Chapter 5 - Risk, Return, and the Historical Record

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Measuring Returns over Different Holding Periods

Holding period return (HPR) is the price increase plus income, all divided by the price paid

$$\text{HPR} = r_1 = \frac{P_1 - P_0 + \text{Income}}{P_0}$$

where P_0 and P_1 are the prices at $t = 0$ and $t = 1$. Bodie, Kane, and Marcus (2023) present this formula in terms of a $100 face value bond, but we will jump right to the general formula.

We typically express a HPR as an effective annual rate (EAR) to compare it with different horizon returns

$$r_T = (1 + \text{EAR})^T - 1$$

where r_T is the HPR over T years, so

$$\text{EAR} = (1 + r_T)^{1/T} - 1$$

Rates on short-term investments are often annualized with simple interest instead of compound interest, called annual percentage rate (APR)

$$1 + \text{EAR} = \left(1 + \frac{\text{APR}}{n}\right)^n$$

where n is the number of compounding periods per year, so

$$\text{APR} = n \times [(1 + \text{EAR})^{1/n} - 1]$$

If we compound continuously, $n \to \infty$

The relation between EAR and APR (here r_{cc}) becomes

$$1 + \text{EAR} = \exp(r_{cc})$$

so

$$\ln(1 + \text{EAR}) = r_{cc}$$

where $\ln(\cdot)$ is the natural log

Interest Rates and Inflation Rates

The *dollar value* of a bank account grows at the *nominal interest rate*, but its *buying power* grows at the *real interest rate*

$$1 + r_{real} = \frac{1 + r_{nom}}{1 + i}$$

where r_{real}, r_{nom}, and i are the real interest, nominal interest, and inflation rates, so

$$r_{real} = \frac{r_{nom} - i}{1 + i}$$

so, if inflation is low,

$$r_{real} \approx r_{nom} - i$$

Inflation is a tax on savings!

- The real after-tax rate is approximately the after-tax nominal rate minus the inflation rate
- Therefore, the after-tax real returns fall by the inflation rate times the tax rate

$$r_{nom}(1 - t) - i = (r_{real} + i)(1 - t) - i = r_{real}(1 - t) - it$$

The Fisher equation predicts that the nominal interest rate should track the (expected) inflation rate, leaving the real rate relatively stable

Risk and Risk Premiums

The realized rate of return depends on price changes and dividends paid

- The realized rate of return is the same as the *holding period return* (HPR) above
- We calculate HPA as

$$\text{HPR} = \frac{\text{Ending price of a share} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

	Average Annual Rates			Standard Deviation		
	T-Bills	Inflation	Real T-Bill	T-Bills	Inflation	Real T-Bill
Full sample	3.30	3.02	0.38	3.10	3.98	3.78
1927–1951	0.95	1.80	-0.48	1.24	6.06	6.34
1952–2021	4.14	3.46	0.68	3.14	2.84	2.27

Source: Annual rates of return from rolling over 1-month T-bills: Kenneth French; annual inflation rates: Bureau of Labor Statistics.

Figure 1: Statistics for T-bill rates, inflation rates, and real rates, 1927-2021 (Bodie, Kane, and Marcus 2023, Table 5.3)

The future is uncertain

- We perform *scenario analysis* to consider HPRs of various economic scenarios and the probabilities of each scenario
- We measure expected or mean rates of return, $E(r)$, as the *probability-weighted average of the rates returns in each scenario*

$$E(r) = \sum_s p(s) r(s)$$

  where $p(s)$ and $r(s)$ are the probability and HPR of each scenario
- We can measure uncertainty as the variance of returns, which is the average *squared* deviation of actual returns from the mean return

$$\text{Var}(r) = \sigma^2 = \sum_s p(s)[r(s) - E(r)]^2$$

- In practice, we use the square root of the variance of returns (i.e., $\sigma = \sqrt{\sigma^2}$), which is the standard deviation of returns and has the same units are our returns (i.e., % instead of $\%^2$)

