MAT334 Practice Problems

§1.1.

#1. Let $z=1+2i$, $w=2-i$, $\zeta=4+3i$.

(a) $z+3w = 1+2i+6-3i = 7-i$

(b). $-2w+\bar{\zeta} = -4+2i+4-3i = -i$

(c). $z^2 = 1-4 = -3$

d). $w^3+w = (4-1)(2-i)+2-i = 4(2-i) = 8-4i$

(e). $Re(\zeta^{-1}) = Re\left(\frac{4-3i}{16+9}\right) = \frac{4}{25}$

(f). $\frac{w}{z} = \frac{(2-i)(1-2i)}{1+4} = \frac{2-i-4i+2}{5} = \frac{4}{5} - i$

(g). $\zeta^2+2\bar{\zeta}+3 = 16+24i-9+2(4-3i)+3 = 18+18i$

#2(b, d)

(b). $2z^2+2z+5=0$

$2(z^2+z+\frac{1}{4}+\frac{9}{4})=0$

$(z+\frac{1}{2})^2 = -\frac{9}{4}$

$z+\frac{1}{2} = \pm\frac{\sqrt{-9}}{2} = \pm\frac{3i}{2}$

$z = -\frac{1}{2} \pm \frac{3i}{2}$

d). $z^2-z=1$

$\cancel{z^2-z+\frac{1}{4} = \frac{5}{4}}$

$\cancel{(z-\frac{1}{2})^2 = \frac{5}{4}}$

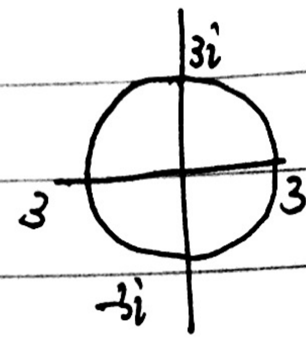
$\cancel{z}$

$z^2-z-1=0$

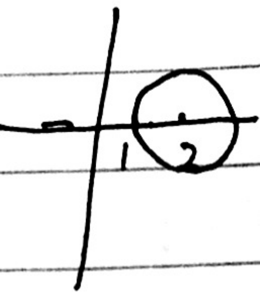
$z = \frac{1\pm\sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$

#3.

