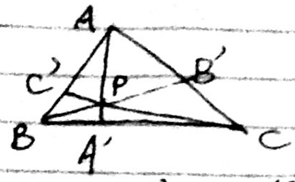
Midterm Review Feb. 20th

I. Prove the following theorems.

Ceva's Theorem.



AA', BB', CC' concurrent

iff $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$

$\Rightarrow$. $\frac{BA'}{A'C} = \frac{\triangle ABA'}{\triangle A'AC}$ ~~so $\frac{\triangle ABA'}{\triangle AA'C} = \triangle AA'B$~~

$= \frac{\triangle APB}{\triangle APC}$

$\frac{CB'}{B'A} = \frac{\triangle BPC}{\triangle APB}$

$\frac{AC'}{C'B} = \frac{\triangle APC}{\triangle BPC}$

So $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = 1$

$\Leftarrow$ Let P be intersection of AA' and BB'

Then connect CP. to AB. let intersection be C''.

so $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC''}{C''B} = 1$

but we assume - - - $\frac{AC'}{C'B} = 1$, so $\frac{AC''}{C''B} = \frac{AC'}{C'B}$

C' is C''.

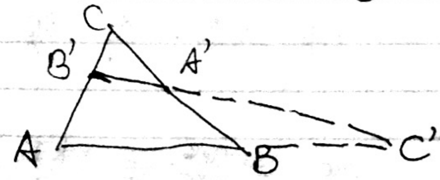
done.

Menelaus's theorem

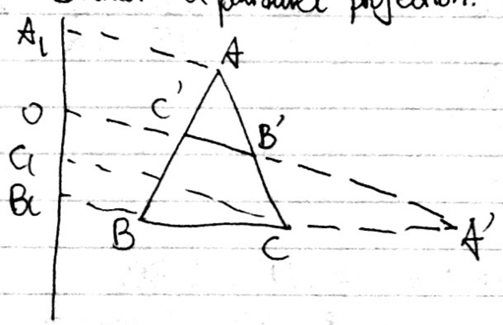
ABC be a triangle and let points A', B', C' on - - -.

Then A', B', C' on a line.

iff $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1$



$\Rightarrow$. consider a parallel projection.



By lemma $\frac{A'B}{A'C} = \frac{OB_1}{OC_1}$ $\frac{B'C}{B'A} = \frac{OC_1}{OA_1}$, $\frac{C'A}{C'B} = \frac{OA_1}{OB_1}$

multiple them $= 1$. done.

$\Leftarrow$ similarly! (like ceva)

sps ——— $= 1$, let C'' be the point on ~~BC~~ AB at which $A'B'$ intersects it.

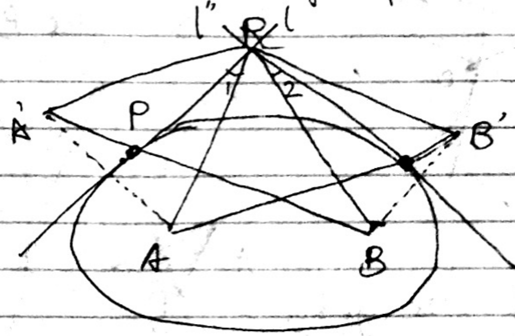
By the proof above we know $\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C''A}{C''B} = 1$

hence $\frac{C''A}{C''B} = \frac{C'A}{C'B}$

So $C'' = C'$. done.

Optical property of ellipse/hyperbola

ellipse: if light ray comes out of a focus A and gets reflected in an elliptical mirror with foci A and B, then its reflection passes through ~~anoth~~ the other ~~foci~~ focus B of the ellipse.



This is a generalization!

$L = AP + BP = A'P + BP$

$\triangle A'RB \cong \triangle ARB'$ (SSS sides)

so $\cancel{\angle 1 = \angle 2}$ $\angle A'RB = \angle ARB'$

$\angle A'RB - \angle ARB$
$= \angle ARB' - \angle ARB$
Both sides divided by 2 $\Rightarrow \angle 1 = \angle 2$

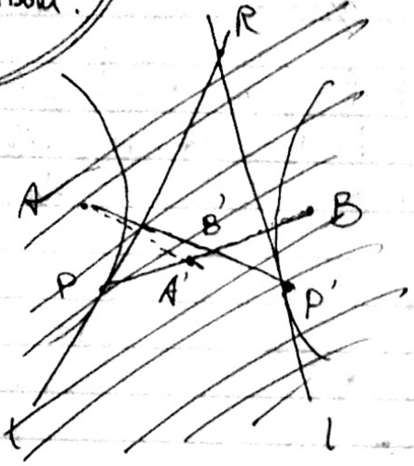
$\Rightarrow$ for any point R outside of the ellipse the angles b/w the segments AR and BR connecting R to the foci & the ~~tangent~~ lines from R to the ellipse are equal.

(Move R to the ellipse, 2 tangent lines $\to$ 1 line)

done.

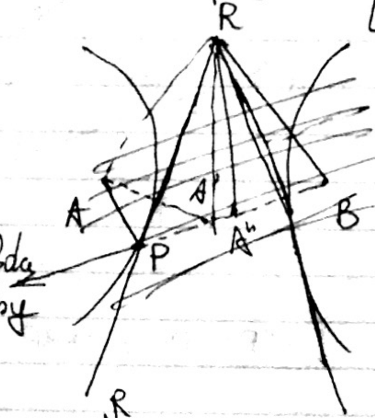
Hyperbola.

see the next few pages.



own property.
light passing one focus A,
reflected by its closest branch,
~~then~~ looks like shine from
the other focus B.

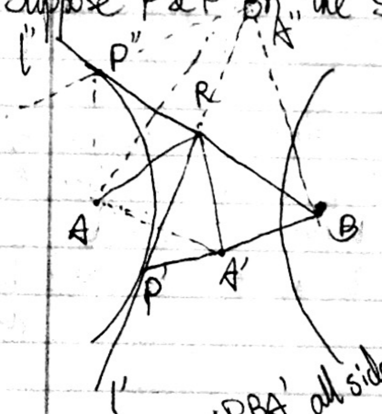
Let $|PA - PB| = L$



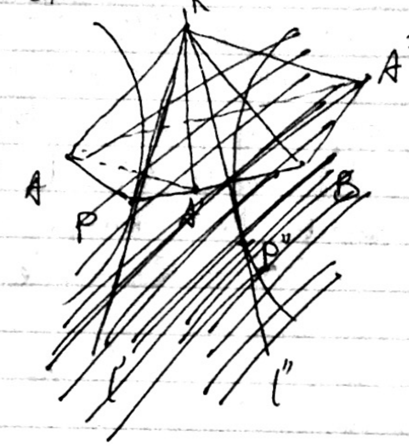
$\triangle RBA'$
$\triangle RBA''$

i.e. for every point P on the hyperbola
the angle $\angle APB$ is bisected by
the tangent line at P.

Suppose P & P' on the same branch.



$\triangle RBA'$ all sides known
$\triangle RBA''$



Separation Theorem:

let p be a point outside a closed convex set δ, then $\exists$ a hyperplane that separates them, i.e. a hyperplane with the property that point p and the set δ lie on different sides of it.

