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Slender-body theory for particles of arbitrary cross-section in Stokes flow

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A rigid body whose length ($2l$) is large compared with its breadth (represented by R_0) is straight but is otherwise of arbitrary shape. It is immersed in fluid whose undisturbed velocity, at the position of the body and relative to it, may be either uniform, corresponding to translational motion of the body, parallel or perpendicular to the body length, or a linear function of distance along the body length, corresponding to an ambient pure straining motion or to rotational motion of the body. Inertia forces are negligible. It is possible to represent the body approximately by a distribution of Stokeslets over a line enclosed by the body; and then the resultant force required to sustain translational motion, the net stresslet strength in a straining motion, and the resultant couple required to sustain rotational motion, can all be calculated. In the first approximation the Stokeslet strength density $\mathbf{F}(x)$ is independent of the body shape and is of order $\mu U \epsilon$, where U is a measure of the undisturbed velocity and $\epsilon = (\log 2l/R_0)^{-1}$. In higher approximations, $\mathbf{F}(x)$ depends on both the body cross-section and the way in which it varies along the length. From an investigation of the 'inner' flow field near one section of the body, and the condition that it should join smoothly with the 'outer' flow which is determined by the body as a whole, it is found that a given shape and size of the local cross-section is equivalent, in all cases of longitudinal relative motion, to a circle of certain radius, and, in all cases of transverse relative motion, to an ellipse of certain dimensions and orientation. The equivalent circle and the equivalent ellipse may be found from certain boundary-value problems for the harmonic and biharmonic equations respectively. The perimeter usually provides a better measure of the magnitude of the effect of a non-circular shape of a cross-section than its area. Explicit expressions for the various integral force parameters correct to the order of ϵ^2 are presented, together with iterative relations which allow their determination to the order of any power of ϵ. For a body which is 'longitudinally elliptic' and has uniform cross-sectional shape, the force parameters are given explicitly to the order of any power of ϵ, and, for a cylindrical body, to the order of ϵ^3.

1. Introduction

Slender-body theory for Stokes flow with negligible inertia forces was initiated by Burgers (1938), but appears to have remained almost unnoticed for many years after his work. The elementary form of the theory suggested by Burgers was improved a little by Broersma (1960*a*, *b*), and the theory in more general form has recently been revived and developed considerably (Tuck 1964, 1970; Taylor 1969; Cox 1970*a*, *b*; Tillett 1970). All this published work is concerned with slender bodies of circular cross-section. Since an important potential application of the results is to naturally occurring particles, it is desirable to be able to assess the consequences of a non-circular shape of the cross-section of the body. It is intuitively evident that, as the body cross-section shrinks to a point, the effect of the cross-sectional shape on integral parameters such as the drag on the body in translational motion will diminish, but the magnitude and nature of this residual effect need examination.

Part of the current interest in slender-body theory for Stokes flow derives from its use in work on the mechanics of suspensions (see, for instance, Goldsmith & Mason 1967). In that subject one is often concerned with the flow due to force-free rigid particles immersed in a pure straining motion and also with the flow due to the rotation of a rigid particle on which a couple is exerted by external means. In this paper we shall therefore consider the implications of slender-body theory for these two kinds of flow (both of which involve a linear variation of the undisturbed velocity over the length of the body), as well as for the more familiar case of a body in translational motion, either parallel or normal to its length, through fluid at rest at infinity.

As might be expected from experience with Stokes flow generally, 'slender-body theory' here involves a consideration of the approximate forms of the flow fields in the regions near to and far from one section of the body, and of the conditions that they be compatible. The version of the theory to be given here is mathematically simple, and may appeal to those who, like myself, tend to get lost in the details of formal inner and outer expansion matching techniques. We include in passing the formulae which describe the effect of the meridional shape of the body, some of which are already available in print.

For the present purpose of examining the consequences of a non-circular cross-section, it is sufficient to take a straight body (that is, a body which reduces to a straight line as the thickness ratio tends to zero). The effect of the local cross-sectional shape can be determined from a consideration of the local 'inner' flow alone, and the generalization of the theory to include curved slender bodies of circular cross-section recently given by Cox (1970*a*) may be adapted to cases of non-circular cross-section.

2. Replacement of the body by a line of Stokeslets

The basic idea in slender-body theory for Stokes flow is that the disturbance motion due to the presence of the body is approximately the same as that due to a suitably chosen line distribution of Stokeslets. A Stokeslet is a singularity in Stokes flow representing the effect of a force applied to the fluid at a point, and the notion is that the main effect of an element of length of the slender body on the surrounding fluid arises from the resultant force exerted across the boundary of that short section of the body. For a force $\mathbf{F}$ applied at the origin in an infinite body of fluid free from boundaries and other sources of motion, the velocity at the point $\mathbf{x}$ in flow with negligible inertia forces is

$$\frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{|\mathbf{x}|}+\frac{x_i x_j}{|\mathbf{x}|^3}\right), \tag{2.1}$$

where μ is the fluid viscosity. The associated vorticity is

$$-\frac{F_j}{4\pi\mu}\epsilon_{ijk}\frac{\partial|\mathbf{x}|^{-1}}{\partial x_k}, \tag{2.2}$$

which, of all the spherical harmonics of negative degree capable of representing a solenoidal axial vector, decreases least rapidly as $|\mathbf{x}|\to\infty$. Thus if Stokeslets are distributed over the portion $-l < x < l$ of the x-axis so that the line density of the applied force is $\mathbf{F}(x)$, the resulting fluid velocity at point $\mathbf{x}$ is

$$v_i(\mathbf{x}) = \frac{1}{8\pi\mu}\int_{-l}^{l}\left[\frac{F_i(x')}{\{(x-x')^2+r^2\}^{\frac{1}{2}}}+\frac{(x_i-x_i')\,(x_j-x_j')\,F_j(x')}{\{(x-x')^2+r^2\}^{\frac{3}{2}}}\right]dx', \tag{2.3}$$

where $r^2 = x_2^2+x_3^2$, $x_2' = x_3' = 0$, and x, x' are written in place of x_1, x_1' whenever possible. The notation is shown in figure 1.

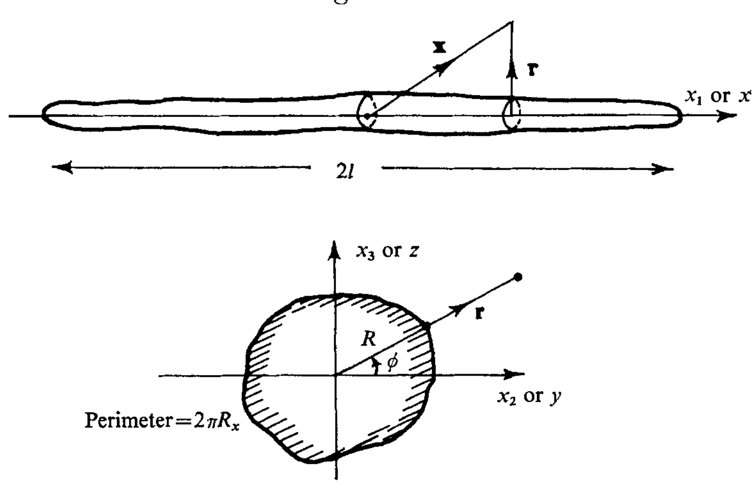

FIGURE 1. Sketch to show notation.

The rigid body of length $2l$ whose disturbance motion is to be represented by this distribution of Stokeslets encloses the portion $-l < x < l$ of the x-axis, and points on the body surface are given by

$$r = R(\phi, x), \tag{2.4}$$

where $\tan\phi = x_3/x_2$ and $R/l \ll 1$. The cross-sectional shape is not necessarily independent of x. We choose the axes of reference so that the body is stationary. The undisturbed fluid velocity (that is, the velocity which the fluid would have in the absence of the body) at cross-section x of the body will be written as $-\mathbf{U}(x)$, where $\mathbf{U}$ is a vector of uniform direction whose magnitude is either uniform or a linear function of x. The case in which $\mathbf{U}$ is uniform corresponds to translational motion of the body with velocity $\mathbf{U}$ through fluid at rest at infinity (with speed U_1 in the longitudinal or end-on direction, and components

U_2, U_3 in the transverse plane); the case $\mathbf{U} = (0, \Omega_3 x, -\Omega_2 x)$ corresponds, with an adequate approximation, to rotation of the body about its midpoint with angular velocity $(0, \Omega_2, \Omega_3)$ relative to the fluid at infinity; and the case

$$\mathbf{U} = (-e_{11}x, 0, 0)$$

corresponds to the body being immersed in a pure straining motion with rate of extension e_{11} in the direction of the body length (extension of the fluid in directions normal to the body length being affected negligibly by the presence of the body). All these different cases are of course superposable.

