MATHEMATICAL ANALYSIS I TUTORING
11^{TH} WEEK (24/05/2016)

IMPROPER INTEGRALS II - DIFFERENTIAL EQUATIONS I

PROPOSED EXERCISES - SOLUTIONS

1. Discuss the convergence properties of the following improper integrals

(a) $\boxed{\displaystyle\int_0^1 \frac{1}{\sqrt[3]{1-x^4}}\,\mathrm{d}x}$

The domain of the function is $D = (-\infty,-1)\cup(-1,1)\cup(1,+\infty)$. In the interval $[0,1)$ the inner function is continuous; thus we have to study only a left neighborhood of 1.
Study the asymptotic behavior of the inner function for $x \to 1^-$.

$$\begin{aligned}\frac{1}{\sqrt[3]{1-x^4}} &= \frac{1}{\sqrt[3]{(1-x)(1+x)(1+x^2)}}\\ &\sim \frac{1}{\sqrt[3]{4(1-x)}}, \quad \text{per } x\to 1\end{aligned}$$

Since $\displaystyle\frac{1}{\sqrt[3]{4}}\int_0^1 \frac{1}{(1-x)^{1/3}}\,\mathrm{d}x$ is convergent, the given integral is convergent too, by asymptotic comparison criteria.

(b) $\boxed{\displaystyle\int_0^\pi \frac{x-\pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x}$

The denominator in $[0,\pi]$ is zero in $x=0$, in $x=\frac{\pi}{2}$ and in $x=\pi$.
We have to study the improper integral in $x=0$, in $x=\frac{\pi}{2}$ and in $x=\pi$, hence we split the integral in 4 summands, in such a way that in each summand we only have to study the behavior of $f(x)=\dfrac{x-\pi/2}{\cos x\sqrt{\sin x}}$ in one boundary point of the integration interval:

$$\int_0^\pi f(x)\,\mathrm{d}x = \int_0^{\pi/4} f(x)\,\mathrm{d}x + \int_{\pi/4}^{\pi/2} f(x)\,\mathrm{d}x + \int_{\pi/2}^{3\pi/4} f(x)\,\mathrm{d}x + \int_{3\pi/4}^{\pi} f(x)\,\mathrm{d}x = I_1+I_2+I_3+I_4$$

Study the first integral: $I_1 = \displaystyle\int_0^{\pi/4}\frac{x-\pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x$
As $x\to 0$, it holds

$$\frac{x-\pi/2}{\cos x\sqrt{\sin x}} \sim \frac{-\pi/2}{(1-\frac{1}{2}x^2)\sqrt{x}} = -\frac{\pi/2}{\sqrt{x}} \asymp \frac{1}{\sqrt{x}}$$

Since $\displaystyle\int_0^{\pi/4}\frac{1}{\sqrt{x}}\,\mathrm{d}x$ converges, by asymptotic comparison, also I_1 is convergent.
Study the second integral:

$$I_2 = \int_{\pi/4}^{\pi/2}\frac{x-\pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x$$

Change variable $x-\dfrac{\pi}{2}=t$, i.e. $x=\dfrac{\pi}{2}+t$, $\cos x = -\sin t$, $\sin x = \cos t$. We have to study the behavior of the function as $t\to 0$

$$g(t) = \frac{t}{-\sin t\sqrt{\cos t}} \sim \frac{t}{-t} = -1$$

Thus it is bounded in $x=\frac{\pi}{2}$, and the integral I_2 converges.
Also I_3 converges, because the behavior in $x=\dfrac{\pi}{2}$ is the same as before.

