Neural Networks

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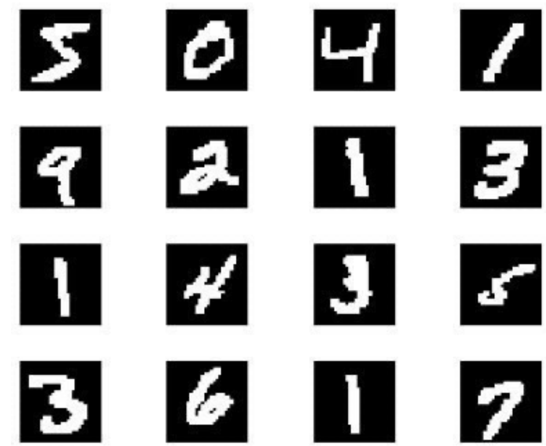
Handwritten Digit Recognition

- Given a collection of handwritten digits and their corresponding labels, we'd like to be able to correctly classify handwritten digits
  - A simple algorithmic technique can solve this problem with 95% accuracy
    - State-of-the-art methods can achieve near 99% accuracy (you've probably seen these in action if you've deposited a check recently)



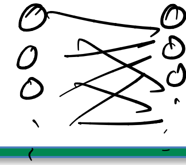
Digits from the MNIST data set

Neural Networks

- The basis of neural networks was developed in the 1940s -1960s
  - The idea was to build mathematical models that might "compute" in the same way that neurons in the brain do
  - As a result, neural networks are biologically inspired, though many of the algorithms developed for them are not biologically plausible

    Learning Algos
  - Perform surprisingly well for the handwritten digit recognition task (and many others)

Neural Networks

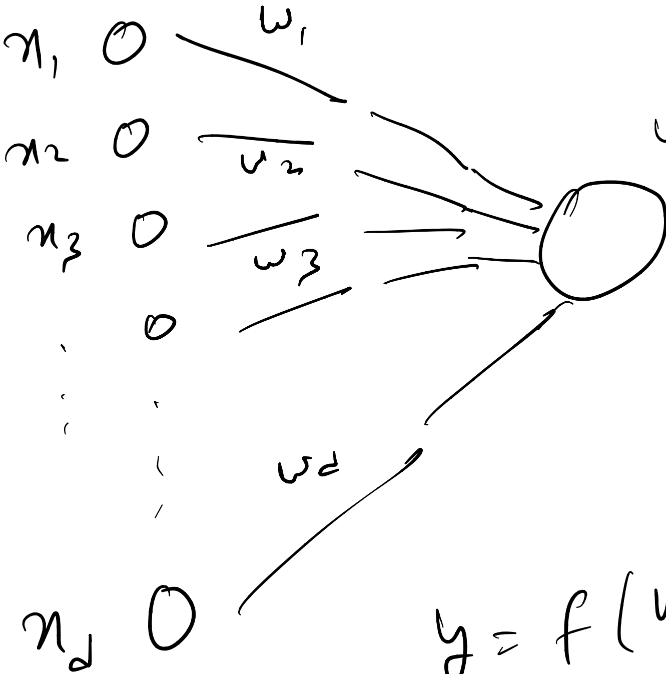


- Neural networks consist of a collection of artificial neurons
- There are different types of neuron models that are commonly studied

  sign.

  - The perceptron (one of the first studied)
  - The sigmoid neuron (one of the most common, but many more)
  - Rectified linear units ← $\max(x, 0)$
- A neural network is a directed graph consisting of a collection of neurons (the nodes), directed edges (each with an associated weight), and a collection of fixed binary inputs

Perceptrons / Logistic Regression



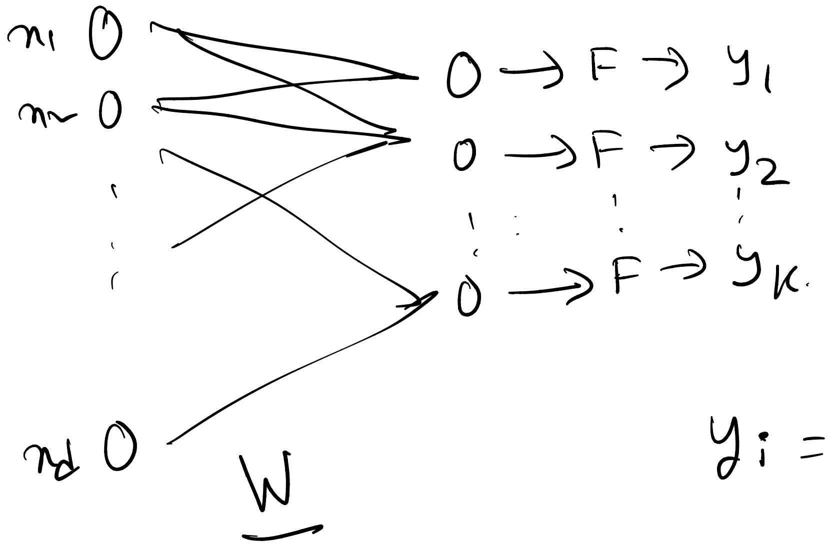
$y = \underline{\text{Sign}}(w^T x + b)$

$\sigma(w^T x + b)$

$\uparrow$

$p(y = 1 \mid x, w, b)$

$y = f(w^T x + b)$



$$y_i = F(w_i^T x + b_i)$$

Non-Linearity

Sigmoid

Sign

⋮

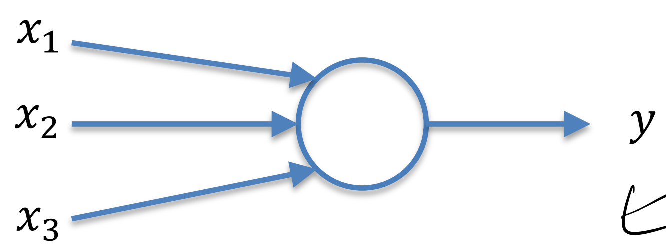
$$\underset{K\times 1}{y} = F(\underset{K\times d}{W^T}\, \underset{d\times 1}{x} + b)$$

The Perceptron

- A perceptron is an artificial neuron that takes a collection of binary inputs and produces a binary output

  ↑ Not necessary

  - The output of the perceptron is determined by summing up the weighted inputs and thresholding the result: if the weighted sum is larger than the threshold, the output is one (and zero otherwise)



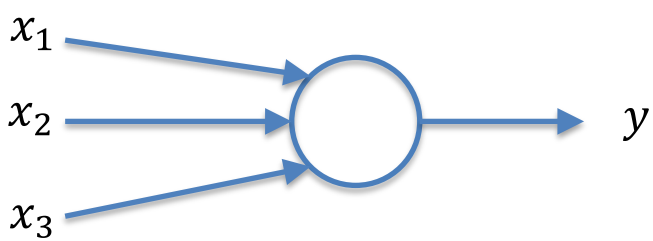
$w^T x + b > 0$

$$y = \begin{cases} 1 & w_1 x_1 + w_2 x_2 + w_3 x_3 > threshold \\ 0 & otherwise \end{cases}$$

