MATHEMATICAL ANALYSIS I TUTORING
5^{TH} Week

DERIVATIVES - DIFFERENTIATION RULES - NON-DIFFERENTIABLE POINTS INVERSE FUNCTIONS AND DIFFERENTIATION

EXERCISES - SOLUTIONS

1. Using the definition, calculate the derivative of the function f in the given points:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

(a) $f_1(x) = (x-2)^2, \quad x = -1$ e $x = 2$

$$\begin{aligned}
f_1'(-1) &= \lim_{x \to -1} \frac{f_1(x) - f_1(-1)}{x - (-1)} = \lim_{x \to -1} \frac{(x-2)^2 - 9}{x+1} \\
&= \lim_{x \to -1} \frac{x^2 - 4x + 4 - 9}{x+1} = \lim_{x \to -1} \frac{x^2 - 4x - 5}{x+1} = \lim_{x \to -1} \frac{(x+1)(x-5)}{x+1} = \lim_{x \to -1} (x-5) = -6 \\
f_1'(2) &= \lim_{x \to 2} \frac{f_1(x) - f_1(2)}{x-2} = \lim_{x \to 2} \frac{(x-2)^2 - 0}{x-2} = \lim_{x \to 2} (x-2) = 0
\end{aligned}$$

(b) $f_2(x) = \sqrt{x^2 - 1}, \quad x = 1$ and $x = 2$

The domain is $D = (-\infty, 1] \cup [1, +\infty)$; thus in $x = 1$ we can only find right derivative, if it exists:

$$f_2'(1) = \lim_{x \to 1^+} \frac{f_2(x) - f_2(1)}{x-1} = \lim_{x \to 1} \frac{\sqrt{x^2-1} - 0}{x-1} = \lim_{x \to 1^+} \frac{\sqrt{x-1}\sqrt{x+1}}{\sqrt{x-1}\sqrt{x-1}} = \lim_{x \to 1^+} \frac{\sqrt{x+1}}{\sqrt{x-1}} = +\infty$$

Since the limit is not finite, the function is not differentiable in $x = 1$.

$$\begin{aligned}
f_2'(2) &= \lim_{x \to 2} \frac{f_2(x) - f_2(2)}{x-2} = \lim_{x \to 2} \frac{\sqrt{x^2-1} - \sqrt{3}}{x-2} \\
&= \lim_{x \to 2} \frac{(\sqrt{x^2-1} - \sqrt{3})(\sqrt{x^2-1} + \sqrt{3})}{(x-2)(\sqrt{x^2-1} + \sqrt{3})} = \lim_{x \to 2} \frac{(x^2-1) - 3}{(x-2)(\sqrt{x^2-1} + \sqrt{3})} \\
&= \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2-1} + \sqrt{3})} = \lim_{x \to 2} \frac{x+2}{\sqrt{x^2-1} + \sqrt{3}} = \frac{2}{\sqrt{3}}
\end{aligned}$$

(c) $f_3(x) = |x^2 - 1|, \quad x = 1$ e $x = 3$

$$f_3(x) = |x^2 - 1| = \begin{cases} x^2 - 1 & \text{se } x \leq -1 \vee x \geq 1 \\ -x^2 + 1 & \text{se } -1 < x < 1 \end{cases}$$

$$f_3'(1) = \lim_{x \to 1} \frac{f_3(x) - f_3(1)}{x-1} = \lim_{x \to 1} \frac{|x^2-1| - 0}{x-1}$$

$$\begin{aligned}
\lim_{x \to 1^+} \frac{(x-1)(x+1)}{x-1} &= \lim_{x \to 1^+} (x+1) = 2 \\
\lim_{x \to 1^-} \frac{-(x-1)(x+1)}{x-1} &= \lim_{x \to 1^-} -(x+1) = -2
\end{aligned}$$

Therefore the function is not differentiable in $x = 1$: it is a corner point.

$$f_3'(3) = \lim_{x \to 3} \frac{f_3(x) - f_3(3)}{x-3} = \lim_{x \to 3} \frac{x^2 - 1 - 8}{x-3} = \lim_{x \to 3} \frac{x^2 - 9}{x-3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \to 3} (x+3) = 6$$

Hence the function is differentiable in $x = 3$ and $f'(3) = 6$.

2. Applying the derivation rules, compute the first derivative of the following functions:

(a) $\boxed{f(x) = 2x^3 - 9x + 7\cos x}$

Recall that $\Big(af(x) + bg(x)\Big)' = af'(x) + bg'(x)$

$$f'(x) = 6x^2 - 9 - 7\sin x$$

(b) $\boxed{f(x) = (2x^3 - x)e^x}$

recall that $\Big(f(x) \cdot g(x)\Big)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

$$f'(x) = (6x^2 - 1)e^x + (2x^3 - x)e^x = (6x^2 - 1 + 2x^3 - x)e^x = (2x^3 + 6x^2 - x - 1)e^x$$

(c) $\boxed{f(x) = \frac{x^3 - 5x^2}{x^2 - x}} = \frac{x(x^2 - 5x)}{x(x-1)} = \frac{x^2 - 5x}{x - 1}$

Recall that $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{f^2(x)}$

$$f'(x) = \frac{(2x-5)(x-1) - (x^2 - 5x)}{(x-1)^2} = \frac{2x^2 - 5x - 2x + 5 - x^2 + 5x}{(x-1)^2} = \frac{x^2 - 2x + 5}{(x-1)^2}$$

(d) $\boxed{f(x) = \frac{x \log x}{1 + \log x}}$

$$f'(x) = \frac{(\log x + x\frac{1}{x})(1 + \log x) - (x\log x)\frac{1}{x}}{(1 + \log x)^2} = \frac{(\log x + 1)(1 + \log x) - \log x}{(1 + \log x)^2} = \frac{\log^2 x + \log x + 1}{(1 + \log x)^2}$$

(e) $\boxed{f(x) = e^{\tan(x^3)}}$

Since $\Big(f(g(x))\Big)' = f'(g(x)) \cdot g'(x)$

$$f'(x) = e^{\tan(x^3)} \frac{1}{\cos(x^3)} 3x^2$$

(f) $\boxed{f(x) = x \arctan x}$

$$f'(x) = \arctan x + x\frac{1}{1 + x^2}$$

(g) $\boxed{f(x) = \log(\log x)}$

$$f'(x) = \frac{1}{\log x} \cdot \frac{1}{x}$$

(h) $\boxed{f(x) = x\log(\sin x)}$

$$f'(x) = 1 \cdot \log(\sin x) + x\frac{1}{\sin x}\cos x = \log(\sin x) + x\frac{\cos x}{\sin x}$$

(i) $\boxed{f(x) = \log(1 + \arctan^2 x)}$

$$f'(x) = \frac{1}{1 + \arctan^2 x} 2\arctan(x) \cdot \frac{1}{1 + x^2}$$

(l) $\boxed{f(x) = 2\log|1 - x| + 3\log^2|1 - x|}$

Note that $D(\log|x|) = \frac{1}{x}$ and $D(\log|f(x)|) = D(\log f(x)) = \frac{f'(x)}{f(x)}$

$$f'(x) = 2\frac{-1}{1-x} + 6\log|1 - x| \cdot \frac{-1}{1-x} = \frac{2}{x-1} + \frac{6\log|1-x|}{x-1} = \frac{2 + 6\log|1-x|}{x-1}$$

