PAST PROBLEMS

Practice questions from past final exams

- $\boxed{\text{GRANGER CAUSALITY}}$

a). Define Granger causality & the two test procedure introduced in class for checking Granger causality.

b). using VAR(p) model.

$$\begin{bmatrix} y_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + \sum_{j=1}^{p} \begin{bmatrix} \phi_{j,11} & \phi_{j,12} \\ \phi_{j,21} & \phi_{j,22} \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} a_{1t} \\ a_{2t} \end{bmatrix}$$

if y_{1t} Granger cause y_{2t}

if all $\phi_{j,21} = 0$, then y_{1t} does not Granger cause y_{2t}.

if one of $\phi_{j,21} \neq 0$, then y_{1t} Granger causes y_{2t}.

c). without VAR approach, can also test Granger causality.

d). Define Granger causality and how to test Granger causality without VAR models.

Answer:

1). Granger causality is defined through the enhancement on the prediction of conditional mean. The Granger causality between $\{X_t\}$ and $\{Y_t\}$ may be defined as.

- If Y_t can be predicted by using the past values of X_t, i.e. $\{X_\tau, \tau<t\}$ than by not doing so, all other relevant information (including the past values of $\{X_t\}$) is the universe being used in either case. (X_t G. causes Y_t)

2). Alternative approach. (Pierce & Haugh 1977)

For simplicity, consider 2 causal and invertible time series $\{X_t\}$ and $\{Y_t\}$. Let both be given by:

$$\Phi_x(B)x_t = \Theta_x(B)u_t, \quad u_t \sim WN(0, \sigma_u^2)$$

$$\Phi_y(B)y_t = \Theta_y(B)v_t, \quad v_t \sim WN(0, \sigma_v^2)$$

where $x_t = X_t - \mu_x$

$y_t = Y_t - \mu_y$

$\phi_x(B), \phi_y(B), \Theta_x(B), \Theta_y(B)$ are polynomials of the backward shift operator B and satisfy all causal and invertible conditions.

cross ~~co~~correlation of lag k between u and v

is $\rho_{uv}(k) = \dfrac{E(u_t, v_{t+k})}{\sqrt{E(u_t^2)E(v_t^2)}}$

There are many possible types of causal interpretation between $\{X_t\}$ & $\{Y_t\}$

which can be characterized by ~~all~~ the properties of residual cross-correlation functions between $\hat{u_t}$ and $\hat{v_t}$.

An overall (portmanteau) test for testing Granger causality (H_0: X_t does not Granger cause Y_t) is given by

$$Q_L = n^2 \sum_{k=0}^{L} (n-k)^{-1} r_{uv}^2(k)$$

The Q_L statistic follows a χ^2_{L+1}, L is a user defined integer. We will reject null hypothesis if the p-value of Q_L is smaller than a predetermined significance level.

$\boxed{\text{Cointegration}}$

e). Define $I(0)$, $I(1)$, $I(d)$ processes.

Answer: 1) A stochastic process $\{X_t\}$ is $I(0)$ if X_t itself is a weakly stationary process.

2). $I(1)$. $\{X_t\}$ is $I(1)$ if $Y_t = (1-B)X_t$ is a weakly stationary process.

3). $\{X_t\}$ is $I(d)$ if $Y_t = (1-B)^d X_t$ is a weakly stationary process.

For B is the backward shift operator with $BX_t = X_{t-1}$ and $d>0$, $d \in \mathbb{Z}$.

f). Define cointegration.

Answer: If two or more series are individually integrated but some linear combination of them has a lower order of integration, then the series are said to be cointegrated.

g). Define $I(d)$ processes ~~ex~~ and explain how to test the cointegration of two $I(1)$ processes using a regression approach.

Answer: 1). A stochastic process $\{X_t\}$ ~~is~~ is said to be an $I(d)$ process if $Y_t = (1-B)^d X_t$ is a weakly stationary process, where B denotes the backward shift operator. $BX_t = X_{t-1}$, $d>0$, $d \in \mathbb{Z}$.

