Chapter 10 - Arbitrage Pricing Theory and Multifactor Models of Risk and Return

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We can decompose stock variability into two sources: market and firm-specific

- Market risk is also known as systematic risk or non-diversifiable risk
- Firm-specific risk is also known as non-systematic risk, diversifiable risk, or idiosyncratic risk
- However, risk premiums may depend on correlations with extra-market risk factors (e.g., inflation, interest rates, and volatility)

Single-factor models allow two sources of uncertainty in asset returns

- *Single-factor* models of excess returns have the form:

$$R_i = E(R_i) + \beta_i F + e_i$$

  where:
  - R_i is the realized excess return on stock i
  - $E(R_i)$ is the expected excess return on stock i
  - β_i is the sensitivity of stock i to factor F
  - F is the *deviation* of the common factor from its expected value
  - e_i is the firm-specific surprise, which is uncorrelated with F
- If $F = 0$ (i.e., no macroeconomic surprises), then $R_i = E(R_i) + e_i$ so the return on stock i is its expected return plus the firm-specific surprise

Multifactor models can better describe uncertainty in asset returns by allowing two or more systematic risk factors, as well as firm-specific surprises

- BKM propose a two-factor model where the systematic risk factors are unexpected growth in gross domestic product (GDP) and unexpected changes in interest rates (IR):

$$R_i = E(R_i) + \beta_{i\text{GDP}}\text{GDP} + \beta_{i\text{IR}}\text{IR} + e_i$$

where $\beta_{i\text{GDP}}$ and $\beta_{i\text{IR}}$ are the *factor loadings* or *factor betas* on GDP and IR, respectively
  - GDP increases are typically good news to asset returns, so we expect $\beta_{i\text{GDP}} > 0$
  - However, interest rate increases are typically bad news for asset returns, so we expect $\beta_{i\text{IR}} < 0$
  - Multifactor models can better capture different responses to different sources of macroeconomic uncertainty

The arbitrage pricing theory (APT) predicts a security market line (SML) that links expected returns to risk

- However, the APT takes a different path to this SML than the capital asset pricing model (CAPM)
- The APT relies on three propositions:
  1. A factor model can describe security returns
  2. There are sufficient securities to diversify away the idiosyncratic risk
  3. Well-functioning security markets do not allow for the persistence of arbitrage opportunities

An arbitrage opportunity exists when an investor can earn a riskless profit without making a net investment

- A simple example would be if shares of a stock sold for different prices on two different exchanges
- The *Law of One Price* states that two assets that are the same in all economically relevant aspects should have the same price
- *Arbitrageurs* enforce the Law of One Price
  - If they observe a violation, they engage in *arbitrage activity* by simultaneously buying the cheap asset and selling the expensive asset
  - These trades bid up (down) the price that is low (high) until the arbitrage opportunity goes away

In a well-diversified portfolio, the firm-specific risk becomes negligible so that only factor risk remains I

- Consider the excess return R_P on an n-stock portfolio:

$$R_P = E(R_P) + \beta_P F + e_P$$

where $E(R_P) = \sum_{i=1}^{n} w_i E(R_i)$, $\beta_P = \sum_{i=1}^{n} w_i \beta_i$, $e_P = \sum_{i=1}^{n} w_i e_i$, and $\sum_{i=1}^{n} w_i = 1$

- F and e_P are two uncorrelated random variables, so portfolio variance σ_P^2 is:

$$\sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma^2(e_P)$$

- If this portfolio were equally weighted, we could rewrite $\sigma^2(e_P)$ as:

$$\sigma^2(e_P) = \sum_{i=1}^{n} w_i^2 \sigma^2(e_i) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma^2(e_i)}{n} = \frac{1}{n}\overline{\sigma}^2(e_i)$$

so the non-systematic variance of the portfolio equals the average non-systematic variance of the stocks divided by n

In a well-diversified portfolio, the firm-specific risk becomes negligible so that only factor risk remains II

- Therefore, non-systematic variance decreases as n increases, which is the effect of diversification
- For a well-diversified portfolio e_P and $\sigma^2(e_P)$ are effectively zero, so:

$$R_P = E(R_P) + \beta_P F$$

In a well-diversified portfolio, the firm-specific risk becomes negligible so that only factor risk remains III
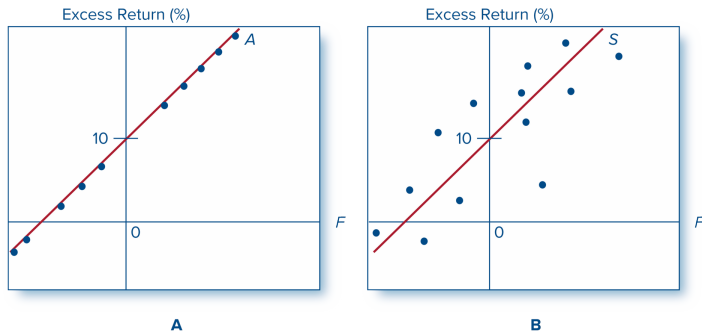

