1258. For a positive integer n, show that

$$\left(\frac{\cos(1)+\cos(2)+\cos(3)+\cdots+\cos(n)}{\sin(1)+\sin(2)+\cdots+\sin(n)}\right)^2 = \frac{2}{1-\cos(n+1)} - 1$$

Solution. Expressing the trigonometric functions as exponentials, we have:

$$\left(\frac{\cos(1)+\cos(2)+\cos(3)+\cdots+\cos(n)}{\sin(1)+\sin(2)+\cdots+\sin(n)}\right)^2 = \left(\frac{\displaystyle\sum_{1\le x\le n} \frac{e^{ix}+e^{-ix}}{2}}{\displaystyle\sum_{1\le x\le n} \frac{e^{ix}-e^{-ix}}{2i}}\right)^2$$

Each geometric sum can be evaluated in separate, leading to:

$$\begin{aligned}
\left(\frac{2i}{2}\right)^2 \left(\frac{\displaystyle\sum_{1\le x\le n} e^{ix}+e^{-ix}}{\displaystyle\sum_{1\le x\le n} e^{ix}-e^{-ix}}\right)^2 &= -\left(\frac{\dfrac{e^i\left(e^{in}-1\right)}{e^i-1}+\dfrac{e^{-in}\left(e^{in}-1\right)}{e^i-1}}{\dfrac{e^i\left(e^{in}-1\right)}{e^i-1}-\dfrac{e^{-in}\left(e^{in}-1\right)}{e^i-1}}\right)^2 \\
&= -\left(\frac{e^i+e^{-in}}{e^i-e^{-in}}\right)^2 \\
&= -\left(\frac{1+e^{-i(n+1)}}{1-e^{-i(n+1)}}\right)^2 \\
&= -\left(\frac{1+2e^{-i(n+1)}+e^{-2i(n+1)}}{1-2e^{-i(n+1)}+e^{-2i(n+1)}}\right) \\
&= -\left(\frac{e^{i(n+1)}+2+e^{-i(n+1)}}{e^{i(n+1)}-2+e^{-i(n+1)}}\right)
\end{aligned}$$

Converting the exponentials back to trigonometric functions:

$$
\begin{aligned}
-\left(\frac{e^{i(n+1)}+2+e^{-i(n+1)}}{e^{i(n+1)}-2+e^{-i(n+1)}}\right) &= -\left(\frac{2+2\left(\dfrac{e^{i(n+1)}+e^{-i(n+1)}}{2}\right)}{-2+2\left(\dfrac{e^{i(n+1)}+e^{-i(n+1)}}{2}\right)}\right) \\
&= \frac{1+\cos(n+1)}{1-\cos(n+1)} \\
&= \frac{1+\cos(n+1)}{1-\cos(n+1)}+1-1 \\
&= \left(\frac{1+\cos(n+1)+1-\cos(n+1)}{1-\cos(n+1)}\right)-1 \\
&= \frac{2}{1-\cos(n+1)}-1
\end{aligned}
$$

$\square$