MATHEMATICAL ANALYSIS I TUTORING
12^{th} WEEK

DIFFERENTIAL EQUATIONS II - CAUCHY PROBLEM

PROPOSED EXERCISES - SOLUTIONS

Preliminary exercise: differential equations classification
Classify the following differential equations

Equation	order	separable variables	linear	homogeneous	constant coefficients	forcing term
$y' = 2xy + x^3$	1		$\surd$			$\surd$
$xy' + y = \sin y$	1	$\surd$				
$y' = \dfrac{y+1}{x}$	1	$\surd$	$\surd$			$\surd$
$y' + y = 2x$	1		$\surd$		$\surd$	$\surd$
$y' + y = \cos x$	1		$\surd$		$\surd$	$\surd$
$y'' - 5y' + 6y = 0$	2		$\surd$	$\surd$	$\surd$	
$y'' + 2y' + y = 0$	2		$\surd$	$\surd$	$\surd$	
$y' - 9y = 0$	1	$\surd$	$\surd$	$\surd$	$\surd$	
$\dfrac{t-y}{y''} = 3$	2		$\surd$		$\surd$	$\surd$
$y'' - 4y = x^2 e^{3x}$	2		$\surd$		$\surd$	$\surd$

1. Solve the following **2nd** order homogeneous differential equations with constant coefficients

$$a)\, y'' - 3y' + 2y = 0 \qquad b)\, y'' - 4y' + 4y = 0 \qquad c)\, y'' + 6y' + 10y = 0$$

(a) $\boxed{y'' - 3y' + 2y = 0}$

The characteristic equation has two real distinct solutions:

$$\lambda^2 - 3\lambda + 2 = 0 \Leftrightarrow (\lambda - 2)(\lambda - 1) = 0 \Leftrightarrow \lambda = 2, \quad \lambda = 1$$

The general integral is in the form:

$$y_0 = c_1 e^{1\cdot x} + c_2 e^{2\cdot x}, \quad c_1, c_2 \in \mathbb{R}$$

(b) $\boxed{y'' - 4y' + 4y = 0}$

The characteristic equation has two real coinciding solutions:

$$\lambda^2 - 4\lambda + 4\lambda = 0 \Leftrightarrow (\lambda - 2)^2 = 0 \Leftrightarrow \lambda = 2$$

The general integral is in the form:

$$y_0 = c_1 e^{2\cdot x} + c_2 x e^{2\cdot x} = e^{2\cdot x}(c_1 + c_2 x), \quad c_1, c_2 \in \mathbb{R}$$

(c) $\boxed{y'' + 6y' + 10y = 0}$

The characteristic equation has two complex solutions:

$$\lambda^2 + 6\lambda + 10 = 0 \;\Leftrightarrow\; \lambda = -3 \pm \sqrt{9 - 10} \;\Leftrightarrow\; \lambda = -3 \pm i$$

The general integral is in the form:

$$y_0 = e^{-3x}(c_1 \cos x + c_2 \sin x), \quad c_1, c_2 \in \mathbb{R}$$

2. For the following $\mathbf{2^{nd}}$ order homogeneous differential equations with constant coefficients, write the family of functions of their particular integral:

> Recall the rule: if the forcing term $g(x)$ is in the form
>
> $$g(x) = p_n(x)e^{\mu x}\cos\theta x \quad \text{or} \qquad g(x) = p_n(x)e^{\mu x}\sin\theta x$$
>
> the particular integral is in the form
>
> $$y_p(x) = x^m e^{\mu x}(q_{1,n}(x)\cos\theta x + q_{2,n}(x)\sin\theta x)$$
>
> where $q_{1,n}(x)$ e $q_{2,n}(x)$ are 2 polynomials with degree n and unknown coefficients.
> If $\Delta > 0$ we have $m = 1$ if $\theta = 0$ and $\mu = \lambda_1$ or $\mu = \lambda_2$ roots of the characteristic polynomial (*resonance*), otherwise $m = 0$.

(a) $\boxed{y'' + y' - 6y = e^{2x}}$

The associated homogeneous equation is: $y'' + y' - 6y = 0$
The characteristic equation $\lambda^2 + \lambda - 6 = 0$ has real distinct solutions

$$\lambda^2 + \lambda - 6 = 0 \Leftrightarrow (\lambda - 2)(\lambda + 3) = 0 \Leftrightarrow \lambda = -3, \lambda = 2$$

Analyze the forcing term
$g(x) = 1 \cdot e^{2x} \cdot \cos 0$, here $p_0(x) = 1$, polynomial with degree 0 $\quad\Rightarrow\quad q_{1,0}(x) = a$.
Since $\mu = 2 = \lambda$ and $\theta = 0$, we have resonance.
Thus the particular integral is in the form $y_p(x) = axe^{2x}$

$\boxed{y'' + y' - 6y = e^{3x}}$

Analyze the forcing term: $\theta = 0$, $\mu = 3$ and 3 is not a solution for the characteristic equation, $p_0(x) = 1$, with degree 0 $\quad\Rightarrow\quad q_{1,0}(x) = a$; thus

$$y_p(x) = ae^{3x}$$

$\boxed{y'' + y' - 6y = xe^{-3x}}$

Analyze the forcing term: $\theta = 0$, $\mu = -3$ and -3 is a solution for the characteristic equation; $p_1(x) = x$,; there is resonance thus

$$y_p(x) = (ax + b) \cdot xe^{-3x}$$

$\boxed{y'' + y' - 6y = x + 3}$

Analyze the forcing term: $\mu = 0$ (it is not a solution for the characteristic equation), $\theta = 0$ and $p_1(x) = x + 3$;therefore

$$y_p(x) = ax + b$$

(b) $\boxed{y'' + 4y' + 4y = e^{-2x}}$

The associated homogeneous equation is: $y'' + 4y' + 4y = 0$
The characteristic equation $\lambda^2 + 4\lambda + 4 = 0$ has real coinciding solutions

$$\lambda^2 + 4\lambda + 4 = 0 \Leftrightarrow (\lambda + 2)^2 = 0 \Leftrightarrow \lambda_1 = \lambda_2 = \lambda = -2$$

