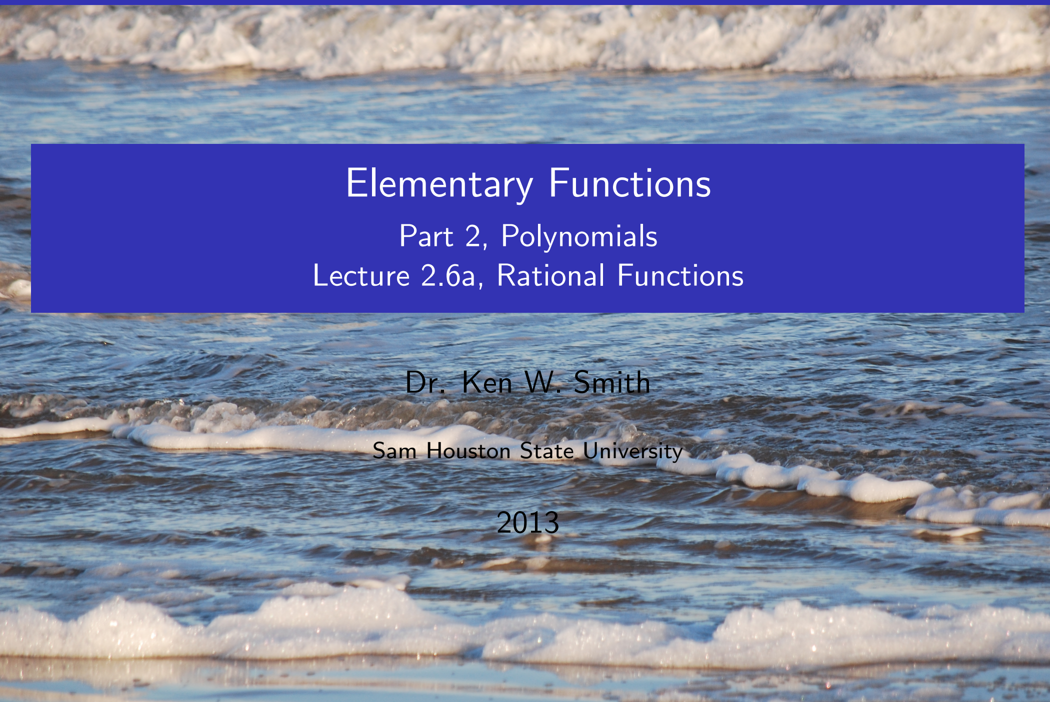

Elementary Functions

Part 2, Polynomials
Lecture 2.6a, Rational Functions

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2013

Rational Functions

A **rational function** $f(x)$ is a function which is the **ratio** of two polynomials, that is,

$$f(x) = \tfrac{n(x)}{d(x)}$$

where $n(x)$ and $d(x)$ are polynomials.

For example, $f(x) = \frac{3x^2-x-4}{x^2-2x-8}$ is a rational function.

In this case, both the numerator and denominator are quadratic polynomials.

Algebra with mixed fractions

Consider the function $g(x)$ which appeared in an earlier lecture:

$$g(x) := \frac{1}{x+2} + \frac{2x-3}{2x+1} + x - 5.$$

This function, g, is a rational function. We can put g into a fraction form, as the ratio of two polynomials, by finding a common denominator.

The least common multiple of the denominators $x+2$ and $2x+1$ is simply their product, $(x+2)(2x+1)$. We may write $g(x)$ as a fraction with this denominator if we multiply the first term by $1 = \frac{2x+1}{2x+1}$, multiply the second term by $1 = \frac{x+2}{x+2}$ and multiply the third term by $1 = \frac{(2x+1)(x+2)}{(2x+1)(x+2)}$. Then

$$g(x) = (\frac{1}{x+2})\frac{(2x+1)}{(2x+1)} + (\frac{2x-3}{2x+1})\frac{(x+2)}{(x+2)} + (x-5)\frac{(2x+1)(x+2)}{(2x+1)(x+2)}.$$

Combine the numerators (since there is a common denominator):

$$g(x) = \frac{(2x+1)+(2x-3)(x+2)+(x-5)(2x+1)(x+2)}{(2x+1)(x+2)}.$$

Algebra with mixed fractions

$$g(x) = \frac{(2x+1)+(2x-3)(x+2)+(x-5)(2x+1)(x+2)}{(2x+1)(x+2)}.$$

The numerator is a polynomial of degree 3 (it can be expanded out to $2x^3 - 3x^2 - 20x - 15$) and the denominator is a polynomial of degree 2.