We compare returns on risky investments to the risk-free rate

- The *risk premium* is the difference between the *expected* rate of return and the risk-free rate
- The *excess return* is the difference between the *realized* rate of return and the risk-free rate
- Investors are risk averse and require positive risk premiums to hold risky investments
- *However, after the fact, excess returns may be negative*

The Sharpe ratio is a reward-to-volatility ratio

- Investors accept risk for the *potential* to earn returns greater than the risk-free rate
- Risk premiums are proportional to the risk of expected excess returns, and we measure risk as the standard deviation of *excess* returns
- The Sharpe ratio is the ratio of the risk premium to the standard deviation of excess returns and commonly used to evaluate investment manager performance

$$\text{Sharpe ratio} = \frac{\text{Risk premium}}{\text{SD of excess returns}}$$

- We multiply the numerator by T and the denominator by $\sqrt{T}$ to annualize the Sharpe ratio (e.g., $T = 12$ for monthly returns, and $T = 252$ for daily returns)

The Normal Distribution

We typically assume that market returns are normally distributed

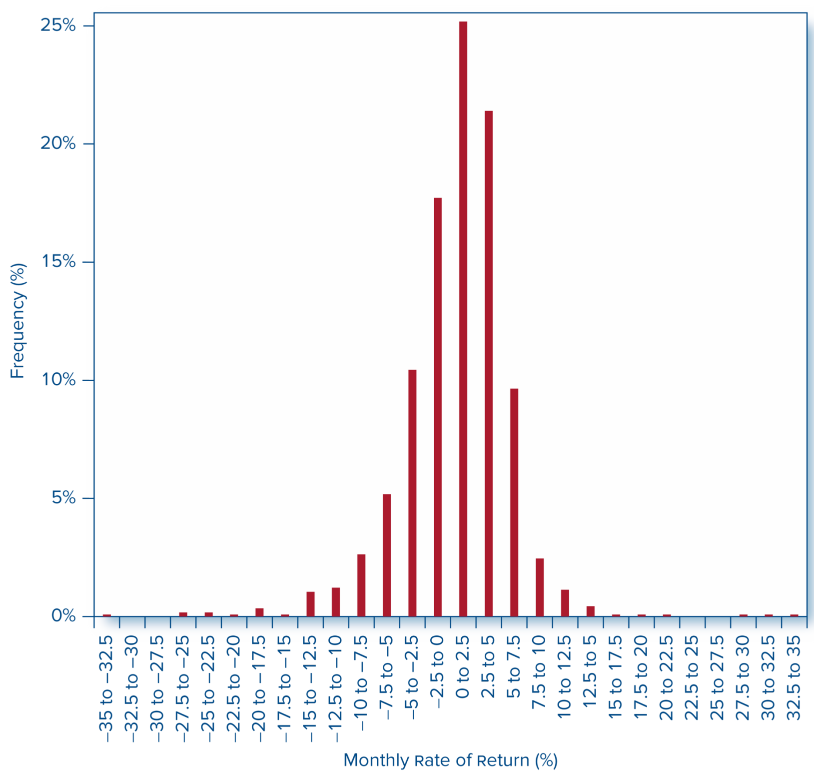

Figure 2: Frequency distribution of monthly rate of return on the broad market index, 1927-2020 (Bodie, Kane, and Marcus 2023, fig. 5.3)

