ratio is 8 per cent. at ordinary temperatures which remains constant till about 120°, and then falls off, rapidly at first and then slowly to about 1·2 per cent. at the critical point. There is no change of imperfection of polarisation on passing through the critical point. The correction due to this in the expression for the intensity of the scattered light is given.

In conclusion, I have great pleasure in recording my indebtedness to Prof. C. V. Raman who suggested the research, and who continued to take an inspiring interest in its progress. The experimental work was carried out in the Physical Laboratory of the Indian Association for the Cultivation of Science, Calcutta.

The Motion of Ellipsoidal Particles Immersed in a Viscous Fluid.

By G. B. Jeffery, M.A., D.Sc., Fellow of University College, London.

(Communicated by Prof. L. N. G. Filon, F.R.S. Received January 30, 1922.)

§ 1. *Introduction.*

In both physical and biological science, we are often concerned with the properties of a fluid, or plasma, in which small particles or corpuscles are suspended and carried about by the motion of the fluid. The presence of the particles will influence the properties of the suspension in bulk, and, in particular, its viscosity will be increased. The most complete mathematical treatment of the problem, from this point of view, has been that given by Einstein,* who considered the case of spherical particles and gave a simple formula for the increase in the viscosity. We have extended this work to the case of particles of ellipsoidal shape.

The second section of the paper is occupied with the requisite solution of the equations of motion of the fluid. The problem of the motion of a viscous fluid, due to an ellipsoid moving through it with a small velocity of translation in a direction parallel to one of its axes, has been solved by Oberbeck,† and the corresponding problem for an ellipsoid rotating about one of its axes by Edwards.‡ In both cases the equations of motion are approximated by neglecting the terms involving the squares of the velocities. It may be seen,

* "Eine neue Bestimmung der Moleküldimensionen," 'Ann. d. Physik,' vol. 19, p. 289 (1896); with a correction in vol. 34, p. 591 (1911).

† 'Crelle,' vol. 81 (1876).

‡ 'Quart. Jour. Math.,' vol. 26 (1892).

a posteriori, that the condition for the validity of this approximation is that the product of the velocity of the ellipsoid by its linear dimensions shall be small compared with the "kinematic coefficient of viscosity" of the fluid. In relation to our present problem, it will therefore be satisfied either for sufficiently slow motions, *or* for sufficiently small particles.

The third section is devoted to the consideration of the forces acting upon the particle as a whole. It is found that these reduce to two couples, one tending to make the particle adopt the same rotation as the surrounding fluid, and the other tending to set the particle with its axes parallel to the principal axes of distortion of the surrounding fluid.

In the fourth section we investigate the motion of a particle which is subject to no forces except those arising from the pressure of the fluid on its surface. It is found that there is a variety of possible motions corresponding to different initial conditions. They are periodic and of a general character somewhat like the motion of a top. There is thus a certain degree of indeterminacy in the problem as treated by the approximation which neglects the squares of the velocities.

Sections five and six are concerned with the dissipation of energy and with the viscosity of suspensions. It is found that the increase of viscosity may still be represented by a formula of Einstein's type with a modified numerical factor. Owing to the indeterminacy revealed in section four, it is not possible to give a definite value for this numerical factor. It is, however, possible to specify an upper and a lower limit for this factor, and the difference between these limits diminishes to zero as the particles approach a spherical shape.

Section seven is devoted to a tentative suggestion for the removal of this indeterminacy. It is, that when a variety of motions is possible for given boundary conditions under the approximated equations of motion, the actual motion will be that which corresponds to minimum dissipation of energy. If this hypothesis be granted, it is possible to state many of the results of the paper with greater precision.

§ 2. *The Solution of the Equations of Motion.*

Consider, in the first instance, the modification produced in the motion of a viscous fluid by the presence of a single ellipsoidal particle. We shall assume that, apart from the disturbance produced in the immediate neighbourhood of the particle, the motion is steady, and varies in space on a scale which is large compared with the dimensions of the particle. It appears from the work of Oberbeck that, if the fluid at a distance from the particle has a velocity of translation relative to the particle, the latter is acted upon by a resultant force. It will appear in the course of the investigation that any other motion of the fluid can only give rise to a resultant couple. The particle will therefore ultimately assume the velocity of translation appropriate to that part of the fluid which it displaces. On the above assumptions this velocity of translation will be sensibly uniform, and we may therefore take the centre of the particle as at rest.

Let x, y, z be rectangular co-ordinates referred to axes fixed in the particle and moving with it, and let the surface of the particle be the ellipsoid

$$x^2/a^2+y^2/b^2+z^2/c^2 = 1. \tag{1}$$

The undisturbed motion of the fluid in the neighbourhood of the particle is then given by

$$\left.\begin{aligned} u_0 &= \mathbf{a}x+\mathbf{h}y+\mathbf{g}z+\eta z-\zeta y \\ v_0 &= \mathbf{h}x+\mathbf{b}y+\mathbf{f}z+\zeta x-\xi y \\ w_0 &= \mathbf{g}x+\mathbf{f}y+\mathbf{c}z+\xi y-\eta x \end{aligned}\right\}, \tag{2}$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$, ξ, η, ζ are the components of distortion and rotation of the fluid. These are taken to be constant in space through a volume which is large compared with the dimensions of the particle. They will, however, vary with the time, since the particle will rotate under the influence of the fluid, and the axes of x, y, z will be rotating axes.

Let the spins of the ellipsoid about its axes be $\omega_1, \omega_2, \omega_3$. Take a set of axes, x', y', z', in fixed directions with their origin at the centre of the particle, and let the components of distortion and rotation of the fluid in its undisturbed motion referred to these axes be $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, ... These are constant, but the corresponding unaccented quantities are linear functions of these, with coefficients which depend upon the direction cosines of the two sets of axes, and which therefore vary with the time in consequence of $\omega_1, \omega_2, \omega_3$.

Neglecting squares and products of the velocities, the equations of motion of an incompressible viscous fluid referred to moving axes are of the type

$$\mu\nabla^2 u-\frac{\partial p}{\partial x} = \rho\left\{\frac{\partial u}{\partial t}-\omega_3 v+\omega_2 w\right\}, \tag{3}$$

where ρ, μ, p are respectively the density, coefficient of viscosity, and mean pressure of the fluid.

The spins $\omega_1, \omega_2, \omega_3$ are produced by the motion of the fluid, and, as will appear explicitly later, they are of the same order as the velocities. Hence to our order of approximation all products like $\omega_3 v$ may be neglected. It is not quite so obvious that the remaining terms on the right-hand side of (3) are of the second order. It will appear later that u, v, w are homogeneous linear functions of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ..., with constant coefficients. They are, therefore, homogeneous linear functions of the constants $\mathbf{a}'$, $\mathbf{b}'$, $\mathbf{c}'$, ..., with coefficients which are functions of the direction cosines of the moving axes. Hence the only terms which can arise in $\partial u/\partial t$ are of the type $\omega_1 \mathbf{a}'$, which is of the second order of small quantities, and is therefore to be neglected.

