12338. Prove:

$$\int_0^\infty \frac{\cos(x) - 1}{x(e^x - 1)}\,dx = \frac{1}{2}\ln(\pi\,\mathrm{csch}(\pi))$$

Solution. We start with the series for $\cos x$:

$$\begin{aligned}
\int_0^\infty \frac{\cos(x) - 1}{x(e^x - 1)}\,dx &= \int_0^\infty \frac{\left(\sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!}\right) - 1}{x(e^x - 1)}\,dx \\
&= \int_0^\infty \sum_{n=1}^\infty \frac{(-1)^n x^{2n-1}}{(2n)!\,(e^x - 1)}\,dx \\
&= \sum_{n=1}^\infty \frac{(-1)^n}{(2n)!} \int_0^\infty \frac{x^{2n-1}}{e^x - 1}\,dx
\end{aligned}$$

We know the Riemann zeta can be expressed as an integral:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1}\,dx$$

and therefore our previous integral can be expressed as $\zeta(2n)\Gamma(2n)$:

$$\begin{aligned}
\sum_{n=1}^\infty \frac{(-1)^n}{(2n)!} \int_0^\infty \frac{x^{2n-1}}{e^x - 1}\,dx &= \sum_{n=1}^\infty \frac{(-1)^n \zeta(2n)\Gamma(2n)}{(2n)!} \\
&= \sum_{n=1}^\infty \frac{(-1)^n \zeta(2n)(2n-1)!}{(2n)!} \\
&= \sum_{n=1}^\infty \frac{(-1)^n \zeta(2n)}{2n}
\end{aligned}$$

The zeta function $\zeta(2n)$ can be expressed in closed form as a function of the Bernoulli numbers:

$$\zeta(2n) = \frac{(-1)^{n+1} B_{2n} (2\pi)^{2n}}{2(2n)!}$$

leading to:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n)}{2n} = \sum_{n=1}^{\infty} \frac{(-1)^n (-1)^{n+1} B_{2n} (2\pi)^{2n}}{4n(2n)!}$$
$$= \sum_{n=1}^{\infty} -\frac{B_{2n}(2\pi)^{2n}}{4n(2n)!}$$

This last series we will solve with a lemma:

Lemma 1. $\ln(k \operatorname{csch} k) = -\sum_{n=1}^{\infty} \frac{B_{2n}(2k)^{2n}}{2n(2n)!}$

Proof. We start with the logaritmic differentiation of $x \operatorname{csch} x$:

$$\frac{d}{dx} \ln(x \operatorname{csch} x) = \frac{1}{x \operatorname{csch} x} \left(\operatorname{csch}(x) - x \coth(x) \operatorname{csch}(x)\right)$$
$$= \frac{1}{x} - \coth(x)$$

Using the series of $\coth x$:

$$\coth x = \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$

we have:

$$\frac{1}{x} - \coth(x) = \frac{1}{x} - \sum_{n=0}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$
$$= -\sum_{n=1}^{\infty} \frac{2^{2n} B_{2n} x^{2n-1}}{(2n)!}$$

At this point we can integrate both sides:

$$
\begin{aligned}
\int_0^k \frac{d}{dx}\ln(x\operatorname{csch}x)\;dx &= -\int_0^k \sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}x^{2n-1}}{(2n)!}\;dx \\
\ln(k\operatorname{csch}k) - \lim_{x\to 0}\ln(x\operatorname{csch}x) &= -\sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!}\int_0^k x^{2n-1}\;dx \\
&= -\sum_{n=1}^{\infty} \frac{2^{2n}B_{2n}}{(2n)!}\left(\frac{k^{2n}}{2n}\right) \\
&= -\sum_{n=1}^{\infty} \frac{B_{2n}(2k)^{2n}}{2n(2n)!}
\end{aligned}
$$

The limit on the left side can be proven to be zero:

$$
\begin{aligned}
\lim_{x\to 0}\ln(x\operatorname{csch}x) &= \ln\left(\lim_{x\to 0} x\operatorname{csch}x\right) \\
&= \ln\left(\lim_{x\to 0}\frac{x}{\sinh x}\right) \\
&= \ln\left(\lim_{x\to 0}\frac{1}{\cosh x}\right) \\
&= 0
\end{aligned}
$$

This completes the proof.

$\square$

We finish the original integral by considering the series for $\ln(\pi\operatorname{csch}\pi)$:

$$
\begin{aligned}
\int_0^\infty \frac{\cos(x)-1}{x\left(e^x-1\right)} &= \sum_{n=1}^{\infty} -\frac{B_{2n}(2\pi)^{2n}}{4n(2n)!} \\
&= \frac{1}{2}\left(-\sum_{n=1}^{\infty}\frac{B_{2n}(2\pi)^{2n}}{2n(2n)!}\right) \\
&= \frac{1}{2}\ln(\pi\operatorname{csch}\pi)
\end{aligned}
$$