IMPROPER INTEGRALS

1. Let $f : \mathbb{R} \to \mathbb{R}$ be locally integrable and such that $f(x) \geq \dfrac{1}{100}$, for all $x \in \mathbb{R}$.

   Prove that the improper integral $\displaystyle\int_0^{+\infty} f(t)\, dt$ is divergent.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be locally integrable and such that $f(x) \sim x^5$ as $x \to +\infty$.

   Prove that the improper integral $\displaystyle\int_1^{+\infty} f(x)e^{-x}\, dx$ is convergent.

3. Let $f : (0, 1] \to \mathbb{R}$ be locally integrable and such that $f(x) \sim \dfrac{1}{x}$ as $x \to 0$.

   Prove that the improper integral $\displaystyle\int_0^{1} f(x)e^{-x}\, dx$ is divergent.

4. Let $f : (0, 1] \to \mathbb{R}$ be an infinite of order α with respect to x, as $x \to 0$.

   Find the values of $\beta \in \mathbb{R}$ such that the improper integral $\displaystyle\int_0^{1} \frac{f(x)}{x^{2\beta}}\, dx$ is convergent.

5. Let

$$f(x) = \begin{cases} -1 & \text{if } \ x \in [2n, 2n+1) \\ \ \ 0 & \text{if } \ x \in [2n+1, 2n+2), n \in \mathbb{N}. \end{cases}$$

   - Prove that the integral function $F(x) = \displaystyle\int_0^{x} f(t)\, dt$ is decreasing.
   - Find out if the improper integral $\displaystyle\int_0^{+\infty} f(t)\, dt$ is convergent, divergent or indeterminate.

6. Let $f : [1, +\infty) \to \mathbb{R}$ be the following function:

$$f(x) = \frac{1}{n} \qquad \text{if } \ x \in [n, n+1), \quad n \in \mathbb{N}, \ n \geq 1.$$

   Find out if $\displaystyle\int_1^{+\infty} f(t)\, dt$ is convergent.

7. Let $f : [1, +\infty) \to \mathbb{R}$ be the following function:

$$f(x) = \frac{1}{(n+1)^2} \qquad \text{if } \ x \in [n, n+1), \ n \in \mathbb{N}, \ n \geq 1.$$

   Find out if $\displaystyle\int_1^{+\infty} f(t)\, dt$ is convergent.