Thm $\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)' = \sum_{n=0}^{\infty} a_n \left(\underbrace{(z-z_0)^n}_{n(z-z_0)^{n-1}}\right)'$

$\| $

$F(z)$

$\longleftarrow$ this claim holds whenever R the radius of conv. for $F(z)$ is $\geq$ or ∞

$$F^{(k)}(z) = \sum_{n=0}^{\infty} \frac{n!}{(n-k)!}(z-z_0)^{n-k}$$

in this case, $F \in C^{\infty}(D)$
$D = \{|z-z_0| < R\}$

can use thm to deduce $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = F(z)$

where does $F(z)$ make sense

$$R = \lim_{n\to\infty} \frac{1}{\left|\frac{a_{n+1}}{a_n}\right|} \quad / \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{1}{(n+1)!} / \frac{1}{n!}\right| = \frac{1}{n+1} \longrightarrow 0$$

$$F'(z) = \sum_{n=1}^{\infty} n\frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \overset{m=0}{=} \sum_{m=0}^{\infty} \frac{z^m}{m!} = F(z)$$

$F'(z) = F(z)$, $F(0) = 1$

$e^z = \cos z + i\sin z$

$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$

Thm (Cauchy's Thm)

$\int_\gamma f(z)dz = 0$

γ : p.w simple closed

$f: D \supset \gamma, \mathring{\gamma}$

f analytic on D.

$\longrightarrow$ D is "simply connected" if $\gamma \subset D$ simple closed, its int. $\gamma \subset D$

Thm: if analytic on a domain has a "primitive" $\exists F' = f$

Thm (Cauchy's Formula)

$$f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz$$

$f(z)$ analytic on a domain $D \supset \gamma, \mathring{\gamma}, z_0 \in \mathring{\gamma}$

γ closed

Applications of Cauchy Formula

$$\int_0^{2\pi} \frac{d\theta}{2+\sin\theta}$$

$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$d\theta = \frac{dz}{iz}$

$e^{i\theta} = \cos\theta + i\sin\theta$

$\cos\theta = \frac{1}{2}(z + z^{-1})$

$\sin\theta = \frac{1}{2i}(z - z^{-1})$

$$\int \frac{dz}{iz} \frac{1}{2 + \frac{1}{2i}(z - z^{-1})}$$

$$= \int \frac{dz}{\frac{1}{2}z(4i + (z - z^{-1}))}$$

parametrize $|z| = 1$

$\Rightarrow \frac{1}{2}(z^2 + 4iz - 1) = \frac{1}{2}(z - r_1)(z - r_2)$

r_1 inside

r_2 outside

$f(\theta)d\theta = \frac{1}{2}\frac{1}{(z-r_1)(z-r_2)}dz$

$h(z) = (z - r_2)^{-1}$

$$\Rightarrow \int_{\gamma=\{|z|=1\}} \frac{h(z)}{z-r_1}dz$$

$= 2\pi i\, h(r_1) = 2\pi i(r_1 - r_2)^{-1}$

$$\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2}\,2\pi i\,h(r_1)$$
$$= \pi i\,(r_1 - r_2)^{-1}$$

June 16th

July 8th

$$\frac{e^{\frac{1}{z-2}}}{z-2} = f(z)$$

← order of the pole

if it was a pole at 2, then $f(z)\cdot(z-2)^{j}$ would have a removable singularity at 2.

