March 6th

Complex numbers

$|z| = \sqrt{a^2+b^2}$

$|z_1 \cdot z_2| = |z_1||z_2|$

$a = |z|\cos\theta \quad b = |z|\sin\theta$

$a+bi = z = |z|(\cos\theta + i\sin\theta)$

De Moivre's Formula

$|z_1||z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = |z_1||z_2|(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$

$= z_1 \cdot z_2$

Ex: $z_1 = 1+i$
$z_2 = i$

$|z_1| = \sqrt{2} \quad \theta_1 = \frac{\pi}{4} \quad 1+i = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$

$|z_2| = 1 \quad \theta_2 = \frac{\pi}{2} \quad i = 1(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$(1+i)\cdot 1 = \sqrt{2}\cdot 1(\cos(\frac{\pi}{4}+\frac{\pi}{2}) + i\sin(\frac{\pi}{4}+\frac{\pi}{2}))$

$= \sqrt{2}(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$

$= \sqrt{2}(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i)$

$= i-1$

$$\boxed{\begin{aligned} z &= |z|(\cos\theta + i\sin\theta) \\ \frac{1}{z} &= \frac{1}{|z|}(\cos(-\theta) + i\sin(\theta)) \end{aligned}}$$

$z^2 = |z|^2(\cos 2\theta + i\sin 2\theta)$

$z^3 = |z|^3(\cos 3\theta + i\sin 3\theta)$

. . .

$z^n = |z|^n(\cos n\theta + \sin n\theta)$

$(1+i)^{100} = (\sqrt{2})^{100}(\cos\frac{100}{4}\pi + i\sin\frac{100}{4}\pi)$

$= 2^{50}(-1 + i\cdot 0)$

$= -2^{50}$

$(1+\sqrt{3}i)^{100} = \sqrt{1+3}^{100}\left(\cos\frac{\pi}{3}\cdot 100 + i\sin\frac{\pi}{3}\cdot 100\right)$

$= 2^{100}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -2^{99}(1+\sqrt{3}i)$

$z = a+bi$

we want to find $\sqrt[n]{a+bi}$

How?

$\sqrt[n]{z} = |z|^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

$\sqrt{i} = ?$ $\quad = \sqrt{1}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{1}{\sqrt{2}}(1+i)$

$z = |z|(\cos\theta + \sin\theta) = |z|(\cos(\theta+2\pi) + i\sin(\theta+2\pi))$

$z^{\frac{1}{n}}$, can take $|z|^{\frac{1}{n}}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

so we can take $|z|^{\frac{1}{n}}\left(\cos\frac{\theta+2\pi}{n} + i\sin\frac{\theta+2\pi}{n}\right)$

$2\pi \longrightarrow 2k\pi \cdots$

$\cos\left(\frac{\theta+2\pi n}{n}\right) = \cos\left(\frac{\theta}{n} + 2\pi\right) = \cos\left(\frac{\theta}{n}\right)$

similarly, $\sin\left(\frac{\theta+2\pi n}{n}\right) = \sin\left(\frac{\theta}{n}\right)$

$1 = \cos\theta + i\sin\theta \qquad k=0 \Rightarrow \cos 0 + i\sin 0 = 1$

$\sqrt[3]{1} = 1\left(\cos\frac{0+2\pi k}{3} + i\sin\frac{0+2\pi k}{3}\right) \qquad k = 0,1,2$

$k=2 \quad \cos\left(0+\frac{4\pi}{3}\right) + i\sin\left(0+\frac{4\pi}{3}\right)$

$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z^3 = 1$ has 3 solutions

$z_0 = 1,\ z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

$z^n = 1$ has n roots

$\sqrt{1+\sqrt{3}i}$

$1+\sqrt{3}i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$\sqrt{1+\sqrt{3}i} = \sqrt{2}\left(\cos\frac{(\frac{\pi}{3}+2k\pi)}{2} + i\sin\frac{(\frac{\pi}{3}+2k\pi)}{2}\right)$

$k=0,1$

$k=0$, $\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}) = \sqrt{2}(\frac{\sqrt{3}}{2} + \frac{1}{2}i)$

$k=1$, $\sqrt{2}(\cos\frac{7}{6}\pi + i\sin\frac{7}{6}\pi) = \sqrt{2}(-\frac{\sqrt{3}}{2} - \frac{1}{2}i)$

$\sqrt[n]{z_0}$ means solving $z^n = z_0$ polynomial equation of degree n.

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

- polynomial of degree n.

How many solutions does $p(z)=0$ have?

$2z^3 + 5z^2 - 2z + 1 = 0$

$(1+i)z^5 + 2\sqrt{2}z^3 - \frac{5}{2} = 0$

$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$

Fundamental Theorem of Algebra.
Let $P(z) = a_n z^n + \cdots + a_1 z_1 + a_0$
polynomial of degree n | $a_n \neq 0$ $n \geq 1$, $a_i \in \mathbb{C}$
then $P(z)=0$ always has a solution

$z^5 - 2z^4 + (i+10)z - \sqrt{2}i = 0$ there is a complex root z.

A: always exactly n roots
"counting multiplicity" $n \geq 1$, $a_n \neq 0$

FTA $\Rightarrow$ if $P(z) = a_n z^n + \cdots + a_0$

$\Rightarrow p(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n) = 0$

$\Rightarrow$ roots are exactly $z_1, \cdots, z_n$

ex: $p(z) = z^3 + 2z^2 + z$ - cubic polynomial

$= z(z^2+2z+1)$

$= (z-0)(z-(-1))(z-(-1))$ multiplicity $=2$

Long division of polynomials

if $P_1 = a_n z^n + \cdots + a_0$

$P_2 = b_m z^m + \cdots + b_0$

$\Rightarrow$ can divide p_1 by P_2 with remainder

$\exists\, Q(z), R(z)$ polynomials s.t.

$P_1(z) = P_2(z)\cdot Q(z) + R(z)$ & $\deg R(z) < \deg P_2(z)$

Proof: Induction in $n = \deg p$

if $\deg p_1 < \deg P_2$ $(n < m)$

$\Rightarrow P_1 = 0\cdot P_2 + P_1$
$Q = 0,\ R = P_1$ work!

for $n = 0, 1, 2, \cdots, m-1 \Rightarrow$ easy to divide with remainder

now suppose $n \geq m$ & theorem is proved for polynomials of deg $\leq n-1$

$P_1(z) = a_n z^n + \cdots + a_0$
$P_2(z) = b_m z^m + \cdots + b_0$

$P_2 \cdot \left(\frac{a_n}{b_m} z^{n-m}\right) = \cdots$

$\widetilde{P_1} = P_1 - \frac{a_n}{a_m} z^{n-m} P_2$ has degree $< n$

$\Rightarrow$ by the induction assumption

can divide $\widetilde{P_1}$ by P_2 $\Rightarrow$

$\widetilde{P_1(z)} = \widetilde{Q(z)}\, P_2(z) + R(z)$

$\deg R(z) < \deg P_2$

$P_1 = z^3 - 2z + 3$ by $P_2 = z - 1$

$$
\begin{array}{r}
z^2 + z - 1 \\
z - 1 \;\big)\; z^3 - 2z + 3 \\
z^3 - z^2 \\
\hline
z^2 - 2z + 3 \\
z^2 - z \\
\hline
-z + 3 \\
-z + 1 \\
\hline
2
\end{array}
$$