Higher order linear equations

$L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$

homogeneous equation: $L[y] = 0$

solution $y_1, \cdots, y_n$ of $L[y]=0$

are fundamental set $\Leftrightarrow W[y_1, \cdots, y_n] \neq 0$

$\Leftrightarrow y_1, \cdots, y_n$ are linearly independent.

in this case, the general solution is

$y = c_1 y_1 + \cdots + c_n y_n$

Reduction of order

Suppose y_1 is a solution of $L[y]=0$

write $y = v y_1$ for some function v.

Then the n-th order equation $L[y]=0$ becomes an $(n-1)$th order linear ODE for $w = v'$

Reason: $L[vy_1] = v\underbrace{(\cdots\cdots)}_{L[y_1]=0} + v'(\cdots) \cdots + v^{(n)}(\cdots)$

Constant coefficient equation

$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$

$y = e^{rt}$ solves $L[y]=0$ iff r is a roots of the char. polynomial

$a_0 r^n + a_1 r^{n-1} + \cdots + a_n$

Thus, if the char. poly has a distinct roots $r_1, \cdots, r_n$, get solutions

$y_1 = e^{r_1 t}, \cdots, y_n = e^{r_n t}$

(If $r = \lambda + i\mu$ is a complex root, then $\bar{r} = \lambda - i\mu$ is also a root, and the pair of solutions $e^{rt}, e^{\bar{r}t}$ can be replaced with the pair $(e^{\lambda t}\cos(\mu t), e^{\lambda t}\sin(\mu t))$

$\Rightarrow$

This is a fundamental set of solutions

$$W[e^{r_1 t},\dots,e^{r_n t}] = \begin{vmatrix} e^{r_1 t} & \cdots & e^{r_n t} \\ r e^{r_1 t} & & r_n e^{r_n t} \\ \vdots & & \vdots \\ r^{n-1} e^{r_1 t} & \cdots & r_n^{n-1} e^{r_n t} \end{vmatrix} \cdot e^{(r_1+\cdots+r_n)t}$$

$$= e^{(r_1+\cdots+r_n)t} \prod_{i>j} (r_i - r_j)$$

$$W[e^{r_1 t}, e^{r_2 t}, e^{r_3 t}] = e^{(r_1+r_2+r_3)(t)} (r_3-r_2)(r_2-r_1)(r_2-r_1)$$

more generally, if root r has multiplicity k, replace e^{rt} with $e^{rt}, te^{rt}, \dots, t^{k-1}e^{rt}$

EXAMPLE: $y^{(4)} + 8y^{(2)} + 16y = 0$

char. equation: $r^4 + 8r^2 + 16 = 0$

$(r^2+4)^2 = 0$

roots are $r = 2i, -2i$ each with multiplicity 2

$\Rightarrow e^{2it}, te^{2it}, e^{-2it}, te^{-2it}$ is a fundamental set of solution as is $\cos(2t)$, $t\cos(2t)$, $\sin(2t)$, $t\sin(2t)$

EXAMPLE: $y^{(3)} + 3y'' + 3y' + y = 0$

char. polynomial: $r^3 + 3r^2 + 3r + 1 = 0$

$(r+1)^3 = 0$

$\Rightarrow r = -1$ is root of multiplicity of 3.

$\Rightarrow y_1 = e^{-t}, y_2 = te^{-t}, y_3 = t^2 e^{-t}$

Inhomogeneous equations $L[y] = y^{(n)} + p_1 y^{(n-1)} + \cdots + p_n y$

$L[y] = g$

Undetermined coeff's Applies to $L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$

with right hand side $g(t)$ of form:

$e^{\lambda t}, \cos(\mu t), \sin(\mu t), P(t)$ polynomials or products of such.

In this case, can use trial solution of similar form

<u>EXAMPLE</u>: $y^{(3)}+3y''+3y'+y=e^{2t}$

Trial solution: $Y=Ae^{2t}$

$L[Y]=Ae^{2t}(8+3\cdot 4+3\cdot 2+1)$

$=Ae^{2t}(27)\overset{!}{=}g(t)=e^{2t}\Rightarrow A=\frac{1}{27}$

$\Rightarrow Y=\frac{1}{27}e^{2t}$ is a particular solution

<u>EXAMPLE</u>: $y^{(3)}+3y''+3y'+y=e^{-t}$

Trial solution: $Y=At^3e^{-t}$ works

$L[Y]=Ae^{-t}(t^3(\dots)+t^2(\cdots)+t(\cdots)+(.4.1))\overset{!}{=}e^{-t}$

$\Rightarrow A=t$

<u>Variation of parameters</u>

$L[Y]=y^{(n)}+p_1y^{(n-1)}+\cdots+p_ny=g$

suppose $y_1,\dots,y_n$ fundamental set of solutions of $L[y]=0$

Trial solution: $Y=v_1y_1+\cdots+v_ny_n$

$Y'=v_1y_1'+\cdots+v_ny_n'+v_1'y_1+\cdots+v_n'y_n$

impose condition: $v_1'y_1+\cdots+v_n'y_n=0$ ①

$Y''=v_1y_1''+\cdots+v_ny_n''+v_1'y_1'+\cdots+v_n'y_n'$

impose condition $v_1'y_1'+\cdots+v_n'y_n'=0$ ②

$Y^{(n)}=v_1y_1^{(n)}+\cdots v_ny_n^{(n)}+v_1'y_1^{(n-1)}+\cdots+v_n'y_n^{(n-1)}$

$L[Y]=v_1L[y_1]+\cdots+v_nL[y_n]+v_1'y_n^{(n)}+\cdots+v_n'y_n^{(n-1)}\overset{!}{=}g(t)$

Get: $v_1'y_1+\cdots+v_n'y_n=0$

$v_1'y_1^{(n-2)}+\cdots=0$

$v_1'y_1^{(n-1)}+\cdots=g(t)$

$$\begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & & \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} v_1' \\ \vdots \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix} \quad \text{OR} \quad \begin{pmatrix} v_1' \\ \vdots \\ \vdots \\ v_n' \end{pmatrix} = \begin{pmatrix} \text{COEEF} \\ \text{MATRIX} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix}$$

Cramer's rule : $v_i' = \dfrac{g(t) W_i(t)}{W(t)}$