Random Projections and Dimension Reduction

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Johnson-Lindenstrauss Lemma

If we have n data points in $\mathbb{R}^d$, there exists a linear map into $\mathbb{R}^k$ such that pairwise distances between data points can be preserved up to an ϵ tolerance, provided $k > C\varepsilon^{-2} \log n$, where $C \approx 24$ [JL84]. The proof follows three steps [Mic09]:

- Define a random linear map $f : \mathbb{R}^d \to \mathbb{R}^k$ by $f(\mathbf{u}) = \frac{1}{\sqrt{k}}\mathbf{R} \cdot \mathbf{u}$, where $R \in \mathbb{R}^{k\times d}$ drawn elementwise from a standard normal distribution
- If $\mathbf{u} \in \mathbb{R}^d$, show $\mathbb{E}[\|f(\mathbf{u})\|_2^2] = \|\mathbf{u}\|_2^2$
- Show that the random variable $\|f(\mathbf{u})\|_2^2$ concentrates around $\|\mathbf{u}\|_2^2$, and construct a union bound over all pairwise distances.

Johnson-Lindenstrauss Lemma: Demonstration

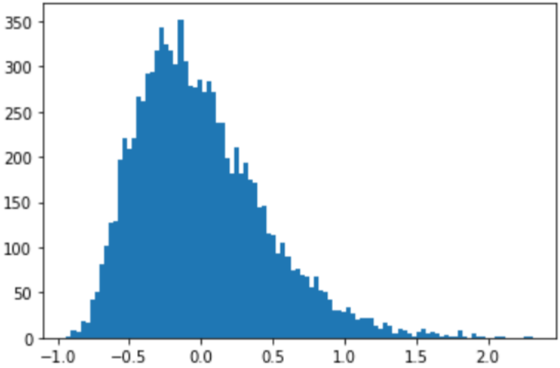

Figure: Histogram of $\|\mathbf{u}\|_2^2 - \|f(\mathbf{u})\|_2^2$ for a fixed $\mathbf{u} \in \mathbb{R}^{1000}$, $f(\mathbf{u}) \in \mathbb{R}^{10}$

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Deterministic Interpolative Decomposition

Given a matrix $A \in \mathbb{R}^{m \times n}$ we can come up with a low-rank matrix approximation that uses $A's$ own columns [Yin+18]. The ID can be computed using the column-pivoted QR factorization :

$$AP = QR\,.$$

To obtain our low-rank approximation we form the submatrix Q_k formed by the first k columns of Q. Thus we have the approximation:

$$A \approx Q_k Q_k^* A$$

which gives us a particular rank k projection of A.

Randomized Interpolative Decomposition

This method constructs a subset S of randomly selected distinct $p > k$ columns from the n columns of A. The algorithm then performs the column-pivoted QR factorization on the p columns of A:

$$A_{(:,S)}P = QR$$

Accordingly we have the following rank k projection of A:

$$A \approx Q_k Q_k^* A\,,$$

where Q_k is the submatrix formed by the first k columns of Q.

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Deterministic Singular Value Decomposition

- Recall the singular value decomposition of a matrix [16]:

$$A_{m\times n} = U_{m\times m}\Sigma_{m\times n}V_{n\times n}^{T},$$

  where U and V are orthogonal matrices, and Σ is a diagonal matrix with positive diagonal entries $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$, where r is the rank of the matrix A.
- The σ_is are called the singular values of A.

Randomized Singular Value Decomposition

Utilizing ideas from [HMT09], our algorithm executes the following steps to compute the randomized SVD:

1. Construct a $n \times k$ random Gaussian matrix Ω
2. Form $Y = A\Omega$
3. Construct a matrix Q whose columns form an orthonormal basis for the column space of Y
4. Set $B = Q^*A$
5. Compute the SVD: $B = U'\Sigma V^*$,
6. Construct the SVD approximation: $A \approx QQ^*A = QB = QU'\Sigma V^*$

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Results - Testing 620×187500 Matrix

