Scalar Valued Line Integral.

$$\int_C \vec{F}\cdot d\vec{x} = \int_C F_1dx_1+F_2dx_2+\cdots+F_ndx_n = \int_a^b \vec{F}(\vec{g}(t))\cdot\vec{g}'(t)dt$$

where C is $\vec{x}=\vec{g}(t)$, $a\le t\le b$

Remark ① This line integral is independent of parametrization as long as the orientation is unchanged, but it acquires a factor of -1, if the orientation is reversed.

② The line integral can be expressed as an integral of a scalar function over C.

Let $\vec{T}(\vec{g}(t)) = \dfrac{\vec{g}'(t)}{|\vec{g}'(t)|}$ ← unit vector in the direction of $\vec{g}'(t)$.
which is the tangent vector of $\vec{g}(t)$.

Define $F_{tang}(\vec{x}) = \vec{F}(x)\cdot\vec{T}(x)$

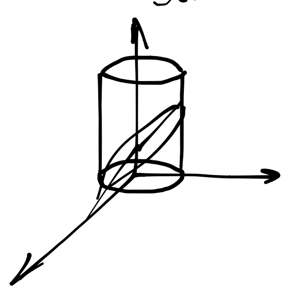
$$\Rightarrow F(\vec g(t))\cdot\vec g'(t)dt = \vec F(\vec g(t))\cdot \frac{\vec g'(t)}{|\vec g'(t)|}|\vec g'(t)|dt$$

$ds=\sqrt{(\frac{dx_1}{dt})^2+\cdots+(\frac{dx_n}{dt})^2}\,dt$

$$=\vec F(\vec g(t))\cdot \vec T(\vec g(t))ds$$
$$= F_{tang}(\vec g(t))ds$$

$$\Rightarrow \int_C F\cdot d\vec x = \int_a^b F(\vec g(t))\vec g'(t)dt = \int_a^b F_{tang}(\vec g(t))ds = \int_a^b F_{tang}(\vec x)ds$$

Eg. Let C be the ellipse formed by the intersection of the circular cylinder $x^2+y^2=1$ and the plane $z=2y+1$, oriented counterclockwise as viewed from above, let $\vec F(x,y,z)=(y,z,x)$.
Calculate $\int_C \vec F\cdot d\vec x = \int_C (ydx+zdy+xdz)$



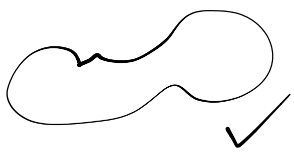
Let $x=\cos t$, $y=\sin t$, $z=2y+1=2\sin t+1$

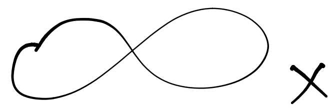
$$\vec g'(t) = (x',y',z') = (-\sin t, \cos t, 2\cos t)dt$$

$$\int_C \vec F\cdot d\vec x = \int_0^{2\pi}(\sin t, 2\sin t+1, \cos t)\cdot(-\sin t,\cos t,2\cos t)dt$$
$$= \int_0^{2\pi} -\sin^2 t + 2\sin t\cos t + \cos t + 2\cos^2 t\, dt$$
$$=\int_0^{2\pi} \cos 2t + \sin 2t + \cos t + \cos^2 t\, dt$$
$$= \frac{\sin(2t)}{2}\Big|_0^{2\pi} - \frac{\cos 2t}{2}\Big|_0^{2\pi} + \sin t\Big|_0^{2\pi} + \int_0^{2\pi}\frac{1+\cos 2t}{2}dt$$
$$=\int_0^{2\pi}\frac12 dt = \frac12\cdot 2\pi = \pi$$

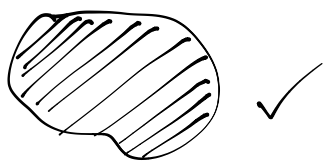
§5.2 Green's Theorem

Def: A simple closed curve in $\mathbb{R}^n$ is a curve whose starting and ending points coincide but that does not intersect itself otherwise.





