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Fundamental singularities of viscous flow

Part I: The image systems in the vicinity of a stationary no-slip boundary

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SUMMARY

The image system for the fundamental singularities of viscous (including potential) flow are obtained in the vicinity of an infinite stationary no-slip plane boundary. The image system for a: stokeslet, the fundamental singularity of Stokes flow; rotlet (also called a stresslet), the fundamental singularity of rotational motion; a source, the fundamental singularity of potential flow and also the image system for a source-doublet are discussed in terms of illustrative diagrams. Their far-fields are obtained and interpreted in terms of singularities. Both the stokeslet and rotlet have similar far field characteristics: for force or rotational components parallel to the wall a far-field of a stresslet type $O(r^{-2})$ is obtained, whereas normal components are of higher order $O(r^{-3})$.

1. Introduction

Many problems in differing branches of theoretical physics are resolved by the distribution of singularities on, along or over some body or shape being investigated. There are many powerful techniques in fluid mechanics using a distribution of singularities especially in potential and slender body theory. Often these problems need to be resolved in the vicinity of a plane boundary, which requires us to obtain an image system in either a closed or integral form. In problems at very small Reynolds number it seems that very few studies have been made on viscous fluid singularities, especially in the vicinity of walls. Notable work in this field has been carried out by Lorentz [8] and Oseen [9]. For a general review of this subject from the chemical engineering context the book of Happel and Brenner [5] can be consulted.

The four singularities we will consider in this paper are (i) stokeslet, (ii) rotlet, (iii) source and (iv) source-doublet. The study was initially instigated to find the influence a wall has on two purely viscous singularities, the stokeslet and rotlet, which correspond to the fundamental singularities of translational and rotational motion respectively.

The fundamental singularity of translational motion at zero Reynolds number, the stokeslet, has been studied in some detail, while the image system in a plane stationary boundary, and resulting far-field has recently been discussed by Blake [2]. In this paper we will briefly recapitulate his results in order to form some resemblance of completeness.

However, on the other hand, the rotlet (the fundamental singularity of rotational motion) has rarely been discussed, except the general solution in spherical harmonics for rotational motion that can be found in Lamb [6]. Landau and Lifschitz [7] briefly discuss the rotlet (although it is not called by this name) in their book. Batchelor [1] obtains and discusses the rotlet singularity (he calls it a stresslet) in some detail when he considers the bulk stress of a suspension of force free particles. Discussion of the rotlet singularity has recently been given a new impetus by Chwang and Wu [4] when they discussed the propulsive mechanisms of flagellated micro-organisms exhibiting helical beating patterns. They correctly realized that it was essential to balance the angular momentum when discussing their propulsive mechanisms. Surprisingly, the angular momentum had not been included in nearly all previous studies. The obvious solution to this problem was to supplement the distribution of stokeslet singularities by rotlet singularities along the centre-line of the flagellum.

The need for a knowledge of the singularities in the presence of a wall is required to account for the interactions which both flagella and cilia have with the wall. Observations of many flagella show that they exhibit a planar beat, and studies are needed to show whether in fact this is an artifact of their motion induced by the presence of a wall (in the form of a slide or coverslip).

The other two singularities in the source and its doublet are included mainly for interest sake. It should be pointed out though that the ease with which we can find other singularities (by differentiating) in an infinite viscous fluid does not hold when a boundary is present. This is amply illustrated by considering these two simple singularities.

In section 2, we discuss the singularities and their image systems, illustrating the mathematical representation by diagrams. In section 3, the results are summarized and applications to real problems are discussed.

2. Singularities and image system

The velocity and pressure fields to be discussed in this paper will satisfy the Stokes flow equations of motion, defined as follows in Cartesian coordinates:

$$\nabla p = \mu \nabla^2 \boldsymbol{u}\,, \quad \nabla \cdot \boldsymbol{u} = 0\,, \tag{1}$$

where p is the pressure, $\boldsymbol{u}$ the velocity vector and μ the dynamic viscosity.