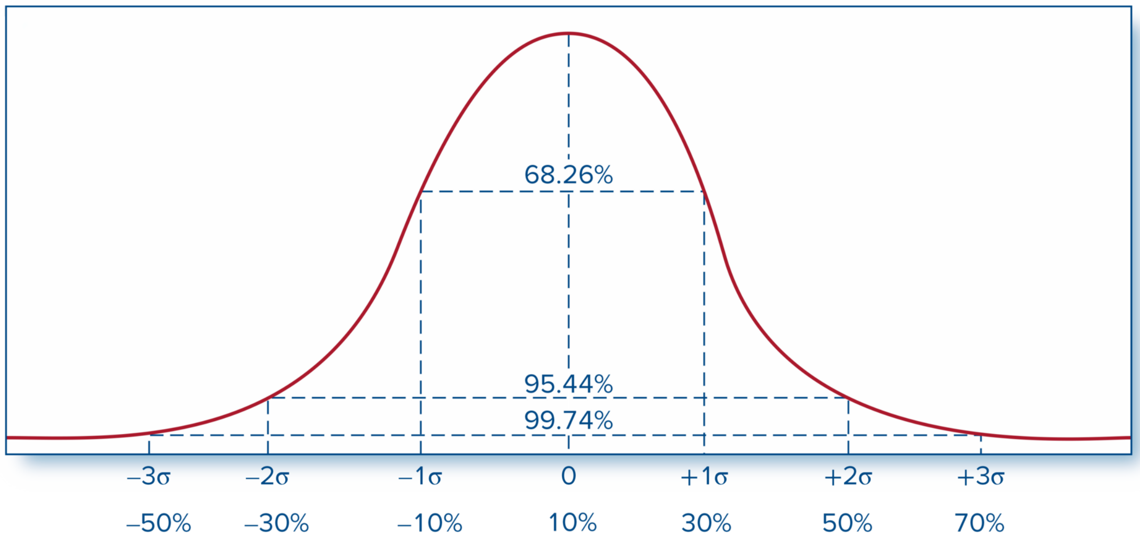

Figure 3: The normal distribution with mean 10% and standard deviation 20% (Bodie, Kane, and Marcus 2023, fig. 5.4)

This normally-distributed returns assumption is not perfect (e.g., stocks returns cannot be less than -100%), but it has a few advantages

1. Normal distribution is symmetric
2. The return on a portfolio of assets with normally distributed returns is also normally distributed
3. We can fully describe a normal distribution with two parameters: mean and standard deviation
4. We can fully describe the relation between two normal distributions with one parameter: correlation coefficient

Deviations from Normality and Tail Risk

Asset returns are negative-skewed and fat-tailed

- *Skew* is the average *cubed* deviation from the mean

$$\text{Skew} = \text{Average}\left[\frac{(r-\bar{r})^3}{\hat{\sigma}^3}\right] = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_t-\bar{r})^3}{\hat{\sigma}^3}$$

- Negative skew indicates that extreme bad outcomes are more likely than extreme good outcomes

- Negative-skewed distributions are left-skewed with left tails that are fatter than right tails
- *When returns are positively-skewed, SD overestimates risk; when returns are negatively-skewed, SD underestimates risk*

