Unit 11: Solving household problems in macroeconomics and finance

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Contents

11 Solving household problems in macroeconomics and finance

In this unit, we explore how to solve simple consumption-savings problems that are common in (heterogeneous-agent) macroeconomics and household finance.

For simplicity, we study infinite-horizon problems even though life-cycle models with finitely-lived agents are becoming more common in macroeconomics and are almost universal in household finance. The implementation is almost identical in both cases, with the major difference being that

1. Infinite-horizon problems are solved by iteration until convergence (of the value or policy functions) starting from some initial guess.
2. Life-cycle problems are solved by backward induction, starting from the terminal period T and iterating backwards through periods $T-1$, $T-2$, etc. until the first period.

Throughout this unit, we will exclusively solve partial equilibrium problems for given prices (interest rates and wages). Solving for general equilibrium would require us to find an additional fixed point in terms of equilibrium prices (in steady-state models such as Aiyagari (1994) or Huggett (1993)), or a equilibrium forecasting rule (for models with aggregate uncertainty such as Krusell-Smith (1998)).

For numerical purposes it is necessary to formulate the household problem in recursive form. You may be more familiar with the sequential formulation of an infinite horizon problem which could look something like the following,

$$
\begin{aligned}
V(a_0) &= \max_{(c_t, a_{t+1})_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & c_t + a_{t+1} = (1+r)a_t + y_t \quad \forall\, t \\
& c_t \geq 0, \; a_{t+1} \geq 0 \quad \forall\, t
\end{aligned}
$$

where the household chooses *sequences* of consumption $(c_t)_{t=0}^{\infty}$ and asset levels $(a_{t+1})_{t=0}^{\infty}$ for some initial assets a_0. The value function V is the maximum over such sequences. However, when we solve a problem numerically on the computer, it is usually more convenient to reformulate it recursively to obtain

$$\begin{aligned} V(a) = &\max_{c,a'} \left\{ u(c) + \beta V(a') \right\} \\ \text{s.t.} \quad & c + a' = (1+r)a + y \\ & c \geq 0, \; a' \geq 0 \end{aligned}$$

where the household picks optimal *scalars* c and a' for given a and parameters.

Note: With recursive formulations we often use primes to denote next-period values such as a' instead of writing $t+1$ since time t does not play any explicit role.

11.1 VFI with deterministic income

11.1.1 Household problem

For starters, consider the following *deterministic* infinite-horizon consumption-savings problem,

$$\begin{aligned} V(a) = &\max_{c,a'} \left\{ \frac{c^{1-\gamma} - 1}{1-\gamma} + \beta V(a') \right\} \\ \text{s.t.} \quad & c + a' = (1+r)a + y \\ & c \geq 0, \; a' \geq 0 \end{aligned}$$

where a are beginning-of-period assets, and r and y are the interest rate and labour earnings, respectively, which are both exogenous. The household has CRRA utility and chooses optimal consumption c and next-period assets a' which are required to be non-negative (i.e., we impose a borrowing constraint at 0).

Note that from a programming perspective we are interested in the case when $y > 0$ as otherwise this problem can be solved analytically (the household consumes a constant fraction of its assets due to CRRA preferences).

For $y > 0$, this problem has to be solved numerically which we do using *value function iteration (VFI)*. VFI takes an initial guess V_0 and solves the above problem repeatedly, generating a sequence of updated guesses $V_n, V_{n+1}, \dots$. We terminate the algorithm once consecutive V_n do not change anymore.

VFI itself does not dictate how the updated V is computed. In the sections below we explore two alternatives:

1. VFI combined with grid search; and
2. VFI combined with interpolation;

11.1.2 VFI with grid search

The simplest way to solve the above problem is to use the so-called *grid search* algorithm where we try all candidate savings levels from a discrete set of choices (the *grid*). Instead of picking an arbitrary $a' \in [0, (1+r)a + y]$, the household is thus constrained to pick an element $a' \in \Gamma_a$ from the pre-determined set of N_a feasible asset levels

$$\Gamma_a = \left\{ a_1, a_1, a_2, \dots, a_{N_a} \right\}$$

For convenience, we often use the grid Γ_a both as the set of points for which we solve the problem numerically as well as the household's choice set for a', thus forcing the household to pick a point on the grid on which the problem is defined. This is not strictly required (the household could in principle choose from an arbitrary finite set), but allows us to skip any interpolation of the continuation value function $V(a')$.

While grid search is generally considered obsolete in modern applications, it has a few distinct advantages:

1. Easy to implement;
2. Does not require computing derivatives (even more, it does not assume differentiability);
3. Fast (unless the grid is very dense or there are many choices);

However, we usually avoid it because of its disadvantages:

1. It's imprecise, and policy function are often not smooth (unless the grid is very dense);
2. Does not scale well to multiple choice dimensions;

You might still have to resort to grid search if your problem is not differentiable or has local maxima which breaks more sophisticated solution methods.

Outline of the algorithm

The grid search algorithm for this problem can be summarized as follows:

1. Create an asset grid $\Gamma_a = (a_1, \dots, a_{N_a})$.
2. Pick an initial guess V_0 for the value function defined on Γ_a.
3. In iteration n, perform the following steps:

   For each asset level a_i at grid point i,

   1. Find all feasible next-period asset levels $a_j \in \Gamma_a$ that satisfy the budget constraint,
   $$a_j \leq (1 + r)a_i + y$$
   2. For each j, compute consumption:
   $$c_j = (1 + r)a_i + y - a_j$$
   3. For each j, compute utility:
   $$U_j = u(c_j) + \beta V_n(a_j)$$
   4. Find the index k that maximises the above expression:
   $$k = \arg\max_j \Big\{ u(c_j) + \beta V_n(a_j) \Big\}$$
   5. Set $V_{n+1,i} = U_k$ and store k as the optimal choice at i.
4. Check for convergence: If $\|V_n - V_{n+1}\| < \epsilon$, for some small tolerance $\epsilon > 0$, exit the algorithm.

We implement this algorithm below. For this level of complexity, we would normally want to store the code as regular Python files (`*.py`) as opposed to writing down the problem in a Jupyter notebook. The complete implementation is therefore available in the files

- `lectures/unit11/main.py`: sets up the problem, calls the VFI solver, plots results; and
- `lectures/unit11/VFI.py`: implements VFI with grid search as well as with interpolation;

The sections below walk you through this implementation in notebook format.

Defining parameters and grids

For our implementation, we use the following parameters which are standard in macroeconomics:

Parameter	Description	Value
β	Discount factor	0.96
γ	Coef. of relative risk aversion	2
r	Interest rate	0.04
y	Labour income	1

In Python, it is convenient to store all these parameters in a single object so we don't have to pass numerous arguments to functions that perform the actual computations. To this end, we define a class which serves as a container object:

[1]:
```
from dataclasses import dataclass

@dataclass
class Parameters:
    """
    Define object to store model parameters and their default values
    """
    beta = 0.96          # Discount factor
    gamma = 1.0          # Risk aversion
    y = 1.0              # Labour income
    r = 0.04             # Interest rate
    grid_a = None        # Asset grid (to be created)
```

The `@dataclass` decorator is a convenient short-hand to define the attributes directly within the class body (we can safely ignore the technical details).

Note: Once our code becomes more complex, it is good practice to document the purpose of a `class` or function using the triple-quote doc strings `""" ... """`. These are ignored by the Python interpreter.

