About final

1. ~~A~~ Affine geometry

1.1 ~~Af~~ - Theorems of Ceva & Menelaus.
- Three heights, three medians & three bisectors in a triangle are concurrent.

1.2 ~~A~~ - center of mass and its property.

1.3 ~~A~~ - Affine transformations. quantities invariant under affine transformations

Linear map: a function $f: V \to W$ ~~is~~ between K-vector spaces V, W is a linear mapping if

$f(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 f(v_1) + \lambda_2 f(v_2)$ for $v_1, v_2 \in V$, λ_1, λ_2 in the field K.

e.g. ① rotations around origin $(x, y) \to (\cos\theta x + \sin\theta y, -\sin\theta x + \cos\theta y)$

② reflections in axis passing through the origin $(x, y) \to (x, -y)$

③ Shearing in direction of x-axis given in coordinates by $(x, y) \to (x + \lambda y, y)$

④ Scaling (aka homothety) ...

⑤ Stretching

⑥ Parallel projection.

...

! notions of length & angles are not defined in affine geometry.

Transformation $L_{A,b}(x) = A \cdot x + b$ from $\mathbb{R}^n$ to $\mathbb{R}^n$.

A is some invertible matrix, b is some vector.

Check for line: line is a set of points: $\{P + \lambda v \mid \lambda \in \mathbb{R}\}$

where P is a point on the line & ~~λ is~~ v is the direction along this line.

so apply $L_{A,b}$: set is $\{L_{A,b}(P + \lambda v) \mid \lambda \in \mathbb{R}\}$

$= \{(A \cdot P + b) + \lambda(A \cdot v) \mid \lambda \in \mathbb{R}\}$

which is the line passing $A \cdot P + b$ with direction vector $A \cdot v$.

- the ~~nat~~ notion of parallel lines preserved.
  since all lines with direction v $\Rightarrow$ lines with direction $A \cdot v$.

- the notion of quotient of directed lengths:
  sps B_1, B_2, B_3 are collinear ~~lines~~ pts on the line $\{P+\lambda v\}$.
  $B_1 = P+\lambda_1 v$
  $B_2 = P+\lambda_2 v$
  $B_3 = P+\lambda_3 v$
  quotient $\frac{B_1B_2}{B_2B_3} = \frac{(P+\lambda_2 v)-(P+\lambda_1 v)}{(P+\lambda_3 v)-(P+\lambda_2 v)} = \frac{\lambda_2-\lambda_1}{\lambda_3-\lambda_2}$
  if we apply $L_{A,b}$, still the same result.

But, lengths, angles, not preserved by affine transformation

2. Convex Geometry.

2.1 Convex hulls.

Def: the convex hull $\mathcal{S}(S)$ of the set S is the smallest convex closed set that contains S.
- i.e. as the intersection of all convex closed sets that contain S,
  or, the intersection of all the half-spaces that contain S.

Some claims about convex hulls.

a point p belongs to the convex hull of the set S iff there ~~exists~~ exist points $x_1, \cdots, x_m \in S$ & numbers $\lambda_1, \cdots, \lambda_m \geq 0$ such that $\lambda_1 + \cdots + \lambda_m = 1$, and $p = \lambda_1 x_1 + \cdots + \lambda_m x_m$ #.

· explicit bound on the number of points one needs in order to represent any pt in the convex hull of the set S as a convex linear combination. The bound is known as Caratheodory bd.

Thm: If S is a subset of $\mathbb{R}^n$, then any point in the convex hull of S can be written as $\lambda_1 x_1 + \cdots + \lambda_{n+1} x_{n+1}$ for some pts $x_1, \ldots, x_{n+1}$ in S and some numbers $\lambda_i \geq 0$, $\lambda_1 + \cdots + \lambda_{n+1} = 1$.

2.2. Simple polyhedra & their h-vectors
Dehn-Sommerville duality
Euler's formula for 3-dim convex polyhedra.

· simple polyhedra if the vectors along the edges of every vertex form a basis of $\mathbb{R}^n$. In particular only n edges meet at every vertex.

f_0: number of vertices
f_1: number of edges
f_2: number of 2-dim faces
f_3: number of 3-dim faces
⋮
f_n: number of n-dim faces

Euler's ~~formula~~ formula: $f(t) = f_0 + f_1 t + \cdots + f_n t^n$
take $t = -1$.

h-vector: ~~$h(t) = f(t-1) = f_0 + (f_1) \ldots - f_1 + f_2 - f_3 + \cdots + (-1)^n f_n$~~

$$h(t) = f(t-1) = f_0 + f_1(t-1) + \cdots + f_n(t-1)^n$$

$$h(t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_n t^n.$$

D-S relation:

$$\begin{cases} h_i = h_{n-i} \\ h_i \geq 1 \end{cases}$$

Euler's formula for 3-dim:

$$f_0 - f_1 + f_2 - f_3 = 1$$

$h(t) = f(t-1)$

...

* For simple convex, in 3dim $f_0 \times \frac{3}{2} = f_1$.

2.3. Helly's Theorem.

Thm If in a finite collection of convex sets every three intersect, then the intersection of all the sets in the collection is not empty.

For $n=3$, it's clear.

For $n=4$.

Let G_1, G_2, G_3 & G_4 be 4 convex sets

Sps the intersections of every 3 sets are nonempty.

& $A_1, \dots, A_4$ are in those intersections.

Say
$$\left.\begin{array}{l} A_1 \in G_2 \cap G_3 \cap G_4 \\ A_2 \in G_1 \cap G_3 \cap G_4 \\ A_3 \in G_1 \cap G_2 \cap G_4 \\ A_4 \in G_1 \cap G_2 \cap G_3 \end{array}\right\}$$
Radon's thm says we can subdivide the collection of A_1, A_2, A_3, A_4 to 2 subsets, whose convex hulls intersect.

① Say $A_1 \in$ convex hull of A_2, A_3 & A_4

then $A_1 \in G_1$

hence any pt in their convex hull is in G_1

Since $A_1 \in G_2 \cap G_3 \cap G_4$

then $A_1 \in G_1 \cap G_2 \cap G_3 \cap G_4$

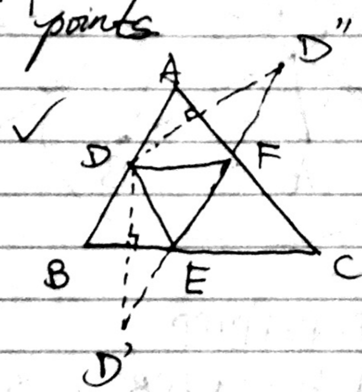
② Similarly if 2 sets, $[A_1, A_2]$ $[A_3, A_4]$ segments intersect at P

since $\left.\begin{array}{l} A_1 \in G_3 \cap G_4 \\ A_2 \in G_3 \cap G_4 \end{array}\right\} \Rightarrow$ every pt in their (A_1, A_2) convex hull is always in $G_3 \cap G_4$.

So $P \in G_3 \cap G_4$
Similary, $P \in G_1 \cap G_2$.
Then $P \in G_1 \cap G_2 \cap G_3 \cap G_4$.

