Topics in Macroeconomics

Value function iteration (VFI)

Richard Foltyn

University of Glasgow

February 2023

Topics covered in this unit

Solving household problems with VFI

We discuss the following models:

1. No uncertainty, no labour income: analytical solution
2. Certain labour income
3. Risky labour income

In all cases we assume CRRA preferences!

Solution methods

1. Grid search: no interpolation
2. "Unrestricted" maximisation:
    1. Linear interpolation
    2. Spline interpolation

Contents

1 VFI with analytical solution (no income)

2 VFI with grid search (constant income)

3 VFI with interpolation (risky income)

4 Appendix: Approximating AR(1) processes with Markov chains

VFI with analytical solution

Analytical solution

- Under some conditions value function has closed-form solution
- Assumptions:
  1. CRRA preferences
  2. No labour income
  3. No uncertainty
- Solution methods:
  1. Iteration on value function (“manual” VFI)
  2. Guess and verify (not covered)

Illustration: manually iterate on closed form, compare with numerical solution.

Household problem

Analytical VFI

Consider infinite-horizon consumption-savings problem with log preferences:

$$
\begin{aligned}
V(a) = \max_{c,\, a'} \; & \left\{ \log(c) + \beta V(a') \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a \\
& c \geq 0, \; a' \geq 0
\end{aligned}
$$

where

- a Beginning-of-period assets
- a' Next-period assets (savings)
- r Constant interest rate
- β Discount factor $\beta \in (0, 1)$

How can we find V using VFI?

- Initial guess:
    - Consume everything: $c = (1+r)a$
    - Continuation value is zero
- Value function in iteration 1:

$$V_1(a) = \log(c) = \log\big((1+r)a\big) = \log(1+r) + \log(a)$$

- HH problem in iteration 2:

$$\begin{aligned} V_2(a) &= \max_{c,\,a'} \left\{\log(c) + \beta V_1(a')\right\} \\ &= \max_{c,\,a'} \left\{\log(c) + \beta\Big[\log(1+r) + \log(a')\Big]\right\} \\ \text{s.t.} \quad & c + a' = (1+r)a \\ & c \geq 0,\ a' \geq 0 \end{aligned}$$

Iteration 2

- First-order conditions:

$$c^{-1} = \lambda \qquad\qquad \beta\,(a')^{-1} = \lambda$$

  where λ is Lagrange multiplier on budget constraint.
- Eliminate Lagrange multiplier: $a' = \beta c$
- Substitute for a' in budget constraint to find policy functions:

$$\begin{aligned} c + \beta c = (1+r)a \quad &\Longrightarrow \quad c = (1+\beta)^{-1}(1+r)a \\ &\Longrightarrow \quad a' = \beta(1+\beta)^{-1}(1+r)a \end{aligned}$$

- Plug policy functions into value function:

$$\begin{aligned} V_2(a) &= \log\left((1+\beta)^{-1}(1+r)a\right) + \beta\Big[\log(1+r) + \log\Big(\beta(1+\beta)^{-1}(1+r)a\Big)\Big] \\ &= \beta\log\beta - (1+\beta)\log(1+\beta) + (1+2\beta)\log(1+r) + (1+\beta)\log(a) \end{aligned}$$

After 2 iterations we have:

$$V_1(a) = \underbrace{\log(1+r)}_{\chi_1} + \underbrace{1}_{\varphi_1} \times \log(a)$$

$$V_2(a) = \underbrace{\beta \log \beta - (1+\beta)\log(1+\beta) + (1+2\beta)\log(1+r)}_{\chi_2} + \underbrace{(1+\beta)}_{\varphi_2} \log(a)$$

- Continue iterating? — Expressions become too complicated!
- Instead conjecture that value function takes the form

$$V_n(a) = \chi_n + \varphi_n \log(a) \tag{1}$$

We have shown this to be true for $n = 1, 2$

Solving via induction

Analytical VFI

- Assume V has functional form given in (1) for some n
- Then V_{n+1} will be given by

$$V_{n+1}(a) = \chi_{n+1} + \varphi_{n+1} \log(a)$$

- Task: find $\chi_{n+1}, \varphi_{n+1}$ given χ_n, φ_n
- We do this by solving

$$\begin{aligned}
V_{n+1}(a) &= \max_{c,\, a'} \left\{ \log(c) + \beta V_n(a') \right\} \\
&= \max_{c,\, a'} \left\{ \log(c) + \beta \Big[\chi_n + \varphi_n \log(a') \Big] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a \\
& c \geq 0, \;\; a' \geq 0
\end{aligned}$$