We use this class definition to create a `Parameters` instance named `par` below. Additionally, we create the asset grid `grid_a` which we add to this object:

[2]:
```
import numpy as np

par = Parameters()

# Start + end point for asset grid
a_min = 0.0
a_max = 10.0
# Number of grid points
N_a = 30

# Create asset grid with more points at the beginning
grid_a  = a_min + (a_max - a_min) * np.linspace(0.0, 1.0, N_a)**1.5

# Store asset grid in Parameters object
par.grid_a = grid_a
```

Note the seemingly unconventional way to create the asset grid where we first create a uniform grid on $[0, 1]$ and then compress the grid at lower asset levels using an exponential transformation.

Alternatively, we could have created the usual uniformly-spaced grid:

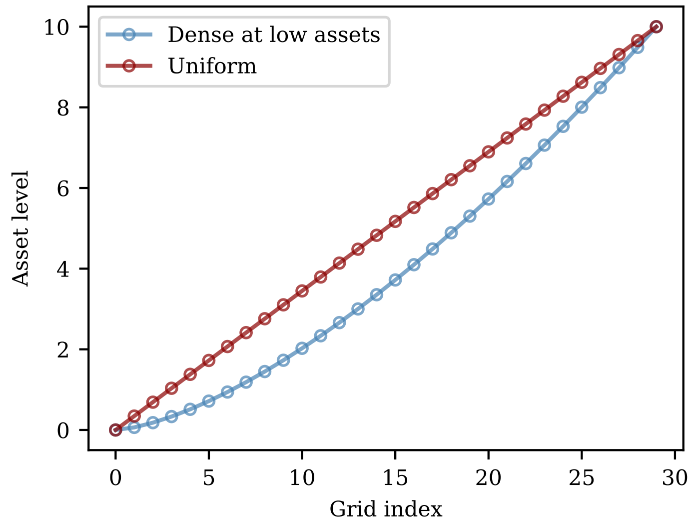
[3]:
```
grid_a_uniform = np.linspace(a_min, a_max, N_a)
```

However, we know from experience that the value and policy functions tend to be nonlinear for low assets, and therefore it is advantageous to put more grid points in that region.

You can see the difference by plotting the resulting grids:

[4]:
```python
import matplotlib.pyplot as plt

# Common arguments controlling plot style
style = dict(marker='o', mfc='none', alpha=0.7, markersize=4)

plt.plot(grid_a, c='steelblue', label='Dense at low assets', **style)
plt.plot(grid_a_uniform, c='darkred', label='Uniform', **style)
plt.xlabel('Grid index')
plt.ylabel('Asset level')
plt.legend(loc='upper left')
```

[4]: <matplotlib.legend.Legend at 0x7ff91ed2b220>



Implementing VFI with grid search

The following code implements the VFI algorithm with grid search. We define the function

```python
def vfi_grid(par, tol=1e-5, maxiter=1000):
    ...
```

which takes as arguments an instance of the `Parameters` class, the termination tolerance `tol` and the maximum number of iterations `maxiter`. The function creates a few arrays to hold the intermediate and final results and then iterates on the value function until convergence or until the maximum number of iterations is exceeded.

Since the optimal savings choice has to lie on the asset grid, we represent the savings policy function as an *integer* array which contains the *indices* of the optimal asset level instead of the asset level itself.

[5]:
```python
def vfi_grid(par, tol=1e-5, maxiter=1000):

    N_a = len(par.grid_a)
    vfun = np.zeros(N_a)
    vfun_upd = np.empty(N_a)
    # index of optimal savings decision (stored in integer array!)
    pfun_ia = np.empty(N_a, dtype=np.uint)

    # pre-compute cash at hand for each asset grid point
    cah = (1 + par.r) * par.grid_a + par.y

    for it in range(maxiter):

        for ia, a in enumerate(par.grid_a):
```

```
            # find all values of a' that are feasible, ie. they satisfy
            # the budget constraint
            ia_to = np.where(par.grid_a <= cah[ia])[0]

            # consumption implied by choice a'
            #   c = (1+r)a + y - a'
            cons = cah[ia] - par.grid_a[ia_to]

            # Evaluate "instantaneous" utility
            if par.gamma == 1.0:
                u = np.log(cons)
            else:
                u = (cons**(1.0 - par.gamma) - 1.0) / (1.0 - par.gamma)

            # 'candidate' value for each choice a'
            v_cand = u + par.beta * vfun[ia_to]

            # find the 'candidate' a' which maximises utility
            ia_to_max = np.argmax(v_cand)

            # store results for next iteration
            v_opt = v_cand[ia_to_max]
            vfun_upd[ia] = v_opt
            pfun_ia[ia] = ia_to_max

        diff = np.max(np.abs(vfun - vfun_upd))

        # switch references to value functions for next iteration
        vfun, vfun_upd = vfun_upd, vfun

        if diff < tol:
            msg = f'VFI: Converged after {it:3d} iterations: dV={diff:4.2e}'
            print(msg)
            break
        elif it == 1 or it % 50 == 0:
            msg = f'VFI: Iteration {it:3d}, dV={diff:4.2e}'
            print(msg)
    else:
        msg = f'Did not converge in {it:d} iterations'
        print(msg)

    return vfun, pfun_ia
```

Running the solver

We are now ready to run the VFI. Note that the second return value is an array of *indices* and hence we first need to recover the associated asset levels to get proper savings policy function. The consumption policy function can then be recovered from the budget constraint.

[6]:
```
vfun, pfun_ia = vfi_grid(par)

# Recover savings policy function from optimal asset indices
pfun_a = par.grid_a[pfun_ia]

# Recover consumption policy function from budget constraint
cah = (1.0 + par.r) * par.grid_a + par.y
pfun_c = cah - pfun_a
```

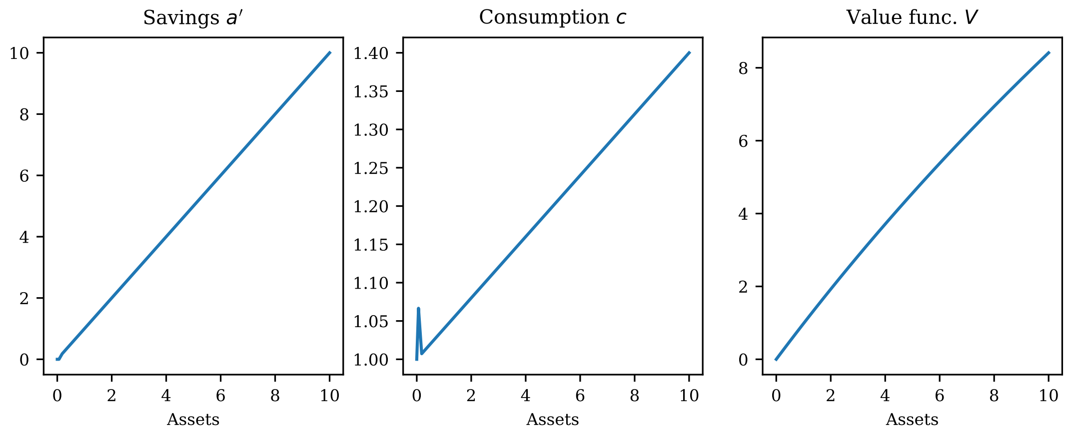
```
VFI: Iteration   0, dV=2.43e+00
VFI: Iteration   1, dV=1.17e+00
```

```
VFI: Iteration  50, dV=8.31e-03
VFI: Iteration 100, dV=1.08e-03
VFI: Iteration 150, dV=1.40e-04
VFI: Iteration 200, dV=1.82e-05
VFI: Converged after 215 iterations: dV=9.87e-06
```

Finally, it i always a good idea to visualise the results to get some economic intuition and spot errors.