δ is a set. $q \in \delta$

first we need to show $dist(q, p) = dist(p, \delta)$

$dist(p, q) =$

q_i : a seq. of pts of δ s.t.

$\lim_{i \to \infty} dist(p, q_i) = dist(p, \delta)$

$dist(p, q_i) < dist(p, \delta) + 1$

hence $\| q_i \| < \| p + dist(p, \delta) + 1 \|$

$\Rightarrow$ converging subsequence

sps $r \in \delta$, $qr \in \delta$, then ~~if~~ $dist(p, r) > dist(p, q)$

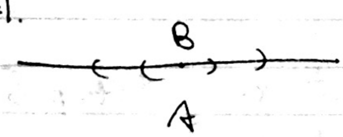
so $\angle pqr$ is obtuse ($> 90°$)

so r is on the same side of H as q.

Radon's theorem

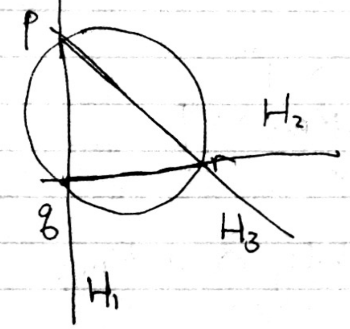
if S is a finite set of at least $d+2$ pts in $\mathbb{R}^d$.
then the set S can be partitioned as a union of 2 subsets
A and B s.t. the convex hulls of A & B intersect non-trivially.

$d=1$.



Convex hull of A concides
Convex hull of B

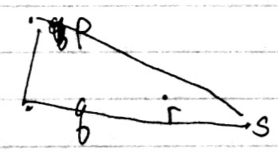
$d=2$, say $p, q, r \in S$.



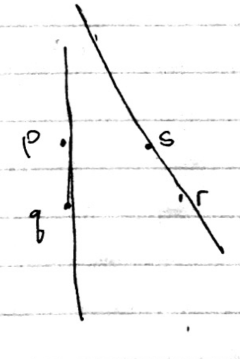
H_1, H_2, H_3 are 3 ~~hyperplanes~~ half-planes.
and Δpqr on the ~~same~~ side of
H_1, H_2, H_3

① $s \in S$ inside H_1, H_2, H_3
So $A = \{p, q, r\}$
$B = A^c = \{s\}$

② $s \in S$ inside one, outside two (WLOG)
$A = \{p, q, s\}$
$B = A^c = \{r\}$



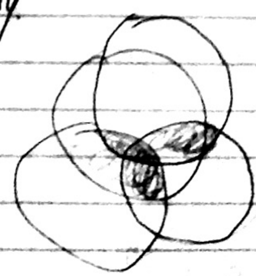
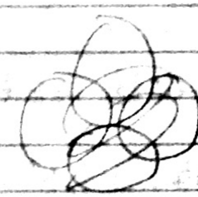
③ $s \in S$ inside two, outside one
(WLOG)



then pq, rs intersect
so. $A = \{p, q\}$
$B = A^c = \{r, s\}$

Helly's theorem. (for 3)

If in a finite collection of convex sets every three intersect, then the intersection of all the sets in the collection is not empty.


$G_1, G_2, G_3, G_4 \to$ convex sets.

Let A_1, A_2, A_3, A_4 be pts in those (3-intersections)

$$\left.\begin{array}{l} A_1 \in G_2 \cap G_3 \cap G_4 \\ A_2 \in G_1 \cap G_3 \cap G_4 \\ A_3 \in G_1 \cap G_2 \cap G_4 \\ A_4 \in G_1 \cap G_2 \cap G_3 \end{array}\right\}$$

by Radon's thm $\Rightarrow$ we can subdivide the collection A_1, A_2, A_3 & A_4 to 2 subsets whose convex hulls intersection

1, 3

① if, say A_1 belongs to the convex hull of A_2, A_3 & A_4

then $A_1 \in G_1$

hence any pt in their convex hull belongs to G_1

since $A_1 \in G_2 \cap G_3 \cap G_4$

hence $A_1 \in G_3 \cap G_2 \cap G_4 \cap G_1$

② similarly if 2 sets, $[A_1, A_2]$ $[A_3, A_4]$ segments intersect at P.

2, 2

$$\left.\begin{array}{l} \text{since } A_1 \in G_3 \cap G_4 \\ A_2 \in G_3 \cap G_4 \end{array}\right\} \Rightarrow \text{every point in their convex hull also in } G_3 \cap G_4.$$

$$\left.\begin{array}{l} \text{So } P \in G_3 \cap G_4 \\ \text{similarly } P \in G_1 \cap G_2 \end{array}\right\} \Rightarrow P \in G_1 \cap G_2 \cap G_3 \cap G_4$$

Sps we proved the thm for every collection of n planar sets & want to prove for n+1
induction. . . .

$\mathbb{R}^3$ simple convex

顶点 × $\frac{3}{2}$ = 棱

$\boxed{f_0 \times \frac{3}{2} = f_1}$

☆ Dehn–Sommerville relations.

$$f(t) = f_0 + f_1 t + \dots + f_n t^n$$

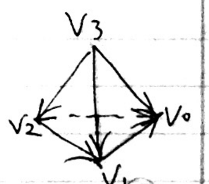
~~$h_0, \dots, h_n$ are coefficients of the~~

$$h(t) = f(t-1) = f_0 + f_1(t-1) + \dots + f_n(t-1)^n$$

~~$h(0) = f(-1) = f_0 - f_1 + \dots + f_n(-1)^n$~~

$$h_0 = f(-1) = 1.$$

Thm: the numbers $h_0, \dots, h_n$ associated to a simple polyhedron satisfy $h_i = h_{n-i}$ (symmetry)
$h_i \geq 1$ (positive)

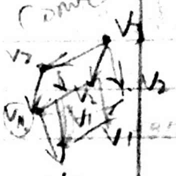


~~Note: def.~~

• every face of P has exactly 1 maximal

$\mathbb{R}^2$ • count pairs (F, v) st. v is the maximal vertex of the m dimensional face F.
so # of pairs is f_m

convex set



The other method is counting # of relevant faces for each vertex first and summing up the results over all vertices.

Sps v is a vertex, has index i, then the number of m-dim faces for which v is the maximal vertex is $\binom{i}{m}$

there are i edges going out of v along L is decreasing & choice of an m-dim face for which v is maximal vertex is equivalent to a choice of m of these edges.

