Elementary functions: solved exercises

<u>WARNING</u>: In some exercise, the global properties of the continuous functions are used; they will be explained later on during the course of Mathematical Analysis I.

It is stated that

$$\mathbb{R}^+ = \{x \in \mathbb{R} : \; x > 0\}, \qquad \mathbb{R}^- = \{x \in \mathbb{R} : \; x < 0\}.$$

Exercise.

1) Determine the domain of the following functions with real variable:

$$(a) \quad f(x) = \sqrt{x^2 - 4} \qquad (b) \quad f(x) = \log(\sqrt{x+2} - x)$$

$$(c) \quad f(x) = \sqrt[4]{|x| - |x+2|} \qquad (d) \quad f(x) = \frac{1}{4^x - 5 \cdot 2^x + 6}$$

2) Having the function f, determine the range $f(X)$ of the set X and the pre-image $f^{-1}(Y)$ of the set Y:

$$(a) \quad f(x) = x^2, X = [1, 2], Y = [1, 4] \qquad (b) \quad f(x) = |x|, X = \{1\}, Y = \{-5\}$$

$$(c) \quad f(x) = x^3, X = (1, 8], Y = \{-1\} \qquad (d) \quad f(x) = \sin x, X = \{0\}, Y = \{0\}$$

3) Determine the domain and sketch the graph of the function $f(x) = |x^2 - 2x|$; then, determine the range $f(X)$ of the set $X = [0, 1)$ and the pre-image $f^{-1}(Y)$ of the set $Y = [0, 1)$.

4) Determine the pre-image $f^{-1}(Y)$ of the set $Y = [0, 1]$, where $f(x) = \frac{x^2-1}{x+5}$

5) Determine the range $f(X)$ of the set $X = [-1, 2]$, where $f(x) = 2x^3 - 1$.

6) Verify that $f(x) = 2x - 1$ is both injective and surjective on $\mathbb{R}$.

7) Determine the domain of $f(x) = 3 + \sqrt{x+1}$ and verify that is injective in $\mathrm{dom}(f)$. Moreover, determine the range of f and verify that f is not surjective in $\mathbb{R}$.

8) Sketch the graphs of the functions $f(x) = |x|$, $g(x) = |x+1|$, $h(x) = |x-1|$, $l(x) = |x| + 1$, $m(x) = |x| - 1$.

9) Sketch the graphs of the functions $f(x) = \sin x$, $g(x) = 2\sin x$, $h(x) = \sin(2x)$, $l(x) = \sin(\frac{x}{2})$, $m(x) = \frac{1}{2}\sin(x)$.

10) Sketch the graphs of the functions $f(x) = \log x$, $g(x) = -\log x$, $h(x) = |\log x|$, $l(x) = \log(-x)$, $m(x) = \log|x|$.

11) Determine the domain and sketch the graph of the function $f(x) = 2 + \log(x+3)$.

12) Determine the domains and sketch the graphs of the following functions:

$$(a) \quad f(x) = |x^2 + x - 2| \qquad\qquad (b) \quad f(x) = 2 - |x+3|$$

$$(c) \quad f(x) = 2 - \sqrt{x+1} \qquad\qquad (d) \quad f(x) = ||x+1| - 1|$$

13) Determine the domain and the range of $f(x) = 3x + 1$. Verify that f is strictly increasing on $\mathbb{R}$. Determine f^{-1}. Sketch the graphs of f and f^{-1}.

14) Determine the domain and sketch the graph of the function $f(x) = \frac{2x+1}{x-1}$. Verify that f is injective and compute f^{-1}.

15) Sketch the graph of the function

$$f(x) = \begin{cases} x^2 & \text{se } 0 \le x < 1 \\ 5 - 2x & \text{se } 1 \le x \le 2. \end{cases}$$

Prove whether f is monotone (in particular, whether it is strictly monotone) on $[0,1)$, on $[1,2]$ and on $[0,2]$. Verify that f is injective in its domain and determine its range. Determine f^{-1}.

16) Verify that the restrictions of $f(x) = x^2$ to $\mathbb{R}^-$ and to $\mathbb{R}^+$ are strictly monotone. Defining these restrictions as $f_1 = f_{|\mathbb{R}^-}$ and $f_2 = f_{|\mathbb{R}^+}$, compute the corresponding inverse functions g_1 and g_2; then, sketch their graphs.

17) Sketch the graph of f, verify that it is invertible, and determine the inverse function f^{-1}, where $f(x) = x|x-2| + 2x$.

18) Determine the ranges where $f(x) = (x-1)|x+2|$ is strictly monotone; then, compute the inverse functions related to the computed restrictions.

19) Verify that the function $f(x) = x^4$ is bounded on $[-2, 3]$, bounded above on $\mathbb{R}$, bounded below on $\mathbb{R}$.

20) Determine the domain and sketch the graph of the function $f(x) = \left|\frac{1}{x} + 1\right|$. Verify that f is bounded below in its domain. Determine $\min\{f(x) : x \in \mathrm{dom}(f)\}$. Verify that f is unbounded above in its domain but that is bounded above in $]-\infty, -1]$. Compute $\sup\{f(x) : x \in]-\infty, -1]\}$ and prove that f does not have a maximum in that interval.

21) Determine the domains of the functions $f(x) = x - \sqrt{x}$ and $g(x) = \sqrt{x-2}$; then, compute the composite functions $g \circ f$ and $f \circ g$.

22) Determine the domains of the functions $f(x) = x^2 + 3x$ and $g(x) = |x|$; then, compute the composite functions $g \circ f$ and $f \circ g$ and sketch their graphs.

23) Determine domain, range and the monotonic sets of the functions $f(x) = 1+x^2$ and $g(x) = \frac{1}{x}$; then, sketch the graph of the composite function $g \circ f$.

24) Determine domain, range and the monotonic sets of the functions $f(x) = -\frac{1}{x}$ and $g(x) = e^x$; then, sketch the graph of the composite function $g \circ f$.

SOLUTIONS

1)

1a) In order to determine the domain of the function $f(x) = \sqrt{x^2-4}$, the inequality $x^2 - 4 \geq 0$ has to be solved: the values of $x \in \mathbb{R}$ which satisfy the inequality are $x \leq -2$ and $x \geq 2$. Therefore, $\mathrm{dom}(f) = (-\infty, -2] \cup [2, +\infty)$.

1b) In order to determine the domain of the function $f(x) = \log(\sqrt{x+2} - x)$, the inequality $\sqrt{x+2} - x > 0$ has to be solved: the inequality is satisfied if and only if $x \in \mathbb{R}$ verifies one of the two following systems:

$$\begin{cases} x < 0 \\ x + 2 \geq 0 \end{cases} \qquad \begin{cases} x \geq 0 \\ x + 2 \geq 0 \\ x + 2 > x^2. \end{cases}$$

The first system is verified for $-2 \leq x < 0$, while the second one for $0 \leq x < 2$. Hence, the analysed inequality is satisfied for $-2 \leq x < 2$. Therefore, $\mathrm{dom}(f) = [-2, 2)$.

1c) In order to determine the domain of the function $f(x) = \sqrt[4]{|x| - |x+2|}$ the inequality $|x| - |x+2| \geq 0$ has to be solved: the inequality is satisfied if and only if $x \in \mathbb{R}$ verifies one of the three following systems:

$$\begin{cases} x < -2 \\ -x \geq -x - 2 \end{cases} \qquad \begin{cases} -2 \leq x \leq 0 \\ -x \geq x + 2 \end{cases} \qquad \begin{cases} x > 0 \\ x \geq x + 2. \end{cases}$$

The first system is verified for $x < -2$, the second one for $-2 \leq x \leq -1$, while the third one does not have any solution. Hence, the analysed inequality is satisfied for $x \leq -1$. Therefore, $\mathrm{dom}(f) = (-\infty, -1]$.

1d) In order to determine the domain of the function $f(x) = \frac{1}{4^x - 5 \cdot 2^x + 6}$, the values of $x \in \mathbb{R}$ which verify the equation $4^x - 5 \cdot 2^x + 6 = 0$ have to be excluded. Using the substitution $2^x = t$ the equation $t^2 - 5t + 6 = 0$ is obtained: its solutions are $t_1 = 2$ and $t_2 = 3$. Since $2^x = t$ is equal to $x = \log_2 t$ for $t > 0$, it is obtained $x_1 = \log_2 2 = 1$ and $x_2 = \log_2 3$. Therefore, $\mathrm{dom}(f) = \mathbb{R} \setminus \{1, \log_2 3\}$.

