MATHEMATICAL ANALYSIS I TUTORING
7^{TH} WEEK

STUDY of FUNCTIONS - Solutions

30 January 2015 - II

Given the function

$$f(x) = \frac{1}{2}x^2 + x + \log|x+3|.$$

1. Find the domain, the limits at boundary points of the domain and asymptotes.
2. Compute the derivative of f, find monotonicity intervals and maxima/minima if there are any.
3. Find inflection points and convexity/concavity intervals.
4. Draw a qualitative graph.
5. Say if the function $g(x) = \arctan f(x)$ admits a continuous prolongation in $x = -3$, justify the answer.

Solution

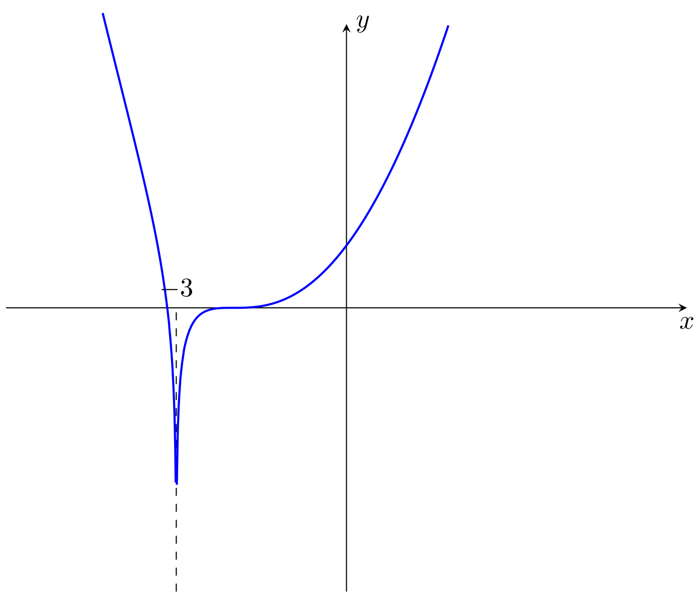
1. Imposing the argument of the logarithm to be positive, we get $\mathrm{dom} f = \mathbb{R} \setminus \{-3\}$.
   At the boundary points of the domain: $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} x^2\left(\frac{1}{2} + o(1)\right) = +\infty$; $\lim_{x\to-3^\pm} f(x) = -\infty$.
   Hence the line $x = -3$ is a vertical asymptote for f.
   Since $\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty}\left(\frac{x}{2} + 1 + o(1)\right) = \pm\infty$, the function does not admit oblique asymptotes.

2. it holds $f'(x) = x+1+\frac{1}{x+3} = \frac{(x+2)^2}{x+3} > 0 \iff x > -3$ and $f'(x) = 0 \iff x = -2$. Therefore we conclude that f strictly decreases in $(-\infty, -3)$ and strictly increases in $(-3, +\infty)$.

   The extremal points must be among stationary points, non differentiable points, and boundary points of the domain. The only stationary point is $x = -2$ and all point are differentiable. From the sign of f' we know that $x = -2$ is not extremal. Moreover, from point 1., the function is unbounded from above and below, thus f does not admit relative/absolute minima or maxima.

3. It holds $f''(x) = \frac{(x+2)(x+4)}{(x+3)^2} \geq 0 \iff x \leq -4$ o $x \geq -2$. Thus f is convex in $(-\infty, 4)$ and $(-2, +\infty)$; concave in $(-4, -3)$ and $(-3, -2)$. In particular, $x = -4$ is a discending inflection point and $x = -2$ is a horizontal tangent ascending inflection point.

4. Here a qualitative graph.



5. Being f continuous in $\mathbb{R} \setminus \{-3\}$, g is continuous in $\mathbb{R} \setminus \{-3\}$. Since $\lim\limits_{x \to -3^{\pm}} g(x) = -\dfrac{\pi}{2}$, it holds that $x = -3$ is a 3-rd kind (removable) singularity point for g. Therefore, g admits continuous prolongation in $x = -3$ defined as follows

$$\widetilde{g}(x) := \begin{cases} g(x) & \text{if } x \neq -3 \\ -\frac{\pi}{2} & \text{if } x = -3\,. \end{cases}$$

13 February 2015 - II

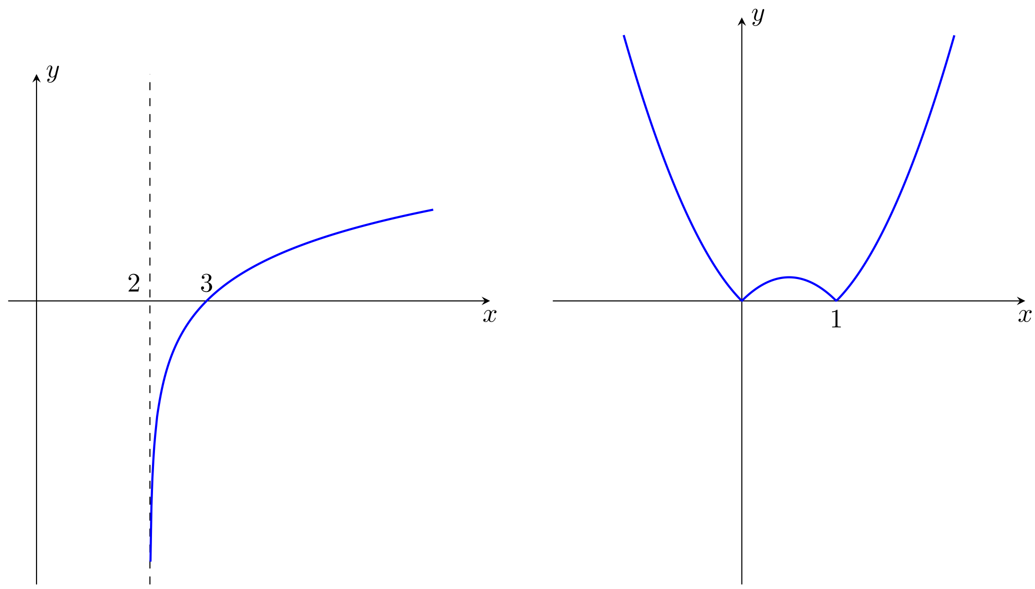
Let

$$f(x) = |\log(x-2) - \log^2(x-2)|.$$

1. Find the domain of f and limits at the boundary points.
2. Compute the derivative of f, and specify non differentiable points.
3. Find monotonicity intervals for f and maxima/minima.
4. Draw a qualitative graph.

Solution

1. Given $g(x) = \log(x-2)$ for $x \in (2, +\infty)$ and $h(y) = |y - y^2|$ for $y \in \mathbb{R}$, it holds $f(x) = h(g(x))$. The graph of g (left) and h (right) are below.



In particular, the domain of f coincides with the domain of g: dom $f = (2, +\infty)$ and
$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{y \to -\infty} h(y) = +\infty$ and $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{y \to +\infty} h(y) = +\infty\,.$
Hence $x = 2$ is a right vertical asymptote, whereas f as $x \to +\infty$ is a lgarithmic infinity, there is no oblique asymptote as $x \to +\infty$.

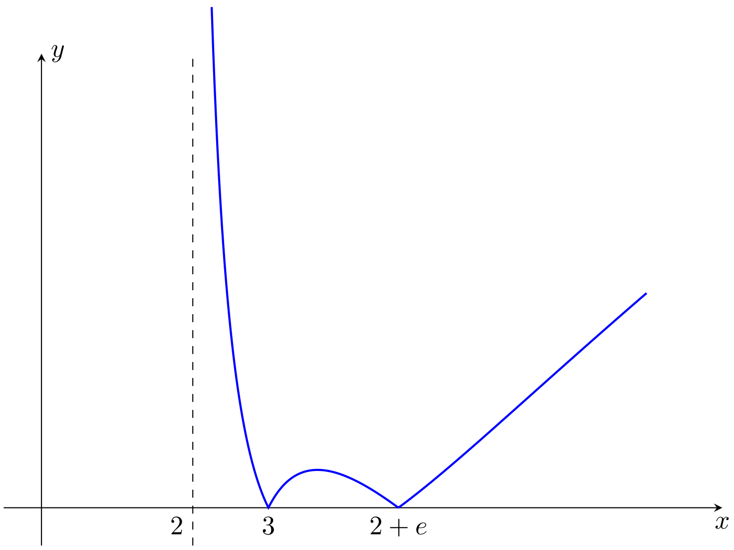
2. The function is continuous by composition of continuous functions. Moreover, g is differentiable for every $x \in (2, +\infty)$ with $g'(x) = \frac{1}{x-2}$ and h is differentiable for $y \neq 0, 1$ while $y = 0$ and $y = 1$ are corner points. Hence, by composition f is differentiable for every $x \in dom f$ such that $g(x) \neq 0$ and $g(x) \neq 1$, i.e. for $x \in dom f \setminus \{3, 2+e\}$ and it holds

$$f'(x) := \begin{cases} g'(x)(1 - 2g(x)) = \frac{1 - 2\log(x-2)}{x-2} & \text{if } 3 < x < 2 + e \\ \\ g'(x)(2g(x) - 1) = \frac{2\log(x-2) - 1}{x-2} & \text{if } 2 < x < 3 \text{ or } x > 2 + e\,. \end{cases}$$

Since $\lim\limits_{x \to 3^-} f'(x) = -1 \neq \lim\limits_{x \to 3^+} f'(x) = 1$ and $\lim\limits_{x \to (2+e)^-} f'(x) = \dfrac{-1}{e} \neq \lim\limits_{x \to (2+e)^+} f'(x) = \dfrac{1}{e}$, we conclude that $x = 3$ and $x = 2 + e$ are corner points for f.

3. Note that $g'(x) > 0$ for every $x \in dom f$ and $1 - 2g(x) = 1 - 2\log(x-2) \geq 0$ if $2 < x \leq 2 + \sqrt{e}$, it follows that $x = 2 + \sqrt{e}$ is the only stationary point for f; moreover f is strictly increasing in $(3, 2 + \sqrt{e})$ and $(2 + e, +\infty)$, f is strictly decreasing in $(2, 3)$ and $(2 + \sqrt{e}, 2 + e)$. Hence $x = 3$ and $x = 2 + e$ are relative minima while $x = 2 + \sqrt{e}$ is a relative maximum. Knowing that f is unbounded from above, there is no absolute maximum, since $f(x) \geq 0$ for every $x \in dom f$ and $f(3) = f(2 + e) = 0$, we conclude that $x = 3$ and $x = 2 + e$ are absolute minima.

4. Below a qualitative graph for f.



17 June 2015 - I

Given

$$f(x) = x\, e^{\frac{2}{\log x}}.$$

a) Find domain and limits at boundary points.

b) Find monotonicity intervals and minimum/maximum points.

c) Draw a qualitative graph of f.

d) Find a continuous prolongation of f on $(-\infty, 1]$. Say if there exists a continuous prolongation for f on $\mathbb{R}$, justify the answer.

