MATHEMATICAL ANALYSIS I TUTORING
3^{rd} WEEK

SEQUENCES - LIMITS OF SEQUENCES
LIMITS OF FUNCTIONS - CONTINUITY

PROPOSED EXERCISES

1. For each of the following properties concerning a generic sequence $(a_n), a_n \in \mathbb{R}$, write the definition and its logical negation:

   (a) (a_n) is indeterminate

   (b) (a_n) is negative

   (c) (a_n) is not upper bounded

   (d) (a_n) is bounded

   (e) (a_n) is regular[1]

   (e) (a_n) is definitely increasing

2. For each of the following sequences

$$a_n = (-1)^n \cos((2n+1)\pi); \quad b_n = \frac{11-n}{3n}; \quad c_n = \sin\left(n\frac{\pi}{2}\right)$$

   say which properties are true or false.

   (a) the terms are definitively less than a certain $k > 0$

   (b) the range of the sequence is $\{-1, 1\}$

   (c) the terms are all negative

   (d) the sequence is regular

   (e) the sequence is increasing

3. Write the definition of limit of a generic sequence $(a_n), a_n \in \mathbb{R}$:

   (a) $\lim_{n\to+\infty} a_n = -\infty$

   (b) $\lim_{n\to+\infty} a_n = l$

   (c) $\lim_{n\to+\infty} a_n = e$

   (d) $\lim_{n\to+\infty} a_n = +\infty$

   (e) $\lim_{n\to+\infty} a_n = 0$

4. Study the asymptotic behaviour of the following sequences:

   (a) $a_n = \left(\frac{1}{3}\right)^{1/n}$

   (b) $b_n = \cos\left(n\frac{\pi}{2}\right) \quad ; \quad c_n = \dfrac{\sin\left(n\frac{\pi}{2}\right)}{n^2 - n}$

   (c) $x_n = \dfrac{3n^4 - 7n^2 - 3}{1 - 3n + n^3}; \quad y_n = \dfrac{2n^5 - 8n^3}{1 - 3n^5}; \quad z_n = \dfrac{n^2 + 2\sqrt{2}n - 3}{n^4 - \pi n + e}$

   (d) $a_n = \left(1 + \frac{1}{3n}\right)^n; \quad b_n = \left(1 + \frac{2}{n}\right)^{n/4}; \quad c_n = \left(1 + \frac{1}{n}\right)^{-n^2}; \quad d_n = \left(1 + \frac{1}{3n}\right)^{3n+5}$

[1] Regular = The limit exists, either finite or infinite.

5. Discuss and compute the following limits (where [ ] denotes the integer part)

$a)\ \lim_{x\to-\infty} \frac{x-4}{\sqrt{x^2+4}}$ $\quad b)\ \lim_{x\to+\infty} \frac{x-4}{\sqrt{x^2+4}}$ $\quad c)\ \lim_{x\to-\infty} \frac{3x+\sin\pi x}{-x-e}$

$d)\ \lim_{x\to+\infty} (3x+2\cos\pi x)$ $\quad e)\ \lim_{x\to+\infty} \frac{M(x^2-\pi x+3)}{x}$ $\quad f)\ \lim_{x\to-\infty} \frac{\sin x}{x}$

$g)\ \lim_{x\to 0} \sinh\frac{1}{x}$ $\quad h)\ \lim_{x\to 0} \sin\frac{1}{x}$ $\quad i)\ \lim_{x\to\pi/2} \frac{x}{1-\sin x}$

$l)\ \lim_{x\to\pi/3} x[3+\cos x]$ $\quad m)\ \lim_{x\to\pi/2^+} x[3+\cos x]$ $\quad n)\ \lim_{x\to\pi/2^-} x[3+\cos x]$

$o)\ \lim_{x\to\pi/2^+} x\ \mathrm{sign}(\cos x)$ $\quad p)\ \lim_{x\to\pi/2^-} x\ \mathrm{sign}(\cos x)$ $\quad q)\ \lim_{x\to\pi} [x\ \mathrm{sign}(\sin^4 x)]$

6. Compute the following limits:

$a)\ \lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2-2}}$ $\quad b)\ \lim_{x\to-\infty} \frac{x^7+8x^5+3x}{-4x^7+x}$

$c)\ \lim_{x\to+\infty} (\sqrt{x+1}-\sqrt{x})$ $\quad d)\ \lim_{x\to-\infty} \sqrt{x^2+x}+x$

$e)\ \lim_{x\to 4} \frac{\sqrt{x}-2}{x^2-5x+4}$ $\quad f)\ \lim_{x\to 2} \frac{x^2-5x+6}{x^2-4x+4}$