2) In order to determine the range $f(X)$ on X and the pre-image $f^{-1}(Y)$ on Y of the function f, it is necessary to refer to the definitions of range and pre-image. In particular:

$$f(X) = \{y \in \mathbb{R} : \exists x \in X \text{tale che } f(x) = y\}$$

$$f^{-1}(Y) = \{x \in \mathrm{dom}(f) : f(x) \in Y\}.$$

2*a*) Examining the graph of the function $f(x) = x^2$, that is a parabola, it is easy to notice that $f(X) = f([1, 2]) = [1, 4]$. It is possible to rigorously prove this result by observing that $f(1) = 1$, $f(2) = 4$: it follows that f is increasing on $[1, 2]$ (hence $1 \le x \le 2 \Rightarrow 1 \le x^2 \le 4$) and it is also continuous (therefore, for the Intermediate Value Theorem, f takes *any* value between 1 and 4). To compute $f^{-1}(Y) = f^{-1}([1, 4])$, the solutions of the two inequalities $1 \le x^2 \le 4$, that correspond to the system

$$\begin{cases} x^2 \ge 1 \\ x^2 \le 4 \end{cases}$$

have to be determined; then, it emerges that $f^{-1}(Y) = [-2, -1] \cup [1, 2]$. The result is confirmed by observing the parabola's graph.

2*b*) The range of a set containing only one element corresponds to the range of such element; therefore, $f(X) = f(\{1\}) = f(1) = |1| = 1$. In order to find the pre-image of $Y = \{-1\}$, the equation $|x| = -5$ has to be solved. However, this equation has no solutions: hence, $f^{-1}(Y) = \emptyset$.

2*c*) Examining the graph of the function $f(x) = x^3$ it is easy to notice that $f(X) = f((1, 8]) = (1, 8^3]$. As in exercise 2a), it is possible to rigorously prove this result by observing that $f(1) = 1$, $f(8) = 8^3$: it follows that f is increasing (hence $1 < x \le 8 \Rightarrow 1 < x^3 \le 8^3$) and continuous (therefore, for the Intermediate Value Theorem, f takes *any* value between 1 and 8^3). To compute $f^{-1}(Y) = f^{-1}(-1)$, the solutions of the equation $x^3 = -1$ has to be determined; then, it emerges that $f^{-1}(Y) = \{-1\}$. The result is confirmed by observing the function's graph.

2*d*) Being $f(x) = \sin x$, it follows that $f(X) = f(\{0\}) = \{0\}$. In order to determine $f^{-1}(Y) = f^{-1}(\{0\})$, the equation $\sin x = 0$. has to be solved. The solution is $f^{-1}(Y) = \{n\pi : n \in \mathbb{Z}\}$.

3) The function $f(x) = |x^2 - 2x|$ holds for each value of $x \in \mathbb{R}$, hence $\text{dom}(f) = \mathbb{R}$.
Using the definition of the absolute value, it is possible to re-write the equation as

$$f(x) = \begin{cases} x^2 - 2x & \text{if } x^2 - 2x \ge 0 \\ -(x^2 - 2x) & \text{if } x^2 - 2x < 0 \end{cases} .$$

In order to sketch the graph, first of all, the parabola $g(x) = x^2 - 2x$ is drawn; then, all the parts of the graph of g belonging to the negative part of the $y - axis$ are "mirrored" about the $x - axis$. In this way, the graph shown in Figure 1 is obtained. Looking at the graph, the range $f(X)$ on $X = [0, 1)$ is $f(X) = [0, 1)$. The result is justified by $f(0) = 0$, $\lim_{x \to 1^-} f(x) = 1$, f is continuous and $f(x) = -(x^2 - 2x)$ is increasing for $x \in X$. In order to determine $f^{-1}(Y) = f^{-1}([0,1))$, the solutions of the two inequalities $0 \leq |x^2 - 2x| < 1$ have to be computed. The first inequality is always true; hence, it is enough to solve the second inequality $|x^2 - 2x| < 1$. According to the definition of absolute value, it is necessary to solve the two following systems

$$\begin{cases} x^2 - 2x \geq 0 \\ x^2 - 2x < 1 \end{cases} \qquad \text{and} \qquad \begin{cases} x^2 - 2x < 0 \\ -(x^2 - 2x) < 1. \end{cases}$$

The solution of the first system is $(1 - \sqrt{2}, 0] \cup [2, 1 + \sqrt{2})$; while that of the second one is $(0, 2) \setminus \{1\}$. The union of these sets gives $f^{-1}(Y) = (1 - \sqrt{2}, 1 + \sqrt{2}) \setminus \{1\}$. The result is confirmed by observing the function's graph.

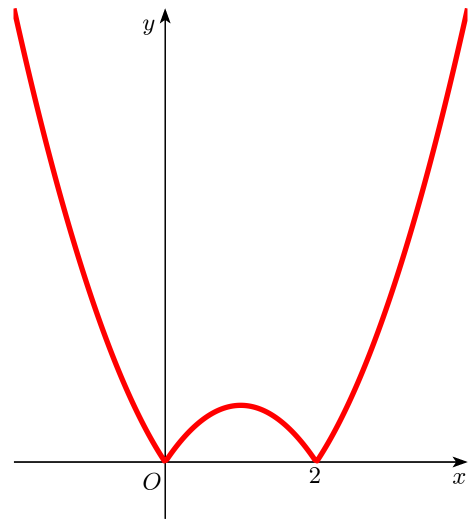

Fig. 1: Graph of $f(x) = |x^2 - 2x|$, (exercise 3)

4) In order to determine the pre- image of $f^{-1}(Y)$ on $Y = [0, 1]$, where $f(x) = \frac{x^2-1}{x+5}$, the solutions of the two inequalities $0 \leq \frac{x^2-1}{x+5} \leq 1$,, that correspond to the system

$$\begin{cases} \frac{x^2-1}{x+5} \geq 0 \\ \\ \frac{x^2-1}{x+5} \leq 1. \end{cases}$$

have to be computed. The solution is $f^{-1}(Y) = [-2, -1] \cup [1, 3]$.

5) In order to determine the range of $f(X)$ in $X = [-1, 2]$, where $f(x) = 2x^3 - 1$, it may be useful drawing the graph of f. It can be easily derived starting from the know graph of $y = x^3$ and considering that the multiplication by the constant 2 gives a vertical expansion (i.e. it corresponds to a scale change on the $y - axis$) and that subtracting 1 gives a downward translation by 1 unit of the graph of $y = 2x^3$. Observing the obtained graph of f suggests

that f is an increasing and continuous function in $\mathbb{R}$, as well as $y = x^3$. The monotony is directly verifiable:

$$x_1 < x_2 \quad \Rightarrow \quad x_1^3 < x_2^3 \quad \Rightarrow \quad 2x_1^3 < 2x_2^3 \quad \Rightarrow \quad 2x_1^3 - 1 < 2x_2^3 - 1$$

The continuity is derived by the consideration that $f(x)$ is a polynomial; hence, for the Intermediate Value Theorem, $f(X) = f([-1, 2]) = [f(-1), f(2)] = [-3, 15]$.

6) In order to verify that $f(x) = 2x-1$ is injective on $\mathbb{R}$, it has to be proven that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. Let's suppose that $2x_1 - 1 = 2x_2 - 1$: then, adding 1 to both sides, it results $2x_1 = 2x_2$; then, dividing by 2, it results $x_1 = x_2$. In order to verify that f is surjective on $\mathbb{R}$, it has to be proven that for every $y \in \mathbb{R}$, the equation $f(x) = y$ has at least one solution. Considering a generic $y \in \mathbb{R}$ and solving the equation $2x - 1 = y$, it easily results that $x = \frac{y+1}{2}$: the surjectivity of f has been proven.

7) In order to determine the domain of the function $f(x) = 3 + \sqrt{x+1}$, the inequality $x + 1 \geq 0$ has to be solved, with $x \in \mathbb{R}$: therefore, it follows that $\mathrm{dom}(f) = [-1, +\infty[$. In order to verify that f is injective in $\mathrm{dom}(f)$, let's suppose that $x_1, x_2 \geq -1$, obtaining that $3 + \sqrt{x_1 + 1} = 3 + \sqrt{x_2 + 1}$. Subtracting 3 to both sides, it results $\sqrt{x_1 + 1} = \sqrt{x_2 + 1}$; then, squaring both sides, it follows that $x_1 + 1 = x_2 + 1$ and so $x_1 = x_2$. It proves that f is injective. In order to verify that f is not surjective on $\mathbb{R}$, it is sufficient to find just one value $y \in \mathbb{R}$ which has no pre-image through f, i.e. a value that does not satisfy the equation $f(x) = y$. It is possible to notice that $3 + \sqrt{x+1} \geq 3$, for $x \geq -1$, since $\sqrt{x+1} \geq 0$. Therefore, $y < 3$ does not have pre-image: it has been proven that f is not surjective on $\mathbb{R}$.