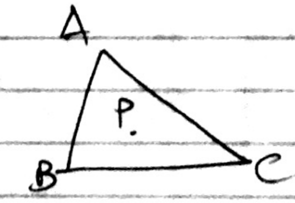
3. Extreme problems in geometry

3.1. Use of reflections for minimizing lengths of broken lines.
Triangle of minimal perimeter inscribed in a given triangle.
A pt which minimizes the sum of the distances from 3 given points

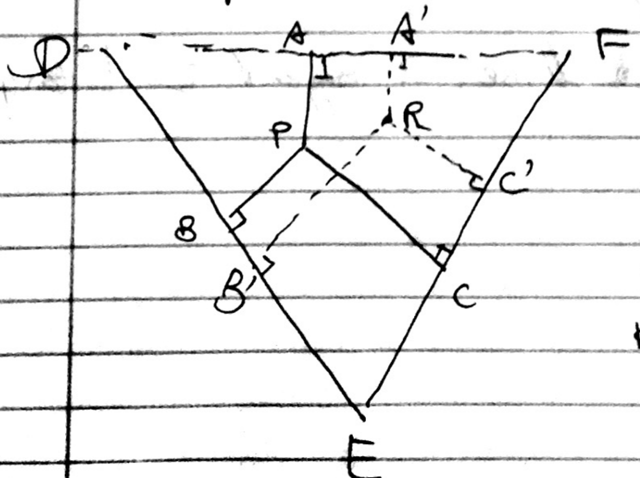


D fixed

Torricelli pt ($\triangle$ whose angles $<120°$).



The pt minimizes $PA+PB+PC$ is where $\angle APB = \angle BPC = \angle CPA = 120°$



we extend $\triangle ABC$ to $\triangle DEF$ which is equalateral,

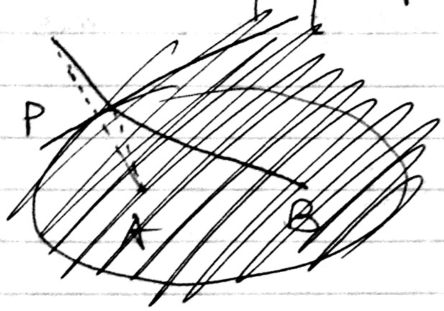
- $PA+PB+PC = RA'+RB'+RC'$!

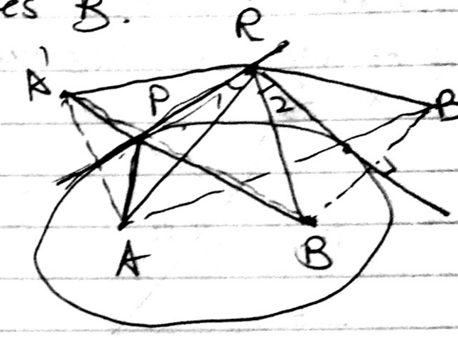
Know $RA > RA'$, $RB > RB'$, $RC > RC'$ $\Rightarrow$ so $RA+RB+RC > PA+PB+PC$.

b/c. RA, RB, RC 是斜边.

3.2. Isoperimetric problem (not covered)
3.3. Optical properties of conic ~~system~~ sections.
Billiard trajectory in an ellipse billiard.

Ellipse: if light ray comes out of a focus A, gets reflected in an ellipse mirror with foci A & B, then the reflection passes B.





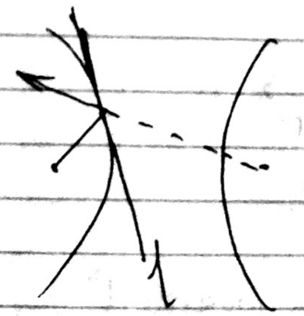
$L = AP + BP = A'P + BP$

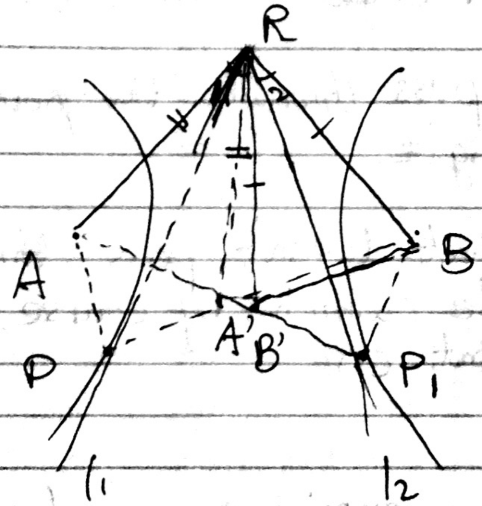
$\triangle A'RB \cong \triangle ARB'$ (SSS)

$\Rightarrow \forall$ any R out of ellipse, the angles between AR & BR connecting R to A & B & the tangent line from R to ellipse are equall

(Move R to ellipse, 2 tangent $\Rightarrow$ 1 tangent)

Hyperbola: if we shine light from one of the foci & it gets reflected by branch, then the reflected light looks like it had been sent from the other foci.





$\angle 1 = \angle 2$

l_1, l_2 tangent lines

$\triangle ARB' \cong \triangle A'RB$

Since $AR = A'R$

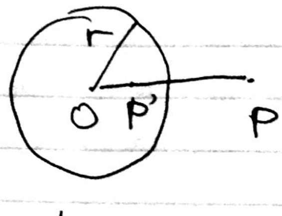
$B'R = BR$

$\angle ARB' = \angle 1 + \angle 1 + \angle A'RB'$

$\angle A'RB = \angle 2 + \angle 2 + \angle A'RB'$

4. Inversion & Möbius transformation

4.1 Properties of inversion.



$OP \cdot OP' = r^2$

① lines passing through O get inverted to themselves.

② lines that do not pass through the point
③ ~~pass~~ preserves angle.

④ maps any circle that doesn't pass through the point O to a circle (which, of course, doesn't pass through the point O).

4.2. Existence of inversions mapping a pair of non-intersecting of circles into a pair of concentric circles.

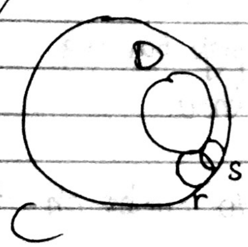
aka. Porism of Steiner
Let C & D be two disjoint circles, for simplicity, say D is inside. C_0 be circle tangent between C & D.
C_1 be tangent to C, D & C_0.
C_2 - - — — C, D, C_1.
C_3 — — — C, D, C_2.
~~after~~ n steps, $C_0 = C_n$.

Proof: by inversion.

Obviously, if C & D concentric. done

Not that obviously, $\boxed{\text{non concentric}}$
Choose an inversion maps them to a pair of concentric circles. (since tangency & circle preserved)

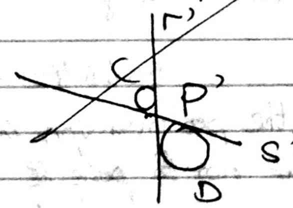
Lemma: for a pair of disjoint circles C, D.
$\exists$ a pair r, s intersecting at two points that are orthogonal to both C & D.