(continued page 1)

Obtain ML (or OLS) estimates of the following equations

$$y_{1t} = \alpha_1 + \sum_{j=1}^{p} \Phi_{j,11} y_{1,t-j} + e_{1t} \quad (5)$$

$$y_{1t} = \alpha_1 + \sum_{j=1}^{p} \Phi_{j,11} y_{1,t-j} + \sum_{j=1}^{p} \Phi_{j,12} y_{2,t-j} + \varepsilon_{1t} \quad (6)$$

Calculate the Log Likelihood functions in (5) & (6). And the LR statistic is given by

$$n(\log|\tilde{\Sigma}| - \log|\hat{\Sigma}|) \sim \chi^2_p$$

where $\tilde{\Sigma}$ & $\hat{\Sigma}$ denote the residual covariance matrix estimated from eqn (5) & (6), respectively.

?

2). Steps to test cointegration of 2 I(1) processes may be given as follows.

i.) Use the unit root tests, such as Augmented Dickey-Fuller test, ~~so~~ to test if the given two time series, $\{X_t\}$ and $\{Y_t\}$, follow I(1) processes.

ii). If both $\{X_t\}$ and $\{Y_t\}$ are ~~stationary~~ I(1), then regress $\{X_t\}$ against $\{Y_t\}$ or $\{Y_t\}$ against $\{X_t\}$. Let $\{\varepsilon_t\}$ denote the corresponding regression residuals.

iii). Apply the unit root test again on the regression residuals $\{\varepsilon_t\}$. If the test result show that $\{\varepsilon_t\}$ is stationary ~~or follows a I(d) process~~. Then we ~~did~~ said $\{X_t\}$ and $\{Y_t\}$ are cointegrated.

h). Define cointegration between two t.s $\{X_t\}$ and $\{Y_t\}$. Describe how to test the existence of the integration.

i). Describe the Engle-Granger method to test ~~in~~ cointegration.

?

Answer:

1). Test if data of interests are I(1) ~~or nonstationary~~ using unit root tests.

2). If the data of interest follows I(1), then run regression using the least square method.

3). Collect the residuals of the aforesaid regression and test if the residuals are stationary using unit root tests.

If the ~~sto stochastic~~ residuals do not contain stochastic trend (unit ~~to~~ root), we say that the data of interest are cointegrated.

Remarks: They may exist multiple cointegrated relationships among a set of variables.

j). Describe Granger representation theorem.

Answer:

If time series $\{X_t\}$ and $\{Y_t\}$ are cointegrated, an ECM (Error correction mechanism) must be included for their modeling process.

Specifically, consider the following bivariate VAR model:

$$\Delta X_t = \alpha_1 + \gamma_1 Z_{t-1} + \sum_i^{m_1} \beta_{1i} \Delta X_{t-i} + \sum_i^{m_2} \beta_{2i} \Delta Y_{t-i} + e_{1t}$$

$$\Delta Y_t = \alpha_2 + \gamma_2 Z_{t-1} + \sum_i^{m_3} \beta_{3i} \Delta X_{t-i} + \sum_i^{m_4} \beta_{4i} \Delta Y_{t-i} + e_{2t}$$

where $(e_{1t}, e_{2t})'$ follows a bivariate white noises, and

$Z_t = X_t - \alpha Y_t$ is an $I(0)$ process,

where $(1, -\alpha)$ is a cointegrated vector between $\{X_t\}$ and $\{Y_t\}$.

Otherwise, there will be a model misspecification in the ~~afore~~ aforesaid bivariate VAR model.

Remark 1 : also require $\gamma_1 < 0$ and $\gamma_2 > 0$ in our model

Remark 2 : cointegration between $\{X_t\}$ and $\{Y_t\}$ is a necessary condition for ECM and vice versa.

k). State the Granger ~~re~~ representation theorem. Discuss its implication on vector autoregressive (VAR) modeling.

OTHER (VAR, forecast accuracy and else)

l). Discuss a method taught in class for removing (or modeling) seasonality of time series data.

- Differencing at lag d to remove.

$$\nabla_d X_t = X_t - X_{t-d} = (1-B^d)X_t$$

m). Describe the Granger-Newbold test for compare forecast accuracy and its assumptions.

Answer:

Granger & Newbold (1976) considers the following transformation

$x_i = e_{1i} + e_{2i}$ and $z_i = e_{1i} - e_{2i}$

$i = 1, \dots, H.$

where e_{ki} stands for the one-step ahead forecast error of model k, $k=1,2$ at time $t+i$.

G&N assumes

① The forecast errors have zero mean and normally distributed.