Iteration $n+1$

- First-order conditions for the $(n+1)$-th iteration:

$$c^{-1} = \lambda \qquad\qquad \beta\varphi_n\,(a')^{-1} = \lambda$$

- Substitute for a' in budget constraint to find policy functions:

$$c + \beta\varphi_n c = (1+r)a \implies c = (1+\beta\varphi_n)^{-1}(1+r)a \tag{2}$$

$$\implies a' = \frac{\beta\varphi_n}{1+\beta\varphi_n}(1+r)a \tag{3}$$

- Plug policy functions into value function:

$$V_{n+1}(a) = \log\Big((1+\beta\varphi_n)^{-1}(1+r)a\Big) + \beta\left[\chi_n + \varphi_n \log\left(\frac{\beta\varphi_n}{1+\beta\varphi_n}(1+r)a\right)\right]$$

Solution for V

Analytical VFI

- Collect terms:

$$V_{n+1}(a) = \underbrace{\beta\chi_n + \beta\varphi_n \log(\beta\varphi_n) - (1+\beta\varphi_n)\Big[\log(1+\beta\varphi_n) + \log(1+r)\Big]}_{\chi_{n+1}} + \underbrace{(1+\beta\varphi_n)}_{\varphi_{n+1}}\log(a)$$

- Pin down sequence of $(\varphi_n)_{n=1}^{\infty}$:
  - Follows first-order linear difference equation

$$\varphi_{n+1} = 1 + \beta\varphi_n$$

  - General solution (φ_0 pinned down by $\varphi_1 = 1$):

$$\varphi_n = \beta^n \left(\varphi_0 - \frac{1}{1-\beta}\right) + \frac{1}{1-\beta}$$

  - Convergence to limit:

$$\varphi = \lim_{n\to\infty} \varphi_n = \frac{1}{1-\beta}$$

- Difference equation for χ_n is much more complicated (depends on φ_n!)
- Compute only limiting value (move $\beta\chi_n$ to l.h.s.):

$$\lim_{n\to\infty} \chi_{n+1} - \beta\chi_n = \lim_{n\to\infty} \beta\varphi_n \log(\beta\varphi_n) - (1+\beta\varphi_n)\Big[\log(1+\beta\varphi_n) + \log(1+r)\Big]$$

- Limiting value given by:

$$\chi = \lim_{n\to\infty} \chi_n = \frac{\beta}{1-\beta}\log\beta + \log(1-\beta) + \frac{1}{1-\beta}\log(1+r)$$

- Converged value function V:

$$V(a) = \chi + \frac{1}{1-\beta}\log(a)$$

Value function convergence

Analytical VFI

Convergence of coefficients χ_n and φ_n in

$$V_n(a) = \chi_n + \varphi_n \log(a)$$

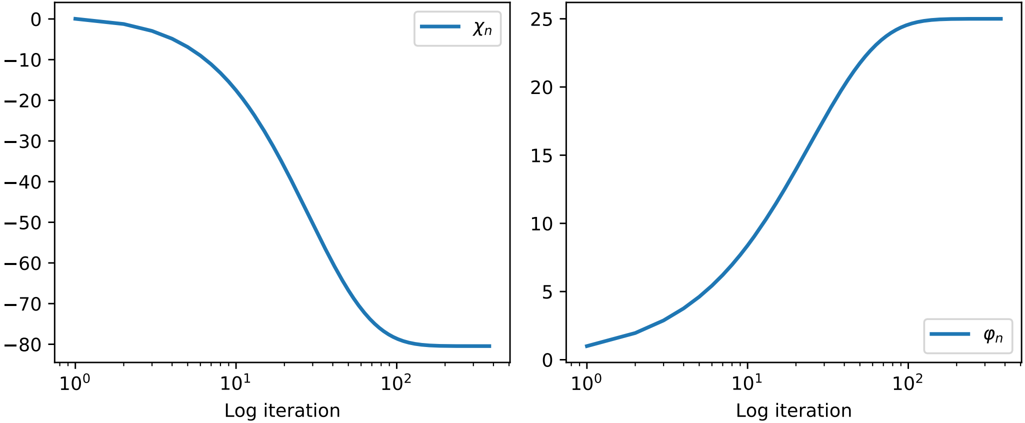

Figure 1: Convergence of analytical value function coefficients.

