Chapter 6 - Capital Allocation to Risky Assets

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Speculation and gambling are different concepts

- Speculation is accepting risk because we perceive a favorable risk-return trade-off
- Gambling is accepting risk because we enjoy it

Historical data clearly show risky assets earn a risk premium and most investors are risk averse

- Risk-averse investors
  - Reject *fair games* or worse, where fair games are risky investments with zero risk premiums
  - Consider only risk-free investments or risky investments with positive risk premiums
- We characterize risk aversion with a "utility function"
  - A commonly-used utility function is $U = E(r) - \frac{1}{2}A\sigma^2$
  - Here A is a risk-aversion coefficient, and risk aversion increases as A increases

An investment is more attractive when its expected return is higher and its risk is lower

Portfolio	Risk Premium	Expected Return	Risk (SD)
L (low risk)	2%	4%	5%
M (medium risk)	4	6	10
H (high risk)	8	10	20

Figure 1: Available risky portfolios (risk-free rate = 2%) (BKM 2023, Table 6.1)

Investor Risk Aversion (A)	Utility Score of Portfolio L [$E(r)$ = .04; σ = 0.05]	Utility Score of Portfolio M [$E(r)$ = .06; σ = 0.10]	Utility Score of Portfolio H [$E(r)$ = .10; σ = 0.20]
2.0	$0.04 - ½ \times 2 \times 0.05^2 = 0.0375$	$0.06 - ½ \times 2 \times 0.1^2 = 0.0500$	**$0.10 - ½ \times 2 \times 0.2^2 = 0.06$**
3.5	$0.04 - ½ \times 3.5 \times 0.05^2 = 0.0356$	**$0.06 - ½ \times 3.5 \times 0.1^2 = 0.0425$**	$0.10 - ½ \times 3.5 \times 0.2^2 = 0.03$
5.0	$0.04 - ½ \times 5 \times 0.05^2 = 0.0338$	**$0.06 - ½ \times 5 \times 0.1^2 = 0.0350$**	$0.10 - ½ \times 5 \times 0.2^2 = 0.00$

Figure 2: Utility scores of alternative portfolios for investors with varying degrees of risk aversion (BKM 2023, Table 6.2)

There are three investor types concerning risk

- Risk-averse investors ($A > 0$) consider risky portfolios only if they have a risk premium
- Risk-neutral investors ($A = 0$) ignore risk and consider only the expected return
- Risk-loving investors ($A < 0$) accept lower expected returns on risky portfolios with higher amounts of risk

The mean-variance (M-V) criterion helps us rank portfolios I

The M-V criterion says that portfolio A dominates B if

$$E(r_A) \geq E(r_B)$$

and

$$\sigma_A \leq \sigma_B$$

with at least one strict inequality

The mean-variance (M-V) criterion helps us rank portfolios II

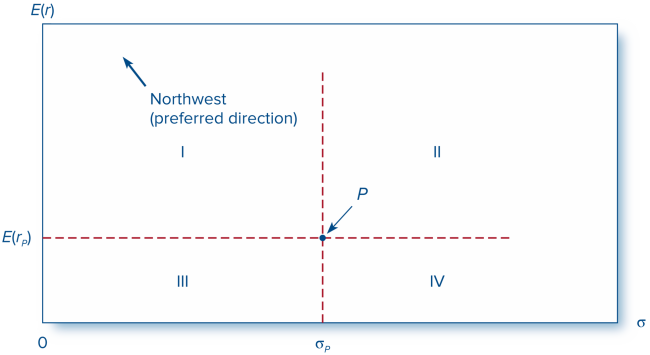

Figure 3: The trade-off between risk and return of a potential investment portfolio, P (BKM 2023, Figure 6.1)

