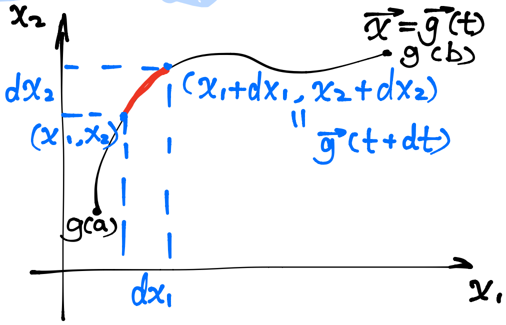
July 23rd

Chapter 5

§5.1 Arc length & Line Integral



$d\vec{x} = (dx_1, dx_2, \cdots, dx_n)$

$d\vec{x} = \vec{g}(t+dt) - \vec{g}(t)$

$\approx g'(t)dt = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \cdots, \frac{dx_n}{dt}\right) \cdot dt$

Def: $dS = |d\vec{x}| = \sqrt{(\ )^2 + (\ )^2 + \cdots + (\ )^2} \cdot dt = |\vec{g}'(t)|\,dt$

Def: Arc Length $= \int_C dS = \int_a^b |g'(t)|\,dt$

curve

Remark: ① The arc length of C is an intrinsic property of the geometric object C and should not depend on the particular parameterization we use.
Indeed. consider a new parameter u and $t = \varphi(u)$, φ is a one-to-one mapping from $[c, d]$ to $[a, b]$, $\varphi(c) = a$, $\varphi(d) = b$, $\vec{g}(t) = \vec{g}(\varphi(u))$

Arc length $= \int_c^d |(\vec{g} \circ \varphi)'(u)|\,du = \int_c^d |\vec{g}'(\varphi(u)) \cdot \varphi'(u)|\,du$

$= \int_c^d |\vec{g}'(\varphi(u))|\,|\varphi'(u)|\,du$

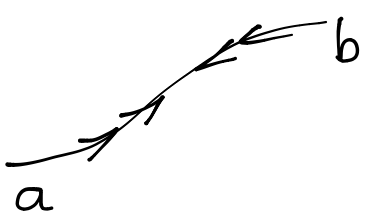
$= \int_{[a,b]} |g'(t)|\,dt$

Remark ② Consider $\int_a^b \vec{g}(t)\,dt$ $(**)$

When $t = \varphi(u)$ goes from another direction. $\varphi(c) = b$, $\varphi(d) = a$

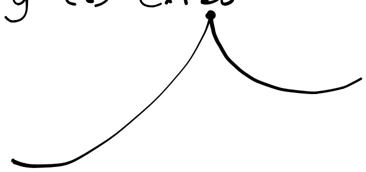
The initial and final points get switched

Then we need to multiply by -1
we call this change of orientation



> This integral $(**)$ depends on the parameterization only in so far as the parameterization determines a choice of orientation.

Remark ③ Precise smooth curve is a function $\vec{g}:[a,b]\longrightarrow \mathbb{R}^n$
(1) it is continuous
(2). its derivative exists and is continuous except perhaps at finite many points $\{t_j\}$, and at t_j, one side limits $\lim_{t\to t_j^{\pm}} \vec{g}'(t)$ exist



Arc length $= \sum_{i=2}^{K} \int_{t_{i-1}}^{t_i} |\vec{g}'(t)|\,dt$

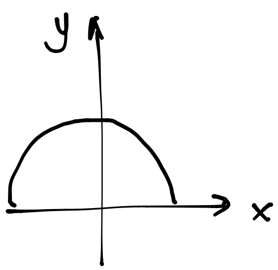
Line Integral

Def: If C is parametrized by $\vec{x}=\vec{g}(t)$, $a\le t\le b$
we define $\int_C f\,ds = \int_a^b f(\vec{g}(t))\,|\vec{g}'(t)|\,dt$ as line integral

Remark: This is independent of parametrization and the orientation by chain rule as the arc length.

$(\bar{x},\bar{y})$ is the average of x, y over C

Eg 1 · What is the centroid of the upper half of the unit circle
$C=\{(x,y) : x^2+y^2=1, y\ge 0\}$



Def: average of f over $C = \dfrac{\int_C f\,ds}{\text{Arc length of } C} = \dfrac{\int_C f\,ds}{\int_C ds}$

$\bar{x}=0$ by symmetry

we need to find $\int_C y\,ds = \int_0^{\pi} \sin\theta\sqrt{\sin^2\theta+\cos^2\theta} = \int_0^{\pi} \sin\theta\,d\theta$
$= -\cos\theta\Big|_0^{\pi} = 2$

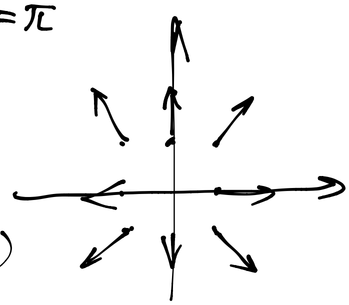
$x=r\cos\theta,\ y=r\sin\theta \overset{r=1}{\Longrightarrow} x=\cos\theta,\ y=\sin\theta$

$dx=-\sin\theta\,d\theta,\ dy=\cos\theta\,d\theta$

$\int_C ds = \int_0^{\pi} \sqrt{(-\sin\theta)^2+\cos^2\theta}\,d\theta = \int_0^{\pi} d\theta = \pi$

Line Integral of vector Fields

Vector Field: $F: \mathbb{R}^n \to \mathbb{R}^n$
Eg. $F(x,y)=(x,y)$



If we write $\vec{F}=(F_1, F_2, \cdots, F_n)$

Line Integrals over a vector field
$\int_C \vec{F}\,ds = (\int_C F_1\,ds, \cdots, \int_C F_n\,ds)$

Prop 5.8: If F is a continuous $\mathbb{R}^n$-valued function on $[a,b]$, then $|\int_a^b F(t)\,dt| \le \int_a^b |F(t)|\,dt$

Pf: $\forall$ unit vector $\vec{u}$, i.e. $|\vec{u}|=1$, we have

$$\left|\left(\int_a^b \vec{F}(t)dt\right)\cdot\vec{u}\right| = \left|\int_a^b \vec{F}(t)\cdot u\,dt\right| \leq \int_a^b |\vec{F}(t)\cdot\vec{u}|dt \overset{\text{Cauchy}}{\leq} \int_a^b |\vec{F}(t)||\vec{u}|dt$$

$$= \int_a^b |\vec{F}(t)|dt$$

Let $\vec{u} = \dfrac{\int_a^b \vec{F}dt}{\left|\int_a^b \vec{F}dt\right|}$, $\left|\int_a^b \vec{F}(t)dt\cdot\vec{u}\right| = \left|\int_a^b \vec{F}dt \, \dfrac{\int_a^b \vec{F}dt}{\left|\int_a^b \vec{F}dt\right|}\right|$

$$= \left|\int_a^b \vec{F}dt\right|$$