Study the last integral, i.e. he behavior of the function $f(x)$ for $x\to\pi^-$:

$$I_4 = \int_{3\pi/4}^{\pi}\frac{x-\pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x$$

Change variable $x - \pi = t$, i.e. $x = \pi + t$, $\cos x = -\cos t$, $\sin x = -\sin t$. We have to study the behavior of the function as $t \to 0$

$$g(t) = \frac{\frac{\pi}{2} + t}{-\cos t\sqrt{-\sin t}} \sim \frac{\frac{\pi}{2}}{-\sqrt{-t}} \asymp \frac{1}{\sqrt{-t}}$$

Since $\displaystyle\int_{-1}^{0} \frac{1}{\sqrt{-t}}\,\mathrm{d}t$ converges, also I_4 converges.
In conclusion: the given integral

$$\int_0^{\pi} \frac{x - \pi/2}{\cos x\sqrt{\sin x}}\,\mathrm{d}x$$

is convergent.

(c) $\boxed{\displaystyle\int_0^{+\infty} x \sin\frac{1}{x}\,\mathrm{d}x}$
The domain is $(0, +\infty)$, hence we split the integral as

$$\int_0^{+\infty} x\,\sin\frac{1}{x}\,\mathrm{d}x = \int_0^{1} x\,\sin\frac{1}{x}\,\mathrm{d}x + \int_1^{+\infty} x\,\sin\frac{1}{x}\,\mathrm{d}x = I_1 + I_2.$$

We start from I_1; since $\displaystyle\lim_{x\to 0} x\,\sin\frac{1}{x} = 0$, the function is bounded in a neighborhood of $x = 0$ and continuous on $(0, 1]$ hence $\displaystyle\int_0^{1} x\,\sin\frac{1}{x}\,\mathrm{d}x$ is finite.

For I_2, we apply the asymptotic comparison for $x \to +\infty$, where $\displaystyle\sin\frac{1}{x} \sim \frac{1}{x}$ hence $\displaystyle x\sin\frac{1}{x} \sim x\,\frac{1}{x} = 1$; since $\displaystyle\int_1^{+\infty} 1\,\mathrm{d}x$ diverges, also I_2 diverges and so does the integral we started with.

(d) $\boxed{\displaystyle\int_0^{+\infty} \frac{\log(x+1)}{\sqrt[3]{x^2}}\,\mathrm{d}x}$

The domain is $(0, +\infty)$;hence we split the integral in 2 parts:

$$\int_0^{+\infty} \frac{\log(x+1)}{\sqrt[3]{x^2}}\,\mathrm{d}x = \int_0^{2} \frac{\log(x+1)}{\sqrt[3]{x^2}}\,\mathrm{d}x + \int_2^{+\infty} \frac{\log(x+1)}{\sqrt[3]{x^2}}\,\mathrm{d}x = I_1 + I_2$$

Consider I_1, and apply the asymptotic comparison; find the behavior of the function as $x \to 0$:

$$\frac{\log(x+1)}{\sqrt[3]{x^2}} \sim \frac{x}{x^{2/3}} = \frac{1}{x^{-1/3}}$$

Since $\displaystyle\int_0^{2} \frac{1}{x^{-1/3}}\,\mathrm{d}x$ converges, also I_1 converges.

Consider I_2, and apply the comparison criteria: as $x \geq 2$, it holds $\displaystyle\frac{\log(x+1)}{\sqrt[3]{x^2}} \geq \frac{1}{\sqrt[3]{x^2}}$ and $\displaystyle\int_2^{+\infty} \frac{1}{\sqrt[3]{x^2}}\,\mathrm{d}x$ diverges, then also I_2 diverges.
Therefore the given integral diverges.

(e) $\boxed{\displaystyle\int_0^{1} \frac{\sqrt{x - x^2}}{\sin \pi x}\,\mathrm{d}x}$

In the interval $[0, 1]$ the denominator is zero in both the boundary points; thus we split the integral as

$$\int_0^{1} \frac{\sqrt{x - x^2}}{\sin \pi x}\,\mathrm{d}x = \int_0^{1/2} \frac{\sqrt{x - x^2}}{\sin \pi x}\,\mathrm{d}x + \int_{1/2}^{1} \frac{\sqrt{x - x^2}}{\sin \pi x}\,\mathrm{d}x = I_1 + I_2$$

Start with I_1; apply the asymptotic comparison as $x \to 0^+$

$$\frac{\sqrt{x-x^2}}{\sin \pi x} \quad = \quad \frac{x^{1/2}\sqrt{1-x}}{\sin \pi x} \sim \frac{x^{1/2}}{\pi x} \asymp \frac{1}{x^{1/2}}$$

Since $\displaystyle\int_0^{1/2} \frac{1}{x^{1/2}}\, \mathrm{d}x$ converges, also I_1 will converge.