The particular solution:
$g(x) = 1 \cdot e^{-2x} \cos 0$, it holds: $p_0(x) = 1$, with degree 0 (and thus $q_{1,0}(x) = a$); $\mu = -2 = \lambda$ and $\theta = 0$, we have resonance.
Find $y_p = ax^2e^{-2x}$

$\boxed{y'' + 4y' + 4y = e^{3x}}$

Analyze the forcing term: $g(x) = 1 \cdot e^{3x}\ \cos 0$, it holds: $p_0(x) = 1$, with degree 0 (and thus $q_{1,0}(x) = a$); $\mu = 3 \neq \lambda$ and $\theta = 0$; there is no resonance, thus

$$y_p(x) = ae^{3x}$$

$\boxed{y'' + 4y' + 4y = xe^{-2x}}$

Analyze the forcing term: there is resonance, thus

$$y_p(x) = (ax + b)x^2 e^{-2x}$$

$\boxed{y'' + 4y' + 4y = x + 3}$
Analyze the forcing term: there is no resonance, thus

$$y_p(x) = ax + b$$

(c) $\boxed{y'' - 2y' + 5y = e^x}$

The associated homogeneous equation is: $y'' - 2y' + 5y = 0$
The characteristic equation $\lambda^2 - 2\lambda + 5 = 0$ has complex solutions

$$\lambda^2 - 2\lambda + 1 + 4 = 0 \ \Leftrightarrow\ \lambda_{1,2} = 1 \pm 2i$$

Analyze the forcing term: $g(x) = e^x = 1 \cdot e^x \cdot \cos 0$; it holds $p_0(x) = 1$, with degree 0 hence $q_{1,0}(x) = a$
Since $\mu = 1 = \alpha$ and $\theta = 0 \neq \beta$, there is no resonance, thus

$$y_p = a \cdot e^x$$

$\boxed{y'' - 2y' + 5y = e^x \cos 2x}$

Analyze the forcing term: there is resonance, thus

$$y_p = xe^x(a \cos 2x + b \sin 2x)$$

$\boxed{y'' - 2y' + 5y = \cos 2x}$

Analyze the forcing term: there is no resonance, thus

$$y_p = a \cos 2x + b \sin 2x$$

$\boxed{y'' - 2y' + 5y = x + 3}$
Analyze the forcing term: there is no resonance, thus

$$y_p = ax + b$$

3. Solve the following $\mathbf{2^{nd}}$ order homogeneous and complete differential equations with constant coefficients:

(a) $\boxed{y'' + y' - 6y = 2x^3 - x^2 + 1}$

The associated homogeneous equation is: $y'' + y' - 6y = 0$.
The characteristic equation $\lambda^2 + \lambda - 6 = 0$ has solutions $\lambda = -3, \lambda = 2$.
The general integral of the homogeneous equation is

$$y_0(x) = c_1 e^{-3x} + c_2 e^{2x}, \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(x) = 2x^3 - x^2 + 1$ there is no resonance, thus

$$y_p(x) = ax^3 + bx^2 + cx + d$$

By substitution of y_p and its derivatives in the initial equation, we have

$$y_p(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \; y_p'(x) = 3ax^2 + 2bx + c \quad \Rightarrow \; y_p''(x) = 6ax + 2b$$

hence

$$6ax + 2b \, + 3ax^2 + 2bx + c \, - 6(ax^3 + bx^2 + cx + d) \equiv 2x^3 - x^2 + 1$$

Impose the equality f the coefficients of x: $\begin{cases} -6a = 2 \\ 3a - 6b = -1 \\ 6a + 2b - 6c = 0 \\ 2b + c - 6d = 1 \end{cases}$.

Compute a, b, c, d; then

$$y_p(x) = -\frac{x^3}{3} - \frac{x}{3} - \frac{2}{9}$$

The general integral for the complete equation is

$$y(x) = y_o(x) + y_p(x) = c_1 e^{-3x} + c_2 e^{2x} - \frac{x^3}{3} - \frac{x}{3} - \frac{2}{9}, \quad c_1, c_2 \in \mathbb{R}$$

(b) $\boxed{x'' + 4x = \cos t}$

The associated homogeneous equation is: $x'' + 4x = 0$.
The characteristic equation $\lambda^2 + 4 = 0$ has solutions $\lambda = 2i, \lambda = -2i$. The general integral of the homogeneous equation is

$$x_0(t) = c_1 \cos(2t) + c_2 \sin(2t), \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(t) = \cos t$ there is no resonance, thus

$$x_p(t) = a \cos t + b \sin t$$

By substitution $x_p(t) = a \cos t + b \sin t$, $x_p' = -a \sin t + b \cos t$, $x_p'' = -a \cos t - b \sin t$ in the initial equation $y_p'' + 4y_p = \cos t$, we have

$$-a \cos t - b \sin t + 4(a \cos t + b \sin t) \equiv \cos t \quad \Leftrightarrow \quad 3a \cos t + 3b \sin t \equiv 1 \cos t + 0 \sin t \quad \Leftrightarrow \quad a = \frac{1}{3} \qquad b = 0$$

The particular integral for the complete equation is

$$x_p(t) = \frac{\cos t}{3}$$

The general integral for the complete equation is

$$x(t) = x_o(t) + x_p(t) = c_1 \cos(2t) + c_2 \sin(2t) + \frac{\cos t}{3}, \quad c_1, c_2 \in \mathbb{R}$$