Figure 1: Excess returns as a function of the systematic factor: Panel A, Well-diversified portfolio A; Panel B, Single stock S (BKM 2023, Figure 10.2)

Only systematic or factor risk should be related to portfolio expected returns
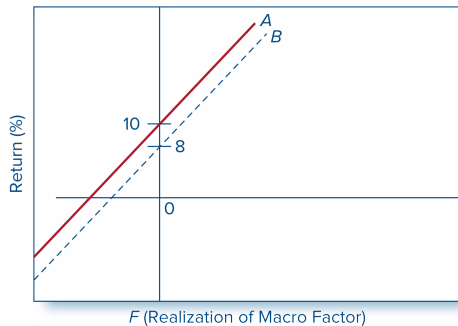

Figure 2: Returns as a function of the systematic factor: An arbitrage opportunity (BKM 2023, Figure 10.3)
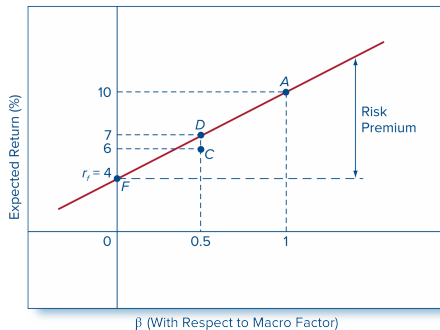
Risk premiums must be proportional to betas

Figure 3: An arbitrage opportunity (BKM 2023, Figure 10.4)

If a market index portfolio is well-diversified, its returns will perfectly reflect the macro factor F

- So, for a well-diversified portfolio P:

$$R_P = \alpha_P + \beta_P R_M$$

and:

$$E(R_P) = \beta_P E(R_M)$$

- This relation holds for *almost all* securities, too

a. Which are assumptions of the APT?

Check all that apply:

- ☐ Security returns can be described by a factor model.
- ☐ There are enough securities to diversify away firm-specific risk.
- ☐ There are no persistent arbitrage opportunities.
- ☐ There is no non-systematic risk.

a. An activity that includes an exploitation of price differences between two or more identical or similar securities in order to achieve riskless profits is called _____.

○ arbitrage
○ diversification
○ factoring
○ market timing
○ optimization

b. Which of the following is true about the arbitrage pricing theory (APT)?

○ APT assumes that all investors are identical in terms of risk aversion.
○ APT assumes that all investors are identical in terms of wealth.
○ APT can be applied to both well-diversified portfolios and individual securities.
○ APT derives the equilibrium rate of return in large capital markets with no arbitrage opportunities.

a. What does the APT say about alpha?

○ Alpha equals the risk-free rate.

○ Alpha is irrelevant.

○ Alpha is zero for all securities.

○ Alpha is zero for well-diversified portfolios.

The APT serves many of the same functions as the CAPM

- Both the CAPM and APT
  - Arrive at the same SML, which gives benchmarks for rates of return for capital budgeting, security valuation, or investment performance evaluation
  - Highlight the important distinction between non-diversifiable risk, which requires a risk premium, and diversifiable risk, which does not
- However, there are important differences between the CAPM and APT
  - CAPM
    - Based on an inherently unobservable market portfolio
    - Provides unequivocal statement on the expected return-beta relation for all securities
  - APT
    - Based on the foundation of well-diversified portfolios
    - Cannot rule out a violation of the expected return-beta relationship for any particular asset
    - Does not assume investors are mean-variance optimizers
    - Uses an observable market index

a. Which of the following statements are true about multifactor models?

Check all that apply:

☐ The coefficients on each factor in the multifactor models are called factor sensitivities.

☐ The coefficients on each factor in the multifactor models are called factor loadings.

☐ The coefficients on each factor in the multifactor models are called factor betas.

☐ The coefficients on each factor in the multifactor models are called factor alphas.

☐ In the multifactor models, the return of the stock responds only to firm-specific influences and not to sources of macro risk.

