

# Risk-Budgeted Mean-Variance Portfolios

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- Portfolio allocation is a central question in Finance;
- Two major issues from the practitioner perspective:
  1. How to model the stochastic process driving returns?
  2. Given a stochastic process for returns, how to allocate money?

**This paper is about the latter, not the former.**

The cornerstone: Mean-Variance (MV)... but nothing comes for free;

## **Mean-Variance (MV)**

- Best trade-off between risk (variance) and expected returns;
- Leads to concentrated portfolios;
- Not very robust to moment estimation error;

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- Explicit guard against portfolio concentration;
- No control over expected returns;

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**This paper**: Risk-Budgeted Mean-Variance (RBMV) portfolio;

- RBMV nests both the RB and MV frameworks;
- Makes **explicit** the tension between expected returns and risk concentration;
- Measures how much concentration you need to accept to get higher returns;

1. MV and RB: a quick overview;
2. Our methodology: theory and simulation results;
3. Empirical application: U.S. equity market using CRSP data;

Empirical takeaway:

- Our methodology indeed delivers portfolios that are less concentrated than MV;
- Higher expected returns than RB;
- No way around it: moment estimation is still key;

- We trade  $d$  assets indexed by  $i = 1, \dots, d$ ;
- Returns  $r_i$  have an expected value  $\mu_{d \times 1}$  and covariance matrix  $\Sigma_{d \times d}$ ;
- An allocation  $\mathbf{v} = (v_1, \dots, v_d)^\top$  is a vector of *dollars* invested in each asset;
- You have  $v_0 > 0$  dollars to invest;
- The dollar return for an allocation  $\mathbf{v}$  is given by:

$$R(\mathbf{v}) \equiv \sum_{i=1}^d v_i \cdot r_i = v_0 \left[ \sum_{i=1}^d w_i \cdot r_i \right], \quad w_i \equiv \frac{v_i}{v_0} \quad (1)$$

## The MV way

- You want to minimize the variance of returns  $\sigma(R(\mathbf{v}))$ ;
- But you request a minimum expected return  $\mu_{\min}^{MV}$ ;
- The (long-only) MV allocation  $\mathbf{v}^{MV}$  solves:

$$\begin{aligned} \min_{\mathbf{v} \geq 0} \quad & \sigma(R(\mathbf{v})) \\ \text{s.t.} \quad & \sum_{i=1}^d v_i = v_0 \\ & \mu(R(\mathbf{v})) \geq \mu_{\min}^{MV} \cdot v_0 \end{aligned} \tag{2}$$

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**Drawback: MV portfolios can be highly concentrated on a few assets!**

- If we increase  $v_i$  by \$1, how much does the portfolio risk  $\sigma(R(\mathbf{v}))$  change?

### Definition

The *risk contribution* of asset  $i$  to the total portfolio risk  $\sigma(R(\mathbf{v}))$ , is given by:

$$\mathcal{RC}_i(\mathbf{v}) \equiv v_i \cdot \frac{\partial \sigma(R(\mathbf{v}))}{\partial v_i}$$

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- $\mathcal{RC}_i(\mathbf{v})$  can be high for two reasons: high exposure or high asset volatility;
- Since  $\sigma(\cdot)$  is homogeneous of degree 1, Euler's theorem implies:

$$\sigma(R(\mathbf{v})) = \sum_{i=1}^d \mathcal{RC}_i(\mathbf{v}).$$

- Risk budgeting (RB) is a framework to limit the risk contributions  $\mathcal{RC}_i(\mathbf{v})$ ;
- We require a *risk budget*  $\mathbf{b} = (b_1, \dots, b_d)^\top$  for total risk and want to invest  $v_0$ ;

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## Definition (The Risk Budgeting Portfolio)

The *risk budgeting* (RB) portfolio is an allocation  $\mathbf{v} \geq 0$  that satisfies  $v_0 = \sum_{i=1}^d v_i$  and

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- No clue from the definition how to *compute*  $\mathbf{v}$ !
- The system in (3) is a non-linear system of  $d$  equations;

## How to find this portfolio?

### Proposition

Given a risk budget  $\mathbf{b}$ , any optimal solution  $\mathbf{v}^*$  to

$$\min_{\mathbf{v} \in \mathbb{R}_+^d} \sigma(R(\mathbf{v})), \quad \text{subject to} \quad \sum_{i=1}^d b_i \cdot \log(v_i) \geq 0 \quad (4)$$

