Lecture 7.

Joint Distribution.

(X, Y) - r.vector

Let X, Y be r.v's,

joint cdf : $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

n-dim'l case :

$$F_{X_1 \cdot X_2 \dots X_n}(x_1, \dots, x_n) = P(X_1 \le x_1, \dots, X_n \le x_n)$$

Discrete Case :

Suppose X, Y are discrete r.v.'s defined on $(\Omega, \mathcal{F}, P)$. The joint probability mass function of two discrete r.v.'s X and Y is the function $p_{X,Y}(x,y)$ defined for all pairs of real numbers x and y by

$$p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$$

$p_{X,Y}(x,y) \ge 0$ for all values of x and y

$$\sum_x \sum_y p_{X,Y}(x,y) = 1$$

Ex. Toss a coin 3 times.

$X = \#$ of heads on 1^{st} toss

$Y =$ total $\#$ of heads

$\Omega = \{TTT, TTH, THT, HTT, THH, HTH, HHT, HHH\}$

$y \backslash x$	0	1
0	1/8	0
1	2/8	1/8
2	1/8	2/8
3	0	1/8

$$p_X(x) = \sum_y p_{X,Y}(x,y) \Rightarrow p_X(0) = \frac{1}{2} = p_X(1)$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) \Rightarrow p_Y(0) = \frac{1}{8} = p_Y(3)$$

$$p_Y(1) = \frac{3}{8} = p_Y(2)$$

Marginal Probability Function.

The marginal probability mass function for X is

$$p_X(x) = \sum_y p_{X,Y}(x,y)$$

for Y: $$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

If $X_1, \dots, X_n$ are discrete r.v's on the same sample space with

$$p_{X_1 \dots X_n}(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$$

Then the marginal probability of X_1 is

$$p_{X_1}(x_1) = \sum_{x_2, \dots, x_n} p_{X_1 \dots X_n}(x_1, \dots, x_n) = \sum_{x_2} \sum_{x_3} \dots \sum_{x_n}$$

The 2-dim'l marginal prob. function of X_1 and X_2 is

$$p_{X_1 X_2}(x_1, x_2) = \sum_{x_3 \dots x_n} p_{X_1 \dots X_n}(x_1, \dots, x_n)$$

Ex. Roll a die twice.

Let X be # of 1's

Y be total of the two dice.

Find $p_{XY}(x, y)$.

$x \backslash y$	2	3	4	...	12
0	0	0	1/36		
1	0	2/36	2/36		
2	1/36	0	0		

$(1, 3)$

$p_X(1) = \frac{10}{36}$

$p_Y(3) = \frac{2}{36}$

$p_{XY}(1,3)$

$= \frac{2}{36} \neq \frac{10}{36} \cdot \frac{2}{36}$

p_X

X	0	1	2
p	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

p_Y

Y	2	3	4		12
p	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$		$\frac{1}{36}$

$P(X \leq 1 \text{ and } Y \leq 4) = ?$

$= \frac{1}{36} + \frac{2}{36} + \frac{2}{36} = \frac{5}{36}$

Independence of Random Variables

Recall: r.v. $X : \Omega \to \mathbb{R}$ s.t $\{\omega \in \Omega : X(\omega) = x\}$ $\in \mathcal{F}$, $x \in \mathbb{R}$

$$(X \in A) = \{\omega : X(\omega) \in A\} \text{ – event}$$

__Def.__ Random variables X and Y are independent if the events $(X \in A)$ and $(Y \in B)$ are independent.

Recall: A, B are independent, if

$$P(A \cap B) = P(A)\,P(B)$$

__Theorem.__ Two discrete r.v.'s X and Y with joint pmf $p_{X,Y}(x,y)$ and marginal mass functions $p_X(x)$ and $p_Y(y)$, are independent iff

$$p_{X,Y}(x,y) = p_X(x)\,p_Y(y)$$

__Proof:__

$$\begin{aligned} p_{X,Y}(x,y) &= P(\{X=x\} \text{ and } \{Y=y\}) \\ &= P(\{X=x\})\,P(\{Y=y\}) \\ &= p_X(x)\,p_Y(y) \end{aligned}$$

$\boxed{\text{B}}$

Q: Are X and Y independent?