8) It is possible to draw the graph of the functions $g(x) = |x+1|$, $h(x) = |x-1|$, $l(x) = |x| + 1$ and $m(x) = |x| - 1$, starting from the graph of the well-known function $f(x) = |x|$. It is possible to notice that $g(x) = f(x+1)$; it means that $(x+1, y)$ is a point belonging to the graph of f if and only if (x, y) is a point belonging to the graph of g. Consequently, the graph of g can be obtained by a leftward translation by one unit of the graph of f. Similarly, since $h(x) = f(x-1)$, the graph of h can be obtained by a rightward translation by one unit of the graph of f. On the other hand, it is possible to notice that $l(x) = f(x)+1$; it means that (x, y) is a point belonging to the graph of l if and only if $(x, y+1)$ is a point belonging to the graph of l. Consequently, the graph of l can be obtained by a upward translation by one unit of the graph of f. f. Similarly, since $m(x) = f(x) - 1$, the graph of m can be obtained by a downward translation by one unit of the graph of f. The graphs are shown in Figure 2.

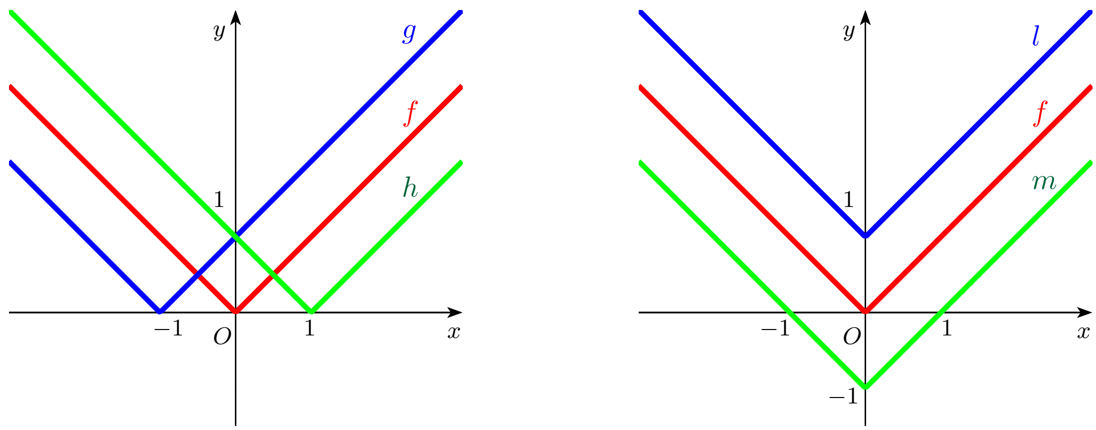

Fig. 2: Graphs of $f(x) = |x|$, $g(x) = |x+1|$, $h(x) = |x-1|$, $l(x) = |x|+1$, $m(x) = |x|-1$ (exercise 8)

9) It is possible to draw the graph of the functions $g(x) = 2\sin x$, $h(x) = \sin(2x)$, $l(x) = \sin(\frac{x}{2})$ and $m(x) = \frac{1}{2}\sin x$, starting from the graph of the well-known function$f(x) = \sin x$. It is possible to notice that $g(x) = 2f(x)$; it means that (x, y) is a point belonging to the graph of f if and only if $(x, 2y)$ is a point belonging to the graph of g. Consequently, the graph of g can be obtained by a vertical expansion by a factor 2 of the graph of f. Similarly, the graph of m can be obtained by a vertical shrinkage by a factor 1/2 of the graph of f. Furthermore, it is possible to notice that $h(x) = f(2x)$; it means that (x, y) is a point belonging to the graph of h if and only if $(2x, y)$ is a point belonging to the graph of f. Consequently, the graph of h can be obtained by a horizontal shrinkage by a factor 1/2 of the graph of f. Similarly, the graph of l can be obtained by a horizontal expansion by a factor 2 of the graph of f. The graphs are shown in Figure 3.

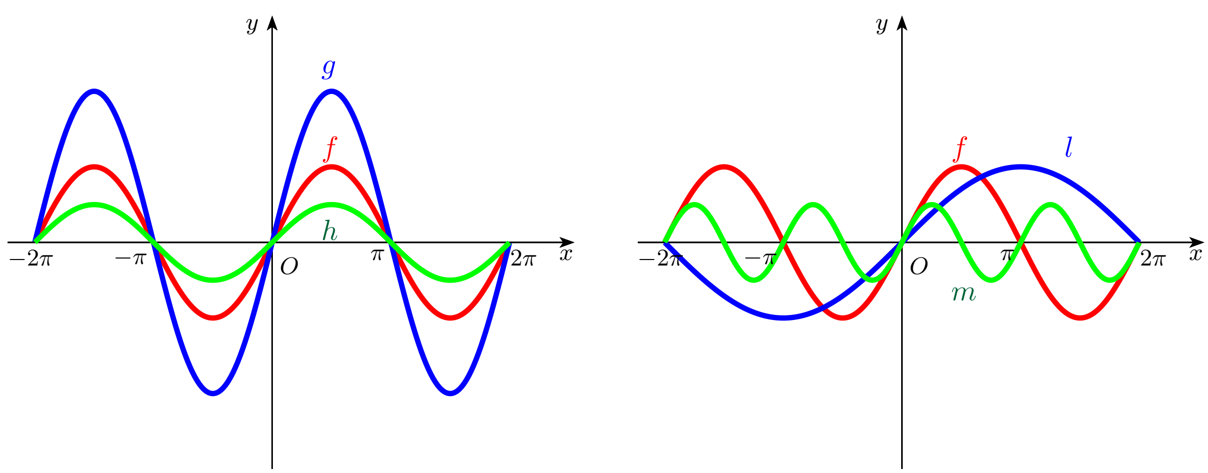

Fig. 3: Graphs of $f(x) = \sin x$, $g(x) = 2\sin x$, $h(x) = \sin(2x)$, $l(x) = \sin(\frac{x}{2})$, $m(x) = \frac{1}{2}\sin(2x)$ (exercise 9)

10) It is possible to draw the graph of the functions $g(x) = -\log x$, $h(x) = |\log x|$, $l(x) = \log(-x)$ and $m(x) = \log|x|$, starting from the graph of the well-known function $f(x) = \log x$. It is possible to notice that $g(x) = -f(x)$; it means that (x, y) is a point belonging to the graph of g if and only if $(x, -y)$ is a point belonging to the graph of f Consequently, the graph of g can be obtained by mirroring the graph of f about the $x-axis$. On the other hand, being $l(x) = f(-x)$, it results that $x \in \mathrm{dom}(l)$ if and only if $-x \in \mathrm{dom}(f)$; moreover, (x, y) is a point belonging to the graph of l if and only if $(-x, y)$ is a point belonging to the graph of f. Therefore, the two domains are symmetrical about the origin and the two graphs are symmetrical about the $y-axis$: consequently, the graph of l can be obtained by mirroring the graph of f about the $y-axis$. In order to draw the graph of $h(x) = |f(x)|$, the definition of absolute value has to be taken into account; so, it follows that:

$$h(x) = \begin{cases} f(x) & \text{se } f(x) \geq 0 \\ -f(x) & \text{se } f(x) < 0 \end{cases}$$

In conclusion, the graph of h corresponds to that of f where the points of f have a positive or zero ordinate value; instead, it is obtained by mirroring the graph of f about the $x-axis$ where the points of f have a negative ordinate value. Regarding $m(x) = f(|x|)$, it is possible to observe that $x \in \mathrm{dom}(m)$ if and only if $|x| \in \mathrm{dom}(f)$; moreover,

$$m(x) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$

Consequently, $\mathrm{dom}(m) = \mathrm{dom}(f) \cup \mathrm{dom}(l)$ and the graph of m can be obtained by merging the graphs of f and l. The graphs are shown in Figure 4 and 5.