V3 3V2, 3V1 V0

? V1

V1 $= h_1 = h_i$

So, sum the numbers $\binom{i}{m}$ over all vertices of index i and over all indices i. So the number of such pairs is equal to $\sum_{i\geq m}\binom{i}{m}\tilde{h}_i$ (note $\tilde{h}_i$ is # of vertices of index i)

So, $f(t)=\sum_{i=0}^{n} f_i t^i = \sum_{i=0}^{n}\sum_{i\geq m}\binom{i}{m}\tilde{h}_i t^i$

$$=\sum_{0\leq m\leq i\leq n}\binom{i}{m}\tilde{h}_i t^i = \sum_{i=0}^{n}\left(\sum_{m=0}^{i}\binom{i}{m}t^m\right)\tilde{h}_i$$

$$=\sum_{i=0}^{n}\tilde{h}_i(1+t)^i$$

Thus $f(t-1)=\sum_{i=0}^{n}\tilde{h}_i t^i$, or $h_i=\tilde{h}_i$

$\tilde{h}_i$: the number of vertices of index i w.r.t. L

h_i : ~~a~~ - - -

If v has index i w.r.t. the functional L, then its index w.r.t. the functional $-L$ is $n-i$.
Thus the number h_i of vertices of index i w.r.t. functional L is equal to the number h_{n-i} of the ~~verl~~ vertices of index $n-i$ for the functional $-L$.

So $h_i=h_{n-i}$

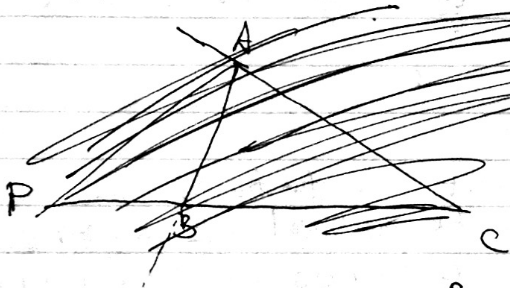
$h_i\geq 1$ is trivial, since at least 1 pt is able to be called as ~~$\tilde{h}$~~ v_i.

II. Solve the following problems

a. Problems on Ceva & Menelaus thm

1). Term test 2008. problem 1

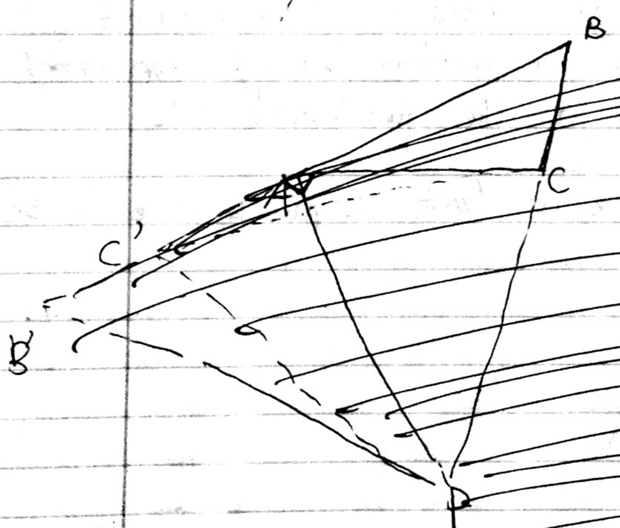
non-isosceles Δ 非等腰.



$\cancel{\frac{PC}{PB}=\frac{AC}{AB}}$ To prove QPR ~~concurrent~~

For P

$\cancel{\frac{PC}{PB}=\frac{AC}{AB}}$



To prove $\frac{PC}{PB}=\frac{AC}{AB}$

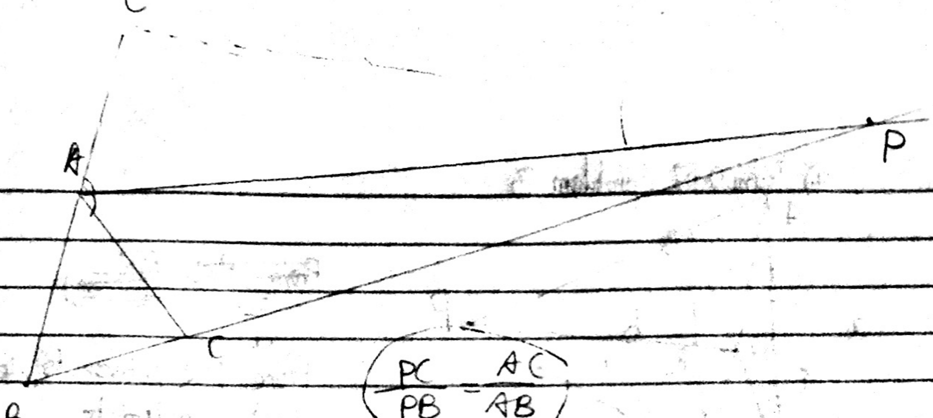
$\frac{PC}{PB}=\frac{AC'}{AB}$

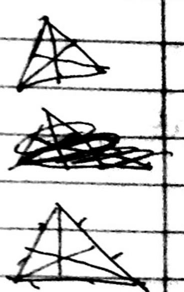
$\frac{PC}{AC'}=\frac{PB}{AB}$

$\cancel{\frac{PC'}{AC'}=\frac{PB}{AB}}$

$\cancel{\frac{PC'}{AC'}=\frac{PB'}{AB'}}$

$\cancel{\frac{PC'}{PB'}=\frac{AC'}{AB'}}$





$$\frac{PC}{PB} = \frac{AC}{AB}$$

$$\frac{PC}{PB} = \frac{\triangle ACP}{\triangle ABP} = \frac{\triangle AC'P}{\triangle ABP} = \frac{AC'}{AB} = \frac{AC}{AB} \quad \text{(WLOG)}$$

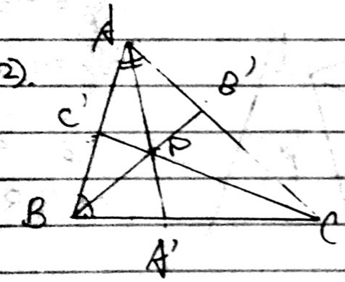
similarly, $\frac{QB}{QA} = \frac{CB}{CA}$

$\frac{RA}{RC} = \frac{BA}{BC}$

So $\frac{PC}{PB} \cdot \frac{QB}{QA} \cdot \frac{RA}{RC} = 1$

So on the same line.

2).



Pf: Suppose P is the intersection of bisectors AA' & BB'

since $dist(p, AB) = dist(p, BC)$

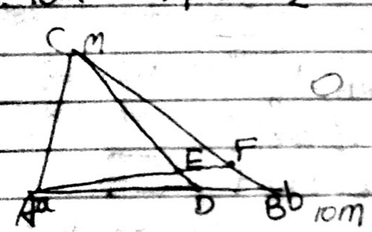
$dist(p, AB) = dist(p, AC)$

So $dist(p, BC) = dist(p, AC)$

Thus P is on the bisector of C.

b. Problems with centre of mass.