An *indifference curve* connects all portfolios with the same utility

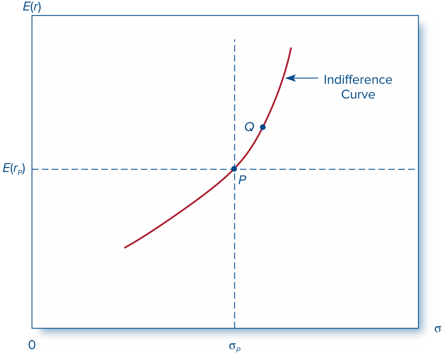

Figure 4: The indifference curve (BKM 2023, Figure 6.2)

The most basic asset allocation choice is between risk-free and risky assets

- We can control risk by changing the portfolio weights on risk-free and risky assets (i.e., $1 - y$ and y)
- Only governments can issue default-free bonds
  - A security is risk-free with a guaranteed real return only if
    1. its price is inflation-indexed and
    2. its maturity equals the investor's holding period
  - Still, we view U.S. Treasury bills as "the" risk-free asset
  - Also, a broad range of money market instruments are effectively risk-free

The capital allocation line (CAL) plots all available risk-return combinations I

- Consider a portfolio of one risky asset and one risk-free asset
- Allocate y to risky asset P and $1-y$ to the risk-free asset F
- The return on our complete portfolio C is:

$$r_C = y r_P + (1-y) r_f$$

- Therefore, the expected return on the complete portfolio C is:

$$E(r_C) = yE(r_p) + (1-y)r_f = r_f + y[E(r_P) - r_f]$$

and its volatility is:

$$\sigma_C = y\sigma_P$$

The capital allocation line (CAL) plots all available risk-return combinations II

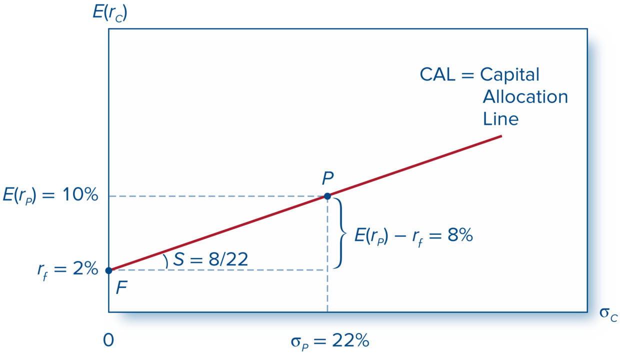

Figure 5: The investment opportunity set with a risk asset and risk-free asset in the expected return-standard deviation plane (BKM 2023, Figure 6.3)

The slope of the CAL is the Sharpe ratio

- The slope S of the CAL is:

$$S = \frac{\text{rise}}{\text{run}} = \frac{E(r_P) - r_f}{\sigma_P - 0} = \frac{E(r_P) - r_f}{\sigma_P}$$

- So we can write the CAL as:

$$E(r_P) = r_f + S\sigma_C = r_f + \frac{E(r_P) - r_f}{\sigma_P}\sigma_C$$

We need to borrow to earn $E(r_C) > E(r_P)$ I

- $E(r_C) > E(r_P)$ requires $y > 0$
- $y > 0$ requires $1 - y < 0$
- $1 - y < 0$ is a short position in F (i.e., borrowing or using leverage)
- The CAL is straight if we borrow at r_f (shown above)
- However, only the government borrows at r_f
- Instead, investors borrow at $r_f^B > r_f$, so the CAL has a kink at $y = 1$ (shown below)

We need to borrow to earn $E(r_C) > E(r_P)$ II

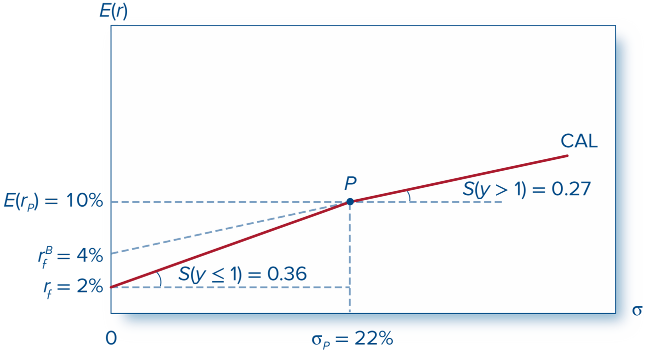

Figure 6: The investment opportunity set with different borrowing and lending rates (BKM 2023, Figure 6.4)