$\mathbb{1}(w^T x + b > 0)$

Perceptrons

- Perceptrons are usually expressed in terms of a collection of input weights and a bias b (which is the negative threshold)

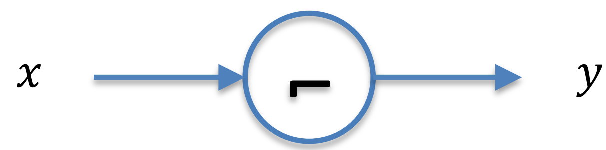


$$y = \begin{cases} 1 & w_1x_1 + w_2x_2 + w_3x_3 + b > 0 \\ 0 & otherwise \end{cases}$$

- A single node perceptron is just a linear classifier

- This is actually where the "perceptron algorithm" comes from

Perceptron for NOT



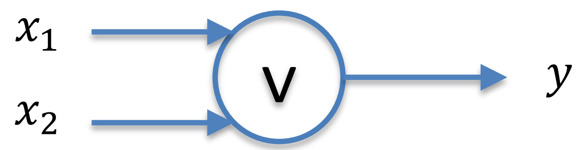
- Choose $w = -1$, threshold $= -.5$

- $y = \begin{cases} 1 & -x > -.5 \\ 0 & -x \leq -.5 \end{cases}$

Not Function = Linearly Sep.

Perceptron for OR

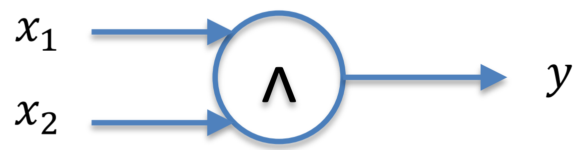
Perceptron for OR



- Choose $w_1 = w_2 = 1$, threshold $= 0$
- $y = \begin{cases} 1 & x_1 + x_2 > 0 \\ 0 & x_1 + x_2 \leq 0 \end{cases}$

Perceptron for AND

Perceptron for AND

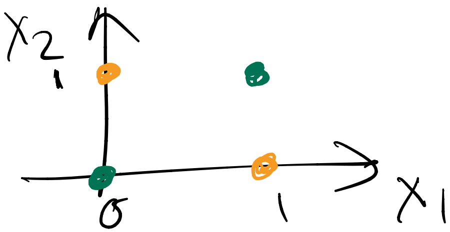


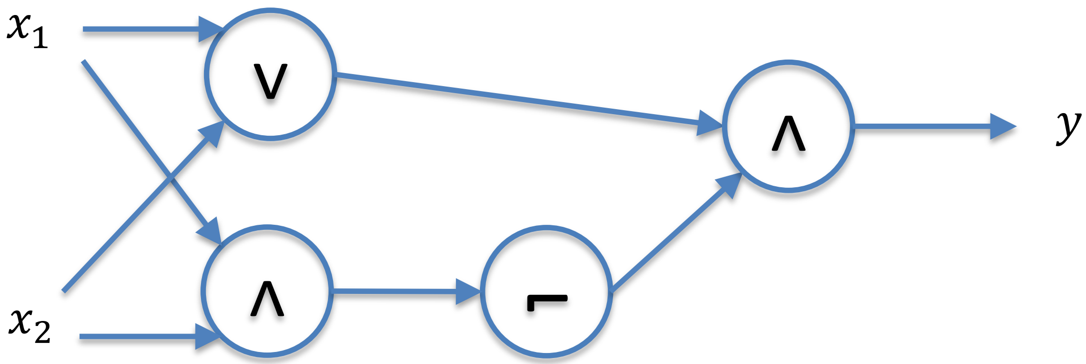
- Choose $w_1 = w_2 = 1$, threshold $= 1.5$
- $y = \begin{cases} 1 & x_1 + x_2 > 1.5 \\ 0 & x_1 + x_2 \leq 1.5 \end{cases}$

Perceptron for XOR

Perceptron for XOR

- Need more than one perceptron!





- Weights for incoming edges are chosen as before

- Networks of perceptrons can encode any circuit!

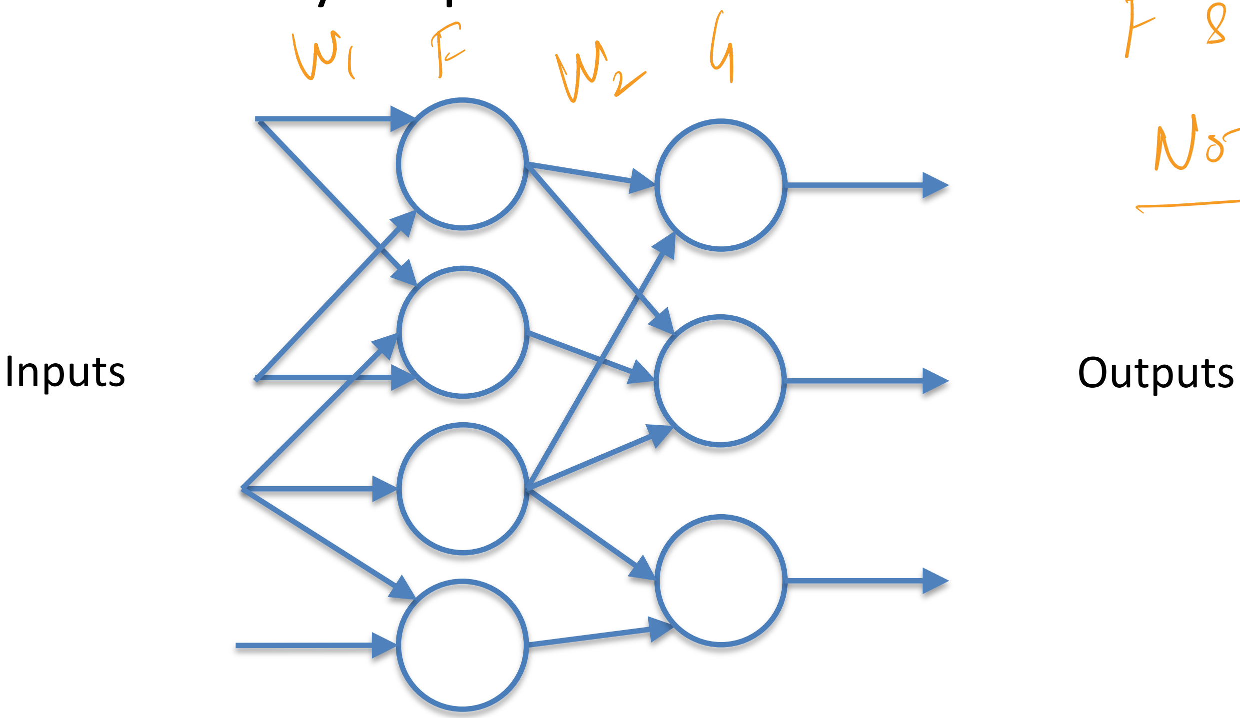
Neural Networks

- Gluing a bunch of perceptrons together gives us a neural network

- In general, neural nets have a collection of binary inputs and a collection of binary outputs

F & G & ...