For I_2, we study the behavior as $x \to 1^-$; by substitution $x-1=t$, that is $x=t+1$, $\sin(x\pi) = -\sin \pi t$, as $t \to 0^-$, we have

$$\frac{\sqrt{(1+t)-(1+t)^2}}{-\sin \pi t} = \frac{\sqrt{-t}\sqrt{1+t}}{-\sin \pi t} \sim \frac{(-t)^{1/2}}{-\pi t} \asymp \frac{1}{(-t)^{1/2}}$$

The integral $\displaystyle\int_{-1/2}^{0} \frac{1}{(-t)^{1/2}}\, \mathrm{d}t$ converges, hence I_2 is convergent; finally the initial integral converges.

(f) $\boxed{\displaystyle\int_0^{+\infty} \frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}}\, \mathrm{d}x}$

The domain is $D=(0,+\infty)$. Study the integral at 0 and $+\infty$. Split the integral as follows

$$\int_0^1 \frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}}\, \mathrm{d}x \ + \int_1^{+\infty} \frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}}\, \mathrm{d}x \ = I_1 + I_2$$

For I_1, study the inner function as $x \to 0$

$$\frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}} \quad \sim \quad \frac{1}{\sqrt{x}}, \quad \text{per } x \to 0$$

Since $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, \mathrm{d}x$ converges, I_1 is also convergent by asymptotic comparison.

For I_2, study the inner function as $x \to +\infty$

$$\frac{\sqrt{x+1}}{(x^2+1)\sqrt{x}} \quad \sim \quad \frac{1}{x^2}, \quad \text{per } x \to +\infty$$

Since $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\, \mathrm{d}x$ converges, I_2 is also convergent by asymptotic comparison.

2. Let $f:(0,1] \to \mathbb{R}$ locally integrable and $f(x) \sim \dfrac{1}{x}$ for $x \to 0$. Prove that $\displaystyle\int_0^1 f(x)e^{-x}dx$ is divergent.

   We study the integrand for $x \to 0^+$, and apply the asymptotic comparison. For $x \to 0^+$ $f(x) \sim \dfrac{1}{x}$, we have $f(x)e^{-x} \sim \dfrac{1}{x}$; since $\displaystyle\int_0^1 \frac{1}{x}\, \mathrm{d}x$ diverges, so does the proposed integral.

3. Let $f:(0,1] \to \mathbb{R}$ infinite of order α for $x \to 0$ w.r.t. the standard sample. Determine for which values of $\beta \in \mathbb{R}$ the integral of $\displaystyle\int_0^1 \frac{f(x)}{x^{2\beta}}dx$ is convergent.

   We study the integrand for $x \to 0^+$; since $f(x)$ is infinite of order α for $x \to 0$ w.r.t. $\dfrac{1}{x}$ for $x \to 0^+$, it means that $f(x) \sim \dfrac{1}{x^\alpha}$;by asymptotic comparison, for $x \to 0$, $\dfrac{f(x)}{x^{2\beta}} \sim \dfrac{1}{x^\alpha x^{2\beta}} = \dfrac{1}{x^{\alpha+2\beta}}$; since $\displaystyle\int_0^1 \frac{1}{x^{\alpha+2\beta}}\, \mathrm{d}x$ converges if and only if $\alpha+2\beta<1$, then $\displaystyle\int_0^1 \frac{f(x)}{x^{2\beta}}\, \mathrm{d}x$ converges if $\beta < \dfrac{1-\alpha}{2}$.

