MATHEMATICAL ANALYSIS I TUTORING
8^{TH} WEEK

TAYLOR EXPANSIONS- Applications to LOCAL STUDY of FUNCTIONS

PROPOSED EXERCISES - SOLUTIONS

1. Compute the Taylor expansion of $f(x) = \sqrt{2x+1}$ in $x_0 = 4$, of order 2.

   Recall $\boxed{\sqrt{1+t} = 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \cdots + \binom{1/2}{n}t^n + o(t^n), \text{for } t \to 0}$

   By substitution $t = x - 4$, for $x \to 4$ it holds $t \to 0$, then the Mac Laurin expansion of the function is $g(t) = \sqrt{2(t+4)+1} = \sqrt{2t+9}$ and therefore:

$$
\begin{aligned}
g(t) &= \sqrt{2t+9} = \sqrt{9\left(1+\frac{2}{9}t\right)} = 3\sqrt{1+\frac{2}{9}t} \\
&= 3\left(1 + \frac{1}{2}\frac{2}{9}t - \frac{1}{8}\left(\frac{2}{9}t\right)^2 + o(t^2)\right) \\
&= 3\left(1 + \frac{1}{9}t - \frac{1}{8}\frac{4}{81}t^2 + o(t^2)\right) \\
&= 3 + \frac{1}{3}t - \frac{1}{54}t^2 + o(t^2)
\end{aligned}
$$

   By substitution $t = x - 4$, we obtain the Taylor expansion:

$$
f(x) = 3 + \frac{1}{3}(x-4) - \frac{1}{54}(x-4)^2 + o((x-4)^2)
$$

2. Compute the Taylor expansion at $x_0 = 1$, of order 3, of the function

$$
f(x) = -2(x-1)^2 + 4\sin(x-1) - \log x^4
$$

   Recall

$$
\boxed{\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + (-1)^{n-1}\frac{t^n}{n} + o(t^n), \text{for } t \to 0}
$$

$$
\boxed{\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n\frac{t^{2n+1}}{(2n+1)!} + o(t^{2n+2}), \text{for } t \to 0}
$$

   as before, if $t = x - 1$, for $x \to 1$ it holds that $t \to 0$, and we can study the Mac Laurin expansion $g(t) = -2t^2 + 4\sin t - \log(1+t)^4$ and thus:

$$
\begin{aligned}
g(t) &= -2t^2 + 4\sin t - \log(1+t)^4 \\
&= -2t^2 + 4\sin t - 4\log(t+1) \\
&= -2t^2 + 4\left(t - \frac{t^3}{3!}\right) - 4\left(t - \frac{t^2}{2} + \frac{t^3}{3}\right) + o(t^3) \\
&= -2t^2 + 4t - \frac{4}{6}t^3 - 4t + \frac{4}{2}t^2 - \frac{4}{3}t^3 + o(t^3) \\
&= -2t^3 + o(t^3)
\end{aligned}
$$

3. Compute Mac Laurin expansions for the following functions:

(a) $\boxed{f_1(x) = e^{x^2} \cdot \sin 2x,\ n = 4}$

Recall

$$\boxed{e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + o(t^n), \text{for } t \to 0}$$

$$\boxed{\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots + (-1)^n \frac{t^{2n+1}}{(2n+1)!} + o(t^{2n+2}), \text{for } t \to 0}$$

$$\begin{aligned}
f_1(x) &= e^{x^2} \cdot \sin 2x \\
&= \left(1 + x^2 + \frac{(x^2)^2}{2!} + o(x^4)\right) \cdot \left(2x - \frac{1}{3!}(2x)^3 + o(x^4)\right) \\
&= \left(1 + x^2 + \frac{x^4}{2} + o(x^4)\right) \cdot \left(2x - \frac{4}{3}x^3 + o(x^4)\right) \\
&= \left(2x - \frac{4}{3}x^3\right) + x^2\left(2x - \frac{4}{3}x^3\right) + \frac{x^4}{2}\left(2x - \frac{4}{3}x^3\right) + o(x^4) \\
&= 2x - \frac{4}{3}x^3 + 2x^3 - \frac{4}{3}x^5 + x^5 - \frac{2}{3}x^7 + o(x^4)
\end{aligned}$$