(m) $\boxed{f(x) = [1 + \log(x - \sin x)]\, e^{2\sin x}}$

$$f'(x) = \frac{1}{(x - \sin x)}(1 - \cos x)e^{2\sin x} + [1 + \log(x - \sin x)]\, e^{2\sin x} 2\cos x$$

(n) $\boxed{f(x) = \sqrt[7]{(2x - \log x)^3}} = (2x - \log x)^{3/7}$

$$f'(x) = \frac{3}{7}(2x - \log x)^{-4/7}\left(2 - \frac{1}{x}\right) = \frac{3}{7}\frac{1}{\sqrt[7]{(2x - \log x)^4}}\left(2 - \frac{1}{x}\right)$$

3. Write the equation of the tangent line to the graph of the following functions, at the given point x_0. Remind that, if $f(x)$ is differentiable in x_0, the tangent line to the graph of f in x_0 has equation:

$$y = f'(x_0)(x - x_0) + f(x_0)$$

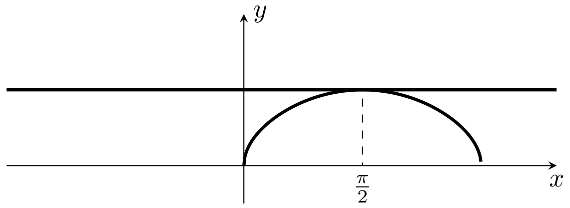
a) $\boxed{f_1(x) = \sqrt{\sin x} \qquad \text{in} \;\; x_0 = \frac{\pi}{2}}$

Compute the function at $x_0 = \frac{\pi}{2}$: $f_1\left(\frac{\pi}{2}\right) = \sqrt{\sin\frac{\pi}{2}} = 1$.

Compute the first derivative and evaluate it at such point:

$$f_1'(x) = \frac{1}{2\sqrt{\sin x}}\cos x \Rightarrow f_1'\left(\frac{\pi}{2}\right) = \frac{1}{2\sqrt{\sin\frac{\pi}{2}}}\cos\frac{\pi}{2} = 0$$

The equation of the tangent line at $x_0 = \frac{\pi}{2}$ is $y = 0(x - \frac{\pi}{2}) + 1$ and thus $y = 1$.



b) $\boxed{f_2(x) = \log|x - 4| \quad \text{in} \;\; x_0 = 1}$

Compute the function at $x_0 = 1$:

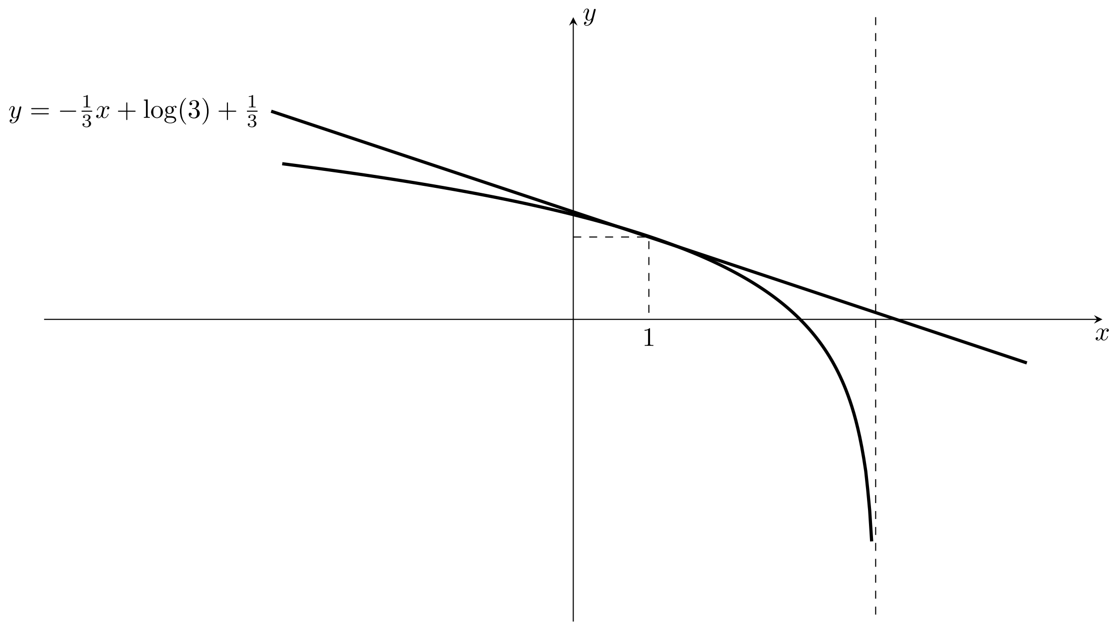
for $x < 4$ it holds $f_2(x) = \log(4 - x) \Rightarrow f_2(1) = \log(4 - 1) = \log(3)$.

Compute the first derivative:

$$f_2'(x) = \frac{1}{4 - x}(-1) \Rightarrow f_2'(1) = \frac{-1}{4 - 1} = -\frac{1}{3}$$

The equation of the tangent line at $x_0 = 1$ is

$$y = -\frac{1}{3}(x - 1) + \log(3) \quad \Rightarrow \quad y = -\frac{1}{3}x + \log(3) + \frac{1}{3}$$

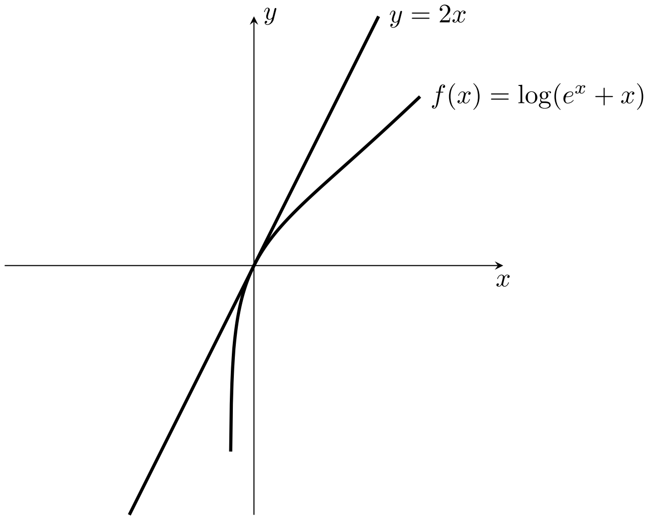


c) $\boxed{f(x) = \log(e^x + x) \quad \text{ in } x_0 = 0}$
Compute the function at $x_0 = 0$: $f(0) = \log(e^0 + 0) = \log(1) = 0$
Compute the first derivative:

$$f'(x) = \frac{e^x + 1}{e^x + x} \Rightarrow f'(0) = \frac{e^0 + 1}{e^0 + 0} = 2$$

The tangent line at $x_0 = 1$ is

$$y = 2(x - 0) + 0 \quad \Rightarrow \quad y = 2x$$



4. Find the non differentiable points of the following functions and trace a qualitative graph in a neighborhood of such points.

