A Study of Locking Phenomena in Oscillators

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Abstract—Impression of an external signal upon an oscillator of similar fundamental frequency affects both the instantaneous amplitude and instantaneous frequency. Using the assumption that time constants in the oscillator circuit are small compared to the length of one beat cycle, a differential equation is derived which gives the oscillator phase as a function of time. With the aid of this equation, the transient process of "pull-in" as well as the production of a distorted beat note are described in detail.

It is shown that the same equation serves to describe the motion of a pendulum suspended in a viscous fluid inside a rotating container. The whole range of locking phenomena is illustrated with the aid of this simple mechanical model.

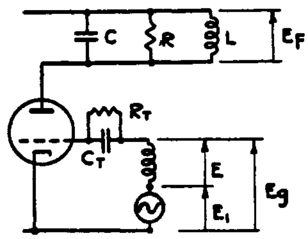

Fig. 1. Oscillator circuit.

I. Introduction

THE BEHAVIOR of a regenerative oscillator under the influence of an external signal has been treated by a number of authors. The case of synchronization by the external signal is of great practical interest; it has been applied to frequency-modulation receivers [1], [2] and carrier-communication systems [3], and formulas, as well as experimental data, have been given for the conditions required for synchronization [4]–[8]. The other case, arising when the external signal is not strong enough to effect synchronization, is of practical importance in beat-frequency oscillators. Here the tendency toward synchronization lowers the beat frequency and produces strong harmonic distortion of the beat note [4]–[8].

It is the purpose of this paper to derive the rate of phase rotation of the oscillator voltage at a given instant from the phase and amplitude relations between the oscillator voltage and the external signal at that instant; in other words, to find a differential equation for the oscillator phase as a function of time. This equation must be expected to describe the case of synchronization where any transient disturbance vanishes in time, giving way to a steady state in which phase difference between oscillator and external signal is constant. It must also give frequency and wave form of the beat note, in case no synchronization occurs. To cover both cases, it must contain a parameter which decides whether or not the transient term will vanish in time, thus producing an equivalent to the criteria for synchronization derived by other methods. Finally, the equation must suggest a mechanical analogy simple enough to give a clear picture of what actually happens in an oscillator when an external signal is impressed upon it.

In the following analysis, it is assumed that the impressed signal and the free oscillation are of similar frequency. Locking effects at submultiple frequencies are analogous in many respects, but the analysis does not apply directly.

II. Conditions for Bandwidth and Time Constants

In attempting to derive the rate of phase rotation at a given instant from no other data but phase and amplitude relations at that same instant, we assume implicitly that there are no aftereffects from different conditions which may have existed in the past. The value of such an assumption lies in the fairly simple analysis which it permits. But our experience with practical oscillators warns us that it may not always be justified. In this section we will study the requirements which an oscillator must meet so that our analysis may be applicable.

If an oscillator is disturbed but not locked by an external signal, we observe a beat note—periodic variations of frequency and amplitude. If these variations are rapid, a sharply tuned circuit in the oscillator may not be able to respond instantaneously, or a capacitor may delay the automatic readjustment of a bias voltage. In either case, our assumption would be invalid. To validate it, we shall have to specify a minimum bandwidth for the tuned circuit and a maximum time constant for the biasing system. To establish these limits, let us study the circuit shown in Fig. 1, with the understanding that the impressed signal is not strong enough to cause locking. We will use the following symbols:

Angular frequencies:

ω_0 = free-running frequency
ω_1 = frequency of impressed signal
$\Delta\omega_0 = \omega_0 - \omega_1$ = "undisturbed" beat frequency
ω = instantaneous frequency of oscillation
$\Delta\omega = \omega - \omega_1$ = instantaneous beat frequency.

Voltages:

E_p = voltage across plate load
E = voltage induced in grid coil
E_1 = voltage of impressed signal
E_g = resultant grid voltage
Q = figure of merit of plate load L, C, R.

If the oscillator were undisturbed, the only frequencies present[1] would be ω_0 and ω_1, producing a beat frequency $\Delta\omega_0$. Actually, a lower beat frequency is observed, so that the value of ω averaged over one complete beat cycle is shifted toward ω_1. We cannot yet predict, however, how large the excursions of the momentary value of ω might be. We may think of ω as of a signal which is frequency modulated with the beat note $\Delta\omega$; this beat note is known to contain strong harmonics if the oscillator is almost locked, so that ω can be represented by a wide spectrum of frequencies extending to both sides of its average value.

