Week 12 Diff. Equations II order
Cauchy Problems

(IMPROPER INTEGRALS:
$x_0 \neq 0$ $\quad f(x) \xrightarrow[x \to x_0]{} \infty$ $\qquad f(x) \sim_{x_0} \dfrac{1}{|x-x_0|^\alpha}$

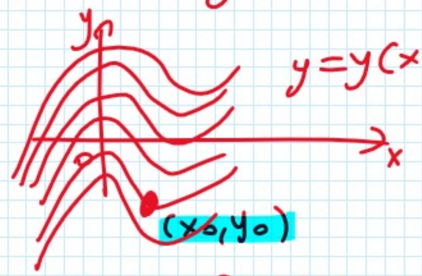
$\alpha \gtrless 1$?

ex $\displaystyle\int_0^{+\infty} \frac{\sin x}{x^2}\,dx$ $\qquad \left|\dfrac{\sin x}{x^2}\right| \leq \dfrac{1}{x^2}$ div.)

- **ODE order n** (order ≠ degree)

$$F(x, y, y', y'', \dots, y^{(n)}) = 0 \qquad \text{Find } y = y(x)$$

- **normal form** $\quad y^{(n)} = G(x, y, y, \dots, y^{(n-1)})$



$y = y(x) + c$ solution (general integral)

- **1st order Cauchy Problem**

$$\text{C.P.} \begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

- **2nd order Cauchy Problem**

$$\text{C.P.} \begin{cases} y'' = f(x,y,y') \to \text{2 constants} \\ y(x_0) = y_0 \\ y'(x_1) = y_1 \end{cases}$$

Existence & UNIQUENESS Theorem (local solution / global solution) for C.P.

Recall these SUFFICIENT conditions

OPEN

- IF $f \in C^0(I)$ $\quad$ f continuous on some interval I
$\implies \exists!$ local solution for C.P.
- IF $f \in C^1(I)$ $\implies \exists!$ global solution for C.P.

II order ODE with constant coefficients

$\Rightarrow$ $y'' + ay' + by = 0$ $\to$ HOMOGENEOUS equation, $a, b \in \mathbb{R}$

- characteristic polynomial $\quad (y = e^{\lambda x})$ $\quad \boxed{\lambda^2 + a\lambda + b = 0}$

(1) $\Delta > 0$ $\quad \lambda_1 \neq \lambda_2 \in \mathbb{R}$ $\qquad y(x, c_1, c_2) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

(2) $\Delta = 0$ $\quad \lambda$ $\qquad y(x, c_1, c_2) = (c_1 + c_2 x) e^{\lambda x}$

(3) $\Delta < 0$ $\quad \sigma \pm i\omega$ $\qquad y(x, c_1, c_2) = e^{\sigma x}(c_1 \cos \omega x + c_2 \sin \omega x)$

$c_1, c_2 \in \mathbb{R}$ $\qquad$ σ → Real part

(3) $\mapsto \Delta < 0$ $\quad \sigma \pm i\omega$ $\quad y(x, c_1, c_2) = e^{\sigma x}(c_1 \cos \omega x + c_2 \sin \omega x)$

$c_1, c_2 \in \mathbb{R}$ Real part

Imaginary part

example $\quad y'' - 16y = 0$

characteristic polynomial $\quad r^2 - 16 = 0 \quad r = \pm 4 \quad (\Delta > 0)$

$r_1 = -4 \quad r_2 = 4 \quad \boxed{y = c_1 e^{-4x} + c_2 e^{4x}}$

$\exists$ bounded solutions on $\mathbb{R}$? $\quad c_1, c_2 \in \mathbb{R}$

$\exists c_1, c_2 \,/\, \lim_{x \to \pm\infty} y(x) \in \mathbb{R}$? $\quad$ general integral

$$\lim_{x \to -\infty} \left(c_1 e^{-4x} + c_2 e^{4x}\right) = c_1 \underbrace{e^{+\infty}}_{+\infty} + c_2 \underbrace{e^{-\infty}}_{0} \Rightarrow \text{Take } c_1 = 0$$

$$\lim_{x \to +\infty} \left(c_1 e^{-4x} + c_2 e^{4x}\right) = c_1 \underbrace{e^{-\infty}}_{0} + c_2 \underbrace{e^{+\infty}}_{+\infty} \Rightarrow \text{Take } c_2 = 0$$

Unique bounded sol. on $\mathbb{R}$!

