Week 4 **Global properties of continuous functions**
Local comparison

$\boxed{\text{es. 1A}}$ f continuous at x_0 $\overset{?}{\Longrightarrow}$ f bounded in EVERY neighborhood of x_0

Hp

$$\lim_{x\to x_0} f(x) = f(x_0) \iff \forall \varepsilon>0\ \exists \delta>0 \ / \ 0<|x-x_0|<\delta,\ x\in \text{Dom} f \implies |f(x)-f(x_0)|<\varepsilon$$

Th

it means that $x \neq x_0$

$f(x)$ is bounded

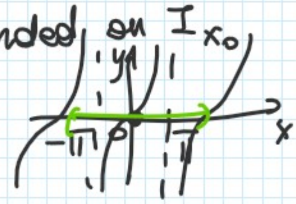
False

True: f continuous at x_0

$\implies \exists I_{x_0}$ s.t. f bounded on I_{x_0}

Counterexample

$f(x) = \tan x$



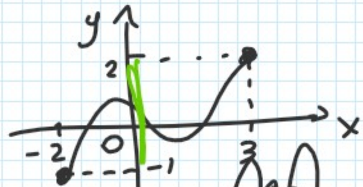
$\text{Dom} f = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$

f contin. in $x_0 = 0$

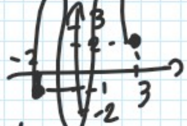
Take $I = (-5, 5)$ f unbounded on I

$\boxed{\text{ex 1C}}$

$$\text{Hp}\begin{cases} f:[-2,3] \longrightarrow \mathbb{R} \quad \text{contin.} \\ f(-2) = -1 \qquad f(3) = 2 \end{cases}$$

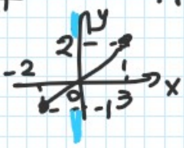


$\overset{?}{\Longrightarrow}$ Th $\{ f(x) = \lambda$ admits solutions for $-2 < \lambda < 3$



By continuity $[-1, 2] \subset \text{Im} f$ Apply Mean Value Theorem

FALSE Counterexample



If $\lambda \in (-2,-1) \cup (2,3)$

$\nexists$ solution for $f(x) = \lambda$

$\boxed{\text{ex 1E}}$

$$f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

f NOT contin. at $x_0 = 0$?

$$\lim_{x\to 0} \underbrace{x}_{\to 0} \underbrace{\sin\frac{1}{x}}_{\text{bounded}} = 0$$

$(A = x \qquad w = \frac{1}{x^2})$

False

$\lim_{x\to 0} f(x) \overset{?}{=} f(0) = 0$

Comparison Th. : $-x \le x \sin\frac{1}{x} \le x$, with $-x \to 0$, $x\sin\frac{1}{x} \to 0$, $x \to 0$

f continuous at $x_0 = 0$.

What if $t = \frac{1}{x}$ $\quad x = \frac{1}{t}$? $\qquad \lim_{t\to\pm\infty} \frac{1}{t}\sin t$ $\quad (x \to 0^{\pm})$

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

1) Comparison Theorems

$x \to 0^{\pm}$

Comparison Theorems:

$\lim_{x\to x_0} f(x) = \ell$ $\quad \lim_{x\to x_0} g(x) = \ell$ $\quad \ell \in \mathbb{R}$

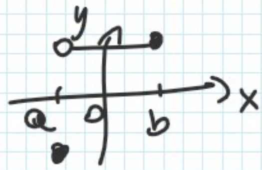
Suppose $f(x) \le h(x) \le g(x)$ then $\lim_{x\to x_0} h(x) = \ell$

Existence of zeros th:

$f:[a,b] \to \mathbb{R}$ continuous on $[a,b]$

$f(a)\cdot f(b) < 0 \implies \exists\, c \in (a,b)$ s.t. $f(c) = 0$

What IF continuity only on (a,b)?
$f(a)\cdot f(b) < 0$



Counterexample: $\nexists$ ZEROS

Recall:

- f and g have the <u>same order</u> AS $x \to x_0$

$\Updownarrow$

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = c \in \mathbb{R} \quad c \neq 0$

$\boxed{x_0 \in \mathbb{R} \text{ or } x_0 = \pm\infty}$

<u>NOTATION:</u> $f(x) \asymp_{x_0} g(x)$

- f is <u>infinite</u> AS $x \to x_0 \iff \lim_{x\to x_0} f(x) = \pm\infty$
- f is <u>infinitesimal</u> AS $x \to x_0 \iff \lim_{x\to x_0} f(x) = 0$
- f and g are <u>asymptotically equivalent</u> AS $x \to x_0 \iff \lim_{x\to x_0} \frac{f(x)}{g(x)} = 1$

<u>NOTATION:</u> $f(x) \sim_{x_0} g(x) \iff f(x) = g(x) + o(g(x))$ AS $x \to x_0$

- f is <u>negligible</u> w.r.t. g AS $x \to x_0 \iff$

$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0$

<u>NOTATION:</u> $f(x) = o(g(x))$ AS $\boxed{x \to x_0}$

Note that by def. $\frac{o(g(x))}{g(x)} \xrightarrow[x\to x_0]{} 0$

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example

$f(x) = 2x^3 + x$

$g(x) = 4x^3 - x^2$

$\boxed{x \to +\infty}$

$f(x) \sim_{+\infty} 2x^3$

$g(x) \sim_{+\infty} 4x^3$

AS $x \to +\infty$

$x = o(x^3)$

$-x^2 = o(x^3)$

$\boxed{x \to 0}$

$f(x) \sim_0 x$

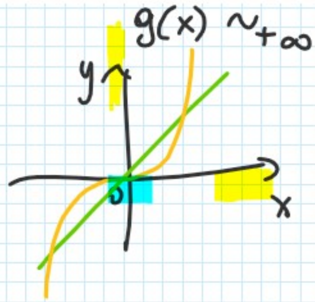
$g(x) \sim_0 -x^2$

as $x \to 0$

$2x^3 = o(x)$

$4x^3 = o(x^2)$

$g(x) \sim_{+\infty} 4x^{2}$ $\quad -x^2 = o(x^3)$ $\quad | \quad g(x) \sim_0 -x^2$ $\quad 4x^3 = o(x^2)$



Recall: COMPARE x^m and x^n, suppose $\boxed{m<n}$

$\boxed{x \to \infty \quad x^m = o(x^n)}$ $\qquad$ $\boxed{x \to 0 \quad x^n = o(x^m)}$

$\boxed{T5}$

5. For $x \to -\infty$, the function $f(x) = \dfrac{\log(1+x^2) - x + \arctan x}{2x^2 + e^x}$

$$\sim_{-\infty} \frac{\log(x^2) - x - \frac{\pi}{2}}{2x^2} \sim_{-\infty} \frac{2\log|x| - x}{2x^2} \sim$$

$+\infty \quad \to 0$

(a) is infinite

(b) is infinitesimal

(c) has limit equal to 1

(d) has limit equal to -1

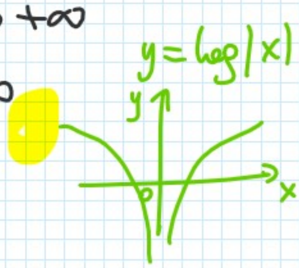
(e) has limit equal to $-\frac{1}{2}$

Recall $\sqrt{x^2} = |x|$

Compare As $x \to -\infty$ $\quad 2\log|x|$ and $-x \to +\infty$

$\to +\infty$

Fundamental limit: $\lim_{x \to +\infty} \dfrac{\log(x^\alpha)}{x^\beta} = 0 \quad \forall \alpha, \beta > 0$