In order to satisfy the no-slip condition the resultant fluid velocity $\mathbf{v} - \mathbf{U}$ must be zero at the body surface. Thus the unknown function $\mathbf{F}(x')$ representing the line density of Stokeslet strength must be chosen so that the disturbance velocity $\mathbf{v}(\mathbf{x})$ given by (2.3) is equal to $\mathbf{U}(x)$ at all points on the body surface defined by (2.4). The procedure suggested by Burgers (1938) (for a body of circular cross-section) was to write $\mathbf{F}(x')$ as a polynomial of low degree and to determine the coefficients in a way which optimized the satisfaction of this boundary condition; and Broersma (1960*a*, *b*) improved this procedure numerically by increasing the degree of the polynomial.

It is impossible in general to satisfy the no-slip condition at all points of the body surface exactly by means of a line distribution of Stokeslets alone on the x-axis. Stokeslet doublets (that is, force doublets) and higher order singularities are also needed, even in the case of a body with axial symmetry (in transverse motion). However, we shall see that the distribution of Stokeslet strength can be determined to a certain order of approximation, when the body is slender, without the need for explicit introduction of the other types of singularity. It also happens that most of the flow parameters of greatest practical interest, such as the total force or couple which the body must exert on the fluid to sustain the specified motion, are determined by the Stokeslet distribution alone. (The period of orbit of a slender body in a simple shearing motion is an exception, as Cox (1970*b*) has pointed out.) For this reason we consider here only the Stokeslet distribution on the x-axis, with the implication that representation of the body can be achieved to a certain order of approximation only.

It is evident that if $F_i(x')$ has the same sign over a large part of the range $-l < x' < l$, the first of the two terms in the integrand of (2.3) may lead to divergence of the integral, as $r/l \to 0$ for fixed x/l, and likewise the second term when $i = 1$; and if $F_i(x')$ were constant, the divergence would be as $\log l/r$. That is to say, important contributions to the induced velocity due to the line distribution of Stokeslets at positions close to the axis are made by both the neighbouring and the distant Stokeslets, at any rate for bodies whose shape is such that the Stokeslet strength is distributed not too unevenly. (This is of course in contrast to the dominance of nearby singularities in slender-body theory for irrotational flow.) It is a feature of this logarithmic behaviour that the induced velocity due to the whole Stokeslet distribution varies by a relatively small amount only, over the perimeter of a cross-section of the body. This is the key to the use of a line distribution of Stokeslets as a means of satisfying the no-slip condition, to the crudest approximation, at points on the surface of a slender body of arbitrary cross-section. It is possible to choose the Stokeslet strength distribution so that the velocity due to the line of Stokeslets cancels the undisturbed velocity approximately, not just at particular points near the axis (for $-l < x < l$), but *everywhere* near the axis.

3. The first approximation to the Stokeslet strength distribution

The integral equation represented by (2.3) with $\mathbf{v} = \mathbf{U}$ at $r = R$ cannot be solved directly, but we perceive the nature of the asymptotic solution (as $R/l \to 0$) for the case of translational motion of the body from a consideration of the integrals

$$I = \int_{-l}^{l} \frac{dx'}{\{(x-x')^2+r^2\}^{\frac{1}{2}}}, \quad I'_n = \int_{-l}^{l} \frac{r^{2-n}(x-x')^n\,dx'}{\{(x-x')^2+r^2\}^{\frac{3}{2}}} \quad (n = 0, 1 \text{ or } 2),$$

all of which can be evaluated by elementary methods. We then find that, when $r/l \ll 1$,

$$\left.\begin{aligned} I &\approx 2\log\frac{2l}{R_0} + 2\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0}, \\ I'_0 &\approx 2, \quad I'_1 \approx 0, \quad I'_2 \approx I - 2, \end{aligned}\right\} \tag{3.1}$$

with errors of order r^2/l^2 in the case of I, I'_0, I'_2 and of order r/l in the case of I'_1. (The errors are larger in a small region near the end of a blunt body where either $r/(l-x)$ or $r/(l+x)$ is not small, but we shall ignore any such exceptions on the intuitive grounds that change of the body shape in this region has negligible effect on the integral force parameters.) Here R_0 is an arbitrary length chosen to be representative in some way of the values of R over the surface of the body; for example $2\pi R_0$ might be the perimeter of the cross-section at $x = 0$. The primary small quantity in the subsequent analysis is

$$\epsilon = (\log 2l/R_0)^{-1}.$$

Hence, for a case in which the Stokeslet strength density is uniform, the induced velocity (2.3) at a position for which $r/l \ll 1$ becomes

$$v_i(\mathbf{x}) \approx \frac{1}{4\pi\mu}\left\{\left(\frac{1}{\epsilon} + \log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0}\right)(F_i + \delta_{i1}F_1) - \delta_{i1}F_1 + \frac{r_i r_j}{r^2}F_j\right\}, \tag{3.2}$$

where $\mathbf{r}$ is the vector in the transverse plane with components (x_2, x_3) and r_i is taken as zero when $i = 1$. The dominant term on the right-hand side is that involving the factor $1/\epsilon$, and

$$v_i(\mathbf{x}) = \frac{1}{4\pi\mu\epsilon}\{F_i + \delta_{i1}F_1 + |\mathbf{F}|\,O(\epsilon)\}. \tag{3.3}$$

It appears then that the choice of a uniform line density of Stokeslet strength, with

$$F_1 = 2\pi\epsilon\mu U_1, \quad F_i = 4\pi\epsilon\mu U_i \quad (i = 2 \text{ or } 3), \tag{3.4}$$

gives longitudinal and transverse velocity components which to the first approximation are uniform over (and near) the body surface and equal to the values corresponding to translational motion of the body. This statement is as valid for bodies of non-circular cross-section as for axisymmetric bodies, and it also holds independently of the way in which the cross-section varies along the length. Additions to the Stokeslet strength density (3.4) of magnitude $\epsilon^2\mu U$ cause changes in the induced velocity of magnitude ϵU, and so the magnitude of the error in the distribution of $\mathbf{F}$ given by (3.4) is $\mu U O(\epsilon^2)$.

Hence, when a slender body of length $2l$ is in translational motion through fluid at rest at infinity with velocity (U_1, U_2, U_3), of magnitude U, the force exerted on the fluid by the body is

$$\int_{-l}^{l} F_i(x')\,dx' = \frac{8\pi\mu l U}{\log 2l/R_0}\left\{\frac{1}{2}\frac{U_1}{U}\delta_{i1} + \frac{U_2}{U}\delta_{i2} + \frac{U_3}{U}\delta_{i3} + O(\epsilon)\right\}. \tag{3.5}$$

Here R_0 is a length which is still arbitrary aside from being comparable with the body thickness; different choices of R_0 affect only the error term in (3.5). We see that the ratio of the drag for motion in any transverse direction to the drag for longitudinal motion of the body at the same speed is

$$2 + O(\epsilon),$$

which extends to bodies of arbitrary cross-section the striking result previously obtained for axisymmetric bodies (obtained in effect first by Burgers (1938)). This ratio 2 is a direct consequence of the fact that the induced velocity due to an isolated Stokeslet is twice as large at a point on the axis of symmetry as at a point at an equal distance in the transverse direction (see (2.1)).

Analogous results can be obtained for the case in which $|\mathbf{U}|$ varies linearly with x. The trial solution is here $F_i(x') \propto x'$. As a preliminary we evaluate the integrals

$$J = \frac{1}{l}\int_{-l}^{l}\frac{x'\,dx'}{\{(x-x')^2+r^2\}^{\frac{1}{2}}}, \quad J'_n = \frac{1}{l}\int_{-l}^{l}\frac{r^{2-n}(x-x')^n\,x'\,dx'}{\{(x-x')^2+r^2\}^{\frac{3}{2}}} \quad (n = 0, 1 \text{ or } 2),$$

and find that when $r/l \ll 1$

$$J \approx \frac{x}{l}\left\{2\log\frac{2l}{R_0} + 2\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0} - 2\right\},$$

$$J'_0 \approx 2x/l, \quad J'_1 \approx 0, \quad J'_2 \approx J - (2x/l),$$

with errors of order r^2/l^2 in the case of J, J'_0, J'_2 and of order r/l in the case of J'_1 (except perhaps in a small neighbourhood of the ends of the range $-l < x < l$). R_0 is again an arbitrary length chosen to be representative of the values of R.

