MATHEMATICAL ANALYSIS I TUTORING
9^{TH} WEEK (10/05/2016)

PRIMITIVES - INDEFINITE INTEGRALS - INTEGRATION TECHNIQUES

PROPOSED EXERCISES - SOLUTIONS

1. Find the generic primitive of the following functions, and indicate an interval where it can be found:

   (a) $\boxed{f(x) = \dfrac{x}{x^2+9}}$
   The set of all the primitives of $f(x)$is given by the indefinite integral
   $$\int \frac{x}{x^2+9}\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+9}\mathrm{d}x = \frac{1}{2}\log(x^2+9) = F(x)$$
   Since $f(x)$ is continuous on $\mathbb{R}$, the primitives $F(x)+c,\ c\in\mathbb{R}$ are defined on $\mathbb{R}$.

   (b) $\boxed{f(x) = e^{-x} - e^{-4x}}$
   $$\int \left(e^{-x} - e^{-4x}\right)\mathrm{d}x = -e^{-x} + \frac{1}{4}e^{-4x} + c$$
   Such primitives are defined on $\mathbb{R}$.

   (c) $\boxed{f(x) = \dfrac{3-\cos x}{3x-\sin x}}$
   $$\int \frac{3-\cos x}{3x-\sin x}\mathrm{d}x = \log|3x-\sin x| + c$$
   Such primitives are defined on $I=(-\infty,0)$ or $J=(0,+\infty)$.

2. Find the primitive such that in x_0 it takes value y_0:

   (a) $\boxed{f(x) = \dfrac{1}{x^2+9},\ x_0=0,\ y_0=2}$
   $$F(x) = \int \frac{1}{x^2+9}\mathrm{d}x = \int \frac{1}{9(1+\frac{x^2}{9})}\mathrm{d}x = \frac{1}{3}\int \frac{\frac{1}{3}}{1+(\frac{x}{3})^2}\mathrm{d}x = \frac{1}{3}\arctan\frac{x}{3} + c$$
   We have to compute c imposing the condition $F(0)=2:$
   $$F(0) = \frac{1}{3}\arctan\frac{0}{3} + c = 2 \Rightarrow c = 2$$
   hence
   $$F(x) = \frac{1}{3}\arctan\frac{x}{3} + 2$$

   (b) $\boxed{f(x) = \dfrac{4}{x}\log^3 x,\ x_0=e,\ y_0=0}$
   $$F(x) = \int \frac{4}{x}\log^3 x\mathrm{d}x = \log^4 x + c$$
   We have to compute c imposing the condition $F(e)=0:$
   $$F(e) = \log^4 e + c = 0 \Rightarrow 1 + c = 0 \Rightarrow c = -1$$
   hence
   $$F(x) = \log^4 x - 1$$

   (c) $\boxed{f(x) = \sin x e^{\cos x},\ x_0=2\pi,\ y_0=0}$
   $$F(x) = \int \sin x e^{\cos x}\mathrm{d}x = -e^{\cos x} + c$$
   We have to compute c imposing the condition $F(2\pi)=0:$
   $$F(2\pi) = -e^{\cos(2\pi)} + c = 0 \Rightarrow -e + c = 0 \Rightarrow c = e$$
   hence
   $$F(x) = -e^{\cos x} + e$$

3. Prove that the functions $F(x) = \sin^2 x + 7$ and $G(x) = -\frac{1}{2}\cos(2x) - 11$ are two primitives of the same function $f(x)$ on $\mathbb{R}$; find $f(x)$ and say which is the constant $F(x) - G(x)$.

   $F(x)$ and $G(x)$ are both differentiable on $\mathbb{R}$. They are both primitives of the same function $f(x)$ if $F'(x) = G'(x) = f(x)$, for every $x \in \mathbb{R}$. Compute the derivatives:

   $F'(x) = 2\sin x \cos x = \sin(2x)$, $G'(x) = -\frac{1}{2}(-2)\sin(2x) = \sin(2x)$.

   Hence $F'(x) = G'(x) = f(x) = \sin(2x)$.

   Being 2 primitives of the same function on the same interval, their difference must be constant:

$$F(x) - G(x) = \sin^2 x + 7 + \frac{1}{2}\cos(2x) + 11 = \sin^2 x + \frac{1}{2}(1 - 2\sin^2 x) + 18 = 18 + \frac{1}{2} = \frac{37}{2}.$$

4. Consider the function

$$f(x) = \sqrt{4 - x^2}$$

   Prove that the function $F(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2}$ is a primitive of $f(x)$ on the interval $(-2, 2)$.

   Prove that the function $G(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2} - \frac{\pi}{3}$ is a primitive of $f(x)$ on the interval $(-2, 2)$ passing through the point P= $(1, \frac{\sqrt{3}}{2})$.

   In order to prove that $F(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2}$ is a primitive of $f(x) = \sqrt{4-x^2}$ on the interval $(-2, 2)$, it is sufficient to prova that $F'(x) = f(x)$, for every $x \in (-2, 2)$.

$$F'(x) = \frac{1}{2}\sqrt{4-x^2} + \frac{x}{2}\frac{-2x}{2\sqrt{4-x^2}} + 2\frac{1/2}{\sqrt{1-x^2/4}} = \frac{1}{2}\sqrt{4-x^2} + \frac{-x^2}{2\sqrt{4-x^2}} + \frac{2}{\sqrt{4-x^2}} =$$

$$= \frac{1}{2}\sqrt{4-x^2} + \frac{-x^2+4}{2\sqrt{4-x^2}} = \frac{1}{2}\sqrt{4-x^2} + \frac{1}{2}\sqrt{4-x^2} = \sqrt{4-x^2} = f(x).$$

   For sure $G(x)$ is a primitive of $f(x)$, since differs from $F(x)$ only by the constant $-\frac{\pi}{3}$.

   Check that $G(1) = \frac{\sqrt{3}}{2}$.

