Instructor: Regina Rotman

Week 1-2: review

§2.3 The least upper bound principle

Continuum Property (the least upper bound principle)

Def: A set $S \subset \mathbb{R}$ is bounded from above if $\exists$ a real number M s.t. $s \le M$ $\forall s \in \mathbb{R}$.

$S = \{2, 4, 6 \cdots\}$ not bounded from above.

$T = \{\frac{1}{n} : n \in \mathbb{N}\}$ bounded from above. We call M an upper bound for S.

Bounded from below (lower bound)

If S is bdd both above and below $\Rightarrow S$ is bdd.

Sps $S \subset \mathbb{R}$ is bdd above, then L is the supreme (sup) or the least upper bound (lub).

If L is an upper bound for S that is smaller than other upper bounds

Similarly, we can define infimum (inf) or the greatest lower bound (glb)

$\sup S = +\infty$ (S not bounded above)
$\inf S = -\infty$ (S not bounded below)

* $\sup \phi = -\infty$
$\inf \phi = +\infty$

Remark: $\sup S = L \in \mathbb{R}$ iff L is an upper bound for S and $\forall K < L$, $\exists x \in S$ s.t. $K < x \le L$.

Maximum of S:

$m \in S$ s.t. $s \le m$, $\forall s \in S$

$(0,1) \subset \mathbb{R}$

minimum . . .

Ex: $A = \{4, -2, 5, 7\}$

$\sup A = 7 = \max A$

$\inf A = -2 = \min A$

$C = \{\frac{\pi}{n} : n \in \mathbb{N}\}$ $\quad \sup C = \pi = \max C$

$\inf C = 0$ but no such point in C is equal to 0.

Least upper bound principle

Continuum property: every nonempty subset $S \subset \mathbb{R}$ that is bdd above has a supremum.

Similarly, every nonempty subset $S \subset \mathbb{R}$ that is bdd below has an infimum.

§2.4 Limits

Limit of a sequence.

$$(a_n)_{n=1}^{\infty} \qquad \lim_{n\to\infty} a_n = L$$

For every $\varepsilon > 0$, $\exists N = N(\varepsilon) \in \mathbb{Z}$ s.t. $|a_n - L| < \varepsilon$, $\forall n \ge N$

Remark: enough to consider $\varepsilon = \frac{1}{10^k}$

Example: $(a_n) = \left(\frac{n}{n+1}\right)_{n=1}^{\infty}$

Claim: $\lim_{n\to\infty} \frac{n}{n+1} = 1$

Let $\frac{1}{10^k}$ be given $\quad |a_n - 1| = \left|\frac{n}{n+1} - 1\right| = \left|\frac{n-(n+1)}{n+1}\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1}$

want to show $\frac{1}{n+1} < \frac{1}{10^k}$ for sufficiently large $n \Rightarrow n+1 > 10^k$

$n > 10^k - 1$

Take $N = 10^k$, then $\forall n \geq N$, we'll have $n+1 > 10^k$ indeed.

$\Downarrow$

$\frac{1}{n+1} < \frac{1}{10^k}$ $\blacksquare$

Ex 2: $(a_n) = (-1)^n$

$-1, 1, -1, 1, \cdots$

$|a_n - a_{n+1}| = 2$

Let L be any real number (no a limit necessarily)

$2 = |a_n - a_{n+1}| = |a_n - L + L - a_{n+1}| \geq |a_n - L| + |a_{n+1} - L|$

so $\max\{|a_n - L|, |a_{n+1} - L|\} \geq 1$

Let $\varepsilon = 1$, if (a_n) converges $\Rightarrow \exists N$ s.t. $|a_n - L| < 1$, $n \geq N$

$|a_{n+1} - L| < 1$ contradiction $\blacksquare$

The Squeeze Theorem:
Sps that 3 sequences $(a_n), (b_n), (c_n)$ satisfy
$a_n \leq b_n \leq c_n, \forall n \geq 1$, & $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

Then $\lim_{n\to\infty} b_n = L$

Proof: Given $\varepsilon > 0$, $\exists N_1$ s.t. $|a_n - L| < \varepsilon$, $\forall n \geq N_1$

