MATHEMATICAL ANALYSIS I TUTORING
6^{TH} WEEK

THEOREMS ON DIFFERENTIABLE FUNCTIONS - STUDY OF FUNCTIONS

PROPOSED EXERCISES - SOLUTIONS

1. Say if the function $f(x) = 1 + x + \sqrt{1-x^2}$ satisfies the hypothesis of Lagrange Theorem. If so, compute the Lagrange points.

   Lagrange Theorem. Let f be defined on a closed and bounded interval $[a, b]$, continuous on $[a, b]$ and differentiable (at least) on (a, b). Then there exists $x_0 \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(x_0)$$

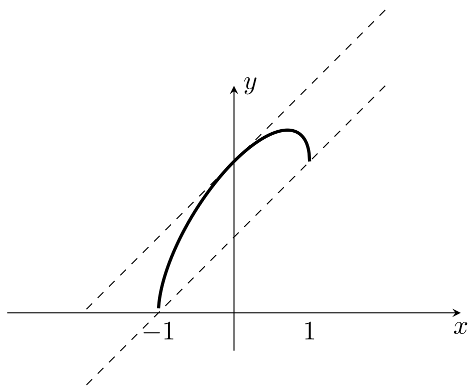
   $D = [-1, 1]$ is the domain and its is closed and bounded, on such domain the function is continuous. Deriving the function $f'(x) = 1 + \frac{-x}{\sqrt{1-x^2}}$, we observe that the function is differentiable on $(-1, 1)$. The hypothesis of Lagrange Theorem are satisfied. Thus there exists at least one point $x_0 \in (-1, 1)$ such that

$$f'(x_0) = \frac{f(1) - f(-1)}{1 - (-1)} = \frac{2 - 0}{2} = 1$$

   In order to find the Lagrange points, we have to solve the equation $f'(x) = 1$, i.e.

$$1 + \frac{-x}{\sqrt{1-x^2}} = 1 \Rightarrow \frac{-x}{\sqrt{1-x^2}} = 0 \Rightarrow x = 0$$

   Thus $x_0 = 0$ is the Lagrange point. Recall that the Lagrange points are the ones such that the tangent to the graph of f in $(x_0, f(x_0))$ is parallel to the line passing through $(-1, f(-1))$ and $(1, f(1))$.



2. Find the maximal interval I containing the point $x = 1$ such that $f(x) = e^{x^2} + x^2$ is invertible. Let $g(y)$ be the inverse function of $f(x)$ on such interval, compute $g'(e + 1)$.

   The function is even and thus not injective. A sufficient condition for inveritibility is (strict) monotonicity. Find a monotonicity interval including $x = 1$:
   $f'(x) = 2x\left(e^{x^2} + 1\right) \quad \Rightarrow \quad I = [0, +\infty)$.
   Apply the Theorem for the derivative of the inverse function, then:
   $g'(e+1) = (f^{-1})'(e+1) = \frac{1}{f'(a)}$, where a is the unique value such that $f(a) = e + 1$.
   Since $f(1) = e + 1$, and $f'(1) = 2(e + 1)$, it holds

$$g'(e+1) = \frac{1}{2(e+1)}$$

3. Compute the proposed limits, if possible, applying De l'Hopital Theorem:

De l'Hopital Theorem. Given f and g defined in a neighborhood of x_0, except in x_0, and such that $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = L$, with $L = 0$ or $+\infty$ or $-\infty$. If f and g are differentiable in a neighborhood of x_0 except in x_0, with $g'(x) \neq 0$, and if there exists (finite or infinite) $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$, then there exists also $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ and $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$.

a) $\boxed{\lim_{x\to 0^+} x\log x}$

$$\lim_{x\to 0^+} x\log x = \lim_{x\to 0^+} \frac{\log x}{\frac{1}{x}} = \frac{-\infty}{+\infty}$$

De l'Hopital Theorem applies; compute the limit of the derivatives quotient

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = 0$$

Such limit exists; then we can conclude that

$$\lim_{x\to 0^+} x\log x = 0$$

b) $\boxed{\lim_{x\to 0^+} \frac{\log \sin x}{\log x}}$ We get the indeterminate form ∞/∞. De l'Hopital Theorem applies; compute the limit of the derivatives quotient

$$\lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{x}{\sin x}\cos x = 1$$

Such limit exists; then we can conclude that

$$\lim_{x\to 0^+} \frac{\log \sin x}{\log x} = 0$$

c) $\boxed{\lim_{x\to 0} \frac{e^{3x} - e^{-2x}}{\sin 2x}}$

We get the indeterminate form $0/0$. De l'Hopital Theorem applies; compute the limit of the derivatives quotient

$$\lim_{x\to 0} \frac{3e^{3x} + 2e^{-2x}}{2\cos 2x} = \frac{5}{2}$$

Such limit exists; then we can conclude that

$$\lim_{x\to 0} \frac{e^{3x} - e^{-2x}}{\sin 2x} = \frac{5}{2}$$

d) $\boxed{\lim_{x\to 0} \frac{1 + 6x - \sqrt{(1+4x)^3}}{2x\sin x}}$ We get the indeterminate form $0/0$. De l'Hopital Theorem applies; compute the limit of the derivatives quotient

$$\lim_{x\to 0} \frac{6 - \frac{6(4x+1)^2}{\sqrt{(1+4x)^3}}}{2\sin x + 2x\cos x}$$

We get the indeterminate form $0/0$, De l'Hopital Theorem applies again:

$$\lim_{x\to 0} \frac{-\frac{12(4x+1)}{\sqrt{(1+4x)^3}}}{2\cos x + 2\cos x - 2x\sin x} = -3$$

