
# Bayesian Methods

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based on the slides of Vibhav Gogate and Nick Rouzzi

# Binary Variables

$$E[X] = \sum_x x \cdot p(X = x) = 0 \cdot (1-\mu) + 1 \cdot \mu = \mu$$

- Coin flipping: heads=1, tails=0 with bias $\mu$ [prob. of Heads]

$$p(X = 1|\mu) = \mu$$

- Bernoulli Distribution

$$Bern(x|\mu) = \mu^x \cdot (1-\mu)^{1-x}$$

$$E[X] = \mu$$

$$var(X) = \mu \cdot (1-\mu)$$

$$Bern(x = 1|\mu) = \mu$$

$$Bern(x = 0|\mu) = 1 - \mu$$

# Variance

$$\text{Var}(X) = \sum_x (x-\mu)^2 P(X=x)$$

$$= E[X-\mu]^2 \rightarrow \mu = E[X]$$

$$= E[X^2] - 2\mu \underbrace{E[X]}_{\mu} + \mu^2$$

$$= E[X^2] - \mu^2 = E[X^2] - (E[X])^2$$

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Coin Flip case:

$$E[X^2] = \mu \cdot 1^2 + (1-\mu)\cdot 0^2 = \mu$$

$$(E[X])^2 = \mu^2$$

$$\Rightarrow \text{Var}(X) = \mu - \mu^2 = \mu(1-\mu)$$

# Binary Variables

- $N$ coin flips: $X_1, \ldots, X_N$ IID Coin Flips → Independent Identically Distributed

$$p(\textstyle\sum_i X_i = m | N, \mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}$$

