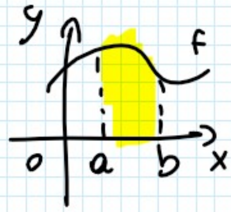
Integral function, Fundamental theorem of Integral Calculus
Improper Integrals

def $\int_a^b f(x)\,dx$

↳ definite integral (Cauchy Riemann construction)

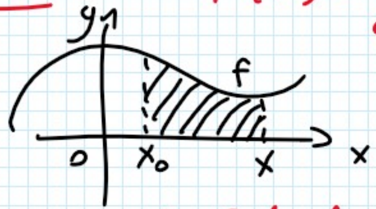
Corollary : f continuous (integrable)

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b)-F(a)$$



def integral function $F(x)=\int_{x_0}^{x} f(t)\,dt$

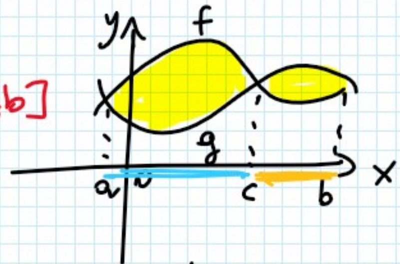
↓ integrand function



Fundamental Th. of Integral Calculus: (Hp...)

$$F'(x) = f(x)$$

$\boxed{\text{AREA}}$ between 2 function in [a,b]



$$A=\int_a^b |f(x)-g(x)|\,dx$$

Note that IF $f(x) \geq g(x)$ $\quad A=\int_a^b (f(x)-g(x))\,dx$

IF $f(x) < g(x)$ $\quad A=-\int_a^b (f(x)-g(x))\,dx$

In this case

$$A=\int_a^c (f(x)-g(x))\,dx + \int_c^b (g(x)-f(x))\,dx$$

$\boxed{\text{AREA}}$ between the graph of f and the x-axis in [a,b]

(y=0)

↓ $g(x)=0$

$$A=\int_a^b |f(x)|\,dx$$

$\boxed{\text{EX1A}}$ $\int_{-1}^{2} x \log(1+|x+1|)\,dx = \int_{-1}^{2} x \log(x+2)\,dx =$

$x+1 \geq 0 \iff x \geq -1$ in $[-1,2]$ $\quad |x+1| = x+1$

STRATEGY to compute integrals:

- compute a primitive (c=0)
- F(b) − F(a)

$$\int x \log(x+2)\,dx = \frac{x^2}{2}\log(x+2) - \frac{1}{2}\int \frac{x^2}{x+2}\,dx = (*)$$

• $F(b) - F(a)$

$$\int x\log(x+2)\,dx = \frac{x^2}{2}\log(x+2) - \frac{1}{2}\int \frac{x^2}{x+2}\,dx = (*)$$

By PARTS

$$\int f'g = fg - \int fg'$$

$f'(x) = x \qquad f(x) = \frac{x^2}{2}$

$g(x) = \log(x+2) \qquad g'(x) = \frac{1}{x+2}$

OR

$x^2 \quad | \; x+2$

$x^2+2x \quad | \; x-2$

$// \; -2x$

$-2x-4$

$// \; 4$

$$\int \frac{x^2+4-4}{x+2}\,dx = \int\left(\frac{x^2-4}{x+2} + \frac{4}{x+2}\right)dx = \int\left(\frac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} + \frac{4}{x+2}\right)dx =$$

$$= \frac{x^2}{2} - 2x + 4\log|x+2) + c \qquad (c=0)$$

$(*)$ $$\int x\log(x+2)\,dx = \frac{x^2}{2}\log(x+2) - \frac{1}{2}\left(\frac{x^2}{2} - 2x + 4\log|x+2|\right) = F(x)$$

$$\int_{-1}^{2} f(x)\,dx = \Big[F(x)\Big]_{-1}^{2} = F(2) - F(-1) =$$

$$= 2\cancel{\log 4} - \cancel{1} + 2 - 2\cancel{\log 4} - \left(-\frac{1}{4} \cancel{-1}\right)$$

$$= 2 + \frac{1}{4} = \frac{9}{4}$$

$$F(x) = \frac{x^2}{2}\log(x+2) - \frac{x^2}{4} + x + - 2\log(x+2)$$

$\boxed{\text{EX 3B}}$ area of B?

