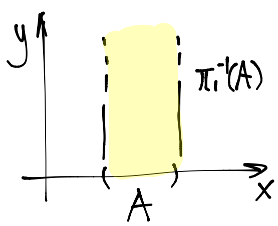
Lecture 10

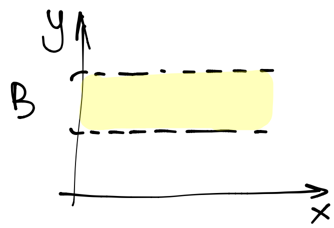
Midterm: July 2nd 4-6pm. SF3202

Recall: A basis for the product topology on $X \times Y$ where $(X,\mathcal{T})$, $(Y,\mathcal{U})$ are top spaces is $\mathcal{T} \times \mathcal{U} := \{A \times B : A \in \mathcal{T}, B \in \mathcal{U}\}$

Def'n: For $(X,\mathcal{T})$, $(Y,\mathcal{U})$ top spaces, define the projection maps:
$\pi_1 : X \times Y \to X$ by $(x,y) \mapsto x$
and $\pi_2 : X \times Y \to Y$ by $(x,y) \mapsto y$

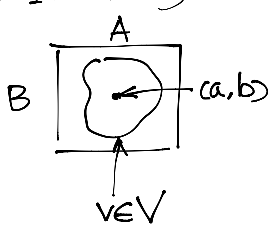
Identity: For $\forall A \subseteq X, B \subseteq Y$. $\pi_1^{-1}(A) = A \times Y$, $\pi_2^{-1}(B) = X \times B$





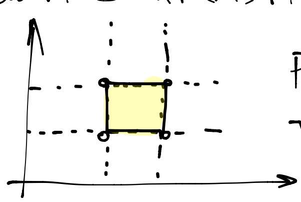
Prop: π_1 & π_2 are continuous function, moreover, the product top. is the smallest topology (coarest) topology where they are continuous.

Proof: Identities give continuity. Let $\mathcal{V}$ be top. on $X \times Y$ where π_1 & π_2 are continuous



Let $(a,b) \in A \times B$

π_1 is continuous $\to \pi^{-1}(A) = A \times Y \in \mathcal{V}$
π_2 is continuous $\to \pi_2^{-1}(B) = X \times B \in \mathcal{V}$
Now $A \times B = \pi_1^{-1}(A) \cap \pi_2^{-1}(B) \in \mathcal{V}$



prop 9 from §2

$\mathcal{V}$ refines the product top.

Prop: $\mathcal{f} := \{\pi_1^{-1}(U) : U \text{ open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ is a subbasis for the product topology on $X \times Y$.

Q: is $f(x) : \mathbb{R} \to \mathbb{R}^2$ defined by $x \mapsto (x^2 - 2\arctan x, \sqrt{x})$ continuous?

Prop: Let $(X,\mathcal{T})$, $(Y_1,\mathcal{U}_1)$, $(Y_2,\mathcal{U}_2)$ be top spaces, Let $f : X \to Y_1 \times Y_2$ be a function. TFAE:
1. f is continuous
2. $\pi_1 \circ f$, $\pi_2 \circ f$ are continuous

Proof: ① $\Rightarrow$ ② obvious

② $\Rightarrow$ ①. Check that pre-image of a sub basic open set is open.

Let $U \times Y_2$ be a subbasic open set.

$f^{-1}(U \times Y_2) = f^{-1}(\pi_1^{-1}(U)) = (\pi_1 \circ f)^{-1}(U)$ open by ②

So our previous Ex is continuous

$$\left.\begin{aligned} \text{as } \pi_1 \circ f(x) &= x^2 - 2 \\ \pi_2 \circ f(x) &= \arctan x \\ \pi_3 \circ f(x) &= \sqrt{x} \end{aligned}\right\} \text{cont.}$$

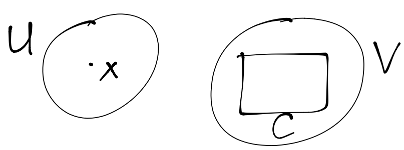
Product things:

- $\mathbb{R}^2 \not\cong \mathbb{R}$
- (cantor) For each $n \in \mathbb{N}$, $\mathbb{Q}^n \cong \mathbb{Q}$
- $\forall n \in \mathbb{N}$: $(\mathbb{R} \setminus \mathbb{Q})^n \cong \mathbb{R} \setminus \mathbb{Q}$ (all with usual subspace topology)

See §9 Separation Axiom

Def'n: $(X, \mathcal{T})$ is Hausdorff if $\forall x \neq y \in X$, $\exists U \ni x, V \ni y$, U, V open and disjoint.

