# Functional Implementation of the CNF Algorithm. Technical Report

Fco. M. García Olmedo* Jesús García Miranda†
Pedro González Rodelas‡

**Abstract**

We define three functions on propositional formulas expressed by conjunction and disjunction which will be understood as the input of the CNF algorithm in its syntactic version. These functions calculate the exact number of iterations necessary for the distributivity and associativity passes by means of which we will obtain from this input formula another one —the output of the algorithmic procedure— logically equivalent but already in conjunctive normal form. The effectiveness of that exact numbers is mathematically proven and carried over to an improved functional implementation of the CNF algorithm in the Haskell language.

## 1 Introduction

The Conjunctive Normal Form (CNF) is famous in the history of thought because it organizes the discourse by normalizing potentially chaotic statements through the conjunction of a series of statements in the form of disjunctions of other simple statements, namely atomic or negation of them.

CNF has proven to be essential in the treatment of the SAT problem, which in turn is at the core of automated theorem proving, thanks to the effectiveness of the resolution rule and what is known about the treatment of Horn clauses.

The other great utility of CNF lies in the minimization of Boolean expressions under the condition of being expressed as a product of sums (POS). It is clear that the POS criterion is the dual concept of the SOP criterion, in fact it is their dual concept.

In general, transforming a formula into CNF is the essence of the Petrick's Method ([5]), an algorithm widely used in various fields: cybernetics, economics sciences, linguistics, philosophy, psychology, etc.

Given a propositional formula (resp. Boolean expression), obtaining an equivalent formula in Conjunctive Normal Form (CNF) can be carried out by constructing and inspecting its so called "truth table" (resp. the exhaustive description of the associated Boolean function). In the case of propositional formulas, this is a tedious task with a significant amount of wasted effort. In the case of commutation theory, the use of a table for obtaining a CNF or DNF (Disjunctive Normal Form) prior the optimisation is only

*Dpto. de Álgebra, UGR; folmedougr.es
†Dpto. de Álgebra, UGR; jesusgm@ugr.es
‡Dpto. de Matemática Aplicada, UGR; prodelas@ugr.es

prescribed when reverse engineering is carried out on a found circuit. In other cases, a syntactic analysis of the Boolean expression is prescribed.

If we focus on the method of syntactic analysis for obtaining CNF, we will consider propositional logic formulas without loss of generality in the appropiate language. In this line of thought, the essence of the algorithm is quite simple: after internalizing negation and eliminating double negation, the main task is to replace subformulas of the form[1] $A\alpha K\beta\gamma$ with their equivalent $KA\alpha\beta A\alpha\gamma$ (distributivity). This work is organized in iterations of recursive commands (cfr. [2, pag. 31]), and the iterative phase is essential.

However, how do we stop the process of consecutive recursive commands, each of which brings the formula closer to its CNF expression? We have not found an answer in the available literature, not even a reference. We consider [2] to be a good and almost exhaustive compilation of techniques and algorithms on propositional logic. In it, we discovered that the iterations should be stopped when the result of an iteration matches the formula that was initially given. Nevertheless, if we had an exact computation based on the (unique) syntax of the formula for the number of necessary iterations, we could save the time spent comparing a formula with the one obtained in the previous iterative step. Actually, what we would need is to define a "distance" from the given formula to the set of formulas in CNF, reducing it by one unit for each recursive substitution command.

In this work, we will perform the exact computation of that number of iterations based on an initial inspection with low computational complexity of the input formula. However, there is more; the formula $A\alpha A\beta\gamma$ (resp. $K\alpha K\beta\gamma$) is equivalent to $AA\alpha\beta\gamma$ (resp. $K\alpha K\beta\gamma$) (associativity). In general, loading on the left side the $K$ connective and then loading on the left side the $A$ connective will provide a highly interesting specialization for our CNF in terms of computation; in fact this is in some ways comparable to something like the prenex form in first-order logic. Once again, we face the need for a new measure to compute the iterations for this new purpose. This work carries out that innovative exact computation, which is computationally inexpensive in all cases.

Finally, as a result of the work, we provide a functional implementation of the algorithm with these effective innovations in the Haskell programming language.

In summary, the sections of this article contain the following. Since the aim of this paper is the manipulation of formulas over a language, Section 2 sets out the rigorous definition of language and formula. Four subsets of formulas generated by a non-empty subset of formulas according to appropriate rules are then defined; essentially they are the support for defining the concept of clause and formula in conjunctive normal forms. The section continues by giving four different concepts of complexity of a formula. The rest of the section is devoted to defining the concept of semantic equivalence of formulas according to classical propositional logic, some examples, and the statement of a classical result on distributivity. Negation is not mentioned because it is not relevant to the theoretical framework of the article. Section 3 is devoted to the treatment of the distributivity that, in a broad sense of the term, classical propositional logic contains between disjunction and conjunction. In the section, the function dak is defined (cfr. Figure 1), which when applied to formulas manages to reduce the alternation in them of the connector $K$ over $A$ in a unit; this alternation is measured in each formula by the function alt. The section concludes by showing that the alternation of each formula is precisely the minimum number of applications of dak to it in order to obtain from it another equivalent formula in conjunctive normal form. Practical laboratory experience in the field of logical deduction indicates that the associativity of the connectives $K$ and $A$ must be taken into account in order to obtain a canonical form which, in Polish notation, accumulates these connectives at the beginning of the formula; the rigorous treatment of this matter is the aim of Section 4. In writing it we have been inspired by the structure of Section 3, however the intrinsic theory is appreciably more complex. It is all based on two "measures" on formulae that essentially express how far the formula is from that canonical form; the maximum of these two values is

[1]For the sake of efficiency, we will use Polish notation here (cfr. [8].

also important. In the section we justify the immense expressive capacity of the aforementioned measures to characterise the membership of the subsets of formulae defined in Section 2. The counterpart for the associativity of the dak function is defined, namely the lasc function (cfr. Figure 2). This time the application of the lasc function decreases the separation measure of the formula to the canonical form by half, in that sense it serves to make big steps. A certain natural value based on the logarithm in base two gives the minimum number of applications of lasc to the formula to obtain its desired canonical form. The last section, Section 5, is for conclusions.

## 2 Basic Definitions and Preliminary Results

In this section we list the definitions of the basic concepts that will be used in the development of this work, as well as the essential results needed in it.

**Definition 2.1.** A *language* is a pair $\langle S, g\rangle$, where $S$ is a non-empty set of *symbols* and $g$ is a map from $S$ that takes natural values (i.e. $g\colon S \longrightarrow \omega$). For all $n \in \omega$, let $S(n) = g^*(n)$. A symbol belonging to $S$ is of ariety $n$ iff, by def., it belongs to $S(n)$.

**Definition 2.2.** The *propositional language of the conjunctive normal form*, abbreviated c.n.f., is the language $\mathbf{L_{KA}} = \langle S, g\rangle$ where:

- $S(0)$ is a numerable set (non-finite in most applications and here) which we will represent by $X$. Its elements are notated by the first lower–case letters of the latin alphabet, subindicating them if necessary: $x$, $y$, $z$, $x_0$, $x_1$, $x_2$, etc.
- $S(2) = \{\mathrm{A}, \mathrm{K}\}$.
- For any natural number $n$ other than 0 and 2, $S(n) = \varnothing$.

**Definition 2.3.** Given a language $\mathbf{S} = \langle S, g\rangle$ let us define for all $k \in \omega$:

$$\begin{aligned}\Phi_0 &= S(0)\\ \Phi_{k+1} &= \Phi_k \cup \{fW_0 \cdots W_{n-1}\colon f \in S \setminus S(0),\ g(f) = n \text{ and } W_0, \ldots, W_{n-1} \in \Phi_k\}\end{aligned}$$

and finally:

$$\mathrm{P}(\mathbf{S}) = \bigcup_{k\in\omega} \Phi_k$$

$P(\mathbf{S})$ is the set of (propositional) *formulas* of $\mathbf{S}$.

In order to abbreviate we could occasionally simply write $\mathrm{P}(\mathbf{L})$ instead of $\mathrm{P}(\mathbf{L_{KA}})$.

**Definition 2.4.** Let $\Delta$ be a non-empty set of formulas of the language $\mathbf{L_{KA}}$. Let us define the following sets:

- $\mathrm{A}(\Delta) = \bigcap\{\Gamma\colon \Delta \subseteq \Gamma \subseteq \mathrm{P}(\mathbf{L}) \text{ and } \Gamma \text{ is closed under } \mathrm{A}\}$
- $\mathrm{K}(\Delta) = \bigcap\{\Gamma\colon \Delta \subseteq \Gamma \subseteq \mathrm{P}(\mathbf{L}) \text{ and } \Gamma \text{ is closed under } \mathrm{K}\}$
- $\Xi_{\mathrm{A}}(\Delta) = \bigcap\{\Gamma\colon \Delta \subseteq \Gamma \subseteq \mathrm{P}(\mathbf{L}) \text{ and } \mathrm{A}\,\alpha\beta \in \Gamma \text{ whenever } \alpha \in \Gamma \text{ and } \beta \in \Delta\}$
- $\Xi_{\mathrm{K}}(\Delta) = \bigcap\{\Gamma\colon \Delta \subseteq \Gamma \subseteq \mathrm{P}(\mathbf{L}) \text{ and } \mathrm{K}\,\alpha\beta \in \Gamma \text{ whenever } \alpha \in \Gamma \text{ and } \beta \in \Delta\}$

**Definition 2.5.** Let $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$. The formula $\alpha$ is a *conjunctive normal form* (resp. *clause*) iff, by definition, $\alpha \in \mathrm{K}(\mathrm{A}(X))$ (resp. $\alpha \in \Xi_{\mathrm{A}}(X)$). Moreover, $\alpha$ is a *left conjunctive normal form* iff, by definition $\alpha \in \Xi_{\mathrm{K}}(\Xi_{\mathrm{A}}(X))$. In the following $\mathrm{cl}(X)$ (resp. $\mathrm{ml}(X)$, $lcnf(X)$) will stand for $\Xi_{\mathrm{A}}(X)$ (resp. $\Xi_{\mathrm{K}}(X)$, $\Xi_{\mathrm{K}}(\Xi_{\mathrm{A}}(X))$).

*Remark* 2.1. The following inclusions are evident:

1. $\mathrm{cl}(X) \subseteq \mathrm{lcnf}(X)$

2. $\mathrm{ml}(X) \subseteq \mathrm{lcnf}(X)$
3. $\mathrm{lcnf}(X) \subseteq \mathrm{K}(\mathrm{A}(X))$

The "complexity" and "length" of formulas will serve as a measure of their volume. Both concepts are expressed as follows.