Solution

(a) $\mathrm{dom} f = (0, 1/2) \cup (1/2, +\infty)$,

$$\lim_{x\to 0^+} f(x) = 0\,, \quad \lim_{x\to(1/2)^-} f(x) = 0\,, \quad \lim_{x\to(1/2)^+} f(x) = +\infty\,, \quad \lim_{x\to+\infty} f(x) = +\infty\,.$$

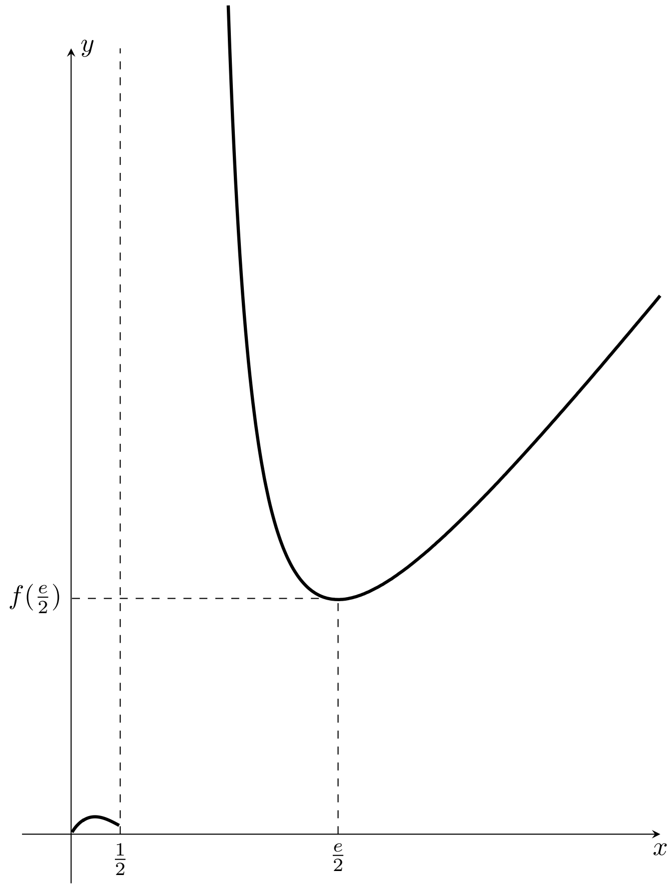
There is no oblique asymptote in $+\infty$, indeed $\lim_{x\to+\infty} f(x)/x = 1$ but $\lim_{x\to+\infty} f(x) - x = \lim_{x\to+\infty} x(e^{\frac{1}{\log 2x}} - 1) = \lim_{x\to+\infty} x(\frac{1}{\log 2x} + o(\frac{1}{\log 2x})) = +\infty$.

(b) It holds:

$$f'(x) = e^{\frac{1}{\log 2x}}\left(1 - \frac{1}{(\log 2x)^2}\right) \geq 0 \Leftrightarrow (\log 2x)^2 - 1 \geq 0 \Leftrightarrow 0 < x \leq e^{-1}/2 \cup x \geq e/2\,.$$

Thus: f increasing on $(0, e^{-1}/2)$ and $(e/2, +\infty)$, decreasing on $(e^{-1}/2, 1/2)$ and $(1/2, e/2)$, $x = e^{-1}/2$ relative max and $x = e/2$ relative min. There are no absolute minima or maxima since f is unbounded from above and $\inf f = 0$ is never reached.

(c) Below a qualitative graph for f.



(d) A continuous prolongation for f on $(-\infty, \frac{1}{2}]$ is

$$\tilde{f}(x) = \begin{cases} f(x) & x \in (0, 1/2) \\ 0 & x \in (-\infty, 0] \cup \{1/2\}\,. \end{cases}$$

Since $\lim_{x \to (1/2)^+} f(x) = +\infty$, there is no continuous prolongation for f on $\mathbb{R}$.

9 September 2015 - I

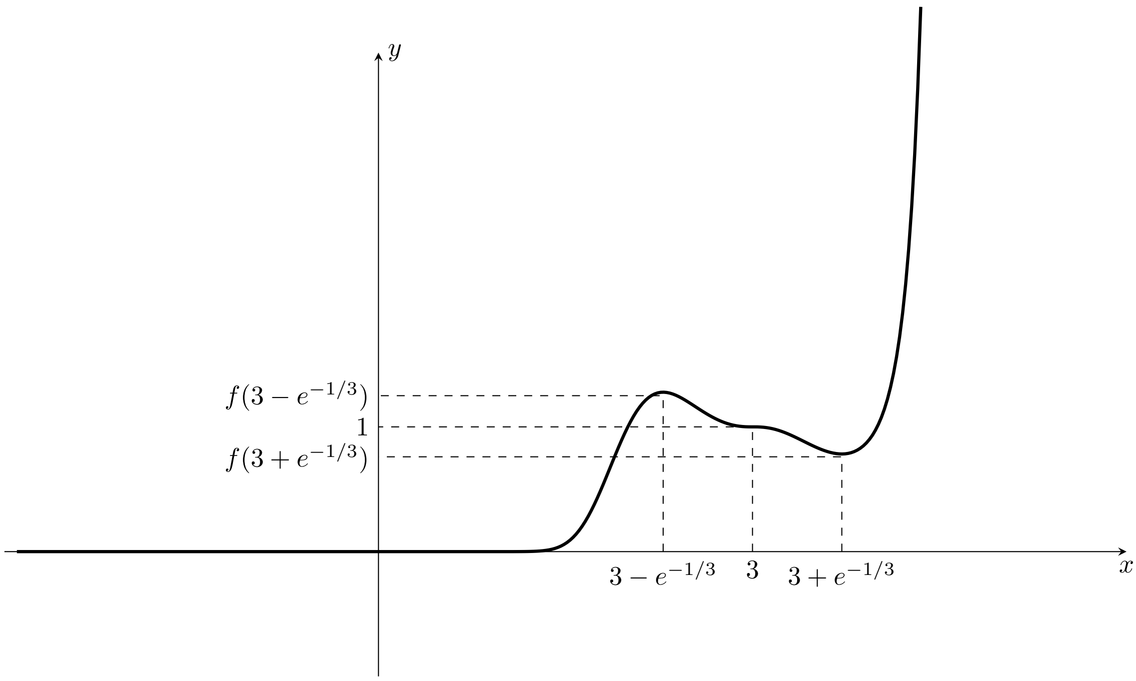
Let

$$f(x) = e^{2(x-3)^3 \log|x-3|}.$$

1. Find domain and asymptotes for f. Show that f admits continuous prolongation in $x = 3$.
2. Denote the continuous prolongation by f, compute the first derivative and study differentiability in $x = 3$.
3. Find monotonicity intervals and maxima/minima for f.
4. Draw a qualitative graph for f.

Solution

1. It holds: $dom f = \mathbb{R} \setminus \{3\}$,
$$\lim_{x\to-\infty} f(x) = 0\,, \quad \lim_{x\to+\infty} f(x) = +\infty \quad \text{e} \quad \lim_{x\to+\infty} \frac{f(x)}{x} = +\infty\,.$$
Thus $y = 0$ is a left horizontal asymptote whereas there are no oblique asymptote. On the other hand,
$$\lim_{x\to3^-} f(x) = 1 = \lim_{x\to3^+} f(x)\,,$$
hence there are no vertical asymptotes and f admits continuous prolongation in $x = 3$. The continuous prolongation (denoted by f) is
$$f(x) := \begin{cases} e^{2(x-3)^3 \log|x-3|} & \text{if } x \neq 3 \\ \\ 1 & \text{if } x = 3\,. \end{cases}$$
2. It holds $f'(x) = 2(x-3)^2(3\log|x-3|+1)e^{2(x-3)^3 \log|x-3|}$ for $x \neq 3$. Since $\lim_{x\to3^-} f'(x) = 0 = \lim_{x\to3^+} f'(x)$, we can conclude (applying de l'Hopital Theorem) that f is differentiable in $x = 3$ and $f'(3) = 0$. It holds
$$f'(x) := \begin{cases} 2(x-3)^2(3\log|x-3|+1)e^{2(x-3)^3 \log|x-3|} & \text{if } x \neq 3 \\ \\ 0 & \text{if } x = 3\,. \end{cases}$$
3. Study the sign of f':
$$f'(x) > 0 \iff 3\log|x-3| + 1 > 0 \iff x \in (-\infty, 3-e^{-1/3}) \cup (3+e^{-1/3}, +\infty)\,.$$
Then f is increasing in $(-\infty, 3-e^{-1/3}]$ and in $[3+e^{-1/3}, +\infty)$, decreasing in $(3-e^{-1/3}, 3+e^{-1/3})$ (using the continuity of f). Finally, $x = 3-e^{-1/3}$ is a relative maximum but absolute because f is unbounded from above while $x = 3+e^{-1/3}$ is a relative minimum that is not absolute because $inf f = 0 < f(3+e^{-1/3})$. In particular, there are no absolute minima or maxima.
4. Below a qualitative graph of f.



28 January 2016 - I

Given

$$f(x) = \arcsin|1-2^x| + 1.$$

(a) Find domain, limits at boundary points and asymptotes.

(b) Compute the derivative, find non differentiable points and say which type they are.

(c) Find monotonicity intervals, minima and maxima: specify if they are local or global.

(d) Draw a qualitative graph.

(e) Say if f admits continuous prolongation on $\mathbb{R}$ such that it is differentiable in $x = 1$, justify the answer.

Solution

(a) Since the argument of cosine must lie between -1 and 1, we have that $\mathrm{dom} f = (-\infty, 1]$.
At boundary points of the domain:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \arcsin|1-2^x| + 1 = \frac{\pi}{2} + 1\ ; \qquad f(1) = \frac{\pi}{2} + 1.$$

Thus, the line $y = \frac{\pi}{2} + 1$ is a (left) horizontal asymptote for the function.

(b) Compute the derivative; write $f(x)$ specifying the two cases for the abosulte value:

$$f(x) = \begin{cases} \arcsin(1-2^x) + 1 & \text{if } x < 0 \\ \arcsin(-1+2^x) + 1 & \text{if } 0 \le x \le 1 \end{cases}$$

Thus:

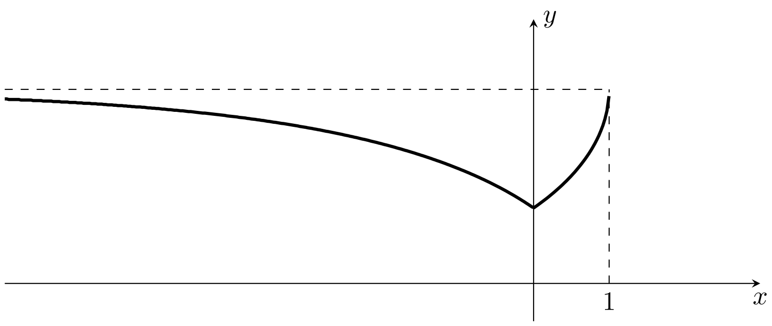
$$f'(x) = \begin{cases} -\dfrac{2^x \log(2)}{\sqrt{2^x(2-2^x)}} & \text{if } x < 0 \\ \\ \dfrac{2^x \log(2)}{\sqrt{2^x(2-2^x)}} & \text{if } 0 < x < 1 \end{cases}$$

Since $\lim_{x\to 0^-} f'(x) = -\log(2) \neq \lim_{x\to 3^+} f'(x) = \log(2)$ we conclude that $x = 0$ is a corner point for f.

