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Consider a stochastic discrete-time system described by the following stochastic difference equation:

$$\mathbf{x}_{k+1} = F\left(\mathbf{x}_k, \mathbf{u}(\mathbf{x}_k)\right) + \mathbf{w}_k \tag{1}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state of the system at time step k and $\mathbf{u}(\mathbf{x}_k) \in U$ is the control taken at time step k. Define $\mathbf{w} \sim \mathcal{N}_n(\mathbf{0}, \Sigma)$ to be the stochasticity in the environment where Σ is a diagonal matrix with entries $\sigma_1^2, ..., \sigma_n^2$ along the diagonal. Define $\boldsymbol{\sigma}$ to be the vector such that $\mathrm{Diag}(\boldsymbol{\sigma}) = \Sigma$. Define the hyper rectangle $\Omega_1 = \{\mathbf{v} \in \mathbb{R}^n | -6\boldsymbol{\sigma} \leq \mathbf{v} \leq 6\boldsymbol{\sigma}\}$ and define $\Omega_2 = \Omega \setminus \Omega_1$. Define $h(\mathbf{x})$ to be the barrier function for which $h(\mathbf{x}) \geq \mathbf{m}$. Then clearly:

$$\begin{aligned}\mathbb{E}\left[h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w})\right] &= \int_{\Omega} h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) p(\mathbf{w})\, d\mathbf{w} \\ &= \int_{\Omega_1} h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) p(\mathbf{w})\, d\mathbf{w} + \int_{\Omega_2} h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) p(\mathbf{w})\, d\mathbf{w}\end{aligned} \tag{2}$$

Now let W_i be hyper rectangles that form a partition over Ω_1 such that $\bigcup_{i\in I} W_i = \Omega_1$ and let U_i form a partition over U such that $\bigcup_{j\in J} U_j = U$. Then given that there exist matrices A_i and vectors $\mathbf{b}_i$ such that for each $\mathbf{w}_i \in W_i$ the following inequality holds:

$$h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) \geq A_i \begin{bmatrix}\mathbf{x} \\ \mathbf{u} \\ \mathbf{w}_i\end{bmatrix} + \mathbf{b}_i \tag{3}$$

Define

$$A_i = \begin{bmatrix}A_i^x & A_i^u & A_i^w\end{bmatrix}. \tag{4}$$

Then the integral can be bounded as such:

$$\begin{aligned}\int_{\Omega_1} h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) p(\mathbf{w})\, d\mathbf{w} &\geq \sum_{i\in I} \int_{W_i} \left(\begin{bmatrix}A_i^x & A_i^u & A_i^w\end{bmatrix} \begin{bmatrix}\mathbf{x} \\ \mathbf{u} \\ \mathbf{w}_i\end{bmatrix} + \mathbf{b}_i\right) p(\mathbf{w}_i) d\mathbf{w}_i \\ &= \sum_{i\in I} \left((A_i^x\mathbf{x} + A_i^u\mathbf{u} + \mathbf{b}_i) \int_{W_i} p(\mathbf{w}_i) d\mathbf{w}_i + A_i^w \int_{W_i} \mathbf{w}_i p(\mathbf{w}_i) d\mathbf{w}_i\right) \\ &= \sum_{i\in I} (A_i^x\mathbf{x} + A_i^u\mathbf{u} + \mathbf{b}_i)\, \mathbb{P}\left(\mathbf{w} \in W_i\right) + \sum_{i\in I} A_i^w \mathbb{E}\left[\mathbf{w} | \mathbf{w} \in W_i\right] \mathbb{P}\left(\mathbf{w} \in W_i\right)\end{aligned} \tag{5}$$

Additionally since $h(\mathbf{x}) \geq \mathbf{m}$:

$$\begin{aligned}\int_{\Omega_2} h(F(\mathbf{x}, \mathbf{u}) + \mathbf{w}) p(\mathbf{w})\, d\mathbf{w} &\geq \mathbf{m} \int_{\Omega_2} p(\mathbf{w}) d\mathbf{w} \\ &= \mathbf{m}\mathbb{P}\left(w \in \Omega_2\right).\end{aligned} \tag{6}$$

Using (5) with (6) a lower bound can be obtained on (2) which is linear in $\mathbf{u}$ and thus can be used for a quadratic programming problem.