**Definition 2.6.** For all $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$ let $\mathrm{comp}(\alpha)$ (*complexity*), $\mathrm{comp}_{\mathrm{k}}(\alpha)$, $\mathrm{comp}_{\mathrm{a}}(\alpha)$ y $\mathrm{lg}(\alpha)$ (*length*) the natural values defined as follows:

$$\mathrm{comp}(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ 1 + \mathrm{comp}(\varphi) + \mathrm{comp}(\psi), & \text{if } \alpha \equiv A\varphi\psi \text{ or } \alpha \equiv K\varphi\psi. \end{cases}$$

$$\mathrm{comp}_{\mathrm{k}}(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ \mathrm{comp}_{\mathrm{k}}(\varphi) + \mathrm{comp}_{\mathrm{k}}(\psi), & \text{if } \alpha \equiv A\varphi\psi, \\ 1 + \mathrm{comp}_{\mathrm{k}}(\varphi) + \mathrm{comp}_{\mathrm{k}}(\psi), & \text{if } \alpha \equiv K\varphi\psi. \end{cases}$$

$$\mathrm{comp}_{\mathrm{a}}(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ \mathrm{comp}_{\mathrm{a}}(\varphi) + \mathrm{comp}_{\mathrm{a}}(\psi), & \text{if } \alpha \equiv K\varphi\psi, \\ 1 + \mathrm{comp}_{\mathrm{a}}(\varphi) + \mathrm{comp}_{\mathrm{a}}(\psi), & \text{if } \alpha \equiv A\varphi\psi. \end{cases}$$

$$\mathrm{lg}(\alpha) = \begin{cases} 0, & \text{if } \alpha \in X, \\ 1 + \max\{\mathrm{lg}(\varphi), \mathrm{lg}(\psi)\}, & \text{if } \alpha \equiv A\varphi\psi \text{ ó } \alpha \equiv K\varphi\psi. \end{cases}$$

**Definition 2.7.** Consider the propositional language $\mathbf{L_{KA}}$ and the details of Definition 2.2. A *valuation* is any map $v\colon X \longrightarrow \{0,1\}$. By the *unique readability principle* any valuation can be uniquely extended to $P(L_{KA})$ verifying (we assume that such an extension is represented by the same letter $v$):

- $v(A\alpha\beta) = v(\alpha)v(\beta) + v(\alpha) + v(\beta)$
- $v(K\alpha\beta) = v(\alpha)v(\beta)$

where addition and multiplication between the elements 0 and 1 which we have considered in the above enumeration are those of $\mathbb{Z}_2$, i.e. they are those given in the following tables:

| $+$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

**Definition 2.8.** Given a set $\Gamma$ of formulas —possibly empty— and a formula $\varphi$, the set $\Gamma$ *semantically implies* $\varphi$, in symbols $\Gamma \models \varphi$, iff by def. for every valuation $v$ occurs $v(\varphi) = 1$ provided that for every formula $\gamma$ in $\Gamma$ holds $v(\gamma) = 1$. If $\Gamma$ consists only of the formulas $\gamma_1, \ldots, \gamma_n$, instead of $\{\gamma_1, \ldots, \gamma_n\} \models \varphi$ we will write $\gamma_1, \ldots, \gamma_n \models \varphi$ and when $\Gamma = \emptyset$ we will simply write $\models \varphi$ instead of $\emptyset \models \varphi$.

**Definition 2.9.** The formula $\alpha$ and $\beta$ are *equivalent*, in symbols $\alpha = \beta$, iff by definition $\alpha \models \beta$ and $\beta \models \alpha$.

*Remark* 2.2. According to Definition 2.9 in the following we will use the symbol $=$ to indicate logical equivalence and the symbol $\equiv$ to indicate syntactic equality.

**Example 2.1.** *Let $\alpha$, $\alpha'$, $\beta$, $\beta'$ and $\gamma$ are formulas. If $\alpha$ is equivalent to $\alpha'$ and $\beta$ is equivalent to $\beta'$, then each subsequent item lists equivalent formulas:*

*1.* $\mathrm{A}\,\alpha\beta$, $\mathrm{A}\,\alpha'\beta'$
*2.* $\mathrm{K}\,\alpha\beta$, $\mathrm{K}\,\alpha'\beta'$

3. $\mathrm{A}\,\alpha\,\mathrm{K}\,\beta\gamma$, $\mathrm{K}\,\mathrm{A}\,\alpha\beta\,\mathrm{A}\,\alpha\gamma$
4. $\mathrm{K}\,\alpha\,\mathrm{A}\,\beta\gamma$, $\mathrm{A}\,\mathrm{K}\,\alpha\beta\,\mathrm{K}\,\alpha\gamma$

**Theorem 2.2.** *Let $\varphi$, $\psi$, $\xi$, $\alpha$, and $\beta$ propositional formulas. If $\alpha$ is (logically) equivalent to $\mathrm{A}\,\varphi\psi$ and $\beta$ is equivalent to $\mathrm{A}\,\varphi\xi$, then $\mathrm{A}\,\varphi\,\mathrm{K}\,\psi\xi$ and $\mathrm{K}\,\alpha\beta$ are equivalent.*

**Theorem 2.3.** *Let $\varphi$, $\psi$, $\xi$, and $\alpha$ propositional formulae. If $\alpha$ is (logically) equivalent to $A\varphi\psi$ (resp. $K\varphi\psi$), then $A\varphi A\psi\xi$ (resp. $K\varphi K\psi\xi$) is equivalent to $A\alpha\xi$ (resp. $K\alpha\xi$).*

# 3 Distributivity

The aim now is, given a formula $\varphi$ belonging to $\mathrm{P}(\mathbf{L_{KA}})$, to find $\varphi_{cnf}$ belonging to $K(\mathrm{A}(X))$ such that both are equivalent. This will be the basis of the algorithm and the first thing to do is to determine how far $\varphi$ is from $\mathrm{K}(\mathrm{A}(X))$. As we shall see shortly, the expected measure is given by the function *alt*, which indicates the alternating in its formula argument of the symbols K and A from the inner to the outside the formula.

**Definition 3.1.** Let $\alpha$ be any formula belonging to $\mathrm{P}(\mathbf{L_{KA}})$. The *alternation* of $\alpha$, in symbols $\mathrm{alt}(\alpha)$, is by definition:

$$\mathrm{alt}(\alpha) = \begin{cases} 0, & \text{if } \mathrm{K} \notin \alpha; \\ \max\{alt(\varphi), alt(\psi)\}, & \text{if } \alpha \equiv \mathrm{K}\,\varphi\psi; \\ 1 + \max\{alt(\varphi), alt(\psi)\}, & \text{if } \alpha \equiv \mathrm{A}\,\varphi\psi \text{ and } \mathrm{K} \in \alpha. \end{cases}$$

In Lemma 3.1 we characterize the meaning of "belong to the set $\mathrm{K}(\mathrm{A}(X))$" by means of the map alt. As we shall see, the formulae for which $\mathrm{alt}(\alpha) = 0$ are exactly those of the set $\mathrm{K}(\mathrm{A}(X))$.

**Lemma 3.1.** *Let $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$. The following statements are equivalent:*

1. $\mathrm{alt}(\alpha) = 0$.
2. $\alpha \in \mathrm{K}(\mathrm{A}(X))$.

*Proof.* Let $\alpha$ be a fórmula such that $\mathrm{alt}(\alpha) = 0$. If $\mathrm{K} \notin \alpha$, then $\alpha \in \mathrm{A}(X)$ and so $\alpha \in \mathrm{K}(\mathrm{A}(X))$. If $\mathrm{K} \in \alpha$, then $\alpha \equiv \mathrm{K}\,\phi\psi$, where $\mathrm{alt}(\phi) = 0 = \mathrm{alt}(\psi)$. By induction hypothesis, $\phi, \psi \in \mathrm{K}(\mathrm{A}(X))$ and so $\alpha \in \mathrm{K}(\mathrm{A}(X))$. The reciprocal statement has immediate proof from Definition 3.1. $\square$

**Definition 3.2 (distributivity).** Consider the following compound rules on binary relationships between formulas of $P(\mathbf{L_{KA}})$:

$$x_i \rightarrow_{dak} x_i \tag{1}$$

$$\frac{\mathrm{A}\,\varphi\xi \rightarrow_{dak} \alpha \quad \mathrm{A}\,\psi\xi \rightarrow_{dak} \beta}{\mathrm{A}\,\mathrm{K}\,\varphi\psi\xi \rightarrow_{dak} \mathrm{K}\,\alpha\beta} \tag{2}$$

$$\frac{\mathrm{A}\,\xi\varphi \rightarrow_{dak} \alpha \quad \mathrm{A}\,\xi\psi \rightarrow_{dak} \beta}{\mathrm{A}\,\xi\,\mathrm{K}\,\varphi\psi \rightarrow_{dak} \mathrm{K}\,\alpha\beta} \tag{3}$$

$$\frac{\varphi \rightarrow_{dak} \varphi' \quad \psi \rightarrow_{dak} \psi'}{\mathrm{A}\,\varphi\psi \rightarrow_{dak} \mathrm{A}\,\varphi'\psi'} \tag{4}$$

$$\frac{\varphi \rightarrow_{dak} \varphi' \quad \psi \rightarrow_{dak} \psi'}{\mathrm{K}\,\varphi\psi \rightarrow_{dak} \mathrm{K}\,\varphi'\psi'} \tag{5}$$

where (2), (3), and (4) are applied with the precedence indicated by the order in which they are given. This being so, we define the following application:

$$\mathrm{dak}\colon\ \mathrm{P}(\mathbf{L_{KA}}) \longrightarrow \mathrm{P}(\mathbf{L_{KA}})$$

by

$$\mathrm{dak}(\varphi) \equiv \psi, \text{ provided that } \varphi \rightarrow_{dak} \psi$$

*Remark* 3.1. dak really is an application because a precedence has been established in the rules on which its definition is based.

$$x_i \rightarrow_{dak} x_i$$

$$\frac{A\varphi\xi \rightarrow_{dak} \alpha \quad A\psi\xi \rightarrow_{dak} \beta}{AK\varphi\psi\xi \rightarrow_{dak} K\alpha\beta} \qquad \frac{A\xi\varphi \rightarrow_{dak} \alpha \quad A\xi\psi \rightarrow_{dak} \beta}{A\xi K\varphi\psi \rightarrow_{dak} K\alpha\beta}$$

$$\frac{\varphi \rightarrow_{dak} \varphi' \quad \psi \rightarrow_{dak} \psi'}{A\varphi\psi \rightarrow_{dak} A\varphi'\psi'} \qquad \frac{\varphi \rightarrow_{dak} \varphi' \quad \psi \rightarrow_{dak} \psi'}{K\varphi\psi \rightarrow_{dak} K\varphi'\psi'}$$

Figure 1: Axiom and rules for the definition of dak.

Moreover, the transformation by dak of a formula argument turns out to be a formula equivalent to that argument, the process does not alter its logical meaning in any way. The proof of Theorem 3.2 consists of a demonstration by routine induction based on Theorem 2.2 and Example 2.1.

**Theorem 3.2.** *For all* $\zeta \in \mathrm{P}(\mathbf{L_{KA}})$*,* $\mathrm{dak}(\zeta) = \zeta$*, that is,* $\mathrm{dak}(\zeta)$ *and* $\zeta$ *are equivalents.*

**Lemma 3.3.** *Let* $\alpha$ *be a formula in* $\mathrm{P}(\mathbf{L_{KA}})$*. If* $\alpha \in \mathrm{A}(X)$ *then* $\mathrm{dak}(\alpha) \equiv \alpha$

*Proof.* Suppose that $\alpha \in \mathrm{A}(X)$ and that $\mathrm{comp}(\alpha) = n$. Reasoning by induction on $\mathrm{comp}(\alpha)$ we will show that $\mathrm{dak}(\alpha) \equiv \alpha$. Suppose that the implication is true for any formula $\beta \in \mathrm{A}(X)$ such that $\mathrm{comp}(\beta) < n$. If $\alpha \in \mathrm{A}(X)$, two situations are possible:

1. $\alpha \in X$; if $x \in X$ and $\alpha \equiv x$, then

$$\begin{aligned} \mathrm{dak}(\alpha) &\equiv \mathrm{dak}(x) \\ &\equiv x && \text{by Rule 3} \\ &\equiv \alpha \end{aligned}$$

2. There exist formulas $\varphi$ y $\psi$ in $\mathrm{A}(X)$ of complexities less than those of $\alpha$ such that $\alpha \equiv \mathrm{A}\,\varphi\psi$. Since $\mathrm{K} \notin \alpha$, only Rule 4 applies for the calculation of $\mathrm{dak}(\alpha)$. We have:

$$\begin{aligned} \mathrm{dak}(\alpha) &\equiv \mathrm{dak}(A\varphi\psi) \\ &\equiv A\,\mathrm{dak}(\varphi)\,\mathrm{dak}(\psi) && \text{by Rule 4} \\ &\equiv A\varphi\psi && \text{by induction hypothesis} \\ &\equiv \alpha \end{aligned}$$

□

*Remark* 3.2. By Lemma 3.3, $\alpha \in \mathrm{A}(X)$ is a sufficient condition for $\mathrm{dak}(\alpha) \equiv \alpha$, but it is not a necessary condition. Lemma 3.4, which is a consequence of Lemma 3.3, gives the necessary and sufficient condition, although this will be fully proved in Corollary 3.6.