Since $x^5 = o(x^4), \quad x^7 = o(x^4)$, for $x \to 0$, the 4 order expansion of $f_1(x)$ is:

$$f_1(x) = 2x + \frac{2}{3}x^3 + o(x^4)$$

(b) $\boxed{f_2(x) = \log(1 - \sin^2 x),\ n = 4}$

Recall

$$\boxed{\log(1+t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \cdots + (-1)^{n-1}\frac{t^n}{n} + o(t^n), \text{for } t \to 0}$$

Consider $t = -\sin^2 x$; write the Mac Laurin expansion of 4^{th} order

$$-\sin^2 x = -\left(x - \frac{x^3}{3!} + o(x^4)\right)^2 = -\left(x^2 - \frac{x^4}{3} + o(x^4)\right) = -x^2 + \frac{x^4}{3} + o(x^4)$$

$$\begin{aligned}
f_2(x) &= \log(1 - \sin^2 x) \\
&= \log\left(1 - x^2 + \frac{x^4}{3} + o(x^4)\right) \\
&= \left(-x^2 + \frac{x^4}{3} + o(x^4)\right) - \frac{1}{2}\left(-x^2 + \frac{x^4}{3} + o(x^4)\right)^2 + \\
&+ \frac{1}{3}\left(-x^2 + \frac{x^4}{3} + o(x^4)\right)^3 - \frac{1}{4}\left(-x^2 + \frac{x^4}{3} + o(x^4)\right)^4 + o(x^4) \\
&= -x^2 + \frac{x^4}{3} - \frac{1}{2}x^4 + o(x^4) \\
&= -x^2 - \frac{x^4}{6} + o(x^4)
\end{aligned}$$

(c) $\boxed{f_3(x) = \log(\cos x),\ n = 4}$

Recall $\boxed{\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots + (-1)^n \frac{t^{2n}}{(2n)!} + o(t^{2n+1}),\ \text{for } t \to 0}$

$$
\begin{aligned}
f_3(x) &= \log(\cos x) \\
&= \log\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\right) \\
&= \left(-\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\right) - \frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + o(x^4)\right)^2 + o(x^4) \\
&= -\frac{x^2}{2} + \frac{x^4}{24} - \frac{1}{2}\frac{x^4}{4} + o(x^4) \\
&= -\frac{x^2}{2} - \frac{x^4}{12} + o(x^4)
\end{aligned}
$$

(d) $\boxed{f_4(x) = e^{x^2},\ n = 6}$

Recall $\boxed{e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + o(t^n), \text{for } t \to 0}$

$$
\begin{aligned}
f_4(x) &= e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + o(x^6) \\
&= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + o(x^6)
\end{aligned}
$$

4. Find principal part and order of infinitesimal, for $x \to 0$, for the following functions:

(a) $\boxed{f(x) = \sin x - x\cos\frac{x}{\sqrt{3}}}$

Find the Mac Laurin expansion of the function; the term with minimum degree is a function equivalent to $f(x)$ for $x \to 0$ and thus it is the principal part; we now have to write the expansion with a sufficiently high order, such that the first term is not null:

$$
\begin{aligned}
f(x) &= \sin x - x\cos\frac{x}{\sqrt{3}} \\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + o(x^5) - x\left(1 - \frac{1}{2!}\left(\frac{x}{\sqrt{3}}\right)^2 + \frac{1}{4!}\left(\frac{x}{\sqrt{3}}\right)^4 + o(x^5)\right) \\
&= x - \frac{x^3}{6} + \frac{x^5}{5 \cdot 4!} - x + \frac{1}{6}x^3 - \frac{1}{4! \cdot 9}x^5 + o(x^5) \\
&= \frac{9-5}{4! \cdot 5 \cdot 9}\,x^5 + o(x^5) \\
&= \frac{4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 9}\,x^5 + o(x^5) \\
&= \frac{1}{270}\,x^5 + o(x^5)
\end{aligned}
$$