$\Rightarrow 2\log|x| = o(-x)$

$$\sim_{-\infty} \frac{-\cancel{x}}{2x^{\cancel{2}}} \sim_{-\infty} \frac{-1}{2x} \longrightarrow 0$$

$\boxed{T6}$

6. Let $f(x) = e^{5x^2} - 1$. Then, for $x \to 0$, it holds:

(a) $f(x) = o(x) \to g(x)$

(b) $f(x) = o(x^2)$

(c) $f(x) = o(x^3)$

(d) $f(x) = o(x^4)$

(e) is not comparable with x

$f = o(g) \iff \dfrac{f}{g} \xrightarrow[x \to x_0]{} 0$ as $x \to x_0$

$(A) \iff \dfrac{e^{5x^2} - 1 \;\; f(x)}{x \;\; g(x)} \xrightarrow[x \to 0]{?} 0$

Recall $\lim_{t \to 0} \dfrac{e^t - 1}{t} = 1 \iff e^t - 1 \sim_0 t$

$\Updownarrow$

$e^t \sim_0 1 + t$

Compute $\dfrac{e^{5x^2} - 1}{x} \sim_0 \dfrac{5x^2}{x} \sim_0 5x \xrightarrow[x \to 0]{} 0$ $\quad$ (A) correct

$e^{5x^2} - 1 = o(x)$ as $x \to 0$ By def.

(B) is True $\iff \dfrac{e^{5x^2} - 1}{x^2} \xrightarrow[x \to 0]{?} 0$

We know that $5 \cdot \dfrac{e^{5x^2} - 1}{5x^2} \xrightarrow[x \to 0]{} 5$ $\quad$ False!

$\to 1$

OR: $t = 5x^2$ $\quad \dfrac{e^{5x^2} - 1}{x^2} \sim_0 \dfrac{5x^2}{x^2} \sim 5$

(C) $\dfrac{e^{5x^2} - 1}{x^3} \sim_0 \dfrac{5\cancel{x^2}}{x^{\cancel{3}}} \sim \dfrac{5}{x} \xrightarrow[x \to 0^{\pm}]{} \pm\infty$

$\boxed{ex6}$

6. As $\alpha \in \mathbb{R}$, compute the limit:

$\lim \dfrac{1 - e^x}{\ldots}$ $\qquad \boxed{\alpha > 0}$

6. As $\alpha \in \mathbb{R}$, compute the limit:

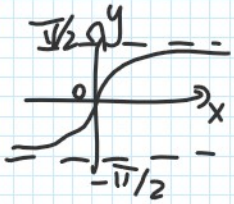
$$\lim_{x\to 0^+} \frac{1-e^x}{\frac{2}{\pi}\arctan x^\alpha}$$

$\boxed{\alpha > 0}$

Recall $\lim_{x\to 0} \frac{\arctan x}{x} = 1 \iff \arctan x \sim_0 x$

$$\frac{1-e^x}{\frac{2}{\pi}\arctan x^\alpha} \sim_{0^+} \frac{-x}{\frac{2}{\pi}x^\alpha} \sim_{0^+} -\frac{\pi}{2}x^{1-\alpha} \xrightarrow[x\to 0]{} \begin{cases} -\pi/2 & \text{IF } \alpha = 1 \\ -\infty & \text{IF } \alpha > 1 \quad (1-\alpha<0) \\ 0 & \text{IF } \alpha < 1 \quad (1-\alpha>0) \end{cases}$$

$\boxed{\alpha = 0}$ $\quad \frac{1-e^x}{\frac{2}{\pi}\arctan 1} \sim_0 \frac{-x}{\frac{2}{\pi}\frac{\pi}{4}} \longrightarrow 0$



$\boxed{\alpha < 0}$ $\quad \lim_{x\to 0^+} \arctan x^\alpha = \lim_{x\to 0^+} \arctan \frac{1}{x^{-\alpha}}, \ -\alpha > 0 \ = \arctan(+\infty) = \frac{\pi}{2}$