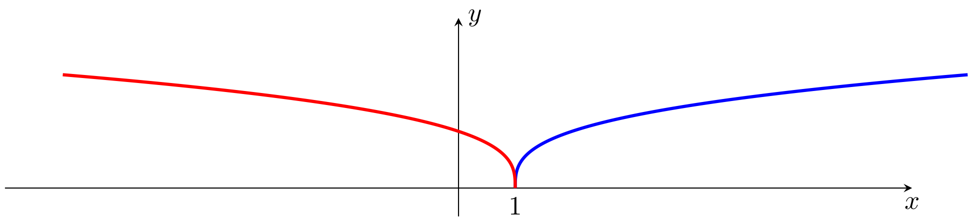
a) $\boxed{f(x) = \sqrt[3]{|x-1|}}$ The function is continuous in $x_0 = 1$, but not differentiable in $x_0 = 1$. Indeed

$$f(x) = \sqrt[3]{|x-1|} = \begin{cases} \sqrt[3]{x-1} & \text{se } x \geq 1 \\ \sqrt[3]{1-x} & \text{se } x < 1 \end{cases}$$

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x-1} = \lim_{x\to1^-} \frac{\sqrt[3]{|x-1|} - 0}{x-1} = \lim_{x\to1^-} \frac{\sqrt[3]{1-x}}{x-1} = \lim_{x\to1^-} \frac{(1-x)^{1/3}}{-(1-x)} = \lim_{x\to1^-} -(1-x)^{-2/3} = \lim_{x\to1^-} -\frac{1}{\sqrt[3]{(1-}}$$

$$\lim_{x\to1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x\to1^+} \frac{\sqrt[3]{|x-1|} - 0}{x-1} = \lim_{x\to1^+} \frac{\sqrt[3]{x-1}}{x-1} = \lim_{x\to1^-} \frac{1}{\sqrt[3]{(x-1)^2}} = +\infty$$

Thus $f(x)$ is not differentiable in $x_0 = 1$, and $x_0 = 1$ is a cusp.

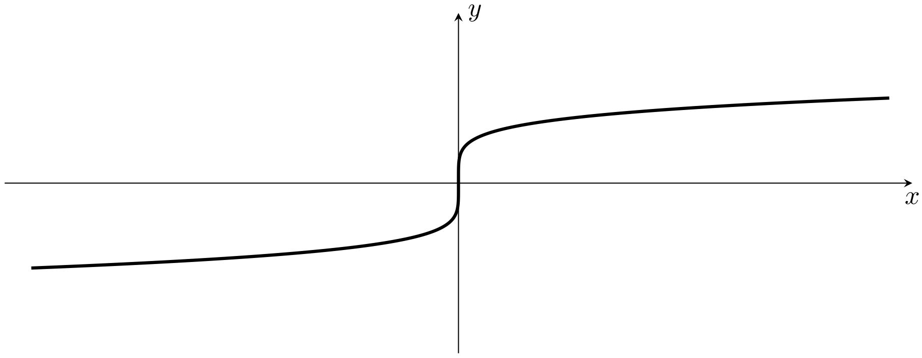


b) $\boxed{f(x) = \sqrt[5]{x}}$ The function is continuous in $x_0 = 0$, but not differentiable in $x_0 = 0$. Indeed:

$$\lim_{x\to0^-} \frac{f(x) - f(0)}{x-0} = \lim_{x\to0^-} \frac{\sqrt[5]{x} - 0}{x} = +\infty$$

$$\lim_{x\to0^+} \frac{f(x) - f(0)}{x-0} = \lim_{x\to0^+} \frac{\sqrt[5]{x} - 0}{x} = +\infty$$

Thus $x_0 = 0$ is an in ection point with vertical tangent.

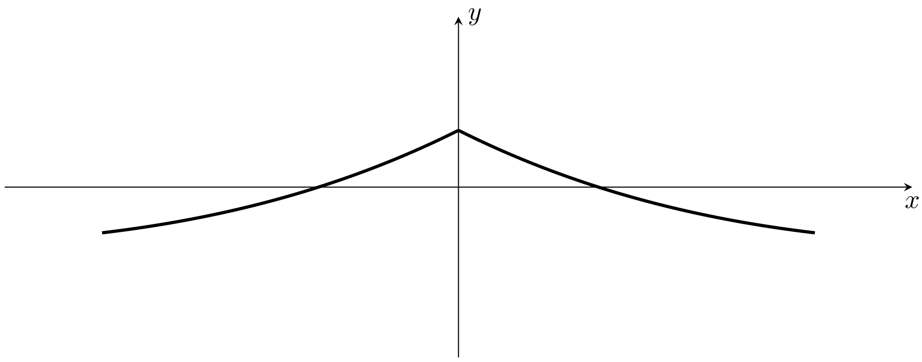


c) $\boxed{f(x) = \cos\sqrt{|x|}}$ The function is continuous in $x_0 = 0$, but not differentiable in $x_0 = 0$. Indeed:

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\cos\sqrt{-x}-1}{x} = \lim_{x\to 0^-} \frac{1-\frac{1}{2}(-x)+o(x)-1}{x} = \lim_{x\to 0^-} \frac{\frac{1}{2}x+o(x)}{x} = \frac{1}{2}$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\cos\sqrt{x}-1}{x} = \lim_{x\to 0^+} \frac{1-\frac{1}{2}(x)+o(x)-1}{x} = \lim_{x\to 0^+} \frac{-\frac{1}{2}x+o(x)}{x} = -\frac{1}{2}$$

Therefore $x_0 = 0$ is a corner point.

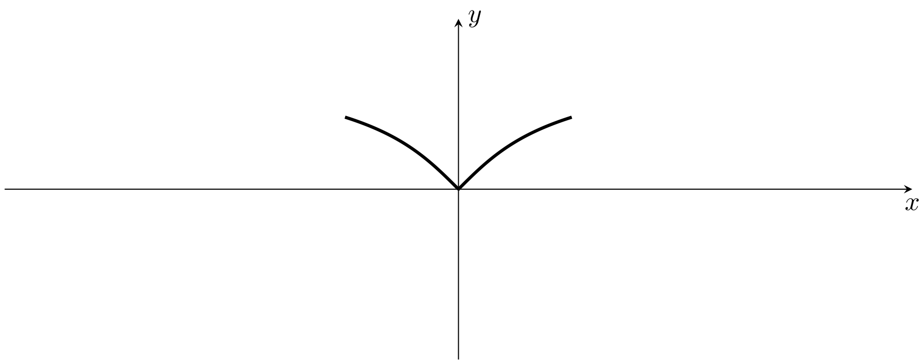


d) $\boxed{f_1(x) = \sqrt{\log(x^2+1)}}$ The function is continuous in $x_0 = 0$, but not differentiable in $x_0 = 0$. Indeed:

$$\lim_{x\to 0^-} \frac{f_1(x)-f_1(0)}{x-0} = \lim_{x\to 0^-} \frac{\sqrt{\log(x^2+1)}-0}{x-0} = \lim_{x\to 0^-} \frac{\sqrt{\log(x^2+1)}}{x} = \lim_{x\to 0^-} \frac{\sqrt{x^2}}{x} = \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x\to 0^+} \frac{f_1(x)-f_1(0)}{x-0} = 1$$

Thus $x_0 = 0$ is a corner point.

