July 15th

Integrable

Def: A set $Z \subset \mathbb{R}^2$ is said to have zero content if for any $\varepsilon > 0$ there is a finite collection of rectangles $R_1, \cdots, R_M$ s.t.

(i). $Z \subset \bigcup_1^M R_m$

(ii). the sum of the areas of R_m's is less than ε

4.18 Thm: sps f is a bdd func on the rectangle R. If the set of pts in R at which f is discontinuous has zero content, then f is integrable on R.

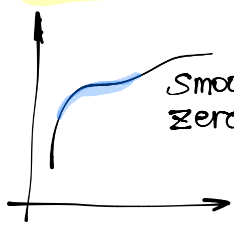
Pf: exercise.

4.19 Prop. (Properties of zero content)

a. If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.

b. If $Z_1, \cdots, Z_k$ have zero content, then so does their union $\bigcup_1^k Z_j$.

c. If $\vec{f}: (a_0, b_0) \to \mathbb{R}^2$ is of class C^1, then $\vec{f}([a,b])$ has zero content whenever $a_0 < a < b < b_0$.



Smooth curves in $\mathbb{R}^2$ have zero content

Proof of (c): Let $P_k = \{t_0, \cdots, t_k\}$ be the partition of $[a,b]$ into k equal sub-intervals of length $\delta = (b-a)/k$

Since f' is cont. on $[a,b]$, we define

$C = \sup\{|\vec{f}'(t)| : t \in [a,b]\} < \infty$

$\sqrt{(x')^2 + (y')^2} \geq |x'|$

$\vec{f}$ is continuous on $[a,b]$. $\vec{f}$ is uniform continuous, we use mean value theorem

let $= \vec{f} = (\vec{x}(t), \vec{y}(t))$

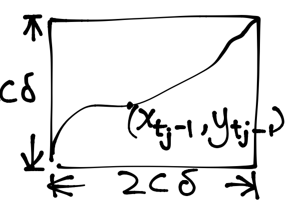
$$|x(t) - x(t_j)| = x'(\xi)|t - t_j| \text{ for } \xi \in (t, t_j)$$
$$\leq C(t_j - t_{j-1}), \text{ for } t \in [t_{j-1}, t_j]$$
$$= C\delta$$

$|y(t) - y(t_j)| \leq C\delta$ for $t \in [t_{j-1}, t_j]$

$\Rightarrow \vec{f}([t_{j-1}, t_j])$ is in the square of side length $2C\delta$.

$\Rightarrow \vec{f}([a,b])$ is inside K such square

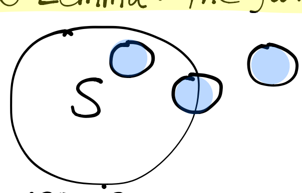
with area $K \cdot (2C\delta)^2 = K \cdot 4C^2\delta^2 = K \cdot 4C^2 \frac{(b-a)^2}{K^2} \downarrow 0$ when $K \uparrow \infty$



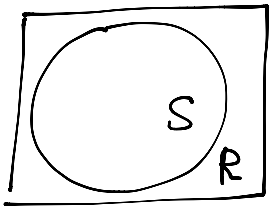
$\Rightarrow \vec{f}([a,b])$ has zero content.

discontinuities of ϕ_S (χ_S)

4.20 Lemma: The function χ_S is discontinuous at $\vec{x}$ iff $\vec{x}$ is in the ∂S



Pf: exercise.



$f\phi_S$: S has two properties

① bdd
② its boundary has zero content.

Jordan measurable (measurable)

4.21 Thm: Let S be a measurable subset of $\mathbb{R}^2$. Sps $f:\mathbb{R}^2 \to \mathbb{R}$ is bdd and the set of pts in S at which f is discontinuous has zero content. Then f is integrable on S.

Pf: exercise.

