INTEGRAL CALCULUS

1. Let f be an even function, locally integrable in $\mathbb{R}$.

   Prove that the integral function $F(x) = \displaystyle\int_0^x f(t)\ dt$ is odd.

2. Let f be an odd function, locally integrable in $\mathbb{R}$.

   Prove that $F(x) = \displaystyle\int_{x_0}^x f(t)\ dt$ is even for all $x_0 \in \mathbb{R}$.

3. Let f be locally integrable in $\mathbb{R}$ and such that, for all $x \in \mathbb{R}$ with $|x| > 10$, $f(x) = 0$.

   Find out if the function $F(x) = \displaystyle\int_{-3}^x f(t)\ dt$ is bounded or unbounded.

4. Let f be locally integrable in $\mathbb{R}$, such that $f(x) \geq 1$. Consider the integral function $F(x) = \displaystyle\int_2^x f(t)\ dt$.

   Verify that $\displaystyle\lim_{x\to+\infty} F(x) = +\infty$ and $\displaystyle\lim_{x\to-\infty} F(x) = -\infty$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be periodic, with period 2, such that $f(x) = -\operatorname{sgn}\left(x - \frac{1}{2}\right)$, for all $x \in \left[-\frac{1}{2}, \frac{3}{2}\right)$.

   - Sketch a graph of $F(x) = \displaystyle\int_0^x f(t)\ dt$.

   - Compute, if it exists, $\displaystyle\lim_{x\to+\infty} F(x)$.

6. Let g be differentiable in $\mathbb{R}$ and f be continuous in $\mathbb{R}$. Let $G(x) = \displaystyle\int_0^{g(x)} f(t)\ dt$.

   Prove that $G(x)$ is differentiable on $\mathbb{R}$ and compute its derivative, using the chain rule.

7. Let f be continuous in $[a, b]$, such that $f(x) \geq 0$, for all $x \in [a, b]$.

   Prove that $\displaystyle\int_a^b f(x)\, dx = 0$ if and only if $f(x) = 0$ for all $x \in [a, b]$.

   [*Hint: argue by contradiction.*]