Basic mathematics: suggested exercises

Exercise 1. Factorize the following polynomials:

(a) $x^3 - 3x^2 - x + 3$ $\qquad [(x-1)(x+1)(x-3)]$

(b) $x^3 + x^2 - 2x$ $\qquad [x(x-1)(x+2)]$

(c) $x^3 - x^2 - x - 2$ $\qquad \left[(x-2)(x^2+x+1)\right]$

Exercise 2. Given the polynomial

$$p(x) = x^4 - 2x^3 - 2x^2 + 9x - 6,$$

Check if $x - 5$ divides $p(x)$ and find the real roots of $p(x)$. Is the inequality $p(x) > 0$ equivalent to $\frac{x-1}{x+2} > 0$? $\qquad$ [No, $-2, 1$, Yes]

Exercise 3. Solve the following equations:

(a) $\left(\dfrac{3x-5}{x}\right)^6 + 9\left(\dfrac{5-3x}{x}\right)^3 + 8 = 0$ $\qquad \left[\dfrac{5}{2};\ 5\right]$

(b) $(-x^2 + 2x - 1)^3 = (-x^2 + 2x - 1)^5$ $\qquad [0;\ 1;\ 2]$

Exercise 4. Determine for which values of $b \in \mathbb{R}$ the equation

$$x^4 + bx^2 + 1 = 0$$

admits:

(a) no solutions; $\qquad [b > -2]$

(*b*) only one solution; $[\nexists b \in \mathbb{R}]$

(*c*) two solutions; $[b = -2]$

(*d*) four solutions. $[b < -2]$

Exercise 5. Determine for which values of $b \in \mathbb{R}$ the equation

$$x^4 + bx^2 - 1 = 0$$

admits:

(*a*) no solutions; $[\nexists b \in \mathbb{R}]$

(*b*) only one solution; $[\nexists b \in \mathbb{R}]$

(*c*) two solutions; $[\forall b \in \mathbb{R}]$

(*d*) four solutions. $[\nexists b \in \mathbb{R}]$

Exercise 6. Solve the following inequalities:

(*a*) $5x^3 - 2x^2 - 5x + 2 < 0$ $\left[x < -1,\ \frac{2}{5} < x < 1\right]$

(*b*) $\dfrac{x^2 - 5x + 6}{x^2 + 1} > 0$ $[x < 2,\ x > 3]$

(*c*) $x^4 - 10x^2 + 9 > 0$ $[x < -3,\ -1 < x < 1,\ x > 3]$

(*d*) $\dfrac{x^2 - 5x + 6}{x^2 - 3x + 10} > 0$ $[x < 2,\ x > 3]$

(*e*) $\dfrac{x^2 + 10x + 16}{x - 1} > 10$ $[x > 1]$

(*f*) $2(x^2 - 5)(x^2 + 4) > 0$ $\left[x < -\sqrt{5},\ x > \sqrt{5}\right]$

(*g*) $(-2x^2 + 7x - 5)^3(x + 1)(4 - x^2)^4 \leq 0$ $\left[x = -2\,, -1 \leq x \leq 1\,, x = 2\,,\ x \geq \frac{5}{2}\right]$

(*h*) $\dfrac{1}{(1 + x^2)^2} - \dfrac{4}{(16 + x^2)^2} < 0$ $\left[x < -\sqrt{14}\,,\ x > \sqrt{14}\right]$

Exercise 7. Let $P(x)$ be a polynomial. Knowing that the solutions of the inequality $P(x) > 0$ is

$$x < -1\,, \quad 0 < x < 1\,, \quad x > 9\,,$$

determine the solutions of the following inequalities:

$$P(-x) > 0\,, \quad P\left(\sqrt{x}\right) > 0\,, \quad P\left(x^2\right) > 0\,, \quad P(3x) > 0\,,$$

$$P\left(\frac{1}{3}x\right) > 0\,, \quad \frac{1}{P(x)} > 0\,, \quad \frac{P(x)}{P(-x)} > 0\,.$$

$$\left[\begin{array}{rl}
P(-x) > 0 & \Longrightarrow x < -9\,, \quad -1 < x < 0\,, \quad x > 1\,, \\
P\left(\sqrt{x}\right) > 0 & \Longrightarrow 0 < x < 1\,, \quad x > 81\,, \\
P\left(x^2\right) > 0 & \Longrightarrow x < -3\,, \quad -1 < x < 0\,, \quad 0 < x < 1\,, \quad x > 3\,, \\
P(3x) > 0 & \Longrightarrow x < -\frac{1}{3}\,, \quad 0 < x < \frac{1}{3}\,, \quad x > 3\,, \\
P\left(\frac{1}{3}x\right) > 0 & \Longrightarrow x < -3\,, \quad 0 < x < 3\,, \quad x > 27\,, \\
\frac{1}{P(x)} > 0 & \Longrightarrow x < -1\,, \quad 0 < x < 1\,, \quad x > 9\,, \\
\frac{P(x)}{P(-x)} > 0 & \Longrightarrow x < -9\,, \quad x > 9\,.
\end{array}\right]$$