$c_1 = c_2 = 0 \qquad y = y(x) = 0$

$\exists$ bounded sol. on $[a, +\infty)$? $\quad a \in \mathbb{R}$

$\lim_{x \to a^+} y(x) \in \mathbb{R}$ ok

$$\lim_{x \to +\infty} \left(c_1 \underbrace{e^{-4x}}_{0} + \underbrace{c_2 e^{4x}}_{+\infty}\right) \quad \text{as before } c_2 = 0$$

$y(x) = c_1 e^{-4x} \qquad (c_2 = 0) \qquad \forall c_1 \in \mathbb{R}$

How many bounded sol. on $[a, +\infty)$? infinite

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example $\quad y'' + 16y = 0$

$r^2 + 16 = 0 \quad r = \pm 4i \qquad$ (3) $\Delta < 0$

$\boxed{\sigma \pm i\omega} \qquad \sigma = 0$ (real part)

$\omega = 4$ (imaginary part)

$$y = y(x) = \underbrace{e^{0x}}_{1} (c_1 \cos 4x + c_2 \sin 4x)$$

$= c_1 \cos 4x + c_2 \sin 4x \qquad \cos x, \sin x$ are BOUNDED!!

How many sol. on $\mathbb{R}$? Infinite: $\forall c_1, c_2 \in \mathbb{R}$

$y(0) = 1 \qquad y'(0) = 2 \qquad \rightsquigarrow$ C.P.

Recap on COMPLETE equations (ODE II order with constant coefficients)

$$y'' + ay' + by = \boxed{g(x)}$$

with constant coefficients)

$$y'' + ay' + by = \boxed{g(x)}$$

FORCING TERM

$y = y_p + y_0 \rightarrow$ sol. for ASSOCIATED HOMOG. EQUATION

y_p: particular solution / integral

$\boxed{g(x) =}$ $P_m(x) \cdot e^{\mu x} \cdot \cos \vartheta x$ (or $\sin \vartheta x$)

P_m: polynomial of degree m

Note that it could be
$m = 0$
$\mu = 0$
$\vartheta = 0$

ex $\quad y'' + 16y = \boxed{3e^{2x}}$ $\qquad m = 0$, $\mu = 2$, $\vartheta = 0$

$$y_p(x) = x^{m} \cdot e^{\mu x} \cdot \left(Q_{1,m}(x) \cdot \cos \vartheta x + Q_{2,m}(x) \cdot \sin \vartheta x \right)$$

x^m: RESONANCE term, $m = 0, 1, 2$

$Q_{1,m}$, $Q_{2,m}$: UNKNOWN POLYNOMIALS of degree m

How to understand m?

(1) $\Delta > 0$ $\quad (y_0 = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x})$ $\quad \vartheta = 0$ and $\mu = \lambda_1$ OR $\mu = \lambda_2$ $\rightarrow \boxed{m = 1}$

(2) $\Delta = 0$ $\quad (y_0 = (C_1 + C_2 x) e^{\lambda x})$
$\vartheta = 0$ and $\mu = \lambda \longrightarrow \boxed{m = 2}$

(3) $\Delta < 0$ $\quad \left(y_0 = e^{\sigma x} \left(C_1 \cos \omega x + C_2 \sin \omega x \right) \right)$ $\quad \vartheta = \omega$ and $\mu = \sigma$
solutions $\sigma \pm i\omega$
$\longrightarrow \boxed{m = 1}$

IN ALL OTHER CASES $\boxed{m = 0}$

$\boxed{T1}$ $\quad (*) \quad y'' - 16y = \underbrace{-32 \sin 4x}_{= g(x)}$ $\qquad$ COMPLETE eq.

$y = y_0 + y_p$

- Associated Homog. eq. $\quad y'' - 16y = 0$ $\qquad \lambda^2 - 16 = 0$, $\lambda = \pm 4$ $\quad$ case (1) $(\Delta > 0)$

$y_0 = C_1 e^{-4x} + C_2 e^{4x}$

- Complete eq. (to find y_p) $\qquad$ look at $g(x) = -32 \sin 4x$

$m = 0$
$\mu = 0$
$\vartheta = 4$

$\vartheta \neq 0$ and also $\mu \neq \pm 4$ $\implies \nexists$ Resonance $m = 0$

$\mu = 0$ and also $\qquad m = 0$

$\vartheta = 4$ $\quad \mu \mp \pm 4$

$$y_p = x^0 \cdot e^{0x} \left(A \cdot \cos 4x + B \cdot \sin 4x \right) = A \cos 4x + B \sin 4x$$

$x^0 = 1$, $e^{0x} = 1$, $A \cos 4x \to q_{1,m}(x)$, $B \sin 4x \to q_{2,m}(x)$

Find A and B! Compute y_p' and y_p'' and plug these in the complete eq.

