Week 8 Taylor expansions - Local study of function

def Recall Taylor expansion for f in x_0 of order m $\left(f \in C^m(I_{x_0})\right)$

$$f(x) = \underbrace{P_m(x)}_{\text{Taylor polynomial}} + \underbrace{R_m(x)}_{\text{Remainder}} \quad \text{in } I_{x_0}$$

Remainder → ① PEANO, ② LAGRANGE

$$P_m(x) = \sum_{k=0}^{m} \underbrace{\boxed{\frac{f^{(k)}(x_0)}{k!}}}_{= a_k \in \mathbb{R}} (x-x_0)^k$$

Peano Remainder $R_m(x) = o\left((x-x_0)^m\right)$

Lagrange Remainder $\left(f \in C^{m+1}(I_{x_0})\right)$

$\exists \bar{x} \in (x, x_0)$ such that

$$R_m(x) = \frac{f^{(m+1)}(\bar{x})}{(m+1)!}(x-x_0)^{m+1}$$

$P_m(x)$ local approx of f in I_{x_0}

def Mac Laurin expansion $x_0 = 0$ (the same as before)

$$f(x) = \underbrace{\sum_{k=0}^{m} \frac{f^{(k)}(0)}{k!} x^k}_{\text{polynomial}} + o(x^m) \quad \to \text{expansion in } I_0$$

PROPERTY

f even $\Longrightarrow$ P_m even (P_m Mac Laurin polynomial)

f odd $\Longrightarrow$ P_m odd

Note that $\cancel{\Leftarrow}$

Note also that $a_k = \dfrac{f^{(k)}(x_0)}{k!}$ Taylor expansion

$$\Longleftrightarrow \boxed{f^{(k)}(x_0) = a_k \cdot k!}$$

EX1 $f(x) = \sqrt{2x+1}$ $\quad x_0 = 4$, $m = 2$

Note that $(1+x)^\alpha = \sum_{k=0}^{m} \binom{\alpha}{k} x^k + o(x^m)$

$\alpha \in \mathbb{R}$

Recall $\binom{\alpha}{k} \overset{def}{=} \dfrac{\overset{0}{\alpha}(\overset{1}{\alpha-1})(\overset{2}{\alpha-2})\dots(\overset{k-1}{\alpha-k+1})}{k!}$ → k factors

$$\sqrt{1+x} = (1+x)^{1/2} \underset{\alpha=\frac{1}{2}}{=} \underbrace{\binom{1/2}{0}}_{1} + \binom{1/2}{1} x + \binom{1/2}{2} x^2 + o(x^2)$$

Recall $\binom{\alpha}{0} = 1$ $\qquad \binom{1/2}{1} = \dfrac{\frac{1}{2}}{1} = \dfrac{1}{2}$

$\binom{\alpha}{1} = \alpha$ $\qquad \binom{1/2}{2} = \dfrac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \dfrac{-\frac{1}{4}}{2} = -\dfrac{1}{8}$

$$\binom{\alpha}{1} = \alpha \qquad \binom{1/2}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$$

$$\sqrt{1+t} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + o(t^2) \quad \text{with } t \to 0$$

$$\sqrt{1+2x} = \sqrt{9+2t} = \sqrt{9\left(1+\frac{2}{9}t\right)} =$$

$x_0 = 4$ $\quad \boxed{\begin{matrix} t = x-4 \\ x = t+4 \end{matrix}}$

In general if $x \to x_0$, $t \to 0$ $\quad \boxed{t = x - x_0}$

$$= 3\sqrt{1+\frac{2}{9}t} = 3\left(1 + \frac{1}{2}\cdot\frac{2}{9}t - \frac{1}{8}\left(\frac{2}{9}t\right)^2\right) + o(t^2)$$

$\frac{2}{9}t \to 0$

$$= 3 + \frac{1}{3}t - \frac{1}{54}t^2 + o(t^2) = 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + o\left((x-4)^2\right)$$

