JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 18, 59-74 (1967)

# Equilibrium States of Elastic Rings

I. TADJBAKHSH AND F. ODEH

*IBM Watson Research Center, Yorktown Heights, New York*

*Submitted by Peter D. Lax*

## 1. INTRODUCTION

This paper is concerned with the problem of determining the possible equilibrium states of buckled elastic rings under pressure. The nonlinear problem with which we deal is based upon the Euler-Bernoulli Theory of the elastica. It was first considered by M. Levy [1] who reduced it to the investigation of an algebraic problem involving elliptic integrals. Later, G. F. Carrier [2] reconsidered the problem, obtained an approximate solution to the algebraic formulation for small deformation, and discussed some other related problems.

Our main concern is to prove the existence of solutions to the full nonlinear problem. For this purpose we formulate the problem along two different lines. The first formulation stated in Section 2 is the classical one and leads to a nonlinear ordinary differential equation for the curvature of the ring. This formulation was used to prove the existence of solutions with small norm (Section 4). Such solutions exist for all modes of deformation with two or more axes of symmetry. It is shown that solutions with only one axis of symmetry do not exist for the linearized problem and numerical search failed to uncover any for the nonlinear problem. A variational form of this formulation is used in Section 5 to prove the existence of solutions with arbitrary norm.

A second and equivalent formulation derived from Hamilton's principle is given in Section 3. It leads to a minimum problem which is used to prove the existence of solutions for loads greater than the first characteristic load of the linearized theory (Section 6). In Section 7 we develop numerical methods for the solution of the problem and plot for different modes of deformation, the shape of the ring for various loads. We also calculate the energy of the ring for various modes.

## 2. The Equilibrium Equations

Consider the equilibrium of an inextensible elastic circular ring of radius unity which is acted upon by a uniform external pressure[1] $p$. The pressure load is assumed constant in magnitude and in the direction of inward drawn normal to the deformed ring. We denote by $\mathbf{x}(s)$ the position vector, in the plane, of a point on the ring which is at the arc length $s$ and let $T$ and $N$ stand for the components of stress $\mathbf{R}$ along the unit tangent $\mathbf{x}'$ and outward unit normal $\mathbf{n} = (x_2', -x_1')$. The equilibrium equation $\mathbf{R}' = p\mathbf{n}$ when resolved in the tangential and normal directions and the moment equilibrium equation are respectively

$$T' + kN = 0 \tag{1}$$

$$N' - kT - p = 0 \tag{2}$$

$$M' + N = 0. \tag{3}$$

Here, $k$ is the curvature and is given by $\theta'$ where $\theta$ is the angle between the tangent $\mathbf{x}'$ and the $x_1$-axis, and $M$ is related to curvature by the constitutive relation $M = 1 - k$. Therefore, $N = \theta''$ and (1) and (2) yield

$$\theta''' - c\theta' + \tfrac{1}{2}\theta'^3 = p, \tag{4}$$

where $c$, an arbitrary constant, appears as a consequence of integrating (1). Equation (4) admits of the trivial solution $\theta = s$, $c = \frac{1}{2} - p$ corresponding to the undeformed ring.

In order to investigate the existence of nontrivial solutions, it is convenient to introduce the new variables

$$v(s) = \theta' - 1 = -M, \qquad \mu = \tfrac{3}{2} - c, \qquad \beta = c + p - \tfrac{1}{2}, \tag{5}$$

in terms of which the pressure $p$ is found from $p = \mu + \beta - 1$ and (4) transforms into

$$v'' + \mu v = \beta - f(v), \tag{6}$$

where

$$f(v) = \tfrac{1}{2}v^2(v + 3). \tag{7}$$

For a closed ring we need only consider periodic solutions of (6). Since the circumference of the ring is $2\pi$, the period of $v(s)$ can only be one of the numbers $2\pi/n$, $n = 1, 2, 3 \ldots$. Moreover, (6) is unaltered under either translation or change in the sign of the independent variable. Therefore, for a fixed

[1] In dimensional terms the radius will be $R$ and the pressure $EIp/R^3$, where $EI$ is the flexural rigidity.