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[1] "Frequency" always means the angular frequency.

If the plate circuit is to reproduce variations of ω without noticeable delay, each half of the pass band must be wide compared to the "undisturbed" beat frequency. For a single tuned circuit we can write

$$\frac{\omega_0}{2Q} \gg \Delta\omega_0. \tag{1}$$

Without reference to any specific type of circuit, we can say that the frequency of the external signal should be near the center of the pass band.

Up to this point, we have assumed that the circuit of Fig. 1 operates as a linear amplifier. But it is well known [9] that some nonlinear element must be present to stabilize the amplitude of any self-excited oscillator. Curved tube characteristics may produce a nonlinear relation between grid voltage and plate current, distorting every individual cycle of oscillation ("instantaneous" nonlinearity); plate-current saturation is an example for this case. On the other hand, a nonlinear element may control the transconductance as the amplitude varies, thus acting like an automatic volume control; the relation between grid voltage and plate current may then remain linear over a period of many cycles. Oscillators stabilized by an inverse-feedback circuit containing an incandescent lamp provide perhaps the best example for this type. The combination of C_T and R_T in the circuit of Fig. 1 functions also as a controlling element of the automatic-volume-control type; at the same time, some nonlinearity of the "instantaneous" type will generally be present in this circuit.

We want the instantaneous amplitudes of the plate current and of the voltage E fed back to the grid to be the same as if the total grid voltage E_r at that instant had been stationary for some time; earlier amplitudes should have no noticeable aftereffects. How fast the amplitudes vary depends on the beat frequency. The amplitude control mechanism should, therefore, have a time constant which is short compared to one beat cycle.[2] (For the circuit of Fig. 1 this time constant would be of the order $T = C_T R_T$.) Since the shortest possible beat cycle corresponds to the "undisturbed" beat frequency $\Delta\omega_0$, we can write

$$T \ll \frac{1}{\Delta\omega_0}\,. \tag{2}$$

If the oscillator contains only amplitude limiting of the "instantaneous" type, this condition is inherently satisfied. An oscillator of the pure automatic-volume-control type will show the same locking and synchronizing effects as long as it fulfills[3] condition (2). But when the amplitude control mechanism acts too slowly to accommodate the beat frequency, phenomena of an entirely different character appear. Such an oscillator would fall outside the scope of the mathematical analysis presented in the following, but its special characteristics merit brief discussion.

In an oscillator of the pure automatic-volume-control type, let us represent all elements outside the tuned circuit L, C, R by a negative admittance connected in parallel with L, C, R. The numerical value of this negative admittance is proportional to the gain in the oscillator tube. Over a long period of time, the automatic-volume-control mechanism will so adjust the gain that the negative admittance becomes numerically equal to the positive loss admittance of L, C, R. At this point the net loss vanishes and the prevailing amplitude is maintained indefinitely, as if the tuned circuit had infinite Q.

Now, let an external signal of slightly different frequency be superimposed upon this oscillation, so that the resulting amplitude varies periodically. Then if the automatic-volume-control mechanism acts so slowly that no substantial gain adjustments can be made within one beat cycle, that value of negative admittance which resulted in zero net loss will be retained. In other words, the system acts as if the Q of the plate circuit were still infinitely large. An external signal E_1 with a frequency very close to ω_0 will then produce a large near-resonant amplitude, increasing further the closer ω_1 approaches ω_0. This magnified signal of frequency ω_1, superimposed on the original signal of frequency ω_0, which is still maintained, produces amplitude modulation of a percentage much greater than would correspond to the ratio E_1/E.

Evidently, similar effects could be observed if the tuned circuit had of itself a Q high enough to violate condition (1). This suggests an alternative way of stating that condition. The tuned circuit will "memorize" phase and amplitude for a period of the order T', its "decay time." This period must be short compared to a beat cycle[2]

$$T' \ll \frac{1}{\Delta\omega_0}\,. \tag{1a}$$

For a simple tuned circuit, $T' = (2Q/\omega_0)$ hence $(\omega_0/2Q) \gg \Delta\omega_0$ which is the same as (1).

