2161. Let x_n denote the number of bitstrings of length n which contain neither 11 nor 000 as substrings. Find a recursive formula for x_n.

Solution.

The desired bitstrings match the following regular expression, where ? denote an optional expression, | is the exclusive or operator, and $*$ is the Kleene star:

$$1?((0|00)1)*(0|00)?$$

Using Flajolet's Analytic Combinatorics, this regular expression can be directly converted into a generating funcion, by replacing concatenation with multiplication, exclusive or with addition, optional expressions with $(1+f(z))$ and the Kleene star with $1/(1-f(z))$.

$$X(z) = (1+z)\,\frac{1}{1-(z+z^2)\,z}\left(1+z+z^2\right) = \frac{(1+z)(1+z+z^2)}{1-z^2-z^3}$$

The desired recursive formula can be extracted directly from the definition of the generating function:

$$\begin{aligned}
X(z) &= \frac{(1+z)(1+z+z^2)}{1-z^2-z^3} \\
\left(1-z^2-z^3\right)\sum_{n\geq 0} x_n z^n &= (1+z)(1+z+z^2) \\
\sum_{n\geq 0} x_n z^n - \sum_{n\geq 0} x_n z^{n+2} - \sum_{n\geq 0} x_n z^{n+3} &= 1+2z+2z^2+z^3 \\
\sum_{n\geq 0} x_n z^n - \sum_{n\geq 2} x_{n-2} z^{n} - \sum_{n\geq 3} x_{n-3} z^{n} &= 1+2z+2z^2+z^3
\end{aligned}$$

The general recurrence can be extracted from the coefficients above when $n \geq 4$:

$$\begin{aligned}
[z^n]\left(\sum_{n\geq 0} x_n z^n - \sum_{n\geq 2} x_{n-2} z^{n} - \sum_{n\geq 3} x_{n-3} z^{n}\right) &= [z^n]\left(1+2z+2z^2+z^3\right) \\
x_n - x_{n-2} - x_{n-3} &= 0 \\
x_n &= x_{n-2} + x_{n-3}
\end{aligned}$$

The initial conditions can be read when n is small. Since the order of the recurrence is 3, we need 3 initial values:

$$
\begin{aligned}
n = 0 &\implies x_0 = 1 \\
n = 1 &\implies x_1 = 2 \\
n = 2 &\implies x_2 - x_0 = 2 \implies x_2 = 3
\end{aligned}
$$

Therefore, the final recurrence formula is:

$$
x_n = \begin{cases} n + 1 & \text{if } n < 3 \\ x_{n-2} + x_{n-3} & \text{if } n \geq 3 \end{cases}
$$