Chapter 7

The reaction wheel pendulum

7.1 Introduction

The reaction wheel pendulum is one of the simplest non-linear underactuated systems. It is a pendulum with a rotating wheel at the end, which is free to spin about an axis parallel to the axis of rotation of the pendulum (see Figure 7.1). The wheel is actuated by a DC-motor, while the pendulum is unactuated. The coupling torque generated by the angular acceleration of the disk can be used to actively control the system. This mechanical system was introduced and studied in [108], where a partial feedback linearization control law was presented.

In [76], Olfati-Saber transformed the reaction wheel (or inertia wheel) pendulum's dynamics into a cascade non-linear system in strict feedback form, using a global change of coordinates in an explicit form. Then, he proposed global asymptotic stabilization of the upright equilibrium point using the standard backstepping procedure. In his approach, contrary to the strategy proposed here, the magnitude of the control input increases with the norm of the state initial condition.

The control objective here will also be to swing the pendulum up and balance it about its unstable inverted position. We will focus our study on the swinging-up control law. The non-linear swinging-up controller will be based on the total energy of the system. The control design will exploit the passivity property of the complete Lagrangian system dynamics. Note that the technique has been presented in [23]. Similar control strategies have been used to control other underactuated mechanical systems in [59] for the cart-pole system, in [24] for the

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pendubot and in [21] for planar manipulators with springs.

In this chapter, we present two approaches based on the total energy stored in the system. We make use of LaSalle's theorem to prove that the system trajectories asymptotically converge to a homoclinic orbit in both approaches. Therefore, asymptotically, after every swing of the pendulum, the system state gets successively closer to the origin.

The first approach proposed here is such that the wheel's angular velocity converges to zero but does not necessarily bring the wheel to a desired angular position. Nevertheless, the control input can be made smaller than any arbitrary upper bound. The second approach is such that the wheel's angular position converges to zero.

In Section 7.2, we develop the equations of motion of the reaction wheel pendulum. In Sections 7.3 and 7.4, two different energy-based control algorithms are presented. Simulation results are given in Section 7.5. The concluding remarks are presented in Section 7.6.

7.2 The reaction wheel pendulum

7.2.1 Equations of motion

The reaction wheel pendulum is a two-degree-of-freedom robot as shown in Figure 7.1. The pendulum constitutes the first link, while the rotating wheel is the second one. The angle of the pendulum is q_1 and is measured clockwise from the vertical. The angle of the wheel is q_2.

The parameters of the system are described in the following table.

m_1	:	Mass of the pendulum
m_2	:	Mass of the wheel
l_1	:	Length of the pendulum
l_{c1}	:	Distance to the center of mass of the pendulum
I_1	:	Moment of inertia of the pendulum
I_2	:	Moment of inertia of the wheel
q_1	:	Angle that the pendulum makes with the vertical
q_2	:	Angle of the wheel
τ	:	Motor torque input applied on the disk

We introduce the parameter $\bar{m} = m_1 l_{c1} + m_2 l_1$, which will be used later.

The kinetic energy of the pendulum is $K_1 = \frac{1}{2}\left(m_1 l_{c1}^2 + I_1\right)\dot{q_1}^2$ and the kinetic energy of the wheel is $K_2 = \frac{1}{2} m_2 l_1^2 \dot{q_1}^2 + \frac{1}{2} I_2 (\dot{q_1} + \dot{q_2})^2$. There-

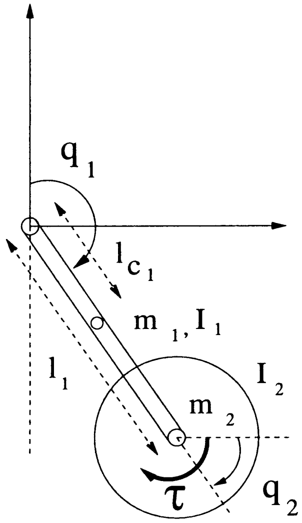

Figure 7.1: The reaction wheel pendulum

fore the total kinetic energy is given by

$$K = K_1 + K_2 = \frac{1}{2}(m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2)\dot{q_1}^2 + I_2\dot{q}_1\dot{q}_2 + \frac{1}{2}I_2\dot{q_2}^2 \tag{7.1}$$

The potential energy of the system is $P = \bar{m}g(\cos(q_1) - 1)$. Finally, the Lagrangian function is given by

$$\begin{aligned} L &= K - P \\ L &= \frac{1}{2}(m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2)\dot{q_1}^2 + I_2\dot{q}_1\dot{q}_2 + \frac{1}{2}I_2\dot{q_2}^2 \\ &\quad -\bar{m}g(\cos(q_1) - 1) \end{aligned} \tag{7.2}$$

Using Euler-Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = \tau \tag{7.3}$$

we therefore have

$$\begin{aligned}
\left(\frac{\partial L}{\partial \dot{q}_1}\right) &= (m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2)\dot{q}_1 + I_2\dot{q}_2 \\
\left(\frac{\partial L}{\partial q_1}\right) &= \bar{m}g\sin(q_1) \\
\left(\frac{\partial L}{\partial \dot{q}_2}\right) &= I_2\dot{q}_1 + I_2\dot{q}_2 \\
\left(\frac{\partial L}{\partial q_2}\right) &= 0
\end{aligned}$$