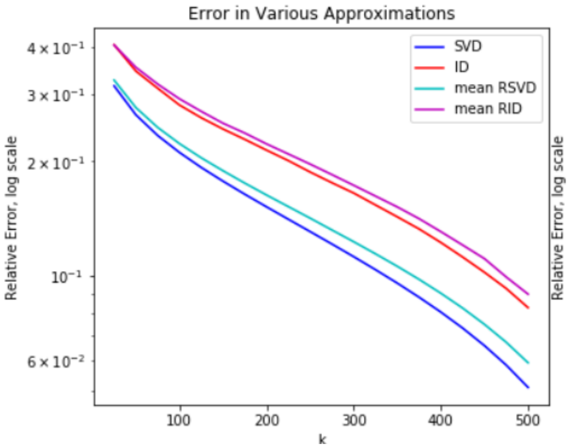

Figure: Error Relative to Original Data

Results - Testing 620 × 187500 Matrix

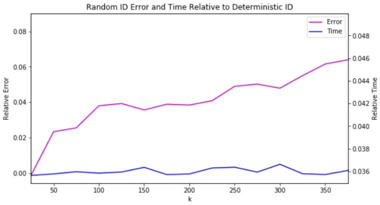

Figure: Random ID Error and Time Relative to Deterministic ID

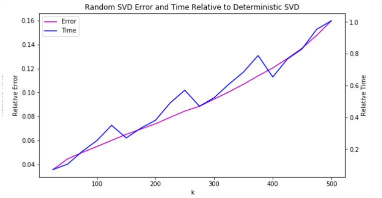

Figure: Random SVD Error and Time Relative to Deterministic SVD

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Eigenfaces

- Using ideas from [BKP15] our eigenfaces experiment tests the LFW dataset [Hua+07]. This dataset contains more then 13,000 RGB images of faces where each image is a 250×250.
- We can flatten each image to represent it as vector of length $250 \cdot 250 \cdot 3 = 187500$.
- In our experiment we will only use 620 images from the LFW dataset. This gives us a data matrix A of size 187500×620.
- We then can perform SVD on the mean-subtracted columns of A.

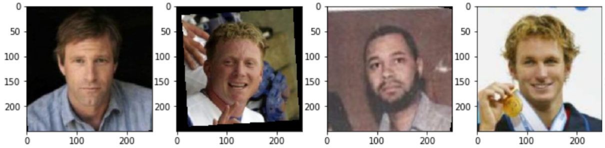

Figure: Original LFW Images

We obtain the following eigenfaces from the columns of the matrix U:

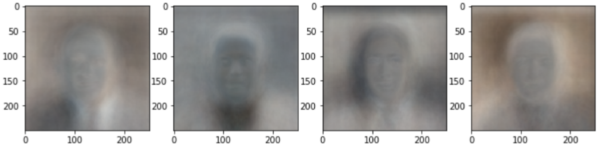

Figure: Eigenfaces Obtained using Deterministic SVD

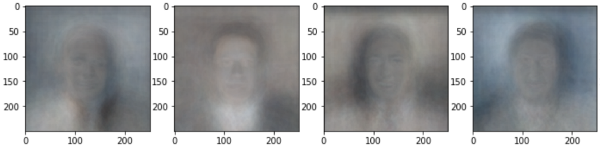

Figure: Eigenfaces Obtained using Randomized SVD

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Kernel Methods

- Kernel methods describe mapping the data into a high-dimensional space to add more structure and encourage linear separability.

$$\phi : \mathbb{R}^n \to \mathbb{R}^m, \quad m > n$$

- The 'kernel trick' allows for only the inner products to be computed, as opposed to the explicit high-dimensional mappings.

$$k(\mathbf{x}, \mathbf{y}) = \langle \phi(\mathbf{x}), \phi(\mathbf{y}) \rangle$$

- Ex. Gaussian/RBF Kernel: $k(\mathbf{x}, \mathbf{y}) = \exp\left(-\gamma \|\mathbf{x} - \mathbf{y}|_2^2\right)$
- Kernel methods include kernel PCA, kernel SVM, and more.

Randomized Fourier Features Kernel

Introduced in [RR08], we can sample random Fourier features to approximate a kernel. Let $k(\mathbf{x}, \mathbf{y})$ denote our kernel, and $p(\mathbf{w})$ the probability distribution corresponding to the inverse Fourier transform of k.

$$
\begin{aligned}
k(\mathbf{x}, \mathbf{y}) &= \int_{\mathbb{R}^d} p(\mathbf{w}) e^{-\mathrm{j}\mathbf{w}^T(\mathbf{x}-\mathbf{y})} \mathrm{d}\mathbf{w} \\
&\approx \frac{1}{m} \sum_{i=1}^{m} \cos(w_i^T \mathbf{x} + b_i) \cos(\mathbf{w_i}^T \mathbf{y} + b_i)
\end{aligned}
$$

where $\mathbf{w_i} \sim p(\mathbf{w})$, $b_i \sim \text{Uniform}(0, 2\pi)$. For a given m, define

$$
z(\mathbf{x}) = \sum_{i=1}^{m} \cos(\mathbf{w_i}^T \mathbf{x} + b_i)
$$

to yield the approximation $k(\mathbf{x}, \mathbf{y}) \approx \frac{1}{m} z(\mathbf{x}) z(\mathbf{y})^T$ [Lop+14].