Then, for a Stokeslet distribution given by

$$F_1(x')/8\pi\mu = -\tfrac{1}{4}e_{11}\epsilon x', \quad F_2 = 0, \quad F_3 = 0, \tag{3.6}$$

and at positions such that $r/l \ll 1$, the right-hand side of (2.3) is a vector approximately parallel to the x-axis and of magnitude

$$-e_{11}x - \epsilon e_{11}x\left\{\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0} - \frac{3}{2}\right\}, \tag{3.7}$$

where $\epsilon = (\log 2l/R_0)^{-1}$ as before. We see that for $R_0/l \ll 1$ the distribution (3.6) is approximately that required to represent a straight slender rigid body, of arbitrary cross-section, embedded in a pure straining motion with rate of extension e_{11} in the direction of the body length, the error in (3.6) being of magnitude

$\epsilon^2 e_{11} x'$. The aspect of the Stokeslet distribution that is of primary interest in this context is in the first integral moment of F_1, giving the net force doublet or dipole moment. Minus this integral moment has been termed the 'stresslet' strength of the body (Batchelor 1970), in view of its connexion with the additional bulk stress due to the presence of such a body in suspension in fluid subjected to a straining motion (the minus sign arising from the usual convention that a normal stress representing a tension is positive), and is found here to be

$$-\int_{-l}^{l} x' F_1(x')\, dx' = \frac{4\pi\mu l^3 e_{11}}{3\log 2l/R_0}\{1+O(\epsilon)\}. \tag{3.8}$$

Likewise, for a Stokeslet distribution given by

$$F_1 = 0, \quad F_i(x')/8\pi\mu = \tfrac{1}{2}x'\Omega_j \epsilon_{1ij}\epsilon \quad (i,j = 2 \text{ or } 3), \tag{3.9}$$

the induced velocity due to the Stokeslets is approximately a vector in the transverse plane with components

$$\epsilon_{1ij}\Omega_j x + \epsilon\epsilon_{1jk}\Omega_k x\left\{\delta_{ij}\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0} - \delta_{ij} + \frac{r_i r_j}{r^2}\right\} \tag{3.10}$$

in the region $r/l \ll 1$. Thus for $R_0/l \ll 1$ the distribution (3.9) is approximately that required to represent a straight rigid body of arbitrary cross-section rotating with angular velocity $(0, \Omega_2, \Omega_3)$ about its mid-point in fluid at rest at infinity, and the couple exerted on the fluid by the body is

$$\int_{-l}^{l} \epsilon_{i1j} x' F_j(x')\, dx' = \frac{8\pi\mu l^3 \Omega_i}{3\log 2l/R_0}\{1+O(\epsilon)\}. \tag{3.11}$$

In both (3.8) and (3.11), different choices of the arbitrary length R_0 affect only the two error terms.

It will be observed that the relations (3.4) represent correctly the first approximation to $F_i(x)$ for $\mathbf{U}(x)$ either constant or a linear function of x.

The relations (3.5), (3.8) and (3.11) are asymptotic results for quantities of practical interest which hold for any shape of the body cross-section and for any type of variation of the shape and size of the cross-section along the length of the body, even for knobbly bodies such as a (straight) necklace of irregularly shaped stones (which may be useful as a model of a macromolecule). This is a significant advance in principle, but the working value is severely limited by the fact that l/R_0 needs to be numerically very large indeed before ϵ is small compared with unity (as is apparent from the formula $\epsilon = (0{\cdot}69 + 2{\cdot}30n)^{-1}$ for $l/R_0 = 10^n$). It is desirable to obtain some information about the error terms in the relations (3.5), (3.8) and (3.11), which is where the effect of the body shape enters. Methods of determining approximately terms of order $\epsilon^2, \epsilon^3, \ldots$ in the expressions (3.5), (3.8) and (3.11) as a function of the meridional shape of an axisymmetric body have already been developed (Tuck 1970; Cox 1970*a*; Tillett 1970). Here we wish to see what additional considerations are required when the cross-section is not circular. The two kinds of shape effect can in fact be treated together, and give rise to correction terms of comparable magnitude.

4. A method of improving the approximation to the Stokeslet strength distribution

In the case of a Stokeslet strength distribution which is either uniform or a linear function of x, the first approximation to the induced velocity at points such that $r/l \ll 1$ has precisely the same dependence on x. This property holds for other smooth forms of the function $F_i(x)$, as we may see by writing

$$\int_{-l}^{l} \frac{F_i(x')\,dx'}{\{(x-x')^2+r^2\}^{\frac{1}{2}}} = F_i(x)\,I + \int_{-l}^{l} \frac{F_i(x')-F_i(x)}{\{(x-x')^2+r^2\}^{\frac{1}{2}}}\,dx', \tag{4.1}$$

where I is the integral found (see (3.1)) to behave as $2\log\{2(l^2-x^2)^{\frac{1}{2}}/r\}$ when $r/l \ll 1$. Tuck (1964) has made a study of the last integral in (4.1), and has established that, for $r/l \ll 1$, the effect of deleting the term r^2 in the denominator of the integrand is to cause an error which is of order

$$|\mathbf{F}|\frac{r^2}{l^2}\log\frac{r}{l}$$

if derivatives of $\mathbf{F}(x)$ of all orders exist for $-l < x < l$ and is certainly as small as $|\mathbf{F}|\,(r/l)^{\frac{1}{3}}$ provided only that $\mathbf{F}(x)$ and its first derivative are bounded and piecewise continuous. We shall assume that at least the latter weaker conditions on $\mathbf{F}(x)$ are satisfied. Hence when $r/l \ll 1$ we may write

$$\int_{-l}^{l} \frac{F_i(x')\,dx'}{\{(x-x')^2+r^2\}^{\frac{1}{2}}} \approx F_i(x)\left\{\frac{2}{\epsilon}+2\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0}\right\} + \int_{-l}^{l} \frac{F_i(x')-F_i(x)}{|x'-x|}\,dx', \tag{4.2}$$

with an error which is at least as small as some positive power of r/l and which is therefore smaller than any power of $(\log 2l/r)^{-1}$.

Similar considerations may be applied to the integral formed from the second term in the integrand of (2.3). By taking the possible values of i in turn, we find

$$\int_{-l}^{l} \frac{(x_i-x_i')(x_j-x_j')F_j(x')}{\{(x-x')^2+r^2\}^{\frac{3}{2}}}\,dx' \approx \delta_{i1}\int_{-l}^{l} \frac{F_1(x')\,dx'}{\{(x-x')^2+r^2\}^{\frac{1}{2}}} - 2\delta_{i1}F_1(x) + 2\frac{r_i r_j}{r^2}F_j(x), \tag{4.3}$$

when $r/l \ll 1$, and the largest error involved in this approximation is of order

$$\left|\frac{d\mathbf{F}}{dx}\right|\frac{r}{l}\log\frac{l}{r}.$$

Hence, with the aid of (4.2) and (4.3) we may write the velocity due to the line of Stokeslets at positions for which $r/l \ll 1$ as

$$v_i(\mathbf{x}) = \frac{1}{4\pi\mu}\left\{\left(\frac{1}{\epsilon}+\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{r/R_0}\right)(F_i+\delta_{i1}F_1) - \delta_{i1}F_1 + \frac{r_i r_j}{r^2}F_j\right\}$$
$$+\frac{1}{8\pi\mu}(\delta_{ij}+\delta_{i1}\delta_{j1})\int_{-l}^{l}\frac{F_j(x')-F_j(x)}{|x'-x|}\,dx', \tag{4.4}$$

correct to the order of any power of $(\log 2l/r)^{-1}$; as before, $\mathbf{r}$ is a vector in the transverse plane with components x_2, x_3, and $x_2' = x_3' = 0$.