NOT LEARNT

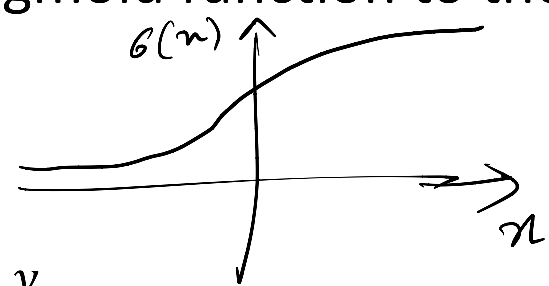


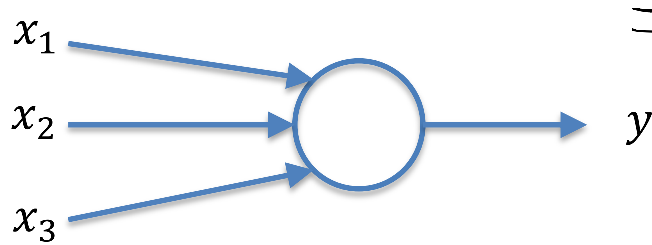
Beyond Perceptrons

- Given a collection of input-output pairs, we'd like to learn the weights of the neural network so that we can correctly predict the output of an unseen input
  - We could try learning via gradient descent (e.g., by minimizing the Hamming loss) 0/1 loss
    - This approach doesn't work so well: small changes in the weights can cause dramatic changes in the output
    - This is a consequence of the discontinuity of sharp thresholding (same problem we saw with perceptron alg.)

The Sigmoid Neuron

- A sigmoid neuron is an artificial neuron that takes a collection of real inputs and produces an output in the interval [0,1]
  - The output is determined by summing up the weighted inputs plus the bias and applying the sigmoid function to the result





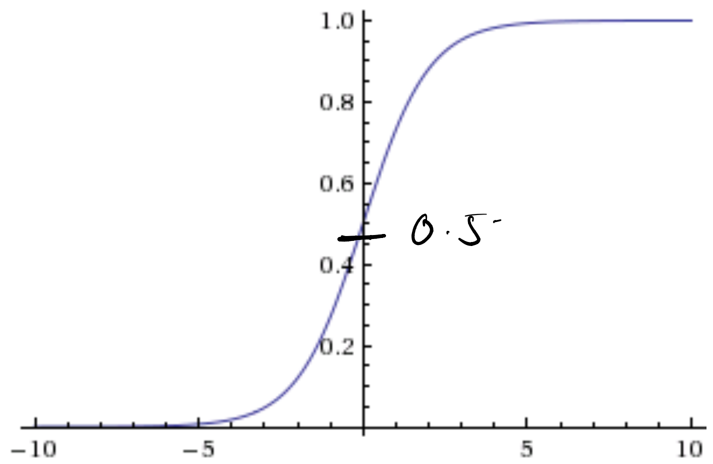
$$y = \sigma(\underbrace{w_1 x_1 + w_2 x_2 + w_3 x_3 + b})$$

where σ is the sigmoid function

The Sigmoid Function

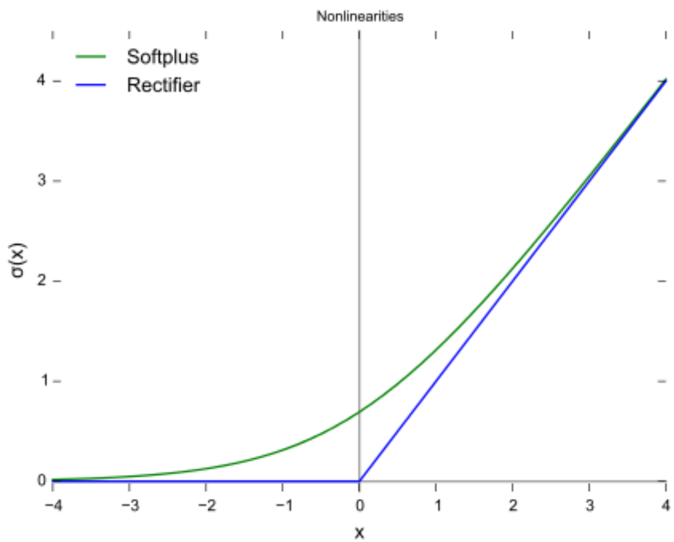
- The sigmoid function is a continuous function that approximates a step function

$$\sigma(z) = \frac{1}{1+e^{-z}}$$



Rectified Linear Units

- The sigmoid neuron approximates a step function as a smooth function
- The relu is given by $\max(0, x)$ which can be approximated as a smooth continuous function $\ln(1 + e^x)$



Softmax

- The softmax function maps a vector of real numbers to a vector of probabilities as

