lecture 1

Probability.

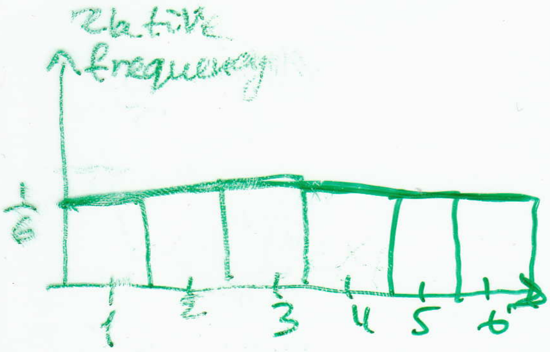
- What is probability?

It is a measure of one's belief in the occurrence of a future event.

Ex. Roll a die  Is it balanced?

{1, 2, 3, 4, 5, 6} - possible outcomes

If a die is balanced and roll it 600 times, then we expect $\frac{1}{6} \cdot 600 = 100$ times to be '1', 100 times to be '2' ...



What if we roll a die 10 times and had '1's all the time?

$\Rightarrow$ conclude: not balanced

If in the beginning we propose the hypothesis that the die is balanced, and we have all 1's, we'll have to reject our hypothesis.

Theory of probability allows us to calculate the probability of observing specified outcomes, assuming that our hypothesized model is correct.

Set Notation.

We use A, B, C to denote sets

Ex: $A = \{1,2,3\}$

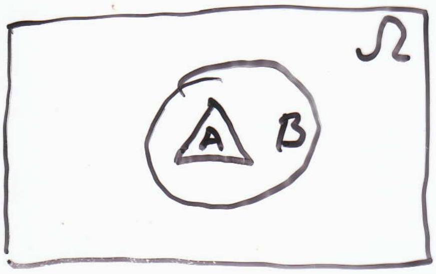
Let Ω denote the set of all elements under consideration; that is, Ω is the universal set.

A is a subset of B, $A \subset B$, if for any $x \in A$, $x \in B$

An empty set, $\emptyset$, is a set that contains no elements

Note: $\emptyset$ is a subset of any set.

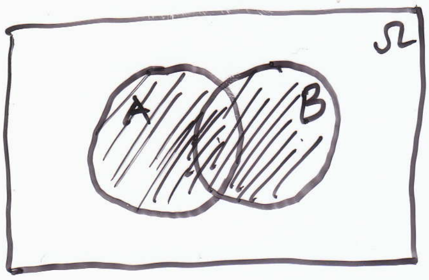
Venn diagrams:



$A \subset B$

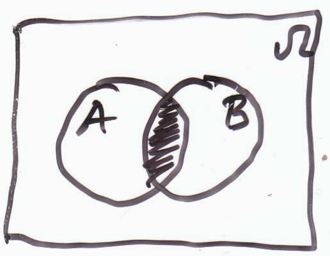
Union:

$$A \cup B = \{x \in \Omega : x \in A \text{ or } x \in B\}$$



Intersection:

$$A \cap B = \{x \in \Omega : x \in A \text{ and } x \in B\}$$

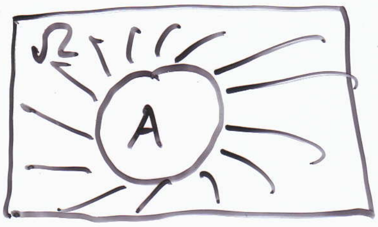


Note: $A \subset B \Rightarrow A \cap B = A$

Complement:

$$\bar{\in}$$

$$\bar{A} = A^c = \{x \in \Omega : x \notin A\}$$

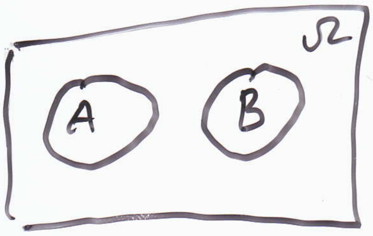


$$A \cup \bar{A} = \Omega$$

$$A \cap \bar{A} = \emptyset$$

Disjoint (mutually exclusive) sets:

A and B are disjoint, if $A \cap B = \emptyset$



Ex: $\Omega = \{1, 2, 3, 4, 5, 6\}$
$A = \{1, 2\}$, $B = \{1, 4\}$, $C = \{2, 5, 6$

$A \cup B = \{1, 2, 4\}$, $A \cap B = \{1\}$,
$\bar{A} = \{3, 4, 5, 6\}$, $B \cap C = \emptyset$

Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof: Use Venn's diagrams

De Morgan's laws:

$$(\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

$$(\overline{A \cup B}) = \bar{A} \cap \bar{B}$$

Proof: use Venn's diagrams

or $x \in \overline{A \cap B} \Leftrightarrow x \in \bar{A} \cup \bar{B}$

Note: For any collection of sets $A_1, A_2, A_3, \ldots$ in any universal set Ω

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c$$

Def. – A set A is called **finite** if it contains a finite number of elements.