The equations of motion therefore reduce to

$$\mu\nabla^2 u = \frac{\partial p}{\partial x}, \qquad \mu\nabla^2 v = \frac{\partial p}{\partial y}, \qquad \mu\nabla^2 w = \frac{\partial p}{\partial z}, \tag{4}$$

with the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0. \tag{5}$$

We have to find a solution of (4) and (5) which agrees with (2) at great distances from the origin, and which gives on the surface of the ellipsoid

$$u = \omega_2 z-\omega_3 y, \qquad v = \omega_3 x-\omega_1 z, \qquad w = \omega_1 y-\omega_2 x. \tag{6}$$

As usual, let λ denote the positive root of

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda} = 1, \tag{7}$$

and

$$\Delta = \{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)\}^{\frac{1}{2}}, \tag{8}$$

and let

$$\alpha = \int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)\,\Delta}, \qquad \beta = \int_\lambda^\infty \frac{d\lambda}{(b^2+\lambda)\,\Delta}, \qquad \gamma = \int_\lambda^\infty \frac{d\lambda}{(c^2+\lambda)\,\Delta}. \tag{9}$$

We will also write

$$\alpha' = \int_\lambda^\infty \frac{d\lambda}{(b^2+\lambda)\,(c^2+\lambda)\,\Delta}, \qquad \alpha'' = \int_\lambda^\infty \frac{\lambda\, d\lambda}{(b^2+\lambda)\,(c^2+\lambda)\,\Delta}, \tag{10}$$

with symmetrical integrals for β', β'', γ', γ'', so that

$$\alpha' = \frac{\gamma-\beta}{b^2-c^2}, \qquad \alpha'' = \frac{b^2\beta-c^2\gamma}{b^2-c^2}. \tag{11}$$

We will denote the corresponding integrals, in which the lower limit of integration has been replaced by zero, by α_0, β_0', ...

We have the following well-known solutions of Laplace's equation :—

$$\Omega = \int_\lambda^\infty \left\{\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1\right\}\frac{d\lambda}{\Delta} \tag{12}$$

and

$$\chi_1 = \alpha' yz, \qquad \chi_2 = \beta' zx, \qquad \chi_3 = \gamma' xy. \tag{13}$$

We shall seek a solution in which u, v, w are expressed as linear functions of the first and second differential coefficients of Ω and the first differential coefficients of χ_1, χ_2, χ_3. These latter can, of course, be expressed in terms of the second differential coefficients of Ω, but as the corresponding terms enter into the solution in a somewhat different way, it will be convenient to keep them separate.

From (7) we have

$$\frac{\partial\lambda}{\partial x}=\frac{2x\mathrm{P}^2}{a^2+\lambda},\qquad \frac{\partial\lambda}{\partial y}=\frac{2y\mathrm{P}^2}{b^2+\lambda},\qquad \frac{\partial\lambda}{\partial z}=\frac{2z\mathrm{P}^2}{c^2+\lambda} \tag{14}$$

where

$$\frac{1}{\mathrm{P}^2}=\frac{x^2}{(a^2+\lambda)^2}+\frac{y^2}{(b^2+\lambda)^2}+\frac{z^2}{(c^2+\lambda)^2}. \tag{15}$$

We then have a number of formulæ of which the following are typical:

$$\frac{\partial\Omega}{\partial x}=2\alpha x,\qquad \frac{\partial^2\Omega}{\partial x^2}=2\alpha-\frac{4x^2\mathrm{P}^2}{(a^2+\lambda)^2\Delta},\qquad \frac{\partial^2\Omega}{\partial y\,\partial z}=-\frac{4yz\mathrm{P}^2}{(b^2+\lambda)(c^2+\lambda)\Delta}, \tag{16}$$

and

$$\frac{\partial\chi_1}{\partial x}=\frac{\partial\chi_2}{\partial y}=\frac{\partial\chi_3}{\partial z}=-\frac{2xyz\mathrm{P}^2}{\Delta^3},$$

$$\frac{\partial\chi_1}{\partial y}=\alpha' z-\frac{2y^2z\mathrm{P}^2}{(b^2+\lambda)^2(c^2+\lambda)\Delta},\qquad \frac{\partial\chi_1}{\partial z}=\alpha' y-\frac{2yz^2\mathrm{P}^2}{(b^2+\lambda)(c^2+\lambda)^2\Delta}. \tag{17}$$

All these differential coefficients tend to zero at infinity.

We assume

$$\begin{aligned}
u=u_0&+\frac{\partial}{\partial x}(\mathrm{R}\chi_1+\mathrm{S}\chi_2+\mathrm{T}\chi_3)+\mathrm{W}\frac{\partial\chi_3}{\partial y}-\mathrm{V}\frac{\partial\chi_2}{\partial z}\\
&+\mathrm{A}\left(x\frac{\partial^2\Omega}{\partial x^2}-\frac{\partial\Omega}{\partial x}\right)+\mathrm{H}\left(x\frac{\partial^2\Omega}{\partial x\,\partial y}-\frac{\partial\Omega}{\partial y}\right)+\mathrm{G}'\left(x\frac{\partial^2\Omega}{\partial x\,\partial z}-\frac{\partial\Omega}{\partial z}\right)\\
&+y\left(\mathrm{H}'\frac{\partial^2\Omega}{\partial x^2}+\mathrm{B}\frac{\partial^2\Omega}{\partial x\,\partial y}+\mathrm{F}\frac{\partial^2\Omega}{\partial x\,\partial z}\right)\\
&+z\left(\mathrm{G}\frac{\partial^2\Omega}{\partial x^2}+\mathrm{F}'\frac{\partial^2\Omega}{\partial x\,\partial y}+\mathrm{C}\frac{\partial^2\Omega}{\partial x\,\partial z}\right),
\end{aligned} \tag{18}$$

$$\begin{aligned}
v=v_0&+\frac{\partial}{\partial y}(\mathrm{R}\chi_1+\mathrm{S}\chi_2+\mathrm{T}\chi_3)+\mathrm{U}\frac{\partial\chi_1}{\partial z}-\mathrm{W}\frac{\partial\chi_3}{\partial x}\\
&+x\left(\mathrm{A}\frac{\partial^2\Omega}{\partial x\,\partial y}+\mathrm{H}\frac{\partial^2\Omega}{\partial y^2}+\mathrm{G}'\frac{\partial^2\Omega}{\partial y\,\partial z}\right)\\
&+\mathrm{H}'\left(y\frac{\partial^2\Omega}{\partial x\,\partial y}-\frac{\partial\Omega}{\partial x}\right)+\mathrm{B}\left(y\frac{\partial^2\Omega}{\partial y^2}-\frac{\partial\Omega}{\partial y}\right)+\mathrm{F}\left(y\frac{\partial^2\Omega}{\partial y\,\partial z}-\frac{\partial\Omega}{\partial z}\right)\\
&+z\left(\mathrm{G}\frac{\partial^2\Omega}{\partial x\,\partial y}+\mathrm{F}'\frac{\partial^2\Omega}{\partial y^2}+\mathrm{C}\frac{\partial^2\Omega}{\partial y\,\partial z}\right),
\end{aligned} \tag{19}$$

and

$$\begin{aligned}
w=w_0&+\frac{\partial}{\partial z}(\mathrm{R}\chi_1+\mathrm{S}\chi_2+\mathrm{T}\chi_3)+\mathrm{V}\frac{\partial\chi_2}{\partial x}-\mathrm{U}\frac{\partial\chi_1}{\partial y}\\
&+x\left(\mathrm{A}\frac{\partial^2\Omega}{\partial x\,\partial z}+\mathrm{H}\frac{\partial^2\Omega}{\partial y\,\partial z}+\mathrm{G}'\frac{\partial^2\Omega}{\partial z^2}\right)\\
&+y\left(\mathrm{H}'\frac{\partial^2\Omega}{\partial x\,\partial z}+\mathrm{B}\frac{\partial^2\Omega}{\partial y\,\partial z}+\mathrm{F}\frac{\partial_2\Omega}{\partial z^2}\right)\\
&+\mathrm{G}\left(z\frac{\partial^2\Omega}{\partial x\,\partial z}-\frac{\partial\Omega}{\partial x}\right)+\mathrm{F}'\left(z\frac{\partial^2\Omega}{\partial y\partial z}-\frac{\partial\Omega}{\partial y}\right)+\mathrm{C}\left(z\frac{\partial^2\Omega}{\partial z^2}-\frac{\partial\Omega}{\partial z}\right).
\end{aligned} \tag{20}$$

It may be seen by inspection that these values of u, v, w satisfy (5), since $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ for an incompressible fluid, and Ω and χ_1, χ_2, χ_3 satisfy Laplace's equation.