If an oscillator fulfills both conditions (1) and (2), the amplitude modulation arising from a given signal E_1 is solely determined by the ratio E_1/E and by the shape of the amplitude-limiting or automatic-volume-control characteristic. Most oscillators operate in a fairly flat part of this characteristic, so that the amplitude actually varies less than in proportion to E_1/E. Keeping this in mind, we further assume a weak external signal

$$E_1 \ll E \tag{3}$$

so that the amplitude variations of E will also be small compared to E itself.

A surprisingly large number of practical cases meet all three conditions.

III. Derivation of the Phase as a Function of Time

Let Fig. 2 be a vector representation of the voltages in the grid circuit as they are found at a given instant. Furthermore, let E_1 be at rest with respect to our eyes; any vector at rest will therefore symbolize an angular frequency ω_1, that of the external signal, and a vector rotating clockwise with an angular velocity $(d\alpha/dt)$ shall represent an angular frequency $\omega_1 + (d\alpha/dt)$, or angular beat frequency of

$$\Delta\omega = \frac{d\alpha}{dt} \tag{4}$$

relative to the external signal.

It is important to keep in mind that this vector diagram shows beat frequency and phase. Many high-frequency oscillations may occur during a small shift of the vectors. We call $(d\alpha/dt)$ the instantaneous angular beat frequency; we would count $(1/2\pi)(d\alpha/dt)$ beats per second if this speed of rotation were maintained. Actually, $(d\alpha/dt)$ may vary and a complete beat cycle may never be accomplished.

With no external signal impressed, E_g and E must coincide: the voltage E returned through the feedback circuit must

[2] $1/\Delta\omega_0$, or the time required for one radian of a beat cycle, is used in the following.

[3] For synchronization on a subharmonic of the impressed signal, nonlinearity of the "instantaneous" type is necessary.

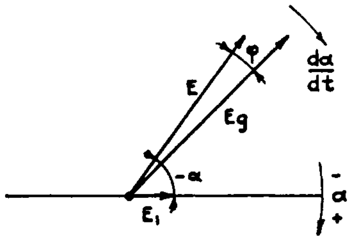

Fig. 2. Vector diagram of instantaneous voltages.

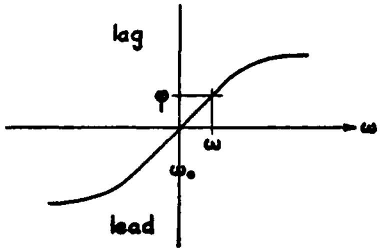

Fig. 3. Phase versus frequency for a simple tuned circuit.

have the same amplitude and phase as the voltage E_g applied to the grid. Those nonlinear elements which limit the oscillator amplitude will adjust the gain so that $|E| = |E_g|$; but the phase can only coincide at one frequency, the free-running frequency ω_0. At any other frequency the plate load would introduce phase shift between E_g and E. Fig. 3 shows a typical curve of phase shift versus frequency for a single tuned circuit as assumed in Fig. 1. The amount of lead or lag of the voltage drop across such a circuit with respect to the current flowing through it is plotted. For our oscillator circuit, we may take the curve to represent the lead or lag of E with respect to E_g as a function of frequency.

Let now an external voltage E_1 be introduced, and let Fig. 2 represent the voltage vectors at a given instant during the beat cycle. Evidently, the voltage E returned through the feedback circuit is now no longer in phase with the grid voltage E_g; the diagram shows E lagging behind E_g by a phase angle ϕ.

No such lag could be produced if the oscillator were still operating at its free frequency ω_0. We conclude that the frequency at this instant exceeds ω_0 by an amount which will produce a lag equal to ϕ in the plate circuit.