Before proceeding with a discussion of the image systems for these singularities, we will briefly define them, as they occur in an infinite viscous fluid. In a cartesian coordinate scheme we suppose the singularity is at $\boldsymbol{x}=\boldsymbol{y}$. We now define the translated coordinates $\boldsymbol{r}=\boldsymbol{x}-\boldsymbol{y}$.

The velocity and pressure field due to a force $\boldsymbol{F}$ (i.e. a stokeslet) is defined as follows:

$$u_i = \frac{F_j}{8\pi\mu}\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right), \quad p = \frac{F_j}{4\pi}\frac{r_j}{r^3}\,. \tag{2}$$

To obtain higher order singularities, corresponding to stokes doublets and quadrupoles, we simply take the gradient of (2) in the chosen direction. Thus from Batchelor [1], Blake [2] we obtain the following for the velocity and pressure field for a stokes doublet

$$\begin{aligned} u_i &= \frac{D_{jk}}{8\pi\mu}\left[\left\{-\frac{r_i\delta_{jk}}{r^3} + \frac{3r_i r_j r_k}{r^5}\right\} + \left\{\frac{r_k\delta_{ij} - r_j\delta_{ik}}{r^3}\right\}\right], \\ p &= \frac{D_{jk}}{4\pi}\left(\frac{3r_j r_k}{r^5} - \frac{\delta_{jk}}{r^3}\right). \end{aligned} \tag{3}$$

The strength of this type of singularity is characterized by a second order tensor D_{jk}. The symmetric term of (3) is called by Batchelor [1], a stresslet, the velocity field corresponding to straining motion. The antisymmetric term in (3) is termed by Batchelor, a couplet, the velocity field corresponding to rotational motion. He also makes the observation that if we define

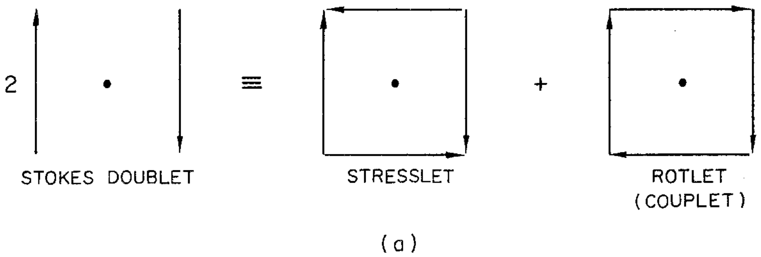
$$\Omega_n = -\frac{\varepsilon_{njk} D_{jk}}{8\pi\mu}\,, \tag{4}$$

then the velocity field due to the couplet may be represented by the rotational vector $\boldsymbol{\Omega}$ as follows,

$$\boldsymbol{u} = \frac{\boldsymbol{\Omega} \times \boldsymbol{r}}{r^3}\,, \quad p = \text{const.} \tag{5}$$

This corresponds to the fundamental singularity for rotational motion (see *e.g.* Lamb [6]). Unfortunately, in the literature, there appears to be two different terms in current use for this singularity; the other from Chwang and Wu [4] who call this singularity a "rotlet". In this paper we will use the term "rotlet", mainly because in the problem we are concerned with, we will be considering rotational motion. These two singularities of Stokes flow can be illustrated diagrammatically (see Figure 1). The streamlines for a stresslet are radial while those for a rotlet are circular.

A source, with mass outflow M in unit time is defined as follows



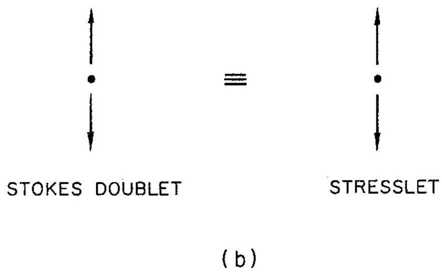


Figure 1. (a) Illustrates the decomposition of a couple producing stokesdoublet into a stresslet (symmetric) and rotlet (or couplet, antisymmetric). The dashed lines represent streamlines. (b) A symmetric stokesdoublet only produces a stresslet.

$$u_i = \frac{M}{4\pi}\frac{r_i}{r^3}, \tag{6}$$

and a source doublet of vector strength and direction $\boldsymbol{D}$ is defined by

$$u_i = \frac{D_j}{4\pi}\left(-\frac{\delta_{ij}}{r^3} + \frac{3r_i r_j}{r^5}\right). \tag{7}$$

The problem of finding the image system can be accomplished by using Fourier transforms similar to that of Blake [2]. We will discuss each singularity separately.