Analytical vs. numerical iteration

Analytical VFI

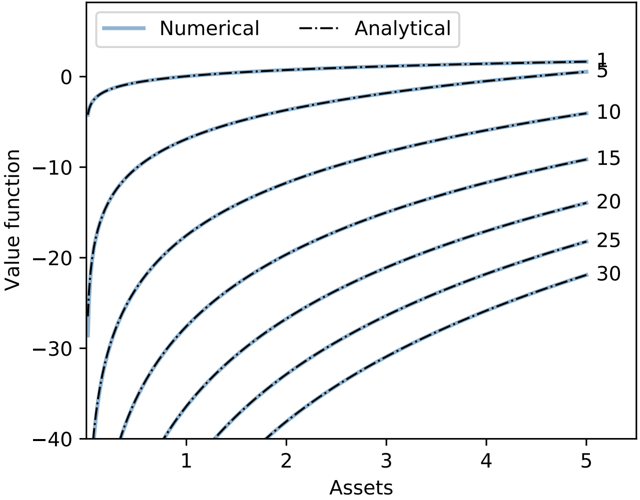

Figure 2: Value function V_n for the first few iterations.

Apply same reasoning to policy functions

- Define MPC as $\kappa_n \equiv (1 + \beta\varphi_n)^{-1}$
- Rewrite policy functions (2) and (3) at iteration $n + 1$ as:

$$
\begin{aligned}
c_{n+1} &= \kappa_n(1 + r)a \\
a'_{n+1} &= (1 - \kappa_n)(1 + r)a
\end{aligned}
$$

- $\lim_{n\to\infty} \kappa_n = 1 - \beta$
- Policy functions usually converge faster than value functions!

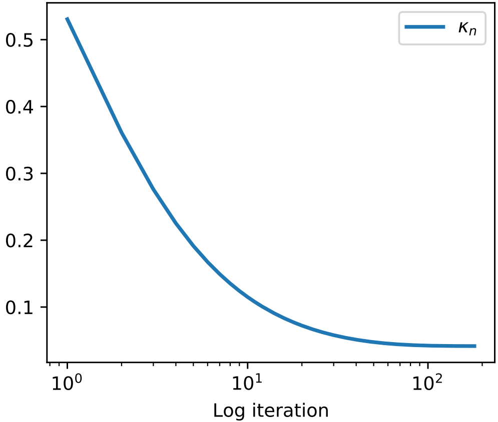

Figure 3: Convergence of policy function coefficient.

Analytical vs. numerical solution

Analytical VFI

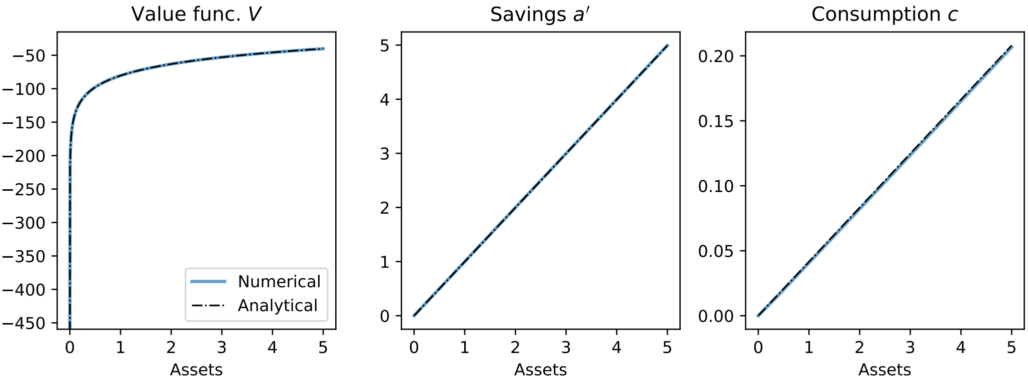

Figure 4: Converged value and policy functions.