$f(x) \sim_0 \frac{-x}{\cancel{\frac{2}{\pi}}\cdot\cancel{\frac{\pi}{2}}} \longrightarrow 0$

Final conclusion

$\boxed{T1}$ $\quad \lim_{n\to\infty} \frac{7n^2+n}{n+\sin(n!)} \cdot \frac{n^n \cdot \cancel{n!}}{(n+2)^n \cancel{(n+1)!}} = \left[\frac{\infty}{\infty}\right] =$

$(n+1)! = (n+1)\cdot n!$ $\qquad (n+1)$

$\left(\frac{n}{n+2}\right)^n = \left(\frac{n+2}{n}\right)^{-n} = \left(1+\frac{2}{n}\right)^{-n} \to e^{-2}$

Recall $\lim_{n\to\infty}\left(1+\frac{\alpha}{n}\right)^{\beta n} = e^{\alpha\beta}$, $\alpha,\beta \in \mathbb{R}$

$$= \lim_{n\to\infty} \frac{7n^2 + \cancel{o(n^2)}}{n+\cancel{o(n)}} \cdot e^{-2} \cdot \frac{1}{\underbrace{n+1}_{\sim n}} = \lim_{n\to\infty} \frac{7\cancel{n^2}}{\cancel{n \cdot n}} e^{-2} = 7e^{-2}$$

$\sin(n!) \in [-1,1]$ bounded

$\sin(n!) = o(n)$ as $n\to\infty \iff \frac{\sin(n!)}{n} \xrightarrow[n\to\infty]{\text{yes}} 0$

Recall • principal part of $f(x)$ as $x \to x_0$ w.r.t. standard TEST FUNCTIONS $u(x)$

Suppose $\lim_{x\to x_0} \frac{f(x)}{(u(x))^\alpha} = c \in \mathbb{R}\setminus\{0\}$

Then $f(x) \sim_{x_0} c\cdot(u(x))^\alpha$ → principal part

order (α)

example

$f(x) = x^4$ as $x \to +\infty$

is INFINITE $\qquad u(x) = x$

$f(x) \sim_{+\infty} x^4$ Principal part $\qquad$ order 4 of ∞

(equals)

What IF $\boxed{u(x) = x^2}$ order?

$f(x) \sim \ x^4 = (x^2)^2$ $\qquad$ order $\alpha = 2$

when if $\boxed{u(x)=x^2}$ order.

$$f(x) \sim_{+\infty} x^4 = (x^2)^2$$

order $\alpha=2$

Note that → order depends on the choice of $u(x)$
→ P.P. DOES NOT depend " "

$\boxed{EX10}$

10. Find the order of infinity and the principal part w.r.t. $\varphi(x) = x$, as $x \to +\infty$, for the functions:

$$f_1(x) = 2x - \sqrt{4x^2 + x^4}; \quad f_2(x) = \frac{x^3 + x + 2}{x^2 - x - 1}; \quad f_3(x) = \frac{x^2+1}{\sqrt{x^2-1}}$$

$[\infty - \infty]$ $\left[\frac{\infty}{\infty}\right]$

$u(x) = \varphi(x)$ Test function

Compute

$$\lim_{x\to+\infty} \frac{f_1(x)}{x^2 + \sin x}; \quad \lim_{x\to+\infty} \frac{x - \sqrt{1+e^{-x}}}{f_2(x)}; \quad \lim_{x\to+\infty} \frac{f_3(x)(2+\cos x)}{x^2+x}$$

$$f_1(x) = \frac{2x - \sqrt{4x^2+x^4}}{2x+\sqrt{4x^2+x^4}}\,(2x+\sqrt{4x^2+x^2}) = \frac{\cancel{4x^2} - (\cancel{4x^2}+x^4)}{2x+\sqrt{4x^2+x^4}} = \frac{-x^4}{2x+\sqrt{\underbrace{4x^2}_{=o(x^4)}+x^4}} \sim_{+\infty}$$