. An investor holds two well-diversified portfolios on US securities. The expected return on portfolio A is 13% and the expected return on portfolio B is 9%. The investor identified that the US economy has only one factor (industrial production), and $\beta_A = 1$ and $\beta_B = 0.7$.

a. What should be the risk-free rate according to the APT?

A well-diversified portfolio P has an expected return of 5% and a beta of 1.2. The risk-free rate is 3.1% and the expected return on the S&P 500 is 8%.

a. What is the portfolio's expected alpha?

b. What should you do?

○ Buy the security

○ Short-sell the security

c. If you trade $2,000 of the portfolio, how much should you trade of the S&P 500 in order to create a risk-free and costless arbitrage opportunity (in $)? Enter a positive number for buying the S&P 500 and a negative number for short-selling it.

The APT can be generalized to accommodate multiple sources of risk

- Assume a two-factor model:

$$R_i = E(R_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$$

- The benchmark portfolios in the APT are *factor portfolios*, which are well-diversified portfolios constructed to have a beta of 1 on one factor and betas of 0 on the other factors
- We can think of these benchmark portfolios as *tracking portfolios* that track one source of macroeconomic risk and are uncorrelated with other sources of risk

The Fama-French (FF) three-factor model uses firm characteristics to identify sources of systematic risk

- The Fama-French three-factor model is:

$$R_{it} = \alpha_i + \beta_{iM}R_{Mt} + \beta_{iSMB}\mathsf{SMB}_t + \beta_{iHML}\mathsf{HML}_t + e_{it}$$

where:
  - SMB = small minus big, which is the return of a portfolio of small stocks minus the return on a portfolio of large stocks
  - HML = high minus low, which is the return of a portfolio of stocks with a high book-to-market ratio minus the return on a portfolio of stocks with a low book-to-market ratio
- Fama and French (1993) chose SMB and HML because size and the book-to-market ratio predict deviations of average stock returns from CAPM levels
- SMB and HML are not obvious candidates for risk factors, but may proxy hard-to-measure fundamental variables

	Single-index model		Three-Factor Model	
	Regression Coefficient	*t*-Statistic	Regression Coefficient	*t*-Statistic
Intercept (alpha)	−2.034	−0.991	−0.745	−0.363
$r_m - r_f$	2.030	4.785	1.771	4.074
SMB			0.369	0.487
HML			1.249	2.303
R-square	0.283		0.352	
Residual std dev (%)	15.171		14.675	

Figure 4: Estimates of single-index and three-factor Fama-French regressions for U.S. Steel, monthly data, 5 years ending in 2021 (BKM 2023, Table 10.1)

- The three-factor model predicts:

$$E(r_X) = 2\% + 1.771 \times 8.8\% + 0.369 \times 3.0\% + 1.249 \times 4.2\% = 23.9\%$$

- In contrast, the CAPM predicts:

$$E(r_X) = 2\% + 2.030 \times 8.8\% = 19.9\%$$

- We ignore α estimates in both models because we expect $\alpha = 0$ for all stocks in equilibrium

Researchers have added *many*, *many* factors to the three-factor model

- Carhart (1997) adds a momentum factor
  - The difference between those that performed well last year and those that performed poorly
  - Often called UMD for "up minus down"
- Fama and French (2015) add a profitability factor and an investment factor
  - The profitability factor, RMW for "robust minus weak," is the difference in returns between a diversified portfolio with strong versus weak profitability
  - The investment factor, CMA for "conservative minus aggressive," is the difference in returns between a diversified portfolio with low versus high investment
- The overwhelming number of other factors are often called the *factor zoo*
- *In the end, there should be a limited number of factors portfolios, each of which proxies for a source of systematic risk*

Multifactor models have two important generalizations

1. Investors should be aware that portfolios have exposure to more than one source of systematic risk, and they should think about how much exposure they want to each factor
2. When they evaluate performance, they should be aware of the risk premium from each factor and consider them when they estimate alphas

- Smart-beta exchange-traded funds (ETFs) are designed to provide exposure to specific characteristics such as value, growth, and volatility
- The most common extra-market risk factors are:
  - The Fama-French five-factor model: Mkt-RF, SMB, HML, RMW, and CMA
  - Momentum
  - Volatility, liquidity, and dividend yield

Suppose that an investor holds a stock which has a rate of return based on a single-factor model. The model equation can be presented as:

$r_i = E(r_i) + \beta_i F + e_i$

where F is the unanticipated growth in GDP. The market consensus about the GDP growth rate is 4%. The investor currently expects to earn a 10.4% return. The stock's β value is 1.1.

  a. Suppose that after some period of time the macroeconomic news suggest that GDP growth will be 5%, which is higher than the market consensus. What will be the revised estimate of the stock's expected rate of return?

  b. What will be the revised estimate of the stock's expected rate of return if the macroeconomic news suggest that GDP will be 2.5%, which is lower than the market expected value?