Properties of integrable functions

Proposition 4.22

Sps $Z \subset \mathbb{R}^2$ has zero content. If $f:\mathbb{R}^2 \to \mathbb{R}$ is bdd then f is integrable on Z and $\iint_Z f\,dA = 0$.

Proof: Z has zero content

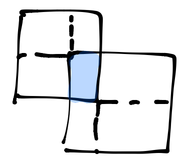
$\Rightarrow \forall \varepsilon > 0, \exists R_1, \cdots, R_M$ s.t. $Z \subset \bigcup^M R_m$ and the sum of the area of $R_m < \varepsilon$.

By subdividing those rectangles, we assume R_m has disjoint interiors. let

$C = \sup_{\vec{x}} |f| \Rightarrow -C < |f(x)| < C$

$$\Rightarrow -C\varepsilon < -C\sum_1^M |R_j| \leq \underline{S} f\phi_Z \leq \overline{S} f\phi_Z \leq C\sum_1^M |R_j| < C\varepsilon$$

area of R_j



ε is arbitrary, let $\varepsilon \to 0 \Rightarrow \iint_Z f\,dA = 0$. ■

4.23 Cor

a. Sps f is integrable on the set $S \subset \mathbb{R}^2$. If $g(\vec{x}) = f(\vec{x})$ except for $\vec{x}$ in a set of zero content, then g is integrable on S and $\iint_S g\,dA = \iint_S f\,dA$

b. Sps f is intgb on S & on T, and $S \cap T$ has zero content. Then f is intgb. on $S \cup T$ and $\iint_{S\cup T} f\,dA = \iint_S f\,dA + \iint_T f\,dA$

Proof: (a) consider $f-g$, Prop 4.22. exercise.

(b). $\iint_{S\cup T} f\,dA = \iint_S f\,dA + \iint_{T\setminus S} f\,dA = \iint_S f\,dA + \iint_T dA - \underbrace{\iint_{T\cap S} f\,dA}_{=0}$. ■

High dimensions

change rectangle R into boxes

$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$

4.24 Thm (The MVT for Integral) Let S be a compact, connected, measurable subset of $\mathbb{R}^n$, and let f and g be continuous functions on S with $g \geq 0$. Then there's a point $\vec{a} \in S$ s.t.

$$\int \cdots \int_S f(\vec{x}) g(\vec{x})\, d^n\vec{x} = f(\vec{a}) \int \cdots \int_S g(\vec{x})\, d^n\vec{x}.$$

Proof: Let $m = \inf\{f(\vec{x})\}, \vec{x} \in S$

$M = \sup\{f(\vec{x})\}, \vec{x} \in S$

since $g \geq 0 \Rightarrow mg \leq fg \leq Mg$

$$\Rightarrow m\int\cdots\int_S g\, d^n\vec{x} \leqslant \int\cdots\int_S f(\vec{x})g(\vec{x})\, d^n\vec{x} \leqslant M\int\cdots\int_S g(\vec{x})\, d^n\vec{x}$$

If $\int\cdots\int_S g\, d^n\vec{x} = 0$, then we have done.

If $\int\cdots\int_S g\, d^n\vec{x} \neq 0$

$$m \leqslant \frac{\int\cdots\int_S f(\vec{x})g(\vec{x})\, d^n\vec{x}}{\int\cdots\int_S g\, d^n\vec{x}} \leqslant M = \sup\{f\}$$

using f is continuous and the intermediate value theorem

$\exists\, \vec{a} \in S$, s.t. $f(\vec{a}) = \int\cdots\int f(\vec{x})g(\vec{x})\, d^n\vec{x} \,/\, \int\cdots\int_S g\, d\vec{x}$

let $g \equiv 1$

4.25 Cor.

$$\int\cdots\int f(\vec{x})\, d^n\vec{x} = f(\vec{a})\cdots\underbrace{\int\cdots\int_S 1\, d\vec{x}}_{\text{Area of } S}$$

Corollary: Let S be a compact measurable subset of $\mathbb{R}^n$ and let f be a cont. function on S then $\exists\, \vec{a} \in S$, s.t.