$$B = \{(x,y) \in \mathbb{R}^2) \quad 0 \le x \le \frac{\pi}{3}, \; 0 \le y \le \sin^3 x \cos^2 x\}$$

Note that: • IF f is ODD $\int_{-a}^{a} f(x)\,dx = 0$

AREA $= \int_{-a}^{a} |f(x)|\,dx = 2\int_0^a f(x)\,dx$

• IF f is EVEN $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

AREA $= \int_{-a}^{a} |f(x)|\,dx = 2\int_0^a f(x)\,dx$

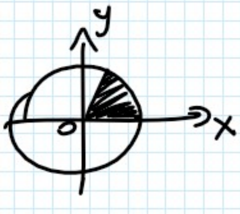
$$F(x) = \int_0^x f(t)\,dt$$

Note that F (integral function): $F'(x) = f(x)$

IF f is ODD $\Longrightarrow$ F is even

IF f is even $\Longrightarrow$ F is odd

IF $F(x) = \int_{x_0}^{x} f(t)\,dt$, $x_0 \neq 0$ → IT'S NOT TRUE



$f(x) = \sin^3 x \cdot \cos^2 x$ in $[0, \frac{\pi}{3}]$

IF $x \in [0, \frac{\pi}{3}]$ $\sin x \ge 0$

$\Longrightarrow f(x) \ge 0$ $\quad \int^{\pi/3}$

if $x \in [0, \frac{\pi}{3}]$ $\sin x \geq 0$

$\Rightarrow f(x) \geq 0$

$$\text{Area } B = \int_0^{\pi/3} |f(x)|\,dx = \int_0^{\pi/3} f(x)\,dx =$$

$$= \int_0^{\pi/3} \underbrace{\sin^3 x}_{\sin x \cdot \sin^2 x} \cdot \cos^2 x \, dx =$$

Note that $\sin^2 x + \cos^2 x = 1$

$$= \int_0^{\pi/3} \sin x \, (1-\cos^2 x)\cos^2 x \, dx =$$

$t = \cos x \qquad dt = -\sin x\,dx$

$$-\int (1-t^2)\,t^2 \, dt =$$

$$= \int (t^4 - t^2)\,dt = \frac{t^5}{5} - \frac{t^3}{3} = \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} = F(x)$$

$c = 0$

$$= \left[\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3}\right]_0^{\pi/3} = \frac{1}{5}\cdot\frac{1}{2^5} - \frac{1}{24} - \left(\frac{1}{5} - \frac{1}{3}\right) = (\ldots) = \frac{47}{480}$$

$\cos \pi/3 = \frac{1}{2}$

$\cos 0 = 1$

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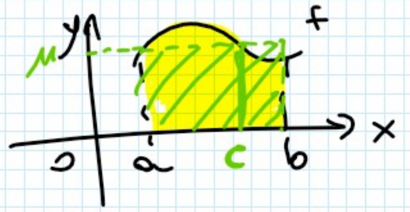
def: integral mean value of f in $[a,b]$

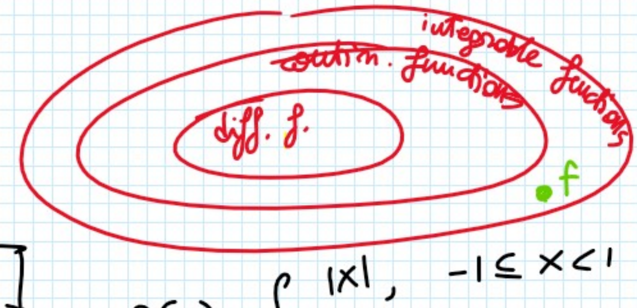
$$\mu(f,a,b) \stackrel{def}{=} \frac{\int_a^b f(x)\,dx}{b-a}$$

