Lecture 10

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Term Test coverage: Lecture 1–9
including simple proofs.
& cantor set...

Test the ability to use definitions & apply simple theorems.

Borel–Lebesgue theorem

Sequential compactness

Every sequence has a convergent subsequence.

Every open cover has a finite subcover.

Total Boundedness

Metric space is totally bounded if for any $\varepsilon > 0$, there are finitely many points $\{x_1, \cdots, x_k\}$ s.t. $\{B_\varepsilon(x_i)\}_{i \in \{1,\dots,k\}}$ is a cover of (X, ρ)

Borel Lebesgue thm. for a metric space X the following are equivalent.

① X is compact

② Every collection of closed subsets of X with the finite intersection property has nonempty intersection

③ X is sequentially compact

④ X is complete & totally bounded.

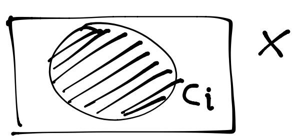
Proof: (1)⇒(2)⇒(3)⇒(4)⇒(1)

(1)⇒(2) Let $\{C_i, i \in \Lambda\}$ be a collection of closed sets that satisfy a finite intersection property.

Suppose $\bigcap_i C_i = \emptyset$

Let $U_i = X - C_i$

$\bigcup_{i \in \Lambda} U_i$



$\bigcup_{i \in \Lambda}(X - C_i) = X - \bigcap_{i \in \Lambda} C_i = X$

$\{U_i, i \in \Lambda\}$ is an open cover of X.

By the def of compactness, any open cover has a finite subcover.

$U_{i_1}, \cdots, U_{i_k}$ $\quad \bigcup_{j=1}^k U_{i_j} = X$

Consider $C_{i_1}, \cdots, C_{i_k}$

$\bigcap_{j=1}^k C_{i_j} = \bigcap_{j=1}^k (X - U_{i_j}) = X - \bigcup_{j=1}^k U_{i_j} = \emptyset$, which is a contradiction to the finite intersection property.

(2)$\Rightarrow$(3)
Let (x_j) be a sequence in X,

$$X_1, X_2, X_3, X_4, \cdots, X_k, \cdots$$
$$X_2, X_3, X_4, \cdots, X_k, \cdots$$
$$X_3, X_4, \cdots, X_k, \cdots$$

$C_n = \overline{\{X_i : i \geq n\}}$

C_n are closed.
Claim. the family of closed sets $\{C_n\}$ satisfy the definite intersection property

$C_{i_1}, \cdots, C_{i_n}$
consider $\max\{i_1, \cdots, i_n\} = N$

$X_N \in C_{i_j} \quad j \in \{1, \cdots, n\}$

By(2) $\bigcap_{i\in\mathbb{N}} C_i \neq 0 \qquad x \in \bigcap_{i\in\mathbb{N}} C_i$

We will construct a subsequence that converges to x.

Let us consider $B_{\frac{1}{k}}(x)$

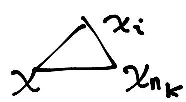
For each k choose x_{n_k}

$n_k > n_{k-1}$ in the $B_{\frac{1}{k}}(x) \cap C_n$ a convergent sequence

(3)$\Rightarrow$(4)
X is sequencial compact $\Rightarrow$ complete & totally bounded.
Completeness.

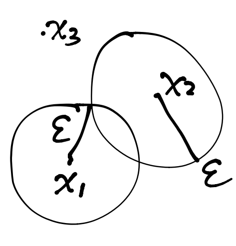
Let (x_i) be a Cauchy seq.
$\Rightarrow$ For any $\varepsilon > 0$, $\exists N$ s.t. $\varphi(x_i, x_j) < \varepsilon/2$ for all $i, j \geq N$
Also, since X is sequentially compact $\exists (x_{n_k})$ s.t. $x_{n_k} \to x$, so far any $\varepsilon > 0$, $\exists K$ s.t. $\varphi(x_{n_k}, x) < \frac{\varepsilon}{2}$ $\forall k \geq K$

$\varphi(x, x_i) \leq \varphi(x_1, x_{n_k}) + \varphi(x_{n_k}, x_i)$
$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$



Now I need to show that X is totally bounded.
Suppose X is not totally bounded.
$\Rightarrow \exists\, \varepsilon > 0$, s.t. there is no finite covering for metric balls of Rad ε

Let $x_1 \in X$
In particular $B_\varepsilon(x_1)$: does not cover x.



Let $x_2 \in X - B_\varepsilon(x_1)$
$B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$ still do not cover X.
Let $x_3 \in X - (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$

$\vdots$

Suppose we have defined. In general, $\{x_1, \cdots, x_{k-1}\}$

$$x_k \in X - \bigcup_{i=1}^{k-1} B_\varepsilon(x_i)$$

(x_k) is a sequence of pts of X.

Claim: (x_k) does not have a convergent subsequence.
Any conv. subsequence has to be, cauchy, however $\varphi(x_{n_i}, x_{n_j}) > \varepsilon$
Contradiction

(4) $\Rightarrow$ (1)
Suppose $\exists$ an open cover that does not have a finite subcover
For each k $\exists$ a finite set of centers X_k s.t. $\bigcup_{k \in X_k} B_{\frac{1}{k}}(x_k) = X$

Now I need to show that X is totally bounded.
Suppose X is not totally bounded.
(1). $\exists y_1 \in X_1$ (for $k=1$, that's X_1)
s.t. $B_1(y_1)$ does not have a finite subcover from sets of $\{U_\alpha\}_{\alpha \in A}$