- Binomial Distribution

$$Bin(m|N,\mu) = \binom{N}{m} \mu^m (1-\mu)^{N-m}$$

HHTTT
HTHTT
⋮
$\binom{5}{2}$

$$E\left[\sum_i X_i\right] = \sum_i E[X_i]$$

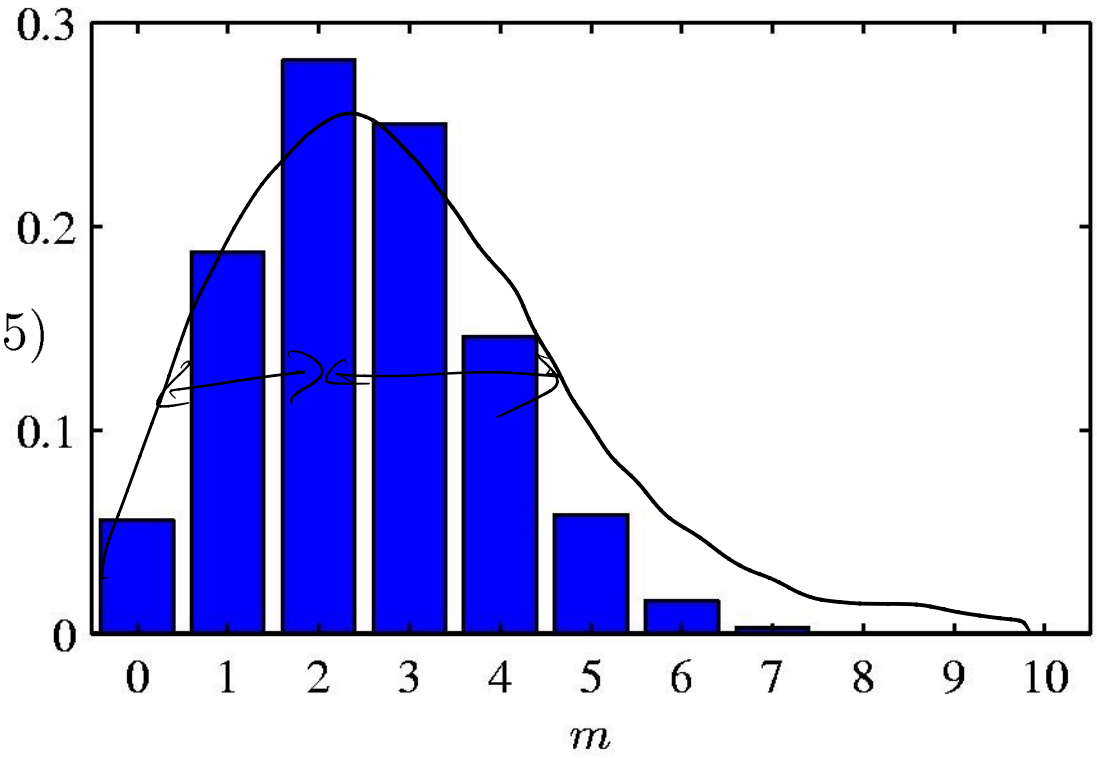
$$E\left[\sum_i X_i\right] = N\mu$$

$$var\left[\sum_i X_i\right] = N\mu(1-\mu)$$

$$\sum_m \binom{N}{m} \mu^m (1-\mu)^{N-m} = (\mu + 1 - \mu)^N = 1.$$

# Binomial Distribution

$\text{Bin}(m|10, 0.25)$




$E[X] = 2.5$

# Estimating the Bias of a Coin ($\mu$)

- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
  - How should we estimate the bias?

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# Estimating the Bias of a Coin

- Suppose that we have a coin, and we would like to figure out what the probability is that it will flip up heads
  - How should we estimate the bias?


  - With these coin flips, our estimate of the bias is: 3/5
    - Why is this a good estimate?

# Coin Flipping – Binomial Distribution


- $P(Heads) = \theta,\ P(Tails) = 1 - \theta$
- Flips are i.i.d.
  - Independent events
  - Identically distributed according to Binomial distribution
- Our training data consists of $\alpha_H$ heads and $\alpha_T$ tails

$$p(D|\theta) = \theta^{\alpha_H} \cdot (1-\theta)^{\alpha_T}$$

$$p(D|\theta) = \prod_{i=1}^{|D|} p(D_i|\theta) \quad \text{--- Independence}$$

$$= \theta^{\alpha_H} (1-\theta)^{\alpha_T}$$

$$p(D_i|\theta) = \theta^{I(D_i=H)} (1-\theta)^{I(D_i=T)}$$

# Maximum Likelihood Estimation (MLE)

- **Data:** Observed set of $\alpha_H$ heads and $\alpha_T$ tails
- **Hypothesis:** Coin flips follow a Bernoulli distribution
- **Learning:** Find the "best" $\theta$
- **MLE:** Choose $\theta$ to maximize probability of $D$ given $\theta$

$$\begin{aligned} \hat{\theta} &= \arg\max_{\theta} P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \ln P(\mathcal{D} \mid \theta) \end{aligned}$$

# First Parameter Learning Algorithm

$$\begin{aligned}\hat{\theta} &= \arg\max_{\theta} \ln P(\mathcal{D} \mid \theta) \\ &= \arg\max_{\theta} \ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\end{aligned}$$

Set derivative to zero, and solve!

$$\begin{aligned}\frac{d}{d\theta}\ln P(\mathcal{D} \mid \theta) &= \frac{d}{d\theta}\left[\ln \theta^{\alpha_H}(1-\theta)^{\alpha_T}\right] \\ &= \frac{d}{d\theta}\left[\alpha_H \ln\theta + \alpha_T \ln(1-\theta)\right] \\ &= \alpha_H \frac{d}{d\theta}\ln\theta + \alpha_T \frac{d}{d\theta}\ln(1-\theta) \\ &= \frac{\alpha_H}{\theta} - \frac{\alpha_T}{1-\theta} = 0\end{aligned}$$