VFI with grid search

VFI with grid search

- Restriction: solution method forces next-period assets to be exactly on discretized grid: $a' \in \mathcal{G}_a$
- Advantages:
  1. Easy to implement
  2. Derivative-free method
  3. Fast (unless grid is very dense)
- Disadvantages:
  1. Imprecise
  2. Policy functions are not smooth (unless grid is very dense)
  3. Does not scale well to multiple dimensions

Example: HH problem with constant labour income

VFI with grid search

- Infinitely-lived HH solves consumption-savings problem

$$\begin{aligned} V(a) = &\max_{c,\, a' \in \mathcal{G}_a} \Big\{ u(c) + \beta V(a') \Big\} \\ \text{s.t.} \quad & c + a' = (1+r)a + y \\ & c \geq 0, \;\; a' \geq 0 \end{aligned}$$

  where

  $\mathcal{G}_a$ Beginning-of-period asset grid
  y Constant labour income

- Preferences are assumed to be CRRA with relative risk aversion γ:

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

- Note that with $\gamma = 1$, $u(c) = \log(c)$ as before

Solution algorithm

VFI with grid search

1. Create asset grid $\mathcal{G}_a = (a_1, \ldots, a_{N_a})$
2. Pick initial guess for value function, V_0
3. In iteration n, perform the following steps
   1. For each asset level a_i, find all feasible next-period asset levels $a_j \leq (1+r)a_i + y$, $a_j \in \mathcal{G}_a$
   2. For each j, compute consumption $c_j = (1+r)a_i + y - a_j$
   3. For each j, compute utility
      $$U_j = u(c_j) + \beta V_n(a_j) \tag{4}$$
   4. Find the index k that maximises (4):
      $$k = \arg\max_j \left\{ u(c_j) + \beta V_n(a_j) \right\}$$
   5. Set $V_{n+1}(a_i) = U_k$ and store k as the optimal choice at a_i

Parametrisation for problem with constant labour income

VFI with grid search

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
y	Labour income	1
N_a	Asset grid size	50, 100, 1000

Table 1: Parameters for HH problem with constant labour income

- Each graph compares three solution methods:
  1. VFI with grid search
  2. VFI with linear interpolation
  3. VFI with cubic spline interpolation

Grid search is quite sensitive to grid size!

Compare results for $N_a = 50$, $N_a = 100$ and $N_a = 1000$.

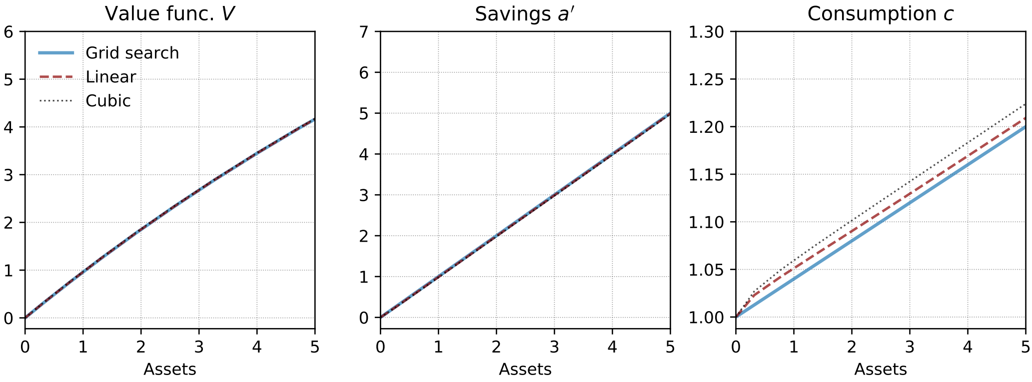

Figure 5: Solution with 50 asset grid points.

Solution for $N_a = 100$

VFI with grid search

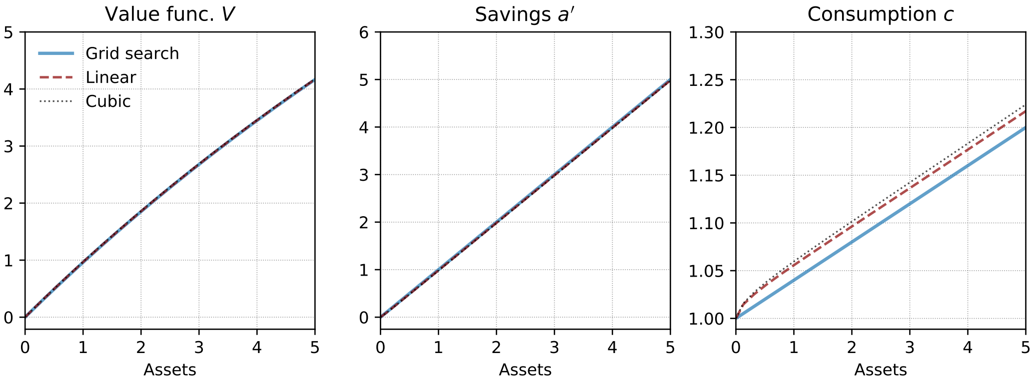

Figure 6: Solution with 100 assets grid points.