$y_p' = -4A \sin 4x + 4B \cos 4x$

$y_p'' = -16A \cos 4x - 16B \sin 4x$

Plug in $(*)$ $\quad y'' - 16y = -32 \sin 4x$

$$\underbrace{-16A\cos 4x - 16B \sin 4x}_{} - 16 \underbrace{(A \cos 4x + B \sin 4x)}_{} = -32 \sin 4x$$

$-32A\cos 4x - 32B \sin 4x = -32 \sin 4x$

$$\begin{cases} -32A = 0 & (\cos) \\ -32B = -32 & (\sin) \end{cases}$$

example $\boxed{m=1 \quad q_{1,1}(x) = Ax+B \quad q_{2,1}(x) = Cx+D}$

$$\begin{cases} A = 0 \\ B = 1 \end{cases} \qquad \boxed{y_p = \sin 4x}$$

$$y = y_0 + y_p = C_1 e^{-4x} + C_2 e^{4x} + \sin 4x$$

$\boxed{\text{EX 1C}}$ $\quad y'' + 6y' + 10y = (x+2)e^{-3x} \cos x = g(x) \qquad (*)$

• y_0? $\quad \lambda^2 + 6\lambda + 10 = 0 \qquad \Delta = -4$

$$\lambda = \frac{-6 \pm 2i}{2} = -3 \pm i \quad \curvearrowright \quad \sigma \pm \omega i$$

• $\sigma = -3$

* $\omega = 1$

$y_0 = e^{-3x} (C_1 \cos x + C_2 \sin x) \qquad C_1, C_2 \in \mathbb{R}$

• y_p? $\quad M = 1$ $\qquad m = ?$ In this case $\mu = \sigma$, $\vartheta = \omega$

• $\mu = -3$

* $\vartheta = 1$ $\qquad$ Yes, we have Resonance $\boxed{m=1}$

$$y_p = x^m \cdot e^{\mu x} \left(Q_{1,m}(x) \cdot \cos \vartheta x + Q_{2,m}(x) \sin \vartheta x \right)$$

$$y_p = x^1 \cdot e^{-3x} \left((Ax+B) \cos x + (Cx+D) \cdot \sin x \right)$$

(IF $M = 2$ $\quad Ax^2 + Bx + C \qquad Dx^2 + Ex + F$)

(IF $m = 2$ Ax^2+Bx+C Dx^2+Ex+F)

compute A, B, C, D in this way: y_p', y_p''
and plug in (*) $\longrightarrow$ 4 equations to find 4 parameters A, B, C, D

$\exists$ bounded solutions? on $\mathbb{R}$ $y = y_0 + y_p$

$y_0 = e^{-3x}(C_1\cos x + C_2 \sin x)$ (IF $x \to +\infty$ $y_0 \to 0$)

$e^{-3x} \to +\infty$

IF $x \to -\infty$ $y_0 \longrightarrow \nexists$

$C_1 = C_2 = 0$

$y_p = \frac{x}{e^{3x}}((Ax+B)\cos x + (Cx+D)\sin x) \xrightarrow[x\to+\infty]{} 0$

$\frac{x}{e^{3x}} \to 0$

$y_p = x e^{-3x}((Ax+B)\cos x + (Cx+D)\sin x) \xrightarrow[x\to+\infty]{} \nexists$

$x e^{-3x} \to \infty$ ok   oscillating

IF $A,B,C,D \neq 0 \implies \nexists$ bounded sol. on $\mathbb{R}$.   On $[a,+\infty)$ $\exists$ bounded solution

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(ex) $y'' - 4y' + 4y = e^{2x} \cdot \frac{x}{2} = g(x)$ $m = 1$, $\mu = 2$, $\vartheta = 0$

Resonance? $y_0 = ?$ $y_p = ?$

$\lambda^2 - 4\lambda + 4 = 0$ $(\lambda-2)^2 = 0$ $\boxed{\lambda = 2}$ $\Delta = 0$

$y_0 = (C_1 + C_2 x)e^{2x}$

In this case $\Delta = 0$:
$\vartheta = 0$ and $\mu = \lambda$ $\implies \exists$ resonance $\boxed{M=2}$