Thus, for $x \to 0, f(x) \sim \frac{1}{270}\,x^5$; hence (w.r.t. the test function $u(x) = x$) the principal part of $f(x)$ is $p(x) = \frac{1}{270}\,x^5$ and the order of infinitesimal is 5

(b) $\boxed{f(x) = \sin^2 4x - \log(1 + 16x^2) - \lambda(x^5 - x^4),\ \text{ as } \lambda \in \mathbb{R}}$

Impose $g(x) = \sin^2 4x - \log(1 + 16x^2)$, and write the Mac Laurin expansion of order 5:

$$
\begin{aligned}
g(x) &= \sin^2 4x - \log(1+16x^2) \\
&= \left(4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} + o(x^5)\right)^2 - \left(16x^2 - \frac{(16x^2)^2}{2} + o(x^5)\right) \\
&= \left(4x - \frac{64x^3}{6} + \frac{4^5x^5}{5!} + o(x^5)\right)^2 - 16x^2 + \frac{4^4x^4}{2} + o(x^5) \\
&= 16x^2 - 2\cdot 4\frac{4^3x^4}{6} - 16x^2 + \frac{4^4x^4}{2} + o(x^5) \\
&= -2\cdot\frac{4^4x^4}{6} + 3\cdot\frac{4^4x^4}{6} + o(x^5) \\
&= \frac{128x^4}{3} + o(x^5)
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
f(x) &= g(x) - \lambda(x^5 - x^4) \\
&= \frac{128}{3}x^4 + o(x^5) + \lambda x^4 - \lambda x^5 \\
&= \left(\frac{128}{3} + \lambda\right)x^4 - \lambda x^5 + o(x^5)
\end{aligned}
$$

If $\frac{128}{3} + \lambda \neq 0$, then the principal part is $\left(\frac{128}{3} + \lambda\right)x^4$ and the order of infinitesimal is 4.
Whereas, if $\lambda = -\frac{128}{3}$, then the principal part is $\frac{128}{3}x^5$ and the order of infinitesimal is 5.

(c) $\boxed{f(x) = e^{\cos(\alpha x)} - e + (\sin^2 x)\log(1+x), \ \text{ as } \alpha \in \mathbb{R}}$

Write the Mac Laurin expansion of order 3:

$$
\begin{aligned}
f(x) &= e^{\cos(\alpha x)} - e + (\sin^2 x)\log(1+x) \\
&= e^{1-\frac{1}{2}(\alpha x)^2 + o(x^3)} - e + \left(x - \frac{1}{3!}x^3 + o(x^3)\right)^2\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)\right) \\
&= e\,e^{-\frac{1}{2}(\alpha x)^2 + o(x^3)} - e + \left(x^2 + o(x^3)\right)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)\right) \\
&= e\left(1 - \frac{1}{2}(\alpha x)^2 + o(x^3)\right) - e + \left(x^3 + o(x^3)\right) \\
&= e - \frac{e}{2}\alpha^2x^2 - e + x^3 + o(x^3) \\
&= -\frac{e}{2}\alpha^2x^2 + x^3 + o(x^3)
\end{aligned}
$$

If $-\frac{e}{2}\alpha^2 \neq 0$, and thus if $\alpha \neq 0$, then the principal part is $p(x) = -\frac{e}{2}\alpha^2x^2$ and the order of infinitesimal is 2.
If $\alpha = 0$, then the principal part is x^3 and the order of infinitesimal is 3.