Such limit exists; then we can conclude that

$$\lim_{x\to 0} \frac{1 + 6x - \sqrt{(1+4x)^3}}{2x\sin x} = -3$$

e) $\boxed{\lim_{x\to+\infty}\frac{2x+\sin x}{3x-\cos x}}$

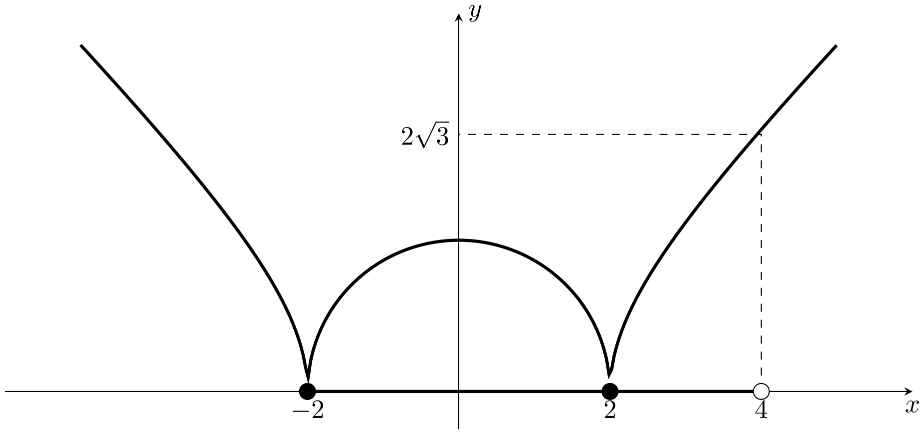
We get the indeterminate form $0/0$. De l'Hopital Theorem applies, but $\lim_{x\to+\infty}\frac{2+\cos x}{3+\sin x}$ does not exist. This does not mean that the original limit does not exist, indeed:

$$\lim_{x\to+\infty}\frac{2x+\sin x}{3x-\cos x}=\lim_{x\to+\infty}\frac{x(2+\frac{\sin x}{x})}{x(3-\frac{\cos x}{x})}=\lim_{x\to+\infty}\frac{(2+\frac{\sin x}{x})}{(3-\frac{\cos x}{x})}=\frac{2}{3}$$

(recall that $\lim_{x\to+\infty}\frac{\sin x}{x}=\lim_{x\to+\infty}\frac{\cos x}{x}=0$.)

4. Find the extremal points (minima/maxima) of the following function: $f:[-2,4)\to\mathbb{R}$, $f(x)=\sqrt{|4-x^2|}$.

- $x=-2$, $x=2$ are absolute minimum points; the absolute minimum of f is 0
- $x=0$ is a relative maximum point
- f has no absolute maximum; $\sup f=2\sqrt{3}$



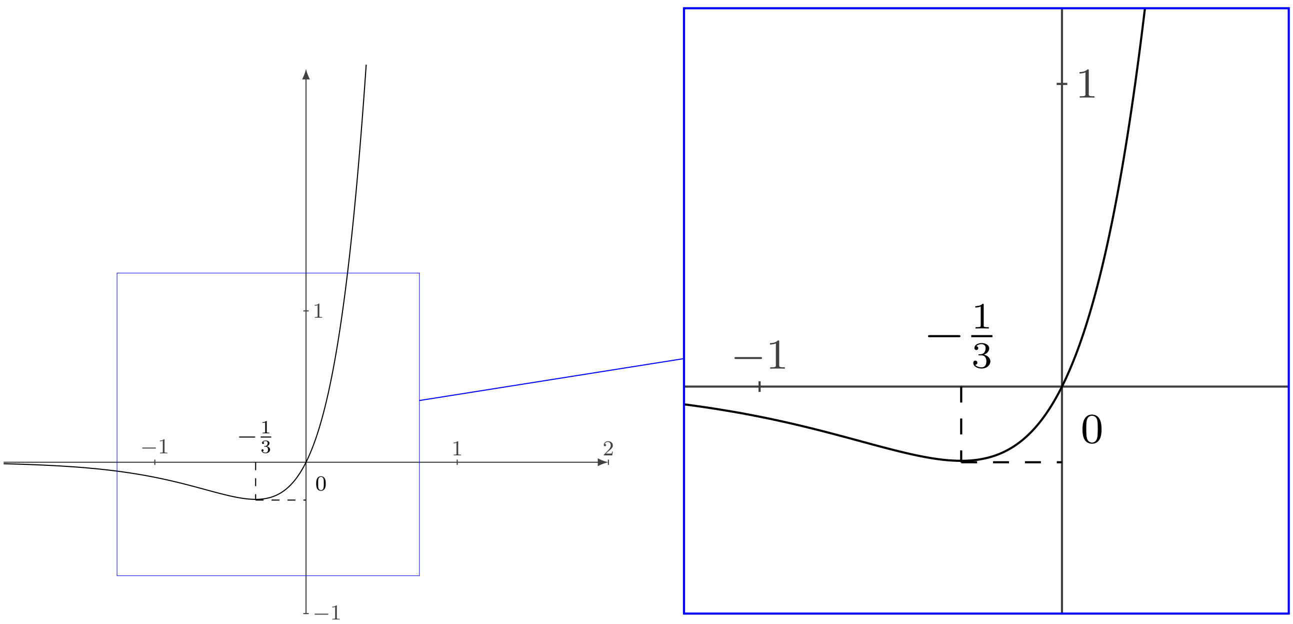
5. Find the stationary points of the following function: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=2xe^{3x}$

$$f'(x)=2e^{3x}+6xe^{3x}=(2+6x)e^{3x}$$

The unique stationary point of f is $x=-\frac{1}{3}$. In order to study its nature, we study the sign of f':

$$f'(x)>0\Leftrightarrow(2+6x)>0\Leftrightarrow x>-\frac{1}{3}$$

Thus f is decreasing on $\left(-\infty,-\frac{1}{3}\right)$ and increasing on $\left(-\frac{1}{3},+\infty\right)$; thus $x=-\frac{1}{3}$ is an absolute minimum for f.



