Lec 9

Consider the system of ODEs

(*) $\begin{cases} \frac{d}{dx} x(t) = M x(t) + N \underbrace{\alpha(t)}_{\text{control}} \\ x_0 = x \end{cases}$

$x(t) \in \mathbb{R}^n$, $M = n \times n$ matrix

$\alpha(t) \in [-1, 1]$, $N = n \times m$ matrix

Ex: railroad rocket car

$\frac{d^2 x}{dt^2} = \alpha(t)$. $\quad |\alpha(t)| \leq 1$, $x(t) \in \mathbb{R}$

rewrite as $\quad \frac{dx_1}{dt} = x_2$, $\quad \frac{dx_2}{dt} = \alpha$

rewrite again: $\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \alpha(t)$

so here $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad N = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Define $C(t)$ reachable set at time t = set of X for which $\exists$ a control $\alpha(\cdot)$ s.t. $x(t) = 0$

First, review of systems of ODEs

a) defn: if $M = n \times n$ matrix, then $e^M = I + M + \frac{1}{2} M^2 + \frac{1}{3!} M^3 + \cdots$

Fact: $\boxed{\frac{d}{dt} e^{tM} = M e^{tM} = e^{tM} \cdot M}$

Idea: $\frac{d}{dt}(e^{tM}) = \frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{t^k M^k}{k!}\right) \overset{\text{should}}{=} \sum_{k=0}^{\infty} \frac{k \cdot t^{k-1} M^k}{k!} = M e^{tM}$ (or $= e^{tM} \cdot M$)

Ex: ① D is diagonal matrix $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ then $e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & \\ & \ddots & \\ & & e^{\lambda_n t} \end{pmatrix}$

$$\sum \frac{(tD)^k}{k!} = \begin{pmatrix} 1 + t\lambda_1 + \frac{(t\lambda_1)^2}{2!} + \cdots & & & \\ & 1 + t\lambda_2 + \frac{t\lambda_2^2}{2!} + \cdots & & \\ & & \cdots & \\ & & & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{t\lambda_1} & & \\ & \ddots & \\ & & e^{t\lambda_n} \end{pmatrix}$$

② $M = QDQ^T$, D-diagonal, $QQ^T = Q^TQ = I$
($\forall$ symmetric matrix can be written like this)
Then

$$e^{tM} = I + t(QDQ^T) + \frac{t^2}{2!}(QD^2Q^T) + \frac{t^3}{3!}(QD^3Q^T) + \cdots$$

$$= Q\left(I + tD + \frac{t^2D^2}{2!} + \frac{t^3D^3}{3!} + \cdots\right)Q^T$$

So, $e^{tM} = Qe^{tD}Q^T$

③ $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad M^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\quad$ So $e^{tN} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$

④ $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad M^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$

$M^3 = M^2 \cdot M = -I \cdot M = -M$
$M^4 = M^3 \cdot M = -M \cdot M = -M^2 = I$

So $e^{tM} = \sum \frac{t^k M^k}{k!} = \cdots$

$$= I\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) - M\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right)$$

$$= (\cos t) \cdot I - (\sin t) \cdot M$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

Examples of ODEs

$$\left.\begin{aligned} \frac{dx}{dt} &= Mx \\ x(0) &= x_0 \end{aligned}\right\} \Rightarrow \text{sol'n} \quad x(t) = e^{tM} x^0$$

Check: $\frac{d}{dt}(e^{tM}x_0) = Me^{tM}x_0 = M(e^{tM}x_0)$

$\frac{dx}{dt} = Mx + f(t)$, $f(t)$ function from $\mathbb{R}$ to $\mathbb{R}^n$

Sol'n: $x(t) = e^{tM}x_0 + e^{tM}\left(\int_0^t e^{-sM} f(s)\,ds\right)$

Idea: rewrite equation, $\frac{dx}{dt} - M(x) = f(t)$

Multiply by e^{-Mt}

$$\underbrace{e^{-Mt}\frac{dx}{dt} - Me^{-Mt}x(t)}_{\frac{d}{dt}(e^{-Mt}x(t))} = e^{-Mt}f(t)$$

$\frac{d}{dt}(e^{-Mt}x(t)) = e^{-Mt}f(t)$ $\quad$ Now integrate both sides & rewrite.

Back to controllability.

Goal : understand "reachable set" for system (*)

Basic computation

$x_0 \in C(t) \Longleftrightarrow \exists$ a control $\alpha(\cdot)$ s.t. $x(t)=0$

$\Longleftrightarrow$ a control $\alpha(\cdot)$ s.t. $0=x(t)=e^{Mt}x_0+e^{Mt}\int_0^t e^{-Mt}N\alpha(s)ds$

$$=e^{Mt}\left(x_0+\int_0^t e^{-Mt}N\alpha(s)ds\right)$$

Multiply by e^{-Mt}

$x_0=-\int_0^t e^{-Mt}N\alpha(s)ds$ for some control α.