The algebra of mixed fractions, including the use of a common denominator, is an important tool when working with rational functions.

Zeroes of rational functions

Given a rational function $f(x) = \frac{n(x)}{d(x)}$ we are interested in the y- and x-intercepts.

The y-intercept occurs where x is zero and it is usually very easy to compute

$$f(0) = \frac{n(0)}{d(0)}.$$

However, the x-intercepts occur where $y = 0$, that is, where

$$0 = \frac{n(x)}{d(x)}.$$

As a first step to solving this equation, we may multiply both sides by $d(x)$ and so concentrate on the zeroes of the numerator, solving the equation

$$0 = n(x).$$

At this point, we have reduced the problem to finding the zeroes of a polynomial, exercises from a previous lecture!

Algebra with mixed fractions

For example, suppose

$$h_1(x) = \frac{x^2-6x+8}{x^2+x-12}.$$

The y-intercept is $(-\frac{2}{3}, 0)$ since $h_1(0) = \frac{8}{-12} = -\frac{2}{3}$.

The x-intercepts occur where $x^2 - 6x + 8 = 0$.
Factoring $x^2 - 6x + 8 = (x-4)(x-2)$ tells us that $x = 4$ and $x = 2$ should be zeroes and so $(4, 0)$ and $(2, 0)$ are the x-intercepts.
(We do need to check that they do not make the denominator zero – but they do not.)

Poles and holes

Since rational functions have a denominator which is a polynomial, we must worry about the domain of the rational function. In particular, any real number which makes the denominator zero can*not* be in the domain.

The domain of a rational function is all the real numbers *except* those which make the denominator equal to zero. For example

$$h_1(x) = \frac{x^2-6x+8}{x^2+x-12} = \frac{(x-4)(x-2)}{(x+4)(x-3)}$$

has domain $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$ since only $x = -4$ and $x = 3$ make the denominator zero.

Poles and holes

The domain of a rational function is all the real numbers *except* those which make the denominator equal to zero.
There are two types of zeroes in the denominator.
One common type is a zero of the denominator which is *not* a zero of the numerator. In that case, the real number which makes the denominator zero is a "pole" and creates, in the graph, a vertical asymptote.

For example, using $h_1(x) = \frac{x^2-6x+8}{x^2+x-12}$ from before, we see that $x^2 + x - 12 = (x+4)(x-3)$ has zeroes at $x = -4$ and $x = 3$.

Since neither $x = -4$ and $x = 3$ are zeroes of the numerator, these values give poles of the function $h_1(x)$ and in the graph we will see vertical lines $x = -4$ and $x = 3$ that are "approached" by the graph.

The lines are called **asymptotes**, in this case we have vertical asymptotes with equations $x = -4$ and $x = 3$.

Poles and holes

We continue to look at

$$h_1(x) = \frac{x^2-6x+8}{x^2+x-12} = \frac{(x-4)(x-2)}{(x-3)(x+4)}$$

The figure below graphs our function in blue and shows the asymptotes. (The graph is in blue; the asymptotes, which are *not* part of the graph, are in red.)

Algebra with mixed fractions

If we change our function just slightly, so that it is

$$h_2(x) = \frac{x^2-6x+8}{x^2-x-12} = \frac{(x-4)(x-2)}{(x-4)(x+3)}$$

something very different occurs.

The rational function $h_2(x)$ here is *still* undefined at $x = 4$. If one attempts to evaluate $h_2(4)$ one gets the fraction $\frac{0}{0}$ which is *undefined.*

But, as long as x is not equal to 4, we can cancel the term $x - 4$ occurring both in the numerator and denominator and write

