MATHEMATICAL ANALYSIS I TUTORING 10TH WEEK

INTEGRAL FUNCTION - FUNDAMENTAL THEOREM OF INTEGRAL CALCULUS- IMPROPER INTEGRALS

PROPOSED EXERCISES - SOLUTIONS

1. Compute the following definite integrals

a) $\boxed{\int_{-1}^{2} x\log(1+|x+1|)\,\mathrm{d}x = \frac{9}{4}}$

Note that the integrand function is continuous in $[-1,2]$, and thus integrable on such interval; moreover

$$x\log(1+|x+1|) = \begin{cases} x\log(1+x+1) & \text{if } x \geq -1 \\ x\log(1-x-1) & \text{if } x < -1 \end{cases} = \begin{cases} x\log(2+x) & \text{if } x \geq -1 \\ x\log(-x) & \text{if } x < -1 \end{cases}$$

Hence:

$$\begin{aligned}
\int_{-1}^{2} x\log(1+|x+1|)\,\mathrm{d}x &= \int_{-1}^{2} x\log(2+x)\,\mathrm{d}x \\
&= \left[\frac{x^2}{2}\log(2+x)\right]_{-1}^{2} - \frac{1}{2}\int_{-1}^{2} x^2\frac{1}{2+x}dx \\
&= \left[\frac{2^2}{2}\log(2+2) - 0\right] - \frac{1}{2}\int_{0}^{2}\frac{x^2-4+4}{2+x}dx \\
&= 2\log 4 - \frac{1}{2}\int_{-1}^{2}\frac{x^2-4}{2+x}dx - \frac{1}{2}\int_{-1}^{2}\frac{4}{2+x}dx \\
&= 2\log 4 - \frac{1}{2}\int_{-1}^{2}(x-2)dx - \frac{1}{2}\int_{-1}^{2}\frac{4}{2+x}dx \\
&= 2\log 4 - \frac{1}{2}\left[\frac{(x-2)^2}{2}\right]_{-1}^{2} - 2\left[\log(x+2)\right]_{-1}^{2} \\
&= 2\log 4 - \frac{1}{2}\left[-\frac{(-3)^2}{2}\right] - 2\log 4 = \frac{9}{4}
\end{aligned}$$

b) $\boxed{\int_{e}^{e^2} \frac{-1+2\log x}{x\ (1+\log x)\ \log x}\,\mathrm{d}x}$

Note that the integrand function is continuous in I=$[e,e^2]$, since the denominator is zero only outside I; thus it is integrable on I.
Apply the substitution $\log x = t$, then $x = e^t$, $dx = e^t dt$.
The integration interval $x \in [e,e^2]$ becomes $t \in [1,2]$. Therefore:

$$\int_{e}^{e^2} \frac{-1+2\log x}{x\ (1+\log x)\ \log x}\,\mathrm{d}x = \int_{1}^{2}\frac{-1+2t}{e^t(1+t)t}e^t dt = \int_{1}^{2}\frac{-1+2t}{(1+t)t}\ dt$$

$$\frac{-1+2t}{(1+t)t} = \frac{A}{(1+t)} + \frac{B}{t}$$

Multiply by $(1+t)$

$$\frac{-1+2t}{(1+t)t}(1+t) = \frac{A}{(1+t)}(1+t) + \frac{B}{t}(1+t) \Rightarrow \frac{-1+2t}{t} = A + \frac{B}{t}(1+t)$$

Compute in $t=-1$, then $A = \frac{-3}{-1} = 3$.
Multiply by t

$$\frac{-1+2t}{(1+t)t}t = \frac{A}{(1+t)}t + \frac{B}{t}t \ \Rightarrow\ \frac{-1+2t}{(1+t)} = \frac{A}{(1+t)}t + B$$

In $t = 0$ we have $B = -1$. Thus:

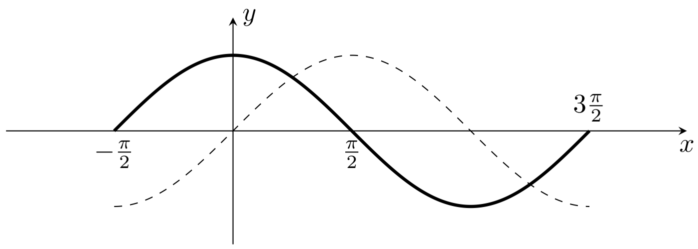
$$\begin{aligned}
\int_1^2 \frac{-1+2t}{(1+t)t}\,dt &= \int_1^2 \frac{3}{(1+t)}\,dt - \int_1^2 \frac{1}{t}\,dt \\
&= \left[3\log|1+t| - \log|t|\right]_1^2 \\
&= 3\log 3 - \log 2 - 3\log 2 = 3\log 3 - 4\log 2 = \log\frac{27}{16}
\end{aligned}$$

c) $\boxed{\int_{-3}^{-1} \frac{2}{x^2+6x+1}\,\mathrm{d}x}$

Note that the integrand function is continuous in $[-3,-1]$ (since the denominator is zero only outside the interval), therefore it is integrable on $[-3,-1]$; applying simple fractions $\frac{2}{x^2+6x+1} = \frac{1}{\sqrt{8}}\left(\frac{1}{x+3-\sqrt{8}} - \frac{1}{x+3+\sqrt{8}}\right)$; hence:

$$\begin{aligned}
\int_{-3}^{-1} \frac{2}{x^2+6x+1}\,\mathrm{d}x &= \frac{1}{\sqrt{8}}\int_{-3}^{-1}\left(\frac{1}{x+3-\sqrt{8}} - \frac{1}{x+3+\sqrt{8}}\right)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{8}}\left[\log\left|\frac{x+3-\sqrt{8}}{x+3+\sqrt{8}}\right|\right]_{-3}^{-1} \\
&= \frac{1}{\sqrt{8}}\log\left|\frac{1-\sqrt{2}}{1+\sqrt{2}}\right| = \frac{\sqrt{2}}{4}\log(\sqrt{2}-1)^2 = \frac{\sqrt{2}}{2}\log(\sqrt{2}-1)
\end{aligned}$$

d) $\boxed{\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2|\cos x|\,dx = -4+4\pi+3\pi^2}$



$$(x+1)^2|\cos x| = \begin{cases} (x+1)^2\cos x & \text{if } -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2} \\[2ex] -(x+1)^2\cos x & \text{if } -\dfrac{\pi}{2} \le x \le -3\dfrac{\pi}{2} \end{cases}$$

Therefore:

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2|\cos x|\,dx &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (x+1)^2\cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2\cos x\,dx \\
&= 2\int_0^{\frac{\pi}{2}} (x+1)^2\cos x\,dx - \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2\cos x\,dx
\end{aligned}$$

Compute the indefinite integral

$$\begin{aligned}
\int (x+1)^2\cos x\,dx &= (x+1)^2\sin x - \int 2(x+1)\sin x\,dx \\
&= (x+1)^2\sin x + \left(2(x+1)\cos x - \int 2\cos x\,dx\right) \\
&= (x+1)^2\sin x + 2(x+1)\cos x - 2\sin x \\
&= (x^2+2x-1)\sin x + 2(x+1)\cos x
\end{aligned}$$

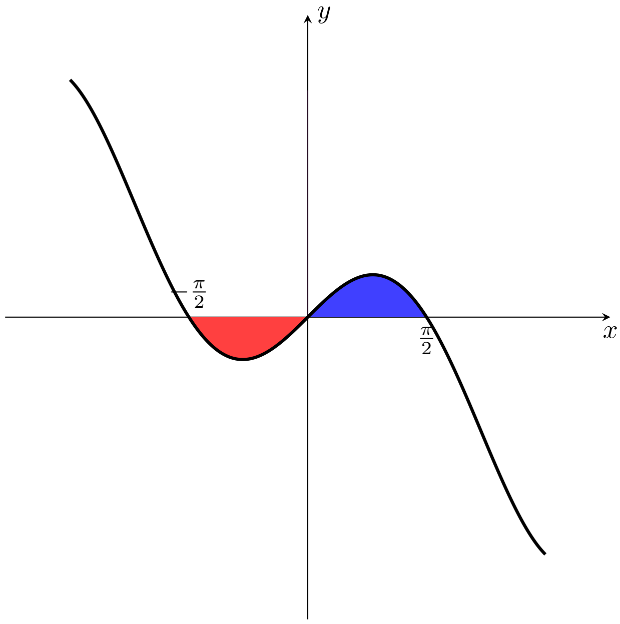
Thus:

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} (x+1)^2 |\cos x| \, dx &= 2\left[(x^2+2x-1)\sin x + 2(x+1)\cos x\right]_0^{\frac{\pi}{2}} - \left[(x^2+2x-1)\sin x + 2(x+1)\cos x\right]_{\frac{\pi}{2}}^{3\frac{\pi}{2}} \\
&= 2\left(\frac{\pi^2}{4} + 2\frac{\pi}{2} - 1 - 2\right) - \left(-\frac{9\pi^2}{4} - 6\frac{\pi}{2} + 1 - \left(\frac{\pi^2}{4} + 2\frac{\pi}{2} - 1\right)\right) \\
&= 3\pi^2 + 6\pi - 8
\end{aligned}
$$

2. Calculate the area between the x-axis and the following functions:

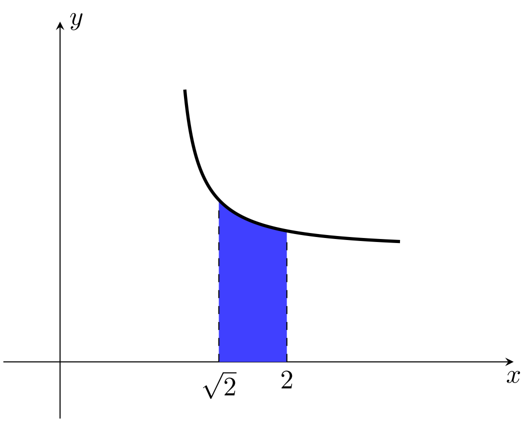
$$f(x) = x\cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \; ; \; g(x) = \frac{x}{\sqrt{x^2-1}}, \quad x \in [\sqrt{2}, 2]$$

$$\boxed{f(x) = x\cos x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}}$$



$$
\begin{aligned}
\text{Area} &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} |x\cos x| \,\mathrm{d}x \; = 2\int_0^{+\frac{\pi}{2}} x\cos x \,\mathrm{d}x \\
&= 2\left[x\sin(x) + \cos(x)\right]_0^{\frac{\pi}{2}} = \pi - 2
\end{aligned}
$$

$$\boxed{f(x) = \frac{x}{\sqrt{x^2-1}}, \quad x \in [\sqrt{2}, 2]}$$



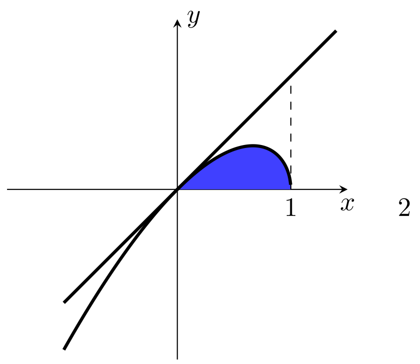
$$\begin{aligned}\text{Area} &= \int_{\sqrt{2}}^{2} \left| \frac{x}{\sqrt{x^2-1}} \right| \, \mathrm{d}x \\ &= \int_{\sqrt{2}}^{2} \frac{x}{\sqrt{x^2-1}} \, \mathrm{d}x \\ &= \left[\sqrt{x^2-1} \right]_{\sqrt{2}}^{2} = \sqrt{3} - 1\end{aligned}$$

3. Calculate the area of the following 2-dimensional sets

$$\mathrm{A} = \left\{ (x,y) \in \mathbb{R}^2, \ 0 \le x \le 1, \ 0 \le y \le x\sqrt{1-x} \right\}$$

$$\mathrm{B} = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le \frac{\pi}{3}, \ 0 \le y \le \sin^3 x \cos^2 x \right\}$$

$\boxed{\mathrm{A} = \left\{ (x,y) \in \mathbb{R}^2, \ 0 \le x \le 1, \ 0 \le y \le x\sqrt{1-x} \right\}}$



Since the integrand function is positive on $[0,1]$, we have:

$$\text{Area}(A) = \int_0^1 |x\sqrt{1-x}| \mathrm{d}x = \int_0^1 x\sqrt{1-x} \mathrm{d}x$$

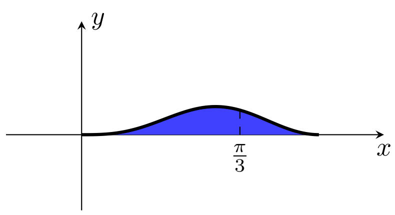
Compute the indefinite integral $\displaystyle\int x\sqrt{1-x}\mathrm{d}x$ with the substitution $\sqrt{1-x} = t$, then $x = 1 - t^2$ and $\mathrm{d}x = -2t \ \mathrm{d}t$; therefore

$$\int x\sqrt{1-x}\mathrm{d}x = \int t(1-t^2)(-2t) \ \mathrm{d}t = \int (2t^4 - 2t^2) \ \mathrm{d}t = \frac{2}{5} t^5 - \frac{2}{3} t^3$$

It follows that

$$\int_0^1 x\sqrt{1-x}\mathrm{d}x = \left[\frac{2}{5} (\sqrt{1-x})^5 - \frac{2}{3} (\sqrt{1-x})^3 \right]_0^1 = \frac{4}{15}$$

$\boxed{\mathrm{B} = \left\{ (x,y) \in \mathbb{R}^2 : 0 \le x \le \frac{\pi}{3}, \ 0 \le y \le \sin^3 x \cos^2 x \right\}}$



$$\begin{aligned}
\text{Area}(B) &= \int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x \mathrm{d}x \\
&= \int_0^{\frac{\pi}{3}} \sin x \sin^2 x \cos^2 x \mathrm{d}x \\
&= \int_0^{\frac{\pi}{3}} \sin x (1-\cos^2 x) \cos^2 x \mathrm{d}x \\
&= \int_0^{\frac{\pi}{3}} \Big(\sin x \cos^2 x \ \ - \sin x \cos^4 x\Big) \mathrm{d}x \\
&= \Big[-\frac{\cos^3 x}{3} \ + \frac{\cos^5 x}{5}\Big]_0^{\frac{\pi}{3}} = -\frac{\frac{1}{8}-1}{3} + \frac{\frac{1}{16}-1}{5} \\
&= \frac{47}{480}
\end{aligned}$$

4. Let $f(x) = \sin^3 x \cos^2 x$. Compute the integral average μ of f on the interval $\left[0, \frac{\pi}{3}\right]$; determine whether there exists a point $c \in \left[0, \frac{\pi}{3}\right]$ such that $f(c) = \mu$.

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By definition $\mu = \dfrac{\int_0^{\pi/3} \sin^3 x \cos^2 x \ \mathrm{d}x}{\pi/3}$. It holds:

$$\int \sin^3 x \cos^2 x \ \mathrm{d}x \ = \int \sin x (1-\cos^2 x) \cos^2 x \ \mathrm{d}x \ = \int (\sin x \cos^2 - \sin x \cos^4 x) \ \mathrm{d}x \ = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + c$$

Thus

$$\mu = \frac{3}{\pi}\left[-\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x\right]_0^{\pi/3} = \frac{47}{160\pi}$$

There exists a point $c \in \left[0, \frac{\pi}{3}\right]$ such that $f(c) = \mu$, since $f(x)$ is continuous on $\left[0, \frac{\pi}{3}\right]$.