This equation (4.4), together with the requirement that $\mathbf{v}(\mathbf{x}) = \mathbf{U}(x)$ at

$r = R(\phi, x)$, provides a basis for the determination of all terms in an expansion of $\mathbf{F}(x)$ in powers of ϵ. The leading term is of the form (3.4), as already noted, and this may be used as the start of a reiteration procedure to determine subsequent terms in turn.

Consider, for instance, the case of longitudinal motion for a body whose cross-section is circular with radius R_x at station x and radius R_0 at $x = 0$. The no-slip condition at the body surface becomes

$$4\pi\mu U_1(x) = \left\{\frac{2}{\epsilon} + 2\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0} - 1\right\}F_1(x) + \int_{-l}^{l}\frac{F_1(x')-F_1(x)}{|x'-x|}dx', \qquad (4.5)$$

and on writing
$$U_1(x) \propto x^n \quad (n = 0 \text{ or } 1), \qquad (4.6)$$
we find immediately that

$$F_1(x) = 2\pi\mu U_1(x)\left[\epsilon + \epsilon^2\left\{n + \tfrac{1}{2} - \log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right\} + \epsilon^3 f_3(x) + \epsilon^4 f_4(x) + \ldots\right]; \qquad (4.7)$$

$f_3(x)$ and further coefficients may be obtained from the kind of recurrence relation given by Tillett (1970) for the case $n = 0$.

However, this straight-forward procedure described for the case of longitudinal motion and a body of circular cross-section cannot be used in more general circumstances, because when the cross-section is non-circular, and also in the case of transverse motion for a body which does have circular cross-section, it is impossible to satisfy the no-slip condition at the body surface by a suitable choice of the Stokeslet strength density $\mathbf{F}(x)$. We see this explicitly by writing (4.4) as

$$v_i(\mathbf{x}) - U_i(x) = \frac{1}{4\pi\mu}\left\{(F_i + \delta_{i1}F_1)\log\frac{R_x}{r} + F_j\frac{r_ir_j}{r^2} + G_i(x)\right\}, \qquad (4.8)$$

where
$$G_i(x) = \left(\frac{1}{\epsilon} + \log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right)(F_i + \delta_{i1}F_1) - \delta_{i1}F_1$$
$$+ \tfrac{1}{2}(\delta_{ij} + \delta_{i1}\delta_{j1})\int_{-l}^{l}\frac{F_j(x')-F_j(x)}{|x'-x|}dx' - 4\pi\mu U_i \qquad (4.9)$$

and $2\pi R_x$ is the perimeter of the body cross-section at station x; at $r = R(\phi, x)$ the right-hand side of (4.8) varies with the azimuthal angle ϕ and no choice of $\mathbf{F}(x)$ can eliminate this variation.

In these circumstances the right way to amend the distribution of singularities is to spread the Stokeslets over the body surface.† A procedure which is equivalent is to add at the x-axis distributions of force doublets, force quadruplets, and so on, the accuracy of the representation being greater as the number of different types of singularity at the axis is increased. Now a force multi-pole of order n (with $n = 1$ corresponding to a Stokeslet) at a point induces a velocity field which falls off as the $(-n)$-power of distance from it, and provided $n \geqslant 2$ a line distribution of such multi-poles at the x-axis gives a velocity which at positions

† It is known from the mathematical theory of Stokes flow described by Oseen (1927) and more recently by Ladyzhenskaya (1963) that, in the case of fluid at rest at infinity and with a single interior boundary, the velocity and pressure at any point in the fluid may each be expressed exactly and uniquely as an integral over the interior boundary representing the effect of a distribution of surface density of force applied there.

such that $r/l \ll 1$ is dominated by the multi-pole strength density at the nearby portion of the x-axis and which diminishes as $(r/R_x)^{1-n}$ as $r/R_x \to \infty$. Consequently, provided the body is so slender that there exists a region in which $r/l \ll 1$ and $r/R_x \gg 1$, the velocity distribution in this region due to the complete set of singularities has the following important properties: (1) it is dominated by the Stokeslet distribution and so has the form given by (4.8), and (2) whatever residual influence of the higher-order multi-poles at the x-axis there may be comes from the nearby portion of the axis.

We shall assume that the body cross-section varies sufficiently slowly with x for the flow within a neighbourhood of many body diameters from one section of the body—the 'inner' flow field—to be approximately independent of x. This cylindrical flow field must take the form (4.8), so far as dependence on r and ϕ is concerned, when $r/R_x \gg 1$. It proves to be possible to consider this inner flow field, as a separate problem of cylindrical flow, on an exact basis, and thereby to avoid the explicit introduction of force multi-poles on the x-axis. We shall see that the requirement that the inner flow field takes the form (4.8) for $r/R_x \gg 1$ determines the function $\mathbf{G}(x)$ uniquely in terms of $\mathbf{F}(x)$ and the shape of the local cross-section.† The expression (4.9) for $\mathbf{G}(x)$ then provides an integral equation for $\mathbf{F}(x)$ which may be solved by the standard reiteration procedure.

We consider now the cylindrical flow in the neighbourhood of one section of the body. To the order of approximation being considered, the cases of longitudinal and transverse motion involve only the longitudinal and transverse components of $\mathbf{F}$ respectively, and since the corresponding inner flow fields have slightly different characters we take them separately.

5. The inner flow field for longitudinal motion

By longitudinal motion we mean cases in which the undisturbed fluid velocity at the position of the body, $-\mathbf{U}(x)$, is parallel to the body length so that $\mathbf{U} = (U_1, 0, 0)$; and U_1 may be either uniform or linear in x. The fluid velocity in the presence of the body is $\mathbf{u}(\mathbf{x})$, and is zero at the body surface. In the neighbourhood of one section of the body, $\mathbf{u}$ is approximately of the form $(u_1, 0, 0)$, with u_1 independent of x. We propose to examine the dependence of u_1 on the co-ordinates (y, z) in the lateral plane for this cylindrical flow, in particular when $r/R_x \gg 1$, where $r^2 = y^2+z^2$ and $2\pi R_x$ is the perimeter of the local cross-section.

The governing equation for this case of Stokes flow is

$$\partial^2\boldsymbol{\omega}/\partial y^2 + \partial^2\boldsymbol{\omega}/\partial z^2 = 0, \tag{5.1}$$

where $\boldsymbol{\omega} = (0, \partial u_1/\partial z, -\partial u_1/\partial y)$ is the fluid vorticity (derivatives with respect to x being neglected). The body surface is the source of vorticity, and the leading term in the series of circular harmonics of negative degree which represents $\boldsymbol{\omega}$ is of degree -1. Thus,

$$\boldsymbol{\omega} \approx \mathbf{F}\times\mathbf{r}/2\pi\mu r^2 \quad \text{for} \quad r/R_x \gg 1, \tag{5.2}$$

† There are interesting resemblances here to slender-body theory for non-axisymmetric bodies in irrotational flow, as described by Ward (1955).

where $\mathbf{r} = (0, y, z)$ and $\mathbf{F} = (F_1, 0, 0)$ is a vector constant (although strictly speaking $\mathbf{F}$, like R_x, is a slowly varying function of x). The corresponding expression for the velocity, relative to axes fixed in the body, is

$$u_1 = \frac{F_1}{2\pi\mu}\left\{\log\frac{R_x}{r} + K + O\left(\frac{R_x}{r}\right)\right\} \tag{5.3}$$

for $r/R_x \gg 1$, where K is a dimensionless constant. In the cylindrical flow field near the body, the local force (in the x-direction) exerted on the fluid by unit length of the body is equal to

$$-\oint \mu(\partial u_1/\partial n)\, ds,$$

where s and n represent distance along and normal to any closed curve in the cross-sectional plane which encloses the body, and on choosing the curve as a circle of radius r for which (5.3) holds we see that this force per unit length is F_1.

