11999. Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor}}{k(k+1)}$$

Solution. Let $k = q^2 + r$, where q, r are integers with $q > 0$ and $0 \leq r < 2q+1$. We claim that:

$$(-1)^{\lfloor \sqrt{k}+\sqrt{k+1} \rfloor} = \begin{cases} 1 & \text{for } 0 \leq r < q \\ -1 & \text{for } q \leq r < 2q+1 \end{cases}$$

We need the four lemmas below to prove this.

Lemma 1. *If $0 \leq r < q$, then $2q < \sqrt{k} + \sqrt{k+1}$.*

Proof. This is trivial since $\sqrt{q^2+r} + \sqrt{q^2+r+1} > \sqrt{q^2} + \sqrt{q^2} > 2q$. $\square$

Lemma 2. *If $0 \leq r < q$, then $\sqrt{k} + \sqrt{k+1} < 2q+1$.*

Proof. We start by noticing that $\left(q + \frac{r}{2q}\right)^2 = q^2 + r + \frac{r^2}{4q^2} \geq q^2 + r$, since the last term is non-negative. Therefore, $q + \frac{r}{2q} \geq \sqrt{q^2+r}$, with equality only when $r = 0$. Now we apply it twice:

$$\begin{aligned} \sqrt{q^2+r} + \sqrt{q^2+r+1} &\leq \sqrt{q^2+q-1} + \sqrt{q^2+q} \\ &< q + \frac{q-1}{2q} + q + \frac{q}{2q} \\ &< 2q + 1 - \frac{1}{2q} \\ &< 2q+1 \end{aligned}$$

$\square$

Lemma 3. *If $q \leq r < 2q+1$, then $2q+1 < \sqrt{k} + \sqrt{k+1}$.*

Proof. Consider the sequences $(q, \sqrt{q}, 1)$ and $(q, \sqrt{q}, 0)$ and apply Minkowski's inequality:

$$\begin{aligned} \sqrt{q^2+r} + \sqrt{q^2+r+1} &\geq \sqrt{q^2+q} + \sqrt{q^2+q+1} \\ &> \sqrt{(2q)^2 + (2\sqrt{q})^2 + 1^2} \\ &> \sqrt{4q^2+4q+1} \\ &> 2q+1 \end{aligned}$$

$\square$

Lemma 4. *If $q \leq r < 2q+1$, then $\sqrt{k} + \sqrt{k+1} < 2q+2$.*

Proof. We can use again the inequality $\sqrt{q^2+r} \leq q + \frac{r}{2q}$ to get:

$$\begin{aligned}\sqrt{q^2+2q} + \sqrt{q^2+2q+1} &< q + \frac{2q}{2q} + q + 1 \\ &< 2q+2\end{aligned}$$ ☐

When $0 \leq r < q$, from Lemmas 1 and 2, we have:

$$\begin{aligned}2q < \sqrt{k} + \sqrt{k+1} < 2q+1 &\implies \left\lfloor \sqrt{k} + \sqrt{k+1} \right\rfloor = 2q \\ &\implies (-1)^{\left\lfloor \sqrt{k}+\sqrt{k+1} \right\rfloor} = 1\end{aligned}$$

When $q \leq r < 2q+1$, from Lemmas 3 and 4, we have:

$$\begin{aligned}2q+1 < \sqrt{k} + \sqrt{k+1} < 2q+2 &\implies \left\lfloor \sqrt{k} + \sqrt{k+1} \right\rfloor = 2q+1 \\ &\implies (-1)^{\left\lfloor \sqrt{k}+\sqrt{k+1} \right\rfloor} = -1\end{aligned}$$

Since we can write every positive integer as q^2+r with $0 \leq r < 2q+1$, then we can write the original summation as:

$$\sum_{k=1}^{\infty} \frac{(-1)^{\left\lfloor \sqrt{k}+\sqrt{k+1} \right\rfloor}}{k(k+1)} = \sum_{q\geq 1} \left(\sum_{q^2 \leq r < q^2+q} \frac{1}{r(r+1)} \right) - \sum_{q \geq 1} \left(\sum_{q^2+q \leq r \leq q^2+2q} \frac{1}{r(r+1)} \right)$$

Let's consider the first summation. After splitting the fractions, the sum telescope:

$$
\begin{aligned}
\sum_{q\geq 1}\left(\sum_{q^2\leq r<q^2+q}\frac{1}{r(r+1)}\right) &= \sum_{q\geq 1}\left(\sum_{q^2\leq r<q^2+q}\frac{1}{r}-\frac{1}{r+1}\right)\\
&= \sum_{q\geq 1}\left(\frac{1}{q^2}-\frac{1}{q^2+q}\right)\\
&= \sum_{q\geq 1}\left(\frac{1}{q^2}-\frac{1}{q}+\frac{1}{q+1}\right)\\
&= \sum_{q\geq 1}\frac{1}{q^2}-\sum_{q\geq 1}\frac{1}{q}+\sum_{q\geq 1}\frac{1}{q+1}\\
&= \sum_{q\geq 1}\frac{1}{q^2}-\sum_{q\geq 1}\frac{1}{q}+\sum_{q\geq 2}\frac{1}{q}\\
&= \sum_{q\geq 1}\frac{1}{q^2}-\sum_{q\geq 1}\frac{1}{q}+\left(\sum_{q\geq 1}\frac{1}{q}\right)-1\\
&= \sum_{q\geq 1}\frac{1}{q^2}-1\\
&= \zeta(2)-1\\
&= \frac{\pi^2}{6}-1
\end{aligned}
$$

The same can be done with the second summation:

$$
\begin{aligned}
\sum_{q\geq 1}\left(\sum_{q^2+q\leq r\leq q^2+2q}\frac{1}{r(r+1)}\right) &= \sum_{q\geq 1}\left(\sum_{q^2+q\leq r\leq q^2+2q}\frac{1}{r}-\frac{1}{r+1}\right)\\
&= \sum_{q\geq 1}\left(\frac{1}{q^2+q}-\frac{1}{q^2+2q+1}\right)\\
&= \sum_{q\geq 1}\left(\frac{1}{q}-\frac{1}{q+1}-\frac{1}{(q+1)^2}\right)\\
&= \sum_{q\geq 1}\frac{1}{q}-\sum_{q\geq 1}\frac{1}{q+1}-\sum_{q\geq 1}\frac{1}{(q+1)^2}\\
&= \sum_{q\geq 1}\frac{1}{q^2}-\left(\sum_{q\geq 1}\frac{1}{q}\right)+1-\left(\sum_{q\geq 1}\frac{1}{q^2}\right)+1\\
&= 2-\sum_{q\geq 1}\frac{1}{q^2}\\
&= 2-\zeta(2)\\
&= 2-\frac{\pi^2}{6}
\end{aligned}
$$

The original summation can now be expressed as the two partial sums, arriving the final result:

$$
\begin{aligned}
\sum_{k=1}^{\infty}\frac{(-1)^{\left\lfloor\sqrt{k}+\sqrt{k+1}\right\rfloor}}{k(k+1)} &= \frac{\pi^2}{6}-1-\left(2-\frac{\pi^2}{6}\right)\\
&= \frac{\pi^2}{3}-3
\end{aligned}
$$