The dynamic equations of the system are finally given by

$$(m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2)\ddot{q}_1 + I_2\ddot{q}_2 - \bar{m}g\sin(q_1) = 0 \qquad (7.4)$$

$$I_2\ddot{q}_1 + I_2\ddot{q}_2 = \tau \qquad (7.5)$$

In compact form, the system can be rewritten as follows

$$D(q)\ddot{q} + g(q) = u \qquad (7.6)$$

where $q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$ is the vector of generalized coordinates, $u = \begin{bmatrix} 0 \\ \tau \end{bmatrix}$ is the vector of joint torques, $D(q)$ is the inertia matrix and is given by

$$D(q) = \begin{bmatrix} m_1 l_{c1}^2 + m_2 l_1^2 + I_1 + I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \qquad (7.7)$$

and

$$g(q) = \begin{bmatrix} -\bar{m}g\sin(q_1) \\ 0 \end{bmatrix} \qquad (7.8)$$

Note that the matrix $D(q)$ is constant and positive definite. The equations of motion are also given by

$$d_{11}\ddot{q}_1 + d_{12}\ddot{q}_2 + g(q_1) = 0 \qquad (7.9)$$

$$d_{21}\ddot{q}_1 + d_{22}\ddot{q}_2 = \tau \qquad (7.10)$$

7.2.2 Passivity properties of the system

The total energy of the reaction wheel pendulum is

$$\begin{aligned}
E &= \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q) \\
&= \frac{1}{2}\dot{q}^T D(q)\dot{q} + \bar{m}g(\cos(q_1) - 1) \qquad (7.11)
\end{aligned}$$

From (7.6)-(7.8) and (7.11), it follows that

$$\dot{E} = \dot{q}^T D(q)\ddot{q} - \bar{m}g\sin(q_1)\dot{q}_1 = \dot{q}_2\tau \tag{7.12}$$

As a consequence, the system with τ as input and $\dot{q}_2$ as output is passive. The reaction wheel pendulum with zero control input ($\tau = 0$) has an unstable equilibrium at $(q_1, \dot{q}_1, \dot{q}_2) = (0,0,0)$ with energy $E(0,0,0) = 0$ and a stable equilibrium at $(q_1, \dot{q}_1, \dot{q}_2) = (\pi, 0, 0)$ with energy $E(0,0,0) = -2\bar{m}g$. The disk position q_2 can be arbitrary, since the energy in (7.11) does not depend on q_2, i.e. q_2 is a cyclic variable. Hence, the equilibrium points described above are not isolated equilibrium points in the four-dimensional state space of the system.

The control objective will be to control the pendulum position q_1, the pendulum velocity $\dot{q}_1$and the disk velocity $\dot{q}_2$ and to leave the disk position q_2 unspecified.

Let us consider the state vector $z = [\cos q_1, \sin q_1, \dot{q}_1, \dot{q}_2]^T$. We will bring the state vector z to $[1,0,0,0]^T$.

7.2.3 Linearization of the system

In this section, we will study the controllability of the linearized system about the origin. The equations of motion are given by (see (7.9) and (7.10))

$$\begin{aligned} \ddot{q}_1 &= \frac{d_{22}}{\det(D)}\bar{m}g\sin(q_1) - \frac{d_{12}}{\det(D)}\tau && (7.13) \\ \ddot{q}_2 &= \frac{-d_{21}}{\det(D)}\bar{m}g\sin(q_1) + \frac{d_{11}}{\det(D)}\tau && (7.14) \end{aligned}$$

where $\det(D) = d_{11}d_{22} - d_{21}d_{12} = (m_1 l_{c1}^2 + m_2 l_1^2 + I_1)I_2$.

Let us consider the vector state $X = [q_1, \dot{q}_1, q_2, \dot{q}_2]$. Differentiating equations (7.13) and (7.14) with respect to the states and evaluating them at the origin leads to the following linear system

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{d_{22}}{\det(D)}\bar{m}g & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-d_{21}}{\det(D)}\bar{m}g & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \end{bmatrix}\tau \\ &= AX + B\tau \end{aligned}$$

We then have

$$B = \begin{bmatrix} 0 \\ \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \end{bmatrix} \qquad AB = \begin{bmatrix} \frac{-d_{12}}{\det(D)} \\ 0 \\ \frac{d_{11}}{\det(D)} \\ 0 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 \\ \frac{-d_{22}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \\ \frac{d_{21}d_{12}}{(\det(D))^2}\bar{m}g \end{bmatrix} \qquad A^3B = \begin{bmatrix} \frac{-d_{22}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \\ \frac{d_{21}d_{12}}{(\det(D))^2}\bar{m}g \\ 0 \end{bmatrix}$$

and $\det\left(B|AB|A^2B|A^3B\right) = \frac{d_{12}^2 m^2 g^2}{(\det(D))^4} = \frac{\bar{m}^2 g^2}{(m_1 l_{c1}^2 + m_2 l_1^2 + I_1)^4 I_2^2}$. The linearized system is controllable. Therefore, a full state feedback controller $\tau = -K^T X$ with an appropriate gain vector K is able to successfully stabilize the system in a neighborhood of the origin.