**Lemma 3.4.** *Let* $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$*. If* $\alpha \in \mathrm{K}(\mathrm{A}(X))$ *then* $\mathrm{dak}(\alpha) \equiv \alpha$*.*

*Proof.* Let us assume that $\alpha \in \mathrm{K}(\mathrm{A}(X))$ and $\mathrm{comp}(\alpha) = n$. Reasoning by induction on $\mathrm{comp}(\alpha)$ we will show that $\mathrm{dak}(\alpha) \equiv \alpha$. As an induction hypothesis, suppose that the implication is true for any formula $\beta \in \mathrm{K}(\mathrm{A}(X))$ such that $\mathrm{comp}(\beta) < n$. If $\alpha \in \mathrm{K}(\mathrm{A}(X))$, then two situations are possible:

1. $\alpha \in \mathrm{A}(X)$; in this case $\mathrm{dak}(\alpha) \equiv \alpha$ is what Lemma 3.3 states.
2. there exist formulas $\varphi$ and $\psi$ in $\mathrm{K}(\mathrm{A}(X))$ of complexities lower than those of $\alpha$ such that $\alpha \equiv K\varphi\psi$. For the calculation of $\mathrm{dak}(\alpha)$ it is only possible to start with Rule 5:

$$
\begin{aligned}
\mathrm{dak}(\alpha) &\equiv \mathrm{dak}(K\varphi\psi) & \\
&\equiv K\,\mathrm{dak}(\varphi)\,\mathrm{dak}(\psi) && \text{by Rule 5} \\
&\equiv K\varphi\psi && \text{induc. hyp.} \\
&\equiv \alpha
\end{aligned}
$$

□

As for Theorem 3.5, in essence its meaning is that in applying dak to a given formula not in $\mathrm{K}(\mathrm{A}(X))$, say $\alpha$, according to the "measure" alt the result is closer to $\mathrm{K}(\mathrm{A}(X))$ than $\alpha$ .

**Theorem 3.5.** *For all $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$,*

$$
\mathrm{alt}(\mathrm{dak}(\alpha)) = \begin{cases} 0, & \text{if } \alpha \in \mathrm{K}(\mathrm{A}(X)); \\ \mathrm{alt}(\alpha) - 1, & \text{otherwise.} \end{cases} \tag{6}
$$

*Proof.* The proof is by induction on the complexity of the formula $\alpha$. Let $\alpha$ be a formula $\mathrm{P}(\mathbf{L_{KA}})$ such that $\mathrm{comp}(\alpha) = n$. Suppose, as an induction hypothesis, that (6) holds for any formula $\beta$ such that $\mathrm{comp}(\beta) < n$. Several cases are possible:

1. $\alpha \equiv x \in X$; in this case $\mathrm{dak}(\alpha) \equiv x \in X$ and since $X \subseteq \mathrm{K}(\mathrm{A}(X))$ we deduce, according to Lemma 3.1, that $\mathrm{alt}(\alpha) = 0$ which proves the result in this case.
2. $\alpha \equiv AK\varphi\psi\xi$; therefore $\alpha \notin \mathrm{K}(\mathrm{A}(X))$ and

$$
\begin{aligned}
\mathrm{alt}(\alpha) &= 1 + \max\{\mathrm{alt}(K\varphi\psi), \mathrm{alt}(\xi)\} \\
&= 1 + \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\psi), \mathrm{alt}(\xi)\}
\end{aligned} \tag{7}
$$

On the other hand, $\mathrm{dak}(\alpha) \equiv K\,\mathrm{dak}(A\varphi\xi)\,\mathrm{dak}(A\psi\xi)$ so that:

$$
\mathrm{alt}(\mathrm{dak}(\alpha)) = \max\{\mathrm{alt}(\mathrm{dak}(A\varphi\xi)), \mathrm{alt}(\mathrm{dak}(A\psi\xi))\} \tag{8}
$$

For short, we will call $\beta$ to $A\varphi\xi$ and $\gamma$ to $A\psi\xi$. Let us bear in mind the following:

(a) $K \in \beta$; then $\mathrm{alt}(\beta) = 1 + \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\xi)\}$ and $\beta \notin \mathrm{K}(\mathrm{A}(X))$. Since $\mathrm{comp}(\beta) < \mathrm{comp}(\alpha)$, the induction hypothesis allows us to establish that:

$$
\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\beta)) &= \mathrm{alt}(\beta) - 1 \\
&= 1 + \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\xi)\} - 1 \qquad (9) \\
&= \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\xi)\} \qquad (10)
\end{aligned}
$$

(b) $K \notin \beta$; then $\varphi, \xi, \beta \in \mathrm{A}(X)$. According to Lemma 3.3 then $\mathrm{dak}(\beta) \equiv \beta$ and, according to Lemma 3.1,

$$
\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\beta)) = \mathrm{alt}(\beta) &= 0 \qquad (11) \\
\mathrm{alt}(\varphi) &= 0 \qquad (12) \\
\mathrm{alt}(\xi) &= 0 \qquad (13)
\end{aligned}
$$

We will now analyse equality 8 on a case–by–case basis:

(a) $K \in \beta$ and $K \in \gamma$; then

$$
\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\alpha)) &= \max\{\mathrm{alt}(\mathrm{dak}(\beta)), \mathrm{alt}(\mathrm{dak}(\gamma))\} && \text{by (8)} \\
&= \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\psi), \mathrm{alt}(\xi)\} && \text{by (10)} \\
&= \mathrm{alt}(\alpha) - 1
\end{aligned}
$$

(b) $K \notin \beta$ and $K \in \gamma$; then

$$\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\alpha)) &= \max\{\mathrm{alt}(\mathrm{dak}(\beta)), \mathrm{alt}(\mathrm{dak}(\gamma))\} && \text{by (8)}\\
&= \max\{0, \mathrm{alt}(\mathrm{dak}(\gamma))\} && \text{by (11)}\\
&= \mathrm{alt}(\mathrm{dak}(\gamma)) &&\\
&= \max\{\mathrm{alt}(\psi), \mathrm{alt}(\xi)\} && \text{by (10)}\\
&= \mathrm{alt}(\psi) && \text{by (13)}\\
&= \max\{0, \mathrm{alt}(\psi), 0\} &&\\
&= \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\psi), \mathrm{alt}(\xi)\} && \text{by (11) y (13)}\\
&= \mathrm{alt}(\alpha) - 1 && \text{by (7)}
\end{aligned}$$

(c) $K \in \beta$ and $K \notin \gamma$; this situation is treated as the case in paragraph 2b).

(d) $K \notin \beta$ and $K \notin \gamma$; in this case $\beta, \gamma \in \mathrm{A}(X)$ and $\alpha \in \mathrm{K}(\mathrm{A}(X))$. By what Lemma 3.4 states, $\mathrm{dak}(\alpha) \equiv \alpha$ and, as Lemma 3.1 states, $\mathrm{alt}(\alpha) = 0$; so $\mathrm{alt}(\mathrm{dak}(\alpha)) = 0$.

3. $\alpha \equiv A\xi K\varphi\psi$; this situation is treated as the case in paragraph 2).
4. $\alpha \equiv A\varphi\psi$; neither $\varphi$ nor $\psi$ begin with $K$ but $K \in \alpha$; without loss of generality suppose that $\mathrm{alt}(\psi) \leq \mathrm{alt}(\varphi)$, whence $K \in \varphi$ and $\varphi$ begins with $A$, i.e. $\varphi \notin \mathrm{K}(\mathrm{A}(X))$. Then $\mathrm{alt}(\alpha) = 1 + \mathrm{alt}(\varphi)$ and

$$\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\alpha)) &= \mathrm{alt}(A\,\mathrm{dak}(\varphi)\,\mathrm{dak}(\psi)) && \text{by Rule 4}\\
&= 1 + \max\{\mathrm{alt}(\mathrm{dak}(\varphi)), \mathrm{alt}(\mathrm{dak}(\psi))\} &&\\
&= 1 + \mathrm{alt}(\mathrm{dak}(\varphi)) &&\\
&= 1 + \mathrm{alt}(\varphi) - 1 && \text{induc. hyp. and conditions of } \varphi\\
&= \mathrm{alt}(\varphi) &&\\
&= \mathrm{alt}(\alpha) - 1 &&
\end{aligned}$$

5. $\alpha \equiv K\varphi\psi$; without loss of generality suppose that $\mathrm{alt}(\psi) \leq \mathrm{alt}(\varphi)$. If $\mathrm{alt}(\varphi) = 0$, then $\mathrm{alt}(\psi) = 0$, $\varphi, \psi, \alpha \in \mathrm{K}(\mathrm{A}(X))$ and so $\mathrm{alt}(\alpha) = 0$ (cfr. Lemma 3.1). If $\mathrm{alt}(\varphi) \neq 0$, i.e. $\varphi \notin \mathrm{K}(\mathrm{A}(X))$, then $\alpha \notin \mathrm{K}(\mathrm{A}(X))$. Thus:

$$\begin{aligned}
\mathrm{alt}(\mathrm{dak}(\alpha)) &= \mathrm{alt}(K\,\mathrm{dak}(\varphi)\,\mathrm{dak}(\psi)) && \text{by Rule 5}\\
&= \max\{\mathrm{alt}(\mathrm{dak}(\varphi)), \mathrm{alt}(\mathrm{dak}(\psi))\} && \text{def. of alt}\\
&= \mathrm{alt}(\mathrm{dak}(\varphi)) &&\\
&= \mathrm{alt}(\varphi) - 1 && \text{induc. hyp. and conditions of } \varphi\\
&= \max\{\mathrm{alt}(\varphi), \mathrm{alt}(\psi)\} - 1 &&\\
&= \mathrm{alt}(\alpha) - 1 && \text{def. of alt}
\end{aligned}$$

□

As a consequence of Theorem 3.5 it follows that the sufficient condition of Lemma 3.4 is also a necessary condition.

