Risk-Budgeted Mean-Variance Portfolios

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Intro

- Portfolio allocation is a central question in Finance;
- Two major issues from the practitioner perspective:
    1. How to model the stochastic process driving returns?
    2. Given a stochastic process for returns, how to allocate money?

This paper is about the latter, not the former.

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The cornerstone: Mean-Variance (MV)... but nothing comes for free;

Mean-Variance (MV)

- Best trade-off between risk (variance) and expected returns;
- Leads to concentrated portfolios;
- Not very robust to moment estimation error;

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- Assets control a maximum fraction of the total portfolio risk;
- Explicit guard against portfolio concentration;
- No control over expected returns;

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This paper: Risk-Budgeted Mean-Variance (RBMV) portfolio;

- RBMV nests both the RB and MV frameworks;
- Makes explicit the tension between expected returns and risk concentration;
- Measures how much concentration you need to accept to get higher returns;

Flight Plan

1. MV and RB: a quick overview;
2. Our methodology: theory and simulation results;
3. Empirical application: U.S. equity market using CRSP data;

Empirical takeway:

- Our methodology indeed delivers portfolios that are less concentrated than MV;
- Higher expected returns than RB;
- No way around it: moment estimation is still key;

Setup

- We trade d assets indexed by $i = 1, \ldots, d$;
- Returns r_i have an expected value $\mu_{d \times 1}$ and covariance matrix $\Sigma_{d \times d}$;
- An allocation $\boldsymbol{v} = (v_1, \ldots, v_d)^\intercal$ is a vector of *dollars* invested in each asset;
- You have $v_0 > 0$ dollars to invest;
- The dollar return for an allocation $\boldsymbol{v}$ is given by:

$$R(\boldsymbol{v}) \equiv \sum_{i=1}^{d} v_i \cdot r_i = v_0 \left[\sum_{i=1}^{d} w_i \cdot r_i \right], \qquad w_i \equiv \frac{v_i}{v_0} \tag{1}$$

The MV way

- You want to minimize the variance of returns $\sigma(R(\boldsymbol{v}))$;
- But you request a minimum expected return μ_{min}^{MV};
- The (long-only) MV allocation $\boldsymbol{v}^{MV}$ solves:

$$\begin{aligned} \min_{\boldsymbol{v} \geq 0} \quad & \sigma(R(\boldsymbol{v})) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{d} v_i = v_0 \\ & \mu(R(\boldsymbol{v})) \geq \mu_{\text{min}}^{MV} \cdot v_0 \end{aligned} \tag{2}$$

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Drawback: MV portfolios can be highly concentrated on a few assets!

Risk Contributions

- If we increase v_i by \$1, how much does the portfolio risk $\sigma(R(\boldsymbol{v}))$ change?

Definition

The *risk contribution* of asset i to the total portfolio risk $\sigma\left(R(\boldsymbol{v})\right)$, is given by:

$$\mathcal{RC}_i(\boldsymbol{v}) \equiv v_i \cdot \frac{\partial \sigma(R(\boldsymbol{v}))}{\partial v_i}$$

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- Since $\sigma(.)$ is homogeneous of degree 1, Euler's theorem implies:

$$\sigma(R(\boldsymbol{v})) = \sum_{i=1}^{d} \mathcal{RC}_i(\boldsymbol{v}).$$

Risk Budgeting

- Risk budgeting (RB) is a framework to limit the risk contributions $\mathcal{RC}_i(\boldsymbol{v})$;
- We require a *risk budget* $\boldsymbol{b} = (b_1, \ldots, b_d)^\intercal$ for total risk and want to invest v_0;

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The *risk budgeting* (RB) portfolio is an allocation $\boldsymbol{v} \geq 0$ that satisfies $v_0 = \sum_{i=1}^{d} v_i$ and

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- No clue from the definition how to *compute* $\boldsymbol{v}$!
- The system in (3) is a non-linear system of d equations;

How to find this portfolio?