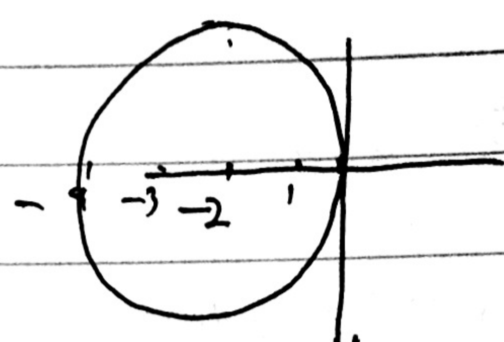
(a). $|w|=3$



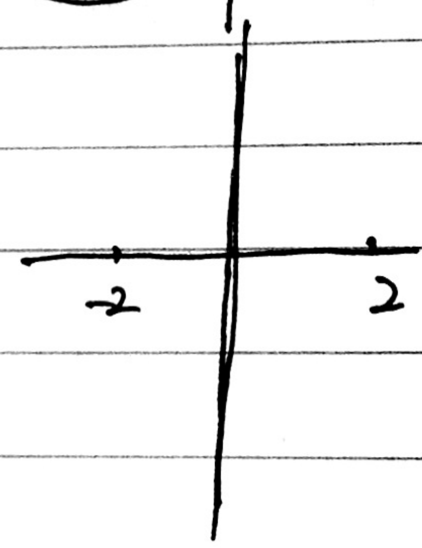
(b). $|w-2|=1$



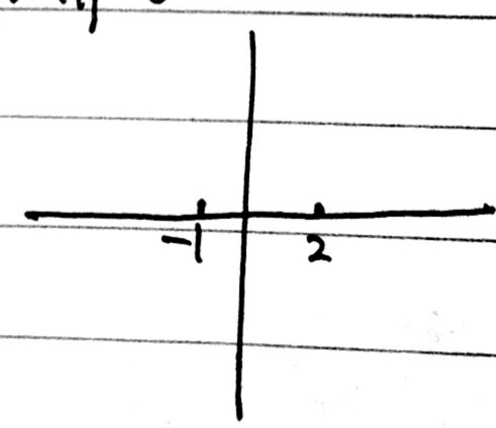
(c). $|w+2|^2=4$



(d). $|w+2|=|w-2|$



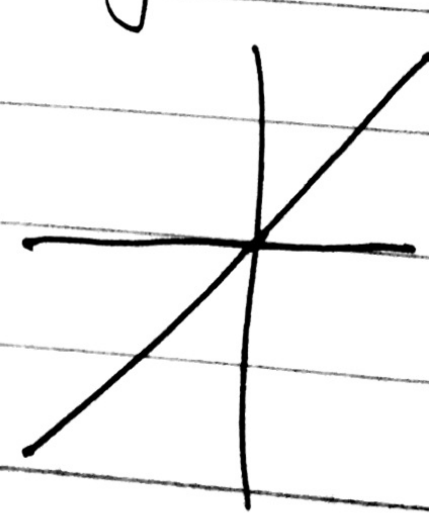
(e). $|w^2-2w-1| = |w-2||w+1|=0$



(f). $\mathrm{Re}[(1-i)\bar{z}]=0$

$(x-iy)(1-i) = x-xi-iy-y = (x-y)+i(-x-y) \Rightarrow$ ~~Re~~

$x=y$

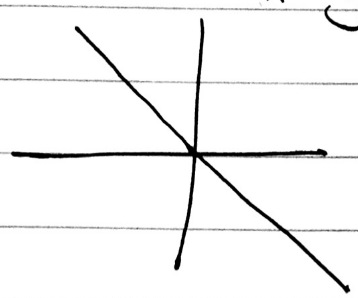


Date No

(g). $Re[z/(1+i)]=0$

$$\frac{(x+iy)(1-i)}{(1+i)(1-i)} = \frac{x+iy-ix+y}{2} \Rightarrow \frac{1}{2}(x+y) + \frac{1}{2}i(y-x)$$

$$x+y=0$$



#4. $z=x+iy$, $Re\left(\frac{1}{z}\right)=Re\left(\frac{x-iy}{x^2+y^2}\right)=\frac{x}{x^2+y^2}$

$Im\left(\frac{1}{z}\right)=\frac{-y}{x^2+y^2}$

$Re(iz)=Re(xi-y)=-y$

$-Im\, z=-y$

$Im(iz)=x=Re\, z$

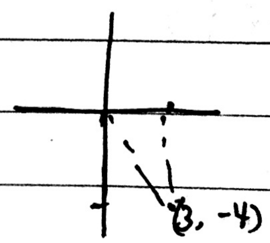
#5. (a). $-1+i=\sqrt{2}\left(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi\right)=\sqrt{2}e^{i\frac{3}{4}\pi}$

(b). $1+i=\sqrt{2}\left(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi\right)=\sqrt{2}e^{i\frac{1}{4}\pi}$

(c). $-i=e^{i\frac{3}{2}\pi}$

(d). $(2-i)^2=4-4i-1=3-4i$

$=5e^{i\arctan\frac{-4}{3}}$



(e). $|4+3i|=5=5e^{i0\pi}$

(f). $\sqrt{5}-i=\sqrt{6}e^{i\arctan\frac{-1}{\sqrt{5}}}$

(g). $-2-2i=2\sqrt{2}e^{i\frac{5}{4}\pi}$

(h). $\sqrt{2}/(1+i)=\frac{\sqrt{2}(1-i)}{2}=\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i=1e^{i\frac{7}{4}\pi}$

(i). $\left[\frac{1+i}{\sqrt{2}}\right]^4 \Rightarrow \frac{1+i}{\sqrt{2}}=\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i=e^{i\frac{1}{4}\pi}$

$$\Rightarrow \left(e^{i\frac{1}{4}\pi}\right)^4=e^{i\pi}=-1$$

#6.