$n$ it is sufficient to consider the solutions of (6) in an interval whose length is one-half the period and with the boundary conditions

$$v' = 0 \qquad x = 0, \quad \frac{\pi}{n}. \tag{8}$$

A further restriction to be imposed is that the angle $\theta$ should change by $2\pi$ for one complete traverse of the ring and hence by $\pi/n$ for each half period. Since $\theta' = v + 1$, this implies

$$\int_0^{\pi/n} v(s)\, ds = 0. \tag{9}$$

Once $v$ is determined in the interval $0 \leqslant x \leqslant \pi/n$, it can be extended for all $s$ continuously and periodically by defining

$$v(-s) = v(s) \qquad \text{and} \qquad v\left(s - \frac{2\pi}{n}\right) = v(s).$$

We are then able to calculate the coordinates of the ring $x_1$, $x_2$, from the relations

$$x_1 + ix_2 = \int_0^s e^{i\theta(\xi)}\, d\xi \qquad 0 \leqslant s \leqslant 2\pi \tag{10}$$

and

$$\theta(s) = s + \int_0^s v(\xi)\, d\xi \qquad 0 \leqslant s \leqslant 2\pi. \tag{11}$$

Finally we note that for a closed ring we must have[2]

$$x_1(2\pi) + ix_2(2\pi) = \int_0^{2\pi} e^{i\theta(s)}\, ds = 0, \tag{12}$$

which will be shown to be satisfied by all periodically extended solutions of (6)-(9) except possibly when $n = 1$. To show this, let $\psi(s) = \int_0^s v(\mu)\, d\mu$. Since $v$ is even, periodic with period $2\pi/n$ and its average vanishes, $\psi$ is odd and periodic with the same period. Hence, $q = e^{i\psi}$ has the same period and has the representation

$$q(s) \sim \sum_{m=-\infty}^{\infty} q_m e^{imns}. \tag{13}$$

[2] This point was not considered in Refs. [1] and [2].

But then for $n \geqslant 2$

$$\int_0^{2\pi} e^{i\theta(s)}\, ds = \int_0^{2\pi} e^{is} e^{i\psi(s)}\, ds = \int_0^{2\pi} e^{is} \sum_{m=-\infty}^{\infty} q_m e^{imns}\, ds = 0 \tag{14}$$

which proves (12).

The case $n = 1$ deserves special attention. In this case we note that since $\psi$ is odd in $s$, $\cos \psi$ is even and $\sin \psi$ is odd. Therefore, again, due to periodicity of these functions, $x_2(2\pi)$ vanishes identically, while vanishing of $x_1(2\pi)$ imposes a further restriction on the solution. This restriction cannot be met if $| v |$ is small enough, i.e., in the linearized problem of Section 4. Numerical calculations for the full nonlinear problem indicate that it cannot be met for any $| v |$.

## 3. The Minimum Problem

In this section we shall use Hamilton's principle of least-energy to derive an alternate formulation of the equations governing the buckled states of an extensible elastic ring. This formulation will be used (Section 6) in proving the existence of buckled states for loads greater than the first buckling load of the linearized theory. For this, we seek the functional

$$V = \int_0^{2\pi} [W(\mathbf{x}', \mathbf{x}'') - P(\mathbf{x}, \mathbf{x}')]\, ds \tag{1}$$

whose stationary points over the class of functions $\mathbf{x}(s)$ satisfying the constraint

$$G(\mathbf{x}') \equiv \mathbf{x}' \cdot \mathbf{x}' - 1 = 0 \tag{2}$$

are the coordinates of the buckled elastic ring. The strain energy $W$ is given by

$$W = \tfrac{1}{2} (k - 1)^2, \tag{3}$$

and $P$ is the potential for the non-conservative[3] applied force $\mathbf{f} = -p\mathbf{n}$. Clearly $P$ must remain invariant under a rotation of the coordinate axes $\mathbf{x}$ and, therefore, must be a function of the invariants of its vector arguments; that is, the three functionally independent scalars $\mathbf{n} \cdot \mathbf{x}$, $\mathbf{x} \cdot \mathbf{x}$, and $\mathbf{t} \cdot \mathbf{n}$. A particular solution is

$$P = -\tfrac{1}{2} p\mathbf{n} \cdot \mathbf{x}. \tag{4}$$

Any other $P$ can differ from the particular solution by an amount whose

[3] There exists no function $G(\mathbf{x}, \mathbf{x}', \dot{\mathbf{x}})$ such that work per particle $\mathbf{f} \cdot \dot{\mathbf{x}}$ equals $dG/dt$.

Euler differential expression vanishes identically. This fact and the regularity of $P$ in its arguments reduce the additional function to an exact differential which changes $V$ simply by a constant which we ignore [3].

With the potential $P$ thus uniquely determined, we proceed to show that the stationary points of $V$ subject to the constraint $G = 0$ are the solutions of Eqs. (2.1)-(2.3) of the previous section with the boundary condition

$$\left[\frac{d^\nu}{ds^\nu}\mathbf{x}\right]_0^{2\pi} = 0, \qquad \nu = 0, 1, 2, 3. \tag{5}$$