$y_p = x^2 e^{2x}((Ax+B)\underbrace{\cos 0x}_{1} + (Cx+D)\cancel{\sin 0x})$ $\sin 0x = 0$

$= x^2 e^{2x}(Ax+B) \to y_p'$ y_p'' plug in the complete eq.
$\to$ find A, B

1^{st} order
Cauchy problem

$\boxed{EX5}$
$$\begin{cases} x' = x^2 - 3x + 2 = f(x,t) \\ x(13) = 5 \end{cases}$$
find $x = x(t)$

initial condition

$\begin{cases} x(13) = 5 \end{cases}$ find $x = x(t)$

initial condition

separable variables

- <u>constant solutions</u> $(x' = 0)$

sum, product

$x^2 - 3x + 2 = 0$

$(x-2)(x-1) = 0$

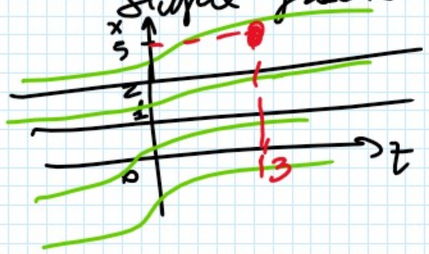
$x = 2$ or $x = 1$

BUT $x(13) = 5 \longrightarrow$ exclude

- <u>general sol.</u>

$$\frac{dx}{dt} = x^2 - 3x + 2 \qquad \int \frac{dx}{(x-2)(x-1)} = \int dt$$

simple fractions



$$\frac{1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$(x-2)\cdot$ and $x \to 2$

$$\underbrace{\frac{1}{x-1}}_{\to 1} = A + \underbrace{(x-2)\frac{B}{x-1}}_{0}$$

$\boxed{A = 1}$

$(x-1)\cdot$ and $x \to 1$

$$\underbrace{\frac{1}{x-2}}_{-1} = \underbrace{(x-1)\frac{A}{x-2}}_{\to 0} + B$$

$\boxed{B = -1}$

$$\int \frac{dx}{x-2} - \int \frac{dx}{x-1} = \int dt$$

Find $x = x(t)$

$$\log|x-2| - \log|x-1| = t + C \qquad C \in \mathbb{R}$$

$$\log\left|\frac{x-2}{x-1}\right| = t + C$$

$$\left|\frac{x-2}{x-1}\right| = e^t \cdot \underbrace{e^C}_{= k}$$

$C \in \mathbb{R}$ $k > 0$

$$\frac{x-2}{x-1} = \underbrace{(\pm k)}_{= a} e^t \qquad a \in \mathbb{R}\setminus\{0\}$$

$$x - 2 = a\, e^t (x-1)$$

$$x(1 - a e^t) = 2 - a e^t$$

$$x = \frac{2 - a e^t}{1 - a e^t} \qquad a \in \mathbb{R},\ a \neq 0$$

general integral

$\boxed{x(13) = 5}$

$$5 = \frac{2 - a e^{13}}{1 - a e^{13}} \quad \text{find } a \qquad 5 - 5 a e^{13} = 2 - a e^{13}$$

$$(\ldots) \quad a = \frac{3}{4 e^{13}}$$

the unique solution that satisfies the CP is

$(\dots)$ $a = \dfrac{3}{4e^{13}}$ the unique solution that satisfies the C.P. is

$$x = x(t) = \frac{2 - \frac{3}{4e^{13}} e^t}{1 - \frac{3}{4e^{13}} e^t}$$

$\boxed{EX\ 7B}$ C.P.

$$\begin{cases} (1) & x'' + x' - 2x = 0 \\ (2) & x(0) = 1 \\ (3) & \lim_{t \to +\infty} x(t) = 0 \end{cases}$$

$(x'' = -x' + 2x)$

$f(t, x, x')$

$\exists!$ solution

sum product

(1) $r^2 + r - 2 = 0$

$(r+2)(r-1) = 0$ $r = 1$ or $r = -2$ $(\Delta > 0)$

$x = c_1 e^t + c_2 e^{-2t}$ $c_1, c_2 \in \mathbb{R}$ → general integral

plug (2) $1 = c_1 e^0 + c_2 e^0$ $\boxed{1 = c_1 + c_2}$

plug (3) $\lim_{t \to +\infty} c_1 \underbrace{e^t}_{+\infty} + \underbrace{c_2 e^{-2t}}_{0} \underset{\text{IMPOSE}}{=} 0$ $\boxed{c_1 = 0}$

$$\begin{cases} (3) & c_1 = 0 \\ (2) & c_1 + c_2 = 1 \to c_2 = 1 \end{cases}$$

$\boxed{x(t) = e^{-2t}}$

$\boxed{T3}$

3. Which of the following statements is satisfied by the differential equation $y'' = e^{2t}$? $= g(t)$

$y_0 + y_p$ $y = y(t)$

(a) There are solutions bounded on $\mathbb{R}$

(b) $y(t) = e^{t^2} + c_1 t + c_2,\ c_1, c_2 \in \mathbb{R}$ is the general integral