With $E_1 \ll E$ according to our third condition, inspection of Fig. 2 yields

$$\phi = \frac{E_1 \sin(-\alpha)}{E} = -\frac{E_1}{E} \sin \alpha. \tag{5}$$

The instantaneous frequency ω follows from Fig. 3. But our first condition implies that the pass band of the plate circuit is so wide that all frequencies are near its center. So we are using only a small central part of the ϕ versus ω curve which approaches a straight line with the slope

$$A = \frac{d\phi}{d\omega}. \tag{6}$$

Then, if ω_0 is the free frequency, the phase angle for another frequency ω close to it will be

$$\phi = A(\omega - \omega_0). \tag{7}$$

The instantaneous beat frequency $\Delta\omega$ is the difference between ω and the impressed frequency ω_1. Setting again $\Delta\omega_0 = \omega_0 - \omega_1$, we have

$$\phi = A(\omega - \omega_0) = A[(\omega - \omega_1) - (\omega_0 - \omega_1)] = A[\Delta\omega - \Delta\omega_0]. \tag{8}$$

Now, substituting (5) on the left and (4) on the right, we find

$$-\frac{E_1}{E} \sin \alpha = A\left[\frac{d\alpha}{dt} - \Delta\omega_0\right] \tag{9a}$$

and substituting

$$B = \frac{E_1}{E} \cdot \frac{1}{A}$$

we obtain

$$\frac{d\alpha}{dt} = -B \sin \alpha + \Delta\omega_0. \tag{9b}$$

Adding the impressed frequency ω_1 on both sides, we may also write

$$\omega = -B \sin \alpha + \omega_0. \tag{9c}$$

This means physically that the instantaneous frequency is shifted from the free-running frequency by an amount proportional to the sine of the phase angle existing at that instant between the oscillator and the impressed signal. The shift is also proportional to the impressed signal E_1, but inversely proportional to the oscillator grid amplitude E and to the phase versus frequency slope A of the tuned system employed.

For a single tuned circuit, textbooks give

$$\tan \phi = 2Q \frac{\omega - \omega_0}{\omega_0} \tag{10}$$

and for small angles we can write

$$\phi = 2Q \frac{\omega - \omega_0}{\omega_0}. \tag{10a}$$

Hence, substituting into (6)

$$A = \frac{2Q}{\omega_0} \tag{10b}$$

and

$$B = \frac{E_1}{E} \cdot \frac{\omega_0}{2Q}. \tag{10c}$$

Equation (9b) reads, therefore, for a single tuned circuit,

$$\frac{d\alpha}{dt} = -\frac{E_1}{E} \frac{\omega_0}{2Q} \sin \alpha + \Delta\omega_0. \tag{11}$$

The possibility of a steady state is immediately apparent; $(d\alpha/dt)$ must then be zero, so that in the steady state

$$0 = -\frac{E_1}{E} \frac{\omega_0}{2Q} \sin \alpha + \Delta\omega_0 \tag{12a}$$

or

$$\sin \alpha = 2Q \frac{E}{E_1} \cdot \frac{\Delta\omega_0}{\omega_0}. \tag{12b}$$

This gives the stationary phase angle between oscillator and impressed signal. Since $\sin \alpha$ can only assume values between $+1$ and -1, no steady state is possible if the right side of (12b) is outside this range. This gives the condition for synchronization

$$\left| 2Q \frac{E}{E_1} \cdot \frac{\Delta\omega_0}{\omega_0} \right| < 1 \tag{13a}$$

or

$$\frac{E_1}{E} > 2Q \left| \frac{\Delta\omega_0}{\omega_0} \right|. \tag{13b}$$

Because of its practical importance for receiver applications, another form of this condition shall be considered. E is the voltage which the oscillator (Fig. 1) produces across its grid coil; but if a locked oscillator is used to replace an amplifier, the voltage E_p across the plate circuit is the one that matters, since (E_p/E_1) represents the total gain. Now the tuned circuit is equivalent to a plate load $R_p = Q\sqrt{(L/C)}$, so that for a given transconductance g_m

$$E_p = E \cdot g_m \cdot Q \sqrt{\frac{L}{C}} \cdot$$

Combining this with (13b), we obtain

$$\frac{E_p}{E_1} < \left| \frac{\omega_0}{2\Delta\omega_0} \right| \cdot g_m \sqrt{\frac{L}{C}} \cdot \tag{13c}$$

It is interesting to note that Q, the only circuit constant entering into (13b) where the grid voltage E is of interest, cancels out in (13c) where the plate voltage E_p is determined.

For an oscillator which contains a plate load other than a simple tuned circuit, the condition for synchronization may be written

$$\frac{E_1}{E} > | A\Delta\omega_0 | \tag{13d}$$

where $A = (d\phi/d\omega)$ for the particular type of plate load.