Thm: (Structure of the reachable set)

① $C(t)$ is convex and symmetric ($x_0\in C$ iff $-x_0\in C$)

② if $x_0\in C(t)$, then $x_0\in C(T)$, for $\forall\, T>t$

③ C is convex & symmetric

Proof: (idea):

① Symmetric

If $x_0\in C(t)$, then $\exists$ a control s.t. $x_0=-\int_0^t e^{-Ms}N\alpha(s)\,ds$ so control $-\alpha(s)$ works for $-x_0$

Assume x_0, $\hat{x}_0\in C(t)$.

There are controls $\alpha(\cdot)$, $\hat{\alpha}(\cdot)$ s.t. $x_0=-\int_0^t e^{-Ms}N\alpha(s)ds$

(also true with $\wedge$)

need $\theta\cdot x_0+(1-\theta)\hat{x}_0\in C(t)$ $\quad 0\le\theta\le1$

Now the control,

$\theta\cdot\alpha(\cdot)+(1-\theta)\hat{\alpha}(\cdot)$ works for this point

② if $x_0\in C(t)$, then $\exists\,\alpha(\cdot)$ s.t. $x_0=-\int_0^t e^{-Ms}N\alpha(s)ds$

Define $\tilde{\alpha}(s)=\begin{cases}\alpha(s), & 0\le s\le t\\ 0, & s>t\end{cases}$

This leaves me at 0 at time t

③ follow from combining ①&② (see notes for detail)

Next:

Thm: $0\in$ interior of C iff the following matrix G has rank n.

$G=[N, MN, M^2N, \cdots, M^{n-1}N]=n\times(mn)$ matrix

Recall: G has rank n iff $b^T\cdot G\neq0$ for $\forall$ nonzero $b\in\mathbb{R}^n$

$\underset{1\times n}{(\quad)}\underset{n\times(mn)}{\left(\quad\quad\right)}=1\times(mn)$ $\Longleftrightarrow$ the n rows are Linearly Independent

Ex: railroad rocket car

$M=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $N=\begin{pmatrix}0\\1\end{pmatrix}$, $MN=\begin{pmatrix}1\\0\end{pmatrix}$

$G \cdot [N, MN] = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ rank 2 ✓

Idea for Thm:
We'll prove half of it: sps rank $G<n$,
wts. the reachable set does not contain a nbhd of 0.

Proof: Rank $G<n \Rightarrow b^TG=0$ for some b. $\Rightarrow b^TN=b^TMN=\cdots=bM^{N-1}N=0$
Fact:
For any k, M^k is a linear combination of $I, M, \cdots, M^{n-1}$
Conclusion, $b^TM^kN=0$ for $\forall k$
Then, $b^Te^{tM}N=b^TIN+tb^TMN+\cdots=0$ for all t.

So $b^T\int_0^t e^{-Ms}N\alpha(s)ds=0$ for every control $\alpha(\cdot)$

It follows that if $x^0\in C(t)$ for some t, then $x^0=-\int_0^t e^{-Ms}N\alpha(s)ds$ for some α.
so $b^Tx=0$ so $C\subseteq\{x^0\in\mathbb{R}^n : b^Tx^0=0\}$
So $0\notin$ interior(C)

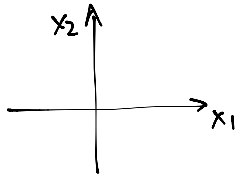
The other half of this property is harder but similar idea

Thm: assume matrix G from above has rank n, and $\forall$ eigenvalue λ of M has $Re(\lambda)\le 0$
Then $C=\mathbb{R}^n$
(When $C=\mathbb{R}^n$ holds, we say that (*) is controllable)

Example of "G"

Ex: $\frac{dx_1}{dt}=x_2+\alpha$ $\quad M=\begin{pmatrix}0&1\\0&0\end{pmatrix}, N=\begin{pmatrix}1\\0\end{pmatrix}$ $\quad MN=\begin{pmatrix}0\\0\end{pmatrix}$

$\frac{dx_2}{dt}=0$ $\quad G=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ rank=1



In fact, we can see that if $x_2(0)\neq 0$, then it can never equal 0
So we cannot get to the origin

Also, for railroad rocket car (again), $M=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $N=\begin{pmatrix}0\\1\end{pmatrix}$

G has rank 2. and eigenvalues of M are $\det(M-\lambda I)=\det\begin{pmatrix}-\lambda&1\\0&-\lambda\end{pmatrix}=\lambda^2$
roots are $\lambda=0\Rightarrow Re(\lambda)\le 0$ ✓

So system is controllable.