(c) $y(t) = \frac{1}{4}e^{2t} - t + 3$ is a particular integral of the equation TRUE

(d) No solution of the equation has asymptotes

(e) $y(t) = \frac{1}{4}e^{2t} + c,\ c \in \mathbb{R}$ is the general integral

$r^2 = 0$ $r = 0$ $(\underline{\Delta = 0})$ $y_0 = (c_1 + c_2 t)e^{0t}$
$= c_1 + c_2 t$

$y_p = ?$ $m = 0$
$\mu = 2$
$\theta = 0$

$m = \begin{cases} 0 \\ 2 \end{cases}$ because $\Delta = 0$

$\theta = 0$ ok $\underset{\downarrow\ 2}{\mu} = \underset{\downarrow\ 0}{r}$? NO

$m = 0$

$y_p = t^0 \cdot e^{2t} (A \cdot \underbrace{\cos 0x}_{1} + \cancel{B \sin 0x}) = Ae^{2t}$ Compute A:

$y_p' = 2Ae^{2t}$

$y_p' = 2Ae^{2t}$

$y_p'' = 4Ae^{2t}$ plug in eq. $4Ae^{2t} = e^{2t}$

$4A = 1 \quad A = \frac{1}{4}$

general integral

$$\boxed{y = \underline{y_0} + \underline{y_p} = \underbrace{C_1 + C_2 t}_{} + \underline{\frac{1}{4} e^{2t}}} \rightarrow (C) \quad \cancel{(B)} \quad \cancel{(E)}$$

NOT bounded

$\lim_{t \to \pm\infty} y(t) = \pm\infty$

EVEN IF we set $C_2 = 0$ $\quad \lim_{t \to +\infty} y_p = +\infty \quad \cancel{(A)}$

$\cancel{(D)} \quad \lim_{t \to -\infty} y(t) = -\infty \qquad y(t) \sim_{-\infty} C_1 + C_2 t$

left oblique asymptote

T5

3. Which of the following statements is satisfied by the differential equation $y'' + 9y = \cos 3x$? $= g(x)$

~~(a) with such forcing term, there is no resonance~~

(b) A particular integral of the complete equation is $y_p(x) = x(\alpha \cos 3x + \beta \sin 3x)$

~~(c) The general integral of the complete equation is in the form $y(x) = \alpha \cos 3x$~~

~~(d) The solutions are periodic functions~~ False

~~(e) The general integral is $y(x) = x(C_1 \cos 3x + C_2 \sin 3x)$~~

$m = 0$
$\mu = 0$
$\vartheta = 3$

$\lambda^2 + 9 = 0 \quad \lambda = \pm 3i \quad (\Delta < 0) \qquad C_1, C_2 \in \mathbb{R}$

$y_0 = \underbrace{e^{0x}}_{=1}(C_1 \cos 3x + C_2 \sin 3x) = C_1 \cos 3x + C_2 \sin 3x$

$m = ?$ $\quad \mu = \delta = 0 \quad$ Yes, $\exists$ resonance $\quad \boxed{m = 1}$

$\vartheta = \omega = 3$

$y_p = x^1 e^{0x}(A \cos 3x + B \sin 3x) =$

$= x(A \cos 3x + B \sin 3x) \rightarrow (B) \qquad y = y_0 + y_p$

NOT periodic $\qquad$ GIVEN

$\exists$ bounded sol. on $\mathbb{R}$? $\quad$ Suppose $A \neq 0, B \neq 0$

$\lim_{x \to \pm\infty} y_p \notin \mathbb{R} \quad (\nexists \lim)$

No $\qquad$ Range of y_p NOT bounded

T4

14. Consider the Cauchy problem:

$$C.P. \begin{cases} (1) & xy'(x) = \sqrt{y} \\ (2) & y\left(\frac{e}{3}\right) = \frac{1}{4} \end{cases}$$

separable variables (1^{ST} order ODE)

- constant sols $(y' = 0)$ $\quad \sqrt{y} = 0 \quad y = 0$ NOT verified by (2)

Which of the following is true?