An investor chooses y to maximize her utility I

- Recall utility is $U = E(r_C) - \frac{1}{2}A\sigma_C^2$
- To maximize her utility, the investor solves the following maximization problem

$$\max_y U = E(r_C) - \frac{1}{2}A\sigma_C^2 = r_f + y[E(r_P) - r_f] - \frac{1}{2}Ay^2\sigma_P^2$$

and finds

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

An investor chooses y to maximize her utility II

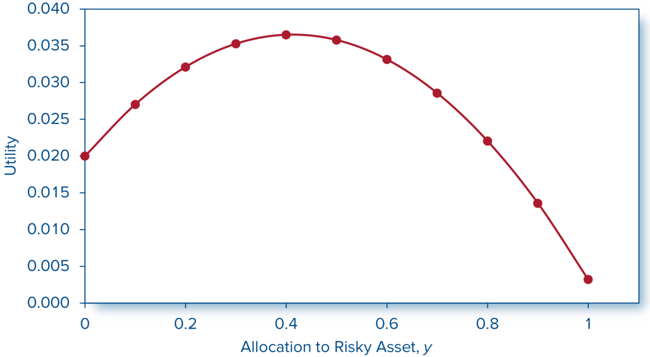

Figure 7: Utility as a function of allocation to the risky asset, y (BKM 2023, Figure 6.5)

An investor chooses y to maximize her utility III

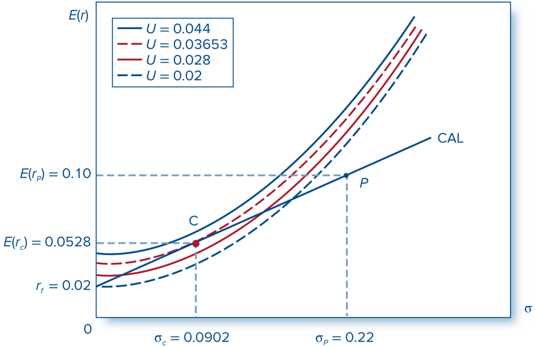

Figure 8: Finding the optimal complete portfolio by using indifference curves (BKM 2023, Figure 6.7)

Our analysis above assumes normality by treating standard deviation as the only measure of risk

- Our analysis above is standard
- Still, we might want also to consider
  - VaR and ES to estimate exposure to extreme losses
  - “Black swan” events (i.e., rare but high-impact events)

The capital market line (CML) plots the opportunity set of *passive* strategies I

- The CML is the CAL based on T-bills and a broad index of common stocks
- We typically use the value-weighted portfolio of all U.S. common stocks as this broad index
- Why are passive strategies reasonable?
  1. Active strategies require costly research to find and maintain the optimal portfolio, while passive strategies require very little research and trading
  2. Passive strategies free ride on the research and trading of active strategies, which keep most securities fairly priced

The capital market line (CML) plots the opportunity set of *passive* strategies II

	Average Annual Returns		U.S. Equity Market		
Period	U.S. equity	1-month T-bills	Excess return	Standard deviation	Sharpe ratio
1927–2021	12.17	3.30	8.87	20.25	0.44
1927–1950	10.26	0.93	9.33	26.57	0.35
1951–1974	10.21	3.59	6.63	20.32	0.33
1975–1998	17.97	6.98	11.00	14.40	0.76
1999–2021	10.16	1.66	8.50	18.85	0.45

Figure 9: Average annual return on stocks and 1-month T-bills; standard deviation and Sharpe ratio of stocks over time (BKM 2023, Table 6.7)

We can use these historical data to estimate A!