$$h_2(x) = \frac{x-2}{x+3}, \quad x \neq 4.$$

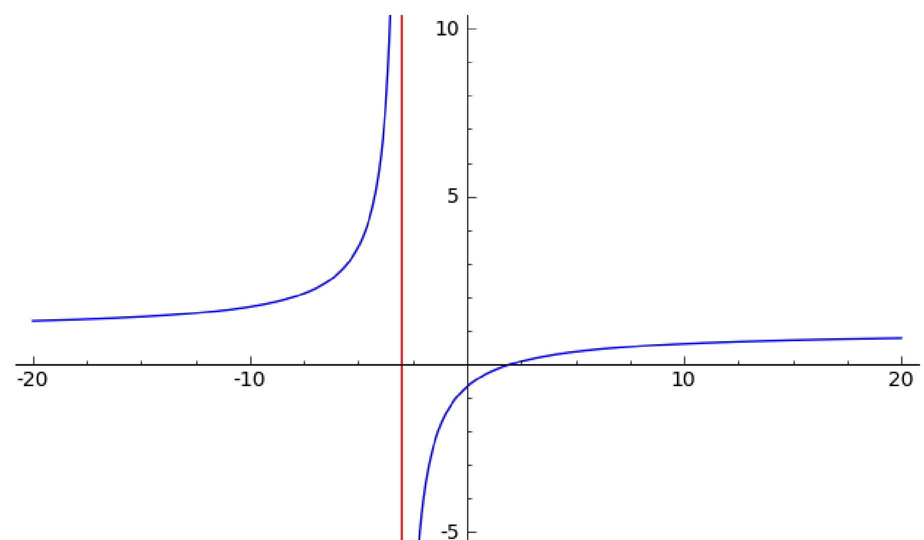
Poles and holes

In this case there is a pole at $x = -3$, represented in the graph by a vertical asymptote (in red) and there is a hole ("removable singularity") at $x = 4$ where, (for just that point) the function is undefined.

Poles and holes

Here is the graph of

$$y = \frac{x-2}{x+3}.$$



When we looked at graphs of polynomials, we viewed the zeroes of the polynomial as dividers or fences, separating regions of the x-axis from one another.

Within a particular region, between the zeroes, the polynomial has a fixed sign, $(+)$ or $(-)$, since changing sign requires crossing the x-axis.

We used this idea to create the **sign diagram** of a polynomial, a useful tool to guide us in the drawing of the graph of the polynomial.

Just as we did with polynomials, we can create a **sign diagram** for a rational function.

In this case, we need to use both the zeroes of the rational function and the vertical asymptotes as our dividers, our "fences" between the sign changes.

To create a sign diagram of rational function, list all the x-values which give a zero or a vertical asymptote. Put them in order. Then between these x-values, test the function to see if it is positive or negative and indicate that by a plus sign or minus sign.

For example, consider the function

$$h_2(x) = \frac{x^2-6x+8}{x^2-x-12} = \frac{(x-4)(x-2)}{(x-4)(x+3)}$$

from before. It has a zero at $x = 2$ and a vertical asymptote $x = -3$.

The sign diagram represents the values of $h_2(x)$ in the regions divided by $x = -3$ and $x = 2$. (For the purpose of a sign diagram, the hole at $x = 4$ is irrelevant since it does not effect the sign of the rational function.) To the left of $x = -3$, $h_2(x)$ is positive. Between $x = -3$ and $x = 1$, $h_2(x)$ is negative. Finally, to the right of $x = 1$, $h_2(x)$ is positive.

So the sign diagram of $h_2(x)$ is

$$\begin{array}{ccccc} (+) & | & (-) & | & (+) \\ \hline & -3 & & 1 & \end{array}$$

Rational Functions

In the next presentation we look at the end behavior of rational functions.

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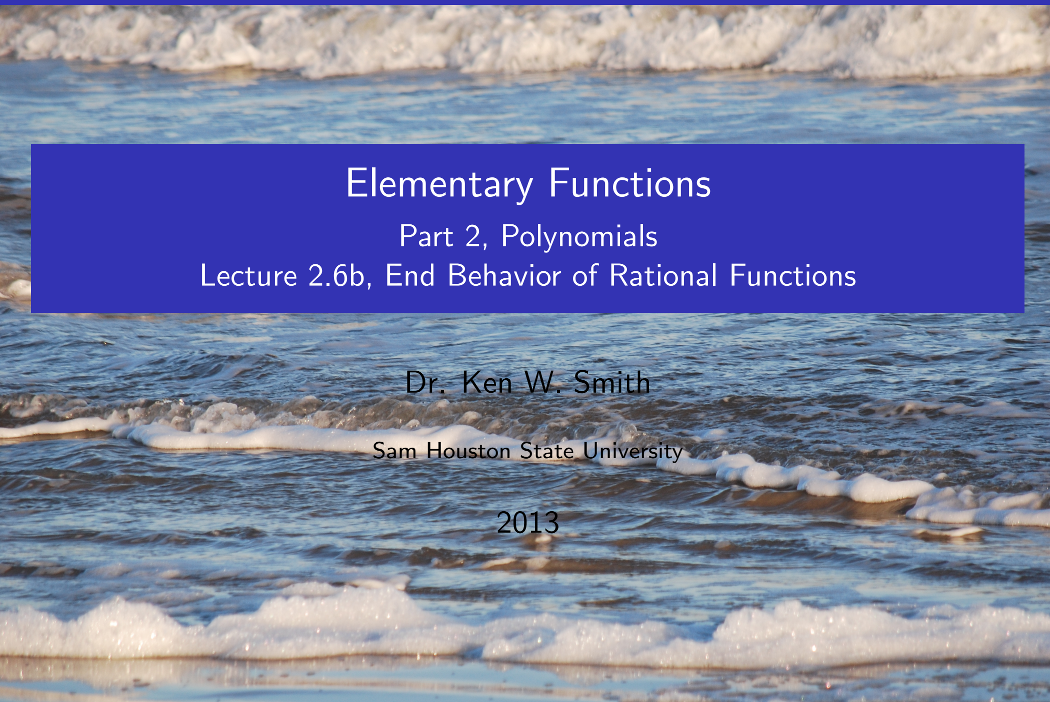