4. Determine the positive number n such that the following improper integral is convergent:

$$\int_1^{+\infty} \frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}}dx$$

The domain is $D = (1, +\infty)$. We have to study it as $x \to 1$ and as $x \to +\infty$. Split the integral as follows

$$\int_1^{+\infty} \frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}} dx = \int_1^{2} \frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}} dx + \int_2^{+\infty} \frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}} dx = I_1 + I_2$$

For I_1:

$$\frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}} \quad \sim \quad \frac{1}{(2)^{n/3}(x-1)^{3/n}} \asymp \frac{1}{(x-1)^{3/n}}, \quad \text{per } x \to 0$$

For I_2:

$$\frac{x^2}{(1+x)^{n/3}(x-1)^{3/n}} \quad \sim \quad \frac{x^2}{x^{n/3}x^{3/n}} dx = \frac{1}{x^{\frac{n}{3}+\frac{3}{n}-2}} \quad \text{for } x \to +\infty$$

The improper integral I_1 converges for $\frac{3}{n} < 1$, i.e. $n > 3$.

The improper integral I_2 converges for $\frac{n}{3} + \frac{3}{n} - 2 > 1$, i.e. $n > \frac{9+\sqrt{69}}{2}$.
In conclusion, the improper integral converges if and only if $n \geq 9$.

5. Study the convergence of the improper integrals, using Taylor expansions for $f(x)$:
a) $f(x)$ continuous in $[4,5]$, $f(x) = -\frac{\pi^2}{128}(x-4)^2 + o((x-4)^2)$; study the convergence of $\int_4^5 \frac{f(x)}{(x-4)^{7/2}}\, \mathrm{d}x$

Study the asymptotic behavior of f as $x \to 4^+$

$$\begin{aligned} \frac{f(x)}{(x-4)^{7/2}} &= \frac{-\frac{\pi^2}{128}(x-4)^2 + o((x-4)^2)}{(x-4)^{7/2}} \\ &\sim \frac{-\frac{\pi^2}{128}}{(x-4)^{7/2-4}} \asymp \frac{1}{(x-4)^{-1/2}}, \quad \text{for } x \to 4 \end{aligned}$$

Since $\int_4^5 \frac{1}{(x-4)^{-1/2}}\, \mathrm{d}x$ is convergent, then the initial integral is also convergent by asymptotic comparison.

b) $f(x) = \sqrt[5]{1+\sin x} - \frac{5}{5-x}$; study the convergence of $\int_0^1 \frac{f(x)}{x^3}\, \mathrm{d}x$
Study the asymptotic behavior of f as $x \to 0$, applying the Mac Laurin expansions:

$$\begin{aligned} \sqrt[5]{1+\sin x} - \frac{5}{5-x} &= \sqrt[5]{1+x+o(x^2)} - \frac{5}{5(1-\frac{x}{5})} \\ &= \sqrt[5]{1+x+o(x^2)} - \left(1 - \left(-\frac{x}{5}\right) + \left(-\frac{x}{5}\right)^2 + o(x^2)\right) \\ &= 1 + \frac{x}{5} - \frac{2}{25}x^2 + o(x^2) - \left(1 + \frac{x}{5} + \frac{x^2}{25} + o(x^2)\right) \\ &= -\frac{2}{25}x^2 - \frac{x^2}{25} + o(x^2) = -\frac{3}{25}x^2 + o(x^2) \end{aligned}$$

Hence, for $x \to 0$

$$\frac{f(x)}{x^3} \quad \sim \quad \frac{-\frac{3}{25}x^2}{x^3} \asymp \frac{1}{x}$$

Since $\int_0^1 \frac{1}{x}\, \mathrm{d}x$ is divergent, then the initial integral is also divergent by asymptotic comparison.

