Chapter 5 - Risk, Return, and the Historical Record

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Holding period return (HPR) is the price increase plus income, all divided by the price paid

$$\text{HPR} = \frac{\text{Ending price of a share} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

We typically express HPRs as effective annual rates (EARs) to simplify comparisons

The HPR r_T over T years is:

$$r_T = (1 + \mathsf{EAR})^T - 1$$

so:

$$\mathsf{EAR} = (1 + r_T)^{1/T} - 1$$

Rates on short-term investments are often annualized with simple interest instead of compound interest, called annual percentage rate (APR)

The relation between EAR and APR is:

$$1 + \mathsf{EAR} = \left(1 + \frac{\mathsf{APR}}{n}\right)^n$$

where n is the number of compounding periods per year, so:

$$\mathsf{APR} = n \times \left[(1 + \mathsf{EAR})^{1/n} - 1\right]$$

If we compound continuously, $n \to \infty$

The relation between EAR and APR (here written as r_{cc}) is:

$$1 + \text{EAR} = \exp(r_{cc})$$

so:

$$\ln(1 + \text{EAR}) = r_{cc}$$

where $\ln(\cdot)$ is the natural log

The quoted interest rate is 3.8% (APR with quarterly compounding).

a. What is the quarterly rate?

b. What is the effective annual rate (EAR)?

c. What is the effective rate for 6 months, i.e., the semiannual rate?

d. What is the effective daily rate?

The *dollar value* of a bank account grows at the *nominal interest rate*, but its *buying power* grows at the *real interest rate*

The Fisher equation is:

$$1 + r_{real} = \frac{1 + r_{nom}}{1 + i}$$

where r_{real}, r_{nom}, and i are the real interest, nominal interest, and inflation rates, so:

$$r_{real} = \frac{r_{nom} - i}{1 + i}$$

so, if inflation is low:

$$r_{real} \approx r_{nom} - i$$

Inflation is a tax on savings!

- The real after-tax rate is approximately the after-tax nominal rate minus the inflation rate
- Therefore, the after-tax real returns fall by the inflation rate times the tax rate:

$$r_{nom}(1-t)-i \approx (r_{real}+i)(1-t)-i = r_{real}(1-t)-it$$

. The expected inflation rate is 1.4%.

a. What is the nominal interest rate if the real rate of interest is 2.1%?

b. What is the real interest rate if the nominal rate of interest is 6.6%?

. You've invested money at an interest rate of 8%. Your tax rate is 33%.

a. What is your after-tax interest rate?

The Fisher equation predicts that the nominal interest rate should track the (expected) inflation rate, leaving the real rate relatively stable

	Average Annual Rates			Standard Deviation		
	T-Bills	Inflation	Real T-Bill	T-Bills	Inflation	Real T-Bill
Full sample	3.30	3.02	0.38	3.10	3.98	3.78
1927–1951	0.95	1.80	-0.48	1.24	6.06	6.34
1952–2021	4.14	3.46	0.68	3.14	2.84	2.27

Source: Annual rates of return from rolling over 1-month T-bills: Kenneth French; annual inflation rates: Bureau of Labor Statistics.

Figure 1: Statistics for T-bill rates, inflation rates, and real rates, 1927-2021 (BKM 2023, Table 5.3)

The realized rate of return depends on price changes and dividends paid

Recall:

$$\text{HPR} = \frac{\text{Ending price of a share} - \text{Beginning price} + \text{Cash dividend}}{\text{Beginning price}}$$

The future is uncertain I

- We perform *scenario analysis* to consider the HPRs of various economic scenarios and the probabilities of each scenario
- We measure expected or mean rates of return $E(r)$ as the *probability-weighted average of the rates returns in each scenario*:

$$E(r) = \sum_{s} p(s) r(s)$$

  where $p(s)$ and $r(s)$ are the probability and HPR of each scenario s
- We can measure uncertainty as the variance of returns, which is the average *squared* deviation of actual returns from the mean return:

$$\mathrm{Var}(r) = \sigma^2 = \sum_{s} p(s) \left[r(s) - E(r) \right]^2$$

The future is uncertain II

- In practice, we use the square root of the variance of returns (i.e., $\sigma = \sqrt{\sigma^2}$), which is the standard deviation of returns and has the same units as returns (i.e., % instead of $\%^2$)