(c) $\boxed{y'' + 2y' + y = t^2}$

The associated homogeneous equation is: $y'' + 2y' + y = 0$.
The characteristic equation $\lambda^2 + 2\lambda + 1 = 0$ has real coinciding solutions $\lambda = -1$. The general integral of the homogeneous equation is

$$y_0(t) = c_1 e^{-t} + c_2 t e^{-t}, \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(t) = t^2$ there is no resonance, thus

$$y_p(t) = at^2 + bt + c$$

By substitution $y_p(t) = at^2 + bt + c$, $y_p' = 2at + b$, $y_p'' = 2a$ in the initial equation $y_p'' + 2y_p' + y_p = x^2$ then

$$2a + 2(2at + b) + at^2 + bt + c \equiv 1\, t^2 + 0\, t + 0 \quad \Leftrightarrow \quad at^2 + (4a + b)t + 2a + 2b + c \equiv t^2 \quad \Leftrightarrow \quad a = 1, \; b = -4, \; c = 6$$

The particular integral for the complete equation is

$$y_p(t) = t^2 - 4t + 6$$

The general integral for the complete equation is

$$y(t) = y_o(t) + y_p(t) = c_1 e^{-t} + c_2 t e^{-t} + t^2 - 4t + 6, \quad c_1, c_2 \in \mathbb{R}$$

(d) $\boxed{y'' + 2y' = x^2 - 3x + 1}$

The associated homogeneous equation is: $y'' + 2y' = 0$.
The characteristic equation $\lambda^2 + 2\lambda = 0$ has real distinct solutions $\lambda = 0, \lambda = -2$. The general integral of the homogeneous equation is

$$y_0(x) = c_1 + c_2 e^{-2x}, \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(x) = x^2 - 3x + 1 = e^{0x}(x^2 - 3x + 1)$ there is no resonance, thus

$$y_p(x) = x(ax^2 + bx + cx + d)$$

By substitution $y_p(x) = x(ax^2 + bx + cx + d)$ and $y_p' = 3ax^2 + 2bx + c$, $y_p'' = 6ax + 2b$ in the initial equation $y_p'' + 2y_p' \equiv x^2 - 3x + 1$, then

$$6ax + 2b + 2(3ax^2 + 2bx + c) \equiv x^2 - 3x + 1$$

$$6ax^2 + (6a + 4b)x + 2b + 2c \equiv x^2 - 3x + 1$$

We get

$$a = \frac{1}{6}, \quad b = -1 \quad c = \frac{3}{2}$$

The particular integral for the complete equation is

$$y_p(x) = \frac{x^3}{6} - x^2 + \frac{3}{2}x$$

The general integral for the complete equation is

$$y(x) = y_o(x) + y_p(x) = c_1 + c_2 e^{-2x} + \frac{x^3}{6} - x^2 + \frac{3}{2}x, \quad c_1, c_2 \in \mathbb{R}$$

(e) $\boxed{x'' - 2x' + 2x = e^t \cos t}$

The associated homogeneous equation is: $x'' - 2x' + 2x = 0$.
The characteristic equation $\lambda^2 - 2\lambda + 2 = 0$ has real distinct solutions $\lambda = 1 + i, \lambda = 1 - i$. The general integral of the homogeneous equation is

$$x_0(t) = c_1 e^t \cos(t) + c_2 e^t \sin(t), \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(t) = e^t \cos t$ there is resonance, thus

$$x_p(t) = te^t(a \cos t + b \sin t)$$

By substitution $x_p(t) = e^t(at \cos t + bt \sin t)$,

$$x_p'(t) = e^t \left[(a + at + bt)\cos t + (b + bt - at)\sin t\right]$$

and

$$x_p''(t) = e^t \left[(2a + 2b + 2bt)\cos t + (2b - 2a - 2at)\sin t\right]$$

in the initial equation, we get

$$e^t \left[(2a + 2b + 2bt)\cos t + (2b - 2a - 2at)\sin t\right] - 2e^t \left[(a + at + bt)\cos t + (b + bt - at)\sin t\right] + 2e^t(at \cos t + bt \sin t) \equiv e^t \cos t$$

Dividing by e^t and collecting $\cos t$ e $\sin t$ we have

$$(2a+2b+2bt-2a-2at-2bt+2at)\cos t+(2b-2a-2at-2b-2bt+2at+2bt)\sin t \equiv 1\cdot\cos t+0\cdot\sin t$$

Simplify and impose the identity on the coefficients of $\cos t$ and $\sin t$, then

$$a=0,\ b=\frac{1}{2}$$

The particular integral for the complete equation is

$$x_p(t)=\frac{1}{2}te^t\sin t$$

The general integral for the complete equation is

$$x(t)=x_o(t)+x_p(t)=c_1e^t\cos(t)+c_2e^t\sin(t)+\frac{1}{2}te^t\sin t,\quad c_1,c_2\in\mathbb{R}$$

(f) $\boxed{y''-2y'+5y=t}$

The associated homogeneous equation is: $y''-2y'+5y=0$.
The characteristic equation $\lambda^2-2\lambda+5=0$ has complex distinct solutions $\lambda=1+2i,\lambda=1-2i$.
The general integral of the homogeneous equation is

$$y_0(t)=c_1e^t\cos(2t)+c_2e^x\sin(2t),\quad c_1,c_2\in\mathbb{R}$$

Analyze the forcing term: $f(t)=t=te^{0t}\cos(0t)$ there is no resonance, thus

$$y_p(t)=at+b$$

By substitution $y_p'(t)=a$ and $y_p''(t)=0$ in the initial equation, we have

$$-2a+5at+5b\equiv t$$

Therefore

$$a=\frac{1}{5},\quad b=\frac{2}{25}$$

The particular integral for the complete equation is

$$y_p(t)=\frac{1}{5}t+\frac{2}{25}$$

The general integral for the complete equation is

$$y(t)=y_o(t)+y_p(t)=c_1e^t\cos(2t)+c_2e^x\sin(2t)+\frac{1}{5}t+\frac{2}{25},\quad c_1,c_2\in\mathbb{R}$$

