12398. Evaluate:

$$\sum_{n=0}^{\infty} \operatorname{csch}\left(2^n\right)$$

.

Solution. First we notice that $\coth(x/2) - \coth(x) = \operatorname{csch}(x)$. Using the definitions:

$$\begin{aligned}
\coth\left(x/2\right) - \coth\left(x\right) &= \frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1} - \frac{\mathrm{e}^{2x} + 1}{\mathrm{e}^{2x} - 1} \\
&= \frac{\mathrm{e}^x + 1}{\mathrm{e}^x - 1} - \frac{\mathrm{e}^{2x} + 1}{\left(\mathrm{e}^x - 1\right)\left(\mathrm{e}^x - 1\right)} \\
&= \frac{\left(\mathrm{e}^x + 1\right)^2 - \mathrm{e}^{2x} - 1}{\left(\mathrm{e}^x - 1\right)\left(\mathrm{e}^x - 1\right)} \\
&= \frac{2\mathrm{e}^x}{\mathrm{e}^{2x} - 1} \\
&= \operatorname{csch}\left(x\right)
\end{aligned}$$

Now the original sum telescopes:

$$\begin{aligned}
\sum_{n=0}^{\infty} \operatorname{csch}\left(2^n\right) &= \lim_{t\to\infty} \sum_{n=0}^{t} \coth\left(2^{n-1}\right) - \coth\left(2^n\right) \\
&= \lim_{t\to\infty} \coth\left(1/2\right) + \sum_{n=1}^{t} \coth\left(2^{n-1}\right) - \sum_{n=0}^{t-1} \coth\left(2^n\right) - \coth\left(2^t\right) \\
&= \lim_{t\to\infty} \coth\left(1/2\right) - \coth\left(2^t\right) \\
&= \coth\left(1/2\right) - 1
\end{aligned}$$

Plugging in the definition we have our closed form:

$$\sum_{n=0}^{\infty} \operatorname{csch}\left(2^n\right) = \coth\left(1/2\right) - 1 = \frac{\mathrm{e} + 1}{\mathrm{e} - 1} - 1 = \frac{2}{\mathrm{e} - 1}$$