5. Consider the function $\quad f(x) = \begin{cases} |x| & \text{se } -1 \le x < 1 \\ 16 - x^2 & \text{se } 1 \le x \le 3. \end{cases}$.

a) Calculate the average value μ of f on the interval $[-1, 3]$.

By definition of integral average

$$\mu = \frac{1}{3-(-1)} \int_{-1}^{3} f(x) \ \mathrm{d}x \ = \frac{1}{4}\left(\int_{-1}^{1} |x| \ \mathrm{d}x \ + \int_1^3 (16 - x^2) \ \mathrm{d}x \ \right) = \frac{1}{4}\left(2\int_0^1 x \ \mathrm{d}x \ + \left[16x - \frac{x^3}{3}\right]_1^3\right) = \frac{73}{12}.$$

b) Determine whether there exists a point $c \in [-1, 3]$ such that $f(c) = \mu$.

Since$f(x)$ is not continuous on $[-1, 3]$, we cannot apply the integral mean value Theorem to state the existence of a point c with the required properties.

Check directly if μ belongs to the image of$f$.

Verify that Im $(f) = [0, 1) \cup [7, 15]$. Since $1 < \frac{73}{12} < 7$, $\mu \notin$Im(f) and there is no $c \in [-1, 3]$ such that $f(c) = \mu$.

6. Given the integral function $\qquad F(x) = \displaystyle\int_1^x e^{-t^2} \ dt$

a) Verify that $F(x)$ is invertible on $\mathbf{R}$; compute $(F^{-1})'(0)$.

$$F'(x) = e^{-x^2} > 0, \forall x \in \mathbb{R}$$

F is strictly monotone increasing and thus invertible on$\mathbb{R}$.

$F(1) = 0 \Rightarrow F^{-1}(0) = 1$
$F'(x) = e^{-x^2} \Rightarrow F'(1) = e^{-1}$

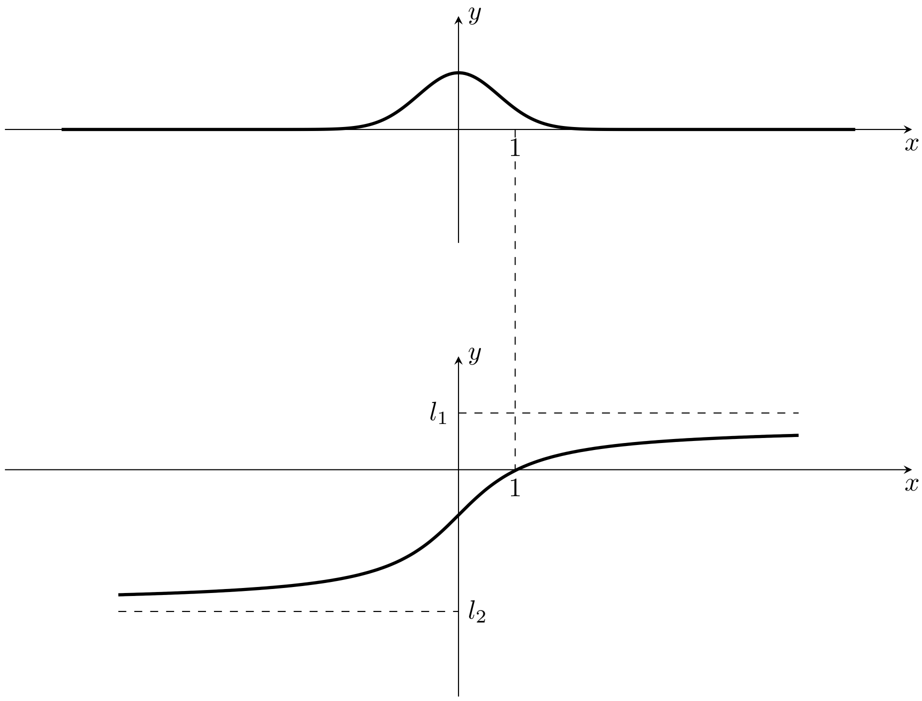
$$(F^{-1})'(0) = \frac{1}{F'(1)} = \frac{1}{e^{-1}} = e$$

b) Draw the graph of $f(x) = e^{-x^2}$ and a qualitative graph for $F(x)$.

- The integrand function if defined on $\mathbb{R}$ and it is integrable on $\mathbb{R}$, thus $DomF = \mathbb{R}$.
- $F(1) = 0$
- for $x > 1$ the definite integral $\displaystyle\int_1^x e^{-t^2}\,dt$ is the area of the region between the x-axis and the positive function; hence the integral is positive and $F(x) > 0$ for $x > 1$.

  for $x < 1$ it holds $\displaystyle F(x) = \int_1^x e^{-t^2}\,dt = -\int_x^1 e^{-t^2}\,dt$ that is $F(x) < 0$ for $x < 1$
- F is strictly monotone increasing on $\mathbb{R}$.

- $F''(x) = f'(x) = -2xe^{-x^2}$
  for $x < 0$ $f' > 0 \Rightarrow f$ increasing $\Rightarrow F$ convex
  for $x > 0$ $f' < 0 \Rightarrow f$ decreasing $\Rightarrow F$ concave
  $f' = 0 \Rightarrow f$ critical point $\Rightarrow F$ has an inflection point in $x = 0$



c) Say if $F(x)$ has horizontal asymptotes.

- $\boxed{\lim_{x\to+\infty} F(x)}$ We study the convergence of the improper integral $\displaystyle\int_1^{+\infty} e^{-t^2}\,dt$: apply Comparison Theorem and note that $e^{-x^2} < \dfrac{1}{x^2}$ for every x.
  The integral $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x$ is convergent, then by comparison also $\displaystyle\int_1^{+\infty} e^{-t^2}\,dt$ is convergent to some $l_1 > 0$: $\lim_{x\to+\infty} F(x) = l_1$.

- $\boxed{\lim_{x\to-\infty} F(x)}$ As before: the integral $\displaystyle\int_1^{-\infty} \frac{1}{x^2}\,\mathrm{d}x = -\int_{-\infty}^1 \frac{1}{x^2}\,\mathrm{d}x$ is convergent, then by comparison also $\displaystyle\int_1^{-\infty} e^{-t^2}\,dt$ converges to some $l_2 < 0$: $\lim_{x\to-\infty} F(x) = l_2$
  Thus $F(x)$ has a right horizontal asymptote (the line $y = l_1$) and a left horizontal asymptote (the line $y = l_2$).

7. Let $\qquad f(x) = \begin{cases} \cos x & \text{if } -\pi \le x \le \pi \\ \\ \frac{1}{x} & \text{if } x > \pi \end{cases}$. Say if f is locally integrable on $[-\pi, +\infty[$.

Let $F(x) = \displaystyle\int_{\frac{\pi}{2}}^{x} f(t)\, dt$. Draw a qualitative graph of $F(x)$.