Elementary Functions

Part 2, Polynomials
Lecture 2.6b, End Behavior of Rational Functions

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2013

End-behavior of Rational Functions

Just as we did with polynomials, we ask questions about the "end behavior" of rational functions: what happens for x-values far away from 0, towards the "ends" of our graph?

In many cases this leads to questions about **horizontal asymptotes** and **oblique asymptotes** (sometimes called "slant asymptotes").

Before we go very far into discussing end-behavior of rational functions, we need to agree on a basic fact.

As long as a polynomial $p(x)$ has degree at least one (and so was not just a constant) then as x grows large $p(x)$ also grows large in absolute value.

So, as x goes to infinity, $\frac{1}{p(x)}$ goes to zero.

End-behavior of Rational Functions

We explicitly list this as a **lemma**, a mathematical fact we will often use.

Lemma 1. Suppose that $p(x)$ is a polynomial of degree at least 1. Then the rational function $\frac{1}{p(x)}$ tends to zero as x gets large in absolute value.

In calculus terms, the limit as x goes to infinity of $\frac{1}{p(x)}$ is zero.

Here is a slight generalization of the fact in Lemma 1:

Lemma 2. Suppose that $f(x) = \frac{n(x)}{d(x)}$ is a rational function where the degree of $n(x)$ is *smaller* than the degree of $d(x)$.

Then the rational function $\frac{n(x)}{d(x)}$ tends to zero as x grows large in absolute value.

In calculus terms, the limit as x goes to infinity of $\frac{n(x)}{d(x)}$ is zero.

End-behavior of Rational Functions

Repeating the previous slide:

Lemma 2. Suppose that $f(x) = \frac{n(x)}{d(x)}$ is a rational function where the degree of $n(x)$ is *smaller* than the degree of $d(x)$.
Then the rational function $\frac{n(x)}{d(x)}$ tends to zero as x grows large in absolute value.

This means that if $f(x) = \frac{n(x)}{d(x)}$ is a rational function where the degree of $n(x)$ is *smaller* than the degree of $d(x)$ then as x gets large in absolute value, the graph approaches the x-axis.

The x-axis, $y = 0$, is a **horizontal asymptote** of the rational function $\frac{n(x)}{d(x)}$.

$\frac{n(x)}{d(x)} \approx q(x)$

A **corollary** of a lemma is a result that follows directly from it.

Corollary. Suppose that $f(x) = \frac{n(x)}{d(x)}$ is a rational function. Divide $d(x)$ into $n(x)$, using the division algorithm, and write

$$\frac{n(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder.
Then the graph of $f(x)$ approaches the graph of $q(x)$ as x grows large in absolute value.

Why? Because, since the degree of $r(x)$ is less than the degree of $d(x)$, the fraction $\frac{r(x)}{d(x)}$ goes to zero and begins to be irrelevant.

We may write

$$\frac{n(x)}{d(x)} \approx q(x)$$

as x gets large in absolute value.

$\frac{n(x)}{d(x)} \approx q(x)$

From the previous slide:
Corollary. Suppose that $f(x) = \frac{n(x)}{d(x)}$ is a rational function. Divide $d(x)$ into $n(x)$, using the division algorithm, and write

$$\frac{n(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where $q(x)$ is the quotient and $r(x)$ is the remainder.
Then

$$\frac{n(x)}{d(x)} \approx q(x)$$

That is, the end behavior of the graph of the rational function $\frac{n(x)}{d(x)}$ is the graph of $q(x)$!