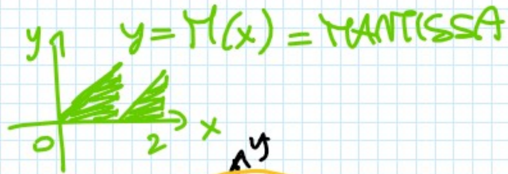


f diff. on $I \Rightarrow f$ continuous on $I \Rightarrow f$ integrable on I

MEAN VALUE THEOREM

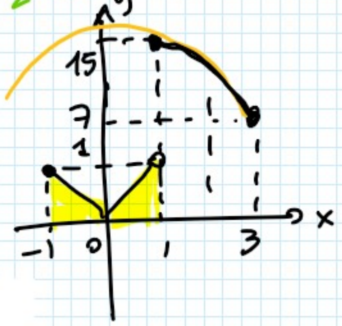
IF f contin. on $[a,b]$ $\Rightarrow \exists\, c \in [a,b] \;/\; f(c) = \mu(f,a,b)$





EXS

$$f(x) = \begin{cases} |x|, & -1 \leq x < 1 \\ 16 - x^2, & 1 \leq x \leq 3 \end{cases}$$



5. Let $f(x) = \begin{cases} |x| & \text{if } -1 \leq x < 1 \\ 16 - x^2 & \text{if } 1 \leq x \leq 3. \end{cases}$

Calculate the average value μ of f on the interval $[-1,3]$. Determine whether there exists a point $c \in [-1,3]$ such that $f(c) = \mu$.

$$\lim_{x \to 1^-} f(x) = 1 \qquad \lim_{x \to 1^+} f(x) = 15 \qquad f(3) = 7$$

f NOT contin. in $[-1,3]$: Mean Value Th. DOES NOT apply

$\lim_{x \to 1^-} f(x) = 1 \qquad \lim_{x \to 1^+}$

f NOT contin. in $[-1,3]$: Mean Value Th. DOES NOT apply

$$\mu(f,-1,3) \overset{def}{=} \frac{\int_{-1}^{3} f(x)\,dx}{4} = \frac{1}{4}\left(\int_{-1}^{1} |x|\,dx + \int_{1}^{3}(16-x^2)\,dx\right)$$

$$= \frac{1}{4}\left(2\int_0^1 x\,dx + \int_1^3 (16-x^2)\,dx\right) =$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{4}\left[16x - \frac{x^3}{3}\right]_1^3 = \frac{1}{4} + \frac{1}{4}\left(48-9-16+\frac{1}{3}\right) = \frac{73}{12}$$

$\frac{73}{12} < 7$

$$f([-1,3]) = [0,1] \cup [7,15]$$

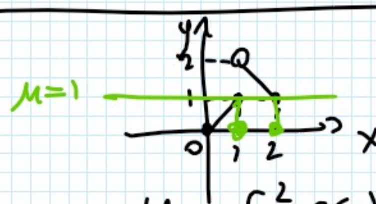
$$1 < \frac{73}{12} < 7 \qquad \frac{73}{12} \notin f([-1,3])$$

$$\nexists\, c \in [-1,3] \;/\; f(c) = \mu$$

Suppose we HAD $\mu = 8 \implies \exists\, c \in [-1,3] \;/\; f(c) = 8$

because $8 \in [0,1] \cup [7,15]$

EX



$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ -x+3, & 1 < x \le 2 \end{cases}$$

