G. HERRMANN
Professor of Civil Engineering,
The Technological Institute,
Northwestern University, Evanston, Ill.
Mem. ASME

R. W. BUNGAY
Department of Civil Engineering,
The Technological Institute,
Northwestern University.
Now Staff Member,
Stanford Research Institute,
Menlo Park, Calif.

On the Stability of Elastic Systems Subjected to Nonconservative Forces[1]

Free motions of a linear elastic, nondissipative, two-degree-of-freedom system, subjected to a static nonconservative loading, are analyzed with the aim of studying the connection between the two instability mechanisms (termed divergence and flutter by analogy to aeroelastic phenomena) known to be possible for such systems. An independent parameter is introduced to reflect the ratio of the conservative and nonconservative components of the loading. Depending on the value of this parameter, instability is found to occur for compressive loadings by divergence (static buckling), flutter, or by both (at different loads) with multiple stable and unstable ranges of the load. In the latter case either type of instability may be the first to occur with increasing load. For a range of the parameter, divergence (only) is found to occur for tensile loads. Regardless of the nonconservativeness of the system, the critical loads for divergence can always be determined by the (static) Euler method. The critical loads for flutter (occurring only in nonconservative systems) can be determined, of course, by the kinetic method alone.

General Considerations

The analysis of the stability of equilibrium of an elastic system was based until rather recently on a concept introduced over two centuries ago by Euler [1],[2] who determined the buckling load of a column in the context of classification and characterization of elastic curves. Euler defined this load as the smallest force under which the column could be in equilibrium not only in its original straight configuration but also in an infinitely close (adjacent) curved configuration (bifurcation of equilibrium). An alternative concept is embodied in the energy method, which, as shown by Pearson [2], is equivalent to the equilibrium method of Euler in the determination of the critical load.

No basic inadequacies were encountered in later years in applying this equilibrium concept to a variety of elastic systems under static compressive loading and even extending it to inelastic systems. It was not until Pflüger [3] investigated statically the problem of a cantilevered column, subjected at the free end to a compressive force which remains tangent to the deformed column, Fig. 1(*a*), that a deficiency of the Euler method became apparent. The result obtained was that there was no possible adjacent equilibrium configuration so that the magnitude of the force could be increased indefinitely without reaching a critical value in the Euler sense. This imposed the paradoxical conclusion that no buckling load existed in the problem considered, contrary to an intuitive expectation.

Later, Ziegler [4] resolved the paradox by demonstrating with the aid of a two-degree-of-freedom model that Pflüger's problem could be treated meaningfully only by applying the kinetic stability criterion. By this criterion the critical load is the smallest load under which a suitable disturbance will result in a motion that does not take place in the vicinity of the equilibrium configuration. The failure of the Euler method in this case was attributed to the fact that the applied force is not derivable from a potential; i.e., the system is nonconservative. Pflüger's problem itself was actually solved by the kinetic method by Beck [5], who found the critical load for such a tangential force to be approximately eight times larger than for the usual case of a force which remains constant in direction as the bar deforms, Fig. 1(*b*).

In two subsequent memoirs, Ziegler [6, 7] elaborated on the various concepts of elastic stability and offered a comprehensive classification both of systems and of methods of analysis. A more recent monograph by Bolotin [8], devoted entirely to nonconservative problems of the theory of elastic stability, presents in a broad manner the fundamental concepts and various applications of this special branch of elastomechanics, incorporating many contributions of the past decade.

It appears, however, that certain aspects of instability in nonconservative systems still merit a further clarification.

For example, the breakdown of the Euler method is not a necessary consequence of the nonconservativeness of the loading. This may be illustrated by the problem of a bar simply supported at both ends and subjected to an axial loading, uniformly distributed along its length, which remains tangential to the deformed elastic curve at every point, Fig. 2. This nonconservative

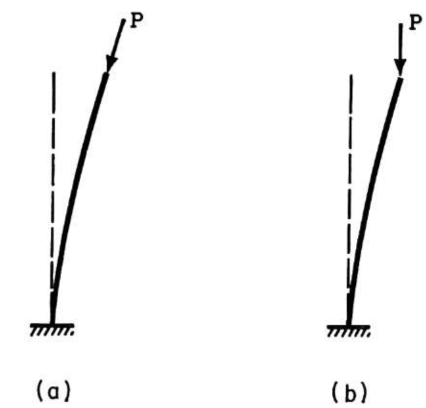

Fig. 1 Cantilevered column subjected to (*a*) tangential and (*b*) constant-directional end loading

Fig. 2 Simply supported bar under uniformly distributed tangential loading

---

[1] This research was supported by the United States Air Force under Grant AF-AFOSR-100-63 monitored by the Air Force Office of Scientific Research.