[7]:
```
fig, axes = plt.subplots(1, 3, sharex=True, sharey=False, figsize=(9, 3.0))

# Plot savings in first column
axes[0].plot(par.grid_a, pfun_a)
axes[0].set_title(r'Savings $a^{\prime}$')
axes[0].set_xlabel('Assets')

# Plot consumption in second column
axes[1].plot(par.grid_a, pfun_c)
axes[1].set_title(r'Consumption $c$')
axes[1].set_xlabel('Assets')

# Plot value function in third column
axes[2].plot(par.grid_a, vfun)
axes[2].set_title('Value func. $V$')
axes[2].set_xlabel('Assets')
```

[7]:
```
Text(0.5, 0, 'Assets')
```



The above plot suggests that there is a jump in the consumption policy function for low assets. There is no economic reason why this should be the case, and in fact this is an undesirable artifact of grid search. Such artifacts may appear for some choices of the asset grid, as is the case here. This is one of the reason why we usually prefer other solution methods that do not exhibit this behaviour, such as VFI with interpolation, which we turn to next.

11.1.3 VFI with interpolation

In the previous section, we saw how grid search can give rise to undesired numerical artifacts that misrepresent the solution. Moreover, unless our grid is extremely dense, the solution is unlikely to satisfy the first-order optimality conditions because the optimal a' need not be on the candidate grid.

A more advanced solution method is to combine VFI with interpolation of the continuation value paired with a maximisation step, which we explore in this section. The advantages of this method are

1. The solution is "exact" in a numerical sense.
2. It is less affected by the curse of dimensionality with many (continuous) choice variables.
3. It is easier to spot mistakes because the resulting policy functions tend to be smooth.

On the other hand, this method

1. Is likely to be slower than grid search; and
2. Is more complex to implement because we need an additional numerical maximisation routine which might require computing (numerical) derivatives.

Outline of the algorithm

To implement VFI with interpolation, we need to modify the original algorithm as follows:

1. Create an asset grid $\Gamma_a = (a_1, \dots, a_{N_a})$.
2. Pick an initial guess V_0 for value function defined on Γ_a.
3. In iteration n, perform the following steps:

   For each asset level a_i at grid point i,

   1. Compute the available resources (cash at hand):
      $$x_i = (1+r)a_i + y$$
   2. Find the maximiser
      $$a^\star = \arg\max_{a' \in [0, x_i]} \Big\{ u\left(x_i - a'\right) + \beta V_n(a') \Big\}$$
      This step is usually performed using a numerical maximisation (or minimisation) routine.
   3. The value $V^\star$ is then given by
      $$V^\star = u\left(x_{ij} - a^\star\right) + \beta V_n(a^\star)$$
   4. Store the updated value $V_{n+1,i} = V^\star$ and the associated savings policy function.
4. Check for convergence: If $\|V_n - V_{n+1}\| < \epsilon$, for some small tolerance $\epsilon > 0$, exit the algorithm.

Note that we still solve the problem on a grid $(a_1, \dots, a_{N_a})$ but we no longer require households to make savings choices exactly on this grid.

Implementing VFI with interpolation

As before, the full implementation is provided in the Python script files `lectures/unit11/main.py` and `lectures/unit11/VFI.py`.

To perform the maximisation step, we need to define an objective function that can be passed to a minimiser such as SciPy's `minimize_scalar()`. For the problem at hand, we define this objective function as

```
def f_objective(sav, cah, par, f_vfun):
    ...
```

where `sav` is the candidate savings level a', `cah` is the cash-at-hand at the current point a_i, `par` is the `Parameters` instance and `f_vfun` is a callable function that can be used to evaluate the continuation value $V_n(a')$ using interpolation. The implementation of this function looks as follows:

[8]:
```
def f_objective(sav, cah, par, f_vfun):

    sav = float(sav)
    if sav < 0.0 or sav >= cah:
        return np.inf

    # Consumption implied by savings level
    cons = cah - sav
```

```python
    # Continuation value interpolated onto asset grid
    vcont = f_vfun(sav)

    # evaluate "instantaneous" utility
    if par.gamma == 1.0:
        u = np.log(cons)
    else:
        u = (cons**(1.0 - par.gamma) - 1.0) / (1.0 - par.gamma)

    # Objective evaluated at current savings level
    obj = u + par.beta * vcont

    # We are running a minimiser, return negative of objective value
    return -obj
```

Because we are going to call this objective function from a minimiser, we need to return the negative utility as shown in the last line.

Now that we have the objective function, VFI with interpolation is implemented by the following function:

[9]:
```python
from scipy.optimize import minimize_scalar

def vfi_interp(par, tol=1e-5, maxiter=1000):

    N_a = len(par.grid_a)
    vfun = np.zeros(N_a)
    vfun_upd = np.empty(N_a)
    # Optimal savings decision
    pfun_a = np.zeros(N_a)

    for it in range(maxiter):

        # Define function that interpolates continuation value
        f_vfun = lambda x: np.interp(x, par.grid_a, vfun)

        for ia, a in enumerate(par.grid_a):
            # Solve maximization problem at given asset level
            # Cash-at-hand at current asset level
            cah = (1.0 + par.r) * a + par.y
            # Restrict maximisation to following interval:
            bounds = (0.0, cah)
            # Arguments to be passed to objective function
            args = (cah, par, f_vfun)
            # perform maximisation
            res = minimize_scalar(f_objective, bracket=bounds, args=args)

            # Minimiser returns NEGATIVE utility, revert that
            vfun_upd[ia] = - res.fun
            # Store optimal savings a'
            pfun_a[ia] = res.x

        diff = np.max(np.abs(vfun - vfun_upd))

        # switch references to value functions for next iteration
        vfun, vfun_upd = vfun_upd, vfun

        if diff < tol:
            msg = f'VFI: Converged after {it:3d} iterations: dV={diff:4.2e}'
            print(msg)
            break
        elif it == 1 or it % 50 == 0:
            msg = f'VFI: Iteration {it:3d}, dV={diff:4.2e}'
```

```
            print(msg)
    else:
        msg = f'Did not converge in {it:d} iterations'
        print(msg)

    return vfun, pfun_a
```

Running the solver

We run the solver in the same way as we did with grid search, except that now the function returns an array of optimal savings *levels*, not the indices on the grid. We therefore no longer need to recover the actual savings policy function.

