Chapter 6 - Capital Allocation to Risky Assets

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Speculation and gambling are different concepts

- Speculation is accepting risk because we perceive a favorable risk-return trade-off
- Gambling is accepting risk because we enjoy it

Historical data show risky assets earn a risk premium and most investors are risk averse

- Risk-averse investors
  - Reject *fair games*, where fair games are risky investments with zero risk premiums
  - Consider only risk-free investments or risky investments with positive risk premiums
- We characterize risk aversion with a utility function
  - A commonly-used utility function is $U = E(r) - \frac{1}{2}A\sigma^2$
  - A is a risk-aversion coefficient, and risk aversion increases as A increases

An investment is more attractive when its expected return is high and its risk is low

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	4%	5%
M (medium risk)	4	6	10
H (high risk)	8	10	20

Figure 1: Available risky portfolios (risk-free rate = 2%) (BKM 2023, Table 6.1)

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r)$ = .04; σ = 0.05]	Utility Score of Portfolio M [$E(r)$ = .06; σ = 0.10]	Utility Score of Portfolio H [$E(r)$ = .10; σ = 0.20]
2.0	$0.04 - ½ \times 2 \times 0.05^2 = 0.0375$	$0.06 - ½ \times 2 \times 0.1^2 = 0.0500$	$\mathbf{0.10 - ½ \times 2 \times 0.2^2 = 0.06}$
3.5	$0.04 - ½ \times 3.5 \times 0.05^2 = 0.0356$	$\mathbf{0.06 - ½ \times 3.5 \times 0.1^2 = 0.0425}$	$0.10 - ½ \times 3.5 \times 0.2^2 = 0.03$
5.0	$0.04 - ½ \times 5 \times 0.05^2 = 0.0338$	$\mathbf{0.06 - ½ \times 5 \times 0.1^2 = 0.0350}$	$0.10 - ½ \times 5 \times 0.2^2 = 0.00$

Figure 2: Utility scores of alternative portfolios for investors with varying degrees of risk aversion (BKM 2023, Table 6.2)

There are three investor types concerning risk

- Risk-averse investors ($A > 0$) only consider risky portfolios if they have risk premiums
- Risk-neutral investors ($A = 0$) ignore risk and only consider expected returns
- Risk-loving investors ($A < 0$) accept lower expected returns on risky portfolios with higher risk

The mean-variance (M-V) criterion ranks portfolios I

The M-V criterion says that portfolio A dominates B if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

with at least one strict inequality

The mean-variance (M-V) criterion ranks portfolios II

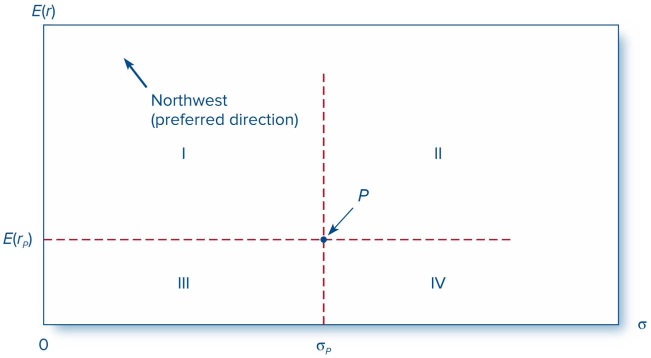

Figure 3: The trade-off between risk and return of a potential investment portfolio, P (BKM 2023, Figure 6.1)

An *indifference curve* connects all portfolios with the same utility

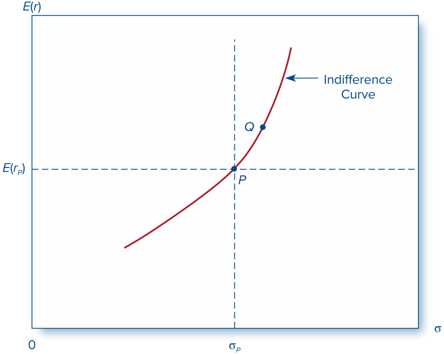

Figure 4: The indifference curve (BKM 2023, Figure 6.2)

a. According to the mean-variance criterion, portfolio A is better than portfolio B for a risk-averse investor whenever _____.