We compare risky investment returns to the risk-free rate

- The *risk premium* is the difference between the *expected* rate of return and the risk-free rate
- The *excess return* is the difference between the *realized* rate of return and the risk-free rate
- Investors are risk averse and require positive risk premiums to hold risky investments
- *However, after the fact, excess returns may be negative*

The Sharpe ratio is a reward-to-volatility ratio

- Investors accept risk for the *potential* to earn returns greater than the risk-free rate
- Risk premiums are proportional to risk, and we measure risk as the standard deviation of *excess* returns
- The Sharpe ratio is commonly used to evaluate investment manager performance:

$$\text{Sharpe ratio} = \frac{\text{Risk premium}}{\text{SD of excess returns}}$$

- We multiply the numerator by T and the denominator by $\sqrt{T}$ to annualize the Sharpe ratio (e.g., $T = 12$ for monthly returns, and $T = 252$ for daily returns)

. We know the following expected returns for stocks A and B, given different states of the economy:

State (s)	Probability	$E(r_{A,s})$	$E(r_{B,s})$
Recession	0.2	-0.03	0.05
Normal	0.5	0.12	0.08
Expansion	0.3	0.2	0.12

a. What is the expected return for stock A?

b. What is the expected return for stock B?

c. What is the standard deviation of returns for stock A?

d. What is the standard deviation of returns for stock B?

. Below are the expected returns for different asset classes for next year:

Asset class	Exp. return
T-bills	1.7%
Corporate bonds	4.7%
Small company stocks	18%
Large company stocks	12.4%

a. What is the risk premium for corporate bonds?

b. What is the risk premium for small company stocks?

c. What is the risk premium for large company stocks?

. A portfolio had an annual return of 6% and an annual standard deviation of 33%. Treasury bills yielded 1% during the same period.

a. What was the Sharpe ratio?

We typically assume that market returns are normally distributed I

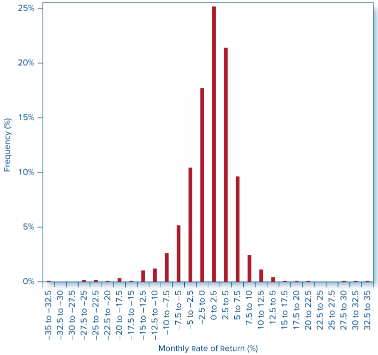

Figure 2: Frequency distribution of monthly rate of return on the broad market index, 1927-2020 (BKM 2023, Figure 5.3)

We typically assume that market returns are normally distributed II

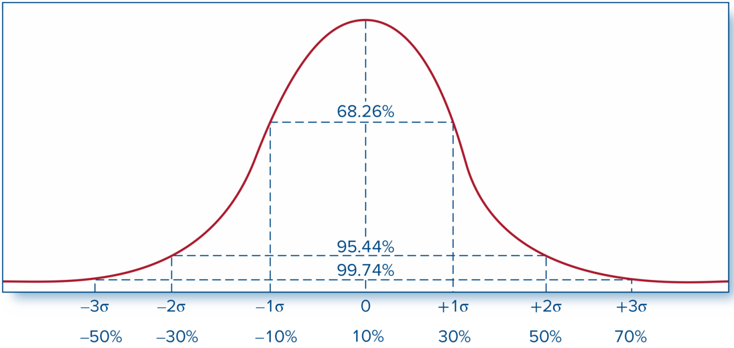

Figure 3: The normal distribution with mean 10% and standard deviation 20% (BKM 2023, Figure 5.4)

This normally-distributed returns assumption is not perfect, but it has a few advantages

1. Normal distribution is symmetric
2. The return on a portfolio of assets with normally distributed returns is also normally distributed
3. We can fully describe a normal distribution with two parameters: mean and standard deviation
4. We can fully describe the relation between two normal distributions with one parameter: the correlation coefficient

Asset returns have negative skew and fat tails I

- *Skew* is the average *cubed* deviation from the mean:

$$\text{Skew} = \text{Average}\left[\frac{(r-\bar{r})^3}{\hat{\sigma}^3}\right] = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_t-\bar{r})^3}{\hat{\sigma}^3}$$

- Negative skew indicates that extremely bad outcomes are more likely than extremely good outcomes
- Negative-skewed distributions are left skewed with left tails that are fatter than right tails
- When returns are:
    - Positively skewed, SD overestimates risk
    - Negatively skewed, SD underestimates risk

