Primitives - Integration Techniques

Integration rules

LINEARITY property

1) $$\int \Big(\alpha f(x) + \beta g(x)\Big)\,\mathrm{d}x = \alpha \int f(x)\,\mathrm{d}x + \beta \int g(x)\,\mathrm{d}x$$

BY PARTS

2) $$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\,\mathrm{d}x$$

$$\int \frac{\varphi'(x)}{\varphi(x)}\,\mathrm{d}x = \log|\varphi(x)| + c$$

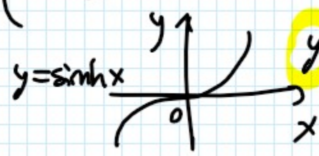
SUBSTITUTION formula

3) $$\int f(\varphi(x))\varphi'(x)\,\mathrm{d}x = \int f(y)\,\mathrm{d}y \quad \text{where} \quad y = \varphi(x)$$

$f(x)$	$\int f(x)\,\mathrm{d}x$
x^α	$\frac{x^{\alpha+1}}{\alpha+1} + c, \quad \alpha \neq -1$
$\frac{1}{x}$	$\log\lvert x\rvert + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$e^x + c$
$\sinh x$	$\cosh x + c$
$\cosh x$	$\sinh x + c$
$\frac{1}{1+x^2}$	$\arctan x + c$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\frac{1}{\sqrt{1+x^2}}$	$\log(x + \sqrt{x^2+1}) + c = \text{sett}\sinh x + c$
$\frac{1}{\sqrt{x^2-1}}$	$\log(x + \sqrt{x^2-1}) + c = \text{sett}\cosh x + c$

(def) F is a primitive for f on I (interval) if F is a differentiable function such that $F'(x) = f(x), \forall x \in I$

($\text{sett}\sinh x = \sinh^{-1}(x)$



$y = \sinh x \overset{\text{def}}{=} \frac{e^x - e^{-x}}{2}$)

FUNDAMENTAL theorem of Integral CALCULUS

Theorem 9.37 *Let f be defined and continuous over a real interval I. Given $x_0 \in I$, let*

$$F(x) = \int_{x_0}^{x} f(s)\,\mathrm{d}s$$

denote an integral function of f on I. Then F is differentiable everywhere over I and

$$F'(x) = f(x), \qquad \forall x \in I.$$

Corollary 9.39 *Let f be continuous on $[a,b]$ and G any primitive of f on that interval. Then*

$$\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a). \tag{9.24}$$

$$F(x) \xrightarrow{\text{derive}} F'(x) = f(x)$$
$$F(x) \xleftarrow{\text{integrate}} f(x)$$

(def) Indefinite integral $\int \boxed{f(x)}\,dx = \{F(x) + c, c \in \mathbb{R}\}$
$= \boxed{F(x) + c}$

$\boxed{\text{EX 2B}}$

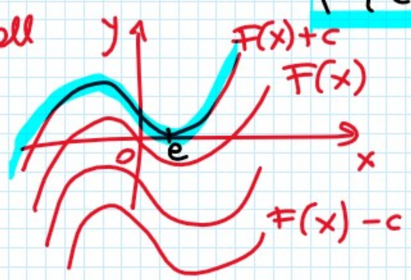
$f(x) = \frac{4}{x}\log^3 x \qquad \text{Dom} f = (0, +\infty)$

$x_0 = e \quad y_0 = 0 \qquad (x_0, y_0) \in \text{graph of } F$

$\Updownarrow$

$\boxed{F(e) = 0}$

Recall



$F'(x) = f(x)$

Note that
f diff. $\Rightarrow$ f continuous
$\Downarrow$
f integrable

$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + c, \quad \boxed{\alpha \neq -1}$

$\int \frac{4}{x}\log^3 x \, dx = \int 4t^3\,dt = t^4 + c = \boxed{(\log x)^4 + c = F(x)}$

Substitution $\log x = t \longrightarrow \frac{1}{x}dx = dt$

$\log x = \log_e x = \ln x$

$x_0 = e \quad y_0 = 0 \qquad (\log e)^4 + c = 0$

$x_0 = e \qquad y_0 = 0 \qquad (\log e)^4 + c = 0$

$1 + c = 0 \qquad c = -1$

$\log x = \log_e x = \ln x$

$F(x) = (\log x)^4 - 1$

Recall $\int \sin^2 x \, dx = ?$

↓ BISECTION FORMULAE:

$$\boxed{\begin{aligned} \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}}$$