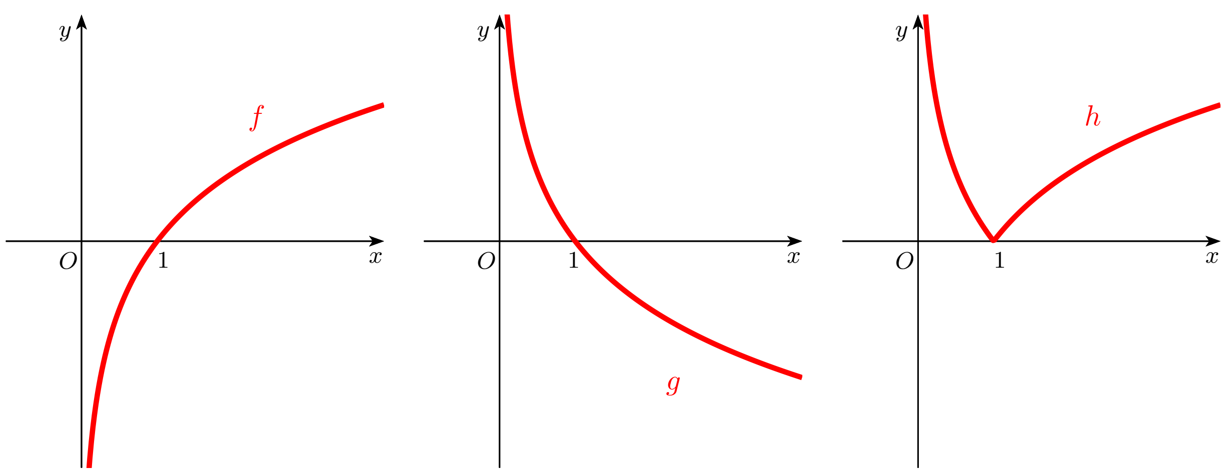

Fig. 4: Graphs of $f(x) = \log x$, $g(x) = -\log x$, $h(x) = |\log x|$ (exercise 10)

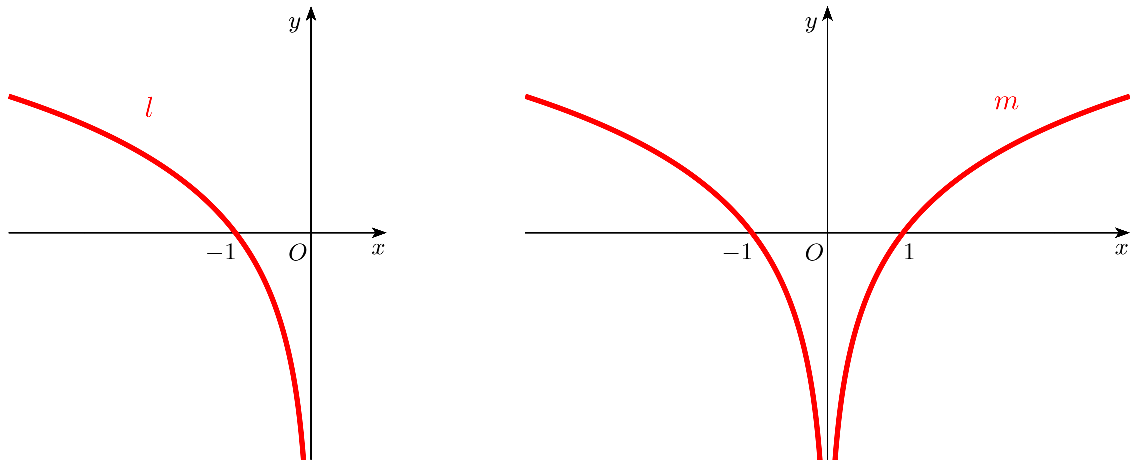

Fig. 5: Graphs of $l(x) = \log(-x)$, $m(x) = \log|x|$ (exercise 10)

11) In order to determine the domain of the function $f(x) = 2+\log(x+3)$, the inequality $x+3 > 0$ has to be solved, with $x \in \mathbb{R}$: therefore, it follows that $\mathrm{dom}(f) =]-3, +\infty[$. In order to draw the graph of f starting from the known graph of $y = \log x$, it is sufficient to translate the latter leftward by 3 units and upward by 2 units. The resulting graph is shown in Figure 6.

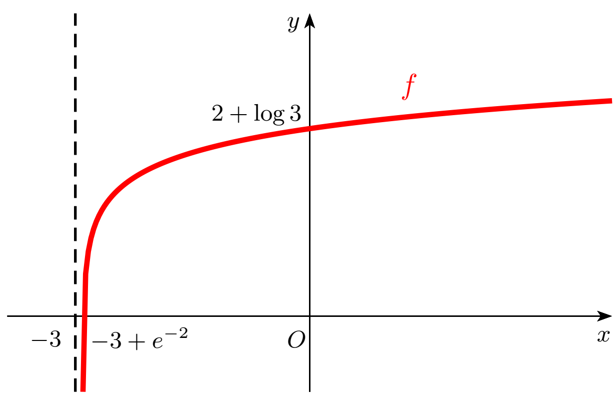

Fig. 6: Graph of $f(x) = 2 + \log(x + 3)$ (exercise 11)

12a) The function $f(x) = |x^2 + x - 2|$ is defined on each point of $\mathbb{R}$. Its graph can be obtained starting from that of the parabola $y = x^2+x-2$: the points of the parabola having a negative ordinate value are mirrored about the $y - axis$. The resulting graph is shown in Figure 7.

12b) The function $f(x) = 2 - |x + 3|$ is defined on each point of $\mathbb{R}$. Its graph can be obtained starting from that of the function $y = |x|$: it has to be leftward translated by 3 units; then, it has to be mirrored about the $y - axis$; finally, it has to be upward translated by 2 units.

Alternatively, considering the definition of the absolute value, it is possible to re-write the function f as follows

$$f(x) = \begin{cases} 2-(x+3) & \text{se } x+3 \geq 0 \\ 2+(x+3) & \text{if } x+3 < 0 \end{cases}$$

that is

$$f(x) = \begin{cases} -x-1 & \text{if } x \geq -3 \\ x+5 & \text{se } x < -3 \end{cases}$$

The resulting graph is shown in Figure 7.

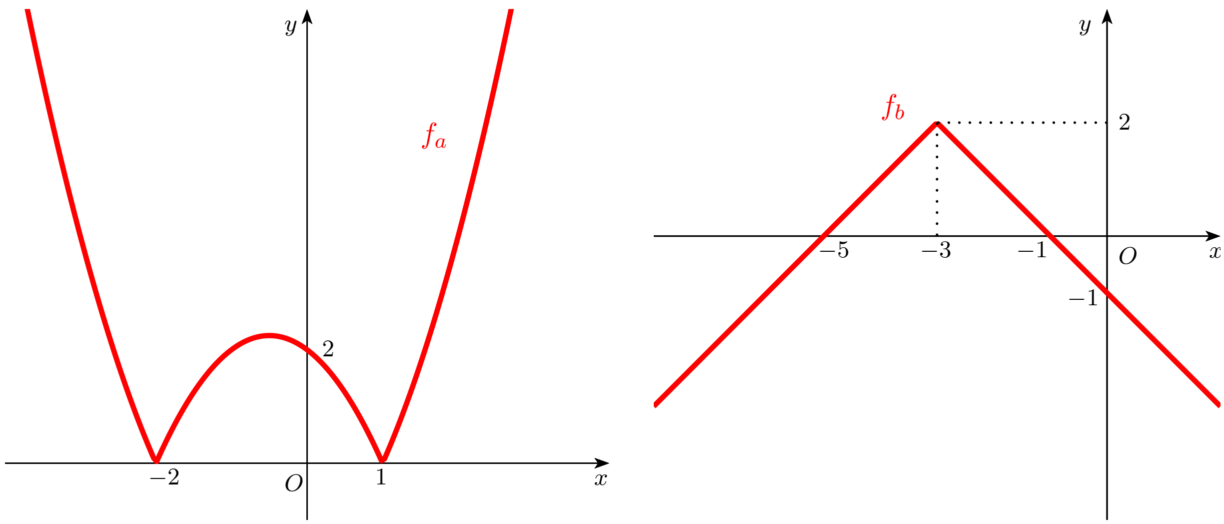

Fig. 7: Graphs of $f_a(x) = |x^2+x-2|$, $f_b(x) = 2-|x+3|$ (exercise 12a, 12b)

12c) In order to determine the domain of the function $f(x) = 2-\sqrt{x+1}$, the inequality $x+1 \geq 0$ has to be solved, with $x \in \mathbb{R}$: therefore, it follows that $\text{dom}(f) = [-1, +\infty[$. In order to draw the graph of f starting from the known graph of $y = \sqrt{x}$, it is possible to leftward translate the latter by 1 unit, then to mirror it about the $x - axis$ and, finally, to upward translate it by 2 units. The resulting graph is shown in Figure 8.

12d) The function $f(x) = ||x+1|-1|$ is defined on each point of $\mathbb{R}$. In order to draw the graph of f starting from the known graph of $y = |x|$, it is possible to leftward translate the latter by 1 unit, then to downward translate it by 1 unit and, finally, to mirror all the points having a negative ordinate about the $y - axis$. Alternatively, it is possible to re-write the function f as follows

$$f(x) = \begin{cases} |x+1|-1 & \text{if } |x+1|-1 \geq 0 \\ 1-|x+1| & \text{if } |x+1|-1 < 0 \end{cases}$$

But, it is important to notice that

$$|x+1|-1 = \begin{cases} x & \text{if } x+1 \geq 0 \\ -x-2 & \text{if } x+1 < 0 \end{cases} = \begin{cases} x & \text{if } x \geq -1 \\ -x-2 & \text{if } x < -1 \end{cases}.$$

Therefore, being $|x+1|-1 \geq 0$ both for $x \leq -2$ and for $x \geq 0$, it results:

$$f(x) = \begin{cases} -x-2 & \text{if } x \leq -2 \\ x+2 & \text{if } -2 < x < -1 \\ -x & \text{if } -1 < x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

The resulting graph is shown in Figure 8.

