MATHEMATICAL ANALYSIS I TUTORING
3^{rd} WEEK

SEQUENCES - LIMITS OF SEQUENCES
LIMITS OF FUNCTIONS - CONTINUITY

PROPOSED EXERCISES - SOLUTIONS

Ex 1

1. For each of the following properties concerning a generic sequence $(a_n), a_n \in \mathbb{R}$, write the definition and its logical negation:

   (a) (a_n) is indeterminate
   A sequence (a_n) is indeterminate if it does not admit limit.
   Negation: A sequence (a_n) is not indeterminate if it does not admit limit, either finite (i.e. convergent sequence), or positively infinite (i.e. positively divergent sequence), or negatively infinite (i.e. negatively divergent sequence).

   (b) (a_n) is negative
   A sequence (a_n) is negative if $a_n \leq 0 \ \forall n \ \in \mathbb{N}$
   Negation: $\exists m \in \mathbb{N}$ such that $a_m > 0$.

   (c) (a_n) is not upper bounded
   A sequence(a_n) is not upper bounded if $\forall A \geq 0 \ \exists \, m \in \mathbb{N}$ such that $a_m > A$.
   Negation: $\exists A > 0$ such that $\forall m \in \mathbb{N}$, $a_m \leq A$.

   (d) (a_n) is bounded

   A sequence (a_n) is bounded if $\exists A > 0$ such that $\forall n \in \mathbb{N} \ |a_n| \leq A$.
   Negation: $\forall A > 0, \exists m \in \mathbb{N}$ such that $|a_m| > A$.

   (e) (a_n) is regular

   A sequence (a_n) is regular if it admits limit, either convergent or divergent (positively or negatively).
   Negation: A sequence (a_n) is not regular if it does not admit limit.

   (f) (a_n) is definitely increasing
   A sequence (a_n) is definitely increasing if $\exists n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 : a_{n+1} \geq a_n$

Ex 2 For each of the following sequences

$$a_n = (-1)^n \cos((2n+1)\pi); \quad b_n = \frac{11-n}{3n}; \quad c_n = \sin\left(n\frac{\pi}{2}\right)$$

say which properties are true or false.

$$\boxed{a_n = (-1)^n \cos((2n+1)\pi)}$$

Since $\forall n \in \mathbb{N}, \cos((2n+1)\pi) = -1$, the sequence is $a_n = (-1)^{n+1}$. Thus:

(a) the terms a_n are definitely less than some $k > 0$ $\boxed{\text{T}}$
(b) its image is $\{-1, 1\}$ $\boxed{\text{T}}$
(c) the terms a_n are all definitely negative $\boxed{\text{F}}$
(d) the sequence is regular $\boxed{\text{F}}$
(e) the sequence is increasing $\boxed{\text{F}}$

$$\boxed{b_n = \frac{11-n}{3n}}$$

It can be written as $b_n = \dfrac{11-n}{3n} = -\dfrac{1}{3} + \dfrac{11}{3n}$
The sequence has positive terms if $n \leq 11$ and negative if $n > 11$.
The sequence is decreasing; the first term is $a_1 = \dfrac{10}{3}$. Moreover $\lim\limits_{n\to+\infty} b_n = -\dfrac{1}{3}$. Thus:

(a) the terms a_n are definitely less than some $k > 0$ $\boxed{\text{T}}$

(b) its image is $\{-1, 1\}$ $\boxed{\text{F}}$

(c) the terms a_n are all definitely negative $\boxed{\text{T}}$

(d) the sequence is regular $\boxed{\text{T}}$

(e) the sequence is increasing $\boxed{\text{F}}$

$$\boxed{c_n = \sin\left(n\frac{\pi}{2}\right)}$$

$n = 0 \;\rightarrow\; c_0 = \sin(0) = 0$

$n = 1 \;\rightarrow\; c_1 = \sin\left(\frac{\pi}{2}\right) = 1$

$n = 2 \;\rightarrow\; c_n = \sin\left(2\frac{\pi}{2}\right) = 0$

$n = 3 \;\rightarrow\; c_n = \sin\left(3\frac{\pi}{2}\right) = -1$

$\vdots$

$n = 2k \;\rightarrow\; c_{2k} = \sin\left(2k\frac{\pi}{2}\right) = \sin(k\pi) = 0$

$n = 2k+1 \;\rightarrow\; c_{2k+1} = \sin\left((2k+1)\frac{\pi}{2}\right) = (-1)^k$

(a) the terms a_n are definitely less than some $k > 0$ $\boxed{\text{T}}$

(b) its image is $\{-1, 1\}$ $\boxed{\text{F}}$

(c) the terms a_n are all definitely negative $\boxed{\text{F}}$

(d) the sequence is regular $\boxed{\text{F}}$

(e) the sequence is increasing $\boxed{\text{F}}$

Ex 3 Write the definition of limit of a generic sequence $(a_n), a_n \in \mathbb{R}$:

(a) $\lim\limits_{n\to+\infty} a_n = -\infty \;\Leftrightarrow\; \forall\, A > 0,\ \exists n_A\ :\ \ n > n_A \;\Rightarrow\; a_n < -A$

(b) $\lim\limits_{n\to+\infty} a_n = l \;\Leftrightarrow\; \forall\, \epsilon > 0,\ \exists n_\epsilon\ :\ n > n_\epsilon \;\Rightarrow\; |a_n - l| < \epsilon$

(c) $\lim\limits_{n\to+\infty} a_n = e \;\Leftrightarrow\; \forall\, \epsilon > 0,\ \exists n_\epsilon\ :\ n > n_\epsilon \;\Rightarrow\; |a_n - e| < \epsilon$

(d) $\lim\limits_{n\to+\infty} a_n = +\infty \;\Leftrightarrow\; \forall\, B > 0,\ \exists n_B\ :\ \ n > n_B \;\Rightarrow\; a_n > B$

(e) $\lim\limits_{n\to+\infty} a_n = 0 \;\Leftrightarrow\; \forall\, \epsilon > 0,\ \exists n_\epsilon\ :\ n > n_\epsilon \;\Rightarrow\; |a_n| < \epsilon$

Ex 4 Study the asymptotic behaviour of the following sequences:

(a) $a_n = \left(\frac{1}{3}\right)^{1/n}$ the sequence is positive and convergent to 1; indeed:

$\lim_{n\to+\infty} \left(\frac{1}{3}\right)^{1/n} = \lim_{n\to+\infty} e^{\frac{1}{n}\log\left(\frac{1}{3}\right)} = e^0 = 1.$

(b) $b_n = \cos\left(n\frac{\pi}{2}\right)$: the sequence has image $\{0, \pm 1\}$ and it is indeterminate;

$c_n = \frac{\sin\left(n\frac{\pi}{2}\right)}{n^2 - n}$ the sequence converges to 0;

The function sin is bounded between -1 and 1

$$-1 \leq \sin\left(n\frac{\pi}{2}\right) \leq 1$$

Notice that $n^2 - n > 0$ for every $n \in \mathbb{N} \setminus \{0\}$, thus we can divide both sides by $n^2 - n >$:

$$-\frac{1}{n^2 - n} \leq \frac{\sin\left(n\frac{\pi}{2}\right)}{n^2 - n} \leq \frac{1}{n^2 - n}$$

Since $\lim_{n\to+\infty} \frac{\pm 1}{n^2 - n} = 0$, from the Comparison Theorem it holds:

$$\lim_{n\to+\infty} \frac{\sin\left(n\frac{\pi}{2}\right)}{n^2 - n} = 0.$$

(c) $x_n = \frac{3n^4 - 7n^2 - 3}{1 - 3n + n^3}$ the sequence diverges to $+\infty$; indeed:

$$\lim_{n\to+\infty} \frac{3n^4 - 7n^2 - 3}{1 - 3n + n^3} = \lim_{n\to+\infty} \frac{n^4(3 - \frac{7}{n^2} - \frac{3}{n^4})}{n^3(\frac{1}{n^3} - \frac{3}{n^2} + 1)} = \lim_{n\to+\infty} \frac{n^4 \cdot 3}{n^3 \cdot 1} = \lim_{n\to+\infty} 3n = +\infty$$

$y_n = \frac{2n^5 - 8n^3}{1 - 3n^5}$ the sequence converges to $-\frac{2}{3}$; indeed:

$$\lim_{n\to+\infty} \frac{2n^5 - 8n^3}{1 - 3n^5} = \lim_{n\to+\infty} \frac{n^5(2 - \frac{8}{n^2})}{n^5(\frac{1}{n^5} - 3)} = \lim_{n\to+\infty} \frac{2n^5}{n^5(-3)} = -\frac{2}{3}$$

$z_n = \frac{n^2 + 2\sqrt{2}n - 3}{n^4 - \pi n + e}$ the sequence converges to 0; indeed:

$$\lim_{n\to+\infty} \frac{n^2 + 2\sqrt{2}n - 3}{n^4 - \pi n + e} = \lim_{n\to+\infty} \frac{n^2(1 + 2\sqrt{2}\frac{1}{n} - \frac{3}{n^2})}{n^4(1 - \frac{\pi}{n^3} + \frac{e}{n^4})} = \lim_{n\to+\infty} \frac{n^2}{n^4} = 0$$

(d) $a_n = \left(1 + \frac{1}{3n}\right)^n$ the sequence converges to $\sqrt[3]{\mathrm{e}}$; indeed:

$$\lim_{n\to+\infty} \left(1 + \frac{1}{3n}\right)^n = \lim_{n\to+\infty} \left(1 + \frac{1}{3n}\right)^{\frac{3n}{3}} = \left(\lim_{n\to+\infty} \left(1 + \frac{1}{3n}\right)^{3n}\right)^{1/3} = \sqrt[3]{\mathrm{e}}$$

$b_n = \left(1 + \frac{2}{n}\right)^{n/4}$ the sequence converges to $\sqrt{\mathrm{e}}$; indeed:

$$\lim_{n\to+\infty} \left(1 + \frac{2}{n}\right)^{n/4} = \lim_{n\to+\infty} \left(1 + \frac{1}{n/2}\right)^{n/2\cdot\ 1/2} = \left(\lim_{n\to+\infty} \left(1 + \frac{1}{n/2}\right)^{n/2}\right)^{1/2} = \sqrt{\mathrm{e}}$$

$c_n = \left(1 + \frac{1}{n}\right)^{-n^2}$ the sequence converges to 0; indeed:

$$\lim_{n\to+\infty} \left(1 + \frac{1}{n}\right)^{n^2} = \lim_{n\to+\infty} \left(\left(1 + \frac{1}{n}\right)^n\right)^n = \lim_{n\to+\infty} \mathrm{e}^n = +\infty$$