6. Verify that the point $x = 0$ is stationary for

$$f(x) = \begin{cases} x^3 \left(4 + \sin \frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Determine whether this point is a maximum, a minimum or a point of inflection with horizontal tangent.

Compute the limit of the difference quotient:

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^3 \left(4 + \sin \frac{1}{x}\right) - 0}{x - 0} == \lim_{x \to 0} x^2 \left(4 + \sin \frac{1}{x}\right) = 0$$

Thus $f'(0) = 0$ and $x = 0$ is a stationary point for $f(x)$.
If $x > 0$, we have $f(x) > 0$, while if $x < 0$, we have $f(x) < 0$; therefore $x = 0$ cannot be maximum nor minimum (otherwise there would be a neighborhood of $x = 0$ where the function is positive or negative; hence it is a horizontal tangent point.

7. Consider the function

$$f(x) = \log(x^2 - |2x - 1| + 3)$$

a) domain, limits at boundary points of the domain and asympototes;

b) monotonicity intervals, non-differentiable points and extrema;

c) find the largest interval where f is invertible, containing the point $x = 1$;

e) trace a qualitative graph of $f(x)$, $f(|x|)$, $|f(x)|$, $e^{f(x)}$, $\log f(x)$.

a) Notice that

$$f(x) = \log(x^2 - |2x - 1| + 3) = \begin{cases} \log(x^2 - 2x + 4) = f_1(x) & \text{if } x \geq \frac{1}{2} \\ \log(x^2 + 2x + 2) = f_2(x) & \text{if } x < \frac{1}{2} \end{cases}$$

Since $x^2 - 2x + 4$ and $x^2 + 2x + 2$ have no real roots, they are always positive, and thus the two functions $f_1(x)$ and $f_2(x)$ are everywhere defined. Thus dom $f = \mathbb{R}$.
Since $\lim\limits_{x \to \frac{1}{2}^-} f_2(x) = \lim\limits_{x \to \frac{1}{2}^+} f_1(x) = f_1(\frac{1}{2}) = \log \frac{13}{4}$, f is continuous $\forall\, x \in \mathbb{R}$ (and there are no vertical asymptotes).
$\lim\limits_{x \to +\infty} f_1(x) = \lim\limits_{x \to -\infty} f_2(x) = +\infty \;\Rightarrow\;$ there are no horizontal asymptotes
$\lim\limits_{x \to +\infty} \dfrac{f_1(x)}{x} = \lim\limits_{x \to -\infty} \dfrac{f_2(x)}{x} = 0 \;\Rightarrow\;$ there are no oblique asymptotes

b)

$$f'(x) = \begin{cases} \dfrac{2x - 2}{x^2 - 2x + 4} = f_1'(x) & \text{if } x > \frac{1}{2} \\[2ex] \dfrac{2x + 2}{x^2 + 2x + 2} = f_2'(x) & \text{if } x < \frac{1}{2} \end{cases}$$

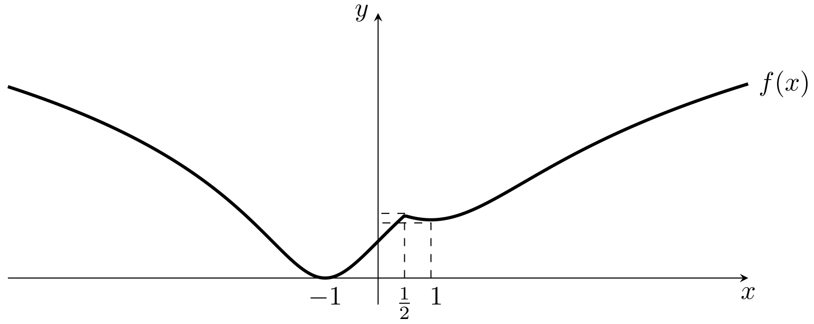
From the study of $f_1'(x)$, it's null in $x = 1$, $f_1'(x) > 0$ if $x > 1$ and $f_1'(x) < 0$ if $x < 1$; thus f_1 is decreasing on $(\frac{1}{2}, 1)$ and increasing on $(1, +\infty)$, and $x = 1$ is a (relative) minimum point.
From the study of $f_2'(x)$, it's null in $x = -1$, $f_2'(x) > 0$ if $x > -1$ and $f_2'(x) < 0$ if $x < -1$; thus f_2 is decreasing on $(-\infty, -1)$ and increasing on $(-1, \frac{1}{2})$; $x = -1$ is a minimum point (absolute, because $f(-1) = 0$ and $f(x) \geq 0, \forall x \in \mathbb{R}$).
Since $\lim\limits_{x \to \frac{1}{2}^-} f_2'(x) = \frac{12}{13}$ while $\lim\limits_{x \to \frac{1}{2}^+} f_1'(x) = \frac{-4}{5}$, it holds that f is not differentiable in $x = \frac{1}{2}$ (corner point)
In conclusion: f is increasing on $\left(-1, \frac{1}{2}\right)$ and $(1, +\infty)$; f is decreasing on $(-\infty, -1)$ and $\left(\frac{1}{2}, 1\right)$; $x = -1$ is absolute minimum point; $x = 1$ is relative minimum point; $x = \frac{1}{2}$ is relative maximum point

c) f is strictly increasing on $[1, +\infty)$; thus the largest interval of invertibility of f containg $x = 1$ is $[1, +\infty)$.