**Corollary 3.6.** *Let $\alpha$ be any formula in $\mathrm{P}(\mathbf{L_{KA}})$. The following statements are equivalent:*

1. $\mathrm{dak}(\alpha) \equiv \alpha$
2. $\alpha \in \mathrm{K}(\mathrm{A}(X))$
3. $\mathrm{alt}(\alpha) = 0$

*Proof.* Let $\alpha$ be any formula in $\mathrm{P}(\mathbf{L_{KA}})$. First let us assume that $\mathrm{dak}(\alpha) \equiv \alpha$. According to Theorem

3.5, if $\alpha \notin \mathrm{K}(\mathrm{A}(X))$ then:

$$\begin{aligned} \mathsf{alt}(\alpha) &= \mathsf{alt}(\mathsf{dak}(\alpha)) \\ &= \mathsf{alt}(\alpha) - 1 \end{aligned}$$

which is absurd and must therefore be $\alpha$ an elemenent of $\mathrm{K}(\mathrm{A}(X))$. If $\alpha \in \mathrm{K}(\mathrm{A}(X))$ then, as Lemma 3.1 states, $\mathsf{alt}(\alpha) = 0$. Finally, if $\mathsf{alt}(\alpha) = 0$ then, as Lemma 3.1 states, $\alpha \in \mathrm{K}(\mathrm{A}(X))$ and as Lemma 3.4 states, $\mathsf{dak}(\alpha) \equiv \alpha$. □

We now know that given any formula $\alpha$ in $\mathrm{P}(\mathbf{L_{KA}})$ it is possible to obtain from it another in conjunctive normal form by iterated application of the dak function. Moreover, the minimum number or iterations required is exactly $\mathsf{alt}(\alpha)$. Theorem 3.2 proves that this other formula in conjunctive normal form is logically equivalent to $\alpha$, the input formula

**Corollary 3.7.** *For all $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$, the natural number $\mathsf{alt}(\alpha)$ is the smallest natural number $m$ satisfying $\mathsf{dak}^m(\alpha) \in \mathrm{K}(\mathrm{A}(X))$.*

*Proof.* The proof is by induction on $n$ according to the predicate $Q(n)$ of the literal content:

If $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$ and $n = \mathsf{alt}(\alpha)$, then $n$ is the natural minor $m$ satisfying $\mathsf{dak}^m(\alpha) \in \mathrm{K}(\mathrm{A}(X))$

The reasoning is as follows:

- $n = 0$; if $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$ and $0 = \mathsf{alt}(\alpha)$, then by Corollary 3.6 we know that $\alpha \in \mathrm{K}(\mathrm{A}(X))$, i.e., $\mathsf{dak}^0(\alpha) \in \mathrm{K}(\mathrm{A}(X))$ since $\mathsf{dak}^0$ is the identity map. Since 0 is the smallest natural number, the set of natural numbers smaller than it is empty, from which we conclude the assertion.
- Suppose that $0 < n$, that $Q(n-1)$ is true and that $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$ is fixed but arbitrary under the condition that $n = \mathsf{alt}(\alpha)$. As we know from Theorem 3.5 and Corollary 3.6, it holds:

$$\mathsf{alt}(\mathsf{dak}(\alpha)) = \mathsf{alt}(\alpha) - 1 = n - 1 \tag{14}$$

  By (14) and the induction hypothesis we have, in particular, that:

$$\mathsf{dak}^n(\alpha) = \mathsf{dak}^{n-1}(\mathsf{dak}(\alpha)) \in \mathrm{K}(\mathrm{A}(X))$$

  On the other hand, let $m$ be a natural number such that $m < n$. Three cases can occur:
  * $m = n - 1$; then:

$$\begin{aligned} \mathsf{alt}(\mathsf{dak}^{n-1}(\alpha)) &= \mathsf{alt}(\alpha) - n + 1 \\ &= n - n + 1 \\ &= 1 \end{aligned}$$

    so (cfr. Corollary 3.6) $\mathsf{dak}^{n-1}(\alpha) \notin \mathrm{K}(\mathrm{A}(X))$.
  * $0 < m < n-1$; by Theorem 3.5 we know that $\mathsf{alt}(\mathsf{dak}(\alpha)) = n-1$, and since $m-1 < m < n-1$, by the induction hypothesis we have

$$\mathsf{dak}^m(\alpha) = \mathsf{dak}^{m-1}(\mathsf{dak}(\alpha)) \notin \mathrm{K}(\mathrm{A}(X))$$

  * $m = 0$; $\mathsf{dak}^0(\alpha) = \alpha$ and since $\mathsf{alt}(\alpha) = n > 0$, we deduce that $\mathsf{dak}^0(\alpha) \notin \mathrm{K}(\mathrm{A}(X))$.

By the principle of finite induction we deduce that $Q(n)$ is true for any natural number $n$. Since the function alt can be applied to any formula, the result is true. □

## 4 Associativity

By iterating dak from any formula we will obtain, as we have seen, a formula logically equivalent to it which is in conjunctive normal form. However, the expression of any formula in conjunctive normal form can be given according to multiple formulas all logically equivalent to each other. From any one of them we will explain in what follows an algorithm to obtain a given one which we will consider canonical and which we will call "conjunctive left normal form".

**Definition 4.1.** Let $\mathrm{ar}\colon \mathrm{P}(\mathbf{L}) \longrightarrow \mathbb{Z}$ be defined as follows:

$$\mathrm{ar}(\alpha) = \begin{cases} -1, & \text{if } \alpha \in X; \\ \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\}, & \text{if } \alpha = \mathrm{K}\,\varphi\psi; \\ \max\{\mathrm{ar}(\varphi), 1 + \mathrm{ar}(\psi)\}, & \text{if } \alpha = \mathrm{A}\,\varphi\psi. \end{cases}$$

and let $\mathrm{kr}\colon \mathrm{P}(\mathbf{L}) \longrightarrow \mathbb{Z}$ be defined as follows:

$$\mathrm{kr}(\alpha) = \begin{cases} -1, & \text{if } \alpha \in X; \\ \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\}, & \text{if } \alpha = \mathrm{K}\,\varphi\psi; \\ \max\{\mathrm{kr}(\varphi), \mathrm{kr}(\psi)\}, & \text{if } \alpha = \mathrm{A}\,\varphi\psi. \end{cases}$$

Also let $\mathrm{her}\colon \mathrm{P}(\mathbf{L}) \longrightarrow \mathbb{Z}$ be defined as follows:

$$\mathrm{her}(\alpha) = \max\{\mathrm{ar}(\alpha), \mathrm{kr}(\alpha)\}$$

The maps ar and kr have the properties given in Lemma 4.1. This lemma gives a characterisation of the elements of $\mathrm{K}(X)$, $\mathrm{A}(X)$, and $X$. As we can see, and we will see, is transcendental the importance of both functions assigning the value $-1$ to the elements of $X$.

**Lemma 4.1.** *For all $\alpha \in \mathrm{P}(\mathbf{L})$:*

1. *$\alpha \in \mathrm{K}(X)$ if, and only if, $\mathrm{ar}(\alpha) = -1$.*
2. *$\alpha \in \mathrm{A}(X)$ if, and only if, $\mathrm{kr}(\alpha) = -1$.*

*Proof.* Let us prove statement 1). First we will reason by induction according to the complexity of $\alpha$ and according to the predicate $Q(n)$ of the literal content:

"for all $\alpha \in \mathrm{P}(\mathbf{L})$, if $\alpha \in \mathrm{K}(X)$ and $\mathrm{comp}(\alpha) = n$ then $\mathrm{ar}(\alpha) = -1$"

Suppose, as an induction hypothesis, that n is a natural number and that for any natural number $k$ such that $k < n$ holds $Q(k)$. We have the following cases:

- $n = 0$; then let be —as the only case of interest— $\alpha \equiv x \in X$. By definition (cfr. Definition 4.1), $\mathrm{ar}(\alpha) = -1$ so $Q(0)$ is true.
- $n > 0$; if $\alpha \in \mathrm{K}(X)$ and $n > 0$, there must exist $\varphi, \psi \in \mathrm{K}(X)$ such that $\alpha \equiv K\varphi\psi$. Then:

$$\begin{aligned} \mathrm{ar}(\alpha) &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} && \text{Definition 4.1} \\ &= \max\{-1, -1\} && \text{hip. of induc.} \\ &= -1 \end{aligned}$$

so $Q(n)$ is true in this case.

By the *second principle of finite induction*, for any natural number $n$ is true $Q(n)$ and hence the implication. Reciprocally, let us now consider the predicate $Q(n)$:

"for all $\alpha \in \mathrm{P}(\mathbf{L})$, if $\mathrm{comp}(\alpha) = n$ and $\mathrm{ar}(\alpha) = -1$, then $\alpha \in \mathrm{K}(X)$"

Suppose, as an induction hypothesis, that n is a natural number and that for any natural number $k$ such that $k < n$ holds $Q(k)$. We have the following cases:

- $n = 0$; then let —as the only case of interest— $\alpha \equiv x \in X$. Since $X \subseteq \mathrm{K}(X)$ it follows that $Q(0)$ es true.
- $n > 0$; let $\alpha \in \mathrm{P}(\mathbf{L})$ such that $\mathrm{comp}(\alpha) = n$ and $\mathrm{ar}(\alpha) = -1$. In principle, the following possibilities are possible:
  - there exist $\varphi, \psi \in \mathrm{P}(\mathbf{L})$ such that $\alpha \equiv K\varphi\psi$; then

$$\begin{aligned} -1 &= \mathrm{ar}(\alpha) \\ &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} \\ &\Rightarrow \mathrm{ar}(\varphi) = -1 = \mathrm{ar}(\psi) \\ &\Rightarrow \varphi, \psi \in \mathrm{K}(X) && \text{hip. inducc.} \\ &\Rightarrow \alpha \in \mathrm{K}(X) \end{aligned}$$

  - there exist $\varphi, \psi \in \mathrm{P}(\mathbf{L})$ such that $\alpha \equiv A\varphi\psi$; then

$$\begin{aligned} -1 &= \mathrm{ar}(\alpha) \\ &= \max\{\mathrm{ar}(\varphi), 1 + \mathrm{ar}(\psi)\} \\ &\geq 0 \\ &\Rightarrow \text{therefore this case is not possible} \end{aligned}$$

  so that $Q(n)$ is true.

By the *second principle of finite induction*, for any natural number $n$, $Q(n)$ holds and hence the implication. The statement 2) can be proved with the same scheme as above. Suppose now that $\mathrm{kr}(\alpha) = -1 = \mathrm{ar}(\alpha)$. Since $\mathrm{ar}(\alpha) = -1$ we have that $\alpha \in \mathrm{K}(X)$ and if there exist $\varphi, \psi \in \mathrm{P}(\mathbf{L})$ such that $\alpha \equiv K\varphi\psi$, then one would have:

$$\begin{aligned} -1 &= \mathrm{kr}(\alpha) && \text{hypothesis} \\ &= \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\} && \text{Definition 4.1} \\ &\geq 0 \end{aligned}$$

which is absurd, so $\alpha \in X$. The reciprocal statement is obviously true and it follows that $\mathrm{K}(X) \cap \mathrm{A}(X) = X$. □

**Lemma 4.2.** *For all $\alpha \in \mathrm{P}(\mathbf{L})$, $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = -1$ if, and only if, $\alpha \in X$. Furthermore, $\mathrm{K}(X) \cap \mathrm{A}(X) = X$.*

*Proof.* If $\mathrm{ar}(\alpha) = -1 = \mathrm{kr}(\alpha)$, this requires (cfr. Lemma 4.1) $\alpha \in \mathrm{K}(X) \cap \mathrm{A}(X) = X$. The reciprocal statement is obviously true. As a consequence of this and Lemma 4.1 we have that $\mathrm{K}(X) \cap \mathrm{A}(X) = X$. □

**Lemma 4.3.** *For all $\alpha \in \mathrm{P}(\mathbf{L})$:*

1. *If $\alpha \in \mathrm{K}(X) \setminus X$ then $0 \leq \mathrm{kr}(\alpha)$.*
2. *If $\alpha \in \mathrm{A}(X) \setminus X$ then $0 \leq \mathrm{ar}(\alpha)$.*

*Proof.* For all $\psi \in \mathrm{P}(\mathbf{L})$, $-1 \leq \mathrm{kr}(\psi)$ (resp. $-1 \leq \mathrm{ar}(\psi)$) so that $0 \leq 1 + \mathrm{kr}(\psi)$ (resp. $0 \leq 1 + \mathrm{ar}(\psi)$). □

*Remark* 4.1. The respective reciprocal statements of Lemma 4.3 are not true. Indeed, $\mathrm{kr}(AKxyx) = 0$ (resp. $\mathrm{ar}(KAxyx) = 0$) and yet $AKxyx \notin \mathrm{K}(X) \setminus X$ (resp. $KAxyx \notin \mathrm{A}(X) \setminus X$).

