Week 11

Improper Integrals II
Diff. Equations (1st order)

$\boxed{T1}$

$$\int_0^a (5x)^{-\frac{1}{3}}\,dx = \lim_{z\to 0^+} \int_z^a (5x)^{-\frac{1}{3}}\,dx = (*)$$

$a\in\mathbb{R}$

(check convergence:

$f(x) = \frac{1}{\sqrt[3]{5x}}$ $\quad \mathrm{Dom} f = \mathbb{R}\setminus\{0\}$

$\lim_{x\to 0^+} f(x) = +\infty$

$$\text{I}\quad \int_0^1 \frac{1}{x^\alpha}\,dx = \begin{cases} \text{CONV. IF } \alpha<1 \\ \text{DIV. IF } \alpha\geq 1\end{cases}$$

Here $\alpha = \frac{1}{3} < 1 \Rightarrow$ convergent!)

$$\text{II}\quad \int_1^{+\infty} \frac{1}{x^\alpha}\,dx = \begin{cases} \text{CONV. IF } \alpha>1 \\ \text{DIV. IF } \alpha\leq 1\end{cases}$$

$t=5x \quad dt = 5\,dx$

$$(*) = \lim_{z\to 0^+} \frac{1}{5}\left[\frac{(5x)^{2/3}}{2/3}\right]_z^a =$$

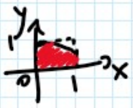
$\boxed{n\neq -1}$ $\int t^n\,dx = \frac{t^{n+1}}{n+1} + c,\ c\in\mathbb{R}$

$$= \lim_{z\to 0^+} \frac{1}{5}(5a)^{2/3}\cdot\frac{3}{2} - \underbrace{\frac{1}{5}(5z)^{2/3}\cdot\frac{3}{2}}_{\to 0} = \frac{3}{10}(5a)^{2/3}$$

$\square$

$\boxed{T3}$

$$\int_0^1 \underbrace{\frac{\sin x}{x}}_{f(x)}\,dx$$



$\mathrm{Dom} f = \mathbb{R}\setminus\{0\}$

$\lim_{x\to 0} f(x) = 1$

$\lim_{x\to 1} f(x) = \sin 1 < 1$

$\Rightarrow$ PROPER (definite) integral

$\boxed{EX\ 1F}$

$$I = \int_0^{+\infty} \frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}}\,dx = \underbrace{\int_0^5 f(x)\,dx}_{I_1} + \underbrace{\int_5^{+\infty} f(x)\,dx}_{I_2}$$

$\mathrm{Dom} f = (0,+\infty)$ $\quad \begin{cases} x+1\geq 0 & x\geq -1 \\ x>0 \\ x^2+1\neq 0 & \text{always TRUE}\end{cases}$ $\quad \boxed{x>0}$

$\lim_{x\to 0} f(x) = \infty$

I_1: $f(x) = \frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}} \sim_{0^+} \frac{1}{\sqrt{x}}$ $\quad \alpha = \frac{1}{2} < 1 \Rightarrow$ conv.

I_2: $f(x) \sim_{+\infty} \frac{\cancel{\sqrt{x}}}{x^2\cdot\cancel{\sqrt{x}}} \sim \frac{1}{x^2}$ $\quad \alpha = 2 > 1 \Rightarrow$ conv.

$\Rightarrow I$ is convergent

Note that $f(x)\geq 0$ on $\mathrm{Dom} f \Rightarrow I = +\infty$ OR $I\in\mathbb{R}^+$

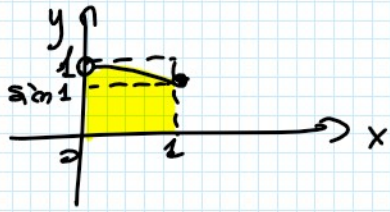
IF either $I_1 = +\infty$ OR $I_2 = +\infty$

$\Longrightarrow$ I is convergent ... $\Longrightarrow$ $I < +\infty$ or $I \in \mathbb{R}^+$