[10]:
```
vfun, pfun_a = vfi_interp(par)

# Recover consumption policy function from budget constraint
cah = (1.0 + par.r) * par.grid_a + par.y
pfun_c = cah - pfun_a
```

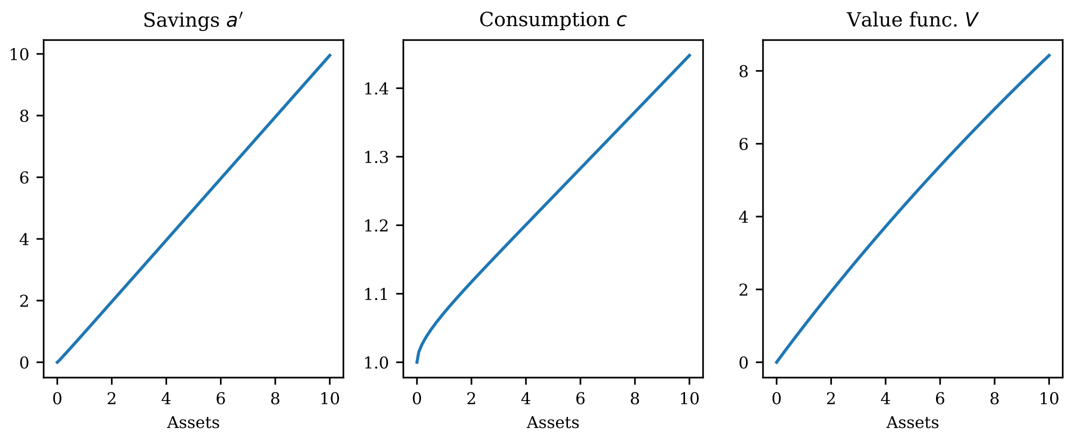
```
/home/richard/.conda/envs/python-intro-PGR/lib/python3.10/site-
packages/scipy/optimize/_optimize.py:2417: RuntimeWarning: invalid value
encountered in scalar multiply
  tmp2 = (x - v) * (fx - fw)
```

```
VFI: Iteration   0, dV=2.43e+00
VFI: Iteration   1, dV=1.17e+00
VFI: Iteration  50, dV=6.39e-03
VFI: Iteration 100, dV=4.35e-04
VFI: Iteration 150, dV=2.96e-05
VFI: Converged after 170 iterations: dV=9.57e-06
```

Visualising the solution, we see that the artifacts in the consumption policy function are no longer present.

[11]:
```
fig, axes = plt.subplots(1, 3, sharex=True, sharey=False, figsize=(9, 3.0))

# Plot savings in first column
axes[0].plot(par.grid_a, pfun_a)
axes[0].set_title(r'Savings $a^{\prime}$')
axes[0].set_xlabel('Assets')

# Plot consumption in second column
axes[1].plot(par.grid_a, pfun_c)
axes[1].set_title(r'Consumption $c$')
axes[1].set_xlabel('Assets')

# Plot value function in third column
axes[2].plot(par.grid_a, vfun)
axes[2].set_title('Value func. $V$')
axes[2].set_xlabel('Assets')
```

[11]:
```
Text(0.5, 0, 'Assets')
```



11.2 VFI with stochastic labour income

11.2.1 Modelling stochastic labour income

So far, we assumed that labour income y was deterministic and constant. In more realistic heterogeneous-agent models in macroeconomics and household finance, we instead model labour income as stochastic and thus risky. One frequent assumption is that labour income follows an AR(1) process in logs, i.e.,

$$\begin{aligned}\log y_{t+1} &= \rho \log y_t + \nu_{t+1} \\ \nu_{t+1} &\stackrel{\text{iid}}{\sim} N(0, \sigma^2)\end{aligned}$$

The corresponding household problem is then given by

$$\begin{aligned}V(y,a) = \max_{c,\,a'} &\left\{u(c) + \beta \mathbb{E}\left[V(y',a') \mid y\right]\right\} \\ \text{s.t.} \quad & c + a' = (1+r)a + y \\ & c \geq 0, \; a' \geq 0\end{aligned}$$

where we added the additional state variable y and need to take expectations over future realisations of y'.

There are two commonly-used approaches to incorporate such an income process into our household problem:

1. Assume that y is a continuous state variable and use Gauss-Hermite quadrature to compute expectations.
2. Discretize y to take on a few selected values and model it as a Markov chain on the discretised state space.

We will following the second approach in this unit. Two common algorithms to "convert" an AR(1) into a Markov chain are the Tauchen and the Rouwenhorst methods. The latter seems to be preferable when dealing with highly persistent processes such as labour income. The discretised labour income process with N_y grid points is then given by the states $\Gamma_y = (y_1, \dots, y_{N_y})$ and the transition matrix Π_y with typical element

$$\Pi_y(i,j) = \Pr\left[y_{t+1} = y_j \mid y_t = y_i\right] .$$

We use the function `rouwenhorst()` defined below to perform this mapping for us. Its arguments are the number of nodes of the discretized state space `n`, the mean of the AR(1) process `mu`, the autocorrelation parameter `rho` and the conditional standard deviation `sigma` (see the file `lectures/unit11/markov.py` for a complete implementation which includes error checking of input arguments). There is no need to understand the details of this function as we are only interested in using its return values: the state space stored in the vector `z` and the transition matrix `Pi`.

[12]:
```python
import numpy as np

def rouwenhorst(n, mu, rho, sigma):

    p = (1+rho)/2
    Pi = np.array([[p, 1-p], [1-p, p]])

    for i in range(Pi.shape[0], n):
        tmp = np.pad(Pi, 1, mode='constant', constant_values=0)
        Pi = p * tmp[1:, 1:] + (1-p) * tmp[1:, :-1] + \
             (1-p) * tmp[:-1, 1:] + p * tmp[:-1, :-1]
        Pi[1:-1, :] /= 2

    fi = np.sqrt(n-1) * sigma / np.sqrt(1 - rho ** 2)
    z = np.linspace(-fi, fi, n) + mu

    return z, Pi
```

Model parameters

Since the household problem with stochastic labour income has a few more parameters, we redefine the `Parameters` class from earlier to include these. The additional parameters are listed in the table below. The values for the AR(1) labour income process are standard and taken from papers such as Aiyagari (1994).

Parameter	Description	Value
β	Discount factor	0.96
γ	Coef. of relative risk aversion	2
r	Interest rate	0.04
ρ	Autocorrelation of labour income	0.95
σ	Conditional std. dev. of labour income	0.20
N_y	Number of states for Markov chain	3

The updated definition of `Parameters` now looks as follows:

[13]:
```python
@dataclass
class Parameters:
    """
    Define object to store model parameters and their default values
    """
    beta = 0.96          # Discount factor
    gamma = 1.0          # Risk aversion
    r = 0.04             # Interest rate
    rho = 0.95           # Autocorrelation of log labour income
    sigma = 0.20         # Conditional standard deviation of log labour income
    grid_a = None        # Asset grid (to be created)
    grid_y = None        # Discretised labour income grid (to be created)
    tm_y = None          # Labour transition matrix (to be created)
```

We can use the above function to create a discretised representation of the risky labour income. For illustrative purposes we choose to discretise y to only three points which would be considered too few for realistic models.

[14]:
```python
par = Parameters()

# Number of labour grid points
N_y = 3

# Discretise labour income process (assume zero mean in logs)
```

```python
states, tm_y = rouwenhorst(N_y, mu=0.0, rho=par.rho, sigma=par.sigma)

# State space in levels
grid_y = np.exp(states)

# Store labour grid and transition matrix
par.grid_y = grid_y
par.tm_y = tm_y
```

Note that because stochastic income is log-normal and has zero mean in logs, due to the properties of the log-normal distribution the average income in *levels* in the cross-section of households is given by

$$\mathbb{E}[y] = \exp\left(\frac{1}{2}\sigma_v^2\right) > 1$$

This is often undesirable as we'd like to normalise average income in the economy to unity since this makes interpreting magnitudes easier. We can achieve this by computing the ergodic (stationary) distribution of labour using the following function (see also `lectures/unit11/markov.py` for a complete implementation). Again, there is no need to understand what this function does, we are only interested in the stationary distribution it returns in the vector `mu`.