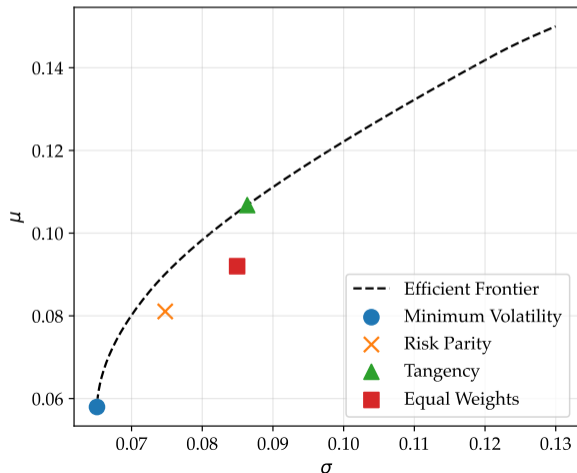
is proportional to the exposure  $\mathbf{v}$  of the RB portfolio for risk budget  $\mathbf{b}$ .

- This problem is strictly convex and easy to solve;
- The FOC's coincide with the RB conditions in (3);
- We can rescale the solution:

$$\mathbf{v}^{RB} \equiv \frac{v_0}{\sum_i v_i^*} \mathbf{v}^*$$

## Calibrated Example

- Let  $d = 5$ , and  $\mathbf{b} = (0.2, 0.2, 0.2, 0.2, 0.2)$  – which denotes *risk parity*;
- With population moments, we compute several portfolios:



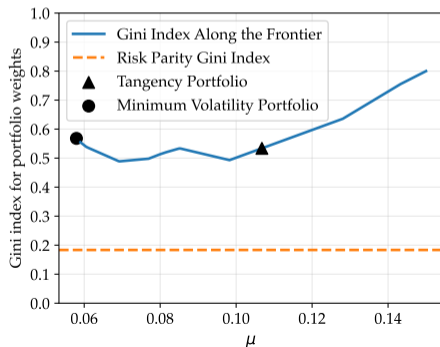
- If all you care is Sharpe Ratio, stick to MV;
- Risk parity  $\neq$  Equal weights;
- You almost always have to visit the interior to get risk budgeting;

► Calibration

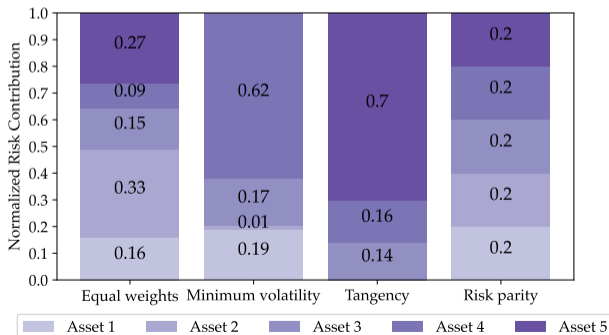
# Calibrated Example: Concentration

- The Gini index is a measure of concentration;
- Given a vector  $\mathbf{x} = (x_1, \dots, x_n)$ , we have  $\text{Gini}(\mathbf{x}) \equiv \frac{\sum_{i=1}^n \sum_{j=1}^n |x_i - x_j|}{2n \sum_{i=1}^n x_i}$ .
- $0 \leq \text{Gini}(\mathbf{x}) \leq 1$ , and  $\text{Gini}(\mathbf{x}) = 0$  if and only if  $\mathbf{x}$  is uniform;
- Brazilian (wealth) Gini: 0.52; South African Gini: 0.63; Swedish Gini: 0.29;