Substituting these expressions into (4), it is easily seen that these are satisfied provided that

$$p = p_0+2\mu\left\{\mathrm{A}\frac{\partial^2\Omega}{\partial x^2}+\mathrm{B}\frac{\partial^2\Omega}{\partial y^2}+\mathrm{C}\frac{\partial^2\Omega}{\partial z^2}+(\mathrm{F}+\mathrm{F}')\frac{\partial^2\Omega}{\partial y\,\partial z}\right.$$
$$\left.+(\mathrm{G}+\mathrm{G}')\frac{\partial^2\Omega}{\partial z\,\partial x}+(\mathrm{H}+\mathrm{H}')\frac{\partial^2\Omega}{\partial x\,\partial y}\right\}, \quad (21)$$

where p_0 is the constant mean pressure at a distance from the ellipsoid. It therefore remains to satisfy the conditions (6) on the surface of the ellipsoid $\lambda=0$. Substituting from (16) and (17) into (18), (19), (20), we have

$$\begin{aligned} u = {} & x\{\mathbf{a}+\gamma'\mathrm{W}-\beta'\mathrm{V}-2(\alpha+\beta+\gamma)\mathrm{A}\} \\ & +y\{\mathbf{h}-\zeta+\gamma'\mathrm{T}-2\beta\mathrm{H}+2\alpha\mathrm{H}'\} \\ & +z\{\mathbf{g}+\eta+\beta'\mathrm{S}-2\gamma\mathrm{G}'+2\alpha\mathrm{G}\} \\ & -\frac{2x\mathrm{P}^2}{(a^2+\lambda)\Delta}\Big[\{\mathrm{R}+2(b^2+\lambda)\mathrm{F}+2(c^2+\lambda)\mathrm{F}'\}yz/(b_2+\lambda)(c^2+\lambda) \\ & \qquad +\{\mathrm{S}+2(c^2+\lambda)\mathrm{G}+2(a^2+\lambda)\mathrm{G}'\}zx/(c^2+\lambda)(a^2+\lambda) \\ & \qquad +\{\mathrm{T}+2(a^2+\lambda)\mathrm{H}+2(b^2+\lambda)\mathrm{H}'\}xy/(a^2+\lambda)(b^2+\lambda) \\ & \qquad +\{\mathrm{W}-2(a^2+\lambda)\mathrm{A}+2(b^2+\lambda)\mathrm{B}\}y^2/(b^2+\lambda)^2 \\ & \qquad -\{\mathrm{V}-2(c^2+\lambda)\mathrm{C}+2(a^2+\lambda)\mathrm{A}\}z^2/(c^2+\lambda)^2\Big], \end{aligned} \quad (22)$$

$$\begin{aligned} v = {} & x\{\mathbf{h}+\zeta+\gamma'\mathrm{T}+2\beta\mathrm{H}-2\alpha\mathrm{H}'\} \\ & +y\{\mathbf{b}+\alpha'\mathrm{U}-\gamma'\mathrm{W}-2(\alpha+\beta+\gamma)\mathrm{B}\} \\ & +z\{\mathbf{f}-\xi+\alpha'\mathrm{R}-2\gamma\mathrm{F}+2\beta\mathrm{F}'\} \\ & -\frac{2y\mathrm{P}^2}{(b^2+\lambda)\Delta}\Big[\{\mathrm{R}+2(b^2+\lambda)\mathrm{F}+2(c^2+\lambda)\mathrm{F}'\}yz/(b^2+\lambda)(c^2+\lambda) \\ & \qquad +\{\mathrm{S}+2(c^2+\lambda)\mathrm{G}+2(a^2+\lambda)\mathrm{G}'\}zx/(c^2+\lambda)(a^2+\lambda) \\ & \qquad +\{\mathrm{T}+2(a^2+\lambda)\mathrm{H}+2(b^2+\lambda)\mathrm{H}'\}xy/(a^2+\lambda)(b^2+\lambda) \\ & \qquad +\{\mathrm{U}-2(b^2+\lambda)\mathrm{B}+2(c^2+\lambda)\mathrm{C}\}z^2/(c^2+\lambda)^2 \\ & \qquad -\{\mathrm{W}-2(a^2+\lambda)\mathrm{A}+2(b^2+\lambda)\mathrm{B}\}x^2/(a^2+\lambda)^2\Big], \end{aligned} \quad (23)$$

and

$$\begin{aligned} w = {} & x\{\mathbf{g}-\eta+\beta'\mathrm{S}-2\alpha\mathrm{G}+2\gamma\mathrm{G}'\} \\ & +y\{\mathbf{f}+\xi+\alpha'\mathrm{R}+2\gamma\mathrm{F}-2\beta\mathrm{F}'\} \\ & +z\{\mathbf{c}+\beta'\mathrm{V}-\alpha'\mathrm{U}-2(\alpha+\beta+\gamma)\mathrm{C}\} \\ & -\frac{2z\mathrm{P}^2}{(c^2+\lambda)\Delta}\Big[\{\mathrm{R}+2(b^2+\lambda)\mathrm{F}+2(c^2+\lambda)\mathrm{F}'\}yz/(b^2+\lambda)(c^2+\lambda) \\ & \qquad +\{\mathrm{S}+2(c^2+\lambda)\mathrm{G}+2(a^2+\lambda)\mathrm{G}'\}zx/(c^2+\lambda)(a^2+\lambda) \\ & \qquad +\{\mathrm{T}+2(a^2+\lambda)\mathrm{H}+2(b^2+\lambda)\mathrm{H}'\}xy/(a^2+\lambda)(b^2+\lambda) \\ & \qquad +\{\mathrm{V}-2(c^2+\lambda)\mathrm{C}+2(a^2+\lambda)\mathrm{A}\}x^2/(a^2+\lambda)^2 \\ & \qquad -\{\mathrm{U}-2(b^2+\lambda)\mathrm{B}+2(c^2+\lambda)\mathrm{C}\}y^2/(b^2+\lambda)^2\Big]. \end{aligned} \tag{24}$$