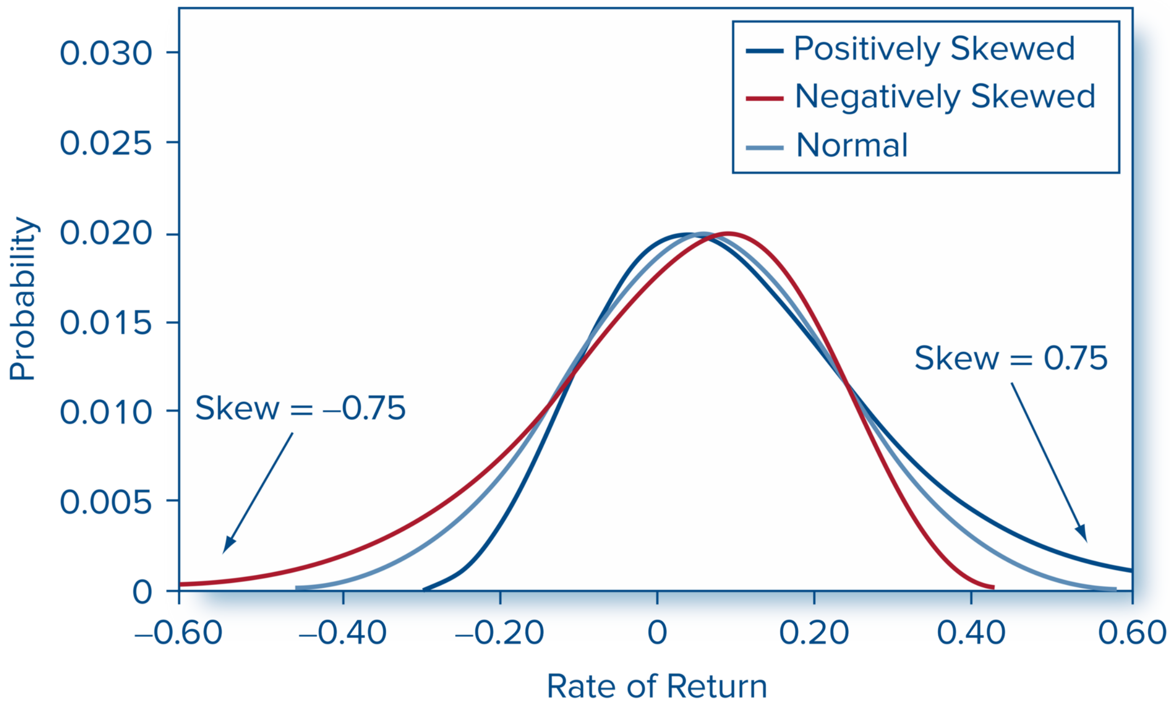

Figure 4: Normal and skewed distributions (mean = 6%, SD = 17%) (Bodie, Kane, and Marcus 2023, fig. 5.5)

- *Kurtosis* is the average deviation from the mean *raised to the fourth power*

$$\text{Kurtosis} = \text{Average}\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3 = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_t-\bar{r})^4}{\hat{\sigma}^4} - 3$$

- Then subtract 3 because the normal distribution has kurtosis of 3
- Deviations are raised to the fourth power, so kurtosis is more sensitive to outliers than variance
- Kurtosis values greater than zero indicate "fat tails", where extreme outcomes are more likely than for normal distributions