7.2.4 Feedback linearization

For completeness purposes, we briefly develop in this section the feedback linearization first presented in [108].

Consider the following output function

$$y = d_{11}q_1 + d_{12}q_2 \tag{7.15}$$

Differentiating (7.15), we obtain

$$\begin{aligned} \dot{y} &= d_{11}\dot{q}_1 + d_{12}\dot{q}_2 && (7.16) \\ \ddot{y} &= \bar{m}g \sin q_1 && (7.17) \\ y^{(3)} &= \bar{m}g \cos(q_1)\dot{q}_1 && (7.18) \\ y^{(4)} &= \bar{m}g \cos(q_1)\ddot{q}_1 - \bar{m}g \sin(q_1)\dot{q}_1^2 && (7.19) \end{aligned}$$

From (7.9) and (7.10), we get

$$\ddot{q}_1 = \frac{d_{22}}{\det(D)}\bar{m}g \sin q_1 - \frac{d_{12}}{\det(D)}\tau \tag{7.20}$$

Introducing (7.20) in (7.19) yields

$$\begin{aligned} y^{(4)} &= \bar{m}g \cos(q_1)\left(-\frac{d_{12}}{\det(D)}\right)\tau \\ &\quad + \bar{m}g \sin(q_1)\left(-\dot{q}_1^2 + \bar{m}g \cos(q_1)\left(\frac{d_{22}}{\det(D)}\right)\right) && (7.21) \end{aligned}$$

Thus, the system has a relative degree of four with respect to the output $d_{11}\dot{q}_1 + d_{12}\dot{q}_2$ in the region $-\frac{\pi}{2} < q_1 < \frac{\pi}{2}$, i.e. when $\cos(q_1) \neq 0$. We can define a controller τ, so the closed-loop system is given by

$$y^{(4)} = -\alpha_3 y^{(3)} - \alpha_2 \ddot{y} - \alpha_1 \dot{y} - \alpha_0 y \tag{7.22}$$

where $s^4 + \alpha_3 s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0$ is a stable polynomial. Therefore, $y^{(i)} \to 0$. Finally from (7.16) and (7.17), it follows that $(q_1, \dot{q}_2) \to (0, 0)$. The above shows that the reaction wheel pendulum is feedback linearizable in the region $|q_1| < \frac{\pi}{2}$, i.e. when the pendulum angle q_1 is above the horizontal.

7.3 First energy-based control design

Define the following Lyapunov function candidate

$$V_1 = \frac{1}{2}k_E E^2 + \frac{1}{2}k_v \left(d_{21}\dot{q}_1 + d_{22}\dot{q}_2\right)^2 \tag{7.23}$$

where k_E and k_v are strictly positive constants. The time derivative of V_1 is given by

$$\begin{aligned} \dot{V}_1 &= k_E E \dot{E} + k_v \left(d_{21}\dot{q}_1 + d_{22}\dot{q}_2\right)\tau \\ &= \left(k_E E \dot{q}_2 + k_v(d_{21}\dot{q}_1 + d_{22}\dot{q}_2)\right)\tau \end{aligned}$$

We propose a controller such that

$$\tau = -k_d \left(k_E E \dot{q}_2 + k_v(d_{21}\dot{q}_1 + d_{22}\dot{q}_2)\right) \tag{7.24}$$

which leads to

$$\dot{V}_1 = -\frac{1}{k_d}\tau^2 \tag{7.25}$$

Equations (7.23) and (7.25) imply that V_1 is a non-increasing function, V_1 converges to a constant and $V_1 \leq V_1(0)$. This implies that the energy E remains bounded as well as $\dot{q}_1$ and $\dot{q}_2$. Therefore the closed-loop state vector $z = \left[\cos q_1, \sin q_1, \dot{q}_1, \dot{q}_2\right]^T$ is bounded and we can thus apply LaSalle's invariance principle.

In order to apply LaSalle's theorem, we are required to define a compact (closed and bounded) set Ω with the property that every solution of the system $\dot{z} = F(z)$ that starts in Ω remains in Ω for all future time. Therefore, the solutions of the closed-loop system $\dot{z} = F(z)$ remain inside a compact set Ω that is defined by the initial value of z.

Let Γ be the set of all points in Ω such that $\dot{V}_1(z) = 0$. Let M be the largest invariant set in Γ. LaSalle's theorem ensures that every solution starting in Ω approaches M as $t \to \infty$. Let us now compute the largest invariant set M in Γ.