$L - \varepsilon < a_n < L + \varepsilon$, $\forall n \geq N_1$

Similarly, $\exists N_2$ s.t. $|c_n - L| < \varepsilon$, $\forall n \geq N_2$

$L - \varepsilon < c_n < L + \varepsilon$, $\forall n \geq N_2$

$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$, $\forall n \geq \max\{N_1, N_2\}$

$\Rightarrow |b_n - L| < \varepsilon$, $\forall n \geq \max\{N_1, N_2\}$ $\blacksquare$

Ex: let $x > 0$, $a_n = x^{\frac{1}{n}}$, $\lim_{n\to\infty} x^{\frac{1}{n}} = 1$

Let $a_1, \dots, a_n \in \mathbb{R}_+$

$A_n = \frac{a_1 + \cdots + a_n}{n}$ mathematical mean

$G_n = (a_1 \cdots a_n)^{\frac{1}{n}}$ geometric mean

$G_n \leq A_n$

Take $1=a_1$
$1=a_2$
$\vdots$
$1=a_{n-1}$
$x=a_n$

$G_n=(x)^{\frac{1}{n}}$

$A_n=\frac{(n-1)+x}{n}$

$x^{\frac{1}{k}} \leq \frac{(n-1)x}{n}$

$x^{\frac{1}{n}}-1 \leq \frac{1}{n}(x-1)$

Case 1: Sps $x \geq 1$
$0 \leq x^{\frac{1}{n}}-1 \leq \frac{1}{n}(x-1)$

Let $C_n=\frac{1}{n}(x-1)$
$\lim_{n\to\infty} C_n=0$

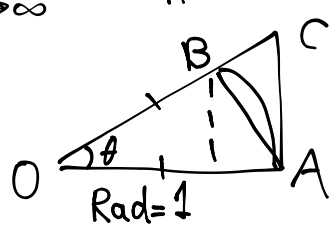
By Squeeze Thm: $\lim_{n\to\infty} x^{\frac{1}{n}}-1=0 \Rightarrow \lim_{n\to\infty} x=1$

Case 2:
$0<x<1$, let $y=\frac{1}{2}$, $\lim_{n\to\infty} y^{\frac{1}{n}}=1$

$\lim_{n\to\infty} x^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{y^{\frac{1}{n}}} = 1$

Ex: $(n\sin\frac{1}{n})_{n=1}^{\infty}$?

$\lim_{n\to\infty} n\sin\frac{1}{n}=1$



Area of ΔOAB is $\frac{\sin\theta}{2}$

Area of sector OAB is $\frac{\theta}{2}$

Area of $\Delta OAC=\frac{\tan\theta}{2}$

$\frac{\sin\theta}{2}<\frac{\theta}{2}<\frac{\tan\theta}{2}$

$\sin\theta<\theta<\tan\theta=\frac{\sin\theta}{\cos\theta}$

$\sin\theta<\theta$

$\frac{\sin\theta}{\theta}<1$ $\qquad \theta<\frac{\sin\theta}{\cos\theta}$

$\cos\theta<\frac{\sin\theta}{\theta}$

$\cos\theta<\frac{\sin\theta}{\theta}<1$

$\theta=\frac{1}{n}$

$\cos\frac{1}{n}<n\sin\frac{1}{n}<1$

$\cos\frac{1}{n}=\sqrt{1-\sin^2\frac{1}{n}}>\sqrt{1-(\frac{1}{n})^2}$
$>1-\frac{1}{n^2}$

$1-\frac{1}{n^2}<n\sin\frac{1}{n}<1$

as $n\to\infty$, $1-\frac{1}{n^2}\to 1$

So $\lim_{n\to\infty} n\sin\frac{1}{n}=1$

§2.5

Basic properties of limits

If $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$, $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = M$, $\alpha \in \mathbb{R}$

Them $\lim\limits_{n\to\infty} a_n + b_n = L + M$

$\lim\limits_{n\to\infty} \alpha a_n = \alpha L$

$\lim\limits_{n\to\infty} a_n b_n = LM$

$\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$

Theorem

Proposition: If (a_n) is a convergent seq. of real numbers, then $\{a_n : n \in \mathbb{N}\}$ is bdd.