Lemma 4.4 gives a characterisation of the elements of $\alpha \in \mathrm{ml}(X) \setminus X$, where $\mathrm{ml}(X)$ was defined in Definition 2.5.

**Lemma 4.4.** *For all $\alpha \in \mathrm{P}(\mathbf{L})$, $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$ if, and only if, $\alpha \in \mathrm{ml}(X) \setminus X$.*

*Proof.* Let $\alpha$ be any formula in $\mathrm{P}(\mathbf{L})$ satisfying $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$. If $\mathrm{ar}(\alpha) = -1$, then $\alpha \in \mathrm{K}(X)$ (cfr. Lemma 4.1) but $\alpha \notin X$ since $\mathrm{kr}(\alpha) \neq -1$ (cfr. Lemma 4.2); therefore $\alpha \in \mathrm{K}(X) \setminus X$ and, in particular, $\mathrm{comp}(\alpha) > 0$. The proof is by induction according to the complexity of $\alpha$ according to the predicate $Q(n)$ of the literal content:

"for all $\alpha$, if $\alpha \in \mathrm{P}(\mathbf{L})$, $\mathrm{comp}(\alpha) = n$, $\mathrm{ar}(\alpha) = -1$, and $\mathrm{kr}(\alpha) = 0$, then $\alpha \in \mathrm{ml}(X)$"

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. We distinguish the following cases:

- $n = 1$; there will exist $x, y \in X$ such that $\alpha \equiv Kxy$ and hence $\alpha \in \mathrm{ml}(X) \setminus X$, thus $Q(1)$ necessarily holds.
- $n > 1$; there will exist $\varphi, \psi \in \mathrm{K}(X)$ such that $\alpha \equiv K\varphi\psi$. As $\mathrm{kr}(\alpha) = 0$, necessarily $\psi \in X$. As for $\varphi$: it is an element of $\mathrm{K}(X) \setminus X$ since $2 \leq n$; $\mathrm{ar}(\varphi) = -1$, because if it were not then $\mathrm{ar}(\alpha) \neq -1$, and $0 \leq \mathrm{kr}(\varphi)$ according to Lemma 4.3, but $\mathrm{kr}(\varphi) \leq 0$ by definition of kr and the condition $\mathrm{kr}(\alpha) = 0$, so $\mathrm{kr}(\varphi) = 0$. By the induction hypothesis, $\varphi \in \mathrm{ml}(X) \setminus X$; thus $\alpha \equiv K\varphi\psi$, with $\varphi \in \mathrm{ml}(X) \setminus X$ and $\psi \in X$, which means that $\alpha \in \mathrm{ml}(X) \setminus X$.

By the *second principle of finite induction*, for every natural number $n$, $Q(n)$ holds. To prove the reciprocal statement we reason by induction about the complexity of the formula $\alpha$ according to the following predicate $Q(n)$:

"for all formula $\alpha$, if $\mathrm{comp}(\alpha) = n$ and $\alpha \in \mathrm{ml}(X) \setminus X$ then $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$"

- $n = 1$; if $\alpha \in \mathrm{ml}(X) \setminus X$ and $\mathrm{comp}(\alpha) = 1$, there exist $x, y \in X$ such that $\alpha \equiv Kxy$. This being so, $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$ and therefore $Q(1)$ holds.
- Suppose, as an induction hypothesis, that $n$ is a non-zero natural number and that $Q(n-1)$ holds.
- If $\alpha \in \mathrm{ml}(X) \setminus X$ and $\mathrm{comp}(\alpha) = n$, there must exist $\varphi \in \mathrm{ml}(X)$ and $\psi \in X$ such that $\alpha \equiv K\varphi\psi$. Since $\mathrm{comp}(\varphi) = n - 1$ there must be $\mathrm{ar}(\varphi) = -1$ and $\mathrm{kr}(\varphi) = 0$. Therefore $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$.

By the *principle of mathematical finite induction* it follows that for any non-zero natural number $n$, $Q(n)$ holds and hence what we want to prove. ☐

**Lemma 4.5.** *For all $\alpha \in \mathrm{P}(\mathbf{L})$, $\mathrm{ar}(\alpha) = 0$ y $\mathrm{kr}(\alpha) = -1$ if, and only if, $\alpha \in \mathrm{cl}(X) \setminus X$.*

*Proof.* The proof of this lemma is the dual reasoning of the one used in the proof of Lemma 4.4. ☐

**Lemma 4.6.** *For all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, the following statements are equivalent:*

1. $\mathrm{ar}(\alpha) = 0$ *and* $\mathrm{kr}(\alpha) = 0$.
2. $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.

*Proof.* Suppose that: $\alpha \in \mathrm{K}(\mathrm{A}(X))$, $\mathrm{ar}(\alpha) = 0$, and $\mathrm{kr}(\alpha) = 0$. As Lemma 4.1 states neither $\alpha \in \mathrm{K}(X)$ nor $\alpha \in \mathrm{A}(X)$, so in particular $\alpha \notin X$, from which it follows that under the assumptions of the lemma necessrily $\mathrm{comp}(\alpha) > 0$. On the other hand, neither can $\alpha \equiv Kxy$, where $x, y \in X$, nor $\alpha \equiv Axy$, where equally $x, y \in X$. Indeed, if $\alpha \equiv Kxy$, $\mathrm{ar}(\alpha) = -1$, and if $\alpha \equiv Axy$, then $\mathrm{kr}(\alpha) = -1$. However,

$\mathrm{kr}(KxAxy) = 0 = \mathrm{ar}(KxAxy)$; therefore, $\mathrm{comp}(\alpha) = n \geq 2$. The proof is by induction according to the complexity of $\alpha$ and according to the predicate $Q(n)$ of the literal content:

"for all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, if $\mathrm{comp}(\alpha) = n$, $\mathrm{ar}(\alpha) = 0$, and $\mathrm{kr}(\alpha) = 0$, then $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$"

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. Let now be $\alpha$ a formula fixed but arbitrary such that $\mathrm{comp}(\alpha) = n$; we distinguish the following cases:

1. $n = 2$; there must exist $x, y, z \in X$ such that $\alpha \equiv KxAyz$ o bien $\alpha \equiv KAxyz$. In either case $\alpha$ is an element of $\mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X)$. Hence $P(2)$ holds.
2. $n > 2$; if $\alpha$ were an element of $\mathrm{A}(X)$ (resp. $\mathrm{K}(X)$), then $\mathrm{kr}(\alpha)$ (resp. $\mathrm{ar}(\alpha)$) would be worth $-1$ (cfr. Lemma 4.1) when it is actually worth 0; therefore $\alpha \in \mathrm{K}(\mathrm{A}(X)) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$. Thus, there exist $\varphi, \psi \in \mathrm{K}(\mathrm{A}(X))$ such that $\alpha \equiv K\varphi\psi$. This being so, and $\mathrm{kr}(\alpha) = 0$, it must be the case that $\mathrm{kr}(\psi) = -1$ or equivalently $\psi \in \mathrm{A}(X)$. But $\mathrm{ar}(\alpha) = 0$, whence $-1 \leq \mathrm{ar}(\psi) \leq 0$. Consider, in general, that if $\mathrm{ar}(\psi) = 0$ and $\mathrm{kr}(\psi) = -1$, then $\psi \in \mathrm{cl}(X) \setminus X$ (cfr. Lemma 4.5); while if $\mathrm{ar}(\psi) = -1$ and $\mathrm{kr}(\psi) = -1$, then $\psi \in X \subseteq \mathrm{cl}(X)$ (cfr. Lemma 4.2). The following cases can occur:
   (a) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = 0$, $\mathrm{ar}(\varphi) = 0$, and $\mathrm{kr}(\varphi) = -1$; since $\mathrm{ar}(\varphi) = 0$ and $\mathrm{kr}(\varphi) = -1$, $\varphi \in \mathrm{cl}(X) \setminus X$ (cfr. Lemma 4.5). Thus $\varphi, \psi \in \mathrm{cl}(X) \setminus X$, from which $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.
   (b) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = 0$, $\mathrm{ar}(\varphi) = 0$ y $\mathrm{kr}(\varphi) = 0$; as $\varphi, \psi \in \mathrm{K}(\mathrm{A}(X))$, $\mathrm{ar}(\varphi) = 0$, $\mathrm{kr}(\varphi) = 0$, and $\mathrm{comp}(\varphi) < n$, the induction hypothesis allows us to conclude that $\varphi \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$ and, ultimately, that $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.
   (c) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = 0$, $\mathrm{ar}(\varphi) = -1$, and $\mathrm{kr}(\varphi) = -1$; in such a case $\varphi \in X \subseteq \mathrm{lcnf}(X)$ (cfr. Lemma 4.2). Similarly, the conclusion is $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.
   (d) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = 0$, $\mathrm{ar}(\varphi) = -1$, and $\mathrm{kr}(\varphi) = 0$; in such a case $\varphi \in X \subseteq \mathrm{ml}(X) \setminus X$ (cfr. Lemma 4.4). Again the conclusion is $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.
   (e) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = -1$, $\mathrm{ar}(\varphi) = 0$, and $\mathrm{kr}(\varphi) = -1$; in such a case $\varphi \in X \subseteq \mathrm{cl}(X) \setminus X$ (cfr. Lemma 4.4). Thus, $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.
   (f) $\mathrm{kr}(\psi) = -1$, $\mathrm{ar}(\psi) = -1$, $\mathrm{ar}(\varphi) = 0$, and $\mathrm{kr}(\varphi) = 0$; as $\varphi, \psi \in \mathrm{K}(\mathrm{A}(X))$, $\mathrm{ar}(\varphi) = 0$, $\mathrm{kr}(\varphi) = 0$, and $\mathrm{comp}(\varphi) < n$, the induction hypothesis allow us to conclude that $\varphi \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$ and, ultimately, that $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.

By the *second principle of finite induction* it follows that for any non-zero natural number $n$, $Q(n)$ holds and hence the validity of the direct implication. For the reciprocal, let us observe that if $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$ then there must exist $\varphi \in \mathrm{lcnf}(X)$ and $\psi \in \mathrm{cl}(X)$ such that $\alpha \equiv K\varphi\psi$; but $\psi \in X$, then $\varphi \in \mathrm{lcnf}(X) \setminus \mathrm{ml}(X)$ and in particular $\mathrm{comp}(\alpha) \geq 2$. We also reason by induction about the complexity of the formula according to the predicate $Q(n)$:

"for all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, if $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$ and $\mathrm{comp}(\alpha) = n$, then $\mathrm{ar}(\alpha) = 0$ and $\mathrm{kr}(\alpha) = 0$"

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. Let $\alpha$ now be fixed but arbitrary such that $\mathrm{comp}(\alpha) = n$; we distinguish the following cases of interest:

1. $\psi \in X$; then $\alpha \equiv K\varphi\psi$ with $\varphi \in \mathrm{lcnf}(X) \setminus \mathrm{ml}(X)$. Two cases can occur:

(a) $\varphi \in \mathrm{cl}(X)$; then $\varphi \in \mathrm{cl}(X) \setminus X$ and one has:

$$\begin{aligned}
\mathrm{kr}(\alpha) &= \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\} \\
&= \max\{-1, 1 - 1\} \\
&= \max\{-1, 0\} \\
&= 0 \\
\mathrm{ar}(\alpha) &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} \\
&= \max\{0, -1\} \\
&= 0
\end{aligned}$$

(b) $\varphi \notin \mathrm{cl}(X)$; then:

$$\begin{aligned}
\varphi &\in (\mathrm{lcnf}(X) \setminus \mathrm{cl}(X)) \cap (\mathrm{lcnf}(X) \setminus \mathrm{ml}(X)) \\
&= \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))
\end{aligned}$$