$d_n = \left(1+\dfrac{1}{3n}\right)^{3n+5}$ the sequence converges to e. Indeed:

$$\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{3n+5} = \lim_{n\to+\infty}\left(\left(1+\frac{1}{3n}\right)^{3n}\left(1+\frac{1}{3n}\right)^{5}\right) = \lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{3n}\cdot\lim_{n\to+\infty}\left(1+\frac{1}{3n}\right)^{5} = \mathrm{e}\cdot 1$$

Ex 5 Discuss and compute the following limits (where [ ] denotes the integer part)

(a) $\boxed{\lim_{x\to-\infty}\dfrac{x-4}{\sqrt{x^2+4}}}$

$$\lim_{x\to-\infty}\frac{x-4}{\sqrt{x^2+4}} = \lim_{x\to-\infty}\frac{x\left(1-\frac{4}{x}\right)}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to-\infty}\frac{x}{|x|} = \lim_{x\to-\infty}\frac{x}{-x} = -1$$

(b) $\boxed{\lim_{x\to+\infty}\dfrac{x-4}{\sqrt{x^2+4}}}$

$$\lim_{x\to+\infty}\frac{x-4}{\sqrt{x^2+4}} = \lim_{x\to+\infty}\frac{x\left(1-\frac{4}{x}\right)}{\sqrt{x^2\left(1+\frac{4}{x^2}\right)}} = \lim_{x\to+\infty}\frac{x}{|x|} = \lim_{x\to+\infty}\frac{x}{x} = 1$$

(c) $\boxed{\lim_{x\to-\infty}\dfrac{3x+\sin\pi x}{-x-e}}$

$$\lim_{x\to-\infty}\frac{3x+\sin\pi x}{-x-e} = \lim_{x\to-\infty}\frac{x\left(3+\frac{\sin\pi x}{x}\right)}{x\left(-1-\frac{e}{x}\right)} = \lim_{x\to-\infty}\frac{3x}{-x} = -3$$

(d) $\boxed{\lim_{x\to+\infty}(3x+2\cos\pi x)}$

$$\lim_{x\to+\infty}(3x+2\cos\pi x) = \lim_{x\to+\infty}x\left(3+\frac{2\cos\pi x}{x}\right) = \lim_{x\to+\infty}x(3+0) = +\infty$$

(e) $\boxed{\lim_{x\to+\infty}\dfrac{M(x^2-\pi x+3)}{x}}$

$$0 \le M(x^2-\pi x+3) < 1$$
$$0 \le \frac{M(x^2-\pi x+3)}{x} < \frac{1}{x}$$

Since $\lim_{x\to+\infty}\dfrac{1}{x}=0$, applying the Comparison Theorem, we have

$$\lim_{x\to+\infty}\frac{M(x^2-\pi x+3)}{x} = 0$$

(f) $\boxed{\lim_{x\to-\infty}\dfrac{\sin x}{x}}$

$$-1 \le \sin x \le 1$$
$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

Since $\lim_{x\to-\infty}\dfrac{1}{x}=0$, applying the Comparison Theorem, we have

$$\lim_{x\to-\infty}\frac{\sin x}{x} = 0$$

(g) $\boxed{\lim_{x\to0}\sinh\dfrac{1}{x}}$

It holds $\lim_{x\to0}\dfrac{1}{x}=\pm\infty$; since the function sinh is odd, it results $\lim_{x\to-\infty}\sinh x=-\infty$ while $\lim_{x\to+\infty}\sinh x = +\infty$; hence $\lim_{x\to\pm\infty}\sinh x$ does not exist.

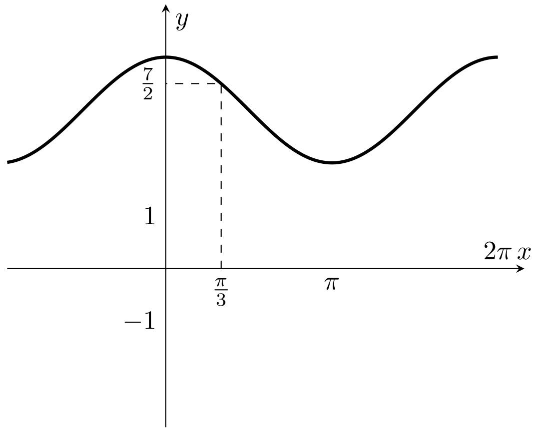
(h) $\boxed{\lim_{x\to 0} \sin\frac{1}{x}}$

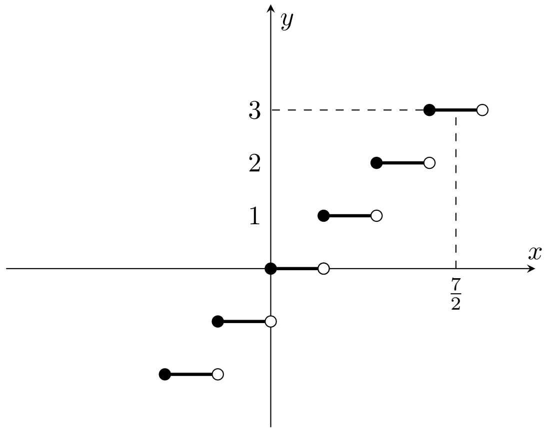
It holds $\lim_{x\to 0}\frac{1}{x} = \pm\infty$; the function $\sin t$ has no limit for $t \to \pm\infty$, hence the given limit does not exist.

(i) $\boxed{\lim_{x\to\pi/2} \frac{x}{1-\sin x}}$

By substitution, we get the ratio of $\frac{\pi}{2}$ and a function tending to 0. From $1 - \sin x \geq 0, \forall x \in \mathbb{R}$, the quotient tends to $+\infty$. Thus $\lim_{x\to\pi/2} \frac{x}{1-\sin x} = +\infty$.

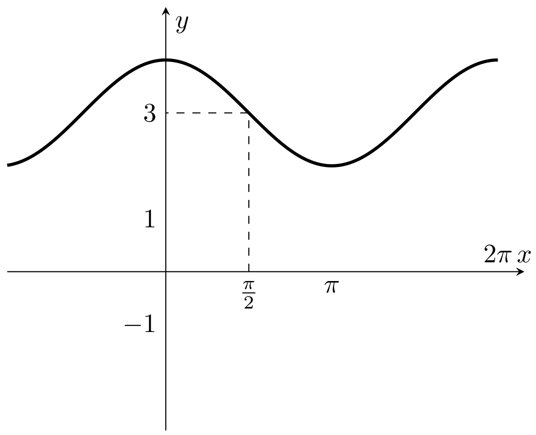
(l) $\boxed{\lim_{x\to\pi/3} x[3+\cos x]}$

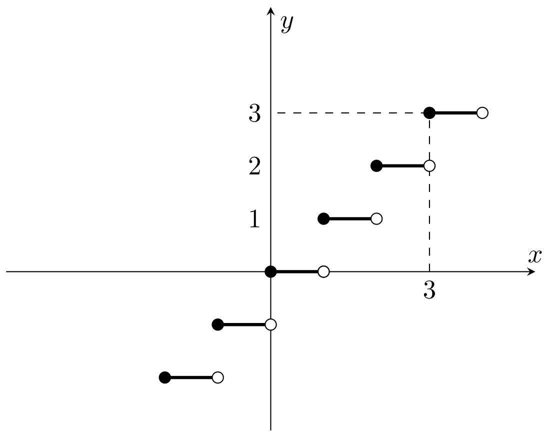




The function $3 + \cos x$ associates to a neighborhood of $\frac{\pi}{3}$, a neighborhood of $\frac{7}{2}$. The integer part in a neighborhood of $\frac{7}{2}$ takes value 3, thus: $\lim_{x\to\pi/3} x[3+\cos x] = \frac{\pi}{3}3 = \pi$.

(m) $\boxed{\lim_{x\to\pi/2^+} x[3+\cos x]}$



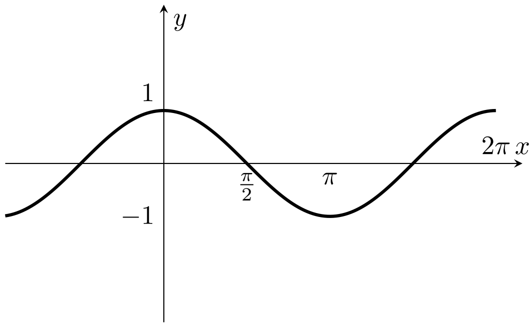


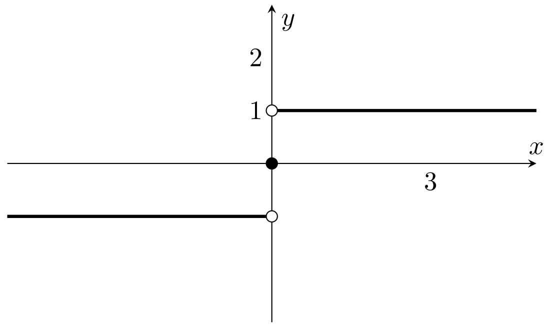
Given a right neighborhood of$\frac{\pi}{2}$, the function $3 + \cos x$ goes in a left neighborhood of 3, where the integer part takes value 2, thus: $\lim_{x\to\pi/2^+} x[3+\cos x] = \frac{\pi}{2}2 = \pi$.

(n) $\boxed{\lim_{x\to\pi/2^-} x[3+\cos x]}$

The function $3 + \cos x$ associates a neighborhood of $\frac{\pi}{2}$ to a right neighborhood of 3. In such neighborhood, the integer part takes value 3. Therefore $\lim_{x\to\pi/2^-} x[3+\cos x] = \frac{\pi}{2}3 = \frac{3}{2}\pi$.

(o) $\boxed{\lim_{x\to\pi/2^+} x\ \mathrm{sign}(\cos x)}$



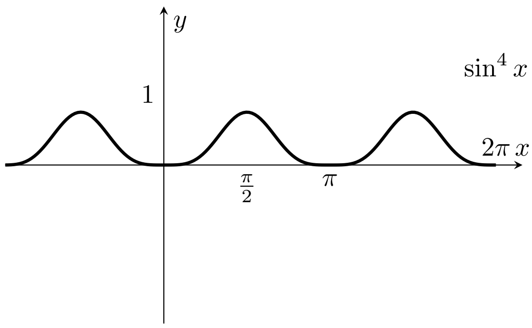


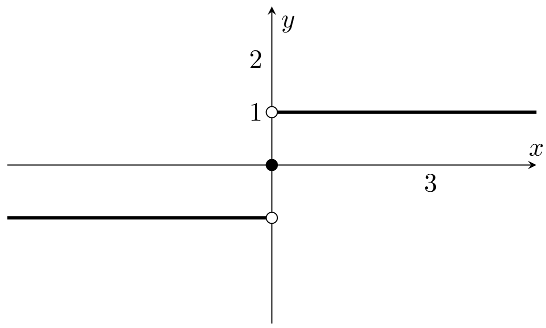
Given a neighborhood of $\dfrac{\pi}{2}$, the function $\cos x$ goes in a right neighborhood of 0. The function *sign* is -1 in such neighborhood of 0, hence: $\displaystyle\lim_{x\to\pi/2^+} x\ \mathrm{sign}(\cos x) = \lim_{x\to\pi/2^+} x(-1) = -\frac{\pi}{2}$.