Proposition

Given a risk budget $\boldsymbol{b}$, any optimal solution $\boldsymbol{v}^*$ to

$$\min_{\boldsymbol{v}\in\mathbb{R}_+^d} \sigma(R(\boldsymbol{v})), \qquad \textit{subject to} \quad \sum_{i=1}^{d} b_i \cdot \log(v_i) \geq 0 \tag{4}$$

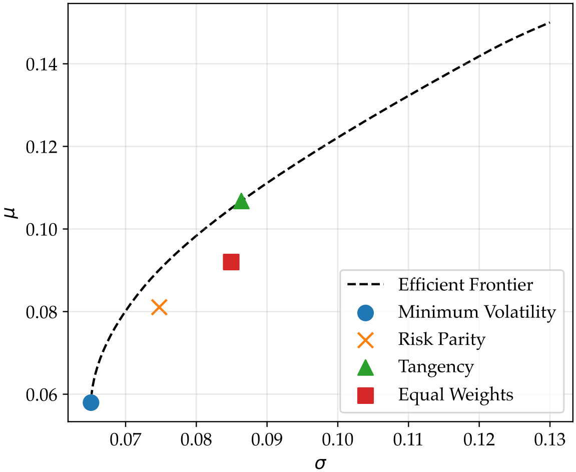
is proportional to the exposure $\boldsymbol{v}$ of the RB portfolio for risk budget $\boldsymbol{b}$.

- This problem is strictly convex and easy to solve;
- The FOC's coincide with the RB conditions in (3);
- We can rescale the solution:

$$\boldsymbol{v}^{RB} \equiv \frac{v_0}{\sum_i^d v_i^*}\boldsymbol{v}^*$$

Calibrated Example

- Let $d = 5$, and $\boldsymbol{b} = (0.2, 0.2, 0.2, 0.2, 0.2)$ – which denotes *risk parity*;
- With population moments, we compute several portfolios:



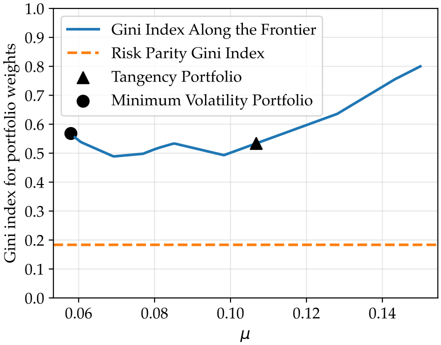
- If all you care is Sharpe Ratio, stick to MV;
- Risk parity $\neq$ Equal weights;
- You almost always have to visit the interior to get risk budgeting;

Calibration

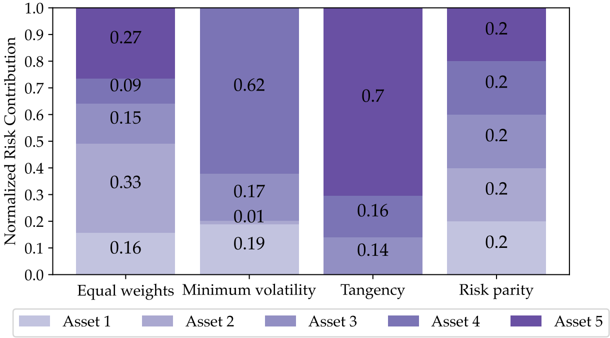
Calibrated Example: Concentration

- The Gini index is a measure of concentration;
- Given a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$, we have $\text{Gini}(\boldsymbol{x}) \equiv \frac{\sum_{i=1}^{n}\sum_{j=1}^{n}|x_i - x_j|}{2n\sum_{i=1}^{n} x_i}$.
- $0 \leq \text{Gini}(\boldsymbol{x}) \leq 1$, and $\text{Gini}(\boldsymbol{x}) = 0$ if and only if $\boldsymbol{x}$ is uniform;
- Brazilian (wealth) Gini: 0.52; South African Gini: 0.63; Swedish Gini: 0.29;