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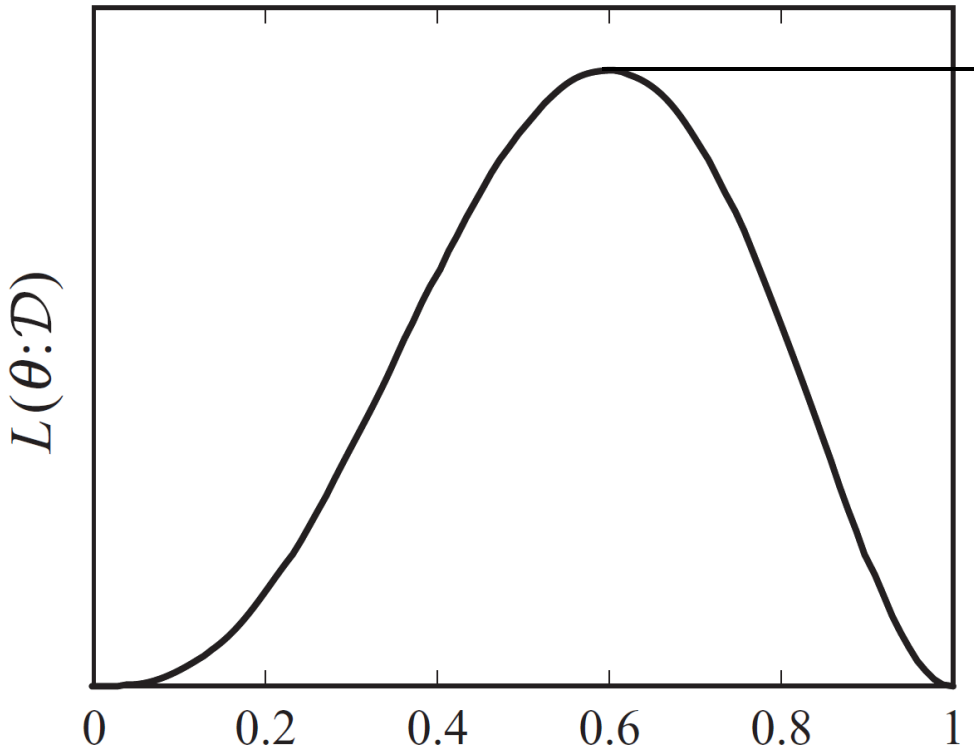
$$\widehat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

# Coin Flip MLE


$$\hat{\theta}_{MLE} = \frac{\alpha_H}{\alpha_H + \alpha_T}$$

# MLE of Dice Roll

$D_1, D_2, \ldots, D_N$

$D_i \in [1, 6]$

$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, 1 - \sum_{i=1}^{5} \theta_i$ ← parameters

$$p(D|\theta) = \prod_{i=1}^{N} \theta_1^{1(D_i=1)} \theta_2^{1(D_i=2)} \ldots$$

$$= \theta_1^{\#1} \theta_2^{\#2} \ldots \left(1 - \sum_{i=1}^{5} \theta_i\right)^{\#6}$$

$$\theta_1 = \frac{\#1}{N}, \theta_2 = \frac{\#2}{N}, \ldots, \theta_6 = \frac{\#6}{N}$$

# Priors


- Suppose we have 5 coin flips all of which are heads
  - Our estimate of the bias is?

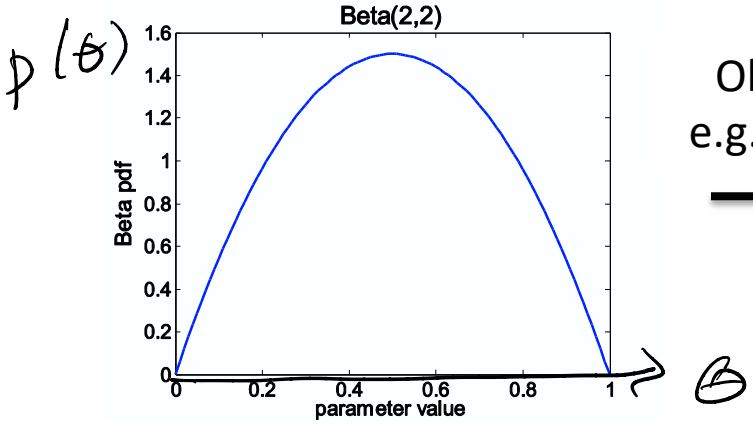
# Priors


- Suppose we have 5 coin flips all of which are heads
  - MLE would give $\theta_{MLE} = 1$
  - This event occurs with probability $\frac{1}{2^5} = \frac{1}{32}$ for a fair coin
  - Are we willing to commit to such a strong conclusion with such little evidence?