$g)\ \lim_{x\to 1} \frac{x^3-1}{x^4-1}$ $\quad h)\ \lim_{x\to a} \frac{x^2-a^2}{(x-a)^3}$

$i)\ \lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 x}$ $\quad l)\ \lim_{x\to 0} \frac{x+\sin 4x}{x+\sin x}$

$m)\ \lim_{x\to\pi} \frac{\sin x}{x-\pi}$ $\quad n)\ \lim_{x\to 0} \frac{2^{2x}-2^{-x}}{2^x-1}$

$o)\ \lim_{x\to 0} \frac{\log(1+xe^x)}{e^{-3x}-1}$ $\quad p)\ \lim_{x\to 0} \frac{1-\log(e+x)}{x}$

7. Compute the following limits:

a) $\lim_{x\to-\infty} (x^3+M(x))e^{5x}$ $\quad$ b) $\lim_{x\to+\infty} \ln(x^3-1)4^{-3x}$

c) $\lim_{x\to 0^+} (x)^{\sin x}$ $\quad$ d) $\lim_{x\to 0} \sqrt{x\frac{2-x}{x-3}(e^x-1)}$

8. Find the values for α, if they exist, such that the following functions are continuous on their domain:

$$f_1(x) = \begin{cases} \alpha x & \text{if } x \le 1 \\ x-\alpha & \text{if } x > 1 \end{cases} \qquad f_2(x) = \begin{cases} \alpha x & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ x-\alpha & \text{if } x > 1 \end{cases} \qquad f_3(x) = \begin{cases} \sin x + \alpha & \text{if } x < 0 \\ \cos(\alpha x) & \text{if } x \ge 0 \end{cases}$$

9. Discuss continuity of the following functions, and in case of discontinuity, find if possible their continuous prolongation.

$$f_1(x) = \begin{cases} \left|\arctan\frac{1}{x}\right| & \text{if } x \ne 0 \\ \pi/2 & \text{if } x = 0 \end{cases} \qquad f_2(x) = \begin{cases} \arctan\left|\frac{1}{x}\right| & \text{if } x \ne 0 \\ -\pi/2 & \text{if } x = 0 \end{cases} \qquad f_3(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$$

10. Discuss continuity and classify singularities in $x_0 = 0$ for the following functions; in case of singularity, find the continuous prolongation where possible:

$$f_1(x) = 2x^2 + \sin x; \quad f_2(x) = \frac{1-\cos x}{x^2}; \quad f_3(x) = \frac{1}{x - x^2}; \quad f_4(x) = M(x)$$

11. Find domain, limits at boundary points of the domain, vertical/horizontal/oblique asymptotes (where they exist) for the following functions:

$$f_1(x) = \sqrt{x^2-1}; \quad f_2(x) = \frac{x^4-2x+1}{x^3-x}; \quad f_3(x) = \frac{x^2-(x-1)\,|x-2|}{2x+3}$$

$$f_4(x) = 2 - e^{-|x|} - x; \quad f_5(x) = xe^{\frac{1}{|x^2-1|}}; \quad f_6(x) = \arctan(x^2+2x) + 3x$$

$$f_7(x) = \log\left(1+e^{2x}\right); \quad f_8(x) = \log(e^x + 2x)$$

FUNCTIONS from WRITTEN EXAMS

1. Find domain, limits at boundary points of the domain and asymptotes for the following functions:

   (a) (30 January 2015 - II)
$$f(x) = \frac{1}{2}x^2 + x + \log|x+3|$$

   (b) (30 January 2015 - III)
$$f(x) = \log|x+1| - 2\arctan x$$

   (c) (13 February 2015 - III)
$$f(x) = \left|x+2-2\sqrt{x+3}\right|$$

   (d) (17 June 2015 - II)
$$f(x) = \arctan\left(\frac{1}{x-1}\right) + \frac{x}{2} + 3$$

   (e) (28 January 2016 - I)
$$f(x) = \arcsin|1-2^x| + 1$$

   (f) (13 February 2015 - III)
$$f(x) = \left|x+2-2\sqrt{x+3}\right|$$

   (g) (30 January 2015 - II)
$$f(x) = \frac{1}{2}x^2 + x + \log|x+3|$$

   (h) (30 January 2015 - III)
$$f(x) = \log|x+1| - 2\arctan x$$

   (i) (10 February 2016 - II)
$$f(x) = (\sinh 2x)^2 - 2\sinh 2x - 3 \quad \text{defined for } \geq 0.$$