(a) Portfolio weight Gini indices



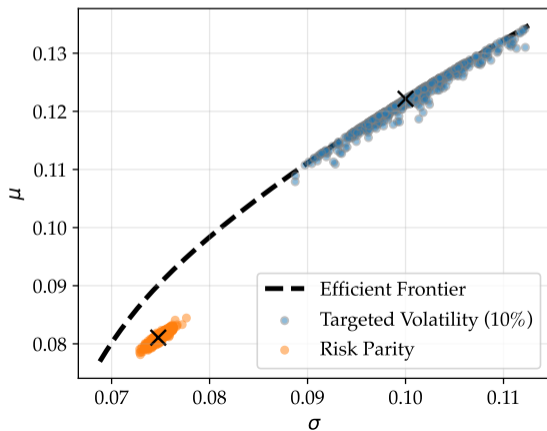
(b) Risk Contributions



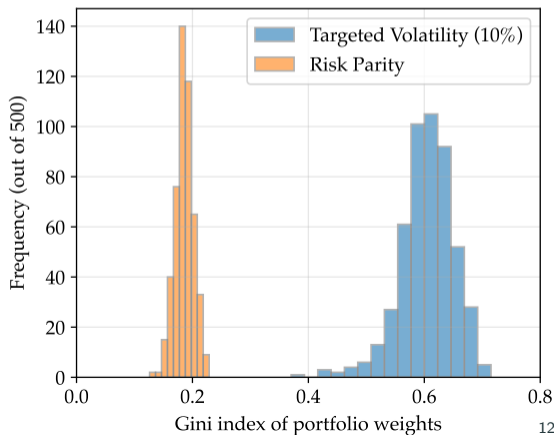
## Calibrated Example: Adding Estimation Error

- Simulate 1 year of daily returns, estimate moments, and compute portfolios;
- Plot expected returns and vols using the *population moments*;

(a) Simulated portfolios in the  $(\sigma, \mu)$ -plane



(b) Distribution of realized Gini indices for  $\mathbf{w}_i$



## Definition (The Risk-Budgeted Mean-Variance Portfolio)

Given a risk budget  $\mathbf{b}$ , an endowment  $v_0$ , a minimum required expected return  $\mu_{\min}$ , and a maximum volatility bound  $\sigma_{\max}$ , the Risk Budgeted Mean-Variance Portfolio (RBMV) is given by  $\mathbf{v} = \frac{v_0}{\sum_{i=1}^d v_i^*} \cdot \mathbf{v}^*$ , where  $\mathbf{v}^*$  is the solution of:

$$\begin{aligned} \min_{\mathbf{v} \in \mathbb{R}_+^d} \quad & \sigma(R(\mathbf{v})) \\ \text{s.t.} \quad & \sum_{i=1}^d b_i \log(v_i) \geq 0 & [\lambda_v] \\ & \mu(R(\mathbf{v})) \geq \mu_{\min} \sum_{i=1}^d v_i & [\lambda_\mu] \\ & \sigma(R(\mathbf{v})) \leq \sigma_{\max} \sum_{i=1}^d v_i, & [\lambda_\sigma] \end{aligned} \tag{5}$$

The corresponding portfolio weights are given by  $\mathbf{w} = \frac{1}{v_0} \mathbf{v}$ .

We adapt this from [Maillard, Roncalli, and Teïletche \(2010\)](#) and [Roncalli \(2014\)](#).

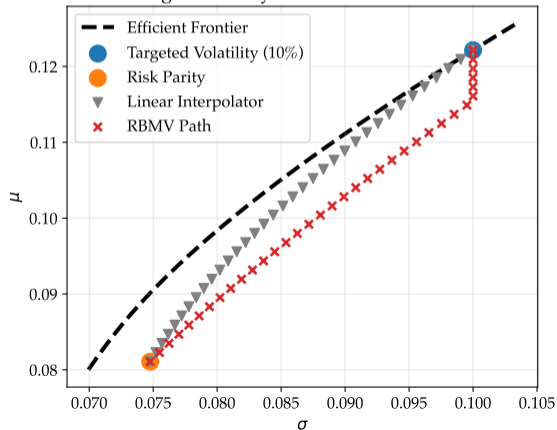
## Our Methodology: The Lagrangian View

$$\frac{\partial \mathcal{L}(\mathbf{v}; \lambda_v, \lambda_\mu, \lambda_\sigma)}{\partial v_i} = \frac{\partial \sigma(R(\mathbf{v}))}{\partial v_i} - \lambda_v \frac{b_i}{v_i} - \lambda_\mu \left( \mu_i - \underbrace{\mu_{\min}}_{\text{moves around}} \right) - \lambda_\sigma \left( \underbrace{\sigma_{\max}}_{=0.1} - \frac{\partial \sigma(R(\mathbf{v}))}{\partial v_i} \right)$$