[2] Numbers in brackets designate References at end of paper.

Presented at the Summer Conference of the Applied Mechanics Division, Boulder, Colo., June 9–11, 1964, of The American Society of Mechanical Engineers.

Discussion of this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N. Y. 10017, and will be accepted until October 10, 1964. Discussion received after the closing date will be returned. Manuscript received by ASME Applied Mechanics Division, June 21, 1963. Paper No. 64—APM-4.

problem was solved by Pflüger by the Euler method, and this solution was verified by Leipholz [9] in a dynamic analysis using a variational method with a two-term approximation. If the distributed loading remains constant in direction (as in a simply supported column buckling under its own weight), the system is conservative, and the critical load, as given by Pflüger [3], is only 2 percent less.

It should be noted further that the presence or absence of an adjacent equilibrium configuration in a nonconservative system is dictated not only by the behavior of the loading but also by the constraints. Thus the distributed tangential loading just considered, if applied to a cantilever, results in a problem similar to that solved by Beck, in which there is again no possible adjacent equilibrium configuration (apparently this problem has not yet been solved).

Thus it appears desirable to gain a deeper insight into the interrelation of nonconservativeness, existence, or absence of adjacent equilibrium configurations and applicability of stability criteria.

Scope of Study

To investigate some of the aforementioned interrelations, the same two-degree-of-freedom system, which simulates a cantilever, analyzed by Ziegler [4] is employed herein. The applied end loading, however, is introduced with an additional parameter α, Fig. 3, which permits study of the stability of the system in continuous transition from the constant-directional (conservative) loading ($\alpha = 0$) to the purely tangential loading ($\alpha = 1$) examined by Ziegler. For completeness, a discussion of certain aspects of the complete range $-\infty < \alpha < +\infty$ will be included.

The analysis, restricted to a linearized formulation, consists in the determination of the two natural frequencies of free vibration as a function of the loading. For sufficiently small loads both frequencies are real and the system is thus stable under an arbitrary small disturbance, exhibiting bounded harmonic oscillations. As the load is increased, instability may occur by either one frequency becoming zero (static buckling) at the critical loading and then in general purely imaginary, or the two frequencies becoming complex, having passed a common real value at the critical loading. The ensuing motion under a supercritical force in the first case is nonoscillatory with the amplitude increasing exponentially (divergent motion), and the critical load can be determined statically by the Euler method. In the second case the ensuing motion is an oscillation with a definite period but with an exponentially increasing amplitude, and the critical load cannot be found by the Euler method because no associated adjacent equilibrium exists. The first case could be called "static instability" in view of the behavior at the critical load, and the second "dynamic instability." In aeroelasticity, however, analogous phenomena have been termed "divergence" and "flutter," respectively [10, 11], and we propose to employ this terminology in the sequel.

The type of loading considered could have been applied to a continuous cantilever, but this would have resulted in a transcendental frequency equation involving unwarranted mathematical complexities. On the other hand, a cantilever with a single degree of freedom as considered by Dzhanelidze, and reported by Leonov and Zorii [12] and by Bolotin [8], cannot exhibit flutter instability since such an instability is marked by a coincidence of two natural frequencies and a phase difference in the generalized coordinates.

The Model

We consider a double pendulum in a vertical plane, Fig. 3, composed of two rigid weightless bars of equal lengths l, which carry concentrated masses $m_1 = 2m$, $m_2 = m$. The configuration of the system is completely specified by the two angles φ_1 and φ_2, assumed to be small, formed between the vertical and each of the two bars, respectively. A load P is applied at the free end at an angle $\alpha\varphi_2$ with respect to the vertical, and at the joints the elastic restoring moments $c\varphi_1$ and $c(\varphi_2 - \varphi_1)$ are induced.