$\sin 0 = 0$ $\qquad$ $\cos 0 = 1$

$\sin \frac{\alpha}{2} = \pm \sqrt{\frac{1-\cos\alpha}{2}}$

"

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx =$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\cdot\frac{2}{2}\int \cos 2x \, dx = \frac{1}{2}x - \frac{1}{4}\int \cos t \, dt =$$

$t = 2x \qquad dt = 2\,dx \qquad dx = \frac{dt}{2}$

$D(\sin t) = \cos t$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + c$$

Integrals of some trascendent functions

1) $\int R(e^x)\,\mathrm{d}x$, with R rational functions
It is useful the following substitution
$$e^x = t \Rightarrow x = \log t$$
The integral becomes the integral of a rational function.

2) $\int \frac{1}{x} R(\log x)\,\mathrm{d}x$, with R rational function
It is useful the following substitution
$$\log x = t \Rightarrow x = e^t$$
The integral becomes the integral of a rational function.

3) $\int R(\cos x, \sin x)\,\mathrm{d}x$, with R rational function
It is useful the following substitution
$$t = \tan\frac{x}{2} \Rightarrow x = 2\arctan t$$
with the parametric formulas:
$$\sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$
The integral becomes the integral of a rational function.

4) $\int R(\sin x)\cos x\,\mathrm{d}x$ or $\int R(\cos x)\sin x\,\mathrm{d}x$, with R rational function
It is useful the following substitution
$$\sin x = t \quad \text{or, in the second case,} \quad \cos x = t$$

$\sin^2 x, \cos^2 x, \cos x \sin x$ (*)

5) $\int R(\tan x)\,\mathrm{d}x$, with R rational function
It is useful the following substitution
$$t = \tan x \Rightarrow x = \arctan x$$

Otherwise USE parametric formula (*)

$f(\sin^2 x, \cos^2 x, \tan x, \sin x \cos x)$ $\qquad$ $\tan x = t$

$x = \arctan t$

$\sin^2 x = \frac{t^2}{1+t^2} \qquad \cos^2 x = \frac{1}{1+t^2} \qquad dx = \frac{dt}{1+t^2}$

TRY TO COMPUTE:

$\int \sqrt{1-x^2}\, dx = ?$ $\qquad$ $x = \sin t$ $\quad (1 - \sin^2 t = \cos^2 t)$

$\int \sqrt{1+x^2}\, dx = ?$ $\qquad$ $x = \sinh t$ $\quad (1 + \sinh^2 t = \cosh^2 t)$

Integrals of some irrational functions

$\int R(x^{r_1/s_1}, \ldots, x^{r_k/s_k})\,\mathrm{d}x$, with R rational function
It is useful the following substitution
$$x = t^n$$
with n least common multiple of $s_1, \ldots, s_k$.

EX5K

$$\int \frac{dx}{\tan^4 x \cdot \cos^2 x} = \int \frac{dt}{t^4} = \int t^{-4}\,dt = \frac{t^{-3}}{-3} + c =$$

$t = \tan x \qquad dt = \frac{1}{\cos^2 x}\,dx$

$$= -\frac{1}{3(\tan x)^3} + c$$

$D(\tan x) = \frac{1}{\cos^2 x} = \tan^2 x + 1$

EX4 $\quad f(x) = \sqrt{4 - x^2}$

Prove that $F(x) = \frac{x}{2}\sqrt{4-x^2} + 2\arcsin\frac{x}{2}$ is a primitive for f on $(-2, 2)$

