$\frac{20}{20}$

MAT246 HW 10 Rui Qiu #999292309

Q1. ~~What is~~ Which of the following is a number field?

(a). the set of all nonnegative rational numbers. NO.

Answer: Since for a $x > 0$ is in the set of nonnegative rational numbers,

but $-x < 0$ is not in such a set.

Hence this set is not a number field.

(b). the set of numbers of the form $a+b\sqrt{2}+c\sqrt{3}$ where $a,b,c \in \mathbb{Q}$. NO

Answer:

~~Since when $a=1, b=2, c=3$, say the set is named S.~~

~~$1+2\sqrt{2}+3\sqrt{3} \in S$.~~

~~but $(1+2\sqrt{2}+3\sqrt{3})(1+2\sqrt{2}+3\sqrt{3})$~~

~~$= 1+2\sqrt{2}+3\sqrt{3}+2\sqrt{2}+8+6\sqrt{6}+3\sqrt{3}+6\sqrt{6}+27$~~

~~$= 36+4\sqrt{2}+6\sqrt{3}+12\sqrt{6}$~~

~~which is not in S.~~

~~Hence S is not a number field.~~

Say the set is S.

for $a+b\sqrt{2}+c\sqrt{3} \in S$

and $a'+b'\sqrt{2}+c'\sqrt{3}$,

$a,b,c,a',b',c' \in \mathbb{Q}$

$(a+b\sqrt{2}+c\sqrt{3})(a'+b'\sqrt{2}+c'\sqrt{3})$

$= aa'+ab'\sqrt{2}+ac'\sqrt{3}+a'b\sqrt{2}$
$+bb'\cdot 2+bc'\sqrt{6}+a'c\sqrt{3}+b'c\sqrt{6}$
$+cc'\cdot 3 = aa'+2bb'+3cc'$
$+(ab'+a'b)\sqrt{2}$
$+(ac'+a'c)\sqrt{3}$
$+(bc'+b'c)\sqrt{6}$

note that $(bc'+b'c)\sqrt{6} \notin S$

Hence S is not a number field.

(c). the set of numbers of the form $a+b\sqrt{2}+c\sqrt[4]{2}+d\sqrt[4]{8}$ where $a,b,c,d \in \mathbb{Q}$. YES

Answer: Basically, we want to construct the following tower of number fields.

$$\mathbb{Q} = F_0 \subset F_1 = \mathbb{Q}(\sqrt{2}) \subset F_2 = F_1(\sqrt[4]{2}) \subset F_3 = F_2(\sqrt[4]{8})$$

For $a,b \in \mathbb{Q}$, it's quite obvious to prove $\underset{F_1=}{\mathbb{Q}(\sqrt{2})}$ is a number field.

Now we want to prove $F_2 = F_1(\sqrt[4]{2})$ is a number field.

say a,b,c,d ~~$\in \mathbb{Q}$~~ F_1

~~then $a+b\sqrt{2} \in F_1 = \mathbb{Q}(\sqrt{2})$~~

~~$c+d\sqrt{2} \in F_1 = \mathbb{Q}(\sqrt{2})$~~

~~$(a+b\sqrt{2})+(c+d\sqrt{2})\sqrt[4]{2}$~~ −

then $a+b\sqrt[4]{2} \in F_2$

$c+d\sqrt[4]{2} \in F_2$

$$(a+b\sqrt[4]{2})(c+d\sqrt[4]{2})$$
$$=ac+(ad+bc)\sqrt[4]{2}+bd\sqrt{2}$$

$$=(ac+bd\sqrt{2})+(ad+bc)\sqrt[4]{2}$$

Since $a,b,c,d\in Q(\sqrt{2})$

then $ac+bd\sqrt{2}\in Q(\sqrt{2})$

$ad+bc\in Q(\sqrt{2})$

thus $(a+b\sqrt[4]{2})(c+d\sqrt[4]{2})\in F_2=F_1(\sqrt[4]{2})$

$$\frac{a+b\sqrt[4]{2}}{c+d\sqrt[4]{2}}=\frac{(a+b\sqrt[4]{2})(c-d\sqrt[4]{2})}{c^2-d^2\sqrt{2}}$$

(Note: $c+d\sqrt[4]{2}\neq 0$ unless c and d are both zero, b/c $\sqrt[4]{2}\notin F_1$. if $c^4-2d^4=0$ and $d\neq 0$ then $(\frac{c}{d})^4=2$, and it would follow that $\sqrt[4]{2}\in F_1$.)

$$=\frac{(ac-ad\sqrt[4]{2}+bc\sqrt[4]{2}-bd\sqrt{2})(c^2+d^2\sqrt{2})}{(c^2-d^2\sqrt{2})(c^2+d^2\sqrt{2})}$$

$$=\frac{ac^3+acd\sqrt{2}-bdc^2\sqrt{2}-2bd^3+(bc-ad)\sqrt[4]{2}(c^2+d^2\sqrt{2})}{c^4-2d^4}$$

Since $Q(\sqrt{2})$ is a number field and $a,b,c,d\in Q(\sqrt{2})$

then $\frac{ac^3+acd\sqrt{2}-bdc^2\sqrt{2}-2bd}{c^4-2d^4}\in Q(\sqrt{2})$

and $(bc-ad)(c^2+d^2\sqrt{2})\in Q(\sqrt{2})$

Hence $\frac{a+b\sqrt[4]{2}}{c+d\sqrt[4]{2}}\in F_2=F_1(\sqrt[4]{2})$

Besides, we can easily prove that $1,0\in F_2=F_1(\sqrt[4]{2})$ and for $x,y\in F_2=F_1(\sqrt[4]{2})$,

$x+y,\ x-y\in F_2=F_1(\sqrt[4]{2})$. (It's obvious to prove ~~that~~)

Therefore, $F_2=F_1(\sqrt[4]{2})$ is a number field.

