STA447 Homework #3 — Rui Qiu #999292509

Problem 1

$p=\frac{1}{2}$, $a=5$, $c=20$. $T=\inf\{n\geq 0; X_n=0 \text{ or } c\}$, $U=T-1$

(a). Solution:

$E(X_T)=c\cdot P(X_T=c)+0\cdot P(X_T=0)=c\cdot P(X_T=c)$

Since when $p=\frac{1}{2}$, Gambler's Ruin has the following equality:

$$S(a)=\frac{a}{c}=P_a(T_c<T_0)=P(X_T=c)=\frac{1}{4}$$

Therefore $E(X_T)=20\cdot\frac{1}{4}=5$

(b). Solution:

If $X_T=c$, we know $U=T-1$; so $X_U=c-1$

If $X_T=0$, $U=T-1$, $X_U=1$

From (a) we have $P(X_T=c)=\frac{1}{4}$, then $P(X_T=0)=1-\frac{1}{4}=\frac{3}{4}$

This is equivalent as $P(X_U=c-1)=\frac{1}{4}$, $P(X_U=1)=\frac{3}{4}$

So $E(X_U)=(c-1)\cdot P(X_U=c-1)+1\cdot P(X_U=1)$

$=19\cdot\frac{1}{4}+\frac{3}{4}$

$=\frac{22}{4}=5.5$

Solution:

(c). The Optional Stopping Corollary states: If $\{X_n\}$ is martingale with stopping time T (i.e. $\exists M<\infty$ with $P(|X_n|1_{n\leq T}\leq M)=1\ \forall n$), and $P(T<\infty)=1$, then $E(X_T)=E(X_0)$.

So in this case, T is stopping time

$$\begin{cases}|X_n|1_{n\leq T}\leq c\\ P(T<\infty)=1 \text{ since } P(T>Mc)\leq(1-p^c)^M\to 0 \text{ as } M\to\infty\end{cases}$$

- Therefore, $E(X_T)=E(X_0)$

- But $E(X_U)\neq E(X_0)$ because $U=T-1$ is not a stopping time. It depends on "future", which is not valid as a stopping time.

Problem 2

$\{X_n\}$ simple symmetric random walk, $X_0=0$, $S=\inf\{n\geq 0 : X_n=-5\}$

Proof: We claim that $T_M=\min(S,M)\leq M<\infty$
is a valid stopping time as T_M is bounded.
(and not looking into "future")

- Also, we know that $\{X_n\}$ is a simple symmetric random walk $\Rightarrow$ $\{X_n\}$ is a martingale.

So now we can apply Optional Stopping Lemma safely:

$E(T_M)=E(X_0)=0$

Then $E(T_M)=E(T_M|S\leq M)\cdot P(S\leq M)+E(T_M|S>M)\cdot P(S>M)=0$

$E(T_M|S\leq M)=-5$

$E(T_M|S>M)=E(X_M|S>M)$

Since $M\to\infty$, $P(S>M)\to 0$

Therefore $\lim_{M\to\infty} E(X_M|S>M)=0$. (if not then $E(T_M)=-5\cdot P(S\leq M)$
≤ 0
where $P(S\leq M)\in[0,1]$,
but we've already know
$E(T_M)=0$, contradiction)

∎

Problem 3

$\{X_n\}$ MC, $S=\{1,2,3,\cdots,100\}$. $\nu_{30}=\nu_{40}=\frac{1}{2}$. $\begin{cases} P_{1,1}=P_{100,100}=1 \\ P_{99,100}=P_{99,98}=\frac{1}{2} \\ P_{i,i-1}=\frac{2}{3}, 2\le i\le 98 \\ P_{i,i+2}=\frac{1}{3} \end{cases}$

$T=\inf\{n\geq 0: X_n=1, \text{ or } 100\}$

(a). Solution:

$P(X_2=41)=\nu_{40}P_{40,39}P_{39,41}+\nu_{40}\cdot P_{40,42}P_{42,41}$

$=\frac{1}{2}\times\frac{2}{3}\times\frac{1}{3}+\frac{1}{2}\times\frac{1}{3}\times\frac{2}{3}$

$=\frac{2}{9}$

② If $i=1$,
$\sum_j P_{1j}=P_{1,1}=1$

③ If $i=100$,
$\sum_j P_{100,j}=\cancel{100}P_{100,100}=100$

④ If $i=99$
$\sum_j P_{99j}=100\cdot\frac{1}{2}+98\cdot\frac{1}{2}=99$.