3). Term 2010, problem 2



$AD : DB = 10$

$\frac{CE}{ED} = 11$

$\frac{CE}{FB} = ?$

A: m
B: 10m
A+B = 11m
C: $11 \cdot 11 = 121m$

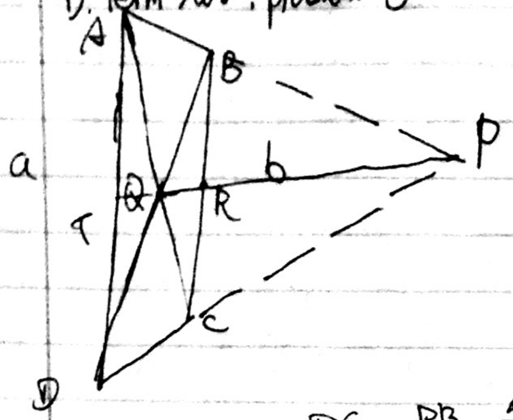
C m

~~$\frac{a}{b} = \frac{10}{1}$, $\frac{c}{a+b} = \frac{1}{11}$, $\frac{c}{b} = ?$~~

~~let b = 1, a = 10, c = 110, $\frac{c}{b} = \frac{110}{1}$ so $\frac{CE}{FB} = 110$~~

~~121m CE = Fb·10m~~ $\Rightarrow \frac{CE}{FB} = \frac{10}{121}$

4). Term 2008, problem 3



~~Prove BC $\frac{BR}{RC} = 1$~~

Prove PQ intersects AD, BC at their midpoints.

$$\frac{DC}{CP} \cdot \frac{PB}{BA} \cdot \frac{AT}{TD} = 1$$

$$\frac{a-b}{b} \cdot \frac{b}{a-b} \cdot \frac{AT}{TD} = 1$$

So $\frac{AT}{TD} = 1$

$$\frac{DC}{CP} = \frac{a-b}{b}$$

$$\frac{PB}{BA} = \frac{\cancel{a-b}\ b}{a-b}$$

So $AT = TD$

Then

$\triangle ATP \sim \triangle BRP$

$\triangle PDT \sim \triangle PCR$

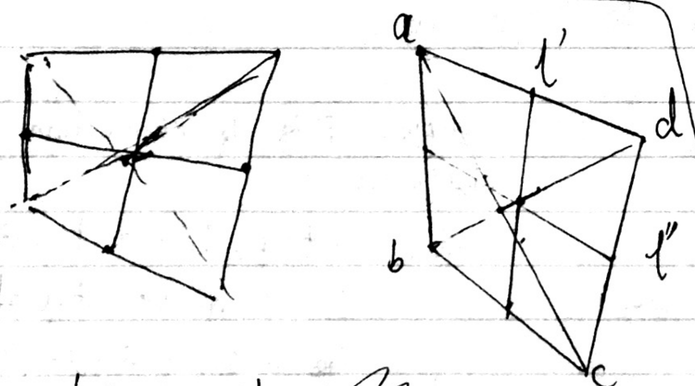
So $\frac{AT}{BR} = \frac{TP}{RP}$

$\frac{DT}{RC} = \frac{TP}{RP}$

So $\frac{AT}{BR} = \frac{DT}{RC}$

$\Rightarrow BR = RC$ done.

5).



l', l'' are two segments joining midpoints

~~$\frac{a}{b} = \frac{?}{?}$~~

~~$\frac{d}{c} = \frac{1}{1}$~~

~~Suppose~~

only 1 center of mass

~~So ... is center of mass~~

~~& a+b = c+d~~

~~a+d = b+c~~

~~a+c~~

~~b+d~~

So ~~a+c, b+d pass through com~~

~~& a+c / b+d = 1~~

~~so it's ... midpoints.~~

ac, bd are diagonal lines

Suppose ~~each~~ each point has same mass m

Then there are 2m on each end of l' and l''

and l intersects with l'' at com with 4m.

Similarly, ac has 2m, bd has 2m.

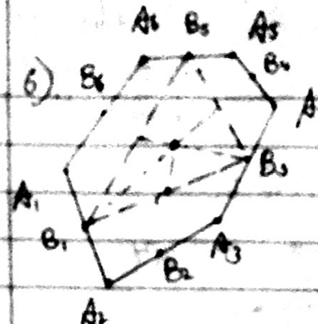
and the mid point of ac is com of ac

— — bd — bd

the third segment from two mid points passes com of 4-gon

so it's also a midpoint of this segment.

6).



Suppose we have equal masses on A_i's

Then

$B_1 = A_1 + A_2$

$B_3 = A_3 + A_4$

$B_5 = A_5 + A_6$

we know $B_1 + B_3 + B_5$ is the center of mass of $\triangle B_1 B_3 B_5$, which is also COM of 6-gon. (since we put all masses on B_1, B_3, B_5)

Similarly it's COM of $\triangle B_2 B_4 B_6$.

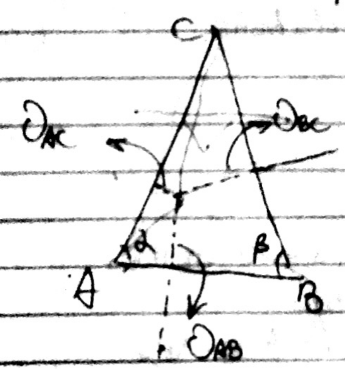
~~is~~ same as ... coincide.

C. Extremal problems.

(7). Sum of distances to some segments with some coefficients.

8. Term 2010, problem 1.

$\triangle ABC$. $\cos\alpha = \cos\beta = \frac{1}{4}$



$\frac{1}{2} O_{AC} \cdot \overline{AC} + \frac{1}{2} O_{BC} \cdot \overline{BC} + \frac{1}{2} O_{AB} \cdot \overline{AB} = \text{fixed} = 1$

want $O_{AC} + 2 O_{AB} + 2 O_{BC}$ big

$O_{AB} = \frac{2 - (O_{AC} + O_{BC})\overline{BC}}{\overline{AB}}$

so want $O_{AC} + \frac{4 - 2(O_{AC} + O_{BC})\overline{BC}}{\overline{AB}} + 2 O_{BC}$

$\cos\alpha = \frac{1}{4}$

$= O_{AC} + \frac{4}{\overline{AB}} - 4(O_{AC} + O_{BC}) + 2 O_{BC}$

So. $\frac{\frac{1}{2}AB}{AC} = \frac{1}{4}$

$= \frac{4}{\overline{AB}} - 3 O_{AC} - 2 O_{BC}$

$2AB = AC$

So O_{AC}, O_{BC} be small, so O is on C.

(nothing to do with O_{AB})

9). ~~Similarly~~, Same idea.

Note: now it's a random $\triangle$ (not isosceles)

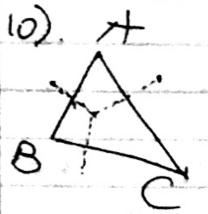
~~Still the same~~!

But note:

$$O_{AB} = \frac{2-(O_{AC}+O_{BC})\overline{BC}}{\overline{AB}}$$

$\Rightarrow$ here we need to split into a few situations ~~that~~ with different values of $\frac{\overline{BC}}{\overline{AB}}$

10). since $O_{AB}+O_{BC}+O_{CA}$ is fixed

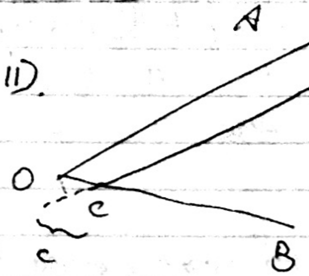


~~sum $4O_{AB}+\dots$ small.~~

~~so $O_{AB}\to$ small~~ $\to 0$.

$$O_A = \frac{2-(O_{AC}+O_{BC})\overline{BC}}{\overline{AB}}$$

11).