(c) Note that $\quad f'(x) < 0 \quad \Longleftrightarrow \quad x < 0 \quad$ and $\quad f'(x) > 0 \quad \Longleftrightarrow \quad 0 < x < 1\,.$
Since f is (strictly) decreasing in $(-\infty, 0)$ and (stricly) increasing in $(0, 1)$.

Extremal points for a function are among stationary points, non differentiable points, and boundary points for tha domain. In our case the non-differentiable point $x = 0$ is an absolute minimum. The function is upper bounded: we have $f(x) \le \frac{\pi}{2} + 1$; moreover $f(1) = \frac{\pi}{2} + 1$. We can conclude that $x = 1$, boundary of the domain, is an absoute maximum point.

(d) Below we have a qualitative graph for f:



(e) Denote by $g(x)$ a generic prolongation of f on $\mathbb{R}$, defined by 'adding' an unknown function $h(x)$ for $x > 1$:

$$g(x) = \begin{cases} f(x) & \text{if } x \le 1 \\ \\ h(x) & \text{if } x > 1 \end{cases}$$

The function g is continuous in $x = 1$ if $g(1) = f(1) = \dfrac{\pi}{2} + 1$, and thus if $\displaystyle\lim_{x\to 1^+} h(x) = \frac{\pi}{2} + 1$.
Therefore, e.g., the constant function $h(x) = \dfrac{\pi}{2} + 1$ makes the prolongation $g(x)$ continuous on $\mathbb{R}$.
Concerning differentiability in $x = 1$, note that, if we compute the limit of the derivative in a left neighborhood of 1, we have:

$$\lim_{x\to 1^-} \frac{2^x \log(2)}{\sqrt{2^x(2-2^x)}} = +\infty$$

This result is sufficient to say that, for every function $h(x)$, g will never be differentiable in $x = 1$.

28 January 2016 - II

Given

$$f(x) = \begin{cases} \dfrac{(x+2)^2}{\log(x+2)} - 3 & x \in (-2,-1) \cup (-1,+\infty) \\ \\ -3 & x \le -2 \end{cases}$$

(a) Study the limits at the boundary points of Dom$f$. Study the continuity of f on its domain.

(b) Study differentiability of f and compute the derivative, where it exists.

(c) Find monotonicity intervals for f and minima/maxima, saying if they are local or global.

(d) Draw a qualitative graph of f.

(e) Consider

$$f_k(x) = \begin{cases} \dfrac{(x+2)^2}{\log(x+2)} - 3 & x \in (-2,-1) \\ \\ -3 + (x+2)^k & x \le -2 \end{cases}$$

(a) Find $k \in \mathbb{N}$ such that f_k is continuous in $(-\infty,-1)$

(b) Find $k \in \mathbb{N}$ such that f_k is in $\mathcal{C}^1$ in $(-\infty,-1)$

Solution

(a) Impose that the argument of the logarithm must be positive and the denominator different from 0, then dom $f = \mathbb{R} \setminus \{-1\}$. Limits at the boundary points of the domain are

$$\lim_{x\to-\infty} f(x) = -3\,; \qquad \lim_{x\to+\infty} f(x) = +\infty\,; \qquad \lim_{x\to-1^\pm} f(x) = \pm\infty\,.$$

Being $\lim_{x\to+\infty} \frac{f(x)}{x} = +\infty$, the function has no right oblique asymptote.
Finally, note that $\lim_{x\to-2^+} f(x) = -3 = f(-2)$ hence the function is continuous in $x = -2$. By composition of continuous functions, we get the continuity of the whole domain.

(b) It holds

$$f'(x) = \frac{(x+2)}{\log^2(x+2)}(2\log(x+2) - 1) \text{ for } x > -2\,, x \ne -1\,.$$

Since $\lim_{x\to-2^+} f'(x) = 0$, we can conclude that f is differentiable in $x = -2$ e

$$f'(x) = \frac{(x+2)}{\log^2(x+2)}(2\log(x+2) - 1) \text{ for } x > -2\,, x \ne -1 \quad \text{and} \quad f'(x) = 0 \text{ for } x \le -2\,.$$

Then f is differentiable on the domain.

(c) For $x > -2$ it holds

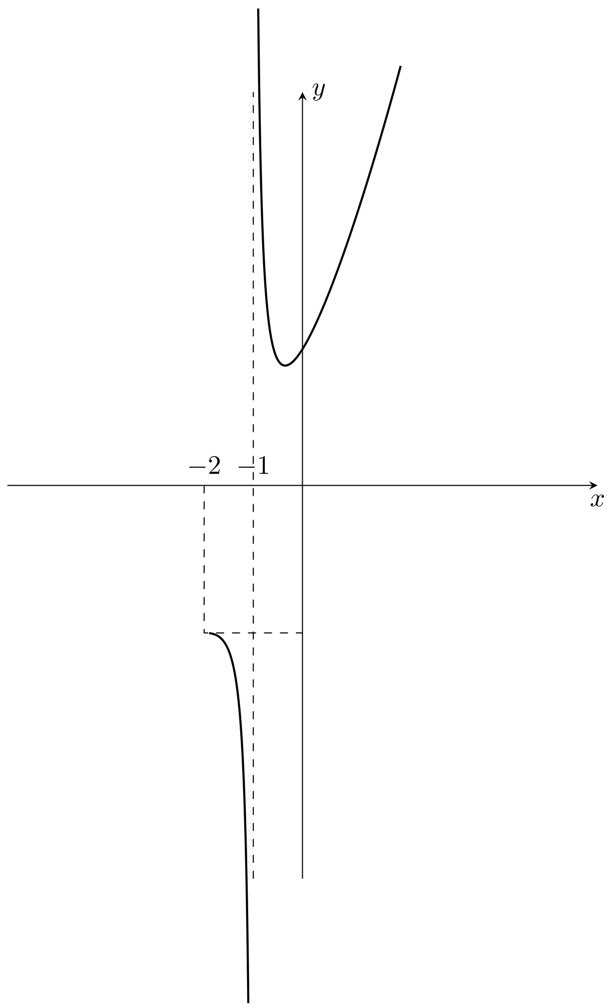
$$f'(x) = \frac{(x+2)}{\log^2(x+2)}(2\log(x+2) - 1) > 0 \quad \Longleftrightarrow \quad x > \sqrt{e} - 2 > -1$$

whereas

$$f'(x) = 0 \quad \Longleftrightarrow \quad x \le -2 \quad \text{and} \quad x = \sqrt{e} - 2\,.$$

Hence, f is (strictly) decreasing in $[-2,-1)$ and in $(-1, \sqrt{e}-2]$, (strictly) increasing in $[\sqrt{e}-2, +\infty)$.
In particular, all points $x \le -2$ are relative maxima, while all points $x < -2$ are relative minima, and $x = \sqrt{e} - 2$ is the unique strict minimum. Finally, from (a), the function is unbounded from above and from below, thus there are no extremal points that are absolute.

(d) Below a qualitative graph for f.



(e) 1) It holds $f_0(-2) = -2$ (as $(x-x_0)^0 = 1$ also in $x = x_0$) while $f_k(-2) = -3$ for $k \geq 1$. Therefore, $\lim_{x\to -2^+} f_k(x) = f(-2) = \lim_{x\to -2^-} f_k(x) = -3$ and the function is continuous in $x = -2$ if and only if $k \geq 1$. In the regions it is continuous by composition of continuous functions.

2) For $k \geq 1$ it holds that $f_k'(x) = k(x+2)^{k-1}$ if $x < -2$. Recall (b), then $\lim_{x\to -2^+} f_k'(x) = 0 = \lim_{x\to -2^-} f_k'(x)$ (and the function is differentiable in $x = -2$) if and only if $k \geq 2$ and $f_k'(-2) = 0$. In particular, the latter limit guarantees continuity for the derivative in $x = -2$. Then f_k is in C^1 for $k \geq 2$.

28 January 2016 - III

Consider

$$f(x) = \sqrt{1-|x|} - \arcsin\sqrt{1-|x|} + 2$$

(a) Find the domain of f, and possible symmetries.

(b) Study the continuity of f on its domain.

(c) Compute the derivative of f. Find non differentiable points, and specify their type.

(d) Find monotonicity intervals for f and maxima/minima, specify if they are local or global.

(e) Draw a qualitative graph for f.

(f) Say if there are constants $a, b \in \mathbb{R}$ such that

$$g(x) = \begin{cases} f(x) & x \in \operatorname{dom} f \cap (0, +\infty) \\ ax + b & x \in (0, +\infty) \setminus \operatorname{dom} f \end{cases}$$

is continuous and differentiable in $(0, +\infty)$.

Solution

(a) Since the argument of the arcsine must lie between -1 and 1, and the root argument must be positive, we have dom$f = [-1, 1]$. It holds $f(\pm 1) = 2$.
The function is even, thus the graph is symmetric with respect to the origin.

(b) The function $f(x)$ is continuous on the domain, by composition of continuous functions.

(c) Since $f(x)$ is even, study the derivative for $x > 0$ for $f_1(x) = \sqrt{1-x} - \arcsin \sqrt{1-x} + 2$, on the domain $\mathrm{D}_1 = [0, 1]$.

$$f_1'(x) = \frac{-1}{2\sqrt{1-x}} - \frac{1}{1-(1-x)} \cdot \frac{-1}{2\sqrt{1-x}} = \frac{-1}{2\sqrt{1-x}}\left(1 - \frac{1}{x}\right) = \frac{1-x}{2x\sqrt{1-x}} = \frac{\sqrt{1-x}}{2x}$$

It follows:

$f_1'(x) > 0 \quad \forall x \in (0, 1)$

$\lim_{x \to 0^+} f_1'(x) = +\infty$

$\lim_{x \to 1^-} f_1'(x) = 0$

Since f is even we have:

$f'(x) < 0 \quad \forall x \in (-1, 0)$

$\lim_{x \to 0^-} f'(x) = -\infty$

$\lim_{x \to -1^+} f'(x) = 0$

Thus $x = 0$ is a cusp, whereas $x = \pm 1$ are horizontal tangent points; $x = 0$ is the unique non differentiable point for f.

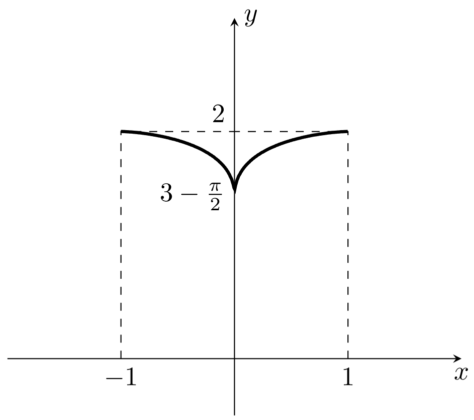
(d) The function f is strictly increasing on $(0, 1)$; strictly decreasing on $(-1, 0)$.

Extrema can be found among stationary points, non differentiable points and boundaries of the domain.
In our case, $x = 0$ and $x = \pm 1$ are extremum points, precisely:

the function f has absolute maxima in $x = -1$ and in $x = 1$ (horizontal tangent points).

the function f has absolute minimum in the cusp $x = 0$ (vertical tangent point).