(b). Solution:

Check for martingale: ① If $i\in[2,98]$

$\sum_j P_{ij}=(i-1)P_{i,i-1}+(i+2)P_{i,i+2}=(i-1)\times\frac{2}{3}+(i+2)\times\frac{1}{3}=i$

so it is a martingale by ①②③④.

(c). Solution:

Since $T=\inf\{n\geq 0: X_n=1 \text{ or } 100\}$, $|X_n|1_{n\leq T}\leq 100$

Let $M=100$, so $P(|X_n|1_{n\leq T}\leq M)=1$

And $P(T<\infty)=1$ b/c by part (b), $\{X_n\}$ is martingale

Then by Optional Stopping corollary,

$E(X_T)=E(X_0)=\frac{1}{2}\times 30+\frac{1}{2}\times 40=35$

Also: $E(X_T)=$ ~~[illegible]~~ $P(X_T=1)\cdot 1+P(X_T=100)\cdot 100$

$=P(X_T=1)+100P(X_T=100)=35$

And we also know there are only two possible stopping time

$P(X_T=1)+P(X_T=100)=1$

Solve this linear equation system we get:

$P(X_T=1)=\frac{100-35}{99}=\frac{65}{99}$

Problem 4

$\{B_t\}_{t\geq 0}$ Brownian motion

~~Proof~~: Solution:

$0 \leq 5 < 8$

$$E(B_5 B_8) = Cov(B_5, B_8) = E(B_5[B_8 - B_5 + B_5])$$
$$= E(B_5[B_8 - B_5]) + E(B_5^2)$$
$$= E(B_5)E[B_8 - B_5] + E(B_5^2)$$
$$= 0 \cdot 0 + 5$$
$$= 5$$

$$(B_5 B_8)^2 = (B_8 - B_5)^2 * B_5^2 + 2(B_8 - B_5)B_5^3 + B_5^4$$

$$E(B_5^4) = E[(\sqrt{5}B_1)^4] = 25E(B_1^4) = 25 \cdot 3 = 75 \quad (\text{Since } E(Z^4) = 3)$$

$$E[(B_8 - B_5)^2 B_5^2] = E[(B_8 - B_5)^2]E(B_5^2)$$
$$= E[(\sqrt{3}B_1)^2]E[(\sqrt{5}B_1)^2]$$
$$= 3 \cdot 5 \quad (\text{since } E(Z^2) = 1)$$
$$= 15$$

$$E[2(B_8 - B_5)B_5^3] = 2E[(B_8 - B_5)B_5^3]$$
$$= 2E[B_8 - B_5]E[B_5^3]$$
$$= 2E[\sqrt{3}B_1]E[(\sqrt{5}B_1)^3]$$
$$= 0 \quad (\text{since } E(Z) = E(Z^3) = 0)$$

Hence, $E[(B_5 B_8)^2] = E[B_5^4] + E[(B_8 - B_5)^2 B_5^2] + E[2(B_8 - B_5)B_5^3]$
$$= 75 + 15 + 0$$
$$= 90$$

Therefore: $Var(B_5 B_8) = E[(B_5 B_8)^2] - [E(B_5 B_8)]^2$
$$= 90 - 25$$
$$= 65$$

Problem 5

$\{B_t\}_{t\geq 0}$ Brownian motion

$\theta \in \mathbb{R}$. $Z_t = \exp(\theta B_t - \theta^2 t/2)$

$\{Z_t\}_{t\geq 0}$ is martingale.