$\sqrt{x^4} = x^2$

$$\sim_{+\infty} \frac{-x^4}{\underbrace{2x}_{=o(x^2)}+x^2} \sim_{+\infty} \frac{-\cancel{x^4}^{\,2}}{\cancel{x^2}} \sim_{+\infty} \boxed{-x^2}$$

P.P. order $\alpha = 2$

$$f_2(x) \sim_{+\infty} \boxed{x} \quad \text{order } \alpha = 1$$

IF you go by def. $\lim\limits_{x\to+\infty} \frac{f(x)}{x^\alpha} = \ldots = c \in \mathbb{R}\setminus\{0\}$

$\alpha = 2$ ⟸

$$\lim_{x\to+\infty} \frac{-x^2}{x^\alpha} = c \in \mathbb{R}\setminus\{0\}$$

I apply def. NOW

$$f_3(x) = \frac{x^2+1}{\sqrt{x^2-1}} = \frac{x^2\left(1+\frac{1}{x^2}\right)}{|x|\sqrt{1-1/x^2}} \sim_{+\infty} \frac{x^2}{|x|}$$

$\boxed{1^{ST}\text{ method}}$ ☹

$\boxed{2^{nd}\text{ method}}$

$$\frac{x^2+1}{\sqrt{x^2-1}} \sim_{+\infty} \frac{x^2+o(x^2)}{\sqrt{x^2+o(x^2)}} \sim_{+\infty} \frac{x^2}{|x|} \sim_{+\infty} \frac{x^{\cancel{2}}}{\cancel{x}} \sim \boxed{x} \quad \text{order } 1$$

- $$\lim_{x\to+\infty} \frac{f_1(x)}{x^2+\underbrace{\sin x}_{=o(x^2)}} = \lim_{x\to+\infty} \frac{-x^2}{x^2} = -1$$

- $$\lim_{x\to+\infty} \frac{x-\sqrt{1-e^{-x}}}{f_2(x)} =$$ ($\sqrt{1-e^{-x}} \to 1$)

Recall $$\lim_{t\to0} \frac{(1+t)^\alpha - 1}{t} = \alpha \qquad \alpha\in\mathbb{R}$$

$t = e^{-x}$ $\quad \alpha = \frac{1}{2}$

⇕

$$(1+t)^\alpha - 1 \sim_0 \alpha t$$

$$\left(\sqrt{1-e^{-x}} \sim_0 1 + \frac{1}{2}e^{-x}\right)$$

$$= \lim_{x\to+\infty} \frac{x+o(x)}{x} = 1$$

- $$\lim_{x\to+\infty} \frac{f_3(x)(2+\cos x)}{x^2+\underbrace{x}_{=o(x^2)}} = \lim_{x\to+\infty} \frac{\cancel{x}\cdot\overbrace{(2+\cos x)}^{\text{bounded}}}{\underbrace{x^{\cancel{2}}}_{\to 0}} = 0$$

$\boxed{EX11}$

$$\lim_{x\to+\infty} \frac{\dots}{x^2 + x} = \lim_{x\to+\infty} \frac{\dots}{x^2} \to 0 \qquad x = o(x^2)$$

EX12

12. Compute the principal part as $x \to +\infty$ and as $x \to 0$ (w.r.t. the standard test functions) for $f(x) = \dfrac{5x^2+2x}{3}$.

Solve

$$\underline{\lim_{x\to+\infty}} \frac{f(x)}{x^3 \sin(\frac{1}{x})}; \quad \lim_{x\to 0} f(x) \frac{\sin^2 x}{1-\cos x}$$

