Jan 23rd

Dimension Theorem

Let $T: V \to W$ is a linear transformation

$Im(T) = \{w \in W \mid w = T(v)$ for some $v \in V\}$

Claim: $Im(T)$ is a subspace of W

Pf: $w = Im(T)$ then $w = T(v)$ for some $v \in V$.

$\lambda w = \lambda T(v) = T(\lambda \cdot v) \in Im(T)$

$w_1, w_2 \in Im(T)$

$w_1 = T(v_1)$ & $w_2 = T(v_2)$

$w_1 + w_2 = T(v_1) + T(v_2) = T(v_1 + v_2) \in Im(T)$

$Ker(T) = \{v \in V \mid T(v) = 0\}$

Thm: $\dim(Ker(T)) + \dim(Im(T)) = \dim(V)$

ex: $a \in \mathbb{R}^2$, $P_a(v) = \frac{\langle a, v\rangle}{\langle a, a\rangle} = a$ "projection onto a"

$Im(P_a) = span_{\mathbb{R}}(a)$ [Pf $P_a(v) \in span_{\mathbb{R}}(a)$ by def'n, $P_a(\lambda a) = \lambda a$]

$Ker(P_a)$ = line which is perpendicular to a.

one-dimension one-dimension

The proof

$\dim(V) = k$, $\dim(W) = l$

Claim: ① sps $\{v_1, \cdots, v_k\}$ is a basis of V.
then $\{T(v_1), \cdots, T(v_k)\}$ spans $Im(T)$

Proof: $w \in Im(T) \Rightarrow w = T(v)$ some $v \in V$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$

$\therefore w = T(v) = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_k T(v_k)$

Note!

A **basis** B of a vector space V over a field F is a linearly independent subset of V that spans V.

In more detail, suppose that $B = \{ v_1, \ldots, v_n \}$ is a finite subset of a vector space V over a field $\mathbf{F}$ (such as the real or complex numbers $\mathbf{R}$ or $\mathbf{C}$). Then B is a basis if it satisfies the following conditions:

- the *linear independence* property,

for all $a_1, \ldots, a_n \in \mathbf{F}$, if $a_1v_1 + \ldots + a_nv_n = 0$, then necessarily $a_1 = \ldots = a_n = 0$; and

- the *spanning* property,

for every x in V it is possible to choose $a_1, \ldots, a_n \in \mathbf{F}$ such that $x = a_1v_1 + \ldots + a_nv_n$.

Definition of basis

Claim: ② now choose a basis $\{v_1, \cdots, v_k\}$ such that $\{v_1, \cdots, v_n\}$ are a basis for $Ker(T) \subseteq V$
(use a basis of $Ker(T)$ can be extended to a basis of V)
Then: $T(v_{n+1}), T(v_{n+2}), \cdots, T(v_k)$ are a basis of $Im(T)$

Proof: a) we use ① to show that $\{T(v_{n+1}), T(v_{n+2}), \cdots T(v_k)\}$ span $Im(T)$:

In fact: $T(v_1) = T(v_2) = \cdots = T(v_n) = 0$

$Span_F\{T(v_1), T(v_2), \cdots, T(v_k)\} = span_F\{T(v_{n+1}), \cdots, T(v_k)\}$

$\therefore \operatorname{span}_F \{T(v_{n+1}), \cdots, T(v_k)\} = Im(T)$

b). $\{T(v_{n+1}), \cdots, T(v_k)\}$ is linearly indep.

So: Let $\beta_{n+1}T(v_{n+1}) + \beta_{n+2}T(v_{n+2}) + \cdots + \beta_k T(v_k) = 0$

$\Rightarrow T(\beta_{n+1}v_{n+1} + \beta_{n+2}v_{n+2} + \cdots + \beta_k v_k) = 0$

By def'n $\Rightarrow \beta_{n+1}v_{n+1} + \beta_{n+2}v_{n+2} + \cdots + \beta_k v_k \in Ker(T)$

BUT : $Ker(T) = \operatorname{span}_F\{v_1, \cdots, v_n\}$

$\Rightarrow \beta_{n+1}v_{n+1} + \beta_{n+2}v_{n+2} + \cdots + \beta_k v_k = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_n v_n$

$\Rightarrow -\beta_1 v_1 - \beta_2 v_2 - \cdots - \beta_n v_n + \beta_{n+1}v_{n+1} + \beta_{n+2}v_{n+2} + \cdots + \beta_k v_k = 0$

$\Rightarrow$ all $\beta_i = 0$

Therefore $\Rightarrow \{T(v_{n+1}), \cdots, T(v_k)\}$ are lin. indep. ■

Claim ③: $\dim(Ker(T)) = n$

$\dim(Im(T)) = k-n$

$\dim(Ker(T)) + \dim(Im(T)) = k = \dim(V)$

Example : $a \in \mathbb{R}^n$, $a \in \mathbb{R}^n$, P_a : projection onto $\operatorname{span}_{\mathbb{R}}(a)$

$P_a(v) = \frac{\langle a, v\rangle}{\langle a, a\rangle} = a$

now: $\dim(Im(P_a)) = 1$ since $Im(P_a) = \operatorname{span}_{\mathbb{R}}(a)$

$\Rightarrow \dim(Ker(P_a)) = n-1$

$Ker(P_a) = \{v \in \mathbb{R}^n \mid \langle a, v\rangle > 0\}$

def'n: $T: V \to W$

① T is injective (or one-to-one) if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$

② T is surjective (or "onto") if $Im(T) = W$.

Claim: T is injective $\Leftrightarrow Ker(T) = 0$

Proof: $\Rightarrow$ If T is injective, $v \in Ker(T)$.

$T(v) = 0 = T(0)$ so injectivity $\Rightarrow v = 0$

$\Leftarrow$ Know $Ker(T) = 0$, suppose $T(v_1) = T(v_2)$

$\Leftrightarrow T(v_1) - T(v_2) = 0 \Leftrightarrow T(v_1 - v_2) = 0$

$\Rightarrow v_1 - v_2 \in Ker(T) = 0$

so $v_1 - v_2 = 0 \Rightarrow v_1 = v_2$ ■

Corallories:

① T inj $\Rightarrow \dim(V) \leq \dim(W)$

② T surj. $\Rightarrow \dim(V) \geq \dim(W)$