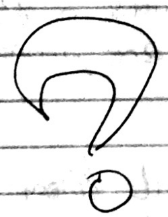
say r intersects s at O & P.

after inversion with ~~O~~ O

$r, s \Rightarrow$ two ~~in inters~~ lines intersecting at P' (inversed P)

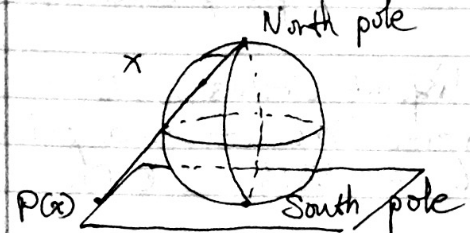


& C and D become 2 circles orthogonal to r' & s'



*Basic idea is: find a center of inversion such that maps C & D to a set of non-intersecting concentric circles.

4.3 Stereographic projection of a sphere onto a plane and its properties (application of inversion)



tangent at SP

connect x & NP, the intersection with plane is the projection we desire to know.

property: ① it maps circle on sphere S to circles on the plane π.
② it preserves angles.

"like inversion"

it actually IS the ~~pe~~ restriction of the inversion in a sphere with center at the north pole & radius = the diameter of the sphere S from which we are projecting.

4.4. Mobius transformation is either fractional linear or composition of conjugation and fractional-linear transformation.

· inverse of complex number z

is $\frac{\bar{z}}{|z|^2}$ where $\frac{\bar{z}}{|z|^2} \cdot z = \frac{|z|^2}{|z|^2} = 1$

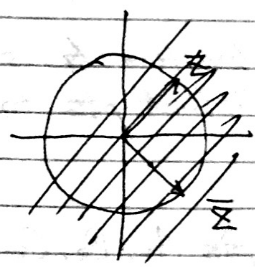
· preserves angles

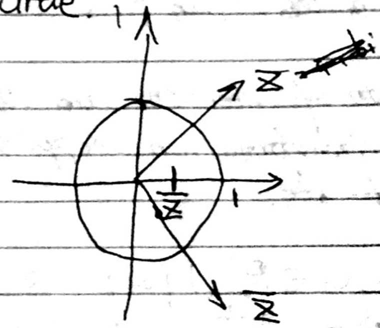
· sends generalized circles to generalized circles.

1. translation $z \to z + c$: distances, angles, shapes preserved.
2. dilation $z \to \lambda z$ with λ real & positive
3. rotation $z \to \lambda z$ λ complex & magnitude 1
4. transformation $z \to \lambda z$ $\lambda \neq 0$ is a composition of dilation by $|\lambda|$ & rotation

about the origin by angle $\arg(z)$)

5. The "inversion" $z \to \frac{1}{\bar{z}}$. composition of reflection in the real line with inversion in unit circle.





We combine the 5 examples above, always have the form

$$z = \frac{az+b}{cz+d} \quad \text{with } a, b, c, d \in \mathbb{C} \ \& \ ad-bc \neq 0.$$

$$\frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} = \frac{a}{c} + \frac{-\frac{ad}{c}+b}{cz+d}$$

$$= A + \frac{B}{cz+d}$$

$$= A + \frac{B}{c(z+\frac{d}{c})}$$

$z \to v = z + \frac{d}{c}$ shift

$v \to w = \frac{1}{z+\frac{d}{c}}$ inversion + symmetry

$w \to \frac{B}{c}\frac{1}{v} = \frac{B}{c(z+\frac{d}{c})}$ scalar

$\to A + \frac{B}{c(z+\frac{d}{c})}$ shift

Thm: Let f be 1-1 transformation of the extended complex line that sends generalized circle to generalized circles and preserves angles. Then f is a factional linear transformation.

4.5 Composition of inversions are Mobius transformations.
Mobius transformation mapping 3 distinct pts to 3 distinct pts.

- $A_1, B_1, C_1 \in \mathbb{R}^2 \cup \{\infty\}$. 3 diff pts A_2, B_2, C_2

$$\exists f(z) = \frac{az+b}{cz+d} \quad \text{s.t.} \quad \begin{aligned} f(A_1) &= A_2 \\ f(B_1) &= B_2 \\ f(C_1) &= C_2 \end{aligned}$$

Plug in $A_1 A_2\ B_1 B_2\ C_1 C_2$. solve for a, b, c, d done.

Thm: Any M trans that fixes 3 pts is just identity.

$$\left.\begin{aligned} M(0) &= 0 \\ M(1) &= 1 \\ M(\infty) &= \infty \end{aligned}\right\} \Rightarrow M \text{ identity}$$

4.6. Mobius Transformation preserves a circle.

5 Ruler & Compass constructions.

5.1 What is possible to construct using ruler & compass.

5.2 Impossibility of construct roots of irreducible cubic equation over $\mathbb{Q}$.

$a_n x^n + \cdots + a_1 x + a_0 = 0$

sps $r = \frac{p}{q}$ is a solution plug it in.

$a_n p^n + \underbrace{\cdots + a_0 q^n}_{\text{divisible by } q} = 0$

has to be divisible by q.

i.e. $a_n = q$.

e.g.

$6y^3 - 2y + 4 = 0$

check $6 = 1 \times 2 \times 3$. p can be $\{\pm 1, \pm 2, \pm 3, \pm 6\}$

$4 = 2 \times 2 \times 1$ q can be $\{\pm 1, \pm 2, \pm 4\}$

$\frac{p}{q}$ can be $\{\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{3}{4}\}$

6. Projective geometry.

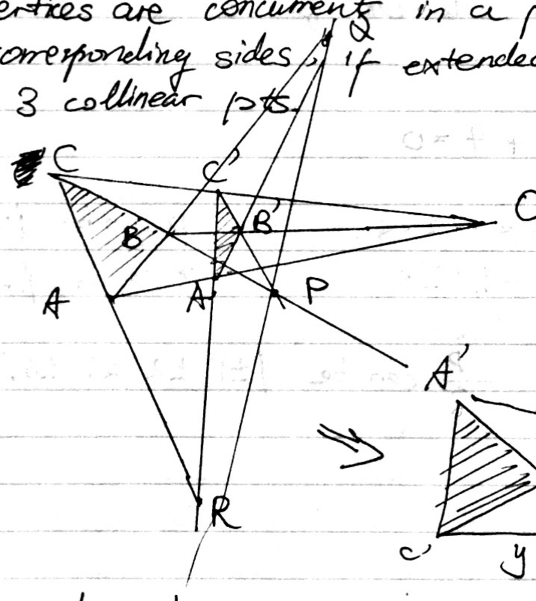
6.1. Projections. (central, parallel proj.)

Desargue's theorem.

Cross ratio of four collinear points & of four concurrent lines on a plane.

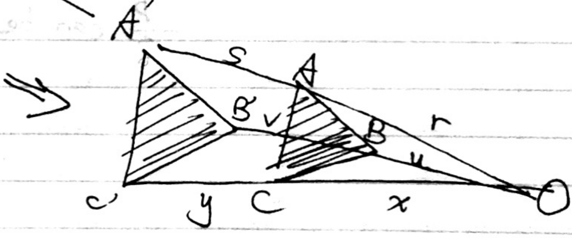
~~Deg~~ Desargue's theorem (in the plane)

If in a plane, 2 △'s ABC & $A'B'C'$ are situated so that the straight lines joining corresponding vertices are concurrent in a point O, then the corresponding sides, if extended will intersect in 3 collinear pts.



PQR collinear !

$\Rightarrow$



Proof: Generalize !

Project the figures so that Q, R go to ∞.

