Lecture 9
(Cantor set continued)

A subset N of $\mathbb{R}$ has null Lebesgue measure (Lebesgue measure zero) iff given any $\varepsilon>0$.
$\exists$ a seq. $\{I_n\}$ of intervals s.t. $N \subset \bigcup e_n$.
Total length of these intervals $< \varepsilon$.

Cantor set has measure 0.

S_n contains 2^n intervals $\left(\frac{1}{3}\right)^n$

Total length $\left(\frac{2}{3}\right)^n$

We are removing $\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1/3}{1-2/3} = 1$

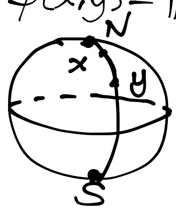
Ch 9. Metric spaces

§ 9.1 def'n & e.g's

Def: Let X be a set. A metric on X is a function $\varphi: X\times X \longrightarrow [0,+\infty)$ that satisfies the following:

① positive definiteness
$\varphi(x,y)=0$ iff $x=y$
② symmetry $\varphi(x,y)=\varphi(y,x)$ $\forall x,y\in X$
③ triangle inequality $\varphi(x,z)=\varphi(x,y)+\varphi(y,z)$ $\forall x,y,z\in X$

(1). Any vector space with $\| \quad \|$
$\varphi(x,y)=\|x-y\|$

(2).


$\varphi(x,y)$ the length of a shortest path that connects x&y

geodesis

(3). The discrete metric
define $d(x,y)=\begin{cases} 0 & x=y \\ 1 & x\neq y \end{cases}$

(i). $d(x,y)=0 \Longleftrightarrow x=y$
(ii) $d(x,y)=d(y,x)$
(iii) $d(x,z)=1$ WLOG assume $d(x,z)>0$, $d(x,y)=1$, $d(y,z)=1$
$1<2$

$(\mathbb{R},d)$
$B_r(x)=\{y: \varphi(x,y)<r\}$

(4). Define a metric on $\mathbb{Z}$.
$\varphi_2(n,n)=0$
$\varphi_2(m,n)=\frac{1}{2^d}$, where d is the largest power of 2 dividing $m-n$

2-adic metric. if you take any prime p instead of $2 \Rightarrow p$-adic.

(1). $\varphi_2(m,n)=0$ iff $m=n$

(2). $\varphi_2(m,n)=\varphi_2(n,m)$

Let $l, m, n \in \mathbb{Z}$

Sps $\varphi_2(l,m)=2^{-d}$

$\varphi_2(m,n)=2^{-e}$

$\varphi_2(l,n)$

$l-n=l-m+m-n$

$\min\{d,e\}$ divides $l-n$.

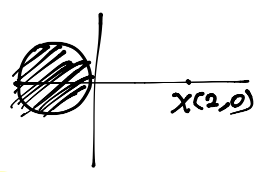
$$\varphi_2(l,n)\le 2^{-\min(d,e)}.$$
$$=\max\{\varphi_2(l,m),\varphi_2(m,n)\}$$
$$\le \varphi_2(l,m)+\varphi_2(m,n)$$

∎

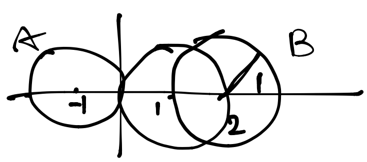
(5). If X is a closed subset of $\mathbb{R}^n$, let $k(X)$ denote the collection of all nonempty compact subsets of X.

If A is a compact subset of X, $x\in X$, $\text{dist}(x,A)=\inf_{a\in A}\|x-a\|$

$\text{dist}(x,A)=2$



Hausdorff metric

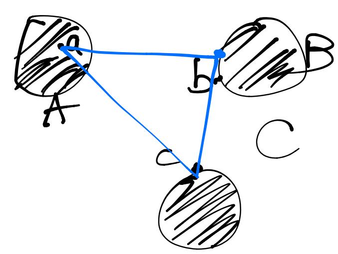


$$d_H(A,B)=\max\{\sup_{a\in A}(\text{dist } a,b),\ \sup_{b\in B}\text{dist}(b,A)\}$$

Claim: A is closed, $\text{dist}(x,A)=0 \Leftrightarrow x\in A$

(1). $d_H(A,B) \Leftrightarrow A=B$

(2). symmetry is easy



Let a be any pt in A. There exists a pt $b\in B$ that is closest to a.

$\exists c\in C$ that is closest to b.

$d(a,C)\le\|a-c\|\le\|a-b\|+\|b-c\|=\text{dist}(a,B)+\text{dist}(b,C)\le d_H(A,B)+d_H(B,C)$

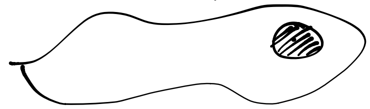
$\sup_{a\in A}\text{dist}(a,C)\le d_H(A,B)+d_H BC$ $\qquad \sup_{c\in C}\text{dist}(c,A)\le d_H(A,B)+d_H(B,C)$

$$\Rightarrow d_H(A,C)\le d_H(A,B)+d_H(B,C)$$

∎

Def: The ball $B_r^{\rho}(x)$ of rad $R>0$ about a pt x.