$$\text{softmax}(z)_j = \frac{e^{z_j}}{\sum_k e^{z_k}}$$

- If there is a dominant value in z, then it will become one under the softmax

- Often used as the final layer of a neural network

Multilayer Neural Networks

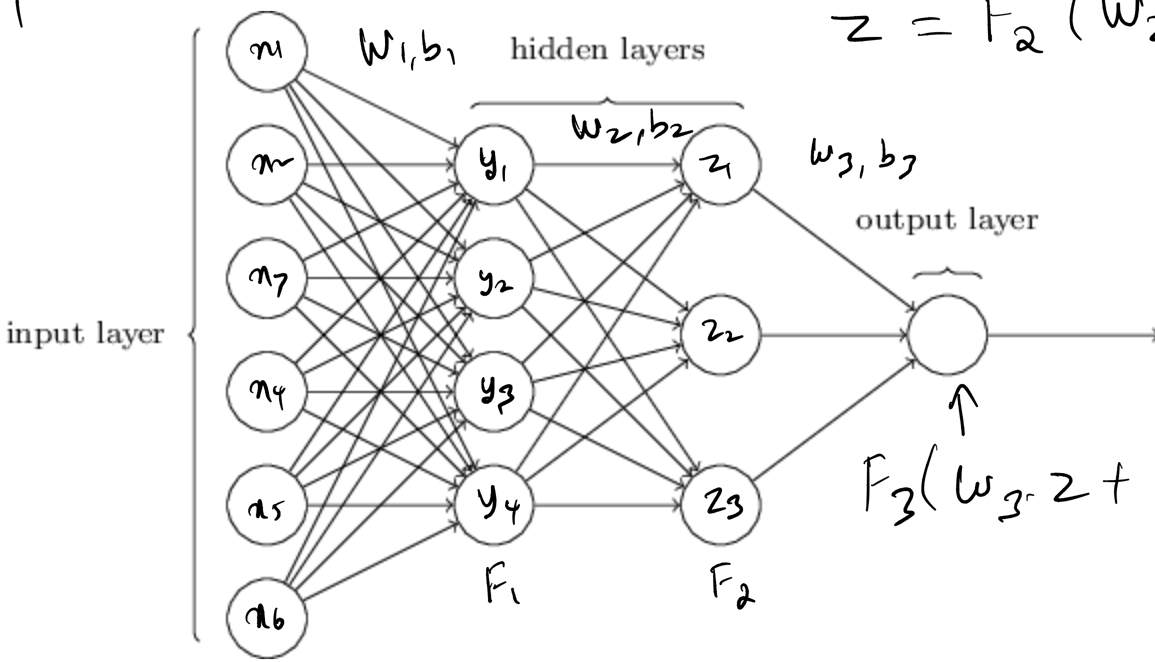
$x \in 6 \times 1$ vector

$W \in 4 \times 6$ Matrix

$y \in 4 \times 1$

$y = F_1(W_1 \cdot x + b_1)$

$z = F_2(W_2 y + b_2)$

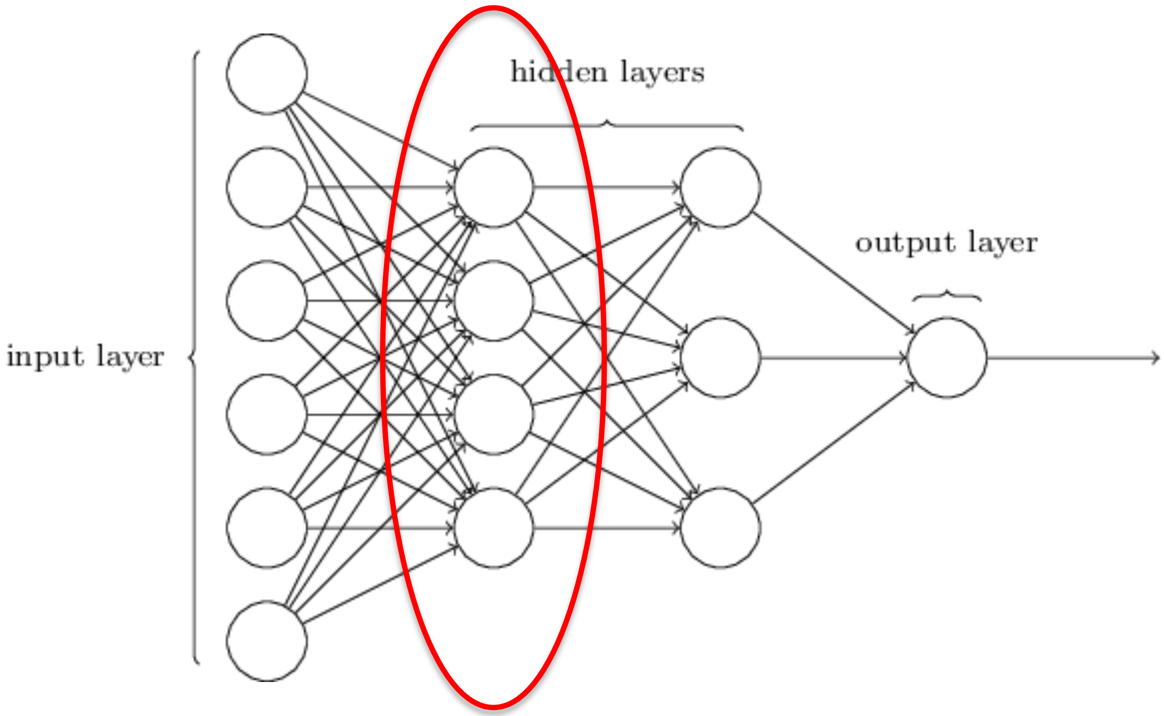


$F_3(W_3 \cdot z + b_3)$

from Neural Networks and Deep Learning by Michael Nielson

Multilayer Neural Networks

NO intralayer connections



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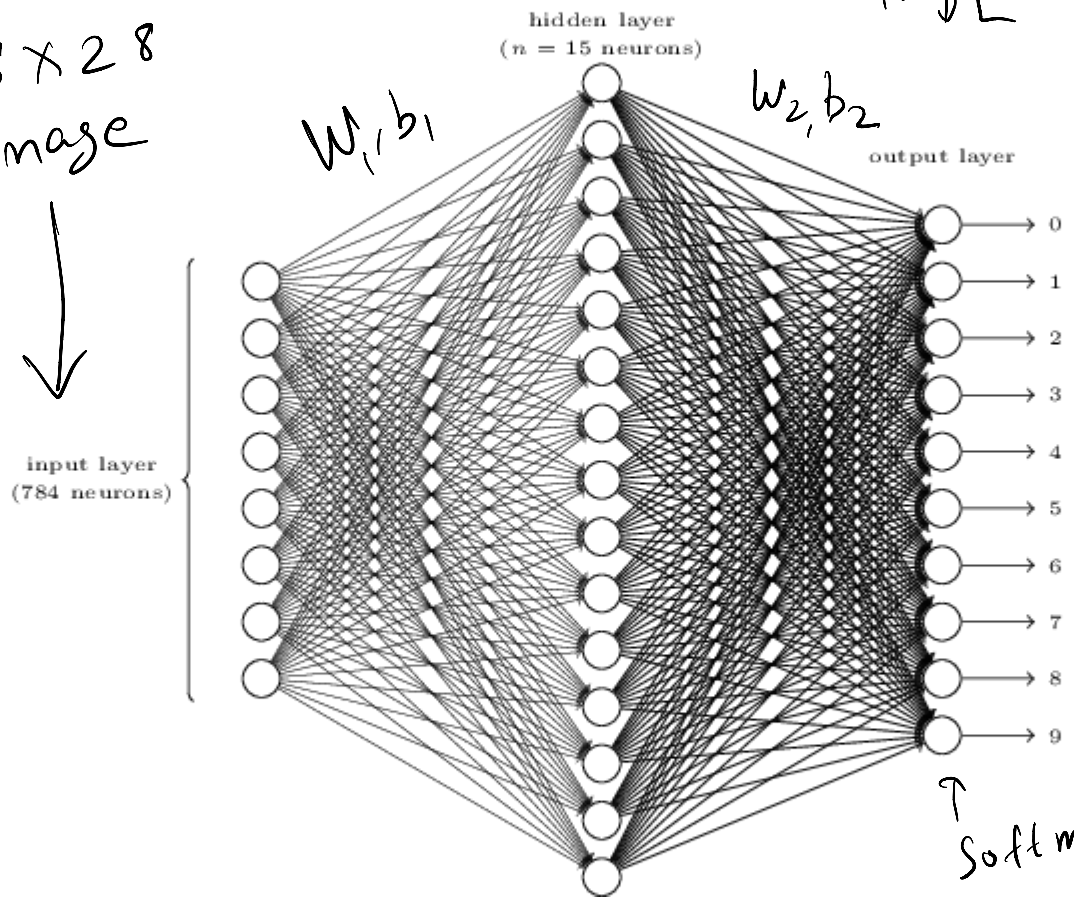
Neural Network for Digit Classification

