MATHEMATICAL ANALYSIS I TUTORING
4^{TH} WEEK

GLOBAL PROPERTIES OF CONTINUOUS FUNCTIONS - LANDAU SYMBOLS - LOCAL COMPARISON INFINITESIMAL, INFINITE FUNCTIONS, ORDER AND PRINCIPAL PART - ASYMPTOTES

PROPOSED EXERCISES

1. For each of the following statements, say if it is trues or false, justifying by means of theoretical references.

   (a) If a function is continuous in x_0, then it is bounded in every neighborhood of x_0.

   A function $f(x)$ is continuous in x_0 if $\lim\limits_{x\to x_0} f(x) = f(x_0)$, that is, if for every neighborhood I of $f(x_0)$ **there exists a neighborhood J of** x_0, such that for every $x \in J$, it holds that $f(x) \in I$; hence $f(x)$ is bounded if $x \in J$.
   It is not guaranteed that f is bounded in **every** neighborhood of x_0: it is sufficient to consider the following counterexample: $f(x) = \tan x$, it is continuous in $x_0 = 0$ but it is not bounded in every neighborhood of x_0 (take for instance a neighborhood with radius 5).
   Therefore the statement is FALSE.

   (b) If a function is continuous in I=$[a,b]$, then its admits maximum in I.

   Weierstrass Theorem - *Let f be a continuous functin in a closed and bounded interval $[a,b]$. Then f is bounded in $[a,b]$ and it admits maximum and minimum.*
   Therefore the statement is TRUE by Weirstrass Theorem.

   (c) Let $f : [-2,3] \subseteq \mathbb{R} \to \mathbb{R}$, continuous and such that $f(-2) = -1$ *and* $f(3) = 2$; then necessarily the equation $f(x) = \lambda$ admits at least a solution in $-2 \leq \lambda \leq 3$.

   This statement applies (in a wrong way) the Mean Values Theorem (Corollary of Existence of Zeros Theorem) that states:

   Mean Values Theorem - *Let f be a continuous function on a closed and bounded interval $[a,b]$. Then $f(x)$ assumes all the values between $f(a)$ and $f(b)$.*

   The equation $f(x) = \lambda$ admits at least a solution if $-2 \leq \lambda \leq 3$ is equivalent to say that $f(x)$ assumes any value λ such that $-2 \leq \lambda \leq 3$.
   Therefore the statement is FALSE, since the function $f(x)$ takes all the values $-1 \leq \lambda \leq 2$, and not necessarily the values $-2 \leq \lambda \leq 3$.

   (d) There exists a function $f : [3,7] \to \mathbb{R}$ continuous and surjective.

   FALSE. If the function is continuous on $[3,7]$ then the function is bounded on such interval (by Weirstrass Theorem, it assumes maximum M and minimum m and thus $\mathrm{Im}(f) = [m,M]$). It follows that the image of the function on such interval cannot be the whole set $\mathbb{R}$, i.e. the function cannot be surjective.

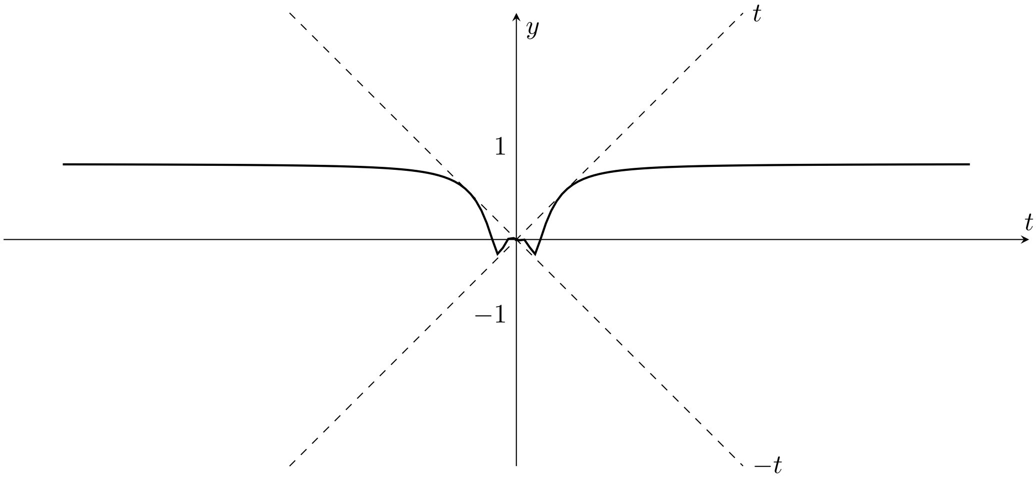
   (e) The function $f(x) = \begin{cases} x \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous in $x = 0$.