5. For every value of α, compute the order of infinitesimal for $x \to 0$ w.r.t. the test function x, of the function:

$$
f(x) = \frac{\sin(2x^2) - \alpha x^2 + \log(1+2x^2)}{\alpha - 4\cosh x}
$$

Recall $\boxed{\cosh t = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \cdots + \frac{t^{2n}}{(2n)!} + o(t^{2n+1}), \ \text{ for } t \to 0}$

Write the Mac Laurin expansion of order 4 for the numerator and order 2 for the denominator:

$$
f(x) = \frac{2x^2 - \alpha x^2 + 2x^2 - \frac{1}{2}4x^4 + o(x^4)}{\alpha - 4(1 + \frac{x^2}{2!} + o(x^2))} = \frac{(4-\alpha)x^2 - 2x^4 + o(x^4)}{\alpha - 4 - 2x^2 + o(x^2)}
$$

If $4-\alpha \neq 0$, for $x \to 0$ we have $f(x) \sim \dfrac{(4-\alpha)x^2}{\alpha-4} = -x^2$; thus, if $\alpha \neq 4$, the principal part is $-x^2$

On the other hand, if $\alpha = 4$, for $x \to 0$ we have $f(x) \sim \dfrac{-2x^4}{-2x^2} = x^2$, and its principal part is x^2.

6. Compute the limits:

$$\boxed{(a) \quad \lim_{x\to 0} \frac{x\sin x - x^2}{\sqrt{1+x^4}-\cos x^4}}$$

Using the Mac Laurin expansions of order 4 for numerator and denominator, it holds:

$$\lim_{x\to 0} \frac{x\sin x - x^2}{\sqrt{1+x^4}-\cos x^4} = \lim_{x\to 0} \frac{\frac{-x^4}{6}+o(x^4)}{\frac{x^4}{2}+o(x^4)} = -\frac{1}{3}$$

$$\boxed{(b) \quad \lim_{x\to 0} \frac{x^2-\sin^2 x}{x^3(e^x-\cos x)}}$$

Using the Mac Laurin expansions of order 4 for numerator and denominator, it holds:

$$\lim_{x\to 0} \frac{x^2-\sin^2 x}{x^3(e^x-\cos x)} = \lim_{x\to 0} \frac{\frac{x^4}{3}+o(x^4)}{x^4+o(x^4)} = \frac{1}{3}$$

$$\boxed{(c) \quad \lim_{x\to 0^+} \frac{\log(1+x\arctan x)+1-e^{x^2}}{\sqrt{1+2x^4}-1}}$$

Using the Mac Laurin expansions of order 4 for numerator and denominator, it holds:

$$\lim_{x\to 0^+} \frac{\log(1+x\arctan x)+1-e^{x^2}}{\sqrt{1+2x^4}-1} = \lim_{x\to 0} \frac{\frac{-4x^4}{3}+o(x^4)}{\frac{1}{2}(2x^4)+o(x^4)} = -\frac{4}{3}$$

7. Compute the third and forth derivative of the following functions evaluated at 0:

$$\boxed{(a) \quad f_1(x) = e^x \sin x}$$

Recall the Mac Laurin expansion of order 4 of a generic function $f(x)$:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + o(x^4)$$

Using the Mac Laurin expansions of order 4 of $f_1(x)$:

$$\begin{aligned}
f_1(x) &= e^x \sin x \\
&= \left(x - \frac{x^3}{3!} + o(x^4)\right)\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)\right) \\
&= x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)\right) - \frac{x^3}{3!}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + o(x^4)\right) + o(x^4) \\
&= x + x^2 + \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^3}{6} - \frac{x^4}{6} + o(x^4) \\
&= x + x^2 + \frac{1}{3}x^3 + o(x^4)
\end{aligned}$$

