12344. An integer is a *one-drop number* if its decimal digits $d_1 \dots d_n$ satisfy $1 \leq d_1 \leq \dots \leq d_i > d_{i+1} \leq \dots \leq d_n$ for some i. For example, 13802 and 49557 are one-drop numbers. For $n \geq 2$, how many n-digit one-drop numbers are there?

Solution. We'll use the analytic combinatorics from Flajolet and Sedgewick. Let $SEQ(z)$ be the set of sequences of the digit z, and $SEQ_{\geq 1}(z)$ be the set of non-empty sequences of z. Their generating functions are given by:

$$SEQ(z) = \frac{1}{1-z}$$
$$SEQ_{\geq 1}(z) = \frac{1}{1-z} - 1 = \frac{z}{1-z}$$

Suppose the digits where the drop occur are a and b, with $b < a$. The generating function of the one-drop numbers with these digits in the drop are:

$$F(a,b) = SEQ(1) \dots SEQ(a-1) SEQ_{\geq 1}(a) SEQ_{\geq 1}(b) SEQ(b+1) \dots SEQ(9)$$

The generating function of all one-drop numbers is, therefore, the sum of the previous generating functions for all valid a, b:

$$
\begin{aligned}
G(z) &= \sum_{1\le a\le 9}\ \sum_{0\le b<a} F(a,b) \\
&= \sum_{1\le a\le 9}\ \sum_{0\le b<a} SEQ(z)^{a-1}\, SEQ_{\ge 1}(z)\, SEQ_{\ge 1}(z)\, SEQ(z)^{9-b} \\
&= \sum_{1\le a\le 9}\ \sum_{0\le b<a} SEQ(z)^{8+a-b}\, SEQ_{\ge 1}(z)^2 \\
&= \sum_{1\le a\le 9}\ \sum_{0\le b<a} \left(\frac{1}{1-z}\right)^{8+a-b} \left(\frac{z}{1-z}\right)^2 \\
&= \frac{z^2}{(1-z)^{10}} \sum_{1\le a\le 9} \left(\frac{1}{1-z}\right)^{a} \sum_{0\le b<a} (1-z)^b \\
&= \frac{z^2}{(1-z)^{10}} \sum_{1\le a\le 9} \left(\frac{1}{1-z}\right)^{a} \left(\frac{1-(1-z)^a}{z}\right) \\
&= \frac{z}{(1-z)^{10}} \left(\sum_{1\le a\le 9} \left(\frac{1}{1-z}\right)^{a} - \sum_{1\le a\le 9} 1 \right) \\
&= \frac{1}{(1-z)^{10}} \left(\frac{1-(1-z)^9}{(1-z)^9} - 9z \right) \\
&= \frac{1}{(1-z)^{19}} - \frac{1}{(1-z)^{10}} - \frac{9z}{(1-z)^{10}}
\end{aligned}
$$

The coefficients can be extracted directly, since $1/(1-z)^{k+1} = \sum_{n\ge 0} \binom{n+k}{k} x^n$ and multiplying by z is a shift in the coefficients. The solution to the problem is:

$$
[z^n]G(z) = \binom{n+18}{18} - \binom{n+9}{9} - 9\binom{n+8}{9}
$$