   $G(1) = \frac{1}{2}\sqrt{4-1} + 2\arcsin\frac{1}{2} - \frac{\pi}{3} = \frac{\sqrt{3}}{2} + 2\frac{\pi}{6} - \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

5. Compute the integrals:

   (a) $\boxed{\int \left(3x^4 + \frac{1}{x} + \sqrt[3]{x^2}\right) \, dx}$

$$\begin{aligned}\int \left(3x^4 + \frac{1}{x} + \sqrt[3]{x^2}\right) dx &= \int 3x^4 \, dx + \int \frac{1}{x} \, dx + \int \sqrt[3]{x^2} \, dx \\ &= 3\frac{x^5}{5} + \ln|x| + \int x^{2/3} \, dx \\ &= 3\frac{x^5}{5} + \ln|x| + \frac{x^{2/3+1}}{2/3+1} + c \\ &= 3\frac{x^5}{5} + \ln|x| + \frac{3}{5}\sqrt[3]{x^5} + c\end{aligned}$$

   (b) $\boxed{\int 4x^3(1+2x^4)^4 \, dx}$

$$\begin{aligned}\int 4x^3(1+2x^4)^4 \, dx &= = \frac{1}{2}\int 8x^3(1+2x^4)^4 \, dx \\ &= \frac{1}{10}(1+2x^4)^5 + c\end{aligned}$$

   (c) $\boxed{\int \frac{dx}{x\log^3 x}}$

$$\int \frac{dx}{x\log^3 x} = = \int \frac{1}{x}\log^{-3} x \, dx = -\frac{1}{2\log^2 x} + c$$

(d) $\boxed{\int x^2 e^{x^3}\ \mathrm{d}x}$

$$\int x^2 e^{x^3}\ \mathrm{d}x \quad = \quad = \frac{1}{3}\int 3x^2 e^{x^3}\ \mathrm{d}x\ = \frac{1}{3}e^{x^3} + c$$

(e) $\boxed{\int \frac{x^3}{1+x^8}\ \mathrm{d}x}$

$$\int \frac{x^3}{1+x^8}\ \mathrm{d}x \quad = \quad \int \frac{x^3}{1+(x^4)^2}\ \mathrm{d}x\ = \frac{1}{4}\int \frac{4x^3}{1+(x^4)^2}\ \mathrm{d}x\ = \frac{1}{4}\arctan x^4 + c$$

(f) $\boxed{\int \frac{x^3+x+1}{x^2+1}\ \mathrm{d}x}$

$$\begin{aligned}\int \frac{x^3+x+1}{x^2+1}\ \mathrm{d}x \quad &= \quad \int \frac{x(x^2+1)}{x^2+1}\ \mathrm{d}x\ + \int \frac{1}{x^2+1}\ \mathrm{d}x \\ &= \quad \int x\ dx + \int \frac{1}{x^2+1}\ \mathrm{d}x \\ &= \quad \frac{x^2}{2} + \arctan x + c\end{aligned}$$

(g) $\boxed{\int \cos^3 x\ \mathrm{d}x}$

$$\begin{aligned}\int \cos^3 x\ \mathrm{d}x \quad &= \quad \int \cos^2 x \cdot \cos x\ \mathrm{d}x \\ &= \quad \int (1-\sin^2 x)\cdot \cos x\ \mathrm{d}x \\ &= \quad \int \cos x\ \mathrm{d}x\ - \int \sin^2 x \cdot \cos x\ \mathrm{d}x \\ &= \quad \sin x - \frac{\sin^3 x}{3} + c\end{aligned}$$

(h) $\boxed{\int \frac{\sin x}{\cos x - 4}\ \mathrm{d}x}$

$$\int \frac{\sin x}{\cos x - 4}\ \mathrm{d}x \quad = \quad -\log|\cos x - 4| + c = log(4-\cos x) + c$$

(i) $\boxed{\int \frac{x^2}{(x^3+5)^4}\ \mathrm{d}x}$

$$\int \frac{x^2}{(x^3+5)^4}\ \mathrm{d}x \quad = \quad \frac{1}{3}\int 3x^2(x^3+5)^{-4}\ \mathrm{d}x\ = -\frac{1}{9}\frac{1}{(x^3+5)^3} + c$$

(j) $\boxed{\int \frac{1}{x(1+\log^2 x)}\ \mathrm{d}x}$

$$\int \frac{1}{x(1+\log^2 x)}\ \mathrm{d}x \quad = \quad \arctan(\log(x)) + c$$

(k) $\boxed{\int \frac{1}{\tan^4 x \cos^2 x}\ \mathrm{d}x}$

$$\int \frac{1}{\tan^4 x \cos^2 x}\ \mathrm{d}x \quad = \quad \int \frac{1}{\cos^2 x}\tan^{-4} x\ \mathrm{d}x\ = -\frac{1}{3\ \tan^3 x} + c$$

(l) $\boxed{\displaystyle\int \frac{\cos x}{\sqrt{3-\sin^2 x}}\ \mathrm{d}x}$

$$\begin{aligned}\int \frac{\cos x}{\sqrt{3-\sin^2 x}}\ \mathrm{d}x &= \int \frac{\cos x}{\sqrt{3}\sqrt{1-\frac{\sin^2 x}{3}}}\ \mathrm{d}x \\ &= \int \frac{\frac{\cos x}{\sqrt{3}}}{\sqrt{1-(\frac{\sin x}{\sqrt{3}})^2}}\ \mathrm{d}x\ = \arcsin\left(\frac{\sin x}{\sqrt{3}}\right)+c\end{aligned}$$

(m) $\boxed{\displaystyle\int (2x+3)^3\ \mathrm{d}x}$

$$\int (2x+3)^3\ \mathrm{d}x = \frac{(2x+3)^4}{8}+c$$

(n) $\boxed{\displaystyle\int \frac{1}{2-x}\ \mathrm{d}x}$

$$\int \frac{1}{2-x}\ \mathrm{d}x = -\log|2-x|+c$$

(o) $\boxed{\displaystyle\int \frac{x^2}{\sqrt{x^3+2}}\ \mathrm{d}x}$

$$\begin{aligned}\int \frac{x^2}{\sqrt{x^3+2}}\ \mathrm{d}x &= \frac{1}{3}\int 3x^2(x^3+2)^{-\frac{1}{2}}\ \mathrm{d}x \\ &= \frac{1}{3}\frac{(x^3+2)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \\ &= \frac{2}{3}(x^3+2)^{\frac{1}{2}} \\ &= \frac{2}{3}\sqrt{x^3+2}+c\end{aligned}$$