Putting $\lambda = 0$, identifying with (6), and equating coefficients, we obtain equations which determine the constants. On solution we find their values to be as follows :—

$$\left.\begin{aligned} \mathrm{A} &= \tfrac{1}{6}\{2\alpha_0''\mathbf{a}-\beta_0''\mathbf{b}-\gamma_0''\mathbf{c}\}/(\beta_0''\gamma_0''+\gamma_0''\alpha_0''+\alpha_0''\beta_0'') \\ \mathrm{B} &= \tfrac{1}{6}\{2\beta_0''\mathbf{b}-\gamma_0''\mathbf{c}-\alpha_0''\mathbf{a}\}/(\beta^{0''}\gamma_0''+\gamma_0''\alpha_0''+\alpha_0''\beta_0'') \\ \mathrm{C} &= \tfrac{1}{6}\{2\gamma_0''\mathbf{c}-\alpha_0''\mathbf{a}-\beta_0''\mathbf{b}\}/(\beta_0''\gamma_0''+\gamma_0''\alpha_0''+\alpha_0''\beta_0'') \end{aligned}\right\}, \tag{25}$$

$$\left.\begin{aligned} \mathrm{F} &= \frac{\beta_0\mathbf{f}-c^2\alpha_0(\xi-\omega_1)}{2\alpha_0'(b^2\beta_0+c^2\gamma_0)}, & \mathrm{F}' &= \frac{\gamma_0\mathbf{f}+b^2\alpha_0(\xi-\omega_1)}{2\alpha_0'(b^2\beta_0+c^2\gamma_0)} \\ \mathrm{G} &= \frac{\gamma_0\mathbf{g}-a^2\beta_0(\eta-\omega_2)}{2\beta_0'(c^2\gamma_0+a^2\alpha_0)}, & \mathrm{G}' &= \frac{\alpha_0\mathbf{g}+c^2\beta_0(\eta-\omega_2)}{2\beta_0'(c^2\gamma_0+a^2\alpha_0)} \\ \mathrm{H} &= \frac{\alpha_0\mathbf{h}-b^2\gamma_0(\zeta-\omega_3)}{2\gamma_0'(a^2\alpha_0+b^2\beta_0)}, & \mathrm{H}' &= \frac{\beta_0\mathbf{h}+a^2\gamma_0(\zeta-\omega_3)}{2\gamma_0'(a^2\alpha_0+b^2\beta_0)} \end{aligned}\right\}, \tag{26}$$

$$\mathrm{R} = -\mathbf{f}/\alpha_0', \qquad \mathrm{S} = -\mathbf{g}/\beta_0', \qquad \mathrm{T} = -\mathbf{h}/\gamma_0'. \tag{27}$$

The values of U, V, W will not be needed for our purpose. They are given by

$$\left.\begin{aligned} \mathrm{U} &= 2b^2\mathrm{B}-2c^2\mathrm{C} \\ \mathrm{V} &= 2c^2\mathrm{C}-2a^2\mathrm{A} \\ \mathrm{W} &= 2a^2\mathrm{A}-2b^2\mathrm{B} \end{aligned}\right\}. \tag{28}$$

Thus the constants in (18), (19), and (20) are all uniquely determined and the velocity of the fluid is known at all points.

For later use we note the following relations :—

$$\mathrm{A}+\mathrm{B}+\mathrm{C} = 0, \tag{29}$$

$$\mathrm{U}+\mathrm{V}+\mathrm{W} = 0, \tag{30}$$

and, using the relation $\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$,

$$\mathrm{A}\mathbf{a}+\mathrm{B}\mathbf{b}+\mathrm{C}\mathbf{c} = \tfrac{1}{2}(\alpha_0''\mathbf{a}^2+\beta_0''\mathbf{b}^2+\gamma_0''\mathbf{c}^2)/(\beta_0''\gamma_0''+\gamma_0''\alpha_0''+\alpha_0''\beta_0''). \tag{31}$$

§ 3. *The Resultant Couple on the Particle.*

For an incompressible fluid the stresses are given, with the usual notation, by

$$\widehat{xx} = -p + 2\mu \frac{\partial u}{\partial x}, \qquad \widehat{yz} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right). \tag{32}$$

The calculation of the values of these stresses on the surface of the ellipsoid is greatly simplified by noting that most of the brackets in (22), (23), and (24) vanish for $\lambda = 0$, and that, therefore, the only terms surviving in the surface values of the differential coefficients of the velocity components are those arising from the differentiation of the brackets.

If we denote the components of force per unit area acting on the surface of the ellipsoid by X, Y, Z, we have

$$\left.\begin{aligned} \mathrm{X} &= \{\widehat{xx}\, x/a^2 + \widehat{xy}\, y/b^2 + \widehat{xz}\, z/c^2\} \\ \mathrm{Y} &= \{\widehat{yx}\, x/a^2 + \widehat{yy}\, y/b^2 + \widehat{yz}\, z/c^2\} \\ \mathrm{Z} &= \{\widehat{zx}\, x/a^2 + \widehat{zy}\, y/b^2 + \widehat{zz}\, z/c^2\} \end{aligned}\right\}. \tag{33}$$

The reduction is straightforward and, after simplifying as far as possible by means of the relations between the constants, we obtain

$$\begin{aligned} \mathrm{X} &= -p_0 \mathrm{P} \frac{x}{a^2} + \frac{8\mu \mathrm{P}}{abc} \left\{ \mathrm{A} \frac{x}{a^2} + \mathrm{H} \frac{y}{b^2} + \mathrm{G}' \frac{z}{c^2} \right\} - 4\mu \mathrm{P} (\alpha_0 \mathrm{A} + \beta_0 \mathrm{B} + \gamma_0 \mathrm{C}) \frac{x}{a^2}, \\ \mathrm{Y} &= -p_0 \mathrm{P} \frac{y}{b^2} + \frac{8\mu \mathrm{P}}{abc} \left\{ \mathrm{H}' \frac{x}{a^2} + \mathrm{B} \frac{y}{b^2} + \mathrm{F} \frac{z}{c^2} \right\} - 4\mu \mathrm{P} (\alpha_0 \mathrm{A} + \beta_0 \mathrm{B} + \gamma_0 \mathrm{C}) \frac{y}{b^2}, \\ \mathrm{Z} &= -p_0 \mathrm{P} \frac{z}{c^2} + \frac{8\mu \mathrm{P}}{abc} \left\{ \mathrm{G} \frac{x}{a^2} + \mathrm{F}' \frac{y}{b^2} + \mathrm{C} \frac{z}{c^2} \right\} - 4\mu \mathrm{P} (\alpha_0 \mathrm{A} + \beta_0 \mathrm{B} + \gamma_0 \mathrm{C}) \frac{z}{c^2}. \end{aligned} \tag{34}$$

Since, apart from P, which is even in x, y, z, only first powers of the co-ordinates appear in these expressions, they will separately vanish on integration over the surface of the ellipsoid. Hence there is no resultant *force* acting on the particle. Denoting the components of the resultant couple acting on the particle by L, M, N, we have on integration over the surface of the ellipsoid

$$\mathrm{L} = \iint (y\mathrm{Z} - z\mathrm{Y})\, d\mathrm{S},$$

which on substitution from (34), omitting terms which obviously vanish on integration, gives

$$\mathrm{L} = \frac{8\mu}{abc} \iint \left(\mathrm{F}' \frac{y^2}{b^2} - \mathrm{F} \frac{z^2}{c^2} \right) d\mathrm{S}.$$

This integral is readily evaluated in terms of the volume of the ellipsoid, and we have from this and two similar expressions

$$L = \tfrac{32}{3}\pi\mu(F'-F), \quad M = \tfrac{32}{3}\pi\mu(G'-G), \quad N = \tfrac{32}{3}\pi\mu(H'-H). \quad (35)$$

Substituting from (26), these give

$$\left.\begin{aligned} L &= \frac{16\pi\mu}{3(b^2\beta_0+c^2\gamma_0)}\{(b^2-c^2)\mathbf{f}+(b^2+c^2)(\xi-\omega_1)\}, \\ M &= \frac{16\pi\mu}{3(c^2\gamma_0+a^2\alpha_0)}\{(c^2-a^2)\mathbf{g}+(c^2+a^2)(\eta-\omega_2)\}, \\ N &= \frac{16\pi\mu}{3(a^2\alpha_0+b^2\beta_0)}\{(a^2-b^2)\mathbf{h}+(a^2+b^2)(\zeta-\omega_3)\}. \end{aligned}\right\} \quad (36)$$

The resultant couple acting upon the ellipsoid may thus be regarded as consisting of two parts: one which vanishes when the ellipsoid has the same resultant spin as the fluid, and one which vanishes when the axes of the ellipsoid coincide with the principal axes of distortion of the fluid.