IF either $I_1 = +\infty$ OR $I_2 = +\infty$

$\Longrightarrow I = +\infty$

$\boxed{T3}$ $A = \int_0^1 \frac{\sin x}{x}\, dx = ?$ (area)



$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 1} f(x) = \sin 1$

By PARTS

$0 < A < 1$

$\boxed{EX5C}$ $f(x) = (e^x - 1)\log(1 + \sin^2 x)$ → Dom f $= \mathbb{R}$

Study convergence

$$\int_0^1 \frac{f(x)}{x^3\sqrt{\tan x}}\, dx \qquad = g(x)$$

$(0, 1] \subseteq$ Dom $g = (0, +\infty) \setminus \left\{\frac{\pi}{2} + k\pi\right\}_{k \in \mathbb{Z}}$

Asymptotic COMPARISON Theorem

$$g(x) \sim_{0^+} \frac{(x + o(x))(x^2 + o(x^2))}{x^3 \cdot \sqrt{x}}$$

$$\sim_{0^+} \frac{\cancel{x^3}}{\cancel{x^3}\sqrt{x}} \sim \frac{1}{\sqrt{x}} \qquad \alpha = \frac{1}{2} < 1 \quad \text{CONV.}$$

$\log(1+t) \sim_0 t$

$e^t - 1 \sim_0 t$

$\sin t \sim_0 t$

$\tan t \sim_0 t$

$\sqrt{x} = x^{1/2}$

$\boxed{T2}$

2. The improper integral $\int_0^1 \frac{1}{(1+x^2)\sqrt{\arctan x}}\, dx$ $\quad t = \arctan x$, $= dt$, $f(x)$

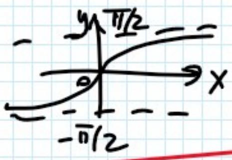
- (a) is 1
- (b) is $\sqrt{\pi}$
- (c) is indeterminate
- (d) has the same behavior of $\int_0^1 \frac{1}{x^2\sqrt{x}}\, dx$
- (e) is equal to $\frac{\pi}{2}$

Dom f $= (0, +\infty)$

$\arctan x > 0$

$\Updownarrow$

$x > 0$



$\lim_{x \to 0} f(x) = \frac{1}{0^+} = +\infty$

Note that $f(x) \geq 0$ in $(0, 1]$ $\quad$ (c)

$f(x) \sim_{0^+} \frac{1}{\sqrt{x}} \qquad \alpha = \frac{1}{2} < 1 \quad$ CONV.

ex $\quad \int_0^5 \frac{dx}{x-3} = \int_0^3 + \int_3^5$

Dom f $= \mathbb{R} \setminus \{3\}$

$\arctan t \sim_0 t$

(d) $\int_0^1 \frac{1}{x^{2+\frac{1}{2}}}\, dx \qquad \alpha = \frac{5}{2} > 1 \qquad$ DIV.

By DIRECT computation $\quad \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{1/2} + c =$

$2\sqrt{\arctan x} + c = F(x)$

By DIRECT comparison $\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt$

$= 2\sqrt{\arctan x} + c = F(x)$

$\int_0^1 f(x)\,dx = \lim_{z\to 0^+} \int_z^1 f(x)\,dx = \lim_{z\to 0^+} \left[2\sqrt{\arctan x}\right]_z^1 =$

$= \lim_{z\to 0^+} 2\sqrt{\arctan 1} - \cancel{2\sqrt{\arctan z}} = 2\sqrt{\pi/4} = \sqrt{\pi}$ (B)

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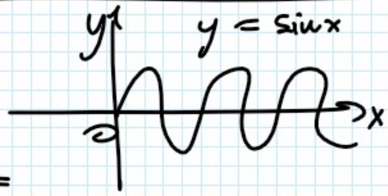
ex $\int_0^{+\infty} \sin x\, dx =$



$= \lim_{z\to+\infty} \left[-\cos x\right]_0^z =$

$= \lim_{z\to+\infty} -\cos z + 1 \quad \nexists$

T4

4. The integral $\int_0^1 \frac{\sin x}{x^2}\, dx$

(a) is not finite

~~(b) has the same behavior of the integral $\int_0^1 \frac{1-\cos x}{x^2}\, dx$~~