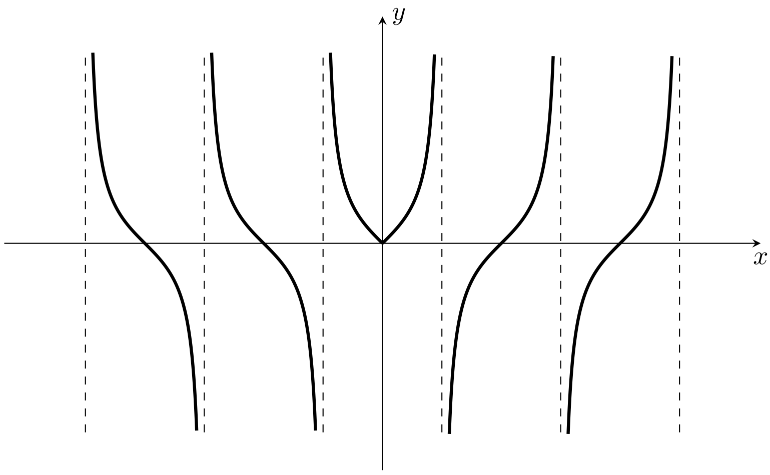


e) $\boxed{f_2(x) = \tan|x|}$ The function is continuous in $x_0 = 0$, but not differentiable in $x_0 = 0$.

Indeed, since $f'(x) = \dfrac{1}{\cos^2|x|}\dfrac{|x|}{x}$, it holds:

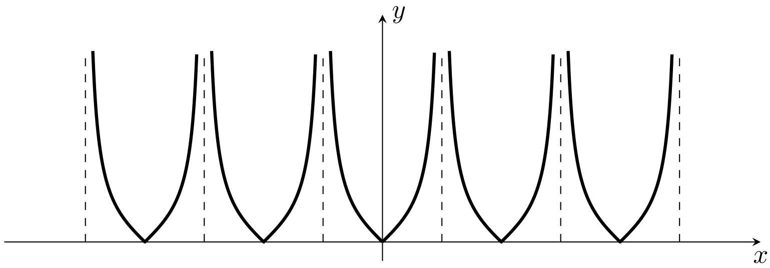
$$\lim_{x\to 0^-} f'(x) = -1 \quad \text{whereas} \quad \lim_{x\to 0^+} f'(x) = 1$$

Hence $x = 0$ is a corner point.



f) $\boxed{f_3(x) = |\tan x|}$ The function is continuous on its domain but not differentiable in the points $k\pi, k \in \mathbb{Z}$. Indeed, from $f'(x) = \dfrac{1}{\cos^2 x}\dfrac{|\tan x|}{\tan x}$, we have:

$$\lim_{x\to k\pi^-} f'(x) = -1 \quad \text{whereas} \quad \lim_{x\to k\pi^+} f'(x) = 1$$



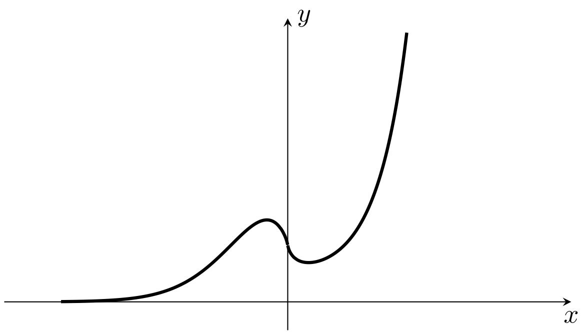
5. Prove that the following functions admit continuous prolongation at $x = 0$; say if the continuous extension is also differentiable, computing the limit of the difference quotient.

a) $\boxed{f(x) = |x|^x}$ $\quad D = (-\infty, 0) \cup (0, +\infty)$

Consider the equality $f(x) = |x|^x = e^{x \log|x|}$.
Since $\lim\limits_{x\to 0} e^{x\log|x|} = 1$, we can argue that it admits continuous prolongation in $x = 0$. The continuous prolongation is:

$$\tilde{f}(x) = \begin{cases} |x|^x & \text{if } x \neq 0 \\ \\ 1 & \text{if } x = 0 \end{cases}$$

Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:
$\lim_{x\to 0} \frac{\tilde{f}(x)-\tilde{f}(0)}{x-0} = \lim_{x\to 0} \frac{|x|^x-1}{x} = \lim_{x\to 0} \frac{e^{x\ln|x|}-1}{x} = \lim_{x\to 0} \frac{x\ln|x|}{x} = \lim_{x\to 0} \ln|x| = -\infty$
Hence $\tilde{f}$ is not differentiable in $x_0 = 0$, that is an inflection point with vertical tangent.



b) $\boxed{f(x) = \dfrac{\sin x}{x}}$ $\quad D = (-\infty, 0) \cup (0, +\infty)$

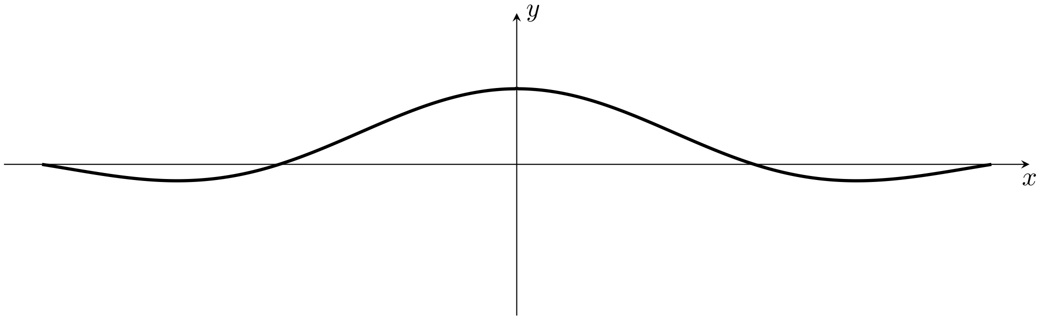
Recall that $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$, we can say that the function admits continuous prolongation in $x = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:

$$\lim_{x\to 0} \frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\sin x}{x} - 1}{x - 0} = \lim_{x\to 0} \frac{\sin x - x}{x^2} = \lim_{x\to 0} \frac{\cos x - 1}{2x} = \lim_{x\to 0} \frac{-\frac{1}{2}x^2}{2x} = 0$$

Hence $\tilde{f}$ is differentiable on $\mathbb{R}$.



c) $\boxed{f(x) = \frac{\arctan x}{x}}$ $D = (-\infty, 0) \cup (0, +\infty)$ Recall that $\lim_{x\to 0} \frac{\arctan x}{x} = 1$ we can say that the function admits continuous prolongation in $x = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{\arctan x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