Since $\mathrm{comp}(\varphi) < \mathrm{comp}(\alpha)$, by the induction hypothesis we deduce that $\mathrm{ar}(\alpha) = 0 = \mathrm{kr}(\alpha) = 0$ and then:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \max\{0, -1\} = 0 \\
\mathrm{kr}(\alpha) &= \max\{0, 1 - 1\} = 0
\end{aligned}$$

2. $\psi \notin X$; since $\psi \in \mathrm{cl}(X) \setminus X$ then $\mathrm{ar}(\psi) = 0$ y $\mathrm{kr}(\psi) = -1$ (cfr. Lemma 4.5). We will have the following cases:

(a) $\varphi \in X$; then:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \max\{-1, 0\} = 0 \\
\mathrm{kr}(\alpha) &= \max\{-1, 1 - 1\} = 0
\end{aligned}$$

(b) $\varphi \notin \mathrm{cl}(X) \setminus X$; then $\mathrm{ar}(\varphi) = 0$ and $\mathrm{kr}(\varphi) = -1$. Therefore:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} = 0 \\
\mathrm{kr}(\alpha) &= \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\} \\
&= \max\{-1, 1 - 1\} = 0
\end{aligned}$$

(c) $\varphi \notin \mathrm{ml}(X) \setminus X$; then $\mathrm{ar}(\varphi) = -1$ y $\mathrm{kr}(\varphi) = 0$ (cfr. Lemma 4.4). Thus:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} \\
&= \max\{-1, 0\} = 0 \\
\mathrm{kr}(\alpha) &= \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\} \\
&= \max\{0, 1 - 1\} = 0
\end{aligned}$$

(d) $\varphi \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$; by the induction hypothesis $\mathrm{ar}(\varphi) = 0 = \mathrm{kr}(\varphi)$ will be satisfied. Thus:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\} \\
&= \max\{0\} = 0 \\
\mathrm{kr}(\alpha) &= \max\{\mathrm{kr}(\varphi), 1 + \mathrm{kr}(\psi)\} \\
&= \max\{0, 1 - 1\} = 0
\end{aligned}$$

By the *second principle of finite induction* it follows that for any non-zero natural number $n$, $Q(n)$ holds and hence the validity of the reciprocal implication and the equivalence. □

*Remark* 4.2. The formula $\alpha \equiv AKxyx$ satisfies $\mathrm{ar}(\alpha) = 0 = \mathrm{kr}(\alpha)$, but $\alpha \notin \mathrm{lcnf}(X)$. Hence the need for the restriction in the statement of Lemma 4.6.

After several technical lemmas in the Theorem 4.7, in which by means of $operatorname her$ we reach to characterise the set $operatorname lcnf(X)$ of formulae in left conjunctive normal form. Once again we will stress the importance of the sign $\leq$ in its statement, highlighting the importance of conveniently and subtly including the value $-1$ in the definition Definition 4.1; this will appear in the demonstration.

**Theorem 4.7.** *For all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, the following statements are equivalent:*

1. $\mathrm{her}(\alpha) \leq 0$.
2. $\alpha \in \mathrm{lcnf}(X)$.

*Proof.* Let us first show that statement 1) is a sufficient condition for 2) to be fulfilled. The following cases are possible:

1. $\mathrm{ar}(\alpha) = -1 = \mathrm{kr}(\alpha)$; as stated in Lemma 4.2, $\alpha \in X$.
2. $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = 0$; as stated in Lemma 4.4, $\alpha \in \mathrm{ml}(X) \setminus X$.
3. $\mathrm{ar}(\alpha) = 0$ and $\mathrm{kr}(\alpha) = -1$; as stated in Lemma 4.5, $\alpha \in \mathrm{cl}(X) \setminus X$
4. $\mathrm{ar}(\alpha) = 0$ and $\mathrm{kr}(\alpha) = 0$; as stated in Lemma 4.6, $\alpha \in \mathrm{lcnf}(X) \setminus (\mathrm{cl}(X) \cup \mathrm{ml}(X))$.

and hence, $\alpha \in \mathrm{lcnf}(X)$. But 1) is a necessary condition for 2), which follows as a consequence of the aforementioned lemmas. □

$$x_i \rightarrow_{lasc} x_i$$

$$\frac{A\varphi\psi \rightarrow_{lasc} \alpha \quad \xi \rightarrow_{lasc} \beta}{A\varphi A\psi\xi \rightarrow_{lasc} A\alpha\beta} \qquad \frac{K\varphi\psi \rightarrow_{lasc} \alpha \quad \xi \rightarrow_{lasc} \beta}{K\varphi K\psi\xi \rightarrow_{lasc} K\alpha\beta}$$

$$\frac{\varphi \rightarrow_{lasc} \varphi' \quad \psi \rightarrow_{lasc} \psi'}{A\varphi\psi \rightarrow_{lasc} A\varphi'\psi'} \qquad \frac{\varphi \rightarrow_{lasc} \varphi' \quad \psi \rightarrow_{lasc} \psi'}{K\varphi\psi \rightarrow_{lasc} K\varphi'\psi'}$$

Figure 2: Axiom and rules for the definition of lasc.

**Definition 4.2 (left associativity).** Consider the following rules:

$$x_i \rightarrow_{lasc} x_i \tag{15}$$

$$\frac{A\varphi\psi \rightarrow_{lasc} \alpha \quad \xi \rightarrow_{lasc} \beta}{A\varphi A\psi\xi \rightarrow_{lasc} A\alpha\beta} \tag{16}$$

$$\frac{K\varphi\psi \rightarrow_{lasc} \alpha \quad \xi \rightarrow_{lasc} \beta}{K\varphi K\psi\xi \rightarrow_{lasc} K\alpha\beta} \tag{17}$$

$$\frac{\varphi \rightarrow_{lasc} \varphi' \quad \psi \rightarrow_{lasc} \psi'}{A\varphi\psi \rightarrow_{lasc} A\varphi'\psi'} \tag{18}$$

$$\frac{\varphi \rightarrow_{lasc} \varphi' \quad \psi \rightarrow_{lasc} \psi'}{K\varphi\psi \rightarrow_{lasc} K\varphi'\psi'} \tag{19}$$

where (16), (17), (18), and (19) shall be applied with the priority from highest to lowest according to the order given. Les us now define the map $\mathrm{lasc}\colon P(\mathbf{L}) \longrightarrow P(\mathbf{L})$ by $\mathrm{lasc}(\varphi) \equiv \psi$ if, and only if, $\varphi \rightarrow_{lasc} \psi$.

As a consequence of the Theorem 2.3 it is clear that for any formula $\varphi$, $\mathrm{lasc}(\varphi)$ is (logically) equivalent to $\varphi$; therefore lasc does not alter the logical meaning of the formulas by acting on them, although it does eventually alter their syntax. What is stated in Lemma 4.8 is obviously true.

**Lemma 4.8.** *For all* $\alpha \in \mathrm{P}(\mathbf{L})$,

1. *If* $\alpha \in A(X)$ *then* $\mathrm{lasc}(\alpha) \in A(x)$.
2. *If* $\alpha \in K(X)$ *then* $\mathrm{lasc}(\alpha) \in K(x)$.

**Lemma 4.9.** *For all* $\alpha \in A(X)$,

$$\mathrm{ar}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor.$$

*Proof.* The proof is by induction on the complexity of $\alpha$ using the predicate $Q(n)$ of the literal content:

$$\text{“for all } \alpha\text{, if } \alpha \in A(X) \text{ and } \mathrm{comp}(\alpha) = n \text{ then } \mathrm{ar}(\mathrm{lasc}(\alpha)) = \left\lfloor \tfrac{\mathrm{ar}(\alpha)}{2} \right\rfloor\text{”}$$

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. We have the following cases:

- $n = 0$; then let —as the only case of interest— $\alpha \equiv x \in X$ and therefore

$$\begin{aligned}\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \mathrm{ar}(\mathrm{lasc}(x)) = \mathrm{ar}(x)\\ &= -1 = \left\lfloor \frac{-1}{2} \right\rfloor\end{aligned}$$

  so $Q(0)$ is true.
- $n > 0$; let $\alpha \equiv A\varphi\rho$ for given $\phi, \rho \in \mathrm{A}(X)$. The following casuistry then arises:
  - $\varphi, \rho \in X$; $\alpha \equiv Ayx$, for certain $y, x \in X$, then:

$$\begin{aligned}\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \mathrm{ar}(\mathrm{lasc}(Ayx))\\ &= \mathrm{ar}(A\,\mathrm{lasc}(y)\,\mathrm{lasc}(x)) = \mathrm{ar}(Ayx)\\ &= \max\{\mathrm{ar}(y), \mathrm{ar}(x)+1\} = \max\{-1, -1+1\}\\ &= 0 = \left\lfloor \frac{0}{2} \right\rfloor = \left\lfloor \frac{\mathrm{ar}(Ayx)}{2} \right\rfloor = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor\end{aligned}$$

  - $\varphi \in \mathrm{A}(X) \setminus X$ and $\rho \in X$; $\alpha \equiv A\varphi x$, for certain $x \in X$, entonces:

$$\begin{aligned}\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \mathrm{ar}(\mathrm{lasc}(A\varphi x))\\ &= \mathrm{ar}(A\,\mathrm{lasc}(\varphi)\,\mathrm{lasc}(x)) = \mathrm{ar}(A\,\mathrm{lasc}(\varphi)x)\\ &= \max\{\mathrm{ar}(\mathrm{lasc}(\varphi)), \mathrm{ar}(x)+1\} = \max\{\mathrm{ar}(\mathrm{lasc}(\varphi)), 0\}\\ &= \max\left\{\left\lfloor \frac{\mathrm{ar}(\varphi)}{2} \right\rfloor, 0\right\} && \text{hyp. of induction}\\ &= \left\lfloor \frac{\mathrm{ar}(\varphi)}{2} \right\rfloor && \text{Lemma 4.3}\\ &= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), 0\}}{2} \right\rfloor = \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \mathrm{ar}(x)+1\}}{2} \right\rfloor\\ &= \left\lfloor \frac{\mathrm{ar}(A\varphi x)}{2} \right\rfloor = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor\end{aligned}$$

– $\alpha \equiv A\varphi A\psi\xi$, for certain $\varphi, \psi, \xi \in \mathrm{A}(X)$; then:

$$\begin{aligned}
\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \max\{\mathrm{ar}(\mathrm{lasc}(A\varphi\psi)), 1+\mathrm{ar}(\mathrm{lasc}(\xi))\} \\
&= \max\left\{\left\lfloor \frac{\mathrm{ar}(A\varphi\psi)}{2} \right\rfloor, 1+\left\lfloor \frac{\mathrm{ar}(\xi)}{2} \right\rfloor\right\} && \text{(hyp. ind.)} \\
&= \max\left\{\left\lfloor \frac{\max\{\mathrm{ar}(\varphi), 1+\mathrm{ar}(\psi)\}}{2} \right\rfloor, \left\lfloor \frac{2+\mathrm{ar}(\xi)}{2} \right\rfloor\right\} \\
&= \left\lfloor \frac{\max\{\max\{\mathrm{ar}(\varphi), 1+\mathrm{ar}(\psi)\}, 2+\mathrm{ar}(\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), 1+\max\{\mathrm{ar}(\psi), 1+\mathrm{ar}(\xi)\}\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), 1+\mathrm{ar}(A\psi\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{ar}(A\varphi A\psi\xi)}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor
\end{aligned}$$

so $Q(n)$ is true.

By the *second principle of finite induction*, for every natural number $n$, $Q(n)$ is true, hence the result. □

**Lemma 4.10.** *For all $\alpha \in A(X)$,*

$$\mathrm{kr}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor.$$

*Proof.* If $\alpha \in A(X)$ then $\mathrm{lasc}(\alpha) \in A(X)$. The proof concludes by applying Lemma 4.1 and considering that:

$$-1 = \left\lfloor \frac{-1}{2} \right\rfloor$$

□

As we can see from the Theorem 4.11 the reductive role of $\mathrm{her}(\alpha)$ is very powerful when applying lasc to the formula $\alpha$; as we can see, it is such that divides by 2 the "complexity of the situation", which inevitably invokes the logarithm in base 2.