Exercise 8. Solve the following equations:

(a) $\quad -\sqrt{x^3} - 4\sqrt{x} + 4x + 1 = 0$ $\qquad \left[1;\ \frac{7 \pm 3\sqrt{5}}{2}\right]$

(b) $\quad 2\sqrt{x^2 - 1} + x\sqrt{4 - x^2} = 0$ $\qquad \left[-\sqrt{2}\right]$

(c) $\quad 2 + \dfrac{1}{(x+1)\sqrt{-(x+1)}} = 0$ $\qquad \left[-1 - 2^{-\frac{2}{3}}\right]$

(d) $\quad \dfrac{2x}{(4 - x^2)^{\frac{3}{2}}} - \dfrac{1}{(x^2 - 1)^{\frac{3}{2}}} = 0$ $\qquad \left[\sqrt{2}\right]$

(e) $\quad 4 - x^{\frac{2}{3}}\left(1 - x^{\frac{1}{3}}\right)^2 = 0$ $\qquad [-1;\ 8]$

Exercise 9. Solve the following inequalities:

(a) $x+3>\sqrt{3x^2+10x+3}$ $\left[-\dfrac{1}{3}\le x<1\right]$

(b) $\sqrt{x^2-3x+2}\ge\sqrt{-x^2-x+6}$ $[-3\le x\le -1\,,\; x=2]$

(c) $2x+1>\sqrt[4]{16x^4+32x^3+24x^2}$ $\left[x>-\dfrac{1}{8}\right]$

(d) $2\sqrt{6x^3+7x^2-9x+2}\ge -x^2$ $\left[-2\le x\le\dfrac{1}{3}\,,\; x\ge\dfrac{1}{2}\right]$

(e) $3x\sqrt{1+x^2}+2\ge 0$ $\left[x\ge-\dfrac{\sqrt{3}}{3}\right]$

(f) $\sqrt{\dfrac{x-2}{x-1}}>2$ $\left[\dfrac{2}{3}<x<1\right]$

(g) $10-x^2+3x\sqrt{5-x^2}\ge 0$ $\left[-\sqrt{5}\le x\le -2\,,\; -\sqrt{\dfrac{5}{2}}\le x\le\sqrt{5}\right]$

(h) $16-3x-8\sqrt{4-x}\ge 0$ $\left[x\le 0\,,\; \dfrac{32}{9}\le x\le 4\right]$

(i) $\dfrac{4x\sqrt{x}-3x-1}{4x(x+1)^2\sqrt{x}}\ge 0$ $[x\ge 1]$

(l) $\dfrac{x}{(4-x^2)^{\frac{3}{2}}}+\dfrac{1}{x^2}\le 0$ $\left[-2<x\le-\sqrt{2}\right]$

(m) $\sqrt{7-2x}\ge x-3$ $\left[x\le 2+\sqrt{2}\right]$

(n) $\sqrt{x+1}<2-x$ $\left[-1\le x<\dfrac{5-\sqrt{13}}{2}\right]$

(o) $\sqrt{(x-1)(3-x)}>-2x+3$ $\left[\dfrac{6}{5}<x\le 3\right]$

(p) $\sqrt{3-2x-x^2}>0$ $[-3<x<1]$

Exercise 10. Solve the following equations:

(a) $|x-1| = |1-x|$ $[\forall x \in \mathbb{R}]$

(b) $|6x-5| = |3-2x|$ $\left[\frac{1}{2},\, 1\right]$

(c) $|x^2-2| = 3-|x|$ $\left[\pm\frac{\sqrt{21}-1}{2}\right]$

Exercise 11. Solve the following inequalities:

(a) $|x| + |-x| \le 2$ $[-1 \le x \le 1]$

(b) $|3x^2-x-2| < 1$ $\left[\frac{1-\sqrt{37}}{6} < x < \frac{1-\sqrt{13}}{6},\ \frac{1+\sqrt{13}}{6} < x < \frac{1+\sqrt{37}}{6}\right]$

(c) $\left|\frac{6x+1}{2x+5} - 3\right| < 1$ $\left[x < -\frac{19}{2}\,,\ x > \frac{9}{2}\right]$

