Part III: Biological Physics and Fluid Dynamics (Michaelmas 2023)
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Example Sheet #1

1. *Interaction of charged surfaces with nonuniform charge density.* Two parallel charged, planar, laterally-infinite membranes are located at $z = \pm d/2$. The upper one has charge density $\sigma_+ = \sigma\cos(kx)$, while the lower has $\sigma_- = \sigma\cos(kx+\theta)$, where θ is a constant phase shift. Within Debye-Hückel theory, find the total electrostatic energy of the pair as a function of θ. Find the value of θ that minimizes the energy, averaged over one wavelength of the charge modulation, and explain the physical content of this result.

2. *Electrostatic contributions to elastic energy of surfaces.*
(a) Using Debye-Hückel theory, calculate the electrostatic potential inside and outside of an infinite cylindrical shell of radius R and infinitesimal thickness, and a spherical shell of radius R and negligible thickness, assuming that the charge densities inside and out are $\sigma_{\rm in} = \sigma(1-\epsilon)$ and $\sigma_{\rm out} = \sigma(1+\epsilon)$, where ϵ is a parameter that determines the charge asymmetry.
(b) By comparing the energies from (a), along with that of a plane with the same charge densities $\sigma_{\rm in}$ and $\sigma_{\rm out}$, in the regime such that $\kappa R \gg 1$, where R is the cylinder or sphere radius and κ is the inverse screening length, deduce the electrostatic contribution to the elastic modulus k_c, Gaussian curvature modulus $\bar{k}_c$, and spontaneous curvature H_0 in the Helfrich elastic energy

$$\mathcal{E} = \frac{1}{2}k_c\int dS\,(H-H_0)^2 + \frac{1}{2}\bar{k}_c\int dSK,$$

where $H = (1/R_1+1/R_2)/2$ is the mean curvature and $K = 1/R_1R_2$ is the Gaussian curvature (with R_1 and R_2 the principal radii of curvature).
(c) Estimate k_c, $\bar{k}_c$, and the spontaneous radius $R_0 = 1/H_0$ for typical values of σ.

3. *Brownian motion with inertia.* Here we generalize the Langevin equation discussed in lecture to a particle with inertia.
Consider the Langevin equation for a single particle of mass m, drag coefficient γ and random forcing $\mathbf{A}'(t)$,

$$m\frac{d\mathbf{u}}{dt} = -\gamma\mathbf{u} + \mathbf{A}'(t). \tag{1}$$

Assume the random force has zero mean and a variance $< \mathbf{A}'(t)\cdot\mathbf{A}'(t') >$ that is a function $\phi(|t-t'|)$ decaying very rapidly with $t-t'$, satisfying $\int_{-\infty}^{\infty} dy\phi(y) = m^2\tau$. If $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ are the initial velocity and position, solve (1) to obtain $\mathbf{U} \equiv \mathbf{u}(t) - \mathbf{u}_0{\rm e}^{-\zeta t}$ formally in terms of $\mathbf{A}$, where $\zeta = \gamma/m$ and $\mathbf{A} = \mathbf{A}'/m$. From this deduce the variance $\langle U^2\rangle$ and thereby determine τ from equipartition.

In order to evaluate higher moments of $\mathbf{U}$, assume that the random process $A(t)$ is Gaussian, so $\langle A(t_1)A(t_2)\cdots A(t_{2n+1})\rangle = 0$, and

$$\langle A(t_1)A(t_2)\cdots A(t_{2n})\rangle = \sum_{\text{all pairs}} \langle A(t_i)A(t_j)\rangle\langle A(t_k)A(t_l)\rangle\cdots.$$

Considering carefully the number of pairs in the above sum, show that the moments satisfy

$$\langle U^{2n+1}\rangle = 0 \qquad \langle U^{2n}\rangle = (2n-1)!!\langle U^2\rangle^n,$$

and hence that the probability distribution of $\mathbf{U}$ is Gaussian,

$$W(\mathbf{u},t;\mathbf{u}_0) = \left[\frac{m}{2\pi k_BT(1-{\rm e}^{-2\zeta t})}\right]^{3/2} \exp\left[-\frac{m|\mathbf{u}-\mathbf{u}_0{\rm e}^{-\zeta t}|^2}{2k_BT(1-{\rm e}^{-2\zeta t})}\right].$$

Integrate the equation for $\mathbf{u}$ to obtain the position vector $\mathbf{r}$. Find the mean and variance of $\mathbf{r}$. Examine the short and long-time behaviour and explain the distinction between the two.

4. *The wormlike chain.* A wormlike polymer of contour length L is subject to an external force f acting at its two ends, directed along the z axis. The configurational energy of the polymer is

$$\mathcal{E} = \frac{1}{2} A \int_0^L ds \kappa^2 - fz,$$

where A is the bending modulus, κ is the curvature, and z is the end-to-end extension.
(a) Consider the high-force limit, where the chain's configuration deviates only slightly from a straight line. Then the tangent vector $\hat{\mathbf{t}}$ fluctuates only slightly around $\hat{\mathbf{z}}$, the unit vector in the z direction. If we take t_x and t_y as independent fluctuating components, the constraint $|\hat{\mathbf{t}}| = 1$ shows that t_z deviates from unity quadratically in the vector $\mathbf{t}_\perp \equiv (t_x, t_y)$. Show that to quadratic order

$$\mathcal{E} \simeq \frac{1}{2} \int ds \left[A(\partial_s \mathbf{t}_\perp)^2 + f \mathbf{t}_\perp^2 \right] - fL.$$

(b) Use equipartition to find the thermal average $\langle \mathbf{t}_\perp^2 \rangle$, being careful to account for the two independent components of $\mathbf{t}_\perp$. From this, show that in this high-force limit the force-extension relation takes the form

$$\frac{z}{L} = 1 - \frac{k_B T}{\sqrt{4fA}}.$$

Compare this asymptotic result with that for the freely-jointed chain composed of N links, each of length b.