No

Conditional probability.

$$P(X=x \mid Y=y) = \frac{P(X=x \text{ and } Y=y)}{P(Y=y)}$$

$$P(Y=y \mid X=x) = \frac{P(X=x \text{ and } Y=y)}{P(X=x)}$$

Back to prev. ex.

$$P(Y=2 \mid X=1) = \frac{P(Y=2 \text{ and } X=1)}{P(X=1)} = 0$$

$$P(Y=3 \mid X=1) = \frac{2/36}{10/36} = \frac{1}{5}$$

$\vdots$

$$P(Y=12 \mid X=1) = 0$$

conditional pmf of Y given $X=1$

Def. Let X, Y be discrete r.v's with joint pmf $p_{X,Y}(x,y)$ and marginal mass functions $p_X(x)$ and $p_Y(y)$. If x is a number s.t. $p_X(x) > 0$, then the conditional pmf of Y given $X=x$ is

$$\boxed{p_{Y|X}(y|x) = p_{Y|X}(y \mid X=x) = \frac{p_{X,Y}(x,y)}{p_X(x)}}$$

Similarly, $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(x)}$

$$p_{X,Y}(x,y) = p_{X|Y}(x|y)\, p_Y(x)$$

$$p_X(x) = \sum_y p_{X|Y}(x|y)\, p_Y(y)$$

Ex. Suppose we roll a fair die; whatever number comes up we toss a coin that many times.
What is the distribution of the # of heads?

$X = \#$ of heads, $Y = \#$ on die

$p_X(x) = ?$

$p_Y(y) = \frac{1}{6}, \; y = 1, 2, \ldots, 6$

$p_{X|Y}(x|y) = \binom{y}{x}\left(\frac{1}{2}\right)^y, \quad x = 0, 1, \ldots, y$

$$p_X(x) = \sum_y p_{X|Y}(x|y)\, p_Y(y)$$

$$= \sum_{y=1}^{6} \binom{y}{x}\left(\frac{1}{2}\right)^y \cdot \frac{1}{6}, \quad x = 0, 1, 2 \ldots 6$$

Multiple Integration.

Recall: $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \underline{f(x_i)\Delta x_i},$

Δx_i = length of the i^{th} subinterval of some partition of $[a,b]$

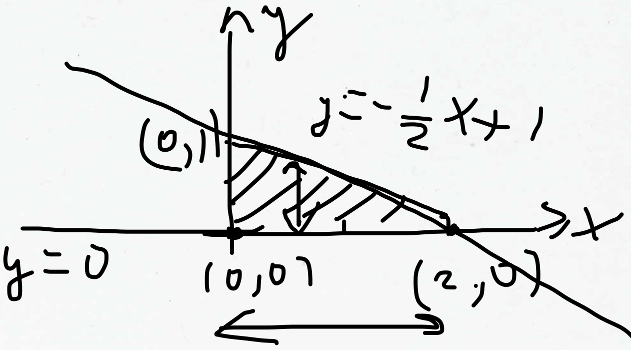
$$\iint_D f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i,y_i)\,\Delta A_i\,,$$

where A_i is the area of the i^{th} rectangle in a decomposition of D into rectangles, D is a finite region in the xy-plane.

__Theorem__. Let D be the subset of $\mathbb{R}^2$ defined by $y_L(x) \leq y \leq y_U(x)$ and $x_L \leq x \leq x_U$. If $f(x,y)$ is continuous on D then

$$\iint_D f(x,y)\,dA = \int_{x_L}^{x_U}\left[\int_{y_L(x)}^{y_U(x)} f(x,y)\,dy\right]dx$$

Ex. Evaluate $\iint_D xy\,dA$, where D is the triangle with vertices $(0,0)$, $(2,0)$ and $(0,1)$)



$$\iint_D xy\,dA = \int_0^2\left[\int_0^{-\frac{1}{2}x+1} xy\,dy\right]dx$$

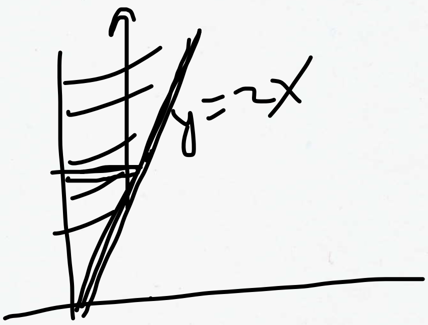
$$= \int_0^2 x\frac{y^2}{2}\Big|_0^{-\frac{1}{2}x+1} dx$$

$$= \frac{1}{8}\int_0^2 x(2-x)^2 dx = \frac{1}{8}\int_0^2 [4x-4x^2+x^3]\,dx$$

$$= \frac{1}{8}\left[2x^2-\frac{4}{3}x^3+\frac{1}{4}x^4\right]_0^2 = \frac{1}{8}\left[8-\frac{32}{3}+4\right]$$

$$= \frac{1}{8}\frac{4}{3} = \frac{1}{6} = \int_0^1\int_0^{2-2y} xy\,dx\,dy$$

Ex. Integrate $f(x,y) = xe^{-x-2y}$ over the set of points in the positive quadrant for which $y > 2x$.