§ 4. *The Motion of the Particle—Laminar Motion.*

If the particle is subject to no forces except those exerted by the fluid upon its surface, then in the slow motions to which our assumptions have already restricted us, the resultant couple on the particle must vanish at every instant. Hence, from (36),

$$\left.\begin{aligned} (b^2+c^2)\omega_1 &= b^2(\xi+\mathbf{f})+c^2(\xi-\mathbf{f}) \\ (c^2+a^2)\omega_2 &= c^2(\eta+\mathbf{g})+a^2(\eta-\mathbf{g}) \\ (a^2+b^2)\omega_3 &= a^2(\zeta+\mathbf{h})+b^2(\zeta-\mathbf{h}) \end{aligned}\right\}, \quad (37)$$

or, from (35),

$$F' = F, \qquad G' = G, \qquad H' = H. \quad (38)$$

These relations will simplify many of our formulæ and, in particular, we may note that (26) are replaced by

$$F = \frac{\mathbf{f}}{2\alpha_0'(b^2+c^2)}, \qquad G = \frac{\mathbf{g}}{2\beta_0'(c^2+a^2)}, \qquad H = \frac{\mathbf{h}}{2\gamma_0'(a^2+b^2)}. \quad (39)$$

It has not been found possible to obtain a solution of equations (37) which would give the motion of the particle for the most general possible undisturbed motion of the fluid. Some light may, however, be thrown on the problem by the consideration of a particular, but important, case.

Take axes x', y', z', fixed in direction, and let the components of the fluid velocity in the undisturbed motion be u_0', v_0', w_0', parallel to these axes. Consider the "laminar motion" given by $u_0' = v_0' = 0$, $w_0' = \kappa y'$, where κ is a constant. Let the direction cosines of the axes of the ellipsoid (*i.e.*, the axes x, y, z) referred to the axes of x', y', z' be (l_1, m_1, n_1), (l_2, m_2, n_2) and

(l_3, m_3, n_3) respectively. Then employing the usual formulæ for the transformation of axes, we obtain

$$\begin{array}{lll} \mathbf{a} = \kappa m_1 n_1, & \mathbf{b} = \kappa m_2 n_2, & \mathbf{c} = \kappa m_3 n_3, \\ \mathbf{f} = \tfrac{1}{2}\kappa (m_2 n_3 + m_3 n_2), & \mathbf{g} = \tfrac{1}{2}\kappa (m_3 n_1 + m_1 n_3), & \mathbf{h} = \tfrac{1}{2}\kappa (m_1 n_2 + m_2 n_1), \\ \xi = \tfrac{1}{2}\kappa (m_2 n_3 - m_3 n_2), & \eta = \tfrac{1}{2}\kappa (m_3 n_1 - m_1 n_3), & \zeta = \tfrac{1}{2}\kappa (m_1 n_2 - m_2 n_1), \end{array} \quad (40)$$

and equations (37) become

$$\left.\begin{aligned} (b^2+c^2)\,\omega_1 &= \kappa\,(b^2 m_2 n_3 - c^2 m_3 n_2) \\ (c^2+a^2)\,\omega_2 &= \kappa\,(c^2 m_3 n_1 - a^2 m_1 n_3) \\ (a^2+b^2)\,\omega_3 &= \kappa\,(a^2 m_1 n_2 - b^2 m_2 n_1) \end{aligned}\right\}. \quad (41)$$

These may be regarded as the equations of motion of the ellipsoid and give its spins for any position of its axes. It may be seen from these that a possible type of motion is that in which any one of the three axes of the ellipsoid lies permanently along the axis of x', *i.e.*, perpendicular both to the direction of the velocity and to that of the velocity gradient in the undisturbed motion, while the ellipsoid rotates about this axis with a variable spin. A typical motion of this class is that for which

$$l_1, m_1, n_1 = 1, 0, 0; \qquad l_2, m_2, n_2 = 0, \cos\chi, \sin\chi;$$
$$l_3, m_3, n_3 = 0, -\sin\chi, \cos\chi; \qquad \omega_1 = \dot{\chi}; \qquad \omega_2 = \omega_3 = 0. \quad (42)$$

Equations (41) then give

$$(b^2+c^2)\,\dot{\chi} = \kappa\,(b^2\cos^2\chi + c^2\sin^2\chi),$$

or, omitting a constant of integration which is obviously immaterial,

$$\tan\chi = \frac{b}{c}\tan\frac{bc\kappa t}{b^2+c^2}. \quad (43)$$

The equations readily admit of integration in the case of an ellipsoid of revolution. Let $b = c$, and introduce the three Euler angles, θ the angle between the axes of x and x', ϕ the angle between the planes of $x'y'$ and $x'x$, ψ the angle between the planes of $x'x$ and xy. Then

$$\omega_1 = \dot{\phi}\cos\theta + \dot{\psi}, \quad \omega_2 = \dot{\theta}\sin\psi - \dot{\phi}\sin\theta\cos\psi, \quad \omega_3 = \dot{\theta}\cos\psi + \dot{\phi}\sin\theta\sin\psi,$$

and

$$\left.\begin{aligned} m_1 &= \sin\theta\cos\phi, \qquad n_1 = \sin\theta\sin\phi \\ m_2 &= -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi \\ n_2 &= \cos\phi\sin\psi + \cos\theta\sin\phi\cos\psi \\ m_3 &= -\sin\phi\cos\psi - \cos\theta\cos\phi\sin\psi \\ n_3 &= \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi \end{aligned}\right\}. \quad (44)$$

Substituting in (41), we have after some reduction,

$$\dot{\phi}\cos\theta + \dot{\psi} = \tfrac{1}{2}\kappa\cos\theta, \tag{45}$$

$$(a^2+b^2)\,\dot{\theta} = \kappa\,(a^2-b^2)\sin\theta\cos\theta\sin\phi\cos\phi, \tag{46}$$

$$(a^2+b^2)\,\dot{\phi} = \kappa\,(a^2\cos^2\phi + b^2\sin^2\phi). \tag{47}$$

From (47) we have, if $t = 0$ when $\phi = 0$,

$$\tan\phi = \frac{a}{b}\tan\frac{\kappa abt}{a^2+b^2}, \tag{48}$$

and dividing (46) by (47) and integrating,

$$\tan^2\theta = \frac{a^2b^2}{k^2(a^2\cos^2\phi + b^2\sin^2\phi)}, \tag{49}$$

where k is a constant of integration.

It appears that the motion of the ellipsoid is periodic. Its axis of revolution describes a cone about the perpendicular to the plane of the undisturbed motion, to which its inclination varies between the limits $\tan^{-1}(a/k)$ and $\tan^{-1}(b/k)$.

This statement requires modification in the degenerate cases of a disc and a thin rod respectively. Putting $a = 0$ in equations (45), (46), and (47), we find that a disc can take any position in which its faces are entirely composed of stream-lines of the undisturbed motion of the fluid. It thus moves through the fluid edge-on with its axis at any inclination to the fluid shear. If we put $b = c = 0$ in the same equations we find what is otherwise obvious, that a thin rod can set itself in any direction in a plane which is a plane of constant velocity for the undisturbed motion of the fluid. In what follows, however, we shall regard rods and discs as spheroids of small but finite thickness.

