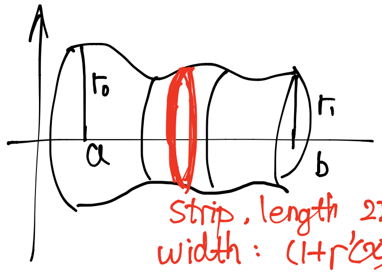
Lecture 8

"Infinite-dimensional minimization problems"

Example: find a surface of revolution around interval $a \le x \le b$ with given radius at $x=a$, $x=b$, and minimal area.



$r(x)$ = radius at x

$\left.\begin{array}{l} r(a)=r_0 \\ r(b)=r_1 \end{array}\right\}$ given

Strip, length $2\pi r(x)$

width: $(1+r'(x)^2)^{\frac{1}{2}} dx$

Goal: find $r(x)$, $a \le x \le b$, such that $r(a)=r_0$, $r(b)=r_1$ and minimizing area

$$A[r()] = \int_a^b 2\pi r(x)(1+r'(x)^2)^{\frac{1}{2}} dx$$

"infinite-dimensional" because not finite-dimensional

General Problem:

given a function $L(x, v)$, x = position, v = velocity

find a function $y(t)$, $a \le t \le b$ minimizing $\boxed{\int_a^b L(y(t), y'(t))\,dt}$

With additional conditions such as $y(a)$, $y(b)$ given

This is an example of "the calculus of variations"

<u>Note</u>: from now on, following online lecture notes of Evans

Basic theorem: (1st order necessary condition)

Assume that $y^*: (a,b) \to \mathbb{R}$ is C^2 and satisfies

$I[y^*(\cdot)] \le I[y(\cdot)]$ for all functions $y(a,b) \to \mathbb{R}$ s.t. $y(a)=y^*(a)$, $y(b)=y^*(b)$

Where $I[y(\cdot)] = \int_a^b L(y(t), y'(t))\,dt$

<u>Then</u> y^* satisfies the equation, so

$$-\frac{d}{dt}\left[L_v(y^*(t), y^{*\prime}(t))\right] + L_x(y^*(t), y^{*\prime}(t)) = 0$$

where $L_v = \frac{\partial L}{\partial v}$, $L_x = \frac{\partial L}{\partial x}$

Idea: reduce to 1st year calculus

Fix C^2 function $z : [a,b] \to \mathbb{R}$ s.t. $z(a) = z(b) = 0$.

Define $i(s) = I[y^*(\cdot) + s z(\cdot)]$, $s \in \mathbb{R}$

We'll show:

① $i'(0) = 0$

② this implies the conclusion

Proof of ①: $\forall s$, $i(s) = I[y^*(\cdot) + s z(\cdot)] \geq I[y^*(\cdot)] = i(0)$

because of optimality of y^*

So: $s=0$ is a global minimum pt of i.

So by calculus $i'(0) = 0$

②a Computation of i'

$$i(s) = \int_a^b L(y^*(t) + s z(t), y^{*\prime}(t) + s z'(t))\, dt$$

$$\text{so } i'(s) = \int_a^b \frac{d}{ds} L(\cdots)\, dt = \int_a^b \left[L_x(\cdots)\, z(t) + L_v(\cdots) z'(t) \right] dt$$

Integration by parts:

$$\int L_v(\cdots) z'(t)\, dt = L_v(\cdots) z \Big|_{t=a}^{t=b} - \int \frac{d}{dt}\left(L_v(\cdots)\right) z(t)\, dt$$

Combine the above to find:

$$0 = i'(0) = \int_a^b \overbrace{\left[L_x(y^*(t), y^{*\prime}(t)) - \frac{d}{dt} L_v(y^*(t), y^{*\prime}(t)) \right]}^{w(t)} \cdot z(t)\, dt$$

This is true for any C^2 function z as above

②b finish the proof:

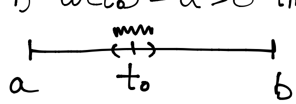
In view of (3), we only need to show that if $w : (a,b) \to \mathbb{R}$ is continuous and

if $\int_a^b w(t) z(t)\, dt = 0$ for all C^2 fcns z, $z(a) = z(b) = 0$

then $w(t) = 0$ for all $t \in [a,b]$.

sketch of proof:

if $w(t_0) = \alpha > 0$ then continuity $\Rightarrow w(t) \geq \frac{\alpha}{2}$ in an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$