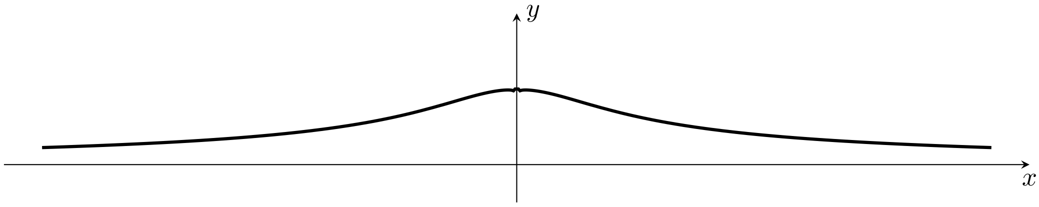
Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:

$$\tilde{f}'(0) = \lim_{x\to 0} \frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\arctan x}{x} - 1}{x - 0} = \lim_{x\to 0} \frac{\arctan x - x}{x^2}$$

Apply De L'Hospital Theorem:

$$\lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{2x} = \lim_{x\to 0} \frac{1 - 1 - x^2}{2x(1+x^2)} = \lim_{x\to 0} \frac{-x^2}{2x(1+x^2)} = \lim_{x\to 0} \frac{-x}{2(1+x^2)} = 0$$

Thus $\tilde{f}$ is differentiable on $\mathbb{R}$, and $\tilde{f}'(0) = 0$.



d) $\boxed{f(x) = e^{-1/x^2}}$ $D = (-\infty, 0) \cup (0, +\infty)$

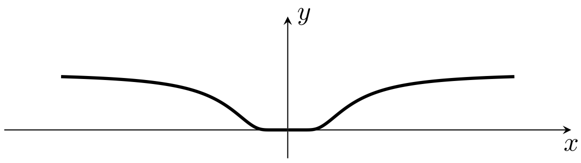
Recall that $\lim_{x\to 0} e^{-1/x^2} = 0$, we can say that the function admits continuous prolongation in $x = 0$:

$$\tilde{f}(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:

$$\lim_{x\to 0} \frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2} - 0}{x - 0} = \lim_{t\to\pm\infty} e^{-t^2} \cdot t = \lim_{t\to\pm\infty} \frac{t}{e^{t^2}} = 0$$

(apply the substitution in the third step $\frac{1}{x} = t$). Hence $\tilde{f}$ is differentiable on $\mathbb{R}$ and $\tilde{f}'(0) = 0$



e) $\boxed{f(x) = x^2 \log|x|}$ $D = (-\infty, 0) \cup (0, +\infty)$

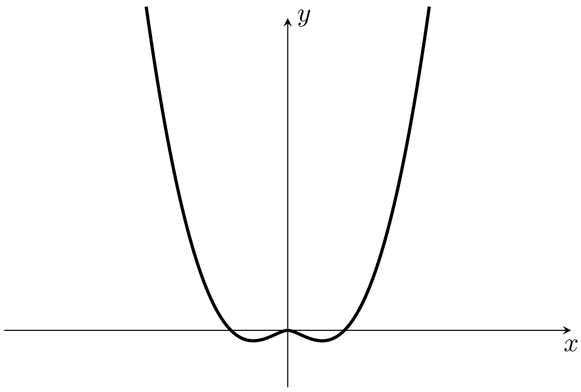
Note that $\lim_{x\to 0} x^2 \log|x| = 0$, we can say that the function admits continuous prolongation in $x = 0$:

$$\tilde{f}(x) = \begin{cases} x^2 \log|x| & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:

$$\lim_{x\to 0} \frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \log|x| - 0}{x - 0} = \lim_{x\to 0} x \log|x| = 0$$

Therefore $\tilde{f}$ is differentiable on $\mathbb{R}$ and $\tilde{f}'(0) = 0$.



f) $\boxed{f(x) = \frac{1 - \cos x}{x^2}}$ $D = (-\infty, 0) \cup (0, +\infty)$

Recall that $\lim_{x\to 0^+} \frac{1 - \cos x}{x^2} = \frac{1}{2}$, we can say that the function admits continuous prolongation in $x = 0$:

$$\tilde{f}(x) = \begin{cases} \frac{1 - \cos x}{x^2} & \text{if } x \neq 0 \\ \frac{1}{2} & \text{if } x = 0 \end{cases}$$

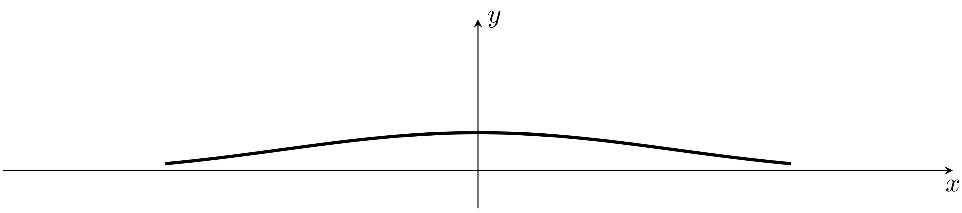
Such function is differentiable for $x \neq 0$ by composition of differentiable functions; we now study the origin:

$$lim_{x\to 0} \frac{\tilde{f}(x) - \tilde{f}(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{1-\cos x}{x^2} - \frac{1}{2}}{x - 0} = \lim_{x\to 0} \frac{2 - 2\cos x - x^2}{2x^3}$$

Apply De L'Hospital Theorem twice:

$$\lim_{x\to 0} \frac{2\sin x - 2x}{6x^2} = \lim_{x\to 0} \frac{2\cos x - 2}{12x} = \lim_{x\to 0} \frac{-2(1 - \cos x)}{12x} = \lim_{x\to 0} \frac{-2(\frac{1}{2}x^2)}{12x} = 0$$

Hence $\tilde{f}$ is differentiable on $\mathbb{R}$ and $\tilde{f}'(0) = 0$.



6. Find (if they exist) the values for the parameters such that the functions are differentiable in their domain.

   Recall the Theorem:

Theorem Given f continuous in x_0 and differentiable for every $x \neq x_0$ in a neighborhood of x_0. If there exists finite limit for $x \to x_0$ for $f'(x)$, then f is differentiable at x_0 and it holds

$$f'(x_0) = \lim_{x \to x_0} f'(x)$$

a) $\boxed{f(x) = \begin{cases} (x-\alpha)^2 & \text{se } x \geq 0 \\ \alpha \sin x & \text{se } x < 0 \end{cases}}$

For $x \neq 0$ the function is differentiable by composition of differentiable functions. We study continuity at the origin, imposing

$$\lim_{x \to 0^+} (x-\alpha)^2 = \lim_{x \to 0^-} \alpha \sin x = (-\alpha)^2 \quad \Leftrightarrow \quad (-\alpha)^2 = 0 = (-\alpha)^2$$

Therefore the function is continuous if and only if $\alpha = 0$.
We impose $\alpha = 0$ and now study differentiability of the continuous function

