Part III: Biological Physics and Fluid Dynamics (Michaelmas 2023)
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Example Sheet #4

1. *Turing instability.* (a) Starting from the standard system of reaction-diffusion equations,

$$u_t = D_u \nabla^2 u + f(u,v), \quad (1)$$
$$v_t = D_v \nabla^2 v + g(u,v), \quad (2)$$

complete the analysis outlined in class, and thereby show that the condition for a Turing instability can be expressed as

$$f_u + d g_v > 2\sqrt{dJ},$$

where $d = D_u/D_v$ and $J = f_u g_v - f_v g_u > 0$ is the determinant of the stability matrix of the homogeneous system, presumed stable. (b) Show that this result implies that a Turing instability at equal diffusivities ($d = 1$) is not possible. (c) Show that the result in (b) also follows directly from the structure of the linearised equations of motion around the homogeneous fixed point in the case $D_u = D_v$. (d) Demonstrate the concept of *fine-tuning* of the Turing instability in the sense that when the stability matrix of the homogeneous system is close to one with a double-zero eigenvalue, for example

$$\begin{pmatrix} -1 & -1 \\ 1+\delta & 1-\delta \end{pmatrix},$$

then the diffusivity ratio can be made as close to unity as desired by taking δ suitably small.

2. *Turing instability and front motion in the FHN model.* Consider the FHN model discussed in lectures,

$$u_t = D\nabla^2 u - u(u-r)(u-1) - \rho(v-u), \quad (3a)$$
$$\epsilon v_t = \nabla^2 v - (v-u), \quad (3b)$$

where $0 \le r \le 1$. (a) Assume the fast-inhibitor limit $\epsilon = 0$ and find the regions in the $\rho - r$ plane where the states $u = 0$ and $u = 1$ are (i) simultaneously linearly stable and (ii) linearly unstable. Provide a quantitative graph for the case $D = 0.01$. (b) Generalise the analysis done in lecture for the motion of a single front between the states $u = 0$ and $u = 1$ to the case of two interacting fronts with a "top-hat" profile. Here, the outer solution for the activator has the form

$$u^0(x) = \begin{cases} 0, & -\infty < x \le -Q(t) \\ 1, & -Q(t) \le x \le +Q(t) \\ 0, & +Q(t) \le x < +\infty. \end{cases}$$

where the fronts are located at $x = \pm Q(t)$. Show that the outer inhibitor field is

$$v^0(x) = \begin{cases} \sinh Q \mathrm{e}^{+x}, & -\infty < x \le -Q(t) \\ 1 - \mathrm{e}^{-Q}\cosh x, & -Q(t) \le x \le +Q(t) \\ \sinh Q \mathrm{e}^{-x}, & +Q(t) \le x < +\infty. \end{cases}$$

Use the solvability condition to show that the front speed is

$$Q_t = -6\sqrt{2D}\left[\Delta F - \frac{\rho}{2}\mathrm{e}^{-2Q}\right],$$

and thus there is a stable *localized state* in the FHN model.

3. *Keller-Segel model of chemotaxis.* A celebrated model of chemotaxis involves the coupled dynamics of the organism concentration n and a chemical species a to which it is attracted. The dynamics includes nonlinear growth of the organisms and motion in response to gradients of a. In one spatial dimension the governing PDEs are

$$n_t = Dn_{xx} + bn\left(1 - \frac{n}{n_0}\right) - (\chi(a)na_x)_x \tag{4a}$$

$$a_t = D_A a_{xx} + hn - ka, \tag{4b}$$

where $\chi(a) = \chi_0 K/(K+a)^2$. Find a scaling such that this reduces to

$$\dot{u} = u'' + u(1-u) - \beta\left[\frac{uv'}{(\alpha+v)^2}\right]' \tag{5a}$$

$$\dot{v} = \delta v'' + \gamma(u - v), \tag{5b}$$

where $\cdot$ and $'$ refer to differentiation with the scaled t and x respectively, and $\alpha, \beta, \gamma, \delta$ are positive constants. Show that the uniform, steady solution $u = v = 1$ is unstable if

$$\frac{\beta\gamma}{(1+\alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2,$$

and find the wavenumber at which the system first becomes unstable as χ_0 is increased from zero, in the case $\alpha = \gamma = \delta = 1$.