This investigation reveals no tendency on the part of the ellipsoid to set its axis in any particular direction with regard to the undisturbed motion of the fluid. The motions corresponding to all values of k are consistent with the boundary conditions and with the approximated equations of motion.

§ 5. *The Dissipation of Energy.*

As the velocity of the fluid is finite at great distances from the ellipsoid, the total rate of dissipation of energy will be infinite. We will accordingly calculate the excess of the rate of dissipation of energy over that which obtains in the same motion of the fluid in the absence of the ellipsoidal particle. Difficulties which arise in connection with the boundary conditions, make it imperative that we should define precisely the two fluid motions which we are thus comparing.

Describe a sphere S, of large but finite radius R, whose centre is at the centre of the particle. Let the velocity of the fluid at points on S be given by equations (2), and let the ellipsoid move freely under the fluid pressures on its surface so that relations (38) are satisfied. We will calculate the excess of the rate of dissipation of energy over that which would occur if the ellipsoid were removed, the sphere S completely filled with fluid, and the velocity at all points on its surface the same as in the first case.

The direct calculation of the rate of dissipation of energy presents very great difficulties. For our purpose, it will be sufficient to consider the rate at which work must be done over the boundary of the sphere S in order to maintain the motion. This, of course, differs from the rate of dissipation of energy by the rate of increase of the kinetic energy of the fluid. For the periodic motions investigated in the last section, however, the average rate of work of maintaining the motion must be the same as the average rate of dissipation of energy.

As we shall ultimately suppose the radius of the sphere S to tend to infinity, we may adopt such a degree of approximation that negative powers of R are neglected in the final result. This implies that the stresses must be correct to order R^{-3} and the velocities to order R^{-2}.

At great distances from the origin we have, neglecting terms of order r^{-3},

$$\Omega = -\tfrac{4}{3} 1/r, \qquad \chi_1 = \chi_2 = \chi_3 = 0. \tag{50}$$

Substituting in (18), (19), (20), and remembering that (38) are satisfied, and writing

$$\Phi \equiv Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy, \tag{51}$$

we have, neglecting terms of order r^{-4},

$$u = u_0-4x\Phi/r^5, \qquad v = v_0-4y\Phi/r^5, \qquad w = w_0-4z\Phi/r^5, \tag{52}$$

while (21) gives

$$p = p_0-8\mu\Phi/r^5. \tag{53}$$

Superpose a motion, finite at all points within the sphere S, and such as to cancel the second terms in the right-hand members of equations (52) at points on its surface. Such a motion is easily found by the ordinary spherical harmonic methods, and when combined with the motion represented by (52), we have,

$$\left.\begin{aligned} u &= u_0-\frac{4x(R^5-r^5)}{R^5r^5}\Phi+\frac{5(R^2-r^2)}{R^5}\frac{\partial\Phi}{\partial x} \\ v &= v_0-\frac{4y(R^5-r^5)}{R^5r^5}\Phi+\frac{5(R^2-r^2)}{R^5}\frac{\partial\Phi}{\partial y} \\ w &= w_0-\frac{4z(R^5-r^5)}{R^5r^5}\Phi+\frac{5(R^2-r^2)}{R^5}\frac{\partial\Phi}{\partial z} \end{aligned}\right\}, \tag{54}$$

which give on the surface of S

$$p = p_0-50\mu\Phi/R^5. \tag{55}$$

These velocities will not accurately satisfy the boundary conditions on the surface of the ellipsoid, but proceeding by successive approximations it is easily seen that the additional terms will be of order R^{-7} on the surface of S, and may therefore be neglected. The stresses at $r = R$ are calculated from (54) and (55), by the aid of (32). Typical results are

$$\widehat{xx} = -p_0 + 2\mu\mathbf{a} + 10\mu\left\{\frac{5}{R^5}\Phi + \frac{4x^2}{R^7}\Phi - \frac{2x}{R^5}\frac{\partial\Phi}{\partial x}\right\}$$

$$\widehat{yz} = 2\mu\mathbf{f} + 10\mu\left\{\frac{4yz}{R^7}\Phi - \frac{z}{R^5}\frac{\partial\Phi}{\partial y} - \frac{y}{R^5}\frac{\partial\Phi}{\partial z}\right\}. \tag{56}$$

From these we have

$$\widehat{xr} = (x\,\widehat{xx} + y\,\widehat{xy} + z\,\widehat{xz}/r)R$$

$$= -\frac{x}{R}p_0 + \frac{2\mu}{R}(\mathbf{a}x + \mathbf{h}y + \mathbf{g}z) + 10\mu\left\{\frac{7x}{R^6}\Phi - \frac{1}{R^4}\frac{\partial\Phi}{\partial x}\right\}, \tag{57}$$

and two similar equations.

The rate at which work is done in maintaining the motion is given by

$$\iint (u_0\,\widehat{xr} + v_0\,\widehat{yr} + w_0\,\widehat{zr})\,dS,$$

where the integration extends over the surface of S. On substitution from (57), this integral is readily evaluated and it gives

$$\tfrac{4}{3}\pi R^3 \,.\, 2\mu(\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2\mathbf{f}^2 + 2\mathbf{g}^2 + 2\mathbf{h}^2)$$
$$+ \tfrac{32}{3}\pi\mu(\mathbf{a}A + \mathbf{b}B + \mathbf{c}C + 2\mathbf{f}F + 2\mathbf{g}G + 2\mathbf{h}H). \tag{58}$$

The first term is the work necessary to maintain the motion in the absence of the particle; the second represents the addition due to its presence. On making the radius of the sphere S tend to infinity, this result becomes exact. On substitution from (31) and (39). this gives for the additional rate of doing work owing to the presence of the particle

$$\frac{dW}{dt} = \tfrac{16}{3}\pi\mu\left\{\frac{\alpha_0''\mathbf{a}^2 + \beta_0''\mathbf{b}^2 + \gamma_0''\mathbf{c}^2}{\beta_0''\gamma_0'' + \gamma_0''\alpha_0'' + \alpha_0''\beta_0''}\right.$$
$$\left. + \frac{2\mathbf{f}^2}{\alpha_0'(b^2+c^2)} + \frac{2\mathbf{g}^2}{\beta_0'(c^2+a^2)} + \frac{2\mathbf{h}^2}{\gamma_0'(a^2+b^2)}\right\}. \tag{59}$$

It is interesting to note that if due regard is not paid to the conditions on the bounding sphere, and if the calculation is made directly from equations (52), the result is only one-fifth of that given above.

In the case of an ellipsoid of revolution ($b = c$) we have γ_0', γ_0'' respectively equal to β_0', β_0'', and it is easy to show that $\beta_0'' + 2\alpha_0'' = 2b^2\alpha_0'$. Equation (59) then takes the simplified form

$$\frac{dW}{dt} = \tfrac{16}{3}\pi\mu\left\{\frac{\alpha_0''\mathbf{a}^2}{2b^2\alpha_0'\beta_0''} + \frac{\mathbf{b}^2 + \mathbf{c}^2 + 2\mathbf{f}^2}{2b^2\alpha_0'} + \frac{2(\mathbf{g}^2 + \mathbf{h}^2)}{\beta_0'(a^2+b^2)}\right\}. \tag{60}$$

If the fluid is moving in laminar motion, this gives on substitution from (40) and (44)

$$\frac{dW}{dt} = \tfrac{4}{3}\pi\mu\kappa^2\left\{\left(\frac{\alpha_0''}{2b^2\alpha_0'\beta_0''}+\frac{1}{2b^2\alpha_0'}-\frac{2}{\beta_0'(a^2+b^2)}\right)\sin^4\theta\sin^2 2\phi\right.$$
$$\left.+\frac{1}{b^2\alpha_0'}\cos^2\theta+\frac{2}{\beta_0'(a^2+b^2)}\sin^2\theta\right\}. \quad (61)$$