The simplest way to demonstrate this is to consider the modified variational problem

$$\delta \int_0^{2\pi} (W - P + QG)\, ds = 0, \tag{6}$$

where $Q$ is the Lagrange multiplier, carry out the Euler differentiation and define the tension $T$ by

$$T = 2[k(k-1) + Q], \tag{7}$$

then the Euler equations become

$$(T\mathbf{x}' + k'\mathbf{n})' - p\mathbf{n} = 0, \tag{8}$$

which is equivalent to Eqs. (2.1)-(2.3). We note now that the parametrization

$$x_1' = \cos\theta(s), \qquad x_2' = \sin\theta(s) \tag{9}$$

transforms $V$ into

$$V(\theta) = -\pi + \tfrac{1}{2}\int_0^{2\pi}\left\{\theta_s^2 + p\int_0^s \sin[\theta(s) - \theta(\xi)]\, d\xi\right\} ds. \tag{10}$$

The appearance of the constant $-\pi$ is due to the fact that

$$\tfrac{1}{2}(k-1)^2\, ds = \tfrac{1}{2}\theta_s^2\, ds - d\theta + \tfrac{1}{2}\, ds.$$

In this new formulation, the inextensibility condition is satisfied and one should minimize $V(\theta)$ over the class of functions $\theta(s)$ which satisfy the constraints

$$\theta(0) = 0 \qquad \theta(2\pi) = 2\pi \tag{11}$$

$$\int_0^{2\pi} \cos\theta(s)\, ds = 0 \qquad \int_0^{2\pi} \sin\theta(s)\, ds = 0. \tag{12}$$

Let us consider the first variation of $V(\theta)$ in an arbitrary admissible "direction" $\eta$. Because of (11) and (12), $\eta$ vanishes at the boundary and satisfies

$$\int_0^{2\pi} \eta \cos \theta \, ds = 0 \qquad \int_0^{2\pi} \eta \sin \theta \, ds = 0. \tag{13}$$

It is easily verified that

$$\delta V(\theta) = \left[\frac{d}{d\epsilon} V(\theta + \epsilon\eta)\right]_{\epsilon=0} = \int_0^{2\pi} \eta \left\{- \theta_{ss} + p \int_0^s \cos [\theta(s) - \theta(\xi)] \, d\xi\right\} ds. \tag{14}$$

Therefore, (13) and (14) imply that the Euler equation for the minimizing function $\theta(s)$ is

$$\theta_{ss} - p \int_0^s \cos [\theta(s) - \theta(\xi)] \, d\xi = \mu_1 \cos \theta + \mu_2 \sin \theta, \tag{15}$$

where $\mu_1$ and $\mu_2$ are undetermined constants. Thus we have a new formulation for the ring buckling problem consisting of the differential-integral equation (15) and the conditions (11) and (12).[4] It is, however, the minimum problem stated by Eqs. (10)-(12) which is of primary interest and which will be used to show the existence of buckled states for the ring for $p > 3$ (see Section 6).

## 4. Existence of Solutions with Small Norm

We now consider the boundary value problem defined by Eqs. (2.6)-(2.9) and prove the existence of solutions in a neighborhood of linearized solutions. Let $\epsilon$ denote the value of the solution $v$ at $s = 0$. Now substitute $v = \epsilon w$, set aside the boundary condition at $s = \pi/n$ and the integral condition (2.9), and consider the resulting initial value problem

$$w'' + \mu w = \gamma - \tfrac{3}{2} \epsilon w^2 - \tfrac{1}{2} \epsilon^2 w^3 \qquad s > 0 \tag{1}$$

$$w' = 0, \qquad w = 1, \qquad s = 0, \tag{2}$$

[4] To show the equivalence of this new formulation with the old, proceed as follows: Eliminate $\mu_1$ and $\mu_2$ from Eq. (15) by using the two equations that result from it by successive differentiation. The resulting equation will be the same as that obtained from (2.4) if the constant $c$ is eliminated there by differentiation.

where $\gamma = \beta/\epsilon$. Among all solutions $W(\mu, \gamma, \epsilon; s)$ we seek those which for any fixed positive integer $n \geqslant 2$ satisfy

$$w_s \,|_{s=\pi/n} = a(\mu, \gamma; \epsilon) = 0 \tag{3}$$

$$\int_0^{\pi/n} w\, ds = b(\mu, \gamma; \epsilon) = 0. \tag{4}$$

Then the solution $v$ of the boundary value problem (2.6)-(2.9) will be $\epsilon w$ for a certain range of values of $\mu(\epsilon)$ and $\beta = \epsilon\gamma(\epsilon)$ as determined parametrically from (3) and (4).

We note that the solution of the initial value problem (1)-(2) is unique, exists for all $s$, and is differentiable with respect to the parameters $\mu$, $\gamma$, and $\epsilon$. The existence for all $s$ of $w$ follows from its uniform boundedness which may be inferred from the first integral of (1)-(2), which is

$$(w_s^2 + \tfrac{1}{4}\,\epsilon^2 w^4) + \epsilon w^3 + \mu w^2 - 2\gamma w = \mu + \tfrac{1}{4}\,\epsilon^2 + \epsilon - 2\gamma. \tag{5}$$

For given values of $\mu$, $\gamma$ and $\epsilon \neq 0$ the right side of (5) is a constant and for large $w$ the parenthesis on the left side of the equation dominates the lower degree terms. Hence, $w$ stays uniformly bounded for all $s$ and its existence, uniqueness and differentiability with respect to parameters in the equation follow by standard arguments [4].