2. Find domain and limits at boundary points of the domain for the following functions

   (a) (30 January 2015 - I)
$$f(x) = xe^{\frac{|x-1|}{x-2}}$$

   (b) (13 February 2015 - I)
$$f(x) = e^{-|(x-2)(x+3)|}$$

   (c) (13 February 2015 - II)
$$f(x) = |\log(x-2) - \log^2(x-2)|$$

   (d) (17 June 2015 - I)
$$f(x) = xe^{\frac{1}{\log 2x}}$$

(e) (9 September 2015 - II)

$$f(x) = (x+1)^2 e^{\frac{x-2}{x+2}} - 2$$

(f) (14 February 2017 - II)

$$f(x) = e^{2x}|x-1|^{1/3}$$

3. (9 September 2015 - I)
   Given the function
   $$f(x) = e^{2(x-3)^3 \log|x-3|}$$
   Find Dom$f$ and the asymptotes of f. Show that f admits continuous prolongation at $x = 3$.

4. (28 January 2016 - III)
   Given the function
   $$f(x) = \sqrt{1-|x|} - \arcsin\sqrt{1-|x|} + 2$$
   Find Dom$f$, and simmetries for f. Study continuity of f on its domain.

5. (28 January 2016 - II)
   Consider the function
   $$f(x) = \begin{cases} \dfrac{(x+2)^2}{\log(x+2)} - 3 & \text{if } x \in (-2,-1) \cup (-1,+\infty) \\ \\ -3 & \text{if } x \leq -2 \end{cases}$$
   Compute limits at the boundary points of the domain. Study continuity of f on its domain.

   $\vdots$

   Given the functions
   $$f_k(x) = \begin{cases} \dfrac{(x+2)^2}{\log(x+2)} - 3 & \text{if } x \in (-2,-1) \\ \\ -3 + (x+2)^k & \text{if } x \leq -2 \end{cases}$$
   Find the values for $k \in \mathbb{N}$ such that f_k is continuous on $(-\infty,-1)$

6. (10 February 2016 - I)
   Given the function
   $$f(x) = \frac{\log|x|}{\log^2|x| - \log|x| + 1}.$$
   Find the domain, the asymptotes and simmetries if they exist. Show that f admits continuous prolongation at $x = 0$.

7. (10 February 2016 - III)
   Given the function
   $$f(x) = 2\log|2^{2x} - 3e^x|$$
   Find Dom$f$ and limits at the boundary points of the domain.
   Find the asymptote equation for f as $x \to +\infty$.

8. (23 June 2016 - I)
   Given the function
   $$f(x) = \log\arctan\frac{|x-1|}{|x-4|}.$$
   (a) Find the domain of f and show that f admits continuous prolongation at $x = 4$.
   (b) Let g be the prolongation of f, find the limits at the boundary points of $Domf$ and find asymptotes, if there are any.

9. (23 June 2016 - II)

Consider

$$f(x) = \sqrt[3]{(2|x| - 1)(x - 2)^2}$$

(a) Find domain, zeros and limits at boundary points of the domain.

(b) Say if f admits oblique asymptotes, and compute them.

ESERCISES from WRITTEN EXAMS

1. (30 January 2015 - I)
   (a) Write the definition of limit: $\lim_{x\to+\infty} h(x) = L \in \mathbb{R}$. Consider the functions $f, g : (0, +\infty) \to \mathbb{R}$ both strictly positive and such that

   $$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = L \quad \text{is finite.}$$

   (b) Show that $L > 1$ implies $f(x) > g(x)$ in a neighborhood of $+\infty$.

2. (28 January 2016 - I)
   (a) Write the definition of strictly increasing function on $A \subseteq \mathbb{R}$
   (b) Consider the functions f and g defined on $\mathbb{R}$. If f is continuous on $[a, b]$ and g is increasing on $\mathbb{R}$, prove that $g \circ f$ admits global maximum and global minimum on $[a, b]$.
   (c) Say if the following statement is true or false. If true, show it; if false, find a counterexample. If f is continuous on $[a, b]$ and g is strictly increasing on $\mathbb{R}$, the set of local (or relative) maxima and minima of f on $[a, b]$ coincides with the set of local (or relative) maxima and minima of g on $[a, b]$.

3. (28 January 2016 - III)
   (a) The following statement is false. Prove it with a counterexample.
   If the function $f(x)$ is defined, strictly positive and strictly increasing on $(0, 1)$, then $\lim_{x\to1^-} f(x) > 0$.
   (b) Show that the following statement is true. if $f(x)$ is defined and strictly increasing on $(0, 1]$, then $\lim_{x\to1^-} f(x) \leq f(1)$
   (c) Discuss if the statement in (b) is true when we take $<$ instead of $\leq$. Motivate the answer.