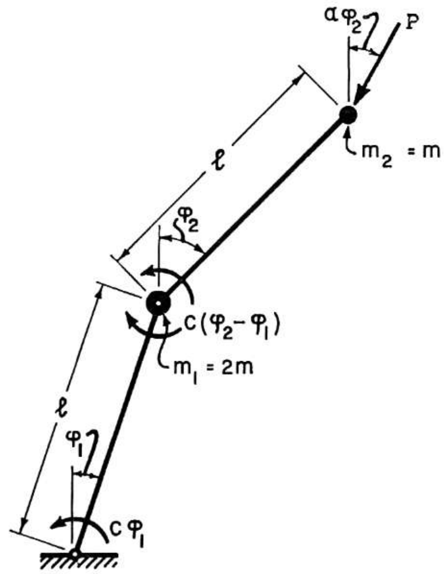

Fig. 3 Two-degree-of-freedom model

The coefficient α, which is a measure of the relative magnitude of the nonconservative component of the applied force, will be considered as ranging from $-\infty$ to $+\infty$. $\alpha = 0$ represents a conservative system; $\alpha = 1$ the case of a tangential force investigated by Ziegler. It is to be noted that, for any $\alpha \neq 0$, the system becomes nonconservative, which may be verified readily by computing the work done by this force along different paths in passing from the straight ($\varphi_1 = \varphi_2 = 0$) to a deformed configuration.

Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}_k}\right) - \frac{\partial L}{\partial \varphi_k} = Q_k \qquad (L = T - V, \quad k = 1, 2)$$

are used to establish the linear equations of motion, in which the kinetic energy T is

$$T = \tfrac{1}{2}ml^2[3\dot{\varphi}_1{}^2 + 2\dot{\varphi}_1\dot{\varphi}_2 + \dot{\varphi}_2{}^2]$$

the potential energy V of the restoring moments is

$$V = \tfrac{1}{2}c[2\varphi_1{}^2 - 2\varphi_1\varphi_2 + \varphi_2{}^2]$$

and the generalized forces Q_k (due to applied loading) are

$$Q_1 = Pl[\varphi_1 - \alpha\varphi_2]$$

$$Q_2 = Pl[(1 - \alpha)\varphi_2]$$

These forms lead to the equations of motion

$$3ml^2\ddot{\varphi}_1 + ml^2\ddot{\varphi}_2 + (2c - Pl)\varphi_1 + (\alpha Pl - c)\varphi_2 = 0$$

and

$$ml^2\ddot{\varphi}_1 + ml^2\ddot{\varphi}_2 - c\varphi_1 + (c - (1 - \alpha)Pl)\varphi_2 = 0$$

which, in turn, stipulating exponential solutions of the form

$$\varphi_k = A_k e^{i\omega t} \qquad (k = 1, 2)$$

yield the frequency equation

$$p_0\omega^4 - p_2\omega^2 + p_4 = 0$$

where

$$p_0 = 2m^2l^4$$

$$p_2 = ml^2[7c - 2(2 - \alpha)Pl]$$

$$p_4 = c^2 - (1 - \alpha)(3cPl - (Pl)^2)$$

Analytical Results

The four characteristic roots will occur in pairs, the positive and negative roots of the two values of ω^2 obtainable directly from the frequency equation. For a negative ω^2 one root describes an exponentially divergent motion; $\omega^2 = 0$ corresponds to neutral equilibrium, the appearance of an adjacent equilibrium configuration (static buckling, divergence). A complex value of ω^2 yields one root describing an oscillation with increasing amplitude (flutter). The system is thus stable only as long as both values of ω^2 are real and positive. We are interested in the manner in which ω^2 varies with P for different values of α. This is accomplished by investigating the curves of P versus real values of ω^2.

Expanding the frequency equation we find that it is a general quadratic equation in ω^2 and Pl, of the form

$$A(\omega^2)^2 + B(\omega^2 Pl) + C(Pl)^2 + D(\omega^2) + E(Pl) + F = 0$$

where the indicator, $B^2 - 4AC$, is

$$4m^2l^4[2(1 - \alpha) + \alpha^2]$$

Since this expression is always positive, the frequency curves (P versus ω^2; P, ω^2, real) are all of the hyperbolic type.

Except for degenerate cases, which shall be noted, there are but two general types of hyperbolas, with regard to orientation in the real ω^2, Pl plane, that may be encountered. These two types, qualitatively, are of "conjugate" orientations.

In the first type, each of the two branches of the hyperbola yields a single (real) value of ω^2 for every load and the two values never coincide. Instability may occur only in the form of divergence or divergent motions.