~~(c) converges, but not absolutely~~

~~(d) is real and greater than 1~~

~~(e) is between $\frac{1}{2}$ and 1~~

$Dom f = \mathbb{R}\setminus\{0\}$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{1}{x} = +\infty$ ($\frac{\sin x}{x}\to 1$, $\frac{1}{x}\to+\infty$)

$f(x) \sim_{0^+} \frac{1}{x}$ DIV. $(\alpha = 1)$

in $(0,1]$, $f(x) \geq 0$

(A) ~~(C)~~ ~~(D)~~ ~~(E)~~

~~(B)~~ $\int_0^1 \frac{1-\cos x}{x^2}\, dx$ = PROPER integral ($\xrightarrow{x\to 0} \frac{1}{2}$)

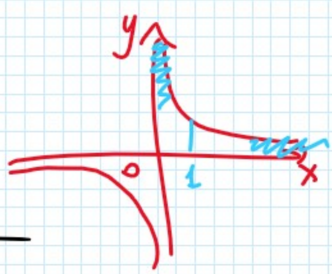
$\hookrightarrow Dom f = \mathbb{R}\setminus\{0\}$

$f(x) \sim_{0^+} \frac{\frac{1}{2}\cancel{x^2}}{\cancel{x^2}} \sim_{0^+} \frac{1}{2}$

def $\left\{ \int_a^{+\infty} f(x)\,dx \right.$ is absolutely conv. $\overset{def.}{\Longleftrightarrow}$ $\int_a^{+\infty} |f(x)|\,dx$ is conv.

Note that $\cos x = 1 - \frac{x^2}{2} + o(x^2)$ as $x \to 0$

$\int_0^1 \frac{1}{x}\,dx = +\infty \qquad \int_1^{+\infty} \frac{1}{x}\,dx = +\infty$



T5

5. The integral $\int_0^3 \frac{1-\cos x}{x^\pi}\, dx$

(a) diverges positively

(b) is indeterminate

(c) diverges negatively

(d) is not computable since π is irrational

(e) is less then 1

By def.

$\int_a^{+\infty} f(x)\,dx \overset{def}{=} \lim_{z\to+\infty} \int_a^z f(x)\,dx$

Asymptotic comparison TH.

$\int_a^{+\infty} f(x)\,dx \qquad f(x) \sim_{+\infty} \frac{1}{x^\alpha}$

$\hookrightarrow$ converges $\Longleftrightarrow \alpha > 1$

in $(0,3]$, $f(x) \geq 0$

$f(x) \sim_{0^+} \frac{\frac{1}{2}x^2}{x^\pi} \sim$

POSITIVELY

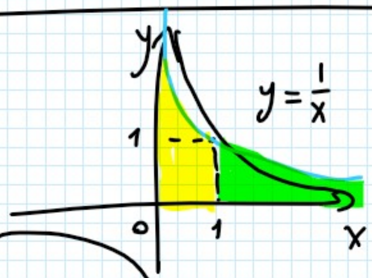
$$f(x) \sim_{0^+} \frac{\frac{1}{2}x^2}{x^\pi} \sim$$

$$\sim_{0^+} \frac{1}{2\, x^{\pi-2}} \qquad \pi - 2 > 1$$

POSITIVELY DIV.

$\boxed{\alpha > 0}$

$f(x) = \frac{1}{x^\alpha}$ $\qquad \alpha < 1$



$$\int_0^1 f(x)\,dx < \infty$$

$$\int_1^{+\infty} f(x)\,dx = +\infty$$

EX 8D week 9

d) $\boxed{\int \frac{1}{1-\sin x+\cos x}\,dx}$

By substitution:

$$\tan\frac{x}{2} = t \;\Rightarrow\; \sin x = \frac{2t}{t^2+1},\; \cos x = \frac{1-t^2}{t^2+1}$$