The average values of $\cos^2\theta$ and $\sin^4\theta\sin^2 2\phi$ are calculated from (48) and (49) as follows:

$$\frac{\kappa ab}{2\pi(a^2+b^2)}\int_0^{2\pi}\cos^2\theta\frac{dt}{d\phi}d\phi = \frac{k^2}{\sqrt{\{(a^2+k^2)(b^2+k^2)\}}}$$

and

$$\frac{\kappa ab}{2\pi(a^2+b^2)}\int_0^{2\pi}\sin^4\theta\sin^2 2\phi\frac{dt}{d\phi}d\phi = \frac{2a^2b^2}{(a^2-b^2)^2}\left\{\frac{a^2+b^2+2k^2}{\sqrt{\{(a^2+k^2)(b^2+k^2)\}}}-2\right\}.$$

If D is the average rate of dissipation of energy, (61) then gives

$$D = \tfrac{4}{3}\pi\mu\kappa^2\left[\frac{2a^2b^2}{(a^2-b^2)^2}\left\{\frac{a^2+b^2+2k^2}{\sqrt{\{(a^2+k^2)(b^2+k^2)\}}}-2\right\}\right.$$
$$\left\{\frac{\alpha_0''}{2b^2\alpha_0'\beta_0''}+\frac{1}{2b^2\alpha_0'}-\frac{2}{\beta_0'(a^2+b^2)}\right\}$$
$$\left.+\frac{k^2}{\sqrt{\{(a^2+k^2)(b^2+k^2)\}}}\left\{\frac{1}{b^2\alpha_0'}-\frac{2}{\beta_0'(a^2+b^2)}\right\}+\frac{2}{\beta_0'(a^2+b^2)}\right]. \quad (62)$$

We will show that this has no maxima or minima for finite, non-zero values of k. Differentiating with regard to k^2, we obtain

$$\frac{dD}{d(k^2)} = -\frac{2\pi\mu\kappa^2[k^2\beta_0''\{2b^2\alpha_0'-(a^2+b^2)\beta_0'\}+a^2b^2\beta_0'(\alpha_0''-\beta_0'')]}{3b^2\alpha_0'\beta_0'\beta_0''\{(a^2+k^2)(b^2+k_2)\}^{3/2}}. \quad (63)$$

From their definition in equations (10), we see that α_0', α_0'', β_0', β_0'' are all positive, while $\alpha_0''-\beta_0''$ is of the same sign as a^2-b^2. Further

$$2b^2\alpha_0'-(a^2+b^2)\beta_0' = (a^2-b^2)\int_0^\infty\frac{(b^2-\lambda)\,d\lambda}{(a^2+\lambda)^{3/2}(b^2+\lambda)^3}$$
$$= \tfrac{3}{2}(a^2-b^2)\int_0^\infty\frac{\lambda d\lambda}{(a^2+\lambda)^{5/2}(b^2+\lambda)^2},$$

after an integration by parts. The latter integral is clearly positive, and hence the left-hand side is of the same sign as a^2-b^2. It follows that the right-hand side of (63) never vanishes for finite values of k, and is of opposite sign to a^2-b^2. Accordingly, the motion which gives minimum average dissipation of energy will correspond to $k=\infty$ for a prolate spheroid, and to $k=0$ for an oblate spheroid. The average rate of dissipation of energy will, in all cases, lie between the values corresponding to $k=0$ and $k=\infty$.

We will write

$$D = \nu v\mu\kappa^2 \quad (64)$$

where ν is a factor and $v = 4\pi ab^2/3$, and is therefore the volume of the particle. Equation (62) then gives, — for $k = \infty$,

$$\nu = 1/ab^4\alpha_0', \tag{65}$$

and for $k = 0$

$$\nu = \frac{1}{ab^3(a+b)^2}\left\{\frac{ab^2}{\beta_0''}+\frac{a}{2\alpha_0'}+\frac{2b}{\beta_0'}\right\}. \tag{66}$$

In the case of spheroids the integrals in (9) and (10) can be evaluated in finite terms. We obtain,

(1) for prolate spheroids, $b = c = a\cos\theta$

$$\begin{aligned} a^5\alpha_0' &= \tfrac{1}{4}\operatorname{cosec}^4\theta\,\{(2-5\cos^2\theta)\sec^4\theta+3\Theta\},\\ a^5\beta_0' &= \operatorname{cosec}^4\theta\,\{2+\sec^2\theta-3\Theta\},\\ a^3\beta_0'' &= \operatorname{cosec}^4\theta\,\{(2+\cos^2\theta)\Theta-3\}, \end{aligned} \tag{67}$$

where $$\Theta \equiv \operatorname{cosec}\theta\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right),$$

(2) for oblate spheroids, $b = c = a\sec\theta$,

$$\begin{aligned} b^5\alpha_0' &= \tfrac{1}{4}\operatorname{cosec}^4\theta\,\{3\Theta-\cos\theta\,(5-2\cos^2\theta)\},\\ b^5\beta_0' &= \operatorname{cosec}^4\theta\,\{(2+\cos^2\theta)\sec\theta-3\Theta\},\\ b^3\beta_0'' &= \operatorname{cosec}^4\theta\,\{(1+2\cos^2\theta)\Theta-3\cos\theta\}, \end{aligned} \tag{68}$$

where $$\Theta \equiv \theta/\sin\theta.$$

The maximum and minimum values of ν for spheroids of different shapes are given in Tables I and II. They are given in terms of the "ellipticity" of the meridian section of the particle, as this is generally more convenient for measurement than the eccentricity. The ellipticity ϵ is defined to be the difference of the greatest and least diameters divided by the greatest, so that $\epsilon = (a-b)/a$ for a prolate spheroid and $\epsilon = (b-a)/b$ for an oblate spheroid.

Table I.—Prolate Spheroids.

Ellipticity $\epsilon = \dfrac{a-b}{a}$.	Minimum ν.	Maximum ν.
0	2·5	2·5
0·1	2·431	2·540
0·2	2·361	2·586
0·3	2·295	2·645
0·4	2·232	2·719
0·5	2·174	2·819
0·6	2·120	2·958
0·7	2·073	3·170
0·8	2·035	3·548
0·9	2·010	4·485
1·0	2·000	∞

Table II.—Oblate Spheroids.

Ellipticity $\epsilon = \dfrac{b-a}{b}$.	Minimum ν.	Maximum ν.
0	2·5	2·5
0·1	2·464	2·582
0·2	2·426	2·683
0·3	2·388	2·818
0·4	2·348	3·003
0·5	2·306	3·267
0·6	2·262	3·670
0·7	2·216	4·354
0·8	2·168	5·744
0·9	2·116	9·960
1·0	2·061	∞

It will be noted that the maximum value of ν tends to infinity in each case as the ellipticity tends to unity. This does not, of course, mean that the dissipation of energy becomes infinite, since the volume of the particle at the same time tends to zero.

§ 6. *The Viscosity of Suspensions.*

The results of the last section may be applied, as Einstein has applied the corresponding results for a sphere, to the determination of the viscosity of a fluid containing solid spheroidal particles in suspension. If the suspension is so dilute that the distances between neighbouring particles are great compared with their dimensions, we may regard each particle as the cause of a certain increase in the dissipation of energy.