(d) Below a qualitative graph of f:



(e) From dom$(f) = [-1, 1]$, we can write the definition of continuous prolongation of f on $(0, +\infty)$ as follows:

$$g(x) = \begin{cases} f(x) & x \in (0,1] \\ ax+b & x \in (1,+\infty) \end{cases}$$

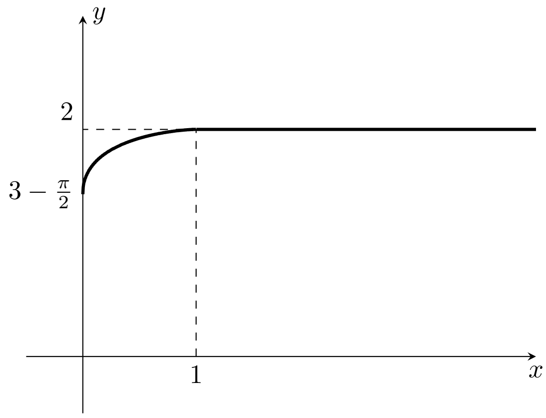
Therefore

$$g'(x) = \begin{cases} f'(x) & x \in (0,1) \\ a & x \in (1,+\infty) \end{cases}$$

The function $g(x)$ is continuous and differentiable on $(0,1)$ and $(1,+\infty)$, because $f(x)$ and the line $ax+b$ are so. Check now $x=1$.
The function $g(x)$ is continuous in $x=1$ if $g(1) = f(1) = 2$, and thus if $\lim_{x\to 1^+}(ax+b) = 2$ that is $a+b=2$.
Concerning differentiability in $x=1$, since $\lim_{x\to 1^-} f'(x) = 0$, we must have $a=0$; therefore

$$g(x) = \begin{cases} f(x) & x \in (0,1] \\ 2 & x \in (1,+\infty) \end{cases}$$

is continuous and differentiable in $(0,+\infty)$.



10 February 2016 - I

Consider

$$f(x) = \frac{\log|x|}{\log^2|x| - \log|x| + 1}.$$

(a) Find the domain, possible symmetries and asymptotes. Show that f admits continuous prolongation in $x=0$.

(b) Denote the prolongation by g, and compute its derivative. Find non differentiable points for g and specify their type.

(c) Find monotonicity intervals for g and minima/maxima, saying if they are local or global.

(d) Draw a qualitative graph for g.

(e) Find the solutions of $g(x) = 1$

Solution

(a) Denominators must be different from 0 therefore dom $f = \mathbb{R} \setminus \{0\}$.

$$f(-x) = \frac{\log|-x|}{\log^2|-x| - \log|-x| + 1} = \frac{\log|x|}{\log^2|x| - \log|x| + 1} = f(x).$$

The function is even.

Limits at boundary points of the domain are:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = 0\,; \quad \lim_{x\to 0} f(x) = 0.$$

Thus the line $y = 0$ is (left and right) horizontal asymptote for f.
Moreover, f admits continuous prolongation on $\mathbb{R}$, defined as:

$$g(x) = \begin{cases} \dfrac{\log|x|}{\log^2|x| - \log|x| + 1} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0. \end{cases}$$

(b) As previously said:

$$g(x) = \begin{cases} \dfrac{\log|x|}{\log^2|x| - \log|x| + 1} & \text{if } x \neq 0 \\ \\ 0 & \text{if } x = 0. \end{cases}$$

In order to check differentiability in $x = 0$, compute the limit of the difference quotient

$$\lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{\frac{\log|x|}{\log^2|x| - \log|x| + 1}}{x} = \lim_{x\to 0} \frac{\log|x|}{x(\log^2|x| - \log|x| + 1)}$$

Compute the right limit and apply the substitution
$\log x = t \Rightarrow x = e^t$

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \frac{\log(x)}{x(\log^2(x) - \log(x) + 1)} = \lim_{t\to -\infty} \frac{t}{e^t(t^2 - t + 1)} = \lim_{t\to -\infty} \frac{1}{e^t(t)} = -\infty$$

Since g is even, we have

$$\lim_{x\to 0^-} g(x) = +\infty$$

Then $x = 0$ is a cusp.

(c) In order to find monotonicity intervals for g and maxima and minima, study the derivative g' for $x > 0$ (g is even):
$g(x) = \dfrac{\log(x)}{\log^2(x) - \log(x) + 1}$

$$\begin{aligned} g'(x) &= \frac{\frac{1}{x}(\log^2(x) - \log(x) + 1) - \log(x)(2\log(x)\frac{1}{x} - \frac{1}{x})}{(\log^2(x) - \log(x) + 1)^2} \\ &= \frac{(\log^2(x) - \log(x) + 1) - \log(x)(2\log(x) - 1)}{x(\log^2(x) - \log(x) + 1)^2} \\ &= \frac{\log^2(x) - \log(x) + 1 - 2\log^2(x) + \log(x)}{x(\log^2(x) - \log(x) + 1)^2} \\ &= \frac{-\log^2(x) + 1}{x(\log^2(x) - \log(x) + 1)^2} \end{aligned}$$

For $x > 0$:

$g'(x) > 0 \Leftrightarrow -\log^2(x) + 1 > 0 \Leftrightarrow \dfrac{1}{e} < x < e \qquad g'(x) < 0 \Leftrightarrow \dfrac{-\log^2(x) + 1}{x} < 0 \Leftrightarrow 0 < x < \dfrac{1}{e} \vee x > e$

The function g is stricty increasing on $\left(\dfrac{1}{e}, e\right)$, and strictly decreasing on $\left(0, \dfrac{1}{e}\right)$ and $(e, +\infty)$.

Since g is even, we have:

g is stricty decreasing on $\left(-e, -e^{-1}\right)$, whereas it is strictky increasing on $\left(-e^{-1}, 0\right)$ and $(-\infty, -e)$.

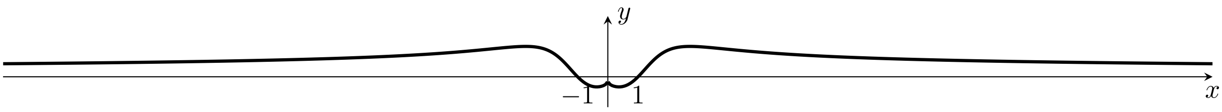
Stationary points for f are $x = -e$, $x = -e^{-1}$, $x = e^{-1}$ and $x = e$ and we have a non differentiable point in $x = 0$. From the sign of g' we know that $x = -e$, $x = -e^{-1}$, $x = e^{-1}$, $x = e$ and $x = 0$ are extrema.

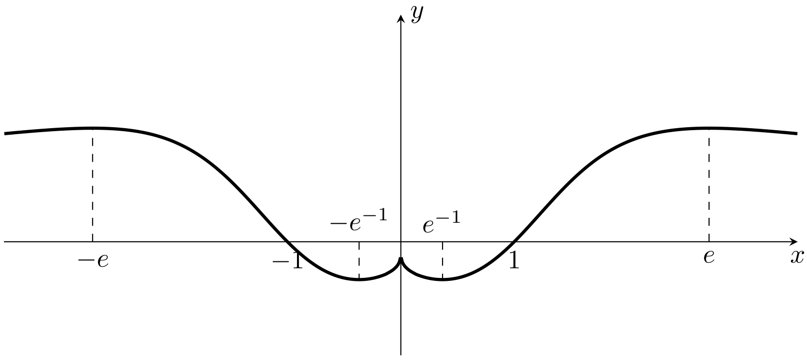
The function g has absolute maxima in $x = -e$ and in $x = e$.

The function g has absolute minima in $x = -e^{-1}$ and in $x = e^{-1}$.

The function g has relative maximum in $x = 0$.

(d) Below a qualitative graph for g.





(e) In order to find the roots of the equation $g(x) = 1$, we have to compute the value of the absolute maxima:

$$f(\pm e) = \frac{\log|\pm e|}{\log^2|\pm e| - \log|\pm e| + 1} = \frac{1}{1} = 1.$$

Therefore $g(x) = 1$ has two solutions: $x = -e$ and $x = e$.

10 February 2016 - II

Consider

$$f(x) = (\sinh 2x)^2 - 2\sinh 2x - 3 \qquad \text{defined for } x \geq 0.$$

(a) Find limits at boundary points of the domain and asymptotes.

(b) Compute the derivative and show that there exists a unique points $x_0 > 0$ such that $f'(x_0) = 0$ (the value of x_0 is not required).

(c) Find monotonicity intervals and say if there are maximum or minimum points.

(d) Draw a qualitative graph.

(e) Draw $g(x) = f(|x|)$ defined for $x \in R$ and investigate non differentiable points.

Solution

(a) The function $f(x)$ is the restriction to $x \geq 0$ of the composite function of the following defined on $\mathbb{R}$

$$f(x) = k(h(x)), \qquad h(x) = \sinh 2x\,, \quad k(t) = t^2 - 2t - 3\,. \tag{1}$$

Hence it is defined and continuous in $[0, +\infty)$. In particular, **it does not have vertical asymptote** and

$$\lim_{x\to 0^+} f(x) = f(0) = -3\,.$$

The Theorem on composite functions shows that $\lim_{x\to+\infty} f(x) = +\infty$.

The function **has no oblique asymptote** since the order of infinity is greater than 1.

(b) Compute the derivative of $f(x)$:

$$f'(x) = 4\,(\sinh 2x)\cosh 2x - 4\cosh 2x = 4\,(\cosh 2x)\,\{\sinh 2x - 1\}\ . \tag{2}$$

Recall thatcosh $2x$ is never zero.
The function $f'(x)$ is zero for $x > 0$ applying the Existence of Zeros Theorem, because $\sinh 2x - 1$ is **continuous** and

$$\lim_{x\to 0^+} \sinh 2x - 1 = -1\,, \qquad \lim_{x\to +\infty} \sinh 2x - 1 = +\infty\,.$$

Note that x_0 can be easily computed, even if it is not required. It suffices to solve

$$\sinh 2x = 1 \quad \text{that is} \quad e^{2x} - e^{-2x} = 2\,.$$

Moltiply both sides by e^{2x} we get

$$y^2 - 2y - 1 = 0 \qquad \text{where } y = e^{2x}.$$

The exponential is never negative and thus the only solution for the equation is

$$y = 1 + \sqrt{5} \quad \text{that implies} \quad x_0 = \frac{1}{2}\log\left(1 + \sqrt{5}\right)\ .$$

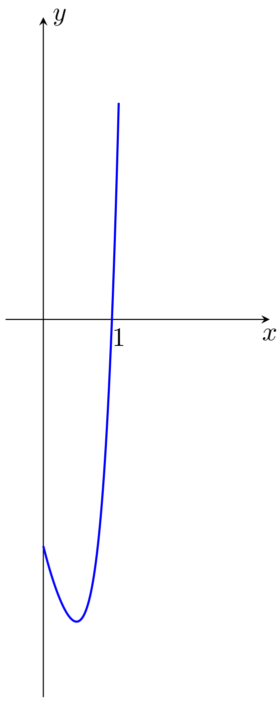
(c) Recall that $\cosh 2x > 0$ and $\sinh 2x$ is monotone, then $f(x)$ is **increasing** for $x > x_0 = \frac{1}{2}\log\left(1 + \sqrt{5}\right)$ and **decreasing** on $(0, x_0)$. The point $x = 0$ is a relative maximum, the point x_0 is an absolute minimum. There is no absolute maximum because the function is unbounded from above.

