Chapter 10 - Arbitrage Pricing Theory and Multifactor Models of Risk and Return

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We can decompose stock variability into two sources: market and firm-specific

- Market risk is also known as systematic risk or non-diversifiable risk
- Firm-specific risk is also known as non-systematic risk, diversifiable risk, or idiosyncratic risk
- However, risk premiums may depend on correlations with extra-market risk factors (e.g., inflation, interest rates, and volatility)

Single-factor models allow two sources of uncertainty in asset returns

- These two sources of uncertainty in asset returns are
  - Common or macroeconomic surprises
  - Firm-specific surprises
- *Single-factor* models of excess returns have the form

$$R_i = E(R_i) + \beta_i F + e_i$$

  where
  - R_i is the realized excess return on stock i
  - $E(R_i)$ is the expected excess return on stock i
  - β_i is the sensitivity of stock i to factor F
  - F is the *deviation* of the common factor from its expected value
  - e_i is the firm-specific surprise, which is uncorrelated with F
- If $F = 0$ (i.e., no macroeconomic surprises), then $R_i = E(R_i) + e_i$ so the return on stock i is its expected return plus the firm-specific surprise

Multifactor models can better describe uncertainty in asset returns by allowing two or more systematic risk factors, as well as firm-specific surprises

- BKM propose a two-factor model where the systematic risk factors are
  - Unexpected growth in gross domestic product (GDP)
  - Unexpected changes in interest rates (IR)
- This two-factor model has the form:

$$R_i = E(R_i) + \beta_{i\mathsf{GDP}}\mathsf{GDP} + \beta_{i\mathsf{IR}}\mathsf{IR} + e_i$$

- We often call $\beta_{i\mathsf{GDP}}$ and $\beta_{i\mathsf{IR}}$ the *factor loadings* or *factor betas* on GDP and IR, respectively
- GDP increases are typically good news to asset returns, so we expect $\beta_{i\mathsf{GDP}} > 0$
- However, interest rate increases are typically bad news for asset returns, so we expect $\beta_{i\mathsf{IR}} < 0$
- Multifactor models can better capture different responses to different sources of macroeconomic uncertainty

Stephen Ross's arbitrage pricing theory (APT) predicts a security market line (SML) that links expected returns to risk...

- But the APT takes a different path to this SML than the capital asset pricing model (CAPM)
- The APT relies on three propositions
    1. A factor model can describe security returns
    2. There are sufficient securities to diversify away the idiosyncratic risk
    3. Well-functioning security markets do not allow for the persistence of arbitrage opportunities

An arbitrage opportunity exists when an investor can earn riskless profits without making a net investment

- A simple example would be if shares of a stock sold for different prices on two different exchanges
- The *Law of One Price* states that two assets that are the same in all economically relevant aspects should have the same price
- *Arbitrageurs* enforce the Law of One Price
- If they observe a violation, they will engage in *arbitrage activity* by simultaneously buying the cheap asset and selling the expensive asset
- These trades bid up (down) the price where it is low (high) until the arbitrage opportunity goes away

In a well-diversified portfolio, the firm-specific risk becomes negligible so that only factor risk remains

- Consider the excess return R_P on an n-stock portfolio with weights w_i where $\sum_{i=1}^{n} w_i = 1$

$$R_P = E(R_P) + \beta_P F + e_P$$

  where $E(R_P) = \sum_{i=1}^{n} w_i E(R_i)$, $\beta_P = \sum_{i=1}^{n} w_i \beta_i$, and $e_P = \sum_{i=1}^{n} e_i$
- There are two uncorrelated random variables on the right-hand side of the equation above, so portfolio variance σ_P^2 is

$$\sigma_P^2 = \beta_P^2 \sigma_F^2 + \sigma^2(e_P)$$

- If this portfolio were equally-weighted, we can rewrite $\sigma^2(e_P)$ as

$$\sigma^2(e_P) = \sum_{i=1}^{n} w_i^2 \sigma^2(e_i) = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \sigma^2(e_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma^2(e_i)}{n} = \frac{1}{n}\overline{\sigma}^2$$

  so the non-systematic variance of the portfolio equals the average non-systematic variance of the stocks, divided by n
- Therefore, non-systematic variance decreases as n increases

Only systematic or factor risk should be related to portfolio expected returns

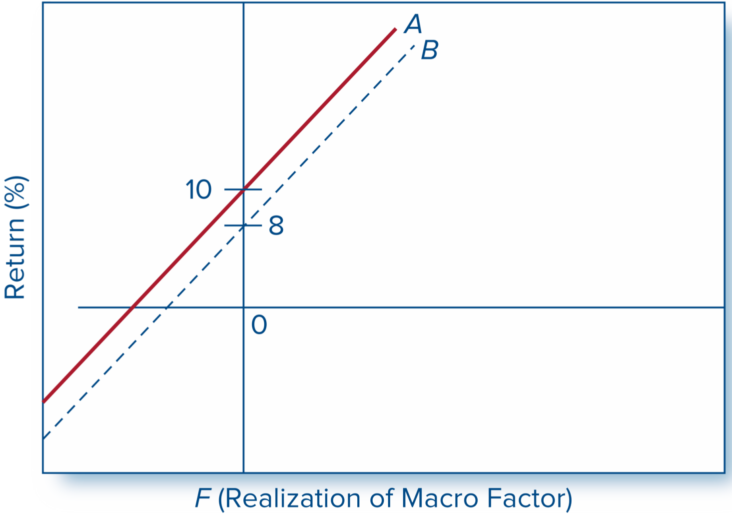

Figure 2: Returns as a function of the systematic factor: An arbitrage opportunity (BKM 2023, Figure 10.3)