Solution for $N_a = 1000$

VFI with grid search

Figure 7: Solution with 1000 assets grid points.

VFI with interpolation

VFI with interpolation

- Grid search is rarely used today
- We prefer solution algorithms which find local maximum for each point on the grid (i.e. solution satisfies first-order conditions)
- Optimal points need not be on the grid, hence we have to interpolate
- Advantages:
  1. "Exact" solution (in a numerical sense)
  2. Less affected by curse of dimensionality in case of multiple choice variables
  3. Easier to spot mistakes since policy functions don't have artificial kinks as in grid search
- Disadvantages:
  1. Likely slower than grid search
  2. More complex to implement:
     - Need maximisation or root-finding routine
     - Need to compute derivatives of objective function or first-order condition, unless we use derivative-free methods or numerical differentiation.

Example: HH problem with risky labour income

VFI with interpolation

- Illustrate VFI with interpolation using standard Bewley/Huggett/Aiyagari problem with risky labour income
- Infinite-lived HH solves consumption-savings problem

$$
\begin{aligned}
V(a,y) = \max_{c,\,a'} \; & \left\{ u(c) + \beta \mathbf{E}\left[V(a',y') \,\middle|\, y \right] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0,\; a' \geq 0
\end{aligned}
$$

where

y Labour income process on state space $\mathcal{G}_y$ with transition probability $\Pr\left(y' = y_j \,|\, y = y_i \right) = \pi_{ij}$

- As before, $u(\bullet)$ is CRRA
- Note that now we have a two-dimensional state space on $\mathcal{G}_a \times \mathcal{G}_y$.

Solution algorithm

VFI with interpolation

1. Create asset grid $\mathcal{G}_a = (a_1, \dots, a_{N_a})$
2. Create discrete labour income process with states $\mathcal{G}_y = (y_1, \dots, y_{N_y})$ and transition matrix Π_y
3. Pick initial guess for value function, V_0
4. In iteration n, perform the following steps
   1. For each point (a_i, y_j) in the state space, find

      $$a^\star = \underset{a' \in [0, x_{ij}]}{\arg\max} \left\{ u\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} V_n\left(a', y_k\right) \right\}$$

      where $x_{ij} = (1+r)a_i + y_j$ is the cash at hand.
   2. Compute value at optimum,

      $$V^\star = u\left(x_{ij} - a^\star\right) + \beta \mathbf{E}\left[V_n\left(a^\star, y'\right) \middle| y_j \right]$$

   3. Set $V_{n+1}(a_i, y_j) = V^\star$ and store $A_{n+1}(a_i, y_j) = a^\star$ as the savings policy function.

How do we find $a^\star$?

1. We use a maximiser that finds the maximum $a^\star \in [0, x_{ij}]$ of the function

$$f\left(a' \mid a_i, y_j\right) = u\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} V_n\left(a', y_k\right)$$

for given (a_i, y_j).

   - Need to interpolate $V_n(\bullet, y_k)$ onto arbitrary a'
   - Need to either use derivative-free maximizer, or differentiate df/da' numerically

2. In principle, we could perform *root-finding* on the FOC

$$-u'\left(x_{ij} - a'\right) + \beta \sum_{k=1}^{N_y} \pi_{jk} dV_n\left(a', y_k\right)/da' = 0$$

This is rarely done since we don't know $dV_n\left(a', y_k\right)/da'$ and fast root-finders *additionally* need the derivative of the FOC!

Parametrisation for problem with risky labour

VFI with interpolation

- Assume labour process follows AR(1),

$$y_{t+1} = \rho y_t + \varepsilon_{t+1} \qquad \varepsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma^2\right)$$

  which we discretise as a Markov chain using the Rouwenhorst (1995) or Tauchen (1986) methods.