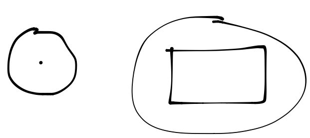
Def'n: A space $(X, \mathcal{T})$ is Regular if $\forall$ closed $C \subseteq X$, $\forall x \in X \setminus C$, $\exists U, V \in \mathcal{T}$ s.t. $x \in U$, $C \subseteq V$, U and V are disjoint (T_3 in Assignment 4)



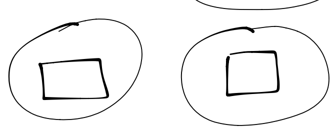
Def'n: $(X, \mathcal{T})$ is Normal if $\forall$ closed, disjoint set, $C, D \subseteq X$, $\exists$ disjoint $U, V \in \mathcal{T}$ s.t. $C \subseteq U$, $D \subseteq V$.

Hausdorff:

Regular:



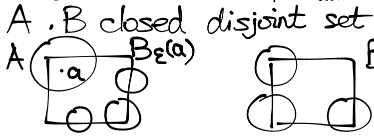
Normal..



Ex: $\mathbb{R}^n_{usual}$: HV, RV, NV

Let's check that $\mathbb{R}^n_{usual}$ is normal.

A, B closed disjoint set in $\mathbb{R}^n$.



Take $a \in A$, $X \setminus B$ is open and $a \in X \setminus B$.
$\exists\, \varepsilon_a$ s.t. $B_{\varepsilon_a}(a) \subseteq X \setminus B$.
Find $\varepsilon_b > 0$, $\forall b \in B$
Want to take $\hat{A} = \bigcup_{a \in A} B_{\varepsilon_a}(a)$ and $\hat{B} = \bigcup_{b \in B} B_{\varepsilon_b}(b)$, but might not be disjoint.

So $A' = \bigcup_{a \in A} B_{\frac{\varepsilon_a}{2}}(a)$ (triangle inequality)

$B' = \bigcup_{b \in B} B_{\frac{\varepsilon_b}{2}}(b)$ instead

$\boxed{\text{A horrible } \mathbb{R}_{\text{indiscrete}}}$

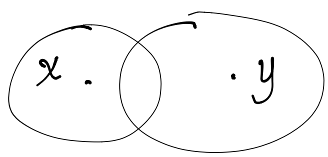
$\mathbb{R}_{\text{indiscrete}}$ is not Hausdorff but is (vacuously) regular & normal (No non-trivial disjoint closed set)

Normal $\Longrightarrow$ regular $\Longrightarrow$ Hausdorff

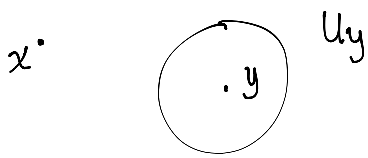
<u>Prop</u>: if $\{x\}$ is closed, $\forall x \in X$. then X is normal $\Longrightarrow X$ is regular $\Longrightarrow X$ is Hausdorff.

<u>Def'n</u>: X is T_2 if Hausdorff & points are closed;
X is T_3 if regular & pts are closed
X is T_4 if normal & pts are closed.

<u>Def'n</u>: X is T_1 if $\forall x \neq y \in X$, $\exists\, U, V$ open. s.t. $x \in U, y \in V$ (not necessarily disjoint)
$x \notin V, y \notin U$.



<u>Prop</u>: X is T_1 iff pts are closed in X "$\Longrightarrow$" fix $x \in X$, $\forall y \neq x$. find an open set U_y
$x \notin U_y$, so $x \in \bigcap_{y \in X \setminus \{x\}} (X \setminus U_y)$



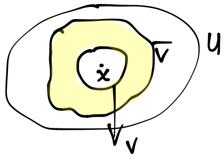
In fact, $x = \bigcap_{y \in X \setminus \{x\}} (X \setminus U_y)$ so $\{x\}$ is closed.