Proof:

By lecture notes, if $0<t<s$,

$$B_s \mid B_t = B_t + (B_s - B_t) \mid B_t = B_t + \text{Normal}(0, s-t)$$
$$\sim \text{Normal}(B_t, s-t)$$

(i.e. given B_t, B_s is normal with mean B_t, variance $s-t$)

So $E[Z_s \mid \{B_r\}_{r\leq t}] = \cancel{E[\exp(\theta B_t - \theta^2 t/2)}$

$$= E[\exp(\theta B_s - \theta^2 s \cdot \tfrac{1}{2}) \mid \{B_r\}_{r\leq t}]$$

$$= E[\exp(\theta B_s) \mid \{B_r\}_{r\leq t}] \cdot \exp(-\theta^2 s \tfrac{1}{2})$$

(using the fact)

$$E[\theta \exp(B_s) \mid \{B_r\}_{r\leq s}] = \exp(\theta B_t + \tfrac{1}{2}(s-t)\theta^2)$$

as $B_s \mid B_t \sim N(B_t, s-t)$ when $t<s$.

So $E[Z_s \mid \{B_r\}_{r\leq s}] = \exp(\theta B_t + \frac{1}{2}(s-t)\theta^2) \cdot \exp(-\theta^2 s \frac{1}{2})$

$$= \exp(\theta B_t - \tfrac{1}{2}\theta^2 t)$$

$$= Z_t$$

Now we apply <u>law of iterated expectations</u> from notes.

$E[Z_s \mid \{Z_r\}_{r\leq t}] = \cancel{E[Z_s \mid \{B_r\}_{r\leq t}]}$

$$= E[E[Z_s \mid \{B_r\}_{r\leq t}] \mid \{Z_r\}_{r\leq t}]$$

$$= E[Z_t \mid \{Z_r\}_{r\leq t}]$$

$= Z_t$ (So $\{Z_t\}_{t\geq 0}$ is a martingale.

Problem 6

(a). Proof:

$\because D_t = e^{-rt} X_t$

$= e^{-rt} x_0 \cdot e^{\mu t + \sigma B_t}$

$\because \mu = r - \frac{\sigma^2}{2}$

$\therefore D_t = e^{-rt} \cdot x_0 \cdot e^{\sigma B_t + rt - \frac{\sigma^2}{2} t}$

$= x_0 \cdot \exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$

Now assume $\theta = \sigma$, then we can rewrite $\exp(\sigma B_t - \frac{1}{2}\sigma^2 t)$ as Z_t defined in the previous problem.

so $D_t = x_0 \cdot Z_t$

Then $E[D_s \mid \{D_r\}_{r \le t}] = E[x_0 Z_s \mid \{x_0 \cdot Z_r\}_{r \le t}] = x_0 \cdot Z_t = D_t$
which is consistent with the previous problem.
Hence $\{D_t\}$ is a martingale as well.

(b). Proof: Before this, we claim that for any t, we transform $B_t \sim N(0,t)$ to $B_t = \sqrt{t} X$, $X \sim N(0,$

$E[e^{-rs} \max(0, X_s - K)]$

$= e^{-rs} \int_{-\infty}^{\infty} \max(0, x_0 \cdot e^{\mu s + \sigma \sqrt{s} x} - K) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$

Then $\max(0, x_0 \cdot e^{\mu s + \sigma\sqrt{s} x} - K)$

$$= \begin{cases} x_0 \cdot e^{\mu s + \sigma \sqrt{s} x} - k & \text{if } x \geq \frac{\log(\frac{K}{x_0}) - \mu s}{\sigma \sqrt{s}} \\ 0 & \text{if } x < \frac{\log(\frac{K}{x_0}) - \mu s}{\sigma\sqrt{s}} \end{cases}$$