- Investors solve:

$$y^* = \frac{E(r_M) - r_f}{A\sigma_M^2}$$

so:

$$A = \frac{E(r_M) - r_f}{y^*\sigma_M^2}$$

- About 69.3% of household net worth is invested in risky assets, so:

$$A = \frac{0.0887}{0.693 \times 0.2025^2} = 3.12$$

Summary from BKM (2023)

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1. Speculation is the undertaking of a risky investment for its risk premium. The risk premium has to be large enough to compensate a risk-averse investor for the risk of the investment.
2. A fair game is a risky prospect that has a zero risk premium. It will not be undertaken by a risk-averse investor.
3. Investors' preferences toward the expected return and volatility of a portfolio may be expressed by a utility function that is higher for higher expected returns and lower for higher portfolio variances. More risk-averse investors will apply greater penalties for risk. We can describe these preferences graphically using indifference curves.
4. The desirability of a risky portfolio to a risk-averse investor may be summarized by the certainty equivalent value of the portfolio. The certainty equivalent rate of return is a value that, if received with certainty, would yield the same utility as the risky portfolio.
5. Shifting funds from the risky portfolio to the risk-free asset is the simplest way to reduce risk. Other methods involve diversification of the risky portfolio and hedging. We take up these methods in later chapters.
6. T-bills provide a perfectly risk-free asset in nominal terms only. Nevertheless, the standard deviation of real returns on short-term T-bills is small compared to that of other assets such as long-term bonds and common stocks, so for the purpose of our analysis we consider T-bills as the risk-free asset. Money market funds hold, in addition to T-bills, short-term relatively safe obligations such as repurchase agreements or bank CDs. These entail some default risk, but again, the additional risk is small relative to most other risky assets. For convenience, we often refer to money market funds as risk-free assets.
7. An investor's risky portfolio (the risky asset) can be characterized by its reward-to-volatility or Sharpe ratio, $S = [E(r_P) - r_f]/\sigma_P$. This ratio is also the slope of the CAL, the line that, when graphed, goes from the risk-free asset through the risky asset. All combinations of the risky asset and the risk-free asset lie on this line. Other things equal, an investor would prefer a steeper-sloping CAL because that means higher expected return for any level of risk. If the borrowing rate is greater than the lending rate, the CAL will be "kinked" at the point of the risky asset.
8. The investor's degree of risk aversion is characterized by the slope of the indifference curve. Indifference curves show, at any level of expected return and risk, the required risk premium for taking on each additional percentage point of standard deviation. More risk-averse investors have steeper indifference curves; that is, they require a greater risk premium for taking on more risk.
9. The optimal position, y^*, in the risky asset is proportional to the risk premium and inversely proportional to the variance and degree of risk aversion:

$$y^* = \frac{E(r_P) - r_f}{A\sigma_P^2}$$

   Graphically, this portfolio represents the point at which the indifference curve is tangent to the CAL.
10. A passive investment strategy disregards security analysis, targeting instead the risk-free asset and a broad portfolio of risky assets such as the S&P 500 stock portfolio. If in 2021 investors took the mean historical return and standard deviation of the S&P 500 as proxies for its expected return and standard deviation, then the values of outstanding assets would imply a degree of risk aversion of about $A = 3.12$ for the average investor. This is in line with other studies, which estimate typical risk aversion in the range of 2.0 through 4.0.

Key equations from BKM (2023)

Utility score: $U = E(r) - ½ A\sigma^2$

Expected return on complete portfolio: $E(r_C) = yE(r_C) + (1 - y)r_f$

Standard deviation of complete portfolio: $\Sigma_C = y\Sigma_P$

Optimal allocation to risky portfolio: $y^* = \dfrac{E(r_P) - r_f}{A\sigma_P^2}$

References I

Bodie, Zvi, Alex Kane, and Allan J. Marcus (2023). *Investments*. 13th ed. New York: McGraw Hill.