$\sin\frac{2\pi}{n}\cdot c$ ✓

$$\frac{8}{\overline{AB}} - \frac{4\overline{BC}}{\overline{AB}}O_{AC} - \frac{4\overline{BC}}{\overline{AB}}O_{BC} + O_{BC} + O_{AC}$$

$$= \frac{8}{\overline{AB}} + \underbrace{\left(1-\frac{4\overline{BC}}{\overline{AB}}\right)}_{M}O_{AC} + \underbrace{\left(1-\frac{4\overline{BC}}{\overline{AB}}\right)}_{N}O_{BC}$$

want small.

so. if $M \geq 0 \Rightarrow \frac{\overline{BC}}{\overline{AB}} \leq \frac{1}{4}$. O_{AC} (O_{BC}) should be as small as possible namely P on AC (BC)

if $M<0 \Rightarrow \frac{\overline{BC}}{\overline{AB}} > \frac{1}{4}$, O_{AC} should be as large as possible, so should O_{BC}.

so P on AB (somewhere)

d. Problems on Dehn-Sommerville relations: (not done here...)

12). Term 2010, problem 4.

~~$h_0 + h_1 t + h_2 t^2 + h_3 t^3 = t$~~

①

we know 2010 edges, we want to know vertices.

also know simplex convex polytope,

so there are 3 edges at each vertex.

~~$\frac{2010}{3}+2 = 670+2 = 672$ vertices.~~

"each edge is calculated twice"

~~$\frac{6}{3}+2=4$~~ $\quad \frac{6\times 2}{3}=4$

~~$\frac{12}{3}+2=$~~ $\quad \frac{12\times 2}{3}=8$

So $\frac{2010\times 2}{3} = \frac{4020}{3} = 1340$ vertices

② $f_0 - f_1 + f_2 - f_3 = 1$

$1340 - 2010 + f_2 - 1 = 1$

$f_2 = 2 - 1340 + 2010$

$= 672$ faces.

13). $\mathbb{R}^3$ convex polyhedron.

① $3f_0 \le 2f_1$

② $\frac{2f_1}{f_2} < 6$

~~for 12). simple convex $\Rightarrow$ $f_0 = \frac{1}{2}f_1$~~

①

at least 3 ~~me~~ edges meet at one vertex.

so $3f_0$ is the edges based on # of vertikes.

have to know all of them are calculated twice.

So $\frac{3}{2}f_0$ is the minimum of ~~vert~~ edges

for some vertex, could have 4 or 5 edges.

so $\frac{3}{2}f_0 < f_1$

② $f_0 - f_1 + f_2 = 2$

$f_0 = 2 + f_1 - f_2 \le \frac{2}{3}f_1$

$2 + \frac{1}{3}f_1 - f_2 \le 0$

$f_1 \le 3f_2 - 6$

$f_1 \le 3f_2 - 6$ $2f_1 \le 6f_2 - 12$

$\frac{2f_1}{f_2} \le 6 - \frac{12}{f_2}$

$4 \ge 2f_2 - f_1$

so $\frac{2f_1}{f_2} < 6$

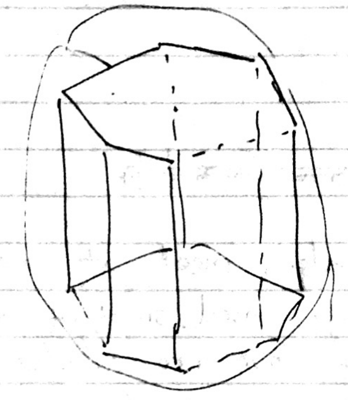
(14) Find.

prism with 2012-gon as a base

So

$f(t)\ 4024 + 6036t + 2014t^2 + t^3$

$h(t) = f(t-1)$



So $f_0 = 4024$

$f_1 = 2012\times 3 = 6036$

$f_2 = 2+2012 = 2014$

$f_3 = 1$

$4024 - 6036 + 2014 - 1 = 1$

$f(t) = 4024 + 6036t + 2014t^2 + t^3$

$h(t) = f(t-1)$

index 1?

for a 2012-gon at most $2012-2=2010$ vertices has index 1

then think about it's a prism.

we have a "higher base" & "lower base" s.t. all higher base has higher index.

hence. $\left.\begin{array}{ll} 1 & V_3, \\ 2010 & V_2 \\ 1 & V_1 \end{array}\right\}$ higher $\left.\begin{array}{ll} 1 & V_2 \\ 2010 & V_1 \\ 1 & V_1 \end{array}\right\}$ lower

$V_1 \rightarrow$ 2011 vertices.

$f(t) = 4024+6036t+2014(t^2)+t^3$

$h(t) = f(t-1)$

$= 4024+6036(t-1)+2014(t-1)^2+(t-1)^3$

$= \cancel{4024+6036t-6036}$

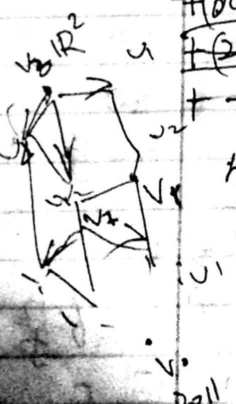
$\cancel{+2014(t^2-2t+1)+(t^2-2t+1)}(t-1)$

$\cancel{=(4024+6036+2014+1)}$ $=(t^3-2t^2+t$ $-t^2+2t-1)$

$\cancel{+(6036-4024+3)}\,t$

$\cancel{+(2014-3)}\,t^2$

$+t^3$



$\cancel{h_1 = h_{3-1} = h_2}$

$h(t) = 4024+6036t-6036+2014t^2-4028t+2014+(t-1)(t^2-2t+1)$

$t^3-t^2-2t^2+2t+t-1$

$= t^3-3t^2+3t-1$

$h_0 = 4024-6036+2014-1$ ✓

$h_1 = (6036-4028+3)$ ✓ $=2011$ $\quad h(t) = h_0 + h_1 t + h_2 t^2 + h_3 t^3$

$h_2 = 2014-3$ ✓ $=2011$

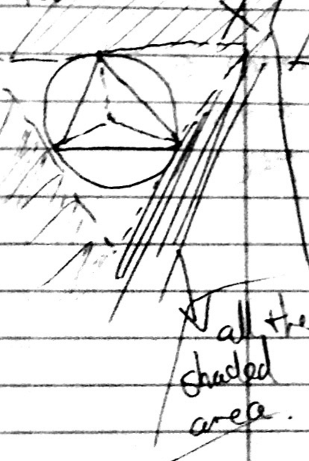
$h_3 = 1$ $\quad$ DS $\quad \tilde{h}_1 = h_1$

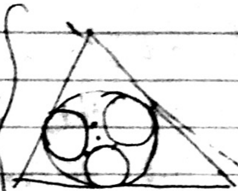
$h_i = h_{n-i}$

e. Inversion.

