$L(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2, \vec{x}_1, \vec{x}_2) = \left( \frac{1}{\sqrt{2\pi\sigma_1^2}} \right)^{n_1} \left( \frac{1}{\sqrt{2\pi\sigma_2^2}} \right)^{n_2} e^{\frac{-\sum_{i=1}^{n_1} (x_{1i} - \mu_1)^2}{2\sigma_1^2} + \frac{-\sum_{j=1}^{n_2} (x_{2j} - \mu_2)^2}{2\sigma_2^2}}$$

$$\lambda = \frac{L(\mu_1 = \mu_2, \sigma_1^2, \sigma_2^2, \vec{x}_1, \vec{x}_2)}{L(\mu_1 = \mu_2 + \delta, \sigma_1^2, \sigma_2^2, \vec{x}_1, \vec{x}_2)} = \exp\left( \frac{1}{2\sigma_2^2} \left( -2\delta \sum x_{2j} + 2n_2 \mu_1 \delta + n_2 \delta^2 \right) \right)$$

$$C = \{ \lambda(x) \leq k \}$$

$$= \left\{ \frac{1}{2\sigma_2^2} \left( -2\delta \sum x_{2j} + 2n_2 \mu_1 \delta + n_2 \delta^2 \right) \leq \log k \right\}$$

$$= \left\{ \sum x_{2j} > \frac{2\sigma_2^2 \log k - n_2 (2\mu_1 \delta + \delta^2)}{-2\delta} \right\}$$

$$= \left\{ \sum x_{2j} \geq \frac{\sigma_2^2 \log k}{\delta} + \frac{n_2 (2\mu_1 \delta + \delta^2)}{2\delta} \right\}$$

$$= \left\{ \sum x_{2j} \geq \frac{\sigma_2^2}{\delta} \log k + n_2 \mu_1 + \frac{n_2 \delta}{2} \right\}$$

Q4.2. When $\sigma_1^2 = \sigma_2^2 = \sigma = 1$, $n_1 = n_2 = n$

(a). $C = \{\Sigma X_i \geq -\log k + n\mu_1 + \frac{n}{2}\}$

$= \{\bar{X} \geq -\frac{\log k}{n} + \mu_1 + \frac{1}{2}\}$

$P_{H_0}(C) = P_{H_0}(\bar{X} \geq k^*) = \alpha = 0.01$

$\Rightarrow P_{H_0}\left(\frac{\bar{X} - \mu}{1/\sqrt{n}} \geq k^{**}\right) = 0.01$

$\Rightarrow P_{H_0}(Z \geq k^{**}) = 0.01$

So $k^{**} = 2.326348$ $\boxed{\text{qnorm}(0.99)}$

$\gamma(\mu) = \text{Power} = P_{H_1}(C) = P_{H_1}(\bar{X} \geq k^*)$

$\frac{k^* - \mu_1}{1/\sqrt{n}} = k^{**}$

$k^* = k^{**} \cdot \frac{1}{\sqrt{n}} + \mu_1$

$= P_{H_1}(\bar{X} \geq k^{**} \frac{1}{\sqrt{n}} + \mu_1)$

$= P_{H_1}(\bar{X} \geq 2.326348 \frac{1}{\sqrt{10}} + \mu_1)$

$= P_{H_1}\left(\frac{\bar{X} - \mu_1 - 1}{\frac{1}{\sqrt{10}}} \geq \left(\frac{2.326348 \frac{1}{\sqrt{10}} + \mu_1 - \mu_1 - 1}{\frac{1}{\sqrt{10}}}\right)\right)$

$= P_{H_1}\left(\frac{\bar{X} - \mu_1 - 1}{1/\sqrt{10}} \geq 2.326348 - \sqrt{10}\right)$

$= 1 - P(Z < 2.326348 - \sqrt{10})$

$= 0.7948$

(b).

$\gamma(\mu) = 1 - P(Z < 2.326348 - \sqrt{n}) \geq 0.95$

$P(Z < 2.326348 - \sqrt{n}) \leq 0.05$

$2.326348 - \sqrt{n} \leq -1.644854$

$\sqrt{n} \geq 15.77045$

$n = 16.$

Q4.3 : $L(\vec{x};\theta) = \frac{\theta^{n\lambda}}{(\Gamma(\lambda))} x^{(\lambda-1)n} e^{-\theta\sum x_i}$

$$\lambda = \frac{L(\theta_0)}{L(\theta_1)} = \left(\frac{\theta_0}{\theta_1}\right)^{n\lambda} e^{(\theta_1-\theta_0)\sum x_i}$$

$$C = \{\lambda \leq c\}$$

$$= \left\{\sum x_i \leq \frac{1}{\theta_1-\theta_0}\left(\log c - n\lambda \log\left(\frac{\theta_0}{\theta_1}\right)\right)\right\}$$

When $n=\lambda=1$, $C = \left\{\sum x_i \leq \frac{1}{\theta_1-\theta_0}\left(\log c - \log\left(\frac{\theta_0}{\theta_1}\right)\right)\right\}$

$$= \left\{x \leq \underbrace{\frac{1}{(\theta_1-\theta_0)}\left(\log - \log\left(\frac{\theta_0}{\theta_1}\right)\right)}_{c^*}\right\}$$

$$P_{H_0}(C) = P_{H_0}(x \leq c^*)$$

$$= \int_0^{c^*} \frac{\theta_0}{\Gamma(1)} e^{-\theta_0 x} \, dx$$

$$= \theta_0 - \frac{1}{\theta_0} e^{-\theta_0 x}\Big|_0^{c^*}$$

$$= -1(e^{-\theta_0 c^*} - 1)$$

$$= 1 - e^{-\theta_0 c^*} = \alpha$$

$$c^* = \frac{\log(1-\alpha)}{-\theta_0}$$

$$P_{H_1}(C) = P_{H_1}(x \leq c^*)$$

$$= \int_0^{c^*} \theta_1 \cdot e^{-\theta_1 x} \, dx$$

$$= 1 - e^{-\theta_1 c^*}$$

$$= 1 - (1-\alpha)^{\theta_1/\theta_0}$$