(g) $\boxed{x''-x=te^{-t}}$

The associated homogeneous equation is: $x''-x=0$.
The characteristic equation $\lambda^2-1=0$ has real distinct solutions $\lambda=-1,\lambda=1$. The general integral of the homogeneous equation is

$$x_0(t)=c_1e^{-t}+c_2e^t,\quad c_1,c_2\in\mathbb{R}$$

Analyze the forcing term: $f(t)=te^{-t}=te^{-t}\cos(0t)$ there is resonance, thus

$$x_p(t)=t(at+b)e^{-t}$$

By substitution

$$x_p(t)=(at^2+bt)e^{-t}$$

and

$$x_p'(t)=\left[-at^2+(2a-b)t+b\right]e^{-t}$$

and

$$x_p''(t)=\left[at^2+(b-4a)t+2a-2b\right]e^{-t}$$

in the initial equation, we have

$$\left[at^2 + (b-4a)t + 2a - 2b\right] e^{-t} - (at^2 + bt)e^{-t} \equiv te^{-t}$$

Divide by e^{-t}:

$$at^2 + (b-4a)t + 2a - 2b - at^2 - bt \equiv t$$

then

$$a = -\frac{1}{4}, \quad b = -\frac{1}{4}.$$

The particular integral for the complete equation is

$$x(t) = \left(-\frac{1}{4}\,t^2 - \frac{1}{4}\,t\right) e^{-t}$$

The general integral for the complete equation is

$$x(t) = x_o(t) + x_p(t) = c_1 e^{-t} + c_2 e^{t} + \left(-\frac{1}{4}\,t^2 - \frac{1}{4}\,t\right) e^{-t}, \quad c_1, c_2 \in \mathbb{R}$$

(h) $\boxed{y'' - y = 2x \sin x}$

The associated homogeneous equation is: $y'' - y = 0$.
The characteristic equation $\lambda^2 - 1 = 0$ has real distinct solutions $\lambda = -1, \lambda = 1$. The general integral of the homogeneous equation is

$$y_0(x) = c_1 e^{-x} + c_2 e^{x}, \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(x) = 2x \sin x = 2x \cdot e^{0x} \sin x$ there is no resonance, thus

$$y_p(x) = (ax + b)\cos x + (cx + d)\sin x$$

By substitution

$$y_p(x) = (ax + b)\cos x + (cx + d)\sin x$$

and

$$y_p'(x) = (cx + a + d)\cos x + (-ax - b + c)\sin x$$

and

$$y_p''(x) = (-ax - b + 2c)\cos x + (-cx - 2a - d)\sin x$$

in the initial equation $y'' - y = 2x \sin x$, we have

$$(-ax - b + 2c)\cos x + (-cx - 2a - d)\sin x - (ax + b)\cos x - (cx + d)\sin x \equiv 2x \sin x$$

$$(-2ax - 2b + 2c)\cos x + (-2cx - 2a - 2d)\sin x \equiv 0 \cdot \cos x + 2x \sin x$$

We get the linear system:

$$\begin{cases} -2a = 0 \\ -2b + 2c = 0 \\ -2c = 2 \\ -2a - 2d = 0 \end{cases}.$$

Therefore

$$a = 0, \quad b = -1, \quad c = -1, \quad d = 0$$

and the particular integral is

$$y_p(x) = -\cos x - x \sin x$$

The general integral for the complete equation is

$$y(t) = y_o(t) + y_p(t) = c_1 e^{-x} + c_2 e^{x} - \cos x - x \sin x, \quad c_1, c_2 \in \mathbb{R}$$

(i) $\boxed{y'' - y = e^{2x}}$

The associated homogeneous equation is: $y'' - y = 0$.
The characteristic equation $\lambda^2 - 1 = 0$ has real distinct solutions $\lambda = -1, \lambda = 1$. The general integral of the homogeneous equation is

$$y_0(t) = c_1 e^{-x} + c_2 e^{x}, \quad c_1, c_2 \in \mathbb{R}$$

Analyze the forcing term: $f(x) = e^{2x} = 1 \cdot e^{2x} \cos(0x)$ there is no resonance, thus

$$y_p(x) = ae^{2x}$$

By substitution $y_p' = 2ae^{2x}$ and $y_p'' = 4ae^{2x}$ in the initial equation $y_p'' - y_p = e^{2x}$ we get

$$4ae^{2x} - ae^{2x} = e^{2x}$$

Divide by e^{2x}, thus $a = \dfrac{1}{3}$, and the particular integral is

$$y_p(x) = \frac{1}{3}\, e^{2x}$$

The general integral for the complete equation is

$$y(t) = y_o(x) + y_p(x) = c_1 e^{-x} + c_2 e^{x} + \frac{1}{3}\, e^{2x}, \quad c_1, c_2 \in \mathbb{R}$$

4. Given the differential equation $y' = 2xy(y-4)$

a) find the constant solutions
$y = 0$ and $y = 4$

b) find the solution verifying the condition $y(0) = 2$
Let us observe that the constant solutions $y(x) = 0$ and $y(x) = 4$ do not satisfy the condition $y(0) = 2$; then we must find solve the Cauchy problem:

$$\begin{cases} y' = 2xy(y-4) \\ y(0) = 2 \end{cases}$$

The separable variables equation can be solved as follows

$$y' = 2xy(y-4) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy(y-4) \Rightarrow \frac{\mathrm{d}y}{y(y-4)} = 2x\mathrm{d}x$$

$$\int \frac{\mathrm{d}y}{y(y-4)} = \int 2x\mathrm{d}x$$

$$\frac{1}{4}\log\left|\frac{y-4}{y}\right| = x^2 + c \quad \Leftrightarrow \quad \log\left|\frac{y-4}{y}\right| = 4x^2 + 4c \quad \Leftrightarrow \quad \left|\frac{y-4}{y}\right| = e^{4x^2+4c}$$

