2186. Evaluate

$$\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\,dx$$

Solution.

Let $x=\sqrt{1-\cos\theta}$. It follows that $x\sqrt{2-x^2}=\sin\theta$ and the integral becomes:

$$\begin{aligned}
\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\,dx &= \int_0^{\pi/2} \frac{\operatorname{arctanh}(\sin\theta)}{\sqrt{1-\cos\theta}}\left(\frac{\sin\theta}{2\sqrt{1-\cos\theta}}\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta\operatorname{arctanh}(\sin\theta)}{1-\cos\theta}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta\operatorname{arctanh}(\sin\theta)}{1-\cos\theta}\left(\frac{1+\cos\theta}{1+\cos\theta}\right)d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\sin\theta(1+\cos\theta)\operatorname{arctanh}(\sin\theta)}{(\sin\theta)^2}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{(1+\cos\theta)\operatorname{arctanh}(\sin\theta)}{\sin\theta}\,d\theta \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\operatorname{arctanh}(\sin\theta)}{\sin\theta}\,d\theta + \frac{1}{2}\int_0^{\pi/2} \frac{\cos\theta\operatorname{arctanh}(\sin\theta)}{\sin\theta}\,d\theta
\end{aligned}$$

For the first integral, let's introduce a parameter a and then differentiate it:

$$\begin{aligned}
I(a) &= \int_0^{\pi/2} \frac{\operatorname{arctanh}(a\sin\theta)}{\sin\theta}\,d\theta \\
I'(a) &= \frac{d}{da}\int_0^{\pi/2} \frac{\operatorname{arctanh}(a\sin\theta)}{\sin\theta}\,d\theta \\
&= \int_0^{\pi/2} \frac{\partial}{\partial a}\frac{\operatorname{arctanh}(a\sin\theta)}{\sin\theta}\,d\theta \\
&= \int_0^{\pi/2} \left(\frac{\sin\theta}{1-a^2\sin^2\theta}\right)\frac{1}{\sin\theta}\,d\theta \\
&= \int_0^{\pi/2} \frac{1}{1-a^2\sin^2\theta}\,d\theta
\end{aligned}$$

With the substitution $u=\cot\theta$, followed by $s=\frac{u}{\sqrt{1-a^2}}$, we have:

$$
\begin{aligned}
I'(a) &= \int_{\infty}^{0} \frac{1}{1-a^2\frac{1}{1+u^2}} \left(\frac{1}{-1-u^2} \right) \, du \\
&= \int_{\infty}^{0} \frac{1}{a^2-1-u^2} \, du \\
&= \frac{1}{1-a^2} \int_{0}^{\infty} \frac{1}{1+\frac{u^2}{1-a^2}} \, du \\
&= \frac{1}{1-a^2} \int_{0}^{\infty} \frac{1}{1+s^2} \left(\sqrt{1-a^2} \right) \, ds \\
&= \frac{1}{\sqrt{1-a^2}} \left(\arctan s \right)_0^{\infty} \\
&= \frac{\pi}{2} \frac{1}{\sqrt{1-a^2}} \\
I(a) &= \frac{\pi}{2} \arcsin a + C
\end{aligned}
$$

By setting $a = 0$, it's clear that $I(0) = 0$, and therefore $C = 0$. The first integral can be evaluated by setting $a = 1$:

$$
I(1) = \frac{\pi}{2} \arcsin 1 = \frac{\pi^2}{4}
$$

For the second integral we again introduce a parameter a:

$$
\begin{aligned}
J(a) &= \int_{0}^{\pi/2} \frac{\cos\theta \operatorname{arctanh}(a \sin\theta)}{\sin\theta} \, d\theta \\
J'(a) &= \int_{0}^{\pi/2} \frac{\cos\theta}{\sin\theta} \left(\frac{\sin\theta}{1-a^2\sin^2\theta} \right) \, d\theta \\
&= \int_{0}^{\pi/2} \frac{\cos\theta}{1-a^2\sin^2\theta} \, d\theta
\end{aligned}
$$

The substitution $u = a\sin\theta$ leads to:

$$
\begin{aligned}
J'(a) &= \frac{1}{a}\int_0^a \frac{1}{1-u^2}\, du \\
&= \frac{1}{a}\left(\operatorname{arctanh} u\right)_0^a \\
&= \frac{\operatorname{arctanh} a}{a} \\
J(a) &= \int \frac{\operatorname{arctanh} a}{a}\, da
\end{aligned}
$$

Replacing arctanh with its Taylor series we get:

$$
\begin{aligned}
J(a) &= \int \frac{1}{a}\sum_{n\geq 1}\frac{a^{2n-1}}{2n-1} \\
&= \sum_{n\geq 1}\int \frac{a^{2n-2}}{2n-1}\, da \\
&= \sum_{n\geq 1}\left(\frac{a^{2n-1}}{(2n-1)^2} + C\right)
\end{aligned}
$$

For $a = 0$ we should again have $J(0) = 0$, which implies $C = 0$. The desired value for the second integral is $J(1)$:

$$
\begin{aligned}
J(1) &= \sum_{n\geq 1}\frac{1}{(2n-1)^2} \\
&= \sum_{n\geq 1}\frac{1}{n^2} - \sum_{n\geq 1}\frac{1}{(2n)^2} \\
&= \sum_{n\geq 1}\frac{1}{n^2} - \frac{1}{4}\sum_{n\geq 1}\frac{1}{n^2} \\
&= \left(1-\frac{1}{4}\right)\zeta(2) \\
&= \frac{3}{4}\left(\frac{\pi^2}{6}\right) \\
&= \frac{\pi^2}{8}
\end{aligned}
$$

The original integral now can be determined by our previous results:

$$
\begin{aligned}
\int_0^1 \frac{\operatorname{arctanh}\left(x\sqrt{2-x^2}\right)}{x}\, dx &= \frac{1}{2}I(1) + \frac{1}{2}J(1) \\
&= \frac{1}{2}\left(\frac{\pi^2}{4} + \frac{\pi^2}{8}\right) \\
&= \frac{3\pi^2}{16}
\end{aligned}
$$