# Priors

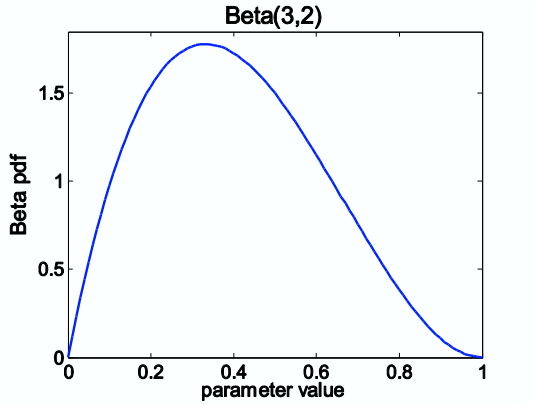
- Priors are a Bayesian mechanism that allow us to take into account "prior" knowledge about our belief in the outcome
- Rather than estimating a single $\theta$, consider a distribution over possible values of $\theta$ given the data
  - Update our prior after seeing data

Our best guess in the absence of any data




Observe flips
e.g.: {tails, tails}

Our estimate after we see some data

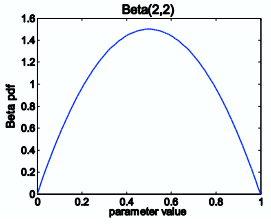


# Bayesian Learning

Apply Bayes rule:

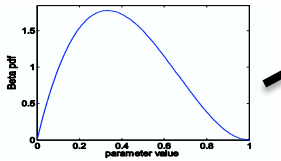
Data Likelihood

Prior




$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

Posterior




Normalization

- Or equivalently: $p(\theta|D) \propto p(D|\theta)p(\theta)$
- For uniform priors this reduces to the MLE objective

$$p(\theta) \propto 1 \quad \Rightarrow \quad p(\theta|D) \propto p(D|\theta)$$

# Picking Priors

- How do we pick a good prior distribution?
  - Could represent expert domain knowledge
  - Statisticians choose them to make the posterior distribution “nice” (conjugate priors)
- What is a good prior for the bias in the coin flipping problem?

$p(D|\theta)$

# Picking Priors

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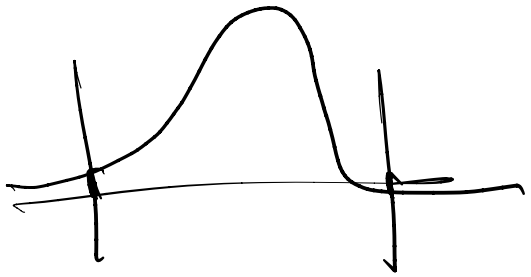
- What is a good prior for the bias in the coin flipping problem?
  - Truncated Gaussian (tough to work with)
  - Beta distribution (works well for binary random variables)



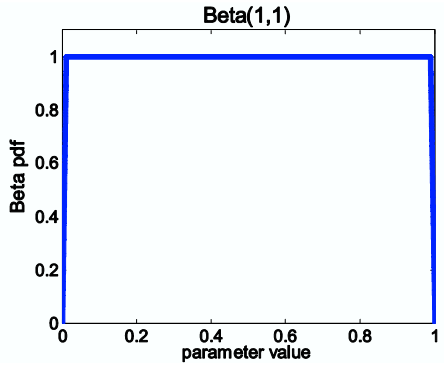
# Coin Flips with Beta Distribution

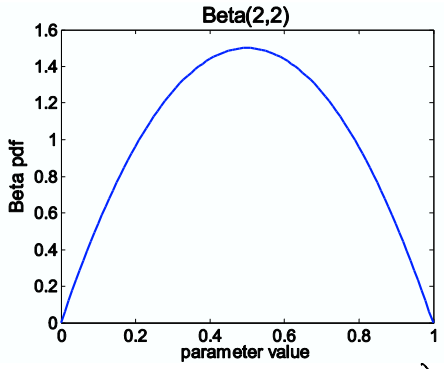
Likelihood function:

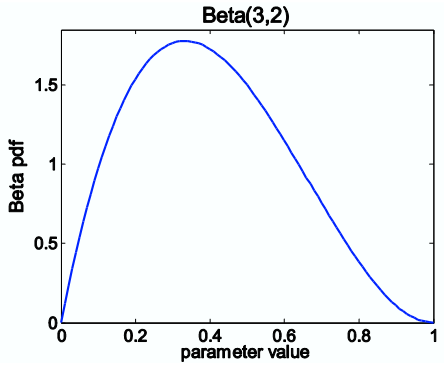
$$P(\mathcal{D} \mid \theta) = \theta^{\alpha_H}(1-\theta)^{\alpha_T}$$

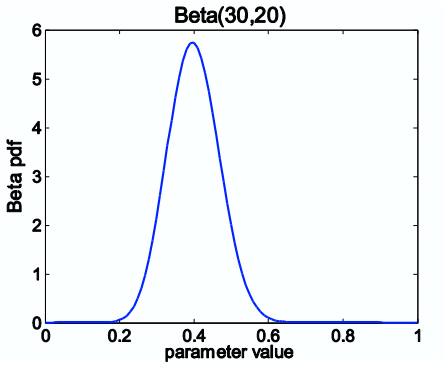
Prior:

$$P(\theta) = \frac{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}{B(\beta_H, \beta_T)} \sim Beta(\beta_H, \beta_T)$$




$$P(\theta \mid \mathcal{D}) \propto \overbrace{\theta^{\alpha_H}(1-\theta)^{\alpha_T}}^{P(\mathcal{D}\mid\theta)} \overbrace{\theta^{\beta_H - 1}(1-\theta)^{\beta_T - 1}}^{P(\theta)}$$

$$= \theta^{\alpha_H + \beta_H - 1}(1-\theta)^{\alpha_T + \beta_T - 1}$$

$$= Beta(\alpha_H + \beta_H, \alpha_T + \beta_T)$$

# MAP Estimation

MLE → $\max_{\theta} p(D|\theta)$

- Choosing $\theta$ to maximize the posterior distribution is called maximum a posteriori (MAP) estimation

$$\theta_{MAP} = \arg\max_{\theta} p(\theta|D)$$

- The only difference between $\theta_{MLE}$ and $\theta_{MAP}$ is that one assumes a uniform prior (MLE) and the other allows an arbitrary prior

# Priors

- Suppose we have 5 coin flips all of which are heads
  - MLE would give $\theta_{MLE} = 1$
  - MAP with a $Beta(2,2)$ prior gives $\theta_{MAP} = \frac{6}{7} \approx .857$
  - As we see more data, the effect of the prior diminishes
    - $\theta_{MAP} = \frac{\alpha_H + \beta_H - 1}{\alpha_H + \beta_H + \alpha_T + \beta_T - 2} \approx \frac{\alpha_H}{\alpha_H + \alpha_T}$ for large # of observations

# MAP for Dice Roll

$$p(D|\theta) = \theta_1^{\#1}\,\theta_2^{\#2}\cdots$$

$$p(\theta) = \theta_1^{\beta_1-1}\,\theta_2^{\beta_2-1}\cdots\theta_5^{\beta_5-1}\,\theta_6^{\beta_6-1}$$

$$p(\theta|D) \propto p(\theta)\,p(D|\theta)$$

$$\theta_1 = \frac{\#1 + \beta_1 - 1}{N + \sum_{i=1}^{6}\beta_i - 6}$$

# Sample Complexity

- How many coin flips do we need in order to guarantee that our learned parameter does not differ too much from the true parameter (with high probability)?
- Can use Chernoff bound
  - Suppose $Y_1, \ldots, Y_N$ are i.i.d. random variables taking values in $\{0, 1\}$ such that $E_p[Y_i] = y$. For $\epsilon > 0$,

$$p\left(\left|y - \frac{1}{N}\sum_i Y_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

# Sample Complexity

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- Can use Chernoff bound
  - For the coin flipping problem with $X_1, \ldots, X_n$ iid coin flips and $\epsilon > 0$,

$$p\left(\left|\theta_{true} - \frac{1}{N}\sum_i X_i\right| \geq \epsilon\right) \leq 2e^{-2N\epsilon^2}$$

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being large.

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$$\delta \geq 2e^{-2N\epsilon^2} \Rightarrow N \geq \frac{1}{2\epsilon^2} \ln \frac{2}{\delta}$$