Proof: Take $\varepsilon = 1$, $\exists N > 0$ s.t. $|a_n - L| < 1$, $\forall n \geq N$

$L - 1 < a_n < L + 1$

$M = \max\{a_1, a_2, \cdots, a_{N-1}, L+1\}$

$m = \min\{a_1, \cdots, a_{N-1}, L-1\}$

So the set is bdd.

§2.6

Monotone sequences

A sequence (a_n) is (strictly) monotone increasing if $a_n \leq a_{n+1}$ $(a_n < a_{n+1})$, $\forall n \geq 1$

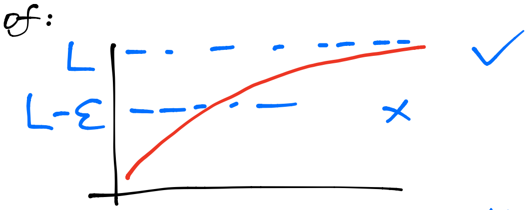
Similarly monotone decreasing ...

Monotone convergence Thm:

① A monotone increasing sequence that is bounded from above converges.

② A monotone decreasing seq. that is bdd from below converges.

Proof:



$L = \sup\{a_n : n \in \mathbb{N}\}$

Let $\varepsilon > 0$ be given

$L - \varepsilon$ is not an upper bound.

$\exists N$ s.t. $L > a_N > L - \varepsilon$

Also $\forall n \geq N$, $L - \varepsilon < a_N \leq a_n < L$

$|a_n - L| < \varepsilon$, $\forall n \geq N$

Sps a_n monotone decreasing and bounded below.

$-a_n$ is monotone increasing and bounded above.

Ex: $a_1 = 1$

$a_{n+1} = \sqrt{2+\sqrt{a_n}}$, $n > 1$

Claim: (a_n) converges

Pf: $1 \le a_n < a_{n+1} < 2$, $\forall n \ge 1$

$n=1$, $a_1 = 1$, $a_2 = \sqrt{3}$, $1 \le 1 < \sqrt{3} < 2$

Sps true for n.

$a_{n+2} = \sqrt{2+\sqrt{a_{n+1}}} > \sqrt{2+\sqrt{a_n}} = a_{n+1} \ge 1$

$a_{n+2} = \sqrt{2+\sqrt{a_{n+1}}} < \sqrt{2+\sqrt{2}} < 2$ ✓

$\lim\limits_{n\to\infty} a_n = L$

$\lim\limits_{n\to\infty} a_{n+1} = L$

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2+\sqrt{a_n}} = \sqrt{2+\sqrt{\lim_{n\to\infty} a_n}} = \sqrt{2+\sqrt{L}}$$

$L = \sqrt{2+\sqrt{L}}$

$L^2 = 2+\sqrt{L}$

$L^2 - 2 = \sqrt{L}$

$L^4 - 4L^2 + 4 = L$

$L^4 - 4L^2 - L + 4 = 0$

$(L-1)(L^3 + L^2 - 3L - 4) = 0$

$L \ne 1$ since it's monotone increasing

so L is in $L^3 + L^2 - 3L - 4 = 0$

you can find it.

$a_1 = 2$, $a_{n+1} = \dfrac{a_n^2+1}{2}$ no

$L = \dfrac{L^2+1}{2}$, $2L = L^2+1$, $L^2 - 2L + 1 = 0$, $L = 1$ (does not make sense)

Theorem: Suppose $\lim\limits_{n\to\infty} x_n = L$

① If $x_n \ge a$, $\forall n \ge N$, then $L \ge a$

② If $x_n \le b$, $\forall n \ge N$, then $L \le b$

not on textbook

Proof: Let $\varepsilon > 0$, $\exists N$ s.t. $|x_n - L| < \varepsilon$, $\forall n \ge N$

$L - \varepsilon < x_n < L + \varepsilon$, $\forall n \ge N$

$a \le x_n < L + \varepsilon$

$a < L + \varepsilon$

$a \le L$

(If $a > L$, then take $\varepsilon = a - L$, then $a < L + \varepsilon = a$ contradiction)

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