(p) $\boxed{\lim_{x\to\pi/2^-} x\ \mathrm{sign}(\cos x)}$

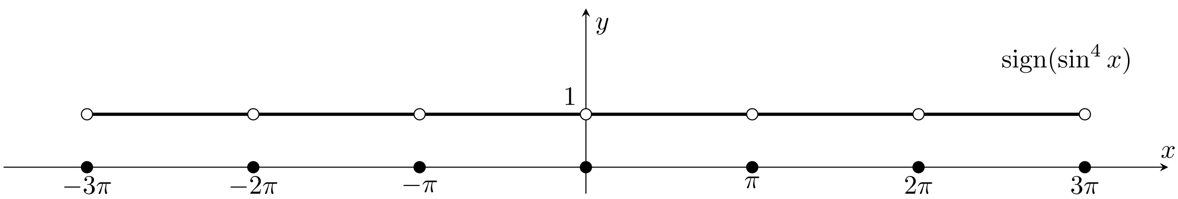
Given a left neighborhood of $\dfrac{\pi}{2}$, the function $\cos x$ goes in right neighborhood of 0. In such interval, the function *sign* takes value 1, thus: $\displaystyle\lim_{x\to\pi/2^-} x\ \mathrm{sign}(\cos x) = \lim_{x\to\pi/2^+} x(1) = \frac{\pi}{2}$

(q) $\boxed{\lim_{x\to\pi} [x\ \mathrm{sign}(\sin^4 x)]}$

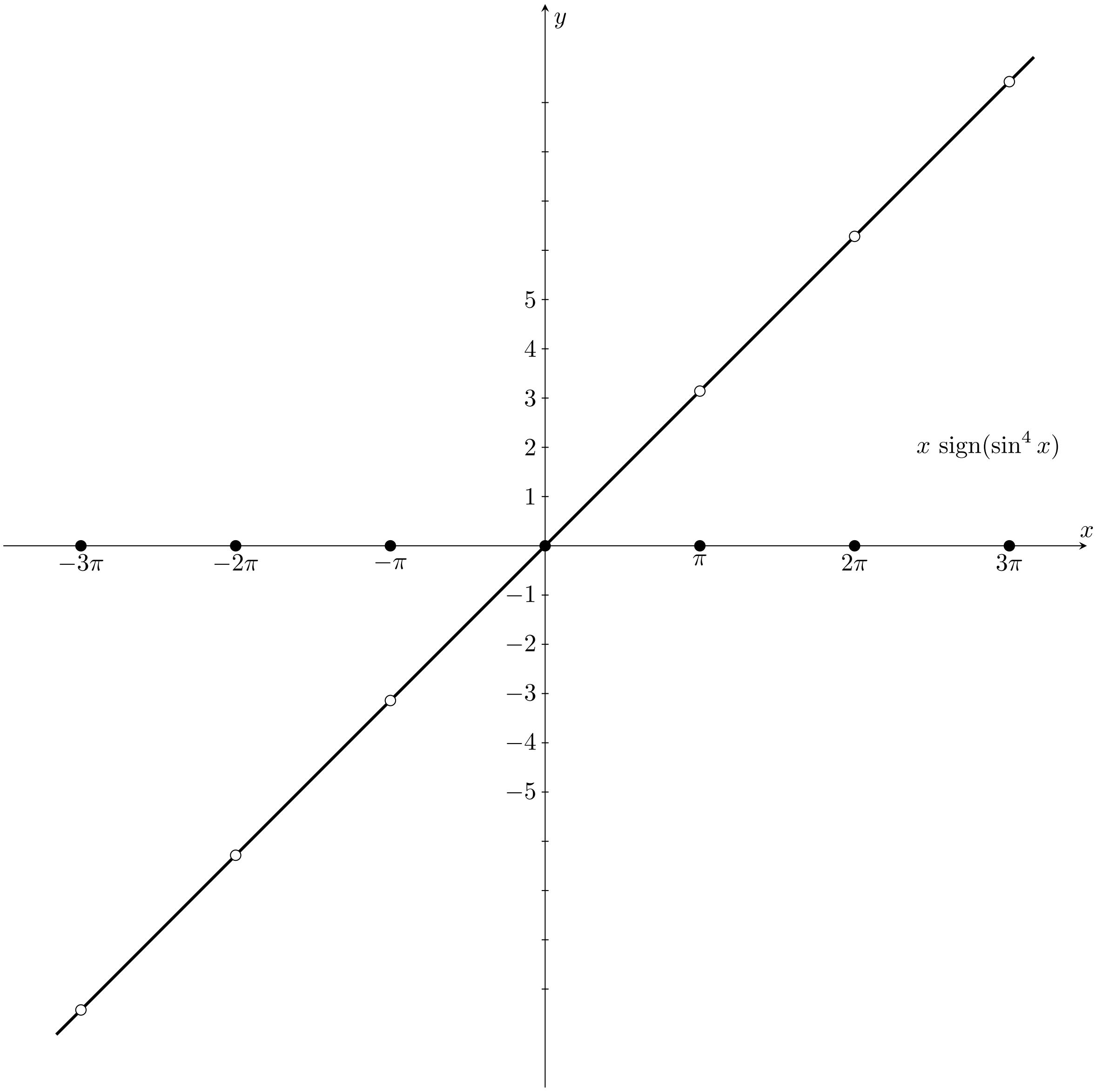




This is the graph of $\mathrm{sign}(\sin^4 x)$: it is the line $y = 1$ *empty* in all the points $x = k\pi$, where the function takes value 0:



The next picture shows the graph of $x\ \mathrm{sign}(\sin^4 x)$: it is the line $y = x$ *empty* in all the points $x = k\pi$, where the function takes value 0:



In $x = \pi$ the function $x\ \mathrm{sign}(\sin^4 x)$ is 0, while

$$\lim_{x\to\pi} x\ \mathrm{sign}(\sin^4 x) = \pi.$$

Since the integer part is continuous in π, we can say that

$$\lim_{x\to\pi} \left[x\ \mathrm{sign}(\sin^4 x)\right] = \left[\lim_{x\to\pi} x\ \mathrm{sign}(\sin^4 x)\right] = [\pi] = 3.$$

Ex 6 Compute the following limits:

(a) $\boxed{\lim_{x\to-\infty} \dfrac{2x-1}{\sqrt{3x^2-2}}}$

By substitution we get the indeterminate form $\frac{-\infty}{+\infty}$. Collect x with the highest powers in both numerator and denominator:

$$\lim_{x\to-\infty} \frac{2x-1}{\sqrt{3x^2-2}} = \lim_{x\to-\infty} \frac{x\left(2-\frac{1}{x}\right)}{\sqrt{x^2\left(3-\frac{2}{x^2}\right)}} = \lim_{x\to-\infty} \frac{2x}{|x|\sqrt{3}} = \lim_{x\to-\infty} \frac{2x}{-x\sqrt{3}} = -\frac{2}{\sqrt{3}}$$

(b) $\boxed{\lim_{x\to-\infty} \dfrac{x^7+8x^5+3x}{-4x^7+x}}$

By substitution we get the indeterminate form $\frac{-\infty}{+\infty}$. Collect x with the highest powers in both numerator and denominator:

$$\lim_{x\to-\infty} \frac{x^7+8x^5+3x}{-4x^7+x} = \lim_{x\to-\infty} \frac{x^7\left(1+8\frac{x^5}{x^7}+3\frac{x}{x^7}\right)}{x^7\left(-4+\frac{x}{x^7}\right)} = \lim_{x\to-\infty} \frac{\left(1+8\frac{1}{x^2}+3\frac{1}{x^6}\right)}{\left(-4+\frac{1}{x^6}\right)} = -\frac{1}{4}$$

(c) $\boxed{\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x})}$

By substitution we get the indeterminate form $+\infty-\infty$. Since we have the difference of 2 radicals, multiply and divide by theier sum:

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x+1}-\sqrt{x} &= \lim_{x\to+\infty}\left(\sqrt{x+1}-\sqrt{x}\right)\frac{(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} = \lim_{x\to+\infty}\frac{(x+1-x)}{\left(\sqrt{x\left(1+\frac{1}{x}\right)}+\sqrt{x}\right)}\\ &= \lim_{x\to+\infty}\frac{1}{(\sqrt{x}+\sqrt{x})}=0.\end{aligned}$$

(d) $\boxed{\lim_{x\to-\infty}(\sqrt{x^2+x}-x)}$

By substitution we get the indeterminate form $+\infty-\infty$, and multiply numerator and denominator by the following sum:

$$\begin{aligned}\lim_{x\to+\infty}\sqrt{x^2+x}-x &= \lim_{x\to+\infty}\left(\sqrt{x^2+x}-x\right)\frac{(\sqrt{x^2+x}+x)}{(\sqrt{x^2+x}+x)} = \lim_{x\to+\infty}\frac{x^2+x-x^2}{\left(\sqrt{x^2\left(1+\frac{x}{x^2}\right)}+x\right)}\\ &= \lim_{x\to+\infty}\frac{x}{(|x|+x)} = \lim_{x\to+\infty}\frac{x}{2x}=\frac{1}{2}\end{aligned}$$

(e) $\boxed{\lim_{x\to4}\frac{\sqrt{x}-2}{x^2-5x+4}}$

By substitution we get the indeterminate form $\frac{0}{0}$; multiply numerator and denominator by the sum $\sqrt{x}+2$:

$$\lim_{x\to4}\frac{\sqrt{x}-2}{x^2-5x+4}=\lim_{x\to4}\frac{\sqrt{x}-2}{(x-1)(x-4)}\frac{\sqrt{x}+2}{\sqrt{x}+2}=\lim_{x\to4}\frac{x-4}{(x-1)(x-4)}\frac{1}{\sqrt{x}+2}=\lim_{x\to4}\frac{1}{(x-1)\sqrt{x}+2}=\frac{1}{12}$$

