2.21. $X_1, \dots, X_n$ n iid obs. $\sim f(x;\theta) = \theta^{-1} e^{-x/\theta}$, $x \geq 0, \theta > 0$.

Tutorial 5 Solutions

STAT 3013/4027/8027

1.

a. 2.21: Let $Y_1, Y_2, \dots, Y_n \overset{\text{iid}}{\sim}$ exponential(β), where $E[Y] = \beta$. Consider the following three estimators for β:

$$
\begin{aligned}
T_1 &= \sum_{i=1}^{n} Y_i/n \\
T_2 &= \sum_{i=1}^{n} Y_i/(n+1) \\
T_3 &= nY_{(1)}, \text{ where } Y_{(1)} = \text{minimum}(Y_1, Y_2, \dots, Y_n).
\end{aligned}
$$

- Let's determine the mean, bias, and variance of T_1 to then determine the MSE:

$$
\begin{aligned}
E[\sum_{i=1}^{n} Y_i/n] &= E[Y] = \beta \\
V[\sum_{i=1}^{n} Y_i/n] &= \frac{1}{n} V[Y] = \frac{\beta^2}{n} \\
Bias[T_1] &= E[T_1] - \beta = \beta - \beta = 0 \\
MSE[T_1] &= V[T_1] + [Bias(T_1)]^2 = \frac{\beta^2}{n}
\end{aligned}
$$

- Let's determine the mean, bias, and variance of T_2 to then determine the MSE:

$$
\begin{aligned}
E[\sum_{i=1}^{n} Y_i/(n+1)] &= \frac{n}{n+1} E[Y] = \frac{n}{n+1}\beta \\
V[\sum_{i=1}^{n} Y_i/(n+1)] &= \frac{1}{(n+1)^2} \sum V[Y] = \frac{n}{(n+1)^2}\beta^2 \\
Bias[T_2] &= \frac{n}{n+1} E[Y] - \beta = \frac{n}{n+1}\beta - \beta = \beta\left(\frac{n}{n+1} - 1\right) = \beta\left(\frac{1}{n+1}\right) \\
MSE[T_2] &= V[T_2] + [Bias(T_2)]^2 = \frac{n}{(n+1)^2}\beta^2 + \left(\beta\left(\frac{1}{n+1}\right)\right)^2 = \frac{\beta^2}{n+1}
\end{aligned}
$$

- For T_3, let's first determine the distribution if $Y_{(1)}$. I will use the CDF method:

$$\begin{aligned}
P(Y_{(1)} \le y) &= 1 - P(Y_{(1)} > y) \\
&= 1 - P(Y_1 > y, Y_2 > y, \ldots, Y_n > y) \\
&= 1 - [P(Y > y)]^n \\
&= 1 - [1 - P(Y \le y)]^n \\
&= 1 - [1 - (1 - exp(-y/\beta))]^n = 1 - exp(-yn/\beta) = F_{Y(1)}(y)
\end{aligned}$$

$$f_{Y(1)}(y) = \frac{d}{dy} F_{Y(1)}(y) = \frac{n}{\beta} exp(-yn/\beta)$$

Based on the density we know: $Y_{(1)} \sim$ exponential($\beta^* = \beta/n$).

$$\begin{aligned}
E[nY_{(1)}] &= nE[Y_{(1)}] = n\frac{\beta}{n} = \beta \\
V[nY_{(1)}] &= n^2 V[Y_{(1)}] = n^2 \frac{\beta^2}{n^2} = \beta^2 \\
Bias[T_3] &= E[nY_{(1)}] - \beta = \beta - \beta = 0 \\
MSE[T_3] &= V[T_3] + [Bias(T_3)]^2 = \beta^2
\end{aligned}$$

- We can show that the density is a member of the one parameter exponential family:

$$\begin{aligned}
f(y;\beta) &= \frac{1}{\beta} exp(-\beta/y) \\
&= exp\left\{-log(\beta) - \beta\frac{1}{y}\right\}
\end{aligned}$$

$$A(\beta) = -\beta; \quad B(x) = x; \quad C(x) = 0; \quad D(\beta) = -log(\beta).$$

This suggests that $\sum_{i=1}^n x_i$ is a complete and minimal sufficient statistic for β. We can see that T_1 and T_2 are functions of this statistic. **In fact since T_1 is also unbiased we know it is a MVUE**.