$\boxed{\text{EX 3A}}$ $\quad f(x) = e^{x^2}\cdot \sin(2x)$

Recall $\quad e^x = \sum_{k=0}^{m} \frac{x^k}{k!} + o(x^m)$

(ODD) $\quad \sin x = \sum_{k=0}^{m} \frac{(-1)^k}{(2k+1)!} x^{2k+1} + o(x^{2m+2})$

MAC LAURIN EXPANSIONS

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^k}{k!} + \ldots + \frac{x^n}{n!} + o(x^n)$$

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^{n-1}\frac{x^n}{n} + o(x^n)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \ldots + (-1)^m \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2m+1}}{(2m+1)!} + o(x^{2m+2})$$

$$\cosh x = 1 + \frac{x^2}{2} + \frac{x^4}{4!} + \ldots + \frac{x^{2m}}{(2m)!} + o(x^{2m+1})$$

$$\arcsin x = x + \frac{x^3}{6} + \frac{3x^5}{40} + \ldots + \left|\binom{-\frac{1}{2}}{m}\right| \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^m \frac{x^{2m+1}}{2m+1} + o(x^{2m+2})$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2}x^2 + \ldots + \binom{\alpha}{n}x^n + o(x^n)$$

$$\frac{1}{1+x} = 1 - x + x^2 - \ldots + (-1)^n x^n + o(x^n)$$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^3)$$

$x_0 = 0 \qquad m = 4$

$$f(x) = \left(1 + x^2 + \frac{x^4}{2} + o(x^4)\right)\cdot\left(2x - \frac{(2x)^3}{3!} + o(x^4)\right)$$

$\frac{(2x)^3}{3!} = \frac{4}{3}x^3$

$$= 2x - \frac{4}{3}x^3 + 2x^3 + o(x^4) =$$

$$= 2x + \frac{2}{3}x^3 + o(x^4)$$

$\boxed{\text{EX 2}}$ $\quad x_0 = 1 \qquad m = 3$

$\forall x \in \text{Dom} f$
$f(-x) = f(x)$ Even
$f(-x) = -f(x)$ odd

$$f(x) = -2(x-1)^2 + 4\sin(x-1) - \log x^4$$

Recall $\quad \log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} + o(t^3)$

$$\sin t = t - \frac{t^3}{6} + \boxed{o(t^3)}$$

$o(t^4)$

$t = x - 1$

$$f(t) = -2t^2 + 4\sin t - 4\log(1+t) = \cancel{-2t^2} + 4\left(\cancel{t} - \frac{t^3}{6} + o(t^3)\right) +$$

$$-4\left(\cancel{t} - \cancel{\frac{t^2}{2}} + \frac{t^3}{3} + o(t^3)\right) = (\ldots) = -2t^3 + o(t^3)$$

$\boxed{\text{J1}}$ $\quad m = 2 \qquad x_0 = 0 \qquad f(x) = e^{\cos x}$

Recall $\quad \cos t = 1 - \frac{t^2}{2} + o(t^2) \quad$ with $t \to 0$

$$f(x) = e^{1 - \frac{x^2}{2} + o(x^2)} = e^1 \cdot e^{-\frac{x^2}{2} + o(x^2)} = e\left(1 - \frac{x^2}{2}\right) + o(x^2)$$

$1 - \frac{x^2}{2} + o(x^2) \to 1$, $\quad -\frac{x^2}{2} + o(x^2) \to 0$

$$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$$

$f(x) = e^{\dots} = e \cdot e^{\dots} = e\left(1 - \frac{\dots}{2}\right) + o(x^{\dots})$