We note that if Eqs. (3) and (4) have a solution $\mu^*$, $\gamma^*$ at $\epsilon = \epsilon^*$ for some $\epsilon^*$, then by the implicit function theorem there will exist a solution in the finite neighborhood $|\,\epsilon - \epsilon^*\,| < \delta_0$, for some $\delta_0 > 0$, if

$$J = \frac{\partial(a, b)}{\partial(\mu, \gamma)} \neq 0 \qquad \text{for} \qquad \epsilon = \epsilon^*. \tag{6}$$

This is so at $\epsilon = 0$. For then the solution of (1) and (2) is

$$\varphi = \varphi(\mu, \gamma; s) = \left(1 - \frac{\gamma}{\mu}\right) \cos \sqrt{\mu}\, s + \frac{\gamma}{\mu}. \tag{7}$$

If one then requires that (3) and (4) be satisfied, then it follows that

$$\mu = n^2, \qquad \gamma = 0 \qquad \text{for} \qquad \epsilon = 0 \tag{8}$$

and

$$\det J \,|_{s=\pi/n} = \Big[\varphi_{s\mu}\left(\mu, \gamma; \frac{\pi}{n}\right) \int_0^{\pi/n} \varphi_\gamma(\mu, \gamma; s)\, ds - \varphi_{s\gamma}\left(\mu, \gamma; \frac{\pi}{n}\right) \int_0^{\pi/n} \varphi_\mu(\mu, \gamma; s)\, ds\Big]_{\substack{\mu=n^2\\ \gamma=0}}$$

$$= \frac{\pi^2}{2n^4}\,. \tag{9}$$

Therefore, a solution exists in some neighborhood $|\epsilon| < \delta_0$. By regular perturbation we find

$$v = \epsilon \cos ns + \frac{\epsilon^2}{4n^2}(\cos 2ns - \cos ns) + 0(\epsilon^2)$$

$$\mu = n^2 - \frac{3\epsilon^2}{8}\left(1 + \frac{1}{n^2}\right) + 0(\epsilon^2)$$

$$\beta = \frac{3}{4}\epsilon^2 - \frac{3\epsilon^2}{4n^2} + 0(\epsilon^3). \tag{10}$$

From the relationship $p = \mu + \beta - 1$ one finds that $p$ is an increasing function of $\epsilon$ for small $\epsilon$ and for $n \geqslant 2$

$$p = (n^2 - 1) + \frac{3\epsilon^2}{8}\left(1 - \frac{1}{n^2}\right) + 0(\epsilon^2). \tag{11}$$

If one proceeds to calculate the vector $\mathbf{x}(s)$ from (2.10) and (2.11) one finds that $x_1(2\pi) \neq 0$ for $n = 1$. Also, numerical results show that, for $0 < p < 14$, $x_1(2\pi)$ starts at zero and monotonically decreases with $\epsilon$. This seems to indicate that there is no buckled state with only one axis of symmetry.

## 5. Existence of Solutions with an Arbitrary Norm

This section is concerned with proving that the problem defined by Eqs. (2.6)-(2.9) of Section 2 admits of at least one solution on every "surface" defined by $\int_0^{\pi/n} v^2(s)\, ds = c$, where $c$ is an arbitrary positive constant. The problem, for a fixed integer $n \geqslant 2$, is

$$v_{ss} + \mu v = \beta - f(v) \qquad 0 < s < \ell = \frac{\pi}{n}$$

$$\int_0^{\ell} v\, ds = 0 \tag{1}$$

$$v_s = 0 \qquad \text{at} \qquad s = 0, \ell.$$

For any $c > 0$, we seek a solution $v(c)$, $\mu(c)$, $\beta(c)$ which satisfies (1) at least in a variational or weak sense. In order to define the problem precisely, we introduce the Hilbert space

$$W_2^{(1)}(\Omega) = \{v \mid v, v_s \in L_2(\Omega)\} \tag{2}$$

(where derivatives are taken in the distribution sense and $\Omega$ denotes the interval $[0, \ell]$) with the scalar product and norm defined by

$$(u, v) = \int_0^\ell (uv + u_s v_s)\, ds \tag{3}$$

$$\| u \| = \int_0^\ell (u^2 + u_s^2)\, ds. \tag{4}$$

We note in passing that $W_2^{(1)}(\Omega)$ is the completion, in the energy norm (4), of differentiable functions whose derivatives belong to $L_2(\Omega)$.