If $h = e^{4c}$ we have

$$\left|\frac{y-4}{y}\right| = he^{4x^2}$$

Impose $y(0) = 2$, then

$$\left|\frac{2-4}{2}\right| = h \quad \Rightarrow \quad h = 1$$

Thus the solution $y(x)$, defined on $(0,4)$, such that

$$\frac{4-y}{y} = e^{4x^2} \quad \Rightarrow \quad 4 - y = ye^{4x^2} \quad \Rightarrow \quad y\left(1 + e^{4x^2}\right) = 4$$

Therefore

$$y\,(x) = \frac{4}{e^{4x^2} + 1}.$$

5. Find, if they exist, the constant solutions for the equation $x' = x^2 - 3x + 2$.
Then, solve the Cauchy problem: $x' = x^2 - 3x + 2,\ x(13) = 5$.

$$x^2 - 3x + 2 = (x-2)(x-1) = 0 \Rightarrow x = 2, x = 1.$$

Then the particular solutions are $x(t) = 1$ and $x(t) = 2$; since they do not satisfy the condition $x(13) = 5$, then we must find solve the Cauchy problem:

$$\begin{cases} x' = x^2 - 3x + 2 \\ x(13) = 5 \end{cases}$$

$$x' = x^2 - 3x + 2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - 3x + 2 \Rightarrow \frac{\mathrm{d}x}{(x-2)(x-1)} = \mathrm{d}t$$

$$\int \frac{1}{(x-2)(x-1)}\,\mathrm{d}x = \int \mathrm{d}t \;\Rightarrow\; \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right)\mathrm{d}x = \int \mathrm{d}t \;\Rightarrow\; \log\left|\frac{x-2}{x-1}\right| = t+c \;\Rightarrow\; \left|\frac{x-2}{x-1}\right| = e^{t+c}$$

Putting $e^c = k > 0$, the solutions $x(t)$ for $x > 2$, are

$$\frac{x-2}{x-1} = ke^t \quad\Rightarrow\quad x - 2 = ke^t x - ke^t \quad\Rightarrow\quad x = \frac{2 - ke^t}{1 - ke^t}$$

Imposing the condition $x(13) = 5$,we get $k = \frac{3}{4}e^{-13}$, then we conclude

$$x(t) = \frac{8 - 3e^{t-13}}{4 - e^{t-13}}.$$

6. Solve the Cauchy Problems:

(a) $\boxed{\begin{cases} y' = \dfrac{y-1}{x\log x} \\ y\,(1/e) = 2 \end{cases}}$

The constant solution $y = 1$ does not satisfy the boundary condition thus it's not a solution for the Cauchy problem.

Solve the separable variables equation

$$y' = \frac{y-1}{x\log x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-1}{x\log x} \Rightarrow \frac{\mathrm{d}y}{y-1} = \frac{\mathrm{d}x}{x\log x}$$

We consider the initial condition by integrating from 2 and y, and on the right hand-side from $\frac{1}{e}$ to x:

$$\int_2^y \frac{\mathrm{d}s}{s-1} = \int_{1/e}^x \frac{\mathrm{d}t}{t\log t} \Rightarrow \log|s-1|\big|_2^y = \log|\log t|\big|_{1/e}^x \Rightarrow \log|y-1| - \log|2-1| = \log|\log x| - \log|\log e^{-1}|$$

From $y\,(1/e) = 2$, we have $y > 1$ and $\log x < 0 \;\Rightarrow\; 0 < x < 1$; thus

$$\Rightarrow \log(y-1) = \log(-\log x) - \log(1) \Rightarrow y - 1 = -\log x \Rightarrow y = 1 - \log x$$

In conclusion

$$y = 1 - \log x, \quad x \in (0,1)$$

(b) $\boxed{\begin{cases} y' = (y+2)^2\cos^3 x \\ y\left(\dfrac{\pi}{2}\right) = 0 \end{cases}}$

The constant solution $y = -2$ does not satisfy the boundary condition thus it's not a solution for the Cauchy problem.

Solve the separable variables equation

$$y' = (y+2)^2\cos^3 x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = (y+2)^2\cos^3 x \Rightarrow \frac{\mathrm{d}y}{(y+2)^2} = \cos^3 x\,\mathrm{d}x$$

As before, integrate from 0 to y, and from $\frac{\pi}{2}$ and x:

$$\int_0^y \frac{\mathrm{d}s}{(s+2)^2} = \int_{\pi/2}^x \cos^3 t \ \mathrm{d}t$$

Recall that

$$\int \cos^3 t \, \mathrm{d}t = \int \left(\cos t \cos^2 t\right) \, \mathrm{d}t = \int \left(\cos t (1-\sin^2 t)\right) \, \mathrm{d}t = \int \left(\cos t - \sin^2 t \cos t\right) \, \mathrm{d}t = \sin t - \frac{\sin^3 t}{3}$$

then

$$\begin{aligned}
\left| -\frac{1}{(s+2)} \right|_0^y = \left| \sin t - \frac{\sin^3 t}{3} \right|_{\pi/2}^x \quad & \Rightarrow \quad -\frac{1}{(y+2)} + \frac{1}{2} = \sin x - \frac{\sin^3 x}{3} - 1 + \frac{1}{3} \\
& \Rightarrow \quad \frac{1}{(y+2)} = -\sin x + \frac{\sin^3 x}{3} + 1 - \frac{1}{3} + \frac{1}{2} \\
& \Rightarrow \quad y + 2 = \frac{6}{-6\sin x + 2\sin^3 x + 7} \\
& \Rightarrow \quad y = \frac{12 \sin x - 4\sin^3 x - 8}{-6\sin x + 2\sin^3 x + 7}
\end{aligned}$$

The solution is

$$y(x) = \frac{12 \sin x - 4\sin^3 x - 8}{-6\sin x + 2\sin^3 x + 7} \ , \quad x \in (-\infty, +\infty)$$

(c) $\boxed{\begin{cases} y' = \dfrac{y}{x(1+9x^2)} \\ y(1) = 0 \end{cases}}$

The constant solution $y = 0$ is a solution for the Cauchy problem.
Since $g(x) = \dfrac{1}{x(1+9x^2)}$ is continuous in a neighborhood of $x_0 = 1$ and $h(y) = y$ belongs to C^1 in a neighborhood of $y_0 = 0$, the problem has a unique solution, that is given by $y(x) = 0$, with $x \in (0, +\infty)$.