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1st method Integrate

$$\int \sqrt{4-x^2}\,dx = \int \sqrt{4\left(1-\left(\frac{x}{2}\right)^2\right)}\,dx$$

$$= 2\int \sqrt{1-\left(\frac{x}{2}\right)^2}\,dx \qquad \frac{x}{2} = \sin t$$

$(\ldots)$

2nd method Derive $F'(x) \overset{?}{=} f(x)$

$\mathrm{Dom} f = [-2,2]$

$$F'(x) = \frac{1}{2}\sqrt{4-x^2} + \frac{x}{2}\frac{1}{\cancel{2}\sqrt{4-x^2}}\cdot(-\cancel{2}x) + \cancel{2}\cdot\frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}}\cdot\frac{1}{\cancel{2}} =$$

$$= \frac{1}{2}\sqrt{4-x^2} - \frac{x^2}{2\sqrt{4-x^2}} + \frac{2}{\sqrt{4-x^2}} =$$

$$\boxed{D(\arcsin x) = \frac{1}{\sqrt{1-x^2}}}$$

$$= \frac{1}{2}\sqrt{4-x^2} + \frac{4-x^2}{2\sqrt{4-x^2}} = \qquad 4-x^2 = \left(\sqrt{4-x^2}\right)^2$$

$$= \frac{1}{2}\sqrt{4-x^2} + \frac{1}{2}\sqrt{4-x^2} = \sqrt{4-x^2} = f(x) \qquad \square$$

Rational functions $\longrightarrow$ PARTIAL FRACTIONS expansion

$\frac{N(x)}{D(x)}$ → polynomials

1 • check degree N < degree D (IF NOT: DIVISION)

2 • Partial fractions: factorize DENOMINATOR in IRREDUCIBLE FACTORS
- 1st degree
- 2nd degree $\Delta < 0$

$y = ax^2+bx+c$

$\Delta = b^2-4ac$

ex $$\int \frac{dx}{(x+1)^2(x^2+1)(x-3)} = ?$$

degree $N = 0$
degree $D = 5$ ok

$$\frac{1}{(x+1)^2(x^2+1)(x-3)} = \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} + \frac{E}{x-3} \quad (\ldots)$$

ex $$\int \frac{5x^4+4}{x^5+4x}\,dx \overset{=dt}{\underset{t = x^5+4x}{=}} \int \frac{dt}{t} = \log|x^5+4x| + c \qquad \square$$

$dt = (5x^4+4)\,dx$

$$\int \frac{D'(x)}{D(x)}\,dx = \log|D(x)| + c$$

ex $\int \frac{dx}{\ldots} = ? \qquad \frac{1}{\ldots} = \frac{A}{(x+5)^1} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3} +$

$\int \frac{\dots}{D(x)}$

ex $\int \frac{dx}{(x+5)^3(x-1)} = ?$

$$\frac{1}{(x+5)^3(x-1)} = \frac{A}{(x+5)^1} + \frac{B}{(x+5)^2} + \frac{C}{(x+5)^3} + \frac{D}{x-1} \quad (\dots)$$

$\boxed{EX\,7A}$ $\int \frac{x}{x^2+4x+3}\,dx$ (x → degree 1; x^2+4x+3 → degree 2)

$x^2+4x+3 = (x+3)(x+1)$ (4: sum, 3: product)

$$\frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

Roots $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$(x-x_1)(x-x_2)$

Compute A and B !!

1st Method

$$\frac{x+0}{\cancel{(x+3)(x+1)}} = \frac{A(x+1)+B(x+3)}{\cancel{(x+3)(x+1)}} = (A+B)x+(A+3B)$$

coeff. of x, constants

$$\begin{cases} 1 = A+B \\ 0 = A+3B \end{cases}$$ linear system

$$\begin{cases} B = 1-A \\ 0 = A+3(1-A) \end{cases} \qquad \begin{cases} B = -1/2 \\ 2A = 3 \to A = 3/2 \end{cases}$$

2nd method

To find A: Multiply by $(x+3)$ both sides and then $x \to -3$

$$\frac{x}{x+1} = A + (x+3)\frac{B}{x+1}$$

Now $x \to -3$ $\quad \frac{-3}{-3+1} = A + 0 \qquad A = \frac{3}{2}$

To find B: Multiply by $(x+1)$ and then $x \to -1$

$$\frac{x}{x+3} = \cancel{(x+1)\frac{A}{x+3}} + B$$

$$B = \frac{-1}{-1+3} = -\frac{1}{2}$$

$$\int \frac{x}{x^2+3x+4}\,dx = \int\left(\frac{3/2}{x+3} + \frac{-1/2}{x+1}\right)dx =$$

$$= \frac{3}{2}\int\frac{dx}{x+3} - \frac{1}{2}\int\frac{dx}{x+1} = \frac{3}{2}\log|x+3| - \frac{1}{2}\log|x+1| + c$$

ex $\int \frac{x^3-x}{x^2-4}\,dx =$ (degree 3 / degree 2)