   The function is continuous for $x \neq 0$ by composition and product of continuous functions. We have to investigate the continuity at the origin, that is, we have to verify the equality $\lim_{x\to 0} f(x) = f(0)$. By direct computation:
   $$\lim_{x\to 0} x \sin \frac{1}{x} = 0$$
   applying the substitution $\dfrac{1}{x} = t$, we reduce to the known limit $\lim\limits_{t\to +\infty} \dfrac{\sin t}{t} = 0.$. Hence the function is continuous on$\mathbb{R}$.

Now we investigate if the function is bounded or not.
Since f is even, we study the function only for $x > 0$.
Observe that

- $-1 \le \sin\frac{1}{x} \le 1,\ \forall x \in \mathbb{R}$
- $\lim_{x\to+\infty} x\sin\frac{1}{x} = \lim_{t\to 0}\frac{\sin t}{t} = 1.$
- We have that $-1 \le x\sin\frac{1}{x} \le 1$

  Indeed, $\forall t \in \mathbb{R},\quad -t \le \sin t \le t$; thus $-\frac{1}{x} \le \sin\frac{1}{x} \le \frac{1}{x}$, and finally $-1 \le x\sin\frac{1}{x} \le 1.$



We can conclude that the function is continuous and bounded, therefore the statement is TRUE.

2. a) Prove that for $x \to 0$, given two powers of x, the negligible one has higher exponent.

$$\lim_{x\to 0}\frac{x^n}{x^m} = \lim_{x\to 0} x^{n-m} = 0 \Leftrightarrow n - m > 0 \Leftrightarrow n > m$$

   b) Prove that for $x \to \pm\infty$, between two powers of x, the negligible one has lower degree.

$$\lim_{x\to\pm\infty}\frac{x^n}{x^m} = \lim_{x\to\pm\infty}\frac{1}{x^{m-n}} = 0 \Leftrightarrow m - n > 0 \Leftrightarrow n < m$$

3. Verificare che per $x \to +\infty$, $2^x + 2^{-x} = o(2^{3x} + 1)$.

   Per $x \to x_0$ $f(x) = o\big(g(x)\big)$ se $\lim_{x\to x_0}\frac{f(x)}{g(x)} = 0.$

   Il limite $\lim_{x\to+\infty}\frac{2^x + 2^{-x}}{2^{3x}+1}$ presenta la forma indeterminata $\frac{\infty}{\infty}$.
   Utilizziamo il principio di eliminazione dei termini trascurabili: poiché, per $x \to +\infty$, si ha $2^{-x} = o(2^x)$ e $1 = o(2^{3x})$, il limite si semplifica:

$$\lim_{x\to+\infty}\frac{2^x + 2^{-x}}{2^{3x}+1} = \lim_{x\to+\infty}\frac{2^x}{2^{3x}} = \lim_{x\to+\infty}\frac{1}{2^{2x}} = 0$$

4. Given $f(x) = x^3$, $g(x) = x^3 - x$, $h(x) = x^3 - 1$.

   Verify that for $x \to +\infty$, $f \sim g \sim h$; say if $f - g \sim f - h$.

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \lim_{x\to+\infty}\frac{x^3}{x^3 - x} = \lim_{x\to+\infty}\frac{x^3}{x^3\left(1 - \frac{1}{x^2}\right)} = 1 \Rightarrow f \sim g$$

$$\lim_{x\to+\infty}\frac{f(x)}{h(x)} = \lim_{x\to+\infty}\frac{x^3}{x^3 - 1} = \lim_{x\to+\infty}\frac{x^3}{x^3\left(1 - \frac{1}{x^3}\right)} = 1 \Rightarrow f \sim h$$

$$\lim_{x\to+\infty}\frac{g(x)}{h(x)}=\lim_{x\to+\infty}\frac{x^3-x}{x^3-1}=\lim_{x\to+\infty}\frac{x^3\left(1-\frac{1}{x^2}\right)}{x^3\left(1-\frac{1}{x^3}\right)}=1\Rightarrow g\sim h$$

$$\lim_{x\to+\infty}\frac{f(x)-g(x)}{f(x)-h(x)}=\lim_{x\to+\infty}\frac{x^3-(x^3-x)}{x^3-(x^3-1)}=\lim_{x\to+\infty}\frac{x}{1}=+\infty\Rightarrow g\not\sim h$$

5. For each couple of functions, check if they are infinite or infinitesimal with the same order:

- $f_1(x)=\sqrt{2x^2+3x+5},\quad f_2(x)=2x-1,\quad \text{for } x\to+\infty$

$$\begin{aligned}
f_1(x) &= \sqrt{2x^2+3x+5}=\sqrt{x^2\left(2+3\frac{x}{x^2}+\frac{5}{x^2}\right)}\\
&= |x|\sqrt{\left(2+3\frac{1}{x}+\frac{5}{x^2}\right)}\\
&\sim \sqrt{2}|x|=\sqrt{2}x,\quad \text{for } x\to+\infty\\
f_2(x) &\sim 2x,\quad \text{for } x\to+\infty
\end{aligned}$$

Hence $f_1(x)$ and $f_2(x)$ are infinite functions with same order (not equivalent), for $x\to+\infty$.