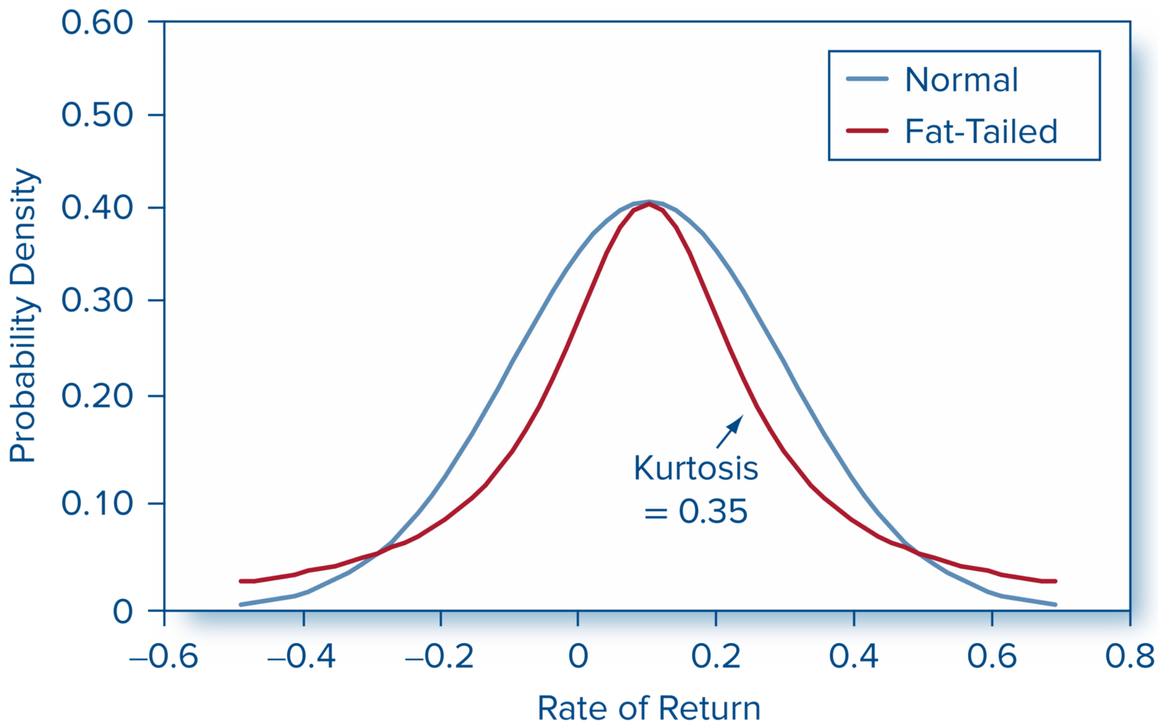

Figure 5: Normal and fat-tailed distributions (mean = 10%, SD = 20%) (Bodie, Kane, and Marcus 2023, fig. 5.6)

There are four more common measures of downside risk

1. Value at risk (VaR) is the loss corresponding to a low percentile of the return distribution (e.g., we expect 99% of returns to exceed the 1% VaR)
2. Expected shortfall (ES) is the expected loss given that we find ourselves in a worst-case scenari (e.g., the 1% ES is the the average of the worst 1% of all returns)
3. Lower partial standard deviation (LPSD) is the standard deviation of *negative* excess returns, and the Sortino ratio is the ratio of the mean excess return to the LPSD (i.e., the Sharpe ratio but with the LPSD)
4. Relative frequency of large, negative 3-sigma returns calculates the percent of returns that fall 3 standard deviations below the mean

Learning from Historical Records

Expected returns and the arithmetic average

- When we do *scenario analysis*, we specify scenarios, each with their own probability and return
- When we use *historical data*
  - We treat each observation as an equally likely scenario, so we use an equal probability of $\frac{1}{n}$ for each $p(s)$
  - We estimate expected return, $E(r)$, as the arithmetic average of sample returns

$$E(r) = \sum_{s=1}^{n} p(s) r(s) = \frac{1}{n} \sum_{s=1}^{n} r(s)$$

Geometric average return versus arithmetic average return

- The *arithmetic* average return is an unbiased estimate of *expected* returns
- The *geometric* average return is a HPR that would compound to the same terminal value as the historical data

$$\text{Geometric average} = \left[\prod_{s=1}^{n} (1 + r_s) \right]^{1/n} - 1$$

- The geometric average return is always less than or equal to the arithmetic average return
- If returns are drawn from a normal distribution

$$E[\text{Geometric average}] = E[\text{Arithmetic average}] - \frac{\sigma^2}{2}$$

Estimating return variance and standard deviation

- Again, with historical data, we treat each observation as an equally likely scenario, so we use an equal probability of $\frac{1}{n}$ for each $p(s)$
- This assumption provides $\bar{r}$ as an unbiased estimate of $E(r)$
- However, this assumption provides downward biased (too low) estimates of σ^2 because we use one degree of freedom to estimate $\bar{r}$
- Therefore, we use an equal probaiblity of $\frac{1}{n-1}$ for each $p(s)$

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{s=1}^{n}[r(s) - \bar{r}]^2}$$

