# Tutorial 9

*STAT 3013/4027/8027*

1. SI Example 4.8. Simply write out the steps as outlined in the example. This is the classic t-test.
2. Consider a Poission regression model using the canonical link function (how do we determine the canonical link function?):

$$\begin{array}{rcl} Y_1, \ldots, Y_n & \overset{\text{indep.}}{\sim} & \text{Poisson}(\lambda_i) \\ \log(\lambda_i) & = & \beta_0 + \beta_1 x_i + \beta_2 x_i^2 \\ & \text{for } i = 1, \ldots, n. & \end{array}$$

- Using `optim()` and the data on the website to find the MLEs: $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$, as well as their estimated asymptotic variances.
- Additionally, using the bootstrap procedure discussed in class, provide the estimated biases and variances for the parameters.
- Data: A sample from a population of 52 female song sparrows was studied over the course of a summer and their reproductive activities were recorded. In particular, the age and number of new offspring were recorded for each sparrow (Arcese et al, 1992). Let $Y$ = fledged (number of offspring), and $X$ = age (age of mother).

3. SI 4.19, 5.1.

Q1. Example 4.8

Sps $X_1, X_2, \cdots, X_n \sim N(\mu, \sigma^2)$, $\sigma^2$ unknown, wts. $H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$.

- MLRT is equivalent to the optimal test
- $L(\theta; \vec{x}) = (2\pi\sigma^2)^{-n/2} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right\}$
- Plug in $\mu = \mu_0$, $\tilde{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (x_i - \mu_0)^2$
- $\max_{\theta \in \omega} L(\theta; \vec{x}) = L(\hat{\theta}; \vec{x}) = (2\pi\tilde{\sigma}^2)^{-n/2} \exp\left\{-\frac{1}{2\tilde{\sigma}^2}\sum_{i=1}^n (x_i - \mu_0)^2\right\} = (2\pi\tilde{\sigma}^2)^{-n/2}\exp\left\{-\frac{1}{2\tilde{\sigma}^2} n\tilde{\sigma}^2\right\}$

$$= (2\pi\tilde{\sigma}^2)^{-n/2}\exp\left\{-\frac{n}{2}\right\}$$

- Under $H_0 \cup H_1$. $\hat{\mu}_{MLE} = \bar{x}$, $\hat{\sigma}^2_{MLE} = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x})^2$

$$\max_{\theta \in \Omega} L(\theta; \vec{x}) = L(\hat{\theta}; \vec{x}) = (2\pi\hat{\sigma}^2)^{-n/2}\exp\left\{-\frac{1}{2\hat{\sigma}^2}\sum_{i=1}^n (x_i - \bar{x})^2\right\}$$

$$= (2\pi\hat{\sigma}^2)^{-n/2}\exp\left\{-\frac{n}{2}\right\}$$

$$\lambda = \left(\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right)^{-n/2}$$

$$= \left[\frac{\sum_{i=1}^n (x_i - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]^{-n/2}$$

$$= \left[\frac{\sum_{i=1}^n (x_i - \bar{x})^2 + n(\bar{x} - \mu_0)^2}{\sum_{i=1}^n (x_i - \bar{x})^2}\right]^{-n/2}$$

Since $(x_i - \mu_0)^2 = [(x_i - \bar{x}) + (\bar{x} - \mu_0)]^2$

$$= \left[1 + \frac{n(\bar{x} - \mu_0)^2}{\sum (x_i - \bar{x})^2}\right]^{-n/2}$$

$$= \left[1 + \frac{t^2}{n-1}\right]^{-n/2}$$ where $t = \frac{(\bar{x} - \mu_0)}{s/\sqrt{n}}$ is the usual t-statistic.

Now MLRT rejects $H_0$ for small $\lambda$ i.e. large $t^2$, $\iff$ 2-sided t-test.

# Tutorial 9

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## Q2

```
data <- read.table('Data.txt', header=T)
set.seed(8027)
x <- data$age
y <- data$fledged

mll <- function(beta, y, x){
    sum(dpois(y, lambda=exp(beta[1]+x*beta[2]+x^2*beta[3]),log=TRUE))
}
(out <- optim(c(1,1,1), mll, hessian = TRUE,
      control=list(fnscale=-1), method="BFGS", y=y, x=x))
```

```
## $par
## [1]  0.2757469  0.6825819 -0.1346644
##
## $value
## [1] -96.38849
##
## $counts
## function gradient
##      152       36
##
## $convergence
## [1] 0
##
## $message
## NULL
##
## $hessian
##             [,1]      [,2]       [,3]
## [1,]  -125.0064  -361.015  -1241.054
## [2,]  -361.0150 -1241.001  -4719.142
## [3,] -1241.0537 -4719.142 -19113.182
```

```
beta0 <- out$par[1]
beta1 <- out$par[2]
beta2 <- out$par[3]
```

So $\hat{\beta}_0 = 0.2757469,\ \hat{\beta}_1 = 0.6825819,\ \hat{\beta}_2 = -0.1346644.$