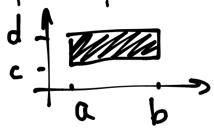
$$\int\cdots\int_S f(\vec{x})\, d^n\vec{x} = f(\vec{a})|S|.$$

§ 4.8 Multiple Integrals and Iterated Integrals

Eg. when $n=2$, consider integral of a function f over a rectangle R

$P = \{x_0, \cdots, x_J, y_0, \cdots, y_K\}$

Pick points $\tilde{x}_j \in [x_{j-1}, x_j]$ and $\tilde{y}_k \in [y_{k-1}, y_k]$



Riemann sum $\sum_{j=1}^{J}\sum_{k=1}^{K} f(\tilde{x}_j, \tilde{y}_k)\,\Delta x_j \Delta y_k$ where

$\Delta x_j = x_j - x_{j-1}$, $\Delta y_k = y_k - y_{k-1}$

when f is integrable on R $\quad \sum\sum f(\tilde{x}_j, \tilde{y}_k)\,\Delta x_j \Delta y_k \approx \iint_R f(x,y)\, dx dy$

On the other hand

for each y $\quad \sum_1^J f(\tilde{x}_j, y)\,\Delta x_j \approx \int_a^b f(x,y)\, dx$ a function of y

Let $g(y) = \int_a^b f(x,y)\, dy$

$$\sum_{k=1}^{K} g(\tilde{y}_k)\Delta y_k \approx \int_c^d g(y)\, dy$$

$$\Rightarrow \iint_R f(x,y)\, dx dy \approx \sum^J\sum^K f(\tilde{x}_j, \tilde{y}_k)\,\Delta x_j \Delta y_k$$
$$\approx \sum^J \int_a^b f(x, \tilde{y}_k)\Delta y_k$$
$$\approx \int_c^d \left[\int_a^b f(x,y)\, dx\right] dy$$

Similarly, switch x and y.

$$\iint_R f(x,y)\, dx dy \approx \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx$$

4.26 Thm: Let $R = \{(x,y) : a \leq x \leq b, c \leq y \leq b\}$ and let f be an integrable function on R. ① Suppose that, for each $y \in [c,d]$ the function f_y defined by $f_y(x) = f(x,y)$ ② is integrable on $[a,b]$, and the function $g(y) = \int_a^b f(x,y)\, dx$ is integrable on $[c,d]$, ③ then

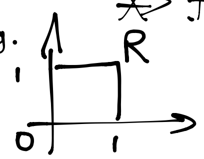
$\iint_R f\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$ $\rightarrow$ from the inside out called iterated integrals.

Likewise if $f^x y = f(x,y)$ is integrable on $[c,d]$ for each $x \in [a,b]$ and $h(x) = \int_c^d f(x,y)\,dy$ is integrable on $[a,b]$, then $\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$.

Remark: f is integrable on R.

$\not\Rightarrow$ for a fixed y_0, $f(x_0, y_0)$ is integrable.

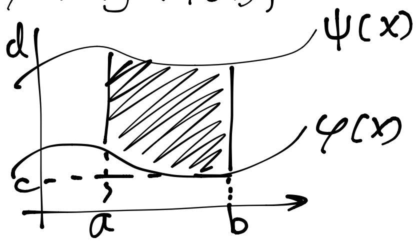
E.g.



$$f(x,y) = \begin{cases} 1 & \text{when } y=0 \text{ and } x \text{ is a irrational \#} \\ 0 & \text{o.w.} \end{cases}$$

f is continuous on R except on x axis thus f is int. on R but when $y=0$, for $f(x,0)$. $\overline{S}_P = 1$, $\underline{S}_P = 0$ for any partition $\Rightarrow f(x,0)$ is not int.

When the region is not a rectangle R, but a subset $S = \{(x,y) : a \le x \le b,\ \varphi(x) \le y \le \psi(x)\}$



$$\Rightarrow \iint_S f\,dA = \int_a^b \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$

Remark: ① when φ and ψ are of class C', f is continuous on S.