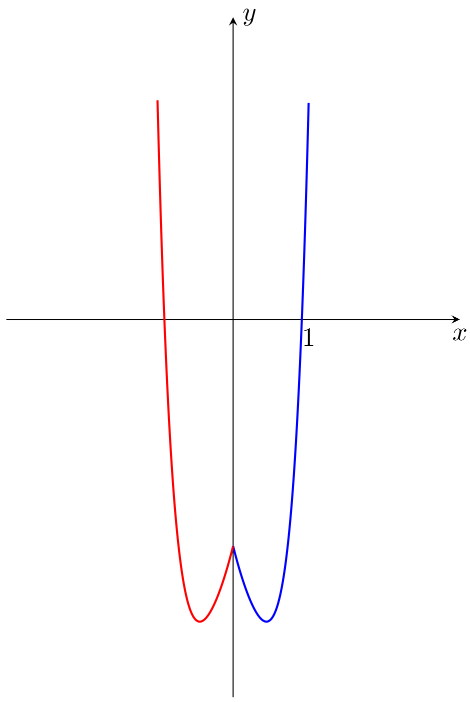
(d) The graph is below (left).

(e) The function $g(x)$ is the even extension of $f(x)$ and its graph is symmetric to $f(x)$ w.r.t. the y-axis. See the graph below (right).
The function $g(x)$ is continuous on $\mathbb{R}$ as it is composition of differentiable functions, and thus it is differentiable for $x \neq 0$.
Note that

$$\lim_{x\to 0^+} f'(x) = -4$$

it follows that $g(x)$ is not differentiable in $x = 0$ ($\lim_{x\to 0^-} g'(x) = 4 \neq \lim_{x\to 0^+} g'(x) = -4$).





10 February 2016 - III
Consider

$$f(x) = 2\log|e^{2x} - 3e^{x}|$$

(a) Find the domain of f, and limits at boundary points.

(b) Find the asymptote for f as $x \to +\infty$

(c) Compute the derivative of f.

(d) Find monotonicity intervals for f and relative/absolute extremal points.

(e) Draw a qualitative graph for f.

(f) Find the number of zeros of f.

Solution

(a) It is sufficient to impose the argument of the logarithm different from 0:
$e^{2x} - 3e^x \neq 0 \iff e^x(e^x - 3) \neq 0 \iff e^x \neq 3 \iff x \neq \log 3$.
Thus $\text{dom} f = (-\infty, \log 3) \cup (\log 3, +\infty)$.
At boundary points of the domain:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} 2\log|e^{2x} - 3e^x| = -\infty$$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} 2\log|e^{2x} - 3e^x| = \lim_{x\to+\infty} 2\log|e^{2x}| = +\infty$$

$$\lim_{x\to\log 3^-} f(x) = \lim_{x\to\log 3^+} f(x) = \lim_{x\to\log 3} 2\log(|e^x||e^x - 3|) = -\infty$$

Therefore, the line $x = \log 3$ is a vertical asymptote for the function. There are no horizontal asymptotes.

(b) As $x \to +\infty$, we can re-write $f(x)$ as follows:

$f(x) = 2\log|e^{2x} - 3e^x| = 2\log(e^{2x} - 3e^x) = 2\log(e^{2x}(1 - 3e^{-x})) = 2\log e^{2x} + 2\log(1 - 3e^{-x}) = 4x + 2\log(1 - 3e^{-x})$

Thus the oblique asymptote has equation $y = mx + q$, with

$$m = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{4x + 2\log(1 - 3e^{-x})}{x} = 4$$

and

$$q = \lim_{x\to+\infty} (f(x) - 4x) = \lim_{x\to+\infty} (4x + 2\log(1 - 3e^{-x}) - 4x) = 0$$

Thus the oblique asymptote has equation $y = 4x$.

(c) Recall that the derivative of $f(x) = \log|h(x)|$ is $f'(x) = \dfrac{h'(x)}{h(x)}$, then:

$$f'(x) = 2\,\frac{2e^{2x} - 3e^x}{e^{2x} - 3e^x} = 2\,\frac{e^x(2e^x - 3)}{e^x(e^x - 3)} = 2\,\frac{2e^x - 3}{e^x - 3}$$

(d) Study the sign of $f'(x)$:

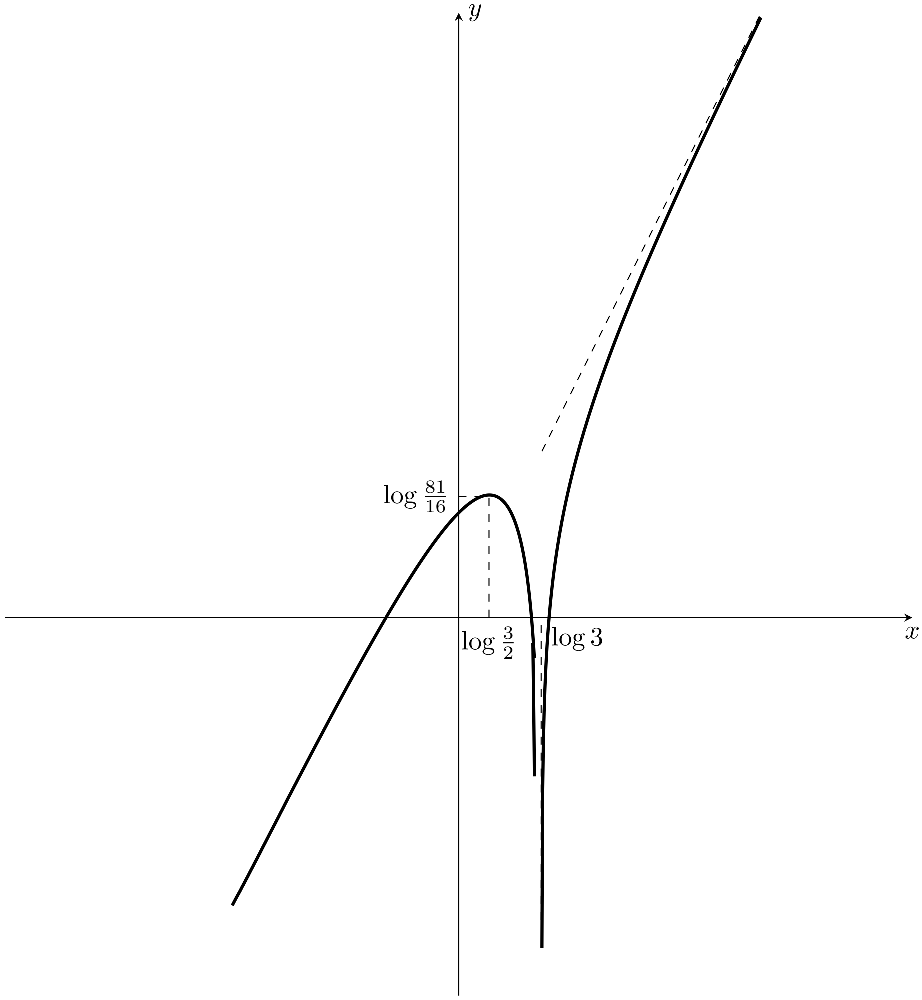
$$f'(x) < 0 \iff \log\frac{3}{2} < x < \log 3; \qquad f'(x) > 0 \iff x < \log\frac{3}{2} \ \vee \ x > \log 3\,.$$

We conclude that f is strictly decreasing in $(\log\frac{3}{2}, \log 3)$ and it is strictly increasing in $(-\infty, \log\frac{3}{2})$ and in $(\log 3, +\infty)$.

The point $x = \log\frac{3}{2}$ is a relative maximum; there are no absolute maxima or minima (because the function is not upper nor lower bounded).

(e) In order to draw a qualitative graph of f, compute the value of the relative maximum:

$$f\left(\log\frac{3}{2}\right) = 2\log\left|\frac{9}{4} - \frac{9}{2}\right| = \log\frac{81}{16} > 0$$



(f) Find the number of zeros of f, applying three times the Zeros Existence Theorem, one for each monotonicity interval:

in $I_1 = (-\infty, \log \frac{3}{2})$ the function is continuous and strictly increasing; $\lim\limits_{x\to-\infty} f(x) < 0$ and $f\left(\log \frac{3}{2}\right) > 0$; thus f has a unique zero in I_1;

in $I_2 = \left(\log \frac{3}{2}, \log 3\right)$ the function is continuous and strictly decreasing; $f\left(\log \frac{3}{2}\right) > 0$ and $\lim\limits_{x\to\log 3^-} f(x) < 0$; thus f has a unique zero in I_2;

in $I_3 = (\log 3, +\infty)$ the function is continuous and strictly increasing; $\lim\limits_{x\to\log 3^+} f(x) < 0$, $\lim\limits_{x\to+\infty} f(x) > 0$; thus f has a unique zero in I_3.

In conclusion $f(x)$ has three zeros.

1 February 2017 - I

Given the function

$$f(x) = 1 - \left|\frac{1}{x^2 - 5x + 6}\right|.$$

(a) Find domain, limits at the boundary points of the domain and asymptotes.

(b) Compute the derivative of f. Study the monotonicity intervals and a qualitative graph for f.

(c) Consider now the composite function $g(x) = e^{f(x)}$. Using the information on f, find the domain of g and the limits at the boundary points. Verify that g admits a continuous prolongation $h(x)$.

(d) Find the monotonicity intervals and maximum or minimum points of h.

(e) Trace a qualitative graph of h, and find upper and lower extrema.

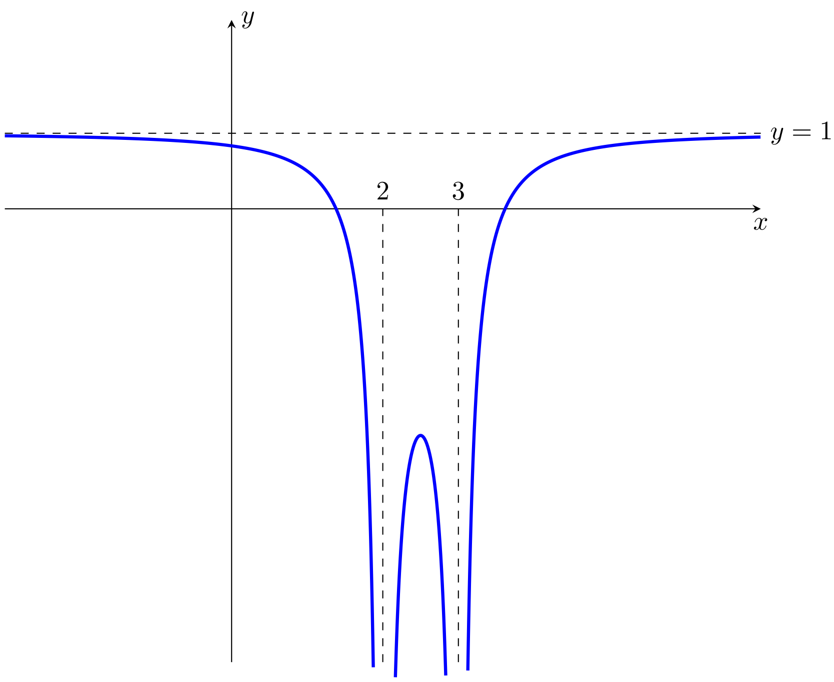
Solution

1. The function is defined if $x^2 - 5x + 6 \neq 0$ and thus
$$\operatorname{dom} f(x) = \mathbb{R} \setminus \{2,3\} = (-\infty, 2) \cup (2,3) \cap (3, +\infty)\,.$$
Moreover
$$\lim_{x \to 2} f(x) = \lim_{x \to 3} f(x) = -\infty$$
and
$$\lim_{x \to \pm\infty} f(x) = 1\,.$$
Thus
   (a) vertical asymptotes: $x = 2$ and $x = 3$;
   (b) (left and right) horizontal asymptote: $y = 1$.
   (c) There are no oblique asymptotes.