- For T_1 and T_2, as $n \to \infty$ the MSE goes to zero, so they are consistent estimators. However, T_3 is not consistent by definition:

$$P(|\hat{\beta} - \beta| > \epsilon) \not\to 0 \text{ as } n \to \infty.$$

b. **See handwritten solutions for** 3.1, 3.11.

2. Based on Question 3.1 (c), using the following data:

```r
set.seed(1001)
n <- 100
x <- rgamma(n, 2, scale=5)
```

a. Calculate the estimate $\hat{\theta}$ analytically.

- From above, we have $\hat{\theta} = \frac{1}{2}\bar{x}$.

```r
theta.hat <- (1/2)*mean(x)
theta.hat
```

```
## [1] 5.158541
```

b. Code up a Newton-Raphson routine to compute $\hat{\theta}$.

- For this we need $U(\theta)$ and $H(\theta)$. We already derived $U(\theta)$.

$$H(\theta) = \frac{2n}{\theta^2} - 2\frac{\sum x_i}{\theta^3}$$

```r
## Starting values - typically we have to be a bit careful
## I just took a rough guess
theta <- 1

## Write some functions for U and H
U <- function(lambda, x){
    n <- length(x)
    out <- -2*n/theta + sum(x)/theta^2
    return(out)
    }

H <- function(lambda, x){
    n <- length(x)
    out <- 2*n/theta^2 - 2*sum(x)/lambda^3
    return(out)
    }
```

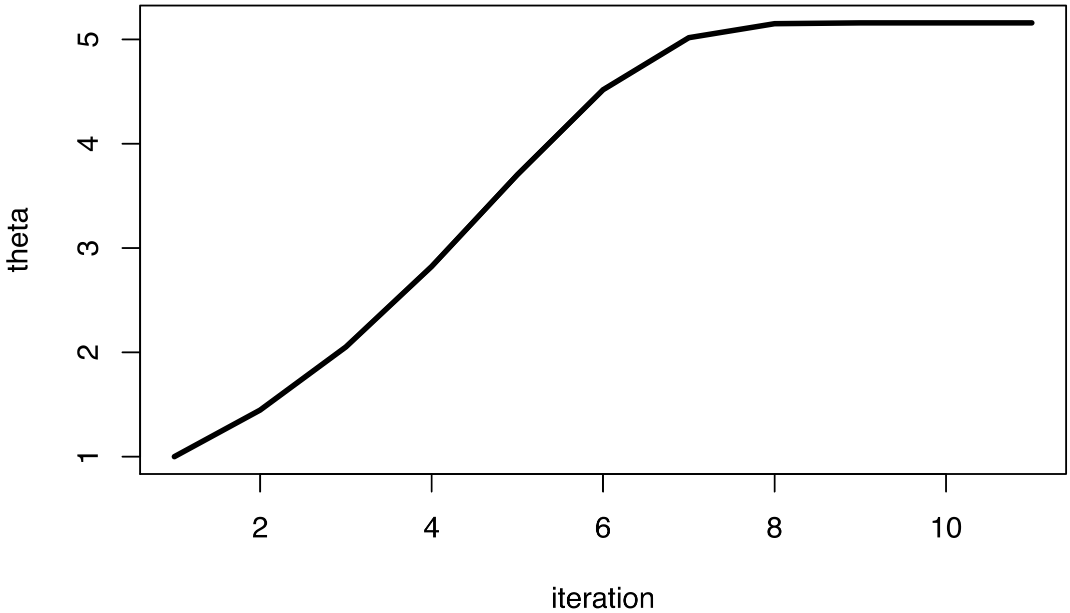
```r
## set a stopping point
eps <- 1e-07
check <- 10

## We are only interested in the final results.
## Why not save them as we go along.
out <- theta

## Run the algorithm
while(check > eps){
theta.new <- theta -   U(theta, x)/H(theta, x)
check <- abs(theta - theta.new)
theta <- theta.new
```

```
out <- c(out, theta)
    }
```

```
plot(out, type="l", lwd=3, xlab="iteration", ylab="theta")
```




```
theta
```

```
## [1] 5.158541
```