- The next slides show solutions for the following parametrisation:

	Description	Value
β	Discount factor	0.96
σ	Coef. of relative risk aversion	2
r	Interest rate	0.04
ρ	Autocorrelation of AR(1) process	0.95
σ	Conditional std. dev. of AR(1) process	0.20
N_y	Number of states for Markov chain	3
N_a	Asset grid size	50, 100, 1000

Table 2: Parameters for HH problem with risky labour income

Solution for $N_a = 50$

VFI with interpolation

Solution for different income levels: low, middle, high

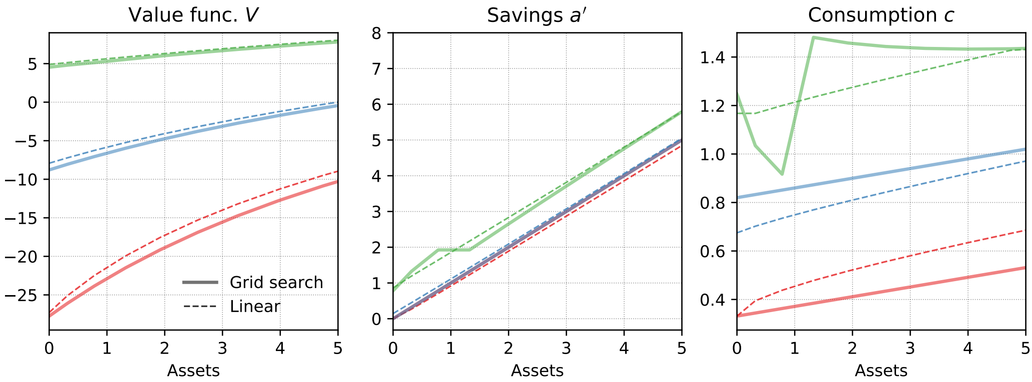

Figure 8: Solution with 50 asset grid points.

Solution for $N_a = 100$

VFI with interpolation

Solution for different income levels: low, middle, high

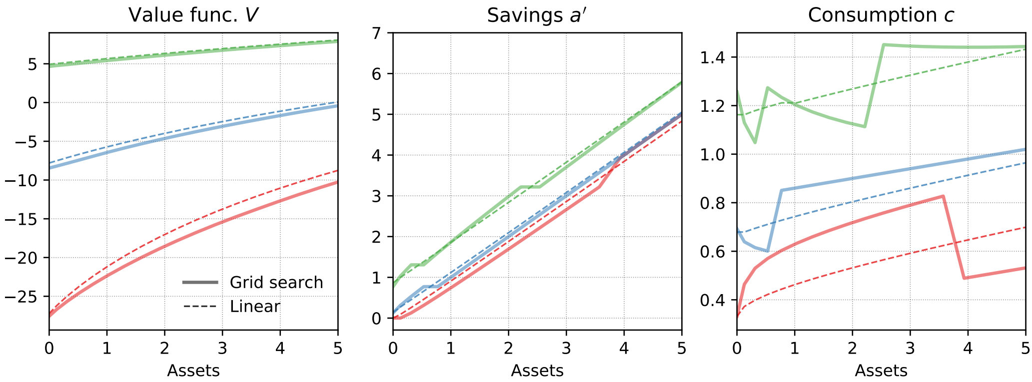

Figure 9: Solution with 100 assets grid points.

Solution for $N_a = 1000$

VFI with interpolation

Solution for different income levels: low, middle, high

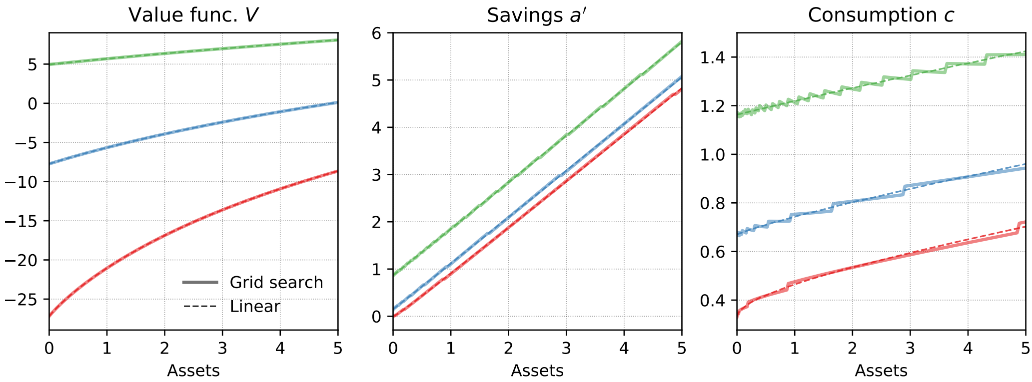

Figure 10: Solution with 1000 assets grid points.

