1132. Determine a closed form of the series

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{n+m}}{(n+m)!}$$

Solution. We start by introducing $p = n+m$, so $m = p-n$ and $m \geq 1 \implies p \geq n+1$:

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{x^{n+m}}{(n+m)!} &= \sum_{n\geq 1}\sum_{m\geq 1}\frac{x^{n+m}}{(n+m)!} \\
&= \sum_{n\geq 1}\sum_{p\geq n+1}\frac{x^{p}}{p!} \\
&= \sum\sum\frac{x^{p}}{p!}[n\geq 1][p\geq n+1]
\end{aligned}$$

Notice that $p \geq n+1 \implies n+1 \leq p \implies n \leq p-1$ and now we can interchange the summations.

$$\begin{aligned}
\sum\sum\frac{x^{p}}{p!}[n\geq 1][p\geq n+1] &= \sum\sum\frac{x^{p}}{p!}[1\leq n\leq p-1] \\
&= \sum\sum\frac{x^{p}}{p!}[1\leq p-1][n\leq p-1] \\
&= \sum\sum\frac{x^{p}}{p!}[p\geq 2][1\leq n\leq p-1] \\
&= \sum_{p\geq 2}\left(\sum_{1\leq n\leq p-1}\frac{x^{p}}{p!}\right)
\end{aligned}$$

This summation can be massaged to bring out the exponential series:

$$
\begin{aligned}
\sum_{p\geq 2}\left(\sum_{1\leq n\leq p-1}\frac{x^p}{p!}\right) &= \sum_{p\geq 2}\frac{p-1}{p!}x^p \\
&= \sum_{p\geq 2}\frac{p-1}{p!}x^p + \frac{x^p}{p!} - \frac{x^p}{p!} \\
&= \sum_{p\geq 2}\frac{p}{p!}x^p - \frac{x^p}{p!} \\
&= \sum_{p\geq 2}\frac{x^p}{(p-1)!} - \frac{x^p}{p!} \\
&= \sum_{p\geq 2}x\frac{x^{p-1}}{(p-1)!} - \sum_{p\geq 2}\frac{x^p}{p!} \\
&= \left(x\sum_{p\geq 1}\frac{x^p}{p!}\right) - \left(-1 - x + \sum_{p\geq 0}\frac{x^p}{p}\right) \\
&= \left(x\left(e^x - 1\right)\right) - \left(-1 - x + e^x\right) \\
&= xe^x - x + 1 + x - e^x \\
&= 1 + (x-1)e^x
\end{aligned}
$$

The desired closed form, therefore, is $1 + (x-1)e^x$.