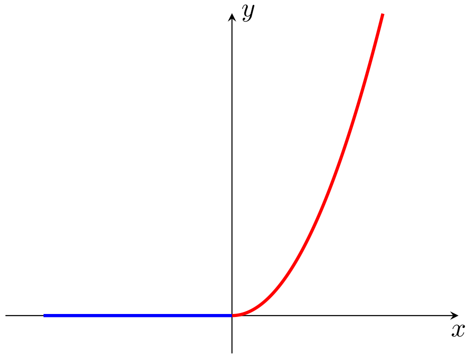
$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Compute the derivative

$$f'(x) = \begin{cases} 2x & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Since $\lim_{x \to 0^+} 2x = 0 = \lim_{x \to 0^-} 0$, we can conclude that f is differentiable at the origin, and $f'(0) = 0$.

$$f(x) = \begin{cases} {\color{red}x^2} & \text{if } x \geq 0 \\ {\color{blue}0} & \text{if } x < 0 \end{cases}$$



b) $\boxed{f(x) = \begin{cases} (x-\beta)^2 + 4 & \text{if } x \geq 0 \\ \alpha \sin x & \text{if } x < 0 \end{cases}}$

For $x \neq 0$ the function is differentiable by composition of differentiable functions. We study continuity at the origin, imposing

$$\lim_{x \to 0^+} (x-\beta)^2 + 4 = \lim_{x \to 0^-} \alpha \sin x = (-\beta)^2 + 4 \quad \Leftrightarrow \quad (-\beta)^2 + 4 = 0 = (-\beta)^2 + 4 \quad \Leftrightarrow \quad \beta^2 + 4 = 0$$

There is no $\beta \in \mathbb{R}$ such that $\beta^2 + 4 = 0$ holds true, thus for no value of α and β the function is continuous at the origin (and thus it's not differentiable at the origin).

c) $\boxed{f(x) = \begin{cases} (x-\beta)^2 - 4 & \text{if } x \geq 0 \\ \alpha \sin x & \text{if } x < 0 \end{cases}}$

For $x \neq 0$ the function is differentiable by composition of differentiable functions. We study continuity at the origin, imposing

$$\lim_{x \to 0^+} (x-\beta)^2 - 4 = \lim_{x \to 0^-} \alpha \sin x = (-\beta)^2 - 4 \quad \Leftrightarrow \quad (-\beta)^2 - 4 = 0 = (-\beta)^2 - 4 \quad \Leftrightarrow \quad \beta^2 - 4 = 0$$

The function is continuous for $\beta = \pm 2$.
We study differentiability at the origin, imposing $\beta = 2$:

$$f(x) = \begin{cases} (x-2)^2 - 4 & \text{if } x \geq 0 \\ \alpha \sin x & \text{if } x < 0 \end{cases}$$

Compute the derivative

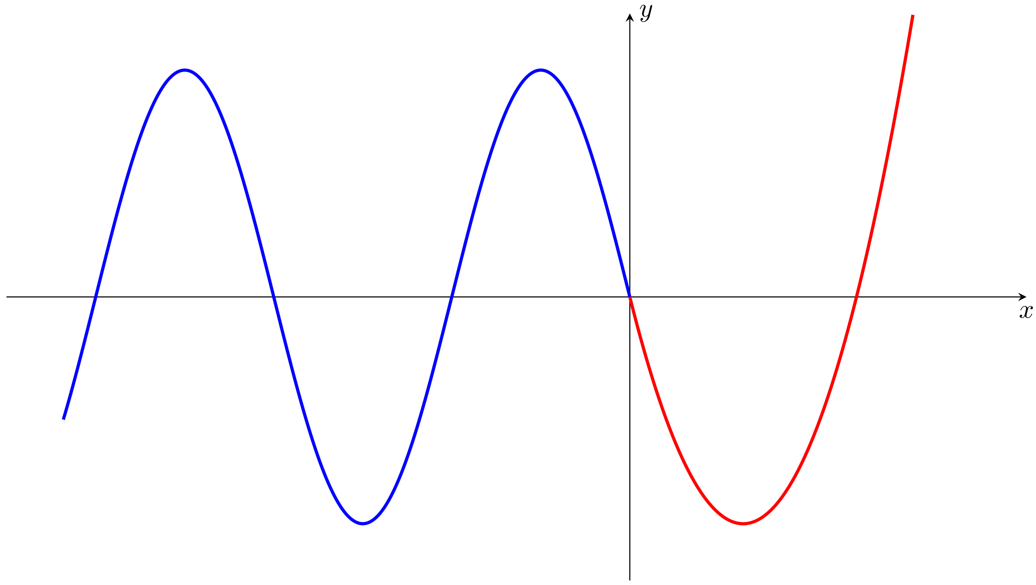
$$f'(x) = \begin{cases} 2(x-2) & \text{if } x > 0 \\ \alpha \cos x & \text{if } x < 0 \end{cases}$$

It must hold:

$$\lim_{x\to 0^+} 2(x-2) = \lim_{x\to 0^-} \alpha \cos x \quad \Leftrightarrow \quad 2(-2) = \alpha \quad \Leftrightarrow \quad \alpha = -4$$

Thus the function is differentiable at the origin if $\alpha = -4$.
Theorefore the function is continuous and differentiable at the origin if $\alpha = -4$ and $\beta = 2$:

$$f(x) = \begin{cases} (x-2)^2 - 4 & \text{if } x \geq 0 \\ -4\sin x & \text{if } x < 0 \end{cases}$$



Analogously, for $\beta = -2$, we can verify that f is differentiable if $\alpha = 4$.

d) $\boxed{f(x) = \begin{cases} \log(x+\beta^2) & \text{if } x > 0 \\ e^{\alpha x} & \text{if } x \leq 0 \end{cases}}$

For $x \neq 0$ the function is differentiable by composition of differentiable functions. We study continuity at the origin, imposing

$$\lim_{x\to 0^+} \log(x+\beta^2) = \lim_{x\to 0^-} e^{\alpha x} = e^0 \quad \Leftrightarrow \quad \log(\beta^2) = 1$$

The function is continuous for $\beta^2 = e$.
For $\beta^2 = e$ we have

$$f(x) = \begin{cases} \log(x+e) & \text{if } x > 0 \\ e^{\alpha x} & \text{if } x \leq 0 \end{cases}$$

Compute the derivative

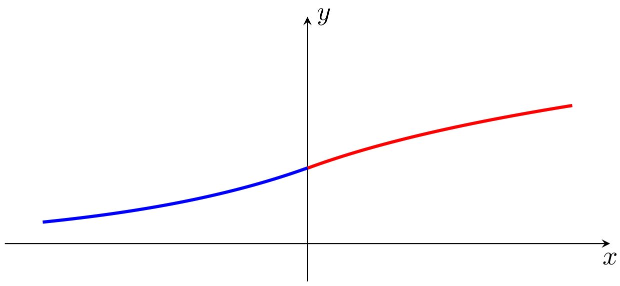
$$f'(x) = \begin{cases} \dfrac{1}{x+e} & \text{if } x > 0 \\ \alpha e^{\alpha x} & \text{if } x < 0 \end{cases}$$