Idea: the previous Thm implies that nbhd of origin $\subseteq$ reachable set C

So need to know: can I reach this nbhd finite time, starting from an arbitrary $x^0 \in \mathbb{R}^n$

Case 1: (easier) $\mathrm{Re}\lambda < 0$ for all nbhd of M.

$$N = \begin{pmatrix} -\mu_1 & & \\ & \ddots & \\ & & -\mu_n \end{pmatrix} \quad \text{then } e^{tM} = \begin{pmatrix} e^{-t\mu_1} & & \\ & \ddots & \\ & & e^{-t\mu_n} \end{pmatrix}$$

everything __?__ twd origin

In ODE, it is proved that $\mathrm{Re}\lambda < 0$, for all eigenvalues, then $|e^{tM}x_0| \to 0$ as $t \to \infty$ for every x^0,

so from any x^2

① Let $u(t)=0$ for a long time until $e^{tM}x_0 \in$ the nbhd of 0 where I know system is controllable

② ______ to reach the origin

Case 2: $\mathrm{Re}\lambda \leq 0 \Rightarrow$ harder (check notes)

Next, consider

Assume (*) is controllable. Can we find starting from $x_0 \in \mathbb{R}^n$, is there a control that reaches 0 as quickly as possible.

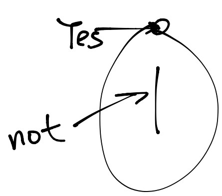
Bang-bang Principle

a control $\alpha(\cdot)$ is a bang-bang control if $|\alpha_i(t)| = 1$ for all $i = 1, \cdots, n$ for all t

Thm: Let $x^0 \in C(t)$. then $\exists$ a bang-bang control that steers x to 0 at time b.

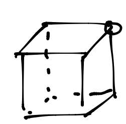
Idea:

Def'n: if K is a convex subset of a vector space, then $z \in K$ is an extreme point of K if



z extreme pt means if $z = \theta x + (1-\theta)\hat{x}$ for $\theta \in (0,1)$ $x, \hat{x} \in K$ then $x = \hat{x} = z$

In particular, if $K = [-1, 1]^n$ extreme pts are just vertice.



i.e. α is an extreme point if $(\alpha_i) = 1$ for all i. (bang-bang condition)

Idea : k

① The set of consists $\alpha(\cdot)$ that steer x_0 to 0 at time is a convex set with nbhd good properties.

② $\forall$ convex set with these properties has at least 1 extreme pt.

③ Any extreme pt of k is a Bang-bang control

Define:

$L^\infty(c, t, \mathbb{R}^m) = \{ \text{fcns } [0,t] \longrightarrow \mathbb{R}^m : \sup_{s\in[0,t]} |\alpha(s)| < \infty \}$

A seq. $\alpha_k(\cdot)$ in L^∞ converges ~~in the~~ in the weak $*$ sense to a limit $\alpha(\cdot)$ if $\int_0^t \alpha_k(t)\cdot v(t)\,dt \to \int_0^t \alpha(t)\cdot v(t)\,dt$ for $\forall$ fcn $V:[0,t]\longrightarrow \mathbb{R}^m$

s.t. $\int_0^t u(s)\,ds < \infty$

Ex: (not obvious) if $m=1$, $\alpha_k(t) = \sin(kt)$ converges in the weak $*$ sense to 0 in $L^\infty(0,1,\mathbb{R})$

$T = \frac{2\pi}{k}$

Famous Theorem (Krein–Milman Thm)

If k is a convex subset of of L^∞ and k is compact w.r.t. weak $*$ convergence then k has at least one extreme pt.

Note: if $k = (-1, 1)^m$, then k has no extreme pts

As stated above, we'll define $k = \{$controls $\alpha(\cdot)$ s.t. sol'n $x(\cdot)$ satisfies $x(t) = 0\}$

Claim: this satisfies hypothesis of Krein–Milman Thm

Need to check: ① convex ② weak $*$ compactness

To check weak $*$ compact, we need: Famous Thm #2. (Alaoglu)

- If $T \subset L^\infty$ and $\sup_{\alpha\in T} \sup_{0\le s\le t} |\alpha(s)| < \infty$

then $\forall$ seq in T has a cvgt subseq. (α_k) that cvg to a limit α to check that $\alpha \in k$. (see textbook for detail).

True b/c $\alpha \in k$, $x_0 = -\int_0^t e^{-Ms} N \alpha_k(s)\,ds$

so for $\forall$ extreme

$$x^0 = -\int_0^t \underbrace{b^T e^{-Ns} N}_{V(s)} \alpha(s)\,ds$$

$$\rightarrow \int v(\cdot)\alpha(\cdot)\,ds = b^T \int_0^t e^{-Ns} N\alpha(s)\,ds$$

Finally, why is an extreme point "bang-bang"? $\alpha(\cdot)$