We can choose a C^2 function z as pictured

$z \geq 0$ everywhere,

$z = 0$ outside interval $(t_0 - \varepsilon, t_0 + \varepsilon)$,

$z > 1$ in sub-interval $(t_0 - \frac{\varepsilon}{2}, t_0 + \frac{\varepsilon}{2})$

$$\text{Then } \int_a^b w(t) z(t)\, dt \geq \int_{t_0 - \frac{\varepsilon}{2}}^{t_0 + \frac{\varepsilon}{2}} \frac{\alpha}{2}\, dt = \frac{\alpha \varepsilon}{2} > 0$$

impossible

Thus $w(t)=0$ for all $t\in(a,b)$

$\blacksquare$

Summary:

If y^* minimizes $I[y(\cdot)]=\int_a^b L(y(t),y'(t))dt$ (among functions w/ same boundary values), then

$y(t),\ a\le t\le b$

$$-\frac{d}{dt}(L_v(y^*,y^{*\prime}))+L_x(y^*,y^{*\prime})=0$$ "Euler-Lagrange equation"

Example: area of surface of revolution for function $r(x)$, $a\le x\le b$. we want to minimize

$$L[r(t)]=\int_a^b L(r(x),r'(x))dt$$

$$L(r,v)=2\pi r(1+v^2)^{\frac{1}{2}}$$

What's Euler-Lagrange Eqn?

$L_r=2\pi(1+v^2)^{\frac{1}{2}}$

$L_v=2\pi r=\dfrac{2\pi r v}{(1+v^2)}$

$$-\frac{d}{dt}(2\pi\cdot k\sinh(\tfrac{t}{k}))+2\pi\cosh(\tfrac{t}{k})\overset{?}{=}0$$

So equation is:

$$-\frac{d}{dt}\left(\frac{2\pi r(t)r'(t)}{\sqrt{1+r'(t)^2}}\right)+2\pi(1+r'(t)^2)^{\frac{1}{2}}=0$$

some solutions

$r(t)=k\cosh(\frac{t}{k})$

Note: $r'(t)=\sinh(\frac{t}{k})$

$$(1+r'(t)^2)^{\frac{1}{2}}=\cosh(t/k)$$

This gives us some candidates for area minimizing surface of revolution

Example 2:

$$I[y(\cdot)]=\int_a^b \frac{1}{2}y'(t)^2\,dt$$

Here $L(y,v)=\frac{1}{2}v^2$

$L_y=0$

$L_v=v$

so E-L eqns are $-\frac{d}{dt}(y'(t))=0$

i.e. $y''=0$

<u>Note also:</u> for any functions $y(\cdot)$ and $z(\cdot)$

$$I[z(\ )+y(\ )]=\int\frac{1}{2}(y'+z')^2\,dt=\int\frac{1}{2}y'^2dt+\int y'z'dt+\frac{1}{2}\int z'^2dt$$

If $z(a)=z(b)=0$, can integrate by parts to get $I[z(\cdot),y(\cdot)]=I[y(\cdot)]+\int -y''(t)z(t)dt$ $+$ "$o(z)$"

looks like $f(z+y)=f(y)+\nabla f(y)z+o(z)$

for a function $f: \mathbb{R}^n \to \mathbb{R}$

Based on this parallel, we interpret " $y'' = -\nabla I(y)$ "

Note: for $f: \mathbb{R}^n \to \mathbb{R}$, $\nabla f(x)$ is a vector in $\mathbb{R}^n$, so for $I[\cdot]$: (funs on (a,b)) $\to \mathbb{R}$

$\nabla I(y)$ is a function on (a,b)

More generally we interpret

$$-\frac{d}{dt}\left(L_v(y,y')\right) + L_x(y,y') \text{ is gradient of } I[\cdot] \text{ at } y.$$

Example 3:

Minimize: $I[y(\cdot)] = \int \frac{1}{2} y'^2(t)\,dt$ for $0 \le t \le \pi$, with constants:

$$\boxed{J[y(\cdot)] = \int_0^\pi \frac{y^2}{2}(t)\,dt = 1}$$

$y(0) = y(1) = 0$

Guess: first-order condition should be " $\nabla I[y^*] + \lambda \nabla J[y^*] = 0$ "

(if this is like a finite dimentional problem with equality constraint)

also, we be believe " $\nabla I[y^*] = -y^{*''}$ "

and we guess that $\nabla J[y^*] = y^*$ based on (4)

So we might guess necessary condition is:

$$\boxed{\begin{array}{l} -y^{*''} + \lambda y = 0 \text{ for some } \lambda \\ \int_0^\pi \frac{y^2}{2} = 1 \end{array}}$$

This is in fact true.

Optimal Control

Setup: we have ① a system of ODEs describing a dynamical process with some parameters we can control.

We also have ② a "payoff function" to be maximized.

Goal: choose "control parameters" to maximize the "payoff function"

Example: We own a factory. $x(t)$ = rate of output at time t

$\alpha(t)$ = fraction of output reinvested at time t.

$1-\alpha(t)$ = fraction of output that we consume.