- ○ $E(r_A) \le E(r_B)$ and $\sigma_A \le \sigma_B$
- ○ $E(r_A) \le E(r_B)$ and $\sigma_A \ge \sigma_B$
- ○ $E(r_A) \ge E(r_B)$ and $\sigma_A \le \sigma_B$
- ○ $E(r_A) \ge E(r_B)$ and $\sigma_A \ge \sigma_B$

. A risky portfolio has an expected return of 16% and a standard deviation of 18%. Short-term T-bills have a return of 7%. An investor has $8,000 to invest and needs to decide between investing in the risky portfolio and T-bills. Suppose that the investor has the utility function $U = E(r_p) - (½ A \sigma^2)$ and a risk-aversion parameter of A = 1.8.

a. What is the utility of the risky portfolio for the investor?

b. What is the utility of the T-bills for the investor?

An investor is considering an investment in a risky portfolio. Based on his expectations, the end-of-year cash flow derived from the portfolio will be either $70,000 with a probability of 50% or $154,000 with the probability of 50%. T-bills have a yield of 5%. The investor requires a risk premium of 11%.

a. How much is the investor willing to pay for the portfolio, based on his required risk premium?

The most basic asset allocation choice is between risky and risk-free assets

- We can control risk by changing the portfolio weights on risky and risk-free assets (i.e., y and $1 - y$)
- Only governments can issue default-free bonds
    - A security is risk-free with a guaranteed real return only if
        1. its price is inflation-indexed and
        2. its maturity equals the investor's holding period
    - Still, we view U.S. Treasury bills as "the" risk-free asset
    - Many money market instruments are effectively risk-free, too

The capital allocation line (CAL) plots all available risk-return combinations I

- Consider a portfolio of one risky asset and one risk-free asset
- Allocate y to risky asset P and $1 - y$ to the risk-free asset F
- The return on complete portfolio C is:

$$r_C = yr_P + (1 - y)r_f$$

- Therefore, the expected return on the complete portfolio C is:

$$E(r_C) = yE(r_p) + (1 - y)r_f = r_f + y[E(r_P) - r_f]$$

and its volatility is:

$$\sigma_C = y\sigma_P$$

The capital allocation line (CAL) plots all available risk-return combinations II

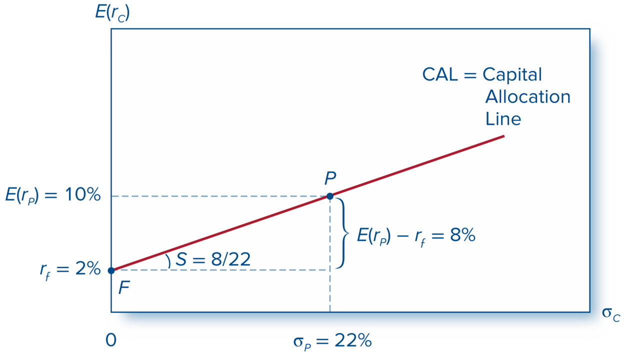

Figure 5: The investment opportunity set with a risk asset and risk-free asset in the expected return-standard deviation plane (BKM 2023, Figure 6.3)

The slope of the CAL is the Sharpe ratio

The slope S of the CAL is:

$$S = \frac{\text{rise}}{\text{run}} = \frac{E(r_P) - r_f}{\sigma_P - 0} = \frac{E(r_P) - r_f}{\sigma_P}$$

so:

$$E(r_P) = r_f + S\sigma_C = r_f + \frac{E(r_P) - r_f}{\sigma_P}\sigma_C$$

We borrow to earn $E(r_C) > E(r_P)$ I

- $E(r_C) > E(r_P)$ requires $y > 0$
- $y > 0$ requires $1 - y < 0$
- $1 - y < 0$ is a short position in F (i.e., borrowing or using leverage)
- The CAL is straight if we borrow at r_f (shown above)
- Only the government borrows at r_f
- Instead, investors borrow at $r_f^B > r_f$, so the CAL has a kink at $y = 1$

We borrow to earn $E(r_C) > E(r_P)$ II

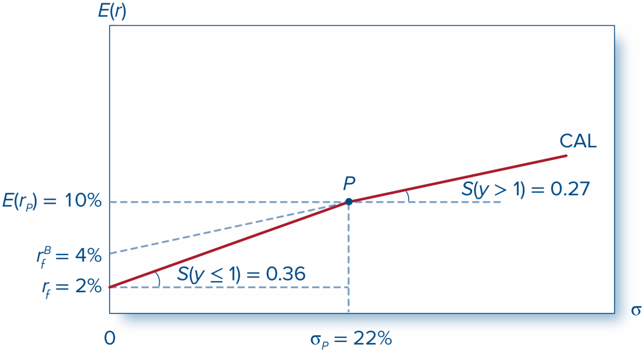

Figure 6: The investment opportunity set with different borrowing and lending rates (BKM 2023, Figure 6.4)

You want to invest in either a stock or Treasury bills (the risk-free asset). The stock has an expected return of 7% and a standard deviation of returns of 46%. T-bills have a return of 1%.

a. If you invest 70% in the stock and 30% in T-bills, what is your expected return for the complete portfolio?

b. What is the standard deviation of returns for such a portfolio?