Division

$$\begin{array}{rr|l} x^3 & -x & x^2-4 \\ \hline x^3 & -4x & x \end{array}$$

x → quotient

degree 3 degree 2

Division

$$\begin{array}{lll} x^3 & -x & \mid x^2-4 \\ x^3 & -4x & \mid x \text{ quotient} \\ \hline // & 3x & \text{Remainder} \end{array}$$

$$= \int \left(x + \frac{3x}{x^2-4} \right) dx =$$

quotient, Remainder

$$= \frac{x^2}{2} + \int \frac{3x}{x^2-4} dx =$$

partial fractions

$$\boxed{\frac{3x}{(x+2)(x-2)} = \frac{A}{x+2} + \frac{B}{x-2}}$$

2nd method $\Rightarrow$ $\boxed{\bullet\ (x+2)}$

$$\frac{3x}{x-2} = A + (x+2)\frac{B}{x-2}$$

$x \to -2$

$$-\frac{6}{-4} = A \qquad A = \frac{3}{2}$$

$\bullet (x-2)$ and $x \to 2$ $\qquad B = \frac{6}{4} = \frac{3}{2}$

OR in this case: $t = x^2-4 \qquad dt = 2x\,dx$

$$= \frac{x^2}{2} + \frac{3}{2} \int \frac{2x\,dx}{x^2-4} = \frac{x^2}{2} + \frac{3}{2} \log|x^2-4| + c \qquad \square$$

$$\int \frac{dt}{t}$$

EX7G

$$\int \frac{x^4 - 3x^2 - 1}{x^3-1} dx = \int \left(x + \boxed{\frac{-3x^2+x-1}{x^3-1}} \right) dx = (*)$$

$$\begin{array}{llll} x^4 & -3x^2 & & -1 \quad \mid x^3-1 \\ x^4 & & -x & \quad\ \mid x \text{ quotient} \\ \hline // & -3x^2 & +x & -1 \end{array}$$

$$\frac{-3x^2+x-1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \qquad A?B?C?$$

$\Delta < 0$

1ST method $\Rightarrow$ $-3x^2+x-1 = A(x^2+x+1) + (Bx+c)(x-1)$

coeff. of x^2 $\to$; coeff. of x $\to$; constants $\to$

$$\begin{cases} -3 = A + B \\ 1 = A - B + C \\ -1 = A - C \end{cases} \qquad \begin{cases} B = -3-A \\ 1 = A + 3+A+A+1 \\ C = A+1 \end{cases}$$

$$\begin{cases} B = -2 \\ A = -1 \end{cases}$$

$$(*) = \frac{x^2}{2} + \int \left(-\frac{1}{x-1} - \frac{2x}{x^2+x+1} \right) dx = \boxed{\begin{array}{l} t = x^2+x+1 \\ dt = (2x+1)dx \end{array}}$$

$$\begin{cases} B = -2 \\ A = -1 \\ C = 0 \end{cases}$$

$$(*) = \frac{x^2}{2} + \int \left(-\frac{1}{x-1} - \frac{2x}{x^2+x+1} \right) dx = \quad \boxed{\begin{aligned} t &= x^2+x+1 \\ dt &= (2x+1)\,dx \end{aligned}}$$

$$= \frac{x^2}{2} - \log|x-1| - \int \frac{2x+1-1}{x^2+x+1} dx =$$

$$= \frac{x^2}{2} - \log|x-1| - \underbrace{\int \frac{2x+1}{x^2+x+1} dx}_{-\log(x^2+x+1)} + \int \frac{dx}{x^2+x+1} = (\dots)$$