– A set A is called **countable infinite** if it can be put into a one-to-one correspondence with the set of positive integers $\mathbb{N}$.

Ex. $A = \{1, 2, 3\}$ – finite

$\mathbb{Z} = \{$all integers$\}$ – countable

-3	-2	-1	0	1	2	3
7	5	3	1	2	4	6

$\mathbb{Q} = \{$all rational~~s~~ numbers$\}$ – countable (prove it)

I, $\mathbb{Q}^c = \{$irrational~~s~~ numbers$\}$ – uncountable set

$(0,1) = \mathbb{Q}_{(0,1)} \cup I_{(0,1)} \rightarrow$ uncountable set

Probability Model.

Def. An experiment is the process by which an observation is made. An experiment can result in one or more outcomes, which are called events.

Ex. We toss a die:

Consider the following events:

A: observe an odd number, i.e. $A = \{1, 3, 5\}$

B: observe a number less than 4, i.e. $B = \{1, 2, 3\}$

C: observe a 4 or a 6, i.e. $C = \{4, 6\}$

E_1: observe a 1, $E_1 = \{1\}$

E_2: observe a 2

⋮

E_6: observe a 6 $E_6 = \{6\}$

Note; $A = E_1 \cup E_3 \cup E_5$

Def. An event A, which can be decomposed into other events, is called a compound event. An event that cannot be decomposed is called a simple event. (e.g. $E_1, E_2, \ldots, E_6$) Each simple event corresponds to one and only one point, called a sample point.

Def. The set consisting of all possible sample points is called the sample space, Ω, associated with an experiment.

Def. A **σ-algebra**, $\mathcal{F}$, is a collection of subsets of Ω satisfying the following properties:

- $\mathcal{F}$ contains $\emptyset$ and Ω
- $\mathcal{F}$ is closed under taking complement, i.e. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$
- $\mathcal{F}$ is closed under taking countable union, i.e.

$$A_1, A_2, A_3, \ldots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$$

Claim: $\mathcal{F}$ is closed under countable intersection.

Proof:

$$A_1, A_2, \ldots \in \mathcal{F} \Rightarrow A_1^c, A_2^c, \ldots \in \mathcal{F}$$

$$\Rightarrow \bigcup_{n=1}^{\infty} A_n^c \in \mathcal{F}$$

$$\left[\bigcap_n A_n\right]^c = \bigcup_n A_n^c \in \mathcal{F}$$

$$\left(\left[\bigcap A_n\right]^c\right)^c = \bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$$

∎

Def. Suppose Ω is a sample space associated with an experiment.
To every event A in $\mathcal{F}$ ($A \subset \mathcal{F}$), we assign a number, $P(A)$, called the probability measure of A, mapping $\mathcal{F} \to [0,1]$, so that the following axioms hold:

axioms of probability

Axiom 1: $P(A) \geq 0$
Axiom 2: $P(\Omega) = 1$
Axiom 3: If $A_1, A_2, \ldots \subset \mathcal{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$,
then $P(A_1 \cup A_2 \cup \ldots) = \sum_{i=1}^{\infty} P(A_i)$ ← countable additivity

Finite case: $P(A_1 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) = P(A_1) + \ldots + P(A_n)$

Ex. We toss 2 balanced coins.

$$\Omega = \{\underset{\omega_1}{HH}, \underset{\omega_2}{TT}, \underset{\omega_3}{HT}, \underset{\omega_4}{TH}\} = \{\omega_i\}_{i=\overline{1,4}}$$

$$P(\omega_i) = \frac{1}{4}$$

$A = \{\text{exactly } 1 \text{ head}\} = \{TH, HT\}$

$P(A) = \frac{1}{2}$

$B = \{\text{at least } 1 \text{ head}\} = \{TH, HT, HH\}$

$P(B) = \frac{3}{4}$

$P(A \cap B) = P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{4} = P(B)$

$P(\bar{A} \cup B) = P(\Omega) = 1$.

Proposition: $P(\emptyset) = 0$

Proof: for any set $A = A \cup \emptyset$

$$A \cap \emptyset = \emptyset$$

$$P(A) = P(A \cup \emptyset) \underset{\uparrow \text{ by additivity}}{=} P(A) + P(\emptyset) \Rightarrow P(\emptyset) = 0 \quad \blacksquare$$

Probability Model: $(\Omega, \mathcal{F}, P)$

Ω - sample set

$\mathcal{F}$ - σ-algebra, - collection of subsets of Ω

P - probability measure, $: \mathcal{F} \to [0, 1]$

Def. A sample space is **discrete** if it contains either a finite or a countable number of distinct sample points.