(f) $\boxed{\lim_{x\to2}\frac{x^2-5x+6}{x^2-4x+4}}$

We get the indeterminate form $\frac{0}{0}$; it means that $x=2$ is a root for both polynomials in numerator and denominator, and thus they can be both factorized as the product of $(x-2)$ and another factor, that can found searching for the other roots:

$$\lim_{x\to2}\frac{x^2-5x+6}{x^2-4x+4}=\lim_{x\to2}\frac{(x-2)(x-3)}{(x-2)(x-2)}=\lim_{x\to2}\frac{(x-3)}{(x-2)}$$

Such limit does not exist since

$$\lim_{x\to2^-}\frac{x^2-5x+6}{x^2-4x+4}=\lim_{x\to2^-}\frac{(x-3)}{(x-2)}=+\infty$$

$$\lim_{x\to2^+}\frac{x^2-5x+6}{x^2-4x+4}=\lim_{x\to2^+}\frac{(x-3)}{(x-2)}=-\infty$$

(g) $\boxed{\lim_{x\to1}\frac{x^3-1}{x^4-1}}$

By substitution we get the indeterminate form $\frac{0}{0}$; apply the fundamental products:

$$\lim_{x\to1}\frac{x^3-1}{x^4-1}=\lim_{x\to1}\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)(x^2+1)}=\lim_{x\to1}\frac{(x^2+x+1)}{(x+1)(x^2+1)}=\frac{3}{4}$$

(h) $\boxed{\lim_{x\to a}\frac{x^2-a^2}{(x-a)^3}}$

By substitution we get the indeterminate form $\frac{0}{0}$; apply the fundamental products:

$$\lim_{x\to a}\frac{x^2-a^2}{(x-a)^3}=\lim_{x\to a}\frac{(x-a)(x+a)}{(x-a)^3}=\lim_{x\to a}\frac{(x+a)}{(x-a)^2}=\begin{cases}-\infty & \text{se } a<0\\ \nexists & \text{se } a=0\\ +\infty & \text{se } a>0\end{cases}$$

(i) $\boxed{\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 x}}$

By substitution we get the indeterminate form $\frac{0}{0}$; given the fundamental products: $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{1-\cos 2x}{\sin^2 x} = \lim_{x\to 0} \frac{1-\cos 2x}{4x^2}\frac{4x^2}{\sin^2 x} = \lim_{x\to 0} 4\cdot \frac{1-\cos 2x}{4x^2}\left(\frac{x}{\sin x}\right)^2 = 4\cdot\frac{1}{2} = 2$$

(j) $\boxed{\lim_{x\to 0} \frac{x+\sin 4x}{x+\sin x}}$

By substitution we get the indeterminate form $\frac{0}{0}$; remind the fundamental product: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\lim_{x\to 0} \frac{x+\sin 4x}{x+\sin x} = \lim_{x\to 0} \frac{x\left(1+\frac{\sin 4x}{x}\right)}{x\left(1+\frac{\sin x}{x}\right)} = \lim_{x\to 0} \frac{\left(1+\frac{\sin 4x}{x}\right)}{\left(1+\frac{\sin x}{x}\right)} = \frac{1+4}{1+1} = \frac{5}{2}$$

(k) $\boxed{\lim_{x\to \pi} \frac{\sin x}{x-\pi}}$

By substitution we get the indeterminate form $\frac{0}{0}$; in order to apply a fundamental limit, observe that, if $x \to \pi$, it holds $x - \pi \to 0$, and thus by substitution $t = x - \pi$, i.e. $x = t + \pi$, as $t \to 0$:

$$\lim_{x\to \pi} \frac{\sin x}{x-\pi} = \lim_{t\to 0} \frac{\sin(t+\pi)}{t} = \lim_{t\to 0} \frac{-\sin(t)}{t} = -1$$

(l) $\boxed{\lim_{x\to 0} \frac{2^{2x}-2^{-x}}{2^x-1}}$

By substitution we get the indeterminate form $\frac{0}{0}$; recall the fundamental limit $\lim_{x\to 0} \frac{a^x-1}{x} = \ln a, \forall a > 0$.

$$\lim_{x\to 0} \frac{2^{2x}-2^{-x}}{2^x-1} = \lim_{x\to 0} \frac{2^{-x}(2^{3x}-1)}{2^x-1} = \lim_{x\to 0} 2^{-x} \lim_{x\to 0} \frac{2^{3x}-1}{3x}\frac{3x}{2^x-1} = 1\cdot \ln 2 \cdot \frac{3}{\ln 2} = 3$$

(m) $\boxed{\lim_{x\to 0} \frac{\log(1+xe^x)}{e^{-3x}-1}}$

By substitution we get the indeterminate form $\frac{0}{0}$. Recall the fundamental limits: $\lim_{x\to 0} \frac{\log(x+1)}{x} = 1$ and $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

$$\lim_{x\to 0} \frac{\log(1+xe^x)}{e^{-3x}-1} = \lim_{x\to 0} \frac{\log(1+xe^x)}{xe^x}\frac{xe^x}{e^{-3x}-1} = \lim_{x\to 0} e^x \lim_{x\to 0} \frac{\log(1+xe^x)}{xe^x}\frac{-3x}{e^{-3x}-1}\frac{1}{(-3)} = -\frac{1}{3}$$

(n) $\boxed{\lim_{x\to 0} \frac{1-\log(e+x)}{x}}$

By substitution we get the indeterminate form $\frac{0}{0}$. Recall $\lim_{x\to 0} \frac{\log(x+1)}{x} = 1$:

$$\lim_{x\to 0} \frac{1-\log(e+x)}{x} = \lim_{x\to 0} \frac{1-\log(e(1+\frac{x}{e}))}{x} = \lim_{x\to 0} \frac{1-\log(e)-\log(1+\frac{x}{e})}{x} = \lim_{x\to 0} \frac{-\log(1+\frac{x}{e})}{e\frac{x}{e}} = -\frac{1}{e}$$

Ex 7 Study the following limits:

a) $\boxed{\lim_{x\to -\infty} (x^3+M(x))e^{5x}}$ Notice that Mantissa is bounded

$$0 \le M(x) < 1 \quad \Rightarrow \quad x^3+0 \le x^3+M(x) < x^3+1 \Rightarrow x^3e^{5x} \le (x^3+M(x))e^{5x} < (x^3+1)e^{5x}$$

Since the limits of external functions coincide, then

$$\lim_{x\to -\infty} x^3e^{5x} = 0 \qquad \lim_{x\to -\infty} (x^3+1)e^{5x} = 0$$

applying Comparison Theorems, we have $\lim_{x\to -\infty} (x^3+M(x))e^{5x} = 0$

b) $\boxed{\lim_{x\to+\infty} \ln(x^3-1)4^{-3x}}$

$$\lim_{x\to+\infty} \ln(x^3-1)4^{-3x} \quad = \quad \lim_{x\to+\infty} \frac{\ln(x^3-1)}{4^{3x}} = 0$$

c) $\boxed{\lim_{x\to0^+} (x)^{\sin x}}$ Given the equivalence $\sin x \sim x$, for $x \to 0$ and $\lim_{x\to 0} x\log x = 0$, it holds:

$$\begin{aligned} \lim_{x\to0^+} (x)^{\sin x} &= \lim_{x\to0^+} e^{\log(x)^{\sin x}} \\ &= \lim_{x\to0^+} e^{\sin x\log(x)} \\ &= \lim_{x\to0^+} e^{x\log(x)} = e^0 = 1 \end{aligned}$$

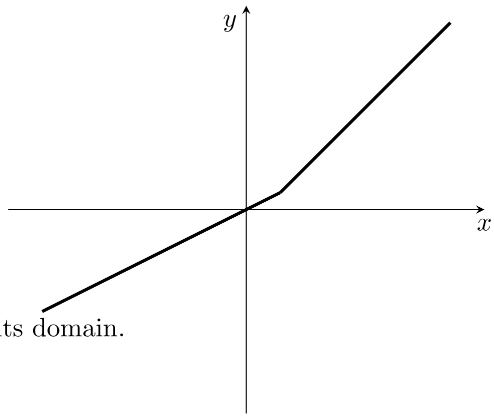
d) $\boxed{\lim_{x\to0}\sqrt{x\frac{2-x}{x-3}(e^x-1)}}$ We check if it makes sense to compute the limit, verifying that $x=0$ is an accumulation point for the domain. Since x and e^x-1 have always the same sign (both positive for $x\geq 0$ and negative for $x\leq 0$), their product is positive. Therefore the domain is:

$$x\frac{2-x}{x-3}(e^x-1)\geq 0 \Rightarrow x=0 \vee \frac{2-x}{x-3}\geq 0 \Rightarrow x=0 \;\vee\; 2\leq x<3 \Rightarrow D=\{0\}\cup[2,3)$$

It follows that $x=0$ is an isolated point for the domain and the limit does not exist.

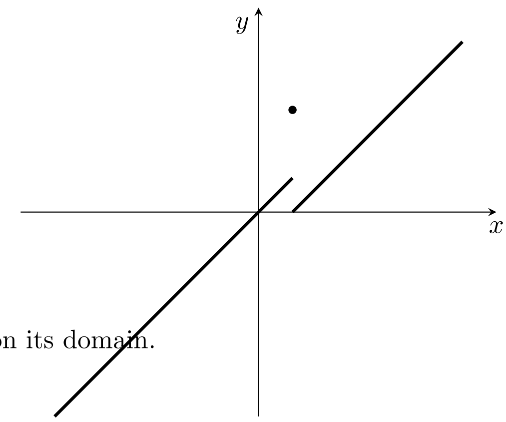
Ex 8 Say if there exists (and find) the values for α such that the following functions are continuous on their domain:

$$f_1(x) = \begin{cases} \alpha x & \text{if} \quad x\leq 1 \\ x-\alpha & \text{if} \quad x>1 \end{cases}$$



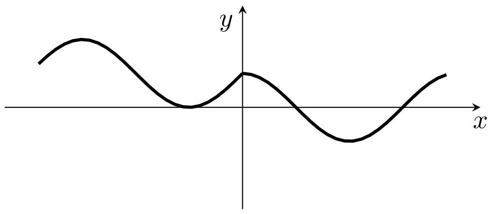
Check continuity in $x=1$
that is $\lim_{x\to1} f_1(x) = f_1(1)$
and thus if $\lim_{x\to1^-}\alpha x = \lim_{x\to1^+}(x-\alpha) = \alpha$.
This is true if and only if: $\alpha = 1-\alpha = \alpha \Leftrightarrow \alpha = \frac{1}{2}$.
Therefore, if $\alpha = \dfrac{1}{2}$, the function $f_1(x)$ is continuous on its domain.

$$f_2(x) = \begin{cases} \alpha x & \text{if} \quad x<1 \\ 3 & \text{if} \quad x=1 \\ x-\alpha & \text{if} \quad x>1 \end{cases}$$



Check continuity in $x=1$
that is $\lim_{x\to1} f_2(x) = f_2(1)$.
It holds : $\lim_{x\to1^-}\alpha x = \alpha$ and $\lim_{x\to1^+}(x-\alpha) = 1-\alpha$.
Now $\alpha = 1-\alpha \Leftrightarrow \alpha = \dfrac{1}{2}$: but $f(1) = 3 \neq \dfrac{1}{2}$.
Thus there is no $\alpha \in \mathbb{R}$. such that $f_2(x)$ is continuous on its domain.

$$f_3(x) = \begin{cases} \sin x + \alpha & \text{if} \quad x < 0 \\ \\ \cos(\alpha x) & \text{if} \quad x \geq 0 \end{cases}$$



Check continuity in $x = 0$
that is $\lim_{x\to 0} f_3(x) = f_3(0)$
and thus if $\lim_{x\to 0^-} (\sin x + \alpha) = \lim_{x\to 0^+} \cos(\alpha x) = f_3(0)$.
It holds: $\lim_{x\to 0^-} (\sin x + \alpha) = \alpha$; $\lim_{x\to 0^+} \cos(\alpha x) = 1$; $f_3(0) = 1$
Therefore, if $\alpha = 1$, the function $f_3(x)$ is continuous on its domain.