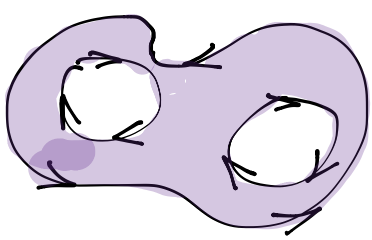
Def: A regular region is a compact set in $\mathbb{R}^n$ that is the closure of its interior



The interior is null

When $n=2$, a regular region $S \subset \mathbb{R}^2$ has a piecewise smooth boundary if the boundary ∂S consists of a finite union of disjoint, piecewise smooth closed curves.

The positive orientation on ∂S is the orientation on each of the closed curves such that the region S is on the left w.r.t. positive direction on the curve.



Green's Thm: Sps S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S, Sps also $\vec{F}=(F_1,F_2)$ is a vector field of class C^1 on $\bar{S}$, then

$$\int_{\partial S} \vec{F}\cdot d\vec{x} = \iint_S \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dA$$

If we set $\vec{F}=(P,Q)$, and $\vec{x}=(x,y)$

$$\int_{\partial S} P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad (*)$$

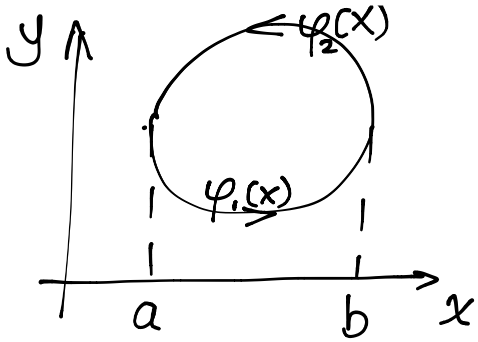
Pf: ① Green's Thm on simple region

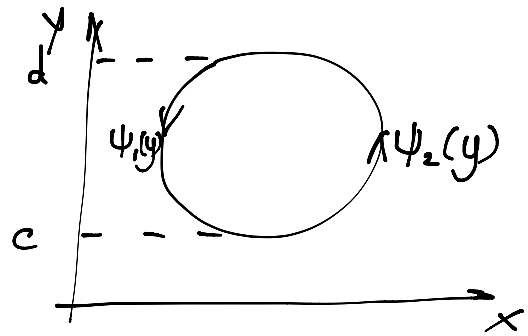
Def: The region S is x-simple if it is the region between the graph of two functions of x.

$S=\{(x,y) : a\le x\le b,\ \varphi_1(x)\le y\le \varphi_2(x)\}$

If y-simple

$S=\{(x,y) : c\le y\le d,\ \psi_1(y)\le x\le \psi_2(y)\}$





Sps S is both x-simple & y-simple

From x-simple $\int_{\partial S} P\,dx = \int_a^b P(x, \varphi_1(x))dx + \int_b^a P(x, \varphi_2(x))dx$

$$= \int_a^b P(x, \varphi_1(x))dx - \int_a^b P(x, \varphi_2(x))dx$$

$$\iint_{\partial S} \frac{\partial P}{\partial y} dA = \int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} \frac{\partial P}{\partial y} dy\,dx \overset{\text{FTC}}{=} \int_a^b P(x, \varphi_2(x)) - P(x, \varphi_1(x))\,dx$$

$$= -\int_{\partial S} P\,dx \quad \cdots\cdots (1)$$

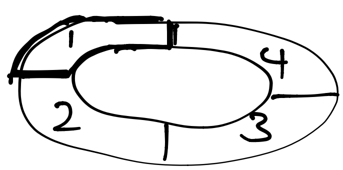
From y-simple $\int_{\partial S} Q\,dy = \int_d^c Q(\psi_1(y), y)dy + \int_c^d Q(\psi_2(y), y)dy$

$$= -\int_c^d Q(\psi_1(y), y)dy + \int_c^d Q(\psi_2(y), y)dy$$

$$\iint_S \frac{\partial Q}{\partial x} dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} \frac{\partial Q}{\partial x} dx\,dy = \int_c^d Q(\psi_2(y), y) - Q(\psi_1(y), y)\,dx = \int_{\partial S} Q\,dy \cdots (2)$$