Comparing the latter with the generic formula, it holds that

$$f_1'''(0) = 3! \cdot \frac{1}{3} = 2, \quad f_1^{(4)}(0) = 0$$

$$\boxed{b) \quad f_2(x) = \frac{\sinh(x^2 + 2\sin^4 x)}{1+x^{10}}}$$

Recall

$$\boxed{\sinh t = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + \cdots + \frac{t^{2n+1}}{(2n+1)!} + o(t^{2n+2}), \text{for } t \to 0}$$

$$\boxed{\frac{1}{1+t} = 1 - t + t^2 - t^3 + \cdots + (-1)^n t^n + o(t^n)}$$

Using the Mac Laurin expansions of order 4 of $f_2(x)$:

$$\begin{aligned} f_2(x) &= \frac{\sinh(x^2 + 2\sin^4 x)}{1 + x^{10}} \\ &= \left(\sinh(x^2 + 2x^4)\right)\frac{1}{1+x^{10}} \\ &= \left((x^2 + 2x^4) + \frac{1}{3!}(x^2 + 2x^4)^3 + o(x^4)\right)(1 - x^{10} + o(x^4)) \\ &= x^2 + 2x^4 + o(x^4) \end{aligned}$$

Comparing the latter with the generic formula, it holds that

$$f_2'''(0) = 0, \quad f_2^{(4)}(0) = 4! \cdot 2 = 48,$$

8. Study the nature of $x_0 = 3$ for the function: $f(x) = -2 - 4(x-3)^3 - 90(x-3)^5 + o((x-3)^5)$

Theorem 1. *Let f be defined and differentiable $n-1$ times in a neighborhood of x_0, and suppose it admits the n-th derivative in x_0, with $n \geq 2$.*
Suppose that $f''(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$.
Then, if n is odd and $f^{(n)}(x_0) > 0$, x_0 is an ascending inflection point f; if n is odd and $f^{(n)}(x_0) < 0$, x_0 is a discending inflection point.
If n is even and $f^{(n)}(x_0) > 0$, f is strictly convex in x_0; if n is even and $f^{(n)}(x_0) < 0$, f is strictly concave in x_0.

Suppose $f \in C^5$; since $f'(3) = f''(3) = 0, f'''(3) = -4 \cdot 4! < 0$ we can apply Theorem 1 and conclude that $x_0 = 3$ is a discending inflection point.

9. Let f be a function in $\mathcal{C}^2(\mathbb{R})$ such that, for $x \to 0$

$$f(x) = 2 + \sin(16x) + 2\alpha x + \beta x^2 + o(x^2)$$

say for which values of the parameters α and β the function has a minimum or maximum in $x = 0$, explain why.
Theorem 2. *Let f be defined and differentiable $n-1$ times in a neighborhood of x_0, and suppose it admits the n-th derivative in x_0, with $n \geq 2$.*
Suppose that $f'(x_0) = 0$, $f''(x_0) = 0, \ldots, f^{(n-1)}(x_0) = 0$, $f^{(n)}(x_0) \neq 0$.
Then, if n is odd and $f^{(n)}(x_0) > 0$, x_0 is an ascending inflection point f; if n is odd and $f^{(n)}(x_0) < 0$, x_0 is a discending inflection point.
par If n is even and $f^{(n)}(x_0) > 0$, f has a relative minimum in x_0; if n is even and $f^{(n)}(x_0) < 0$, f has a relative maximum in x_0.

Recall the Mac Laurin expansion of order 2 for $f(x)$:

$$f(x) = 2 + (16x + o(x^2)) + 2\alpha x + \beta x^2 = 2 + (16 + 2\alpha)x + \beta x^2 + o(x^2)$$

Then, applying Theorem 2, in order to have a maximum in $x_0 = 0$ we impose

$$16 + 2\alpha = 0, \ \wedge \ \beta < 0$$

whereas to get a minimum in $x_0 = 0$ we impose

$$16 + 2\alpha = 0, \ \wedge \ \beta > 0$$

10. For $x \to 0$, it holds:
$$f(x) = 2x + 4x^2 + o(x^2)$$
Compute the first and second derivative in $x = 0$ of the following functions:
$$a) \quad a(x) = f(\sin x), \quad b) \quad b(x) = f(e^x - 1), \quad c) \quad c(x) = e^{f(x)}$$