(p) $\boxed{\displaystyle\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x}}\ \mathrm{d}x}$

$$\begin{aligned}\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x}}\ \mathrm{d}x &= \int \frac{x^{1/2}+x^{1/3}}{x^{1/4}}\ \mathrm{d}x \\ &= \int (x^{1/2-1/4}+x^{1/3-1/4})\ \mathrm{d}x \\ &= \int (x^{1/4}+x^{1/12})\ \mathrm{d}x \\ &= \frac{x^{1/4+1}}{1/4+1}+\frac{x^{1/12+1}}{1/12+1}+c \\ &= \frac{4}{5}(x^{5/4}+\frac{12}{13}x^{13/12})+c\end{aligned}$$

(q) $\boxed{\displaystyle\int \frac{e^{2x}}{3+e^{2x}}\ \mathrm{d}x}$

$$\int \frac{e^{2x}}{3+e^{2x}}\ \mathrm{d}x = \frac{1}{2}\log(3+e^{2x})+c$$

(r) $\boxed{\displaystyle\int \frac{e^{2+\sqrt{x}}}{\sqrt{x}}\ \mathrm{d}x}$

$$\int \frac{e^{2+\sqrt{x}}}{\sqrt{x}}\ \mathrm{d}x = 2\int \frac{e^{2+\sqrt{x}}}{2\sqrt{x}}\ \mathrm{d}x\ = 2e^{2+\sqrt{x}}+c$$

(s) $\boxed{\displaystyle\int \frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x}$

$$\int \frac{x}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \int \frac{2x}{2\sqrt{x^2+a^2}}\,\mathrm{d}x = \sqrt{x^2+a^2}+c$$

(t) $\boxed{\displaystyle\int \frac{e^x}{\sqrt{2e^x+1}}\,\mathrm{d}x}$

$$\int \frac{e^x}{\sqrt{2e^x+1}}\,\mathrm{d}x = \sqrt{2e^x+1}+c$$

(u) $\boxed{\displaystyle\int \frac{e^x}{4+e^{2x}}\,\mathrm{d}x}$

$$\int \frac{e^x}{4+e^{2x}}\,\mathrm{d}x = \frac{1}{4}\int \frac{e^x}{1+(\frac{e^x}{2})^2}\,\mathrm{d}x = \frac{1}{2}\arctan\frac{e^x}{2}+c$$

(v) $\boxed{\displaystyle\int \frac{\cos x}{\sqrt{3-\sin^2 x}}\,\mathrm{d}x}$

$$\begin{aligned}\int \frac{\cos x}{\sqrt{3-\sin^2 x}}\,\mathrm{d}x &= \int \frac{\cos x}{\sqrt{3}\sqrt{1-\frac{\sin^2 x}{3}}}\,\mathrm{d}x \\ &= \int \frac{\frac{\cos x}{\sqrt{3}}}{\sqrt{1-(\frac{\sin x}{\sqrt{3}})^2}}\,\mathrm{d}x = \arcsin\left(\frac{\sin x}{\sqrt{3}}\right)+c\end{aligned}$$

(w) $\boxed{\displaystyle\int \frac{\cos\log x}{x}\,\mathrm{d}x}$

$$\int \frac{\cos\log x}{x}\,\mathrm{d}x = \sin\log x+c$$

(z) $\boxed{\displaystyle\int \sqrt[4]{(x-2)^3}\,\mathrm{d}x}$

$$\int \sqrt[4]{(x-2)^3}\,\mathrm{d}x = \int (x-2)^{3/4}\,\mathrm{d}x = \frac{4}{7}(x-2)^{7/4}+c$$

6. Compute the integrals by parts:

a) $\boxed{\displaystyle\int \sqrt{x}\log x\,\mathrm{d}x}$

$$\begin{aligned}\int \sqrt{x}\log x\,\mathrm{d}x &= \frac{x^{1/2+1}}{1/2+1}\log x - \int \frac{x^{1/2+1}}{1/2+1}\frac{1}{x}\,\mathrm{d}x \\ &= \frac{x^{3/2}}{3/2}\log x - \int \frac{x^{1/2}}{3/2}\,\mathrm{d}x \\ &= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\int x^{1/2}\,\mathrm{d}x \\ &= \frac{2}{3}x^{3/2}\log x - \frac{2}{3}\frac{2}{3}x^{3/2}+c \\ &= \frac{2}{3}x^{3/2}\left[\log x - \frac{2}{3}\right]+c\end{aligned}$$

b) $\boxed{\int e^{2x}\sin(3x)\,\mathrm{d}x}$

$$\begin{aligned}\int e^{2x}\sin(3x)\,\mathrm{d}x &= \frac{1}{2}\,e^{2x}\sin(3x) - \int \frac{1}{2}\,e^{2x}\cdot 3\cos(3x)\,\mathrm{d}x = \\ &= \frac{1}{2}\,e^{2x}\sin(3x) - \frac{3}{2}\,\left(\frac{1}{2}\,e^{2x}\cos(3x) - \int \frac{1}{2}\,e^{2x}\cdot(-3)\sin(3x)\,\mathrm{d}x\right) = \\ &= \frac{1}{2}\,e^{2x}\sin(3x) - \frac{3}{4}\,e^{2x}\cos(3x) - \frac{9}{4}\,\int e^{2x}\sin(3x)\,\mathrm{d}x\end{aligned}$$