2.1. *Stokeslet*

The problem of a stokeslet in the presence of a stationary plane boundary has recently been discussed by Blake [3]: his results being briefly repeated here. The exact solution for a force singularity in the presence of a stationary plane boundary is as follows

$$\begin{aligned} u_i &= \frac{F_j}{8\pi\mu}\left[\left(\frac{\delta_{ij}}{r} + \frac{r_i r_j}{r^3}\right) - \left(\frac{\delta_{ij}}{R} + \frac{R_i R_j}{R^3}\right) + 2h(\delta_{j\alpha}\delta_{\alpha k} - \delta_{j3}\delta_{3k})\frac{\partial}{\partial R_k}\left\{\frac{hR_i}{R^3} - \left(\frac{\delta_{i3}}{R} + \frac{R_i R_3}{R^3}\right)\right\}\right], \\ p &= \frac{F_j}{4\pi}\left[\frac{r_j}{r^3} - \frac{R_j}{R^3} - 2h(\delta_{j\alpha}\delta_{\alpha k} - \delta_{j3}\delta_{3k})\frac{\partial}{\partial R_k}\left(\frac{R_3}{R^3}\right)\right], \end{aligned} \tag{8}$$

where $\alpha = 1, 2$; $y = (y_1, y_2, h)$ and $r = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - h)^2]^{\frac{1}{2}}$, $R = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + h)^2]^{\frac{1}{2}}$.

The tensor $(\delta_{j\alpha}\delta_{\alpha k} - \delta_{j3}\delta_{3k})$ is non zero only when $j = k$; its value is $+1$ for $j = 1$ or 2, and -1 for $j = 3$. The image system can be illustrated as follows in Figure 2 (a) and (b).

The velocity in the far-field is

$$u_i = \frac{F_k}{8\pi\mu}\left[\frac{12hx_i x_\alpha x_3\delta_{k\alpha}}{|\boldsymbol{x}|^5} + h^2\delta_{k3}\left(-\frac{(12 + 6\delta_{i3})x_i x_3}{|\boldsymbol{x}|^5} + \frac{30x_i x_3^3}{|\boldsymbol{x}|^7}\right)\right]. \tag{9}$$

Thus there is a fundamental difference in the far-fields for the force components parallel to the plane and those normal to it, being of $O(r^{-2})$ and $O(r^{-3})$ respectively. In the propulsion of