(d) $\boxed{\begin{cases} y' + 2y \cos x - \sin 2x = 0 \\ y(\pi) = 1 \end{cases}}$

Solve the differential equation

$$y' + 2y\cos x = \sin 2x$$

using the formula

$$\begin{aligned}
y(x) \quad &= \quad e^{-2\int \cos x \mathrm{d}x} \left\{ \int e^{\int 2\cos x} \sin 2x \mathrm{d}x + C \right\} \\
&= \quad e^{-2\sin x} \left\{ \int e^{2\sin x} \sin 2x \mathrm{d}x + C \right\} \\
&= \quad e^{-2\sin x} \left\{ \frac{1}{2} e^{2\sin x}(2\sin x - 1) + C \right\} \\
&= \quad \sin x - \frac{1}{2} + C\, e^{-2\sin x}
\end{aligned}$$

Impose $y(\pi) = 1$, then

$$1 = \sin \pi - \frac{1}{2} + C\, e^{-2\sin \pi} \quad \Rightarrow \quad 1 = -\frac{1}{2} + C \quad \Rightarrow \quad C = \frac{3}{2}$$

The solution is

$$y\,(x) = \sin x - \frac{1}{2} + \frac{3}{2} e^{-2\sin x}, \quad x \in \mathbb{R}$$

(e) $\boxed{\begin{cases} y' - \dfrac{1}{\sqrt{x}}\, y + e^{\sqrt{x}} = 0 \\ y(0) = 1 \end{cases}}$

Solve the differential equation

$$y' - \frac{1}{\sqrt{x}}\, y = -e^{\sqrt{x}}$$

using the solving formula, and recall that $\displaystyle\int e^{-\sqrt{x}} \mathrm{d}x$ can be computed by substitution $\sqrt{x} = t \Rightarrow x = t^2 \Rightarrow \mathrm{d}x - 2t\mathrm{d}t$:

$$\begin{aligned} y(x) &= e^{\int \frac{1}{\sqrt{x}} \mathrm{d}x} \left\{ \int e^{-\int \frac{1}{\sqrt{x}} x} (-e^{\sqrt{x}}) \mathrm{d}x + C \right\} \\ &= e^{2\sqrt{x}} \left\{ - \int e^{-2\sqrt{x}} e^{\sqrt{x}} \mathrm{d}x + C \right\} \\ &= e^{2\sqrt{x}} \left\{ - \int e^{-\sqrt{x}} \mathrm{d}x + C \right\} \\ &= e^{2\sqrt{x}} \left\{ 2e^{-\sqrt{x}} \left(\sqrt{x} + 1 \right) + C \right\} \\ &= 2e^{\sqrt{x}} \left(\sqrt{x} + 1 \right) + Ce^{2\sqrt{x}} \end{aligned}$$

Impose $y(0) = 1$, then $\quad 1 = 2 + C \quad \Rightarrow \quad C = -1$. The solution is

$$y\,(x) = 2e^{\sqrt{x}} \left(\sqrt{x} + 1 \right) - e^{2\sqrt{x}}, \quad x \in (0, +\infty)$$

(f) $\boxed{\begin{cases} y' = \sqrt{x} \\ y\,(0) = 1 \end{cases}}$

There are no constant solutions, thus solve the separable variables equation

$$y' = \sqrt{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x} \Rightarrow \mathrm{d}y = \sqrt{x}\mathrm{d}x$$

As before, integrate between 1 and y, and in the right hand-side between 0 and x:

$$\int_1^y \mathrm{d}s = \int_0^x \sqrt{t}\mathrm{d}t \Rightarrow |s|_1^y = \left| \frac{2}{3} t^{3/2} \right|_0^x \Rightarrow y - 1 = \frac{2}{3}\, x^{3/2} - 0$$

The solution is

$$y\,(x) = 1 + \frac{2}{3}\, \sqrt{x^3}, \quad x \in (0, +\infty)$$

7. Solve the following problems:

(a) $\boxed{\begin{cases} y'' - 4y' + \dfrac{7}{4} y = 0 \\ y(0) = 0 \\ y'(0) = 3 \end{cases}}$

The characteristic equation is

$$\lambda^2 - 4\lambda + \frac{7}{4} = 0 \quad \Rightarrow \quad \left(\lambda - \frac{1}{2} \right) \left(\lambda - \frac{7}{2} \right) = 0$$

The solution is

$$y(x) = c_1 e^{\frac{1}{2} x} + c_2 e^{\frac{7}{2} x}$$

thus

$$y'(x) = c_1 \frac{1}{2} e^{\frac{1}{2} x} + c_2 \frac{7}{2} e^{\frac{7}{2} x}$$

Impose

$y(0)=0 \quad \Rightarrow \quad c_1+c_2=0$
$y'(0)=3 \quad \Rightarrow \quad c_1\frac{1}{2}+c_2\frac{7}{2}=3.$
The system

$$\begin{cases} c_1+c_2=0 \\ c_1+7c_2=6 \end{cases}$$

has solution $\quad c_2=1 \quad c_1=-1.$
The solution for the Cauchy problem is

$$y(x)=e^{\frac{7}{2}x}-e^{\frac{1}{2}x}$$

(b) $\boxed{\begin{cases} x''+x'-2x=0 \\ x(0)=1 \\ \lim\limits_{t\to+\infty} x(t)=0 \end{cases}}$

