Chapter 7 - Efficient Diversification

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There are two broad sources of uncertainty: Market risk and firm-specific risk

- Market risk
  - Attributable to market-wide risk factors
  - Remains after diversification
  - Also called non-diversifiable risk or systematic risk
- Firm-specific risk
  - Eliminated by diversification
  - Also called idiosyncratic risk, diversifiable risk, or nonsystematic risk

Portfolio risk decreases as we increase the number of (diversified) stocks in the portfolio

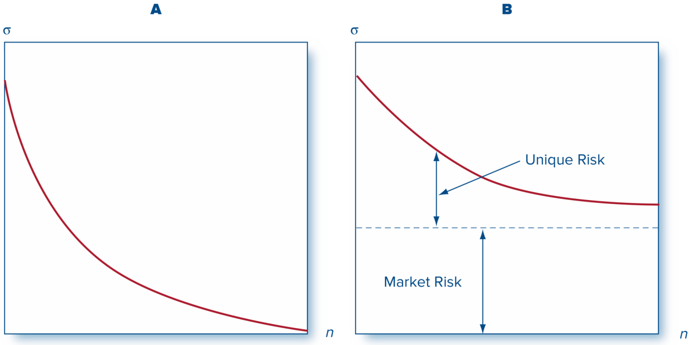

Figure 1: Portfolio risk as a function of the number of stocks in the portfolio. *Panel A:* All risk is firm-specific. *Panel B:* Some risk is systematic of marketwide (BKM 2023, Figure 7.1)

The *insurance principle* reduces risk by spreading exposure across many independent risk sources

- "An insurance company depends on such diversification when it writes many policies insuring against many uncorrelated sources of risk, each policy being a small part of the company's overall portfolio."
- We will revisit this concept later

Consider a portfolio of two risky assets

- The return on a portfolio is the weighted average return on its risky assets

$$r_P = w_D r_D + w_E r_E$$

where ws are weights, rs are returns, and subscripts D and E indicate debt and equity

- This portfolio's expected return is

$$E(r_P) = w_D E(r_D) + w_E E(r_E)$$

and its variance is

$$\sigma_P^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \mathsf{Cov}(r_D, r_E)$$

We can rewrite σ_P^2 as a *border-multiplied* covariance matrix

- $\text{Cov}(r_D, r_D) = \sigma_D^2$
- So we can rewrite

$$\sigma_P^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \text{Cov}(r_D, r_E)$$

as

$$\sigma_P^2 = w_D^2 \text{Cov}(r_D, r_D) + w_E^2 \text{Cov}(r_E, r_E) + 2 w_D w_E \text{Cov}(r_D, r_E)$$

which is a *border-multiplied* covariance matrix

The *border-multiplied* covariance matrix version is easily applied to three or more assets

A. Bordered Covariance Matrix		
Portfolio Weights	$\boldsymbol{w_D}$	$\boldsymbol{w_E}$
w_D	$\text{Cov}(r_D, r_D)$	$\text{Cov}(r_D, r_E)$
w_E	$\text{Cov}(r_E, r_D)$	$\text{Cov}(r_E, r_E)$
B. Border-Multiplied Covariance Matrix		
Portfolio Weights	$\boldsymbol{w_D}$	$\boldsymbol{w_E}$
w_D	$w_D w_D \text{Cov}(r_D, r_D)$	$w_D w_E \text{Cov}(r_D, r_E)$
w_E	$w_E w_D \text{Cov}(r_E, r_D)$	$w_E w_E \text{Cov}(r_E, r_E)$
$w_D + w_E = 1$	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D)$	$w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$
Portfolio variance	$w_D w_D \text{Cov}(r_D, r_D) + w_E w_D \text{Cov}(r_E, r_D) + w_D w_E \text{Cov}(r_D, r_E) + w_E w_E \text{Cov}(r_E, r_E)$	

Figure 2: Computation of portfolio variance from the covariance matrix (BKM 2023, Table 7.2)

We can rewrite σ_P^2 in terms of correlations

- $\rho_{D,E} = \frac{\text{Cov}(r_D, r_E)}{\sigma_D \sigma_E} \Rightarrow \text{Cov}(r_D, r_E) = \rho_{D,E}\sigma_D\sigma_E$
- So we can rewrite

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \text{Cov}(r_D, r_E)$$

as

$$\sigma_P^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E \rho_{D,E}\sigma_D\sigma_E$$

Correlations fall between -1 and +1 (inclusive), making them easier to interpret than covariances