. You are managing a portfolio with a standard deviation of 40% and an expected return of 22%. The Treasury bill rate is 4%. A client wants to invest 16% of his investment budget in a T-bill money market fund and 84% in your portfolio.

a. What is the expected rate of return on your client's complete portfolio?

b. What is the standard deviation for your client's complete portfolio?

c. What is the reward-to-volatility (Sharpe) ratio of your client's complete portfolio?

d. What is the Sharpe ratio of your portfolio?

An investor chooses y to maximize utility I

- Recall $U = E(r_C) - \frac{1}{2}A\sigma_C^2$
- To maximize utility, the investor solves the following maximization problem:

$$\max_y U = E(r_C) - \frac{1}{2}A\sigma_C^2 = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$$

so:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

An investor chooses y to maximize utility II

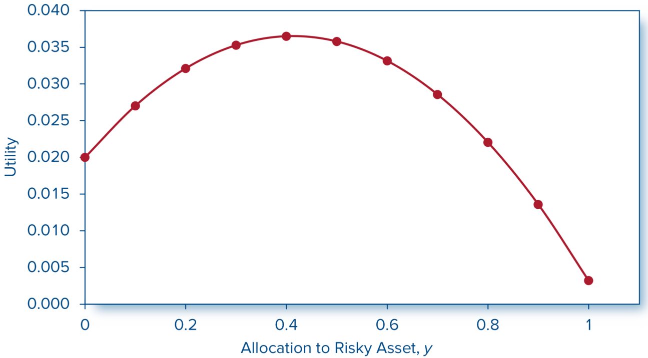

Figure 7: Utility as a function of allocation to the risky asset, y (BKM 2023, Figure 6.5)

An investor chooses y to maximize utility III

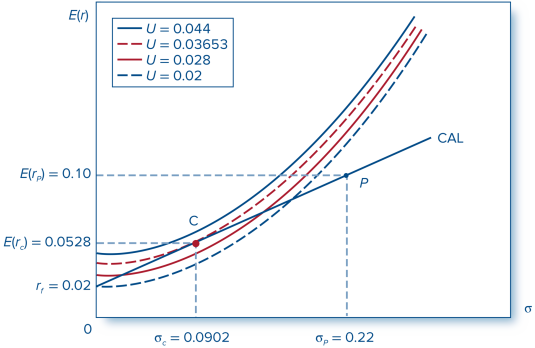

Figure 8: Finding the optimal complete portfolio by using indifference curves (BKM 2023, Figure 6.7)

The normality assumption treats standard deviation as the only measure of risk

- The analysis above is standard
- Still, we might want also to consider
  - VaR and ES to estimate exposure to extreme losses
  - "Black swan" events (i.e., rare but high-impact events)

The capital market line (CML) plots the opportunity set of *passive* strategies I

- The CML is the CAL based on T-bills and a broad index of common stocks, typically the value-weighted portfolio of all U.S. common stocks
- Why are passive strategies reasonable?
  1. Active strategies require costly research and trading to find and maintain the optimal portfolio
  2. Passive strategies free ride on the research and trading of active strategies, which keep securities fairly priced

	Average Annual Returns		U.S. Equity Market		
Period	**U.S. equity**	**1-month T-bills**	**Excess return**	**Standard deviation**	**Sharpe ratio**
1927–2021	12.17	3.30	8.87	20.25	0.44
1927–1950	10.26	0.93	9.33	26.57	0.35
1951–1974	10.21	3.59	6.63	20.32	0.33
1975–1998	17.97	6.98	11.00	14.40	0.76
1999–2021	10.16	1.66	8.50	18.85	0.45

Figure 9: Average annual return on stocks and 1-month T-bills; standard deviation and Sharpe ratio of stocks over time (BKM 2023, Table 6.7)

We can use these historical data to estimate A!

- Investors solve:
$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2}$$
so:
$$A = \frac{E(r_M) - r_f}{y^*\sigma_M^2}$$
- About 69.3% of household net worth is invested in risky assets, so:
$$A = \frac{0.0887}{0.693 \times 0.2025^2} = 3.12$$

a. Which of the following statements are true about passive vs active investment strategies?