28 x 28 image



W_1, b_1

W_2, b_2

Softmax

784

15

W_1

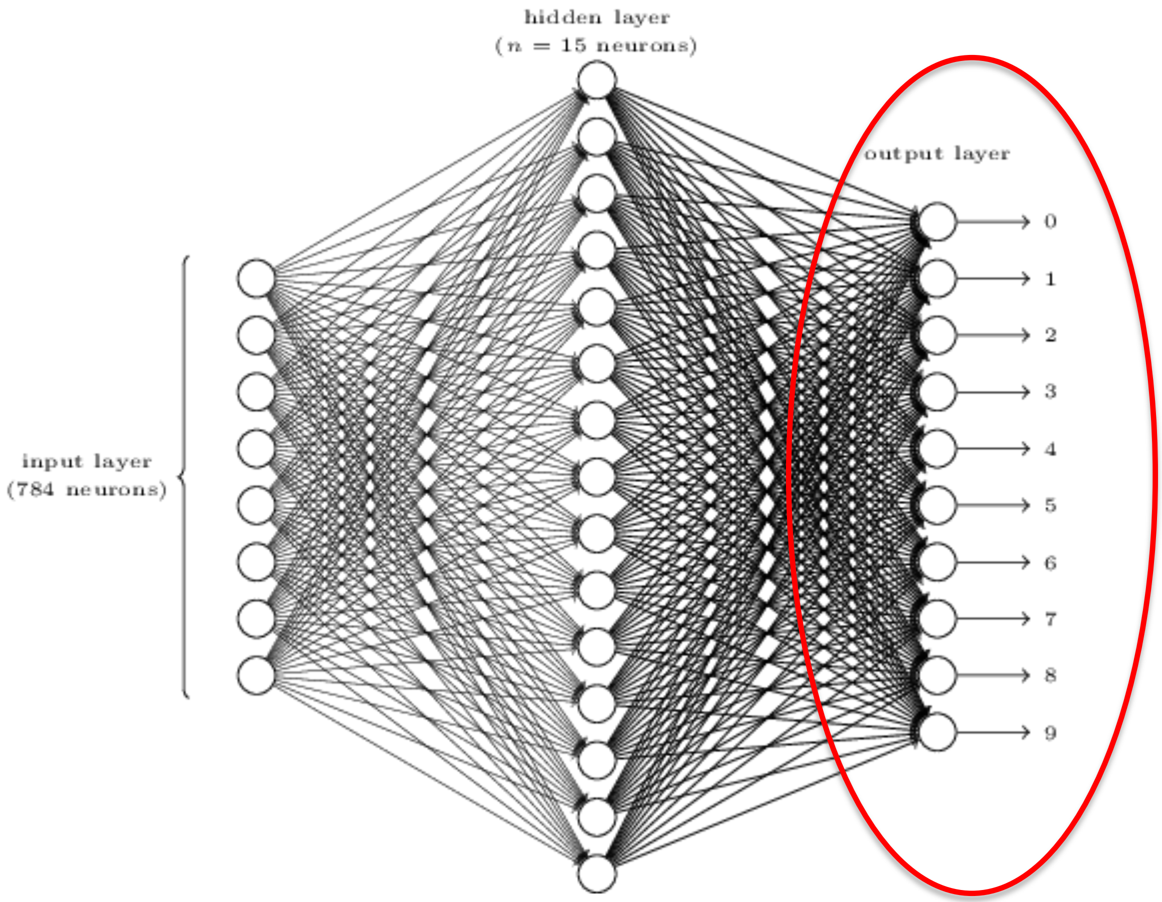
784

x

$y = F_1(W_1 x + b_1)$

from Neural Networks and Deep Learning by Michael Nielson

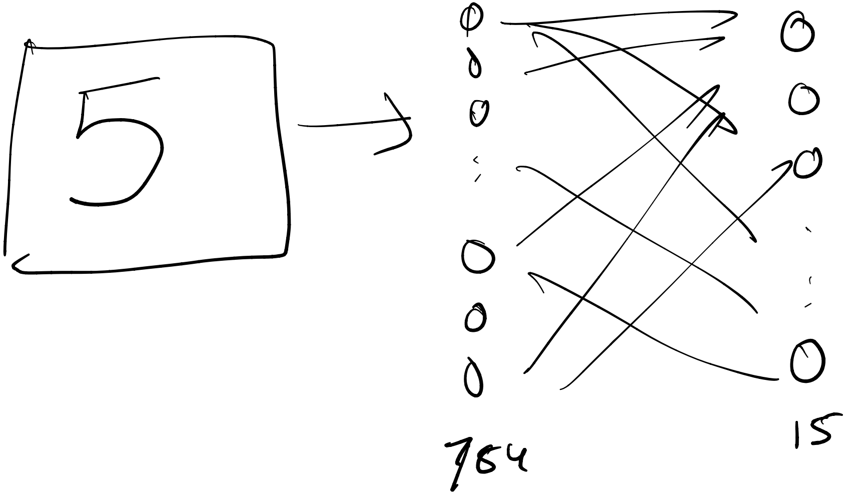
Neural Network for Digit Classification



10 classes
for 10
Digits
0 → 9.