Recall $\int \frac{dx}{x^2+1} = \arctan x + c$

SQUARE COMPLETION: $(a+b)^2 = a^2 + 2ab + b^2$

$$x^2+x+1 = \left(x^2 + x + \frac{1}{4}\right) - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4} =$$

$$= \frac{3}{4}\left[\frac{4}{3}\left(x+\frac{1}{2}\right)^2 + 1 \right] = \frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1 \right]$$

$$\int \frac{dx}{x^2+x+1} = \frac{4}{3}\int \frac{dx}{\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right)^2 + 1} =$$

$$t = \frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}$$

$$dt = \frac{2}{\sqrt{3}}dx \qquad dx = \frac{\sqrt{3}}{2}dt$$

$$= \frac{4}{3}\frac{\sqrt{3}}{2}\int \frac{dt}{t^2+1} = \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}}x + \frac{1}{\sqrt{3}}\right) + c \qquad \square$$

EX 8B

$$\int \frac{1}{\sqrt{x} + \sqrt[3]{x}} dx =$$

least common multiple of indices: $2, 3 \longrightarrow 6$

$$t = \sqrt[6]{x} \qquad x = t^6 \qquad dx = 6t^5\,dt$$

$$\sqrt{x} = t^3$$

$$\sqrt[3]{x} = t^2$$

$$= \int \frac{6t^5\,dt}{t^3+t^2} \quad (\dots)$$

- Division
- $t^3 + t^2 = t^2(t+1)$

$$\frac{\text{Remainder}}{t^2(t+1)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+1} \qquad (\dots)$$

$$\frac{6t^{\cancel{5}\,3}}{\cancel{t^2}(t+1)} = \frac{6t^3}{t+1}$$

Division $(\dots)$

$$\frac{e^x - e^{-x}}{2} + 1 = \frac{e^x + e^{-x} + 2}{2}$$

$$\cancel{t}(t+1)$$

$$\frac{e^x-e^{-x}}{2}+1=\frac{e^x+\ldots}{2}$$

EX 8H

$$\int \frac{\sinh x+1}{\cosh x-1}\,dx=$$

Note that
$t=\cosh x-1$
$dt=\sinh x\,dx$

$$\sinh x \overset{\text{def}}{=} \frac{e^x-e^{-x}}{2} \qquad \cosh x \overset{\text{def}}{=} \frac{e^x+e^{-x}}{2}$$

$$=\int \frac{e^x-e^{-x}+2}{e^x+e^{-x}-2}\,dx \underset{\downarrow}{=} \int \frac{e^{2x}-1+2e^x}{e^{2x}+1-2e^x}\,dx=$$

$$e^{-x}=\frac{1}{e^x}$$

$t=e^x$

$x=\log t$
$dx=\frac{1}{t}\,dt$

degree 2

$$=\int \frac{t^2+2t-1}{t^2-2t+1}\cdot\frac{dt}{t}=$$

degree 3

$$=\int \frac{t^2+2t-1}{t(t-1)^2}\,dt$$

Partial FRACTIONS: $\dfrac{t^2+2t-1}{t(t-1)^2}=\dfrac{A}{t}+\dfrac{B}{t-1}+\dfrac{C}{(t-1)^2}$

2^{nd} Method

- t and $t\to 0$ (to get A)

$$\frac{t^2+2t-1}{(t-1)^2}=A+\cancel{t\frac{B}{t-1}}+\cancel{t\frac{C}{(t-1)^2}}$$

$$\boxed{A=-1}$$

- $(t-1)^2$ and $t\to 1$ (to get C)

$$\frac{t^2+2t-1}{t}=\cancel{(t-1)^2\frac{A}{t}}+\cancel{(t-1)B}+C$$

$$\boxed{C=2}$$

- $(t-1)$ and $t\to\infty$ (to get B)

$$\underbrace{\frac{t^2+2t-1}{t(t-1)}}_{1}=\underbrace{(t-1)\frac{A}{t}}_{A}+B+\underbrace{\frac{C}{t-1}}_{0}$$

$$\begin{cases} A+B=1 \longrightarrow B=1-A=2\\ A=-1\\ C=2\end{cases}$$

$$\int\left(\frac{-1}{t}+\frac{2}{t-1}+\frac{2}{(t-1)^2}\right)dt=-\log|t|+2\log|t-1|+2\int(t-1)^{-2}\,dt$$

$$\underset{\boxed{t=e^x}}{=}-x+2\log|e^x-1|+\underbrace{2\frac{(e^x-1)^{-1}}{-1}}_{=\,\frac{-2}{e^x-1}}+c$$

EX3

duplication formula $\cos 2x=2\cos^2 x-1$

3. Prove that the functions $F(x)=\sin^2 x+7$ and $G(x)=-\frac{1}{2}\cos(2x)-11$ are two primitives of the same function $f(x)$ on $\mathbb{R}$; find $f(x)$ and say which is the constant $F(x)-G(x)$.

duplication formula $\cos 2x = 2\cos^2 x - 1$

3. Prove that the functions $F(x) = \sin^2 x + 7$ and $G(x) = -\frac{1}{2}\cos(2x) - 11$ are two primitives of the same function $f(x)$ on $\mathbb{R}$; find $f(x)$ and say which is the constant $F(x) - G(x)$.