In order to get differentiability:

$$\lim_{x\to 0^+} \frac{1}{x+e} = \lim_{x\to 0^-} \alpha e^{\alpha x} \quad \Leftrightarrow \quad \frac{1}{e} = \alpha$$

Therefore the function is differentiable for $\beta^2 = e$and $\alpha = \dfrac{1}{e}$:

$$f(x) = \begin{cases} \log(x+e) & \text{if } x \geq 0 \\ e^{e^{-1}x} & \text{if } x < 0 \end{cases}$$



e) $\boxed{f(x) = \begin{cases} e^x + \alpha\cos x & \text{se } x \geq 0 \\ \beta(x^2+3x+1) & \text{se } x < 0 \end{cases}}$

For $x \neq 0$ the function is differentiable by composition of differentiable functions. We study continuity at the origin, imposing

$$\lim_{x\to 0^+}(e^x + \alpha\cos x) = \lim_{x\to 0^-}\beta(x^2+3x+1) = 1+\alpha \quad \Leftrightarrow \quad 1+\alpha = \beta = 1+\alpha$$

In order to get differentiability, given $\beta = 1+\alpha$:

$$f(x) = \begin{cases} e^x + \alpha\cos x & \text{if } x \geq 0 \\ (1+\alpha)(x^2+3x+1) & \text{if } x < 0 \end{cases}$$

Compute the first derivative

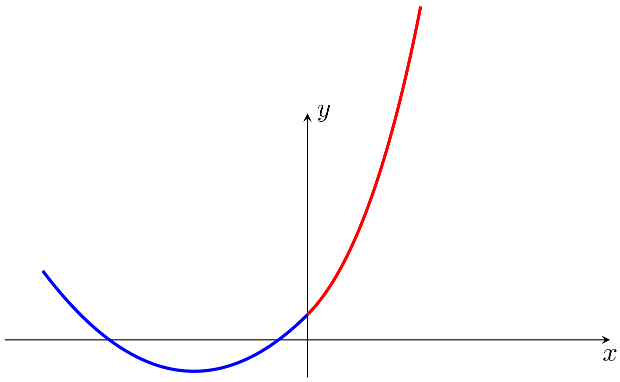
$$f'(x) = \begin{cases} e^x - \alpha\sin x & \text{if } x > 0 \\ (1+\alpha)(2x+3) & \text{if } x < 0 \end{cases}$$

The function is differentiable if and only if

$$\lim_{x\to 0^+}(e^x - \alpha\sin x) = \lim_{x\to 0^-}(1+\alpha)(2x+3) \quad \Leftrightarrow \quad 1 = 3(1+\alpha) \quad \Leftrightarrow \quad \alpha = -\frac{2}{3}$$

Thus the function is differentiable if $\alpha = -\frac{2}{3}$, and hence $\beta = 1 - \frac{2}{3} = \frac{1}{3}$, that is

$$f(x) = \begin{cases} e^x - \frac{2}{3}\cos x & \text{if } x \geq 0 \\ \frac{1}{3}(x^2+3x+1) & \text{if } x < 0 \end{cases}$$



7. Given the function $f(x) = x\,e^x$; compute

$$(f^{-1})'(0), \quad (f^{-1})'(e)$$

Recall the Theorem of the derivative for inverse function:

Theorem Let $f(x)$ be a continuous and invertible function is a neighborhood of $x_0 \in \mathbb{R}$: moreover, let $f(x)$ be differentiable in x_0, with $f'(x_0) \neq 0$. Then the inverse function $f^{-1}(y)$ is differentiable in $y_0 = f(x_0)$ and it holds

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

If $f(x) = x\,e^x$, it holds $f'(x) = e^x + x\,e^x = (1+x)e^x$.

Find the preimage of 0: $x\,e^x = 0 \Leftrightarrow x = 0$; moreover $f'(0) = 1 \neq 0$; thus

$$(f^{-1})'(0) = \frac{1}{f'(0)} = 1$$

Find the preimage of e: $x\,e^x = e \Leftrightarrow x = 1$: since $f'(1) = 2e \neq 0$, we have

$$(f^{-1})'(e) = \frac{1}{f'(1)} = \frac{1}{2e}$$

8. Verify that $f(x) = x^7 + x$ is invertible of $\mathbb{R}$ and its inverse is differentiable on $\mathbb{R}$. Moreover, compute

$$(f^{-1})'(0), \quad (f^{-1})'(2)$$

Since $f'(x) = 7x^6 + 1 \neq 0 \ \forall x \in \mathbb{R}$, the function $f(x) = x^7 + x = x(x^6 + 1)$ is strictly increasing, and thus injective. Therefore f is invertible.

Find the preimage of 0: $x^7 + x = 0 \Leftrightarrow x = 0$; moreover $f'(0) = 1 \neq 0$. Thus

$$(f^{-1})'(0) = \frac{1}{f'(0)} = 1$$

Find the preimage of 2: $x^7 + x = 2 \Leftrightarrow x = 1$; moreover $f'(1) = 8 \neq 0$. Thus

$$(f^{-1})'(2) = \frac{1}{f'(1)} = \frac{1}{8}$$

9. Given a differentiable function $f(x)$, prove that if $f(x)$ is even, then $f'(x)$ is odd and viceversa, if $f(x)$ is odd, then $f'(x)$ is even.

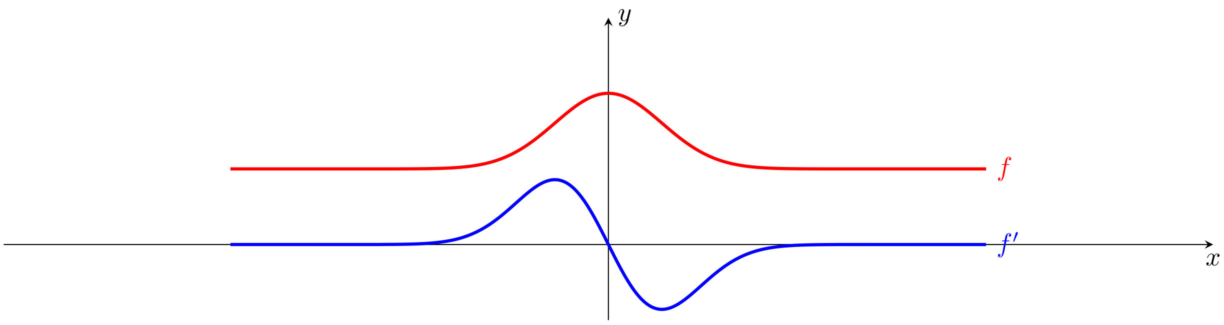
Notice that $f'(-x) = [f(-x)]' \cdot (-1) = -[f(-x)]'$.

a) If $f(x)$ is even, that is $f(-x) = f(x)$, then

$$f'(-x) = -[f(-x)]' = -[f(x)]' = -f'(x)$$

and thus f' is odd.