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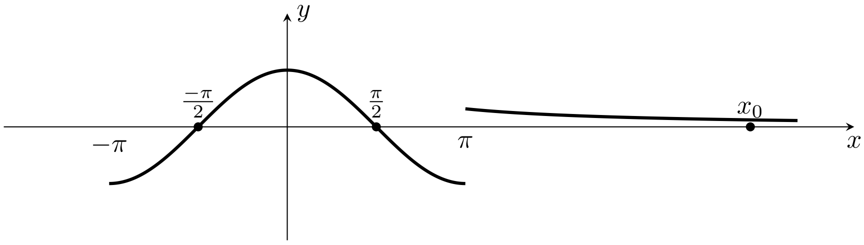
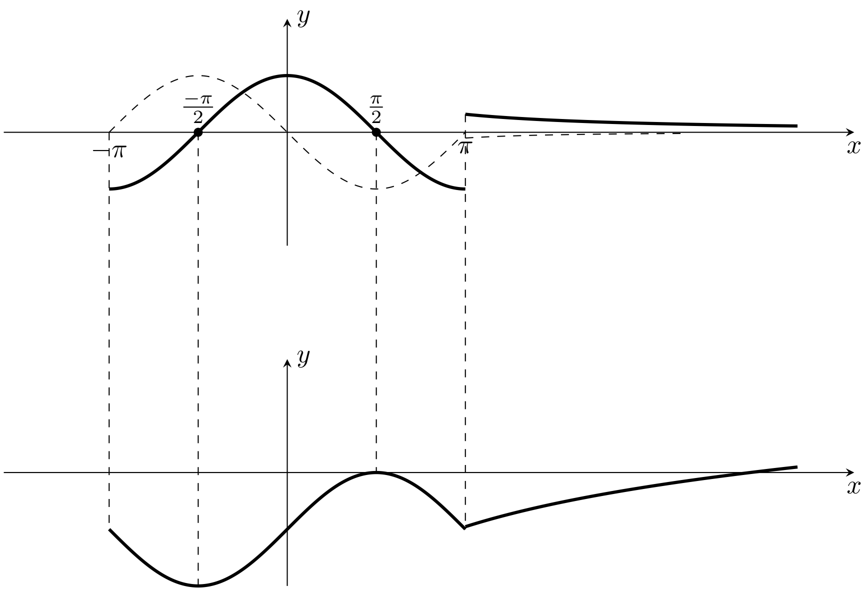
The integrand function f is integrable on $[-\pi, +\infty[$ and thus F is continuous on $\mathbb{R}$.



- $F\left(\frac{\pi}{2}\right) = 0$
- $$F'(x) = f(x) = \begin{cases} \cos x & \text{if } -\pi \le x \le \pi \\ \\ \dfrac{1}{x} & \text{if } x > \pi \end{cases}$$

  for $-\pi < x < -\dfrac{\pi}{2}$ it holds $f < 0 \Rightarrow F$ decreasing
  for $-\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$ it holds $f > 0 \Rightarrow F$ increasing
  for $\dfrac{\pi}{2} < x < \pi$ it holds $f < 0 \Rightarrow F$ decreasing
  for $x > \pi$ it holds $f > 0 \Rightarrow F$ increasing
  $\lim\limits_{x\to\pi^-} F(x) = \lim\limits_{x\to\pi^-} \cos(x) = -1,\ \lim\limits_{x\to\pi^+} F(x) = \lim\limits_{x\to\pi^+} \dfrac{1}{x} = \dfrac{1}{\pi} \Rightarrow F$ has a corner point in $x = \pi$.
- $$F''(x) = f'(x) = \begin{cases} -\sin x & \text{if } -\pi < x < \pi \\ \\ -\dfrac{1}{x^2} & \text{if } x > \pi \end{cases}$$

  for $-\pi < x < 0$ it holds $f' > 0 \Rightarrow f$ decreasing $\Rightarrow F$ convex
  for $0 < x < \pi$ it holds $f' < 0 \Rightarrow f$ decreasing $\Rightarrow F$ concave
  for $x > \pi$ it holds $f' < 0 \Rightarrow f$ decreasing $\Rightarrow F$ concave
  for $x = 0$ it holds $f' = 0 \Rightarrow f$ critical point $\Rightarrow F$ has an inflection point in $x = 0$



8. Consider the function $\quad F(x) = \displaystyle\int_0^x (\cos t + 1)e^{-5t^2}\,\mathrm{d}t.$

   a) Study the simmetries of $F(x)$.

   b) Study monotonicity and say if there are stationary points; classify them.

   c) Find the order of infinitesimal of $F(x)$, as $x \to 0$ and its Mac Laurin expansion of order 1. Say if the function has constant sign in a neighborhood of $x = 0$.

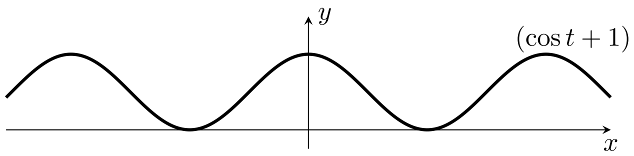
   d) Say if it exists and it is finite $\displaystyle\int_0^{+\infty} (\cos t + 1)e^{-5t^2}\,\mathrm{d}t.$

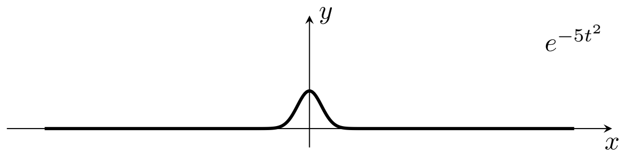
   e) Say if $F(x)$ admits horizontal asymptotes.

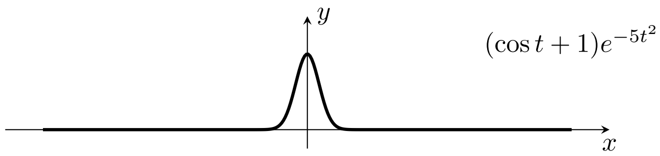
- -

   a) $f(t) = (\cos t + 1)e^{-5t^2}$ is even. Since $F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$, it holds $F(x)$ is odd.

   In order to get the graph of $f(t)$, we first draw $(\cos t + 1)$ and then we multiply by e^{-5t^2}







   b) $F'(x) = f(x)$.
   Since $f(x) = (\cos x + 1)e^{-5x^2} \geq 0,\ \forall x \in \mathbb{R}$, we have that $F(x)$ is always increasing. Moreover $f(\pi + 2k\pi) = 0$, hence the points $x_k = \pi + 2k\pi,\ k \in \mathbb{Z}$ are inflection points with horizontal tangent for $F(x)$.

   c) The Mac Laurin expansion of order 1 for $F(x)$ is $F(x) = F(0) + F'(0)\,x + o(x)$. It holds:
   $F(x) = \displaystyle\int_0^x (\cos t + 1)e^{-5t^2}\,\mathrm{d}t \Longrightarrow F(0) = 0$
   $F'(x) = (\cos x + 1)e^{-5x^2} \Longrightarrow F'(0) = 2$
   Therefore: $F(x) = 2x + o(x)$ and the order of infinitesimal for $F(x)$ for $x \to 0$ is 1.
   Since the princial part of $F(x)$ as $x \to 0$ is $2x$, in a neighborhood of $x = 0$, $F(x)$ changes sign: it is positive for $x > 0$ and negative for $x < 0$.

   d) We study convergence of $\displaystyle\int_0^{+\infty} (\cos t + 1)e^{-5t^2}\,\mathrm{d}t$: apply Comparison Theorem and note that $0 \leq (\cos x + 1)e^{-5x^2} < \dfrac{1}{5x^2}$ for every x.
   The integral $\displaystyle\int_0^{+\infty} \frac{1}{5x^2}\,\mathrm{d}x$ is convergent, then $\displaystyle\int_0^{+\infty} (\cos t + 1)e^{-5t^2}\,\mathrm{d}t$ is convergent to some $l > 0$: hence $\displaystyle\lim_{x \to +\infty} F(x) = l \in \mathbb{R}$.

   e) Since $\displaystyle\int_0^{+\infty} (\cos t + 1)e^{-5t^2}\,\mathrm{d}t = l \in \mathbb{R}$ it holds $\displaystyle\lim_{x \to +\infty} F(x) = l \in \mathbb{R}$. Then the line $y = l$ is a right horizontal asymptote for $F(x)$.

9. Compute the order of infinitesimal and the principal part (w.r.t. the standard test function) for $x \to \pi$ of the function $\qquad F(x) = \displaystyle\int_\pi^x (e^{\sin t} - 1)\ dt.$

Prove that $F(x)$ has a relative maximum point in $x = \pi$.