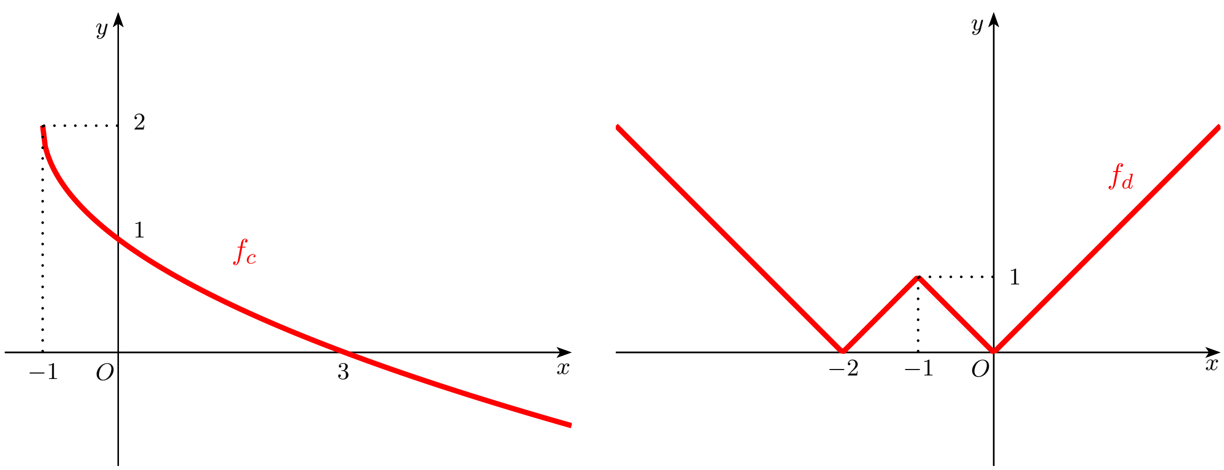

Fig. 8: Graphs of $f_c(x) = 2 - \sqrt{x+1}$, $f_d(x) = ||x+1|-1|$ (exercise 12c, 12d)

13) The function $f(x) = 3x+1$ is defined on each point of $\mathbb{R}$. In order to calculate the range, it is sufficient to determine for which values $y \in \mathbb{R}$ the equation $f(x) = y$ holds, i.e. the equation $3x+1 = y$ has to have at least one solution. Since the equation has a (single) solution for any y, $x = \frac{y-1}{3}$, it follows that $\mathrm{Im}\,(f) = \mathbb{R}$. To verify that f is strictly increasing on $\mathbb{R}$, considering $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$, it possible to observe that $3x_1 < 3x_2$ and that $3x_1 + 1 < 3x_2 + 1$: it means that $f(x_1) < f(x_2)$. Since f is strictly monotone, it follows that it is injective too. In order to determine f^{-1}, bearing in mind the definition of the inverse function, it is sufficient to impose $f^{-1}(x) = y$ if and only if $f(y) = x$, that is $3y+1 = x$. Solving the latter equation, it results $y = \frac{x-1}{3} = f^{-1}(x)$. The graphs of f and f^{-1} are symmetrical about the straight line $y = x$ and are shown in Figure 9.

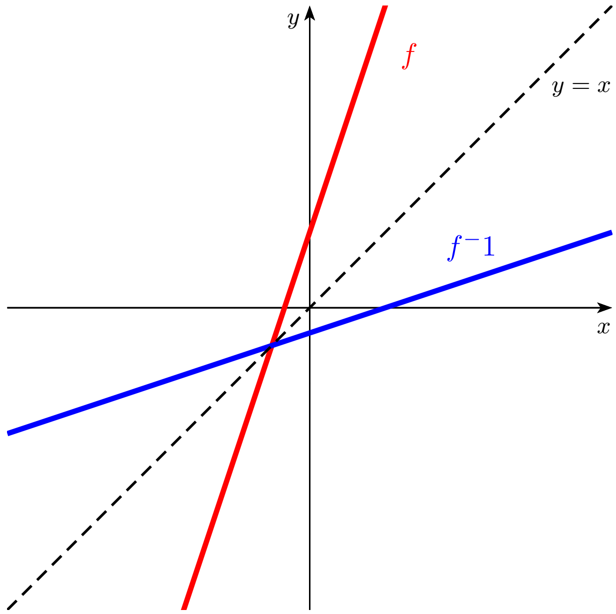

Fig. 9: Graphs of $f(x) = 3x + 1$, $f^{-1}(x) = \frac{x-1}{3}$ (exercise 13)

14) The domain of the function $f(x) = \frac{2x+1}{x-1}$ is $\text{dom}(f) = \mathbb{R} \setminus \{1\}$. In order to verify that f is injective, let's suppose $x_1, x_2 \in \text{dom}(f)$ such that $f(x_1) = f(x_2)$, that is

$$\frac{2x_1 + 1}{x_1 - 1} = \frac{2x_2 + 1}{x_2 - 1}.$$

It follows that

$$(2x_1 + 1)(x_2 - 1) = (2x_2 + 1)(x_1 - 1).$$

Doing the math, it results $3x_2 = 3x_1$ and so $x_2 = x_1$: the injectivity has been proven. In order to determine the inverse function f^{-1}, it is possible to impose $f^{-1}(x) = y$ if and only if $f(y) = x$, that is

$$\frac{2y + 1}{y - 1} = x$$

Solving for y as a function of x, it results

$$y = f^{-1}(x) = \frac{x + 1}{x - 2}$$

The graph of f is shown in Figure 10.

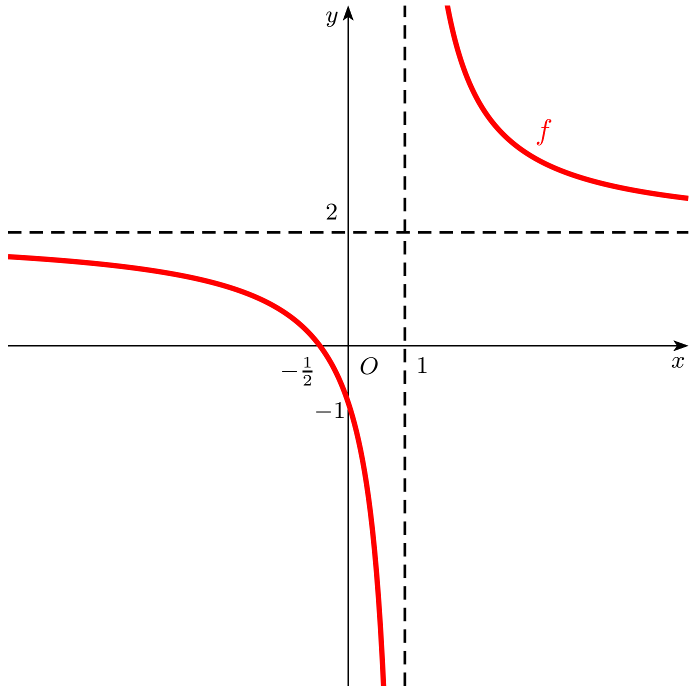

Fig. 10: Graph of $f(x) = \frac{2x+1}{x-1}$ (exercise 14)

15) The graph of the function

$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x < 1 \\ 5 - 2x & \text{if } 1 \leq x \leq 2 \end{cases}$$

is shown in Figure 11.

The function f is strictly increasing on $[0, 1)$; indeed, considering $0 \leq x_1 < x_2$, if the inequalities are multiplied by x_1, it follows that $0 \leq x_1^2 < x_1 \cdot x_2$; whereas, if the inequalities are multiplied by x_2, it follows that $0 \leq x_1 \cdot x_2 < x_2^2$. Overall, it results that $0 \leq x_1^2 < x_2^2$. Moreover, f is strictly decreasing on $[1, 2]$, since $0 \leq x_1 < x_2$ implies that $-x_2 < -x_1$ whence it follows that $-2x_2 < -2x_1$ and that $5 - 2x_2 < 5 - 2x_1$: it means that $f(x_1) > f(x_2)$. Obviously, f is not monotone on $[0, 2]$. Nevertheless, f is injective in its domain. In order to verify that, let's suppose that $f(x_1) = f(x_2)$. If either $x_1, x_2 \in [0, 1)$ or $x_1, x_2 \in [1, 2]$, it is easy to obtain that $x_1 = x_2$, thanks to the strict monotony of f on those intervals. On the other hand, supposing, for instance, $x_1 \in [0, 1)$ and $x_2 \in [1, 2]$, it follows that $f(x_1) = f(x_2)$ (i.e. $x_1^2 = 5 - 2x_2$) cannot occur, since $x_1^2 \in [0, 1)$, and $5 - 2x_2 \in [1, 3]$. Indeed $f([0, 1) = [0, 1)$ and $f([1, 2]) = [1, 3]$. Moreover, it is possible to observe that $\mathrm{Im}\,(f) = [0, 3]$.