In the set Γ, $\dot{V}_1 = 0$ and from (7.25) it follows that $\tau = 0$ in Γ. Note that $\dot{V}_1 = 0$ also at the stable equilibrium point $(q_1, \dot{q}_1, \dot{q}_2) = (\pi, 0, 0)$. Recall that the pendulum's energy is $E(\pi, 0, 0) = -2\bar{m}g$ at the stable equilibrium point. A way to avoid this undesired convergence point is to constrain the initial conditions. Indeed, if the initial state is such that

$$V_1(0) < 2k_E\bar{m}^2g^2 \tag{7.26}$$

then, in view of (7.23) and given that $V_1 \leq V_1(0)$, the energy E will never reach the value $-2\bar{m}g$, which characterizes the stable equilibrium point $(q_1, \dot{q}_1, \dot{q}_2) = (\pi, 0, 0)$. The inequality (7.26) mainly imposes upper bounds on $|\dot{q}_1|$ and $|\dot{q}_2|$.

Since $\dot{E} = \dot{q}_2\tau$ (see (7.12)) and $\tau = 0$ in the invariant set, then E is constant. From (7.23), it follows that

$$d_{21}\dot{q}_1 + d_{22}\dot{q}_2 = K \tag{7.27}$$

for some constant K. Then, from (7.24), we get $E\dot{q}_2 = -\frac{k_v K}{k_E}$. We will consider two cases: a) $E = 0$ and b) $E = \bar{K} \neq 0$, for some constant $\bar{K}$.

- Case a: $E = 0$. From (7.24), we have

$$d_{21}\dot{q}_1 + d_{22}\dot{q}_2 = 0 \tag{7.28}$$

Introducing (7.28) in (7.11), with $E = 0$, it then follows that

$$\begin{aligned} E &= \dot{q}^T D\dot{q} + \bar{m}g(\cos q_1 - 1) = 0 \\ &= \frac{\det(D)}{d_{22}}\dot{q}_1^2 + \bar{m}g(\cos q_1 - 1) = 0 \end{aligned} \tag{7.29}$$

Since $\det(D)$ is a constant, equation (7.29) defines a particular trajectory called a homoclinic orbit of the pendulum in the $(q_1, \dot{q}_1)$ phase plane, which is a two-dimensional subspace of the four-dimensional state space of the complete system. (see [24, 59]). It means that $\dot{q}_1 = 0$ only when $q_1 = 0$. The pendulum moves clockwise or counter-clockwise until it reaches the equilibrium point $(q_1, \dot{q}_1) = (0, 0)$. Then, from (7.28), it follows that $\dot{q}_2 = 0$ also.

- Case b: $E = \bar{K} \neq 0$. Since $E\dot{q}_2$ is constant, then $\dot{q}_2$ is also constant. Thus from (7.27), $\dot{q}_1$ is also constant. From (7.7)-(7.10) and since $\tau = 0$, we obtain

$$\ddot{q}_1 = \frac{d_{22}}{\det(D)}\bar{m}g\sin q_1 \tag{7.30}$$

Since $\dot{q}_1$ is constant, we have $\ddot{q}_1 = 0$ and from (7.30), we conclude that $q_1 = 0[\pi]$. Note that the case when $q_1 = \pi[2\pi]$ has been excluded by imposing the constraint (7.26). Suppose now that $\dot{q}_2 \neq 0$. From (7.24), since $\tau = 0$ and $\dot{q}_1 = 0$, it follows that

$$E = -\frac{k_v d_{22}}{k_E} < 0 \tag{7.31}$$

However, since $q_1 = 0$, the energy (7.11) becomes

$$E = d_{22}\dot{q}_2^2 > 0 \tag{7.32}$$

Therefore equations (7.31) and (7.32) lead to a contradiction, which proves that the assumption $\dot{q}_2 = 0$ is false. We finally conclude that $\dot{q}_2 = 0$. Moreover, when $\dot{q}_2 = 0$, $q_1 = 0$ and $\dot{q}_1 = 0$, E is also zero, which contradicts the assumption $E \neq 0$.

Finally, the largest invariant set is given by the homoclinic orbit (7.29) together with the kinematic constraint (7.28). LaSalle's invariance principle guarantees that the system trajectories asymptotically converge to this invariant set, provided that the initial conditions are such that (7.26) is satisfied.

7.4 Second energy-based controller

In this section, we will propose another approach based on the total energy of the system, using a similar strategy developed in previous works [59] and [24]. Contrary to the algorithm proposed in the previous section, the controller presented next will be such that the wheel angular position q_2 will also converge to zero. Define the following Lyapunov function candidate

$$V_2 = \frac{1}{2}k_E E^2 + \frac{1}{2}k_v\dot{q}_2^2 + \frac{1}{2}k_p q_2^2 \tag{7.33}$$

where k_E, k_v and k_p are strictly positive constants. From (7.7)-(7.10), we get

$$\ddot{q}_2 = -\frac{d_{21}}{\det(D)}\bar{m}g\sin q_1 + \frac{d_{11}}{\det(D)}\tau \tag{7.34}$$