$F'(x) = G'(x) = f(x)$ check

T3

3. Consider f defined on an interval I. $F(x)$ is a primitive of f on I if:

(a) F is differentiable and there exists $x_0 \in I$ such that $F'(x_0) = f(x_0)$

(b) F is differentiable on I and $F'(x) = f(x), \forall x \in I$

(c) $F(x) = \int f(x)\,dx, \forall x \in I$ set of primitives

(d) F is twice differentiable on I and $F''(x) = f'(x)$

(e) f is differentiable in I and $F(x) = f'(x), \forall x \in I$

$F'(x) = f(x) \quad \forall x \in I$

$\int f(x)\,dx = \{F(x) + c, c \in \mathbb{R}\} = F(x) + c$

$F'(x) = f(x)$

T17

17. Given the function $f(x) = \frac{3x^2 + x - 4}{x^3 + 5x^2 + 9x + 5}$, which of the following statements is NOT true?

degree 2

degree 3

(a) $\int f(x)\,dx = -\int \frac{1}{x+1}\,dx + \int \frac{4x+1}{x^2+4x+5}\,dx$ TRUE

(b) $\int f(x)\,dx = -\ln|x+1| + 2\ln(x^2+4x+5) - 7\arctan(x+2) + c$ TRUE

(c) $\int f(x)\,dx = -\ln|x+1| - 7\arctan(x+2) + c$ False

(d) $\int f(x)\,dx = -\int \frac{1}{x+1}\,dx + 2\int \frac{2x+4}{x^2+4x+5}\,dx - 7\int \frac{1}{1+(x+2)^2}\,dx$ TRUE

(e) $f(x) = \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}$ TRUE

$D(x) = x^3 + 5x^2 + 9x + 5$

$D(-1) = -1 + 5 - 9 + 5 = 0$

is a ROOT

$(x+1)$ is a factor of $D(x)$

$$\begin{array}{l|l} x^3 + 5x^2 + 9x + 5 & x+1 \\ x^3 + x^2 & x^2 + 4x + 5 \\ \hline // \quad 4x^2 + 9x + 5 & \\ \quad 4x^2 + 4x & \\ \hline \quad // \quad 5x + 5 & \\ \quad 5x + 5 & \\ \hline \quad // \quad // & \end{array}$$

$\Delta < 0$

$D(x) = (x+1)(x^2+4x+5)$

$f(x) = \frac{A}{x+1} + \frac{Bx+C}{x^2+4x+5}$

Derivative for (c)

$$-\frac{1}{x+1} - 7\frac{1}{1+(x+2)^2} = -\frac{1}{x+1} - \frac{7}{x^2+4x+5} =$$

$$= \frac{-x^2-4x-5-7x-7}{(x+1)(x^2+4x+5)} = \frac{-x^2-11x-12}{//} \neq f(x)$$

(c) FALSE

T18

18. Which of the following statements is NOT true for the function $f(x) = \frac{x^2-3x+3}{x^3-2x^2+x}$?

(a) $f(x) = \frac{A}{x} + \frac{B}{x-1} + \boxed{\frac{Cx+D}{(x-1)^2}}$ FALSE

(b) $\int f(x)\,dx = 3\ln|x| - 2\ln|x-1| - \frac{1}{x-1} + c$

(c) $\int f(x)\,dx = \int \left(\frac{3}{x} - \frac{2}{x-1} - \frac{d}{dx}\left(\frac{1}{x-1}\right)\right)\,dx$

(d) $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{d}{dx}\left(\frac{C}{x-1}\right)$