The characteristic equation is

$$\lambda^2+\lambda-2=0 \quad \Rightarrow \quad (\lambda-1)(\lambda+2)=0$$

The solution is

$$x(t)=c_1e^{-2t}+c_2e^t$$

Impose

$$x(0)=1 \quad \Rightarrow \quad c_1e^{-2\cdot 0}+c_2e^{\cdot 0}=1 \quad \Rightarrow \quad c_1+c_2=1$$

$$\lim_{t\to+\infty} x(t)=0 \quad \Rightarrow \quad \lim_{t\to+\infty}\left(c_1e^{-2t}+c_2e^t\right)=0 \quad \Rightarrow \quad c_2=0$$

The solution for the Cauchy problem is

$$x(t)=e^{-2t}$$

(c) $\boxed{\begin{cases} x''+2x'-3x=3-3e^{-3t} \\ x(0)=0 \\ x(t) \text{ limitata su } [0,+\infty) \end{cases}}$

The characteristic equation is

$$\lambda^2+2\lambda-3=0 \quad \Rightarrow \quad (\lambda-1)(\lambda+3)=0 \quad \Rightarrow \quad \lambda_1=1\ , \quad \lambda_2=-3$$

Hence the general integral for the associated homogeneous equation is

$$x_o(t)=c_1e^{-3t}+c_2e^t$$

There is resonance, thus

$$x_p(t)=a_1+a_2te^{-3t}$$

$$x_p'=a_2e^{-3t}-3a_2te^{-3t}=a_2e^{-3t}(1-3t)$$

and therefore

$$x_p''=-3a_2e^{-3t}(1-3t)+a_2e^{-3t}(-3)=a_2e^{-3t}(-6+9t)$$

and

$$x_p''+2x_p'-3x_p=3-3e^{-3t}$$

Impose that $x_p(t)$ satisfies the initial equation

$$a_2e^{-3t}(-6+9t)+2a_2e^{-3t}(1-3t)-3(a_1+a_2te^{-3t})\equiv 3-3e^{-3t}$$

Simplify and get

$$a_2e^{-3t}(-4)-3a_1\equiv 3-3e^{-3t}$$

thus

$$a_1=-1 \quad a_2=\frac{3}{4}$$

Therefore the particular integral is

$$x_p(t)=-1+\frac{3}{4}te^{-3t}$$

The general integral is

$$x(t) = c_1 e^{-2t} + c_2 e^{t} - 1 + \frac{3}{4} t e^{-3t}$$

Impose the initial conditions. In order to have a bounded $x(t)$ on $[0, +\infty)$ we need $c_2 = 0$. Hence

$$x(t) = c_1 e^{-2t} - 1 + \frac{3}{4} t e^{-3t}$$

Impose $x(0) = 0$ thus $c_1 = 1$. Finally

$$x(t) = \frac{3}{4} t e^{-3t} + e^{-3t} - 1$$

(d) $\boxed{\begin{cases} y'' + 9y = x^2 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}}$

The solution for the differential equation is

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{x^2}{9} - \frac{2}{81}$$

thus

$$y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x) + \frac{2x}{9}$$

Imposing the initial conditions we have

$$\begin{cases} y(0) = c_1 + c_2 - \dfrac{2}{81} = 0 \\ y'(0) = 3c_2 = 0 \end{cases} \quad \Rightarrow \; c_1 = \frac{2}{81}, \quad c_2 = 0$$

Finally the solution is

$$y(x) = \frac{2}{81} \cos(3x) + \frac{x^2}{9} - \frac{2}{81}$$

(e) $\boxed{\begin{cases} y'' - 7y' + 6y = 1 \\ y(0) = 0 \\ y'(0) = 0 \end{cases}}$

The solution for the differential equation is

$$y(x) = c_1 e^{x} + c_2 e^{6x} + \frac{1}{6}$$

thus

$$y'(x) = c_1 e^{x} + 6c_2 e^{6x}$$

Imposing the initial conditions we have

$$\begin{cases} y(0) = c_1 + c_2 + \dfrac{1}{6} = 0 \\ y'(0) = c_1 + 6c_2 = 0 \end{cases} \quad \Rightarrow \; c_1 = -\frac{1}{5}, \quad c_2 = \frac{1}{30}$$

Finally the solution is

$$y(x) = -\frac{1}{5} e^{x} + \frac{1}{30} e^{6x} + \frac{1}{6}$$

(f) $\boxed{\begin{cases} y'' + 5y' + 6y = \cos 2x \\ y(0) = 1 \\ y'(0) = 0 \end{cases}}$

The solution for the differential equation is

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x} + \frac{5}{52} \sin(2x) + \frac{1}{52} \cos(2x)$$

thus

$$y'(x) = -3c_1 e^{-3x} - 2c_2 e^{-2x} + 2\frac{5}{52} \cos(2x) - 2\frac{1}{52} \sin(2x)$$

Imposing the initial conditions we have

$$\begin{cases} y(0) = c_1 + c_2 + \dfrac{1}{52} = 1 \\ \\ y'(x) = -3c_1 - 2c_2 + \dfrac{5}{26} = 0 \end{cases} \quad \Rightarrow \; c_1 = -\frac{23}{13}, \quad c_2 = \frac{11}{4}$$

Finally the solution is

$$y(x) = -\frac{23}{13}\, e^{-3x} + \frac{11}{4}\, e^{-2x} + \frac{5}{52}\, \sin(2x) + \frac{1}{52}\, \cos(2x)$$

EXERCISES from WRITTEN EXAMS

1. (2017 February 14-th - I)