(a). $\sqrt{3}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})$

$=\sqrt{3}\cdot(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})$

$=\frac{\sqrt{6}}{2}+i\frac{\sqrt{6}}{2}$

(b). $(\frac{1}{\sqrt{2}},\pi)$

$=\frac{\sqrt{2}}{2}(\cos\pi+i\sin\pi)$

$=-\frac{\sqrt{2}}{2}$

(c). $(4,-\frac{\pi}{2})$

$=4(\cos-\frac{\pi}{2}+i\sin\frac{\pi}{2})$

$=-4i$

(d). $(2,-\frac{\pi}{4})$

$=2(\cos-\frac{\pi}{4}+i\sin-\frac{\pi}{4})$

$=2(\frac{\sqrt{2}}{2}+i(-\frac{\sqrt{2}}{2}))$

$=\sqrt{2}+(-\sqrt{2}i)$

(e). $(1,4\pi)$

$=1\cdot(1+0)$

$=1$

(f). $(\sqrt{2},\frac{9\pi}{4})$

$=\sqrt{2}(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2})$

$=1+i$

#15. $0, z, w$

$|z-0| = |z|$

$|w-0| = |w|$

$|z-w| = |z| = |w|$!

~~Square them!~~

$\cancel{|z|^2 = |w|^2 = |z-w|^2}$

~~Let $z = a+ib$~~

~~$w = c+id$~~

$\cancel{|z-w|^2}$ $\cancel{(a-c)^2+(b-d)^2 = a^2+b^2 = c^2+d^2}$

$\cancel{a^2-2ac+c^2+b^2-2bd+d^2 = a^2+b^2 = c^2+d^2}$

$\cancel{c=0}$

$\Rightarrow$ if equalateral,

$|z-w| = |z|$

$|z-w|^2 = |z|^2$

$|(a+bi)-(c+di)|^2 = |a+bi|^2$

$(a+c)^2+(b+d)^2 = a^2+b^2$

$a^2+b^2+c^2+d^2+2ac+2bd = a^2+b^2$

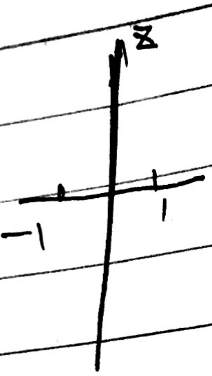
~~so $c=d=0$~~.

$2\,\mathrm{Re}(z\bar{w}) = 2\,\mathrm{Re}((a+bi)\cdot(c-di)) = 2\,\mathrm{Re}(ac+ibc-iad+bd)$

$= 2(ac+bd) =$ ✓

§1.2.

#1. $|z+1| = |z-1|$



$x=0$

#3. $\mathrm{Re}[(4+i)z+6] = 0$

$z = x+iy$

$4x+ix+iy-y+6$

$\Rightarrow (4x-y+6) = 0$

$y = 4x+6$. line

#7. $\mathrm{Re}(z^2) = 4$

$z = x+iy$, $z^2 = x^2+2xyi-y^2 \Rightarrow x^2-y^2 = 4$ hyperbola

#9. $|z^2-1| = 0$

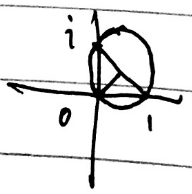
$z^2 = 1$

$z = \pm 1$

#11. $|z-(4+i)| = 2$

#13. $\mathrm{Re}(z+3) = 0$

#15.