$$\mu(f,0,2) \overset{def}{=} \frac{\int_0^2 f(x)\,dx}{2} = \frac{1}{2}\left(\int_0^1 x\,dx + \int_1^2 (-x+3)\,dx\right) =$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_0^1 + \frac{1}{2}\left[-\frac{x^2}{2} + 3x\right]_1^2 = \frac{1}{4} + \frac{1}{2}\Big(\underbrace{-2+6+\tfrac{1}{2}-3}_{\frac{3}{2}}\Big) =$$

$$= \frac{1}{4} + \frac{3}{4} = 1 \qquad \mu = 1$$

f NOT contin. $[0,2]$ Th. Does Not apply

$\exists\, c \in [0,2] \;/\; f(c) = \mu$? YES

$\operatorname{Im} f = f([0,2]) = [0,2] \qquad 1 \in \operatorname{Im} f = [0,2]$

$\downarrow$

$\operatorname{Dom} f = [0,2] \qquad \exists\, c \in [0,2] \;/\; f(c) = 1$

Both $c=1$ and $c=2$ work!

IMPROPER INTEGRALS

1^{ST} UNBOUNDED INTEGRANDS

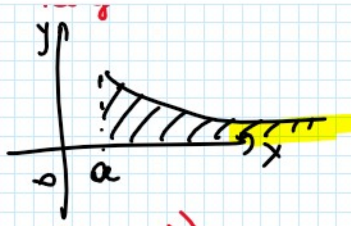
$\left(\lim_{x \to a^+} f(x) = +\infty\right)$

2^{nd} UNBOUNDED DOMAINS of integration

$\left(\lim_{x\to a^+} f(x) = +\infty\right)$

$f \in R_{loc}((a,b])$ (f is locally integrable on $(a,b]$)

$$\int_a^b f(x)\,dx \overset{def}{=} \lim_{z\to a^+} \int_z^b f(x)\,dx$$



$f \in R_{loc}([a,+\infty))$

$$\int_a^{+\infty} f(x)\,dx \overset{def}{=} \lim_{z\to+\infty} \int_a^z f(x)\,dx$$

IMPROPER Integrals: DIV. / CONV. / INDET.

ex.

$$\int_0^1 \frac{dx}{x}$$ IMPROPER 1^{st} kind

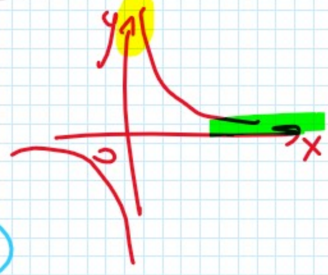
Dom f $= \mathbb{R}\setminus\{0\}$

$\lim_{x\to 0^+} f(x) = +\infty$

TOOLS TO check CONVERGENCE

1. Comparison TH.
2. ASYMPTOTIC COMPARISON TH.
3. Absolute convergence
4. some FUNDAMENTAL INTEGRALS $(\alpha > 0)$

$$\int_0^1 \frac{dx}{x^\alpha} = \begin{cases} \text{CONV. IFF } \alpha < 1 \\ \text{DIV. IFF } \alpha \geq 1 \end{cases}$$



$$\int_1^{+\infty} \frac{dx}{x^\alpha} = \begin{cases} \text{CONV. IFF } \alpha > 1 \\ \text{DIV. IFF } \alpha \leq 1 \end{cases}$$

(3) def $\int_a^{+\infty} f(x)\,dx$ is absolutely conv. IFF $\int_a^{+\infty} |f(x)|\,dx$ is CONV.