Suppose $\Omega = \{\omega_1, \omega_2, \dots, \omega_n\}$

any A, $P(A) = \sum_{\omega_i \in A} P(\omega_i)$

If the outcomes are equally likely, then

$P(\omega_1) = P(\omega_2) = \dots = P(\omega_n) = \frac{1}{n}$

$$P(A) = \frac{\#\text{ of outcomes for which } A \text{ occurs}}{n} = \frac{n_A}{n}$$

Ex. roll a die, $A = \{\text{even number}\} = \{2, 4, 6\}$

$$P(A) = \frac{3}{6} = \frac{1}{2}$$

Combinatorics.

Multiplication Principle:

Suppose we are to make a series of decisions. Suppose there are c_1 choices for decision 1 and for each of these there are c_2 choices for decision 2, etc. Then the number of ways the series of decisions can be made is $c_1 \cdot c_2 \cdot c_3 \cdot \ldots$

Ex. Suppose you have 2 pairs of jeans, 3 shirts and 2 pairs of shoes. In how many ways you can choose an outfit?

Sol'n:

$$2 \cdot 3 \cdot 2 = 12$$

Ex. How many 3 letter words can be composed from the English Alphabet s.t.

(1) No limitation;

(2) The words have 3 different letters.

Sol'n: (1) 26^3

(2) $26 \cdot 25 \cdot 24$

Ex. How many 3-digit # are there?

$$\underline{9}\ \underline{10}\ \underline{10} = 900$$

All digits different: $\underline{9} \cdot \underline{9} \cdot \underline{8} = 648$

Combinations:

Def. The number of combinations of n objects taken k at a time is the number of subsets, each of size k, that can be formed from the n objects

Notation: C_k^n or $\binom{n}{k}$ ("n choose k")

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \qquad n! = n(n-1)\cdot\ldots\cdot 2\cdot 1$$
$$0! = 1$$

Facts:

① $\binom{n}{0} = 1 = \binom{n}{n}$ $\qquad \binom{n}{n-1} = n$

② $\binom{n}{1} = n = \binom{n}{n-1}$

③ $\binom{n}{k} = \binom{n}{n-k}$

Ex. Two cards are drawn from a standard 52-card playing deck. Find P(ace & face card).

Sol'n: $\Omega = \{$two cards$\}$, $\text{card}(\Omega) = \{$# of elements in $\Omega\}$
cardinality

$\text{card}(\Omega) = \binom{52}{2} = 1326$

$A = \{$ace & face card$\}$, $\text{card}(A) = \binom{4}{1}\binom{12}{1} = 4\cdot 12 = 48$

$$P(A) = \frac{48}{1326} = 0.0362$$

<u>Fact</u>: The number of subsets of a set of size n is 2^n.

<u>Ex</u>. $A = \{a_1, a_2\}$ there are $2^2 = 4$ subsets
$\emptyset, A, \{a_1\}, \{a_2\}$

Permutation:

<u>Def</u>. An ordered arrangement of n distinct objects is called a permutation.

The number of ordered arrangements or permutations of k subjects selected from n distinct objects is $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1)$.

Notation: $P_k^n = n \cdot (n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$

<u>Ex</u>. $n = 3$, $k = 2$ $\{1, 2, 3\}$
$(1,2), (2,3), (1,3), (2,1), (3,2), (3,1) \leftarrow \boxed{P_2^3 = \frac{3!}{1!} = 6}$

<u>Note</u>: The number of ordered arrangements of k subjects selected with replacement from n objects is n^k. $(1,1)\ (2,2)\ (3,3)$ $3^2 = 9$

<u>Ex</u>. (a) How many 2-digit numbers are there?
(b) digits must be different.

<u>Sol'n</u>: (a) $9 \cdot 10 = 90$

(b) $9 \cdot 9 = 81$

The Binomial Theorem:

For any numbers a, b, and any positive integer n

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$\binom{n}{k}$ is called a binomial coefficient.

Multinomial Coefficients:

The number of ways to partition n distinct objects into k distinct groups containing $n_1, n_2, \dots, n_k$ objects respectively, where each object appears in exactly one group and $\sum_{i=1}^{k} n_i = n$, is

$$\binom{n}{n_1\, n_2 \dots n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k!}$$

$$(a_1 + a_2 + \dots + a_k)^n = \sum_{n_i} \underbrace{\binom{n}{n_1\, n_2 \cdots n_k}}_{\text{multinomial coefficient}} a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}$$

Ex. A committee of 10 members is to be divided into 3 subcommittees of size 4, 3, and 3. In how many ways can it be done?

Sol'n: $n = 10$ $\quad n_1 = 4,\ n_2 = 3,\ n_3 = 3$

$n_1 + n_2 + n_3 = n$

$$\frac{10!}{4!\,3!\,3!} = 4200$$

■

Prove that # of subsets of a set of size n is 2^n.

Pf:

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

\# of all subsets = # of subsets with 0 elements

+ \# - - - 1 element

+ \# - - - 2 elements

⋮

+ \# n elements

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}$$

$a = b = 1$

$$\sum_{i=0}^{n} \binom{n}{i} = \sum_{i=0}^{n} \binom{n}{i} 1^i 1^{n-i} = (1+1)^n = 2^n \quad ∎$$