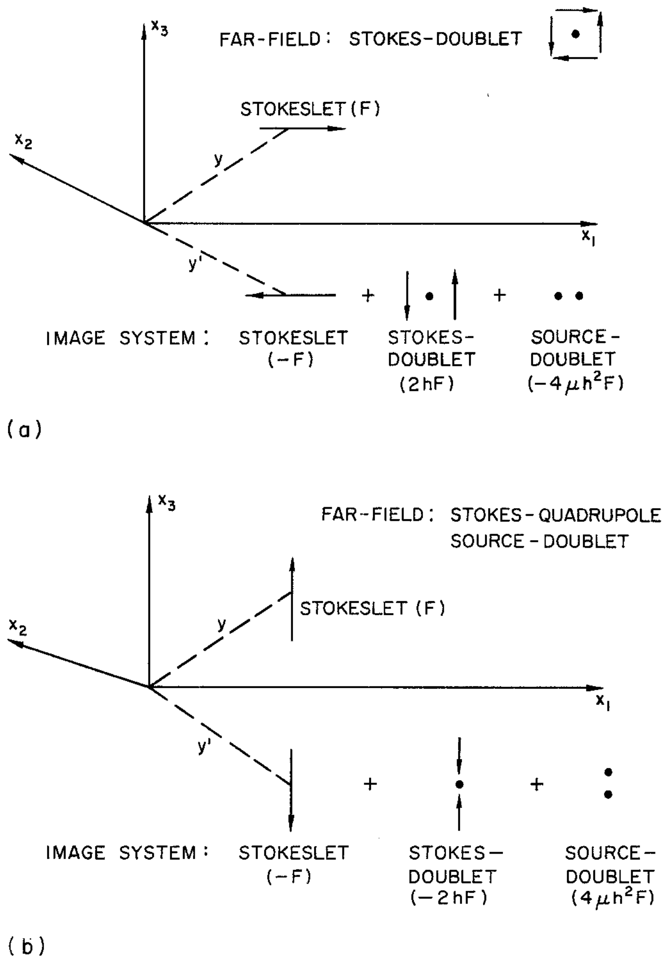


Figure 2. (a) Diagram illustrating image system and far-fields for a stokeslet with $j=1$. (b) The image system for $j=3$, the strength of the components being given in brackets.

ciliated micro-organisms this implies that the motion of cilia parallel to the organisms surface is far more important in producing general fluid motion than the movements normal to its surface. For a further discussion of this theoretical application, the reader is referred to Blake [3].

2.2. *Rotlet* (or Couplet)

The exact solution for a rotlet, characterized by the rotational vector $\boldsymbol{\Omega}$, in the vicinity of a stationary plane boundary is

$$\left.\begin{aligned} u_i &= \frac{\varepsilon_{ijk}\Omega_j r_k}{r^3} - \frac{\varepsilon_{ijk}\Omega_j R_k}{R^3} + 2h\varepsilon_{kj3}\Omega_j\left(\frac{\delta_{ik}}{R^3} - \frac{3R_iR_k}{R^5}\right) + 6\varepsilon_{kj3}\frac{\Omega_j R_i R_k R_3}{R^5}, \\ p &= -4\mu\frac{\partial}{\partial R_k}\left(\frac{\varepsilon_{kj3}\Omega_j R_3}{R^3}\right) \end{aligned}\right\} \tag{10}$$

with the same definitions for r and R that were used previously. Again it is probably easiest to consider the components of $\boldsymbol{\Omega}$ parallel and normal to the wall (see Figure 3).

For the components of the rotation vector $\boldsymbol{\Omega}$ parallel to the wall the image system consists of an opposite rotlet, a stresslet of strength $(16\pi\mu\Omega_1)$ and a source doublet of strength $(8\pi h\Omega_1)$.

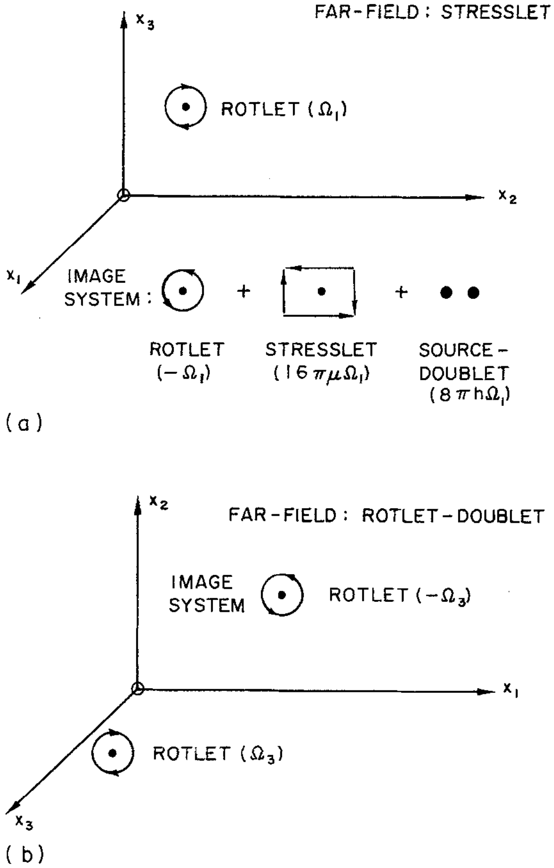


Figure 3. (a) The image system and far-fields for a rotlet with components tangential to, and (b) normal to the wall.