If we zoom out far enough, the graph of $\frac{n(x)}{d(x)}$ looks like the graph of $q(x)$.

Horizontal asymptotes

Horizontal asymptotes: some worked examples.
Example 1. Consider the rational function

$$f(x) = \frac{x^2-9}{x^3-4x}.$$

Since the numerator has degree 2 and the denominator has degree 3 then as x gets large in absolute value (say x is equal to one million ... or x is equal to negative one million) then the denominator is much larger in absolute value than the numerator and so $f(x)$ is close to zero.

This means that as $x \to \infty$ or $x \to -\infty$, $f(x) \to 0$.
So $y = 0$ is a horizontal asymptote of $f(x)$.

Horizontal asymptotes

Example 2. If instead the degree of $n(x)$ is equal to the degree of $d(x)$, then the highest power terms dominate.

For example consider the rational function

$$h_1(x) = \frac{x^2-6x+8}{x^2+x-12}.$$

As x gets large in absolute value, the quadratic terms x^2 begin to dominate. For example, if $x = 1,000,000$ then the denominator $x^2 + x - 12$ is equal to
$1,000,000,000,000 + 1,000,000 - 12 = 1,000,000,099,988$, which for all practical purposes can be approximated by $1,000,000,000,000$.
Similarly, if x is a million, the numerator is equal to
$1,000,000,000,000 - 6,000,000 + 8 = 999,999,400,008$ which can also be approximated by $1,000,000,000,000$.
Thus

$$f(1000000) = \frac{1{,}000{,}000{,}099{,}988}{999{,}999{,}400{,}008} \approx 1.0000007 \approx 1.$$

Horizontal asymptotes

Continuing with

$$h_1(x) = \frac{x^2-6x+8}{x^2+x-12}.$$

The same result occurs if we set x equal to negative numbers which are large in absolute value, such as $x = -1000000$. More generally, as x gets large in absolute value,

$$\frac{x^2-6x+8}{x^2+x-12}$$

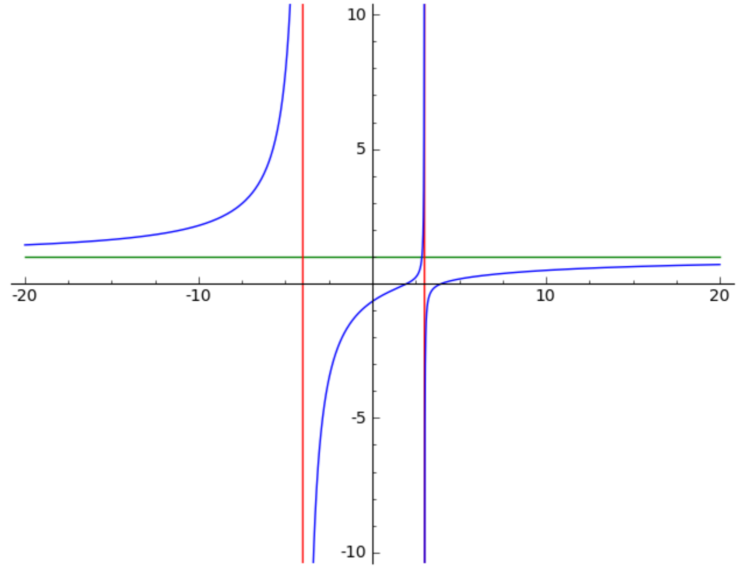
begins to look like

$$\frac{x^2}{x^2} = 1.$$

We conclude then that as x gets large in absolute value, $f(x)$ approaches 1 and so $y = 1$ is a horizontal asymptote of $f(x)$.

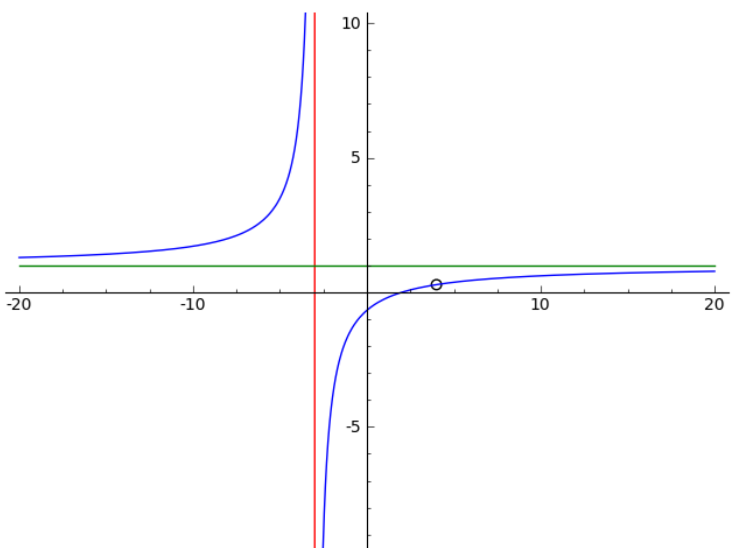
Horizontal asymptotes

Below, is a graph of $y = h_1(x) = \frac{x^2-6x+8}{x^2+x-12}$, with the function drawn in blue and the various asymptotes drawn in green or red.