② The forecast errors are serially uncorrelated

Under the above two assumptions and under the assumption of equal forecast accuracy (H_0), x_i and z_i should be uncorrelated

$(\because \rho_{xz} = E(xz) = E(e_1^2 - e_2^2) = 0)$

We can therefore use the sample correlation coefficient between $\{x_i\}$ and $\{z_i\}$, denoted as r_{xz}, to evaluate the accuracy between model 1 and model 2. In particular, Granger and Newbold (1976) showed that

$$\frac{r_{xz}}{\sqrt{\frac{(1-r_{xz}^2)}{H-1}}} \sim t_{H-1}$$

if assumption ① and ② hold.

Thus, if r_{xz} is not zero, reject H_0.

Specifically if $r_{xz} > 0$, model 1 has larger MPSE (less accuracy)

if $r_{xz} < 0$, model 2 has larger MPSE.

n) Consider a two dimensional vector autoregressive process of order one.

$$\begin{bmatrix} r_{2t} \\ r_{1t} \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix} + \begin{bmatrix} 1.1 & -0.6 \\ 0.3 & 0.2 \end{bmatrix} \begin{bmatrix} r_{2,t-1} \\ r_{1,t-1} \end{bmatrix} + \begin{bmatrix} a_{2t} \\ a_{1t} \end{bmatrix}$$

where $\Sigma = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ is the covariance matrix between a_{2t} and a_{1t}.

Whether VAR(1) is stationary?

Answer: Let $\Phi = \begin{bmatrix} 1.1 & -0.6 \\ 0.3 & 0.2 \end{bmatrix}$.

need the eigenvalues of this matrix ~~greater~~ less than 1 in modulus.

$\det[\Phi - \lambda I]$

$= \cancel{1+\lambda}\ (1.1-\lambda)(0.2-\lambda) + 0.6 \times 0.3 = 0$

$0.22 - 1.3\lambda + \lambda^2 + 0.18 = 0$

$\lambda^2 - 1.3\lambda + 0.4 = 0$

$\lambda = \frac{1.3 \pm \sqrt{1.69 - 1.6}}{2} = \frac{1.3 \pm 0.3}{2} = 0.8 \text{ or } 0.5$

So stationary.

FORECAST (ACCURACY) EVALUATION

ARMA(1,1) model

$$(1-0.5B)(X_t-4)=(1+0.5B)a_t, \quad a_t \sim NID(0,1)$$

Its one-step ~~at~~ forecast at $t=99$ is $\hat{X}_{99}(1)=2.09$

and $\{X_{99}, X_{100}, X_{101}, X_{102}, X_{103}, X_{104}, X_{105}\}$

$=\{2.11, 1.39, 2.57, 4.11, 6.28, 4.89, 5.94\}$ model A

a). Calculate the l step ahead forecast $\hat{X}_{100}(l)$ for $l=1,2,3$

$$X_t - 0.5X_{t-1} - 4 + 2 = a_t + 0.5a_{t-1}$$

$$X_t = 0.5X_{t-1} + 2 + a_t + 0.5a_{t-1}$$

$$X_{n+l} = 0.5X_{n+l-1} + 2 + a_{n+l} + 0.5a_{n+l-1}$$

Using the conditional expectation given filtration at time n.

$$\hat{X}_n(1) = 0.5X_n + 2 + 0.5\hat{a}_n, \quad \hat{a}_n = X_n - \hat{X}_{n-1}(1)$$

and

$$\hat{X}_n(l) = 0.5\hat{X}_n(l-1) + 2, \quad l \geq 2$$

$$\hat{a}_{100} = X_{100} - \hat{X}_{99}(1) = 1.39 - 2.09 = -0.7$$

$$\hat{X}_{100}(1) = 0.5\times 1.39 + 2 + 0.5\times(-0.7) = 2.345$$

$$\hat{X}_{100}(2) = 0.5\times 2.345 + 2 = 3.1725$$

$$\hat{X}_{100}(3) = 0.5\times 3.1725 + 2 = 3.58625$$

b). Forecast error variance for $l=1,2,3$

since $\phi = 0.5 < 1$.

$$(1-\phi B)(1+\psi_1 B + \psi_2 B^2 + \cdots) = 1 - \theta B$$

$$1 = 1$$

$$-\phi + \psi_1 = -0.5 \quad \psi_1 = 1$$

$$\psi_2 + (-0.5\psi_1) = 0$$

$$\psi_2 = 0.5$$

$$\psi_3 = 0.25 \ \cdots$$

$$var(e_n(l)) = \sigma_a^2 \sum_{j=0}^{l-1} \psi_j^2$$

$$var(e_{100}(1)) = 1^2 = 1$$

$$var(e_{100}(2)) = 1^2 + (0.5^0)^2 = 2$$

$$var(e_{100}(3)) = 1^2 + (0.5^0)^2 + (0.5^1)^2 = 2.25$$

n)

c. Describe the Granger-Newbold test for compare forecast accuracy and its assumption.

d. Model B:

a non-nested model B

$\{e_{100}(1), e_{101}(1), e_{102}(1), e_{103}(1), e_{104}(1)\} = \{0.3, 0.9, 2, -1.5, 1.8\}$

Answe.