The image system for components of the rotation vector $\boldsymbol{\Omega}$ normal to the wall consists solely of an opposite image rotlet.

The far-fields for a rotlet, like that of a stokeslet, are extremely interesting to study. The displacement of two opposite rotlets produce a rotational doublet (or a rotlet doublet) which has a velocity field of $O(r^{-3})$. However, the stresslet is of $O(r^{-2})$ and is the dominating feature in the far-field for the components of $\boldsymbol{\Omega}$ parallel to the wall. This is perhaps a curious result as we have pure rotational motion near the plane producing a characteristic straining motion in the far-field, which has radial streamlines; the exact opposite to that which occurs in an infinite domain. The velocity far-fields is as follows

$$u_i = \frac{6\varepsilon_{kj3}\Omega_j x_i x_k x_3}{|\boldsymbol{x}|^5} + \frac{6h\varepsilon_{ik3} x_k x_3 \Omega_3}{|\boldsymbol{x}|^5}. \tag{11}$$

This, of course, has many similarities to the far-fields due to a stokeslet. In both cases the force and rotation components parallel to the axis produce stresslet far-fields which fall off as $O(r^{-2})$ whereas those normal to the wall fall off as $O(r^{-3})$. However, the important feature to note is that a stokeslet field is $O(r^{-1})$ which placed in the presence of a wall is reduced to $O(r^{-2})$ or $O(r^{-3})$ in the far-field, whereas a rotlet is $O(r^{-2})$ initially, the presence of the wall only changes the characteristics of the far-field, but not changing the order of magnitude of fall-off except for the case of the normal component. This may have some fascinating repercussions in the movements of micro-organisms near walls, especially for ciliated bodies. Biologists have noted an apparent twisting of a cilium during its beating cycle which may contribute to its propulsive thrust if it is an active movement, or alternatively it could be due to its passive response to stress exerted on it by the fluid.

2.3. *Source*

The exact solution for a source in the vicinity of a stationary plane boundary is as follows

$$\begin{aligned} u_i &= \frac{M}{4\pi}\left[\frac{r_i}{r^3} + \frac{R_i}{R^3} - 2\left(\frac{R_i}{R^3} - \frac{3R_iR_3R_3}{R^5}\right) + 2h\left(\frac{\delta_{i3}}{R^3} - \frac{3R_iR_3}{R^5}\right)\right], \\ p &= \frac{\mu M}{\pi}\left(\frac{3R_3^2}{R^5} - \frac{1}{R^3}\right). \end{aligned} \tag{12}$$

Thus the image system for a source consists of an equal source at the image point, the same as in potential flow (which one would expect on mass conservation grounds anyway), a stresslet of strength $-4\mu M$ and a source-doublet of strength $2Mh$. This is illustrated schematically in Figure 4. The far-field behaves as

$$u_i \sim \frac{3M}{2\pi}\frac{x_i x_3^2}{|\boldsymbol{x}|^5}. \tag{13}$$

Perhaps it is easier to compare the components of the far-field to that of an isolated source if we consider the velocity field in terms of spherical polar coordinates, in which case

$$\begin{aligned} u_r &= \frac{M}{4\pi r^2} &&: \text{ source, infinite fluid} \\ u_r &= \frac{M}{2\pi r^2} &&: \text{ source, image in wall: potential flow} \\ u_r &= \frac{3M\cos^2\theta}{2\pi r^2} &&: \text{ source, no-slip wall} \end{aligned} \tag{14}$$

where θ measures the angle between the vertical and $\boldsymbol{x}$. Thus the mass flow is concentrated in a conical region about $\theta=0$. In fact, in the vertical direction the mass outflow is six times that for a source in an infinite fluid and three times that of potential flow which only requires zero normal velocities on the plane.