Q18. Show that the set of all polynomials with rational coefficients is countable.

Proof: We are going prove this by induction.
Define P_n to be the set of polynomials of degree n with coefficients

(Base Case): $n=0$. the set of degree n polynomials is just the set of rational numbers.

(Inductive assumption): Assume this is true for polynomials of degree n and we will prove it for degree n+1.

(Inductive step): Define $f: P_{n+1} \to \mathbb{Q} \times P_n$ be

$$f(a_{n+1}x^{n+1}+\cdots+a_1x+a_0)=(a_{n+1}x^{n+1}+\cdots+a_1, a_0)$$

This map is 1-1 & onto.

Sine by the induction P_n is countable and we already know $\mathbb{Q}$ is countable.
So $\mathbb{Q}\times P_n$ is countable as a product of two countable sets.

Therefore the set of all polynomials with rational coefficients is countable.

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Q23. Sps that S & T each have cardinality c. Show that $S \cup T$ also has cardinality c.

Proof: Suppose $x \in S$, $y \in T$.

The $S \cap T = \{(x, y) \mid x \in S, y \in T\}$

$$
\begin{aligned}
|S \cap T| = |S \times T| &= |S| \times |T| \\
&= |R| \times |R| \\
&= |R \times R| \\
&= |R| = c
\end{aligned}
$$

$$
\begin{aligned}
|S \cup T| &= |S| + |T| - |S \cap T| \\
&= c + c - c \\
&= c
\end{aligned}
$$

∎

Q24. What is the cardinality of the set of all finite subsets of R?

Solution: Let S be the set of all finite subsets of R and S_n be the set of subsets of R with n elements in it.

We have $S = \bigcup_{n=1}^{\infty} S_n$

Obviously $S_1 = R$ and $|S| \geq |R|$.

Given any $A \in S_n$ we have $A = \{x_1 < \cdots < x_n\}$

Hence the map $S_n \to R^n$ is one-to-one, since it maps from $\{x_1 < \cdots < x_n\}$ to $(x_1, x_2, \cdots, x_n)$

Therefore $|S_n| \leq |R^n| = |R|$

Hence $|S| = |\bigcup_{n=1}^{\infty} S_n| \leq |\bigcup_{n=1}^{\infty} (\{n\} \times R)| = |N \times R| \leq |R \times R| = |R|$

By Schroeder--Bernstein Theorem, $|S| = |R| = c$.

Here the cardinality of S is c.

Q27. Let S be the set of all real ~~finite~~ numbers that have a decimal representation using only the digits 2 and 6. Show that the cardinality of S is c.

Proof: ~~$S=\{a_0.a_1a_2\cdots a_n\cdots \mid a_i = 2 \text{ or } 6 \text{ for } i=0,1,2,\cdots\}$~~
~~e.g. S contains 2.26, 6.6, 6.2662666, etc.~~

$S = \{\cdots a_n \cdots a_2 a_1 . a_1' a_2' \cdots a_n' \cdots \mid a_i = 2 \text{ or } 6, a_i' = 2 \text{ or } 6 \text{ for } i = 1,2,3,\cdots\}$

e.g. S contains 2.626, 66.222, 262.6266 etc.

It is obviously clear that ~~|S|~~ $S \subset R$ since each element in S is in R as well.

So $|S| \leq |R| = c$

Now we need to construct the reverse inequality.
will will construct a one-to-one function
$f: R \to S$

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Recall the conversion from base-10 ~~integers~~ real numbers to base-2 real numbers (binary).
Note that we consider 2, 6 as 0, 1 in binary respectively

e.g. ~~$32.15_{(10)} = 100000.1111$~~

and or
Binary/decimal representations are not unique

$3.125_{(10)} = 11.001_{(2)} = 66.226_{(S)}$

$\sqrt{2}_{(10)} = 1.4142135\cdots_{(10)} = 1.0110101000001\cdots_{(2)}$
$= 6.2662626222226\ldots$

So f is a function that first converts the base-10 real numbers into binary, then for every a_i or a_i' $\in\{2,6\}$.

$f(a_i) = 0$ if $a_i = 2$ $(f(a_i') = 0$ if $a_i' = 2)$

$f(a_i) = 1$ if $a_i = 6$ $(f(a_i') = 1$ if $a_i' = 6)$

for all $i \in N^+$

$1 = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} \cdots$

$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \cdots$

Now if ~~[illegible]~~

$$f(x) = f(y)$$

then $$x = y$$

So f is one-to-one

Hence $c = |R| \leq |S|$

By Shroeder-Beranstein Theorem.

$$c = |R| = |S|$$

$\blacksquare$