General 1-step ahead forecast

$$\hat{X}_n(1) = 2 + 0.5X_n + 0.5\hat{a}_n, \quad \hat{a}_n = X_n - \hat{X}_{n-1}(1)$$

$\hat{X}_{100}(1) = 2.345$, $\hat{a}_{100} = -0.7$

① i. $\hat{X}_{101}(1) = 2 + 0.5X_{101} + 0.5\hat{a}_{101}$, $\hat{a}_{101} = X_{101} - \hat{X}_{100}(1) = 0.225$

ii $\hat{X}_{102}(1) = 2 + 0.5X_{102} + 0.5\hat{a}_{102}$, $\hat{a}_{102} = X_{102} - \hat{X}_{101}(1) = 0.7125$

iii $\hat{X}_{103}(1) = 2 + 0.5X_{103} + 0.5\hat{a}_{103}$, $\hat{a}_{103} = X_{103} - \hat{X}_{102}(1) = 1.86875$

iv $\hat{X}_{104}(1) = 2 + 0.5X_{104} + 0.5\hat{a}_{104}$, $\hat{a}_{104} = X_{104} - \hat{X}_{103}(1) = -1.184375$

~~$\hat{X}_{104}(1) = 3.5175$~~

~~② Let $x_i = \hat{a}_i + e_i(1)$ $i = 100, 101, 102, 103, 104$~~

~~$z_i = \hat{a}_i - e_i(1)$~~

$\hat{X}_{101}(1) = 2 + 0.5 \times 2.57 + 0.5 \times 0.225 = 3.3975$

$\hat{X}_{102}(1) = 2 + 0.5 \times 4.11 + 0.5 \times 0.7125 = 4.41125$

$\hat{X}_{103}(1) = 2 + 0.5 \times 6.28 + 0.5 \times 1.86875 = 6.074375$

$\hat{X}_{104}(1) = 2 + 0.5 \times 4.89 + 0.5 \times (-1.184375) = 3.8528125$

② Let $x_i = \hat{a}_i + e_i(1)$

$z_i = \hat{a}_i - e_i(1)$ $\quad i = 100, 101, 102, 103, 104 \quad H = 5$

i. sample correlation $\hat{r}_{xz} = \dfrac{\sum(x_i - \bar{x})(z_i - \bar{z})}{\sqrt{\sum(x_i - \bar{x})^2}\sqrt{\sum(z_i - \bar{z})^2}} =$

$x_{100} = -0.4$, $x_{101} = 1.125$, $x_{102} = 2.7125$, $x_{103} = -0.36875$, $x_{104} = 0.615625$

$z_{100} = -1$, $z_{101} = -0.675$, $z_{102} = -1.2875$, $z_{103} = 3.36875$, $z_{104} = -2.984375$

$\bar{x} = 0.884375 \approx 0.88$

$\bar{z} = -0.515625 \approx -0.52$

TRANSFER FUNCTION NOISE MODEL

TFN
Practice ~~##~~ questions.

1. TFN model is

$$y_t = \frac{-(0.53+0.37B+0.51B^2)}{1-0.57B}x_t + \frac{1}{1-0.53B+0.63B^2}\varepsilon_t$$

a). What are the values of b, s and r for this model?

lag $b=0$,

$$y_t = \frac{-0.53B^{b} - 0.37B^{b+1} - 0.51B^{b+2}}{1-0.57B^{r}}x_t \quad b+s$$

$s=2$
$r=1$

b). Form of ARIMA for the errors?

error:

$$e_t = \frac{1}{1-0.53B+0.63B^2}\varepsilon_t \sim WN$$

$$(1-0.53B+0.63B^2)e_t = \varepsilon_t$$

so AR(2)

c). lag=0 so ~~st sim~~ simultaneously happening.