$\cancel{|z-p| = \rho|z-q|}$

~~plug in~~

$\cancel{|0-p| = \rho|0-q| \Rightarrow |p| = \rho|q|}$

$\cancel{|1-p| = \rho|1-q|}$

$\cancel{|i-p| = \rho|i-q|}$

$|z - z_0| = \frac{1}{2}$

$\left|z - \left(\frac{1}{2} + \frac{1}{2}i\right)\right| = \frac{1}{2}$

#23. $z^5 = i$

$z^5 = e^{i\frac{\pi}{2}}$

$z = e^{i\frac{\pi}{10}}$

$e^{i\frac{\pi}{10} + \frac{2}{5}\pi}$

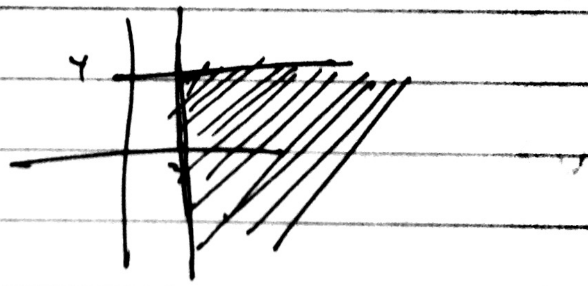
$e^{i\frac{\pi}{10} + \frac{4}{5}\pi}$

$e^{i\frac{\pi}{10} + \frac{6}{5}\pi}$

$e^{i\frac{\pi}{10} + \frac{8}{5}\pi}$

#24. In Hw 1.

§1.3.

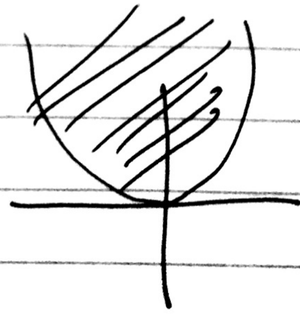


#1. $A = \{z = x + iy : x \geq 2 \,\&\, y \leq 4\}$.

$\text{int} A = \{z = x + iy : x > 2 \,\&\, y < 4\}$

$\partial A = \{z = x + iy : (x = 2 \text{ and } y < 4) \text{ and } (x > 2 \text{ and } y = 4)\}$

closed. connected.



#3. $C = \{z = x + iy : x^2 < y\}$

$\text{int}\, C = C$

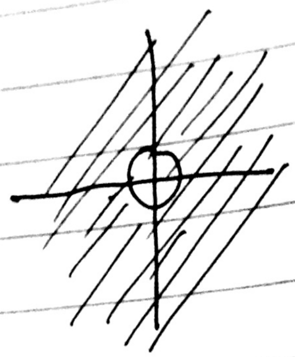
$\partial C = \emptyset$

open, connected.

#5. $E=\{z : z\bar{z}-2\geq 0\}$

$(x+iy)(x-iy)=x^2+y^2-2\geq 0$

$x^2+y^2\geq 2$

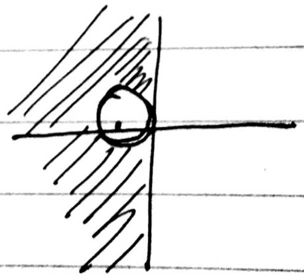


closed, connected

#7.

$G=\{z=x+iy : |z+1|\geq 1 \,\&\, x<0\}$

$(x+1)^2+y^2\geq 1,\ x<0$



~~clos~~ neither closed nor open.

#11. bdd, unbdd

Date No

#13. (a). ~~A, B open~~

~~$A\cup B$,~~

~~$\forall a\in A$, $\exists\varepsilon$ s.t. $B_\varepsilon(a)\in A$~~

~~$\forall b\in B$, $\exists\varepsilon'$ s.t. $B_\varepsilon(b)\in B$~~

~~So $\forall a,b\in A\cup B$, $\exists\varepsilon$ s.t. $B_\varepsilon(a)\in A\cup B$ open.~~
~~$B_\varepsilon(b)\in A\cup B$~~

~~closed: that's to prove the $\overline{A\cup B}=A\cup B$ which is open.~~

~~$\mathbb{C}\setminus(A\cup B)$~~

Sps D_1, D_2 open, let $p\in D_1\cap D_2$. $\exists r_1>0$ s.t. $\{|z-p|<r_1\}\subset D_1$

likewise $\exists r_2>0$ s.t. $\{|z-p|<r_2\}\subset D_2$

Let $r=\min\{r_1,r_2\}$. So $\{|z-p|<r\}\subset D_1\cap D_2$

Let C_1, C_2 closed. Then $\mathbb{C}\setminus(C_1\cup C_2)=(\mathbb{C}\setminus C_1)\cap(\mathbb{C}\setminus C_2)\Rightarrow$ open.