Risk premiums must be proportional to betas

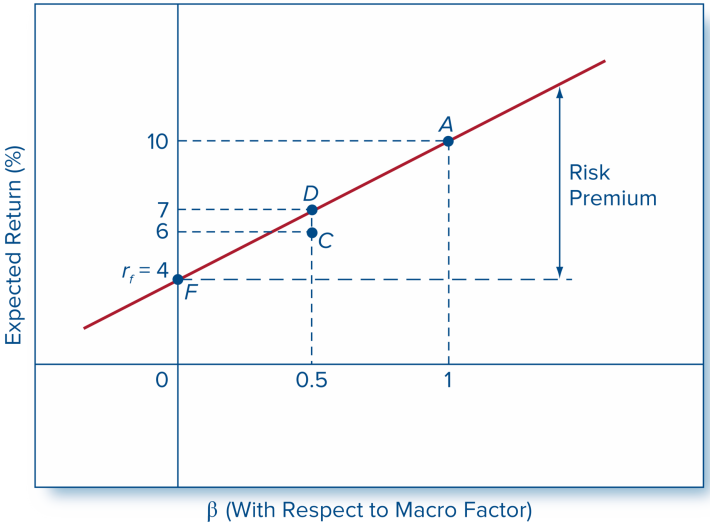

Figure 3: An arbitrage opportunity (BKM 2023, Figure 10.4)

The APT serves many of the same functions as the CAPM

- Both the CAPM and APT
  - Arrive at the same SML, which gives benchmarks for rates of return for capital budgeting, security valuation, or investment performance evaluation
  - Highlight the important distinction between non-diversifiable risk, which requires a reward in the form of a risk premium, and diversifiable risk, which does not
- However, there are important differences between the CAPM and APT
  - CAPM
    - Model is based on an inherently unobservable "market" portfolio
    - Provides unequivocal statement on the expected return-beta relationship for all securities
  - APT
    - Built on the foundation of well-diversified portfolios
    - Cannot rule out a violation of the expected return-beta relationship for any particular asset
    - Does not assume investors are mean-variance optimizers
    - Uses an observable market index

The APT can be generalized to accommodate multiple sources of risk

- Assume a two-factor model

$$R_i = E(R_i) + \beta_{i1}F_1 + \beta_{i2}F_2 + e_i$$

- The benchmark portfolios in the APT are *factor portfolios*, which are well-diversified portfolios constructed to have a beta of 1 on one factor and betas of 0 on the other factors
- We can think of these benchmark portfolios as *tracking portfolios* that track one source of macroeconomic risk and are uncorrelated with other sources of risk

The Fama-French (FF) three-factor model uses firm characteristics to identify sources of systematic risk

- The Fama-French three-factor model is:

$$R_{it} = \alpha_i + \beta_{iM} R_{Mt} + \beta_{iSMB} \mathsf{SMB}_t + \beta_{iHML} \mathsf{HML}_t + e_{it}$$

  where
  - SMB = small minus big, which is the return of a portfolio of small stocks minus the return on a portfolio of large stocks
  - HML = high minus low, which is the return of a portfolio of stocks with a high book-to-market ratio minus the return on a portfolio of stocks with a low book-to-market ratio
- Fama and French (1993) chose SMB and HML because size (market capitalization) and the book-to-market ratio (book value per share divided by stock price) predict deviations of average stock returns from levels consistent with the CAPM
- SMB and HML are not obvious candidates for risk factors, but Fama and French (1993) argue that they may proxy hard-to-measure fundamental variables

Estimating and implementing a three-factor SML for U.S. Steel

	Single-Index model		Three-Factor Model	
	Regression Coefficient	***t*-Statistic**	**Regression Coefficient**	***t*-Statistic**
Intercept (alpha)	−2.034	−0.991	−0.745	−0.363
$r_m - r_f$	2.030	4.785	1.771	4.074
SMB			0.369	0.487
HML			1.249	2.303
R-square	0.283		0.352	
Residual std dev (%)	15.171		14.675	

Figure 4: Estimates of single-index and three-factor Fama-French regressions for U.S. Steel, monthly data, 5 years ending in 2021 (BKM 2023, Table 10.1)

