Jan 29th

$[V]_\alpha$ What does this mean?

V vector space over field F with basis $\alpha = \{v_1, \dots, v_n\}$

$v \in V$, the coordinates of v w.r.t. α is

$$[V]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n \quad \text{where } V = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

Ex: $V = \mathbb{R}^2$

$\alpha = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_\alpha = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Ex: $V = M_2(\mathbb{R})$

$\alpha = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

$v = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

Q: $[V]_\alpha =$

$$\begin{cases} 1 = a + c \\ 2 = a + b \\ 2 = c + d \\ -1 = b + d \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$T: V \to W$ lin trans.

V has basis $\alpha = \{v_1, \dots, v_n\}$

W has basis β = has $\beta = \{w_1, \dots w_m\}$

The matrix associated to T is $[T]_\alpha^\beta$ or $[T]_{\alpha\beta}$

Given by: $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$

Then $[T]_\alpha^\beta = [a_{ij}]$ an $m \times n$ mature.

EX. $V = C''(K)$ = diff. function $\mathbb{R}^2 \to \mathbb{R}^3$

$T: V - V$: $T(f) = f'$

$W \subset V$, $W = \text{span}\{\cos\theta, \sin\theta\}$

W is a 2 digit space w/ basis given by $\{\cos\theta, \sin\theta\} = \alpha$

$T|_W = W \to W$

$T|_W \to W$

Q: $[T|_W] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$T(\cos\theta) = -\sin\theta$

$T(\sin\theta) = \cos\theta$

EX: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

rotation by θ

$[R_\theta]_\alpha^\alpha = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\alpha = \{e_1, e_2\}$, $R_\theta(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2$

$R_\theta(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2$

<u>Thm</u> : $T: V \longrightarrow W$ lin. trans.

$\alpha = \{v_1, \cdots, v_n\}$ basis of V

$\beta = \{w_1, \cdots, w_m\}$ basis of W

$v \in V$ $\qquad \underset{(F^m)}{[T(v)]_\beta} = \underset{[T]^{\gamma}_{\delta}}{[T]^\beta_\alpha} \underset{(F^n)}{[v]_\alpha}$

$v \in V$ $v = a_1 v_1 + \cdots + a_n v_n$

Evaluate $[v]_\alpha = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in F^n$

$[T]^\beta_\alpha = [b_{ij}]$; $T(v_i) = \sum_{i=1}^{m} b_{ij} w_i$

$T(v) + a_1 T(v_1) + \cdots + a_n (T(v_n)). = \sum_{k=1}^{n} a_k T(v_k)$

$= \sum_{k=1}^{n} a_k\, T(v_k) = \sum_{k=1}^{n} a^k \left(\sum_{l=1}^{m} b_{ek}\, w_e \right) = \sum_{l=1}^{m} \left(\sum_{k=1}^{n} b_{el}\, a_k w_l \right)$

$[T(v)]_\beta = [b_{ij}] \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = [T]^\beta_\alpha [v]_\alpha$

Dim Theorem $\qquad$ Assuming finite dim.

$T: V \to W$, $\dim V = \dim \ker(T) + \dim \operatorname{im} T$.

In matrix language: A $m \times n$

$n = \operatorname{rank} A + \operatorname{nullity} A$

Corollaries

V, W finite dimensional vector space $T: V \to W$. If T is surjective then $\dim V \geq \dim W$

On the other hand, if T is injective $\dim V \leq \dim W$

§ 2.5 Composition

$$\begin{tikzcd} U \arrow[r, "S"] \arrow[rr, bend right, "T\circ S"'] & V \arrow[r, "T"] & W \end{tikzcd}$$

TS

DEF: $TS(u) = T(S(u))$

"composition of lin. trans."

Lemma : $S\,T$ lin. trans. then TS is also.