(a) Portfolio weight Gini indices



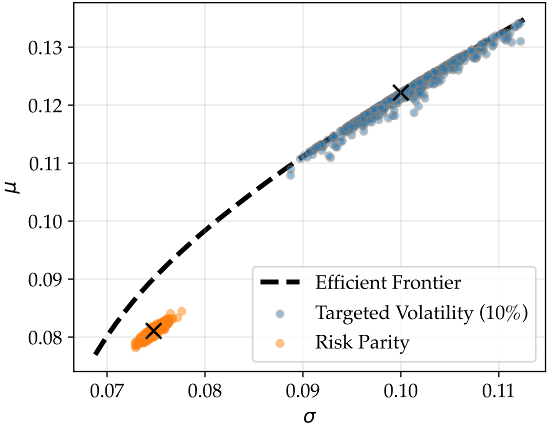
(b) Risk Contributions



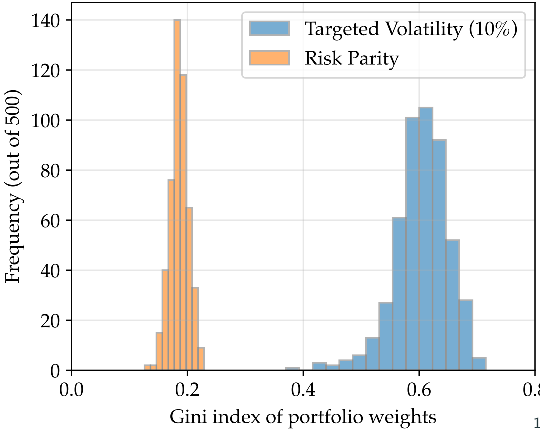
Calibrated Example: Adding Estimation Error

- Simulate 1 year of daily returns, estimate moments, and compute portfolios;
- Plot expected returns and vols using the *population moments*;

(a) Simulated portfolios in the (σ, μ)-plane



(b) Distribution of realized Gini indices for $\boldsymbol{w}_i$



Our Methodology

Definition (The Risk-Budgeted Mean-Variance Portfolio)

Given a risk budget $\boldsymbol{b}$, an endowment v_0, a minimum required expected return $\mu_{\min}$, and a maximum volatility bound $\sigma_{\max}$, the Risk Budgeted Mean-Variance Portfolio (RBMV) is given by $\boldsymbol{v} = \frac{v_0}{\sum_{i=1}^{d} v_i^*} \cdot \boldsymbol{v}^*$, where $\boldsymbol{v}^*$ is the solution of:

$$
\begin{aligned}
\min_{\boldsymbol{v}\in\mathbb{R}_+^d} \quad & \sigma(R(\boldsymbol{v})) & \\
\text{s.t.} \quad & \sum_{i=1}^{d} b_i \log(v_i) \geq 0 & [\lambda_v] \\
& \mu(R(\boldsymbol{v})) \geq \mu_{\min} \textstyle\sum_{i=1}^{d} v_i & [\lambda_\mu] \\
& \sigma(R(\boldsymbol{v})) \leq \sigma_{\max} \textstyle\sum_{i=1}^{d} v_i \,, & [\lambda_\sigma]
\end{aligned}
\tag{5}
$$

The corresponding portfolio weights are given by $\boldsymbol{w} = \frac{1}{v_0}\boldsymbol{v}$.

We adapt this result from Maillard, Roncalli, and Teïletche (2010) and Roncalli (2014).

Our Methodology: The Lagrangian View

$$\frac{\partial \mathcal{L}(\boldsymbol{v}; \lambda_v, \lambda_\mu, \lambda_\sigma)}{\partial v_i} = \frac{\partial \sigma(R(\boldsymbol{v}))}{\partial v_i} - \lambda_v \frac{b_i}{v_i} - \lambda_\mu \left(\mu_i - \underbrace{\mu_{\text{min}}}_{\text{moves around}} \right) - \lambda_\sigma \left(\underbrace{\sigma_{\text{max}}}_{=0.1} - \frac{\partial \sigma(R(\boldsymbol{v}))}{\partial v_i} \right)$$

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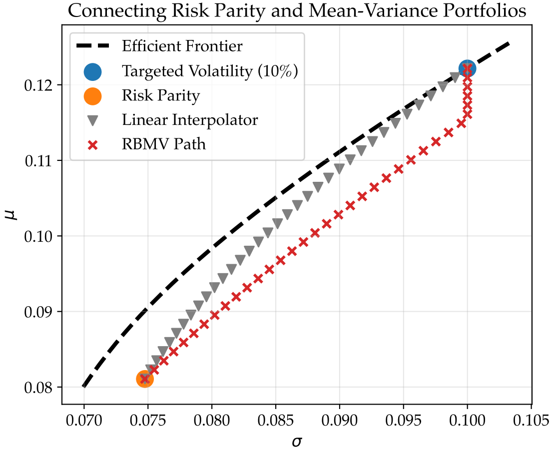
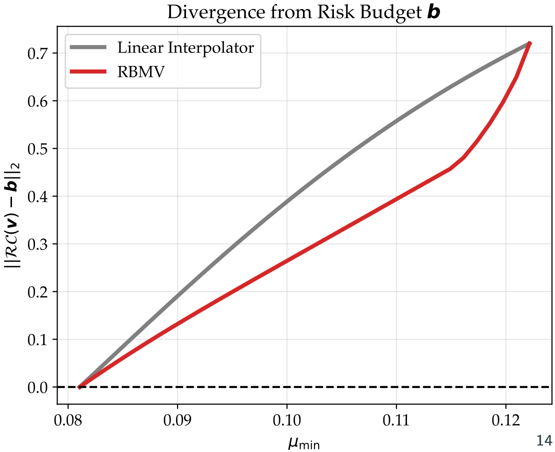