$\boxed{(a) \quad a(x) = f(\sin x)}$

Suppose $f(x)$ is differentiable twice $x_0 = 0$, the given expression is the Mac Laurin excpansion of $f(x)$ (Theorem of uniqueness) and thus:
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + o(x^2)$$
Hence: $\quad f(0) = 0,\ f'(0) = 2,\ f''(0) = 8.$
Finally
$$a(x) = f(\sin x) \quad \Rightarrow \quad a'(x) = f'(\sin x)\cos x \quad \Rightarrow \quad a''(x) = f''(\sin x)\cos^2 x - f'(\sin x)\sin x$$
and thus
$$a(0) = f(\sin 0) = f(0) = 0,\ a'(0) = f'(\sin 0)\cos 0 = f'(0) = 2,\ a''(0) = f''(0)\cos^2 0 - f'(0)\sin 0 = f''(0) = 8$$

$\boxed{(b) \quad b(x) = f(e^x - 1)}$

As before: $\quad f(0) = 0,\ f'(0) = 2,\ f''(0) = 8.$
Therefore
$$b(x) = f(e^x - 1) \quad \Rightarrow \quad b'(x) = f'(e^x - 1)e^x \quad \Rightarrow \quad b''(x) = f''(e^x - 1)(e^x)^2 + f'(e^x - 1)e^x$$
and thus
$$b(0) = f(e^0 - 1) = f(0) = 0,\ b'(0) = f'(0)e^0 = f'(0) = 2,\ b''(0) = f''(0)(e^0)^2 + f'(0)e^0 = f''(0) + f'(0) = 10$$

$\boxed{(c) \quad c(x) = e^{f(x)}}$

As before $\quad f(0) = 0,\ f'(0) = 2,\ f''(0) = 8.$
Therefore
$$c(x) = e^{f(x)} \quad \Rightarrow \quad c'(x) = e^{f(x)}f'(x) \quad \Rightarrow \quad c''(x) = e^{f(x)}(f'(x))^2 + e^{f(x)}f''(x)$$
and
$$c(0) = e^{f(0)} = e^0 = 1,\quad c'(0) = e^{f(0)}f'(0) = e^0 \cdot 2 = 2,\quad c''(0) = e^{f(0)}(f'(0))^2 + e^{f(0)}f''(0) = 4 + 8 = 12$$

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function in $C^{(\infty)}(\mathbb{R})$ such that $f(x) = -x^4 + \frac{1}{16}x^5 + o(x^5)$, for $x \to 0$.

Suppose $g(x) = f(2x)$.

a) compute the derivatives $g^k(0)$ for $k = 1, 2, 3, 3, 4, 5$.
Recall the Mac Laurin expansion of $g(x)$:
$$g(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2 + \frac{g'''(0)}{3!}x^3 + \frac{g^{(4)}(0)}{4!}x^4 + \frac{g^{(5)}(0)}{5!}x^5 + o(x^5)$$
The expansion of $g(x) = f(2x)$ is:
$$g(x) = f(2x) = -16x^4 + \frac{32}{16}x^5 + o(x^5) = -16x^4 + 2x^5 + o(x^5)$$
Comparing the coefficient, we get:
$$g'(0) = 0 \quad , \quad g''(0) = 0 \quad , \quad g'''(0) = 0 \quad , \quad g^{(4)}(0) = -16 \cdot 4! = -324 \quad , \quad g^{(5)}(0) = 2 \cdot 5! = 240$$

b) Establish if $x_0 = 0$ is an extremal point for g and, if so, study its nature.
From the expansion: $g'(0) = g''(0) = g'''(0) = 0, g^{(4)}(0) = -324 < 0$; thus $x = 0$ is a relative maximum for $g(x)$.

c) Can we say something about the behavior of g in $x_0 = \dfrac{3}{4}$?
It is not possible, because we only have information on the behavior of g in $x_0 = 0$.