The figure shows an example of a graph of an even function f and its f' that is odd:

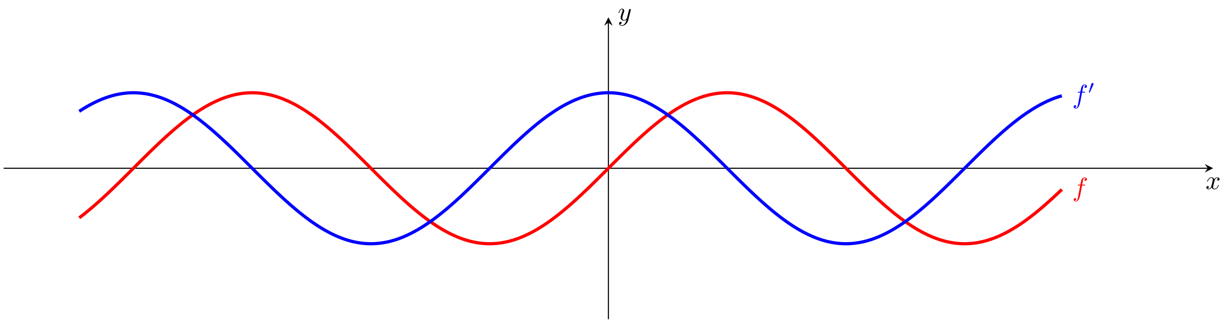


b) If $f(x)$ is odd, that is $f(-x) = -f(x)$, then

$$f'(-x) = -[f(-x)]' = -[-f(x)]' = f'(x)$$

and thus f' is even.

The figure shows an example of a graph of an odd function $f$odd and its f' that is even:



10. Let $f(x) \in \mathcal{C}^\infty(\mathrm{I\!R})$ such that $f'(x) = f^2(x)$. Moreover $f(0) = 3$. Compute $f'(0),\ f''(0),\ f'''(0)$

From $f'(x) = f^2(x)$ and $f(0) = 3$, we have $f'(0) = f^2(0) = 3^2 = 9$.
Compute $f''(x) = 2f(x)\ f'(x)$; since $f(0) = 3$ e $f'(0) = 9$, it holds

$$f''(0) = 2f(0)\ f'(0) = 2 \cdot 3 \cdot 9 = 54$$

Compute now $f'''(x) = 2f'(x)\ f'(x) + 2f(x)\ f''(x)$; since $f(0) = 3,\ f'(0) = 9,\ f''(0) = 54$, we have

$$f'''(0) = 2f'(0)\ f'(0) + 2f(0)\ f''(0) = 2 \cdot 9 \cdot 9 + 2 \cdot 3 \cdot 54 = 162 + 324 = 486$$

11. Let $f(x)$ be a differentiable function on $\mathrm{I\!R}$ such that $f'(x) = x^2 f(x),\quad f(0) = 3$.
Compute the derivative in $x = 0$ of the following functions:

$$g(x) = e^{f(x)\sin x}; \quad h(x) = \cos(f(x)); \quad k(x) = f(\cos x)$$

Note that, from $f'(x) = x^2 f(x)$ and $f(0) = 3$, we have $f'(0) = 0 \cdot 3 = 0$.

$\boxed{g(x) = e^{f(x)\sin x}}$

$g'(x) = e^{f(x)\sin x}(f'(x)\sin x + f(x)\cos x)$ dunque $g'(0) = e^{f(0)\sin 0}(f'(0)\sin 0 + f(0)\cos 0) = e^0(0 \cdot \sin 0 + 3\cos 0) = 1 \cdot 3 = 3$

$\boxed{h(x) = \cos(f(x))}$

$$h'(x) = -\sin(f(x))\ f'(x) \quad \text{thus} \quad h'(0) = -\sin(f(0))\ f'(0) = -\sin 3 \cdot 0 = 0$$

$\boxed{k(x) = f(\cos x)}$

$$k'(x) = f'(\cos x)\ (-\sin x) \quad \text{therefore} \quad k'(0) = f'(\cos 0)\ (-\sin 0) = f'(1) \cdot 0 = 0$$

EXERCISES FROM WRITTEN EXAMS

1. (2/1/2017 - II)

   (a) *State Lagrange Theorem, and provide an example of function that is NOT satisfying the thesis of the theorem.*
   See the textbook for the statement.
   An example of function that does NOT satisfy the thesis can be $f(x) = M(x)$ (mantissa), with $x \in [0,1]$. Indeed $\dfrac{f(1)-f(0)}{1-0} = \dfrac{0-0}{1} = 0$, but there exists no $c \in (0,1)$ such that $f'(c) = 0$.

   (b) *show that, if f is differentiable and $f'(x) < -3$ for every $x \in \mathbb{R}$, then $\lim_{x\to+\infty} f(x) = -\infty$.*
   Since f is differentiable (on $\mathbb{R}$), we can apply Lagrange Theorem to $f(x)$, with x in any closed and bounded interval; choose $[0,x]$, for any $x > 0$.
   Hence there exists $c \in (0,x)$ such that $\dfrac{f(x)-f(0)}{x-0} = f'(c) < -3$.
   Then $f(x) < f(0) - 3x$. Apply the first Comparison Theorem (infinite limits): since $\lim_{x\to+\infty}(f(0) - 3x) = -\infty$, also $\lim_{x\to+\infty} f(x) = -\infty$.

2. (9/9/2015 - II)

   (a) *State the definition of derivative of a function f in a point x_0 of its domain.* See the textbook.

   (b) *Consider the function*
   $$f(x) = \begin{cases} 3\sin^2 x \sin\frac{2}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$
   Is it differentiable in $x = 0$? Explain.
   By definition
   $$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{3\sin^2 x \sin\frac{2}{x}}{x} = \lim_{x\to 0} \frac{3x^2 \sin\frac{2}{x}}{x} = \lim_{x\to 0} 3x \sin\frac{2}{x} = 0.$$
   Thus $f(x)$ is differentiable in $x = 0$.

3. (2/18/2013)

   Let $f(x)$ and $g(x)$ two continuos and differentiable function in a neighborhood of a point x_0. Moreover, assume:

   - $f(x) < g(x)$ for all $x \in I, x \neq x_0$

   - $f(x_0) = g(x_0)$.

   Show that $f'(x_0) = g'(x_0)$.

   Show that $f'(x_0) - g'(x_0) = 0$, i.e. $(f-g)'(x_0) = 0$.
   Since $f(x)$ and $g(x)$ are differentiable at x_0, the same holds for $(f-g)(x)$, therefore we have a finite limit
   $$(f-g)'(x_0) = \lim_{x\to x_0} \frac{(f-g)(x)-(f-g)(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-g(x)-f(x_0)+g(x_0)}{x-x_0} = \lim_{x\to x_0} \frac{f(x)-g(x)}{x-x_0}$$
   Consider the sign of $\dfrac{f(x)-g(x)}{x-x_0}$: numerator is always negative, whereas denominator is positive if $x > x_0$, and negative if $x < x_0$. Therefore, as $x \to x_0$, we have
   $$\lim_{x\to x_0^+} \frac{f(x)-g(x)}{x-x_0} \leq 0, \text{ whereas } \lim_{x\to x_0^-} \frac{f(x)-g(x)}{x-x_0} \geq 0.$$
   Such limit does not exist, therefore the only common value they may have is 0; hence $(f-g)'(x_0) = 0$.