An investor analyzed 20 stocks listed on the FTSE and found that half of them have an alpha of 2.3% and the other half an alpha of -2.3%. All stocks listed on the FTSE have the market return as the common factor, and all of them have a beta of 1 on the market index. Also, firm-specific returns have a standard deviation of 35%. The investor decides to invest $5 million in an equally-weighted portfolio of 10 positive alpha stocks, and to sell short $5 million in an equally-weighted portfolio of 10 negative alpha stocks.

a. What would be the expected profit for the investor?

b. What is the standard deviation of the profit calculated in the previous part?

An investor holds two well-diversified portfolios. The risk-free rate is 1.4% and the investor identified two macroeconomic factors: industrial production (X) and consumer confidence (Y). Factor betas and expected rates of return are given below:

Portfolio	Beta on factor X	Beta on factor Y	Expected return
A	0.8	1.1	18%
B	1.1	1.3	25%

a. According to the APT, what should be the expected return for a well-diversified portfolio with β_X of 1.2 and β_Y of 0.4?

Summary and Key Equations I

1. Multifactor models seek to improve the explanatory power of single-factor models by explicitly accounting for the various components of systematic risk. These models use indicators intended to capture a wide range of macroeconomic risk factors.

2. Once we allow for multiple risk factors, we conclude that the security market line also ought to be multidimensional, with exposure to each risk factor contributing to the total risk premium of the security.

3. A (risk-free) arbitrage opportunity arises when two or more security prices enable investors to construct a zero-net-investment portfolio that will yield a sure profit. The presence of arbitrage opportunities will generate a large volume of trades that puts pressure on security prices. This pressure will continue until prices reach levels that preclude such arbitrage.

4. When securities are priced so that there are no risk-free arbitrage opportunities, we say that they satisfy the no-arbitrage condition. Price relationships that satisfy the no-arbitrage condition are important because we expect them to hold in real-world markets.

5. Portfolios are called "well diversified" if they include a large number of securities and the investment proportion in each is sufficiently small. The proportion of any particular security in a well-diversified portfolio is small enough so that, for all practical purposes, a reasonable change in that security's rate of return will have a negligible effect on the portfolio's rate of return.

6. In a single-factor security market, all well-diversified portfolios have to satisfy the expected return–beta relationship of the CAPM to satisfy the no-arbitrage condition. If all well-diversified portfolios satisfy the expected return–beta relationship, then individual securities also must satisfy this relationship, at least approximately.

7. The APT does not require the restrictive assumptions of the CAPM and its (unobservable) market portfolio. The price of this generality is that the APT does not guarantee this relationship for all securities at all times.

8. A multifactor APT generalizes the single-factor model to accommodate several sources of systematic risk. The multidimensional security market line predicts that exposure to each risk factor contributes to the security's total risk premium by an amount equal to the factor beta times the risk premium of the factor portfolio that tracks that source of risk.

9. The multifactor extension of the single-factor CAPM, the ICAPM, predicts the same multidimensional security market line as the multifactor APT. The ICAPM suggests that priced extra-market risk factors will be the ones that lead to significant hedging demand by a substantial fraction of investors. Other approaches to the multifactor APT are more empirically based, where the extra-market factors are selected based on past ability to predict risk premiums.

Figure 5: Chapter 10 summary from BKM (2023)

Summary and Key Equations II

Single-factor model: $R_i = E(R_i) + \beta_i F + e_i$

Multifactor model (here, 2 factors, F_1 and F_2): $R_i = E(R_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$

Single-index model: $R_i = \alpha_i + \beta_i R_M + e_i$

Multifactor SML (here, 2 factors, labeled 1 and 2):

$$\begin{aligned}
E(r_i) &= r_f + \beta_{i1}\left[E(r_1) - r_f\right] + \beta_{i2}\left[E(r_2) - r_f\right] \\
&= r_f + \beta_{i1}E(R_1) + \beta_{i2}E(R_2)
\end{aligned}$$

where β_{i1} and β_{i2} measure the stock's typical response to returns on each factor portfolio and the risk premiums on the two factor portfolios are $E(R_1)$ and $E(R_2)$.

Figure 6: Chapter 10 key equations from BKM (2023)

References I

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