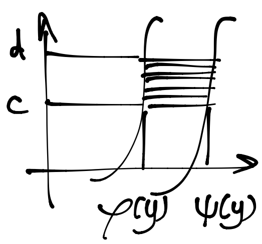
Consider $f\phi_S$ on $R \supset S$

$f\phi_S$ is continuous except on the bdy curves, and curves have zero contents, thus $f\phi_S$ is integrable on R

for each $x \in [a,b]$, $f(x,y)$ is integrable w.r.t. y on $[\varphi(x), \psi(x)]$ using $\varphi, \psi \in C'$

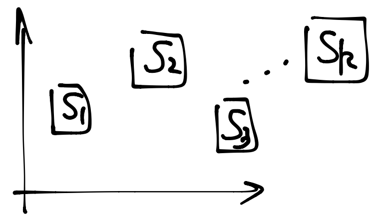
$\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy$ is continuous with respect to x, thus it is integrable $\Rightarrow$ we can apply Fubini.

②



$$\iint_S f\,dA = \int_c^d \int_{\varphi(y)}^{\psi(y)} f(x,y)\,dx\,dy$$

③

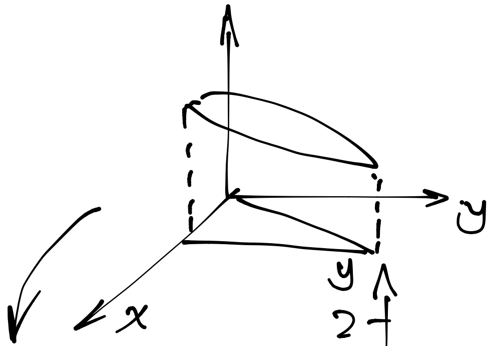


$S = S_1 \cup \cdots \cup S_k$

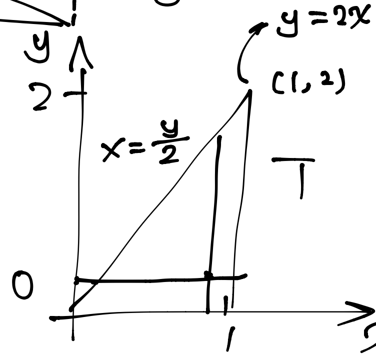
$$\iint_S f\,dA = \sum_{i=1}^{k} \iint_{S_i} f\,dA$$

Eg 2: Find the volume of the region in $\mathbb{R}^3$ above the triangle T in the xy-plane with vertices $(0,0)$, $(1,0)$ and $(1,2)$ and below the surface $z = xy + y^2$.

Soln: $\iint_T xy + y^2\,dA$



on xy-plane :

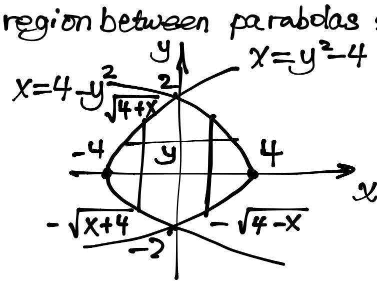


$$\int_0^2\int_{\frac{y}{2}}^1 xy+y^2\,dx\,dy = \int_0^2 \left(\tfrac12 x^2 y + xy^2\Big|_{\frac{y}{2}}^1\right) dy = \int_0^2 \left(\tfrac12 y + y^2\right) - \left(\tfrac{y^3}{8} + \tfrac{y^3}{2}\right) dy$$

$$= \int_0^2 (\tfrac12 y + y^2 - \tfrac58 y^3\, dy = \tfrac14 y^2 + \tfrac13 y^3 - \tfrac{5}{32} y^4 \Big|_0^2 = \tfrac76$$

$$\int_0^1\int_0^{2x} xy+y^2\,dy\,dx = \int_0^1 \left(x\tfrac{y^2}{2} + \tfrac13 y^3\Big|_0^{2x}\right) dx = \int_0^1 2x^3 + \tfrac83 x^3\, dx = \tfrac12 x^4 + \tfrac23 x^4\Big|_0^1 = \tfrac76$$