Let $H$ denote the closed subspace of $W_2^{(1)}(\Omega)$ orthogonal to constants; i.e.,

$$H = \left\{ v \in W_2^{(1)}(\Omega) \mid \int_0^\ell v\, ds = 0 \right\} \tag{5}$$

and define, for $v \in H$, the functionals

$$\begin{aligned} F(v) &= \int_0^v f(u)\, du = \tfrac{1}{8} v^4 + \tfrac{1}{2} v^3 \\ \Phi(u, v) &= \tfrac{1}{2} \int_0^\ell u_s^2\, ds - \int_0^\ell F(v(s))\, ds \\ E(v) &= \Phi(v, v) \\ g(v) &= \int_0^\ell v^2\, ds. \end{aligned} \tag{6}$$

Our object is to prove that the energy functional $E(v)$, where $v \in H$ is constrained to lie on the surface $g(v) = c > 0$, achieves its absolute minimum at some point $v = v_0(c)$ in the interior of some large sphere.

A particularly convenient theorem, which may be used to prove the existence of such conditional minima, was given by Browder [5]. Using his terminology, we state:

LEMMA 1. *Let the functionals $\Phi$, $E$ and $g$ be defined by (6). Then* (i) *$\Phi$ is a semi-convex real differentiable function on $H \times H$, and $E$ is differentiable on $H$.* (ii) *$g$ is weakly continuous and differentiable on $H$.*

The proof consists in simply checking that the smoothness and monotonicity conditions imposed in Browder's theorem [5] are satisfied in the present case. However, we give an indication of the proof:

For each fixed $v$, $\Phi(u, v)$ is clearly convex in $u$. Next we need to verify that $F(v_n) \to F(v)$ (converges strongly to $v$) as $v_n \rightharpoonup v$ (converges weakly to $v$). But then $\{v_n\}$ is uniformly bounded in $H$ with an upper bound $M$. By

Rellich's lemma there exists a subsequence, call it again $v_n$, which converges strongly (in $L_2$) to $v$. Since by Sobolev's lemma the norm (4) dominates the maximum norm, one has

$$\begin{aligned} | F(v_n) - F(v) | &= \left| \int_0^\ell (\tfrac{1}{8} v^4 + \tfrac{1}{2} v^3)\, ds - \int_0^\ell (\tfrac{1}{8} v_n^4 + \tfrac{1}{2} v_n^3)\, ds \right| \\ &\leqslant \tfrac{1}{8} \int_0^\ell | v^4 - v_n^4 |\, ds + \tfrac{1}{2} \int_0^\ell | v^3 - v_n^3 |\, ds \\ &= \tfrac{1}{8} \int_0^\ell | (v^2 + v_n^2)(v^2 - v_n^2) |\, ds + \cdots . \end{aligned}$$

Hence

$$| F(v_n) - F(v) | \leqslant KM^2 \int_0^\ell | v^2 - v_n^2 |\, ds, \tag{7}$$

where $K$ is some constant related to $\ell$. Hence $F(v_n) \to F(v)$ as $v_n \rightharpoonup v$. This calculation and the strong continuity of $\Phi$ in $u$ for fixed $v$ proves the semi-convexity of $\Phi$. (In fact, the calculation shows that $E(v)$ is weakly lower-semi-continuous, since $\int_0^\ell v_s^2\, ds$ clearly is.) The weak continuity of $g$, and the differentiability of $\Phi$, $E$ and $g$ may be proved in a similar manner.

Lemma 1 shows that on the set $S_R = \{v \in H \mid g(v) = c, \| v \| \leqslant R\}$, where $R$ is given, the functional $E(v)$ takes on its absolute minimum at a point $v = v_0$. Before one can deduce from this fact that $dE(v_0)$, the Frechet derivative, in $H$, of $E$ at $v_0$, and $dg(v_0)$ are proportional (by the use of Lagrange multiplier method), we need to prove that $v_0$ is not a boundary point of $S_R$. This is a consequence of

LEMMA 2. *For all $v$ in $H$, such that $g(v) = c > 0$, the energy $E(v) \to \infty$ as $\| v \| \to \infty$.*

PROOF. It is sufficient to prove the lemma for smooth $v$. To estimate $\int_0^\ell F(v)\, ds$ consider first

$$\int_0^\ell v^4\, ds \leqslant \max v^2 \int_0^\ell v^2\, ds = c \max (v^2).$$

Since $v$ is orthogonal to constants, there exists a point $\xi$, $0 < \xi < \ell$ such that $v(\xi) = 0$. Hence

$$v^2(s) = 2 \int_\xi^s v v_s\, ds \leqslant 2c^{1/2} \left( \int_0^\ell v_s^2\, ds \right)^{1/2}.$$

