2147. Evaluate:

$$\prod_{n=2}^{\infty} \frac{n^4+4}{n^4-1}$$

Solution. We start by factoring and completing squares:

$$\begin{aligned}
\prod_{n=2}^{\infty} \frac{n^4+4}{n^4-1} &= \prod_{n=2}^{\infty} \frac{n^4+4n^2+4-4n^2}{(n^2+1)(n^2-1)} \\
&= \prod_{n=2}^{\infty} \frac{\left(n^2+2\right)^2-(2n)^2}{(n^2+1)(n+1)(n-1)} \\
&= \prod_{n=2}^{\infty} \frac{\left(n^2+2n+2\right)\left(n^2-2n+2\right)}{(n^2+1)(n+1)(n-1)} \\
&= \prod_{n=2}^{\infty} \frac{\left(n^2+2n+1+1\right)\left(n^2-2n+1+1\right)}{(n^2+1)(n+1)(n-1)} \\
&= \prod_{n=2}^{\infty} \frac{\left((n+1)^2+1\right)\left((n-1)^2+1\right)}{(n^2+1)(n+1)(n-1)}
\end{aligned}$$

At this point we can split the expression into five products, and change their indexes:

$$\prod_{n=2}^{\infty} \frac{\left((n+1)^2+1\right)\left((n-1)^2+1\right)}{(n^2+1)(n+1)(n-1)} = ABCDE$$

$$A = \prod_{n=2}^{\infty} (n+1)^2+1 = \prod_{n=3}^{\infty} n^2+1 = \frac{1}{(1^2+1)(2^2+1)} \prod_{n=1}^{\infty} n^2+1 = \frac{1}{10} \prod_{n=1}^{\infty} n^2+1$$

$$B = \prod_{n=2}^{\infty} (n-1)^2+1 = \prod_{n=1}^{\infty} n^2+1$$

$$C = \prod_{n=2}^{\infty} \frac{1}{n^2+1} = \left(1^2+1\right) \prod_{n=1}^{\infty} \frac{1}{n^2+1} = 2 \prod_{n=1}^{\infty} \frac{1}{n^2+1}$$

$$D = \prod_{n=2}^{\infty} \frac{1}{n+1} = \prod_{n=3}^{\infty} \frac{1}{n} = (1)(2) \prod_{n=1}^{\infty} \frac{1}{n} = 2 \prod_{n=1}^{\infty} \frac{1}{n}$$

$$E = \prod_{n=2}^{\infty} \frac{1}{n-1} = \prod_{n=1}^{\infty} \frac{1}{n}$$

Joining the products again, we obtain:

$$
\begin{aligned}
ABCDE &= \frac{2}{5}\prod_{n=1}^{\infty}\frac{\left(n^2+1\right)\left(n^2+1\right)}{\left(n^2+1\right)(n)(n)}\\
&= \frac{2}{5}\prod_{n=1}^{\infty}\frac{n^2+1}{n^2}\\
&= \frac{2}{5}\prod_{n=1}^{\infty}1+\frac{1}{n^2}
\end{aligned}
$$

This expression can be simplified with the well-known formula for sinh:

$$
\begin{aligned}
\sinh(x) &= x\prod_{n=1}^{\infty}1+\frac{x^2}{n^2\pi^2}\\
\sinh(\pi x) &= \pi x\prod_{n=1}^{\infty}1+\frac{(\pi x)^2}{n^2\pi^2}\\
\frac{\sinh(\pi x)}{\pi x} &= \prod_{n=1}^{\infty}1+\frac{x^2}{n^2}
\end{aligned}
$$

By using $x = 1$, we arrive at the final expression:

$$
\begin{aligned}
\frac{2}{5}\prod_{n=1}^{\infty}1+\frac{1}{n^2} &= \frac{2}{5}\left(\frac{\sinh(\pi)}{\pi}\right)\\
&= \frac{2\sinh(\pi)}{5\pi}
\end{aligned}
$$