In order to determine the inverse function f^{-1}, the restrictions of f to the intervals $[0, 1)$ and $[1, 2]$ have to be separately inverted; from now on, for the sake of clarity, the two restrictions are called f_1 and f_2.

For any $x \in [0,1)$, it results that $f^{-1}(x) = f_1^{-1}(x) = y$ if and only if $f_1(y) = x$, with $y \in [0,1[$, that is $y^2 = x$; solving for y, it follows that $f^{-1}(x) = f_1^{-1}(x) = \sqrt{x}$.
For any $x \in [1,3]$ it results that $f^{-1}(x) = f_2^{-1}(x) = y$ if and only if $f_2(y) = x$, with $y \in [1,2]$, that is $5 - 2y = x$; solving for y, it follows that $f^{-1}(x) = f_2^{-1}(x) = \frac{5-x}{2}$. Therefore,

$$f^{-1}(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0,1) \\ \frac{5-x}{2} & \text{if } x \in [1,3] \end{cases}$$

The graph of f is shown in Figure 11.

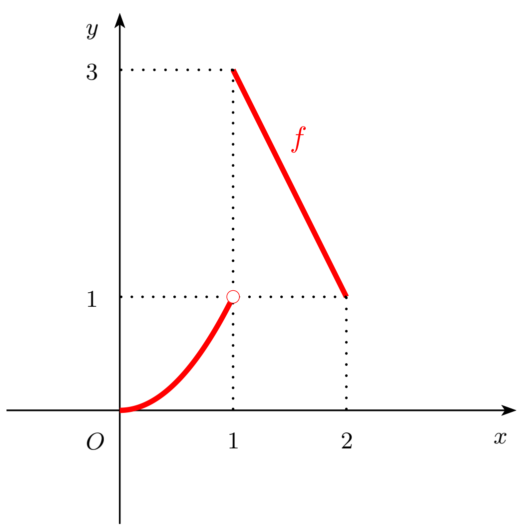

Fig. 11: Graph of f (exercise 15)

16) Let's consider the function $f(x) = x^2$ and its restrictions $f_1 = f_{|\mathbb{R}^-}$ and $f_2 = f_{|\mathbb{R}^+}$. In order to verify that f_2 is strictly increasing on $\mathbb{R}^+$ it is possible to proceed as in the exercise 15. Similarly, it is possible to verify that f_1 is strictly decreasing on $\mathbb{R}^+$. Moreover, $\mathrm{Im}\,(f_1) = \mathrm{Im}\,(f_2) = \mathbb{R}^+$.
Let's calculate the inverse functions g_1 and g_2. By definition, for any $x \in \mathbb{R}^+$, it follows that $g_1(x) = y$ if and only if $f_1(y) = x$, with $y \in \mathbb{R}^-$, that is $y^2 = x$, $y \leq 0$. Solving for y as a function of x, it results $y = g_1(x) = -\sqrt{x}$. Similarly, it is possible to determine $y = g_2(x) = \sqrt{x}$, for any $x \in \mathbb{R}^+$. The graphs of the inverse functions are symmetrical to those of the given function, about the straight line $y = x$. They are shown in Figure 12.

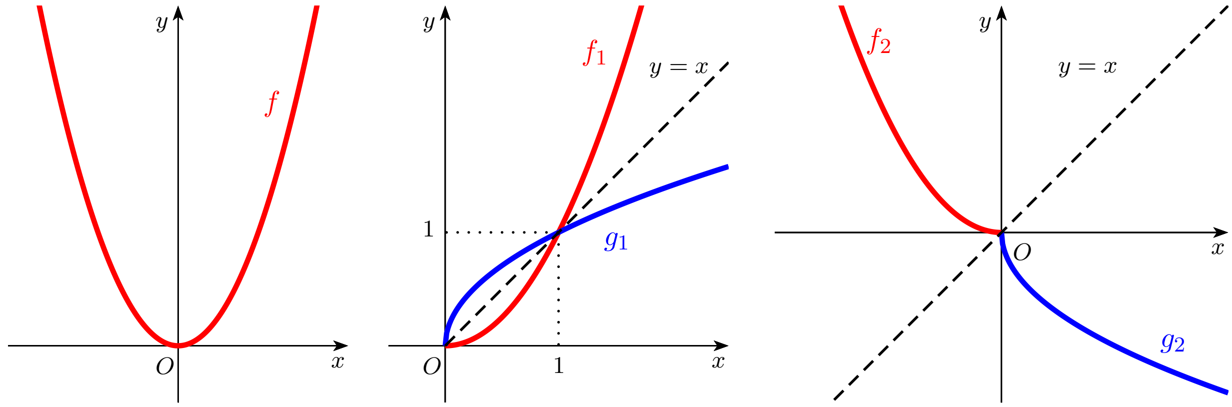

Fig. 12: Graphs of $f(x) = x^2$, f_1, g_1, f_2, g_2 (exercise 16)

17) It is possible to explicitly re-write the function $f(x) = x|x-2| + 2x$ as

$$f(x) = \begin{cases} -x^2 + 4x & \text{se } x < 2 \\ x^2 & \text{se } x \geq 2 \end{cases}$$

The graph of f is shown in Figure 13.

In order to verify whether f is invertible, it is possible to observe that the restriction $f_1(x) = -x^2 + 4x = 4 - (x-2)^2$ of f to $(-\infty, 2)$ is strictly increasing. Similarly, the restriction $f_2(x) = x^2$ of f to $[2, +\infty)$ is strictly increasing. Moreover $\text{Im}\,(f_1) = (-\infty, 4)$ and $\text{Im}\,(f_2) = [4, +\infty)$. Hence, f is injective, surjective and strictly increasing.

In order to determine the inverse function f^{-1}, the two restrictions are separately managed. For any $x \in (-\infty, 4)$, it has to be satisfied that $f_1^{-1}(x) = y$ if and only if $f_1(y) = x$, with $y \in (-\infty, 2)$, that is $-y^2 + 4y = x$, $y \in (-\infty, 2)$. Solving the second order equation for y, the two solutions $y_{1,2} = 2 \pm \sqrt{4-x}$ are found; however, only $y = 2 - \sqrt{4-x}$ satisfies the condition $y \in (-\infty, 2)$. Hence, it follows that

$$f_1^{-1} : (-\infty, 4) \to (-\infty, 2) \quad , \quad y = f_1^{-1}(x) = 2 - \sqrt{4-x}$$

In order to determine the function $f_2^{-1}(x)$, for any $x \in [4, +\infty)$, it has to be satisfied that $f_2^{-1}(x) = y$ if and only if $f_2(y) = x$, with $y \in [2, +\infty)$, that is $y^2 = x$, $y \in [2, +\infty)$. It easily results $y = f_2^{-1}(x) = \sqrt{x}$, for any $x \in [4, +\infty)$. Hence,

$$f_2^{-1} : [4, +\infty) \to [2, +\infty) \quad , \quad y = f_2^{-1}(x) = \sqrt{x}$$

Overall, it is

$$f^{-1}(x) = \begin{cases} 2 - \sqrt{4-x} & \text{se } x \in (-\infty, 4) \\ \\ \sqrt{x} & \text{se } x \in [4, +\infty) \end{cases}$$

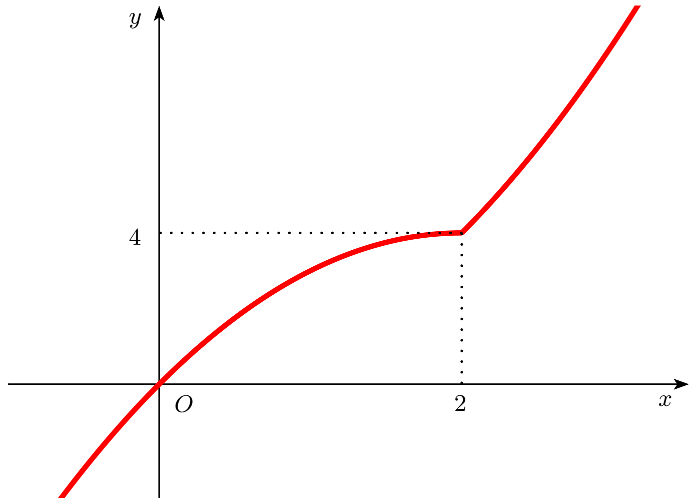

Fig. 13: Graph of $f(x) = x|x-2| + 2x$ (exercise 17)

18) It is possible to explicitly re-write the function $f(x) = (x-1)|x+2|$ as

$$f(x) = \begin{cases} -x^2 - x + 2 & \text{se } x < -2 \\ \\ x^2 + x - 2 & \text{se } x \geq -2 \end{cases}$$

let's consider the two functions $f_1(x) = -x^2 - x + 2$ and $f_2(x) = x^2 + x - 2$. It emerges that

$$f_1(x) = \frac{9}{4} - \left(x + \frac{1}{2}\right)^2.$$

It follows that f_1 is strictly increasing in $(-\infty, -1/2]$ and strictly decreasing in $[-1/2, +\infty)$. Since $f_2 = -f_1$, obviously f_2 is strictly decreasing in $(-\infty, -1/2]$ and strictly increasing in $[-1/2, +\infty)$.