- $g_1(x)=\sqrt{x+3}-\sqrt{3},\quad g_2(x)=2x,\quad \text{for } x\to 0$

$$\lim_{x\to0}\frac{g_1(x)}{g_2(x)}=\lim_{x\to0}\frac{\sqrt{x+3}-\sqrt{3}}{2x}=\lim_{x\to0}\frac{x+3-3}{2x(\sqrt{x+3}+\sqrt{3})}=\lim_{x\to0}\frac{x}{2x(\sqrt{x+3}+\sqrt{3})}=\lim_{x\to0}\frac{1}{2(\sqrt{x+3}+\sqrt{3})}=\frac{1}{2\sqrt{3}}$$

Therefore $g_1(x)$ and $g_2(x)$ have the same order, and thus they are infinitesimal functions with same order, for $x\to 0$.

6. As $\alpha\in\mathbb{R}$, compute the limit:

$$\lim_{x\to0^+}\frac{1-e^x}{\frac{2}{\pi}\arctan x^\alpha}$$

Recall that, $e^t-1\sim t$ and $\arctan t\sim t$, for $t\to0$, by substitution:

$$\lim_{x\to0^+}\frac{1-e^x}{\frac{2}{\pi}\arctan x^\alpha}=\lim_{x\to0^+}\frac{-x}{\frac{2}{\pi}x^\alpha}=-\frac{\pi}{2}\lim_{x\to0^+}x^{1-\alpha}=\begin{cases}-\frac{\pi}{2} & \text{if } \alpha=1\\ -\infty & \text{if } \alpha>1\\ 0 & \text{if } \alpha<1\end{cases}$$

7. Compute the following limits:

a) $\boxed{\lim_{x\to0}\frac{1-\cos 5x}{\sin^2 3x}}$

Recall that $\sin t\sim t$ and $1-\cos t\sim\frac{1}{2}t^2$, for $t\to0$. By substitution:

$$\lim_{x\to0}\frac{1-\cos 5x}{\sin^2 3x}=\lim_{x\to0}\frac{\frac{1}{2}(5x)^2}{(3x)^2}=\lim_{x\to0}\frac{\frac{1}{2}25x^2}{9x^2}=\frac{25}{18}$$

b) $\boxed{\lim_{x\to0}\frac{2x\cos x-2x}{x\sin^2 x}}$

Recall that $\sin t\sim t$ and $1-\cos t\sim\frac{1}{2}t^2$, for $t\to0$. By substitution:

$$\lim_{x\to0}\frac{2x\cos x-2x}{x\sin^2 x}=\lim_{x\to0}\frac{2x(\cos x-1)}{x\cdot x^2}=\lim_{x\to0}\frac{2x\left(-\frac{1}{2}x^2\right)}{x^3}=\lim_{x\to0}\frac{-x^3}{x^3}=-1$$

c) $\boxed{\lim_{x\to 0} \frac{1-\cos^4 x}{x\tan x}}$

Recall that $\tan t \sim t$ and $1-\cos t \sim \frac{1}{2}t^2$, for $t \to 0$. By substitution:

$$\lim_{x\to 0} \frac{1-\cos^4 x}{x\tan x} = \lim_{x\to 0} \frac{(1-\cos x)(1+\cos x)(1+\cos^2 x)}{x \cdot x} = 4\lim_{x\to 0} \frac{(1-\cos x)}{x^2} = 4\lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = 2$$

d) $\boxed{\lim_{x\to 0} \frac{\tan x - \sin x}{x^3}}$

Recall that $\sin t \sim t$ and $1-\cos t \sim \frac{1}{2}t^2$, per $t \to 0$. By substitution:

$$\begin{aligned}
\lim_{x\to 0} \frac{\tan x - \sin x}{x^3} &= \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} \\
&= \lim_{x\to 0} \frac{\sin x\left(\frac{1}{\cos x} - 1\right)}{x^3} \\
&= \lim_{x\to 0} \frac{\sin x(1-\cos x)}{\cos x\ x^3} \\
&= \lim_{x\to 0} \frac{x \cdot \frac{1}{2}x^2}{x^3} \\
&= \lim_{x\to 0} \frac{x\frac{1}{2}x^2}{x^3} = \frac{1}{2}
\end{aligned}$$

e) $\boxed{\lim_{x\to 0} \frac{(1+\sin x)^5 - 1}{1-\cos\sqrt{x}}}$

Recall that $(1+t)^\alpha - 1\ \sim \alpha t$, $1-\cos t \sim \frac{1}{2}t^2$ and $\sin t \sim t$, for $t \to 0$. By substitution:

$$\begin{aligned}
\lim_{x\to 0} \frac{(1+\sin x)^5 - 1}{1-\cos\sqrt{x}} &= \lim_{x\to 0} \frac{\frac{1}{5}\sin x}{\frac{1}{2}(\sqrt{x})^2} \\
&= \lim_{x\to 0} \frac{\frac{1}{5}x}{\frac{1}{2}x} = 10
\end{aligned}$$

f) $\boxed{\lim_{x\to 1} \frac{\sqrt{2-x^2}-1}{(x-1)^2}}$

If $x \to 1$, then $x-1 \to 0$; we apply the substitution $t = x-1$ (and thus $x = t+1$), and we recall that $(1+t)^\alpha - 1\ \sim \alpha t$, for $t \to 0$, then:

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{2-x^2}-1}{(x-1)^2} &= \lim_{t\to 0} \frac{\sqrt{2-(t+1)^2}-1}{t^2} \\
&= \lim_{t\to 0} \frac{\sqrt{2-t^2-2t-1}-1}{t^2} \\
&= \lim_{t\to 0} \frac{(1+(-t^2-2t))^{1/2}-1}{t^2} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}(-t^2-2t)}{t^2} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}(-2t)}{t^2} \\
&= \lim_{t\to 0} \frac{-1}{t}
\end{aligned}$$

The limit does not exist.

g) $\boxed{\lim_{x\to \frac{1}{2}} \frac{3^{2x-1}-1}{\sin(2x-1)}}$

If $x \to \frac{1}{2}$, then $x - \frac{1}{2} \to 0$; imposing $t = x - \frac{1}{2}$, (and thus $x = t + \frac{1}{2}$), we have that $t \to 0$; therefore:

$$\begin{aligned}
\lim_{x\to\frac{1}{2}} \frac{3^{2x-1}-1}{\sin(2x-1)} &= \lim_{x\to\frac{1}{2}} \frac{3^{2(x-\frac{1}{2})}-1}{\sin(2(x-\frac{1}{2}))} \\
&= \lim_{t\to 0} \frac{3^{2t}-1}{\sin(2t)} \\
&= \lim_{t\to 0} \frac{2t\log(3)}{2t} = \log(3)
\end{aligned}$$

(we applied the equivalences, for $t \to 0$, $a^t - 1 \sim t\log a$ and $\sin t \sim t$)

h) $\boxed{\lim_{x\to 0} \frac{\sqrt{2x^3} - x^6}{4x^6 - \sqrt{x^4 + x^3}}}$

It holds $x^6 = o(x^{3/2})$ for $x \to 0$ and $x^4 = o(x^3)$ for $x \to 0$; therefore we eliminate the negligible terms as follows:

$$\lim_{x\to 0} \frac{\sqrt{2x^3} - x^6}{4x^6 - \sqrt{x^4 + x^3}} = \lim_{x\to 0} \frac{\sqrt{2}x^{3/2}}{4x^6 - \sqrt{x^3}} = \lim_{x\to 0} \frac{\sqrt{2}x^{3/2}}{-x^{3/2}} = -\sqrt{2}$$

i) $\boxed{\lim_{x\to 0} \frac{\sin 3x + x^4}{6x + 5\log(1+x^2)}}$

It holds $x^4 = o(x)$ for $x \to 0$, and $\log(1+5x^2) = o(x)$ for $x \to 0$ (since $\log(1+5x^2) \sim 5x^2$); therefore

$$\lim_{x\to 0} \frac{\sin 3x + x^4}{6x + 5\log(1+x^2)} = \lim_{x\to 0} \frac{3x}{6x} = \frac{1}{2}$$

j) $\boxed{\lim_{x\to 0} \frac{1-\cos(x^3+2x^2)}{x^4 - x^7}}$

It holds $x^7 = o(x^4)$ for $x \to 0$ and $x^3 = o(x^2)$ for $x \to 0$; moreover $1 - \cos t \sim \frac{1}{2}t^2$ for $x \to 0$; hence:

$$\lim_{x\to 0} \frac{1-\cos(x^3+2x^2)}{x^4 - x^7} = \lim_{x\to 0} \frac{1-\cos(2x^2)}{x^4} = \lim_{x\to 0} \frac{\frac{1}{2}(4x^4)}{x^4} = 2$$

k) $\boxed{\lim_{x\to 0} \frac{\log(2-\cos x)}{\sin^2 x}}$

Recall that $\sin t \sim t$, $\log(1+t) \sim t$ and $1 - \cos t \sim \frac{1}{2}t^2$, for $t \to 0$. It holds:
$\log(2-\cos x) = \log(1+(1-\cos x)) \sim (1-\cos x)$ (set $t = 1-\cos x$, then if $x \to 0$ then $t \to 0$); thus:

$$\begin{aligned}
\lim_{x\to 0} \frac{\log(2-\cos x)}{\sin^2 x} &= \lim_{x\to 0} \frac{\log\big(1+(1-\cos x)\big)}{x^2} \\
&= \lim_{x\to 0} \frac{1-\cos x}{x^2} \\
&= \lim_{x\to 0} \frac{\frac{1}{2}x^2}{x^2} = \frac{1}{2}
\end{aligned}$$

l) $\boxed{\lim_{x\to +\infty} 2x\left(\log(x+1) - \log x\right)}$ Recall that $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x = e$, it holds:

$$\lim_{x\to +\infty} 2x\left(\log(x+1) - \log x\right) = \lim_{x\to +\infty} 2x\log\left(\frac{x+1}{x}\right) = \lim_{x\to +\infty} 2\log\left(1+\frac{1}{x}\right)^x = 2$$

8.

9. Verify that $f(x)=\sqrt{x+5}-\sqrt{5}$ and $g(x)=\sqrt{x+7}-\sqrt{7}$ are infinitesimal with the same order for $x\to 0$ and find $k\in\Re : f\sim kg \quad (x\to 0)$. Find the principal part of $f(x)$ for $x\to 0$, with respect to the infinitesimal test function $\varphi(x)=x$.

Remember that for $t\to 0$, it holds $(1+t)^\alpha-1\sim\alpha t$, and thus $\sqrt{1+t}-1\sim\dfrac{1}{2}t$, we have:

$$f(x)=\sqrt{x+5}-\sqrt{5}=\sqrt{5\left(1+\frac{x}{5}\right)}-\sqrt{5}=\sqrt{5}\left(\sqrt{1+\frac{x}{5}}-1\right)\sim\sqrt{5}\left(\frac{1}{2}\frac{x}{5}\right),\quad \text{for } x\to 0$$

Therefore $\quad f(x)\sim\dfrac{\sqrt{5}}{10}\,x,\quad$ for $x\to 0$

$$g(x)=\sqrt{x+7}-\sqrt{7}=\sqrt{7\left(1+\frac{x}{7}\right)}-\sqrt{7}=\sqrt{7}\left(\sqrt{1+\frac{x}{7}}-1\right)\sim\sqrt{7}\left(\frac{1}{2}\frac{x}{7}\right),\quad \text{for } x\to 0$$

Hence $\quad g(x)\sim\dfrac{\sqrt{7}}{14}\,x,\quad$ for $x\to 0$.

$$f\sim kg\quad (x\to 0)\ \Leftrightarrow\ \lim_{x\to 0}\frac{f(x)}{kg(x)}=1\ \Leftrightarrow\ \lim_{x\to 0}\frac{\frac{\sqrt{5}}{10}\,x}{k\frac{\sqrt{7}}{14}\,x}=1\ \Leftrightarrow\ \frac{\frac{\sqrt{5}}{10}}{k\frac{\sqrt{7}}{14}}=1\ \Leftrightarrow\ k=\frac{\sqrt{5}}{5}\frac{7}{\sqrt{7}}=\frac{\sqrt{7}}{\sqrt{5}}$$

The principal part of $f(x)$ as $x\to 0$ w.r.t. the test function $\varphi(x)=x$ is $p(x)=\dfrac{\sqrt{5}}{10}\,x$

Find the order of infinitesimal and the principal part w.r.t. (with respect to) the test function $\varphi(x)=\dfrac{1}{x}$, for $x\to+\infty$, for the functions:

$\boxed{f_1(x)=\dfrac{3x^2+\sqrt[3]{x}}{2x^3}}$

$$f_1(x)=\frac{3x^2+\sqrt[3]{x}}{2x^3}\sim\frac{3x^2}{2x^3},\quad \text{for}\quad x\to+\infty$$

Therefore $f_1(x)\sim\frac{3}{2x},\quad$ for $\quad x\to+\infty$

W.r.t. (with respect to) the infinitesimal test function $\varphi(x)=\dfrac{1}{x}$, for $x\to+\infty$, the order of infinitesimal of $f_1(x)$ is 1 and its principal part is $p(x)=\dfrac{3}{2}\dfrac{1}{x}$.