- When $\rho = 1$
$$\sigma_P^2 = (w_D\sigma_D + w_E\sigma_E)^2$$
so $\sigma_P =$ Absolute value$(w_D\sigma_D + w_E\sigma_E)$
- When $\rho < 1$, we reduce risk through *diversification*
- When $\rho < 0$, we *substantially* reduce risk through *hedging*
- When $\rho = -1$
$$\sigma_P^2 = (w_D\sigma_D - w_E\sigma_E)^2$$
so $\sigma_P = 0$ when $w_D\sigma_d - w_E\sigma_E = 0$

Portfolio expected return is the weighted average of asset expected returns, but the same is *not* true of portfolio standard deviation I

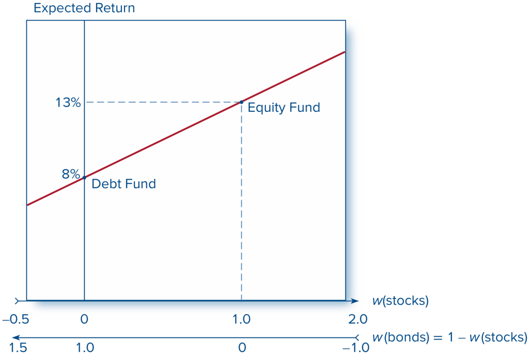

Figure 3: Portfolio expected return as a function of investment proportions (BKM 2023, Figure 7.3)

Portfolio expected return is the weighted average of asset expected returns, but the same is *not* true of portfolio standard deviation II

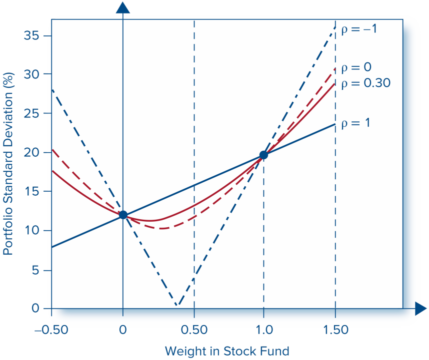

Figure 4: Portfolio standard deviation as a function of investment proportions (BKM 2023, Figure 7.4)

Portfolio expected return is the weighted average of asset expected returns, but the same is *not* true of portfolio standard deviation III

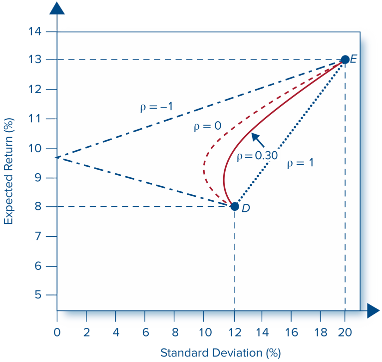

Figure 5: Portfolio expected return as a function of standard deviation (BKM 2023, Figure 7.5)

When allocating capital between risky and risk-free portfolios, investors want the risky portfolio with the highest Sharpe ratio I

- The higher the Sharpe ratio, the higher the expected return corresponding to any level of volatility
- Recall the Sharpe ratio is the slope of the capital allocation line (CAL)
- The steepest CAL intersects that optimal risky portfolio

When allocating capital between risky and risk-free portfolios, investors want the risky portfolio with the highest Sharpe ratio II

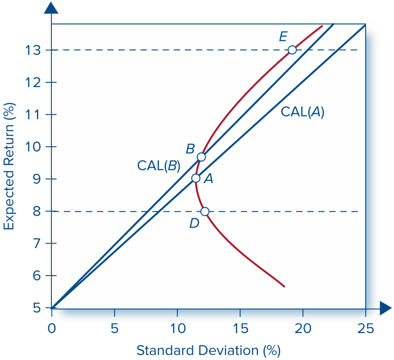

Figure 6: The opportunity set of the debt and equity funds and two feasible CALs (BKM 2023, Figure 7.6)

When allocating capital between risky and risk-free portfolios, investors want the risky portfolio with the highest Sharpe ratio III

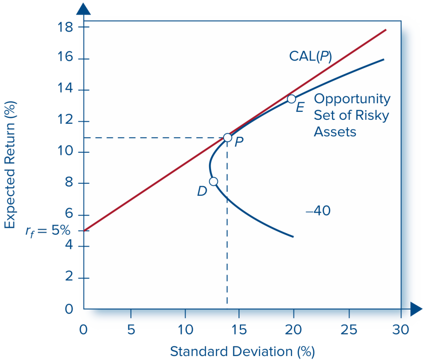

Figure 7: The opportunity set of the debt and equity funds with the optimial CAL and the optimal risky portfolio (BKM 2023, Figure 7.7)