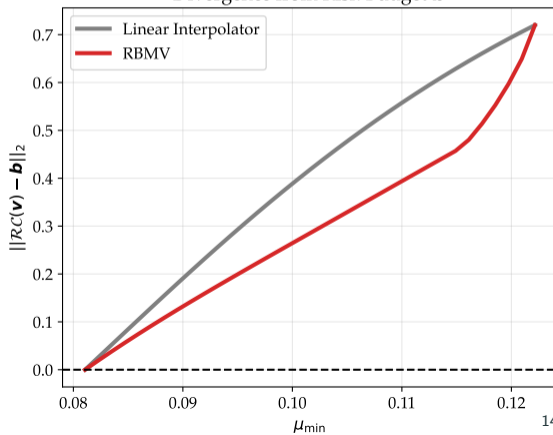
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Connecting Risk Parity and Mean-Variance Portfolios



Divergence from Risk Budget  $\mathbf{b}$



# Empirical Application

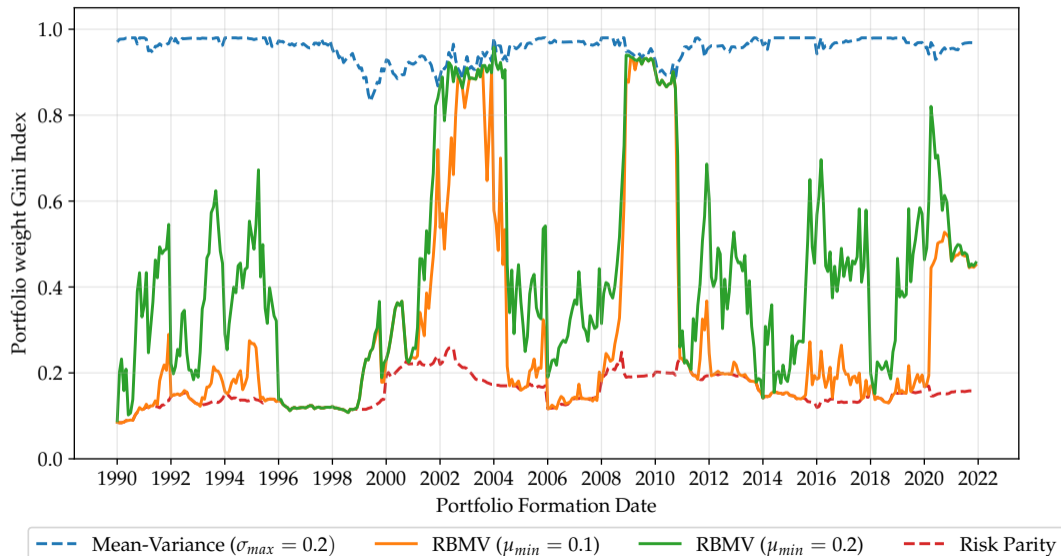
- Every month, we build a long-only portfolio with the 50 largest stocks in the U.S.;
- Sample: 1990-2023;
- Estimate population moments  $\hat{\mu}$  and  $\hat{\Sigma}$  with 2 years of historical data (CRSP);
- Solve for the minimum vol portfolio and set  $\sigma_{max} = \min\{\sigma_{MinVol} + 0.02, 0.2\}$ ;
- Find the MV portfolio with the highest returns, given  $\sigma_{max}$ ;
- Set two possible values for  $\mu_{min}$ :

$$\mu_{min, \text{ conservative}} \equiv \min\{\mu_{MV} - 0.05, 0.1\}$$

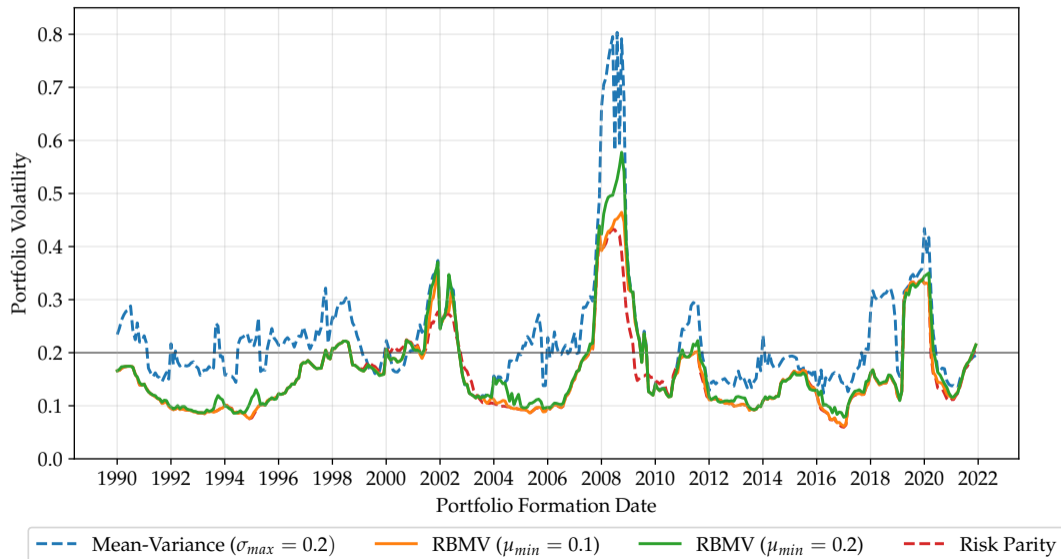
$$\mu_{min, \text{ greedy}} \equiv \min\{\mu_{MV} - 0.05, 0.2\}$$