~~By projecting from a center O onto a plane π' parallel to the plane of O, A, B~~

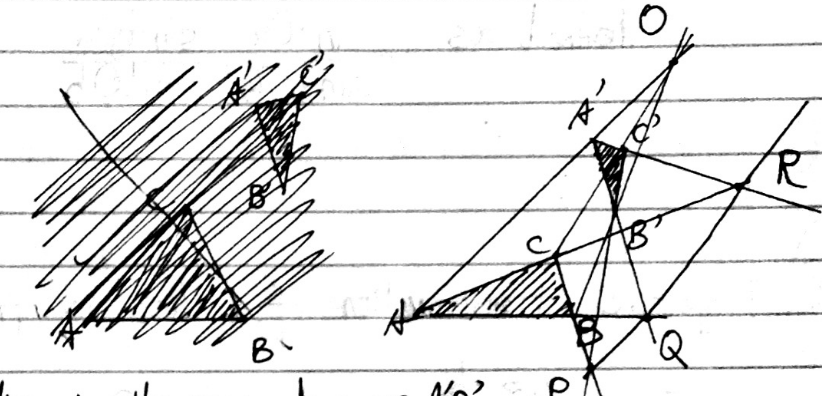
$AB /\!/ A'B' \Rightarrow \frac{u}{v} = \frac{r}{s}$

$AC /\!/ A'C' \Rightarrow \frac{x}{y} = \frac{r}{s}$

$\therefore \frac{u}{v} = \frac{x}{y} \Rightarrow BC /\!/ B'C'$

P, Q, R collinear (meet at infinity)

Theorem (Desargues Thm in space) If 2 triangles ABC and A'B'C' ~~are~~ lie in 2 different (non-parallel) planes & they situated so that the straight line joining corresponding vertices are concurrent in a point, then the corresponding sides, if extended will intersect in three collinear pts.

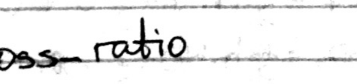


Proof: AB lies in the same plane as A'B',
so that these 2 lines intersect at some Q.
.. same for ... R
... P

Since P & R are on extension of sides of $\triangle ABC$ & $\triangle A'B'C'$.
They lie in the same plane with each of these 2 triangles,

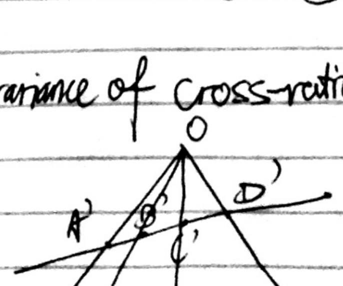
& must consequently lie on the line of intersection of these 2 planes. Therefore P, Q & R collinear. ■

Cross-ratio



$$\frac{C-A}{C-B} : \frac{D-A}{D-B}$$

invariance of cross-ratio



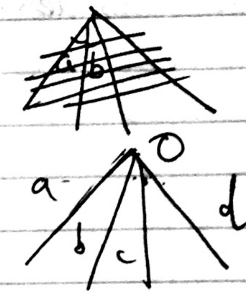
cross-ratio only depends on "angle"

$$\frac{\frac{C-A}{C-B}}{\frac{D-A}{D-B}} = \frac{\frac{C'-A'}{C'-B'}}{\frac{D'-A'}{D'-B'}}$$

Proof by area:

cross-ratio of 4 lines (concurrent). this is, dually speaking.

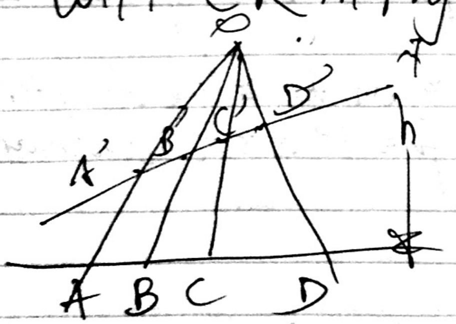
defined as $\dfrac{\sin cOa}{\sin cOb} : \dfrac{\sin dOa}{\sin dOb}$



6.2 Projective transformation of a line is fractional-linear. (§9.5)

$T(z) = \dfrac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$. $ad-bc \neq 0$.

WHY C-R in Proj. invariant.



$\triangle OCA = \frac{1}{2} h \cdot CA = \frac{1}{2} OA \cdot OC \cdot \sin\angle COA$

$\triangle OCB = \frac{1}{2} h\, BC = \frac{1}{2} OB \cdot OC \cdot \sin\angle BOC$

$\triangle ODA = \frac{1}{2} h\, DA = \frac{1}{2} OA \cdot OD \cdot \sin\angle AOD$

$\triangle ODB = \frac{1}{2} h\, BD = \frac{1}{2} OB \cdot OD \cdot \sin\angle BOD$

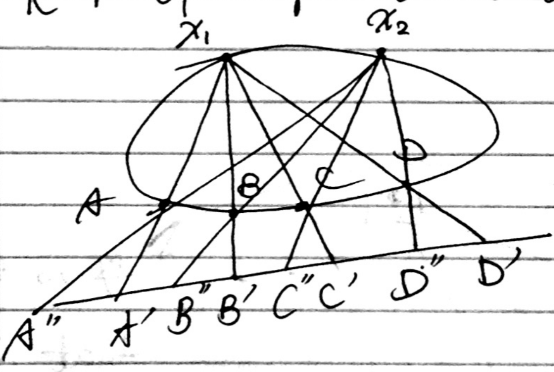
§6.3. Coordinate formulas of projective transformations on a line and on a plane

6.4 Projective transformation between two ~~linear~~ lines in a plane.

A projective mapping of a space to itself preserves the c-r of any 4 collinear pts.

def: a mapping $f: l \to l'$ from a projective l to a projective line l' is projective if it preserves c-r: for any 4 pts. A, B, C, D on l we have $(A,B,C,D) = (f(A), f(B), f(C), f(D))$

6.5. C-r of 4 pts on a conic section (direct & dual)



$(A', B', C', D') = (A'', B'', C'', D'')$

by proj.

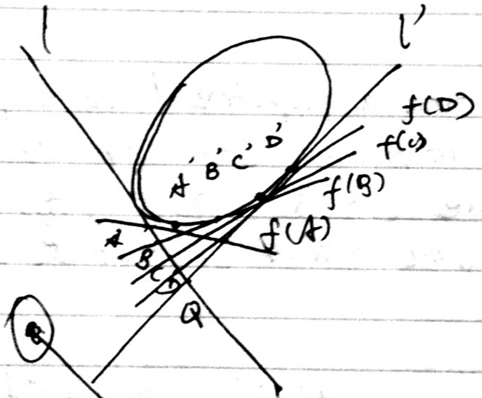
Idea: ① ellipse (or other conic) $\Rightarrow$ circle
② cross-ratio $\Rightarrow$ angle $\to$ angle invariant

done.

Duality: Thm: $f: l_1 \to l_2$ be a projective mapping from line l_1 & l_2 in the plane. O be intersection of l_1 & l_2.
If $f(O) = O$, $\exists$ C in plane s.t. f coincides with the central projection with center C.
If $f(O) \neq O$, then $\exists$ a non-degenerate conic C tangent to l_1 & l_2 s.t. f sends every X on l_1 to pts on l_2 lying on the line passing through X & tangent to C, which is different from l_1.