In the second general type of hyperbola, the two values of ω^2, for any load producing real values of ω^2, lie on the same branch of the curve. For each branch there is one critical load at which the two values coincide. Regardless of the behavior indicated by the real values of ω^2 on the branches, these two critical loads always bracket a single, limited range of the load "between" the two branches of the hyperbola, for which the values of ω^2 are complex and the free motions are of the flutter type. Since the system must be stable for sufficiently small loads, these critical loads must be of the same sign for any given value of α.

The solution of the frequency equation is

$$\omega_{1,2}^2 = \frac{\left\{\begin{matrix} 7c - 2(2 - \alpha)Pl \\ \mp [4(Pl)^2(2(1 - \alpha) + \alpha^2) - 4cPl(8 - \alpha) + 41c^2]^{1/2} \end{matrix}\right\}}{4ml^2}$$

from which P versus $\omega_{1,2}^2$ can be plotted for any α. We determine, first, the critical loads corresponding to coincidence of frequencies (occurring in the second type of hyperbola) by setting the discriminant equal to zero in the equation for $\omega_{1,2}^2$. This yields, for the critical loads, in nondimensional form, the equation

$$\frac{Pl}{c} = \frac{(8 - \alpha) \mp [(8 - \alpha)^2 - 41(2(1 - \alpha) + \alpha^2)]^{1/2}}{2(2(1 - \alpha) + \alpha^2)}$$

Real values of these critical loads are associated with the second type of hyperbola, complex values with the first. We wish to determine the transitional values of α. Thus, setting this discriminant equal to zero,

$$(8 - \alpha)^2 - 41(2(1 - \alpha) + \alpha^2) = 0$$

yields the roots $\alpha_{tr} = 0.345, 1.305$.

Substituting this equation into that for $\omega_{1,2}^2$ yields

$$[\omega_{1,2}^2]_{\alpha_{tr}} = \frac{7c - 2Pl(2 - \alpha) \mp \dfrac{1}{\sqrt{41}}(2Pl(8 - \alpha) - 41c)}{4ml^2}$$

with $\alpha = \alpha_{tr} = 0.345, 1.305$.

Thus two transitional values of α are obtained, at each of which the hyperbolas degenerate into two intersecting straight lines. Between these values of α the second type of hyperbola is found to occur, and the phenomenon of flutter is thus limited to this range of α. The corresponding critical loads are all positive (compressive).

Consider next the constant term p_4 in the frequency equation. The Euler method is equivalent to setting $p_4 = 0$ ($\omega^2 = 0$), corresponding to intercepts of the hyperbolas on the P-axis. Setting $p_4 = 0$ we obtain for the Euler buckling loads, in nondimensional form

$$\frac{Pl}{c} = \frac{1}{2}\left[3 \mp \left(\frac{5 - 9\alpha}{1 - \alpha}\right)^{1/2}\right]$$

For real values of the load we must have $\alpha \leq \frac{5}{9}$ or $\alpha > 1$. Thus there are critical values of $\alpha(\alpha_{cr} = \frac{5}{9}, 1)$, marking the limits of a range in which no adjacent equilibrium position occurs in the system for any value of the load.

We note from the form of p_4 that the lower critical value of α is a function of the elastic and geometric parameters of the system, and under a variation of these parameters might increase indefinitely, approaching one as a limit. Thus, for $\alpha < 1$ there is a class of systems or loadings wherein the absence of an adjacent equilibrium configuration for any value of the load is a function of the elastic and geometric parameters. However, for $\alpha = 1$, the terms in p_4 involving P drop out entirely, leaving a positive definite expression which contains the elastic parameter alone. Therefore, in this special case alone, we may say that it is the specification of the loading itself which results in the absence of any possible adjacent equilibrium configuration, a point of view which might also be adopted in regard to Beck's problem. We note here that in the case of a uniform continuous cantilever subjected to a load characterized by the same type of parameter, the Euler method reveals a similar critical value of the parameter, which in that case is one half.

A third set of values of α of interest is denoted by α', and is associated with a coincidence of an Euler load with a critical load for flutter. This occurs when a value of ω^2, at which $\omega_1^2 = \omega_2^2$, is zero. Thus we set $\omega_{1,2}^2 = 0$, which is equivalent to setting $p_2 = 0$, $p_4 = 0$, simultaneously; i.e.,

$$[7c - 2(2 - \alpha)Pl] = 0$$

and

$$c^2 - (1 - \alpha)(3cPl - (Pl)^2) = 0$$

Solving the first equation for Pl and substituting into the second yields a quadratic equation in α, the roots of which are found to be $\alpha' = 0.423, 1.182$.