When allocating capital between risky and risk-free portfolios, investors want the risky portfolio with the highest Sharpe ratio IV

- With two risky assets

$$E(r_P) = w_D E(r_D) + w_E E(r_E)$$

and

$$\sigma_P = \sqrt{w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \mathsf{Cov}(r_D, r_E)}$$

- The investor solves

$$\max_w S_P = \frac{r_P - r_f}{\sigma_P}$$

subject to

$$w_D + w_E = 1$$

When allocating capital between risky and risk-free portfolios, investors want the risky portfolio with the highest Sharpe ratio V

- The solution to this maximization problem is

$$w_D = \frac{E(R_D)\sigma_E^2 - E(R_E)\mathsf{Cov}(R_D, R_E)}{E(R_D)\sigma_E^2 + E(R_E)\sigma_D^2 - [E(R_D) + E(R_E)]\mathsf{Cov}(R_D, R_E)}$$

where Rs indicate *excess* returns and $w_E = 1 - w_D$

The optimal *complete* portfolio, given the optimal *risky* portfolio and its CAL, depends on investor risk aversion

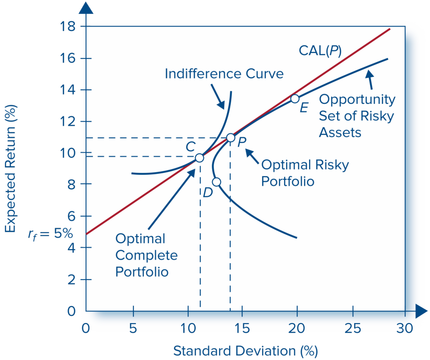

Figure 8: Determination of the optimal complete portfolio (BKM 2023, Figure 7.9)

Putting it all together: How to find the complete portfolio

1. Specify the return characteristics of all securities
2. Establish the risky portfolio (asset allocation)
   - Calculate the optimal risky portfolio P
   - Calculate its properties
3. Allocate funds between the risky portfolio and the risk-free asset (capital allocation)
   - Calculate the fraction of the complete portfolio allocated to P
   - Calculate the share of the complete portfolio invested in each risky and risk-free asset

Markowitz provides a more general solution to finding the complete portfolio

1. Find the risk-return combinations of the risky assets
2. Find the optimal risky portfolio, which has the steepest CAL
3. Find the appropriate complete portfolio by mixing the risk-free asset with the optimal risky portfolio

The *minimum-variance frontier* of risky assets plots the portfolio with the lowest variance for a given expected return I

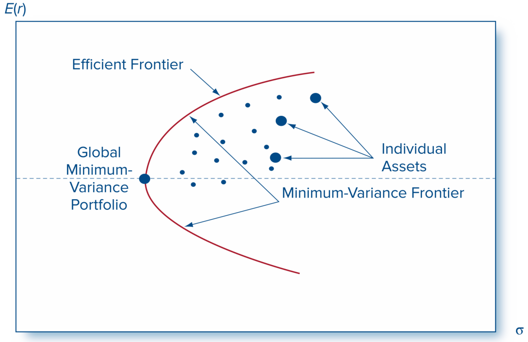

Figure 9: The minimum-variance frontier of risky-assets (BKM 2023, Figure 7.10)

The *minimum-variance frontier* of risky assets plots the portfolio with the lowest variance for a given expected return II

- All individual risky assets plot inside the minimum-variance frontier (if we allow short sales)
- Therefore, portfolios of one risky asset are inefficient because diversification lets us build portfolios with lower standard deviations
- The bottom portion of the minimum-variance frontier is inefficient because the portfolios directly above it have the same standard deviations but higher expected returns
- The *efficient frontier* of risky assets is the portion of the minimum-variance frontier above the *global minimum-variance portfolio*

The *minimum-variance frontier* of risky assets plots the portfolio with the lowest variance for a given expected return III

- We can find the portfolio expected return as

$$E(r_P) = \sum_{i=1}^{n} w_i E(r_i)$$

  and portfolio variance as

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \text{Cov}(r_i, r_j)$$

- Therefore, to find the minimum-variance frontier of n risky assets, we need
  - n forecasts of $E(r_i)$
  - n forecasts of $\text{Var}(r_i)$
  - $\frac{n^2-n}{2}$ forecasts of $\text{Cov}(r_i, r_j)$

