12003. Given an odd positive n, compute:

$$\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2 \frac{k\pi}{n}}$$

Solution. We start with a helping lemma:

Lemma 1. *If n is odd, then $\sum_{k=1}^{n} \frac{1}{\cos^2 \frac{k\pi}{n}} = n^2$.*

Proof. Let's rewrite the sum to use tangents:

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{\cos^2 \frac{k\pi}{n}} &= \sum_{k=1}^{n} \sec^2 \frac{k\pi}{n} \\ &= \sum_{k=1}^{n} 1 + \tan^2 \frac{k\pi}{n} \\ &= n + \sum_{k=1}^{n} \tan^2 \frac{k\pi}{n}\end{aligned}$$

Consider now the following application of Euler's identity, which is valid for all integer k:

$$\left(\cos \frac{k\pi}{n} + i \sin \frac{k\pi}{n}\right)^n = \left(\exp\left(\frac{ik\pi}{n}\right)\right)^n = \exp\left(ik\pi\right) = (-1)^k$$

We can open this expression using the Binomial theorem. Let's also replace $n = 2m+1$ since we have the requirement of n odd.

$$\sum_{p=0}^{2m+1} \binom{2m+1}{p} \left(i \sin \frac{k\pi}{2m+1}\right)^p \left(\cos \frac{k\pi}{2m+1}\right)^{2m+1-p} = (-1)^k$$
$$\sum_{p=0}^{2m+1} \binom{2m+1}{p} \left(i \tan \frac{k\pi}{2m+1}\right)^p \left(\cos \frac{k\pi}{2m+1}\right)^{2m+1} = (-1)^k$$

Now we'll keep only the imaginary part on both sides. The left side is purely imaginary when p is odd, so we'll rewrite the index as $p = 2q+1$, and extract the constants:

$$\sum_{q=0}^{m}\binom{2m+1}{2q+1}\left(i\tan\frac{k\pi}{2m+1}\right)^{2q+1}\left(\cos\frac{k\pi}{2m+1}\right)^{2m+1}=0$$

$$i\tan\frac{k\pi}{2m+1}\left(\cos\frac{k\pi}{2m+1}\right)^{2m+1}\sum_{q=0}^{m}\binom{2m+1}{2q+1}\left(i\tan\frac{k\pi}{2m+1}\right)^{2q}=0$$

We can again divide everything by the constants, provided the tangent is never zero (the cosine will never be zero because we're dividing π by an odd number). The previous expression is valid for all k, the next expression is only valid when k is not a multiple of $2m+1$:

$$\sum_{q=0}^{m}\binom{2m+1}{2q+1}\left(i\tan\frac{k\pi}{2m+1}\right)^{2q}=0$$

$$\sum_{q=0}^{m}\binom{2m+1}{2q+1}\left(-\tan^2\frac{k\pi}{2m+1}\right)^{q}=0$$

At this point, notice that $-\tan^2\frac{k\pi}{2m+1}$ is a root of the following polynomial equation, for every k that is not a multiple of $2m+1$:

$$P(x)=\sum_{q=0}^{m}\binom{2m+1}{2q+1}x^q=0$$

We can use Vieta's formula to get the sum of the roots of this equation. The degree is m so we'll have m unique roots. These roots correspond exactly to $-\tan^2\frac{k\pi}{2m+1}$ for k from 1 to m. Remember we have $k=0$ excluded due to the division early on; and also notice that roots for k from $m+1$ to $2m$ are repeated, since $\tan^2\frac{(2m+1-k)\pi}{2m+1}=\tan^2\left(\pi-\frac{k\pi}{2m+1}\right)=\tan^2\frac{k\pi}{2m+1}$.