d)

$$f''(x) = \begin{cases} \dfrac{-2(x^2-2x-2)}{(x^2-2x+4)^2} = f_1''(x) & \text{se } x > \frac{1}{2} \\ \\ \dfrac{-2x(x+2)}{(x^2+2x+2)^2} = f_2''(x) & \text{se } x < \frac{1}{2} \end{cases}$$

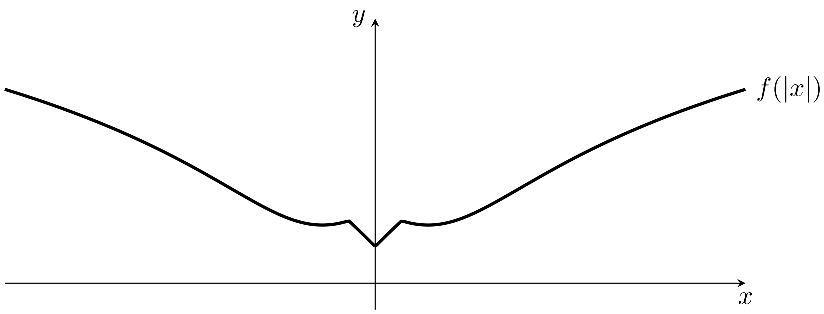
From the study of f_1'', it's null in $x = 1 \pm \sqrt{3}$, but only $x = 1+\sqrt{3} > \dfrac{1}{2}$ and thus it's an inflection point for f_1; moreover $f_1''(x) > 0$ if $x > 1+\sqrt{3}$ while $f_1''(x) < 0$ if $x < 1+\sqrt{3}$; thus f_1 is concave on $(\frac{1}{2}, 1+\sqrt{3})$ and convex on $(1+\sqrt{3}, +\infty)$: the point $x = 1+\sqrt{3}$ is an inflection point.
From the study of f_2'', it's null in $x = 0$ and in $x = -2$; moreover $f_2''(x) > 0$ if $-2 < x < 0$ while $f_2''(x) < 0$ if $x < -2$ and $0 < x < \dfrac{1}{2}$; thus f_2 is concave on $(-\infty, -2)$ and $(0, \frac{1}{2})$ and convex on $(-2, 0)$: $x = -2$ and $x = 0$ are inflection points.
In conclusion: f is convex on $(-2, 0)$ and $\left(\frac{1}{2}, 1+\sqrt{3}\right)$; f is concave on $(-\infty, -2)$, $\left(0, \frac{1}{2}\right)$ and $(1+\sqrt{3}, +\infty)$.
The point $x = -2$ is a descending inflection point; the points $x = 0$ are $x = 1+\sqrt{3}$ are ascending inflection points.



$\boxed{f(|x|)}$

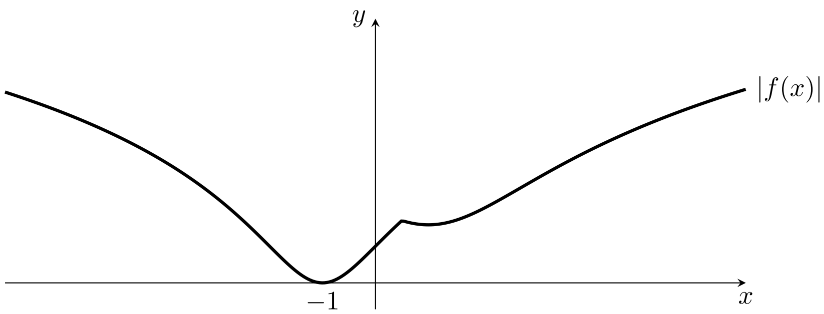
By definition $f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$.

The function $f(|x|)$ is even and coincides with $f(x)$ on $\mathbb{R}_+$; on $\mathbb{R}_-$, it coincides with $f(-x)$ and thus its graph is the symmetric of $f(x), x \in \mathbb{R}_+$ w.r.t. the y axis.



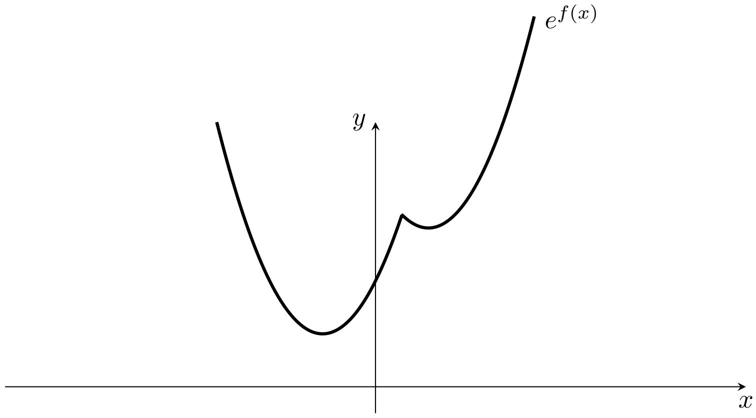
$\boxed{|f(x)|}$

Since $f(x) \geq 0, \forall x \in \mathbb{R}$, it holds $|f(x)| = f(x)$.