Asset returns have negative skew and fat tails II

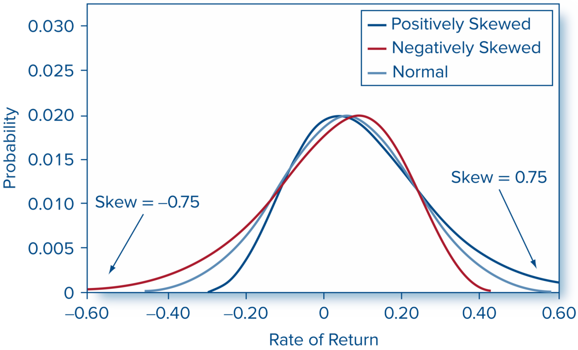

Figure 4: Normal and skewed distributions (mean = 6%, SD = 17%) (BKM 2023, Figure 5.5)

Asset returns have negative skew and fat tails III

- *Kurtosis* is the average deviation from the mean *raised to the fourth power*

$$\text{Kurtosis} = \text{Average}\left[\frac{(r-\bar{r})^4}{\hat{\sigma}^4}\right] - 3 = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_t-\bar{r})^4}{\hat{\sigma}^4} - 3$$

- We subtract 3 because the normal distribution has a kurtosis of 3
- Kurtosis values greater than zero indicate "fat tails," where extreme outcomes are more likely than for normal distributions
- Kurtosis raises deviations to the fourth power, so kurtosis is more sensitive to outliers than the variance

Asset returns have negative skew and fat tails IV

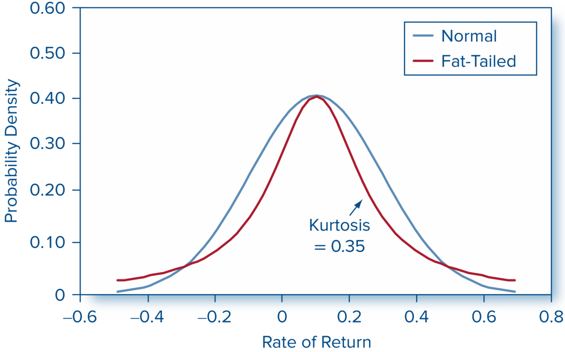

Figure 5: Normal and fat-tailed distributions (mean = 10%, SD = 20%) (BKM 2023, Figure 5.6)

There are four common measures of downside risk

1. Value at risk (VaR) is the loss corresponding to a low percentile of the return distribution (e.g., we expect 99% of returns to exceed the 1% VaR)
2. Expected shortfall (ES) is the expected loss given that we find ourselves in a worst-case scenario (e.g., the 1% ES is the average of the worst 1% of all returns)
3. Lower partial standard deviation (LPSD) is the standard deviation of *negative* excess returns
    - The Sortino ratio is the ratio of the mean excess return to the LPSD (i.e., the Sharpe ratio with the LPSD instead of the SD)
4. Relative frequency of large, negative 3-sigma returns calculates the percent of returns more than 3 SDs below the mean

Expected returns and the arithmetic average

- When we do *scenario analysis*, we specify a probability $p(s)$ and return $r(s)$ for each scenario s, so:

$$E(r) = \sum_{s=1}^{n} p(s) r(s)$$

- When we use *historical data*, we treat each observation as an equally likely scenario, so $p(s) = \frac{1}{n}$ for each s, so:

$$E(r) = \sum_{s=1}^{n} p(s) r(s) = \frac{1}{n} \sum_{s=1}^{n} r(s)$$

which is the *arithmetic* average

Arithmetic average versus geometric average

- The *arithmetic* average is an unbiased estimate of the *expected* return
- The *geometric* average is a HPR that compounds to the same terminal value as the historical data:

$$\text{Geometric average} = \left[\prod_{s=1}^{n}(1+r_s)\right]^{1/n} - 1$$

- The geometric average is always less than or equal to the arithmetic average
- With normally-distributed returns:

$$E[\text{Geometric average}] = E[\text{Arithmetic average}] - \frac{\sigma^2}{2}$$

