ODE's

$F(t, y, y', \cdots, y^{(n)}) = 0$

$y' = \frac{dy}{dt}, \cdots, y^{(n)} = \frac{dy^n}{dt^n}$ n = order of the ODE

t = independent variable

y = dep. v.

Other conventions

t = indep. x = dep. $x = \varphi(t)$ $(= x(t))$

x = indep, y = dep. $y = \varphi(x)$ $(= y(x))$

t = indep. u = dep. $u = u(t)$

Also common : $y' = \frac{dy}{dt}$

$- F(t, u_0, \cdots, u_n)$

eg: $F(t, u_0, u_1) = t + u_1^2 - u_0$ is $t + (\frac{dy}{dt})^2 - y = 0$

The DE is linear if F is linear in y and its derivative (but not necessarily in t). Thus linear DE's have form

$a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \cdots + a_n(t) y = g(t)$

where $a_0, \cdots a_n, g$ are function of t.

linear homogeneous if $g = 0$, otherwise inhomogeneous.

Example. linear or not.

1. $y' + e^t y = 1$ linear

2. $yy' = t + y$ not

3. $y'' + \sin(t) y = t^2$ linear

4. $\frac{y''-y}{y'+y}=4$ linear, after simple manipulation

5. $y''=t^3y'$ linear

6. $y'''+y''y'=3y$ not.

First order ODE's:

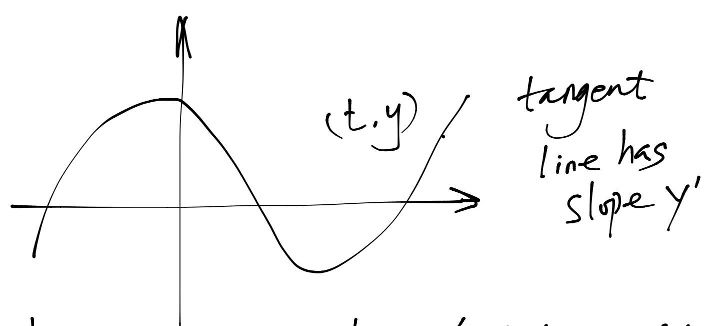
$F(t,y,y')=0$

Let's assume this can be solved for y'

$y'=f(t,y)$ standard form of 1st order ODE.

Geometric interpretation

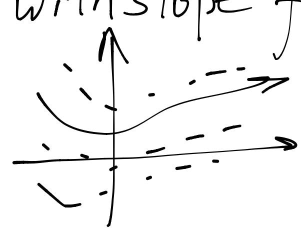
Recall $\frac{dy}{dt}$ is the slope at time t.



Thus, $y=\varphi(t)$ solves $y'=f(t,y)$ iff for all t the slope of $y=\varphi(t)$ is given by $f(t,\varphi(t))$

$f(t,y)$ interpreted as slope as (t,y)

Direction field (slope field). Draw little line segments at (t,y) with slope $f(t,y)$



Example: $f(t,y)=y-t+1$

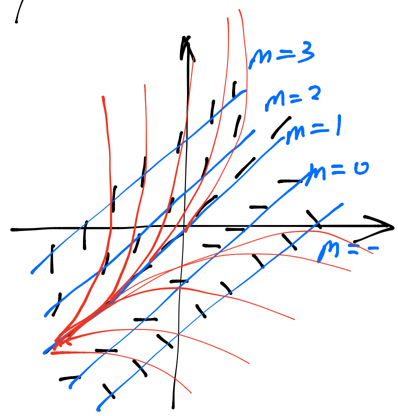
Let's draw the direction field.

convinient to first draw "isolines"

i.e. linear where slope is constant.

$y-t+1=m$

$y=t+m-1$



Note: Solution lines cannot cross:

because at point of intersection you'd get two slope.

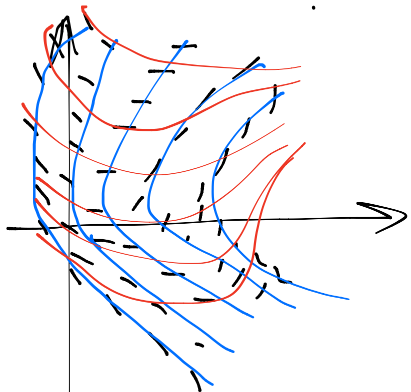
less: They cannot even touch.

under a small technical assumption

$\frac{\partial f}{\partial y}$ is continuous.

Example 2. $\frac{dy}{dt}=t-y^2$ Draw direction field.

Find isoclines: $t-y^2=m$ m is a slope



We looked at $\frac{dy}{dt} = y\sin(t)$

$$\frac{dy}{dt} = t^2 - y^2$$

The first equation can be solved:

$$\frac{dy}{dt} = y\sin(t) \Rightarrow \frac{1}{y}\frac{dy}{dt} = \sin(t) \Rightarrow \frac{d}{dt}\ln|y| = \sin(t)$$

$\Rightarrow \ln|y| = -\cos(x) + C \Rightarrow |y| = e^{-\cos(t)+C} = e^{c}\cdot e^{-\cos(t)}$

$\Rightarrow y = Ae^{-\cos(t)}$ $(e^{c} = A)$ with $A \in \mathbb{R}$

or $\frac{dy}{dt} = y\sin(t) \Rightarrow \frac{1}{y}dy = \sin(t)\,dt$

$$\Rightarrow \int \frac{1}{y}dt = \int \sin(t)\,dt$$

$$\Rightarrow \ln|y| = -\cos(t) + C$$

Technique for solving 1st order ODE's

We first consider linear ODE's

$$y' + p(t)y = g(t)$$

Let's first consider p, g are constant.

$\frac{dy}{dt} = ry + k$, r, k constants.

Example. $\frac{dy}{dt} = 4y = 1 \Rightarrow \int \frac{1}{4y-1}dy = \int dt$

$$\Rightarrow \frac{1}{4}\ln\left|y - \frac{1}{4}\right| = t + C_1$$

$$\Rightarrow y - \frac{1}{4} = \pm e^{4t + C_2} = C_3 e^{4t}$$

$$\Rightarrow y = Ae^{4t} + \frac{1}{4},\ A = C_3$$

So $\frac{dy}{dt} = ry + k$, r, k constants

$$\frac{1}{ry+k}\,dy = dt$$

$$\int \frac{1}{ry+k}\,dy = \int dt$$

$$\frac{1}{r}\int \frac{1}{y+\frac{k}{r}}\,dy = t + c$$

$$\frac{1}{r}\ln\left|y + \frac{k}{r}\right| = t + c$$

$$\left|y + \frac{k}{r}\right| = e^{rt + c_2}$$

$$y = Ae^{rt} - \frac{k}{r}$$

The solution for $A = 0$, $y = -\frac{k}{r}$ is "equilibrium solution"

Remember: if $r = 0$ $\frac{dy}{dt} = k$

have general solution $y = kt + c$