Main take-aways

- Avoid grid search if you can!
- Test sensitivity of your solution to chosen grid size:
    - Check policy functions, value function almost always looks smooth!

Appendix:
Approximating AR(1) processes with Markov chains

AR(1) processes

Consider the following AR(1) process:

$$x_{t+1} = \mu + \rho x_t + \epsilon_{t+1} \qquad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_\epsilon^2\right)$$

This process has the following conditional and unconditional moments:

	Conditional	**Unconditional**
Mean	$\mathbf{E}\left[\, x_{t+1} \mid x_t \,\right] = \mu + \rho x_t$	$\mathbf{E}\left[\, x_t \,\right] = \frac{\mu}{1-\rho}$
Variance	$\operatorname{Var}\left(\, x_{t+1} \mid x_t \,\right) = \operatorname{Var}\left(\, \epsilon_{t+1} \,\right) = \sigma_\epsilon^2$	$\operatorname{Var}\left(\, x_t \,\right) = \frac{\sigma_\epsilon^2}{1-\rho^2}$
Autocorrelation	–	$\operatorname{Corr}\left(x_{t+1}, x_t\right) = \rho$

Unconditional moments:

- Reflect long-run behaviour of a single process
- With a large cross-section of individuals, they also represent the cross-sectional mean and variance of the stationary distribution

Approximating AR(1) processes

- Any Markov chain approximation of an AR(1) needs to provide:
  1. The discrete state space $\boldsymbol{x} = (x_1, x_2, \ldots, x_N)$
  2. The transition matrix $\boldsymbol{\Pi}$ where the element (i, j) is the probability $\Pr\left(x_{t+1} = x_j \mid x_t = x_i \right)$

  Using these, we can find the ergodic (invariant, stationary) distribution $\boldsymbol{\lambda}$ over states $\boldsymbol{x}$ which satisfies

$$\boldsymbol{\lambda}' = \boldsymbol{\lambda}'\boldsymbol{\Pi}$$

- Approximation should match conditional / unconditional moments reasonably well!
- Frequently-used methods:
  1. Tauchen (1986)
  2. Rouwenhorst (1995): much better for processes with high persistence

Example: Income process

Assume that log income follows an AR(1) process:

$$\log y_{t+1} = \rho \log y_t + \epsilon_{t+1} \qquad \epsilon_{t+1} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_\epsilon^2\right)$$

with $\mu = 0$ (omitted), $\rho = 0.95$, $\sigma_\epsilon^2 = (0.2)^2$

Discretized Markov chain (Rouwenhorst method)

- State space in logs, transition matrix and ergodic distribution:

$$\log \boldsymbol{y} = \begin{bmatrix} -0.9058 \\ 0 \\ 0.9058 \end{bmatrix} \qquad \boldsymbol{\Pi} = \begin{bmatrix} 0.9506 & 0.0488 & 0.0006 \\ 0.0244 & 0.9512 & 0.0244 \\ 0.0006 & 0.0488 & 0.9506 \end{bmatrix} \qquad \boldsymbol{\lambda} = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.25 \end{bmatrix}$$

- State space in levels:

$$\boldsymbol{y} = \begin{bmatrix} 0.4042 \\ 1.0000 \\ 2.4740 \end{bmatrix}$$

- Unconditional average income: $\mathrm{E}y_t = \boldsymbol{\lambda}'\boldsymbol{y} = 1.2195$

References I

Rouwenhorst, Geert K. (1995). "Asset Pricing Implications of Equilibrium Business Cycle Models". In: **Frontiers of Business Cycle Research**. Ed. by Thomas F. Cooley. Vol. 10. Princeton University Press. Chap. 10, pp. 294–330.

Tauchen, George (1986). "Finite state markov-chain approximations to univariate and vector autoregressions". In: **Economics Letters** 20.2, pp. 177–181.