The problem of determining the inner flow, near a section of the body where the cross-sectional perimeter is $2\pi R_x$ and the longitudinal force per unit length exerted by the body is F_1, is thus as follows. With neglect of x-derivatives again, the equation for the fluid velocity is

$$\partial^2 u_1/\partial y^2 + \partial^2 u_1/\partial z^2 = 0, \tag{5.4}$$

and the boundary conditions are (1) $u_1 = 0$ at the body surface and (2) u_1 takes the form (5.3) for $r/R_x \gg 1$. This is a two-dimensional potential problem in which F_1/μ plays the part of a cyclic constant for the doubly-connected region outside the body surface and which is over-determined if K is given. In other words, K is found as a part of the solution and depends only on the shape of the body cross-section. In the particular case of a circular cross-section, $u_1 \propto \log R_x/r$ over the whole of the inner flow region, and the boundary condition $u_1 = 0$ at $r = R_x$ gives $K = 0$. Thus, for any non-circular cross-section we may write

$$u_1 = \frac{F_1}{2\pi\mu}\left\{\log\frac{kR_x}{r} + O\left(\frac{R_x}{r}\right)\right\} \tag{5.5}$$

for $r/R_x \gg 1$, where $\log k = K$, and regard kR_x as the radius of the circle which is equivalent to this cross-section in the sense that a given total longitudinal force at the surface of a circular cylinder of this radius produces the same flow field in the region $r/R_x \gg 1$.

This type of inner flow field and the associated idea of an equivalent circular section have already been described, in the context of time-dependent flow due to a rigid cylinder of great length moving parallel to a generator with steady speed in fluid at rest at infinity (Batchelor 1954). The value of k can be found numerically for any cross-sectional shape by determining the conformal transformation that converts the boundary into a circle without distortion of the region of the complex plane far from the origin, and the values for several special cases are given in that paper. The most useful result is that for an ellipse with semi-diameters b and c the radius of the equivalent circle is

$$kR_x = \tfrac{1}{2}(b+c). \tag{5.6}$$

The value of k here varies monotonically between $\frac{1}{4}\pi(= 0{\cdot}785)$ for a flat plate and 1·0 for a circle as c/b increases from 0 to 1. Exactly the same range of values of k is traversed monotonically in the case of a regular polygon of n sides as n varies from $n = 2$ (again a flat plate) to $n \to \infty$, the value for a circle being reached in the limit. It seems likely that for any cross-sectional shape not too different from a regular figure the value of k lies between 0·785 and 1·0 (the corresponding range of values of K being from $-0·242$ to 0). The smallness of this range of values of k is a consequence of our choice of the perimeter (divided by 2π) as the reference length. If we had chosen the square root of the area of the cross-section as the reference length, k would have been infinite for a flat plate.

A cross-sectional shape which has many deep indentations is in a different class. An example for which an appropriate conformal transformation can be found with a little trouble is a 'star' consisting of n flat plates all of length a which have a common end point and equal angular separation $2\pi/n$, the result being that the radius of the equivalent circle is

$$kR_x = 2^{-2/n}a.$$

The result for a single flat plate is recovered when $n = 1$ or 2, and then as n increases the value of kR_x gradually approaches a; when n is large, the fluid in the space between neighbouring arms of the star is stagnant and the equivalent circle is approximately the circumscribing circle. The wetted perimeter divided by 2π is here na/π and is less appropriate as the reference length. A better choice for $2\pi R_x$ would be the 'convex perimeter', that is, the length of a taut string passing round the cross-section, equal to $2na \sin \pi/n$, which gives $k \to 1$ as $n \to \infty$; adoption of this definition of R_x would of course not change any of the above results for wholly convex plane figures.†

The matching of the outer section of the inner flow field, given by (5.3) or (5.5), with the inner section of the outer flow field, represented by (4.8) (with $i = 1$), is now straight-forward. The right-hand side of (4.8) represents the velocity, relative to the body, given by the undisturbed flow together with the whole line of Stokeslets, and is identical, when $i = 1$, with the right-hand side of (5.3) provided we choose

$$G_1(x) = 2F_1 K = 2F_1 \log k. \tag{5.7}$$

The two expressions for $G_1(x)$, (4.9) and (5.7), now yield an integral equation for $F_1(x)$ which is correct to the order of any power of ϵ, and which is equivalent to satisfaction of the no-slip condition at the circle $r = kR_x$ rather than at the actual surface of the body.

It appears that, as a consequence of the smoothing action of vorticity diffusion, the velocity distribution in the fluid near one section of the body tends to an axisymmetric form with increasing distance from the x-axis and, before the outer flow region is reached, becomes the same as if the body were of circular cross-section. For this reason, the total Stokeslet strength per unit length of the x-axis, and the radius of the circular section over whose surface this Stokeslet strength may be regarded as being spread, are all that matter so far as the matching of the inner and outer flows is concerned; and higher approximations to the

† I am indebted to Dr J. R. A. Pearson for this suggestion.

Stokeslet strength distribution, correct to some power of ϵ, depend on the local cross-section only through the dependence on the radius kR_x of the local equivalent circle.

6. The inner flow field for transverse motion

Similar considerations apply to the approximately two-dimensional flow, in the neighbourhood of one section of the body, associated with the components of velocity and force in the cross-sectional plane. The vorticity equation in the inner region is again of the Laplacian form (5.1), although the vorticity vector is now $(\omega_1, 0, 0)$, with

$$\omega_1 = \partial u_3/\partial y - \partial u_2/\partial z.$$

The nearby body surface is again the dominant source of vorticity, and the asymptotic form of the vorticity distribution is like (5.2) although in the present case the constant $\mathbf{F}$ has components $(0, F_2, F_3)$ so that

$$\omega_1 \approx \frac{F_2 z - F_3 y}{2\pi\mu r^2} \quad \text{for} \quad r/R_x \gg 1.$$

The two-dimensional velocity vector which is solenoidal and whose curl has the above form is

$$u_i = \frac{F_j}{4\pi\mu}\left\{\delta_{ij}\left(\log\frac{R_x}{r} - \frac{1}{2}\right) + \frac{r_i r_j}{r^2} + K_{ij} + O\left(\frac{R_x}{r}\right)\right\} \quad (i, j = 2 \text{ or } 3), \tag{6.1}$$

for $r/R_x \gg 1$, where $r_2 = y$, $r_3 = z$ as before, and K_{ij} is a dimensionless tensor constant (which may of course be a function of x). The equation of motion shows that the corresponding expression for the pressure is

$$p = p_0 + \frac{\mathbf{F}.\mathbf{r}}{2\pi r^2}$$

for $r/R_x \gg 1$. The corresponding expression for the stress tensor is $-r_i r_j r_k F_k/\pi r^4$, from which we find that the total force, per unit length in the x-direction, exerted across a circle of radius sufficiently large for this asymptotic relation to be applicable is $\mathbf{F}$; thus the local force per unit length exerted by the body on the fluid is $\mathbf{F}$. Any couple which is exerted on the fluid by the body affects only the term of order R_x/r in the expression for $\mathbf{u}$.

The equation describing the inner flow, which states simply that the x-component of vorticity is a harmonic function in two dimensions, is here most conveniently expressed in terms of a stream function ψ. We have

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0, \tag{6.2}$$

and the inner boundary conditions are

$$\psi = 0, \quad \mathbf{n}.\nabla\psi = 0 \tag{6.3}$$

on the closed curve in the (y, z)-plane representing the body surface. The outer boundary condition, obtained from (6.1), is

$$4\pi\mu\psi \approx (zF_2 - yF_3)\{\log(R_x/r) + \tfrac{1}{2}\} + (zK_{2j} - yK_{3j})F_j \tag{6.4}$$

for $r/R_x \gg 1$. According to the theory of two-dimensional slow viscous motion, a solution to this problem for a given shape of the inner boundary may be found only when the two-dimensional tensor K_{ij} has a certain value.

We may prove that K_{ij} is a symmetric tensor by the kind of argument used for the tensor coefficient in the relation between force and translational velocity of a body finite in all three dimensions. The integral

$$\int (u_i \sigma'_{ij} - u'_i \sigma_{ij})\, n_j \, dA$$

has the same value for any curve enclosing the body in the (y, z)-plane, where u_i, σ_{ij} and u'_i, σ'_{ij} are the velocity and stress at any point in the fluid corresponding to the values $\mathbf{F}$ and $\mathbf{F}'$ of the force on (unit length of) the body. With axes fixed in the body, the integral is zero for the case of a curve coinciding with the body surface; and so is zero for all closed curves, including one at a large distance from the origin on which the velocity and stress have the above asymptotic forms. The symmetry of K_{ij} then follows after substitution of these forms in the integrand.