Dunque

$$\left(1+\frac{9}{4}\right)\int e^{2x}\sin(3x)\,\mathrm{d}x = e^{2x}\left(\frac{1}{2}\,\sin(3x) - \frac{3}{4}\,\cos(3x)\right)$$

e finalmente

$$\int e^{2x}\sin(3x)\,\mathrm{d}x = \frac{1}{13}\,e^{2x}(2\sin(3x) - 3\cos(3x)) + c$$

c) $\boxed{\int \arcsin x\ \mathrm{d}x}$

$$\begin{aligned}\int \arcsin x\ \mathrm{d}x &= x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}\ \mathrm{d}x \\ &= x\arcsin x + \int \frac{-2x}{2\ \sqrt{1-x^2}}\ \mathrm{d}x \\ &= x\arcsin x +\ \sqrt{1-x^2} + c\end{aligned}$$

d) $\boxed{\int \log^2 x\,\mathrm{d}x}$

$$\begin{aligned}\int \log^2 x\,\mathrm{d}x &= x\log^2 x - \int x\log x\frac{1}{x}\ \mathrm{d}x \\ &= x\log^2 x - \int \log x\ \mathrm{d}x \\ &= x\log^2 x - 2x\log x + 2x + c\end{aligned}$$

e) $\boxed{\int (x+2)^2 e^x\,\mathrm{d}x}$

$$\begin{aligned}\int (x+2)^2 e^x\,\mathrm{d}x &= (x+2)^2 e^x - \int 2(x+2)e^x\,\mathrm{d}x \\ &= (x+2)^2 e^x - \left(2(x+2)e^x - \int 2e^x\,\mathrm{d}x\right) \\ &= (x+2)^2 e^x - 2(x+2)e^x + 2e^x + c\end{aligned}$$

f) $\boxed{\int \arctan x\,\mathrm{d}x}$

$$\begin{aligned}\int \arctan x\,\mathrm{d}x &= x\ \arctan x - \int \frac{x}{1+x^2}\ \mathrm{d}x \\ &= x\ \arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}\ \mathrm{d}x \\ &= x\ \arctan x - \frac{1}{2}\ \log(1+x^2) + c\end{aligned}$$

g) $\boxed{\int x \arctan x \, \mathrm{d}x}$

$$\begin{aligned}
\int x \arctan x \, \mathrm{d}x &= \frac{1}{2}\, x^2 \arctan x - \int \frac{1}{2}\, x^2 \frac{1}{1+x^2} \, \mathrm{d}x \\
&= \frac{1}{2}\, x^2 \arctan x - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} \, \mathrm{d}x \\
&= \frac{1}{2}\, x^2 \arctan x - \frac{1}{2} \int \left(1 - \frac{1}{1+x^2}\right) \, \mathrm{d}x \\
&= \frac{1}{2}\, x^2 \arctan x - \frac{1}{2}\, x + \frac{1}{2} \arctan x + c \\
&= \frac{x^2+1}{2} \arctan x - \frac{1}{2}\, x + c
\end{aligned}$$

h) $\boxed{\log(1+x^2)\, \mathrm{d}x}$

$$\begin{aligned}
\int \log(1+x^2)\, \mathrm{d}x &= x \log(1+x^2) - \int \frac{2x^2}{1+x^2} \, \mathrm{d}x \\
&= x \log(1+x^2) - 2 \int \frac{x^2+1-1}{1+x^2} \, \mathrm{d}x \\
&= x \log(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2}\right) \, \mathrm{d}x \\
&= x \log(1+x^2) - 2x + 2 \arctan x + c
\end{aligned}$$

7. Compute the following **integrals of rational functions**

a) $\boxed{\int \frac{x}{x^2+4x+3} \, \mathrm{d}x}$

$$\int \frac{x}{x^2+4x+3} \, \mathrm{d}x = \int \frac{x}{(x+1)(x+3)} \, \mathrm{d}x$$

$$\frac{x}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)}$$

Multiplying both sides by $(x+1)$

$$\frac{x}{(x+1)(x+3)}(x+1) = \frac{A}{(x+1)}(x+1) + \frac{B}{(x+3)}(x+1) \Rightarrow \frac{x}{(x+3)} = A + \frac{B}{(x+3)}(x+1)$$

Evalueting in $x = -1$ we get $\frac{-1}{2} = A \Rightarrow A = -\frac{1}{2}$

$$\frac{x}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)}$$

Multiplying both sides by $(x+3)$

$$\frac{x}{(x+1)(x+3)}(x+3) = \frac{A}{(x+1)}(x+3) + \frac{B}{(x+3)}(x+3) \Rightarrow \frac{x}{(x+1)} = \frac{A}{(x+1)}(x+3) + B$$

Evalueting in $x = -3$ we get $\frac{-3}{-2} = B \Rightarrow B = \frac{3}{2}$

Thus:

$$\begin{aligned}
\int \frac{x}{x^2+4x+3} \, \mathrm{d}x &= -\frac{1}{2} \int \frac{1}{(x+1)} \, \mathrm{d}x + \frac{3}{2} \int \frac{1}{(x+3)} \, \mathrm{d}x \\
&= -\frac{1}{2} \log(x+1) + \frac{3}{2} \log(x+3) + c
\end{aligned}$$

b) $\boxed{\displaystyle\int \frac{x}{(x^2+1)(x-1)}\,\mathrm{d}x}$

$$\frac{x}{(x^2+1)(x-1)} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)}$$

Multiplying both sides by $(x-1)$

$$\frac{x}{(x^2+1)(x-1)}(x-1) = \frac{A}{(x-1)}(x-1) + \frac{Bx+C}{(x^2+1)}(x-1) \Rightarrow \frac{x}{(x^2+1)} = A + \frac{Bx+C}{(x^2+1)}(x-1)$$

In $x = 1$ we get $A = \frac{1}{2}$

Multiplying both sides by (x^2+1)

$$\frac{x}{(x^2+1)(x-1)}(x^2+1) = \frac{A}{(x-1)}(x^2+1) + \frac{Bx+C}{(x^2+1)}(x^2+1) \Rightarrow \frac{x}{(x-1)} = \frac{A}{(x-1)}(x^2+1) + Bx + C$$