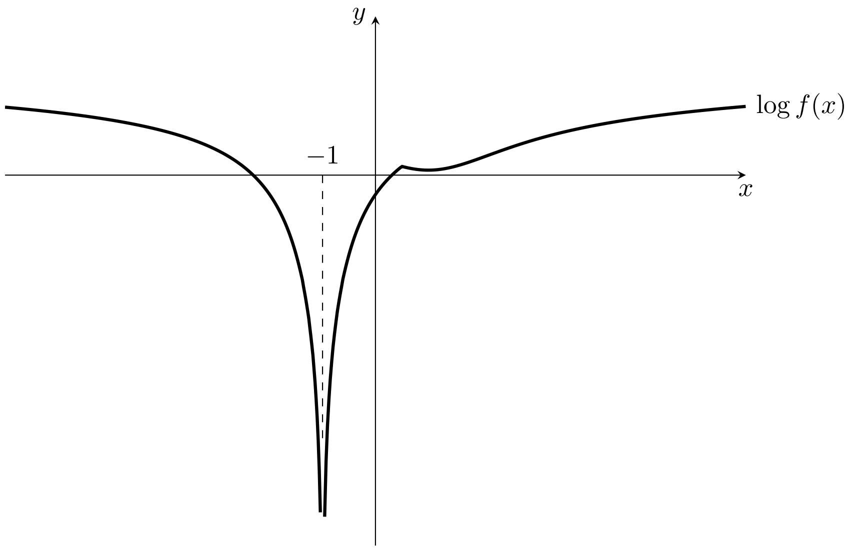
$\boxed{e^{f(|x|)}}$

$$e^{f(|x|)} = e^{\log(x^2-|2x-1|+3)} = x^2 - |2x-1| + 3 == \begin{cases} x^2 - 2x + 4 & \text{if } x \geq \frac{1}{2} \\ x^2 + 2x + 2 & \text{if } x < \frac{1}{2} \end{cases}$$



$\boxed{\log f(x)}$

The function $\log(x)$ is always increasing, thus $\log f(x)$ has the same monotonicity of $f(x)$; $f(x)$ is positive for $x \neq -1$, in a neighborhood of such point $\log f(x)$ tends to $-\infty$.



8. Given the following function, find:

   a) domain, limits at boundary points of the domain and asympototes;
   b) zeros and sign;
   c) monotonicity intervals and (relative/absolute) extremal points;
   d) concavity and inflection points;
   e) trace a qualitative graph.

$$\boxed{f_1(x) = \frac{x^2+2x+4}{2x}}$$

a) $\text{dom} f_1 = \mathbb{R} \setminus \{0\}$, $\quad f_1$ is not even nor odd
$\lim_{x\to-\infty} f_1(x) = -\infty, \quad \lim_{x\to+\infty} f_1(x) = +\infty \quad \Rightarrow$ there are no horizontal asymptotes
$\lim_{x\to 0^\pm} f_1(x) = \mp\infty \;\Rightarrow x = 0$ vertical asymptote
$\lim_{x\to\pm\infty} \frac{f_1(x)}{x} = \frac{1}{2}, \quad \lim_{x\to\pm\infty}\left(f_1(x) - \frac{1}{2}x\right) = 1 \quad \Rightarrow y = \frac{1}{2}x + 1$ oblique asymptote (right and left)

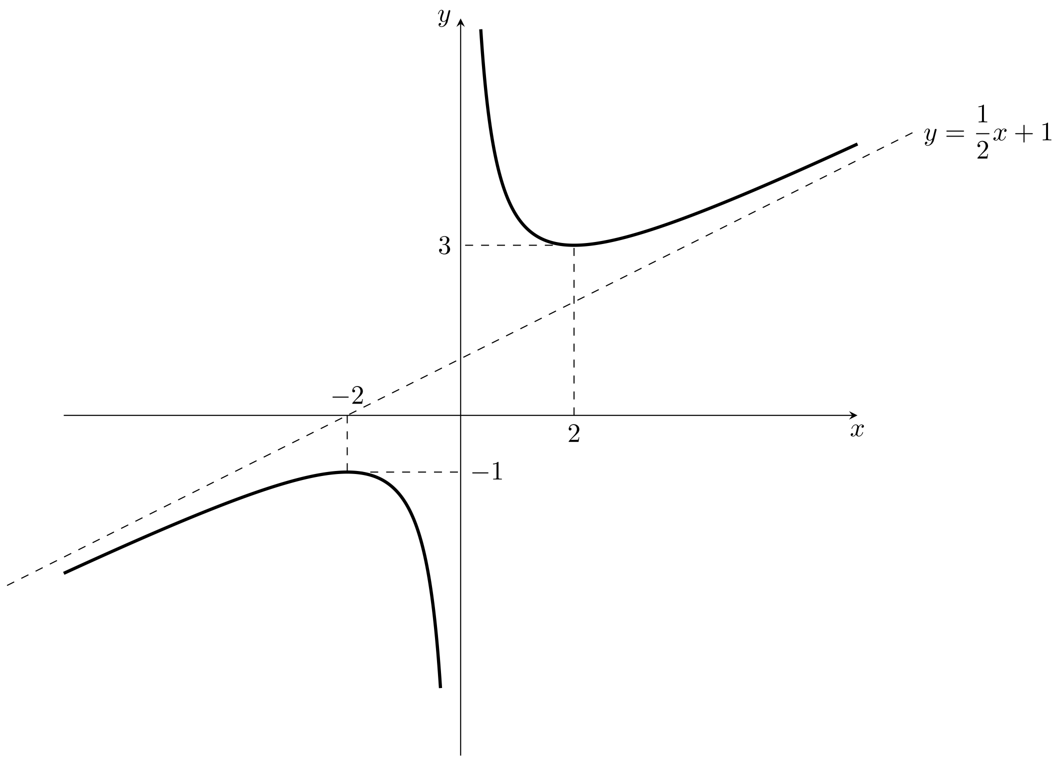
b) $f_1(x)$ has no zeros, because the numerator is strictly positive for every $x \in \mathbb{R}$.
$f_1(x) > 0 \Leftrightarrow x > 0$; thus $f_1(x)$ is positive if $x > 0$ and negative if $x < 0$.