$e^t = 1 + t + \frac{t^2}{2} + o(t^2)$ with $t \to 0$

$\boxed{\text{EX9}}$

x_0 is a STATIONARY POINT

Recall <u>Theorem</u> Suppose $f \in C^{(n-1)}(I_{x_0})$ such that $f'(x_0) = 0$

LET k be the first order $(k > 1)$ such that

$$f^{(k)}(x_0) \neq 0$$

IF $\to$ k even $\to$ $f^{(k)}(x_0) > 0$ x_0 MIN.; $f^{(k)}(x_0) < 0$ x_0 MAX

$\to$ k odd $\to$ x_0 inflection point

Before ex 9. See these easy examples

$\boxed{x_0 = 0}$

	Classify $x_0 = 0$	
1) $f(x) = 3 - x^2 + o(x^5)$	$f'(0) = 0$	MAX $f''(0) < 0$
2) $f(x) = 10 + x^6 + o(x^6)$	$f'(0) = 0$	MIN $f^{(6)}(0) > 0$
3) $f(x) = \frac{1}{2}x^7 + o(x^9)$	$f'(0) = 0$	INFLECTION point $f^{(7)}(0) \neq 0$
4) $f(x) = x - x^3 + o(x^3)$	$f'(0) = 1$	NOT stationary

1) $\boxed{f^{(k)}(0) = a_k \cdot k!}$

$a_0 = 3$ $\quad a_1 = 0$ $\quad a_2 = -1$

$f(0) = 3$ $\quad f'(0) = 0$ $\quad f''(0) = a_2 \cdot 2! = -1 \cdot 2! < 0$

$\boxed{\text{EX9}}$

9. Let f be a function in $\mathcal{C}^2(\mathbb{R})$ such that, for $x \to 0$

$$f(x) = 2 + \sin(16x) + 2\alpha x + \beta x^2 + \boxed{o(x^2)}$$

say for which values of the parameters α and β the function has a minimum or maximum in $x = 0$, explain why.

$$f(x) = 2 + 16x + 2\alpha x + \beta x^2 + o(x^2) =$$

$$= 2 + (16 + 2\alpha)x + \beta x^2 + o(x^2)$$

$x_0 = 0$ Stationary point $\iff f'(0) = 0 \iff a_1 = 0 \iff 16 + 2\alpha = 0$

$\boxed{\alpha = -8}$

IF $\boxed{\alpha = -8}$ $\quad f(x) = 2 + \beta x^2 + o(x^2)$

$\beta \neq 0$ $\to$ $\beta > 0$ MIN; $\beta < 0$ MAX

IF $\beta = 0$ $\quad f(x) = 2 + o(x^2)$ $\quad$??

EX 11

IF $\beta=0$ $f(x) = 2 + o(x^2)$!!

EX 11

11. Let $f:\mathbb{R}\to\mathbb{R}$ be a function in $C^{(\infty)}(\mathbb{R})$ such that $f(x) = -x^4 + \frac{1}{16}x^5 + o(x^5)$, for $x \to 0$. Suppose $g(x) = f(2x)$.

a) compute the derivatives $g^k(0)$ for $k = 1,2,3,3,4,5$.

b) Establish if $x_0 = 0$ is an extremal point for g and, if so, study its nature.

c) Can we say something about the behavior of g in $x_0 = \frac{3}{4}$? NO

For f : $a_0 = a_1 = a_2 = a_3 = 0$ $a_4 = -1$ $a_5 = \frac{1}{16}$

$f^{(4)}(0) = a_4 \cdot 4! = -1\cdot 4! < 0$ $f^{(5)}(0) = \frac{1}{16}\cdot 5!$

$x_0 = 0$ stationary for f

K EVEN $f^{(4)}(0) < 0$ $\cap$ MAX

$g(x) = f(2x) = -16x^4 + 2x^5 + o(x^5)$

2x instead of x

$\overset{\text{def}}{=} g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g^{(3)}(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \frac{g^{(5)}(0)}{5!}x^5 + o(x^5)$