TH IF $\int_a^{+\infty} |f(x)|\,dx \boxed{< \infty}$ (is convergent) $\Longrightarrow$ $\int_a^{+\infty} f(x)\,dx \boxed{< \infty}$

ABSOLUTE conv. $\Longrightarrow$ convergence

ex $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$ Study ABSolute conv. $\displaystyle\int_1^{+\infty} \left|\frac{\sin x}{x^2}\right| dx$

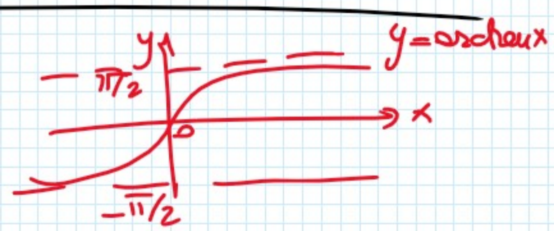
Now I apply (1) $\left|\frac{\sin x}{x^2}\right| \leq \frac{1}{x^2}$

Tool (4) $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx < \infty$ $\quad \alpha = 2 > 1$

By Comparison Th., I HAVE absolute conv. $\Longrightarrow$ conv.

By Comparison Th., I HAVE absolute conv. $\Longrightarrow$ conv.

EX 11F $\displaystyle\int_0^{+\infty} \frac{\arctan x}{1+x^2}\,dx$, $= f(x)$



$f(x) \geq 0$ on $[0,+\infty)$

Dom $f = \mathbb{R}$

Tool (2) $\quad f(x) \sim_{+\infty} \dfrac{\pi/2}{x^2}$

From Tool (4) $(\alpha > 1)$ $\quad \displaystyle\int_1^{+\infty} \frac{dx}{x^2} < \infty$

$\longrightarrow$ Converges

To be precise $\displaystyle\int_0^{+\infty} f(x)\,dx = \underbrace{\int_0^1 f(x)\,dx}_{\text{PROPER def. integral}} + \int_1^{+\infty} f(x)\,dx \in \mathbb{R}$

what's the value for ?

$$\int_0^{+\infty} \frac{\arctan x}{1+x^2}\,dx \overset{def}{=} \lim_{z \to +\infty} \int_0^z \frac{\arctan x}{1+x^2}\,dx = (*)$$

FIRST $\displaystyle\int \frac{\arctan x}{1+x^2}\,dx = \int t\,dt = \frac{t^2}{2} = \frac{\arctan^2 x}{2}$ $\;/\!/\; F(x)$ $(c=0)$

$t = \arctan x \qquad dt = \dfrac{1}{1+x^2}\,dx$

Then $\displaystyle\int_0^z f(x)\,dx = \Big[F(x)\Big]_0^z = \left[\frac{\arctan^2 x}{2}\right]_0^z = \frac{\arctan^2 z}{2} - 0$

$$(*) = \lim_{z \to +\infty} \frac{\arctan^2 z}{2} = \frac{(\pi/2)^2}{2} = \frac{\pi^2}{8}$$

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(b) + \cancel{c} - (F(a) + \cancel{c})$$

$\boxed{F(x) + c}$

EX 12C $\displaystyle\int_3^{+\infty} \frac{1}{\sqrt{|1-x^2|}}\,dx$, $= f(x)$

Dom $f = \mathbb{R} \setminus \{\pm 1\}$

$|1-x^2| > 0 \qquad x \neq \pm 1$

$f(x) \geq 0$ on $[3,+\infty)$

Tool (2) ASYMP. COMP. TH.

$f(x) \sim_{+\infty} \dfrac{1}{\sqrt{|x^2|}} = \dfrac{1}{x}$ $\quad$ DIV.

Tool (4) $\displaystyle\int^{+\infty}$

$f(x) \sim_{+\infty} \sqrt{|x^2|} \quad x$ DIV.

Tool (4) $\int_1^{+\infty} \frac{1}{x}\,dx = +\infty$

EX 12E $\int_1^{+\infty} \frac{dx}{x^2 + \sqrt[3]{x^4+1}}$ Dom $f = \mathbb{R}$

$f \geq 0$ Tool (2) $f(x) \sim_{+\infty} \frac{1}{x^2 + x^{4/3}} \sim \frac{1}{x^2}$

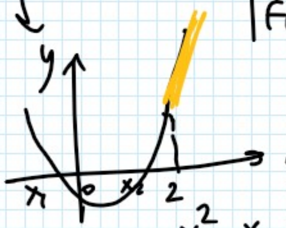
$\alpha = 2 > 1$

CONV. (Tool (4))