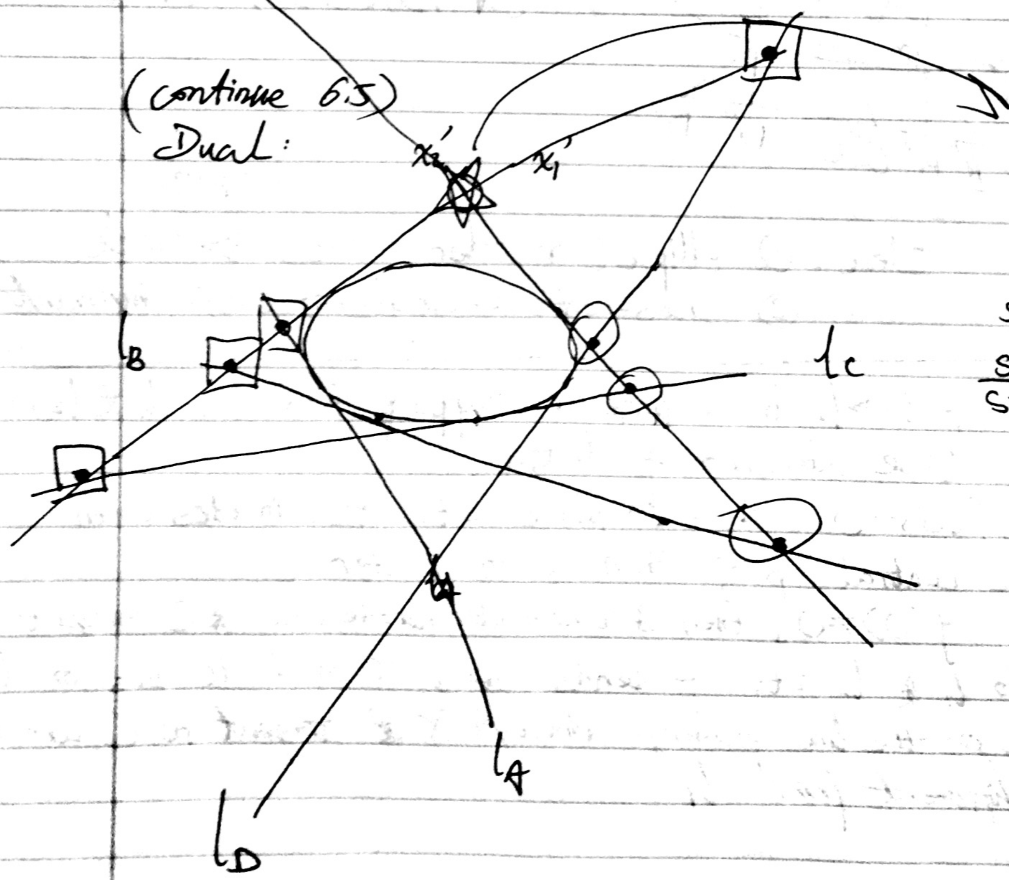
~~(6.5 continued.)~~

(Thm)
6.4. Let l & l' be two lines in proj. plane & let $f: l \to l'$ be a proj. trans. from l to l' which maps point of intersection to itself. Then $\exists$ E in the plane s.t. the f is just the central projection of l onto l' with center E.



Proof skipped.

(continue 6.5)
Dual:



☆ is the ∞ line that $AA'BB'CC'DD'$ used to be on.
so, C-r is

$\frac{\sin 0°}{\sin 0°} : \frac{\sin 180°}{\sin 180°} = \frac{\sin 0°}{\sin 0°} : \frac{\sin 180°}{\sin 180°}$

6.6. Theorems of Pascal & Brianchon. including degenerate cases.

Pappus thm & dual

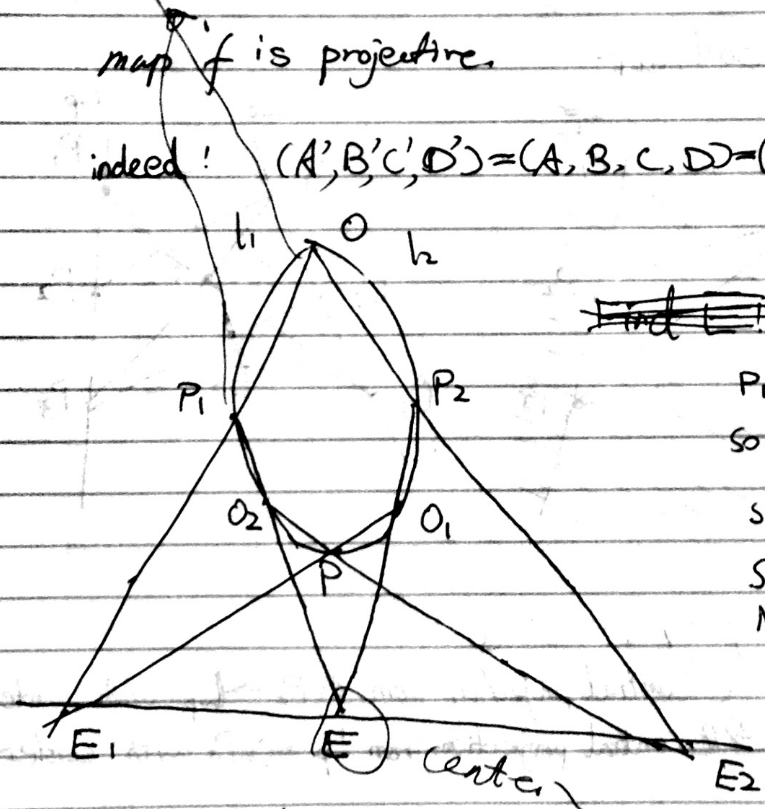
l_1 & l_2 in the plane, E any conic. O_1, O_2 on E.

$f: l_1 \to l_2$ by ① proj. from l_1 to E with center at O_1

② proj. from conic to l_2 with center at O_2

map f is projective.

indeed! $(A', B', C', D') = (A, B, C, D) = (f(A), f(B), f(C), f(D))$



~~Find E!~~

$P_1 \xrightarrow{①} P_1 \xrightarrow{②} O_2P_1 \cap l_2$

so $f(P_1)$ on O_2P_1

similarly $f(P_2)$ on O_1P_2

So E is $O_1P_2 \cap O_2P_1$!

Now let P be any pt

$E_1 \to P \to E_2$. collinear.

What we proved: E be ellipse, $OP_1O_2PO_1P_2O$ be any hexagon inscribed in it. E, E_1, E_2 be points of interaction of opposite sides of the hexagon, $E_1 = OP_1 \cap O_1P$

$E_2 = OP_2 \cap O_2P$

$E = P_1O_2 \cap P_2O_1$

Then E, E_1, E_2 collinear!