$g(0) = 0$ $g''(0) = 0$ $\boxed{g^{(4)}(0) = -16\cdot 4!}$ $\cap$

$\boxed{g'(0) = 0}$ $g'''(0) = 0$ $g^{(5)}(0) = 2\cdot 5!$

$x_0 = 0$ is MAX

EX 14

14. Given the function $f(x) = \sin x\sqrt[5]{\cos x^2} - x$:

a) compute the Mac Laurin expansion of $\boxed{\text{order 6}}$

b) find the principal part and order of infinitesimal of $f(x)$, for $x \to 0$

c) compute $f^{(4)}(0)$ and $f^{(5)}(0)$

d) which is the nature of $x = 0$?

e) study the sign of $f(x)$ in a neighborhood of $x = 0$.

f) compute the limit: $\lim_{x\to 0}\frac{f(x)}{\sin^3 x}$

Recall $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^6)$

$\cos x^2 = 1 - \frac{x^4}{2} + o(x^6)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6)$

$\sqrt[5]{1+t} = \sum_{k=0}^{6}\binom{1/5}{k}t^k + o(t^6) = 1 + \frac{1}{5}t + o(t)$

$\alpha = \frac{1}{5}$

$f(x) = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\sqrt[5]{1 - \frac{x^4}{2}} + o(x^6) - x =$

$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\left(1 - \frac{1}{10}x^4\right) + o(x^6) - x =$

$= \cancel{x} - \frac{1}{10}x^5 - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^6) \cancel{-x} = (\dots) = \underbrace{-\frac{x^3}{6}}_{\text{Principal part}} - \frac{11}{120}x^5 + o(x^6)$

$P(x) = -\frac{x^3}{6}$ order of infinitesimal is 3

$a_0 = a_1 = a_2 = 0 = a_4$ $a_3 = -\frac{1}{6}$ $a_5 = -\frac{11}{120}$

$f^{(4)}(0) = a_4\cdot 4! = 0$ $f^{(5)}(0) = a_5 \cdot 5! = -\frac{11}{120}\cdot 5!$

$f'(0) = 0$ stationary p. $f^{(3)}(0) \neq 0$ 3 odd $\Rightarrow$ $x_0 = 0$ is inflection point

Sign of f in I_0 $f(x) \sim_0 -\frac{x^3}{6}$

IF $x>0$ $f(x)<0$

sign of f in I_0 $\quad f(x) \sim_0 -\frac{x^3}{6}$

IF $x>0$ $\quad f(x)<0$

IF $x<0$ $\quad f(x)>0$

example: IF $f(x) = 3 - x^5 + o(x^5)$ as $x \to 0$

$f(0)=3$

$x_0=0$ inflection p.

IF f is infinitesimal and we have Mc Laurin expansion $\Rightarrow f(0)=0$

$$\lim_{x\to 0} \frac{f(x)}{\sin^3 x} = \lim_{x\to 0} \frac{-x^3/6}{x^3} = -\frac{1}{6}$$

T4

4. If $f \in C^{(\infty)}(\mathbb{R})$ is infinitesimal for $x \to 0$ and it has a minimum in $x = 0$, then necessarily: $\to f(0)=0$ $\quad \hookrightarrow f'(0)=0$

~~(a)~~ $f(x) = kx^2 + o(x^2),\ k \in (0, +\infty)$ $\quad k$ even

~~(b)~~ $f(x) = h + kx^2 + o(x^2),\ h, k \in (0, +\infty)$ $\quad h=0$

~~(c)~~ $f(x) = kx^4 + o(x^4),\ k \in (0, +\infty)$ $\quad k$ even

(d) the first non-null derivative, in $x = 0$ has even order. $\quad$ k even $\hookrightarrow$ means $\quad f^{(k)}(0)>0$ $\quad h>0,\ k>0$

~~(e)~~ $f'(0) = 0$, e $f'(x) \geq 0$ $\to f$ always increasing

$\downarrow$ false

T7

7. If the Mac Laurin expansion of $f \in C^{(\infty)}(\mathbb{R})$ is : $f(x) = -\frac{2}{3}x^2 + 3x^4 + o(x^4)$, which of the following is NOT necessarily TRUE?