Therefore

$$\begin{aligned}\dot{V}_2 &= k_E E\dot{E} + k_v \dot{q}_2 \ddot{q}_2 + k_p \dot{q}_2 q_2 \\ &= \dot{q}_2 \left(k_E E\tau - k_1 \sin q_1 + k_2 \tau + k_p q_2\right) \qquad (7.35)\end{aligned}$$

where $k_1 = \frac{k_v d_{21} \bar{m} g}{\det(D)}$; $k_2 = \frac{k_v d_{11}}{\det(D)}$. We propose a controller τ such that

$$\tau(k_E E + k_2) - k_1 \sin q_1 + k_p q_2 = -k_d \dot{q}_2 \qquad (7.36)$$

which leads to

$$\dot{V}_2 = -k_d \dot{q}_2^{\,2} \qquad (7.37)$$

In view of (7.11), we have

$$E \geq -2\bar{m}g \qquad (7.38)$$

In order to avoid singularity in (7.36), it suffices to choose k_E and k_v such that for some $\epsilon > 0$

$$k_E E + k_2 \geq k_E(-2\bar{m}g) + k_v \frac{d_{11}}{\det(D)} \geq \epsilon > 0 \qquad (7.39)$$

The stabilizing controller is of the form

$$\tau = \frac{-k_d \dot{q}_2 - k_p q_2 + k_1 \sin q_1}{k_E E + k_2} \qquad (7.40)$$

From (7.33) and (7.37), we conclude that $V_2 \leq V_2(0)$. This implies that the energy E remains bounded as well as $\dot{q}_1$, q_2 and $\dot{q}_2$. Thus the closed-loop state vector $z = [\cos q_1, \sin q_1, \dot{q}_1, q_2, \dot{q}_2]^T$ is bounded and we can thus apply LaSalle's invariance principle, as has been done in the previous section.

Defining a compact (closed and bounded) set Ω and Γ the set of all points in Ω such that $\dot{V}_2(z) = 0$. Let M be again the largest invariant set in Γ. LaSalle´s theorem ensures that every solution starting in Ω approaches M as $t \to \infty$. Let us now compute the largest invariant set M in Γ.

In the set Γ, $\dot{V}_2 = 0$ and from (7.37) it follows that $\dot{q}_2 = 0$ and thus q_2 is constant in Γ. Note that $\dot{V}_2 = 0$ also at the stable equilibrium point $(q_1, \dot{q}_1, \dot{q}_2) = (\pi, 0, 0)$ and that $\tau = 0$ at this point (see (7.40)). Recall that the pendulum's energy is $E(\pi, 0, 0) = -2\bar{m}g$ at the stable equilibrium point. We will avoid this undesired convergence point by imposing a constraint on the initial conditions. Indeed, if the initial state is such that

$$V_2(0) < 2k_E\bar{m}^2g^2 \tag{7.41}$$

then, in view of (7.33) and given that $V_2 \leq V_2(0)$, the energy E will never reach the value $-2\bar{m}g$, which characterizes the stable equilibrium point for $\dot{q}_2$. The inequality (7.41) imposes upper bounds on $|\dot{q}_1|$ and $|\dot{q}_2|$. Note that since τ in (7.40) is bounded and $\dot{q}_2 = 0$ in the invariant set Γ, then E is constant (see (7.12)). Equation (7.34) can be rewritten as

$$\ddot{q}_2 = -\frac{k_1}{k_v}\sin q_1 + \frac{k_2}{k_v}\tau \tag{7.42}$$

Therefore, since $\dot{q}_2 = 0$ we have

$$k_2\tau = k_1 \sin q_1 \tag{7.43}$$

Introducing the above in (7.36) and since $\dot{q}_2 = 0$, we obtain

$$k_E E\tau + k_p q_2 = 0 \tag{7.44}$$

We conclude that $E\tau$ is constant in Γ. Since E is also constant, we either have a) $E = 0$ or b) $E \neq 0$.

- Case a: If $E = 0$, then from (7.44) $q_2 = 0$. Note that τ in (7.40) is bounded since $|E| < 2\bar{m}g$. Moreover, in view of (7.7), (7.8) and (7.11), since $\dot{q}_2 = 0$ and $E = 0$, (7.11) reduces to

$$E = \frac{1}{2}d_{11}\dot{q}_1^{\,2} + \bar{m}g(\cos(q_1) - 1) = 0 \tag{7.45}$$

  which defines a homoclinic orbit of the pendulum in the $(q_1, \dot{q}_1)$ phase plane. In this case, we conclude that q_2, $\dot{q}_2$ and E converge to zero. Note that τ does not necessarily converge to zero.

- Case b: If $E \neq 0$ and since $E\tau$ is constant, then the control input τ is also constant. We will prove next that in this case $\tau = 0$ in Γ.