$\boxed{f_2(x)=\sqrt{x^2+2}-x}$

$$f_2(x)=\sqrt{x^2+2}-x=\sqrt{x^2\left(1+\frac{2}{x^2}\right)}-x=|x|\sqrt{1+\frac{2}{x^2}}-x=x\left(\sqrt{1+\frac{2}{x^2}}-1\right)\sim x\left(\frac{1}{2}\frac{2}{x^2}\right),\quad \text{as } x\to+\infty$$

Thus $f_2(x)\sim\frac{1}{x},\quad$ as $\quad x\to+\infty$.

W.r.t. the infinitesimal test function $\varphi(x)=\dfrac{1}{x}$, as $x\to+\infty$, the order of infinitesimal of $f_2(x)$ is 1 and its principal part is $p(x)=\dfrac{1}{x}$.

$\boxed{f_3(x)=\log\left(1+\sin\dfrac{2}{x}\right)}$

$$f_3(x)=\log\left(1+\sin\frac{2}{x}\right)\sim\sin\frac{2}{x}\sim\frac{2}{x},\quad \text{for}\quad x\to+\infty$$

Thus $f_3(x)\sim\frac{2}{x},\quad$ for $\quad x\to+\infty$

W.r.t. the infinitesimal test function $\varphi(x)=\dfrac{1}{x}$, as $x\to+\infty$, the order of infinitesimal of $f_3(x)$ is 1 and its principal part is $p(x)=2\,\dfrac{1}{x}$.

$\boxed{g(x)=\tan\dfrac{3}{x^2}}$

$$g(x)=\tan\frac{3}{x^2}\sim\frac{3}{x^2},\quad \text{for}\quad x\to+\infty$$

W.r.t. the infinitesimal test function $\varphi(x) = \dfrac{1}{x}$, per $x \to +\infty$, the order of infinitesimal of $g(x)$ is 2 and its principal part is $p(x) = 3\,\dfrac{1}{x^2}$.

$\boxed{\text{Which of them is } o(\tfrac{1}{x}) \quad (\text{ as } x \to +\infty)\ ?}$

The functions $f_1(x), f_2(x), f_3(x)$ have the same order as $\dfrac{1}{x}$, for $x \to +\infty$.

On the other hand $g(x) = o(\frac{1}{x})$: indeed $\displaystyle\lim_{x\to+\infty} \frac{g(x)}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{3}{x^2}}{\frac{1}{x}} = \lim_{x\to+\infty} \frac{3}{x} = 0.$

Find the order of infinity and the principal part w.r.t. $\varphi(x) = x$, as $x \to +\infty$, for the functions:

$\boxed{f_1(x) = 2x - \sqrt{4x^2 + x^4}}$

$$f_1(x) = 2x - \sqrt{4x^2 + x^4} \sim 2x - \sqrt{x^4}, \quad \text{for} \quad x \to +\infty \sim 2x - x^2, \quad \text{for} \quad x \to +\infty \sim -x^2, \quad \text{for} \quad x \to +\infty$$

W.r.t. the infinite test function $\varphi(x) = x$, as $x \to +\infty$, the order of infinity of $f_1(x)$ is 2 and its principal part is $P(x) = -x^2$.

$\boxed{f_2(x) = \dfrac{x^3 + x + 2}{x^2 - x - 1}}$

$$f_2(x) = \frac{x^3 + x + 2}{x^2 - x - 1} = \frac{x^3\left(1 + \frac{1}{x^2} + \frac{2}{x^3}\right)}{x^2\left(1 - \frac{1}{x} - \frac{1}{x^2}\right)} = \frac{x^3}{x^2}, \quad \text{as} \quad x \to +\infty \sim x, \quad \text{as} \quad x \to +\infty$$

W.r.t. the infinite test function $\varphi(x) = x$, as $x \to +\infty$, the order of infinity of $f_2(x)$ is 1 and its principal part is $P(x) = x$.