   Note that
   - $f(x) < 1$ for every x and thus the function is upper bounded.
   - The function $f(x)$ is continuous and therefore, by *Weierstrass Theorem*, $f(x)$ has at least a maximum point in $(2,3)$.

2. Note that
$$f(x) = \begin{cases} 1 - \dfrac{1}{x^2 - 5x + 6} & \text{if} \quad x < 2 \text{ e } x > 3\,; \\ 1 + \dfrac{1}{x^2 - 5x + 6} & \text{if} \quad 2 < x < 3\,. \end{cases}$$
hence
$$f'(x) = \begin{cases} \dfrac{2x - 5}{(x^2 - 5x + 6)^2} & \text{if} \quad x < 2 \text{ e } x > 3\,; \\ -\dfrac{2x - 5}{(x^2 - 5x + 6)^2} & \text{if} \quad 2 < x < 3 \end{cases}$$
Since $2x - 5$ has constant sign for $x < 2$ and $x > 3$ but the sign changes in $(2,3)$. We have:
   - the function **decreases** for $x < 2$ and **increases** for $x > 3$;
   - the function **increases** on $(2, 5/2)$ and **decreases** on $(5/2, 3)$. The point $x_0 = 5/2$ is a maximum point.

   From $f(5/2) < 1$, it is a relative maximum. The graph is the following



3. (a) The function $g(x)$ has the same domain of $f(x)$ because the exp function is always defined;

(b) From Composite Functions Theorem we have

- $g(x)$ is continuous, by composition of continuous functions;
- $$\lim_{x\to\pm\infty} g(x) = e\,.$$

  Hence, $y = e$ is (left and right) horizontal asymptote.
- $g(x)$ tends to 0 when $f(x)$ tends to $-\infty$, hence for $x \to 2$ and $x \to 3$.
- Therefore, in both values, $g(x)$ admits continuous prolongation $h(x)$, defined as

  $$h(2) = 0\,, \qquad h(3) = 0\,.$$

  The function h is continuous on $\mathbb{R}$.

(c) The function e^y is increasing, and thus $h(x)$ is increasing (or decreasing) in the same intervals of $f(x)$. In particular, the point $5/2$ is a relative maximum for $h(x)$. We have $h(x) = 0$ if $x = 2$ and $x = 3$, whereas $h(x) > 0$ on $\mathbb{R} \setminus \{2\,,\ 3\}$. Hence the points 2 and 3 are absolute minima for h. Moreover, h is differentiable in $x = 2$ and in $x = 3$, and it holds $h'(3) = 0 = h'(2)$.

(d) The graph is below . The lower bound for h is 0 (it is absolute minimum), the upper bound is e. There is no absolute maximum.

4 **July 2017**

Consider the function

$$f(x) = (x+1)\cdot e^{\frac{1}{|x+1|-2}}\,.$$

(a) *Find domain, symmetry properties, limits at boundary points of the domain and asymptotes, if there are any.*

Since $|x+1| = 2 \Leftrightarrow x = -3\ \vee x = 1$, the domain of f is
D=dom$(f) = (-\infty, -3) \cup (-3, 1) \cup (1, +\infty)$.
It holds:

$$f(x) = \begin{cases} (x+1)e^{\frac{-1}{x+3}}, & \text{if } x < -1, x \neq -3 \\ \\ (x+1)e^{\frac{1}{x-1}}, & \text{if } x > -1, x \neq 1 \end{cases}$$

$\lim_{x\to-3^-} f(x) = -\infty$; $\lim_{x\to-3^+} f(x) = 0$: the line $x = -3$ is a left vertical asymptote.
$\lim_{x\to1^-} f(x) = 0$; $\lim_{x\to1^-+} f(x) = +\infty$: the line $x = 1$ is a right vertical asymptote.
$\lim_{x\to-\infty} f(x) = -\infty$; $\lim_{x\to+\infty} f(x) = +\infty$; since there are no horizontal asymptotes, we look for oblique asymptotes, starting from right:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = 1$$

$$\lim_{x\to+\infty}(f(x) - x) = \lim_{x\to+\infty} x\left(e^{\frac{1}{x-1}} - 1\right) + \lim_{x\to+\infty} e^{\frac{1}{x-1}} = \lim_{t\to0^+}\left(1 + \frac{1}{t}\right)(e^t - 1) + 1 = 2$$

(apply the substitution $\frac{1}{x-1} = t$, thus $x = 1 + \frac{1}{t}$); therefore the line $y = x + 2$ is a right oblique asymptote for f.
Analogously (apply the substitution $\frac{-1}{x+3} = t$), the line $y = x - 1$ is a left oblique asymptote for f.

(b) *Study differentiability of $f(x)$ in each point of the domain, and compute the derivative.*

It holds:

$$f'(x) = \begin{cases} e^{\frac{-1}{x+3}}\ \dfrac{x^2 + 7x + 10}{(x+3)^2}, & \text{if } x < -1, x \neq -3 \\ \\ e^{\frac{1}{x-1}}\ \dfrac{x^2 - 3x}{(x-1)^2}, & \text{if } x > -1, x \neq 1 \end{cases}$$

$f(x)$ is differentiable in every point of the domain, also in $x = -1$, since it holds $\lim_{x\to-1^-} f'(x) = \lim_{x\to-1^+} f'(x) = \dfrac{1}{\sqrt{e}}$. Therefore $f'(-1) = \dfrac{1}{\sqrt{e}}$.

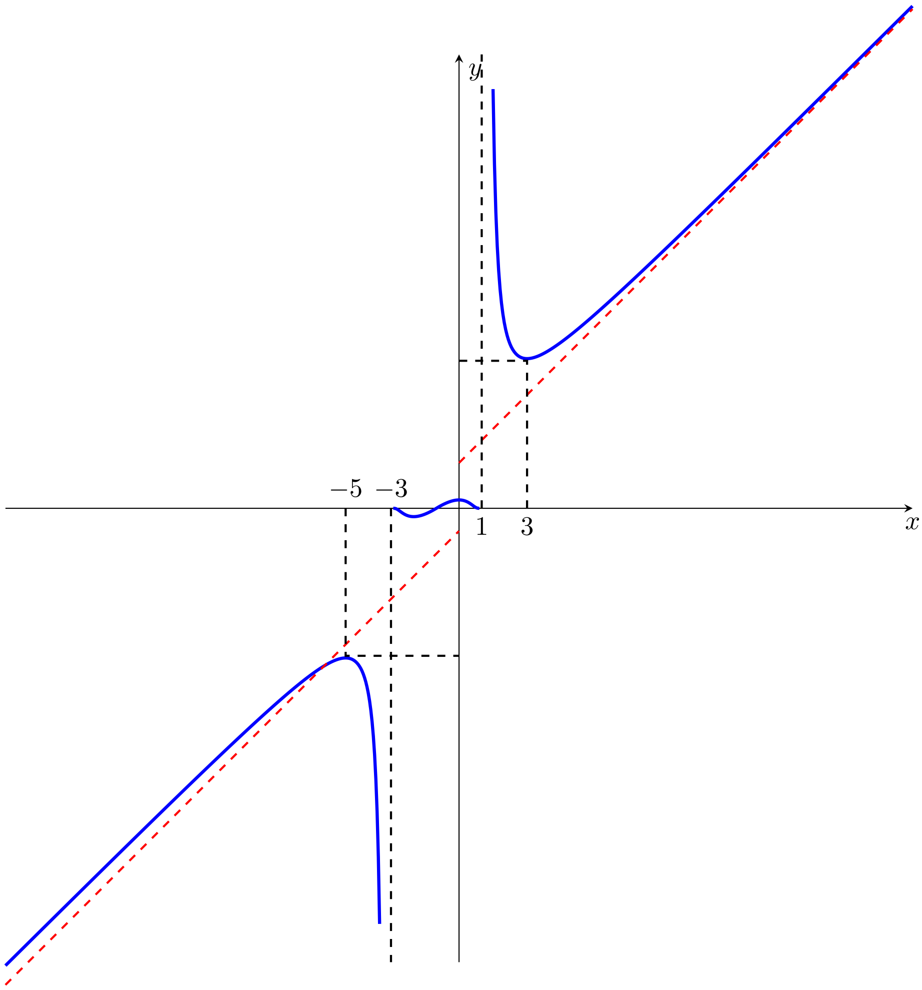
(c) *Find monotonicity intervals and maximum/minimum points. Say if they are relative or absolute.*

It holds:
- $f'(x) = 0 \Leftrightarrow x = -5 \ \vee \ x = -2; \quad x = 0 \ \vee \ x = 3$
- $f'(x) > 0 \Leftrightarrow x < -5 \ \vee -2 < x < -1; \quad -1 < x < 0 \ \vee x > 3$
- $f'(x) < 0 \Leftrightarrow -5 < x < -3 \ \vee -3 < x < -2; \quad 0 < x < 1 \ \vee \ 1 < x < 3$

Hence:
- f is strictly increasing in the intervals $(-\infty, -5)$, $(-2, 0)$ e $(3, +\infty)$
- f is strictly decreasing in the intervals $(-5, -3)$, $(-3, -2)$, $(0, 1)$ e $(1, 3)$
- the points $x = -5$, $x = 0$ are relative maxima
- the points $x = -2$, $x = 3$ are relative minima

(d) *Trace a qualitative graph.*

In order to plot f, compute the ordinates of the extremum points:
$f(-5) = -4\sqrt{e}, \quad f(-2) = -\frac{1}{e}, \ f(0) = \frac{1}{e}, \quad f(3) = 4\sqrt{e}$.
Note the symmetry w.r.t. (with respect to) the point $(-1, 0)$ (if we translate the origin in that point, f would be odd); finally note that
$\lim_{x \to 1^-} f'(x) = \lim_{x \to -3^+} f'(x) = 0.$



(e) *Say if there exists a continuous prolongation of f in the interval $[-3, 1]$.*

Define

$$\tilde{f}(x) = \begin{cases} (x+1) \cdot e^{\frac{1}{|x+1|-2}} & \text{if } -3 < x < 1 \\ 0 & \text{if } x = -3 \vee x = 1 \end{cases}$$

the function $\tilde{f}$ is continuous on $[-3, 1]$ (and differentiable too).

19 September 2017

Consider the function

$$f(x) = \frac{|x|^3}{x^2-16}.$$

(a) *Find domain, symmetry properties, limits at boundary points of the domain and asymptotes, if there are any.*

D=dom$(f) = \{x \in \mathbb{R} : x^2 - 16 \neq 0\} = (-\infty, -4) \cup (-4, 4) \cup (4, +\infty)$.