IV. Approximation for the Pull-In Process

Turning now to the transient solution of the differential equation (9b), we examine first the case $\Delta\omega_0 = 0$. This means that the free-running frequency equals that of the impressed signal and that locking will eventually occur for any combination of voltages and circuit constants as evidenced by all forms of (13).

The equation

$$\frac{d\alpha}{dt} = -B \sin \alpha \tag{14a}$$

shows what happens when the external signal E_1 is suddenly switched on with an initial lag α_1 behind the free-running oscillator. Equation (14a) is quite similar to

$$\frac{d\alpha}{dt} = -B\alpha \tag{14b}$$

and actually goes over into this form when α is small. Equation (14b) has the familiar solution

$$\alpha = \alpha_1 \epsilon^{-Bt} \tag{14c}$$

and this means physically that the oscillator phase "sinks" toward that of the impressed signal, first approximately, and later accurately as a capacitor discharges into a resistor. The speed of this process, according to (10c) which defines B, is proportional to the ratio of impressed voltage to oscillator voltage and to the bandwidth of the tuned circuit.

If the free-running frequency is not equal to that of the impressed signal, but close enough to permit locking for a given combination of constants according to (13), the manner in which the steady state is reached must still resemble a capacitor discharge. It is particularly worth noting that the final value α_∞ is always approached from one side in an aperiodic fashion. The accurate solution for this case will be given later.

V. Phenomena Outside the Locking Range

To obtain a general solution giving α as a funcion of time, it is necessary to integrate (9b). We first substitute

$$K = \frac{\Delta\omega_0}{B} \tag{15a}$$

which means for a single tuned circuit

$$K = 2Q \frac{E}{E_1} \frac{\Delta\omega_0}{\omega_0} \cdot \tag{15b}$$

By comparing with (13a) and (13d), we find that the condition for synchronization can now be written

$$| K | < 1. \tag{15c}$$

Substituting into (9b) we obtain

$$\frac{d\alpha}{dt} = -B(\sin \alpha - K). \tag{16}$$

Integration gives

$$\tan \frac{\alpha}{2} = \frac{1}{K} + \frac{\sqrt{K^2 - 1}}{K} \tan \frac{B(t - t_0)}{2} \sqrt{K^2 - 1} \tag{17a}$$

or

$$\alpha = 2 \tan^{-1} \left[\frac{1}{K} + \frac{\sqrt{K^2 - 1}}{K} \tan \frac{B(t - t_0)}{2} \sqrt{K^2 - 1} \right] \tag{17b}$$

wherein t_0 is an integration constant.

Let us now assume that the condition for synchronization is not fulfilled, so that $|K| > 1$. This makes $\sqrt{K^2 - 1}$ real. With continually increasing t, the term $[B(t-t_0)/2]\sqrt{K^2-1}$ will pass through $\pi/2$, $3\pi/2$, etc., and the tangent on the right side of (17a) will become $+\infty$, $-\infty$, etc., in succession; at these instants $\alpha/2$ must also be $\pi/2$, $3\pi/2$, etc., although it will assume values different from $[B(t-t_0)/2]\sqrt{K^2-1}$ during the intervals.

So, while $[B(t-t_0)/2]\sqrt{K^2-1}$ increases uniformly with time, $\alpha/2$ will grow at a periodically varying rate; but the total length of a period must be the same for both. The average angular beat frequency—the actual number of beats in 2π seconds—is therefore

$$\overline{\Delta\omega} = B\sqrt{K^2 - 1} \tag{18a}$$

or, substituting from (15a),

$$\overline{\Delta\omega} = \Delta\omega_0 \frac{\sqrt{K^2 - 1}}{K} \cdot \tag{18b}$$

$\Delta\omega_0$ is that beat frequency which would appear if the oscillator

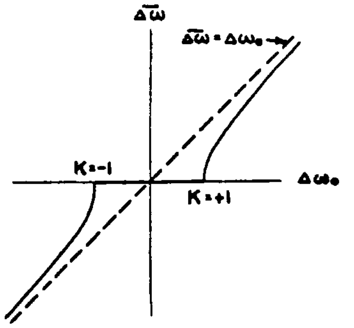

Fig 4. Reduction of beat frequency due to locking.