Estimating return variance and standard deviation

- With historical data, we treat each observation as an equally likely scenario, so we use an equal probability of $p(s) = \frac{1}{n}$ for each scenario s
- This assumption provides $\bar{r}$ as an unbiased estimate of $E(r)$
- However, this assumption provides downward biased (too low) estimates of σ^2 because we use one degree of freedom to estimate $\bar{r}$
- Therefore, we use an equal probability of $p(s) = \frac{1}{n-1}$ for each scenario s:

$$\hat{\sigma} = \sqrt{\frac{1}{n-1}\sum_{s=1}^{n}[r(s) - \bar{r}]^2}$$

. You've collected the following historical rates of return for stocks A and B:

Year (t)	$r_{A,t}$	$r_{B,t}$
2016	0.02	0.03
2015	0.08	0.05
2014	0.12	0.07

a. What was the average annual return for stock A

b. What was the average annual return for stock B?

c. What was the standard deviation of returns for stock A?

d. What was the standard deviation of returns for stock B?

A stock delivered the following annual returns over the last 4 years:

Year	1	2	3	4
Return	8%	3%	-8%	8%

a. What was the arithmetic average return over the 4 years?

b. What was the geometric average return over the 4 years?

c. If you invested $1,000 at the beginning, how much would you have had at the end?

d. What is the standard deviation of the stock's return based on the historical data?

. You want to have $500,000 in today's (real) dollars when you retire in 40 years. The expected inflation rate is 2.9% and the nominal return on your investments is 7.6%.

a. How much money do you have to save now if you can't make any additional deposits?

a. Which statements are correct? The geometric average return _____.

Check all that apply:

- [ ] takes into account compounding
- [ ] is usually less than the arithmetic mean
- [ ] is better for forecasting returns over many periods
- [ ] is the simple average of the individual returns

Historic returns on risky portfolios I

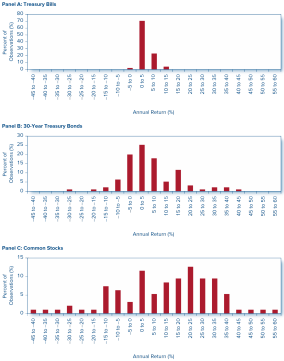

Figure 6: Frequency distribution of annual returns on U.S. Treasury bills, Treasury bonds, and common stocks (BKM 2023, Figure 5.7)

Historic returns on risky portfolios II

	Market Index	Big/Growth	Big/Value	Small/Growth	Small/Value
A. 1927–2021					
Mean excess return (annualized)	8.86	8.79	12.02	9.60	15.54
Standard deviation (annualized)	18.52	18.35	24.83	25.97	28.23
Sharpe ratio	0.48	0.48	0.48	0.37	0.55
Lower partial SD (annualized)	19.62	18.85	23.61	25.86	26.33
Skew	0.17	-0.11	1.50	0.59	2.06
Kurtosis	7.61	5.47	17.58	7.10	21.60
VaR (1%) actual (monthly) returns	-13.58	-14.40	-19.93	-20.10	-20.68
VaR (1%) normal distribution	-11.75	-11.64	-15.75	-16.69	-17.78
% of monthly returns more than 3 SD below mean	0.88%	0.79%	0.88%	0.79%	0.62%
Expected shortfall (monthly)	-19.60	-19.86	-24.61	-24.64	-26.00
B. 1952–2021					
Mean excess return (annualized)	8.40	8.47	10.59	8.21	13.51
Standard deviation (annualized)	14.95	15.50	17.08	22.39	19.02
Sharpe ratio	0.56	0.55	0.62	0.37	0.71
Lower partial SD (annualized)	16.17	15.78	17.69	23.72	20.23
Skew	-0.52	-0.35	-0.46	-0.33	-0.45
Kurtosis	1.90	1.75	3.03	2.17	3.83
VaR (1%) actual (monthly) returns	-10.89	-10.84	-12.48	-17.22	-15.24
VaR (1%) normal distribution	-9.39	-9.75	-10.66	-14.40	-11.75
% of monthly returns more than 3 SD below mean	0.71%	0.60%	0.83%	0.95%	1.07%
Expected shortfall (monthly)	-14.47	-13.96	-18.08	-22.76	-20.27

Statistics for monthly excess returns on the market index and four "style" portfolios

Source: Authors' calculations using data from Prof. Kenneth French's web site: http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