- Note: We computed the same value for the MLE as we did analytically.

c. Use `optim()` in R to compute $\hat{\theta}$.

```
## likelihood function
log.lik <- function(theta){
  theta <- theta[1]
  out <- sum(dgamma(x, shape=2, scale=theta, log=TRUE))
  return(out)
  }

## starting values
theta.start <- 1

##
out <- optim(theta.start, log.lik, hessian = TRUE,
             control = list(fnscale=-1), method="BFGS")
```

```
## Warning in dgamma(x, shape = 2, scale = theta, log = TRUE): NaNs produced
```

```
out
```

```
## $par
```

```
## [1] 5.158541
##
## $value
## [1] -322.5051
##
## $counts
## function gradient
##       26       15
##
## $convergence
## [1] 0
##
## $message
## NULL
##
## $hessian
##           [,1]
## [1,] -7.515819
```

- Note that the convergence code is zero, suggesting the algorithm did converge. And the parameter value is the same as the other two approaches.

Find MLE for θ in each case, where $x_1, \cdots, x_n$.

(a). Geo with $\theta(1-\theta)^{x-1}$, $x=1,2,\cdots$

(b). Unif on $(-\frac{\theta}{2}, \frac{\theta}{2})$

(c). Gamma $(2,\theta) \sim$ pdf $f(x|\theta) = \frac{1}{\theta^2} x e^{-x/\theta}$, $x>0$.

(d). Pois(θ)

GJJ Q 3.1

a.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta)$

$$f(x;\theta) = \theta(1-\theta)^{x-1} \; ; \; x = 1, 2, 3, \ldots$$

$$L(\theta; x) = \prod_{i=1}^{n} \theta (1-\theta)^{x_i - 1}$$

$$= \theta^n (1-\theta)^{\Sigma x_i - n}$$

$$\ell(\theta) = n \log(\theta) + (\Sigma x_i - n) \log(1-\theta)$$

$$\frac{d\ell}{d\theta} = \frac{n}{\theta} - \frac{(\Sigma x_i - n)}{1-\theta} = 0$$

$$\Rightarrow \frac{n}{\theta} = \frac{\Sigma x_i - n}{1-\theta}$$

$$\Rightarrow n - \cancel{n\theta} = \theta \Sigma x_i - \cancel{n\theta}$$

$$\hat{\theta} = \frac{n}{\Sigma x_i}$$

- I leave it to you to check that it is a maximum!

b.) $X_1, \ldots, X_n \overset{iid}{\sim} \text{Uniform}(\underset{\text{"a}}{-\theta/2}, \underset{\text{"b}}{\theta/2})$

$$L(\theta; \underset{\sim}{x}) = \prod_{i=1}^{n} \frac{1}{b-a} = \prod_{i=1}^{n} \frac{1}{\frac{\theta}{2} - (-\frac{\theta}{2})} = \prod_{i=1}^{n} \frac{1}{\theta}$$

$$= \frac{1}{\theta^n}$$

- To maximize $L(\theta)$, we want θ to be a small as possible. However, θ is constrained by $\underset{\sim}{x}$.

$$-\frac{\theta}{2} \leq x_{(1)} \leq x_{(n)} \leq \frac{\theta}{2}$$

Note: As $\frac{\theta}{2} \downarrow \Rightarrow -\frac{\theta}{2} \uparrow$

$\therefore$ We are constrained by both $x_{(1)}$ and $x_{(n)}$

This suggests that:

$$\frac{\theta}{2} \geq \max(|x_{(1)}|, |x_{(n)}|)$$

$$\therefore \hat{\theta} = 2 \max(|x_{(1)}|, |x_{(n)}|)$$

c.) $X_1, \ldots, X_n \overset{iid}{\sim} f(x;\theta) = \frac{1}{\theta^2} x \exp(-x/\theta)$

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i \exp(-x_i/\theta)$$

$$= \frac{1}{\theta^{2n}} \left[\prod_{i=1}^{n} x_i\right] \exp\left(-\Sigma x_i/\theta\right)$$

$$\ell(\theta) = -2n\log(\theta) + \sum_{i=1}^{n} \log(x_i) - \Sigma x_i/\theta$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{2n}{\theta} + \frac{\Sigma x_i}{\theta^2} = 0$$

$$\Rightarrow \hat{\theta} = \frac{\Sigma x_i}{2n} = \frac{1}{2}\bar{X}$$

- Is this a max? Make sure to check.

d.) The Poisson case was done in the lecture notes.