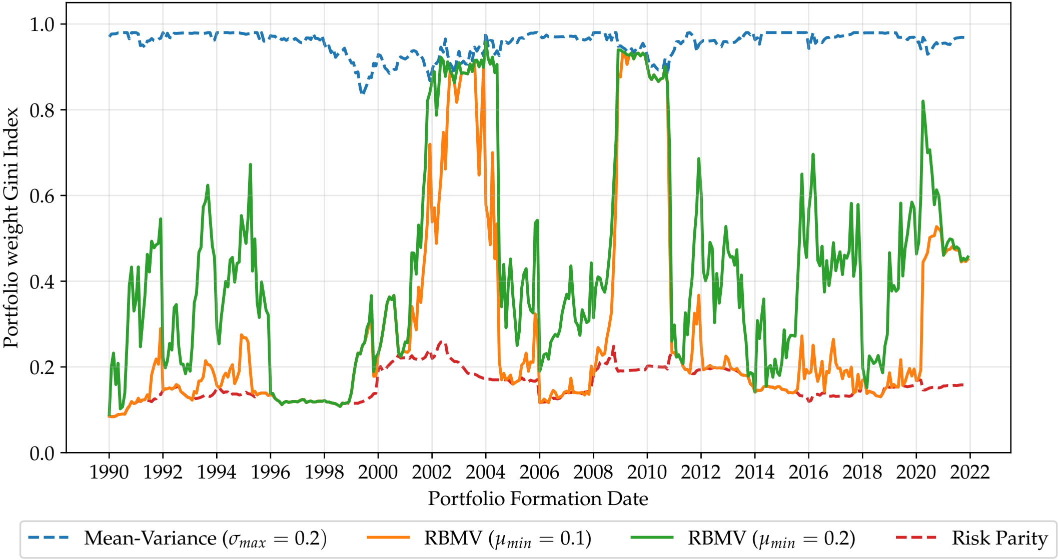
Empirical Application

- Every month, we build a long-only portfolio with the 50 largest stocks in the U.S.;
- Sample: 1990-2023;
- Estimate population moments $\hat{\mu}$ and $\hat{\Sigma}$ with 2 years of historical data (CRSP);
- Solve for the minimum vol portfolio and set $\sigma_{max} = \min\{\sigma_{MinVol} + 0.02, 0.2\}$;
- Find the MV portfolio with the highest returns, given σ_{max};
- Set two possible values for μ_{min}:

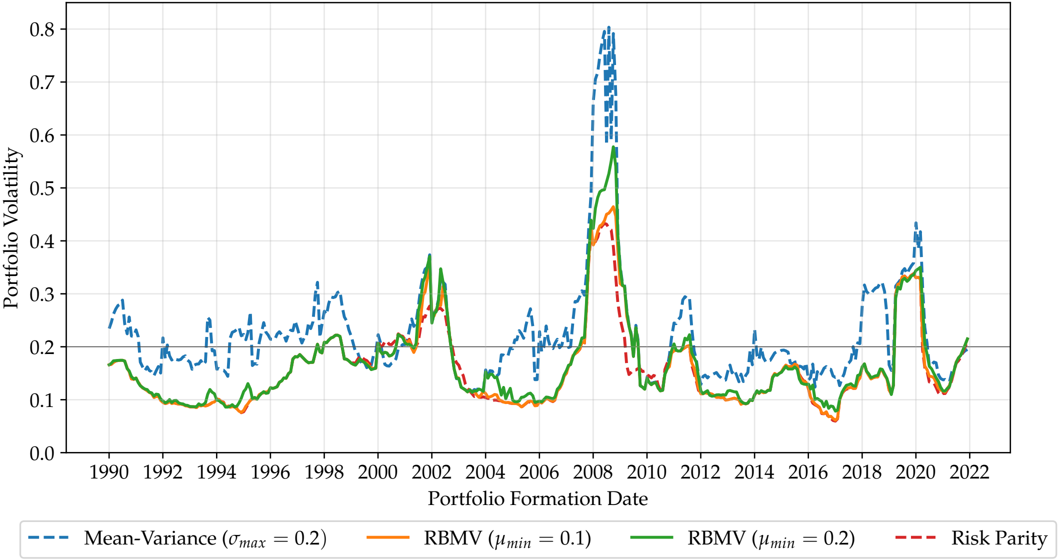
$$\mu_{min,\text{ conservative}} \equiv \min\{\mu_{MV} - 0.05, 0.1\}$$
$$\mu_{min,\text{ greedy}} \equiv \min\{\mu_{MV} - 0.05, 0.2\}$$