Eg 3. Let S be the region between parabolas $x=4-y^2$ and $x=y^2-4$



$$\int_{-2}^2\int_{y^2-4}^{4-y^2} f(x,y)\,dx\,dy$$

$$\int_{-4}^0\int_{-\sqrt{x+4}}^{\sqrt{x+4}} f(x,y)\,dy\,dx + \int_0^4\int_{-\sqrt{4-x}}^{\sqrt{4-x}} f(x,y)\,dy\,dx$$

Higher dimension

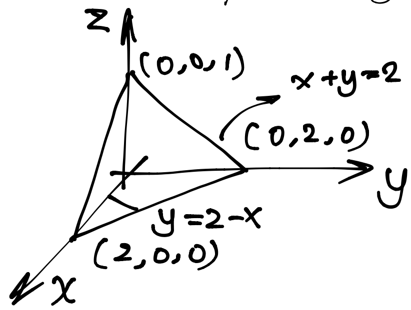
$$\iint\cdots\int_R f\,dv = \int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}\cdots\int_{a_1}^{b_1} f(x_1,\dots,x_n)\,dx_1\cdots dx_n$$

$n=3$ $\quad S=\{(x,y,z):(x,y)\in U,\ \varphi(x,y)\leq z\leq \psi(x,y)\}$

$\quad U=\{(x,y): a\leq x\leq b,\ \sigma(x)\leq y\leq \tau(x)\}$

$$\Rightarrow \iiint_S f\,dv = \int_a^b\int_{\sigma(x)}^{\tau(x)}\int_{\varphi(x,y)}^{\psi(x,y)} f(x,y,z)\,dz\,dy\,dx$$

Eg. Find the mass of tetrahedron T formed by the three coordinate planes and the plane $x+y+z=2$, if density $\rho(x,y,z)=e^{-z}$



$$z=1-\frac{x+y}{2}$$

$$\iiint_T \rho\, dv=\int_0^2\int_0^{2-x}\int_0^{1-\frac{x+y}{2}} e^{-z}\,dz=dydx$$

$$=\int_0^2\int_0^{2-x}\left(-e^{-z}\Big|_0^{1-\frac{x+y}{2}}\right)dy\,dx$$

$$=\int_0^2\int_0^{2-x}\left(1-e^{\frac{x+y}{2}-1}\right)dy\,dx$$

$$=\int_0^2 \left(y-2e^{\frac{x+y}{2}-1}\right)\Big|_0^{2-x} dx$$

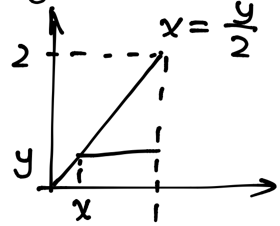
$$=\int_0^2 2-x-2e^0-\left(-2e^{\frac{x}{2}-1}\right)dx$$

$$=\int_0^2 2e^{\frac{x}{2}-1}-x\,dx$$

$$=4e^{\frac{x}{2}-1}-\frac{x^2}{2}\Big|_0^2$$

$$=4-2-4e^{-1}$$
$$=2-4e^{-1}$$

Eg. $\int_0^2\int_{y/2}^1 ye^{-x^3}dx\,dy$. No explicit form for the antiderivative of e^{-x^3}.



$$\int_0^1\int_0^{2x} ye^{-x^3}dy\,dx$$

$$=\int_0^1\left[e^{-x^3}\frac{y^2}{2}\Big|_0^{2x}\right]dx=\int_0^1 e^{-x^3}2x^2dx$$

$$=-\frac{2}{3}e^{-x^3}\Big|_0^1=\frac{-2}{3}e^{-1}+\frac{2}{3}$$