Ex 9

2. Discuss continuity of the following functions and, if they are discontinuous, write the continuous prolongation when possible.

$$f_1(x) = \begin{cases} \left|\arctan \dfrac{1}{x}\right| & \text{if} \quad x \neq 0 \\ \\ \pi/2 & \text{if} \quad x = 0 \end{cases}$$

Notice that $f_1(x)$ is continuous if:

$$\lim_{x\to 0} f_1(x) = f_1(0)$$

and thus, if:

$$\lim_{x\to 0} \left|\arctan \frac{1}{x}\right| = \frac{\pi}{2}$$

Compute:

$$\lim_{x\to 0} \left|\arctan \frac{1}{x}\right| = \left|\lim_{x\to 0} \arctan \frac{1}{x}\right| = \begin{cases} \left|\dfrac{\pi}{2}\right| = \dfrac{\pi}{2} & \text{if } x \to 0^+ \\ \\ \left|-\dfrac{\pi}{2}\right| = \dfrac{\pi}{2} & \text{if } x \to 0^- \end{cases}$$

Hence $f_1(x)$ is continuous on $\mathbb{R}$ (it coincides with its continuous prolongation).

$$\boxed{f_2(x) = \begin{cases} \arctan \left|\dfrac{1}{x}\right| & \text{if} \quad x \neq 0 \\ \\ -\pi/2 & \text{if} \quad x = 0 \end{cases}}$$

Check if $f_2(x)$ is continuous, that is, if:

$$\lim_{x\to 0} f_2(x) = f_2(0)$$

and thus:

$$\lim_{x\to 0} \arctan \left|\frac{1}{x}\right| = -\frac{\pi}{2}$$

but:

$$\lim_{x\to 0} \left|\frac{1}{x}\right| = +\infty \text{ and } \lim_{t\to +\infty} \arctan t = \frac{\pi}{2}$$

Therefore $f_2(x)$ is not continuous in $x = 0$, but we can find a continuous prolongation defing the function

$$\tilde{f}_2(x) = \begin{cases} \arctan \left|\dfrac{1}{x}\right| & \text{if} \quad x \neq 0 \\ \\ \pi/2 & \text{if} \quad x = 0 \end{cases}$$

$$\boxed{f_3(x) = \begin{cases} e^{-1/x^2} & \text{if} \quad x \neq 0 \\ \\ 0 & \text{if} \quad x = 0 \end{cases}}$$

The function $f_3(x)$ is continuous if:

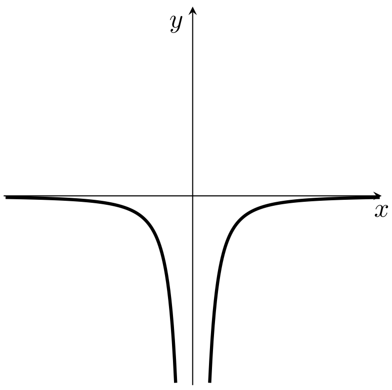
$$\lim_{x \to 0} f_3(x) = f_3(0)$$

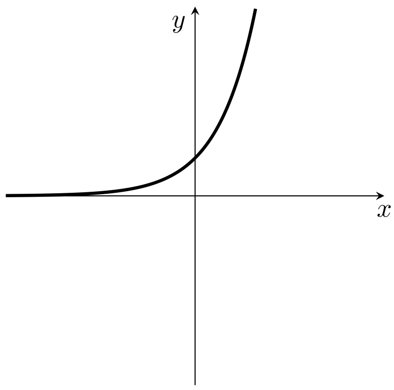
and thus, if:

$$\lim_{x \to 0} e^{-1/x^2} = 0$$

Since $\lim_{x \to 0} e^{-1/x^2} = e^{\lim_{x \to 0} -1/x^2} = 0$, we can conclude that f_3 is continuous in $x = 0$ (and therefore everywhere in $\mathbb{R}$).

We can observe that for every neighborhood of 0 (without 0), the function $-\dfrac{1}{x^2}$ goes in a neighborhood of $-\infty$, and the exponential function goes in a neighborhood of 0.





Ex 10 Discuss continuity and classify the singularities in $x_0 = 0$ for the following functions; if they are discontinuous, write the continuous prolongation when possible:

$\boxed{f_1(x) = 2x^2 + \sin x;}$ the function is continuous on $\mathbb{R}$, since it is a sum of continuous functions on $\mathbb{R}$.

$\boxed{f_2(x) = \dfrac{1 - \cos x}{x^2};}$

The function is not continuous on $\mathbb{R}$ since it is not defined in $x = 0$; but we can find a continuous prolongation in $x = 0$, indeed:

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$$

The continuous prolongation of $f_2(x)$ is:

$$\tilde{f}_2(x) = \begin{cases} \dfrac{1 - \cos x}{x^2} & x \neq 0 \\ \\ \dfrac{1}{2} & x = 0 \end{cases}$$

$\boxed{f_3(x) = \dfrac{1}{x - x^2};}$

The domain is $D = (-\infty, 0) \cup (0, 1) \cup (1, +\infty)$:

Compute the limits at the boundary points of the domain:

$$\lim_{x \to -\infty} \frac{1}{x - x^2} = 0$$

$$\lim_{x \to 0^-} \frac{1}{x - x^2} = -\infty$$

$$\lim_{x \to 0^+} \frac{1}{x - x^2} = +\infty$$

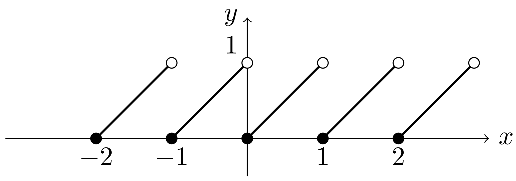
$$\lim_{x\to 1^-} \frac{1}{x-x^2} = +\infty$$

$$\lim_{x\to 1^+} \frac{1}{x-x^2} = -\infty$$

$$\lim_{x\to +\infty} \frac{1}{x-x^2} = 0$$

In $x=0$ e $x=1$ the function has singularities of II kind, and thus it cannot be extended by continuity.

$\boxed{f_4(x) = M(x)}$ The function Mantissa is not continuous, it has infinite discontinuity points of I kind: $x_k = k, k \in \mathbb{Z}$, and thus it cannot be extended by continuity.



Ex 11 Find domain, limits at boundary points of the domain, vertical/horizontal/oblique asymptotes (where they exist) for the following functions:

- $\boxed{f_1(x) = \sqrt{x^2-1}}$

  Domain: $D = (-\infty, -1] \cup [1, +\infty)$

  Symmetry: $f_1(-x) = \sqrt{(-x)^2-1} = \sqrt{x^2-1} = f_1(x)$, la funzione è pari.

  Limits at boundary points of the domain:

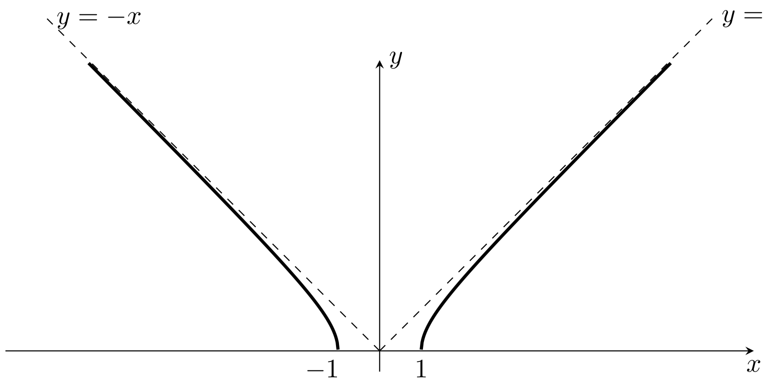
$$f(\pm 1) = 0$$

$$\lim_{x\to\pm\infty} \sqrt{x^2-1} = \lim_{x\to\pm\infty} \sqrt{x^2\left(1-\frac{1}{x^2}\right)} = \lim_{x\to\pm\infty} |x|\sqrt{1-\frac{1}{x^2}} = +\infty$$

  $\Longrightarrow$ f there are no horizontal asymptotes.
  Oblique asymptotes: $y = mx+q$: note the asymptotic equivalence for $x\to\pm\infty$

$$\sqrt{x^2-1} = \sqrt{x^2\left(1-\frac{1}{x^2}\right)} \sim |x|, \quad \text{per } x\to\pm\infty$$

  Thus $m = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \pm 1$; moreover $\lim_{x\to-\infty}(f(x)+x) = \lim_{x\to+\infty}(f(x)-x) = 0$; therefore there is a left oblique asymptote $y=-x$ and a right one $y=x$.