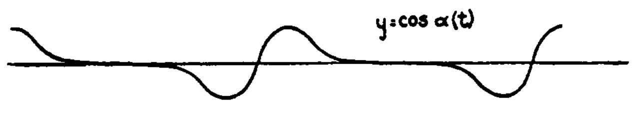

Fig. 5. Wave form of beat note for $\cos \alpha(t)$.

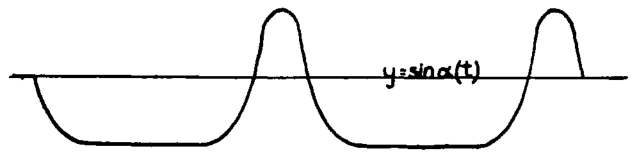

Fig. 6. Wave form of beat note for $\cos [\alpha(t)+\pi/2]$.

maintained its free frequency; $\sqrt{K^2-1}/K$ approaches unity for large values of K, far from the point where locking occurs; but it drops toward zero when this point ($K=1$) is approached.

Fig. 4 shows a plot of the average beat frequency $\overline{\Delta\omega}$ versus the undisturbed beat frequency $\Delta\omega_0$ as computed from (18b).

In the intervals between the arguments $\pi/2$, $3\pi/2$, etc., the two angles in (17a) cannot be the same because of the factor $\sqrt{K^2-1}/K$ with which one tangent is multiplied, and the addition of $1/K$. For large values of K, $1/K$ vanishes and the factor approaches unity, so that the rate of increase of $\alpha/2$ with time will vary by a smaller percentage as the beat frequency increases; but (16) shows that $d\alpha/dt$ must still vary between $B(K-1)$ and $B(K+1)$. Now, $BK=\Delta\omega_0$, according to (15), and B represents the highest difference $\Delta\omega_{\max}$ for which locking can occur ($K=1$ for $B=\Delta\omega_0$). So the instantaneous beat frequency $\Delta\omega$ will vary periodically between $\Delta\omega_0-\Delta\omega_{\max}$ and $\Delta\omega_0+\Delta\omega_{\max}$ as long as $\Delta\omega_0$ exceeds $\Delta\omega_{\max}$.

$\Delta\omega_{\max}$ itself is determined by (13). It is

$$\Delta\omega_{\max} = \frac{\omega_0}{2Q}\,\frac{E_1}{E} \tag{19a}$$

or

$$\Delta\omega_{\max} = \frac{\omega_0}{2}\,\frac{E_1}{E_p}\, g_m \sqrt{\frac{L}{C}} \tag{19b}$$

for a single-tuned circuit, and

$$\Delta\omega_{\max} = \frac{1}{A}\cdot\frac{E_1}{E} \tag{19c}$$

for any type of plate load for which $A = d\phi/d\omega$.

If K is only slightly above unity, the factor $\sqrt{K^2-1}/K$ falls far below unity, and the phase angle between E_1 and E increases at an extremely nonuniform rate. Inspection of the vector diagram in Fig. 2 gives the resultant grid voltage $E_g = E - E_1 \cos\alpha$. To illustrate the wave form of the resultant beat note the function $\cos\alpha(t)$ is plotted in Fig. 5. Operation very close to locking is assumed. Other wave forms are possible in beat-frequency oscillators where the beat note is produced in a separate detector; a constant phase shift may then be added to α on the way to the detector. Fig. 6 shows an example with a phase shift of $\pi/2$: the function plotted is $\cos[\alpha(t)+\pi/2]$ which equals $-\sin\alpha(t)$.

VI. Accurate Analysis of the Pull-In Process

To make the discussion of (17a) complete, we may finally apply it to the case of an oscillator pulling into the locked condition, $|K|<1$. The term $\sqrt{K^2-1}$ then becomes $j\sqrt{1-K^2}$. By use of the relation $\tanh x = -j \tan jx$, equation (17a) is transformed[4] into

$$\tan\frac{\alpha}{2} = \frac{1}{K} - \frac{\sqrt{1-K^2}}{K}\tanh\frac{B(t-t_0)}{2}\sqrt{1-K^2}. \tag{20a}$$

The integration constant t_0 permits one to fit the equation to the initial phase difference α_1, which exists when the external signal is switched on.