15) Term 2008 problem ~~4~~ 5.

False understanding



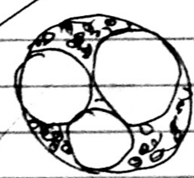


The triangle (lines) ⇒ 3 circle tangent inside of big circle.
and any two of them are tangent to each other.

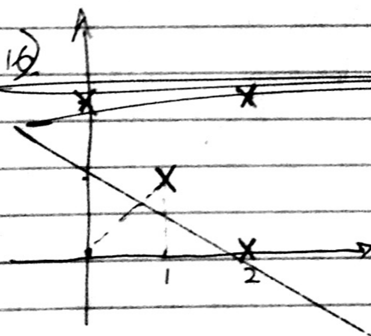
int of Δ ⇒ small circles inside of large circle.

all the shaded area.

Basically.



see last 16 page.



~~let the inversion circle centered~~ at (x,y).
and has radius r.

$$\frac{r^4}{(x-2)^2+y^2} = a_1^2+b_1^2 \quad : A$$

$$\frac{r^4}{(x-2)^2+(y-2)^2} = a_2^2+b_1^2 \quad : B$$

Then say ~~there must~~

be ~~a~~ relations:

say $A-B=A-C$

Then $2(A-B)=\cancel{2}(C-D)$

$$\frac{r^4}{x^2+(y-2)^2} = a_3^2+b_1^2 \quad : C$$

$$\frac{r^4}{(x-1)^2+(y-1)^2} = a_4^2+b_4^2 \quad : D$$

we guess $2(B-D) = A-C$

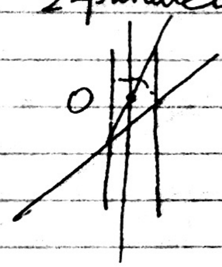
17). case ①

3-parallel

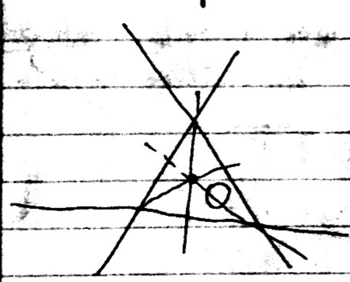
=> ~~T~~ impossible.

3 equal radii circle? that means the ~~near~~ smallest distance ~~≤~~ from three lines to O is the same

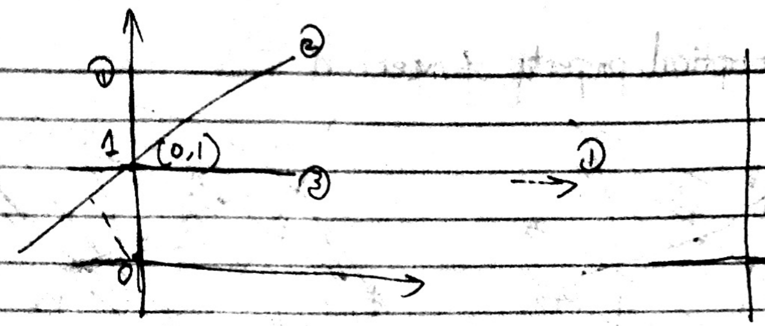
2-parallel.



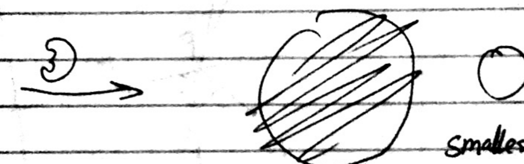
~~3~~ non-parallel.



18).

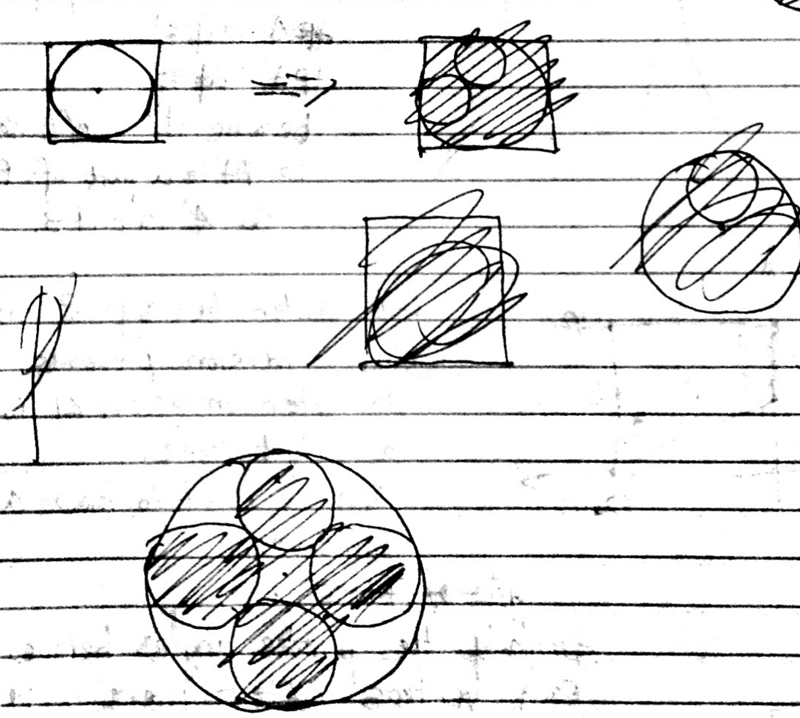


line closer to origin, the radius larger

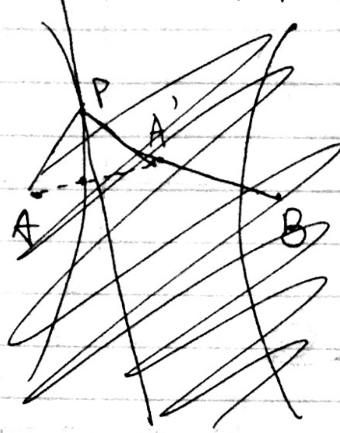


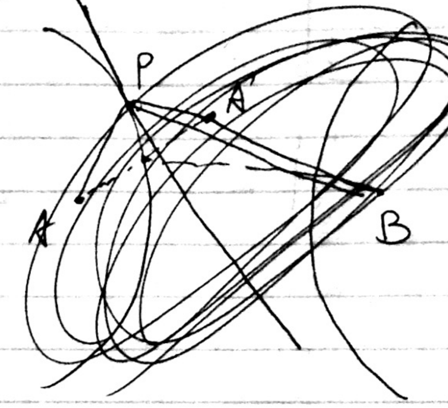
smallest after inversion.

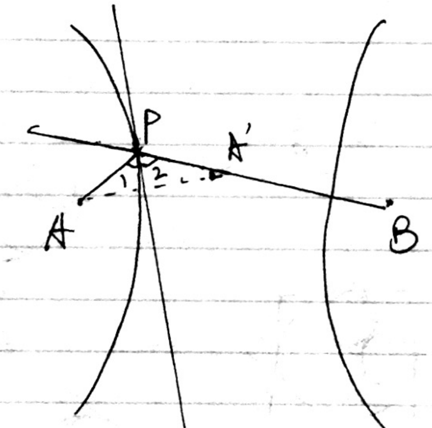
19).