The CAL with the optimal portfolio P is tangent to the efficient frontier I

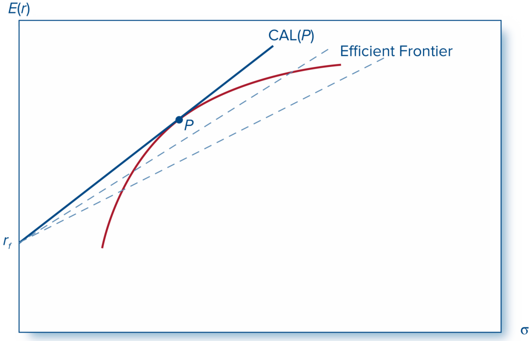

Figure 10: The efficient frontier of risky assets with the optimal CAL (BKM 2023, Figure 7.11)

All investors choose their appropriate complete portfolios as before

- Recall
$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$
- Here y^* is the optimal weight on the optimal risky portfolio and A is the risk aversion index

All investors choose the same optimal risky portfolio, *P*, regardless of their risk aversion

- Risk aversion affects capital allocation, which is choosing y
- Risk aversion *does not* affect finding the optimal risky portfolio, which is choosing w_is
- This result is a *separation property*, which separates the portfolio choice into two *independent* tasks
    1. Find the optimal risky portfolio, which is the same, regardless of risk aversion
    2. Allocate capital, which depends on risk aversion

The risk of a highly diversified portfolio depends on the covariance of asset returns I

- Portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \mathsf{Cov}(r_i, r_j)$$

- For an equally-weighted portfolio, portfolio variance is

$$\sigma_p^2 = \sum_{i=1}^{n} \frac{1}{n^2} \sigma_i^2 + \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{1}{n^2} \mathsf{Cov}(r_i, r_j)$$

The risk of a highly diversified portfolio depends on the covariance of asset returns II

- We can simplify the variance term as

$$\overline{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\sigma_i^2$$

  and the covariance term as

$$\overline{\mathsf{Cov}} = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathsf{Cov}(r_i, r_j)$$

- So, for an equally-weighted portfolio, portfolio variance is

$$\sigma_p^2 = \frac{1}{n}\overline{\sigma}^2 + \frac{n-1}{n}\overline{\mathsf{Cov}}$$

- These simplifications highlight the importance of the covariance of asset returns

The risk of a highly diversified portfolio depends on the covariance of asset returns III

- If $\overline{\text{Cov}} = 0$, $\sigma_p^2 \to 0$ as $n \to \infty$
- If $\overline{\text{Cov}} > 0$, $\frac{1}{n}\overline{\sigma}^2 \to 0$ but $\frac{n-1}{n}\overline{\text{Cov}} \to \overline{\text{Cov}}$ as $n \to 0$
- If all assets have standard deviation σ and all asset-pairs have correlation ρ, we can express portfolio variance as

$$\sigma_p^2 = \frac{1}{n}\sigma^2 + \frac{n-1}{n}\rho\sigma^2$$

and draw the same conclusions

Risk sharing complements risk pooling

- An insurance company that sells n identical fire insurance policies, each with random payoff x with variance σ^2, has total payoff of n policies is $\sum_{i=1}^{n} x_i$
- The variance of total payoff increases as n increases because

$$\text{Var}\left(\sum_{i=1}^{n} x_i\right) = n\sigma^2$$

- However, the variance of the *average* payoff decreases as n increases because

$$\text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$$

- Therefore, true diversification requires allocating a given investment budget across a large number of different assets, limiting exposure to any one asset

Longer horizons alone do not reduce risk

	Investment Horizon (years)			
	1	10	30	Comment
1. Mean of average return	0.050	0.050	0.050	= .05
2. Mean of total return	0.050	0.500	1.500	$= .05^*T$
3. Standard deviation of total return	0.300	0.949	1.643	$= .30\sqrt{T}$
4. Standard deviation of average return	0.300	0.095	0.055	$= .30/\sqrt{T}$
5. Prob(return > 0)	0.566	0.701	0.819	From normal distribution
6. 1% VaR total return	−0.648	−1.707	−2.323	Continuously compounded cumulative return
7. Cumulative loss at 1% VaR	0.477	0.819	0.902	= 1 − exp(cumulative return from line 6)
8. 0.1% VaR total return	−0.877	−2.432	−3.578	Continuously compounded return
9. Cumulative loss at 0.1% VaR	0.584	0.912	0.972	= 1 − exp(cumulative return from line 8)

Figure 11: Investment risk for different horizons (BKM 2023, Table 7.5)

Summary from BKM (2023)