  From (7.9) and (7.10), we get

$$d_{11}\ddot{q}_1 = \bar{m}g \sin q_1 \tag{7.46}$$
$$d_{21}\ddot{q}_1 = \tau \tag{7.47}$$

Introducing (7.47) in (7.46), we obtain

$$\bar{m}g \sin q_1 = \frac{d_{11}}{d_{21}} \tau \tag{7.48}$$

Differentiating (7.48), it follows

$$\bar{m}g \cos q_1 \dot{q}_1 = 0 \tag{7.49}$$

We conclude that either $\dot{q}_1 = 0$ or $\cos(q_1) = 0$. If $\dot{q}_1 = 0$, then from (7.47) we get $\tau = 0$. If $\cos(q_1) = 0$, then q_1 is constant and we conclude also that $\tau = 0$. We therefore conclude that $\tau = 0$ in Γ. Since $\tau = 0$, then from (7.48) $\sin(q_1) = 0$. This implies that $q_1 = 0, \pm 2\pi, \ldots$. Note that the points $q_1 = \pi, 3\pi \ldots$ have been avoided by imposing the constraint (7.41). From (7.44) we get $q_2 = 0$.

We finally conclude that the largest invariant set is $M = \{q_2 = 0, E = 0\}$. LaSalle's invariance principle guarantees that the system trajectories asymptotically converge to this invariant set, provided that the initial conditions are such that (7.41) is satisfied.

7.5 Simulation results

In this section, we present the simulation results using MATLAB and SIMULINK. In the model, we used the real system parameters of the reaction wheel pendulum at the University of Illinois at Urbana-Champaign. $m_1 = 0.02$ kg, $m_2 = 0.3$ kg, $l_1 = 0.125$ m, $l_{c1} = 0.063$ m, $I_1 = 47 \times 10^{-6}$ kg.m^2, $I_2 = 32 \times 10^{-6}$ kg.m^2 and $g = 9.804$. The initial conditions are

$$q_1 = 0.8\pi \qquad \dot{q}_1 = 0$$

$$q_2 = 0 \qquad \dot{q}_2 = 0$$

For the first approach, we chose the gains $k_E = 0.01$, $k_v = 200$ and $k_d = 0.1$ and the results are shown in Figure 7.2. For the second approach, we chose the gains $k_E = 400$, $k_v = 0.01$, $k_p = 0.1$ and $k_d = 0.05$. Note that the gains k_E and k_v satisfy the condition (7.39). The results are shown in Figures 7.3 and 7.4.

In both cases, the Lyapunov function decreases as expected. The energy converges monotonically to zero, but note this is not necessarily

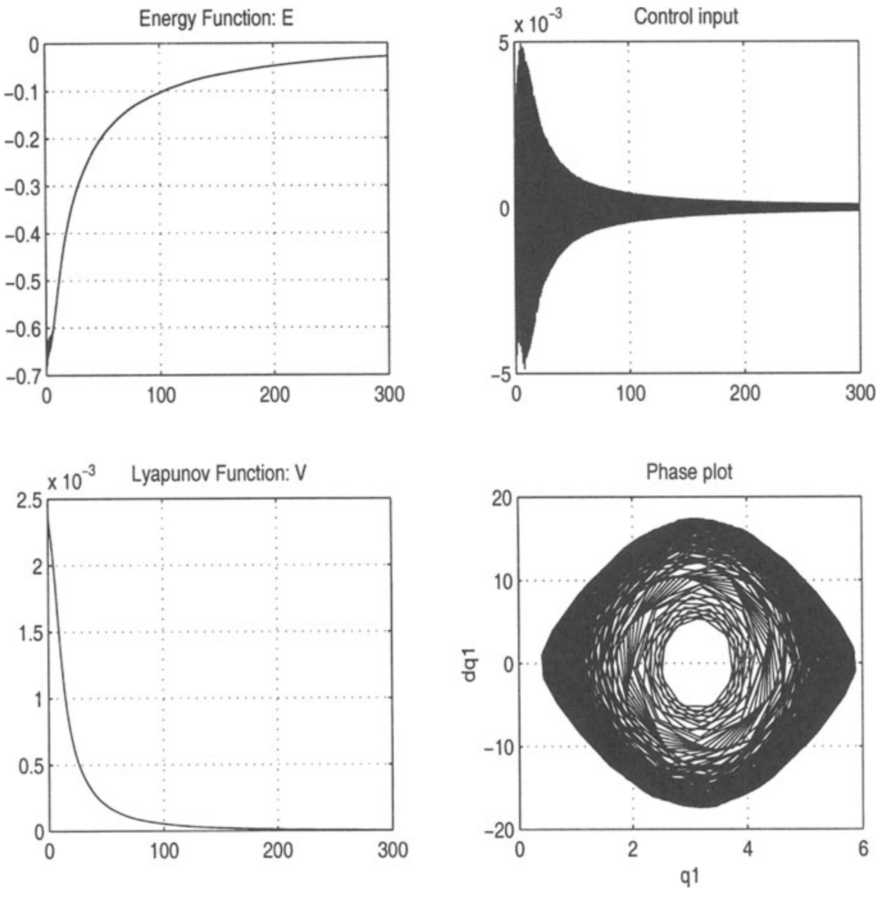

Figure 7.2: Simulation results using the first controller (7.24)