$(1) + (2) \Longrightarrow (*)$

② On general regular region



we cut the region into finitely many sub-regions say $S = S_1 \cup S_2 \cup \cdots \cup S_K$ s.t.
a). the S_j's may intersect along common edges but disjoint interiors
b). each S_j has a piecewise smooth boundary and is both x-simple & y-simple

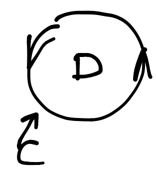
$$\iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \sum_{i=1}^{K} \iint_{S_i} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \sum_{i=1}^{K} \iint_{\partial S_j} P\,dx + Q\,dy = \int_{\partial S} P\,dx + Q\,dy$$

The line integrals and one the opposite direction for common edges.

Eg. Let C be the unit circle $x^2+y^2=1$ oriented counterclockwise

$$\int_C \underbrace{[\sqrt{1+x^2}-ye^{xy}+3y]}_{P}dx+\underbrace{[x^2-xe^{xy}+\log(1+y^4)]}_{Q}dy$$



$$=\iint_D (2x-e^{xy}-xye^{xy})-(-e^{xy}-xye^{xy}+3)\,dA$$

$$=\iint_D (2x-3)\,dA=\int_0^1\int_0^{2\pi}(2r\cos\theta-3)\,r\,d\theta\,dr$$

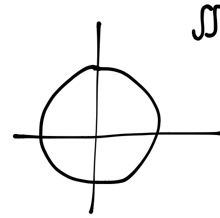
$$=\int_0^1\left[2r^2\int_0^{2\pi}\cos\theta\,d\theta-3r\int_0^{2\pi}d\theta\right]dr=\int_0^1\left[2r^2\sin\theta\Big|_0^{2\pi}-6\pi r\right]dr$$

$$=-6\pi\int_0^1 r\,dr=-6\pi\frac{1}{2}r^2\Big|_0^1=-3\pi$$

odd function

$\iint_D x\,dA=0$

symmetric with y



Eg. $\int_{\partial S} x\,dy=\iint_S dA=\text{Area}(S)$

$\int_{\partial S} y\,dx=-\iint_S dA=-\text{Area}(S)$

$\Rightarrow \frac{1}{2}\int_{\partial S} x\,dy-y\,dx=\text{Area}(S)$

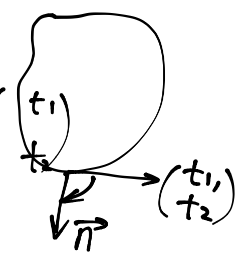
Cor. Sps S is a regular region in $\mathbb{R}^2$ with piecewise smooth boundary ∂S and let $\vec{n}(x)$ be the unit outward normal vector to ∂S at $\vec{x}\in\partial S$. Sps also that $\vec{F}$ is a vector field of class C^1 on $\bar{S}$, then

$$\int_{\partial S}\vec{F}\cdot\vec{n}\,dS=\iint_S\left(\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}\right)dA$$

Proof:

Let $\vec{t}=\begin{pmatrix}t_1\\t_2\end{pmatrix}$ be the unit tangent vector then $\vec{n}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}t_1\\t_2\end{pmatrix}$

$\theta=-\frac{\pi}{2}$



$$=\begin{pmatrix}t_2\\-t_1\end{pmatrix}$$

Let $\vec{F}=\begin{pmatrix}F_1\\F_2\end{pmatrix}$

Def $\tilde{\vec{F}}=\begin{pmatrix}\cos\pi/2 & -\sin\pi/2\\ \sin\pi/2 & \cos\pi/2\end{pmatrix}\begin{pmatrix}F_1\\F_2\end{pmatrix}=\begin{pmatrix}-F_2\\F_1\end{pmatrix}$

tangent vector $=\begin{pmatrix}dx_1\\dx_2\end{pmatrix}=|d\vec{x}|\begin{pmatrix}t_1\\t_2\end{pmatrix}=\sqrt{(dx_1)^2+(dx_2)^2}\begin{pmatrix}t_1\\t_2\end{pmatrix}=\vec{t}\,ds$