Hence

$$\int_0^\ell v^4\, dx \leqslant 2c^{3/2} \left( \int_0^\ell v_s^2 \right)^{1/2}. \tag{8}$$

Moreover

$$\int_0^\ell v^3\,ds = \int_0^\ell v \cdot v^2\,ds \leqslant \left(\int_0^\ell v^2\right)^{1/2} \left(\int_0^\ell v^4\right)^{1/2}$$

$$\leqslant 2c^{5/4} \left(\int_0^\ell {v_s}^2\right)^{1/4}. \tag{9}$$

Now, for $v$ in $H$, $\| v \| \to \infty$ only if $\int_0^\ell (v_s^2)\,dx \to \infty$. Combining this fact with (8)-(9) and the definition of $E$ we obtain the lemma. Now we state

THEOREM 1. *For every $c > 0$ there exists a smooth function $v = v(c)$ and two constants $\mu = \mu(c)$, $\beta = \beta(c)$ such that (6) is satisfied.*

PROOF. Lemma 1 implies that $E(v)$ achieves its absolute minimum on each $S_R$. Taking $R$ large enough, using Lemma 2 and the rule of Lagrange multipliers (Theorem 4 in Ref. 5), we find that at this minimum, call it $v$,

$$d(E + \mu g)\,|_v = 0 \tag{10}$$

for some $\mu$. Since $H$ is orthogonal to constants, this implies that $v$ is a variational (or weak) solution to (1), for some constant $\beta$. The regularity of the solution follows from the general regularity theory for variational problems connected with mildly nonlinear elliptic equations (see, for example, Morrey [6]).

## 6. EXISTENCE OF BUCKLED STATES FOR $p > 3$

In this section, the minimum principle formulation of the problem (Section 3) will be used to prove the existence of nontrivial weak solutions for values of the pressure $p > 3$. This critical value is the first buckling load of the linearized theory. We start with the functional $V(\theta)$ defined in (3.10) and introduce the dependent variable $\psi(s) = \theta(s) - s$. Then $V$ takes the form

$$V(\psi) = \tfrac{1}{2} \int_0^{2\pi} \left\{ {\psi_s}^2 + p \int_0^s N[\psi(\xi); s]\,d\xi \right\} ds, \tag{1}$$

where

$$N[\psi; s] = \sin\,[(\psi(s) - \psi(\xi)) + (s - \xi)]. \tag{2}$$

The function $\psi$ satisfies the boundary conditions

$$\psi(0) = \psi(2\pi) = 0 \tag{3}$$

and the constraints

$$\int_0^{2\pi} \cos[\psi(s) + s]\, ds = \int_0^{2\pi} \sin[\psi(s) + s]\, ds = 0. \tag{4}$$

Consider the Sobolev space $W_2^{(1)}(0, 2\pi)$ introduced in Section 5. Let $H$ be the subspace corresponding to the Dirichlet boundary conditions (3). We shall prove that $V$, subject to the constraints (4) achieves its minimum at some interior point $\psi_0$. To this end define

$$V_1(\psi) = \int_0^{2\pi} \psi_s{}^2\, ds$$

$$V_2(\psi) = p \int_0^{2\pi} \int_0^s N[\psi]\, d\xi\, ds, \tag{5}$$

then we have:

PROPOSITION 1. *The set of functions in $H$ which satisfy the constraints (4) is weakly closed. The functional $V_2$ defined in (5) is sequentially weakly continuous in $H$.*

PROOF. Consider the second statement. Let $\psi_n \rightharpoonup \psi$. Then $\{\psi_n\}$ is uniformly bounded in $H$, and hence by Sobolev's embedding lemma, is compact in the maximum norm. Therefore, there exists a subsequence, call it again $\psi_n$, such that $\psi_n \to \psi$ in the maximum norm. Now consider the difference

$$\begin{aligned} | V_2(\psi_n) - V_2(\psi) | &\leqslant | p | \int_0^{2\pi} \left[\int_0^s | N(\psi_n) - N(\psi) |\, d\xi\right] ds \\ &\leqslant | p | \int_0^{2\pi} \int_0^s | (\psi_n(s) - \psi(s)) + (\psi_n(\xi) - \psi(\xi)) |\, d\xi\, ds \\ &\leqslant (2\pi + 1) | p | \int_0^{2\pi} | \psi_n(s) - \psi(s) |\, ds \to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{6}$$

Hence, $V_2$ is weakly continuous. The proof of the first statement follows from a similar argument.