c)
$$f_1'(x) = \frac{(2x+2)2x - 2(x^2+2x+4)}{4x^2} = \frac{x^2-4}{2x^2}$$

$f_1'(x) = 0 \Leftrightarrow x = \pm 2$; $\quad f_1'(x) > 0 \Leftrightarrow x \in (-\infty,-2)\cup(2,+\infty)$; $\quad f_1'(x) < 0 \Leftrightarrow x \in (-2,2)$.
Thus f_1 is increasing on $(-\infty,-2)$ and $(2,+\infty)$; $\quad f_1$ is decreasing on $(-2,0)$ and $(0,2)$;
$x = -2$ is a relative maximum point; $x = 2$ is a relative minimum point.

d)
$$f_1''(x) = \frac{4}{x^3}$$

$f_1''(x)$ is never null; $\quad f_1''(x) > 0 \Leftrightarrow x \in (0,+\infty) : \; f_1''(x) < 0 \Leftrightarrow x \in (-\infty, 0)$.
Hence f_1 has no inflection points, it is concave in $(-\infty,0)$ and convex in $(0,+\infty)$.



$$\boxed{f_2(x) = \frac{|x^2-1|}{x^2}}$$

a) $\text{dom} f_2 = \mathbb{R} \setminus \{0\}$, $\quad f_2$ is even
$\lim_{x\to\pm\infty} f_2(x) = 1 \;\Rightarrow\; y = 1$ horizontal asymptote
$\lim_{x\to 0^\pm} f_2(x) = +\infty \;\Rightarrow\; x = 0$ vertical asymptote
There are no oblique asymptotes, because there is a horizontal (left and right) asymptote.

b) $f_2(x) = 0 \;\Leftrightarrow x = \pm 1$; f_2 is always positive.

c)

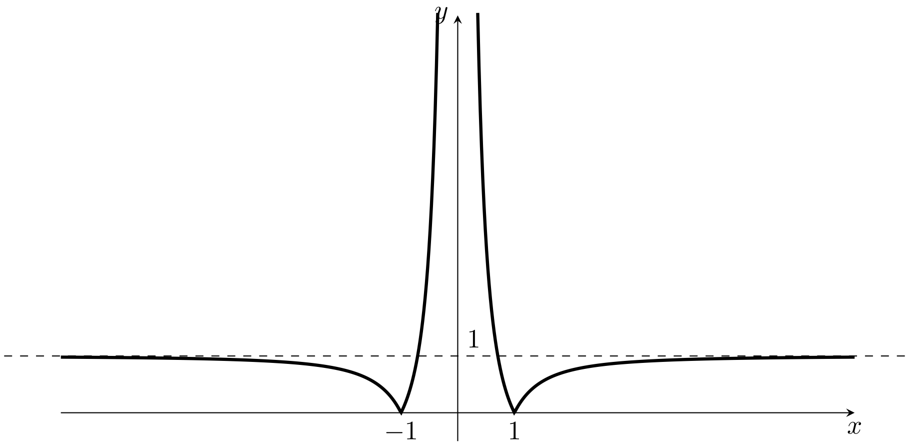
$$f_2'(x) = \frac{2x \cdot x^2 - 2x(x^2-1)}{x^4} \frac{|x^2-1|}{x^2-1} = \frac{2}{x^3} \frac{|x^2-1|}{x^2-1}$$

f_2' is never null, thus there are no stationary points; f_2 is increasing on $(-1, 0)$ and $(1, +\infty)$; f_2 is decreasing on $(-\infty, -1)$ and $(0, 1)$.
The points $x = -1$ and $x = 1$ are absolute minima (corner points).

d)

$$f_2''(x) = \frac{-6}{x^4} \frac{|x^2-1|}{x^2-1}$$

f_2'' is never null, thus there are no inflection points; f_2 is convex in $(-1, 0)$ e in $(0, 1)$; f_2 is concave in $(-\infty, -1)$ and $(1, +\infty)$.



$\boxed{f_3(x) = |x-2|e^x}$

a) $\text{dom} f_3 = \mathbb{R}$ f_3 is not even nor odd
$\lim_{x\to-\infty} f_3(x) = 0, \quad \lim_{x\to+\infty} f_3(x) = +\infty \;\Rightarrow\;$ the line $y = 0$ is left horizontal asymptote $\Rightarrow$ there is no left oblique asymptote
there is no left oblique asymptote, since $\lim_{x\to+\infty} \frac{f_3(x)}{x} = +\infty$

b) $f_3(x) = 0 \iff x = 2; \quad f_3(x) > 0 : \ \forall x \in \mathbb{R} \setminus \{2\}$

c)

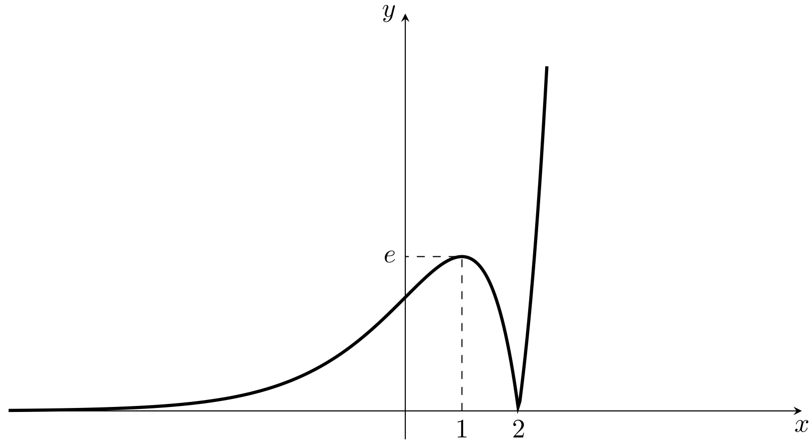
$$f_3'(x) = \frac{|x-2|}{x-2} e^x + |x-2|e^x = (x-1)e^x \frac{|x-2|}{x-2}$$