12. Consider the function $f(x) = \cos 2x - 1 - \log(2 - e^{x^2})$

a) compute the MacLaurin expansion of order 5 of $f(x)$

We have
$$f(x) = \cos 2x - 1 - \log(2 - e^{x^2}) = -x^2 + \frac{5}{3}x^4 + o(x^5)$$

b) Compute the order of infinitesimal of $f(x)$, for $x \to 0$

Since, for $x \to 0$, it holds $f(x) \sim -x^2$, the order of infinitesimal of $f(x)$ is $n = 2$

c) Compute the principal part of $f(x)$, for $x \to 0$.

Since $f(x) \sim -x^2$, the principal part of $f(x)$ is $p(x) = -x^2$

d) Say if$f(x)$ has constant sign in a neighborhood of $x = 0$.

Since, for $x \to 0$, it holds $f(x) \sim -x^2$, in a neighborhood of zero the function is negative, as the parabola $y = -x^2$.

e) Say if $x = 0$ is a stationary point for $f(x)$.

The coeffiecint of x in the expansion is null, thus $f'(0) = 0$ and the point $x = 0$ is stationary.

f) Compute the order of infinity of $g(x) = \dfrac{f(x)}{(2x^2)^5}$, for $x \to 0$ (w.r.t. the test function $u(x) = 1/x$).

It holds:
$$g(x) = \frac{f(x)}{(2x^2)^5} \sim \frac{-x^2}{32x^{10}} = \frac{-1}{32}\frac{1}{x^8}$$
Thus the order of infinity is $n = 8$.

13. Consider the function $f(x) = \log(\cos 2x) + 1 - \sqrt{1 + 4x^2}$.

a) Find the Mac Laurin expansion of order 4, principal part and order of infinitesimal of $f(x)$, for $x \to 0$.

It holds
$$f(x) = \log(\cos 2x) + 1 - \sqrt{1 + 4x^2}2x - 1 - \log(2 - e^{x^2}) = -4x^2 + \frac{2}{3}x^4 + o(x^4)$$

b) Compute $f^{(4)}(0)$.

Recall the meaning of the Mac Laurin coefficients, then
$$f^{(4)}(0) = \frac{2}{3} \cdot 4! = 16$$

c) Compute the order of infinity of $g(x) = \dfrac{f(x)}{x^3 \sin x}$, for $x \to 0$ (w.r.t. $u(x) = 1/x$).

It holds:
$$g(x) = \frac{f(x)}{x^3 \sin x} \sim \frac{-4x^2}{x^3 \cdot x} = -4\,\frac{1}{x^2}$$
Hence the order of infinity of $g(x)$, for $x \to 0$ (w.r.t. $u(x) = 1/x$) is $n = 2$.

14. Given the function $f(x) = \sin x \sqrt[5]{\cos x^2} - x$:

a) compute the Mac Laurin expansion of order 6
It holds

$$f(x) = \sin x \sqrt[5]{\cos x^2} - x = -\frac{x^3}{6} - \frac{11}{120}x^5 + o(x^6)$$

b) find the principal part and order of infinitesimal of $f(x)$, for $x \to 0$

Since, for $x \to 0$, it holds $f(x) \sim -\dfrac{x^3}{6}$, the order of infinitesimal of $f(x)$ is 3 and its principal part is $p(x) = -\dfrac{x^3}{6}$

c) compute $f^{(4)}(0)$ and $f^{(5)}(0)$

Recall the meaning of the Mac Laurin coefficients, then

$$f^{(4)}(0) = 4! \cdot \frac{-1}{6} = -4, \quad f^{(5)}(0) = 5! \cdot \frac{-11}{120} = -11$$

d) which is the nature of $x = 0$?