$\boxed{f_3(x) = \frac{x^2+1}{\sqrt{x^2-1}}}$

$$f_3(x) = \frac{x^2 + 1}{\sqrt{x^2 - 1}} = \frac{x^2\left(1 + \frac{1}{x^2}\right)}{\sqrt{x^2\left(1 - \frac{1}{x^2}\right)}} \sim \frac{x^2}{|x|}, \quad \text{as} \quad x \to +\infty \sim x, \quad \text{as} \quad x \to +\infty$$

W.r.t. the infinite test function $\varphi(x) = x$, for $x \to +\infty$, the order of infinity of $f_3(x)$ is 1 and its principal part is $P(x) = x$.

Compute now

$$\lim_{x\to+\infty} \frac{f_1(x)}{x^2 + \sin x} = \lim_{x\to+\infty} \frac{-x^2}{x^2\left(1 + \frac{\sin x}{x^2}\right)} = -1$$

$$\lim_{x\to+\infty} \frac{x - \sqrt{1 + e^{-x}}}{f_2(x)} = \lim_{x\to+\infty} \frac{x - 1}{x} = 1$$

$$\lim_{x\to+\infty} \frac{f_3(x)(2 + \cos x)}{x^2 + x} = \lim_{x\to+\infty} \frac{x(2 + \cos x)}{x(x + 1)} = \lim_{x\to+\infty} \frac{(2 + \cos x)}{(x + 1)} = 0$$

10. Let $a_n = \dfrac{(n+4)! - n!}{(n+2)!}$.

a) Verify that $a_n \sim n^2, \quad n \to +\infty$

$$\begin{aligned} a_n &= \frac{(n+4)! - n!}{(n+2)!} \\ &= \frac{(n+4)(n+3)(n+2)(n+1)n! - n!}{(n+2)(n+1)n!} \\ &= \frac{(n+4)(n+3)(n+2)(n+1) - 1}{(n+2)(n+1)} \sim \frac{n^4}{n^2}, \quad \text{as } n \to +\infty \end{aligned}$$

Thus $a_n \sim n^2$, as $n \to +\infty$.

b) Compute $\displaystyle\lim_{n\to\infty} \frac{a_n}{\left(1 + \frac{1}{n}\right)^{n^2}}$

$$\lim_{n\to\infty} \frac{a_n}{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty} \frac{n^2}{\left(\left(1+\frac{1}{n}\right)^n\right)^n} = \lim_{n\to\infty} \frac{n^2}{e^n} = 0$$

11. Compute the principal part as $x \to +\infty$ and as $x \to 0$ (w.r.t. the standard test functions) for $f(x) = \dfrac{5x^2+2x}{3}$.
Given the previous computations, solve

$$\lim_{x\to+\infty} \frac{f(x)}{x^3\sin(\frac{1}{x})} \quad ; \quad \lim_{x\to 0} f(x)\frac{\sin^2 x}{1-\cos x}$$

It holds:

$$\frac{5x^2+2x}{3} \sim \frac{5}{3}\,x^2, \quad \text{as } x \to +\infty, \qquad\qquad \frac{5x^2+2x}{3} \sim \frac{2}{3}\,x, \quad \text{as } x \to 0$$

Hence the principal part of $f(x)$ as $x \to +\infty$ is $P(x) = \dfrac{5}{3}\,x^2$, while the principal part of $f(x)$ as $x \to 0$ is the function $p(x) = \dfrac{2}{3}\,x$.
Using these results, compute the limits:

$$\lim_{x\to+\infty} \frac{f(x)}{x^3\sin(\frac{1}{x})} = \lim_{x\to+\infty} \frac{\frac{5}{3}\,x^2}{x^3\,\frac{1}{x}} = \lim_{x\to+\infty} \frac{\frac{5}{3}\,x^2}{x^2} = \frac{5}{3}$$

$$\lim_{x\to 0} f(x)\frac{\sin^2 x}{1-\cos x} = \lim_{x\to 0} \frac{2}{3}\,x\,\frac{x^2}{\frac{1}{2}\,x^2} = 0$$

SOLUTIONS - EXERCISES from WRITTEN EXAMS

1. (19 September 2017)

   (a) *Suppose that a function assumes value with oppsite sign in the two points a and b. State which are the conditions to have* 0 *in the range of the function.* If the function is continuous on the interval I $= [a, b]$, by the Zeros Existence Theorem, f necessarily takes value 0 in I.