~~(a) The function $y = \frac{1}{4}$ is a solution of the problem~~

~~(b) The function $y = 0$ is a solution of the problem~~

~~(c) If $f(x)$ is a solution of the problem, then $f(x)$ is not differentiable on $(0, +\infty)$~~ y

(d) There are no constant solutions for the problem

$y > 0$

(b) The function $y = 0$ is a solution of the problem

(c) If $f(x)$ is a solution of the problem, then $f(x)$ is not differentiable on $(0, +\infty)$

(d) There are no constant solutions for the problem

(e) The function $f(x) = \log^2 3x$ is a solution for the Cauchy problem

verified by (2)

$y > 0$

IF So: $y = \log^2 3x$ (2) $\log^2\left(3 \cdot \frac{e}{3}\right) = 1 \neq \frac{1}{4}$ No

(check also (1) $y' = \dots$)

Solve in general • general sol.

$$x \frac{dy}{dx} = \sqrt{y} \qquad \int \frac{dy}{\sqrt{y}} = \int \frac{dx}{x} \qquad (\dots)$$

$y = y(x, C)$ ← plug (2) and find C

Past written exams 3

(b) Find the solution oh the following Cauchy problem :

$$\begin{cases} y' = \underbrace{te^{-t^2}}_{h(t)} \underbrace{(y+1)^3}_{g(y)} = f(t, y) \\ y(0) = -\frac{1}{2} \end{cases}$$

Separable variables

• constant solutions $(y' = 0)$ $(y+1) = 0$ $\boxed{y = -1}$ Not verified

No constant solutions

• general sol.

$$\int \frac{dy}{(y+1)^3} = \int t e^{-t^2}\, dt \implies \frac{(y+1)^{-2}}{-2} = -\frac{1}{2} e^{-t^2} + k$$

Q. $\int \frac{dx}{\sqrt{1+x^2}} = \int dt = \text{sett}\,\text{senh}(x) + c$ $\quad x = \sinh t \quad dx = \cosh t\, dt$ $\quad \text{senh}^{-1}(x)$

$$(y+1)^{-2} = e^{-t^2} + k \qquad k \in \mathbb{R}$$

$$\frac{1}{(y+1)^2} = \frac{1 + ke^{t^2}}{e^{t^2}} \qquad (y+1)^2 = \frac{e^{t^2}}{1 + ke^{t^2}} \qquad \boxed{y+1 = \pm\sqrt{\frac{e^{t^2}}{1+ke^{t^2}}}}$$

Note that $\sqrt{x^2} = |x|$

$y(0) = -\frac{1}{2}$ $\quad (y+1)(0) = \frac{1}{2} > 0$ choose $+$

$$y = \sqrt{\frac{e^{t^2}}{1+ke^{t^2}}} - 1$$

$$-\frac{1}{2} = \sqrt{\frac{1}{1+k}} - 1 \qquad (\dots) \qquad k = 3$$

$$\boxed{y = \sqrt{\frac{e^{t^2}}{1+3e^{t^2}}} - 1}$$

$$y = \sqrt{\frac{\cdots}{1+3e^{t^2}} - 1}$$

(ex) Suppose the general integral is $\to$ $y(x) = C_1 e^{2x} + C_2 e^{3x}$

$(\Delta > 0 \quad r_1 = 2 \quad r_2 = 3)$

(1) $\exists$ bounded sol. on $\mathbb{R}$? How many? 1 bounded sol. $y=0$

(2) $\exists$ bounded on on $(-\infty, a)$? " ? ∞ bounded sols.

(3) " $(a, +\infty)$? " ? 1 "

(1) $\lim_{x \to +\infty} y(x) = +\infty$ UNLESS $C_1 = 0$, $C_2 = 0$

(2) $\lim_{x \to -\infty} y(x) = 0$ always bounded $\forall C_1, C_2$

(3) REASON AS IN (1)

Suppose $y = y_0 + \underbrace{y_p} = C_1 e^{2x} + C_2 e^{3x} + 4 e^x$

$\nexists$ bounded sol. on $(a, +\infty)$ $\to 0 \quad 0 \quad 0$ IF $x \to -\infty$

$\nexists$ " $\mathbb{R}$

(IF $x \to -\infty$ $y \to 0$

THEREFORE $\exists$ bounded solutions on $(-\infty, a)$ $a \in \mathbb{R}$)