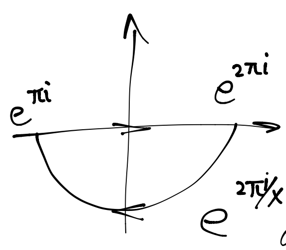
if you have an essential singularity, you cannot get rid of it by multiplying ...

$$\frac{e^{\frac{1}{z-2}}\,(z^2-2)^2}{(z-2)(z+2)}$$

August 5th

§3.1.10

There is no entire function F with $F(x) = 1 - \exp[2\pi i/x]$ for $1 \le x \le 2$.



look at
arg $F(x)$ from $x=1$ to 2
the change in arg F is π

extend F to $\mathbb{C}$
$F(z)$ $1 \le |z| \le 2$
change in arg $F(z)$ over 1st quadrant

· $z = Re^{i\theta}$ as θ goes from 0 to $\frac{\pi}{2}$
$\arg(e^{2\pi i/z}) = \frac{2\pi}{z} = \frac{2\pi}{Re^{i\theta}}$ goes from $\frac{2\pi}{R}$ to $\frac{2\pi}{Re^{i\frac{\pi}{2}}}$

Find that there is a change in argument of $F(z)$ over some closed curve containing the interval $[1,2]$ ⟹ not analytic ⟹ not entire

3.1.10

(P171) If F is analytic on some domain D, there is $\{z_n\} \subset D$ s.t. $f(z_n) = 0$ all n. and $z_n \to z \in D \Rightarrow f(z) \equiv 0$ on D.

Towards a contradiction. If F entire and *
consider $G(z) = F(z) - \{1 - e^{2\pi i/z}\}$
it's analytic on $\mathbb{C}^* = \mathbb{C} \setminus 0$
By (171) ⟹ $G(z) \equiv 0$ on $D = \mathbb{C} \setminus 0$
$F(z) = 1 - e^{2\pi i/z}$ on D
But we assumed F entire
Must be analytic continuation of F to $z = 0$

idea: $H(z) = 1 - e^{2\pi i/z}$ analytic on D

$z_n = \frac{1}{n} \to 0$

$H(z_0) = 0$

$F|_D = H$

$F(0) = 0$ By (171) $F \equiv 0$

$\longrightarrow$ But $F \not\equiv 0$

so $\Rightarrow\!\Leftarrow$

3.1.12

\# zeros of $f(z) = z^3 - 3z + 1$ in $1 < |z| < 2$

$|z| = 1$

$|f + 3z| \leq 1 + 1 = 2 < 3 = |3z|$

Rouché's $\Rightarrow f(z)$ & $3z$ have the same $\underbrace{\#\text{ zeros in } |z|=1}_{=1}$

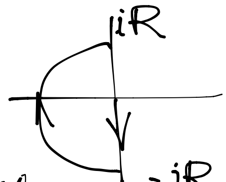
$|z| = 2$

$|f - z^3| \leq 3(2) + 1 = 7 < 8 = |z|^3$

Rouché's $\Rightarrow f(z)$ & z^3 have the same $\underbrace{\#\text{ zeros in } |z|=2}_{=3}$

How many zeros in $Re(z) < 0$. $(0 < \lambda < 1)$

$h(z) = z + e^{-z} - \lambda$



$\cos\theta < 0$

as $z = x + iy$ goes from iR to $-iR$

$h(iy) = iy + e^{-iy} - \lambda$

$= iy + \cos y - i\sin y - \lambda$

$= \underbrace{(\cos y - \lambda)}_{(-2,1)} + i\underbrace{(y - \sin y)}$

$y = R \Rightarrow$ passitive

$y = -R \Rightarrow$ neg

iR to $-iR$: $\Delta \arg h \approx \pi$

along $z = Re^{i\theta}$, $(\frac{3\pi}{2}, \frac{\pi}{2})$

$\cancel{\text{Look at } |e^{-Re^{i\theta}}| = |e^{-R\cos\theta - Ri\sin\theta}| = |e^{-R\cos\theta}| \to \infty}$ as $R \to \infty$

$\cancel{h(Re^{i\theta}) \sim Re^{i\theta} + e^{R}}$

$\Delta \arg(z) = \theta$

$Re^{i\theta_1} \to Re^{i\theta_2}$

$\theta = \theta_2 - \theta_1$

$\arg(e^{z}) = \arg(e^{Re^{i\theta}}) = R(e^{i\theta_2} - e^{i\theta_1})$