Part III: Biological Physics and Fluid Dynamics (Michaelmas 2023)
R E Goldstein (05/06/23)

Example Sheet #2

1. *Elastic equilibria.* An elastic filament of length L, bending modulus A, and linear mass density λ is clamped at an angle ψ relative to the horizontal and hangs under gravity (acceleration g). The angle ψ may be of either sign. Using the tangent angle representation, find a differential equation defining the equilibrium shape of the filament. Examine the strong-gravity limit and find a simplified solution.

2. *Wiggling filaments.* In the lectures we considered a semi-infinite elastic filament with bending modulus A whose left end was oscillated with amplitude $h_0 \cos \omega t$ and was torque-free. Develop a perturbative expansion for finite filaments of length L in the regime of slow oscillations, so that $L/\ell(\omega) \ll 1$, where $\ell(\omega)$ is the elastohydrodynamic penetration length. Assume as before that the distal end is torque-free and force-free. Show how the first contributions that break time-reversal invariance lead to net propulsion.

3. *Fluctuations of membrane tethers.* In lecture we considered the basic features of membrane tethers in the limit of high tension. (a) In this limit, compute the spectrum of thermal fluctuations in the cylindrical region of a tether of length L and radius determined by the force balance equation. (b) Set up (at least formally) the calculation of thermal fluctuations in the catenoidal region away from the tether itself. (c) If very motivated, determine numerically the appropriate eigenfunctions and eigenvalues that govern the spectrum.

4. *A particle forced by a moving optical trap.* A microsphere of radius a and drag coefficient ζ is constrained to move along the x-axis, and is acted on by an optical trap that is moving in the positive x-direction at velocity v_T. When the trap is located at a point x_0 it exerts a force $F(x - x_0)$, so the overdamped dynamics of the particle is

$$\zeta \dot{x} = F(x - v_T t).$$

Suppose that the trap has compact support, so that $F(x) = 0$ for $x < -X_L$ and for $x > X_R$. If the trap starts to the left of the particle, find the particle's net displacement Δx after the trap has passed it by, and the time Δt spent by the particle interacting with the trap. What is the condition that assures that the particle does not remain trapped as $t \to \infty$? Assuming this is the case, show that whatever the form of $F(y)$ the net displacement is always in the direction of the trap motion, and suggest a heuristic explanation for this result. Find the asymptotic behaviour of Δx for large trap velocities.

The trap is now moved around a circle of radius $R \gg a$. Derive the particle's net rotational frequency f_p as a function of the trap angular frequency $f_T = v_T/(2\pi R)$, the displacement Δx in each kick, the interaction time Δt and the potential width $2X_0 = X_R - X_L$. Confirm that in the regime of suitably large trap velocity, which you should define precisely, one obtains the intuitive result $f_p \simeq (\Delta x/2\pi R) f_T$. Specializing to the case of a triangular trapping potential, with $F(y) = F$ for $-X_0 < x < 0$ and $F(y) = -F$ for $0 < x < X_0$, obtain an explicit expression for f_p/f_c as a function of the two quantities $\alpha = X_0/(\pi R)$ and $\beta = f_T/f_c$, where $2\pi R f_c = F/\zeta$.