PROPOSITION 2. *Let $S$ denote the set of functions, in $H$, which satisfy the constraints (4). Then $V(\psi)$ achieves its minimum on $S$ at some point $\psi_0$ in the interior of some sphere.*

PROOF. Because of the Dirichlet conditions (3) the norm in $H$ is equivalent to the semi-norm $\int_0^{2\pi} \psi_s{}^2\, ds$. Moreover $V_2(\psi)$ is bounded for all $\psi$. Hence,

$V(\psi) \to \infty$ as $\| \psi \| \to \infty$ and it is sufficient to consider $V$ defined on bounded sets in $H$. But then $V_1(\psi)$ is weakly lower semi-continuous and $V_2$ is weakly continuous. Since $S$ is weakly closed the proposition follows.

THEOREM 2. *For all $p > 3$ there exists a nontrivial (i.e., a buckled state) variational solution of equations (3.15, 3.11, 3.12).*

PROOF. Let $\psi_0$ be the minimizing function of Proposition 2 above. Since $\psi_0$ is in the interior of some sphere, the rule of Lagrange multipliers [5] proves that $\psi_0$ is a variational solution of (3.15). It is sufficient then to check that $\psi_0 \not\equiv 0$ for $p > 3$. But the second variation of $V$ at $\psi = 0$ in the direction $\eta$, calculated from (3.10), is

$$\delta^2 V \,|_{\psi=0} = \int_0^{2\pi} \eta(s) \left(-\frac{d^2}{ds^2}\right) \eta(s)\, ds - p \int_0^{2\pi} \eta^2(s)\,(1 - \cos s)\, ds + p \int_0^{2\pi} \eta(s) \int_0^s \sin (s - \xi)\, \eta(\xi)\, d\xi\, ds. \tag{7}$$

The variation $\eta$ must satisfy the Dirichlet condition (3) as well as the constraints induced by (4); i.e.,

$$\int_0^{2\pi} \eta(s) \begin{Bmatrix} \sin s \\ \cos s \end{Bmatrix} ds = 0 \qquad \int_0^{2\pi} \eta^2(s) \begin{Bmatrix} \sin s \\ \cos s \end{Bmatrix} ds = 0. \tag{8}$$

Therefore,

$$\delta^2 V \,|_{\psi=0} = \langle \eta, (A - p + pG)\, \eta \rangle \tag{9}$$

where

$$A\eta = -\frac{d^2}{ds^2}\eta, \qquad G\eta = \int_0^s \sin (s - \xi)\, \eta(\xi)\, d\xi = (I - A)^{-1}\eta \tag{10}$$

for all admissible variations $\eta$.

Choosing $\eta$ to be the *second* eigenfunction of $A$, i.e., $\eta = \cos 2s$, all the required constraints are satisfied, and we find

$$\delta^2 V \,|_{\psi=0} = \left\langle \eta, \left(4 - p - \frac{p}{3}\right) \eta \right\rangle < 0 \qquad \text{for all} \qquad p > 3. \tag{11}$$

Hence, $\psi = 0$ is not even a local minimum of $V$ for $p > 3$ and the theorem is proved.

## 7. Numerical Calculations

The existence proof of Section 4 can be used to develop a numerical procedure which solves the algebraic equations (4.3) and (4.4), for a fixed $\epsilon$, by the Newton-Raphson iteration method. For this purpose we introduce the vector $Y = (Y_1, Y_2)$ by

$$(Y_1, Y_2) = \left(\frac{\partial}{\partial\mu}, \frac{\partial}{\partial\gamma}\right) \int_0^s w(\mu, \gamma, \epsilon; t)\, dt \tag{1}$$

and note that, from (4.1, 4.2)

$$\begin{gathered} Y''' + [\mu + f_v(\epsilon w)]\, Y' = D \qquad s > 0 \\ Y(0) = Y'(0) = Y''(0) = 0, \end{gathered} \tag{2}$$

where $f_v(\epsilon w) = 3\epsilon w + 3\epsilon^2 w^2/2$ and $D$ is the vector $(-w, 1)$. The iteration procedure consists of solving successively the system,

$$J^\nu\left(\frac{\pi}{n}\right)(\lambda^{\nu+1} - \lambda^\nu) = -A^\nu \qquad \nu = 0, 1, 2 \ldots, \tag{3}$$

where

$$\lambda = \begin{bmatrix}\mu \\ \gamma\end{bmatrix}, \qquad A = \begin{bmatrix}a \\ b\end{bmatrix} \tag{4}$$

with $a$, $b$ defined by (4.3)-(4.4) and

$$J(s) = \begin{bmatrix} Y_1 & Y_2 \\ Y_1'' & Y_2'' \end{bmatrix}. \tag{5}$$

The implicit function theorem of Section 4 proves that $J(\pi/n)$, for small $\epsilon$, is nonsingular and, hence, (3) is solvable. The first term in the iteration is determined from the perturbation solution (4.10) and the $\nu$th iterate of $Y$, which determines $J$, is found from (2) with coefficients determined by the $(\nu - 1)$th iterate. Upon convergence, the process is repeated for a neighboring value of $\epsilon$ with the first term determined by linear extrapolation.