In $x = i$ it holds

$$\frac{i}{i-1} = Bi + C$$

$$\frac{i}{i-1}\frac{i+1}{i+1} = Bi + C$$

$$\frac{1-i}{2} = Bi + C$$

$$\frac{1}{2} - \frac{1}{2}i = Bi + C \Rightarrow C = \frac{1}{2}, B = -\frac{1}{2}$$

$$\begin{aligned}
\int \frac{x}{(x^2+1)(x-1)}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{(x-1)}\,\mathrm{d}x + \frac{1}{2}\int \frac{x-1}{(x^2+1)}\,\mathrm{d}x \\
&= \frac{1}{2}\int \frac{1}{(x-1)}\,\mathrm{d}x + \frac{1}{4}\int \frac{2x}{(x^2+1)}\,\mathrm{d}x - \frac{1}{2}\int \frac{1}{(x^2+1)}\,\mathrm{d}x \\
&= \frac{1}{4}\big(+2\log(1-x) - \log(x^2+1) + 2\arctan x\big) + c
\end{aligned}$$

c) $\boxed{\displaystyle\int \frac{x+2}{x^2+3x+5}\;\mathrm{d}x}$

$$\begin{aligned}
\int \frac{x+2}{x^2+3x+5}\;\mathrm{d}x &= \frac{1}{2}\int \frac{2x+4}{x^2+3x+5}\;\mathrm{d}x \\
&= \frac{1}{2}\int \frac{2x+3}{x^2+3x+5}\;\mathrm{d}x \;+ \frac{1}{2}\int \frac{1}{x^2+3x+5}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{2}\int \frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}+5}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{2}\int \frac{1}{\left(x+\frac{3}{2}\right)^2+\frac{11}{4}}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{2}\int \frac{1}{\frac{11}{4}\left(\left(\frac{x+\frac{3}{2}}{\frac{\sqrt{11}}{2}}\right)^2+1\right)}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{2}\frac{4}{11}\int \frac{1}{\left(\frac{2x+3}{\sqrt{11}}\right)^2+1}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{2}\frac{4}{11}\frac{\sqrt{11}}{2}\int \frac{\frac{2}{\sqrt{11}}}{\left(\frac{2x+3}{\sqrt{11}}\right)^2+1}\;\mathrm{d}x \\
&= \frac{1}{2}\log(x^2+3x+5) + \frac{1}{\sqrt{11}}\arctan\frac{2x+3}{\sqrt{11}} + c
\end{aligned}$$

d) $\boxed{\displaystyle\int \frac{x}{(x-1)^2}\ \mathrm{d}x}$

$$\begin{aligned}\int \frac{x}{(x-1)^2}\ \mathrm{d}x &= \int \frac{x+1-1}{(x-1)^2}\ \mathrm{d}x \\ = &= \int \left(\frac{1}{x-1}+\frac{1}{(x-1)^2}\right)\ \mathrm{d}x\ = \\ &= \log|x-1| - \frac{1}{x-1} + c\end{aligned}$$

e) $\boxed{\displaystyle\int \frac{3x^2-x}{(x+1)^2(x+2)}\ \mathrm{d}x}$

$$\frac{3x^2-x}{(x+1)^2(x+2)} = \frac{A}{(x+2)} + \frac{B}{(x+1)} + \frac{D}{(x+1)^2}$$

In $x=0$ we have $0 = \dfrac{A}{2} + B + D$

Multiplying both sides by

$$\frac{3x^2-x}{(x+1)^2(x+2)}(x+2) = \frac{A}{(x+2)}(x+2) + \frac{B}{(x+1)}(x+2) + \frac{D}{(x+1)^2}(x+2)$$

$$\Rightarrow\quad \frac{3x^2-x}{(x+1)^2} = A + \frac{B}{(x+1)}(x+2) + \frac{D}{(x+1)^2}(x+2)$$

In $x=-2$ we get $A = \dfrac{3(-2)^2-(-2)}{(-2+1)^2} = \dfrac{3\cdot 4+2}{(-1)^2} = 14$

Multiplying both sides by $(x+1)^2$

$$\frac{3x^2-x}{(x+1)^2(x+2)}(x+1)^2 = \frac{A}{(x+2)}(x+1)^2 + \frac{B}{(x+1)}(x+1)^2 + \frac{Cx+D}{(x+1)^2}(x+1)^2$$

$$\Rightarrow\quad \frac{3x^2-x}{(x+2)} = \frac{A}{(x+2)}(x+1)^2 + B(x+1) + D$$

In $x=-1$ it holds $D = \dfrac{3(-1)^2-(-1)}{(-1+2)} = \dfrac{3+1}{1} = 4$

Thus we solve the system

$$\begin{cases} \frac{A}{2}+B+D=0 \\ A=14 \\ D=4 \end{cases} \quad\Rightarrow \begin{cases} A=14 \\ B=-11 \\ D=4 \end{cases}$$

$$\begin{aligned}\int \frac{3x^2-x}{(x+1)^2(x+2)}\ \mathrm{d}x &= \int \frac{14}{x+2}\ \mathrm{d}x\ - \int \frac{11}{x+1}\ \mathrm{d}x\ + \int \frac{4}{(x+1)^2}\ \mathrm{d}x \\ &= 14\log|x+2| - 11\log|x+1| - \frac{4}{x+1} + c\end{aligned}$$

f) $\boxed{\displaystyle\int \frac{x^2-2x-1}{x^2-4x+4}\ \mathrm{d}x}$

Compute the polynomial division:

$$\frac{x^2-2x-1}{x^2-4x+4} = 1 + \frac{2x-5}{x^2-4x+4}$$

and

$$\frac{2x-5}{x^2-4x+4} = \frac{2x-4-1}{x^2-4x+4} = \frac{2x-4}{x^2-4x+4} - \frac{1}{x^2-4x+4} = \frac{2(x-2)}{(x-2)^2} - \frac{1}{(x-2)^2} = \frac{2}{x-2} - \frac{1}{(x-2)^2}$$