Hence $C_1\cup C_2$ closed.

likewise ~~$\mathbb{C}\setminus(C_1\cup C_2)=$~~

$\mathbb{C}\setminus(C_1\cap C_2)=(\mathbb{C}\setminus C_1)\cup(\mathbb{C}\setminus C_2)$ is open.

So $C_1\cap C_1$ is closed.

~~Open~~

#28.

D is a set. E is closed and $D \subseteq E$

then $\partial D \subseteq E$.

~~Sps not $\partial D \not\subseteq E$~~

~~i.e. $\exists d \in \partial D$~~

~~such that $B_\varepsilon(d)$ contains points outside E for~~

~~some ε.~~

① if D is open, no boundary, or say $\partial D = \emptyset$, and $\emptyset \subseteq E$

done.

② if D has $\partial D \neq \emptyset$.

So $D \subseteq E$, sps $d \in D$, then $d \in E$.

E is closed that means E has all of it boundary point

say ~~d' is on~~ $d' \in \partial D$.

① $d' \in \text{int} E$, then $\partial D \in E$ for sure.

② $d' \in \partial E$, then $\partial D \in E$ b/c, $\partial E \subseteq E$.

③ d' is outside E,

impossible since $d' \in E$.

2 cases ① + ②

so $\partial D \subseteq E$.

§1.4.

#1

$$Z_n=\left(\frac{1+i}{\sqrt{3}}\right)^n=\left(\frac{1}{\sqrt{3}}\right)^n\cdot(1+i)^n \Rightarrow \lim_{n\to\infty} z_n=0$$

#5.

$Z_n=n+\frac{i}{n}$ diverges.

#7. $Z_n=\text{Arg}(1+\frac{\alpha}{n})$ $\lim_{n\to\infty} Z_n=\text{Arg}\,1$

#9. $f(z)=|1-z|^2$

at $z_0=i$ $|1-i|^2=\sqrt{2}$

$$\lim_{z\to z_0} f(z)=f(z_0)$$

#11. $f(z)=(1-\text{Im}\,z)^{-1}$ at $z_0=8$ & $z_0=8+i$

$\cancel{f(z)=}$ $\cancel{f(z_0)=}$ $f(8)=1$

$f(8+i)=$ does not exist

#13. $f(z)=\frac{|z|^2}{z}$, $z\neq 0$, at $z_0=0$.

$$\cancel{\frac{|z|^2}{|z|e^{i\theta}}=\frac{|z|}{e^{i\theta}}}$$

does not exist

b/c it has a jump when

z_0 is approached from

1st & 3rd quadrant.

Date No

#2| $f(z) = \frac{1}{|z|-1} \longrightarrow 0$ as $z \to \infty$

#3| $\sum_{n=1}^{\infty} \left(\frac{1+2i}{\sqrt{6}}\right)^n$

geo series ~~Comparison~~ test : $\left|\frac{1+2i}{\sqrt{6}}\right| = \frac{\sqrt{5}}{\sqrt{6}} < 1$ conv.

#4|. Sps $\sum_{n=1}^{\infty} a_n$ cvg. $|z|<1$,

show $\sum_{n=1}^{\infty} a_n z^n$ is convg.

$\sum_{n=1}^{\infty} a_n$ cvg $\Rightarrow \lim_{n\to\infty} a_n = 0$

$|z|<1$. & $\lim_{n\to\infty} a_n = 0$ so $\sum |a_n z^n|$ convg $\Rightarrow \sum a_n z^n$ convg.