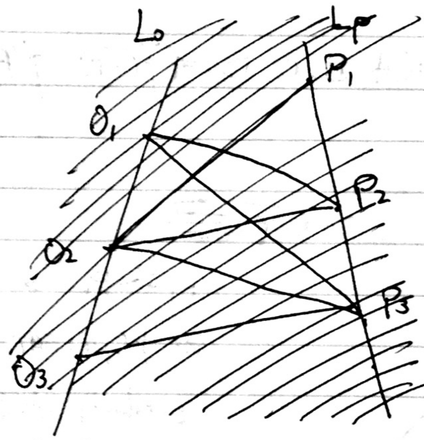
Pappus Theorem:

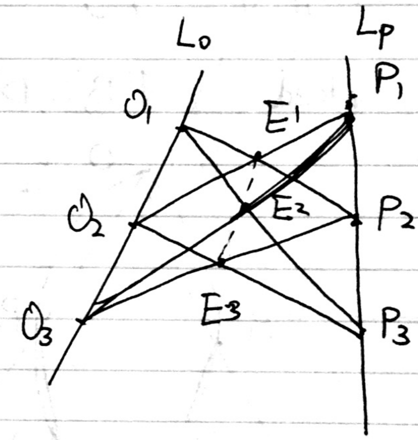
Let L_o, L_p be 2 lines. O_1, O_2, O_3 3 pts on L_o & P_1, P_2, P_3 on L_p. E_1 be pt of intersection of O_2P_3 & O_3P_2

$E_2 = O_1P_3 \cap O_3P_1$

$E_3 = O_1P_2 \cap O_2P_1$

$E_1E_2E_3$ collinear





consider $f: O_1P_2 \to O_1P_3$

(① central projection from $O_1P_2 \to L_p$ with center at O_2

② & central projection from $L_p \to O_1P_3$ with center at O_3)

Projective trans.

maps O_1 to O_1

It's a central projection with some center.

Find it!

Consider P_2. $\xrightarrow{①} P_2 \to f(P_2)$ lies on O_3P_2

Consider P_3 $\xrightarrow{②} P_3 \to f(P_3)$ lies on O_2P_3

So center of central projection on $O_3P_2 \cap O_2P_3$, is E_3

~~E_3~~ $\Rightarrow$ Let's Look at $E_1 \xrightarrow{①} P_1 \xrightarrow{②} E_2$

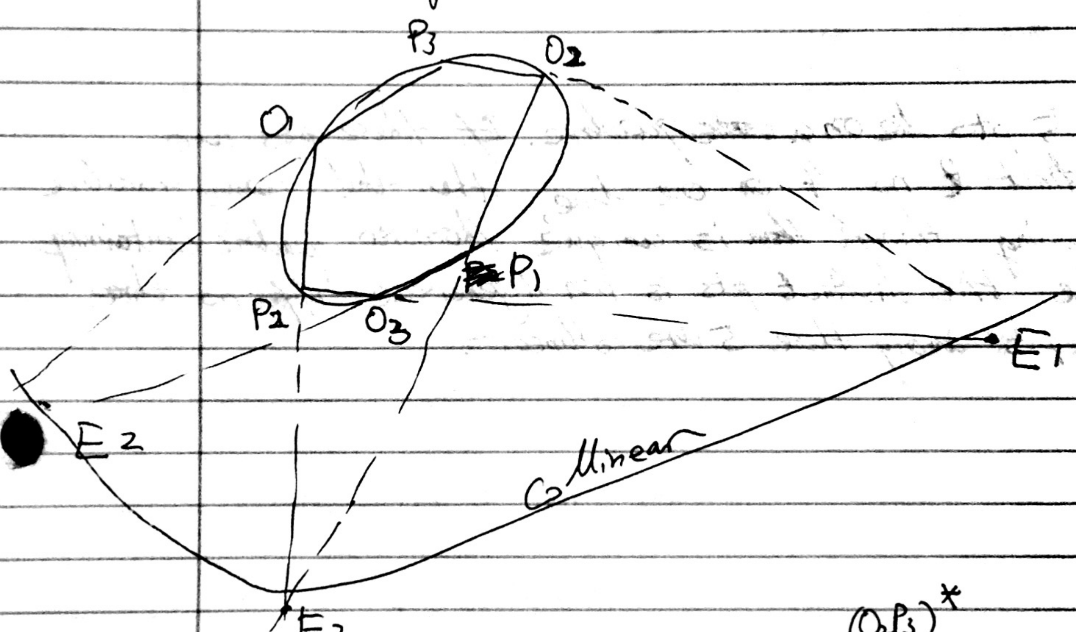
$f(E_3) = E_2$

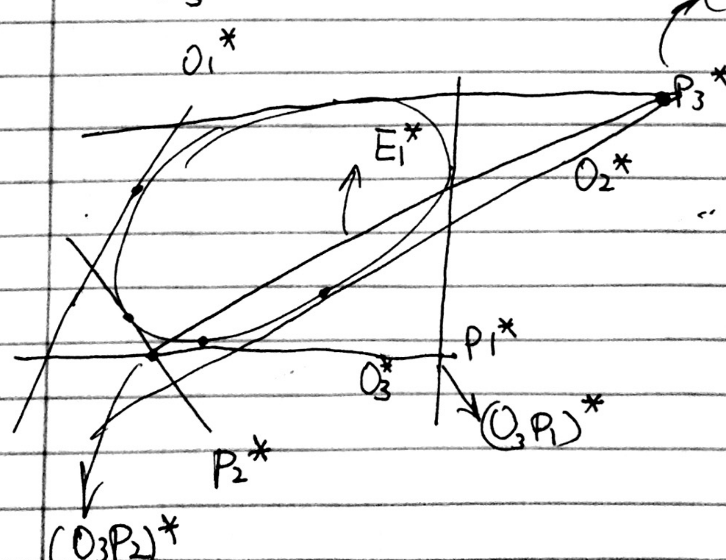
since f is central proj. with E_1, the pts $E_1 E_2 E_3$ collinear

Brianchon's thm (Pascal's dual)

Pascal: $O_1, O_2, O_3, P_1, P_2, P_3$ pts on a quadric E, E_1, E_2, E_3 are intersections of $O_2P_3 \cap O_3P_2$, $O_1P_3 \cap O_3P_1$, $O_2P_1 \cap O_1P_2$ then E_1, E_2, E_3 are collinear.

Brianchon: E^* is a quadric, lines $O_1^*, O_2^*, O_3^*, P_1^*, P_2^*, P_3^*$ are tangent to E^*





Then E_1^*, E_2^*, E_3^* concurrent.

"aka":

"The diagonals connecting opposite ~~sid~~ vertices of a hexagon circumscribed around a conic are concurrent.

Proof skipped.

7.7. General duality principle. Homogeneous coordinates.

· A projective space on dim n is the space of lines passing through the origin in a vector space of dim $n+1$.

duel: ~~The~~ curve $\Rightarrow$ curve
point $\Rightarrow$ line
line $\Rightarrow$ point

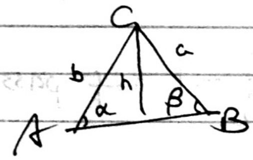
$\left.\begin{array}{l} \text{curve} \Rightarrow \text{curve} \\ \text{point} \Rightarrow \text{line} \\ \text{line} \Rightarrow \text{point} \end{array}\right\}$ in $\mathbb{R}^2$

* Any 5 pts lie on a ~~conic~~ quadric. If these pts are distinct & no 4 on one line, then the ~~conic~~ quadric passing through them is unique. Moreover, quadric containing the five distinct pts is non-degenerate iff no three pts among these 5 are collinear.