Horizontal asymptotes

Example 3. Earlier we considered the rational function $h_2(x) = \frac{(x-4)(x-2)}{(x-4)(x+3)}$. Like $h_1(x)$, this function has a horizontal asymptote $y = 1$ (drawn in green.)



Horizontal asymptotes

Example 4. Find the zeroes and vertical asymptotes of the rational function

$$g(x) = \frac{3(x+1)(x-2)}{4(x+3)(x-1)}$$

and draw the sign diagram. Then find the horizontal asymptotes.

Solution.
Looking at the numerator $3(x+1)(x-2)$ of $g(x)$ we see that the zeroes occur at $\boxed{x=-1}$ and $\boxed{x=2}$.

Looking at the denominator $4(x+3)(x-1)$ of $g(x)$ we can see that the vertical asymptotes of $g(x)$ are the lines $\boxed{x=-3}$ and $\boxed{x=1.}$

Horizontal asymptotes

Example 4, continued Find the zeroes and vertical asymptotes of the rational function

$$g(x) = \frac{3(x+1)(x-2)}{4(x+3)(x-1)}$$

and draw the sign diagram. Then find the horizontal asymptotes.

The sign diagram is

$$\begin{array}{ccccccccc} (+) & | & (-) & | & (+) & | & (-) & | & (+) \\ \hline & -3 & & -1 & & 1 & & 2 & \end{array}$$

There is one horizontal asymptote found by considering the end behavior of $g(x)$. As x goes to infinity, $g(x) = \frac{3(x+1)(x-2)}{4(x+3)(x-1)}$ approaches $\frac{3x^2}{4x^2} = \frac{3}{4}$ so the horizontal asymptote is the line $\boxed{y = \frac{3}{4}.}$

Oblique asymptotes

Consider a rational function we saw earlier:

$$g(x) := \frac{1}{x+2} + \frac{2x-3}{2x+1} + x - 5 = \frac{2x^3-3x^2-20x-15}{2x^2+5x+2}.$$

If we do long division we get

Summary

Asymptotes of a function are lines that approximate a rational function "in the large", as we zoom out and look at global behavior of the rational function.

Vertical asymptotes occur where the denominator is zero and the numerator is *not* zero.

If the degree of the numerator of a rational function is *less than* the degree of the denominator then the rational function has horizontal asymptote $y = 0$.

If the degree of the numerator of a rational function is *equal* to the degree of the denominator then the rational function has a horizontal asymptote which can be found either by doing long division or by focusing on the leading coefficients of the numerator and denominator.

If the degree of the numerator of a rational function is *one more* than the degree of the denominator then the rational function has an oblique asymptote which can be found as the quotient after long division.

Summary

Remember! Asymptotes are lines! When asked for the asymptotes of a rational function, make sure to give equations of lines!

In the next presentation we work through the "Six Steps" to graphing a rational function.