(d) $\sqrt[3]{x^3-x} \ge |x|$ $\left[-\frac{\sqrt{2}}{2} \le x \le 0\right]$

(e) $\sqrt[3]{x^3-|x|} \ge |x|$ $[x = 0]$

(f) $\sqrt[3]{|x|^3+|x|} \ge |x|$ $[\forall x \in \mathbb{R}]$

(g) $\sqrt{-x} + |x| \ge 2$ $[x \le -2\,,\ -1 \le x \le 0]$

(h) $1 + |x-1| \le 1 + |x|$ $\left[x \ge \frac{1}{2}\right]$

(i) $\sqrt{x+1} < -|x^2-3x+7|$ $[\nexists x \in \mathbb{R}]$

(l) $\frac{1}{1+x^2} + \frac{4|x|}{x(16+x^2)} \ge 0$ $[x \ge -2]$

(m) $\quad \dfrac{|x|}{x}\,(2|x|-2)^{\frac{2}{3}} \geq (x+1)^{\frac{2}{3}}$ $\qquad [x \geq 3]$

Exercise 12. Determine for which values of $a \in \mathbb{R}$ the equation

$$\Big||x|-1\Big| = a$$

admits:

(a) no solutions; $\qquad [a < 0]$

(b) only one solution; $\qquad [\nexists a \in \mathbb{R}]$

(c) two solutions; $\qquad [a = 0\,,\ a > 1]$

(d) three solutions; $\qquad [a = 1]$

(e) four solutions. $\qquad [0 < a < 1]$

Exercise 13. Determine for which values of $x \in \mathbb{Z}$ the number

$$\Big|(x^2+x-1)(x^2-7x+11)\Big|$$

is prime. $\qquad \Big[\{x \in \mathbb{Z} :\ -2 \leq x \leq 5\}\Big]$

Exercise 14. Determine how many points $P(x,y)$ of the plane have integer coordinates x, y such that $|x|+|y| \leq 3$. Determine how many points $P(x,y)$ of the plane have integer coordinates x, y such that $|x|+|y| \leq n$, as n varies in $\mathbb{N}$.

$[25;\ 2n^2+2n+1]$

Exercise 15. Solve the following equations:

(a) $\quad (\log x - 1)^{\frac{2}{3}} = \dfrac{1}{3}$ $\qquad \Big[e^{1 \pm \frac{\sqrt{3}}{9}}\Big]$

(b) $\quad \log(2x+3) - \log(x+2) = \log(1-x)$ $\qquad \left[\dfrac{-3+\sqrt{5}}{2}\right]$

(c) $\quad \log \dfrac{x+2}{x-2} - \log x = 0$ $\qquad \left[\dfrac{3+\sqrt{17}}{2}\right]$

Exercise 16. Solve the following inequalities:

(a) $\log_{\frac{1}{2}}(x^2+3x+2) < \log_{\frac{1}{2}}(x^2+x-2)$ $[x > 1]$

(b) $\log x - 2\log\left(\sqrt{2}\right) < \log(3x^2-1)$ $\left[x > \dfrac{2}{3}\right]$

(c) $\log_x(2-x^2) \geq 0$ $[\nexists x \in \mathbb{R}]$

(d) $\log_{\frac{1}{3}}\sqrt{x+1} < 1 + \log_{\frac{1}{3}}\sqrt{4-x^2}$ $\left[\dfrac{\sqrt{61}-9}{2} < x < 2\right]$

(e) $\dfrac{\log(|x|-1)}{x} < 0$ $[x < -2\,,\ 1 < x < 2]$

(f) $\dfrac{\log(-x)}{1-x} < 0$ $[-1 < x < 0]$

(g) $\dfrac{|\log x|}{(\log x - 1)^2} \leq \dfrac{1}{2}$ $\left[0 < x \leq e^{2-\sqrt{3}}\,,\ x \geq e^{2+\sqrt{3}}\right]$

(h) $\log_{x^2+2}(x+1) \leq \dfrac{1}{2}$ $\left[-1 < x \leq \dfrac{1}{2}\right]$

Exercise 17. Plot in the Cartesian plane (O, x, y) the solutions of the following equations:

(a) $\log(xy) = 0$ $\left[y = \dfrac{1}{x}\right]$

(b) $\log x + \log y = 0$ $\left[y = \dfrac{1}{x}\,, x > 0\right]$

(c) $\log|x| + \log|y| = 0$ $\left[y = \pm\dfrac{1}{x}\right]$

(d) $\log x + \log y = 1$ $\left[y = \dfrac{e}{x}\,, x > 0\right]$

(e) $\log(-x) - \log y = \log 2$ $\left[y = -\dfrac{x}{2}\,, x < 0\right]$

Exercise 18. Solve the following equations:

(a) $\quad 2 \cdot 4^{2x} - 3 \cdot 4^{x} - 1 = 0$ $\qquad \left[\log_4 \dfrac{3+\sqrt{17}}{4}\right]$

(b) $\quad \dfrac{\sqrt[x]{5^{8-3x}}}{\sqrt[2-x]{5^{2x}}} = \sqrt[3-x]{5^{x+5}}$ $\qquad [1]$

(c) $\quad 9^{x} + 3^{x+1} - 4 = 0$ $\qquad [0]$

Esercizio 19. Solve the following inequalities:

(a) $\quad \dfrac{e^{x} - e^{-x}}{2} > 1$ $\qquad \left[x > \log\left(1+\sqrt{2}\right)\right]$

(b) $\quad (e^{x} - 1)\left(e^{2x} - 5e^{x} + 6\right) \leq 0$ $\qquad [x \leq 0\,,\ \log 2 \leq x \leq \log 3]$

(c) $\quad 2 - e^{x}\,(e^{x} - 1)^{\frac{2}{3}} \geq 0$ $\qquad [x \leq \log 2]$

(d) $\quad \left(\dfrac{1}{4}\right)^{4x^2-1} < \left(\dfrac{1}{4}\right)^{3x^2+2x+2}$ $\qquad [x < -1\,,\ x > 3]$

(e) $\quad \left(\dfrac{1}{2}\right)^{2x-3} > 8$ $\qquad [x < 0]$

(f) $\quad 5^{x} + 5^{\frac{x}{2}} < 2$ $\qquad [x < 0]$

Exercise 20. Solve the following equations:

(a) $\quad \sin^3 x - 1 = 0$ $\qquad \left[\dfrac{\pi}{2} + 2k\pi\,,\ \forall k \in \mathbb{Z}\right]$

(b) $\quad \sin(3x - 2) = \frac{1}{2}$ $\qquad \left[(-1)^{k}\dfrac{\pi}{18} + \dfrac{2}{3} + k\dfrac{\pi}{3}\,,\ \forall k \in \mathbb{Z}\right]$

(c) $\quad \sin x - \cos 2x = 2$ $\qquad \left[\dfrac{\pi}{2} + 2k\pi\,,\ \forall k \in \mathbb{Z}\right]$

Exercise 21. Solve the following inequalities:

(a) $\cos\dfrac{\pi}{2}x > -\dfrac{1}{2}$ $\left[-\dfrac{4}{3}+4k < x < \dfrac{4}{3}+4k\,,\ \forall k\in\mathbb{Z}\right]$

(b) $\cot\left(\dfrac{3x+\pi}{2}\right) > -1$ $\left[(2k-1)\dfrac{\pi}{3} < x < \dfrac{\pi}{2}+(2k-1)\dfrac{\pi}{3}\,,\ \forall k\in\mathbb{Z}\right]$

(c) $\cos x(1-2\sin x) > 0$

$$\left[\frac{\pi}{2}+2k\pi < x < \frac{5}{6}\pi+2k\pi\,,\ \frac{3}{2}\pi+2k\pi < x < \frac{13}{6}\pi+2k\pi\,,\ \forall k\in\mathbb{Z}\right]$$

(d) $1+2\sin 2x \geq 0\,,\qquad -\pi\leq x\leq\pi$

$$\left[-\pi\leq x\leq -\frac{5}{12}\pi\,,\ -\frac{\pi}{12}\leq x\leq\frac{7}{12}\pi\,,\ \frac{11}{12}\pi\leq x\leq\pi\right]$$

(e) $\dfrac{\cos x}{\sqrt{2\cos x-1}} > \dfrac{\sqrt{2}}{2}\,,\qquad -\pi<x<\pi$ $\left[-\dfrac{\pi}{3} < x < \dfrac{\pi}{3}\right]$

(f) $\sqrt{5-2\sin x}\geq 6\sin x-1\,,\qquad 0\leq x\leq 2\pi$ $\left[0\leq x\leq\dfrac{\pi}{6}\,,\ \dfrac{5}{6}\pi\leq x\leq 2\pi\right]$

(g) $\dfrac{1-2\sin x}{1+2\cos x}\leq 0\,,\qquad 0\leq x\leq 2\pi$ $\left[\dfrac{\pi}{6}\leq x<\dfrac{2}{3}\pi\,,\ \dfrac{5}{6}\pi\leq x<\dfrac{4}{3}\pi\right]$

(h) $\tan\left(\dfrac{1}{1+x^2}\right)\geq 1$ $\left[-\sqrt{\dfrac{4-\pi}{\pi}}\leq x\leq\sqrt{\dfrac{4-\pi}{\pi}}\right]$