The function space analogue of the Newton-Raphson method for the direct solution of the boundary value problem consisting of Eqs. (2.6)-(2.9) proved more effective in practice. Here, no $\epsilon$ parametrization is necessary. For a fixed $\mu$, the problem is linearized about an approximate solution $\mathring{v}(s)$ and $\mathring{\beta}$. This linearized problem gives a correction $\dot{w}$, $\dot{\alpha}$, and leads to the next approximation $w^1 = \mathring{v} + \dot{w}$ and $\beta^1 = \mathring{\beta} + \dot{\alpha}$. The process is repeated until it converges. Again, the first approximate solution was obtained from the perturbation solution.

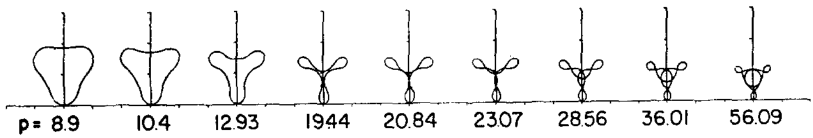


FIG. 1

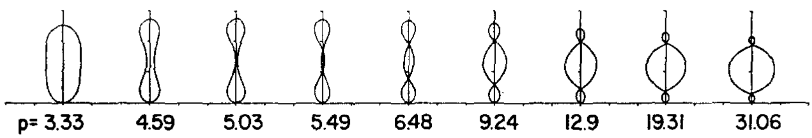


FIG. 2

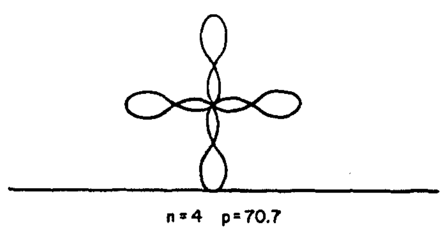


FIG. 3

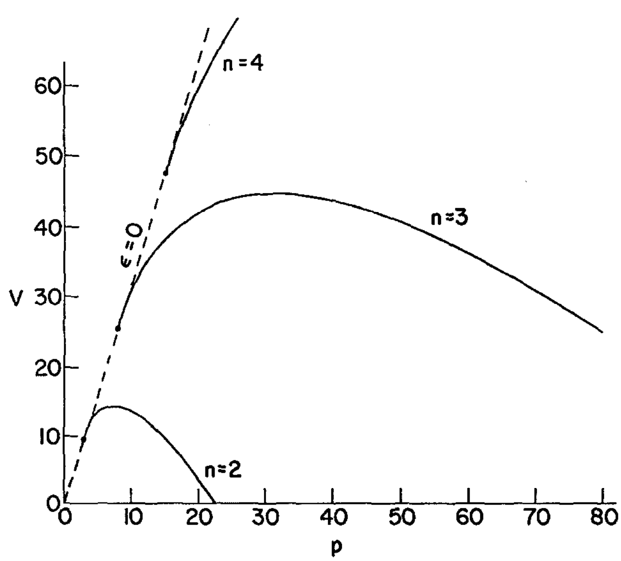


FIG. 4

The results of these calculations are shown in Figs. 1-4. Figures 1-3 show the deformation of the ring for various modes and for increasing load. The ring can cross itself for large loads, for we have assumed that the pressure always acts on one side (the original outside wall) of the ring. Figure 4 depicts the energy $V$ as a function of pressure for various modes of deformation. It has been conjectured by Friedrich [7] that a mechanical system, for a fixed load, chooses among all possible modes of deformation, the one with minimum energy. Assuming this, Fig. 4 indicates that as $p$ is slowly (quasistatically) increased from 3, the ring always assumes the shape of the lowest mode $n = 2$.

## References

1. M. Levy. Mémoire sur un nouveau cas intégrable du problème de l'élastique et l'une de ces applications. *J. Math. (Liouville)*, *Ser.* 3 (1884), 7.
2. G. F. Carrier. On the buckling of elastic rings. *J. Math. Phys.* **26** (1947), 94-103.
3. R. Courant and D. Hilbert. "Methods of Mathematical Physics," Vol. I, p. 193. Wiley (Interscience), New York, 1953.
4. P. Hartman. "Ordinary Differential Equations." Wiley, New York, 1964.
5. F. Browder. Variational methods for non-linear elliptic eigenvalue problems. *Bull. Amer. Math. Soc.* **71** (1965), 176-183.
6. C. B. Morrey. On the analyticity of analytic nonlinear elliptic systems of partial differential equations. I, II. *Am. J. Math.* **80** (1958), 198-237.
7. K. O. Friedrichs. "On the Minimum Load for Spherical Shells." pp. 258-272. von Karman Anniversary volume, California Institute of Technology, Pasadena, 1941.