Part III: Biological Physics and Fluid Dynamics (Michaelmas 2023)
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Example Sheet #3

1. Complete Taylor's calculation of the fluid dynamics of two nearby waving sheets separated by a distance $2h$ to obtain the dissipation rate as a function of the phase shift θ between the two traveling waves of deformation. Show that it is minimised for $\theta = 0$.

2. Using Blake's calculation of the fluid velocity for a Stokeslet near a no-slip wall, consider the half-space below a wall in the $x - y$ plane, and take z increasing downwards. Calculate the flow field at points (x, y, R) associated with a particle of radius R that is denser than the surrounding fluid and hovers infinitesimally below the wall, and whose effect on the fluid is modelled as a downward-pointing force F located at $(0, 0, R)$. Suppose a second particle is placed at position $(x_0, 0, R)$. Find the equations of motion for the two particles as they are each advected by the flow due to the other. Show that they are attracted to one another and collide in finite time.

3. As discussed in lectures, the stochastic Adler equation for the dynamics of the phase difference between two coupled oscillators is

$$\dot{\theta} = \delta\omega - \gamma \sin\theta + \xi(t), \tag{1}$$

where $\delta\omega$ is the intrinsic frequency difference between the oscillators, $\gamma > 0$ is a coupling constant and the noise has the properties $\langle \xi(t) \rangle = 0$, $\langle \xi(t)\xi(t') \rangle = 2T_{\mathrm{eff}}\delta(t - t')$, where T_{eff} is an effective temperature that characterises the biochemical noise. Assuming that the coupling strength is sufficiently large to yield synchrony, show how measurement of (i) the autocorrelation function of the fluctuations in θ around the synchronised state and (ii) the relative probability of positive and negative phase slips can be used to determine the three unknown quantities $\delta\omega$, γ, and T_{eff}.

4. Consider a model for a planar, ciliated tissue with long-range orientational order in which the action of each cilium is represented by a point force $\mathbf{F} = F\hat{\mathbf{e}}_x$ acting on the fluid a distance $z = h$ above a no-slip surface in the $x - y$, where a fluid of viscosity μ occupies the half-space $z \geq 0$. Suppose the cilia reside on some regular lattice with a typical spacing d, so that the lateral density of cilia is $P = 1/d^2$. Assuming a coarse-grained description in which the discreteness of the cilia is neglected in favour of a continuous distribution with density P, use the asymptotic form of the Blake tensor in the far-field limit $\rho, z \gg h$ (where $\rho^2 = x^2 + y^2$), to obtain a scaling form for the x-component of the fluid velocity u as a function of height z, assuming some large-scale cutoff Λ. Examine the asymptotic behaviour in the various limits $z \to \infty$, $\Lambda \to \infty$ and comment on the significance.