$f_3'(x) = 0 \Leftrightarrow x = 1; \quad f_3'(x) > 0 \Leftrightarrow x < 1 \vee x > 2; \quad f_3'(x) < 0 \Leftrightarrow 1 < x < 2$. Thus:
f_3 is strictly increasing in $(-\infty, 1)$ and $(2, +\infty)$; $\quad f_3$ is strictly decreasing in $(1, 2)$;
$x = 1$ is a stationary point for f_3 and it's a relative maximum; $f_3(1) = e$ is a relative maximum for f_3.
$x = 2$ is an absolute minimum point for f_3 and it's a corner point; $f_3(2) = 0$ is the absolute minimum for f_3

d)

$$f_3''(x) = e^x \frac{|x-2|}{x-2} + (x-1)e^x \frac{|x-2|}{x-2} + 0 = xe^x \frac{|x-2|}{x-2}$$

$f_3''(x) = 0 \Leftrightarrow x = 0; f_3''(x) > 0 \Leftrightarrow x < 0 \vee x > 2; f_3''(x) < 0 \Leftrightarrow 0 < x < 2$. Hence:
f_3 is convex in $(-\infty, 0)$ e in $(2, +\infty)$; $\quad f_3$ is concave in $(0, 2)$; $\quad x = 0$ is an inflection point.



$$\boxed{f_4(x) = \log \left| \frac{x+3}{x-1} \right|}$$

a) $\mathrm{dom} f_4 = \mathbb{R} \setminus \{-3, 1\}$; f_4 is not even nor odd
$\lim\limits_{x\to\pm\infty} f_4(x) = 0 \;\Rightarrow\; y = 0$ horizontal asymptote, therefore there are no oblique asymptotes
$\lim\limits_{x\to -3^\pm} f_4(x) = -\infty; \quad \lim\limits_{x\to 1^\pm} f_4(x) = +\infty; \;\Rightarrow\; x = -3$ and $x = 1$ are vertical asymptotes

b) $f_4(x) = 0 \Leftrightarrow \left| \dfrac{x+3}{x-1} \right| = 1 \Leftrightarrow \dfrac{x+3}{x-1} = \pm 1 \Leftrightarrow x = -1$
$f_4(x) > 0 \Leftrightarrow \left| \dfrac{x+3}{x-1} \right| > 1 \Leftrightarrow |x+3| > |x-1| \Leftrightarrow x > -1$
Thus $f_4(x) > 0$ on $(-1, 1)$ and $(1, +\infty)$; $\quad f_4(x) < 0$ on $(-\infty, -3)$ and $(-3, -1)$

c)

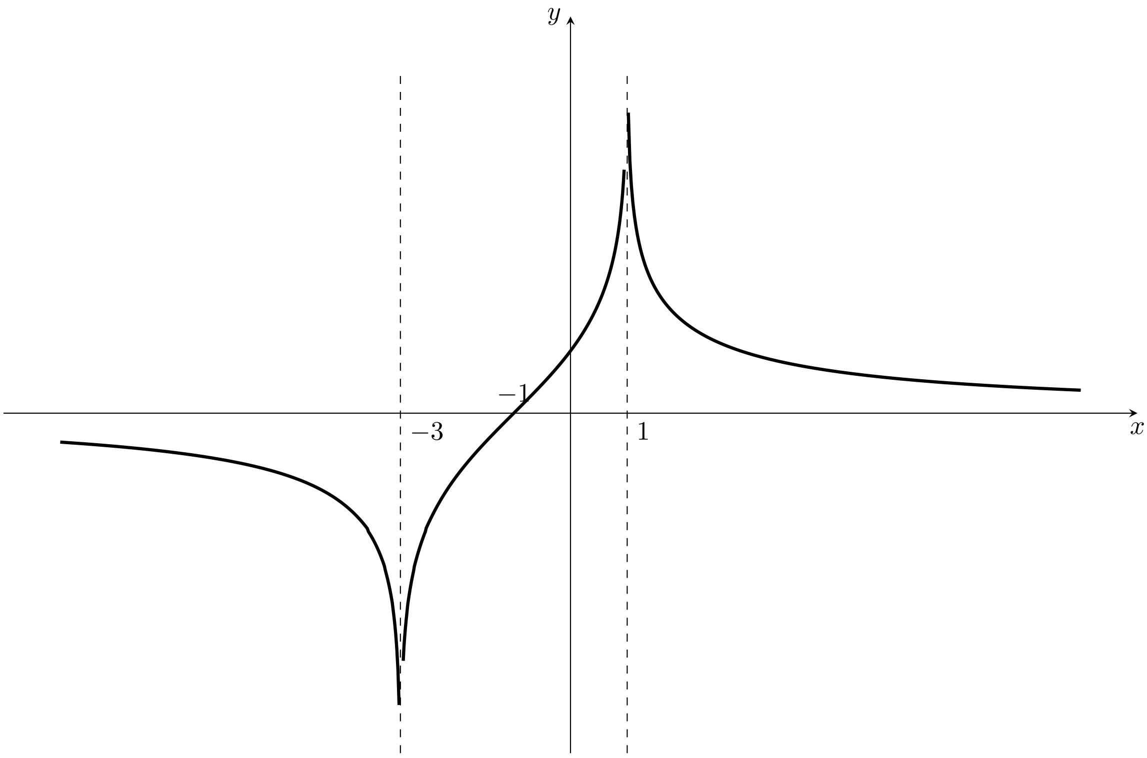
$$f_4'(x) = \frac{x-1}{x+3} \frac{(x-1)-(x+3)}{(x-1)^2} = \frac{-4}{(x+3)(x-1)}$$