Why 10
instead of 4?

from Neural Networks and Deep Learning by Michael Nielson



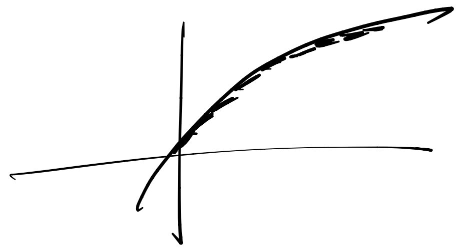
$$w_i \in \mathbb{R}^{784}$$

$$W = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{15} \end{bmatrix}$$

Expressiveness of NNs

ε $O\left(\frac{1}{\varepsilon}\right)$

- Boolean functions
  - Every Boolean function can be represented by a network with a single hidden layer consisting of possibly exponentially many hidden units



- Continuous functions
  - Every bounded continuous function can be approximated up to arbitrarily small error by a network with one hidden layer
  - Any function can be approximated to arbitrary accuracy with two hidden layers

Expressiveness of NNs

- **Theorem [Zhang et al. 2016]:** There exists a two-layer neural network with ReLU activations and $2n + d$ weights that can represent any function on a sample of size n in d dimensions

  - This should mean that it is very easy to overfit with neural networks

    Fit very well

  - Generalization performance of networks is difficult to assess theoretically

    Resillient to overfitting

Training Neural Networks

- To do the learning, we first need to define a loss function to minimize

$$C(w,b) = \frac{1}{2M}\sum_{m} \|y^m - a(x^m, w, b)\|^2$$

- The training data consists of input output pairs $(x^1, y^1), \dots, (x^M, y^M)$
- $a(x^m, w, b)$ is the output of the neural network for the m^{th} sample
- w and b are the weights and biases

Gradient of the Loss

- The derivative of the loss function is calculated as follows

$$\frac{\partial C(w,b)}{\partial w_k} = \frac{1}{M}\sum_m [y^m - a(x^m,w,b)]\frac{\partial a(x^m,w,b)}{\partial w_k}$$

  - To compute the derivative of a, use the chain rule and the derivative of the sigmoid function

$\sigma(z) = \frac{1}{1+e^{-z}}$

$$\frac{d\sigma(z)}{dz} = \sigma(z)\cdot(1-\sigma(z))$$

  - This gets complicated quickly with lots of layers of neurons

Stochastic Gradient Descent

- To make the training more practical, stochastic gradient descent is used instead of standard gradient descent

- Recall, the idea of stochastic gradient descent is to approximate the gradient of a sum by sampling a few indices and averaging

$$\nabla_x \sum_{i=1}^{n} f_i(x) \approx \frac{1}{K} \sum_{k=1}^{K} \nabla_x f_{i^k}(x)$$

here, for example, each i^k is sampled uniformly at random from $\{1, \dots, n\}$

Computing the Gradient

- We'll compute the gradient for a single sample

$$C(w,b) = \frac{1}{2}\|y - a(x,w,b)\|^2$$

- Some definitions:

$a_j^\ell = \sigma(z_j^\ell)$

  - L is the number of layers
  - a_j^l is the output of the j^{th} neuron on the l^{th} layer
  - z_j^l is the weighted input of the j^{th} neuron on the l^{th} layer

$$z_j^l = \sum_k w_{jk}^l a_k^{l-1} + b_j^l$$

  - δ_j^l is defined to be $\frac{\partial C}{\partial z_j^l}$

Computing the Gradient

For the output layer, we have the following partial derivative

Last Layer

$$\frac{\partial C}{\partial z_j^L} = -(y_j - a_j^L)\frac{\partial a_j^L}{\partial z_j^L}$$

$$= -(y_j - a_j^L)\frac{\partial \sigma(z_j^L)}{\partial z_j^L}$$

$$= -(y_j - a_j^L)\,\sigma(z_j^L)\left(1 - \sigma(z_j^L)\right)$$

$$= \delta_j^L$$

C = Loss Function

Square Loss

L = # Layers

$a_j^L = \sigma(z_j^L)$

- For simplicity, we will denote the vector of all such partials for each node in the l^{th} layer as δ^l

Computing the Gradient

For the $L-1$ layer, we have the following partial derivative

$$\frac{\partial C}{\partial z_k^{L-1}} = \sum_j (a_j^L - y_j) \frac{\partial a_j^L}{\partial z_k^{L-1}}$$

$$= \sum_j (a_j^L - y_j) \boxed{\frac{\partial \sigma(z_j^L)}{\partial z_k^{L-1}}}$$

$$= \sum_j (a_j^L - y_j)\, \sigma(z_j^L)\left(1 - \sigma(z_j^L)\right) \frac{\partial z_j^L}{\partial z_k^{L-1}}$$

$$= \sum_j (a_j^L - y_j)\, \sigma(z_j^L)\left(1 - \sigma(z_j^L)\right) \frac{\partial \sum_{k'} w_{jk'}^L a_{k'}^{L-1} + b_j^L}{\partial z_k^{L-1}}$$

$$= \sum_j \boxed{(a_j^L - y_j)\, \sigma(z_j^L)\left(1 - \sigma(z_j^L)\right)}\, \sigma(z_k^{L-1})\left(1 - \sigma(z_k^{L-1})\right) w_{jk}^L$$

$$= \left((\delta^L)^T w_{*k}^L\right)\left(1 - \sigma(z_k^{L-1})\right)\sigma(z_k^{L-1})$$

δ_k^{L-1}

$$C(w,b) = \|y - a(x,w,b)\|^2 = \sum_j \|y_j - a_j^L(x,w,b)\|^2$$

$$\frac{\partial \sigma(z_j^L)}{\partial z_j^L} \cdot \frac{\partial z_j^L}{\partial z_k^{L-1}}$$