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The Taylor expansion of order 2 of $F(x)$ centered in $x_0 = \pi$ is:

$$F(x) = F(\pi) + F'(\pi)(x-\pi) + F''(\pi)\frac{(x-\pi)^2}{2!} + o((x-\pi)^2)$$

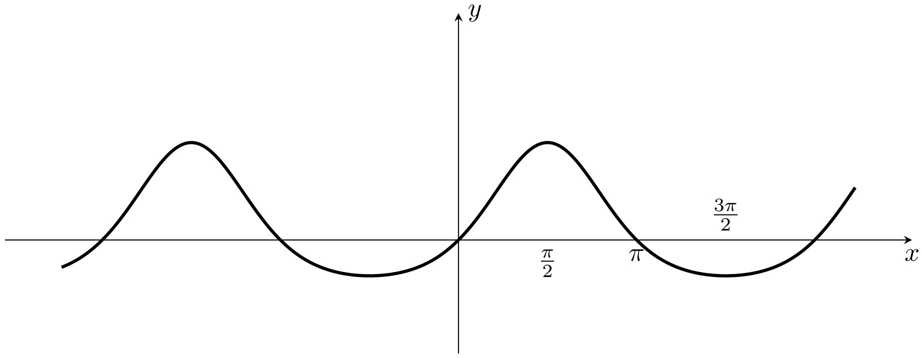
We have:

- $F(\pi) = 0$
- $F'(x) = (e^{\sin x} - 1) \;\Rightarrow\; F'(\pi) = 0$
- $F''(x) = \cos x e^{\sin x} \;\Rightarrow\; F''(\pi) = 1$

Thus

$$F(x) = \frac{(x-\pi)^2}{2} + o((x-\pi)^2),\ \text{for } x \to \pi$$

the principal part is $\dfrac{(x-\pi)^2}{2}$, the order of infinitesimal is 2.



- The integrand function is defined on $\mathbb{R}$ and it is integrable on $\mathbb{R}$, thus $DomF = \mathbb{R}$.
- $F(\pi) = 0$
- Since $F'(x) = f(x) = e^{\sin x} - 1 > 0 \Leftrightarrow \sin x > 0$, for $\dfrac{\pi}{2} < x < \pi$ it holds $f > 0$, hence F is increasing, whereas for $\pi < x < 3\dfrac{\pi}{2}$ we have $f < 0$, that is F is decreasing; then F haa a relative maximum in $x = \pi$.

10. Compute $\displaystyle\lim_{x\to 0} \frac{\int_0^x e^{-t^2}\, dt}{\sin x}$ (*Suggestion: use De l'Hopital Theorem*)

- -

The integral function $F(x) = \displaystyle\int_0^x e^{-t^2}\, dt$ is differentiable and $F(0) = 0$, then the linit is in the form $\dfrac{0}{0}$ and we can apply De l'Hospital Theorems:
Derive numerator and denominator:

$$\lim_{x\to 0} \frac{F'(x)}{\cos x} = \lim_{x\to 0} \frac{e^{-x^2}}{\cos x} = 1$$

Since such limit exists and is 1, also the initial limit is 1.

11. Compute the following improper integrals

(a) $\boxed{\displaystyle\int_1^{+\infty} \frac{x}{\sqrt{(x^2+5)^3}}\,\mathrm{d}x}$

The domain is $D = (-\infty, +\infty)$. Study the improper integral for $x \to +\infty$. Apply the definition:

$$\begin{aligned}
\int_1^{+\infty} \frac{x}{\sqrt{(x^2+5)^3}}\,\mathrm{d}x &= \lim_{t\to+\infty} \frac{1}{2}\int_1^t 2x(x^2+5)^{-3/2}\,\mathrm{d}x \\
&= \lim_{t\to+\infty} \frac{1}{2}\frac{(x^2+5)^{-\frac{1}{2}}}{-\frac{1}{2}}\Bigg|_1^t \\
&= \lim_{t\to+\infty}\left(-\frac{1}{\sqrt{(t^2+5)}} + \frac{1}{\sqrt{(1^2+5)}}\right) = \frac{1}{\sqrt{6}}
\end{aligned}$$

The improper integral is convergent.

(b) $\boxed{\displaystyle\int_2^{+\infty} \frac{4x-3}{2x^2-3x+1}\,\mathrm{d}x}$

The domain is $D = (-\infty, 1/2) \cup (1/2, 1) \cup (1, +\infty)$. Since $\dfrac{1}{2} \notin (2, +\infty)$ and $1 \notin (2, +\infty)$ we study the convergence only for $x \to +\infty$. By definition:

$$\begin{aligned}
\int_2^{+\infty} \frac{4x-3}{2x^2-3x+1}\,\mathrm{d}x &= \lim_{t\to+\infty}\int_2^t \frac{4x-3}{2x^2-3x+1}\,\mathrm{d}x \\
&= \lim_{t\to+\infty} \log(2x^2-3x+1)\Big|_2^t \\
&= \lim_{t\to+\infty} \log(2t^2-3t+1) - \log(3) = +\infty
\end{aligned}$$

The improper integral is divergent.

(c) $\boxed{\displaystyle\int_5^{+\infty} \left(\frac{1}{3x^2-4x} - \frac{5}{x\sqrt{x}}\right)\mathrm{d}x}$

The domain is $D = (0, 4/3) \cup (4/3, +\infty)$. Since $\dfrac{4}{3} \notin (5, +\infty)$ we study the convergence only for $x \to +\infty$.

$$\begin{aligned}
& \lim_{t\to+\infty}\int_5^t \left(\frac{1}{3x^2-4x} - \frac{5}{x\sqrt{x}}\right)\mathrm{d}x \\
=\ & \lim_{t\to+\infty}\int_5^t \left(\frac{3}{4(3x-4)} - \frac{1}{4x} - 5x^{-1/2-1}\right)\mathrm{d}x \\
=\ & \lim_{t\to+\infty}\left(\frac{1}{4}\log(3x-4) - \frac{1}{4}\log(x) + \frac{10}{\sqrt{x}}\right)_5^t \\
=\ & \lim_{t\to+\infty}\left(\frac{1}{4}\log\left(\frac{3x-4}{x}\right) + \frac{10}{\sqrt{x}}\right)_5^t \\
=\ & \lim_{t\to+\infty}\left(\frac{1}{4}\log\left(\frac{3t-4}{t}\right) + \frac{10}{\sqrt{t}}\right) - \left(\frac{1}{4}\log\frac{11}{5} + \frac{10}{\sqrt{5}}\right) \\
=\ & \left(\frac{1}{4}\log(3) - \frac{1}{4}\log\frac{11}{5} + \frac{10}{\sqrt{5}}\right)
\end{aligned}$$

The improper integral is convergent.

(d) $\boxed{\displaystyle\int_{1/2}^{+\infty} \frac{1}{\sqrt{2x}(2x+1)}\,\mathrm{d}x}$

The domain is $D = (0, +\infty)$. Since $0 \notin (\frac{1}{2}, +\infty)$ we study the convergence only at $+\infty$.

$$\begin{aligned}
\int_{1/2}^{+\infty} \frac{1}{\sqrt{2x}(2x+1)}\,\mathrm{d}x &= \lim_{t\to+\infty} \Big(\arctan(\sqrt{2x})\Big)_{1/2}^{t} \\
&= \lim_{t\to+\infty} \Big(\arctan(\sqrt{2t})\Big) - \Big(\arctan(1)\Big) \\
&= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}
\end{aligned}$$

The improper integral is convergent.