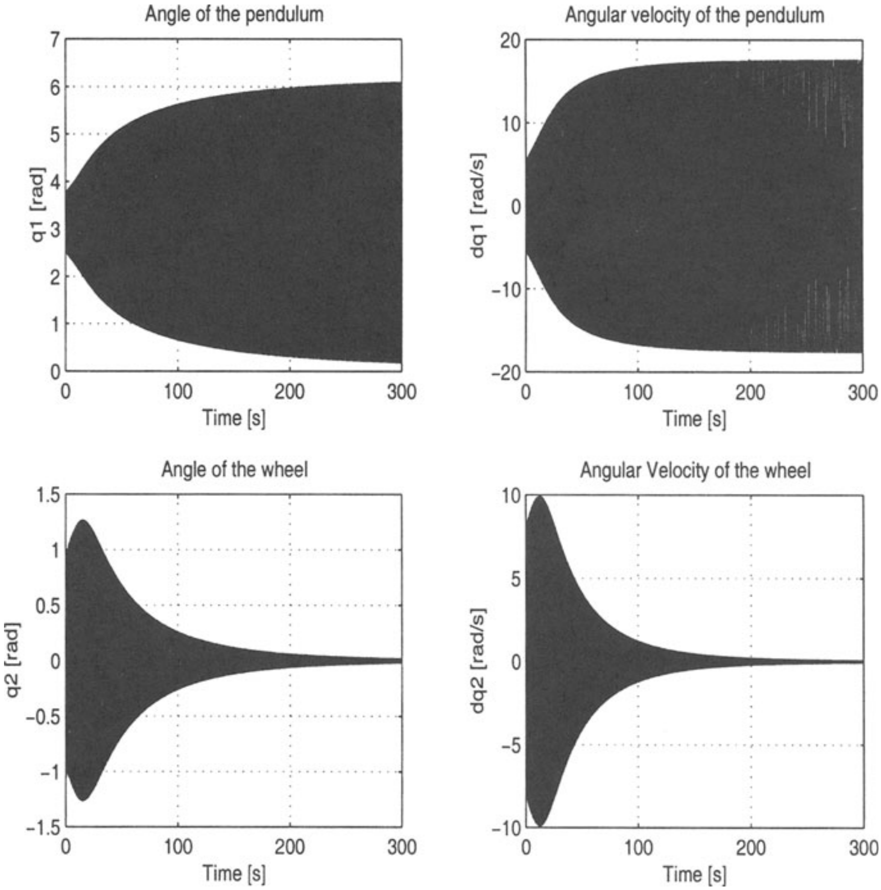

Figure 7.3: Simulation results using the second controller (7.40)

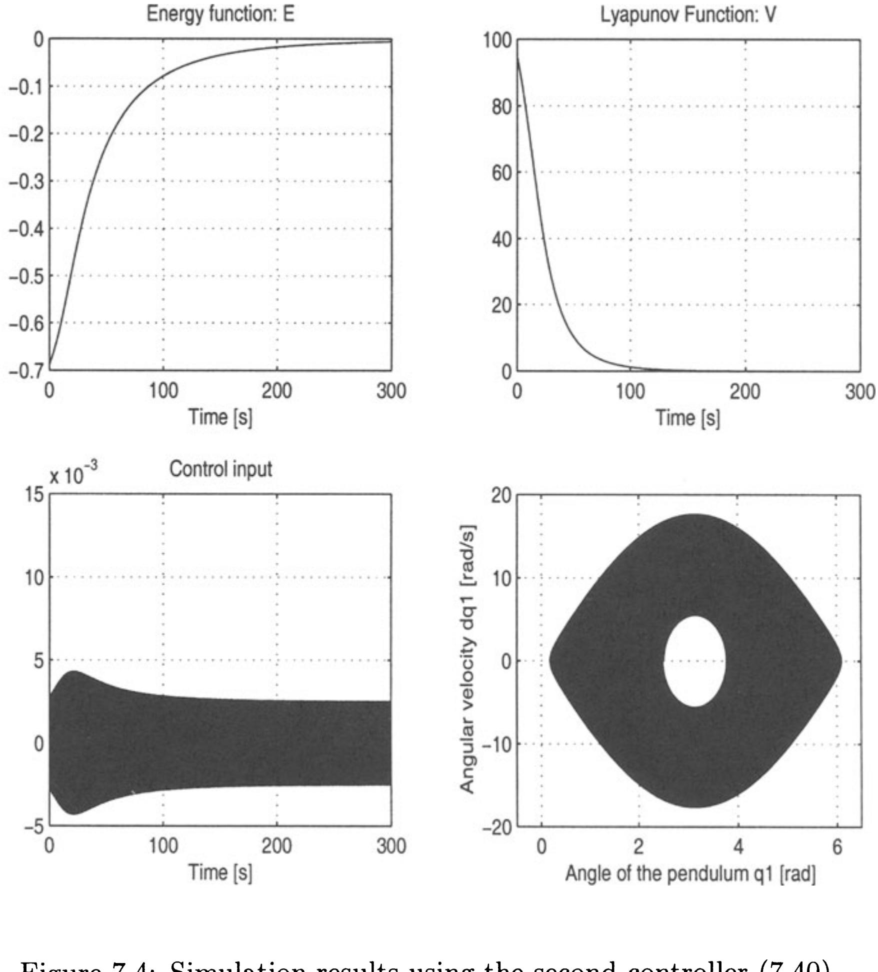

Figure 7.4: Simulation results using the second controller (7.40)

always the case. The control input τ converges to zero for the first controller (see Figure 7.2) but does not converge to zero for the second controller (see Figure 7.4). In both cases, the control input magnitude is acceptable. The phase plots show convergence to the homoclinic orbit in both cases. We have observed that the convergence rate is larger for the first controller. The closed-loop behavior strongly depends on the controller parameters k_E, k_v and k_d. The parameters used in the simulations have been selected by trying different values.