7. Spherical Geometry

7.1. Law of sines ~~of~~ for spherical triangle.

Recap: <u>sine rule</u>

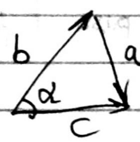


$h = b\sin\alpha$

$h = a\sin\beta$

$\therefore \quad \frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

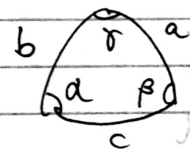
<u>cosine rule</u>

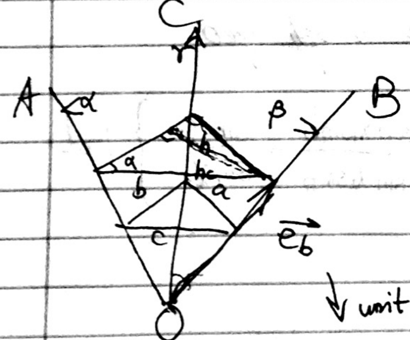


$a^2 = b^2 + c^2 - 2bc\cos\alpha$

if $\alpha = 90°$, we have Pythagorean thm $a^2 = b^2 + c^2$

Assume 3 pts on unit sphere





↓ unit vector

Let $\sin\angle AOB = \sin C$

$\sin\angle BOC = \sin A$

$\sin\angle COA = \sin B$

$h_c = 1 \cdot \sin C$

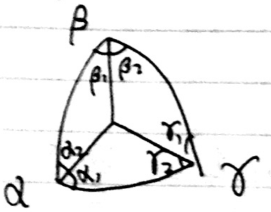
$h = \sin\alpha \cdot h_c = \sin\alpha \sin C$

Similarly $h = \sin\gamma \sin A$

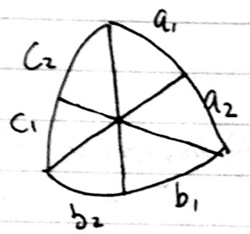
So: $\frac{\sin\beta}{\sin B} = \frac{\sin\gamma}{\sin C} = \frac{\sin\alpha}{\sin A}$

7.2. Ceva's thm for spherical triangle.

Spherical Ceva theorem

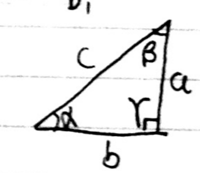


① $\frac{\sin\alpha_1 \sin\beta_1 \sin\gamma_1}{\sin\alpha_2 \sin\beta_2 \sin\gamma_2} = 1$ iff pass 1 point



② $\frac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = 1$ iff pass 1 point

recall

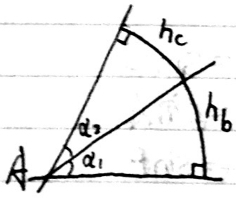


$a = c\sin\alpha$

In sph. geo $\frac{\sin c}{\sin\gamma} = \frac{\sin c}{\sin 90^\circ} = \frac{\sin a}{\sin\alpha}$

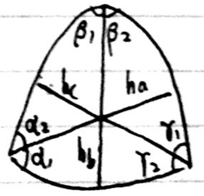
$\sin a = \sin c \sin\alpha$

Proof of ① statement of Ceva in sph. geo.



$\left.\frac{\sin h_b}{\sin h_c} = \frac{b\sin\alpha_1 \sin b}{b\sin\alpha_2 \sin b} = \frac{\sin\alpha_1}{\sin\alpha_2}\right\}$ Tool

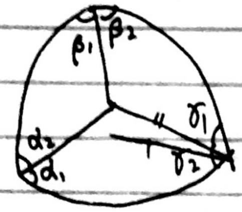
Assume 3 lines do pass 1 pt.



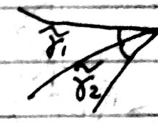
$\frac{\sin\alpha_1}{\sin\alpha_2} = \frac{h_b}{h_c} \cdot \frac{\sin\beta_1}{\sin\beta_2} = \frac{h_c}{h_a} \cdot \frac{\sin\gamma_1}{\sin\gamma_2} = \frac{h_a}{h_b}$.

$\cdots \cdot \underline{\qquad} = 1$ (1 direction done)

The other direction:



prove $\perp$ & $\parallel$ coincide

say  & 

$$\gamma_1 + \gamma_2 = \tilde{\gamma}_1 + \tilde{\gamma}_2 = \gamma$$

$$\& \quad \frac{\sin \tilde{\gamma}_1}{\sin \tilde{\gamma}_2} = \frac{\sin \gamma_1}{\sin \gamma_2} \longrightarrow \text{The guy is fixed.}$$

So ~~γ_2~~ $\tilde{\gamma}_1, \tilde{\gamma}_2$ fixed.

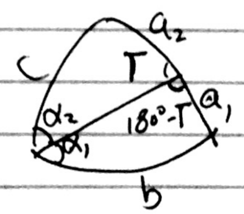
So $\tilde{\gamma}_1 = \gamma - \tilde{\gamma}_2$ $\qquad \dfrac{\sin(\gamma - \tilde{\gamma}_2)}{\sin \tilde{\gamma}_2} = \dfrac{\sin(\gamma - \gamma_2)}{\sin \gamma_2}$

$\gamma_1 = \gamma - \gamma_2$

expand $\Rightarrow \gamma_2 = \tilde{\gamma}_2$ done.

$\blacksquare$

Statement ②



$$\frac{\sin \alpha_1}{\sin a_1} = \frac{\sin(180 - \Gamma)}{\sin b} \Rightarrow \sin\alpha_1 = \frac{\sin(180 - \Gamma)\sin a_1}{\sin b}$$

$$\frac{\sin \alpha_2}{\sin a_2} = \frac{\sin \Gamma}{\sin c} \Rightarrow \sin\alpha_2 = \frac{\sin\Gamma \sin a_2}{\sin c}$$

$$\frac{\sin \alpha_1}{\sin \alpha_2} = \frac{\sin c}{\sin b}\frac{\sin a_1}{\sin a_2}$$

similarly, $\dfrac{\sin \beta_1}{\sin \beta_2} = \dfrac{\sin a}{\sin c}\dfrac{\sin b_1}{\sin b_2}$

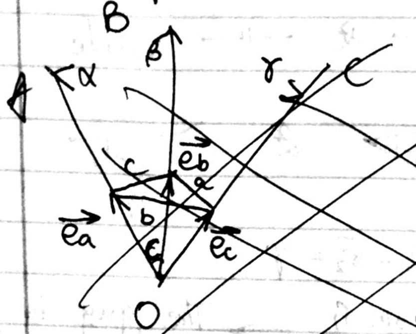
$$\frac{\sin \gamma_1}{\sin \gamma_2} = \frac{\sin b}{\sin a}\frac{\sin c_1}{\sin c_2}$$

$$\frac{\sin a_1 \sin b_1 \sin c_1}{\sin a_2 \sin b_2 \sin c_2} = \frac{\sin\alpha_1}{\sin\alpha_2}\frac{\sin\beta_1}{\sin\beta_2}\frac{\sin\gamma_1}{\sin\gamma_2} = 1 \text{ iff concurrent.}$$

The other direction is easy.

$\blacksquare$

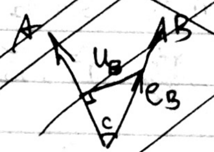
7.3 Law of cosine & triangle inequality for sph. Δ.