   (b) *Consider the function* $g(x) = \mathrm{sgn}((x-1)(x+3))$, *where*

$$\mathrm{sgn}(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}$$

   Say if the previous functions annihilates in the interval $[0, 4]$ *and say if, in such interval, it satisfies conditions given at point (a).*

Study the sign of $(x-1)(x+3)$, then

$$\begin{cases} (x-1)(x+3) > 0 & \text{if } x < -3 \ \vee x > 1 \\ (x-1)(x+3) = 0 & \text{if } x = -3 \ \vee x = 1 \\ (x-1)(x+3) < 0 & \text{if } -3 < x < 1 \end{cases}$$

hence:

$$\mathrm{sgn}((x-1)(x+3)) = \begin{cases} 1 & \text{if } x < -3 \ \vee x > 1 \\ 0 & \text{if } x = -3 \ \vee x = 1 \\ -1 & \text{if } -3 < x < 1 \end{cases}$$

The function is zero in $x = 1$, the belongs to the interval $[0, 4]$.
Since $g(x)$ is not continuous in $[0, 4]$, we cannot apply the Zeros Existence Theorem; nevertheless, we have that the function annihilates: indeed continuity is a sufficient condition, but not necessary for the existence of zeros.

2. (1 February 2017 - I)

   (a) *Write the definition for*

$$f(x) = o(2x^2 - 1) \quad \textit{as } x \to +\infty,$$

   and show the latter with an example.

   The definition is: $\displaystyle \lim_{x \to +\infty} \frac{f(x)}{2x^2 - 1} = 0\,.$.

   Consider the examples

   Example 1:

$$f(x) = x\,.$$

   In this case, both the functions $f(x) = x$ and $g(x) = 2x^2 - 1$ are infinite (as $x \to +\infty$) and **it holds** $f(x) = o(2x^2 - 1)$.

   Example 2:

$$f(x) \equiv 1\,.$$

   In this case $g(x) = 2x^2 - 1$ is infinite for $x \to +\infty$, whereas $f(x)$ is not and **it holds** $f(x) = o(2x^2 - 1)$.

   (b) *Establish if, from the hypothesis* $f(x) = o(2x^2 - 1)$ *for* $x \to +\infty$, *it follows that*

$$e^{f(x)} = o(e^{2x^2 - 1}) \qquad \text{as} \qquad x \to +\infty$$

   If it is true, prove it, otherwise provide a counterexample.

   Let

$$g(x) = 2x^2 - 1\,.$$

   Note that

$$f(x) - g(x) = g(x)\left(\frac{f(x)}{g(x)} - 1\right)\,.$$

   Given $f = \mathrm{o}(g)$ we have that $f/g \to 0$ and thus

$$\left(\frac{f(x)}{g(x)} - 1\right) \to -1\,.$$

Given that $\lim_{x\to+\infty} g(x) = +\infty$ it holds

$$\lim_{x\to+\infty} g(x)\left(\frac{f(x)}{g(x)} - 1\right) = -\infty \text{ and therefore } \lim_{x\to+\infty} \frac{e^{f(x)}}{e^{g(x)}} = \lim_{x\to+\infty} e^{g(x)\left(\frac{f(x)}{g(x)}-1\right)} = 0\,.$$

Then, $e^{f(x)} = \mathrm{o}\left(e^{g(x)}\right)$ for $x \to +\infty$.

3. (28 January 2016 - II)

(a) *Write the definition of order of infinitesimal of $f(x)$ with respect to $\frac{1}{x}$, for $x \to +\infty$.*

See the textbook.

(b) *Prove that*

$$a_n = \sqrt[7]{4n - \frac{2}{\sqrt{n}\log^3 n}} - \sqrt[7]{4n} = o\left(\frac{1}{n}\right), \qquad \textit{for } n \to +\infty.$$

It holds

$$\sqrt[7]{4n}\left[\left(1 - \frac{1}{2n^{3/2}\log^3 n}\right)^{1/7} - 1\right] = \sqrt[7]{4n}\left[-\frac{1}{14n^{3/2}\log^3 n} + o\left(\frac{1}{n^{3/2}\log^3 n}\right)\right]$$

$$= -\frac{\sqrt[7]{4}}{14\,n^{19/14}\log^3 n} + o\left(\frac{1}{n^{19/14}\log^3 n}\right) \qquad \textit{for } n \to +\infty\,.$$

Therefore

$$\frac{a_n}{\frac{1}{n}} = a_n n = -\frac{\sqrt[7]{4}}{14\,n^{5/14}\log^3 n} + o\left(\frac{1}{n^{5/14}\log^3 n}\right) = o(1) \qquad \textit{as } n \to +\infty\,.$$

(c) *Say if the following statement is true or false. If it is true, prove it; otherwise provide a counterexample.* If a_n is such that $1 \le a_n \le 3$, for every $n \in \mathbb{N}$, then $\lim_{n\to+\infty} a_n$ exists and it is a real number $l \in [1,3]$.

The statement is false. Take the following counterexample: $a_n = (-1)^n + 2$ such that $1 \le a_n \le 3$ but it does not admit limit as $n \to +\infty$.