Historic Returns on Risky Portfolios

Historic returns on risky portfolios

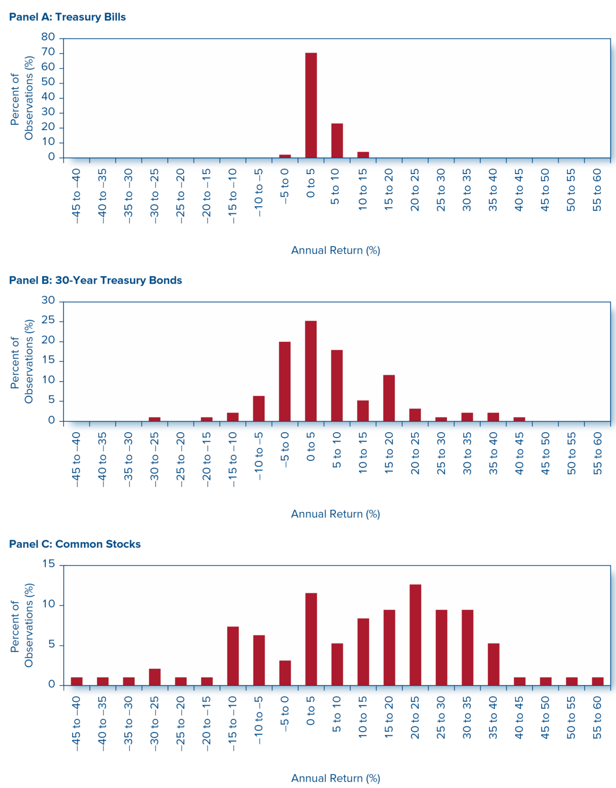

Figure 6: Frequency distribution of annual returns on U.S. Treasury bill, Treasury bonds, and common stocks (Bodie, Kane, and Marcus 2023, fig. 5.7)

	Market Index	Big/Growth	Big/Value	Small/Growth	Small/Value
A. 1927–2021					
Mean excess return (annualized)	8.86	8.79	12.02	9.60	15.54
Standard deviation (annualized)	18.52	18.35	24.83	25.97	28.23
Sharpe ratio	0.48	0.48	0.48	0.37	0.55
Lower partial SD (annualized)	19.62	18.85	23.61	25.86	26.33
Skew	0.17	-0.11	1.50	0.59	2.06
Kurtosis	7.61	5.47	17.58	7.10	21.60
VaR (1%) actual (monthly) returns	-13.58	-14.40	-19.93	-20.10	-20.68
VaR (1%) normal distribution	-11.75	-11.64	-15.75	-16.69	-17.78
% of monthly returns more than 3 SD below mean	0.88%	0.79%	0.88%	0.79%	0.62%
Expected shortfall (monthly)	-19.60	-19.86	-24.61	-24.64	-26.00
B. 1952–2021					
Mean excess return (annualized)	8.40	8.47	10.59	8.21	13.51
Standard deviation (annualized)	14.95	15.50	17.08	22.39	19.02
Sharpe ratio	0.56	0.55	0.62	0.37	0.71
Lower partial SD (annualized)	16.17	15.78	17.69	23.72	20.23
Skew	-0.52	-0.35	-0.46	-0.33	-0.45
Kurtosis	1.90	1.75	3.03	2.17	3.83
VaR (1%) actual (monthly) returns	-10.89	-10.84	-12.48	-17.22	-15.24
VaR (1%) normal distribution	-9.39	-9.75	-10.66	-14.40	-11.75
% of monthly returns more than 3 SD below mean	0.71%	0.60%	0.83%	0.95%	1.07%
Expected shortfall (monthly)	-14.47	-13.96	-18.08	-22.76	-20.27

Statistics for monthly excess returns on the market index and four “style” portfolios

Source: Authors' calculations using data from Prof. Kenneth French's web site: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

Figure 7: Statistics for monthly excess returns on the market index and four “style” portfolios (Bodie, Kane, and Marcus 2023, Table 5.5)