Then $x = 2\arctan t$, $dx = \frac{2}{t^2+1}\,dt$ and it follows

$$\begin{aligned}\int \frac{1}{1-\sin x+\cos x}\,dx &= \int \frac{1}{1-\frac{2t}{t^2+1}+\frac{1-t^2}{t^2+1}}\,\frac{2}{1+t^2}\,dt\\ &= \int \frac{1}{\frac{t^2+1-2t+1-t^2}{t^2+1}}\,\frac{2}{1+t^2}\,dt\\ &= \int \frac{1}{2-2t}\,2\,dt \qquad z = 1-t \quad dz = -dt\\ &= \int \frac{1}{1-t}\,dt = -\ln|1-t| + c = -\ln\left|1-\tan\frac{x}{2}\right| + c\end{aligned}$$

$f(\sin x, \cos x)$

$f(\sin^2 x, \cos^2 x, \tan x, \sin x \cdot \cos x)$

FORMULA: $t = \tan x$

ex IS IT CONVERGENT?

$$\int_0^5 \frac{dx}{(x-1)^2\sqrt[3]{x-2}} = \qquad \text{Dom} f = \mathbb{R}\setminus\{1,2\}$$

$x \neq 1$, $x \neq 2$

$x \to 1$: $\sqrt[3]{-1}$

$f(x) \xrightarrow[x\to 1^\pm]{} -\infty$

$f(x) \xrightarrow[x\to 2]{} \infty$

$$= \underbrace{\int_0^1 f(x)\,dx}_{I_1} + \underbrace{\int_1^{3/2} f(x)\,dx}_{I_2} + \underbrace{\int_{3/2}^2 f(x)}_{I_3} + \underbrace{\int_2^5 f(x)\,dx}_{I_4} = -\infty$$

I_1, I_2 $\qquad f(x) \sim_1 -\frac{1}{(x-1)^2} \qquad \alpha = 2 > 1 \qquad$ DIV.

I_3, I_4 $\qquad f(x) \sim_2 \frac{1}{\sqrt[3]{x-2}} \qquad \alpha = \frac{1}{3} < 1 \qquad$ CONV.

Differential equations

$$F(x, y, y', y'', \dots, y^{(m)}) = 0$$

order m

independent variable $\qquad y = y(x) \qquad$ dependent variable

ORDINARY $\qquad$ AN 2 $\to$ PARTIAL

independent variable $\quad y = y(x) \quad$ dependent variable

$$y^{(n)} = f(x, y, y', \dots, y^{(n-1)})$$

NORMAL form

ORDINARY DIFF. EQ. — PARTIAL DIFF. equations

order (ODE) ⇝ (PDE)

linear (in y)

↳ at most degree 1

(ex. $y' = \cos y$) NOT linear

homogeneous

(LINE : $y = mx + q$ linear in x)

I separable variables 1ST order and (2nd order) $\quad y' = f(x) \cdot g(y)$

II LINEAR 1ST order $\quad y' + \underline{a(x)} \cdot y = \underline{b(x)}$

III 2nd order with CONSTANT coefficients $\quad y'' + a y' + b y = 0$ (HOMOGENEUS)

$y'' + a y' + b y = f(x)$ — FORCING TERM

IV Cauchy problems < 1ST order, 2nd order

C.P. $\begin{cases} y' = f(x,y) & \text{on } I \text{ interval} \\ y(x_0) = y_0 \end{cases}$

given numbers

I

$y' = f(x)\, g(y)$ — We look for $y = y(x)$

↳ particular integral $\quad y' = 0 \quad g(y) = 0$ solve

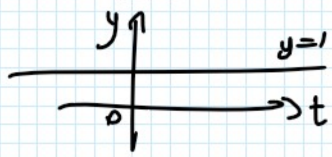
↳ general integral

ex

$$y' = \underbrace{(t-5)}_{f(t)}\underbrace{(y-1)}_{g(y)}$$

1ST ORDER separable variable (look for $y = y(t)$)

- particular integral : $y' = 0 \qquad y - 1 = 0 \qquad y = 1 \quad \forall t \in \mathbb{R}$