Discussion of Results

In the sequel we restrict our detailed attention to $0 \leq \alpha \leq 1$, as this range is somewhat more meaningful physically and is sufficient to demonstrate a connection between the various instability phenomena. Fig. 4 shows the frequency curves for the various values of α of particular interest in this range. From these curves, in which both branches of the hyperbolas and their asymptotes are shown for completeness, we can determine by inspection the particular character of the frequency curve for any α in the range $0 \leq \alpha \leq 1$, and we proceed now to a discussion of the behavior of the individual curves in this range and the various stability phenomena that they illustrate.

For $0 \leq \alpha < \alpha_{tr}$ the hyperbolas are all of the first type, and the behavior is as previously discussed. The frequency curves and the characteristic behavior of the nonconservative systems are qualitatively indistinguishable from the conservative case. Obviously, the Euler method would yield the lowest buckling load, which here marks the boundary between the single stable and unstable ranges of the loading. A kinetic analysis would yield nothing additional. With increasing values of α in this range, the hyperbolas draw closer to their asymptotes and finally degenerate into two straight lines at $\alpha = \alpha_{tr}$, as previously noted.

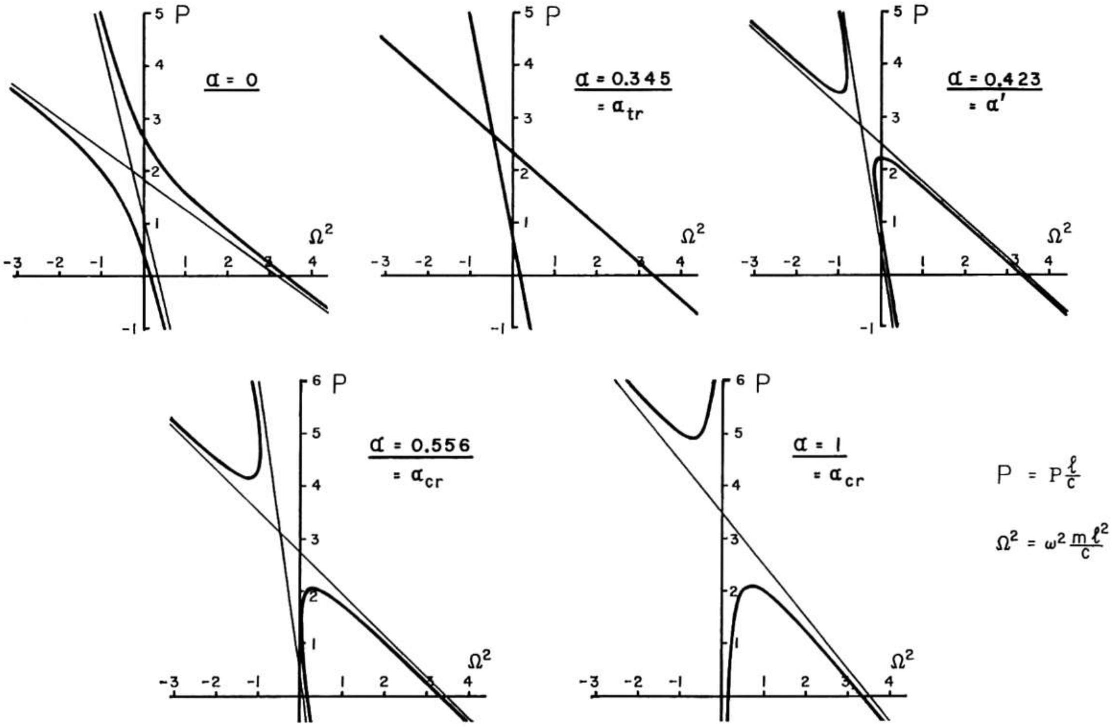

Fig. 4 Load versus frequency curves for particular values of parameter α in the range $0 \leq \alpha \leq 1$

This case marks the first occurrence of a coincidence of the characteristic roots.

For values of α greater than α_{tr} in this range, hyperbolas of the second type, with the conjugate orientation, occur and pull away from their asymptotes with increasing α. The upper branch lies entirely in the second quadrant, corresponding to divergent motion, and such an instability follows flutter with increasing load.