~~$\cos\angle BOC = \cos a$~~

~~b/c $\langle e_B, e_C\rangle = |e_B||e_C|\cos a = \cos a$~~

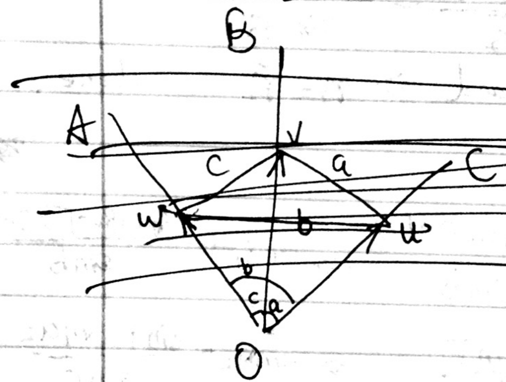
~~project e_B onto OA~~



~~$\sin c = |\vec{u}_B|$~~

~~$e_B = \cos c\, e_B + u_B$~~

~~sin~~

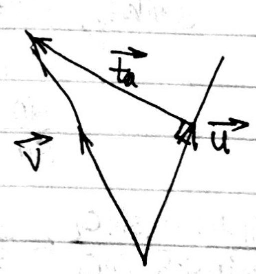


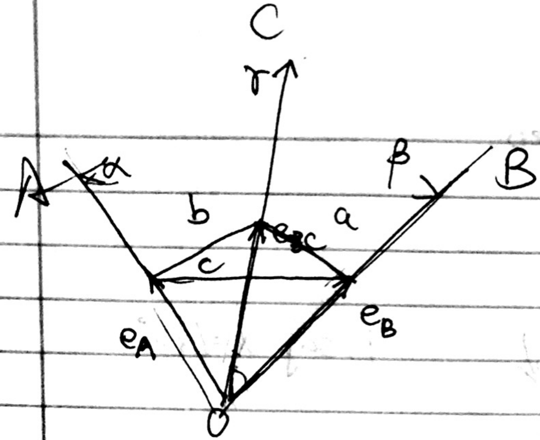
~~$\cos a = \vec{u}\cdot\vec{v}$~~

~~$\cos b = \vec{u}\cdot\vec{w}$~~

~~$\cos c = \vec{v}\cdot\vec{w}$~~

~~To get angle C need tangent vectors $\vec{t}_a$ & $\vec{t}_b$ at $\vec{u}$ along~~ sides a ~~& b. e.g. $\vec{t}_a$ is the unit vector $\perp \vec{u}$ in $\vec{u}$-$\vec{v}$ plane~~



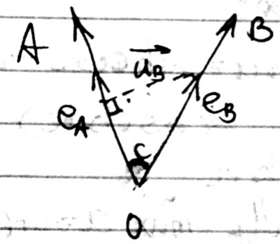


$\cos\angle BOC = \cos a$

~~$\cos\angle BOC$~~

$\langle e_B, e_C\rangle = \cos a$

$\boxed{\langle e_B, e_C\rangle = |e_B||e_C|\cos a = \cos a}$



$\sin c = |u_B|$

$e_B = u_B + \cos c\, e_A$

$\sin b = |u_C|$

$e_C = u_C + \cos b\, e_A$

$\langle e_B, e_C\rangle = \cos a$

$= \langle u_B + \cos c\, e_A,\ u_C + \cos b\, e_A\rangle$

$= \langle u_B, u_C\rangle + \langle u_B, \cos b\, e_A\rangle + \langle \cos c\, e_A, u_C\rangle + \langle \cos c\, e_A, \cos b\, e_A\rangle$

$= \cancel{\sin}\ |u_B||u_C|\cos\alpha + 0 + 0 + \cos c \cos b |\cos 0°|$

$= \sin c \sin b \cos\alpha + \cos c \cos b$

$\Rightarrow \cos a = \sin c \sin b \cos\alpha + \cos c \cos b$

$$\cos\alpha = \frac{\cos a - \cos c \cos b}{\sin c \sin b}$$

if $\cancel{\cos}\ \alpha = 90°$, $\cos a - \cos c \cos b = 0$

if this Pythagorean?

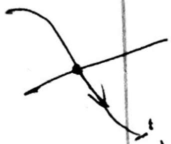
use Taylor expansion:

$$\left(1 - \frac{a^2}{2}\right) = \left(1 - \frac{b^2}{2}\right)\left(1 - \frac{c^2}{2}\right)$$

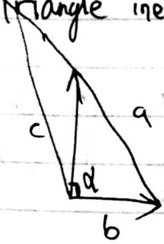
$$= 1 - \frac{b^2 + c^2}{2} + \cancel{\frac{b^2c^2}{4}} \quad \text{omit}$$

$a^2 = b^2 + c^2$

done ✓



⊥ Triangle inequality in sph. Geo.



want $a \uparrow$

so $\cos\alpha \downarrow$ ⊥, say $\alpha = 180^\circ$

cosine law goes:

$$\cos a = \cos b \cos c - \sin b \sin c$$
$$= \cos(b+c)$$

$a = b+c$

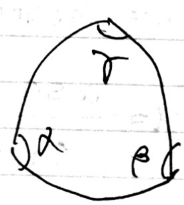
actually max $a = b+c$

since $a \leq b+c$ ✓

7.4. Area of sph. Δ.

$\alpha+\beta+\gamma = \pi$

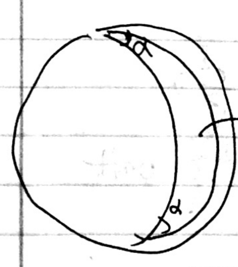
For sph. Δ



$\alpha+\beta+\gamma > \pi$

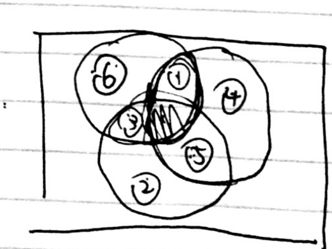
$\alpha+\beta+\gamma-\pi = R^2 S_{\Delta\alpha\beta\gamma}$

we assume $R=1$

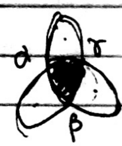


$$S_\alpha = \cancel{\frac{S_{2\pi}}{\frac{\alpha}{2\pi}}} = \cancel{\frac{4\pi R^2}{\frac{\alpha}{2\pi}}} = S_{2\pi} \cdot \frac{\alpha}{2\pi} = 4\pi R^2 \cdot \frac{\alpha}{2\pi} = 2R^2\alpha = 2\alpha$$

do projection:



8 pieces



$S_\alpha, S_\beta, S_\gamma$

$$\left.\begin{array}{l} ①-② \\ ③-④ \\ ⑤-⑥ \end{array}\right\} \text{are sph. symmetries}$$

$$S_\beta = S_{\Delta\alpha\beta\gamma} + ①$$

$$S_\alpha = S_{\Delta\alpha\beta\gamma} + ⑤$$

$$S_\gamma = S_{\Delta\alpha\beta\gamma} + ③$$

$$S_\alpha + S_\beta + S_\gamma = \frac{1}{2} \text{ Area of sphere } + 2\Delta$$

$$2\alpha + 2\beta + 2\gamma = \frac{1}{2} 4\pi + 2\Delta$$

$$\alpha + \beta + \gamma = 1\pi + \Delta$$

$$\Delta = \alpha + \beta + \gamma - \pi.$$

done.