EX 13 Absolute conv. of

$$\int_2^{+\infty} \underbrace{\frac{\sin x + \cos x}{x^2 - x - 1}}_{f(x)}\,dx$$

$x^2 - x - 1 \neq 0 \quad x \neq \frac{1 \pm \sqrt{5}}{2} < 2 \qquad f$ contin. on $[2, +\infty)$



$x^2 - x - 1 > 0$

$$|f(x)| = \left| \frac{\sin x + \cos x}{x^2 - x - 1} \right| \leq \frac{2}{|x^2 - x - 1|} = \frac{2}{x^2 - x - 1}$$

Tool (1)

Study $\int_2^{+\infty} \frac{2}{x^2 - x - 1}\,dx$

Tool (2) $\frac{2}{x^2 - x - 1} \sim_{+\infty} \frac{2}{x^2} \qquad \alpha = 2 > 1$

CONV.

We have absolute conv.

EX 8

8. Consider the function $F(x) = \int_0^x \underbrace{(\cos t + 1) e^{-5t^2}}_{f(t)}\,dt.$

a) Study the simmetries of $F(x)$.

b) Study monotonicity and say if there are stationary points; classify them.

c) Find the order of infinitesimal of $F(x)$, as $x \to 0$ and its Mac Laurin expansion of order 1. Say if the function has constant sign in a neighborhood of $x = 0$.

d) Say if it exists and it is finite $\int_0^{+\infty} (\cos t + 1) e^{-5t^2}\,dt$.

e) Say if $F(x)$ admits horizontal asymptotes.

(a) $F'(x) = f(x) \qquad f$ even $\Longrightarrow$ F odd : check

$$F(-x) = \int_0^{-x} f(t)\,dt = -\int_{-x}^{0} \underbrace{f(t)}_{\text{even}}\,dt = -F(x)$$

(b) $F'(x) \geq 0$

$f(x) \geq 0 \qquad (\cos x + 1) e^{-5x^2} \geq 0$ always $\forall x \in \mathbb{R}$

(Dom $f = \mathbb{R}$)

$-1 \leq \cos x \leq 1$

$0 \leq \cos x + 1 \leq 2 \quad \longrightarrow \quad F$ increasing because

$-1 \leq \cos x \leq 1$

$0 \leq \cos x + 1 \leq 2$ $\Longrightarrow$ F increasing because $F' = f \geq 0$

ⓒ Mac Laurin expansion for F

$$F(x) = F(0) + F'(0)\,x + o(x)$$

$$F(0) = \int_0^0 f(t)\,dt = 0$$

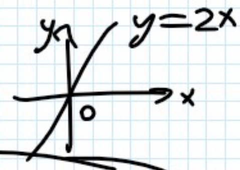
$$F'(0) = f(0) = (\cos 0 + 1)e^0 = 2$$

$F(x) = 2x + o(x)$ $\qquad F(x) \sim_0 2x$ $\qquad$ order 1

Sign of F around 0?

IF $x > 0$ $\quad F(x) > 0$

IF $x < 0$ $\quad F(x) < 0$



ⓓ $\displaystyle\int_0^{+\infty} f(t)\,dt \overset{\text{def}}{=} \lim_{x\to+\infty} \underbrace{\int_0^x f(t)\,dt}_{= F(x)} = \lim_{x\to+\infty} F(x)$

$\exists$ Horizontal asymptote for F?
(RIGHT)

$$f(x) = \frac{(\cos x + 1)}{e^{5x^2}} < \frac{2}{e^{5x^2}}$$

$(f \geq 0)$ $\qquad$ COMPARISON (Tool ①)

Asymptotic COMP. (Tool ②) $\quad \dfrac{2}{e^{5x^2}} = o\left(\dfrac{1}{x^\alpha}\right) \quad \forall \alpha > 0$

I choose $\alpha > 1$ $\Longrightarrow$ CONV.