~~opertical~~ optical property of hyperbola







why $\angle 1 = \angle 2$?

reflection

why P, A', B concurrent?

(looks like ray from B?)

~~$|PA - PB| = L$~~

~~$|PA' - PB| = L$~~

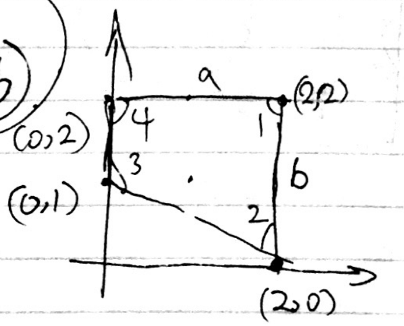
because it's a reflection

so PA is a part of PB

So A' is on PB.

?

16)



Let center be (x, y) & radius r,

since inversion preserves angle.

So after inversion $\angle 1$ is still a right angle.

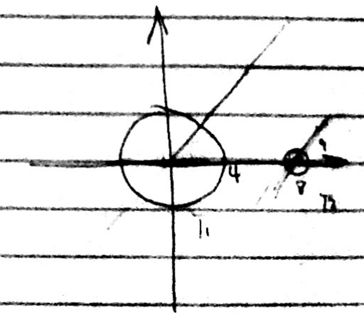
then we want to know $a = b$ after inversion.

~~$r^2 = \sqrt{x^2 + (y-2)^2}$~~ .

similarly, the ~~points~~ angles between each vertices after inversion remain the same,

i.e. $\angle 3 \neq 90°$. contradicts the condition?

$x^2+y^2=16$ $(x-8)^2+y^2=1$



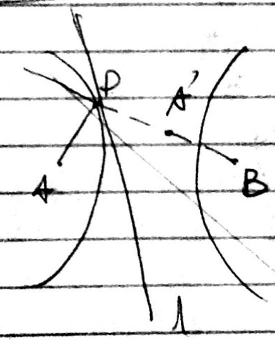
① x-axis

② ~~line~~ circle $\xrightarrow{T_1}$ line passing (8,0) $\xrightarrow{T_2}$ same line

③ ~~circle~~ line $\xrightarrow{T_1}$ circle $\xrightarrow{T_2}$ line

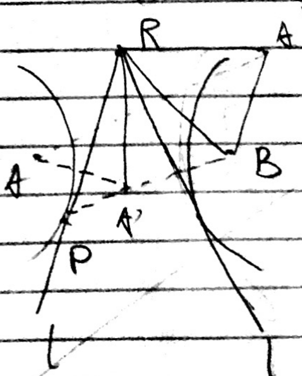
(not passing (0,0)) (not passing (8,0))

☆ Optical property of hyperbola proof

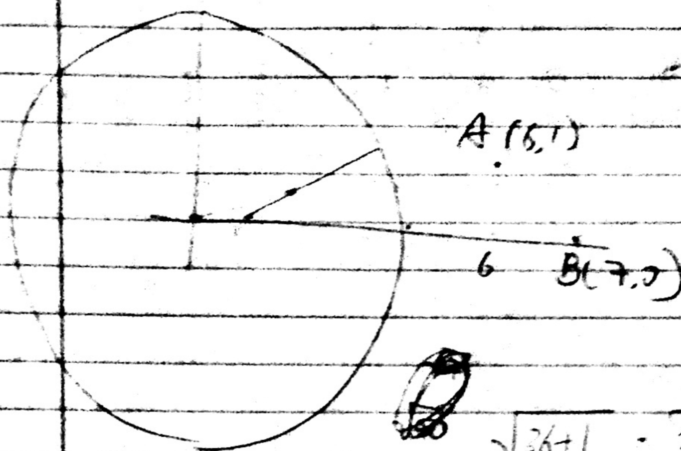


$L=|AP-BP|$ attains its max on line l

l bisector $\angle APB$



For $\triangle RBA'$, $RA'=RA$, $A'B=L$, RB given



$\sqrt{36+1} \cdot ? = 25$

$? = \frac{25}{\sqrt{37}}$

x

$(6y)^2 + y^2 = \frac{(25)^2}{37}$

$37y^2 = \frac{625}{37}$

A' : $y = \frac{25}{37}$ $\quad x = \frac{25 \times 6}{37}, \ y = \frac{25}{37}$

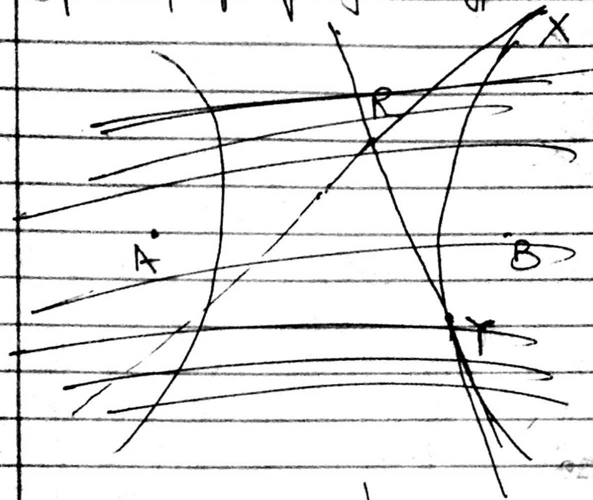
B' : $7 \cdot ? = 25$ $\quad B' = (\frac{25}{7}, 0)$

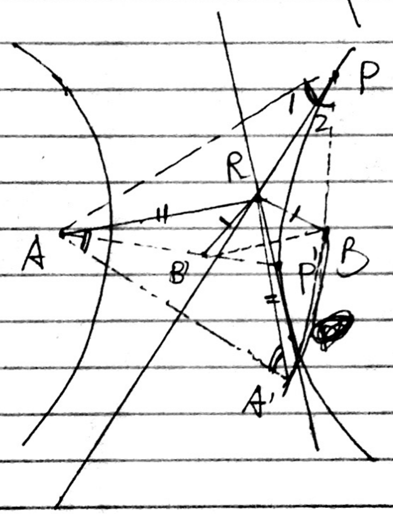
$? = \frac{25}{7}$

$(\frac{25}{7}, 0)$ $\quad (\frac{150}{37}, \frac{25}{37})$

$$\frac{\frac{25}{37}}{\frac{150}{37} - \frac{25}{7}}$$

Optical property of a hyperbola

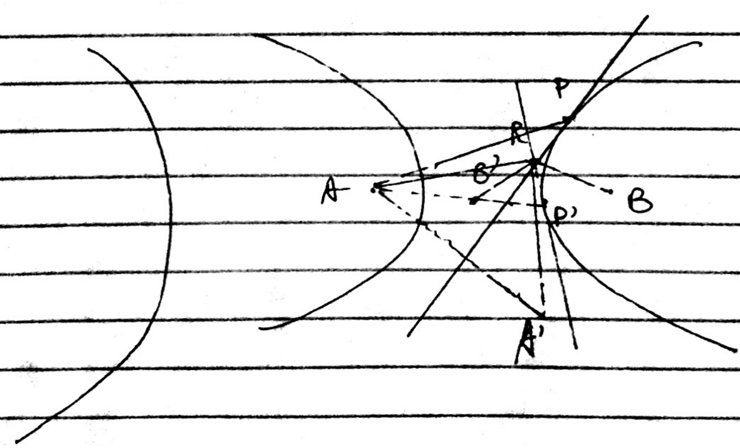




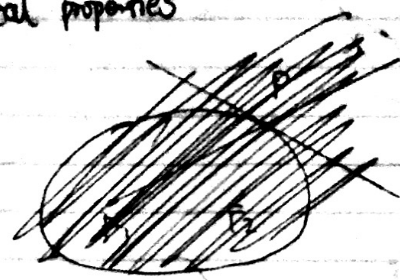
$\triangle RBA' \cong \triangle RB'A$ (three sides equal)