$\Rightarrow \vec{F}\cdot\vec{n}=\begin{pmatrix}F_1\\F_2\end{pmatrix}\begin{pmatrix}t_2\\-t_1\end{pmatrix}=t_2F_1-F_2t_1=\begin{pmatrix}-F_2\\F_1\end{pmatrix}\begin{pmatrix}t_1\\t_2\end{pmatrix}=\tilde{\vec{F}}\cdot\vec{t}$

$$\Rightarrow \int_{\partial S} \vec{F}\cdot\vec{n}\,ds = \int_{\partial S} \tilde{\vec{F}}\cdot\vec{t}\,ds = \int_{\partial S} \tilde{\vec{F}}\cdot d\vec{x} = \int_{\partial S} \begin{pmatrix}-F_2\\ F_1\end{pmatrix}\begin{pmatrix}dx_1\\ dx_2\end{pmatrix}$$

$$= \int_{\partial S} -F_2\,dx_1 + F_1\,dx_2 \underset{\text{Green's Thm}}{=} \iint_S \frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_1}\,dA$$

E.g. Consider $F_1 = \partial_1 f$, $F_2 = \partial_2 f$

$$\int_{\partial S} \nabla f\cdot d\vec{x} = \iint_S \partial_1\partial_2 f - \partial_2\partial_1 f\,dA = 0$$ when S is piecewise smooth closed curve. The line integral of a gradient over any closed curves vanishes

$$\underbrace{\int_{\partial S} \vec{F}\cdot\vec{n}\,ds}_{\frac{\partial f}{\partial \vec{n}}} = \iint_S \underbrace{\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2}}_{\to \Delta f}\,dA$$

$\frac{\partial f}{\partial \vec{n}}$ called normal derivative

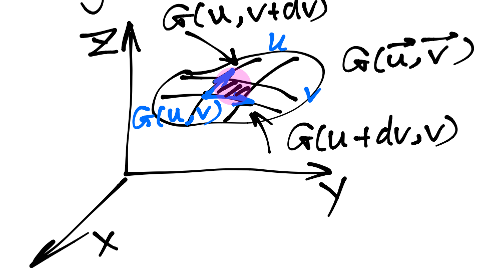
$\Delta f \leftarrow$ Laplaician of f

§5.3 Surface, area and surface integral

Orientation of the surface

If a surface forms part of the boundary of a regular region in $\mathbb{R}^3$, it is always orientable usually we use the one pointing out as the positive normal vector.

Surface Arc



$$\vec{G}(u+du, v) - \vec{G}(u,v) \approx \frac{\partial \vec{G}}{\partial u}\cdot du$$

$$\vec{G}(u, v+dv) - \vec{G}(u,v) \approx \frac{\partial \vec{G}}{\partial v}\cdot dv$$

$$dA = \left|\frac{\partial \vec{G}}{\partial u}\cdot\frac{\partial \vec{G}}{\partial v}\right| du\,dv$$

def: Area of $\vec{G}(R) = \iint_R \left|\frac{\partial \vec{G}}{\partial u}\times\frac{\partial \vec{G}}{\partial v}\right| du\,dv$ $\quad R = \{(u,v)\}$ $\quad$ (**)

Consider $\vec{G}(u,v) = (x, y, z)$

$$\frac{\partial \vec{G}}{\partial u}\times\frac{\partial \vec{G}}{\partial v} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\end{pmatrix} = \frac{\partial(y,z)}{\partial(u,v)}\vec{i} + \frac{\partial(z,x)}{\partial(u,v)}\vec{j} + \frac{\partial(x,y)}{\partial(u,v)}\vec{k}$$

$$dA=\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2}\;du\,dv$$

consider $z=\varphi(x,y)$, i.e. $\vec{G}$ is graph of (x,y)