It is not as easy to find K_{ij} explicitly for a given cross-section of the body as it was to determine the analogous scalar constant K in the case of longitudinal motion. Moreover, it is evident that, since K_{ij} must take the special form const. $\times\, \delta_{ij}$ in the case of a circular cross-section for reasons of symmetry, it is not possible here to represent an arbitrary cross-section by a circle of suitably chosen radius. We need as a standard some cross-section whose shape and size are specified by as many scalar quantities as the number of independent components of K_{ij}, that is, an ellipse. It so happens that an explicit solution to the above problem for the case of an elliptic boundary is available (Berry & Swain 1923). For an ellipse whose semi-diameters are b and c (with $b > c$), and with y' and z' axes parallel to the principal diameters, the stream function is

$$4\pi\mu\psi(y', z') = (z'F'_2 - y'F'_3)(\xi_0 - \xi) + \frac{b \tanh \xi - c}{b^2 - c^2} \left(\frac{z'F'_2 c}{\tanh \xi} - y'F'_3 b \right), \tag{6.5}$$

where $y' = (b^2 - c^2)^{\frac{1}{2}} \cosh \xi \cos \eta$, $z' = (b^2 - c^2)^{\frac{1}{2}} \sinh \xi \sin \eta$, and the elliptic boundary is given by

$$\xi = \xi_0 = \tanh^{-1} \frac{c}{b} = \tfrac{1}{2} \log \frac{b+c}{b-c}. \tag{6.6}$$

This expression for ψ obviously satisfies the inner boundary conditions (6.3). Since

$$e^{2\xi} = \frac{4r^2}{b^2 - c^2} + O\left(\frac{b^2 - c^2}{r^2}\right), \quad \tanh \xi = 1 + O\left(\frac{b^2 - c^2}{r^2}\right)$$

when $r/(b^2 - c^2)^{\frac{1}{2}} \gg 1$, we see that at large distances from the ellipse (6.5) becomes

$$4\pi\mu\psi(y', z') \approx (z'F'_2 - y'F'_3) \log \frac{b+c}{2r} + \frac{z'F'_2 c - y'F'_3 b}{b+c},$$

or, with general axes such that the direction of the larger principal diameter of the ellipse $(2b)$ is given by the unit vector $\boldsymbol{\beta}$,

$$4\pi\mu\psi(y, z) \approx (zF_2 - yF_3) \log \frac{b+c}{2r} + (yF_2 - zF_3) \frac{\beta_2 \beta_3 (b-c)}{b+c} - yF_3 \frac{b\beta_2^2 + c\beta_3^2}{b+c} + zF_2 \frac{b\beta_3^2 + c\beta_2^2}{b+c}. \tag{6.7}$$

This is of the expected form (6.4), and it appears that, for an ellipse characterized by b, c and the unit vector $\boldsymbol{\beta}$,

$$K_{ij} = \delta_{ij}\left(\log\frac{b+c}{2R_x}+\frac{1}{2}\frac{b-c}{b+c}\right)-\beta_i\beta_j\frac{b-c}{b+c}. \tag{6.8}$$

The components of the tensor K_{ij} for an arbitrary cross-section may now be regarded as corresponding to values of b, c and $\boldsymbol{\beta}$ according to the relation (6.8). The value of K_{ij}, and so of the major and minor semi-diameters and the directions of the principal diameters of the equivalent ellipse, is to be determined from an investigation of a solution of the biharmonic equation for the inner boundary shape in question. For many simple cross-sectional shapes it may be sufficient to estimate these quantities on crude geometrical grounds. For a circular cross-section $K_{ij} = 0$.

Again the matching of the r and ϕ-dependences of the asymptotic or outer section of the inner flow field, given by (6.1), with the inner section of the outer flow field, given by (4.8) (with $i = 2$ or 3), is straightforward. Matching is achieved if we choose

$$G_i(x) = F_j(K_{ij}-\tfrac{1}{2}\delta_{ij}) = F_i\left(\log\frac{b+c}{2R_x}-\frac{c}{b+c}\right)-\beta_i\beta_jF_j\frac{b-c}{b+c} \quad (i,j = 2 \text{ or } 3), \tag{6.9}$$

where the quantities b, c and $\boldsymbol{\beta}$ refer to the ellipse that is equivalent to the local cross-section.

A significant feature of the case of transverse motion is that, in any approximation to $\mathbf{F}$ better than that given in § 3, $\mathbf{F}$ is not necessarily parallel to $\mathbf{U}$ for a body of non-circular cross-section.

7. The expression for F for a non-axisymmetric body

We may now return to the question of calculation of the function $\mathbf{F}(x)$ representing the force density on the x-axis. The relation (4.9) together with the matching results (5.7) and (6.9) provide an integral equation for $\mathbf{F}(x)$ which is correct to the order of any power of ϵ and which can be solved for $\mathbf{F}$ reiteratively as a power series in ϵ. The tensorial character of the various terms in this integral equation is a little clearer if we continue to take separately the cases of longitudinal and transverse motion, whence

$$4\pi\mu U_1 = 2F_1\left\{\frac{1}{\epsilon}+\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}-K-\tfrac{1}{2}\right\}+\int_{-l}^{l}\frac{F_1(x')-F_1(x)}{|x'-x|}dx' \tag{7.1}$$

and

$$4\pi\mu U_i = F_j\left\{\frac{\delta_{ij}}{\epsilon}+\delta_{ij}\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}-K_{ij}+\tfrac{1}{2}\delta_{ij}\right\}$$
$$+\frac{1}{2}\int_{-l}^{l}\frac{F_i(x')-F_i(x)}{|x'-x|}dx' \quad (i,j = 2 \text{ or } 3). \tag{7.2}$$

In these equations $K = \log k$, kR_x is the radius of the equivalent circle for longitudinal motion, and K_{ij} is given in terms of the properties of the equivalent ellipse for transverse motion by (6.8). K, K_{ij}, R_x and $\mathbf{U}$ may all be functions of x.

We shall confine attention to undisturbed velocity distributions of the kind

$$U_i \propto x^n \quad (n = 0 \text{ or } 1).$$

A useful result which is relevant in these circumstances is that

$$\int_{-l}^{l} \frac{x'^n - x^n}{|x'-x|} dx' = -2nx^n \tag{7.3}$$

if (and only if) $n = 0$ or 1.

So far as the case of longitudinal motion is concerned, the integral equation is identical with that for a body whose cross-section is circular with radius kR_x (for compare (4.5) and (7.1)), and so the details of the recurrence relation for the higher coefficients can be taken directly from the paper by Tillett (1970). There is however a possible numerical improvement on a series of the form (4.7) which was not noticed by Cox (1970) or Tillett (1970) and which may be pointed out here. In the particular case of a body which is longitudinally elliptic in the sense that

$$\frac{R_x}{R_0} = \left(1 - \frac{x^2}{l^2}\right)^{\frac{1}{2}}$$

and for which K is independent of x (notice that neither of these conditions places any restriction on the shape of the cross-section), equation (7.1) can be satisfied *exactly* by the choice $F_1(x) \propto U_1(x)$ (this is where (7.3) is used); there is thus no need for reiteration in this case and we have

$$F_1(x) = \frac{2\pi\mu U_1(x)\,\epsilon}{1-\epsilon(n+\frac{1}{2}+K)} = \frac{2\pi\mu U_1(x)}{\log(2l/kR_0) - n - \frac{1}{2}} \tag{7.4}$$

correct to the order of any power of ϵ. This suggests that as an alternative to (4.7) we should expand $F_1(x)$ for a general body in the form†

$$F_1(x) = 2\pi\mu U_1(x)\,\epsilon\left\{\frac{1-\epsilon\log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1-\epsilon(n+\frac{1}{2}+K)}\right\} + 2\pi\mu U_1\{\epsilon^3 g_3(x) + \epsilon^4 g_4(x) + \ldots\}, \tag{7.5}$$

the advantage of which is that the unknown functions $g_3(x), g_4(x), \ldots$ all vanish in the case of a body for which $(1-x^2/l^2)^{\frac{1}{2}}/R_x$ and K are both independent of x and may be expected to be small when the body shape is nearly of this kind. The expression for $g_3(x)$ is

$$g_3(x) = -P\left\{n + \log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right\} - \frac{1}{2U_1}\int_{-l}^{l}\frac{U_1'P' - U_1P}{|x'-x|}dx', \tag{7.6}$$

and the recurrence relation is

$$g_{m+1} = g_m(P-n) - PQ^{m-2}\left\{n + \log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right\} - \frac{1}{2U_1}\int_{-l}^{l}\frac{U_1'(g_m' + P'Q'^{m-2}) - U_1(g_m + PQ^{m-2})}{|x'-x|}dx' \tag{7.7}$$

† The position of the log term, which vanishes for a longitudinally elliptic body, is arbitrary, and since this term can have large magnitude, and be of either sign, near each end of the body, we place it in the numerator to avoid the non-integrable singularity in $F_1(x)$ which would arise from the vanishing of the denominator at two points near the ends of the range $|x| \leqslant l$.

for $m \geqslant 3$, where

$$P(x) = Q - \log \frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}, \quad Q = n + \tfrac{1}{2} + K, \tag{7.8}$$

and a prime to a symbol denotes the value at x'.