The estimated asymptotic variances are:

```
diag(solve(-out$hessian))
```

```
## [1] 0.195518108 0.114552207 0.003346105
```

```
n <- nrow(data)
B <- 1000
```

```r
beta0.hat <- rep(0, B)
beta1.hat <- rep(0, B)
beta2.hat <- rep(0, B)

for (b in 1:B){
    data.star <- data[sample(1:n,replace=TRUE),]
    x <- data.star$age
    y <- data.star$fledged
    mll <- function(beta, y, x){
        sum(dpois(y, lambda=exp(beta[1]+x*beta[2]+x^2*beta[3]),log=TRUE))
        }
    out <- optim(c(1,1,1), mll, hessian = TRUE,
      control=list(fnscale=-1), method="BFGS", y=y, x=x)
    beta0.hat[b] <- out$par[1]
    beta1.hat[b] <- out$par[2]
    beta2.hat[b] <- out$par[3]
}

(bias.beta0.hat <- mean(beta0.hat)-beta0)
```

```
## [1] -0.06317356
```

```r
(bias.beta1.hat <- mean(beta1.hat)-beta1)
```

```
## [1] 0.03495636
```

```r
(bias.beta2.hat <- mean(beta2.hat)-beta2)
```

```
## [1] -0.005032503
```

```r
var(beta0.hat)
```

```
## [1] 0.2829929
```

```r
var(beta1.hat)
```

```
## [1] 0.1393406
```

```r
var(beta2.hat)
```

```
## [1] 0.003514416
```

Q3

$$f(x;\theta)=\frac{\theta^x e^{-\theta}}{x!}$$

§4.19 Using a random sample of size $n$ from a Poisson dist. with mean $\theta$, it is required to test $H_0: \theta=\theta_0$ vs $H_1: \theta\neq\theta_0$. Find the test statistic for

a). a score test of $H_0$ vs $H_1$

b). a Wald test of $H_0$ vs $H_1$.

Compare these stats with that of MLRT for the same ~~pair~~ $H_0$ vs $H_1$.

Solution:

$$l(\theta,\vec{x})=-n\theta+\left(\sum_{i=1}^n x_i\right)\ln\theta+\ln\left(\prod_{i=1}^n x_i!\right)$$

$$\frac{\partial l}{\partial\theta}=-n+\frac{\sum_{i=1}^n x_i}{\theta}$$

$$I_\theta=\frac{n}{\theta}\quad,\quad \hat{\theta}=\frac{\sum_{i=1}^n x_i}{n}$$

$\theta=\theta_0$

- score test statistic is $\left(\frac{\partial l}{\partial\theta_0}\right)^2\Big/I_{\theta_0}=\frac{-n+\frac{\sum x_i}{\theta_0}}{\frac{n}{\theta_0}}=-\theta_0+\frac{\sum x_i}{n}=\hat{\theta}-\theta_0$

- Wald test statistic is $(\hat{\theta}-\theta_0)^2\cdot I_{\theta_0}=(\hat{\theta}-\theta_0)^2\cdot\frac{n}{\theta_0}$

$$\text{so}\quad(\hat{\theta}-\theta_0)\frac{n}{\theta_0}\approx 1$$

$$n\hat{\theta}-n\theta_0\approx\theta_0$$

$$n\hat{\theta}\approx(1+n)\theta_0$$

$$\hat{\theta}\approx\frac{1+n}{n}\theta_0\qquad \text{as } n\to\infty,\ \hat{\theta}=\theta_0$$

- MLRT:

$$\lambda=\frac{L(\theta_0,\vec{x})}{L(\theta_1,\vec{x})}=\frac{\prod\frac{\theta_0^x e^{-\theta_0}}{x!}}{\prod\frac{\theta_1^x e^{-\theta_1}}{x!}}=\frac{\theta_0^{\Sigma x}}{\theta_1^{\Sigma x}}\frac{e^{-n\theta_0}}{e^{-n\theta_1}}=\left(\frac{\theta_0}{\theta_1}\right)^{\Sigma x}\cdot\exp(-n\theta_0+n\theta_1)$$

$$C=\{\lambda\le k\}$$
$$=\left\{\Sigma x\log\left(\frac{\theta_0}{\theta_1}\right)+(-n\theta_0+n\theta_1)\le\log k\right\}$$

§5.1. $X_1, X_2, \cdots, X_n$ are a random sample from the exponential dist. with p.d.f. $f(x;\theta) = \frac{1}{\theta} e^{-x/\theta}$, $x > 0$.

Using the results that $Y = \frac{2\sum_{i=1}^n X_i}{\theta}$ has a $\chi^2_{(2n)}$ dist, construct a CI for $\theta$ based on the pivital quantity $Y$.

$$1-\alpha = \Pr(L < Y < U) = \Pr\left(L < \frac{2\sum X_i}{\theta} < U\right)$$

$$= \Pr\left(U^{-1} < \frac{\theta}{2\sum X_i} < L^{-1}\right)$$

$$= \Pr\left(2\sum X_i \cdot U^{-1} < \theta < 2\sum X_i \cdot L^{-1}\right)$$

where $U = \chi^2_{1-\frac{\alpha}{2}, 2n}$

$L = \chi^2_{\frac{\alpha}{2}, 2n}$

$2n\bar{X}$

Hence $1-\alpha = \Pr\left(\frac{2\sum X_i}{\chi^2_{1-\frac{\alpha}{2}, 2n}} < \theta < \frac{2\sum X_i}{\chi^2_{\frac{\alpha}{2}, 2n}}\right)$

Why $Y \sim \chi^2_{2n}$ ?

$f(x) = \frac{1}{\theta} e^{-\theta/x}$, $x > 0$

$Y = \frac{2\sum_{i=1}^n X_i}{\theta} = \frac{2n\bar{X}}{\theta} \sim \chi^2_{2n}$ (shown by mgf)

what is a pivot?

a function of both parameters & sample/observations whose distribution is known & independent of parameters.

Sometimes, if $\theta$ is scalar, $g$ needs to be monotonic.