As t increases, the functions tanh and coth go asymptotically toward unity. The steady state must therefore be given by

$$\tan\frac{\alpha}{2(\infty)} = \frac{1-\sqrt{1-K^2}}{K}. \tag{20b}$$

Using (16) we identify K with $\sin\alpha_\infty$ for the steady state. Hence $\sqrt{1-K^2}=\cos\alpha_\infty$ and (20b) becomes $(1-\cos\alpha_\infty)/\sin\alpha_\infty$, which is indeed equal to $\tan(\alpha_\infty/2)$ by a trigonometrical identity.

VII. A Mechanical Model

In conclusion, let us construct a mechanical model to illustrate the processes which we have derived. To provide a full analogy, the model must follow the same differential equation (9b)

$$\frac{d\alpha}{dt} = -B\sin\alpha + \Delta\omega_0.$$

Let us forget $\Delta\omega_0$ for the moment. A pendulum in a viscous fluid would follow the remaining equation if α is taken to mean the angle between the pendulum and a vertical line. If we assume the viscosity of the fluid to be so great that we need not consider the inertia of the pendulum, the angular speed of the pendulum $d\alpha/dt$ is proportional to the force which causes it to move. We may shape the pendulum so that one unit of force will produce one unit of speed. Now, if B is the weight of the pendulum, the force acting to return it to its rest position will indeed be $-B\sin\alpha$.

To include the term $\Delta\omega_0$, we must add a constant force. We may also bring $\Delta\omega_0$ over to the left side of the equation; since $d\alpha/dt$ stands for angular speed, $-\Delta\omega_0$ on the left would mean a constant backward rotation of the pendulum with respect to the liquid. Constant forward rotation of the liquid with respect to the pendulum would produce the same force, and we choose this interpretation for our model shown in Fig. 7.

[4] Equation (20a) holds for $\sin\alpha_1 > K$. Otherwise, substitute coth for tanh.

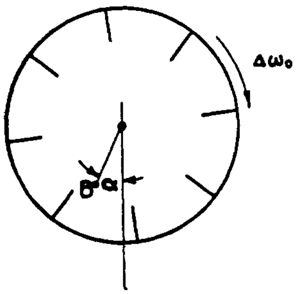

Fig. 7. Mechanical model: pendulum in a rotating container filled with viscous liquid.

The viscous liquid is enclosed in a drum rotating with an angular speed $\Delta\omega_0$. Again we assume that the viscosity of the liquid is so great that it will follow the rotation of the drum completely. Let us also assume that the rotation of the liquid is not noticeably affected by inserting the pendulum.

Remembering now that the vertical direction represents the phase of the impressed signal, while the position of the pendulum indicates the relative phase of the oscillator grid voltage, we can go through the whole range of phenomena by rotating the drum with various speeds, corresponding to the undisturbed beat frequencies $\Delta\omega_0$.

At low drum speed, the pendulum will come to rest at a definite angle α_∞ which will increase as the drum speed rises. If disturbed, the pendulum will "sink" back; it will never go past the rest position since inertia effects are absent.

If we lift the pendulum clockwise to any point below $\alpha_1=\pi-\alpha_\infty$, it will come back counterclockwise; but if we lift it past this limit, or over to the right, it will return clockwise. This is the reason why there are two different transient solutions for (20a).

At a certain critical drum speed $\Delta\omega_{max}=B$ the pendulum will stand horizontal; if the drum is further accelerated, it will "unlock" and begin to go around, moving fast on the right but very slow on the left and completing a much smaller number of revolutions than the liquid.

But as we increase the speed further, the fast whirling fluid takes the pendulum along, irrespective of the weight. The motion appears much more uniform, and the speed of the pendulum becomes nearly equal to that of the drum: the average beat frequency $\overline{\Delta\omega}$ is approaching the undisturbed value $\Delta\omega_0$.

Acknowledgment

C. W. Carnahan and H. P. Kalmus, in the course of their work on locked oscillators for frequency-modulation receivers [1], assembled a great deal of information regarding the behavior of such oscillators inside and outside the locking range. To study these phenomena further, they built a 1000-cycle oscillator which permitted direct observation of phase and amplitude variations on the oscilloscope. They investigated the influence of time constants and, among other effects mentioned in this paper, observed the large amplitude modulation which occurs when the time constant of the grid bias is large (case of "infinite Q" noted in Section II). Discussion of these experiments laid the groundwork for the analysis presented here, and the author gratefully acknowledges this important contribution.

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