System of ODEs

$$\frac{dx}{dt} = k\alpha(t)x(t)$$

$x(0) = x^0$ initial rate of output

I get to choose $\alpha(t)$

Payoff function

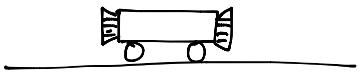
Fix time T and maximize

$$P[\alpha(\cdot)] = \int_0^T (1-\alpha(t))x(t)\,dt = \text{total amount consumed up to time } T$$

depends on choice of α

= total amount consumed up to time T.

Example: 2 rocket railroad car



rockets can apply force $\alpha(t) \in E[-1, 1]$ (in units of force we have chosen)

ODE description:

$q(t)$ = position at time t

$v(t)$ = velocity at time t = $\dot{q}(t)$

Newton's law : $f = ma$

a = acceleration = $\ddot{q}(t) = \dot{v}(t)$

$f = \alpha(t)$ from rockets

So ODE is:

$$\dot{q}(t) = v(t)$$
$$\dot{v}(t) = \frac{\alpha(t)}{m}$$

I'm free to choose $\alpha(t)$

Sps my goal is: get to $\boxed{\begin{matrix} q=0 \\ v=0 \end{matrix}}$ as fast as possible.

If we want to state a maximization problem:

goal: choose $\alpha(\cdot)$ to maximize $P[\alpha] = -\tau$, τ = first time when $q = v = 0$

Geometric analysis of this problem

only consider what happens when $\alpha(t) = 1$ for t in some sub-interval

or

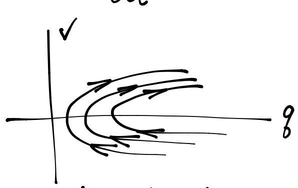
$\alpha(t) = -1$ ... ... (Let's assume $m=1$ for simplicity)

case 1 $\alpha = 1 \Rightarrow$ ODEs are $\dot{q} = v$ (1)

$\dot{v} = 1$ (2)

multiply (1) by (2), $v \cdot \dot{v} = q\dot{v} = \dot{q}$ since $\dot{v} = 1$

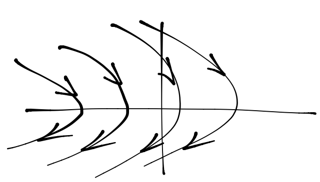
rewrite $\frac{d}{dt}\left(q - \frac{1}{2}v^2\right) = 0$, so $q(t) - \frac{v^2(t)}{2}$ constant as long as $\alpha = 1$



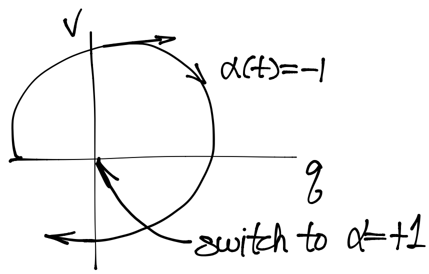
$q - \frac{1}{2}v^2 = \text{const.}$

$v > 0 \Rightarrow \dot{q} = v > 0$

Similarly, $\alpha = -1 \Rightarrow \frac{d}{dt}\left(q + \frac{1}{2}v^2\right) = 0 \Rightarrow q + \frac{1}{2}v^2 = \text{const.}$ as long as $\alpha = -1$



What's a strategy to reach $(q,v)=(0,0)$.



We will see later on that this is in fact optimal.
Also, fact: in investment problem: an optimal strategy is $\alpha(t)=1$ for $0\le t\le \bar{t}$
$\alpha(t)=0$ for $\bar{t}<t\le T$ for some $\bar{t}$
i.e. reinvest all output to time $\bar{t}$, then consume all output after $\bar{t}$.
These are examples of "bang-bang controls"
— common for many problems

We'll start by considering "controllability"
"can I choose control parameters to reach a given destination?"
All these problems involve systems of ODEs. For simplicity, we'll linear ODEs.

ODE $\quad \dfrac{dx(t)}{dt}=Mx(t)+N\alpha(t)$

$\quad x(0)=x^0\in\mathbb{R}^n$

Here $x:[0,\infty)\to\mathbb{R}^n$
$M=n\times n$ matrix
$\alpha:[0,\infty)\to A$: admissible set
for us, $A=[-1,1]^m=\{\alpha\in\mathbb{R}^m : |\alpha_i|\le 1 \text{ for all } i\}$
$N=n\times m$ matrix

I'll always assume my goal is to reach the origin in $\mathbb{R}^n$.

Def: $C(t)$ = "reachable set at time t" = initial conditions x^0 for which there exists a control α s.t. $x(t)=0$ (with α in ODE)
$C=\bigcup_{t\ge 0} C(t)$ = intial conditions x^0 from which I can reach the origin eventually.

So I want to understand can I determine C from knowing M & N.