Figure 7: Statistics for monthly excess returns on the market index and four "style" portfolios (BKM 2023, Table 5.5)

Investments in risky portfolios do not become safer in the long run

	Investment Horizon			
	1	10	30	Comment
Mean total return	0.050	0.500	1.500	= 0.05*T
Mean average return	0.050	0.050	0.050	= 0.05
Std dev total return	0.300	0.949	1.643	$= 0.30^* \sqrt{T}$
Probability return < 0	0.434	0.299	0.181	assuming normal distribution
1% VaR total return	−0.648	−1.707	−2.323	continuously compounded return
Implies final wealth relative of:	0.523	0.181	0.098	= exp(VaR total return)
0.1% VaR total return	−0.877	−2.432	−3.578	continuously compounded return
Implies final wealth relative of:	0.416	0.088	0.028	= exp(VaR total return)

Figure 8: Investment risk at different horizons (BKM 2023, Table 5.6)

Summary and Key Equations I

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1. The economy's equilibrium level of real interest rates depends on the willingness of households to save, as reflected in the supply curve of funds, and on the expected profitability of business investment in plant, equipment, and inventories, as reflected in the demand curve for funds. It depends also on government fiscal and monetary policy.
2. The nominal rate of interest is the equilibrium real rate plus the expected rate of inflation. In general, we can directly observe only nominal interest rates; from them, we must infer expected real rates, using inflation forecasts. Assets with guaranteed nominal interest rates are risky in real terms because the future inflation rate is uncertain.
3. The equilibrium expected rate of return on any security is the sum of the equilibrium real rate of interest, the expected rate of inflation, and a security-specific risk premium.
4. Investors face a trade-off between risk and expected return. Historical data confirm our intuition that assets with low degrees of risk should provide lower returns on average than do those of higher risk.
5. Historical rates of return over the last century in other countries suggest the U.S. history of stock returns is may be a positive outlier compared to other countries. This would suggest that its historical experience may overstate realistic projections for future performance as well as the risk premium demanded by stock market investors.
6. Historical returns on stocks exhibit somewhat more frequent large deviations from the mean than would be predicted from a normal distribution. However, the discrepancies from the normal distribution tend to be minor and inconsistent across various measures of tail risk, and have declined in recent years. The lower partial standard deviation (LPSD), skew, and kurtosis of the actual distribution quantify the deviation from normality.
7. Widely used measures of tail risk are value at risk (VaR) and expected shortfall or, equivalently, conditional tail expectation. VaR measures the loss that will be exceeded with a specified probability such as 1% or 5%. Expected shortfall (ES) measures the expected rate of return conditional on the portfolio falling below a certain value. Thus, 1% ES is the expected value of the outcomes that lie in the bottom 1% of the distribution.
8. Investments in risky portfolios *do not* become safer in the long run. On the contrary, the longer a risky investment is held, the greater the risk. The basis of the argument that stocks are safe in the long run is the fact that the probability of an investment shortfall becomes smaller. However, probability of shortfall is an incomplete measure of the safety of an investment because it ignores the magnitude of possible losses.

Figure 9: Chapter 5 summary from BKM (2023)

Summary and Key Equations II

Arithmetic average of n returns: $(r_1 + r_2 + \cdots + r_n) / n$

Geometric average of n returns: $[(1 + r_1)(1 + r_2) \cdots (1 + r_n)]^{1/n} - 1$

Continuously compounded rate of return, $r_{cc} = \ln(1 + \text{Effective annual rate})$

Expected return: $\sum [\text{prob(Scenario)} \times \text{Return in scenario}]$

Variance: $\sum[\text{prob(Scenario)} \times (\text{Deviation from mean in scenario})^2]$

Standard deviation: $\sqrt{\text{Variance}}$

Sharpe ratio: $\dfrac{\text{Portfolio risk premium}}{\text{Standard deviation of excess return}} = \dfrac{E(r_P) - r_f}{\sigma_P}$

Real rate of return: $\dfrac{1 + \text{Nominal return}}{1 + \text{Inflation rate}} - 1$

Real rate of return (continuous compounding): $r_{\text{nominal}} - \text{Inflation rate}$

Figure 10: Chapter 5 key equations from BKM (2023)

References I

Bodie, Zvi, Alex Kane, and Allan J. Marcus (2023). *Investments*. 13th ed. New York: McGraw Hill.