c) $f(x) = (e^x - 1)\log(1+\sin^2 x)$; study the convergence of $\int_0^1 \frac{f(x)}{x^3\sqrt{\tan x}}\, \mathrm{d}x$

$$\begin{aligned} f(x) = (e^x-1)\log(1+\sin^2 x) &= (1+x+o(x)-1)\log(1+x^2+o(x^2)) \\ &= (x+o(x))(x^2+o(x^2)) = x^3 + o(x^3) \end{aligned}$$

Therefore, for $x \to 0$:

$$\frac{f(x)}{x^3\sqrt{\tan x}} \sim \frac{x^3}{x^3\sqrt{x}} = \frac{1}{x^{1/2}}$$

Since $\displaystyle\int_0^1 \frac{1}{x^{1/2}}\,\mathrm{d}x$ is convergent, then the initial integral is also convergent by asymptotic comparison.

6. Classify the following differential equations

Equation	order	separable variables
$y' = 2xy + x^3$		X
$xy' + y = \sin y$	X	
$y' = \dfrac{y+1}{x}$	X	X
$y' + y = 2x$		X
$y' + y = \cos x$		X
$x' = x^2 - 3x + 2$	X	
$(1+64t^2)y' = y\log^2 y^4$	X	
$y' - 9y = 0$	X	X
$\dfrac{t-y}{y'} = 3$		X
$(1+x^3)y' - x^2y = 0$	X	X

7. Given the differential equation $x' = x^2 - 3x - 2$:

   a) find the constant solutions
      $x^2 - 3x + 2 = 0 \Leftrightarrow (x-2)(x-1) = 0 \Leftrightarrow x = 2, x = 1$

   b) find the set of all the solutions
      For $x \neq 2, x \neq 1$:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 - 3x + 2 \quad &\Leftrightarrow \quad \frac{\mathrm{d}x}{(x-2)(x-1)} = \mathrm{d}t \\
&\Leftrightarrow \quad \int \frac{\mathrm{d}x}{(x-2)(x-1)} = \int \mathrm{d}t \\
&\Leftrightarrow \quad \frac{1}{3}\int\left(\frac{1}{x-2} - \frac{1}{x-1}\right)\mathrm{d}x = \int \mathrm{d}t \\
&\Leftrightarrow \quad \log\left|\frac{x-2}{x-1}\right| = 3t + 3c
\end{aligned}$$

If $x < 1 \lor x > 2$: $\log\left(\dfrac{x-2}{x-1}\right) = 3t + 3c$

$$\begin{aligned}
\frac{x-1-1}{x-1} = e^{3t+3c} \quad &\Leftrightarrow \quad 1 - \frac{1}{x-1} = e^{3t+3c} \\
&\Leftrightarrow \quad \frac{1}{x-1} = 1 - e^{3t+3c} \\
&\Leftrightarrow \quad x - 1 = \frac{1}{1-e^{3t+3c}} \\
&\Leftrightarrow \quad x = \frac{1}{1-e^{3t+3c}} + 1 \\
&\Leftrightarrow \quad x = \frac{2-e^{3t+3c}}{1-e^{3t+3c}}
\end{aligned}$$

Finally, if $e^{3c} = k$, it holds

$$x(t) = \frac{2-ke^{3t}}{1-ke^{3t}}, \quad k > 0, \quad x \in (-\infty, 1) \cup (2, +\infty)$$

If $1 < x < 2$: $\log\left(-\dfrac{x-2}{x-1}\right) = 3t + 3c$

$$
\begin{aligned}
-\frac{x-1-1}{x-1} = e^{3t+3c} \quad &\Leftrightarrow \quad -1+\frac{1}{x-1} = e^{3t+3c}\\
&\Leftrightarrow \quad \frac{1}{x-1} = 1+e^{3t+3c}\\
&\Leftrightarrow \quad x-1 = \frac{1}{1+e^{3t+3c}}\\
&\Leftrightarrow \quad x = \frac{1}{1+e^{3t+3c}}+1\\
&\Leftrightarrow \quad x = \frac{2+e^{t+3c}}{1+e^{t+3c}}
\end{aligned}
$$

Finally, if $e^{3c} = k$, it holds

$$x(t) = \frac{2+ke^{3t}}{1+ke^{3t}}, \quad k > 0, \quad x \in (1,2)$$

c) find, if they exist, solutions defined on $\mathbb{R}$
The constant solutions $x = 1$ and $x = 2$ are defined on $\mathbb{R}$.
Moreover, since $k > 0$, also the solutions for $2 < x < 3$, i.e. $x(t) = \dfrac{2+ke^{3t}}{1+ke^{3t}}$, are defined on $\mathbb{R}$.