**Theorem 4.11.** *For all $\alpha \in \mathrm{K}(\mathrm{A}(X))$:*

*1.* $\mathrm{ar}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor$

*2.* $\mathrm{kr}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor$

*3.* $\mathrm{her}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{her}(\alpha)}{2} \right\rfloor$

*Proof.* To prove 1) we will reason by induction about the complexity of $\alpha$ using the predicate $Q(n)$ of the literal content:

"for all $\alpha$, if $\alpha \in K(A(X))$ and $\mathrm{comp}_{\mathrm{k}}(\alpha) = n$ then $\mathrm{ar}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor$"

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$ $Q(k)$ holds. We have the following cases:

- $n = 0$; must be $\alpha \in \mathrm{A}(X)$, formula for which $\mathrm{ar}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor$ as set out in Lemma 4.9.
- $n > 0$; let —as the only case of interest— $\alpha \equiv K\varphi\rho$ for certain $\varphi, \rho \in \mathrm{K}(\mathrm{A}(X))$. Let us distinguish the following cases:
  - $\rho \in A(X)$; then:

$$\begin{aligned}
\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \mathrm{ar}(\mathrm{lasc}(K\varphi\rho)) = \mathrm{ar}(K\,\mathrm{lasc}(\varphi)\,\mathrm{lasc}(\rho)) \\
&= \max\{\mathrm{ar}(\mathrm{lasc}(\varphi)), \mathrm{ar}(\mathrm{lasc}(\rho))\} = \max\left\{\left\lfloor \frac{\mathrm{ar}(\varphi)}{2} \right\rfloor, \left\lfloor \frac{\mathrm{ar}(\rho)}{2} \right\rfloor\right\} \quad \text{(h.i. and Lemma 4.9)} \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \mathrm{ar}(\rho)\}}{2} \right\rfloor = \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor
\end{aligned}$$

  - $\rho \notin A(X)$; then $\alpha \equiv K\varphi K\psi\xi$ for certain $\psi, \xi \in \mathrm{K}(\mathrm{A}(X))$. In this case:

$$\begin{aligned}
\mathrm{ar}(\mathrm{lasc}(\alpha)) &= \mathrm{ar}(\mathrm{lasc}(K\varphi K\psi\xi)) = \mathrm{ar}(K\,\mathrm{lasc}(K\varphi\psi)\,\mathrm{lasc}(\xi)) \\
&= \max\{\mathrm{ar}(\mathrm{lasc}(K\varphi\psi)), \mathrm{ar}(\mathrm{lasc}(\xi))\} \\
&= \max\left\{\left\lfloor \frac{\mathrm{ar}(K\varphi\psi)}{2} \right\rfloor, \left\lfloor \frac{\mathrm{ar}(\xi)}{2} \right\rfloor\right\} \quad \text{(hyp. induc.)} \\
&= \max\left\{\left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi)\}}{2} \right\rfloor, \left\lfloor \frac{\mathrm{ar}(\xi)}{2} \right\rfloor\right\} \\
&= \max\left\{\left\lfloor \frac{\mathrm{ar}(\varphi)}{2} \right\rfloor, \left\lfloor \frac{\mathrm{ar}(\psi)}{2} \right\rfloor, \left\lfloor \frac{\mathrm{ar}(\xi)}{2} \right\rfloor\right\} \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \mathrm{ar}(\psi), \mathrm{ar}(\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \max\{\mathrm{ar}(\psi), \mathrm{ar}(\xi)\}\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{ar}(\varphi), \mathrm{ar}(K\psi\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{ar}(K\varphi K\psi\xi)}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor
\end{aligned}$$

By the *second principle of finite induction*, for every natural number $n$, $Q(n)$ holds and hence the validity of the statement 1). To prove 2) let us reason by induction about the complexity of $\alpha$ according to the predicate $Q(n)$ of the literal content:

"for all $\alpha$, si $\alpha \in K(A(X))$ and $\mathrm{comp}_{\mathrm{k}}(\alpha) = n$ then $\mathrm{kr}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor$

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. We have the following cases:

- $n = 0$; must be $\alpha \in \mathrm{A}(X)$, formula for which $\mathrm{kr}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor$ as set out in Lemma 4.10.
- $n > 0$; let —as the only case of interest— $\alpha \equiv K\varphi\rho$ for certain $\varphi, \rho \in \mathrm{K}(\mathrm{A}(X))$. Let us distinguish the following cases:

- $\rho \in A(X)$; then:

$$\begin{aligned}
\mathrm{kr}(\mathrm{lasc}(\alpha)) &= \mathrm{kr}(\mathrm{lasc}(K\varphi\rho)) = \mathrm{kr}(K\,\mathrm{lasc}(\varphi)\,\mathrm{lasc}(\rho)) \\
&= \max\{\mathrm{kr}(\mathrm{lasc}(\varphi)), 1+\mathrm{kr}(\mathrm{lasc}(\rho))\} = \max\left\{\left\lfloor \frac{\mathrm{kr}(\varphi)}{2} \right\rfloor, 0\right\} \quad \text{(h.i. and Lemma 4.1)} \\
&= \left\lfloor \frac{\max\{\mathrm{kr}(\varphi), 0\}}{2} \right\rfloor = \left\lfloor \frac{\max\{\mathrm{kr}(\varphi), 1+\mathrm{kr}(\rho)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{kr}(K\varphi\rho)}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor
\end{aligned}$$

- $\rho \notin A(X)$; then $\alpha \equiv K\varphi K\psi\xi$ for certain $\psi, \xi \in \mathrm{K}(\mathrm{A}(X))$. In this case:

$$\begin{aligned}
\mathrm{kr}(\mathrm{lasc}(\alpha)) &= \mathrm{kr}(\mathrm{lasc}(K\varphi K\psi\xi)) = \mathrm{kr}(K\,\mathrm{lasc}(K\varphi\psi)\,\mathrm{lasc}(\xi)) \\
&= \max\{\mathrm{kr}(\mathrm{lasc}(K\varphi\psi)), 1+\mathrm{kr}(\mathrm{lasc}(\xi))\} \\
&= \max\left\{\left\lfloor \frac{\mathrm{kr}(K\varphi\psi)}{2} \right\rfloor, 1+\left\lfloor \frac{\mathrm{kr}(\xi)}{2} \right\rfloor\right\} \quad \text{(hyp. induc.)} \\
&= \max\left\{\left\lfloor \frac{\max\{\mathrm{kr}(\varphi), 1+\mathrm{kr}(\psi)\}}{2} \right\rfloor, \left\lfloor \frac{2+\mathrm{kr}(\xi)}{2} \right\rfloor\right\} \\
&= \left\lfloor \frac{\max\{\max\{\mathrm{kr}(\varphi), 1+\mathrm{kr}(\psi)\}, 2+\mathrm{kr}(\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{kr}(\varphi), \max\{1+\mathrm{kr}(\psi), 2+\mathrm{kr}(\xi)\}\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{kr}(\varphi), 1+\max\{\mathrm{kr}(\psi), 1+\mathrm{kr}(\xi)\}\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\max\{\mathrm{kr}(\varphi), 1+\mathrm{kr}(K\psi\xi)\}}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{kr}(K\varphi K\psi\xi)}{2} \right\rfloor \\
&= \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor
\end{aligned}$$

By the *second principle of finite induction*, for every natural number $n$, $Q(n)$ holds and hence the validity of the statement 2). Statement 3) is immediate from 1) and 2) given that:

$$\max\left\{\left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor, \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor\right\} = \left\lfloor \frac{\max\{\mathrm{ar}(\alpha), \mathrm{kr}(\alpha)\}}{2} \right\rfloor = \left\lfloor \frac{\mathrm{her}(\alpha)}{2} \right\rfloor$$

□

**Lemma 4.12.** *Let $\alpha \in \mathrm{A}(X)$. The following statements are equivalent:*

1. $\alpha \in \mathrm{cl}(X)$
2. $\mathrm{lasc}(\alpha) \equiv \alpha$
3. $\mathrm{ar}(\alpha) \leq 0$

*Proof.* To show that statetement 1) implies statement 2) we will reason by induction about the complexity of $\alpha$ according to the predicate $Q(n)$ of the literal content:

"for all $\alpha$, if $\alpha \in \Xi_{\mathrm{A}}(X)$ and $\mathrm{comp}(\alpha) = n$ then $\mathrm{lasc}(\alpha) \equiv \alpha$

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. We distinguish the following cases:

- $n = 0$; must be $\alpha \equiv x \in X$ and then $\mathrm{lasc}(\alpha) \equiv \mathrm{lasc}(x) \equiv x \equiv \alpha$.
- $n > 0$; let —as the only case of interest— $\alpha \equiv A\varphi x$, where $\varphi \in \Xi_{\mathrm{A}}(X)$ and $x \in X$. Then:

$$\begin{aligned}
\mathrm{lasc}(\alpha) &\equiv \mathrm{lasc}(A\varphi x) \\
&\equiv A\,\mathrm{lasc}(\varphi)\,\mathrm{lasc}(x) \\
&\equiv A\varphi x && \text{(hyp. induction and Definition 4.2)} \\
&\equiv \alpha
\end{aligned}$$

  hence $Q(n)$ holds.

By the *second principle of finite induction*, for every natural number $n$ $Q(n)$ holds, and hence the validity of the statement 2). Let us now suppose that 2) is true, i.e. that $\alpha \in \mathrm{A}(X)$ and that $\mathrm{lasc}(\alpha) \equiv \alpha$; then one has:

$$\begin{aligned}
\mathrm{ar}(\alpha) &= \mathrm{ar}(\mathrm{lasc}(\alpha)) \\
&= \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor && \text{(Lemma 4.9)}
\end{aligned}$$

from which we deduce that $\mathrm{ar}(\alpha) \in \{-1, 0\}$, i.e., that statement 3) holds. Finally, suppose that $\alpha \in \mathrm{A}(X)$ and that $\mathrm{ar}(\alpha) \leq 0$. By Lemma 4.1 it must be $\mathrm{kr}(\alpha) = -1$, so that exactly the following cases fit:

1. $\mathrm{ar}(\alpha) = -1$ and $\mathrm{kr}(\alpha) = -1$; then $\alpha \in X \subseteq \mathrm{cl}(X)$ (cfr. Lemma 4.2).
2. $\mathrm{ar}(\alpha) = 0$ and $\mathrm{kr}(\alpha) = -1$; then $\alpha \in \mathrm{cl}(X) \setminus X$ (cfr. Lemma 4.5).

This proves that under the assumption of statement 3) the fact $\alpha \in \mathrm{cl}(X)$ is satisfied, as we sought to prove. □

In the Theorem 4.13, the effects of alt and alt are finally combined to characterise the formulas of $\mathrm{lcnf}(X)$. As expected, the minimum values of these functions are involved, again including $-1$ in our discourse in an essential way.

**Theorem 4.13.** *For all $\alpha \in \mathrm{P}(\mathbf{L_{KA}})$, the following statements are equivalent:*

1. $\alpha \in \mathrm{lcnf}(X)$
2. $\mathrm{dak}(\alpha) \equiv \alpha$ *and* $\mathrm{lasc}(\alpha) \equiv \alpha$
3. $\mathrm{alt}(\alpha) = 0$ *and* $\mathrm{her}(\alpha) \leq 0$.