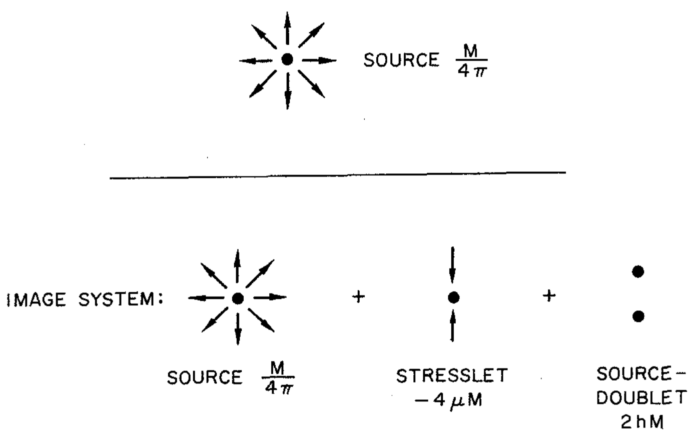


Figure 4. Image system for a source in a no-slip wall.

2.4. *Source-doublet*

A source-doublet is included in this discussion for two main reasons: One is that when considering a closed surface the source-doublet singularity will be found in preference to the source type singularity and secondly we discuss this singularity to illustrate the fact that we cannot simply derive the doublet by taking the gradient of a source when in the presence of a wall. The solution for a source doublet in the presence of a stationary plane boundary is as follows

$$u_i = D_j\left[\left(\frac{\delta_{ij}}{r^3} - \frac{3r_ir_j}{r^5}\right) - \left(\frac{\delta_{ij}}{R^3} - \frac{3R_iR_j}{R^5}\right)\right] - D_3\delta_{i\alpha}\frac{6R_\alpha R_3}{R^5} - \frac{D_\alpha\delta_{i3}6R_\alpha R_3}{R^5}$$
$$+ 2D_3\delta_{i\alpha}\left(-\frac{3R_\alpha x_3}{R^5} + \frac{15R_3^2x_3R_\alpha}{R^7}\right) + 2D_3\delta_{i3}\left(-\frac{9x_3R_3}{R^5} + \frac{15x_3R_3^3}{R^7}\right) \tag{15}$$
$$- 2D_\alpha\delta_{i\beta}\left(-\frac{3R_3\delta_{\alpha\beta}x_3}{R^5} + \frac{15R_3R_\alpha R_\beta}{R^7}\right) - 2D_\alpha\delta_{i3}\left(-\frac{3x_3R_\alpha}{R^5} + \frac{15R_3^2x_3R_\alpha}{R^7}\right),$$

$$p = 4\mu D_k\left[\delta_{k3}\left(-\frac{9R_3}{R^5} + \frac{15R_3^3}{R^7}\right) - \delta_{k\alpha}\left(-\frac{3R_\alpha}{R^5} + \frac{15R_\alpha R_3^2}{R^7}\right)\right].$$

The image system in this case is obviously more complex, although it can be represented in terms of a finite number of known singularities. It appears that the higher the order of the singularity, the more image singularities are needed to satisfy the no-slip condition.

3. Summary and applications

We have found the image systems for the four singularities: stokeslet, rotlet, source and source doublet, in the neighborhood of a stationary no-slip plane boundary. Perhaps the most important finding in this paper is the influence of the walls on the far-fields of both the stokeslet and rotlet. In an infinite viscous fluid there is a power difference in the decay of vorticity for a stokeslet $O(r^{-2})$ and a rotlet $O(r^{-3})$, but in the presence of a stationary no-slip boundary the vorticity penetration of both decays as $O(r^{-3})$. This may lead to some interesting results for motions in the vicinity of walls.

An important influence of a no-slip boundary on a source is the concentration of mass outflow about the vertical coordinate. There do not seem to be any immediate applications of this singularity, but there may be possibilities for its use in modelling finite shaped bodies with a point or line distribution of other singularities, obviously including a sink distribution of equal strength.

In conclusion we can add that the wall exerts an equal and opposite force, but no couple, in the case of a stokeslet, an equal and opposite couple with no force in the case of a rotlet while for all other cases both the force and couple on the wall are zero.

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