May 14th

Instructor: Trefor Bazett

- Quiz 1 (linear Algebra) on May 21st
- problem sessions TBA (optional)

Def: A set is a collection of objects called elements.

Ex: $S=\{1,2,7\}$, $7\in S$

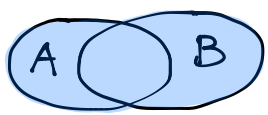
$S=\{x \mid P(x) \text{ is true}\}$

$S=\{x\in\mathbb{R}, x>0\}=\mathbb{R}^+$

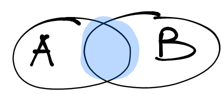
Def: A subset $A\subset B$ if $\forall x\in A$, then $x\in B$.

Read: "qualifier" doc on portal

Def: if A, B are sets, the union $A\cup B=\{x \mid x\in A \text{ or } x\in B\}$

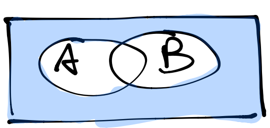


Def: $A\cap B=\{x \mid x\in A \text{ and } x\in B\}$ "intersection"



Def: If a U is known, $A^c=\{x\in U \mid x\notin A\}$ "complement"

$(A\cup B)^c$



Def: Empty set $\emptyset=\{\ \}$

Def: The Cartesian Product

$A\times B=\{\underbrace{(x,y)}_{\text{"2-tuple"}} \mid x\in A, y\in B\}$

Note: As $x\in A$ or $x\in A^c$, $A\cup A^c=U$, $A\cap A^c=\emptyset$.

$f: A\to B$ A is called domain, B is called codomain.
B is not called "range" b/c range is $f(A)$.

Define $f: A \to B$ is a subset of $A \times B$ s.t. $\forall x \in A$, $\exists$ exactly one pair $(x,y) = (x, f(x))$.

Def: A <u>binary operation</u> is a function $f: S \times S \to S$

Ex: $+$ $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$
$\cdot$ $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$

Def: A <u>binary relation</u> is a subset A of $S \times S$.
Two elements are related if $(x,y) \in A$.

Def: A <u>total ordering</u> "$\leq$" is a binary relation s.t.

1). if $a \leq b$, $b \leq a$ then $a = b$ (antisymmetry)

2). if $a \leq b$, $b \leq c$ then $a \leq c$ (transitivity)

3). Either $a \leq b$ or $a \geq b$ (totality)

The real numbers have two binary operations:

$+$ (addition) and $\cdot$ (multiplication). which go from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

satisfy

Field
Abelian Group
1). $(a+b)+c = a+(b+c)$ — Associativity
2). $a+b = b+a$ — Commutativity
3). $\exists 0$ s.t. $a+0 = 0+a = a$ — Additive Identity
4). $\forall a$, $\exists y$ s.t. $a+y = y+a = 0$ — Addtive Inverse
5). $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ — Associativity
6). $a \cdot b = b \cdot a$ — Commutativity
7). $\exists 1$ s.t. $a \cdot 1 = 1 \cdot a = a$ — Multiplicative Identity
8). $\forall a \neq 0$, $\exists y$ s.t. $ay = ya = 1$. — Multiplicative Inverse

substraction, division($\neq 0$) defined as above.

9). $(a+b) \cdot c = a \cdot c + b \cdot c$ — Distributivity
$a \cdot (b+c) = a \cdot b + a \cdot c$

An ordering $\geq$ s.t.
10). if $a \leq b$, $a+c \leq b+c$
11). $x \leq 0$, $y \geq 0$, $xy \leq 0$.

Totally order Field.

Def: A non-empty set A is <u>bounded above</u> if $\exists M$ s.t. $x < M \ \forall x \in A$. (not just for $M \in \mathbb{R}$)

such an M is an <u>upper bound</u>

Def: A <u>least upper bound (lub)</u> a is one where any other upper bound b has $a \leq b$.

12). If a non-empty set has an upper bound it has a lub.

Ex: $\{x \mid x \leq \sqrt{2}\}$

Def: $\mathbb{R}$ has prop. 1–12 $\leftarrow$ one can prove that there is only one complete totally ordered field

To $\mathbb{R}^n$: $\mathbb{R}^n$ is the cartesian product of n $\mathbb{R}$s

i.e. $\mathbb{R}^n$ is $\underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n\text{-times}}$ which is an n-tuple.