Thus:

$$\int \frac{x^2-2x-1}{x^2-4x+4}\ \mathrm{d}x = x + \frac{1}{x-2} + 2\log|x-2| + c$$

g) $\boxed{\displaystyle\int \frac{x^4-3x^2-1}{x^3-1}\ \mathrm{d}x}$

Compute the polynomial division and the partial fraction expansion:

$$\begin{aligned}\frac{x^4-3x^2-1}{x^3-1} &= x+\frac{-3x^2+x-1}{x^3-1}= \\ &= x+\frac{-1}{x-1}+\frac{-2x}{x^2+x+1}= \\ &= x-\frac{1}{x-1}-\frac{2x+1-1}{x^2+x+1}= \\ &= x-\frac{1}{x-1}-\frac{2x+1}{x^2+x+1}+\frac{1}{x^2+x+1}\end{aligned}$$

Integrate the last summand: complete the square; observe that $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$, thus:

$$\begin{aligned}\int \frac{1}{x^2+x+1}\ \mathrm{d}x &= \int \frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\ \mathrm{d}x \\ &= \frac{4}{3}\int \frac{1}{1+\frac{4}{3}\left(x+\frac{1}{2}\right)^2}\ \mathrm{d}x \\ &= \frac{2}{\sqrt{3}}\int \frac{\frac{2}{\sqrt{3}}}{1+\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right]^2}\ \mathrm{d}x \\ &= \frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+c\end{aligned}$$

Therefore

$$\int \frac{x^4-3x^2-1}{x^3-1}\ \mathrm{d}x = \frac{x^2}{2}-\log|x-1|-\log(x^2+x+1)+\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+c$$

or:

$$\int \frac{x^4-3x^2-1}{x^3-1}\ \mathrm{d}x = \frac{x^2}{2}-\log|x^3-1|+\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+c$$

h) $\boxed{\displaystyle\int \frac{1}{x^2(x^2+1)}\ \mathrm{d}x}$

By partial fraction expansion:

$$\frac{1}{x^2(x^2+1)}=\frac{1}{x^2}-\frac{1}{x^2+1}$$

Thus:

$$\int \frac{1}{x^2(x^2+1)}\ \mathrm{d}x = -\frac{1}{x}-\arctan x+c$$

i) $\boxed{\displaystyle\int \frac{1}{(1-x^2)^2}\ \mathrm{d}x}$

By partial fraction expansion:

$$\frac{1}{(1-x^2)^2}=\frac{1}{4(x+1)}+\frac{1}{4(x+1)^2}-\frac{1}{4(x-1)}+\frac{1}{4(x-1)^2}$$

Thus:

$$\int \frac{1}{(1-x^2)^2}\ \mathrm{d}x = \frac{1}{4}\log|x+1|-\frac{1}{4}\log|x-1|-\frac{1}{4(x+1)}-\frac{1}{4(x-1)}+c$$

j) $\boxed{\displaystyle\int \frac{dx}{x^4-1}\ \mathrm{d}x}$

By partial fraction expansion:

$$\frac{1}{x^4-1}=\frac{1}{(x-1)(x+1)(1+x^2)}=\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(1+x)^2}$$

Hence:

$$\int \frac{dx}{x^4-1}\ \mathrm{d}x \quad = \quad \frac{1}{4}\log|1-x|-\frac{1}{4}\log|x+1|-\frac{1}{2}\arctan x+c$$

8. Compute by substitution the following **integrals of trascendent and irrational functions**:

a) $\boxed{\displaystyle\int \frac{1+x+\sqrt{x}}{1+x\sqrt{x}}\mathrm{d}x}$

Apply the substitution $\sqrt{x}=t$ and thus $x=t^2$, deriving both members $\mathrm{d}x \ = 2t\ \mathrm{d}t$ and thus

$$\int \frac{1+x+\sqrt{x}}{1+x\sqrt{x}}\mathrm{d}x=2\int \frac{t^3+t^2+t}{t^3+1}\ \mathrm{d}t$$

Compute the polynomial division and the partial fraction expansion

$$\begin{aligned}\frac{t^3+t^2+t}{t^3+1} &= 1+\frac{t^2+t-1}{(t+1)(t^2-t+1)}\\ &= 1-\frac{1}{3(t+1)}+\frac{4t-2}{3(t^2-t+1)}\\ &= 1-\frac{1}{3}\frac{1}{1+t}+\frac{2}{3}\frac{2t-1}{t^2-t+1}\end{aligned}$$

Hence:

$$\begin{aligned}\int \frac{t^3+t^2+t}{t^3+1}\ \mathrm{d}t &= \int\left(1-\frac{1}{3}\frac{1}{1+t}+\frac{2}{3}\frac{2t-1}{t^2-t+1}\right)\ \mathrm{d}t\\ &= t-\frac{1}{3}\log|1+t|+\frac{2}{3}\log(t^2-t+1)+c\end{aligned}$$

Finally

$$\int \frac{1+x+\sqrt{x}}{1+x\sqrt{x}}\mathrm{d}x=2\sqrt{x}-\frac{2}{3}\log(1+\sqrt{x})+\frac{4}{3}\log(x-\sqrt{x}+1)+c$$

b) $\boxed{\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\mathrm{d}x}$

By substitution $\sqrt[6]{x}=t$ i.e. $x=t^6$, deriving both sides $\mathrm{d}x \ = 6t^5\ \mathrm{d}t$ and thus

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\mathrm{d}x=\int \frac{6t^5}{t^3+t^2}\ \mathrm{d}t \ =\int \frac{6t^3}{t+1}\ \mathrm{d}t$$

Compute the polynomial division and the partial fraction expansion

$$\frac{6t^3}{t+1} \quad = \quad 6\left(t^2-t+1-\frac{1}{t+1}\right)$$

Hence:

$$\begin{aligned}\int \frac{6t^3}{t+1} &= 6\int\left(t^2-t+1-\frac{1}{t+1}\right)\ \mathrm{d}t\\ &= 6\left(\frac{t^3}{3}-\frac{t^2}{2}+t-\log|1+t|+c\right)\end{aligned}$$

