2171. Evaluate the following sums in closed form.

(a) $\displaystyle\sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1}\,\frac{x^{2n}}{(2n)!}\right)$

(b) $\displaystyle\sum_{n=0}^{\infty}\left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^{n-1}\,\frac{x^{2n+1}}{(2n+1)!}\right)$

Solution.

For the item (a), we start by writing the infinite sum in a more concise format, and then open the cos in a series:

$$
\begin{aligned}
A(x) &= \sum_{n=0}^{\infty}\left(\cos x - 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \cdots + (-1)^{n-1}\,\frac{x^{2n}}{(2n)!}\right) \\
&= \sum_{n=0}^{\infty}\left(\cos x - \sum_{k=0}^{n}(-1)^k\,\frac{x^{2k}}{(2k)!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}(-1)^k\,\frac{x^{2k}}{(2k)!} - \sum_{k=0}^{n}(-1)^k\,\frac{x^{2k}}{(2k)!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=n+1}^{\infty}(-1)^k\,\frac{x^{2k}}{(2k)!}\right) \\
&= \sum_{k\geq 1}\left(\sum_{0\leq n<k}(-1)^k\,\frac{x^{2k}}{(2k)!}\right) \\
&= \sum_{k\geq 1}\left(k\,(-1)^k\,\frac{x^{2k}}{(2k)!}\right)
\end{aligned}
$$

At this point we can multiply by $2/x$ and introduce $q = k - 1$ to get the result:

$$
\begin{aligned}
A(x) &= \sum_{k\geq 1} \left(k\,(-1)^k \, \frac{x^{2k}}{(2k)!} \right) \\
\frac{2}{x} A(x) &= \sum_{k\geq 1} \left(2k\,(-1)^k \, \frac{x^{2k-1}}{(2k)!} \right) \\
&= \sum_{k\geq 1} \left((-1)^k \, \frac{x^{2k-1}}{(2k-1)!} \right) \\
&= \sum_{k\geq 1} (-1)^k \, \frac{x^{2k-1}}{(2k-1)!} \\
&= \sum_{q\geq 0} (-1)^{q+1} \, \frac{x^{2q+1}}{(2q+1)!} \\
&= -\sin x \\
A(x) &= -\frac{x \sin x}{2}
\end{aligned}
$$

The calculations for item (b) are similar:

$$
\begin{aligned}
B(x) &= \sum_{n=0}^{\infty} \left(\sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + (-1)^{n-1} \, \frac{x^{2n+1}}{(2n+1)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!} - \sum_{k=0}^{n} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=n+1}^{\infty} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= \sum_{k\geq 1} \left(\sum_{0\leq n<k} (-1)^k \, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= \sum_{k\geq 1} \left(k\,(-1)^k \, \frac{x^{2k+1}}{(2k+1)!} \right)
\end{aligned}
$$

This time we multiply by 2 and split the sum into two parts:

$$
\begin{aligned}
B(x) &= \sum_{k\geq 1} \left(k\,(-1)^k\, \frac{x^{2k+1}}{(2k+1)!} \right) \\
2B(x) &= \sum_{k\geq 1} \left(2k\,(-1)^k\, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= \sum_{k\geq 1} \left((2k+1-1)\,(-1)^k\, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= \sum_{k\geq 1} \left((-1)^k\, \frac{x^{2k+1}}{(2k)!} \right) - \sum_{k\geq 1} \left((-1)^k\, \frac{x^{2k+1}}{(2k+1)!} \right) \\
&= x \sum_{k\geq 1} \left((-1)^k\, \frac{x^{2k}}{(2k)!} \right) - (\sin x - x) \\
&= x(\cos x - 1) - (\sin x - x) \\
&= x\cos x - \sin x \\
B(x) &= \frac{x\cos x - \sin x}{2}
\end{aligned}
$$