[15]:
```python
import numpy.linalg

def markov_ergodic_dist(transm):
    transm = transm.transpose()
    m = transm - np.identity(transm.shape[0])
    m[-1] = 1
    m = np.linalg.inv(m)
    mu = np.ascontiguousarray(m[:, -1])
    assert np.abs(np.sum(mu) - 1) < 1e-9
    mu /= np.sum(mu)
    return mu
```

We use this function to compute the ergodic distribution `edist` implied by the Markov chain transition matrix. We can then use this distribution to compute expectation $\mathbb{E}[y]$ and we normalise the state space by this value.

[16]:
```python
# Ergodic distribution of labour income
edist = markov_ergodic_dist(tm_y)

# Non-normalised mean of labour income
mean = np.dot(edist, par.grid_y)

# Normalise states such that unconditional expectation is 1.0
par.grid_y /= mean
```

Finally, we complete the setup of the problem by re-creating the asset grid in the same way we did above.

[17]:
```python
# Start + end point for asset grid
a_min = 0.0
a_max = 10.0
# Number of grid points
N_a = 30

# Create asset grid with more points at the beginning
grid_a  = a_min + (a_max - a_min) * np.linspace(0.0, 1.0, N_a)**1.5

# Store asset grid in Parameters object
par.grid_a = grid_a
```

11.2.2 VFI with grid search

Before turning to the actual implementation, it is instructive to look at the modified grid search algorithm for the case of risky labour income. The main difference is that we now have an additional state variable y and need to take care of expectations.

Outline of the algorithm

1. Create an asset grid $\Gamma_a = (a_1, \dots, a_{N_a})$.
2. Create a discretised labour income process with states $\Gamma_y = (y_1, \dots, y_{N_y})$ and transition matrix Π_y.
3. Pick an initial guess V_0 for the value function defined on $\Gamma_y \times \Gamma_a$.
4. In iteration n, perform the following steps:

   For each labour income y_i at index i,

   1. Compute the expectation over labour income realisations y' given y_i,

      $$\widetilde{V}(a') = \mathbb{E}\big[V_n(y', a') \mid y_i\big]$$

      using the i-th row of the transition matrix Π_y.
   2. For each asset level a_j at grid point j,
      1. Find all feasible next-period asset levels $a_k \in \Gamma_a$ that satisfy the budget constraint

         $$a_k \leq (1+r)a_j + y_i$$

      2. For each k, compute consumption

         $$c_k = (1+r)a_j + y_i - a_k$$

      3. For each k, compute utility

         $$U_k = u(c_k) + \beta\widetilde{V}(a_k)$$

      4. Find the index ℓ that maximises the above expression:

         $$\ell = \arg\max_k \Big\{u(c_k) + \beta\widetilde{V}(a_k)\Big\}$$

      5. Set $V_{n+1,ij} = U_\ell$ and store ℓ as the optimal choice at (i,j)
5. Check for convergence: If $\|V_n - V_{n+1}\| < \epsilon$, for some small tolerance $\epsilon > 0$, exit the algorithm.

Implementing VFI with grid search

The following code implements VFI with risky labour income using grid search (the code can also be found in `lectures/unit11/VFI_risk.py`).

[18]:
```python
def vfi_grid(par, tol=1e-5, maxiter=1000):

    N_a, N_y = len(par.grid_a), len(par.grid_y)
    shape = (N_y, N_a)
    vfun = np.zeros(shape)
    vfun_upd = np.empty(shape)
    # index of optimal savings decision
    pfun_ia = np.empty(shape, dtype=np.uint)

    # pre-compute cash at hand for each (asset, labour) grid point
    cah = (1 + par.r) * par.grid_a[None] + par.grid_y[:,None]

    for it in range(maxiter):
```

```
        # Compute expected continuation value E[V(y',a')|y] for each (y,a')
        EV = np.dot(par.tm_y, vfun)

        for iy in range(N_y):
            for ia, a in enumerate(par.grid_a):

                # find all values of a' that are feasible, ie. they satisfy
                # the budget constraint
                ia_to = np.where(par.grid_a <= cah[iy, ia])[0]

                # consumption implied by choice a'
                #   c = (1+r)a + y - a'
                cons = cah[iy, ia] - par.grid_a[ia_to]

                # Evaluate "instantaneous" utility
                if par.gamma == 1.0:
                    u = np.log(cons)
                else:
                    u = (cons**(1.0 - par.gamma) - 1.0) / (1.0 - par.gamma)

                # 'candidate' value for each choice a'
                v_cand = u + par.beta * EV[iy, ia_to]

                # find the 'candidate' a' which maximizes utility
                ia_to_max = np.argmax(v_cand)

                # store results for next iteration
                v_opt = v_cand[ia_to_max]
                vfun_upd[iy, ia] = v_opt
                pfun_ia[iy, ia] = ia_to_max

        diff = np.max(np.abs(vfun - vfun_upd))

        # switch references to value functions for next iteration
        vfun, vfun_upd = vfun_upd, vfun

        if diff < tol:
            msg = f'VFI: Converged after {it:3d} iterations: dV={diff:4.2e}'
            print(msg)
            break
        elif it == 1 or it % 50 == 0:
            msg = f'VFI: Iteration {it:3d}, dV={diff:4.2e}'
            print(msg)
    else:
        msg = f'Did not converge in {it:d} iterations'
        print(msg)

    return vfun, pfun_ia
```

We can now run and plot the solution as we did in the case of deterministic income.

[19]:
```
vfun, pfun_ia = vfi_grid(par)

# Recover savings policy function from optimal asset indices
pfun_a = par.grid_a[pfun_ia]

# Recover consumption policy function from budget constraint
cah = (1.0 + par.r) * par.grid_a + par.grid_y[:, None]
pfun_c = cah - pfun_a
```

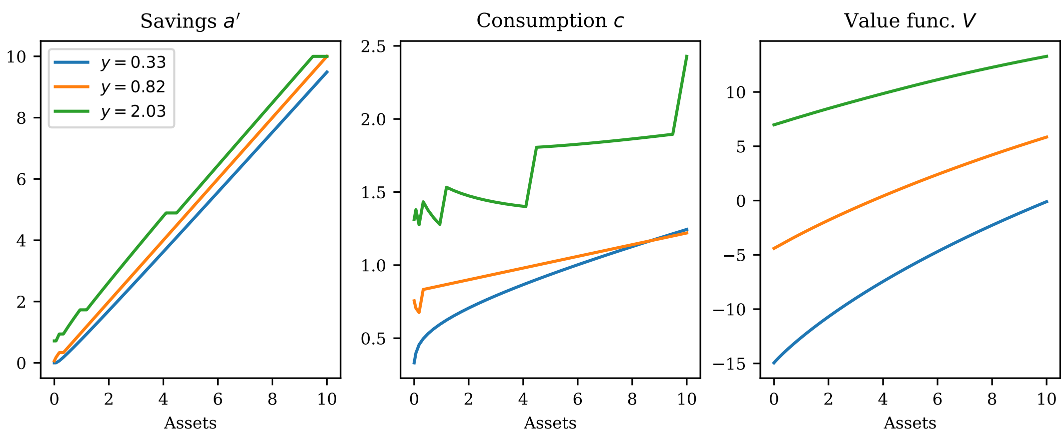
```
VFI: Iteration   0, dV=2.52e+00
VFI: Iteration   1, dV=1.38e+00
```

```
VFI: Iteration  50, dV=3.02e-02
VFI: Iteration 100, dV=2.07e-03
VFI: Iteration 150, dV=2.29e-04
VFI: Iteration 200, dV=2.87e-05
VFI: Converged after 226 iterations: dV=9.88e-06
```

Finally, we plot the solution for all levels of labour income. Note that we need to transpose the result arrays because Matplotlib's `plot()` expects input arrays to have the same number of elements along the first dimension.