$\vec{x} \in \mathbb{R}^n$, $\vec{x} = (x_1, \cdots, x_n)$, $x_i \in \mathbb{R}$ $\leftarrow$ vector

$\vec{0}$ "zero vector" $= (0, \cdots, 0)$

$\boxed{\text{"set \& functions" doc on portal.}}$

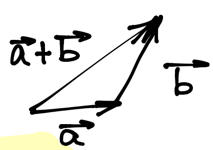
1). There is an operation called vector addition
$\vec{x} + \vec{y} = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

satisfying properties

$\vec{x} + \vec{y} = (x_1 + y_1, \cdots, x_n + y_n) = (y_1 + x_1, \cdots, y_n + x_n) = \vec{y} + \vec{x}$ (commutativity)

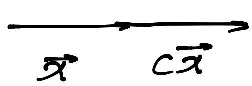
(You can prove other properties yourself ^_^)

Geometric meaning :



2). Scalar Multiplication :
for $c \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$, $c\vec{x} = (cx_1, \cdots, cx_n)$

Geo meaning



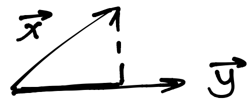
1) & 2) $\Rightarrow$ vector space
Exercise : Verify these defs satisfy vector space axioms

Note : Scalar Multi. $\mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

3). Dot Product

$$\vec{x}\cdot\vec{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$$

Geo meaning :



length of the projection is $\vec{x}\cdot\vec{y}$

1), 2) & 3) $\Rightarrow$ Inner Product Space.

- No canonical ordering in $\mathbb{R}^n$!
  Since you cannot compare (0,1,0) & (0,0,1), which one is larger?

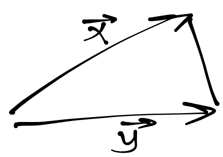
4). Cross Product (talk it later)

Def : "norm" $|\vec{x}| = \sqrt{\vec{x}\cdot\vec{x}} = \sqrt{x_1^2+\cdots+x_n^2}$ where $\vec{x} = (x_1, \cdots, x_n)$

We use either double-bar $\|\ \ \|$ or single-bar $|\ \ |$ to represent norm.

Define distance as $d(\vec{x},\vec{y}) = |\vec{x}-\vec{y}|$

Geo meaning :



$d(\vec{x},\vec{y})$

Cauchy–Schwarz Inequality

For an i.p.s V (Ex: $\mathbb{R}^n$)

$\forall a,b \in \mathbb{R}^n$, $|\vec{a}\cdot\vec{b}| \leq |\vec{a}|\cdot|\vec{b}|$

Proof: if $\vec{b}=0$, done.

Let's have $\vec{b} \neq \vec{0}$

Define a function $f(t) = |\vec{a}-t\vec{b}|^2 = (\vec{a}-t\vec{b})\cdot(\vec{a}-t\vec{b})$
$= |\vec{a}|^2 + t^2|\vec{b}|^2 - 2t\,\vec{a}\cdot\vec{b}$

It's a quadratic func. $2t|\vec{b}|^2 - 2\vec{a}\cdot\vec{b} = 0 \Rightarrow t_m$

so $f'(t_m) = 0$ $\leftarrow$ minimum

$\Rightarrow t_m = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$ $\qquad f(t_m) \leq f(t)$

$$f(t) \geqslant f(t_m) = |\vec{a}|^2 + \frac{(\vec{a}\cdot\vec{b})^2 |\vec{b}|^2}{|\vec{b}|^4} - \frac{2\,\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$$

$$= |\vec{a}|^2 - \frac{(\vec{a}\cdot\vec{b})^2}{|\vec{b}|^2} \geqslant 0$$

$$|\vec{a}|^2 |\vec{b}|^2 \geqslant (\vec{a}\cdot\vec{b})^2$$

Take $\sqrt{}$:

$$|\vec{a}|\cdot|\vec{b}| \geqslant |\vec{a}\cdot\vec{b}|$$

$\blacksquare$

Note: Equality when $f=0 \Rightarrow \vec{a} = t\vec{b}$ "linear"

Def: $\vec{x}$ is <u>orthogonal</u> to $\vec{y}$ if $\vec{x}\cdot\vec{y} = 0$

Triangle Inequality

$\forall \vec{a}, \vec{b} \in V$, $|\vec{a}+\vec{b}| \leqslant |\vec{a}| + |\vec{b}|$

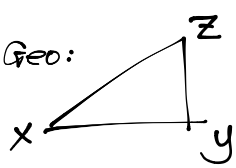
Proof: $|\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2$

$\leqslant |\vec{a}|^2 + |\vec{b}|^2 + 2|\vec{a}||\vec{b}|$ (by Cauchy Ineq.)