(by solving $x_0 \cdot e^{\mu s + \sigma\sqrt{s} x} - k \geq 0$)

So $E[e^{-rs}\max(0, X_s-K)]$

$$=e^{-rs}\int_{\frac{\log(\frac{K}{X_0})-\mu s}{\sigma\sqrt{s}}}^{\infty}(X_0\cdot e^{\mu s+\sigma\sqrt{s}x}-K)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx$$

let this be m

$$=e^{-rs}X_0\int_m^\infty e^{\mu s+\sigma\sqrt{s}x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx-Ke^{-rs}\int_m^\infty\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \quad (*)$$

Since $\Phi(u)=\int_{-\infty}^{u}\frac{1}{\sqrt{2\pi}}e^{-v^2/2}dv$ is cdf of standard normal.

$$\text{So } (*)=X_0\int_{-\infty}^{-m}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\cdot e^{(\mu-r)s+\sigma\sqrt{s}x} \quad ①$$

$$-e^{-rs}K\int_{-\infty}^{-m}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx \quad ② \quad (**)$$

Also $-m=\dfrac{\mu s-\log(\frac{K}{X_0})}{\sigma\sqrt{s}}$, $\mu=r-\dfrac{\sigma^2}{2}$

$$\text{So } -m=\frac{(r-\frac{\sigma^2}{2})s-\log(\frac{K}{X_0})}{\sigma\sqrt{s}}$$

So ② $=-e^{-rs}K\Phi(-m)$

① $=\cancel{X_0\int_{-\infty}^{-m}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}dx\cdot e^{-\frac{\sigma^2}{2}s+\sigma\sqrt{s}x}}$

$$=e^{(\frac{1}{2}\sigma^2+\mu-r)s}X_0\int_{-\infty}^{-m+\sqrt{s}\sigma}e^{-x^2/2}dx$$

$$=\cancel{e^{(\frac{1}{2}\sigma^2+\mu-r)s}}X_0\int_{-\infty}^{-m+\sigma\sqrt{s}}e^{-x^2/2}dx\cdot e^{(\frac{1}{2}\sigma^2+\mu-r)s}$$

$$=X_0\Phi(-m+\sigma\sqrt{s})$$

Now we combine ①+②:

$$E[e^{-rs}\max(0,X_s-K)]=X_0\Phi\left(\frac{(r+\frac{\sigma^2}{2})s-\log(\frac{K}{X_0})}{\sigma\sqrt{s}}\right)-e^{-rs}K\Phi\left(\frac{(r-\frac{\sigma^2}{2})s-\log(\frac{K}{X_0})}{\sigma\sqrt{s}}\right)$$

Problem 7

(a). Solution: Goal: HTHT

$S = \{0, 1, 2, 3, 4\}$

transition probability matrix

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix}$$

$\{\pi_i\}$ is

Assume π_i, $i = \{0,1,2,3,4\}$ stationary:

then $\sum_{i \in S} \pi_i P_{ij} = P_j$

$$\begin{cases} \frac{1}{2}\pi_0 + \frac{1}{2}\pi_2 + \frac{1}{2}\pi_4 = \pi_0 \\ \frac{1}{2}\pi_0 + \frac{1}{2}\pi_1 + \frac{1}{2}\pi_3 + \frac{1}{2}\pi_4 = \pi_1 \\ \frac{1}{2}\pi_1 = \pi_2 \\ \frac{1}{2}\pi_2 = \pi_3 \\ \frac{1}{2}\pi_3 = \pi_4 \end{cases}$$

Solve this system we get $\pi_0 = \frac{1}{4}$, $\pi_1 = \frac{2}{5}$, $\pi_2 = \frac{1}{5}$, $\pi_3 = \frac{1}{10}$, $\pi_4 = \frac{1}{20}$

So expected value of the number of flips before seeing "HTHT" is $\frac{1}{\frac{1}{20}} = 20$.

