Lecture 4

§3.2 Convergence tests for series

E.g. (r_n)

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$$

$\limsup_{n\to\infty} r_n = 1$ $\qquad$ $\liminf_{n\to\infty} r_n = 0$

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots$ $\qquad$ $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

E.g. $(a_n) = (-1)^n(1+\frac{1}{n})$

$\limsup a_n = 1$

$\liminf a_n = -1$

Remark: Let (a_n) be bounded, a_n converges to l iff $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = l$

The Root Test

Sps $a_n \geq 0$ for all n, let $l = \limsup_{n\to\infty} \sqrt[n]{a_n}$

If $l<1$, then $\sum_{n=1}^{\infty} a_n$ converges

If $l>1$, then $\sum_{n=1}^{\infty} a_n$ diverges

If $l=1$, nothing to be said.

(we know comparison test and geometric series, so we can prove this)

Proof:

Sps $\limsup_{n\to\infty} a_n = l < 1$

choose any r s.t. $l<r<1$

Let $\varepsilon = r - l > 0$

$\exists N>0$ s.t. $a_n^{1/n} < l+\varepsilon = r$?

(Remark: if not $\Rightarrow \forall N, \exists a_n, n\geq N$ s.t. $a_n \geq l+\varepsilon$

Pick $N_2 > N_1$ be a_{n_2}

then subseq. $a_{n_1}, a_{n_2}, a_{n_3}, \dots$

so $a_{n_k} \geq l+\varepsilon$)

$a_n^{1/n} < l+\varepsilon = r$

$a_n < r^n$, for all $n \geq N$

$b_n = a_n \quad 1 \leq n < N$

$b_n = r^n \quad n \geq N$

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{N-1} a_n + \sum_{n=N}^{\infty} r^n = \sum_{n=1}^{N-1} a_n + \frac{r^N}{1-r}$$

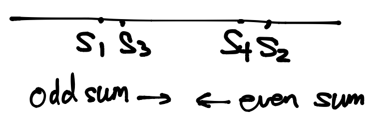
By comparison $\sum_{n=1}^{\infty} a_n$ converges. ∎

Definition: Alternating seq.
$((-1)^n a_n)$ or $((-1)^{n+1} a_n)$, $a_n \geq 0, \forall n \geq 1$.
$\sum_{n=1}^{\infty} (-1)^n a_n$ or $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges?

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Leibniz Alternating series test.
Sps $(a_n)_{n=1}^{\infty}$ is a monotone decreasing $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$
Sps $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

Proof: Let's look at partial sums first,
$S_1 = -a_1$
$S_2 = -a_1 + a_2$
$S_3 = -a_1 + a_2 - a_3$
$S_4 = -a_1 + a_2 - a_3 + a_4$



Note:
(1). $S_2 \geq S_4 \geq S_6 \geq S_8, \cdots$
(2). $S_1 \leq S_3 \leq S_7 \leq S_9$
(3). $S_{2m-1} \leq S_{2n}$, $\forall m, n \geq 1$

Proof of (1)(2)(3)
① $S_{2n} - S_{2n-2} = a_{2n} - a_{2n-1} \leq 0$
② $S_{2n+1} - S_{2n-1} = a_{2n} - a_{2n+1} \geq 0$
③ Let $m, n \in \mathbb{Z}_+$, $N = \max\{m, n\}$
$S_{2m-1} \leq S_{2N-1} \leq S_{2N} \leq S_{2n}$
$(S_2, S_4, S_6, \cdots)$ bounded from below, monotone nonincreasing $\Rightarrow$ converges to L.
$(S_1, S_3, S_5, \cdots)$ bounded from above, monotone nondecreasing $\Rightarrow$ converges to M.
$L - M = \lim_{n\to\infty} S_{2n} - \lim_{n\to\infty} S_{2n-1} = \lim_{n\to\infty} a_{2n} = 0$ ■

$x = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$

$x = (1 - \frac{1}{2}) - \frac{1}{4} + (\frac{1}{3} - \frac{1}{6}) - \frac{1}{8} + (\frac{1}{5} - \frac{1}{10}) - \frac{1}{12} + \cdots$
$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \cdots$
$= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots)$

so $x = \frac{1}{2}x$, $x = 0$ But we know the answer should be $\ln 2$. Why is it wrong here?

§ 3.3 Absolute and Conditional Convergence

Conditional convergence!

Absolute and conditional convergence.
Def: A series $\sum_{n=1} a_n$ is called absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges,
A series that converges, but not absolutely converges is called conditionally convergent

Proposition: An absolutely convergent series is convergent (comparison test).

Def: A rearrangement of $\sum_{n=1}^{\infty} a_n$ is another series with the same term but different order $\sum_{n=1}^{\infty} a_{\pi(n)}$, π = a permutation of $\mathbb{N}$.

Theorem: Every rearrangement of an absolutely convergent series converges to the same limit.

Proof: $\sum_{n=1}^{\infty} a_n = L$, π is a permutation of $\mathbb{N}$. Let $\varepsilon > 0$ be given.

By definition of a limit, $\exists N$ s.t. $\sum_{k=N+1} |a_n| < \frac{\varepsilon}{2}$

$a_{\pi(1)}, a_{\pi(2)}, \cdots$ somewhere we find a term that $= a_1$

a_2

$\vdots$

we find all of them in finitely many steps.

$a_1, \cdots a_N$ occur in first M terms of the rearranged series for some $M \in \mathbb{Z}_+$

We want to show that for any $\varepsilon > 0$, $\exists N$ s.t. $|\tilde{S}_m - L| < \varepsilon$

we want to show

$$
\begin{aligned}
\left| \sum_{k=1}^{m} a_{\pi(k)} - L \right| &\le \left| \sum_{k=1}^{m} a_{\pi(k)} - \sum_{k=1}^{N} a_k + \sum_{k=1}^{N} a_k - L \right| \\
&\le \left| \sum_{k=1}^{m} a_{\pi(k)} - \sum_{k=1}^{N} a_k \right| + \left| \sum_{k=1}^{N} a_k - L \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}
$$