In the case of transverse motion, the solution for $F_i(x)$ is a little more complicated as a consequence of the difference between the directions of the vectors $\mathbf{U}$ and $\mathbf{F}$ in the transverse plane. The difference in direction is zero for a circular cross-section, and so the results of Tillett (1970) and Cox (1970) cannot be taken over so directly as in the case of longitudinal motion. We again take advantage of the fact that an exact solution of the integral equation can be obtained when the body is longitudinally elliptic and K_{ij} is independent of x, the solution of (7.2) in that case being

$$F_i(x) = 4\pi\mu U_j \epsilon T_{ij} \quad (i,j = 2 \text{ or } 3), \tag{7.9}$$

correct to the order of any power of ϵ, where T_{ij} is an inverse tensor defined by the relation

$$T_{ij}\{\delta_{jk} - \epsilon(n\delta_{jk} - \tfrac{1}{2}\delta_{jk} + K_{jk})\} = \delta_{ik}. \tag{7.10}$$

The expansion corresponding to (7.5) for a general body is then

$$F_i(x) = 4\pi\mu U_j \epsilon \left\{1 - \epsilon \log \frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right\} T_{ij} + 4\pi\mu U_j \{\epsilon^3 g_{ij}^{(3)}(x) + \epsilon^4 g_{ij}^{(4)}(x) + \ldots\}, \tag{7.11}$$

where T_{ij} is still defined by the relation (7.10) and is now a function of x; the functions $g_{ij}^{(3)}$, $g_{ij}^{(4)}$, ... all vanish in the case of a body for which $(1-x^2/l^2)^{\frac{1}{2}}/R_x$ and K_{jk} are both independent of x, and may be expected to be small for a body shape nearly of this kind. Substitution of (7.11) in (7.2) and expansion of T_{ij} in powers of ϵ then gives an expression for $g_{ij}^{(3)}$ and also a recurrence relation between $g_{ij}^{(m+1)}$ and $g_{ij}^{(m)}$ for $m \geqslant 3$ similar to (7.7), but these are probably too complicated to be useful and will not be displayed here. The complexity is mainly a consequence of the inclusion of slender bodies with twist. In a case in which the directions of the principal axes of the (two-dimensional) tensor K_{ij} are independent of x, the choice of these axes as the axes of reference (so that $\beta_2 = 1$, $\beta_3 = 0$) reduces to zero the non-diagonal components of T_{ij} and $g_{ij}^{(m)}$. The expansion then becomes

$$F_2(x) = 4\pi\mu U_2 \epsilon \left\{\frac{1 - \epsilon \log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1 - \epsilon(n - \frac{1}{2} + K_{22})}\right\} + 4\pi\mu U_2 \{\epsilon^3 g_{22}^{(3)}(x) + \epsilon^4 g_{22}^{(4)}(x) + \ldots\}, \tag{7.12}$$

with a similar expression for $F_3(x)$, and the expressions for $g_{22}^{(3)}$, $g_{22}^{(m+1)}$ and $g_{33}^{(3)}$, $g_{33}^{(m+1)}$ are of exactly the same form as those for g_3 and g_{m+1} given by (7.6), (7.7) and (7.8) except that $K + \frac{1}{2}$ in the latter relations should be replaced by $K_{22} - \frac{1}{2}$ or $K_{33} - \frac{1}{2}$.

8. The integral force parameters

It remains to give expressions for the various integral force parameters, to note their values for some particular body shapes, and to check their consistency with the exact results available for an ellipsoidal body.

The following expressions (8.1)–(8.6) for force parameters are approximations with an error of order ϵ^3 in general, where $\epsilon = (\log 2l/R_0)^{-1}$, and correction terms of the order of ϵ^3 and of any higher power of ϵ may be calculated from the formulae given in §7. All such correction terms are zero in the case of a body for which the following conditions are satisfied:

(i) $(1-x^2/l^2)^{\frac{1}{2}} R_0/R_x = 1$ for all x (a 'longitudinally elliptic' body),

(ii) K and K_{ij} are independent of x (a sufficient condition for which is that the cross-section has the same shape and orientation, although not the same size, for all x).

The force that must be applied to a body in order to sustain translational motion of the body with velocity $(U_1, 0, 0)$ through fluid at rest at infinity is found from (7.5), with $n = 0$, to be

$$\mathscr{F}_1 = \int_{-l}^{l} F_1(x)\,dx = 2\pi\mu U_1\epsilon \int_{-l}^{l} \frac{1-\epsilon\log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1-\epsilon(K+\frac{1}{2})}\,dx, \tag{8.1}$$

where $K = \log k$, kR_x is the radius of the circle equivalent to the cross-section at station x (see §5), and R_x is the perimeter of the cross-section at station x.

The force required for translational motion with velocity $(0, U_2, U_3)$ may be found from (7.11) with $n = 0$. Provided that the directions of the principal axes of K_{ij} are independent of x we may adopt these as the axes of reference and use (7.12) to find

$$\mathscr{F}_2 = \int_{-l}^{l} F_2(x)\,dx = 4\pi\mu U_2\epsilon \int_{-l}^{l} \frac{1-\epsilon\log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1-\epsilon(K_{22}-\frac{1}{2})}\,dx, \tag{8.2}$$

with a similar expression for $\mathscr{F}_3$, where K_{ij} is given in terms of the geometry of the ellipse equivalent to the cross-section at station x by (6.8) (with $\beta_2 = 1$, $\beta_3 = 0$).

The stresslet strength corresponding to a body immersed in a pure straining motion with (undisturbed) rate of extension e_{11} in the x-direction is found from (7.5), with $U_1 = -e_{11}x$ and $n = 1$, to be

$$\mathscr{S}_{11} = -\int_{-l}^{l} xF_1(x)\,dx = 2\pi\mu e_{11}\epsilon \int_{-l}^{l} \frac{1-\epsilon\log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1-\epsilon(K+\frac{3}{2})}\,x^2\,dx. \tag{8.3}$$

In the case of a body which is not symmetrical about the transverse plane at $x = 0$, it may happen that, when the body is suspended freely in a pure straining motion, the point of the body that moves with the velocity of the undisturbed fluid is not the mid-point but is at $x = \alpha l$. The number α must be of order ϵ and may be calculated from the requirement that $\int_{l}^{l} F_1(x)\,dx = 0$. In these circumstances the expression for the Stokeslet strength density contains an additional term arising from a superposed translational velocity of the body of magnitude $\alpha l e_{11}$ in the x-direction. However, this additional term affects the expression for $\mathscr{S}_{11}$ only to order ϵ^3 and so (8.3) remains correct to the order of ϵ^2.

The couple about the mid-point (at $x = 0$) which must be applied to a body to sustain rotational motion with angular velocity $(0, \Omega_2, \Omega_3)$, with the mid-point held fixed, may be found from (7.11) with $U_i = \epsilon_{1ij}\Omega_j x$ and $n = 1$. Provided again that the directions of the principal axes of K_{ij} are independent of x, we may use (7.12) to find

$$\mathscr{L}_2 = -\int_{-l}^{l} xF_3(x)\,dx = 4\pi\mu\Omega_2\epsilon\int_{-l}^{l}\frac{1-\epsilon\log[(1-x^2/l^2)^{\frac{1}{2}}/(R_x/R_0)]}{1-\epsilon(K_{33}+\frac{1}{2})}x^2\,dx, \quad (8.4)$$

with a similar expression for $\mathscr{L}_3$.