$$\sum_{k=1}^{m}-\tan^2\frac{k\pi}{2m+1}=-\frac{[x^{m-1}]P(x)}{[x^m]P(x)}$$

$$\sum_{k=1}^{m}\tan^2\frac{k\pi}{2m+1}=\frac{\binom{2m+1}{2(m-1)+1}}{\binom{2m+1}{2m+1}}$$

$$\sum_{k=1}^{m}\tan^2\frac{k\pi}{2m+1}=(2m+1)m$$

Since the tangents are repeated with k from $m+1$ to $2m$, we have:

$$\begin{aligned}\sum_{k=1}^{2m+1} \tan^2 \frac{k\pi}{2m+1} &= \sum_{k=1}^{m} \tan^2 \frac{k\pi}{2m+1} + \sum_{k=m+1}^{2m} \tan^2 \frac{k\pi}{2m+1} + \left(\tan^2 \frac{(2m+1)\pi}{2m+1}\right) \\ &= (2m+1)m + (2m+1)m + 0 \\ &= 2m(2m+1) \\ &= n(n-1)\end{aligned}$$

And finally, returning to our original claim:

$$\sum_{k=1}^{n} \frac{1}{\cos^2 \frac{k\pi}{n}} = n + \sum_{k=1}^{n} \tan^2 \frac{k\pi}{n} = n + n(n-1) = n^2$$

$\square$

We will now make use of the following well-known identity relating the gcd to Euler's totient function:

$$\gcd(a,b) = \sum_{d|a \text{ and } d|b} \varphi(d)$$

Substituting in our original summation and making use of Iverson's notation:

$$\begin{aligned}\sum_{k=1}^{n} \frac{\gcd(k,n)}{\cos^2 \frac{k\pi}{n}} &= \sum_{k=1}^{n} \frac{1}{\cos^2 \frac{k\pi}{n}} \sum_{d|k \text{ and } d|n} \varphi(d) \\ &= \sum_{k=1}^{n} \frac{1}{\cos^2 \frac{k\pi}{n}} \sum_{d\geq 1} \varphi(d)\ [d|k]\ [d|n] \\ &= \sum_{d\geq 1} \varphi(d)\ [d|n] \sum_{1\leq k\leq n} \frac{[d|k]}{\cos^2 \frac{k\pi}{n}} \\ &= \sum_{d|n} \varphi(d) \sum_{1\leq q\leq n/d} \frac{1}{\cos^2 \frac{q\pi}{n/d}} \\ &= \sum_{d|n} \varphi(d) \left(\frac{n}{d}\right)^2\end{aligned}$$

I am unsure if this last expression has a closed form. We can notice, however, that it corresponds to the Dirichlet convolution of $\varphi(n)$ and n^2, and therefore the Dirichlet generating function of our expression is given by the product of each Dirichlet g.f.:

$$\sum_{n\geq 1}\sum_{d|n}\varphi(d)\left(\frac{n}{d}\right)^2 n^{-s} = \left(\sum_{n\geq 1}\varphi(n)\, n^{-s}\right)\left(\sum_{n\geq 1} n^2\, n^{-s}\right)$$
$$= \frac{\zeta(s-1)\zeta(s-2)}{\zeta(s)}$$

Also, since both $\varphi(n)$ and n^2 are multiplicative, the solution we found before also is multiplicative. Let's call it $g(n)$, then $g(ab) = g(a)g(b)$ and it can be completely defined by its value at powers of primes:

$$\begin{aligned}
g(n) &= \sum_{d|n}\varphi(d)\left(\frac{n}{d}\right)^2 \\
g(p^k) &= \sum_{p^i|p^k}\varphi(p^i)\left(\frac{p^k}{p^i}\right)^2 \\
&= p^{2k} + \sum_{1\leq i\leq k}\varphi(p^i)\left(\frac{p^k}{p^i}\right)^2 \\
&= p^{2k} + p^{2k}\sum_{1\leq i\leq k}\frac{p^i - p^{i-1}}{p^{2i}} \\
&= p^{2k} + p^{2k}p^{-k-1}\left(p^k - 1\right) \\
&= p^{k-1}\left(p^{k+1} + p^k - 1\right) \\
g(n) &= \prod_{p^k|n} p^{k-1}\left(p^{k+1} + p^k - 1\right)
\end{aligned}$$

This last product is taken for every p^k dividing n, where p is prime and k is max.