4. (23 June 2016 - I - A)
   (a) Let $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = \ell \in \mathbb{R}$ and $a_n \leq b_n \leq c_n$, for every $n \geq 10^5$. Explain which Theorem can be applied to compute $\lim_{n\to\infty} b_n$
   (b) State the previous Theorem.
   (c) Let $a_n = n \log n$. Compute, if possible, the following limits. Motivate the answers:

   $$\lim_{n\to\infty} \frac{1}{a_n} (2e + \cos n\pi)$$

   $$\lim_{n\to\infty} a_n (-1 + \cos n\pi)$$

5. (23 June 2016 - I - B)
   (a) Given $\lim_{n\to\infty} b_n = +\infty$ and $a_n \geq b_n$, for every $n \geq 10^3$. Explain which Theorem can be applied to compute $\lim_{n\to\infty} a_n$
   (b) State the previous Theorem.
   (c) Let $a_n = n \arctan n$. Compute, if possible, the following limits. Motivate the answers:

   $$\lim_{n\to\infty} a_n \left(e + \sin \frac{\pi}{2} n\right)$$

   $$\lim_{n\to\infty} a_n \left(1 + \sin \frac{\pi}{2} n\right)$$

6. (31 January 2018 - Ith - A)
   (a) State one of the Comparison Theorems for functions with finite limit.

(b) Show that, if $f(x)$ is such that $\lim_{x\to+\infty} f(x) = 0$ and $M(x)$ is the mantissa function, then $\lim_{x\to+\infty} f(x)M(x) = 0$.

(c) Compute, if they exist, the following limits

$$\lim_{x\to-\infty} \frac{4^{-x}+1}{M(4^{-x})+1}, \qquad \lim_{x\to+\infty} \frac{4^{-x}+1}{M(4^{-x})+1}$$

where $M(x)$ denotes the mantissa function.

7. (31 January 2018 - Ith - B)

(a) State one of the Comparison Theorems for functions with limit $+\infty$.

(b) Show that, if $f(x)$is such that $\lim_{x\to+\infty} f(x) = +\infty$ and $M(x)$ is the mantissa function, then $\lim_{x\to+\infty} (f(x) + M(x)) = +\infty$.

(c) Compute, if they exist, the following limits

$$\lim_{x\to-\infty} \frac{2^{x}+1}{M(2^{x})+1}, \qquad \lim_{x\to+\infty} \frac{2^{x}+1}{M(2^{x})+1}$$

where $M(x)$ denotes the mantissa function.

8. (13 February 2018 - Ith - A)

(a) Write the definition of increasing function on a subset $A \subseteq \mathbb{R}$.

(b) Given f increasing on A, and g decreasing on $\mathbb{R}$, study monotonicity of the composite function $g \circ f$ on A.

(c) Suppose $A = (1, 5)$ and f increasing on A. Show that $\lim_{x\to1^+} f(x)$ cannot be equal to $+\infty$.

9. (13 February 2018 - Ith - B)

(a) Write the definition of decreasing function on a subset $A \subseteq \mathbb{R}$.

(b) Given f decreasing on A, and g increasing on $\mathbb{R}$, study monotonicity of the composite function $g \circ f$ on A.

(c) Suppose $A = (2, 5)$ and f decreasing on A. Show that $\lim_{x\to5^-} f(x)$ cannot be equal to $+\infty$.

QUESTIONS - THEORY EXERCISES

(a) Do the sequences $a_n = \dfrac{1+n(-1)^n}{1+n}$ and $b_n = \dfrac{|1+n(-1)^n|}{1+n}$ admit limit? Justify the answer.

(b) Is the sequence (a_n) with $a_n = n + \dfrac{100}{n}$ monotone? Can we apply the theorem on monotone sequences?

(c) Let (a_n) be a decreasing sequence such that $a_n \in \mathbb{N}$ for every $n \in \mathbb{N}$. Show that is admits finite limit.

(d) Let $b_k = \frac{1}{k} - \frac{1}{k+1}$, $k \in \mathbb{N}$, and $a_n = b_1 + b_2 + ... + b_n$. Say if (a_n) admits limit, and compute such limit.

(e) Let (a_n) be a sequence such that if n is even $a_n > 0$ and if n is odd $a_n < 0$. Show that: if there exists $\lim a_n = l$ then $l = 0$.

(f) Discuss the statement: if (a_n) and (b_n) are two increasing sequences, then $(a_n b_n)$ is increasing. Is it true or false? Which are the hypothesis?

(g) Do Local boundedness Theorem and Sign and limit Theorem hold for one-sided limits? Try to state them and prove them.