8. Given the differential equation

$$y' = \frac{x(y^2+2y+10)}{x+2}, \quad x \in (-2,+\infty)$$

say if it has constant solutions and find all the solutions.

Separate the variables and integrate:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{x+2}(y^2+2y+10)\\
\frac{\mathrm{d}y}{(y^2+2y+10)} &= \frac{x}{x+2}\mathrm{d}x\\
\int \frac{\mathrm{d}y}{(y^2+2y+10)} &= \int \frac{x+2-2}{x+2}\mathrm{d}x\\
\frac{1}{3}\arctan\frac{y+1}{3} &= x - 2\log(2+x)+c\\
\arctan\frac{y+1}{3} &= 3x - 6\log(2+x)+3c
\end{aligned}
$$

that is, impose $3c = k$:

$$\frac{y+1}{3} = \tan\Big(3x-6\log(2+x)+k\Big)$$

finally, the solutions are

$$f(x) = 3\tan\Big(3x-6\log(2+x)+k\Big) - 1, \quad k \in \mathbb{R}, \ x \in (-2,+\infty).$$

9. Given the differential equation

$$y' = \frac{y\log^2 y^4}{1+64t^2}$$

a) find the constant solutions
$y = -1$, $y = 1$.

b) find the set of all the solutions
Notice that $\log^2 y^4 = 16\log^2|y|$; thus:

$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= \frac{y\log^2 y^4}{1+64t^2}\\
\frac{\mathrm{d}y}{16y\log^2|y|} &= \frac{1}{1+64t^2}\mathrm{d}t
\end{aligned}
$$

$$\int \frac{\mathrm{d}y}{y\log^2|y|} = \int \frac{16}{1+64t^2}\mathrm{d}t$$
$$-\frac{1}{\log|y|} = 2\arctan(8t) + c$$
$$\log|y| = \frac{-1}{2\arctan(8t)+c}$$
$$|y| = e^{-1/(2\arctan(8t)+c)}$$

Therefore the solutions are:

$$y = \pm 1; \quad y = \pm e^{-1/(2\arctan(8t)+c)}, \quad c \in \mathbb{R}$$

10. Solve the following separable variable differential equations:

(a) $\boxed{(1+x^3)y' - x^2 y = 0, \quad x \in (-1,+\infty)}$

$$(1+x^3)y' - x^2y = 0 \;\Leftrightarrow\; y' = \frac{x^2}{1+x^3}\,y$$

Note the constant solution $y = 0$; find now the other solutions:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{1+x^3}\,y$$
$$\int \frac{\mathrm{d}y}{y} = \int \frac{x^2}{1+x^3}\mathrm{d}x$$
$$\log|y| = \frac{1}{3}\log(1+x^3) + c,\; c \in \mathbb{R}$$

Imposing $c = \log k,\; k > 0$, we get

$$\log|y| = \log\sqrt[3]{1+x^3} + \log k \;\Leftrightarrow\; \log|y| = \log k\sqrt[3]{1+x^3} \;\Leftrightarrow\; |y| = k\sqrt[3]{1+x^3}$$

and thus $h = \pm k,\; h \in \mathbb{R}\setminus\{0\}$:

$$y = h\sqrt[3]{1+x^3},\; h \in \mathbb{R}\setminus\{0\}$$

With $h = 0$ we get the constant solution, therefore the general solution is

$$y = C\sqrt[3]{1+x^3}, \quad C \in \mathbb{R}, \quad x \in (-1,+\infty)$$

(b) $\boxed{xy' = y^2 - 4y + 3, \quad x \in (0,+\infty)}$

Since $x \in (0,+\infty)$, the equation becomes:

$$y' = \frac{1}{x}(y^2 - 4y + 3)$$

and it has separable variables with constant solutions $y = 1$ and $y = 3$.
Separate the variables and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}(y^2 - 4y + 3)$$
$$\int \frac{\mathrm{d}y}{(y-1)(y-3)} = \int \frac{1}{x}\mathrm{d}x$$
$$\frac{1}{2}\int \left(\frac{1}{y-3} - \frac{1}{y-1}\right)\mathrm{d}y = \int \frac{1}{x}\mathrm{d}x$$
$$\log\left|\frac{y-3}{y-1}\right| = 2\log x + c,\; c \in \mathbb{R}$$

With $c = \log k,\; k > 0$, it holds:

$$\log\left|\frac{y-3}{y-1}\right| = \log kx^2 \;\Leftrightarrow\; \left|\frac{y-3}{y-1}\right| = kx^2$$

Suppose $h = \pm k,\ h \in \mathbb{R}\setminus\{0\}$, we find

$$\frac{y-3}{y-1} = hx^2 \;\Leftrightarrow\; y-3 = hx^2y - hx^2 \;\Leftrightarrow\; y(1-hx^2) = 3-hx^2 \;\Leftrightarrow\; y = \frac{3-hx^2}{1-hx^2}$$

For $h = 0$, we get the constant solution $y = 3$, we can then conclude:

$$y = 1;\ y = \frac{3-hx^2}{1-hx^2},\ h \in \mathbb{R},\ x \in (0,+\infty)$$

(c) $\boxed{y' = 2x\sqrt{1-y^2}}$

The constant solutions are $y = 1$ and $y = -1$. Separate the variables and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x\sqrt{1-y^2}$$

$$\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} = \int 2x\mathrm{d}x$$

$$\arcsin y = x^2 + c$$

$$y = \sin\left(x^2 + c\right)$$

Thus the general integral is

$$y = 1;\ y = -1;\quad y = \sin(x^2 + c),\ c \in \mathbb{R}$$

(d) $\boxed{y' = -\frac{\log^2 x}{2xy}, \quad x \in (0,+\infty)}$

Since $\frac{1}{y}$ is never zero, there are no constant solutions.
Separate the variables and integrate:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\log^2 x}{2xy}$$

$$\int 2y\mathrm{d}y = -\int \frac{\log^2 x}{x}\mathrm{d}x$$

$$y^2 = -\frac{\log^3 x}{3} + c$$

$$y = \pm\sqrt{-\frac{\log^3 x}{3} + c}$$

Therefore the solution is

$$f(x) = \pm\sqrt{-\frac{\log^3 x}{3} + c}, \quad x \in (0,+\infty)$$

11. Solve the following linear differential equations:

$a)\ y' = \frac{1}{x}y + \frac{1}{x}e^{\frac{1}{x}}, \quad x \in (0,+\infty)$ $\qquad$ $b)\ y' = \frac{xy}{x+1} + e^{4x}, \quad x \in (-1,+\infty)$

$d)\ xy' - 2y = x\arctan x, \quad x \in (0,+\infty)$ $\qquad$ $e)\ y' = \frac{1}{x}y - \frac{3x+2}{x^3} \quad x \in (0,+\infty)$

EXERCISE FROM PAST EXAMS

1. (10 February 2016 - I^o)

   (a) Let f a continuous function on $(a, b]$ not bounded on $[a, b]$. Write the definition of convergence and absolute convergence for the improper integral $\int_a^b f(x)\,dx$.
   see textbook.