$$\int_0^\infty\int_{2x}^\infty \underbrace{xe^{-x}}\,e^{-2y}\,dy\,dx$$

$$= \int_0^\infty xe^{-x}\cdot -\frac{1}{2}e^{-2y}\Big|_{2x}^\infty dx$$

$$= \frac{1}{2}\int_0^\infty xe^{-5x}\,dx = \left[\begin{array}{ll} u=x & du=dx \\ e^{-5x}dx=dv & v=-\frac{1}{5}e^{-5x}\end{array}\right.$$

$$= \frac{1}{2}\left(-\frac{1}{5}\right)\underbrace{xe^{-5x}\Big|_0^\infty}_{=0} + \frac{1}{2}\int_0^\infty \frac{1}{5}e^{-5x} = \frac{-1}{10}\frac{1}{5}e^{-5x}\Big|_0^\infty$$

$$= \frac{1}{50}$$

Continuous case:

Def. Random variables X and Y are (jointly) continuous if there is a non-negative function $f_{X,Y}(x,y)$ s.t.

$$P[(X,Y)\in A] = \iint_A f_{X,Y}(x,y)\,dx\,dy$$

where A is a 2-dimensional set.

$f_{X,Y}(x,y)$ is a joint df for (X,Y)

If $A = \{(X,Y): X\le x, Y\le y\}$, then

the joint cdf of X,Y is

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,dv$$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x\,\partial y} F_{X,Y}(x,y) = \frac{\partial^2}{\partial y\,\partial x} F_{X,Y}(x,y)$$

Properties of $f_{XY}(x,y)$:

$f_{XY}(x,y) \ge 0$ for all $x,y \in \mathbb{R}$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

Ex.

$$f_{X,Y}(x,y) = \begin{cases} \frac{12}{7}(x^2+xy), & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & , \text{ow} \end{cases}$$

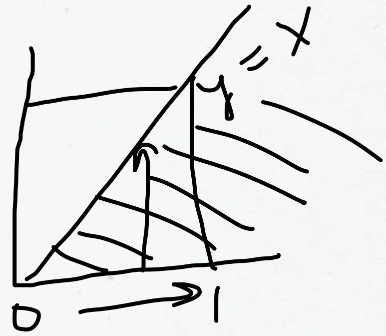
Is it valid?

$$\int_0^1\int_0^1 \frac{12}{7}(x^2+xy)\,dx\,dy = \int_0^1 \frac{12}{7}\left(\frac{x^3}{3}+\frac{x^2}{2}y\right)\Big|_0^1 dy$$

$$= \frac{12}{7}\int_0^1 \left(\frac{1}{3}+\frac{y}{2}\right)dy = \frac{12}{7}\left(\frac{1}{3}y+\frac{1}{4}y^2\right)\Big|_0^1$$

$$= \frac{12}{7}\left(\frac{1}{3}+\frac{1}{4}\right) = \frac{12}{7}\cdot\frac{7}{12} = 1$$

$$P(X>Y) = \int_0^1\int_0^X \frac{12}{7}(X^2+xy)\,dy\,dx$$



$$= \int_0^1 \frac{12}{7}\left(X^2y+\frac{X}{2}y^2\right)\Big|_0^X dX$$

$$= \frac{12}{7}\int_0^1 \left(X^3+\frac{X^3}{2}\right)dX$$

$$= \frac{12}{7}\,\frac{3}{2}\,\frac{1}{4}X^4\Big|_0^1 = \frac{\cancel{12}}{7}\cdot\frac{3}{\cancel{8}_2} = \frac{9}{14}$$

Properties of Joint Dist'n Function

For r.v's X, Y, $F_{XY}: \mathbb{R}^2 \to [0,1]$ is given by $F_{XY}(x,y) = P(X \le x, Y \le y)$

$$\lim_{\substack{x \to -\infty \\ y \to -\infty}} F_{XY}(x,y) = 0$$

$$\lim_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = 1$$

$F_{XY}(x,y)$ is right cont. and non-decreasing in each variable, i.e.

$F_{XY}(x_1,y_1) \le F_{XY}(x_2,y_2)$ if $x_1 \le x_2$, $y_1 \le y_2$

$$\lim_{x \to \infty} F_{XY}(x,y) = F_Y(y), \quad \lim_{y \to \infty} F_{XY}(x,y) = F_X(x)$$

Marginal Densities and Dist'n Functions.