7.6 Conclusions

We have proposed two alternative approaches to swing up the reaction wheel pendulum. Both approaches are based on the total energy of the system and guarantee convergence of the pendulum to a homoclinic orbit. The first controller is such that the torque input can be saturated to any arbitrary value. The second controller is such that the wheel reaches a desired position. Simulations have shown good performance of both controllers.

We will give below a generalization of the main ideas developed for the examples of pendulum systems for a class of Euler-Lagrange systems, which possesses particular properties. It is related to the work of Shiriaev et al. [100].

7.7 Generalization for Euler-Lagrange systems

In this section, we propose to formalize the results developed for pendulum systems and give some general conditions under which the technique can be applied. The Lagrangian function L of a system having a vector of generalized variables $q^T = [x, y]$ in a configuration space $S = X \times Y$ is given by

$$L(q,\dot{q}) = K(q,\dot{q}) - P(q) \tag{7.50}$$

$$L(q,\dot{q}) = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q) \tag{7.51}$$

where $P(q)$ is the potential energy of the system. The corresponding equations of motion are derived using Euler-Lagrange's equations

$$\frac{d}{dt}\nabla_{\dot{x}}L - \nabla_x L = \tau \tag{7.52}$$

$$\frac{d}{dt}\nabla_{\dot{y}}L - \nabla_y L = 0 \tag{7.53}$$

The control input vector is given by $U = [\tau, 0]^T$.

The equations of motion of the system can also be written in standard form, as follows

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = U \tag{7.54}$$

with the following properties: $D(q)$ is symmetric, positive definite and $\dot{D} - 2C$ is a skew-symmetric matrix. Moreover, P is related to $G(q)$ as follows

$$G(q) = \nabla_q P \tag{7.55}$$

The total energy of the system is given by

$$\begin{aligned} E(q,\dot{q}) &= K(q,\dot{q}) + P(q) && (7.56)\\ &= \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q) && (7.57) \end{aligned}$$

Then, using (7.54), (7.55), (7.52) and (7.53), the time derivative of E is given by

$$\frac{d}{dt}E(q(t),\dot{q}(t)) = \dot{x}(t)^T\tau(t) \tag{7.58}$$

Therefore, the total energy satisfies the passivity property. Let us consider the desired energy E_d and the desired vector x_d.

We propose the following Lyapunov function candidate

$$V(q,\dot{q}) = \frac{k_E}{2}(E(q,\dot{q}) - E_d)^2 + \frac{k_v}{2}|\dot{x}|^2 + \frac{k_x}{2}|x - x_d|^2 \tag{7.59}$$

Differentiating V along the solutions of system (7.54), we obtain

$$\dot{V} = \dot{x}^T\left[k_E(E - E_d)\tau + k_v\ddot{x} + k_x(x - x_d)\right] \tag{7.60}$$

$$\dot{V} = \dot{x}^T\left[\left(k_E(E - E_d)I + k_v[I\ 0]D(q)^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}\right)\tau + F(q,\dot{q})\right] \tag{7.61}$$

where $F(q,\dot{q})$ is a function that depends on L, x_d, the parameters k_E, k_v, k_x and is given by

$$F(q,\dot{q}) = k_v[I\ 0]D(q)^{-1}\left[-C\dot{q} - G\right] + k_x(x - x_d) \tag{7.62}$$

We propose a control law such that

$$\left(k_E(E - E_d)I + k_v[I\ 0]D(q)^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}\right)\tau + F(q, \dot{q}) = -k\dot{x} \tag{7.63}$$

which will lead to

$$\dot{V} = -k\dot{x}^T\dot{x} \leq 0 \tag{7.64}$$

if the following matrix is invertible

$$k_E(E - E_d)I + k_v[I\ 0]D(q)^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} \tag{7.65}$$

We assume that the energy function is bounded from below, which is normally the case for the pendulums. Therefore, there exist some positive parameters k_E and k_v such that the matrix (7.65) is strictly positive definite and thus invertible. Then, the stability analysis will be based on LaSalle's invariance principle. The objective is to prove that along the closed-loop system solutions $(q(t), \dot{q}(t))$, $\lim_{t\to+\infty} E = E_d$ and $\lim_{t\to+\infty} x = x_d$. On the other hand, this part is in general not straightforward and has actually only been developed for each particular system. Further studies on the subject are underway.