(e) $\boxed{\displaystyle\int_0^{+\infty} \left[x^3\left(8+x^4\right)^{-5/3} + 2xe^{-x}\right]\,\mathrm{d}x}$

The domain is $D = (-\infty, +\infty)$. We study the convergence at $+\infty$.

$$\begin{aligned}
\int_0^{+\infty} \left[x^3\left(8+x^4\right)^{-5/3} + 2xe^{-x}\right]\,\mathrm{d}x &= \lim_{t\to+\infty} \int_0^{t} \left[x^3\left(8+x^4\right)^{-5/3} + 2xe^{-x}\right]\,\mathrm{d}x \\
&= \lim_{t\to+\infty} \left[\frac{1}{4}\frac{\left(8+x^4\right)^{-5/3+1}}{-5/3+1} - 2(x+1)e^{-x}\right]_0^t \\
&= \lim_{t\to+\infty} \left[\frac{1}{4}\frac{\left(8+t^4\right)^{-2/3}}{-2/3} - 2(t+1)e^{-t} - \frac{1}{4}\frac{(8)^{-2/3}}{-2/3} + 2(+1)e^0\right] \\
&= \frac{3}{32} + 2 = \frac{67}{32}
\end{aligned}$$

The improper integral is convergent.

(f) $\boxed{\displaystyle\int_0^{+\infty} \frac{\arctan x}{1+x^2}\,\mathrm{d}x}$

The domain is $D = (-\infty, +\infty)$. We study the convergence at $+\infty$.

$$\begin{aligned}
\lim_{t\to+\infty} \int_0^t \frac{\arctan x}{1+x^2}\,\mathrm{d}x &= \lim_{t\to+\infty} \left.\frac{\arctan^2 x}{2}\right|_0^t \\
&= \lim_{t\to+\infty} \frac{\arctan^2 t}{2} = \frac{\pi^2}{8}
\end{aligned}$$

The improper integral is convergent.

12. Study the convergence of the following improper integrals

$a)\ \displaystyle\int_2^{+\infty} \frac{1}{\sqrt[3]{x^5+x-2}}\,\mathrm{d}x \qquad b)\ \int_1^{+\infty} \sqrt{\frac{x^2+x+2}{x+1}}\,\mathrm{d}x \qquad c)\ \int_3^{+\infty} \frac{1}{\sqrt{|1-x^2|}}\,\mathrm{d}x$

$d)\ \displaystyle\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,\mathrm{d}x \qquad e)\ \int_1^{+\infty} \frac{1}{x^2+\sqrt[3]{x^4+1}}\,\mathrm{d}x \qquad f)\ \int_0^{+\infty} \frac{x}{(x+1)^3}\,dx$

Recall that:

$$\boxed{\int_a^b \frac{1}{(x-a)^\alpha}\,dx = \begin{cases} \text{converges} & \text{if } \alpha < 1 \\ \text{diverges} & \text{if } \alpha \geq 1 \end{cases}}$$

$$\boxed{\int_c^{+\infty} \frac{1}{x^\alpha}\,dx \qquad \begin{cases} \text{converges} & \text{if } \alpha > 1 \\ \text{diverges} & \text{if } \alpha \leq 1 \end{cases}}$$

(a) $\boxed{\displaystyle\int_2^{+\infty} \frac{1}{\sqrt[3]{x^5+x-2}}\,\mathrm{d}x}$

The domain is $D=(-\infty,1)\cup(1,+\infty)$. We study the convergence at $+\infty$. Study the asymptotic behavior of the integrand function at $+\infty$

$$\frac{1}{\sqrt[3]{x^5+x-2}} \quad\sim\quad \frac{1}{x^{5/3}}, \quad \text{for } x\to+\infty$$

Since $\displaystyle\int_2^{+\infty} \frac{1}{x^{5/3}}\,\mathrm{d}x$ converges, by asymptotic comparison also the initial integral converges.

(b) $\boxed{\displaystyle\int_1^{+\infty} \sqrt{\frac{x^2+x+2}{x+1}}\,\mathrm{d}x}$

The domain is $D=(-1,+\infty)$. We study convergence at $+\infty$. Study the asymptotic behavior of the integrand function at $+\infty$

$$\sqrt{\frac{x^2+x+2}{x+1}} \quad\sim\quad \frac{1}{x^{-1/2}}, \quad \text{per } x\to+\infty$$

Since $\displaystyle\int_2^{+\infty} \frac{1}{x^{-1/2}}\,\mathrm{d}x$ diverges, by asymptotic comparison also the initial integral diverges.

(c) $\boxed{\displaystyle\int_3^{+\infty} \frac{1}{\sqrt{|1-x^2|}}\,\mathrm{d}x}$

The domain is $D=(-\infty,-1)\cup(-1,1)\cup(1,+\infty)$. Since in $[3,+\infty)$ the integrand function is continuous, we study convergence at $+\infty$.
Study the asymptotic behavior of the integrand function at $+\infty$:

$$\frac{1}{\sqrt{|1-x^2|}} \quad\sim\quad \frac{1}{x}, \quad \text{for } x\to+\infty$$

Since $\displaystyle\int_3^{+\infty} \frac{1}{x}\,\mathrm{d}x$ diverges, by asymptotic comparison also the initial integral diverges.

(d) $\boxed{\displaystyle\int_4^{+\infty} \frac{1}{\sqrt{x}(\sqrt{x}-1)}\,\mathrm{d}x}$

The domain is $D=(0,1)\cup(1,+\infty)$. Since in $[4,+\infty)$ the integrand function is continuous, we study convergence at $+\infty$.
Study the asymptotic behavior of the integrand function at $+\infty$:

$$\frac{1}{\sqrt{x}(\sqrt{x}-1)} \quad\sim\quad \frac{1}{x}, \quad \text{per } x\to+\infty$$

Since $\displaystyle\int_4^{+\infty} \frac{1}{x}\,\mathrm{d}x$ diverges, by asymptotic comparison also the initial integral diverges.

(e) $\boxed{\displaystyle\int_1^{+\infty} \frac{1}{x^2+\sqrt[3]{x^4+1}}\,\mathrm{d}x}$

The domain is $D=(-\infty,+\infty)$. We study convergence at $+\infty$. Study the asymptotic behavior of the integrand function at $+\infty$:

$$\frac{1}{x^2+\sqrt[3]{x^4+1}} \quad\sim\quad \frac{1}{x^2}, \quad \text{per } x\to+\infty$$

Since $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,\mathrm{d}x$ converges, by asymptotic comparison also the initial integral converges.

(f) $\boxed{\displaystyle\int_0^{+\infty} \frac{x}{(x+1)^3}\,\mathrm{d}x}$

The domain is $D = (-\infty, -1) \cup (-1, +\infty)$. Since in $[0, +\infty)$ the integrand function is continuous, we study convergence at $+\infty$.
Study the asymptotic behavior of the integrand function at $+\infty$:

$$\frac{x}{(x+1)^3} \sim \frac{1}{x^2}, \quad \text{per } x \to +\infty$$

Since $\displaystyle\int_2^{+\infty} \frac{1}{x^2}\, \mathrm{d}x$ is convergent at $+\infty$, by asymptotic comparison also the initial integral converges.

13. Study the absolute convergence of the improper integral $\displaystyle\int_2^{+\infty} \frac{\sin x + \cos x}{x^2 - x - 1}\, \mathrm{d}x$.

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The integrand function $f(x) = \dfrac{\sin x + \cos x}{x^2 - x - 1}$ is continuous on $[2, +\infty)$, thus we study its behavior only at $x \to +\infty$; apply comparison criteria to study absolute convergence:

$$|f(x)| = \left|\frac{\sin x + \cos x}{x^2 - x - 1}\right| \leq \left|\frac{2}{x^2 - x - 1}\right| = \frac{2}{x^2 - x - 1}, \quad \text{se } x \in [2, +\infty)$$

From $\dfrac{2}{x^2 - x - 1} \sim \dfrac{2}{x^2}$ and $\displaystyle\int_2^{+\infty} \frac{1}{x^2}\, \mathrm{d}x$ converges, it holds that $\displaystyle\int_2^{+\infty} \frac{2}{x^2 - x - 1}\, \mathrm{d}x$ converges; thus the given integral is absolutely convergent, and thus convergent.