[20]:
```
fig, axes = plt.subplots(1, 3, sharex=True, sharey=False, figsize=(9, 3.0))

# Plot savings in first column
axes[0].plot(par.grid_a, pfun_a.T)
axes[0].set_title(r'Savings $a^{\prime}$')
axes[0].set_xlabel('Assets')
# Insert legend
labels = [f'$y={y:.2f}$' for y in par.grid_y]
axes[0].legend(labels, loc='upper left')

# Plot consumption in second column
axes[1].plot(par.grid_a, pfun_c.T)
axes[1].set_title(r'Consumption $c$')
axes[1].set_xlabel('Assets')

# Plot value function in third column
axes[2].plot(par.grid_a, vfun.T)
axes[2].set_title('Value func. $V$')
axes[2].set_xlabel('Assets')
```

[20]:
```
Text(0.5, 0, 'Assets')
```



With stochastic labour income, the numerical artifacts produced by grid search are even more pronounced which speaks against using this method for this type of problem, if possible.

11.2.3 VFI with interpolation

We can adjust the interpolation algorithm to deal with stochastic labour income in the same way as we augmented the grid search.

Outline of the algorithm

To implement VFI with interpolation, we need to modify the original algorithm as follows:

1. Create an asset grid $\Gamma_a = (a_1, \dots, a_{N_a})$.
2. Create a discretised labour income process with states $\Gamma_y = (y_1, \dots, y_{N_y})$ and transition matrix Π_y.
3. Pick an initial guess V_0 for value function defined on $\Gamma_y \times \Gamma_a$.
4. In iteration n, perform the following steps:

   For each labour income y_i at index i,

   1. Compute the expectation over labour income realisations y' given y_i,
   $$\widetilde{V}(a') = \mathbb{E}\left[V_n(y', a') \mid y_i\right]$$
   using the i-th row of the transition matrix Π_y.
   2. For each asset level a_j at grid point j,
      1. Compute the available resources (cash at hand):
      $$x_{ij} = (1+r)a_i + y_i$$
      2. Find the maximiser
      $$a^\star = \underset{a' \in [0, x_{ij}]}{\arg\max} \left\{ u\left(x_{ij} - a'\right) + \beta \widetilde{V}(a') \right\}$$
      This step is usually performed using a numerical maximisation (or minimisation) routine.
      3. The value $V^\star$ is then given by
      $$V^\star = u\left(x_{ij} - a^\star\right) + \beta \widetilde{V}(a^\star)$$
      4. Store the updated value $V_{n+1,ij} = V^\star$ and the associated savings policy function.
5. Check for convergence: If $\|V_n - V_{n+1}\| < \epsilon$, for some small tolerance $\epsilon > 0$, exit the algorithm.

Implementing VFI with interpolation

The following code implements VFI with risky labour income using interpolation (the code can also be found in `lectures/unit11/VFI_risk.py`).

Note that the objective function `f_objective()` remains unchanged from before because we adapted the argument `f_vfun` to interpolate over the pre-computed expected value of $\widetilde{V}(a')$ instead of $V_n(a')$ as in the original implementation.

[21]:
```
def vfi_interp(par, tol=1e-5, maxiter=1000):

    N_a, N_y = len(par.grid_a), len(par.grid_y)
    shape = (N_y, N_a)
    vfun = np.zeros(shape)
    vfun_upd = np.empty(shape)
    # Optimal savings decision
    pfun_a = np.zeros(shape)

    for it in range(maxiter):

        # Compute expected continuation value E[V(y',a')|y] for each (y,a')
        EV = np.dot(par.tm_y, vfun)

        for iy, y in enumerate(par.grid_y):

            # function to interpolate continuation value
            f_vfun = lambda x: np.interp(x, par.grid_a, EV[iy])

            for ia, a in enumerate(par.grid_a):
                # Solve maximization problem at given asset level
```

```
                # Cash-at-hand at current asset level
                cah = (1.0 + par.r) * a + y
                # Restrict maximisation to following interval:
                bounds = (0.0, cah)
                # Arguments to be passed to objective function
                args = (cah, par, f_vfun)
                # perform maximisation
                res = minimize_scalar(f_objective, bracket=bounds, args=args)

                # Minimiser returns NEGATIVE utility, revert that
                v_opt = - res.fun
                sav_opt = float(res.x)

                vfun_upd[iy, ia] = v_opt
                pfun_a[iy, ia] = sav_opt

        diff = np.max(np.abs(vfun - vfun_upd))

        # switch references to value functions for next iteration
        vfun, vfun_upd = vfun_upd, vfun

        if diff < tol:
            msg = f'VFI: Converged after {it:3d} iterations: dV={diff:4.2e}'
            print(msg)
            break
        elif it == 1 or it % 50 == 0:
            msg = f'VFI: Iteration {it:3d}, dV={diff:4.2e}'
            print(msg)
    else:
        msg = f'Did not converge in {it:d} iterations'
        print(msg)

    return vfun, pfun_a
```

The following code is used to perform the VFI. Note that once as in the deterministic setting, the savings policy function returns savings *levels*, even though now all return values are 2-dimensional arrays.

[22]:
```
vfun, pfun_a = vfi_interp(par)

# Recover consumption policy function from budget constraint
cah = (1.0 + par.r) * par.grid_a + par.grid_y[:, None]
pfun_c = cah - pfun_a
```

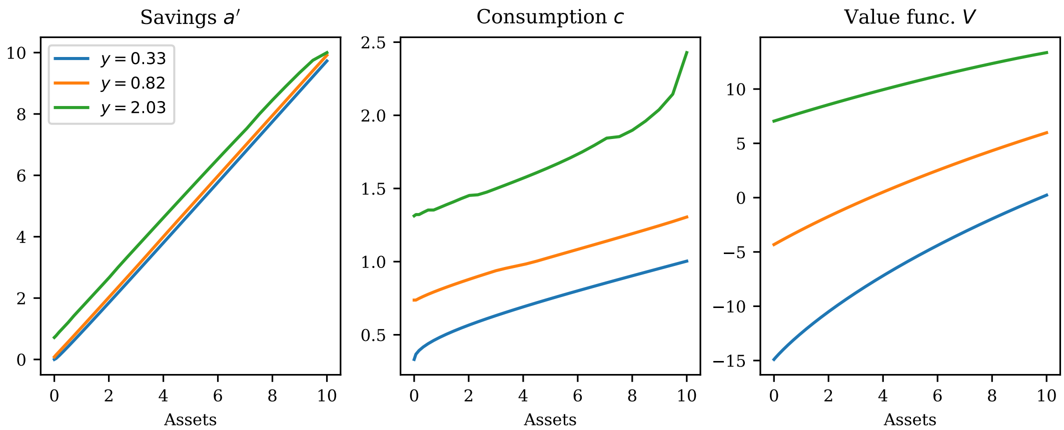
```
/home/richard/.conda/envs/python-intro-PGR/lib/python3.10/site-
packages/scipy/optimize/_optimize.py:2417: RuntimeWarning: invalid value
encountered in scalar multiply
  tmp2 = (x - v) * (fx - fw)

VFI: Iteration   0, dV=2.52e+00
VFI: Iteration   1, dV=1.38e+00
VFI: Iteration  50, dV=2.95e-02
VFI: Iteration 100, dV=1.95e-03
VFI: Iteration 150, dV=2.16e-04
VFI: Iteration 200, dV=2.73e-05
VFI: Converged after 225 iterations: dV=9.80e-06
```