In the case of spherical particles the motion is steady and D is given accurately, without averaging, by the second term in (58). It is then easy to show from (10), (25) and (39), that, when $a = b = c$,

$$\mathrm{A}/\mathbf{a} = \mathrm{B}/\mathbf{b} = \mathrm{C}/\mathbf{c} = \mathrm{F}/\mathbf{f} = \mathrm{G}/\mathbf{g} = \mathrm{H}/\mathbf{h} = 5a^3/8. \qquad (69)$$

If v is the volume of the particle, so that $v = 4\pi a^3/3$, we have,

$$\mathrm{D} = 5v\mu\,(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+2\mathbf{f}^2+2\mathbf{g}^2+2\mathbf{h}^2).$$

Hence, if v is the total volume of the particles in unit volume of the suspension, the total rate of dissipation of energy is

$$2\mu\,(1+2{\cdot}5\mathrm{V})\,(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+2\mathbf{f}^2+2\mathbf{g}^2+2\mathbf{h}^2).$$

From the point of view of the dissipation of energy, we may, therefore, regard the suspension as a homogeneous fluid whose coefficient of viscosity is μ,* where

$$\mu^* = \mu\,(1+2{\cdot}5\mathrm{V}). \qquad (70)$$

This agrees with Einstein's well-known result. It does not require the particles to be equal in size, so long as they are all spherical in shape. It applies only to suspensions which are sufficiently dilute and in which there is no tendency to form aggregates. With these restrictions it is true for any steady motion of the suspension, subject only to the condition that, if $\mathbf{a}, \mathbf{b}, \ldots$ vary in space, they shall do so only on a scale which is large compared with the dimensions of the particles, and that this variation shall not promote a variation in the concentration of the particles. For example, in applying this formula to measurements of viscosity by capillary tube methods, we should have to be assured: (1) that the diameter of the tube is large compared with that of the particles; (2) that there is no tendency for the particles to select particular paths, say, along the axis of the tube.

Some further light may be thrown upon this last point. It is easily seen that, within the limitations of the present theory, there is no resultant force on the particle in a direction perpendicular to the axis of the tube. If the axis of the tube is taken as axis of z, the undisturbed motion is given by $u = v = 0$, $w = \mathrm{A}(a^2 - x^2 - y^2)$, where a is the radius of the capillary tube and A is a known constant. Referred to parallel axes through the centre of a particle (h, k), this gives $u = v = 0, w = a^2 - h^2 - k^2 - x^2 - y^2 - 2hx - 2ky$. This may be regarded as the superposition of two motions: (1) $u = v = 0$, $w = a^2 - h^2 - k^2 - x^2 - y^2$; (2) $u = v = 0$, $w = -2hx - 2ky$. Neither of these motions can give rise to any force on the particle perpendicular to the axis of z—the motion (1), for reasons of symmetry, and the motion (2), by the investigations of the present paper. It follows that the combined motion can give rise to no such force.

In the case of laminar motion, we can extend Einstein's result to ellipsoidal particles. We have then from (64) for the average total dissipation of energy per unit volume of the suspension

$$\mu\kappa^2(1+\nu\mathrm{V})$$

giving in place of (70)

$$\mu^* = \mu(1+\nu\mathrm{V}). \tag{71}$$

The values of ν for prolate and oblate spheroids of different ellipticity of meridian section are given in Tables I and II respectively.

It will be noted that the minimum value of ν is always less than the value (2·5) for spherical particles. In this connection it may be remarked that, although in some cases Einstein's formula agrees well with experiment, in others it agrees only if ν has a value greater than 2·5.* Smoluchowski† proposes a formula of the type (71) for non-spherical particles, and, with regard to the factor ν, he remarks: "For the sphere this value ($\nu = 2·5$) is evidently a minimum, since by virtue of its rolling motion it influences the shearing motion of the least volume of the fluid." This argument is not valid, for we have shown that for any spheroid there are possible motions for which the increase in the dissipation of energy is less than it is for a sphere of the same volume.

§ 7. *A Hypothesis on the Dissipation of Energy.*

It appears from the foregoing paragraphs that we are able only to specify limits for the numerical factor in Einstein's formula. It is obviously undesirable to leave a problem, which is physically quite determinate, in this indeterminate form.

* For an account of the many experimental investigations of this problem, see Arrhenius, 'Biochem. Jour.,' vol. 11, p. 112 (1917).

† "Theoretische Bemerkungen über die Viskosität der Kolloide," 'Kolloid Zeitsch.,' vol. 18, p. 191 (1916).

We might assume that at any instant the axes of the particles are in random directions. On this hypothesis it is easy to calculate the rate of dissipation of energy. It is found that this gives a value for Einstein's factor which is greater than the maximum values shown in Tables I and II. The explanation is that random directions for the axes of the particles do not correspond to a permanent steady state.

Again, we can determine ν if we know the number of particles for which the k of (49) lies between k and $k+dk$, say $f(k)\,dk$. This, however, is just what our present theory fails to reveal: according to it, all values of k are possible, and there is no indication of their relative probability. It would appear that this failure is due to the limitations of the theory of the slow motion of a viscous fluid.

It will be remembered that the only stable direction of permanent translation of an ellipsoid in a *perfect* fluid is that of its least axis. If an attempt is made to solve the corresponding problem for a viscous fluid by Overbeck's method, it is found that no couple is exerted upon the ellipsoid by the fluid if its motion is one of pure translation in any direction whatever. A heavy ellipsoid falling through a viscous fluid would fall with different velocities for different directions of its axes with regard to the vertical, but apparently it would not tend to set its axes in any particular directions. This is due to the fact that in the case of the perfect fluid the resultant couples arise from those terms in the pressure which depend upon the square of the velocity, terms which are neglected in the ordinary viscous theory.

It seems not improbable that, in our present problem, a more complete investigation would reveal the fact that the particles do tend to adopt special orientations with respect to the motion of the surrounding fluid. Our suggestion is that *the particles will tend to adopt that motion which, of all the motions possible under the approximated equations, corresponds to the least dissipation of energy*. If this be true, it reconciles the perfect fluid theory, the viscous theory, and experiment in the case of an ellipsoid falling under gravity through a fluid.

Our assumption is reminiscent of the theorems of Helmholtz and Korteweg on the rate of dissipation of energy in the motion of a viscous fluid. A little reflection, however, will show that these theorems are not really applicable to our present problem. For one thing, except in the case of spheroidal particles, no steady motion is possible. Further, these theorems are themselves deduced from the approximate equations of motion obtained by neglecting the squares and products of the velocities, and our investigations have been based upon exact solutions of these same approximate equations. It is impossible that a general theorem, based upon a set of differential equations, should give, with regard to a particular case, information which is not contained in the corresponding exact solution of the equations.

If this hypothesis be granted, we are able to assert the following propositions:—

1. Prolate spheroidal particles immersed in a fluid in laminar motion will tend to set themselves with their axes perpendicular to the plane of the undisturbed motion of the fluid. They will rotate about their axes with constant angular velocity, and the whole motion will be steady.

2. Oblate spheroidal particles immersed in a fluid in laminar motion will tend to set themselves with an equatorial diameter perpendicular to the plane of the undisturbed motion of the fluid. They will rotate about this diameter with a variable angular velocity, and the whole motion will be periodic but not steady.

3. For motion in a capillary tube, the particles will tend to concentrate along the axis of the tube.

4. The effective value of the numerical factor in Einstein's formula will be the minimum ν, as shown in Tables I and II. This is always less than the 2·5 appropriate to spherical particles. Thus, for particles of given volume, the increase of the viscosity is greatest when the particles are spherical.

An experimental investigation of some of these provisional conclusions would be valuable.