$\hat{\theta} = \bar{X}$.

A random sample of n obs. is taken on a r.v. X which $\sim Pois(\theta)$
Sps $\phi = \theta^2$. Find MLE, $\hat{\phi}$ for ϕ, show $\hat{\phi}$ is biased but consistent estimator.
(Note that $E[X^4] = \theta^4 + 6\theta^3 + 7\theta^2 + \theta$)

GJJ Q 3.11

- $X_1, \ldots, X_n \overset{iid}{\sim} \text{Poisson}(\theta)$

- We know $\hat{\theta} = \bar{X}$. What is the MLE for $\phi = \theta^2$?

- Reparameterize the likelihood!

$$\theta^2 = \phi \Rightarrow \theta = \phi^{1/2}$$

$$\therefore \quad f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}$$

$$f(x;\phi) = \frac{e^{-(\phi^{1/2})}(\phi^{1/2})^x}{x}$$

$$L(\phi) = \prod_{i=1}^{n} \frac{e^{-(\phi^{1/2})}(\phi^{1/2})^{x_i}}{x_i!} = \frac{e^{-n(\phi^{1/2})}(\phi^{1/2})^{\Sigma x_i}}{\Pi x_i!}$$

$$\ell(\phi) = -n\phi^{1/2} + \frac{\Sigma x_i}{2}\log(\phi) - \sum_{i=1}^{n}\log(x_i!)$$

$$\frac{\partial \ell}{\partial \phi} = \frac{\Sigma x_i}{2\phi} - \frac{n}{2\phi^{1/2}} = 0$$

$$\Rightarrow \frac{\Sigma x_i}{2\phi} = \frac{n}{2\phi^{1/2}} \Rightarrow \frac{\Sigma x_i}{n} = \phi^{1/2}$$

$\therefore \hat{\phi} = \bar{x}^2$

$\therefore \hat{\phi} = f(\hat{\theta}) = (\bar{x})^2$

This is the invariance property of the MLE!

- What is $E(\hat{\phi})$?

$$E(\hat{\phi}) = E(\bar{x}^2) = V(\bar{x}) + [E(\bar{x})]^2$$

$$= \frac{\theta}{n} + \theta^2 \neq \theta^2 \quad \therefore \text{biased}$$

- What is the $V(\hat{\phi})$?

$$V(\bar{x}^2) = E(\bar{x}^4) - \left[E[\bar{x}^2]\right]^2$$

$$\Rightarrow E(\bar{x}^4) = E\left(\left(\tfrac{1}{n}\Sigma x_i\right)^4\right)$$

$$= \frac{1}{n^4} E((\Sigma x_i)^4)$$

- Let $Y = \Sigma x_i$ by Moment Generating Functions we can show $Y \sim \text{Poisson}(\gamma = n\theta)$

$$\therefore \quad E(\bar{x}^4) = \frac{1}{n^4} E(Y^4)$$

$$= \frac{1}{n^4}\left[\gamma^4 + 6\gamma^3 + 7\gamma^2 + \gamma\right]$$

$$= \frac{1}{n^4}\left[(n\theta)^4 + 6(n\theta)^3 + 7(n\theta)^2 + n\theta\right]$$

$$\Rightarrow V(\hat{\phi}) = E(\bar{x}^4) - \left[\theta/n + \theta^2\right]^2$$

$$= \frac{\theta}{n^3} + \frac{6\theta^2}{n^2} + \frac{4\theta^3}{n}$$

- As $n \to \infty$ Bias $\to 0$ $\therefore$ MSE $\to 0$

  Var $\to 0$

$\therefore$ Consistent