We use the same code as above to visualise the results:

[23]:
```
fig, axes = plt.subplots(1, 3, sharex=True, sharey=False, figsize=(9, 3.0))

# Plot savings in first column
axes[0].plot(par.grid_a, pfun_a.T)
axes[0].set_title(r'Savings $a^{\prime}$')
```

```
axes[0].set_xlabel('Assets')
# Insert legend
labels = [f'$y={y:.2f}$' for y in par.grid_y]
axes[0].legend(labels, loc='upper left')

# Plot consumption in second column
axes[1].plot(par.grid_a, pfun_c.T)
axes[1].set_title(r'Consumption $c$')
axes[1].set_xlabel('Assets')

# Plot value function in third column
axes[2].plot(par.grid_a, vfun.T)
axes[2].set_title('Value func. $V$')
axes[2].set_xlabel('Assets')
```

[23]: Text(0.5, 0, 'Assets')



The policy functions are much smoother compared to those produced by grid search. However, as you can see there is another artifact of our numerical implementation that did not stand out as much earlier: the consumption policy for the highest labour state is suspiciously upward-sloping towards the end! This is a consequence of not permitting extrapolation of the continuation value, and hence any savings that would take the household beyond the grid upper bound are wasted. It is therefore optimal to consume all such "excess" savings, which gives rise to the increase in consumption at the right end of the asset grid. We could address this problem by allowing extrapolation, but linear extrapolation of value functions is undesirable for other reasons.

11.3 EGM with stochastic labour income

While value function iteration is a robust and long-established method to solve dynamic problems, it tends to be slow when combined with interpolation, while it produces numerical artifacts when combined with grid search. The endogenous grid-point method (EGM) due to Carroll (2005) addresses both the issue of speed and at the same time produces smooth and accurate solutions.

To fix ideas, let's revisit the stochastic labour income we studied above:

$$
\begin{aligned}
V(y,a) = \max_{c,a'} & \left\{ u(c) + \beta \mathbb{E}\left[V(y',a') \mid y \right] \right\} \\
\text{s.t.} \quad & c + a' = (1+r)a + y \\
& c \geq 0, \ a' \geq 0
\end{aligned}
$$

The optimality conditions for this problem are the standard Euler equation combined with the budget constraint:

$$
\begin{aligned}
c^{-\gamma} &= \beta(1+r)\mathbb{E}\left[(c')^{-\gamma} \middle| y \right] \\
c + a' &= (1+r)a + y
\end{aligned}
$$

So far, we haven't used the Euler equation at all, but it can be leveraged to invert the problem and speed up the solution. Suppose that instead of trying to determine the optimal savings level a' for an exogenously given beginning-of-period asset level a, we ask the following: if we exogenously impose that the household optimally saves a', what is the beginning-of-period asset level a that rationalizes this savings choice given the household's optimality conditions?

To answer this question, assume we have a guess for the consumption policy function $C(y,a)$; for example, we could initially impose that the household chooses to consume everything, i.e., $C(y,a) = a$. For a given a', from the Euler equation we know that

$$(c^*)^{-\gamma} = \beta(1+r)\mathbb{E}\left[C(y',a')^{-\gamma} \mid y\right]$$

where we use the notation c^* to indicate that this consumption level is a function of the exogenously imposed a'. Consequently, optimal consumption today must be given by

$$c^* = \left(\beta(1+r)\mathbb{E}\left[C(y',a')^{-\gamma} \mid y\right]\right)^{-\frac{1}{\gamma}}$$

With c^* and a' in hand, the only unknown in the budget constraint are the beginning-of-period assets a^* which we again denote using $*$ to stress that it is a function of the exogenous a':

$$a^* = \frac{c^* + a' - y}{1+r}$$

Note that the beginning-of-period assets a^* thus arise *endogenously* as a function of the exogenously-imposed optimal savings a', hence the name *endogenous* grid-point method.

To summarize, every point on the exogenous savings $\Gamma_{a'} = \{a'_1, a'_2, \dots, a'_{N_{a'}}\}$ gives rise to an endogenous tuple of values (c^*, a^*). After applying this procedure to all $N_{a'}$ exogenous savings levels, we have the collection of values $(c_i^*)_{i=1}^{N_{a'}}$ and $(a_i^*)_{i=1}^{N_{a'}}$. We are usually not interested in characterising the solution to the household problem as a function of the endogenous grid of $\{a_1^*, a_2^*, \dots\}$, so the final step is to use these points to interpolate the consumption policy function back onto our standard beginning-of-period asset grid $\Gamma_a = \{a_1, a_2, \dots, a_{N_a}\}$. This yields an updated consumption policy function $C(y,a)$ mapping a into c which we can use in the next iteration.

Outline of the algorithm

To solve the household problem with stochastic labour income with EGM, we proceed as follows:

1. Create an asset grid $\Gamma_a = (a_1, \dots, a_{N_a})$ and optionally and exogenous savings grid $\Gamma_{a'} = (a'_1, \dots, a'_{N_{a'}})$. For simplicity we use the same grid for both purposes.
2. Create a discretised labour income process with states $\Gamma_y = (y_1, \dots, y_{N_y})$ and transition matrix Π_y.
3. Pick an initial guess $C_0(y,a)$ for the consumption policy function defined on $\Gamma_y \times \Gamma_a$. One possibility is to assume that $C_0(y,a) = a$.
4. In iteration n, perform the following steps:

   For each labour income y_i at index i,

   1. Compute the vector of *expected marginal utilities* for all points a'_j on the exogenous savings grid with typical element
   $$m_{ij} \equiv \mathbb{E}\left[C_n\left(y', a'_j\right)^{-\gamma} \;\middle|\; y_i\right]$$
   2. Invert the Euler equation to get a vector of today's consumption levels,
   $$c_{ij}^* = \left[\beta(1+r)m_{ij}\right]^{-\frac{1}{\gamma}}$$

3. Use the budget constraint to find the vector of required beginning-of-period asset levels

$$a_{ij}^* = \frac{c_{ij}^* + a_j' - y_i}{1+r}$$

4. Use the collection of values $(a_{ij})_{j=1}^{N_{a'}}$ and $(c_{ij})_{j=1}^{N_{a'}}$ to interpolate optimal consumption onto the beginning-of-period asset grid Γ_a. This yields an updated guess for $C_{n+1}(y_i, a)$.
5. Finally, note that the Euler equation only holds for *interior* solutions but does not hold when the household is borrowing constraint. The endogenous grid point a_{i1}^* associated with the first savings grid-point $a_1' = 0$ exactly identifies the beginning-of-period asset level at which the household is no longer borrowing constraint. For all asset levels below this point, we know that the household will choose not to save, so we set

$$C_{n+1}(y_i, a) = (1+r)a + y_i \quad \forall a < a_{i1}^*$$

5. Check for convergence: If $\|C_n - C_{n+1}\| < \epsilon$, for some small tolerance $\epsilon > 0$, exit the algorithm.