(b). Solution: Similarly, write P first.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \end{pmatrix} \qquad \sum_{i \in S} \pi_i p_{ij} = P_j \Rightarrow \begin{cases} \pi_0 = \frac{1}{6} \\ \pi_1 = \frac{4}{9} \\ \pi_2 = \frac{2}{9} \\ \pi_3 = \frac{1}{9} \\ \pi_4 = \frac{1}{18} \end{cases}$$

$\therefore$ Expected value of number of flips before seeing "HTTH" is $1/\frac{1}{18} = 18$.

Problem 8

$\{N(t)\}_{t\geq 0}$ Poisson process. $\lambda = 3$

(a). Solution:

$$P[N(6)=2 \mid N(8)=4]$$

$$= \frac{P[N(6)=2,\ N(8)-N(6)=2]}{P[N(8)=4]}$$

$$= \frac{e^{-6\lambda}\frac{(6\lambda)^2}{2!} \cdot e^{-2\lambda}\frac{(2\lambda)^2}{2!}}{e^{-8\lambda}\frac{(8\lambda)^4}{4!}}$$

$$= \frac{\frac{36\times 4}{2\cdot 2}}{\frac{64^2}{24}} = \frac{36\times 4}{2\cdot 2} \times \frac{24}{64^2} = \frac{27}{128}$$

(b). Solution:

$$P[N(6)=2 \mid N(8)=4,\ N(3)=1]$$

$$= \frac{P[N(6)=2,\ N(8)=4,\ N(3)=1]}{P[N(8)=4,\ N(3)=1]}$$

$$P[N(8)=4, N(3)=1] = P[N(3)=1, N(8)-N(3)=3]$$

$$= e^{-3\lambda}\cdot\frac{(3\lambda)^1}{1!}\cdot e^{-5\lambda}\cdot\frac{(5\lambda)^3}{3!} = A$$

$$P[N(3)=1, N(6)=2, N(8)=4] = P[N(3)=1, N(6)-N(3)=1, N(8)-N(6)=2]$$

$$= e^{-3\lambda}\cdot\frac{3\lambda}{1!}\cdot e^{-3\lambda}\cdot\frac{3\lambda}{1!}\cdot e^{-2\lambda}\cdot\frac{(2\lambda)^2}{2!} = B$$

So $P[N(6)=2 \mid N(8)=4, N(3)=1] = \dfrac{B}{A} = \dfrac{e^{-8\lambda}(3\lambda\cdot 3\lambda\cdot 4\lambda^2\cdot\frac{1}{2})}{e^{-8\lambda}(3\lambda\cdot 125\lambda^3\cdot\frac{1}{6})}$

$$= \frac{36}{125}$$

(c). Solution:

$$E([N(8)-N(5)][N(7)-N(2)])$$
$$= E([N(8)-N(5)][N(7)-N(5)+N(5)-N(2)])$$
$$= E([N(8)-N(5)][N(7)-N(5)]) + \underbrace{E([N(8)-N(5)][N(5)-N(2)])}$$

But note that $N(5)-N(6)$ and $N(5)-N(2)$ are independent.

$$E([N(8)-N(5)][N(7)-N(5)])$$
$$= E([N(8)-N(7)+N(7)-N(5)][N(7)-N(5)])$$
$$= \underbrace{E([N(8)-N(7)][N(7)-N(5)])} + E([N(7)-N(5)]^2)$$

Similarly, $N(8)-N(7)$ and $N(7)-N(5)$ are independent

~~Therefore,~~ $\cancel{E([N(8)-N(5)][N(7)-N(2)])}$

$$\cancel{= E([N(7)-N(5)]^2)}$$

$\cancel{\Rightarrow}$

So now we know $Y \sim Poisson(m)$, $E(Y) = Var(Y) = m$.

