2154. Let $f(n)$ denote the number of ordered partitions of a positive integer n such that all of the parts are odd. For example, $f(5) = 5$ since 5 can be written as 5, $3+1+1$, $1+3+1$, $3+1+1$, and $1+1+1+1+1$. Determine $f(n)$.

(There is a typo in the problem, $3+1+1$ is repeated, one of them should be $1+1+3$.)

Solution. The generating function for the odd integers is:

$$ODD(z) = \sum_{n \geq 0} z^{2n+1} = z \sum_{n \geq 0} z^{2n} = \frac{z}{1-z^2}$$

The ordered partitions of a integer into odd parts can be seen as a non-empty sequence of odd integers. From the theory of analytic combinatorics, a non-empty sequence of objects $SEQ_{\geq 0}(G)$ has a generating function $1/(1-G)-1$. Combining the results, the generating function for the ordered partitions into odd parts is:

$$F(z) = SEQ_{\geq 0}(ODD(z)) = \frac{1}{1-\frac{z}{1-z^2}} - 1 = \frac{z}{1-z-z^2}$$

This expression is the well-known generating function of the Fibonacci numbers. Therefore, the solution are the Fibonacci numbers themselves: $f(n) = F_n$.