- Hold 4 portfolios for 1 year: Risk Parity, MV(σ_{max}), RBMV($\mu_{min,\text{conservative}}; \sigma_{max}$), and RBMV($\mu_{min,\text{greedy}}; \sigma_{max}$);

Realized Gini Index



Annualized Standard Deviation of Daily Returns



Average results

Full Sample (1990-2022)

Portfolio	**Return (%)**	**Volatility (%)**	**Sharpe Ratio**	**Gini Index ($\boldsymbol{w}_i$)**
Risk Parity	9.68	15.69	0.84	0.16
RBMV ($\mu_{min} = 0.1$, $\sigma_{max} = 0.2$)	10.29	15.98	0.86	0.29
RBMV ($\mu_{min} = 0.2$, $\sigma_{max} = 0.2$)	10.50	16.70	0.81	0.43
Mean-Variance ($\sigma_{max} = 0.2$)	10.20	22.28	0.58	0.96

- We were able to tilt the Risk Parity portfolio towards higher returns;
- Just a bit more of concentration led to higher returns and slightly higher SR;
- The MV portfolio was invested in just a few assets!

Alternative Samples | Realized Sharpe Ratio | Realized Returns | σ_{max} | μ^{MV}

Conclusion

Wrap-Up:

- MV is the cornerstone of portfolio allocation, but leads to high concentration;
- RB tackles concentration at the expense of expected returns;
- RBMV nests both, and allows you to control this trade-off in a transparent way;

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Going forward:

- Some extensions: short-selling, transaction costs, ...;
- What kind of other markets can benefit from this methodology? What examples are interesting?

Appendix and References

Calibration Details

We assume $d = 5$ and use the following population moments for the mean returns μ, the individual standard deviations s and correlation matrix C:

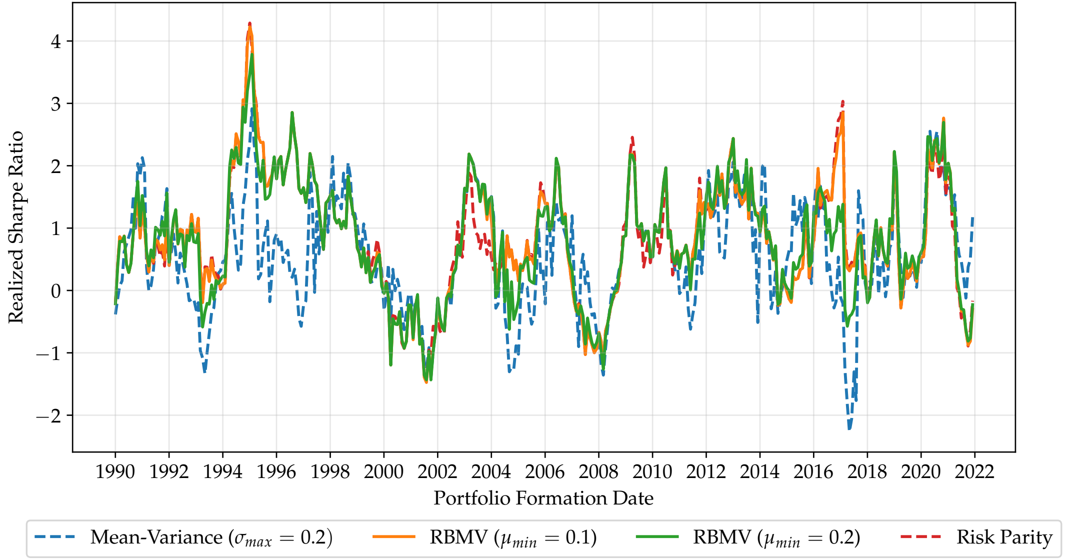
$$\mu = \begin{bmatrix} 0.5 \\ 0.12 \\ 0.09 \\ 0.05 \\ 0.15 \end{bmatrix}, \quad s = \begin{bmatrix} 0.10 \\ 0.20 \\ 0.15 \\ 0.08 \\ 0.13 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0.20 & 0.40 & 0.25 & 0.50 \\ 0.20 & 1 & -0.20 & 0.40 & 0.6 \\ 0.40 & -0.20 & 1 & -0.10 & 0.30 \\ 0.25 & 0.40 & -0.10 & 1 & 0.30 \\ 0.50 & 0.60 & 0.30 & 0.30 & 1 \end{bmatrix} \tag{6}$$