(a) *Find the general integral of the differential equation*

$$y'' + 4y = 2x.$$

The characteristic equation $\lambda^2 + 4 = 0$ admits pure imaginary solutions $\lambda_\pm = \pm 2i$, thus the general integral of the associated homogeneous equation is $y_0(x) = c_1 \cos(2x) + c_2 \sin(2x)$ with $c_1, c_2 \in \mathbb{R}$. The forcing term is the form $f(x) = e^{0x} p_1(x)$, with $p_1(x) = 2x$ (polynomial of degree 1). Since $\lambda = 0$ is not a root of the characteristic equation, a particular integral is $y_p(x) = ax + b$ con $a, b \in \mathbb{R}$ (polynomial of degree 1).
If we subtitute in the equation we get: $a = 1/2$ and $b = 0$. Finally, the general integral is

$$y(x) = y_0(x) + y_p(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{2}\, x \quad c_1, c_2 \in \mathbb{R}\,.$$

(b) *Let $y(x)$ be a solution of the following Cauchy problem, in a neighborhood of $x_0 = 1$:*

$$\begin{cases} y' = 3e^{xy} \\ y(1) = 2 \end{cases}$$

Compute the equation of the tangent line to the graph of $y(x)$ at the point $(1, 2)$.

The tangent line equation is $y = y(1) + y'(1)(x - 1)$. Fronm the Cauchy problem, we have $y'(1) = 3e^{1 \cdot y(1)} = 3e^2$. Thus the equation is

$$y = 2 + 3e^2(x - 1)\,.$$

2. (2016 June 23-th - II)

(a) *Write the definition of solution for the linear differential equation*

$$y'' + ay' + by = f(x)$$

with $a, b \in \mathbb{R}$ and $f(x)$ is a continuous function on $\mathbb{R}$. See the textbook.

(b) *Given the linear differential equation*

$$y'' + 3y' = 3x$$

(i) Say if there are bounded solutions on $\mathbb{R}$ for the associated homogeneous differential equation.

The characteristic equation is $\lambda^2 + 3\lambda = 0$ and the roots are real $\lambda_1 = 0, \lambda_2 = -3$; thus the general integral of the associated homogeneous equation is $y_0(x) = c_1 + c_2 e^{-3x}$, with $c_1, c_2 \in \mathbb{R}$. If $c_2 = 0$ we have infinite constant solutions $y = c_1$, bounded on $\mathbb{R}$; if $c_2 \neq 0$ the solutions are unbounded for $x \to -\infty$, for every value of c_1.

(*ii*) *Compute the general integral of the given equation.*

The forcing term is in the form $f(x) = e^{0x}p_1(x)$, with $p_1(x) = 3x$ (polynomial of degree 1). Since $\lambda = 0$ is a root, a particular integral is $y_p(x) = x(ax + b)$ with $a, b \in \mathbb{R}$.
If we subtitute in the equation we get: $a = 1/2$ e $b = -1/3$. Finally, the general integral is

$$y(x) = y_0(x) + y_p(x) = c_1 + c_2e^{-3x} + \frac{1}{2}x^2 - \frac{1}{3}x \quad c_1, c_2 \in \mathbb{R}$$

3. (2016 February 10th - III^o)

(a) Given two functions f, g continuous on $\mathbb{R}$, write the definition of a Cauchy problem solution

$$\begin{cases} y' = f(t)g(y) \\ y(t_0) = y_0 \end{cases}$$

See the textbook

(b) Find the solution oh the following Cauchy problem :

$$\begin{cases} y' = te^{-t^2}(y+1)^3 \\ y(0) = -\frac{1}{2} \end{cases}$$

The separable variables equation admits the constant solution $y = -1$, but this does not solve the Cauchy problem; the other solution can be solved as follows

$$\frac{dy}{dt} = te^{-t^2}(y+1)^3 \Longrightarrow \int (y+1)^{-3}\,dy = \int te^{-t^2}\,dt \Longrightarrow -\frac{1}{2}(y+1)^{-2} = -\frac{1}{2}e^{-t^2}+c \Longrightarrow (y+1)^{-2} = e^{-t^2}+k.$$

dove $k \in \mathbb{R}$. Imposing the initial conditions in the implicit solution we have $k = 3$.
From $(y + 1)^{-2} = e^{-t^2} + 3$, we explicit y:

$$(y+1)^2 = \frac{e^{t^2}}{1+3e^{t^2}} \Longrightarrow y + 1 = \pm\sqrt{\frac{e^{t^2}}{1+3e^{t^2}}}$$

Since $y(0) = -1/2$ and so $(y + 1)(0) = 1/2 > 0$, we must choose the positive sign for the root; then the solution of the Cauchy problem is

$$y(t) = -1 + \pm\sqrt{\frac{e^{t^2}}{1+3e^{t^2}}}$$

4. (2015 June 17th - I^o)

(a) Given two functions f, g continuous on $\mathbb{R}$, write the definition of a Cauchy problem solution

$$\begin{cases} y' = a(t)b(y) \\ y(t_0) = y_0 \end{cases}$$

See the textbook

(b) Find the solution oh the following Cauchy problem :

$$\begin{cases} y' = \frac{1}{t}(3e^y)\log t \\ y(1) = 0 \end{cases}$$

We can separate the variables; the equation has no constant solutions. We find:

$$\frac{dy}{dt} = \frac{1}{t}(3e^y)\log t \Longleftrightarrow \int e^{-y}\,dy = 3\int \frac{\log t}{t}\,dt \Longleftrightarrow e^{-y} = -(3/2)\log^2 t - c \quad c \in \mathbb{R}.$$

Imposing the initial conditions in the implicit solution we have $c = -1$. From $e^{-y} = -(3/2)\log^2 t + 1$, making explicit the y we find the solution of the given Cauchy problem

$$y(t) = -\log(1 - (3/2)\log^2 t).$$