In this range, for $\alpha_{tr} < \alpha < \alpha'$ the coincidence of frequencies on the lower branch occurs at negative values of ω^2, with divergent motion already characterizing both modes. Thus in this class of systems the boundary between the single stable and unstable ranges of the loading parameter is marked solely by the appearance of an adjacent equilibrium configuration in the first mode, and is obtainable by the equilibrium approach. The system is unstable for all higher loads. The critical loads corresponding to coincidence of frequencies do not mark any bound between stability and instability.

Thus, for such systems, the Euler method would yield the critical load with regard to stability, even though the phenomenon of flutter is possible at some higher loadings. Conversely, the sole use of the kinetic method, if employed so as to determine merely the critical loads corresponding to the coincidence of frequencies, would lead to erroneous conclusions.

For $\alpha = \alpha'$, ω^2 at the coincidence of the frequencies on the lower branch is zero. The sequence of instabilities with increasing load is the same as in the preceding range of α.

For $\alpha > \alpha'$ the coincidence of frequencies occurs at positive values of ω^2 and this critical point now marks the bound between a stable and unstable range of the load. However, for $\alpha' < \alpha < \alpha_{cr}$ the lower branch still intersects the load axis, and the two corresponding critical loads, both occurring in the first mode, now bracket a separate range of instability through divergent motion. Such a system is rather remarkable in that it displays, for different loads, losses of stability by both divergence and flutter.

Thus for $\alpha' < \alpha < \alpha_{cr}$ we have a rather interesting sequence of free motions with increasing load, resulting in multiple regions of stability and instability. This is illustrated in Fig. 5 by the frequency curve for the arbitrary value of $\alpha = 0.5$. Such a system has characteristic free motions which include successively stable oscillations, divergent motion, stable oscillations, flutter, and then divergent motion again for all higher loads. In such a situation the lowest critical load marking the appearance of an instability would still be a buckling load, obtainable by the Euler method. However, the existence of the second range of stability, above the second "buckling" load, as well as its upper limit, would be revealed only by a detailed kinetic analysis.

For $\alpha = \alpha_{cr}$ the two buckling loads, bracketing the lower region of instability, coincide and the frequency curve is tangent to the load axis. Thus in this case there is a divergence instability at that isolated load, with no associated divergent motion for neighboring loads. The sequence of instabilities is otherwise the same as for $\alpha' < \alpha < \alpha_{cr}$.

For $\alpha \leq \alpha_{cr}$ the lowest critical load was always a buckling load, obtainable by the equilibrium approach, and varied continuously with α. Above α_{cr} no adjacent equilibrium configurations occur and the lowest critical load is that corresponding to the coincidence of the frequencies. There is thus a discontinuity (jump) in the magnitude of the lowest critical load, at $\alpha = \alpha_{cr}$. Systems in the class $\alpha_{cr} < \alpha \leq 1$ possess a single range of stability and of instability, with a sequence of characteristic free motions of

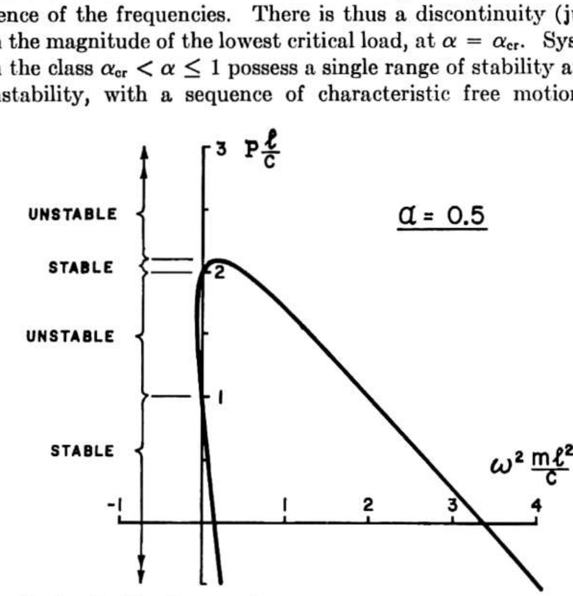

Fig. 5 Detail of load versus frequency curve for $\alpha = 0.5$, illustrating multiple ranges of stability and instability