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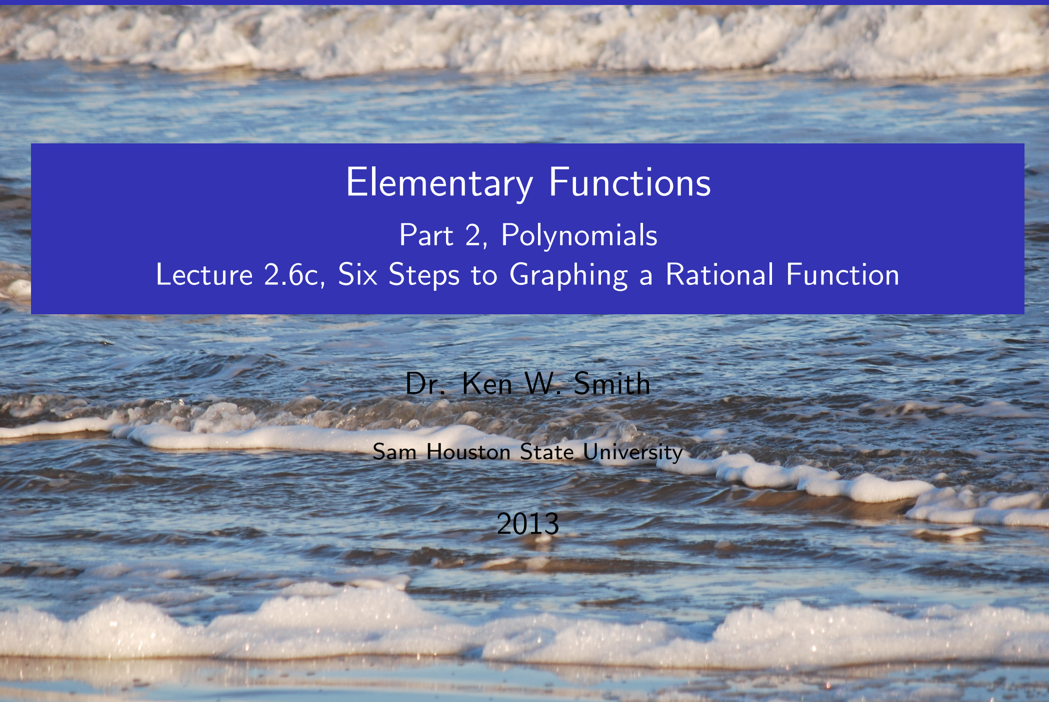

Elementary Functions

Part 2, Polynomials

Lecture 2.6c, Six Steps to Graphing a Rational Function

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2013

Putting it all together – the six steps

The textbook *Precalculus, by Stitz and Zeager* suggests six steps to graphing a rational function $f(x)$. Here (from page 321 in the third edition) are the six steps.

1. Find the domain of the rational function $f(x)$:
2. Reduce $f(x)$ to lowest terms, if applicable.
3. Find the x- and y-intercepts of the graph of $y = f(x)$, if they exist.
4. Determine the location of any vertical asymptotes or holes in the graph, if they exist. Analyze the behavior of $f(x)$ on either side of the vertical asymptotes, if applicable.
5. Analyze the end behavior of $f(x)$. Find the horizontal or slant asymptote, if one exists.
6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = f(x)$.

The sign diagram of a rational function

Two worked examples.

Example 1. Use the six steps, above, to graph the rational function $h(x) = \frac{10x^2-250}{x^2+6x+8}$.

Solution.

1. We factor the numerator and denominator to rewrite
$$h(x) = \frac{10x^2-250}{x^2+6x+8} = \frac{10(x-5)(x+5)}{(x+2)(x+4)}.$$
The domain is the set of all real numbers except $x = -2$ and $x = -4$. In interval notation this is $(-\infty, -4) \cup (-4, -2) \cup (-2, \infty)$.
2. Since the numerator and the denominator have no common factors then $h(x)$ is in lowest terms. This means that there are no holes (removable singularities) in the graph.
3. The y-intercept is when $y = h(0) = \frac{-250}{8} = -\frac{125}{4} = -31.25$. To find the x-intercepts we set the numerator equal to zero:
$$0 = 10x^2 - 250.$$
Divide both sides by 10 and factor
$$0 = x^2 - 25.$$

The sign diagram of a rational function

4. Since $h(x)$ is in lowest terms, there are no holes. Since the denominator factors as $x^2 + 6x + 8 = (x + 4)(x + 2)$ then the denominator is zero when $x = -4$ or $x = -2$. So the vertical asymptotes are $x = -4$ and $x = -2$.
5. As x gets large in absolute value (and so the graph is far away from the y-axis), $h(x)$ begins to look like
$$h(x) = \frac{10x^2}{x^2} = 10.$$
So the horizontal asymptote is $y = 10$.
6. To draw the sign diagram we use zeroes $x = -5$ and $x = 5$ and vertical asymptotes $x = -4$ and $x = -2$ to create our "fences" and then test values between the "fences". Here is the result:

$$\begin{array}{ccccccccc} (+) & \mid & (-) & \mid & (+) & \mid & (-) & \mid & (+) \\ \hline & -5 & & -4 & & -2 & & 5 & \end{array}$$

Graphing a rational function

The graph of $h(x) = \frac{10x^2-250}{x^2+6x+8}$ is given below. The graph is in blue; the vertical asymptotes are in red and the horizontal asymptote is in green.