Finally

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\mathrm{d}x \quad = \quad 2\sqrt{x}-3\sqrt[3]{x}+6\sqrt[6]{x}-6\log(1+\sqrt[6]{x})+c$$

c) $\boxed{\displaystyle\int \frac{\sin x}{1+\cos x+\sin^2 x}\,\mathrm{d}x}$

Observe that $\dfrac{\sin x}{1+\cos x+\sin^2 x} = \dfrac{\sin x}{2+\cos x-\cos^2 x}$; thus the integral is in the form $\displaystyle\int R(\cos x)\sin x\,\mathrm{d}x$ where R denotes a rational function; by substitution $\cos x = t$ and thus $\sin x\ \mathrm{d}x\ = -\ \mathrm{d}t$:

$$\int \frac{\sin x}{1+\cos x+\sin^2 x}\,\mathrm{d}x = \int \frac{1}{2+\cos x-\cos^2 x}\ \sin x\,\mathrm{d}x = \int \frac{1}{t^2-t-2}\ \mathrm{d}t$$

By partial fraction expansion

$$\begin{aligned}\int \frac{1}{t^2-t-2}\ \mathrm{d}t &= \int \frac{1}{(t-2)(t+1)}\ \mathrm{d}t\\ &= \frac{1}{3}\int\left(\frac{1}{t-2}-\frac{1}{t+1}\right)\ \mathrm{d}t\\ &= \frac{1}{3}\log|t-2|-\frac{1}{3}\log|t+1|+c\end{aligned}$$

Finally

$$\int \frac{\sin x}{1+\cos x+\sin^2 x}\ \mathrm{d}x = \frac{1}{3}\log|\cos x-2|-\frac{1}{3}\log|\cos x+1|+c$$

d) $\boxed{\displaystyle\int \frac{1}{1-\sin x+\cos x}\,\mathrm{d}x}$

By substitution:

$$\tan\frac{x}{2}=t \quad\Rightarrow\quad \sin x=\frac{2t}{t^2+1},\ \ \cos x=\frac{1-t^2}{t^2+1}$$

Then $\ x=2\arctan t$, $\quad \mathrm{d}x\ = \dfrac{2}{t^2+1}\ \mathrm{d}t\ $ and it follows

$$\begin{aligned}\int \frac{1}{1-\sin x+\cos x}\ \mathrm{d}x &= \int \frac{1}{1-\frac{2t}{t^2+1}+\frac{1-t^2}{t^2+1}}\ \frac{2}{1+t^2}dt\\ &= \int \frac{1}{\frac{t^2+1-2t+1-t^2}{t^2+1}}\ \frac{2}{1+t^2}\ \mathrm{d}t\\ &= \int \frac{1}{2-2t}\,2\ \mathrm{d}t\\ &= \int \frac{1}{1-t}\ \mathrm{d}t\ = -\ln|1-t|+c = -\ln\left|1-\tan\frac{x}{2}\right|+c\end{aligned}$$

e) $\boxed{\displaystyle\int \frac{\tan^2 x+1}{\tan x+1}\ \mathrm{d}x}$

By substitution $\tan x+1=t$, deriving both sides we have $(\tan^2 x+1)\ \mathrm{d}x\ = \ \mathrm{d}t\ $ and thus:

$$\int \frac{\tan^2 x+1}{\tan x+1}\ \mathrm{d}x\ = \int \frac{1}{t}\ \mathrm{d}t\ = \ln|t|+c = \ln|\tan x+1|+c$$

f) $\boxed{\displaystyle\int \frac{1}{x(\log^2 x-1)}\ \mathrm{d}x}$

By substitution $\log x=t$, deriving both sides we have $\dfrac{1}{x}\ \mathrm{d}x\ = \ \mathrm{d}t\ $ and thus

$$\begin{aligned}\int \frac{1}{x(\log^2 x-1)}\ \mathrm{d}x &= \int \frac{1}{t^2-1}\ \mathrm{d}t\\ &= \int \frac{1}{(t-1)(t+1)}\ \mathrm{d}t\end{aligned}$$

By partial fraction expansion

$$\frac{1}{(t-1)(t+1)} = \frac{A}{(t-1)}+\frac{B}{(t+1)}$$

Multiplying both sides by $t+1$:

$$\frac{1}{(t-1)(t+1)}(t+1) = \frac{A}{(t-1)}(t+1) + \frac{B}{(t+1)}(t+1) \Rightarrow \frac{1}{t-1} = \frac{A}{t-1}(t+1) + B$$

In $t=-1$, we have:

$$\frac{1}{(-1-1)} = \frac{A}{(-1-1)}(-1+1) + B \Rightarrow B = -\frac{1}{2}$$

Multiplying both sides by $t-1$:

$$\frac{1}{(t-1)(t+1)}(t-1) = \frac{A}{(t-1)}(t-1) + \frac{B}{(t+1)}(t-1) \Rightarrow \frac{1}{t+1} = A + \frac{B}{t+1}(t-1)$$

In $t=1$, we have:

$$\frac{1}{1+1} = A + \frac{B}{1+1}(1-1) \Rightarrow A = \frac{1}{2}$$

Hence:

$$\int \frac{1}{(t-1)(t+1)}\ \mathrm{d}t\ = \frac{1}{2}\int \frac{1}{t-1}dt - \frac{1}{2}\int \frac{1}{t+1}\ \mathrm{d}t\ = \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| + c$$

Finally

$$\int \frac{1}{x(\log^2 x - 1)}\ \mathrm{d}x = \frac{1}{2}\log|\log x - 1| - \frac{1}{2}\log|\log x + 1| + c$$

g) $\boxed{\int \frac{\log^3 x + 2}{x(\log^2 x + 1)}\ \mathrm{d}x}$

By substitution $\log x = t$, deriving both members we have $\frac{1}{x}\ \mathrm{d}x\ =\ \mathrm{d}t$ and thus the integral becomes:

$$\int \frac{\log^3 x + 2}{x(\log^2 x + 1)}\,\mathrm{d}x = \int \frac{t^3+2}{t^2+1}\ \mathrm{d}t$$