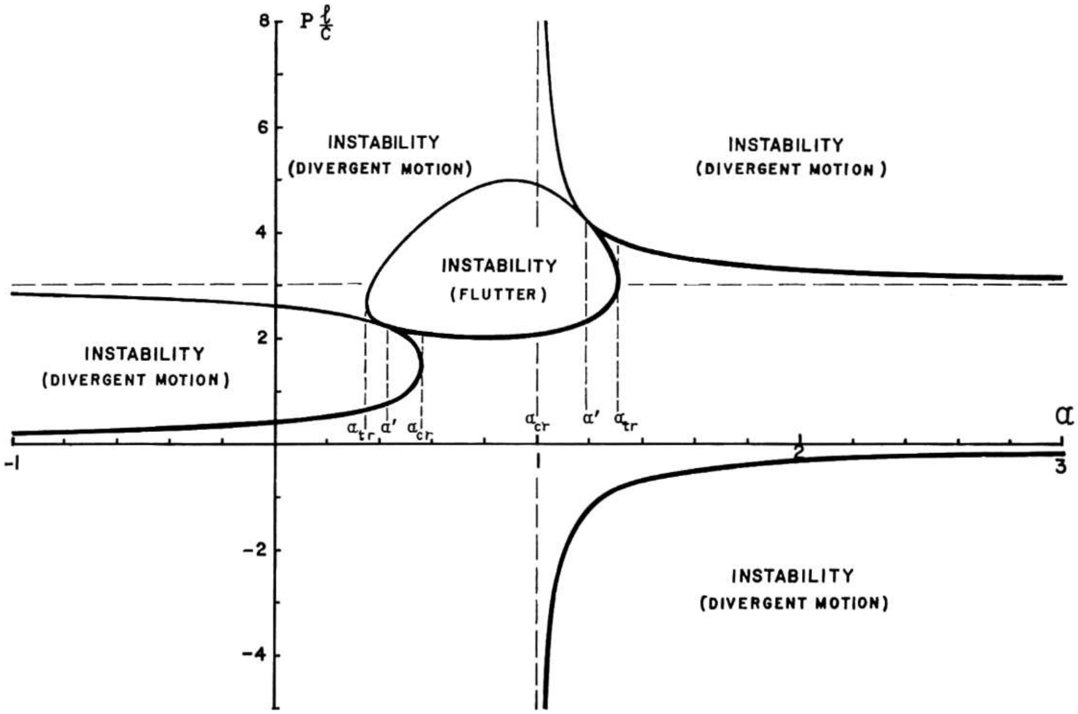

Fig. 6 Critical loads versus parameter α

stable oscillations, flutter, and finally divergence for all higher loads.

The foregoing discussion could be extended to the values of α outside the range $0 \leq \alpha \leq 1$, but is omitted here for the sake of brevity. However, a plot showing the variation in all the critical loads for a wider range of α, including the entire region of flutter instability, and with the asymptotic behavior of the critical loads for divergence clearly indicated for extreme values of α, is given in Fig. 6.

Considering systems corresponding to given values of α, this plot illustrates that in systems displaying multiple regions (and types) of instability under compressive loading, the lowest critical load may correspond to either divergence or flutter. Also illustrated is the existence of systems displaying instability by divergence for both compressive and tensile loads, another example of which was pointed out by Barta [13].

For $0 \leq \alpha \leq 1$, this plot bears a remarkably close analogy to that given by Leonov and Zorii, and Bolotin, for a single-degree-of-freedom system in that range. These investigators, however, do not seem to note that the discontinuity in the magnitude of the (lowest) critical load at our α_{cr}, corresponds to a change in the nature of the instability mechanism occurring at that critical load (in a multi-degree-of-freedom system) nor do they note the possibility of the second range of stability in the range $\alpha' < \alpha \leq \alpha_{cr}$.

Concluding Remarks

With the aid of the parameter α in the simple model analyzed here, we have attempted to show a connection between instability phenomena of divergence and flutter by demonstrating a generic relationship between such disparate frequency curves as those characterizing $\alpha = 0$ and $\alpha = 1$. Thus, such curves (and systems characterized by them) may be seen to be not of a singular or isolated nature, but part of a continuous "spectrum" of frequency curves.