In the case of a body which is not symmetrical about the transverse plane through $x = 0$, translational motion in the transverse direction will require the application of a couple of order ϵ^2; and we find from (7.11), with U_i constant and $n = 0$, that the couple about the mid-point of the body is

$$\mathscr{L}'_i = \int_{-l}^{l}\epsilon_{i1j}xF_j(x)\,dx = 4\pi\mu U_k\epsilon_{i1j}\epsilon\int_{-l}^{l}\left\{1-\epsilon\log\frac{(1-x^2/l^2)^{\frac{1}{2}}}{R_x/R_0}\right\}T_{jk}(x)\,dx, \quad (8.5)$$

where $T_{ij}(x)$ is defined by (7.10). Likewise in these circumstances there is a non-zero force required to sustain rotation of the body with angular velocity $(0, \Omega_2, \Omega_3)$ about the mid-point, which may be calculated from (7.11), with $U_i = \epsilon_{1ij}\Omega_j x$ and $n = 1$; but no new calculation is needed, because there is a general theorem (Happel & Brenner 1965, ch. 5) which says that, if the relation (8.5) be written as

$$\mathscr{L}'_i = Q_{ij}U_j \quad (i,j = 2 \text{ or } 3),$$

the force required to sustain a pure rotation is simply

$$\mathscr{F}'_i = Q_{ji}\Omega_j \quad (i,j = 2 \text{ or } 3). \quad (8.6)$$

The component of this force in the direction of the angular velocity vector is non-zero in general, showing that the rotating body is capable of producing thrust (albeit inefficiently) like a propeller.

In the case of a body with circular cross-section, $K = 0$ and $K_{ij} = 0$. The expressions (8.1) and (8.2) for the total force in translational motion then agree, to the order of ϵ^2, with results obtained by Cox (1970*a*) and Tillett (1970); and the expression (8.4) for the couple required to sustain rotational motion agrees with a result given by Cox (1970*b*). These authors express their results as power series in ϵ, but, as indicated above, the forms (8.1), etc., are likely to give more accurate results than a series expansion terminating in the term of order ϵ^2.

A particular local cross-sectional shape of importance is an ellipse, which gives several useful special cases as the ratio of the semi-diameters b and c is varied. For an ellipse

$$K = \log k = \log\frac{b+c}{2R_x}, \quad R_x = \frac{2b}{\pi}E\left(\frac{(b^2-c^2)^{\frac{1}{2}}}{b}\right),$$

where E is the complete elliptic integral of the second kind and whose values are available in tables. (A simpler formula for R_x which is accurate to within a few per cent when $\frac{1}{2} \leqslant c/b \leqslant 1$, and which is worst at $c/b = 0$ where it is not needed but would give a value 11 per cent too large, is $R_x \approx (\frac{1}{2}b^2+\frac{1}{2}c^2)^{\frac{1}{2}}$.) The expression (6.8) for K_{ij} is already in terms of the equivalent ellipse and so is immediately applicable to this case. A cross-section in the form of a flat plate of width $2b$ is obtained by putting

$$c = 0, \quad R_x = 2b/\pi.$$

It is worth noticing that, contrary to what might be inferred from formulae for slender bodies with circular cross-section, none of the force integrals vanishes in the case of a body whose cross-section is everywhere simply a line and whose volume is zero; as remarked earlier, the inner flow field is determined more by the perimeter of the cross-section than by its area.

When the body is an ellipsoid with semi-diameters l, b_0, c_0 ($l \gg b_0, c_0$), the formulae (8.1), (8.2), (8.3) and (8.4) are correct to the order of any power of ϵ and become

$$\mathscr{F}_1 = 4\pi\mu l U_1 \Big/ \left(\log\frac{4l}{b_0+c_0} - \tfrac{1}{2}\right), \tag{8.7}$$

$$\mathscr{F}_2 = 8\pi\mu l U_2 \Big/ \left(\log\frac{4l}{b_0+c_0} + \frac{b_0}{b_0+c_0}\right), \tag{8.8}$$

$$\mathscr{S}_{11} = \tfrac{4}{3}\pi\mu l^3 e_{11} \Big/ \left(\log\frac{4l}{b_0+c_0} - \frac{3}{2}\right), \tag{8.9}$$

$$\mathscr{L}_2 = \tfrac{8}{3}\pi\mu l^3 \Omega_2 \Big/ \left(\log\frac{4l}{b_0+c_0} - \frac{b_0}{b_0+c_0}\right), \tag{8.10}$$

with similar expressions for $\mathscr{F}_3$ and $\mathscr{L}_3$; it will be recalled that the axes of reference are parallel to the principal axes of the ellipsoid.

A complete set of exact results is available for this case of an ellipsoidal body and may be used to check the expressions obtained from slender-body theory. The formula for the force required to translate an ellipsoid is given by Lamb (1932, § 339) in terms of ellipsoidal potential functions, and it may be shown by straight-forward algebra that when $l \gg b_0, c_0$ the components of this force are approximated by (8.7) and (8.8) with an error which is of smaller order than any power of ϵ. The flow due to a stationary rigid ellipsoid immersed in infinite fluid whose undisturbed velocity is a linear function of position and vanishes at the position of the centre of the ellipsoid has been calculated by Jeffery (1922), in terms of the same ellipsoidal potential functions. Formulae for the net stresslet strength in a pure straining motion and for the couple required to maintain relative rotation can be extracted from Jeffery's results, and have been given explicitly in a recent paper (Batchelor 1970). The formulae are rather lengthy, but their asymptotic forms for $l/(b_0+c_0) \to \infty$ may be obtained without much difficulty and are found to confirm the above relations (8.9) and (8.10).

A cylindrical body is also of special interest in view of its convenience for experimental purposes. In this case $R_x/R_0 = 1$, K and K_{ij} are independent of x, and the above expressions for the integral force parameters can be evaluated in closed form. It also proves to be possible to evaluate the integrals of the ϵ^3-terms in (7.5) and (7.12), and since this appears not to have been noticed by previous writers we give below expressions correct to the order of ϵ^3:

$$\mathscr{F}_1 = 4\pi\mu U_1 l \left[\frac{\epsilon(1-\epsilon H_{01})}{1-\epsilon(K+\frac{1}{2})} + \epsilon^3\{H_{02} - (K+\tfrac{1}{2})\, H_{01}\}\right], \tag{8.11}$$

$$\mathscr{F}_2 = 8\pi\mu U_2 l \left[\frac{\epsilon(1-\epsilon H_{01})}{1-\epsilon(K_{22}-\frac{1}{2})} + \epsilon^3\{H_{02} - (K_{22}-\tfrac{1}{2})\, H_{01}\}\right], \tag{8.12}$$

$$\mathscr{S}_{11} = \tfrac{4}{3}\pi\mu e_{11} l^3 \left[\frac{\epsilon(1-\epsilon H_{21})}{1-\epsilon(K+\frac{3}{2})} + \epsilon^3\{H_{22} - (K+\tfrac{3}{2})\, H_{21}\}\right], \tag{8.13}$$

$$\mathscr{L}_2 = \tfrac{8}{3}\pi\mu\Omega_2 l^3 \left[\frac{\epsilon(1-\epsilon H_{21})}{1-\epsilon(K_{33}+\frac{1}{2})} + \epsilon^3\{H_{22} - (K_{33}+\tfrac{1}{2})\, H_{21}\}\right], \tag{8.14}$$

with expressions for $\mathscr{F}_3$ and $\mathscr{L}_3$ similar to those for $\mathscr{F}_2$ and $\mathscr{L}_2$, and again the axes of reference in the transverse plane coincide with the principal axes of K_{ij}. The symbol H_{mn} ($m = 0$ or 2; $n = 1$ or 2) is a number defined by

$$H_{mn} = (m+1)\int_0^1 \xi^m \{\log(1-\xi^2)^{\frac{1}{2}}\}^n \, d\xi,$$

and has the following values (obtained analytically in the case of $n = 1$, and by numerical integration for $n = 2$):

$$H_{01} = -0{\cdot}307, \quad H_{21} = -0{\cdot}640,$$
$$H_{02} = 0{\cdot}272, \qquad H_{22} = 0{\cdot}699.$$

The formulae (8.11)–(8.14) emphasize that, for a body as far from being longitudinally elliptic as is a cylinder, the numerical accuracy of the approximations correct to the order of ϵ^2 developed by Burgers (1938) and subsequent authors (for a circular cross-section) is rather limited unless l/R_0 is exceeding large.

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