$\sim \frac{1}{x}$ as $x \to +\infty$ $\qquad \sim_0 x^2$

Recall $1-\cos x \sim_0 \frac{1}{2}x^2$

$\sin x \sim_0 x$

$\lim_{x\to+\infty} f(x) = +\infty \qquad \varphi(x) = x$ AS $x \to +\infty$

$\lim_{x\to 0} f(x) = 0 \qquad \varphi(x) = x$ AS $x \to 0$

$f(x) \sim_{+\infty} \boxed{\dfrac{5x^2}{3}}$ $\underline{\text{order } 2}$ $\qquad f(x) \sim_0 \boxed{\dfrac{2}{3}x}$ $\underline{\text{order } 1}$

$$\lim_{x\to+\infty} \frac{5/3\,\cancel{x^2}}{\cancel{x^3}\,\frac{1}{\cancel{x}}} = 5/3 \qquad \lim_{x\to 0} \frac{2}{3}x \cdot \frac{\cancel{x^2}}{\frac{1}{2}\cancel{x^2}} = \frac{4}{3} \cdot 0 = 0$$

IF $x \to +\infty$ $\quad \sin x \sim_{+\infty}$? $\quad \nexists$ because $\nexists \lim_{x\to+\infty} \sin x$

IF $x \to +\infty$ $\quad \sin\left(\frac{1}{x}\right) \sim_{+\infty} \frac{1}{x}$ $\qquad \frac{1}{x} \to 0$

$(\sin t \sim_0 t)$

$t = \frac{1}{x}$

T16

16. The functions $\sin\frac{1}{x}$ and $\cos\frac{1}{x}$

NO (a) are equivalent for $x \to 0$

(b) do not admit limit for $x \to 0$

NO (c) are equivalent for $x \to +\infty$

NO (d) are equivalent to $\frac{1}{x}$ for $x \to +\infty$

NO (e) have the same order of infinitesimal for $x \to +\infty$

IF $x \to +\infty$ $\quad \frac{1}{x} \to 0$

IF $x \to 0$ $\quad \frac{1}{x} \to \infty$

IF $x \to +\infty$ $\quad \sin\frac{1}{x} \sim_{+\infty} \boxed{\frac{1}{x}} \to 0 \qquad \cos\frac{1}{x} \sim_{+\infty} \boxed{1} \left(-\frac{1}{2}x^2\right)$

IT'S NOT infinitesimal

$\cancel{C}$ $\cancel{D}$ $\cancel{E}$

IF $x \to 0$ $\quad \nexists \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \qquad \nexists \lim_{x\to 0} \cos\left(\frac{1}{x}\right)$ $\quad \cancel{A}$

T18

18. The principal part as $x \to 0$ for the function $e^{x^3+3} - e^3$ is $\to 0$

(a) e^3x^3

(b) $\dfrac{x^3}{e^3}$

(c) $e^3 - 1$

(d) e^3x

(e) $x^3/3$

Recall $e^t - 1 \sim_0 t$

$$e^3\left(e^{x^3} - 1\right) \sim_0 e^3x^3$$

T20

20. The function $f(x) = \dfrac{x^a + \sin x}{x^b - \cos x}$, $a, b \in \mathbb{R}$, $\underline{a > 0, b > 0}$:

NO (a) does not admit principal part as $x \to +\infty$, w.r.t any test function

NO (b) has no horizontal asymptotes for any value of a, b $\quad$ IF $a = b$

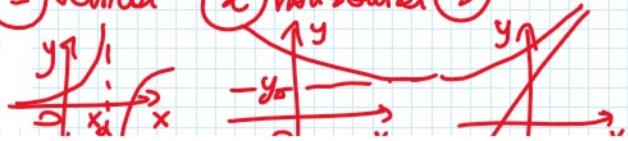
(c) for every a, b, has principal part x^{a-b} for $x \to +\infty$, w.r.t. the standard test function

