Tmr Term test #2

6:10 – 7:00 pm

3 problems

higher order linear ODE's

Everything from 2nd order theory go

- Existence and uniqueness for IVP
- Wronskians
- constant coefficient equation
- reduction of order
- undetermined coefficients
- Variation of parameters

Notation: $y^{(k)} = \frac{d^k y}{dt^k}$ k-th derivatives

$L[y] = y^{(n)} + P_1 y^{(n-1)} + \cdots + P_n y$

($P_1, \cdots, P_n$ function) n-th order linear diff. operation

Inhomogeneous equation $L[y] = g$ (given function)

Homogeneous equation : $L[y] = 0$

The general solution of $L[y] = g$ is the sum of a particular solution of $L[y] = g$ with the general solution of $L[y] = 0$.

(If $L[y] = g$ and $L[\tilde{y}] = g$

then $L[y - \tilde{y}] = L[y] - L[\tilde{y}] = g - g = 0$)

General solution of $L[y] = 0$ will be given by n independent solutions $y_1, \cdots, y_n$.

Need to make this precise.

~~Def~~: The Wronskian of function $f_1, \cdots, f_n$ is the function

$$W[f_1, \cdots, f_n] = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \end{vmatrix}$$

$$W[f_1, \cdots, f_n] = \begin{vmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & & \vdots \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

When is the Wronskian zero?

If there exists $c_1, \cdots, c_n$, not all 0, with $c_1 f_1 + \cdots + c_n f_n = 0$ (for all f) then the Wronskian is zero:

$$c_1 f_1' + \cdots + c_n f_n' = 0$$
$$\vdots \qquad \vdots$$
$$c_1 f_1^{(n-1)} + \cdots + c_n f_n^{(n-1)} = 0$$

Converse is also true:

$W[f_1, \cdots, f_n] = 0 \iff f_1, \cdots f_n$ are linearly dependent i.e. there exist $c_1, \cdots, c_n$ not all zero, with $c_1 f_1 + \cdots + c_n f_n = 0$.

Thus, $W \neq 0 \iff f_1, \cdots, f_n$ linearly independent.

<u>Def'n</u>: If $y_1, \cdots, y_n$ are solution of $L[y] = 0$, then they are a <u>fundamental set</u> of solution if $W[y_1, \cdots, y_n] \neq 0$ (i.e. they are lin. indep.).

If $y_1, \cdots, y_n$ are a fund. set of solution of $L[y] = 0$, then the general solution of $L[y] = 0$ is $y = c_1 y_1 + \cdots + c_n y_n$

Further properties

- Abel's formula

for $W = W[y_1, \cdots, y_n]$

$$W(t) = W(t_0) \exp\left(-\int_{t_0}^{t} p_1(s)\, ds\right)$$

$$(L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y)$$

In particular, $W(t) \neq 0$ for all t

$\iff W(t_0) \neq 0$ for some t_0

If $y_1, \cdots, y_n$ a solution of the general $y = c_1$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

- Initial value problem (IVP)

If $W[y_1, \cdots, y_n] \neq 0$ then for all $y_0, \cdots, y_0^{(n-1)}$ there exists constants $c_1, \cdots, c_n$ $y = c_1 y_1 + \cdots + c_n y_n$ solves the IVP

$$L[y] = 0, y(t_0) = y_0, y'(t_0) = y_0', \cdots, y^{(n-1)}(t_0) = y_0^{(n-1)}.$$

$W[f, g] = e^{6t}$, $f = e^{3t}$, find g.

Constant coefficient equations

$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$

Hom. equations $L[y]=0$ $(a_0, \dots, a_k \in \mathbb{R})$

Trial solution: $y(t) = e^{rt}$

$y'(t) = r e^{rt}$

$y^{(k)}(t) = r^k e^{rt}$.

Plug into $L[y]=0$

$e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_n)$

Thus e^{rt} is a solution $\Longleftrightarrow$ r is a root

of the char. equation

$a_0 r^n + a_1 r^{n-1} + \cdots + a_n = 0$

This equ. has n roots (some are complex), (counted with multiplication)

If r is a root, then e^{rt} is a solution. In case of $r = \lambda + i\mu$, e^{rt} is a complex solution, then $\bar{r} = \lambda - i\mu$ is also a root.

Can use $e^{rt}, e^{\bar{r}t}$ complex solution or $e^{\lambda t}\cos(\mu t)$, $e^{\lambda t}\sin(\mu t)$.

$e^{rt} = e^{(\lambda + i\mu)t} = e^{\lambda t} + e^{i\mu t}$

$e^{\bar{r}t} = \cdots$

If all roots are distinct, get fundamental set of solution $y_1(t) = e^{r_1 t}, \dots, y_n(t) = e^{r_n t}$

In some of multiplications, multiply by powers of t.

E.g. if root r has mult k, get solutions $e^{rt}, te^{rt}, \dots, t^{k-1}e^{rt}$ are solution.

E.g. $y^{(3)} + y = 0$

char. eqn: $r^3 + 1 = 0$ ($r = -1$ is a root)

$r^3 + 1 = (r+1)(r^2 - r + 1)$, $r_2, r_3 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - 1} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$