The justification for considering the entire range of $-\infty < \alpha < +\infty$ may be made clearer through the following observation. The type of loading specified may be considered as the result of a superposition of two component loads, corresponding to constant-directional vertical loading ($\alpha = 0$) and tangential loading ($\alpha = 1$), the two being kept in a constant ratio as the loading is varied. In such a perspective, $0 < \alpha < 1$ corresponds to these component loads having the same sense. Then, $\alpha < 0$ and $\alpha > 1$ corresponds to these component loads having opposite senses, with their relative magnitudes determined by the magnitude and sign of α, and with positive load always corresponding to a resultant compressive loading.

The effect of weights of the masses has not been included here, but our investigations indicate that for small such constant loads the principal effect consists in shifting the frequency curves in the positive (negative) direction of the abscissa for a suspended (inverted, Fig. 3) model. Referring to the frequency curve in Fig. 4 for $\alpha = 1$, we can see that the shift caused by stabilizing constant forces would result in an intercept of the upper branch of the hyperbola on the $\omega^2 = 0$ coordinate axis. This is, in fact, the case analyzed by Ziegler [4], in which the Euler method yielded a higher critical load than the kinetic method, and which has contributed to the general discrediting of the applicability of the static approach in nonconservative problems. This particular case is somewhat equivalent to the situation occurring herein for $1 < \alpha < \alpha'$, in which, under compressive loading, the system becomes unstable through flutter, with the higher critical load, for divergence, of no consequence.

Also excluded were dissipative forces, which are known to have destabilizing effects in nonconservative systems, as well as the effect of a variation in mass distribution. These effects will be considered in later studies.

Finally, it should be emphasized that the instability phenomena with which we have been concerned here are a consequence of linearized treatments. While the mathematical concept of neutral or adjacent equilibrium in linear theory has a firm basis in experiment and bifurcation theory, the phenomenon of flutter (or the significance of the critical load) in such systems (as opposed to aeroelastic problems) still requires some analytical verification by means of a nonlinear treatment, as well as experimental corroboration.

References

1 L. Euler, "Abhandlungen über das Gleichgewicht und die Schwingungen der ebenen elastischen Kurven," *Ostwald's Klassiker der*

exakten Wissenschaften, Nr. 175, W. Engelmann, Leipzig, Germany, 1910.

2 C. E. Pearson, "General Theory of Elastic Stability," *Quarterly of Applied Mathematics*, vol. 14, 1956, pp. 133–144.

3 A. Pflüger, *Stabilitätsprobleme der Elastostatik*, Julius Springer, Berlin, Germany, 1950.

4 H. Ziegler, "Die Stabilitätskriterien der Elastomechanik," *Ingenieur-Archiv*, vol. 20, 1952, pp. 49–56.

5 M. Beck, "Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes," *Zeitschrift für angewandte Mathematik und Physik*, vol. 3, 1952, pp. 225–228.

6 H. Ziegler, "Linear Elastic Stability," *Zeitschrift für angewandte Mathematik und Physik*, vol. 4, 1953, pp. 89–121, 168–185.

7 H. Ziegler, "On the Concept of Elastic Stability," *Advances in Applied Mechanics*, vol. 4, edited by H. L. Dryden and T. von Karman, Academic Press Inc., New York, N. Y., 1956, pp. 351–403.

8 V. V. Bolotin, *Nonconservative Problems of the Theory of Elastic Stability*, Moscow, 1961; English translation published by Pergamon Press, Inc., New York, N. Y., 1963.

9 H. Leipholz, "Anwendung des Galerkinschen Verfahrens auf nichtkonservative Stabilitätsprobleme des elastischen Stabes," *Zeitschrift für angewandte Mathematik und Physik*, vol. 13, 1962, pp. 359–372.

10 R. L. Bisplinghoff, H. Ashley, and R. L. Halfman, *Aeroelasticity*, Addison-Wesley Publishing Company, Inc., Cambridge, Mass., 1955.

11 Y. C. Fung, *An Introduction to the Theory of Aeroelasticity*, John Wiley & Sons, Inc., New York, N. Y., 1955.

12 M. Ia. Leonov and L. M. Zorii, "Effect of Friction on the Critical Load of a Compressed Rod," *Soviet Physics—Doklady*, vol. 7, 1963, pp. 611–613; English translation published by the American Institute of Physics, New York, N. Y., from *Doklady Akademii Nauk SSSR*, vol. 145, 1962, pp. 295–297.

13 J. Barta, "Das Becksche Stabilitätsproblem und verwandte Probleme," *Acta Technica*, Academiae Scientiarum Hungaricae, vol. 31, 1960, pp. 241–259.