Considering f, it results that it is strictly decreasing on $[-2, -1/2]$ and strictly increasing on $(-\infty, -2]$ and on $[-1/2, +\infty)$. Hence, it is possible to determine 3 invertible restrictions, which are called: $F_1 = f_{|(-\infty,-2]}$, $F_2 = f_{|[-2,-1/2]}$, $F_3 = f_{|[-1/2,+\infty)}$. The corresponding ranges are

$$\operatorname{im}(F_1) = f((-\infty, -2]) = (-\infty, 0]$$
$$\operatorname{im}(F_2) = f([-2, -1/2]) = \left[-\frac{9}{4}, 0\right]$$
$$\operatorname{im}(F_3) = f([-1/2, +\infty)) = \left[-\frac{9}{4}, +\infty\right)$$

To compute the corresponding inverse functions, bearing in mind that $f^{-1}(x) = y$ if and only if $f(y) = x$, it follows that

$$F_1^{-1} : (-\infty, 0] \to (-\infty, -2], \quad F_1^{-1}(x) = -\frac{1}{2} - \sqrt{\frac{9}{4} - x}$$

$$F_2^{-1} : \left[-\frac{9}{4}, 0\right] \to [-2, -1/2], \quad F_2^{-1}(x) = -\frac{1}{2} - \sqrt{x + \frac{9}{4}}$$

$$F_3^{-1} : \left[-\frac{9}{4}, +\infty\right) \to [-1/2, +\infty), \quad F_3^{-1}(x) = -\frac{1}{2} + \sqrt{x + \frac{9}{4}}$$

19) In order to verify that the function $f(x) = x^4$ is bounded on $[-2, 3]$, it is possible to observe that $x^4 \geq 0$ for any $x \in \mathbb{R}$; it means that f is bounded below on $\mathbb{R}$. Moreover, f is strictly decreasing on $\mathbb{R}^-$ and strictly increasing on $\mathbb{R}^+$. Consequently, for $x \in [-2, 0]$, it results that $0 \leq x^4 \leq 16$; for $x \in [0, 3]$, it results that $0 \leq x^4 \leq 81$: overall, for $x \in [-2, 3]$, it follows that $0 \leq x^4 \leq 81$. To verify that f is unbounded above on $\mathbb{R}$, it is possible to observe that $\mathrm{Im}\,(f) = [0, +\infty)$, since for any $y \in [0, +\infty)$ the equation $f(x) = y$ (i.e. $x^4 = y$) has at least one solution; in general, this equation has two solutions $x = \pm\sqrt[4]{y}$.

20) The domain of the function $f(x) = \left|\frac{1}{x} + 1\right|$ is $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$. To sketch its graph is useful to explicitly re-write the function, that is

$$f(x) = \begin{cases} \frac{1}{x} + 1 & \text{if } x \leq -1 \text{ or } x > 0 \\ \\ -\frac{1}{x} - 1 & \text{if } -1 < x < 0 \end{cases}$$

The graph of $y = \frac{1}{x} + 1$ is a rectangular hyperbola. The graph of f is shown in Figure 14.
In order to verify that f is bounded below on its domain, it is sufficient to observe that $f(x) \geq 0$ for any $x \in \mathrm{dom}(f)$. To determine $\min\{f(x) : x \in \mathrm{dom}(f)\}$ it is possible to observe that $f(-1) = 0 \leq f(x)$ for any $x \in \mathrm{dom}(f)$; hence, $x = -1$ is the absolute minimum of f.
In order to verify that f is unbounded above on its domain, it is sufficient to notice that $\mathrm{Im}\,(f) = [0, +\infty)$, since for any $y \geq 0$, it will exist $x \in \mathrm{dom}(f)$ such that $f(x) = y$.
Finally, it is possible to observe that $\sup\{f(x) : x \in (-\infty, -1]\} = 1$, since for $x \in (-\infty, -1]$, it results $\frac{1}{x} < 0$ and so $f(x) = \frac{1}{x} + 1 < 1$: it means that 1 is an upper bound (majorant) of f on the set $(-\infty, -1]$. To verify whether 1 is the upper bound it is possible to observe that $f((-\infty, -1]) = [0, 1)$: indeed, for any $y \in [0, 1)$, the equation $f(x) = y$ (i.e. $\frac{1}{x} + 1 = y$) has at least a solution with $x \in (-\infty, -1]$. The solution of that equation is $x = \frac{1}{y-1}$.Finally, it is possible to state that 1 is not a maximum of f on the interval $(-\infty, -1]$, because the equation $\frac{1}{x} + 1 = 1$ has no solution.

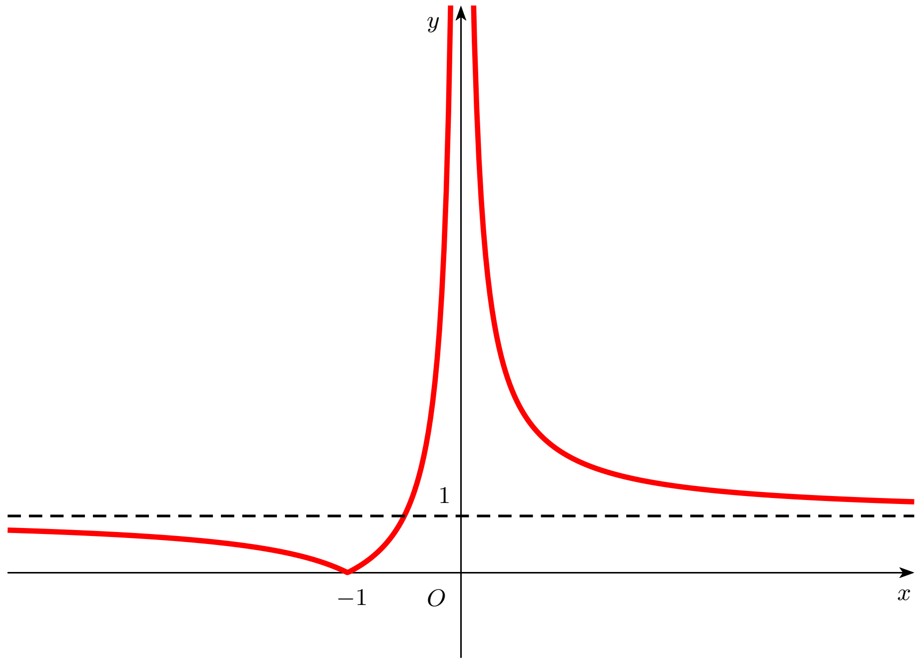

Fig. 14: Graph of $f(x) = \left|\frac{1}{x} + 1\right|$ (exercise 20)

21) The domain of the functions $f(x) = x - \sqrt{x}$ and $g(x) = \sqrt{x-2}$ are respectively

$$\mathrm{dom}(f) = [0, +\infty[\quad , \quad \mathrm{dom}(g) = [2, +\infty[$$

The domain of $g \circ f$ is $\mathrm{dom}(g \circ f) = \{x \in \mathrm{dom}(f) : f(x) \in \mathrm{dom}(g)\} = \{x \in [0, +\infty) : x - \sqrt{x} \geq 2\} = [4, +\infty)$ by definition. Similarly, the domain of $f \circ g$ is $\mathrm{dom}(f \circ g) = \{x \in \mathrm{dom}(g) : g(x) \in \mathrm{dom}(f)\} = \{x \in [2, +\infty) : \sqrt{x-2} \geq 0\} = [2, +\infty)$. Moreover, $(g \circ f)(x) = \sqrt{x - \sqrt{x} - 2}$ and $(f \circ g)(x) = \sqrt{x-2} - \sqrt[4]{x-2}$.