- Hold 4 portfolios for 1 year: Risk Parity,  $MV(\sigma_{max})$ ,  $RBMV(\mu_{min, conservative}; \sigma_{max})$ , and  $RBMV(\mu_{min, greedy}; \sigma_{max})$ ;

# Realized Gini Index



# Annualized Standard Deviation of Daily Returns



## Average results

Full Sample (1990-2022)				
Portfolio	Return (%)	Volatility (%)	Sharpe Ratio	Gini Index ( $w_i$ )
Risk Parity	9.68	15.69	0.84	0.16
RBMV ( $\mu_{min} = 0.1, \sigma_{max} = 0.2$ )	10.29	15.98	0.86	0.29
RBMV ( $\mu_{min} = 0.2, \sigma_{max} = 0.2$ )	10.50	16.70	0.81	0.43
Mean-Variance ( $\sigma_{max} = 0.2$ )	10.20	22.28	0.58	0.96

- We were able to tilt the Risk Parity portfolio towards higher returns;
- Just a bit more of concentration led to higher returns and slightly higher SR;
- The MV portfolio was invested in just a few assets!

► Alternative Samples

► Realized Sharpe Ratio

► Realized Returns

►  $\sigma_{max}$

►  $\mu^{MV}$

## Wrap-Up:

- MV is the cornerstone of portfolio allocation, but leads to high concentration;
- RB tackles concentration at the expense of expected returns;
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## Going forward:

- Some extensions: short-selling, transaction costs, ...;
- What kind of other markets can benefit from this methodology? What examples are interesting?

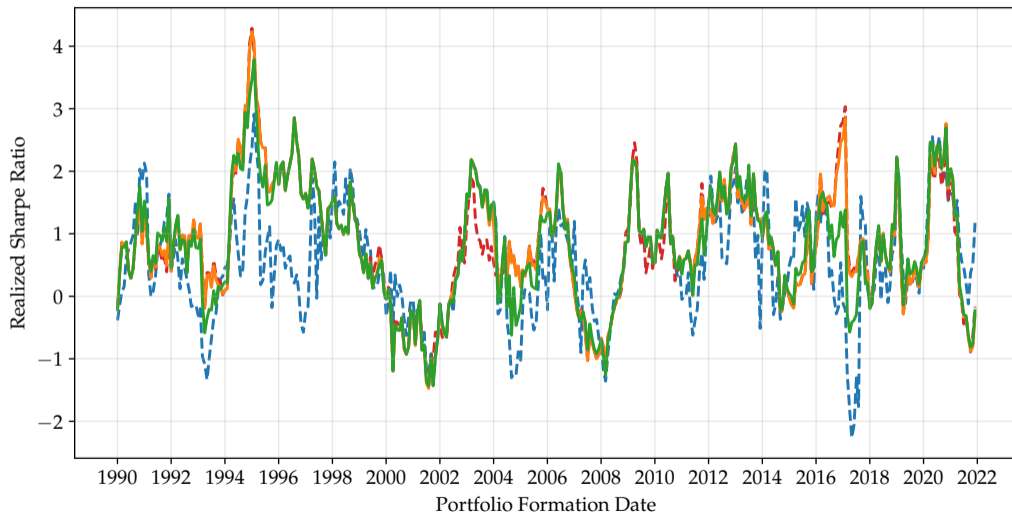
# Appendix and References

We assume  $d = 5$  and use the following population moments for the mean returns  $\mu$ , the individual standard deviations  $s$  and correlation matrix  $C$ :