*Proof.* Let $\alpha$ any formula belonging to $\mathrm{P}(\mathbf{L_{KA}})$. Assume what 1) states, i.e. that $\alpha \in \mathrm{lcnf}(X)$. We will reason by induction about the complexity of $\alpha$ according to the predicate $Q(n)$ of the literal content:

"for all $\alpha$, if $\alpha \in \mathrm{lcnf}(X)$ and $\mathrm{comp}(\alpha) = n$ then $\mathrm{dak}(\alpha) = \alpha$ and $\mathrm{lasc}(\alpha) = \alpha$"

Suppose, as an induction hypothesis, that $n$ is a natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. We distinguish the following cases:

- $n = 0$; must be $\alpha \equiv x \in X$ and then $\mathrm{dak}(\alpha) \equiv \mathrm{dak}(x) \equiv x \equiv \alpha$ and similarly $\mathrm{lasc}(\alpha) \equiv \alpha$. It follows that $Q(0)$ is true.

- $n > 0$; let be —as the only case of interest— $\alpha \equiv K\varphi\psi$, where $\varphi \in \mathrm{lcnf}(X)$ and $\psi \in \mathrm{cl}(X)$. Then:

$$\begin{aligned}
\mathrm{dak}(\alpha) &\equiv \mathrm{dak}(K\varphi\psi) \\
&\equiv K\,\mathrm{dak}(\varphi)\,\mathrm{dak}(\psi) \\
&\equiv K\varphi\,\mathrm{dak}(\psi) && \text{(hyp. induc.)} \\
&\equiv K\varphi\psi && \text{(Remark 2.1 and hyp. induc.)} \\
&\equiv \alpha \\
\mathrm{lasc}(\alpha) &\equiv \mathrm{lasc}(K\varphi\psi) \\
&\equiv K\,\mathrm{lasc}(\varphi)\,\mathrm{lasc}(\psi) \\
&\equiv K\varphi\,\mathrm{lasc}(\psi) && \text{(hyp. induc.)} \\
&\equiv K\varphi\psi && \text{(Lemma 4.12)} \\
&\equiv \alpha
\end{aligned}$$

  so we know that $Q(n)$ is true.

By the *second principle of finite induction*, for every natural number $n$ is true $Q(n)$ and hence the validity of assertion 2). If we now assume that 2) is true, that $\mathrm{alt}(\alpha) = 0$ and that $\alpha \in \mathrm{K}(\mathrm{A}(X))$ is ensured by the Corollary 3.6. In particular, as a consequence of Theorem 4.11, we have:

$$\begin{aligned}
\mathrm{ar}(\alpha) = \mathrm{ar}(\mathrm{lasc}(\alpha)) &= \left\lfloor \frac{\mathrm{ar}(\alpha)}{2} \right\rfloor \\
\mathrm{kr}(\alpha) = \mathrm{kr}(\mathrm{lasc}(\alpha)) &= \left\lfloor \frac{\mathrm{kr}(\alpha)}{2} \right\rfloor
\end{aligned}$$

and therefore, $\mathrm{ar}(\alpha) \leq 0$ and $\mathrm{kr}(\alpha) \leq 0$; thus, we have proved 3). Let us finally assume 3) to be true and show that $\alpha \in \mathrm{lcnf}(X)$. Since $\mathrm{alt}(\alpha) = 0$ and again using Corollary 3.6, we know that $\alpha \in \mathrm{K}(\mathrm{A}(X))$. According to Theorem 4.7, since $\mathrm{her}(\alpha) \leq 0$, $\alpha \in \mathrm{lcnf}(X)$ must necessarily hold and this is what statement 1) establishes. □

**Definition 4.3.** For all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, $\mathrm{hre}(\alpha)$ is the natural number defined by equality:

$$\mathrm{hre}(\alpha) = \begin{cases} 0, & \text{if } \mathrm{her}(\alpha) \leq 0, \\ \lfloor log_2(\mathrm{her}(\alpha)) \rfloor + 1, & \text{otherwise.} \end{cases}$$

*Remark* 4.3. Understanding the evaluation of the expressions from a "lazy" point of view, it should be accepted the following equality:

$$\mathrm{hre}(\alpha) = (1 - \chi_{\{-1,0\}}(\mathrm{her}(\alpha)))(\lfloor log_2(\mathrm{her}(\alpha)) \rfloor + 1)$$

where, of course, $\chi_{\{-1,0\}}$ is the characteristic function on the set $\{-1, 0\}$. Let it also be noted that in the case where for the formula $\alpha$ one has $0 < \mathrm{her}(\alpha)$, then $\mathrm{hre}(\alpha)$ is the number of digits in the (single) binary expression of $\mathrm{her}(\alpha)$ when it is greater than 0 and 0 otherwise.

Finally the Corollary 4.14 informs that for any formula $\alpha$ belonging to $\mathrm{K}(\mathrm{A}(X))$, $\mathrm{lasc}^{\mathrm{hre}(\alpha)}(\alpha)$ is a formula in left conjunctive normal form (equivalent to $\alpha$, of course) and that per iteration of lasc, the number $\mathrm{hre}(\alpha)$ of iterations is the smallest number of those that achieve it. This gives us an estimate of the complexity of this part of our algorithm: it is logarithmic.

**Corollary 4.14.** *For all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, $\mathrm{hre}(\alpha)$ is the smallest of the natural numbers $m$ satisfying $\mathrm{lasc}^m(\alpha) \in \mathrm{lcnf}(X)$.*

*Proof.* The proof is by induction on $n$ according to the predicate $Q(n)$ of the literal content:

“for all $\alpha \in \mathrm{K}(\mathrm{A}(X))$, if $n = \mathrm{her}(\alpha)$ then $\mathrm{hre}(\alpha)$ is the smallest natural $m$ fulfilling $\mathrm{lasc}^m(\alpha) \in \mathrm{lcnf}(X)$”

Suppose, as an induction hypothesis, that $n$ is an natural number and that for any natural number $k$ such that $k < n$, $Q(k)$ holds. Now let $\alpha$ be, fixed but arbitrary, such that $\mathrm{her}(\alpha) = n$; we shall distinguish the following cases:

1. $n \leq 0$; by Theorem 4.7 we know that $\alpha \in \mathrm{fncl}(X)$, i.e. $\mathrm{lasc}^0(\alpha) \in \mathrm{fncl}(X)$ since $\mathrm{lasc}^0$ is the identity map. Since 0 is the smallest natural number, the set of natural numbers smaller than it is empty, from which we conclude the assertion.
2. $n > 0$; as we know from Theorem 4.11 must be satisfied:

$$\mathrm{her}(\mathrm{lasc}(\alpha)) = \left\lfloor \frac{\mathrm{her}(\alpha)}{2} \right\rfloor < \mathrm{her}(\alpha) \tag{20}$$

   and since $n > 0$ if follows that $\mathrm{her}(\mathrm{lasc}(\alpha)) < n$. Then:

$$\begin{aligned} \mathrm{lasc}^{\mathrm{hre}(\alpha)}(\alpha) &= \mathrm{lasc}^{\mathrm{hre}(\mathrm{lasc}(\alpha))+1}(\alpha) \\ &= \mathrm{lasc}^{\mathrm{hre}(\mathrm{lasc}(\alpha))}(\mathrm{lasc}(\alpha)) \end{aligned} \tag{21}$$

   From (21), (20) and the induction hypothesis we deduce that $\mathrm{lasc}^{\mathrm{hre}(\alpha)}(\alpha) \in \mathrm{lcnf}(X)$.

Suppose now that $m < \mathrm{hre}(\alpha)$; we will have the following cases:

- $m = 0$; since $0 < \mathrm{her}(\alpha)$, $\mathrm{lasc}^0(\alpha) \equiv \alpha \notin \mathrm{lcnf}(X)$ (cfr. Theorem 4.7).
- $0 < m$; taking into account (20), $\mathrm{hre}(\mathrm{lasc}(\alpha)) = \mathrm{hre}(\alpha) - 1$ will be fulfilled and therefore $m - 1 < \mathrm{hre}(\mathrm{lasc}(\alpha))$. In virtue of this and the induction hypothesis (let (20) be considered again):

$$\mathrm{lasc}^m(\alpha) = \mathrm{lasc}^{m-1}(\mathrm{lasc}(\alpha)) \notin \mathrm{lcnf}(X)$$

By the *second principle of finite induction*, for any natural number $n$, $Q(n)$ is true hence the corollary.

□

# 5 Conclusions

In this article we have developed an algorithm that is correct, complete and efficient; we will explain this.

The notion of *correctness* has to do with the fact that every application of dak (resp. lasc), as detailed in Definition 3.2 (resp. Definition 4.2), provides a (logically) equivalent formula to its argument, equivalence being a transitive relation. This statement rests on Theorem 2.2 (resp. Theorem 2.3) which has been stated and not proved because it is well known in the field of classical logic.

Here the notion of *completeness* means that the iteration in an appropriate number of times of dak (resp. lasc) concludes by providing a formula belonging to $\mathrm{K}(\mathrm{A}(X))$ (resp. $\mathrm{lcnf}(X)$). This has been clearly proved in Corollary 3.7 (resp. Corollary 4.14).

The notion of *efficiency* is also uncontroversial since both Corollary 3.7 and Corollary 4.14 state the minimum number of times the aforementioned functions have to be iterated to obtain their objectives.

Thus, combined dak and lasc and iterated the appropriate number of times as detailed, all this provides an optimal algorithm for the transformation of a formula in the language of this article into an equivalent one in conjunctive normal form; that was the goal and it has been achieved. We have thus contributed to improving the expositions and technical applications as far as we know them.

The content of this article will be used in the future to act in the field of the classical SAT problem, e.g. to optimally prepare the application of the Davis&Putnam algorithm. We intend to compare this algorithm, based on the optimisation of this article and others, with deduction by sequents.

Although both dak and lasc are used in iterations, the definition of both (cfr. Definition 3.2 and Definition 4.2) is clearly based on the recursion. This makes them suitable for use in a functional implementation in Haskell-like languages. In order to show the feasibility and effectiveness of the theoretical advances presented in this article, we have elaborated a functional implementation for the Haskell language of the algorithm developed. We also give a preview of the implementation of the Davis&Putnam algorithm. The code can be consulted at https://github.com/ringstellung/CNF. This justifies the fundamental objective of the work, which is set out in its title.

# References


[1] Bourbaki, N. *Théorie des Ensembles*. Eléments de Mathématique. Hermann, 1977.

[2] Büning, H.K. and Lettmann, T. *Propositional Logic: Deduction and Algorithms*. Cambridge University Press, 1999.

[3] Gill, A. *Applied Algebra for the Computer Sciences*. Prentice-Hall, 1976.

[4] Gorbátov, V.A. *Fundamentos de la Matemática Discreta*. Mir, Moscú, 1988.

[5] Hill, F.J. and Peterson, G.R. *Introduction to Switching Theory and Logical Design*. John Wiley & Sons, 1981.

[6] Jackson, P. and Sheridan, D. Clause form conversions for boolean circuits. In Hoos, H.H. and Mitchell, D.G., editors, *SAT 2004, LNCS 3542*, pages 183–198. Springer–Verlag Berlin Heidelberg, 2005.

[7] Kunen, K. *The Foundations of Mathematics*, volume 19 of *Mathematical Logic and Foundations*. College Publications, 2009.

[8] Łukasiewicz, J. *Elements of Mathematical Logic*. Pergamon Press, 1966.

[9] Tseitin, G.S. On the complexity of derivation in propositional calculus. In Bolç, L., Bundy, A., Siekmann, J., and Sloman, A., editors, *Automation of Reasoning. 2 Classical Papers on Computational Logic 1967–1970*, pages 466–483. Springer-Verlag, 1983.

[10] Whitesitt, J.E. *Boolean Algebra and its Applications*. Addison-Wesley Publishing Company, 1962.