22) Both the functions $f(x) = x^2 + 3x$ and $g(x) = |x|$ are defined on any point of $\mathbb{R}$. Moreover, $\mathrm{dom}(g \circ f) = \mathbb{R}$ and $\mathrm{dom}(f \circ g) = \mathbb{R}$. The explicit expressions of the composite functions are $(g \circ f)(x) = |x^2 + 3x|$ and $(f \circ g)(x) = x^2 + 3|x|$.
The graph of $g \circ f$ can be derived from the graph of $y = x^2 + 3x$ by mirroring the points of y having a negative ordinate value about the $x - axis$.
The graph of $f \circ g$ can be derived from the graph of $y = x^2 + 3x$, considering that the two graphs coincide for $x \geq 0$; then, it is symmetrical about the $y - axis$.
The graphs of $g \circ f$ and $f \circ g$ are shown in Figure 15.

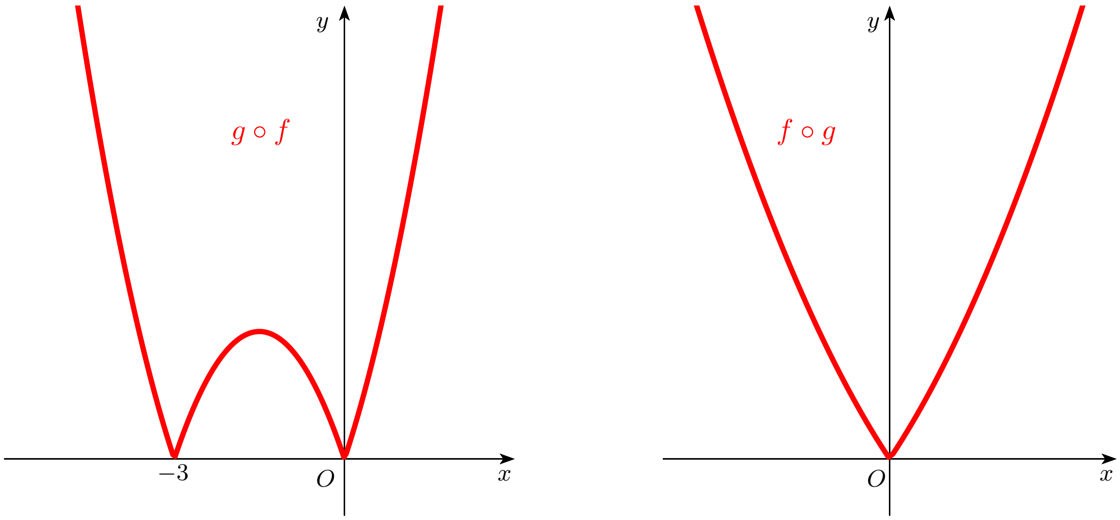

Fig. 15: Graph of $g \circ f$ and $f \circ g$ (exercise 22)

23) Having the functions $f(x) = 1 + x^2$ and $g(x) = \frac{1}{x}$, it is easy to obtain that $\mathrm{dom}(f) = \mathbb{R}$, $\mathrm{dom}(g) = \mathbb{R} \setminus \{0\}$, $\mathrm{im}(f) = [1, +\infty)$, $\mathrm{im}(g) = \mathbb{R} \setminus \{0\}$.

The function f is strictly decreasing on $\mathbb{R}^-$ and strictly increasing on $\mathbb{R}^+$.

The function g is strictly decreasing on $(-\infty, 0)$ and on $(0, +\infty)$.

The domain of the composite function $g \circ f$ is $\mathrm{dom}(g \circ f) = \mathbb{R}$; the range is $\mathrm{im}(g \circ f) = g([1, +\infty)) = (0, 1]$.

Finally, using the properties about the composition of monotonic functions, it results that $g \circ f$ is strictly increasing on $\mathbb{R}^-$ and strictly decreasing on $\mathbb{R}^+$.

The graph of $g \circ f$ is shown in Figure 16.

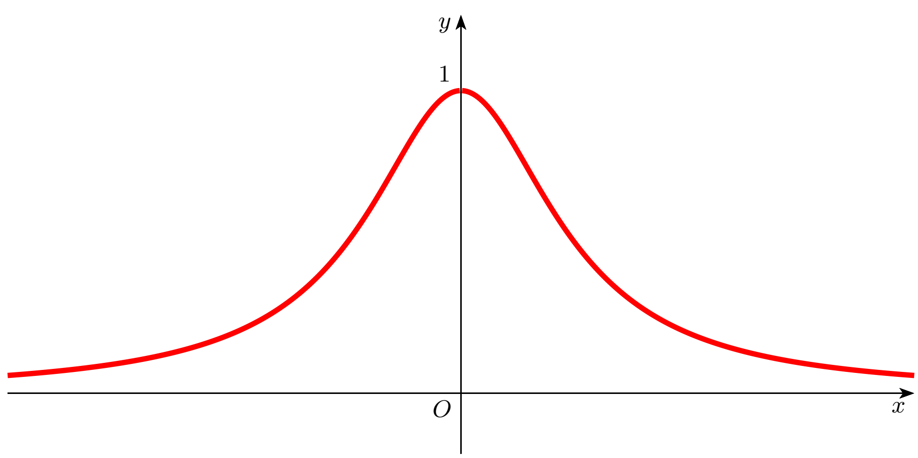

Fig. 16: Graph of $g \circ f$ (exercise 23)

24) Having the functions $f(x) = -\frac{1}{x}$ and $g(x) = e^x$, it is easy to obtain that $\mathrm{dom}(f) = \mathbb{R} \setminus \{0\}$, $\mathrm{dom}(g) = \mathbb{R}$, $\mathrm{im}(f) = \mathbb{R} \setminus \{0\}$, $\mathrm{im}(g) = (0, +\infty)$.

    The function f is strictly decreasing on $(-\infty, 0)$ and on $(0, +\infty)$.

    The function g is strictly decreasing on $\mathbb{R}$.

    The domain of the composite function $g \circ f$ is $\mathrm{dom}(g \circ f) = \mathbb{R} \setminus \{0\}$; the range is $\mathrm{im}(g \circ f) = g(\mathbb{R} \setminus \{0\}) = (0, +\infty) \setminus \{1\}$.

    Finally, using the properties about the composition of monotonic functions, it results that $g \circ f$ is strictly increasing on $(0, 1)$ and on $(1, +\infty)$.

    The graph of $g \circ f$ is shown in Figure 17.

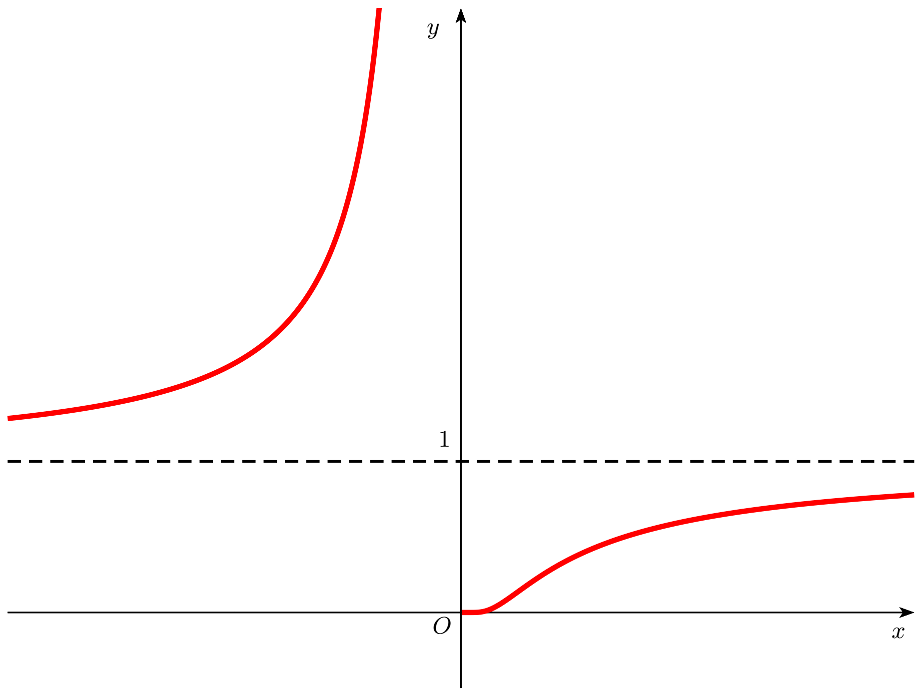

Fig. 17: Graph of $g \circ f$ (exercise 24)