$= (|\vec{a}| + |\vec{b}|)^2$

Take $\sqrt{}$: $|\vec{a}+\vec{b}| \leqslant |\vec{a}| + |\vec{b}|$

Geo:



$$|\vec{x} - \vec{z}| \leqslant |\vec{x} - \vec{y}| + |\vec{y} - \vec{z}|$$

Def:

$$\theta = \arccos\left(\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}\right)$$

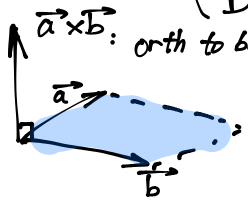
$\hookrightarrow$ this part $\in [-1, 1] \leftarrow$ arccos is defined.

Back to Cross Product:

$\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$, $\hat{k} = (0,0,1)$

$$\vec{a}\times\vec{b} = \det\begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} = \hat{i}(a_2b_3 - a_3b_2) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_3 - a_3b_1)$$

$\vec{a}\times\vec{b}$: orth to both



$\rightarrow$ and $|\vec{a}\times\vec{b}|$ = area of parallelogram generated by $\vec{a}, \vec{b}$

properties of Cross Product

* distributes
* anticommutativity $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$
* not associative $\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$
* But $\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$

Jacobi Identity

-The End of § 1.1

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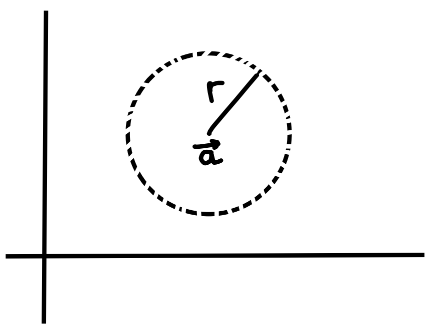
§ 1.2

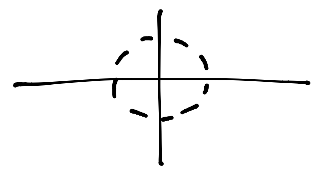
Def: "OPEN BALL"

$B^n(r,a) = \{x \in \mathbb{R}^n \mid |x-a| < r\}$

Note: "n" is usually ignored.



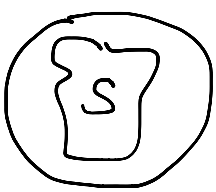
Ex: $B^2(1,0)$



$B^1(1,3) = (2,4)$
open interval

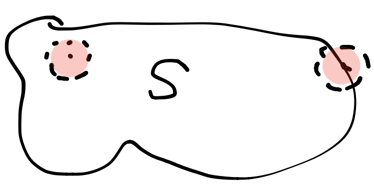
Def: A set is <u>bounded</u> if $\exists R>0$ s.t. $|x|<R, \forall x \in S$. $\Leftrightarrow S \subset B(R,0)$

Ex:



S is bounded since we have larger ball outside.

Def: $S^{int} = \{x \in S \mid \exists r>0, \text{ s.t. } B(r,x) \subset S\}$



$x \in S^{int}$ is an "interior point".

$x \in \partial S$ is a "boundary point".

Def: $\partial S = \{x \in \mathbb{R}^n \mid \forall r>0, B(r,x)\cap S \neq \emptyset \text{ and } B(r,x)\cap S^c \neq \emptyset\} = \partial S^c$

none-trivial intersection with both S & S^c.

either $x \in S$ or $x \in S^c$.

take $x \in S$, $x \notin S^{int} \Longleftrightarrow \forall r > 0$, $B(r, x) \not\subset S$

$\Longleftrightarrow \forall r > 0$, $B(r, x) \cap S^c \neq \emptyset$

clearly $B(r, x) \cap S \neq \emptyset$ as x is in it.

$\Longleftrightarrow x \in \partial S$

Do the same for $x \in S^c$, $x \notin (S^c)^{int} \Longleftrightarrow x \in \partial S^c$

Thus, 3 cases: $x \in S^{int}$, $x \in \partial S$, $x \in S^c$. (disjoint cases)