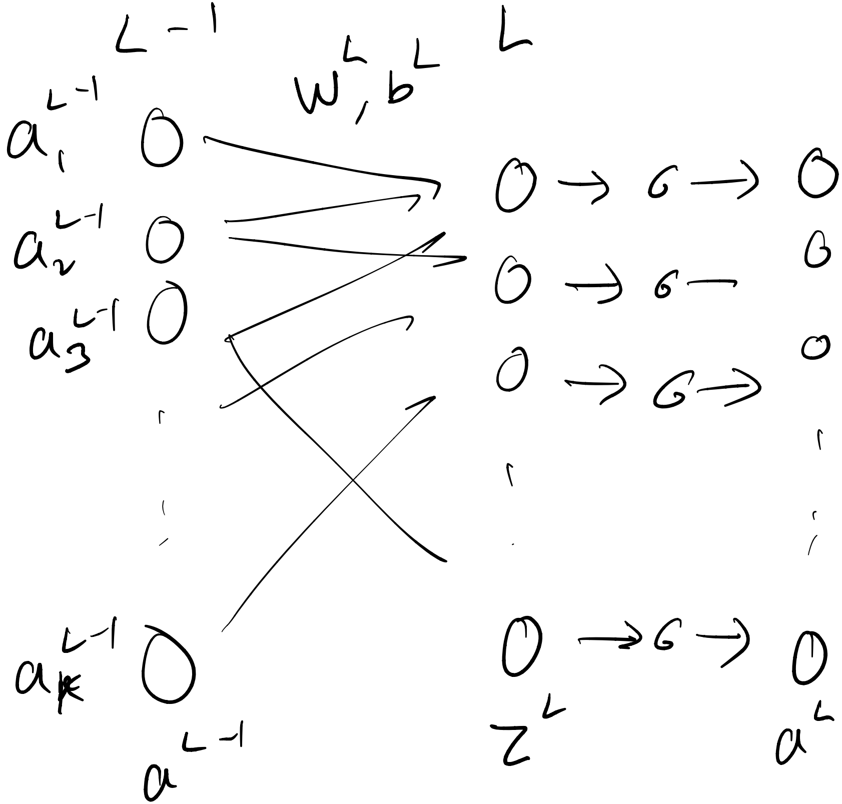
$$a_j^L = \sigma(z_j^L)$$

$$a_{k'}^{L-1} = \sigma(z_{k'}^{L-1})$$

$$\frac{\partial (w_{jk}^L a_k^{L-1} + b_j^L)}{\partial z_k^{L-1}}$$

0

$$\frac{\partial a_k^{L-1}}{\partial z_k^{L-1}}$$


$L-1$
L
W^L, b^L
a_1^{L-1}
a_2^{L-1}
a_3^{L-1}
a_k^{L-1}
a^{L-1}
z^L
a^L

Computing the Gradient

- We can think of w^l as a matrix
- This allows us to write

$$\delta^{L-1} = \left((\delta^L)^T w^L\right)^T \circ \left(1 - \sigma(z^{L-1})\right) \circ \sigma(z^{L-1})$$

where $\sigma(z^{L-1})$ is the vector whose k^{th} component is $\sigma\left(z_k^{L-1}\right)$

- Applying the same strategy, for $\underline{l < L}$

$$\delta^l = \left(\left(\delta^{l+1}\right)^T w^{l+1}\right)^T \circ \left(1 - \sigma\left(z^l\right)\right) \circ \sigma\left(z^l\right)$$

$$\delta^L \longrightarrow \delta^{L-1} \longrightarrow \delta^{L-2} \longrightarrow \cdots \longrightarrow \delta^1$$

Computing the Gradient

- Now, for the partial derivatives that we care about

$$\frac{\partial C}{\partial b_j^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial b_j^l} = \delta_j^l$$

$$\frac{\partial C}{\partial w_{jk}^l} = \frac{\partial C}{\partial z_j^l} \cdot \frac{\partial z_j^l}{\partial w_{jk}^l} = \delta_j^l a_k^{l-1}$$

- We can compute these derivatives one layer at a time!

Backpropagation

- Compute the inputs/outputs for each layer by starting at the input layer and applying the sigmoid functions
- Compute δ^L for the output layer

$$\delta_j^L = -(y_j - a_j^L)\,\sigma(z_j^L)\left(1 - \sigma(z_j^L)\right)$$

- Starting from $l = L - 1$ and working backwards, compute

$$\delta^l = \left(\left(\delta^{l+1}\right)^T w^{l+1}\right)^T \circ \sigma(z^l) \circ \left(1 - \sigma(z^l)\right)$$

- Perform gradient descent

$$b_j^l = b_j^l - \gamma \cdot \delta_j^l$$

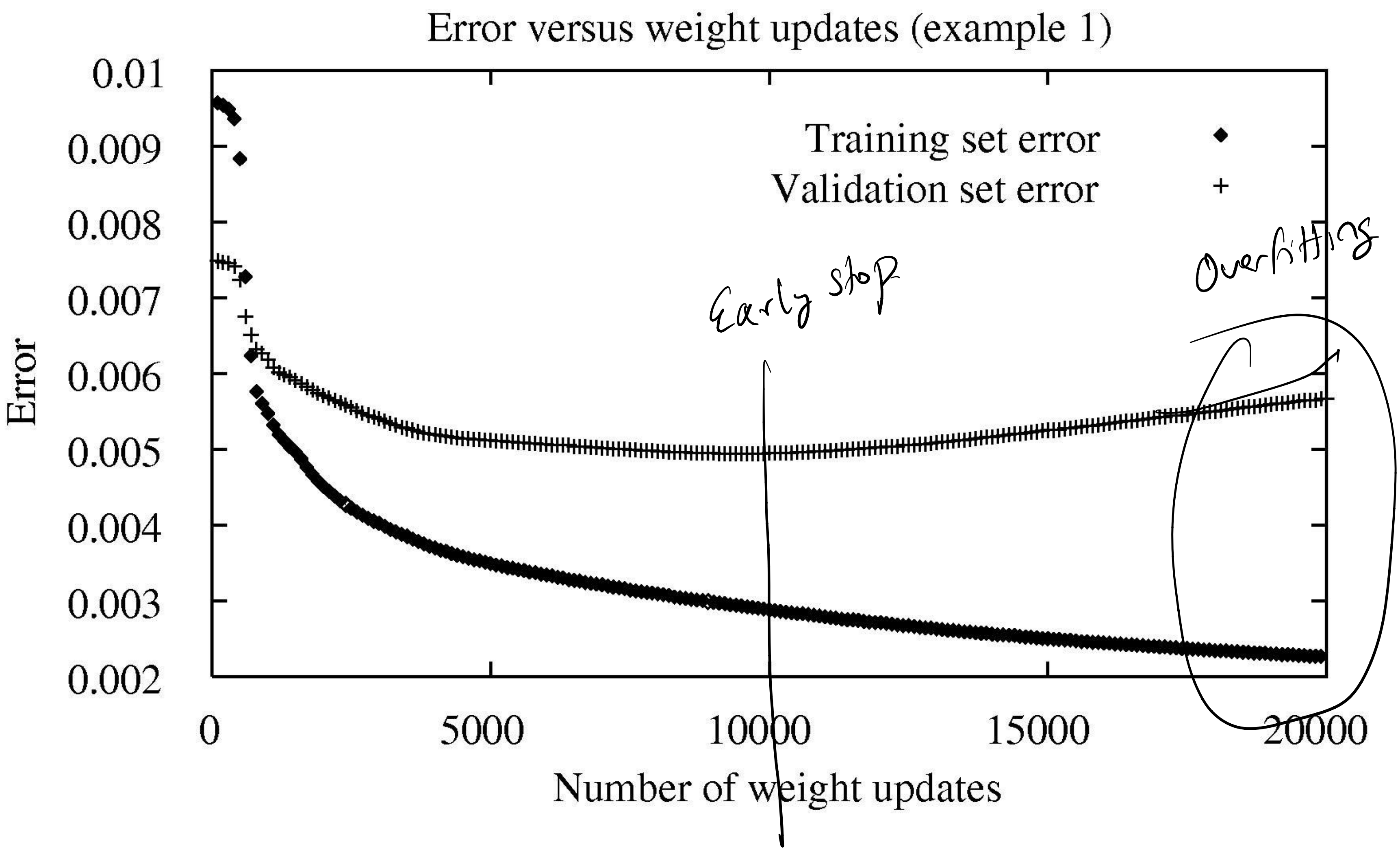
$$w_{jk}^l = w_{jk}^l - \gamma \cdot \delta_j^l a_k^{l-1}$$