$f(0)=0$ $\quad f'(0)=0$

$f''(0) = -\frac{2}{3}\cdot 2! = -\frac{4}{3} > 0$

~~(a)~~ The function is infinitesimal for $x \to 0$. TRUE

(b) The function is even. Take $f(x) = -\frac{2}{3}x^2 + 3x^4 + x^5$

~~(c)~~ The function has a stationary point in $x = 0$. TRUE

~~(d)~~ $f''(0) = -\frac{4}{3}$ TRUE

~~(e)~~ There exists a neighborhood of $x = 0$ where the function is negative. TRUE $\quad f(x) \sim_0 -\frac{2}{3}x^2$

Recall $C^k(I) = \{ f: I \to \mathbb{R} \mid f$ is diff. k TIMES s.t. all DERIVATIVES are continuous $\}$

T12

12. Given the function $f(x) = x^2 \log x$, which of the following is TRUE?

I guess we need $n=3$

~~(a)~~ $f(x)$ has a horizontal tangent point at $x_0 = 1$ $\quad$ False

~~(b)~~ the tangent line to the graph of $f(x)$ at $(1, 0)$ has equation $y = x + 1$ $\quad y = x-1$

~~(c)~~ the Taylor expansion of f of degree 3 centered in $x_0 = 1$ is $f(x) = \frac{3}{2}(x-1)^2 + \frac{1}{6}(x-1)^3 + o((x-1)^3)$

(d) $f'''(1) = 2$

~~(e)~~ $f''(0) = 3$ $\quad x_0=1$

$x_0=1$

$t = x-1 \qquad x = 1+t$ $\qquad \boxed{\text{IF } x\to 1 \quad t \to 0}$

$f(t) = (1+t)^2 \log(1+t) =$

$= (1+2t+t^2)\cdot\left(t - \frac{t^2}{2} + \frac{t^3}{3}\right) + o(t^3) =$

$= t - \frac{t^2}{2} + \frac{t^3}{3} + 2t^2 \cancel{-t^3} + o(t^3) + \cancel{t^3} = t + \frac{3}{2}t^2 + \frac{1}{3}t^3 + o(t^3)$

$f(x) = (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + o((x-1)^3)$ $\quad$ ~~C~~

$a_0 = 0 \qquad a_1 = 1 \qquad a_2 = \frac{3}{2} \qquad a_3 = \frac{1}{3}$

$f(1)=0 \qquad f'(1) = 1$ $\quad$ ~~A~~

$f''(1) = a_2 \cdot 2 = 3 \qquad f'''(1) = \frac{1}{3}\cdot 3! = 2$ $\quad$ D

Tangent line in $x_0 \longleftrightarrow 1^{ST}$ order Taylor POLYNOMIAL

$y = f(x_0) + f'(x_0)\cdot(x-x_0)$

T13

13. Given $f(x) = \sin^2 2x$, which of the following is TRUE? $\quad = (\sin(2x))^2$

~~(a)~~ The Mac Laurin expansion of order 4 is $f(x) = 4x^2 - \frac{16}{3}x^4$ $\quad + o(x^4)$ is missing