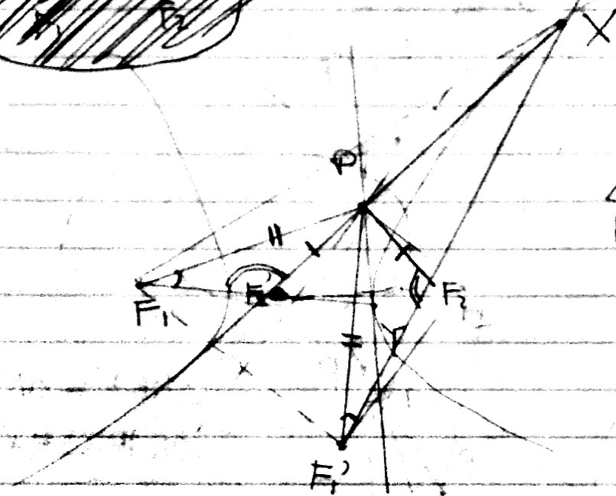
$\angle RAP = \angle RA'B = \angle RAB' = \angle RAP'$

($\angle 1 = \angle 2$ done)



optical properties





Y, X are tangency pt

$\triangle PF_1F_2' \cong \triangle PF_1'F_2$

b/c $PF_1 = PF_1'$

$PF_2' = PF_2$

$F_1F_2' = F_2F_1' = L$

(hyperbola, L fixed)

So $\angle PF_1F_2' = \angle PF_1'F_2$

$\angle PF_2'F_1 = \angle PF_2F_1'$

So $\angle PF_1X = \angle PF_1'F_2$

$= \angle PF_1F_2' = \angle PF_1Y$

past

Mobius problem from midterm

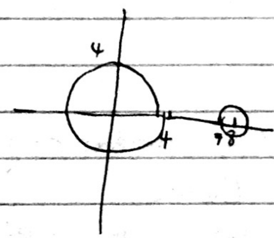
① $z \to \frac{z+1}{z-1}$

$z \to \frac{z-1+2}{z-1} = 1 + \frac{2}{z-1}$

~~$z=a+bi$~~ ~~circles that become lines after inversion about~~ unit circle

~~will be lines after transformation~~

②



line (find state)

before T_2, it passes $(8,0)$ (either as a circle, or a line)

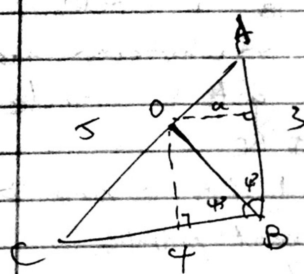
① if it's a circle after T_1

it pass $(0,0)$

so ....,

② if it's a line

which line do T_1 still a line? $y=0$!



~~$OB = \frac{12}{5}$~~

~~$OC = \sqrt{4^2 - (\frac{12}{5})^2} = \sqrt{16 - \frac{144}{25}} = \sqrt{\frac{256}{25}} = \frac{16}{5}$~~

~~$OA = \sqrt{3^2 - (\frac{12}{5})^2} = \sqrt{9 - \frac{144}{25}} = \sqrt{\frac{81}{25}} = \frac{9}{5}$~~

~~So $\frac{OA}{OC} = \frac{9}{16}$~~

$\frac{a}{4} = \frac{3-a}{3} = \frac{OA}{AC}$

$3a = 12 - 4a$

$a = \frac{12}{7}$

$\Rightarrow \frac{OA}{AC} = \frac{\frac{12}{7}}{4} = \frac{3}{7}$

$\frac{OA}{OC} = \frac{3}{4}$

Proj. Geometry ~~past~~ practice problems

①
A B C D

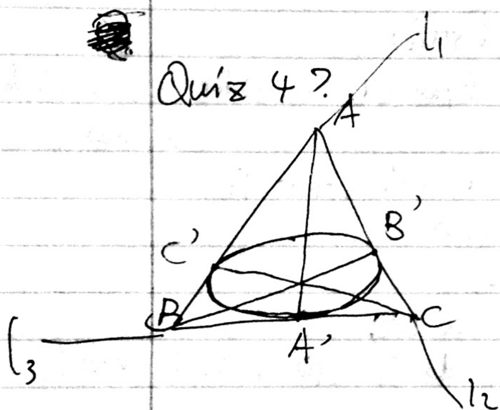
a b c

$$(A,B,C,D) = \frac{C-A}{D-A} : \frac{C-B}{D-B} = \frac{a+b}{a+b+c} \cdot \frac{b+c}{b} = \frac{ab+ac+b^2+bc}{ab+b^2+cb}$$

$$= 1 + \frac{ac}{ab+b^2+cb}$$

$$\left.\begin{array}{l} (B,A,D,C) = \cdots \\ (D,C,B,A) = \cdots \\ (C,D,A,B) = \cdots \end{array}\right\} \Longrightarrow$$

Quiz 4?



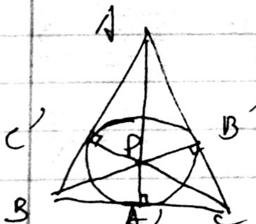
AA', BB', CC' ~~concur~~ congruent?

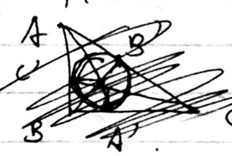
proved by Ceva.

namely. $\frac{AC'}{C'B} \cdot \frac{BA'}{A'C} \cdot \frac{CB'}{B'A} = 1$ (∗)

$\exists$ a point O (not on the line of l_1, l_2, l_3) such that the ellipse after central projection about point O is a circle.

And AB, BC, CA for whatever their lengths are, we get a circle inscribed in a triangle $\triangle ABC$





and A', B', C' are still tangent points.

then

3 heights, AA', BB', CC'

So $\frac{AC'}{C'B} = \frac{\triangle ACC'}{\triangle BCC'} = \frac{\triangle APC}{\triangle BPC}$

$=$

$=$

$\Rightarrow$ (∗) $\Rightarrow$

$$\frac{APC}{BPC} \cdot \frac{APB}{APC} \cdot \frac{BPC}{APC} = 1$$