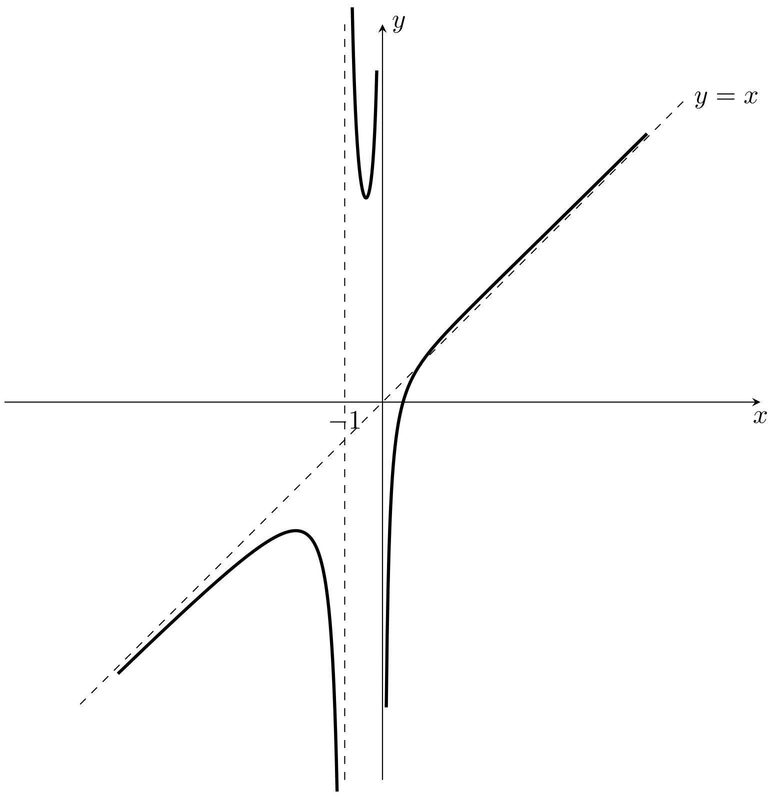
- $\boxed{f_2(x) = \dfrac{x^4 - 2x + 1}{x^3 - x}}$

Domain: $D = (-\infty, -1) \cup (-1, 0) \cup (0, 1) \cup (1, +\infty)$

Limits at boundary points of the domain:
$\displaystyle\lim_{x\to-\infty} \frac{x^4 - 2x + 1}{x^3 - x} = \lim_{x\to-\infty} \frac{x^4\left(1 - \frac{2}{x^3} + \frac{1}{x^4}\right)}{x^3\left(1 - \frac{1}{x^2}\right)} = \lim_{x\to-\infty} \frac{x^4}{x^3} = -\infty \implies$ there are no left oblique asymptotes.
$\displaystyle\lim_{x\to-1^-} \frac{x^4 - 2x + 1}{x(x-1)(x+1)} = -\infty \implies$ the line $x = -1$ is a left vertical asymptote $\displaystyle\lim_{x\to-1^+} \frac{x^4 - 2x + 1}{x(x-1)(x+1)} = +\infty \implies$ the line $x = -1$ is a right vertical asymptote
$\displaystyle\lim_{x\to0^-} \frac{x^4 - 2x + 1}{x(x-1)(x+1)} = +\infty \implies$ the line $x = 0$ is a left vertical asymptote
$\displaystyle\lim_{x\to0^+} \frac{x^4 - 2x + 1}{x(x-1)(x+1)} = -\infty \implies$ the line $x = 0$ is a right vertical asymptote
$\displaystyle\lim_{x\to1} \frac{(x-1)(x^3 + x^2 + x - 1)}{x(x-1)(x+1)} = \lim_{x\to1} \frac{(x^3 + x^2 + x - 1)}{x(x+1)} = 1$
$\displaystyle\lim_{x\to+\infty} \frac{x^4 - 2x + 1}{x^3 - x} = \lim_{x\to+\infty} \frac{x^4\left(1 - \frac{2}{x^3} + \frac{1}{x^4}\right)}{x^3\left(1 - \frac{1}{x^2}\right)} = \lim_{x\to+\infty} \frac{x^4}{x^3} = +\infty \implies$ there are no right vertical asymptotes.
Oblique asymptotes: note that

$$\frac{x^4 - 2x + 1}{x^3 - x} = \frac{x^4 - x^2 + x^2 - 2x + 1}{x^3 - x} = \frac{x^4 - x^2}{x^3 - x} + \frac{x^2 - 2x + 1}{x^3 - x} = x + \frac{x^2 - 2x + 1}{x^3 - x} \sim x, \quad \text{for } x \to \pm\infty$$

Therefore
$\displaystyle m = \lim_{x\to\mp\infty} \frac{f_2(x)}{x} = \lim_{x\to\mp\infty} \frac{x}{x} = 1$
$\displaystyle q = \lim_{x\to\mp\infty} (f_2(x) - mx) = \lim_{x\to\mp\infty} \left(\frac{x^3 + x^2 + x - 1}{x^2 + x} - x\right) == \lim_{x\to\mp\infty} \left(\frac{x - 1}{x^2 + x}\right) = 0$
The line $y = x$ is a left and right oblique asymptote.



- $\boxed{f_3(x) = \dfrac{x^2 - (x-1)\,|x-2|}{2x+3}}$

Domain: $D = (-\infty, -3/2) \cup (-3/2, +\infty)$

Limits at boundary points of the domain:

$$\lim_{x\to-\infty} f_3(x) = \lim_{x\to-\infty} \frac{x^2 + (x-1)(x-2)}{2x+3} = \lim_{x\to-\infty} \frac{x^2 + (x^2-3x+2)}{2x+3} = \lim_{x\to-\infty} \frac{2x^2-3x+2}{2x+3} = -\infty$$

Hence f_3 has no left horizontal asymptotes.

$\lim_{x\to-3/2^-} \frac{x^2+(x-1)(x-2)}{2x+3} = \frac{11}{0^-} = -\infty \implies$ the line $x = -\dfrac{3}{2}$ is a left vertical asymptote

$\displaystyle\lim_{x\to-3/2^+} \frac{x^2 + (x-1)(x-2)}{2x+3} = \frac{11}{0^+} = +\infty \implies$ the line $x = -\dfrac{3}{2}$ is a right vertical asymptote

$$\lim_{x\to+\infty} f_3(x) = \lim_{x\to+\infty} \frac{x^2 - (x-1)(x-2)}{2x+3} = \lim_{x\to+\infty} \frac{x^2 - (x^2-3x+2)}{2x+3} = \lim_{x\to+\infty} \frac{3x-2}{2x+3} = \frac{3}{2}$$

Therefore $y = \dfrac{3}{2}$ is a right horizontal asymptote

Oblique asymptote: note that there is no right oblique asymptote since there is a horizontal one. Moreover for $x \to -\infty$

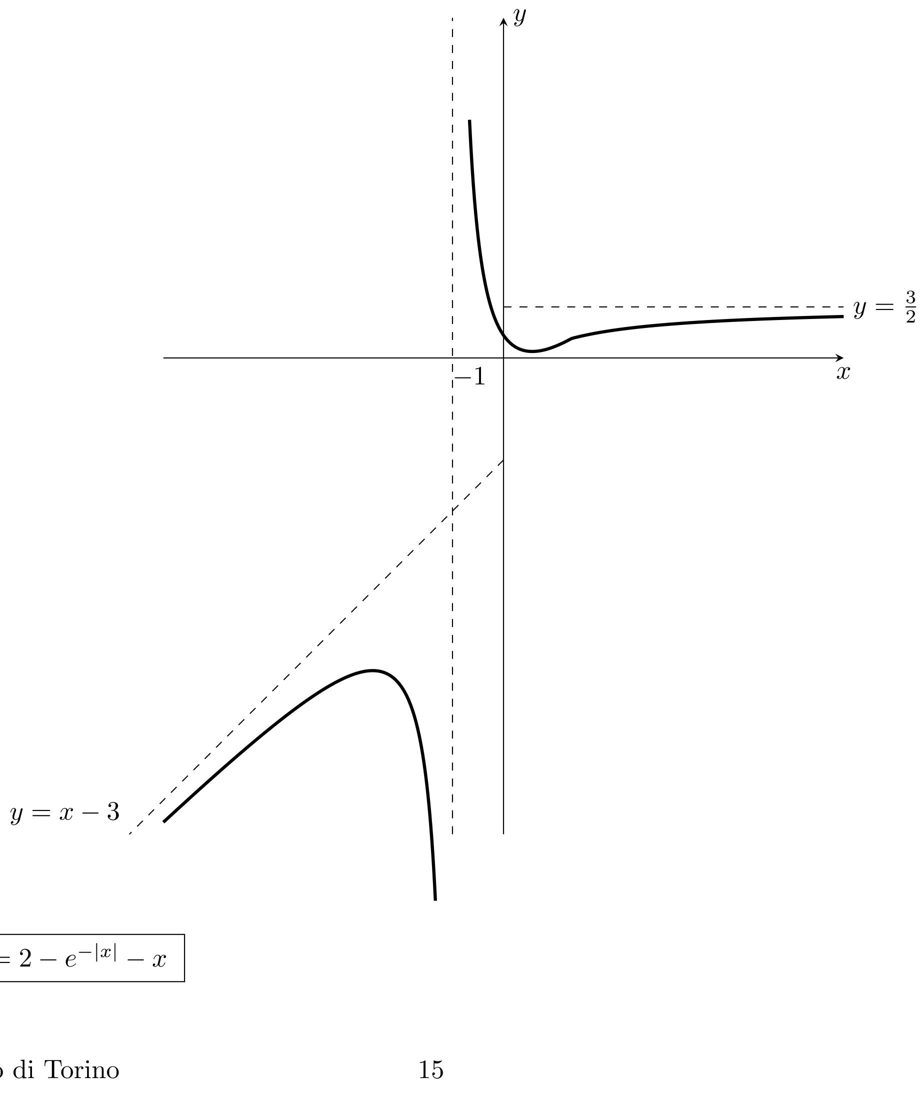
$$f_3(x) = \frac{x^2 + (x-1)(x-2)}{2x+3} = \frac{2x^2-3x+2}{2x+3} = x + \frac{-6x+2}{2x+3}$$

Thus

$m = \displaystyle\lim_{x\to-\infty} \frac{f_3(x)}{x} = \lim_{x\to-\infty} \left(\frac{x}{x} + \frac{-6x+2}{x(2x+3)}\right) = 1$

$q = \displaystyle\lim_{x\to-\infty} (f_3(x) - mx) = \lim_{x\to-\infty} \left(x + \frac{-6x+2}{2x+3} - x\right) = -3$

There is a left oblique asymptote $y = x - 3$.