$\displaystyle\lim_{x\to+\infty} F(x) = \ell \in \mathbb{R}$ $\qquad \boxed{y = \ell}$ HORIZ. asymptote (right) for F

PAST exams (13/2/2015 – II)

(c) *Study the behavior of the improper integral*

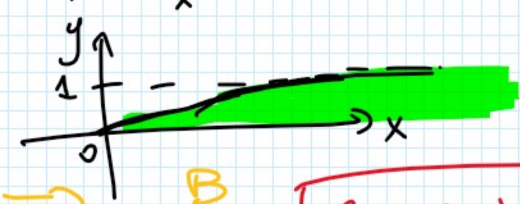
$$\int_0^{+\infty} \frac{x^2 + \sin x}{x^2+1}\,dx.$$

$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{f(x)}$

Tool ②

$$f(x) \sim_{+\infty} \frac{x^2}{x^2} \sim 1 \qquad \int_0^{+\infty} 1\,dx = +\infty$$

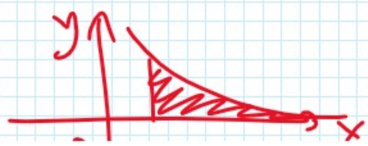
Dom $f = \mathbb{R}$

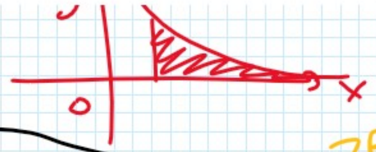


DIV.

Note this $\displaystyle\int_a^{+\infty} f(x)\,dx < \infty$ $\overset{A \Rightarrow B}{\Longrightarrow}$ $\boxed{\lim_{x\to+\infty} f(x) = 0}$

necessary condition for convergent integral





condition for convergent integral

$\neg B \implies \neg A$

EX 10

10. Compute $\quad \lim_{x\to 0} \frac{\int_0^x e^{-t^2}\,dt}{\sin x} = \left[\frac{0}{0}\right]$ (*Suggestion: use De l'Hopital Theorem*)

$F(x)$ NOT integrable

$F'(x) = f(x) = e^{-x^2}$

$$= \lim_{x\to 0} \frac{e^{-x^2}}{\cos x} = 1$$

T11

11. Given $f:\mathbb{R}\to\mathbb{R}$, continuous on $\mathbb{R}$ and such that $f(0)=2$, let $F(x)=\int_0^x f(t)\,dt$. Then:

(a) $\lim_{x\to 0} \frac{F(x)}{x^2} = 2$

(b) $\lim_{x\to 0} \frac{F(x)}{2xe^x} = 1$

(c) $\lim_{x\to 0} \frac{F(x)}{x}$ does not exist

(d) $\lim_{x\to 0} \frac{F(x)}{\sin x} = 1$

(e) The principal part of $F(x)$ w.r.t. the standard test function, for $x\to 0$, is $p(x) = 2$

$\nearrow \int_0^0$

$$F(x) = F(0) + F'(0)x + o(x) =$$
$$= 0 + f(0)x + o(x) =$$
$$= 2x + o(x)$$

MAC Laurin expansion

$$F(x) \sim_0 2x$$

$$F(0) = \int_0^0 f(t)\,dt = 0$$

(A) $\lim_{x\to 0} \frac{2x}{x^2} = \lim_{x\to 0} \frac{2}{x} = \pm\infty \quad \nexists$

(B) $\lim_{x\to 0} \frac{1}{e^x} = 1$

(C) $\lim_{x\to 0} \frac{2x}{x} = 2$

(D) $\lim_{x\to 0} \frac{2x}{x} = 2$

T12

12. $F(x) = \int_{-2}^x \underbrace{\left(\arctan t^5 + \sqrt[3]{t}\right)}_{=f(t)} dt$