Since, for $x \to 0$, it holds $f(x) \sim -\frac{x^3}{6}$, the point $x = 0$ is an inflection point.

e) study the sign of $f(x)$ in a neighborhood of $x = 0$.

Since, for $x \to 0$, it holds $f(x) \sim -\frac{x^3}{6}$, in a left neighborhood of 0 the function is positive, while in a right one, it is negative.

f) compute the limit: $\displaystyle\lim_{x\to 0} \frac{f(x)}{\sin^3 x}$

$$\lim_{x\to 0} \frac{f(x)}{\sin^3 x} = \lim_{x\to 0} \frac{\frac{-x^3}{6}}{x^3} = \frac{-1}{6}$$

15. Consider the function $f(x) = \sin \dfrac{2\pi}{x} - 1$.

a) Find the Taylor expansion of order 2 of $f(x)$, with $x_0 = 4$.
Recall that:

$$f(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 + o((x-4)^2)$$

Compute the coefficients:

$$\begin{aligned}
f(x) &= \sin \frac{2\pi}{x} - 1 \\
f(4) &= \sin \frac{2\pi}{4} - 1 = \sin \frac{\pi}{2} - 1 = 0 \\
f'(x) &= -\frac{2\pi}{x^2} \cos \frac{2\pi}{x} \\
f'(4) &= -\frac{2\pi}{16} \cos \frac{2\pi}{4} = -\frac{\pi}{8} \cos \frac{\pi}{2} = 0 \\
f''(x) &= \frac{4\pi}{x^3} \cos \frac{2\pi}{x} - \frac{2\pi}{x^2}\frac{2\pi}{x^2} \sin \frac{2\pi}{x} \\
&= \frac{4\pi}{x^3} \cos \frac{2\pi}{x} - \frac{4\pi^2}{x^4} \sin \frac{2\pi}{x} \\
f''(4) &= \frac{4\pi}{4^3} \cos \frac{2\pi}{4} - \frac{4\pi^2}{4^4} \sin \frac{2\pi}{4} = \\
&= \frac{4\pi}{4^3} \cos \frac{\pi}{2} - \frac{\pi^2}{64} \sin \frac{\pi}{2} = -\frac{\pi^2}{64}
\end{aligned}$$

Thus

$$f(x) = -\frac{\pi^2}{128}(x-4)^2 + o\big((x-4)^2\big)$$

b) Compute the order of infinitesimal, for $x \to 4$, of $f(x)$.

Since, for $x \to 4$, it holds $f(x) \sim -\frac{\pi^2}{128}(x-4)^2$, the order of infinitesimal is $n = 2$.

c) Compute the principal part of $f(x)$, for $x \to 4$.

Since, for $x \to 4$, it holds $f(x) \sim -\frac{\pi^2}{128}(x-4)^2$, the principal part is $p(x) = -\dfrac{\pi^2}{128}(x-4)^2$.

d) Say if $f(x)$ has constant sign in a neighborhood of $x = 4$.

Since, for $x \to 4$, it holds $f(x) \sim -\frac{\pi^2}{128}(x-4)^2$, in a neighborhood of $x = 4$ it has negative sign

e) Say if $x = 4$ is a stationary point for $f(x)$.

The first derivative is null in $x = 4$, thus $x = 4$ is a stationary point.

f) Compute the order of infinity of $g(x) = \dfrac{f(x)}{(x-4)^{7/2}}$, for $x \to 4$ (w.r.t. $u(x) = 1/(x-4)$).

It holds:

$$g(x) = \frac{f(x)}{(x-4)^{7/2}} \sim \frac{-\frac{\pi^2}{128}(x-4)^2}{(x-4)^{7/2}} = -\frac{\pi^2}{128}\frac{1}{(x-4)^{3/2}}$$

Thus the order of infinity as $x \to 4$ w.r.t. $u(x) = 1/(x-4)$ is $3/2$.