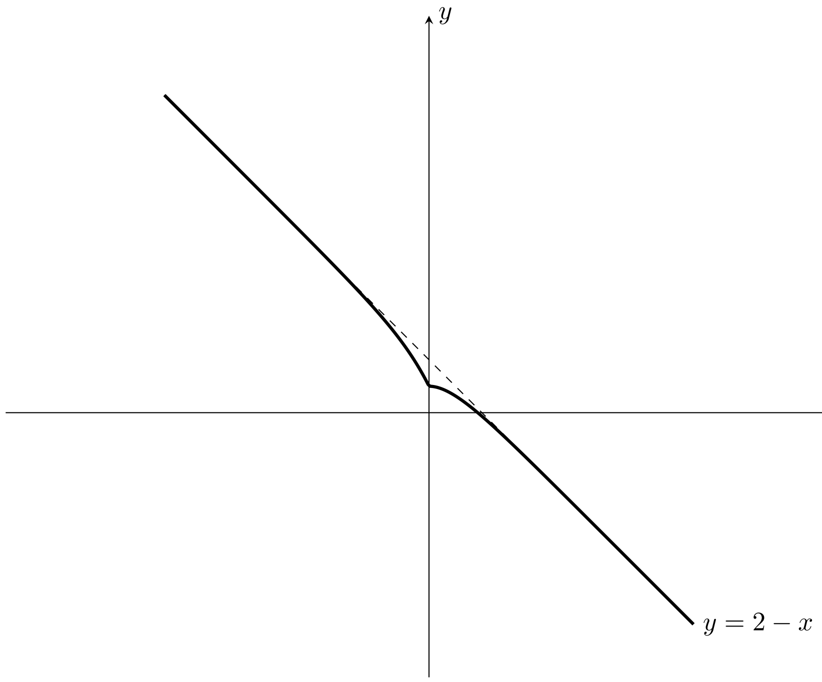


- $\boxed{f_4(x) = 2 - e^{-|x|} - x}$

Domain: $D = (-\infty, +\infty)$

Limits at boundary points of the domain:
$\lim_{x\to-\infty} f_4(x) = \lim_{x\to-\infty} 2 - e^{x} - x = +\infty$
$\lim_{x\to+\infty} f_4(x) = \lim_{x\to+\infty} 2 - e^{-x} - x = -\infty$

There are no horizontal asymptotes.
Oblique asymptotes: they may axist for $x \to \pm\infty$
$m = \lim_{x\to\pm\infty} \frac{f_4(x)}{x} = \lim_{x\to\pm\infty} \frac{2 - e^{-|x|} - x}{x} = -1$
$q = \lim_{x\to\pm\infty} (f_4(x) - mx) = \lim_{x\to\pm\infty} (2 - e^{-|x|} - x - (-1)x) = \lim_{x\to\pm\infty} (2 - e^{-|x|}) = 2$
The line $y = -x + 2$ is a right and left oblique asymptote.



- $\boxed{f_5(x) = xe^{\frac{1}{|x^2-1|}}}$

Domain: $D = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$

Symmetry: $f_5(-x) = -xe^{\frac{1}{|(-x)^2-1|}} = -xe^{\frac{1}{|x^2-1|}} = -f_5(x)$, the function is odd.

Limits at boundary points of the domain:

$\lim_{x\to-\infty} xe^{\frac{1}{|x^2-1|}} = -\infty$

$\lim_{x\to-1} xe^{\frac{1}{|x^2-1|}} = -\infty$

Given f_5 odd, it holds:

$\lim_{x\to1} xe^{\frac{1}{|x^2-1|}} = +\infty$

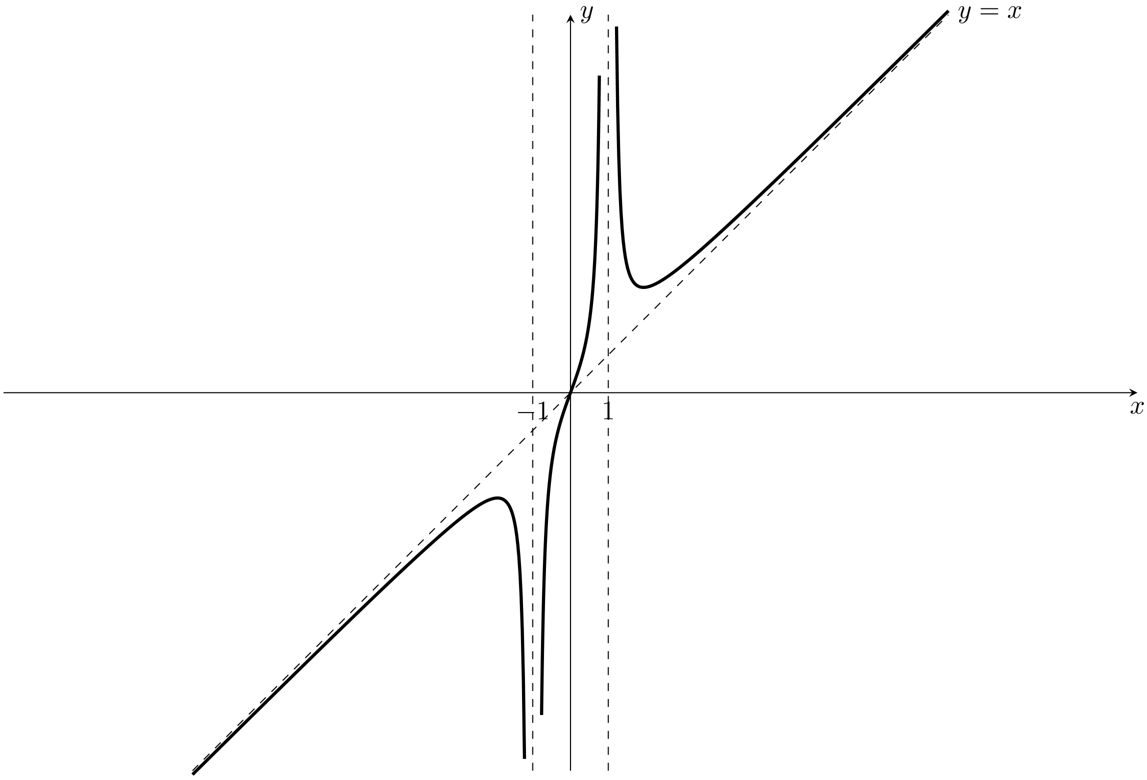
$\lim_{x\to+\infty} xe^{\frac{1}{|x^2-1|}} = +\infty$

Therefore f_5 has no horizontal asymptotes; the lines $x = -1$ and $x = 1$ are vertical asymptotes.
Oblique asymptotes: they may exist on the left and on the right.
$m = \lim_{x\to\pm\infty} \frac{f_5(x)}{x} = \lim_{x\to\pm\infty} e^{\frac{1}{|x^2-1|}} = e^0 = 1$

$$q = \lim_{x\to\pm\infty} \left(xe^{\frac{1}{|x^2-1|}} - x \right) = \lim_{x\to\pm\infty} x \left(e^{\frac{1}{|x^2-1|}} - 1 \right) = \lim_{x\to\pm\infty} \frac{x}{|x^2-1|} = 0$$

We have the e qui equivalence for $x \to \pm\infty$: $e^{\frac{1}{|x^2-1|}} - 1 \sim \dfrac{1}{|x^2-1|}$

Hence the line $y = x$ is an oblique asymptote.



- $\boxed{f_6(x) = \arctan(x^2 + 2x) + 3x}$

  Domain: $D = (-\infty, +\infty)$

  Limits at boundary points of the domain:

  $$\lim_{x\to-\infty} f_6(x) = \lim_{x\to-\infty} \arctan(x^2+2x) + 3x = \lim_{x\to-\infty} 3x \left(\frac{\arctan(x^2+2x)}{3x} + 1 \right) = -\infty$$

  $$\lim_{x\to+\infty} f_6(x) = \lim_{x\to+\infty} \arctan(x^2+2x) + 3x = \lim_{x\to-\infty} 3x \left(\frac{\arctan(x^2+2x)}{3x} + 1 \right) = +\infty$$

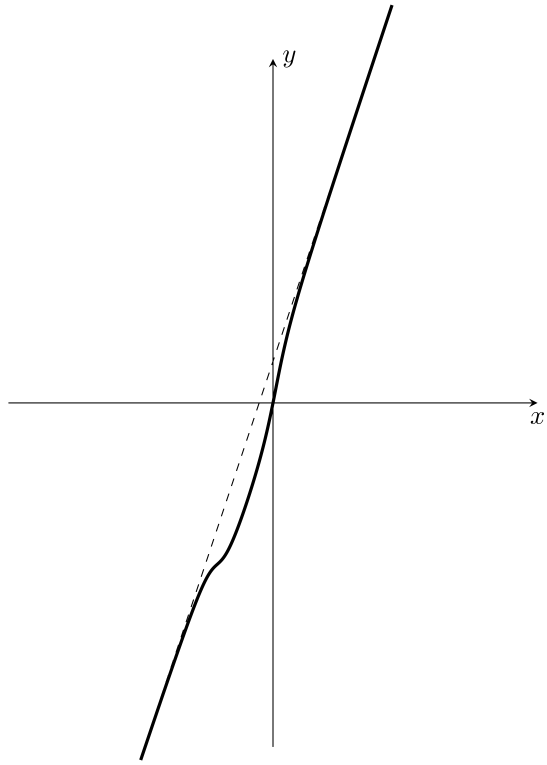
  There are no horizontal asymptotes.

  Oblique asymptotes: they may exist on the left and on the right.

  $$m = \lim_{x\to\pm\infty} \frac{f_6(x)}{x} = \lim_{x\to\pm\infty} \frac{\arctan(x^2+2x) + 3x}{x} = 3$$

  $$q = \lim_{x\to\pm\infty} \big(\arctan(x^2+2x) + 3x - 3x \big) = \frac{\pi}{2}$$

  Hence the line $y = \frac{\pi}{2} + 3x$ is an oblique asymptote.