then

$$E([N(8)-N(5)][N(5)-N(2)])$$
$$= E(N(8)-N(5)) \cdot E(N(5)-N(2))$$
$$= 3\lambda \cdot 3\lambda$$
$$= 9\lambda^2 = 81$$

$$E([N(8)-N(7)][N(7)-N(5)])$$
$$= E(N(8)-N(7))\, E(N(7)-N(5))$$
$$= \lambda \cdot 2\lambda$$
$$= 2\lambda^2 = 18$$

Therefore. $E([N(8)-N(5)][N(7)-N(2)])$

$$= 9\lambda^2 + 2\lambda^2 + E([Pois(2\lambda)]^2)$$
$$= 11\lambda^2 + 4\lambda^2 + 2\lambda$$
$$= 15\lambda^2 + 2\lambda$$
$$= 141$$

Problem 9

$\{X(t)\}_{t\geq 0}$ continuous ~~...~~-time Markov process

$S=\{1,2,3\}$, $t\downarrow 0$

$$P^{(t)}=\begin{pmatrix} 1-7t & 7t & 0 \\ 0 & 1-3t & 3t \\ t & 2t & 1-3t \end{pmatrix}+o(t)$$

(a). Solution:

$$p_{ij}^{(t)}=\delta_{ij}+t\,g_{ij}$$

where $g_{ij}=\lim_{t\downarrow 0}\dfrac{p_{ij}^{(t)}-\delta_{ij}}{t}=p'_{ij}(0)$. $\delta_{ij}=p_{ij}^{(0)}=\begin{cases}1 & ,i=j\\ 0 & ,i\neq j\end{cases}$

So $g_{11}=-7$, $g_{12}=7$, $g_{13}=0$,

$g_{21}=0$, $g_{22}=-3$, $g_{23}=3$

$g_{31}=1$, $g_{32}=2$, $g_{33}=-3$

$$\therefore G=\begin{pmatrix} -7 & 7 & 0 \\ 0 & -3 & 3 \\ 1 & 2 & -3 \end{pmatrix}$$

(b). Solution:

We want to show $\sum_{i\in S}\pi_i g_{ij}=0 \quad \forall j\in S$

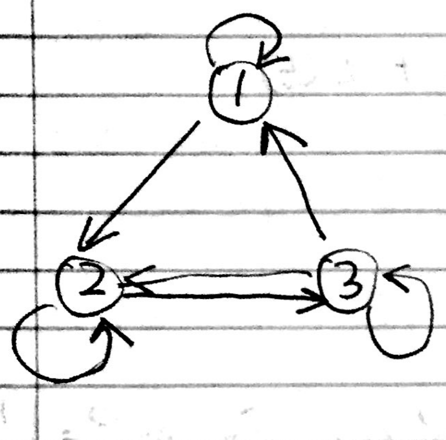
$$\text{So}\begin{cases} -7\pi_1+\pi_3=0 \\ 7\pi_1-3\pi_2+2\pi_3=0 \\ 3\pi_2+(-3\pi_3)=0 \\ \pi_1+\pi_2+\pi_3=1 \end{cases} \Rightarrow \begin{cases} \pi_1=\frac{1}{15} \\ \pi_2=\pi_3=\frac{7}{15} \end{cases}$$

So $\pi=\{\frac{1}{15},\frac{7}{15},\frac{7}{15}\}$ is a stationary dist'n ~~process~~ for this Markov process.

(c) Proof:

Irreducible:

As $t \downarrow 0$ but $t>0$, we can know from $P=\begin{pmatrix} 1-7t & 7t & 0 \\ 0 & 1-3t & 3t \\ t & 2t & 1-3t \end{pmatrix}$



Although t is small, but probabilities like $t, 2t \cdots$ are still not 0, so $i \to j$ for all $i,j \in S$. Then this continuous-time MC is still irreducible.

Stationary dist'n: This is proved in part (b).

By continuous-time Markov convergence Theorem:

$$\lim_{t\to\infty} p_{ij}^{(t)} = \pi_j \quad \forall i,j \in S.$$