"$\Longleftarrow$" $x \neq y$. $x \in X \setminus \{y\}$, $y \in X \setminus \{x\}$

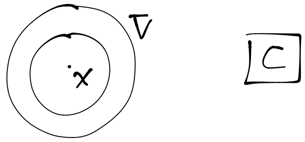
Note: $T_4 + T_1 \Longrightarrow T_3 + T_1 \Longrightarrow T_2 + T_1$
normal $+ T_1 \Longrightarrow$ regular $+ T_1 \Longrightarrow$ Hausdorff $+ T_1$

Note: X is $T_2 \Longrightarrow X$ is T_1.

Prop: Let X be a T_1 space, X is regular iff $\forall x, \forall U$ open $U \ni x\ \exists V$ open s.t. $x \in V \subseteq \overline{V} \subseteq U$

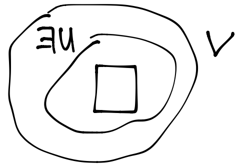


"$\Leftarrow$" C closed & take $x \in X \setminus C$. Find V s.t. $x \in V \subseteq \overline{V} \subseteq X \setminus C$



Since V is open, find a U s.t. $x \in U \subseteq \overline{U} \subseteq V$
So $x \in U$ open, $C \subseteq X \setminus \overline{V}$. U & $(X \setminus \overline{V})$ are disjoint.

Prop: X be T_1: X is normal iff $\forall$ closed C, open V, $C \subseteq V$. $\exists$ open U s.t. $C \subseteq U \subseteq \overline{U} \subseteq V$

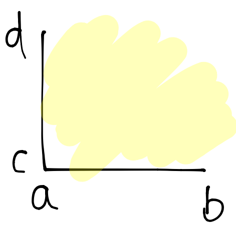


Fact: each T_i axioms are topological invariants.
Fact: T_1, T_2, T_3 are Hereditary but T_4 not.
Fact: if X is T_4, and $C \subseteq X$, closed, then $(C, \mathcal{T}_{sub})$ is T_4.

Proof: straight forward.

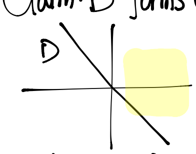
Finitely Productive?
Fact: T_2, T_3 are productive.
(see previous PS). but T_4 need not be productive.

Ex: Take X the sorgenfrey line, X is normal (Exercise), but $X \times X$ is not normal.
basic open sets look like $[a,b) \times [c,d)$



Take $D = \{(x,-x) : x \in \mathbb{R}\}$
the anti-diagonal

Claim: D forms a discrete subspace.



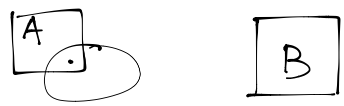
For $(x,-x) \in D$, take $A_x := [x, x+1) \times [-x, -x+1)$
Note $A_x \cap D = \{x\}$. So any subset of D is closed.

Claim: $C = \{(x,-x) : x \in \mathbb{Q}\}$. $D = \{(x,-x) : x \in \mathbb{R} \setminus \mathbb{Q}\}$ They are disjoint, closed and can't separate by open sets. (Munkres)

Prop: if X is T_3 & 2nd countable, X is T_4.

Proof: Let A, B be disjoint closed set.

$\forall a \in A$. find an open U_a st. $a \in U_a$, $\overline{U_a} \cap B = \emptyset$



Assume also that U_a is from $\mathcal{B}$, a fixed ctb basis.

Repeat finding V_b s.t. $\overline{V_b} \cap A = \emptyset$

Assume $\bigcup_{b \in B} V_b = \bigcup_{n \in \mathbb{N}} V_n$, and $\bigcup_{a \in A} V_a = \bigcup_{n \in \mathbb{N}} U_n$

define $U_n' = U_n \setminus \left(\bigcup_{i=1}^{n} \overline{V_i} \right)$

$V_n' = V_n \setminus \left(\bigcup_{i=1}^{n} U_i \right)$

Check $A \subseteq \bigcup U_n'$, $B \subseteq \bigcup V_n'$ and disjoint.