$$\mu = \begin{bmatrix} 0.5 \\ 0.12 \\ 0.09 \\ 0.05 \\ 0.15 \end{bmatrix}, \quad s = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.15 \\ 0.08 \\ 0.13 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.20 & 0.40 & 0.25 & 0.50 \\ 0.20 & 1 & -0.20 & 0.40 & 0.6 \\ 0.40 & -0.20 & 1 & -0.10 & 0.30 \\ 0.25 & 0.40 & -0.10 & 1 & 0.30 \\ 0.50 & 0.60 & 0.30 & 0.30 & 1 \end{bmatrix} \quad (6)$$

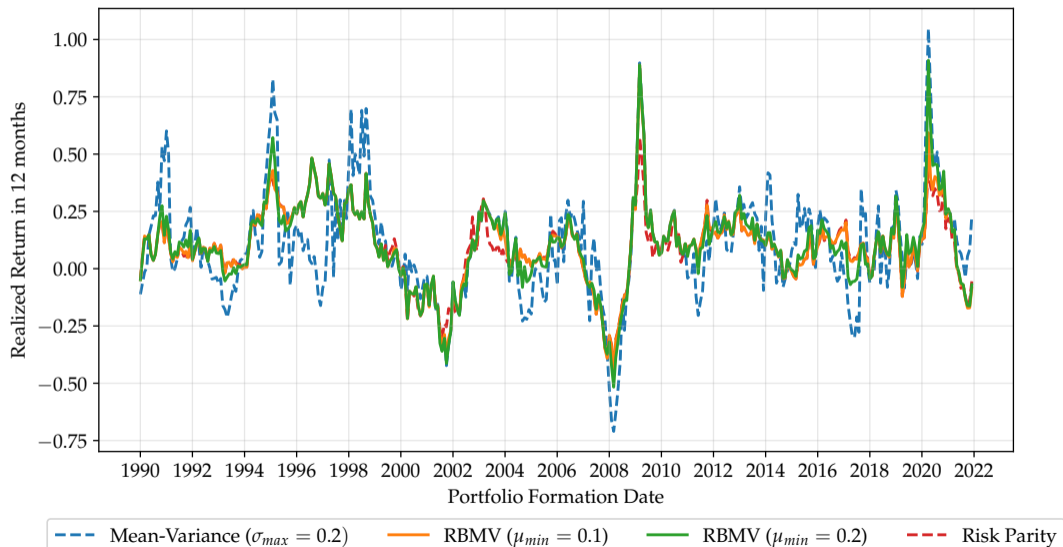
We further define  $\Sigma \equiv s^T C s$ .

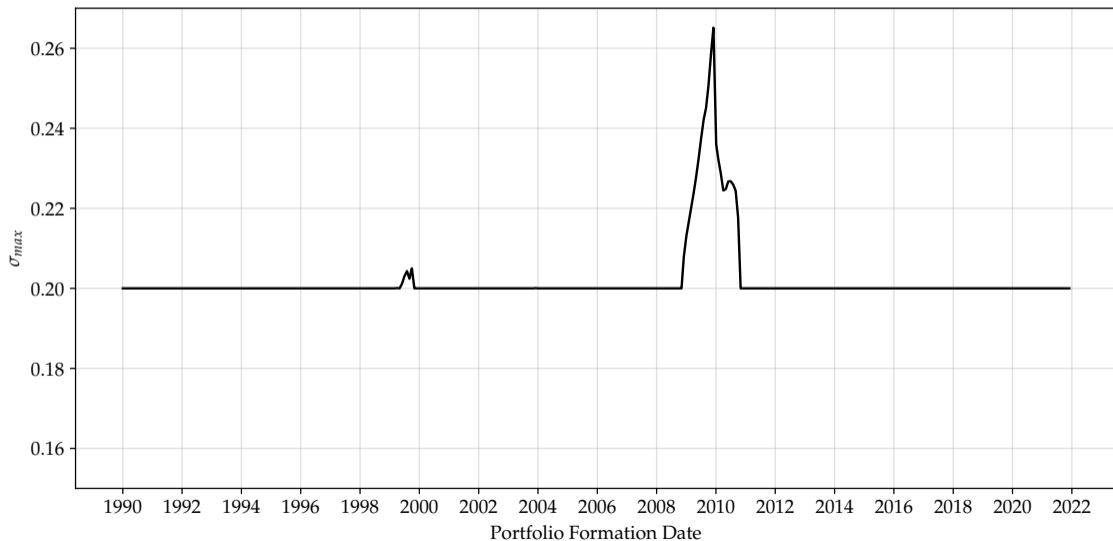
# Realized Sharpe Ratio

[► Back to Average Results](#)