   (b) Study the behavior of the following integral discussing each step

$$\int_0^3 \frac{x}{\sqrt[3]{x-3}}\,dx.$$

The function $\dfrac{x}{\sqrt[3]{x-3}}$ is continuous hence locally integrable in $[0, 3)$; moreover its sign is definite (alway negative) in $(0, 3)$. We can apply the asymptotic comparison theorem for $x \to 3^-$, that yields $\dfrac{x}{\sqrt[3]{x-3}} \sim \dfrac{3}{(x-3)^{1/3}}$. Since $\displaystyle\int_0^3 \frac{3}{(x-3)^{1/3}}\,dx$ converges to a negative number since it is an integral of the type $\displaystyle\int_a^b \frac{1}{(x-b)^\alpha}\,dx$, with $\alpha < 1$. Hence the given integral converges to a negative number as well.

2. (13 February 2015 - II^o) Let $f : [0, 1) \to \mathbb{R}$ a continuous function such that $\displaystyle\lim_{x\to 1^-} f(x) = +\infty$.

   (a) Write the definition of convergence of the following improper integral

$$\int_0^1 f(x)\ dx.$$

   see textbook.

   (b) Prove that if $f(x) \geq 0$ on $[0, 1)$, the integral $\int_0^1 f(x)\ dx$ it can not be determined. *see textbook.*

   (c) Study the behavior of the improper integral

$$\int_0^1 \frac{\sin(x-1)}{(x-1)^2}\ dx.$$

The function $\dfrac{\sin(x-1)}{(x-1)^2}$ is continuous hence locally integrable on $[0, 1)$; moreover is (alway) negative in $(0, 1)$. Applying the asymptotic comparison theorem for $x \to 1^-$, we have $\dfrac{\sin(x-1)}{(x-1)^2} \sim \dfrac{(x-1)}{(x-1)^2} = \dfrac{1}{(x-1)}$; since $\displaystyle\int_0^1 \frac{1}{x-1}\ dx$ diverges negatively, also the given integral diverges to $-\infty$.

3. (30 January 2015 - III^o) Consider the differential equation

$$y' = 3y - y^2.$$

   (a) Find the constant solutions (if any) We can separate the variables and the equation admits $y = 0$ and $y = 3$ as constant solutions

   (b) Calculate all the solutions

   Separating the variables of integration, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(3-y)$$

$$-\int \frac{\mathrm{d}y}{y(y-3)} = \int \mathrm{d}x$$

$$\frac{1}{3}\int \left(\frac{1}{y} - \frac{1}{y-3}\right)\mathrm{d}y = \int \mathrm{d}x$$

$$\log\left|\frac{y}{y-3}\right| = 3x + c \implies \left|\frac{y}{y-3}\right| = e^{3x+c},\ c \in \mathbb{R}$$

   Set $k = e^c,\ k > 0$, we have $\left|\dfrac{y}{y-3}\right| = ce^{3x}$. Define $h = \pm c,\ h \in \mathbb{R} \setminus \{0\}$, then we obtain

$$\frac{y}{y-3} = he^{3x} \implies y = he^{3x}y - 3he^{3x} \implies y(he^{3x}-1) = 3he^{3x} \implies y = \frac{3he^{3x}}{he^{3x}-1}$$

Since the constant solution $y = 0$ can be obtained setting $h = 0$, while $y = 3$ is not of that type for any values of h, the general solution is

$$y = 3; \; y = \frac{3he^{3x}}{he^{3x}-1}, \; h \in \mathbb{R}$$

(c) find the maximum domain of the solution that satisfies $y(0) = 4$.

The solution for which $y(0) = 4$ is obtained for $h = 4$, i.e. $f(x) = \dfrac{12e^{3x}}{4e^{3x}-1}$.
Since the denominator is zero for $x = -\dfrac{1}{3}\log 4$, the domain of f is union of two intervals $\left(-\infty, -\dfrac{1}{3}\log 4\right)$ and $\left(-\dfrac{1}{3}\log 4, +\infty\right)$. Since the solution of the ODE has to be defined inside an interval that contains $x = 0$, the maximal domain of the solution is $\left(-\dfrac{1}{3}\log 4, +\infty\right)$.