Exercises from previous exams

1. (31 January 2018 - II - A)
   (a) *State the Integral Average Theorem.*

   See textbook.

   (b) *Given $a > 0$, find $M \in \mathbb{R}$ such that* $\displaystyle\int_0^a e^{-x^2}\, dx \leq M$.

   By the integral average Theorem, let i be the lower bound and S the upper bound for the function e^{-x^2} in the interval $[0, a]$, it holds:

   $$i \leq \frac{1}{a}\int_0^a e^{-x^2}\, dx \leq S$$

   Therefore $ai \leq \displaystyle\int_0^a e^{-x^2}\, dx \leq aS$. Since $S = 1$, we have $\displaystyle\int_0^a e^{-x^2}\, dx \leq a$. Thus, it is sufficient to choose $M \geq a$ in order to have $\displaystyle\int_0^a e^{-x^2}\, dx \leq M$.

   (c) *Show that the integral function*

   $$F(x) = \int_0^x e^{-t^2}\, dt$$

   has horizontal asymptotes.

   We have to prove that $\lim\limits_{x\to+\infty} F(x)$ and $\lim\limits_{x\to-\infty} F(x)$ exist and they are finite. Since $F(x)$ is odd, it suffices to prove (for example), that $\lim\limits_{x\to+\infty} F(x) = \lambda \in \mathbb{R}$, and therefore also the other limit exists and is finite (it equals $-\lambda$).
   By definition, $\lim\limits_{x\to+\infty} F(x) = \displaystyle\int_0^{+\infty} e^{-t^2}\, dt$; the improper integral $\displaystyle\int_0^{+\infty} e^{-t^2}\, dt$ converges: indeed we can write $\displaystyle\int_0^{+\infty} e^{-t^2}\, dt = \int_0^1 e^{-t^2}\, dt + \int_1^{+\infty} e^{-t^2}\, dt$ and the second integral converges by comparison criterium (we have $e^{-t^2} \leq \dfrac{1}{t^2}$ and $\displaystyle\int_1^{+\infty} \frac{1}{t^2}\, dt$ converges).

2. (4 July 2017)
   (a) *State the Integral Average Theorem for continuous functions.*

   See textbook.

(b) *Consider the function*

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ x & \text{if } x \geq 0. \end{cases}$$

Compute the integral average for the function in the intervals $[-1, 1]$ *and* $\left[-1, \frac{1}{2}\right]$, *and say if the integral average is assumed by the function in such intervals.*

The integral average μ_1 on the interval $[-1, 1]$ is given by the value of the following definite integral:

$$\mu_1 = \frac{\int_{-1}^{1} f(x)\,dx}{1-(-1)} = \frac{\int_{-1}^{0} 1\,dx + \int_0^1 x\,dx}{2} = \frac{3}{4}$$

The function takes value $\frac{3}{4}$ at $x = \frac{3}{4}$; hence the value of the integral average in the interval $[-1, 1]$ is in $f([-1, 1])$.

The integral average μ_2 on the interval $\left[-1, \frac{1}{2}\right]$ is given by the value of the following definite integral:

$$\mu_2 = \frac{\int_{-1}^{1/2} f(x)\,dx}{\frac{1}{2}-(-1)} = \frac{\int_{-1}^{0} 1\,dx + \int_0^{1/2} x\,dx}{\frac{3}{2}} = \frac{3}{4}$$

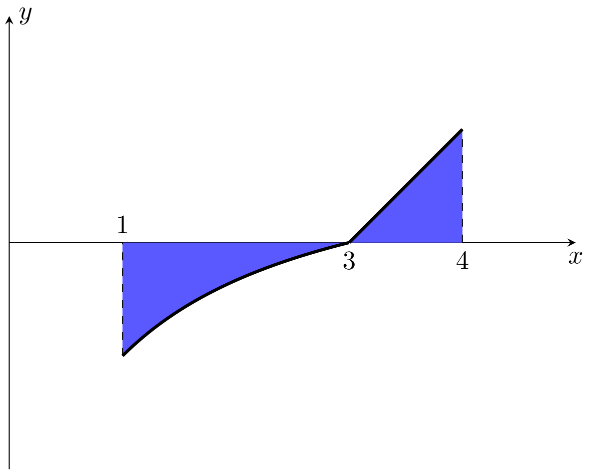
The function takes value $\frac{3}{4}$ at the point $x = \frac{3}{4}$, but $x = \frac{3}{4}$ does not belong to the interval $\left[-1, \frac{1}{2}\right]$; hence the value of the integral average in the interval $\left[-1, \frac{1}{2}\right]$ is not assumed by the function in such interval.

3. (21 September 2016)

Let

$$h(x) = \begin{cases} \dfrac{x-3}{x+1} & \text{if } 1 \leq x \leq 3 \\ \\ x-3 & \text{if } \ 3 < x \leq 4. \end{cases}$$

(a) *Compute the area between the* x*-axis, the lines* $x = 1$, $x = 4$ *and the graph of* $h(x)$*.*



Let $h_1(x) = \dfrac{x-3}{x+1}$, $\quad x \in [1, 3]$ and $h_2(x) = x - 3$, $\quad x \in [3, 4]$.

If $x \in [1, 3]$ we have $h_1(x) \leq 0$, while if $x \in [3, 4]$ we have $h_2(x) \geq 0$, then the area A of the region between the x-axis and the lines $x = 1$, $x = 4$ and the graph of $h(x)$, is the following:

$$A = \int_1^3 (-h_1(x))\,\mathrm{d}x + \int_3^4 h_2(x)\,\mathrm{d}x = \int_1^3 \frac{3-x}{x+1}\,\mathrm{d}x + \int_3^4 (x-3)\,\mathrm{d}x = \int_1^3 \left(-1 + \frac{4}{x+1}\right)\mathrm{d}x + \int_3^4 (x-3)\,\mathrm{d}x =$$

$$= [-x + 4\ln|x+1|]_1^3 + \left[\frac{x^2}{2} - 3x\right]_3^4 = (-3 + 4\ln 4 + 1 - 4\ln 2) + \left(8 - 12 - \frac{9}{2} + 9\right) = \ln 16 - \frac{3}{2}$$

(b) *State the Fundamental Theorem of the Integral Calculus.* See textbook.

(c) *Verify if the previous function (a) satisfies the hypothesis of the Fundamental Theorem of the Integral Calculus on the interval* $[1,4]$.

The function $h(x)$ is continuous on $[1,4]$, since

- $h_1(x)$ is continuous in $[1,3]$

- $h_2(x)$ is alsways continuous, in particular in $[3,4]$

- in $x=3$ it holds $\lim\limits_{x\to 3^-} h_1(x) = h(3) = 0 = \lim\limits_{x\to 3^+} h_2(x)$.

Hence $h(x)$ satisfies the hypothesis of the Fundamental Theorem of integral calculus in the interval $[1,4]$.

4. (10 February 2016 - I)

(a) *Given the continuous function* $f : [1,+\infty) \to \mathbb{R}$. *Write the definition of convergence and the absolute convergence for the improper integral* $\int_1^{+\infty} f(x)\,dx$.

See textbook.

(b) *Study the behavior of the following improper integral discussing each step*

$$\int_1^{+\infty} \frac{\sin x}{x^2}\,dx.$$

The function $\dfrac{\sin x}{x^2}$ is continuous and thus locally integrable in $[1,+\infty)$.

Study absolute convergence: since $0 \le \left|\dfrac{\sin x}{x^2}\right| \le \dfrac{1}{x^2}$ and $\displaystyle\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, by Comparison theorem the given integral is absolutely convergent, and therefore $\displaystyle\int_1^{+\infty} \frac{\sin x}{x^2}\,dx$ converges.

5. (17 June 2015 - II)

(a) *State the Fundamental Theorem of the Integral Calculus.*

See textbook.