Now:

$$\begin{aligned}
\int \frac{t^3+2}{t^2+1}\ \mathrm{d}t &= \int \frac{t^3+t-t+2}{t^2+1}\ \mathrm{d}t \\
&= \int \frac{t^3+t}{t^2+1}\ \mathrm{d}t\ + \int \frac{-t+2}{t^2+1}\ \mathrm{d}t \\
&= \int \frac{t(t^2+1)}{t^2+1}\ \mathrm{d}t\ - \int \frac{t}{t^2+1}\ \mathrm{d}t\ + \int \frac{2}{t^2+1}\ \mathrm{d}t \\
&= \int t\ \mathrm{d}t\ - \frac{1}{2}\int \frac{2t}{t^2+1}\ \mathrm{d}t\ + \int \frac{2}{t^2+1}\ \mathrm{d}t \\
&= \frac{t^2}{2} - \frac{1}{2}\log(t^2+1) + 2\arctan t + c
\end{aligned}$$

Finally:

$$\int \frac{\log^3 x + 2}{x(\log^2 x + 1)}\,\mathrm{d}x = \frac{\log^2 x}{2} - \frac{1}{2}\log(\log^2 x + 1) + 2\arctan(\log x) + c$$

h) $\boxed{\int \frac{\sinh x + 1}{\cosh x - 1}\,\mathrm{d}x}$

Notice that

$$\frac{\sinh x + 1}{\cosh x - 1} = \frac{e^x - e^{-x} + 2}{e^x + e^{-x} - 2} = \frac{e^{2x} - 1 + 2e^x}{e^{2x} + 1 - 2e^x}$$

By substitution $e^x = t$, and so $x = \ln t$ and $\mathrm{d}x\ = \frac{1}{t}\ \mathrm{d}t$. The integral becomes

$$\begin{aligned}
\int \frac{\sinh x + 1}{\cosh x - 1}\,\mathrm{d}x &= \int \frac{e^{2x} - 1 + 2e^x}{e^{2x} + 1 - 2e^x}\ \mathrm{d}x \\
&= \int \frac{t^2+2t-1}{t^2-2t+1}\,\frac{1}{t}\ \mathrm{d}t\ = \int \frac{t^2+2t-1}{t(t-1)^2}\ \mathrm{d}t \\
&= \int \left(\frac{2}{t-1} + \frac{2}{(t-1)^2} - \frac{1}{t}\right)\ \mathrm{d}t \\
&= 2\log|t-1| - \frac{2}{t-1} - \log|t| + c \\
&= 2\log|e^x-1| - \frac{2}{e^x-1} - x + c
\end{aligned}$$

i) $\boxed{\displaystyle\int \frac{\sin 2x}{6\sin x - \cos 2x + 5}\ \mathrm{d}x}$

Notice that $\dfrac{\sin 2x}{6\sin x - \cos 2x + 5} = \dfrac{2\sin x \cos x}{6\sin x - 1 + 2\sin^2 x + 5} = \dfrac{\sin x}{\sin^2 x + 3\sin x + 2}\ \cos x$; the integral is in the form $\displaystyle\int R(\sin x)\cos x\,\mathrm{d}x$ where R indicates a rational function; by substitution $\sin x = t$ and thus $\cos x\ \mathrm{d}x\ =\ \mathrm{d}t$:

$$\int \frac{\sin 2x}{6\sin x - \cos 2x + 5}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x + 3\sin x + 2}\ \cos x\,\mathrm{d}x = \int \frac{t}{t^2 + 3t + 2}\ \mathrm{d}t$$

By partial fraction expansion

$$\begin{aligned}\int \frac{t}{t^2+3t+2}\ \mathrm{d}t \quad &= \quad \int \frac{t}{(t+2)(t+1)}\ \mathrm{d}t\ = \int \left(\frac{2}{t+2} - \frac{1}{t+1}\right)\ \mathrm{d}t \\ &= \quad 2\log|t+2| - \log|t+1| + c\end{aligned}$$

Finally

$$\int \frac{\sin 2x}{6\sin x - \cos 2x + 5}\ \mathrm{d}x\ = 2\log|\sin x + 2| - \log|\sin x + 1| + c$$

j) $\boxed{\displaystyle\int \frac{2x+5}{x+\sqrt{x-3}}\,\mathrm{d}x}$

By substitution $\sqrt{x-3} = t$ and thus $x = 3 + t^2$, deriving both members $\mathrm{d}x\ = 2t\ \mathrm{d}t$:

$$\int \frac{2x+5}{x+\sqrt{x-3}}\,\mathrm{d}x = \int \frac{4t^3+22t^2}{t^2+t+3}\ \mathrm{d}t$$

Computing the polynomial division and the partial fraction expansion

$$\begin{aligned}\frac{4t^3+22t^2}{t^2+t+3} \quad &= \quad 4t - 4 + \frac{14t+12}{t^2+t+3} \\ &= \quad 4t - 4 + 7\,\frac{2t+1}{t^2+t+3} + \frac{5}{t^2+t+3}\end{aligned}$$

Complete the square: $t^2+t+3 = \left(t+\frac{1}{2}\right)^2 + \frac{11}{4} = \frac{11}{4}\left[1 + \left(\frac{2t+1}{\sqrt{11}}\right)^2\right]$. Hence:

$$\begin{aligned}\int \frac{4t^3+22t^2}{t^2+t+3}\ \mathrm{d}t \quad &= \quad \int \left(4t - 4 + 7\,\frac{2t+1}{t^2+t+3} + \frac{5}{t^2+t+3}\right)\ \mathrm{d}t \\ &= \quad 2t^2 - 4t + 7\log(t^2+t+3) + \frac{20}{11}\int \frac{1}{1+\left(\frac{2t+1}{\sqrt{11}}\right)^2} \\ &= \quad 2t^2 - 4t + 7\log(t^2+t+3) + \frac{10}{\sqrt{11}}\arctan\frac{2t+1}{\sqrt{11}} + c\end{aligned}$$

Finally

$$\int \frac{2x+5}{x+\sqrt{x-3}}\,\mathrm{d}x \quad = \quad 2x - 6 - 4\sqrt{x-3} + 7\log(x+\sqrt{x-3}) + \frac{10}{\sqrt{11}}\arctan\frac{2\sqrt{x-3}+1}{\sqrt{11}} + c$$