$\{y \in X : \rho(x,y) < r\}$



A subset U is open if $\forall x \in U\ \exists\, r>0$ s.t. $B_r(x) \subset U$.
The Interior of a set A, $\operatorname{int} A$ is the largest open set that contains A.

A seq (x_n) of pts of (X, ρ) is said to converge to x if
$\lim_{n\to\infty} \rho(x_1, x_n) = 0$

A set C is closed if it contains all of its limit pts.

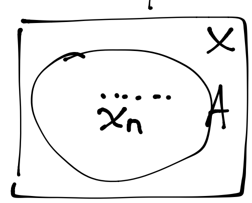
$\bar{A}$ = closure of A is the set of all limit points of A.

A sequence $(x_n)_{n=1}^{\infty}$ in (X, ρ) is a Cauchy seq if $\forall \varepsilon > 0, \exists N$ s.t.
$\rho(x_i, x_j) < \varepsilon,\ \forall i, j \geq N$

A metric space is complete if every Cauchy sequence converges in X.

Theorem: a subset of a complete metric space is complete if and only if it is closed.

Proof: Let A be a closed subset of a complete metric space (X, ρ)
Let (x_n) be a Cauchy seq. of pts of A.
$\Rightarrow (x_n)$ is a Cauchy seq. of pts o X
Since X is complete, (x_n) converges to a pt $x \in X$
$\Rightarrow x$ is a limit pt of A.
A is closed, so $x \in A \Rightarrow A$ is complete.



Suppose A is complete
$\Rightarrow$ every cauchy seq. of pts of A converges to a pt at A.
Let a be a limit pt of A. $\exists$ a convergent seq. $(a_n) \in A$ s.t $a_n \to a$
$\Rightarrow (a_n)$ is Cauchy
$\Rightarrow a \in A$
$\Rightarrow A$ closed ∎

Ex. (X, d), d = discrete metric, $(\mathbb{R}, d)$
$1, 1, 1, \cdots$
$0, 1, 1, \cdots$
$0, \frac{1}{2}, 1, \cdots$

$\boxed{\text{every convergent seq. is eventually constant}}$

Ex: $(2^n)_{n=1}^{\infty} \in \mathbb{Z}$

$2, 2^2, 2^3, 2^4, \cdots$

$\rho_2(2^n, 0) = \frac{1}{2^n} \to 0$ as $n \to \infty$

$\rho_2(n, m)$ converges to zero

A Cauchy seq. that does not converge:

$$a_n = \frac{(1-(-2)^n)}{3} \qquad n \geq 1$$

$a_1 = 1$

$a_2 = -1$

$a_3 = 3$

$a_4 = -5$ etc.

The sequence is Cauchy

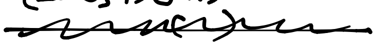
$n > m \qquad a_n - a_m = \frac{(1-(-2)^n)}{3} - \frac{(1-(-2)^m)}{3} = \frac{(-2)^m}{3}(1-(-2)^{n-m}) = (-2)^m(a_{n-m})$

$\varphi(a_m, a_n) = 2^{-m} \leq \frac{1}{2^N} \to 0 \quad n > m \geq N$

§9.2 Compact Metric space

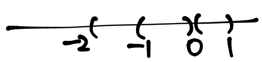
Def: A collection of open sets $A = \{U_\alpha : \alpha \in A\}$ in X is called an open cover of $Y \subseteq X$ if $Y \subseteq \bigcup_{\alpha \in A} U_\alpha$

$(-\infty, 1) \cup (1, \infty)$



$\mathbb{R}$

$(n-1, n+1) \quad n \in \mathbb{R}$



A subcover of Y in $\{U_\alpha, \alpha \in A\}$ is a subcollection $\{U_\alpha, \alpha \in B\}$ for some $B \subseteq A$ that is still a cover of Y.

If B is finite $\Rightarrow$ It is called a finite subcover.

$\{(n-1, n+1) : n \in \mathbb{N}\}$ is a cover of $\mathbb{R}$

Def: A metric space is compact if every open cover has a finite subcover.

Def: A collection of closed sets $\{C_\alpha : \alpha \in A\}$ has the finite intersection property if every finite subcollection has nonempty intersection.

Def: A metric space is sequentially compact if every sequence in X has a convergent subsequence.

Def: A metric space is totally bounded if $\forall \varepsilon > 0, \exists$ finitely many pts $x_1, \cdots x_k \in X$ s.t. $\{B_\varepsilon(x_i) : 1 \leq i \leq k\}$ is an open cover.

Borel–Lebesgue Thm:

Let (X, ρ) be a metric space, then the following are equivalent

(1). X is compact

(2). every collection of closed subsets of X with the finite intersec-
tion property has nonempty intersection.
(3). X is sequentially compact.
(4). X is complete & totally bounded.