1. The expected return of a portfolio is the weighted average of the component-security expected returns with investment proportions as weights.
2. The variance of a portfolio is the weighted sum of the elements of the covariance matrix using the products of the investment proportions as weights. Thus, the variance of each asset is weighted by the square of its investment proportion. The covariance of each pair of assets appears twice in the covariance matrix; thus, the portfolio variance includes twice each covariance weighted by the product of the investment proportions of each pair of assets.
3. Even if the covariances are positive, the portfolio standard deviation is less than the weighted average of the component standard deviations, as long as the assets are not perfectly positively correlated. Thus, portfolio diversification is beneficial as long as assets are less than perfectly correlated.
4. The greater an asset's covariance with the other assets in the portfolio, the more it contributes to portfolio variance. An asset that is perfectly negatively correlated with a portfolio can serve as a perfect hedge. That perfect hedge asset can reduce the portfolio variance to zero.
5. The efficient frontier shows the set of portfolios that maximize expected return for each level of portfolio risk. Rational investors will choose a portfolio on the efficient frontier.
6. A portfolio manager identifies the efficient frontier by first establishing estimates for asset expected returns and the covariance matrix. This input list is then fed into an optimization program that produces as outputs the investment proportions, expected returns, and standard deviations of the portfolios on the efficient frontier.
7. In practice, portfolio managers will arrive at different efficient portfolios because of differences in methods and quality of security analysis. Managers compete on the quality of their security analysis relative to their management fees.
8. If a risk-free asset is available and input lists are identical, all investors will choose the same portfolio on the efficient frontier of risky assets: the portfolio tangent to the CAL. All investors with identical input lists will hold an identical risky portfolio, differing only in how much each allocates to this optimal portfolio versus the risk-free asset. This result is characterized as the separation principle of portfolio construction.
9. Diversification is based on the allocation of a portfolio of fixed size across several assets, limiting the exposure to any one source of risk. Adding additional risky assets to a portfolio, thereby increasing the total amount invested, does not reduce dollar risk, even if it makes the *rate* of return more predictable. This is because that uncertainty is applied to a larger investment base. Nor does investing over longer horizons reduce risk. Increasing the investment horizon is analogous to investing in more assets. It increases total risk. Analogously, the key to the insurance industry is risk sharing–the spreading of many independent sources of risk across many investors, each of whom takes on only a small exposure to any particular source of risk.

Key equations from BKM (2023)

Expected portfolio return: $E\left(r_p\right) = \sum_{\delta=1}^{n} \Pr(s)\, \mathrm{r_p}(s)$ [with n scenarios, indexed by s]

The expected rate of return on a two-asset portfolio: $E(r_p) = w_D\, E(r_D) + w_E\, E(r_E)$

Variance of portfolio return: $\mathrm{Var}\left(r_p\right) = \sum_{\delta=1}^{n} \Pr(s)\left[r_p(s) - E\left(r_p\right)\right]^2$

Covariance between portfolio returns: $\mathrm{Cov}(r_E, r_D) = \sum_{\delta=1}^{n} \Pr(s)\, [r_E(s) - E(r_E)]\, [r_D(s) - E(r_D)]$

Covariance and correlation: $\mathrm{Cov}(r_E, r_D) = \rho_{ED}\sigma_E\sigma_D$

The variance of the return on a two-asset portfolio: $\sigma_p^2 = \left(\mathrm{w}_D\, \sigma_D\right)^2 + \left(\mathrm{w}_E\, \sigma_E\right)^2 + 2\left(\mathrm{w}_D\, \sigma_D\right)\left(\mathrm{w}_E\, \sigma_E\right)\rho_{DE}$

Variance of n-asset portfolio: $\mathrm{Var}\left(r_p\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i\, w_j\, \mathrm{Cov}\left(r_i, r_j\right)$

The Sharpe ratio of a portfolio: $S_p = \dfrac{E\left(r_p\right) - r_f}{\sigma_p}$

Sharpe ratio maximizing portfolio weights with two risky assets (D and E) and a risk-free asset:

$$
\begin{aligned}
w_D &= \frac{\left[E(r_D) - r_f\right]\sigma_E^2 - \left[E(r_E) - r_f\right]\sigma_D\sigma_E\rho_{DE}}{\left[E(r_D) - r_f\right]\sigma_E^2 + \left[E(r_E) - r_f\right]\sigma_D^2 - \left[E(r_D) - r_f + E(r_E) - r_f\right]\sigma_D\sigma_E\rho_{DE}} \\
w_E &= 1 - w_D
\end{aligned}
$$

Optimal capital allocation to the risky asset: $y = \dfrac{E\left(r_p\right) - r_f}{A\sigma_p^2}$

References I

Bodie, Zvi, Alex Kane, and Allan J. Marcus (2023). *Investments*. 13th ed. New York: McGraw Hill.