Note that, $\forall x \in D$, $f(-x) = f(x)$, and thus f is even, and $f(x) = 0 \Leftrightarrow x = 0$.
Study $f(x)$ only for $x \geq 0$, i.e. study $h(x) = \dfrac{x^3}{x^2-16}$, $x \geq 0$.
For $x < 0$, we have $f(x) = \dfrac{-x^3}{x^2-16} = -h(x)$.
$\lim\limits_{x\to+\infty} h(x) = +\infty$: there are no horizontal asymptotes.
$\lim\limits_{x\to4^-} h(x) = -\infty$, $\lim\limits_{x\to4^+} h(x) = +\infty$: hence the line $x = 4$ is a vertical asymptote.
$\lim\limits_{x\to+\infty} \dfrac{h(x)}{x} = 1$, $\lim\limits_{x\to+\infty} (h(x) - x) = 0$: therefore the line $y = x$ is a right oblique asymptote.

(b) *Study differentiability of $f(x)$ in each point of the domain, and compute the derivative.*

It holds $h'(x) = \dfrac{x^2(x^2-48)}{(x^2-16)^2}$; then:

$$f'(x) = \begin{cases} -\dfrac{x^2(x^2-48)}{(x^2-16)^2}, & \text{if } x < 0 \\ \\ \dfrac{x^2(x^2-48)}{(x^2-16)^2}, & \text{if } x > 0 \end{cases}$$

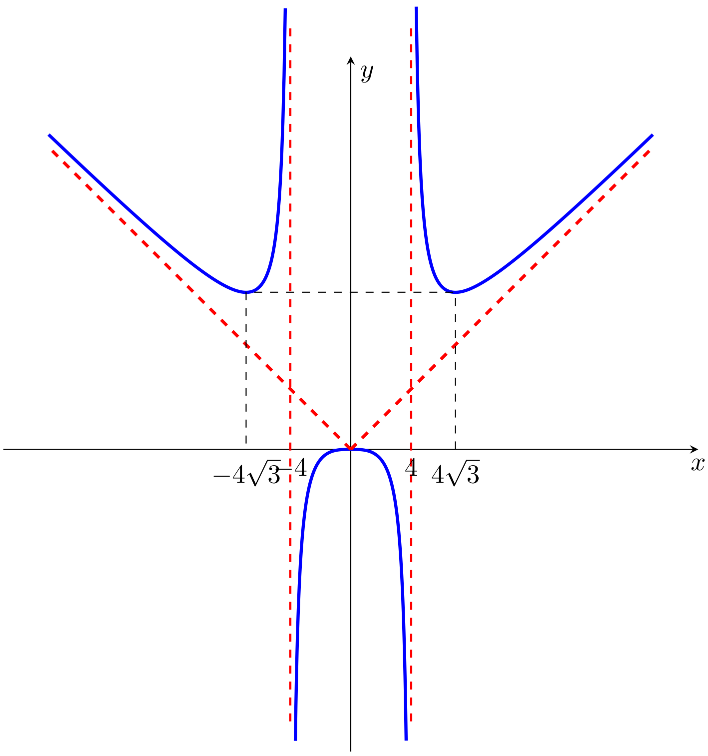
f is differentiable in $x = 0$, because it is continuous and $\lim\limits_{x\to0^-} f'(x) = \lim\limits_{x\to0^+} f'(x) = 0$; then $f'(0) = 0$; ut follows that f is differentiable on its domain.

(c) *Find monotonicity intervals and maximum/minimum points. Say if they are relative or absolute.*

Critical points of f are $x = 0$ and $x = \pm 4\sqrt{3}$. Observing the two expressions of f', and knowing the symmetry of f, we have that:
- f is decreasing in the intervals $(-\infty, -4\sqrt{3})$, $(0, 4)$ and $(4, 4\sqrt{3})$
- f is decreasing in the intervals $(-4\sqrt{3}, -4)$, $(-4, 0)$ and $(4\sqrt{3}, +\infty)$
- the points $x = \pm 4\sqrt{3}$ are relative minima
- the point $x = 0$ is a relative maximum.

(d) *Trace a qualitative graph of $f(x)$.*

Compute the ordinates at the critical points:
$f(0) = 0$, $f(\pm 4\sqrt{3}) = 6\sqrt{3}$.
Here we have a qualitative graph for f.



(e) *Let $g(x)$ be defined as $g(x) = \dfrac{|x|^n}{x^2-16}$, with n integer. Say for which values of $n \in \mathbb{Z}$, the function $g(x)$ is differentiable in $x = 0$.*

Note that:
- if $n < 0$, the function is not defined, and thus it is not differentiable in $x = 0$
- if $n = 0$, the function $\dfrac{1}{x^2-16}$ is differentiable in $x = 0$.
- if $n > 0$, it holds

$$f'(x) = \begin{cases} -\dfrac{x^{n-1}((n-2)x^2-16n)}{(x^2-16)^2}, & \text{if } x < 0 \\ \dfrac{x^{n-1}((n-2)x^2-16n)}{(x^2-16)^2}, & \text{if } x > 0 \end{cases}$$

- if $n = 1$ we have $f'(x) = \pm\dfrac{x^2+16}{(x^2-16)^2}$ and thus $f(x)$ is not differentiable in $x = 0$
- if $n \geq 2$, f is differentiable in $x = 0$, since it is continuous and $\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$; then $f'(0) = 0$.

31 **January 2018** - I

Consider the function

$$f(x) = e^3 - e^{4\sqrt{|x|}-x}.$$

(a) *Find domain, symmetry properties, limits at boundary points of the domain and asymptotes, if there are any.*

The domain of f is the whole $\mathbb{R}$.
Zeros of f:

$$f(x) = 0 \quad \Leftrightarrow e^{4\sqrt{|x|}-x} = e^3 \ \Leftrightarrow 4\sqrt{|x|} - x = 3 \ \Leftrightarrow 4\sqrt{|x|} = x + 3$$

If $x > 0$, we solve the equation $4\sqrt{x} = x + 3 \ \Leftrightarrow 16x = (x+3)^2 \Leftrightarrow x = 1 \vee x = 9$.
If $x < 0$, we have $4\sqrt{-x} = x + 3 \ \ \Leftrightarrow \ \ 16(-x) = (x+3)^2 \ \wedge \ (x+3 > 0) \Leftrightarrow x = -11 + 4\sqrt{7}$ (the solution $x = -11 - 4\sqrt{7}$ is not acceptable because $x + 3 < 0$).
In conclusion, the zeros of $f(x)$ are $x = 4\sqrt{7} - 11$, $\ x = 1$ and $x = 9$.
Since

$$\lim_{x\to-\infty}(4\sqrt{-x} - x) = +\infty, \quad \lim_{x\to+\infty}(4\sqrt{x} - x) = -\infty$$

it holds:

$$\lim_{x\to-\infty} f(x) = -\infty, \quad \lim_{x\to+\infty} f(x) = \ e^3$$

Then the line $y = e^3$ is a right horizontal asymptote; there are no left oblique asymptotes, because the function tends to $-\infty$ exponentially.

(b) *Study differentiability of $f(x)$ on its domain, and establish the nature of its non differentiable points. Compute the derivative $f'(x)$.*

We have that:

$$f'(x) = \begin{cases} \left(1 + \dfrac{2}{\sqrt{-x}}\right) e^{4\sqrt{-x}-x}, & \text{if } x < 0 \\ \left(1 - \dfrac{2}{\sqrt{x}}\right) e^{4\sqrt{x}-x}, & \text{if } x > 0 \end{cases}$$

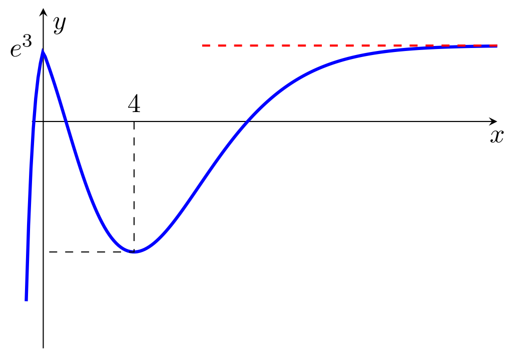
$f(x)$ is not differentiable in $x = 0$, because $\lim\limits_{x\to0^-} f(x) = +\infty$ and $\lim\limits_{x\to0^+} f(x) = -\infty$. Then the point $x = 0$ is a cusp for f.

(c) *Find monotonicity intervals and maximum/minimum points. Say if they are relative or absolute.*

If $x < 0$, we have $f'(x) > 0$ and thus $f(x)$ is strictly increasing.
If $x > 0$, it holds $f'(x) = 0 \Leftrightarrow x = 4$; $f'(x) > 0 \Leftrightarrow x > 4$; $f'(x) < 0 \Leftrightarrow 0 < x < 4$.
Therefore:
- f is increasing in $(-\infty, 0)$ and in $(4, +\infty)$
- f is decrasing in $(0, 4)$
- the point $x = 4$ is a relative minimum for f (indeed $inf(f) = -\infty$)
- the point $x = 0$ is a relative maximum (cusp) for f (indeed $sup(f) = e^3$)

(d) *Trace a qualitative graph.*

Compute the ordintes at extrema:
$f(0) = e^3$, $f(4) = e^3 - e^4$.
In the figure we depict the qualitative graph of f.



(e) *Find the largest interval in the form $(k, +\infty)$, $k \in \mathbb{R}$, such that the restriction of f on such interval, is invertible. Find the domain and study the monotonicity of the inverse function.*

The function is strictly monotone, and thus invertible on $A = [4, +\infty)$.
Let $g = f/A$ and let h be the inverse function of g; since the range of g is $B = [e^3 - e^4, +\infty)$, the domain of h coincides with B. Since g is increasing on A, also h is increasing on B.
Indeed, consider $x_1, x_2 \in A$ such that $x_1 < x_2$ and let $y_1 = g(x_1), y_2 = g(x_2)$; as g is strictly increasing on A, it holds $y_1 < y_2$.
We have that $h(y_1) = g^{-1}(g(x_1)) = x_1$ and $h(y_2) = g^{-1}(g(x_2)) = x_2$, therefore $h(y_1) < h(y_2)$ and thus h is strictly increasing on B.

31 January 2018 - II *Consider the function*

$$f(x) = \frac{|x|}{\sqrt{x^2 - x - 2}}.$$

(a) *Find domain, symmetry properties, limits at boundary points of the domain and asymptotes, if there are any.*

D=dom$(f) = \{x \in \mathbb{R} : x^2 - x - 2 > 0\} = (-\infty, -1) \cup (2, +\infty)$.
Note that $\forall x \in D$, $f(x) \geq 0$, and $f(x) = 0 \Leftrightarrow x = 0$; since $0 \notin D$, f is never zero, then it is strictly positive.
$\lim_{x\to\pm\infty} f(x) = 1$: hence the line $y = 1$ is a complete horizontal asymptote for f.
$\lim_{x\to-1^-} f(x) = +\infty = \lim_{x\to 2^+} f(x)$: hence the lines $x = -1$ and $x = 2$ are vertical asymptotes.