(b) *Show that* f *is a continuous function on* $[a,b]$ *and* F *is a primitive, the following formula holds*

$$\int_a^b f(x)dx = F(b) - F(a).$$

See textbook.

6. (13 February 2015 - I)
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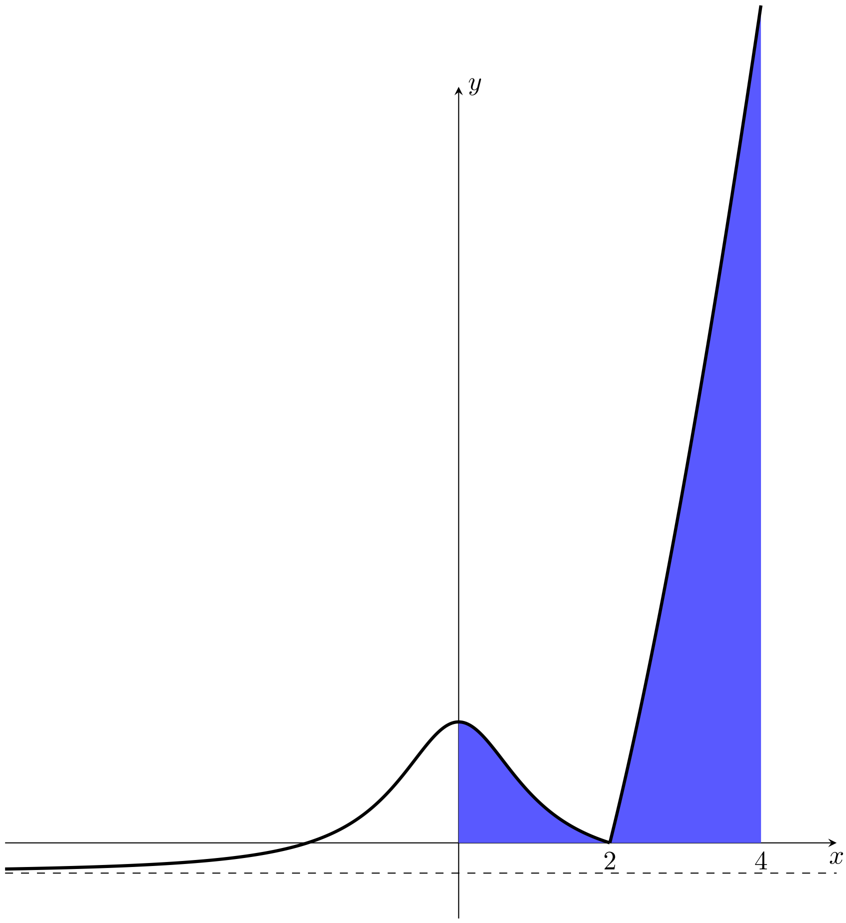
$$g(x) = \begin{cases} \dfrac{2}{1+x^2} - \dfrac{2}{5} & \text{if } x \leq 2 \\ 4x \log \dfrac{x}{2} & \text{if } x > 2\,. \end{cases}$$

(a) *Compute* $\int_0^4 g(x)\mathrm{d}x$.



Let$g_1(x) = \dfrac{2}{1+x^2} - \dfrac{2}{5}$, $\quad x \in [0,2]$ and $g_2(x) = 4x \log \dfrac{x}{2}$, $\quad x \in (2,4]$.
The function $g_1(x)$ is always continuous, in particular in $[0,2]$, hence it is integrable in $[0,2]$; $g_2(x)$ is continuous if $x > 0$, in particular in $[2,4]$, hence it is integrable in $[2,4]$. Thus:

$$\int_0^4 g(x)\ \mathrm{d}x \ = \int_0^2 \left(\frac{2}{1+x^2} - \frac{2}{5}\right)\ \mathrm{d}x \ + \int_2^4 4x \log \frac{x}{2}\ \mathrm{d}x\ .$$

Compute by parts the indefinite integral

$$\int x \ln \frac{x}{2}\ \mathrm{d}x \ = \frac{x^2}{2} \ln \frac{x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x}\ \mathrm{d}x \ = \frac{x^2}{2} \ln \frac{x}{2} - \frac{x^2}{4} + c.$$

Therefore

$$\int_0^4 g(x)\ \mathrm{d}x \ = \left[2 \arctan x - \frac{2}{5}x\right]_0^2 + \left[2x^2 \ln \frac{x}{2} - x^2\right]_2^4 = 2 \arctan 2 + 32 \ln 2 - \frac{64}{5}.$$

(b) *Say if there exists* $c \in [0,4]$ *such that* $g(c)$ *equals the integral average of* g *on* $[0,4]$*. Giustify the answer.*
The function $g(x)$ is continuous in $[0,4]$, since:
- $g_1(x)$ is always continuous, in particular in $[0,2]$
- $g_2(x)$ is continuous if $x > 0$, in particular in $[2,4]$
- in $x = 2$ we have $\lim\limits_{x \to 2^-} g_1(x) = g(2) = 0 = \lim\limits_{x \to 2^+} g_2(x)$.

Hence $g(x)$ satisfies the hypothesis of the Integral average Theorem in $[0,4]$, thus there exists $c \in [0,4]$ such that $g(c)$ equals the integral average of g on $[0,4]$.

7. (13 February 2015 - II)
Let $f : [0, +\infty) \to \mathbb{R}$ be a continuous function.

   (a) *Write the definition of convergence of the improper integral*

$$\int_0^{+\infty} f(x)\, dx.$$

   See textbook.

   (b) *Prove that if $f(x) \geq 0$ on $[0, +\infty)$, the integral $\int_0^{+\infty} f(x)\, dx$ cannot be indeterminate.*
   See textbook.

   (c) *Study the behavior of the improper integral*

$$\int_0^{+\infty} \frac{x^2 + \sin x}{x^2 + 1}\, dx.$$

   The function $\dfrac{x^2 + \sin x}{x^2 + 1}$ is continuous and thus locally integrable in $[0, +\infty)$. Moreover $\dfrac{x^2 + \sin x}{x^2 + 1} \sim 1$ for $x \to +\infty$, then the given integral diverges at $+\infty$ by Asymptotic Comparison theorem.

QUESTIONS - THEORY

1. Let $f : \mathbb{R} \to \mathbb{R}$ even and locally integrable and let $F(x) = \displaystyle\int_0^x f(t)dt$. Can we determine whether F is even or odd?

2. Let $f : \mathbb{R} \to \mathbb{R}$ locally integrable such that for $|x| > 10$ then $f(x) = 0$. The function $F(x) = \displaystyle\int_{-3}^x f(t)dt$ is bounded or not?

3. Let $f : \mathbb{R} \to \mathbb{R}$ locally integrable such that $f(x) \geq 1$ for every $x \in \mathbb{R}$, and let $F(x) = \displaystyle\int_0^x f(t)dt$. Verify that $\lim_{x\to+\infty} F(x) = +\infty$ and that $\lim_{x\to-\infty} F(x) = -\infty$. Deduce that $\int_0^{+\infty} f(t)dt$ is divergent.

4. Let $f : \mathbb{R} \to \mathbb{R}$ locally integrable such that $f(x) \sim x^5$ as $x \to +\infty$. Show that $\int_1^{+\infty} f(x)e^{-x}dx$ is convergent.

5. Let $f : [1, +\infty) \to \mathbb{R}$ be defined as $f(x) = \dfrac{1}{n}$ if $x \in [n, n+1), n \in \mathbb{N} \setminus \{0\}$. Say if $\int_1^{+\infty} f(t)dt$ is convergent.

6. Let $f : [1, +\infty) \to \mathbb{R}$ be defined as $f(x) = \dfrac{1}{(n+1)^2}$ if $x \in [n, n+1), n \in \mathbb{N} \setminus \{0\}$. Say if $\int_0^{+\infty} f(t)dt$ is convergent or not.