Normality and Long-Term Investments

Investments in risky portfolios do not become safer in the long run

	Investment Horizon			
	1	10	30	Comment
Mean total return	0.050	0.500	1.500	$= 0.05^{*}T$
Mean average return	0.050	0.050	0.050	= 0.05
Std dev total return	0.300	0.949	1.643	$= 0.30^{*}\sqrt{T}$
Probability return < 0	0.434	0.299	0.181	assuming normal distribution
1% VaR total return	−0.648	−1.707	−2.323	continuously compounded return
Implies final wealth relative of:	0.523	0.181	0.098	= exp(VaR total return)
0.1% VaR total return	−0.877	−2.432	−3.578	continuously compounded return
Implies final wealth relative of:	0.416	0.088	0.028	= exp(VaR total return)

Figure 8: Investment risk at different horizons (Bodie, Kane, and Marcus 2023, Table 5.6)

Appendix

Summary from Bodie, Kane, and Marcus (2023)

1. The economy's equilibrium level of real interest rates depends on the willingness of households to save, as reflected in the supply curve of funds, and on the expected profitability of business investment in plant, equipment, and inventories, as reflected in the demand curve for funds. It depends also on government fiscal and monetary policy. Page 160
2. The nominal rate of interest is the equilibrium real rate plus the expected rate of inflation. In general, we can directly observe only nominal interest rates; from them, we must infer expected real rates, using inflation forecasts. Assets with guaranteed nominal interest rates are risky in real terms because the future inflation rate is uncertain.
3. The equilibrium expected rate of return on any security is the sum of the equilibrium real rate of interest, the expected rate of inflation, and a security-specific risk premium.
4. Investors face a trade-off between risk and expected return. Historical data confirm our intuition that assets with low degrees of risk should provide lower returns on average than do those of higher risk.
5. Historical rates of return over the last century in other countries suggest the U.S. history of stock returns is may be a positive outlier compared to other countries. This would suggest that its historical experience may overstate realistic projections for future performance as well as the risk premium demanded by stock market investors.
6. Historical returns on stocks exhibit somewhat more frequent large deviations from the mean than would be predicted from a normal distribution. However, the discrepancies from the normal distribution tend to be minor and inconsistent across various measures of tail risk, and have declined in recent years. The lower partial standard deviation (LPSD), skew, and kurtosis of the actual distribution quantify the deviation from normality.
7. Widely used measures of tail risk are value at risk (VaR) and expected shortfall or, equivalently, conditional tail expectation. VaR measures the loss that will be exceeded with a specified probability such as 1% or 5%. Expected shortfall (ES) measures the expected rate of return conditional on the portfolio falling below a certain value. Thus, 1% ES is the expected value of the outcomes that lie in the bottom 1% of the distribution.
8. Investments in risky portfolios *do not* become safer in the long run. On the contrary, the longer a risky investment is held, the greater the risk. The basis of the argument that stocks are safe in the long run is the fact that the probability of an investment shortfall becomes smaller. However, probability of shortfall is an incomplete measure of the safety of an investment because it ignores the magnitude of possible losses.

Key equations from Bodie, Kane, and Marcus (2023)

Arithmetic average of n returns: $(r_1 + r_2 + \cdots + r_n) / n$

Geometric average of n returns: $[(1 + r_1)(1 + r_2) \cdots (1 + r_n)]^{1/n} - 1$

Continuously compounded rate of return, $r_{cc} = \ln(1 + \text{Effective annual rate})$

Expected return: $\sum$ [prob(Scenario) × Return in scenario]

Variance: $\sum$[prob(Scenario) × (Deviation from mean in scenario)^2]

Standard deviation: $\sqrt{\text{Variance}}$

Sharpe ratio: $\dfrac{\text{Portfolio risk premium}}{\text{Standard deviation of excess return}} = \dfrac{E(r_P) - r_f}{\sigma_P}$

Real rate of return: $\dfrac{1 + \text{Nominal return}}{1 + \text{Inflation rate}} - 1$

Real rate of return (continuous compounding): $r_{\text{nominal}} - \text{Inflation rate}$

Bodie, Zvi, Alex Kane, and Allan J. Marcus. 2023. *Investments*. 13th ed. New York: McGraw Hill.