Since $f_4'(x)$ is never null, there are no stationary points. Moreover:
$f_4'(x) > 0 \Leftrightarrow -3 < x < 1$, while $f_4'(x) < 0 \Leftrightarrow x < -3 \vee x > 1$.
Thus there are no extremal points (relative or absolute); f_4 is strictly increasing on $(-3, 1)$, and strictly decreasing on $(-\infty, -3)$ and $(1, +\infty)$

d)

$$f_4''(x) = \frac{0 + 4(x - 1 + x + 3)}{(x+3)^2(x-1)^2} = \frac{8(x+1)}{(x+3)^2(x-1)^2}$$

$f_4''(x) = 0 \Leftrightarrow x = -1, \quad f_4'(x) > 0 \Leftrightarrow x > -1, \quad f_4'(x) < 0 \Leftrightarrow x < -1.$
Hence f_4 is convex on $(-1, 1)$ and $(1, +\infty)$; f_4 is concave on $(-\infty, -3)$ and $(-3, -1)$; $x = -1$ is an ascending inflection point.

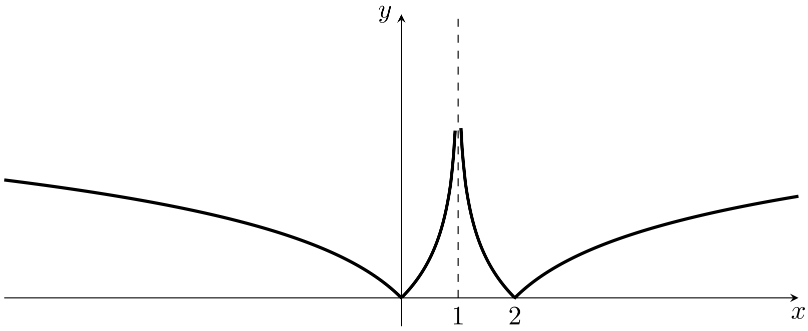


$$\boxed{f_5(x) = \sqrt{\log^2 |x-1|}}$$

Notice that $\sqrt{\log^2 |x-1|} = |\log |x-1||$; thus

$$f_5(x) = |\log |x-1||$$

and the graph can be found from the graph of $\log x$:



Thus:

a) $\mathrm{dom} f_5 = \mathbb{R} \setminus \{1\}$, $\quad f_5$ is not even nor odd
$\lim\limits_{x\to\pm\infty} f_5(x) = +\infty \;\Rightarrow$ there are no horizontal asymptotes
$\lim\limits_{x\to 1^\pm} f_5(x) = +\infty \;\Rightarrow\; x = 1$ is vertical asymptote
$\lim\limits_{x\to\pm\infty} \dfrac{f_5(x)}{x} = 0 \;\Rightarrow$ there are no oblique asymptotes

b) $f_5(x)$ is always positive; $f_5(x) = 0 \Leftrightarrow |x-1| = 1 \Leftrightarrow x = 0 \vee x = 2$

c) f_5 is increasing on $(0,1)$ and $(2,+\infty)$; $\quad f_5$ is decreasing on $(-\infty,0)$ and $(1,2)$
There are no stationary points
The points $x = 0$ and $x = 2$ are absolute minima (corner points).

SOLUTIONS - PAST EXAMS

(9 September 2015 - I)

(a) State the Intermediate value theorem (Bolzano's Theorem)
See the textbook.

(b) Determine whether the following equation

$$29^x + 3^x + 1 = 26,$$

has a solution in $\mathbb{R}$. Discuss your answer and show an interval of length one around the solution.

Define $g(x) := 29^x + 3^x + 1 - 26$, $g : \mathbb{R} \to \mathbb{R}$ of class C^∞ (differentiable infinite times with continuity). Finding the solutions of the equation is equivalent to find the zeros of g. Since

$$g \in C^0[0,1] \quad and \quad g(0)g(1) < 0\,,$$

we can apply the Existence of zeros Theorem and there exists $x_0 \in (0,1)$ such that $g(x_0) = 0$, that is a solution of the initial equation.

(c) Is the solution unique? Discuss.

We can verify that $g'(x) > 0$ for every $x \in \mathbb{R}$, thus g is strictly increasing on $\mathbb{R}$. It follows that there a unique zero for g in $\mathbb{R}$, i.e. the equation has a unique solution in $\mathbb{R}$: $x_0 \in (0,1)$.

(30 January 2015 - II)

(a) State Rolle's Theorem.

See textbook.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a C^2 function, such that the equation$f(x) = 5x + 13$ has at least three different (real) solutions. Prove that there exists c such that $f''(c) = 0$.

Define $h : \mathbb{R} \to \mathbb{R}$, $h(x) := f(x) - 5x - 13$. By hypothesis, h is in C^2 and there are $x_1 < x_2 < x_3$ such that $h(x_1) = h(x_2) = h(x_3) = 0$. Applying Rolle Theorem to h in $[x_1, x_2]$ and $[x_2, x_3]$, it follows that there are $a \in (x_1, x_2)$ and $b \in (x_2, x_3)$ such that $h'(a) = 0 = h'(b)$. Apply now Rolle Theorem to $h' \in C^1(\mathbb{R})$ in $[a, b]$, thus there is $c \in (a, b)$ such that $h''(c) = 0$. Since $h''(x) = f''(x)$ for every $x \in \mathbb{R}$, then $f''(c) = 0$.

(c) Draw a qualitative draft of this.

With the same previous notations: $h'(x) = f'(x) - 5$. Therefore, $f'(a) = 5 = f'(b)$, where 5 is also the slope of $y = 5x + 13$. See the plot below for the geometric meaning.