Graphs of Rational Functions

The graphs of rational functions very often have *vertical asymptotes*, which correspond to those points (if there are any) where the denominator becomes zero. If we want to sketch the graph of a rational function, the main things to do are

(i) to locate these vertical asymptotes by finding the values of x for which the denominator is zero;

(ii) to locate the points at which the curve crosses the x-axis, by finding the values of x for which the *numerator* is zero;

(iii) to find the point where the curve crosses the y-axis, by setting $x = 0$.

(iv) to consider the sign of the function on either side of its zeros and its asymptotes

(v) to consider the behaviour of the function as $x \to \pm\infty$, and hence characterise any other asymptotes, horizontal or oblique, that the curve may possess.

For example, consider the rational function

$$y = \frac{x-1}{(x-2)(x+3)}.$$

It has vertical asymptotes at $x = 2$ and $x = -3$. It crosses the x-axis at $x = 1$. It crosses the y-axis at

$$y = \frac{-1}{(-2) \times 3} = -\frac{1}{6}.$$

If $x < -3$, then $(x-2)$, $(x-1)$ and $(x+3)$ are all negative, meaning that y is negative.

If $-3 < x < 1$, then $(x+3)$ is positive, but the other two are negative, so y is positive.

If $1 < x < 2$, only $(x-2)$ is negative, so y is negative again.

For $x > 2$, everything is positive, so y is positive.

Finally, the function may be rewritten as

$$y = \frac{1/x - 1/x^2}{(1-2/x)(1+3/x)},$$

meaning that it tends to zero as $x \to \pm\infty$, and therefore has a horizontal asymptote at $y = 0$.

This is enough for us to sketch the graph of the function:

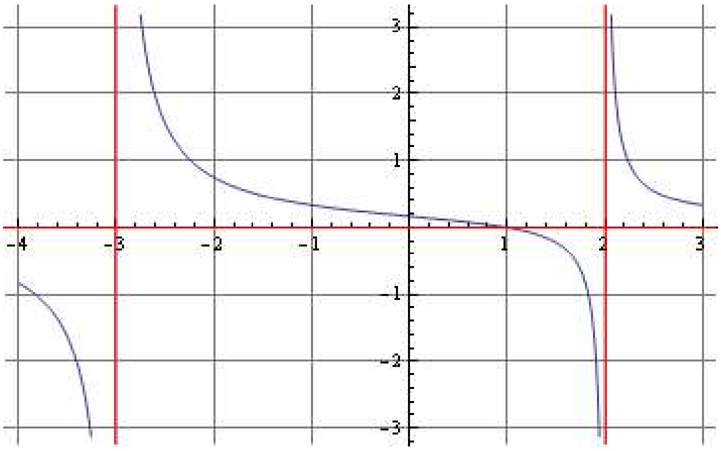

Figure 1: graph of the function $y = (x-1)/[(x-2)(x+3)]$

Repeated factors

A squared factor in the numerator means that the curve just touches the x-axis. For example, consider

$$y = \frac{(x+2)^2}{(x+3)(x-1)}.$$

There are vertical asymptotes at $-3$and 1. The x-axis is tangent to the curve at $x = -2$. At $x = 0$,

$$y = \frac{2^2}{3 \times (-1)} = -\frac{4}{3}.$$

The function's values are positive for $x < -3$ and $x > 1$, and negative for $-3 < x < 1$.

Finally, the function may be rewritten as

$$y = \frac{(1+2/x)^2}{(1+3/x)(1-1/x)},$$

meaning that as $x \longrightarrow \pm\infty$, y tends to 1.

This is enough for us to sketch the graph of the function:

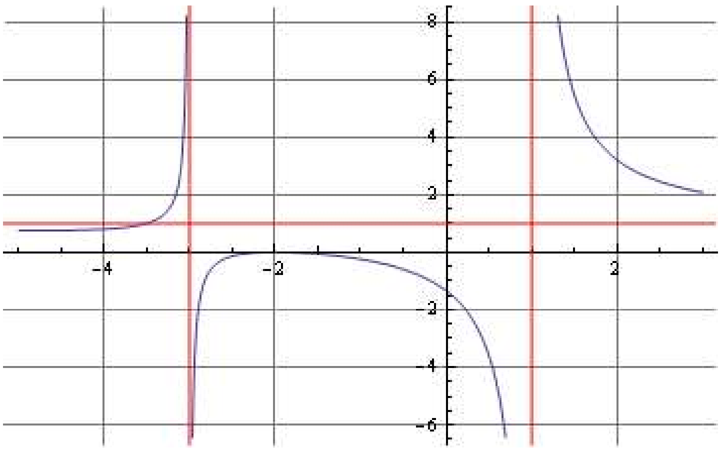

Figure 2: graph of the function $y = (x+2)^2/[(x+3)(x-1)]$

Notice that the graph has a *non-zero* horizontal asymptote. That's what happens when the degree of the numerator and that of the denominator are the same; when the degree of the numerator is less than that of the denominator, as in the previous example, the horizontal asymptote is always zero.

Oblique asymptotes

Consider the function

$$y = \frac{(x+1)(x-3)}{x-4}.$$

This is of the form "quadratic over linear". Rational functions of this type have *oblique asymptotes*. We can tell that by expressing them in terms of partial fractions: in the case of our example, we get

$$y = x + 2 + \frac{5}{x-4}.$$

From this we can see that as $x \to \pm\infty$, y approaches $(x+2)$. The graph looks like this:

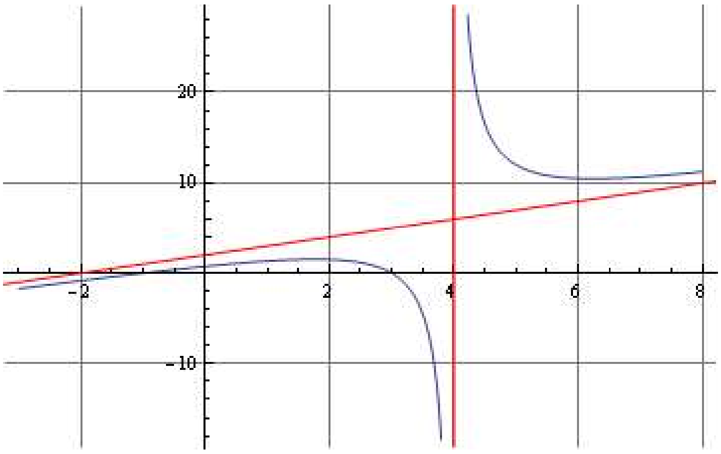

Figure 3: graph of the function $y = [(x+1)(x-3)]/(x-4)$