Graphing a rational function

Example 2. Let $f(x) = \frac{3x^3+x^2-12x-4}{x^2-2x-8}$ Find all intercepts, zeroes and then graph this function, displaying the features found.

Solution. We will work through the six steps.

1. The denominator factors as $x^2 - 2x - 8 = (x-4)(x+2)$. So the domain is the set of real numbers where the denominator is *not* zero, that is,
$$(-\infty, -2) \cup (-2, 4) \cup (4, \infty).$$
2. Notice that when we evaluate the numerator at $x = -2$, we get zero. So $(x+2)$ is a factor of both the numerator and the denominator.

   Recognizing that $x + 2$ is a factor of the numerator, we can further factor the numerator using the techniques we learned in the sections on polynomial zeroes. The numerator factors as
   $(x+2)(x-2)(3x+1)$.
   So
   $$f(x) = \frac{(x+2)(x-2)(3x+1)}{(x+2)(x-4)}.$$
   The point where $x = -2$ is a hole (removable singularity.) If we put the rational function into lowest terms, it becomes

The sign diagram of a rational function

Continuing with Example 2 where
$$f(x) = \frac{3x^3+x^2-12x-4}{x^2-2x-8}$$

3. The y-intercept is when $y = f(0) = \frac{-2}{-4} = 2$.
   The x-intercepts occur when we set $(x-2)(3x+1)$ equal to zero and so these occur when $x = 2$ and when $x = -\frac{1}{3}$.
4. A hole occurs when $x = -2$. Looking at the reduced form, we see that the hole has y-value
   $$\frac{(-2-2)(3(-2)+1)}{(-2-4)} = \frac{(-4)(-5)}{-6} = -\frac{10}{3}.$$

   There is one vertical asymptote; it is $x = 4$.

Graphing a rational function

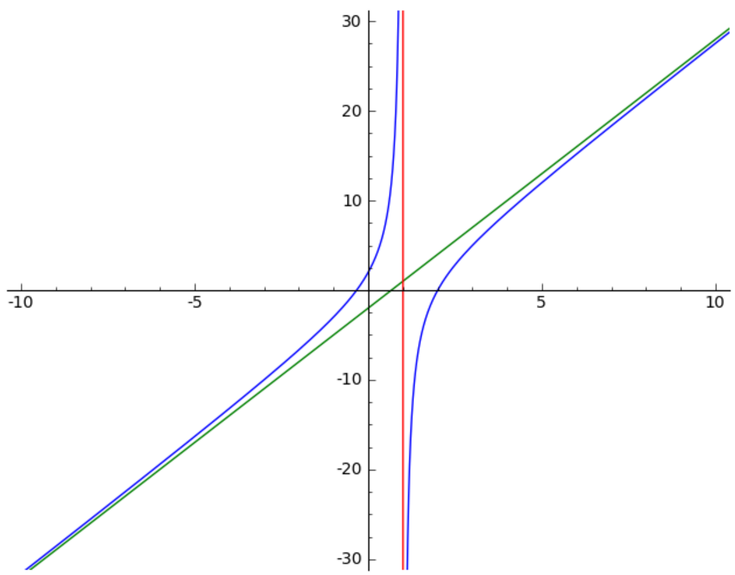
5. Analyze the end behavior of the rational function. Find the horizontal or slant asymptote, if one exists. To analyze the end behavior, we do long division:
$$\begin{array}{r|r}
 & 3x - 2 \\ \hline
x - 1 & 3x^2 - 5x - 2 \\
 & -\ 3x^2 + 3x \\ \hline
 & -\ 2x - 2 \\
 & 2x - 2 \\ \hline
 & -\ 4
\end{array}$$
   and write $f(x) = 3x - 2 - \frac{4}{x-1}$ and so there is an oblique (slant) asymptote at $y = 3x - 2$.

Continuing with Example 2 where $f(x) = \frac{3x^3+x^2-12x-4}{x^2-2x-8}$

6. Use a sign diagram and plot additional points, as needed, to sketch the graph of $y = f(x)$.

The sign diagram is $\begin{array}{ccccccc} (-) & | & (+) & | & (-) & | & (+) \\ \hline & -\frac{1}{3} & & 2 & & 4 & \end{array}$

The graph is drawn below in blue (with asymptotes in colors red and green.)



The graph of $f(x)$ (in blue) with vertical asymptotes (red) and horizontal asymptote (green). The hole at $(-2, -\frac{10}{3})$ is not shown.

In the next series of lectures, we move on to a new topic, exponential functions and their inverse functions (logarithms.)

(END)