-- Mean-Variance ( $\sigma_{max} = 0.2$ )    — RBMV ( $\mu_{min} = 0.1$ )    — RBMV ( $\mu_{min} = 0.2$ )    -- Risk Parity

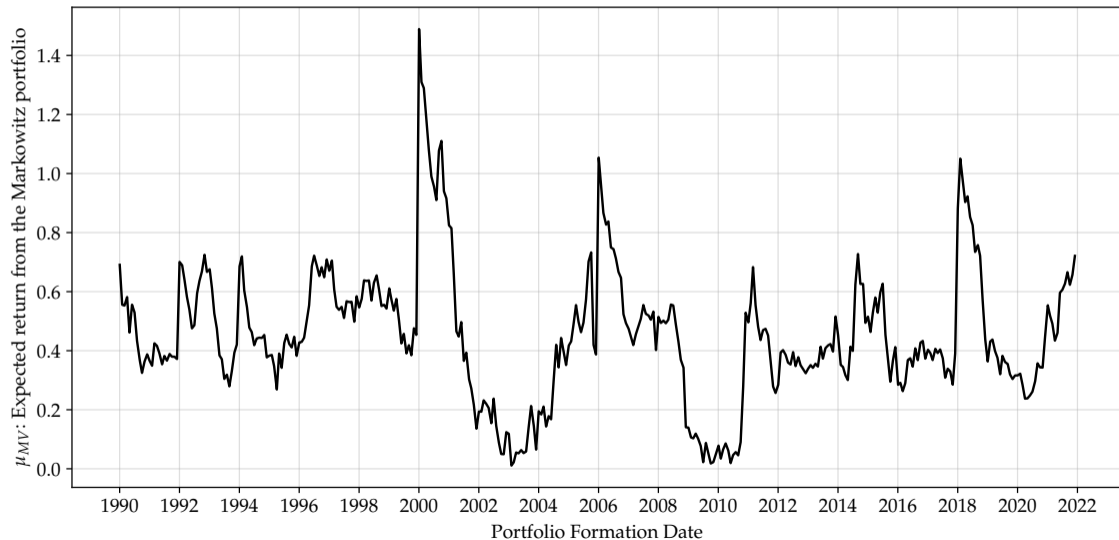
# Realized Returns

[► Back to Average Results](#)



# Expected Return From The MV Approach

[▶ Back to Average Results](#)



Panel B: Before the Great Financial Crisis (1990-2006)



Portfolio	Return (%)	Volatility (%)	Sharpe Ratio	Gini Index ( $w_i$ )
Risk Parity	10.79	14.42	0.89	0.15
RBMV ( $\mu_{min} = 0.1, \sigma_{max} = 0.2$ )	10.90	14.45	0.91	0.26
RBMV ( $\mu_{min} = 0.2, \sigma_{max} = 0.2$ )	10.74	14.99	0.85	0.39
Mean-Variance ( $\sigma_{max} = 0.2$ )	9.92	20.84	0.50	0.95

Panel C: After 2020 (2021-2022)

Portfolio	Return (%)	Volatility (%)	Sharpe Ratio	Gini Index ( $w_i$ )
Risk Parity	0.07	15.65	0.21	0.16
RBMV ( $\mu_{min} = 0.1, \sigma_{max} = 0.2$ )	0.93	16.00	0.27	0.46
RBMV ( $\mu_{min} = 0.2, \sigma_{max} = 0.2$ )	1.52	16.01	0.30	0.47
Mean-Variance ( $\sigma_{max} = 0.2$ )	12.62	16.31	0.84	0.96

### References

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-  Maillard, Sébastien, Thierry Roncalli, and Jérôme Teïletche (2010). “The properties of equally-weighted risk contributions portfolios”. In: *The Journal of Portfolio Management*.
-  Roncalli, Thierry (2014). “Introducing expected returns into risk parity portfolios: A new framework for asset allocation”. In: *Available at SSRN 2321309*.