- general integral

$$y' = \frac{dy}{dt} \qquad \frac{dy}{dt} = (t-5)(y-1)$$

$$\int \frac{dy}{y-1} = \int (t-5)\, dt$$

$$\log|y-1| = \frac{t^2}{2} - 5t + c \qquad c \in \mathbb{R}$$

$$|y-1| = e^{\frac{t^2}{2} - 5t} \cdot e^c \qquad e^c = C \qquad C > 0$$

$$y - 1 = \pm C \cdot e^{\frac{t^2}{2} - 5t}$$

Note that $|f(x)| = k \quad (k \geq 0)$

$$y - 1 = \underbrace{\overset{\overset{\shortparallel}{c}}{\pm c}}_{\overset{\shortparallel}{k}} \cdot e^{\frac{t^2}{2} - 5t}$$

$k \neq 0$

Note that
$|f(x)| = k$ $(k \geq 0)$
$f(x) = \pm k$

$$\boxed{y = k e^{\frac{t^2}{2} - 5t} + 1 \qquad k \neq 0}$$ general integral

$y = 1$ IF $k = 0$

in conclusion
$y = k e^{t^2/2 - 5t} + 1, \; k \in \mathbb{R}$

$\boxed{\text{ex 10D}}$

(d) $y' = -\dfrac{\log^2 x}{2xy}, \quad x \in (0, +\infty)$

$y' = f(x) \cdot g(y)$

$f(x) = -\dfrac{\log^2 x}{2x}$

$g(y) = \dfrac{1}{y}$

$\boxed{\begin{matrix} x > 0 \\ y \neq 0 \end{matrix}}$

$\dfrac{1}{y} \neq 0$ always

- particular integral : $y' = 0 \iff \dfrac{1}{y} = 0$

$\nexists$ constant solution

- general integral

$$\frac{dy}{dx} = -\frac{\log^2 x}{2xy}$$

$\Rightarrow \log^2 x = (\log x)^2$
$\neq$
$\log(x^2) = 2 \log x$

$$2 \int y \, dy = -\int \frac{\log^2 x}{x} dx$$

$t = \log x$
$dt = \dfrac{dx}{x}$

$$y^2 = -\frac{\log^3 x}{3} + c, \quad c \in \mathbb{R}$$

$$\boxed{y = \pm \sqrt{-\frac{\log^3 x}{3} + c}, \quad c \in \mathbb{R}, \; x > 0}$$

Note that
$\sqrt{y^2} = |y|$

$\boxed{\text{T14}}$

14. The differential equation $y' - 3xy = 2e^{2x}$

(a) ~~has as particular integral the function $y(x) = -xe^{2x}$~~
(b) ~~is a separable variables equation~~
(c) is linear
(d) ~~admits constant solutions~~
(e) ~~the general integral is $y(x) = e^{2x} + c, \; c \in \mathbb{R}$~~

y'

y and y'
LINEAR
1^{st} order
NOT sep. variables
NOT homog.

II
1^{st} order
LINEAR
case

$y' + a(x) y = b(x)$

general integral
$$\boxed{y = e^{-A(x)} \int e^{A(x)} \cdot b(x) \, dx}$$

with $A(x) = \int a(x) \, dx$

$a(x) = -3x$
$b(x) = 2e^{2x}$

$$A(x) = \int -3x \, dx = -\frac{3}{2} x^2$$

$b(x) = 2e^{2x}$ $\quad A(x) = \int -3x\,dx = -\frac{3}{2}x^2$

$$y = e^{+\frac{3}{2}x^2} \cdot \int e^{-\frac{3}{2}x^2} \cdot 2e^{2x}\,dx$$

~~(E)~~

suppose $\exists$ constant solution

~~(D)~~ $y = k$ $(k \in \mathbb{R})$ $\quad y' = 0 \longrightarrow$ plug in ODE

$0 - 3xk = 2e^{2x} \longrightarrow$ Not always TRUE on $\mathbb{R}$

$\Rightarrow \nexists$ constant solution

T18

18. Which of the following statements is satisfied by the differential equation $x' = 3x + t$?

~~(a)~~ there exist bounded solutions

$x(t) =$ (b) $-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + k,\ \ k \in \mathbb{R}$ is the set of all the solutions