- The three-factor model predicts an expected return of

  $$E(r_X) = r_f + \beta_{X,M}[E(r_M) - r_f] + \beta_{X,SMB}E(r_{SMB}) + \beta_{X,HML}E(r_H$$

  which is

  $$23.9\% = 2\% + 1.771 \times 8.8\% + 0.369 \times 3.0\% + 1.249 \times 4.2\%$$

Researchers have added *many*, *many* factors to the FF three-factor model

- The two most widely-accepted additions to the three-factor model are:
  - Carhart (1997) adds a momentum factor
    - The difference between those that performed well last year and those that performed poorly
    - Often called UMD for "up minus down"
  - Fama and French (2015) add a profitability factor and an investment factor
    - The profitability factor, RMW for "robust minus weak", is the difference in returns between a diversified portfolio with strong versus weak profitability
    - The investment factor, CMA for "conservative minus aggressive", is the difference in returns between a diversified portfolio with low versus high investment
- The overwhelming number of other factors are often called the *factor zoo*
  - Many may be redundant and minor variations on the same theme
  - Many may be spurious and the result of "data snooping"

Multifactor models have two important generalizations

1. Investors should be aware that portfolios have exposure to more than one source of systematic risk, and they should think about how much exposure they want to each factor
2. When they evaluate performance, they should be aware of the risk premium from each factor and consider these when they estimate alphas

- Smart-beta exchange-traded funds (ETFs) are designed to provide exposure to specific characteristics such as value, growth, and volatility
- The most common extra-market risk factors are:
  - The Fama-French five-factor model: Size (SMB), value (HML), profitability (RMW), and investment (CMA)
  - Momentum
  - Volatility, liquidity, and dividend yield

Summary from BKM (2023)

1. Multifactor models seek to improve the explanatory power of single-factor models by explicitly accounting for the various components of systematic risk. These models use indicators intended to capture a wide range of macroeconomic risk factors.
2. Once we allow for multiple risk factors, we conclude that the security market line also ought to be multidimensional, with exposure to each risk factor contributing to the total risk premium of the security.
3. A (risk-free) arbitrage opportunity arises when two or more security prices enable investors to construct a zero-net-investment portfolio that will yield a sure profit. The presence of arbitrage opportunities will generate a large volume of trades that puts pressure on security prices. This pressure will continue until prices reach levels that preclude such arbitrage.
4. When securities are priced so that there are no risk-free arbitrage opportunities, we say that they satisfy the no-arbitrage condition. Price relationships that satisfy the no-arbitrage condition are important because we expect them to hold in real-world markets.
5. Portfolios are called "well diversified" if they include a large number of securities and the investment proportion in each is sufficiently small. The proportion of any particular security in a well-diversified portfolio is small enough so that, for all practical purposes, a reasonable change in that security's rate of return will have a negligible effect on the portfolio's rate of return.
6. In a single-factor security market, all well-diversified portfolios have to satisfy the expected return–beta relationship of the CAPM to satisfy the no-arbitrage condition. If all well-diversified portfolios satisfy the expected return–beta relationship, then individual securities also must satisfy this relationship, at least approximately.
7. The APT does not require the restrictive assumptions of the CAPM and its (unobservable) market portfolio. The price of this generality is that the APT does not guarantee this relationship for all securities at all times.
8. A multifactor APT generalizes the single-factor model to accommodate several sources of systematic risk. The multidimensional security market line predicts that exposure to each risk factor contributes to the security's total risk premium by an amount equal to the factor beta times the risk premium of the factor portfolio that tracks that source of risk.
9. The multifactor extension of the single-factor CAPM, the ICAPM, predicts the same multidimensional security market line as the multifactor APT. The ICAPM suggests that priced extra-market risk factors will be the ones that lead to significant hedging demand by a substantial fraction of investors. Other approaches to the multifactor APT are more empirically based, where the extra-market factors are selected based on past ability to predict risk premiums.

Key equations from BKM (2023)

Single-factor model: $R_i = E(R_i) + \beta_i F + e_i$

Multifactor model (here, 2 factors, F_1 and F_2): $R_i = E(R_i) + \beta_{i1} F_1 + \beta_{i2} F_2 + e_i$

Single-index model: $R_i = \alpha_i + \beta_i R_M + e_i$

Multifactor SML (here, 2 factors, labeled 1 and 2):

$$\begin{aligned} E(r_i) &= r_f + \beta_{i1}\left[E(r_1) - r_f\right] + \beta_{i2}\left[E(r_2) - r_f\right] \\ &= r_f + \beta_{i1} E(R_1) + \beta_{i2} E(R_2) \end{aligned}$$

where β_{i1} and β_{i2} measure the stock's typical response to returns on each factor portfolio and the risk premiums on the two factor portfolios are $E(R_1)$ and $E(R_2)$.

References I

Bodie, Zvi, Alex Kane, and Allan J. Marcus (2023). *Investments*. 13th ed. New York: McGraw Hill.

Carhart, Mark M. (1997). "On persistence in mutual fund performance". In: *The Journal of Finance* 52.1, pp. 57–82.

Fama, Eugene F. and Kenneth R. French (1993). "Common risk factors in the returns on stocks and bonds". In: *Journal of Financial Economics* 33.1, pp. 3–56.

— (2015). "A five-factor asset pricing model". In: *Journal of Financial Economics* 116.1, pp. 1–22.