~~(b)~~ $f(x)$ has a relative ~~maximum~~ MIN in $x = 0$

~~(c)~~ $f(x)$, for $x \to 0$, is infinitesimal of order 4 $\to 2$

(d) its principal part for $x \to 0$, (w.r.t. x), is $p(x) = 4x^2$

~~(e)~~ $f''(0) = 4$

$\sin x = x - \frac{x^3}{3!} + o(x^4)$

~~(c) $f(x)$, for $x \to 0$, is infinitesimal of order 4~~ $\to 2$

(d) its principal part for $x \to 0$, (w.r.t. x), is $p(x) = 4x^2$

~~(e) $f''(0) = 4$~~

$\sin x = x - \frac{x^3}{3!} + o(x^4)$

$m = 4$ $\quad \sin(2x) = 2x - \frac{4}{3}x^3 + o(x^4)$

$f(x) = (\quad)^2 = 4x^2 - 2\cdot 2x \cdot \frac{4}{3}x^3 + o(x^4) =$

$= 4x^2 - \frac{16}{3}x^4 + o(x^4)$

$\Rightarrow f'(0) = 0 \quad f(0) = 0 \quad f(x) \sim_0 4x^2$ order 2

$f''(0) = a_2 \cdot 2! = 4 \cdot 2 = 8$ $\quad \cancel{(E)}$

$f^{(k)}(0) = a_k \cdot k!$

T10

10. If $\quad f(x) = \log(\cos x)$, then:

~~(a) the Mac Laurin expansion of f of degree 2 is $\quad f(x) = -x^2 + o(x^2)$~~

~~(b) $f(x)$ has an inflection point in $x = 0$~~

(c) the tangent parabola to the graph of $f(x)$ at $(0,0)$ has equation $y = -\frac{x^2}{2}$

~~(d) $f(x)$ does not admit Taylor expansion at $x_0 = 2\pi$~~ false

~~(e) $f(x)$ does admit Taylor expansion at $x_0 = \pi$~~

$m = 2$

$\cos x = 1 - \frac{x^2}{2} + o(x^2)$

$f(x) = \log\left(1 - \frac{x^2}{2} + o(x^2)\right) =$

$= -\frac{x^2}{2} + o(x^2)$ $\quad \cancel{(A)}$

$\log(1+t) = t - \frac{t^2}{2} + o(t^2)$

$f'(0) = 0 \quad f''(0) < 0$ MAX $\quad \cancel{(B)}$

$\cos(2\pi) = 1 \quad f(x) \xrightarrow{x \to 2\pi} \log 1 = 0$

$\cos(\pi) = -1 \quad \cancel{\log(-1)}$

$f(x) \xrightarrow[x \to \pi]{} \ ?$ $\quad \pi \notin \mathrm{Domf}$

EX 7B

⚠ We cannot divide 2 expansions ...

$f(x) = \frac{\sinh(x^2 + 2\sin^4 x)}{1 + x^{10}}$ $\quad \boxed{\begin{array}{c} m = 4 \\ x_0 = 0 \end{array}}$ $\quad f^{(4)}(0)?\quad f^{(3)}(0)?$

$= \frac{1}{1 + x^{10}} \cdot \sinh(x^2 + 2\sin^4 x) =$

Recall

$\frac{1}{1+t} = (1+t)^{-1} = 1 - t + t^2 - t^3 + t^4 - \dots + (-1)^n t^n + o(t^n)$

$\alpha = -1 \qquad t \to 0$

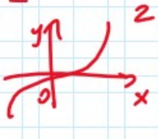
Recall odd $\to \sinh t = t + \frac{t^3}{3!} + o(t^4)$

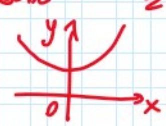
Recall $\boxed{e^t = \sinh t + \cosh t}$

$(\cosh^2 t - \sinh^2 t = 1)$

$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \dots$

$\sinh t \stackrel{\text{def}}{=} \frac{e^t - e^{-t}}{2}$ $\quad \cosh t \stackrel{\text{def}}{=} \frac{e^t + e^{-t}}{2}$





$= (1 - x^{10} + o(x^4)) \cdot \sinh(x^2 + 2x^4 + o(x^4)) =$

$= 1 \cdot (x^2 + 2x^4 + o(x^4)) = x^2 + 2x^4 + o(x^4)$

$f^{(3)}(0) = 0 \qquad f^{(4)}(0) = 2 \cdot 4!$