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} \int_{-\infty}^{\infty} f_{X,Y}(s,y)\, dy\, ds$$

$$f_X(x) = F_X'(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

Ex. $f_{X,Y}(x,y) = \begin{cases} 6xy^2, & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & ow \end{cases}$

Is it valid?

$$\int_0^1\int_0^1 6xy^2\,dx\,dy = \int_0^1 3x^2\Big|_0^1 y^2\,dy$$

$$= y^3\Big|_0^1 = 1$$

$$P\left(X \le \tfrac{1}{2},\ Y \le \tfrac{1}{2}\right) = ?$$

$$F(x,y) = \int_0^x\int_0^y 6ts^2\,dt\,ds, \quad 0 \le x \le 1,\ 0 \le y \le 1$$

$$P\left(X \le \tfrac{1}{2},\ Y \le \tfrac{1}{2}\right) = F\left(\tfrac{1}{2},\tfrac{1}{2}\right) = \int_0^{1/2}\int_0^{1/2} 6ts^2\,dt\,ds$$

$$= \int_0^{1/2} 3s^2 \cdot \frac{1}{4}\,ds = \frac{1}{4}\cdot\frac{1}{8} = \frac{1}{32}$$

$$P\left(X \le 2,\ Y \le \tfrac{1}{2}\right) = \int_0^1\int_0^{1/2} 6ts^2\,dt\,ds = \frac{1}{8}$$

$$= P\left(Y \le \tfrac{1}{2}\right)$$

$$f_X(x) = \int_0^1 6xy^2\,dy = 2x, \quad 0 \le x \le 1$$

$$f_Y(x) = \int_0^1 6xy^2\,dx = 3y^2, \quad 0 \le y \le 1$$

$$P\left(Y \le \tfrac{1}{2}\right) = \int_0^{1/2} 3y^2\,dy = y^3\Big|_0^{1/2} = \frac{1}{8}$$

Generalization to higher dimensions.

Let X, Y, Z ~~be~~ jointly continuous r.v's with density $f(x,y,z)$ then

- Marginal density of X is given by:

$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dy\,dz$$

- Marginal density of X, Y is given by:

$$f_{X,Y}(x,y) = \int_{-\infty}^{\infty} f(x,y,z)\,dz$$

Ex. Given $F_{X,Y}(x,y) = x^2y + y^2x - x^2y^2$, $\begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}$

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) =$$

$$= \frac{\partial^2}{\partial x \partial y}\left(\underline{x^2y + y^2x - x^2y^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\underline{x^2 + 2yx - 2x^2y}\right)$$

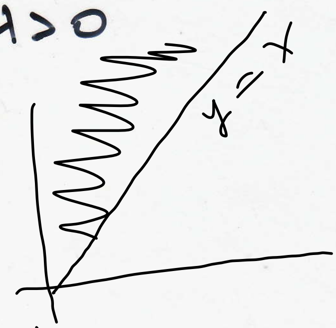
$$= 2x + 2y - 4xy, \quad \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix}$$

$$f_X(x) = \int_0^1 (2x + 2y - 4xy)\,dy = \left[2xy + y^2 - 2xy^2\right]_0^1$$
$$= 1, \quad 0 \le x \le 1$$

$$f_Y(y) = \int_0^1 [2x + 2y - 4xy]\,dx = (x^2 + 2yx - 2x^2y)\Big|_0^1$$
$$= 1, \quad 0 \le y \le 1$$

Ex.

$$f_{X,Y}(x,y)=\begin{cases}\lambda^2 e^{-\lambda y}, & 0\le x\le y\\ 0, & ow\end{cases}, \quad \lambda>0$$



Is it valid?

$$\int_0^\infty\int_0^y \lambda^2 e^{-\lambda y}\,dx\,dy=\int_0^\infty \lambda^2 y e^{-\lambda y}\,dy$$

$$=\begin{bmatrix} u=y & e^{-\lambda y}dy=dv\\ du=dy & v=-\frac{1}{\lambda}e^{-\lambda y}\end{bmatrix}$$

$$=\underbrace{-\lambda y e^{-\lambda y}\Big|_0^\infty}_{=0}+\lambda\int_0^\infty e^{-\lambda y}\,dy$$

$$=\lambda\left(-\frac{1}{\lambda}\right)e^{-\lambda y}\Big|_0^\infty=1$$

$$f_X(x)=\int_x^\infty \lambda^2 e^{-\lambda y}\,dy=-\lambda e^{-\lambda y}\Big|_x^\infty$$

$$=\lambda e^{-\lambda x},\ x\ge 0$$

$\exp(\lambda)$

$$f_Y(y)=\int_0^y \lambda^2 e^{-\lambda y}\,dx=\lambda^2 y e^{-\lambda y},\ y\ge 0$$

$\text{Gamma}(2,\lambda)$