Implementing EGM with stochastic labour income

The `lectures/unit11` folder contains two EGM implementations,

1. `lectures/unit11/EGM.py`: implements infinite-horizon EGM for the problem with deterministic labour income;
2. `lectures/unit11/EGM_risk.py`: implements infinite-horizon EGM for the problem with stochastic labour income.

Setting up the problem is identical to the case of VFI, see `lectures/unit11/main.py` for the problem with deterministic labour and `lectures/unit11/main_risk.py` for the stochastic version. For completeness, the implementation is shown below. Note that here we use SciPy's interpolation routine `interp1d()` since it supports extrapolation, unlike its NumPy counterpart `np.interp()`.

[24]:
```
from scipy.interpolate import interp1d

def egm(par, tol=1.0e-8, maxiter=10000):

    N_a, N_y = len(par.grid_a), len(par.grid_y)
    shape = (N_y, N_a)

    # Cash-at-hand at every asset/savings level
    cah = (1.0 + par.r) * par.grid_a[None] + par.grid_y[:, None]

    # Initial guess for consumption policy function
    pfun_c = np.copy(cah)
    pfun_c_upd = np.zeros(shape)

    # Extract parameters from par object
    beta, gamma, r = par.beta, par.gamma, par.r

    for it in range(maxiter):

        # Iterate over all labour income states
        for iy, y in enumerate(par.grid_y):

            # Expected marginal utility tomorrow
            mu = np.dot(par.tm_y[iy], pfun_c**(-gamma))
            # Compute right-hand side of Euler equation (EE)
            ee_rhs = beta * (1.0 + r) * mu

            # Invert EE to get consumption as a function of savings today
            cons_sav = ee_rhs**(-1.0/gamma)
```

```python
            # Use budget constraint to get beginning-of-period assets
            assets_sav = (cons_sav + par.grid_a - y) / (1.0 + r)

            # Interpolate back onto exogenous savings grid
            f_cons = interp1d(
                assets_sav, cons_sav,
                copy=False, assume_sorted=True,
                bounds_error=False, fill_value='extrapolate'
            )
            pfun_c_upd[iy] = f_cons(par.grid_a)

            # Fix consumption in region where HH does not save
            amin = assets_sav[0]
            idx = np.where(par.grid_a <= amin)[0]
            # HH consumes entire cash-at-hand
            pfun_c_upd[iy, idx] = cah[iy, idx]

        # Make sure that consumption policy satisfies constraints
        assert np.all(pfun_c_upd >= 0.0) and np.all(pfun_c_upd <= cah)

        # Compute max. absolute difference to policy function from previous
        # iteration.
        diff = np.max(np.abs(pfun_c - pfun_c_upd))

        # switch references to policy functions for next iteration
        pfun_c, pfun_c_upd = pfun_c_upd, pfun_c

        if diff < tol:
            # Convergence achieved, exit loop
            msg = f'EGM: Converged after {it:d} iterations: d(c)={diff:4.2e}'
            print(msg)
            break
        elif it == 1 or it % 50 == 0:
            msg = f'EGM: Iteration {it:4d}, dV={diff:4.2e}'
            print(msg)
    else:
        msg = f'Did not converge in {it:d} iterations'
        print(msg)

    pfun_a = cah - pfun_c

    return pfun_a, pfun_c
```

The following code runs and plots the results. It assumes that the parameters have been initialised as shown in the previous sections. Note that the EGM method does not return the value function since it is not needed for the algorithm. Once the policy functions have been computed, it is however possible to construction the value function from these by iteration (without having to perform maximisation since we already know the optimal choices).

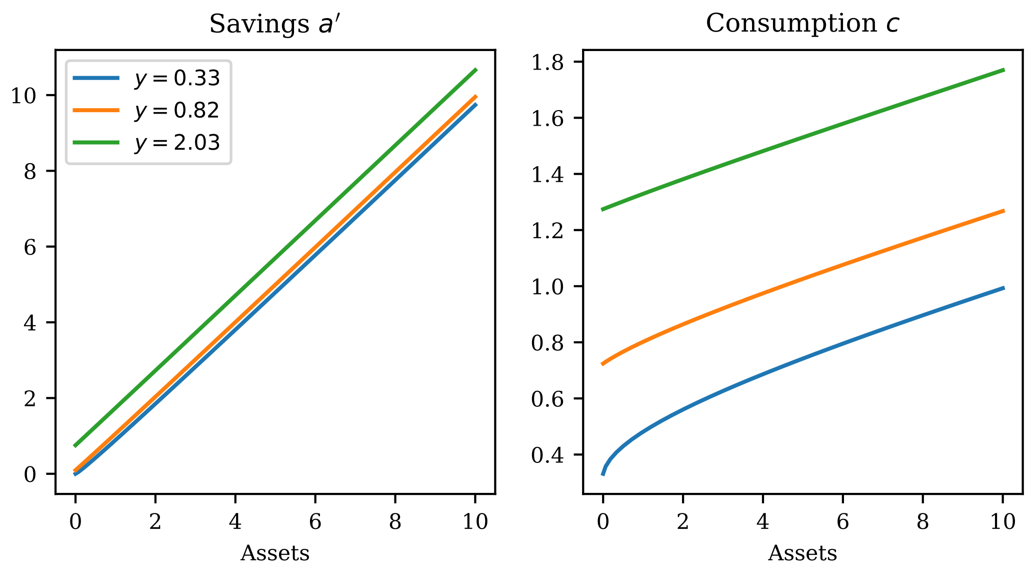
[25]:
```python
# Run EGM, store savings and consumption policy functions.
pfun_a, pfun_c = egm(par)
```

```
EGM: Iteration    0, dV=5.13e+00
EGM: Iteration    1, dV=1.73e+00
EGM: Iteration   50, dV=4.54e-03
EGM: Iteration  100, dV=3.29e-04
EGM: Iteration  150, dV=2.47e-05
EGM: Iteration  200, dV=1.88e-06
EGM: Iteration  250, dV=1.44e-07
EGM: Iteration  300, dV=1.10e-08
EGM: Converged after 302 iterations: d(c)=9.95e-09
```

[26]:

```python
# Visualise the EGM solution
fig, axes = plt.subplots(1, 2, sharex=True, sharey=False, figsize=(6.5, 3.0))

# Plot savings in first column
axes[0].plot(par.grid_a, pfun_a.T)
axes[0].set_title(r'Savings $a^{\prime}$')
axes[0].set_xlabel('Assets')
# Insert legend
labels = [f'$y={y:.2f}$' for y in par.grid_y]
axes[0].legend(labels, loc='upper left')

# Plot consumption in second column
axes[1].plot(par.grid_a, pfun_c.T)
axes[1].set_title(r'Consumption $c$')
axes[1].set_xlabel('Assets')
```

[26]: Text(0.5, 0, 'Assets')



We close this section with a comment on the relative run time of the methods we have studied above to solve the problem with stochastic labour income. The following table shows the relative run times of each solution method, with the run time for VFI with grid search normalised to one. The problem is solved on 50 asset grid points and risky labour income is discretised to three grid points.

Method	Relative run time
VFI + grid search	1.00
VFI + linear interp.	22.01
EGM	0.21

As you can see, EGM is not only more accurate but also 5 times faster than VFI with grid search, and 100 times faster than VFI with linear interpolation which on top yields worse results.