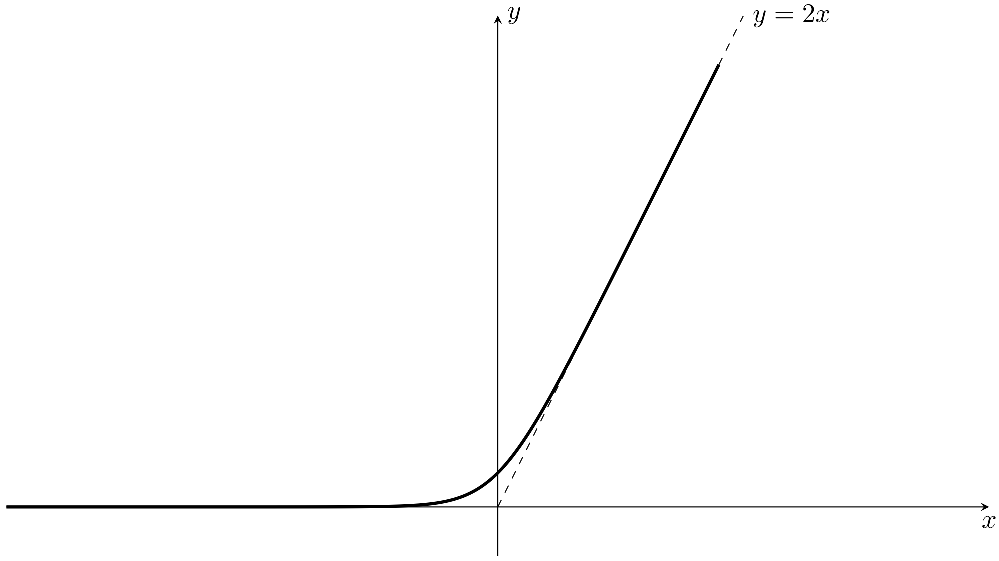


- $\boxed{f_7(x) = \log\left(1+e^{2x}\right)}$

  Domain: $D = (-\infty, +\infty)$

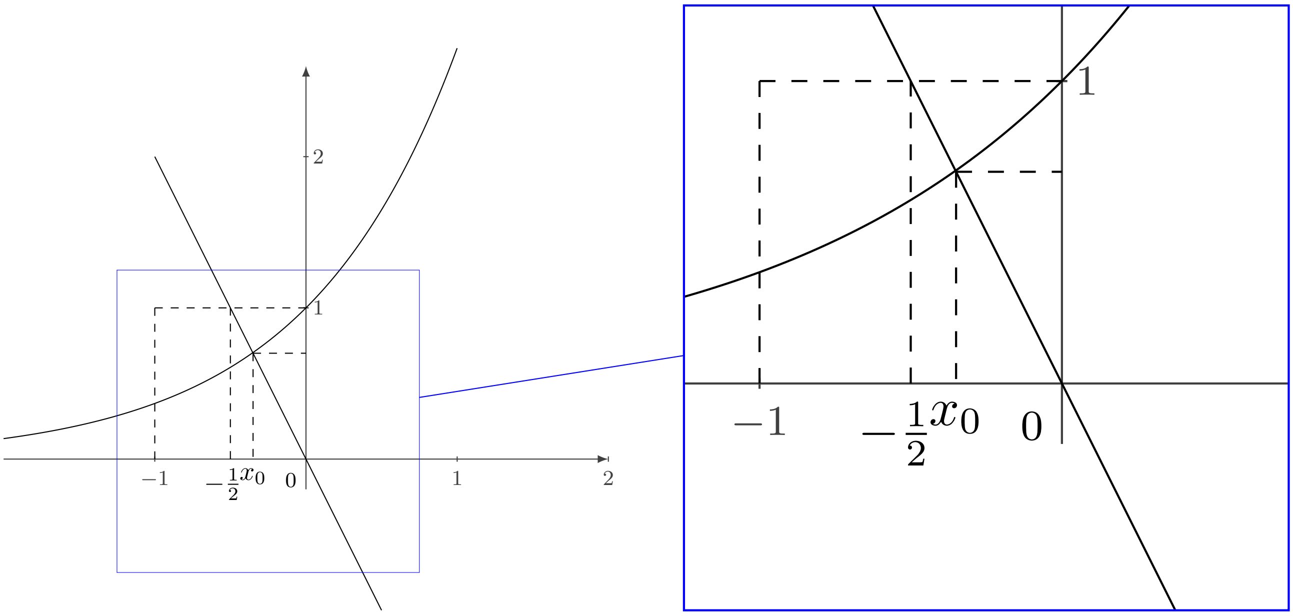
  Limits at boundary points of the domain:
  $\lim_{x\to-\infty} f_7(x) = \lim_{x\to-\infty} \log\left(1+e^{2x}\right) = 0 \implies$ the line $y=0$ is a left horizontal asymptote
  $\lim_{x\to+\infty} f_7(x) = \lim_{x\to+\infty} \log\left(1+e^{2x}\right) = +\infty \implies f_7$ has no right horizontal asymptote

  Oblique asymptotes: it may exist only on the right.
  $m = \lim_{x\to+\infty} \dfrac{f_7(x)}{x} = \lim_{x\to+\infty} \dfrac{\log\left(1+e^{2x}\right)}{x} = \lim_{x\to+\infty} \dfrac{\log\left(e^{2x}\right)}{x} = \lim_{x\to+\infty} \dfrac{2x}{x} = 2$
  $q = \lim_{x\to+\infty} \left(\log e^{2x}\left(1+e^{-2x}\right) - 2x\right) = \lim_{x\to+\infty} \left(2x + \log\left(1+e^{-2x}\right) - 2x\right) = \lim_{x\to+\infty} \log\left(1+e^{-2x}\right) = 0$
  Thus the line $y = 2x$ is a right oblique asymptote.



- $\boxed{f_8(x) = \log(e^x + 2x)}$

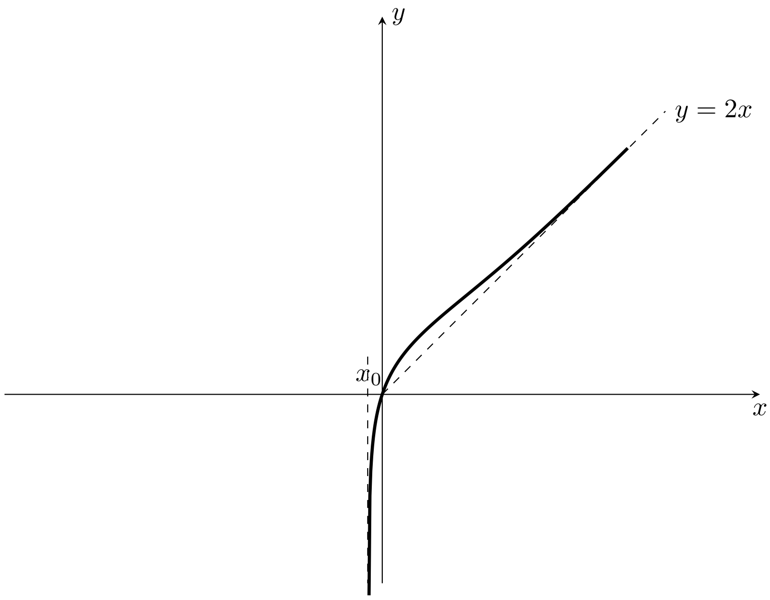
  Domain: ask for
  $$e^x + 2x > 0 \Leftrightarrow e^x > -2x$$
  The inequality $e^x > -2x$ can be graphically solved, finding the intersection between the 2 graphs $x_0 \in (-\frac{1}{2}, 0)$.



  Domain: $D = (x_0, +\infty)$

  Limits at boundary points of the domain:
  $\lim\limits_{x \to x_0^+} f_8(x) = \lim\limits_{x \to x_0^+} \log(e^x + 2x) = -\infty \implies$ the line $y = x_0$ is a right vertical asymptote
  $\lim\limits_{x \to +\infty} f_8(x) = \lim\limits_{x \to +\infty} \log(e^x + 2x) = +\infty \implies$ there are no horizontal asymptotes

  Oblique asymptotes: note that
  $$f_8(x) = \log\left(e^x + 2x\right) = \log\left(e^x\left(1 + \frac{2x}{e^x}\right)\right) = x + \log\left(1 + \frac{2x}{e^x}\right)$$
  Since $\lim\limits_{x \to +\infty} \log\left(1 + \frac{2x}{e^x}\right)$, we conclude that $y = x$ is a right oblique asymptote.



3. (31 January 2018 - Ist - A)

(a) *State one of the Comparison Theorems for functions with finite limit.*

See textbook.

(b) *Show that, if $f(x)$ is such that $\lim_{x\to+\infty} f(x) = 0$ and $M(x)$ is the mantissa function, then $\lim_{x\to+\infty} f(x)M(x) = 0$.*

The mantissa function is bounded (it holds that $\forall x \in \mathbb{R},\ 0 \le M(x) < 1$). We know that, $\forall x_0 \in \mathbb{R} \cup \{\pm\infty\}$, an infinitesimal function as $x \to x_0$ multiplied by a bounded functions is always infinitesimal, for $x \to x_0$.
Thus $\lim_{x\to+\infty} f(x) = 0 \ \Rightarrow \ \lim_{x\to+\infty} f(x)M(x) = 0$

(c) *Compute, if they exist, the following limits*

$$\lim_{x\to-\infty} \frac{4^{-x}+1}{M(4^{-x})+1}, \qquad \lim_{x\to+\infty} \frac{4^{-x}+1}{M(4^{-x})+1}$$

where $M(x)$ denotes the mantissa function.

The denominator is bounded; it holds that $\forall x \in \mathbb{R},\ 1 \le M(4^{-x}) + 1 < 2$. Thus
$\forall x \in \mathbb{R},\ \frac{1}{2} < \frac{1}{M(4^{-x})+1} \le 1$.
Regarding the first limit, since $\lim_{x\to-\infty} \big(4^{-x}+1\big) = +\infty$, by Comparison Theorems we have that $\lim_{x\to-\infty} \frac{4^{-x}+1}{M(4^{-x})+1} = +\infty$.
The second limit takes value 1; indeed, $\lim_{x\to+\infty} \big(4^{-x}+1\big) = 1$: therefore numerator tends to 1. Also the denominator tends to 1, because $\lim_{x\to+\infty} 4^{-x} = 0^+$ and $\lim_{t\to0^+} M(t) = 0$.

4. (13 February 2018 - Ist - A)

(a) *Write the definition of increasing function on a subset $A \subseteq \mathbb{R}$.*

See textbook.

(b) *Given f increasing on A, and g decreasing on $\mathbb{R}$, study monotonicity of the composite function $g \circ f$ on A.*

Since f is increasing on A, we have, $\forall x_1, x_2 \in A,\ \ x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$; being g decreasing on $\mathbb{R}$, $\forall t_1, t_2 \in \mathbb{R},\ \ t_1 \le t_2 \Rightarrow g(t_1) \ge g(t_2)$; therefore $g(f(x_1)) \ge g(f(x_2))$.
Hence $g \circ f$ is decreasing on A

(c) *Suppose $A = (1, 5)$ and f increasing on A. Show that $\lim_{x\to1^+} f(x)$ cannot be equal to $+\infty$.*

Given f increasing on $A = (3, 5)$, apply the Theorem on limit existence for monotone functions, then the following limit exists $L = \lim_{x\to3^+} f(x) = inf\{f(x)\}_{x\in A}$, where L may be finite or $L = +\infty$.
Consider any $a \in A :\ \ 3 < a < 5$; since $f(x)$ is defined at $x = a$, then $f(a) \in \mathbb{R}$; being $L = inf\{f(x)\}_{x\in A}$, by definition $L \le f(a)$, and therefore L cannot be equal to $+\infty$.