~~(c)~~ there is at least a solution with horizontal asymptote

$x(t) =$ (d) $-\frac{1}{3}t - \frac{1}{9} + ke^{3t},\ \ k \in \mathbb{R}$ is the set of the solutions

~~(e)~~ there are no solutions with oblique asymptotes

$\to$ 1^{st} order

$\to$ LINEAR

$x = x(t)$

$\hookrightarrow \forall t \in \mathbb{R}$

$x' - 3x = t$

$a(t) = -3$

$b(t) = t$

$A(t) = \int -3\,dt = -3t$

$$y = e^{+3t} \int e^{-3t} \cdot t\,dt$$

$\hookrightarrow$ By parts $\quad \int f'g = fg - \int fg'$

OR check B or D

~~(B)~~ $x' = \cancel{-\frac{1}{3}e^{-3t}} + \frac{1}{\cancel{3}}(\cancel{3})te^{-3t} + \cancel{\frac{1}{3}e^{-3t}} = te^{-3t}$

PLUG x and x' in ODE

$$te^{-3t} = 3\left(-\frac{1}{3}te^{-3t} - \frac{1}{9}e^{-3t} + k\right) + t$$

Not an identity

$\Rightarrow$ (D)

OR By Parts $\quad f'(t) = e^{-3t} \quad g(t) = t \quad (\ldots)$

(D) $x(t) = -\frac{1}{3}t - \frac{1}{9} + \cancel{k}e^{3t}$ $\quad k \in \mathbb{R}$ $\quad (\forall t \in \mathbb{R})$

$\exists$ bounded solutions? $\lim_{t \to \pm\infty} x(t) = ?$

$\lim_{t \to +\infty} x(t) = -\infty$ ($k=0$)

$\nexists$ bounded sol.

$\lim_{t \to -\infty} \left(-\frac{1}{3}t - \frac{1}{9} + ke^{3t}\right) = +\infty$ ($ke^{3t} \to 0$)

(ex. IF Domain is $(0, +\infty)$

$\lim_{t \to 0^+} x(t) = ?$)

$\lim_{t \to +\infty} x(t) = ?$

~~(A)~~

$\lim_{t\to+\infty} x(t) = ?$

A

$x(t) \sim_{+\infty} -\frac{1}{3}t - \frac{1}{9}$ with $\boxed{k=0}$

$x(t) \sim_{-\infty} -\frac{1}{3}t - \frac{1}{9}$ $\boxed{\forall k}$

C E

$\boxed{\text{EX 11B}}$ $y' = \frac{xy}{x+1} + e^{4x}$ $x \in (-1, +\infty)$ $\boxed{x \neq -1}$

1st order

LINEAR

$y = y(x)$

$y' + a(x)y = b(x)$

$a(x) = -\frac{x}{x+1}$

$b(x) = e^{4x}$

$A(x) = -\int \frac{x^{+1-1}}{x+1}\,dx = -\int\left(1 - \frac{1}{x+1}\right)dx =$

$= -x + \log|x+1| = -x + \log(x+1)$ $x > -1$

$y = e^{x-\log(x+1)} \int e^{-x+\log(x+1)} \cdot e^{4x}\,dx =$

$= e^{x} \cdot \underbrace{e^{-\log(x+1)}}_{e^{\log\frac{1}{x+1}} = \frac{1}{x+1}} \int \frac{e^{-x}}{x+1} \cdot e^{4x}\,dx = \frac{e^{x}}{x+1}\int \frac{e^{3x}}{x+1}\,dx$

by parts (...)

Mac Laurin $e^{\cos x} = e^{1-\frac{x^2}{2}+o(x^2)} =$

$= e \cdot e^{-\frac{x^2}{2}+o(x^2)} = e\left(-\frac{x^2}{2} + o(x^2)\right)$