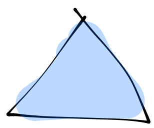
Today – §1.3 (end) and §1.4

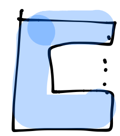
Def'n: A subset $S \subseteq \mathbb{R}^n$ is convex provided, for any $x_1 \in S$ and $x_2 \in S$, the line segment joining x_1 and x_2 is a subset of S.

Remark: This means, for any $x_1 \in S$ and $x_2 \in S$ and $\lambda \in [0,1]$, we have $x_1 + \lambda(x_2 - x_1) \in S$

Eg. In $\mathbb{R}^2$



is convex



is not convex

(because line isn't completely inside the slope.

See page 79.

Theorem 1.1 A closed half-space is convex.

Proof: Let S be a closed half-space in $\mathbb{R}^n$.

There exist $c \in \mathbb{R}^n$, $c \neq 0 \in \mathbb{R}^n$ and a real number k such that $x \in S$ iff $c^T x \leq k$.

Now fix $x_1 \in S$ and $x_2 \in S$, and λ in $[0,1]$ to show $x_1 + \lambda(x_2 - x_1) = (1-\lambda)x_1 + \lambda x_2 \in S$, that is, $c^T((1-\lambda)x_1 + \lambda x_2) \leq k$

But $c^T((1-\lambda)x_1 + \lambda x_2) = \underbrace{(1-\lambda)}_{\geq 0} c^T x_1 + \underbrace{\lambda}_{\geq 0} c^T x_2 \leq (1-\lambda)k + \lambda k = k$

Theorems 1.3 The intersection of finitely many convex sets is convex.

Proof: Let $S_1, \cdots, S_m$ be convex subsets of $\mathbb{R}^n$, to show $S_1 \cap \cdots \cap S_m$ is convex. Pick x_1 and x_2 in $S_1 \cap \cdots \cap S_m$ and pick $\lambda \in [0,1]$. For any $i = 1, \cdots, m$, $x_1 + \lambda(x_2 - x_1) \in S_i$ (S_i is convex) Thus $x_1 + \lambda(x_2 - x_1) \in S_1 \cap \cdots \cap S_m$.

Theorem 1.2 A hyperplane is convex.

Proof: Any hyperplane is the intersection of 2 half-spaces.

(The hyperplane given by $c^T x = k$ is the intersection of the half-spaces given by $c^T x \leq k$ and $c^T x \geq k$.)

Eg. The empty set is convex.

Theorem 1.4 (generalized)

The feasible region of a linear programming problem is convex.

Proof: The feasible region is the intersection of finitely many closed half-spaces.

Definition A set S in $\mathbb{R}^n$ is banded provided there exists M so large that for any $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ we have $-M \leq x_i \leq M$ for any $i = 1, \cdots, n$.

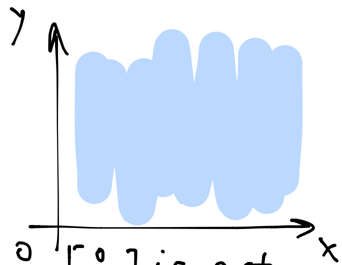
Remark: If a problem is unbounded (has no optimal solution, although the feasible region is non-empty), its feasible region is unbounded. The converse of this is false: certain problems are unbounded (have an optional solution) even though they have unbounded feasible regions.

§ 1.4

Definition: Let S be a convex subset of $\mathbb{R}^n$ and let $u \in S$. u is an extreme point of S provided for any $x_1 \in S$ and $x_2 \in S$ and any $\lambda \in [0,1]$ with $(1-\lambda)x_1 + \lambda x_2 = u$ we have either $u = x_1$ or $u = x_2$.

Remark: This means: if u lies on a line segment which is entirely in S, then u is an endpoint of the line segment.

Eg: In $\mathbb{R}^2$, let S be the solution set of $x \geq 0, y \geq 0$



$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is extreme

Suppose $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in S$ and suppose $\lambda \in [0,1]$ so that

$$(1-\lambda)\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \lambda \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We may assume $0 < \lambda < 1$

$$(1-\lambda)\overset{\geq 0}{x_1} + \lambda \overset{\geq 0}{x_2} = 0$$
$$(1-\lambda)\underset{\geq 0}{y_1} + \lambda \underset{\geq 0}{y_2} = 0$$

So $x_1 = 0, x_2 = 0, y_1 = 0, y_2 = 0$.