NO (d) has principal part $2x$ for $x \to 0$

(e) has no vertical asymptotes $\quad$ where $x^b = \cos x$ there may exist V.A

Recall ASYMPTOTES

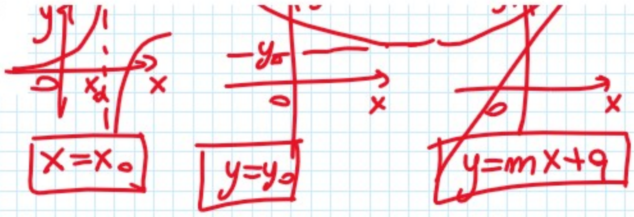
(1) vertical (2) horizontal (3) oblique



(c) for every a, b, has principal part $x^{\dots}$ for $x \to +\infty$, w.r.t. the standard test function

NO (d) has principal part $2x$ for $x \to 0$

(e) has no vertical asymptotes — where $x^b = \cos x$ there may exist V.A.



① x_0 is boundary point of Dom f

$$\lim_{x \to x_0} f(x) = \infty$$

② $$\lim_{x \to \pm\infty} f(x) = y_0 \in \mathbb{R}$$

③ $$m = \lim_{x \to \infty} \frac{f(x)}{x} \quad (\text{OR } f(x) \sim_\infty mx)$$

$$q = \lim_{x \to \infty} (f(x) - mx) \qquad m, q \in \mathbb{R}$$

As $x \to +\infty$ find P.P. $(a>0, b>0)$

$$f(x) \sim_{+\infty} \frac{x^a}{x^b} = x^{a-b}$$

IF $a = b$ $\quad f(x) \sim_{+\infty} 1$, $y = 1$ H.A.

As $x \to 0$

$$f(x) \sim_0 \frac{x^a + x}{x^b - 1} \sim_0 \frac{x^a + x}{-1} \longrightarrow 0$$

$b > 0 \qquad x^b = o(1)$

IF $a > 1$ $\quad f(x) \sim_0 -x$

IF $a \in (0,1)$ $\quad f(x) \sim_0 -x^a$

IF $a = 1$ $\quad f(x) \sim_0 -2x$

$\Longrightarrow$ ~~D~~

$\boxed{\text{T13}}$ $$f(x) = \frac{2x^2 + e^{-x}}{4x+5} \qquad e^{-x} \xrightarrow[x\to+\infty]{} 0$$

Find all possible asymptotes

① V.A. $\quad \text{Dom} f = \mathbb{R} \setminus \{-\frac{5}{4}\} \qquad 4x + 5 \neq 0$

$$\lim_{x \to -\frac{5}{4}} f(x) = \frac{\text{number}}{0} = \infty \qquad \boxed{x = -\frac{5}{4}}$$

② H.A. $$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} \left(\frac{2x^2}{4x}\right) = +\infty \qquad f(x) \sim_{+\infty} \frac{1}{2}x \qquad m = \frac{1}{2}$$

$(y = mx + q)$

$$q = \lim_{x \to +\infty} (f(x) - mx) = \lim_{x \to +\infty} \left(f(x) - \frac{1}{2}x\right)$$

compute P.P.

$$\frac{2x^2 + e^{-x}}{4x+5} - \frac{1}{2}x = \frac{\cancel{4x^2} + 2e^{-x} - \cancel{4x^2} - 5x}{2(4x+5)} \sim_{+\infty} \frac{-5\cancel{x}}{8\cancel{x}} \sim -\frac{5}{8} = q$$

$2e^{-x} \to 0$

$$\boxed{y = \frac{1}{2}x - \frac{5}{8}} \quad \text{RIGHT O.A.}$$

What IF $x \to -\infty$?

$$\frac{2x^2 + e^{-x}}{4x+5} \sim_{-\infty} \frac{e^{-x}}{4x} \longrightarrow -\infty$$

NOT comparable with $x = \varphi(x)$

$\nexists$ oblique (LEFT) A.

$2x^2 = o(e^{-x})$ as $x \to -\infty$

$\Updownarrow$

$\nexists\, c \in \mathbb{R} \setminus \{0\}$ s.t. $\displaystyle\lim_{x \to x_0} \frac{f(x)}{g(x)} = c$

$\hookrightarrow$ could be ∞, 0, $\nexists$