(b) *Study differentiability of $f(x)$ in each point of the domain, and compute the derivative.*

It holds:

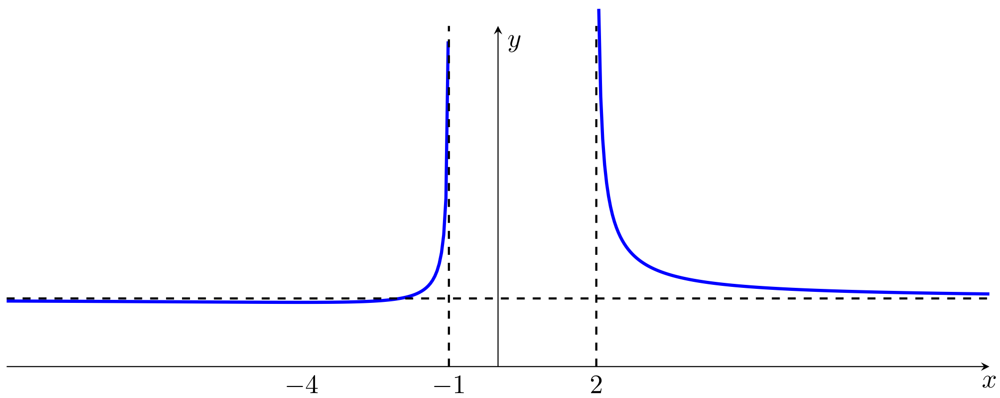
$$f'(x) = \begin{cases} \dfrac{x+4}{2\sqrt{(x^2-x-2)^3}}, & \text{if } x < -1 \\[2ex] -\dfrac{x+4}{2\sqrt{(x^2-x-2)^3}}, & \text{if } x > 2 \end{cases}$$

f is differentiable on its domain.

(c) *Find monotonicity intervals and maximum/minimum points. Say if they are relative or absolute.*

Observing the two expressions of f', we have that in both cases, denominator is strictly positive on D.
If $x > 2$, numerator is also strictly positive, and thus the fraction is negative; that is, for $x > 2$, $f(x)$ is strictly decreasing.
On the other hand, if $x < -1$numerator is zero in $x = -4$ and $f'(x) < 0$, for $x < -4$. Then $f(x)$ decreases if $x < -4$, whereas it increases if $-4 < x < -1$. Since $f(-4) = \dfrac{4}{\sqrt{18}} < 1$, the point $x = -4$ is an absolute minimum point for f.
In conclusion:
- f is decreasing in the intervals $(-\infty, -4)$ and $(2, +\infty)$
- f is increasing in the interval $(-4, -1)$
- the point $x = -4$ is an absolute minimum point
- there are no maxima, because $sup(f) = +\infty$

(d) *Trace a qualitative graph.*



(e) *Represent in the plane, the following set*

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : \; y^2(x^2 - x - 2) - x^2 = 0\}.$$

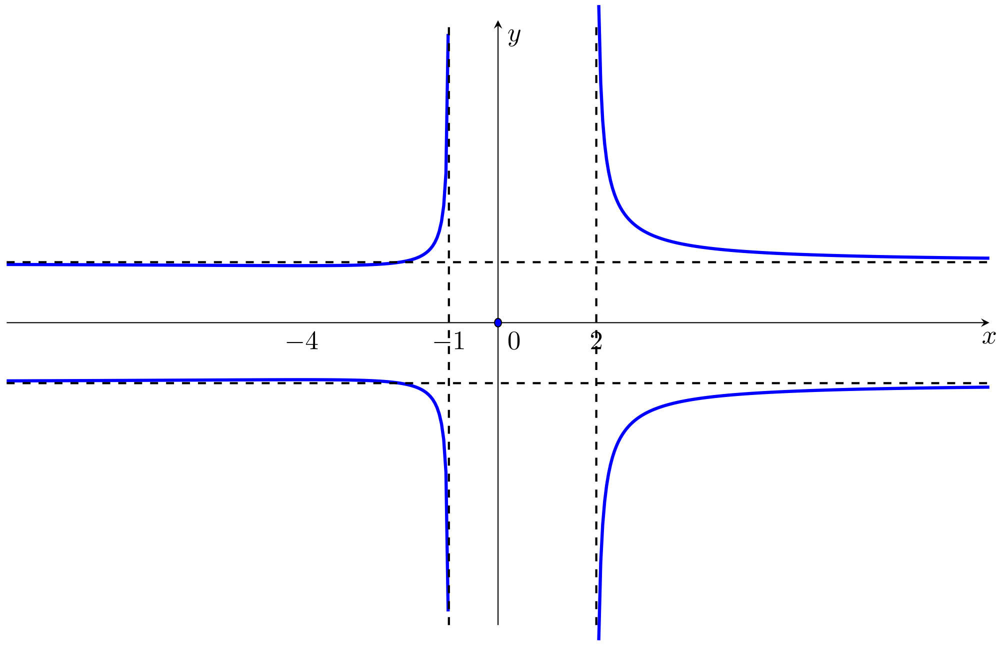
The point $(0, 0)$ belongs to the set, since if satisfies the equation $y^2(x^2 - x - 2) - x^2 = 0$.
Solve w.r.t. y and suppose $x \neq -1$ and $x \neq 2$, then:

$$y^2 = \frac{x^2}{x^2 - x - 2}.$$

Note that $x^2 - x - 2 > 0$, i.e. $x < -1 \; \vee x > 2$; therefore we can find y:

$$y = \pm\frac{|x|}{\sqrt{x^2 - x - 2}}.$$

The solutions for the equation $y^2(x^2 - x - 2) - x^2 = 0$ verify the condition
$y = \pm f(x)$.
The considered set is made of the union of the points in the graph of $f(x)$, the ones in the graph of $-f(x)$, and the origin
Here we depict qualitatively the set.



13 February 2018 - I

Consider the function

$$f(x) = \log|\sqrt[3]{x} + 2| + \sqrt[3]{x}.$$

(a) *Find domain, symmetry properties, limits at boundary points of the domain and asymptotes, if there are any.*
Since $\sqrt[3]{x} + 2 = 0 \Leftrightarrow x = -8$, the domain of f is the set $D = \mathbb{R}\backslash\{-8\}$.
Since $f(x) = \sqrt[3]{x} + o(\sqrt[3]{x})$, for $x \to \pm\infty$, it holds $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.
Therefore, there are no horizontal asymptotes, nor oblique ones, because the order of infinity of f is $1/3$.
Note that $\lim_{x\to-8^\pm} f(x) = -\infty$, thus the line $x = -8$ is a vertical asymptote for f.

(b) *Study differentiability of $f(x)$ in each point of the domain, and compute the derivative.*
Remember that $\log|g(x)|$ coincides with the derivative of $\log g(x) = \frac{g'(x)}{g(x)}$, we have:

$$f'(x) = \frac{1}{3}x^{-2/3}\frac{1}{\sqrt[3]{x}+2} + \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}\frac{\sqrt[3]{x}+3}{\sqrt[3]{x}+2}$$

Note that $\lim_{x\to0^\pm} f(x) = +\infty$, hence the function is not differentibali at the point $x = 0$ (vertical tangent point); the function is differentiable elsewhere.

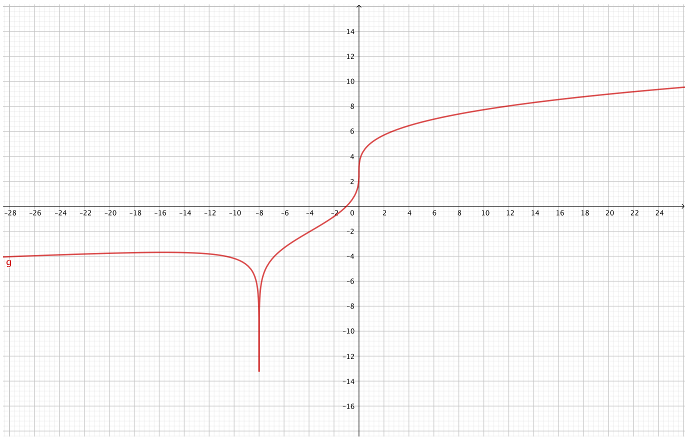
(c) *Find monotonicity intervals and maximum/minimum points. Say if they are relative or absolute.*
The function $f'(x) = 0$ iff $x = -27$, f' is strictly positive in the outer values of the interval $(27, -8)$ and strictly negative in the inner values. Therefore:
- f is strictly increasing in the intervals $(-\infty, -27)$ and $(-8, +\infty)$
- f is strictly decreasing in the interval $(-27, -8)$
- the point $x = -27$ is a relative maximum point (because $sup(f) = +\infty$)
- the point $x = 0$ is a vertical tangent inflection point
- there are no relative nor absolute minima ($inf(f) = -\infty$)

(d) *Trace a qualitative graph.*

Compute $f(-27) = -3$ and $f(0) = \log 2$. The function has a unique zero β, with $-8 < \beta < 0$; indeed:
- f is continuous on its domain

- $f(x) < 0$ if $x < -8$ and $f(x) > 0$ if $x > 0$
- f is strictly increasing in $(-8, 0)$
- $\lim_{x \to -8^+} f(x) < 0$ and $\lim_{x \to 0^-} f(x) = f(0) > 0$.

Here we depict a qualitative graph of f:



(e) *Define*

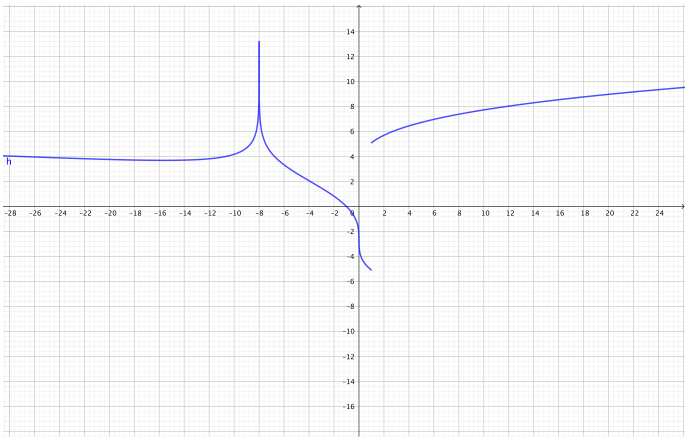
$$g(x) = \frac{x - \alpha}{|x - \alpha|} f(x)$$

with $\alpha \in \mathbb{R}$.from the graph of $f(x)$ (see point (d)), trace a qualitative graph for $g(x)$ with $\alpha = 1$. Say if there exist values of α, such that the function $g(x)$ admits continuous prolongation in $x = \alpha$.

Let $\alpha = 1$; represent graphically the function $g_1(x) = \dfrac{x-1}{|x-1|} f(x)$:

$$g_1(x) = \begin{cases} -f(x), & \text{if } x < 1 \\ f(x), & \text{if } x > 1 \end{cases}$$

For $x > 1$ its graph coincides with the graph of f, whereas for $x < 1$ it coincides with the symmetric one w.r.t. the x-axis.



Note that the function has a jump point in $x = 1$ and thus there exists no continuous prolongation in $x = 1$.

Such prolongation would exist only if the function is zero in $x = \alpha$, as in that case: $\lim_{x \to \alpha^{\pm}} f(x) = 0$ and we can define $f(\alpha) = 0$.

Therefore $g(x)$ has a continuous prolongation in $x = \alpha$ if and only if α coincides with the unique zero of f, i.e. iff $\alpha = \beta$.