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Data for Kernel PCA Experiments

To test kernel PCA methods, we used a dataset that is not linearly separable– a cloud of points surrounded by a circle:

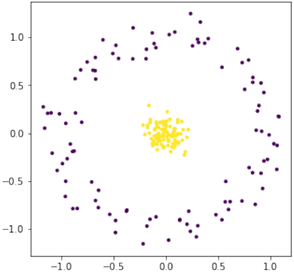

Figure: Data used to test kernel PCA methods

Randomized Kernel PCA Results

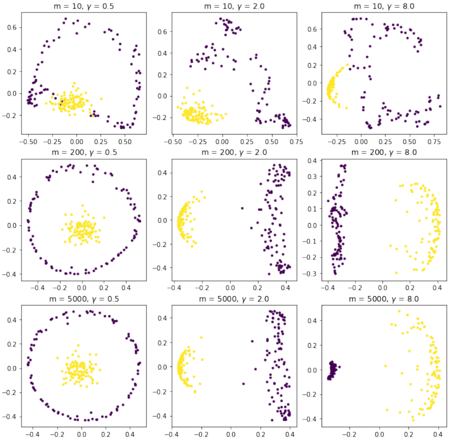

Figure: Random Fourier features KPCA results

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Kernel SVM

- We may also use kernel methods for support vector machines (SVM).
- The goal of an SVM is to find the $(d-1)$-hyperplane that best separates two clusters of d-dimensional data points.
- In two dimensions, this is a line separating two clusters of points in a plane.
- Using the kernel trick, we can project inseparable points into a higher dimension and run an SVM algorithm on the resulting points.

Randomized Kernel SVM

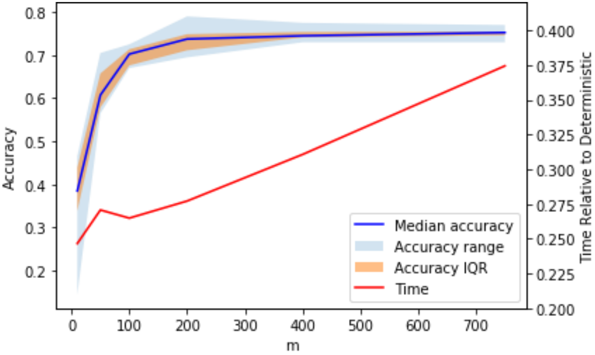


Figure: Randomized Kernel SVM Accuracy and time results as m varies

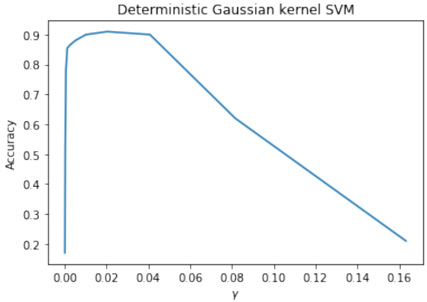
Comparison of Deterministic and Randomized Kernel SVM

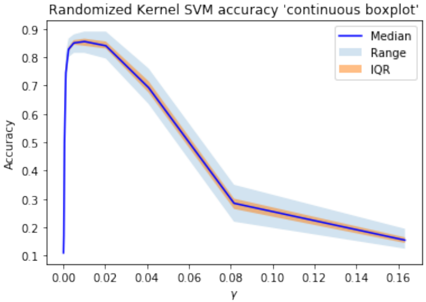
Using the MNIST dataset [LC10] we test 10000 images (784 features), for a **fixed** γ:

- Deterministic Kernel
  - Accuracy: 0.9195
  - Time: 37.99s
- Randomized Kernel
  - Accuracy: Mean: 0.891, St. dev. 0.0042
    Min: 0.881, Max: 0.9005
  - Mean Time: 2.14s

Comparison of Deterministic and Randomized Kernel SVM

On 1000 MNIST images, we plot the accuracies of the deterministic and random kernel SVMs as γ varies:





Application of Randomized Kernel SVM: Grid Search

Testing 100 γ values to identify the best one:

- Deterministic Kernel, Series: 133.03s
- Randomized Kernel, Series: 78.97s
- Randomize Kernel, Parallel: 41.18s
- Best γ value obtained from randomized method corresponds with either best or second best deterministic γ (3 trials)

$$\hat{\mathbf{K}} = \frac{1}{m} z(\mathbf{X}) z(\mathbf{X})^T$$

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Website

To explore more visit our website at the following link:
`https://rishi1999.github.io/random-projections/`