$\vec{G}(x,y)=(x,y,\varphi(x,y))$

$\Rightarrow \vec{G_x}=(1,0,\partial_x\varphi)$, $\vec{G_y}=(0,1,\partial_y\varphi)$

$$\partial_x\vec{G}\cdot\partial_y\vec{G}=\det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & \partial_x\varphi\\ 0 & 1 & \partial_y\varphi\end{pmatrix}=-\partial_x\varphi\,\vec{i}-\partial_y\varphi\,\vec{j}+\vec{k}$$

$$dA=\sqrt{(\partial_x\varphi)^2+(\partial_y\varphi)^2+1}\;dx\,dy$$

(**) is independent of the parametrization

consider a change of variable $(u,v)=\vec{\phi}(s,t)$

$\Rightarrow du\,dv=\left|\frac{\partial(u,v)}{\partial(s,t)}\right|ds\,dt$

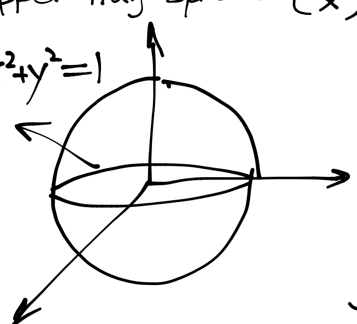
From (**) $\Rightarrow$ $$dA=\sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(s,t)}\right)^2+\left(\frac{\partial(z,x)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}\right)^2+\left(\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(s,t)}\right)^2}\;ds\,dt$$

$$=\sqrt{\left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2+\left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2+\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2}\;ds\,dt$$

Surface area of unit sphere $x^2+y^2+z^2=1$

Upper half sphere $(x,y,\sqrt{1-x^2-y^2})$

$x^2+y^2=1$



$$\varphi_x=\frac{1}{2}\frac{-2x}{\sqrt{1-x^2-y^2}}=\frac{-x}{\sqrt{1-x^2-y^2}}$$

$$\varphi_y=-\frac{y}{\sqrt{1-x^2-y^2}}$$

$$\iint_S\sqrt{\frac{x^2}{1-x^2-y^2}+\frac{y^2}{1-x^2-y^2}+1}\;dx\,dy=\iint_S\sqrt{\frac{1}{1-x^2-y^2}}\,dx\,dy$$

$x=r\cos\theta$
$y=r\sin\theta$

$$\int_0^1\int_0^{2\pi}\sqrt{\frac{1}{1-r^2}}\,r\,d\theta\,dr=\int_0^1\int_0^{2\pi}\frac{r}{\sqrt{1-r^2}}\,d\theta\,dr=2\pi\int_0^1\frac{r}{\sqrt{1-r^2}}\,dr$$

$$=2\pi\left(-\sqrt{1-r^2}\right)\Big|_0^1=2\pi$$

Surface Area of unit sphere $=2\cdot 2\pi=4\pi$

Soln (2) Consider
$$\begin{cases} x=\sin\varphi\cos\theta & \theta\in[0,2\pi]\\ y=\sin\varphi\sin\theta & \varphi\in[0,\pi]\\ z=\cos\varphi \end{cases}$$

$$\left|\frac{\partial \vec{G}}{\partial \varphi} \times \frac{\partial \vec{G}}{\partial \theta}\right| = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{pmatrix} = \sin^2\varphi\cos\theta\,\vec{i} + \sin^2\varphi\sin\theta\,\vec{j} + (\sin\varphi\cos\varphi\cos^2\theta + \sin\varphi\cos\varphi\sin^2\theta)\vec{k}$$

$$dA = \sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = \sqrt{\sin^2\varphi} = \sin\varphi$$

by $\varphi \in [0, \pi]$

$$\int_0^{\pi}\int_0^{2\pi} \sin\varphi \, d\theta \, d\varphi = 2\pi\int_0^{\pi} \sin\varphi \, d\varphi = 2\pi(-\cos\varphi)\Big|_0^{\pi} = 4\pi$$