Backpropagation

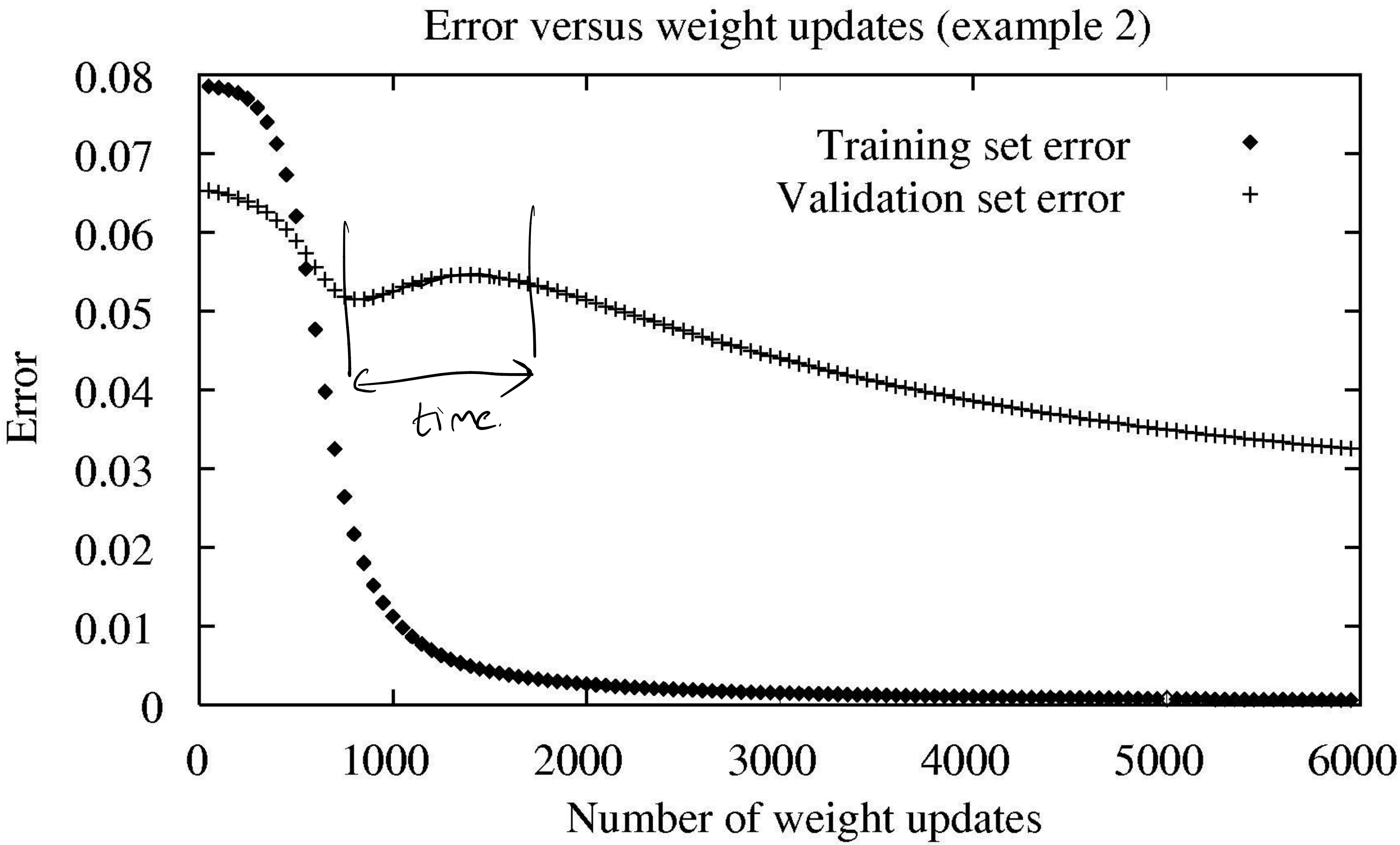
- Backpropagation converges to a local minimum (loss is not convex in the weights and biases)
  - k means
  - Like EM, can just run it several times with different initializations
  - Training can take a very long time (even with stochastic gradient descent)
  - GPU
  - Prediction after learning is fast
  - Sometimes include a momentum term α in the gradient update

$$w(t) = w(t-1) - \gamma \cdot \nabla_w C(t-1) + \alpha(-\gamma \cdot \nabla_w C(t-2))$$

Overfitting



Overfitting



Neural Networks in Practice

- Many ways to improve weight learning in NNs
  - Use a regularizer! (better generalization?)
  - Try other loss functions, e.g., the cross entropy

    $$-y \log a(x, w, b) - (1 - y) \log(1 - a(x, w, b))$$

  - Initialize the weights of the network more cleverly
    - Random initializations are likely to be far from optimal
- The learning procedure can have numerical difficulties if there are a large number of layers

Regularized Loss

- Penalize learning large weights

$$C'^{(w,b)} = \frac{1}{2M} \sum_m \|y^m - a(x^m, w, b)\|^2 + \frac{\lambda}{2} \|w\|_2^2$$

- Can still use the backpropagation algorithm in this setting
- ℓ_1 regularization can also be useful
- Regularization can help with convergence, λ should be chosen with a validation set

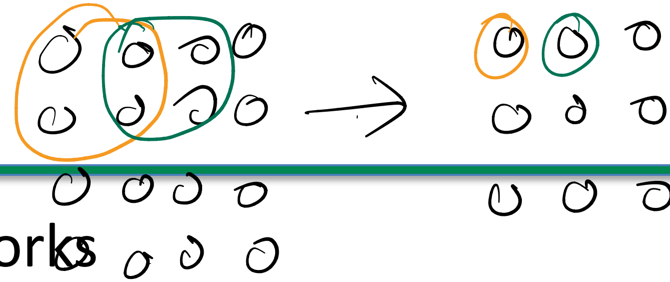
Dropout

- A heuristic bagging-style approach applied to neural networks to counteract overfitting
  - Randomly remove a certain percentage of the neurons from the network and then train only on the remaining neurons
  - The networks are recombined using an approximate averaging technique (keeping around too many networks and doing proper bagging can be costly in practice)

Other Techniques

- Early stopping
  - Stop the learning early in the hopes that this prevents overfitting
- Parameter tying
  - Assume some of the weights in the model are the same to reduce the dimensionality of the learning problem
  - Also a way to learn “simpler” models
  - Can lead to significant compression in neural networks (i.e., >90%)

Other Ideas



- Convolutional neural networks
  - Instead of the output of every neuron at layer l being used as an input to every neuron at layer $l+1$, the edges between layers are chosen more locally
  - Many tied weights and biases (i.e., convolution nets apply the same process to many different local chunks of neurons)
  - Often combined with pooling layers (i.e., layers that, say, half the number of neurons by replacing small regions of neurons with their maximum output)
  - Used extensively for image classification tasks