(e) $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$ TRUE

$x(x^2-2x+1) = x(x-1)^2$

$f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

T9

9. A primitive of $\frac{\sqrt{x^2-1}}{x}$ is $= f(x)$

(a) $\sqrt{x^2-1} - x$ $= F(x)$

(b) $\sqrt{x^2-1} - \arctan x$

(c) $\sqrt{x^2-1} - \arctan\sqrt{x^2-1} - 2$ TRUE

(d) does not exist: the function is not integrable

(e) $\sqrt{x^2-1} - \arctan\sqrt{x^3-1}$

check : False (A) $F'(x) \stackrel{?}{=} f(x)$

$y = ax^2 + bx + c$

$\Delta = b^2 - 4ac < 0$

~~(d)~~ does not exist: the function is not integrable

~~(e)~~ $\sqrt{x^2-1}-\arctan\sqrt{x^3-1}$ check: False

$$\int \frac{\sqrt{x^2-1}}{x}\,dx = ?!?$$

(A) $F'(x) \overset{?}{=} f(x)$

$$F'(x) = \frac{1}{\cancel{2}\sqrt{x^2-1}}\cdot\cancel{2}x - 1 \qquad \text{NO}$$

(B) $F'(x) = \dfrac{x}{\sqrt{x^2-1}} - \dfrac{1}{1+x^2}$ NO

(C) $$F'(x) = \frac{x}{\sqrt{x^2-1}} - \frac{1}{\cancel{1+}(x^2\cancel{-1})}\cdot\frac{x}{\sqrt{x^2-1}} = \frac{x}{\sqrt{x^2-1}} - \frac{1}{x\sqrt{x^2-1}} =$$

$$= \frac{x^2-1}{x\sqrt{x^2-1}} = \frac{\sqrt{x^2-1}}{x} = f(x)$$

$x^2-1 = \left(\sqrt{x^2-1}\right)^2$

T10

10. Given the function $f(x) = \frac{1}{x} - \frac{1}{x-2}$ which of the following is FALSE?

(a) $\int f(x)\mathrm{d}x = \log|x| - \log|x-2| + c$

(b) The primitive of $f(x)$ that is zero in $x=1$ is $F(x) = \log\frac{x}{2-x}$

(c) The primitive of $f(x)$ that is zero in $x=1$ is $F(x) = \log\frac{x}{x-2}$

(d) The primitive of $f(x)$ that is zero in $x=1$ is $F(x) = -\log\frac{2-x}{x}$

(e) $F(x) = \log\frac{x}{2-x}$ and $G(x) = \log\frac{2x}{2-x}$ are two primitives of $f(x)$ defined on I$=(0,2)$

$1 \notin \operatorname{Dom}F$

$F(1) = \log\frac{1}{-1} = \cancel{\log(-1)}$

$F(1) = 0$

$1 \in \operatorname{Dom}F$

(C) $\operatorname{Dom}F = ?$ $\quad \dfrac{x}{x-2} > 0$

(N) $x > 0$

(D) $x > 2$

$x < 0 \lor x > 2$

0 2

$+ \ - \ +$

$\operatorname{Dom}F = (-\infty, 0) \cup (2, +\infty)$

(D) $\dfrac{2-x}{x} > 0$

(N) $2-x > 0$ $\quad x < 2$

(D) $x > 0$

0 2

$- \ + \ -$

$\operatorname{Dom}F = (0, 2)$

EX3

3. Prove that the functions $F(x) = \sin^2 x + 7$ and $G(x) = -\frac{1}{2}\cos(2x) - 11$ are two primitives of the same function $f(x)$ on $\mathbb{R}$; find $f(x)$ and say which is the constant $F(x) - G(x)$.

$$F'(x) = 2\sin x \cdot \cos x = \sin 2x$$

$$G'(x) = -\frac{1}{\cancel{2}}\cdot(-\sin(2x))\cdot\cancel{2} = \sin 2x = f(x) \quad \text{ok}$$

$$F(x) - G(x) = \sin^2 x + 7 + \frac{1}{2}\cos 2x + 11 =$$

$$= \sin^2 x + 7 + \frac{1}{2}\left(2\cos^2 x - 1\right) + 11 =$$

$$= \underbrace{\sin^2 x + \cos^2 x}_{} + 7 - 2 + 11 = 1 + 7 - 2 + 11 = 17 = c \in \mathbb{R}$$