We further define $\Sigma \equiv s^\intercal C s$.

Back to Example

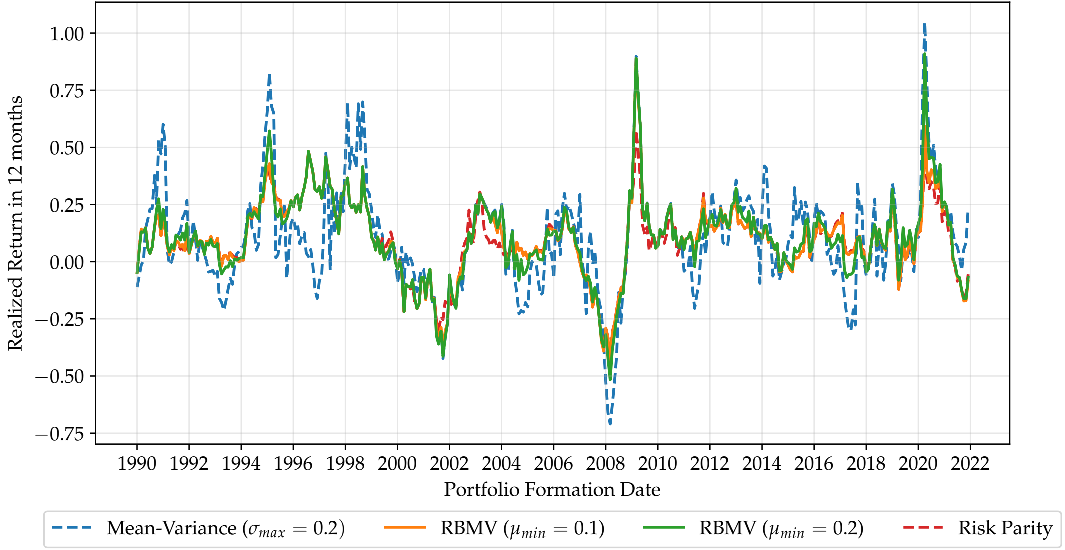
Realized Sharpe Ratio

▸ Back to Average Results



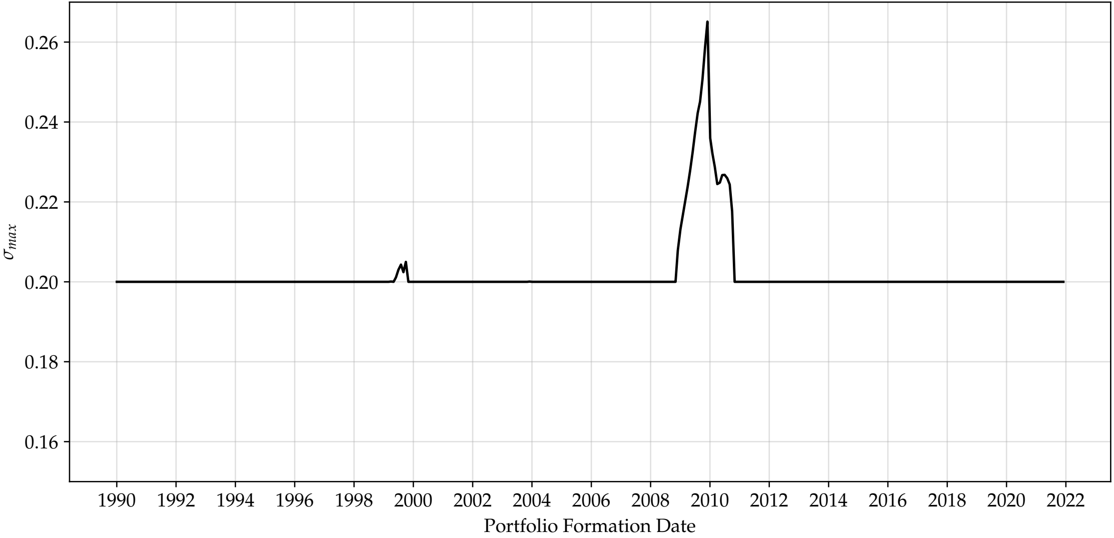
Realized Returns

Back to Average Results



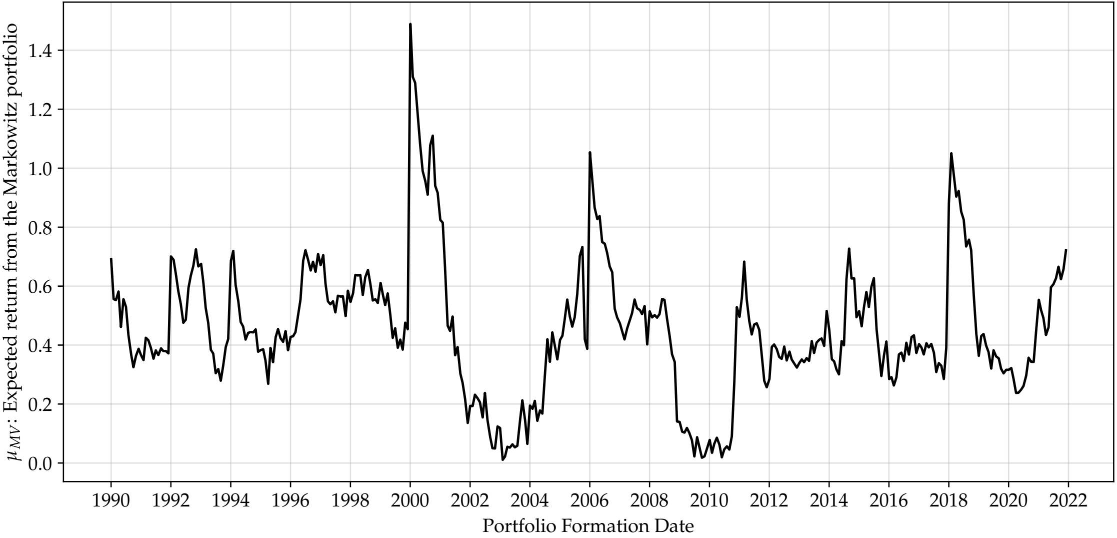
Actual Maximal Volatility

▸ Back to Average Results



Expected Return From The MV Approach

▸ Back to Average Results



Alternative Samples

Back to Average Results

Panel B: Before the Great Financial Crisis (1990-2006)

Portfolio	Return (%)	Volatility (%)	Sharpe Ratio	Gini Index ($\boldsymbol{w}_i$)
Risk Parity	10.79	14.42	0.89	0.15
RBMV ($\mu_{min} = 0.1$, $\sigma_{max} = 0.2$)	10.90	14.45	0.91	0.26
RBMV ($\mu_{min} = 0.2$, $\sigma_{max} = 0.2$)	10.74	14.99	0.85	0.39
Mean-Variance ($\sigma_{max} = 0.2$)	9.92	20.84	0.50	0.95

Panel C: After 2020 (2021-2022)

Portfolio	Return (%)	Volatility (%)	Sharpe Ratio	Gini Index ($\boldsymbol{w}_i$)
Risk Parity	0.07	15.65	0.21	0.16
RBMV ($\mu_{min} = 0.1$, $\sigma_{max} = 0.2$)	0.93	16.00	0.27	0.46
RBMV ($\mu_{min} = 0.2$, $\sigma_{max} = 0.2$)	1.52	16.01	0.30	0.47
Mean-Variance ($\sigma_{max} = 0.2$)	12.62	16.31	0.84	0.96

References

- Maillard, Sébastien, Thierry Roncalli, and Jérôme Teïletche (2010). "The properties of equally-weighted risk contributions portfolios". In: *The Journal of Portfolio Management*.
- Roncalli, Thierry (2014). "Introducing expected returns into risk parity portfolios: A new framework for asset allocation". In: *Available at SSRN 2321309*.