Proof: $TS(u_1 + u_2) = T(S(u_1 + u_2))$

$= T(S(u_1) + S(u_2))$

$= T(S(u_1)) + T(S(u_2))$

$= TS(u_1) + TS(u_2)$ also scalar multi

EX: ① $R_\theta \cdot R_\tau = R_{\theta+\tau}$

② $\mathrm{Proj}_x : \mathbb{R}^2 \to \mathbb{R}^2$

$\mathrm{Proj}_y : \mathbb{R}^2 \to \mathbb{R}^2$

$\mathrm{Proj}_y \cdot \mathrm{Proj}_x = 0$

③ $T: V \to V$

$I \cdot T = T$

④ $T = R_{\pi/2}$ S = reflection about y-axis

$S \cdot T(e_1) = S(e_2) = e_2$

$T \cdot S(e_1) = T(-e_1) = -e_2$

$S \cdot T \neq T \cdot S$

⑤ If S, T injective, is ST also inj.?

$V \xrightarrow{T} W \xrightarrow{S} U$

Look at $\ker ST$

$v \in \ker ST \Rightarrow S(T(v)) = 0$

$\Rightarrow T(v) = 0$

$\Rightarrow v = 0$

$\Rightarrow \ker ST = \{0\}$

Properties of compositions

① $T(SR) = (TS)R$

$U \xrightarrow{R} V \xrightarrow{S} W \xrightarrow{T} X$

② $U \xrightarrow{R,S} V \xrightarrow{T} W$

$T(R+S) = TR + TS$

③ $U \xrightarrow{R} V \xrightarrow{S,T} W$

$(S+T)R = SR + TR$

Thm: $U \xrightarrow{S} V \xrightarrow{T} W$

U has basis $\alpha = \{u_1, \dots, u_l\}$

V - - - $\beta = \{v_1, \dots, v_m\}$

W - - - $\gamma = \{w_1, \dots, w_n\}$

$[TS]_\alpha^\gamma = [T]_\beta^\gamma [S]_\alpha^\beta$

composition matrix mult.

Proof: $S(u_j) = \sum_{i=1}^{m} a_{ij} v_i$

so $[S]_\alpha^\beta = [a_{ij}]$

$\cdot T(v_s)=\sum_{r=1}^{n} b_{rs} w_r$ so $[T]_\beta^\gamma = [b_{rs}]$

$\cdot TS(u_j)=\sum_{r=1}^{n} c_{rj} w_r$

$[TS]_\alpha^\gamma = [c_{rj}]$

WANT: $[c_{rj}]=[b_{rs}][a_{ij}]$

$$\boxed{\text{i.e. } c_{rj}=\sum_{i=1}^{m} b_{ri}\, a_{ij}}$$

$$TS(u_j) = T(S(u_j)) = T\left(\sum_{i=1}^{m} a_{ij} v_i\right) = \sum_{i=1}^{m} a_{ij}\, T(v_i) = \sum_{i=1}^{m} a_{ij}\left(\sum_{r=1}^{n} b_{ri} w_r\right)$$

$$= \sum_{r=1}^{n}\left(\sum_{i=1}^{m} b_{ri}\, a_{ij}\right) w_r$$

$$\Longrightarrow c_{rj} = \sum_{i=1}^{m} b_{ri}\, a_{ij} \quad \blacksquare$$

Ex: V vector space over F of dim n. $\alpha=\{v_1,\cdots,v_n\}$

Def: The dual of V is $V^*=L(V,F)=\{T: V\to F\}$

Recall: given $T: V\to W$ is determined by $T(v_i)$ for $i=1,2,\cdots,n$. in particular when $W=F$ then T is determined by n numbers $T(v_1),\cdots,T(v_n)$

Q: $\dim V^*$? Basis?

$v_i^*: V\to F \quad v_i^*(v_j)=\begin{cases} 1 & i=j \\ 0 & \text{else}\end{cases}$

Claim: $\{v_1^*,\cdots,v_n^*\}$ is a basis of V^*.

Proof: Sps $T\in V^*$, i.e. $T: V\to F$. and let $a_i=T(v_i)$. Then $T=a_1v_1^*+\cdots+a_nv_n^*$ since $(a_1v_1^*+\cdots+a_nv_n^*)(v_i)=a_1v_1^*(v_i)+\cdots+a_nv_n^*(v_i)=a_i$

This shows $\{v_1^*,\cdots,v_n^*\}$ span V^*. For lin. ind. sps $a_1v_1^*+\cdots+a_nv_n^*=0$

Apply v_i to both sides

$a_i=0$

$\Longrightarrow$ indep.