2. 2500 observations

AR(2)

prewhitened input series

$$y_t = 0.4y_{t-1} + 0.2y_{t-2} + a_t$$

have.

$\hat{\sigma}_\alpha = 0.3$

$\hat{\sigma}_\beta = 0.35$

a). approximate se of cross-correlation $\rho_{\alpha\beta}$ is

$$\frac{1}{\sqrt{N_{obs}}} = \frac{1}{\sqrt{2500}} = 0.02$$

b). Which spikes on $\rho_{\alpha\beta}$ appear to be significant?

- -

c). b, s, r = ?

3. 300 observations:

AR(2) $y_t = 0.5y_{t-1} + 0.2y_{t-2} + a_t$

$\hat{\sigma}_\alpha = 0.2$, $\hat{\sigma}_\beta = 0.4$.

lag	0	1	2	3	4	5	6	7	8	9	10
$\rho_{\alpha\beta}$	0.01	0.03	-0.03	-0.25	-0.35	-0.51	-0.3	-0.15	-0.02	0.07	-0.02

$$\hat{v}_{\alpha\beta}(j) = \rho_{\alpha\beta} \frac{se(\beta)}{se(\alpha)} = \rho_{\alpha\beta} \cdot 2$$

$\hat{v}_{\alpha\beta}$	0.02	0.06	-0.06	-0.5	-0.7	-1.02	-0.6	-0.3	-0.04	0.14	-0.04

approx. se of $\rho_{\alpha\beta} = \frac{1}{\sqrt{300}} = 0.058$

95% CI.

$$\hat{v}_{\alpha\beta} \pm 1.96\,se = \hat{v}_{\alpha\beta} \pm 0.11368$$

Lower CI

Upper CI

lag	0	1	2	3	4	5	6	7	8	9	10
Lower CI	-0.09	-0.05	-0.17	-0.61	-0.81	-1.13	-0.71	-0.41	-0.15	0.03	-0.15
Upper CI	0.13	0.17	0.05	-0.39	-0.59	-0.91	-0.49	-0.19	0.07	0.25	0.07

a). which spikes on the cross-corr appear to be significant?

3, 4, 5, 6, 7, 9

b). Suppose $r = s = 2$

provide preliminary est. on w_0, w_1, w_2, δ_1 and δ_2 for:

$$v(B) = \sum_{i=0}^{\infty} v_i B^i$$

$$= \frac{w_0 - w_1 B - w_2 B^2}{1 - \delta_1 B - \delta_2 B^2} B^b$$

$b = 3$. lag 3 (easy to see)

$$y_t = \frac{w_0 - w_1 B - w_2 B^2}{1 - \delta_1 B - \delta_2 B^2} x_{t-3} = v(B) x_t \qquad vB^3$$

$v_0 = v_1 = v_2 = 0$

$v_3 = w_0 = -0.5$

$\cancel{v_4 = \delta_1 v_4}$

$\cancel{v_4 = \delta_1 v_3 - \delta_2 v_2}$

$$B^3(w_0 - w_1 B - w_2 B^2) = (1 - \delta_1 B - \delta_2 B^2)(v_0 + v_1 B + v_2 B^2 + \cdots)$$

$$B^3(w_0 - w_1 B - w_2 B^2)$$

$$= (1 - \delta_1 B - \delta_2 B^2)(v_3 B^3 + v_4 B^4 + \cdots)$$

$$= v_3 B^3 + (v_4 - v_3\delta_1)B^4 + (v_5 - v_3\delta_2 - v_4\delta_1)B^5 + \cdots$$

$\cancel{v_3}$ $\cancel{1 - \delta_1 B}$

$$\text{so } w_0 - w_1 B - w_2 B^2$$

$$= v_3 + (v_4 - v_3\delta_1)B + (v_5 - v_3\delta_2 - v_4\delta_1)B^2 + \cdots$$

$\cancel{v_3}$ $v_0 = v_1 = v_2 = 0$

$v_3 = w_0 = -0.5$

$v_4 - v_3\delta_1 = -w_1$

$v_5 - v_3\delta_2 - v_4\delta_1 = -w_2$

$v_6 - \cancel{v_3\delta_3} - v_4\delta_2 - v_5\delta_1 = 0$

$v_7 - \cancel{v_3\delta_4} \cancel{\cdots} v_6\delta_1 - v_5\delta_2 = 0$

solve.