(a) has domain $[-2,+\infty)$

(b) is concave

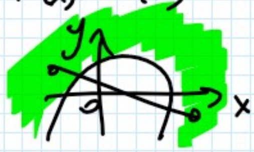
(c) is increasing

(d) $F'(0)=0$

(e) $F'(-2)=0$

$\mathrm{Dom} f = \mathbb{R} = \mathrm{dom} F$

$F'(x) = f(x)$



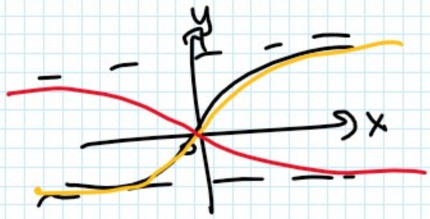
$$F''(x) \le 0$$
$$F''(x) = f'(x) =$$
$$= \frac{5t^4}{1+t^{10}} + \frac{1}{3\sqrt[3]{t^2}} \ge 0$$

F convex

(C) F increasing IF $F' = f \ge 0$

$f(x) \ge 0 \qquad \arctan x^5 \ge -\sqrt[3]{x}$

$\iff x \ge 0$



(D) $F'(0) = f(0) = 0$

T13

13. Let f locally integrable on $[0,+\infty)$; we say that the improper integral $\int_0^{+\infty} f(x)\,dx$ is divergent if:

(a) $\lim_{t\to+\infty} f(x)$ is not finite

(b) $\lim_{t\to+\infty} \int_0^t f(x)\,dx$ does not exists

(c) $\int_0^{+\infty} f(x)\,dx = +\infty$

(d) $\lim_{t\to+\infty} \int_0^t f(x)\,dx$ exists and it is not finite

By def.

(c) $\int_0^{+\infty} f(x)\ dx\ = +\infty$

By def.

(d) $\lim_{t\to+\infty} \int_0^t f(x)\ dx$ exists and it is not finite

(e) $\lim_{t\to+\infty} \int_0^t |f(x)|\ dx$ exists and it is $+\infty$

T5

5. Which one of the following statements is not true for the function $f(x) = \dfrac{e^{-x}}{x^2-1}$, con $x \in \left[-\frac{1}{2}, 0\right]$?

~~(a)~~ Given μ average value of the function f, there exist at least a point $c \in [-\frac{1}{2}, 0]$: $\mu = f(c)$ TRUE

~~(b)~~ The function $[-\frac{1}{2}, 0]$ admits absolute maximum and minimum By Weierstrass

~~(c)~~ $\dfrac{\min f(x)}{2} \le \int_{-\frac{1}{2}}^{0} f(x)\ dx \le \dfrac{\operatorname{Max} f(x)}{2}$ ✓ TRUE

~~(d)~~ Given μ average value of the function f, $\int_{-\frac{1}{2}}^{0} f(x)\,dx = \frac{\mu}{2}$ TRUE

(e) Let A be the area between the function and the x-axis in the given interval. Then, $A = \int_{-\frac{1}{2}}^{0} f(x)\,dx$

$\mathrm{Dom} f = \mathbb{R} \setminus \{\pm 1\}$

$\cup|$

$[-\frac{1}{2}, 0]$

f contin. on $[-\frac{1}{2}, 0]$

(A)

$$\mu = \frac{\int_{-\frac{1}{2}}^{0} f(x)\,dx}{\frac{1}{2}} = \boxed{2\int_{-\frac{1}{2}}^{0} f(x)\,dx} \Longrightarrow \text{(D)}$$

$$\inf \mathrm{Im} f \le \mu \le \sup \mathrm{Im} f \quad \text{(C)}$$

(E) $f(x) \ge 0$ in $[-\frac{1}{2}, 0]$? NO

$\Updownarrow$

$x^2 - 1 > 0 \iff x < -1 \vee x > 1$

in $[-\frac{1}{2}, 0]$ $f(x) < 0$