COMO 401 Assignment Four: Automated 4D Kinematics

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June 21, 2009

Running Instructions

The script `animatedp` animates a double pendulum with massive rods and masses on the ends of the rods. The pendulum is described by the state variables θ, the angle between the first rod and the z-axis, and ϕ, the angle between the first and second rods . Inputs are initial conditions $\begin{pmatrix}\theta_0\\ \theta'_0\\ \phi_0\\ \phi'_0\end{pmatrix}$, simulation time, rod lengths $\begin{pmatrix}r_1\\ r_2\end{pmatrix}$, and masses $\begin{pmatrix}m_1\\ m_2\end{pmatrix}$. To change wheel parameters, edit the first few lines of the script. To adjust secondary parameters (rod radius, etc.), edit the first few lines of `makedp`.

Explanation of Equations of Motion

Since the details of pose matrices, animation, and linkages have been discussed in earlier projects, we will focus on explaining the method of obtaining the equations of motion in a way that is applicable to large class of systems (those using the Denavitt-Hartenburg convention).

Basic Kinematic Matrices

(a). Velocity Ratio Matrix

The velocity ratio matrix L describes the ratios of angular and regular velocities of a link in different directions. Using the DH conventions, only two velocity ratio matrices need to be defined:

$$L_{(i)} = \begin{pmatrix}0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix} \qquad \text{for revolute joints}$$

$$L_{(i)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{for prismatic joints}$$

The pose composition rule gives the translation rule:

$$L_{a(b)} = P_{b,c} L_{a(c)} P_{b,c}^{-1}$$

where $L_{a(b)}$ is the velocity ratio matrix of a in the reference frame of b.

(b). Inertia Matrix

The inertia matrix of link i in its own reference frame is given by:

$$J_{i(i)} = \begin{pmatrix} I_p & mr \\ mr^T & m \end{pmatrix}$$

where m is the mass of the link, r is the vector from the link's origin to its center of mass, and I_p is the pseudo-inertia matrix of the object:

$$I_p = \begin{pmatrix} \int x^2 dm & \int xydm & \int xzdm \\ \int yxdm & \int y^2 dm & \int yzdm \\ \int zxdm & \int zydm & \int z^2 dm \end{pmatrix}$$

Legnani's Equation

Using the pose scheme and 4D kinematics matrices, Legnani et. al discovered the equations of motion for an arbitrary linkage. This equation is

$$M\ddot{q} + C(q, \dot{q}) = Q(t)$$

where M is the mass matrix of the system, C is the inertial and conservative forces, and Q is the external forces. The form of this equation allows it to be used by differential equation solvers as $\ddot{q} = M^{-1}\left(Q(t) - C(q, \dot{q})\right)$

(a). Mass Matrix

The $N \times N$ (N is the number of state variables q_i) mass matrix describes the resistence of each link to the a force upon one of the links. The ij^{th} component can be calculated with 4D kinematics using the formula

$$M_{ij} = \sum_{k=max(i,j)}^{N} \text{Trace}\left(L_{i(0)} J_{k(0)} L_{j(0)}^T\right)$$

(b). Inertial and Conservative Forces

The N-vector C is given by:

$$C(n) = \sum_{k=1}^{N} \text{Trace}\left(\tilde{H}_{0,k} J_{k(0)} L_{i(0)}^T\right)$$

where $\tilde{H}_{0,k}$ is the acceleration of link i in the world frame when $\ddot{q} = 0$. This is given by

$$\tilde{H}_{0,k} = \sum_{i=1}^{k} L_{i(0)}^2 \dot{q_i}^2 + 2 \sum_{i=2}^{k} \sum_{j=1}^{i-1} L_{j(0)} L_{i(0)} \dot{q}_i \dot{q}_j$$

The first sum is to account for centripetal acceleration, and the second sum is to account for the coriolis effect. We can treat gravity as additional acceleration on the ith link, so that we get the final expression

$$\tilde{H}_{0,k} = \sum_{i=1}^{k} L_{i(0)}^2 \dot{q_i}^2 + 2 \sum_{i=2}^{k} \sum_{j=1}^{i-1} L_{j(0)} L_{i(0)} \dot{q}_i \dot{q}_j + H_{g_k(0)}$$

where $H_{g_k(0)}$ is the force matrix $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -981 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ using cgs units

(c). External Forces

External forces must be projected along the external screw axis. If we have forces $F_1, ...F_n$ in 4D form, then the kth compent $Q(k)$ of the force is given by

$$Q(k) = \left(\sum_{i=1}^{N} F_{i(0)}\right) \otimes L_{k(0)}$$

where

$$A \otimes B = A(3,2) \cdot B(3,2) + A(1,3) \cdot B(1,3) + A(2,1) \cdot B(2,1) + A(1,4) \cdot B(1,4) + A(2,4) \cdot B(2,4) + A(3,4) \cdot B(3,4)$$