Check all that apply:

- ☐ An investment strategy that avoids any security analysis is called a passive strategy.
- ☐ The capital market line is the capital allocation line that represents an active investment strategy.
- ☐ A passive strategy is an investment in a stock fund that matches a broad market index and an investment in T-bills.
- ☐ A passive strategy is an investment in a portfolio with a reward-to-volatility ratio of 1.
- ☐ The capital market line represents the investment opportunity set of a passive investment strategy.

An investor wants to invest money in Treasury bills and a risky fund managed by Infinity Capital. The investor wants to achieve an expected return of 7% on his complete portfolio. Infinity Capital has an expected return of 15% and a standard deviation of returns of 28%. T-bills have a return of 3%.

a. What proportion of his total investment should he invest in the risky fund in order to achieve the expected return?

b. What is the standard deviation of the complete portfolio?

. Suppose that you manage a portfolio with a standard deviation of 49% and an expected return of 18%. An investor wants to invest a proportion of his investment budget in your fund. The remaining proportion of his investment budget will be invested in Treasury bills. The T-bill rate is 7%.

a. If the investor wants to maximize their expected return on their complete portfolio subject to the constraint that the complete portfolio's standard deviation will not exceed 18%, what fraction should they invest in your fund?

b. Now suppose that the investor has a degree of risk aversion of A = 2.9. What fraction should they invest in your fund?

Summary and Key Equations I

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1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero risk premium. It will not be undertaken by a risk-averse investor.
3. Investors' preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if received with certainty, would yield the same utility as the risky portfolio.
5. Shifting funds from the risky portfolio to the risk-free asset is the simplest way to reduce risk. Other methods involve diversification of the risky portfolio and hedging. We take up these methods in later chapters.
6. T-bills provide a perfectly risk-free asset in nominal terms only. Nevertheless, the standard deviation of real returns on short-term T-bills is small compared to that of other assets such as long-term bonds and common stocks, so for the purpose of our analysis we consider T-bills as the risk-free asset. Money market funds hold, in addition to T-bills, short-term relatively safe obligations such as repurchase agreements or bank CDs. These entail some default risk, but again, the additional risk is small relative to most other risky assets. For convenience, we often refer to money market funds as risk-free assets.
7. An investor's risky portfolio (the risky asset) can be characterized by its reward-to-volatility or Sharpe ratio, $S = [E(r_P) - r_f]/\sigma_P$. This ratio is also the slope of the CAL, the line that, when graphed, goes from the risk-free asset through the risky asset. All combinations of the risky asset and the risk-free asset lie on this line. Other things equal, an investor would prefer a steeper-sloping CAL because that means higher expected return for any level of risk. If the borrowing rate is greater than the lending rate, the CAL will be "kinked" at the point of the risky asset.
8. The investor's degree of risk aversion is characterized by the slope of the indifference curve. Indifference curves show, at any level of expected return and risk, the required risk premium for taking on each additional percentage point of standard deviation. More risk-averse investors have steeper indifference curves; that is, they require a greater risk premium for taking on more risk.
9. The optimal position, y^*, in the risky asset is proportional to the risk premium and inversely proportional to the variance and degree of risk aversion:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

   Graphically, this portfolio represents the point at which the indifference curve is tangent to the CAL.
10. A passive investment strategy disregards security analysis, targeting instead the risk-free asset and a broad portfolio of risky assets such as the S&P 500 stock portfolio. If in 2021 investors took the mean historical return and standard deviation of the S&P 500 as proxies for its expected return and standard deviation, then the values of outstanding assets would imply a degree of risk aversion of about $A = 3.12$ for the average investor. This is in line with other studies, which estimate typical risk aversion in the range of 2.0 through 4.0.

Figure 10: Chapter 6 summary from BKM (2023)

Summary and Key Equations II

Utility score: $U = E(r) - \frac{1}{2} A\sigma^2$

Expected return on complete portfolio: $E(r_C) = yE(r_C) + (1 - y)r_f$

Standard deviation of complete portfolio: $\Sigma_C = y\Sigma_P$

Optimal allocation to risky portfolio: $y^* = \dfrac{E(r_P) - r_f}{A\sigma_P^2}$

Figure 11: Chapter 6 key equations from BKM (2023)

References I

Bodie, Zvi, Alex Kane, and Allan J. Marcus (2023).
Investments. 13th ed. New York: McGraw Hill.